Nonreciprocity as a generic route to traveling states

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We examine a non-reciprocally coupled dynamical model of a mixture of two diffusing species. We demonstrate that nonreciprocity, which is encoded in the model via antagonistic cross diffusivities, provides a generic mechanism for the emergence of traveling patterns in purely diffusive systems with conservative dynamics. In the absence of non-reciprocity, the binary fluid mixture undergoes a phase transition from a homogeneous mixed state to a demixed state with spatially separated regions rich in one of the two components. Above a critical value of the parameter tuning non-reciprocity, the static demixed pattern acquires a finite velocity, resulting in a state that breaks both spatial and time translational symmetry, as well as the reflection parity of the static pattern. We elucidate the generic nature of the transition to traveling patterns using a minimal model that can be studied analytically. Our work has direct relevance to nonequilibrium assembly in mixtures of chemically interacting colloids that are known to exhibit non-reciprocal effective interactions [1, 2], as well as to mixtures of active and passive agents where traveling states of the type predicted here have been observed in simulations [3]. It also provides insight on transitions to traveling and oscillatory states seen in a broad range of nonreciprocal systems with non-conservative dynamics, from reaction-diffusion and prey-predators models to multispecies mixtures of microorganisms with antagonistic interactions.

Traveling patterns occur ubiquitously in nature. Examples range from oscillating chemical reactions [4–6], waves of metabolic synchronization in yeast [7], to the spatial spread of epidemics [8–11]. Most mathematical models that capture such spatio-temporal dynamics, including reaction-diffusion equations [4, 12–15], excitable systems [16, 17], collections of coupled oscillators [18, 19], and prey-predator equations [20–22] are unified by the fact that the dynamical variables are non-conserved fields [23]. In this case the coupling to birth-death or to other reaction processes provides a promoter-inhibitor mechanism that sets up oscillatory states. In this paper we demonstrate that traveling patterns can arise in multi-component systems described by purely diffusive conserved fields from non-reciprocal interactions between species. The appearance of traveling or sustained oscillatory states in a purely diffusive system with no apparent external forcing is unexpected and defies intuition. Our work suggests that non-reciprocity provides a generic mechanism for the establishment of traveling states in the dynamics of conserved scalar fields.

The third law of Newtonian mechanics establishes that interactions are reciprocal: for every action there is an equal and opposite reaction. While of course this remains true at the microscopic level, non-reciprocal effective interactions can occur ubiquitously on mesoscopic scales when interactions are mediated by a nonequilibrium environment [24–26]. A striking physical example is realized in diffusiophoretic colloidal mixtures [1, 2, 27]. Non-reciprocal interactions are also the norm in the living world. Examples are promoter-inhibitor interactions among different cell types [28] and the antagonistic interactions among species in bacterial suspensions [29–32]. Social forces that control the behavior of human crowds [33–35] and collective animal behavior [36, 37] are other important examples as well.

To highlight the role of non-reciprocal couplings in driving time-dependent phases, we examine a minimal model of the dynamics of two interdiffusing species, each described by a scalar field $\phi_\mu$, for $\mu = A, B$. The evolution of each concentration field is governed by a $\phi^4$ field theory that allows for a spinodal instability according to Model B dynamics [38]. When decoupled, each phase field can undergo a Hopf bifurcation describing the transition from a homogeneous state to a phase-separated state composed of dilute and dense phases. The two fields are coupled via cross-diffusion terms with diffusivities $\kappa_{\mu\nu}$. When these couplings are reciprocal, the interaction between the two fields leads to a transition between a mixed state where both fields are homogeneous to a demixed state with distinct regions of high $A$ and low $B$. Non-reciprocity is introduced by allowing the two cross-diffusivity to have opposite signs and is quantified by $\delta = (\kappa_{BA} - \kappa_{AB})/2$. Non-reciprocal cross-diffusivities drive a second transition through a drift bifurcation to a time-dependent state that breaks parity, where the domains of the demixed regions travel at a constant drift velocity. This transition is closely related to ones previously reported in specific models of prey-predator and reaction-diffusion dynamics [11–14, 20, 21, 39], but occurs here from the coupling of two conserved fields. We demonstrate that the transition to traveling states is a parity and time-reversal (PT) symmetry breaking bifurcation that arises generically from non-reciprocal couplings. The phase diagram obtained from a numerical solution of a one-dimensional realization of this minimal model in the simplest case where only field A is supercritical, while B is subcritical, i.e., the ground state value of field B is simply $\phi_B^0 = 0$, is shown in Fig. 1a. Tuning the
control parameter that drives phase separation of species
A ($\chi_A$) and the measure of non-reciprocity $\delta$, we observe
three distinct states: a mixed state where both fields are
homogeneous, a static demixed state that breaks trans-
lational symmetry with out-of-phase spatial modulations
of the two fields, and a time-dependent state that addi-
tionally breaks reflection and time-translational sym-
metry, where the spatial modulation of the demixed state
travels at constant velocity. The solid lines are obtained
from a one-mode approximation to the continuum model
that can be solved analytically and provides an excellent
fit to the numerics. Within this one-mode approxima-
tion, the transition from the stationary to the traveling
state can be understood as an instability of the relative
phase of the first Fourier harmonic of the fields. The
instability arises because non-reciprocity allows pertur-
ations in the two fields to travel in the same direction,
promoting a “run-and-catch” scenario that stabilizes the
traveling pattern. While the spatial pattern in the static
demixed phase is even in the relative displacement of
the two phase fields, non-reciprocity breaks this reflec-
tion symmetry in the traveling state, mediating a PT-
symmetry-breaking transition. Note that the transition
to a PT-broken phase occurs at finite value of $\delta$, hence
requires sufficiently strong non-reciprocity. Finally, the
phase boundary separating the static and traveling pat-
terns in Fig. 1a corresponds to a so-called “exceptional
point” where the eigenmodes of the matrix controlling
the dynamical stability of the system coalesce [40–42].

A microscopic model that displays the phenomenology
captured by Fig. 1a is a mixture of active and passive
Brownian particles, where the active component exhibits
motility-induced phase separation and fluctuations in the
density of passive particles can enhance fluctuations in
the density of the active fraction via an effective nega-
tive cross diffusivity [43, 44]. The connection between
the active-passive mixture and the dynamics embodied
by our model is unfolded in the SI. Another realization
of this macrodynamics is a binary suspension of colloidal
particles where species A attracts species B, but species B
repels species A. Such competing interactions have been
studied in simple models [45, 46] and can be realized
in mixtures of self-catalytic active colloids, where the lo-
cal chemistry mediates non-reciprocal interactions among
the two species, as demonstrated for instance in [1, 2] via
numerical simulations.

I. CONTINUUM MODEL

We consider a binary mixture described by two con-
served phase fields $\phi_A$ and $\phi_B$ with Cahn-Hillard dynam-
isics [47–49] augmented by cross-diffusion,

$$\partial_t \phi_\mu = \nabla \cdot \left[ (\chi_\mu + \phi_\mu^2 - \gamma_\mu \nabla^2) \nabla \phi_\mu + \kappa_{\mu\nu} \nabla \phi_\nu \right] ,$$

where $\mu, \nu = A, B$ and no summation is intended. In
the absence of cross-diffusive couplings ($\kappa_{\mu\nu} = 0$), the
fields are decoupled, with ground states $\phi_\mu^0 = 0$ for $\chi_\mu >$
0, describing homogeneous states, and $\phi_\mu^0 = \pm \sqrt{-3\chi_\mu}$
when $\chi_\mu < 0$, corresponding to phase separated states.

The cross-diffusivities control interspecies interaction,
allowing phase gradient of one species to drive currents of
the other species. Equal cross-diffusivities, $\kappa_{AB} = $
$\kappa_{BA} = \kappa$, yield an effective repulsion between the two
fields. When sufficiently strong to overcome the entropy of
mixing, such a repulsion results in the formation of spatial
domains of high/low $\phi_A/\phi_B$, i.e., a demixed state.
Here, in contrast, we introduce non-reciprocity by allow-
ing these two quantities to have opposite signs [50], as can
for instance be achieved in mixtures of active and passive
Brownian particles (see SI section VI.A) or in mixtures of
colloids with competing repulsive and attractive in-
teractions (see SI section VI.B). We tune the degree of
non-reciprocity $\delta > 0$ by letting

$$\kappa_{AB} = \kappa - \delta ,$$
$$\kappa_{BA} = \kappa + \delta .$$

As shown below, this non-reciprocity breaks PT symme-
try and gives rise to spatio-temporal patterns of $\phi_A$ and
$\phi_B$ that break both spatial and temporal translational
symmetry.

We have studied numerically (1) in a one-dimensional
box of length $L = 2\pi$, for the case where $\chi_A < 0$ and
$\chi_B > 0$. The results are easily generalized to the case
where both components are supercritical ($\chi_A < 0$ and
$\chi_B < 0$) and to higher dimensions (see SI), but remain
qualitatively unchanged. We have integrated (1) with
a fourth-order central difference on a uniform grid with
spacing $h = 2\pi/64$. To march in time, we use a second-
order, 128-stage Runge-Kutta-Chebyshev scheme with a
time step $\Delta t = 0.1$ [51, 52]. All simulations start from
nearly uniform phase fields, where weak random fluctua-
tions are added on top of the initial compositions $\phi_\mu^0 = 0$
and $\phi_\nu^0 = 0$. We fix the values of the parameters as:
$\gamma_A = 0.04$, $\gamma_B = 0$, $\chi_B = 0.005$, $\kappa = 0.005$, and study
how the system dynamics changes with $\chi_A$ and $\delta$.

We find three distinct states by varying $\chi_A$ and $\delta$, as
summarized in Fig. 1a. When the cross-diffusivities are
reciprocal ($\delta = 0$), by increasing $|\chi_A|$ the system un-
dergoes a Hopf bifurcation from a homogeneous state
(gray circles) to a demixed state (cyan rectangles) where
the two fields are spatially modulated with alternating
regions of high $\phi_A$/low $\phi_B$ (Figs. 1b,c). This state is
stabilized by the cubic term in (1), as in conventional
Cahn-Hillard models. Above a critical value of $\delta$, the
demixed state undergoes a second bifurcation to a state
where the domains of high $\phi_A$/low $\phi_B$ travel at a con-
stant speed (pink triangles in Fig. 1a and Figs. 1b,d). The velocity of the traveling pattern provides an order
parameter for this transition, and the direction of motion is
picked spontaneously. The opposite signs of the cross-
diffusivities provide effective antagonistic repulsive and
attractive interactions between the two fields. The drift
bifurcation is triggered by the nucleation of a phase shift
in the spatial modulation of the two fields that allows
species A to outrun B, while B tries to catch up with A.
of non-reciprocity and of the “run-an-catch” mechanism that controls the dynamics in 1D. Both traveling and oscillating states appear to be stable and coexist at high $\delta$, with the state selection being controlled by initial conditions. This suggests that it would be interesting to go beyond the deterministic model considered here to examine the role of noise. A full study of 2D systems will be reported elsewhere.

II. ONE-MODE APPROXIMATION

To uncover the physics behind the PT-breaking bifurcation, we expand the fields $\phi_\mu$ in a Fourier series as $\phi_\mu(x, t) = \sum_{j=-\infty}^{\infty} \hat{\phi}_\mu^j(t)e^{iq_jx}$, where $\hat{\phi}_\mu^j = (2\pi)^{-1} \int_0^{2\pi} dx \phi_\mu e^{-iq_jx}$ is the amplitude of mode $j$. Substituting this in (1), and apply the Galerkin method [53], one obtains a set of coupled ODEs for the Fourier amplitudes. For the one dimensional model described above, we have verified numerically that only the first Fourier mode $q_1 = 1$ is activated. We can then replace the original PDEs with a single-mode approximation, given by

\[
\frac{d\hat{\phi}_A^j}{dt} = -\left(\alpha_A + |\hat{\phi}_A^j|^2\right)\hat{\phi}_A^j - (\kappa - \delta)\hat{\phi}_B^j, \quad (3a)
\]

\[
\frac{d\hat{\phi}_B^j}{dt} = -\alpha_B\hat{\phi}_B^j - (\kappa + \delta)\hat{\phi}_A^j, \quad (3b)
\]

where $\alpha_A = \chi_A + \gamma_A + (\phi_0^A)^2$ can be negative and $\alpha_B = \chi_B > 0$. When $\chi_B > 0$ the cubic term in the dynamics of $\phi_B$ simply provides a higher order damping and can be neglected. Writing the complex amplitudes in terms of amplitudes and phases as $\hat{\phi}_\mu^j = \rho_\mu e^{i\theta_\mu}$, (3) can be written as

\[
\dot{\rho}_A = - (\alpha_A + \rho_0^2)\rho_A - (\kappa - \delta)\rho_B \cos \theta, \quad (4a)
\]

\[
\dot{\rho}_B = - \alpha_B \rho_B - (\kappa + \delta)\rho_A \cos \theta, \quad (4b)
\]

\[
\dot{\theta} = [(\kappa - \delta)\rho_B/\rho_A + (\kappa + \delta)\rho_A/\rho_B] \sin \theta, \quad (4c)
\]

\[
\dot{\Phi} = [(\kappa - \delta)\rho_B/\rho_A - (\kappa + \delta)\rho_A/\rho_B] \sin \theta \quad (4d)
\]

where $\theta \equiv \theta_A - \theta_B$ and $\Phi \equiv \theta_A + \theta_B$ are the difference and sum of the two phases. Note that the sum phase $\Phi$ is slaved to the other quantities. A broken PT pattern traveling at constant velocity corresponds to $\dot{\rho}_A = \dot{\rho}_B = 0$ and $\Phi$ = constant, which requires $\sin \theta \neq 0$ and $(\kappa - \delta)\rho_B/\rho_A + (\kappa + \delta)\rho_A/\rho_B = 0$, or equivalently $\rho_A \kappa_A \rho_B = - \rho_B \kappa_B \rho_A$, hence the two cross-diffusivities must have opposite signs. As we will see below, this is a necessary, but not sufficient condition for the existence of the traveling state. Next, we examine the fixed points of (4) and their stability.

a. Fixed points. There are three fixed points: a trivial fixed point ($F_H$) with $\rho_A = \rho_B = 0$ (and $\Phi$ are undetermined), corresponding to a homogeneous mixed state, and two non-trivial fixed points, corresponding to static ($F_S$) and traveling ($F_T$) demixed states. The state

![Fig. 1](image_url)
describes out-of-phase spatial variations of the two phases, with \( \theta^* = \pi \) and

\[
\rho_A^\pm = \left( \frac{k^2 - \delta^2 - \alpha_A \alpha_B}{\alpha_A \alpha_B} \right)^{1/2},
\]

\[
\rho_B^\pm = (\kappa + \delta) \rho_A^\pm / \alpha_B,
\]

while \( \Phi \) remains undetermined. This solution of course only exists provided \( \alpha_A \alpha_B < k^2 - \delta^2 \). Since \( \alpha_B > 0 \), the onset of the static demixed state requires \( \alpha_A < 0 \) to drive the growth of \( \rho_A \), which is then saturated by the cubic damping in (4a). Interspecies interactions modulate the pattern, resulting in out-of-phase spatial variations of \( \phi_A \) and \( \phi_B \), while \( \theta_A = \theta_B \) remains zero, i.e., the modulation is static. Note that in this state the two fields, although out of phase, have the same parity, either both even or both odd functions of \( x \).

The \( F_T \) state is a spatial modulation traveling at constant speed

\[
v = \Phi^t = \pm \sqrt{\delta^2 - \delta_c^2} \sim (\delta - \delta_c)^{1/2},
\]

with \( \delta_c = \sqrt{k^2 + \alpha_B^2} \) the critical value of nonreciprocity required for the establishment of the traveling pattern, and

\[
\rho_A^t = (-\alpha_A - \alpha_B)^{1/2},
\]

\[
\rho_B^t = \sqrt{\delta + \kappa} / (\delta - \kappa) \rho_A^t,
\]

\[
\theta^t = \arccos \left( -\sqrt{\frac{\alpha_B^2}{\delta^2 - \kappa^2}} \right).
\]

As we will see below, the speed \( v \) provide the order parameter for the transition form the static to the traveling state. This latter of course only exists when \( \kappa - \delta < 0 \), or more specifically it requires both \( -\alpha_A > \alpha_B \) and \( \delta^2 \geq \kappa^2 + \alpha_B^2 \), i.e., strong enough non-reciprocity. It arises because a solution with \( \sin \theta \neq 0 \) allows each field to travel at a finite velocity \( v_\mu = \theta_\mu \). The direction of each \( v_\mu \) is set by fluctuations or initial conditions. As shown in Fig. 2, the velocity of the traveling modulation and the spatial profiles of the two fields obtained from the one-mode approximation provide an excellent fit to those extracted from numerical solution of the equations. As discussed below, the traveling pattern breaks the reflection symmetry (parity) of the static one, as well as time translational invariance.

b. Linear stability analysis. A linear stability analysis of the fixed points yields the boundaries between the various phases shown in Fig. 1a and provides a clear understanding of the mechanism of the drift instability. Linearizing (3) about the homogeneous state reveals that in this phase the dynamics of fluctuations is controlled by two eigenvalues given by

\[
\lambda_{\pm} = -\frac{1}{2} (\alpha_A + \alpha_B) \pm \frac{1}{2} \sqrt{(\alpha_A - \alpha_B)^2 + 4(\kappa^2 - \delta^2)}. \tag{8}
\]

If \( \delta^2 < \kappa^2 + (\alpha_A - \alpha_B)^2 / 4 \), the eigenvalues are real. The largest eigenvalue \( \lambda_+ \) becomes positive, signaling an instability, when \( \delta^2 = \kappa^2 - \alpha_A \alpha_B \). This diffusive instability is displayed as a blue line in Fig. 1a. It is a super-critical pitchfork bifurcation, where the trivial steady state \( F_H \) undergoes spontaneous breaking of translational symmetry leading to the transition to the static phase-separated state \( F_S \). Conversely, when If \( \delta^2 > \kappa^2 + (\alpha_A - \alpha_B)^2 / 4 \) the eigenvalues are complex conjugate. The state \( F_H \) can still become unstable when \( -\alpha_A > \alpha_B \), albeit now via an oscillatory instability shown as a dashed blue line in Fig. 1a.

Further insight is gained by examining the stability of \( F_S \). This requires the analysis of the eigenvalues of the \( 4 \times 4 \) matrix obtained by linearizing (4). Details are given in the SI. Note that the matrix is block diagonal, coupling separately the two amplitudes and the two phases. One finds that the instability is driven by the growth of fluctuations in the relative phase \( \theta \) that become unstable when \( \delta > \delta_c \). This boundary \( \delta = \delta_c \) corresponds to the appearance of \( F_T \) and is shown as a black line in Fig. 1a. The instability of the relative phase is associated with the “run-and-catch” scenario described earlier and signals the transition to state where the two fields sets into a state with a constants phase lag (different from \( \pi \)), while traveling with a common speed.

To highlight the mechanism responsible for the traveling pattern, note that the velocity of the fields modulations \( v_\mu = \theta_\mu \) are given by \( v_A = \kappa_{AB} (\rho_B / \rho_A) \sin \theta \) and \( v_A = -\kappa_{AB} (\rho_A / \rho_B) \sin \theta \), hence are identically zero in the static state \( F_S \) where \( \theta_s = \pi \). Now consider the effect of a small fluctuation in the relative phase by letting \( \theta_s = \pi + \psi \), as shown in Fig. 3a. Evaluating the amplitudes at the steady state values, the velocities are then given by \( v_A = -\kappa_{AB} \rho_B \alpha_B \psi \) and \( v_B = \alpha_B \psi \) (see Fig. 3b). If the cross-diffusivities \( \kappa_{AB} \) and \( \kappa_{BA} \) have the same sign, the two species move in opposite directions (black and
blue arrows in Fig. 3a), exerting reciprocal driving forces on each other, and the perturbation \( \psi \) decays. On the other hand, if \( \kappa_{AB} \) and \( \kappa_{BA} \) have opposite signs, the two species travel in the same direction (black and red arrows in Fig. 3a) and can play catch-and-run with each other. To establish the precise condition for the onset of the traveling state, it is useful to examine the ratio of the two velocity, which is well defined even in the static demixed state and is given by

\[
\frac{v_A}{v_B} = -\frac{\kappa_{AB}\rho_B^2}{\kappa_{BA}\rho_A^2} = \frac{(\kappa - \delta)\rho_B^2}{(\kappa + \delta)\rho_A^2}. \tag{9}
\]

In the stationary demixed state, where \( \rho_B^2/\rho_A^2 = (\kappa + \delta)/\alpha_B \), we find \( v_A^2/v_B^2 = (\kappa^2 - \delta^2)/\alpha_B^2 \). This quantity is shown in Fig. 3b. When \( v_A^2/v_B^2 < 0 \) (blue portion of the curve) a small fluctuation \( \psi = \theta - \pi \) of the relative phase yields opposite field velocities (blue arrows), while when \( v_A^2/v_B^2 > 0 \) the velocities are in the same direction (green portion of the curve and green arrows).

Only when \( v_A^2/v_B^2 > 1 \), however, nonreciprocity is strong enough to destabilize the static pattern (red line and arrows in Fig. 3b). The onset of the traveling state corresponds to \( v_A^2/v_B^2 = \delta = \delta_s \), as obtained from the linear stability analysis. The condition \( v_A = v_B \) provides a general necessary condition for the onset of traveling patterns of two interacting scalar fields.

The equality of the velocities is not, however, sufficient to stabilize the traveling pattern as the perturbation \( \psi \) will keep increasing if \( v_A > v_B \) persists. Non-reciprocal interactions come again to the rescue by facilitating the “redistribution” of amplitude growth. Specifically, as \( \psi \) increases, both the damping of \( \rho_A \) and the activation of \( \rho_B \) originating from the non-reciprocal nature of the cross couplings become weaker (last terms in (4a)-(4b)). Consequently, the amplitude ratio \( \rho_A/\rho_B \) increases and suppresses the velocity ratio until \( v_A = v_B \), allowing the development of a steady traveling pattern, as shown in Fig. 3c. We have validated this simple picture displayed in Fig. 3d by examining numerically the mechanisms of stabilization of the traveling state \( F_T \) for \( \delta \) slightly larger than \( \delta_s \).

III. STATIC-TO-TRAVELING AS A PT-BREAKING TRANSITION

The static-to-traveling transition described in this work belongs to a more generic class of PT-breaking transitions [23], which has been studied in optical and quantum systems [54–56] and more recently in polar active fluids with non-reciprocal interactions [42]. This type of transition is known to occur at a so-called exceptional point, which is simply a point where the eigenvalues of the matrix that governs the linear stability of a fixed point become equal and its eigenvectors are co-linear. While not uncommon in hydrodynamics when fluids are driven by external forces or in systems described by non-conserved fields, the occurrence of such a transitions giving rise to nontrivial traveling structures in conserved systems is unexpected.

The dynamics of our coupled fields can be written in a compact form as

\[
\partial_t \begin{pmatrix} \phi_A \\ \phi_B \end{pmatrix} = \mathbf{M} \cdot \begin{pmatrix} \phi_A \\ \phi_B \end{pmatrix}, \tag{10}
\]

where the \( 2 \times 2 \) matrix operator \( \mathbf{M} = \mathbf{M}[\phi_A^2, \phi_B^2, \nabla^2] \) can be inferred from (1). In the static, spatially modulated solution corresponding the demixed state, the two fields \( \phi_A \) and \( \phi_B \) are out of phase, but have the same parity under spatial inversion, \( x \rightarrow -x \), as required by the symmetry of \( \mathbf{M} \). The domains become traveling by acquiring a component of the opposite parity that breaks the relative parity of the two fields, as described in Ref. [23]. Hence the transition to the traveling state breaks both parity and time translational (or Galileian) invariance.

This is most easily understood in the context of the one-mode approximation by considering a static \( F_S \) so-
lution of the form $\phi_B = \rho_B \cos(x)$ and $\phi_A = \rho_A \cos(x + \pi)$. Both fields are even and are out of phase, but have different amplitudes. A perturbation $\psi$ in the phase difference yields $\phi_A = \rho_A \cos(x + \pi + \psi) = \rho_A [\cos \psi \cos x + \sin \psi \sin x]$, breaking parity as $\phi_A$ now acquires an odd component. The response to such a perturbation is governed by (4c) linearized about the steady state for $\delta \to \delta_c$, which is given by

$$\dot{\psi} \simeq \frac{2\delta_c (\delta - \delta_c)}{\alpha_B} \psi.$$  

For $\delta < \delta_c$, the odd component of $\phi_A$ proportional to $\psi$ decays, restoring the parity of the static solution. For $\delta > \delta_c$, $\psi$ grows to a finite value, destabilizing the static state. As a result, $\phi_A$ acquires a finite odd component, breaking the parity of the static solution. Meanwhile, near the transition (4d) gives $\dot{\Phi}_T \simeq -2\alpha_B \psi$, resulting in a finite $\dot{\Phi}$ for $\delta > \delta_c$ and breaking time translational symmetry.

IV. DISCUSSION AND OUTLOOK

We have shown that non-reciprocal effective interactions in a minimal model of conserved coupled fields with purely diffusive dynamics lead to a PT-breaking transition to traveling spatially modulated states. While the emergence of traveling spatio-temporal patterns is well known in reaction-diffusion, prey-predators and related models, its appearance in the dynamics of conserved fields without external forcing is surprising. Although the work presented here is limited to a minimal model in one dimension, preliminary results shown in the SI indicate that the same mechanism is at play in two dimensions, as well as in mixtures of active and passive particles and of particles interacting via competing repulsive and attractive interactions, as may be realized in phoretic colloidal mixtures. We speculate therefore that the mechanism described here through which non-reciprocal effective couplings grant motility to static spatial modulations may be a generic property of multispecies systems described by scalar fields.

The type of static-to-traveling transition described here occurs in Mullins-Sekerka models of crystal growth [ ], Keller-Segel, prey-predator and reaction-diffusion models of population dynamics and general systems described by non-conserved dynamical fields, where it has been referred to as a drift bifurcation [21, 23, 39]. It occurs in these systems when a stationary or standing wave pattern generated through a conventional Hoff bifurcation undergoes a second instability to a traveling state. The drift bifurcation can be understood using amplitude equations as arising from the antagonistic coupling of at least two leading modes [39]. Here we show that a similar mechanism can be at play in multispecies systems with dynamics described by two conserved scalar fields coupled by sufficiently strong nonreciprocal interactions. When sufficiently strong, nonreciprocity leads to an effective antagonistic repulsion/atraction between the two fields, resulting in the chase-and-run mechanism described here that yields a PT-symmetry breaking transition. Our one-mode approximation provides a minimal analytic description of this generic mechanism, where $\nu = \dot{\Phi}$ serves as the order parameter for the transition.

A scenario similar to the one described here was recently identified in a binary Vicsek model with non-reciprocal interactions [42]. The mechanisms promoting the onset of a phase with broken PT are the same in both models, but the outcomes are distinct due to the different symmetry of the two systems. In Ref. [42] it is suggested that non-reciprocal interaction in a polar system may generically result in macroscopically chiral phases. Here, in contrast, we consider a scalar model with conserved dynamics and demonstrate that in this case non-reciprocity generically yields spatially inhomogeneous traveling states through the same type of PT-breaking transition. Together, these works pave the way to the study of the interplay of non-reciprocity and spontaneously-broken symmetry, suggesting a path to the classification of a new type of PT-breaking transitions.

Understanding and quantifying the role of non-reciprocity in controlling nonequilibrium pattern formation has direct implication to the assembly of chemically interacting colloids, where different particles naturally produce different chemicals mediating nonreciprocal couplings that can induce the type of chasing behavior seen in our work. It also provides a general framework for understanding the nature of wave and oscillatory behavior seen ubiquitously in systems with non-conserved field, from diffusion reaction to prey-predator and population dynamics models. Our predictions can be tested in detailed simulations of active-passive colloidal mixtures or of particles with antagonistic interactions, as well as experiments in mixtures of chemically driven microswimmers.

Our work opens up many new directions of inquiry. Obvious extensions are to higher dimensions where we expect a richer phase diagram and to systems with birth and death processes that select a scale of spatial patterns [57]. The exploration of the role of nonreciprocal interactions in active matter systems with broken orientational symmetry, either polar or nematic, is only beginning [42] and promises to reveal a rich phenomenology. Chemically mediated or other nonequilibrium couplings can often be time-delayed, which can provide an additional, possibly competing mechanism for the emergence of oscillatory behavior. Finally, an important open problem is understanding how nonreciprocity arises as an emergent property in systems with microscopic reciprocal interactions, such as active-passive mixtures.
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We note that in a binary mixture of diffusing particles, the cross diffusivities would differ as each $\kappa_{\mu\nu}$ would depend on the concentration of the two species as required to maintain detailed balance, but they would always have the same sign.

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