Regularization approaches for quantitative photoacoustic tomography using the radiative transfer equation

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Abstract:
Quantitative Photoacoustic tomography (QPAT) is an emerging medical imaging modality that offers the possibility of combining the high resolution of acoustic waves and large contrast of optical waves by quantifying the molecular concentration in biological tissue.

In this paper, we prove properties of the forward operator that associate optical parameters to the measurements of a reconstructed Photoacoustic image. This is often referred to as the optical inverse problem, that is nonlinear and ill-posed. The proved properties of the forward operator provide sufficient conditions to show regularized properties of approximated solutions obtained by Tikhonov-type approaches. The proposed Tikhonov-type approaches analyzed in this contribution are concerned with physical and numerical issues as well as with a priori information on the smoothness of the optical coefficients for with (PAT) is particularly well-suited imaging modality.

Key words: Quantitative Photoacoustic tomography, Tikhonov-type regularization, Convergence, Stability.
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1 Introduction

Photoacoustic tomography (PAT) is an emerging medical imaging modality that combines the high contrast of optical waves with the large resolution of the acoustic waves by a laser-generated ultrasound. From the practical point of view, a PAT image is reconstructed from temporal surface measurements of propagated photoacoustic waves. Such photoacoustic waves are generated by illuminating an optically absorbing and scattering medium with short pulses of variable or near-infrared light. As the optical radiation propagates, a fraction of its energy is absorbed by the chromophores within the tissue and generates a small and localized heating and pressure of the underlying medium. Due the elasticity of soft tissue, the given perturbation of the physical conditions produces a spatial dependent ultrasound signal that propagated through the domain of interest. This physical phenomenon is often called the photoacoustic effect. The resulting emitted pressure wave is measured by ultrasonic transducers located at the surface of the interesting domain as a function of time for which one tries to recover the acoustic source that gives us information about the rate of absorption at each point within the body, creating an image.

The image reconstruction in (PAT) embraces the solution of two inverse problem: First one should reconstruct the amount of deposit energy from surface measurements of the propagated acoustic wave pulses. In this issue, there was many advances both theoretical and numerical,
1.1 The forward model: Radiative transfer equation

Recently, many advances and several inversion methods have been proposed for (QPAT) \([2, 3, 4, 5, 17]\). However, the proposed model typically assumes the diffusion approximation to the radiative transfer equation (RTE), i.e. it assumes that the propagation of light throughout the tissue is near-isotropic. But, light propagation in a turbid regions is highly anisotropic in regions close to light sources. Therefore, does not behave diffusively until travel away to the source location. Hence, the
diffusion approximation does not provide a suitable accurate model in a significant portion of the image that often contain information of great interest \cite{21,23}.

By modeling the physical process involved in the generation of a photoacoustic signal and performing the image reconstruction of PAT images, consider a region of interest $\Omega \subset \mathbb{R}^n$, with $n = 2, 3$. Light transport in a turbid medium may be modeled analytically using the radiative transfer equation (RTE)

$$ (s \cdot \nabla + \mu_a(x) + \mu_s(x))u(x, s) - \mu_s(x) \int_{S^{n-1}} \Theta(s, s') u(x, s') ds' = q(x, s). $$

(1)

The integro-differential equation (1) represents the conservation of energy in a particular control volume. The physical interpretation of equation (1) can be read as follows: light travels throughout a region in a particular direction $s \in S^{n-1}$, energy can be lost through the absorption and scattering of a photon out of the direction of interest or the net outflow of the region due to the gradient, and can be gained via the scattering of a photon into the direction of interest or from any light sources in the medium. The probability per unit length of an absorbing and scattering event are represented, respectively, by the absorption coefficient $\mu_a(x)$ and the scattering coefficient $\mu_s(x)$, at a point $x \in \Omega$. $\Theta(s, s')$ is the scattering phase function, a probability density function with describes the probability that a photon traveling in a direction $s$ will be scattered into a direction $s'$, $q$ is the light source. Since light propagates faster than sound, the optical propagation and absorption can be treated as instantaneous on an acoustics timescale, the quantity of interest is the time- integrated radiance $u(x, s)$, that is the energy per unity of area at a point $x \in \Omega$ in a direction $s \in S^{n-1}$. Assuming that there is no photons travel in an inward direction at the boundary $\partial \Omega$ except at the source position $\Gamma_s \subset \partial \Omega$, we can complete the (RTE) equation (1) with the given boundary condition

$$ u(x, s) = \begin{cases} u_0(x, s), & x \in \Gamma_s \quad s \times \eta < 0 \\ 0, & x \in \partial \Omega - \Gamma_s \quad s \times \eta < 0 \end{cases} $$

(2)

where $u_0$ is the boundary source and $\eta$ is a unitary vector normal to $\partial \Omega$. In this paper, we will assume that $\cup \Gamma_s$ has a compact support in $\partial \Omega$. This property is necessary for the existence of the trace operator in appropriated spaces (see Proposition 2 below).

The total energy at a point $x \in \Omega$ shall be equal to the integral of the total energy per unit of area $u(x, s)$ over all directions, i.e.,

$$ U(x) = \int_{S^{n-1}} u(x, s) ds, $$

(3)

often called the fluence.

From thermodynamic considerations we are allowed to write the initial pressure $p_0$ arising from this optical absorption as

$$ p_0(x) = \Pi(x) F(x), $$

(4)

where

$$ F(x) = F(\mu_a(x), \mu_s(x)) := \mu_a(x) U(\mu_a(x), \mu_s(x)). $$

(5)
is the amount of optical energy absorbed per unit volume in Ω. Π is the Grüneisen parameter, a dimensionless, tissue-specific property responsible for the photoacoustic efficiency, i.e., representing the conversion efficiency of the heat energy into pressure.

The first inverse problem in PAT is recover the initial pressure $p_0(x)$, from measurements of the acoustic pressure $p(x, t)$ over some arbitrary measurement surface. When the sound speed $c_s$ and the density are uniform and the optical excitation is regarded as instantaneous, the acoustic propagation may be well described by initial value problem for the homogeneous wave equation

$$p_{tt} - c_s^2 \Delta p = 0,$$

(6)

whose the initial conditions are given by

$$p(x, t)|_{t=0} = p_0(t), \quad p_t(x, t)|_{p=0} = 0.$$  

(7)

When the sound speed $c_s$ is constant, explicit formulas for recovering $p_0$ have been obtained for a large class of geometries of interest, e.g. [2, 3, 4, 5, 18, 17, 20, 22, 25, 23] and references therein. When the sound speed is not constant, but known, and non-trapping conditions are assuming, time reversal algorithms perform well as demonstrated in [25, 22].

In this paper, we assume that the first inverse problem is solved and that $p_0$ is known. Normally, precise estimations for $\Pi$ are known from recorded experiments and therefore, we will assume $\Pi$ known throughout the domain. Hence, it is straightforward to obtain a measured observed energy map

$$E(x) = p_0(x)/\Pi(x).$$

(8)

Since chromophore concentration is linearly related to the optical coefficient via the chromophores’ molar absorption coefficient, they can be obtained straightforwardly from $\mu_a$ provided that all contributing chromophore types are known. Therefore, we seek to determine a quantitative accurate estimate of $\mu_a$ from measurements of the absorbed energy map $E$. Resuming, this is the second inverse problem in QPAT. However, the dependence of $U$ on $\mu_a$ and $\mu_s$ means that $E$ (and hence $p_0$) is nonlinear related to the absorption and scattering coefficients. Since neither of them are likely to be known or easily measured, it means that we need to seek to recover quantitative estimates of both of the coefficient simultaneously.

**Notations:** Throughout this presentation, we assume that $\Omega \subset \mathbb{R}^n$, with $n = 2, 3$ is a bounded domain with $C^1$ boundary $\partial \Omega$. We shall define the product domain $D := \Omega \times S$, where $S := S^{n-1}$ is the sphere in $\mathbb{R}^n$. $C$ will denote a generic constant, whose values may depend on the context.

In the product space, we have the boundary $\Gamma := \partial \Omega \times S$ can be decomposed into an inflow part $\Gamma_- = \{(x, s) \in \Gamma : s \cdot \eta < 0\}$, an outflow part $\Gamma_+ = \{(x, s) \in \Gamma : s \cdot \eta > 0\}$, and a remainder tangential part $\Gamma_0 = \Gamma - (\partial D_- \cup \Gamma_+)$. For $L^p(\Omega)$ we denote the Lebesgue space of real functions on $\Omega$ such that $\int_\Omega |f(x)|^p dx < \infty$ if $1 \leq p < \infty$ and $\text{ess sup} |f(x)| \leq \infty$ for $p = \infty$. We also denote by $W^{k,p}(\Omega)$ the Sobolev space of all functions whose all the derivatives up to the order $k$ belongs to $L^p(\Omega)$. In particular, for $p = 2$ we have the Hilbert spaces $W^{k,2}(\Omega) = H^k(\Omega)$.

In this contribution we will consider the parameter space belongs to the subset

$$D(F) := \{(\mu_a, \mu_s) : 0 < \underline{\mu} \leq \mu_a, \mu_s \leq \overline{\mu}\},$$

where $\underline{\mu}$ and $\overline{\mu}$ are the lower and upper bounds for $\mu_a$ and $\mu_s$, respectively.
under different topologies.

Aware of possible confusions, we shall introduce also the Banach space $L^p(D)$ $(1 \leq p < \infty)$ defined on the space of Lebesgue function for the product measure $dxds$ such that $\|f\|_{L^p(D)} = \int_\Omega \int_S |f(x,s)|^p dxds < \infty$. Moreover, $W^p(D) := \{ f \in L^p(D) : s \cdot \nabla f \in L^p(D) \}$ denotes the Banach space where the integro-differential operator in equation (1) will be well-posed.

As we will see, for physical reasons, the natural spaces for the radiances and for the fluence are $L^1(D)$ and $L^1(\Omega)$, respectively. Indeed, we can define the so-called transport operator $T$ such that

$$Tu(x,s) = (s \cdot \nabla + \mu_a(x) + \mu_s(x))u(x,s) - \mu_s(x) \int_S \theta(s,s')u(x,s')ds',$$

which is naturally defined in the space $L^1(D)$ of integrable functions in $(x,s)$ and its domain $D(T)$ is

$$D(T) := \{ u \in L^1(D) : Tu \in L^1(D) \text{ and } u(x,s) = 0, \text{ a.e. } (x,s) \in \partial \Omega \times S, s \cdot \eta < 0 \}.$$

Therefore, for absorption and scattering coefficient belongs to $D(F)$, is easy to see that $D(T) \subset W^p(D)$. Of course, the trace operator must make sense in such topology. It will be guarantee in Lemma 3 below.

However, for numerical as well as theoretical reasons, other $L^p$-spaces play an important role in the game. In particular, the development of computational schemes in a Hilbert space makes $L^2(D)$ with the inner product

$$(u,v)_{L^2(D)} := \int_\Omega \int_S u(x,s)v(x,s)dxds$$

a very suitable candidate.

2 On the existence and regularity of a solution of RTE equation

In this section we will revisit some well-known results of existence and regularity for the (RTE) equation (1)-(2), for which we suggest the reference [6]. Such a collection of results and the respective techniques of proofs will be mimicked for proving properties of the forward operator in Section 3.

The first result in this direction is concerned with the trace operator and the well posed of the boundary condition (2).

**Lemma 1.** The inflow and the outflow boundaries $\Gamma_-$ and $\Gamma_+$ are open subsets of $\Gamma$, and $\Gamma_0$ is a closed subset of $\Gamma$ with $(2n-2)$-dimensional zero measure.

**Proof.** The $C^1$ regularity of $\partial \Omega$ implies that: (i) map $(x,s) \mapsto s \cdot \eta$ is continuous and (ii) $\partial \Omega$ is locally diffeomorphic to a subset of $\mathbb{R}^{n-1}$. From (i) we have that $\Gamma_0$ is closed and $\Gamma_-$ and $\Gamma_+$ are open subsets of $\Gamma$. From (ii) and the standard product structure of $\Gamma_0, \Gamma_-, \Gamma_+$ we have the assertions. \( \square \)

Previews lemma allows to identify measurable functions on $\Gamma$ with functions defined in $\Gamma_- \cup \Gamma_+$. However, it is worth notice that if $u \in W^p(D)$, it is not true that the trace $u|_{\Gamma_-}$ (respectively $u|_{\Gamma_+}$) satisfies

$$\int_{\Gamma_-} s \cdot \eta |u|^p dxds < \infty$$
even for $p = 2$, see [6]. But the result is true if $u$ has a compact support in $\Gamma_-$ as shown by the following proposition.

**Proposition 2.** Let $K$ be a compact subset of $\Gamma_-$ (resp. $\Gamma_-$). Then the trace map $u \mapsto u|_K$ defined in $\mathcal{D}(\mathcal{D})$ is extended by continuity to a bounded linear operator from $W^p(\mathcal{D})$ to $L^p(K)$.

In particular the boundary condition (2) is well-defined.

**Proof.** The proof is given by [6, Theorem 1, pp 220].

The well-definition of the boundary condition (2) follows from the assumption that the support of $u_0$ is compact embedding in $\Gamma_-$. \hfill \Box

We will prove the next lemma in details, since similar techniques will be used later. A similar result is given in [6, Lemma 1, pp. 227]

**Lemma 3.** The scattering operator

$$Ku := \mu_a \int_S \Theta(x, s, s') u(x, s) ds'$$

is linear and continuous from $L^p(\mathcal{D})$ in itself, for $p$ in $[1, \infty]$.

**Proof.** The linearity of $K$ follows immediately.

Since $\Theta$ is a probability kernel, it follows that $\Theta \geq 0$ and $\int_S \Theta(x, s, s') \leq 1$. Hence, given the uniformly bounded of the coefficient, the assertion for $p = 1$ and $p = \infty$ follows. Let we consider the other cases. Using the Hölder inequality ($p^{-1} + p'^{-1} = 1$) we have

$$\|Ku\|_{L^p(\mathcal{D})}^p = \int_D \left| \mu_a \int_S \Theta(x, s, s') u(x, s') ds' \right|^p dx ds$$

$$\leq \overline{p}^p \int_D \left( \int_S \Theta(x, s, s')^{1-1/p} \Theta(x, s, s')^{p'} |u(x, s')| ds' \right)^p dx ds$$

$$\leq \overline{p}^p \int_D \left( \int_S \Theta(x, s, s') ds \right)^{p/p'} \left( \int_S \Theta(x, s, s') |u(x, s')| ds' \right)^p dx ds$$

$$\leq \overline{p}^p \int_D \int_S \Theta(x, s, s') ds |u(x, s')|^p dx ds' \leq \overline{p}^p \int_D |u(x, s')|^p dx ds' \leq \overline{p}^p \int_D |u(x, s')|^p dx ds'.$$

\hfill \Box

Next we will present the regularity of solutions of the (RTE) equation (1).

**Theorem 4.** Let $q \in L^p(\mathcal{D})$ and the coefficient $(\mu_a, \mu_s) \in D(F)$, such that $u_0 \in L^p(\Gamma_-)$ (meaning that $u_0$ belong to the spaces of a trace of a function $v \in W^p(\mathcal{D})$ as in Proposition 2). Then, there exists a unique $u \in L^p(\mathcal{D})$ solution of (1)-(2), for $p \in [1, \infty]$.

Moreover, we have the bound

$$\|u\|_{L^p(\mathcal{D})} \leq C \left( \|q\|_{L^p(\mathcal{D})} + \|u_0\|_{L^p(\Gamma_-)} \right),$$

with $C$ depending only of the boundedness of the coefficients and on $\mathcal{D}$. 

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Proof. Since \((\mu_a, \mu_s) \in D(F)\) and \(\int_S \Theta(x, s, s') \leq 1\), it follows that
\[
\mu_a + \mu_s - \mu_a \int_S \Theta(x, s, s') \geq \mu_s \geq \underline{\mu} > 0 \quad \text{and} \quad \mu_a \int_S \Theta(x, s, s') \leq \mu_a \leq \beta (\mu_a + \mu_s) \text{ for some } 0 \leq \beta < 1.
\]
Therefore, the assumptions on Theorem 4, Proposition 5 and Proposition 6 in \([6, \text{Chapter XXI}]\) are satisfied. They guarantee, respectively, the existence of a unique solution \(u \in L^p(D)\), with \(1 < p < \infty\), \(p = 1\) and \(p = \infty\) for equation (1) with absorbing boundary condition \((u_0 = 0)\), satisfying \(\|u\|_{L^p(D)} \leq C\|q\|_{L^p(D)}\).

Now, by using the lifting of the boundary condition \(u_0\), the linearity of (1) and the superposition principle, the existences of a unique solution \(u \in L^p(\Omega)\) for the non-homogeneous boundary condition equation (1)-(2) follows from the absorbing boundary condition results. \(\square\)

Although the natural space of definition of the transport operator \(T\) in (9) is \(L^1(D)\), Theorem 4 shows that the transport operator is also well defined in \(W^p(D)\). Moreover, we can see that the operator \(T\) as an adjoint in \(L^{p'}(D)\), given by
\[
T^*v = (-s \cdot \nabla + \mu_a + \mu_s)v - \mu_s \int_S \Theta(s')v(s')ds',
\]
such that the integro-differential equation
\[
T^*v = \tilde{\theta}, \quad v|_{\Gamma_+} = g
\]
has a unique solution in \(L^{p'}(D)\), for \(1/p + 1/p' = 1\), for any \(\tilde{\theta} \in L^{p'}(\Omega)\) and \(g \in L^{p'}(\Gamma_+)\) which compact support. A detailed proof can be found in \([6, \text{Section 3.3}]\).

3 Properties of the forward operator

In PAT, the nonlinear operator equation \(D(F) \ni (\mu_a, \mu_s) \mapsto F(\mu_a, \mu_s)\) naturally maps the absorbed energy given by equation (5) into \(L^1(\Omega)\). However, as we show in Theorem 4, the radiance \(u\) may belong to \(L^p(D)\) if source \(q\) and the boundary condition \(u_0\) are smooth enough. Hence, it makes sense (from the numerical as well as from the theoretical point of view) to looking for the operator equation
\[
F : D(F) \to L^p(\Omega) \quad 1 \leq q \leq \infty
\]
\[
(\mu_a, \mu_s) \mapsto F(\mu_a, \mu_s)
\]
(12)

In this section, we shall to show the properties of the operator equation (12) by consider \(D(F)\) in different topologies. Among those, we are interested in proving continuity, compactness and Fréchet differentiability that allows prove convergence and stability of different regularization approaches in Section 4.
Let \((\mu_a, \mu_s), (\tilde{\mu}_a, \tilde{\mu}_s) \in D(F)\) and \(u = u(\mu_a, \mu_s)\) and \(v = u(\tilde{\mu}_a, \tilde{\mu}_s)\) the respectively unique solutions of (1)-(2), with source \(q \in L^p(D)\) and boundary condition \(u_0 \in L^p(\Gamma_-)\). By linearity of (1)-(2), we have that \(w = u - v\) satisfies

\[
Tw = ((\mu_a - \tilde{\mu}_a) + (\mu_s - \tilde{\mu}_s))v + (\mu_s - \tilde{\mu}_s) \int_S \Theta(s, s')v(s')ds'.
\]  

(13)

with absorbing boundary condition. Notice that, from Theorem 4 and Lemma 3 there exists a unique solution \(w \in L^p(\Omega)\) for the integro-differential equation (13).

Let us consider \(p' = 2\) for a while. Then, from the proof of Theorem 4, pp. 241 in [6], we have

\[
\mu \|w\|_{L^2(D)}^2 \leq \langle ((\mu_a + \mu_s)w - \mu_s \int_S \Theta(s, s')w(s')ds', w \rangle_{L^2(D)}
\]

\[
\leq \langle (s \cdot \nabla + \mu_a + \mu_s)w - \mu_s \int_S \Theta(s, s')w(s')ds', w \rangle_{L^2(D)}.
\]

(14)

By multiplying equation (13) by \(w\), integrate over \(D\) in both sides and uses (14) we get

\[
\mu \|w\|_{L^2(D)}^2 \leq \int_D (|\mu_a - \tilde{\mu}_a| + |\mu_s - \tilde{\mu}_s|)|v|\|w\|dxds
\]

\[
+ \int_D (|\mu_s - \tilde{\mu}_s| \int_S \Theta(s, s')|v(s')|ds')|w|dxds.
\]

(15)

Using the Hölder inequality for \(1/p + 1/r + 1/p' = 1\) and Lemma (3), it follows from (15) that

\[
\mu \|w\|_{L^2(D)}^2 \leq C \left( \|\mu_a - \tilde{\mu}_a\|_{L^r(\Omega)} + \|\mu_s - \tilde{\mu}_s\|_{L^r(\Omega)} \right) \|v\|_{L^p(D)} \|w\|_{L^2(D)}.
\]

(16)

Now, from Theorem 4 we conclude that

\[
\|w\|_{L^2(D)} \leq C(\|q\|, \|u_0\|) \left( \|\mu_a - \tilde{\mu}_a\|_{L^r(\Omega)} + \|\mu_s - \tilde{\mu}_s\|_{L^r(\Omega)} \right).
\]

(17)

**Remark 1.** The coercivity of the bilinear form \(\langle (s \cdot \nabla + \mu_a + \mu_s)w - \mu_s \int_S \Theta(s, s')w(s')ds', w \rangle_{L^p(D)}\) follows from shown that \(-\langle (\mu_a + \mu_s)w - \mu_s \int_S \Theta(s, s')w(s')ds', w \rangle_{L^p(D)} \leq -\mu \|w\|_{L^p(D)}^{p'} \) for \(p' \in [1, \infty[\).

It is proved analogous to Theorem 4, pp. 241 [6], with the help of the Hölder inequality replacing the Cauchy-Schwarz inequality in the case \(p' = 2\).

Now, with the same arguments in equations (15)-(17) we deduce that, for \(1/p + 1/r + 1/p' = 1\),

\[
\|w\|_{L^p(D)} \leq C(\|q\|_{L^p(D)}, \|u_0\|_{L^p(D)}) \left( \|\mu_a - \tilde{\mu}_a\|_{L^r(\Omega)} + \|\mu_s - \tilde{\mu}_s\|_{L^r(\Omega)} \right).
\]

(18)

**Remark 2.** Notice that equation (18) reflect the amount of regularity that we should expect on the coefficient and the solutions of (1) in order to get continuity of the forward operator. In particular, from Theorem 4, if the source and boundary conditions are in \(L^p(D)\) and \(p \to \infty\), that we can expect continuity of the forward operator in \(L^r(\Omega)\) for \(r \to 1\).

Indeed, we have
**Theorem 5.** Let $p' \in [1, \infty]$. Then the forward operator $F$ defined in (12) is continuous from $D(F)$ to $L^{p'}(\Omega)$, with $D(F)$ consider in the $L^r(\Omega) \times L^r(\Omega)$-topology, for $1/p + 1/p' + 1/r = 1$.

**Proof.** As before, let $\left(\mu_a, \mu_s\right), \left(\tilde{\mu}_a, \tilde{\mu}_s\right) \in D(F)$ and $u, \tilde{u} \in L^p(\mathcal{D})$ the respective solution of (1)-(2) (from Theorem 4, the $L^p$ regularity of $u, \tilde{u}$ is reflected into the regularity of the source and the boundary condition).

Notice that

$$F(\mu_a, \mu_s) - F(\tilde{\mu}_a, \tilde{\mu}_s) = \mu_a \int_S u(\cdot; s)d\mathbf{s} - \tilde{\mu}_a \int_S \tilde{u}(\cdot; s)d\mathbf{s}$$

$$= (\mu_a - \tilde{\mu}_a) \int_S u(\cdot; s)d\mathbf{s} - \tilde{\mu}_a \int_S (\tilde{u}(\cdot; s) - u(\cdot, s))d\mathbf{s}. \quad (19)$$

Therefore, using the same arguments in the proof of Lemma 3, we have

$$\int_{\Omega} |F(\mu_a, \mu_s) - F(\tilde{\mu}_a, \tilde{\mu}_s)|^{p'} dx \leq \int_{\Omega} \left( |\mu_a - \tilde{\mu}_a| \int_S |u(\cdot; s)|d\mathbf{s} + |\tilde{\mu}_a| \int_S |\tilde{u}(\cdot; s) - u(\cdot, s)|d\mathbf{s} \right)^{p'} dx \quad (20)$$

$$\leq C \left( \|\mu_a - \tilde{\mu}_a\|_{L^\infty(\Omega)}^{p'/p} \|u\|_{L^{p'}(\mathcal{D})}^{p'} + \|\tilde{u} - u\|_{L^{p'}(\mathcal{D})} \right).$$

Now, Theorem 4 and the inequality (18) conclude the assertion. \qed

**Remark 3.** There are some cases that we would like to point out in Theorem 5.

**Case $p' = 1$:** As commented before, $L^1(\Omega)$ is the natural topology for the fluence $U$ and so is the natural topology for the range of the operator $F$.

It is easy to see that from equation (20) that the inequality

$$\int_{\Omega} |F(\mu_a, \mu_s) - F(\tilde{\mu}_a, \tilde{\mu}_s)| dx \leq \|\mu_a - \tilde{\mu}_a\|_{L^1(\Omega)} \|u\|_{L^\infty(\mathcal{D})} + |\tilde{\mu}_a| \|\tilde{u} - u\|_{L^1(\mathcal{D})}, \quad (21)$$

holds true, if the respective solution of equation (1) - (2) are in $L^\infty(\mathcal{D})$. Remember that, from Theorem (4), a sufficient condition for the $L^\infty$ regularity of a solution of (1) - (2) is that $q, u_0 \in L^\infty(\mathcal{D})$. Since $\mathcal{D}$ is bounded, we have $L^s(\mathcal{D})$ is continuously embedding in $L^1(\mathcal{D})$ and $\|\cdot\|_{L^1(\mathcal{D})} \leq C\|\cdot\|_{L^s(\mathcal{D})}$, for any $s \leq 1$. Using this in (21), for $s = p'$ w.r.t. the norm of $u - \tilde{u}$ and $s = r$ w.r.t. the norm of the coefficient and (18), we have

$$\int_{\Omega} |F(\mu_a, \mu_s) - F(\tilde{\mu}_a, \tilde{\mu}_s)| dx \leq C \left( \|\mu_a - \tilde{\mu}_a\|_{L^r(\Omega)} + \|\mu_s - \tilde{\mu}_s\|_{L^r(\Omega)} \right). \quad (22)$$

**Case $p' = 2$:** For the numerical point of view is important to have an inner product to help in the computational implementation.

A closer look at the proof of Theorem 5 imply in the inequality

$$\int_{\Omega} |F(\mu_a, \mu_s) - F(\tilde{\mu}_a, \tilde{\mu}_s)|^2 dx \leq C \left( \|\mu_a - \tilde{\mu}_a\|_{L^2(\Omega)} \|u\|_{L^2(\mathcal{D})} + \|\tilde{u} - u\|_{L^2(\mathcal{D})} \right);$$

Now, we can use (17) to have

$$\int_{\Omega} |F(\mu_a, \mu_s) - F(\tilde{\mu}_a, \tilde{\mu}_s)|^2 dx \leq C \left( \|\mu_a - \tilde{\mu}_a\|_{L^2(\Omega)}^{2} + \|\mu_a - \tilde{\mu}_a\|_{L^r(\Omega)} + \|\mu_s - \tilde{\mu}_s\|_{L^r(\Omega)} \right). \quad (23)$$

Hence, for longs as we take $p = +\infty$ in the deduction of (17) (this means that the respective solutions of (1) - (2) are in $L^\infty(\mathcal{D})$), we have the continuity of the operator $F$ in $L^2(\Omega)$ into itself.
The next result shows the continuity of the operator $F$ for piecewise constant coefficients.

**Corollary 6.** Assume that the admissible coefficient in $D(F)$ by piecewise constant in $\Omega$ and $q, u_0 \in L^\infty(\mathcal{D})$. Then the forward operator $F : D(F) \to L^2(\Omega)$ defined in (12) is continuous in $L^1(\Omega) \times L^1(\Omega)$.

**Proof.** Notice that, for the assumption on the source and boundary conditions of equation (1)- (2), we can take $p = \infty$. Taking $p' > 2$ (and $p = +\infty$) and follows the same arguments in Remark 3 we easily obtain

$$\int_\Omega |F(\mu_a, \mu_s) - F(\tilde{\mu}_a, \tilde{\mu}_s)|^2 dx \leq C \left( \|\mu_a - \tilde{\mu}_a\|_{L^r(\Omega)} + \|\mu_s - \tilde{\mu}_s\|_{L^r(\Omega)} \right), \tag{24}$$

for $r = p'/(p' - 1) > 1$. Therefore, there exists a $s > 0$ such that $r = 1 + s$.

Without loss of generality, assume that the coefficient assumes only two distinct values, let say $\mu_a(x), \mu_s(x) \in \{c_1, c_2\}, x \in \Omega$. Hence,

$$\int_\Omega |\mu_a - \tilde{\mu}_a|^r dx = \int_\Omega |\mu_a - \tilde{\mu}_a||\mu_a - \tilde{\mu}_a|^s dx \leq 2 \max\{c_1, c_2\}^s \int_\Omega |\mu_a - \tilde{\mu}_a|dx.$$

The same inequality is true for the scattering coefficient. Therefore, the assertion follows. \[\square\]

In the following we will prove that the inverse problem is ill-posed in appropriated topologies.

**Theorem 7.** Assume that the solution of (1) - (2) are in $L^p(\mathcal{D})$ (see Theorem 4 for such conditions), $\in ]1, \infty$ if $n = 2$ or $r < 6$ if $n = 3$ and $1/p + 1/p' + 1/r = 1$. Moreover, let the operator $F : D(F) \to \mathcal{L}'(\Omega)$ as defined in (12), with $D(F)$ equipped with the $H^1(\Omega) \times H^1(\Omega)$-norm. Then $F$ is completely continuous and weak sequentially closed in $\mathcal{L}'(\Omega)$.

**Proof.** Let $\{(\mu_a^k, \mu_s^k)\}$ be a sequence in $D(F)$ weakly convergent to $(\mu_a, \mu_s)$. Since $D(F)$ is convex and closed, it is weakly closed. Hence the weak limit $(\mu_a, \mu_s) \in D(F)$. Since $H^1(\Omega)$ is compact embedding in $L^r(\Omega)$ for $r$ as in the assumption [1], there exist a subsequence (that we denote with the same index) that strongly converge in $L^r(\Omega)$. From Theorem 5 we have $F(\mu_a^k, \mu_s^k) \to F(\mu_a, \mu_s)$ in $\mathcal{L}'(\Omega)$. \[\square\]

**Remark 4.** The assertions of Corollary 6 remained true for $D(F)$ embedding in any space that is compact embedding in $L^r(\Omega) \times L^r(\Omega)$. In particular, since $\BV(\Omega)(\Omega)$ is compact embedding in $L^r(\Omega)$ for $1 \leq r \leq 3/2$ (see [14]), one can consider $D(F)$ with the $\BV(\Omega)(\Omega) \times \BV(\Omega)(\Omega)$-norm.

We will see that the presented results on continuity and compactness allows us to prove regularizing properties of approximated solutions for the inverse problem in Section 4. Let us move forward and proves the differentiability of the forward operator in suitable topologies. Differentiability is a key property for the convergence of the iterative algorithm employed to obtain the approximated solution of the nonlinear operator equation (12).

**Theorem 8.** Let $(\mu_a, \mu_s) \in D(F)$ and $(\Delta \mu_a, \Delta \mu_s) \in H^1(\Omega) \times H^1(\Omega)$ such that $(\mu_a + t \Delta \mu_a, \mu_s + t \Delta \mu_s) \in D(F)$ for all $t \in \mathbb{R}$ with $|t|$ sufficiently small. Then the directional derivative of $F$ in the direction $(\Delta \mu_a, \Delta \mu_s)$ is given by

$$F'(\mu_a, \mu_s)[\Delta \mu_a, \Delta \mu_s] = \Delta \mu_a U(\mu_a, \mu_s) + \mu_a \int_S u'(\mu_a, \mu_s; s)[\Delta \mu_a, \Delta \mu_s] ds \tag{25}$$
where \( u'(\mu_a, \mu_s; s) \) satisfies
\[
(s \cdot \nabla + \mu_a + \mu_s)u - \mu_s \int_S \Theta(s, s')u'(s')ds' = [\Delta \mu_a + \Delta \mu_s]u + \delta \mu_s \int_S \Theta(s, s')u(s')ds',
\]
with absorbing boundary conditions, and \( u \) denotes the unique solution of (1)-(2).

Proof. By linearity of equation (1), it follows that the directional derivative \( u'(\mu_a, \mu_s; s)[\Delta \mu_a, \Delta \mu_s] \) := \( \lim_{t \to 0} \frac{1}{t}(u(\mu_a + t \Delta \mu_a, \mu_s + t \Delta \mu_s) - u(\mu_a, \mu_s)) \) satisfies (26).

Now, the linearity and continuity of the multiplication for \( \mu_a \) in the definition of \( F \) imply the assertion.

\[ \square \]

Lemma 9. The directional derivative \( F'(\mu_a, \mu_s) \) given in (25) satisfies the uniform estimate
\[
\|F'(\mu_a, \mu_s)[\Delta \mu_a, \Delta \mu_s]\|_{L^2(\Omega)} \leq C(\|\Delta \mu_a\|_{H^1(\Omega)} + \|\Delta \mu_s\|_{H^1(\Omega)})\left(\|q\|_{L^2(D)} + \|u_0\|_{L^2(D)}\right),
\]
were the constant \( C \) depends only on \( D \) and the bounds of the coefficients.

Proof. The result follows similarly to Theorem 5 and Remark (3).

Since \( D(F) \) has no interior point in the \( H^1(\Omega) \times H^1(\Omega) \)-topology, the directional derivative is not Gateaux differentiable. However, we will prove that it defines a linear operator that can be extended continuously to \( H^1(\Omega) \times H^1(\Omega) \).

Theorem 10. Let the assumptions of Theorem 8 holds. Then \( F'(\mu_a, \mu_s)[\Delta \mu_a, \Delta \mu_s] \) has a linear and bounded extension to \( H^1(\Omega) \times H^1(\Omega) \).

Proof. Consider the ball \( B_\rho(\mu_a, \mu_s) := \{(\tilde{\mu}_a, \tilde{\mu}_s) : \|\mu_a - \tilde{\mu}_a\|_{H^1(\Omega)}^2 + \|\mu_s - \tilde{\mu}_s\|_{H^1(\Omega)}^2 \leq \rho\} \). It is easy to see that the set \( B_\rho(\mu_a, \mu_s) \cap D(F) \) is dense in \( B_\rho(\mu_a, \mu_s) \) with the \( H^1 \)-topology. Hence, \( F'(\mu_a, \mu_s) \) is densely defined by the directional derivatives with satisfying the uniform bound (27).

The uniform boundedness principle \([27]\) imply the existence of a unique continuous extension to \( H^1(\Omega) \times H^1(\Omega) \), that we will denote again by \( F'(\mu_a, mu_s) \).

\[ \square \]

As observed before, \( D(F) \) has no interior points when equipped with the \( H^1(\Omega) \times H^1(\Omega) \)-norm. Because of that, \( F \) is not necessarily differentiable in every direction \( (\Delta \mu_a, \Delta \mu_s) \in H^1(\Omega) \times H^1(\Omega) \). In other words, \( F \) is not Gateaux differentiable. This will not affect the convergence analysis that follows. In fact, for such analysis we only need that the operator \( F \) attains a one-sided directional derivative at \( (\mu_a, \mu_s) \) in the directions \( (\Delta \mu_a, \Delta \mu_s) \), for all \( (\Delta \mu_a, \Delta \mu_s) \in D(F) \). The sufficient condition for this to happen is \( D(F) \) to be star-like with respect \( (\mu_a, \mu_s) \). That is, for every \( (\mu_a, \mu_s) \in D(F) \) there exists \( t_0 > 0 \) such that \( (\mu_a, \mu_s) + t(\Delta \mu_a, \Delta \mu_s) = t(\Delta \mu_a, \Delta \mu_s) + (1-t)(\mu_a, \mu_s) \in D(F) \) for \( 0 \leq t \leq t_0 \). Since \( D(F) \) has been convex, the requirement above follows. Moreover, the bounded linear operator \( F'(\mu_a, \mu_s) \) has properties that mimic the Gateaux derivative.
4 Regularization approaches

Notwithstanding that we are assuming the first inverse problem in PAT is solved, in practice, using any of the well-known methods, e.g. [25, 22] and references therein, it is very unlikely that one can get the exact solution $p_0$. In other words, the solution of the first inverse problem is relayed in the necessity of recovering the initial pressure $p_0$, from measurements of the pressure on the surface of the domain. Many sources of noise can affect the measurements in practical applications, e.g. thermal noise in the detectors. Therefore, we assume to know a measured absorbed energy map $E^\delta \in L^2(\Omega)$, satisfying

$$\|E - E^\delta\|_{L^2(\Omega)} \leq \delta,$$

where $\delta$ is a bounded for the noise level.

Hence, the second inverse problem in PAT can be rewritten as follows: finds $(\mu_a, \mu_s) \in D(F)$ which correspond to the measurements $E^\delta$. Mathematically, it means to solve the nonlinear operator equation

$$F(\mu_a, \mu_s) = E^\delta, \quad \text{s.t. } (\mu_a, \mu_s) \in D(F) \text{ and } E^\delta \text{ satisfying (28)}.$$

From physical reasons, is natural to assume that there exists $(\mu_a, \mu_s) \in D(F)$ such that $F(\mu_a, \mu_s) = E$. It is, that the inverse problem has a solution. However, the forward operator $F$ is compact (see Theorem 7). Hence, the inverse problem is ill-posed and some regularization method has to be used to guarantee the existence of stable approximated solutions. In this contribution we will consider Tikhonov-type regularization strategies for obtaining a stable approximated solution for the QPAT inverse problem.

Since PAT is particularly used for imaging different tissue regions, it is common to see different requirements for the smoothness of the structures. Such an information is crucial for proposing appropriated the regularization term in the Tikhonov-type approaches that reflect the expected smoothness of the coefficients. Hence, we will analyze Tikhonov approaches that reflect the a priori smoothness of the coefficients.

4.1 Tikhonov-type regularization: Smooth coefficients

In the following, we consider the standard Tikhonov regularization, i.e., we define an approximated solution $(\mu_a^{\alpha, \delta}, \mu_s^{\alpha, \delta})$ as a minimizer of the Tikhonov functional

$$\mathcal{J}_\alpha(\mu_a, \mu_s) := \frac{1}{p} \|F(\mu_a, \mu_s) - E^\delta\|^p_{L^p(\Omega)} + \alpha \left(\|\mu_a - \mu_{a,0}\|_{H^1(\Omega)}^2 + \|\mu_s - \mu_{s,0}\|_{H^1(\Omega)}^2\right)$$

subject to $(\mu_a, \mu_s) \in D(F) \cap H^1(\Omega)$ and $1 \leq p \leq 2$. The element $(\mu_{a,0}, \mu_{s,0}) \in H^1(\Omega)$ serves as an a-priori guess for the unknown parameters and $\alpha > 0$ is the regularization parameter.

Remark 5. The restriction on $p \in [1, 2]$ reflect the following estimate: Since $\Omega$ is bounded, $L^2(\Omega)$ is continuous embedding in $L^p(\Omega)$, for $p \in [1, 2]$ and $\|\cdot\|_{L^p(\Omega)} \leq \|\cdot\|_{L^2(\Omega)}$. Assume that $(\mu_a^1, \mu_s^1) \in$
D(F) ∩ H¹(Ω) is a solution of (12). Then, for any minimizer \((μ^α_0, μ^δ_0)\) of \(J_α\) we have

\[
J_α(μ^α_0, μ^δ_0) ≤ J_α(μ^α_0, μ^δ_0) = \frac{1}{p}||E - E^δ||_{L^p(Ω)}^p + α\left(||μ^α_0 - μ_0||_{H^1(Ω)}^2 + ||μ^δ_0 - μ_0||_{H^1(Ω)}^2\right)
\]

\[
≤ \frac{1}{p}||E - E^δ||_{L^p(Ω)}^p + α\left(||μ^α_0 - μ_0||_{H^1(Ω)}^2 + ||μ^δ_0 - μ_0||_{H^1(Ω)}^2\right)
\]

\[
≤ \frac{1}{p}||E - E^δ||_{L^p(Ω)}^p + α\left(||μ^α_0 - μ_0||_{H^1(Ω)}^2 + ||μ^δ_0 - μ_0||_{H^1(Ω)}^2\right)
\].

This estimated imply that, if the perturbation in the measurements goes to zero and the regularization parameter \(α\) is chosen appropriately, then the regularized solutions can be shown to converge to a solution of the inverse problem.

The estimate in Remark 5, the continuity and compactness of the forward operator \(F\) in Theorem 5 and Theorem 7, and some minor modifications from the standard Tikhonov regularization theory for nonlinear inverse problems is all that we need show stability and convergence of an approximated solution. Below we will present them without a proof. For details of the proofs see [13, Chapter 10]. The next result states, existence, stability and convergence of the approximated solutions of the inverse problem.

**Theorem 11.** Let the Tikhonov functional \(J_α\) defined in (30), then:

**[Existence of a minimizer]** For any \(α > 0\), the Tikhonov functional \(J_α\) has a minimizer in \(D(F) ∩ H^1(Ω)\).

**[Stability]** Let \(α > 0\), and let \(\{E^k\}\) be a sequence of data that converges strongly to \(E\) in \(L^2(Ω)\). Let \(\{μ^k_0, μ^k_δ\}\) be the respective sequence of minimizers of \(J_α\) with \(E^δ\) replaced by \(E^k\). Then, \(\{μ^k_0, μ^k_δ\}\) has a convergent subsequence, and the limit of every convergent subsequence is a minimizer of \(J_α\) in \(D(F) ∩ H^1(Ω)\).

**[Convergence]** Let \(\{E^k\}\) be a sequence of data satisfying (28), with \(δ\) replaced by \(δ_k\). If \(δ_k → 0\) and the regularization parameter is chosen such that \(\delta_k^2/α(δ_k) → 0\), then any sequence of minimizers of the Tikhonov functional \(J_α\) with \(E^δ\) replaced by \(E^k\) has a convergent subsequence. Moreover, the limit of every convergent subsequence is compatible with the data and has a minimum distance to the a priori guess \((μ_0, μ_0, s)\). This limit is called an \((μ_0, μ_0, s)\)-minimum-norm solution and denoted by \((μ^†_0, μ^†_δ)\).

It is possible to obtain quantitative convergence results is some a priori smoothness of the solution is required. It is known as source condition as read as follows: Let \((μ^†_0, μ^†_δ)\) a \((μ_0, μ_0, s)\)-minimum-norm solution. Assume that \(F\) is Fréchet differentiable at \((μ^†_0, μ^†_δ)\) and denote the adjoint of \(F\) by \(F^*(μ^†_0, μ^†_δ)\). Moreover, assume that there exists an element \(w ∈ L^2(Ω)\) such that

\[
(μ^†_0, μ^†_δ) - (μ_0, μ_0, s) = F^*(μ^†_0, μ^†_δ)w, \text{ whith } C||w||_{L^2(Ω)} ≤ 1,
\]

where \(C\) is a constant that depends only on the boundedness of the coefficients, the source of (1), then the classical convergence rate result [13, Theorem 10.4] holds

\[
\|F(μ^δ_0, μ^δ_0) - E^δ\|_{L^p(Ω)} = O(δ^{1/p}) \quad \text{and} \quad \|F(μ^δ_0, μ^δ_0) - (μ^†_0, μ^†_δ)\|_{H^1(Ω) × H^1(Ω)} = O(\sqrt{δ}). \tag{31}
\]

However, the source condition is hard to be verified in practice.
4.2 Piecewise constant coefficient: A level set regularization approach

PAT is particularly well-suited to imaging the non-smooth structure of the blood vasculature. In this case, the absorption and scattering coefficients are well approximated by piecewise constant functions.

For easiness of notation, in this article we will assume that the pair of absorption and scattering parameters \((\mu_a, \mu_s)\) are assuming two distinct unknown values, i.e. \(\mu_a(x) \in \{a^1, a^2\}\) and \(\mu_s(x) \in \{c^1, c^2\}\) a.e. in \(\Omega \subset \mathbb{R}^n\). Therefore, we can assume the existence of open and measurable sets \(\mathbb{A}_1 \subset \subset \Omega\) and \(\mathbb{C}_1 \subset \subset \Omega\), with \(\mathcal{H}^1(\partial \mathbb{A}_1) < \infty\) and \(\mathcal{H}^1(\partial \mathbb{C}_1) < \infty\), s.t. \(\mu_a(x) = a^1, x \in \mathbb{A}_1, \mu_a(x) = c^1, x \in \mathbb{C}_1\) and \(\mu_s(x) = a^2, x \in \mathbb{A}_2 := \Omega - \mathbb{A}_1, \mu_s(x) = c^2, x \in \mathbb{C}_2 := \Omega - \mathbb{C}_1\). Hence, the pair of piecewise constant absorption and scattering coefficients can be written as

\[
(\mu_a(x), \mu_s(x)) = (a^2 + (a^1 - a^2)\chi_{\mathbb{A}_1}(x), c^2 + (c^1 - c^2)\chi_{\mathbb{C}_1}(x)),
\]

where \(\chi_S\) is the indicator function of the set \(S\).

In order to model the space of admissible parameters (the pair of piecewise constant function \((\mu_a(x), \mu_s(x))\)), we use a standard level set (sls) approach proposed in [15, 9, 7, 8, 11]. According to this representation strategy, level set functions \(\phi_a, \phi_s : \Omega \to \mathbb{R}\), in \(H^1(\Omega)\), are chosen in such a way that its zero level-set \(\{x \in \Omega; \phi_a(x) = 0\}\) and \(\{x \in \Omega; \phi_s(x) = 0\}\) defines connected curves within \(\Omega\) and that the discontinuities of the parameters are located 'along' the zero level set of \(\phi_a\) and \(\phi_s\), respectively.

The piecewise constant requirement for the pair of coefficients \((\mu_a, \mu_s)\) is obtained by introducing the Heaviside projector

\[
H(t) := \begin{cases} 
1, & \text{if } t > 0 \\
0, & \text{if } t \leq 0 
\end{cases},
\]

that allows to represent the absorption and scattering coefficients as

\[
(\mu_a(x), \mu_s(x)) = (a^1H(\phi_a) + a^2(1 - H(\phi_a)), c^1H(\phi_s) + c^2(1 - H(\phi_s))) =: P(\phi_a, \phi_s, b_{ij}),
\]

where \(b_{ij}\) represents the vector of constant values \(b_{ij} := (a^1, a^2, c^1, c^2) \in \mathbb{R}^4\).

Within this framework, the inverse problem in (12), with data given by equation (28), can be written in the operator equation form

\[
F(P(\phi_a, \phi_s, b_{ij})) = E^S.
\]

Notice that, if an approximate solution \((\phi_a, \phi_s, b_{ij})\) of (34) is calculated, a corresponding approximate solution of (12) is obtained in a straightforward way: \((\mu_a, \mu_s) = P(\phi_a, \phi_s, b_{ij})\).

Let us remark that the analysis of level set approach for the pair of parameter that have many piecewise components follows essentially from the techniques derived in this approach with the multi-level framework approach in [8]. Therefore we do not go through the details here.

For guarantee a stable approximate solution for the operator equation (34) we introduce the energy functional

\[
\mathcal{F}_\alpha(\phi_a, \phi_s, b_{ij}) := \|F(P(\phi_a, \phi_s, b_{ij})) - E^S\|_{L^2(\Omega)}^2 + \alpha f(\phi_a, \phi_s, b_{ij}),
\]

1Here \(\mathcal{H}^1(S)\) denotes the one-dimensional Hausdorff-measure of the set \(S\).
where $\alpha > 0$ plays the role of a regularization parameter and $f(\phi_a, \phi_s, b_{ij}) = |H(\phi_a)|_{BV(\Omega)} + |H(\phi_s)|_{BV(\Omega)} + \|\phi_a - \phi_a,0\|_{H^1(\Omega)}^2 + \|\phi_s - \phi_s,0\|_{H^1(\Omega)}^2 + \|b_{ij}\|_{\mathbb{R}^2}^2$ is the regularization functional. This approach is based on TV-$H^1$ penalization. The $H^1$-terms act simultaneously as a control on the size of the norm of the level set function and as a regularization on the space $H^1(\Omega)$. The BV($\Omega$)-seminorm terms are well known for penalizing the length of the Hausdorff measure of the boundary of the sets \{x $\in \Omega : \phi_a(x) > 0$\}, \{x $\in \Omega : \phi_s(x) > 0$\} (see [14]). Other level set approaches have been applied to recover piecewise constant function in the literature, e.g. [12, 11, 9, 26] and references therein.

In general, variational minimization techniques involve compact embedding arguments on the set of admissible minimizers and continuities of the operator in such set to guarantee the existence of minimizers. The Tikhonov functional in (35) does not allow such characteristic, since the Heaviside operator $H$ and consequently the operator $P$ are discontinuous. Therefore, given a minimizing sequence $(\phi^k_a, \phi^k_c, b^k_{ij})$ for $F_\alpha$ we cannot prove existence of a (weak-*) convergent subsequence. Consequently, we cannot guarantee the existence of a minimizer in $[H^1(\Omega)]^2 \times \mathbb{R}^2$.

To overcome this difficulty we follow [15, 7, 8] and introduce the concept of generalized minimizers in order to guarantee the existence of minimizers of the Tikhonov functional (35).

First we introduce and smooth approximation of the Heaviside projection given by

$$H_\varepsilon(t) := \begin{cases} 1 + t/\varepsilon & \text{for } t \in [-\varepsilon, 0] \\ H(t) & \text{for } t \in \mathbb{R}/[-\varepsilon, 0] \end{cases}$$

and the corresponding operator

$$P_\varepsilon(\phi_a, \phi_s, b_{ij}) := (a^1 H_\varepsilon(\phi_a) + a^2(1 - H_\varepsilon(\phi_a)), c^1 H_\varepsilon(\phi_s) + c^2(1 - H_\varepsilon(\phi_s))) \quad (36)$$

for each $\varepsilon > 0$. Then:

**Definition 1.** Let the operators $H$, $P$, $H_\varepsilon$ and $P_\varepsilon$ be defined as above.

a) A vector $(z_1, z_2, \phi_a, \phi_s, b_{ij}) \in [L^\infty(\Omega)]^2 \times [H^1(\Omega)]^2 \times \mathbb{R}^2$ is called admissible when there exists sequences $\{\phi^k_a\}$ and $\{\phi^k_c\}$ of $H^1(\Omega)$-functions satisfying

$$\lim_{k \to \infty} \|\phi^k_a - \phi_a\|_{L^2(\Omega)} = 0, \quad \lim_{k \to \infty} \|\phi^k_c - \phi_s\|_{L^2(\Omega)} = 0$$

and there exists a sequence $\{\varepsilon_k\} \in \mathbb{R}^+$ converging to zero such that

$$\lim_{k \to \infty} H_{\varepsilon_k}(\phi^k_a) - z_1 \|_{L^1(\Omega)} = 0 \quad \text{and} \quad \lim_{k \to \infty} H_{\varepsilon_k}(\phi^k_c) - z_2 \|_{L^1(\Omega)} = 0.$$

b) A generalized minimizer of the Tikhonov functional $F_\alpha$ in (35) is considered to be any admissible vector $(z_1, z_2, \phi_a, \phi_s, b_{ij})$ minimizing

$$G_\alpha(z_1, z_2, \phi_a, \phi_s, b_{ij}) := \|F(q(z_1, z_2, b_{ij})) - E^\delta\|_{L^2(\Omega)}^2 + \alpha R(z_1, z_2, \phi_a, \phi_s, b_{ij}) \quad (37)$$

over the set of admissible vectors, where

$$q : [L^\infty(\Omega)]^2 \times \mathbb{R}^2 \ni (z_1, z_2, b_{ij}) \mapsto (a^1 z_1 + a^2(1 - z_2), c^1 z_2 + c^2(1 - z_2)) \in [L^\infty(\Omega)]^2,$$
and the functional $R$ is defined by
\[ R(z_1, z_2, \phi_a, \phi_s, b_{ij}) := \rho(z_1, z_2, \phi_a, \phi_s) + \| b_{ij} \|^2, \]
with
\[ \rho(z_1, z_2, \phi_a, \phi_s) := \inf \left\{ \liminf_{k \to \infty} \left( |H_{\varepsilon_k}(\phi_a^k)|_{\BV(\Omega)} + |H_{\varepsilon_k}(\phi_c^k)|_{\BV(\Omega)} + \| (\phi_a^k, \phi_c^k) - (\phi_{a,0}, \phi_{s,0}) \|_{H^1(\Omega) \times H^1(\Omega)}^2 \right) \right\}. \]

Here the infimum is taken over all sequences $\{\varepsilon_k\}$ and $\{\phi_a^k, \phi_c^k\}$ characterizing $(z_1, z_2, \phi_a, \phi_s, b_{ij})$ as an admissible vector.

Theorem 5 we can follow the proofs in [7, 8] to guarantee the existence, stability and convergence of approximated solutions for (34). For the sake of completeness, we collect the results without proving.

**Theorem 12.** The following assertions hold true.

**Existence:** The functional $G_\alpha$ in (35) attains minimizers on the set of admissible vectors.

**Convergence for exact data:** Assume that we have exact data, i.e. $E^k = E$. For every $\alpha > 0$ denote by $(z_{\alpha_1}^1, z_{\alpha_2}^2, \phi_{a\alpha_1}, \phi_{s\alpha_2}, b_{ij,\alpha})$ a minimizer of $G_\alpha$ on the set of admissible vectors. Then, for every sequence of positive numbers $\{\alpha_k\}$ converging to zero there exists a subsequence, denoted again by $\{\alpha_k\}$, such that $(z_{\alpha_k}^1, z_{\alpha_k}^2, \phi_{a\alpha_k}, \phi_{s\alpha_k}, b_{ij,\alpha_k})$ is strongly convergent in $L^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^2$. Moreover, the limit is a solution of (34).

**Convergence for noise data:** Let $\alpha = \alpha(\delta)$ be a function satisfying $\lim_{\delta \to 0} \alpha(\delta) = 0$ and $\lim_{\delta \to 0} \delta^2 \alpha(\delta)^{-1} = 0$. Moreover, let $\{\delta_k\}$ be a sequence of positive numbers converging to zero and $\{E^{\delta_k}\} \in L^2(\Omega)$ be corresponding noise data satisfying (28). Then, there exists a subsequence, denoted again by $\{\delta_k\}$, and a sequence $\{\alpha_k := \alpha(\delta_k)\}$ such that $(z_{\alpha_k}^1, z_{\alpha_k}^2, \phi_{a\alpha_k}, \phi_{s\alpha_k}, b_{ij,\alpha_k})$ converges in $[L^1(\Omega)]^2 \times [L^2(\Omega)]^2 \times \mathbb{R}^2$ to solution of (34).

**Remark 6.**
1. The set of admissible vector is to be considered as a topological space, namely a subset of $(L^\infty(\Omega))^2 \times (H^1(\Omega))^2 \times \mathbb{R}^2$ endowed with the topology of $(L^1(\Omega))^2 \times (L^2(\Omega))^2 \times \mathbb{R}^2$. In order to guarantee the existence of generalized minimizers of $F_\alpha$ one interesting properties is the closedness of this extended parameter space. It is analyzed in [15, 8].

2. We also remark that the definition of admissible vector (see Definition 1) is constructed in a non-standard manner. However, such definition implies in the closedness of the graph of the Tikhonov functional defined in (35) and hence the existence of a generalized minimizer of the Tikhonov functional $F_\alpha$ in (35).

3. It is worth noticing that, for each $\varepsilon > 0$ the $L^\infty$-functions $H_\varepsilon(\phi_a)$ and $H_\varepsilon(\phi_c)$ in Definition 1 are elements of $D(F)$. Moreover, from the smoothness of the level set functions, they are also in $H^1(\Omega)$. Hence, the Fréchet derivative of the forward operator $F$ in Theorem 8 holds.
4.2.1 Numerical realization of the Tikhonov approach

We remark that the Tikhonov functional $G_{\alpha}$ defined in the previous section is not suitable for computing numerical approximations to the solution of (34). This becomes obvious when one observes the definition of the penalization term $\rho$ in Definition 1.

In this section we introduce the functional $G_{\varepsilon,\alpha}$, which can be used for the purpose of numerical implementations. This functional is defined in such a way that its minimizers are 'close' to the generalized minimizers of $G_{\alpha}$ in a sense that will be made clear later (see Proposition 14). For each $\varepsilon > 0$ we define the functional

$$
G_{\varepsilon,\alpha}(\phi, \phi_s, b_{ij}) := \|F(P_{\varepsilon}(\phi, \phi_s, b_{ij})) - E^s\|_{L^2(\Omega)}^2 + \alpha R_{\varepsilon}(\phi, \phi_s, b_{ij}),
$$

(40)

where

$$
R_{\varepsilon}(\phi, \phi_s, b_{ij}) := \left(||H_{\varepsilon}(\phi)||_{BV(\Omega)} + ||H_{\varepsilon}(\phi_s)||_{BV(\Omega)} + ||(\phi, \phi_s) - (\phi_{a,0}, \phi_s, b_{ij})||_{H^1(\Omega) \times H^1(\Omega)}^2 + ||b_{ij}||_2^2\right)
$$

(41)

and $P_{\varepsilon}(\phi, \phi_s, b_{ij}) := q(H_{\varepsilon}(\phi), H_{\varepsilon}(\phi_s), b_{ij})$ is the functional defined in (36). This functional is well-posed as the following lemma shows:

Lemma 13. [8, Lemma 10] Given $\alpha, \varepsilon > 0$ and $\phi_{a,0}, \phi_s, b_{ij}$ in $H^1(\Omega)$, then the functional $G_{\varepsilon,\alpha}$ in (40) attains a minimizer on $[H^1(\Omega)]^2 \times \mathbb{R}^2$.

The next result guarantees that, for $\varepsilon \to 0$, the minimizers of $G_{\varepsilon,\alpha}$ approximate a generalized minimizer of $G_{\alpha}$.

Proposition 14. [8, Theorem 11] Let $\alpha$, be given. For each $\varepsilon > 0$ denote by $(\phi_{a,\varepsilon,\alpha}, \phi_s, b_{ij,\varepsilon})$ a minimizer of $G_{\varepsilon,\alpha}$. There exists a sequence of positive numbers $\{\varepsilon_k\}$ converging to zero such that $(H_{\varepsilon_k}(\phi_{a,\varepsilon_k,\alpha}), H_{\varepsilon_k}(\phi_s), \phi_{a,\varepsilon_k,\alpha}, \phi_s, b_{ij,\varepsilon_k})$ converges strongly in $[L^1(\Omega)]^2 \times [L^2(\Omega)]^2 \times \mathbb{R}^2$ and the limit is a generalized minimizer of $G_{\alpha}$ in the set of admissible vectors.

Proposition 14 justifies the use functionals $G_{\varepsilon,\alpha}$ in order to obtain numerical approximations to the generalized minimizers of $G_{\alpha}$. It is worth noticing that, differently from $G_{\alpha}$, the minimizers of $G_{\varepsilon,\alpha}$ can be actually computed.

5 A note on the numerical realization

In order to develop a computational scheme to minimize the proposed Tikhonov functionals (30)-(35), one may look for the first order optimality condition, for which gradient or Newton-type algorithms can be implemented, see [21] and references therein. Here we will concentrate our attention to the inner product structure of $L^2(\Omega)$. For the proposed $L^p$ approaches with $p \in [1,2]$ some sub-gradient type algorithm may be used, see [19] and references therein.

Let we first provide a formal derivative for the least square term $J(\mu_a, \mu_s) := \frac{1}{2}\|E^s - F(\mu_a, \mu_s)\|_{L^2(\Omega)}^2$ in the functionals (30)-(35).

Using the same notation of Theorem 8, the derivative of the least square term of $J$ at $(\mu_a, \mu_s) \in D(F) \cap H^1(\Omega)$ in the direction $(\Delta \mu_a, \Delta \mu_s) \in H^1(\Omega)$, can be written as

$$
D J[(\Delta \mu_a, \Delta \mu_s)] = -\langle E^s - F(\mu_a, \mu_s), F'(\mu_a, \mu_s)[(\Delta \mu_a, \Delta \mu_s)] \rangle_{L^2(\Omega)}
$$

(42)
By substituting (25) in the first term in the right hand side of (42), we have

\[
D J[\Delta \mu_a, \Delta \mu_s] = - \langle U(\mu_a, \mu_s)(E^\delta - F(\mu_a, \mu_s)), \Delta \mu_a \rangle_{L^2(\Omega)}
- \langle \mu_a(E^\delta - F(\mu_a, \mu_s)), \int_S u'(\mu_a, \mu_s; s)[\Delta \mu_a, \Delta \mu_s] ds \rangle_{L^2(\Omega)}
\]

(43)

Let \( v \) be the solution of the adjoint problem

\[
T^*v = \mu_a(E^\delta - F(\mu_a, \mu_s)),
\]

(44)

with absorbing boundary conditions.

The substitution of (44) into equation (43) produces

\[
D J[\Delta \mu_a, \Delta \mu_s] = - \langle U(\mu_a, \mu_s)(E^\delta - F(\mu_a, \mu_s)), \Delta \mu_a \rangle_{L^2(\Omega)}
- \langle T^*v, \int_S u'(\mu_a, \mu_s; s')[\Delta \mu_a, \Delta \mu_s] ds' \rangle_{L^2(\Omega)}
\]

(45)

Since \( T^*v \) does not depend on the direction \( s' \), equation (46) can be written equivalently as

\[
D J[\Delta \mu_a, \Delta \mu_s] = - \langle U(\mu_a, \mu_s)(E^\delta - F(\mu_a, \mu_s)), \Delta \mu_a \rangle_{L^2(\Omega)}
- \langle T^*v, u'(\mu_a, \mu_s; s')[\Delta \mu_a, \Delta \mu_s] \rangle_{L^2(D)}
\]

(46)

Assume that \( u'(\mu_a, \mu_s) = 0 \) on \( \partial\Omega \). From the divergent theorem, we can state that

\[
0 = \int_{\partial \Omega} (s \cdot \eta)v(x,s)u'(x,s)[\Delta \mu_a, \Delta \mu_s] dx = \int_{\Omega} (s \cdot \nabla)(v(x,s)u'(x,s)[\Delta \mu_a, \Delta \mu_s]) dx
= \int_{\Omega} v(x,s)(s \cdot \nabla) u'(x,s)[\Delta \mu_a, \Delta \mu_s] + u'(x,s)[\Delta \mu_a, \Delta \mu_s](s \cdot \nabla)v(x,s) dx
= \langle v, Tu' \rangle_{L^2(D)} - \langle T^*v, u' \rangle_{L^2(D)}
\]

(47)

From, (26) we have

\[
\langle v, Tu' \rangle_{L^2(D)} = \int_D v(x,s) \left( (\Delta \mu_a + \Delta \mu_s)u(x,s) - \Delta \mu_s \int_S \Theta(s,s')u(x,s') ds' \right) dx ds
= \langle uv, \Delta \mu_a + \Delta \mu_s \rangle_{L^2(D)} - \langle u \int_S \Theta(s,s')v(\cdot, s') ds', \Delta \mu_s \rangle_{L^2(D)}
\]

(48)

Putting (43) - (48) together we have

\[
D J[\Delta \mu_a, \Delta \mu_s] = - \langle U(E^\delta - F(\mu_a, \mu_s)), \Delta \mu_a \rangle_{L^2(\Omega)} + \langle uv, \Delta \mu_a \rangle_{L^2(D)}
+ \langle uv, \Delta \mu_s \rangle_{L^2(D)} - \langle u \int_S \Theta(s,s')v(\cdot, s') ds', \Delta \mu_s \rangle_{L^2(D)}.
\]

(49)

A quickly calculation shows that the derivative of the regularization term satisfies

\[
\langle (I - \Delta)(\mu_a - \mu_{a,0}), \Delta \mu_a \rangle_{L^2(\Omega)} + \langle (I - \Delta)(\mu_s - \mu_{s,0}), \Delta \mu_s \rangle_{L^2(\Omega)}
= - \int_{\partial\Omega} \frac{\partial}{\partial \eta} (\mu_a - \mu_{a,0}) \Delta \mu_a dx - \int_{\partial\Omega} \frac{\partial}{\partial \eta} (\mu_s - \mu_{s,0}) \Delta \mu_s dx.
\]

(50)
Hence, the gradient of $J_{\alpha}$ with respect to the absorption and scattering coefficients can be formally written as
\[
\frac{\partial J_{\alpha}}{\partial \mu_a}(\mu_a, \mu_s) = -U(\mu_a, \mu_s)(E^a - F(\mu_a, \mu_s)) + \int_S u(\cdot, s)v(\cdot, s)ds + 2\alpha(I - \Delta)(\mu_a - \mu_{a,0}), \quad (51)
\]
\[
\frac{\partial J_{\alpha}}{\partial \mu_s}(\mu_a, \mu_s) = \int_S u(\cdot, s)v(\cdot, s)ds - \int_S \int_S u(\cdot, s)\Theta(s, s')v(\cdot, s)dsds' + 2\alpha(I - \Delta)(\mu_s - \mu_{s,0}), \quad (52)
\]
subject to the homogeneous Neumann boundary conditions
\[
\frac{\partial}{\partial \eta}(\mu_a - \mu_{a,0}) = 0, \quad \frac{\partial}{\partial \eta}(\mu_s - \mu_{s,0}) = 0, \quad (53)
\]
respectively.

In [21] limited-memory BFGS was used to solve (51) while a finite element model of the (RTE)-equation (1) was then used to determine the optical absorption and scattering coefficients.

## 5.1 Optimality conditions for the Tikhonov functional $G_{\varepsilon, \alpha}$

For the numerical implementation of the level set approach is necessary to derive the first order optimality conditions for a minimizer of the functionals $G_{\varepsilon, \alpha}$. With this finality, we consider $G_{\varepsilon, \alpha}$ in (40) and we look for the Gâteaux directional derivatives with respect to $\phi_a, \phi_s$. For easiness of presentation, we assume here that the constant values $b_{ij}$ are known. An algorithm for unknown constant values was implemented in [7].

Given the composition of $P_s$ with the forward operator $F$ in the level set approach and the self-adjointness$^2$ of $H'_\varepsilon(\varphi)$, the optimality conditions for a minimizer of the functional $G_{\varepsilon, \alpha}$ can be written in the form of the system of equations
\[
\frac{\partial}{\partial \phi_a}G_{\varepsilon, \alpha}(\phi_a, \phi_s) = L_{\varepsilon, \alpha}^1(\phi_a, \phi_s) + \alpha(I - \Delta)(\phi_a - \phi_{a,0}), \quad (54a)
\]
\[
\frac{\partial}{\partial \phi_a}G_{\varepsilon, \alpha}(\phi_a, \phi_s) = L_{\varepsilon, \alpha}^2(\phi_a, \phi_s) + \alpha(I - \Delta)(\phi_s - \phi_{s,0}), \quad (54b)
\]
with the homogeneous Neumann boundary condition
\[
\frac{\partial}{\partial \eta}(\phi_a - \phi_{a,0}) = 0, \quad \frac{\partial}{\partial \eta}(\phi_s - \phi_{s,0}) = 0 \quad (55)
\]
where
\[
L_{\varepsilon, \alpha}^1(\phi_a, \phi_s) = (a_1 - a_2)H'_{\varepsilon}(\phi_a) \frac{\partial J}{\partial \phi_a}(P(\phi_a, \phi_s, b_{ij})) + \alpha \left[ H'_{\varepsilon}(\phi_a) \nabla \cdot \left( \frac{\nabla H_{\varepsilon}(\phi_a)}{|\nabla H_{\varepsilon}(\phi_a)|} \right) \right] \quad (56a)
\]
\[
L_{\varepsilon, \alpha}^2(\phi_a, \phi_s) = (c_1 - c_2)H'_{\varepsilon}(\phi_s) \frac{\partial J}{\partial \phi_s}(P(\phi_a, \phi_s, b_{ij})) + \alpha \left[ H'_{\varepsilon}(\phi_s) \nabla \cdot \left( \frac{\nabla H_{\varepsilon}(\phi_s)}{|\nabla H_{\varepsilon}(\phi_s)|} \right) \right] \quad (56b)
\]
and $\frac{\partial J}{\partial \phi_a}(P(\phi_a, \phi_s, b_{ij})) \frac{\partial J}{\partial \phi_a}(P(\phi_a, \phi_s, b_{ij}))$ are obtained analogous to (49) for absorption and scattering coefficients parameterized by $P_\varepsilon(\phi_a, \phi_s, b_{ij})$.

$^2$Notice that $H'_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon} & t \in (-\varepsilon, 0) \\ 0 & \text{else} \end{cases}$.
5.2 The real case: Multiple illumination positions

As it is well known for the diffusive approximation for (PAT), nonuniqueness may be encountered when both absorption and scattering coefficients are to be recovered, without include additional information into the problem [5, 3]. One additional information generally used to avoid the nonuniqueness is using a multiple-illumination approach, whereby a set of images are obtained using sources placed at different positions around the image domain. It was also reported in [3] that the multiple-illumination approach improves the ill-posedness of the (QPAT) inverse problem.

A few theoretical modification of the presented approach imply the stated results. Indeed, if \( N_m \) is the number of source positions, then one alternative is looking for the Tikhonov-type functionals (30) - (35) with the misfit replaced by

\[
\sum_{m=1}^{N_m} \frac{1}{p} \left\| E_m^\delta - F_m(\mu_a, \mu_s) \right\|^p_{L^p(\Omega)} \quad \text{and} \quad \sum_{m=1}^{N_m} \frac{1}{2} \| E_m^\delta - F_m(\phi_a, \phi_s, b_{ij}) \|^2_{L^2(\Omega)},
\]

respectively. Another alternative is write the problem as a system of nonlinear operator equations

\[
F_m(\mu_a, \mu_s) = E_m^\delta, \quad m = 1 \cdots N_m,
\]

and then uses a Kaczmarz-type strategy [10] for regularize the problem.

6 Conclusions and future directions

Existence and stability of approximated solution have been shown to successfully determine the absorption and scattering coefficients in the (RTE) model for (QPAT), using Tikhonov-type regularization approaches. Sufficient conditions to obtain the regularization properties of the approximated solution has been shown by proving properties of the forward problem. The results concern with different topologies that includes the physical and numerical issues. Although we do not present any implementation, our results imply in the theoretical guarantee of regularization properties of the approach presented in [21]. Using a priori information of the regularity of the coefficient, we propose different Tikhonov-type regularization. We are confident that the level set approach for piecewise continuous coefficient can improve significatively the results in [21].

Since PAT is a new imaging modality, many theoretical and computational issues are still open, e.g. [18, 21, 24, 25, 22, 5, 4, 3]. The numerical implementation of the regularization approaches that has been proposed in this contribution will be the subject of future work. In particular, it will bring the discussion on the numerical issue in a real PAT image [21]. As far as the authors known, iterative regularization [16, 13] was not attempted in the PAT context. It shall be considered in future works.

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