Commuting quantum transfer matrix approach to intrinsic Fermion system: Correlation length of a spinless Fermion model

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The quantum transfer matrix (QTM) approach to integrable lattice Fermion systems is presented. As a simple case we treat the spinless Fermion model with repulsive interaction in critical regime. We derive a set of non-linear integral equations which characterize the free energy and the correlation length of \( \langle c_j^\dagger c_i \rangle \) for arbitrary particle density at any finite temperatures. The correlation length is determined by solving the integral equations numerically. Especially in low temperature limit this result agrees with the prediction from conformal field theory (CFT) with high accuracy.

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I. INTRODUCTION

Exact evaluations of physical quantities at finite temperatures pose serious difficulties even for integrable models. One has to go much beyond mere diagonalization of a Hamiltonian; summation over the eigenspectra must be performed.

The string hypothesis\[1\] brought the first breakthrough and success. It yields a systematic way to evaluate several bulk quantities including specific heats, susceptibilities and so on.

More recently, the quantum transfer matrix (QTM) method has been proposed to overcome some difficulties to which the standard approach is not applicable.\[2\] One reduces the original problem to finding the largest eigenvalue of the QTM which acts on a fictitious system of size \( N \) (referred to as the Trotter number), which should be sent \( N \to \infty \) (Trotter limit). As this procedure is sometimes difficult, we integrate its procedure with another ingredient, the integrable structure of the underlying model. It allows for introduction of the commuting QTMs which are labeled by complex parameter \( x \). A set of auxiliary functions, including the QTM itself, satisfy certain functional relations. We shall choose these functions such that they have a nice analytical property called ANZC (Analytic, NonZero, and Constant asymptotics, see Sec. III) in a certain strip on the complex \( x \) plane. This admits the transformation of the functional relations into a closed set of the integral equations. For all cases known up to now, the Trotter limit \( N \to \infty \) can be taken analytically in the integral equations. We thus have seen a remarkable reduction from the problem of combinatorics (summation over the eigenspectra) to the study of analytic structures of suitably chosen auxiliary functions.

This novel scenario has been applied to many models of physical interest\[3\]. Especially, the correlation lengths, of which calculation have been one of the major difficulty in the string hypothesis, are explicitly evaluated in the spin models. For example of the success, we refer to recent analysis on the quantum-classical crossover phenomena in the massless XXZ model in “attractive” regime.\[4,5\]

We extend these studies to lattice Fermion systems. Our formulation is fully general for the 1D Fermion systems which are integrable in the sense of the Yang-Baxter (YB) equation. As a concrete example, we take the spinless Fermion model with repulsive interactions in gapless regime. This simple example already manifests some fundamental differences from the spin models, and yields a sound basis for the future studies on more realistic Fermion systems such as the Hubbard model.

As in Refs.\[6\],\[7\],\[8\], first one may perform the Jordan-Wigner (JW) transformations to the Fermion models and further convert the resultant quantum spin models into 2D classical vertex models. These procedures have been successful in studies of the bulk quantities. In evaluating correlation lengths, however, this is no longer true. As an example, which will be discussed in the main body of this paper, let us take Fermion one-particle Green’s function \( \langle c_j^\dagger c_i \rangle \) and its correspondent \( \langle \sigma_j^+ \sigma_i^- \rangle \) in the spin model. Obviously they are related, but quite different by nonlocal terms due to the JW transformation. At zero temperature \( (T = 0) \), using the conformal mapping, one evaluates the scaling dimensions from the finite size corrections to the energy spectra. As the Hamiltonians are equivalent through the JW transformations, it is normally difficult to discriminate between the energy spectra of the Fermions and those of the spins. The difference lies only in the boundary conditions. Nevertheless, even after JW transformation one can explicitly calculate the correct scaling dimensions only by incorporating the proper Fermion statistics at the very last stage (see Appendix B). At finite temperature \( (T > 0) \), the QTM approach gives the correlation function in the spectral decomposition form as \( \sum_k |A_k|^2 (\Lambda_k/\Lambda_1)^z \). Here \( \Lambda_k \) de-
notes the $k$-th largest eigenvalue of the QTM and $A_k$ is a certain matrix element. Once the JW transformation is performed, it is difficult to trace the difference in the integrability and other algebraic structures of the Fermion systems have been discussed successfully recently.

The formulation, however, has a severe problem in applying to the finite temperature case. We must treat the quantum and the auxiliary spaces on the same footing when constructing the QTM. On the contrary, in the graded YB relation the quantum space is the Fermion Fock space, while the auxiliary space is the (graded) vector space.

To overcome these difficulties, we adopt another approach to the Fermion systems, which was invented quite recently. In this method, we consider an $R$-operator consisting of the Fermion operators alone, together with its “super-transposition”. This time both quantum and auxiliary spaces are Fermion Fock spaces. Therefore we can, for instance, exchange their roles with no difficulty. Actually, by careful introduction of the super-trace and interchange of it with the normal trace to the partition function, we can derive the commuting QTM for the Fermion systems.

The resultant QTM preserves genuine Fermion statistics. In other words, the selection rule is already built-in algebraically. This proper treatment for the statistics results in a change of the analytic structure for the QTM. In the “physical strip”, the QTM has only one additional zero which characterizes “excited free energy” at finite $T$, while in the corresponding spin model there appear two such zeros. Consequently, one observes a $T$-dependent oscillating behavior of one-particle Green’s function, as well as the difference in the correlation length between the Fermion model and the corresponding spin model. These are smoothly connected to the expected values at the CFT limit, $T \to 0$ (see Appendix B).

This paper is organized as follows. In the next section, we will present the commuting QTM formulation of the spinless Fermion model at $T > 0$. The Fermionic $R$-operator, together with its “super-transposition” $\tilde{R}$, play fundamental roles. The analytic structure of the QTM and the auxiliary functions are discussed in Sec. III, which leads to the nonlinear integral equations (NLIE) characterizing the correlation length. The limit $T \to 0$ is treated analytically at the “half-filling” ($n_e = 0.5$), which recovers the prediction from CFT. We also perform numerical investigations on NLIE and the correlation length for one-particle Green’s function. To our knowledge, this is the first exact computation of the correlation length for various interaction strengths, electron filling and for wide range of temperatures. In Sec. IV, we comment on alternative forms of NLIE derived from different choice of the auxiliary functions. They are akin to the standard “thermodynamic Bethe ansatz (TBA) equations” from the string hypothesis, thus may be of their own interest. Details of calculations and supplementary knowledge on CFT are summarized in appendices.

II. COMMUTING QUANTUM TRANSFER MATRIX FOR THE SPINLESS FERMION MODEL

In this section we formulate the commuting QTM for the spinless Fermion model. The formulation is based on the recent developments in the study of the integrability of the lattice Fermion systems. The central role is played by an operator solution of the YB equation called the Fermionic $R$-operator. The “transfer matrix” can be constructed from the $R$-operator, which generates the left-shift operator, the Fermionic Hamiltonian and other conserved operators. Here and in Sec. II A, we briefly describe the method.

To extend the method to the finite temperature case utilizing the Trotter formula, it is necessary to look for another transfer matrix which generates right-shift operator and the Hamiltonian. In Sec. II B, we shall show how to construct the desired transfer matrix by considering the super-transposition of the $R$-operator.

Based on these two kinds of the transfer matrices, we devise the QTM for the Fermion model in Sec. II C. The QTM constitutes a one-parameter commuting family, which is a consequence of the global YB relation. The YB relation also enables us to diagonalize the QTM by means of the algebraic Bethe ansatz. The free-energy and the correlation length are expressed in terms of the eigenvalues of the QTM.

A. Fermionic $R$-Operator

We define the spinless Fermion model by the Hamiltonian

$$\mathcal{H} := \sum_{j=1}^{L} \mathcal{H}_{j,j+1}$$

$$\mathcal{H}_{j,j+1} := \frac{t}{2} \left\{ c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j + 2\Delta \left( n_j - \frac{1}{2} \right) \left( n_{j+1} - \frac{1}{2} \right) \right\}, \quad (2.1)$$

where $c_j^\dagger$ and $c_j$ are the Fermionic creation and annihilation operators at the $j$-th site satisfying the canonical anti-commutation relations

$$\{c_j, c_k\} = \{c_j^\dagger, c_k^\dagger\} = 0, \quad \{c_j^\dagger, c_k\} = \delta_{jk}. \quad (2.2)$$
We assume the periodic boundary condition (PBC) on the Fermion operators,
\[
c_i^{\dagger}_{L+1} = c_1, \quad c_{L+1} = c_1. \tag{2.3}
\]
The parameters \( t, \Delta \) are real coupling constants. In the present paper we consider the repulsive critical region \( 0 \leq \Delta < 1, \quad 0 < t \) and introduce the parametrization
\[
\Delta := \cos 2\eta, \quad 0 < 2\eta \leq \frac{\pi}{2}. \tag{2.4}
\]
In the subsequent sections, we shall also use the parameter \( p_0 \) defined by
\[
p_0 := \frac{\pi}{2\eta}. \tag{2.5}
\]
Hereafter we set \( t = 1 \) for simplicity.

The model (2.1) is exactly solved by the Bethe ansatz method. Since the Hamiltonian (2.1) preserves the number of the particles, we can add the “chemical potential” term without breaking the integrability
\[
\mathcal{H}_{\text{chemical}} := \mu \sum_{j=1}^{L} \left( n_j - \frac{1}{2} \right). \tag{2.6}
\]
However we consider only the case \( \mu = 0 \) for a while.

The several physical properties including the integrability of the Fermion model (2.1) has been discussed by transforming it into the XXZ model
\[
H = \frac{1}{4} \sum_{j=1}^{L} \left\{ \sigma_j^x \sigma_j^x + \sigma_j^y \sigma_j^y + \Delta \sigma_j^z \sigma_{j+1}^z \right\}, \tag{2.7}
\]
through the JW transformation. However it was recently discovered that we can treat the Fermion model (2.1) only with the Fermion operators. We shall summarize the method in what follows.

First let us consider a two-dimensional Fermion Fock space \( V_j \), a basis of which is given by
\[
|0\rangle_j, \quad |1\rangle_j := c_j^\dagger |0\rangle_j, \quad c_j|0\rangle_j = 0. \tag{2.8}
\]

Define the Fermionic \( R \)-operator acting on the tensor product of the Fermion Fock spaces \( V_j \otimes V_k \) by
\[
R_{jk}(v) := a(v) \left\{ -n_jk + (1 - n_j)(1 - n_k) \right\} + b(v) \left\{ n_j(1 - n_k) + (1 - n_j)n_k \right\} + c(v) (c_j^\dagger c_k - c_j c_k^\dagger), \tag{2.9}
\]
where
\[
a(v) := \frac{\sin \eta (v + 2)}{\sin 2\eta}, \quad b(v) := \frac{\sin \eta v}{\sin 2\eta}, \quad c(v) := 1. \tag{2.10}
\]
A basis of \( V_j \otimes V_k \) is given by
\[
|0\rangle_j \otimes |0\rangle_k := |0\rangle, \quad |1\rangle_j \otimes |0\rangle_k := c_j^\dagger |0\rangle, \quad |0\rangle_j \otimes |1\rangle_k := c_k^\dagger |0\rangle, \quad |1\rangle_j \otimes |1\rangle_k := c_j^\dagger c_k^\dagger |0\rangle, \tag{2.11}
\]
and we can calculate the matrix elements of (2.9) if necessary. However we keep the operator form (2.9) as much as possible and avoid the use of the matrix elements, because the former is more transparent. The \( R \)-operator (2.9) satisfies the following YB equation
\[
R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v). \tag{2.12}
\]
The equation (2.12) is an operator identity and one should carefully use the anti-commutation relations (2.2) to confirm its validity.

It is one of the fundamental properties of the \( R \)-operator \( R_{ij}(v) \) that \( R_{ij}(0) = P_{ij} \) is the permutation operator for the Fermion operators,
\[
P_{jk} := (1 - n_j)(1 - n_k) - n_j n_k + c_j^\dagger c_k - c_j c_k^\dagger, \quad P_{jk} x_j = x_k P_{jk}, \quad (x_j = c_j \text{ or } c_j^\dagger). \tag{2.13}
\]
We can define an analog of the transfer matrix by
\[
T(v) := \text{Str}_a \{ R_{aL}(v) \cdots R_{a1}(v) \}. \tag{2.14}
\]
Here the super-trace of an arbitrary operator \( X \) is defined by
\[
\text{Str}_a X := a(0|X|0) - a(1|X|1), \tag{2.15}
\]
where the dual Fermion Fock space is spanned by \( a(0| \) and \( a(1| \) with
\[
a(0|c_a^\dagger = 0, \quad a(1| := a(0|c_a. \tag{2.16}
\]
We also assume
\[
a(0| = a(1| = 1. \tag{2.17}
\]
The super-trace (2.15) corresponds to the PBC for the Fermion operators (2.3) satisfies the property
\[
\text{Str}_a \{ R_{aL}(v) \cdots R_{a1}(v) \} = \text{Str}_a \{ R_{a1}(v)R_{aL}(v) \cdots R_{a2}(v) \}. \tag{2.18}
\]
Hereafter we call (2.14) the transfer matrix for simplicity.

As in the case with the integrable spin models, the YB equation (2.12) ensures the commutativity of the transfer matrices (2.13)
\[
[T(v), T(v')] = 0. \tag{2.19}
\]
The expansion of the transfer matrix (2.13) with respect to the spectral parameter \( v \) is given by
\[
T(v) = T(0) \left\{ 1 + \frac{2\eta}{\sin 2\eta} \left( \mathcal{H} + \frac{L}{4} \Delta \right) v + \mathcal{O}(v^2) \right\}. \tag{2.20}
\]
which follows from the relationship
\[
\frac{d\mathcal{R}_{aj}(v)}{dv} \bigg|_{v=0} \mathcal{P}_{a,j}^{-1} = \frac{2\eta}{\sin 2\eta} \mathcal{P}_{aj} \mathcal{P}_{a,j-1} \left( \mathcal{H}_{j-1,j} + \frac{1}{4} \Delta \right). \tag{2.21}
\]

Note that the operator \( T(0) = \text{Str}_a \{ \mathcal{P}_{al} \cdots \mathcal{P}_{a1} \} \) is the left-shift operator
\[
T(0)x_j = x_{j+1}T(0), \quad (x_j = c_j \text{ or } c_j^\dagger). \tag{2.22}
\]

One can easily prove the relation (2.23) utilizing the property of the permutation operator,
\[
\mathcal{P}_{a,j+1} \mathcal{P}_{aj} x_j = x_{j+1} \mathcal{P}_{a,j+1} \mathcal{P}_{aj}, \quad (x_j = c_j \text{ or } c_j^\dagger). \tag{2.23}
\]

**B. Super-Transposed Fermionic R-Operator**

In this section, we shall consider another transfer matrix which generates the right-shift operator. For this purpose we first define the super-transposition \( st_j \) for an arbitrary operator \( X_j(v) \) in the form
\[
X_j(v) = A(v)(1 - n_j) + D(v)n_j + B(v)c_j + C(v)c_j^\dagger, \tag{2.24}
\]

by
\[
X_j^{st_j}(v) := A(v)(1 - n_j) + D(v)n_j + B(v)c_j^\dagger - C(v)c_j. \tag{2.25}
\]

Here \( A(v) \) and \( D(v) \) (\( B(v) \) and \( C(v) \)) are assumed to be Grassmann even (odd) operators.

Now applying the super-transposition \( st_j \) to both sides of the YB equation (2.12), we obtain
\[
\mathcal{R}_{13}^{st_1}(u) \mathcal{R}_{12}^{st_1}(u - v) \bar{\mathcal{R}}_{23}(v) = \mathcal{R}_{23}(v) \mathcal{R}_{12}^{st_1}(u - v) \mathcal{R}_{13}^{st_1}(u), \tag{2.26}
\]

where we have used a property of the super-transposition
\[
(\mathcal{R}_{jk}(u) \mathcal{R}_{jl}(v))^{st_l} = \mathcal{R}_{jl}^{st_l}(v) \mathcal{R}_{jk}^{st_l}(u), \quad (k \neq l). \tag{2.27}
\]

Then changing suffixes and spectral parameters as
\[
1 \rightarrow 3, \quad 2 \rightarrow 1, \quad 3 \rightarrow 2, \quad u \rightarrow -v, \quad v \rightarrow u - v, \tag{2.28}
\]
we get the following new type of the YB equation
\[
\bar{\mathcal{R}}_{12}(u - v) \bar{\mathcal{R}}_{13}(u) \bar{\mathcal{R}}_{23}(v) = \bar{\mathcal{R}}_{23}(v) \bar{\mathcal{R}}_{13}(u) \bar{\mathcal{R}}_{12}(u - v), \tag{2.29}
\]

where
\[
\bar{\mathcal{R}}_{jk}(v) := \mathcal{R}_{jk}^{st_k}(v) = a(v) \{ -n_jn_k + (1 - n_j)(1 - n_k) \} \tag{2.30}
\]
\[
+ b(v) \{ n_j(1 - n_k) + (1 - n_j)n_k \} \tag{2.30}
\]
\[
- c(v)(c_j^\dagger c_k^\dagger + c_j c_k). \tag{2.30}
\]

Although the new \( R \)-operator \( \bar{\mathcal{R}}_{jk}(v) \) is not symmetric (\( \bar{\mathcal{R}}_{jk}(v) \neq \bar{\mathcal{R}}_{kj}(v) \)), it is still possible to prove the relation
\[
\bar{\mathcal{R}}_{12}(u - v) \bar{\mathcal{R}}_{13}(u) \bar{\mathcal{R}}_{23}(v) = \mathcal{R}_{13}(u) \bar{\mathcal{R}}_{12}(u - v). \tag{2.31}
\]

Using \( \mathcal{R}_{aj}(v) \), we define another transfer matrix by
\[
\bar{T}(v) := \text{Str}_a \left\{ \bar{\mathcal{R}}_{al}(v) \cdots \mathcal{R}_{a1}(v) \right\}. \tag{2.32}
\]

Then the commutative properties of the transfer matrices follow from the YB equations (2.29) and (2.31),
\[
\left[ T(v), \bar{T}(v') \right] = \left[ \bar{T}(v), \bar{T}(v') \right] = 0. \tag{2.33}
\]

The following remarkable relations hold
\[
\bar{\mathcal{P}}_{aj} \mathcal{P}_{a,j-1} x_j = x_{j-1} \mathcal{P}_{a,j} \mathcal{P}_{a,j-1}, \quad (x_j = c_j \text{ or } c_j^\dagger), \tag{2.34}
\]

where
\[
\bar{\mathcal{P}}_{jk} := \mathcal{R}_{jk}(0) \tag{2.35}
\]
\[
= (1 - n_j)(1 - n_k) - n_jn_k - (c_j^\dagger c_k^\dagger + c_j c_k). \tag{2.35}
\]

Using the relations (2.34), one can confirm that the operator \( \bar{T}(0) \) provides the right-shift operator, i.e.,
\[
\bar{T}(0)x_j = x_{j-1} \bar{T}(0), \quad (x_j = c_j \text{ or } c_j^\dagger). \tag{2.36}
\]

In other words, \( \bar{T}(0) \) is the inverse of \( T(0) \)
\[
\bar{T}(0) \bar{T}(0) = 1. \tag{2.37}
\]

Furthermore, from the relationship
\[
\mathcal{P}_{a,j+1} \frac{d\bar{\mathcal{R}}_{aj}(v)}{dv} \bigg|_{v=0} = -\frac{2\eta}{\sin 2\eta} \mathcal{P}_{a,j+1} \mathcal{P}_{aj} \left( \mathcal{H}_{j+1,j} + \frac{1}{4} \Delta \right), \tag{2.38}
\]

the expansion of the transfer matrix \( \bar{T}(v) \) with respect to the spectral parameter \( v \) is given by
\[
\bar{T}(v) = \bar{T}(0) \left\{ 1 - \frac{2\eta}{\sin 2\eta} \left( \mathcal{H} + \frac{L}{4}\Delta \right) v + \mathcal{O}(v^2) \right\}. \tag{2.39}
\]
C. Commuting Quantum Transfer Matrix

The expansions (2.26) and (2.39) with the relation (2.37) are combined into a formula

\[
T(u)\bar{T}(-u) = 1 + \frac{4\eta}{\sin 2\eta} \left( \mathcal{H} + \frac{L}{4} \Delta \right) u + \mathcal{O}(u^2).
\]  

(2.40)

This facilitates the investigation of the finite temperature properties of the spinless Fermion model (2.1) via the Trotter formula,

\[
\exp \left( -\beta \left( \mathcal{H} + \frac{L}{4} \Delta \right) \right) = \lim_{N \to \infty} \left( T(u_N)\bar{T}(-u_N) \right)^{N/2},
\]

\[
u_N = -\frac{\beta \sin 2\eta}{2\eta N}.
\]  

(2.41)

Here an (even) integer \(N\) called the Trotter number, represents the number of sites in the fictitious Trotter direction and \(\beta\) is the inverse temperature \(\beta = 1/T\).

The free energy per site, for instance, is given by

\[
f = -\lim_{L \to \infty} \lim_{N \to \infty} \frac{1}{L\beta} \ln \text{Tr} \left( T(u_N)\bar{T}(-u_N) \right)^{N/2} - \frac{1}{4}\Delta.
\]  

(2.42)

However, as is the case with the corresponding spin model, the eigenvalues of \(T(u_N)\bar{T}(-u_N)\) are infinitely degenerate in the limit \(N \to \infty\).

Therefore it is a formidable task to take the trace in this limit. To avoid this difficulty, we transform the term

\[
\text{Tr} \left( T(u_N)\bar{T}(-u_N) \right)^{N/2}
\]

in (2.42) as follows:

\[
\text{Tr} \left( T(u_N)\bar{T}(-u_N) \right)^{N/2} = \text{Str} \prod_{m=1}^{N/2} \text{Str} a_{2m} a_{2m-1} \left[ \mathcal{R}_{a_{2m}, L}(u_N) \cdots \mathcal{R}_{a_{2m},1}(u_N) \right.
\]

\[
\times \mathcal{R}_{a_{2m-1}, L}(-u_N) \cdots \mathcal{R}_{a_{2m-1},1}(-u_N) \left. \right],
\]

\[
= \text{Str} \prod_{j=1}^{L} \prod_{m=1}^{N/2} \mathcal{R}_{a_{2m},j}(u_N) \mathcal{R}_{a_{2m-1},j}(-u_N).
\]  

(2.43)

We now introduce a fundamental object in the present approach called the quantum transfer matrix (QTM)

\[
T_{\text{QTM}}(u_N, v) := \text{Tr}_j T_j(u_N, v),
\]  

(2.44)

where the monodromy operator \(T_j(u_N, v)\) is defined by

\[
T_j(u_N, v) := \prod_{m=1}^{N/2} \mathcal{R}_{a_{2m},j}(v+u_N) \mathcal{R}_{a_{2m-1},j}(v-u_N).
\]  

(2.45)

Using the YB equations (2.12) and (2.29), we can show that the monodromy operator satisfies the global YB relation

\[
\mathcal{R}_{21}(v-v')\mathcal{R}_{1}(u_N, v)\mathcal{R}_{2}(u_N, v') = \mathcal{R}_{2}(u_N, v')\mathcal{R}_{1}(u_N, v)\mathcal{R}_{21}(v-v').
\]  

(2.46)

Accordingly the QTM constitutes a commuting family

\[
[T_{\text{QTM}}(u_N, v), T_{\text{QTM}}(u_N, v')] = 0.
\]  

(2.47)

We remark that the trace in the definition of the QTM (2.44) implies the anti-periodic boundary condition for the Fermion operators in the Trotter direction, i.e.,

\[
c_{a_{N+1}} = -c_{a_1}, \quad c_{a_{N+1}}^\dagger = -c_{a_1}^\dagger.
\]  

(2.48)

The free energy per site (2.42) is then represented in terms of the QTM as

\[
f = -\lim_{L \to \infty} \lim_{N \to \infty} \frac{1}{L\beta} \ln \text{Str} \left( T_{\text{QTM}}(u_N, 0)L \right)^{N/2} - \frac{1}{4}\Delta.
\]  

(2.49)

Since the two limits in (2.49) are exchangeable, we take the limit \(L \to \infty\) first. Because there is a finite gap between the first and the second largest eigenvalue of the QTM for finite temperature, we can write

\[
f = -\frac{1}{\beta} \lim_{N \to \infty} \ln \Lambda_1 - \frac{1}{4}\Delta.
\]  

(2.50)

where \(\Lambda_1\) is the first largest eigenvalue of the QTM \(T_{\text{QTM}}(u_N, 0)\). From now on \(\Lambda_k\) denotes the \(k\)-th largest eigenvalue of the QTM. The correlation length \(\xi\) of the correlation function \(\langle c_i^\dagger c_k \rangle\) can also be represented in terms of the first and the second largest eigenvalues \(\Lambda_2\) as

\[
\xi^{-1} = -\lim_{N \to \infty} \ln \left| \frac{\Lambda_2}{\Lambda_1} \right|.
\]  

(2.51)

In this way the calculation of certain thermal quantities reduces to the evaluation of the eigenvalues of the QTM in the Trotter limit \((N \to \infty)\).

For \(N\) finite, it is possible to diagonalize the QTM (2.44) by means of the algebraic Bethe ansatz (see Appendix A). The eigenvalue is then given by

\[
\Lambda(x) = \lambda_1(x) + \lambda_2(x),
\]

\[
\lambda_1(x) := \phi_+(x)\phi_-(x-2i)\frac{Q(x+2i)}{Q(x)} e^{\beta \mu/2},
\]

\[
\lambda_2(x) := (-1)^{N/2+N_e} \phi_-(x)\phi_+(x+2i)\frac{Q(x-2i)}{Q(x)} e^{-\beta \mu/2},
\]  

(2.52)

where

\[
\phi_\pm(x) := \left( \frac{\sinh \eta(x \pm i u_N)}{\sin 2\eta} \right)^{N/2},
\]

\[
Q(x) := \prod_{j=1}^{N_e} \sinh \eta(x - x_j).
\]  

(2.53)
Here we have changed the spectral parameter from $v$ to $x$ defined by $v = ix$ for later convenience. Note that we have also included the contribution from the chemical potential term $2i\hbar$ in the expression (2.43).

The associated Bethe ansatz equation (BAE) is given by

$$
\left( \frac{\phi_+(x)\phi_-(x-2i)}{\phi_-(x)\phi_+(x+2i)} \right)^{N/2} = -(-1)^{N/2+N_e}e^{-\beta\mu} \prod_{j=1}^{N_e} \frac{Q(x_j-2i)}{Q(x_j+2i)}
$$

Compared with the XXZ model, we observe an extra factor $(-1)^{N/2+N_e}$ in (2.52) and (2.54) which reflects the Fermionic nature of the present system. In particular, if $N/2 + N_e \equiv 1 \pmod{2}$, the Eqs. (2.52) and (2.54) are clearly different from the corresponding ones for the XXZ model. Actually the second largest eigenvalue lies in the sector $N_e = N/2 - 1$, while the first largest one is in the sector $N_e = N/2$. Therefore the correlation length $\xi$ (2.51) exhibits the manifest difference between the Fermion system (2.1) and the spin system (2.7).

III. NLIE AND THE EXACT ENUMERATION OF CORRELATION LENGTH

A. Analyticities of Auxiliary Functions and NLIE

In order to proceed further, one needs to clarify the analytic property of the QTM. For this purpose, we perform numerical investigations by fixing the Trotter number $N$ finite.

First we give the description for the largest eigenvalue sector, which is naturally identical to the corresponding XXZ model. There are $N_e = N/2$ BAE roots. Only at the “half-filling”, they distribute exactly on the real axis and become symmetric with respect to the imaginary axis. The QTM has $N$ zeros in $3\pi x \in [-2\eta_0, 2\eta_0]$: $N/2$ zeros locate on the smooth curve $3\pi x \sim 2$, and the other $N/2$ zeros are on the curve $3\pi x \sim -2$. Thus there is a strip $3\pi x \in [-1, 1]$ where the QTM is analytic and nonzero. We call this “physical strip”.

Next consider the excited state relevant to the second largest eigenvalue. In contrast to the XXZ model, we find that two complex eigenvalues are degenerate in magnitude. Both of them are characterized by $N_e = N/2 - 1$ BAE roots located on a smooth curve near the real axis. The distribution of the BAE roots for the one and that for the other are symmetric with respect to the imaginary axis. As to the zeros of the QTM, $N - 2$ zeros are on the smooth curves $3\pi x \sim \pm 2$.

The locations of the two “missing zeros” are vital in the evaluation of the excited states. For the XXZ model, both of them enter into the physical strip. Especially, with vanishing external field $h$, they are on the real axis and are symmetric with respect to the imaginary axis. With the increase of $h$, they are away from the real axis, but still stay in the physical strip preserving the symmetry.

We find a different situation for the Fermion model. At the half-filling, corresponding to $h = 0$ in the XXZ model, one of them is located at $\theta_0$ on the real axis, while the other is at $\theta_0 + ip_0$, $\theta_0 \sim \theta'_0$. Namely only one zero appears in the physical strip. Away from the half-filling, the zero in the physical strip (we call it $\theta$) moves upward while the other ($\theta'$) moves downward. Nevertheless, we find that $\theta$ remains in the physical strip while $\theta'$ never comes in. From now on we consider the case $\Re\theta > 0$ ($\Re\theta' > 0$). Then the trajectories of $\theta'$, for example, are depicted in FIG. 1.

![FIG. 1. The trajectories of the additional zero $\theta'$ are depicted in the case $p_0 = 3$, $N = 100$. With the decrease of $T$, $\theta'$ moves downward, whereas it never comes into the physical strip.](image)

We assume all these features are valid in the Trotter limit $N \rightarrow \infty$. Then a set of nonlinear integral equations (NLIE) can be derived as in the case of the XXZ model. We define auxiliary functions

$$
a(x) := \frac{\lambda_1(x + i - i\gamma_1)}{\lambda_2(x + i + i\gamma_1)} \quad \overline{a}(x) := 1 + a(x),$$

$$\overline{a}(x) := \frac{\lambda_2(x - i + i\gamma_2)}{\lambda_1(x - i - i\gamma_2)} \quad \overline{\overline{a}}(x) := 1 + \overline{a}(x).
$$

where $\gamma_1, \gamma_2$ are small positive quantities introduced for the convenience in numerical calculations. Note these functions have asymptotic values

$$a(x) = \begin{cases} 
\exp((\pi - 4\eta)i + \beta\mu) & \text{for } x \rightarrow -\infty, \\
\exp((\pi - 4\eta)i - \beta\mu) & \text{for } x \rightarrow \infty, 
\end{cases}
$$

$$\overline{a}(x) = \begin{cases} 
\exp((\pi - 4\eta)i - \beta\mu) & \text{for } x \rightarrow -\infty, \\
\exp((\pi - 4\eta)i + \beta\mu) & \text{for } x \rightarrow \infty.
\end{cases}
$$
Immediately seen from the above analyticity argument, \( a(x) \), \( \mathfrak{A}(x) \) (\( \mathfrak{m}(x), \mathfrak{b}(x) \)) are Analytic, NonZero and have Constant asymptotic values (ANZC) in a certain strip in the lower (upper) half plane including real axis. The above definitions, together with the knowledge of zeros for \( \Lambda(x) \), fix the NLIE among these auxiliary functions. We defer the detail derivation to Appendix C. The resultant expressions allow for taking the Trotter limit analytically. Thereby one arrives at the final expressions totally independent of fictitious parameter \( N \),

\[
\ln a(x) = -\frac{\pi \beta \sin 2\eta}{4\eta \cosh \frac{\pi}{2}(x - i\gamma_1)} + F \ast \ln \mathfrak{A}(x)
\]

\[
-F \ast \ln \mathfrak{A}(x+2i-i(\gamma_1+\gamma_2))
\]

\[
+2\pi i F(x - \theta + i(1 - \gamma_1)) + \frac{\beta \mu \rho}{2(p_0-1)}.
\]

\[
\ln \mathfrak{a}(x) = -\frac{\pi \beta \sin 2\eta}{4\eta \cosh \frac{\pi}{2}(x + i\gamma_2)} + F \ast \ln \mathfrak{A}(x)
\]

\[
-F \ast \ln \mathfrak{A}(x-2i+i(\gamma_1+\gamma_2))
\]

\[
-2\pi i F(x - \theta - i(1 - \gamma_2)) - \frac{\beta \mu \rho}{2(p_0-1)}.
\]

(3.3)

where

\[ A * B(x) := \int_{-\infty}^{\infty} A(x-y)B(y)dy, \]

\[ F(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sinh(p_0 - 2)k}{2 \cosh k \sinh(p_0 - 1)k} e^{-ikx} dk, \]

\[ F(x) := \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{\sinh(p_0 - 2)k}{2 \cosh k \sinh(p_0 - 1)k} e^{-ikx} dk. \]

(3.4)

The location of zero \( \theta \) satisfies a subsidiary condition,

\[ a(\theta - i + i\gamma_1) = -1. \]

(3.5)

Taking the Trotter limit \( N \to \infty \) after setting \( x = 0 \) in \([3.13]\) we derive the “first excited free energy” per site \( f_2 \) is

\[ f_2 = -\frac{1}{\beta} \ln \Lambda_2(0) - \frac{1}{4} \Delta \]

\[ = \epsilon_0 - \frac{1}{\beta} K \ast \ln \mathfrak{A}(i\gamma_1) - \frac{1}{\beta} K \ast \ln \mathfrak{b}(-i\gamma_2), \]

\[ -\frac{1}{\beta} \ln \tanh \frac{\pi \theta}{4} - i \frac{\pi}{2 \beta}. \]

(3.6)

where \( \epsilon_0 \) is the ground state energy defined in \([3.20]\) and

\[ K(x) := \frac{1}{4 \cosh \frac{\pi x}{2}}. \]

(3.7)

Together with the NLIE for the largest eigenvalue, summarized in Appendix C, these relations characterize the correlation length \( \xi \) of one-particle Green’s function \( \langle \psi_j^* \psi_i \rangle \) at \( T > 0 \) completely (see Eq. \([2.51]\)).

We remark that in derivations of above relations one does not need precise information like roots distributions of the BAE. Only ANZC properties of the QTM and the auxiliary functions are sufficient. Thus the structure is rather robust, and permits to introduce small free parameters \( \gamma_1 \) and \( \gamma_2 \).

In the next two subsections, we present analytical and numerical studies on these equations and the correlation length of one-particle Green’s function, which are main results in this paper.

B. Low temperature property of NLIE (\( \mu = 0 \))

We study the low temperature behavior for the half-filling case \( \mu = 0 \) utilizing the Dilogarithm trick\([34]\) which enables us to obtain the first low temperature correction without solving NLIE. As in the case of the largest eigenvalue sector, \( |a(x)| \) and \( |\mathfrak{b}(x)| \) exhibit a crossover behavior,

\[ \begin{aligned}
& |a(x)|, |\mathfrak{b}(x)| \ll 1 \quad \text{for} \ |x| < \mathcal{K}, \\
& |a(x)|, |\mathfrak{b}(x)| \sim 1 \quad \text{for} \ |x| > \mathcal{K},
\end{aligned} \]

where

\[ \mathcal{K} := \frac{2}{\pi} \ln \frac{\pi \beta \sin(2\eta)}{2 \eta}. \]

(3.9)

Thus one carefully takes into account of contributions near “Fermi-surfaces” \( \pm \mathcal{K} \). For this purpose, we introduce following shifted variables and scaling functions,

\[ la_+ (x) := \ln a \left( \frac{\pm}{\pi} x + \mathcal{K} \right), \]

\[ \tilde{la}_\pm (x) := \ln \mathfrak{b} \left( \frac{\pm}{\pi} x + \mathcal{K} \right), \]

(3.10)

and similarly for capital functions \( \mathfrak{A}, \mathfrak{b}, A_\pm, \mathfrak{A}_\pm \). In \( T \to 0 \), they satisfy truncated equations,

\[ \begin{aligned}
& la_+ (x) = -e^{-x+\frac{\pi}{2}i\gamma_1} + F_1 \ast lA_+ (x) - F_2 \ast l\mathfrak{A}_+ (x) \\
& + 2\pi i F \left( \frac{2}{\pi} (x - \theta) + i(1 - \gamma_1) \right), \quad \text{(3.11a)}
\end{aligned} \]

\[ \begin{aligned}
& \tilde{la}_+ (x) = -e^{-x+\frac{\pi}{2}i\gamma_2} + F_1 \ast l\mathfrak{A}_+ (x) - F_2 \ast lA_+ (x) \\
& - 2\pi i F \left( \frac{2}{\pi} (x - \theta) - i(1 - \gamma_2) \right), \quad \text{(3.11b)}
\end{aligned} \]

\[ \begin{aligned}
& la_- (x) = -e^{-x+\frac{\pi}{2}i\gamma_1} + F_1 \ast lA_-(x) - F_2 \ast l\mathfrak{A}_-(x) \\
& + 2\pi i F \left( -\infty \right), \quad \text{(3.11c)}
\end{aligned} \]

\[ \begin{aligned}
& \tilde{la}_- (x) = -e^{-x+\frac{\pi}{2}i\gamma_2} + F_1 \ast l\mathfrak{A}_-(x) - F_2 \ast lA_-(x) \\
& - 2\pi i F \left( -\infty \right), \quad \text{(3.11d)}
\end{aligned} \]

where

\[ F_1(x) := \frac{2}{\pi} F \left( \frac{2x}{\pi} \right), \]

\[ F_2(x) := \frac{2}{\pi} F \left( \frac{2}{\pi} x + 2i - i(\gamma_1 + \gamma_2) \right). \]

(3.12)
and $\overline{F_1}, \overline{F_2}$ are their complex conjugates. In this limit, the finite $T$ correction part, $\ln \Lambda_n(x)$ (see (3.11d)) reads

$$
\ln \Lambda_n(x) \sim \frac{\pi}{2} i + \frac{2\eta}{\pi^2 \beta \sin 2\eta} \left( -2\pi e^{\frac{i\pi}{2} x - \eta} 
+ e^{\frac{i\pi}{2} x} \int_{-\infty}^{\infty} e^{-y} \left( e^{\frac{i\pi}{2} \gamma_l \ln A_+(y)} + e^{\frac{i\pi}{2} \gamma_l \ln \overline{A_+}(y)} \right) dy 
+ e^{-\frac{i\pi}{2} x} \int_{-\infty}^{\infty} e^{-y} \left( e^{-\frac{i\pi}{2} \gamma_l \ln A_-(y)} + e^{-\frac{i\pi}{2} \gamma_l \ln \overline{A_-}(y)} \right) dy \right).
$$

(3.13)

Thanks to the subsidiary condition for the additional zero $\theta$, we have

$$
e^{-\frac{i\pi}{2} x} = \pi - \frac{2i}{\pi} \left( \int_{-\infty}^{\infty} F \left( \frac{2}{\pi} (z - \theta) + i(1 - \gamma_l) \right) iA_+(z) dz 
- \int_{-\infty}^{\infty} F \left( \frac{2}{\pi} (z - \theta) - i(1 - \gamma_l) \right) i\overline{A_+}(z) dz \right).
$$

(3.14)

For further simplification, we define $D_\pm$ by,

$$
D_\pm := \int_{-\infty}^{\infty} \left( \frac{d}{dx} \ln \Lambda_\pm(x) \right) dx 
\pm \int_{-\infty}^{\infty} \left( \frac{d}{dx} \ln \overline{\Lambda_\pm}(x) \right) dx
$$

$$= \int_{a_\pm(\infty)}^{a_\pm(-\infty)} \left( \frac{\ln(1 + a)}{a} - \frac{\ln a}{1 + a} \right) da
+ \int_{\pi_\pm(\infty)}^{\pi_\pm(-\infty)} \left( \frac{\ln(1 + \pi)}{\pi} - \frac{\ln \pi}{1 + \pi} \right) d\pi.
$$

(3.15)

Obviously, they are equal to special values of Roger’s Dilogarithm $\mathcal{L}$,

$$
D_\pm = 2\mathcal{L} \left( \frac{a_\pm(\infty)}{1 + a_\pm(\infty)} \right) + 2\mathcal{L} \left( \frac{\pi_\pm(\infty)}{1 + \pi_\pm(\infty)} \right)
- 2\mathcal{L} \left( \frac{a_\pm(-\infty)}{1 + a_\pm(-\infty)} \right) - 2\mathcal{L} \left( \frac{\pi_\pm(-\infty)}{1 + \pi_\pm(-\infty)} \right),
$$

$$\mathcal{L}(x) := -\frac{1}{2} \int_0^1 dy \left[ \frac{\ln(1 - y)}{y} + \frac{\ln y}{1 - y} \right].
$$

(3.16)

We then apply the dilogarithm trick to (3.11a)–(3.11d). For example, we take the first two equations. After differentiating, we multiply them by $i\Lambda_+(x), \overline{\Lambda_+(x)}$ respectively and take the summation. We call resultant equality (A). Next multiply (3.11a), (3.11b) by $(\overline{\Lambda_+(x)})', (\Lambda_+(x))^'$ respectively and take the summation. Let us call the outcome as (B). Finally we subtract (B) from (A) and integrate over $x$. The lhs of the equality is nothing but $D_+$. Remarkably in the rhs, most complicated terms like

$$
- \int i\Lambda_+(x) \frac{dF_2(x - y)}{dx} \overline{\Lambda_+(y)} dx dy
= - \int i\Lambda_+(x) F_2(x - y) \frac{d\overline{\Lambda_+(y)}}{dy} dx dy,
$$

(3.17)

and

$$
\int \frac{d\overline{\Lambda_+}(x)}{dx} F_2(x - y) i\Lambda_+(y) dx dy,
$$

(3.18)

cancel with each other. After rearrangement we obtain,

$$
D_+ + 2\pi i \mathcal{F}(\infty) \ln \frac{\Lambda_+(\infty)}{\Lambda_-(\infty)}
= \int_{-\infty}^{\infty} 2e^{-y} \left( e^{\frac{i\pi}{2} \gamma_l \ln A_+(y)} + e^{\frac{i\pi}{2} \gamma_l \ln \overline{A_+}(y)} \right) dy
+ 8i \int_{-\infty}^{\infty} F \left( \frac{2}{\pi} (x - \theta) + i(1 - \gamma_l) \right) iA_+(x) dx
- 8i \int_{-\infty}^{\infty} F \left( \frac{2}{\pi} (x - \theta) - i(1 - \gamma_l) \right) i\overline{A_+}(x) dx,
$$

(3.19)

where $a_+(\infty) = \pi_+(\infty) = 0$ is used. Similarly, from (3.11a), (3.11b), we have

$$
D_- + 2\pi i \mathcal{F}(\infty) \ln \frac{\Lambda_-(\infty)}{\Lambda_-(-\infty)}
= \int_{-\infty}^{\infty} 2e^{-x} \left( e^{\frac{i\pi}{2} \gamma_l \ln A_-(x)} + e^{\frac{i\pi}{2} \gamma_l \ln \overline{A_-(x)} \right) dx,
$$

(3.20)

Applying (3.19), (3.20), together with (3.14) to (3.13),

$$
\ln \Lambda_n(x) \sim \frac{\pi}{2} i + \frac{2\eta}{\pi^2 \beta \sin 2\eta} \times \left\{ e^{\frac{i\pi}{2} x} \left( -4\pi^2 + D_+ + 2\pi i \mathcal{F}(\infty) \ln \frac{\Lambda_+(\infty)}{\Lambda_-(\infty)} \right)
+ e^{-\frac{i\pi}{2} x} \left( D_- + 2\pi i \mathcal{F}(\infty) \ln \frac{\Lambda_-(\infty)}{\Lambda_-(\infty)} \right) \right\}. \tag{3.21}
$$

Now that the asymptotic values are easily found,

$$\mathcal{F}(\infty) = \mathcal{F}(\infty) = \frac{\pi - 4\eta}{4(\pi - 2\eta)},
$$

$$a_+(\infty) = \pi_-(\infty) = e^{(\pi - 4\eta)i},
$$

$$a_-(\infty) = \pi_+(\infty) = e^{(-\pi + 4\eta)i}, \tag{3.22}
$$

we can explicitly evaluate (3.21) at $x = 0$,

$$
\ln \Lambda_n(x = 0) = \frac{\pi}{6\beta v_F} - \frac{\pi}{3\beta v_F} \left( \frac{1}{4} + \frac{\alpha}{4} \right) + \frac{\pi}{2} i, \tag{3.23}
$$

where $\mathcal{L}(x) + \mathcal{L}(1 - x) = \pi^2/6$ is also applied. Here $\alpha$ is introduced in (3.13) and the Fermi velocity $v_F$ is also derived in (3.13) for $n_e = 0.5$. The first term is identical to the largest eigenvalue sector, and it reproduces conformal anomaly term with $c = 1$. Comparing them, one concludes

$$\frac{\Lambda_2}{\Lambda_1} \sim e^{ik_F - 1/\xi}. \tag{3.24}$$
where $k_F$ denotes the “Fermi momentum”. Note that $k_F = \pi/2$ in the half-filling case. Consequently the inverse correlation length is given as

$$\xi^{-1} = \frac{\pi T}{v_F} \left( \frac{1}{\alpha} + \frac{\alpha}{4} \right), \quad (3.25)$$

These are nothing but the expected results from CFT (see (3.14)). This fact represents the consistency of both our result and validity of CFT mapping in the finite temperature problem at low temperatures.

C. Numerical Analyses on NLIE

Having verified consistency at the specific limits, we now perform numerical analyses on the NLIE for a wide range of temperatures, electron fillings and interaction strengths.

To keep the electron filling constant, we adopt the temperature dependent chemical potential which are determined by the curve,

$$\frac{d(n_e(T, \mu(T)))}{dT} = \frac{d}{dT} \left( \frac{\theta f}{\theta \mu} \right)_T = 0. \quad (3.26)$$

The NLIE are numerically solved by the iteration method. In each iteration steps, convolution parts are treated by the Fast Fourier Transformation (FFT). As a technical remark, we call an attention to proper re-scaling of auxiliary functions for the FFT; one needs to modify the integrands such that these asymptotic values vanish. From the asymptotics in (3.23) and (3.24), we introduce

$$\mathfrak{B}(x) := \begin{cases} \mathfrak{A}(x)/\mathfrak{A}(\infty) & \text{for } x \geq 0, \\ \mathfrak{A}(x)/\mathfrak{A}(-\infty) & \text{for } x < 0, \end{cases} \quad (3.27)$$

and similarly for others. We also rewrite NLIE in terms of $\mathfrak{B}(x)$, which has now zero asymptotic values. For example,

$$\ln a(x) = -\frac{\pi \beta \sin(2\eta)}{4\eta \cosh \frac{\pi}{2} (x - i\gamma_1)} + F \ast \ln \mathfrak{B}(x)$$

$$-F \ast \ln \mathfrak{B}(x + 2i - i(\gamma_1 + \gamma_2)) + \mathcal{F}(x) \ln \frac{\mathfrak{A}(\infty)}{\mathfrak{A}(-\infty)}$$

$$-\mathcal{F}(x + 2i - i(\gamma_1 + \gamma_2)) \ln \frac{\mathfrak{A}(\infty)}{\mathfrak{A}(-\infty)}$$

$$+2\pi i \mathcal{F}(x - \theta + i(1 - \gamma_1)) + \beta \mu. \quad (3.28)$$

In addition, one must be careful in the branch cuts of the logarithms. In the above, $\mathfrak{A}(\infty)/\mathfrak{A}(\infty)$ and so on must be understood as

$$\ln \frac{\mathfrak{A}(\infty)}{\mathfrak{A}(\infty)} = \ln \left( -\frac{\sinh(\beta \mu/2 - 2i\eta)}{\sinh(\beta \mu/2 + 2i\eta)} \right) + (\pi - 4\eta)i. \quad (3.29)$$

Under these arrangements, the iteration method works in a stable manner.

We plot the temperature dependence of the correlation length $\xi$ in FIG. 2 for various fillings keeping the interaction strength constant $\Delta = \cos(\pi/6)$.

FIG. 2. The temperature dependence of the correlation length $\xi$ for $n_e=6$.

The extrapolated values $T \rightarrow 0$ agree with the predictions from CFT within few percents even far away from “half-filling” ($n_e = 0.5$). The curves are going down gradually with the decrease of electron density $n_e$. As further information, chemical potential $\mu(T)$ determined by Eq. (3.26) and the location of the additional zero $\theta$ are depicted in FIG. 3 and FIG. 4 respectively. The zero $\theta$ moves on a smooth curve and its curvature increases with the decrease of $n_e$. In fact, we find that it moves to $\theta = i$ when $n_e, T \rightarrow 0$. (See also the analytic argument for non-interacting Fermion case in FIG. 10 for $\mu = 1.0$.) We also calculate the “Fermi momentum” $k_F = 3 \ln \Lambda_2/\Lambda_1$ (cf. Eq. (3.24)). (Here the inverse period of oscillatory behavior at arbitrary $T$ is referred to as $k_F$ as in the case of $T = 0$.) The figure clearly shows the temperature dependency of $k_F$. In the low temperature limit $T \rightarrow 0$, it converges to the expected value, $k_F = n_e \pi$, which indicates the significance of the Fermi surface for one-particle excitations in the Luttinger liquid at $T = 0$. With the increase of $T$, the auxiliary functions cease to exhibit a sharp crossover behavior (3.8), which roughly corresponds to broadening of the Fermi distribution at $T > 0$. The particle excitations are enhanced within the wide range near the Fermi surface, which yield the shift of $k_F$. We remark that such $T$ dependent oscillatory behavior has been reported for the longitudinal correlation function of ferromagnetic Heisenberg model. Although the physical origins are different for these two cases, the explicit determination of $T$ dependency is important.
FIG. 3. The temperature dependence of the chemical potential $\mu$ for $p_0=6$.

FIG. 4. The trajectory of the additional zero $\theta$ inside the physical strip.

FIG. 5. The temperature dependence of the Fermi momentum $k_F$ for $p_0=6$.

FIG. 5.5 presents the temperature dependence of the correlation length for various interaction strengths for fixed $n_e$. Naturally in the limit, $n_e, T \to 0$, $\xi T$ does not depend significantly on the interaction strength; it merely behaves as $\xi T \sim v_F/\pi \sim n_e$ (see Appendix B). This fact is typical for non-interacting cases. Although our model inherits strong correlations, FIG. 6 indicates that $n_e = 0.1$ is already well described by “non-interacting approximation” and also shows this approximation is applicable in the wide range of $T$. On the other hand, data for $n_e = 0.4$ show strong dependency on $\Delta$, therefore it belongs to proper “interacting class” (see FIG. 7). It seems these crossover occurs near $n_e \sim 0.25$ but it is not yet conclusive. We hope to clarify this in a future communication.

FIG. 6. The temperature dependence of the correlation length for $n_e = 0.1$.

FIG. 7. The temperature dependence of the correlation length for $n_e=0.4$.

Finally we plot the correlation length of transverse spin-spin correlation $\langle \sigma_j^+ \sigma_j^- \rangle$ without external field (FIG. 8) for comparison with $n_e = 0.5$ of spinless Fermion models (FIG. 9). Besides the difference between their
limiting values at $T \rightarrow 0$, one clearly sees the difference in the dependence of $\xi T$ on $T$.

![FIG. 8. The correlation length for $\langle \sigma_j^+ \sigma_i^- \rangle$ of the corresponding XXZ model with zero magnetic field.](image)

![FIG. 9. The temperature dependence of the correlation length at half-filling.](image)

**IV. SUMMARY AND DISCUSSION**

We have proposed the QTM approach to the integrable lattice Fermion systems at any finite temperatures. The Fermionic $R$-operator, together with its super-transposition $\hat{R}$, where the Fermion statistics is embedded naturally, play the crucial role in this approach. Consequently, we have observed the significant difference between the Fermion model and that of the spin model. In principle, we can apply this approach to any integrable 1D Fermion systems. The application to the Hubbard model is under progress.

Here we comment on the “attractive regime” $t > 0$, $\Delta < 0$ in [2,1], which we have not been concerned with in this paper. In the XXZ model without external magnetic field, one may recall the remarkable difference between the repulsive (anti-ferromagnetic) case and the attractive (ferromagnetic) one [2,1]. In the repulsive regime, the eigenvalues related to the correlation $\langle \sigma_j^+ \sigma_i^- \rangle$ or $\langle \sigma_j^z \sigma_i^z \rangle$ is characterized by two real additional zeros which are symmetric with respect to the imaginary axis. This symmetry is never broken at any temperatures. On the other hand in the attractive regime, “level crossing” occurs successively. One may attribute it to the change of the distribution patterns of the additional zeros. It will be interesting to see if similar phenomena happens for the spinless Fermion model in the attractive regime.

Finally we refer to another formulation of NLIE derived from the different choice of the auxiliary functions. The NLIE have a close connection with the “TBA” or “excited states TBA” equations from the standard “string hypothesis”.

The idea is as follows. First we embed the QTM itself into a more general family called $T$-functions and explore functional relations among them ($T$-system). Then we define the $Y$-functions by certain ratio of the $T$-functions and also derive functional relations for them ($Y$-system). The analytical properties of these functions leads to the NLIE which determine the free energy and the correlation length. As concerns the largest eigenvalue sector, the $T$-functions coincide with those in Ref. [21]. Therefore the derived NLIE for the free energy are identical to the TBA equations of the XXZ model [2,1]. In contrast, for the second largest eigenvalue sector we find the essential difference between the Fermion model and the corresponding spin model. For example, we write explicitly the NLIE (excited state TBA equation) for $p_0 = 5$, $\mu = 0$ as

$$
\begin{align*}
\ln \eta_1(x) &= -\frac{5\beta \sin \frac{\pi}{2} x}{2 \cosh \frac{\pi}{2}} + K * \ln(1 + \eta_2)(x) + \pi i, \\
\ln \eta_2(x) &= K * \ln(1 - \eta_1)(1 - \eta_3)(x) \\
&\quad + \ln \left( \tanh \frac{\pi}{4}(x - \theta_1) \tanh \frac{\pi}{4}(x - \theta_2) \right) + \pi i, \\
\ln \eta_3(x) &= K * \ln(1 + \eta_2)(1 - \kappa^2)(x) \\
\ln \kappa &= K * \ln(1 - \eta_2)(x) + \ln \left( \tanh \frac{\pi}{4}(x - \theta_2) \right) \\
&\quad + \frac{\pi}{2} i,
\end{align*}
$$

(4.1)

where $\theta_1$ and $\theta_2$ are determined from

$$
\begin{align*}
\frac{5\beta \sin \frac{\pi}{2} \theta_1}{2 \sinh \frac{\pi}{2} \sigma_1} + K * \ln(1 + \eta_2)(\theta_1 + i) - \pi i &= 0, \\
K * \ln(1 + \eta_2)(1 - \kappa^2)(\theta_2 + i) &= 0.
\end{align*}
$$

(4.2)

The meaning of the functions $\eta_j$ and the quantities $\theta_j$ are similar to those in Ref. [21]. Although the above expressions are quite different from those in Sec. III, the numerical result shows a good agreement. The detailed derivations of above equations will be described in a separate communication [4].
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APPENDIX A: DIAGONALIZATION OF THE QUANTUM TRANSFER MATRIX

Here we shall diagonalize the QTM $\mathcal{R}_{21}(v - v')\mathcal{T}_1(u_N, v)\mathcal{T}_2(u_N, v')$ satisfies the global Yang-Baxter relation

$$\mathcal{R}_{21}(v - v')\mathcal{T}_1(u_N, v)\mathcal{T}_2(u_N, v') = \mathcal{T}_2(u_N, v')\mathcal{T}_1(u_N, v)\mathcal{R}_{21}(v - v'). \quad (A1)$$

Writing the monodromy operator as

$$\mathcal{T}_j(u_N, v) = A(v) (1 - n_j) + B(v) c_j + C(v) c_j^\dagger + D(v) n_j, \quad j = 1, 2, \quad (A2)$$

and substituting them into (A1), we get the commutation relations among the operators $A(v), \ldots, D(v)$,

$$A(v)B(v') = \frac{a(v' - v)}{b(v' - v)} B(v')A(v) - \frac{c(v' - v)}{b(v' - v)} B(v)A(v'), \quad (A3a)$$

$$D(v)B(v') = \frac{a(v - v')}{b(v - v')} B(v')D(v) + \frac{c(v - v')}{b(v - v')} B(v)D(v'), \quad (A3b)$$

$$B(v)B(v') = B(v')B(v). \quad (A3c)$$

To derive these relations, one should pay attention to the fact that $B(v)$ and $C(v)$ anti-commute with the Fermion operators $c_j$ and $c_j^\dagger$.

The commutation relations (A3a)–(A3c) are quite similar to the corresponding ones for the XXX model. In fact the relations (A3a) and (A3c) are identical. The second relation (A3b), however, is different: there appears an overall “minus” sign on the rhs.

Now we define the reference state by

$$|\Omega\rangle := \prod_{m=1}^{N/2} |0\rangle_{a_{2m}} \otimes |1\rangle_{a_{2m-1}},$$

$$|1\rangle_{a_{2m-1}} := c_{a_{2m-1}}^\dagger |0\rangle_{a_{2m-1}}. \quad (A4)$$

Then using the relations,

$$\mathcal{R}_{a_{2m-1}, j}(v - u_N)|1\rangle_{a_{2m-1}} = -a(v - u_N) n_j |1\rangle_{a_{2m-1}} + b(v - u_N)(1 - n_j) |1\rangle_{a_{2m-1}} + c_j |0\rangle_{a_{2m-1}}, \quad (A5)$$

and

$$\mathcal{R}_{a_{2m}, j}(v + u_N) |0\rangle_{a_{2m}} = a(v + u_N)(1 - n_j) |0\rangle_{a_{2m}} + b(v + u_N)n_j |0\rangle_{a_{2m}} - c_j |1\rangle_{a_{2m}}, \quad (A6)$$

we find that

$$A(v)|\Omega\rangle = (a(v + u_N)b(-v + u_N))^{N/2} |\Omega\rangle, \quad (A7)$$

$$D(v)|\Omega\rangle = (-b(v + u_N)a(-v + u_N))^{N/2} |\Omega\rangle. \quad (A8)$$

Hence the state $|\Omega\rangle$ is an eigenstate of the QTM $\mathcal{R}_{21}$ with the eigenvalue

$$\Lambda_0(v) = \left( \frac{\sin \eta(v + u_N + 2) \sin \eta(-v + u_N)}{\sin^2 2\eta} \right)^{N/2} \times \left( \frac{\sin \eta(v + u_N) \sin \eta(-v + u_N + 2)}{\sin^2 2\eta} \right)^{N/2}. \quad (A9)$$

An eigenstate with $N_c$ “particles” can be constructed by multiplying the operators $B(v_j)$ to the reference state

$$|\Psi\rangle := \prod_{j=1}^{N_c} B(v_j)|\Omega\rangle. \quad (A10)$$

Indeed, using the standard argument of the algebraic Bethe ansatz, we can show that the state (A10) becomes the eigenstate of the QTM if the spectral parameters $v_j$ fulfill the Bethe ansatz equations

$$\frac{\sin \eta(-v_j + u_N) \sin \eta(v_j + u_N + 2)}{\sin \eta(v_j + u_N) \sin \eta(-v_j + u_N + 2)} = (-1)^{N/2 + N_c} \prod_{k=1}^{N_c} \frac{\sin \eta(v_j - v_k + 2)}{\sin \eta(v_j - v_k)}. \quad (A11)$$

The corresponding eigenvalue of the QTM $\mathcal{U}_{21}$ is given by

$$\mathcal{T}_{QTM}(u_N, v)|\Psi\rangle = \Lambda(v)|\Psi\rangle \quad (A12)$$

is given by

$$\Lambda(v) = \left( \frac{\sin \eta(v + u_N + 2) \sin \eta(-v + u_N)}{\sin^2 2\eta} \right)^{N/2} \times \prod_{j=1}^{N_c} \frac{\sin \eta(v - v_j - 2)}{\sin \eta(v - v_j)} + (-1)^{N/2 + N_c} \prod_{j=1}^{N_c} \frac{\sin \eta(v - v_j + 2)}{\sin \eta(v - v_j)}. \quad (A13)$$
APPENDIX B: $T \ll 1$ BEHAVIOR AND PREDICTION FROM CFT

We summarize the known results of the correlation function at $T = 0$ and its $T \ll 1$ behavior predicted from CFT.

Let us start with the zero temperature case. One-particle Green’s function shows an oscillatory behavior due to the Fermi surface,

$$\langle c^\dagger(x)c(0) \rangle \sim \cos(k_F x)/x^{2\Delta} \quad (B1)$$

The scaling dimension $\Delta$ is evaluated from the energy spectra in the finite size system,

$$\Delta = \frac{1}{4Z(K_F)^2}(\Delta N)^2 + Z(K_F)^2(\Delta D)^2. \quad (B2)$$

Here $Z(K_F)$ is the dressed charge and $K_F$ denotes the “Fermi surface” satisfying

$$Z(x) + \frac{1}{2\pi} \int_{-K_F}^{K_F} R(x-y)Z(y)dy = 1, \quad (B3)$$

$$R(x) = \frac{2\sin 4\eta}{\cosh 2x - \cos 4\eta}. \quad (B3)$$

$\Delta D$ and $\Delta N$ are (half-)integers constrained by a selection rule, $\Delta D = \Delta N/2 \mod 1$. For one-particle Green’s function, they are given by $\Delta D = 1/2$ and $\Delta N = 1$. Thus the critical exponent $\eta_F$ is defined as

$$\eta_F := 2\Delta = \frac{1}{2}\left( Z(K_F)^2 + \frac{1}{Z(K_F)^2} \right). \quad (B4)$$

The dressed charge $Z(K_F)$ is explicitly evaluated for two special cases:

$$Z(K_F) = \begin{cases} \frac{1}{\sqrt{\alpha/2}} & \text{for } n_e = 0 \ (K_F = 0), \\ \frac{\pi}{\pi - 2\eta} & \text{for } n_e = 0.5 \ (K_F = \infty), \end{cases} \quad (B5)$$

where $\alpha$ is

$$\alpha = \frac{\pi}{\pi - 2\eta}. \quad (B6)$$

Then the critical exponent $\eta_F$ is given by

$$\eta_F = \begin{cases} 1 & \text{for } n_e = 0, \\ 1/\alpha + \alpha/4 & \text{for } n_e = 0.5. \end{cases} \quad (B7)$$

In the scaling limit where CFT is valid, the correlation functions at $T \ll 1$ are recovered by the replacement

$$x \rightarrow \frac{v_F}{\pi T} \sinh \frac{\pi T x}{v_F} \quad (B8)$$

in the denominator in (B1). Here $v_F$ denotes the Fermi velocity

$$v_F := \frac{1}{2\pi \rho(x)} \left. \frac{\partial \varepsilon(x)}{\partial x} \right|_{x = K_F}. \quad (B9)$$

Note that $\rho(x)$ and $\varepsilon(x)$ are the density function and the dressed energy defined by

$$\rho(x) + \frac{1}{2\pi} \int_{-K_F}^{K_F} R(x-y)\rho(y)dy = \frac{\sin 2\eta}{\pi(\cosh 2x - \cos 2\eta)},$$

$$\varepsilon(x) + \frac{1}{2\pi} \int_{-K_F}^{K_F} R(x-y)\varepsilon(y)dy = -\frac{\sin^2 2\eta}{\cosh 2x - \cos 2\eta} + \mu. \quad (B10)$$

Thus the long distance behavior of one-particle Green’s function is given by

$$\langle c^\dagger(x)c(0) \rangle \sim \cos(k_F x)x^{-\pi \eta_F |x| T/v_F}. \quad (B11)$$

Consequently the correlation length at $T \ll 1$ is identified with

$$\xi = \frac{v_F}{\pi \eta_F T}. \quad (B12)$$

The Fermi velocity (B9) is analytically calculated for the cases $n_e \ll 1$ and $n_e = 0.5$

$$v_F = \begin{cases} \sim n_e/\pi & \text{for } n_e \ll 1, \\ \pi \sin 2\eta/4\eta & \text{for } n_e = 0.5. \end{cases} \quad (B13)$$

Therefore we get the explicit correlation length (B12) for these two special cases:

$$\xi T = \begin{cases} \sim n_e & \text{for } n_e \ll 1, \\ \sin 2\eta/(4\eta(1/\alpha + \alpha/4)) & \text{for } n_e = 0.5. \end{cases} \quad (B14)$$

We have also verified the extrapolations from the NLIE agree with the prediction (B13).

Finally we remark on the spin correlation. The main contribution to the transverse correlation function simply decays algebraically,

$$\langle \sigma^+(x)\sigma(0) \rangle \sim 1/x^{2\Delta'}, \quad (B15)$$

that has no oscillation term. Here $\Delta'$ takes the identical form (B2). However we have to use $\Delta N = 1$ and $\Delta D = 0$ this time. The difference in selection rules for these integers, which originates from the difference in statistics, leads to a conclusion

$$\Delta \neq \Delta' = \frac{1}{4Z(K_F)^2}. \quad (B16)$$

The corresponding correlation length is given by (B12), replacing $\eta_F$ by $\eta_S = 1/2(2Z(K_F))^2$. One thus obtain different correlation lengths simply according to the selection rules.
APPENDIX C: DERIVATION OF NLIE

For simplicity in notation we define
\[ \epsilon(x) := a(x + i\gamma_1), \quad \mathcal{C}(x) := 1 + \epsilon(x), \]
\[ \Theta(x) := a(x - i\gamma_2), \quad \mathcal{C}(x) := 1 + \Theta(x). \]  
(C1)

That is, we forget additional shifts for a moment.

We identify the analytic strips,
\[ Q(x) : \quad \exists x \in (-2p_0, 0) \]
\[ \phi_-(x) : \quad \exists x \in [0, 2p_0) \]
\[ \phi_+(x) : \quad \exists x \in (-2p_0, 0]. \]  
(C2)

The following identities are direct consequence of the definitions,
\[ \Lambda(x + i) = \mathcal{C}(x) \frac{Q(x - i)}{Q(x - (2p_0 - 1)i)} \times \phi_-(x + i) \phi_+(x - i(2p_0 - 3))e^{-\beta\mu/2} \]
\[ \Lambda(x - i) = (-1)^{N/2 + N} \epsilon(x) \frac{Q(x - (2p_0 - 1)i)}{Q(x - i)} \times \phi_+(x - i) \phi_-(x + i(2p_0 - 3))e^{\beta\mu/2}. \]  
(C3)

Now we consider the second largest eigenvalue case \( N = N/2 - 1 \). We are in position to utilize the knowledge of zeros of \( \Lambda_2(x) \).

Consider the integral,
\[ \int_{\mathcal{C}} \frac{d}{dz} \ln \Lambda_2(z)e^{ikz}dz, \]
where \( \mathcal{C} \) encircles the edges of “square”: \([z_1, z_2]\cup[z_3, z_4]\cup[z_3, z_1] \) in the counterclockwise manner, where \( z_1 = -\infty - i, z_2 = \infty - i, z_3 = \infty + i, z_4 = -\infty + i \).

There is one zero of \( \Lambda_2(x) \) in the region inside \( \mathcal{C} \). Thus Cauchy’s theorem is applied,
\[ 2\pi ie^{ik\theta} = \int_{-\infty}^{\infty} \frac{d}{dx} \ln \Lambda_2(x - i)e^{ik(x - i)}dx \]
\[ + \int_{\infty}^{-\infty} \frac{d}{dx} \ln \Lambda_2(x + i)e^{ik(x + i)}dx. \]  
(C4)

One substitutes Eq. (C3) into the above equation and derives identities among the Fourier components of logarithmic derivatives of \( Q, \mathcal{C} \) and \( \Theta \). Explicitly we have,
\[ \tilde{\Lambda}Q[k] = -e^{k(2p_0 - 1)}\tilde{\Lambda}[k] = -e^{k(2p_0 - 1)}\tilde{\mathcal{C}[k]} \]
\[ + e^{k(2p_0 - 1)}\tilde{\phi_+}[k] + e^{k\tilde{\phi_+}[k]} \]
\[ - 2\pi ie^{ik\mu/2} - 2\sinh(p_0 - 1)\cosh k \]  
(C5)

In the above we adopt a notation
\[ \tilde{\Lambda}[k] := \int_{-\infty}^{\infty} \frac{d}{dx} \ln \mathcal{C}(x)e^{ikx}dx, \]
\[ \tilde{\Lambda}_2(x) := \int_{-\infty}^{\infty} \frac{d}{dx} \ln \mathcal{C}(x)e^{ikx}dx, \]
eq 0 etc as the Fourier component of the logarithmic derivatives.

On the other hand, from the definition \([C1]\), we have
\[ \tilde{\Lambda}Q[k] = -2\pi i \cosh k - 2\pi i \sinh k - 2\pi i \tanh k \]
\[ - 2\pi i e^{ik\mu/2} - 2\sinh(p_0 - 1)\cosh k \]  
(C7)

and similarly for \( \tilde{\Lambda}_2 \).

One substitutes (C3) into (C7) to obtain a closed equation among the Fourier modes of the auxiliary functions. Using the explicit form for \( \phi_\pm \), we get,
\[ \tilde{\Lambda}Q[k] = -2\pi i \cosh k \]
\[ - 2\pi i e^{ik\mu/2} - 2\sinh(p_0 - 1)\cosh k \]  
(C8)

By the inverse transformation and integration over \( x \) we arrive at NLIE. Note that the integration constant is determined by the asymptotic values in (3.2a) and (3.21).

After introducing the shifts \( \gamma_1, 2 \), one obtains the identical NLIE in the main text, except for “driving terms” as we have not yet taken the Trotter limit \( N \to \infty \). To be precise the driving term for \( \ln a(x) \) is
\[ \frac{N}{2} \int_{-\infty}^{\infty} \frac{\sinh u_N k}{\cosh k} e^{ik(x - i\gamma_1)}dk. \]  
(C9)

Due to the combination of \( u_N = -\beta \sin 2\eta/2\eta N \) and \( N \) entering above, the Trotter limit is carried out analytically. Then one ends up with \([3.3]\).

The expression for the eigenvalue is derived in a similar way. One first notes the “inversion identity”,
\[ \tilde{\Lambda}_2(x + i)\tilde{\Lambda}_2(x - i) = -\psi(x)\mathcal{C}(x)\mathcal{C}(x), \]  
(C10)

where
\[ \psi(x) := \frac{\phi_+(x - i)\phi_-(x + i)}{\phi_+(x + i)\phi_-(x - i)}, \]  
(C11)

and
\[ \tilde{\Lambda}_2(x) = \frac{\Lambda_2(x)}{\tanh \frac{\pi}{2}(x - \theta)\phi_+(x + 2i)\phi_-(x - 2i)}. \]  
(C12)

is introduced to exclude the zeros of \( \Lambda_2(x) \) and to compensate the divergence of \( \Lambda_2(x) \) at \( x \to \pm \infty \).

Then the lhs is ANZC in a strip \( \exists x \in [-1, 1] \) and also rhs is ANZC in a narrow strip including the real axis. One thus can solve (C10) and we get the expression
\[ \ln \Lambda_2(x) = \ln \Lambda_{g}\Lambda(x) + \ln \Lambda_{fn}(x), \]  
(C13)
where

\[
\ln A_{gs}(x) := -\frac{N}{2} \int_{-\infty}^{\infty} \frac{\sinh k u_N \sinh (p_0 - 1)k}{k \cosh k \sinh p_0 k} e^{-ikx} dk \\
+ \ln \phi_+(x + 2i) \phi_-(x - 2i),
\]

\[
(\text{C14a})
\]

\[
\ln A_{in}(x) := K * \ln \Phi(x + i\gamma_1) + K * \ln \Phi(x - i\gamma_2) \\
+ \ln \tanh \frac{\pi}{4}(x - \theta) - \frac{\pi i}{2}.
\]

\[
(\text{C14b})
\]

Taking the Trotter limit \( N \to \infty \) after setting \( x = 0 \) and using the identity

\[
\lim_{N \to \infty} \ln \phi_+(2i)\phi_-(2i) = -\frac{\beta}{2} \Delta,
\]

\[
(\text{C15})
\]

we derive the first excited free energy as \( (3.6) \).

Next we consider the largest eigenvalue sector \( N_e = N/2 \). In this case, the spinless Fermion model shares same equations with the \( XXZ \) model. Then the following NLIE have been already derived in Ref. 15.

\[
\frac{\partial}{\partial s} \Lambda(x) = \phi_+(x + 2i) \phi_-(x - 2i) \Lambda(x),
\]

\[
(\text{D1})
\]

where

\[
\phi_+(x + 2i) = e^{\frac{2}{2}\beta \mu}
\]

\[
(\text{D2})
\]

We can easily show that

\[
\Lambda(x + i) \Lambda(x - i) = (-1)^{N_e} \phi(x + i) \phi(x - i).
\]

\[
(\text{D3})
\]

This rhs is a known function, which is a distinct feature of the free Fermion model. It is convenient to modify the function \( \Lambda(x) \) as

\[
\tilde{\Lambda}(x) = \frac{\Lambda(x)}{\phi_+(x + 2i) \phi_-(x - 2i)},
\]

\[
(\text{D4})
\]

satisfying

\[
\tilde{\Lambda}(x + i) \tilde{\Lambda}(x - i)
\]

\[
= (-1)^{\frac{N_e}{2} + 1} \left( \psi(x) + \psi(x)^{-1} + 2 \cosh(\beta \mu) \right),
\]

\[
(\text{D5})
\]

where \( \psi(x) \) has been already defined in \( (\text{3.1}) \).

First we consider the free energy characterized by the largest eigenvalue \( \Lambda_1(x) \). It lies in the sector \( N_e = N/2 \). The Bethe ansatz \( \{x_j^{(1)}\}_{j=1}^{N/2} \) are symmetric with respect to the imaginary axis. The function \( \phi(x) \) in \( (\text{D2}) \) has \( N \) zeros in \( \{x_j\} \). \( N/2 \) zeros \( \{x_j^{N/2}\} \) are in the physical strip \( \{x \in [-1,1] \} \) and the others \( \{x_j^{N/2}\} \) are out of the strip. As \( \phi(x) \) has a property

\[
\phi(x + 2i)|_{\mu} = (\Lambda(x + 2i)|_{\mu} = (-1)^{\frac{N}{2}} \phi(x)|_{-\mu},
\]

\[
(\text{D6})
\]

we have

\[
x_j^{N/2} = x_j^{N/2} - 2i.
\]

\[
(\text{D7})
\]

From the BAE \( (\text{2.53}) \), \( \{x_j^{(1)}\} \) are completely equivalent to \( \{x_j\} \). Thereby one can shows from \( (\text{D1}) \), \( \Lambda_1(x) \) does not possess any zeros in the physical strip.

Since the function \( \tilde{\Lambda}_1(x) \) is ANZC \( \{x \in [-1,1] \} \), we have

\[
\ln \tilde{\Lambda}_1(x) = \left[ K * \ln \left( X + X^{-1} + 2 \cosh(\beta \mu) \right) \right](x).
\]

\[
(\text{D8})
\]
Using the relations
\[
\lim_{N \to \infty} \psi(x) = \exp\left(\frac{-\beta}{\cosh \frac{\pi x}{2}}\right),
\]
\[
\lim_{N \to \infty} \Lambda(0) = \lim_{N \to \infty} \Lambda(0), \quad (D9)
\]
we obtain the free energy per site \( f \) as
\[
f = -\frac{1}{\beta} \lim_{N \to \infty} \ln (A_1(0))
= -\frac{1}{\pi \beta} \int_0^\infty \ln (2 \cosh(\beta \cos \theta) + 2 \cosh(\beta \mu)) \, d\zeta, \quad (D10)
\]
in agreement with Ref. [14].

Next we consider the correlation length \( \xi \) for \( \langle c_i^\dagger c_i \rangle \).
The BAE roots \( \{x_j^{(2)}\}_{j=1}^{N/2-1} \) relevant to the second largest eigenvalue are identical with \( \{x_j^{(1)}\}_{j=1}^{N/2} \) except that the largest magnitude one \( x_j^{(1)} = \theta \) is absent. Then the \( \Lambda_2(x) \) possesses the additional zero \( \theta \) in the physical strip. In the Trotter limit, \( \theta \) is given by
\[
\theta = \frac{2}{\pi} \sinh^{-1} \left( \frac{-\beta}{\pi - i \beta \mu} \right). \quad (D11)
\]
The corresponding zero \( \theta' \) through the property (D7) is
\[
\theta' = \frac{2}{\pi} \sinh^{-1} \left( \frac{-\beta}{\pi + i \beta \mu} \right) + 2i. \quad (D12)
\]
The zero \( \theta (\theta') \) never goes over (never comes into) the physical strip (see FIG. 10).

Thus we have correlation length \( \xi \) for \( \langle c_i^\dagger c_i \rangle \) as
\[
\frac{1}{\xi} = -\ln \left| \tanh \left( \frac{1}{2} \sinh^{-1} \left( \frac{-\beta}{\pi - i \beta \mu} \right) \right) \right|
= \frac{1}{2} \left( \sinh^{-1} \left( \frac{\pi + i \beta \mu}{\beta} \right) + \sinh^{-1} \left( \frac{\pi - i \beta \mu}{\beta} \right) \right). \quad (D14)
\]

In FIG. 11 we plot the results (D14) for some fixed particle densities.

FIG. 11. The temperature dependence of the correlation length for the free Fermion model.

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