\( \mathcal{H}^2 \)-matrices for translation-invariant kernel functions

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Boundary element methods for elliptic partial differential equations typically lead to boundary integral operators with translation-invariant kernel functions. Taking advantage of this property is not straightforward if general unstructured meshes and general basis functions are used, since we need the supports of these basis functions to be contained in a hierarchy of subdomains with translational symmetry.

In this article, we present a modified construction for \( \mathcal{H}^2 \)-matrices on unstructured quasi-uniform meshes that uses translation-invariance to significantly reduce the storage requirements for the farfield representation.

We construct a nested hierarchy of axis-parallel boxes so that translational symmetry is preserved and prove optimal-order complexity estimates under moderate assumptions. In particular, we need only one weak assumption for proving that the entire farfield requires only \( O(\log(n)) \) coefficients.

It should be mentioned that, since we are working with an unstructured mesh and general basis functions, the nearfield of the matrix still requires \( O(n) \) units of storage.

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1 Introduction

Boundary element methods are an attractive technique for handling homogeneous linear partial differential equations, e.g., the Laplace, Lamé, Helmholtz, or Maxwell equations \cite{24, 27, 30}, on a domain \( \Omega \subseteq \mathbb{R}^d \) with \( d \in \{2, 3\} \). A Galerkin discretization of related boundary integral operators based on test functions \( (\varphi_i)_{i\in I} \) and trial functions \( (\psi_j)_{j\in J} \) leads to matrices \( G \in \mathbb{R}^{I \times J} \) given by

\[
G_{ij} := \int_{\Gamma} \varphi_i(x) \int_{\Gamma} g(x, y) \psi_j(y) \, dy \, dx \quad \text{for all } i \in I, \ j \in J,
\]  

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where $\Gamma := \partial \Omega$ denotes the boundary of $\Omega$ and $g: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a kernel function.

In typical applications, the kernel function $g$ is non-zero almost everywhere, and the matrix $G$ is therefore densely populated. In order to reduce the storage requirements and computational complexity, compression techniques are employed, e.g., the panel-clustering technique [26], the fast multipole method [28, 21, 1, 22, 33], interpolation [18, 10, 12, 13, 17], algebraic approximations [31, 32, 3, 4, 14], or hybrid methods [19, 9, 8].

All of these techniques split the matrix $G$ into submatrices that can belong either to the nearfield or the farfield. Nearfield submatrices are small and can be stored directly, while farfield submatrices can be large and have to be approximated, e.g., by low-rank matrices that can be stored efficiently in factorized form. Unfortunately, even the factorized form still requires a large amount of storage, and while recompression techniques [11, 20, 7] can help, they require additional computational work.

In this article, we pursue an alternative approach frequently used in the context of fast multipole methods for particle systems: if the kernel function $g$ is invariant under translation, i.e., if

$$g(x, y) = g(x + c, y + c) \quad \text{for all } x, y, c \in \mathbb{R}^d,$$

we can modify interpolation and certain hybrid methods in a way that reduces the storage requirements for the farfield matrices to $O(k^2 \log(n))$, where $k$ denotes the rank of the approximation and $n := \max\{|I|, |J|\}$ the maximal dimension of the matrix $G$.

This task is straightforward for particle systems with approximately uniformly distributed particles, since each particle is represented by just one point in space, but it is significantly more challenging for Galerkin discretizations, since every matrix entry depends on the supports of the basis functions $\varphi_i$ and $\psi_j$, and the supports of different basis functions may overlap or have a non-trivial shape. In some implementations, this problem is circumvented by replacing the integrals in eq. (1) by quadrature, thus reducing the computation to a sum of kernel evaluations closely related to particle methods [5]. While this is certainly an elegant approach and allows BEM codes to take advantage of highly sophisticated implementations of the fast multipole method, the number of quadrature points may grow excessively large if higher quadrature orders are used to keep up with the discretization error as the grid is refined.

In this article, we present a different approach: Initially, we construct a hierarchy of axis-parallel boxes that cover representative points of the supports of the basis functions, and we ensure that all boxes on a given level are translations of the same reference box with a fixed displacement step size. By suitably enlarging the reference boxes and applying the aforementioned translations to these enlarged reference boxes we then construct a corresponding hierarchy of axis-parallel boxes that cover the entire supports.

If we apply standard techniques like interpolation [18, 10] or Green quadrature [8] to this structure, matrices corresponding to translation-equivalent pairs of boxes are identical and therefore have to be stored only once. As long as the enlargement of the reference boxes is sufficiently bounded, this allows us to reduce the corresponding storage requirements to $O(k^2)$ for each level. No further assumption is needed for proving this result.
This approach also reduces the assembly time, since the matrices have to be set up only once. The run-time for the matrix-vector multiplication may benefit, too, since fewer data has to be moved between main memory and the processor, although the number of arithmetic operations usually grows since guaranteeing translational symmetry limits our flexibility when constructing the boxes.

That is why a major part of this article is devoted to proving that our proposed construction does not change the asymptotically optimal complexity of the $H^2$-matrix method.

In section 2 we introduce $H^2$-matrices, a suitable representation of our matrix approximation, and outline the modifications required to take full advantage of translation-invariance. Section 3 contains the key result of this article: we prove that the entire farfield of the modified $H^2$-matrices requires only $O(k^2 \log(n))$ coefficients if the enlargement of the reference boxes is sufficiently bounded. The nearfield matrices and the leaf matrices are directly connected to the unstructured surface mesh and therefore cannot take advantage of the translation-invariance property, so we store these matrices explicitly using $O(nk)$ units of storage. Section 4 illustrates the advantage of the modified representation in a series of numerical experiments.

2 $H^2$-matrices for translation-invariant kernel functions

In order for our approximation technique to achieve the desired accuracy, the kernel function $g$ needs to be asymptotically smooth, i.e., there have to be constants $C_{\text{as}} \in \mathbb{R}_{\geq 0}$, $c_0 \in \mathbb{R}_{> 0}$ and a singularity degree $\sigma \in \mathbb{N}$ such that for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ with $x \neq y$ the following condition is fulfilled:

$$|\partial_y^\nu g(x, y)| \leq C_{\text{as}} \frac{(\sigma - 1 + \nu)! c_0^\nu}{\|x - y\|^{\nu + \sigma}}$$

for all $\nu \in \mathbb{N}_0$, $\nu \in \{1, \ldots, 2d\}$. (3)

Kernel functions occurring in typical applications are known to be asymptotically smooth, see for example [25, Appendix E] and [2]. Additionally, we assume that $g$ satisfies the translation-invariance property [2].

Definition 1 (Tree notations) Let $T$ be a tree. We use the notation $t \in T$ for "$t$ is a node in $T$". For each node $t \in T$ we denote the set of its sons by $\text{sons}_T(t)$. The set of leaves of $T$ is denoted by $\mathcal{L}_T := \{t \in T : \text{sons}_T(t) = \emptyset \}$. We denote the root of $T$ by $\mathcal{R}_T$ and define the level of a node by

$$\text{level}_T(t) := \begin{cases} 0 & \text{if } t = \mathcal{R}_T, \\ \text{level}_T(\tilde{t}) + 1 & \text{if } t \text{ has a father } \tilde{t} \in T \end{cases} \text{ for all } t \in T.$$ 

For all $\ell \in \mathbb{N}_0$ we define $T^{(\ell)} := \{t \in T : \text{level}_T(t) = \ell \}$.

The first phase of our approximation procedure relies on characteristic points $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$ satisfying $x_i \in \text{supp}(\varphi_i)$ for all $i \in I$ and $y_j \in \text{supp}(\psi_j)$ for all $j \in J$. 3
These points are split hierarchically into a hierarchy of boxes that will then give rise to a decomposition of the matrix into submatrices.

We choose a maximal level \( \ell_{\text{max}} \in \mathbb{N}_{\geq d} \) and construct trees \( T_I \) and \( T_J \) of closed axis-parallel boxes in \( \mathbb{R}^d \) satisfying

\[
t = r^{(\ell)} + \delta^{(\ell)} \odot p_t = r^{(\ell)} + m_t \quad \text{for all } t \in T_I^{(\ell)} \cup T_J^{(\ell)}
\]

(4)
on every level \( \ell \in \{0, \ldots, \ell_{\text{max}}\} \), where \( (r^{(\ell)})_{\ell=0}^{\ell_{\text{max}}} \) is a family of reference boxes, \( \delta^{(\ell)} \in \mathbb{R}^d \) consists of the interval lengths of \( r^{(\ell)} \) for each \( \ell \in \{0, \ldots, \ell_{\text{max}}\} \) and \( (p_t)_{t \in T_I \cup T_J} \) are suitable integer vectors.

We start with computing an axis-parallel box \( r^{(0)} \subseteq \mathbb{R}^d \) containing \( \Gamma \). In order to create the isotropic boxes required by our complexity analysis, we then compute \( (r^{(\ell)})_{\ell=1}^{\ell_{\text{max}}} \) recursively by consecutive splitting in each coordinate direction. The corresponding cyclic sequence of splitting directions is given by

\[
\ell^{(0)} := 1, \quad \ell^{(\ell)} := \begin{cases} 
\ell^{(\ell-1)} + 1 & \text{if } \ell^{(\ell-1)} < d, \\
1 & \text{if } \ell^{(\ell-1)} = d
\end{cases} \quad \text{for all } \ell \in \mathbb{N}.
\]

(5)
Assuming that \( r^{(\ell)} = [a_1^{(\ell)}, b_1^{(\ell)}] \times \ldots \times [a_d^{(\ell)}, b_d^{(\ell)}] \) on a level \( \ell \in \{0, \ldots, \ell_{\text{max}} - 1\} \) is already given, we compute

\[
r^{(\ell+1)} := \{ z \in r^{(\ell)} : z_i^{(\ell)} \leq c_i^{(\ell)} \} \quad \text{with} \quad c_i^{(\ell)} := \frac{a_i^{(\ell)} + b_i^{(\ell)}}{2}.
\]

(6)
If \( \ell + 1 < \ell_{\text{max}} \) holds, we proceed with \( r^{(\ell+1)} \) by recursion. Otherwise, we stop.

Next, we create the tree \( T_J \) by shifting the reference boxes \( r^{(\ell)} \) in a regular pattern. In this stage, we also construct subsets \( (I_t)_{t \in T_J} \) of \( I \) such that

\[
x_i \in t \quad \text{for all } i \in I_t.
\]

(7)
Beginning with \( r^{(0)} \) as the root \( \mathcal{T}_I \) of \( T_I \), \( p_{r^{(0)}} := 0 \) and \( I_{r^{(0)}} := I \), we construct \( T_J \) also recursively: Assuming that a box \( t = [a_{t,1}, b_{t,1}] \times \ldots \times [a_{t,d}, b_{t,d}] \in T_J^{(\ell)} \) on a level \( \ell \in \{0, \ldots, \ell_{\text{max}} - 1\} \) with \( t = r^{(\ell)} + \delta^{(\ell)} \odot p_t \) for a vector \( p_t \in \mathbb{N}^d \) and a corresponding non-empty subset \( I_t \) of \( I \) with (7) are already given, we compute the midpoint \( c_{t,i^{(\ell)}} := (a_{t,i^{(\ell)}} + b_{t,i^{(\ell)}})/2 \) and

\[
t_1 := \{ z \in \mathbb{R}^d : z_i^{(\ell)} \leq c_{t,i^{(\ell)}} \} = r^{(\ell+1)} + \delta^{(\ell+1)} \odot p_{t_1},
\]

\[
t_2 := \{ z \in \mathbb{R}^d : z_i^{(\ell)} \geq c_{t,i^{(\ell)}} \} = r^{(\ell+1)} + \delta^{(\ell+1)} \odot p_{t_2}
\]
with \( p_{t_1}, p_{t_2} \in \mathbb{N}^d \) given by

\[
p_{t_1,i^{(\ell)}} := \begin{cases} 
2p_{t,i^{(\ell)}} & \text{if } i = i^{(\ell)}, \\
p_{t,i^{(\ell)}} & \text{if } i \neq i^{(\ell)}
\end{cases}, \quad p_{t_2,i^{(\ell)}} := \begin{cases} 
2p_{t,i^{(\ell)}} + 1 & \text{if } i = i^{(\ell)}, \\
p_{t,i^{(\ell)}} & \text{if } i \neq i^{(\ell)}
\end{cases} \quad \text{for all } i \in \{1, \ldots, d\}.
\]

(8)
For every axis-parallel box \( Q \)
and index sets created before with tensor interpolation. For this purpose we choose a
then lead to
we construct the smallest closed axis-parallel box \( B \)
supports (supp(\( \psi \)) \( C \)) we construct the smallest closed axis-parallel box
supports (supp(\( \phi \)) \( i \)) by recursion. Otherwise, we stop the recursion.
By replacing \( I \) with \( J \) and \((x_i)_{i \in I} \) with \((y_j)_{j \in J} \) we analogously compute the tree \( T_J \)
along with a corresponding hierarchical family \((J_s)_{s \in T_J} \) of non-empty subsets of \( J \).
In conjunction with the families \((I_t)_{t \in T_I} \) and \((J_s)_{s \in T_J} \), our trees \( T_I \) and \( T_J \) form cluster
trees for \( I \) and \( J \), respectively, which implies (cf. [4, Corollary 3.9])
Furthermore, by construction, we have
In the second phase we create corresponding support bounding boxes \((B_t)_{t \in T_I} \) and
\((C_s)_{s \in T_J} \) with (see fig. 1) a property based on eq. (4): For every level \( \ell \in \{0, \ldots, \ell_{\text{max}} \} \)
we construct the smallest closed axis-parallel box \( B^{(\ell)} \) containing \( r^{(\ell)} \) and the shifted
supports (supp(\( \varphi_i \)) - \( m_t \)) \( i \in I_t \) for all \( t \in T_I^{(\ell)} \). Likewise, for every level \( \ell \in \{0, \ldots, \ell_{\text{max}} \} \)
we construct the smallest closed axis-parallel box \( C^{(\ell)} \) containing \( r^{(\ell)} \) and the shifted
supports (supp(\( \psi_j \)) - \( m_s \)) \( j \in J_s \) for all \( s \in T_J^{(\ell)} \). According to [4], the properties
then lead to
In the third phase we construct an approximation of \( G \) by combining the trees, boxes
and index sets created before with tensor interpolation. For this purpose we choose a
degree \( \theta \in \mathbb{N} \) and define
\[ \Theta := \{0, \ldots, \theta \}^d. \]
For every axis-parallel box \( Q \subset \mathbb{R}^d \) let
\[ \mathcal{J}_Q : C(Q) \to \Pi_\theta, \quad f \mapsto \sum_{\nu \in \Theta} f(\xi_{Q,\nu}) \varepsilon_{Q,\nu}, \]
be the corresponding tensor Chebyshev interpolation operator mapping to the set $\Pi_\theta$ of tensor polynomials of degree $\theta$ with associated interpolation points $(\xi_{Q,\nu})_{\nu \in \Theta}$ and Lagrange polynomials $(L_{Q,\nu})_{\nu \in \Theta}$.

We use the admissibility condition

$$A_\eta: T_I \times T_J \to \{\text{true, false}\},$$

$$(t,s) \mapsto \begin{cases} 
\text{true} & \text{if } \max\{\text{diam}(B_t), \text{diam}(C_s)\} \leq \eta \text{dist}(B_t, C_s) \\
\text{false} & \text{if } \max\{\text{diam}(B_t), \text{diam}(C_s)\} > \eta \text{dist}(B_t, C_s) 
\end{cases} \quad (14)$$

with a given parameter $\eta \in \mathbb{R}_{>0}$ to decide whether the kernel function $g$ can be approximated in a domain $B_t \times C_s$. For every pair $(t,s) \in T_I \times T_J$ with $A_\eta(t,s) = \text{true}$ the corresponding tensor interpolation error satisfies

$$\|g - (\mathcal{I}_{B_t} \otimes \mathcal{I}_{C_s})[g]\|_{\infty, B_t \times C_s} \leq \frac{C_{in}}{\text{dist}(B_t, C_s)^{q_\eta + 1}}$$

for a constant $C_{in} \in \mathbb{R}_{>0}$ and $q_\eta := \min \left\{ \frac{c_0}{c_0 + 1}, \frac{c_0}{c_0 + 2}, \frac{c_0}{c_0 + 3} \right\}$ (cf. [6, Remark 4.23]) and therefore converges exponentially to zero with respect to the degree $\theta$. The actual convergence rates that appear in practice are often considerably better than $q_\eta$.

For every $t \in T_I$ let now $V_t \in \mathbb{R}^{I_t \times \Theta}$ be given by

$$(V_t)_{i\nu} := \int_{\Gamma} \varphi_i(x) \mathcal{L}_{B_t,\nu}(x) \, dx \quad \text{for all } i \in I_t, \nu \in \Theta,$$

and for every $s \in T_J$ let $W_s \in \mathbb{R}^{J_s \times \Theta}$ be given by

$$(W_s)_{j\mu} := \int_{\Gamma} \psi_j(y) \mathcal{L}_{C_s,\mu}(y) \, dy \quad \text{for all } j \in J_s, \mu \in \Theta.$$

Then for every pair $(t,s) \in T_I \times T_J$ with $A_\eta(t,s) = \text{true}$ we get, by using (11) and (13),
for all $i \in I_t, j \in J_s$

$$G_{ij} \approx \int_{\Gamma} \varphi_i(x) \int_{\Gamma} (\mathcal{J}_{B_t} \otimes \mathcal{J}_{C_s}) [g](x,y) \psi_j(y) \, dy \, dx$$

$$= \sum_{\nu \in \Theta} \sum_{\mu \in \Theta} \int_{\Gamma} \varphi_i(x) \mathcal{L}_{B_t,\nu}(x) \, dx \, g(\xi_{B_t,\nu}, \xi_{C_s,\mu}) \int_{\Gamma} \mathcal{L}_{C_s,\mu}(y) \psi_j(y) \, dy,$$  

which leads to

$$G|_{I_t \times J_s} \approx V_t S_{(t,s)} W_s^*$$  

with $S_{(t,s)} \in \mathbb{R}^{\Theta \times \Theta}$ given by

$$(S_{(t,s)})_{\nu \mu} := g(\xi_{B_t,\nu}, \xi_{C_s,\mu}) \text{ for all } \nu \in \Theta, \mu \in \Theta,$$

i.e., we have a factorized low-rank approximation of admissible submatrices $G|_{I_t \times J_s}$.

We compute a tree $T_{I \times J} \subseteq T_I \times T_J$ defined by the following properties:

- $\mathcal{R}_{T_{I \times J}} = (\mathcal{R}_{T_I}, \mathcal{R}_{T_J}) = ((r(0), r(0)))$;
- For all $(t, s) \in T_{I \times J}$ the sons are given by

$$\text{sons}_{T_{I \times J}}(t, s) = \begin{cases} \emptyset & \text{if } A_{\eta}(t, s) = \text{true}, \\ \text{sons}_{T_I}(t) \times \text{sons}_{T_J}(s) & \text{otherwise}. \end{cases}$$

Note that this implies $\text{sons}_{T_{I \times J}}(t, s) = \emptyset$ if $\text{sons}_{T_I}(t) = \emptyset$ or $\text{sons}_{T_J}(s) = \emptyset$.

Its admissible leaves and inadmissible leaves are denoted by

$$\mathcal{L}^+_{T_{I \times J}} := \{ (t, s) \in \mathcal{L}_{T_{I \times J}} : A_{\eta}(t, s) = \text{true} \},$$

$$\mathcal{L}^-_{T_{I \times J}} := \{ (t, s) \in \mathcal{L}_{T_{I \times J}} : A_{\eta}(t, s) = \text{false} \}.$$  

Drawing on the properties of the tree $T_I$ and $T_J$, we find (cf. [K Corollary 3.15])

$$I \times J = \bigcup_{(t, s) \in \mathcal{L}^+_T} (I_t \times J_s) = \bigcup_{(t, s) \in \mathcal{L}^+_{T_{I \times J}}} (I_t \times J_s) \cup \bigcup_{(t, s) \in \mathcal{L}^-_{T_{I \times J}}} (I_t \times J_s).$$

We can therefore use (10) to construct an approximation $\tilde{G} \in \mathbb{R}^{t \times J}$ of $G$ by

$$\tilde{G}|_{I_t \times J_s} := V_t S_{(t,s)} W_s^* \quad \text{for all } (t, s) \in \mathcal{L}^+_{T_{I \times J}},$$

$$\tilde{G}|_{I_t \times J_s} := G|_{I_t \times J_s} \quad \text{for all } (t, s) \in \mathcal{L}^-_{T_{I \times J}}.$$

This means that for admissible blocks $(t, s) \in \mathcal{L}^+_{T_{I \times J}}$, only a small coupling matrix $S_{(t,s)} \in \mathbb{R}^{\Theta \times \Theta}$ has to be stored.
The **row cluster basis** \((V_t)_{t \in T_J}\) and the **column cluster basis** \((W_s)_{s \in T_J}\) can be stored efficiently using the identity theorem for polynomials: we have

\[
\Sigma_{B,t,\nu} = \sum_{\mu \in \Theta} \Sigma_{B,t,\nu}(\xi_{B,t,\mu}) \Sigma_{B,t,\mu} \quad \text{for all } t \in T_J \setminus \mathcal{L}_{T_J}, t' \in \text{sons}_{T_J}(t), \nu \in \Theta.
\]

Consequently, for every \(t \in T_J \setminus \mathcal{L}_{T_J}, t' \in \text{sons}_{T_J}(t), i \in \mathcal{I}_{t'}, \nu \in \Theta\) we get

\[
(V_{t,\nu})_{i,T} = \int_{\mathcal{T}} \varphi_i(x) \Sigma_{B,t,\nu}(x) \, dx = \int_{\mathcal{T}} \varphi_i(x) \sum_{\mu \in \Theta} \Sigma_{B,t,\nu}(\xi_{B,t,\mu}) \Sigma_{B,t,\mu}(x) \, dx
\]

\[
= \sum_{\mu \in \Theta} \int_{\mathcal{T}} \varphi_i(x) \Sigma_{B,t,\nu}(x) \, dx \Sigma_{B,t,\nu}(\xi_{B,t,\mu}) = \sum_{\mu \in \Theta} (V_{t,\mu})_{i,\mu} \Sigma_{B,t,\nu}(\xi_{B,t,\mu})
\]

and therefore

\[
V_{t,\nu} = V_{t,\mu} E_{\mu} \tag{17}
\]

with \(E_{\mu} \in \mathbb{R}^{\Theta \times \Theta}\) given by

\[
(E_{\mu})_{\nu,\mu} := \Sigma_{B,t,\nu}(\xi_{B,t,\mu}) \quad \text{for all } \mu \in \Theta, \nu \in \Theta.
\]

Since we also have \(\mathcal{I}_t = \bigcup_{t' \in \text{sons}_{T_J}(t)} I_{t'}\) for all \(t \in T_J \setminus \mathcal{L}_{T_J}\) by construction, we do not have to store all the matrices \((V_t)_{t \in T_J}\) explicitly. It is sufficient to store only the **leaf matrices** \((V_t)_{t \in \mathcal{L}_{T_J}}\) and the **transfer matrices** \((E_{\mu})_{t \in T_J \setminus \mathcal{L}_{T_J}}\) instead.

So far, we have not taken advantage of the translation-invariance eq. \((2)\) of the kernel function. It allows us to avoid storing all the coupling matrices \((S_0)_{b \in \mathcal{L}_{T_J}^+}^\perp\) individually: eq. \((13)\) implies

\[
\xi_{B,t,\nu} = \xi_{B(0),\nu} + m_t \quad \text{for all } t \in T_J^{(f)}, \nu \in \Theta, \tag{18a}
\]

\[
\xi_{C,s,\mu} = \xi_{C(0),\mu} + m_s \quad \text{for all } s \in T_J^{(f)}, \mu \in \Theta \tag{18b}
\]

on every level \(\ell \in \{0, \ldots, \ell_{\max}\}\) while, due to eq. \((11)\), the tree \(T_{I \times J}\) satisfies

\[
\text{level}_{T_{I \times J}}(t, s) = \text{level}_{T_I}(t) = \text{level}_{T_J}(s) \quad \text{for all } (t, s) \in T_{I \times J}. \tag{19}
\]

By combining eq. \((19)\) with eq. \((18)\) and eq. \((2)\) we get

\[
(S_{(t,s)})_{\nu,\mu} = g(\xi_{B,t,\nu}, \xi_{C,s,\mu}) = g(\xi_{B(0),\nu} + m_t, \xi_{C(0),\mu} + m_s)
\]

\[
= g(\xi_{B(0),\nu} + m_t - m_t, \xi_{C(0),\mu} + m_s - m_t)
\]

\[
= g(\xi_{B(0),\nu}, \xi_{C(0),\mu} - (m_t - m_s)) \quad \text{for all } (t, s) \in \mathcal{L}_{T_I \times J}^+ \cap T_{I \times J}^{(f)}, \nu, \mu \in \Theta
\]

on every level \(\ell \in \{0, \ldots, \ell_{\max}\}\). Hence, on every level \(\ell \in \{0, \ldots, \ell_{\max}\}\) the implication

\[
m_t - m_s = m_{\tilde{t}} - m_{\tilde{s}} \implies S_{(t,s)} = S_{(\tilde{t},\tilde{s})} \tag{20}
\]

holds for all \((t, s), (\tilde{t}, \tilde{s}) \in \mathcal{L}_{T_I \times J}^+ \cap T_{I \times J}^{(f)}\).
Moreover, due to our regular construction of the cluster tree, we also do not have to store all the transfer matrices individually: On every level $\ell \in \{0, ..., \ell_{\max} - 1\}$ we have, according to eq. (18a),

$$S_{B_{t}, \nu}(x) = S_{B(t), \nu}(x - m_{t})$$

for all $t \in T^{(\ell)}_{I}, \nu \in \Theta, x \in \mathbb{R}^{d}$,

which implies

$$(E_{t})_{\mu \nu} = S_{B_{t}, \nu}(\xi_{t}, \mu) = S_{B(t), \nu}(\xi_{t}, \mu) - m_{t}$$

$= S_{B(t), \nu}(\xi_{t+1}, \mu + m_{t} - m_{t})$$

$= S_{B(t), \nu}(\xi_{t+1}, \mu - (m_{t} - m_{t}'))$ for all $t \in T^{(\ell)}_{I}, t' \in \text{sons}_{T_{I}}(t), \nu, \mu \in \Theta$.

Hence, on every level $\ell \in \{0, ..., \ell_{\max} - 1\}$, we have

$$m_{t} - m_{t'} = m_{t} - m_{t'} \implies E_{t} = E_{t'}$$

(21)

for all $t, t' \in T^{(\ell)}_{I}, t' \in \text{sons}_{T_{I}}(t), t' \in \text{sons}_{\tilde{T}_{I}}(\tilde{t})$. The column cluster basis $(W_{s})_{s \in T_{J}}$ and the corresponding transfer matrices $(F_{s})_{s \in T_{J}}$ have similar properties.

The implications eq. (20) and eq. (21) allow us to significantly reduce the storage requirements of the $\mathcal{H}^{2}$-matrix representation by taking advantage of the translation-invariance of the kernel function.

## 3 Complexity estimates

In the following, let

$$n := \max\{|I|, |J|\}, \quad k := |\Theta| = (\theta + 1)^{d}.\tag{22}$$

For every $i \in \{1, ..., d\}$ let $e_{i} \in \mathbb{R}^{d}$ denote the $i$-th canonical unit vector.

**Lemma 1 (Storage requirements of the leaf matrices)** Storing the leaf matrices $(V_{l})_{l \in \mathcal{L}_{T_{I}}}$ and $(W_{s})_{s \in \mathcal{L}_{T_{J}}}$ requires not more than $2kn$ units of storage.

**Proof.** Combining eq. (22) with eq. (10) implies that storing the matrices $(V_{l})_{l \in \mathcal{L}_{T_{I}}}$ and $(W_{s})_{s \in \mathcal{L}_{T_{J}}}$ directly requires not more than

$$\sum_{t \in \mathcal{L}_{T_{I}}} |I_{t}| |\Theta| + \sum_{s \in \mathcal{L}_{T_{J}}} |J_{s}| |\Theta| = k \left( \sum_{t \in \mathcal{L}_{T_{I}}} |I_{t}| + \sum_{s \in \mathcal{L}_{T_{J}}} |J_{s}| \right) = k(|I| + |J|) \leq 2kn$$

units of storage. \hfill \square

**Lemma 2** On every level $\ell \in \{0, ..., \ell_{\max} - 1\}$ we have

$$\{m_{t} - m_{t'} : t \in T^{(\ell)}_{I}, t' \in \text{sons}_{T_{I}}(t)\} \subseteq \{0, -\delta_{e_{(\ell)}}\},$$

$$\{m_{s} - m_{s'} : s \in T^{(\ell)}_{J}, s' \in \text{sons}_{T_{J}}(s)\} \subseteq \{0, -\delta_{e_{(\ell)}}\},$$

where $\delta_{e_{(\ell)}}$ is the $e_{(\ell)}$-th canonical unit vector.
Proof. Let \( \ell \in \{0, ..., \ell_{\text{max}} - 1\} \), let \( t \in T^{(\ell)}_I \) and let \( t' \in \text{sons}_{T_{\ell}}(t) \). Then \( p_{\nu} \in \mathbb{N}^d_0 \) satisfies
\[
p_{\nu, t} = p_{\nu, t'} \quad \text{for all} \quad \ell \in \{1, ..., d\} \setminus \{\ell(t)\} \quad \text{and} \quad p_{\nu, t'} \in \{2p_{\nu, t} + 1\},
\]
according to eq. (8). Due to eq. (4) and eq. (6) we therefore get, as illustrated in Figure 2, \((m_t - m_{\nu})_t = 0\) for all \( \ell \in \{1, ..., d\} \setminus \{\ell(t)\} \) and
\[
(m_t - m_{\nu})_{t(\ell)} = p_{t(\ell)} \delta_{t(\ell)} - p_{t'(\ell)} \delta_{t'(\ell)}
\]
\[
\in \{p_{t(\ell)} \delta_{t(\ell)} - 2p_{t(\ell)} \delta_{t(\ell)}', p_{t(\ell)} \delta_{t(\ell)} - 2p_{t(\ell)} \delta_{t(\ell)}' - \delta_{t(\ell)}'\}
\]
\[
= \{p_{t(\ell)} \delta_{t(\ell)} - p_{t(\ell)} \delta_{t(\ell)}', p_{t(\ell)} \delta_{t(\ell)} - p_{t(\ell)} \delta_{t(\ell)}' - \delta_{t(\ell)}\}
\]
\[
= \{0, -\delta_{t(\ell)}'\},
\]
which proves the first statement. The second statement can be proven similarly.

Figure 2: The translations of the boxes of \( T_I \) and the translations of their sons.

**Theorem 1 (Storage requirements of the transfer matrices)** The transfer matrices \((E_t)_{t \in T_I \setminus \pi T_I}\) and \((F_s)_{s \in T_J \setminus \pi T_J}\) require not more than \(4k^2 \ell_{\text{max}}\) units of storage.

Proof. Combining the implication eq. (21) with lemma 2 and eq. (22) directly implies that for representing \((E_t)_{t \in T_I \setminus \pi T_I}\) and \((F_s)_{s \in T_J \setminus \pi T_J}\) not more than
\[
|\Theta|^2 \sum_{t=0}^{\ell_{\text{max}} - 1} \left| \{m_t - m_{\nu} : t \in T^{(\ell)}_I, t' \in \text{sons}_{T_{\ell}}(t)\} \right|
\]
\[
+ |\Theta|^2 \sum_{s=0}^{\ell_{\text{max}} - 1} \left| \{m_s - m_{s'} : s \in T^{(\ell)}_J, s' \in \text{sons}_{T_{\ell}}(s)\} \right|
\]
\[
\leq |\Theta|^2 \sum_{t=0}^{\ell_{\text{max}} - 1} 4 = 4k^2 \ell_{\text{max}}
\]
units of storage are needed.
In order to obtain further results, we now consider not only the boxes appearing as nodes in our trees $T_I$ and $T_J$, but the infinitely many boxes belonging to $\hat{T} := \bigcup_{\ell=0}^{\ell_{\text{max}}} \hat{T}^{(\ell)}$ given by (cf. eq. (14))

$$\hat{T}^{(\ell)} := \{r^{(\ell)} + \delta^{(\ell)} \circ p : p \in \mathbb{Z}^d\} \quad \text{for all } \ell \in \{0, ..., \ell_{\text{max}}\}. \quad (23)$$

For all $\ell \in \{0, ..., \ell_{\text{max}}\}$ let $\left( p_t \right)_{t \in \hat{T}^{(\ell)}}$ be defined by

$$p_t \in \mathbb{Z}^d, \quad t = r^{(\ell)} + \delta^{(\ell)} \circ p_t = r^{(\ell)} + m_t \quad \text{for all } t \in \hat{T}^{(\ell)} \quad (24)$$

and let (cf. eq. (13))

$$B_t := B^{(\ell)} + \delta^{(\ell)} \circ p_t = B^{(\ell)} + m_t \quad \text{for all } t \in \hat{T}^{(\ell)}, \quad (25a)$$

$$C_s := C^{(\ell)} + \delta^{(\ell)} \circ p_s = C^{(\ell)} + m_s \quad \text{for all } s \in \hat{T}^{(\ell)}. \quad (25b)$$

Based on (14), let

$$\hat{A}_\eta : \hat{T} \times \hat{T} \rightarrow \{\text{true}, \text{false}\}, \quad (t, s) \mapsto \begin{cases} \text{true} & \text{if } \max\{\text{diam}(B_t), \text{diam}(C_s)\} \leq \eta \text{ dist}(B_t, C_s) \\ \text{false} & \text{if } \max\{\text{diam}(B_t), \text{diam}(C_s)\} > \eta \text{ dist}(B_t, C_s) \end{cases}. \quad (26)$$

We base the following analysis on one key assumption.

**Assumption 1 (Diameters of supports)** We assume that there is a constant $C_{bb} \in \mathbb{R}_{\geq 1}$ satisfying

$$\text{diam}(B^{(\ell)}) \leq C_{bb} \text{ diam}(r^{(\ell)}) \quad \text{and} \quad \text{diam}(C^{(\ell)}) \leq C_{bb} \text{ diam}(r^{(\ell)}) \quad (27)$$

for all $\ell \in \{0, ..., \ell_{\text{max}}\}$.

We will see later on that under normal circumstances, due to our choice of $\ell_{\text{max}}$, this condition is actually fulfilled since we do not allow the boxes to become too small (see Remark 1).

Let $\lambda_d$ denote the $d$-dimensional Lebesgue measure.

**Lemma 3** There is a constant $C_{dv} \in \mathbb{R}_{>1}$ such that

$$\text{diam}(r^{(\ell)})^d \leq C_{dv} \lambda_d(r^{(\ell)}) \quad \text{for all } \ell \in \{0, ..., \ell_{\text{max}}\}. \quad (28)$$

**Proof.** According to eq. (5) and eq. (6), we have

$$\text{diam}(r^{(\ell)}) = \frac{1}{2} \text{ diam}(r^{(\ell-d)}) \quad \text{and} \quad \lambda_d(r^{(\ell)}) = \frac{1}{2^d} \lambda_d(r^{(\ell-d)}) \quad \text{for all } \ell \in \{d, ..., \ell_{\text{max}}\}$$

and therefore

$$\frac{\text{diam}(r^{(\ell)})^d}{\lambda_d(r^{(\ell)})} = \frac{\text{diam}(r^{(\ell-d)})^d}{\lambda_d(r^{(\ell-d)})} \quad \text{for all } \ell \in \{d, ..., \ell_{\text{max}}\}. \quad (29)$$
This implies
\[
\left\{ \frac{\text{diam}(r^{(\ell)})^d}{\lambda_d(r^{(\ell)})} : \ell \in \{0, \ldots, \ell_{\text{max}}\} \right\} = \left\{ \frac{\text{diam}(r^{(\ell)})^d}{\lambda_d(r^{(\ell)})} : \ell \in \{0, \ldots, d-1\} \right\} =: R.
\]
Hence, \(C_{dv} := \max R\) satisfies eq. \((28)\). \(\square\)

**Lemma 4 (Sparsity)** We define
\[
C_{sp} := 2^{-d} \left(3 + 2\eta^{-1}\right)^d C_{dv}^d \omega_d C_{dv},
\]
where \(\omega_d := \lambda_d(\{s \in \mathbb{R}^d : \|s\|_2 \leq 1\})\) is the volume of the \(d\)-dimensional unit ball.

For every \(\ell \in \{0, \ldots, \ell_{\text{max}}\}\) we have
\[
\begin{align*}
\left\{ s \in \hat{T}^{(\ell)} : \hat{A}_0(t, s) = \text{false} \right\} &\leq C_{sp} \quad \text{for all } t \in \hat{T}^{(\ell)}, \\
\left\{ t \in \hat{T}^{(\ell)} : \hat{A}_0(t, s) = \text{false} \right\} &\leq C_{sp} \quad \text{for all } s \in \hat{T}^{(\ell)}.
\end{align*}
\]

**Proof.** Let \(\ell \in \{0, \ldots, \ell_{\text{max}}\}\), let \(t \in \hat{T}^{(\ell)}\), let \(H_t := \{ s \in \hat{T}^{(\ell)} : \hat{A}_0(t, s) = \text{false} \}\) and let \(\beta_t\) denote the midpoint of \(B_t\). For every \(s \in H_t\) we have according to eq. \((29)\)
\[
\text{dist}(B_t, C_s) < \eta^{-1} \max\{\text{diam}(B_t), \text{diam}(C_s)\}.
\]
Thus for every \(s \in H_t\) there is at least one pair \((x_s, y_s) \in B_t \times C_s\) satisfying \(\|x_s - y_s\|_2 < \eta^{-1} \max\{\text{diam}(B_t), \text{diam}(C_s)\}\), which, according to \((26)\), implies
\[
\begin{align*}
\|z - \beta_t\|_2 &\leq \|z - y_s\|_2 + \|y_s - x_s\|_2 + \|x_s - \beta_t\|_2 \\
&< \text{diam}(C_s) + \eta^{-1} \max\{\text{diam}(B_t), \text{diam}(C_s)\} + \frac{1}{2} \text{diam}(B_t) \\
&\leq \frac{1}{2}(3 + 2\eta^{-1}) \max\{\text{diam}(B_t), \text{diam}(C_s)\} \\
&= \frac{1}{2}(3 + 2\eta^{-1}) \\text{diam}(B^{(\ell)}), \text{diam}(C^{(\ell)})\} =: \rho^{(\ell)} \quad \text{for all } z \in C_s,
\end{align*}
\]
which, combined with eq. \((12)\), eq. \((21)\) and eq. \((25)\), implies (cf. Figure 3)
\[
s \subseteq C_s \subset B_2[\beta_t, \rho^{(\ell)}] := \{ z \in \mathbb{R}^d : \|z - \beta_t\|_2 \leq \rho^{(\ell)} \}. \tag{29}
\]
Moreover, according to eq. \((23)\), we have (cf. Figure 3)
\[
\lambda_d(\{ z \in B_2[\beta_t, \rho^{(\ell)}] : \{s \in H_t : B_t \in C_s \} > 1 \}) = 0. \tag{30}
\]
By using eq. \((28)\), eq. \((23)\), eq. \((29)\), eq. \((30)\) and eq. \((27)\) we now get
\[
\begin{align*}
\text{diam}(r^{(\ell)})^d|H_t| &\leq C_{dv} \lambda_d(r^{(\ell)})|H_t| = C_{dv} \sum_{s \in H_t} \lambda_d(s) = C_{dv} \sum_{s \in H_t} \int_{B_2[\beta_t, \rho^{(\ell)}]} 1_s(z) \, dz \\
&= C_{dv} \int_{B_2[\beta_t, \rho^{(\ell)}]} \sum_{s \in H_t} 1_s(z) \, dz \leq C_{dv} \int_{B_2[\beta_t, \rho^{(\ell)}]} 1_{B_2[\beta_t, \rho^{(\ell)}]}(z) \, dz
\end{align*}
\]
which, in view of eq. (23), eq. (24), eq. (25) and eq. (26), implies
\[
= C_{d_0} \lambda_d(B_{\beta_\ell, \rho(\ell)}) = C_{d_0} \omega_d(\rho(\ell))^d
\]
\[
= C_{d_0} \omega_d 2^{-d}(3 + 2\eta^{-1})^d \max\{\text{diam}(B(\ell)), \text{diam}(C(\ell))\}^d
\]
\[
\leq C_{d_0} \omega_d 2^{-d}(3 + 2\eta^{-1})^d C_{\beta_\ell} \text{diam}(r(\ell))^d,
\]
which proves the first statement. The second statement can be proven similarly. □

**Lemma 5 (Difference bound for \(T_{\ell \times \mathcal{J}}\))** On every level \(\ell \in \{0, \ldots, \ell_{\text{max}}\}\) we have
\[
\left| \{m_t - m_s : (t, s) \in T_{\ell \times \mathcal{J}}^{(0)} \} \right| \leq 3C_{\beta_\ell}.
\]

**Proof.** By definition, we have \(T_{\ell \times \mathcal{J}}^{(0)} = \{(\mathcal{R}_{T_\ell}, \mathcal{R}_{T_\ell})\} = \{(r(0), r(0))\}\) and therefore
\[
\{m_t - m_s : (t, s) \in T_{\ell \times \mathcal{J}}^{(0)} \} = \{m_{r(0)} - m_{r(0)}\} = \{0\}.
\]
Let \(\ell \in \{1, \ldots, \ell_{\text{max}}\}\) and let \((t, s) \in T_{\ell \times \mathcal{J}}^{(0)}\). Then, by the definition of \(T_{\ell \times \mathcal{J}}\) and due to eq. (19), there is a pair \((\check{t}, \check{s}) \in T_{\ell \times \mathcal{J}}^{(\ell - 1)} \subseteq T_{\ell_1}^{(\ell - 1)} \times T_{\ell_2}^{(\ell - 1)}\) satisfying \(t \in \text{sons}_{T_\ell}(\check{t})\), \(s \in \text{sons}_{T_\ell}(\check{s})\) and \(A_\eta(\check{t}, \check{s}) = 0\). According to eq. (13) and eq. (14), we therefore get
\[
\max\{\text{diam}(B^{(\ell - 1)}), \text{diam}(C^{(\ell - 1)})\} = \max\{\text{diam}(B_\check{t}), \text{diam}(C_\check{s})\}\eta\text{dist}(B_\check{t}, C_\check{s})
\]
\[
= \eta \inf\{\|x + m_\check{t} - (y + m_\check{s})\|_2 : x \in B^{(\ell - 1)}, y \in C^{(\ell - 1)}\}
\]
\[
= \eta \inf\{\|x - (y - (m_\check{t} - m_\check{s}))\|_2 : x \in B^{(\ell - 1)}, y \in C^{(\ell - 1)}\}
\]
\[
= \eta \text{dist}(B^{(\ell - 1)}, C^{(\ell - 1)} - (m_\check{t} - m_\check{s}))
\]
which, in view of eq. (23), eq. (24), eq. (25) and eq. (26), implies
\[
r^{(\ell - 1)} - (m_\check{t} - m_\check{s}) = r^{(\ell - 1)} + \delta^{(\ell - 1)} \odot (p_\check{t} - p_\check{s}) \in \left\{ u \in \hat{T}^{(\ell - 1)} : \hat{A}_\eta(r^{(\ell - 1)}, u) = \text{false} \right\}.
\]
Moreover, lemma 2 implies
\[
m_t - m_s \in \{m_\check{t} - m_\check{s}, m_\check{t} - m_\check{s} + \delta^{(\ell - 1)} e_{\ell - 1}, m_\check{t} - m_\check{s} - \delta^{(\ell - 1)} e_{\ell - 1}\},
\]
which, combined with eq. (31) and eq. (24), leads to
\[ m_t - m_s \in \bigcup_{u \in H^{(\ell-1)}} \{-m_u, -m_u + \delta_{i(\ell-1)}^{(\ell)} e_{i(\ell-1)}, -m_u - \delta_{i(\ell)}^{(\ell)} e_{i(\ell-1)}\}. \]

Since we have chosen \((t, s) \in T_{I \times J}^{(\ell)}\) arbitrarily, this leads to
\[ |\{m_t - m_s : (t, s) \in T_{I \times J}^{(\ell)}\}| \leq 3|H^{(\ell-1)}|. \]
Furthermore, due to lemma 4, we have \(|H^{(\ell-1)}| \leq C_{sp}\). □

**Theorem 2 (Storage requirements of the coupling matrices)** The coupling matrices \((S_b)_{b \in \mathcal{L}_{T_{I \times J}}^+}\) require not more than \(3C_{sp}k^2(\ell_{\max} - d)\) units of storage.

**Proof.** On every level \(\ell \in \{0, \ldots, d\}\) all boxes \(t \in T_{I}^{(\ell)}\) and \(s \in T_{J}^{(\ell)}\) contain, by construction, the midpoint of \(r^{(0)}\), which implies \(\text{dist}(B_t, C_s) = 0\) for all \((t, s) \in T_{I \times J}^{(\ell)}\) and therefore \(\mathcal{L}_{T_{I \times J}}^+ \cap T_{I \times J}^{(\ell)} = \{(t, s) \in T_{I \times J}^{(\ell)} : A_{\eta}(t, s) = \text{true}\} = \emptyset\). Combining this with the implication eq. (20), lemma 5 and eq. (22) directly implies that for representing all coupling matrices \((S_b)_{b \in \mathcal{L}_{T_{I \times J}}^+}\) not more than
\[ |\Theta|^2 \sum_{\ell=d+1}^{\ell_{\max}} \left|\{m_t - m_s : (t, s) \in \mathcal{L}_{T_{I \times J}}^+ \cap T_{I \times J}^{(\ell)}\}\right| \leq k^2 \sum_{\ell=d+1}^{\ell_{\max}} 3C_{sp} = 3C_{sp}k^2(\ell_{\max} - d) \]
units of storage are needed. □

In order for arithmetic operations like the matrix-vector multiplication to have the desired complexity, \(T_{I \times J}\) has to be sparse. This property is guaranteed by our construction.

**Lemma 6 (Sparsity of \(T_{I \times J}\))** We have
\[ |\{s \in T_{J} : (t, s) \in T_{I \times J}\}| \leq 2C_{sp} \quad \text{for all } t \in T_{I}, \]
\[ |\{t \in T_{I} : (t, s) \in T_{I \times J}\}| \leq 2C_{sp} \quad \text{for all } s \in T_{J}. \]

**Proof.** The statements are implied by the definition of \(T_{I \times J}\), eq. (19), lemma 4 and the fact that every node of \(T_{I}\) or \(T_{J}\) has at most two sons. □

For the sake of completeness, we also have to consider the storage requirements of the nearfield matrices
\[ N_{(t, s)} := G|_{I \times J_s} \quad \text{for all } (t, s) \in \mathcal{L}_{T_{I \times J}}^- \]
that capture the part of the matrix that cannot be compressed. Since we want to maintain the ability to work with unstructured surface meshes, we cannot avoid storing all of these matrices explicitly.

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Lemma 7 (Storage requirements of the nearfield matrices) We define the resolutions of the cluster trees $T_I$ and $T_J$ by

$$\gamma_I := \max \{|I_t| : t \in L_{T_I}\}, \quad \gamma_J := \max \{|J_s| : s \in L_{T_J}\}.$$

Storing $(N_b)_{b \in \mathcal{L}_{T_I} \times T_J}$ requires not more than $C_{sp} \min\{\gamma_I, \gamma_J\} n$ units of storage.

**Proof.** By construction and due to (11), (4) and (23) we have

$$L_{T_I} \times L_{T_J} \subseteq L_{T_I} \times L_{T_J} = T_I^{(\ell_{max})} \times T_J^{(\ell_{max})} \subseteq \hat{T}(\ell_{max}) \times \hat{T}(\ell_{max}).$$

Lemma 4 and (11) therefore imply that storing all nearfield matrices $(N_b)_{b \in \mathcal{L}_{T_I} \times T_J}$ requires not more than

$$\sum_{(t,s) \in \mathcal{L}_{T_I} \times T_J} |I_t| |J_s| = \sum_{t \in \mathcal{L}_{T_I}} \sum_{s \in \mathcal{L}_{T_J}} |I_t| |J_s| \leq \sum_{t \in \mathcal{L}_{T_I}} \sum_{s \in \mathcal{L}_{T_J}} \gamma_J |I_t| \leq \sum_{t \in \mathcal{L}_{T_I}} C_{sp} \gamma_J |I_t| = C_{sp} \gamma_J \sum_{t \in \mathcal{L}_{T_I}} |I_t| = C_{sp} \gamma_J |I|$$

units of storage and similarly not more than $C_{sp} \gamma_J |J|$ units of storage, which, in view of (22), implies the statement. \(\square\)

Theorem 1, lemma 7 and theorem 2 suggest that we still have to discuss how to properly choose the maximal level $\ell_{max}$. On the one hand, we have to choose $\ell_{max}$ large enough to keep at least one of the resolutions $\gamma_I$ and $\gamma_J$ sufficiently small. On the other hand, since the complexity of the matrix-vector multiplication mainly depends on $|T_I|$ and $|T_J|$ (cf. [6, Theorem 3.42]), we have to choose $\ell_{max}$ small enough to prevent $|T_I|$ and $|T_J|$ from getting too large.

In order to be able to analyze $\gamma_I$ and $\gamma_J$, we assume that the relative number of characteristic points assigned to a box (cf. (7) and (9)) can essentially be estimated by its size, i.e., we assume that there is a constant $C_{nd} \in \mathbb{R}_{>0}$ satisfying

$$|I_t| \leq C_{nd} \operatorname{diam}(t)^{d-1} |I|, \quad |J_s| \leq C_{nd} \operatorname{diam}(s)^{d-1} |J| \quad \text{for all } t \in T_I, s \in T_J. \quad (32)$$

This condition is, for example, usually fulfilled on shape-regular and quasi-uniform meshes (cf. [30, Remark 4.1.14]). Due to (15) and (16), we have

$$\operatorname{diam}(r^{(\ell+d)}) = \frac{1}{2} \operatorname{diam}(r^{(\ell)}) \quad \text{for all } \ell \in \{0, \ldots, \ell_{max} - d\}. \quad (33)$$

For the sake of simplicity, we therefore choose $\ell_{max} \in d \cdot N$. According to (32), (4) and (33), we thus get

$$|I_t| \leq C_{nd} \operatorname{diam}(t)^{d-1} |I| = C_{nd} \operatorname{diam}(r^{(\ell_{max})})^{d-1} |I|$$

\(15\)
and likewise
\[ |J_s| \leq C_{nd} 2^{-\frac{d-1}{d} \ell_{\text{max}}} \text{diam}(r(0))^{d-1} |J| \quad \text{for all } s \in T^{(\ell_{\text{max}})} = \mathcal{T}_I. \]

Hence, with \( C_{rs} := C_{nd} \text{diam}(r(0))^{d-1} \) we have
\[
\gamma_I \leq C_{rs} 2^{-\frac{d-1}{d} \ell_{\text{max}}} |I|, \quad \gamma_J \leq C_{rs} 2^{-\frac{d-1}{d} \ell_{\text{max}}} |J|. \tag{34}
\]

We choose a constant \( C_{rk} \in \mathbb{R}_{>0} \) satisfying \( C_{rk} k \leq \min\{|I|, |J|\} \) and observe
\[
2^{-\frac{d-1}{d} \ell_{\text{max}}} |I| \leq C_{rk} k \iff \ell_{\text{max}} \geq \frac{d}{d-1} \left( \log_2(|I|) - \log_2(C_{rk}) \right),
\]
\[
2^{-\frac{d-1}{d} \ell_{\text{max}}} |J| \leq C_{rk} k \iff \ell_{\text{max}} \geq \frac{d}{d-1} \left( \log_2(|J|) - \log_2(C_{rk}) \right).
\]

Based on this, we choose
\[
\ell_{\text{max}} := \min \left\{ \ell \in d \cdot \mathbb{N} : \ell \geq \frac{d}{d-1} \left( \min\{|\log_2(|I|)|, |\log_2(|J|)| \} - \log_2(C_{rk}) \right) \right\} \tag{35}
\]
to gain, according to eq. (34),
\[
\min\{\gamma_I, \gamma_J\} \leq C_{rs} C_{rk} k. \tag{36}
\]

In order to be able to analyze \( |T_I| \) and \( |T_J| \), we assume that the number of boxes per level which contain only a relatively small part of the boundary \( \Gamma \) does not grow too fast, i.e., we assume that there are constants \( C_{da}, C_{sa} \in \mathbb{R}_{>0} \) satisfying
\[
|\{t \in T_I^{(\ell)} : \text{diam}(t)^{d-1} > C_{da} |t \cap \Gamma| \}| \leq C_{sa} \text{diam}(r(\ell))^{-(d-1)} \tag{37a}
\]
\[
|\{s \in T_J^{(\ell)} : \text{diam}(s)^{d-1} > C_{da} |s \cap \Gamma| \}| \leq C_{sa} \text{diam}(r(\ell))^{-(d-1)} \tag{37b}
\]
on every level \( \ell \in \{0, ..., \ell_{\text{max}}\} \), where
\[
|X| := \int_{\Gamma} 1_X(y) \, dy \quad \text{for all measurable subsets } X \subseteq \Gamma.
\]

**Lemma 8** On every level \( \ell \in \{0, ..., \ell_{\text{max}}\} \) we have
\[
|\{t \in T_I^{(\ell)} : \text{diam}(t)^{d-1} \leq C_{da} |t \cap \Gamma| \}| \leq 2C_{da} |\Gamma| \text{diam}(r(\ell))^{-(d-1)},
\]
\[
|\{s \in T_J^{(\ell)} : \text{diam}(s)^{d-1} \leq C_{da} |s \cap \Gamma| \}| \leq 2C_{da} |\Gamma| \text{diam}(r(\ell))^{-(d-1)}.
\]

**Proof.** Let \( \ell \in \{0, ..., \ell_{\text{max}}\} \) and let \( T^{(\ell)}_{I, C_{da}} := \{t \in T_I^{(\ell)} : \text{diam}(t)^{d-1} \leq C_{da} |t \cap \Gamma| \} \). Due to (4), the surface area of the set \( \{x \in \Gamma : |\{t \in T_I^{(\ell)}_{I, C_{da}} : x \in t\}| > 2\} \) equals 0. Thus
\[
\text{diam}(r(\ell))^{d-1} |T^{(\ell)}_{I, C_{da}}| = \sum_{t \in T^{(\ell)}_{I, C_{da}}} \text{diam}(t)^{d-1} \leq \sum_{t \in T^{(\ell)}_{I, C_{da}}} C_{da} |t \cap \Gamma|
\]
By using \((38)\), \((33)\) and \((39)\) we get

\[
\begin{align*}
\sum_{t \in T_i^{(\ell)} \cap C_{da}} I_{t \cap \Gamma(y)} dy &= C_{da} \int_{\Gamma} \sum_{t \in T_i^{(\ell)} \cap C_{da}} I_{t \cap \Gamma(y)} dy \\
&\leq C_{da} \int_{\Gamma} 2 \cdot 1_{\Gamma(y)} dy = 2C_{da} |\Gamma|
\end{align*}
\]

holds, which proves the first statement. The second statement can be proven similarly. \(\square\)

Defining \(C_{nb} := 2C_{da} |\Gamma| + C_{sa}\) and combining assumption \((37)\) with Lemma \(8\) yield

\[
\begin{align*}
|T_i^{(\ell)}| &\leq C_{nb} \text{diam}(r^{(\ell)})^{-(d-1)} \quad \text{for all } \ell \in \{0, \ldots, \ell_{\max}\}, \quad (38a) \\
|T_j^{(\ell)}| &\leq C_{nb} \text{diam}(r^{(\ell)})^{-(d-1)} \quad \text{for all } \ell \in \{0, \ldots, \ell_{\max}\}. \quad (38b)
\end{align*}
\]

Moreover, according to \((35)\), we also have

\[
\ell_{\max} \leq \frac{d}{\ell_{\max}} \left( \min\{\log_2(|I|), \log_2(|J|)\} - \log_2(C_{rk}k) \right) + d. \quad (39)
\]

By using \((38a)\), \((38b)\) and \((39)\) we get

\[
|T_i| = \sum_{\ell=0}^{\ell_{\max}} |T_i^{(\ell)}| \leq \sum_{\ell=0}^{\ell_{\max}} C_{nb} \text{diam}(r^{(\ell)})^{-(d-1)} = C_{nb} \sum_{\ell=0}^{\ell_{\max}} \text{diam}(r^{(\ell)})^{-(d-1)}
\]

\[
= C_{nb} \left( \text{diam}(r^{(0)})^{-(d-1)} + \sum_{\ell=0}^{\ell_{\max}-1} \sum_{u=1}^{d} \text{diam}(r^{(d(\ell+u))})^{-(d-1)} \right)
\]

\[
< C_{nb} \left( \text{diam}(r^{(0)})^{-(d-1)} + \sum_{\ell=0}^{\ell_{\max}-1} d \cdot \text{diam}(r^{(d(\ell+1))})^{-(d-1)} \right)
\]

\[
= C_{nb} \left( \text{diam}(r^{(0)})^{-(d-1)} + d \sum_{\ell=0}^{\ell_{\max}-1} 2^{\ell+1}(d-1) \text{diam}(r^{(0)})^{-(d-1)} \right)
\]

\[
< C_{nb} \left( d \cdot \text{diam}(r^{(0)})^{-(d-1)} + d \sum_{\ell=0}^{\ell_{\max}-1} 2^{\ell+1}(d-1) \right)
\]

\[
= C_{nb}d \text{diam}(r^{(0)})^{-(d-1)} \left( 1 + \sum_{\ell=1}^{\ell_{\max}} (2^{d-1})^\ell \right)
\]

\[
= C_{nb}d \text{diam}(r^{(0)})^{-(d-1)} \sum_{\ell=0}^{\ell_{\max}} (2^{d-1})^\ell = C_{nb}d \text{diam}(r^{(0)})^{-(d-1)} \left( 1 - \frac{(2^{d-1})^{\ell_{\max}+1}}{1 - 2^{d-1}} \right)
\]

\[
< C_{nb}d \text{diam}(r^{(0)})^{-(d-1)} (2^{d-1} - 1)^{-1} 2^{d-1} \ell_{\max} + d - 1
\]

\[
\leq C_{nb}d \text{diam}(r^{(0)})^{-(d-1)} (2^{d-1} - 1)^{-1} 2^{d-1} \left( \frac{d}{\ell_{\max}} \left( \min\{\log_2(|I|), \log_2(|J|)\} - \log_2(C_{rk}k) \right) + d \right) + d - 1
\]

\[
= C_{nb}d \text{diam}(r^{(0)})^{-(d-1)} (2^{d-1} - 1)^{-1} 2 \log_2(|I|) - \log_2(C_{rk}k) + 2(d-1)
\]
\[ C_{\text{nb}}d \text{diam}(r(0))^{-(d-1)}(2^{d-1} - 1)^{-1}2^{2(d-1)}C_{rk}^{-1}k^{-1}|I|. \]

Due to \( x^2 = \frac{x^2(x+2)}{(x-1)(x+2)} = \frac{x^2(x+2)}{x^2-x-2} \leq x + 2 \) for \( x \in \mathbb{R}_{\geq 2} \) we therefore have

\[ |T_I| < C_{\text{nb}}d \text{diam}(r(0))^{-(d-1)}(2^{d-1} + 2)C_{rk}^{-1}k^{-1}|I|. \]

and analogously

\[ |T_J| < C_{\text{nb}}d \text{diam}(r(0))^{-(d-1)}(2^{d-1} + 2)C_{rk}^{-1}k^{-1}|J|. \]

Finally, based on our choice of the maximal level \( \ell_{\text{max}} \), we can now specify the storage requirements for our \( H^2 \)-matrix approximation:

**Theorem 3 (Total storage requirements)** \( \tilde{G} \) requires not more than

- 2\( kn \) units of storage for the leaf matrices,
- 4\( k^2 (\frac{d}{d} \log_2(n) - \log_2(C_{rk}k)) + d \) units of storage for the transfer matrices,
- \( C_{sp}C_{rs}C_{rk}kn \) units of storage for the nearfield matrices,
- 3\( C_{sp}k^2 \frac{d}{d} \left( \log_2(n) - \log_2(C_{rk}k) \right) \) units of storage for the coupling matrices.

**Proof.** The statement is a direct consequence of Lemma 1, Theorem 1, Lemma 7, Theorem 2, (36) and (39).

Compared to the standard \( H^2 \)-approach, these storage complexity bounds lead to a significant reduction of the total storage requirements, as we will see and discuss in chapter 4.

**Remark 1 (Bounded enlargement)** Due to (33) and (39), we have

\[ \text{diam}(r(\ell)) \geq \text{diam}(r(\ell_{\text{max}})) \sim 2^{-\frac{\ell_{\text{max}}}{a}} \geq 2^{-\frac{\log_2(n)}{a}} = n^{-\frac{1}{a}} \quad \text{for all } \ell \in \{0, \ldots, \ell_{\text{max}}\}. \]

In many situations, the diameters of the supports behave like \( n^{-\frac{1}{a}} \), for example on shape-regular and quasi-uniform meshes. In these cases, the condition (27) is therefore fulfilled.

**4 Numerical experiments**

In the following, let

\[ g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, \quad (x, y) \mapsto \begin{cases} \frac{1}{2\pi} \frac{1}{\|x-y\|^2} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \]

and for all \( y \in \Gamma = \partial \Omega \) let \( n_y \) denote the corresponding outward-pointing unit normal vector. Under suitable conditions, a solution \( u \) of the *interior Dirichlet boundary value problem*

\[ -\Delta u = 0 \quad \text{in } \Omega, \]
\[ u = g_D \quad \text{on } \Gamma \quad (40) \]
is given by Green’s representation formula (cf. [23] Theorem 2.2.2, [30] Theorem 3.1.6), which reduces obtaining $u$ to obtaining the corresponding Neumann boundary values $g_N : \Gamma \rightarrow \mathbb{R}, y \mapsto \partial_n \psi(y)$. Depending on the situation, $g_N$ as well as the given Dirichlet boundary values $g_D$ might only exist in a generalized sense. The Neumann values $g_N$ can be obtained by solving a boundary integral equation resulting from combining Green’s representation formula with suitable trace operators (cf. [30] Section 3.4.2.1). Applying a Galerkin discretization to this integral equation on a triangulation of $\Gamma$ with discontinuous piecewise constant basis functions $(\varphi_i)_{i \in I}$ and replacing $g_D$ with the corresponding $L^2$-orthogonal projection $\tilde{g}_D$ into a space spanned by continuous piecewise linear nodal basis functions $(\psi_j)_{j \in J}$ leads to a linear system

$$Gx = \left( K + \frac{1}{2}M \right) b,$$

where $b \in \mathbb{R}^I$ contains the coefficients of $\tilde{g}_D = \sum_{j \in J} b_j \psi_j$, $x \in \mathbb{R}^I$ contains the coefficients of the approximate Neumann values $\tilde{g}_N := \sum_{i \in I} x_i \varphi_i$, and where $G \in \mathbb{R}^{I \times I}$, $K \in \mathbb{R}^{I \times J}$ and $M \in \mathbb{R}^{J \times J}$ are given by

$$G_{ij} := \int_\Gamma \varphi_i(x) \int_\Gamma g(x,y) \varphi_j(y) \, dy \, dx \quad \text{for all } i, j \in I,$$

$$K_{ij} := \int_\Gamma \varphi_i(x) \int_\Gamma \frac{\partial}{\partial n_y}(x,y) \psi_j(y) \, dy \, dx \quad \text{for all } i \in I, j \in J,$$

$$M_{ij} := \int_\Gamma \varphi_i(x) \psi_j(x) \, dx \quad \text{for all } i \in I, j \in J.$$

The derivative $\frac{\partial}{\partial n_y}$ is applied with respect to the $y$-variable.

We construct an $H^2$-matrix approximation $\tilde{G}$ of $G$ as described in section [2]. In order to approximate $K$, we replace $\frac{\partial}{\partial n_y}g(x,y)$ with $\frac{\partial}{\partial n_y}(\mathcal{J}_{B_t} \otimes \mathcal{J}_{C_s})[g](x,y)$ for all $(x,y) \in B_t \times C_s$, $(t,s) \in \mathcal{L}_{I_t \times I_s}^+$ (cf. eq. [15]). This approach leads to an $H^2$-matrix approximation $\tilde{K}$ of $K$ which differs from the $H^2$-matrix described in section [2] only in terms of the leaf matrices for the column cluster basis, which are given by

$$W_s \in \mathbb{R}^{J \times \Theta}, \quad (W_s)_{j\mu} := \int_\Gamma \psi_j(y) \frac{\partial}{\partial n_y} \Sigma_{C_s,\mu}(y) \, dy \quad \text{for all } s \in \mathcal{L}_{T_j}, j \in J_s, \mu \in \Theta.$$

Hence, all results from section [3] also hold for $\tilde{K}$. Moreover, we can expect a similar convergence behavior as for $\tilde{G}$ (cf. [3] Chapter 4). The sparse matrix $\frac{1}{2}M$ can simply be added to the nearfield entries of $\tilde{K}$ and therefore does not require any additional storage.

In this way, we approximate the Neumann values of the function $u_0 := 4\pi g(\cdot, y_0)$ with $y_0 := (1.2, 1.2, 1.2)^T$ on an approximation $\Gamma$ of the unit sphere $S^2 := \{ x \in \mathbb{R}^3 : ||x||_2 = 1 \}$ consisting of plane triangles. $S^2$ and its approximation are both contained in the box $[-1, 1]^3$, which we use as our starting box $r(0)$. We choose the centers of gravity of the triangles as characteristic points for $(\varphi_i)_{i \in I}$, the vertices of the triangles as characteristic points for $(\psi_j)_{j \in J}$, $C_{rk} = 2$ and $\eta = 2$. We denote the
maximal level of the trees used for the construction of \( \tilde{G} \) by \( \ell_{\text{max}}^{I \times I} \) and the maximal level of the trees used for the construction of \( \tilde{K} \) by \( \ell_{\text{max}}^{I \times J} \). We compute the matrix entries by tensor Gauss quadrature as described in [16, 29, 30, Chapter 5] with 3 quadrature points per dimension for the regular integrals and 5 quadrature points per dimension for the singular integrals. In order to preserve the convergence behavior of the standard Galerkin solution, we increase the interpolation degree \( \theta \) by 1 whenever the mesh width \( h \) is halved, i.e., whenever the number \( n = \max\{|I|, |J| = |I| \} \) of triangles is quadrupled (cf. [6, Section 10.1 and Section 10.2]). We solve the linear system using the cg method with a relative residual accuracy of \( 10^{-6} \) (cf. [30, Section 6.1]).

It is clearly visible in the Tables 1, 2 and 3 that the storage requirements of our \( \mathcal{H}^2 \)-matrices \( \tilde{G} \) and \( \tilde{K} \) behave as predicted in Chapter 3: The storage for the leaf matrices grows like \( \mathcal{O}(nk) \) (cf. Lemma 1), the storage for the transfer matrices grows like \( \mathcal{O}(\ell_{\text{max}}^{I \times I} k^2) \) (cf. Theorem 1), the storage for the nearfield matrices grows like \( \mathcal{O}(2^{-\frac{3}{2} \ell_{\text{max}} n^2}) \) (cf. Lemma 7, (34)) and the storage for the coupling matrices grows like \( \mathcal{O}((\ell_{\text{max}} - 3) k^2) \) (cf. Theorem 2).

| \( n \) | \( \theta \) | \( k \) | \( \ell_{\text{max}}^{I \times I} \) | \( \ell_{\text{max}}^{I \times J} \) | \( \epsilon_{L^2} \) |
|---|---|---|---|---|---|
| 8192 | 4 | 125 | 9 | 9 | 1.194-2 |
| 32768 | 5 | 216 | 12 | 9 | 5.683-3 |
| 131072 | 6 | 343 | 12 | 12 | 2.870-3 |
| 524288 | 7 | 512 | 15 | 15 | 1.412-3 |
| 2097152 | 8 | 729 | 18 | 15 | 7.050-4 |

Table 1: Parameters and resulting \( L^2 \)-error \( \epsilon_{L^2} := \|g_N - \tilde{g}_N\|_{L^2(\Gamma_S)} \).

| \( n \) | leaf matrices | transfer matrices | nearfield matrices | coupling matrices |
|---|---|---|---|---|
| 8192 | 15.6 | 4.3 | 74.4 | 51.8 |
| 32768 | 108.1 | 17.1 | 303.8 | 218.6 |
| 131072 | 686.1 | 43.1 | 3913.3 | 509.9 |
| 524288 | 4096.3 | 120.0 | 16090.5 | 1496.1 |
| 2097152 | 23329.2 | 291.9 | 65051.2 | 3762.8 |

Table 2: Storage requirements of \( \tilde{G} \) on \( \Gamma_S \) in MB.

| \( n \) | leaf matrices | transfer matrices | nearfield matrices | coupling matrices |
|---|---|---|---|---|
| 8192 | 11.7 | 4.3 | 38.8 | 54.2 |
| 32768 | 81.0 | 12.8 | 476.1 | 143.9 |
| 131072 | 514.6 | 43.1 | 1956.3 | 524.3 |
| 524288 | 3072.3 | 120.0 | 8047.3 | 1528.1 |
| 2097152 | 17496.3 | 243.3 | 128996.6 | 2765.3 |

Table 3: Storage requirements of \( \tilde{K} \) on \( \Gamma_S \) in MB.
Figure 4 and Figure 5 show that the storage requirements of the matrix components can be bounded as predicted by Theorem 3. The expected theoretical convergence rate of $O(h)$ for the Neumann values in the $L^2$-norm (cf. [30, Theorem 4.1.33] and [15, Theorem 4.6]) is preserved (see Table 1).

We want to compare these results with the corresponding results obtained by conventional $H^2$-matrix approximations $\tilde{G}_{\text{conv}}$ and $\tilde{K}_{\text{conv}}$, which are based on different cluster trees and support bounding boxes (cf. [6, Section 3.3]): The construction is focused on reducing the diameters of the boxes as fast as possible instead of keeping them uniform on every level. Hence, one has to store an individual transfer matrix for almost every
box and an individual coupling matrix for every admissible leaf of the block cluster tree. Since approximating a block with less than \( k^2 \) entries would therefore be inefficient, the recursion of the algorithm for the cluster tree construction stops when it reaches a box with not more than \( 2k \) characteristic points. On the finest mesh \( (n = 2097152) \), \( \tilde{G}_{\text{conv}} \) requires 23328.1 MB for the leaf matrices, 33718.1 MB for the transfer matrices, 241218.8 MB for the nearfield matrices and 206819.6 MB for the coupling matrices. \( \tilde{K}_{\text{conv}} \) requires 17496.1 MB for the leaf matrices, 25187.2 MB for the transfer matrices, 200100.7 MB for the nearfield matrices and 126874.0 MB for the coupling matrices. The resulting \( L^2 \)-error is 7.022\( -4 \).

In order to test our translation-invariant compression method on a more challenging boundary containing edges and corners, we approximate the Neumann values of the function \( u_1: \mathbb{R}^3 \rightarrow \mathbb{R}, x \mapsto x_1^2 - x_3^2 \) on the boundary \( \Gamma_C := \partial([-1, 1] \times [-\frac{3}{4}, \frac{3}{4}] \times [-\frac{1}{2}, \frac{1}{2}]) \) of a rectangular cuboid by the same method as before, i.e., we replace the matrices \( G \) and \( K \) appearing in (41) with their respective translation-invariant \( H^2 \)-matrix approximations \( \tilde{G} \) and \( \tilde{K} \) and then solve the resulting perturbed linear system with the cg method. This time, we use \( r^{(0)} = [-1, 1] \times [-\frac{3}{4}, \frac{3}{4}] \times [-\frac{1}{2}, \frac{1}{2}] \) as the starting box, \( C_{rk} = 1 \) for the determination of \( \ell_{max}^{I \times J} \) and increasing quadrature orders. The number of quadrature points per dimension for the regular integrals is denoted by \( q_{\text{reg}} \) and the number of quadrature points per dimension for the singular integrals is denoted by \( q_{\text{sing}} \). The characteristic points, \( \eta \), \( \ell_{max}^{I \times J} \) and the accuracy of the cg method are chosen as before.

| \( n \) | \( q_{\text{reg}} \) | \( q_{\text{sing}} \) | \( \theta \) | \( k \) | \( \ell_{max}^{I \times I} \) | \( \ell_{max}^{I \times J} \) | \( \epsilon_{L^2} \) |
|---|---|---|---|---|---|---|---|
| 8112 | 4 | 6 | 4 | 125 | 9 | 9 | 3.895\( -2 \) |
| 32448 | 4 | 7 | 5 | 216 | 12 | 12 | 1.484\( -2 \) |
| 129792 | 4 | 8 | 6 | 343 | 12 | 12 | 7.408\( -3 \) |
| 519168 | 5 | 9 | 7 | 512 | 15 | 15 | 2.626\( -3 \) |
| 2076672 | 5 | 10 | 8 | 729 | 18 | 18 | 1.332\( -3 \) |

Table 4: Parameters and resulting \( L^2 \)-error \( \epsilon_{L^2} := \| g_N - \tilde{g}_N \|_{L^2(\Gamma_C)} \).

| \( n \) | \( \text{leaf matrices} \) | \( \text{transfer matrices} \) | \( \text{nearfield matrices} \) | \( \text{coupling matrices} \) |
|---|---|---|---|---|
| 8112 | 15.5 | 4.3 | 55.9 | 48.0 |
| 32448 | 107.0 | 17.1 | 192.4 | 215.8 |
| 129792 | 679.4 | 43.1 | 1906.6 | 484.8 |
| 519168 | 4056.4 | 120.0 | 6915.7 | 1488.1 |
| 2076672 | 23101.6 | 291.9 | 26143.3 | 3843.8 |

Table 5: Storage requirements of \( \tilde{G} \) on \( \Gamma_C \) in MB.

As in the previous experiment, the storage requirements of the translation-invariant \( H^2 \)-matrix approximations and the resulting \( L^2 \)-error of the Neumann values behave as predicted by the theory (cf. Tables 4, 5, 6 and Figures 6, 7).
| $n$   | leaf matrices | transfer matrices | nearfield matrices | coupling matrices |
|-------|---------------|-------------------|-------------------|------------------|
| 8112  | 11.6          | 4.3               | 31.0              | 59.0             |
| 32448 | 80.3          | 17.1              | 106.7             | 269.2            |
| 129792| 509.6         | 43.1              | 1237.3            | 545.8            |
| 519168| 3042.4        | 120.0             | 4480.0            | 1624.1           |
| 2076672| 17326.6      | 291.9             | 16913.0           | 4119.6           |

Table 6: Storage requirements of $\tilde{K}$ on $\Gamma_C$ in MB.

Figure 6: Storage requirements of $\tilde{G}$ on $\Gamma_C$ in MB.

Figure 7: Storage requirements of $\tilde{K}$ on $\Gamma_C$ in MB.
We again want to compare the results of the translation-invariant $\mathcal{H}^2$-matrix compression with the corresponding results of the conventional $\mathcal{H}^2$-matrix compression on the finest mesh ($n = 2076672$): $\tilde{G}_{\text{conv}}$ requires 23100.3 MB for the leaf matrices, 26630.6 MB for the transfer matrices, 247629.2 MB for the nearfield matrices and 167992.2 MB for the coupling matrices. $\tilde{K}_{\text{conv}}$ requires 17325.2 MB for the leaf matrices, 20321.7 MB for the transfer matrices, 193189.7 MB for the nearfield matrices and 115508.9 MB for the coupling matrices. The resulting $L^2$-error is $1.503_{-3}$.

As the complexity bounds suggest and the results demonstrate, the translation-invariant approach reduces the storage requirements for the transfer and coupling matrices drastically. Furthermore, the translation-invariant $\mathcal{H}^2$-matrix approximations require remarkably less storage for the nearfield matrices than their respective conventional counterparts. This is due to the fact that the leaves of the cluster trees used in the translation-invariant approach contain significantly less characteristic points than the leaves of the conventional cluster trees. The resulting increase of the farfield can be handled very efficiently, since no individual but only very few coupling matrices have to be stored.

We conclude that by properly exploiting the translation-invariance property (2) of the kernel function the storage requirements of $\mathcal{H}^2$-matrix approximations can be greatly reduced without affecting the desired accuracy.

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