Invariance, Symmetry and Irreversibility in Coherence Theory

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Abstract. The intrinsic degrees of coherence characterize several properties of partially polarized and partially coherent light. In particular, their properties of invariance by the multiplication of the electric fields by non singular deterministic Jones matrices is reviewed and their ability to exhibit different possible symmetry configurations of the second order statistical properties is discussed. Their evolution when the electric fields of the light are multiplied by random Jones matrices is also analyzed and it is shown that two kinds of irreversible behavior can be exhibited.

1. Introduction
The analysis of the coherence properties of partially polarized electromagnetic waves is the subject of interesting new developments which emphasize different physical aspects[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. In particular, the intrinsic degrees of coherence[5, 6] are quantities invariant by the multiplication of the fields by non singular deterministic Jones matrices and it is proposed here to discuss two consequences of this invariance property. The first one is related to the symmetry properties of partially coherent and partially polarized light. The second one concerns the irreversible behavior that can be observed when partially coherent and partially polarized light is randomly modulated (i.e. when they are multiplied by random Jones matrices).

2. Background
The second order statistical properties of stationary electromagnetic fields are generally analyzed in the space-frequency domain[12, 13]. Let \( \mathbf{E}(\mathbf{r}_1, \omega) \) and \( \mathbf{E}(\mathbf{r}_2, \omega) \) be the spectral component at frequency \( \omega \) of random electromagnetic fields vectors, respectively, at point \( \mathbf{r}_1 \), and at point \( \mathbf{r}_2 \). The electromagnetic field is assumed two dimensional and can thus be written \( \mathbf{E}(\mathbf{r}, \omega) = (E_x(\mathbf{r}, \omega), E_y(\mathbf{r}, \omega))^T \) where \( \mathbf{a}^T \) is the transpose of \( \mathbf{a} \) and where \( E_x(\mathbf{r}, \omega), E_y(\mathbf{r}, \omega) \) represent the components of the electromagnetic field vector in orthogonal directions that are perpendicular to the z direction of the beam propagation. The cross-spectral density matrix is defined by

\[
\mathbf{W}(\mathbf{r}_1, \mathbf{r}_2, \omega) = \langle \mathbf{E}(\mathbf{r}_2, \omega)\mathbf{E}^\dagger(\mathbf{r}_1, \omega) \rangle, \tag{1}
\]

where \( \langle \cdot \rangle \) denotes ensemble averaging and where \( \mathbf{a}^\dagger \) is the transpose conjugate of \( \mathbf{a} \). The spectral density matrix[13] \( \mathbf{S}(\mathbf{r}, \omega) \) at point \( \mathbf{r} \) is thus obtained with \( \mathbf{S}(\mathbf{r}, \omega) = \mathbf{W}(\mathbf{r}, \mathbf{r}, \omega) \).
3. Visibility and invariance

The degree of coherence discussed by E. Wolf in [1] is defined by

$$\mu_W(r_1, r_2, \omega) = \frac{tr [W(r_1, r_2, \omega)]}{\sqrt{I(r_1, \omega) I(r_2, \omega)}},$$

(2)

where $I(r_1, \omega) = tr [S(r_1, \omega)]$ and where $tr [S(r_1, \omega)]$ denotes the trace of $S(r_1, \omega)$. This degree of coherence is directly related to the maximal visibility of interference fringes that can be observed between the fields $E(r_1, \omega)$ and $E(r_2, \omega)$ when their spectral intensities $I(r_i, \omega)$ are optimized.

Indeed, following [1, 13], let us consider a Young interferometric experiment as schematically described in figure 1, the total field at the detector located at point $r$ in the plane $P_3$ is

$$E_T(r, \omega) = \alpha_1 d_1 E(r_1, \omega) e^{i k d_1} + \alpha_2 d_2 E(r_2, \omega) e^{i k d_2},$$

(3)

where $k = 2\pi c/\omega$, $i^2 = -1$, $c$ is the speed of light in vacuum, $\alpha_1$ and $\alpha_2$ do not depend on $d_i$ which is defined by $d_i = ||r_i - r||$ where $|| ||$ is the euclidian norm. The visibility of the interference fringes is thus [1, 13]

$$V(\omega) = 2 \sqrt{I(r_1, \omega) I(r_2, \omega)} \frac{|\mu_W(r_1, r_2, \omega)|}{I(r_1, \omega) + I(r_2, \omega)}.$$

(4)

It is first interesting to note that the modulus $|\mu_W(r_1, r_2, \omega)|$ is invariant by transformations of the form

$$E(r_i, \omega) \rightarrow \rho_i U E(r_i, \omega),$$

(5)

where $\rho_i$ is any nonzero complex number and $U$ is any unitary matrix. The visibility of the interference fringes is thus equal to $|\mu_W(r_1, r_2, \omega)|$ when the electric fields have been multiplied by scalar numbers $a_1$ and $a_2$ so that $|a_1|^2 I(r_1, \omega) = |a_2|^2 I(r_2, \omega)$. More precisely, $|\mu_W(r_1, r_2, \omega)|$ is the maximal visibility of the interference fringes of the fields $a_1 E(r_1, \omega)$ and $a_2 E(r_2, \omega)$ which intensities have been optimized in order to maximize the visibility. This situation is schematically shown in figure 2.

Other quantities have been proposed in order to characterize coherence properties of partially polarized light [2, 9, 10, 11]. In [2, 10], different quantities that are invariant by unitary
transformations have been defined. In[9, 11] the maximal value of the modulus of the Wolf degree of coherence is analyzed when the electric fields can be modified with any unitary transformations. This value is thus the maximal value of the visibility of the interference fringes when the interfering fields have been optimized with a succession of optical densities and birefringent plates (see figure 3). This quantity is thus invariant by transformations of the initial fields of the form

$$E(r_1, \omega) \rightarrow \rho_i \, U_i \, E(r_1, \omega), \quad (6)$$

where $\rho_i$ is any nonzero complex number and $U_i$ is any unitary matrix, which is also the case of the measures considered in [2, 4, 14]. The main difference with Eq.(5) is that now different unitary transformations can be applied to each field.

The intrinsic degrees of coherence introduced in[5, 6] are the singular values of the normalized mutual coherence matrix $M(r_1, r_2, \omega) = S^{-\frac{1}{2}}(r_2, \omega) \, W(r_1, r_2, \omega) \, S^{-\frac{1}{2}}(r_1, \omega)$. In other words, the singular value decomposition of the normalized mutual coherence matrix can be written $M(r_1, r_2, \omega) = N_2(r_1, r_2, \omega) \, D(r_1, r_2, \omega) \, N_1^T(r_1, r_2, \omega)$ where $N_2(r_1, r_2, \omega)$ and $N_1(r_1, r_2, \omega)$ are unitary matrices and where $D(r_1, r_2, \omega)$ is a diagonal matrix whose elements are the intrinsic degrees of coherence $\mu_S(r_1, r_2, \omega)$ and $\mu_I(r_1, r_2, \omega)$ with $\mu_S(r_1, r_2, \omega) \geq \mu_I(r_1, r_2, \omega) \geq 0$. It has been shown[5, 15] that the intrinsic degrees of coherence are invariant by transformations of the form

$$E(r_1, \omega) \rightarrow J_i \, E(r_1, \omega), \quad (7)$$

where $J_i$ is any nonsingular deterministic Jones matrix. The largest intrinsic degree of coherence is equal to the maximal fringes visibility that can be observed in a Young’s interference experiment if the fields $E(r_1, \omega)$ and $E(r_2, \omega)$ are modified with transformations that correspond to the multiplication of the fields by any Jones matrices $J_i$ (see figure 4)[5, 16].

The transformations defined by Eqs. (5), (6) and (7), where $\rho_i$ is any deterministic nonzero complex number, $U$ and $U_i$ are any deterministic unitary matrices and $J_i$ is any deterministic nonsingular Jones matrix, define groups of transformations that will be respectively noted $G_H$, $G_U$ and $G_J$. The group $G_H$ is a subgroup of $G_U$ which is a subgroup of $G_J$. Consequently, a quantity which is invariant by Eq. (7) is also invariant by Eq. (6) and a quantity which is invariant by Eq. (6) is also invariant by Eq. (5) but the opposite is not necessarily true. This property implies that the maximal visibility of interference fringes which can be obtained with

**Figure 3.** Fringes visibility optimization in Young’s interference experiment with modification of the intensity and polarization state adjustment with unitary transformation $U_i$ applied to each interfering beam.

**Figure 4.** Fringes visibility optimization in Young’s interference experiment with polarization state adjustment with general Jones matrix transformation applied to each electric field.
transformations of the form of Eq. (7), where $J_i$ is any Jones matrix, cannot be smaller than the maximal visibility obtained when the interfering fields have been optimized with a succession of optical densities and birefringent plates (i.e. with Eq. (6)). Similarly, the maximal visibility of interference fringes which can be obtained with transformations of the form of Eq. (6) cannot be smaller than the maximal visibility of the interference fringes of the fields obtained when transformations of the form of Eq. (5) are considered.

In the following, two properties that are related to the group $G_J$ will be analyzed and it is thus interesting to discuss briefly its relation with the factorization condition at order one which is an important concept in the theory of coherence. This factorization condition is satisfied when the cross-spectral density matrix can be written

$$W(r_1, r_2, \omega) = \Psi(r_2, \omega) \Psi^\dagger(r_1, \omega),$$

(8)

where $\Psi(r, \omega)$ is a deterministic vectorial function, which implies that the light has to be totally polarized [17]. Thus, if $E(r_i, \omega) \rightarrow J_i E(r_i, \omega)$ then $W(r_1, r_2, \omega) \rightarrow \Phi(r_2, \omega) \Phi^\dagger(r_1, \omega)$ with $\Phi(r_i, \omega) = J_i \Psi(r_i, \omega)$. It can thus be observed that transformations of the group $G_J$ preserve the factorization condition at order one.

4. Symmetry

Let us now analyze symmetry properties of the second order statistical characteristics when the fields can be modified by non singular deterministic Jones matrices. These symmetry properties thus correspond to the determination of the subgroups of $G_J$ that do not modify the second order statistical quantities $S(r_1, \omega)$, $S(r_2, \omega)$, and $W(r_1, r_2, \omega)$. It has been shown in [15] that different situations can appear according to the values of the intrinsic degrees of coherence $\mu_S$ and $\mu_I$ (the dependency on $(r_1, r_2, \omega)$ is not written for brevity reasons). The most symmetrical situation (i.e. the case where the subgroup of transformations that does not modify the second order statistical properties is the largest) corresponds to $\mu_S = \mu_I = 0$. In that case, the second order statistical properties $S(r_1, \omega)$, $S(r_2, \omega)$, and $W(r_1, r_2, \omega)$ are invariant by the transformations of a large subgroup of $G_J$. The precise form of this subgroup is discussed in [15] and is dependent on $S(r_1, \omega)$, $S(r_2, \omega)$, and $W(r_1, r_2, \omega)$ and will not be analyzed further here. The less symmetrical situation is obtained when $\mu_S > \mu_I > 0$. In that case, the second order statistical properties are invariant by transformations of a small subgroup of $G_J$. There exist two intermediate situations, the first one corresponds to $\mu_S > \mu_I = 0$ and the second to $\mu_S = \mu_I > 0$. This situation is schematically described in figure 5.

![Figure 5](image1.png)

Figure 5. The situations $\mu_S = \mu_I = 0$, $\mu_S > \mu_I = 0$, $\mu_S = \mu_I > 0$, and $\mu_S > \mu_I > 0$, correspond to different symmetry properties.

![Figure 6](image2.png)

Figure 6. Possible evolution of the intrinsic degrees of coherence when random modulations described by Eq. 11 are applied.
The concept of symmetry is related to the general group of invariance that has been considered (i.e. $G_j$). It can be shown with a simple generalization of [18] that different physical situations, that correspond to different subgroups of symmetry, can lead to different experimental behaviors. Indeed, if $\mu_S \neq \mu_I$, each interfering field $E(r_1, \omega)$ and $E(r_2, \omega)$ has to be totally polarized in order to optimize the fringes visibility in a Young’s interference experiment while if $\mu_S = \mu_I$ the maximal visibility can be observed with partially polarized light. Furthermore, if $\mu_S = \mu_I = 0$, no interference fringes can be observed whatever the linear transformations applied to each interfering field. Other practical consequences have been also reported in [19], where the visibility of interference fringes is analyzed when the optimization of the polarization state is performed on a single beam. Such a practical case can appear in homodyne like detection. Furthermore, the special case where $\mu_S = \mu_I = 1$ is also of theoretical and practical interest [18] since any partially polarized light with a Wolf degree of coherence of modulus one satisfy this relation.

5. Irreversibility
Irreversibility can be related to different physical phenomena in coherence theory. The first one is the consequence of the energy absorption. For instance, considering a transformation of the form $E(r_i, \omega) \rightarrow \rho_i E(r_i, \omega)$, where $\rho_i$ can be any nonzero complex number, can correspond to an irreversible transformation if $|\rho_i| < 1$. This is also the case in general with transformations of the form $E(r_i, \omega) \rightarrow J_i E(r_i, \omega)$. The situation is different when unitary transformations are considered. Indeed, if $A(r_i, \omega) = U_i E(r_i, \omega)$ where $U_i$ is a unitary transformation, then it is easy to see that $|A(r_i, \omega)| = |E(r_i, \omega)|$, which corresponds to transformations without energy absorption. The optimization of the Young’s fringes visibility through such energy reversible optical transformations has been analyzed in [9, 11].

The introduction of disorder is the second physical phenomenon related to irreversibility. It is interesting to first analyze that point in the case of a scalar model of light. Let us consider the action of random modulations of the scalar fields $E(r_1, \omega)$ and $E(r_2, \omega)$ such as

$$A(r_j, \omega) = J_\lambda(r_j, \omega) E(r_j, \omega),$$

where $J_\lambda(r_j, \omega)$ are complex valued random numbers. The general definition of the standard degree of coherence between two scalar fields $E(r_1, \omega)$ and $E(r_2, \omega)$ is

$$\mu_{EE}(r_1, r_2, \omega) = \frac{\langle E(r_2, \omega) \ E^*(r_1, \omega) \rangle}{\sqrt{\langle |E(r_1, \omega)|^2 \rangle \langle |E(r_2, \omega)|^2 \rangle}},$$

where $a^*$ is the complex conjugate of $a$. Assuming $J_\lambda(r_j, \omega)$ and $E(r_j, \omega)$ statistically independent, it is easy to see that $|\mu_{AA}(r_1, r_2, \omega)| \leq |\mu_{EE}(r_1, r_2, \omega)|$. The decrease of the modulus of the standard degree of coherence is the consequence of the disorder introduced with random modulation by $J_\lambda(r_j, \omega)$ in Eq. (9). Indeed, it is simple to check that if $J_\lambda(r_j, \omega)$ are nonzero and deterministic (i.e. equal to deterministic complex numbers) then $|\mu_{AA}(r_1, r_2, \omega)| = |\mu_{EE}(r_1, r_2, \omega)|$ although energy can have been absorbed, which corresponds to the first kind of irreversibility (energy irreversibility) mentioned above. Random modulations described with Eq. (9) thus introduce a second kind of irreversibility (disorder irreversibility) which is characterized by the decrease of the modulus of the degree of coherence.

It has recently been shown [20] in the spatial time domain that an analogous behavior can be obtained with complex two dimensional electric fields. This result can be generalized in the spatial-frequency domain where the random modulations are described by

$$A(r_j, \omega) = J_\lambda(r_j, \omega) E(r_j, \omega),$$

where $J_\lambda(r_j, \omega)$ are complex valued random Jones matrices statistically independent to $E(r_j, \omega)$. Since the generalization of the multiplication of scalar fields by random complex numbers is the
multiplication of two dimensional complex vectors by random complex Jones matrices, Eq.(11) is a direct generalization of Eq.(9). The generalization of the result obtained in[20] to the space-frequency domain shows that the maximal visibility of the interference fringes, obtained when the polarization states of fields are optimized, cannot increase with random modulations (i.e. with transformations of the form of Eq. (11)). In other words, since the value of the maximal visibility is equal to the largest intrinsic degree of coherence, the largest intrinsic degree of coherence $\mu^{(A)}_{S}(r_1, r_2, \omega)$ between $A(r_1, \omega)$ and $A(r_2, \omega)$ cannot be larger than the largest intrinsic degree of coherence $\mu^{(E)}_{S}(r_1, r_2, \omega)$ between $E(r_1, \omega)$ and $E(r_2, \omega)$. This property is illustrated in figure 6.

This result shows that the maximal intrinsic degree of coherence describes the effect of the irreversible transformations due to random modulations of the electric fields. This behavior is analogous to the one of the modulus of the standard degree of coherence in the scalar case and characterizes the disorder irreversibility. It has also been shown[20] that the ability of the largest intrinsic degree of coherence to describe the disorder irreversibility is a consequence of its invariance property by transformations of the group $G_J$.

6. Conclusion
Invariance, symmetry and irreversibility appear to be related concepts for the analysis of different aspects of partially polarized and partially coherent light. Some of them have been discussed in this communication. This is in particular the case of the symmetry characteristics of the second order statistical properties and of the analysis of the disorder irreversibility which is different and complementary to energy irreversibility. Nevertheless there are still several directions of fruitful investigations such as the relation of these concepts with information theory and with some different optical experiments.

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