A NOTE ON MIXING TIMES OF PLANAR RANDOM WALKS

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Abstract. We present an infinite family of finite planar graphs \{X_n\} with degree at most five and such that for some constant \(c > 0\),
\[
\lambda_1(X_n) \geq c \left( \frac{\log \text{diam}(X_n)}{\text{diam}(X_n)} \right)^2,
\]
where \(\lambda_1\) denotes the smallest non-zero eigenvalue of the graph Laplacian. This significantly simplifies a construction of Louder and Souto.

We also remark that such a lower bound cannot hold when the diameter is replaced by the average squared distance: There exists a constant \(c > 0\) such that for any family \{X_n\} of planar graphs we have
\[
\lambda_1(X_n) \leq c \left( \frac{1}{|X_n|^2} \sum_{x,y \in X_n} d(x,y)^2 \right)^{-1},
\]
where \(d\) denotes the path metric on \(X_n\).

Recently, Louder and Souto [3] answered negatively a question of Benjamini and Curien [1] by showing that there is an infinite family of bounded-degree planar graphs for which the mixing time is asymptotically less than the square of the diameter. Their construction uses expander graphs to construct a surface which they triangulate to arrive at a planar graph. We present a simple family of graphs exhibiting the same result.

1. The Laplacian

For a finite, undirected graph \(G = (V, E)\) with edge and vertex weights \(w : E \to [0, \infty)\) and \(\pi : V \to (0, \infty)\), one defines the combinatorial Laplacian \(L : \ell^2(V, \pi) \to \ell^2(V, \pi)\) by
\[
Lf(x) = \sum_{y : (x, y) \in E} \frac{w(x, y)}{\pi(x)} (f(x) - f(y)).
\]
Here, \(\ell^2(V, \pi)\) is equipped with the inner product \(\langle f, g \rangle_{\ell^2(V, \pi)} = \sum_{x \in V} \pi(x) f(x) g(x)\). This coincides with the unweighted combinatorial Laplacian when \(w \equiv 1\) and \(\pi \equiv 1\).

Recall that the smallest non-zero eigenvalue of \(L\) as an operator on \(\ell^2(V, \pi)\) is given by
\[
\lambda_1(G) = \min_{f \in \ell^2(V, \pi)} \left\{ \frac{\sum_{(x,y) \in E} w(x,y) (f(x) - f(y))^2}{\sum_{x \in V} \pi(x) f(x)^2} : \sum_{x \in V} \pi(x) f(x) = 0 \right\}.
\]

The Cheeger constant of \(G\) is defined by
\[
h(G) = \min_{S \subseteq V} \left\{ \frac{\sum_{(x,y) \in E} w(x,y) |1_S(x) - 1_S(y)|}{\pi(S)} : \pi(S) \leq \pi(V)/2 \right\},
\]
where \(1_S\) denotes the characteristic function of \(S\) and \(\pi(S) = \sum_{v \in V} \pi(v)\) for a subset \(S \subseteq V\).

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We recall the discrete Cheeger inequality

$$\lambda_1(G) \geq \frac{h(G)^2}{2d_{\text{max}}},$$

where $d_{\text{max}} = \max_{v \in V} \left( \pi(x)^{-1} \sum_{y \in E} w(x, y) \right)$ is the maximum “degree” in $G$.

For a graph $G$, we use $V(G)$ and $E(G)$ to denote the vertex and edge sets of $G$, respectively. Unless otherwise stated, the edge and vertex weights on a graph $G$ are taken to be uniform.

## 2. The Construction

Let $T_h$ denote the complete, rooted binary tree of height $h$. Let $T_{h,k}$ denote the result of subdividing every edge of $T_h$ by $k$. If $T_{h,k}$ has root $r$, we write

$$V_\ell = \{ v \in V(T_{h,k}) : \text{dist}(v, r) = \ell \}$$

for the set of nodes at distance $\ell$ from $r$. Fix an in-order traversal of $T_{h,k}$. For each $\ell = 0, 1, \ldots, h k$, we add a path $P_\ell$ on the nodes at depth $\ell$. Specifically, edges go between consecutive nodes of $V_\ell$ in the in-order traversal. Call this final graph $\widehat{T}_{h,k}$. It is straightforward to verify that $\widehat{T}_{h,k}$ is planar.

**Theorem 1.** For every $h \geq 1$ and $k = 2^h$, the following bounds hold:

i) $\text{diam}(\widehat{T}_{h,k}) \geq h k$, and

ii) $\lambda_1(\widehat{T}_{h,k}) \geq \frac{(\log_2 \text{diam}(\widehat{T}_{h,k}))^2}{6 \text{diam}(\widehat{T}_{h,k})}$.

**Proof.** The diameter bound is (i) clear, so we focus on (ii).

Let $V = V(\widehat{T}_{h,k})$. Consider any $f : V \to \mathbb{R}$ with $\sum_{x \in V} f(x) = 0$. Define $\bar{f} : V \to \mathbb{R}$ as follows: For $x \in V_\ell$,

$$\bar{f}(x) = \frac{1}{|V_\ell|} \sum_{x \in V_\ell} f(x).$$

Observe that $\sum_{x \in V} \bar{f}(x) = 0$ holds as well.

Now, from (1) applied to the graph $P_\ell$, which has $h(P_\ell) \geq 2/|V_\ell|$, we have

$$\sum_{\{x, y\} \in P_\ell} (f(x) - f(y))^2 \geq \frac{1}{|V_\ell|^2} \sum_{x \in V_\ell} (f(x) - \bar{f}(x))^2.$$

Using $|V_\ell| \leq 2^h$ and summing over all $\ell$ yields

$$\sum_{\ell=1}^{h} \sum_{\{x, y\} \in E(P_\ell)} (f(x) - f(y))^2 \geq 2^{-2h} \|f - \bar{f}\|^2 \geq \frac{1}{k^2} \|f - \bar{f}\|^2. \quad (2)$$

That covers the “horizontal” edges of $\widehat{T}_{h,k}$. We claim that the following bound holds for the “vertical” edges:

$$\sum_{\{x, y\} \in E(\widehat{T}_{h,k})} (\bar{f}(x) - \bar{f}(y))^2 \geq \frac{1}{6k^2} \|\bar{f}\|^2. \quad (3)$$

Since $\sum_{x \in V} \bar{f}(x) = 0$ and $\bar{f}$ is constant on the level sets $\{V_\ell\}$, this is implied by a lower bound on the spectral gap of the weighted quotient graph $Q_{h,k}$ of $\widehat{T}_{h,k}$, where each level set $V_\ell$ is identified to a single vertex of weight $|V_\ell|$. 

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But \(Q_{h,k}\) is simply a \(k\)-subdivision of the weighted graph \(Q_h\) which has vertex set \(\{0, 1, \ldots, h\}\), edge weights \(w(j, j + 1) = 2^{j+1}\), and vertex weights \(\pi(j) = 2^j\). In particular, \(\lambda_1(Q_{h,k}) = \frac{1}{k} \lambda_1(Q_h)\). Hence (3) is implied by the lower bound

\[
\lambda_1(Q_h) = \min_{g: \{0, 1, \ldots, h\} \to \mathbb{R}} \left\{ \frac{\sum_{j=0}^{h-1} 2^{j+1}(g(j) - g(j + 1))^2}{\sum_{j=0}^{h} 2^j g(j)^2} : \sum_{j=0}^{h} 2^j g(j) = 0 \right\} \geq -\frac{1}{6}.
\]

But this follows from (1), which is easily verified: Any \(S\) of weight at most half cannot contain \(h\), which has \(\pi(h) = \pi(V(Q_h))/2\). Thus for any such \(S \subseteq \{0, 1, \ldots, h - 1\}\), if \(j = \max(S)\), then \(\pi(S) \leq 2^j + 2^{j-1} + \cdots + 1 \leq 2^{j+1}\), while the edge \((j, j + 1)\) leaves \(S\) and has \(w(j, j + 1) = 2^{j+1}\).

Finally, we claim that

\[
\sum_{\{x,y\} \in E(T_{h,k})} (f(x) - f(y))^2 \geq \sum_{\{x,y\} \in E(T_{h,k})} (f(x) - f(y))^2.
\]

This follows by applying Jensen’s inequality to the edges from \(V_\ell\) to \(V_{\ell+1}\) for each value of \(\ell\). Now summing lines (2) and (4) and using (3) yields

\[
\sum_{\{x,y\} \in E(\hat{T}_{h,k})} (f(x) - f(y))^2 \geq \frac{1}{k^2} \left( \frac{\|f\|^2}{6} + \|\bar{f} - \bar{f}\|^2 \right) \geq \frac{1}{k^2\|f\|^2},
\]

completing the proof. \(\square\)

3. Conclusion

As Yuval Peres pointed out to us, one can check that the mixing time of simple random walk on \(\hat{T}_{h,k}\) is on the order of \(hk^2\) for \(k = 2^h\), which is a \(\log |V(\hat{T}_{h,k})|\) factor larger than the relaxation time (i.e., the inverse spectral gap). Observe that although the diameter of \(\hat{T}_{h,k}\) is \(hk\), the average distance between a uniformly random pair of vertices is bounded by \(O(k)\). And indeed, it is true in general that in a bounded-degree planar graph, the mixing time is at least the average of the squared distance.

**Theorem 2 (2).** For some constant \(c > 0\), the following holds. Let \(G = (V, E)\) be a planar graph with path metric \(d\). Then,

\[
\lambda_1(G) \leq c \left( \frac{1}{|V|^2} \sum_{x,y \in V} d(x, y)^2 \right)^{-1}.
\]

**Proof.** By [2] Thm 4.4, there exists a universal constant \(C > 0\) and a 1-Lipschitz mapping \(f : V \to \mathbb{R}\) such that

\[
\sum_{x,y \in V} |f(x) - f(y)|^2 \geq C \sum_{x,y \in V} d(x, y)^2.
\]

By a standard calculation using the fact that \(|E| \leq 3|V|\), one has

\[
\lambda_1(G) \leq \frac{\sum_{\{x,y\} \in E} |f(x) - f(y)|^2}{2|V| \sum_{x,y \in V} |f(x) - f(y)|^2} \leq \frac{(3|V|)(2|V|)}{C \sum_{x,y \in V} d(x, y)^2},
\]

completing the proof. \(\square\)

We remark that a similar bound holds for any graph of bounded genus, or more generally, for any graph excluding a fixed minor; see [2]. For families of graphs of unbounded degree, one should
consider the normalized Laplacian \( L f(x) = \sum_{y \in \{x,y\} \in E} (\pi(x)\pi(y))^{-1/2} (f(x) - f(y)) \), where \( \pi(x) \) denotes the stationary probability of \( x \in V \). A similar argument shows that for some \( c > 0 \),

\[
\lambda_1(L) \leq c \left( \sum_{x,y \in V} \pi(x)\pi(y)d(x,y)^2 \right) ^{-1},
\]

implying that, in planar graphs, the mixing time is at least the average squared distance when points are chosen uniformly from the stationary measure.

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