Barrett-Crane model from a Boulatov-Ooguri field theory over a homogeneous space

Roberto De Pietri \textsuperscript{a,1}, Laurent Freidel \textsuperscript{b,c,2}, Kirill Krasnov \textsuperscript{b,d,3}, Carlo Rovelli \textsuperscript{a,d,e,4}

\textsuperscript{a} Centre de Physique Théorique - CNRS, Case 907, Luminy, F-13288 Marseille, France
\textsuperscript{b} Center for Gravitational Physics and Geometry, Penn State University, Pa 16802, USA
\textsuperscript{c} Laboratoire de Physique, Ecole Normale Supérieure de Lyon 46, allée d’Italie, 69364 Lyon Cedex 07, France
\textsuperscript{d} Institute for Theoretical Physics, UCSB, Santa Barbara, Ca 93106, USA
\textsuperscript{e} Physics Department, University of Pittsburgh, Pittsburgh, Pa 15260, USA

\texttt{1 depietri@cpt.univ-mrs.fr, 2 freidel@phys.psu.edu, 3 krasnov@phys.psu.edu, 4 rovelli@pitt.edu}

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Boulatov and Ooguri have generalized the matrix models of 2d quantum gravity to 3d and 4d, in the form of field theories over group manifolds. We show that the Barrett-Crane quantum gravity model arises naturally from a theory of this type, but restricted to the homogeneous space \( S^3 = \text{SO}(4)/\text{SO}(3) \), as a term in its Feynman expansion. From such a perspective, 4d quantum spacetime emerges as a Feynman graph, in the manner of the 2d matrix models. This formalism provides a precise meaning to the “sum over triangulations”, which is presumably necessary for a physical interpretation of a spin foam model as a theory of gravity. In addition, this formalism leads us to introduce a natural alternative model, which might have relevance for quantum gravity.

I. INTRODUCTION

The Barrett-Crane relativistic model \cite{1} is an intriguing proposal for addressing the problem of constructing a quantum theory of gravity. The model is related to the covariant, or spin-foam, formulation \cite{2} of loop quantum gravity \cite{3}, and has recently received much attention. Barrett and Crane have defined this model as a state sum defined over a fixed triangulation \( \Delta \) of 4d spacetime. Here, we show that the \( q \to 1 \) limit of the Barrett-Crane model can be derived from the Feynman expansion of a field theory over four copies of the homogeneous space \( S^3 = \text{SO}(4)/\text{SO}(3) \). More precisely, the Barrett-Crane model is defined as a sum over colorings \( c \) (assignments of spins to faces) of the triangulation \( \Delta \)

\[
Z_{BC}[\Delta] = \sum_c A_{BC}(\Delta, c),
\]

where \( A_{BC}(\Delta, c) \) is the Barrett-Crane amplitude of the colored triangulation \( (\Delta, c) \). In the Feynman expansion

\[
Z = \int D\phi e^{-S[\phi]} = \sum_{\Gamma} \frac{\lambda^{v[\Gamma]}}{\text{sym}[\Gamma]} Z[\Gamma]
\]

of the field theory we consider, the Feynman graphs \( \Gamma \) turn out to have the structure of a 2-complex. Here \( S[\phi] \) is the action, which depends on a coupling constant \( \lambda \), while \( v[\Gamma] \) and \( \text{sym}[\Gamma] \) are the number of vertices and the order of symmetries of \( \Gamma \). Every triangulation \( \Delta \) determines a 2-complex \( \Gamma = \Gamma(\Delta) \): the 2-skeleton of its dual cellular complex. We show in this paper that the Barrett-Crane model over the triangulation \( \Delta \) is precisely the amplitude of the Feynman graph \( \Gamma(\Delta) \):

\[
Z_{BC}[\Delta] = Z[\Gamma(\Delta)].
\]

More remarkably, the single terms of the sum \([\text{2}]\) match. In fact, first of all, the space over which our field theory is defined is compact and the Feynman integrals are replaced by Feynman sums.

\[1\] Barrett and Crane define two versions of their model. We recover here their “first version.”
Barrett and Crane realized that this constraint can be implemented within an SO(4) Crane-Yetter state sum model. The constraint that reduces BF theory to general relativity has an appealing geometrical interpretation. For a model of quantum euclidean general relativity as a modification of the SO(4) Ooguri-Crane-Yetter state sum model, BF theory can be seen as an SO(4) BF theory with an added constraint, see e.g. [20], it was then natural to search

\[
Z[\Gamma] = \sum_c A(\Gamma, c).
\]  

(4)

Next, we show that the discrete “momenta” \( c \) in (4) correspond precisely to the colorings \( c \) of the triangulation in (3). That is, they are spins attached to (the 2-cells of the 2-complex \( \Gamma(\Delta) \) which are dual to) the faces of \( \Delta \). And, finally, we show that the Barrett-Crane amplitude is precisely equal to the Feynman amplitude of the corresponding “colored” 2-complex \((\Gamma, c)\)

\[
A_{BC}(\Delta, c) = A(\Gamma(\Delta), c).
\]  

(5)

The Feynman expansion of field theory we consider generates many more terms than a Barrett-Crane model over a fixed triangulation. The additional summation is precisely what we need in order to address the main difficulty in developing a physical theory of quantum gravity starting from the Barrett-Crane model (or from spin foam models in general [4]). In these models, indeed, the choice of a fixed triangulation breaks the full covariance of the theory and unphysically restricts the number of the degrees of freedom. Therefore some sort of “sum over all triangulations” is needed on physical grounds. The difficulty is to choose a precise characterization of the objects over which to sum, and to fix the relative weights. This is precisely done by the full Feynman expansion of our field theory. Therefore, the field theory provides a precise implementation of the loose idea of “summing over triangulations”. In detail, one first notices that the Barrett-Crane amplitude \( A_{BC}(\Delta, c) \) does not depend on the full (combinatorial) information contained in the triangulation \( \Delta \), but only on a subset of this. In fact, it depends only on the 2-complex \( \Gamma(\Delta) \), its dual 2-skeleton. Thus, the Barrett-Crane state sum (3) is actually a sum over all colorings of a fixed 2-complex. The expansion (4) extends the Barrett-Crane state sum (3) to a sum over colored 2-complexes, and fixes the relative weight of these. Thus, what emerges in the Feynman expansion is not a sum over actual triangulations, as one might have naively expected, but rather a sum over certain different objects: colored 2-complexes \( s = (\Gamma, c) \)

\[
Z = \sum_s A(s).
\]  

(6)

Now, colored 2-complexes \( s \) are precisely spin foams. Spin foams were derived (under the name branched colored surfaces) in describing the dynamics of loop quantum gravity [2]. Indeed, the covariant formulation of loop quantum gravity yields a partition function which is precisely a sum over spin foams of the form (3). Intuitively, a spin foam describes the time evolution of the spin networks, the states of loop quantum gravity [3]. The idea that quantum spacetime could be described in terms of objects of this kind has been proposed earlier, in particular by Baez [4] and Reisenberger [7] (see also Iwasaki [8]). In [4], Baez has defined and analyzed the general notion of spin foam model (and has introduced the term “spin foam”); we refer to Baez’ papers [4] and [9] for an introduction and a discussion of these models. See also [4],[7],[11].

The idea of obtaining a “sum over triangulations” from a Feynman expansion was successfully implemented some time ago, in the context of the matrix models of 2d dimensional quantum gravity, or “zero dimensional string theory” [12]. In that context, one views a triangulated spacetime as a term in a Feynman expansion of the matrix model. Here, we essentially show that the same strategy works in 4d: the terms in the Barrett-Crane state sum can be loosely interpreted as a “quantized spacetime geometry”, and we show here that these “spacetime geometries” are generated as Feynman graphs, as in the 2d quantum gravity models. The convergence with the 2d matrix models is not accidental, since matrix models are at the root of one of the lines of development that lead to the Barrett-Crane theory. Indeed, an extension of the matrix models to 3d was obtained by Boulatov in [13]. Boulatov showed that the Feynman expansion of a certain field theory over three copies of SU(2) generates triangulations, colorings and amplitudes of the Ponzano-Regge formulation of 3d quantum gravity [14] (and, taking the \( q \) deformation of SU(2), the amplitudes of the Turaev-Viro model [15]). Ooguri has extended the Boulatov construction to 4d in [16]. The Ooguri theory is a field theory over 4 copies of SU(2). Its Feynman expansion determines a state sum for a triangulated 4d spacetime, in which the sum is over the irreducible representations of SU(2). Replacing SU(2) with the quantum group SU(2)\(_{q}\) (\( q \) root of 1), yields a finite and well defined sum, the Ooguri-Crane-Yetter model [17], which was shown by Crane, Kauffman and Yetter [18] to be triangulation independent, and therefore to be an invariant of the 4-manifold. The Ooguri-Crane-Yetter invariant can be shown to be the partition function of BF theory [13]. As euclidean general relativity can be seen as an SO(4) BF theory with an added constraint, see e.g. [20], it was then natural to search for a model of quantum euclidean general relativity as a modification of the SO(4) Ooguri-Crane-Yetter state sum model. The constraint that reduces BF theory to general relativity has an appealing geometrical interpretation. Barrett and Crane realized that this constraint can be implemented within an SO(4) Crane-Yetter state sum model.
by (appropriately) restricting the sum to simple representations only.\(^2\) Results in [11, 22] support the idea that the Barrett-Crane model is indeed related to quantum general relativity.

Recently, it was pointed out that harmonic analysis on the homogeneous space \(S^3 = \text{SO}(4)/\text{SO}(3)\) yields precisely the simple representations of \(\text{SO}(4)\) [23]. It was then tempting to conjecture that by suitably restricting the 4d Boulatov-Ooguri field theory to this homogeneous space one could get (the \(q \to 1\) limit of) the Barrett-Crane model. Here we show that this is indeed the case, with an additional bonus: while for the Ponzano-Regge and for the Ooguri-Crane-Yetter models the field theory generates a redundant sum over terms—all equal to each other by triangulation independence—, in the Barrett-Crane case the additional summation cures the breaking of covariance introduced with the choice of a triangulation, as we have discussed above. This, in a certain sense, closes a circle, bringing the recent developments back to the original matrix model idea that spacetime can be viewed as a Feynman graph of a quantum theory. The interpretation vividly emphasizes the background independence of these formulations of quantum gravity.

Finally, the formalism we develop, naturally suggest another model, different from Barrett-Crane one. Indeed, in the field theory we consider, \(\text{SO}(4)\) invariance can be imposed in two ways: by the left or the right action of \(\text{SO}(4)\) on \(\text{SO}(4)/\text{SO}(3)\). These turn out to be inequivalent. The second alternative (Case B, below) yields the Barrett-Crane model (first version). The first alternative (Case A, below), is far more natural from the group theoretical point of view. It yields a different model, somewhat closer to, but distinct from the second version of the Barrett-Crane model. We argue in the conclusion that this other model has interesting properties. In particular, it represents another solution to the problem of quantizing the constraints that transform BF theory into general relativity.

In section II we define our field theory, and state our main results in detail. In section III we find the mode expansion of the field. Section IV studies the Feynman expansion for the two theories. In section V we discuss some comments. In the two Appendices, we recall some elements of \(\text{SO}(N)\) representation theory.

II. DEFINITION OF THE MODELS AND MAIN RESULT

We consider the group \(G = \text{SO}(4)\) and we fix an \(\text{SO}(3)\) subgroup \(H\) of \(G\). In particular, we may consider \(\text{SO}(4)\) in its fundamental representation and choose a vector \(v^0\) in the representation space; we then let \(H\) be the \(\text{SO}(3)\) subgroup which leaves \(v^0\) invariant. Right multiplication by an element of \(H\) defines an equivalence relation in \(G\): \(g \sim g'\) if there is \(h \in H\) such that \(gh = g'\). The space of the equivalence classes, \(\text{SO}(4)/\text{SO}(3)\), is diffeomorphic to the 3-sphere \(S^3\). We denote the elements of \(S^3\) as \(x\). Note that the invariant normalized measures \(dx, dh\) and \(dg\) on \(S^3\), \(\text{SO}(3)\) and \(\text{SO}(4)\) respectively are related as \(dg = dx \, dh\).

We consider a real field \(\tilde{\phi}(g_1, g_2, g_3, g_4)\) over the cartesian product of four copies of \(G = \text{SO}(4)\). We take \(\tilde{\phi}\) square integrable with respect to each argument. We require that \(\tilde{\phi}\) is constant along the equivalence classes.

\[
\tilde{\phi}(g_1, g_2, g_3, g_4) = \tilde{\phi}(g_1 h_1, g_2 h_2, g_3 h_3, g_4 h_4), \quad (\forall h_1, h_2, h_3, h_4 \in H); \tag{7}
\]

so that \(\tilde{\phi}\) is in fact a function over the homogeneous space \((S^3)^4\), which we can write as \(\tilde{\phi}(x_1, x_2, x_3, x_4)\). We will employ both notations. We require \(\tilde{\phi}\) to be invariant under any permutation \(\sigma\) of its four arguments

\[
\phi(g_1, g_2, g_3, g_4) = \tilde{\phi}(g_{\sigma(1)}, g_{\sigma(2)}, g_{\sigma(3)}, g_{\sigma(4)}). \tag{8}
\]

(In Section V, we will consider alternative symmetry requirements as well.) Finally, we project \(\tilde{\phi}\) on its \(\text{SO}(4)\) invariant part. This can be done in two ways.

- **Case A.** The natural action of \(\text{SO}(4)\) on the right coset \(S^3 = \text{SO}(4)/\text{SO}(3)\) is the left action of the group. Thus we define

\[
\phi(g_1, g_2, g_3, g_4) = \int dg \, \tilde{\phi}(g g_1, g g_2, g g_3, g g_4). \tag{9}
\]

\(^2\)Irreducible representations of \(\text{SO}(4)\) are labeled by two half integers (two spins) \((j', j'') : j' + j'' = \text{integer}\). A representation is simple if \(j' = j'' := j\). Thus, simple representations are labeled by just one spin \(j\). Here, following group-theory literature conventions, we use the “color” \(N = 2j\), instead of the spin \(j\), for labeling the simple representation.
obtain (the one-skeleton of) a 4-simplex, see Fig. 1. This completes the definition of our field theory.

The potential (fifth order) term has the structure of a 4-simplex. That is, if we represent each of the five fields in the product as a node with 4 legs –one for each $g_i$ and connect pair of legs corresponding to the same argument, we obtain the one-skeleton of) a 4-simplex, see Fig. 1. This completes the definition of our field theory.

- **Case B.** Alternatively, we define

$$\phi(g_1, g_2, g_3, g_4) = \int dg \tilde{\phi}(g_1g_2g_3g_4).$$

The two different invariance properties, (9) and (10) define two different theories. In both cases, the dynamics is defined by the action

$$S[\phi] = \frac{1}{2} \int \prod_{i=1}^{4} dg_i \phi^2(g_1, g_2, g_3, g_4) + \frac{\lambda}{5!} \int \prod_{i=1}^{10} dx_i \phi(g_1, g_2, g_3, g_4)$$

$$\phi(g_4, g_5, g_6, g_7) \phi(g_7, g_3, g_8, g_9) \phi(g_9, g_6, g_2, g_{10}) \phi(g_{10}, g_8, g_5, g_1).$$

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In the following sections, we show that the Feynman expansion (3) of this field theory is a sum over spin foams $s$ as in (3). A spin foam $s = (\Gamma, c)$ is here a colored combinatorial 2-dimensional cellular complex, or colored 2-complex. A 2-complex $\Gamma$ is formed by three abstract sets $V,E,F$, whose elements are respectively denoted “vertices” $v$, “edges” $e$, and “faces” $f$, and boundary relations that assign two vertices to each edge, and a cycle of edges (a cyclic sequence of edges in which any two consecutive edges are distinct and share a vertex) to each face. The number of edges that a vertex bounds (“adjacent” to the vertex) is called the valence of the vertex, and the number of faces that an edge bounds (adjacent to the edge) is called the valence of the edge. A coloring $c = (\{N_f\}, \{i_e\})$ of the 2-complex $\Gamma$ is an assignment of an irreducible representations $N_f$ of some group $G$ to each face $f$ and of an “intertwiner” $i_e$ to each edge $e$. An intertwiner $i_e$ is a vector in $K_{N_e}$, the invariant subspace of the tensor product of the Hilbert spaces of the representations $N_e = (N_{f_1}, \ldots, N_{f_n})$ assigned to the faces $f_1, \ldots, f_n$ adjacent to the edge $e$. See the appendix for details on $K_{N_e}$, and (33) for details on these models in general.

A generic spin foam model is defined by a sum over spin foams:

$$Z = \sum_{\Gamma} w(\Gamma) Z[\Gamma]$$

$$Z[\Gamma] = \sum_{\{N_f\}, \{i_e\}} \prod_f A_f(N_f) \prod_e A_e(N_e, i_e) \prod_v A_v(N_v, \vec{i}_v).$$

The first sum is over the 2-complexes and the second over the colorings. $w(\Gamma)$ is a weight factor for each 2-complex. We have indicated by $\vec{N}_v$ the set of the colors of the faces adjacent to the edge $e$, and by $\vec{i}_v$ the set of the colors of the faces and the edges adjacent to the vertex $v$. The factors $A_f, A_e$ and $A_v$ are called the amplitudes of the faces, edges and vertices. Since every edge connects exactly two vertices, one can always absorb the amplitude of the edge $A_e(\vec{N}_e, \vec{i}_e)$ into the definition of the vertex amplitude $A_v(N_v, \vec{i}_v)$. This is not possible for the amplitude of the face $A_f(N_f)$, since faces can be shared by an arbitrary number of vertices.

In this paper, we consider only 2-complexes $\Gamma$ with 4-valent edges and 5-valent vertices (for any other $\Gamma$, we have $w(\Gamma) = 0$). Notice that the 2-skeleton of the cellular complex dual to a triangulation $\Delta$ of a 4-manifold is a 2-complex.
of this kind. Indeed, vertices are 5-valent because they are dual to 4-simplices, which are bounded by precisely five tetrahedra; and edges are 4-valent because they are dual to the tetrahedra, which are bounded by four triangles.

The Feynman expansion of our field theory yields a spin foam model. The 2-complex weight factor \( w(\Gamma) \) vanishes unless \( \Gamma \) has 4-valent edges and 5-valent vertices only; in this case, it is given by \( w(\Gamma) = \lambda^{|\Lambda(\Gamma)|}/\text{sym}(\Gamma) \). Here \( n(\Gamma) \) is the number of vertices and \( \text{sym}(\Gamma) \) the order of the symmetries of the graph underlying the two-complex \( \Gamma \), defined as in standard Feynman graph theory, see for instance [24], page 93. The amplitudes are as follows.

- **Case A.** We color faces of by simple representation \( N_f \) of \( SO(4) \), and edges by intertwiners \( i_e \), forming an orthonormal basis in \( K_{\vec{N}_e} \). The amplitude of the face is \( A_f = 1 \). The amplitude of the edge is \( A_e = 1 \). The vertex amplitude \( A_v(\vec{N}_v, \vec{i}_v) \) is the \( 15-j \) Wigner symbol constructed from the ten simple representations \( \vec{N}_v \) and five intertwiners \( \vec{i}_v \). The amplitude of a 2-complex is thus

\[
Z_A[\Gamma] = \sum_{\{N_f\},\{i_e\}} \prod_v A(\vec{N}_v, \vec{i}_v). \tag{14}
\]

- **Case B.** In this case, as in case A, faces are colored by simple representation \( N_f \) of \( SO(4) \). Each edge, however, is colored by a single, fixed, intertwiner \( i_{BC} \), the (normalized) Barrett-Crane intertwiner [1]. The amplitude of the faces is \( A_f(N_f) = \text{dim} N_f \) –for simple representations, \( \text{dim} N_f = (N + 1)^2 \). The amplitude of the vertex \( T_v(\vec{N}_v, \vec{i}_v) \) is the \( 15-j \) Wigner symbol constructed from the ten simple representations \( \vec{N}_v \) and the five Barrett-Crane intertwiners \( \vec{i}_{BC} \). Thus, the amplitude of a 2-complex is

\[
Z_B[\Gamma] = \sum_{\{N_f\}} \prod_f \text{dim} N_f \prod_v A(\vec{N}_v, i_{BC}). \tag{15}
\]

This is precisely the Barrett-Crane amplitude for a triangulation \( \Delta \) whose dual 2-skeleton is \( \Gamma \).

### III. Mode Expansion

Consider a square integrable function \( \phi(g) \) over \( SO(4) \), invariant under the right action of \( SO(3) \). Using Peter-Weyl theorem, one can expand it in the matrix elements \( D^{(\Lambda)}_{\alpha\beta}(g) \) of the irreducible representations \( \Lambda \)

\[
\phi(g) = \sum_{\Lambda} \phi^{\Lambda}_{\alpha\beta} D^{(\Lambda)}_{\alpha\beta}(g) \tag{16}
\]

The indices \( \alpha, \beta \) label basis vectors in the corresponding representation space, and the sum over repeated indices is understood. In other words, we choose a basis in the representation space such that the metric in this basis is just the Kronecker delta, which is the standard choice in representation theory literature. The requirement of invariance under the right \( SO(3) \) action can be written as

\[
\phi(g) = \int_{SO(3)} dh \, \phi(gh). \tag{17}
\]

Expanding this into the modes, we have

\[
\phi(g) = \sum_{\Lambda} \phi^{\Lambda}_{\alpha\beta} D^{(\Lambda)}_{\alpha\beta}(g) = \int_{SO(3)} dh \, \phi(gh) = \sum_{\Lambda} \int_{SO(3)} dh \, \phi^{\Lambda}_{\alpha\beta} D^{(\Lambda)}_{\alpha\gamma}(g) D^{(\Lambda)}_{\gamma\beta}(h). \tag{18}
\]

\(^3\)Reference [1] is a bit obscure concerning the amplitudes of the lower dimensional simplices of the triangulation, for which it refers to [17], where a different normalization is used. A careful analysis of the requirements put on these factors, and in particular of the requested invariance under change of basis in \( K_{\vec{N}_e} \) (see [1], section IV), implies that the correct factors (using our normalization) are precisely as in [17].
The integral in the last term projects the \( \beta \) index over a (normalized) \( SO(3) \) invariant vector, which we denote by \( w_\beta \). As this exists only in simple representations (and is unique, see the appendix), the sum over representations \( \Lambda \) reduces to a sum over the simple representations \( N \) only, and thus

\[
\phi(g) = \sum_N \phi^N_\alpha D^{(N)}_{\alpha \beta}(g) \, w_\beta.
\]  

(19)

The quantities \( H^N_\alpha(g) = D^{(N)}_{\alpha \beta}(g) w_\beta \) are invariant under the (right) action of \( SO(3) \), and, thus, can be thought of as functions on the three-sphere: \( H^N_\alpha(g) = H^N_\alpha(x) \). In fact, they form an orthogonal basis for \( S^3 \) spherical harmonics.

Since \( \phi \) is real, we have

\[
\sum_N \phi^N_\alpha D^{(N)}_{\alpha \beta}(g) \, w_\beta = \sum_N \phi^N_\alpha D^{(N)}_{\alpha \beta}(g) \, w_\beta.
\]  

(20)

In the appendix, we show that the invariant vectors \( w_\beta \) are real, and that the matrix elements can also be chosen to be real. Thus, the reality condition simply requires \( \phi^N_\alpha \) to be real.

Let us now consider the field \( \tilde{\phi}(g_1, g_2, g_3, g_4) \). From the observations above, it follows that the property \( \tilde{\phi}(g_1, \ldots, g_4) \) implies that the field can be expanded as

\[
\tilde{\phi}(g_1, \ldots, g_4) = \sum_{N_1 \ldots N_4} \phi^{N_1 \ldots N_4}_{\alpha_1 \ldots \alpha_4} D^{(N_1)}_{\alpha_1 \beta_1}(g_1) \cdots D^{(N_4)}_{\alpha_4 \beta_4}(g_4) \, w_{\beta_1} \cdots w_{\beta_4}.
\]  

(21)

As before, the sum over repeated internal indices is understood. Finally, the symmetry of \( \tilde{\phi}(g_1, \ldots, g_4) \) under permutation of its four arguments implies that

\[
\phi^{N_1 \ldots N_4}_{\alpha_1 \ldots \alpha_4} = \phi^{N_{\sigma(1)} \ldots N_{\sigma(4)}}_{\alpha_{\sigma(1)} \ldots \alpha_{\sigma(4)}},
\]  

(22)

where \( \sigma \) is any permutation of \( \{1, 2, 3, 4\} \).

Let us now find the effect of the \( SO(4) \) invariance property on the modes, and find the expression for the action in terms of the modes. The analysis is different for the two cases A and B.

**Case A.** Substituting the mode expansion (21) into the definition of \( \phi \)

\[
\phi(g_1 \ldots g_4) = \int_{SO(4)} dg \, \phi(gg_1, \ldots, gg_4),
\]  

(23)

and using formula (B8) for integral of the product of four group elements, one obtains

\[
\phi(g_1, \ldots, g_4) = \sum_{N_1 \ldots N_4} \sum_\Lambda \left( \phi^{N_1 \ldots N_4}_{\alpha_1 \ldots \alpha_4} C^{N_1 \ldots N_4 \Lambda}_{\alpha_1 \ldots \alpha_4} \right) \left( C^{N_1 \ldots N_4 \Lambda}_{\beta_1 \ldots \beta_4} D^{(N_1)}_{\beta_1 \gamma_1}(g_1) \cdots D^{(N_4)}_{\beta_4 \gamma_4}(g_4) w_{\gamma_1} \cdots w_{\gamma_4} \right),
\]  

(24)

where, notice, the sum over \( \Lambda \) is over all —not necessarily simple— representations. Here \( C^{N_1 \ldots N_4 \Lambda}_{\alpha_1 \ldots \alpha_4} \) are four-valent intertwiners defined in the appendix. It is clear from the above formula that the modes \( \phi^{N_1 \ldots N_4}_{\alpha_1 \ldots \alpha_4} \) enter the mode decomposition only contracted with the intertwiner \( C \). Defining

\[
\phi^\Lambda_{\Lambda} := \phi^{N_1 \ldots N_4}_{\alpha_1 \ldots \alpha_4} C^{N_1 \ldots N_4 \Lambda}_{\alpha_1 \ldots \alpha_4} \frac{1}{\sqrt{\dim N_1 \cdots \dim N_4}},
\]  

(25)

the mode expansion takes the form

\[
\phi(g_1, \ldots, g_4) = \sum_{N_1 \ldots N_4} \sum_\Lambda \phi^\Lambda_{\Lambda} \sqrt{\dim N_1 \cdots \dim N_4} \left( C^{N_1 \ldots N_4 \Lambda}_{\beta_1 \ldots \beta_4} D^{(N_1)}_{\beta_1 \gamma_1}(g_1) \cdots D^{(N_4)}_{\beta_4 \gamma_4}(g_4) w_{\gamma_1} \cdots w_{\gamma_4} \right).
\]  

(26)

We now write the action in terms of the modes \( \phi^\Lambda \). Using the result (B1) for the integral of the product of two matrix elements, the kinetic term becomes

\[
\frac{1}{2} \int \prod_{i=1}^4 dg_i \, \phi^2(g_1, \ldots, g_4) = \frac{1}{2} \sum_{N_1 \ldots N_4} \sum_\Lambda \phi^\Lambda_{\Lambda} \phi^{N_1 \ldots N_4 \Lambda}.
\]  

(27)
For the interaction term, we obtain
\[
\frac{\lambda}{5!} \int \prod_{i=1}^{10} d\gamma_i \phi(g_1, g_2, g_3, g_4) \phi(g_4, g_5, g_6, g_7) \phi(g_7, g_3, g_8, g_9) \phi(g_9, g_6, g_2, g_{10}) \phi(g_{10}, g_8, g_5, g_1)
\]
\[
= \frac{\lambda}{5!} \sum_{N_1 \ldots N_{10}} \sum \phi_A^{N_1} N_2 N_3 N_4 A_1 \phi_A^{N_3} N_4 N_5 N_6 A_2 \phi_A^{N_6} N_7 N_8 N_9 A_3 \phi_A^{N_9} N_2 N_10 A_4 \phi_A^{N_{10}} N_8 N_5 N_1 A_5
\]
\[
A_{N_1 \ldots N_{10}, A_1 \ldots A_5}.
\]
Here \(A_{N_1 \ldots N_{10}, A_1 \ldots A_5}\) is a \((SO(4)\) analog of the \((15 - j)\)-symbol, defined as
\[
A_{N_1 \ldots N_{10}, A_1 \ldots A_5} = C^{N_1}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} C^{N_4}_{\alpha_4 \alpha_5 \alpha_6 \alpha_7} C^{N_6}_{\alpha_6 \alpha_7 \alpha_8 \alpha_9} C^{N_9}_{\alpha_9 \alpha_6 \alpha_2 \alpha_{10}} C^{N_{10}}_{\alpha_{10} \alpha_4 \alpha_5 \alpha_1},
\]
where summation over repeated internal indices is understood. Notice that the indices are connected as in Figure 1, or Figure 3. In (23), the intertwiners are written in the basis \(B_3\). For a generic choice of basis elements \(i_v\) in the Hilbert spaces \(K_{\alpha \beta}\),
\[
i_v^{N_1 N_2 N_3 N_4 (a)} = \sum_\Lambda M_{\alpha \beta}^{(a)} C^{N_1 N_2 N_3 N_4 \Lambda},
\]
we can write the vertex amplitude as
\[
A(\vec{N}, i_v) = i_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} i_{\alpha_4 \alpha_5 \alpha_6 \alpha_7} i_{\alpha_6 \alpha_7 \alpha_8 \alpha_9} i_{\alpha_9 \alpha_6 \alpha_2 \alpha_{10}} i_{\alpha_{10} \alpha_4 \alpha_5 \alpha_1}.
\]
**Case B.** Substituting the mode expansion (21) into the definition of \(\phi\)
\[
\phi(g_1 \ldots g_4) = \int_{SO(4)} dg \phi(g_1 g \ldots g_4).
\]
one obtains:
\[
\phi(g_1, \ldots, g_4) = \sum_{N_1 \ldots N_4} \sum_\Lambda \left(\phi^{N_1 \ldots N_4}_{\alpha_1 \ldots \alpha_4} D^{(N_1)}_{\alpha_1 \beta_1}(g_1) \ldots D^{(N_4)}_{\alpha_4 \beta_4}(g_4) C^{N_4 \ldots N_1 \Lambda}_{\beta_1 \ldots \beta_4} \phi^{N_4 \ldots N_1}_{\beta_1 \ldots \beta_4} \right) (N_{\gamma_1} \ldots N_4 w_{\gamma_1} \ldots w_{\gamma_4}).
\]
The quantity in the second parenthesis is the scalar product of two \(SO(3)\) invariant vectors in the representation \(\Lambda\). Since invariant vectors exist only in simple representations, this quantity is non-vanishing only when \(\Lambda\) is a simple representations \(N\). In this case its value is given by (31)
\[
C^{N_1 \ldots N_4}_{\gamma_1 \ldots \gamma_4} w_{\gamma_1} \ldots w_{\gamma_4} = \frac{1}{\sqrt{\dim N_1 \ldots \dim N_4}}
\]
This suggests to redefine the mode expansion in term of the new fields
\[
\phi g^{N_1 \ldots N_4}_{\alpha_1 \ldots \alpha_4} = \phi^{N_1 \ldots N_4}_{\alpha_1 \ldots \alpha_4} \sqrt{\dim N_1 \ldots \dim N_4}
\]
Substituting this into the mode expansion, we get
\[
\phi(g_1, \ldots, g_4) = \sum_{N_1 \ldots N_4} \phi g^{N_1 \ldots N_4}_{\alpha_1 \ldots \alpha_4} D^{(N_1)}_{\alpha_1 \beta_1}(g_1) \ldots D^{(N_4)}_{\alpha_4 \beta_4}(g_4) S^{N_1 \ldots N_4}_{\beta_1 \ldots \beta_4}.
\]
Here \(S^{N_1 \ldots N_4}_{\beta_1 \ldots \beta_4}\) is the normalized Barrett-Crane intertwiner\(^4\)

\(^4\)Barrett and Crane use a non-normalized expression in \([5]\). Notice also that they call this intertwiner “vertex”. Here we reserved the expression “vertex” for the five-valent vertices of the 2-complex (the dual to the 4-simplices). This choice is consistent with standard Feynman diagrammatic and with \([4]\).
\[ S_{\beta_1 \ldots \beta_4}^{N_1 \ldots N_4} := \frac{\sum_{N} C_{\beta_1 \ldots \beta_4}^{N_1 \ldots N_4} N}{\sqrt{\sum_{N} C_{\beta_1 \ldots \beta_4}^{N_1 \ldots N_4} C_{\beta_1 \ldots \beta_4}^{N_1 \ldots N_4}}} . \]  

The normalization, given by the denominator in the above expression, is in the scalar product of \( K_{N_e} \). Since the quantities \( C_{\beta_1 \ldots \beta_4}^{N_1 \ldots N_4} N \) are normalized, the denominator is the square root of the dimension of the subspace of \( K_{N_e} \) spanned by the intertwiners having an intermediate simple representation (See Appendix).

Let us now find the mode expansion of the action. Using the result for the integral of the product of two matrix elements, the kinetic term becomes

\[ \frac{1}{2} \int \prod_{i=1}^{4} dg_i \phi^2(g_1, \ldots, g_4) = \frac{1}{2} \sum_{N_1 \ldots N_4} \phi_{B\alpha_1 \ldots \alpha_4} N_{1}^{N_1} \phi_{B\alpha_1 \ldots \alpha_4} N_{1}^{N_4} . \]  

The potential term gives

\[ \frac{\lambda}{5!} \int \prod_{i=1}^{10} dg_i \phi(g_1, g_2, g_3, g_4) \phi(g_4, g_5, g_6, g_7) \phi(g_7, g_8, g_9) \phi(g_9, g_6, g_2, g_10) \phi(g_{10}, g_8, g_5, g_1) = \frac{\lambda}{5!} \sum_{N_1 \ldots N_{10}} \sum_{\Lambda_1 \ldots \Lambda_5} \phi_{B\alpha_1 \ldots \alpha_4} N_{1}^{N_1} N_{2}^{N_2} N_{3}^{N_3} N_{4}^{N_4} \phi_{B\alpha_1 \ldots \alpha_4} N_{1}^{N_1} N_{2}^{N_2} N_{3}^{N_3} N_{4}^{N_4} \phi_{B\alpha_1 \ldots \alpha_4} N_{1}^{N_1} N_{2}^{N_2} N_{3}^{N_3} N_{4}^{N_4} \phi_{B\alpha_1 \ldots \alpha_4} N_{1}^{N_1} N_{2}^{N_2} N_{3}^{N_3} N_{4}^{N_4} \phi_{B\alpha_1 \ldots \alpha_4} N_{1}^{N_1} N_{2}^{N_2} N_{3}^{N_3} N_{4}^{N_4} . \]

Here \( B_{N_1 \ldots N_{10}} \) is the Barrett-Crane vertex-amplitude, which is a \((15 - j)\)-symbol with Barrett-Crane intertwiners

\[ B_{N_1 \ldots N_{10}} := S_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} N_{1}^{N_1} S_{\alpha_4 \alpha_5 \alpha_6 \alpha_7} N_{2}^{N_2} S_{\alpha_6 \alpha_7 \alpha_8 \alpha_9} N_{3}^{N_3} S_{\alpha_8 \alpha_9 \alpha_10} N_{4}^{N_4} S_{\alpha_10 \alpha_2 \alpha_3 \alpha_4} N_{1}^{N_1} . \]

### IV. FEYNMAN GRAPHS

**Case A.** The partition function is given by the integral over modes

\[ Z = \int [D\phi_{A}^{N_1 \ldots N_4} \Lambda] e^{-S[\phi_A]} . \]

We expand \( Z \) in powers of \( \lambda \). The Gaussian integrals are easily computed, giving the propagator

\[ P_{A}^{N_1 \ldots N_4, N'_1 \ldots N'_4} := \langle \phi_{A}^{N_1 \ldots N_4} \phi_{A}^{N'_1 \ldots N'_4} \rangle = \frac{1}{4!} \sum_{\sigma} \delta^{N_1 N'_1} \delta^{N_2 N'_2} \delta^{N_3 N'_3} \delta^{N_4 N'_4} M_{\sigma} . \]

\( \sigma \) are the permutations of \( \{1, 2, 3, 4\} \). The matrix \( M_{\sigma} \), defined in the appendix, gives the change of basis in \( K_{N_e} \) and corresponds to a permutation of the four representations (see appendix). There is a single vertex, of order five, which is

\[ \langle \phi_{A}^{N_1 N_2 N_3 N_4 N_5} \rangle \ldots \phi_{A}^{N_3 N_4 N_5 N_1 N_2} \rangle = \lambda \ A_{N_1 \ldots N_5} . \]

**Case B.** The partition function in this case is given by

\[ Z = \int [D\phi_{B}^{N_1 \ldots N_4}] e^{-S[\phi_B]} . \]

The propagator is

\[ P_{B}^{N_1 \ldots N_4, N'_1 \ldots N'_4} := \langle \phi_{B}^{N_1 \ldots N_4} \phi_{B}^{N'_1 \ldots N'_4} \rangle = \frac{1}{4!} \sum_{\sigma} \delta^{N_1 N'_1} \delta^{N_2 N'_2} \delta^{N_3 N'_3} \delta^{N_4 N'_4} . \]

The vertex is given by

\[ \langle \phi_{B}^{N_1 N_2 N_3 N_4} \phi_{B}^{N_1 N_2 N_3 N_4} \phi_{B}^{N_1 N_2 N_3 N_4} \phi_{B}^{N_1 N_2 N_3 N_4} \phi_{B}^{N_1 N_2 N_3 N_4} \rangle = \lambda \ \delta^{N_1 N'_1} \delta^{N_2 N'_2} \delta^{N_3 N'_3} \delta^{N_4 N'_4} . \]
The set of Feynman rules one gets is similar in the two cases. First, we obtain the usual overall factor $\lambda^{\nu(\Gamma)/\text{sym}(\Gamma)}$ (see for instance [24], page 93). Next, we represent each of the terms in the right hand side of the definitions (42, 45) of the propagator by four parallel strands, as in Fig. 2, carrying the indices at their end. Then, we can represent the propagator itself by the symmetrization of the four strands. In addition, in case A, edges $e$ are labeled by a representation $\Lambda_e$ (not necessarily simple).

The Feynman graphs we get are all possible “4-strand” 5-valent graphs, where a “4-strand graph” is a graph whose edges are collections of four strands, and whose vertices are the ones shown in Fig. 3. Each strand of the propagator can be connected to a single strand in each of the five “openings” of the vertex. Orientations in the vertex and in the propagators should match (this can always be achieved by changing a representation to its conjugate). Each strand of the 4-strand graph goes through several vertices and several propagators, and then closes to itself, forming a cycle. Note that a particular strand can go through a particular edge of the 4-strand graph more than once. Cycles get labeled by the simple representations of the indices. For each graph, the abstract set formed by the vertices, the edges, and the cycles forms a 2-complex, in which the faces are the cycles. The labeling of the cycles by simple representations of $SO(4)$ determines a coloring of the faces by spins. Thus, we obtain a colored 2-simplex, namely a spin foam.

In Case A, edges $e$ are labeled by an intertwiner with index $\Lambda_e$ (in the basis (B3)). Vertices $v$ contribute a factor $\lambda$ times $A$, (see (29)), which depends on ten simple representations labeling the cycles that go through the vertex, and on five intertwiners, basis elements in $K_{\vec{N}}$, labeling the edges that meet at $v$. Each edge contracts two vertices, say $v$ and $v'$. It contributes a matrix $M_\sigma$. This is the matrix of the change of basis from the basis in $K_{\vec{N}}$ used at $v$ and the basis used at $v'$. If we fix a basis of intertwiners $i_e$ in $K_{\vec{N}}$ for every $e$, once and for all for each fixed 2-complex –that is, for each assigned permutation in (45)– and use vertex amplitude $A(\vec{N}, \vec{i})$ given in (31), the matrix $M_\sigma$ is absorbed into the vertex amplitude and the propagator is the identity. The weight of 2-complex $\Gamma$ is then given by (14).

In Case B, there are no labeling of edges, and we gets an additional contribution from the summing of the $\alpha'$ indices of the Kroneker deltas $\delta^{\alpha\alpha'}$ around each cycle. This gives a factor $\dim N_f$ for every face $f$. Thus, faces $f$ are labeled by simple representations $N_f$ and contribute a factor $\dim N_f$. Vertices contribute a factor $\lambda$ times $B$, where $B$ is the Barrett-Crane weight (15), which depends on the ten simple representations of the faces adjacent to the vertex. This yields the result (15).

V. TRIANGULATIONS, ORIENTATION AND SYMMETRIES OF $\phi$

In this section we comment on the relation between 2-complexes and triangulations, and we discuss some possible modification of our models, which leads to oriented 2-complexes.

• Given a triangulation $\Delta$, the (abstract combinatorial structure of its) dual 2-skeleton, $\Gamma(\Delta)$ is a 2-complex. In general, not every 2-complex comes from a triangulation triangulation of a 4-manifold in this way. In 3d, one
can nicely characterize the 2-complexes that are derived from triangulations. Since this characterization introduces an interesting technology, we mention it here, adapted from [25].

As we have seen, a 2-complex corresponds to a 4-strand graph. Consider a sequence of edges forming a closed loop in the graph, starting with an edge \( e \). Label the strands of \( e \), say with integers (1,2,3 in 3d, and 1,2,3,4 in 4d). The labeling can be carried over to the next edge in the loop, say \( e' \) by noticing that the strands of \( e' \) are naturally paired with the strands of \( e \): a strand in \( e \) and a strand in \( e' \) are paired either if they are the same strand that continues across the vertex, or if they both continue across the vertex into an edge \( e'' \). Thus, the labeling can be carried over the loop. In closing the loop, we obtain a new labeling of the strands of \( e \). If the new labeling is an even (respectively odd) permutation of the original labeling, we say that the loop is even (respectively odd).

In 3d we have the following result [23]. If, and only if, each cycle (a loop that bounds a face) of the 2-complex is even, we can “thicken” faces, edges and vertices of the 2-complex, obtaining a 3 manifold with boundaries, whose spine is the 2-complex. If, in addition, each component of the boundary of this manifold is a sphere, we can then fill the sphere with a ball, and obtain a triangulated manifold, whose dual 2-skeleton is the 2-complex. We are not aware of related results in 4d.

Next, let us now consider the notion of orientation. The notion of orientability of a manifold can be extended to abstract, \( n \)-dimensional triangulations. In \( n \) dimensions, an \( n \)-simplex \( S \) has \( n+1 \) points (0-simplices) in its boundary. An orientation of \( S \) is a choice of an ordering of these \( n+1 \) points, up to even permutations. An \( n \)-simplex is bounded by \( n+1 \) \((n-1)\)-simplices. Each of these is bounded by \( n-2 \) points, obtained by dropping one of the points of the \( n \)-simplex. Each \((n-1)\)-simplex inherits an (“outgoing”) orientation from the \( n \)-simplex. This can be obtained by dropping the missing point in an even ordering of the points of the \((n-1)\)-simplices in which this missing point is first. Two \( n \)-simplices that share an \((n-1)\)-simplex have consistent orientation if the shared \((n-1)\)-simplex inherits opposite orientations from the two \( n \)-simplices it bounds. A triangulation is orientable if it admits a consistent orientations of all its \( n \)-simplices. Clearly, the triangulation of an orientable manifold is orientable.

Now, the notion of orientability extends to 2-complexes with 4-valent edges and 5-valent vertices. In fact, notice that in the boundary of an \( n \)-simplex of a triangulation, the \((n-1)\)-simplices are paired with the points (the point that does not belong to the \((n-1)\)-simplex). Thus, the ordering of the points corresponds to an ordering of the \((n-1)\)-simplices. Consequently, we can simply define the orientation of a vertex of a 2-complex as an ordering of its adjacent edges, up to even permutations. In turn, the orientation of a vertex induces an (outgoing) orientation of its adjacent edges, namely an ordering of the faces, obtained as above. The orientation of two vertices separated by an edge is consistent if the edge inherits opposite orientations from the two vertices. The 2-complex is orientable if it admits a consistent orientation of all its vertices. Clearly, a 2-complex is orientable if all its loops (not just the cycles) are even. A 2-complex derived from a triangulation of an orientable manifold, is orientable.

It is not difficult to modify our field theory in such a way that it generates only orientable 2-complexes. To do that, it is sufficient to replace the requirement that \( \phi \) is symmetric under any permutation of its arguments, eq. (8) with

\[
\phi(g_1, g_2, g_3, g_4) = \phi(g_{\sigma_E(1)}, g_{\sigma_E(2)}, g_{\sigma_E(3)}, g_{\sigma_E(4)}),
\]

where \( \sigma_E \) is an even permutation, and to rewrite the action as

\[
S[\phi] = \frac{1}{2} \int \prod_{i=1}^{4} dg_i \phi(g_1, g_2, g_3, g_4) \phi(g_3, g_2, g_1, g_4) + \frac{\lambda}{5!} \int \prod_{i=1}^{10} dg_i \phi(g_1, g_2, g_3, g_4) \phi(g_4, g_5, g_6, g_7) \phi(g_7, g_3, g_8, g_9) \phi(g_9, g_6, g_2, g_10) \phi(g_{10}, g_8, g_5, g_1).
\]

(This action is the same as (11) if \( \phi \) is symmetric under any permutation of its arguments.) If we do so, the propagators (12) and (15) contain odd permutations only. A moment of scrutiny of the vertex shows that the pairing of the strands discussed above (in defining the parity of the loops) determines always an odd permutation. If we consider a loop of edges in the Feynman diagram, we cross an equal number of vertices and edges. Therefore the strands undergo an even number of odd permutations. Therefore in closing the loop we obtain an even permutation. Therefore all loops are even and the 2-complex is orientable.

Other versions of the model can be defined by considering more complicated symmetry properties of the function \( \phi \). In particular, there are 5 irreducible representation of the permutation group of 4 elements \( \Sigma_4 \). Two of these
are one-dimensional (the trivial and the signature), and the other three have dimensions $2, 3$ and $5$. If we denote $\chi_i(g), i = 1, \ldots, 5$ the character of these representations, we can define five different fields

$$\phi_i(g_1, g_2, g_3, g_4) = \frac{1}{4!} \sum_{\sigma \in \Sigma_4} \chi_i(\sigma) \phi_i(g_{\sigma(1)}, g_{\sigma(2)}, g_{\sigma(3)}, g_{\sigma(4)}).$$

(49)

Using the orthonormality of the characters, the original field is totally determined by the $\phi_i$

$$\phi = \sum_i \chi_i(1) \phi_i.$$

(50)

Taking the field $\phi$ to be one of the $\phi_i$ leads to five different models having slightly different Feynman rules. The propagator associated with permutation gets multiplied by the character of this permutation in the given representation, and so on. Finally, one can perhaps add the actions of these five models, and use the coupling constants to try to control the topologies appearing in the Feynman expansion.

VI. CONCLUSION

We close with some general comments.

- The results of this paper generalize easily to higher dimensions. In dimension $D$, we replace the group $SO(4)$ with $SO(D)$, the sphere $S^3$ with $S^D = SO(D) / SO(D - 1)$. The field $\phi$ is defined over $D$ copies of the $SO(D)$. The interaction term is of order $D + 1$ and has the structure of a $D$-simplex. The simple representations of $SO(D)$ are discussed in details in [23]. In any dimension, Case B yields the higher-dimensional generalizations of Barrett-Crane model discussed in [24].

- The sum over spin foam we have obtained is presumably divergent. There are two sources of divergences. The first is the sum over the colorings. This divergence can be regulated by replacing the Clebsch-Gordan coefficients of the group with the ones of the corresponding quantum group, with $q^k = 1$, as in the Barrett-Crane model. This procedure cuts off the sum over representations, and thus makes this sum finite. In this regard, we think that it would be interesting to explore the construction of the field theory presented here, but over a non-commutative 3-sphere.

The second source of divergences is the sum over 2-complexes. The number $N$ of diff-inequivalent triangulations of a given four dimensional manifold grows exponentially only with the number $n$ of simplices [26]. That is $N(n) \sim exp(an)$ for large $n$. This fact suggests that the sum might converge for $\lambda$ smaller than a critical value, determined by $\alpha$ [27]. On the other hand, the Feynman expansion generates triangulations of arbitrary manifolds, and this might destroy the convergence.

- The Barrett-Crane model has been obtained by “quantizing a geometric 4-simplex” [1]. That is, by realizing certain geometrical quantities, as the “bivectors” that can be associated to the faces of the tetrahedron, as operators. Simple representations emerge in this process: in the quantum theory, the condition on bivectors that they describe planes in $R^4$ translates into the the simplicity condition (A1) on the representation. In turn, the condition that these planes intersect along lines determines the Barrett-Crane intertwiner (see also [28]). The model described here as Case A can be seen as a different solution to this intersection constraint. In detail, consider a single tetrahedron of a triangulation. The tetrahedron belongs to two different 4-simplices. In the Barrett-Crane model, one “quantizes each 4-simplex” separately. The corresponding intersection constraint is imposed two times. If, alternatively, we view the tetrahedron as a single element of the triangulation, we may expect a single constraint corresponding to the requirement that the two faces intersect on a line. The vertex of the model A satisfies the intersection constraint in this sense. Indeed, it satisfies the identity pictured in Fig. 4. Thus, the model of case A provides another solution to the intersection constraints.

- We close with a comment on the physical interpretation of the theory we have described. Consider the model of case A. The field is $\phi^{N_1 \cdots N_4}$. The indices corresponds to the assignment of a (simple) representations to each face of a tetrahedron, and of an (arbitrary) intertwiner to the bulk of the tetrahedron. This assignment can be thought as the assignment of metric properties to the tetrahedron: indeed, we know from loop quantum gravity [2, 29] and from other approaches [24], that the $N_i$’s are naturally related to the area of the faces, and their intertwiner to the volume. [More precisely, the intrinsic geometry of a classical tetrahedron in 4d is determined
FIG. 4. The identity satisfied by the 4-valent vertices of model A. The sum over all representations \( \Lambda \) is implicit.

by six numbers (4x4 coordinates of the vertices, minus ten dimensions from the Poincaré group), but two of these quantities do not commute as quantum operators and therefore only five quantum numbers fix the state. Therefore, we can view \( \phi^{N_1 \ldots N_4} \) as a quantum amplitude for a certain geometry of a tetrahedron or, more generally, of an elementary “chunk” of space, with a certain volume, and bounded by four elementary surfaces with a certain area. Correspondingly, we can view the field \( \phi(x_1 \ldots x_4) \) as the “Fourier transform” of such amplitude. From this perspective, the quantum theory defined by the action (11) is the second quantization of the quantum theory of the geometry of an elementary chunk of space. As a second quantized theory, it is a “multiparticle theory”, in which particles are created and destroyed. Four dimensional spacetime can thus be viewed as a Feynman history of creations, destruction and interactions of these elementary quanta of space.

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APPENDIX A: SOME FACTS ON REPRESENTATION THEORY

We collect here some facts on the representation theory of \( SO(D) \). We label finite dimensional irreducible representation of \( SO(D) \) by their highest weight \( \Lambda \). \( \Lambda \) is a vector of length \( n = [D/2] \) \([\cdot]\) is the integral part): \( \Lambda = (N_1, \ldots, N_n) \), where \( N_i \) are integer and \( N_1 \geq \cdots \geq N_n \). If we are interested in representations of \( \text{Spin}(D) \) we let the \( N_i \) be half integers. The representation labeled by the highest weight \( \Lambda = (N, 0, \cdots, 0) \) are called simple or spherical. Let \( X_{ij}, 1 \leq i,j \leq D \) be a basis of the Lie algebra of \( SO(D) \). The simple representation are the ones for which the “simplicity” relations

\[
X_{ij} X_{ij} \cdot V_N = 0 \quad (A1)
\]

are satisfied. Here, \([\cdot]\) denotes the antisymmetrisation. The representation space \( V_N \) of a simple representation can be realized as a space of spherical harmonics, that is, harmonic homogeneous polynomial on \( R^D \). Any \( L^2 \) function on the sphere can be uniquely decomposed in terms of these spherical harmonics

\[
L^2(S^{D-1}) = \bigoplus_{N=0}^{\infty} V_N. \quad (A2)
\]

In the case of \( SO(4) \), since \( \text{Spin}(4) = SU(2) \times SU(2) \), there is an alternate description of the representation as products of two representations \( j' \) and \( j'' \) of \( SU(2) \). The relation with the highest weight presentation is given by

\[
N_1 = j' + j'', \quad N_2 = j' + j''. \quad (A3)
\]

the simple representation are therefore the representation in which \( j' = j'' := j \). Thus, we can label simple representations with a half integer spin \( j \). Notice that the integer “color” \( N = 2j \) is also the (nonvanishing component of the) highest weight of the representation.
An elementary illustration of these simple representations can be given as follows. The vectors $v^\alpha$ of the representation $\Lambda = (j', j'')$ can be written as spinors $\psi^{A_1 \ldots A_j'} B_1 \ldots B_j''$ with $j'$ “undotted” symmetrized indices $A_i$ and $j''$ “dotted” antisymmetrized indices $B_i$, transforming under one of the $SU(2)$, and $j''$ “dotted” antisymmetrized indices $B_i = 1, 2$ transforming under the other. Consider the particular vector $w^\alpha$ which (in the spinor notation, and in a given basis) has components $\psi^{A_1 \ldots A_j'} B_1 \ldots B_j'' = \epsilon(A_1 B_1 \ldots A_j B_j)$, where $\epsilon^{AB}$ is the unit antisymmetric tensor, and the symmetrization is over the $A_i$ indices only. The subgroup of $SO(4)$ that leaves this vector invariant is a $SO(3)$ subgroup of $SO(4)$ (which depends on the basis chosen). Clearly, since $\epsilon^{AB}$ is the only object invariant under this $SU(2)$, a normalized $SO(3)$ invariant vector (and only one) exists in these simple representations only.

Equivalently, the simple representations of $SO(4)$ are the ones defined by the completely symmetric traceless 4d tensors of rank $N$. The invariant vector $w$ is then the traceless part of the tensor with (in the chosen basis) all component vanishing except $w^{A_1 \ldots A_j}$. The $SO(3)$ subgroup is given by the rotations around the fourth coordinate axis. The relation between the vector and spinor representation is obtained contracting the spinor indices with the (four) Pauli matrices: $v_{\mu_1 \ldots \mu_j} = \psi^{A_1 \ldots A_j'} B_1 \ldots B_j'' \sigma^{\mu_1}_{A_1 B_1} \cdots \sigma^{\mu_j}_{A_j B_j}$.

In the main text we have used the following properties of the simple representations

- Let $V_\Lambda$ be a representation of $SO(D)$, we say that $\omega \in V_\Lambda$ is a spherical vector if it is invariant under the action of $SO(D - 1)$. Such a vector exists if and only if the representation is simple. In that case this vector is unique up to normalization.

Let $\omega$ be a vector of $V_\Lambda$, and consider an orthonormal basis $v_i$ of $V_\Lambda$. We can construct the following functions on $G$:

$$\Theta_i(g) = \langle \omega | D^{-1}(g) | v_i \rangle. \quad (A4)$$

These functions span a subspace of $L^2(G)$. The group acts on this subspace by the right regular representation, and the corresponding representation is equivalent to the representation $V_\Lambda$. If $\omega$ is spherical, then these functions are in fact $L^2$ functions on the quotient space $SO(D)/SO(D - 1) = S^{D-1}$ and $V_\Lambda$ is therefore a spherical representation. On the other hand, if the representation is spherical then we can construct a spherical vector: $\Theta_\omega(g) = \sum_i \Theta_i(g) \Theta_i(1)$. When $\omega$ is spherical, the spherical function $\Theta_\omega$ is a function on the double quotient space $SO(D - 1)/SO(D)/SO(D - 1) = U(1)$. It is now a standard exercise to show that there is a unique harmonic polynomial on $R^D$ invariant by $SO(D - 1)$ of a given degree, hence there is a unique spherical function.

- The space of intertwiner of three representations of $SO(4)$ is at most one dimensional.

The dimension $n_{\Lambda_1, \Lambda_2, \Lambda_3}$ of the space of intertwiner between three representations $\Lambda_1, \Lambda_2, \Lambda_3$ is given by the integral

$$n_{\Lambda_1, \Lambda_2, \Lambda_3} = \int dg \chi_{\Lambda_1}(g) \chi_{\Lambda_2}(g) \chi_{\Lambda_3}(g) \quad (A5)$$

where $\chi_\Lambda$ are the characters of the representation $\Lambda$. For $SO(4)$, since any representation is the product of two representation of $SU(2)$ the intertwining number of three $SO(4)$ representation is the product of $SU(2)$ intertwining numbers. It is well known that this number is 0 or 1 for $SU(2)$.

- The representation of $SO(N)$ are real. This means that it is always possible to choose a basis of $V_\Lambda$ such that the representation matrices are real. If we are interested by representation of half integer spin of $Spin(N)$ it is still true that $\Lambda$ is equivalent to its complex conjugate or dual. However the isomorphism is non trivial. The difference between this two case can be characterized by the value of the following integral

$$I_\Lambda = \int dg \chi_\Lambda(g^2). \quad (A6)$$

This is $+1$ if $\Lambda$ is an integer spin representation and $-1$ if $\Lambda$ is a half-integer spin representation. This is easily seen in the case of $Spin(4)$, since any representation is the product of two representations of $SU(2)$ the integral reduces to the product of two $SU(2)$ integrals.
APPENDIX B: INTERTWINERS AND THEIR SPACES

The integral of the product of two unitary matrix elements is given by:

$$\int_{SO(4)} dg \overline{D}^{(A)}_{\alpha \beta} (g) D^{(A')}_{\alpha' \beta'} (g) = \frac{1}{\dim N} \delta^{\Lambda \Lambda'} \delta_{\alpha \alpha'} \delta_{\beta \beta'}.$$  \hfill (B1)

Since we choose a real basis, the bar can be dropped. The integral of the product of three group elements is:

$$\int_{SO(4)} dg D^{(N_1)}_{\alpha_1 \beta_1} (g) D^{(N_2)}_{\alpha_2 \beta_2} (g) D^{(A)}_{\alpha_3 \beta_3} (g) = C^{N_1 N_2 A}_{N_1 N_2 A} C^{N_1 N_2 A}_{N_1 N_2 A}.$$  \hfill (B2)

Here $C^{N_1 N_2 A}_{N_1 N_2 A}$ are normalized $(3-j)$-symbols for $SO(4)$. Several normalization conditions are used in the literature. We follow here the most common one, in which the value of the $\theta$-symbol is one. That is $C^{N_1 N_2 A}_{\alpha_1 \alpha_2 \alpha_3} C^{N_1 N_2 A}_{\alpha_3 \alpha_2 \alpha_1} = 1$. The intertwiner from the tensor product of two representations $N_1, N_2$ to a representation $\Lambda$, if it exists is unique.

Next, consider four representations $N_1 \ldots N_4$. They are defined on the Hilbert spaces $H_1 \ldots H_4$. Consider the tensor product $H_{N_1 \ldots N_4} = H_1 \otimes \ldots \otimes H_4$. This space decomposes into irreducibles. In particular, it contains the trivial representation, with a certain multiplicity $m$. We denote the $m$ dimensional subspace of $H_{N_1 \ldots N_4}$ formed by the trivial representations, that is, the $SO(4)$ invariant subspace of $H_{N_1 \ldots N_4}$ as $K_{N_1 \ldots N_4}$. The space $K_{N_1 \ldots N_4}$, its scalar product, and its subspaces play a key role in the spin foam models. When the representations $N_1 \ldots N_4$ are associated to the four edges adjacent to the edge $e$, we write $K_{N_1 \ldots N_4}$ also as $K_{\Lambda}$. The vectors in $K_{N_1 \ldots N_4}$ are the “intertwiners” between the representations $N_1 \ldots N_4$. They are $SO(4)$ invariant tensors with four indices, one in each representation $H_i$. We write them as $\Lambda_{\gamma_1 \ldots \gamma_4}$. An orthonormal basis in $K_{N_1 \ldots N_4}$ can be obtained as follows. We pair the representations as $(N_1, N_2), (N_3, N_4)$. Then we define

$$C^{N_1 \ldots N_4 \Lambda}_{\gamma_1 \ldots \gamma_4} = \sqrt{\dim N} C^{N_1 N_2 \Lambda}_{\gamma_1 \gamma_2} C^{N_3 N_4 \Lambda}_{\gamma_3 \gamma_4}.$$  \hfill (B3)

As $\Lambda$ runs over the finite number of representations for which the $(3-j)$-symbols do not vanish, the $C^{N_1 \ldots N_4 \Lambda}_{\gamma_1 \ldots \gamma_4}$ form an orthonormal basis of $K_{N_1 \ldots N_4}$. The factor $\sqrt{\dim N}$ normalizes these vectors in $K_{N_1 \ldots N_4}$, $\Lambda$. Clearly, there are other bases of this kind, obtained by pairing the indices in a different manner. For example, we can we pair the indices as $(N_1, N_2), (N_3, N_4)$ and define the basis

$$\tilde{C}^{N_1 N_2 N_3 N_4 \Lambda}_{\gamma_1 \gamma_2 \gamma_3 \gamma_4} = C^{N_1 N_2 N_3 N_4 \Lambda}_{\gamma_1 \gamma_2 \gamma_3 \gamma_4}.$$  \hfill (B4)

Since both the $C$'s and the $\tilde{C}$'s are orthonormal bases, the transformation matrix $M$ is immediately given by linear algebra

$$\tilde{C}^{N_1 N_2 N_3 N_4 \Lambda}_{\gamma_1 \gamma_2 \gamma_3 \gamma_4} = \sum_{\Lambda'} M^{N_1 N_2 N_3 N_4 \Lambda'}_{\Lambda'} C^{N_1 N_2 N_3 N_4 \Lambda'}_{\gamma_1 \gamma_2 \gamma_3 \gamma_4}$$  \hfill (B5)

and

$$M^{N_1 N_2 N_3 N_4 \Lambda}_{\Lambda'} = C^{N_1 N_2 N_3 N_4 \Lambda'}_{\gamma_1 \gamma_2 \gamma_3 \gamma_4}.$$  \hfill (B6)

In fact, $M^{N_1 N_2 N_3 N_4 \Lambda}_{\Lambda'}$ is a $6-j$ symbol for $SO(4)$. For a generic permutation $\sigma$ of four elements, we have a basis

$$\sigma C^{N_1 N_2 N_3 N_4 \Lambda}_{\gamma_1 \gamma_2 \gamma_3 \gamma_4} = C^{N_{\sigma(1)} N_{\sigma(2)} N_{\sigma(3)} N_{\sigma(4)} \Lambda}_{\gamma_{\sigma(1)} \gamma_{\sigma(2)} \gamma_{\sigma(3)} \gamma_{\sigma(4)}}.$$  \hfill (B7)

and a corresponding matrix of change of basis $M^{N_{\sigma(1)} N_{\sigma(2)} N_{\sigma(3)} N_{\sigma(4)} \Lambda}_{\Lambda'}$.

Using this technology, the integral of the product of four group elements is simply a resolution of the identity in $K_{N_1 \ldots N_4}$ and can be written (for any choice of basis) as

$$\int_{SO(4)} dg D^{(N_1)}_{\alpha_1 \beta_1} (g) \ldots D^{(N_4)}_{\alpha_4 \beta_4} (g) = \sum_{\Lambda} C^{N_1 \ldots N_4 \Lambda}_{\alpha_1 \ldots \alpha_4} C^{N_1 \ldots N_4 \Lambda}_{\beta_1 \ldots \beta_4}.$$  \hfill (B8)

Finally, we derive equation (84) used in the main text. Using the ambiguity in the definition of the $(3-j)$-symbol, which so far is defined up to a phase factor, we can choose it so that the quantity

$$C^{N_1 N_2 N_3 N_4}_{\gamma_1 \gamma_2 \gamma_3 \gamma_4} w_{\gamma_1} w_{\gamma_2} w_{\gamma_3}$$
is real and positive. Then its square is given by the following integral

\[
(C^{N_1 \ldots N_4}_{\gamma_1 \ldots \gamma_4} w_{\gamma_1} w_{\gamma_2} w_{\gamma_3} w_{\gamma_4})^2 = \int dg D^{(N_1)}_{00} D^{(N_2)}_{00} D^{(N)}_{00}.
\] (B9)

This integral is computed for the general case of SO(\(N\)) in [30], Chapter 9.4.11. In the case of SO(4) it is given by:

\[
\frac{1}{(N_1 + 1)(N_2 + 1)(N + 1)}.
\] (B10)

It follows

\[
C^{N_1 \ldots N_4}_{\gamma_1 \ldots \gamma_4} w_{\gamma_1} \ldots w_{\gamma_4} = \sqrt{\text{dim}_N} \left( C^{N_1 N_2}_{\gamma_1 \gamma_2} w_{\gamma_1} w_{\gamma_2} \right) \left( C^{N_1 N_2}_{\gamma_1 \gamma_2} w_{\gamma_1} w_{\gamma_2} \right) = \frac{1}{\sqrt{\text{dim}_N \cdots \text{dim}_N}}.
\] (B11)
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