Epicyclic orbital oscillations in Newton’s and Einstein’s dynamics

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Abstract

We apply Feynman’s principle, “The same equations have the same solutions”, to Kepler’s problem and show that Newton’s dynamics in a properly curved 3-D space is identical with that described by Einstein’s theory in the 3-D optical geometry of Schwarzschild’s spacetime. For this reason, rather unexpectedly, Newton’s formulae for Kepler’s problem, in the case of nearly circular motion in a static, spherically spherical gravitational potential accurately describe strong field general relativistic effects, in particular vanishing of the radial epicyclic frequency at $r = r_{\text{ms}}$.

1. Introduction: observed QPOs as a test of strong gravity

The standard accretion disk theory (Shakura and Sunyaev, 1973) assumes that matter in accretion disks around black holes and neutron stars moves on nearly circular, nearly Keplerian (i.e., nearly geodesic) orbits. Theory predicts several strong field general relativistic effects that should follow directly from this assumption, but none of them has so far been clearly detected. Most recently, we have found a new effect of this type (Kluźniak and Abramowicz, 2000, 2001, 2002). It concerns the QPOs, quasi periodic oscillations with kilohertz frequencies, observed as variations in the X-ray luminosity of accreting neutron stars and black holes. QPOs often occur in coupled pairs manifested as characteristic double peaks in the variability power spectra. According to us, observed frequencies at the double peaked QPOs are directly related to orbital epicyclic frequencies, vertical and radial, which are in a 3 : 2 parametric resonance. The resonance occurs in super strong gravity, just several gravitational radii outside the central black hole or neutron star, at a precisely determined radius. Knowing that, one may sharply constrain the global parameters of the source, as it was done, for example, for the Kerr angular momentum parameter in the black hole “candidate” GRO J1655-40 (Abramowicz and Kluźniak, 2001) that shows two QPOs at 450 Hz and 300 Hz (Strohmayer, 2001).

In strong Einstein’s gravity the radial epicyclic frequency $\Omega_r$ is smaller than the Keplerian orbital frequency $\Omega_K$, and at the radius of the marginally stable orbit $\Omega_r = 0$. In weak Newton’s gravity one has $\Omega_r = \Omega_K \neq 0$. These very different behaviours are attributed by many authors to a “non-linearity” of Einstein’s gravity. We discuss here a simpler and more proper explanation: the difference is due only to the curvature of the three dimensional space.
Our explanation consists of three steps. First, we recall the relevant Einstein’s equations. We write them in a particular form, consistent with the optical geometry of space. Second, we re-derive well-known, standard Newton’s equations using a particular notation, and prove that they are identical in form with the corresponding Einstein’s equations in optical geometry. Third, we apply Feynman’s principle, “The same equations have the same solutions” (e.g. Feynman et al., 1989), and derive the formula for the epicyclic radial frequency that is valid in both Einstein’s and Newton’s gravity.

2. Optical geometry in Schwarzschild spacetime

The general static, spherically symmetric metric can be written in a particular form,

\[ ds^2 = e^{2\psi} \left\{ -c^2 dt^2 + \left[ dr^2 + \tilde{r}^2 \left( d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right) \right] \right\}. \] (2.1)

In the specific case of Schwarzschild geometry one has,

\[ e^{2\psi} = 1 - \frac{r_G}{r}, \quad dr_* = e^{-2\psi} dr, \quad \tilde{r}^2 = e^{-2\psi} r^2, \] (2.2)

where \( r_G = 2GM/c^2 \) is the gravitational radius of the central body with the mass \( M \). The 3-D metric of optical geometry was introduced by Abramowicz, Carter and Lasota, (1988). It corresponds to the part of (2.1) in square brackets,

\[ ds^2_{\text{optical}} = dr_*^2 + \tilde{r}^2 \left( d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right), \] (2.3)

In terms of optical geometry the Schwarzschild quantities (2.2) have a very clear geometrical meaning. Obviously, \( r_* \) is the geodesic radius, and \( \tilde{r} \) is the circumferential radius of the nested, concentric spheres \( r = \text{const} \) that generate the spherical symmetry of the metrics (2.1) and (2.3).

Let us introduce the covariant derivative operator \( \nabla^i \) in the optical geometry (2.3). From the \( R_{tt} = 0 \) component of Einstein’s field equations one easily derives,

\[ \nabla^2 \Phi = \nabla^i \nabla_i \Phi = 0, \quad \Phi = \frac{c^2}{2} \left( e^{2\psi} - 1 \right) \] (2.4)

This optical geometry equation for \( \Phi \) is linear and identical with Laplace’s equation that gravitational potential obeys in Newton’s theory. This, together with the asymptotic behaviour of \( \Phi \) for large \( r \), namely \( \Phi = -GM/r \), suggests that \( \Phi \) should be called the gravitational potential. We shall see later more reasons to do so.

Equation of geodesic motion along a great circle on a \( r = \text{const} \) sphere, and with a constant speed \( v = c\beta\gamma \), takes the form, valid both in the optical geometry, and in the full 4-D spacetime (Abramowicz, Carter and Lasota, 1988),

\[ a^i = c^2 \nabla^i \psi - \frac{v^2}{R} \lambda^i = 0. \] (2.5)

Here \( a^i \) is the four-acceleration, \( \gamma = \left( 1 - \beta^2 \right)^{-1/2} \) is the gamma Lorentz factor, \( \lambda^i = \nabla^i r_* \) is the first normal to the circle (orthogonal vector, of unit length in the metric of eq. [2.3]), and \( R \) is the curvature radius of the sphere. (For simplicity, one may consider a particular great circle, located at the equatorial plane, \( \vartheta = \pi/2 \), but our arguments are valid in a general case).
The conserved angular momentum equals (Abramowicz, Carter and Lasota, 1988),

$$L = \tilde{v} \tilde{r} e^{\Psi}. \quad (2.6)$$

We use (2.6) to write the final formula,

$$a^i = e^{-2\Psi} \left[ \nabla^i \Phi - \frac{\mathcal{L}^2}{\mathcal{R}\mathcal{F}} \lambda^i \right] = 0. \quad (2.7)$$

In these calculations one may use a convenient relation,

$$\frac{d\tilde{r}}{d\tau_*} = \tilde{r} / \mathcal{R}. \quad (2.8)$$

For light trajectories \( v = \infty \), and from (2.5) it follows that also \( \mathcal{R} = \infty \), which means that light may go round a circular trajectory if and only if this trajectory is a geodesic circle in optical geometry — a conclusion that follows also from Fermat’s principle: light rays coincide with geodesic trajectories in optical geometry (Abramowicz, Carter and Lasota, 1988; Abramowicz, 1994).

Equations (2.4) and (2.7) that we have recalled from Einstein’s theory will be compared with corresponding Newton’s equation that we derive next.

### 3. Newton’s dynamics in a curved 3-D space

Newton’s dynamics is usually considered in 3-D Euclidean space, but its generalization to a curved space is trivial. Indeed, the only issue that is important in the present context is a careful distinction between the three radii of a sphere: geodesic radius \( r_* \), circumferential radius \( \tilde{r} \), and curvature radius \( \mathcal{R} \). In Newton’s theory, with Euclidean geometry assumed, one has \( r_* = \tilde{r} = \mathcal{R} \), but assuming Euclidean geometry is not necessary in Newton’s dynamics, and one could easily distinguish the three different radii in all calculations.

Indeed, let us consider a curve in space defined by,

$$x^i = x^i(s), \quad (3.1)$$

where \( x^i \) are coordinates in a coordinate system, \( s \) is the length along the curve, and Latin indices run through 1, 2, 3. The velocity \( v^i \) is defined by,

$$v^i = \frac{dx^i}{dt}, \quad (3.2)$$

where \( t \) is the absolute time. From this definition it follows that

$$v^i = \frac{dx^i}{dt} = \frac{dx^i}{ds} \frac{ds}{dt} = \tau^i v, \quad (3.3)$$

where \( \tau^i = dx^i/ds \) is a unit tangent vector to the curve (3.1), and \( v = ds/dt \) is the speed along the curve.

Acceleration is defined as
\[ a^i = \frac{dv^i}{dt}, \quad (3.4) \]

and from this definition it follows

\[ a^i = \frac{dv^i}{dt} = \frac{d}{dt} (\tau^i v) = \frac{d}{dt} \frac{d}{ds} (\tau^i v) = v^2 \frac{d\tau^i}{ds} + v \tau^i \frac{dv}{ds} = -\frac{v^2}{\mathcal{R}} \lambda^i + \tau^i \frac{d\mathcal{E}_K}{ds}, \quad (3.5) \]

where \(-\lambda^i\) is the first normal to the curve (3.1), \(\mathcal{R}\) is the curvature radius of the curve (3.1) and \(\mathcal{E}_K\) is the kinetic energy per unit mass. Because \(\mathcal{E}_K\) is obviously constant for a circular motion with a constant speed, we may write that for such motion,

\[ a^i = -\frac{v^2}{\mathcal{R}} \lambda^i. \quad (3.6) \]

Newtonian dynamics is based on the second law,

\[ F^i = ma^i, \quad (3.7) \]

where \(F^i\) is the applied force and \(m\) the mass. Because we are interested here in a “Keplerian” motion, with

\[ F^i = -m \nabla^i \Phi \quad (3.8) \]

being the gravitational force (equal to the gradient \(\nabla^i\) of gravitational potential \(\Phi\)) the second law takes the form,

\[ \nabla^i \Phi = \frac{v^2}{\mathcal{R}} \lambda^i. \quad (3.9) \]

Using Newton’s formula for angular momentum,

\[ \mathcal{L} = v \tilde{r}, \quad (3.10) \]

we write finally,

\[ \nabla^i \Phi = \frac{\mathcal{L}^2}{\mathcal{R} \mathcal{L}} \lambda^i. \quad (3.11) \]

This is identical with Einstein’s equation (2.7).

The gravitational potential obeys Laplace’s equation,

\[ \nabla^i \nabla_i \Phi = 0, \quad (3.12) \]

which is identical with Einstein’s equation (2.4). We thus have completed the second step, showing that for circular motion in spherical potential, Einstein’s equations in optical geometry, and Newton’s equations are the same.
4. The same equations have the same solutions

Einstein’s and Newton’s equations for Kepler’s circular motion in a spherical potential have the same form,

\[ \nabla^i \Phi = \frac{L^2}{R \tilde{r}} \lambda^i, \quad (4.1) \]

\[ \nabla^i \nabla_i \Phi = 0, \quad (4.2) \]

and the physical and geometrical meaning of all quantities appearing in them is the same.

Let us integrate Laplace’s equation (4.2) in the volume \( V \) between two equipotential surfaces, \( S_1 \), defined by \( \Phi_1 = \text{const} \), and \( S_2 \), defined by \( \Phi_2 = \text{const} \), and use the Gauss theorem,

\[ \int_V \nabla^i (\nabla_i \Phi) dV = \int_{S_1} (\nabla_i \Phi) \lambda^i dS = \int_{S_1} (\nabla_i \Phi) \lambda^i dS - \int_{S_2} (\nabla_i \Phi) \lambda^i dS = 0, \quad (4.3) \]

where \( \lambda^i \) is the vector orthogonal to the surface \( S \) — obviously, \( \lambda^i_s = \lambda^i \) at \( S_1 \) and \( \lambda^i_s = -\lambda^i \) at \( S_2 \).

Let us fix a position of \( S_1 \), and denote

\[ \int_{S_1} (\nabla_i \Phi) \lambda^i dS = C. \quad (4.4) \]

If we now change the position of \( S_2 \) then, because (4.3) holds independently of the position of \( S_2 \), and \( (\nabla_i \Phi) \lambda^i \) is constant on each equipotential surface, one must conclude that

\[ (\nabla_i \Phi) \lambda^i \int_S dS = C = \text{const}, \quad (4.5) \]

for any equipotential surface \( S \). Taking into account that \( \tilde{r} \) is the circumferential radius of the sphere, and therefore

\[ \int_S dS = 4\pi \tilde{r}^2, \quad (4.6) \]

one concludes that

\[ (\nabla_i \Phi) \lambda^i = \frac{C}{4\pi \tilde{r}^2} = \frac{GM}{\tilde{r}^2}, \quad (4.7) \]

with \( C = 4\pi GM \) following from the asymptotic behaviour at \( \tilde{r} \to \infty \).

Thus, finally, the second law takes the form,

\[ \frac{GM}{\tilde{r}^2} = \frac{L^2}{\tilde{r}^2 R}, \quad (4.8) \]

and from this we derive the formula for the Keplerian angular momentum distribution,
\[ \mathcal{L}^2 = GM\mathcal{R}. \] (4.9)

The above formula allows a novel and interesting interpretation of the Keplerian angular momentum, as the geometrical mean of the gravitational radius of the gravitating center \( r_G = 2GM/c^2 \), and of the curvature radius \( \mathcal{R} \) of the particle trajectory,

\[ \mathcal{L} = c\sqrt{2r_G\mathcal{R}}. \] (4.10)

One knows that the circular photon trajectory is located at \( r = (3/2)r_G \). This means that \( \mathcal{R} = \infty \) there. But \( \mathcal{R} = \infty \) also for \( r = \infty \). This means, that somewhere in the range \( [(3/2)r_G, \infty] \) the curvature radius \( \mathcal{R} \) must have a minimum. Because \( \mathcal{L}^2 = GM\mathcal{R} \), the angular momentum \( \mathcal{L} \) also has a minimum at the same radius. It must be the radius \( r = r_{ms} = 3r_G \) of the marginally stable orbit.

To see this in a more quantitative way, let us now consider a small radial perturbation of Einstein’s equation (2.7), which is equivalent to (4.1). We assume that a particle with a fixed angular momentum \( \mathcal{L} \) is displaced from its original orbit at \( r^* \) to a new one, at \( r^* + \delta r^* \). The perturbation introduces an unbalanced force and radial acceleration,

\[ e^{2\Psi} \lambda \alpha^i = e^{4\Psi} \frac{d^2}{ds^2} \delta r_* = e^{4\Psi} A^2 \frac{d^2}{dt^2} \delta r_* = -\frac{1}{\tilde{r}^2 \mathcal{R}} \frac{d\mathcal{L}^2}{dr_*} \delta r_*, \] (4.11)

where \( A \) denotes the total redshift factor, connected to the previously introduced Lorentz gamma factor by

\[ A^2 = \gamma^2 e^{-2\Psi} = \left( 1 - \frac{3r_G}{2r} \right)^{-1}. \] (4.12)

From (4.11) we derive, finally,

\[ \ddot{\delta r}_* + \Omega^2_K \left( \frac{d\mathcal{R}}{dr_*} \frac{\tilde{r}^2}{\mathcal{R}^2} \right) \delta r_* = 0. \] (4.13)

Here, each dot denotes a differentiation with respect to time of the observer at infinity, and

\[ \Omega_K = \mathcal{L}e^{-\Psi} \tilde{r}^{-2} A^{-1} = \left( \frac{GM}{\tilde{r}^3} \right)^{1/2} \] (4.14)

is the Keplerian orbital frequency, also observed at infinity.

It is quite remarkable that equation (4.13) has identical form in Einstein’s and Newton’s theories — the well-known equation for a harmonic oscillator. Its eigenfrequency,

\[ \Omega^2_r = \left( \frac{d\mathcal{R}}{dr_*} \frac{\tilde{r}^2}{\mathcal{R}^2} \right) \Omega^2_K \] (4.15)

is obviously equal to the epicyclic frequency of small radial oscillations. When \( \Omega^2_r > 0 \) the radial epicyclic oscillations are stable, and when \( \Omega^2_r < 0 \), they are unstable. Thus, marginally stability occurs at the radius where \( d\mathcal{L}/dr = 0 = d\mathcal{R}/dr \) which, in Schwarzschild geometry is at \( r = r_{ms} \).
In Newton’s case the geometry is flat (Euclidean) and the geodesic, circumferential and curvature radii are equal,

\[ r_* = \tilde{r} = R. \] (4.16)

From these expression and (4.13) one derives,

\[ \Omega_r^2 = \Omega_K^2 \quad \text{in Newton’s theory.} \] (4.17)

This means that the radial epicyclic frequency equals to the Keplerian orbital frequency, which is why Newton’s orbits are closed ellipses.

In Einstein’s gravity, the geometry is curved and the geodesic, circumferential and curvature radii are all different,

\[ r_* = \int \left(1 - \frac{rG}{r}\right)^{-1} dr, \quad \tilde{r} = r \left(1 - \frac{rG}{r}\right)^{-1/2}, \quad R = r \left(1 - \frac{3rG}{2r}\right)^{-1}. \] (4.18)

From these expressions and (4.13) one recovers the well-known formula, to our best knowledge first derived by Kato and Fukue (1980),

\[ \Omega_r^2 = \Omega_K^2 \left(1 - \frac{3rG}{r}\right), \quad \text{in Einstein’s theory.} \] (4.19)

This completes our point: Einstein’s and Newton’s formulae (4.1) and (4.2) that describe dynamics in static, spherically symmetric gravity are the same in both theories (and in both theories linear). For this reason, the formula for radial epicyclic motion (4.13), derived directly from these equations, is also the same in both theories. However, this formula depends on geodesic, circumferential and curvature radii of circular trajectories. In the flat, Euclidean, geometry of Newton’s space these radii are equal, and in the curved geometry of Einstein’s space they are not. This geometrical difference alone, and not an often mentioned (but not present in this case) “non-linearity” of Einstein’s equations, is the reason for the distinctively different Newton’s and Einstein’s predictions for the physical behaviour of small radial oscillations around Keplerian circular orbits in static spherically symmetric gravity.

The same conclusion was previously reached by Abramowicz, Lanza, Miller and Sonego (1997) for weak gravity, who considered the perihelion of Mercury advance according to Newton’s gravity in a properly curved space. It is rather surprising that the conclusion holds also for arbitrarily strong static spherically symmetric gravity.

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