Maupertuis principle, Wheeler’s superspace and an invariant criterion for local instability in general relativity

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Abstract

It is tempting to raise the issue of (metric) chaos in general relativity since the Einstein equations are a set of highly nonlinear equations which may exhibit dynamically very complicated solutions for the space-time metric. However, in general relativity it is not easy to construct indicators of chaos which are gauge-invariant. Therefore it is reasonable to start by investigating - at first - the possibility of a gauge-invariant description of local instability.

In this paper we examine an approach which aims at describing the dynamics in purely geometrical terms. The dynamics is formulated as a geodesic flow through the Maupertuis principle and a criterion for local instability of the trajectories may be set up in terms of curvature invariants (e.g. the Ricci scalar) of the manifold on which geodesic flow is generated. We discuss the relation of such a criterion for local instability (negativity of the Ricci scalar) to a more standard criterion for local instability and we emphasize that no inferences can be made about global chaotic behavior from such local criteria.

We demonstrate that the Maupertuis principle implemented in the case of the Hamiltonian (ADM) formulation of General Relativity is equivalent to the construction of Wheeler’s superspace on which the development of three-metrics occurs along geodesics. Obstructions for using the superspace metric as a distance measure on the space of three-metrics are pointed out. The discussion is illustrated in the case of a particular toy-model of a gravitational collapse.


1 Introduction

Due to the strong nonlinearity of Einstein’s equations one is tempted to raise the issue of chaos in general relativistic context especially in situations where this nonlinearity is particularly pronounced - for example near a generic gravitational collapse to a spacetime singularity. The property of chaos is usually understood in terms of (exponential) divergence in “time” of “nearby orbits” starting from slightly modified initial conditions. However one faces in the context of general relativity some fundamental problems which make the straightforward application of the machinery developed in the theory of nonlinear dynamical systems difficult. At first, general relativity, i.e. Einstein’s theory of gravitation, is not even a dynamical theory in the usual sense. It does not from the very beginning provide us with a set of parameters (describing the gravitational degrees of freedom) evolving in “time”. Time looses here its absolute meaning as opposed to the classical dynamical theories where the Newtonian time is taken for granted and upon which all our experience in dealing with chaos is based. The division between space and time in general relativity comes through foliating the space-time manifold $M$ into spacelike hypersurfaces $\Sigma_t$. The metric $g_{\mu\nu}$ on $M$ induces a metric $g_{ij}$ on $\Sigma_t$ (the first fundamental form of $\Sigma_t$) and can be parametrized in the form

$$g_{\mu\nu} = \begin{pmatrix} N_i N^i - N^2 N_j & N_j \\ N_i & g_{ij} \end{pmatrix}$$

where $N$ and $N_i$ are called lapse function and shift vector respectively. Only after splitting the space-time into space and time (the 3+1 ADM splitting) \cite{1, 2} we yield the possibility of mapping the Einstein equations to a dynamical system which resembles ordinary Hamiltonian dynamical systems. In general this 3+1 split is quite an arbitrary procedure and there is a priori no preferred time coordinate.

The question we shall address in this paper is in particular the following:

**What is meant by a “nearby orbit” in general relativity?**

Let us suppose that we have performed a 3 + 1 split of the spacetime metric into a three-metric “$^{(3)g}$” evolving in a time coordinate “$t$”. What is then a natural distance measure $||^{(3)g} - ^{(3)g^*}||$ between two three-metrics $^{(3)g}$ and $^{(3)g^*}$ at some given moment $t$ of time? In the case of non-relativistic Hamiltonian dynamical systems, a standard indicator of “chaos” such like a principal Lyapunov exponent (cf., e.g., discussion in \cite{3, 4, 5}) is calculated by using an Euclidean distance measure which is naturally induced from the structure of the kinetic energy term appearing in the non-relativistic Hamiltonian. In general relativity it is, however, not obvious why we should use an Euclidean distance measure to assign a distance between two space-time metrics.

In the class of spatially homogeneous spacetime metrics

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(t) \omega^i(x) \omega^j(x)$$

(1.1)
calculations of principal (maximal) Lyapunov exponents have previously been reported \cite{6, 7, 8, 9} for the dynamically interesting case of the vacuum mix master gravitational

$^1$For a review of spatially homogeneous space-time metrics, see e.g. \cite{10, 11}
collapse. In fact, the mixmaster toy-model collapse is a useful laboratory to test ideas about how to characterize chaos in general relativity (see also e.g. discussions in Rugh [17]). In the case of the mixmaster metric \( \omega^i(x) \) in (1.1) denote the SU(2) invariant one-forms of the mixmaster space [1] and the three-metric is parametrized by three scale factors \( a = e^\alpha, b = e^\beta, c = e^\gamma \),

\[
\gamma_{ij}(t) = \text{diag}(a^2(t), b^2(t), c^2(t)) = \text{diag}(e^{2\alpha}(t), e^{2\beta}(t), e^{2\gamma}(t)).
\]  

(1.2)

The Einstein equations for the mixmaster model are given by three coupled non-linear second order differential equations supplemented by a first integral constraint (cf. e.g. Landau and Lifshitz [3]). It is tempting to treat the mixmaster collapse like any other dynamical system (with few degrees of freedom), integrate (numerically) the 6-dimensional state vector \((\alpha, \beta, \gamma, \alpha', \beta', \gamma')(\tau)\) and extract “Lyapunov exponents” in a standard way.

In all previous studies [9, 10, 12, 13] Lyapunov exponents were calculated by used an Euclidean distance measure (between two “nearby” solutions) of the form

\[
||\delta^{(3)}g_{ij}||^2 = (\delta \alpha)^2 + (\delta \beta)^2 + (\delta \gamma)^2
\]

(1.3)

Such choice of a distance measure is directly inspired from the study of ordinary (non-relativistic) Hamiltonian systems. In general relativity there is however a priori no reason why one should use such a distance measure to give out the distance between \( (3)g \) and the “nearby” three-metric \((3)g + \delta(3)g\). This distance measure has been put in artificially in hand and is not supported by the structure of general relativity. A distance measure which is induced by the structure of the (ADM) Hamiltonian formulation of general relativity is

\[
||\delta^{(3)}g_{ij}||^2 = ds^2 = G_{AB}\delta g^A\delta g^B = G_{ijkl}\delta g^{ij}\delta g^{kl}
\]

(1.4)

where \( G_{AB} = G_{ijkl} \) is the so-called “(mini)superspace” metric. We will here critically examine the idea of using the “superspace” metric \( G_{AB} \) (or a conformally rescaled super-metric) as a distance measure between two three-metrics. In particular we discuss the idea of extracting an invariant from it like the Ricci scalar in order to give invariant (local) statements about the divergence of “nearby” orbits. Thus, in this paper we examine and extend in various ways the discussion of an approach [18] - [24] which has proposed a criterion for instability of the mixmaster gravitational collapse. Szydowski and Lapeta [19], Szydowski and Biesiada [20] proposed to look at the Ricci scalar of the manifold on which the mixmaster dynamics generates a geodesic flow and investigate if one could extract (coordinate invariant) information about instability properties and chaos of the gravitational collapse in this way.

We show in sec.2.2 that for the case of non-relativistic Hamiltonian dynamics the instability criterion of negativity of the Ricci scalar (of the manifold on which the dynamics
generates a geodesic flow) is related to a more standard local instability criterion, which examines local instability in terms of the eigenvalues of the Jacobian of the Hamiltonian flow. We will stress the nontrivial interplay between local and global instability criteria (since several authors draw false conclusions from local instability criteria). In particular, we do not believe in expressions directly relating the average negative curvature to Lyapunov exponents (see, e.g., Szydłowski and Lapeta [19] and Szydłowski and Biesiada [20]).

We note in sec.3 that application of the Maupertuis principle to the Hamiltonian formulation of the mixmaster dynamics - and the associated induction of a natural distance measure on the three-metrics - is exactly to implement the dream by Wheeler, DeWitt and others (described e.g. in [25] for the mixmaster metric) to have a superspace metric “\(G_{ijkl}\)” (Wheeler’s superspace) - with respect to which the three-metrics move along geodesics - and use this as a natural distance measure on the space of mixmaster three-geometries. In a sense such a superspace metric would be the most natural to impose on the configuration space of three-geometries (and appear more natural than introducing a completely arbitrary Euclidean metric on the space of three-geometries - as has been done previously in calculating Lyapunov exponents for the mixmaster collapse dynamics).

Thus, the Ricci scalar calculated in Szydłowski and Biesiada [21] from the Hamiltonian introduced by Bogoyavlenskii [26] is exactly to calculate the Ricci scalar of Wheeler’s superspace metric and the criterion \(R < 0\) imply that we have local instability at least in one direction when we look at the geodesic deviation equation for two nearby mixmaster metrics investigated with Wheeler’s superspace metric as distance measure.

An obstacle to this approach is the accompanying introduction of a host of singularities of the superspace metric which makes the dream of Wheeler troublesome to achieve (see also discussion in C.W. Misner [25]) - even in the restricted class of mixmaster three-geometries. Invariants calculated from the superspace metric, e.g. the Ricci scalar, inherit the singularities of the Wheeler superspace metric. In particular, the Ricci scalar calculated in Szydłowski and Biesiada [20] has such singularities. This obstacle has also most recently been emphasized by Burd and Tavakol [27]. However, as a local instability criterion, \(R < 0\) indicates - at nonsingular points where it is defined - the local exponential instability in some directions of the configuration space.

As a major obstacle, we emphasize that the distance measure induced by using the Wheeler superspace metric is not positive definite. The situation is somewhat similar to the indefiniteness of the line element in special relativity (zero distances along the lightcone). But in the case of Wheeler’s superspace metric in general relativity there is nothing which prevents a trajectory (a three-metric) from crossing the null-surface. Thus, one may contemplate situations where a “zero” distance (between two three-metrics which need not be identical) evolves into a finite (positive or negative) distance. This corresponds formally to a (local) “Lyapunov exponent” which is “\(\infty\)”.

The investigation is to be viewed in the light of the following question: Can one assign an invariant meaning to “chaos” in the general relativistic context?

There is a problem of transferring standard indicators of chaos, e.g. the spectrum of Lyapunov exponents, to the general relativistic context, since they are highly gauge dependent objects. This fact was pointed to and emphasized in Rugh [12, 14] - and was
also discussed in Pullin [13]. (See also more recent discussions in Rugh [17]).

One should try to develop indicators of chaos which capture chaotic properties of the gravitational field (“metric chaos”) in a way which is invariant under spacetime diffeomorphisms - or prove that this can not be done! It may very well be that there is a “no go” theorem, for the most ambitious task of constructing a generalization of a Lyapunov exponent (extracted from the continuous evolution equations) which is meaningful and invariant under the full class of spacetime diffeomorphisms (H.B. Nielsen and S.E. Rugh). One should therefore, at first, try to modify the intentions and construct instability measures which are invariant under a smaller class of diffeomorphisms, e.g. coordinate transformations on the spacelike hypersurfaces $\Sigma_t$ which do not involve transformations of the time coordinate.

“Per time” indicators - like a Lyapunov exponent - have only a little chance of remaining invariant under diffeomorphisms. Even if we consider the restricted class of Lorentz transformations, i.e. space-time transformations within the special theory of relativity, it appears, that the time evolution of a dynamical system is slowed down if we observe it from a frame of reference $S'$, which moves (is boosted) with some velocity $v$ relative to the frame of reference $S$ in which the dynamical system is at rest. Hence, the Lorentz $\gamma$-factor will inevitably appear in the Lyapunov exponent (seen from the moving frame of reference $S'$). One notes, however, that we are able to discriminate qualitatively, in any frame of reference, whether or not the dynamical system is chaotic: The requirement that a Lyapunov exponent is greater than zero will be valid in any frame of reference if it is valid in one. That is, “$\lambda > 0$” is a statement which is invariant under Lorentz-transformations - although the numerical value of the Lyapunov exponent $\lambda$ is not. In general relativity, however, “$\lambda > 0$” is not an invariant statement under diffeomorphisms more general than Lorentz-transformations [12, 13, 14, 15].

A route of progress may lie in pointing to indicators of purely geometrical nature (“fractal dimensions”, etc). We shall here explore into a picture which is purely geometrical and stems from an old construction known since times of Maupertuis and Jacobi — namely that in which Hamiltonian dynamical systems are mapped into geodesic flows. In this approach the entire complicated dynamics comes out from purely geometrical properties of a single “object” — the manifold on which the geodesic flow is generated. One may consequently dream that also such basic properties like integrability or chaos may be somehow captured in purely geometrical (and hence timeless and invariant) terms. For example one may hope to describe local instability of trajectories in terms of curvature invariants (e.g. the Ricci scalar) since it is curvature that determines the behavior of close geodesics (through the geodesic deviation equation). We shall however also point out several obstructions one has to face when implementing such a line of thinking to general relativistic problems.

2 The dynamics as geodesic motion by application of Maupertuis principle

It is indeed a remarkable fact that a large class of Hamiltonian systems can be presented in such a way that Hamilton’s equations acquire formal resemblance to geodesic equations in Riemannian geometry.
The phase space trajectories $x(t) = (p(t), q(t))$ with respect to time parameter $t$ and corresponding to the Hamiltonian

$$H = H(p, q) = \sum_{i,j} \frac{1}{2} a^{ij} p_i p_j + V(q)$$  \hspace{1cm} (2.1)$$

may be mapped into a geodesic motion on the configuration space manifold. It is accomplished in a way which displays a complete analogy with the mapping of the evolution of three-metrics into that of geodesic motion in Wheeler’s superspace (see next section).

The equations of motion for the $q$ coordinates may be written as

$$\ddot{q}^j + \tilde{\Gamma}^j_{ks} \dot{q}^s \dot{q}^k = -a^{ji} \frac{\partial V(q)}{\partial q^i}$$  \hspace{1cm} (2.2)$$

where $\tilde{\Gamma}^j_{ks}$ is the Christoffel symbol calculated with respect to $a_{ij}$ metric. Due to the force term this is, obviously, not a geodesic equation. It is simply the Newton’s second law restated. The momentum variables are just linear combinations of velocities $p_i = a_{ij} \dot{q}^j$.

Transformation to a geodesic motion (i.e. free motion in a curved space) is accomplished in two steps: (1) conformal transformation of the metric $a_{ij}$, and (2) change of the time parameter along the orbit. In fact only the first step is crucial the second one merely introduces the affine parameter along the geodesics. More explicitly we equip the configuration space with the (super)metric

$$g_{ij} = 2(E - V(q)) a_{ij}$$  \hspace{1cm} (2.3)$$

(see that $a_{ij}$ is read off from the kinetic energy term in the Hamiltonian and (in general) is allowed to vary as a function of the configuration space variable, $a_{ij} = a_{ij}(q)$).

With respect to the metric (2.3) and the time parameter $t$ it is not easy to see that the orbits are geodesics since there is a term appearing on the right hand side of the equation,

$$\frac{d^2}{dt^2} q^i + \Gamma^i_{jk} \frac{d}{dt} q^j \frac{d}{dt} q^k = -\frac{1}{E - V(q)} \frac{d}{dt} q^i \frac{\partial}{\partial q^k} V(q) \frac{d}{dt} q^k$$  \hspace{1cm} (2.4)$$

where $\Gamma^i_{jk}$ now denote the Christoffel symbols associated with the (super)metric (2.3). However, if we (re)parametrize the orbit $q^i = \dot{q}^i(s)$ in terms of the parameter $s$ defined via

$$s = \int 2 \left( E - V \right) dt$$  \hspace{1cm} (2.5)$$

the orbits will become affinely parametrized geodesics, i.e. the configuration space variables $q = (q^i)$ satisfy the well known geodesic equation

$$\frac{d^2}{ds^2} q^i + \Gamma^i_{jk} \frac{d}{ds} q^j \frac{d}{ds} q^k = 0$$  \hspace{1cm} (2.6)$$

\*Let us recall from differential geometry that the most general form of the geodesic equation (when the geodesic is parametrized by $v$ parameter) reads $q^{\prime \prime} + \Gamma^i_{jk} q^{\prime j} q^{\prime k} = h(v)q^{\prime i}$ (prime denotes differentiation with respect to $v$) where $h(v)$ is an arbitrary function of $v$. Then by solving the differential equation $\frac{d^2 v}{dv^2} + h(v) \frac{dv}{dv} = 0$ one can construct a new parameter — the so called affine parameter $s$ such that $\ddot{q}^i + \Gamma^i_{jk} \dot{q}^j \dot{q}^k = 0$ (overdot here denotes differentiation with respect to $s$). It is evident that this new parameter is defined up to linear transformations $s \rightarrow As + B (A, B$ are constants) what justifies its name.
with no force term on the right hand side.

There is no “magic” in this, it is merely a rewriting of the Hamiltonian equations. The information about the original force acting on the particle (as described by the potential $V(q)$ in the Hamiltonian (2.1)) has been encoded entirely in the definition of the (super)metric (2.3) and the definition of the new parameter $s$ in (2.5) parametrizing the orbit. The entire procedure involves the following principal steps:

1. At first a metric $a_{ij}$ is read off from structure of kinetic term in the Hamiltonian.
2. We observe that the trajectory $\{q(t)\}$ is not a geodesic with respect to this metric.
3. Transformation of the non-geodesic motion to affinely parametrized geodesic motion is obtained via two steps:
   (1) Conformal transformation of the metric $a_{ij} \rightarrow 2(E - V(q))a_{ij}$
   (2) Rescaling of the time parameter $t \rightarrow s = \int 2(E - V(q))dt$

It is, however, not often that this description is used in the discussion of Hamiltonian dynamical systems, for example in the discussion of instability properties and chaotic behavior of such systems. We contemplate several reasons why this approach is so rarely used:

1. The metric $g_{ij}$ in (2.3) and the parameter $s$ in (2.5) are singular in “turning” points where the kinetic energy term $\sum \frac{1}{2}a_{ij}p_ip_j$ is zero and we thus have $E - V(q) = 0$.

Therefore we have to consider the motion away from the “turning” points. This may be a more severe restriction for systems with few degrees of freedom, e.g. the harmonic oscillator with a single degree of freedom, than for systems with many degrees of freedom, e.g. for $N$ self-gravitating bodies in a gravitational field, to which the virial theorem applies, and thus $\sum \frac{1}{2}a_{ij}p_ip_j = 0$ is rather unlikely.

2. The procedure of mapping to geodesic motion (via the Maupertuis principle) is applicable to any time independent Hamiltonian system of the form (2.1) where the potential $V(q)$ is not allowed to depend on momenta $p$. I.e. the application of Maupertuis principle is not straightforward if we for example have a mechanical system coupled to an electromagnetic field with the Hamiltonian

$$H = \frac{1}{2m}(p - \frac{e}{c}A(q))^2 + V(q).$$

Note that the above picture comes quite naturally from the Maupertuis-Jacobi least action principle

$$\delta S = \delta \int_{q'}^{q''} \sqrt{E-V(q)} \sqrt{a_{ij} dq_i dq_j} = 0$$

and its formal resemblance to the variational formulation of geodesics in Riemannian geometry as curves extremalizing the distance. We have sketched the step-by-step derivation of the geodesic motion in order to see the step-by-step analogy with the derivation of three-metrics as geodesics in Wheeler’s superspace in the next section.
The same objection holds for mechanical problems in noninertial frames, for example the problem of motion of a star in the field of a rotating galaxy [23].

In spite of these principal limitations of the Maupertuis approach, it has been explored in various contexts. In his seminal work [28] on foundations of statistical mechanics Krylov has suggested that viewing the dynamics of an N-body system as a geodesic flow on the appropriate manifold may provide a universal (applicable to a very large class of interaction potentials) tool for discussing relaxation processes. Inspired by works of Hedlund and Hopf [30] Krylov conjectured that the average separation of trajectories evolves according to the sign of the Ricci scalar $R$ on the configuration space accessible for the system. An implementation of this viewpoint have also been attempted in the context of general relativity by Szydłowski, Biesiada et al. [18, 19, 20]. As we shall demonstrate, Wheeler’s original dream to construct a superspace in which the three metrics move along geodesics is exactly to implement the old idea of Maupertuis principle in general relativistic context. We shall however first - in the next two subsections - comment on the status of the $R < 0$ criterion and its relation to more standard criteria for local instability.

2.1 The Ricci scalar $R$ for the “super” metric and $R < 0$ as a (strong) local instability criterion.

Contemplating the local instability properties, i.e. how nearby orbits behave, it is natural to consider the geodesic deviation equation which describes the behavior of nearby geodesics (2.7).

This can be derived in the usual manner by subtracting the equations for the geodesics $q^i(s)$ and $q^i(s)+\xi^i(s)$ respectively or simply by disturbing the fiducial trajectory $(p_i(t), q^i(t))$,

$$\tilde{p}_i(t) = p_i(t) + \eta_i(t),$$
$$\tilde{q}^i(t) = q^i(t) + \xi^i(t)$$

and substituting this directly into Hamilton’s equations. In this way we also arrive momentarily, though tediously, at the geodesic deviation equation for the configuration space variable $\xi$ (the separation vector),

$$\frac{D^2 \xi^i}{D s^2} = -R^{ijkl} u^j \xi^k u^l$$

(2.8)

Here $u = Dq/Ds$ is the tangent vector to the geodesic, $\xi$ is the separation vector orthogonal to $u$. Note, that the covariant derivative $D/Ds$ and Christoffel symbols are calculated with respect to the (super)metric (2.3). This (super)metric induces the natural distance measure

$$||\xi||^2 = g_{ij} \xi^i \xi^j = 2(E - V(q)) a_{ij} \xi^i \xi^j$$

on the configuration space. From the geodesic deviation equation (2.8) one arrives at the inequality (see e.g. [24])

$$\frac{d^2}{ds^2} ||\xi||^2 \geq -2K_w \xi||^2$$

(2.9)
where the $K_{u\xi}$ denotes the sectional curvature in the two-direction $u \wedge \xi$.

Noting that the Ricci scalar may be expressed as a sum of the principal sectional curvatures,

$$R = Tr K = \sum \left( \text{Principal Sectional Curvatures} \right)$$

(2.10)

the inequality (2.9) especially implies that if

$$R < 0$$

(2.11)

then a principal direction $\xi$ exists along which we have local exponential instability.

By first calculating the Christoffel symbols $\Gamma^k_{ij}$ from the metric (2.3) and subsequently substituting this into the standard formula for the Ricci scalar $R$ one obtains the following general expression,

$$R(\text{Maupertuis}) = \frac{(n - 1)}{2(E - V)^3} \sum_{i,j=1}^{n} \left\{ (E - V) \frac{\partial^2 V}{\partial q^i \partial q^j} a^{ij} - \frac{(n - 6)}{4} \frac{\partial V}{\partial q^i} \frac{\partial V}{\partial q^j} a^{ij} \right\}$$

(2.12)

corresponding to a Hamiltonian system with $n$ degrees of freedom.

N.S. Krylov [28] was - as far as we know - the first to emphasize the use of $R < 0$ as an instability criterion in the context of an $N$ body system (a gas) interacting via Van der Waals forces, with the ultimate hope to understand the relaxation processes in a gas. In his toy-model study he found that indeed $R < 0$ in the accessible domain of the configuration space and concluded that this provides an explanation for the relaxation as a consequence of dynamical mixing.[2] Another investigation in the same spirit, have been performed by Gurzadyan and Savvidy [29] for the case of collisionless stars interacting via gravitational forces, also with the purpose of understanding the relaxation properties of such a stellar system.[10]

The system of collisionless stars which was investigated by V.Gurzadyan and G.K. Savvidy [29] and Kandrup [33] had $R < 0$ whenever the number of stars in the system exceeded two but not on a domain which was compact. Kandrup [33], however, argues

7The principal sectional curvatures are defined, as is well known, as the extrema of the sectional curvature $K_{u\xi}$ as function of the wedge product $u \wedge \xi$.

8Note that if there is one direction $\xi$ (for a given $u$) in which there is exponential instability then there is also a continuum of nearby exponentially unstable trajectories. Due to smoothness of $K_{u\xi}$ as a function of two-direction the sectional curvature for nearby directions $\xi$ is also negative.

9The investigations by Krylov was strongly influenced by seminal results of Hedlund and Hopf [30] where the geodesic motion on the Lobachevsky space was proved to be mixing. Whereas the approach by Krylov is intuitively appealing it is however not allowed to transfer the results obtained for the Lobachevsky space to the case where sectional curvatures are negative only in average and not at every point and in every two-direction $u \wedge \xi$. Moreover, Krylov disregarded the problem of compactness of the configuration space manifold which is important for making inferences about mixing.

10Physical motivation of this study comes from the observation that stellar systems (globular clusters and galaxies) are apparently in a well relaxed state that is reflected in regularity of their shapes, velocity dispersions, surface luminosities etc. On the other hand the most obvious relaxation process - namely via binary encounters, provides a relaxation time that is greater than the Hubble time [31]. As a possible way out of this puzzle Lynden-Bell [32] has proposed the so called violent relaxation mechanism. Gurzadyan and Savvidy attempted at looking at the relaxation process in stellar systems from Krylov’s viewpoint. The idea was that if they could show that a self-gravitating N-body system was a K-system it would indicate that the system approached the equilibrium exponentially fast.
that the self-gravitating system is “effectively bound” i.e. the phase-space is compact at least for sufficiently small times. If one waits long enough the effects of “evaporation” of stars from the system come into play, so the phase-space is actually infinite. 11 Let us emphasize that the negativity of the Ricci scalar provides only a sufficient condition for local instability and it is thus not possible to make any rigorous statements about chaos from local instability properties alone as it will be discussed in more detail in the next subsection.

2.2 Remarks on the instability criterion $R < 0$ and its relation to more “standard” instability criteria.

According to Szebehely [36] there are more than fifty distinct criteria of instability! Many of these are related to each other (though not all connections are worked out in a transparent way). Some of the criteria are even in conflict with each other.

Let us emphasize a relation between two important local criteria, the criterion of having a negative Ricci scalar $R = R(Maupertuis) < 0$ (corresponding to the formulation via the Maupertuis principle) and the more usually applied criteria connected to the eigenvalues of the Jacobian of a Hamiltonian flow.

$R < 0$ is a sufficient criterion for local instability — but it is not a necessary criterion and very often it fails to say anything of interest about our dynamical system. For instance, in the case of a simple class of Hamiltonian flows

$$H = E = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \frac{1}{2}x^2y^2$$

(2.13)

it fails to mirror the transition from almost integrable to almost completely chaotic (almost K-system) as the energy $E$ increases. A calculation of the Ricci scalar, according to formula (2.12) or (2.17) shows that it is always positive irrespective of the Energy $E$ (since $\Delta V(q) = 2 + x^2 + y^2 > 0$). Despite this, the model is almost a K-system in the limit for large energies (i.e. when the nonlinear term $\frac{1}{2}x^2y^2$ becomes very large), see discussions in P. Dahlqvist and G. Russberg [37] and references therein.

A standard criterion for local instability is to look at the eigenvalues of the Jacobian of the Hamiltonian flow. (However, wrong conclusions as regards global chaos are often drawn from it.) The line of thinking starts at disturbing the fiducial trajectory $(p_i(t), q^i(t))$: \n
$$\dot{p}_i(t) = p_i(t) + \eta_i(t),$$

11 Had the manifold been compact it would - of course - still not be possible to deduce that the system is a K-system. The K-system property holds for geodesic flows on compact manifolds of negative curvature, i.e. manifolds whose sectional curvature $K_{u\xi}$ is negative at every point and in every two-direction $u \wedge \xi$. On the other hand it is a general property of Hamiltonian systems that sufficiently close to the physical boundary of motion there exist both positive and negative sectional curvatures [33]. This means, in particular, that mapping a Hamiltonian flow to a geodesic flow via Maupertuis principle can never be used to show that the system is a K-system. We note, however, that to be a K-system is a very strong, highly non-generic property to satisfy. If a system is “less” than a K-system, it may - of course - nevertheless be “highly” chaotic (and unstable almost all over in the phase space).

12 In an attempt to predict the onset of dynamical chaos (a key ingredient of which is the exponential instability of adjacent trajectories) Toda [38] and Duff and Brumer [39] proposed the criterion based on evaluation of the Gaussian curvature of the potential $V(q)$. 

10
\[ \ddot{q}^i(t) = q^i(t) + \dot{\xi}^i(t) \] (2.14)

and subsequently investigating the evolution of the disturbance vectors \( \xi \) and \( \eta \). The stability of motion is determined by integrating along the trajectory (see later) the time dependent matrix (the Jacobian matrix of the Hamiltonian flow):

\[
J = \begin{pmatrix}
0 & 1 \\
-Hess(V(q(t))) & 0 \\
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-(\partial^2 V \partial^2 q)_{ij}(t) & 0 \\
\end{pmatrix}
\]

(2.15)

where the Hessian of the potential \( V(q) \) is evaluated along the reference trajectory making the \( J \)-matrix time dependent. The inconvenience of the time dependence can be overcome by replacing the time-dependent phase-point \( q(t) \) (moving along the trajectory) by a fixed phase-space coordinate \( q \) resulting in

\[
\dot{\xi}^i = \eta^i = a^{ij} \eta_j \\
\dot{\eta}_i = -\sum_{ij} \frac{\partial^2 V(q)}{\partial q^i \partial q^j} \xi^j 
\]

(2.16)

This step is sometimes claimed “mathematically dubious” (cf. also, e.g., M.Tabor [40]). Anyway, the \( J \)-matrix is now time-independent and the stability of the autonomous system (2.16) is determined from the eigenvalue problem

\[
\text{det}[J - \lambda I] = 0
\]

If at least one eigenvalue has a positive real part we have exponential growth (locally) of the disturbance vector in one direction of the phase space. For example, in the case of a 2-dimensional Hamiltonian we have the two pairs of roots

\[
\lambda_{\pm} = \pm \frac{1}{\sqrt{2}} \left\{ -\frac{\partial^2 V}{\partial^2 q^1} + \frac{\partial^2 V}{\partial^2 q^2} \pm \sqrt{\left(\frac{\partial^2 V}{\partial^2 q^1} + \frac{\partial^2 V}{\partial^2 q^2}\right)^2 - 4\left(\frac{\partial^2 V}{\partial^2 q^1} \frac{\partial^2 V}{\partial^2 q^2} - \left(\frac{\partial^2 V}{\partial q^1 \partial q^2}\right)^2\right)} \right\}^{1/2}
\]

In particular, if \( \text{det}(\frac{\partial^2 V}{\partial q^i \partial q^j}) = \frac{\partial^2 V}{\partial^2 q^i} \frac{\partial^2 V}{\partial^2 q^j} - \left(\frac{\partial^2 V}{\partial q^i \partial q^j}\right)^2 < 0 \) (i.e. negative Gaussian curvature of the potential \( V \)) a pair of roots becomes real and we have local instability.

The \( R < 0 \) criterion implies the J-instability above\[1\] (whereas the converse is not true). In this sense the \( R < 0 \) is a stronger instability criterion than the J-instability. For example, for the two dimensional Hamiltonian system (with \( a_{ij} = \delta_{ij} \)) the Ricci scalar (2.12) reads:

\[
R(\text{Maupertuis}) = -\frac{1}{2(E - V(q))} \triangle \ln(E - V(q))
\]

\[
= -\frac{1}{2(E - V(q))^2} \left( \triangle V(q) + \frac{(\nabla V(q))^2}{E - V(q)} \right)
\]

(2.17)

where \( \triangle = \sum_i \partial^2 / \partial (q^i)^2 \) is the Laplacian (if \( a_{ij} = \delta_{ij} \)). Thus from \( R < 0 \) follows that \( \triangle V < 0 \) which in turn implies the existence of eigenvalues \( \lambda \)

\[1\] This is not surprising from simple considerations: If we have instability along some direction in the configuration space (as captured by the \( R < 0 \) criterion) then we also have instability at least in one direction in phase space.
with a positive real part irrespective of the sign of the Gaussian curvature of the potential. On the other hand the J-instability criterion (sometimes called the Toda-Brumer criterion) may capture the change of local instability properties of the Hamiltonian flow even in cases when $\Delta V > 0$ and hence $R > 0$. For example, the instability criterion $R < 0$ fails to capture the transition from KAM-integrability to chaos (as the energy $E$ increases) in the class of Hamiltonian flows (2.13), whereas the study of the Jacobian of the flow (the Toda-Brumer-Duff criterion) gives a threshold value $E = E_c$ (onset of “local instability”) for the energy. Thus, a small calculation will verify that for energies

$$E > E_c = 3/2$$

there are eigenvalues of the Jacobian $J(x)$ with a positive real part. Let us emphasize, what is well known, that it is not possible to make conclusions about chaos from this local criterion. Global instability (chaos) is defined by looking at the spectrum of Lyapunov exponents: Consider the set of first order differential equations, $\dot{x} = f(x)$, $x = (p, q) \in R^n$ (the Hamiltonian flow). The Jacobian $J$ of the flow $\tilde{f}$ is the $n \times n$ matrix $J = (\partial \tilde{f}/\partial x)$. In order to define the Lyapunov spectrum of characteristic exponents connected to a given trajectory $\tilde{x}(t)$ with initial conditions $\tilde{x}(0)$ one first define the stability matrix,

$$M = M_{ij}(\tilde{x}(0), t) \equiv \frac{\partial x_i(t)}{\partial x_j(0)}$$

which satisfies the differential equation

$$\dot{M}_{ij} = J(\tilde{x}(t))_{ik} M_{kj}, \quad M(0) = 1$$

(2.19)

The solution to this differential equation is called the exponential of the Jacobian under the flow, symbolically denoted by

$$M(t) = \hat{T} \exp(\int_0^t J(\tilde{x}(t))dt) M(0)$$

(2.20)

where $\hat{T}$ is the time ordering operator (the $J$’s do not commute, being $n \times n$ matrices). In order to determine whether a trajectory is stable (regular) or not, one investigates the growth of the stability matrix $M = M(\tilde{x}(0), t) = M(t)$ with time. More explicitly, consider $\Lambda(t) = (M^t(t) M(t))^{1/2}$ (cf., e.g., Eckmann and Ruelle, [4] p. 630). Let the $n$ eigenvalues of the $n \times n$ matrix $\Lambda(t)$ (which are real and positive) be denoted $d_1(t) \geq d_2(t) \geq ... \geq d_n(t)$. The Lyapunov functions are defined as $\lambda_j(t) = t^{-1} \log d_j(t)$ and the associated spectrum of Lyapunov exponents $\lambda_j$.

$$\lambda_j = \lim_{t \to \infty} \lambda_j(t) = \lim_{t \to \infty} \left\{ \frac{1}{t} \log d_j(t) \right\}.$$  

(2.21)

\[14\] $\hat{T} \exp(\int_0^t J(\tilde{x}(t))dt)$ may be calculated as $\hat{T} \lim_{\Delta t \to 0} \prod (1 + J(\tilde{x}(t) \Delta t) M(0)$ the product being taken along the integrated trajectory $\{\tilde{x}(t)\}$ in time steps $\Delta t \to 0$.

\[15\] Theoretical criteria for convergence of this limit, which holds under rather general assumptions of the flow $\tilde{f}$, are discussed in e.g. Eckmann and Ruelle [4]. The maximal characteristic exponent $\lambda_{max}$ is the easiest to find. But this is also sufficient to detect the exponential divergence of nearby orbits, called “sensitive dependence on initial conditions”. Per definition, a dynamical system displays a stochastic chaotic behavior in a region of the phase space if its maximal Lyapunov characteristic exponent is positive for trajectories in this region, cf. Benettin et al. [3].
In the very special case where the Jacobian is constant in time, \( J(\vec{x}(t)) = J_0 \) we get \( M(t) = \exp(J_0 \cdot t) \). If \( \lambda \) is an eigenvalue of \( J_0 \) we have that \( e^{\lambda t} \) is an eigenvalue of \( M(t) \). It follows, that \( \lambda \) is one of the Lyapunov exponents and local J-instability is equivalent with global instability (Lyapunov exponents).

In general, however, no implication exists between local and global instabilities. This is illustrated by a simple model example. Consider, e.g., a two-dimensional map

\[
x \rightarrow (ABABABAB\ldots)x \, , \, x \in \mathbb{R}^2
\]

where a \( 2 \times 2 \) matrix \( A \) is multiplied on the vector every second time, and the \( 2 \times 2 \) matrix \( B \) every second time.

If we choose \( A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \) and \( B = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \), \( \lambda \in \mathbb{R} \), both are locally stable, since the eigenvalues are equal to 1. \( AB \) has eigenvalues \( 1 \pm \lambda \) and globally the system is unstable.

On the other hand if we choose \( A = \begin{pmatrix} \Lambda & 0 \\ 0 & \lambda \end{pmatrix} \) and \( B = A^{-1} = \begin{pmatrix} 1/\Lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \), where \( \Lambda > 1 \) and \( 0 < \lambda < 1 \), both \( A \) and \( B \) gives local instability while globally the system is stable (\( AB = 1 \)).

The same non-trivial relationship between local and global instability criteria holds for continuous Hamiltonian flows: Orbits \( \gamma \) may build up a “global instability” (as captured by the exponentiated Jacobian along the orbit, cf. formula (2.20)) despite they are stable everywhere in the phase space with respect to the local instability criterion. For example, if we consider the simple Hamiltonian toy-model (2.13), for energies \( E < E_c = 3/2 \) there are still many orbits which are globally unstable, despite the fact that there are no positive, real eigenvalues of the Jacobian in any accessible phase space point for the trajectory.

Moreover, we may have (globally) stable orbits (Dahlqvist and Russberg [37]) which are everywhere locally unstable. (Consider the Hamiltonian (2.13) in the limit \( E \to \infty \). The Jacobian has real, positive eigenvalues in any point of the accessible phase space. Nevertheless, there is an orbit of period 11 which is globally stable, cf. Dahlqvist and Russberg [37]).

Thus there is a non trivial interplay between the local instability (as mirrored in the eigenvalues of the stability matrix (2.18)) and the global, exponentiated Jacobian (2.20), yielding the “Lyapunov exponents” along the orbit. These remarks are well known. We repeat them here to comment on what one may conclude (or not conclude) from local criteria of instability like

\[
R(Maupertuis) < 0.
\]

There has been some confusion concerning this point in the past.

Especially, we can not from \( R < 0 \) (local instability) deduce that a Lyapunov exponent is positive (global instability).

As already noticed in the introduction there were attempts [13, 20, 21] to relate the \( R < 0 \) criterion with Lyapunov exponents. In particular it was argued that the quantity

\[
\lambda = \lim_{\tau \to \infty} \sqrt{-\frac{R}{n(n-1)}} \int_0^\tau 2(E - V(q(t))) dt
\]  

(2.22)

played a role of the principal Lyapunov exponent. The Ricci scalar \( R \) is meant here as an average measure of local divergence of trajectories. The motivation of such claims (see
had its roots in the following facts: If we take the geodesic deviation equation (2.8) and rewrite it as a gradient system (see appendix 1 in Arnold [34]), i.e.

\[
\frac{D^2 \xi}{D s^2} = -\nabla_\xi [V_u(\xi)] = -\nabla_\xi \left( \frac{1}{2} R_{ijkl} \xi^i u^j \xi^k u^l \right) = -\nabla_\xi \left[ \frac{1}{2} K_{wz} g(\xi, \xi) \right]
\]

where the “potential” \( V_u(\xi) \) has been expressed by sectional curvature: \( V_u(\xi) = \frac{1}{2} K_{wz} g(\xi, \xi) \).

Then by averaging the geodesic deviation equation over the orientation of the geodesic (i.e. at a given point we take all possible geodesics passing through this point, average over the bivector \( u \wedge \xi \) and denote this average by \( < > \)) one arrives at the formula [21]:

\[
\frac{D^2 < \xi^i >}{D s^2} = -\frac{R}{n(n-1)} < \xi^i >,
\]

from which one may argue for the quantity (2.22) bearing in mind that the \( s \)-parameter is related to the \( t \)-time by (2.3). The reasoning is flawed, however, in several aspects. One cannot replace the integration of a full Jacobian matrix, cf. equation (2.20), with the integration of a Ricci-scalar which is just some average quantity (trace over the sectional curvatures). The negativity of the Ricci scalar, \( R < 0 \), which is an initial assumption for (2.22) to apply is only a sufficient criterion of local instability. Hence in the case of the Hamiltonian (2.13), for example, we have \( R > 0 \) whereas the Lyapunov exponent \( \lambda > 0 \) for most trajectories. Then apart from the fact that local instability cannot be simply translated into global chaos also the meaning of “averaged deviation equation” is questionable. Even if the Ricci scalar \( R \) is traced (numerically) along trajectories one may give examples of systems with positive Kolmogorov entropy (understood, here, as the sum over positive Lyapunov exponents) for which the quantity (2.12) is either positive or negative (see e.g. discussion in [23]). The above critique apply to some earlier papers [13, 20, 21] and also to Burd and Tavakol who adopted the expression (2.22) in [27].

We would also like to question the validity of the formula (38) in Gurzadyan and Savvidy [29] in which the Ricci scalar is related to a relaxation time of the dynamical system consisting of collisionless stars interacting via gravitational forces.

In order not to close this section in too pessimistic mood let us stress that the Maupertuis-Jacobi procedure of reducing the dynamics to geodesic flows is a powerful tool for geometrizing the dynamical systems (provided one is bearing in mind its limitations, see also [23]).

A serious problem in general relativity is to define a metric space in which our space-time metric (which is itself a metric space) is a phase space point and where the divergence of trajectories makes sense. The next section shall reveal the intimate connection between such metric structure called the superspace and the Maupertuis-Jacobi principle implemented to the General Relativity in its Hamiltonian (ADM) formulation.

\[16\] The geometrical setting arising from the Maupertuis-Jacobi principle makes this picture suited for investigating integrability of Einstein’s equations as a consequence of existence of Killing vectors and tensors in the minisuperspace [14]. Also, the fact that the Maupertuis principle involves variations at a fixed energy makes this formulation useful for the microcanonical ensemble and proves useful in deriving thermodynamical properties of gravitating systems as advocated by Brown and York [4]. These possible applications of the Maupertuis principle are beyond the scope of the present paper and the interested reader is referred to a review paper [23] and the references therein.
3 Wheeler’s superspace: Maupertuis principle implemented in the context of general relativity.

An original dream by Wheeler, DeWitt and others is to consider the dynamics of the three metrics $(3)g$ as geodesics on some manifold called superspace equipped with the metric tensor $G_{ijkl}$ which is named the supermetric. This supermetric $G_{ijkl}$ induces a norm on the space of three metrics $(3)g$,

$$||\delta g_{ab}||^2 = \int d^3 x \sqrt{g} G^{ijkl} \delta g_{ij} \delta g_{kl}$$

(3.1)

and one may measure the distance between two three-metrics $(3)g$ and $(3)\tilde{g}$ with respect to the $G_{ijkl}$ tensor. (In practice it may be difficult to find such global distances. Like in the case of finding distances between Copenhagen and Warszawa, say, one has to minimize over all paths connecting the given two points).

What can one choose for the distance measure $G_{ijkl}$? At first there are many possible choices of metrics on the space of three metrics. For example (cf. DeWitt [42]) one could choose a metric like

$$G^{ijkl} = \frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) \delta(x, x')$$

(3.2)

which has the property that the distance between two three-metrics is zero if and only if the three-metrics are identical. However, another distance measure (supermetric) is suggested from the ADM Hamiltonian formulation of general relativity. It is read off from the kinetic term in the Hamiltonian constraint,

$$\pi^i_k \pi^i_k - \frac{1}{2}(\pi^i_k)^2 - g^{(3)R} = 0$$

(3.3)

(cf. e.g. Misner [25] and references therein) associated with the constrained Hamiltonian formalism which may be set up for the Einstein equations (see e.g. discussion and references in Teitelboim [41]). More explicitly,

$$\pi^i_k \pi^i_k - \frac{1}{2}(\pi^i_k)^2 = \frac{1}{2}(g^{ik}g^{jl}\pi_{ij}\pi_{kl} + g^{il}g^{jk}\pi_{ij}\pi_{kl} - g^{ij}g^{kl}\pi_{ij}\pi_{kl}) = G^{ijkl}\pi_{ij}\pi_{kl}$$

and we thus arrive at

$$G^{ijkl} = \frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk} - g^{ij}g^{kl})$$

(3.4)

Note that the supermetric, in this notation differs by a conformal factor ($\sqrt{g}$) from DeWitt’s expression [42]. This is allowed since the ADM action is invariant with respect to conformal transformations.

Let us now see how we may arrive at the geodesic equation for the three-metric $(3)g$. It turns out that it can be done in a way which is completely analogous, step-by-step, to the previous section dealing with the similar question for an ordinary (nonrelativistic) Hamiltonian system $H = \frac{1}{2}a_{ij}p^i p^j + V(q)$.

The procedure involves the following principal steps:

\[15\]

Abstractly the “superspace” is meant to be the space of all three-metrics modulo diffeomorphisms on the three-space, i.e. the quotient space $S(\mathcal{M}) = \text{Riem}(\mathcal{M})/\text{Diff}(\mathcal{M})$.\]
1. The metric $G_{AB}$ is read off from the ADM Hamiltonian.

2. Observe that we do not have geodesic equation for $\{g\}$, with respect to metric $G_{AB}$.

3. Transformation to (affinely parametrized) geodesic motion of $\{g\}$ is obtained in two steps:
   (1) Conformal transformation of the metric to $\tilde{G}_{AB} = \mathcal{R} G_{AB}$.
   (2) Rescaling of the $\lambda$ parameter, $\lambda \rightarrow \tilde{\lambda}$.

1. As we have already noted, from the structure of the Hamiltonian

   $H = \frac{1}{2} G_{ijkl} \pi^{ij} \pi^{kl} - \frac{1}{2} g^{(3)} R - \frac{1}{2} G_{AB} \pi^{A} \pi^{B} - \frac{1}{2} g^{(3)} R = 0$  \hspace{1cm} (3.5)

   we read off the first candidate for a metric on the configuration space (the space of three-metrics)

   $G_{AB} \equiv G_{ijkl} = \frac{1}{2} (g_{ik} g_{jl} + g_{il} g_{jk} - 2 g_{ij} g_{kl})$.  \hspace{1cm} (3.6)

   This step corresponds to reading off the “metric” $a^{ij}$ from the Hamiltonian

   $H = \frac{1}{2} a^{ij} p_i p_j + V(q) = E$

   in the previous section.

2. We observe that we do not have a geodesic equation for the trajectory $\{g(\lambda)\}$. There is a “force term” on the right hand side:

   $\frac{d^2 g^A}{d\lambda^2} + \Gamma_{BC}^A \frac{dg^B}{d\lambda} \frac{dg^C}{d\lambda} = \frac{1}{2} G_{AB} \frac{\partial \mathcal{R}}{\partial g^B} = \frac{1}{2} G_{AB} \frac{\partial (g^{(3)} R)}{\partial g^B}$  \hspace{1cm} (3.7)

   where

   $\Gamma_{BC}^A = \frac{1}{2} G^{AD} \left( \frac{\partial G_{BD}}{\partial g^C} + \frac{\partial G_{DC}}{\partial g^B} - \frac{\partial G_{BC}}{\partial g^D} \right)$  \hspace{1cm} (3.8)

   Equation (3.7) may be one-to-one translated to the similar expression (2.2) from sec.2, where we also have a force term, $-a^{ij} \partial V(q)/\partial q^i$, in the form of the metric (read off from the kinetic term) multiplied by the gradient of the potential. The role of the potential in this general relativistic context is played by the quantity

   $V = -\frac{1}{2} g^{(3)} R \equiv -\frac{1}{2} \mathcal{R}$.  \hspace{1cm} (3.9)

   In analogy with section 2 (Maupertuis principle) we want to make the three-metrics move along geodesics!

---

18. The covariant form of the supermetric $G_{ijkl}$ actually differs from its contravariant counterpart. This is the consequence of a natural duality of these two tensors:

   $G^{ijkl} G_{mnkl} = \frac{1}{2} (\delta_i^m \delta_j^n + \delta_j^m \delta_i^n)$

19. By $\lambda$ we denote the parameter which parametrizes the evolution of three-metrics — the so-called supertime. For more detailed discussion see C.W.Misner [25]
3. Transformation of the non-geodesic motion (above) of \((3)g\) to geodesic motion is obtained in two steps:

(1) First we make a conformal transformation of the metric

\[
\tilde{G}_{AB} = R G_{AB} , \quad \tilde{G}^{AB} = \frac{1}{R} G^{AB}
\]

This conformal transformation of the metric (3.10) is analogous to equation (2.3) in section 2, i.e. the conformally rescaled metric,

\[
g_{ij} = 2(E - V(q))a_{ij}
\]

since

\[
\tilde{G}_{AB} = R G_{AB} = (-2V)G_{AB} = 2(E - V)G_{AB}
\]

(note that the energy \(E\) is zero in the general relativistic context by virtue of the Hamiltonian constraint).

Analogously to sec. 2 conformal rescaling of the metric alone is not sufficient to map the evolution of three-metrics into \textit{affinely parametrized} geodesics.

(2) However, by rescaling of the \(\lambda\) parameter,

\[
d\tilde{\lambda} = 2(E - V)d\lambda = -2Vd\lambda = Rd\lambda = g^{(3)}R d\lambda
\]

we obtain now the important result (cf. also e.g. Misner [25]) that with respect to this new parameter \(\tilde{\lambda}\) and the conformally rescaled metric \(\tilde{G}_{AB}\) the three-metric \((3)g\) is now an affinely parametrized geodesic (in this “Superspace”)

\[
\frac{d^2 g^A}{d\tilde{\lambda}^2} + \tilde{\Gamma}_{BC} \frac{dg^B}{d\tilde{\lambda}} \frac{dg^C}{d\tilde{\lambda}} = 0.
\]

Like in the previous section there is no “magic” involved (we have “magic without magic”). Information about the potential \((3)R\) has simply been encoded completely in the mathematical definition of the Superspace-metric \(\tilde{G}\) with respect to which the evolution of \((3)g\) then becomes geodesic motion (affinely parametrized if one uses the rescaled parameter \(\tilde{\lambda}\)).

Note, that instead of Misner’s analogy with a free particle in special relativity, with Hamiltonian \(H = \frac{1}{2}(\eta_{\mu\nu}p_{\mu}p_{\nu} + m^2)\) (cf. Misner [25], p. 451) we have rather emphasized here the complete one-to-one correspondence between construction of the Superspace and dynamics of a non-relativistic particle with the Hamiltonian \(H = \frac{1}{2}a^{ij}p_ip_j + V(q)\) reduced to geodesic flow by virtue of the Maupertuis principle.

Whereas we would not be surprised if the analogy above is to some extent well known for those who understand the construction of Wheeler’s superspace, we have not seen the analogy stated so clearly and it puts the previous investigations [18, 19, 20, 21, 24] along this route in a somewhat new perspective. Namely they can be viewed as exploring the geometric structure of the superspace aimed at investigating local instabilities in the evolution of three-geometries. Dreaming of this \textit{purely geometrical} picture of dynamics one has nevertheless to face several problems:
1. There may exist (and in fact it is not of rare occurrence) points where the potential term \( R \) is zero. At such points the Supermetric (3.10) as well as the new parameter \( \bar{\lambda} = \int R\,d\lambda \) are not well defined and present at first an obstruction to this beautiful idea of mapping the evolution of three-metrics into geodesic motion w.r.t. this Superspace. Of course as already mentioned in sec.2 it is merely an artifact which stems from the conformal transformation (3.10) and nothing singular happens at those points to the dynamics. This fact has been well known already to Misner [25]. It has also recently been emphasized by Burd and Tavakol [27].

2. If we are to uphold the dream in the strongest version of having the three-metrics as geodesics in superspace it should be the conformally rescaled “super” metric \( \bar{G}_{AB} \) which should be called the “super metric” (since that is the one which generates geodesic motion of the \( ^3g \)) and not the \( G_{AB} = G_{ijkl} \) metric, which is more frequently used as a “superspace metric”. This imply, however, that the (mini)superspace metric would differ, for example, from one minisuperspace model of some Bianchi type to a minisuperspace model of another Bianchi type - since the conformal rescaling depends on the specific space-time metric considered.

3. The Supermetric \( \bar{G}_{AB} \) induces a natural distance measure on the three-geometries in the relevant superspace. However, it should be noted, what is very well known, that \( \bar{G}_{AB} \) represents a pseudo-Riemannian geometry (with signature \((-++++)\)), therefore we may have distances \( ds^2 \) which are space-like, null or time-like. (This indefiniteness of the distance measure is particular to the application of Maupertuis principle in general relativity and the natural distance measure induced by this procedure. Application of Maupertuis principle to non-relativistic Hamiltonian systems give rise to positive definite distance measures). Therefore in general relativity we may have the unfamiliar situation that two three-metrics which are at zero distance with respect to the Wheeler’s superspace metric \( \bar{G}_{AB} \) may evolve into a finite (positive or negative) distance. This corresponds “formally” to a (local) “Lyapunov exponent” which is \( \infty \).

In analogy to the non-relativistic case but bearing in mind all the limitations we may contemplate the behavior of close trajectories (mapped into geodesics) starting from the geodesic deviation equation:

\[
\frac{D^2 \xi^A}{D\bar{\lambda}^2} = -R^A_{BCD} \frac{dg^B}{d\bar{\lambda}} \xi^C \frac{dg^D}{d\bar{\lambda}}
\]

where \( \xi = \delta^{(3)}g \) is the deviation vector between three-metrics, and the Riemann Christoffel tensor \( R^A_{BCD} \) is calculated with respect to the supermetric \( \bar{G}_{AB} \).

The idea is to use

\[
R = -\frac{(n-1)}{R^3} \sum_{A,B} \left\{ \partial^2 R \frac{\partial R}{\partial g^A \partial g^B} G^{AB} + \frac{n-6}{4} \frac{\partial R}{\partial g^A} \frac{\partial R}{\partial g^B} G^{AB} \right\} < 0
\]

as a geometrical (and hence coordinate invariant) local instability criterion for the three-metrics evolving according to the Einstein’s equations. If \( R < 0 \) then there exist at least one direction of perturbing the three-metric \( g^A \rightarrow g^A + \delta g^A \) so we have a local exponential amplification of the metric perturbation in that direction - measured with respect to the Wheeler’s superspace metric as a distance measure on the three-metrics.
In the next section we shall apply this in a concrete example of the orbits of the mixmaster minisuperspace gravitational collapse.
### Supermetrics (Maupertuis principle) and local instability criteria in non-rel. systems and in general relativity

| A Hamiltonian for a non-relativistic mechanical system | ADM-Hamiltonian in General Relativity (say, the mixmaster collapse) |
|--------------------------------------------------------|-------------------------------------------------------------------|
| 1. Read off the first candidate of a supermetric “$a_{ij}$” from the structure of the kinetic term in the Hamiltonian $H = \sum \frac{1}{2} a^{ij} p_i p_j + V(q)$ | 1. Read off the first candidate of a supermetric “$G_{AB}$” from the structure of the kinetic term in the ADM-Hamiltonian $H = \sum \frac{1}{2} G_{AB} \pi^A \pi^B - \frac{1}{2} g^{(3)} R = 0$ |
| 2. We observe that the trajectory $\{q(t)\}$ do not obey a geodesic equation of motion w.r.t. metric $a^{ij}$. There is a force term $-a^{ij} \partial V(q)/\partial q^i$ on the right hand side of (2.2). | 2. We observe that the trajectory $\{g^A\}$ do not obey a geodesic equation of motion w.r.t. metric $G^{AB}$. There is a force term $G^{AB}(\partial (\frac{1}{2} g^{(3)} R)/\partial g^A)$ on the right hand side of (3.7). |
| 3. Transformation to geodesic motion is obtained in two steps:  
(1) Conformal transformation of the metric $a_{ij} \rightarrow 2(E - V(q)) a_{ij}$  
(2) Rescaling of the time parameter $dt \rightarrow ds = 2(E - V(q)) dt$ and we obtain the equation (2.6). | 3. Transformation to geodesic motion is obtained in two steps:  
(1) Conformal transformation of the metric $\tilde{G}_{AB} = 2(E - V) G_{AB} = (g^{(3)} R) G_{AB}$  
(2) Rescaling of the $\lambda$ parameter $d\tilde{\lambda} = 2(E - V) d\lambda = (g^{(3)} R) d\lambda$ and we obtain the equation (3.13). |
| The Ricci scalar may be calculated from the resulting “supermetric” and the criterion $R < 0$ is used as a local instability criterion (N-body systems etc., Krylov etc.) | The Ricci scalar may be calculated from the resulting “supermetric” and the criterion $R < 0$ has been attempted as a local instability criterion in the context of general relativity (Szydłowski, Biesiada etc.) |

Table 1: summarizing the analogy between the Maupertuis principle for non-relativistic Hamiltonian systems and Wheeler’s superspace (Maupertuis principle) for general relativity.
3.1 Application to the mixmaster gravitational collapse

For the mixmaster homogeneous toy-model collapse with the three-metric,

\[ (3) g_{iX} = \gamma_{ij}(t) \omega^i(x) \omega^j(x) \]

where \( \gamma_{ij}(t) = \text{diag}(a^2(t), b^2(t), c^2(t)) \) and \( a, b, c \) are the scale factors of the metric, we have in the ADM-variables,

\[
\begin{pmatrix}
\Omega \\
\beta_+ \\
\beta_-
\end{pmatrix} = 
\begin{pmatrix}
-\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\
\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \\
\frac{\sqrt{2}}{6} & -\frac{\sqrt{2}}{6} & 0
\end{pmatrix} 
\begin{pmatrix}
\ln a \\
\ln b \\
\ln c
\end{pmatrix}
\]

the following form of the Hamiltonian

\[
\mathcal{H} = \frac{1}{2}(G_{AB} p_A p_B - \mathcal{R}) = \frac{1}{2}(-p_\Omega^2 + p_+^2 + p_-^2 + e^{-4\Omega}(V(\beta_+, \beta_-) - 1))
\]

where the mixmaster three-curvature potential reads

\[
\mathcal{R} = -e^{-4\Omega}(V(\beta_+, \beta_-) - 1)
\]

The (mini)superspace variables are thus \( g^A = (g^\Omega, g^+, g^-) = (\Omega, \beta_+, \beta_-) \) \((A, B = 1, 2, 3)\) and the momenta are \( p_A = G_{AB} d\omega^B / d\lambda \) giving \( p_\Omega = -d\Omega / d\lambda \), \( p_+ = d\beta_+ / d\lambda \), \( p_- = d\beta_- / d\lambda \). We immediately read off the first candidate for a “super” metric from the kinetic term in the Hamiltonian

\[
G_{AB} = G^{AB} = \text{diag}(-1, +1, +1).
\]

The Christoffel symbol corresponding to this flat “supermetric” vanishes, \( \Gamma^A_{BC} = 0 \), and the Hamiltonian equations translate into

\[
\frac{d^2 g^A}{d\lambda^2} = \frac{1}{2} G^{AB} \frac{\partial \mathcal{R}}{\partial g^B}.
\]

I.e. it is a kind of Newton’s second law with non-vanishing force term appearing on the right hand side, \( d^2 \Omega / d\lambda^2 = -\frac{1}{2} \partial \mathcal{R} / \partial \Omega \), \( d^2 \beta_+ / d\lambda^2 = -\frac{1}{2} \partial \mathcal{R} / \partial \beta_+ \), \( d^2 \beta_- / d\lambda^2 = -\frac{1}{2} \partial \mathcal{R} / \partial \beta_- \).

However, the configuration space variables \( g^A = (\Omega, \beta_+, \beta_-) \) may be mapped to a (affinely parametrized) geodesic flow on the configuration space manifold equipped with the (mini)superspace metric

\[
\tilde{G}_{AB} = \mathcal{R} G_{AB} = -e^{4\Omega}(V(\beta_+, \beta_-) - 1) \text{diag}(-1, 1, 1)
\]

with the parametrization

\[
d\tilde{\lambda} = \mathcal{R} d\lambda = -e^{4\Omega}(V(\beta_+, \beta_-) - 1) d\lambda
\]

along the trajectory of the mixmaster collapse. Thus, we have

\[
\frac{d^2 g^A_{IX}}{d\lambda^2} + \tilde{\Gamma}^A_{BC} \frac{dg^B_{IX}}{d\lambda} \frac{dg^C_{IX}}{d\lambda} = 0
\]
This procedure introduces a host of singularities at points where \( R = 0 \) i.e. where \( V = 1 \) (see also an interesting discussion in Misner [25] pp. 453-454).

The fact, recently emphasized also by Burd and Tavakol [27] that the singularities introduced by the supermetric \( G_{ijkl} \) prevents us from mapping the entire mixmaster collapse orbit into one single, unbroken and simple geodesic orbit - is of course an important obstacle to this approach and was, in fact, previously emphasized by one of us (S.E.R.). Despite these troubles, let us suppose that we nevertheless carry out these transformations at points in the mixmaster configuration space \( g^A = (\Omega, \beta_+, \beta_-) \) where the superspace metric \( \tilde{G}_{AB} = \tilde{G}_{ijkl} \) do not introduce artificial singularities, i.e. at those points where \( V(\beta_+, \beta_-) \neq 1 \). With this supermetric \( \tilde{G}_{ijkl} \) one may thus construct non-trivial intervals in which the mixmaster metric is mapped into geodesics (in Wheeler’s (mini)superspace).

We shall start with some remarks on the properties of the distance measure between two nearby mixmaster metrics \((3)g^A_{IX}\) and \((3)g^A_{IX} + \delta(3)g^A_{IX}\). Let us first note, that the distance between \( g^A = (\Omega, \beta_+, \beta_-) \) and \( g^A + \delta g^A = (\Omega + \delta \Omega, \beta_+ + \delta \beta_+, \beta_- + \delta \beta_-) \) is of Lorentzian signature. This indefiniteness is a property of the supermetric, both in its original form
\[
||\delta g^A||^2 = ds^2 = G_{AB} \delta g^A \delta g^B = -d\Omega^2 + d\beta_+^2 + d\beta_-^2
\]
and in the conformally rescaled form
\[
||\delta \tilde{g}^A||^2 = d\tilde{s}^2 = \tilde{G}_{AB} \delta \tilde{g}^A \delta \tilde{g}^B = -e^{-4\Omega} (V(\beta_+, \beta_-) - 1)(-d\Omega^2 + d\beta_+^2 + d\beta_-^2).
\]

Thus the distance can take positive, zero or negative values and one has the unfamiliar situation (relative to positive definite Euclidean distance measures implemented in the context of non-relativistic Hamiltonian dynamical systems) - as previously stressed - that two configuration space points (mixmaster three-metrics) which is not identical may have zero distance with respect to these naturally induced distance measures.

These distance measures have on the other hand the good property of being invariant under canonical coordinate transformations. Thus, if we have a change of coordinates (configuration space variables), \( g^A \rightarrow g^*A \) the distance measure \( ds^2 = G_{AB} \delta g^A \delta g^B \) is invariant
\[
||\delta g^A||^2 = d\tilde{s}^2 = G_{AB} \delta g^A \delta g^B = G_{AB} \delta g^*A \delta g^*B = ||\delta g^*A||^2
\]
under such transformations, since \( G_{AB} \) transforms properly as a tensor,
\[
G_{AB}^* = \frac{\partial g^C}{\partial g^*_A} \frac{\partial g^D}{\partial g^*_B} G_{CD}.
\]

One can easily see this by recalling that the coordinate transformation \( g^A \rightarrow g^*A \) induces the canonical transformation of momenta
\[
p_A \rightarrow p_A^* = \frac{\partial g^B}{\partial g^*A} p_B.
\]

Therefore the Hamiltonian \( \mathcal{H} \) reads
\[
\mathcal{H} = \frac{1}{2}(G^{AB} p_A p_B - R) = \frac{1}{2}(G^{AB} \frac{\partial g^*C}{\partial g^A} \frac{\partial g^*D}{\partial g^B} p_C p_D - R) = \frac{1}{2}(G^{*AB} p^*_A p^*_B - R)
\]
thus justifying claims of the formula (3.22).
A quantity which is invariant under a large class of coordinate reparametrizations (canonical transformations in the Hamiltonian formulation) was considered in works by Biesiada, Lapeta, Szczęsny and Szydłowski [19, 20, 21] where the Ricci scalar of the manifold on which the mixmaster model acts as a geodesic flow has been extracted. As we have seen here this is precisely to extract the Ricci scalar for the Wheeler’s superspace metric $\tilde{G}_{AB}$ (in the conformally rescaled version where it describes the three-metrics as geodesics) and negative values $R < 0$ are naturally interpreted as a local measure of exponential instability via the geodesic deviation equation with respect to the conformally rescaled distance measure - of Lorentzian signature - on the space of mixmaster three-metrics.

Despite the problems with unwanted singularities induced by this approach (as noted by e.g. Misner [25]) we believe that insufficient attention has been paid towards the use of distance measures naturally induced by the structure of general relativity, e.g. the Wheeler’s superspace metric as a distance measure on the space of mixmaster collapses (in the context of, for example, discussing the chaotic properties) - instead of using artificial and completely arbitrary Euclidean distance measures [9, 10, 12] on the solution space (which are not supported by the structure of general relativity).

As concerns our discussion of the mixmaster gravitational collapse, the (mini)superspace Ricci scalar $R$ may in principle be calculated directly from the conformally rescaled Wheeler’s (mini)superspace metric,

$$\tilde{G}_{AB} = -e^{-4\Omega}(V(\beta_+, \beta_-) - 1) \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

However, ADM variables are not very suitable for such calculations due to the complicated form of the potential term. Since our aim is to extract the Ricci scalar $R(Maupertuis)$, which is an invariant under canonical coordinate transformations, it should not matter in which representation we work. The ADM variables and the Bogoyavlenskii variables are related by canonical coordinate transformations.

Therefore, as in [20], we rather consider the Bogoyavlenskii Hamiltonian [26] in $(a, b, c)$ variables

$$H = 2(p_0 p_b a^2 b^2 + p_0 p_c b^2 c^2 + p_0 p_c a^2 c^2) - p_0^2 a^4 - p_b^2 b^4 - p_c^2 c^4 + \frac{1}{2}(a^2 b^2 + b^2 c^2 + a^2 c^2) - \frac{1}{4}(a^4 + b^4 + c^4)$$

(for the mixmaster collapse) in which case the potential is less complicated at the expense that the metric read off from the kinetic energy is no longer diagonal,

$$a_{ij} = \frac{1}{2a^2 b^2 c^2} \begin{bmatrix} 0 & c^2 & b^2 \\ c^2 & 0 & a^2 \\ b^2 & a^2 & 0 \end{bmatrix}.$$  

More precisely, one may transform the ADM variables to $\alpha, \beta, \gamma$ variables (cf. equation (3.14)) and make a corresponding canonical coordinate transformation of the ADM momentum variables to $p_\alpha, p_\beta, p_\gamma$. These are then the Bogoyavlenskii momentum variables, cf. Bogoyavlenskii [26], p. 40, and one regains the Bogoyavlenskii Hamiltonian from the ADM Hamiltonian up to a conformal factor $a^2 b^2 c^2 = e^{2(\alpha + \beta + \gamma)}$ which is not important because the Hamiltonians are equal to zero, $H = 0$. 

20 More precisely, one may transform the ADM variables to $\alpha, \beta, \gamma$ variables (cf. equation (3.14)) and make a corresponding canonical coordinate transformation of the ADM momentum variables to $p_\alpha, p_\beta, p_\gamma$. These are then the Bogoyavlenskii momentum variables, cf. Bogoyavlenskii [26], p. 40, and one regains the Bogoyavlenskii Hamiltonian from the ADM Hamiltonian up to a conformal factor $a^2 b^2 c^2 = e^{2(\alpha + \beta + \gamma)}$ which is not important because the Hamiltonians are equal to zero, $H = 0$. 

23
Explicit evaluation of the Ricci scalar (2.12), corresponding to the conformally rescaled metric, induced by the kinetic term in the Bogoyavlenski Hamiltonian, yields

\[ R_{IX} = -\frac{1}{8(-V)^3}(a^8 + b^8 + c^8 - 2a^4b^4 - 2a^4c^4 - 2b^4c^4 + 16a^2b^2c^2(a^2 + b^2 + c^2)) \] (3.23)

where

\[ -V = \frac{1}{2}g^\lambda(\partial\lambda) = \frac{1}{4}(a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2) \] (3.24)

As concerns the aforementioned instability criterion applied to the mixmaster three-geometries the idea is now to find regions in the configuration space where \( R < 0 \) and see whether a typical trajectory is “confined” to such regions or just traverse them quickly. Clearly, \( R \) cannot be negative all over the entire minisuperspace. For example, it is easy to see that \( R > 0 \) in the case of isotropy, \( a = b = c \). On the other hand one observes [20] that \( R < 0 \) in the asymptotical BKL regime, i.e. when \( a \gg b, c \) (or cyclic permutations thereof).

For sake of illustration we performed a numerical experiment in which we have integrated the vacuum mixmaster field equations [3]

\[ 2\alpha_{\tau\tau} = \frac{d^2}{d\tau^2}(\ln a^2) = (b^2 - c^2)^2 - a^4 \]
\[ 2\beta_{\tau\tau} = \frac{d^2}{d\tau^2}(\ln b^2) = (c^2 - a^2)^2 - b^4 \] (3.25)
\[ 2\gamma_{\tau\tau} = \frac{d^2}{d\tau^2}(\ln c^2) = (a^2 - b^2)^2 - c^4 \]

supplemented by a first integral constraint

\[ I = \alpha_{\tau}\beta_{\tau} + \alpha_{\tau}\gamma_{\tau} + \beta_{\tau}\gamma_{\tau} - \frac{1}{4}(a^4 + b^4 + c^4) - 2(a^2b^2 - a^2c^2 - 2b^2c^2) = 0 \]

Here \( \tau = \int dt/abc \) denotes the standard time variable, [3] and subscript \( \tau \) means differentiation with respect to the time variable \( \tau \). We used a numerical code from S.E. Rugh [12] based on the fourth-order Runge-Kutta integrator with check for the first integral constraint (for details see [12]).

The figure displays the temporal (in \( \tau \)-time) behavior of the minisuperspace Ricci scalar as felt by a trial trajectory. We have selected a set of reference initial conditions as in A. Zardecki [11] but have adjusted the value of \( c' \) to make the first integral vanish to machine precision. (Such an adjustment is indeed necessary. Cf. discussions in S.E. Rugh [12] and D. Hobill [14]). This yields the starting conditions

\[ a = 1.85400.. \quad b = 0.438500.. \quad c = 0.085400.. \]
\[ a' = -0.429200.. \quad b' = 0.135500.. \quad c' = 2.964843279..... \] (3.26)

On the figure the minisuperspace Ricci scalar is displayed. For comparison also the evolution of (logarithmic) scale factors \( \alpha, \beta, \gamma \) is superimposed. It turns out that \( R \) is negative all the time but varies over many orders of magnitude. The latter is connected with the singularities generated by a vanishing conformal factor at points where the potential is zero. At these points \( R \) goes to (minus) infinity and a numerical cure against
it was to provide an arbitrary cut off. This means that our trial trajectory of the mixmaster collapse stays in the regions of minisuperspace where “local instability” occurs. Of course such an illustration cannot be representative for any larger set of trajectories. For example, the minisuperspace Ricci scalar calculated along a Taub axisymmetric solution was found to be positive. However, even in cases when $R < 0$ it is not possible to claim, cf. our previous discussion about local and global properties, that this fact accounts for global chaotic behavior - even though some may be tempted to make such an inference.

4 Concluding remarks.

The possibility of mapping the dynamics of a wide class of Hamiltonian systems to geodesic flows on Riemannian manifolds by application of Maupertuis principle, and extracting coordinate invariant information about local instability of the trajectories (from the corresponding Riemann tensor) has been applied in various contexts, ranging from N-body systems (a gas) interacting via Van der Waals forces (Krylov [28], the instability (relaxation) properties of a collisionless gas interacting via gravitational forces (Gurzadyan and Savvidy [29]) and N-body systems interacting via the Debye-Hückel potential given by $V(r) \sim e^{-\kappa r}/r$ (a model for a hot dilute plasma, Van Velsen [35]).

In this paper we have examined virtues and drawbacks of applying this idea to general relativity [18]-[22]. Coordinate invariant information about local instability properties is in this case - as in the applications to the aforementioned non-relativistic Hamiltonian systems - imagined to be extracted either from negativity of the Ricci scalar (which is the sum over principal sectional curvatures) or by considering the (principal) sectional curvatures themselves [24].

We have shown that the Maupertuis principle as a way of geometrizing the Hamiltonian dynamics when implemented to general relativity is actually reproducing Wheeler’s idea of a superspace in which three-geometries evolve along geodesic lines. We have examined this superspace in order to get an insight into the construction of gauge invariant (geometrical) criteria for local instability in general relativity. The original dream of having Einstein’s equations acting as a geodesic flow has been abandoned by its inventors a long time ago because of artificial singularities created by the conformal rescaling of the metric in points where the ADM potential equals zero. In so far as for its original purpose i.e. to provide a suitable state-space for quantization of gravity, people could (and actually did) retreat to more secure ground of the weaker form of the superspace (giving up the geodesic picture) the conformally rescaled superspace is crucial for our proposal. The idea is simple: If one can cast the dynamics into a geodesic flow then the sensitive dependence on initial conditions (a key ingredient of instability and chaos) may be captured at the level of Jacobi equation for geodesic deviation. The original intention was to point to a certain reasonable approach rather than provide an ultimate solution (if any exist) to the problem of invariant description of chaos. We have noticed some obvious dangers when one makes global conclusions from local considerations. This issue is far from being trivial however, since the history of differential geometry teaches that starting from Gauss’ theorema egregium a lot of effort (sometimes spectacularly successful) has been paid to the question of how to derive global characteristics from local ones.

We have shown that expressions - used previously - such like (2.22) relating directly the
Ricci scalar with Lyapunov exponents \[19, 20, 21\] are not true. Therefore, unfortunately, one cannot base a short time average (STA) Lyapunov exponent on this expression either, cf. Burd and Tavakol \[27\]. The STA approach \[45\] as a tool for studying non-uniform dynamics is based on evaluation of the Lyapunov exponents (extracted in a standard way from the Jacobian) over a finite time interval. The mixmaster toy-model gravitational collapse examined here as an example of a spacetime metric with complex (chaotic) behavior is in fact rather well understood — the BKL-description \[16\] is derived under assumptions which are very good and indeed numerically confirmed in Rugh \[12\] and by Berger \[48\]. Nevertheless, if we want to characterize - or rather to develop measures of - chaos in the highly non-linear Einstein equations, the mixmaster collapse is a suitable toy-model laboratory to address such questions: If we are not even able to invent useful indicators of chaos in the moderately simple example of the mixmaster space-time metric, how are we ever going to be able to deal with these issues for more general and more complicated space-time metrics?

The observation of gauge variance of standard indicators like that of a Lyapunov exponent in the context of general relativity was emphasized by Rugh \[12, 14\] and also by Pullin \[15\]. (See also recent discussion in Rugh \[17\]). If we apply standard techniques of extracting the Lyapunov exponents we at first sight have chaos in some gauges and in other gauges there seems to be no chaos. Apparently, this situation arises because we are looking at the problem in the wrong way - we have not posed a “gauge invariant question”!

Whereas Rugh \[14\] was expressing hope of constructing indicators which are invariant under a large class of coordinate transformations, i.e. indicators which do not refer to a “particular gauge” (a problem which is easier to point out than to solve) it appears that Pullin \[15\] was resorting to the Poincaré disc as a kind of “selected set of coordinates” — a similar viewpoint is shared by Misner \[49\]. It appears to us that no “gauge” is better than others (is chaos in general relativity a concept which should only be defined in certain selected frames of coordinates?) - yet some gauges may make things particularly simple or elegant thereby expressing the explicit or hidden symmetry (geometric structure) of the problem. The geometric approach (via Maupertuis principle) discussed in the present paper also points toward a certain structure uncovered from the Hamiltonian formulation of general relativity (which turned out to be equivalent to Wheeler’s superspace) as a “natural” one. The problem is however not with “preferred” gauges but rather whether the criteria used to establish certain properties (such like chaoticity) are gauge-invariant themselves. Therefore we ought to seek measures of chaos (if they are possible to construct) which are invariant under some large class of gauge transformations.

Here we have examined an approach which points to a more gauge invariant construction involving Wheeler’s superpace. However, this approach does (also) have several obstacles, e.g. when implemented in the toy-model context of the mixmaster gravitational collapse. Hence it is safe to conclude that how to characterize chaos most elegantly in general relativity (even in the context of simple toy-models) remains an open question.
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Figure 1: An illustration of the mixmaster minisuperspace Ricci scalar for a trial trajectory. Flat minima denote that the depth is out of range of the figure. The Ricci scalar is negative all the time (but rapid changes over many orders of magnitude makes it difficult to perceive this fact on the figure). Indeed the artificial singularities introduced by the conformally rescaled superspace metric (see the text) make the Ricci scalar $R$ go to $-\infty$ while traversing the null-surface on which the potential is zero.

**Figure caption**
This figure "fig1-1.png" is available in "png" format from:

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