Finite orbits of the braid group action on sets of reflections

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1 Introduction

The problem of finding all finite orbits of the braid group action on tuples of reflections appeared in the classification of semi-simple Frobenius manifolds. These orbits correspond to algebraic solutions to equations of isomonodromic deformations.

Suppose there is given a Fuchsian system of complex ordinary differential equations

$$\frac{dy}{dz} = \sum_{i=1}^{n} \frac{A_i}{z-x_i} y,$$

where $y(z)$ is a column vector of $m$ functions and $A_i$ are constant matrices. Around each point $x \neq x_i$ there exist $m$ linearly independent solutions $y_1, y_2, \ldots, y_m$ and all solutions can be expressed by linear combinations of them. It is known for linear systems that these solutions can be continued analytically along any path, which doesn’t pass through singularities of the coefficients. For convenience such $m$ linear independent solutions are arranged into a square matrix called the fundamental system of solutions. For two fundamental systems $Y_1$ and $Y_2$ the product $Y_1^{-1}Y_2$ is constant, whence $Y_2 = Y_1G$ for some $G \in GL(m, \mathbb{C})$. The result of the analytic continuation of a fundamental system of solutions $Y(z)$ along a loop $\gamma$ based at $x$ will be another fundamental system of solutions

$$Y_\gamma = YM_\gamma,$$

where $M_\gamma$ is an invertible matrix depending only on the homotopy class of $\gamma$. This gives us a linear representation of the fundamental group

$$\pi_1(\mathbb{C} \setminus \{x_1, \ldots, x_n\}) \to GL(m, \mathbb{C}),$$

called the monodromy representation. Because of the freedom in the choice of the fundamental system $Y_2 = Y_1G$, $Y_2 = Y_2M_2(\gamma)$, $M_2(\gamma) = G^{-1}M_1(\gamma)G$, the monodromy representation is fixed by the Fuchsian system only up to conjugation.

Deformations of the singularity points $x_i = x_i(t)$ and the matrix residues $A_i = A_i(t)$ preserving the monodromy up to conjugation are called isomonodromic deformations. These obey Schlesinger’s equations

$$\frac{\partial A_i}{\partial x_j} = \frac{[A_i, A_j]}{x_i - x_j}, \quad i \neq j \quad \frac{\partial A_i}{\partial x_i} = -\sum_{j \neq i} \frac{[A_i, A_j]}{x_i - x_j}. \quad (4)$$
In a geometric language, as it is explained in [4, 17], the above equations define a non-linear flat connection on the fibre bundle

\[ M^* := (O_1 \times \cdots \times O_n \times O_\infty)/GL_m(\mathbb{C}) \times B \rightarrow B \]  

over \( B := \mathbb{C}^n \setminus \{x_i = x_j, i \neq j\} \), where \( O_i \) is the adjoint orbit of \( A_i \), which is preserved by the equations and \( A_\infty = -\sum_{i=1}^n A_i \) is the residue at infinity. On the other hand the deformed equations (1) define the fibre bundle

\[ M := \text{Hom}(\pi_1(\mathbb{C} \setminus \{x_1, \ldots, x_n\}), GL_m(\mathbb{C}))/GL_m(\mathbb{C}) \rightarrow B, \]  

equipped with a complete flat connection, defined locally by identifying representations taking the same values on a fixed set of generators of the fundamental group. The isomonodromy connection of Schlesinger’s equations is the pull-back of the natural bundle map \( M^* \rightarrow M \) from the Fuchsian system to its monodromy representation. The monodromy of the connection on \( M \) and correspondingly on \( M^* \) amounts to an action of the fundamental group of \( B \) on the fibres, which is the braid group action on the monodromy data

\[ \sigma_i : (M_1, \ldots, M_n) \mapsto (M_1, \ldots, M_{i-1}, M_i M_{i+1} M_i^{-1}, M_{i+2}, \ldots, M_n), \]  

where the \( \sigma_i \) are the standard generators of the braid group (see [2]).

The solutions to isomonodromic deformation equations of two dimensional Fuchsian systems with four singularities on the Riemannian sphere \( \mathbb{P}^1 \) can be expressed through solutions of the Painlevé VI equation. In [10] were found all algebraic solutions to one-parameter family of Painlevé VI equations, which correspond to the finite orbits of the braid group action on triples of reflections. It was shown that such orbits correspond to pairs of reciprocal regular polyhedra or star-polyhedra (see [7]).

Algebraic functions have finite number of branches, therefore, in order to find all algebraic solutions of the isomonodromic deformation equations one must find all finite orbits of the braid group action on tuples of linear transformations under the equivalence of simultaneous conjugation. One class of transformations for which this action is particularly simple is that of reflections, since a generic \( n \)-tuple of reflections can be specified by a square matrix, called here the arrangement matrix. It is the Gram matrix of the normed eigenvectors with eigenvalue \(-1\), provided there is a non-degenerate symmetric bilinear form invariant under all reflections. It is known also as the Cartan matrix when these vectors are simple roots in a root system. The action of the braid group on the entries of these matrices, however, is non-linear. It was conjectured by Dubrovin [8] that all finite orbits of the braid group on \( n \)-tuples of reflections come from finite Coxeter groups. The orbits on non-redundant generating reflections in finite Weyl groups were found in [18] and it was shown that these are in one-to-one correspondence with the conjugacy classes of quasi-coxeter elements in these groups.

In the present article Dubrovin’s conjecture is proved. Moreover, it is shown that all finite orbits on singular matrices come from redundant generators in
finite Coxeter groups. Such matrices, however, represent non-unique equivalence classes of $n$-tuples of reflections, and if the corank is greater than one some of these equivalence classes depend on additional continuous parameters. The question when the orbits on these parameters are finite is not considered here.

The approach in this article is combinatorial. There are found universal sets of generating reflections in each finite Coxeter group. These sets possess maximal symmetry, and, fortunately, all conjugacy classes of quasicoxeter elements are obtained from their products taken with the possible inequivalent orderings. The universal sets allow inductive construction of all symmetric arrangement matrices in finite orbits of the braid group using only the classification of finite orbits on triples of reflections. In the course of this construction we recover all finite Coxeter groups without using the standard generators corresponding to reflections on the walls of Weyl chambers.

Another viewpoint for our construction are Schwarz triangles and their higher dimensional analogues. The elements of the finite orbits on triples of reflections with invertible Gram matrix are reflections, whose reflecting planes intersect the sphere $S^2$ in Schwarz triangles. In the same way the reflecting hyperplanes of $n$-tuples of reflections in finite orbits of the braid group action intersect the sphere $S^{n-1}$ in Schwarz simplexes, which fit on a finite covering of the sphere by reflections on their sides. The most symmetric Schwarz simplexes correspond to our universal sets of generating reflections. For example the universal Schwarz simplex for the group $A_n$ is the projection of a face of the regular simplex on the circumsphere and the angle between any two of its sides is $\frac{2\pi}{3}$ in contrast to the spherical simplex of the Weyl chamber of $A_n$, in which the sides can be ordered so that the consecutive sides to meet at angle $\frac{\pi}{3}$ and the non-consecutive to be orthogonal. This combinatorial information is read directly from Coxeter-Dynkin diagram. We will widely use diagrams to represent Schwarz simplexes and sets of reflections.

The classification of finite orbits of the braid group action on $n$-tuples of reflections with invertible Gram matrix in the real Euclidean space was done by Humphreys in [14]. It coincides with ours, except for the group $D_n$ for which the correct answer for the number of orbits is the whole part of $n/2$ instead of $n - 2$. After the first preprint appearance of the present article, another very short proof was found in [15] and a flaw in the proof of Humphreys was pointed out. Our assumptions are weaker than both these works as we consider not only positive definite but arbitrary symmetrizable Gram matrices with complex entries. In this way we treat simultaneously all linear groups generated by reflections. This includes all Coxeter groups and also some non-Coxeter groups whenever the corank of the Gram matrix is greater than one. The last can be interpreted as groups of quasi-symmetries of almost periodic structures, or as unfaithful representations of Coxeter groups satisfying additional non-coxeter relations.

We require finiteness of the orbits of the braid group on only the equivalence classes of ordered sets of reflections

\[(r_1, \ldots, r_n) \sim (Gr_1G^{-1}, \ldots, Gr_nG^{-1})\]  

(8)
and not on the reflections themselves.

We show that all such orbits can be obtained from the (possibly redundant) generators in finite Coxeter groups, provided the equivalence classes can be specified by the arrangement matrix without additional continuous parameters. This is the case for corank less than 2 arrangement matrices, and for two extremal realizations of the matrices with greater corank namely those in which the eigenvectors with eigenvalue \(-1\) span a subspace of dimension equal either to the rank of the arrangement matrix or to its size.

The actual classification of these orbits is done only for the invertible Gram matrices. The characteristic polynomials of quasicoxeter elements are calculated for each orbit, which is a new result for the icosahedral groups. In another article [16] we classify the orbits in the other extremal case of rank 2 arrangement matrices. In this case the action can be linearized and the obtained linear representation of the braid group coincides with the one considered by Arnol’d in [1] for the odd number of reflections, while for even number our representation is reducible and the nontrivial irreducible component of it coincides with Arnol’d representation.

The result obtained here is not restricted only to sets of reflections. By simple multiplication by \(-1\) the reflections turn to half-turns to which the same result applies. More subtle is the connection with tuples of transvections, which are the nontrivial linear unipotent transformations preserving point-wise hyperplanes of codimension one. An ordered set of such transformations can be specified again by an arrangement matrix with zeroes on the diagonal. If these transvections preserve a non-degenerate alternating form, the arrangement matrix can be taken antisymmetric. The action of the braid group on these antisymmetric matrices coincides with the action on symmetric ones. This duality was used in [10, 9, 4] to switch from one picture to the other using Laplace transformation to convert the Fuchsian system with monodromy generated by transvections into a system with one regular and one irregular singularity

\[
\frac{dY}{dz} = \left( U + \frac{V}{z} \right) Y, \quad z \in \mathbb{C}
\]  

(9)

then applying a scalar shift and converting back to a Fuchsian system with monodromy generated by reflections. The monodromy data for the system (9) is given by Stokes matrices, which relate analytic solutions having the same asymptotic expansion in different sectors centered at the irregular singular point \(z = \infty\). In this case there is essentially one Stokes matrix, and, in an appropriate basis, it is upper triangular with ones on the diagonal. For this system there is a notion of isomonodromic deformation, in which the parameters of deformation are the pairwise distinct eigenvalues of the matrix \(U\). The fundamental group of the space of parameters of the isomonodromic deformation is again the braid group with \(n\) strands, which gives an action of this group on the Stokes matrices. The action of the braid group on Stokes matrices is the same nonlinear action as for the symmetric and antisymmetric matrices. Yet another place where the same action appears is the helix theory, where it is called the braid group
action on semi-orthonormal bases [12]. In the last case, the entries of the upper triangular matrices are integer because it is a cohomology theory.

2 Action of the braid group

Let’s consider the free group $F_n$ with $n$ generators as the fundamental group $\pi_1(\mathbb{C} \setminus \{p_1, p_2, ..., p_n\}, O)$ of the (complex) plane with $n$ points removed, with some fixed base point $O$. The generators of $F_n$ are elementary cycles around points $p_k$.

The braid group $B_n$ on $n$ strands can be defined as the group of homotopic classes of diffeomorphisms of the plane with $n$ points removed $Diff(\mathbb{C} \setminus \{p_1, p_2, ..., p_n\})$. It is generated by $n - 1$ standard generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ subject to the following generating relations (see [2])

$$
\begin{align*}
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \\
\sigma_i \sigma_j &= \sigma_j \sigma_i, \quad j \neq i \pm 1.
\end{align*}
$$

In this context, it is evident that each braid transforms the cycles $\gamma : S^1 \to \mathbb{C} \setminus \{p_1, p_2, ..., p_n\}$, and homotopically equivalent diffeomorphisms transform a given cycle into homotopically equivalent cycles. Therefore we have a natural inclusion $B_n \subset Aut(F_n)$

$$
B_n = \pi_0(Diff(\mathbb{C} \setminus \{p_1, p_2, ..., p_n\})) : F_n \to F_n = \pi_1(\mathbb{C} \setminus \{p_1, p_2, ..., p_n\}).
$$

Figure 1: Hurwitz action on the fundamental cycles
Under convention, the generators of $F_n$ and $B_n$ to be as in Fig.1, we have

\[
\sigma_k(g_k) = g_k g_k+1 \sigma_k^{-1} g_k^{-1} \\
\sigma_k(g_{k+1}) = g_k \\
\sigma_k(g_l) = g_l \quad l \neq k, k+1.
\] (12)

This action can be considered also over any ordered set of elements of a group. Our aim is to classify all the finite orbits of $B_n$ on groups generated by reflections. Throughout this paper the braids act from the left $(\sigma_1 \sigma_2)(g_1, \ldots, g_n) = \sigma_1(\sigma_2(g_1, \ldots, g_n))$.

2.1 Braid group action on arrangements of reflections

Definition 1. Reflection in a linear space $V$ of arbitrary dimension over the field of complex numbers $\mathbb{C}$, is a linear transformation of period 2 fixing pointwise hyperplane of codimension 1.

The general form of a reflection is

\[
\mathbf{r} = I - v \otimes v^\vee, \quad v \in V \setminus \{0\}, \quad v^\vee \in V^* \setminus \{0\} \quad \langle v, v^\vee \rangle = v^\vee(v) = 2,
\] (13)

where the elements of the tensor product $V \otimes V^*$ are naturally identified with endomorphisms of $V$, and $I$ is the identity operator.

The pair $(v, v^\vee)$ is unique for the reflection up to the change $(v, v^\vee) \mapsto (\lambda v, \lambda^{-1} v^\vee)$. Given $n$ reflections $r_i = I - v_i \otimes v_i^\vee$, $v_i^\vee(v_i) = 2$, their relative position can be characterized by the arrangement matrix

\[
B_{ij} = v_i^\vee(v_j), \quad r_i = I - v_i \otimes v_i^\vee, \quad B_{ii} = v_i^\vee(v_i) = 2.
\] (14)

The same reflections can be characterized by different matrices $B$ and $B'$ if

\[
B_{ij} = \lambda_i \lambda_j^{-1} B'_{ij}
\] (15)

for some nonzero numbers $\{\lambda_i\}_{i=1}^n$. In (15) appear only ratios of the numbers $\lambda_i$ so one may always fix $\lambda_1 = 1$. The equation (15) defines an equivalence relation on the arrangement matrices. An arrangement matrix $B$ will be called symmetrizable if it is equivalent to a symmetric matrix. If the reflections preserve a non-degenerate quadratic form, the matrix $B_{ij}$ is symmetrizable. We will consider only symmetrizable arrangement matrices but will not assume the existence of invariant symmetric bilinear form. The arrangement matrices of simple reflections in Coxeter groups are known as Cartan matrices.

Let’s remark that we do not set any restrictions on these reflections. Usually it is required that the group of reflections should act properly discontinuously on some geometric space. For vector spaces only the finite groups of reflections act in this way. The affine and hyperbolic Coxeter systems act properly discontinuously on affine and hyperbolic spaces correspondingly. Allowing this greater freedom in the arrangement of generating reflections permits us to include some non-Coxeter groups. Moreover this gives us a uniform way to deal with Coxeter
groups, because the $n$-dimensional affine or hyperbolic spaces can be embedded in a $(n + 1)$-dimensional vector space, where the reflections of the one are reflections of the other.

The braid group transforms equivalence classes of arrangement matrices $B$ as well as ordered sets of reflections. So we have an action of the braid group on an ordered $n$-tuple of reflections and on the equivalence classes of arrangement matrices. Finite orbits of reflections imply finite orbits of configuration matrices called onward $B$-orbits, but the opposite isn’t necessarily true. As it was stated before we will consider only symmetric configurations $B_{ij} = B_{ji}$. The freedom (15) for symmetric matrices is restricted:

$$B'_{ij} = \frac{\lambda_i}{\lambda_j} B_{ij} = B'_{ji} = \frac{\lambda_j}{\lambda_i} B_{ij} \Rightarrow \lambda_i^2 = \lambda_j^2 = \lambda_1^2 = 1 \Rightarrow \lambda_i = \pm 1 \quad (16)$$

The action of the braid group on the ordered sets of reflections induces action on the space of symmetric arrangement matrices given by

$$[\sigma_i(B)]_{ij} = B_{i+1,j} - B_{i,i+1} B_{i,j}, \quad j \neq i, i + 1$$
$$[\sigma_i(B)]_{i+1,j} = B_{i,j}, \quad j \neq i$$
$$[\sigma_i(B)]_{i,i+1} = -B_{i,i+1}$$
$$[\sigma_i(B)]_{jj} = B_{jj}, \quad k \neq i, i + 1. \quad (17)$$

These transformations can be written in a compact form

$$\sigma(B) = K_\sigma(B) \cdot B \cdot K_\sigma(B), \quad (18)$$

where the symmetric matrices $K_\sigma(B)$ for the canonical generators of the braid group are

$$(K_\sigma(B))_{jk} = \delta_{jk} - \delta_{ij} \delta_{jk} (1 + B_{i,i+1}) - \delta_{i+1} \delta_{ij} + \delta_{ij} \delta_{i+1,k} + \delta_{i,k} \delta_{i+1,j}. \quad (19)$$

The same action (18) can be defined on upper triangular matrices with ones on the diagonal and it coincides with the action of the braid group on Stokes matrices. Indeed the (anti-)symmetrization of Stokes matrices and the action of the braid group commute

$$\sigma(S \pm S^T) = (S \pm S^T) \quad (20)$$

The antisymmetrized Stokes matrix $A = S - S^T$ can be interpreted as an arrangement matrix of “symplectic” pseudo-reflections, preserving an antisymmetric form of highest rank. A symplectic pseudo-reflection, called also transvection, can be defined as a linear transformation, fixing point-wise a hyperplane of codimension 1 and having all eigenvalues equal to one. These requirements imply that the Jordan canonical form of such transformation is

$$p = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & 1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \quad (21)$$
and it can be written as
\[ p = \mathbb{1} + v \otimes v^\vee \quad v \in V \setminus \{0\}, v^\vee \in V^\ast \setminus \{0\} \quad v^\vee(v) = 0. \] (22)

The relative positions of \( n \) such pseudo-reflections is given by their arrangement matrix
\[ A_{ij} = v_i^\vee(v_j), \quad p_i = \mathbb{1} + v_i \otimes v_i^\vee, \quad A_{ii} = v_i^\vee(v_i) = 0. \] (23)

If there is a preserved antisymmetric bilinear form, the matrix \( A \) can be taken antisymmetric. Note that for odd-dimensional cases, the antisymmetric form is always degenerate so there is a subspace invariant under all pseudo-reflections. This allows a reduction by one of the dimension of a space in which an odd number of symplectic pseudo-reflections stay in generic position. The transformations in the reduced space will be no more pseudo-reflections.

3 Presentations of arrangements of reflections

As we have seen \( n \) reflections in a linear space \( V \) are determined by \( n \) pairs \((v_i,v_i^\vee)\). To such ordered sets of vectors and covectors we associate an arrangement matrix \( B_{ij} = v_j^\vee(v_i) \). Here we will reconstruct the reflections from the arrangement matrix. We call this procedure a realization of the matrix, and we aim to examine how many essentially different realizations as arrangements of reflections allows a given matrix \( B \).

3.1 Construction of an arrangement from its matrix

We introduce notions of reducibility and decomposability of arrangement matrices in a similar fashion to the theory of group representations.

**Definition 2.** Arrangement matrix \( B \) is called decomposable if there is a permutation matrix \( \Lambda \) such that
\[ \Lambda B \Lambda^{-1} = \begin{pmatrix} B_{1}^{(1)}_{k \times k} & 0 \\ 0 & B_{n-k \times n-k}^{(2)} \end{pmatrix} \] (24)

Otherwise the matrix is called indecomposable.

**Definition 3.** Arrangement matrix \( B \) is called reducible if there is a permutation matrix \( \Lambda \) such that
\[ \Lambda B \Lambda^{-1} = \begin{pmatrix} B_{1}^{(1)}_{k \times k} & 0 \\ B_{n-k \times k}^{(3)} & B_{n-k \times n-k}^{(2)} \end{pmatrix} \] (25)

Otherwise the matrix is called irreducible.
For symmetrizable matrices both notions coincide. It is easy to see that an arrangement of reflections \( r_1, \ldots, r_n \) will have decomposable matrix if and only if there is a proper subset \( S \subset \{ r_1, \ldots, r_n \} \) of reflections, commuting with the remaining ones
\[
    r_i \in S, r_j \notin S \Rightarrow r_i r_j = r_j r_i.
\]
(26)
It is clear that the action of the braid group (12) will preserve this property. In the classification of finite \( B \)-orbits we may restrict our attention only to indecomposable matrices because a decomposable matrix will have finite orbit only if its building indecomposable blocks have finite orbits.

Given \( n \) reflections as above we consider the group \( G \) they generate. As a linear group its presentation will be indecomposable if the arrangement is, provided that there is not a subspace, on which the group \( G \) acts trivially. We will fix the space \( V \) of the representation to be the minimal possible i.e. we will avoid as much as possible the existence of a subspace on which \( G \) acts trivially without changing the reflections.

**Remark.** Although for a symmetric arrangement matrix the properties irreducible and indecomposable coincide it isn’t necessarily true for the linear group \( G \), which the reflections generate. For example
\[
    r_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad r_2 = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}
\]
(27)
are reflections and the group they generate consists of matrices of the form
\[
    g = \begin{pmatrix} \pm 1 & 0 \\ k & 1 \end{pmatrix}, \quad k \in \mathbb{Z},
\]
(28)
and it is isomorphic to the infinite dihedral group \( I_2(\infty) \). Therefore the presentation is indecomposable but reducible. The arrangement matrix for \( r_1, r_2 \) is also indecomposable
\[
    B = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.
\]
(29)

With this example in mind we define minimality for \( V \) by:

**Definition 4.** The linear space \( V \) is minimal for the set of reflections \( r_1, \ldots, r_n \), \( r_i = I - v_i \otimes v_i^\vee \) specified by \( n \) pairs \((v_i, v_i^\vee)\), \( v_i \in V, v_i^\vee \in V^*, v_i^\vee(v_i) = 2 \), if the following holds true for all vectors \( v \in V \)
\[
    \forall i \quad v_i^\vee(v) = 0 \Rightarrow v \in \text{span}(v_1, \ldots, v_n).
\]
(30)
In other words every vector in \( V \) must either be moved by at least one of the reflections or be a linear combination of the given \( n \) vectors.

To any given arrangement of reflections in the space \( V \) there is a naturally associated dual arrangement in the dual space \( V^* \) obtained by exchanging the places of \( v_i \)-s and \( v_i^\vee \)-s.
Lemma 5. $V$ is minimal for $(v_i, v_j)_{i,j \in N}$ if and only if $V^*$ is minimal for $(v_i^*, v_j)_{i,j \in N}$

Proof. Assume $V$ is not minimal so $\{v_j\}_{j \in J} \cup \{w_k\}_{k \in K}$ is a basis in $V$ and there is some $w_k$ such that $v_i^t w_k \equiv 0$. Now let $\{u_j^i\}_{j \in J} \cup \{w_k^i\}_{k \in K}$ be the dual basis in $V^*$.

$$v_i^t w_k \equiv 0 \Rightarrow v_i^t \in \text{span}(\{u_j^i\}, \{w_k^i\} \setminus k) \Rightarrow w_k^i \not\in \text{span}(v_i^t).$$

$w_k^i v_j \equiv 0$ hence $V^*$ is not minimal for $(v_i^*, v_j)$. \hfill \qed

We will do all calculations for arrangements of reflections in their minimal space to avoid unnecessary complications.

Definition 6. Two arrangements $\{v_i \in V\}, \{v_i^\prime \in V^\prime\}_{i=1,..,n}$ and $\{v_i^\prime \in V^\prime\}_{i=1,..,n}$ are isomorphic if there is an isomorphism $i : V \to V^\prime$ mapping $v_i$ to $v_i^\prime$ while the pull-back $i^*: V^* \to V^\prime$ maps $v_i^\prime$ to $v_i^\prime$.

Lemma 7. A non-degenerate matrix $B$ allows up to isomorphism exactly one realization as an arrangement of reflections.

Proof. As $B_{ij} = v_i^t(v_j)$ is invertible, the vectors $v_j$ are linearly independent and so are $v_i^t$. Let $N = \{1, \ldots, n\}$, and $\{v_i^t\}_{i \in N} \cup \{w_k^t\}_{k \in K}$ be a basis in $V^*$. Denote the dual basis in $V$ by $\{u_i\}_{i \in N} \cup \{t_k\}_{k \in K}$.

$$v_j = \sum_{i \in N} a_{ji} u_i + \sum_{k \in K} a'_{jk} t_k$$

$$v_j^t(v_j) = \sum_{i \in N} a_{ji} v_j^t(u_i) + \sum_{k \in K} a'_{jk} v_j^t(t_k) = a_{ji} = B_{ij}$$

$$\Rightarrow v_j = \sum_{i \in N} B_{ij} v_i + \sum_{k \in K} a'_{jk} t_k$$

$$\sum_{j \in N} B_{ji}^{-1} v_j = \sum_{i \in N} B_{ji}^{-1} B_{ij} v_i + \sum_{j \in N, k \in K} B_{ji}^{-1} a'_{jk} t_k$$

$$\Rightarrow u_i = \sum_{j \in N} B_{ji}^{-1} v_j - \sum_{j \in N, k \in K} B_{ji}^{-1} a'_{jk} t_k$$

therefore $\{v_j\}_{j \in N}, \{t_k\}_{k \in K}$ is also basis in $V$. By definition $v_i^t(t_k) \equiv 0$ and from the minimality of $V$ follows $v_j \in \text{span}(v_i)$ which is impossible. We conclude $K = \emptyset$. \hfill \qed

For degenerate matrices there appear several possibilities for non-isomorphic realizations as $v_j$ could be linearly independent and $v_j^t$ linearly dependent with rank equal to the rank of $B$, or the opposite, or $v_j$-s could be linearly dependent with greater rank than $B$.

The formal treatment in the remaining part of this section will be without assuming symmetrizability of the arrangement matrices. Let’s denote by

$$B_i = (B_{i1}, B_{i2}, \ldots, B_{in}) \quad B_j = (B_{1j}, B_{2j}, \ldots, B_{nj})$$
the rows and columns of $B$. Let $\{B_i\}_{i\in I}$ be a basis in $\text{span}(B_i)$ for some $I \subset \{1,2,\ldots,n\}$. This subset $I$ is non-unique for $0 < \text{rank}(B) < n$. In the same way $\{B_j\}_{j\in J}$ is a basis in the span of columns of $B$ for some $J \subset N := \{1,2,\ldots,n\}$. $|I| = |J| = r := \text{rank}(B)$, $\{v_i^\gamma\}_{i\in I}$ and $\{v_j\}_{j\in J}$ are linearly independent.

$$B_i = \sum_{i_1 \in I} a_{i_1i} B_{i_1}, \quad \text{if } i \in N \setminus I \quad B_j = \sum_{j_1 \in J} b_{j_1j} B_{j_1}, \quad \text{if } j \in N \setminus J$$  \hspace{1cm} (36)

**Theorem 8.** Any degenerate matrix $B$ of rank $r$ allows non-unique realization as an arrangement of reflections. To specify a unique (up to isomorphism) arrangement one must say which of the vectors $\{v_j\}_{j\in J}$ and the covectors $\{v_i^\gamma\}_{i\in I}$ are linearly independent. These sets $I''$ and $J''$ must include sets $I$ and $J$ of indices of rows and columns of $B$ forming bases in the span of all rows and columns. Such subsets $I''$, $J''$ may be chosen in $2^{2(n-r)}$ different ways.

Additionally one must fix $(n - |I''|)(|I''| - r) + (n - |J''|)(|J''| - r)$ arbitrary constants in order to specify a unique realization of $B$. The dimension of the minimal space for this arrangement is $|I''| + |J''| - r$.

We have already chosen the sets $I, J$. As $\{v_i^\gamma\}_{i\in I}$ are linear independent we may complement them by $\{v_i^\gamma\}_{i \in I'}$ to a basis in $\text{span}(v_i^\gamma)$. There are $2^{n-r}$ possibilities for the set $I'$. Analogously let $\{v_j\}_{j \in J'}$ form a basis in $\text{span}(v_j)$.

The remaining vectors and covectors are expressed through these

$$v_i^\gamma = \sum_{i_1 \in I \cup J'} a_{i_1i} v_i^{\gamma 1}, \quad i \in N \setminus (I \cup J') \quad v_j = \sum_{j_1 \in J \cup J'} b_{j_1j} v_j, \quad j \in N \setminus (J \cup J').$$  \hspace{1cm} (37)

The coefficients $a_{i_1i}$ must satisfy

$$B_{ij} = v_i^{\gamma} (v_j) = \sum_{i_1 \in I \cup J'} a_{i_1i} v_i^{\gamma} (v_j) = \sum_{i_1 \in I \cup J'} a_{i_1i} B_{i_1j} = \sum_{i_1 \in I} a_{i_1i} B_{i_1j}$$  \hspace{1cm} (38)

for $i \in N \setminus (I \cup J') j \in J$.

The matrix $\tilde{B} = (B_{ij})_{i \in I, j \in J}$ is invertible and for ease of notation we will write $B_{ji}^{-1}$ instead of $\tilde{B}_{ji}^{-1}$. Care must be taken as $B_{ji}^{-1}$ is defined only for $i \in I, j \in J$ and

$$\sum_{j \in J} B_{i_1j} B_{ji_2}^{-1} = \left\{ \begin{array}{ll} a_{i_1i_2} B_{i_1j} B_{ji_2}^{-1} = a_{i_1i_2} & i_1 \in I \\ \sum_{j \in J, j_3 \in I} a_{i_1i_3} B_{i_3j} B_{ji_2}^{-1} = a_{i_1i_2} & i_1 \in N \setminus I \end{array} \right. \hspace{1cm} (39)$$

$$\sum_{i \in I} B_{ji_1}^{-1} B_{ji_2} = \left\{ \begin{array}{ll} \delta_{ji_1} B_{ji_2}^{-1} B_{ji_3} B_{ji_3}^{-1} = b_{ji_2} & j_2 \in J \\ \sum_{i \in I, j_3 \in J} B_{ji_1}^{-1} B_{ji_3} B_{ji_3}^{-1} = b_{ji_2} & j_2 \in N \setminus J \end{array} \right. \hspace{1cm} (40)$$

After multiplying (38) by $B_{ji_2}^{-1}$ and summing over $j \in J$

$$a_{i_2i} = \sum_{i_1 \in I \cup J'} a'_{i_1i} B_{i_1j} B_{ji_2}^{-1} = a'_{i_2i} + \sum_{i_1 \in I'} a'_{i_1i} a_{i_1i_2}$$  \hspace{1cm} (41)
so

\[ a'_{i i_2} = a_{i i_2} - \sum_{i_1 \in I'} a'_{i_1 i} a_{i_1 i_2}. \]  

(42)

The coefficients \( a'_{i i_1} i_1 \in I' \) are independent and the remaining \( a'_{i i_2} i_2 \in I \) are calculated from them and the matrix \( B \).

Let \( \{v_i^\vee\}_{i \in I \cup J'} \cup \{w_k\}_{k \in K} \) be a basis in \( V^* \). Denote \( \{u_i\}_{i \in I \cup J'} \cup \{w_k\}_{k \in K} \) the dual basis in \( V \).

\[ v_j = \sum_{i_1 \in I \cup J'} c'_{i_1 j} u_{i_1} + \sum_{k \in K} c_{kj} w_k \quad \Rightarrow v_j = \sum_{i_1 \in I \cup J'} B_{i_1 j} u_{i_1} + \sum_{k \in K} c_{kj} w_k \]  

(43)

(44)

Multiplying by \( B_{j i_2}^{-1} \) and summing over \( j \in J \)

\[ u_{i_2} = \sum_{j \in J} B_{j i_2}^{-1} v_j - \sum_{i_1 \in I'} a_{i_1 i_2} u_{i_1} - \sum_{k \in K, j \in J} c_{kj} B_{j i_2}^{-1} w_k \]  

(45)

therefore \( \{v_j\}_{j \in J}, \{u_i\}_{i \in I'}, \{w_k\}_{k \in K} \) is also a basis in \( V \).

After substituting (45) in (44) for \( j \in J' \) we obtain

\[ v_j = \sum_{j_1 \in J} b_{j_1 j} v_{j_1} + \sum_{k \in K} d_{kj} w_k \]  

(46)

where the coefficients \( d_{kj} \) are obtained from \( c_{kj} \). The vectors \( w_k \) were not fixed up to now so we may use any other basis in \( \text{span}(w_k) \). As \( v_j, j \in J \cup J' \) are linearly independent by assumption, \( \tilde{w}_j = v_j - \sum_{j_1 \in J} b_{j_1 j} v_{j_1}, j \in J' \) are linearly independent and belong to \( \text{span}(w_k) \) by (46). We choose another basis in \( \text{span}(w_k) \) so that \( w_\ell = \tilde{w}_j, j \in J' \) identifying a subset of \( K \) with \( J' \). Remark that nothing was said about the set \( K \) till now, and we may assume it is a superset of \( J' \). Complete the basis in \( \text{span}(w_k) \) with \( \{w_k\}_{k \in K \setminus J'} \). By assumption \( v_i^\vee(w_k) \equiv 0 \) but \( w_k \notin \text{span}(v_j) \) for \( k \in K \setminus J' \) contradicting the minimality condition on \( V \) hence \( K \setminus J' = \emptyset \).

**Corollary 9.** Given the matrix \( B \) of size \( n \) and rank \( r \), the sets \( I'', J'' \) and the constants \( a'_{i i_1}, b'_{j_1 j}, c_{kj} \) for \( i_1 \in I'', i \in N \setminus I'', j_1 \in J'', j \in N \setminus J'' \), where \( I'' = I \cup J', J'' = J \cup J' \); the reflection arrangement can be build in the following way:

Let the basis vectors in the space \( V^* \) be \( \{v_i^\vee\}_{i \in I'} \) and \( \{w_k^\vee\}_{k \in J'} \). Denote the vectors of the dual basis in \( V \) by \( \{u_i\}_{i \in I'}, \{w_k\}_{k \in J'} \). The vectors in the
\( v_j = \sum_{i \in I} B_{ij} u_i, \quad j \in J \)  
\( v_j = \sum_{j_1 \in J} b_{j_1 j} v_{j_1} + w_j, \quad j \in J' \)  
\( v_j = \sum_{j_1 \in J''} u'_{j_1 j} v_{j_1}, \quad j \in N \setminus J'' \)  
\( v_i^\vee = \sum_{i_1 \in I''} a'_{i i_1} v_{i_1}^\vee, \quad i \in N \setminus I'' \)  

Example. The simplest example allowing demonstration of the above construction with most of the features is the 3 \( \times \) 3 configuration matrix of rank 1:

\[
B = \begin{pmatrix}
2 & 2 & 2 \\
2 & 2 & 2 \\
2 & 2 & 2 \\
\end{pmatrix}
\]

We fix \( I = J = \{1\} \). There are 16 variants to choose subsets \( I', J' \subset \{2, 3\} \). If we let \( I' = J' = \{2\} \) there will be 2 constants to determine completely an arrangement. Calling them \( a, b \) we have

\[
v_3^\vee = (1 - a) v_1^\vee + a v_2^\vee, \quad v_3 = (1 - b) v_1 + b v_2.
\]

The dimension of the minimal space is \( |I''| + |J''| - r = 3 \). We may take the standard basis vectors in \( \mathbb{R}^3 \) to be \( v_1, v_2, w \)

\[
v_1 = (1, 0, 0)^T, \quad v_2 = (0, 1, 0)^T, \quad w = (0, 0, 1)^T
\]

The covectors \( v_1^\vee, v_2^\vee \) must obey the arrangement matrix. We take

\[
v_1^\vee = (2, 2, 0), \quad v_2^\vee = (2, 2, 1)
\]

The three reflections in this arrangement are

\[
r_1 = \begin{pmatrix}
-1 & -2 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad r_2 = \begin{pmatrix}
1 & 0 & 0 \\
-2 & -1 & -1 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
r_3 = \begin{pmatrix}
2b - 1 & 2b - 2 & ab - a \\
-2b & 1 - 2b & -ab \\
0 & 0 & 1
\end{pmatrix}
\]

This gives also an example of non-Coxeter group. One may easily check that \( r_i r_j \) has infinite period if \( i \neq j \), but whenever \( a = b \), the product \( r_1 r_2 r_3 \) is a reflection therefore having period 2. Further analysis shows that all elements in this group are either reflections or have infinite period. As an abstract group it has a presentation

\[
r_1^2 = r_2^2 = r_3^2 = (r_1 r_2 r_3)^2 = 1
\]
which is clearly not Coxeter group (see [5, 13]). It is isomorphic to the group of congruence transformations on \( \mathbb{R} \), preserving a set of points with coordinates \( x + y\sqrt{2}, \quad x, y \in \mathbb{Z} \). This is easy to see as the congruence transformations on a line are only reflections and translations; and three reflections about points with coordinates 0, 1, \( \sqrt{2} \) generate all such transformations.

**Remark.** The dimension of the minimal space of the reflection arrangements having a given arrangement matrix \( B \) of size \( n \) and rank \( r \) can be any integer from \( r \) to \( 2n - r \). We call the **minimal realization** this arrangement, whose minimal space has dimension \( r \). Analogously we call the **maximal realization** of the matrix \( B \) this arrangement of reflections, whose minimal space has dimension \( 2n - r \). The minimal and maximal realizations are unique for every matrix \( B \).

In the applications we are interested, there is a non-degenerate bilinear form, preserved by all reflections. As we have said in this case the matrix \( B \) is symmetric.

**Lemma 10.** Any the reflection \( r \) preserving a non-degenerate symmetric bilinear form \(( , )\) has the form \( r = \mathbb{I} - (v, \cdot) \), where \( v \) is a vector satisfying \((v, v) = 2\).

**Proof.** Writing as before \( r = \mathbb{I} - v \otimes v^\vee \) the invariance means
\[
(u, w) = (r(u), r(w)) = (u, w) - (v^\vee(u)v, w) - (u, v^\vee(w)v) + (v^\vee(u)v, v^\vee(w)v)
\]
for any pair of vectors \( u, w \). If \( v^\vee(u) = 0 \) and \( v^\vee(w) \neq 0 \), it must hold \((u, v^\vee(w)v) = 0 = (u, v)\). Let us now substitute \( w = u \) for a vector \( u \), for which \( v^\vee(u) \neq 0 \). In this case \((v^\vee(u))^2(v, v) = 2v^\vee(u)(v, u)\) therefore \( v^\vee(u) = \frac{2(v, u)}{(v, v)} \) provided \((v, v) \neq 0\). But if \((v, v) = 0\), it must hold \((v, u) = 0\) for any \( u \) which contradicts the non-degeneracy of the form \(( , )\). It follows that always \( v^\vee = \frac{2(v, \cdot)}{(v, v)} \) and we may rescale \( v \) to make the denominator equal to 2.

In this case the arrangement matrix \( B_{ij} = v_i^\vee(v_j) = (v_i, v_j) \) is the Gram matrix of the vectors \( v_i \). To recover the arrangement from this matrix we may use the above construction, taking into account that now we have only \( n \) vectors and a natural isomorphism between \( V \) and \( V^* \).

\[
v_i = \sum_{j \in I, I'} a'_{ij} v_j \quad B_{ik} = \sum_{j \in I} B_{jk}
\]

The following identities must hold
\[
a'_{ij} = a_{ij} - \sum_{k \in I'} a'_{ik} a_{kj}, \quad i \notin I \cup I', \quad j \in I
\]
so there are \((n - |I| - |I'|)|I'|\) independent parameters \( a'_{ij} \) which must be specified along with the matrix \( B \) and the subsets \( I, I' \) to fix a unique set of reflections up to simultaneous conjugation. There must be some additional vectors \( \{w_j\}_{j \in I'} \), which together with \( \{v_i\}_{i \in I \cup I'} \) form a basis in the minimal space \( V \). A convenient choice of \( w_j \) is one, for which \((v_i, w_j) = \delta_{ij}\).
3.2 Representation of the arrangement matrix by a graph

At this point arises the question how transforms the arrangement matrix under the action of the braid group in different realizations.

\[
\sigma_i(B)_{ij} = B_{i+1,j} - B_{ij}
\]

hence

\[
\sigma_i(B)_{i+1,j} = B_{ij}
\]

\[
\sigma_i(B)_{i,i+1} = -B_{i,i+1}
\]

\[
\sigma_i(B)_{k,j} = B_{kj} \text{ for } k \neq i, i+1.
\]

We see that although if \( B \) is degenerate it can have different non-isomorphic realizations as reflection arrangements, it transforms uniformly by the braid group. This key observation will allow us later to classify the finite orbits arising from finite groups as well as those arising from infinite groups.

For the sake of visualization we will represent the matrix \( B \) by a graph \( \Gamma \) with vertices \( \nu_i, i = 1, \ldots, n \) and labeled edges \( (\nu_i, \nu_j) \in \text{Edge}(\Gamma) \) with labels \( g(\nu_i, \nu_j) = \pm \frac{n}{k} \) if \( B_{ij} = \pm 2 \cos \frac{\pi k}{n}, 0 < \frac{k}{n} < \frac{1}{2} \). This restriction on the possible values in \( B_{ij} \) is necessary when we consider matrices from finite orbits of the braid group as we will see later. Indecomposable arrangement matrices have connected graphs. In analogy with Dynkin diagrams we omit the positive signs and write only the negative ones. Remember that the angles between simple roots are non-acute therefore all non-diagonal entries in a Cartan matrix are non-positive. We should always take into account the identification (15) of graphs and matrices representing the same reflection configuration. In particular it makes redundant the signs when the graph is a tree or more than one negative signs when the graph contains one cycle. In graphs we always abbreviate \( \frac{5}{2} \) to \( 5' \) as it is the only fraction to appear.

When investigating reflection arrangements generating given Coxeter group there are considered certain “universal” graphs without indexing of the vertices which will be called unindexed:

\[
\Gamma = \{V, E, g\}, V = \{v_1, \cdots, v_n\}, E \subseteq \{v_i, v_j\}, v_i, v_j \in V, \quad g : E \rightarrow \{\pm \frac{n}{k}\}_{0 < 2k < n}.
\]

Indexing of the vertices is equivalent to their linear ordering:

\[
\Gamma = \{V, E, g, \prec\}
\]

Two graphs, which differ only on ordering of their vertices will be called similar. By a subgraph \( \Gamma' \) of the graph \( \Gamma \) will be understood

\[
\Gamma' = \{V', E', g', \prec'\}, V' \subseteq V, E' = \{v_i, v_j\} \in E, v_i, v_j \in V', \quad g' = g|_{E'} , \prec' = \prec|_{V'}.
\]
We call the graph $\Gamma$ an extension of $\Gamma'$ by $#V - #V'$ vertices and $#E - #E'$ edges.

Graphs corresponding to invertible arrangement matrices will be called non-degenerate and those corresponding to singular arrangement matrices – degenerate.

### 3.3 Invariants of the action of the braid group

As it is seen in (12) the element

$$C = g_1 g_2 \cdots g_k g_{k+1} \cdots g_n = g_1 g_2 \cdots g_k g_{k+1}^{-1} g_k \cdots g_n$$  \hspace{1cm} (66)

is an invariant of the action. For the canonical generators of Coxeter groups $C$ is called Coxeter element and in our case of arbitrary set of reflections it will be called quasicoxeter element. As the matrix $B$ specifies only the relative positions of the reflections $r_1, r_2, \ldots r_n$ we see that the conjugation class of $C$ discriminates the different orbits of $B_n$ acting on the matrix $B$.

We proceed with expressing $C$ by $B$

$$C = \prod_{i=1}^{n} (1 - v_i \otimes v_i^\vee) = \mathbb{1} - \sum_{k=1}^{n} \sum_{1 \leq i_1 < i_2 < \cdots i_k \leq n} (-1)^{k+1} v_{i_1} \otimes v_{i_1}^\vee (v_{i_2}) v_{i_2}^\vee (v_{i_3}) \cdots v_{i_{k-1}}^\vee (v_{i_k}) v_{i_k}^\vee$$

$$= \mathbb{1} - \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \cdots i_k \leq n} B_{i_1 i_2} B_{i_2 i_3} \cdots B_{i_{k-1} i_k} v_{i_1} \otimes v_{i_k}^\vee.$$  \hspace{1cm} (67)

The last expression may be simplified by the following trick

$$\sum_{i < i_1 < i_2 < \cdots < i_k < j} B_{i_1 i_2} \cdots B_{i_k j} = (U^{k+1})_{ij} \text{ where } U_{ij} = \begin{cases} B_{ij} & i < j \\ 0 & i \geq j \end{cases}.$$  \hspace{1cm} (68)

The matrix $U$ is nilpotent so

$$C = \mathbb{1} - \sum_{i,j=1}^{n} \sum_{k=0}^{\infty} (-1)^{k} U^{k+1} v_j \otimes v_i^\vee = \mathbb{1} - \sum_{i,j=1}^{n} (\delta + U)^{-1} v_j \otimes v_i^\vee.$$  \hspace{1cm} (69)

If the matrix $B$ is non-degenerate $\{v_j\}$ form a basis in $V$. Denoting the dual basis in $V^*$ by $\{u_j^\vee\}$ we have

$$C = \sum_{i,j=1}^{n} \left[ \delta - (\delta + U)^{-1} B \right]_{ji} v_j \otimes u_i^\vee.$$  \hspace{1cm} (70)

Introducing $V = B - U$ and expressing $B$ through $U, V$ we obtain

$$C = \sum_{i,j=1}^{n} \left[ (\delta + U)^{-1} (\delta - V) \right]_{ji} v_j \otimes u_i^\vee.$$  \hspace{1cm} (71)
Whenever \( \text{rank}(B) = r < n \) the basis in \( V^* \) is \( \{ v_i^\vee \}_{i \in I \cup I'} \cup \{ w_j^\vee \}_{j \in J'} \) and the dual basis in \( V \) is \( \{ u_i \} \cup \{ w_j \} \).

\[
v_i^\vee = \sum_{i \in I \cup I'} a_{i_1}^i v_{i_1}^\vee \quad v_j = \begin{cases} \sum_{i \in I \cup I'} B_{i_1 j} u_{i_1} & j \in J \\ w_j + \sum_{i \in I \cup I'} B_{i_1 j} u_{i_1} & j \in J' \\ \sum_{i \in I \cup I'} B_{i_1 j} b'_{j_1 j} u_{i_1} + \sum_{j_1 \in J \cup J'} b'_{j_1 j} w_{j_1} & \text{else.} \end{cases}
\]

(72)

To write a compact formula it is best to extend the definition of \( a'_{i_1 i} \), \( b'_{j_1 j} \) to all subscripts by

\[
a'_{i_1 i} = \begin{cases} \delta_{i_1 i} & i \in I \cup I' \\ 0 & i_1 \notin I \cup I' \end{cases} \quad b'_{j_1 j} = \begin{cases} \delta_{j_1 j} & j \in J \cup J' \\ 0 & j_1 \notin J \cup J'. \end{cases}
\]

(73)

Substituting \( v_i^\vee, v_j \) in (69) we finally obtain

\[
C = \sum_{i_1, i_2 \in I \cup I'} (\delta - Bb'(-1)\delta + U)^{-1} a'_{i_1 i_2} u_{i_1} \otimes v_{i_2}^\vee + \sum_{j \in J'} w_j \otimes w_j^\vee
\]

\[- \sum_{j \in J'} (Bb'(-1)\delta + U)^{-1} a'_{j_1 j} w_j \otimes v_j^\vee.
\]

(74)

4 Two and three reflections

From this point on we will consider only symmetric arrangement matrices. The first non-trivial case to study are the orbits of the braid group action on configuration matrices of three reflections as the action is trivial on \( 2 \times 2 \) matrices.

\[
B = \begin{pmatrix} 2 & a & b \\ a & 2 & c \\ b & c & 2 \end{pmatrix}
\]

(75)

The canonical generator \( \sigma_1 \) of the braid group act on this matrix by

\[
\sigma_1(B) = \begin{pmatrix} 2 & a & ab - c \\ a & 2 & b \\ ab - c & b & 2 \end{pmatrix}.
\]

(76)

This is a linear transformation on the pair \( b, c \) and must have finite period if the orbit of \( B_3 \) is finite. This will take place only if the eigenvalues of \( \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} \) are roots of unity so expressing \( a \) we obtain

\[
a = 2 \cos \alpha, \quad \alpha \in \pi \mathbb{Q}.
\]

(77)

As it can be seen in (12), the braid \( \sigma_1 \sigma_2 \cdots \sigma_{n-1} \) permutes cyclically \( g_1, g_2, \ldots, g_n \) and conjugates them by \( g_1 \). The configuration matrix \( B \) remains unchanged.
upon simultaneous conjugation of the reflections therefore the braid \( \sigma_1 \sigma_2 \) permutes \( a, b, c \) cyclically. It follows that all \( a, b, \) and \( c \) must be twice the cosines of rational parts of the straight angle.

If the matrix \( B \) is degenerate it must have the form

\[
B = \begin{pmatrix}
2 \cos \alpha & 2 \cos \alpha & 2 \cos \beta \\
2 \cos \beta & 2 \cos \beta & 2 \cos(\alpha + \epsilon \beta) \\
2 \cos \beta & 2 \cos(\alpha + \epsilon \beta) & 2
\end{pmatrix}, \quad \epsilon = \pm 1. \tag{78}
\]

Changing \( \beta \mapsto -\beta \) we may assure \( \epsilon = -1 \). The generators of the braid group transform parameters \( \alpha, \beta \) by

\[
\sigma_1 : \begin{cases}
\alpha \mapsto \alpha \\
\beta \mapsto \alpha + \beta
\end{cases} \quad \sigma_2 : \begin{cases}
\alpha \mapsto \alpha - \beta \\
\beta \mapsto \alpha
\end{cases}. \tag{79}
\]

Returning to the initial parameters \( a, b, c \) we must identify \( \alpha, \beta \) modulo \( 2\pi \).

One may see that the condition \( \alpha, \beta \in \pi \mathbb{Q} \) is sufficient for the orbit of \( B_3 \) to be finite. Such matrix represents, in the minimal realization, a redundant set of generators of some finite dihedral group \( I_2(k) \).

The case of non-degenerate symmetric \( 3 \times 3 \) matrix \( B \) is considered in [10] where it was proved that the only finite orbits come from matrices representing configurations of reflections generating finite three-dimensional Coxeter groups.

There is a one-to-one correspondence between the conjugacy classes of quasicoxeter elements in them and orbits of the braid group. Even more appealing is the correspondence between the orbits and the pairs of reciprocal regular polyhedra and star-polyhedra in the three dimensional space ([6]).

Summarizing, the finite orbits of the braid group on \( 3 \times 3 \) configuration matrices represent reflections, generating finite groups. In case of degenerate matrix there is a realization of it where one of the reflections belongs to the group generated by the other two.

In the classification of all finite orbits of the braid group action on arrangement matrices we will follow an inductive procedure for which the next lemma is essential. We always identify a graph with the arrangement matrix it represents.

**Lemma 11.** A graph \( \Gamma \) containing a subgraph \( \Gamma' \) which has infinite orbit under the action of the braid group has an infinite orbit itself.

**Proof.** Let the vertices of \( \Gamma \) be numbered \( 1, 2, \ldots, n \) and those of \( \Gamma' \) when ordered \( 1 \leq i(1) < i(2) < \cdots < i(k) \leq n \). The braid \( \sigma_{i-1} \sigma_{i-2} \cdots \sigma_{i(k)} \) moves the \( i(k) \)-th reflection to the last position. Acting by \( \sigma_{i-2} \sigma_{i-3} \cdots \sigma_{i(k-1)} \), will bring the \( i(k-1) \)-th reflection to next to the last position leaving the last unchanged. Continuing in the same manner we may bring all vertices of \( \Gamma' \) to consecutive numbers without changing the subgraph \( \Gamma' \). It is clear that a subgroup of \( B_n \) will have an infinite orbit when acting on the obtained graph.

In the remaining part we will investigate of the finite orbits along the following lines:
First are found the orbits of the braid group on non-redundant sets of generators in finite Coxeter groups. These are necessarily finite because there are only finite number of combinations of generators in a finite group. As it is not obvious how to find all possible such sets we use an inductive argument: choosing a special configuration of generators in one orbit of arrangements, generating a given group $G_n$ with $n$ generators we find how can be added one reflection to obtain a bigger group $G_{n+1}$. There are found “universal” sets of generators in each group. These are very symmetric reducing the number of ways an additional reflection can be added. They also allow to obtain representatives in all orbits coming from the given group by simple permutations of the reflections in them. Such universal graphs exist for all groups except $H_3$, $H_4$, $E_8$ and $I_2(k)$, $k = 5$ or $k \geq 7$.

The extensions by one vertex of the universal graphs are studied. For the groups without universal graphs are used sufficient samples of quasi-universal graphs. It is shown that if the extension does not contain a degenerate subgraph, in order to stay in a finite orbit it must either be degenerate or represent generators in a finite Coxeter group. For the remaining extensions there is given a sequence of braid transformations which produce a subgraph not belonging to a finite orbit. For each degenerate extension it is demonstrated that in the minimal realization $\dim(V) = \text{rank}(B)$ the extended graph represents a redundant set of generators in a finite Coxeter group.

Using these results it is proved that every arrangement graph in a finite orbit of the braid group represents, in its minimal realization, a set of (possibly redundant) generators in a finite Coxeter group provided there is not a number $k$ such that every subgraph with $k$ vertices to be degenerate, but the rank of $B$ to be higher than $k$. It is shown that such property is unstable under the action of the braid group which concludes the classification of arrangements in finite orbits.

The orbits themselves are classified only in the extremal case of invertible arrangement matrices. It is shown that the conjugacy class of the quasi-coxeter element determines completely the orbit of the braid group for non-degenerate configurations. The other extremal case of maximally singular rank=2 arrangement matrices is treated in [16] where additional invariants are introduced in order to distinguish orbits with the same quasi-coxeter element. In the case of arrangement matrices of intermediate rank this work gives a criterion of appurtenance to finite orbits.

5 Orbits on the generators of Coxeter groups

In this chapter are classified all orbits of the braid group action on non-redundant generating reflections in finite Coxeter groups. All the linear spaces will be sup-
plied with non-degenerate symmetric bilinear form for which the basis \( \varepsilon_1, \varepsilon_2 \ldots \varepsilon_n \) is orthonormal. This form gives a natural isomorphism between \( V \) and \( V^* \) therefore each reflection can be given by a nonzero vector \( v_r(u) = u - 2\frac{(v, u)}{(v, v)}v \). We will denote the root systems \( A_n, B_n, \ldots \) and the corresponding Weyl groups by the same letter and the meaning should be clear from the context. Sometimes, for distinction, we will denote by \( W(A_n), \ldots \) the Weyl groups and extend the same notation \( W(H_{3,4}) \) for the non-crystallographic Coxeter groups as well.

5.1 Orbits on the generators of the classical Coxeter groups \( A_n, B_n, D_n \)

The root system \( A_n = \{\varepsilon_i - \varepsilon_j, 1 \leq i \neq j \leq n+1\} \) generates a group \( W(A_n) \) isomorphic to the symmetric group \( S_{n+1} \) of permutations of the basis vectors \( \varepsilon_1 \ldots \varepsilon_{n+1} \). Reflections in \( W(A_n) \) correspond to transpositions in \( S_{n+1} \). In this way the question which reflections generate the group \( W(A_n) \) transforms to the question which transpositions generate the whole symmetric group \( S_{n+1} \).

We may represent a set of \( n \) transpositions on the \( n + 1 \) basis vectors by a graph \( \gamma \) with \( n + 1 \) edges, numbered \( 1, 2 \ldots n + 1 \) and edges corresponding to transpositions \( (i_k j_k) \). We call it the permutation graph in contrast to the arrangement graph \( \Gamma \).

**Lemma 12.** The necessary and sufficient condition transpositions \( (i_k j_k), k = 1 \ldots n \) to generate the whole symmetric group \( S_{n+1} \) is not to exist two disjoint proper subsets \( A, B \subset \{1 \ldots n+1\} \) such that both \( i_k, j_k \) to be either in \( A \) or in \( B \) for all \( k \). This is equivalent to connectedness of the permutation graph \( \gamma \).

**Proof.** The equivalence of the two conditions is immediate. If there exist two such subsets then any product of transpositions will permute \( A \) and \( B \) without mixing them. On the other hand if the graph is connected there always exist a path joining any pair of vertices. As \( (i_1 i_2)(i_2 i_3) \cdots (i_{l-1} i_l) = (i_1 i_l) \) every transposition can be expressed by the given transpositions hence they generate the whole symmetric group.

A connected graph with \( n + 1 \) vertices and \( n \) edges is a tree. The numeration of the vertices of this graph of transpositions is irrelevant to the relative positions of these transpositions. Therefore we have a correspondence between trees with numbered edges and the arrangements of reflections, generating the group \( W(A_n) \).

**Lemma 13.** The product of \( n \) transpositions generating the group \( S_{n+1} \) is a cycle of length \( n + 1 \).

**Proof.** For \( n = 1 \) the claim is trivial. Assume true for \( n \). We have

\[
(i_1 j_1)(i_2 j_2) \cdots (i_n j_n) = (k_1 k_2 \ldots k_{n+1}), \quad \{k_1 \ldots k_{n+1}\} = \{1 \ldots n + 1\} \quad (80)
\]

\[
(k_1 k_2 \ldots k_{n+1})(k_l n + 2) = (k_1 \ldots k_{l-1} k_{n+2} k_{l+1} \ldots k_{n+1} k_l) \quad (81)
\]
**Theorem 14.** There is only one orbit of the braid group action on non-redundant sets of reflections generating $W(A_n)$.

**Proof.** We will show that a suitable braid transforms any set of generators to a canonical one, whose permutation graph is linear, with edges numbered consecutively $1, 2, \ldots, n$. For $n = 1, 2$ the claim is trivial. Assume true for $n$. After ordering the first $n$ transpositions of the given graph one obtains the graph in Fig.2.

![Figure 2: Induction hypothesis](image)

Now acting with the braid $\sigma_k^2 \sigma_{k+1} \sigma_{k+2} \cdots \sigma_{n-1} \sigma_n$ is obtained a canonical graph with $n + 1$ edges. Proof follows by induction. \hfill \Box

Next group to be considered is $W(B_n)$. As it is known this is the group of permutations and sign changes of the basis vectors in $n$-dimensional Euclidean space so it is the semi-direct product $\mathbb{Z}^n \rtimes S_n$. Each reflections corresponds to a sign change of one basis vector $\varepsilon_i \mapsto -\varepsilon_i$, or a transposition of two basis vectors $\varepsilon_i \leftrightarrow \varepsilon_j$ or a transposition with change of sign $\varepsilon_i \leftrightarrow -\varepsilon_j$. These reflections fall into two classes of conjugacy under the reflection group generated by them. Call the transpositions with or without sign change the class $A$ and the sign changes the class $B$. These correspond to the long and short roots in the root system $B_n$ or the opposite in $C_n$.

The group $W(B_n)$ acts transitively on the set of pairs $\{\varepsilon_i, -\varepsilon_i\}$. The reflections of class $B$ act trivially on this set so the group $W(B_n)$ is generated by at least $n - 1$ reflections of class $A$. The last generator must be in the other conjugacy class $B$. We will present such a set of generators by a permutation graph with $n$ vertices, corresponding to the pairs $\{\varepsilon_i, -\varepsilon_i\}$ with $n - 1$ numbered edges and one selected numbered vertex corresponding to the generator of class $B$. From Lemma 12 the graph should be a tree. Such graph describes completely the relative positions of generating reflections.

**Theorem 15.** The braid group action on generators of $W(B_n)$ has one orbit.

**Proof.** Let the $k$-th generator be of class $B$. Acting with $\sigma_{n-1} \sigma_{n-2} \cdots \sigma_k$ we change its number to $n$. The remaining reflections generate $W(A_n)$ and by Theorem 14 there is a braid which brings them to canonical configuration. The obtained graph is shown on Fig.3.

A close inspection of the action (12) on the graph convinces that the braid $\sigma_{n-1}^2 \cdots \sigma_k^2 \sigma_{k+1}^2 \sigma_{k+2}^2 \cdots \sigma_{n-2}^2 \sigma_{n-1}$ transforms this graph to that of Fig.4 which is the canonical arrangement of reflections generating $W(B_n)$.

\hfill \Box
The last classical family of reflection groups is $W(D_n)$, $n = 4, 5, \ldots$. The group $W(D_n)$ acts on the $n$-dimensional Euclidean space by permutations of the basis vectors and even number of sign changes. Reflections form only one conjugacy class in the group and each of them transpose two basis vectors with or without sign change $\varepsilon_i \leftrightarrow \pm \varepsilon_j$. Again considering their transitive action on the pairs $\{\varepsilon_i, -\varepsilon_i\}$ we see that $n - 1$ of them must generate the symmetric group, permuting these pairs. By an isomorphism of the Euclidean space changing directions of the basis vectors these reflections can be made transpositions without sign changes of the basis vectors which forces the last reflection to be a transposition with sign change.

Once more such arrangement will be presented by a permutation graph with $n$ vertices corresponding to the pairs $\{\varepsilon_i, -\varepsilon_i\}$ and indexed edges corresponding to the generating reflections. As $n - 1$ edges form a tree and there are $n$ vertices it follows that the graph is connected, containing one cycle. Here one must allow two vertices of the graph to be connected by two different edges, forming a cycle.

To count the orbits of the braid group on such arrangements we transform them to a canonical form. As in the case of $W(B_n)$ we make the first $n - 1$ to generate the symmetric group and order them to obtain a permutation graph (Fig.5).

The product $r_1 r_2 \cdots r_n$ which is invariant of the action of the braid group seen as a permutation on the set of pairs $\{\varepsilon_i, -\varepsilon_i\}$ decomposes into two cycles of lengths $k$, $n - k$ so there are at least $\left\lfloor \frac{n}{2} \right\rfloor$ orbits. The graph of Fig.5 is transformed to a similar one with $k' = n - k$ by the braid

$$\sigma_{n-1}^{-1} \sigma_{n-k-1} \sigma_{n-k} \cdots \sigma_{n-2}^{-1} \sigma_3^{-1} \sigma_{k+1}^{-1} \sigma_2^{-1} \sigma_{k-1} \sigma_k \sigma_1 \cdots \sigma_{k-2} \sigma_{k-1}$$

and so there are exactly $\left\lfloor \frac{n}{2} \right\rfloor$ orbits of the braid group acting on $n$-tuples of
reflections generating $W(D_n)$.

The graph on Fig.6 can also be used as canonical for the orbits on generators of $W(D_n)$. It is unique for each orbit and universal in the sense that all the

$$ k + 1 $$

orbits are obtained by different numberings of its edges.

The graph of transpositions is unique for every arrangement except for the generators of $W(D_4)$, where many isomorphisms appear. The equivalent trans-

$$ 1 2 3 4 $$

$$ 1 2 3 4 $$

$$ 1 2 3 4 $$

Figure 7: Permutation graphs, corresponding to the same arrangement

position graphs of these generators are shown in the rows of Fig.7, while in the last column are shown the arrangement graphs, corresponding to them. Written explicitly it is easy to see that these isomorphisms are reflections in the Euclidean space. If these reflections are added to the group $W(D_4)$ one obtains the group $W(B_4)$. In fact our presentation with graphs of transpositions, which do not give information whether these transpositions of basis vectors include sign changes or not, is loose enough to hide all the isomorphisms, which if added to $W(D_n)$ give the group $W(B_n)$. In the case of $W(D_4)$ the outer isomorphisms coming from Fig.7 and those from sign changes of the basis vectors are independent and when both added the group obtained is $W(F_4)$.

5.2 Orbits on the generators of exceptional Coxeter groups

There is no obvious interpretation of these groups as permutation groups. Because of that, we will use the arrangement graphs. As it is seen from the definition (12) the action of $i$-th elementary braid coincides with the result of
conjugation of the \( i + 1 \)-th reflection by \( i \)-th \( r_{i+1} \mapsto r_i r_{i+1} r_i \) followed by their transposition \( r_i \leftrightarrow r_{i+1} \). In terms of the graph, conjugation of the reflection corresponding to \( i + 1 \)-th vertex will affect only edges incident with this vertex. Moreover, the resulting edge \( g(i+1,k) \) will depend only on \( g(i,i+1), g(i+1,k) \), and \( g(i,k) \). This dependence is given in Table A.

New arrangements are built inductively by adding one vertex to an arrangement which is known to be in finite orbit. In order to minimize the possibilities of such extensions it is convenient to pick up the most uniform arrangement in every orbit. As a byproduct these uniform arrangements are also universal i.e. by changing only the ordering of their vertices are obtained arrangements in all orbits of the braid group on generators of a given group.

The braids

\[
\sigma_i(r_1, r_2, \ldots, r_n) = (r_1, \ldots, r_{i+1}, r_i, \ldots, r_n) \quad \text{if} \quad r_i r_{i+1} = r_{i+1} r_i \quad (82)
\]

\[
\sigma_{n-1}\sigma_{n-2} \cdots \sigma_1(r_1, r_2, \ldots, r_n) = (r_1 r_2 r_1, r_1 r_3 r_1, \ldots, r_1 r_n r_1, r_1 r_1 r_1) \quad (83)
\]

preserve the graph, which represents only relative positions of the reflections; but they affect the ordering. We will find which orderings are obtained by these transformations.

**Lemma 16.** All permutations of the vertices of an arrangement graph, which is a tree are obtained by the action of the braid group.

**Proof.** Any numbering can be transformed by the braids (82),(83) to a fixed one with the property that the first vertex is a leaf in the tree, and the induced subgraphs on vertices 1, 2, \ldots, \( k \) for \( k = 1, 2, \ldots, n \) are trees. Arbitrary numbering is defined by a permutation \( i : \{1, \ldots, n\}/\to\{1, \ldots, n\} \) of the vertices. By a cyclic permutation from the braid (83) it is always possible to make \( i(1) = 1 \). Assume the vertices to be ordered up to the number \( k \) and \( i(k+1) = p > k + 1 \). If \( i^{-1}(p) \) and \( i^{-1}(p-1) \) are not joined, \( \sigma_{p-1} \) lowers the index of \( k+1 \). If they are joined lower first the \( i^{-1}(p-1) \). This process will continue until either \( i(k+1) = k + 1 \) or \( i(k+1) = p \) and \( i^{-1}(p), i^{-1}(p-1), \ldots, i^{-1}(k+1) \) form a path in the tree. In the last case the braid \( \sigma_{k-1}\sigma_{k-2} \cdots \sigma_1 \) effectively rises the numbers of the first \( k \) vertices and conjugates the remaining ones with \( r_1 \), the last being trivial if \( k \geq 2 \) as 1 is joined only with 2 by assumption on the fixed numbering. After cyclic permutation the first \( k \) numbers are restored and the path obtains the numbers \( p-1, p-2, \ldots, k+1, n \). If this happens with \( k = 1 \), the braid \( (\sigma_{n-1}\sigma_{n-2} \cdots \sigma_2)^{-1} \) preserves the graph and the path obtains the numbers \( p-1, p-2, \ldots, 2, n \). It is always possible to lower the index of \( i(k+1) > k+1 \) eventually bringing it to \( i(k+1) = k+1 \). By induction all the vertices can be ordered so that \( \forall k, i(k) = k \). \( \square \)

**Lemma 17.** There exist \( n-1 \) orbits of the transformations (82)-(83) of a graph with \( n \) vertices, which is a cycle.

**Proof.** Fix a linear ordering of the vertices such that \( k, k+1 \mod n \) to be joined
for all $k \in \mathbb{Z}_n$. Arbitrary ordering will be denoted by $i(k)$. The quantities
\begin{align*}
q_< &= \#\{k \in \mathbb{Z}_n, i(k) < i(k + 1 \mod n)\} \\
q_> &= \#\{k \in \mathbb{Z}_n, i(k) > i(k + 1 \mod n)\}
\end{align*}
are invariants of the transformations (82)-(83). We may order the vertices so that $i(k) = k$ for $k \leq q_<$ in the manner of the previous lemma hence these invariants are the only obstruction. As $q_+ + q_> = n$, $1 \leq q_\leq n - 1$ there are $n - 1$ orbits.

The quantities (84) are preserved also when the cycle is part of the graph. Such invariants, associated to each directed cycle, will classify the orbits of (82)-(83). These invariants are not independent. If $C = C_1 + C_2$ in the first homology group, the quantities $(q_<, q_>)$ associated to $C$ are expressed through $(q_1<, q_2>)$, associated to $C_1, C_2$
\[(q_<, q_>) = (q_{1<} + q_{2<} - k, q_{1>} + q_{2>} - k) \quad (85)\]
where $k$ denotes the number of edges, common to $C_1$ and $C_2$.

Let directed cycles $C^1, \ldots, C^k$ form a basis in the first homology group of an unindexed graph and $l(C^i)$ be the length of the cycle $C^i$. This graph generates at most $(l(C^1) - 1)(l(C^2) - 1) \cdots (l(C^k) - 1)$ orbits. However, due to symmetries between these cycles and restrictions on some of the invariants by fixing the others the number of orbits is usually much lower.

**Lemma 18.** A graph $\Gamma$ can be transformed by (82)-(83) to a similar one with consecutive indices on a pair of vertices $A$ and $B$ if and only if there is not a cycle in which $A$ and $B$ are not neighboring, and with one of its invariants $q_<, q_>$ equal to 1.

**Proof.** We proceed as in Lemma 16. Let the vertices are labeled by letters and indexed by
\[i : \{A, B, \ldots\} \to \{1, 2, \ldots, n\}.\]
For easier notation the operations on indices are done in $\mathbb{Z}_n$. The index of $B$ is lowered by $\sigma_{i(B) - 1}$ if $i^{-1}(i(B) - 1)$ is not connected to $B$. If $B$, $i^{-1}(i(B) - 1)$, $i^{-1}(i(B) - 2)$, $\ldots$, $i^{-1}(i(B) - l)$ is a path $\sigma_{i(B) - l - 1}$ shortens it. If $i^{-1}(i(B) - l) = A$, the index of $B$ cannot be lowered, and there is a cycle $A, B, C, \ldots, D$ with one rising of the index. In such a case we repeat the procedure with rising the index of $B$. Eventually, either $i(B) = i(A) - 1$ or there is a cycle $B, A, E, \ldots, F$ with one rising of the index. In the second case the cycle $A, E, \ldots, F, B, C, \ldots, D$ has one of the invariants equal to 1 and the transformations (82)-(83) alone cannot make $A$ and $B$ with consecutive indices. \qed
5.3 Orbits on generators of the groups $E_6, E_7, E_8$

The most symmetric arrangement in the orbit of generators of $W(A_n)$ is the complete graph $\Gamma_0(A_n)$ corresponding to the matrix

$$B(A_n) = \begin{pmatrix}
2 & 1 & \ldots & 1 & 0 \\
1 & 2 & \ldots & 1 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
1 & \ldots & 2 & 1 \\
0 & \ldots & 1 & \ldots & 2
\end{pmatrix}. \quad (86)$$

Indeed its symmetry group is the group of all permutations of the vertices which is much bigger than the group $\mathbb{Z}_2$ of symmetries of the Dynkin diagram of $W(A_n)$.

An extension of this graph by one vertex and edges labeled $\pm 3$ is determined by the number $k$ of these edges, and the difference between the number of positive and negative labels. Postponing consideration of extensions of configurations with degenerate matrices we see that all edges must have equal sign which can be taken positive.

The arrangement matrix of one vertex extension of $A_n$ is

$$B(A_n, k) = \begin{pmatrix}
2 & 1 & \ldots & 1 & 0 \\
1 & 2 & \ldots & 1 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
1 & \ldots & 2 & 1 \\
0 & \ldots & 1 & \ldots & 1 2
\end{pmatrix}, \quad (87)$$

where on the last row and column there are $k$ 1s. The determinant of this matrix is calculated using

$$\det(B(A_n)) = \begin{vmatrix}
2 & 1 & \ldots & 1 \\
1 & \ddots & \vdots & \vdots \\
\vdots & \ddots & 2 & 1 \\
1 & \ldots & 1 & 2
\end{vmatrix} = n + 1 \quad \det(B(A_n, k)) = 2(n + 1) - k(n - k + 1). \quad (88)$$

Identities (88) are proved by induction. The requirement of non-degeneracy reads $2(n + 1) - k(n - k + 1) \neq 0$. Solving for $n$ the opposite condition

$$n = \frac{k^2 - k + 2}{k - 2} \quad k = 0, 1, 2, 3, 4 \quad \rightarrow \quad n = -1, -2, \infty, 8, 7. \quad (90)$$

One sees that $\det(B(A_n, k)) \neq 0$ for any $n$ if $k = 1, 2, n - 1, n$. It is also satisfied for $k = 3, 5, 6$ if $n < 8$ and $k = 4, n < 7$. Any extension out of these
restrictions will contain a degenerate sub-graph and is not considered here. The extensions by \( k = 1, n \) give an \( A_{n+1} \) arrangement, those with \( k = 2, n - 1 \) give a \( D_{n+1} \) and \( k = 3, 5 < n < 8; k = 4, n = 6; k = 5, n = 7 \) give an \( E_{n+1} \) arrangement.

We continue with the extensions of \( D_n \) arrangements. The most uniform and universal \( D_n \) arrangement is the extension of the symmetric arrangement of \( A_n \) by \( n - 1 \) edges. It will be denoted \( \Gamma_0(D_n) \) and corresponds to a complete graph with one edge deleted. If the ends of this edge are \( v_a, v_b \) and the vertices are indexed \( i : V \to \{1, 2, \cdots, n\} \) according to their ordering, the difference \( |i(v_a) - i(v_b)| \mod n \) determines the different orbits of the braid group.

Let \( \Gamma_0(D_n) \) be extended to \( \Gamma' \) with the vertex \( v' \). The non-degeneracy does not depend on the order of vertices therefore the most general extension of \( \Gamma_0(D_n) \) is one of the following:

1. Extension with \( k \) edges for which \( \{v_a, v'\}, \{v_b, v'\} \notin E' \)

\[
\det(B(D_n, k)_1) = \begin{vmatrix}
2 & 1 & \cdots & 1 & 0 & 0 \\
1 & 2 & 1 & \cdots & 1 & 1 \\
 & \ddots & \ddots & \ddots & \ddots & \ddots \\
 & & \ddots & \ddots & \ddots & \ddots \\
1 & \cdots & \cdots & \cdots & 1 & 1 \\
0 & 1 & \cdots & 1 & 2 & 0 \\
0 & \cdots & 1 & \cdots & 1 & 0 & 2 \\
\end{vmatrix} = 8 - 4k. \quad (91)
\]

It is non-degenerate only for \( k = 1 \) and the obtained arrangement is \( D_{n+1} \).

2. Extension with \( k + 1 \) edges for which \( \{v_a, v'\}, \{v_b, v'\} \notin E' \)

\[
\det(B(D_n, k)_2) = \begin{vmatrix}
2 & 1 & \cdots & 1 & 0 & 0 \\
1 & 2 & 1 & \cdots & 1 & 1 \\
 & \ddots & \ddots & \ddots & \ddots & \ddots \\
 & & \ddots & \ddots & \ddots & \ddots \\
1 & \cdots & \cdots & \cdots & 1 & 1 \\
0 & 1 & \cdots & 1 & 2 & 1 \\
0 & \cdots & 1 & \cdots & 1 & 1 & 2 \\
\end{vmatrix} = 8 - n. \quad (92)
\]

It is non-degenerate for \( n < 8 \) and the obtained arrangement is \( E_{n+1} \) if \( n \geq 5, D_5 \) if \( n = 4 \), and \( A_4 \) if \( n = 3 \). Call such arrangement \( \Gamma(D_{n-1} \subset E_n, k) \).

3. Extension with \( k + 2 \) edges for which \( \{v_a, v'\}, \{v_b, v'\} \in E' \). Up to now, in order to avoid degenerate sub-graphs, it was assumed that all the new edges had positive signs. Here it is possible only for \( k = n - 2 \). If \( 0 < k < \)
n − 2 there exist vertices \( v_c, v_d \) such that \( \{v_c, v'\} \in E', \{v_d, v'\} \notin E' \). The subgraph on vertices \( v_a, v_d, v_b, v'_c, v'_b \) is a cycle as are the triangles \( v_a, v_c, v' \), and \( v_b, v_c, v' \). Non-degenerate cycles have odd number of negative signed edges. It is easy to see that the three cycles cannot be simultaneously non-degenerate. The only permitted extensions are those with \( k = 0, n − 2 \) giving in both cases \( D_{n+1} \).

By Lemma 18 the extension \( A_n \subset E_{n+1} \) can always be braid transformed to make the indices of the ends of one of the new edges consecutive say \( i, i + 1 \). Applying the appropriate braid \( \sigma_i \) or \( \sigma_i^{-1} \) it is transformed to a graph \( \Gamma(D_5) \subset E_{n+1}, n − 1 \). For \( E_6 \) this is actually an universal graph. We will prove that \( \Gamma(D_5 \subset E_7, 2) \) is universal for \( E_7 \) i.e. all extensions \( \Gamma(D_6 \subset E_7, k), \Gamma(E_6 \subset E_7) \) and \( \Gamma(A_6 \subset E_7) \) can be braid transformed to it. The group \( E_8 \) does not have an universal graph so we will use of two unordered graphs of its generators with the property that every orbit contains at least one of them.

Every connected subgraph of \( \Gamma(E_6) \) with 5 vertices is either \( \Gamma(A_5) \) or \( \Gamma(D_5) \). Our approach to finding the orbits of the braid group on arrangements generating \( W(E_6) \) will be to consider the graphs \( \Gamma(D_5 \subset E_6) \), in which the first 5 vertices belong to the subgraph \( \Gamma(D_5) \). The braid

\[
\tau_4^5 = (\sigma_4\sigma_3\sigma_2\sigma_1)^5
\]

preserves the subgraph \( \Gamma(D_5) \) and conjugates the last reflection

\[
r_6 \mapsto r_1r_2r_3r_4r_5r_6r_5r_4r_3r_2r_1.
\]

To identify different \( E_6 \) orbits and arrangements \( \Gamma(D_5 \subset E_6) \) in them one may use the following procedure. First are listed all extensions of \( \Gamma_0(D_5^{(1)}) \) and \( \Gamma_0(D_5^{(2)}) \), and grouped into sets of transitive action of the braid \( \tau^5 \). Then, in each member of these sets are considered other \( D_5 \) subgraphs. Appropriate braid will make the indices of these subgraphs to take the values \( 1, \ldots, 5 \) obtaining a new graph in the given class. When these new graphs fall in different sets, the sets are unified. At the end one obtains a list of sets of \( \Gamma(D_5 \subset E_6) \) graphs representing different orbits. The result is that the graph \( \Gamma_0(E_6) = \Gamma(D_5 \subset E_6, 4) \) is universal for \( W(E_6) \) where the orbit depend on the indices \( i, j, k \). This graph is symmetric with respect to \( j, k \) but actually any permutation of the indices \( i, j, k \) yields a graph in the same orbit. Using the fact that the braid \( \tau_6 = \sigma_5\sigma_4\cdots\sigma_1 \) permutes cyclically the indices we see that the orbit depend on the relative positions of \( i, j, k \) in \( Z_6 \) or in other words the orbits correspond to different inscribed triangles in the regular hexagon (Fig.9).

The subgraphs of arrangements generating \( W(E_7) \) are \( A_6, D_6^{(k)}, E_6^{(k)} \). A detailed inspection shows that all extensions of \( \Gamma_0(E_6) \) to \( \Gamma(E_7) \) contain subgraphs generating \( W(D_6) \) which allows us to proceed in the same way as with \( E_6 \). There are four orbits coming from different inscribed triangles in the regular heptagon (Fig.10) and one more orbit which does not have graph \( \Gamma(D_6 \subset E_7, 5) \). One graph \( \Gamma(D_6 \subset E_7, 4) \) in the last orbit is shown on Fig.11, where the three vertices in the center, with respect to which the graph is symmetric, have indices \( 1, 3, 6 \).
The last group $W(E_8)$ can be generated by a reflection configuration, in which all subgraphs with 7 vertices generate $W(E_7)$. Aside from that, there are 5 orbits coming from the graph $\Gamma(D_7 \subset E_8, 6)$ with indexing of the vertices corresponding to the 5 different inscribed triangles in the regular octagon (Fig. 12). There are also 3 orbits which have graph $\Gamma(D_7 \subset E_8, 5)$, and one more orbit in which all graphs contain only $\Gamma(E_7)$ subgraphs. A very symmetric representative in the last one is shown on Fig. 13.

As we have seen, the quasicoxeter element is invariant under the action of the braid group. It can be computed for every reflection arrangement using (71). The eigenvalues of quasicoxeter elements of reflection arrangements, generating finite Coxeter groups must be roots of unity, moreover, in case of simply-laced groups $A_n, D_n, E_6, E_7, E_8$ the characteristic polynomial factors into cyclotomic polynomials. Recall that the $n$-th cyclotomic polynomial is given by

$$\Phi_n(x) = \prod_{1 \leq k \leq n \atop \gcd(k, n) = 1} (x - e^{2\pi ik/n}).$$

(93)
The characteristic polynomials of quasicoxeter elements corresponding to the different orbits of arrangements, generating $W(E_n)$ up to constant factors are

| orbit  | $\det(C - Ix)$ |
|--------|----------------|
| $E_6^{(1)}$ | $\Phi_3(x)\Phi_{12}(x)$ |
| $E_6^{(2)}$ | $\Phi_9(x)$ |
| $E_6^{(3)}$ | $\Phi_3(x)\Phi_6(x)^2$ |
| $E_7^{(1)}$ | $\Phi_2(x)\Phi_{14}(x)$ |
| $E_7^{(2)}$ | $\Phi_2(x)\Phi_6(x)\Phi_{12}(x)$ |
| $E_7^{(3)}$ | $\Phi_2(x)\Phi_{18}(x)$ |
| $E_7^{(4)}$ | $\Phi_2(x)\Phi_6(x)\Phi_{10}(x)$ |
| $E_7^{(5)}$ | $\Phi_2(x)\Phi_6(x)^3$ |
| $E_8^{(1)}$ | $\Phi_{30}(x)$ |
| $E_8^{(2)}$ | $\Phi_{24}(x)$ |
| $E_8^{(3)}$ | $\Phi_{20}(x)$ |
| $E_8^{(4)}$ | $\Phi_6(x)\Phi_{18}(x)$ |
| $E_8^{(5)}$ | $\Phi_{15}(x)$ |
| $E_8^{(6)}$ | $\Phi_{12}(x)^2$ |
| $E_8^{(7)}$ | $\Phi_{10}(x)^2$ |
| $E_8^{(8)}$ | $\Phi_6(x)^2\Phi_{12}(x)$ |
| $E_8^{(9)}$ | $\Phi_6(x)^4$ |

5.4 Orbits on generators of the groups $F_4, H_4$

In view of their shortness these orbits can be computed manually using Table A and Lemma 18. Canonical generators of $F_4$ lie in the orbit with graphs on Fig.14, where the invariants $(84)$ of the squares in graphs $A, B$ are $(q_-, q_+) = (2, 2)$ and $(1, 3)$ correspondingly. The graphs $A$ and $B$ with invariants $(1, 3)$ and $(2, 2)$ form another orbit of the braid group and these two orbits contain all the arrangements of reflections, generating $W(F_4)$.

![Figure 14: The two orbits of $F_4$](image)

For all crystallographic Coxeter groups, considered up to here, there was a uniform procedure for finding all the configurations, generating a given group. Starting with the canonical generators and acting by the braid group and permuting the indices, it was possible to obtain all configurations generating a
given group. This procedure fails for any group which has an arrangement of generating reflections involving matrix elements 2\cos \frac{2\pi}{n}, n = 5, n \geq 7. The reason is that for an abstract group with two generating reflections \(r_1, r_2\), \(r_1^2 = r_2^2 = (r_1r_2)^n = 1\) there exist more than one linearly non-isomorphic realizations if \(n = 5\) or \(n \geq 7\). More precisely, the number of such realizations is equal to the number of regular star-polygons with \(n\) sides plus one, or the whole part of half the Euler’s totient function \(\left\lfloor \frac{\varphi(n)}{2} \right\rfloor\).

In order to obtain all arrangements generating \(W(H_4)\), one must allow transformations

\[
(r_i, r_j) \mapsto (r_j, r_i) \quad (94)
\]
\[
(r_i, r_j) \mapsto (r_i, r_ir_jr_i) \quad (95)
\]
\[
(r_i, r_j) \mapsto (r_i, r_jr_ir_j) \quad \text{if } (r_i r_j)^5 = 1 \quad (96)
\]

These arrangements split into families such that arrangements from the same family are obtained by transformations not involving (96). Triples of generating reflections of the group \(H_3\) form 3 families containing one orbit each. In every orbit there is a linear graph corresponding to a pair of reciprocal regular polyhedra or star-polyhedra of Kepler-Poinsot [6].

The group \(W(H_4)\) has five families of generating arrangements and in each family there is at least one arrangement, whose graph is linear. These linear graphs correspond to pairs of reciprocal regular star-polyhedra in the four-dimensional space. In each family of arrangements there are two or three orbits.

The list of orbits according to their family is given in Table B. There are given also the characteristic polynomials of quasicoxeter elements in each orbit. Notice that the transformations (94)-(95) preserve \(\det(B)\), while (96) does not, therefore the families can be characterized by \(\det(B)\):

| family | \(H^A\) | \(H^B\) | \(H^C\) | \(H^D\) | \(H^E\) |
|--------|--------|--------|--------|--------|--------|
| \(\det(B)\) | \(\frac{7-3\sqrt{5}}{2}\) | \(\frac{7+3\sqrt{5}}{2}\) | \(\frac{3+\sqrt{5}}{2}\) | \(\frac{3-\sqrt{5}}{2}\) | 1 |

(97)

We conclude with the remark that in each family of orbits there are universal graphs:

![Figure 15: Universal graphs for the families of orbits of \(H_4\)](image-url)
6 Extensions by one vertex of the universal graphs

Here will be considered arrangements in which every subarrangement generates finite Coxeter group. According to [10] and Lemma 11 only such configurations may belong to finite orbits of the braid group. As it is always possible to bring the subconfiguration to its universal graph we will consider only extensions of the universal graphs. By so doing, the task is simplified in two ways: only one graph is considered for all orbits of the braid group on configurations generating particular Coxeter group; and the universal graphs are deliberately chosen to have big symmetry groups to reduce the number of possible extensions.

Definition 19. An admissible extension of a graph \( \Gamma \) is an extension by one vertex, such that the obtained graph does not contain degenerate subgraphs, nor subgraphs with infinite orbit under the braid group action.

Only extensions which do not contain degenerate subarrangements are considered. The remaining extensions are treated in the next section. When talking about realization of a degenerate arrangement matrix it is always understood the minimal realization \( \text{rank}(B) = \dim(V) \) as only in this case the arrangements have a simple meaning of redundant generators in finite Coxeter group. As a demonstration for redundancy will be given expressions for one of the reflections through the others.

6.1 Extensions of \( H_3, H_4 \) configurations

First we consider extensions of the graphs on Fig.16, which are arrangements in the three orbits of \( B_3 \), generating \( W(H_3) \).

![Figure 16: Representatives in the orbits of \( H_3 \)](image)

All degenerate admissible extensions of the graphs in Fig.16 are shown in fig. 17. They represent redundant generators of \( W(H_3) \). The explicit expressions of one of the reflections through the others is given in Table C.

![Image with diagrams](image)

The extensions, which do not generate \( W(H_4) \) e.g. Fig.18 can always be transformed by braids (82)-(83) according to Lemma 18 to a new indexing of
the vertices, in which \( A, B \) have indices 1, 2. Then the braid \( \sigma_1 \) if \( A = 1, B = 2 \) or \( \sigma_1^{-1} \) if \( A = 2, B = 1 \) transforms the graph to a new one with a subgraph not generating finite three-dimensional Coxeter group and according to [10] does not stay in a finite orbit of the braid group. The same argument applies to all non-degenerate admissible extensions of the three graphs in Fig.16, which are enlisted in table D together with determinants of the arrangement matrices. All the remaining admissible extensions represent arrangements of reflections, generating \( W(H_4) \).

For generators of the group \( W(H_4) \) one may use the universal graphs on Fig.19. There are two graphs for the last family of orbits for uniformity. Most of the extensions of these graphs by one vertex, in which all subgraphs represent non-redundant generators of finite Coxeter groups are degenerate. These degenerate extensions are given in Table E, with explicit formulas for one of the reflections through others. In order to save space the numbering of the vertices is not given in the table. The convention is to index the vertices counterclockwise beginning with the upper left vertex; the central vertex has index 5. Apart from that, the non-degenerate admissible extensions are shown in Fig.20. All these graphs, when transformed in a way analogous to that of extensions of \( H_3 \) obtain subgraphs, which do not belong to finite orbits.

### 6.2 Extensions of the universal graphs of the Weyl groups

There is no need to consider admissible extensions with edges labelled \( \pm 5, 5' \) as they are also extensions of \( H_3 \) or \( H_4 \) arrangements.
6.2.1 Extensions of $B_n, F_4$

The admissible extensions of the universal arrangement of $W(F_4)$ are only two (Fig.21). They are degenerate. The reflection corresponding to the fifth vertex is equal to $r_3r_4r_1r_2r_1r_4r_3$ for the first extension and to $r_4r_1r_2r_1r_4$ for the second one.

![Figure 21: Admissible extensions of $\Gamma_0(F_4)$](image)

The extensions of $\Gamma_0(B_n)$, in which all subgraphs with three vertices are non-degenerate and have finite orbits, fall in the following three cases:

$$
\begin{pmatrix}
B_1 & b_1 & a_1 \\
b_1' & 2 & 0 \\
a_1' & 0 & 2
\end{pmatrix}, \quad
\begin{pmatrix}
B_1 & b_1 & b_2 \\
b_1' & 2 & 1 \\
b_2' & 1 & 2
\end{pmatrix}, \quad
\begin{pmatrix}
B_1 & b_1 & a_2 \\
b_1' & 2 & \sqrt{2} \\
a_2' & \sqrt{2} & 2
\end{pmatrix}
$$

(98)

where the submatrices $B_1, b_1, b_2, a_1, a_2$ are

$$
B_1 = \begin{pmatrix}
2 & 1 & \ldots & 1 \\
1 & 2 & 1 & \ldots \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 1 \\
1 & \ldots & 1 & 2
\end{pmatrix}, \quad
b_1' = (\sqrt{2}, \ldots, \sqrt{2})_p, \quad
b_2' = (\sqrt{2}, \ldots, \sqrt{2}, 0, \ldots, 0)
$$

(99)

$$
a_1' = (0, \ldots, 0, -1, \ldots, -1, 1, \ldots, 1), \quad
a_2' = (1, 1, \ldots, 1)
$$

(100)

Determinants of the matrices of the three extensions respectively, are

$$
2(2 - p - q), \quad 4 - n, \quad 2
$$

(101)
The first extension is degenerate for \( p = 2, q = 0 \) or \( p = 0, q = 2 \), or \( p = 1, q = 1 \).

The first two possibilities coincide as arrangements using the identification (15). They are realized by \( r_{n+1} = r_{n-1}r_{n-2}r_{n}r_{n-1} \), while the third – by \( r_{n+1} = r_{n-2}r_{n-1}r_{n} \). The only non-degenerate extensions without degenerate subgraphs are for \( p = 1, q = 0 \) or \( p = 0, q = 1 \) which are equivalent. In this case the extended arrangement generates \( B_{n+1} \).

The extension of the second case is degenerate for \( n = 4 \). It can be realized by
\[
r_2 = g_1 r_2 g_1^{-1} r g_1^{-1}
\]
where \( g_1 = r_4 r_1 r_3 r_1, g_2 = r_1 r_4 \), and \( r \) is a reflection depending on the number \( p \):
\[
\begin{align*}
    r &= r_5 & \text{if } p &= 0 \\
    r &= r_4 r_1 r_5 r_1 r_4 & \text{if } p &= 1 \\
    r &= r_4 r_1 r_4 r_4^{-1} r_1 r_4 & \text{if } p &= 2 \\
    r &= r_4 r_5 r_4 & \text{if } p &= 3
\end{align*}
\]
These are redundant generators in the group \( F_4 \) which explains why we have expressed \( r_2 \) instead of \( r_5 \). The non-degenerate extensions when \( n = 1, 2, 3 \) generate the groups \( B_2, B_3, F_4 \) correspondingly.

The third case is non-degenerate and by exchanging the last two reflections it becomes \( \Gamma_0(B_{n+1}) \).

### 6.2.2 Extensions of the simply-laced graphs

No admissible extensions with edges labelled \( \pm 5, 5', 4 \) need to be considered as these are also extensions of already examined graphs. We begin with the extensions of \( \Gamma_0(A_n) \).

\[
\det\begin{pmatrix}
    B_1 & a_1 \\
    a_1^t & 2
\end{pmatrix} = (p - q)^2 - (p + q)(n + 1) + 2(n + 1),
\]
\[
\text{where } a_1 \text{ is the column vector (100). It is convenient to assume } p > q \text{ as the expression is symmetric with respect to } p \text{ and } q. \text{ To analyze when this determinant vanishes it is convenient to introduce new variables}
\]
\[
p - q = u \\
p + q = v.
\]

Solving for \( v \)
\[
v - 2 = \frac{u^2}{n + 1}.
\]

We are looking for solutions in whole numbers for which \( 0 \leq u \leq v \leq n \). We expand \( n + 1 \) into a product of prime factors and group the square part of it:
\[
n + 1 = a^2 b, \text{ so that } b \text{ to have not repeated prime factors. As } n + 1 \text{ divides } u^2 \text{ it follows that } u = abc. \text{ We obtain the following inequalities}
\]
\[
0 \leq abc \leq bc^2 + 2 \leq a^2 b - 1, \quad a, b, c \geq 0,
\]
which can be rewritten as

\[
\begin{vmatrix}
bc(a - c) & \leq 2 \\
b(a^2 - c^2) & \geq 3 \\
a^2b & \geq 2.
\end{vmatrix}
\]

(108)

The last implies \(b > 0, a > c\). There must be considered two cases:

1. \(c = 0\). This is a solution with \(p = q = 1\) for arbitrary \(n\). Such an extension can be realized by \(r_{n+1} = r_{n} r_{n-1} r_{n}\).

2. \(c > 0\). We have the following system of inequalities

\[
0 < c < a \leq \frac{2}{bc} + c, \quad b > 0
\]

(109)

\(a, b, c \in \mathbb{Z}_+\) therefore \(\frac{2}{bc} \geq 1\). We obtain the following solutions:

(a) \(b = c = 1, a = 2\). It yields \(p = \frac{5}{2}, q = \frac{1}{2}\), which is not a solution in whole numbers.

(b) \(b = c = 1, a = 3\). This is a solution with \(p = 3, q = 0, n = 8\). This arrangement represents redundant generators of \(E_8\) (in the minimal realization). The fifth reflection can be expressed through the others

\[r_5 = gr_{9}g^{-1}, \quad g = r_6 r_3 r_6 r_7 r_6 r_4 r_6 r_9 r_6 r_3 r_7 r_2 r_8 r_1 r_6.\]

(c) \(b = 1, c = 2, a = 3\). This solution gives \(p = 6, q = 0, n = 8\). Again the obtained graph represent redundant generators of \(E_8\), which is seen by the identity

\[r_6 = gr_{9}g^{-1}, \quad g = r_9 r_1 r_5 r_2 r_4 r_9 r_3 r_1 r_8 r_2 r_7.\]

(d) \(b = 2, c = 1, a = 2\). This gives \(p = 4, q = 0, n = 7\) and the extension is a degenerate graph of \(E_7\). One of the reflections can be expressed through the remaining ones

\[r_5 = gr_{9}g^{-1}, \quad g = r_8 r_4 r_3 r_6 r_7 r_2 r_7 r_1 r_4.\]

The above results imply that all non-degenerate extensions of \(\Gamma_0(A_n)\) without degenerate subgraphs must have \(q = 0\). If \(p = 1\) or \(p = n\) the extension generates \(A_{n+1}\), if \(p = 2\) or \(p = n - 1\) it generates \(D_{n+1}\), and if \(p = 3\) or \(p = n - 2\) \((n < 8)\) it generates \(E_{n+1}\).

Next we consider the extensions of the universal graph of \(D_n\). Using the same block matrices the following cases must be examined

\[
\begin{vmatrix}
2 & a_3 & 0 & 0 \\
a_3 & B_1 & a_3 & a_1 \\
0 & a_3 & 2 & 0 \\
0 & a_3 & 0 & 2
\end{vmatrix} = 4(2 - p - q), \quad a_3 = (1, \ldots, 1)
\]

(110)

\[
\begin{vmatrix}
2 & a_3 & 0 & 0 \\
a_3 & B_1 & a_3 & a_1 \\
0 & a_3 & 2 & 1 \\
0 & a_3 & 1 & 2
\end{vmatrix} = 8 - n - 8q,
\]

(111)
\[
\begin{pmatrix}
2 & a_3^t & 0 & 1 \\
a_3 & B_1 & a_3 & a_3 \\
0 & a_3^t & 2 & 0 \\
1 & a_1^t & 0 & 2
\end{pmatrix}
\]

\[
\text{det} = 8 - n - 8p, \quad (112)
\]

\[
\begin{pmatrix}
2 & a_3^t & 0 & 1 \\
a_3 & B_1 & a_3 & a_3 \\
0 & a_3^t & 2 & 1 \\
1 & a_1^t & 1 & 2
\end{pmatrix}
\]

\[
\text{det} = 4(3 + p - n - 3q), \quad (113)
\]

\[
\begin{pmatrix}
2 & a_3^t & 0 & -1 \\
a_3 & B_1 & a_3 & a_3 \\
0 & a_3^t & 2 & 1 \\
-11 & a_1^t & 1 & 2
\end{pmatrix}
\]

\[
\text{det} = 4(1 - p - q), \quad (114)
\]

\[
\begin{pmatrix}
2 & a_3^t & 0 & -1 \\
a_3 & B_1 & a_3 & a_3 \\
0 & a_3^t & 2 & -1 \\
-11 & a_1^t & -1 & 2
\end{pmatrix}
\]

\[
\text{det} = 4(3 + q - n - 3p). \quad (115)
\]

The extension (110) is degenerate if \( p = 2, q = 0 \) or \( p = q = 1 \). The first case is realized with \( r_{n+1} = r_{n-1}r_2r_n^2r_{n-2}r_3^2r_4^2r_{n-1} \) and the second with \( r_{n+1} = r_{n-1}r_{n-2}r_{n-1} \). It is non-degenerate and doesn’t contain degenerate subgraphs only if \( p = 0, q = 1 \) or \( p = 1, q = 0 \) giving a \( D_{n+1} \) arrangement.

The extension (111) is degenerate only for \( n = 8, q = 0 \). In the same way (112) is degenerate only for \( n = 8, p = 0 \). These two coincide as reflection arrangements under permutation (1n) of the indices of their reflections. The first can be realized by \( r_7 = fgr_8g^{-1}f^{-1} \), where

\[
g = r_1hrg^{-1}r_2r_3r_4r_5rh^{-1}r_5r_1r_6 \quad (116)
\]

and

\[
\begin{align*}
f &= \mathbb{I}, & h &= r_8 & \text{if } p &= 0 \\
f &= r_8r_1, & h &= r_8 & \text{if } p &= 1 \\
f &= \mathbb{I}, & h &= r_5r_1r_4r_8r_3r_1r_2r_8 & \text{if } p &= 2 \\
f &= r_8r_1, & h &= r_5r_1r_5 & \text{if } p &= 3 \\
f &= \mathbb{I}, & h &= r_3r_1r_2r_8 & \text{if } p &= 4 \\
f &= r_8r_1, & h &= r_5r_1r_5r_8r_4r_1r_3 & \text{if } p &= 5 \\
f &= \mathbb{I}, & h &= \mathbb{I} & \text{if } p &= 6 
\end{align*}
\]

The extension (111) is admissible and non-degenerate if \( q = 0, n < 8 \) giving \( D_{n+1}(n < 5) \) or \( E_{n+1}(5 \leq n \leq 7) \).

The extension (113) is degenerate for \( q = 0, p = n - 3 \). It can be realized by \( r_{n+1} = r_1r_2r_n^2r_2r_1 \). It is non-degenerate without degenerate principal minors if \( q = 0, p = n - 2 \) giving \( \Gamma_0(D_{n+1}) \).

The extension (114) is degenerate for \( p = 0, q = 1 \) or \( p = 1, q = 0 \). The first case is realized by \( r_{n+1} = r_{n-1}r_n^2r_{n-1} \) and the second by \( r_{n+1} = r_{n-1}r_1r_{n-1} \). It is non-degenerate without degenerate principal minors if \( p = q = 0 \) giving a \( D_{n+1} \) arrangement.
The last extension (115) is equivalent to (113).

We come to extensions of graphs, representing generators of the exceptional groups $E_6, E_7, E_8$. Non-degenerate admissible extensions of $\Gamma_0(E_6)$ fall in the orbits $E_7^{(k)}$ as we have seen, while degenerate ones are realized by $r_7 = g r_5 g^{-1}$, where $g = r_1 r_4 r_6 r_2 r_1 r_3, r_4 r_6 r_2 r_1 r_3, r_6 r_2 r_1 r_3, r_2 r_1 r_3$ respectively for the graphs in Fig.22.

![Figure 22: Degenerate extensions of $\Gamma_0(E_6)$]

Figure 22: Degenerate extensions of $\Gamma_0(E_6)$

![Figure 23: Degenerate extensions of $\Gamma(D_6 \subset E_7, 5)$]

Figure 23: Degenerate extensions of $\Gamma(D_6 \subset E_7, 5)$

The orbits of the braid group on non-degenerate configurations of generating reflections in the group $W(E_7)$ have two "universal graphs" $\Gamma(D_6 \subset E_7, 5)$ and $\Gamma(D_6 \subset E_7, 4)$. All their extensions by one vertex are either degenerate or represent reflections, generating $W(E_8)$. The admissible degenerate extensions of $\Gamma(D_6 \subset E_7, 5)$ can be realized by $r_8 = g r_7 g^{-1}$, where $g = r_6 r_4 r_7 r_3 r_6 r_2 r_5 r_1$ for the first graph and $g = r_4 r_7 r_3 r_6 r_2 r_5 r_1$ for the second graph in Fig.23. The graphs are symmetric with respect to the unindexed vertices, which must be indexed by the remaining numbers from 1 to 8. The other "universal graph" $\Gamma_0'(E_7) = \Gamma(D_6 \subset E_7, 4)$ allow only non-degenerate admissible extensions which generate $E_8$.

For the group $W(E_8)$ there are three "universal" graphs $\Gamma(D_7 \subset E_8, 6), \Gamma(D_7 \subset E_8, 5)$, and the graph $\Gamma_0(E_8^{(9)})$ from Fig.13. The admissible extensions of $\Gamma(D_7 \subset E_8, 6)$ are degenerate and can be realized by $r_9 = g r_8 g^{-1}$, where

\[
g = r_1 r_7 r_5 r_6 r_4 r_8 r_3 r_7 r_2 r_6 r_1 \quad \text{for the first,}
\]
\[
g = r_8 r_7 r_6 r_4 r_8 r_3 r_7 r_2 r_6 r_1 \quad \text{for the second,}
\]
\[
g = r_7 r_5 r_6 r_4 r_8 r_3 r_7 r_2 r_6 r_1 \quad \text{for the third}
\]
Figure 24: Degenerate extensions of $\Gamma(D_7 \subset E_8, 6)$

All admissible extensions of $\Gamma(D_7 \subset E_8, 5)$ are degenerate. They can be realized by $r_9 = gr_7g^{-1}$, where

$$
g = r_8r_7r_4r_8r_3r_5r_2r_6r_1 \quad \text{for the first,}
$$
$$
g = r_7r_4r_8r_3r_5r_2r_6r_1 \quad \text{for the second,}
$$
$$
g = r_7r_5r_4r_8r_3r_5r_2r_6r_1 \quad \text{for the third}
$$

graphs in Fig.25.

Figure 25: Degenerate extensions of $\Gamma(D_7 \subset E_8, 5)$

The last “universal graph” $\Gamma_0(E_8^{(9)})$ from Fig.13 does not allow admissible extensions.

**Corollary 20.** In the minimal realization, all admissible extensions of the universal arrangements in finite Coxeter groups represent reflections in finite Coxeter groups.
7 General arrangement matrix in a finite orbit

7.1 Degenerate arrangements in finite orbits

An obvious way to obtain degenerate arrangements with finite orbits of the braid group is to take the generators of a finite group and append reflections from the same group. As the group is finite these sets of overdetermined generators are finite. We know from corollary 9 that there are non-isomorphic realizations of degenerate arrangement matrices, which gives us a method to construct infinite reflection groups with finite $B$-orbits. A stronger statement that all groups with the property of having finite $B$-orbits are obtained in this way is also valid.

Theorem 21. Any arrangement with positive semi-definite matrix in a finite orbit of the braid group can be realized as an overdetermined system of generators of finite Coxeter group.

Proof. A positive semi-definite matrix may have only non-negative principal minors. For an arrangement matrix this means that all subarrangements must have positive semi-definite matrices. As stated in lemma 11 the orbit of an arrangement can be finite only if all of its subarrangements have finite orbits.

Let $Ar = \{r_{i_1}, r_{i_2}, \cdots, r_{i_k}\}$ form a non-degenerate subsystem of maximal rank with ordered indices $i_1 < i_2 < \cdots < i_k$. One may consider $\tilde{Ar} = \{r_1, r_2, \cdots, r_n\}$ as an extension of $Ar$ by $n-k$ reflections.

The braid $\sigma_j^{-1}$ decreases the index of $r_{j+1}$ by one leaving $r_l$, $l > j + 1$ unchanged, hence

$$\sigma_k^{-1} \sigma_{k+1}^{-1} \cdots \sigma_{i_k-1}^{-1} \sigma_{i_k}^{-1} \cdots \sigma_{i_1-1}^{-1} \sigma_{i_1}^{-1} \cdots \sigma_{i_k}^{-1} \cdots \sigma_{i_1}^{-1} (118)$$

will transform $\tilde{Ar}$ to an arrangement in which the maximal non-degenerate subsystem is formed from the first $k$ reflections. Considering all possible extensions by one reflection of the universal arrangements in every finite Coxeter group we proved that all arrangements with only non-degenerate sub-arrangements and finite $B$-orbits generate finite Coxeter groups at least in their minimal realization in the sense of corollary 9. In most cases of degenerate extensions $r_{k+1}$ was expressed through $r_1, r_2, \cdots, r_k$. In the case of extensions of $\Gamma_0(D_n)$ to degenerate configurations of $B_n$ or $\Gamma_0(D_4), \Gamma_0(B_4)$ to $F_4$, or $\Gamma_0(D_8)$ to $E_8$, or $\Gamma_0(A_8)$ to $E_8$ and $\Gamma_0(A_7)$ to $E_7$ $r_i$ for some $i < k + 1$ was expressed through $r_1, r_2, \cdots, \hat{r}_i, \cdots, r_{k+1}$. This difference reflects the following inclusions of Coxeter systems from the same dimension:

$$D_n \subset B_n, \quad D_4 \subset B_4 \subset F_4, \quad D_8 \subset E_8 \supset A_8, \quad A_7 \subset E_7 \quad (119)$$

These are the only inclusions of irreducible finite Coxeter systems from the same dimension.

We showed that for all degenerate extensions one of the reflections belongs to the group generated by the others in the minimal realization. Now we may take the subarrangement $Ar$ with reflections, generating the whole group and all other $n-k$ reflections will belong to the same group.
The same argument applies also to reducible arrangements. As for irreducible extensions of reducible configurations we may always take another irreducible subsystem and consider its extension. The following inclusions of reducible Coxeter systems in finite irreducible systems of the same dimension appear:

\[ A_1 \times A_1 \subset B_2, \quad D_k \times B_{n-k} \subset B_n, \quad (120) \]
\[ A_1 \times A_1 \times A_1 \subset B_3, \quad B_k \times B_{n-k} \subset B_n, \quad (121) \]
\[ B_n \times A_1 \subset B_{n+1}, \quad A_1^4 \times D_4, \quad D_k \times D_{n-k} \subset D_n, \quad (122) \]
\[ A_1 \times A_5 \subset E_6, \quad A_2 \times A_2 \times A_2 \subset E_6 \quad (123) \]
\[ A_1 \times A_3 \times A_3 \subset E_7, \quad A_2 \times A_5 \subset E_7, \quad A_1 \times D_5 \subset E_7 \quad (124) \]
\[ A_1 \times A_2 \times A_5 \subset E_8, \quad A_1 \times A_7 \subset E_8, \quad (125) \]
\[ A_1^{\times 2} \subset E_8, \quad A_3 \times D_5 \subset E_8, \quad A_1 \times E_7 \subset E_8 \quad (126) \]
\[ A_1 \times A_1 \times A_1 \subset H_3, \quad A_1 \times H_3 \subset H_4 \quad (127) \]
\[ I_2(5) \times I_2(5) \subset H_4, \quad A_2 \times A_2 \subset H_4 \quad (128) \]

The easiest way to obtain these inclusions is to take minimally connected graphs of degenerate configurations and remove one vertex in all possible ways. For the Weyl groups these graphs are the extended Dynkin diagrams of affine Coxeter groups. For the non-crystallographic systems \( H_3, H_4 \) can be used the graphs in Fig.26, which represent degenerate configurations.

![Figure 26: Minimally connected degenerate arrangements for \( H_3, H_4 \)](image)

Although the theorem describes how to be obtained finite orbits of the braid group on degenerate systems of reflections the actual determination of these orbits is far from complete. These orbits may hide additional invariant foliated symplectic structure as in the case of rank 2 matrices discussed in [16].

### 7.2 The main theorem

In order to determine all the symmetrized Stokes matrices with finite orbits under the action of the braid group one needs to consider apart from positive semidefinite also the indefinite arrangement matrices. Our attempt to build inductively matrices with finite orbits by adding one row and column to matrices with proved finite orbit may fail because there are some invertible \( n \times n \) matrices
whose all principal minors of size \( n - 1 \times n - 1 \) are degenerate. Examples are

\[
\begin{pmatrix}
2 & -2 & -2 \\
-2 & 2 & -2 \\
-2 & -2 & 2
\end{pmatrix}
\]  \hspace{1cm} (129)

\[
\begin{pmatrix}
2 & 2 \cos \frac{p \pi}{d} & 2 \cos \frac{(p-q) \pi}{d} & 2 \cos \frac{q \pi}{d} \\
2 \cos \frac{p \pi}{d} & 2 & 2 \cos \frac{(p-q) \pi}{d} & 2 \cos \frac{q \pi}{d} \\
2 \cos \frac{(p-q) \pi}{d} & 2 \cos \frac{q \pi}{d} & 2 & 2 \cos \frac{p \pi}{d} \\
2 \cos \frac{(p-q) \pi}{d} & 2 \cos \frac{q \pi}{d} & 2 \cos \frac{p \pi}{d} & 2
\end{pmatrix}
\]  \hspace{1cm} (130)

These matrices are indefinite and do not have finite orbits but all their principal minors, which are the arrangement matrices of their subarrangements are degenerate and have finite orbits. The following lemma is essential.

**Lemma 22.** An invertible arrangement matrix \( B \) with \( n \) rows for which all principal minors of degree \( n - 1 \) are degenerate can always be transformed by a suitable braid to a matrix without this property.

**Proof.** Recall that a principal minor is a submatrix obtained by deleting rows and columns with the same numbers. We have

\[
A_{ij} = B^{-1}_{ij} = \frac{\det(B_{p,q})_{p \neq i, q \neq j}}{\det(B)},
\]  \hspace{1cm} (131)

so the above property implies \( A_{ii} = 0 \) for every \( i \).

The canonical generators of the braid group transform the matrix \( B \) in the following way

\[
\sigma_i(B) = K_i(B) \cdot B \cdot K_i(B),
\]  \hspace{1cm} (132)

\[
K_i(B) = \begin{pmatrix}
\mathbb{I}_{i-1,i-1} & 0 & 0 & 0 \\
0 & -B_{i,i+1} & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & \mathbb{I}_{n-i-1,n-i-1}
\end{pmatrix}
\]  \hspace{1cm} (133)

The inverse matrix \( A \) is transformed correspondingly

\[
\sigma_i(A) = K_i(B)^{-1} \cdot A \cdot K_i(B)^{-1}.
\]  \hspace{1cm} (134)

Assume that for any braid transformation the matrix \( B \) preserves its property of having only degenerate principal minors. The generators of the braid group transform the entries \( B_{i+1,i+k}, A_{i+1,i+k} \) for \( k \geq 1 \) as

\[
\begin{align*}
\sigma_i(B)_{i+1,i+k} &= B_{i,i+k} \\
\sigma_i(A)_{i+1,i+k} &= A_{i,i+k} + B_{i,i+1}A_{i+1,i+k}
\end{align*}
\]  \hspace{1cm} (135)

\[
\begin{align*}
\sigma_i^{-1}(B)_{i+1,i+k} &= B_{i,i+k} - B_{i,i+1}B_{i+1,i+k} \\
\sigma_i^{-1}(A)_{i+1,i+k} &= A_{i,i+k}
\end{align*}
\]  \hspace{1cm} (136)
The diagonal entries of $A$ are zero and must remain zero after the action of any braid. We will prove by induction that this implies $A_{ij}B_{ij} = 0$. We have $Aii = 0$. Assume it is true that $A_{i,i+k-1}B_{i+k-1,i} = 0$ must hold for every $i$ in order $A_{ii}$ to remain zero under any braid. Acting with $\sigma_i$ we obtain

\[
\begin{align*}
A_{i+1,i+k}B_{i+1,i+k} &= 0 \\
B_{i,i+k}A_{i,i+k} + B_{i,i+1}B_{i+1,i+k} &= 0 \\
B_{i,i+k}A_{i,i+k} - B_{i,i+1}B_{i+1,i+k} &= 0
\end{align*}
\] (137)

which implies $B_{i,i+k}A_{i,i+k} = 0$. By induction on $k$ we find that for diagonal entries of $A$ to remain zero under any braid it is necessary to have $A_{ij}B_{ij} = 0$ which is absurd as

\[
\det(B) = \sum_{j=1}^{n} (-1)^{i+j} B_{ij} \det(B_{pq})_{p\neq i, q\neq j} = \sum_{j=1}^{n} (-1)^{i+j} B_{ij} \det(B)_{A_{ij}} = 0. \quad (138)
\]

We conclude that either the matrix $B$ is degenerate or there is a braid transforming it such that there is a non-degenerate principal minor of degree $n - 1$.

Theorem 23. If in the chain constructed above exists a number $i$ such that $s_i > s_{i-1} + 1$, there is a braid transforming $B$ to $B'$ for which $s_i' = s_{i-1}' + 1$.

Proof. We consider the matrix $B_{s_i}$ which is non-degenerate by definition. It is contained in $B_j$, $j \geq s_i$ therefore $i - 1 < s_i$. On the other hand $s_i \leq i$ which implies $s_i = i$. We have $s_i - s_i = i - 1$ therefore while the matrix $B_i$ is non-degenerate all its principal minors of size $i - 1$ must be degenerate. The previous lemma concludes the proof.

We obtain that $s_i$ takes only values $s_{i-1}$ or $s_{i-1} + 1$ for some matrix in the same orbit of the braid group.

Theorem 24. All arrangement matrices with finite orbits are either non-degenerate corresponding to reflections generating finite groups or their extensions with reflections from the same group.

Proof. We proceed by induction. It is proved for the case of 3 by 3 matrices. Assume true for $n \times n$ matrices. Any $n + 1 \times n + 1$ matrix $B$ which is non-degenerate contains an $n \times n$ non-degenerate subarrangement or can be made
so by a suitable braid. There were considered all extensions of non-degenerate arrangements generating finite Coxeter groups and it was shown that they must generate again finite reflection group in order to have finite orbit. Now let the matrix $B$ be degenerate with maximal non-degenerate principal minor $B_s$. By induction hypothesis $B_s$ is an arrangement generating finite Coxeter group. All degenerate extensions by one reflection of $B_s$ must have the new reflection in the group generated by the other or otherwise the orbit of the extended matrix will be infinite. It follows that all reflections in $B$ must belong to the group generated by $B_s$.

8 Conclusion

The classification of the orbits is unfinished. It will be interesting to find if it is possible to linearize in a uniform way the action of the braid group as it was done for the rank two arrangements. One may expect that there will be some hidden structures in analogy with the symplectic structure, which was found in the studied rank two case.

It is appealing how far can be extended the interpretation of configurations with higher degeneracy. Whether these can be used for classification of the quasi-periodic tilings? How must the definition of abstract presentation of Coxeter groups be extended to include groups generated by reflections with such arrangement matrices?

The action of the braid group on pseudoreflections generating finite unitary groups is considered in [4, 3]. The combinatorics of these complex reflections is not well understood. One way to tackle the problem of absence of notion about simple roots is to consider all possible $n$-tuples of pseudoreflections generating finite groups, where the results of the present work would be helpful.

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Table A
Transformations of graphs under the change $r_j \mapsto r_i r_j r_i$

Only the edges ending to the $j$-th vertex are affected. The transformation has period 2. In the table are given pairs of interchanging graphs. The $i$-th vertex is the upper-left corner of the triangle and the $j$-th is the upper-right. The changes from this table must be applied to all pairs $\{j, k\}$ for which the $k$-th vertex is joined to the $i$-th or $j$-th.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
Family & Orbit & Graphs & \( \text{det}(C - \mathbb{I}x) \) \\
\hline
A & 1 & \begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\end{tikzpicture} & \((x^2 - 2x \cos(\frac{7\pi}{15}) + 1) \left( x^2 - 2x \cos(\frac{11\pi}{15}) + 1 \right) \) \\
2 & \begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\end{tikzpicture} & q_\prec = 2 & \((x^2 - 2x \cos(\frac{7\pi}{9}) + 1)^2 \) \\
\hline
B & 1 & \begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\end{tikzpicture} & \((x^2 - 2x \cos(\frac{7\pi}{15}) + 1) \left( x^2 - 2x \cos(\frac{13\pi}{15}) + 1 \right) \) \\
2 & \begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\end{tikzpicture} & q_\prec = 2 & \((x^2 - 2x \cos(\frac{4\pi}{9}) + 1)^2 \) \\
\hline
C & 1 & \begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\end{tikzpicture} & \((x^2 - 2x \cos(\frac{3\pi}{10}) + 1) \left( x^2 - 2x \cos(\frac{7\pi}{10}) + 1 \right) \) \\
2 & \begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\end{tikzpicture} & q_\prec = 2 & \((x^2 - 2x \cos(\frac{4\pi}{15}) + 1)^2 \) \\
\hline
D & 1 & \begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\end{tikzpicture} & \((x^2 - 2x \cos(\frac{7\pi}{15}) + 1) \left( x^2 - 2x \cos(\frac{14\pi}{15}) + 1 \right) \) \\
2 & \begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\end{tikzpicture} & q_\prec = 2 & \((x^2 - 2x \cos(\frac{8\pi}{15}) + 1)^2 \) \\
\hline
E & 1 & \begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\end{tikzpicture} & \Phi_{12}(x) \\
2 & \begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\end{tikzpicture} & q_\prec = 2 & \Phi_{10}(x) \\
\hline
3 & \begin{tikzpicture}
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1);
\end{tikzpicture} & q_\prec = 2 & \Phi_6(x)^2 \\
\hline
\end{tabular}
\end{table}

5' = \frac{5}{2}
Table C

Minimal realizations of degenerate extensions of $H_3$ configurations.

| Configuration | Equation                                           | Configuration | Equation                                           |
|---------------|----------------------------------------------------|---------------|----------------------------------------------------|
| ![Configuration 1](image1.png) | $r_4 = r_2r_1r_3r_2r_1r_2$                        | ![Configuration 2](image2.png) | $r_4 = r_2r_1r_3r_2r_1r_2$                        |
| ![Configuration 3](image3.png) | $r_4 = g^{-1}r_2g$                                 | ![Configuration 4](image4.png) | $r_4 = g^{-1}r_2g$                                 |
| ![Configuration 5](image5.png) | $r_4 = r_3r_1r_2r_1r_3$                           | ![Configuration 6](image6.png) | $r_4 = r_3r_2r_1r_2r_3$                           |
| ![Configuration 7](image7.png) | $r_4 = r_1r_2r_3r_2r_1$                           | ![Configuration 8](image8.png) | $r_4 = r_3r_2r_1r_2r_3$                           |
| ![Configuration 9](image9.png) | $r_4 = r_1r_2r_3r_2r_1$                           | ![Configuration 10](image10.png) | $r_4 = r_1r_2r_3r_2r_1$                           |
| ![Configuration 11](image11.png) | $r_4 = r_3r_1r_2r_3r_1r_3$                        | ![Configuration 12](image12.png) | $r_4 = r_3r_2r_3r_1$                             |
Table D

Admissible non-degenerate extensions of universal graphs of $H_4$

\[
\begin{align*}
A & \quad 5 & B & \quad \frac{-7\sqrt{5} + 5}{4} & B & \quad 5 & A & \quad \frac{2\sqrt{5} - 3}{2} \\
\begin{array}{c}
\begin{array}{c}
A \\
5 \\
\end{array}
\end{array} & \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B \\
5 \\
\end{array}
\end{array}
\end{array} & \quad \frac{-7\sqrt{5} + 5}{4} & \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
A \\
5 \\
\end{array}
\end{array}
\end{array} & \quad \frac{2\sqrt{5} - 3}{2} \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B \\
5 \\
\end{array}
\end{array}
\end{array}
\end{array} & \quad \frac{-7\sqrt{5} + 5}{4} & \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B \\
5 \\
\end{array}
\end{array}
\end{array}
\end{array} & \quad \frac{2\sqrt{5} - 3}{2} \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B \\
5 \\
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} & \quad \frac{-7\sqrt{5} + 5}{4} & \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
B \\
5 \\
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} & \quad \frac{2\sqrt{5} - 3}{2} \\
\end{align*}
\]
Table E

Minimal realizations of degenerate extensions of $H_4$ configurations.

| Diagram 1 | Diagram 2 | Equation |
|-----------|-----------|----------|
| ![Diagram] | ![Diagram] | $r_5 = gr_4 g^{-1}$ |
|           |           | $g = r_3 r_2 r_1$ |
| ![Diagram] | ![Diagram] | $r_5 = gr_4 g^{-1}$ |
|           |           | $g = r_2 r_3 r_2 r_1$ |
| ![Diagram] | ![Diagram] | $r_5 = gr_4 g^{-1}$ |
|           |           | $g = r_2 r_1 r_3 r_3 r_1$ |
| ![Diagram] | ![Diagram] | $r_5 = gr_4 g^{-1}$ |
|           |           | $g = r_1 r_2 r_3 r_2 r_1$ |
| ![Diagram] | ![Diagram] | $r_5 = gr_4 g^{-1}$ |
|           |           | $g = r_4 r_1 r_2 r_3 r_2 r_1$ |

continues on next page...
Table E

\[
\begin{array}{ccc}
5 & 5' & 5'\\
5' & 5 & 5
\end{array} - \begin{array}{ccc}
5' & 5 & 5
5 & 5' & 5'
\end{array}
\]
\[r_5 = gr_3g^{-1}\]
\[g = r_1r_2r_4\]

\[
\begin{array}{ccc}
5 & 5' & 5'\\
5' & 5 & 5
\end{array} - \begin{array}{ccc}
5' & 5 & 5
5 & 5' & 5'
\end{array}
\]
\[r_5 = gr_3g^{-1}\]
\[g = r_1r_2r_3r_1r_2r_4\]

\[
\begin{array}{ccc}
5' & 3' & 5'\\
5' & 5 & -5
\end{array} - \begin{array}{ccc}
5' & 3' & 5'\\
5' & 5 & -5
\end{array}
\]
\[r_5 = gr_3g^{-1}\]
\[g = r_2r_1r_2r_4\]

\[
\begin{array}{ccc}
5' & -5' & 5'\\
5' & 5 & -5
\end{array} - \begin{array}{ccc}
5' & -5' & 5'\\
5' & 5 & -5
\end{array}
\]
\[r_5 = gr_3g^{-1}\]
\[g = r_4r_1r_2r_4\]

\[
\begin{array}{ccc}
5' & -5' & 5'\\
5' & 5 & -5
\end{array} - \begin{array}{ccc}
5' & -5' & 5'\\
5' & 5 & -5
\end{array}
\]
\[r_5 = gr_3g^{-1}\]
\[g = r_2r_1r_2r_3r_1r_2r_4\]

\[
\begin{array}{ccc}
5' & 5 & -5'\\
5' & 5' & 5
\end{array} - \begin{array}{ccc}
5' & 5 & -5'\\
5' & 5' & 5
\end{array}
\]
\[r_5 = gr_3g^{-1}\]
\[g = r_3r_1r_2r_4\]

continues on next page...
Table E

\[
\begin{array}{c|c}
\text{5'} & 5' \\
5 & 5 \\
\end{array}
\]

\[
r_5 = gr_3g^{-1} \\
g = r_4r_2r_1r_2r_4
\]

\[
\begin{array}{c|c}
\text{5'} & 5' \\
5 & 5 \\
\end{array}
\]

\[
r_5 = gr_3g^{-1} \\
g = r_2r_3r_1r_2r_4
\]

\[
\begin{array}{c|c}
\text{5'} & 5' \\
5 & 5 \\
\end{array}
\]

\[
r_5 = gr_3g^{-1} \\
g = r_3r_1r_2r_3r_1r_2r_4
\]

\[
\begin{array}{c|c}
\text{5'} & 5' \\
5 & 5 \\
\end{array}
\]

\[
r_5 = gr_3g^{-1} \\
g = r_1r_2r_1r_2r_4
\]

\[
\begin{array}{c|c}
\text{5'} & 5' \\
5 & 5 \\
\end{array}
\]

\[
r_5 = gr_3g^{-1} \\
g = r_1r_4r_1r_2r_4
\]

\[
\begin{array}{c|c}
\text{5'} & 5' \\
5 & 5 \\
\end{array}
\]

\[
r_5 = gr_3g^{-1} \\
g = r_3r_4r_2r_1r_2r_4
\]

continues on next page...
Table E

\[
\begin{align*}
  r_5 &= g r_3 g^{-1} \\
  g &= (r_2 r_3 r_1)^2 r_2 r_4
\end{align*}
\]

\[
\begin{align*}
  r_5 &= g r_3 g^{-1} \\
  g &= r_3 (r_2 r_3 r_1)^2 r_2 r_4
\end{align*}
\]

\[
\begin{align*}
  r_5 &= g r_3 g^{-1} \\
  g &= r_4 r_3 (r_2 r_3 r_1)^2 r_2 r_4
\end{align*}
\]

\[
\begin{align*}
  r_5 &= g r_3 g^{-1} \\
  g &= r_4 r_3 r_2 r_3 r_1 r_2 r_4
\end{align*}
\]

\[
\begin{align*}
  r_5 &= g r_3 g^{-1} \\
  g &= r_3 r_2 r_3 r_1 r_2 r_4
\end{align*}
\]

continues on next page ...
Table E

\[
\begin{align*}
5' & \quad r_5 = gr_4g^{-1} \\
& \quad g = r_3r_1r_2r_1r_3 \\
5' & \quad -r_5 = gr_4g^{-1} \\
& \quad g = r_2r_3r_1r_2r_1r_3 \\
-5' & \quad r_5 = gr_4g^{-1} \\
& \quad g = r_2r_3r_1r_2r_1r_3 \\
-5' & \quad -r_5 = gr_4g^{-1} \\
& \quad g = r_3r_2r_3r_1r_2r_1r_3 \\
5' & \quad r_5 = gr_4g^{-1} \\
& \quad g = r_1r_3r_2r_3r_1r_2r_1r_3 \\
5' & \quad -r_5 = gr_4g^{-1} \\
& \quad g = r_3r_1r_4r_3r_2 \\
-5' & \quad r_5 = gr_4g^{-1} \\
& \quad g = r_1r_3r_2r_3r_1r_2r_1r_3 \\
-5' & \quad -r_5 = gr_4g^{-1} \\
& \quad g = r_2r_3r_4r_1r_2 \\
5' & \quad r_5 = gr_4g^{-1} \\
& \quad g = r_1r_4r_3r_2 \\
5' & \quad -r_5 = gr_4g^{-1} \\
& \quad g = r_2r_3r_4r_1r_2 \\
-5' & \quad r_5 = gr_4g^{-1} \\
& \quad g = r_3r_4r_1r_2
\end{align*}
\]

continues on next page...
| Diagram | Equation | Diagram | Equation |
|---------|----------|---------|----------|
| ![Diagram](image1.png) | $r_5 = g r_4 g^{-1}$ \[ g = r_3 r_2 r_1 r_3 \] | ![Diagram](image2.png) | $r_5 = g r_4 g^{-1}$ \[ g = r_3 r_4 r_3 r_4 r_1 r_2 \] |
| ![Diagram](image3.png) | $r_5 = g r_4 g^{-1}$ \[ g = r_1 r_2 r_3 r_1 \] | ![Diagram](image4.png) | $r_5 = g r_4 g^{-1}$ \[ g = r_1 r_4 r_4 r_3 r_2 \] |
| ![Diagram](image5.png) | $r_5 = g r_4 g^{-1}$ \[ g = r_4 r_1 r_2 r_1 r_3 \] | ![Diagram](image6.png) | $r_5 = g r_4 g^{-1}$ \[ g = r_2 r_1 r_2 r_3 r_1 r_2 r_1 r_3 \] |
| ![Diagram](image7.png) | $r_5 = g r_4 g^{-1}$ \[ g = r_4 r_3 r_2 r_3 r_1 \] | ![Diagram](image8.png) | $r_5 = g r_4 g^{-1}$ \[ g = r_2 r_3 r_2 r_1 r_3 r_2 r_3 r_1 \] |
| ![Diagram](image9.png) | $r_5 = g r_4 g^{-1}$ \[ g = r_3 r_2 r_1 r_3 r_4 r_1 r_2 \] | ![Diagram](image10.png) | $r_5 = g r_4 g^{-1}$ \[ g = r_2 r_1 r_3 r_4 r_1 r_2 \] |
| ![Diagram](image11.png) | $r_5 = g r_4 g^{-1}$ \[ g = r_1 r_2 r_3 r_1 r_4 r_3 r_2 \] | ![Diagram](image12.png) | $r_5 = g r_4 g^{-1}$ \[ g = r_2 r_3 r_1 r_4 r_3 r_2 \] |

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Table E

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