THE ONE-DIMENSIONAL SPINLESS RELATIVISTIC COULOMB PROBLEM

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Abstract

Motivated by a recent analysis which presents explicitly the general solution, we consider the eigenvalue problem of the spinless Salpeter equation with a (“hard-core amended”) Coulomb interaction potential in one dimension. We prove the existence of a critical coupling constant (which contradicts the assertions of the previous analysis) and give analytic upper bounds on the energy eigenvalues. These upper bounds seem to disprove the previous explicit solution.

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1 Introduction: The Spinless Salpeter Equation in One Dimension

The spinless Salpeter equation arises either as a standard reduction of the well-known Bethe–Salpeter formalism \cite{1} for the description of bound states within the framework of relativistic quantum field theory or as a straightforward relativistic generalization of the nonrelativistic Schrödinger equation. This semirelativistic equation of motion with a static interaction described by the Coulomb potential (originating, for instance, from the exchange of a massless particle between the bound-state constituents) defines what we call, for short, the “spinless relativistic Coulomb problem.”

Recently, confining the configuration space to the positive half-line (and mimicking thereby the effect of a “hard-core”), the relativistic Coulomb problem has been studied in one dimension \cite{5}. This one-dimensional case may serve as a toy model which might prove to be instructive for the analysis of the still unsolved three-dimensional problem. In view of its potential importance, we re-analyze this nontrivial and delicate problem.

The spinless Salpeter equation may be regarded as the eigenvalue equation

\[ H|\chi_k\rangle = E_k|\chi_k\rangle, \quad k = 1, 2, 3, \ldots , \]

for the complete set of Hilbert-space eigenvectors \( |\chi_k\rangle \) and corresponding eigenvalues

\[ E_k \equiv \frac{\langle \chi_k|H|\chi_k\rangle}{\langle \chi_k|\chi_k\rangle} \]

of a self-adjoint operator \( H \) of Hamiltonian form, consisting of a momentum-dependent kinetic-energy operator and a coordinate-dependent interaction-potential operator:

\[ H = T + V, \tag{1} \]

where \( T \) is the “square-root” operator of the relativistic kinetic energy of some particle of mass \( m \) and momentum \( p \),

\[ T = T(p) \equiv \sqrt{p^2 + m^2}, \tag{2} \]

and \( V = V(x) \) is an arbitrary, coordinate-dependent, static interaction potential. The action of the kinetic-energy operator \( T \) on an element \( \psi \) of \( L^2(R) \), the Hilbert space of square-integrable functions on the real line, \( R \), is defined by (cf. also Eq. (3) of Ref. \cite{4})

\[ (T\psi)(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} dy \sqrt{p^2 + m^2} \exp[ip(x-y)]\psi(y). \tag{3} \]

In Ref. \cite{4}, the domain of \( H \) is restricted to square-integrable functions \( \Psi(x) \) with support on the positive real line, \( R^+ \), only, vanishing at \( x = 0 \) (cf. Eq. (28) of Ref. \cite{4}):

\[ \Psi(x) = 0 \quad \text{for} \quad x \leq 0 . \]

This restriction may be interpreted as due to the presence of a “hard-core” interaction potential effective for \( x \leq 0 \). For \( x > 0 \), the interaction potential \( V \) is chosen to be of Coulomb type, its strength parametrized by a positive coupling constant \( \alpha \), i.e., \( \alpha > 0 \):

\[ V(x) = V_C(x) = -\frac{\alpha}{x} \quad \text{for} \quad x > 0 . \]

Let the Coulomb-type semirelativistic Hamiltonian \( H_C \) be the operator defined in this way.\footnote{The present state-of-the-art of the three-dimensional relativistic Coulomb problem has been reviewed, for instance, in Refs. \cite{2,3,4}.}
2 Concerns — Dark Clouds Appear at the Horizon

Now, according to the analysis of Ref. [5], the point spectrum of the Hamiltonian $H_C$ consists of the set of eigenvalues (cf. Eq. (33) of Ref. [5])

$$\tilde{E}_n = \frac{m}{\sqrt{1 + \frac{\alpha^2}{n^2}}} , \quad n = 1, 2, 3, \ldots . \tag{4}$$

The corresponding eigenfunctions $\Psi_n(x)$ must be of the form (cf. Eq. (28) of Ref. [5])

$$\Psi_n(x) = \psi_n(x) \Theta(x) , \quad n = 1, 2, 3, \ldots , \tag{5}$$

where $\Theta(x)$ denotes the Heaviside step function, defined here by

$$\Theta(x) = 1 \quad \text{for} \quad x > 0 ,$$

$$\Theta(x) = 0 \quad \text{for} \quad x \leq 0 .$$

In particular, the (not normalized) eigenfunctions $\psi_n(x), n = 1, 2, 3$, corresponding to the lowest energy eigenvalues $\tilde{E}_n$ are explicitly given by (cf. Eqs. (37)–(40) of Ref. [5])

$$\psi_1(x) = x \exp(-\beta_1 x) ,$$

$$\psi_2(x) = x \left( x - \frac{m^2}{S_2^2 \beta_2} \right) \exp(-\beta_2 x) ,$$

$$\psi_3(x) = x \left[ x^2 - \frac{3 m^2}{S_3^2 \beta_3} x + \frac{3 m^2 (\beta_3^2 + m^2)}{2 S_3^4 \beta_3^2} \right] \exp(-\beta_3 x) , \tag{6}$$

with (cf. Eq. (32) of Ref. [5])

$$\beta_n \equiv \frac{m \alpha}{n \sqrt{1 + \frac{\alpha^2}{n^2}}} = \frac{\alpha}{n} \tilde{E}_n , \quad n = 1, 2, 3, \ldots ,$$

and the abbreviation (cf. Eq. (26) of Ref. [5])

$$S_n \equiv \sqrt{m^2 - \beta_n^2} , \quad n = 1, 2, 3, \ldots .$$

However, there are some facts which cause severe doubts about the validity of this solution:

**Boundedness from below:** For coupling constants $\alpha$ larger than some critical value $\alpha_c$ (which has yet to be determined), the operator $H_C$ is not bounded from below. This may be seen, for instance, already from the expectation value of $H_C$ with respect to the (normalized) trial state $|\Phi\rangle$ defined by the configuration-space trial function

$$\Phi(x) = \varphi(x) \Theta(x)$$

with

$$\varphi(x) = 2 \mu^{3/2} x \exp(-\mu x) , \quad \mu > 0 ,$$
and satisfying the normalization condition

\[ \|\Phi\|^2 \equiv \langle \Phi | \Phi \rangle = \int_0^\infty dx \, |\varphi(x)|^2 = 1. \]

Apart from the arbitrariness of the variational parameter \( \mu \), this trial function \( \Phi \) coincides, in fact, with the ground-state solution \( \Psi \) as given in Eqs. \((3)\), \((4)\). The expectation value of the Coulomb interaction-potential operator \( V_C \) with respect to the trial state \( |\Phi\rangle \) reads

\[ \langle \Phi | V_C | \Phi \rangle = -\alpha \int_0^\infty dx \, \frac{1}{x} |\varphi(x)|^2 = -\mu \alpha. \]

There is a trivial (but nevertheless fundamental) inequality for the expectation values of a self-adjoint (but otherwise arbitrary) operator \( O = O^\dagger \) and its square, taken with respect to an arbitrary Hilbert-space state \( |\psi\rangle \) in the domain \( \mathcal{D}(O) \) of this operator \( O \):

\[ |\langle \psi | O | \psi \rangle| \leq \sqrt{\langle \psi | O^2 | \psi \rangle / \langle \psi | \psi \rangle} \quad \text{for all} \quad |\psi\rangle \in \mathcal{D}(O). \]

Application of this inequality to the kinetic-energy operator \( T \) of Eq. \((2)\) allows to get rid of the troublesome square-root operator:

\[ \langle \Phi | T | \Phi \rangle \leq \sqrt{\langle \Phi | T^2 | \Phi \rangle} \equiv \sqrt{\langle \Phi | p^2 | \Phi \rangle + m^2}. \]

The expectation value of \( p^2 \) required here reads

\[ \langle \Phi | p^2 | \Phi \rangle = \mu^2. \]

Thus the expectation value of the Coulomb-like semirelativistic Hamiltonian \( H_C \) with respect to the trial state \( |\Phi\rangle \) is bounded from above by

\[ \langle \Phi | H_C | \Phi \rangle = \langle \Phi | T + V_C | \Phi \rangle \leq \sqrt{\mu^2 + m^2} - \mu \alpha. \quad \text{(7)} \]

When inspecting this inequality in the limit of large \( \mu \), that is, for \( \mu \to \infty \), one realizes that, for \( \alpha \) large enough, the operator \( H_C \) is not bounded from below. In fact, the expectation value of the kinetic-energy operator \( T \) with respect to the trial state \( |\Phi\rangle \),

\[ \langle \Phi | T | \Phi \rangle = \int_{-\infty}^{+\infty} dx \, \Phi^*(x) (T \Phi)(x) = \frac{4 \mu^3}{\pi} \int_0^\infty dp \, \sqrt{\frac{p^2 + m^2}{(p^2 + \mu^2)^2}}, \quad \text{(8)} \]

is simple enough to be investigated explicitly. For \( \mu \gg m \), this expectation value simplifies to

\[ \langle \Phi | T | \Phi \rangle = \frac{2 \mu}{\pi} \quad \text{for} \quad \mu \gg m. \]

Consequently, in the (ultrarelativistic) limit \( \mu \to \infty \), the expectation value of \( H_C \) behaves like

\[ \lim_{\mu \to \infty} \frac{\langle \Phi | H_C | \Phi \rangle}{\mu} = \frac{2}{\pi} - \alpha. \]
This clearly indicates that for the Hamiltonian $H_C$ to be bounded from below the Coulomb coupling constant $\alpha$ has to be bounded from above by the critical value

$$\alpha_c \leq \frac{2}{\pi}.$$ 

(This upper bound on $\alpha_c$ is, in fact, identical to the critical coupling constant $\alpha_c$ found in the case of the three-dimensional spinless relativistic Coulomb problem \[6\].)

**Upper bound on lowest eigenvalue:** As rather trivial consequence of the famous minimum–maximum principle \[7\], the expectation value

$$\frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle}$$

of a self-adjoint operator $H$ bounded from below, with respect to some arbitrary state $|\psi\rangle$ in the domain of $H$, $D(H)$, is always larger than or equal to the lowest eigenvalue $E_1$ of $H$:

$$E_1 \leq \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} \text{ for all } |\psi\rangle \in D(H).$$

Accordingly, minimizing the expression on the right-hand side of inequality (7) with respect to the variational parameter $\mu$ yields a simple analytic upper bound $\hat{E}_1$ on the ground-state energy eigenvalue $E_1$ of the Coulomb-like semirelativistic Hamiltonian $H_C$:

$$E_1 \leq \hat{E}_1$$

with

$$\hat{E}_1 = m \sqrt{1 - \alpha^2}. \quad \text{(9)}$$

The same analytic upper bound on the ground-state energy $E_1$ has been found in the case of the three-dimensional spinless relativistic Coulomb problem \[8, 9, 4\]. Reality of this latter expression requires again the existence of a critical coupling constant $\alpha_c$ and indicates that this critical value of $\alpha$ is less than or equal to 1:

$$\alpha_c \leq 1.$$ 

Moreover, at least for the energy eigenvalue $E_1$ corresponding to the ground state of the Hamiltonian $H_C$, the supposedly *exact* value of Eq. (4),

$$\tilde{E}_1 = \frac{m}{\sqrt{1 + \alpha^2}}, \quad \text{(10)}$$

is in clear conflict with the naive upper bound $\hat{E}_1$ of Eq. (9):

$$\frac{\hat{E}_1}{E_1} = \sqrt{1 - \alpha^4}$$

and therefore

$$\hat{E}_1 < \tilde{E}_1 \quad \text{for } \alpha > 0.$$ 

\[2\] This statement constitutes what is sometimes simply called “Rayleigh’s principle.”
For larger values of the Coulomb coupling constant $\alpha$, the upper bound (3) on the ground-state energy can be easily improved by fixing in the expectation value (8) of the kinetic-energy operator $T$ the variational parameter $\mu$ to the value $\mu = m$.

In this case, this expectation value reads

$$\langle \Phi | T | \Phi \rangle = \frac{4 m}{\pi}.$$ 

Accordingly, the ground-state energy eigenvalue $E_1$ is bounded from above by

$$E_1 \leq \left( \frac{4}{\pi} - \alpha \right) m .$$

(11)

For the Coulomb coupling constant $\alpha$ in the range

$$\frac{2}{\pi} - \sqrt{\frac{1}{2} - \frac{4}{\pi^2} < \alpha \leq \frac{2}{\pi} ,$$

the above expression represents a genuine improvement of the upper bound (3).

### Eigenstate expectation values vs. eigenvalues:

The expectation value (8) of the kinetic-energy operator $T$ with respect to the trial state $|\Phi \rangle$ may be written down explicitly:

$$\langle \Phi | T | \Phi \rangle = \frac{2 m}{\pi} \left[ \frac{\mu}{m} + \frac{\arccos \frac{\mu}{m}}{\sqrt{1 - \frac{\mu^2}{m^2}}} \right].$$

Now, for

$$\mu = \beta_1 = \frac{m \alpha}{\sqrt{1 + \alpha^2}},$$

the trial function $\Phi$ coincides with the normalized ground-state eigenfunction $\Psi_1$. In this case, the corresponding expectation value of the Hamiltonian $H_C$ becomes

$$\langle \Psi_1 | H_C | \Psi_1 \rangle = \frac{m}{\sqrt{1 + \alpha^2}} \left[ \frac{2}{\pi} \left( \alpha + (1 + \alpha^2) \arccot \alpha \right) - \alpha^2 \right].$$

(12)

Unfortunately, the above expectation value does not agree with the ground-state energy (10) deduced from Eq. (1):

$$\langle \Psi_1 | H_C | \Psi_1 \rangle \neq \tilde{E}_1 .$$

### Orthogonality of eigenstates:

Eigenstates $|\chi_i \rangle$, $i = 1, 2, 3, \ldots$, of some self-adjoint operator $H$ corresponding to distinct eigenvalues of $H$ are mutually orthogonal:

$$\langle \chi_i | \chi_k \rangle \propto \delta_{ik} , \quad i, k = 1, 2, 3, \ldots .$$

This feature is definitely not exhibited by the overlaps

$$\langle \Psi_i | \Psi_k \rangle = \int_{-\infty}^{+\infty} dx \, \psi_i^* (x) \psi_k (x) = \int_{-\infty}^{+\infty} dx \, \psi_i^* (x) \psi_k (x) , \quad i, k = 1, 2, 3, \ldots ,$$

of the lowest eigenfunctions $\Psi_i (x), i = 1, 2, 3$, given in Eqs. (4), (5). For instance, the overlap $\langle \Psi_1 | \Psi_2 \rangle$ of the ground state $|\Psi_1 \rangle$ and the first excitation $|\Psi_2 \rangle$ is given by

$$\langle \Psi_1 | \Psi_2 \rangle = \frac{2 \left[ 3 S^2 \beta_2 - m^2 (\beta_1 + \beta_2) \right]}{(\beta_1 + \beta_2)^4 S^2 \beta_2} ,$$

revealing thus, beyond doubt, the non-orthogonality of the vectors $|\Psi_1 \rangle$ and $|\Psi_2 \rangle$. 
3 Exact Analytic Upper Bounds on Energy Levels

In view of the above, let us try to collect unambiguous results for the one-dimensional spinless relativistic Coulomb problem. With the help of the definition (3) of the action of a momentum-dependent operator in coordinate space, it is easy to convince oneself of the validity of the operator inequality

\[ T \leq T_{NR} \equiv m + \frac{p^2}{2m}; \]

the relativistic kinetic-energy operator \( T \) is bounded from above by its nonrelativistic counterpart \( T_{NR} \): when introducing the Fourier transform \( \tilde{\psi}(p) \) of the coordinate-space representation \( \psi(x) \) of the Hilbert-space vector \( |\psi\rangle \),

\[ \tilde{\psi}(p) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \exp[-ipx] \psi(x), \]

one finds

\[ \langle \psi | T_{NR} - T | \psi \rangle = \int_{-\infty}^{+\infty} dx \psi^*(x) [(T_{NR}\psi)(x) - (T\psi)(x)] = \int_{-\infty}^{+\infty} dp |\tilde{\psi}(p)|^2 \left( m + \frac{p^2}{2m} - \sqrt{p^2 + m^2} \right) \geq 0. \]

Hence, adding the Coulomb interaction potential \( V_C \), the semirelativistic Hamiltonian \( H_C \) is, of course, bounded from above by the corresponding nonrelativistic Hamiltonian \( H_{C,NR} \):

\[ H_C \leq H_{C,NR} \equiv T_{NR} + V_C. \]

Now, upon invoking the minimum–maximum principle \[7\] (which requires the operator \( H_C \) to be both self-adjoint and bounded from below) and combining this principle with the above operator inequality, we infer that every eigenvalue \( E_n \), \( n = 1, 2, 3, \ldots, \) of \( H_C \) is bounded from above by a corresponding eigenvalue \( E_{n,NR} \), \( n = 1, 2, 3, \ldots, \) of \( H_{C,NR} \):

\[ E_n \leq E_{n,NR} \quad \text{for } n = 1, 2, 3, \ldots. \]

It is a simple and straightforward exercise to calculate the latter set of eigenvalues:

\[ E_{n,NR} = m \left( 1 - \frac{\alpha^2}{2n^2} \right), \quad n = 1, 2, 3, \ldots. \]

These upper bounds on the energy eigenvalues \( E_n \) may be easily improved by the same reasoning as before. Introducing an arbitrary real parameter \( \eta \) (with the dimension of mass), we find a set of operator inequalities for the kinetic energy \( T \) \[9, 4], namely,

\[ T \leq \frac{p^2 + m^2 + \eta^2}{2\eta} \quad \text{for all } \eta > 0. \]

\footnote{The line of arguments leading to the general form of this statement may be found, for instance, in Refs. \[9, 4\]. It is summarized in Appendix \[4\]. For a rather brief account of the application of these ideas to the three-dimensional spinless relativistic Coulomb problem, see, e.g., Ref. \[4\].}
and, consequently, a set of operator inequalities for the Coulomb-type semirelativistic Hamiltonian $H_C$:

$$H_C \leq \hat{H}_C(\eta) \equiv \frac{p^2 + m^2 + \eta^2}{2\eta} + V_C$$

for all $\eta > 0$.

Accordingly, every eigenvalue $E_n$, $n = 1, 2, 3, \ldots$, of $H_C$ is bounded from above by the minimum, with respect to the mass parameter $\eta$, of the corresponding eigenvalue of $\hat{H}_C$:

$$\hat{E}_{n,C}(\eta) = \frac{1}{2} \left[ m^2 + \eta^2 \left( 1 - \frac{\alpha^2}{n^2} \right) \right], \quad n = 1, 2, 3, \ldots,$$

of $\hat{H}_C(\eta)$:

$$E_n \leq \min_{\eta > 0} \hat{E}_{n,C}(\eta) = m \sqrt{1 - \frac{\alpha^2}{n^2}} \quad \text{for all } \alpha \leq \alpha_c.$$

For $n = 1$, this (variational) upper bound coincides with the previous upper bound (9).

It goes without saying that these upper bounds are violated by the energy eigenvalues $\tilde{E}_n$ given in Eq. (4):

$$\frac{1}{E_n} \min_{\eta > 0} \tilde{E}_{n,C}(\eta) = \sqrt{1 - \frac{\alpha^4}{n^4}} < 1 \quad \text{for } \alpha \neq 0, \quad \text{for all } n = 1, 2, 3, \ldots$$

means

$$\min_{\eta > 0} \tilde{E}_{n,C}(\eta) < \tilde{E}_n \quad \text{for } \alpha \neq 0, \quad \text{for all } n = 1, 2, 3, \ldots!$$

Moreover, for $\mu = m \alpha$, our generic trial state $|\Phi\rangle$ becomes the lowest eigenstate of the nonrelativistic Hamiltonian $H_{C, NR}$, corresponding to the ground-state eigenvalue $E_{1, NR}$:

$$E_{1, NR} = m \left( 1 - \frac{\alpha^2}{2} \right),$$

which may be easily seen:

$$(T_{NR}\varphi)(x) = \left( m - \frac{1}{2m} \frac{d^2}{dx^2} \right) \varphi(x) = \left( m - \frac{\mu^2}{2m} + \frac{\mu}{m} \frac{1}{x} \right) \varphi(x) \quad \text{for } x > 0$$

implies (with $\mu = m \alpha$)

$$H_{C, NR}|\Phi\rangle = E_{1, NR}|\Phi\rangle.$$

It appears rather unlikely that the same functional form represents also the eigenstate of the semirelativistic Hamiltonian $H_C$.

4 Summary, Further Considerations, Conclusions

This work is devoted to the study of the one-dimensional spinless relativistic Coulomb problem on the positive half-line. Assuming a (dense) domain in $L_2(R^+)$ such that the semirelativistic Coulombic Hamiltonian $H_C$ defined in the Introduction is self-adjoint, analytic upper bounds on the energy eigenvalues $E_k$, $k = 1, 2, 3, \ldots$, have been derived:

$$E_k \leq m \sqrt{1 - \frac{\alpha^2}{k^2}} \quad \text{for all } k = 1, 2, 3, \ldots. \quad \text{(13)}$$

Surprisingly, the explicit solution presented in Ref. 5 does not fit into these bounds.

\footnote{The Coulomb problem involves no dimensional parameter other than the particle mass $m$. Therefore, both the energy eigenvalues $E_n$ and the parameter(s) $\mu$ have to be proportional to $m$.}
In order to cast some light into this confusing situation, let us inspect the action of the kinetic-energy operator $T$ in more detail. Consider not normalized Hilbert-space vectors $|\Phi_n\rangle$, $n = 0, 1, 2, \ldots$, defined, as usual, by the coordinate-space representation

$$\Phi_n(x) = x^n \exp(-\mu x) \Theta(x) , \quad \mu > 0 , \quad n = 0, 1, 2, \ldots .$$

These vectors certainly belong to the Hilbert space $L_2(R)$ for all $n = 0, 1, 2, \ldots$, since

$$\|\Phi_n\|^2 = \langle \Phi_n | \Phi_n \rangle = \int_{-\infty}^{+\infty} dx |\Phi_n(x)|^2 = \int_0^{+\infty} dx x^{2n} \exp(-2\mu x) = \frac{\Gamma(2n+1)}{(2\mu)^{n+1}} < \infty .$$

**However:** The norm $\|T|\Phi_n\|$ of the vectors $T|\Phi_n\rangle$, $n = 0, 1, 2, \ldots$, may be found from

$$\|T|\Phi_n\|^2 = \int_{-\infty}^{+\infty} dx |(T|\Phi_n\rangle(x)|^2 = \frac{[\Gamma(n+1)]^2}{2\pi} \int_{-\infty}^{+\infty} dp \frac{m^2 + p^2}{(m^2 + p^2)^{n+1}} .$$

This observation might be a hint that the vector $|\Phi_0\rangle$, that is, $\Phi_0(x) = \exp(-\mu x) \Theta(x)$, does not belong to the domain of the kinetic-energy operator $T$. If this is indeed true, it is by no means obvious how to make sense of Eq. (16) of Ref. [5] for the case $n = 0$.

Trivially, if Eq. (16) of Ref. [5] is correct for $n = 0$, all these relations for arbitrary $n = 1, 2, \ldots$ may be obtained by a simple differentiation of the relation for $n = 0$ with respect to the (generic) parameter $\mu$, taking advantage of

$$T x^n \exp(-\mu x) = \left( -\frac{d}{d\mu} \right)^n T \exp(-\mu x) .$$

Similarly, it is somewhat hard to believe that Eq. (16) of Ref. [5] holds for $n = 1$. In our notation, Eq. (16) of Ref. [5] would read for $n = 1$

$$(T|\Phi_1\rangle(x) = S + \frac{\mu}{S x} )|\Phi_1\rangle(x)$$

with

$$S \equiv \sqrt{m^2 - \mu^2} .$$

Considering merely the norms of the vectors on both sides of this equation, we find, for the norm of the vector on the left-hand side,

$$\|T|\Phi_1\|^2 = \frac{m^2 + \mu^2}{4 \mu^3}$$

but, for the norm of the vector on the right-hand side,

$$\left( S + \frac{\mu}{S x} \right) |\Phi_1\rangle \|^2 = \frac{m^4 + \mu^4}{4 \mu^3 S^2} .$$

These two expressions for the norms become equal only for the—excluded—case $\mu = 0$.

Unfortunately, precisely the above relation forms the basis for the assertion in Ref. [5] that $\Phi_1(x)$ with $\mu = \beta_1$ is the ground-state eigenfunction of the (“hard-core amended”) one-dimensional spinless relativistic Coulomb problem as defined in the Introduction.

In conclusion, let us summarize our point of view as follows: The energy eigenvalues $E_k$, $k = 1, 2, 3, \ldots$, of the one-dimensional spinless relativistic Coulomb problem (with hard-core interaction on the nonpositive real line) are bounded from above by Eq. (13).

For the ground-state energy eigenvalue $E_1$, this upper bound may be improved to some extent, by considering appropriately the minimum of the bounds of Eq. (11), or Eq. (12), or Eq. (13) for $k = 1$, that is, Eq. (9). To our knowledge, these upper bounds represent the only information available at present about the exact location of the energy levels of the ("hard-core amended") one-dimensional spinless relativistic Coulomb problem.
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A Combining Minimum–Maximum Principle with Operator Inequalities [9, 4]

There exist several equivalent formulations of the well-known “min–max principle” [7]. For practical purposes, the most convenient one is perhaps the following:

- Let $H$ be a self-adjoint operator bounded from below.
- Let $E_k$, $k = 1, 2, 3, \ldots$, denote the eigenvalues of $H$, defined by
  \[ H|\chi_k\rangle = E_k|\chi_k\rangle, \quad k = 1, 2, 3, \ldots, \]
  and ordered according to
  \[ E_1 \leq E_2 \leq E_3 \leq \ldots. \]
- Consider only the eigenvalues $E_k$ below the onset of the essential spectrum of $H$.
- Let $D_d$ be some $d$-dimensional subspace of the domain $\mathcal{D}(H)$ of $H$: $D_d \subset \mathcal{D}(H)$.

Then the $k$th eigenvalue $E_k$ (when counting multiplicity) of $H$ satisfies the inequality

\[
E_k \leq \sup_{|\psi\rangle \in D_k} \frac{\langle \psi|H|\psi\rangle}{\langle \psi|\psi\rangle} \quad \text{for } k = 1, 2, 3, \ldots.
\]

The min–max principle may be employed in order to compare eigenvalues of operators:

- Assume the validity of a generic operator inequality of the form
  \[ H \leq \mathcal{O}. \]
  Then
  \[
  E_k = \frac{\langle \chi_k|H|\chi_k\rangle}{\langle \chi_k|\chi_k\rangle} \leq \sup_{|\psi\rangle \in D_k} \frac{\langle \psi|H|\psi\rangle}{\langle \psi|\psi\rangle} \leq \sup_{|\psi\rangle \in D_k} \frac{\langle \psi|\mathcal{O}|\psi\rangle}{\langle \psi|\psi\rangle}.
  \]
- Assume that the $k$-dimensional subspace $D_k$ in this inequality is spanned by the first $k$ eigenvectors of the operator $\mathcal{O}$, that is, by precisely those eigenvectors of $\mathcal{O}$ that correspond to the first $k$ eigenvalues $\hat{E}_1, \hat{E}_2, \ldots, \hat{E}_k$ of $\mathcal{O}$ if the eigenvalues of $\mathcal{O}$ are ordered according to
  \[ \hat{E}_1 \leq \hat{E}_2 \leq \hat{E}_3 \leq \ldots. \]

Then
  \[
  \sup_{|\psi\rangle \in D_k} \frac{\langle \psi|\mathcal{O}|\psi\rangle}{\langle \psi|\psi\rangle} = \hat{E}_k.
  \]

Consequently, every eigenvalue $E_k$, $k = 1, 2, 3, \ldots$, of $H$ is bounded from above by the corresponding eigenvalue $\hat{E}_k$, $k = 1, 2, 3, \ldots$, of $\mathcal{O}$:

\[
E_k \leq \hat{E}_k \quad \text{for } k = 1, 2, 3, \ldots.
\]
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