CONJUGACIES BETWEEN P- HOMEOMORPHISMS WITH SEVERAL BREAKS

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Abstract

Let \( f_i, i = 1, 2 \) be orientation preserving circle homeomorphisms with a finite number of break points, at which the first derivatives \( Df_i \) have jumps, and with identical irrational rotation number \( \rho = \rho_{f_1} = \rho_{f_2} \). The jump ratio of \( f_i \) at the break point \( b \) is denoted by \( \sigma_{f_i}(b) \), i.e. \( \sigma_{f_i}(b) := \frac{Df_i(b+0)}{Df_i(b-0)} \). Denote by \( \sigma_{f_i}, i = 1, 2 \), the total jump ratio given by the product over all break points \( b \) of the jump ratios \( \sigma_{f_i}(b) \) of \( f_i \). We prove, that for circle homeomorphisms \( f_i, i = 1, 2 \), which are \( C^{2+\varepsilon}, \varepsilon > 0 \), on each interval of continuity of \( Df_i \) and whose total jump ratios \( \sigma_{f_1} \) and \( \sigma_{f_2} \) do not coincide, the conjugacy between \( f_1 \) and \( f_2 \) is a singular function.

1 Introduction

Let \( f \) be an orientation preserving homeomorphism of the circle \( S^1 \equiv \mathbb{R}/\mathbb{Z}^1 \) with lift \( F : \mathbb{R} \to \mathbb{R} \), which is continuous, strictly increasing and fulfills \( F(x + 1) = F(x) + 1, x \in \mathbb{R} \). The circle homeomorphism \( f \) is then defined by \( f(x) = F(x) \ (mod \ 1), x \in S^1 \).

Denjoy’s classical theorem [5] states, that a circle diffeomorphism \( f \) with irrational rotation number \( \rho = \rho_f \) and such that \( \log Df \) is of bounded variation, is conjugate to the linear rotation \( f_{\rho} \), that is, there exists a homeomorphism \( \varphi \) of the circle with \( f = \varphi^{-1} \circ f_{\rho} \circ \varphi \).

It is well known that a circle homeomorphisms \( f \) with irrational rotation number \( \rho \) is strictly ergodic, i.e. it has a unique \( f \)-invariant probability measure \( \mu_f \). A remarkable fact is then that the conjugacy \( \varphi \) can be defined by \( \varphi(x) = \mu_f([0, x]) \) (see [3]), which shows, that the regularity properties of this conjugacy \( \varphi \) imply the corresponding properties of the density of the absolutely continuous invariant measure \( \mu_f \). The problem of smoothness of the conjugacy of smooth diffeomorphisms is by now very well understood (see for instance [1], [17], [11], [12], [14], [18]). Notice, that for a sufficiently smooth circle diffeomorphism with a typical irrational rotation number its invariant measure is absolutely continuous with respect to Lebesque measure (see [12], [14]).

A natural extension of circle diffeomorphisms are piecewise smooth homeomorphisms with break points or shortly, the class of P-homeomorphisms.

This class of P-homeomorphisms consists of orientation preserving circle homeomorphisms \( f \) which are differentiable except at a finite number of break points, at which the one-sided positive derivatives \( Df_+ \) and \( Df_- \) exist, which do not coincide and for which there exist constants \( 0 < c_1 < c_2 < \infty \), such that

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\[ 1 \text{ MSC: 37E10, 37C15, 37C40} \]

\[ 2 \text{ Keywords and phrases: circle homeomorphism, break point, rotation number, conjugation} \]

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• $c_1 < Df_-(x) < c_2$ and $c_1 < Df_+(x) < c_2$ for all $x \in B(f)$, the set of break points of $f$ in $S^1$;
• $c_1 < Df(x) < c_2$ for all $x \in S^1 \setminus B(f)$;
• $\log Df$ has bounded variation in $S^1$ i.e. $v := \text{var}_{S^1} \log Df < \infty$.

The ratio $\sigma_f(x) = \frac{Df_+(x)}{Df_-(x)}$ is called the \textbf{jump ratio} of $f$ at $x$, or, for short, the \textbf{$f$-jump}. The product of all jump ratios is called the \textbf{total jump} of $f$ and denoted by $\sigma_f$. Notice, that Denjoy’s result can be extended to $P$-homeomorphisms with irrational rotation numbers, its precise formulation will be given later.

The rigidity problem for break point equivalent homeomorphisms $f$ with irrational rotation number $\rho$. Denote by $M_0 = \{ \rho : \exists C > 0, \forall n \in \mathbb{N}, k_{2n-1} \leq C \}$, $M_e = \{ \rho : \exists C > 0, \forall n \in \mathbb{N}, k_{2n} \leq C \}$.

Then K. Khanin and A. Teplinskii proved in [15]:

**Theorem 1.1.** Let $f_i \in C^{2+\alpha}(S^1 \setminus \{b_i\})$, $i = 1, 2$, $\alpha > 0$, be circle homeomorphisms with one break point, the same jump ratio $\sigma$ and the same irrational rotation number $\rho \in (0,1)$. If either $\sigma > 1$ and $\rho \in M_e$ or $\sigma < 1$ and $\rho \in M_o$, then the map $h$ conjugating the homeomorphisms $f_1$ and $f_2$ is a $C^1$-diffeomorphism.

In the case of homeomorphisms with different jump ratios the following theorem was proved by A. Dzhaliilov, H. Akin and S. Temir in [10]:

**Theorem 1.2.** Let $f_i \in C^{2+\alpha}(S^1 \setminus \{b_i\})$, $i = 1, 2$, $\alpha > 0$, be circle homeomorphisms with one break point and different jump ratio but the same irrational rotation number $\rho \in (0,1)$. Then the map $h$ conjugating the homeomorphisms $f_1$ and $f_2$ is a singular function.

Now consider two piecewise-smooth circle homeomorphisms $f_1$ and $f_2$ with $m$ ($m \geq 2$) break points and the same irrational rotation number. Denote by $B(f_1)$ and $B(f_2)$ the set of break points of $f_1$ and $f_2$ respectively.

**Definition 1.3.** The homeomorphisms $f_1$, $f_2$ are said to be \textbf{break point equivalent} if there exists a topological conjugacy $\psi_0$ such that

1. $\psi_0(B(f_1)) = B(f_2)$;
2. $\sigma_{f_2}(\psi_0(b)) = \sigma_{f_1}(b)$, for all $b \in B(f_1)$.

The rigidity problem for break point equivalent $C^{2+\alpha}$-homeomorphisms $f$ with trivial total jumps $\sigma_f = 1$ was studied by K. Cunha and D. Smania in [4]. It was proven
there that any two such homeomorphisms fulfilling certain combinatorial conditions are $C^1$-conjugated. The main idea of their proof is to consider piecewise-smooth circle homeomorphisms as generalized interval exchange transformations. The case of non break point equivalent homeomorphisms with two break points was studied by H. Akhadkulov, A. Dzhalilov and D. Mayer in [2]. Their main result is the following theorem:

**Theorem 1.4.** Let $f_i \in C^{2+\alpha}(S^1 \setminus \{a_i, b_i\}), i = 1, 2$ be circle homeomorphisms with two break points $a_i, b_i$. Assume that

1. their rotation numbers $\rho_{f_i}, i = 1, 2$ are irrational and coincide i.e. $\rho_{f_1} = \rho_{f_2} = \rho$;
2. there exists a bijection $\psi$ such that $\psi(B(f_1)) = B(f_2)$;
3. $\sigma_{f_1} = \sigma_{f_1}(a_1)\sigma_{f_1}(b_1) \neq \sigma_{f_2} = \sigma_{f_2}(a_2)\sigma_{f_2}(b_2)$.

Then the map $h$ conjugating $f_1$ and $f_2$ is a singular function.

In the present paper we study the conjugacy $h$ of two piecewise smooth circle homeomorphisms $f_1$ and $f_2$ with an arbitrary finite number of break points.

Our main result is the following theorem:

**Theorem 1.5.** Let $f_i, i = 1, 2$, be $P$-homeomorphisms with the same irrational rotation number $\rho = \rho_{f_1} = \rho_{f_2}$. Assume, that

1. $f_i, i = 1, 2$ is $C^{2+\alpha}, \alpha > 0$, on each interval of continuity of $Df_i$;
2. the total jumps of $f_1$ and $f_2$ do not coincide i.e. $\sigma_{f_1} = \prod_{b \in B(f_1)} \sigma_{f_1}(b) \neq \sigma_{f_2} = \prod_{b \in B(f_2)} \sigma_{f_2}(b)$.

Then the map $h$ conjugating $f_1$ and $f_2$ is a singular function.

2 Preliminaries and Notations

Let $f$ be an orientation-preserving circle homeomorphism with lift $F$. The important characteristic of homeomorphism $f$ is the rotation number defined by

$$\rho_f := \lim_{n \to \infty} \frac{F^n(x)}{n} \pmod{1}. $$

Here and below, $F^n$ denotes the $n$-th iteration of $F$. Suppose the rotation number $\rho_f$ is irrational. Then it can be uniquely represented as a continued fraction i.e. $\rho_f := [k_1, k_2, ..., k_n, ...]$. Define $q_n := [k_1, k_2, ..., k_n], n \geq 1$ the convergent of $\rho_f$. Their denominators $q_n$ satisfy the recurrence relation: $q_{n+1} = k_{n+1}q_n + q_{n-1}, n \geq 1$, with the initial conditions $q_0 = 1$ and $q_1 = k_1$.

Fix a point $x_0 \in S^1$. Its positive orbit $\{x_i = f^i(x_0), i = 0, 1, 2, ...\}$ defines a sequence of natural partitions of the circle: denote by $\Delta_0^{(n)}(x_0)$ the closed interval in $S^1$ with endpoints $x_0$ and $x_{q_{n}} = f^{q_{n}}(x_0)$. Notice, that for $n$ odd the point $x_{q_{n}}$ is to the left of $x_0$, and for $n$ even it is to its right. Denote by $\Delta_i^{(n)}(x_0) = f^i(\Delta_0^{(n)}(x_0)), i \geq 1$, the iterates of the interval $\Delta_0^{(n)}(x_0)$ under $f$. It is well known, that the set $q_n(x_0)$ of intervals with mutually disjoint interiors defined as

$$q_n(x_0) = \{\Delta_i^{(n-1)}(x_0), 0 \leq i \leq q_n - 1\} \cup \{\Delta_j^{(n)}(x_0), 0 \leq j \leq q_{n-1} - 1\}$$
determines a partition of the circle for any \( n \). The partition \( \eta_n(x_0) \) is called the \( n \)-th **dynamical partition** of the point \( x_0 \). Proceeding from \( \eta_n(x_0) \) to \( \eta_{n+1}(x_0) \) all the intervals \( \Delta_j^{(n)}(x_0) \), \( 0 \leq j \leq q_{n-1} - 1 \), are preserved, whereas each of the intervals \( \Delta_i^{(n-1)}(x_0) \) is partitioned into \( k_n + 1 \) subintervals belonging to \( \eta_{n+1}(x_0) \) such that

\[
\Delta_i^{(n-1)}(x_0) = \Delta_i^{(n+1)}(x_0) \cup \bigcup_{s=0}^{k_n+1} \Delta_i^{(n)}(x_0).
\]

Obviously one has \( \eta_1(x_0) \leq \eta_2(x_0) \leq \ldots \leq \eta_{n}(x_0) \leq \ldots \)

**Definition 2.1.** Let \( K > 1 \) be a constant. We call two intervals \( I_1 \) and \( I_2 \) of \( S^1 \) \( K \)-comparable, if the inequality \( K^{-1} \ell(I_2) \leq \ell(I_1) \leq K \ell(I_2) \) holds.

Following [12] we recall definition.

**Definition 2.2.** An interval \( I = [\tau, t] \subset S^1 \) is said to be \( q_n \)-small, and its endpoints \( q_n \)-close, if the intervals \( f^i(I) \), \( 0 \leq i \leq q_n - 1 \), are pairwise disjoint (except for endpoints).

It follows from the structure of the dynamical partitions that an interval \( I = [\tau, t] \) is \( q_n \)-small if and only if either \( \tau < t \leq f^{q_n-1}(\tau) \), or \( f^{q_n-1}(t) \leq \tau < t \).

**Lemma 2.3.** Let \( f \) be a class \( P \)-homeomorphism with a finite number of break points and irrational rotation number \( \rho = \rho_f \). If the interval \( I = (x, y) \subset S^1 \) is \( q_n \)-small and \( f^s(x), f^s(y) \notin B(f) \) for all \( 0 \leq s < q_n \), then for any \( k \in [0, q_n) \) the following inequality holds:

\[
e^{-v} \leq \frac{Df^k(x)}{Df^k(y)} \leq e^v,
\]

where \( v \) is total variation of \( \log Df \) in \( S^1 \).

**Proof of Lemma 2.3.** Take any two \( q_n \)-close points \( x, y \in S^1 \) and \( 0 \leq k \leq q_n - 1 \). Denote by \( I \) the open interval with endpoints \( x \) and \( y \). Because the intervals \( f^i(I) \), \( 0 \leq i \leq q_n - 1 \) are disjoint, we obtain

\[
|\ln Df^k(x) - \ln Df^k(y)| \leq \sum_{j=0}^{k-1} |\ln Df(f^j(x)) - \ln Df(f^j(y))| \leq v,
\]

from which inequality (2.1) follows immediately.

Using Lemma 2.3 the following lemma can be proven which plays a key role in the study of the metrical properties of homeomorphisms.

**Lemma 2.4.** Suppose the circle homeomorphism \( f \) satisfies the conditions of Lemma 2.3.

Then for any \( y_0 \) with \( y_s := f^s(y_0) \notin B(f) \), for all \( s \in [0, q_n) \) the following inequality holds:

\[
e^{-v} \leq \prod_{s=0}^{q_n-1} Df(y_s) \leq e^v.
\]

Inequality (2.2) is called the Denjoy’s inequality. It follows from Lemma 2.4 that the intervals of the dynamical partition \( \eta_n(x_0) \) have exponentially small lengths. Indeed one finds
Corollary 2.5. Let $\Delta^{(n)}$ be an arbitrary element of the dynamical partition $\eta_n(x_0)$. Then

$$(2.3) \quad \ell(\Delta^{(n)}) \leq \text{const}\lambda^n$$

where $\lambda = (1 + e^{-\nu})^{-\frac{1}{2}} < 1$.

Definition 2.6. Two homeomorphisms $f_1$ and $f_2$ of the circle are said to be topologically equivalent if there exists a homeomorphism $\varphi : S^1 \to S^1$ such that $\varphi(f_1(x)) = f_2(\varphi(x))$ for any $x \in S^1$.

The homeomorphism $\varphi$ is called a conjugacy. Corollary 2.5 implies the following generalization of the classical Denjoy theorem:

Theorem 2.7. Suppose that a homeomorphism $f$ satisfies the conditions of Lemma 2.3. Then the homeomorphism $f$ is topologically conjugate to the linear rotation $f_\rho$.

In the proof of our main theorem the tool of cross-ratio plays a key role.

Definition 2.8. The cross-ratio of four numbers $(z_1, z_2, z_3, z_4)$, $z_1 < z_2 < z_3 < z_4$, is the number

$$\text{Cr}(z_1, z_2, z_3, z_4) = \frac{(z_2 - z_1)(z_4 - z_3)}{(z_3 - z_1)(z_4 - z_2)}.$$

Definition 2.9. Given four real numbers $(z_1, z_2, z_3, z_4)$ with $z_1 < z_2 < z_3 < z_4$ and a strictly increasing function $F : \mathbb{R}^1 \to \mathbb{R}^1$. The distortion of their cross-ratio under $F$ is given by

$$\text{Dist}(z_1, z_2, z_3, z_4; F) = \frac{\text{Cr}(F(z_1), F(z_2), F(z_3), F(z_4))}{\text{Cr}(z_1, z_2, z_3, z_4)}.$$

For $m \geq 3$ and $z_i \in S^1$, $1 \leq i \leq m$, suppose that $z_1 < z_2 < \ldots < z_m < z_1$ (in the sense of the ordering on the circle). Then we set $\hat{z}_1 := z_1$ and

$$\hat{z}_i := \begin{cases} z_i, & \text{if } z_1 < z_i < 1, \\ 1 + z_1, & \text{if } 0 < z_i < z_1. \end{cases}$$

for $2 \leq i \leq m$.

Obviously, $\hat{z}_1 < \hat{z}_2 < \ldots < \hat{z}_m$. The vector $(\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_m)$ is called the lifted vector of $(z_1, z_2, \ldots, z_m) \in (S^1)^m$.

Let $f$ be a circle homeomorphism with lift $F$. We define the cross-ratio distortion of $(z_1, z_2, z_3, z_4)$, $z_1 < z_2 < z_3 < z_4 < z_1$ with respect to $f$ by $\text{Dist}(z_1, z_2, z_3, z_4; f) = \text{Dist}(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4; F)$, where $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ is the lifted vector of $(z_1, z_2, z_3, z_4)$. We need the following

Lemma 2.10. (see [6]). Let $z_i \in S^1$, $i = 1, 2, 3, 4$, $z_1 < z_2 < z_3 < z_4$. Consider a circle homeomorphism $f$ with $f \in C^{2+\varepsilon}([-1, 1])$, $\varepsilon > 0$, and $Df(x) \geq \text{const} > 0$ for $x \in [z_1, z_4]$. Then there is a positive constant $C_1 = C_1(f)$ such that

$$|\text{Dist}(z_1, z_2, z_3, z_4; f) - 1| \leq C_1 |\hat{z}_4 - \hat{z}_1|^{1+\varepsilon},$$

where $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ is the lifted vector of $(z_1, z_2, z_3, z_4)$. 

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We next consider the case where the interval \([z_1, z_4]\) contains a break point \(b\) of the homeomorphism \(f\). More precisely, suppose \(b \in [z_1, z_2]\). Let \(\sigma_f(b)\) be the jump of \(f\) at \(b\). We define numbers \(\alpha, \beta, \tau, \xi\) and \(z\) as follows:

\[
\alpha := \hat{z}_2 - \hat{z}_1, \quad \beta := \hat{z}_3 - \hat{z}_2, \quad \tau := \hat{z}_2 - \hat{b}, \quad \xi := \frac{\beta}{\alpha}, \quad z := \frac{\tau}{\alpha},
\]

where \((\hat{z}_1, \hat{b}, \hat{z}_2, \hat{z}_3)\) is the lifted vector of \((z_1, b, z_2, z_3)\).

In what follows we shall need the following lemma.

**Lemma 2.11.** (see [10]). For the circle homeomorphism \(f\) with \(f \in C^2([z_1, z_4]\setminus\{b\})\), and \(Df(x) \geq \text{const} > 0\) for \(x \in [z_1, z_4]\setminus\{b\}\) one has

\[
\left| \text{Dist}(z_1, z_2, z_3, z_4; f) - \frac{\sigma_f(b) + (1 - \sigma_f(b))z(1 + \xi)}{\sigma_f(b) + (1 - \sigma_f(b))z + \xi} \right| \leq C_2|\hat{z}_4 - \hat{z}_1|,
\]

where the constant \(C_2 > 0\) depends only on \(f\).

### 3 On \(q_n\)-preimages of break points

Let \(f_1\) and \(f_2\) be \(P\)-homeomorphisms with identical irrational rotation numbers \(\rho = \rho_{f_1} = \rho_{f_2}\). Denote by \(B(f_1) = \{b_1^{(i)}, 1 \leq i \leq m_1\}\) and \(B(f_2) = \{b_2^{(j)}, 1 \leq j \leq m_2\}\) the sets of all break points of \(f_1\) and \(f_2\), respectively. Take two copies of the circle on which the homeomorphisms \(f_1\) and \(f_2\) act respectively. Denote by \(\varphi_i, i = 1, 2\) the conjugacies between \(f_1\) and \(f_2\) i.e. \(\varphi_1 \circ f_1 = f_2 \circ \varphi_1\) and \(\varphi_2 \circ f_2 = f_2 \circ \varphi_2\). It is easy to check that the homeomorphisms \(f_1\) and \(f_2\) are then conjugated by \(h = \varphi_2 \circ \varphi_1^{-1}\) i.e. \(h \circ f_1(x) = f_2 \circ h(x), \forall x \in S^1\). For \(x_0 \in S^1\) let \(\eta_n(x_0)\) be its \(n\)-th dynamical partition. Put \(t_0 := h(x_0)\) and consider the dynamical partition \(\tau_n(t_0)\) of \(t_0\) on the second circle determined by the homeomorphism \(f_2\) i.e.

\[
\tau_n(t_0) = \{C_i^{(n-1)}(t_0), 0 \leq i \leq q_n - 1\} \cup \{C_j^{(n)}(t_0), 0 \leq j \leq q_{n-1} - 1\}.
\]

with \(C_i^{(n)}(t_0)\) the closed interval with endpoints \(t_0\) and \(f_2^n(t_0)\). Chose an odd natural number \(n = n(f_1, f_2)\) such that the \(n\)-th renormalization neighborhoods \([x_{q_n}, x_{q_{n-1}}]\) and \([t_{q_n}, t_{q_{n-1}}]\) do not contain any break point of \(f_1\) and \(f_2\) respectively. Since the identical rotation number \(\rho\) of \(f_1\) and \(f_2\) is irrational, the order of the points on the orbit \(\{f_1^k(x_0), k \in \mathbb{Z}\}\) on the first circle will be precisely the same as the one for the orbit \(\{f_2^k(t_0), k \in \mathbb{Z}\}\) on the second one. This together with the relation \(h(f_1(x)) = f_2(h(x))\) for \(x \in S^1\) implies that

\[
(3.1) \quad h(D^{(n-1)}_i) = C^{(n-1)}_i, \quad 0 \leq i \leq q_n - 1, \quad h(D^{(n)}_i) = C^{(n)}_j, \quad 0 \leq j \leq q_{n-1} - 1.
\]

The structure of the dynamical partitions implies that \(\overline{b}^{(i)}_1(n) = f_1^{-l_1^{(i)}}(b_1^{(i)}) \in [x_{q_n}, x_{q_{n-1}}], 1 \leq i \leq m_1\), where \(l_1^{(i)} \in (0, q_{n-1})\) if \(\overline{b}^{(i)}_1(n) \in [x_{q_n}, x_0]\), and \(l_1^{(i)} \in (0, q_n)\) if \(\overline{b}^{(i)}_1(n) \in [x_0, x_{q_{n-1}}]\).

Also \(\overline{b}^{(j)}_2(n) = f_2^{-l_2^{(j)}}(b_2^{(j)}) \in [t_{q_n}, t_{q_{n-1}}], 1 \leq j \leq m_2\) where \(l_2^{(j)} \in (0, q_{n-1})\) if \(\overline{b}^{(j)}_2(n) \in [t_{q_n}, t_0]\), and \(l_2^{(j)} \in (0, q_n)\) if \(\overline{b}^{(j)}_2(n) \in [t_0, t_{q_{n-1}}]\). The points \(\overline{b}^{(i)}_1(n)\) and \(\overline{b}^{(j)}_2(n)\) are called the \(q_n\)-preimages of the break points \(b_1^{(i)}\) and \(b_2^{(j)}\). Denote by \(\overline{B}^{(n)}(f_i), i = 1, 2\), the sets of \(q_n\)-preimages in the renormalization intervals \([x_{q_n}, x_{q_{n-1}}]\) and \([t_{q_n}, t_{q_{n-1}}]\) of the
sets $B(f_1)$ and $B(f_2)$, respectively. Then the number of points in $\overline{B}^{(n)}(f_i), i = 1, 2$, is not greater than $m_i$. Next, consider the set $h^{-1}(\overline{B}^{(n)}(f_2))$. Using the relations \[ (3.3) \] we find that $h^{-1}(\overline{B}^{(n)}(f_2)) \subset [x_{q_n}, x_{q_n-1}]$. Notice, that the number of elements of the set $\overline{B}^{(n)}(f_1) \cup h^{-1}(\overline{B}^{(n)}(f_2))$ is bounded by $m_1 + m_2$.

We set

\[ B^{(n)}_{m_1, m_2} = \{ x_{q_n}, x_{0}, x_{q_n-1} \} \cup \overline{B}^{(n)}(f_1) \cup h^{-1}(\overline{B}^{(n)}(f_2)), \quad d_n = \ell([x_{q_n}, x_{q_n-1}]). \]

Let $m_0 \in \mathbb{N}, m_0 > m_1 + m_2 + 3$. For every $l \geq 0$ we define a partition $D^{(n)}_l$ of the interval $[x_{q_n}, x_{q_n-1}]$ using the points $t_s = x_{q_n} + m_0^{-(l+1)} d_n s, s = 0, 1, ..., m_0^{l+1}$. Obviously, the length of every such interval $I^{(n)}_l$ of $D^{(n)}_l$ is equal to $m_0^{-(l+1)} d_n$. When passing from $D^{(n)}_l$ to $D^{(n)}_{l+1}$, every interval of $D^{(n)}_l$ is divided into $m_0$ intervals $D^{(n)}_{l+1}$.

**Definition 3.1.** An interval of $D^{(n)}_l$ that does not contain any elements of $B^{(n)}_{m_1, m_2}$ is called an $l^{(n)}$-empty interval. Otherwise it is called an $l^{(n)}$-occupied interval.

Since the number of intervals of $D^{(n)}_0$ is greater than the number of elements of $B^{(n)}_{m_1, m_2}$, there exists at least one $0^{(n)}$-empty interval. Furthermore, every $l^{(n)}$-occupied interval of $D^{(n)}_l$ contains at least one $(l + 1)^{(n)}$-empty interval of $D^{(n)}_{l+1}$. Note that the leftmost and rightmost intervals of $D^{(n)}_l$ contain $x_{q_n}$ and $x_{q_n+1}$, respectively. This means that these extreme intervals are $l^{(n)}$-occupied for any $l \geq 0$. Removing all $l^{(n)}$-empty intervals from the interval $[x_{q_n}, x_{q_n+1}]$, we obtain a natural partition of $B^{(n)}_{m_1, m_2}$ into non-empty disjoint parts. Denote this partition by $\Gamma^{(n)}_l$. Between two elements of the partition $\Gamma^{(n)}_l$ lies at least one $l^{(n)}$-empty interval. Removing all $l^{(n)}$-occupied intervals from the interval $[x_{q_n}, x_{q_n+1}]$ we obtain the set $\mathcal{V}^{(n)}_l$ of intervals.

The structure of the set $\overline{B}^{(n)}(f_1) \cup h^{-1}(\overline{B}^{(n)}(f_2))$ in $[x_{q_n}, x_{q_n+1}]$ is given by

**Theorem 3.2.** Let $f_1, f_2$ be $C$-homeomorphisms with a finite number of break points and with identical irrational rotation numbers. Suppose that their total jumps do not coincide. For any positive integer $r$ there exists a number $s_0 = s_0(r, n), 0 \leq s_0 \leq r(m_1 + m_2 + 1)$, such that

1. $\max_{x, y \in E^{(n)}_{s_0}, x \prec y} \ell([x, y]) \leq 2m_0^{-(s_0 + r)} d_n$ for every $E^{(n)}_{s_0} \in \Gamma^{(n)}_{s_0}$;

2. $\ell(I) \geq m_0^{-s_0} d_n$, for all $I \in \mathcal{V}^{(n)}_{s_0}$;

3. there exists at least one element $E^{(n)}_{s_0}$ of the partition $\Gamma^{(n)}_{s_0}$ such that

\[ \prod_{b_1 \in E^{(n)}_{s_0}} \sigma_{f_1}(b_1) \neq \prod_{b_2 \in E^{(n)}_{s_0}} \sigma_{f_2}(b_2). \]

**Proof of Theorem 3.2.** Consider the partitions $D^{(n)}_0, D^{(n)}_1, ..., D^{(n)}_{r(m_1 + m_2 + 1)}$ and the partitions $\Gamma^{(n)}_0, \Gamma^{(n)}_1, ..., \Gamma^{(n)}_{r(m_1 + m_2 + 1)}$ of the set $B^{(n)}_{m_1, m_2}$ generated by them. Let $|\Gamma^{(n)}_l|$ denote the number of elements of the partition $\Gamma^{(n)}_l$. For proving the first two assertions of the theorem it is sufficient to show that $|\Gamma^{(n)}_{s_0}| = |\Gamma^{(n)}_{s_0 + r}|$, for some $s_0$. It follows from the structure of the partitions $\Gamma^{(n)}_l$ that $|\Gamma^{(n)}_l| \leq |\Gamma^{(n)}_{l+1}|$ for any $l \geq 0$. In particular, $|\Gamma^{(n)}_0| \leq |\Gamma^{(n)}_1| \leq ... \leq |\Gamma^{(n)}_{r(m_1 + m_2 + 1)}|$. Then two cases are possible: either
\[ \Gamma_{r_0}^{(n)} = \Gamma_{r(t_0+1)}^{(n)} \] for some \( t_0 = t_0(n), 0 \leq t_0 \leq m_1 + m_2, \) or \( \left| \Gamma_{r_0}^{(n)} \right| < \left| \Gamma_{r}^{(n)} \right| < \ldots < \left| \Gamma_{r(m_1+m_2+1)}^{(n)} \right| \). In the first case we set \( s_0 = r t_0 \). If \( \left| \Gamma_{r_0}^{(n)} \right| < \left| \Gamma_{r_1}^{(n)} \right| < \ldots < \left| \Gamma_{r(m_1+m_2+1)}^{(n)} \right| \), then, because \( \left| \Gamma_{r_0}^{(n)} \right| \geq 2 \), we obtain \( \left| \Gamma_{r(m_1+m_2+1)}^{(n)} \right| \geq m_1 + m_2 + 3 \). But on the other hand \( \left| \Gamma_{r(m_1+m_2+1)}^{(n)} \right| = \left| B_{m_1,m_2}^{(n)} \right| = m_1 + m_2 + 3 \). Consequently, \( \left| \Gamma_{r(m_1+m_2+1)}^{(n)} \right| = m_1 + m_2 + 3 \), and hence the number of elements of \( \Gamma_{r(m_1+m_2+1)}^{(n)} \) coincides with the number of elements of \( B_{m_1,m_2}^{(n)} \). In other words, every element of the partition \( \Gamma_{r(m_1+m_2+1)}^{(n)} \) contains only one element of \( B_{m_1,m_2}^{(n)} \). Hence, \( \left| \Gamma_{s}^{(n)} \right| = m_1 + m_2 + 3 \) for all \( s \geq r(m_1 + m_2 + 1) \). We can take \( r(m_1 + m_2 + 1) = s_0 \). It follows from the construction that the number \( s_0 \) depends on \( n \) but does not exceed \( r(m_1 + m_2 + 1) \). The first and second claims of Theorem 3.2 are therefore proved. Assume, that for every element \( E_{s_0}^{(n)} \) of the partition \( \Gamma_{s_0}^{(n)} \) the relation
\[
\prod_{b_1 \in B(f_1)} \sigma_{f_1}(b_1) = \prod_{b_2 \in B(f_2)} \sigma_{f_2}(b_2),
\]
holds. In this case,
\[
\prod_{b_1 \in B(f_1)} \sigma_{f_1}(b_1) = \prod_{b_2 \in B(f_2)} \sigma_{f_2}(b_2).
\]
in contradiction to the assumption in Theorem 3.2. This concludes the proof of Theorem 3.2.

4 Jump coverings of the circle homeomorphisms with break points

We consider two \( P \)-homeomorphisms \( f_1 \) and \( f_2 \) with identical irrational rotation number \( \rho_1 = \rho_2 \). Suppose that \( f_1 \) and \( f_2 \) has \( m_1 \) respectively \( m_2 \) break points. Denote by \( B(f_i), i = 1, 2 \) the sets of all break points of \( f_i : B(f_1) = \{b_1^{(i)} \mid 1 \leq i \leq m_1 \} \) and \( B(f_2) = \{b_2^{(i)} \mid i = 1, m_2 \} \).

Next we introduce the notion of a "regular" cover of \( B(f_1) \cup h^{-1}(B(f_2)) \), that is the union of the set of break points of \( f_1 \) and the \( h \)-preimage of the set of break points of \( f_2 \).

Let \( z_i \in S^1, i = 1, 2, 3, 4, z_1 < z_2 < z_3 < z_4 < z_1 \) and let \( r_n \) take values in the set \( \{q_{n-1}, q_n, q_{n-1} + q_n\} \). Suppose furthermore that the interval \([z_1, z_4]\) is \( r_n \)-small, i.e. the intervals \( \{f_i([z_1, z_4]), 0 \leq j \leq r_n - 1\} \), are pairwise disjoint. Suppose that the system of intervals \( \{f_i([z_1, z_4]), 0 \leq j \leq r_n - 1\} \), covers the elements of some non-empty subset \( \tilde{B}_1 \subset B(f_1) \) with \( \tilde{B}_1 = \{b_1^{(i,s)} \mid 1 \leq s \leq p_1 \} \). For every element \( b_1^{(i,s)} \in \tilde{B}_1 \) there exists then a number \( l_1^{(i,s)} \), \( 0 \leq l_1^{(i,s)} \leq r_n - 1 \), such that \( \tilde{b}_1^{(i,s)}(n) = f_1^{l_1^{(i,s)}}(b_1^{(i,s)}) \in [z_1, z_4] \). The point \( \tilde{b}_1^{(i,s)}(n) \) is called the \( r_n \)-preimage of the element \( b_1^{(i,s)} \) in \([z_1, z_4]\). The set of \( r_n \)-preimages of elements of \( \tilde{B}_1 \) consists then of the elements \( \tilde{b}_1^{(i_1,s_1)}(n), \tilde{b}_1^{(i_2,s_2)}(n), \ldots, \tilde{b}_1^{(i_{p_1},s_{p_1})}(n) \).

Define
\[
\xi_{f_1}(j) := \frac{\ell([f_1^{j}(z_2), f_1^{j}(z_3)])}{\ell([f_1^{j}(z_1), f_1^{j}(z_2)])}, \quad z_{f_1}(j) := \frac{\ell([f_1^{j}(\tilde{b}_1^{(i,s)}(n)), f_1^{j}(z_2)])}{\ell([f_1^{j}(z_1), f_1^{j}(z_2)])}, \quad 1 \leq s \leq p_1, 0 \leq j \leq r_n.
\]
It follows easily from Lemma 2.2 that
\[
ev^{-\nu}\xi_{f_1}(0) \leq \xi_{f_1}(j) \leq e^{\nu}\xi_{f_1}(0), \quad e^{-\nu}z_{f_1}(j) \leq z_{f_1}(j) \leq e^{\nu}z_{f_1}(0), \quad 1 \leq s \leq p_1,
\]
and all $1 \leq j \leq r_n - 1$. For the further discussion we introduce some definitions.

**Definition 4.1.** Let $K > M \geq 1$, $\zeta \in (0, 1)$, $\delta > 0$ be constant numbers, let $n$ be a positive integer, and let $x_0 \in S^1$. We say that a triple of intervals $([z_1, z_2], [z_2, z_3], [z_3, z_4])$, $z_i \in S^1$, $i = 1, 2, 3, 4$, covers the break points in a subset $\hat{B}_1$, "$(K, M, \delta, \zeta; x_0)$-regularly", if for some $r_1 \in \{q_0, q_1, q_2, q_3 + q_4 - 1\}$ the following conditions hold:

1) $[z_1, z_4] \subset (x_0 - \delta, x_0 + \delta)$, and the system of intervals $\{f^1_j([z_1, z_4]), 0 \leq j \leq r_1 - 1\}$ covers every point in $\hat{B}_1$ only once;

2) $\hat{B}_1^{(i_s)}(n) \in [z_1, z_2]$, $1 \leq s \leq p_1$;

3) $M\ell([z_1, z_2]) \leq \ell([z_2, z_3]) \leq K\ell([z_1, z_2])$, $K^{-1}\ell([z_3, z_4]) \leq \ell([z_2, z_3]) \leq K\ell([z_3, z_4])$;

4) The lengths of the intervals $f^n_1([z_1, z_2])$, $f^n_1([z_2, z_3])$, and $f^n_1([z_3, z_4])$ are pairwise $K$-comparable;

5) $\max\{\ell([f^n_1(z_i), x_0]), \ell([z_i, x_0]), i = \overline{1, 4}\} \leq K\ell([z_1, z_2])$;

6) $\max_{1 \leq s \leq p_1} \{s^{(i_s)}(0)\} < \zeta$.

**Definition 4.2.** Let $\hat{B}_1$ and $\hat{B}_2$ be subsets of the break points of the homeomorphisms $f_1$ and $f_2$, respectively. The subsets $\hat{B}_1$ and $\hat{B}_2$ are said to be "not jump-coinciding", or for short "not coinciding", if

$$\prod_{b_1 \in \hat{B}_1} \sigma_{f_1}(b_1) \neq \prod_{b_2 \in \hat{B}_2} \sigma_{f_2}(b_2).$$

Otherwise we call them "coinciding".

It is clear, that if $\hat{B}_1$ and $\hat{B}_2$ are "not jump-coinciding" subsets, then one of subsets $\hat{B}_1$ and $\hat{B}_2$ is non-empty. For instance, if $\hat{B}_1$ is empty, then we put $\prod_{b_1 \in \hat{B}_1} \sigma_{f_1}(b_1) := 1$.

**Definition 4.3.** Let $\hat{B}_1$ and $\hat{B}_2$ be subsets of the break points of $f_1$ and $f_2$, respectively. We say that the triples of intervals $([z_s, z_{s+1}], s = 1, 2, 3)$ and $([h(z_s), h(z_{s+1})], s = 1, 2, 3)$ cover the subsets $\hat{B}_1$ and $\hat{B}_2$ "$(K, M, \delta, \zeta; x_0, h(x_0))$-regularly" with $r_1 \in \{q_n - 1, q_n, q_n + q_n - 1\}$, if

- the triples of intervals $([z_s, z_{s+1}], s = 1, 2, 3)$ and $([h(z_s), h(z_{s+1})], s = 1, 2, 3)$ cover the points of $\hat{B}_1$ respectively $\hat{B}_2$ "$(K, M, \delta, \zeta; x_0)$-regularly" respectively "$(K, M, \delta, \zeta; h(x_0))$-regularly" if $\hat{B}_1 \neq \emptyset$, $\hat{B}_2 \neq \emptyset$;

- the triple of intervals $([z_s, z_{s+1}], s = 1, 2, 3)$ covers the points of $\hat{B}_1$ "$(K, M, \delta, \zeta; x_0)$-regularly" if $\hat{B}_1 \neq \emptyset$, $\hat{B}_2 = \emptyset$;

- the triple intervals $([h(z_s), h(z_{s+1})], s = 1, 2, 3)$ covers the points of $\hat{B}_2$ "$(K, M, \delta, \zeta; h(x_0))$-regularly" if $\hat{B}_1 = \emptyset$, $\hat{B}_2 \neq \emptyset$.

Next we formulate our main Theorem on the covering of intervals which plays a key role in the proof of Theorem 1.3.
Theorem 4.4. Suppose that the homeomorphisms $f_1$ and $f_2$ satisfies the conditions of Theorem 1.2. Let $Dh(x_0) = \omega_0 > 0$ for some $x_0 \in S^1$ and let $M \geq 1, \zeta, \delta \in (0,1)$ be constants. Then there exist a constant $K = K(f_1, f_2, M, \zeta) > M$ and for any sufficiently large $n$ "not jump-coinciding" subsets $\tilde{B}_1$ and $\tilde{B}_2$, points $z_i \in S^1$, $1 \leq i \leq 4$ with $z_1 < z_2 < z_3 < z_4 < z_1$ and a number $r_n = r_n(z_1, z_2, z_3, z_4) \in \{q_{n-1}, q_n, q_n + q_{n-1}\}$, such that the triples of intervals $([z_s, z_{s+1}], s = 1, 2, 3)$ and $([h(z_s), h(z_{s+1})], s = 1, 2, 3)$ cover the subsets $\tilde{B}_1$ and $\tilde{B}_2$ "$(K, M, \delta, \zeta, x_0, h(x_0))$-regularly" for $r_n$.

Proof of Theorem 4.4. Suppose that the homeomorphisms $f_1$ and $f_2$ satisfy the conditions of Theorem 1.5. Let $B(f_1) = \{b_1(i), 1 \leq i \leq m_1\}$ and $B(f_2) = \{b_2(j), 1 \leq j \leq m_2\}$ be the sets of break points of $f_1$ and $f_2$ respectively. By assumption $Dh(x_0) = \omega_0 > 0$ for some $x_0 \in S^1$. Consider the dynamical partition $\eta_n(x_0)$ of the point $x_0$ under $f_1$. Suppose $n$ to be odd. Let $B_m(n)$ and $d_n$ be defined as in (3.2).

Define the number $m_0$ by using the constants $M > 1$, $\zeta \in (0,1)$ and the total variation $v_1$ of the functions in $Df_i$, $i = 1, 2$ as follows:

$$m_0 := \max\{m_1 + m_2 + 4, [M\zeta^{-1}] + 1, [e^{v_1}] + 1, [e^{v_2}] + 1\},$$

where $[\cdot]$ denotes the integer part, and consider the partition $D_l^{(n)}$ of the interval $[x_{q_n}, x_{q_n-1}]$. It is sufficient to use the assertion of Theorem 3.2 with $r = 9$ and set $s_0 = s_0(9, n)$. By this assertion there exists at least one element $E^{(n)}_{s_0} \in \Gamma^{(n)}_{s_0}$ such that

$$\prod_{b_1, b_1(n \in E^{(n)}_{s_0})} \sigma_{f_1}(b_1) \neq \prod_{b_2, b_2(n \in h(E^{(n)}_{s_0}))} \sigma_{f_2}(b_2).$$

Set $\tilde{B}_1 := \{b_1(i) : \tilde{b}_1(n) \in E^{(n)}_{s_0}\}$ and $\tilde{B}_2 := \{b_2(j) : \tilde{b}_2(n) \in h(E^{(n)}_{s_0})\}$. Then the following cases are possible: $\tilde{B}_1 \neq \emptyset$, $\tilde{B}_2 \neq \emptyset$ or $\tilde{B}_1 \neq \emptyset$, $\tilde{B}_2 = \emptyset$ or $\tilde{B}_1 = \emptyset$, $\tilde{B}_2 \neq \emptyset$. If $\tilde{B}_1 \neq \emptyset$ and $\tilde{B}_2 \neq \emptyset$ we’ll construct a triple of regular covering intervals $[z_s, z_{s+1}]$, $s = 1, 2, 3$, in the other cases the construction of regular covering intervals is analogous.

Let $\tilde{B}_1 \neq \emptyset$ and $\tilde{B}_2 \neq \emptyset$. Then three cases are possible for the set $\tilde{E}^{(n)}_{s_0}$:

$$(c_1) \tilde{E}^{(n)}_{s_0} \text{ does not contain any elements of } \{x_{q_n}, x_0, x_{q_n-1}\};$$

$$(c_2) \tilde{E}^{(n)}_{s_0} \text{ contains only one element of the set } \{x_{q_n}, x_0, x_{q_n-1}\};$$

$$(c_3) \tilde{E}^{(n)}_{s_0} \text{ contains the elements } x_{q_n}, x_0 \text{ of the set } \{x_{q_n}, x_0, x_{q_n-1}\}.$$ 

The case $\{x_{q_n-1}, x_0\} \in \tilde{E}^{(n)}_{s_0}$ turns out to be impossible.

We prove the assertion of the theorem in each of the cases separately.

$$(c_1) \text{ Let } \tilde{E}^{(n)}_{s_0} \cap \{x_{q_n}, x_0, x_{q_n-1}\} = \emptyset. \text{ Then either } \tilde{E}^{(n)}_{s_0} \subset (x_{q_n}, x_0), \text{ or } \tilde{E}^{(n)}_{s_0} \subset (x_0, x_{q_n-1}). \text{ Suppose for definiteness that } \tilde{E}^{(n)}_{s_0} \subset (x_{q_n}, x_0). \text{ The case } \tilde{E}^{(n)}_{s_0} \subset (x_0, x_{q_n-1}) \text{ can be treated in a similar way.}$$

One can deduce from the assertion of Theorem 3.2 that the subset $\tilde{E}^{(n)}_{s_0}$ is covered by one or two intervals of the partition $D_{s_0}^{(n)}$. The union of the intervals of the partition $D_{s_0}^{(n)}$ which cover $\tilde{E}^{(n)}_{s_0}$ is denoted by $I_{s_0}^{(n)}$. In the same way we can define intervals $I_{s_0+p}^{(n)}$, for $0 < p < 9$. Clearly $I_{s_0}^{(n)} \supset I_{s_0+1}^{(n)} \supset \ldots \supset I_{s_0+9}^{(n)}$. It follows from the assertion of Theorem 3.2 that the interval $I_{s_0}^{(n)}$ is adjacent on the left and right to two $s_0^{(n)}$-empty intervals of $D_{s_0}^{(n)}$ contained in the interval $(x_0, x_{q_n-1})$. These two intervals are denoted by $L_{s_0}^{(n)}$ and $R_{s_0}^{(n)}$, respectively.
We now define the points \( z_i, 1 \leq i \leq 4 \), as follows:

\[
z_2 = \max\{y : y \in \tilde{E}_s^{(n)}(1)\}, \quad z_1 = z_2 - d_n m_0^{-(s_0+7)}, \quad z_3 = z_2 + d_n m_0^{-(s_0+6)},
\]

\[
z_4 = z_2 + d_n m_0^{-(s_0+6)} + d_n m_0^{-(s_0+7)}.
\]

We now verify that the triples of intervals \([z_s, z_{s+1}]\), \(s = 1, 2, 3\) and \([h(z_s), h(z_{s+1})]\), \(s = 1, 2, 3\) satisfy the conditions of Definition 4.3. The length of the interval \(I_{s_0+9}^{(n)}\) covering the subset \(\tilde{E}_s^{(n)}\) does not exceed \(2d_n m_0^{-(s_0+9)}\), and the lengths of the intervals \(L_s^{(n)}\) and \(R_s^{(n)}\) adjacent to \(I_s^{(n)}\) are equal to \(d_n \cdot m_0^{s_0}\). Using the definition of the points \(z_i, 0 \leq i \leq 4\) we obtain \(\ell([z_1, z_2]) = m_0^{-7} \ell(L_0^{(n)})\), \(\ell([z_2, z_4]) = (m_0 + 1) m_0^{-7} \ell(R_0^{(n)})\). Hence, \([z_1, z_4] \subseteq L_{s_0}^{(n)} \cup f_{s_0}^{(n)} \cup R_{s_0}^{(n)} \subseteq (x_0, x_{q_n-1})\). Since the interval \([x_0, x_{q_n-1}]\) is \(q_n\)-small, the intervals \(f_j^{(n)}([z_1, z_4]), 0 \leq j \leq q_n - 1\) are pairwise disjoint and cover each point of \(\tilde{B}_1\) only once. One can easily verify that the intervals \([z_s, z_{s+1}]\), \(s = 1, 2, 3\) satisfy condition 2 of Definition 4.4. By Denjoy’s inequality the intervals \([x_{q_n}, x_{q_n-1}]\) and \([x_0, x_{q_n-1}]\) are \(1 + e^{n_1}\)-comparable. Hence, using the fact that \(\ell([z_1, z_4]) = (m_0 + 2) m_0^{-(s_0+7)} d_n\), we obtain that \([z_1, z_4]\) and \([x_0, x_{q_n-1}]\) are \((m_0 + 2)^{-1} m_0^{s_0+7}(1 + e^{n_1})^{-1}\)-comparable. Set \(K := \max\{m_0 e^{2m_1 + n_2}, m_0^{s_0+7}, n = 1, 2, ..., \}\), where \(s_0 = s_0(9, n)\). By the assertion of Theorem 3.2 we have \(s_0 = s_0(9, n) \leq (m_1 + m_2 + 1), \forall n \in N\). Consequently, \(K = \max\{m_0 e^{2m_1 + n_2}, m_0^{s_0+16}\}\). Taking into account that \(m_0 > M\) we conclude that \(m_0 > M\) and the intervals \([z_s, z_{s+1}]\), \(s = 1, 2, 3\), satisfy condition 3 of Definition 4.4 with the constant \(K\). By Denjoy’s inequality the intervals \([z_s, z_{s+1}]\) and \([f_{s_0}^{q_n}(z_s), f_{s_0}^{q_n}(z_{s+1})]\) are \(e^{n_1}\)-comparable for every \(s = 1, 2, 3\). Since \(\ell([z_1, z_{s+1}]) = d_n \cdot m_0^{-(s_0+7)}, s = 1, 3\) and \(\ell([z_2, z_4]) = d_n \cdot m_0^{-(s_0+6)}\), it follows that the intervals \([f_{s_0}^{q_n}(z_s), f_{s_0}^{q_n}(z_{s+1})], s = 1, 2, 3\), satisfy condition 4 of Definition 4.4 with the constant \(K\).

Obviously,

\[
\max_{1 \leq i \leq 4} \ell([z_i, x_0]), \quad \max_{1 \leq i \leq 4} \ell([f_{s_0}^{q_n}(z_i), x_0]) \leq \ell([x_{q_n}, x_{q_n-1}]) = d_n.
\]

It follows from the explicit form of the length of the interval \(\ell([z_1, z_2])\) that \(d_n = m_0^{s_0+7}\). \(\ell([z_1, z_2])\). Hence it is evident that \([z_s, z_{s+1}], s = 1, 2, 3\), satisfy condition 5 of Definition 4.4 with the constant \(K\). We now verify that the triple of intervals \([z_s, z_{s+1}], s = 1, 2, 3\), satisfy condition 6 of Definition 4.4. It follows from the definition of the points \(z_s, s = 1, 2, 3, 4\), that

\[
\max_{i: \tilde{E}_s^{(n)}(i) \subseteq \tilde{E}_s^{(n)}(n)} \frac{\ell([\tilde{B}_1^{(n)}(i), z_2])}{\ell([z_1, z_2])} \leq \frac{d_n \cdot m_0^{-(s_0+8)}}{d_n \cdot m_0^{-(s_0+7)}} = m_0^{-1} < \zeta.
\]

Next we show that the intervals \([h(z_s), h(z_{s+1})], s = 1, 2, 3\), define a regular cover of the elements of the set \(\tilde{B}_2\). The definition of the points \(z_i, 1 \leq i \leq 4\) implies that \([h(z_1), h(z_4)] \subseteq [h(x_0), f_{s_0}^{q_n-1}(h(x_0))] \subseteq (h(x_0) - \delta, h(x_0) + \delta)\) for sufficiently large \(n\) and the system of intervals \(f_j^{(n)}([h(z_1), h(z_4)]), 0 \leq j < q_n\) are pairwise disjoint and cover each point of the subset \(\tilde{B}_2\) only once. Since \(\tilde{E}_s^{(n)}(n) \subseteq [z_1, z_4]\), the elements of the set \(\tilde{B}_2\) are covered by \(f_2^{(n)}\) iterations of the interval \([h(z_1), h(z_2)]\). By the assumption of Theorem 4.4 \(Dh(x_0) = \omega_0 > 0\) at the point \(x_0 \in S^1\).
Let $H(x)$ be the lift of $h$. By the definition of the derivative, for any $\varepsilon > 0$ there exists $\delta_1 = \delta_1(x_0, \varepsilon) > 0$ such that for all $x \in (x_0 - \delta_1, x_0 + \delta_1)$ the inequality

$$
(4.4) \quad \omega_0 - \varepsilon < \frac{H(x) - H(x_0)}{x - x_0} < \omega_0 + \varepsilon.
$$

holds. For $x = \hat{z}_i$, $1 \leq i \leq 4$, it follows from (4.4) that

$$
(5.5) \quad (\omega_0 - \varepsilon)(x_0 - \hat{z}_i) < H(x_0) - H(\hat{z}_i) < (\omega_0 + \varepsilon)(x_0 - \hat{z}_i),
$$

from which one can easily derive the inequalities

$$
\omega_0 - \varepsilon \frac{(x_0 - \hat{z}_{i+1}) + (x_0 - \hat{z}_i)}{\hat{z}_{i+1} - \hat{z}_i} < H(\hat{z}_{i+1} - H(\hat{z}_i) < \omega_0 + \varepsilon \frac{(x_0 - \hat{z}_{i+1}) + (x_0 - \hat{z}_i)}{\hat{z}_{i+1} - \hat{z}_i}, \quad i = 1, 2, 3,
$$

where $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ is the lifted vector of $(z_1, z_2, z_3, z_4)$. Using the definition of the points $z_i$, $1 \leq i \leq 4$, we obtain

$$
(5.7) \quad \max_{1 \leq i \leq 3} (\frac{\tilde{z}_1 - \tilde{z}_i}{\tilde{z}_{i+1} - \tilde{z}_i}) < K.
$$

Using the definition of $m_0$ and the bounds (4.6), (4.7) it can be easily shown that the intervals $[h(z_s), h(z_{s+1})]$, $s = 1, 2, 3$ satisfy the other conditions of Definition 4.1 with the same constant $K$.

(c2). We denote by $E_{s_0}^{(n)}(x_{q_n})$, $E_{s_0}^{(n)}(x_0)$ and $E_{s_0}^{(n)}(x_{q_{n-1}})$ the elements of the partition $\Gamma_{s_0}$ containing just one of the points $x_{q_n}$, $x_0$ or $x_{q_{n-1}}$, and which are pairwise disjoint. By assumption, $\tilde{E}_{s_0}^{(n)}$ coincides with one of them and these subsets are pairwise disjoint. Suppose, that for any element $E_{s_0}^{(n)} \in \Gamma_{s_0}$, which does not contain the points $x_{q_n}$, $x_0$, $x_{q_{n-1}}$ the following relation hold:

$$
\prod_{b_1 \in E_{s_0}^{(n)}} \sigma_f_1(b_1) = \prod_{b_2 \in E_{s_0}^{(n)}} \sigma_f_2(b_2).
$$

Otherwise we arrive again at case (c1). Then the sets $\tilde{B}_1 = \{b_1^{(i)} : b_1^{(i)}(n) \in E_{s_0}^{(n)}(x_{q_n}) \cup E_{s_0}^{(n)}(x_0) \cup E_{s_0}^{(n)}(x_{q_{n-1}})\}$ and $\tilde{B}_2 = \{b_2^{(i)} : h^{-1}(b_2^{(i)}(n)) \in E_{s_0}^{(n)}(x_{q_n}) \cup E_{s_0}^{(n)}(x_0) \cup E_{s_0}^{(n)}(x_{q_{n-1}})\}$ are ”not coinciding”.

Denote by $I_{s_0+p}^{(n)}(x_{q_n})$, $I_{s_0+p}^{(n)}(x_0)$ and $I_{s_0+p}^{(n)}(x_{q_{n-1}})$ be the intervals of partition $D_{s_0+p}^{(n)}$, $1 \leq p \leq 9$, covering $E_{s_0}^{(n)}(x_{q_n})$, $E_{s_0}^{(n)}(x_0)$ and $E_{s_0}^{(n)}(x_{q_{n-1}})$, respectively. By Theorem 3.2. the interval $I_{s_0}^{(n)}(x_0)$ is adjacent on both sides to $s_0^{(n)}$-empty intervals of length $d_{m_0}^{-s_0}$. Consider then the subset $E_{s_0}^{(n)}(x_{q_n}, x_0, x_{q_{n-1}}) = f_1^{-q_n}(E_{s_0}^{(n)}(x_{q_n})) \cup E_{s_0}^{(n)}(x_0) \cup f_1^{-q_{n-1}}(E_{s_0}^{(n)}(x_{q_{n-1}}))$. From Denjoy’s inequality we get $(f_1^{-q_{n}}(I_{s_0+9}(x_{q_n}))) \leq e^{\varepsilon_1} d_{m_0}^{-s_0}$. Hence, $E_{s_0}^{(n)}(x_{q_n}, x_0, x_{q_{n-1}}) \subset (x_0 - 2e^{\varepsilon_1} d_{m_0}^{-s_0+9}, x_0 + 2e^{\varepsilon_1} d_{m_0}^{-s_0+9})$.

We can define now the points $z_i$, $1 \leq i \leq 4$, as follows

$$
z_2 = \max E_{s_0}^{(n)}(x_{q_n}, x_0, x_{q_{n-1}}), \quad z_1 = z_2 - d_{m_0}^{-s_0+6}, \quad z_3 = z_2 + d_{m_0}^{-s_0+3},
$$

\[ z_4 = z_2 + d_n m_0^{-(s_0+3)} + d_n m_0^{-(s_0+6)}. \]

We claim, the triple of intervals \([z_s, z_{s+1}], s = 1, 2, 3\) covers the point of the subset \(\tilde{B}_1^{(K, M, \delta, \zeta; x_0)}\) regularly with \(r_n = q_n + q_{n-1}\).

We verify only that the system of intervals \(\{f_1([z_1, z_4]), 0 \leq i \leq q_n + q_{n-1} - 1\}\) covers each point of the subset \(E_{s_0}^{(n)}(x_{q_n}, x_0, x_{q_{n-1}})\) only once. The other conditions in Definition 4.1 concerning the lengths of the intervals can be verified as in the case (c) by simple calculations. We divide the interval \([z_1, z_4]\) into \([z_1, x_0] \cup (x_0, z_4]\). We claim, the intervals \(f_1([z_1, x_0]), 0 \leq i \leq q_n + q_{n-1} - 1\) cover the break points of \(f_1\) with \(q_{n-1}\)-preimages in \(E_{s_0}^{(n)}(x_0) \cap [z_1, x_0]\) and with \(q_n\)-preimages in \(E_{s_0}^{(n)}(x_{q_{n-1}})\) only once. Since \([z_1, x_0] \subset [x_{q_n}, x_0]\), the intervals \(f_1([z_1, x_0]), 0 \leq i \leq q_{n-1} - 1\) cover each break point of \(f_1\) with \(q_n\)-preimage in \([z_1, x_0] \cap E_{s_0}^{(n)}(x_0)\) only once. It follows from the assertion of Theorem 3.2 that there is an \(s_0\)-empty interval to the left of \(I_{s_0}^{(n)}(x_{q_n})\). By Denjoy’s inequality the length of the interval \(f_1^{q_{n-1}}([z_1, x_0]) = \{f_1^{q_{n-1}}([z_1, x_0])\} = \ell([z_1, x_0])\) is at most \(e^{q_1} \ell([z_1, x_0])\). It is easy to verify that this number is less than the sum of the lengths of the intervals \(I_{s_0}^{(n)}(x_{q_{n-1}})\) and the adjacent \(s_0\)-empty one. In other words, \(\{f_1^{q_{n-1}}([z_1, x_0]), x_{q_{n-1}}\}\) is covered by the interval \(I_{s_0}^{(n)}(x_{q_{n-1}})\) and an \(s_0\)-empty adjacent interval. Clearly, the subset of the \(q_n\)-preimages of the break points of \(f_1\) contained in \(\{f_1^{q_{n-1}}([z_1, x_0])\} = \{f_1^{q_{n-1}}([z_1, x_0])\}\) coincides with \(E_{s_0}^{(n)}(x_{q_{n-1}})\). Since \(\{f_1^{q_{n-1}}([z_1, x_0])\} \subset [x_0, x_{q_n}]\), the intervals \(f_1([f_1^{q_{n-1}}([z_1, x_0]), x_{q_{n-1}}])\), \(0 \leq i \leq q_n - 1\), cover the break points with \(q_n\)-preimages in \(\{f_1^{q_{n-1}}([z_1, x_0])\}, x_{q_{n-1}}\) only once.

It can be shown in a similar way that the intervals \(f_1^{j,s}([x_0, z_4]), 0 \leq i \leq q_n + q_{n-1} - 1\), cover the break points of \(f_1\) with \(q_n\)-preimages in \(E_{s_0}^{(n)}(x_0) \cap (x_0, z_4]\) respectively \(q_{n-1}\)-preimages in \(E_{s_0}^{(n)}(x_{q_n})\) only once. Remember, \(\varepsilon\) to be an arbitrary positive number. Using this and the bounds (4.6), (4.7) it can be proved that the intervals \([h(z_s), h(z_{s+1})]\), \(s = 1, 2, 3\) satisfy all the conditions of Definition 4.1 with the same constant \(K\).

(c_3). First we show that the subset \([x_0, x_{q_{n-1}}]\) cannot be a part of \(\tilde{E}_{s_0}^{(n)}\). Suppose on the contrary \(\{x_0, x_{q_{n-1}}\} \subset \tilde{E}_{s_0}^{(n)}\). Then \(\tilde{E}_{s_0}^{(n)}\) is covered by the interval \(I_{s_0}^{(n)}\) of the partition \(D_{s_0}^{(n+9)}\). Consequently \(\Delta_{s_0}^{(n-1)} \subset I_{s_0}^{(n)}\). Clearly, \(\ell(\Delta_{s_0}^{(n-1)}) \leq m_0^{-(s_0+9)} d_n\). Hence, \(\ell(\Delta_{s_0}^{(n)}) \geq (m_0^{-(s_0+9)} - 1)\ell(\Delta_{s_0}^{(n-1)})\). From Denjoy’s inequality we get \(\ell(\Delta_{s_0}^{(n)}) \leq e^{q_1}\ell(\Delta_{s_0}^{(n-1)})\). Using the definition of \(m_0\), one can easily show that \(m_0^{-(s_0+9)} - 1 > e^{q_1}\). Consequently, \(\ell(\Delta_{s_0}^{(n)}) > \ell(\Delta_{s_0}^{(n)})\), a contradiction.

By assumption (c_3) \(x_{q_n}, x_0 \in \tilde{E}_{s_0}^{(n)}\). The definition of the subsets \(E_{s_0}^{(n)}(x_{q_n})\) and \(E_{s_0}^{(n)}(x_0)\) imply that \(\tilde{E}_{s_0}^{(n)} = E_{s_0}^{(n)}(x_0) = E_{s_0}^{(n)}(x_{q_n})\).

We divide the set \(E_{s_0}^{(n)}(x_{q_n}) \setminus \{x_{q_{n-1}}\}\) into two subsets \(A_1\) and \(A_2\):

1) \(A_1 = \{\tilde{B}_1, h^{-1}(\tilde{B}_2) : \tilde{B}_1, h^{-1}(\tilde{B}_2) \in E_{s_0}^{(n)}(x_{q_n})\}\), \(f_1(\tilde{B}_1)\) and \(f_2(\tilde{B}_2)\) break points of \(f_1\) and \(f_2\), respectively for some \(j, s\) with \(q_n - q_{n-1} \leq j, s < q_{n-1}\);

2) \(A_2 = \{\tilde{B}_1, h^{-1}(\tilde{B}_2) : \tilde{B}_1, h^{-1}(\tilde{B}_2) \in E_{s_0}^{(n)}(x_{q_n})\}, f_1(\tilde{B}_1)\) and \(f_2(\tilde{B}_2)\) break points of \(f_1\) and \(f_2\), respectively for some \(j, k\) with \(0 < i, k \leq q_n - q_{n-1} - 1\).

Let \(I_{s_0+p}^{(n)}(x_0)\) and \(I_{s_0+p}^{(n)}(x_{q_{n-1}}), 0 \leq p \leq 9\), be the intervals of partition \(D_{s_0+p}^{(n)}\) covering \(E_{s_0}^{(n)}(x_0)\) and \(E_{s_0}^{(n)}(x_{q_{n-1}})\), respectively. By the definition of the partition \(D_i^{(n)}\), we have

\[ \ell(I_{s_0+p}^{(n)}(x_0)) = \ell(I_{s_0+p}^{(n)}(x_{q_{n-1}})) = d_n m_0^{-(s_0+p)}, 0 \leq p \leq 9. \]

From Denjoy’s lemma it follows that

\[ \ell(f^{q_n-q_{n-1}}(I_{s_0+p}^{(n)}(x_{q_{n-1}}))) < d_n m_0^{-(s_0+7)}. \]
We set \( E_{s_0}^{(n)}(x_0, x_{q_n-1}) := E_{s_0}^{(n)}(x_0) \cup f_1^{-q_n-1}(A_1) \cup f_1^{-q_n-1}(A_2) \). Obviously, \( E_{s_0}^{(n)}(x_0, x_{q_n-1}) \subset H_{s_0+9}^{(n)} := f_{s_0+9}^{(n)}(x_0) \cup f_1^{-q_n-1}(I_{s_0+9}^{(n)}(x_{q_n-1})) \cup f_1^{-q_n-1}(I_{s_0+9}^{(n)}(x_{q_n-1})). \) This fact together with relations (4.9) and (4.10), implies that \( \ell(H_{s_0+9}^{(n)}) < 2d_n m_0^{-(s_0+7)} \). We can now define points \( z_i, i = 1, 2, 3, 4 \) as follows:

\[
z_2 = \max E_{s_0}^{(n)}(x_0, x_{q_n-1}), \quad z_1 = z_2 - 2d_n m_0^{-(s_0+6)}, \quad z_3 = z_2 + 2d_n m_0^{-(s_0+4)},
\]

(4.11)

\[
z_4 = z_2 + d_n m_0^{-(s_0+4)} + d_n m_0^{-(s_0+6)}.
\]

We claim that the triples of intervals \([z_s, z_{s+1}], s = 1, 2, 3 \) and \([h(z_2), h(z_3)], s = 1, 2, 3 \) cover the points of the "not coinciding" sets \( B_1 = \{ b_1^{(i)} : b_1^{(i)}(n) \in E_{s_0}^{(n)}(x_0) \cup E_{s_0}^{(n)}(x_{q_n-1}) \} \) and \( B_2 = \{ b_2^{(i)} : h^{-1}(b_2^{(i)}(n)) \in E_{s_0}^{(n)}(x_0) \cup E_{s_0}^{(n)}(x_{q_n-1}) \} \) "(K, M, \( \delta, \xi, z_0, h(x_0) \))-regularly" with \( r_n = q_n \).

We shall only show that condition 1) in Definition 4.1 holds, the other conditions can easily be verified as in case (c1). We again divide the interval \([z_1, z_4]\) up into \([z_1, z_4] = [z_1, x_0] \cup [z_0, z_4]\) and decompose the system of intervals \( \{ f_1([z_1, x_0]), 0 \leq i \leq q_n-1 \} \) into two subsystems: \( \{ f_1([z_1, x_0]), 0 \leq i \leq q_n-1 \} \) and \( \{ f_1([z_1, x_0]), q_n-1 \leq i \leq q_n-1 \} \). Clearly, the first subsystem covers those break points of \( f_1 \) whose \( q_n-1 \) pre-images are contained in the interval \([x_{q_n}, x_0]\), as well as those whose \( q_n \) pre-images form the subset \( A_1 \), only once. Note, that the break points of \( f_1 \) with \( q_n \)-pre-images in \( A_2 \) are covered by the second system of intervals only once. Consider then the system of intervals \( \{ f_1^i([x_0, z_4]), 0 \leq i \leq q_n-1 \} \). It follows from the definition of \( z_4 \) that \([x_0, z_4] \subset [x_0, x_{q_n-1}].\) Then two cases are possible for the point \( z_2 \): either \( z_2 > x_0 \) or \( z_2 \leq x_0 \). If \( z_2 > x_0 \), then the break points with \( q_n \)-pre-images contained in \([x_0, z_2]\) are covered by the system of intervals \( \{ f_1^i([x_0, z_4]), 0 \leq i \leq q_n-1 \} \) only once. In the case \( z_2 < x_0 \), the interval \([x_0, z_4]\) does not contain any \( q_n \)-pre-images of break points of \( f_1 \). Notice that the orbits \( \{ f_1^k(x_0), k \in \mathbb{Z} \} \) and \( \{ f_2^k(h(x_0)), k \in \mathbb{Z} \}, \) of any point \( x_0 \in S^1 \) has the same order on the circle. This together with (4.6) and (4.7) implies that the intervals \([h(z_2), h(z_{s+1})], s = 1, 2, 3 \) also satisfy the conditions of Definition 4.1 with the same constant \( \delta \). Hence the theorem is completely proved.

5 Proof of the Theorem 1.5

Consider two copies of the circle \( S^1 \), and homeomorphisms \( f_1 \) and \( f_2 \) with \( m_1, m_2 \geq 2 \) break points respectively and identical irrational rotation number \( \rho \). Denote by \( B(f_1) = \{ b_1^{(i)}, 1 \leq i \leq m_1 \} \) and \( B(f_2) = \{ b_2^{(j)}, 1 \leq j \leq m_2 \} \) the set of break points of \( f_1 \) respectively \( f_2 \). Assume that \( f_1 \) and \( f_2 \) satisfy the conditions of Theorem 1.5. Let \( h \) be the conjugacy between \( f_1 \) and \( f_2 \). For the proof of Theorem 1.5 we need several lemmas.

**Lemma 5.1.** (see [20].) Assume, that the conjugating homeomorphism \( h(x) \) has a positive derivative \( Dh(x_0) = \omega_0 \) at some point \( x_0 \in S^1 \), and that the following conditions hold for the points \( z_i \in S^1, i = 1, \ldots, 4 \), with \( z_1 < z_2 < z_3 < z_4 \), and some constant \( R_1 > 1 \):

(a) the intervals \([z_1, z_2], [z_2, z_3], [z_3, z_4]\) are pairwise \( R_1 \)-comparable;

(b) \( \max_1 \leq i \leq 4 \ell([z_i, x_0]) \leq R_1 \ell([z_1, z_2]). \)
Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

\begin{equation}
|\text{Dist}(z_1, z_2, z_3, z_4; h) - 1| \leq C_3 \varepsilon,
\end{equation}

if $z_i \in (x_0 - \delta, x_0 + \delta)$ for all $i = 1, 2, 3, 4$, where the constant $C_3 > 0$ depends only on $R_1$, $\omega_0$ and not on $\varepsilon$.

We define the following functions on the domain $\{(x, y) : x > 0, 0 \leq y \leq 1\}$:

\[
G^{(i)}_{f_1}(x, y) = \frac{[\sigma^{(i)}_{f_1} + (1 - \sigma^{(i)}_{f_1})y](1 + x)}{\sigma^{(i)}_{f_1} + (1 - \sigma^{(i)}_{f_1})y + x}, \quad 1 \leq i \leq m_1,
\]

\[
G^{(j)}_{f_2}(x, y) = \frac{[\sigma^{(j)}_{f_2} + (1 - \sigma^{(j)}_{f_2})y](1 + x)}{\sigma^{(j)}_{f_2} + (1 - \sigma^{(j)}_{f_2})y + x}, \quad 1 \leq j \leq m_2,
\]

where the $\sigma^{(i)}_{f_1}$ and $\sigma^{(j)}_{f_2}$ are the jumps of $f_1$ and $f_2$ at the points $b^{(i)}_1$ and $b^{(j)}_2$ respectively.

Let $\hat{B}_1 \subset B(f_1)$ and $\hat{B}_2 \subset B(f_2)$. Denote

\[
\sigma_{f_1}(\hat{B}_1) := \prod_{b^{(i)}_1 \in \hat{B}_1} \sigma_{f_1}(b^{(i)}_1), \quad \sigma_{f_2}(\hat{B}_2) := \prod_{b^{(j)}_2 \in \hat{B}_2} \sigma_{f_2}(b^{(j)}_2),
\]

\[
\widehat{\Lambda}_{1,2} := \min\{\sigma_{f_1}(\hat{B}_1), \sigma_{f_2}(\hat{B}_2), |\sigma_{f_1}(\hat{B}_1) - \sigma_{f_2}(\hat{B}_2)|\}.
\]

**Lemma 5.2.** Let $\hat{B}_1 = \{b^{(i_1)}_1, b^{(i_2)}_1, \ldots, b^{(i_{p_1})}_1\}$ and $\hat{B}_2 = \{b^{(j_1)}_2, b^{(j_2)}_2, \ldots, b^{(j_{p_2})}_2\}$ be arbitrary "not coinciding" subsets of the break points of $f_1$ and $f_2$. Then there exist constants $\Omega_0 > 1$ and $\vartheta_0 \in (0, 1)$ such that for arbitrary $x^{(s)}_{f_1}, x^{(s)}_{f_2} \geq \Omega_0$, $y^{(s)}_{f_1}, y^{(s)}_{f_2} \in [0, \vartheta_0)$, $1 \leq s \leq p_1$, $1 \leq t \leq p_2$ the following inequality holds:

\begin{equation}
\frac{1}{8} \prod_{s=1}^{p_1} G^{(i_s)}_{f_1}(x^{(s)}_{f_1}, y^{(s)}_{f_1}) - \sigma_{f_1}(\hat{B}_1) \leq \frac{\widehat{\Lambda}_{1,2}}{8}
\end{equation}

\begin{equation}
\frac{1}{8} \prod_{t=1}^{p_2} G^{(j_t)}_{f_2}(x^{(t)}_{f_2}, y^{(t)}_{f_2}) - \sigma_{f_2}(\hat{B}_2) \leq \frac{\widehat{\Lambda}_{1,2}}{8}.
\end{equation}

where $\Omega_0$ and $\vartheta_0$ only depend on $\sigma^{(i_s)}_{f_1}$, $1 \leq s \leq p_1$ and $\sigma^{(j_t)}_{f_2}$, $1 \leq t \leq p_2$.

**Proof of Lemma 5.2.** Assume $\hat{B}_1 = \{b^{(i_s)}_1, 1 \leq s \leq p_1\}$ and $\hat{B}_2 = \{b^{(j_t)}_2, 1 \leq t \leq p_2\}$ are "not coinciding" subsets of the break points of $f_1$ and $f_2$, respectively. We rewrite

\[
\prod_{s=1}^{p_1} G^{(i_s)}_{f_1}(x^{(s)}_{f_1}, y^{(s)}_{f_1}) = \prod_{s=1}^{p_1} \frac{[\sigma^{(i_s)}_{f_1} + (1 - \sigma^{(i_s)}_{f_1})y^{(s)}_{f_1}](1 + x^{(s)}_{f_1})}{\sigma^{(i_s)}_{f_1} + (1 - \sigma^{(i_s)}_{f_1})y^{(s)}_{f_1} + x^{(s)}_{f_1}} =
\]

\[
\prod_{s=1}^{p_1} \left[\frac{\sigma^{(i_s)}_{f_1} + (1 - \sigma^{(i_s)}_{f_1})y^{(s)}_{f_1}}{\sigma^{(i_s)}_{f_1} + (1 - \sigma^{(i_s)}_{f_1})y^{(s)}_{f_1} + x^{(s)}_{f_1}}\right] \times \prod_{s=1}^{p_1} \frac{1 + x^{(s)}_{f_1}}{\sigma^{(i_s)}_{f_1} + (1 - \sigma^{(i_s)}_{f_1})y^{(s)}_{f_1} + x^{(s)}_{f_1}} =
\]

\[
\prod_{s=1}^{p_1} \left[\frac{\sigma^{(i_s)}_{f_1} + (1 - \sigma^{(i_s)}_{f_1})y^{(s)}_{f_1}}{\sigma^{(i_s)}_{f_1} + (1 - \sigma^{(i_s)}_{f_1})y^{(s)}_{f_1} + x^{(s)}_{f_1}}\right] \times \prod_{s=1}^{p_1} \frac{1 + x^{(s)}_{f_1}}{\sigma^{(i_s)}_{f_1} + (1 - \sigma^{(i_s)}_{f_1})y^{(s)}_{f_1} + x^{(s)}_{f_1}} =
\]

\[
15
\]
\( (5.4) \quad \equiv \Phi_{f_1}^{(1)}(y_{f_1}^{(1)}, \ldots, y_{f_1}^{(p_1)}) \times \Phi_{f_1}^{(2)}(y_{f_1}^{(1)}, \ldots, y_{f_1}^{(p_1)}, x_{f_1}, \ldots, x_{f_1}). \)

Obviously
\[
\lim_{y_{f_1}^{(s)} \to 0, s=1,p_1} \Phi_{f_1}^{(1)}(y_{f_1}^{(1)}, \ldots, y_{f_1}^{(p_1)}) = \sigma_{f_1}(\hat{B}_1),
\]
\[
\lim_{x_{f_1} \to \infty, s=1,p_1} \Phi_{f_1}^{(2)}(y_{f_1}^{(1)}, \ldots, y_{f_1}^{(p_1)}, x_{f_1}, \ldots, x_{f_1}) = 1.
\]

When the variables \( y_{f_1}^{(s)}, 1 \leq s \leq p_1, \) are uniformly close to zero, the function \( \Phi_{f_1}^{(1)} = \Phi_{f_1}^{(1)}(y_{f_1}^{(1)}, \ldots, y_{f_1}^{(p_1)}) \) hence is close to \( \sigma_{f_1}(\hat{B}_1) \), while the function \( \Phi_{f_1}^{(2)} = \Phi_{f_1}^{(2)}(y_{f_1}^{(1)}, \ldots, y_{f_1}^{(p_1)}, x_{f_1}, \ldots, x_{f_1}) \) is close for large values of \( x_{f_1}^{(s)}, 1 \leq s \leq p_1 \). Taking these remarks into account and using the explicit form of the functions \( \Phi_{f_1}^{(1)} \) and \( \Phi_{f_1}^{(2)} \) we can now estimate \( |\Phi_{f_1}^{(1)} \cdot \Phi_{f_1}^{(2)} - \sigma_{f_1}(\hat{B}_1)| \). To estimate \( \Phi_{f_1}^{(1)} \), suppose that \( 0 \leq y_{f_1}^{(s)} \leq \vartheta_0^{(1)} < 1 \), where we shall choose the constant \( \vartheta_0^{(1)} \) later. It is easy to see that
\[
|\Phi_{f_1}^{(1)} - \sigma_{f_1}(\hat{B}_1)| = \sigma_{f_1}(\hat{B}_1) \prod_{s=1}^{p_1} \left(1 + \frac{(1 - \sigma_{f_1}^{(i_s)})}{\sigma_{f_1}^{(i_s)}} y_{f_1}^{(s)} \right) - 1| \leq C_4 \vartheta_0^{(1)},
\]
where the constant \( C_4 > 0 \) depends only on the \( \sigma_{f_1}^{(i_s)}, 1 \leq s \leq p_1 \).

We set \( \vartheta_0^{(1)} = \min\{\frac{\hat{\Lambda}_{1,2}}{16C_4}, 1\} \). Then
\[
(5.5) \quad |\Phi_{f_1}^{(1)} - \sigma_{f_1}(\hat{B}_1)| < \frac{\hat{\Lambda}_{1,2}}{16},
\]
for all \( 0 \leq y_{f_1}^{(s)} \leq \vartheta_0^{(1)}, 1 \leq s \leq p_1 \).

We next estimate \( |\Phi_{f_1}^{(2)} - 1| \) for large values of \( x_{f_1}^{(s)}, 1 \leq s \leq p_1 \). Using the explicit form of the function \( \Phi_{f_1}^{(2)} \), we see that the inequality
\[
(5.6) \quad |\Phi_{f_1}^{(2)} - 1| < R_2 \sum_{s=1}^{p_1} \frac{1}{x_{f_1}^{(s)}},
\]
holds for all \( y_{f_1}^{(s)}, 0 \leq y_{f_1}^{(s)} \leq 1, \) and \( x_{f_1}^{(s)} > 0, 1 \leq s \leq p_1, \) where the constant \( R_2 > 0 \) depends only on \( \sigma_{f_1}^{(i_s)}, 1 \leq s \leq p_1 \).

Suppose now, that \( x_{f_1}^{(s)} \geq \Omega_0^{(1)}, 1 \leq s \leq p_1. \) By (5.6) we have \( |\Phi_{f_1}^{(2)} - 1| < R_3 p_1 \frac{1}{\Omega_0^{(1)}}. \) It follows from this together with (5.5) that \( |\Phi_{f_1}^{(1)} \cdot \Phi_{f_1}^{(2)} - \sigma_{f_1}(\hat{B}_1)| \leq |\Phi_{f_1}^{(1)} - \sigma_{f_1}(\hat{B}_1)| + |\Phi_{f_1}^{(2)} - 1| \) hence is close to \( \sigma_{f_1}(\hat{B}_1) \). It follows from this that the assertion (5.2) of the lemma holds. Analogously it can be shown that with
\[
(5.7) \quad \vartheta_0^{(2)} := \min\{\frac{\hat{\Lambda}_{1,2}}{16C_5}, 1\}, \quad \Omega_0^{(2)} := \frac{16 \sigma_{f_1}(\hat{B}_2) \Lambda_{1,2} R_4 p_2}{\hat{\Lambda}_{1,2}},
\]
and $0 \leq y_{f_2}^{(t)} \leq \vartheta_{0}^{(2)}$ and $x_{f_2}^{(t)} \geq \Omega_{0}^{(2)}$, $1 \leq t \leq p_2$, the assertion (5.3) of Lemma 5.2 holds. In (5.4), the constants $C_5 > 0$ and $R_4 > 0$ depend on the $\sigma_{(j')}^{(s)}$, $1 \leq t \leq p_2$. If we finally set 
$\vartheta_0 := \min\{\vartheta_0^{(1)}, \vartheta_0^{(2)}\}$ and $\Omega_0 := \max\{\Omega_0^{(1)}, \Omega_0^{(2)}\}$ Lemma 5.2 holds for $x_{f_1}$, $x_{f_2} \geq \Omega_0$ and $y_{f_1}^{(s)}, y_{f_2}^{(t)} \in [0, \vartheta_0)$, $1 \leq s \leq p_1$, $1 \leq t \leq p_2$.

Define

\begin{equation}
\Omega_0 := \max \Omega_0 \left(\sigma_{f_1}^{(i_1)}, \ldots, \sigma_{f_1}^{(i_{p_1})}, \sigma_{f_2}^{(j_1)}, \ldots, \sigma_{f_2}^{(j_{p_2})}\right)
\end{equation}

\begin{equation}
\vartheta_0 := \min \vartheta_0 \left(\sigma_{f_1}^{(i_1)}, \ldots, \sigma_{f_1}^{(i_{p_1})}, \sigma_{f_2}^{(j_1)}, \ldots, \sigma_{f_2}^{(j_{p_2})}\right)
\end{equation}

where the minimum and maximum are taken over all "not coinciding" subsets $\hat{B}_1$ and $\hat{B}_2$ of the break points of $f_1$ and $f_2$ and $v_1, v_2 > 0$ are the total variations of $\ln Df_1$ and $\ln Df_2$ over $S^1$ respectively. Next we define the following constants $M_0$ and $\zeta_0$:

$$M_0 := \Omega_0 e^{\max\{v_1, v_2\}}, \quad \zeta_0 := \vartheta_0 e^{-\min\{v_1, v_2\}}.$$ 

Let $K_0 = K_0(f_1, f_2, M_0, \zeta_0) > M_0 > 1$ be the constant $K$ as defined in Theorem 4.3.

Lemma 5.3. Suppose the circle homeomorphisms $f_1$ and $f_2$ satisfy the conditions of Theorem 4.3 and $Dh(x_0) = \omega_0 > 0$ at some point $x_0 \in S^1$. Let be $\delta > 0$ and assume the triples intervals $([z_s, z_{s+1}], s = 1, 2, 3)$ and $([h(z_s), h(z_{s+1})], s = 1, 2, 3)$ cover the "not coinciding" subsets $\hat{B}_1$ and $\hat{B}_2$ of break points of $f_1$ and $f_2$ ".(K_0, M_0, \delta, \zeta_0; x_0, h(x_0))"-regularly" for some $r_n \in \{q_n-1, q_n, q_n + q_n-1\}$. Then for sufficiently large $n$ the following inequality holds:

\begin{equation}
|\frac{\text{Dist}(z_1, z_2, z_3, z_4; f_1^n)}{\text{Dist}(h(z_1), h(z_2), h(z_3), h(z_4); f_2^n)} - 1| \geq R_5 > 0,
\end{equation}

where the constant $R_5$ depends only on $f_1$ and $f_2$.

Proof of Lemma 5.3: Suppose, the triples of intervals $([z_s, z_{s+1}], s = 1, 2, 3)$ and $([h(z_s), h(z_{s+1})], s = 1, 2, 3)$ cover the "not coinciding" subsets $\hat{B}_1$ and $\hat{B}_2$ of break points of $f_1$ and $f_2$ "(K_0, M_0, \delta, \zeta_0; x_0, h(x_0))"-regularly" for some $r_n \in \{q_n-1, q_n, q_n + q_n-1\}$. To be definite, assume $\hat{B}_1 = \{b_1^{(i_1)}, b_1^{(j_1)}, \ldots, b_1^{(j_{p_1})}\} \neq \emptyset, \hat{B}_2 = \{b_2^{(j_1)}, b_2^{(j_2)}, \ldots, b_2^{(j_{p_2})}\} \neq \emptyset$. By Definition 4.3 the triples of intervals $([z_s, z_{s+1}], s = 1, 2, 3)$ and $([h(z_s), h(z_{s+1})], s = 1, 2, 3)$ cover the subsets $\hat{B}_1$ respectively $\hat{B}_2$ "(K_0, M_0, \delta, \zeta_0; x_0, h(x_0))"-regularly" respectively "(K_0, M_0, \delta, \zeta_0; x_0, h(x_0))"-regularly".

By Definition 4.1 we have $b_1^{(i_s)} \in [z_1, z_2], 1 \leq s \leq p_1$ and $b_2^{(j_t)} \in [h(z_1), h(z_2)], 1 \leq t \leq p_2$. Notice that the intervals $f_1^{(i_s)}([z_1, z_2]), 1 \leq s \leq p_1$ cover the break points $b_1^{(i_s)}$, $1 \leq s \leq p_1$. Similarly, the intervals $f_2^{(j_t)}([h(z_1), h(z_2)]), 1 \leq t \leq p_2$ cover the break points $b_2^{(j_t)}$, $1 \leq t \leq p_2$. Next we want to compare the distortion $\text{Dist}(z_1, z_2, z_3, z_4; f_1^n)$ and $\text{Dist}(h(z_1), h(z_2), h(z_3), h(z_4); f_2^n)$. We estimate only the first distortion, the second one can be estimated analogously. Obviously

$$\text{Dist}(z_1, z_2, z_3, z_4; f_1^n) = \frac{Cr(f_1^n(z_1), f_1^n(z_2), f_1^n(z_3), f_1^n(z_4))}{Cr(z_1, z_2, z_3, z_4)} =$$

$$= \frac{Cr(f_1(z_1), f_1(z_2), f_1(z_3), f_1(z_4))}{Cr(z_1, z_2, z_3, z_4)} \times \frac{Cr(f_2^n(z_1), f_2^n(z_2), f_2^n(z_3), f_2^n(z_4))}{Cr(f_2^n(z_1), f_2^n(z_2), f_2^n(z_3), f_2^n(z_4))} \times ...$$

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\[
\frac{Cr(f_1^{(n)}(z_1), f_1^{(n)}(z_2), f_1^{(n)}(z_3), f_1^{(n)}(z_4))}{Cr(f_1^{(n-1)}(z_1), f_1^{(n-1)}(z_2), f_1^{(n-1)}(z_3), f_1^{(n-1)}(z_4))} = \prod_{i=0}^{r_n-1} Dist(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1).
\]

We rewrite \(Dist(z_1, z_2, z_3, z_4; f_1^{(n)})\) in the form
\[
Dist(z_1, z_2, z_3, z_4; f_1^{(n)}) = \prod_{s=1}^{p_1} Dist(f_1^{(i_s)}(z_1), f_1^{(i_s)}(z_2), f_1^{(i_s)}(z_3), f_1^{(i_s)}(z_4); f_1) \times \\
\times \prod_{i=0, i\neq f_1^{(i_s)}, s=1}^{r_n-1} Dist(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1).
\]

(5.11)

To estimate the first factor in (5.11) we use Lemma 2.11 and the definition of the functions \(G_{f_1}(x, y)\) to get
\[
Dist(f_1^{(i_s)}(z_1), f_1^{(i_s)}(z_2), f_1^{(i_s)}(z_3), f_1^{(i_s)}(z_4); f_1) = \\
\frac{\sigma_{f_1}^{(i_s)} + (1 - \sigma_{f_1}^{(i_s)})z_{f_1}^{(i_s)}(l_1^{(i_s)})}{\sigma_{f_1}^{(i_s)} + (1 - \sigma_{f_1}^{(i_s)})z_{f_1}^{(i_s)}(l_1^{(i_s)}) + \xi_{f_1}(l_1^{(i_s)})} + \chi_{s}^{(1)} = \\
G_{f_1}(\xi_{f_1}(l_1^{(i_s)}), z_{f_1}^{(i_s)}(l_1^{(i_s)})) + \chi_{s}^{(1)}, \quad 1 \leq s \leq p_1,
\]

where \(|\chi_{s}^{(1)}| \leq C_2\ell([f_1^{(i_s)}(z_1), f_1^{(i_s)}(z_4)])\), \(1 \leq s \leq p_1\). By construction the interval \([z_1, z_4]\)
(see (4.3), (4.8), (4.11)) is \(r_n\)-small and therefore the intervals \(f_1^{(i_s)}([z_1, z_4])\), \(0 < i < r_n\) are pairwise disjoint. Hence, using Corollary 2.5 we obtain
\[
\ell(f_1^{(i_s)}([z_1, z_4])) \leq const\lambda_{1}^{n}, \quad i = 0, r_n - 1,
\]

where \(\lambda_{1} = (1 + e^{-\epsilon_{0 n}})^{-\frac{1}{2}} < 1\).

Fix now some \(\epsilon > 0\). There exists \(N = N(\epsilon) > 1\) such that
\[
|\chi_{s}^{(1)}| \leq C_0\epsilon, \quad 1 \leq s \leq p_1
\]
holds for \(n > N\), where the constant \(C_0 > 0\) depends only on \(f_1\).

Suppose, \(\xi_{f_1}(0)\) and \(z_{f_1}^{(i_s)}(0), 1 \leq s \leq p_1\) satisfy the following conditions: \(\xi_{f_1}(0) > M_0\) and \(z_{f_1}^{(i_s)}(0) < \zeta_0\) for \(1 \leq s \leq p_1\). Then, using relations (4.1) we get \(\xi_{f_1}(l_1^{(i_s)}) > M_0\) and \(z_{f_1}^{(i_s)}(l_1^{(i_s)}) < \zeta_0\), \(1 \leq s \leq p_1\), where \(M_0\) and \(\zeta_0\) defined in (4.8) and (5.9). Since \(\zeta_0\) is the minimum of \(\vartheta_0\), the assertion of Lemma 5.2 is true for \(\vartheta_0\) also. It follows from the assertion of Lemma 5.2 that
\[
|\prod_{s=1}^{p_1} G_{f_1}(\xi_{f_1}(l_1^{(i_s)}), z_{f_1}^{(i_s)}(l_1^{(i_s)})) - \sigma_{f_1}(\hat{B}_1)| \leq \frac{\Lambda_{12}}{8}.
\]

(5.15)

By combining (5.12), (5.15) we obtain
\[
|\prod_{s=1}^{p_1} G_{f_1}(\xi_{f_1}(l_1^{(i_s)}), z_{f_1}^{(i_s)}(l_1^{(i_s)})) + \chi_{s}^{1} - \sigma_{f_1}(\hat{B}_1)| \leq \frac{\Lambda_{12}}{6}
\]

(5.16)
for sufficiently small $\varepsilon > 0$.

Next we estimate the second factor in (5.11). Applying Lemma 2.10 we obtain

\[
\prod_{i=0, i \neq i_1^{(s)}, s=1, p_1}^{r_n-1} \text{Dist}(f^1_i(z_1), f^1_i(z_2), f^1_i(z_3), f^1_i(z_4); f_1) = \exp\left\{ \sum_{i=0, i \neq i_1^{(s)}, s=1, p_1}^{r_n-1} \log(1 + O((\ell([f^1_i(z_1), f^1_i(z_4)]))^{1+\alpha}))) \right\}
\]

Using the bound (5.13) and that the interval $[z_1, z_4]$ is $r_n$-small, we obtain from (5.17)

\[
|\prod_{i=0, i \neq i_1^{(s)}, s=1, p_1}^{r_n-1} \text{Dist}(f^1_i(z_1), f^1_i(z_2), f^1_i(z_3), f^1_i(z_4); f_1) - 1| \leq \text{const} \lambda_1^{n\alpha} \sum_{i=0, i \neq i_1^{(s)}, s=1, p_1}^{r_n-1} (\ell([f^1_i(z_1), f^1_i(z_4)])) \leq \text{const} \lambda_1^{n\alpha}.
\]

The relations (5.16) and (5.18) imply that for sufficiently large $n$

\[
|\text{Dist}(z_1, z_2, z_3, z_4; f^1_1) - \sigma_{f_1}(\hat{B}_1)| < \frac{\Lambda_{1,2}}{4}
\]

holds.

The same way it can be shown that for the triple of intervals $([h(z_s), h(z_{s+1})], s = 1, 2, 3)$ covering the set $\hat{B}_2$ "$(K_0, M_0, \delta, \zeta_0; h(x_0))$-regularly" the following inequality

\[
|\text{Dist}(h(z_1), h(z_2), h(z_3), h(z_4); f^1_1) - \sigma_{f_2}(\hat{B}_2)| < \frac{\Lambda_{1,2}}{4}
\]

holds for sufficiently large $n$. The inequalities (5.19) and (5.20) show that

\[
\frac{\text{Dist}(z_1, z_2, z_3, z_4; f^1_1) - \sigma_{f_1}(\hat{B}_1)}{\text{Dist}(h(z_1), h(z_2), h(z_3), h(z_4); f^1_1)} - 1 \geq \frac{4(\sigma_{f_1}(\hat{B}_1) - \sigma_{f_2}(\hat{B}_2) - 2\Lambda_{1,2})}{4\sigma_{f_2}(\hat{B}_2) + \Lambda_{1,2}} > 0,
\]

if $\sigma_{f_1}(\hat{B}_1) > \sigma_{f_2}(\hat{B}_2)$, and

\[
\frac{\text{Dist}(z_1, z_2, z_3, z_4; f^1_1) - \sigma_{f_1}(\hat{B}_1)}{\text{Dist}(h(z_1), h(z_2), h(z_3), h(z_4); f^1_1)} - 1 \leq \frac{4(\sigma_{f_1}(\hat{B}_1) - \sigma_{f_2}(\hat{B}_2) + 2\Lambda_{1,2})}{4\sigma_{f_2}(\hat{B}_2) - \Lambda_{1,2}} < 0
\]

if $\sigma_{f_1}(\hat{B}_1) < \sigma_{f_2}(\hat{B}_2)$. If we set

\[
R_5 := \min\{\frac{|4(\sigma_{f_1}(\hat{B}_1) - \sigma_{f_2}(\hat{B}_2)) - 2\Lambda_{1,2}|}{4\sigma_{f_2}(\hat{B}_2) + \Lambda_{1,2}}, \frac{|4(\sigma_{f_1}(\hat{B}_1) - \sigma_{f_2}(\hat{B}_2)) + 2\Lambda_{1,2}|}{4\sigma_{f_2}(\hat{B}_2) - \Lambda_{1,2}}\},
\]

where the minimum is taken over all "not coinciding" subsets $\hat{B}_1$ and $\hat{B}_2$ of break points $f_1$ and $f_2$, then it follows from (5.21)-(5.23) that the assertion of the lemma holds.
Proof of Theorem 1.5  Let \( f_1 \) and \( f_2 \) be circle homeomorphisms satisfying the conditions of Theorem 1.5. The lift \( H(x) \) of the conjugating map \( h(x) \) is a continuous and monotone increasing function on \( R^1 \). Hence \( H(x) \) has a finite derivative \( DH(x) \) for almost all \( x \) with respect to Lebesgue measure. We claim that \( Dh(x) = 0 \) at all the points \( x \) where the finite derivative exists. Suppose \( Dh(x_0) > 0 \) for some point \( x_0 \in S^1 \). We fix \( \varepsilon > 0 \). Let \( K_0 = K_0(f_1, f_2, M_0, \zeta_0) > M_0 > 1 \) be the constant defined in the assertion of Theorem 1.3. Theorem 1.5 is therefore completely proved.

By the assertion of Theorem 1.3 for sufficiently large \( n \) there exist "not coinciding" subsets \( \hat{B}_1 \) and \( \hat{B}_2 \) of break points of \( f_1 \) and \( f_2 \), points \( z_i \in S^1 \), \( 1 \leq i \leq 4 \), with \( z_1 < z_2 < z_3 < z_4 \) and a number \( r_n \in \{ q_n^{-1}, q_n, q_n + q_n^{-1} \} \) such that the triples of intervals \([z_s, z_{s+1}] \), \( s = 1, 2, 3 \), and \([h(z_s), h(z_{s+1})] \), \( s = 1, 2, 3 \), cover the points of \( \hat{B}_1 \) and \( \hat{B}_2 \) \((K_0, M_0, \delta, \zeta_0; x_0, h(x_0))-\)regularly" for \( r_n \). By Definition 4.1 of a regularly covering each of the systems of intervals \([z_s, z_{s+1}] \), \( s = 1, 2, 3 \) and \(([f_1^{r_n}(z_s), f_1^{r_n}(z_{s+1})] \), \( s = 1, 2, 3 \) satisfies the conditions of Lemma 5.1 with constant \( R_1 = K_0 \).

Using the assertion of Lemma 5.1 we obtain

\[
(5.24) \quad |\text{Dist}(z_1, z_2, z_3, z_4; h) - 1| \leq C_3\varepsilon,
\]

\[
(5.25) \quad |\text{Dist}(f_1^{r_n}(z_1), f_1^{r_n}(z_2), f_1^{r_n}(z_3), f_1^{r_n}(z_4); h) - 1| \leq C_3\varepsilon.
\]

Hence

\[
(5.26) \quad \frac{|\text{Dist}(z_1, z_2, z_3, z_4; h) - \text{Dist}(f_1^{r_n}(z_1), f_1^{r_n}(z_2), f_1^{r_n}(z_3), f_1^{r_n}(z_4); h)|}{\text{Dist}(z_1, z_2, z_3, z_4; h)} - 1 \leq C_7\varepsilon,
\]

where the constant \( C_7 > 0 \) does not depend on \( \varepsilon \) and \( n \).

Since \( h \) is conjugating \( f_1 \) and \( f_2 \) we can readily see that

\[
Cr(h(f_1^{r_n}(z_1)), h(f_1^{r_n}(z_2)), h(f_1^{r_n}(z_3)), h(f_1^{r_n}(z_4))) = Cr(f_2^{r_n}(h(z_1)), f_2^{r_n}(h(z_2)), f_2^{r_n}(h(z_3)), f_2^{r_n}(h(z_4))).
\]

Hence we obtain

\[
\frac{\text{Dist}(f_1^{r_n}(z_1), f_1^{r_n}(z_2), f_1^{r_n}(z_3), f_1^{r_n}(z_4); h)}{\text{Dist}(z_1, z_2, z_3, z_4; h)} = \frac{Cr(h(f_1^{r_n}(z_1)), h(f_1^{r_n}(z_2)), h(f_1^{r_n}(z_3)), h(f_1^{r_n}(z_4))))}{Cr(f_1^{r_n}(z_1), f_1^{r_n}(z_2), f_1^{r_n}(z_3), f_1^{r_n}(z_4))}
\]

\[
\times \frac{Cr(z_1, z_2, z_3, z_4)}{Cr(h(z_1), h(z_2), h(z_3), h(z_4))} = \frac{Cr(f_2^{r_n}(h(z_1)), f_2^{r_n}(h(z_2)), f_2^{r_n}(h(z_3)), f_2^{r_n}(h(z_4))))}{Cr(h(z_1), h(z_2), h(z_3), h(z_4))}
\]

\[
\times \frac{Cr(f_1^{r_n}(z_1), f_1^{r_n}(z_2), f_1^{r_n}(z_3), f_1^{r_n}(z_4))}{Cr(z_1, z_2, z_3, z_4)} = \frac{\text{Dist}(h(z_1), h(z_2), h(z_3), h(z_4); f_2^{r_n})}{\text{Dist}(z_1, z_2, z_3, z_4; f_1^{r_n})}.
\]

This, together with (5.26) obviously implies that

\[
|\frac{\text{Dist}(z_1, z_2, z_3, z_4; f_1^{r_n})}{\text{Dist}(h(z_1), h(z_2), h(z_3), h(z_4); f_2^{r_n})} - 1| \leq C_8\varepsilon.
\]

where the constant \( C_8 > 0 \) does not depend on \( \varepsilon \) and \( n \). But this contradicts equation (5.10). Theorem 1.5 is therefore completely proved.
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