Multi-type spatial branching models for local self-regulation
I: Construction and an exponential duality

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Abstract

We consider a spatial multi-type branching model in which individuals migrate in geographic space according to random walks and reproduce according to a state-dependent branching mechanism which can be sub-, super- or critical depending on the local intensity of individuals of the different types. The model is a Lotka-Volterra type model with a spatial component and is related to two models studied in [BEM07] as well as to earlier work in [Eth04] and in [NP99]. Our main focus is on the diffusion limit of small mass, locally many individuals and rapid reproduction. This system differs from spatial critical branching systems since it is not density preserving and the densities for large times do not depend on the initial distribution but mainly on the carrying capacities.

We prove existence of the infinite particle model and the system of interacting diffusions as solutions of martingale problems or systems of stochastic equations. In the exchangeable case in which the parameters are not type dependent we show uniqueness of the solutions. For that purpose we establish a new exponential duality.

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1 Introduction

In this paper we consider interacting stochastic processes indexed by $\mathbb{Z}^d$ or some countable Abelian group which are obtained as the high density fast branching limit of particle models in which particles are assigned a type and the number of particles of each type at a site is changing due to branching and migration. While particles migrate independently of each other, we will choose the branching mechanism state-dependent with local self-regulation for the offspring distribution reflecting competition among subpopulations such that the model obtained may exhibit even in low dimensions

- stable populations, and
- coexistence of different types.

For a few decades the spatial distribution of biological populations with a geographic structure has been modeled by the spatial Dawson-Watanabe superprocess or super random walk in which infinitesimally small particles are supposed to migrate through space and branch independently. However, it is well-known that in the subcritical case we have local extinction, in the supercritical case local explosion. In the critical case in dimensions $d \leq 2$ the interaction due to migration is not strong enough to counteract local extinction due to branching. Worse, at rare sites where the system is not extinct, populations grow without any bound, building up very high peaks. This also holds for multi-type versions of these models for which in addition in the critical case in $d = 1, 2$ the high peaks are build up by mono-type populations.

A way out could be to condition the population to stay constant in the colonies. This results in a system of spatially interacting Fleming-Viot processes. This is a process where particles are assigned a type from a possibly continuous type space and the local frequency of types changes due to a resampling mechanism. Unfortunately, we still have to deal with two competing mechanisms, here migration and resampling, which once more yield a dichotomy to the effect that in dimensions $d \leq 2$ the model tends locally to mono-type configurations (see e.g. [CG94]) and only in high dimension $d \geq 3$ allows for local coexistence of different types.

The extinction and formation of mono-type clusters turn out to be a drawback since most biological populations happen to live in two (or even one) dimensions, while at the same time locally stable populations with local coexistence of different types are observed.

To model the latter features we have to give up the martingale property of a single colony arising from critical branching or resampling. The reason is that we have to take into account that resources are limited. Therefore the key task is to extend the well-studied “neutral” models by new mechanisms inducing a self-regulation of the population size and population type decomposition under the constraints of bounded resources.

In physics such a model is referred to as a model for (local) self-organized criticality. So one can hope for such a model that it does not exhibit the dichotomy of either locally unbounded growth or local extinction and furthermore that coexistence of different types is possible.

There have been previous attempts to introduce a self-regulative mechanism on the level of one-type branching processes without a spatial structure (compare [JK02]).
we will use a set-up which includes a geographic structure of the population. In this framework we will observe, depending on the set of parameters chosen, various behaviors which are observable in nature, namely local extinction versus non-extinction, or locally the coexistence of different types versus the formation of mono-type clusters depending on the parameters of the model.

A starting point for a spatial mono-type model with local self-regulation in one dimension is the following \textit{logistic stochastic partial differential equation} which arises in the large local population limit combined with a spatial rescaling of \(Z\) to \(\mathbb{R}\):

\begin{equation}
\frac{\partial u}{\partial t} = \frac{1}{6} \frac{\partial^2 u}{\partial x^2} + (K - \theta u)u + \sqrt{u} \dot{W},
\end{equation}

where \(\dot{W}\) is white-noise. The function \(u(t,x)\) describes the density of a population in colony \(x \in \mathbb{R}\) at time \(t \geq 0\). In \[MT94\] the existence of a critical capacity \(K_c\) is shown such that for \(K < K_c\), \(\tau^{\text{ext}} := \inf\{t \geq 0 : u(t,0) = 0\} < \infty\), while for \(K > K_c\), \(\tau^{\text{ext}} = \infty\).

This continuous site model cannot be extended to higher dimensions. However, in \[BP97\] and \[BP99\] a similar model is introduced for a discrete geographic space. Bolker and Pacala propose by simulations that equilibria exist even in low dimensions for certain values of the parameters. The latter is proved in \[Eth04\].

On the other hand predator-prey models like the classical Lotka-Volterra model have been shown to be able to explain the coexistence of species. In \[NP99\] the Lotka-Volterra model is studied with a very specific spatial structure having properties in common with a model known in mathematics as the voter model. In particular for certain parameters coexistence of both types even in low dimensions, more precisely \(d = 2\), are established.

This work \[NP99\] by Neuhauser and Pacala is also the starting point of two more recent papers \[HW07\] and \[CP05\]. In \[CP05\], it is shown that suitable rescaling in space and time of the density of one population leads to a limit that is described by super-Brownian motion with a drift. The form of the drift is related to coexistence and survival of a rare type in the original Lotka-Volterra model in \[NP99\].

Whereas in these models exactly one individual inhabits a site in the geographic state space the present paper will continue the theory with a multi-type spatial branching system in a geographic space consisting of different colonies. This means that in our model the possible number of individuals in each colony is not restricted, even though the colony has in some sense a \textit{carrying capacity} due to limited availability of resources at this location which are necessary for the fecundity of the different types in a varying degree.

A technical point which makes this model harder to investigate than other spatial multi-type branching models treated so far is that we face here non-linearities in the drift terms. In particular, the system is not density preserving. Moreover, in the long run the density will be only dependent on the carrying capacities but not very much on the initial state which is the key parameter in the neutral models.

The main goal of the present first part of the paper is to establish the existence of the particle and the diffusion models on infinite geographic space, and to give an analytical characterisation as solutions of martingale problems and systems of stochastic equations. In the particular case where the parameters do not depend on the particles’ type, we establish a new exponential duality which allows to prove uniqueness.
2 Construction and characterization of the models

The model we shall study describes the masses of \( M \in \mathbb{N} \) different types of individuals in a population distributed over colonies in a geographic space \( G \) with generally countably many – thus possibly infinitely many – components.

The diffusion systems studied here arise as the limit of suitable particle models. Due to the fact that we also consider a (countably) infinite geographic space the concepts of the state spaces as well as of solutions to martingale problems and respective SDEs require some modifications compared to the case of a single or finitely many components. Therefore we begin in Subsection 2.1 with an introduction and discussion of these concepts. We then introduce the particle system in Subsection 2.2, the diffusion limit in Subsection 2.3 and results on the approximation of the diffusion limit via particle systems in Subsection 2.4. Finally, in Subsection 2.5 we present an exponential duality relation for a special exchangeable case of our model, which allows us in particular to make stronger statements about convergence to and uniqueness of the limit diffusion.

2.1 Preliminaries on systems in infinite geographic space

In this subsection we discuss the adaptations needed to define a Markov process with countably infinitely many interacting components. We assume that the location of colonies in geographic space is given by some countable Abelian group \( G \). Thus the processes considered here will have state spaces which are subsets of \((\mathbb{R}^M)\). Two problems arise here:

1. the components live in the unbounded set \(\mathbb{R}^+\) and
2. the geographic space is infinite.

This combination makes it more difficult to establish the existence and uniqueness of the stochastic processes in general and in particular given the additional complication in our model with a nonlinear interaction.

For every system with state space contained in \((\mathbb{R}^M)\), the problem arises to set up the state space in such a way that the dynamics can be well-defined and no influence from infinity occurs at specific sites rendering the process unspecified.

To keep the process locally finite we choose as the state space the Liggett-Spitzer space (first introduced in [LS81]).

Let for a countable Abelian group \( G \), \( a(\cdot, \cdot) \) be a random walk kernel from \( G \) to \( G \), i.e.,

\[
(2.1) \quad a(\eta, \xi) = a(0, \xi - \eta) \quad \text{and} \quad \sum_{\xi \in G} a(0, \xi) = 1,
\]

which we use later to model migration on \( G \).

Next we choose a weight function \( \rho \) as follows

\[
(2.2) \quad \rho(\xi) := \sum_{n=0}^{\infty} \sum_{\eta \in G} (R/2)^{-n} \hat{a}^{(n)}(\eta, \xi) \beta(\eta)
\]

with \( R > 2 \), \( \beta(\eta) > 0 \) for all \( \eta \in G \), \( \sum_{\eta \in G} \beta(\eta) < \infty \) and

\[
(2.3) \quad \hat{a}(\xi, \eta) := \frac{1}{2} (a(\xi, \eta) + a(\eta, \xi)).
\]
Note that $\rho$ is positive and summable and for $\eta \in G$,

\[(2.4) \sum_{\xi \in G} \hat{a}(\xi, \eta) \rho(\xi) \leq \frac{2}{\rho}(\eta).\]

As state space we consider

\[(2.5) \mathcal{E}^G := \{x \in (\mathbb{R}_+^M)^G : \sum_{\xi \in G} \rho(\xi) \bar{x}_\xi < \infty\},\]

where

\[(2.6) \bar{x}_\xi := \sum_{m=1}^{M} x^m_{\xi},\]

and write $\|x\|$ for the $\rho$-weighted $l^1$-norm, i.e., for all $x \in (\mathbb{R}_+^M)^G$,

\[(2.7) \|x\| := \sum_{\xi \in G} \rho(\xi) \bar{x}_\xi.\]

We equip the state space $\mathcal{E}$ with the product topology of $(\mathbb{R}_+^M)^G$.

We do not choose the norm topology since we cannot expect to find a solution with regular paths in the state space $\mathcal{E}^G$ equipped with the norm topology.

Finally, as a state space for the approximating particle systems we consider the subset

\[(2.8) \mathcal{E}_{\text{par},G} := \mathcal{E}^G \cap (\mathbb{N}_0^M)^G.\]

In the following and throughout the paper we will denote by $B(E)$, respectively $B(E)$ the set of all measurable, respectively measurable and bounded real valued functions on a topological space $E$. We further denote by $C(E, F)$ and $C_b(E, F)$ the space of continuous and continuous bounded functions from a space $E$ to another space $F$. If $F := \mathbb{R}$, we simply write $C(E)$ or $C_b(E)$. In the case of $E = \mathbb{R}_+$ we use $D(\mathbb{R}_+, F)$ for the Skorohod space of càdlàg functions with values in $F$.

In order to make precise what we mean by the solution to a martingale problem we formulate:

**Definition 2.1 (Martingale problem)** Let a state space $\mathcal{E}$, a set $\mathcal{F} \subset C_b(\mathcal{E}, \mathbb{R})$ and a linear operator $\Omega_X$ with domain including $\mathcal{F}$ be given. Furthermore, let $\nu$ be a distribution on $\mathcal{E}$. Then, solutions to the $(\Omega_X, \mathcal{F}, \nu)$ martingale problem are processes $X$ with paths in $D(\mathbb{R}_+, \mathcal{E})$ such that

\[(2.9) (f(X(t)) - \int_0^t (\Omega_X f)(X(s))\,ds)_{t \geq 0}\]

is a martingale for every $f \in \mathcal{F}$ and $\mathcal{L}[X_0] = \nu$. Uniqueness holds for the martingale problem if there is at most one $P \in D(\mathbb{R}_+, \mathcal{E})$ such that under $P$, $\mathcal{E}$ is a martingale for all $f \in \mathcal{F}$. The martingale problem is well-posed if there exists exactly one such $P$. 


As we shall see later, uniqueness for the martingale problem with infinitely many components can be verified a priori only in very particular cases, for example, via duality relations in the exchangeable case. In order to address the uniqueness problem we will therefore, in general cases only consider solutions which allow an approximation by spatially finite systems.

We shall use approximations by populations that live in finite geographic spaces \((G_L)_{L \in \mathbb{N}}, \ G_L \uparrow G, \ |G_L| < \infty.\)

For example \(G_L = [-L, L]^d \cap \mathbb{Z}^d\) for \(G = \mathbb{Z}^d.\)

**Definition 2.2 (Approximation property)** A solution \(X\) of the \((\Omega_X, \mathcal{F}, \nu)\)-martingale problem has the approximation property (with respect to \(\{G_L, L \in \mathbb{N}\}\)) if there exists \(\{X_L, L \in \mathbb{N}\}\) of \(\mathcal{L}^G\)-valued strong Markov processes with

\[(2.11) \ \mathcal{L}[(X^L(t)_{t \geq 0})_{L \to \infty}] \Rightarrow \mathcal{L}[(X(t)_{t \geq 0})],\]

and for each \(L \in \mathbb{N}\), \(X^L\) solves an \((\Omega_X, \mathcal{F}, \nu)\)-martingale problem for an operator \(\Omega^L\) such that \(\Omega^L f(x) = 0\) whenever \(f \in \mathcal{F}\) and the restriction of \(f\) to \(G_L\) is a constant function.

### 2.2 The underlying particle system

In this subsection we introduce the approximating particle systems. These systems will also give an intuitive meaning to the parameters that are used in the description of self-regulating population models.

We consider particles (individuals) that are assigned a location in the geographic space \(G\) and a type \(m \in \{1, \ldots, M\}\). If not stated otherwise we consider \(M \geq 2\). These particles are migrating in the space \(G\) and they are also branching (reproducing) in an environment of limited resources.

The branching is state-dependent meaning that due to bounded resources the mean number of offspring varies as a function of the current state, although we will assume for simplicity that the branching is binary. In order to describe the state dependence we need parameters that quantify the carrying capacity for type \(m \in \{1, \ldots, M\}\) and the influence of type \(n\) on type \(m\).

For each \(m \in \{1, \ldots, M\}\) the carrying capacity, \(K^m \in (0, \infty)\), of a colony for the \(m^{th}\) type arise as follows: Assume there are \(J\) different resources of respective size \((2.12)\)

\[R_j, \ j = 1, \ldots, J,\]

which the individuals have to share. Abundance and shortage of resources cause additional births and deaths, respectively, which can be quantified by the \(m^{th}\) type sensitivity \(s_{j,m}\) to (abundance and shortage of) resource \(j \in \{1, \ldots, J\}\). Moreover, each resource may be utilized by all types. If \(\tilde{\lambda}_{j,n}\) denotes the amount which the \(n^{th}\) type uses the resource
$j \in \{1, \ldots, J\}$, then given the population $z^m \in \mathbb{N}^G$ of type $m$ it is reasonable to choose the mean deviations from critical offspring according to the penalty term

$$
(2.13) \sum_{j=1}^{J} s_{j,m} \left( R^j - \sum_{n=1}^{M} \tilde{\lambda}_{j,n} z^n \right).
$$

We therefore define the carrying capacity by

$$
(2.14) K^m := \sum_{j=1}^{J} s_{j,m} R^j,
$$

the competition matrix, $(\lambda_{m,n})_{m,n \in \{1, \ldots, M\}}$, by

$$
(2.15) \lambda_{m,n} := \sum_{j=1}^{J} s_{j,m} \tilde{\lambda}_{j,n},
$$

and the natural branching rates by

$$
(2.16) \gamma^m.
$$

The dynamics of the particle system

$$
(2.17) Z^G = \{(z^m_\xi(t))_{t \geq 0}; m = 1, \ldots, M, \xi \in G\}
$$

is given by the following two independent mechanisms:

- Migration. Each particle migrates in $G$ according to a continuous time rate 1 random walk with transition probabilities $a(\xi, \eta)$ from $\xi$ to $\eta$.

- State dependent branching. For each $m = 1, \ldots, M$ and $\xi \in G$, given the current population $z_\xi = (z^1_\xi, \ldots, z^M_\xi)$ the following transitions occur:

- Birth. Each particle of type $m$ at site $\xi$ gives birth to a new particle at rate

$$
(2.18) \gamma^m \cdot \gamma^m_{\text{birth}}(z_\xi) = \gamma^m \left( \frac{1}{2} + K^m \right).
$$

- Death. Each particle of type $m$ at site $\xi$ dies at rate

$$
(2.19) \gamma^m \cdot \gamma^m_{\text{death}}(z_\xi) = \gamma^m \left( \frac{1}{2} + \sum_{n=1}^{M} \lambda_{m,n} z^n_\xi \right).
$$

Define the interaction function of type $m$ with the environment and the other populations by setting for all $m \in \{1, \ldots, M\}$, $\xi \in G$, and $y \in \mathbb{R}^M_+$,

$$
(2.20) \Gamma^m(y_\xi) := \gamma^m_{\text{birth}}(y_\xi) - \gamma^m_{\text{death}}(y_\xi) = K^m - \sum_{n=1}^{M} \lambda_{m,n} y^n_\xi.
$$

Some statements we can prove only in special cases. We therefore introduce the following sets of assumptions.
Set of Assumptions 1 (Constant branching rate) All types reproduce at the same rate, i.e., there is a constant \( \gamma > 0 \) such that

\[
\gamma^m = \gamma, \quad \text{for all } m \in \{1, \ldots, M\}.
\]

Set of Assumptions 2 (Type-non-sensitive resources and capacities) Resources are not sensitive to different types, i.e., there is a constant \( \lambda \geq 0 \) such that

\[
\lambda_{m,n} = \lambda, \quad \text{for all } m, n \in \{1, \ldots, M\}.
\]

All types have the same carrying capacity, i.e., there is a constant \( K \geq 0 \) such that

\[
K^m = K, \quad \text{for all } m \in \{1, \ldots, M\}.
\]

Definition 2.3 (Exchangeable models) If Assumptions 1 and 2 are satisfied, then the corresponding class of models is called the exchangeable model.

We want to characterize the particle model introduced above analytically as a solution of a system of stochastic equations and a martingale problem. First consider the following system of stochastic equations: for all \( m \in \{1, 2, \ldots, M\} \), \( \xi \in G \), and \( t > 0 \),

\[
z^m_\xi(t) = z^m_\xi(0) + \sum_{\eta \in G, \eta \neq \xi} \left[ \int_{[0,t] \times \mathbb{R}_+} 1(z^m_\eta(s-) \geq u) N^m_{\eta}(ds \, du) \right. \\
- \left. \int_{[0,t] \times \mathbb{R}_+} 1(z^m_\xi(s-) \geq u) N^m_{\xi}(ds \, du) \right] + \int_{[0,t] \times \mathbb{R}_+} 1(z^m_\xi(s-) \gamma^m_{\text{birth}}(z^m_\xi(s-)) \geq u) N^{m,+}_{\xi}(ds \, du) \\
- \int_{[0,t] \times \mathbb{R}_+} 1(z^m_\xi(s-) \gamma^m_{\text{death}}(z^m_\xi(s-)) \geq u) N^{m,-}_{\xi}(ds \, du).
\]

Here \( \{N^{m,\eta}_{\xi} : \xi, \eta \in G, \xi \neq \eta, 1 \leq m \leq M\} \) are independent Poisson processes on \( [0, \infty) \times \mathbb{R}_+ \) and \( \{N^{m,+}_{\xi}, N^{m,-}_{\xi} : \xi \in G, 1 \leq m \leq M\} \) are independent Poisson processes on \( [0, \infty) \times \mathbb{R}_+ \), all independent of \( Z(0) \). \( N^{m,\eta}_{\xi} \) has intensity measure \( a(\xi, \eta) \, dt \otimes du \), \( N^{m,+}_{\xi}, N^{m,-}_{\xi} \) have intensity measure \( \gamma^m \, dt \otimes du \) (\( dt, du \) are Lebesgue measures).

Now we formulate the martingale problem. Define for any countable Abelian group \( G \) the domain

\[
\mathcal{D}(\Omega^G) := \{ f \in B(\mathcal{E}^{\text{par},G}), \Omega^G f \in B(\mathcal{E}^{\text{par},G}) \},
\]
where the action of the operator $\Omega^G_Z$ acting on $\mathcal{D}(\Omega^G_Z)$ is given by

$$\Omega^G_Z f(z) := \sum_{m=1}^{M} \sum_{\xi, \eta \in G} z_m^m a(\xi, \eta) \{ f(z + e(m, \eta) - e(m, \xi)) - f(z) \}$$

(2.26) $$+ \sum_{m=1}^{M} \gamma_m^m \sum_{\xi \in G} \gamma^m_{\text{birth}}(z_\xi) z_\xi^m \{ f(z + e(m, \xi)) - f(z) \}$$

$$+ \sum_{m=1}^{M} \gamma_m^m \sum_{\xi \in G} \gamma^m_{\text{death}}(z_\xi) z_\xi^m \{ f(z - e(m, \xi)) - f(z) \},$$

and $e(m, \xi) \in (\mathbb{N}^M)_G$ is defined by $(e(m, \xi))(n, \eta) := \delta_{(m, \xi), (n, \eta)}$.

Notice that if $G'$ is finite and $z(0) \in \mathcal{E}^{\text{par}, G'}$ (recall from 2.13), existence and uniqueness of the solution $Z^{G'}$ with values in $D(\mathbb{R}_+, \mathcal{E}^{\text{par}, G'})$ of the $(\Omega_Z, \mathcal{D}(\Omega^G_Z), z(0))$ martingale problem follow by standard theory on jump processes. Compare also \[ABP05\] and \[ABBP02\]. Moreover, $Z^{G'}$ is also the unique solution of the system (2.24) of stochastic equations.

Moreover, if $G' \subset G$, $z(0) \in \mathcal{E}^{\text{par}, G}$ and $Z^{G'}$ is a solution of the $(\Omega^G_Z, \mathcal{D}(\Omega^G_Z), z_G(0))$-martingale problem, with the restricted initial state $Z_{G'}(0) := \{ z_m^m(0); \xi \in G' \}$ then $Z^{G'}$ can be extended to a $\mathcal{E}^{\text{par}, G^*}$-valued process, $Z^{G', G}$, whose components $\{ z_m^m, \xi \in G \setminus G' \}$ outside of $G'$ are frozen, i.e.

(2.27) $$(z^{G', G})^m_\xi(t) := \begin{cases} \frac{z_m^m(t)}{z_\xi^0(0)} , & \text{if } \xi \in G' \\ z_\xi^0(0) , & \text{if } \xi \in G \setminus G' \end{cases}$$

Fix now a countable Abelian group $G$ and approximating finite Abelian groups $\{ G_L; L \in \mathbb{N} \}$ as in (2.10).

**Theorem 1 (Existence of the particle system)** Fix $z(0) \in \mathcal{E}^{\text{par}, G}$. Let for each $L \in \mathbb{N}$, $Z^{G_L, G}$ be the extended unique solution of the $(\Omega^G_Z, \mathcal{D}(\Omega^G_Z), z_{G_L}(0))$-martingale problem. Then the following hold:

(i) The family of processes $\{ Z^{G_L, G}; L \in \mathbb{N} \}$ is tight in $D(\mathbb{R}_+, \mathcal{E}^{\text{par}, G})$ for each initial condition $z(0) \in \mathcal{E}^{\text{par}, G}$.

(ii) Any limit point $Z^G$ of $\{ Z^{G_L, G}; L \in \mathbb{N} \}$ is a solution of the $(\Omega^G_Z, \mathcal{D}(\Omega^G_Z), z(0))$-martingale problem and a weak solution of (2.24).

(iii) The laws of (subsequences of) $\{ Z^{G_L, G}; L \in \mathbb{N} \}$ converge in $D(\mathbb{R}_+, \mathcal{E}^{\text{par}, G})$, i.e., there are only limit points $Z^G$ of $\{ Z^{G_L, G}; L \in \mathbb{N} \}$ in $D(\mathbb{R}_+, \mathcal{E}^{\text{par}, G})$.

We prove this theorem in Section 3.1.
2.3 A system of interacting branching diffusions

In this section we introduce the candidate for the continuous mass limit. Put for any countable Abelian group $G$,

(2.28) $\bar{a}(\xi, \eta) := a(\eta, \xi), \quad \xi, \eta \in G$.

We consider the following system of differential equations: for all $m \in \{1, \ldots, M\}$, and $\xi \in G$,

(2.29) $d x^m_\xi(t) = \sum_{\eta \in G} \bar{a}(\xi, \eta)(x^m_\eta(t) - x^m_\xi(t))dt + \gamma^m x^m_\xi(t) \Gamma^m(x_\xi(t))dt$

$+ \sqrt{\gamma^m x^m_\xi(t)} dw^m_\xi(t)$,

where $(w^m_\xi)_{\xi \in G, m \in \{1, \ldots, M\}}$ is a family of independent standard Brownian motions.

For all $k \in \mathbb{N}_0$, denote by

(2.30) $C^k(E^G) := \{ f \in C(E^G) : f$ is $k$-times continuously differentiable $\}$.

Put $C^k_b(E^G) := C^k(E) \cap B(E^G)$, the space of bounded functions with derivatives up to the $k$-th order. Consider

(2.31) $\Omega^G_X f(x) = \sum_{m=1}^M \sum_{\xi \in G} \left( \sum_{\eta \in G} \bar{a}(\xi, \eta)(x^m_\eta(t) - x^m_\xi(t)) + \gamma^m x^m_\xi(t) \Gamma^m(x_\xi(t)) \right) \frac{\partial f}{\partial x^m_\xi}(x)$

$+ \frac{1}{2} \sum_{m=1}^M \sum_{\xi \in G} \gamma^m \sum_{\xi \in G} (x^m_\xi) \frac{\partial^2 f}{\partial x^m_\xi^2}(x)$.

acting on

(2.32) $\mathcal{D}(\Omega^G_X) := \{ f \in C^3_b(E^G), \Omega^G_X f \in B(E^G) \}$.

Once more, if $X^{G'}$ solves the $(\Omega^G_X, \mathcal{D}(\Omega^G_X), x(0))$-martingale problem for some $G' \subset G$, we consider it as an $E^G$-valued process starting in $x(0)$ such that $(x^{G'}_\xi)_{\xi \in G'} = x^m_\xi(0)$ for all $\xi \in G'$ and $m = 1, \ldots, M$, and freezing the dynamics outside of $G'$.

**Remark 2.4 (Separating class)** Note that the family

(2.33) $\mathcal{H} := \{ f_{\mu, \kappa}(x) := e^{-\sum_{\xi \in G} \mu_\xi x^\xi} \prod_{\xi \in G} (x^m_\xi)^{\kappa^m_\xi} : \mu \in [0, \infty)^G, \kappa \in \mathbb{N}^G$ with $\sum_{\xi} \kappa_\xi < \infty$,

$\mu_\xi > 0$ if $\kappa^m_\xi > 0$ for some $m \in \{1, \ldots, M\}$,

$\mu_\xi > 0$ only for finitely many $\xi \in G$ \}$

is a subset of $\mathcal{D}(\Omega^G_X)$ which separates points.
The next theorem states existence and uniqueness of the solution of \((2.29)\).

**Theorem 2 (Interacting diffusion well-defined)** Let \(G\) and \(\{G^L; L \in \mathbb{N}\}\) as in \((2.10)\), \(X(0) \in \mathcal{E}^G\), and \(\Omega^G_X: \mathcal{D}(\Omega_X) \to C_b(\mathcal{E}^G)\) be as in \((2.31)\).

(i) The \((\Omega^G_X, \mathcal{D}(\Omega^G_X), X(0))\)-martingale problem has a solution
\[
(2.34) \quad X^G = \{(x^m_\xi(t))_{t \geq 0}; \xi \in G, m = 1, \ldots, M\}
\]
with paths in \(C(\mathbb{R}_+, \mathcal{E}^G)\) that arises as the subsequential limit of the laws of the finite approximations \(X^{G,L}\).

If we are in the exchangeable case, the process \(X\) has a unique solution and this solution is the limit of the finite approximations.

(ii) Any solution of the system of SDEs given in \((2.29)\) solves the \((\Omega^G_X, \mathcal{D}(\Omega^G_X), X(0))\)-martingale problem.

(iii) Any solution to the \((\Omega^G_X, \mathcal{D}(\Omega^G_X), X(0))\)-martingale problem is a weak solution to the system of SDEs given in \((2.29)\).

**Corollary 2.5** The system of SDEs given in \((2.29)\) has a weak solution and this solution has continuous sample paths. In the exchangeable case it is a strong Markov and Feller process.

The proof of Theorem 2 can be found in Section 3.2. The proof uses a diffusion approximation which is the subject of the next section.

If we make stronger assumptions on the initial states we can say more about the states of the process \(X\) later on, and in fact about a proper state space of \(X\). So let for \(p \in [1, \infty)\),

\[
(2.35) \quad l^p(\rho) = \mathcal{E}^G_p := \{x \in (\mathbb{R}^M_+)^G : \|x\|_p < \infty\},
\]
where
\[
(2.36) \quad \|x\|_p := \|x\|_{p,\rho} = \left(\sum_{\xi \in G} (\bar{x}_\xi)^p \rho(\xi)\right)^{\frac{1}{p}}.
\]

Concerning a state space for the multi-type logistic branching diffusion \(X\) we have the following result.

**Proposition 2.6 (State space)** Let \(X\) be a solution of \((2.29)\) with initial condition \(x(0) \in l^p(\rho)\) for some \(p \geq 2\). Then the paths of \(X\) lie in the space \(l^p(\rho)\), almost surely. In fact, for every \(T \geq 0\) we have the bound
\[
(2.37) \quad E\left[\sup_{0 \leq t \leq T} \|x(t)\|_p\right] < \infty.
\]

The proof of Proposition 2.6 is given in Subsection 3.3.
2.4 The diffusion limit

Here we show that the continuous mass population model, $X^G$, can be indeed approximated by the particle system $Z^G$ using a small mass, many particles and rapid reproduction limit.

For that, consider a family $\{Z^{G,\varepsilon}; \varepsilon > 0\}$ with

$$Z^{G,\varepsilon} = \left\{ (z^{m,\varepsilon}_m(t))_{t \geq 0}, m \in \{1, \ldots, M\}, \xi \in G \right\}$$

where $Z^{G,\varepsilon}$ is a solution of the $(\Omega^G_{Z^\varepsilon}, D(\Omega^G_{Z^\varepsilon}))$-martingale problem in which we rescaled the parameters $\gamma^m \mapsto \frac{\gamma^m}{\varepsilon}$, $\gamma^m_{\text{birth}} \mapsto \gamma^m_{\text{birth}} \left(1 + \varepsilon K^m\right)$, $\gamma^m_{\text{death}}(z) \mapsto \frac{\gamma^m}{\varepsilon} \left(1 + \varepsilon \sum_{n=1}^M \lambda_{m,n} z^n_\xi\right)$, (and thus $\Gamma^m \mapsto \varepsilon \Gamma^m$). That is, in the $\varepsilon$-approximation the particles have mass $\varepsilon$, the initial number of particles is blown up by a factor of $\varepsilon^{-1}$, and the branching is speeded up by the factor $\varepsilon^{-1}$ and in addition we replace $\Gamma^m$ by $\varepsilon \Gamma^m$ (limit of small perturbation of criticality of the branching). As a consequence $Z^{G,\varepsilon}$ solves the $(\Omega^G_{Z^\varepsilon}, D(\Omega^G_{Z^\varepsilon}))$-martingale problem where

$$\Omega^G_{Z^\varepsilon} f(z) = \sum_{m=1}^M \sum_{\xi \in G} \frac{z^m_\xi}{\varepsilon} \left\{ \sum_{\eta \in G} a(\xi, \eta) \left( f \left( z + \varepsilon (m, \eta) - \varepsilon (m, \xi) \right) - f(z) \right) + \frac{\gamma^m}{\varepsilon} \left(1 + \varepsilon K^m\right) \left( f \left( z + \vee (m, \xi) \right) - f(z) \right) + \frac{\gamma^m}{\varepsilon} \left(1 + \varepsilon \sum_{n=1}^M \lambda_{m,n} z^n_\xi\right) \left( f \left( z - \vee (m, \xi) \right) - f(z) \right) \right\}.$$  

Theorem 3 (The diffusion limit) Let $X(0)$ in $\mathcal{E}^G$ be random such that $E[X(0)] \in \mathcal{E}^G$. Define $Z^\varepsilon (0)$ by letting for all $\varepsilon > 0$, $\xi \in G$ and $m \in \{1, 2, \ldots, M\}$, $z^m_\xi (0) := \varepsilon \left[ x^m_\xi (0) \right]$, and start all $Z^{G,\varepsilon}$ in $Z^\varepsilon (0)$. Then the family $\{Z^{G,\varepsilon}; \varepsilon > 0\}$ is relatively compact and any limit point $X$ satisfies the $(\Omega^G_{X^\varepsilon}, D(\Omega^G_{X^\varepsilon})), (X(0))$-martingale problem.

If the model is exchangeable, the $(\Omega^G_{X}, D(\Omega^G_{X})), (X(0))$-martingale problem has a unique solution, $X$, and

$$L[Z^\varepsilon] \xrightarrow{\varepsilon \to 0} L[X],$$

where here $\Rightarrow$ means weak convergence in $D(\mathbb{R}_+, \mathcal{E})$ with respect to the Skorohod topology.

We shall give the fairly standard proof of Theorem 3 in Section 3.2.

2.5 Exponential duality in the exchangeable model

In this section we focus on the exchangeable models only. In order to verify uniqueness of the martingale problem of the interacting diffusion process, we shall use the following duality.
Let $X \in \mathcal{E}$ be the weak solution of (2.29) in the exchangeable case, i.e. the solution of the following system of stochastic differential equations:

\[
dx^m_\xi(t) = \sum_{\eta \in G} \bar{a}(\xi, \eta)(x^m_\eta(t) - x^m_\xi(t)) dt + \gamma x^m_\xi(t)(K - \lambda \bar{x}_\xi(t)) dt + \sqrt{\gamma} x^m_\xi(t) dw^m_\xi(t),
\]

where $\lambda > 0$, $K > 0$, and the Brownian motions \{w^m_\xi; m \in \{1, \ldots, M\}, \xi \in G\} are independent.

As we will see this process is dual to the Markov process with state space \(\mathcal{E}^{\text{dual}} := \{(\alpha, \kappa) \in (\mathbb{R}_+)^G \times (\mathbb{N}_0^M)^G : \sum_{\xi \in G} \alpha_\xi \bar{x}_\xi < \infty, \sum_{\xi \in G} \bar{\kappa}_\xi \bar{x}_\xi < \infty, \forall x \in \mathcal{E}\}\),

where \(\bar{\kappa}_\xi := \sum_{m=1}^M \kappa^m_\xi\),

and with the generator acting on \(\mathcal{D}(\Omega_{(\alpha, \kappa)})\)

\[
\Omega_{(\alpha, \kappa)} f(\alpha, \kappa) := \sum_{m=1}^M \sum_{\xi, \eta \in G} \kappa^m_\xi \bar{a}(\xi, \eta) \left( f(\alpha, \kappa + e(m, \eta) - e(m, \xi)) - f(\alpha, \kappa) \right) \\
+ \gamma \sum_{m=1}^M \sum_{\xi \in G} \left( \frac{\kappa^m_\xi}{2} \right) \left( f(\alpha, \kappa - e(m, \xi)) - f(\alpha, \kappa) \right) \\
+ \sum_{\xi, \eta \in G} a(\xi, \eta)(\alpha_\eta - \alpha_\xi) \frac{\partial}{\partial \alpha_\xi} f(\alpha, \kappa) \\
+ \sum_{\xi \in G} \alpha_\xi \left( K - \frac{1}{2} \alpha_\xi \right) \frac{\partial}{\partial \alpha_\xi} f(\alpha, \kappa) + \gamma \lambda \sum_{\xi \in G} \alpha_\xi \frac{\partial^2}{\partial (\alpha_\xi)^2} f(\alpha, \kappa), \\
+ \gamma \lambda \sum_{\xi \in G} \bar{\kappa}_\xi \frac{\partial}{\partial \alpha_\xi} f(\alpha, \kappa).
\]

This means that the process $\kappa$ describes an autonomous spatial Kingman coalescent in which particles migrate independently on $G$ according to $a$ and particles at the same site coalesce according to a Kingman coalescent mechanism with rate $\gamma$. The process $\alpha$ follows a diffusion of a form analogous to that of the total mass process $\bar{x}$ with the exception that we have an additional nonnegative immigration term for $\alpha_\xi$ which is given by $\gamma \lambda \bar{\kappa}_\xi$. This implies that the process $\alpha$ depends on $\kappa$ whenever $\kappa$ is nonzero. In this case, $\alpha$ cannot die out completely as the following lemma states.
Lemma 2.7 Let $\alpha(0) \in l^2(\rho)$, $p \geq 2$, and $\kappa(0) \in (\mathbb{N}_0^M)^G$ with $\sum_{\xi \in G} \bar{\kappa}_\xi < \infty$. Then there exists a unique solution for the $(\Omega_{(\alpha,\kappa)}, D(\Omega_{(\alpha,\kappa)}), (\alpha(0),\kappa(0)))$-martingale problem on $D([0,\infty), (\mathbb{R}_+)^G \times (\mathbb{N}_0^M)^G)$. Furthermore we have continuous paths with exception of finitely many jump points in finite time intervals.

It turns out that $\mathcal{E}^{\text{dual}}$ can be chosen as a state space for the dual process started a.s. in a configuration from $\mathcal{E}^{\text{dual}}_f$ where

\begin{equation}
(2.46) \quad \mathcal{E}^{\text{dual}}_f := \left\{ (\alpha, \kappa) \in (\mathbb{R}_+)^G \times (\mathbb{N}_0^M)^G : (\alpha, \kappa) \text{ has finite support} \right\}.
\end{equation}

This is justified by the following lemma whose proof can be found in Section 4.

Lemma 2.8 For $(\alpha(0), \kappa(0)) \in \mathcal{E}^{\text{dual}}_f$, $(\alpha(t), \kappa(t)) \in \mathcal{E}^{\text{dual}}_f$ a.s. for all $t > 0$.

Consider the duality function

\begin{equation}
(2.47) \quad H((\alpha, \kappa), x) := \exp \left( - \sum_{\xi \in G} \alpha_\xi \bar{x}_\xi \right) x^\kappa,
\end{equation}

for $\alpha \in (\mathbb{R}_+)^G$, $x \in ((\mathbb{R}_+)^M)^G$ and $\kappa \in (\mathbb{N}_0^M)^G$, where $x^\kappa := \prod_{\xi \in G} \prod_{m=1}^M (x^m_\xi)^{\kappa_\xi}$. We then show the following:

Proposition 2.9 (A duality for the exchangeable case) Let $(\alpha(t), \kappa(t))$ be a Markov process with generator $\Omega_{(\alpha,\kappa)}$ defined in (2.45) with $(\alpha(0), \kappa(0)) \in \mathcal{E}^{\text{dual}}_f$ such that

\begin{equation}
(2.48) \quad \mathbb{E} \left[ \left( \sum_{\xi \in G} \alpha_\xi(0) \right)^3 \right] < \infty \quad \text{and} \quad n = \sum_{\xi \in G} \bar{\kappa}_\xi(0) < \infty.
\end{equation}

Let $X$ be a solution to (2.44), that is independent of $(\alpha, \kappa)$, started from $\bar{X}(0)$ which is bounded above by a translation invariant $\bar{X}^{\text{inv}}$ with $\mathbb{E}[(\bar{x}^{\text{inv}})^{n+2}] < \infty$. Then, for all $t \geq 0$,

\begin{equation}
(2.49) \quad \mathbb{E} \left[ H((\alpha(0), \kappa(0)), x(t)) \right] = \mathbb{E} \left[ H((\alpha(t), \kappa(t)), x(0)) \exp \left( \gamma \int_0^t \left\{ \sum_{m,\xi} \left( \bar{\kappa}_\xi^m(s) - \frac{1}{2} \right) + \sum_\xi (K - \alpha_\xi(s)) \bar{\kappa}_\xi(s) \right\} ds \right) \right]
\end{equation}

The proof can be found in Section 4. We will present some applications in Section 2.50.

2.6 Exponential duality and coexistence

We will now use the duality in order to investigate conditions for coexistence. In order to prove that there is long-term coexistence we would like to show that for some $\xi \in G$,

\begin{equation}
(2.50) \quad \liminf_{t \to \infty} \mathbb{E} \left[ x^1_\xi(t) \cdot x^2_\xi(t) \right] > 0.
\end{equation}
We see from the duality function that it suffices to consider $\alpha(0) = 0$ and $\kappa(0)$ to be the configuration with a type 1 and a type 2 particle at $\xi$ in order to then show that

(2.51) $\liminf_{t \to \infty} E[H((\alpha(0), x(0)), x(t))] > 0.$

From the duality we obtain the following monotonicity property for coexistence in the initial condition.

**Proposition 2.10** Let $X^\theta$ be started from a constant initial state, namely $x^\theta_\xi(0) = \theta > 0$ for all $\xi \in G$, $m = 1, \ldots, M$. If $0 < \tilde{\theta} \leq \theta$ and coexistence as in (2.50) holds for $X^\theta$ then it also holds for $X^{\tilde{\theta}}$. Vice versa, if coexistence does not hold for $X^\theta$ then it also does not hold for $X^{\tilde{\theta}}$ for any $\tilde{\theta} > \theta$.

**Proof** By the duality of Proposition 2.9 we have that

\[
E[\tilde{x}_\xi^1(t) \cdot \tilde{x}_\xi^2(t)] = E[\exp(-\tilde{\theta}\tilde{\alpha}(t))\tilde{\theta}^2 \exp\left(\gamma \int_0^t \left\{ \sum_\xi (K - \alpha_\xi(s))\tilde{\kappa}_\xi(s) \right\} ds \right)] \\
\geq \frac{\tilde{\theta}^2}{\theta^2} E[\exp(-\theta\tilde{\alpha}(t))\theta^2 \exp\left(\gamma \int_0^t \left\{ \sum_\xi (K - \alpha_\xi(s))\tilde{\kappa}_\xi(s) \right\} ds \right)] \\
= \frac{\tilde{\theta}^2}{\theta^2} E[\tilde{x}_\xi^1(t) \cdot \tilde{x}_\xi^2(t)].
\]

Taking $\liminf_{t \to \infty}$ on both sides now implies the statement. \hfill \Box

The following lemma may be helpful to establish coexistence results in forthcoming work.

**Lemma 2.11 (Total dual mass diverges)** Suppose that the parameters are such that the total mass process $\bar{X}$ started in a translation invariant nontrivial initial condition survives. Then for any $(\alpha, \kappa)$ process with generator (2.45) such that $\alpha(0)$ has finite support and $\sum_{\xi \in G} \bar{\kappa}_\xi(0) \geq 1$ we obtain

(2.53) $P\left(\sum_{\xi \in G} \alpha_\xi(t) \underset{t \to \infty}{\longrightarrow} \infty \right) = 1.$

**Proof** Let $\bar{X}$ be started from the constant configuration $\bar{x}_\xi(0) = 1$ for all $\xi \in G$. We let $(\alpha^0, \kappa^0)$ be the process with generator (2.45) started from $(\alpha(0), 0)$. Then by duality, for $t \geq 0$,

(2.54) $E\left[\exp\left(-\sum_{\xi \in G} \alpha^0_\xi(t)\right)\right] = E\left[\exp\left(-\sum_{\xi \in G} \alpha_\xi^0(0)\bar{x}_\xi(t)\right)\right].$
By Lemma 8.1 of [HW07] we have

\[ (2.55) \ P \left( \sum_{\xi} \alpha_{\xi}^{0}(t) \xrightarrow{t \to \infty} \infty \right) \text{ or } \exists t' < \infty : \sum_{\xi} \alpha_{\xi}^{0}(t) = 0 \ \forall t \geq t' = 1. \]

On the other hand, by Theorem 5 of [HW07] we have \( \tilde{X}(t) \Rightarrow \tilde{X}(\infty) \) for \( t \to \infty \), where \( \tilde{X}(\infty) \) is translation invariant and also nontrivial by our assumption of survival. Thus, letting \( t \to \infty \) in \( \text{(2.54)} \) we obtain with \( \text{(2.55)} \)

\[ 1 - P \left( \sum_{\xi} \alpha_{\xi}^{0}(t) \xrightarrow{t \to \infty} \infty \right) = P \left( \exists t' < \infty : \sum_{\xi} \alpha_{\xi}^{0}(t) = 0 \ \forall t \geq t' \right) = \mathbb{E} \left[ \exp \left( - \sum_{\xi} \alpha_{\xi}(0) \tilde{x}_{\xi}(\infty) \right) \right]. \]

This implies that for any \( \varepsilon > 0 \),

\[ (2.57) \ \inf_{\alpha(0) \text{ s.t. } \exists \xi \alpha_{\xi}(0) \geq \varepsilon} P \left( \sum_{\xi} \alpha_{\xi}^{0}(t) \xrightarrow{t \to \infty} \infty \right) \geq 1 - \mathbb{E} \left[ e^{-\varepsilon \tilde{x}_{\xi}(\infty)} \right] > 0. \]

Note that we may apply Proposition 3.11 to \( \alpha \) as the migration and drift terms fulfill the assumptions. Thus, due to the monotonicity in the immigration term stated there we obtain also for \( (\alpha, \kappa) \) with any \( \kappa(0) \) that

\[ (2.58) \ \inf_{\alpha(0) \text{ s.t. } \exists \xi \alpha_{\xi}(0) \geq \varepsilon} P \left( \sum_{\xi} \alpha_{\xi}(t) \xrightarrow{t \to \infty} \infty \right) > 0. \]

It now remains to show that there exists an \( \varepsilon > 0 \) such that for any \( (\alpha, \kappa) \) process

\[ (2.59) \ \inf_{(\alpha(0), \kappa(0)) \text{ s.t. } \sum_{\xi} \tilde{\kappa}_{\xi}(0) \geq 1} P \left( \exists t < \infty, \xi \in G : \alpha_{\xi}(t) \geq \varepsilon \right) > 0. \]

Note that if \( \sum_{\xi} \tilde{\kappa}_{\xi}(0) \geq 1 \) then there exists \( \zeta \in G \) such that \( \tilde{\kappa}_{\zeta}(t) \geq 1 \) at least for \( t \leq T \sim \exp(1) \). Let \( 0 < \delta < \gamma \lambda \) be arbitrary. Then there exists an \( \varepsilon > 0 \) such that for any \( \alpha \in \mathbb{R}^{2d} \) with \( 0 \leq \alpha_{\zeta} \leq \varepsilon \),

\[ (2.60) \ \sum_{\eta \in G} a(\zeta, \eta)(\alpha_{\eta} - \alpha_{\zeta}) + \gamma \alpha_{\zeta} \left( K - \frac{1}{2} \alpha_{\zeta} \right) + \gamma \lambda \geq \gamma \lambda - \delta > 0. \]

Since the left hand side of \( (2.60) \) is a lower bound for the drift term of \( \alpha_{\zeta} \) up to time \( T \wedge S \), where \( S = \inf\{ t \geq 0 : \alpha_{\zeta}(t) \geq \varepsilon \} \) this implies that according to Proposition 3.11 we can couple \( \alpha_{\zeta} \) to \( \tilde{\alpha} \) which solves

\[ (2.61) \ \tilde{d}\tilde{\alpha}(t) = (\gamma \lambda - \delta) \, dt + \sqrt{\gamma \lambda \tilde{\alpha}(t)} \, dw_{t} \]

such that \( \tilde{\alpha}(0) = \alpha_{\zeta}(0) \) and \( \tilde{\alpha}(t) \leq \alpha_{\zeta}(t) \) for \( t \leq T \wedge S \). Setting also \( \tilde{S} = \inf\{ t \geq 0 : \tilde{\alpha}(t) \geq \varepsilon \} \) it follows immediately that \( S \leq \tilde{S} < \infty \) a.s. and in particular that

\[ (2.62) \ \inf_{(\alpha(0), \kappa(0)) \text{ s.t. } \sum_{\xi} \tilde{\kappa}_{\xi}(0) \geq 1} P \left( \exists t < \infty, \xi \in G : \alpha_{\xi}(t) \geq \varepsilon \right) \geq P(\tilde{S} \leq T) > 0, \]
which shows (2.59). Taking (2.58) and (2.59) together with the Markov property of \((\alpha, \kappa)\) now implies that for all \(a \in \mathbb{R}\),

\[
(2.63) \quad \inf_{(\alpha(0), \kappa(0)) \text{ s.t. } \sum_{\xi \in \mathcal{G}} \kappa(0) \geq 1} P \left( \exists t' < \infty : \sum_{\xi \in \mathcal{G}} \alpha(0) \geq a \forall t \geq t' \right) =: \varepsilon > 0.
\]

Therefore, for initial conditions with \(\sum_{\xi \in \mathcal{G}} \kappa(0) \geq 1\) and so \(\sum_{\xi \in \mathcal{G}} \kappa(s) \geq 1\) for all \(s \geq 0\), we obtain by the Markov property and martingale convergence that

\[
(2.64) \quad \varepsilon \leq P \left( \exists t' < \infty : \sum_{\xi \in \mathcal{G}} \alpha(t) \geq a \forall t \geq t' \bigg| \alpha(s), \kappa(s) \right) = P \left( \exists t' < \infty : \sum_{\xi \in \mathcal{G}} \alpha(t) \geq a \forall t \geq t' \bigg| F_s \right) \\
\quad \to 1 \{ \exists t' < \infty \text{ s.t. } \sum_{\xi \in \mathcal{G}} \alpha(t) \geq a \forall t \geq t' \} \quad \text{a.s.}
\]

as \(s \to \infty\). This implies the result. \(\square\)

### 3 Proofs of Theorem 1, 2 and 3

Fix \(G\) and \(\{G_L; L \in \mathbb{N}\}\) as in (2.10). Recall \(Z^{G_L,G,\varepsilon}\) and \(X\) from (2.27), (2.39) and (2.29). In this section we give the proofs of Theorems 1, 2 and 3. In particular, we verify weak convergence of \(Z^{G_L,G,\varepsilon}\) along a subsequence as first \(L \to \infty\) and then \(\varepsilon \to 0\), and first \(\varepsilon \to 0\) and then \(L \to \infty\), and show that in both rescaling regimes possible limit points agree.

In Subsection 3.1 we give bounds on the first moments of the supremum of a component of \(Z^{G_L,G,\varepsilon}\) which are uniform in \(L \in \mathbb{N}\) and \(\varepsilon > 0\). We apply them with \(\varepsilon = 1\) to verify tightness of the family indexed by \(L \in \mathbb{N}\). In Subsection 3.2 we first verify tightness of a family of rescaled limit points indexed by \(\varepsilon > 0\). We also show that any limit point is a weak solution of (2.29).

#### 3.1 First moment bounds and proof of Theorem 1

The main goal of this subsection is to prove Theorem 1. Recall \(\{G_L; L \in \mathbb{N}\}\) and \(G\) from (2.10), the spaces \(\mathcal{E}^{\text{par},G}\) from (2.8), the unique solution \(Z^{G_L}\) of the \((\Omega^G, \mathcal{D}(\Omega^G))\)-martingale problem from (2.25), (2.26), and (2.17), as well as its extension to a \((\mathbb{R}_M^G)\)-valued process, \(Z^{G_L,G}\), obtained by freezing all components outside of \(G_L\).

We start by showing the tightness claimed in (i) of Theorem 1. Here, we want to apply Lemma 4.5.1 combined with Remark 4.5.2 in [EK86]. We therefore need to verify the compact containment condition and uniform convergence of generators. As a preparation we verify moment bounds. We proceed in three steps.

**Step 1 (Uniform first moment bounds)** Fix \(\varepsilon > 0\), and recall from (2.38) the re-scaled particle process

\[
(3.1) \quad Z^{G_L,G,\varepsilon}_t = \{ z^m_{\xi,L,\varepsilon}; m \in \{1, \ldots, M\}, \xi \in G \}
\]
obtained by assigning particles individual mass $\varepsilon$, blowing up the initial number of particles by a factor $\varepsilon^{-1}$, speeding up the branching rate by a factor $\varepsilon^{-1}$, and letting $\Gamma^m := \varepsilon \Gamma^m$.

The following applies to $Z^{G^{L^{\varepsilon}}}$ if we let $\varepsilon = 1$, but will be applied with a general $\varepsilon > 0$ in the next section.

**Lemma 3.1 (First moment bounds)** Let $Z(0)$ be a random element in $\mathcal{E}^{par,G}$ such that $\sum_{z \in G} \rho(z) E[z(z(0))] < \infty$. Define $Z^{L^{\varepsilon}}(0)$ by letting for all $\varepsilon > 0$, $\xi \in G$ and $m \in \{1, 2, \ldots, M\}$, $z^m_{\varepsilon, \xi}(0) := \varepsilon [z^m_{\varepsilon, \xi}(0)]$, and start all $Z^{G^{L^{\varepsilon}}} in Z^\varepsilon(0)$. Then the following hold:

(i) For each $1 \leq m \leq M$, $T > 0$ and $\varepsilon > 0$, there is a constant $C(T, \varepsilon)$ such that

$$(3.2) \sup_{t \in \mathbb{N}} \sum_{\xi \in G} \rho(\xi) E[\sup_{0 \leq t \leq T} (z^m_{\varepsilon, L, \xi}(t))] \leq C(T, \varepsilon) \sum_{\xi \in G} \rho(\xi) E[z^m_{\varepsilon, \xi}(0)].$$

(ii) For each $1 \leq m \leq M$ and $T > 0$, there is a constant $C(T)$ such that

$$(3.3) \sup_{\varepsilon > 0} \sup_{t \in \mathbb{N}} \sum_{\xi \in G} \rho(\xi) E[\sup_{0 \leq t \leq T} (z^m_{\varepsilon, L, \xi}(t))] \leq C(T) \sum_{\xi \in G} \rho(\xi) E[(z^m_{\varepsilon, \xi}(0))^2].$$

**Proof** Applying the generator $\Omega^G_{Z^{L^{\varepsilon}}}$ (recall from (2.39)) to $f^p_{m_0, \xi_0}(z) = (z^m_{m_0})^p$, $p \in \{1, 2\}$, yields for all $\varepsilon > 0$ and for $\xi_0 \in G_L$, and $m_0 \in \{1, \ldots, M\}$,

$$(3.4) \Omega^{G_{L^{\varepsilon}}} f^p_{m_0, \xi_0}(z) = \sum_{m=1}^M \sum_{\xi \in G_L} \frac{a(m, \xi)}{\varepsilon} \left\{ \sum_{\eta \in G} a(\xi, \eta)(z^m_{\varepsilon, L, \xi_0} + \varepsilon e(m, \eta)(m_0, \xi_0) - \varepsilon e(m, \xi)(m_0, \xi_0))^p - (z^m_{\varepsilon, L, \xi_0})^p \right\}$$

$$+ \sum_{n=1}^M \sum_{\eta \in G} \varepsilon e(m, \xi)(z^m_{\varepsilon, L, \xi_0} + \varepsilon e(m, \xi)(m_0, \xi_0))^p - (z^m_{\varepsilon, L, \xi_0})^p$$

$$+ \gamma^m \sum_{n=1}^M \sum_{\eta \in G} \varepsilon e(m, \xi)(z^m_{\varepsilon, L, \xi_0} + \varepsilon e(m, \xi)(m_0, \xi_0))^p - (z^m_{\varepsilon, L, \xi_0})^p \right\}.$$ 

Thus

$$(3.5) \Omega^G_{Z^{L^{\varepsilon}}} f^1_{m_0, \xi_0}(z)$$

$$= \sum_{m=1}^M \sum_{\xi \in G_L} \frac{a(m, \xi)}{\varepsilon} \left\{ \sum_{\eta \in G} a(\xi, \eta)(e(m, \eta)(m_0, \xi_0) - e(m, \xi)(m_0, \xi_0))^p \right\}$$

$$+ \gamma^m \Gamma^m (z^m_{\varepsilon, L, \xi_0} + \delta(\xi_0, \eta)) + \gamma^m \Gamma^m (z^m_{\varepsilon, L, \xi_0} - z^m_{\varepsilon, L, \xi_0})$$

$$= \sum_{\eta \in G_L} z^m_{\varepsilon, L, \eta} (a(\xi_0, \eta) - \delta(\xi_0, \eta)) + \gamma^m \Gamma^m \cdot z^m_{\varepsilon, L, \xi_0}$$

$$\leq \sum_{\eta \in G_L} z^m_{\varepsilon, L, \eta} (a(\xi_0, \eta) - \delta(\xi_0, \eta)) + \gamma^m K^m \cdot z^m_{\varepsilon, L, \xi_0}$$
and
\[
\Omega_{Z_{L^2}}^{G_L, \varepsilon} f_{m_0, \xi_0}^2(z) = \sum_{m=1}^{M} \sum_{\xi \in G} z_{\varepsilon, L, \xi}^m \left\{ \sum_{\eta \in G} a(\xi, \eta) (e(m, \eta)(m_0, \xi_0) - e(m, \xi)(m_0, \xi_0)) 2(z_{\varepsilon, L, \xi_0}^m) + \varepsilon \sum_{\eta \in G} a(\xi, \eta) (e(m, \eta)(m_0, \xi_0) - e(m, \xi)(m_0, \xi_0))^2 \right\}
\]
\[
+ 2 \gamma_{m_0} \Gamma_{m_0} (z_{\varepsilon, L, \xi_0})^2 + \gamma_{m_0} (1 + \varepsilon K_{m_0} + \varepsilon \sum_{n=1}^M \lambda_{m_0, n} z_{\varepsilon, L, \xi_0}^n) z_{\varepsilon, L, \xi_0}^m
\]
\[
= 2 \sum_{\eta \in G_L} (z_{\varepsilon, L, \eta}^m) (z_{\varepsilon, L, \xi_0}^m) (a(\xi_0, \eta) - \delta(\xi_0, \eta)) + 2 \gamma_{m_0} \Gamma_{m_0} (z_{\varepsilon, L, \xi_0}) (z_{\varepsilon, L, \xi_0}^m)
\]
\[
+ \varepsilon \sum_{m=1}^{M} \sum_{\xi \in G} z_{\varepsilon, L, \xi}^m \sum_{\eta \in G} a(\xi, \eta) (e(m, \eta)(m_0, \xi_0) - e(m, \xi)(m_0, \xi_0))^2
\]
\[
+ \gamma_{m_0} (1 + \varepsilon K_{m_0} + \varepsilon \sum_{n=1}^M \lambda_{m_0, n} z_{\varepsilon, L, \xi_0}^n) z_{\varepsilon, L, \xi_0}^m.
\]

Put
\[
(3.7) \quad \eta := \max_{1 \leq m \leq M} \gamma^m; \quad K := \max_{1 \leq m \leq M} K^m; \quad \lambda := \max_{1 \leq m, n \leq M} \lambda_{m, n}.
\]

Then
\[
(3.8) \quad \Omega_{Z_{L^2}}^{G_L, \varepsilon} f_{m_0, \xi_0}^1(z) \leq \sum_{\eta \in G_L} z_{\varepsilon, L, \eta} (a(\xi_0, \eta) - \delta(\xi_0, \eta)) + \eta K z_{\varepsilon, L, \xi_0}
\]

and
\[
(3.9) \quad \Omega_{Z_{L^2}}^{G_L, \varepsilon} f_{m_0, \xi_0}^2(z)
\]
\[
\leq \sum_{\eta \in G_L} (z_{\varepsilon, L, \eta}^m)^2 (a(\xi_0, \eta) - \delta(\xi_0, \eta)) + 2 \eta K (z_{\varepsilon, L, \xi_0}^m)^2
\]
\[
+ \varepsilon z_{\varepsilon, L, \xi_0}^m + 2 \varepsilon \sum_{\xi \in G} \hat{a}(\xi_0, \xi) z_{\varepsilon, L, \xi}^m + \eta (1 + \varepsilon K + \varepsilon \lambda z_{\varepsilon, L, \xi_0}) z_{\varepsilon, L, \xi_0}^m.
\]

Notice that neither $f_{m_0, \xi_0}^p$ nor $\Omega_{Z_{L^2}}^{G_L, \varepsilon} f_{m_0, \xi_0}^p$ are bounded functions. However, we can make all the coming arguments work by replacing $f_{m_0, \xi_0}^p$ by $f_{m_0, \xi_0}^p := f_{m_0, \xi_0}^p \cdot e^{-\mu z_{\varepsilon, L, \xi_0}}$ and then use monotone convergence (as $\mu \downarrow 0$). Such calculations are quite involved but standard, so we omit them here and rather work here directly with $f_{m_0, \xi_0}^p$.

Thus for $\xi_0 \in G_L$, and $m_0 \in \{1, \ldots, M\}$,
\[
(3.10) \quad \frac{d}{dt} E[z_{\varepsilon, L, \xi_0}(t)] \leq \sum_{\eta \in G_L} E[z_{\varepsilon, L, \eta}(t)] (a(\xi_0, \eta) - \delta(\xi_0, \eta)) + \eta K E[z_{\varepsilon, L, \xi_0}(t)],
\]

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while \( \frac{d}{dt}E[z_{\varepsilon,L,\xi_0}(t)] = 0 \) for \( \xi_0 \in G \setminus G_L \).

Put

\[
C := \gamma K < \infty.
\]

Then for \( \xi_0 \in G_L \),

\[
(3.11) \quad C := \gamma K < \infty.
\]

(3.12)

\[
\frac{d}{dt}E[z_{\varepsilon,L,\xi_0}(t)] < \sum_{\eta \in G_L} E[z_{\varepsilon,L,\eta}(t)] (\tilde{a}(\xi_0, \eta) - \delta(\xi_0, \eta)) + (1 + C)E[z_{\varepsilon,L,\xi_0}(t)],
\]

and the right hand side of (3.12) is non-negative for all \( \xi_0 \in G \). Consequently, for all \( \xi_0 \in G_L \),

\[
(3.13) \quad E[z_{\varepsilon,L,\xi_0}(t)] \leq e^{(C+1)t} \sum_{\eta \in G} \rho(\xi_0, \eta)E[z_{\varepsilon,L,\eta}(0)].
\]

In particular, by (2.4), for all \( L \in \mathbb{N} \),

\[
(3.14) \quad \sum_{\xi \in G} \rho(\xi)E[z_{\varepsilon,L,\xi}(t)] \leq e^{(C+1)t} \sum_{\xi \in G} \rho(\xi)E[z_{\varepsilon,L,\xi}(0)] = e^{(C+1)t} \sum_{\xi \in G} \rho(\xi)E[z_{\xi}(0)].
\]

Moreover, for \( \xi_0 \in G_L \), and \( m_0 \in \{1, ..., M\} \),

\[
(3.15) \quad \frac{d}{dt}E[(\tilde{z}_{\varepsilon,L,\xi_0}(t))^2] \leq \sum_{\eta \in G_L} E[(\tilde{z}_{\varepsilon,L,\eta}(t))^2] (\tilde{a}(\xi_0, \eta) - \delta(\xi_0, \eta)) + (2C + \varepsilon K)E[(\tilde{z}_{\varepsilon,L,\xi_0}(t))^2]
\]

\[+ \varepsilon \sum_{\xi \in G} \tilde{a}(\xi_0, \xi)E[z_{\varepsilon,L,\xi}(t)] + \varepsilon (1 + \tilde{\gamma} + \varepsilon C)E[z_{\varepsilon,L,\xi_0}(t)].
\]

It is standard to conclude from here - using (3.14) that we can find a constant \( \tilde{C} < \infty \) such that by (2.4), for all \( L \in \mathbb{N} \),

\[
(3.16) \quad \sum_{\xi \in G} \rho(\xi)E[(\tilde{z}_{\varepsilon,L,\xi}(t))^2] \leq e^{Ct} \sum_{\xi \in G} \rho(\xi)E[(\tilde{z}_{\varepsilon,L,\xi}(0))^2]
\]

\[\leq e^{Ct} \sum_{\xi \in G} \rho(\xi)E[(\tilde{z}_{\xi}(0))^2] < \infty.
\]

We will now use (3.14) to get the stronger result stated in the lemma. The process \( Z^{GL,\varepsilon} \) with initial condition \( Z^{GL,\varepsilon}(0) \in \mathcal{E}^{par,\varepsilon,GL} \) can be constructed as the unique solution
to

\[ z_{\varepsilon,L,\xi}^m(t) = z_{\varepsilon,L,\xi}^m(0) + \sum_{\eta \in G_L, \eta \neq \xi} \left[ \int_{[0,t] \times \mathbb{R}_+} \varepsilon \mathbf{1}(z_{\varepsilon,L,\eta}(s-) \geq \varepsilon u) N_{L,\eta}^{m,\xi}(ds \, du) \right. \\
\left. - \int_{[0,t] \times \mathbb{R}_+} \varepsilon \mathbf{1}(z_{L,\xi}(s-) \geq \varepsilon u) N_{L,\xi}^{m,\eta}(ds \, du) \right] \]

(3.17)

\[ + \int_{[0,t] \times \mathbb{R}_+} \varepsilon \mathbf{1}(z_{\varepsilon,L,\xi}^m(s-)(\frac{1}{2} + \varepsilon K^m) \geq \varepsilon u) N_{L,\xi}^{m,+}(ds \, du) \]

\[ - \int_{[0,t] \times \mathbb{R}_+} \varepsilon \mathbf{1}(z_{\varepsilon,L,\xi}^m(s-)(\frac{1}{2} + \sum_{n=1}^M \lambda_{m,n,z_{\varepsilon,L,\xi}^m(s-)}) \geq \varepsilon u) N_{L,\xi}^{m,-}(ds \, du) \]

for all \( \xi \in G_L \), and \( t \geq 0 \). Here \( \{ N_{L,\xi}^{m,\eta} : \xi, \eta \in G_L, \xi \neq \eta, 1 \leq m \leq M \} \) are independent Poisson processes on \( [0, \infty) \times \mathbb{R}_+ \) and \( \{ N_{L,\xi}^{m,+}, N_{L,\xi}^{m,-} : \xi \in G_L, 1 \leq m \leq M \} \) are independent Poisson processes on \( [0, \infty) \times \mathbb{R}_+ \), all independent of \( Z(0) \). \( N_{L,\xi}^{m,\eta} \) has intensity measure \( a(\eta,\xi) \, dt \otimes du \), \( N_{L,\xi}^{m,+}, N_{L,\xi}^{m,-} \) have intensity measure \( (\gamma^m \varepsilon) \, dt \otimes du \) (\( dt, du \) are Lebesgue measures). For fixed \( Z(0), (Z_{\varepsilon,G_L}(t)) \) is adapted to the filtration generated by these Poisson processes.

(i) Hence

\[ E[ \sup_{0 \leq t \leq T} z_{\varepsilon,L,\xi}^m(t)] \leq E[z_{\varepsilon,\xi}^m(0)] + \int_0^T \sum_{\eta \in G_L} a(\xi,\eta) E[z_{\varepsilon,L,\eta}^m(s)] \, ds + \frac{2}{\varepsilon} \int_0^T E[z_{\varepsilon,L,\xi}^m(s)] \, ds. \]

Then by (3.14),

(3.19)

\[ \sum_{\xi \in G_L} \rho(\xi) E[ \sup_{0 \leq t \leq T} \bar{z}_{\varepsilon,L,\xi}(t)] \leq \sum_{\xi \in G_L} \rho(\xi) E[\bar{z}_{\varepsilon}(0)] + R \int_0^T \sum_{\eta \in G_L} \rho(\eta) E[\bar{z}_{\varepsilon,L,\eta}(s)] \, ds + \frac{2}{\varepsilon} \int_0^T \sum_{\xi \in G_L} \rho(\xi) E[\bar{z}_{\varepsilon,L,\xi}(s)] \, ds \]

We therefore can find \( C(T, \varepsilon) < \infty \) such that for all \( L \in \mathbb{N} \),

(3.20)

\[ \sum_{\xi \in G} \rho(\xi) E[ \sup_{0 \leq t \leq T} \bar{z}_{\varepsilon,L,\xi}(t)] \leq C(T, \varepsilon) \sum_{\xi \in G} \rho(\xi) E[\bar{z}_{\varepsilon}(0)], \]

which proves (3.3).

(ii) To get a bound uniform in \( \varepsilon \), we need to take cancellations due to birth and death
into account, i.e.,

\[
\sup_{t \in [0, T]} z_{\varepsilon, L, \xi}^m(t) 
\]

(3.21) \[ \leq z_{\varepsilon, L, \xi}^m(0) + \sum_{\eta \in G_L, \eta \neq \xi} \left[ \int_{[0,T] \times \mathbb{R}^+} \varepsilon 1(z_{\varepsilon, L, \eta}^m(s-), \varepsilon u) N_{\varepsilon, L, \eta}^m(ds \, du) \right] 
\]

+ \frac{z_{\varepsilon}^m}{\varepsilon} \int_0^T \varepsilon \Gamma_m(z_{\varepsilon, L, \xi}(s)) \cdot z_{\varepsilon, L, \xi}^m(s) ds + \sup_{t \in [0, T]} |M_{G, L, \varepsilon}^m(t)|,

where \((M_{G, L, \varepsilon}^m := M_{G, L, \varepsilon}^m(t))_{t \geq 0}\) defined by

\[
M_{G, L, \varepsilon}^m(t) := \int_{[0,T] \times \mathbb{R}^+} \varepsilon 1(z_{\varepsilon, L, \xi}^m(s-), \varepsilon K^m) \geq \varepsilon u) N_{\varepsilon, L, \xi}^m(ds \, du)
\]

(3.22) \[ - \int_{[0,T] \times \mathbb{R}^+} \varepsilon 1(z_{\varepsilon, L, \xi}^m(s-), \varepsilon \sum_{n=1}^M \lambda_{n,m, L} z_{\varepsilon, L, \xi}^m(s-)) \geq \varepsilon u) N_{\varepsilon, L, \xi}^m(ds \, du) 
\]

\[ - \frac{z_{\varepsilon}^m}{\varepsilon} \int_0^T \varepsilon \Gamma_m(z_{\varepsilon, L, \xi}(s)) \cdot z_{\varepsilon, L, \xi}^m(s) ds \]

is a (local) martingale (compare, [BZ07 Lemma 2.1]).

We obtain

\[
E\left[ \sup_{0 \leq t \leq T} z_{\varepsilon, L, \xi}^m(t) \right] 
\]

\[ \leq E\left[ z_{\varepsilon}^m(0) \right] + \int_0^T \sum_{\eta \in G_L} a(\xi, \eta) E\left[ z_{\varepsilon, L, \eta}^m(s) \right] ds
\]

(3.23) \[ + \frac{z_{\varepsilon}^m}{\varepsilon} \int_0^T E\left[ \varepsilon \Gamma_m(z_{\varepsilon, L, \xi}(s)) z_{\varepsilon, L, \xi}^m(s) \right] ds + E\left[ \sup_{0 \leq t \leq T} |M_{G, L, \varepsilon}^m(t)| \right]
\]

\[ \leq E\left[ z_{\varepsilon}^m(0) \right] + \int_0^T \sum_{\eta \in G_L} a(\xi, \eta) E\left[ z_{\varepsilon, L, \eta}^m(s) \right] ds
\]

\[ + \tau(1 + \frac{1}{\varepsilon} \kappa(\xi)) \int_0^T E\left[ z_{\varepsilon, L, \xi}^m(s) \right] ds + E\left[ \sup_{0 \leq t \leq T} |M_{G, L, \varepsilon}^m(t)| \right].
\]

By a Burkholder-Davis-Gundy inequality and Cauchy-Schwart’s inequality, there is a \(C < \infty\) such that

\[
E\left[ \sup_{t \in [0, T]} |M_{G, L, \varepsilon}^m(t)| \right] \leq C \cdot E\left[ (M_{G, L, \varepsilon}^m(\cdot))^2 \right] \leq C \cdot (E\left[ (M_{G, L, \varepsilon}^m(\cdot))^2 \right])^{\frac{1}{2}}
\]

(3.24) \[ \leq C \cdot (1 + E\left[ (M_{G, L, \varepsilon}^m(\cdot))^2 \right]) \]

\[ \leq (\bar{\tau} + \varepsilon C) \int_0^T E\left[ z_{\varepsilon, L, \xi}(s) \right] ds + \varepsilon \bar{\tau} \int_0^T E\left[ (z_{\varepsilon, L, \xi}(s))^2 \right] ds
\]

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since

\[
\langle M^G_{t}, \varepsilon(\cdot) \rangle_t \nabla_{t} = \gamma_m \int_0^t \varepsilon^2 \left( \frac{1}{2} + \frac{K^m}{\varepsilon} \right) z^m_{\varepsilon, L, \xi}(s) z^m_{\varepsilon, L, \xi}(s) \, ds + \varepsilon \lambda_m \sum_{n=1}^{M} z^m_{\varepsilon, L, \xi}(s) \sum_{n=1}^{M} \lambda_{m,n} \zeta_{m,n, \xi} \left( z^m_{\varepsilon, L, \xi}(s) \right)\]

Combining (2.4), (3.14), (3.23) and (3.24) we can find

\[(3.26) \sum_{\xi \in \mathcal{G}} \rho(\xi) \mathbb{E} \left[ \sup_{0 \leq t \leq T} \bar{z}_{\varepsilon, L, \xi}(t) \right] \leq C(T) \sum_{\xi \in \mathcal{G}} \rho(\xi) \mathbb{E} \left[ \left( \bar{z}_{\varepsilon}(0) \right)^2 \right] \]

This completes the proof. \(\square\)

**Step 2 (Uniform convergence of generators)** The next step is to show the following:

**Lemma 3.2 (Convergence of generators)** Let \( f \in \mathcal{D}(\Omega_G^Z) \), and denote by \( f_{\mid L} \) its restriction to \((\mathbb{R}^M_+)_{\xi}\). Then

\[(3.27) \lim_{L \to \infty} \sup_{z \in (\mathbb{R}^M_+)_{\xi}} |\Omega_G^Z f_{\mid L}(z) - \Omega_G^Z f(z)| = 0.\]

**Proof** Fix \( f \in \mathcal{D}(\Omega_G^Z) \). Then

\[
\sup_{z \in (\mathbb{R}^M_+)_{\xi}} |\Omega_G^Z f(z) - \Omega_G^Z f(z)| = \sum_{(\xi, \eta) \in G \setminus \{0\}^2} \sup_{z \in (\mathbb{R}^M_+)_{\xi}} \sum_{m=1}^{M} \gamma^m z^m_{\xi} \left\{ f(z + e(m, \eta) - e(m, \xi)) - f(z) \right\}
\]

\[(3.28) + \sum_{\xi \in G \setminus \{0\} \in (\mathbb{R}^M_+)_{\xi}} \sup_{z \in (\mathbb{R}^M_+)_{\xi}} \sum_{m=1}^{M} \lambda_m \gamma^m \left\{ f(z + e(m, \xi)) - f(z) \right\}
\]

\[
\sup_{z \in (\mathbb{R}^M_+)_{\xi}} \sum_{m=1}^{M} \lambda_m \gamma^m \left\{ f(z - e(m, \xi)) - f(z) \right\} \to 0 \text{ as } L \to \infty,
\]

where we used that \( \sup_{z \in (\mathbb{R}^M_+)_{\xi}} \Omega_G^Z f(z) < \infty. \) \(\square\)

**Step 3 (Compact containment)** The final step in establishing convergence to a solution of the martingale problem is the compact containment condition. First we identify the compact sets.
Lemma 3.3 (Compact sets in $\mathcal{E}$) Let $A$ be a subset of $\mathcal{E}$. The set $A$ is compact in $\mathcal{E}$ equipped with the product topology if

\begin{equation}
\sup_{x \in A} \sum_{\xi \in G} \bar{x}_\xi \rho(\xi) = c < \infty.
\end{equation}

Remark 3.4 (Compact sets in $\mathcal{E}_{\text{par},G}$) Since $\mathcal{E}_{\text{par},G}$ is a closed subset of $\mathcal{E}$ the same statement holds for $\mathcal{E}_{\text{par},G}$.

Proof Let $x^{(n)}$ be a sequence in $A \subset \mathcal{E}$, and let $\varepsilon > 0$. We have for each $m = 1, ..., M$, and $\xi \in G$, $(x^{m}_{\xi})^{(n)} \leq \frac{c}{\rho(\xi)} < \infty$ for all $n$ by (i). Therefore, for each choice of $m = 1, ..., M$, and $\xi \in G$, there exists a subsequence $n_{\xi,m}$ such that $(x^{m}_{\xi})^{(n_{\xi,m})}$ converges to some $\bar{x}^{m}_{\xi}$ as $i \to \infty$. In fact, by a diagonalization argument we can find a common subsequence $n_{i}$ such that for all $m = 1, ..., M$, and $\xi \in G$,

\begin{equation}
(x^{m}_{\xi})^{(n_{i})} \to \bar{x}^{m}_{\xi}
\end{equation}

By Fatou’s lemma, $\sum_{\xi \in G} \rho(\xi) \sum_{m=1}^{M} \bar{x}^{m}_{\xi} < \infty$ if (3.29) holds. We have therefore have constructed a subsequence convergent in $\mathcal{E}$, and hence shown that $A$ is compact.

Lemma 3.5 (Compact containment) For all $\varepsilon > 0$ and $T > 0$ there exists a compact set $A_{\varepsilon,T} \subset \mathcal{E}$ such that

\begin{equation}
\inf_{L \in \mathbb{N}} P(X_{L}(t) \in A_{\varepsilon,T}, \forall 0 \leq t \leq T) \geq 1 - \varepsilon.
\end{equation}

Proof Fix $\varepsilon > 0$ and $T > 0$. Set

\begin{equation}
K_{\varepsilon,T} := \frac{1}{\varepsilon} \cdot C(T) \sum_{\xi \in G} \rho(\xi) E[\bar{x}_{\xi}(0)]
\end{equation}

with $C(T)$ as in (3.3), and put

\begin{equation}
A_{\varepsilon,T} := \{z \in \mathcal{E}_{\text{par},G} : \sum_{\xi \in G} \rho(\xi) \bar{z}_{\xi} \leq K_{\varepsilon,T}\}.
\end{equation}

Then $A_{\varepsilon,T}$ is compact by Lemma 3.3 and for all $L \in \mathbb{N}$,

\begin{equation}
P(\sup_{0 \leq t \leq T} \sum_{\xi \in G} \rho(\xi) \bar{z}_{L,\xi}(t) > K_{\varepsilon,T}) \leq \frac{1}{K_{\varepsilon,T}} E[\sup_{0 \leq t \leq T} \sum_{\xi \in G} \rho(\xi) \bar{z}_{L,\xi}(t)] = \varepsilon,
\end{equation}

by Lemma 3.1.

We conclude this subsection with the
Proof of Theorem 1. (i) Uniform convergence of the generator as given in (3.27) together with the compact containment condition (3.31) imply that the family \( \{ \mathcal{Z}^{G_L, G}; L \in \mathbb{N} \} \) is relatively compact by [EK86, Remark 4.5.2].

(ii) Moreover, any limit point \( \mathcal{Z}^G \) satisfies the \((\Omega_Z^G, \mathcal{D}(\Omega_Z^G), z(0))\)-martingale problem by [EK86, Lemma 4.5.1]. It also satisfies (2.24). This establishes existence.

(iii) Recall the uniform first moment bound stated in (3.3). We claim that this implies the following for any limit point \( \mathcal{Z}^G \), i.e.,

\[
(3.35) \sum_{\xi \in G} \rho(\xi)E\left[ \sup_{t \in [0,T]} \bar{z}_\xi(t) \right] \leq C(T) \sum_{\xi \in G} \rho(\xi)E\left[ \sup_{t \in [0,T]} \bar{z}_\xi(0) \right].
\]

Indeed, applying the Skorohod representation theorem, we can define \( \{ \mathcal{Z}^{G_L}; L \in \mathbb{N} \} \) and \( \mathcal{Z}^G \) on one and the same probability space such that \( \mathcal{Z}^{G_L} \to \mathcal{Z}^G \) in Skorohod topology almost surely, as \( L \to \infty \). Thus also for each \( T > 0 \), \( \sup_{t \in [0,T]} z^m_{L, \xi} \to \sup_{t \in [0,T]} z^m_\xi \) almost surely, \( L \to \infty \). We therefore have by Fatou’s lemma, for all \( \xi \in G \) and \( m = 1, \ldots, M \), and \( (L_k) \uparrow \infty \),

\[
(3.36) \sum_{m=1}^M \sum_{\xi \in G} \rho(\xi)E\left[ \sup_{0 \leq t \leq T} z^m_{L_k, \xi}(t) \right] \leq \lim inf_{k \to \infty} \sum_{m=1}^M \sum_{\xi \in G} \rho(\xi)E\left[ \sup_{0 \leq t \leq T} z^m_{L_k, \xi}(t) \right]
\]

by (3.26).

3.2 Proof of Theorem 2 and Theorem 3

We begin by proving Theorem 3, which then will give the existence in Theorem 2. Here we proceed similarly as in the proof of Theorem 1. First we find a solution to the \((\Omega_X^G, X(0))\) martingale problem as the diffusion limit of the properly rescaled particle system. Here we use again Lemma 4.5.1 and Remark 4.5.2 of [EK86] to first show uniform convergence of the generators and then establish a compact containment condition.

Lemma 3.6 (Convergence of generators) Let \( f \in \mathcal{D}(\Omega_X^G) \). Then

\[
(3.37) \lim_{\varepsilon \to 0} \sup_{x \in \mathcal{G}_G} |\Omega_Z^G, f(x) - \Omega_X^G f(x)| = 0.
\]

Proof Fix \( f \in \mathcal{D}(\Omega_X^G) \). By the Taylor expansion, for all \( \xi, \eta \in G, m = 1, \ldots, M \),

\[
(3.38) f(x \pm \varepsilon^m_\xi) = f(x) \pm \varepsilon \frac{\partial}{\partial x_\xi} f(x) + \frac{1}{2} \varepsilon^2 \frac{\partial^2}{\partial x_\xi^2} f(x) + R^1(\pm \varepsilon, \xi, m),
\]

where

\[
(3.39) R^1(\pm \varepsilon, \xi, m) = \varepsilon^3 \frac{\partial^3}{\partial x_\xi^3} f(x'(\varepsilon, \xi))
\]
for some \( x'(\varepsilon, \xi) \in [x - \varepsilon^m, x + \varepsilon^m] \), and

\[
(3.40) \quad f(x + \varepsilon^m - \varepsilon^m) = f(x) + \varepsilon \left( \frac{\partial}{\partial x^m} - \frac{\partial}{\partial x^m} \right) f(x) + R^2(\varepsilon, \eta, \xi, m),
\]

where

\[
(3.41) \quad R^2(\varepsilon, \eta, \xi, m) = \frac{\varepsilon^2}{2} \left\{ \frac{\partial^2}{\partial (x^m)^2} f(x'(\varepsilon, \xi, \eta)) + \frac{\partial^2}{\partial (x^m)^2} f(x''(\varepsilon, \xi, \eta)) \right\},
\]

where \( x''(\varepsilon, \xi, \eta) \in [x + \varepsilon^m - \varepsilon^m, x + \varepsilon^m] \) and \( x''(\varepsilon, \xi, \eta) \in [x, x + \varepsilon^m] \).

Notice that if \( \Omega^G \) and \( \Omega^X \) are bounded, then \( R(\varepsilon, f; \cdot) \) is also bounded, where

\[
R(\varepsilon, f; x) := \Omega^G f(x) - \Omega^X f(x)
\]

\[
(3.42) \quad R(\varepsilon, f; x) = \sum_{m=1}^{M} \sum_{\varepsilon \in G} \frac{x^m}{\varepsilon} \left[ \sum_{\eta \in G} a(\xi, \eta) R^2(\varepsilon, \eta, \xi, m) + \sum_{m=1}^{M} \sum_{\xi \in G} \frac{x^m}{\varepsilon} (\frac{1}{2} + \varepsilon K^m) R^4(\varepsilon, \xi, m) \right.
\]

\[
+ \sum_{m=1}^{M} \frac{x^m}{\varepsilon} \left( \frac{1}{2} + \varepsilon \sum_{m=1}^{M} \lambda_{m,n} x^m \right) R^4(-\varepsilon, \xi, m).
\]

Moreover, by (3.40) and (3.41),

\[
(3.43) \quad \lim_{\varepsilon \to 0} \sup_{x \in G} |R(\varepsilon, f; x)| = 0,
\]

which proves the statement.

The next lemma shows that the processes with mass-\( \varepsilon \)-particles do not leave compact sets in the special case when \( \lambda = 0 \) (state-independent supercritical branching), i.e., when they solve the \( (\Omega_{Z^\varepsilon,0}, \mathcal{L}_b) \) martingale problem corresponding to the operator

\[
\Omega^G_{Z^\varepsilon,0} f(y) = \sum_{m=1}^{M} \sum_{\varepsilon \in G} \frac{x^m}{\varepsilon} \left( \sum_{\eta \in G} a(\xi, \eta) \left( f(z + \varepsilon e(m, \eta) - \varepsilon e(m, \xi)) - f(z) \right) \right.
\]

\[
+ \frac{\gamma}{\varepsilon} K^m (f(z + \varepsilon e(m, \xi)) - f(z))
\]

\[
+ \frac{\gamma}{\varepsilon} K^m (f(z - \varepsilon e(m, \xi)) - 2f(z)) \right\}.
\]

applied to functions in

\[
(3.45) \quad \mathcal{D}(\Omega^G_{Z^\varepsilon,0}) := \{ f \in B(\mathcal{E}^G), \Omega^G_{Z^\varepsilon,0} f \in B(\mathcal{E}^G) \},
\]

**Lemma 3.7 (Compact containment)** Let \( x(0) \) be a random element in \( \mathcal{E}_{\text{par},G} \) such that \( \sum_{\varepsilon \in G} \rho(\xi) E[\varepsilon^m(0)] < \infty \). Put for all \( \varepsilon > 0 \), \( \xi \in G \) and \( m \in \{1, 2, \ldots, M\} \),

\[
(3.46) \quad \varepsilon^m_{\varepsilon,\xi}(0) := \varepsilon^m \left[ 1_{\xi^m(0)} \right].
\]
Let $Z_{ε,0} := (z_{ε,0,η}^m)_{m=1,...,M, ξ ∈ G}$ be a solution of the $(Ω^G_{Z,ε}, D(Ω^G_{Z,ε}), (z_{ε,0,η}(0))_{m=1,...,M, ξ ∈ G})$ martingale problem. For all $δ > 0$ and $T > 0$ there exists a compact set $A_{δ,T} ⊂ E_{par,G}$ such that

\[ (3.47) \inf_{ε>0} P\{ Z_{ε,0}(t) ∈ A_{δ,T} \forall 0 \leq t \leq T \} ≥ 1 - δ. \]

**Proof** By Lemma 3.3 it suffices to show that

\[ (3.48) \sup_{ε>0} P\left( \sup_{0≤t≤T} z_{ε,0,η}^m(t) ≥ L \right) \xrightarrow{L→∞} 0. \]

Indeed, assume (3.48) and take $δ > 0$. We may also take a enumeration $(ξ_i)_{i}$ of $G$ and numbers $(L_i)_{i}$ with

\[ (3.49) \sup_{ε>0} P\{ \exists 0 ≤ t ≤ T : x_{ε,ξ_i}^m(t) ≥ L_i \} \leq \frac{1}{T} \delta 2^{-i}, \]

for all $i ∈ N$ and $m ∈ \{1, 2, ..., M\}$. The set $A_{δ,T} := \bigotimes_{m=1}^M \bigotimes_{i=1}^{∞} [0, L_i]$ is compact in the product topology. Moreover,

\[ (3.50) \sup_{ε>0} P\{ \exists 0 ≤ t ≤ T : Z_{ε,0}(t) ∉ A_{δ,T} \} \leq \sum_{m=1}^M \sum_{i=1}^{∞} \sup_{ε>0} P\{ \exists 0 ≤ t ≤ T : z_{ε,0,ξ_i}^m(t) > L_i \} ≤ δ. \]

Since for each $W ∈ N$,

\[ (3.51) M_{ε,0,ξ,W} := \left( W \land z_{ε,0,ξ}^m(t) - \int_0^t Ω_{Z,ε,0}^G(s) \land z_{ε,0,ξ}(s) ds \right)_{t≥0} \]

is a martingale,

\[ (3.52) \sup_{ε>0} \left\{ \sup_{0≤t≤T} W \land z_{ε,0,ξ}^m(t) ≥ 2L \right\} ≤ \sup_{ε>0} \left\{ \sup_{0≤t≤T} M_{ε,0,ξ,W}(t) ≥ L \right\} + \sup_{ε>0} \left\{ \sup_{0≤t≤T} \int_0^t (Ω_{Z,ε,0}^G \land z_{ε,0,ξ}(s))^+ ds ≥ L \right\}. \]

We shall show separately for both terms on the right hand side to converge to 0 as $L \rightarrow ∞$ uniformly in $W ∈ N$. First for all $ε > 0$, $W ∈ N$

\[ P\left\{ \sup_{0≤t≤T} \int_0^t (Ω_{Z,ε,0}^G z_{ε,0,ξ}(s))^+ ds ≥ L \right\} \leq \frac{1}{L} E\left[ \int_0^T Ω_{Z,ε,0}^G W \land z_{ε,0,ξ}(s) ds \right] \]

\[ ≤ \frac{1}{L} E\left[ \int_0^T \left\{ \left( \sum_{η∈G} a(η, ξ)z_{ε,0,η}(s) \right) + γKz_{ε,0,ξ}(s) ds \right\} \right] \]

\[ (3.53) \leq \frac{1}{L^{ρξ(1)}} E\left[ \int_0^T \left( \sum_{η, η'∈G} a(η, η')ρ(η')z_{ε,0,η}(s) \right) + γKz_{ε,0,ξ}(s) ds \right] \]

\[ ≤ \frac{R_{ξ,ξ'}}{L^{ρξ(1)}} \int_0^T \sum_{η∈G} ρ(η)E[z_{ε,0,η}(s)] ds \]

\[ ≤ \frac{R_{ξ,ξ'}}{L^{ρξ(1)}} \int_0^T \sum_{η∈G} ρ(η)E[z_{η}^m(0)], \]
for $C$ as defined in (3.11) by (3.14). Second, by the maximal inequality for martingales,
\[
P \left\{ \sup_{0 \leq t \leq T} M_{\varepsilon,0,\xi,W}^m(t) \geq L \right\} \leq \frac{1}{L} \mathbb{E} \left[ (M_{\varepsilon,0,\xi,W}^m(T))^+ \right] \leq \frac{1}{L} \mathbb{E}_0 \left[ z^m_{\varepsilon,0,\xi}(T) \right] + \frac{1}{L} \mathbb{E} \left[ \int_0^T (\Omega_{Z^\varepsilon,0} z^m_{\varepsilon,0,\xi}(s))^+ ds \right]
\]
(3.54)

Here we have used that for all $a, b \in \mathbb{R}$, $(a - b)^+ \leq a^+ - b^-$ and that in the special case where $\lambda \equiv 0$ (and only in this special case!), $(\Omega_{Z^\varepsilon,0} W \wedge z^m_\xi)^- \leq (\Omega_{Z^\varepsilon,0} W \wedge z^m_\xi)^+$. Combining (3.53) and (3.54) shows (3.48). \hfill \Box

**Proof of Theorem 3** By explicit construction we can couple two finite geographic space particle systems $Z^G$ and $Z^{G,0}$ corresponding respectively to the generator $(\Omega^G_{Z^\varepsilon}, \mathcal{D}(\Omega^G_{Z^\varepsilon}))$ defined in (2.29) and (2.30) as well as to the generator $(\Omega^{G,0}_{Z^\varepsilon}, \mathcal{D}(\Omega^{G,0}_{Z^\varepsilon}))$ defined in (3.14) and (3.15) such that $z^m_{\varepsilon,0,\xi}(0) = z^m_{Z^\varepsilon,0,\xi}(0)$ and $z^m_{\varepsilon,0,\xi}(t) \leq z^m_{Z^\varepsilon,0,\xi}(t)$ for all $t \geq 0$. By Theorem 1(i) there exists limit points of such systems as $L \to \infty$ in the product topology. Hence there exist solutions to the corresponding infinite geographic space particle systems that obey the same ordering.

Thus for any particle solution $Z^\varepsilon$ to the $(\Omega^G_{Z^\varepsilon}, \mathcal{D}(\Omega^G_{Z^\varepsilon}), x(0))$-martingale problem constructed in Theorem 1 there exists a particle solution $Z^0_\varepsilon$ of the $(\Omega^{G,0}_{Z^\varepsilon}, \mathcal{D}(\Omega^{G,0}_{Z^\varepsilon}), x(0))$-martingale problem such that again $z^m_{\varepsilon,0,\xi}(t) \leq z^m_{Z^\varepsilon,0,\xi}(t)$ for all $t \geq 0$. Since by Lemma 3.5 the compact containment containment condition (3.47) holds for the family $\{Z^\varepsilon, \varepsilon > 0\}$ we can conclude the same for the family $\{Z^\varepsilon, \varepsilon > 0\}$. Combining this with Lemma 3.6 we can again apply Lemma 4.5.1 and Remark 4.5.2 of [EK86] to complete the proof of Theorem 3. \hfill \Box

We first give the proof of Theorem 2. Existence of a weak solution follows from the diffusion approximation stated in Theorem 3 together with Lemma 4.5.1 in [EK86].

### 3.3 Moment estimation and comparison results

Let $G$ be a countable Abelian group, and $X^G = \{(x^m_\xi(t))_{t \geq 0}; \xi \in G, m = 1, ..., M\}$ a solution of (2.29). Recall from (2.6) the total mass $\bar{x}_\xi$ in $\xi \in G$. In this subsection we state and prove some moment estimates for the total mass process. The following implies Proposition 2.6.

**Proposition 3.8 (Moment estimates)** Assume that the initial condition $X(0)$ has translation invariant total mass with $\mathbb{E}[(\bar{x}_\xi(0))^n] < \infty$ for all $\xi \in G, n \in \mathbb{N}$. Then for all $T \geq 0$,
\[
\text{sup}_{\xi \in G} \sup_{0 \leq t \leq T} (\bar{x}_\xi(t))^n < \infty.
\]
(3.55)

Furthermore there exists a $\delta > 0$ depending on the parameters of the dynamics such that if $\mathbb{E}[(\exp(\lambda \bar{x}_\xi(0)))^{\delta < 0}$ for all $t \geq 0$:
\[
\text{for all } \lambda < \delta.
\]
(3.56)
Remark 3.9 Note that this means that all mixed moments of the process are measure-determining provided some exponential moment exists initially. This implies for our purposes that we can proceed via moment calculations since we can by truncation always achieve approximating processes for which the assumptions are satisfied. □

Proof of Proposition 3.8 We choose \((\theta^m, c^m)_{m=1,...,M}\) in such a way that

\[(3.57) \quad \theta^m - c^m a_m > K^m a_m - \lambda_{m,m} a_m^2,\]

for any \((a_m)_{m=1,...,M} \in \mathbb{R}_+^M\) and all \(m \in \{1, ..., M\}\). Hence

\[(3.58) \quad \theta^m - c^m a_m > a_m \left( K^m - \sum_{n=1}^{M} \lambda_{m,n} a_n \right), \quad m = 1, \ldots, M.\]

Then by Itô’s formula for each \(m \in \{1, \ldots, M\}\), and \(\xi \in G\),

\[
(x^m_\xi(t))^n = (x^m_\xi(0))^n + \int_0^t \sum_{\eta \in G} a(\xi, \eta)(x^m_\eta(s) - x^m_\xi(s)) n (x^m_\xi(s))^{n-1} ds \\
+ \gamma^m \int_0^t x^m_\xi(s) \left( K^m - \sum_{k=1}^{M} \lambda_{m,k} x^k_\xi(s) \right) n (x^m_\xi(s))^{n-1} ds \\
+ \frac{\gamma^m}{2} n(n-1) \int_0^t (x^m_\xi(s))^{n-1} ds + \int_0^t \sqrt{\gamma^m x^m_\xi(s) n (x^m_\xi(s))^{n-1}} dw_\xi(s) \\
\leq (x^m_\xi(0))^n + \int_0^t \sum_{\eta \in G} a(\xi, \eta)(x^m_\eta(s) - x^m_\xi(s)) n (x^m_\xi(s))^{n-1} ds \\
+ \gamma^m \int_0^t (\theta^m - c^m x^m_\xi(s)) n (x^m_\xi(s))^{n-1} ds \\
+ \frac{\gamma^m}{2} n(n-1) \int_0^t (x^m_\xi(s))^{n-1} ds + \int_0^t \sqrt{\gamma^m x^m_\xi(s) n (x^m_\xi(s))^{n-1}} dw_\xi(s).
\]

Due to translation invariance the distribution of \(x^m_\xi(t)\) is identical to the distribution of \(x^m_\eta(t)\) for any \(\xi, \eta \in G\), and \(m = 1, \ldots, M\). An application of Hölder’s inequality implies therefore that

\[
\mathbb{E}[x^m_\eta(t)(x^m_\xi(t))^{n-1}] \leq \mathbb{E}[(x^m_\xi(t))^n].
\]

Hence,

\[
\mathbb{E}[(x^m_\xi(t))^n] \leq \mathbb{E}[(x^m_\xi(0))^n] + \int_0^t \mathbb{E}[(\theta^m - c^m x^m_\xi(s)) n (x^m_\xi(s))^{n-1}] ds \\
+ \frac{\gamma^m}{2} n(n-1) \int_0^t \mathbb{E}[(x^m_\xi(s))^{n-1}] ds.
\]

Due to the positivity we obtain

\[
\mathbb{E}[(x^m_\xi(t))^n] \leq \mathbb{E}[(x^m_\xi(0))^n] + \gamma^m \left( n \theta^m + \left( \frac{n}{2} \right) \right) \int_0^t \mathbb{E}[(x^m_\xi(s))^{n-1}] ds.
\]

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Hence, again by translation invariance and the moment conditions at time \( t = 0 \), we obtain that for any \( T > 0 \), and \( n \in \mathbb{N} \),

\[
(3.62) \quad \sup_{\xi \in G} \sup_{0 \leq t \leq T} E \left[ (x_\xi(t))^{n-1} \right] < \infty \quad \Rightarrow \quad \sup_{\xi \in G} \sup_{0 \leq t \leq T} E \left[ (x_\xi(t))^n \right] < \infty
\]

provided that \( E \left[ (x_\xi(0))^n \right] < \infty \). But since

\[
(3.63) \quad \frac{d}{dt} E \left[ x_\xi^n(t) \right] = \gamma^m E \left[ x_\xi^m(t) \left( K^m - \sum_{k=1}^{M} \lambda_{m,k} x_\xi^k(t) \right) \right] < \gamma^m (\theta^m - c^m E[x_\xi^m(t)]) ,
\]

we have \( E \left[ x_\xi(t) \right] < u(t) \) where \( u(t) \) is the solution to

\[
(3.64) \quad \frac{d}{dt} u(t) = \gamma^m (\theta^m - c^m u(t)) , \quad u(0) = E \left[ x_\xi(0) \right],
\]

which is

\[
(3.65) \quad u(t) = \frac{\theta^m}{c^m} + \left( E \left[ x_\xi^m(0) \right] - \frac{\theta^m}{c^m} \right) e^{-\gamma^m c^m t}.
\]

Hence

\[
(3.66) \quad E \left[ x_\xi(t) \right] < \frac{\theta^m}{c^m} + \left( E \left[ x_\xi^m(0) \right] - \frac{\theta^m}{c^m} \right) e^{-\gamma^m c^m t}.
\]

It now follows from (3.62) by induction on \( n \) that

\[
(3.67) \quad \sup_{\xi \in G} \sup_{0 \leq t \leq T} E \left[ (x_\xi(t))^n \right] < \infty
\]

if \( \sup_{\xi \in G} E[\bar{x}_\xi(0)] < \infty \).

In order to improve (3.67) we observe that by Jensen’s inequality, for some constant \( c(n,T) \), and all \( 0 \leq t \leq T \),

\[
(3.68) \quad (x_\xi^m(t))^n \leq c(n,T) \left( (x_\xi^m(0))^n + \int_0^t \sum_{\eta \in G} a(\xi,\eta) |x_\eta^m(s) - x_\xi^m(s)|^n ds \right. \\
\left. + \int_0^t \gamma^n |\theta^m - c^m x_\xi^m(s)|^n ds + \left( \int_0^t \sqrt{\gamma^n c^m} dw_\xi(s) \right)^n \right).
\]

By Burkholder’s inequality, translation invariance and using the bound \( |a - b|^n \leq (2a)^n + (2b)^n \) for \( a, b \geq 0 \) it now follows that

\[
(3.69) \quad \sup_{\xi \in G} E \left[ \sup_{0 \leq t \leq T} (x_\xi(t))^n \right] \leq c(n,T) \left( \sup_{\xi \in G} E \left[ (x_\xi^m(0))^n \right] \\
+ (2^{n+1} + (2\gamma^mc_m)^n) \int_0^T \sup_{\xi \in G} E \left[ (x_\xi^m(s))^n \right] ds \\
+ (2\gamma^m \theta^m c^n T + \sup_{\xi \in G} \left[ \left( \int_0^T \gamma^m c^m x_\xi^m(s) ds \right)^n \right]^\frac{1}{n} \right).
\]
Combining this with (3.67) implies (3.55). For the exponential moments use that an exponential moment exists if this is true for the non-spatial part. Notice that (3.61) implies that in the non-spatial case,

\[(3.70) \quad \mathbb{E}[(x(t))^n] \leq \mathbb{E}[(x(0))^n] + \gamma(2\theta + 1) \frac{n}{2} \int_0^t \mathbb{E}[(x(s))^{n-1}] \, ds.\]

Notice that if we replace \( \leq \) by \( = \), we obtain the moments of the Feller branching diffusion with branching rate \( \gamma(2\theta + 1) \), which is known to have exponential moments for suitably small exponents.

We next focus on the exchangeable model only.

**Proposition 3.10** In the exchangeable case we also have for all \( n \in \mathbb{N} \) and \( 0 < s \leq T < \infty \)

\[(3.71) \quad \sup_{\xi \in G} \sup_{t \geq s} \mathbb{E}[(\bar{x}_\xi(t))^n] < \infty, \quad \sup_{\xi \in G} \left[ \sup_{s \leq t \leq T} (\bar{x}_\xi(t))^n \right] < \infty, \]

and in particular if \( \bar{X} \) is started from an initial condition \( \bar{X}(0) \) bounded above by a translation invariant \( \bar{X}^{inv}(0) \) with \( \mathbb{E}[(\bar{x}_\xi^{inv}(0))^n] < \infty \) then (3.57) holds and so does

\[(3.72) \quad \sup_{\xi \in G} \sup_{t \geq 0} \mathbb{E}[(\bar{x}_\xi(t))^n] < \infty.\]

We will also need a comparison result for solutions to the total mass process in the exchangeable case or equivalently the one type model with immigration. We will also apply the result to the \( \alpha \) process in (2.45). We therefore state the comparison result in some generality.

**Proposition 3.11** Let \( \gamma^{(i)} \) for \( i = 1, 2 \) be two stochastic processes with values in \( \mathcal{E} \) such that \( y^{m(1)}_\xi(t) \leq y^{m(2)}_\xi(t) \) for all \( \xi \in G, m = 1, \ldots, M \) and \( t \geq 0 \). Let \( f^{(i)} : \mathcal{E} \to \mathcal{E} \) be continuous functions for \( i = 1, 2 \) and assume that for \( x^{(1)}, x^{(2)} \in (\mathbb{R}^+)^G \times \{1, \ldots, M\} \)

\[(3.73) \quad \sum_{\xi \in G} \sum_{m=1}^M m \left( f^{m(1)}_\xi(x^{(1)}) - f^{m(2)}_\xi(x^{(2)}) \right) \mathbb{1}_{\{x^{m(1)}_\xi \leq x^{m(2)}_\xi \}} \rho(\xi) \leq c \sum_{\xi \in G} \sum_{m=1}^M \left( x^{m(1)}_\xi - x^{m(2)}_\xi \right) \mathbb{1}_{\{x^{m(1)}_\xi \leq x^{m(2)}_\xi \}} \rho(\xi).\]

Suppose that \( x^{(i)} \) take values in \( \mathcal{E} \) for \( i = 1, 2 \) and are solutions to

\[(3.74) \quad dx^{m(i)}_\xi(t) = f^{m(i)}_\xi(x^{(i)}(t)) \, dt + \sqrt{\gamma m_{x^{m(i)}_\xi(t)}} \, dw^m(t) + y^{m(i)}_\xi(t) \, dt\]

with respect to the same family of independent Brownian motions \( \{w^m : \xi \in G, m = 1, \ldots, M\} \) and such that \( x^{m(1)}_\xi(0) \leq x^{m(2)}_\xi(0) \) for all \( \xi \in G \). Then, we have that \( x^{m(1)}_\xi(t) \leq x^{m(2)}_\xi(t) \) a.s. for all \( t \geq 0 \) and \( \xi \in G, m = 1, \ldots, M.\)
Remark Let $M = 1$. Condition (5.73) is satisfied if for $i = 1, 2$

\[(3.75) \quad f^{(i)}(x) = \sum_{\eta \in G} a(\xi, \eta)(x_\eta - x_\xi) + \gamma x_\xi (K - \lambda x_\xi)\]

This is due to (2.4) and the fact that we are considering $x^{(i)} \in (\mathbb{R}_+)^G$.

Proof of Proposition 3.10 From Theorem 2 of [HW07] we know that there exists a translation invariant maximal process $\bar{X}^{(\infty)} = (\bar{x}_\xi^{(\infty)}(t))_{t \in G, t > 0}$, also a solution to (2.29) for $M = 1$ such that for all $t > 0$ and $\xi \in G$, $\bar{x}_\xi(t)$ is stochastically smaller than $\bar{x}^{(\infty)}(t)$.

In order to prove the first part of (3.71) it therefore suffices to consider the process $\bar{X}^{(\infty)}$ which decreases stochastically as $t \to \infty$ and which satisfies $E[\bar{x}_\xi^{(\infty)}(t)] < \infty$ for any $t > 0, \xi \in G$, again by Theorem 2 of [HW07]. Due to the translation invariance this implies (3.71) immediately for $n = 1$. For $n \geq 1$, we proceed by induction. We calculate with Itô’s formula,

\[(3.76) \quad d(\bar{x}_\xi^{(\infty)}(t))^n = \sum_{\eta \in G} a(\xi, \eta)(\bar{x}_\eta^{(\infty)}(t) - \bar{x}_\xi^{(\infty)}(t)) n(\bar{x}_\xi^{(\infty)}(t))^{n-1} dt + n(\bar{x}_\xi^{(\infty)}(t))^{(n-1)} dt\]

Due to translation invariance the distribution of $\bar{x}_\xi^{(\infty)}(t)$ is identical to the distribution of $\bar{x}_\eta^{(\infty)}(t)$ for any $\eta \in G$. As before, an application of Hölder’s inequality implies therefore that $E[(\bar{x}_\eta^{(\infty)}(t)(\bar{x}_\xi^{(\infty)}(t))^{n-1}] \leq E[(\bar{x}_\xi^{(\infty)}(t))^n]$. Hence,

\[(3.77) \quad dE[(\bar{x}_\xi^{(\infty)}(t))^n] \leq nKE[(\bar{x}_\xi^{(\infty)}(t))^n] dt - \gamma n\lambda E[(\bar{x}_\xi^{(\infty)}(t))^{n+1}] dt + \frac{\gamma}{2} n(n-1)E[(\bar{x}_\xi^{(\infty)}(t))^{(n-1)}] dt\]

or, more precisely, for $0 \leq s \leq t < \infty$,

\[(3.78) \quad \int_s^t E[(\bar{x}_\xi^{(\infty)}(u))^{n+1}] du \leq \frac{1}{\gamma n \lambda} \left( E[(\bar{x}_\xi^{(\infty)}(s))^n] - E[(\bar{x}_\xi^{(\infty)}(t))^n] \right) + \frac{K}{\lambda} \int_s^t E[(\bar{x}_\xi^{(\infty)}(u))^n] du + \frac{n-1}{2 \lambda} \int_s^t E[(\bar{x}_\xi^{(\infty)}(u))^{(n-1)}] du\]

Thus, if (3.71) is true for $n$ and $n - 1$ then the left hand side is finite. Using that $u \mapsto (\bar{x}_\xi^{(\infty)}(u))^n$ is stochastically decreasing thus implying that $u \mapsto E[(\bar{x}_\xi^{(\infty)}(u))^n]$ is decreasing as well as the translation invariance of $X^{(\infty)}$ the first statement of (3.71) now follows for $n + 1$. For the second statement of (3.71) we use that due to the positivity
of the solutions and Hölder’s inequality there exists a constant $c = c(n, T)$ such that for $0 \leq \xi \leq T$,

$$
E\left[\sup_{s \leq t \leq T} (\bar{x}(t))^n\right] 
\leq c \left( E[(\bar{x}(s))^n] + 2^n \sum_{\eta \in G} a(\xi, \eta) \int_s^T E[(\bar{x}(t))^n] dt + (2^n + (\gamma K)^n) \int_s^T E[(\bar{x}(t))^n] dt + \gamma \frac{c}{2} \left( \int_s^T E[(\bar{x}(t))^n] dt \right)^{\frac{1}{2}} \right).
$$

(3.79)

where we have also used that by Burkholder’s inequality

$$
E\left[\sup_{s \leq t \leq T} \left( \int_s^T \sqrt{\bar{x}(u)} \, dw_u(u) \right) \right]^{\frac{2}{n}} 
\leq E\left[\left( \int_s^T \bar{x}(t) \, dt \right)^n \right]^{\frac{1}{n}}.
$$

(3.80)

This means that

$$
\text{sup}_{\xi \in G} \text{sup}_{s \leq t \leq T} E[(\bar{x}(t))^n] < \infty \quad \text{implies} \quad \text{sup}_{\xi \in G} \text{sup}_{s \leq t \leq T} (\bar{x}(t))^n < \infty,
$$

thus completing the proof of (3.71). In order to prove (3.72) we first note that due to monotonicity in the initial condition (see Proposition 3.11 and the following remark) it suffices to consider a process with translation invariant initial conditions. Thus (3.72) follows from Proposition 3.8. Combining this fact with (3.71) finishes the proof. \[ \square \]

**Proof of Proposition 3.11** This result follows from an adaptation of fairly standard methods of Yamada and Watanabe [YW71], see also Shiga and Shimizu [SS80]. We want to show that $X = X^{(1)} - X^{(2)}$ is not positive. Here, $X$ solves

$$
dx(t) = \left( f_{\xi}(1)(x^{(1)}(t)) - f_{\xi}(2)(x^{(2)}(t)) \right) \, dt + \left( \sqrt{\gamma_{x^{(1)}}} - \sqrt{\gamma_{x^{(2)}}} \right) \, dw(t) 
\quad + \left( y^{(1)} - y^{(2)} \right) \, dt
$$

(3.82)

Let $g^+(x) = x \vee 0, x \in \mathbb{R}$ and let $g^{+, n}$ be an appropriate smoothing of $g^+$ as in [YW71]. We can choose $g^{+, n}$ such that $g^{+, n} \uparrow g^+$ uniformly as $n \to \infty$ as well as $0 \leq (g^{+, n})' \uparrow 1_{(0, \infty)}$ and $0 \leq (g^{+, n})''(x) \leq \frac{2}{n^2}. We apply Itô’s formula to $x_{\xi}^n$ with the function $g^{+, n}$ and consider the result stopped at time $T_N = \inf \{ t \geq 0 : \sum_{\xi \in G} \sum_{m=1}^M |x_{\xi}(t)| \rho(\xi) \geq N \}$. Using that $g^{+, n}(x_{\xi}(0)) = 0$ and $y^{n(1)}(t) - y^{n(2)}(t) \leq 0$ by assumption we get after taking expectations that

$$
\text{(3.83)}
$$
We also have that

\[
\sum_{\xi}
\]\n
formal generator a unique Markov process can use an explicit construction with exponential random variables to see that there exists a unique solution to the nature of the transitions, the number of particles may only decrease over time.

Let \( \rho \) be the initial number of particles, and due to assumption the initial number of particles is finite and thus also the transition rates. Due to dominated convergence theorem that

\[
\lim_{N \to \infty} \int_0^t \mathbb{E} \left[ g^+(x^n_{\xi}^{(m)}(t \wedge T_N)) \right] \leq \mathbb{E} \left[ \int_0^t \mathbb{E} \left[ g^+(x^n_{\xi}^{(m)}(s)) \right] \right]
\]

We multiply the above by \( \rho(\xi) \) and sum over \( \xi \) and \( m \). Using (3.73) we arrive at

\[
\int_0^t \mathbb{E} \left[ \sum_{\xi} \sum_{m=1}^M g^+(x^n_{\xi}^{(m)}(s \wedge T_N)) \rho(\xi) \right] ds 
\]

An application of Gronwall’s lemma implies now that

\[
\mathbb{E} \left[ \sum_{\xi} \sum_{m=1}^M g^+(x^n_{\xi}^{(m)}(t \wedge T_N)) \rho(\xi) \right] = 0.
\]

Since \( T_N \uparrow \infty \) a.s. as \( N \to \infty \) and application of Fatou’s lemma proves our result. \( \square \)

4 Proof of exponential duality in the exchangeable model

Proof of Lemma 2.7 For each \( \kappa = (\kappa^m_{\xi})_{m=1,\ldots,M,\xi \in G} \in (\mathbb{N}_0^M)^G \) with \( \sum_{\xi \in G} \bar{\kappa}_\xi < \infty \) we can use an explicit construction with exponential random variables to see that there exists a unique Markov process \( \kappa(t) \in D(\mathbb{R}_+, (\mathbb{N}_0^M)^G) \) such that \( \kappa(0) = \kappa \) corresponding to the formal generator

\[
\Omega_{\alpha} f(\kappa) := \sum_{m=1}^M \sum_{\xi,\eta \in G} \kappa^m_{\xi}(\xi, \eta) \left( f(\kappa + e(\xi, \eta) - e(m, \xi)) - f(\kappa) \right) \\
+ \gamma \sum_{m=1}^M \sum_{\xi \in G} \left( \kappa^m_{\xi} \right) \left( f(\kappa - e(\xi)) - f(\kappa) \right).
\]

We also have that \( \sum_{\xi \in G} \bar{\kappa}_\xi(t) \leq \sum_{\xi \in G} \bar{\kappa}_\xi < \infty \). This follows immediately since by assumption the initial number of particles is finite and thus also the transition rates. Due to the nature of the transitions, the number of particles may only decrease over time.

Let \( k_\xi \in \mathbb{N}^G \) such that \( \sum_{\xi} k_\xi < \infty \). Then, for any \( \alpha(0) \in L^p(\rho) \) for \( p \geq 4 \) there exists a unique solution \( \alpha \) with initial condition \( \alpha(0) \) and such that \( \alpha(t) \in L^p(\rho) \) for all \( t \geq 0 \) a.s.
corresponding to the generator

\[
\Omega_\alpha^k f(\alpha) := \sum_{\xi, \eta \in G} \bar{a}(\xi, \eta)(\alpha_\eta - \alpha_\xi) \frac{\partial}{\partial \alpha_\xi} f(\alpha, \kappa) \\
+ \gamma \sum_{\xi \in G} \alpha_\xi (K - \frac{1}{2} \alpha_\xi) \frac{\partial}{\partial \alpha_\xi} f(\alpha, \kappa) + \gamma \lambda \sum_{\xi \in G} \alpha_\xi \frac{\partial^2}{\partial (\alpha_\xi)^2} f(\alpha, \kappa) \\
+ \gamma \sum_{\xi \in G} \lambda k_\xi \frac{\partial}{\partial \alpha_\xi} f(\alpha, \kappa),
\]

To see this we may apply Theorem 2.3 in [Stu03], see also the proof of Theorem 5.1 in [BEM07] who apply this theorem in a setting similar to ours. The linear growth condition on the diffusion term needed in Theorem 2.3 in [Stu03] is immediate. Due to (2.4) and [BEM07] who apply this theorem in a setting similar to ours. The weak uniqueness of the process follows with a Girsanov argument as

\[
\sum_{\xi \in G} |\sum_{\eta \in G} \bar{a}(\xi, \eta)(\alpha_\eta - \alpha_\xi) + \gamma \alpha_\xi (K - \frac{1}{2} \alpha_\xi) + \gamma \lambda k_\xi \|\rho(\xi)\| < C(1 + \|\alpha\|_{2p, \rho}).
\]

In fact, we could have bounded the left hand side by \((1 + \|\alpha\|_{2p, \rho})\) if it were not for the quadratic term which only allows for the above bound. The arguments in the proof of Theorem 2.3 of [Stu03] then imply the existence of a process \(\alpha\) to the system of stochastic differential equations corresponding to \(\Omega_\alpha^k\) for each initial condition \(\alpha(0) \in L^{2p}(\rho)\) with \(p \geq 2\). The solution is continuous in each component and satisfies \(\mathbb{E}(\sup_{0 \leq t \leq T} \|\alpha\|_{2p, \rho}) < \infty\) for each \(T \geq 0\). The weak uniqueness of the process follows with a Girsanov argument as in the proof of Theorem [2]. But in fact, by a slight modification of Proposition [2.6] and its proof to accommodate the extra immigration term we can now conclude that \(\alpha(t) \in L^{2p}(\rho)\) for all \(t \geq 0\) a.s. and that \(\mathbb{E}(\sup_{0 \leq t \leq T} \|\alpha\|_{2p, \rho}) < \infty\). This yields the claimed result.

It is therefore also immediate that for any sequence of times \(0 = t_0 \leq t_1 \leq t_2 \leq \ldots\) and sequence \(k^{(i)} \in \mathbb{N}^G\) with \(\sup_i \sum_{\xi \in G} k^{(i)}_\xi < \infty\) and \(\alpha(0) \in L^p(\rho)\), for \(p \geq 4\) there exists a unique process \(\alpha\) with initial condition \(\alpha(0) = \alpha(0)\) such that \(\alpha(t) \in L^p(\rho)\) for all \(t \geq 0\) a.s. that is a solution of the system of stochastic differential equations corresponding to the generator \(\Omega_\alpha^{k^{(i)}}\) on the time interval \([t_{i-1}, t_i]\). Each realization of an independent process \(\kappa\) provides such a sequence of (jump) times and states. Thus, for each \(\alpha(0) \in L^p(\rho)\) for \(p \geq 4\) and \(\kappa(0) \in \{N_0^M\}^G\) with \(\sum_{\xi \in G} \bar{g}_\xi(0) < \infty\) we can define an \((\alpha, \kappa)\) process on the joint probability space that has \(\Omega_\alpha^{(\alpha, \kappa)}\) as its generator, and this process is unique in law. As before we have that the \(\alpha\) process is continuous in each component with \(\alpha(t) \in L^p(\rho)\) for all \(t \geq 0\) a.s. and \(\mathbb{E}(\sup_{0 \leq t \leq T} \|\alpha\|^p_{2p, \rho}) < \infty\) as well as that \(\kappa(t) \in D(\mathbb{R}_+, \{N_0^M\}^G)\) with \(\sum_{\xi \in G} g_\xi(t) \leq \sum_{\xi \in G} \bar{g}_\xi(0) < \infty\).

\[\square\]

**Proof of Lemma 2.8** Fix \(t \geq 0\). Note that in the process \(\kappa(t)\) particles only coalesce or migrate. Since \(\kappa(0)\) has finite support, it follows immediately that \(\sum_{\xi \in G} \bar{g}_\xi(t) < \infty\) for all \(t \geq 0\), a.s.

For the process \(\alpha(t)\) we prove the stronger statement

\[(4.1) \quad \mathbb{E}\left[\sum_{\xi \in G} \alpha_\xi(t) \bar{g}_\xi\right] < \infty \quad \forall x \in \mathcal{E}, (\alpha(0), \kappa(0)) \in \mathcal{E}_t^{\text{dual}}.\]
Note that this term can be compared with the corresponding term for a system of interacting supercritical Feller diffusions with immigration. Namely, by Proposition 3.11,

\[(4.2) \quad \mathbb{E} \left[ \sum_{\xi \in G} \alpha_\xi(t) \bar{x}_\xi \right] \leq \mathbb{E} \left[ \sum_{\xi \in G} y_\xi(t) \bar{x}_\xi \right],\]

where \((y(t), \pi(t))_{0 \leq t \leq T}\) is a Markov process with \((y(0), \pi(0)) = (\alpha(0), \bar{\kappa}(0))\) and with the following generator,

\[(4.3) \quad \Omega_{y,\pi} f(y, \pi) := \sum_{\xi, \eta \in G} \pi_{\xi} a_{\xi}(\xi, \eta) \left( f(y, \pi + \delta_\eta - \delta_\xi) - f(y, \pi) \right) + \sum_{\xi, \eta \in G} \bar{\alpha}_t(\xi, \eta) (y_\eta - y_\xi) \frac{\partial f(y, \pi)}{\partial y_\xi} + \gamma K \sum_{\xi \in G} y_\xi \frac{\partial f(y, \pi)}{\partial y_\xi} \]

\[
+ \gamma \lambda \sum_{\xi \in G} \bar{\alpha}_t(\xi, \eta) \frac{\partial^2 f(y, \pi)}{(\partial y_\xi)^2} + \gamma \lambda \sum_{\xi \in G} \pi_{\xi} \frac{\partial^2 f(y, \pi)}{(\partial y_\xi)^2} \].

Since \(\pi\) is a system of independent random walks, we have

\[(4.4) \quad \mathbb{E}[\pi_\xi(t)] = \sum_{\eta \in G} \pi_{\eta}(0) a_t(\eta, \xi).\]

Then we observe that the first moment \(h_\xi(t) = \mathbb{E}[y_\xi(t)]\) has to fulfill the following system of differential equations, \(\forall \xi \in G, y_\xi(0) = \alpha_\xi(0)\)

\[(4.5) \quad \frac{d}{dt} h_\xi(t) = \sum_{\eta \in G} (\bar{\alpha}(\xi, \eta) - \delta_\xi(\xi, \eta)) h_\eta(t) + \gamma K h_\xi(t) + \gamma \lambda \sum_{\eta \in G} \pi_\eta(0) a_t(\eta, \xi),\]

which is solved by

\[(4.6) \quad h_\xi(t) = e^{\gamma K t} \sum_{\eta \in G} \bar{a}_t(\xi, \eta) \left( \alpha_\eta(0) + \frac{\lambda}{K} (1 - e^{-\gamma K t}) \pi_\eta(0) \right).\]

Now we calculate the right hand side of (4.2) as

\[(4.7) \quad \mathbb{E} \left[ \sum_{\xi \in G} y_\xi(t) \bar{x}_\xi \right] = e^{\gamma K t} \sum_{\xi, \eta \in G} \bar{a}_t(\xi, \eta) \left( \alpha_\eta(0) + \frac{\lambda}{K} (1 - e^{-\gamma K t}) \pi_\eta(0) \right) \bar{x}_\xi.\]

The first term on the r.h.s. of (4.7) can be estimated as follows

\[(4.8) \quad e^{\gamma K t} \sum_{\xi, \eta \in G} \bar{a}_t(\xi, \eta) \alpha_\eta(0) \bar{x}_\xi \leq e^{\gamma K t} \left( \sup_{\xi \in E, \eta \in \text{supp } \alpha(0)} \frac{a_t(\eta, \xi)}{\rho(\xi)} \right) \sum_{\eta \in G} \alpha_\eta(0) \sum_{\xi \in G} \rho(\xi) \bar{x}_\xi.\]

Since \(x \in E\) and \(\alpha(0)\) has finite support, this leaves us with proving

\[(4.9) \quad \sup_{\xi \in G} \frac{a_t(\eta, \xi)}{\rho(\xi)} < \infty, \quad \forall t > 0, \eta \in G.\]
To verify this note that

\[(4.10) \sup_{\xi \in G} a_t(\eta, \xi) \leq \sup_{\xi \in G} \frac{e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} a^{(n)}(\eta, \xi)}{\sum_{n=0}^{\infty} \frac{C^n}{n!} a^{(n)}(\eta, \xi) \beta(\eta)} \leq \frac{1}{\beta(\eta)} e^{Ct} < \infty.\]

Hence we have proved that the first term on the r.h.s. of (4.7) is finite, namely

\[(4.11) e^{-\gamma Kt} \sum_{\xi, \eta \in G} a_t(\xi, \eta) \beta(\eta) \leq e^{\gamma Kt + Ct} \left( \max_{\eta \in \text{supp} \alpha(0)} \frac{1}{\beta(\eta)} \right) \sum_{\eta \in G} \alpha(0) \sum_{\xi \in G} \rho(\xi) \rho(\xi) < \infty.\]

Now we come to the second term on the r.h.s. of (4.7) which we estimate as follows (using (4.10))

\[(4.12) \frac{\lambda K}{e^{\gamma Kt} - 1} \sum_{\xi, \eta \in G} a_t(\xi, \eta) \pi(0) \rho(\xi) \leq \frac{\lambda K}{e^{\gamma Kt} - 1} e^{Ct} \left( \max_{\eta \in \text{supp} \pi(0)} \frac{1}{\beta(\eta)} \right) \sum_{\eta \in G} \pi(0) \sum_{\xi \in G} \rho(\xi) \rho(\xi) < \infty.\]

This completes the proof. \(\square\)

**Proof of Proposition 2.9** Here we want to apply Theorem 4.11 in Chapter 4 of [EK86]. First we check condition (4.42) in Chapter 4 of [EK86] in order to see that the r.h.s. of (4.54) in [EK86] is zero. That is, we show that

\[(4.13) \Omega_X H((\alpha, \kappa), x) = \Omega_{(\alpha, \kappa)} H((\alpha, \kappa), x) + \beta(\alpha, \kappa) H((\alpha, \kappa), x)\]

with

\[(4.14) \beta(\alpha, \kappa) := \gamma \sum_{\xi \in G} \left( \sum_{m=1}^{M} \left( \kappa_\xi^m - \alpha_\xi \kappa_\xi + K \kappa_\xi \right) \right).\]

Since for \(m = 1, \ldots, M,\)

\[(4.15) \frac{\partial}{\partial x_m^\xi} H((\alpha, \kappa), x) = -\alpha_\xi H((\alpha, \kappa), x) + \kappa_\xi^m H((\alpha, \kappa - e(m, \xi), x),\]

and

\[(4.16) \frac{\partial^2}{\partial (x_m^\xi)^2} H((\alpha, \kappa), x) = (\alpha_\xi)^2 H((\alpha, \kappa), x) - 2\alpha_\xi \kappa_\xi^m H((\alpha, \kappa - e(m, \xi), x)\]

\[+ \kappa_\xi^m (\kappa_\xi^m - 1) H((\alpha, \kappa - 2e(m, \xi), x),\]

37
\[
\Omega_X H((\alpha, \kappa), x) = \sum_{m=1}^{M} \sum_{\xi, \eta \in G} (\bar{a}(\xi, \eta) - \delta(\xi, \eta)) x_{\eta}^m \left( -\alpha \xi H((\alpha, \kappa), x) + \kappa_{\xi}^m H((\alpha, \kappa - e(m, \xi), x) \right)
\]

\[
(4.17) \quad \Omega_X H((\alpha, \kappa), x) = \sum_{m=1}^{M} \sum_{\xi, \eta \in G} (\bar{a}(\xi, \eta) - \delta(\xi, \eta)) x_{\eta}^m \left( -\alpha \xi H((\alpha, \kappa), x) + \kappa_{\xi}^m H((\alpha, \kappa - e(m, \xi), x) \right)
\]

\[
+ \gamma \sum_{m=1}^{M} \sum_{\xi \in G} x_{\xi}^m (K - \lambda \bar{x}_{\xi}) \left( -\alpha \xi H((\alpha, \kappa), x) + \kappa_{\xi}^m H((\alpha, \kappa - e(m, \xi), x) \right)
\]

\[
+ \gamma \sum_{m=1}^{M} \sum_{\xi \in G} \left( (\alpha \xi)^2 H((\alpha, \kappa), x) - 2\alpha \xi \kappa_{\xi}^m H((\alpha, \kappa - e(m, \xi), x) \right)
\]

\[
+ \gamma \sum_{\xi \in G} \left( (\alpha \xi)^2 \bar{x}_{\xi} H((\alpha, \kappa), x) - \gamma \sum_{\xi \in G} \alpha \xi \kappa_{\xi} H((\alpha, \kappa), x) \right)
\]

\[
+ \gamma \sum_{m=1}^{M} \sum_{\xi \in G} \left( \frac{\kappa_{\xi}^m}{2} \right) H((\alpha, \kappa - e(m, \xi), x) \right).
\]

Noticing that \( x_{\xi}^m H((\alpha, \kappa), x) = H((\alpha, \kappa + e(m, \xi), x) \) yields that

\[
\Omega_X H((\alpha, \kappa), x) = \sum_{\xi \in G} \left( \frac{\kappa_{\xi}^m}{2} \right) H((\alpha, \kappa - e(m, \xi), x) \right).
\]
Moreover, since \( \partial / \partial \alpha \xi H = -\bar{x}_\xi H \) and \( \partial^2 / \partial (\alpha \xi)^2 H = (\bar{x}_\xi)^2 H \),

\[
\Omega_{XH}(\alpha, \kappa, x) = \sum_{\xi \in G} \alpha_\xi \sum_{\eta \in G} (\bar{a}(\xi, \eta) - \delta(\xi, \eta)) \frac{\partial}{\partial \alpha} H((\alpha, \kappa), x) \\
+ \sum_{m=1}^M \sum_{\xi, \eta \in G} \kappa^m_\xi \bar{a}(\xi, \eta) \left( H((\alpha, \kappa + e(m, \eta) - e(m, \xi), x) - H((\alpha, \kappa), x) \right) \\
+ \gamma \sum_{\xi \in G} \left( K \bar{\kappa}_\xi H((\alpha, \kappa), x) + (K \alpha_\xi + \lambda \bar{\kappa}_\xi) \frac{\partial}{\partial \alpha} H((\alpha, \kappa), x) \right) \\
- \frac{1}{2} \gamma \sum_{\xi \in G} (\alpha_\xi)^2 \frac{\partial}{\partial \alpha} H((\alpha, \kappa), x) - \gamma \sum_{\xi \in G} \alpha_\xi \bar{\kappa}_\xi H((\alpha, \kappa), x) \\
+ \gamma \sum_{m=1}^M \sum_{\xi \in G} \left( \frac{\kappa^m_\xi}{2} \right) H((\alpha, \kappa - e(m, \xi), x) \\
= \Omega(\alpha, \kappa) H((\alpha, \kappa), x) + \gamma \sum_{\xi \in G} \left( \sum_{m=1}^M \left( \frac{\kappa^m_\xi}{2} \right) - \alpha_\xi \bar{\kappa}_\xi + K \bar{\kappa}_\xi \right) H((\alpha, \kappa), x),
\]

which is (4.13).

A careful inspection of the proof of Theorem 4.11 and Corollary 4.13 in Chapter 4 of [EK86] shows that the following conditions are now sufficient to establish the duality:

\[
\sup_{s,t \leq T} E \left[ (|\beta(\alpha(s), \kappa(s))| + 1) |H((\alpha(s), \kappa(s)), X(t))| + 1 \right] < \infty,
\]

(4.20)

\[
\sup_{r,s,t \leq T} E \left[ (|\beta(\alpha(r), \kappa(r))| + 1) \Omega_{XH}((\alpha(s), \kappa(s)), X(t)) \right] < \infty,
\]

(4.21)

\[
\beta(\alpha(r), \kappa(r)) \leq C \quad \forall r \leq T,
\]

(4.22)

where \( C \) is a constant. Indeed, from the proof of Theorem 4.11 in Chapter 4 of [EK86] we
see that it suffices to check that for $0 \leq s, s + h, t \leq T$

\begin{equation}
\int_s^{s+h} E \left[ \left( \int_r^{s+h} \Omega_{(\alpha, \kappa)} H((\alpha(v), \kappa(v)), X(t)) \, dv \right) \beta(\alpha(r), \kappa(r)) \right] \cdot \exp \left( \int_0^r \beta(\alpha(u), \kappa(u)) \, du \right) \, dr \leq C(T)h^2
\end{equation}

\begin{equation}
\int_s^{s+h} E \left[ \Omega_{(\alpha, \kappa)} H((\alpha(r), \kappa(r)), X(t)) \right] \cdot \left\{ \exp \left( \int_0^s \beta(\alpha(u), \kappa(u)) \, du \right) - \exp \left( \int_0^r \beta(\alpha(u), \kappa(u)) \, du \right) \right\} \, dr \leq C(T)h^2
\end{equation}

\begin{equation}
\int_0^s E \left[ \Omega X H((\alpha(t), \kappa(t)), X(r)) \cdot \exp \left( \int_0^t \beta(\alpha(u), \kappa(u)) \, du \right) \right] \, dr \leq C(T)
\end{equation}

\begin{equation}
\int_0^s E \left[ \left\{ H((\alpha(r), \kappa(r)), X(t)) \cdot \beta(\alpha(r), \kappa(r)) + \Omega_{(\alpha, \kappa)} H((\alpha(r), \kappa(r)), X(t)) \right\} \cdot \exp \left( \int_0^r \beta(\alpha(u), \kappa(u)) \, du \right) \right] \, dr \leq C(T)
\end{equation}

Here, (4.23) and (4.25) follow immediately from (4.13) to (4.22). Also, due to (4.22) the expression on the l.h.s. in (4.23) is bounded by

\begin{equation}
\exp(C(s+h)) \int_s^{s+h} \int_r^{s+h} E \left[ \left| \Omega_{(\alpha, \kappa)} H((\alpha(v), \kappa(v)), X(t)) \right| \right] \, dv \, dr \leq e^{C(s+h)}h^2 \sup_{s \leq r \leq s+h} E \left[ \left| \beta(\alpha(r), \kappa(r)) \right| \cdot \Omega_{(\alpha, \kappa)} H((\alpha(v), \kappa(v)), X(t)) \right] \right] \leq C(T)h^2,
\end{equation}
where we have used (4.21). The expression on the l.h.s. in (4.24) can be bounded as follows

\[
\left| \int_s^{s+h} \mathbb{E} \left[ \Omega_{(\alpha,\kappa)}((\alpha(r),\kappa(r)), X(t)) \right] \cdot \exp \left( \int_0^s \beta(\alpha(u),\kappa(u)) \, du \right) \cdot \left\{ 1 - \exp \left( \int_s^r \beta(\alpha(u),\kappa(u)) \, du \right) \right\} \, dr \right|
\]

\[
\leq e^{Cs} \int_s^{s+h} \mathbb{E} \left[ \Omega_{(\alpha,\kappa)} H((\alpha(r),\kappa(r)), X(t)) \right] \cdot \left| \int_s^r \beta(\alpha(u),\kappa(u)) \, du \right| \cdot \exp \left( 0 \vee \sup_{s \leq v \leq s+h} \int_s^v \beta(\alpha(u),\kappa(u)) \, du \right) \, dr
\]

\[
\leq e^{Cs} \exp \left( 0 \vee \sup_{s \leq v \leq s+h} \int_s^v \beta(\alpha(u),\kappa(u)) \, du \right) \cdot \int_s^{s+h} \int_s^r \mathbb{E} \left[ \left| \Omega_{(\alpha,\kappa)} H((\alpha(r),\kappa(r)), X(t)) \right| \cdot \beta(\alpha(u),\kappa(u)) \right] \, du \, dr
\]

\[
\leq C(T)h^2,
\]

where we first applied (4.22) and we used the fact \(|1 - e^x| \leq |x|e^{0|x|}\) and then we applied (4.24).

The remainder of the proof is therefore concerned with showing the three integrability conditions (4.26) to (4.28). We first note that condition (4.22) is naturally fulfilled since we start with \(n = \sum_{\xi \in G} \bar{\kappa}_\xi(0) < \infty\), and \(\sum_{\xi \in G} \bar{\kappa}_\xi(t) \leq n\) for all \(t \geq 0\). In order to show (4.24) and (4.25) we define

\[
A_\kappa(t) := \{ \xi \in G : \bar{\kappa}_\xi(t) > 0 \} \quad \text{and} \quad \bar{A}_\kappa(T) = \bigcup_{0 \leq t \leq T} A_\kappa(t)
\]

as the set of sites occupied by \(\kappa\) particles at time \(t\) and up to time \(T\) respectively. We write \(|\bar{A}_\kappa(T)|\) for the cardinality of the latter set. We let \(\mathcal{F}_t^\kappa = \sigma((\kappa(s))_{s \leq t})\) be the \(\sigma\)-algebra generated by the process \(\kappa\) up to time \(t\). The following estimates will be used repeatedly. The constant \(C\) may change from line to line. First note that

\[
|\beta(\alpha(r),\kappa(r))| \leq \gamma \left( n^2 + Kn + \sum_{\xi \in G} \alpha(\xi) \bar{\kappa}_\xi(r) \right) \leq C \left( 1 + \sum_{\xi \in G} \alpha(\xi) \right).
\]

We also have for any fixed \((m_\xi)_{\xi \in G} \in \mathbb{N}^G\) with \(m = \sum_{\xi \in G} m_\xi\) and \(A = \{ \xi \in G : m_\xi \geq 1 \}\) finite that

\[
\prod_{\xi \in G} \bar{x}_\xi(t)^{m_\xi} x(t)^{\kappa(s)} \leq \prod_{\xi \in G} \bar{x}_\xi(t)^{m_\xi + \bar{\kappa}_\xi(s)} \leq \prod_{\xi \in G} \left( \max_{\xi \in A_\kappa(s) \cup A} \bar{x}_\xi(t) \right)^{m_\xi + \bar{\kappa}_\xi(s)} \leq \left( 1 \vee \max_{\xi \in A_\kappa(s) \cup A} \bar{x}_\xi(t) \right)^{m+n} \leq 1 + \sum_{\xi \in A_\kappa(s) \cup A} \bar{x}_\xi(t)^{m+n}
\]
where $a \lor b$ denotes the maximum of $a$ and $b$. Therefore,

$$
E \left[ \sup_{s,t \leq T} \prod_{\xi \in G} \tilde{x}_\xi(t)^{m \xi} x(t)^{\kappa(s)} \bigg| \mathcal{F}_T^s \right] \leq E \left[ \sup_{s,t \leq T} \left( 1 + \sum_{\xi \in \tilde{A}_\kappa(T) \cup A} \tilde{x}_\xi(t)^{m+n} \right) \bigg| \mathcal{F}_T^s \right]
$$

(4.32)

\[
\leq 1 + \left( |\tilde{A}_\kappa(T)| + |A| \right) \sup_{\xi \in G} E \left[ \sup_{t \leq T} \tilde{x}_\xi(t)^{m+n} \right] \leq C \left( 1 + |\tilde{A}_\kappa(T)| + |A| \right)
\]

by Proposition 3.10 provided that $\tilde{x}(0)$ is bounded above by a translation invariant distribution with bounded $(m + n)$-th moment at each site. We will use (4.32) with $m \leq 2$.

To check (4.20) we condition on $\mathcal{F}_T^s$ and use the conditional independence of $\alpha$ and $X$ as well as (4.30) and (4.32) to arrive at

$$
\sup_{s,t \leq T} E \left[ \beta(\alpha(s), \kappa(s)) \cdot |H((\alpha(s), \kappa(s), X(t)) \bigg| \mathcal{F}_T^s \right]
$$

(4.33)

\[
\leq C \sup_{s,t \leq T} E \left[ E \left[ 1 + \sum_{\xi \in G} \alpha_\xi(s) \bigg| \mathcal{F}_T^s \right] \cdot E \left[ x(t)^{\kappa(s)} \bigg| \mathcal{F}_T^s \right] \right]
\]

\[
\leq C E \left[ \sup_{s,t \leq T} E \left[ 1 + \sum_{\xi \in G} \alpha_\xi(s) \bigg| \mathcal{F}_T^s \right] \cdot \left( 1 + |\tilde{A}_\kappa(T)| \right) \right]
\]

By Proposition 3.11 we can couple $\sum_{\xi \in G} \alpha_\xi$ to a supercritical nonspatial branching process $\tilde{\alpha}$ with immigration solving

$$
d\tilde{\alpha}(t) = \gamma K \tilde{\alpha}(t) \, dt + \sqrt{\gamma \lambda \tilde{\alpha}(t)} \, dw(t) + \gamma \lambda n \, dt,
$$

(4.34)

such that for $\sum_{\xi \in G} \alpha_\xi(0) = \tilde{\alpha}(0)$ we have $\sum_{\xi \in G} \alpha_\xi(t) \leq \tilde{\alpha}(t)$ for $t \geq 0$. We note that $\tilde{\alpha}$ is independent of $\kappa$. Since for any $m \in \mathbb{N}$, $E[\tilde{\alpha}(0)^m] < \infty$ implies $\sup_{t \leq T} E[\tilde{\alpha}(t)^m] < \infty$ we have

$$
\sup_{r \leq T} E \left[ \left( \sum_{\xi \in G} \alpha_\xi(r) \right)^m \bigg| \mathcal{F}_T^s \right] \leq \sup_{r \leq T} E \left[ \tilde{\alpha}(r)^m \right] < \infty
$$

(4.35)

provided that $E\left[ \left( \sum_{\xi \in G} \alpha_\xi(0) \right)^m \right] < \infty$. Hence, we can bound (4.33) by $C(1 + E[|\tilde{A}_\kappa(T)|])$.

But since each $\kappa$ particle performs an independent random walk (until coalescence) at rate one the number of sites in the set $\tilde{A}_\kappa(T)$, $|\tilde{A}_\kappa(T)|$, can be bounded by a Poisson random variable with parameter $nT$ and so we arrive at (4.20).

We now turn to showing (4.21) with similar means. Recall the form of the generator in (2.45), which consists of six terms. The first term we bound as follows (using $\tilde{\kappa}_\xi \leq n$...
and (4.31),
\[
|B_1(s, t)| := \left| \sum_{m=1}^{M} \sum_{\xi, \eta \in G} \kappa_{\xi}^{m}(s) a(\xi, \eta) \exp \left( - \sum_{\zeta \in G} \alpha_{\zeta}(s) \bar{\xi}(t) \right) \left( x(t)^{\kappa(\cdot)} - e_{\eta}^{m} - x(t)^{\kappa(\cdot)} \right) \right|
\]
(4.36)
\[
\leq \sum_{m=1}^{M} \sum_{\xi, \eta \in G} \kappa_{\xi}^{m}(s) a(\xi, \eta) \left| x(t)^{\kappa(\cdot)} - e_{\eta}^{m} - x(t)^{\kappa(\cdot)} \right|
\]
\[
\leq 2n \sum_{\xi \in \bar{A}_{\kappa}(T)} \sum_{\eta \in G} a(\xi, \eta) \left( 1 + \sum_{\zeta \in \bar{A}_{\kappa}(T) \cup \{\eta\}} \bar{\xi}(t)^{\eta} \right)
\]
which depends on \((\alpha, \kappa)\) only through \(\bar{A}_{\kappa}(T)\). Hence, by (4.30) and (4.36) and conditioning on \(F_T^{\kappa}\),
\[
\sup_{r, s, t \leq T} E \left[ \left( |\beta(\alpha(r), \kappa(r))| + 1 \right) |B_1(s, t)| \right] \leq C \sup_{r, t \leq T} \left[ E \left[ 1 + \sum_{\zeta \in G} \alpha_{\zeta}(r) \right] |F_T^{\kappa}| \cdot E \left[ \sum_{\xi \in \bar{A}_{\kappa}(T)} \sum_{\eta \in G} a(\xi, \eta) \left( 1 + \sum_{\zeta \in \bar{A}_{\kappa}(T) \cup \{\eta\}} \bar{\xi}(t)^{\eta} \right) \right] \right]
\]
(4.37)
\[
\leq C \sup_{t \leq T} \left[ \sum_{\xi \in \bar{A}_{\kappa}(T)} \sum_{\eta \in G} a(\xi, \eta) E \left[ 1 + \sum_{\zeta \in \bar{A}_{\kappa}(T) \cup \{\eta\}} \bar{\xi}(t)^{\eta} \right] \right] \leq C E \left[ \sum_{\xi \in \bar{A}_{\kappa}(T)} \sum_{\eta \in G} a(\xi, \eta) \left( 1 + |\bar{A}_{\kappa}(T)| \right) \right] \leq C E \left[ (1 + |\bar{A}_{\kappa}(T)|)^2 \right] < \infty,
\]
where we have used (4.35) and (4.32) in the second and third inequality respectively. Finally we used again that \(|\bar{A}_{\kappa}(T)|\) can be bounded by a Poisson random variable with parameter \(nT\).

For the second term we use (4.31) to obtain
\[
|B_2(s, t)| := \left| \gamma \sum_{m=1}^{M} \sum_{\xi \in G} \left( \kappa_{\xi}^{m}(s) / 2 \right) \exp \left( - \sum_{\zeta \in G} \alpha_{\zeta}(s) \bar{\xi}(t) \right) \left( x(t)^{\kappa(\cdot)} - e_{\eta}^{m} - x(t)^{\kappa(\cdot)} \right) \right|
\]
(4.38)
\[
\leq 2\gamma n^2 \left( 1 + \sum_{\xi \in \bar{A}_{\kappa}(T)} \bar{\xi}(t)^{\eta} \right),
\]
Hence by (4.30) and (4.38), we obtain
\[
\sup_{r, s, t \leq T} E \left[ \left( |\beta(\alpha(r), \kappa(r))| + 1 \right) |B_2(s, t)| \right] \leq C \sup_{r, s, t \leq T} \left[ E \left[ 1 + \sum_{\zeta \in G} \alpha_{\zeta}(r) \right] |F_T^{\kappa}| \cdot E \left[ 1 + \sum_{\xi \in \bar{A}_{\kappa}(T)} \bar{\xi}(t)^{\eta} \right] \right]
\]
(4.39)
\[
\leq C \sup_{r \leq T} E \left[ 1 + \bar{\alpha}(r) \right] \cdot E \left[ 1 + \bar{A}_{\kappa}(T) \right] < \infty,
\]
where we have used (4.32) and (4.35).

For the third term we bound

$$|B_3(s, t)| := \left| \sum_{\xi, \eta \in \mathcal{G}} \tilde{a}(\xi, \eta)(\alpha_\eta(s) - \alpha_\xi(s)) \exp \left( -\sum_{\zeta \in \mathcal{G}} \alpha_\zeta(s) \bar{x}_\zeta(t) \right) \right| \left( -\bar{x}_\xi(t) x(t)^{\alpha(s)} \right) \tag{4.40}$$

$$\leq \sum_{\xi, \eta \in \mathcal{G}} \tilde{a}(\xi, \eta)(\alpha_\eta(s) + \alpha_\xi(s)) \bar{x}_\xi(t) x(t)^{\alpha(s)}$$

By conditioning on $\mathcal{F}_T^r$ and using (4.30) as well as (4.40) we obtain

$$\sup_{r, s, t \leq T} E \left[ \left| \beta(\alpha(r), \alpha(r)) \right| + 1 \right] \left| B_3(s, t) \right|$$

$$\leq C \sup_{r, s, t \leq T} E \left[ \left( 1 + \sum_{\xi \in \mathcal{G}} \alpha_\xi(r) \right) \left( \sum_{\xi, \eta \in \mathcal{G}} \tilde{a}(\xi, \eta)(\alpha_\eta(s) + \alpha_\xi(s)) \bar{x}_\xi(t) x(t)^{\alpha(s)} \right) \right]$$

$$\leq C \sup_{r, s \leq T} E \left[ \left( 1 + \sum_{\xi \in \mathcal{G}} \alpha_\xi(r) \right) \left( \sum_{\xi, \eta \in \mathcal{G}} \tilde{a}(\xi, \eta)(\alpha_\eta(s) + \alpha_\xi(s)) \right) \left| \mathcal{F}_T^r \right| \right]$$

$$\leq C E \left[ \sup_{r, s \leq T} \sum_{\xi, \eta \in \mathcal{G}} \tilde{a}(\xi, \eta) \cdot E \left[ \left( 1 + \sum_{\xi \in \mathcal{G}} \alpha_\xi(r) \right) (\alpha_\eta(s) + \alpha_\xi(s)) \left| \mathcal{F}_T^r \right| \right] \right] \cdot \left( 1 + \tilde{A}_\kappa(T) \right),$$

where we have used (4.32) in the last inequality. But we may bound

$$\sup_{r, s \leq T} \sum_{\xi, \eta \in \mathcal{G}} \tilde{a}(\xi, \eta) E \left[ \left( 1 + \sum_{\xi \in \mathcal{G}} \alpha_\xi(r) \right) \left( \alpha_\eta(s) + \alpha_\xi(s) \right) \left| \mathcal{F}_T^r \right| \right]$$

$$\leq 2 \sup_{r, s \leq T} E \left[ \sum_{\xi \in \mathcal{G}} \alpha_\xi(s) + \sum_{\xi, \eta \in \mathcal{G}} \alpha_\xi(r) \alpha_\xi(s) \left| \mathcal{F}_T^r \right| \right]$$

$$\leq 4 \sup_{s \leq T} E \left[ \sum_{\xi \in \mathcal{G}} \alpha_\xi(s) + \left( \sum_{\xi \in \mathcal{G}} \alpha_\xi(s) \right)^2 \left| \mathcal{F}_T^r \right| \right] \leq 4 \sup_{s \leq T} E \left[ \tilde{a}(s) + \tilde{a}(s)^2 \right] < \infty$$

by (4.33) provided that $E[\left( \sum_{\xi \in \mathcal{G}} \alpha_\xi(0) \right)^2] < \infty$. Hence, the expression in (4.41) is finite as well.
For the fourth term we use that $x \exp(-x) \leq C$ for $x \geq 0$ and obtain

$$
|B_4(s, t)| := \Bigg| \gamma \sum_{\xi \in G} \alpha_\xi(s) \left(K - \frac{1}{2} \alpha_\xi(s)\right) \exp \left(- \sum_{\xi \in G} \alpha_\xi(s) \bar{\xi}(t)\right) \left(-\bar{\xi}(t) t \kappa(s)\right) \Bigg|
$$

$$
\leq \gamma K \sum_{\xi \in G} \alpha_\xi(s) \bar{\xi}(t) \exp \left(- \sum_{\xi \in G} \alpha_\xi(s) \bar{\xi}(t)\right) t \kappa(s)
$$

$$
+ \frac{1}{2} \gamma \sum_{\xi \in G} \alpha_\xi(s)^2 \bar{\xi}(t) t \kappa(s)
$$

$$
\leq C \left( t \kappa(s) + \sum_{\xi \in G} \alpha_\xi(s)^2 \bar{\xi}(t) t \kappa(s)\right)
$$

(4.43)

Therefore, by (4.30) and (4.43)

$$
\sup_{r, s, t \leq T} E \left[ \left( |\beta(\alpha(r), \kappa(r))| + 1 \right) |B_4(s, t)| \right]
$$

$$
\leq C \sup_{r, s, t \leq T} E \left[ \left( 1 + \sum_{\xi \in G} \alpha_\xi(r)\right) \cdot \left( t \kappa(s) + \sum_{\xi \in G} \alpha_\xi(s)^2 \bar{\xi}(t) t \kappa(s)\right) \right]
$$

(4.44)

$$
\leq C \left( 1 + \sup_{r, s, t \leq T} \sum_{\xi \in G} E \left[ \left( 1 + \sum_{\xi \in G} \alpha_\xi(r)\right) \alpha_\xi(s)^2 \bar{\xi}(t) t \kappa(s) \right] \right)
$$

$$
\leq C \left( 1 + \sup_{s \leq T} E \left[ \sum_{\xi \in G} \left( 1 + \sum_{\xi \in G} \alpha_\xi(r)\right) \alpha_\xi(s)^2 \bar{\xi}(t) t \kappa(s) \right] \cdot \left( 1 + \bar{A}_\kappa(T)\right) \right)
$$

$$
\leq C \left( 1 + 2 \sup_{s \leq T} E \left[ \alpha(s)^2 + \bar{\alpha}(s)^3 \right] \cdot \left( 1 + \bar{A}_\kappa(T)\right) \right)
$$

where we have used (4.32) and (4.33) in the second inequality, then (4.32) again in the third inequality. The last calculation is similar to that in (4.42). Hence, the term in (4.44) is finite provided that $E\left[(\sum_{\xi \in G} \alpha_\xi(0))^3\right] < \infty$.

For the fifth term we need to bound

$$
|B_5(s, t)| := \left| \gamma \lambda \sum_{\xi \in G} \alpha_\xi(s) \exp \left(- \sum_{\xi \in G} \alpha_\xi(s) \bar{\xi}(t)\right) \left(\bar{\xi}(t)\right)^2 t \kappa(s) \right|
$$

(4.45)

$$
\leq C \sum_{\xi \in G} \alpha_\xi(s) \left(\bar{\xi}(t)\right)^2 t \kappa(s)
$$
Therefore, again by (4.30), (4.32), and (4.35)

\[
\sup_{r,s,t \leq T} E \left[ \left| \beta(\alpha(r), \kappa(r)) \right| + 1 \right] |B_5(s, t)| 
\leq C \sup_{r,s,t \leq T} E \left[ \left( \sum_{\xi \in G} \alpha_\xi(r) \cdot \left( \sum_{\xi \in G} \alpha_\xi(s) \bar{\xi}(t) x(t)^{\kappa(s)} \right) \right) \right] 
\leq C \left[ \sup_{r,s,t \leq T} E \left[ \left( \sum_{\xi \in G} \alpha_\xi(s) \bar{\xi}(t) x(t)^{\kappa(s)} \right) \right] \right] \cdot E \left[ \bar{\xi}(t)^2 x(t)^{\kappa(s)} \right] 
\leq C \sup_{s \leq T} E \left[ \hat{\alpha}(s) + \hat{\alpha}(s)^2 \right] \cdot E \left[ 1 + \hat{A}_\kappa(T) \right] < \infty.
\]

For the sixth term we need to bound

\[
|B_6(s, t)| := \left| \gamma A \sum_{\xi \in G} \bar{\xi}(s) \exp \left(- \sum_{\xi \in G} \alpha_\xi(s) \bar{\xi}(t) \right) \left( - \bar{\xi}(t) x(t)^{\kappa(s)} \right) \right| 
\leq C \sum_{\xi \in G} \bar{\xi}(s) \bar{\xi}(t) x(t)^{\kappa(s)}
\]

Hence, by (4.30), (4.32), and (4.35) we obtain that

\[
\sup_{r,s,t \leq T} E \left[ \left| \beta(\alpha(r), \kappa(r)) \right| + 1 \right] |B_6(s, t)| 
\leq C \sup_{r,s,t \leq T} E \left[ \left( \sum_{\xi \in G} \alpha_\xi(s) \bar{\xi}(t) x(t)^{\kappa(s)} \right) \right] 
\leq C \left[ \sup_{r,s,t \leq T} E \left[ \left( \sum_{\xi \in G} \alpha_\xi(s) \bar{\xi}(t) x(t)^{\kappa(s)} \right) \right] \right] \cdot E \left[ \bar{\xi}(t)^2 x(t)^{\kappa(s)} \right] 
\leq C \sup_{s \leq T} E \left[ \hat{\alpha}(s) \sup_{r \leq T} E \left[ 1 + \hat{\alpha}(r) \right] \cdot \left( 1 + \hat{A}_\kappa(T) \right) \right] 
\leq C n \sup_{r \leq T} E \left[ 1 + \hat{\alpha}(r) \right] \cdot E \left[ 1 + \hat{A}_\kappa(T) \right] < \infty.
\]

This completes the proof of (4.21) and so establishes the duality. \(\square\)

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