SM gauge fields localized on non-Abelian vortices in 6 dimensions

Masato Arai\textsuperscript{a}, Filip Blaschke\textsuperscript{bc}, Minoru Eto\textsuperscript{de}, Masaki Kawaguchi\textsuperscript{d}, Norisuke Sakai\textsuperscript{e}

\textsuperscript{a}Faculty of Science, Yamagata University, Kojirakawa-machi 1-4-12, Yamagata, Yamagata 990-8560, Japan
\textsuperscript{b}Research Centre for Theoretical Physics and Astrophysics, Institute of Physics, Silesian University in Opava, Bezručovo nám. 1150/13, CZ-746 01 Opava, Czech Republic
\textsuperscript{c}Institute of Experimental and Applied Physics, Czech Technical University in Prague, Husova 240/5, 110 00 Prague 1, Czech Republic
\textsuperscript{d}Department of Physics, Yamagata University, Kojirakawa-machi 1-4-12, Yamagata, Yamagata 990-8560, Japan
\textsuperscript{e}Department of Physics, and Research and Education Center for Natural Sciences, Keio University, 4-1-1 Hiyoshi, Yokohama, Kanagawa 223-8521, Japan

E-mail: arai(at)sci.kj.yamagata-u.ac.jp, filip.blaschke(at)physics.slu.cz, meto(at)sci.kj.yamagata-u.ac.jp, ddwbb.daigaku(at)gmail.com, norisuke.sakai(at)gmail.com

ABSTRACT: A brane-world $SU(5)$ GUT model with global non-Abelian vortices is constructed in six-dimensional spacetime. We find a solution with a vortex associated to $SU(3)$ separated from another vortex associated to $SU(2)$. This $3-2$ split configuration achieves a geometric Higgs mechanism for $SU(5) \to SU(3) \times SU(2) \times U(1)$ symmetry breaking. A simple deformation potential induces a domain wall between non-Abelian vortices, leading to a linear confining potential. The confinement stabilizes the vortex separation moduli, and assures the vorticity of $SU(3)$ group and of $SU(2)$ group to be identical. This dictates the equality of the numbers of fermion zero modes in the fundamental representation of $SU(3)$ (quarks) and of $SU(2)$ (leptons), leading to quark-lepton generations. The standard model massless gauge fields are localized on the non-Abelian vortices thanks to a field-dependent gauge kinetic function. We perform fluctuation analysis with an appropriate gauge fixing and obtain a four-dimensional effective Lagrangian of unbroken and broken gauge fields at quadratic order. We find that $SU(3) \times SU(2) \times U(1)$ gauge fields are localized on the vortices and exactly massless. Complications in analyzing the spectra of gauge fields with the nontrivial gauge kinetic function are neatly worked out by a vector-analysis like method.
1 Introduction

One of the most interesting paradigms of unified theories beyond the standard model is the brane-world scenario in which we live on a brane in higher dimensional spacetime [1–3]. As candidates for the brane, topological solitons have a number of attractive features in contrast with a thin delta-function like branes, which may be regarded as somewhat artificial idealizations. Traditional reasons repeatedly emphasized in the literature are:

1. The topological solitons are dynamically generated as a consequence of spontaneously broken symmetries. This phenomenon is referred to as dynamical compactification [4].

2. They trap chiral fermions on their world volumes which are topological states and therefore inevitably appear irrespective of any details [5, 6].

The models using the topological solitons are often called fat brane-world models, and enjoy an additional merit, which is common to brane-world scenario. The hierarchy problem of fermion masses can be naturally explained as a consequence of overlapping of wave functions of the localized fermions and the Higgs field [7].

In order to realize the standard model (SM) on a brane, a serious obstacle has been localization of massless gauge fields on the brane. Suppose the gauge symmetry $H$ is broken in the bulk (Higgs phase) and is restored only in the vicinity of the topological soliton. One might expect that massless $H$ gauge bosons will appear inside the soliton. However, this is
not true, and in general, they get masses of the order of the inverse width of the topological solitons [8]. This is because the bulk except for the soliton core is a sort of non-Abelian superconductor. Therefore, though the gauge symmetry is restored in the soliton, electric fluxes from a probe electric charge put inside the soliton are immediately absorbed into the superconducting bulk. Hence, the gauge fields can only propagate for a distance about the width of the soliton, namely they are massive [2, 8]. This argument is quite reasonable, and therefore localizing massless gauge bosons on the soliton generally seem to be quite difficult.

The key idea getting over the difficulty was first proposed by Ref. [8]. It employs a sort of electromagnetic duality. If we put a probe magnetic charge inside the soliton in the superconducting bulk, the magnetic flux is entirely repelled from the bulk by the Meissner effect, and because of conservation of the flux, the field lines extend to infinity along the soliton [2, 8]. In Ref. [8] a dual picture of this, namely replacing the Higgs phase by a confining phase under the assumption that magnetic charges are condensed in the bulk, was proposed: A massless \( H \) gauge field is localized inside the soliton if the \( H \) gauge group is unbroken inside the soliton and is enhanced to a large non-Abelian group \( G \) in the confining bulk. This mechanism, the so-called Dvali-Shifman (DS) mechanism, was rigorously proven in the superYang-Mills theory in four dimensions in Ref. [8], but the higher dimensional version has not yet successfully been proven because we do not know how confinement works in higher dimensions. Then, a lot of papers followed [8] and investigated fat brane scenarios but most of them needed to assume the DS mechanism to work.

Ref. [2] is one of the earliest works which proposes a simple phenomenological model for the DS mechanism. There, the Abrikosov-Nilsen-Olsen vortex is considered as a fat 3-brane in six dimensions. In addition to the SM fields, the model includes an extra scalar field \( T \) which condenses only inside the vortex core. Furthermore, the model has a phenomenological dress factor to the gauge kinetic term of the form \( \Lambda^{-2} \text{Tr} T^2 G_{\mu\nu} G^{\mu\nu} \) where \( \Lambda \) is a cut-off scale. Then it was assumed that the model meets the following three conditions: 1) Outside the vortex the SM gauge group \( H \) is extended into a larger non-Abelian gauge group \( G \) (Note that \( H \) is not broken everywhere, and it is included as a part of the unbroken large group \( G \) in the bulk). 2) There is no light matter in the bulk. 3) The tree-level gauge coupling (corresponding to the factor \( T^2 \) in the above example) becomes large away from the vortex core. Ref. [2] proposed that when these conditions are satisfied the localization of massless \( H \) gauge fields on the vortex takes place.

We should note that, although the seminal work of Ref. [8] gave the basic idea for the localization of massless gauge fields on fat branes, the detailed analysis on how to get the physical mass spectrum on vortices in six dimensions has not yet been given in the literature. The purpose of this paper is to provide concrete phenomenological models for the fat brane-world scenario using topological vortices in six dimensions along the line of Ref. [2] and to give a systematic analysis on the physical mass spectra including not only massless but also massive modes.

A classical realization of a confining vacuum in the bulk can be given in terms of generic nonlinear kinetic term, namely a field-dependent gauge kinetic function [10–17] of
the form

$$-\frac{B^2}{4} F_{MN} F^{MN}, \quad (1.1)$$

where $M, N$ are spacetime indices. If $B$ is a constant, then this is a usual minimal gauge kinetic term with a coupling constant $1/B$, but we allow $B$ to be a function of scalar fields, such as $B(T) = T$ in the above example. The scalar fields are not necessarily constant in the extra dimensions, but can be a non-trivial function of coordinates of extra dimensions as a consequence of dynamics of the system under consideration. If this is the case, the inverse effective gauge coupling $B$ is no longer constant and depends on extra-dimensional coordinates nontrivially. With a series of our previous works [11–17], we have established a general criterion to obtain massless gauge fields on the topological solitons: if $B^2$ is square integrable with respect to the integration over the whole extra dimensions, the massless four-dimensional gauge fields are localized on the topological soliton. This corresponds to the third criterion 3) of Ref. [2] mentioned above. Since the square-integrability does not depend on details of the model, this localization mechanism is robust. We verified this statement in various concrete models in five dimensional spacetime [11–15], and also gave a formal proof in generic spacetime dimensions [16, 17]. With the nontrivial kinetic function, we also found an interesting localization mechanism of massless scalar fields [18] on domain walls similarly to Eq. (1.1) for gauge fields. The mechanism is found to have a topological nature similarly to the Jackiw-Rebbi's topological mechanism [5] for localization of massless fermions on domain walls. The fat-brane scenario has been much discussed mostly with other contexts, and has produced various different models with their own advantages/disadvantages [19–41].

Since models in five dimensional spacetime is simplest to analyze, there have been many concrete brane-world models using domain walls to obtain the symmetry breaking of grand unified theories (GUT) down to the SM, such as in [42–48], without an explicit mechanism to localize the gauge fields. With our mechanism for localization of gauge fields, we have constructed a concrete model for the $SU(5)$ GUT on domain walls [13]. The GUT symmetry breaking is determined by positions of domain walls. This geometric Higgs mechanism is a characteristic feature of the brane-world model with topological solitons. By introducing a moduli stabilization potential we obtained the $SU(5) \to SU(3) \times SU(2) \times U(1)$ leading to the SM. As an alternative possibility, we have also obtained a five-dimensional model for the SM, where the condensation of the SM Higgs field $\Phi$ is driven by the formation of domain walls which localizes the SM. It predicts a new contribution to the $\Phi \to \gamma \gamma$ decay and the possible experimental accessibility of heavy monopoles [15].

To explain quark lepton generations naturally, however, we need an additional idea in the brane-world models with topological solitons. One such mechanism is to use vortices in the extra dimensions. If there is a vortex with $k$ vorticity ($k$ coincident vortices), the index theorem gives $k$ zero modes for fermions. This mechanism has been proposed previously to explain fermion generations in the brane-world scenario [49, 50], although the model was without the localization mechanism for gauge fields.

The purpose of this paper is to propose a class of models in six-dimensional spacetime which is a non-Abelian generalization of the simple model considered in Ref. [2]. Our model
is based on a GUT-inspired $G = SU(5)$ gauge theory with a field-dependent gauge kinetic function in six spacetime dimensions. It contains complex scalar fields in a singlet and an adjoint representation of $SU(5)$, which are combined into a $5 \times 5$ matrix-valued field $T$. The model admits topologically stable non-Abelian global vortices. We find that the massless gauge fields of $H = SU(3) \times SU(2) \times U(1)$ are localized on the four-dimensional world volume of the non-Abelian global vortices with multiple winding numbers.\footnote{An application of the non-Abelian local vortices to the brane-world physics without localization of the gauge fields was considered in Ref. [51].}

By a simple potential with the $5 \times 5$ matrix field $T$ we can obtain vortex equations for each diagonal components $T = \text{diag}(T_1, T_2, T_3, T_4, T_5)$ independent of each other. Then we obtain five different species of vortices $I = 1, \cdots, 5$ corresponding to vortices in the $I$-th diagonal component $T_I$. We find two important phenomena due to the vortices:

1. Geometric Higgs mechanism

The gauge symmetry $G = SU(5)$ is partially broken when the topological vortices are present. Interestingly, the breaking pattern of the $SU(5)$ gauge symmetry depends on positions of vortices. Namely, dynamics of the non-Abelian vortices determines which subgroup of $SU(5)$ remains unbroken. We are primarily interested in the configuration where vortices in the three diagonal components, say $T_1, T_2, T_3$, of the matrix field $T$ are coincident at a point in the extra dimensional plane, whereas vortices in the remaining two diagonal components $T_4, T_5$ are coincident at another point. This provides a symmetry breaking pattern $SU(5) \to SU(3) \times SU(2) \times U(1)$. It is important to notice that the origin of the symmetry breaking resides only locally near the non-Abelian vortices. This is the reason why the SM gauge fields are localized around the vortices. We call this vortex solution as the $3 \to 2$ splitting configuration.

Note that the first criterion 1) of Ref. [2] mentioned above is naturally satisfied by the generation of the non-Abelian vortices.

2. Confinement of non-Abelian global vortices

The separation between the position of one group of vortices in $T_1, T_2, T_3$ and position of another group in $T_4, T_5$ is a moduli that depends on the details of the scalar potential, namely on a particular ratio of potentials for singlet and adjoint in parts of $T$. If we perturb this ratio, we find that separation is no longer a moduli. Namely a potential energy is induced and becomes constant for asymptotically large separations. Thus a confining force emerges between non-Abelian global vortices. The confining force can only end at another non-Abelian global vortex. This process continues until they finally form an $SU(5)$ singlet combination of non-Abelian vortices. Namely only when each diagonal component has the same number of vortices, the configuration becomes stable. Any vortices which cannot form a singlet combination are dynamically removed to spatial infinity by the confining energy density (a domain wall) extending to infinity. This is important once fermions couple to the vortices, because the number of fermion zero modes is identical to the winding number. Hence, the confinement of non-Abelian global vortices ensures that the number of fermion
zero modes is common for different representations of the SM, namely leptons and quarks should come in generations.

We also obtain a low energy effective theory on the background of the $3 - 2$ splitting configuration with a particular focus on the problem of localization of the unbroken $SU(3) \times SU(2) \times U(1)$ gauge fields. Deriving the effective Lagrangian can be quite complicated even at the quadratic order of small fluctuations, because of a number of reasons.

1. The background solution is non-trivial, namely non-Abelian vortices.
2. The gauge kinetic Lagrangian in Eq. (1.1) is not in the canonical form.
3. Apart from $SU(3) \times SU(2) \times U(1)$ four-dimensional vector fields, we have gauge fields corresponding to the broken generators of $SU(5)$. The extra-dimensional components of gauge fields are also present, and mix among themselves and with the matrix-valued complex scalar fields.
4. We have to take care of these issues by choosing an appropriate gauge fixing.

In order to organize the calculations, we develop an effective and a compact formula generalizing the usual three-dimensional vector analysis. It turns out that this approach is useful to clean up complicated calculations and make things transparent. Furthermore, our method allows us to treat both unbroken and broken parts in a very similar manner. Armed with our vector-analysis-like method, we show most importantly that the $3-2$ splitting configuration of non-Abelian global vortices localizes the massless degrees of freedom corresponding to the $SU(3) \times SU(2) \times U(1)$ SM gauge fields on the four-dimensional world-volume of the vortices. We also show that other fields, except for a mixing part of the extra-dimensional component of the broken gauge fields and the scalar fields, are either massive or unphysical in the sense that they are absorbed by massive Kaluza-Klein (KK) towers of the gauge fields. Hence, our $3-2$ splitting background configuration of the non-Abelian global vortices in six dimensions is a promising platform for a fat brane-world scenario with GUT.

This paper is organized as follows. In Sec. 2.1, we present our model admitting non-Abelian global vortices in six dimensions, and study multiple vortices, especially the $3-2$ splitting configuration that leads to the desired symmetry breaking $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$. In Sec. 2.2, the confinement of non-Abelian global vortices by domain walls are described. Sec. 3 is devoted to derive a four-dimensional low energy effective Lagrangian of the gauge fields. We develop the useful vector-analysis-like technique and examine which fields provide massless/massive, physical/unphysical, and normalizable/non-normalizable modes under the $3-2$ splitting background. We give proofs of some theorems resembling the Helmholtz’s theorem for our vector-analysis-like method in Appendix A.
2 A brane-world model with non-Abelian vortices

2.1 Non-Abelian global vortices for $SU(5) \to SU(3) \times SU(2) \times U(1)$

Let us consider an $SU(5)$ non-Abelian gauge theory in 6 dimensions

$$\mathcal{L} = \text{Tr} \left[ -\frac{B(T)B^\dagger(T)}{2} \mathcal{F}_{MN} \mathcal{F}^{MN} + D_M T (D^M T)^\dagger \right] - V, \quad (2.1)$$

where $M = 0, 1, \cdots, 5$ denotes spacetime indices, $\mathcal{F}_{MN} = \partial_M A_N - \partial_N A_M + i [A_M, A_N]$ is a field strength of $SU(5)$ gauge fields $A_M$. A $5 \times 5$ matrix-valued complex scalar field $T$ contains a singlet $T_0$ and an adjoint $\hat{T}$ representation of $SU(5)$

$$T_0 = \text{Tr} T, \quad \hat{T} = T - \frac{T_0}{5} 1_5, \quad (2.2)$$

with $\text{Tr} \hat{T} = 0$. Then a covariant derivative is defined by

$$D_M T = \partial_M T + i [A_M, T]. \quad (2.3)$$

The Lagrangian (2.1) has a peculiar factor $BB^\dagger$ in front of the gauge kinetic term $\mathcal{F}_{MN}^2$. If we take a constant $B = 1/g$, the Lagrangian is just standard one. Instead, we allow $B$ to be a generic function of $T$, and call it a gauge kinetic function. We require $B$ to be at least a Hermitian matrix and invariant under the $SU(5)$ gauge transformation. Otherwise, we leave it arbitrary for now, since it does not play any role in the rest of this section.\(^2\) It will play an important role in the next section, and we will clarify concrete conditions on $B$ for a physically meaningful brane-world GUT model, namely the conditions for having massless gauge bosons localized on non-Abelian vortices.\(^3\) In the next section we will conclude that $BB^\dagger$ has to asymptotically go to zero far away from the solitons, and it implies that the effective gauge coupling $g \sim 1/\sqrt{BB^\dagger}$ diverges. Therefore, the Lagrangian (2.1) is not suitable for describing physics in a homogeneous vacuum. Instead, we interpret it as a phenomenological model which is suitable to describe non-trivial confinement physics (the DS mechanism) under the presence of topological solitons in higher dimensions, as proposed in Ref. [2].

There also seems to be no a priori condition for the scalar potential $V$ of $T$ except for the gauge invariance and reality condition. Hence, it can be an arbitrary function of gauge invariant quantities such as $T_0^2, \text{Tr} \left[ \hat{T} \hat{T}^\dagger \right], \text{Tr} \left[ (\hat{T} \hat{T}^\dagger)^2 \right], T_0^2 \text{Tr} \left[ \hat{T}^2 \hat{T}^\dagger \right] + \text{h.c.}, T_0^2 \text{Tr} \left[ \hat{T}^2 \right] + \text{h.c.}$ and so on. Instead of surveying such vast possibilities, we concentrate on a simple potential and its deformations to obtain a platform for the brane-world and GUT

$$V = \frac{\lambda^2}{2} \text{Tr} \left[ (TT^\dagger - v^2 1_5)^2 \right]. \quad (2.4)$$

The vacuum configuration of $V$ up to symmetry transformation is obviously

$$T = v 1_5. \quad (2.5)$$

\(^2\) One can jump to Eq. (3.87) to see a concrete example for $B$.

\(^3\) The conditions for $B$ are given in Eqs. (3.2) and (3.53).
This is the $SU(5)$ preserving vacuum, but it is important to realize that the $U(1)$ global symmetry $T \rightarrow e^{i\alpha}T$ is spontaneously broken. Hence, the vacuum manifold is isomorphic to $S^1$ which is not simply connected space, and the fundamental homotopy group is nontrivial as $\pi_1(S^1) = \mathbb{Z}$. This gives rise to global vortices (three-branes in six-dimensional spacetime) which are topologically stable. The Euler-Lagrange equations can be solved by consistently setting all the off-diagonal components of $T$ and gauge fields to vanish. This leads to five decoupled Euler-Lagrange equations for the diagonal elements of $T = \text{diag}(T_1, T_2, T_3, T_4, T_5)$.

Assuming $T_I$ depends only on extra-dimensional coordinates $x^a, a = 4, 5$, we obtain

$$\partial_a^2 T_I - \lambda^2 \left( |T_I|^2 - v^2 \right) T_I = 0, \quad I = 1, \cdots, 5 \tag{2.6}$$

Each equation is identical to the familiar equation for a global vortex, which can be derived from a one scalar model

$$\mathcal{L}' = |\partial_M \phi|^2 - \frac{\lambda^2}{2} \left( |\phi|^2 - v^2 \right)^2, \tag{2.7}$$

if we identify $\phi$ with $T_I$.

To obtain the $k_I \in \mathbb{Z}$ coincident vortices at the origin for the $I$-th diagonal field $T_I$, we can solve the angular part of the equation by expressing $T_I$ as

$$T_I = v f_I(r) e^{ik_I \theta}, \tag{2.8}$$

in terms of the polar coordinates $x_4 + ix_5 = re^{i\theta}$. The remaining radial equation is given as

$$f_I'' + \frac{f_I'}{r} - \frac{k_I^2}{r^2} f_I - \lambda^2 v^2 (f_I^2 - 1) f_I = 0. \tag{2.9}$$

This should be solved under the following boundary condition

$$f_I(0) = 0, \quad f_I(\infty) = 1. \tag{2.10}$$

Although these vortex solutions are mere five copies of the standard global vortex solutions embedded into the diagonal components of the matrix field $T$, they actually have a characteristic property of non-Abelian vortices. For example, take the simplest case with $k_1 = 1$ and all the others zeros ($k_2, k_3, k_4, k_5 = 0$). In terms of the matrix field $T$, the vortex solution is

$$T = v \text{diag}(f_1 e^{i\theta}, 1, 1, 1, 1) = e^{i\frac{\theta}{2}} v \text{diag} \left( f_1 e^{i\frac{\theta}{2}}, e^{-i\frac{\theta}{2}}, e^{-i\frac{\theta}{2}}, e^{-i\frac{\theta}{2}}, e^{-i\frac{\theta}{2}} \right). \tag{2.11}$$

At spatial infinity ($f_I \rightarrow 1$) we have

$$T \big|_{r \rightarrow \infty} = ve^{i\frac{\theta}{2}} e^{i\frac{\theta}{2} \lambda_{14}}, \tag{2.12}$$

with an $SU(5) \text{ generator } \lambda_{14} = \text{diag} \left( 1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3} \right)$. This decomposition shows that the winding number in the overall $U(1)$ group is $\frac{1}{5}$ when we go around the vortex once. In
this sense, the vortex is called $\frac{1}{5}$ fractionally quantized global vortex. Let us turn our eyes on the opposite limit $r = 0$. We have

$$T\rvert_{r \to 0} = v \text{diag}(0,1,1,1,1). \quad (2.13)$$

Namely symmetry breaking $SU(5) \to SU(4) \times U(1)$ occurs at the very center of the vortex. This implies that the vortices transforms nontrivially under the non-Abelian global $SU(5)$ transformations, and such vortices are called non-Abelian vortices. In summary, the vortex solution in this model is the $\frac{1}{5}$ fractionally quantized non-Abelian global vortex. Note that the $SU(5)$ gauge fields do not play any role at all, and therefore the above vortex solutions are called non-Abelian global vortices as found in models without gauge fields [52–56].

Since the equations in Eq. (2.6) for five diagonal fields $T_I$ are decoupled, we can freely choose positions of the vortices in each diagonal component $T_I$. Namely, they are moduli of the solutions. Let us consider a solution where the first three diagonal components, say $T_1, T_2, T_3$, have a common number of vortices at a single position, whereas the remaining two diagonal components ($T_4, T_5$) have also the same number of the vortices at another position

$$T = v \begin{pmatrix} f_3(r_3) e^{i k_3 \theta_3} & 0 \\ 0 & f_2(r_2) e^{i k_2 \theta_2} \end{pmatrix}, \quad (2.14)$$

where $(r_3, \theta_3)$ and $(r_2, \theta_2)$ are the polar coordinates whose origins are at the respective vortex centers. We have $T \to v \text{diag}(0,0,0,e^{i \theta_2},e^{i \theta_2})$ at $r_3 \to 0$, and $T \to v \text{diag}(e^{i \theta_3}, e^{i \theta_3}, e^{i \theta_3}, 0, 0)$ at $r_2 \to 0$. Since vortex positions of $T_1, T_2, T_3$ and that of $T_4, T_5$ are distinct, the solution breaks $SU(5)$ down to $SU(3) \times SU(2) \times U(1)$. We can identify this gauge symmetry breaking as the breaking of GUT to the SM. We should emphasize that this symmetry breaking occurs only locally in the vicinity of the vortex centers. This fact is important for having massless gauge fields localized on the vortices as we will see later.

We note that tensions of the non-Abelian global vortices in our model are logarithmically divergent similarly to usual Abelian global vortices. However, it is not harmful for construction of the effective theory of four-dimensional fields as we will explain in the subsequent sections.

### 2.2 Moduli stabilization through confinement of vortices

Though we are satisfied with the vortex solution (2.14), of course it is not a perfect solution. Firstly, the vortex positions are moduli. Therefore, they in general scatter around, and then breaking pattern of the gauge symmetry becomes $SU(5) \to U(1)^4$ which is not acceptable with respect to the brane world model. Secondly, the vortex numbers $k_a$ are arbitrary. Indeed, the vortex number is related to the number of generation of fermions coupled to the $T$ field. Therefore, we would like to have $k_1 = k_2 = \cdots = k_5$ at least, and, if possible, we further want $k_a = 3$ for all $a$. Unfortunately we cannot solve for all these conditions at once, but we can at least solve them partially.

Let us begin with extracting quadratic terms from the potential (2.4)

$$V \ni -\lambda^2 v^2 \text{Tr} [T T^I] = -\lambda v^2 \left( \frac{|T_0|^2}{5} + \text{Tr} [\hat{T} T^I] \right). \quad (2.15)$$
The particular ratio $\frac{1}{5}$ between two coefficients of $|T_0|^2$ and $\text{Tr} \left[ \hat{T} \hat{T}^\dagger \right]$ is important for having the decoupled five equations (2.6). However, there is no a priori reason to choose this specific ratio from the symmetry ground. We can single out, for example, the traceless part and modify the potential adding

$$\delta V = \alpha^2 \text{Tr} \left[ \hat{T} \hat{T}^\dagger \right].$$

(2.16)

To see what happens by this additional term, let us plug the minimal vortex solution (2.11) with $k_1 = 1$ and $k_{2,3,4,5} = 0$ into the additional term. It reads

$$\delta V = \frac{22v^2\alpha^2}{25} \left( 1 + f_1^2 - 2f_1 \cos \theta \right) \to \frac{44v^2\alpha^2}{25} \left( 1 - \cos \theta \right),$$

(2.17)

as $r \to \infty$. This depends on the angular coordinate $\theta$ via $\cos \theta$. Therefore, when we traverse around the vortex, we inevitably across the potential barrier once. Namely, the minimal vortex is attached by the domain wall, as familiar axion cosmic strings. For the domain wall pulls the vortex towards the spatial infinity, we cannot retain the vortex. However, the domain wall can end on the different vortex, say, with $k_2 = 1$ and $k_{1,3,4,5} = 0$ at some other point. When we are sufficiently far from both of the vortices, we have $f_1 \sim f_2 \to 1$ and $\theta_1 \sim \theta_2 \equiv \theta$. Therefore, the additional potential asymptotically reduces to

$$\delta V \to \frac{12v^2\alpha^2}{5} (1 - \cos \theta).$$

(2.18)

There is still the domain wall. To eliminate the domain wall completely, we need the same number of the vortices in all the diagonal components. For example, when $k_{1,2,...,5} = 1$, we have $\hat{T} \to 0$ and

$$\delta V \to 0.$$

(2.19)

Now, the $\theta$ dependence asymptotically disappears. This can be understood as follows. The domain walls may exist but they are completely terminated by the five vortices. In other words, the vortices are confined to form the singlet (a set of five different vortices) by the linear force of domain wall.\textsuperscript{4} Note that the domain wall appears by adding $|T_0|^2$ instead of $\text{Tr} \left[ \hat{T} \hat{T}^\dagger \right]$ in Eq. (2.16). Only when they appear together with a particular ratio so that they are unified as $\text{Tr} \left[ T T^\dagger \right]$, no domain walls appear even in the single vortex with $k_1 = 1$ with $k_{2,3,4,5} = 0$. Namely, the vortices are deconfined when the scalar potential can be described by $T$ only (without $T_0$ and/or $\hat{T}$) as Eq. (2.4).

The vortex confinement is definitely an important piece of solving our problems described at the beginning of this subsection. Firstly, it provides the attractive confining force among the asymptotically separated vortices. This lifts the position moduli. Secondly, the domain walls discard unconfined constituent vortices to the spatial infinity. Namely, the vortex winding numbers $k_1$, $k_2$, $\cdots$, $k_5$ are automatically adjusted to a same integer. Hence, this ensures unification of generations of the localized fermions.

\textsuperscript{4} This singlet is a sort of baryonic type with $k_I > 0$ or ($k_I < 0$) for all $I$. We could have a mesonic singlet of $k_1 = 1$ and $k_1 = -1$ with $k_{2,3,4,5} = 0$. But it is topologically trivial and it would annihilate.
All is good news so far. However, there are still unclear points. a) we cannot explain the reason why the unified generation becomes 3. b) We cannot predict very well what kind of the singlet configuration remains under the presence of the confining force. If there is only attractive force by domain wall, all the vortices would completely coincide. This is not good for the brane-world scenario since $T \propto 1_5$ and $SU(5)$ is never broken. We can show this via a concrete numerical simulation. In Fig. 1 we show typical configurations including five non-Abelian vortices. We numerically constructed these by making use of a standard relaxation method [add a dissipation to the EOMs for the potential $V$ in (2.4) with $\delta V$ in (2.16)]. We first prepare an appropriate initial configuration with 5 vortices separately placed at vertices of a pentagon. Then, we run the relaxation simulation. We exhibit two snap shots at an early [(a) and (b) of Fig. 1] and a late [(c) and (d) of Fig. 1] stages. The panel (b) shows the amplitude $|\det T|$ for well separated five non-Abelian vortices. The panel (a) shows the additional potential density $\delta V$ which is nothing but the domain wall.

Figure 1: The two snap shots of the relaxation process starting with the five separate vortices. We take $\lambda_0 = 1$, $v = 1$, and $m^2 = 0.3$. The upper two panels are at the early stage and the lower two panels are at the late stage.
Figure 2: A 3-2 splitting of the five vortices. The vortex centers are on the $x^4$ axes and the cross section of the absolute values $|T_I|$ are shown. The parameters are taken as $\lambda = 15$, $\alpha = \frac{3}{2}\sqrt{2}$, and $v = 1$. The blue curves corresponds to $|T_{1,2,3}|$ and the red dashed lines to $|T_{4,5}|$. The blue dotted curve shows $T_{32} = \text{Tr} [T\lambda_{32}]$.

which is in charge of the confinement. The panels (c) and (d) show the same informations as (a) and (b) but at much later stage. The pentagon is completely squashed, and we are left with an integer quantized Abelian vortex. This occurs because there is only confining force among the non-Abelian vortices.

We should introduce a repulsive force which can compensate the domain wall force. A candidate is

$$\tilde{V} = \frac{\lambda^2}{2} \left( \text{Tr} \left[ T T^\dagger - v^2 1_5 \right] \right)^2 + \alpha^2 \text{Tr} \left[ \hat{T} T^\dagger \right].$$

(2.20)

When $\alpha^2 > 0$, the vacuum is again given by $T = v1_5$. Note that the first terms looks very similar to Eq. (2.4) but is different. We can decompose this

$$\tilde{V} = \frac{\hat{\lambda}^2}{2} \left( \frac{1}{5} |T_0|^2 - 5v^2 \right)^2 + \frac{\hat{\lambda}^2}{2} \left( \text{Tr} \left[ \hat{T} T^\dagger \right] \right)^2 + \hat{\lambda}^2 \left( \frac{1}{5} |T_0|^2 - 5v^2 + \frac{\alpha^2}{\hat{\lambda}^2} \right) \text{Tr} \left[ \hat{T} T^\dagger \right].$$

(2.21)

We should pay attention on the coefficient of $\text{Tr} \left[ \hat{T} T^\dagger \right]$ in the third term. It is positive ($\alpha^2 > 0$) in the bulk, but becomes negative in the vicinity of the vortex center since some of the diagonal component of $T \tilde{T}$ vanishes and $|T_0|^2$ is smaller than $25v^2$. When the sign of the coefficient flip to negative, then $\tilde{T}$ locally condense around the vortices. This leads to a short range repulsive force. Fig. 2 shows a numerical solution where the five vortices split into triply ($T_{1,2,3}$) and doubly ($T_{4,5}$) coincident parts at a finite distance. As can be clearly seen from the figure, the condensations of $T_{4,5}$ increase at the center of the triply coincident vortices whereas those of $T_{1,2,3}$ also increase near the doubly coincident vortices. This is a direct consequence of the condensation of $T_{32} \equiv \text{Tr} [T\lambda_{32}]$ with $\lambda_{32} = \text{diag} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right)$ which is also depicted by the black dotted line in Fig. 2. The blue (red) curves must touch zero at the center of the $T_{1,2,3}$ ($T_{4,5}$) vortex but at the same time it tends to increase
near the adjacent vortex cores $T_{4,5}$ ($T_{1,2,3}$). This opposite tendency conflicts to each other, and generates a desired short range inter-vortex repulsion. Similar mechanism for having vortex molecules plays an important role in a very different context, such as in coherently coupled multicomponent BECs of cold atoms [57–68], and also in the dense QCD [69].

A remark: Though we found one particular numerical solution with the desired 3-2 splitting structure, we have not performed any systematic survey on the parameter space. To figure out how common this configuration is is an important future work. Furthermore, we do not establish stability of our numerical solution. We also leave this as another future work. For this paper, we satisfy ourself by the fact that we succeeded in obtaining the 3-2 splitting solutions in the models with the potentials in Eqs. (2.4) and (2.20).

In the next section, we will present a formal analysis on localization of the $SU(5)$ gauge fields. For that purpose, any concrete solutions are not needed. So we will assume that the 3-2 splitting vortex configuration is obtained by wisely setting a model up somehow.

3 Localization of the gauge fields on vortices

3.1 Quadratic Lagrangian for gauge fields and gauge-fixing

In this section we study small fluctuations of the $SU(5)$ gauge fields $A_M$ in the presence of the non-Abelian global vortices, in order to clarify massless and massive modes. As mentioned above, we are primarily interested in the 3-2 splitting background in which the gauge symmetry is spontaneously broken as $SU(5)\rightarrow SU(3)\times SU(2)\times U(1)$. Our starting point is to divide the small fluctuations as

$$A_M = \begin{pmatrix} G_M & X_M/\sqrt{2} \\ X_M^\dagger/\sqrt{2} & W_M \end{pmatrix} + \mathcal{Y}_M \frac{1}{\sqrt{60}} \begin{pmatrix} 2I_3 \\ -3I_2 \end{pmatrix}$$

(3.1)

where $G_M$ is an $SU(3)$ gauge field, $W_M$ is an $SU(2)$ gauge field and $\mathcal{Y}_M$ is a $U(1)$ gauge field. The off-diagonal gauge field $X_M$ is a 2 by 3 rectangular complex matrix. Finding out physical spectra of these gauge fields $G_M$, $W_M$, $\mathcal{Y}_M$, and $X_M$ involves complicated calculation due to the following factors:

1. Our background is a nontrivial configuration of vortices.

2. We need a separate treatment for the gauge fields with the four-dimensional indices $A_\mu$ and with the extra-dimensional indices $A_a$.

3. We also have to distinguish the unbroken gauge fields $\{G_M, W_M, \mathcal{Y}_M\}$, and the broken gauge field $X_M$, which absorb the fluctuations from the scalar field $T$.

4. We have to clarify the distinction between the physical modes from the unphysical modes by taking account into gauge invariance.

Despite these obstacles, we will provide a reasonably simple scheme with which we handle complicated fluctuation analysis. We can also apply our scheme to a wide class of models, since it does not depend on details of the background.
Our main goal in this section is to examine whether massless gauge fields corresponding to the SM gauge group $SU(3) \times SU(2) \times U(1)$ are localized on the 3-2 splitting vortex background or not. The field dependent gauge kinetic function $B(T)$ in Eq. (2.1), which did not play any role in the previous section, will play a crucial role here. We assume $B$ is a function of $T$ and $T^\dagger$ (more generically a function of $T_0$, $T_0^\dagger$, and $\hat{T}$), although we do not fix a concrete $B$ for now. When the background solution $T$ is a diagonal matrix, we obtain a diagonal $5 \times 5$ matrix $B$:

$$T|_{bg} = \begin{pmatrix} \tau_3(x^a)1_3 \\ \tau_2(x^a)1_2 \end{pmatrix}, \quad B|_{bg} = \begin{pmatrix} \beta_3(\tau_3)1_3 \\ \beta_2(\tau_2)1_2 \end{pmatrix}. \quad (3.2)$$

For later convenience, let us define

$$\beta_1 = \sqrt{\frac{3|\beta_2|^2 + 2|\beta_3|^2}{5}}, \quad \beta_X = \sqrt{\frac{|\beta_2|^2 + |\beta_3|^2}{2}}, \quad \beta_\phi = \tau_3 - \tau_2. \quad (3.3)$$

Note that $\beta_3$, $\beta_2$, and $\beta_\phi$ are in general complex but $\beta_1$ and $\beta_X$ are real by definition.

In addition to the fluctuations of the gauge fields (3.1), we now introduce small fluctuations of the scalar field $T$. Since $T$ is a 5 by 5 complex matrix, there are 50 real fluctuations. We can separate them in three parts ($\Gamma$, $\Psi$, $\Phi$) as

$$T = e^{i\Phi} \left( \tilde{T} + \Gamma + \left[ \Psi, \tilde{T} \right] \right) e^{-i\Phi}$$

$$= \tilde{T} + \Gamma + \left[ \Psi + i\Phi, \tilde{T} \right] + \cdots, \quad (3.4)$$

with

$$\Gamma = \begin{pmatrix} \gamma_3 & 0 \\ 0 & \gamma_2 \end{pmatrix}, \quad \Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \psi \\ \psi^\dagger & 0 \end{pmatrix}, \quad \Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \varphi \\ \varphi^\dagger & 0 \end{pmatrix}. \quad (3.5)$$

Here $\gamma_3$ is a 3 by 3 complex matrix and $\gamma_2$ is a 2 by 2 complex matrix, whereas $\psi$ and $\varphi$ are 3 by 2 rectangular complex matrices. Real degrees of freedom included in $\gamma_3$, $\gamma_2$, $\psi$, and $\varphi$ are 18, 8, 12, and 12, respectively. Summing all them up, we have the correct real degrees of freedom, namely 50. By construction $e^{i\Phi}$ can be regarded as a gauge transformation with the broken generator by an amount $\varphi$. Hence the fluctuation field $\varphi$ contains the Nambu-Goldstone modes corresponding to the broken generators. In the following, we keep $\varphi$ and $\psi$ only, and ignore $\gamma_2$ since they are decoupled from the gauge sector or they have masses of the order of the GUT scale and they do not appear in the low energy effective Lagrangian of the gauge fields at the quadratic order.

Let us next substitute $A_M$ given in Eq. (3.1) into the first term in the square bracket in Eq. (2.1), and pick up only terms of the quadratic order. Those for the unbroken gauge

---

5 The constraint on $B$ for having localized massless non-Abelian gauge fields inside the vortices is given in Eq. (3.53), and a concrete example of $B$ is given in Eq. (3.87).
fields can be expressed as
\[
\mathcal{L}_\alpha = \text{Tr} \left[ A_\mu^\alpha \left\{ \beta_\alpha |^2 (\eta^{\rho\mu} \Box - \partial^\rho \partial^\mu) - \eta^{\rho\mu} \partial_\alpha |^2 \partial_\alpha \right\} A_\rho^\nu \right. \\
\left. - 2(\partial^\rho A_\mu^\alpha) \partial_\alpha (|^2 A_\rho^\alpha) \right. \\
\left. - A_\mu^\alpha \left( \beta_\alpha |^2 \delta_{ab} \Box - \delta_{ab} \partial_\alpha |^2 \partial_\alpha \right) \notag \right] ,
\]
which is valid regardless of details of the gauge kinetic function $B$, provided (3.2) holds. Here $\mu, \nu$ are the four-dimensional spacetime indices, $a, b, c$ are used for the extra-dimensional space indices, and $\alpha = 1, 2, 3$ are introduced to distinguish $SU(3), SU(2),$ and $U(1)$ as $A_M^{a=3} = G_M, A_M^{a=2} = W_M,$ and $A_M^{a=1} = Y_M$, respectively. Note that the trace in Eq. (3.6) for $A_M^{a=1} = Y_M$ should be understood as replacing Tr by $\frac{1}{2}$.

The quadratic Lagrangian for the broken gauge field $X_M$ can be also obtained in the same way
\[
\mathcal{L}_X = \text{Tr} \left[ A_\mu^\nu \left\{ \beta_X |^2 (\eta^{\rho\mu} \Box - \partial^\rho \partial^\mu) - \eta^{\rho\mu} \partial_\alpha |^2 \partial_\alpha \right\} X_\nu \right. \\
\left. - 2(\partial^\rho A_\mu^\nu) \partial_\alpha (|^2 X_\rho^\alpha) \right. \\
\left. - X_\mu^\nu \left( \beta_X |^2 \delta_{ab} \Box - \delta_{ab} \partial_\alpha |^2 \partial_\alpha \right) \notag \right] .
\]

Lastly we write down the scalar Lagrangian which can be obtained by substituting $T$ given in Eq. (3.4) into $K_\phi = \text{Tr} \left[ D_M T (D^M T)^\dagger \right]$ in Eq. (2.1) as
\[
K_\phi = \text{Tr} \left\{ \left[ A_M + \partial_M \Phi, \notag \bar{T} \right] \left[ A_M + \partial_M \Phi, \notag \bar{T} \right]^\dagger \right. \\
\left. - \left[ \Psi, \notag \bar{T} \right] \left( \Box - \partial^2 \right) \left[ \Psi, \notag \bar{T} \right]^\dagger + \left( 2i \left[ A_M + \partial_M \Phi, \notag \bar{T} \right] \left[ \partial^M \bar{T}, \Psi \right] + \text{h.c.} \right) \right\} \\
= \text{Tr} \left[ - \varphi^\dagger \left( |^2 \beta_\phi \Box - \partial_\alpha |^2 \partial_\alpha \right) \varphi + \notag \varphi^\dagger \partial_\alpha \left( |^2 X_\mu^\alpha \partial_\alpha \varphi \right) - |^2 \beta_\phi |^2 X_\mu^\alpha |^2 \partial_\alpha \right. \\
\left. - 2(\partial^\rho X_\mu^\nu) |^2 \beta_\phi |^2 \varphi + X_\mu^\nu \eta^{\rho\mu} |^2 \beta_\phi |^2 X_\nu - X_\mu^\nu |^2 \delta_{ab} |^2 \beta_\phi |^2 X_b \right. \\
\left. - \psi^\dagger \left( |^2 \beta_\phi \Box - \beta_\phi \partial_\alpha |^2 \partial_\alpha \right) \psi + \left( J_a (X_\alpha^\dagger + \partial_\alpha \psi) \right) \psi + \text{h.c.} \right] ,
\]
where we defined
\[
J_a = i \left\{ \beta_\phi^\dagger (\partial_\alpha \beta_\phi) - \beta_\phi (\partial_\alpha \beta_\phi^\dagger) \right\} .
\]
We retained $X_\mu, X_\alpha, \varphi$ and $\psi$, since they are mixed with gauge fields. However, we omitted $\Gamma$, since it decouples at the quadratic order. We also need to denote one more quadratic term from the potential
\[
V_\phi = U \text{Tr} \left[ \psi^\dagger \psi \right] ,
\]
where $U$ is a certain function of the background solution $\bar{T}$. Here, we also omitted $\Gamma$, since it decouples with $\psi$. Hence the quadratic Lagrangian is
\[
\mathcal{L}_\phi = K_\phi - V_\phi.
\]
Finally we introduce a gauge-fixing Lagrangian. For the unbroken generators, we introduce the following gauge fixing Lagrangian

\[ L^{\text{gf}}_\alpha = -\frac{|\beta_\alpha|^2}{\xi} \text{Tr} \left[ \left( \partial^\mu A^\mu_\alpha - \frac{\xi}{|\beta_\alpha|^2} \partial_a (\beta_\alpha^2 A_a^\alpha) \right)^2 \right]. \]  

(3.12)

As before \( \text{Tr} \) is understood to be replaced by \( \frac{1}{2} \) for \( \alpha = 1 \). For the broken generators, we introduce another gauge fixing term

\[ L^{\text{gf}}_{X\phi} = -\frac{\beta_\phi^2}{\xi} \text{Tr} \left[ \left( \partial^\mu X^\mu - \frac{\xi}{\beta_X^2} \left( \partial_a (\beta_X^2 X_a^\alpha) + 2|\beta_\phi|^2 \phi \right) \right)^2 \right]. \]  

(3.13)

Here \( \xi \) is an arbitrary gauge fixing constant similarly to the \( R_\xi \) gauge fixing condition.

3.2 Compact formulae for the unbroken gauge fields

3.2.1 Canonically normalized gauge fields

The above quadratic Lagrangian is complicated and far from the standard expression due to the extra \( \beta^2 \) factor. In order to bring it to a more familiar form, let us define

\[ A^\alpha_M \equiv |\beta_\alpha| A^\alpha_{4M}, \quad (\alpha = 1, 2, 3). \]  

(3.14)

Below we will also use the expression \( A^{\alpha-3}_M = G_M, A^{\alpha-2}_M = W_M, \) and \( A^{\alpha-1}_M = Y_M. \) In the following we need to deal with the extra-dimensional components of gauge field \( A^\alpha_a \) differently from the four-dimensional fields \( A^\alpha_\mu \) due to the fact that \( \beta_\alpha(x^a) \) depends not on \( x^\mu \) but on \( x^a \). The following vector notation turns out to be convenient for describing the low energy effective Lagrangian:

\[ \vec{A}^\alpha \equiv \left( \begin{array}{c} A^\alpha_4 \\ A^\alpha_5 \end{array} \right), \quad (\alpha = 1, 2, 3). \]  

(3.15)

3.2.2 Vector-analysis-like method

We now introduce differential operators useful to perform a vector-analysis-like method for analyzing mass spectra of gauge fields

\[ \vec{D}^\alpha \equiv \left( \begin{array}{c} D^\alpha_4 \\ D^\alpha_5 \end{array} \right), \quad (\alpha = 1, 2, 3) \]  

(3.16)

\[ D^\alpha_a \equiv -\frac{1}{|\beta_\alpha|} \partial_a (\frac{1}{|\beta_\alpha|}) = -\partial_a + \left( |\beta_\alpha|^{-1} \partial_a |\beta_\alpha| \right), \quad (\alpha = 1, 2, 3), \]  

(3.17)

where no sum is taken for the index \( \alpha \) in the middle and the right-most equations. An adjoint operator of the above differential operator is defined by

\[ \vec{D}^{\alpha\dagger} = \left( \begin{array}{c} D^\alpha_4 \dagger \\ D^\alpha_5 \dagger \end{array} \right), \]  

(3.18)

The analysis in this section is a generalization of that in Refs. [11–15], for the fat brane-world scenario with the domain wall in five dimensions, and is a refinement of that in Refs. [16, 17] which also studied similar problems in higher dimensions.
\[
D_a^{\alpha \dagger} = |\beta_\alpha|^{-1} \partial_a |\beta_\alpha| = \partial_a + (|\beta_\alpha|^{-1} \partial_a |\beta_\alpha|), \quad (\alpha = 1, 2, 3),
\]

where we do not sum in \(\alpha\).

To develop an analogue of the usual vector analysis in three spatial dimensions, we introduce analogues of gradient, divergence, and rotation in the following way. In order to avoid inessential complications, we will suppress the index \(\alpha\) in the following.

1) gradient
\[
\text{grad} f(x^a) \equiv \vec{D} \circ f(x^a) = \left( \frac{D_4 f(x^a)}{D_5 f(x^a)} \right).
\]

2) divergence
\[
\text{div} \vec{f}(x^a) \equiv \vec{D}^\dagger \cdot \vec{f}(x^a) = D_4^1 f_4(x^a) + D_5^1 f_5(x^a).
\]

3) vector rotation
\[
\text{rot}_v \vec{f}(x^a) \equiv \vec{D} \times \vec{f}(x^a) = D_5 f_4(x^a) - D_4 f_5(x^a).
\]

4) scalar rotation
\[
\text{rot}_s f(x^a) \equiv \vec{D}^\dagger \otimes f(x^a) = \left( \frac{D_4^1 f(x^a)}{-D_5^1 f(x^a)} \right).
\]

5) Laplacian
\[
\triangle f \equiv \text{div} \text{grad} f = \vec{D}^\dagger \cdot \vec{D} \circ f = \sum_{a=4,5} D_a D_a f.
\]

6) dual scalar Laplacian
\[
\tilde{\triangle}_s f \equiv \text{rot}_v \text{rot}_s f = \vec{D} \times \vec{D}^\dagger \otimes f = \sum_{a=4,5} D_a D_a^1 f.
\]

7) dual vector Laplacian
\[
\tilde{\triangle}_v \vec{f} \equiv \text{rot}_v \text{rot}_v \vec{f} = \vec{D}^\dagger \otimes \vec{D} \times \vec{f} = \left( \begin{array}{cc} D_4^1 D_5 - D_5^1 D_4 \\ -D_4^1 D_5 \end{array} \right) \vec{f}.
\]

Since \([D_4, D_5] = 0\) implies \(\text{rot}_v \text{grad} f = \vec{D} \times \vec{D} \circ f = D_5 D_4 f - D_4 D_5 f = 0\), we find
\[
\text{rot}_v \text{grad} f = 0.
\]

Similarly \(\text{div} \text{rot}_s f = \vec{D}^\dagger \cdot \vec{D}^\dagger \otimes f = D_4^1 D_5^1 f - D_5^1 D_4^1 f = 0\) gives
\[
\text{div} \text{rot}_s f = 0.
\]
We can define the adjoint of div acting on a vector function \( \vec{f}(x^a) \) in terms of the inner product between the scalar function \( h(x^a) \) as

\[
\int dx^4 dx^5 h^*(x^a) \text{div} \vec{f}(x^a) = \int dx^4 dx^5 \left( h^* D_4 f_4 + h^* D_5 f_5 \right) = \int dx^4 dx^5 \left( D_4^* h^* \right) f_4 + \left( D_5^* h^* \right) f_5 = \int dx^4 dx^5 \left( \text{grad} h(x^a) \right)^\dagger \cdot \vec{f}(x^a),
\]

(3.29)

where we assumed \( \beta(x^a)h(x^a) \) goes to zero at infinity \( x^a \to \infty \), and the dagger \( \dagger \) in the last line means the standard Hermite conjugation of a complex vector. Thus we observe the adjoint of div is given by grad and vice versa

\[
\text{grad}^\dagger = \text{div} \quad \text{or} \quad (\vec{D} \circ)^\dagger = \vec{D}^\dagger ,
\]

(3.30)

\[
\text{div}^\dagger = \text{grad} \quad \text{or} \quad (\vec{D}^\dagger \circ)^\dagger = \vec{D}^\circ .
\]

(3.31)

Similarly, the adjoint of rot acting on a scalar function \( f(x^a) \) can be defined in terms of an inner product with a vector function \( \vec{g}(x^a) \) as

\[
\int dx^4 dx^5 \vec{g}^\dagger \cdot \text{rot} f = \int dx^4 dx^5 \left( g_4^* D_5 f - g_5^* D_4 f \right) = \int dx^4 dx^5 \left( D_5 g_4^* - D_4 g_5^* \right) f = \int dx^4 dx^5 \left( \text{rot}_v \vec{g} \right)^\dagger \vec{f},
\]

(3.32)

where the dagger \( \dagger \) in the last line means the standard Hermite conjugation of a complex vector. Thus we find

\[
\text{rot}_v^\dagger = \text{rot}_v \quad \text{or} \quad (\vec{D} \times)^\dagger = \vec{D}^\circ ,
\]

(3.33)

\[
\text{rot}_s^\dagger = \text{rot}_v \quad \text{or} \quad (\vec{D}^\circ \times)^\dagger = \vec{D} \times .
\]

(3.34)

We find also that rot, grad and div rot are adjoint of each other

\[
(\text{rot}_v \text{grad})^\dagger = \text{div} \text{rot}_s \quad \text{or} \quad (\vec{D} \times \vec{D}^\circ)^\dagger = \vec{D}^\dagger \cdot \vec{D}^\dagger \circ .
\]

(3.35)

The final piece of our vector-analysis-like method is a decomposition formula for a (two-extra-dimensional) vector field. Let \( \vec{A} \) be an arbitrary two component vector field. There exist scalar fields \( B \) and \( C \) with which \( \vec{A} \) is decomposed as

\[
\vec{A} = \text{grad} B + \text{rot}_s C.
\]

(3.36)
With the identities in Eqs. (3.27) and (3.28), this theorem implies that a vector field can be decomposed into a rotation-free part and a divergence-free part. The theorem is proved in the Appendix A, and it is an analogue of the Helmholtz’s theorem for a three-dimensional vector field. Taking divergence and rotation leads to Poisson-like equations

\[ \triangle B = \text{div} \tilde{A}, \quad \tilde{\triangle} C = \text{rot}_v \tilde{A}. \]  

(3.37)

By solving these equations under appropriate boundary conditions, we can determine \( B \) and \( C \) for a given \( \tilde{A} \). Note, however, \( B \) and \( C \) are not uniquely determined. We can determine \( B \) and \( C \) only up to solutions of Laplace-like equations

\[ \triangle B_0 = 0, \quad \tilde{\triangle} C_0 = 0. \]  

(3.38)

Solutions to the above Laplace-like equations are given by the kernels of \( \text{grad} = \tilde{D} \circ \) and \( \text{rot}_\nu = \tilde{D}_\nu \), respectively. From Eqs. (3.17) and (3.19), we find explicitly that the kernels \( B_0, C_0 \) are given by

\[ \text{grad} B_0 = 0 \Rightarrow B_0 \propto |\beta|, \]  

(3.39)

\[ \text{rot}_\nu C_0 = 0 \Rightarrow C_0 \propto |\beta|^{-1}. \]  

(3.40)

(Remember we have suppressed the index \( \alpha = 1, 2, 3 \).) Hence, when \( B_0 \) is normalizable, \( C_0 \) is non normalizable, and vice versa. On the other hand, our Laplacian and the dual Laplacian are positive semi definite, as given in Eqs. (3.24) and (3.25).

### 3.2.3 Quadratic Lagrangian for unbroken gauge fields

We can now rewrite the effective Lagrangians (3.6) in terms of the differential operators of the vector-analysis-like method

\[ \mathcal{L} = \text{Tr} \left[ A_\mu (\eta^{\mu\nu} \Box - \partial^\mu \partial^\nu + \eta^{\mu\nu} \Delta) A_\nu - 2 \left( \partial^\mu A_\mu \right) (\text{div} \tilde{A}) - \tilde{A}^\dagger \left( \Box + \tilde{\Delta}_v \right) \tilde{A} \right], \]

(3.41)

after performing a partial integration. Next we rewrite the gauge fixing Lagrangians as

\[ \mathcal{L}^{(gf)} = \frac{1}{\xi} \text{Tr} \left[ \left( \partial^\mu A_\mu - \xi \text{div} \tilde{A} \right)^2 \right]. \]

(3.42)

Summing these two up, we get

\[ \mathcal{L} + \mathcal{L}^{(gf)} = \text{Tr} \left[ A_\mu \left( \eta^{\mu\nu} \Box - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu + \eta^{\mu\nu} \Delta \right) A_\nu - \tilde{A}^\dagger \left( \Box + \tilde{\Delta}_v + \xi \text{grad} \text{div} \right) \tilde{A} \right]. \]

(3.43)

\[ \text{The theorem states that a three-vector field } \tilde{A} \text{ can be decomposed into rotation-free and divergence-free components as } \]

\[ \tilde{A} = \nabla B + \nabla \times C. \]
where we used Eq. (3.31) to obtain

\[(\text{div}\,\vec{A})^\dagger \text{div}\,\vec{A} = \vec{A}^\dagger \text{grad}\,\text{div}\,\vec{A}.\] (3.44)

We decompose \(\vec{A} = \text{grad}\,B + \text{rot}_s\,C\) using the generalized Helmholtz’s theorem (3.36). Note that we can freely exclude the zero mode \(B_0 \propto |\beta| (\text{grad}\,B_0 = 0)\) from \(B\) and the zero mode \(C_0 \propto |\beta|^{-1} (\text{rot}_s\,C_0 = 0)\) from \(C\) without loss of any informations contained in \(\vec{A}\), because they never change \(\vec{A}\). Plugging the decomposition into Eq. (3.43), we arrive at the following simple formula

\[L + L^{(gf)} = \text{Tr}\left[A_\mu \left(\eta^{\mu\nu} \Box - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu + \eta^{\mu\nu}\Box\alpha\right)A_\nu\right.\]
\[- C^\dagger \tilde{\Delta}_s \left(\Box + \tilde{\Delta}_s\right)C - B^\dagger \Delta \left(\Box + \xi\Delta\right)B\]. \tag{3.45}

Since \(\Delta\) and \(\tilde{\Delta}_s\) are semi-positive definite and zero modes are excluded from \(B\) and \(C\), we can redefine the divergence-free and rotation-free parts by

\[b^\alpha \equiv \sqrt{\Delta} B, \quad c^\alpha \equiv \sqrt{\tilde{\Delta}_s} C.\] (3.46)

Substituting these into the last expression, we obtain the final formula with the omitted index \(\alpha = 1, 2, 3\) retained

\[(L + L^{(gf)})_{\alpha=1,2,3} = \text{Tr}\left[A_\mu^\alpha \left(\eta^{\mu\nu} \Box - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu + \eta^{\mu\nu}\Box\alpha\right)A_\nu^\alpha\right.\]
\[- c^\alpha \left(\Box + \tilde{\Delta}_s^\alpha\right)C^\alpha - b^\alpha \left(\Box + \xi\Delta\right)B^\alpha\]. \tag{3.47}

Note that \(b^\alpha\) and \(c^\alpha\) are Hermitian matrix fields of the same size as \(A_\mu^\alpha\), since they arise as rotation-free and divergence-free parts of \(A_\mu^\alpha\). We do not take sum over \(\alpha\) on the right hand side.

In order to make the Kaluza-Klein (KK) expansion more explicit, let us define eigenvalues \(m_n (\tilde{m}_n)\) and eigenfunctions \(B_n(x^a) (C_n(x^a))\) of \(\Delta\) (\(\tilde{\Delta}_s\)) as\(^8\)

\[\Delta B_n = m_n^2 B_n, \quad \tilde{\Delta}_s C_n = \tilde{m}_n^2 C_n.\] \tag{3.48}

As mentioned before, \(\Delta\) and \(\tilde{\Delta}_s\) are positive semi definite. We denote the possible zero eigenvalue as \(m_{n=0} = \tilde{m}_{n=0} = 0\), and positive eigenvalues as \(n > 0\). We set the normalizations of the mode functions as usual:

\[\int dx^4 dx^5 B_m B_n = \delta_{mn}, \quad \int dx^4 dx^5 C_m C_n = \delta_{mn}.\] \tag{3.49}

Of course, these are meaningful only for normalizable modes.\(^9\) If \(|\beta| (|\beta|^{-1})\) is square integrable, \(B_0 (C_0)\) is normalizable but \(C_0 (B_0)\) is not normalizable.

\(^8\) Again, we omit the subscript \(\alpha = 1, 2, 3\) from now on.

\(^9\) More precisely, mode functions should be bounded, so that they can be delta-function normalizable for continuum spectra.
Now, we decompose $B$ ($C$) as

$$
B(x^M) = \sum_n f_n(x^\mu) B_n(x^a), \quad C(x^M) = \sum_n g_n(x^\mu) C_n(x^a),
$$

(3.50)

where expansion coefficients $f_n$ and $g_n$ are effective fields in four dimensions, and the sum is taken only for the normalizable modes. Eq. (3.46) implies that expansions for $b$ and $c$ do not have zero modes

$$
b(x^M) = \sum_{n>0} b^{(n)}(x^\mu) B_n(x^a), \quad c(x^M) = \sum_{n>0} c^{(n)}(x^\mu) C_n(x^a),
$$

(3.51)

with $b^{(n)} = f_n m_n$ and $c^{(n)} = g_n \tilde{m}_n$, since zero modes are excluded in defining $B$ and $C$ from $\vec{A}$.

In contrast, the four-dimensional components do have the zero mode of $\Delta$ because there is no reason to eliminate it

$$
A_\mu(x^M) = \sum_{n \geq 0} A_\mu^{(n)}(x^\mu) B_n(x^a).
$$

(3.52)

Since we wish to have the massless gauge fields localized on the vortices, we should impose the square integrable condition on $\beta$ (This is one of the most important results of this work) as

$$
\frac{1}{g^2} \equiv \int dx^4 dx^5 |\beta|^2 < \infty.
$$

(3.53)

This gives the normalization of the mode function for $n = 0$ as

$$
B_0(x^a) = g |\beta(x^a)|.
$$

(3.54)

Substituting the expansions in Eqs. (3.51) and (3.52) into Eq. (3.47), and integrating it over the $x^4-x^5$ plane, we obtain the effective Lagrangians for the KK towers.

The massless mode ($n = 0$) only appears in the sector of the four-dimensional gauge fields,

$$
\mathcal{L}_\alpha^{(0)} = \text{Tr} \left[ A_\alpha^{(0)} \left( i \eta^{\mu\nu} \square - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu \right) A_\alpha^{(0)} \right] .
$$

(3.55)

This is nothing but the quadratic Lagrangian of massless vector fields. In order to recover self-interaction terms of non-Abelian gauge fields, we return to Eq. (3.14). We have

$$
A_\alpha^\mu(x^M) = B_0^\alpha(x^a) A_\mu^{(0)}(x^\mu) + \cdots ,
$$

therefore

$$
A_\mu^\alpha(x^M) = \beta_\alpha^{-1}(x^a) A_\mu(x^M) = g_\alpha A_\mu^{(0)}(x^\mu) + \cdots.
$$

(3.56)

It is remarkable that the mode function of the zero mode with respect to the original field $A_\mu$ is constant namely $g_\alpha$. This ensures the universality of the gauge coupling constant.
Nevertheless, the zero mode effective Lagrangian is well-defined because of the extra factor $B(T)B(T)\dagger$ in Eq. (2.1) gives the necessary suppression factor $|\beta(x^a)|^2$ in the extra dimensions. Keeping only zero mode in Eq. (3.56), the field strength reads

$$F_{\mu\nu}^\alpha = g_\alpha F_{\mu\nu}^{\alpha(0)}, \quad (3.57)$$

$$F_{\mu\nu}^{\alpha(0)} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + ig_\alpha \left[ A^{\alpha(0)}_\mu, A^{\alpha(0)}_\nu \right]. \quad (3.58)$$

This should be compared with the original field strength $F_{MN} = \partial_M A_N - \partial_N A_M + i [A_M, A_N]$ in which gauge coupling constant is absorbed in the gauge field $A_M$. The four-dimensional effective gauge coupling appears in Eq. (3.58) because of the normalization condition in (3.53). One can also understand this result as a consequence of unbroken four-dimensional local gauge invariance[10]. Now, including the self interactions of the non-Abelian gauge fields, the zero mode effective Lagrangian is found to be the standard one as

$$\int dx^4 dx^5 \text{Tr} \left[ -\frac{\beta^2}{2} g_\alpha^2 F_{\mu\nu}^{\alpha(0)} F^{\alpha(0)\mu\nu} \right] = -\frac{1}{2} \text{Tr} \left[ F_{\mu\nu}^{\alpha(0)} F^{\alpha(0)\mu\nu} \right]. \quad (3.59)$$

We would like to emphasize a new important feature of our model compared to many previous works. The gauge kinetic function $BB\dagger$ in Eq. (2.1) is not a scalar but a matrix as a nontrivial representation of gauge group. A similar mechanism of localizing massless gauge fields on topological solitons have been studied, which utilize a conformal factor. However, it is usually a singlet of the gauge group, since it usually originates from a spacetime metric or dilaton. Such a singlet conformal factor cannot distinguish various components of $SU(5)$ gauge fields, unlike our model.

As for the higher KK modes with $n > 0$, we have two separated parts. The one is for massive vector fields $A_\mu^{\alpha(n)}$

$$L_{1,\alpha}^{(n>0)} = \text{Tr} \left[ A^{\alpha(n)}_\mu \left( \eta^{\mu\nu} \Box - \left( 1 - \frac{1}{2} \right) \partial^\mu \partial^\nu + \eta^{\mu\nu} (m_n^\alpha)^2 \right) A^{\alpha(n)}_\nu \right]
- \frac{i}{\sqrt{2}} \alpha^{\alpha(n)\dagger} \left( \Box + \xi (m_n^\alpha)^2 \right) b^{\alpha(n)}. \quad (3.60)$$

Note that the vector field $A_\mu^{\alpha(n)}$ has two components, with the mass squared $(m_n^\alpha)^2$ and $\xi (m_n^\alpha)^2$. On the other hand, the scalar $b$ is originated from the rotation-free part of $A^\alpha_\mu$, and has the mass squared $\xi (m_n^\alpha)^2$. It should be combined with the above components of vector field $A_\mu^{\alpha(n)}$ with the same mass squared together with the ghost and anti-ghost fields to become unphysical, as can be recognized from their gauge dependent mass. This situation is analogous to the usual situation in the $R_\xi$ gauge. Thus we see that the rotation free part plays a role of an (unphysical) Nambu-Goldstone field to give a mass to the KK tower of vector fields. The other part includes the divergence free part $c$:

$$L_{2,\alpha}^{(n>0)} = -\text{Tr} \left[ c^{\alpha(n)\dagger} \left( \Box + (m_n^\alpha)^2 \right) c^{\alpha(n)} \right]. \quad (3.61)$$

This should be a physical scalar fields with the mass squared $(m_n^\alpha)^2$. 

This should be a physical scalar fields with the mass squared $(m_n^\alpha)^2$. 

This should be a physical scalar fields with the mass squared $(m_n^\alpha)^2$. 

---
Summary of the unbroken sector $SU(3) \times SU(2) \times U(1)$ is the following: The six-dimensional gauge fields $A_{\alpha}^\mu$ provide one massless gauge field $A_{\alpha}^{(0)}$, KK tower of massive vector fields $A_{\alpha}^{(n>0)}$, and KK tower of massive scalar fields $c_{\alpha}^{(n>0)}$ to the four-dimensional effective theory.

### 3.3 Compact formulae for the broken gauge fields

We move to the broken sector with $X_M$ and $\varphi$ which is more complicated than the unbroken sector in the previous subsection. We will make the effective Lagrangians (3.7), (3.11), and (3.13) as simple as possible.

Firstly, we define canonically normalized fields by

$$X_M \equiv \beta_X X_M, \quad X_6 \equiv |\beta_\varphi| \varphi, \quad X_7 \equiv |\beta_\psi| \psi.$$  \hspace{1cm} (3.62)

We will also use the notation $X_6 = \phi$ later. We can treat the NG boson $\varphi$ and $\psi$ as if they are a sixth and a seventh components of a massive gauge field, by defining a four-component vector as

$$X \equiv \begin{pmatrix} X_4 \\ X_5 \\ X_6 \\ X_7 \end{pmatrix},$$  \hspace{1cm} (3.63)

which will bring a benefit of simplification in many expressions below. Assigning the NG boson $X_6 = \phi$ to be the sixth gauge field is somehow natural because would-be eaten NG bosons can be regarded as a part of the massive vector field in general. As before, let us introduce the differential operators by

$$\bar{D}^X \equiv -\beta_X \bar{\partial} \frac{1}{\beta_X} = -\bar{\partial} + \left( \beta_X^{-1} \beta_X \right),$$  \hspace{1cm} (3.64)

$$\bar{D}^\phi \equiv -|\beta_\varphi| \bar{\partial} \frac{1}{|\beta_\varphi|} = -\bar{\partial} + \left( |\beta_\varphi|^{-1} \beta_\varphi \right).$$  \hspace{1cm} (3.65)

Note that $\bar{D}^X$ can be expressed by $\bar{D}^\phi$ by

$$\bar{D}^X = M^{-1} \bar{D}^\phi \circ M, \quad \bar{D}^\phi = M \bar{D}^X \circ M^{-1}, \quad M \equiv \frac{|\beta_\varphi|}{\beta_X}.$$  \hspace{1cm} (3.66)

Then we define a four-component (effectively three-component) operator by

$$D \equiv \begin{pmatrix} \bar{D}^X \\ M \\ 0 \end{pmatrix}.$$  \hspace{1cm} (3.67)

From this we can define

1) gradient

$$\text{Grad } f \equiv D \circ f = \begin{pmatrix} \bar{D}^X \\ M \\ 0 \end{pmatrix} \circ f = \begin{pmatrix} \bar{D}^X \circ f \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ Mf \\ 0 \end{pmatrix}.$$  \hspace{1cm} (3.68)
2) divergence
\[
\text{Div } f \equiv D^\dagger \cdot f = \left(\bar{D}^X, M, 0\right) \cdot \begin{pmatrix} f_6 \\ f_7 \end{pmatrix} = \bar{D}^X \cdot \bar{f} + Mf_6. \tag{3.69}
\]

3) Laplacian
\[
\Delta f \equiv \text{Div Grad } f = D^\dagger \cdot D \circ f = \Delta^X f + M^2 f \tag{3.70}
\]
with \(\Delta^X = \text{div}^X \text{grad}^X = \bar{D}^X \cdot \bar{D}^X \circ\). Note that \(\Delta\) is positive definite since \(\Delta^X\) is non-negative definite and \(M^2\) is positive definite.

4) dual vector Laplacian
\[
\Delta_v \equiv \left( \begin{array}{ccc}
\Delta_v^X + M^2 1_2 & -M\bar{D}^\phi \circ & -\frac{1}{|\beta^\phi|^2}M\bar{J} \\
-M\bar{D}^\phi \circ & \Delta^\phi & \bar{D}^\phi \cdot \bar{J} \frac{1}{|\beta^\phi|^2} \\
-\frac{1}{|\beta^\phi|^2}M\bar{J} & \frac{1}{|\beta^\phi|^2} \bar{J} \bar{D}^\phi \circ & \beta^\phi \bar{D}^\phi \circ \cdot \frac{1}{|\beta^\phi|^2} \bar{D}^\phi \circ \beta^\phi + U
\end{array} \right). \tag{3.71}
\]
with the 2 by 2 Laplacian \(\Delta_v^X = \bar{D}^X \otimes \bar{D}^X \circ\) and the 1 by 1 Laplacian \(\Delta^\phi = \bar{D}^\phi \cdot \bar{D}^\phi \circ\). The elements of two component vector \(\bar{J}\) are given by Eq. (3.9). Note that \(\Delta_v\) itself is 4 by 4.

It would be nice if we could factorize \(\Delta_v\) as \(\Delta_v = \text{Rot}_s \text{Rot}_v\) with appropriate rotation operators \(\text{Rot}_s\) and \(\text{Rot}_v\). Unfortunately, this does not seem very easy, so we abandon it. Nevertheless, we still have identities corresponding to \(\text{Rot}_v \text{Grad } = 0\) and \(\text{Div } \text{Rot}_s = 0\):
\[
\Delta_v \text{ Grad } = 0, \tag{3.72}
\]
and
\[
\text{Div } \Delta_v = 0. \tag{3.73}
\]

The former is verified as
\[
\Delta_v \text{ Grad } f = \Delta_v \left( \begin{array}{c}
\bar{D}^X \circ f \\
Mf \\
0
\end{array} \right) = \left( \begin{array}{c}
(\Delta_v^X + M^2)\bar{D}^X \circ f - M\bar{D}^\phi \circ Mf \\
\bar{D}^\phi \circ M\bar{D}^X \circ f + \bar{D}^\phi \circ \bar{D}^\phi \circ Mf \\
\frac{1}{|\beta^\phi|^2}M\bar{J} \bar{D}^X \circ f + \frac{1}{|\beta^\phi|^2} \bar{J} \bar{D}^\phi \circ Mf
\end{array} \right) = 0, \tag{3.74}
\]
where we used a similar identity \(\Delta_v^X \bar{D}^X \circ = \bar{D}^X \otimes (\bar{D}^X \times \bar{D}^X \circ) = 0\) to Eq. (3.27) for \(\bar{D}^X\) and Eq. (3.66). The latter can also be verified as
\[
\text{Div } \Delta_v f = \text{Div } \left( \begin{array}{c}
(\Delta_v^X + M^2)\bar{f} - M\bar{D}^\phi \circ f_6 - \frac{1}{|\beta^\phi|^2}M\bar{J}f_7 \\
-\bar{D}^\phi \circ M\bar{f} + \Delta^\phi f_6 + \bar{D}^\phi \circ \bar{D}^\phi \circ \frac{1}{|\beta^\phi|^2}f_7
\end{array} \right)
\]
\[
= \bar{D}^X \cdot \left( (\Delta_v^X + M^2)\bar{f} - M\bar{D}^\phi \circ f_6 - \frac{1}{|\beta^\phi|^2}M\bar{J}f_7 \\
+ M \left( -\bar{D}^\phi \circ M\bar{f} + \Delta^\phi f_6 + \bar{D}^\phi \circ \bar{D}^\phi \circ \frac{1}{|\beta^\phi|^2}f_7 \right)\right)
\]
\[
= 0, \tag{3.75}
\]
where we used a similar identity $\tilde{D}^X \cdot \tilde{\Delta}^X = (\tilde{D}^X \cdot \tilde{D}^X \otimes) \tilde{D}^X \times = 0$ to Eq. (3.28) for $\tilde{D}^X$ and Eq. (3.66).

Since we did not succeed to define appropriate rotation operators in the case of broken generators, we cannot introduce a Helmholtz-like decomposition for the four-vector $X$. Instead, we introduce a projection operator

\[
P = \text{Grad} \Delta^{-1} \text{Div}.
\]

This satisfies $P^2 = P$, and is well-defined because $\Delta$ is positive definite. We decompose $X$ as

\[
X = PX + (1 - P)X.
\]

The first term is “rotation-free” because

\[
PX = \text{Grad} Y, \quad \Delta Y = \text{Div} X.
\]

The second term is divergence-free because

\[
\text{Div} (1 - P)X = \text{Div} X - \text{Div} (\text{Grad} \Delta^{-1} \text{Div}) X = 0.
\]

The Hermitian conjugate of this is $(1 - P) \text{Grad} = 0$.

Now, we are ready to rewrite the Lagrangians (3.7), (3.11), and (3.13) in more compact forms. Let us first rewrite (3.7) and (3.11)

\[
\mathcal{L}_X + \mathcal{L}_\phi = \text{Tr} \left[ X^\dagger_\mu \left( \eta^{\mu\nu} \Box - \partial^\mu \partial^\nu + \eta^{\mu\nu} \Delta \right) X_\nu - 2 \left( \partial^\mu X^\dagger_\mu \right) \text{Div} X - X^\dagger \left( \Box + \tilde{\Delta}_v \right) X \right].
\]

Remarkably, unification of $\bar{X}$ originated from the extra-dimensional component of $X_M$, the NG boson $X_6 = \phi$ and the additional scalar $X_7$ into the four-component vector $X$ is essential to have the quadratic Lagrangian in a compact form. Furthermore, (3.80) for $X_\mu$ and $X$ is formally identical to Eq. (3.41) for the unbroken gauge field $A_\mu$ and $\vec{A}$.

Similarly, the gauge fixing Lagrangian (3.13) can be expressed as

\[
\mathcal{L}^{(gf)}_{X,\phi} = -\frac{1}{\xi} \text{Tr} \left[ \left( \partial^\mu X_\phi - \xi \text{Div} X \right)^\dagger \left( \partial^\mu X_\mu - \xi \text{Div} X \right) \right].
\]

This is a counterpart of Eq. (3.42).

Adding Eqs. (3.80) and (3.81), we have the following quadratic Lagrangian

\[
\mathcal{L}_X + \mathcal{L}_\phi + \mathcal{L}^{(gf)}_{X,\phi} = \text{Tr} \left[ X^\dagger_\mu \left( \eta^{\mu\nu} \Box - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu + \eta^{\mu\nu} \Delta \right) X_\nu - X^\dagger \left( \Box + \tilde{\Delta}_v + \xi \text{Grad} \text{Div} \right) X \right],
\]

where we used

\[
(\text{Div} X)^\dagger (\text{Div} X) = X^\dagger \text{Grad} \text{Div} X.
\]
This is clearly a counterpart of Eq. (3.43).

Thus we accomplished obtaining much simpler formula compared to the initial one. But the most important point is that Eq. (3.82) for the broken gauge field with the NG fields is expressed in the form which is formally identical in fashion found in Eqs. (3.43) for the unbroken gauge fields. Since we have developed the way to extract physical spectra for the latter, we just formally but partially repeat the same procedures.

Therefore, what we have to do next is to decompose $X$ as $X = \text{Grad} Y + (1 - P)X$. Substituting this into Eq. (3.82) and using the identities Eqs. (3.72) and (3.73), we get

$$\mathcal{L}_X + \mathcal{L}_\phi + \mathcal{L}^{(gf)}_{X\phi} = \text{Tr} \left[ X^\dagger_\mu \left( \eta^{\mu\nu} \Box - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu + \eta^{\mu\nu} \Delta \right) X_\nu \right]$$

$$- (1 - P)X^\dagger \left( \Box + \Delta_v \right) (1 - P)X - Y \Delta \left( \Box + \xi \Delta \right) Y \right].$$

The last treatment is making this canonical by redefining $Y$ and $(1 - P)X$ by

$$y = \sqrt{\Delta} Y, \quad x = (1 - P)X.$$  

Substituting this into the above expression, we get the final form of the quadratic Lagrangian of the broken sector

$$\mathcal{L}_X + \mathcal{L}_\phi + \mathcal{L}^{(gf)}_{X\phi} = \text{Tr} \left[ X^\dagger_\mu \left( \eta^{\mu\nu} \Box - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu + \eta^{\mu\nu} \Delta \right) X_\nu \right]$$

$$- x^\dagger \left( \Box + \Delta_v \right) x - y \left( \Box + \xi \Delta \right) y \right].$$

We have seen remarkable similarities between the unbroken and broken sectors. From now on, we will shed light on differences between them. Firstly, the Laplacian $\Delta$ given in Eq. (3.70) is positive definite. (This should be compared with $\Delta^\alpha$ in Eq. (3.47) which is only non-negative.) This leads to an important physical consequence that there are no massless modes in the four-dimensional component of the broken gauge field $X_\mu$. This is, of course, expected from the beginning because $X_\mu$ corresponds to the gauge fields associated with the broken generators. What is not totally trivial here is to clarify which modes are eaten by $X_\mu$. The answer is $y$. We are lead to this conclusion from the fact that $X_\mu$ and $y$ share the same mass operator $\Delta$ except for the extra constant factor $\xi$ for $y$. Again, we come across a generalized form of the $R^\xi$ gauge (This should be compared to the relation between $\tilde{A}_\mu$ and $b$ in Eq. (3.47)). Appearance of $\xi$ implies that $y$ plays a role of infinite tower of the NG modes which are absorbed by infinite tower of $X_\mu$ including the bottom of the tower. Remembering the definition given in Eq. (3.78), the source for $y$ is $\text{Div} X = \tilde{D}^X \cdot \tilde{X} + M \phi$. Therefore, the broken gauge field eats not pure NG modes $\phi$ but a mixture of the rotation-free part $\tilde{D}^X \cdot \tilde{X}$ and the NG modes $\phi$. Specifying the lowest mass of $\Delta$ is in general difficult except for a very special case where $M$ is a constant. If $M$ is a constant, $\Delta = \Delta^X + M^2$ and $\Delta^X$ differ by just a constant $M^2$. We know that the zero mode of the non-negative definite operator $\Delta^X$ is given by $\beta \chi$. Therefore, the lowest mass is the constant $M$ itself. However, in general $M$ is not constant, so we are only sure that there is a finite mass gap but explicit spectrum of $\Delta$ depends on $\mathcal{B}$. 

\[ - 25 - \]
Our final comment is on the divergence-free component $\mathbf{x}$. As we mentioned above, we were not able to factorize the Laplacian $\Delta_v$ as a product of rotation operators. Remembering the unbroken part where we decomposed $\tilde{A}$ by the Helmholtz-like decomposition, and we succeeded in splitting the two component vector into the two scalar $b$ and $c$ by replacing $\tilde{\Delta}_v = \text{rot}_s \text{rot}_v \mathbf{x}$ by $\Delta_s = \text{rot}_v \text{rot}_s$ as in Eq. (3.47). Furthermore, this rewriting allows us to conclude that the divergence-free component $c$ does not have the zero mode of the non-negative operator $\Delta_s$. At present unfortunately, we cannot make a similar statement for $\Delta_v$. We are sure that there are no negative eigenvalues, but we cannot exclude a zero eigenvalue. It is an important open problem. But if we could do, it would not so useful in the following sense. Recall $\Delta_v$ acts on the four-component vector $(1 - P)\mathbf{x}$. Precisely speaking, since the divergence part is projected out, the degrees of freedom of the four-component vector $(1 - P)\mathbf{x}$ is not four but three. So if we succeeded in dualizing $\Delta_v$, we are still left with the three-component vector. It is a complicated problem to find spectrum of the 4 by 4 matrix operator $\Delta_v$ given in Eq. (3.71), since it highly depends on the details of models. We leave the spectrum of this operator as one of future problems.

### 3.4 A typical example

Let us illustrate our models more explicitly by a concrete choice of $\mathcal{B}$ as an example. One of the simplest choice of $\mathcal{B}$ is

$$BB^\dagger = \left(v^2 1_5 - T^\dagger T\right)^2.$$  

(3.87)

Note that this is just a single mere one possibility. One can consider, say, $BB^\dagger = \left(v^2 1_5 - T^\dagger T\right)^s$ with $s \geq 2$. The crucial is if the condition (3.53) is satisfied or not. We will verify that for the above simplest choice it is. $T$ is with the 3-2 splitting vortex background solution (see also Eq. (2.14))

$$T = \begin{pmatrix} \tau_3 1_3 \\ \tau_2 1_2 \end{pmatrix}, \quad \tau_2 = v f_2(r_2)e^{i\theta_2}, \quad \tau_3 = v f_3(r_3)e^{i\theta_3}.$$  

(3.88)

For simplicity, we consider the minimal winding number, namely each diagonal component has a single winding number. Here $(r_2, \theta_2)$ and $(r_3, \theta_3)$ are two dimensional polar coordinates defined as $x^1 - a_2 + i(x^2 - b_2) = r_2e^{i\theta_2}$ and $x^4 - a_3 + i(x^5 - b_3) = r_3e^{i\theta_3}$ where $(a_2, b_2)$ is the position of the vortex associated to $SU(3)$ and $(a_3, b_3)$ is that of the vortex associated to $SU(2)$. We impose the profile functions to satisfy the boundary conditions $f_{2,3}(0) = 0$ and $f_{2,3}(\infty) = 1$. Solutions of the vortex equation have been obtained only numerically, but we do not need precise solutions for $f_2$ and $f_3$ in order to understand qualitative aspects of gauge field localization. Hence, we use the following approximation

$$f_\alpha(r) = \frac{r_\alpha}{\sqrt{1 + r^2_\alpha}}, \quad (\alpha = 2, 3).$$  

(3.89)

Note that these satisfy not only the correct boundary condition but also exhibit a good asymptotic behavior $f_{2,3} \rightarrow r_{2,3}$ ($r_{2,3} \rightarrow 0$) and $f_{2,3} \rightarrow 1 - 1/(2r_{2,3}^2)$ ($r_{2,3} \rightarrow \infty$) as a global
vortex. Then we have
\[
|\beta_\alpha(r)| = v^2 (1 - f_\alpha(r_a)^2) = \frac{v^2}{1 + r_a^2}, \quad (\alpha = 2, 3),
\]
(3.90)
which satisfy the square integrability condition (3.53) because their asymptotic behaviors are $|\beta_{2,3}| \to v^2/r^2$ as $r_{2,3} \sim r \to \infty$.\(^{10}\) Now the corresponding $\Delta^\alpha$'s for $\alpha = 2, 3$ are given by
\[
\Delta^\alpha = \sum_{a=4,5} \left(-\partial^2_a + \frac{\partial^2_a|\beta_a|}{|\beta_a|}\right) = -\nabla^2 + \mathcal{V}_\alpha(r), \quad \mathcal{V}_\alpha(r) = \frac{4(r_a^2 - 1)}{(1 + r_a^2)^2},
\]
(3.91)
\[
\tilde{\Delta}^\alpha = \sum_{a=4,5} \left(-\partial^2_a + \frac{\partial^2_a|\beta_a|^{-1}}{|\beta_a|^{-1}}\right) = -\nabla^2 + \tilde{\mathcal{V}}_\alpha(r), \quad \tilde{\mathcal{V}}_\alpha(r) = \frac{4}{1 + r_a^2}.
\]
(3.92)
The eigenvalue problems for these operators are mere two-dimensional Schrödinger equations with the axially symmetric potentials $\mathcal{V}_\alpha$ and $\tilde{\mathcal{V}}_\alpha$. The normalizable zero mode of $\mathcal{V}_\alpha$ is $\beta_\alpha(r)$ itself and it is the only discrete spectrum\(^{11}\) of $\Delta^\alpha$. On the other hand, there is no discrete spectrum such as normalizable zero modes for $\tilde{\Delta}^\alpha$ since $\tilde{\mathcal{V}}_\alpha$ is a positive convex function. This means that massless four-dimensional gauge fields $G_\mu$ and $W_\mu$ exist only in the $SU(3)$ and $SU(2)$ gauge group. For this particular choice of $\mathcal{B}$, there exists a massive localized modes as well in the four-dimensional vector component in the adjoint representation of the $SU(3)$ and $SU(2)$ gauge group. However, together with the continuum spectra in both four-dimensional and extra-dimensional components, all these massive modes have an energy gap of order the GUT scale.

The analysis for the $U(1)$ part goes almost parallel to those above. Firstly, we have from Eq. (3.3)
\[
\beta_1(r) = v^2 \sqrt{\frac{2}{5(1 + r_a^2)^4}} + \frac{3}{5(1 + r_a^2)^4}.
\]
(3.93)
This again leads to Schrödinger problems with the potentials
\[
\mathcal{V}_{\alpha=1} = \frac{\partial^2 \beta_1}{\beta_1^2}, \quad \tilde{\mathcal{V}}_{\alpha=1} = \frac{\partial^2 \beta_1^{-1}}{\beta_1^{-1}}.
\]
(3.94)
Instead of showing form, we just display $\mathcal{V}_\alpha$ and its zero mode $\beta_\alpha$ in Fig. 3. Reflecting the fact that $\beta_1$ is a weighted average of $|\beta_2|$ and $|\beta_3|$, the corresponding potential $\mathcal{V}_1$ has two attractive valleys around the vortices at $(x^4, x^5) = (\pm 2, 0)$. Hence the zero mode wave

\(^{10}\) One might naively expect that the unbroken $H = SU(3) \times SU(2) \times U(1)$ gauge fields remain massless everywhere for the scalar field profile of the global vortex only reduces as a power low. However, this is not correct. Localization of the $H$ gauge fields does not depend on the scalar profile itself but the gauge kinetic function $\beta$ (4). Even though our background solution is the global vortex, it can localize the massless non-Abelian gauge fields by the square integrable $\beta$. Of course, if one considers a local vortex instead of the global one, satisfying the square integrability condition becomes much easier, so that range of possibilities for $\mathcal{B}$ would become larger. We will investigate the case of a local vortex elsewhere.

\(^{11}\) Of course, this is merely a property of the specific choice of $\mathcal{B}^2$ in Eq. (3.87), and not a generic property for other possible choices.
function of $Y_\mu$ is concentrated around both the vortex associated to $SU(3)$ and $SU(2)$. This results in an interesting difference between wave functions of $Y_\mu$ compared to $G_\mu$ and $W_\mu$. The spectra of the gluons and W bosons do not depend on the separation between vortices. The gluon (W boson) has a single massless mode localized around the vortex associated to $SU(3)$ ($SU(2)$). On the other hand, the spectrum of $Y_\mu$ depends on the vortex separations, though the single zero mode always exists. Suppose that the vortices associated to $SU(3)$ and $SU(2)$ are infinitely separated, then two potential wells are also separated infinitely. Each of the isolated well has a localized zero mode. Hence the whole spectrum should include two massless modes for $U(1)$ gauge field. If we, however, bring the vortices again together the degenerate zero modes now split (level repulsion) by a quantum tunneling effect between the two wells. The lowest one remains massless while the other one is lifted by an exponentially suppressed nonperturbative tunneling effect. Note also that if the vortices associated to $SU(3)$ and $SU(2)$ are completely coincident, the $SU(5)$ is unbroken and $\beta_1$ becomes identical to $\beta_3$ and $\beta_2$. As we saw above, $\beta_3$ ($\beta_2$) admits only a single discrete spectrum which is massless, so is $\beta_1$ at the coincident limit. Therefore, the lifted zero mode that is normalizable for well-separated 3-2 splitting background should enter into continuum spectrum at a particular value of the separation. Concrete examples are given in Fig. 4. Hence, an extra massive mode can exist in $Y_\mu$ unlike in $G_\mu$ and $W_\mu$. This extra massive vector particle can be an evidence for the underlying background of vortices. It may be observable if the 3-2 split configuration with a large separation of the vortices is realized as the stabilized configuration.

To see mass spectra of the broken sector, we next need the other $\beta$’s from Eq. (3.3)

$$\beta_X = \nu^2 \frac{1}{2(1 + r_3^2)^4} + \frac{1}{2(1 + r_2^2)^4}, \quad \beta_\phi = \nu \left( \frac{r_3 e^{i\theta_3}}{\sqrt{1 + r_3^2}} - \frac{r_2 e^{i\theta_2}}{\sqrt{1 + r_2^2}} \right).$$

(3.95)

We can compute the Laplacian $\Delta$ for the four-dimensional component $X_\mu$

$$\Delta = \Delta^X + \mathcal{M}^2 = \sum_{a=4,5} \left(-\partial_a^2 + \frac{\partial_a^2 \beta_X}{\beta_X} \right) + \mathcal{M}^2, \quad \mathcal{M}^2 = \frac{|\beta_\phi|^2}{\beta_X^2}.$$  

(3.96)
The persistent zero mode $\beta_1$ (blue) and the lifted zero mode (orange). The positions of the 3- and 2-vortices are set to be $(a_3, b_3) = (d/2, 0)$ and $(a_2, b_2) = (-d/2, 0)$, and the left figure is with $d = 8$ and the right one is with $d = 5$. The lifted zero mode in the left graph is normalizable while the one in the right graph is non-normalizable.

This is also a bit complicated, so we do not show the explicit expression. If we omit $M$, $\beta_X$ is quite similar to $\beta_1$ except for the weights for taking average. Then, there exists a single normalizable zero mode which is proportional to $\beta_X$. However, this zero mode is completely swept out by the additional term $M^2$ which dominates the potential. Fig. 5 shows a concrete example of the potential with and without $M^2$. The vortex potential wells are clearly visible for $M^2 = 0$, but it is almost washed out for $M^2 \neq 0$. Although we do not find eigenvalues and eigenfunctions exactly, we are sure that there are no massless modes and a mass gap of order GUT scale exists when $M^2$ is of the order of GUT scale. Showing the absence of the additional massless vector field rigorously is an important problem from the phenomenological viewpoint and it is left as a future work.

The potential $V_X = \frac{\partial^2 X}{X^2} + M^2$. The panel (a) shows $V_X$ without $M^2$ and (b) with $M^2$. 

Figure 4: The persistent zero mode $\beta_1$ (blue) and the lifted zero mode (orange). The positions of the 3- and 2-vortices are set to be $(a_3, b_3) = (d/2, 0)$ and $(a_2, b_2) = (-d/2, 0)$, and the left figure is with $d = 8$ and the right one is with $d = 5$. The lifted zero mode in the left graph is normalizable while the one in the right graph is non-normalizable.

Figure 5: The potential $V_X = \frac{\partial^2 X}{X^2} + M^2$. The panel (a) shows $V_X$ without $M^2$ and (b) with $M^2$. 

Downloaded from https://academic.oup.com/ptep/advance-article/doi/10.1093/ptep/ptab144/6415217 by guest on 08 November 2021
4 Conclusions and discussion

In this work we examined issues associated to the SM gauge fields, gluons, weak bosons, and hypercharge $U(1)$ gauge fields specifically their localization on the non-Abelian vortices as 3-branes in six-dimensional spacetime. Our six-dimensional Lagrangian has $SU(5)$ gauge symmetry, so the model has an aspect of $SU(5)$ GUT in addition to a fat brane-world scenario. The model also have additional global $U(1)$ symmetry, which is broken spontaneously in the bulk and gives rise to topologically stable vortices of the non-Abelian kind. On the other hand, the $SU(5)$ gauge symmetry is unbroken in the bulk, but breaks only locally near the cores of vortices. Hence, to which subgroup the $SU(5)$ gauge symmetry breaks depends on how many vortices are generated and where they are located.

In Sec. 2 we studied two specific models which exhibit the desired symmetry breaking $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$ by the non-Abelian vortices. The first model in Sec. 2.1 admits the embedding of usual $U(1)$ global vortices into diagonal elements of the 5 by 5 matrix field $T$. The embedded vortex is a $\frac{1}{5}$ fractionally quantized non-Abelian vortex. There are five different species of the fractional non-Abelian vortex associated with each diagonal element of $T$. The desired configuration can be constructed by putting a vortex with a common vorticity $k_3$ in the first three entries and another vortex with a common vorticity $k_2$ in the remaining two entries of the diagonal elements of $T$, given as $(k_3, k_3, k_3, k_2, k_2)$. However, the position (moduli) of vortices can be freely changed in this first model, leading to unwanted patterns of symmetry breaking such as $SU(5) \rightarrow U(1)^4$. We need to remove the zero modes of such a deformation. This moduli stabilization problem was solved in the second model by adding simple deformation potentials in Sec. 2.2. The key point is to deal with the traceless part $\hat{T}$ (adjoint representation) of $T$ separately from the trace part $T_0$, since the vortex separation corresponds to nonvanishing adjoint fields. Our simple potential induces a domain wall attached to the fractional non-Abelian vortex. The domain walls glue the fractional non-Abelian vortices, and we found that the fractional non-Abelian vortices are confined to form $SU(5)$ singlet. This dynamical process aligns the vorticity, so that a stable configuration has to have identical vorticities in all diagonal entries of $T$. Thus we found that the domain walls tend to fully confine the fractional non-Abelian vortices, but this is not our desired solution since the $SU(5)$ gauge symmetry is unbroken. This point is resolved by a short range repulsive interaction between the vortices. Competition between the long range attraction by the domain wall and the short range repulsion results in a singlet combination clustering into a 3-2 splitting molecule. Namely, the five vortices with identical vorticity split into a molecule configuration with three vortices at a point (associated to $SU(3)$) and two vortices (associated to $SU(2)$) at another point separated by a small distance. This is phenomenologically important once fermions are introduced though we did not explicitly deal with them in this paper. The fact that there are identical vorticities associated to $SU(3)$ and to $SU(2)$ ensures the same number of fermion zero modes in the fundamental representation of $SU(3)$ (quark) and of $SU(2)$ (lepton) are localized.

In Sec. 3 we turned to clarify the localization of the SM gauge fields on a 3-2 splitting background. We performed a standard fluctuation study by introducing small fluctuations...
of all the fields and expanded the Lagrangian to the quadratic order of fluctuations. We can then extract physical informations such as mass spectra of fluctuations. The procedure has a number of complications:

1. The background configuration is inhomogeneous vortex background, which is not axially symmetric because of the 3-2 splitting molecule.

2. The $SU(5)$ gauge symmetry is locally broken to $SU(3) \times SU(2) \times U(1)$, and we need to treat the gauge fields in the unbroken and broken sectors differently.

3. We need a gauge fixing suitable for the vortex molecule background.

4. We need to treat four-dimensional and extra-dimensional components of the gauge fields differently.

We succeeded in showing that the massless gauge fields corresponding to the SM gauge group are localized on the non-Abelian vortex molecule, thanks to a field-dependent gauge kinetic function $BB^\dagger$ in Eq. (2.1). In order to perform the fluctuation analysis efficiently, we developed a vector-analysis like method using derivative operators $\vec{D}$ of the two extra dimensions ($x^4, x^5$) in Eq. (3.16). The method helps not only to make the expressions compact, but also to distinguish physical and unphysical modes. The Helmholtz-like theorem turned out to be important to decompose the extra-dimensional gauge field $\vec{A}$ of the unbroken sector into the rotation-free and divergence-free parts. We found that the rotation-free part of $\vec{A}$ (b’s in Eq. (3.47)) is unphysical, playing the role of NG bosons for the KK towers (except for the bottom) of the four-dimensional gauge fields $A_\mu$ (the SM gauge fields: gluons, W bosons, and hypercharge gauge fields). On the other hand, the divergence-free part of $\vec{A}$ (c’s in Eq. (3.47)) is physical but massless modes are absent. In short, the vortex effective theory contains the SM gauge fields as the only massless modes, and an infinite towers of massive KK modes. The extra-dimensional components ($A_4$ and $A_5$) provide only one (five-dimensional) scalar field degree of freedom whose spectrum does not have massless modes. We applied the same kind of technique to the unbroken sector, although the analysis becomes more complicated because of the mixing with $T$ field. Although analogy to the three-dimensional vector analysis is not as complete as in the unbroken sector, we found the vector-analysis like method is still useful. Similarly to the unbroken sector, the “rotation-free” component of $X$ is found to be unphysical because it is absorbed by $X_\mu$. The important difference from the unbroken sector is that all the KK modes of $X_\mu$ become massive, so that there are no massless vector modes in the unbroken sector. The remaining three degrees of freedom, the divergence-free components, of $X$ were only partially understood. We gave the formal expression of the 4 by 4 Laplacian $\hat{\Delta}_v$ in Eq. (3.71), whose eigenvalues depend on details of the model. We also demonstrated the low lying mass spectrum in a concrete model and pointed out the possibility that an exotic heavy (but not too heavy) vector field in the KK tower of the $U(1)$ hypercharge vector field may be observed.

Thus we provided a class of new models suitable for the gauge sector of the fat brane-world scenario with GUT in six spacetime dimensions by the non-Abelian vortices. As a
future work, firstly, we need to include fermions to complete our mission. As usual in $SU(5)$ models, it is natural to have 5 and 10 representations of $SU(5)$ as fermions. The most important advantage of using the vortices as the host 3-branes is that the six-dimensional theory gives the same number of fermion zero modes for the fundamental representation of $SU(3)$ (quarks) and $SU(2)$ (leptons) forming generations. Namely, the number of the fermion generations is identical to vorticity (the topological number) of the background solution. Secondly, we also need to include a Higgs field which breaks the electroweak symmetry. There is a longstanding issue known as the double-triplet splitting problem in $SU(5)$ GUT models. Whether the 3-2 split non-Abelian vortices configuration can account for the doublet-triplet problem without fine tunings is a challenging future problem. Other phenomenologically challenging issues include hierarchy in quark/lepton masses and mixing matrices, possible right-handed neutrinos, and proton decay.

Acknowledgements

This work is supported in part by JSPS Grant-in-Aid for Scientific Research KAKENHI Grant No. JP21K03565 (M. A.), JP19K03839 (M. E.) and 18H01217 (N. S.). The work of M. E. is supported in part by MEXT KAKENHI Grant-in-Aid for Scientific Research on Innovative Areas “Discrete Geometric Analysis for Materials Design” No.JP17H06462 from the MEXT of Japan. F. Blaschke would like to express his acknowledgment for the institutional support of the Research Centre for Theoretical Physics and Astrophysics, Institute of Physics, Silesian University in Opava.

A Generalized Helmholtz’s decomposition

Here we give a proof that a vector field $\vec{A}$ can be always decomposed into the rotation- and divergence-free components as Eq. (3.36). Firstly, we define a projection operator which projects out $\vec{A}$ onto the rotation-free component

$$P = \bar{D} \circ \left( \bar{D}^\dagger \cdot \bar{D} \right)^{-1} \bar{D} \circ = \text{grad} \triangle^{-1} \text{div}. \quad (A.1)$$

This is a projection operator since it satisfies $P^2 = P$ as

$$P^2 = \left( \bar{D} \circ \left( \bar{D}^\dagger \cdot \bar{D} \right)^{-1} \bar{D}^\dagger \right)^2 = P. \quad (A.2)$$

Therefore, we can always decompose the vector $\vec{A} = P \vec{A} + (1 - P) \vec{A}$. However, $P$ is well-defined only when the inverse Laplacian $\triangle^{-1}$ is well-defined. A dangerous case occurs if $\triangle^{-1}$ acts on its zero mode $\triangle B_0 = 0$ ($\text{grad} B_0 = 0$) as is given in Eq. (3.48). However, this is not the case for our case because $\triangle^{-1}$ acts right after the divergence $\text{div}$. Let $\vec{E}$ as an arbitrary two-component vector, and take an inner product as

$$\left( B_0, \text{div} \vec{E} \right) = \left( \text{grad} B_0, \vec{E} \right) = 0, \quad (A.3)$$
where we defined the inner product by \((A, B) = \int dx^4 dx^5 A^* B\) and \((\vec{A}, \vec{B}) = \int dx^4 dx^5 \vec{A}^\dagger \vec{B}\).

Hence, we conclude that \(\text{div} \vec{E}\) does not include \(B_0\), and therefore \(\Delta^{-1} \text{div}\) is always well-defined, so is \(P\).

Next, we show \(P\vec{A}\) is rotation-free. This is trivial by definition because

\[ P\vec{A} = \text{grad} \left( \Delta^{-1} \text{div} \vec{A} \right), \quad (A.4) \]

and from Eq. (3.27) we always have \(\text{rot}_v P\vec{A} = 0\).

Let us next treat \((1 - P)\vec{A}\). We expand it by eigenfunctions of the non-negative definite Hermitian operator \(\hat{\Delta}_v = \text{rot}_s \text{rot}_v\)

\[ \hat{\Delta}_v \vec{C}_n = \text{rot}_s \text{rot}_v \vec{C}_n = \tilde{m}_n^2 \vec{C}_n, \quad (A.5) \]

\[ (1 - P)\vec{A} = \sum_{n \geq 0} d_n \vec{C}_n. \quad (A.6) \]

Note that index \(n\) of the eigenvalue starts at \(n = 0\) which corresponds to the lowest eigenvalue. Indeed, the lowest eigenvalue is \(\tilde{m}_0 = 0\) and the corresponding eigenfunction is

\[ \text{rot}_v \vec{C}_0 = 0 \quad \Rightarrow \quad \vec{C}_0 \propto \text{grad} \xi, \quad (A.7) \]

with \(\xi\) being an arbitrary function. We now show \(d_0\) is zero. This is because

\[
\left( (1 - P)\vec{A}, \vec{C}_0 \right) \propto \left( (1 - P)\vec{A}, \text{grad} \xi \right) \\
= \left( \text{div} (1 - P)\vec{A}, \xi \right) \\
= \left( \text{div} \vec{A} - \text{div} \text{grad} \Delta^{-1} \text{div} \vec{A}, \xi \right) = 0. \quad (A.8)
\]

Therefore, the above expansion is modified as

\[ (1 - P)\vec{A} = \sum_{n > 0} d_n \vec{C}_n. \quad (A.9) \]

Note also that all the positive eigenvalues \(\tilde{m}_n^2 (n > 0)\) of \(\hat{\Delta}_v\) correspond one-to-one to those of the \(\tilde{\Delta}_v\) defined in Eq. (3.48). Actual correspondence is given by

\[ \vec{C}_n \propto \text{rot}_s C_n, \quad C_n \propto \text{rot}_v \vec{C}_n. \quad (A.10) \]

This can be verified as follows for \(n > 0\) as

\[ \hat{\Delta}_v (\text{rot}_s C_n) = (\text{rot}_s \text{rot}_v) \text{rot}_s C_n = \text{rot}_s \hat{\Delta}_s C_n = \tilde{m}_n^2 (\text{rot}_s C_n). \quad (A.11) \]

It is straightforward to show the other equation. Hence, we now arrive at the desired expression

\[ (1 - P)\vec{A} = \text{rot}_s \left( \sum_{n > 0} d'_n \vec{C}_n \right), \quad (A.12) \]

that implies \((1 - P)\vec{A}\) is divergence free.
References

[1] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, “The Hierarchy problem and new dimensions at a millimeter,” Phys. Lett. B 429, 263-272 (1998) doi:10.1016/S0370-2693(98)00466-3 [arXiv:hep-ph/9803315 [hep-ph]].

[2] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, “New dimensions at a millimeter to a Fermi and superstrings at a TeV,” Phys. Lett. B 436, 257-263 (1998) doi:10.1016/S0370-2693(98)00860-0 [arXiv:hep-ph/9804398 [hep-ph]].

[3] L. Randall and R. Sundrum, “A Large mass hierarchy from a small extra dimension,” Phys. Rev. Lett. 83, 3370-3373 (1999) doi:10.1103/PhysRevLett.83.3370 [arXiv:hep-ph/9905221 [hep-ph]].

[4] G. R. Dvali and M. A. Shifman, “Dynamical compactification as a mechanism of spontaneous supersymmetry breaking,” Nucl. Phys. B 504, 127-146 (1997) doi:10.1016/S0550-3213(97)00420-3 [arXiv:hep-th/9612128 [hep-th]].

[5] R. Jackiw and C. Rebbi, “Solitons with Fermion Number 1/2,” Phys. Rev. D 13, 3398 (1976). doi:10.1103/PhysRevD.13.3398

[6] V. A. Rubakov and M. E. Shaposhnikov, “Do We Live Inside a Domain Wall?,” Phys. Lett. 125B, 136 (1983). doi:10.1016/0370-2693(83)91253-4.

[7] N. Arkani-Hamed and M. Schmaltz, “Hierarchies without symmetries from extra dimensions,” Phys. Rev. D 61, 033005 (2000) doi:10.1103/PhysRevD.61.033005 [arXiv:hep-ph/9903417 [hep-ph]].

[8] G. R. Dvali and M. A. Shifman, “Domain walls in strongly coupled theories,” Phys. Lett. B 396, 64-69 (1997) [erratum: Phys. Lett. B 407, 452 (1997)] doi:10.1016/S0370-2693(97)00131-7 [arXiv:hep-th/9612128 [hep-th]].

[9] N. Maru and N. Sakai, “Localized gauge multiplet on a wall,” Prog. Theor. Phys. 111 (2004) 907 [arXiv:hep-th/0305222].

[10] K. Ohta and N. Sakai, “Non-Abelian Gauge Field Localized on Walls with Four-Dimensional World Volume,” Prog. Theor. Phys. 124, 71-93 (2010) [erratum: Prog. Theor. Phys. 127, 1133 (2012)] doi:10.1143/PTP.124.1143/PTEP.124.71 [arXiv:1004.4078 [hep-th]].

[11] M. Arai, F. Blaschke, M. Eto and N. Sakai, “Matter Fields and Non-Abelian Gauge Fields Localized on Walls,” PTEP 2013, 013B05 (2013) doi:10.1093/ptep/pts050 [arXiv:1208.6219 [hep-th]].

[12] M. Arai, F. Blaschke, M. Eto and N. Sakai, “Stabilizing matter and gauge fields localized on walls,” PTEP 2013, no.9, 093B01 (2013) doi:10.1093/ptep/ptt064 [arXiv:1303.5212 [hep-th]].

[13] M. Arai, F. Blaschke, M. Eto and N. Sakai, Phys. Rev. D 96, no.11, 115033 (2017) doi:10.1103/PhysRevD.96.115033 [arXiv:1703.00351 [hep-th]].

[14] M. Arai, F. Blaschke, M. Eto and N. Sakai, “Non-Abelian Gauge Field Localization on Walls and Geometric Higgs Mechanism,” PTEP 2017, no.5, 053B01 (2017) doi:10.1093/ptep/ptx047 [arXiv:1703.00427 [hep-th]].

[15] M. Arai, F. Blaschke, M. Eto and N. Sakai, “Localization of the Standard Model via the Higgs mechanism and a finite electroweak monopole from non-compact five dimensions,” PTEP 2018, no.8, 083B04 (2018) doi:10.1093/ptep/pty083 [arXiv:1802.06649 [hep-ph]].
[16] M. Arai, F. Blaschke, M. Eto and N. Sakai, “Localized non-Abelian gauge fields in non-compact extra-dimensions,” PTEP 2018, no.6, 063B02 (2018) doi:10.1093/ptep/pty057 [arXiv:1801.02498 [hep-th]].

[17] M. Eto and M. Kawaguchi, “Localization of gauge bosons and the Higgs mechanism on topological solitons in higher dimensions,” JHEP 10, 098 (2019) doi:10.1007/JHEP10(2019)098 [arXiv:1907.04573 [hep-th]].

[18] M. Arai, F. Blaschke, M. Eto and N. Sakai, “Massless bosons on domain walls: Jackiw-Rebbi-like mechanism for bosonic fields,” Phys. Rev. D 100, no.9, 095014 (2019) doi:10.1103/PhysRevD.100.095014 [arXiv:1811.08708 [hep-th]].

[19] I. Oda, “Localization of matters on a string - like defect,” Phys. Lett. B 496 (2000), 113-121 doi:10.1016/S0370-2693(00)01284-3 [arXiv:hep-th/0006203 [hep-th]].

[20] G. R. Dvali, G. Gabadadze and M. A. Shifman, “(Quasi)localized gauge field on a brane: Dissipating cosmic radiation to extra dimensions?,” Phys. Lett. B 497, 271 (2001) doi:10.1016/S0370-2693(00)01329-0 [hep-th/0010071].

[21] A. Kehagias and K. Tamvakis, “Localized gravitons, gauge bosons and chiral fermions in smooth spaces generated by a bounce,” Phys. Lett. B 504, 38 (2001) doi:10.1016/S0370-2693(01)00274-X [hep-th/0010112].

[22] I. Oda, “Localization of bulk fields on AdS(4) brane in AdS(5),” Phys. Lett. B 508 (2001), 96-102 doi:10.1016/S0370-2693(01)00376-8 [arXiv:hep-th/0112013 [hep-th]].

[23] I. Oda, “A new mechanism for trapping of photon,” [arXiv:hep-th/0103052 [hep-th]].

[24] I. Oda, “Trapping of nonAbelian gauge fields on a brane,” [arXiv:hep-th/0103257 [hep-th]].

[25] S. L. Dubovsky and V. A. Rubakov, “On models of gauge field localization on a brane,” Int. J. Mod. Phys. A 16, 4331 (2001) doi:10.1142/S0217751X0101176-5 [hep-th/0010243].

[26] K. Ghoroku and A. Nakamura, “Massive vector trapping as a gauge boson on a brane,” Phys. Rev. D 65, 084017 (2002) doi:10.1103/PhysRevD.65.084017 [hep-th/0106145].

[27] E. K. Akhmedov, “Dynamical localization of gauge fields on a brane,” Phys. Lett. B 521, 79 (2001) doi:10.1016/S0370-2693(01)01176-5 [hep-th/0107223].

[28] I. I. Kogan, S. Mouslopoulos, A. Papazoglou and G. G. Ross, “Multilocalization in multibrane worlds,” Nucl. Phys. B 615, 191 (2001) doi:10.1016/S0550-3213(01)00424-2 [hep-ph/0107307].

[29] H. Abe, T. Kobayashi, N. Maru and K. Yoshioka, “Field localization in warped gauge theories,” Phys. Rev. D 67, 045019 (2003) doi:10.1103/PhysRevD.67.045019 [hep-ph/0205344].

[30] M. Laine, H. B. Meyer, K. Rummukainen and M. Shaposhnikov, “Localization and mass generation for nonAbelian gauge fields,” JHEP 0301, 068 (2003) doi:10.1088/1126-6708/2003/01/068 [hep-ph/0211149].

[31] B. Batell and T. Gherghetta, “Yang-Mills Localization in Warped Space,” Phys. Rev. D 75, 025022 (2007) doi:10.1103/PhysRevD.75.025022 [hep-th/0611305].

[32] R. Guerrero, A. Melfo, N. Pantoja and R. O. Rodriguez, “Gauge field localization on brane worlds,” Phys. Rev. D 81, 086004 (2010) doi:10.1103/PhysRevD.81.086004 [arXiv:0912.0463 [hep-th]].
[33] W. T. Cruz, M. O. Tahim and C. A. S. Almeida, “Gauge field localization on a dilatonic deformed brane,” Phys. Lett. B 686, 259 (2010). doi:10.1016/j.physletb.2010.02.064

[34] A. E. R. Chumbes, J. M. Hoff da Silva and M. B. Hott, “A model to localize gauge and tensor fields on thick branes,” Phys. Rev. D 85, 085003 (2012) doi:10.1103/PhysRevD.85.085003 [arXiv:1108.3821 [hep-th]].

[35] C. Germani, “Spontaneous localization on a brane via a gravitational mechanism,” Phys. Rev. D 85, 055025 (2012) doi:10.1103/PhysRevD.85.055025 [arXiv:1109.3718 [hep-ph]].

[36] T. Delsate and N. Sawado, “Localizing modes of massive fermions and a U(1) gauge field in the inflating baby-skyrmion branes,” Phys. Rev. D 85, 065025 (2012) doi:10.1103/PhysRevD.85.065025 [arXiv:1112.2714 [gr-qc]].

[37] W. T. Cruz, A. R. P. Lima and C. A. S. Almeida, Phys. Rev. D 87, no. 4, 045018 (2013) doi:10.1103/PhysRevD.87.045018 [arXiv:1211.7355 [hep-th]].

[38] A. Herrera-Aguilar, A. D. Rojas and E. Santos-Rodriguez, “Localization of gauge fields in a tachyonic de Sitter thick braneworld,” Eur. Phys. J. C 74, no. 4, 2770 (2014) doi:10.1140/epjc/s10052-014-2770-1 [arXiv:1401.0999 [hep-th]].

[39] Z. H. Zhao, Y. X. Liu and Y. Zhong, “U(1) gauge field localization on a Bloch brane with Chumbes-Holf da Silva-Hott mechanism,” Phys. Rev. D 90, no. 4, 045031 (2014) doi:10.1103/PhysRevD.90.045031 [arXiv:1402.6480 [hep-th]].

[40] C. A. Vaquera-Araujo and O. Corradini, “Localization of abelian gauge fields on thick branes,” Eur. Phys. J. C 75, no. 2, 48 (2015) doi:10.1140/epjc/s10052-014-3251-2 [arXiv:1406.2892 [hep-th]].

[41] G. Alencar, R. R. Landim, M. O. Tahim and R. N. Costa Filho, “Gauge Field Localization on the Brane Through Geometrical Coupling,” Phys. Lett. B 739, 125 (2014) doi:10.1016/j.physletb.2014.10.040 [arXiv:1409.4396 [hep-th]].

[42] R. Davies, D. P. George and R. R. Volkas, “The Standard model on a domain-wall brane,” Phys. Rev. D 77, 124038 (2008) doi:10.1103/PhysRevD.77.124038 [arXiv:0705.1584 [hep-ph]].

[43] A. Davidson, D. P. George, A. Kobakhidze, R. R. Volkas and K. C. Wali, “SU(5) grand unification on a domain-wall brane from an E(6)-invariant action,” Phys. Rev. D 77, 085031 (2008) doi:10.1103/PhysRevD.77.085031 [arXiv:0710.3432 [hep-ph]].

[44] J. E. Thompson and R. R. Volkas, “SO(10) domain-wall brane models,” Phys. Rev. D 80, 125016 (2009) doi:10.1103/PhysRevD.80.125016 [arXiv:0908.4122 [hep-ph]].

[45] B. D. Callen and R. R. Volkas, “Fermion masses and mixing in a 4+1-dimensional SU(5) domain-wall brane model,” Phys. Rev. D 83, 056004 (2011) doi:10.1103/PhysRevD.83.056004 [arXiv:1008.1855 [hep-ph]].

[46] B. D. Callen and R. R. Volkas, “Large lepton mixing angles from a 4+1-dimensional SU(5) x A(4) domain-wall braneworld model,” Phys. Rev. D 86, 056007 (2012) doi:10.1103/PhysRevD.86.056007 [arXiv:1205.3617 [hep-ph]].

[47] N. Okada, D. Raut and D. Villalba, “Domain-Wall Standard Model in non-compact 5D and LHC phenomenology,” Mod. Phys. Lett. A 34, no.10, 1950080 (2019) doi:10.1142/S0217732319500809 [arXiv:1712.09323 [hep-ph]].

[48] N. Okada, D. Raut and D. Villalba, “Fermion Mass Hierarchy and Phenomenology in the 5D Domain Wall Standard Model,” JHEP 10, 259 (2019) doi:10.1007/JHEP10(2019)259 [arXiv:1904.10308 [hep-ph]].
[49] M. V. Libanov and E. Y. Nugaev, “Properties of the Higgs particle in a model involving a single unified fermion generation,” Phys. Atom. Nucl. 70, 864-870 (2007) doi:10.1134/S1063778807050092 [arXiv:hep-ph/0512223 [hep-ph]].

[50] J. M. Frere, M. V. Libanov and S. V. Troitsky, “Three generations on a local vortex in extra dimensions,” Phys. Lett. B 512, 169-173 (2001) doi:10.1016/S0370-2693(01)00696-7 [arXiv:hep-ph/0012306 [hep-ph]].

[51] M. Eto, M. Nitta and N. Sakai, “Effective theory on non-Abelian vortices in six dimensions,” Nucl. Phys. B 701, 247-272 (2004) doi:10.1016/j.physletb.2004.09.003 [arXiv:hep-ph/0204262 [hep-ph]].

[52] A. P. Balachandran and S. Digal, “NonAbelian topological strings and metastable states in linear sigma model,” Phys. Rev. D 66, 034018 (2002) doi:10.1103/PhysRevD.66.034018 [arXiv:hep-ph/0204262 [hep-ph]].

[53] M. Nitta and N. Shiliki, “Non-Abelian Global Strings at Chiral Phase Transition,” Phys. Lett. B 658, 143-147 (2008) doi:10.1016/j.physletb.2007.10.055 [arXiv:0708.4091 [hep-ph]].

[54] E. Nakano, M. Nitta and T. Matsuura, “Interactions of non-Abelian global strings,” Phys. Lett. B 672, 61-64 (2009) doi:10.1016/j.physletb.2008.11.049 [arXiv:0708.4092 [hep-ph]].

[55] M. Eto, E. Nakano and M. Nitta, “Non-Abelian Global Vortices,” Nucl. Phys. B 821, 129-150 (2009) doi:10.1016/j.nuclphysb.2009.06.013 [arXiv:0903.1528 [hep-ph]].

[56] M. Eto, Y. Hirono and M. Nitta, “Domain Walls and Vortices in Chiral Symmetry Breaking,” PTEP 2014, no.3, 033B01 (2014) doi:10.1093/ptep/ptu013 [arXiv:1309.4550 [hep-ph]].

[57] D. T. Son and M. A. Stephanov, “Domain walls in two-component Bose-Einstein condensates,” Phys. Rev. A 65, 063621 (2002) doi:10.1103/PhysRevA.65.063621 [arXiv:cond-mat/0103451 [cond-mat.soft]].

[58] K. Kasamatsu, M. Tsubota and M. Ueda, “Vortex molecules in coherently coupled two-component Bose-Einstein condensates,” Phys. Rev. Lett. 93, no.25, 250406 (2004) doi:10.1103/PhysRevLett.93.250406 [arXiv:cond-mat/0406150 [cond-mat.mes-hall]].

[59] M. Eto, K. Kasamatsu, M. Nitta, H. Takeuchi and M. Tsubota, “Interaction of half-quantized vortices in two-component Bose-Einstein condensates,” Phys. Rev. A 83, 063603 (2011) doi:10.1103/PhysRevA.83.063603 [arXiv:1103.6144 [cond-mat.quant-gas]].

[60] M. Eto and M. Nitta, “Vortex trimer in three-component Bose-Einstein condensates,” Phys. Rev. A 85, 053645 (2012) doi:10.1103/PhysRevA.85.053645 [arXiv:1201.0343 [cond-mat.quant-gas]].

[61] M. Eto and M. Nitta, “Vortex graphs as N-omers and CP(N-1) Skyrmions in N-component Bose-Einstein condensates,” EPL 103, no.6, 60006 (2013) doi:10.1209/0295-5075/103/60006 [arXiv:1303.6048 [cond-mat.quant-gas]].

[62] M. Nitta, M. Eto and M. Cipriani, “Vortex molecules in Bose-Einstein condensates,” J. Low Temp. Phys. 175, 177-188 (2013) doi:10.1007/s10909-013-0925-3 [arXiv:1307.4312 [cond-mat.quant-gas]].

[63] K. Kasamatsu, M. Eto and M. Nitta, “Short-range intervortex interaction and interacting dynamics of half-quantized vortices in two-component Bose-Einstein condensates,” Phys. Rev. A 93, no.1, 013615 (2016) doi:10.1103/PhysRevA.93.013615 [arXiv:1510.00139 [cond-mat.quant-gas]].
[64] M. Tylutki, L. P. Pitaevskii, A. Recati and S. Stringari, “Confinement and precession of vortex pairs in coherently coupled Bose-Einstein condensates,” Phys. Rev. A 93, no.4, 043623 (2016) doi:10.1103/PhysRevA.93.043623 [arXiv:1601.03695 [cond-mat.quant-gas]].

[65] M. Eto and M. Nitta, “Confinement of half-quantized vortices in coherently coupled Bose-Einstein condensates: Simulating quark confinement in a QCD-like theory,” Phys. Rev. A 97, no.2, 023613 (2018) doi:10.1103/PhysRevA.97.023613 [arXiv:1702.04892 [cond-mat.quant-gas]].

[66] M. Kobayashi, M. Eto and M. Nitta, “Berezinskii-Kosterlitz-Thouless Transition of Two-Component Bose Mixtures with Intercomponent Josephson Coupling,” Phys. Rev. Lett. 123, no.7, 075303 (2019) doi:10.1103/PhysRevLett.123.075303 [arXiv:1802.08763 [cond-mat.stat-mech]].

[67] M. Eto, K. Ikeno and M. Nitta, “Collision dynamics and reactions of fractional vortex molecules in coherently coupled Bose-Einstein condensates,” Phys. Rev. Res. 2, no.3, 033373 (2020) doi:10.1103/PhysRevResearch.2.033373 [arXiv:1912.09014 [cond-mat.quant-gas]].

[68] W. C. Yang, C. Y. Xia, M. Nitta and H. B. Zeng, “Fractional and Integer Vortex Dynamics in Strongly Coupled Two-component Bose-Einstein Condensates from AdS/CFT Correspondence,” Phys. Rev. D 102, no.4, 046012 (2020) doi:10.1103/PhysRevD.102.046012 [arXiv:2003.09423 [cond-mat.quant-gas]].

[69] M. Eto and M. Nitta, “Minimum non-Abelian vortices and their confinement in three flavor dense QCD,” [arXiv:2103.13011 [hep-ph]].