Unitarity Triangle from CP invariant quantities

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Abstract

We construct the CKM unitarity triangle from CP invariant quantities, using the coupling constant of weak decays with flavor change from $b$ to $u$, and the particle - antiparticle mixing probabilities in the $B_s^0$ and $B_d^0$ systems. Also included are new measurements of the coupling $V_{us}$ in Kaon decays. Of the two solutions, one agrees perfectly with the triangle constructed from CP violating processes in the K and B meson systems. The common solution yields a triangle with an area of $J/2 = (1.51 \pm 0.09) \times 10^{-5}$ and a CP violating phase $\delta = 63.1^\circ \pm 4.0^\circ$.

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The Discovery of direct CP violation in the Kaon system and the confirmation in the neutral B meson system showed that CP violation is due to the weak interaction \[1\]. In the CKM scheme for three generations of quarks, one phase in the complex mixing matrix can cause a difference between CP conjugate transitions. However, if this phase vanishes, this scheme cannot account for CP violation. The CKM mixing matrix parametrizes transitions between three generations of quarks, most of the relevant observables are derived from CP invariant processes, in particular weak decays of quarks.

In this letter we point out that new measurements of the mixing between \(B_s^0\) and \(\bar{B}_s^0\) \[2\] \[3\] mesons together with data on mixing between \(B_d^0\) and \(\bar{B}_d^0\) and on transitions between \(b\) and \(u\) quarks allow a determination of the phase \(\delta\) in the CKM scheme in spite of the fact that both processes are CP invariant. We then compare this double-valued result for \(\delta\) with the value obtained from measurements of CP violating observables in the K meson and B meson system.

The matrix relating the quark mass eigenstates and the weak eigenstates for six quarks was introduced with an explicit parametrization by Kobayashi and Maskawa \[4\] in 1973. By convention, the mixing is expressed in terms of a \(3 \times 3\) unitary matrix \(V\) operating on the charge \(-e/3\) quark mass eigenstates \((d, s, \text{and} b)\):

\[
\begin{pmatrix}
  d' \\
  s' \\
  b'
\end{pmatrix} =
\begin{pmatrix}
  V_{ud} & V_{us} & V_{ub} \\
  V_{cd} & V_{cs} & V_{cb} \\
  V_{td} & V_{ts} & V_{tb}
\end{pmatrix}
\begin{pmatrix}
  d \\
  s \\
  b
\end{pmatrix}.
\]

We use the “standard” parametrization \[5\] of \(V\) that utilizes angles \(\theta_{12}, \theta_{23}, \theta_{13}\), and a phase, \(\delta\)

\[
V =
\begin{pmatrix}
  c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
  -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
  s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix},
\]

with \(c_{ij} = \cos \theta_{ij}\) and \(s_{ij} = \sin \theta_{ij}\) for the “generation” labels \(i, j = 1, 2, 3\). This parametrization is exact to all orders, and has four parameters; the real angles \(\theta_{12}, \theta_{23}, \theta_{13}\) can all be made to lie in the first quadrant by an appropriate redefinition of quark field phases.

Information on the smallest matrix elements of the CKM matrix can be summarized in terms of the “unitarity triangle,” one of six such triangles. Unitarity applied to the first and third columns yields

\[
V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0.
\]
The unitarity triangle is just a geometrical presentation of this equation in the complex plane [8].

The angles of the triangle $\alpha$, $\beta$ and $\gamma$ are also referred to as $\phi_2$, $\phi_1$, and $\phi_3$, respectively, with $\beta$ and $\gamma$ being the phases of the CKM elements $V_{td}$ and $V_{ub}$ as per

$$V_{td} = |V_{td}|e^{-i\beta}, V_{ub} = |V_{ub}|e^{-i\gamma}$$

(4)
to a precision of better than a tenth of a degree.

Rescaling the triangle so that the base is of unit length, the coordinates of the vertices become:

$$\rho = \text{Re}(V_{ud} V_{ub}^*) / |V_{cd} V_{cb}^*|, \quad \eta = \text{Im}(V_{ud} V_{ub}^*) / |V_{cd} V_{cb}^*|, \quad (1, 0), \quad (0, 0).$$

(5)

$CP$-violating processes involve the phase $\delta$ in the CKM matrix, assuming that the observed $CP$ violation is solely related to a nonzero value of this phase. A necessary and sufficient condition for $CP$ violation with three generations is then that the determinant $J$ of the commutator of the mass matrices for the charge $2e/3$ and charge $-e/3$ quarks is non-zero [7].

$CP$-violating amplitudes or differences of rates are all proportional to the product of CKM factors in this quantity, namely

$$J = s_{12}s_{13}s_{23} c_{12} c_{23} \sin \delta.$$  

This is just twice the area of the unitarity triangle.

We now proceed to determine the unitarity triangle from $CP$ invariant quantities alone. For the following input data we refer to our last review [9] and the references therein:

$$|V_{ud}| = 0.9738 \pm 0.0005, \quad |V_{cd}| = 0.224 \pm 0.012, \quad |V_{cs}|^2 = (2.039 \pm 0.026) - |V_{ud}|^2 - |V_{us}|^2 - |V_{ub}|^2 - |V_{cd}|^2 - |V_{cb}|^2,$$

$$|V_{us}|^2 = 0.961 \pm 0.024.$$

(6)

$|V_{us}|$—New results from NA48 [10], KLOE [11], and KTEV [12] on the branching ratio of $K_L \to \pi e\nu$ and on the $K_L$ lifetime have been used to extract $V_{us} \times f^+(0, K^0) = 0.2171 \pm 0.0004$. Two additional results from $K^+ \to \pi^0 e\nu$ yield an average $V_{us} \times f^+(0, K^+) = 0.2233 \pm 0.0011$. Two independent theoretical evaluations of the form factor $f^+(0)$ have been published. Chiral perturbation theory at the $p^6$ approximation gives $f^+(0, K^0) = 0.981 \pm 0.010$ for $K^0$ and $1.002 \pm 0.010$ for $K^+$ [13]. Lattice calculations yield $f^+(0, K^0) = 0.961 \pm 0.009$ [14]. We take the average of both and obtain $V_{us} = 0.2238 \pm 0.0004 \pm 0.0023$ from $K^0$ decays and $0.2252 \pm 0.0022$ from $K^+$ decays. The overall average is

$$|V_{us}| = 0.2244 \pm 0.0022,$$

where the error comes chiefly from the theoretical uncertainty.
Measurements of the exclusive decay $B \to \bar{D}^* \ell^+ \nu_\ell$ have been used to extract a value of $|V_{cb}|$ using corrections based on HQET \cite{15}. Exclusive $B \to \bar{D} \ell^+ \nu_\ell$ decays give a consistent, but less precise result. Analysis of inclusive decays depends on going from the quark to the hadron level and involves an assumption on the validity of quark-hadron duality. The results for $|V_{cb}|$ from exclusive and inclusive decays generally are in good agreement. A more detailed discussion and references are found in a mini-review in the Review of Particle Physics \cite{16}. We take the average of the exclusive result $|V_{cb}| = (41.3 \pm 1.0 \pm 1.8) \times 10^{-3}$ \cite{17} and inclusive result $|V_{cb}| = (41.4 \pm 0.6 \pm 0.1) \times 10^{-3}$ \cite{20} with theoretical uncertainties combined linearly, weighted with their contribution to the average, and obtain

$$|V_{cb}| = (41.4 \pm 0.8) \times 10^{-3}. \quad (7)$$

$|V_{ub}|$ – A compilation of the most recent results on $|V_{ub}|$ has been published by the "Heavy Flavour Averaging Group" recently \cite{21}, averaging results obtained from the CLEO, Babar and BELLE experiments. The quoted uncertainties for the inclusive determination is much smaller than for the exclusive ones, we therefore choose to use their inclusive average.

$$|V_{ub}| = (4.39 \pm 0.46) \times 10^{-3}. \quad (8)$$

The new element in this analysis is the measurement of $B_s^0 - \bar{B}_s^0$ - mixing by the $D0$ \cite{2} and $CDF$ \cite{3} collaborations. We assume that the data shown by these experiments are evidence for such mixing. Then the mixing parameter is determined by $D0$ to be

$$\Delta M_{B_s} = (19 \pm 1) \text{ ps}^{-1} \quad (9)$$

This finding has been confirmed by the $CDF$ collaboration \cite{3}, they measure the mixing parameter to be

$$\Delta M_{B_s} = (17.33^{+0.42}_{-0.21}(\text{stat}) \pm 0.07(\text{syst})) \text{ ps}^{-1} \quad (10)$$

In the ratio of $B_s$ to $B_d$ mass differences, many common factors (such as the QCD correction and dependence on the top-quark mass) cancel, and we have

$$\frac{\Delta M_{B_s}}{\Delta M_{B_d}} = \frac{M_{B_s} \hat{B}_{B_s} f_{B_s}^2 |V_{tb}^* \cdot V_{ts}|^2}{M_{B_d} \hat{B}_{B_d} f_{B_d}^2 |V_{tb}^* \cdot V_{td}|^2}. \quad (11)$$
With the experimentally measured masses, the well measured quantity $\Delta M_{B_d}$ \cite{23}, $\hat{B}_{B_d}f_{B_d}/(\hat{B}_{B_d}f_{B_d})^2 = (1.210^{+0.047}_{-0.035})^2$ \cite{25}, these new measurements when averaged to $\Delta M_{B_s} = (17.50^{+0.40}_{-0.22})$ ps$^{-1}$, transform into

$$|V_{ts}|/|V_{td}| = 4.78 \pm 0.17.$$  \hspace{1cm} (12)

A detailed statistical analysis of all available measurements, including those from Tevatron, LEP and SLD, yields a significance of the $B_s$ mixing signal to be 3.8 $\sigma$ \cite{24}. We insert equation (12) in our fit of the CKM matrix and obtain contours for the allowed range of the top of the unitarity triangle, at one and two standard deviation confidence level, as shown in Figure 1. The $\chi^2_{min}$ found is 3.3 with 4 degrees of freedom, this or a higher value is expected to appear with 50% probability. We define a one standard deviation confidence level to be the multitude of all points in the unitarity plane to have a solution in the four dimensional parameter space with a $\chi^2 < \chi^2_{min} + 1$, and the two standard deviation confidence to fulfil $\chi^2 < \chi^2_{min} + 4$ respectively. This is a slightly different approach than the one used by other groups \cite{18}, \cite{19}, leading to similar results given the precision at which the larger angles are known.

Both measurements of $|V_{ub}|$ and $|V_{ts}|/|V_{td}|$ yield a circle-shaped range of allowed values. The two circles overlap at two positions corresponding to $\delta = 66^\circ \pm 4^\circ$ and $\delta = 294^\circ \pm 4^\circ$. The measurements of CP invariant quantities alone therefore allow for the first time the prediction that CP violation exists in the framework of the CKM scheme. If the two circles would overlap at $\delta = 0$ or $\delta = 180^\circ$, no CP violation would be expected. This would correspond to values of $|V_{ts}|/|V_{td}| = 7.7$ resp. $|V_{ts}|/|V_{td}| = 3.1$ Only the latter of these solutions was excluded by earlier measurements, the first was still allowed within two standard deviations from the measurement of $\Delta M_{B_d}$ and lattice QCD calculations.

We now turn to CP violating effects. Three well understood independent measurements are available by now, the direct measurements of $\gamma$ in $B \to D^{(+)}K$ still have too large uncertainties. Just the added constraint from CP violation in the neutral kaon system, taken together with the restrictions above on the magnitudes of the CKM matrix elements, is tight enough to restrict considerably the range of angles and the phase of the CKM matrix. For example, the constraint obtained from the CP-violating parameter $\epsilon_K$ in the neutral $K$ system corresponds to the vertex A of the unitarity triangle lying on a hyperbola for fixed values of the (imprecisely known) hadronic matrix elements \cite{31}, \cite{32}.
In the B-meson system, for CP-violating asymmetries of neutral B mesons decaying to CP eigenstates, the interference between mixing and a single weak decay amplitude for certain final states directly relates the asymmetry in a given decay to \( \sin 2\phi \), where \( \phi = \alpha, \beta, \gamma \) is an appropriate angle of the unitarity triangle \([8]\). A new generation of experiments has established a non-vanishing asymmetry in the decays \( B_d(\bar{B}_d) \to \psi K_S \) and in other \( B_d \) decay modes where the asymmetry is given by \( \sin 2\beta \). The present experimental results from BaBar \([34]\) and Belle \([35]\), when averaged yield

\[
\sin 2\beta = 0.687 \pm 0.032 .
\]

A non-vanishing asymmetry in the decays \( B_d(\bar{B}_d) \to \rho\rho \) from BaBar \([36]\) and Belle \([37]\), \( B_d(\bar{B}_d) \to \rho\pi \) and \( B_d(\bar{B}_d) \to \pi\pi \), when averaged, can be used to constrain the angle \( \alpha \).

The confidence level contours from the \( \pi\pi \) and \( \rho\rho \) isospin analyses as well as the \( \rho\pi \) Dalitz plot yield \([24]\).
\[ \alpha = (99^{+12}_{-9}[1\sigma]^{+22}_{-16}[2\sigma])^\circ. \]  

These three constraints, together with those on the large angles, applied to the unitarity plane, overlap only in one region at the one or two standard deviation level, as shown in figure 2. The resulting CP-phase \( \gamma = 56.7^\circ \pm 8.0^\circ \), as measured from CP violating effects, can be compared with the one predicted from the CP conserving measurements, as shown in Figure 3.

An overall fit of the CKM matrix yields values of the sines of the angles of \( s_{12} = 0.2261 \pm 0.0014 \), \( s_{23} = 0.04125 \pm 0.00070 \), and \( s_{13} = 0.00362 \pm 0.00018 \) and to a value of the CP-phase \( \gamma = 63.1^\circ \pm 4.0^\circ \), with \( \chi^2_{\text{min}} = 7.1 \) for 7 degrees of freedom, this or a higher value expected at the 42% level. Finally, \( V_{td} = (0.00774 \pm 0.00027 - (0.00314 \pm 0.00018)i \). The area of the unitarity triangle is \( J/2 = (1.51 \pm 0.09) \times 10^{-5} \).

The two allowed regions in the upper half-plane of Figure 3 overlap at bet-

Figure 2: Unitarity triangle constrained by CP violating quantities only. Bold lines indicate one standard deviation, thin lines two standard deviations.
Figure 3: Comparison of unitarity triangle apex obtained from CP conserving amplitudes (horizontally dashed) to the one obtained from CP violating quantities (vertically dashed). Also shown is the best fit unitarity triangle.

The combined fit has 7 degrees of freedom, 3 more than the fit to CP conserving quantities alone. The $\chi^2$ value increases from 3.3 to 7.1 by adding the constraints from CP violating quantities. In conclusion, we have shown that after the observation of $B_s$ mixing, the unitarity triangle can be constructed from CP conserving quantities alone, and one of the resulting solutions agrees at the 50% confidence level with the triangle constructed from CP conserving and CP violating quantities.

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