Comments on Takahashi-Tanimoto’s scalar solution

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Abstract

We study the identity-based solution of Witten’s cubic bosonic open string field theory constructed by Takahashi and Tanimoto, which is claimed to describe the tachyon vacuum. We argue that the observables of the solution coincide with those of the tachyon vacuum using the method proposed by Kishimoto and Takahashi. We also discuss how to treat the kinetic term of the string field theory expanded around it.

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1 Introduction

Since the discovery of the tachyon vacuum solution by Schnabl [1], various kinds of analytic solutions of the equation of motion of the cubic bosonic open string field theory [2] have been constructed (for reviews, see [3, 4, 5]). It is now possible to construct a solution corresponding to any known open string background [6].

Most of the solutions found since [1] are so-called regular solutions which consist mainly of wedge states with non-vanishing width with operator insertions. There exist some solutions which are not of this kind. An example is the solution

$$\Psi_{TT} = \left[ \int_{C_{\text{left}}} \frac{d\xi}{2\pi i} (e^{h_a} - 1) j_B(\xi) - \int_{C_{\text{left}}} \frac{d\xi}{2\pi i} (\partial h_a)^2 e^{h_a} c(\xi) \right]|I\rangle,$$  \hspace{1cm} (1.1)

given by Takahashi and Tanimoto [7], which is called the scalar solution. Here $C_{\text{left}}$ is a contour in the upper half plane depicted in Fig. 1, $j_B$ is the BRST current

$$j_B(\xi) = \left[ cT + b c\partial c + \frac{3}{2} \partial^2 c \right](\xi),$$

$|I\rangle$ is the identity string field and $h_a(\xi)$ is a function taken to be

$$h_a(\xi) = \ln \left( 1 + \frac{a}{2} \left( \xi + \frac{1}{\xi} \right)^2 \right),$$  \hspace{1cm} (1.2)

for $a \geq -\frac{1}{2}$. Takahashi and Tanimoto claim that while the solution is a pure gauge solution for $a > -\frac{1}{2}$, it is a tachyon vacuum solution for $a = -\frac{1}{2}$.

![Figure 1: C_{left}](image)

The solution (1.1) is expressed as an identity state with local operator insertions. The solutions of such a form are called identity-based solutions. It is difficult to calculate observables like energy or Ellwood invariant of identity-based solutions. These quantities correspond to correlation functions of operators on a strip with vanishing width in the worldsheet theory and naive regularizations fail to yield definite values [8, 9, 10].

On the other hand, the identity-based solutions have some advantages. In general, the string field action expanded around a classical solution $\Psi_{\text{cl}}$ can be given as

$$S'[\Psi] = -\frac{1}{g^2} \int \left[ \frac{1}{2} \Psi Q' \Psi + \frac{1}{3} \Psi \Psi \Psi \right],$$  \hspace{1cm} (1.3)
where
\[ Q'A = QA + \Psi_{cl}A - (-1)^{|\Psi_{cl}|A} A\Psi_{cl}. \]

In the case of regular solutions, \( \Psi_{cl} \) involves wedge states with finite width and it will be very difficult to study the string field theory action (1.3) with the kinetic operator \( Q' \). However, if \( \Psi_{cl} \) is an identity-based solution, the \( Q' \) can be expressed by local operators on the worldsheet. For example, if \( \Psi_{cl} \) is the Takahashi-Tanimoto solution (1.1), the \( Q' \) becomes
\[ \oint \frac{d\xi}{2\pi i} \left[ e^{h+}j_B(\xi) - (\partial h_a)^2 e^{h+}c(\xi) \right]. \] (1.4)

With \( Q' \) being an operator like this, we expect it is relatively easy to deal with the string field theory action (1.3).

Although the observables are not available, there are many evidences indicating that the Takahashi-Tanimoto solution (1.1) with \( a = -\frac{1}{2} \) is a tachyon vacuum solution:

- There are no physical open string excitations around the background corresponding to \( a = -\frac{1}{2} \). This fact has been shown by studying the BRST cohomology [11] or by constructing the homotopy operator [12].
- The open string amplitudes around the background can be shown to vanish [13].
- Solving the equation of motion in the background corresponding to \( a = -\frac{1}{2} \) numerically, an unstable solution which is supposed to correspond to the perturbative vacuum can be found [14, 15, 16].

All these evidences imply that the solution corresponds to the tachyon vacuum. It should be interesting to explore the string field theory around such a background and see whether or not the closed string amplitudes can be reproduced from it. Since the solution is an identity-based solution, the string field theory expanded around the solution will have a tractable kinetic term.

In this paper, we would like to study the Takahashi-Tanimoto solution (1.1) with \( a = -\frac{1}{2} \) and the string field theory expanded around it. What we will do first is to evaluate the observables of the solution in a rather indirect manner. In a recent paper [17], the authors consider the Erler-Schnabl solutions in the string field theory expanded around the identity-based marginal solutions found in [18, 7]. Since the Erler-Schnabl solutions will correspond to the tachyon vacuum, by calculating the observables of these solutions, they are able to evaluate the observables of the identity-based marginal solutions. We here apply this method to the scalar solution (1.1) with \( a = -\frac{1}{2} \) and see what we can say about the observables of it. By doing so, we will get further evidences for the claim that the solution is a tachyon vacuum solution. In the latter half of the paper, we will discuss the string field theory expanded around the the solution. We will show how we should treat the kinetic operator (1.4) in order for the solution to correspond to the tachyon vacuum.

The organization of this paper is as follows. In section 2, we evaluate the observables of the Takahashi-Tanimoto solution by calculating those of the Erler-Schnabl solution in the string field theory expanded around it. In section 3, we consider the string field theory around the Takahashi-Tanimoto solution and discuss how we should treat the kinetic operator. Section 4 is devoted to conclusions and discussions. In appendix A, we discuss the method proposed recently by Maccaferri [19] to construct regular solutions gauge equivalent to identity-based solutions. We explain what we can get by applying the method to the solution (1.1). In appendix B, we derive some identities concerning the operators \( U, U^{-1} \) which play important roles in the main text.
Note added

In the workshop “String field theory and related aspects VI, SFT2014” (July 28 -August 1, 2014, SISSA Italy), where this work is presented [20], we have learned that Kishimoto, Masuda and Takahashi work on the same problem from a different point of view [21][22]. Their results have some overlap with those in section 2.

While this paper was being typed, a paper [23] appeared on the arXiv, which also treat the same problem. There is some overlap with the contents of appendix A but the identity-based solution they deal with is different from ours.

2 The Erler-Schnabl solution in the string field theory expanded around the Takahashi-Tanimoto solution

2.1 The Erler-Schnabl solution

The Erler-Schnabl solution [24]

\[ \Psi_{ES} = \frac{1}{1 + K} \left( c + Q(Bc) \right), \]  

satisfies the equation of motion of the cubic string field theory. Here \( K, B, c \) are the string fields defined by

\[ B = \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \frac{dz}{2\pi i} b(z) |I\rangle, \]

\[ c = c(z) \big|_{z = \frac{1}{2}} |I\rangle, \]

\[ K = QB = \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \frac{dz}{2\pi i} T(z) |I\rangle, \]

and the product of them is the star product. \( z \) is the sliver frame coordinate which is expressed by the upper half plane coordinate \( \xi \) in (1.1) as

\[ z = \frac{2}{\pi} \arctan \xi. \]

\( K, B, c \) and \( Q \) satisfy the so-called \( KBc \) algebra [25, 26] and one can show that \( \Psi_{ES} \) is a solution by using the algebra. The Erler-Schnabl solution \( \Psi_{ES} \) describes the tachyon vacuum. This fact can be shown by calculating the observables or by showing that

\[ A = B \frac{1}{1 + K}, \]

gives the homotopy operator for the background \( \Psi_{ES} \), i.e. \( QA + \Psi_{ES} A + A \Psi_{ES} = 1 \) [27]. The existence of the homotopy operator implies that there exists no physical open string states around the background \( \Psi_{ES} \).

As was pointed out in [28], it is straightforward to construct the Erler-Schnabl solution in the string field theory (1.3) expanded around an identity-based solution. \( Q' \) is a nilpotent operator and acts on string fields as a derivation. It is easy to see that

\[ \Psi_{ES}' = \frac{1}{1 + K'} \left( c + Q' (Bc) \right). \]
with
\[ K' = Q'B, \]
satisfies the equation of motion derived from the string field action (1.3), because the \( K', B, c \) and \( Q' \) satisfy the same algebra as the \( KBc \) and \( Q \) do. Moreover, the homotopy operator for the solution \( \Psi'_{ES} \) can be constructed as
\[ A' = B \frac{1}{1 + K'}. \]
Therefore one can argue that the solution \( \Psi'_{ES} \) describes the tachyon vacuum, provided \( \frac{1}{1 + K'} \) is a regular quantity.

Let us consider the Erler-Schnabl solution \( \Psi'_{ES} \) in the string field theory expanded around the Takahashi-Tanimoto solution given in (1.1) with \( a = -\frac{1}{2} \). In this case, \( Q' \) is expressed by a contour integral
\[ \oint \frac{dz}{2\pi i} \left[ -\frac{\sin^2 \pi z}{\cos^2 \pi z} j_B(z) + \frac{4\pi^2}{\cos^4 \pi z} c(z) \right], \]
in the silver frame and \( K' \) becomes
\[ K' = K + J, \]
\[ J \equiv \int_{\frac{1}{2}+i\infty}^{\frac{1}{2}+i\infty} \frac{dz}{2\pi i} \left[ -\frac{1}{\cos^2 \pi z} T'(z) + \frac{4\pi^2}{\cos^4 \pi z} \right] |I|, \]
\[ T'(z) \equiv T_{\text{matter}}(z) - b\partial c(z). \]

\( \frac{1}{1 + K'} \) can be expressed as
\[ \frac{1}{1 + K'} = \int_0^\infty dL e^{-L(1+K')}, \]
in the usual way and we need to define \( e^{-LK'} \) to make sense of such quantities. In this section, we expand \( e^{-LK'} \) as
\[ e^{-LK'} = e^{-L(K+J)} \]
\[ = \sum_{n=0}^\infty (-1)^n \lim_{\delta \to +0} \int_\delta^{\infty} dL_1 \cdots \int_\delta^{\infty} dL_{n+1} \delta \left( \sum_{i=1}^{n+1} L_i - L \right) e^{-L_1 K} J e^{-L_2 K} J \cdots J e^{-L_{n+1} K}. \]
and consider the right hand side as the definition of \( e^{-LK'} \). From the point of view of the worldsheet theory, we define \( e^{-LK'} \) perturbatively treating \( J \) as perturbation. Since \( J \) is a chiral quantity on the worldsheet, we do not encounter any ultraviolet divergences and the expression is well-defined\(^1\). However, there is still a room for finite renormalizations. A prescription for such renormalization is fixed by introducing a cut-off \( \delta \).

Now let us consider the observables of the Erler-Schnabl solution \( \Psi'_{ES} \). The observables we consider are the action and the Ellwood invariant [29, 30, 31]. The action of \( \Psi'_{ES} \) in the string field theory (1.3) is equal to the difference of the energy between the background corresponding to \( \Psi_{TT} \) and that corresponding to \( \Psi'_{ES} \). The Ellwood invariant of \( \Psi'_{ES} \) becomes the difference of the 1-point function of a closed string vertex

\(^1\)Notice that the normalization of \( J \) is fixed by the equation of motion and there is no reason to expect that the higher order terms in the expansion (2.5) are small in any sense. We will treat the operator \( K' \) without using such an expansion in section 3.
operator $V$ between these backgrounds. Thus they can be expressed as

\[ S[\Psi'_{ES}] = E_{TT} - E_{\Psi'_{ES}}, \]
\[ \text{Tr}_V \Psi'_{ES} = \langle V \rangle_{\Psi'_{ES}} - \langle V \rangle_{TT}, \]

where $E, \langle V \rangle$ denote the energy and the one-point function of each background respectively. In the following, we will show $S[\Psi'_{ES}] = \text{Tr}_V \Psi'_{ES} = 0$, which implies

\[ E_{\Psi'_{ES}} = E_{TT}, \]
\[ \langle V \rangle_{\Psi'_{ES}} = \langle V \rangle_{TT}. \]

Since we assume that $\Psi'_{ES}$ corresponds to the tachyon vacuum, we can see that the observables $E_{TT}, \langle V \rangle_{TT}$ of the identity-based solution $\Psi_{TT}$ coincide with those of the tachyon vacuum. Therefore showing $S[\Psi'_{ES}] = \text{Tr}_V \Psi'_{ES} = 0$ gives evidences for the claim that the Takahashi-Tanimoto solution $\Psi_{TT}$ describes the tachyon vacuum.

In this section, we use this indirect way proposed in [17] to calculate the observables $E_{TT}, \langle V \rangle_{TT}$ of the identity-based solution $\Psi_{TT}$. Recently there are somewhat more direct ways to calculate these quantities [28, 32]3[19]. Especially Maccaferri [19] uses the so-called Zeze map [33] to construct regular solutions gauge equivalent to identity-based ones and calculate the observables of the regular ones. Moreover, the calculations eventually reduce to those of the $S[\Psi'_{ES}], \text{Tr}_V \Psi'_{ES}$. In appendix A, we explain how we can apply Maccaferri’s method to the Takahashi-Tanimoto solution (1.1) with $a = -\frac{1}{2}$.

### 2.2 Observables of $\Psi'_{ES}$

Now let us calculate the observables $S[\Psi'_{ES}], \text{Tr}_V \Psi'_{ES}$ and show that both of them vanish4. From the expression (2.2), we obtain

\[ S[\Psi'_{ES}] = -\frac{1}{6g^2} \text{Tr} \left[ \frac{1}{1+K'}c \frac{1}{1+K}Q'c \right], \]
\[ \text{Tr}_V \Psi'_{ES} = \text{Tr}_V \left[ \frac{1}{1+K'}c \right]. \]

Therefore what we will prove are

\[ \text{Tr}_V \left[ \frac{1}{1+K'}c \right] = 0, \]
\[ \text{Tr} \left[ \frac{1}{1+K'}c \frac{1}{1+K}Q'c \right] = 0. \]

These can be proved by using the following identities:

\[ Q \left( \frac{1}{\pi^2}b \right) = 1, \]
\[ Q'c = 0. \]

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2The Ellwood invariant $\text{Tr}_V \Phi$ is given as

\[ \text{Tr}_V \Phi = \langle I | V (i, -i) | \Phi \rangle, \]

where $V (i, -i) = cV^m (i, -i)$ is a closed string vertex operator.

3Kishimoto, Masuda and Takahashi [22] generalize the method of [28, 32] to the case of the scalar solutions.

4Kishimoto, Masuda and Takahashi [22] obtain the same results using a different method, considering more general solutions made from $K'Bc$. 

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Here
\[ \frac{1}{\pi^2} b \equiv \frac{1}{\pi^2} b(z) \bigg|_{z = \xi} = b(\xi) |_{\xi = 1} | I \, , \]
and (2.9) suggests that the $\frac{1}{\pi^2} b$ works as a homotopy operator of the BRST charge $Q'$. To be precise, one can show (2.9)(2.10) in the situation where we have some worldsheet around $\frac{1}{\pi^2} b, c$ without any local operator insertions. Namely we should consider
\[
e^{-\epsilon K} Q' \left( \frac{1}{\pi^2} b \right) e^{-\epsilon K} = e^{-2\epsilon K}, \tag{2.11}
\]
\[
e^{-\epsilon K} Q' ce^{-\epsilon K} = 0, \tag{2.12}
\]
in which we attach $e^{-\epsilon K}$'s to generate worldsheet as is depicted in Fig. 2. 5. With the worldsheet, one can express the action of $Q'$ by the contour integral (2.3) and get
\[
e^{-\epsilon K} Q' \left( \frac{1}{\pi^2} b \right) e^{-\epsilon K} = e^{-\epsilon K} \left( \int_0^{2\pi i} \frac{dz}{2\pi i} \left( \frac{\sin^2 \pi z}{\cos^2 \pi z} b(z) + \frac{4\pi^2}{\cos^4 \pi z} c(z) \right) \frac{1}{\pi^2} b(0) \right) e^{-\epsilon K}
= e^{-2\epsilon K}. \tag{2.13}
\]
(2.12) can be derived in the same way.

Using (2.11)(2.12), we obtain (2.7) as follows. Inserting (2.9) into Trv $\left[ \frac{1}{1 + K'} c \right]$, we get
\[
\text{Tr}_V \left[ \frac{1}{1 + K'} \right] = \text{Tr}_V \left[ \frac{1}{1 + K'} Q' \left( \frac{1}{\pi^2} b \right) \frac{1}{\sqrt{1 + K'}} c \right].
\]
Here we use the definition
\[
\frac{1}{\sqrt{1 + K'}} = \frac{1}{\Gamma \left( \frac{1}{2} \right)} \int_0^{\infty} dLL^{\frac{1}{2}} e^{-L} e^{-LK'},
\]

5In [19], such a prescription is used for the equation of motion. One can show that the $\Psi_{TT}$ in (1.1) satisfies the equation of motion in the same way.
where $e^{-LK'}$ expressed as (2.5). With the cutoff $\delta$, (2.9)(2.10) can be safely used because there are some worldsheets with no operator insertions around $b, c$. Thus we obtain

$$\text{Tr}_V \left[ \frac{1}{1 + K' c} \right] = \text{Tr}_V \left[ \frac{1}{\sqrt{1 + K'}} \frac{1}{\pi^2} b \frac{1}{\sqrt{1 + K'}} Q' c \right] = 0.$$ 

(2.8) can be derived from (2.12) in the same way.

Before closing this section, one comment is in order. Using $Q' c = 0$, one can see that from (2.2)

$$\Psi'_{ES} = c.$$ 

Thus actually the $\Psi'_{ES}$ itself is an identity-based solution$^6$, although we do not have any trouble in calculating the right hand sides of (2.6). One can avoid this by replacing $c$ by

$$c_y \equiv c \left( \frac{1}{2} + i y \right) | I \rangle \quad (y \neq 0, \ y \in \mathbb{R}).$$

$K', B, c_y$ satisfy the $KBC$ algebra and one can construct the Erler-Schnabl solution

$$\Psi'_{ES, y} = \frac{1}{1 + K'} (c_y + Q'(Bc_y)),$$

which is not identity-based. The observables to be calculated become

$$S \left[ \Psi'_{ES, y} \right] = -\frac{1}{6g^2} \text{Tr} \left[ \frac{1}{1 + K'} c_y \frac{1}{1 + K'} Q' c_y \right],$$

$$\text{Tr}_V \Psi'_{ES, y} = \text{Tr}_V \left[ \frac{1}{1 + K'} c_y \right]. \quad (2.14)$$

One can show that these quantities are actually independent of $y$. Indeed, using the $KBC$ identity

$$\{ B, c_y \} = 1,$$

the formulas given in [19] (eqs.(3.4),(3.10)-(3.19)) imply

$$\frac{1}{6} \text{Tr} \left[ \frac{1}{1 + K'} c_y \frac{1}{1 + K'} Q' c_y \right] \quad = \frac{1}{6} \text{Tr} \left[ \frac{1}{1 + K} c_y \frac{1}{1 + K} Q c_y \right] - \frac{1}{6} \text{Tr} \left[ \frac{1}{1 + K} \Psi_{TT} \frac{1}{1 + K} \Psi_{TT} \frac{1}{1 + K'} \Psi_{TT} \frac{1}{1 + K'} \right],$$

$$\text{Tr}_V \left[ \frac{1}{1 + K'} c_y \right] = \text{Tr}_V \left[ \frac{1}{1 + K} c_y \right] - \text{Tr}_V \left[ \frac{1}{1 + K} \Psi_{TT} \frac{1}{1 + K'} \right],$$

and the right hand sides are independent of $y$. Therefore evaluating them at $y = 0$, we can see that the observables (2.14) all vanish.

$^6$One may be able to calculate the observables for such a solution following [34] or [23].
3 String field theory expanded around the Takahashi-Tanimoto solution

The derivation in the previous section uses the perturbative definition (2.5) of $e^{-LK'}$. Since $\Psi_{TT}$ is an identity-based solution, the kinetic term $Q'$ is given by an integral of local operators on the worldsheet and we should be able to treat $K'$ more directly. In this section, we will examine if we can derive the results in section 2 by doing so.

In the calculation of the observables in the previous section, the following relations were essential:

$$e^{-\epsilon K} Q' \left( \frac{1}{\pi^2} b \right) e^{-\epsilon K} = e^{-2\epsilon K'},$$
$$e^{-\epsilon K} Q' ce^{-\epsilon K} = 0.$$

These relations hold for the perturbative definition of $e^{-LK'}$. In the treatment here, it will be more appropriate to consider

$$e^{-\epsilon K'} Q' \left( \frac{1}{\pi^2} b \right) e^{-\epsilon K'} = e^{-2\epsilon K'},$$
$$e^{-\epsilon K'} Q' ce^{-\epsilon K'} = 0,$$

where the $e^{-\epsilon K'}$'s are expected to provide worldsheet with no operator insertions.

Actually, as we will see, the definition of $e^{-LK'}$ is very subtle and we need some regularization to define quantities involving it. There seem to be many ways to treat it, which should be related to the choice of the prescription of renormalization in the perturbative definition of $e^{-LK'}$ (2.5). Here we use the identities (3.1)(3.2) and their consequences (2.7)(2.8) as the guiding principle to find the definition of $e^{-LK'}$ so that the string field action (1.3) should describe the tachyon vacuum.

3.1 Similarity transformation

The $K'$ given in (2.4) involves $T' (z)$ which is a twisted energy momentum tensor with central charge $c = 24$. Therefore we need to take care of the conformal anomaly on the worldsheet to deal with the correlation functions on surfaces generated by $e^{-LK'}$ and the calculations will become cumbersome. Here we would like to use an alternative way of dealing with $K'$ to do calculations.

As was pointed out by Kishimoto and Takahashi [11], the kinetic operator $Q'$ of the string field theory expanded around the solution (1.1) with $a = -\frac{1}{2}$ can be expressed as

$$Q' = e^{-q(\lambda)} \left( -\frac{1}{4} Q_2 + c_2 \right) e^{q(\lambda)},$$

where

$$\lambda (\xi) = -\ln \left( 1 - \frac{1}{\xi^2} \right)^2,$$
$$q (\lambda) = \oint \frac{d\xi}{2\pi i} \lambda (\xi) (-bc) (\xi),$$
$$Q_k = \oint \frac{d\xi}{2\pi i} \xi^k B (\xi),$$
$$c_k = \oint \frac{d\xi}{2\pi i} \xi^{k-2} C (\xi).$$
Using the mode expansion of the ghost number current
\[-bc (\xi) = \sum_n q_n \xi^{n-1},\]
the \(q (\lambda)\) is expressed as
\[q (\lambda) = 2 \sum_{n=1}^{\infty} \frac{1}{n} q_{-2n} .\]

**bc-shift operation**

Eq. (3.3) can be rewritten by using the bc-shift operation \[11\] defined for \(k \in \mathbb{Z}\) as
\[c_n \rightarrow c_n^{(k)} = c_{n+k},\]
\[b_n \rightarrow b_n^{(k)} = b_{n-k},\]
\[|0\rangle \rightarrow |0\rangle^{(k)} = \begin{cases} b_{-k-1}b_{-k} \cdots b_{-2} |0\rangle & k > 0 \\ c_{k+2}c_{k+3} \cdots c_1 |0\rangle & k < 0 \end{cases},\]
\[(0) \rightarrow (k)(0) = \begin{cases} |0\rangle c_{-1}c_0 \cdots c_{k-2} & k > 0 \\ |0\rangle b_2b_3 \cdots b_{-k+1} & k < 0 \end{cases},\]
and \(\phi \rightarrow \phi^{(k)} = \phi\) if \(\phi\) involves only matter fields. A state
\[|a\rangle = \phi_{-n_1} \cdots b_{-m_1} \cdots c_{-l_1} \cdots |0\rangle ,\]
in the Fock space is mapped to
\[|a\rangle^{(k)} = (k)|a\rangle^{(k)} = \phi_{-n_1}^{(k)} \cdots b_{-m_1}^{(k)} \cdots c_{-l_1}^{(k)} \cdots |0\rangle^{(k)} ,\]
under this operation. \(c_n^{(k)}, b_m^{(k)} |0\rangle^{(k)}, (k)|0\rangle\) satisfy
\[\{c_n^{(k)}, b_m^{(k)}\} = \delta_{n+m,0},\]
\[\{c_n^{(k)}, c_m^{(k)}\} = 0,\]
\[b_n^{(k)} |0\rangle^{(k)} = 0 \ (n \geq -1),\]
\[c_n^{(k)} |0\rangle^{(k)} = 0 \ (n \geq 2),\]
\[(k)|0\rangle b_n^{(k)} = 0 \ (n \leq 1),\]
\[(k)|0\rangle c_n^{(k)} = 0 \ (n \leq -2),\]
and
\[(k)|0\rangle c_1^{(k)} c_0^{(k)} |0\rangle^{(k)} = \langle 0| c_{-1}c_0c_1 |0\rangle = 1 .\]

Since we can evaluate all the correlation functions of the bc system using the relations (3.4)(3.5), we can see that for any states \(|a\rangle, |b\rangle,\)
\[(k)|a\rangle b^{(k)} = \langle a|b \rangle .\]
Under the $bc$-shift operation, the BRST charge is transformed as

$$Q \rightarrow Q^{(k)} = Q_k - k^2 c_k.$$ 

Therefore (3.3) can be written as

$$Q' = -\frac{1}{4} e^{-q(\lambda)} Q^{(2)} e^{q(\lambda)}. \quad (3.6)$$

It is convenient to introduce operators $U_k$ ($k \in \mathbb{Z}$) which are defined so that

$$U_k |a\rangle = |a\rangle^{(k)}, \quad \langle a| U_k = (-k) \langle a|.$$ 

$U_k$ satisfies

$$U_k^{-1} = U_{-k}, \quad U_k \mathcal{O} U_k^{-1} = \mathcal{O}^{(k)},$$

for any operator $\mathcal{O}$ and (3.6) can be written as

$$Q' = -\frac{1}{4} UQU^{-1}. \quad (3.7)$$

where

$$U = e^{-q(\lambda)} U_2, \quad U^{-1} = U_{-2} e^{q(\lambda)}. \quad (3.8)$$

Notice that $U, U^{-1}$ are of ghost number $-2, 2$. Another thing to be noticed is that the BPZ conjugates of $U, U^{-1}$ do not coincide with either $U$ or $U^{-1}$.

Therefore, the $Q'$ is related to the original kinetic operator $Q$ by a similarity transformation (3.7), which implies that the solution $\Psi_{TT}$ is formally in the pure gauge form. By the similarity transformation, $K'$ is turned into an operator made from $T = \{Q, b\}$ and thus it is possible to evaluate quantities involving $K'$ without dealing with the twisted energy momentum tensor $T'$.

### 3.2 $U, U^{-1}$

We need some identities satisfied by $U, U^{-1}$ to perform calculations using the relation (3.7). From the definition (3.8) we obtain

$$U c(\xi) U^{-1} = \frac{(\xi^2 - 1)^2}{\xi^2} c(\xi) = -4 e^{h - \frac{1}{2} \xi} c(\xi), \quad (3.9)$$

$$U^{-1} c(\xi) U = \frac{\xi^2}{(\xi^2 - 1)^2} c(\xi) = -\frac{1}{4} e^{h - \frac{1}{2} \xi} c(\xi), \quad (3.10)$$

$$U b(\xi) U^{-1} = \frac{\xi^2}{(\xi^2 - 1)^2} b(\xi) = -\frac{1}{4} e^{h - \frac{1}{2} \xi} b(\xi), \quad (3.11)$$

$$U^{-1} b(\xi) U = \frac{(\xi^2 - 1)^2}{\xi^2} b(\xi) = -4 e^{h - \frac{1}{2} \xi} c(\xi). \quad (3.12)$$
It is also possible to derive how $U, U^{-1}$ act on the states $|0\rangle, |I\rangle, \langle 0|, \langle I|:

$$U |0\rangle = \frac{1}{16} \partial b b (1) \partial b b (-1) c_0 c_1 |0\rangle,$$

$$U^{-1} |0\rangle = \frac{1}{16} \partial c c (1) \partial c c (-1) b_{-3} b_{-2} |0\rangle,$$

$$\langle 0| U = \langle 0| b_2 b_3 ,$$

$$\langle 0| U^{-1} = \langle 0| c_{-1} c_0 .$$

$$U |I\rangle = \frac{1}{32} \partial b b (1) |I\rangle ,$$

$$U^{-1} |I\rangle = 2 \partial c c (1) |I\rangle ,$$

$$\langle I| U = 0 ,$$

$$\langle I| U^{-1} = 0 .$$

These identities are proved in appendix B.

Here let us comment on two things concerning the operators $U, U^{-1}$, which will become important in the subsequent discussions.

- The pure gauge form (3.7) apparently contradicts the existence of the homotopy operator (2.9), as was pointed out in [12, 35]. Indeed, one can see from (3.7) that the representatives of the BRST cohomology of $Q'$ are given by the states of the form [11]

$$U c V^m (0) |0\rangle : \text{gh#} = -1 ,$$

$$U \partial c c V^m (0) |0\rangle : \text{gh#} = 0 .$$

where $V^m$ is a primary field made from the matter fields with weight 1. Therefore one can conclude that there exist no physical open string excitations because they correspond to the states with ghost number 1. On the other hand, the existence of the homotopy operator $b (1)$ implies that the states (3.21)(3.22) should be written in a BRST exact form

$$U c V^m (0) |0\rangle = Q' b (1) U c V (0) |0\rangle ,$$

$$U \partial c c V^m (0) |0\rangle = Q' b (1) U \partial c c V (0) |0\rangle .$$

Actually these do not hold. Indeed, using eqs.(3.10)(3.13), we obtain

$$U c V^m (0) |0\rangle = \frac{1}{16} \partial b b (1) \partial b b (-1) c_{-1} c_0 c_1 V^m (0) |0\rangle ,$$

$$U \partial c c V^m (0) |0\rangle = \frac{1}{16} \partial b b (1) \partial b b (-1) c_{-2} c_{-1} c_0 c_1 V^m (0) |0\rangle ,$$

and $b (1) U c V (0) |0\rangle = b (1) U \partial c c V (0) |0\rangle = 0$. The reason for this apparent contradiction is that the relation (2.9) holds only when there is some worldsheet around $b (1)$ without any local operator insertions, as we mentioned below eq.(2.12). Therefore, for the states (3.23)(3.24) which involve $\partial b b (1)$, $b (1)$ does not work as a homotopy operator of $Q'$.

- Eqs.(3.19)(3.20) imply that the operators $U, U^{-1}$ act on $\langle I|$ in a nonassociative way because

$$\langle I| U U^{-1} \neq \langle I| U (U U^{-1}) ,$$

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although they do not on \(|I\rangle\). The nonassociativity may be related to the nonassociativity of the boundary condition changing operators pointed out in [6]. We will not encounter such a nonassociativity, if the combinations \(|I\rangle U, |I\rangle U^{-1}\) are away from other operators on the worldsheet. We need some worldsheet without any local operator insertions around \(|I\rangle U, |I\rangle U^{-1}\) to avoid nonassociativity.

### 3.3 Calculations of the observables

Now we would like to discuss how we can evaluate the observables (2.6) using the expression (3.7). In order to facilitate the calculation using eq.(3.7), we rewrite everything in terms of the first-quantized operators, rather than string fields. Here let us introduce \(B^+, L^+\) such that [36, 1]

\[
B^+ = \oint \frac{dc}{2\pi i} (1 + \xi^2) \left( \tan^{-1} \xi + \tan^{-1} \left( \frac{1}{\xi} \right) \right) b(\xi)
\]

\[
L^+ \equiv \{Q', B^+\}.
\]  

(3.25)

\(L^+\) is the translation operator with respect to the sliver frame coordinate \(z\) for the left and right half of the string and can be used to express the action of \(K'\) in our setup. Using \(L^+\), the left hand sides of eqs.(2.7)(2.8) are expressed as

\[\int_0^\infty dLe^{-\xi} \langle I| e^{-\frac{\xi}{4}L''^+} c(1) V (i, -i, e^{-\frac{\xi}{4}L''^+} |I\rangle, \]

(3.26)

and eqs. (3.1)(3.2) are rewritten as

\[e^{-\epsilon L'^+} Q'b(1) |I\rangle = e^{-\epsilon L'^+} |I\rangle, \]

(3.28)

\[e^{-\epsilon L'^+} Q'c(1) |I\rangle = 0. \]

(3.29)

Let us check if one can prove (3.28)(3.29) by using the expression (3.7). Substituting (3.7) into the left hand side of (3.28), we get

\[\int_0^\infty dLe^{-\xi} \langle I| e^{-\frac{\xi}{4}L''^+} c(1) V (i, -i, e^{-\frac{\xi}{4}L''^+} |I\rangle, \]

In order to avoid the singularity which appears in moving the operator \(U^{-1}\) to the right, we shift the position of \(b\) for regularization. Thus we consider

\[-\frac{1}{4} \lim_{\xi \to 1} e^{-\epsilon L'^+} UQU^{-1} b(1) |I\rangle.
\]

(3.30)
where
\[
\tilde{\mathcal{L}}^{t^+} = U^{-1} \mathcal{L}^{t^+} U = \left\{ Q, \frac{\pi}{2} \int \frac{d\xi}{2\pi i} (1 + \xi^2) \epsilon(\text{Re}\xi) e^{h - \frac{1}{4}\xi^2} b(\xi) \right\} = \frac{\pi}{2} \int \frac{d\xi}{2\pi i} (1 + \xi^2) \epsilon(\text{Re}\xi) e^{h - \frac{1}{4}\xi^2} T(\xi),
\]
(3.31)

Instead of \( K' \) or \( \mathcal{L}^{t^+} \), \( \tilde{\mathcal{L}}^{t^+} \) is the fundamental translation operator to deal with in the subsequent calculation. Contrary to \( K' \), \( \tilde{\mathcal{L}}^{t^+} \) is made from \( T(\xi) \) and we do not have to worry about the conformal anomaly. If the operator \( e^{-\epsilon \tilde{\mathcal{L}}^{t^+}} \) should generate worldsheet around \( \{ Q, b(\xi) \} \) in (3.30), we could express \( Q \) by a contour integral and proceed further.

The operator of the form (3.31) can be analyzed by the methods explained in [37]. Here it is convenient to go to the silver frame and rewrite (3.31) as
\[
\tilde{\mathcal{L}}^{t^+} = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} e^{h(z + \frac{1}{2})} T\left(\frac{1}{2} + z\right) + \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} e^{h(-\frac{1}{2} + z)} T\left(-\frac{1}{2} + z\right),
\]
where
\[
e^{h(z)} = \frac{\cos^2 \pi z}{\sin^2 \pi z}.
\]

We introduce a new coordinate \( w \) such that
\[
\frac{\partial z}{\partial w} = e^{h(z)},
\]
which is integrated as
\[
w(z) = z - \frac{1}{\pi} \text{sin} \pi z.
\]
(3.32)

Using these, \( \tilde{\mathcal{L}}^{t^+} \) is expressed as
\[
\tilde{\mathcal{L}}^{t^+} = \int_{-\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \frac{dz}{2\pi i} \frac{\partial z}{\partial w} T(w),
\]
and \( \tilde{\mathcal{L}}^{t^+} \) generates translations with respect to the coordinate \( w \). The map \( w(z) \) (3.32) maps the region \( 0 < \text{Im}z < \infty \) to \( -\infty < \text{Im}w < \infty \) and the region \( -\infty < \text{Im}z < 0 \) to \( -\infty < \text{Im}w < \infty \) for \( \text{Re}z = \pm \frac{1}{2} \)
and \( z = \pm \frac{1}{2} \) are singular points. \( z = \pm \frac{1}{2} \) are mapped to \( w = \pm \infty \) and do not move under the translation generated by \( \tilde{\mathcal{L}}^{t^+} \). Therefore the operator \( e^{-\epsilon \tilde{\mathcal{L}}^{t^+}} \) acting on the identity state \( |I\rangle \) generates the worldsheet of the form depicted in Fig. 3. Hence \( e^{-\epsilon \tilde{\mathcal{L}}^{t^+}} \) in (3.30) do not generate worldsheet around \( Qb(\xi) \) and we cannot proceed from (3.30). The correlation functions which appear on the right hand sides of eqs.(3.26)(3.27) correspond to cylinders of the form \( w \sim w + L \). Such a cylinder is mapped to two spheres whose coordinates are given by \( e^{\frac{2\pi i}{\text{Im}z}w} \).

**Regularization**

The operator \( e^{-\epsilon \tilde{\mathcal{L}}^{t^+}} \) generates apparently singular surfaces, which should be defined as a limit of regular surfaces. There are problems in performing calculations on such singular surfaces. We are not able to prove the homotopy relation (3.29) on such surfaces because no worldsheet is generated around the point on the boundary. If one tries to calculate quantities such as (3.26)(3.27) using the relation (3.7), we eventually
encounter combination $\langle I \vert U \rangle$. As we mentioned at the end of previous subsection, we need some worldsheet to separate $\langle I \vert U \rangle$ from other operators to avoid the nonassociativity. With the $\tilde{L}'^+$ as the Hamiltonian, it seems impossible.

We would like to define the string field theory so that it describes the tachyon vacuum. Therefore what we need to do is to regularize the $\tilde{L}'^+$, while preserving the relations (3.28)(3.29). The regularization we propose is to replace $\tilde{L}'^+$ by

$$
\tilde{L}'_a^+ = \frac{\pi}{2} \oint \frac{d\xi}{2\pi i} (1 + \xi^2) \epsilon (\text{Re} \xi) e^{h_a(\xi) T (\xi)} .
$$

(3.33)

($a > -\frac{1}{2}$) with $h_a (\xi)$ given in (1.2). We define $e^{-L\tilde{L}_a'^+}$ as

$$
\lim_{a \rightarrow -\frac{1}{2}} e^{-L\tilde{L}_a'^+} .
$$

(3.34)

For $a > -\frac{1}{2}$, the surface generated by $e^{-L\tilde{L}_a'^+}$ is of the form depicted in Fig. 4 and we realize $e^{-L\tilde{L}_a'^+}$ as a singular limit of $e^{-L\tilde{L}_a'^+}$.
With such a regularization, the right hand side of (3.30) becomes

\[
-\frac{1}{4} \lim_{a \to -\frac{1}{2}} \lim_{\xi \to 1} \left[ \frac{(\xi^2 - 1)^2}{\xi^2} U e^{-\xi \bar{L}_a^+} Q_b(\xi) 2\partial cc(1) |I\rangle \right]
\]

\[
= U \lim_{a \to -\frac{1}{2}} e^{-\xi \bar{L}_a^+} 2\partial cc(1) |I\rangle ,
\]

which can be rewritten as

\[
U \lim_{a \to -\frac{1}{2}} e^{-\xi \bar{L}_a^+} 2\partial cc(1) |I\rangle = U \lim_{a \to -\frac{1}{2}} e^{-\xi \bar{L}_a^+} U^{-1} |I\rangle = e^{-\xi \bar{L}_a^+} |I\rangle ,
\]

and we eventually get (3.28). (3.29) can be proved in the same way:

\[
e^{-\xi \bar{L}_a^+} Q'c(1) |I\rangle = -\frac{1}{4} \lim_{\xi \to 1} e^{-\xi \bar{L}_a^+} UQU^{-1}c(\xi) |I\rangle
\]

\[
= -\frac{1}{4} \lim_{a \to -\frac{1}{2}} \lim_{\xi \to 1} \left[ \frac{\xi^2}{(\xi^2 - 1)^2} U e^{-\xi \bar{L}_a^+} Qc(\xi) 2\partial cc(1) |I\rangle \right]
\]

\[
= 0 .
\]

(3.35)

One can immediately show that the terms on the right hand side of (3.27) vanish by using (3.36). In order to show that the right hand side of (3.26) vanishes, we use (3.35) to get

\[
\langle I| e^{-\frac{1}{4} \bar{L}_a^+} c(1) V (i, -i) e^{-\frac{1}{4} \bar{L}_a^+} |I\rangle
\]

\[
= \lim_{a \to -\frac{1}{2}} \lim_{\xi \to 1} \langle I| e^{-\frac{1}{4} \bar{L}_a^+} c(\xi) V (i, -i) U e^{-\frac{1}{4} \bar{L}_a^+} 2\partial cc(1) |I\rangle
\]

\[
= \lim_{a \to -\frac{1}{2}} \lim_{\xi \to 1} \langle I| U e^{-\frac{1}{4} \bar{L}_a^+} \frac{(\xi^2 - 1)^2}{\xi^2} c(\xi) V (i, -i) e^{-\frac{1}{4} \bar{L}_a^+} 2\partial cc(1) |I\rangle
\]

\[
= 0 .
\]

Here, with the regularization, \(\langle I| U\) is away from the other operators \(c(\xi)\), \(\partial cc(1)\) and we can use the identity (3.19) on the last line to get the result. Thus we have shown how to regularize and define the operator \(e^{-\bar{L}_a^+}\) so that we can derive (3.28)(3.29)(2.7)(2.8). These formulas imply that the string field theory describes the tachyon vacuum.

4 Conclusions and discussions

In this paper, we have evaluated the observables of the Takahashi-Tanimoto’s scalar solution (1.1) with \(a = -\frac{1}{2}\), by studying the Erler-Schnabl solution in the string field theory expanded around it. The results are consistent with the claim that the solution corresponds to the tachyon vacuum. In the calculations, the string field \(K'\) or its worldsheet operator counterpart plays crucial roles. In the latter half of this paper, we study the operator \(K'\) using the similarity transformation proposed by Kishimoto and Takahashi. We discuss how we should treat it in order to be consistent with the claim that the background is the tachyon vacuum.
The relation (3.7) will be useful to evaluate various other quantities in the string field theory expanded around $\Psi_{TT}$. Since the solution is supposed to describe the tachyon vacuum, we expect all the amplitudes involving open string states vanish. On the other hand, we may be able to calculate closed string amplitudes using the string field theory [30, 38, 39]. In order to do such calculations, we should take Siegel gauge for example and construct the propagators. We will need some regularization like (3.34) to define the propagator. We leave it as a future problem.

The operator $U, U^{-1}$ in (3.7) should be related to the boundary condition changing operators which play crucial roles in [6, 40]. Suppose that we formally\(^7\) divide the operators $U, U^{-1}$ into the left and right piece $U_L, U_R, (U^{-1})_L, (U^{-1})_R$ so that the operator $U, U^{-1}$ acts on a string field $A$ as

$$
UA = U_L A U_R, \quad U^{-1}A = (U^{-1})_L A (U^{-1})_R.
$$

$U_L, U_R, (U^{-1})_L, (U^{-1})_R$ may be regarded as some kind of boundary condition changing operators and eqs.(3.17)-(3.20) imply the OPE’s of them. In this way, the nonassociativity mentioned in section 3 may be more directly related to the nonassociativity of the boundary condition changing operators encountered in [6]. It would be inspiring to study the Takahashi-Tanimoto background from the point of view of these operators.

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**A Maccaferri’s method**

In a recent paper [19], Maccaferri considered a special case of Zeze map [33], which maps an identity-based solution to a regular solution. In the case of the Takahashi-Tanimoto solution (1.1) with $a = -\frac{1}{2}$, one obtains

$$
\Psi_{TT} \rightarrow \Psi_{reg.} = \left( 1 + B \frac{1 - F(K)}{K} \Psi_{TT} \right) (Q + \Psi_{TT}) \left( 1 + B \frac{1 - F(K)}{K} \Psi_{TT} \right)^{-1}. \quad (A.1)
$$

The Zeze map (A.1) is a gauge transformation and we can get a regular solution gauge equivalent to $\Psi_{TT}$ by choosing $F \left( K \right)$ appropriately. A convenient choice is $F \left( K \right) = \frac{1}{1+K}$ and we get

$$
\Psi_{reg.} = \frac{1}{1+K} \Psi_{TT} \frac{1}{1+K^2} - Q \left( \frac{1}{1+K} \Psi_{TT} \frac{1}{1+K^2} \right). \quad (A.2)
$$

\(^7\)Since $U, U^{-1}$ involve operators like $U_2, U_{-2}$, we are not so sure if we could do such a decomposition.
which appears to be a regular solution. From the expression (A.2), it is straightforward to calculate the energy and the Ellwood invariant and one obtains [19]

\[
S_{\Psi_{\text{reg}}} = -\frac{1}{6g^2} \text{Tr} \left[ \frac{1}{1 + K} \frac{1}{1 + K} Qc \right] + \frac{1}{6g^2} \text{Tr} \left[ \frac{1}{1 + K} \frac{1}{1 + K} Q'c \right],
\]

\[
\text{Tr}_V \Psi_{\text{reg}}. = \text{Tr}_V \left[ \frac{1}{1 + K} \right] - \text{Tr}_V \left[ \frac{1}{1 + K'} \right].
\] (A.3)

The right hand sides of eq. (A.3) can be written as

\[
S_{\Psi_{\text{reg}}} = S_{\Psi_{\text{ES}}} - S_{\Psi'_{\text{ES}}},
\]

\[
\text{Tr}_V \Psi_{\text{reg}}. = \text{Tr}_V \Psi_{\text{ES}} - \text{Tr}_V \Psi'_{\text{ES}},
\]

where \(\Psi_{\text{ES}}, \Psi'_{\text{ES}}\) are the Erler-Schnabl solutions given in (2.1)(2.2). Thus the observables of \(\Psi_{\text{reg}}\) are obtained from those of the Erler-Schnabl solution \(\Psi'_{\text{ES}}\). Using \(S_{\Psi'_{\text{ES}}} = \text{Tr}_V \Psi'_{\text{ES}} = 0\) derived in section 2, we can see that the observables of \(\Psi_{\text{reg}}\) coincide with those of the tachyon vacuum solution \(\Psi_{\text{ES}}\).

**Singularities**

Actually, the calculation of the observables above suffers from singularities discussed by Maccaferri [19]. In calculating the action, one typically encounters quantities of the form

\[
\langle c(z) c(0) \rangle_{CL} = -\left( \frac{L}{\pi} \right)^2 \sin^2 \frac{\pi z}{L},
\] (A.4)

where \(\langle \cdot \rangle_{CL}\) denotes the correlation function on a semi-infinite cylinder with circumference \(L\). (A.4) diverges in the limit \(\text{Im} z \to \pm \infty\) for small enough \(L > 0\) or in the limit \(L \to 0\) with \(\text{Im} z \neq 0\). Since the Takahashi-Tanimoto solution (1.1) involves an integral of the ghost \(c\) up to \(\text{Im} z = \pm \infty\), we have trouble in calculating the action\(^8\).

Therefore we need to find a good regularization to calculate the action\(^9\). In [19], a solution with

\[
F(K) = F_\epsilon(K) = \frac{e^{-\epsilon K}}{1 + (1 - \epsilon) K},
\] (A.5)

\((0 \leq \epsilon \leq 1)\) in (A.1) is considered as a regularization. Let \(\Psi_\epsilon\) denote the \(\Psi_{\text{reg}}\) with this choice of \(F(K)\). It is easy to see that

\[
\Psi_\epsilon = \frac{1}{1 + K_\epsilon} \left( \Psi_{\text{TT}} - \Psi_{\text{TT}} B_\epsilon \frac{1}{1 + K'_\epsilon} \Psi_{\text{TT}} \right),
\]

where

\[
c_\epsilon = c \frac{K B}{G_\epsilon(K)} c, \quad B_\epsilon = B \frac{G_\epsilon(K)}{K}, \quad K_\epsilon = Q B_\epsilon = G_\epsilon(K), \quad K'_\epsilon = K_\epsilon + J_\epsilon, \quad J_\epsilon = \{B_\epsilon, \Psi_{\text{TT}} \},
\] (A.6)

\(^8\)We do not encounter such divergences in the calculation of the Ellwood invariant or the overlap of \(\Psi_{\text{reg}}\) with Fock space states.

\(^9\)In [23], the author modifies the form of the solution (1.1) as was done in [19] and avoid the singularity.
\[
\frac{1}{1 + K_\epsilon} = \frac{1}{1 + G_\epsilon(K)} = \frac{e^{-\epsilon K}}{1 + (1 - \epsilon) K}.
\]

The \( \Psi_\epsilon \) consists of wedge states of width not smaller than \( \epsilon \) with operator insertions and we can avoid the above-mentioned divergences taking \( \epsilon > \frac{1}{2} \).

\( K_\epsilon, B_\epsilon, c_\epsilon \) in (A.6) satisfy the \( KBC \) algebra [41, 42, 34] and it is straightforward to show that the observables for the solution \( \Psi_\epsilon \) coincide with the shift in those of the modified Erler-Schnabl solutions [19]

\[
\Psi_{ES,\epsilon} = \frac{1}{1 + K_\epsilon} (c_\epsilon + Q (B_\epsilon c_\epsilon)) ,
\]

\[
\Psi'_{ES,\epsilon} = \frac{1}{1 + K'_\epsilon} (c_\epsilon + Q' (B_\epsilon c_\epsilon)) ,
\]

namely

\[
S[\Psi_\epsilon] = S[\Psi_{ES,\epsilon}] - S[\Psi'_{ES,\epsilon}] = -\frac{1}{6g^2} \text{Tr} \left[ \frac{1}{1 + K_\epsilon} c_\epsilon + \frac{1}{1 + K_\epsilon} Qc_\epsilon \right] + \frac{1}{6g^2} \text{Tr} \left[ \frac{1}{1 + K'_\epsilon} c_\epsilon + \frac{1}{1 + K'_\epsilon} Q'c_\epsilon \right].
\]

\[
\text{Tr}_V \Psi_\epsilon = \text{Tr}_V \Psi_{ES,\epsilon} - \text{Tr}_V \Psi'_{ES,\epsilon} = \text{Tr}_V \frac{1}{1 + K_\epsilon} c_\epsilon - \text{Tr}_V \frac{1}{1 + K'_\epsilon} c_\epsilon,
\]

(A.9)

Now we can use (A.9) to calculate the observables. As is pointed in [19], although \( \Psi_\epsilon \) itself may involve singularities for small \( \epsilon \), \( \Psi'_{ES,\epsilon} \) is regular for all \( 0 \leq \epsilon \leq 1 \). Moreover one can show

\[
\frac{\partial}{\partial \epsilon} \Psi_{ES,\epsilon} = QA + \Psi_{ES,\epsilon} A - A \Psi_{ES,\epsilon},
\]

\[
\frac{\partial}{\partial \epsilon} \Psi'_{ES,\epsilon} = Q'A' + \Psi'_{ES,\epsilon} \Lambda' - \Lambda' \Psi'_{ES,\epsilon},
\]

where

\[
\Lambda = B_\epsilon \frac{1}{1 + K_\epsilon} \frac{\partial}{\partial \epsilon} \Psi_{ES,\epsilon},
\]

\[
\Lambda' = B_\epsilon \frac{1}{1 + K'_\epsilon} \frac{\partial}{\partial \epsilon} \Psi'_{ES,\epsilon}.
\]

Since the observables \( \text{Tr}_V \Psi_{ES,\epsilon}, \text{Tr}_V \Psi'_{ES,\epsilon}, S[\Psi_{ES,\epsilon}], S[\Psi'_{ES,\epsilon}] \) are gauge invariant quantities, they are independent of \( \epsilon \) provided the gauge parameters \( \Lambda, \Lambda' \) are regular string fields. Thus we can evaluate them choosing \( \epsilon \) for which the calculation is easy. The most convenient choice is \( \epsilon = 0 \) and we get

\[
\text{Tr}_V \Psi_\epsilon = \text{Tr}_V \Psi_{ES} - \text{Tr}_V \Psi'_{ES},
\]

(A.10)

\[
S[\Psi_\epsilon] = S[\Psi_{ES}] - S[\Psi'_{ES}],
\]

(A.11)

From (2.7)(2.8), we can see that the observables of \( \Psi_\epsilon \) coincide with those of the tachyon vacuum solution \( \Psi_{ES} \).

Thus, by using the Maccaferri’s method, it is possible to construct regular solutions gauge equivalent to \( \Psi_{TT} \), calculate the observables of them and show that they coincide with those of the tachyon vacuum. In a sense, this gives a more direct derivation of the observables of the identity-based solutions than the one given.
in section 2. On the other hand, since the gauge transformation (A.1) transforms an identity-based solution into a regular solution, the transformation itself might be somewhat singular. Therefore if the observables (A.3) can be identified with those of \( \Psi_{TT} \) may be debatable.

Before closing this appendix, one comment is in order. The string field theory expanded around the Takahashi-Tanimoto solution possesses a classical solution \(-\Psi_{TT}\) corresponding to the perturbative vacuum. Although the solution itself is an identity-based solution, one can construct a solution gauge equivalent to it
\[
- \frac{1}{1 + K'} \Psi_{TT} \frac{1}{1 + K} + Q' \left( \frac{1}{1 + K^2} \Psi_{TT} \frac{1}{1 + K} \right),
\]
by Maccaferri’s method. The observables can be calculated at least formally and they coincide with those of the perturbative vacuum.

**B Properties of \( U, U^{-1} \)**

In this appendix, we explain how to derive the following identities satisfied by \( U, U^{-1} \):

\[
U |0\rangle = \frac{1}{16} \partial_{bb} (1) \partial_{bb} (-1) c_0 c_1 |0\rangle ,
\]
\[
U^{-1} |0\rangle = \frac{1}{16} \partial_{cc} (1) \partial_{cc} (-1) b_{-3} b_{-2} |0\rangle ,
\]
\[
\langle 0 | U = \langle 0 | b_2 b_3 ,
\]
\[
\langle 0 | U^{-1} = \langle 0 | c_{-1} c_0 ,
\]
\[
U |I\rangle = \frac{1}{32} \partial_{bb} (1) |I\rangle ,
\]
\[
U^{-1} |I\rangle = 2 \partial_{cc} (1) |I\rangle ,
\]
\[
\langle I | U = 0 ,
\]
\[
\langle I | U^{-1} = 0 .
\]

Let us first consider eqs.(B.1)-(B.4). Since \( q(\lambda) = 2 \sum_{n=1}^{\infty} \frac{1}{n} q_{-2n} \),
\[
e^{\pm q(\lambda)} |0\rangle = \exp \left[ \pm 2 \sum_{n=1}^{\infty} \frac{1}{n} q_{-2n} \right] |0\rangle .
\]

On the other hand, we have the bosonization formula
\[
c(\xi) = \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} q_{-n} \xi^n \right] e^{\sigma_0} e^{q_0 \ln \xi} \exp \left[ - \sum_{n=1}^{\infty} \frac{1}{n} q_n \xi^{-n} \right] ,
\]
\[
b(\xi) = \exp \left[ - \sum_{n=1}^{\infty} \frac{1}{n} q_{-n} \xi^n \right] e^{-\sigma_0} e^{-q_0 \ln \xi} \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} q_n \xi^{-n} \right] ,
\]
where \( \sigma_0 \) is the canonical conjugate of \( q_0 \) satisfying
\[
[q_0, \sigma_0] = 1 .
\]

Eqs.(B.9)(B.10) imply
\[
\partial_{bb} (1) \partial_{bb} (-1) c_{-2} c_{-1} c_0 c_1 |0\rangle = 16 e^{-q(\lambda)} |0\rangle ,
\]
\[
\partial_{cc} (1) \partial_{cc} (-1) b_{-5} b_{-4} b_{-3} b_{-2} |0\rangle = 16 e^{q(\lambda)} |0\rangle .
\]
From these, we get

\[
U |0\rangle = e^{-q(\lambda)} U_2 |0\rangle
\]

\[
= e^{-q(\lambda)} \lim_{\varepsilon \to 0} \partial bb (\varepsilon) |0\rangle
\]

\[
= \lim_{\varepsilon \to 0} e^{2q(\varepsilon)} \partial bb (\varepsilon) \frac{1}{16} \partial bb (1) \partial bb (-1) c_{-2} c_{0} c_1 |0\rangle
\]

\[
= \frac{1}{16} \partial bb (1) \partial bb (-1) c_{0} c_1 |0\rangle ,
\]

\[
U^{-1} |0\rangle = U e^{q(\lambda)} |0\rangle
\]

\[
= U e^{\frac{1}{16} \partial cc (1) \partial cc (-1) b_{-3} b_{-2} b_{-2} |0\rangle}
\]

\[
= \frac{1}{16} \partial cc (1) \partial cc (-1) b_{-3} b_{-2} |0\rangle .
\]

(B.3)(B.4) are obtained from

\[
\langle 0 | e^{\pm q(\lambda)} |0\rangle = \langle 0 | .
\]

Now let us turn to eqs.(B.5)-(B.8). All these are shown by using the defining relation [43, 8, 36] of \[I\]

\[
\langle I | \phi (0) |0\rangle = \langle f \circ \phi (0) \rangle_{UHP} ,
\]

where

\[
f (\xi) = \frac{2\xi}{1 - \xi^2}
\]

and \(\langle \cdot \rangle_{UHP}\) denotes the correlation function on the upper half plane. In order to derive (B.7), what we should do is to calculate

\[
\langle I | U \phi (0) |0\rangle ,
\]

and show that it vanishes for any \(\phi (0)\). Since \(U\) only changes on the ghost part of \(\langle I\), what we should calculate are the quantities of the form

\[
\langle I | U \prod_i c (\xi_i) \prod_j b (\xi'_j) |0\rangle = \langle 0 | f \circ (\partial bb) (1) f \circ (\partial bb) (-1) \prod_i \left( \frac{\xi_i^2 - 1}{\xi_i^2} c (\xi_i) \right) \prod_j \left( \frac{\xi'_j^2 - 1}{\xi'_j^2} b (\xi'_j) \right) |0\rangle .
\]

Using (3.10)(3.12)(B.1), it is straightforward to get

\[
U \prod_i c (\xi_i) \prod_j b (\xi'_j) |0\rangle
\]

\[
= \frac{1}{16} \prod_i \left( \frac{\xi_i^2 - 1}{\xi_i^2} c (\xi_i) \right) \prod_j \left( \frac{\xi'_j^2 - 1}{\xi'_j^2} b (\xi'_j) \right) \partial bb (1) \partial bb (-1) c_{0} c_1 |0\rangle .
\]

Strictly speaking, this expression is defined for \(|\xi_i|, |\xi'_j| > 1\) and the contours in (B.12) should be taken accordingly. However, the possible singularities at \(\xi_i, \xi'_j = \pm 1\) are absent in this combination and we can deform the contours to be around 0. Now using (B.11), we obtain

\[
\langle I | U \prod_i c (\xi_i) \prod_j b (\xi'_j) |0\rangle
\]

\[
= \langle 0 | f \circ (\partial bb) (1) f \circ (\partial bb) (-1) \prod_i \left( \frac{\xi_i^2 - 1}{\xi_i^2} c (\xi_i) \right) \prod_j \left( \frac{\xi'_j^2 - 1}{\xi'_j^2} f \circ b (\xi'_j) \right) |0\rangle .
\]

(B.13)
However, since
\[
 f \circ (\partial bb) (\pm 1) = \lim_{\varepsilon \to 0} \left( \frac{\partial f}{\partial \xi} \right)^5 \partial bb \left( \frac{2\xi}{1 - \xi^2} \right) \bigg|_{\xi = \pm 1 + \varepsilon} = \lim_{\varepsilon \to 0} \varepsilon^{-10} \partial bb \left( -\frac{1}{\varepsilon} \right) \sim b_2 b_3,
\]
acting on \( \langle 0| \), (B.13) vanishes provided none of \( \xi, \xi'_j \) coincides with \( \pm 1 \). Eq.(B.8) can be shown in the same way.

Eq.(B.6) can be shown as
\[
\langle 0| \prod_i c(\xi_i) \prod_j b(\xi'_j) U^{-1} | I \rangle
= \langle I| \prod_i \left( \frac{(\xi_i^2 - 1)^2}{\xi_i^2} I \circ c(\xi_i) \right) \prod_j \left( \frac{\xi_j^2}{(\xi_j^2 - 1)^2} I \circ b(\xi'_j) \right) c_0 c_1 | 0 \rangle
= \langle 0| \prod_i \left( \frac{2}{f(\xi_i)} \right)^2 f \circ I \circ c(\xi_i) \prod_j \left( \frac{f(\xi'_j)}{2} \right)^2 f \circ I \circ b(\xi'_j) \right) \frac{1}{2} c_0 c_1 | 0 \rangle
= 2 \langle 0| U_{-2} \prod_i f \circ I \circ c(\xi_i) \prod_j f \circ I \circ b(\xi'_j) | 0 \rangle
= 2 \langle 0| c_{-1} c_0 \prod_i f \circ I \circ c(\xi_i) \prod_j f \circ I \circ b(\xi'_j) | 0 \rangle
= 2 \langle 0| f \circ I \circ (\partial cc) (1) \prod_i f \circ I \circ c(\xi_i) \prod_j f \circ I \circ b(\xi'_j) | 0 \rangle
= \langle 0| \prod_i c(\xi_i) \prod_j b(\xi'_j) 2\partial cc (1) | I \rangle ,
\]
where
\[
 I : \xi \to -\frac{1}{\xi},
\]
is the inversion map. Eq.(B.5) can be shown in the same way.

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