NONLINEARIZABLE CR AUTOMORPHISMS FOR POLYNOMIAL MODELS IN $\mathbb{C}^N$.

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ABSTRACT. We classify polynomial models for real hypersurfaces in $\mathbb{C}^N$, which admit nonlinearizable infinitesimal CR automorphisms. As a consequence, this provides an optimal 1-jet determination result in the general case. Further we prove that such automorphisms arise from one common source, by pulling back via a holomorphic mapping a suitable symmetry of a hyperquadric in some complex space.

1. Introduction

One of the fundamental questions in the theory of several complex variables, going back to the seminal work of H. Poincaré ([26]), is how to classify domains up to biholomorphic equivalence. In the complex plane, the classical Riemann mapping theorem asserts that domains possess only topological invariants. As was realized already by Poincaré, there is no analogous statement in higher dimension, and smooth boundaries of domains have infinitely many local biholomorphic invariants.

The main topic of this paper concerns the classification of polynomial models of real hypersurfaces according to their infinitesimal CR automorphisms. The first motivation for this study comes from the fact that, as has been shown in [16], the classical Chern-Moser theory can be extended to the case of singular Levi form - hypersurfaces with polynomial models of finite Catlin multitype. In particular, it has been shown in [16] that the kernel of the generalized Chern-Moser operator, which is in one to one correspondence with the Lie algebra of infinitesimal CR automorphisms of the polynomial model of such a hypersurface, gives a precise description of the derivatives needed to characterize an element of its stability group.

In order to develop this approach further, towards a complete normal form construction and solution of the Poincaré local biholomorphic equivalence problem for such manifolds, we need to classify the polynomial models according to the Lie algebra of infinitesimal CR automorphisms.

The second motivation comes from the study of possible sources and forms of symmetries (infinitesimal CR automorphisms) of CR manifolds. Since linear symmetries are relatively well understood, the main interest lies in understanding the possible existence and origin of nonlinearizable symmetries, provided by vector fields with vanishing linear

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part. This research has long history, starting with the classical case of Levi nondegenerate hypersurfaces \([9, 28]\). In this case, nonlinearizable automorphisms of the model hyperquadrics are determined by their two jets.

We consider this problem in the singular Levi form case. In this paper, we study systematically nonlinear infinitesimal CR automorphisms of polynomial models in complex dimension \(N > 3\), the \(\mathbb{C}^2\) and \(\mathbb{C}^3\) cases being completely understood (see \([21, 18]\)). The results provide a description of hypersurfaces of finite Catlin multitype in \(\mathbb{C}^N\) whose polynomial models admit such symmetries. In combination with the results of \([16]\), we prove a sharp 1-jet determination result for the holomorphic automorphism groups in general.

Moreover, we identify the common source of such symmetries. In the case of homogeneous polynomial models, they arise via suitable holomorphic mappings into hyperquadrics in some complex spaces, as “pull-back” of symmetries of the hyperquadrics.

We first recall the sharp results of \([16]\) which describe explicitly the possible structure of the Lie algebra of infinitesimal CR automorphisms.

Let us consider a holomorphically nondegenerate weighted homogeneous model of finite Catlin multitype in \(\mathbb{C}^{n+1}\), and denote

\[
M_H := \{ \text{Im } w = P_C(z, \bar{z}) \}, \quad (z, w) \in \mathbb{C}^n \times \mathbb{C},
\]

where \(P_C\) is a weighted homogeneous polynomial of degree one with respect to the multitype weights in the sense of Catlin (for precise definitions, see Section 2).

It has been proved in \([16]\), that the Lie algebra of infinitesimal CR automorphisms \(\mathfrak{g} = \text{aut}(M_H, 0)\) of \(M_H\) admits the following weighted decomposition,

\[
\mathfrak{g} = \mathfrak{g}^{-1} \oplus \bigoplus_{j=1}^s \mathfrak{g}_{-\mu_j} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_c \oplus \mathfrak{g}_{nc} \oplus \mathfrak{g}_1,
\]

where \(\mathfrak{g}_c\) contains vector fields commuting with \(W = \partial_w\) and \(\mathfrak{g}_{nc}\) contains vector fields not commuting with \(W\), arising by (possible) integration of nontransversal shifts. In both cases, the weights of such vector fields lie in the interval \((0, 1]\). Note that apriori, \(\mathfrak{g}_{nc}\) is only determined modulo \(\mathfrak{g}_c\). However, it will be shown in the proof of Theorem 1.2 that there always exists a canonical representation for \(\mathfrak{g}_{nc}\). Note that \(\mathfrak{g}_c = \{0\}\) in the case of a Levi nondegenerate hypersurface. Recall that the elements of \(\mathfrak{g}_1\) are the (possible) 2–integrations of \(W\), and that \(\dim \mathfrak{g}_1 = 0\) or 1; vector fields in \(\mathfrak{g}_j\) with \(j < 0\) are regular and vector fields in \(\mathfrak{g}_0\) are linear (see \([16]\)).

We introduce the following definition.

**Definition 1.1.** Let \(R\) be a weighted homogeneous polynomial and \(S\) be a holomorphic weighted homogeneous polynomial. We say that \(R\) admits an \(S\)–reproducing field, if there exists a holomorphic weighted homogeneous vector field \(Z\) such that

\[
Z(R) = SR.
\]

Note that for \(S = 1\) we obtain the definition of a complex reproducing field, used in \([16]\).
Theorem 1.2. Let $P_C(z, \bar{z})$ be a weighted homogeneous polynomial of degree $1$ with respect to the multitype weights, such that the hypersurface

$$M_H := \{ \text{Im } w = P_C(z, \bar{z}) \}, \quad (z, w) \in \mathbb{C}^n \times \mathbb{C},$$

is holomorphically nondegenerate. Let $\mathfrak{g}_{nc}$ in (1.2) satisfy

$$\dim \mathfrak{g}_{nc} > 0.$$

Then $M_H$ is biholomorphically equivalent to

$$\text{Im } w = |z_1|^2 + S(z', \bar{z}') \quad \text{or} \quad \text{Im } w = \text{Re}(z_1Q_1(z', \bar{z}')) + S(z', \bar{z}'),$$

where $Q_1$ is a holomorphic polynomial.

Moreover, when $P_C$ is a weighted homogeneous polynomial given by (1.5), then (1.5) admits a nontrivial $\mathfrak{g}_{nc}$ if and only if $S$ admits a reproducing field.

Similarly, when $P_C$ is a homogeneous polynomial given by (1.6) (i.e. the multitype weights are all equal), then (1.6) admits a nontrivial $\mathfrak{g}_{nc}$ if and only if $Q_1$ and $S$ admit a common $Q_1$-reproducing field $Y$, hence $Y(S) = Q_1S$ and $Y(Q_1) = Q_2$.

The explicit description of the second case will be given in Proposition 4.5 in the case of $\mathbb{C}^3$, we have $S = 0$ in the singular Levi form case. (See [18].) Note also that the classical Levi nondegenerate case is covered by Theorem 1.2.

We will show in Section 4 by an example (see Example (4.6)) that the last part of the claim does not hold in the case of unequal weights.

In order to describe hypersurfaces with nontrivial $\mathfrak{g}_{nc}$ (which occur only in the singular Levi form case), we introduce the following definition.

Definition 1.3. Let $Y$ be a weighted homogeneous vector field. A pair of finite sequences of vector valued holomorphic weighted homogeneous polynomials of dimension $s \{U^{(1)}, \ldots, U^{(l)}\}$ and $\{V^{(1)}, \ldots, V^{(l)}\}$ is called a symmetric pair of $Y_s$-chains if

$$Y(U^{(j)}) = 0, \quad Y(U^{(j)}) = A_j U^{(j+1)}, \quad j = 1, \ldots, l - 1,$$

$$Y(V^{(j)}) = 0, \quad Y(V^{(j)}) = B_j V^{(j+1)}, \quad j = 1, \ldots, l - 1,$$

where $A_j$ and $B_j$ are invertible $s \times s$ matrices, which satisfy

$$A_j = -\overline{B_{l-j}}.$$

If the two sequences are identical, we say that $\{U^{(1)}, \ldots, U^{(l)}\}$ is a symmetric $Y_s$-chain.

The following result shows that in general the elements of $\mathfrak{g}_{nc}$ arise from pairs of chains.
Theorem 1.4. Let $M_H$ be a holomorphically nondegenerate model given by (1.3) admitting a nontrivial $Y \in \mathfrak{g}_c$. Then $P_C$ can be decomposed in the following way:

$$P_C = \sum_{j=1}^{M} T_j,$$

where each $T_j$ is given by

$$T_j = \text{Re} \left( \sum_{k=1}^{N_j} \langle U_j^{(k)}, V_j^{(N_j-k+1)} \rangle \right),$$

where $\{U_j^{(1)}, \ldots, U_j^{(N_j)}\}$ and $\{V_j^{(1)}, \ldots, V_j^{(N_j)}\}$ are symmetric pairs of $Y_j$-chains, and $\langle , \rangle$ is the usual scalar product in $\mathbb{C}^{s_j}$. Conversely, if $Y$ and $P_C$ satisfy (1.7) – (1.11), then $Y \in \mathfrak{g}_c$.

Definition 1.5. If $P_C$ satisfies (1.7) – (1.11), the associated hypersurface $M_H$ is called a chain hypersurface.

The description of the remaining component $\mathfrak{g}_1$ is a consequence of Theorem 4.7 in [16] (see section 2 for the notations).

Definition 1.6. We say that $P_C$ given by (1.3) is balanced if it can be written as

$$P_C(z, \bar{z}) = \sum_{|\alpha|_\Lambda = |\bar{\alpha}|_\Lambda = 1} A_{\alpha, \bar{\alpha}} z^\alpha \bar{z}^{\bar{\alpha}},$$

for some nonzero $n$-tuple of real numbers $\Lambda = (\lambda_1, \ldots, \lambda_n)$, where

$$|\alpha|_\Lambda := \sum_{j=1}^{n} \lambda_j \alpha_j.$$ 

The associated hypersurface $M_H$ is called a balanced hypersurface.

Note that $P_C$ is balanced if and only if there exists a complex reproducing field $Y$ in the terminology of [16], i.e., $Y(P_C) = P_C$. Indeed, it can be shown that such a $Y$ is of the form

$$Y = \sum_{j=1}^{n} \lambda_j z_j \partial_{\bar{z}_j}.$$

(See Lemma 4.6 in [16].)

As a consequence of Theorem 1.1 in [16], we obtain the following result.

Theorem 1.7. The component $\mathfrak{g}_1$ satisfies $\dim \mathfrak{g}_1 > 0$ if and only if in suitable multitype coordinates $M_H$ is a balanced hypersurface.

The following theorem gives the number of derivatives needed to uniquely determine the elements of the stability group of a class of smooth hypersurfaces in terms of their model hypersurfaces.
Theorem 1.8. Let $M$ be a smooth hypersurface and $p \in M$ be a point of finite Catlin multitype with holomorphically nondegenerate model, where $P_C$ is a homogeneous polynomial. If its model at $p$ is neither a balanced hypersurface nor a chain hypersurface, then its automorphisms are determined by the 1-jets at $p$.

Our last result is the following theorem

Theorem 1.9. Let $M_H$ be a holomorphically nondegenerate hypersurface given by (1.3), where $P_C$ is a homogeneous polynomial and $Y$ be a vector field of strictly positive weight. Then $Y \in \text{aut}(M_H, 0)$ if and only if there exists an integer $K \geq n + 1$ and a holomorphic mapping $f$ from a neighbourhood of the origin in $C^{n+1}$ into $C^K$ which maps $M_H$ into a Levi nondegenerate hyperquadric $Q \subseteq C^K$ such that the following holds:

1. $Y$ is $f$-related with a 1-integration of a nontransversal shift of $Q$ if $Y \in g_{nc}$,
2. $Y$ is $f$-related with a 2-integration of a transversal shift of $Q$ if $Y \in g_1$,
3. $Y$ is $f$-related with a rotation of $Q$ if $Y \in g_c$.

The already mentioned example (4.6) suggests that in the case of unequal weights, (1) in Theorem 1.9 fails, although we do not prove this.

Let us remark that mappings of CR manifolds into hyperquadrics have been studied intensively in recent years (see e.g. [1], [10]). Here we ask in addition that the mapping be compatible with a symmetry of the hyperquadric.

The paper is organized as follows. Section 2 contains the necessary definitions used in the rest of the paper. Section 3 deals with the $g_{nc}$ component of the Lie algebra $g$. Section 4 deals with the $g_c$ component while Section 5 contains the proofs of Theorem 1.2, Theorem 1.4, Theorem 1.7, Theorem 1.8 and Theorem 1.9.

2. Preliminaries

In this section we recall the notion of Catlin multitype and some definitions needed in the sequel.

Let $M \subseteq C^{n+1}$ be a smooth hypersurface, and $p \in M$ be a point of finite type $m$ in the sense of Kohn and Bloom-Graham ([5]). We will consider local holomorphic coordinates $(z, w)$ vanishing at $p$, where $z = (z_1, z_2, ..., z_n)$. The hyperplane $\{\text{Im} w = 0\}$ is assumed to be tangent to $M$ at $p$, hence $M$ is described near $p$ as the graph of a uniquely determined real valued function

\[(2.1) \quad \text{Im} w = \psi(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n, \text{Re} w), \quad d\psi(p) \neq 0.\]

Using a result of [5], we may assume that

\[(2.2) \quad \psi(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n, \text{Re} w) = P_m(z, \bar{z}) + o(|\text{Re} w| + |z|^m),\]

where $P_m(z, \bar{z})$ is a nonzero homogeneous polynomial of degree $m$ with no pluriharmonic terms.

The definition of multitype involves rational weights associated to the variables $w, z_1, \ldots z_n$ in the following way.
The variables \( w, \Re w, \) and \( \Im w \) are given weight one, reflecting our choice of variables given by (2.1). The complex tangential variables \((z_1, \ldots, z_n)\) are treated according to the following definitions (for more details, see [20], [16]).

**Definition 2.1.** A weight is an \( n \)-tuple of nonnegative rational numbers \( \Lambda = (\lambda_1, \ldots, \lambda_n) \), where \( 0 \leq \lambda_j \leq \frac{1}{2} \), and \( \lambda_j \geq \lambda_j + 1 \).

Let \( \Lambda = (\lambda_1, \ldots, \lambda_n) \) be a weight, and \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( \beta = (\beta_1, \ldots, \beta_n) \) be multi-indices. The weighted degree \( \kappa \) of a monomial \( q(z, \bar{z}, \Re w) = c_{\alpha, \beta} z^\alpha \bar{z}^\beta (\Re w)^l, \ l \in \mathbb{N}, \) is defined as \( \kappa := l + \sum_{i=1}^{n} (\alpha_i + \beta_i) \lambda_i. \)

A polynomial \( Q(z, \bar{z}, \Re w) \) is \( \Lambda \)-homogeneous of weighted degree \( \kappa \) if it is a sum of monomials of weighted degree \( \kappa \).

For a weight \( \Lambda \), the weighted length of a multiindex \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is defined by \( |\alpha|_\Lambda := \lambda_1 \alpha_1 + \cdots + \lambda_n \alpha_n. \)

Similarly, if \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \hat{\alpha} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_n) \) are two multiindices, the weighted length of the pair \( (\alpha, \hat{\alpha}) \) is \( |(\alpha, \hat{\alpha})|_\Lambda := \lambda_1 (\alpha_1 + \hat{\alpha}_1) + \cdots + \lambda_n (\alpha_n + \hat{\alpha}_n). \)

The weighted order \( \kappa \) of a differential operator is defined in a similar way.

**Definition 2.2.** A weight \( \Lambda \) will be called distinguished for \( M \) if there exist local holomorphic coordinates \((z, w)\) in which the defining equation of \( M \) takes form

\[
\Im w = P(z, \bar{z}) + o_\Lambda(1),
\]

where \( P(z, \bar{z}) \) is a nonzero \( \Lambda \)-homogeneous polynomial of weighted degree 1 without pluriharmonic terms, and \( o_\Lambda(1) \) denotes a smooth function whose derivatives of weighted order less than or equal to one vanish.

The following definition is due to D. Catlin ([7]).

**Definition 2.3.** ([7]) Let \( \Lambda_M = (\mu_1, \ldots, \mu_n) \) be the infimum of all possible distinguished weights \( \Lambda \) with respect to the lexicographic order. The multitype of \( M \) at \( p \) is defined to be the \( n \)-tuple \( (m_1, m_2, \ldots, m_n) \), where \( m_j = \begin{cases} \frac{1}{r_j} & \text{if} \ \mu_j \neq 0 \\ \infty & \text{if} \ \mu_j = 0. \end{cases} \)

Furthermore, if none of the \( m_j \) is infinity, we say that \( M \) is of finite multitype at \( p \).

Coordinates corresponding to the multitype weight \( \Lambda_M \), in which the local description of \( M \) has form (2.3), with \( P_C := P \) being \( \Lambda_M \)-homogeneous, are called multitype coordinates.

Note that if \( n = 1 \), \( M \) is of finite type at \( p \) if and only if \( M \) is of finite multitype at \( p \). In this case, the type of \( M \) at \( p \) is equal to the multitype of \( M \) at \( p \).

From now on, we assume that \( p \in M \) is a point of finite multitype, with \( M \) defined locally by

\[
\Im w = P_C(z, \bar{z}) + o_{\Lambda_M}(1).
\]
Definition 2.4. \cite{20} Let $M$ be given by (2.4). We define a model hypersurface $M_H$ associated to $M$ at $p$ by

\begin{equation}
M_H = \{(z,w) \in \mathbb{C}^{n+1} \mid \text{Im} w = PC(z, \bar{z})\}.
\end{equation}

Note that multitype coordinates $(z,w)$ are not unique. Nevertheless it is shown in \cite{20} that all models are biholomorphically equivalent (in fact by a polynomial transformation).

We recall that $\text{Aut}(M,p)$ is the stability group of $M$, that is, the set of those germs at $p$ of biholomorphisms mapping $M$ into itself and fixing $p$, and that $\text{aut}(M,p)$ is the set of germs of holomorphic vector fields in $\mathbb{C}^{n+1}$ whose real part is tangent to $M$. If $M$ admits a holomorphic vector field $X$ in $\text{aut}(M,p)$ such that $\text{Im} X$ is also tangent (i.e. $X$ is complex tangent), then $\text{aut}(M,p)$ is of infinite dimension \cite{27}.

Definition 2.5. \cite{27} A real-analytic hypersurface $M \subset \mathbb{C}^{n+1}$ is \emph{holomorphically nondegenerate at} $p \in M$ if there is no germ at $p$ of a holomorphic vector field $X$ tangent to $M$.

Definition 2.6. We say that the vector field

$$Y = \sum_{j=1}^{n} F_j(z,w) \partial_{z_j} + G(z,w) \partial_w$$

has homogeneous weight $\mu (\geq -1)$ if $F_j$ is a weighted homogeneous polynomial of weighted degree $\mu + \mu_j$, and $G$ is a homogeneous polynomial of weighted degree $\mu + 1$.

Definition 2.7. Let $X \in \text{aut}(M_P, p)$ be a rigid weighted homogeneous vector field, that is, a vector field whose coefficients do not depend on the $w$ variable. $X$ is called

1. a \textit{shift} if the weighted degree of $X$ is less than zero;
2. a \textit{rotation} if the weighted degree of $X$ is equal to zero;
3. a \textit{generalized rotation} if the weighted degree of $X$ is bigger than zero and less than one.

Definition 2.8. \cite{16} We say that $X \in \text{aut}(M_P, p)$ is an $l$-integration of a rigid vector field $Y$ (necessarily in $\text{aut}(M_P, p)$) if the string of brackets given by $[\ldots [[X; \partial_w]; \partial_w]; \ldots]; \partial_w] = Y$ is of length $l$.

3. Computing $\mathfrak{g}_c$

Recall that the component $\mathfrak{g}_c$ of $\text{aut}(M_H, 0)$ is by definition the set of (possible) generalized rotations, that is, the set of (possible) rigid fields of weight strictly bigger than 0. (See \cite{17}). As recalled in the introduction, this component is not trivial only in the singular Levi form case. We refer the reader to \cite{22} for a ”model” example that illustrates this phenomenon. In this section we derive an explicit description of all model hypersurfaces which admit a nontrivial generalized rotation. We start with the following lemmas.
Lemma 3.1. Let \( a_j(z), j = 1, \ldots, n, \) be \( n \) \( \mathbb{C} \)– linearly independent holomorphic polynomials in the variable \( z \in \mathbb{C}^N \). Then there exist \( z_1, \ldots, z_n \in \mathbb{C}^N \) such that the determinant of the matrix

\[
\begin{pmatrix}
a_1(z_1) & a_2(z_1) & \cdots & a_n(z_1) \\
\vdots & & & \\
a_1(z_n) & a_2(z_n) & \cdots & a_n(z_n)
\end{pmatrix}
\]

is non zero.

Proof. Let \( V_z \) be the complex hyperplane in \( \mathbb{C}^n \) given by

\[ V_z = \{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n | \sum \lambda_j a_j(z) = 0 \} \]

Consider the intersection \( V \) of those varieties \( V_z \) indexed by \( z \). Then, for each compact \( Q \), we have

\[ V \cap Q = (V_{z_1} \cap \cdots \cap V_{z_k}) \cap Q \]

for some \( k \). Since \( V = \{0\} \),

\[ (V_{z_1} \cap \cdots \cap V_{z_k}) \cap Q = \{0\} \]

for \( Q \) containing 0. (See Theorem 9C, page 100 in [29]). Since \( V_z \) is a complex hyperplane, we then obtain

\[ V_{z_1} \cap \cdots \cap V_{z_k} = \{0\} \]

Then there need to exist \( n \) of the \( z_j \) with this property. This achieves the proof of the lemma. \( \square \)

Lemma 3.2. Let \( V_k, k \in \mathbb{N} \), be the space

\[ V_k = \{ X | Y^k(X) = 0 \}, \]

where \( X \) is a weighted homogeneous holomorphic polynomial of a given weighted degree and \( Y \) is a weighted homogeneous holomorphic vector field. Suppose that \( V_1 \) is not trivial. Then there are strictly positive integers \( d_k \leq k \) and \( g_k \leq \dim V_k \), and a basis of the form

\[ \{ F^k_s \in V_k, s = 1, \ldots, \dim V_k, \} \]

such that \( \{ Y^{d_k-1}(F^k_s) \}^{g_k}_{s=1} \) are linearly independent.

Proof. We prove the lemma by induction. Since \( V_1 \) is not trivial, the case \( k = 1 \) is clear, with \( d_1 = 1 \). Suppose the lemma true for \( k \). We have \( 0 = Y^{k+1}(X) = Y^k(Y(X)) = 0 \). The conclusion follows. \( \square \)

Theorem 3.3. Let \( M_H \) be given by admitting a generalized rotation \( Y \). Then \( P_C \) can be decomposed in the following way

\[ P_C = \sum_{j=1}^{M} T_j, \]

(3.3)
where each $T_j$ is given by

$$
T_j = \text{Re} \left( \sum_{k=1}^{N_j} \langle U_j^{(k)}, V_j^{(N_j-k+1)} \rangle \right),
$$

where $\{U_j^{(1)}, \ldots, U_j^{(N_j)}\}$ and $\{V_j^{(1)}, \ldots, V_j^{(N_j)}\}$ are a symmetric pair of $Y_s$-chains, and $\langle , \rangle$ is the usual scalar product in $\mathbb{C}^{s_j}$.

Proof. Let

$$
P_C = \sum_{k=1}^{l} P_{c_k},
$$

where $P_{c_1} \neq 0, P_{c_l} \neq 0$, be the bihomogeneous expansion of $P_C$. Each $P_{c_j}$ is weighted homogeneous with respect to $z$ of weighted degree $c_j$ where $c_1 < c_2 < \cdots < c_l$.

We may write

$$
P_{c_1} = \sum_{j=1}^{r_1} Q_{c_1}^j \overline{Q}_{c_1}^j,
$$

with $r_1$ minimal. Since $Y$ is a generalized rotation, we must have

$$
Y(\sum_{j=1}^{r_1} Q_{c_1}^j \overline{Q}_{c_1}^j) = \sum_{j=1}^{r_1} Q_{c_1}^j Y(\overline{Q}_{c_1}^j) = 0.
$$

Since $r_1$ is minimal, we have that $\{Q_{c_1}^j\}_{j=1}^{r_1}$ are linearly independent and hence

$$
Y(\overline{Q}_{c_1}^j) = 0
$$

for all $j$. We may assume that, after a possible linear transformation

$$
\{Y(\overline{Q}_{c_1}^j), Y(\overline{Q}_{c_1}^j) \neq 0\}
$$

are linearly independent. Let $J_1 = \{j|Y(\overline{Q}_{c_1}^j) \neq 0\}$, and $J_2 = \{j|Y(\overline{Q}_{c_1}^j) = 0\}$. We may then rewrite $P_{c_1}$ as

$$
P_{c_1} = \sum_{j \in J_1} Q_{c_1}^j \overline{Q}_{c_1}^j + \sum_{j \in J_2} Q_{c_1}^j \overline{Q}_{c_1}^j
$$

We consider the following subset of the set $\{P_{c_k}\}$, namely $P_k := P_{c_1+(k-1)\mu}$, where $\mu > 0$ is the weight of $Y$. We claim that there exists $N \leq l$, such that $P_k, \ k \leq N$, can be written as

$$
P_k = \sum_{j=1}^{R_k} Q_{c_k}^j \overline{Q}_{c_k}^j + \tilde{P}_k
$$

so that

- $Y(\overline{Q}_{c_k}^j) = 0$,
- $\{Y(\overline{Q}_{c_k}^j)| Y(\overline{Q}_{c_k}^j) \neq 0\}$ are linearly independent,
\( \{ Q^e_k \}_{j=1}^{r_k} \) are linearly independent,

- there is \( d_k \) such that \( \overrightarrow{Y^{dk}}(Q^e_k) = 0 \), \( \{ \overrightarrow{Y^{dk-1}}(Q^e_k) | \overrightarrow{Y^{dk-1}}(Q^e_k) \neq 0 \} \) are linearly independent, and \( \overrightarrow{Y^{dk-1}}(\hat{P}_k) = 0 \).

Note that \( N \) is well defined since \( Y \) is a generalized rotation. We prove the claim by induction. The case \( k = 1 \) has just been proved. Suppose then by induction that (3.10) holds for \( k < N \). We write

\[
P_{k+1} = \sum_{j=1}^{r_{k+1}} S^e_{j} S^{\hat{c}_{k+1}}
\]

with \( r_{k+1} \) minimal.

Since \( Y \) is a generalized rotation, we have

\[
\sum_{j=1}^{r_k} Y(Q^e_k)Q^{\hat{e}_j} + Y(\hat{P}_k) + \sum_{j=1}^{r_{k+1}} S^{\hat{c}_{k+1}} Y(S^{\hat{e}_{k+1}}) = 0.
\]

Applying \( \overrightarrow{Y^{d_k}} \) to (3.12), we get

\[
\sum_{j=1}^{r_{k+1}} S^{\hat{c}_{k+1}} Y(Y^{d_{k+1}}(S^{\hat{e}_{k+1}})) = 0.
\]

Since \( r_{k+1} \) is minimal,

\[
\overrightarrow{Y^{d_{k+1}}}(S^{\hat{e}_{k+1}}) = 0
\]

for all \( j \). Using (3.12), we may then rewrite \( P_{k+1} \) in the form given by (3.10), with \( d_{k+1} \leq d_k + 1 \). The claim is then proved.

We consider the following set \( E \) given by

\[
E := \{ Q^e_j Q^{\hat{e}_j}, j = 1, \ldots, R_k, k = 1, \ldots, N \}.
\]

We claim that the following holds for every element of \( E \).

1. \( d_{k+1} = d_k + 1 \),
2. \( Y(Q^{(c_k)}) = A_k Q^{(c_{k+1})} \),
3. \( Y(Q^{(\hat{c}_{k+1})}) = B_{k+1} Q^{(\hat{c}_k)} + R_k \), where \( Y^{d_k-1}(R_k) = 0 \).

Suppose that this is true for \( k < N - 1 \) and show that it is also true for \( k + 1 \). Using the fact that \( Y \) is a generalized rotation, we have as in (3.12)

\[
\sum_{j=1}^{r_k} Y(Q^e_j)Q^{\hat{e}_j} + Y(\hat{P}_k) + \sum_{j=1}^{r_{k+1}} (Q^{\hat{c}_{k+1}} Y(Q^{\hat{e}_{k+1}}) + Y(\hat{P}_{k+1}) = 0.
\]
Applying $\nabla^{d_k-1}$ to (3.16), we obtain, since $d_{k+1} \leq d_k + 1$,

$$\sum Y(Q_j^{c_k}) \nabla^{d_k-1}(\overline{Q_j^{c_k}}) + \sum (Q_j^{c_k+1}) \nabla^{d_k}(\overline{Q_j^{c_k+1}}) = 0.$$  \hspace{1cm} (3.17)

Hence, using (3.17), $d_{k+1} = d_k + 1$, and therefore, using (3.10) and Lemma 3.1

$$Y(Q^{(c_k)}) = A_k Q^{(c_{k+1})}. \hspace{1cm} (3.18)$$

$$Y^{d_k}(Q^{(c_{k+1})}) = B_{k+1} Y^{d_k-1} Q^{(c_k)}, \hspace{1cm} (3.19)$$

which implies

$$Y^{d_k}(Y(Q^{(c_{k+1})}) - B_{k+1} Q^{(c_k)}) = 0, \hspace{1cm} (3.20)$$

and hence

$$Y(Q^{(c_{k+1})}) = B_{k+1} Q^{(c_k)} + R_k, \hspace{1cm} (3.21)$$

where $Y^{d_k-1}(R_k) = 0$. This achieves the proof of the claim. Using (3.21) and (3.10), we may then assume without loss of generality that $R_k = 0$. We define the chains by putting

$$\begin{aligned}
U^{(k)}_1 &:= Q^{(c_k)}, \\
V^{(k)}_1 &:= Q^{(c_{N-k+1})},
\end{aligned} \hspace{1cm} (3.22)$$

It follows from the above properties of $E$ that $U^{(k)}_1$ and $V^{(k)}_1$ form a chain. In other words, we may write

$$P = \Re \left( \sum_{k=1}^N <U^{(k)}_1, V_1^{(N-k+1)}> \right) + \hat{P}. \hspace{1cm} (3.23)$$

It follows from (3.17) that $Y$ is a generalized rotation for

$$\Im w = \Re \left( \sum_{k=1}^N <U^{(k)}_1, V_1^{(N-k+1)}> \right).$$

It follows from (3.16) that $A_k = -iT_{b_{k+1}}$, which means that the $U$ and $V$ are a pair of symmetric chains. Hence $Y$ is a generalized rotation also for $\hat{P}$. We can repeat the above argument for $\hat{P}$ and in a finite number of steps we reach the conclusion of the theorem.

\[\Box\]

As noticed in [17], symmetric chains and pairs of chains of any length can arise.
4. Computing $\mathfrak{g}_{nc}$

Recall that the component $\mathfrak{g}_{nc}$ of $\text{aut}(M_H,0)$ is by definition the set of (possible) 1-integration of (possible) nontransversal shifts; they are of weight strictly bigger than 0, defined up to $\mathfrak{g}_c$. In [18], we consider the case $n = 2$, and show that "only" two model hypersurfaces occur for which $\mathfrak{g}_{nc} \neq 0$, one being the "model" example studied in [22] given by $\text{Im} \ w = \text{Re} \ z_1 z_2^{d-1}$. In this section we derive an explicit description of all model hypersurfaces which admit a nontrivial 1-integration and show that there exists a canonical representation of such a vector field.

Let $M_H$ of finite Catlin multitype in $\mathbb{C}^{n+1}$ be given by

$$M_H := \{ \text{Im} \ w = P_C(z, \bar{z}) \}, \quad (z, w) \in \mathbb{C}^n \times \mathbb{C},$$

where $P_C$ is a weighted homogeneous polynomial of degree one with respect to the multitype weights $\mu_1, \mu_2, \ldots, \mu_n$. Suppose that $M_H$ has a (nontrivial) $\mathfrak{g}_{-\mu_l}$, with $\mu_l$ chosen to be minimal such that there exists a (nontrivial) $X \in \mathfrak{g}_{-\mu_l}$ that admits a nontrivial 1-integration. By Lemma 6.1 in [16] there exist local holomorphic coordinates preserving the multitype (with pluriharmonic terms allowed), such that

$$X = i \partial_{z_l}.$$  

Hence we may write $P_C$ in the following form

$$P_C(z, \bar{z}) = \sum_{j=0}^{m} (\text{Re} \ z_l)^j P_j(z', \bar{z}'),$$

for some homogeneous polynomials $P_j$ in the variables $z' = (z_1, \ldots, \hat{z}_l, \ldots, z_n)$, with $P_m \neq 0$.

**Proposition 4.1.** Let $M_H$ be a holomorphically nondegenerate model with $\mathfrak{g}_{nc} \neq 0$. Let $X$ be given by (4.2), and $P_C$ and $m$ be given by (4.3). Then

- $m \leq 2$
- If $m = 2$, $P_2$ is a real constant.

**Proof.** Splitting $Y$ with respect to the powers of $z_l$, we obtain

$$Y = i w \partial_{z_l} + \sum_{j=-m}^{k} Y_j,$$

where $Y_j$ is of the form

$$Y_j = \varphi_l^j(z') z_l^{j+1} \partial_{z_l} + < \varphi^j(z') z_l^j, \partial_{z'} > + \psi^j(z') z_l^{j+m} \partial_w.$$

(The coefficients are zero when the power of $z_l$ is negative and $Y_k \neq 0$).

We claim that $m - 1 \leq k$. Indeed, if not, applying $\text{Re} \ Y$ to the first term of the right handside of (4.3), we obtain a term which contains the maximal nonzero power in $z_l$,
namely
\[-\frac{m}{2}P^2_m(\text{Re} z_i)^{2m-1}\]
while all other terms are of maximal power $m + k$ with respect to $z_i$, which gives the contradiction. Suppose by contradiction that $m > 2$. We define $Z \in \text{aut}(M_H, 0)$ by
\begin{equation}
Z := [[Y, X], Y].
\end{equation}
$Z$ is nonzero non rigid vector field since $k \geq 2$, and its weighted homogeneous degree is $2 - 2m$. By Theorem 1.3 in [16], it implies that $2 - 3\mu_t \leq 1 - \mu_t$ by minimality of $\mu_t$, since $Z$ is not a 2–integration of $\partial w$. Hence $\mu_t = \frac{1}{2}$, which is a contradiction with $m > 2$. This achieves the proof that $m \leq 2$.

We now prove that $P_2$ is constant. Let $k$ be given by (4.4). Let us denote the middle term of (4.5)
\begin{equation}
Y_j' := \langle \varphi^j(z'), z_l^j, \partial z_l' \rangle
\end{equation}
and analogously
\begin{equation}
Y' := \sum_j Y_j'.
\end{equation}

Note that by weighted homogeneity of $Y$, each coefficient of $Y'$ is weighted homogeneous in $z'$.

First assume $k = 1$. From coefficients of degree three with respect to $z_i$, we obtain,
\begin{equation}
2x_i^3 P_2^2 = 2x_i P_2 \text{Re} \varphi_i^1 z_i^2 + 2x_i^2 \text{Re} Y'_i(P_2) - \text{Im} \psi^3 z_i^3.
\end{equation}

If $\psi^3(z') \neq 0$ in (4.9), then it is a constant, by comparing terms in $y_i^3$. Hence, by homogeneity, $P_2$ is constant. Next, assume that $\psi^3(z') = 0$. Comparing degrees in $z'$, we see that $\varphi_i^1$ and $P_2$ have the same degree, or $\varphi_i^1 = 0$. If $\varphi_i^1 \neq 0$, then from the coefficients of $x_i y_i^2$ we obtain that $\varphi_i^1$ is a constant, hence $P_2$ is a constant. On the other hand, $\varphi_i^1 = 0$ implies
\begin{equation}
x_i^3 P_2^2 = x_i^2 \text{Re} Y'_1(P_2)
\end{equation}
which is impossible, unless $P_2 = 0$, since by positivity of $P_2^2$, the left hand side contains a non zero diagonal term in $z'$, while the right hand side has no such terms.

Now assume that $k \geq 2$ (note that $k = 0$ is impossible, since $k \geq m - 1$). $Z$ given by (4.6) is then a nonzero nonrigid vector field, and then $\mu_t = \frac{1}{2}$, which means that $P_2$ is constant. This achieves the proof of the proposition.

We have the following lemma.
Lemma 4.2. Let \( m \in \mathbb{N}, \ m \geq 1 \). There exist uniquely determined nonzero complex numbers \( \alpha_0, \ldots, \alpha_{m-1} \) such that for every \( z \in \mathbb{C} \),

\[
(\Re z)^{2m-1} = \sum_{j=0}^{m-1} (\Re z)^j \Re (\alpha_j z^{2m-1-j}).
\]

Proof. Indeed, by comparing coefficients of \( z^{m-1} \overline{z}^m \) we obtain the value of \( \alpha_{m-1} \). Continuing this way, from the coefficients of \( z^{m-1-j} \overline{z}^{m+j} \) we obtain the uniquely determined values of \( \alpha_{m-1-j} \).

Proposition 4.3. Let \( M_H \) be a holomorphically nondegenerate model, with \( P_C \) given by (4.3), and \( m = 2 \). Let \( X = i \partial_z \) be in \( \text{aut}(M_H, 0) \). Then there is a vector field \( Y \) in \( \text{aut}(M_H, 0) \) such that \( [Y, W] = X \), if and only if \( P_C \) is biholomorphically equivalent, by a change of multitype coordinates, to

\[
P_C(z, \overline{z}) = x_0^2 + P_0(z', \overline{z}'),
\]

where \( P_0(z', \overline{z}') \) is a balanced polynomial without pluriharmonic terms.

Moreover \( Y \) can be chosen canonically as

\[
Y = iw \partial_{z_1} + az_1^2 \partial_{z_1} + z_1 S + bz_3 \partial_w,
\]

where \( a \) and \( b \) are uniquely determined nonzero constants, \( S = \langle \varphi(z'), \partial_z' \rangle \) uniquely determined by the condition \( S(P_0) = P_0 \).

Proof. Let \( Y \) be given by (4.4) and (4.5). Without loss of generality, we may assume that both \( P_1 \) and \( P_0 \) contain no pluriharmonic terms. Indeed pluriharmonic terms in \( P_1 \) can be eliminated by a change of variables \( z_1^* = z_1 + S(z') \), where \( S \) is a holomorphic polynomial in \( z' \), using the fact that \( P_2 \) is constant. Then to eliminate pluriharmonic terms in \( P_0 \), we perform a change of coordinates of the form \( w^* = w + H(z') \), where \( H \) is a holomorphic polynomial in \( z' \). We claim that \( P_1 = 0 \). Applying \( \Re Y \) to \( P_C - v \) gives

\[
-(2x_l + P_1)(x_l^2 + x_l P_1 + P_0) + 2 \Re (\varphi_l \frac{\partial P_C}{\partial z_l}) + 2 \Re \left( \sum_{j \neq l} \varphi_j \frac{\partial P_C}{\partial z_j} \right) - \Im \psi = 0.
\]

Let \( k \) be as in (4.4). Assume first that \( k = 1 \). For the third order terms in \( z_l \) we obtain

\[
2x_3^2 = 2x_3 \Re (\varphi_l^* z_l^2) - \Im (\psi^l z_l^3).
\]

By Lemma (4.2), \( \varphi_l^* \) and \( \psi^l \) are unique non zero constants, \( \varphi_1^* \in \mathbb{R}^* \). Looking at terms of second order in \( z_l \) we obtain from (4.3) and (4.4.1)

\[
-3x_l^2 P_1 + x_l \Re (\varphi_l^* z_l) + 2x_l \Re \left( \sum_{j \neq l} \varphi_j^* z_l \frac{\partial P_1}{\partial z_j} \right) + \Re (\varphi_1^* z_l^2 P_1) - \Im \psi^0 z_l^2 = 0.
\]

Looking at coefficients of \( y_l^2 \), we obtain that \( P_1 \) is pluriharmonic, since \( \varphi_1^* \in \mathbb{R}^* \). Hence \( P_1 = 0 \). Observe that this implies

\[
Y_0 \in \mathfrak{g}_c.
\]
Next, let $k > 1$. Then $k = 2$ since $\mu_l = \frac{1}{2}$. We may assume that after a linear change of coordinates,

$$Y_2 = Y_2' = z_l^2 \frac{\partial}{\partial z_s}, \ s \neq l.$$  

We obtain, as before, for the third order terms in $z_l$

$$2x_l^3 = 2x_l \Re (\varphi_l^1 z_l^2) + 2x_l \Re (z_l^2 \frac{\partial P_1}{\partial z_s}) - \Im (\psi_l^1 z_l^3).$$

By Lemma (4.2) and the fact that $P_1$ does not contain harmonic terms, we obtain $\frac{\partial P_1}{\partial z_s} = 0$, and hence $\varphi_l^1$ is a non zero real constant. In fact one may compute directly that $\varphi_l^1 = \frac{3}{2}$.

For the terms of second order in $z_l$, we get

$$-3x_l^2 P_1 + x_l \Re (\varphi_l^0 z_l) + 2x_l \Re \left( \sum_{j \neq l} \varphi_j^l z_l \frac{\partial P_1}{\partial z_j} \right)$$

$$+ \Re (\varphi_l^1 z_l^2 P_1) + 2\Re (z_l^2 \frac{\partial P_0}{\partial z_s}) - \Im \psi_l^0 z_l^2 = 0.$$ 

Looking at coefficients of $y_l^2$, we obtain that $P_0$ contains a non zero term of the form $z_s H$, where $H$ does not depend on the variable $z_s$ and is of the same degree as $P_1$ with

$$\frac{3}{2} P_1 + H + \bar{H} = 0.$$ 

By definition of the weights, since $z_s$ has weight $\mu_s = \frac{1}{2}$, that forces $H$ to be linear, and hence, using (4.21), $P_1$ should contain the harmonic term $H$, which is impossible. It shows that $k = 2$ is impossible.

Returning to the only possible case, $k=1$, and looking at the linear terms in $z_l$, we obtain

$$-2x_l P_0 + 2\Re (z_l \sum_{j \neq l} \varphi_j^1 \frac{\partial P_0}{\partial z_j}) + x_l \Re (\varphi_l^{-1}) - \Im (\psi_l^{-1} z_l) = 0$$

which gives equations for coefficients of $x_l$ and $y_l$. Namely

$$-2P_0 + 2\Re \left( \sum_{j} \varphi_j^1 \frac{\partial P_0}{\partial z_j} \right) + \Re (\varphi_l^{-1}) - \Im (\psi_l^{-1}) = 0$$

and

$$-2\Im \sum_{j=2}^n \varphi_j^1 \frac{\partial P_0}{\partial z_j} - \Re \psi_l^{-1} = 0.$$
Using the fact that $P_0$ contains no pluriharmonic terms, and that the weight of $\psi^{-1}$ is equal to one, it follows that $\psi^{-1} = 0$. By the same argument, $\varphi_1^{-1} = 0$. Hence we obtain

\[(4.25) \quad \text{Im} \sum_{j=2}^{n} \varphi_j \frac{\partial P_0}{\partial z'} = 0,\]

hence

\[(4.26) \quad P_0 = \sum_{j=2}^{n} \varphi_j \frac{\partial P_0}{\partial z'}.\]

It follows that $P_0$ has a complex reproducing field, hence $P_0$ is a balanced polynomial, as claimed. That finishes the proof. \qed

Now we consider the case $m = 1$. Let us write $P_0$ as

\[(4.27) \quad P_0 = \sum_{j=2}^{s} S_j(z') Q_j(z'),\]

where $s$ is minimal. Without loss of generality, we may assume that $P_0$ contains no pluriharmonic terms by performing local holomorphic change of coordinates.

**Definition 4.4.** For a $(n-1)$-tuple of functions $R = (R_1, \ldots, R_{n-1})$ depending on $n-1$ complex variables $Z_1, \ldots, Z_{n-1}$, we denote by $\Delta(R_1, \ldots, R_{n-1})$ the determinant of the Jacobi matrix of $R_1, \ldots, R_{n-1}$

\[\Delta(R_1, \ldots, R_{n-1}) = \text{det}(\frac{\partial R}{\partial Z_1}, \ldots, \frac{\partial R}{\partial Z_{n-1}})\]

and set

\[\Delta_j^H(R_1, \ldots, R_{n-1}) = \text{det}(\frac{\partial R}{\partial Z_1}, \ldots, \frac{\partial R}{\partial Z_{j-1}}, H, \frac{\partial R}{\partial Z_{j+1}}, \ldots, \frac{\partial R}{\partial Z_{n-1}}).\]

**Proposition 4.5.** Let $X = i\partial_{z_l}$ be in $\text{aut}(M_H, 0)$ and $P_C$ a homogeneous polynomial of degree $d$ of the form \((1.3)\) with $m = 1$. Assume that $P_0$ is given by \((1.27)\). Then there is a vector field $Y$ in $\text{aut}(M_H, 0)$ such that $[Y, W] = X$ if and only if

1. $P_1 = \text{Re}(Q_1)$ where $Q_1$ is a holomorphic polynomial,
2. for every choice of $Q := (Q_{j_1}, \ldots, Q_{j_{n-1}})$ such that $\Delta(Q_{j_1}, \ldots, Q_{j_{n-1}}) \neq 0$, there exist homogenous functions $g_k$, $k = 1, \ldots, s$, of degree one, holomorphic outside a analytic set, such that $Q_k = g_k(Q_{j_1}, \ldots, Q_{j_{n-1}})$, $k = 1, \ldots, s$
3. the polynomials
\[\frac{1}{2} Q_1 \Delta_j^Q(Q_{j_1}, \ldots, Q_{j_{n-1}})\]
are divisible by $\Delta(Q_{j_1}, \ldots, Q_{j_{n-1}})$ in the ring of holomorphic polynomials.
Moreover $Y$ can be chosen canonically as

$$Y = iw\partial_z + \varphi_0 z_l \partial_{z_l} + \psi z_l \partial_w + S$$

where $\varphi_0$ and $\psi$ are homogeneous polynomials in the variables $z'$ uniquely determined and satisfying the condition $i\varphi_0 \partial_{z_l} + i\psi \partial_w \in g_c$, $S = \langle \varphi(z'), \partial_{z'} \rangle$ uniquely determined by the condition $2S(P) = Q_1 P$.

Let us remark that in complex dimension three, the previous proposition implies that $P_0 = 0$, which was already proved in [18].

**Proof.** Integrating $X$, we obtain the same form of $Y$ as before,

$$Y = iw\partial_z + \sum_{j=1}^n \varphi_j \partial_{z_j} + \psi \partial_w.$$ 

From $\text{Re} Y(P - v) = 0$, using $\text{Re} X(P) = 0$, we obtain

$$P_0 P_1 + x_l P_1^2 = 2x_l \text{Re} \sum_{j \neq l} \varphi_j \partial_{P_1} + \text{Re} \varphi_1 P_1 + 2\text{Re} \sum_{j \neq l} \varphi_j \partial_{z_j} - \text{Im} \psi.$$

Let $k$ be again as in (4.4). For the constant and linear terms in $z_l$ we have

$$P_0 P_1 = 2\text{Re} \sum_{j \neq l} \varphi_j \partial_{z_j} + \text{Re} \varphi_1^{-1} P_1 - \text{Im} \psi^{-1}$$

and

$$x_l P_1^2 = 2x_l \text{Re} \sum_{j \neq l} \varphi_j \partial_{P_1} + \text{Re} \varphi_1^0 z_l P_1 + 2\text{Re} z_l \sum_{j \neq l} \varphi_j \partial_{z_j} - \text{Im} \psi^0 z_l.$$

Let first $k = 2$. By (4.30), $\mu_l = \frac{1}{2}$, and hence $P_1 = \text{Re} H$, where $H$ is holomorphic linear, using the definition of weights. After performing local holomorphic changes of coordinates, we may assume that $H = z_k$ for some $k \neq l$. Because of minimality in (4.27), we can normalize $P_0$ and assume that $H = z_k$ is not in the $\mathbb{C}$ linear span of the $S_j$ by absorbing such a term into $P_1$. We get

$$0 = \text{Re} \left( \varphi_1^2 z_l^3 \left( \frac{z_k + \bar{z}_k}{2} \right) \right) + x_l \text{Re} \left( z_l^2 \varphi_1^2 \right) - \text{Im} \left( \psi^2 z_l^3 \right).$$

From the coefficient of $z_l z_l^2$, we obtain that $\varphi_1^2 = 0$. If $\varphi_1^2 \neq 0$, it follows that $\text{Re} \varphi_1^2 z_l^3 P_1$ is pluriharmonic, hence $P_1$ is constant, which is impossible. We then assume $\varphi_1^2 = 0$. Looking for terms of second degree in $z_l$, we obtain

$$0 = \text{Re} \left( \varphi_1^1 z_l^2 \left( \frac{z_k + \bar{z}_k}{2} \right) \right) + x_l \text{Re} \left( z_l \varphi_k^1 \right) - \text{Im} \left( \psi^1 z_l^2 \right) + 2\text{Re} \left( z_l^2 \sum_{j \neq l} \varphi_j \partial_{P_0} \right).$$
Looking at the term $z_l\bar{z}_l$ in (4.34), we obtain that $\varphi_k^1 = 0$, since $\varphi_k^1$ has weight $\frac{1}{2}$. If $\varphi_l^1 \neq 0$, then we get a contradiction since $z_k$ is not in the $\mathbb{C}$ linear span of the $S_j$. If $\varphi_l^1 = 0$, then
\begin{equation}
(4.35) \quad \operatorname{Re} (z_l^2 \sum_{j \neq l} \varphi_j^2 \frac{\partial P_0}{\partial z_j}) = 0,
\end{equation}
and hence
\begin{equation}
(4.36) \quad \sum_{j \neq l} \varphi_j^2 \frac{\partial P_0}{\partial z_j} = 0,
\end{equation}
Using the fact that $M_H$ is holomorphically nondegenerate, we obtain that
\begin{equation*}
\sum_{j \neq l} \varphi_j^2 \frac{\partial}{\partial z_j} = 0,
\end{equation*}
since $\varphi_k^2 = 0$. This means that $k = 2$ is impossible.

Now let $k = 1$. First of all, observe that $Y_1(P_l) = 0$.

Since $[Y, X]$ is a generalized rotation ($d > 2$), $P$ has a chain structure.

By the results of Section 3, we can write (in the scalar product notation),
\begin{equation*}
P = \sum Re < U_j, \bar{U}_{n-j+1}>,
\end{equation*}
where
\begin{equation*}
[Y, X](U_n) = 0.
\end{equation*}

Since the degree of $[Y, X]$ is $d - 2$, we obtain that the maximal length of a chain is two, hence the first element in those maximal chains is linear, while the second one has degree $d - 1$. That means that nonlinear terms could exist but are killed by $[Y, X]$. We also notice that $[Y, X] = -i\varphi_0^0 \partial_{z_l} - i\psi^0 \partial_w - i Y_1$, which means that $[Y, X](P_0) = -i Y_1(P_0)$.

Write
\begin{equation*}
P_1(z', \bar{z}') = \operatorname{Re} H(z') + \tilde{P}(z', \bar{z}'),
\end{equation*}
where $H$ is harmonic.

We can write
\begin{equation}
(4.37) \quad P_0(z', \bar{z}') = \operatorname{Re} \left( \sum_{j=1}^r L_j(z') \bar{S}_j(\bar{z}') \right) + \tilde{P}_0(z', \bar{z}'),
\end{equation}
where $Y(\tilde{P}_0(z', \bar{z}')) = 0$, and where $L_j$ are linear functions and $r$ is minimal. Because of minimality, we can normalize $P_0$ and assume that $H$ is not in the $\mathbb{C}$ linear span of the $S_j$, by absorbing such a term into $P_1$.

Let
\begin{equation*}
Y_1 = \sum_{j \neq l} f_j \partial_{z_j}.
\end{equation*}
From the coefficients of \(y_l\) in (4.32) we obtain
\[(4.38) \quad \text{Im} \, \phi_0^l P_1 + 2 \text{Im} \, Y_1(P_0) + \text{Re} \, \psi^0 = 0.\]

We first assume that \(\tilde{P}(z', \bar{z}') = 0\), that is, \(P_1\) is harmonic. Consider the diagonal terms in this equation and in the real part of (4.32). From the real part, on the l.h.s we obtain just \(H \bar{H}\). \(P_0\) cannot produce such a term, since \(H\) is not in the \(\mathbb{C}\) linear span of the \(S_j\); we get \(\phi_0^l = cH, \quad c \in \mathbb{R}\). Now looking at the imaginary part, the first term compensates with the third one, hence
\[\text{Im} \, Y_1(P_0) = 0.\]

Since there are no more diagonal terms on the r.h.s of the real part of (4.32), we obtain
\[\text{Re} \, Y_1(P_0) = 0.\]

It implies \(Y_1(P_0) = 0\), hence \(M_P\) is holomorphically degenerate, which is a contradiction. Assume now that \(\tilde{P}(z', \bar{z}') \neq 0\). Then (4.38) holds if \(\phi_0^l = 0\). Using (4.32), we obtain that \(H = 0\). Applying \(Y_1\) to (4.31), and using the fact that \(Y_1(P_1) = 0\) we obtain a contradiction. It follows that \(k = 1\) is impossible.

Now, let \(k = 0\). We get
\[(4.39) \quad x_l P_1^2 = 2 x_l \text{Re} \left( \sum_{j \neq l} \phi_j \frac{\partial P_1}{\partial z_j} \right) + \text{Re} \left( \phi_0^l \bar{z}_l P_1 \right) - \text{Im} \, \psi^0 z_l.\]

From the coefficients of \(y_l\), we get
\[(4.40) \quad - \text{Im} \, \phi_0^l P_1 - \text{Re} \, \psi^0 = 0.\]

This implies that \(P_1\) is pluriharmonic, namely \(P_1 = c \text{Re} \, \phi_0^l\). Notice that \(\phi_0^l = 0\) leads to contradiction. Indeed, if \(\phi_0^l = 0\), then \(\psi^0 = 0\), since \(P_1\) cannot be constant. It follows that
\[(4.41) \quad P_1^2 = 2 \text{Re} \sum_{j \neq l} \phi_j \frac{\partial P_1}{\partial z_j}\]

which is impossible, since the left hand side contains a nonzero balanced term in \(z'\), while the right hand side has no such terms. That gives the contradiction.

Next consider the equation for the coefficients of \(x_l\) in (4.39).
\[(4.42) \quad P_1 = 2 \text{Re} \, Y_0(P_1) + \text{Re} \, \phi_0^l P_1 - \text{Im} \, \psi^0.\]

Substituting \(P_1 = c \text{Re} \, \phi_0^l\), from the mixed terms we obtain \(c = 1\).

For terms of order zero we obtain
\[(4.43) \quad P_0 P_1 = \text{Re} \, \phi_0^{-1} P_1 + 2 \text{Re} \, Y_0'(P_0) - \text{Im} \, \psi^{-1}.\]

Let us write now \(P_0\) as
\[(4.44) \quad P_0 = \sum_{j=2}^s S_j(z) Q_j(z),\]
where \( s \) is minimal as above. From now on, we denote \( Q_1 = \varphi_l^0 \). From equations (4.42), (4.43) we get the following equations

\[
(4.45) \quad Y_0'(Q_1) = \frac{1}{2} Q_1^2,
\]

and

\[
(4.46) \quad Y_0'(Q_j) = \frac{1}{2} Q_1 Q_j.
\]

By holomorphic nondegeneracy and reality, the polynomials \( Q_1, Q_2, \ldots, Q_s \) are generating, i.e. their gradients span \( \mathbb{C}^{n-1} \) at a generic point. This gives \( s \geq n-1 \). We may then choose \( Q := (Q_{j_1}, \ldots, Q_{j_{n-1}}) \) such that \( \Delta(Q_{j_1}, \ldots, Q_{j_{n-1}}) \neq 0 \). For every \( k = 1, \ldots, s \), there exist holomorphic functions \( g_k \) of \( n-1 \) variables in a neighborhood of a generic point, such that \( Q_k = g_k(Q_{j_1}, \ldots, Q_{j_{n-1}}) \). Substituting for \( Q_k \) into (4.45), (4.46), we obtain

\[
Y_0'(Q_k) = \nabla g_k(Y_0'(Q_{j_1}), \ldots, Y_0'(Q_{j_{n-1}})) = \frac{1}{2} \nabla g_k(Q_1 Q_{j_1}, \ldots, Q_1 Q_{j_{n-1}}).
\]

On the other hand,

\[
(4.47) \quad Y_0''(Q_k) = \frac{1}{2} Q_1 Q_k.
\]

Hence

\[
(4.48) \quad g_k(Q) = <\nabla g_k, Q>.
\]

It follows that \( g_k \) is homogeneous of degree one. Now, in order to determine the component of \( Y_0'' \), we use Cramer’s rule. This leads to

\[
\varphi_j^0 = \frac{1}{2} Q_1 \frac{\Delta Q(Q_{j_1}, \ldots, Q_{j_{n-1}})}{\Delta(Q_{j_1}, \ldots, Q_{j_{n-1}})}
\]

This implies the statement of the proposition. \( \square \)

We will now show that there is no analogous statement in the case of unequal weights.

**Example 4.6.** Consider a model in \( \mathbb{C}^4 \), given by

\[
(4.49) \quad P(z, \bar{z}) = x_1 \text{Re} z_2^l + S(z_3, \bar{z}_3) \text{Re} z_2^l,
\]

where \( S \) is a homogeneous real valued polynomial in \( z_3 \) of degree bigger than one and \( l \) is an integer. Note that \( P_0 \) is not balanced for suitable \( S \). More concretely, let us take \( l = 3 \) and

\[
(4.50) \quad S(z_3, \bar{z}_3) = \text{Re} z_3 \bar{z}_3^3.
\]

The multitype weights become \( (\frac{1}{4}, \frac{1}{4}, \frac{1}{16}) \).

Taking

\[
(4.51) \quad Y'' = \frac{i}{6} z_2 \partial_{z_2},
\]
we obtain a symmetry which is not of the form described in the previous result.

5. Proofs of the main results

In this section we complete the proofs of the results stated in the introduction.

Proof of Theorem 1.2. We apply Propositions 4.1, 4.3 and 4.5.

Proof of Theorem 1.4. This is Theorem 3.3.

Proof of Theorem 1.9. Let \( Y \) be a generalized rotation. In the notation of Theorem 3.3 we set

\[
K = 2 \sum_{j=1}^{M} s_j N_j + 1.
\]

We define a hyperquadric in \( \mathbb{C}^{K+1} \) by

\[
\text{Im } \eta = \text{Re} \sum_{j=1}^{M} \sum_{k=1}^{N_j} < \zeta_{j,(k)}, \zeta'_{j,(N_j-k+1)}> ,
\]

and consider the mapping \( \mathbb{C}^{n+1} \to \mathbb{C}^{K+1} \) given by \( \eta = w \) and

\[
\zeta_{j,(k)} = U_j^{(k)}(z),
\]

and

\[
\zeta'_{j,(k)} = V_j^{(k)}(z).
\]

It is immediate to verify that the automorphism \( Y \) of \( M_\mathcal{P} \) is \( f \)-related to the automorphism of this hyperquadric, defined by

\[
Z = \sum_{j=1}^{M} \sum_{k=2}^{N_j} < A_{k-1,j} \zeta_{j,(k)}, \partial^{\alpha} \zeta_{j,(k)} > + < t A_{k-1,j} \zeta'_{j,(k)}, \partial^{\alpha} \zeta'_{j,(k)} >.
\]

Indeed, the condition for \( f \)-related vector fields becomes exactly the chain condition (1.7)-(1.9).

If \( g_1 \neq 0 \), then by Theorem 1.7

\[
P(z, \bar{z}) = \sum_{|\alpha\lambda|=|\beta\bar{\lambda}|=1} A_{\alpha,\bar{\lambda}} z^\alpha \bar{z}^{\bar{\lambda}},
\]

where \( A_{\alpha,\bar{\lambda}} \neq 0 \). We order the multiindices and write \( P \) as

\[
P(z, \bar{z}) = \text{Re} \left( \sum_{j=1}^{R} \sum_{k=1}^{N_j} A_{j,k} z^{\alpha_j/k} \bar{z}^{\alpha_j} \right).
\]

Consider the hyperquadric in \( \mathbb{C}^{R+\sum_{j=1}^{R} N_j+1} \) defined by

\[
\text{Im } \eta = \text{Re} \left( \sum_{j=1}^{R} \sum_{k=1}^{N_j} \zeta_{j,k} \zeta'_{j,k} \right),
\]
and the mapping \( f : \mathbb{C}^3 \to \mathbb{C}^{R+(\sum_{j=1}^{R} N_j)+1} \) given by \( \eta = w \) and \( \zeta_j = z^{\alpha_j} \) for \( j = 1, \ldots, R, \) \( \zeta_{k,j} = z^{\alpha_{k,j}}, \) \( k = 1, \ldots, N_j. \)

It is immediate to verify that the vector field in \( \text{aut}(M_H,0) \)

\[
Y = \left( \sum_{j=1}^{n} \lambda_j z^j \partial z^j \right) w + w^2 \partial_w,
\]

is \( f \)-related to the infinitesimal automorphism of the above hyperquadric given by

\[
Z = \eta(\sum_{j=1}^{R} \zeta_j \partial_{\zeta_j} + \sum_{j=1}^{R} (\sum_{k=1}^{N_j} \zeta_{k,j} \partial_{\zeta_{k,j}})) + \eta^2 \partial_{\eta}.
\]

The case \( g_{nc} \neq 0 \) is completely analogous.

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