Gelfand–Tsetlin bases of representations for super Yangian and quantum affine superalgebra

Kang Lu

Abstract
We give explicit actions of Drinfeld generators on Gelfand–Tsetlin bases of super Yangian modules associated with skew Young diagrams. In particular, we give another proof that these representations are irreducible. We study irreducible tame $Y(gl_{m|n})$-modules and show that a finite-dimensional irreducible $Y(gl_{m|n})$-module is tame if and only if it is thin. We also give the analogous statements for quantum affine superalgebra of type $A$.

Keywords Gelfand–Tsetlin bases · Super Yangian · Quantum affine superalgebra

Mathematics Subject Classification 17B37 · 81R10

1 Introduction
Yangians and quantum affine algebras and their representations have been extensively studied since 1980. Many striking results are produced. Though super Yangian $Y(gl_{m|n})$ of general linear Lie superalgebra $gl_{m|n}$ was introduced in [16] and its finite-dimensional irreducible modules were classified in [30,31], only few works were done for $Y(gl_{m|n})$ and representations of $Y(gl_{m|n})$ are still far from being well understood. Continuing [12], we study further skew representations of super Yangian which were introduced in [3] and intensively studied in [17,20–22] for the even case.

Inside of the super Yangian, there is a distinguished maximal commutative subalgebra $A(gl_{m|n})$ generated by the Cartan currents of $Y(gl_{m|n})$ which we call the Gelfand–Tsetlin algebra. We say that a finite-dimensional $Y(gl_{m|n})$-module $M$ is tame if the action of the subalgebra $A(gl_{m|n})$ on $M$ is semi-simple. We call $M$ thin if $M$ is tame and the spectrum of $A(gl_{m|n})$ on $M$ is simple.
Skew representations are a certain family of finite-dimensional $Y(\mathfrak{gl}_{m|n})$-modules including evaluation covariant (polynomial) modules. They have bases parameterized by Gelfand–Tsetlin patterns (or semi-standard Young tableaux of the associated skew Young diagrams) and hence are called Gelfand–Tsetlin bases. It turns out that these bases are indeed eigenbases of the Gelfand–Tsetlin algebra. Therefore, skew representations are tame. Moreover, the eigenvalues can be computed explicitly and it is not hard to see that skew representations are actually thin. According to [28, Proposition 3.1], the action of the non-Cartan currents of Drinfeld generators on an eigenvector of $A(\mathfrak{gl}_{m|n})$ in a thin module is essentially determined by the action of the first coefficients of the non-Cartan currents. Combining with [26, Theorem 7], we give the matrix elements of each currents acting on skew representations with respect to Gelfand–Tsetlin bases. In particular, it describes explicitly the poles of the currents acting on skew representations. It would be interesting to determine the set of poles of the currents acting on an arbitrary finite-dimensional irreducible module, cf. [7]. It is also interesting to generalize [13,19] to the supersetting where the main obstacle is the absence of polynomial action of Drinfeld-type currents.

As a corollary, we show that skew representations of $Y(\mathfrak{gl}_{m|n})$ are irreducible. Note that the irreducibility of skew representations is obtained in [12, Theorem 4.9] using the general fact that the Drinfeld functor maps a finite-dimensional irreducible module of degenerate affine Hecke algebra to a finite-dimensional irreducible module of super Yangian, see [12, Proposition 4.8]. Here, we provide another independent proof of the irreducibility of skew representations. Note that the irreducibility should also follow from the superanalogue of the centralizer construction in [15].

The result of this paper is a step toward understanding tame modules of $Y(\mathfrak{gl}_{m|n})$. Let $t_{ij}(u)$ be the R-matrix presentation generating series of the super Yangian $Y(\mathfrak{gl}_{m|n})$, where $t_{ij}(u)$ are series in $u^{-1}$ with $\delta_{ij}$ as the constant term and certain generators of $Y(\mathfrak{gl}_{m|n})$ as other coefficients, see Sect. 2.3. Given $\xi(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$, let $C_{\xi(u)}$ be the one-dimensional module spanned by a nonzero vector $v$ satisfying $t_{ii}(u)v = \xi(u)v$ and $t_{ij}(u)v = 0$ for $1 \leq i \neq j \leq m + n$. For any $z \in \mathbb{C}$ and any skew Young diagram $\lambda/\mu$, denote by $L_z(\lambda/\mu)$ the skew representation corresponding to the skew Young diagram $\lambda/\mu$ with evaluation parameter $z$. It was conjectured in [3] and classified in [20] that all finite-dimensional irreducible tame modules of the (non-super) Yangian $Y(\mathfrak{gl}_N)$ are, up to isomorphism, of the form

$$C_{\xi(u)} \otimes L_{z_1}(\lambda_1/\mu_1) \otimes \cdots \otimes L_{z_k}(\lambda_k/\mu_k).$$

(1.1)

where $\xi(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$, $k \in \mathbb{Z}_{\geq 0}$, and $z_i - z_j \notin \mathbb{Z}$ for all $1 \leq i < j \leq k$. Here, $\lambda_i/\mu_i$ is a skew Young diagram for each $1 \leq i \leq k$. Hence, skew representations are the elementary but also fundamental objects among tame modules in the even case which motivates the study in the supersymmetric setting.

We take the opportunity to list a few open problems about irreducible tame $Y(\mathfrak{gl}_{m|n})$-modules. First, it would be interesting to generalize the classification of irreducible tame modules to supercase.

**Open Problem** Classify all finite-dimensional irreducible tame $Y(\mathfrak{gl}_{m|n})$-modules.
One might suggest again that up to a one-dimensional module, finite-dimensional irreducible tame modules are given by tensor products of skew representations with evaluation parameters in distinct \( \mathbb{Z} \)-cosets. Note that skew representations are direct sums of covariant representations of \( \mathfrak{gl}_{m|n} \) when restricted as \( \mathfrak{gl}_{m|n} \)-modules. It is not hard to see that \( \mathcal{Y}(\mathfrak{gl}_{m|n}) \)-modules of the form in (1.1) are thin and hence are tame. However, they do not cover all finite-dimensional irreducible tame \( \mathcal{Y}(\mathfrak{gl}_{m|n}) \)-modules as there are finite-dimensional irreducible tame modules that are not subquotients of tensor powers of evaluation vector representations. The simplest examples are two-dimensional evaluation \( \mathcal{Y}(\mathfrak{gl}_{1|1}) \)-modules of non-integral weights. Even if we consider only the finite-dimensional irreducible tames modules whose restrictions are direct sums of covariant representations of \( \mathfrak{gl}_{m|n} \), there are still such finite-dimensional irreducible tame modules that are not of the form (1.1). An example of such case will be given in Sect. 3.4.

It was shown for \( \mathcal{Y}(\mathfrak{gl}_N) \) in [20] and for quantum affine algebras of type B in [1] that a finite-dimensional irreducible module is tame if and only if it is thin. We believe the same statement also holds for \( \mathcal{Y}(\mathfrak{gl}_{m|n}) \).

Conjecture 1.1 A finite-dimensional irreducible \( \mathcal{Y}(\mathfrak{gl}_{m|n}) \)-module is tame only if it is thin.

One of our main results is to prove Conjecture 1.1 for the case \( m = n = 1 \) in Sect. 3.4.

The paper is organized as follows. We recall the Gelfand–Tsetlin bases for covariant representations of \( \mathfrak{gl}_{m|n} \) and prepare basic facts of super Yangian in Sect. 2. In Sect. 3, we give our main results for super Yangian and their proofs. We give analogous results for quantum affine superalgebra \( U_q(\hat{\mathfrak{gl}}_{m|n}) \) with generic \( q \) in Sect. 4.

2 Super Yangian \( \mathcal{Y}(\mathfrak{gl}_{m|n}) \)

2.1 Lie superalgebra \( \mathfrak{gl}_{m|n} \)

Throughout the paper, we work over \( \mathbb{C} \). A vector superspace \( W = W_0 \oplus W_1 \) is a \( \mathbb{Z}_2 \)-graded vector space. We call elements of \( W_0 \) even and elements of \( W_1 \) odd. We write \( |w| \in \{ \bar{0}, \bar{1} \} \) for the parity of a homogeneous element \( w \in W \). Set \((-1)^\bar{0} = 1 \) and \((-1)^\bar{1} = -1 \).

Fix \( m, n \in \mathbb{Z}_{\geq 0} \). Set \( I := \{ 1, 2, \ldots, m + n - 1 \} \) and \( \bar{I} := \{ 1, 2, \ldots, m + n \} \). We also set \( |i| = \bar{0} \) for \( 1 \leq i \leq m \) and \( |i| = \bar{1} \) for \( m < i \leq m + n \). Define \( s_i = (-1)^{|i|} \) for \( i \in \bar{I} \).

The Lie superalgebra \( \mathfrak{gl}_{m|n} \) is generated by elements \( e_{ij}, i, j \in \bar{I} \), with the supercommutator relations

\[
[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - (-1)^{(|i|+|j|)(|k|+|l|)} \delta_{il} e_{kj},
\]

where the parity of \( e_{ij} \) is \( |i| + |j| \). Set \( e_i := e_{i,i+1} \) and \( f_i := e_{i+1,i} \) for \( i \in I \). Denote by \( U(\mathfrak{gl}_{m|n}) \) the universal enveloping superalgebra of \( \mathfrak{gl}_{m|n} \).
The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{gl}_{m|n}$ is spanned by $e_{ij}, i \in \tilde{I}$. Let $e_i, i \in I$, be a basis of $\mathfrak{h}^*$ (the dual space of $\mathfrak{h}$) such that $\epsilon_i(e_{jj}) = \delta_{ij}$. There is a bilinear form $(\ , \ )$ on $\mathfrak{h}^*$ given by $(\epsilon_i, \epsilon_j) = s_i \delta_{ij}$. Define the simple roots $\alpha_i := \epsilon_i - \epsilon_{i+1}$, for $i \in I$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m+n})$ be a tuple of complex numbers. We call $\lambda$ a $\mathfrak{gl}_{m|n}$-weight. Denote $L(\lambda)$ the irreducible module of $\mathfrak{gl}_{m|n}$ generated by a nonzero vector $v$ satisfying the conditions

$$e_{ii}v = \lambda_i v, \ e_{jk}v = 0,$$

for $i \in \tilde{I}$ and $1 \leq j < k \leq m + n$.

Let $\mathcal{V} := \mathbb{C}^{m|n}$ be the vector superspace with a basis $v_i, i \in \tilde{I}$, such that $|v_i| = |i|$. Let $E_{ij} \in \text{End}(\mathcal{V})$ be the linear operators such that $E_{ij}v_k = \delta_{jk}v_i$. The map $\rho_{\mathcal{V}} : \mathfrak{gl}_{m|n} \to \text{End}(\mathcal{V}), \ e_{ij} \mapsto E_{ij}$ defines a $\mathfrak{gl}_{m|n}$-module structure on $\mathcal{V}$. We call it the vector representation of $\mathfrak{gl}_{m|n}$. The highest weight of $\mathcal{V}$ is the tuple $(1, 0, \ldots, 0)$.

We call $\lambda$ a covariant $\mathfrak{gl}_{m|n}$-weight if $\lambda$ satisfies: All $\lambda_1, \ldots, \lambda_{m+n}$ are nonnegative integers, and the number $l$ of nonzero components among $\lambda_{m+1}, \ldots, \lambda_{m+n}$ does not exceed $\lambda_m$, see [2,25]; moreover, $\lambda_1 \geq \cdots \geq \lambda_m$ and $\lambda_{m+1} \geq \cdots \geq \lambda_{m+n}$.

We call $L(\lambda)$ a covariant module if $\lambda$ is a covariant $\mathfrak{gl}_{m|n}$-weight. Note that in this case $L(\lambda)$ is a submodule of $\mathcal{V}^{\otimes |\lambda|}$, where $|\lambda| = \sum_{i=1}^{m+n} \lambda_i$. Fix $r \geq 0$. For $k \in \mathbb{Z}_{\geq 0}$, we set $k' = r + k$.

Define similar notations for the Lie algebra $\mathfrak{gl}_r := \mathfrak{gl}_{r|0}$ and the Lie superalgebra $\mathfrak{gl}_{m'|n}$.

### 2.2 Gelfand–Tsetlin tableaux

We identify $\mathfrak{gl}_r$ as a Lie subalgebra of $\mathfrak{gl}_{m'|n}$ via the natural embedding $e_{ij} \mapsto e_{ij}$ and $\mathfrak{gl}_{m|n}$ as a Lie subalgebra of $\mathfrak{gl}_{m'|n}$ via the embedding $e_{ij} \mapsto e_{ij'}$. It is clear that $\mathfrak{gl}_r$ commutes with $\mathfrak{gl}_{m|n}$.

Let $\lambda = (\lambda_1, \ldots, \lambda_{m'+n})$ be a covariant $\mathfrak{gl}_{m'|n}$-weight and $\mu = (\mu_1, \ldots, \mu_r)$ a covariant $\mathfrak{gl}_r$-weight. Let $L(\lambda)$ be the corresponding irreducible $\mathfrak{gl}_{m'|n}$-module. Regard $L(\lambda)$ as a $\mathfrak{gl}_r$-module. Let $L(\lambda/\mu)$ be the subspace of $L(\lambda)$ given by

$$L(\lambda/\mu) := \{ v \in L(\lambda) \mid e_{ii}v = \mu_i v, e_{jk}v = 0, \ \text{for} \ 1 \leq i \leq r, 1 \leq j < k \leq r \}.$$

Clearly, $L(\lambda/\mu)$ is a $\mathfrak{gl}_{m|n}$-module and a $\mathfrak{U}(\mathfrak{gl}_{m'|n})^{\mathfrak{gl}_{r}}$-module.

Our main combinatorial device is an array of complex numbers $\Lambda = (\Lambda_{ij})$ presented in the following form:

$$\begin{array}{cccccccc}
\lambda_{m'+1,1} & \cdots & \lambda_{m'+n,m'} & \lambda_{m'+n,m'+1} & \cdots & \lambda_{m'+n,m'+n-1} & \lambda_{m'+n,m'+n} \\
\lambda_{m'+n-1,1} & \cdots & \lambda_{m'+n-1,m'} & \lambda_{m'+n-1,m'+1} & \cdots & \lambda_{m'+n-1,m'+n-1} & \\
\vdots & & \vdots & & \vdots & \\
\lambda_{r+1,1} & \cdots & \lambda_{r+1,r} & \lambda_{r+1,r+1} \\
\lambda_{r1} & \cdots & \lambda_{rr} \\
\end{array} \quad (2.1)$$

 Springer
We call $\Lambda$ a Gelfand–Tsetlin tableau (GT tableau for short). Given $\Lambda = (\lambda_{ij})$, we set
\[
l_{ki} = \lambda_{k'} + r - i + 1, \quad (1 \leq i \leq m); \quad l_{kj} = -\lambda_{k'j} + r + j - 2m', \quad (m' + 1 \leq j \leq k').
\] (2.2)

A GT tableau is $\lambda/\mu$-admissible if the following conditions are satisfied:

1. $\lambda_{m'+i,n} = \lambda_i$ and $\lambda_{m'} = \mu_j$ for $1 \leq i \leq m' + n$ and $1 \leq j \leq r$;
2. $\theta_{k-1,i} := \lambda_{ki} - \lambda_{k-1,i} \in \{0, 1\}, \quad 1 \leq i \leq m', \quad m' + 1 \leq k \leq m' + n$;
3. $\lambda_{km'} \geq \#\{i : \lambda_{ki} > 0, \quad m' \leq i \leq k\}, \quad m' + 1 \leq k \leq m' + n$;
4. if $\lambda_{m'+1,m'} = 0$, then $\theta_{m',m'} = 0$;
5. $\lambda_{ki} - \lambda_{k,i+1} \in \mathbb{Z}_{\geq 0}, \quad 1 \leq i \leq m' - 1, \quad m' + 1 \leq k \leq m' + n$;
6. $\lambda_{k+1,i} - \lambda_{ki} \in \mathbb{Z}_{\geq 0}$ and $\lambda_{ki} - \lambda_{k+1,i+1} \in \mathbb{Z}_{\geq 0}, \quad 1 \leq i \leq k \leq m' - 1$ or $m' + 1 \leq i \leq k \leq m' + n - 1$.

We recall the following theorem from [26]. Here, we adopt the renormalized version from [6, Theorem 6.1]. Note that our $\Lambda$ corresponds to $\Lambda$ in [6] with $\lambda_{ki} = \mu_i$ for $1 \leq i \leq k \leq r$ as we consider the subspace of singular vectors of $\mathfrak{gl}_r$-weight $\mu$ in $L(\lambda)$.

**Theorem 2.1** ([26, Theorem 7]) The $\mathfrak{gl}_{m|n}$-module $L(\lambda/\mu)$ admits a basis $\xi_\Lambda$ parameterized by all $\lambda/\mu$-admissible GT tableaux $\Lambda$. The actions of the generators of $\mathfrak{gl}_{m|n}$ are given by the formulas

\[
e_{kk'}\xi_\Lambda = \left( \sum_{i=1}^{k'} \lambda_{k'j} - \sum_{j=1}^{k'-1} \lambda_{k'-1,j} \right) \xi_\Lambda, \quad 1 \leq k \leq m + n;
\]
\[
e_{k}\xi_\Lambda = -\sum_{i=1}^{k'} \frac{\prod_{j=1, j \neq i}^{k'+1} (l_{k+1,j} - l_{ki})}{\prod_{j=1}^{k'} (l_{kj} - l_{ki})} \xi_{\Lambda + \delta_{ki}}, \quad 1 \leq k \leq m - 1;
\]
\[
f_{k}\xi_\Lambda = \sum_{i=1}^{k'} \frac{\prod_{j=1, j \neq i}^{k-1} (l_{k-1,j} - l_{ki})}{\prod_{j=1}^{k'} (l_{kj} - l_{ki})} \xi_{\Lambda - \delta_{ki}}, \quad 1 \leq k \leq m - 1;
\]
\[
e_{m'}\xi_\Lambda = \sum_{i=1}^{m'} \theta_{m'i} (-1)^{i-1} (-1)^{\theta_{m'+1} + \cdots + \theta_{m',i-1}} \times \frac{\prod_{1 \leq j < i \leq m'} (l_{mj} - l_{mi} - 1)}{\prod_{1 \leq j < i \leq m'} (l_{mj} - l_{mi}) \prod_{j \neq i, j=1}^{m'} (l_{m+1,j} - l_{mi} - 1)} \xi_{\Lambda + \delta_{mi}},
\]
\[
f_{m'}\xi_\Lambda = \sum_{i=1}^{m'} (1 - \theta_{m'i}) (-1)^{i-1} (-1)^{\theta_{m'+1} + \cdots + \theta_{m',i-1}} \times \frac{(l_{mi} - l_{m+1,m'+1}) \prod_{1 \leq j < i \leq m'} (l_{mj} - l_{mi} + 1) \prod_{j=1}^{m'-1} (l_{m-1,j} - l_{mi})}{\prod_{1 \leq j < i \leq m'} (l_{mj} - l_{mi})} \xi_{\Lambda - \delta_{mi}},
\]
and for \( m + 1 \leq k \leq m + n - 1 \),

\[
e_k \xi_\Lambda = \sum_{i=1}^{m'} \theta_{k,i} (-1)^{\vartheta_{k,i}} (1 - \theta_{k',-1,i}) \times \prod_{j \neq i, j=1}^{m'} \left( \frac{l_{k,j} - l_{k,i} - 1}{l_{k+1,j} - l_{k,i} - 1} \right) \xi_{\Lambda + \delta_{ki}}
- \sum_{i=m'+1}^{k'} \prod_{j=1}^{m'} \left( \frac{l_{k,j} - l_{k,i} + 1}{l_{k+1,j} - l_{k,i}} \right) \prod_{j \neq i, j=m'+1}^{k'} \left( l_{k,j} - l_{k,i} + 1 \right) \xi_{\Lambda + \delta_{ki}}.
\]

\[
f_k \xi_\Lambda = \sum_{i=1}^{m'} \theta_{k',-1,i} (-1)^{\vartheta_{k,i}} (1 - \theta_{k',i}) \times \prod_{j \neq i, j=1}^{m'} \left( \frac{l_{k,j} - l_{k,i} + 1}{l_{k+1,j} - l_{k,i}} \right) \prod_{j \neq i, j=m'+1}^{k'} \left( l_{k,j} - l_{k,i} + 1 \right) \xi_{\Lambda - \delta_{ki}} + \sum_{i=m'+1}^{k'} \prod_{j \neq i, j=m'+1}^{k'} \left( l_{k,j} - l_{k,i} \right) \xi_{\Lambda - \delta_{ki}}.
\]

Here, \( \vartheta_{k,i} = \vartheta_{k,1} + \cdots + \vartheta_{k,i-1} + \vartheta_{k,-1,i+1} + \cdots + \vartheta_{k,-1,m'} \). The arrays \( \Lambda \pm \delta_{ki} \) are obtained from \( \Lambda \) by replacing \( \lambda_{k;i} \) with \( \lambda_{k';i} \pm 1 \). We assume that \( \xi_\Lambda = 0 \) if the GT tableau \( \Lambda \) is not \( \lambda/\mu \)-admissible.

We use the shorthand notations \( \xi_{\pm}^{\pm}_{\lambda,ki} \) for the matrix elements involved in Theorem 2.1 as follows,

\[
e_k \xi_\Lambda = \sum_{i=1}^{k'} \xi_{\lambda,ki}^{+} \xi_{\Lambda + \delta_{ki}}, \quad f_k \xi_\Lambda = \sum_{i=1}^{k'} \xi_{\lambda,ki}^{-} \xi_{\Lambda - \delta_{ki}}.
\] (2.3)

Note that a different Gelfand–Tsetlin-type basis for covariant representations of \( \mathfrak{gl}_{m|n} \) is given in [14, Theorem 4.18].

### 2.3 Super Yangian \( \mathcal{Y}(\mathfrak{gl}_{m|n}) \)

We recall the definition of super Yangian \( \mathcal{Y}(\mathfrak{gl}_{m|n}) \) from [16], see also [9,18] for some basic properties of \( \mathcal{Y}(\mathfrak{gl}_{m|n}) \).

Let \( \mathcal{P} \in \text{End}(\mathcal{Y}^\otimes 2) \) be the \( \mathbb{Z}_2 \)-graded flip operator,

\[
\mathcal{P} = \sum_{i,j \in \tilde{I}} s_i E_{ij}^{(1)} E_{ji}^{(2)}, \quad \text{where} \quad E_{ij}^{(1)} = E_{ij} \otimes 1, \quad E_{ij}^{(2)} = 1 \otimes E_{ij}.
\] (2.4)

The super Yangian \( \mathcal{Y}(\mathfrak{gl}_{m|n}) \) is the \( \mathbb{Z}_2 \)-graded unital associative algebra with generators \( \{ t_{ij}^{(a)} \mid i, j \in \tilde{I}, a \geq 1 \} \) where the generators \( t_{ij}^{(a)} \) have parities \( |i| + |j| \). The defining relations of super Yangian \( \mathcal{Y}(\mathfrak{gl}_{m|n}) \) are as follows.

Define the Yang R-matrix \( R(u) \in \text{End}(\mathcal{Y}^\otimes 2) \) by \( R(u) = 1 - \mathcal{P}/u \). Define the generating series \( t_{ij}(u) \in \mathcal{Y}(\mathfrak{gl}_{m|n})[[u^{-1}]] \) and \( T_k(u) \in \mathcal{Y}(\mathfrak{gl}_{m|n})[[u^{-1}]] \otimes \text{End}(\mathcal{Y}^\otimes 2) \) by

\[
t_{ij}(u) = \delta_{ij} + \sum_{k=1}^\infty t_{ij}^{(k)} u^{-k}, \quad T_k(u) = \sum_{i,j \in \tilde{I}} (-1)^{|i||j|+|j|} E_{ij}^{(k)} \otimes t_{ij}(u), \quad k = 1, 2.
\]
Then, defining relations of \( Y(\mathfrak{gl}_{m|n}) \) are written as
\[
R(u_1 - u_2) T_1(u_1) T_2(u_2) = T_2(u_2) T_1(u_1) R(u_1 - u_2). \tag{2.5}
\]
In terms of generating series, defining relations (2.5) are equivalent to
\[
(u_1 - u_2)[t_{ij}(u_1), t_{kl}(u_2)] = (-1)^{|i||j|+|i|+|j||k|} (t_{kj}(u_1) t_{il}(u_2) - t_{kj}(u_2) t_{il}(u_1)). \tag{2.6}
\]
The super Yangian \( Y(\mathfrak{gl}_{m|n}) \) is a Hopf superalgebra with the coproduct given by
\[
\Delta : t_{ij}(u) \mapsto \sum_{k\in\overline{I}} t_{ik}(u) \otimes t_{kj}(u). \tag{2.7}
\]
For \( z \in \mathbb{C} \), there exists an isomorphism of Hopf superalgebras,
\[
\tau_z : Y(\mathfrak{gl}_{m|n}) \to Y(\mathfrak{gl}_{m|n}), \quad t_{ij}(u) \mapsto t_{ij}(u - z). \tag{2.8}
\]
For any \( \mathfrak{gl}_{m|n} \)-module \( M \), denote by \( M_z \) the \( Y(\mathfrak{gl}_{m|n}) \)-module obtained by pulling back \( M \) through the isomorphism \( \tau_z \).

The universal enveloping superalgebra \( U(\mathfrak{gl}_{m|n}) \) is a subalgebra of \( Y(\mathfrak{gl}_{m|n}) \) via the embedding \( e_{ij} \mapsto s_i t_{ij}^{(1)} \). The left inverse of this embedding is the \textit{evaluation homomorphism} \( \pi_{m|n} : Y(\mathfrak{gl}_{m|n}) \to U(\mathfrak{gl}_{m|n}) \) given by
\[
\pi_{m|n} : t_{ij}(u) \mapsto \delta_{ij} + s_i e_{ij} u^{-1}. \tag{2.9}
\]
For any \( \mathfrak{gl}_{m|n} \)-module \( M \), it is naturally a \( Y(\mathfrak{gl}_{m|n}) \)-module obtained by pulling back \( M \) through the evaluation homomorphism \( \pi_{m|n} \). We denote the corresponding \( Y(\mathfrak{gl}_{m|n}) \)-module by the same letter \( M \) and call it an \textit{evaluation module}.

### 2.4 Gauss decomposition and \( \ell \)-weights

The Gauss decomposition of \( Y(\mathfrak{gl}_{m|n}) \), see [9], gives generating series
\[
e_{ij}(u) = \sum_{a \geq 1} e_{ij}^{(a)} u^{-a}, \quad f_{ji}(u) = \sum_{a \geq 1} f_{ji}^{(a)} u^{-r}, \quad d_k(u) = 1 + \sum_{a \geq 1} d_{k,a} u^{-a},
\]
where \( 1 \leq i < j \leq m + n \) and \( k \in \overline{I} \), such that
\[
t_{ii}(u) = d_i(u) + \sum_{k<i} f_{ik}(u) d_k(u) e_{ki}(u), \tag{3.11a}
\]
\[
t_{ij}(u) = d_i(u) e_{ij}(u) + \sum_{k<i} f_{ik}(u) d_k(u) e_{kj}(u). \tag{3.11b}
\]
$$t_{ji}(u) = f_{ji}(u)d_i(u) + \sum_{k<i} f_{jk}(u)d_k(u)e_{ki}(u).$$

For $i \in I$, let

$$x^+_i(u) = \sum_{a \geq 1} x^+_{i,a}u^{-a} := e_{i,i+1}(u), \quad x^-_i(u) = \sum_{a \geq 1} x^-_{i,a}u^{-a} := f_{i+1,i}(u).$$

The parities of $x^\pm_{i,a}$ are the same as that of $t_{i,i+1}^{(a)}$, while all $dk,a$ are even. The super Yangian $Y(gl_{m|n})$ is generated by $x^\pm_{i,a}$ and $dk,a$, where $i \in I, k \in \bar{I}$, and $a \geq 1$. The full defining relations are described in [9, Lemma 4 or Theorem 3]. Here, we only need the following relations in $Y(gl_{m|n})[[u^{-1}, v^{-1}]]$.

\begin{align}
[d_i(u), d_k(v)] &= 0, & (2.10a) \\
(u - v)[d_i(u), x^+_j(v)] &= (s_i \delta_{ij} - s_i \delta_{i,j+1})d_i(u)(x^+_j(v) - x^+_j(u)), & (2.10b) \\
(u - v)[d_i(u), x^-_j(v)] &= (s_i \delta_{ij} - s_i \delta_{i,j+1})(x^-_j(u) - x^-_j(v))d_i(u), & (2.10c) \\
[x^+_i(u), x^+_j(v)] &= 0 \text{ for } |j - l| > 1, & (2.10d)
\end{align}

for $i, k \in \bar{I}$ and $j, l \in I$.

We call the commutative subalgebra generated by coefficients of $d_i(u)$, for all $i \in \bar{I}$, the Gelfand–Tsetlin algebra, and denote it by $A(gl_{m|n})$. It is not hard to see that $A(gl_{m|n})$ is maximal commutative in $Y(gl_{m|n})$. We do not need this fact for the present paper.

Let $\gamma_1 = 0$ if $m > 0$ and $\gamma_1 = -1$ if $m = 0$. Define $\gamma_k = \gamma_{k-1} + (s_{k-1} + s_k)/2$ recursively for $2 \leq k \leq m + n$.

**Lemma 2.2** ([8]) The coefficients of the series $\prod_{j \in \bar{I}} (d_j(u - \gamma_j))^{s_j}$ are central in $Y(gl_{m|n})$. \hfill \Box

Set $\mathcal{B} := 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ and $\mathfrak{B} := \mathcal{B}^{\bar{I}}$. We call an element $\zeta \in \mathfrak{B}$ an $\ell$-weight. We write $\ell$-weights in the form $\zeta = (\zeta_i(u))_{i \in \bar{I}}$, where $\zeta_i(u) \in \mathcal{B}$ for all $i \in \bar{I}$.

Clearly, $\mathfrak{B}$ is an abelian group with respect to the point-wise multiplication of the tuples. Let $\mathbb{Z}[\mathfrak{B}]$ be the group ring of $\mathfrak{B}$ whose elements are finite $\mathbb{Z}$-linear combinations of the form $\sum a_\zeta(\zeta_i)$, where $a_\zeta \in \mathbb{Z}$.

Let $M$ be a $Y(gl_{m|n})$-module. We say that a nonzero vector $v \in M$ is of $\ell$-weight $\zeta$ if $d_i(u)v = \zeta_i(u)v$ for $i \in \bar{I}$. We say that a vector $v \in M$ is a highest $\ell$-weight vector of $\ell$-weight $\zeta$ if $v$ is of $\ell$-weight $\zeta$ and $x^+_i(u)v = 0$ for all $i \in I$. By the Gauss decomposition, one can deduce that $v$ is a highest $\ell$-weight vector of $\ell$-weight $\zeta$ if and only if

\begin{align}
t_{ij}(u)v = 0, \quad t_{kk}(u)v = \zeta_k(u)v, \quad 1 \leq i < j \leq m + n, k \in \bar{I}. & \quad (2.11)
\end{align}
Let $M$ be a finite-dimensional $\mathbb{Y}(\mathfrak{gl}_{m|n})$-module and $\xi \in \mathcal{B}$ an $\ell$-weight. Let
\[
\xi_i(u) = 1 + \sum_{j=1}^{\infty} \xi_{i,j} u^{-j}, \quad \xi_{i,j} \in \mathbb{C}.
\]
Denote by $M_\xi$ the \textit{generalized $\ell$-weight space} corresponding to the $\ell$-weight $\xi$,
\[
M_\xi := \{ v \in M \mid (d_{i,j} - \xi_{i,j}) \dim M v = 0 \text{ for all } i \in \bar{I}, \ j \in \mathbb{Z}_{>0} \}.
\]
We call $M$ \textit{thin} if $\dim(M_\xi) \leq 1$ for all $\xi \in \mathcal{B}$. We call $M$ \textit{tame} if the joint action of the Gelfand–Tsetlin algebra on $M$ is diagonalizable. In particular, if $M$ is thin, then $M$ is tame.

For a finite-dimensional $\mathbb{Y}(\mathfrak{gl}_{m|n})$-module $M$, define the $q$-character (or Gelfand–Tsetlin character) of $M$ by the element
\[
\chi(M) := \sum_{\xi \in \mathcal{B}} \dim(M_\xi)[\xi] \in \mathbb{Z}[\mathcal{B}].
\]

Let $\mathcal{C}$ be the category of finite-dimensional $\mathbb{Y}(\mathfrak{gl}_{m|n})$-modules. Let $\mathcal{R}(\mathcal{C})$ be the Grothendieck ring of $\mathcal{C}$, then $\chi$ induces a $\mathbb{Z}$-linear map from $\mathcal{R}(\mathcal{C})$ to $\mathbb{Z}[\mathcal{B}]$.

**Lemma 2.3** ([12, Lemma 2.8]) \textit{The map $\chi : \mathcal{R}(\mathcal{C}) \to \mathbb{Z}[\mathcal{B}]$ is a ring homomorphism.}

### 2.5 Skew representations

Let $\psi_r : \mathbb{Y}(\mathfrak{gl}_{m|n}) \to \mathbb{Y}(\mathfrak{gl}_{m'|n})$ be the embedding given by
\[
\psi_r : d_i(u) \mapsto d_i'(u), \quad x_j^\pm(u) \mapsto x_j^\pm(u),
\]
see [9, Lemma 2].

Regard $\mathbb{Y}(\mathfrak{gl}_r)$ as a subalgebra of $\mathbb{Y}(\mathfrak{gl}_{m'|n})$ via the natural embedding $t_{ij}(u) \mapsto t_{ij}(u)$, for $1 \leq i, j \leq r$. Clearly, the subalgebra $\mathbb{Y}(\mathfrak{gl}_r)$ of $\mathbb{Y}(\mathfrak{gl}_{m'|n})$ supercommutes with the image of $\mathbb{Y}(\mathfrak{gl}_{m|n})$ under the map $\psi_r$, see (2.10). Therefore, the image of the homomorphism
\[
\pi_{m'|n} \circ \psi_r : \mathbb{Y}(\mathfrak{gl}_{m|n}) \to \mathcal{U}(\mathfrak{gl}_{m'|n})
\]
supercommutes with the subalgebra $\mathcal{U}(\mathfrak{gl}_r)$ in $\mathcal{U}(\mathfrak{gl}_{m'|n})$. This implies that the subspace $L(\lambda/\mu)$ is invariant under the action of the image of $\pi_{m'|n} \circ \psi_r$. Therefore, $L(\lambda/\mu)$ is a $\mathbb{Y}(\mathfrak{gl}_{m|n})$-module. We call $L(\lambda/\mu)$ a \textit{skew representation}, see [12, Section 3] for more detail. The $q$-character of $L(\lambda/\mu)$ is computed in terms of semi-standard Young tableaux, see [12, Theorem 3.4]. In the rest of this section, we recompute the $q$-character of $L(\lambda/\mu)$ in terms of GT tableaux.
Define the series $\mathcal{A}_k(u)$, for $0 \leq k \leq m + n$, in $\mathfrak{Y}(\mathfrak{gl}_{m|n})[[u^{-1}]]$, by

$$\mathcal{A}_k(u) := \prod_{i=1}^{r} d_i (u + r - i + 1) \prod_{j=1}^{k} (d_j (u - \gamma_j))^{s_j}.$$  

For a $\lambda/\mu$-admissible Gelfand–Tsetlin tableau $\Lambda$, define rational functions $\mathfrak{Y}_{\Lambda,k}(u)$, for $0 \leq k \leq m + n$, by

$$\mathfrak{Y}_{\Lambda,k}(u) = \prod_{i=1}^{r} \left( \frac{u + \lambda'_k + r - i + 1}{u + r - i + 1} \right) \prod_{j=1}^{k} \left( \frac{u + s_j \lambda'_k - \gamma_j}{u - \gamma_j} \right)^{s_j}.$$  

and set $\xi_{\Lambda} = (\xi_{\Lambda,1}(u), \ldots, \xi_{\Lambda,m+n}(u)).$

Lemma 2.4 We have $\mathcal{A}_k(u) \xi_{\Lambda} = \mathfrak{Y}_{\Lambda,k}(u) \xi_{\Lambda}$ for $0 \leq k \leq m + n$.

Proof The proof is similar to [20, Lemma 2.1] and [5, Lemma 4.7] using Lemma 2.2. \hfill \Box

Define rational functions $\zeta_{\Lambda,k}(u)$, for $1 \leq k \leq m + n$, by

$$\zeta_{\Lambda,k}(u) = \left( \frac{\mathfrak{Y}_{\Lambda,k}(u + \gamma_k)}{\mathfrak{Y}_{\Lambda,k-1}(u + \gamma_k)} \right)^{s_k}.$$  

(2.13)

Lemma 2.5 The vector $\xi_{\Lambda} \in L(\lambda/\mu)$ is of $\ell$-weight $\xi_{\Lambda}$, namely

$$d_k(u) \xi_{\Lambda} = \zeta_{\Lambda,k}(u) \xi_{\Lambda}, \quad 1 \leq k \leq m + n.$$

Proof Using the fact that

$$\psi_r(d_k(u)) = (\mathcal{A}_k(u + \gamma_k)/\mathcal{A}_{k-1}(u + \gamma_k))^{s_k},$$

the statement follows from Lemma 2.4. \hfill \Box

Let $\beta = (\beta_1, \ldots, \beta_{m+n})$ be a $\mathfrak{gl}_{m|n}$-weight. Define a rational function $\mathfrak{Y}_{\beta}(u)$ by

$$\mathfrak{Y}_{\beta}(u) = \prod_{i=1}^{m+n} \left( \frac{u + s_i \beta_i - \gamma_i}{u - \gamma_i} \right)^{s_i}.$$  

Lemma 2.6 Let $\beta$ and $\beta'$ be covariant $\mathfrak{gl}_{m|n}$-weights. If $\mathfrak{Y}_{\beta}(u) = \mathfrak{Y}_{\beta'}(u)$, then $\beta = \beta'$.  

\begin{itemize}
  \item \textcopyright Springer
\end{itemize}
Proof If \( mn = 0 \), then the sequence \( (s_i \beta_i - \gamma_i)^{m+n} \) is either strictly increasing or strictly decreasing. The statement is hence clear. We assume that \( mn > 0 \).

We claim that if \( y_\beta(u) = 1 \), then \( \beta = (0, \ldots, 0) \); if \( y_\beta(u) \neq 1 \), then the smallest root of \( y_\beta(u) \) is \( -\beta_1 \). Indeed, let \( a \) be the smallest nonnegative integer such that \( \beta_{a+1} = 0 \), then \( \beta_i = 0 \) for all \( i > a \) and

\[
y_\beta(u) = \prod_{i=1}^{a} \left( \frac{u + s_i \beta_i - \gamma_i}{u - \gamma_i} \right)^{s_i}.
\]

1. If \( a = 0 \), it is clear that \( \beta = (0, \ldots, 0) \) and \( y_\beta(u) = 1 \).
2. If \( 1 \leq a \leq m \), then \( (s_i \beta_i - \gamma_i)^{a+1} \) is strictly decreasing. Moreover, \( s_1 \beta_1 - \gamma_1 > -\gamma_i \) and \( s_i = 1 \) for \( 1 \leq i \leq a \). Hence, \( -\beta_1 = (\gamma_1 - \beta_1 \text{ as } \gamma_1 = 0) \) is the smallest root of \( y_\beta(u) \).
3. If \( a > m \), then to show that the smallest root of \( y_\beta(u) \) is \( -\beta_1 \) it reduces to show that \( s_1 \beta_1 - \gamma_1 > s_1 \beta_1 - \gamma_i \) and \( s_1 \beta_1 - \gamma_1 > -\gamma_i \) for \( i > 1 \). This is clear for \( 1 \leq i \leq m \), see part (2). For \( m+1 < i \leq a \), note that \( (s_i \beta_i - \gamma_i)^{a+1} \) is strictly increasing, it suffices to check that \( \beta_1 > -\beta_a - \gamma_a \) and \( \beta_1 > -\gamma_a \). Because \( \beta \) is covariant, we have

\[
\beta_1 \geq \beta_m \geq \#\{\beta_j > 0| j > m + 1\} = a - m > a - 2m = -\gamma_a.
\]

Now, the claim follows. In particular, \( y_\beta(u) \) uniquely determines \( \beta_1 \) and the rational function

\[
\prod_{i=2}^{m+n} \left( \frac{u + s_i \beta_i - \gamma_i}{u - \gamma_i} \right)^{s_i}.
\]

Since \( (\beta_2, \ldots, \beta_{m+n}) \) is also a covariant \( gl_{m-1|m} \)-weight, the above rational function uniquely determines \( \beta_2 \). Repeating this procedure, we conclude that \( y_\beta(u) \) determines \( \beta \) uniquely. \( \square \)

Remark 2.7 The strategy of proof is clear in terms of [12, Theorem 3.4 and Lemma 3.6] since the smallest zero of \( y_\beta(u) \) corresponds to the largest content of the hook Young diagram corresponding to weight \( \beta \) which is always given by the last box of the first row (assuming \( m > 0 \)). \( \square \)

Corollary 2.8 The skew representation \( L(\lambda/\mu) \) is thin.

Proof Let \( \Lambda_1 \) and \( \Lambda_2 \) be \( \lambda/\mu \)-admissible. We show that if \( \zeta_{\Lambda_1,k} = \zeta_{\Lambda_2,k} \) for all \( 1 \leq k \leq m+n \), then \( \Lambda_1 = \Lambda_2 \). Note that \( y_{\Lambda,0} \) is independent of \( \Lambda \), therefore we have \( y_{\Lambda_1,k} = y_{\Lambda_2,k} \) for all \( 0 \leq k \leq m+n \). Note that each row of a \( \lambda/\mu \)-admissible GT tableau corresponds to a covariant weight of a certain general Lie superalgebra. The statement follows from Lemma 2.6 with suitable choices of \( m \) and \( n \). Here, we only remark that \( \gamma_i \) should change correspondingly with respect to each choice. \( \square \)
3 Main results for super Yangian

3.1 Main results

Our main result is the explicit matrix elements of Drinfeld generating series $x_i^\pm(u)$ and $d_i(u)$ with respect to the basis $\xi_\Lambda$ for all $\lambda/\mu$-admissible Gelfand–Tsetlin tableaux $\Lambda$.

Define the integers $s_{ki}^{\pm}$ for $1 \leq k \leq m + n$ and $1 \leq i \leq k'$ by

$$s_{ki}^+ = \begin{cases} \frac{s_k + 1}{2}, & \text{if } i \leq m'; \\ -1, & \text{if } i > m' \end{cases}, \quad s_{ki}^- = \begin{cases} \frac{s_k - 1}{2}, & \text{if } i \leq m'; \\ 0, & \text{if } i > m'. \end{cases}$$

(3.1)

Theorem 3.1 We have

$$d_k(u)\xi_\Lambda = \xi_{\Lambda,k}(u)\xi_\Lambda,$$

$$x_k^+(u)\xi_\Lambda = s_k \sum_{i=1}^{k'} E_{\Lambda,ki}^+ \frac{\xi_{\Lambda+\delta_{ki}}}{u + l_{ki} + s_{ki}^+ + \gamma_k},$$

$$x_k^-(u)\xi_\Lambda = s_{k+1} \sum_{i=1}^{k'} E_{\Lambda,ki}^- \frac{\xi_{\Lambda-\delta_{ki}}}{u + l_{ki} + s_{ki}^- + \gamma_k},$$

where $l_{ki}, E_{\Lambda,ki}^\pm$, and $\xi_{\Lambda,k}(u)$ are defined in (2.2), (2.3), and (2.13), respectively.

Theorem 3.2 Every skew representation of $Y(\mathfrak{gl}_{m|n})$ is irreducible.

Theorem 3.3 Conjecture 1.1 is true for $m = n = 1$.

We prove these theorems in the next three subsections. Note that Theorem 3.2 is obtained in [12, Theorem 4.9]. Here, we give an independent proof using Theorem 3.1.

3.2 Proof of Theorem 3.1

We prepare the $Y(\mathfrak{gl}_{m|n})$ version of [28, Proposition 3.1]. For each $i \in I$ and $a \in \mathbb{C}$, define the simple $\ell$-root $A_{i,a} \in \mathcal{B}$ by

$$(A_{i,a})_j(u) = \frac{u - a}{u - a - (\alpha_i, \epsilon_j)}, \quad j \in \tilde{I}.$$

The following proposition established the property that all $x_i^\pm(u)$ acts on finite-dimensional representations of $Y(\mathfrak{gl}_{m|n})$ in a rather specific way.

Proposition 3.4 Let $V$ be a finite-dimensional $Y(\mathfrak{gl}_{m|n})$-module. Pick and fix any $i \in I$. Let $(\mu, \nu)$ be a pair of $\ell$-weights of $V$ such that $x_i^\pm(V_{\mu}) \cap V_{\nu} \neq \{0\}$ for some $j \geq 1$. Then:

1. $\nu = \mu A_{i,a}^{\pm 1}$ for some $a \in \mathbb{C}$,
(2) there exist bases $(v_k)_{1 \leq k \leq \dim(V_\mu)}$ of $V_\mu$ and $(w_l)_{1 \leq l \leq \dim(V_\nu)}$ of $V_\nu$, and complex polynomials $P_{k,l}(z)$ of degree $\leq k + l - 2$ such that

$$
(x_i^\pm(z)v_k)_\nu = \sum_{l=1}^{\dim(V_\nu)} w_l P_{k,l}(\partial_z)\left(\frac{1}{z - a}\right),
$$

where $(x_i^\pm(z)v_k)_\nu$ is the projection of $x_i^\pm(z)v_k$ onto the generalized $\ell$-weight subspace $V_\nu$.

**Proof** The proof is very similar to that of [28]. We give it here for completeness.

Let $(v_k)_{1 \leq k \leq \dim(V_\mu)}$ be a basis of $V_\mu$ such that all $d_{j,r}$ act upper-triangularly. More precisely, for all $j \in I$ and $1 \leq k \leq \dim(V_\mu)$, we have

$$(d_j(u) - \mu_j(u))v_k = \sum_{k' < k} v_k \xi_j^{k,k'}(u),$$

where $\xi_j^{k,k'}(u)$ are certain elements in $u^{-1}\mathbb{C}[[u^{-1}]]$. Similarly, let $(w_l)_{1 \leq l \leq \dim(V_\nu)}$ be a basis of $V_\nu$ such that all $d_{j,r}$ act lower-triangularly, namely for all $j \in I$ and $1 \leq l \leq \dim(V_\nu)$,

$$(d_j(u) - v_j(u))w_l = \sum_{l' < l} w_l \zeta_j^{l,l'}(u),$$

for certain $\zeta_j^{l,l'}(u) \in u^{-1}\mathbb{C}[[u^{-1}]]$.

We only show that statement for the case of $x_i^+(z)$. The case of $x_i^-(z)$ is similar.

For all $1 \leq k \leq \dim(V_\mu)$, there exist formal series $\lambda_{k,l}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$ for every $l$, $1 \leq l \leq \dim(V_\nu)$, such that

$$(x_i^+(z)v_k)_\nu = \sum_{l=1}^{\dim(V_\nu)} \lambda_{k,l}(z)w_l.$$

It follows from (2.10b) that

$$(u - z)x_i^+(z)(d_j(u) - \mu_j(u))v_k = (u - z - (\epsilon_j, \alpha_i))d_j(u)x_i^+(z)v_k + (\epsilon_j, \alpha_i)d_j(u)x_i^+(u)v_k - (u - z)x_i^+(z)\mu_j(u)v_k.$$}

Projecting the equation to $V_\nu$ and taking the $w_l$ component, we obtain

$$
(u - z) \sum_{k=1}^{k-1} \xi_j^{k,k'}(u)\lambda_{k',l}(z) = (u - z - (\epsilon_j, \alpha_i))\left(\lambda_{k,l}(z)v_j(u) + \sum_{l' = 1}^{l-1} \lambda_{k,l'}(z)\zeta_j^{l',l}(u)\right) + (\epsilon_j, \alpha_i)\left(\lambda_{k,l}(u)v_j(u) + \sum_{l' = 1}^{l-1} \lambda_{k,l'}(u)\zeta_j^{l',l}(u)\right) - (u - z)\mu_j(u)\lambda_{k,l}(z).
$$
Since \((x^+_i(z) V_{\mu})_p \neq 0\), there exists a smallest \(k_0\) such that \((x^+_i(z) v_{k_0})_p \neq 0\) and hence a smallest \(l_0\) such that \(\lambda_{k_0,l_0}(z) \neq 0\). Then, (3.2) implies

\[
0 = (u - z - (\epsilon_j, \alpha_i))\lambda_{k_0,l_0}(z)v_j(u) + (\epsilon_j, \alpha_i)\lambda_{k_0,l_0}(u)v_j(u) - (u - z)\mu_j(u)\lambda_{k_0,l_0}(z), \tag{3.3}
\]

for all \(j \in \tilde{I}\). Let

\[
\lambda_{k_0,l_0}(z) = a_m z^{-m} + a_{m+1} z^{-m-1} + \ldots,
\]

where \(m \geq 1\) and \(a_m \neq 0\). Considering the coefficients of \(z^{-m}\) in (3.3), we have

\[
(u - (\epsilon_j, \alpha_i))a_m v_j(u) - a_{m+1} v_j(u) - a_m \mu_j(u) + a_{m+1} \mu_j(u) = 0.
\]

Thus,

\[
v_j(u) = \mu_j(u) \frac{u - a}{u - a - (\epsilon_j, \alpha_i)} = \mu_j(u)(A_{i,a})_j(u),
\]

where \(a = a_{m+1}/a_m\) and part (1) follows.

Using part (1), Eq. (3.2) becomes

\[
(\epsilon_j, \alpha_i)\mu_j(u)((u-a)\lambda_{k,l}(u) - (z-a)\lambda_{k,l}(z)) = (u - z) \left( \sum_{k'=1}^{k-1} \xi_{j,k'}(u)\lambda_{k',l}(z) - \sum_{l'=1}^{l-1} \lambda_{k,l'}(z)\xi_{j,l'}(u) \right)
\]

\[
+ (\epsilon_j, \alpha_i) \left( \sum_{l'=1}^{l-1} \lambda_{k,l'}(z)\xi_{j,l'}(u) - \sum_{l'=1}^{l-1} \lambda_{k,l'}(u)\xi_{j,l'}(u) \right). \tag{3.4}
\]

We show part (2) by induction on \(k + l\). For the base case \(k + l = 2\), it follows from (3.4) that

\[
(u - a)\lambda_{1,1}(u) - (z-a)\lambda_{1,1}(z) = 0.
\]

Note that \(\lambda_{1,1}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]\), it follows from direct computations that

\[
\lambda_{1,1}(z) = p_{1,1} \sum_{k \geq 1} a^k z^{-k-1} = \frac{p_{1,1}}{z-a}
\]

for some \(p_{1,1} \in \mathbb{C}\) which shows the base case. Suppose part (2) holds for all \((k', l')\) such that \(k' + l' < k + l\). The case of \((k, l)\) follows directly by considering the coefficient of \(u^{-1}\) in (3.4) and the inductive data. \(\square\)
Let us finish the proof of Theorem 3.1. We apply Proposition 3.4 for $V = L(\lambda/\mu)$, $\mu = \xi\Lambda$, and $\nu = \xi\Lambda_{\pm\delta_{ki}}$. Hence, it suffices to find all polynomials $P_{k,l}$ and the corresponding $a$.

The numbers $a$ are computed directly since the expressions for $\xi\Lambda$ and $\xi\Lambda_{\pm\delta_{ki}}$ are known explicitly. More precisely, we have

$$(A_{k,a})_k(u) = \frac{u - a}{u - a - s_k} = \frac{\xi_{\Lambda_{\pm\delta_{ki}},k}(u)}{\xi_{\Lambda,k}(u)} = \begin{cases} \frac{(u + l_{ki} + 1 + \gamma_k)^{s_k}}{(u + l_{ki} + \gamma_k)}, & \text{if } 1 \leq i \leq m'; \\ \frac{(u + l_{ki} - 1 + \gamma_k)^{-s_k}}{(u + l_{ki} + \gamma_k)}, & \text{if } i > m'; \\ \end{cases}$$

$$(A_{k,a}^{-1})_k(u) = \frac{u - a - s_k}{u - a} = \frac{\xi_{\Lambda_{-\delta_{ki}},k}(u)}{\xi_{\Lambda,k}(u)} = \begin{cases} \frac{(u + l_{ki} + 1 + \gamma_k)^{s_k}}{(u + l_{ki} + \gamma_k)}, & \text{if } 1 \leq i \leq m'; \\ \frac{(u + l_{ki} - 1 + \gamma_k)^{-s_k}}{(u + l_{ki} + \gamma_k)}, & \text{if } i > m'; \\ \end{cases}$$

It is straightforward that $a = -l_{ki} - s_{ki}^{\pm} - \gamma_k$.

By Corollary 2.8, we know that $k = 1$ and $l = 1$. Hence, $P_{1,1}$ in this case is a constant. Moreover, the constant $P_{1,1}$ is determined by the coefficient of $\xi_{\Lambda_{\pm\delta_{ki}}}$ in $x_{k,1}^{\pm}\xi_{\Lambda}$. It is easy to see from the Gauss decomposition that $x_{k,1}^{\pm} = i_{k,k+1}^{(1)}$ and $x_{k,1}^{-} = i_{k+1,k}^{(1)}$. Hence, under our identification of $\mathfrak{gl}_{m|n}$ as a subalgebra of $\mathfrak{gl}_{m'|n}$ and using the evaluation map, we see that the action of $x_{k,1}^{\pm}$ and $x_{k,1}^{-}$ corresponds to that of $s_k e_k$ and $s_{k+1} f_k$, respectively, completing the proof of Theorem 3.1.

### 3.3 Proof of Theorem 3.2

Before starting the proof, we prepare several lemmas.

For a covariant $\mathfrak{gl}_{m|n}$-weight $\beta$, let $l$ be the number of nonzero components among $\beta_{m+1}, \ldots, \beta_{m+n}$. We have $l \leq \beta_m$. There is an associated Young diagram $\Gamma_{\beta}$ whose first $m$ rows are $\beta_1, \ldots, \beta_m$, while the first $l$ columns are $\beta_{m+1} + m, \ldots, \beta_{m+l} + m$. Moreover, $\Gamma_{\beta}$ has no other boxes outside of its first $m$ rows and first $l$ columns. Note that the condition that $l \leq \beta_m$ ensures that $\Gamma_{\beta}$ corresponds to a partition. We say that $\Gamma_{\beta}$ is the $(m|n)$-hook Young diagram associated with $\beta$.

Let $\Gamma_{\lambda}$ and $\Gamma_{\mu}$ be the corresponding $(m'|n)$-hook and $(r|0)$-hook Young diagrams, respectively. If $L(\lambda/\mu)$ is non-trivial, we must have $\Gamma_{\mu} \subset \Gamma_{\lambda}$. Hence, we assume further that $\Gamma_{\mu} \subset \Gamma_{\lambda}$. Let $\Gamma_{\lambda/\mu}$ be the skew Young diagram $\Gamma_{\lambda}/\Gamma_{\mu}$.

A **semi-standard Young tableau** of shape $\Gamma_{\lambda/\mu}$ is the skew Young diagram $\Gamma_{\lambda/\mu}$ with an element from $\{1, 2, \ldots, m + n\}$ inserted in each box such that the following conditions are satisfied:

1. the numbers in boxes are weakly increasing along rows and columns;
2. the numbers from $\{1, 2, \ldots, m\}$ are strictly increasing along columns;
3. the numbers from $\{m + 1, m + 2, \ldots, m + n\}$ are strictly increasing along rows.

We also need the bijection between $\lambda/\mu$-admissible GT tableaux and semi-standard Young tableaux of shape $\Gamma_{\lambda/\mu}$. Let $A$ be a $\lambda/\mu$-admissible GT tableau. Let $\lambda^{(k)} =$
(λ_{k+1}, \ldots, λ_{k'+k'}), then λ^{(k)} is a covariant g_k-weight, where g_k = gl_{k'} if $k \leq m$ and $g_k = gl_{m'|k-m}$ if $k > m$. We have a chain of Young diagrams

$$
\Gamma_\lambda = \Gamma_\lambda^{(m+n)} \supset \Gamma_\lambda^{(m+n-1)} \supset \cdots \supset \Gamma_\lambda^{(k)} \supset \cdots \supset \Gamma_\lambda^{(1)} \supset \Gamma_\lambda^{(0)} = \Gamma_\mu.
$$

One obtains a semi-standard Young tableau $\Omega_\lambda$ by inserting the number $k$ to the skew Young subdiagram $\Gamma_\lambda^{(k)}/\Gamma_\lambda^{(k-1)}$ for $1 \leq k \leq m + n$. It is well known that the map $\Lambda \to \Omega_\Lambda$ defines a bijection between $\lambda/\mu$-admissible GT tableaux and semi-standard Young tableaux of shape $\Gamma_\lambda/\mu$, see [2,26].

Given a $\lambda/\mu$-admissible GT tableau $\Lambda$, if $\Lambda \pm \delta_{ki}$ is also admissible, then we call the transformation from $\Lambda$ to $\Lambda \pm \delta_{ki}$ an admissible transformation. In terms of semi-standard Young tableaux, it means the semi-standard Young tableau corresponding to $\Lambda + \delta_{ki}$ (resp. $\Lambda - \delta_{ki}$) is obtained from the semi-standard Young tableau corresponding to $\Lambda$ by replacing one $k + 1$ with $k$ (resp. one $k$ with $k + 1$).

**Lemma 3.5** Let $\Lambda$ and $\Lambda'$ be $\lambda/\mu$-admissible, then one can obtain $\Lambda'$ from $\Lambda$ by several admissible transformations.

**Proof** It is easy to explain the proof using semi-standard Young tableaux instead of GT tableaux. It suffices to show that one can obtain all semi-standard Young tableaux of shape $\Gamma_\lambda/\mu$ from the semi-standard Young tableau $\Omega^+$ corresponding to the highest weight appearing in $L(\lambda/\mu)$. This could be easily done by transforming the columns of $\Omega^+$ to the desired ones from right to left and from bottom to top. \hfill $\square$

**Lemma 3.6** If $\Lambda$ and $\Lambda \pm \delta_{ki}$ are $\lambda/\mu$-admissible, then $e_{\Lambda,ki}^\pm$ is nonzero.

**Proof** The lemmas follows from a case-by-case computation. We give several examples for the case $e_{\Lambda,ki}^\pm$ when $m + 1 \leq k \leq m + n - 1$.

Suppose that $\Lambda$ and $\Lambda + \delta_{ki}$ are $\lambda/\mu$-admissible.

1. If $i \leq m'$, then by A(2) we have $\theta_{k'i} = 1$ and $\theta_{k'-1,i} = 0$. Now, it reduces to show that $l_{kj} - l_{ki} - 1 \neq 0$ for $j \leq m'$. It follows from A(5) that the sequence $(l_{kj})_{j=1}^{m'}$ is strictly decreasing. Hence, we only need to check that $l_{k,i-1} - l_{ki} - 1 \geq 1$, namely $\lambda_{k',i-1} \geq \lambda_{k'+1}$. This follows from A(5) as $\Lambda + \delta_{ki}$ is admissible.

2. If $i > m'$, we show that $l_{kj} - l_{ki} \neq 0$ for $j \leq m'$. Since $\Lambda + \delta_{ki}$ is admissible, it follows from A(3) that $\lambda_{k,m'} \geq i - m'$. Hence,

$$
l_{kj} - l_{ki} = \lambda_{k'} - j + 1 - (\lambda_{k'} + i - 2m') \\
\geq i - m' + j + 1 - i + 2m' = m' + 1 - j > 0.
$$

Suppose that $\Lambda$ and $\Lambda - \delta_{ki}$ are $\lambda/\mu$-admissible. We elaborate more for the case $e_{ki}^-$ for the factor

$$
\frac{\prod_{j=m'+1}^{k'+1} (l_{k+1,j} - l_{ki}) \prod_{j=m'+1}^{k'-1} (l_{k-1,j} - l_{ki} + 1)}{\prod_{j=m'+1}^{k'} (l_{kj} - l_{ki})(l_{kj} - l_{ki} + 1)},
$$

(3.5)

where $i \leq m'$.

Let $\lambda_{k'+1,m'} = a, \lambda_{k',m'} = b, \lambda_{k'-1,m'} = c$, then $b + 1 \geq a \geq b$ and $c + 1 \geq b \geq c$.
(1) If \( i < m' \), then by A(3), the factor (3.5) is equal to
\[
\frac{\prod_{j=m'+1}^{m'+a} (l_{k+1,j} - l_{ki}) \prod_{j=m'+a+1}^{k'+1} (j' - 2m' - l_{ki})}{\prod_{j=m'+1}^{m'+b} (l_{kj} - l_{ki}) \prod_{j=m'+b+1}^{k'} (j' - 2m' - l_{ki})} \times \frac{\prod_{j=m'+1}^{m'+c} (l_{k-1,j} - l_{ki} + 1) \prod_{j=m'+c+2}^{k'} (j' - 2m' - l_{ki})}{\prod_{j=m'+1}^{m'+b} (l_{kj} - l_{ki} + 1) \prod_{j=m'+b+1}^{k'+1} (j' - 2m' - l_{ki})}.
\]

We only care about the part
\[
\frac{\prod_{j=m'+a+1}^{k'+1} (j' - 2m' - l_{ki}) \prod_{j=m'+c+2}^{k'} (j' - 2m' - l_{ki})}{\prod_{j=m'+b+1}^{k'+1} (j' - 2m' - l_{ki})} = \frac{\prod_{j=m'+b+1}^{m'+a} (j' - 2m' - l_{ki})}{\prod_{j=m'+b+1}^{m'+a} (j' - 2m' - l_{ki})},
\]
(3.6)

since one easily checks similarly to the previous cases that the rests are nonzero. If \( a + c = 2b \), then everything in (3.6) cancels out. If \( b = c + 1 \) and \( a = b \), then (3.6) becomes
\[
m' + b' + 1 - 2m' - l_{ki} < m' + b' + 1 - 2m' - l_{km'} = 0.
\]

The case of \( a = b + c \) and \( b = c \) is similar.

(2) If \( i = m' \), then \( a = b = c + 1 \). By A(2) and A(3), the factor (3.5) is equal to
\[
\frac{\prod_{j=m'+1}^{m'+a} (l_{k+1,j} - l_{ki}) \prod_{j=m'+a+1}^{k'+1} (j' - 2m' - l_{ki})}{\prod_{j=m'+1}^{m'+b} (l_{kj} - l_{ki}) \prod_{j=m'+b+1}^{k'} (j' - 2m' - l_{ki})} \times \frac{\prod_{j=m'+1}^{m'+c} (l_{k-1,j} - l_{ki} + 1) \prod_{j=m'+c+2}^{k'} (j' - 2m' - l_{ki})}{\prod_{j=m'+1}^{m'+b} (l_{kj} - l_{ki} + 1) \prod_{j=m'+b+1}^{k'+1} (j' - 2m' - l_{ki})}.
\]

Again, we only care about the part
\[
\frac{\prod_{j=m'+a+1}^{k'+1} (j' - 2m' - l_{ki}) \prod_{j=m'+c+2}^{k'} (j' - 2m' - l_{ki})}{\prod_{j=m'+b}^{k'+1} (j' - 2m' - l_{ki}) \prod_{j=m'+b+1}^{k'+1} (j' - 2m' - l_{ki})} = \frac{1}{m' + b' - 2m' - l_{ki}} = -1.
\]

The rest cases follow from similar straightforward computations.

Now, we will prove Theorem 3.2. Since the actions of all \( d_1(u) \) on \( L(\lambda/\mu) \) are simultaneously diagonalizable and have simple spectrum, see Corollary 2.8, it suffices to show that we can get all possible \( \xi_\Lambda \) for \( \lambda/\mu \)-admissible \( \Lambda' \) from \( \xi_\Lambda \) for any given \( \lambda/\mu \)-admissible \( \Lambda \).

By Theorem 3.1 and Lemma 3.5, it suffices to show that all \( \xi_{\Lambda + \delta_{ki}} \) with \( \lambda/\mu \)-admissible \( \Lambda \pm \delta_{ki} \) are elements in \( Y(gl_{n|m})\xi_\Lambda \) for \( j \in \mathbb{Z}_{>0} \).
By Theorem 3.1, we have
\[ \sum \mathcal{E}_{A,ki}^\pm \frac{\xi_{A}^{\pm \delta_{ki}}}{u + l_{ki} + s_{ki}^{\pm} + \gamma_k} \in \mathbb{C} x_k^\pm (u) \xi_{A}, \]
where the summation is over all \( i \) with \( \lambda/\mu \)-admissible \( \Lambda \pm \delta_{ki} \). Since \( L(\lambda/\mu) \) is thin by Corollary 2.8, namely all \( \xi_{A}^{\pm \delta_{ki}} \) correspond to different \( \ell \)-weights, and it follows from Lemma 3.6 that \( \xi_{A}^{\pm \delta_{ki}} \in Y(\mathfrak{gl}_{m|n}) \xi_{A} \), completing the proof of Theorem 3.2.

### 3.4 Tame modules of \( Y(\mathfrak{gl}_{1|1}) \)

In this section, we study tame modules of \( Y(\mathfrak{gl}_{1|1}) \) and prove Theorem 3.3. We start by collecting some equalities in \( Y(\mathfrak{gl}_{1|1}) \) that will be used, see, e.g., [10, Appendix].

By (2.6), we have
\[ t_{11}(u)t_{21}(x) = \frac{u - x - 1}{u - x} t_{21}(x)t_{11}(u) + \frac{1}{u - x} t_{21}(u)t_{11}(x). \] (3.7)

Differentiating both sides with respect to \( x \), we obtain
\[ t_{11}(u)t'_{21}(x) = \frac{u - x - 1}{u - x} t'_{21}(x)t_{11}(u) + \frac{1}{u - x} t_{21}(u)t'_{11}(x) \]
\[ - \frac{1}{(u - x)^2} (t_{21}(x)t_{11}(u) - t_{21}(u)t_{11}(x)). \] (3.8)

By Gauss decomposition, we have \( d_{1}(u) = t_{11}(u) \) and
\[ d_{2}(u) = t_{22}(u) - t_{21}(u)(t_{11}(u))^{-1}t_{12}(u). \]

The coefficients of \( d_{1}(u)(d_{2}(u))^{-1} \) are central in \( Y(\mathfrak{gl}_{1|1}) \), see Lemma 2.2.

**Lemma 3.7** We have \( t_{12}(u+1)t_{21}(u) = -t_{22}(u+1)t_{11}(u) + d_{2}(u)(d_{1}(u))^{-1}t_{11}(u)t_{11}(u+1) \).

**Proof** By (2.5), we have \( t_{12}(u)t_{11}(u + 1) = t_{11}(u)t_{12}(u + 1) \) and
\[ [t_{12}(u + 1), t_{21}(u)] = -t_{22}(u + 1)t_{11}(u) + t_{22}(u)t_{11}(u + 1). \]

Therefore,
\[ t_{12}(u + 1)t_{21}(u) = -t_{22}(u + 1)t_{11}(u) + t_{22}(u)t_{11}(u + 1) - t_{21}(u)t_{12}(u + 1) \]
\[ = -t_{22}(u + 1)t_{11}(u) + (t_{22}(u) - t_{21}(u)(t_{11}(u))^{-1}t_{12}(u))t_{11}(u + 1) \]
\[ = -t_{22}(u + 1)t_{11}(u) + d_{2}(u)(d_{1}(u))^{-1}t_{11}(u)t_{11}(u + 1). \]

---

\(^1\) Note that \( t_{ij}(u) \) corresponds to \( \mathcal{L}_{ji}(u) \) therein.
We recall some basic facts about representations of $\text{Y}(\mathfrak{gl}_{1|1})$.

Let $\zeta = (\zeta_1(u), \zeta_2(u))$, where $\zeta_1(u), \zeta_2(u) \in B = 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. Denote $L(\zeta)$ the irreducible $\text{Y}(\mathfrak{gl}_{1|1})$-module generated by a highest $\ell$-weight vector $v^+$ of $\ell$-weight $\zeta$. It is known from [30, Theorem 4] that $L(\zeta)$ is finite-dimensional if and only if

$$\frac{\zeta_1(u)}{\zeta_2(u)} = \frac{\varphi(u)}{\psi(u)}, \quad (3.9)$$

where $\varphi$ and $\psi$ are relatively prime polynomials in $u$ of the same degree. Set $\deg \varphi = k$, then it also known that $\dim L(\zeta) = 2^k$, see [30, Theorem 4].

Let

$$\varphi(u) = \prod_{i=1}^{k} (u + a_i), \quad \psi(u) = \prod_{j=1}^{k} (u - b_j), \quad (3.10)$$

where $a_i, b_j \in \mathbb{C}$. Then, $a_i + b_j \neq 0$ for all $1 \leq i, j \leq k$.

For a subset $J$ of $\{1, \ldots, k\}$, set $\varphi_J = \prod_{i \in J} (u + a_i).$ By convention, $\varphi_\emptyset = 1.$

Lemma 3.8  We have

$$\chi(L(\zeta)) = \sum_{J \subset \{1, \ldots, k\}} \chi \left[ \left. \frac{\varphi_J(u-1)}{\varphi_J(u)} \right| \frac{\varphi_J(u-1)}{\varphi_J(u)} \right] \cdot [\zeta],$$

where the summation is over all subsets of $\{1, \ldots, k\}$.

Proof  Recall that $L(a_i, b_i)$ is the two-dimensional irreducible $\mathfrak{gl}_{1|1}$-module with the highest weight $(a_i, b_i).$ We also have the evaluation $\text{Y}(\mathfrak{gl}_{1|1})$-module $L(a_i, b_i)$. Clearly, up to a one-dimensional module, we have

$$L(\zeta) \cong \bigotimes_{i=1}^{k} L(a_i, b_i),$$

where the order for the tensor product is not important as the tensor product is irreducible. Note that under the condition $a_i + b_j \neq 0$ for all $1 \leq i, j \leq k$, the irreducibility of the tensor product $\bigotimes_{i=1}^{k} L(a_i, b_i)$ follows from either [30, Theorem 5] or [29, Theorem 4.2]. Hence, it suffices to consider irreducible tensor products of the form above.

Using the Gauss decomposition, one has

$$\chi(L(a_i, b_i)) = \left[ \left( u + \frac{a_i}{u} \right) - \left( u + \frac{b_i}{u} \right) \right] \left[ \left( u + \frac{a_i - 1}{u} \right) - \left( u + \frac{a_i - 1}{u} \right) \right].$$

Now, the statement follows from the fact that the $q$-character map is a homomorphism from $C$ to $\mathbb{Z}[\mathcal{B}]$, see Lemma 2.3.

Corollary 3.9  The $\text{Y}(\mathfrak{gl}_{1|1})$-module $L(\zeta)$ is thin if and only if $\varphi$ has no multiple roots.
Indeed, the condition that $\varphi$ has no multiple roots is also a necessary condition for $L(\xi)$ being tame.

**Proposition 3.10** The $Y(\mathfrak{gl}_{1|1})$-module $L(\xi)$ is tame if and only if $\varphi$ has no multiple roots.

**Proof** Again it suffices to show “only if” part for the case of $\bigotimes_{i=1}^{k} L(a_i, b_i)$. We adopt the method from [20]. Note that in this case, a finite-dimensional irreducible module is tame if and only if the action of $t_{11}(u)$ on it is semi-simple.

Suppose $-a_1$ is a multiple root of $\varphi$, we show that $\bigotimes_{i=1}^{k} L(a_i, b_i)$ is not tame. Without loss of generality, we assume there exists $\eta$ such that $a_i = a_1 - 1$ for $\eta < i \leq k$ and $a_j \neq a_1 - 1$ for $1 \leq j \leq \eta$.

By Lemma 3.8, there exists an $\ell$-weight vector $v$ of $\ell$-weight

\[
\left( u^{-k} \prod_{i=1}^{\eta} (u + a_i) \prod_{i=\eta+1}^{k} (u + a_i - 1), u^{-k} \prod_{i=1}^{\eta} (u - b_i) \prod_{i=\eta+1}^{k} \frac{u + a_i - 1}{u + a_i} \right).
\]

Denote the polynomial $\prod_{i=1}^{\eta} (u + a_i) \prod_{i=\eta+1}^{k} (u + a_i - 1)$ by $\varphi(u)$. We have $\varphi(-a_1) = \varphi'(-a_1) = 0$ by assumption.

Let $t_{ij}(u)$ be the linear operators corresponding to $u^k t_{ij}(u)$ in $\text{End}(\bigotimes_{i=1}^{k} L(a_i, b_i))$ ($(u^{-1})$), respectively. Then, all $\tilde{t}_{ij}(u)$ are polynomials in $u$, see (2.7). In particular, we have $\tilde{t}_{11}(u)v = \varphi(u)v$.

Set $w = \tilde{t}_{21}(-a_1)v$ and $w' = \tilde{t}_{21}'(-a_1)v$, where $\tilde{t}_{21}(-a_1)$ is the derivative of $\tilde{t}_{21}(u)$ with respect to $u$. It follows from (3.7) and $\tilde{t}_{11}(-a_1)v = \varphi(-a_1)v = 0$ that

\[
\tilde{t}_{11}(u)w = \tilde{t}_{11}(u)\tilde{t}_{21}(-a_1)v = \frac{u + a_1 - 1}{u + a_1} \varphi(u)w.
\]

Similarly, by (3.8) we have

\[
\tilde{t}_{11}(u)w' = \tilde{t}_{11}(u)\tilde{t}_{21}'(-a_1)v = \varphi(u) \left( \frac{u + a_1 - 1}{u + a_1} w' - \frac{1}{(u + a_1)^2} w \right).
\]

Hence, it suffices to show that $w \neq 0$.

By Lemmas 2.2 and 3.7, we have

\[
\tilde{t}_{12}(u + 1)\tilde{t}_{21}(u)v = -\tilde{t}_{22}(u + 1)\varphi(u)v + \left( \prod_{i=1}^{k} \frac{u - b_i}{u + a_i} \right) \varphi(u)\varphi(u + 1)v
\]

\[
= -\tilde{t}_{22}(u + 1)\varphi(u)v + \eta \prod_{i=1}^{\eta} (u + a_i + 1) \prod_{i=\eta+1}^{k} (u + a_i - 1) \prod_{i=1}^{k} (u - b_i)v.
\]

Setting $u = -a_1$, one obtains

\[
\tilde{t}_{12}(-a_1 + 1)v = \prod_{i=1}^{\eta} (-a_1 + a_i + 1) \prod_{i=\eta+1}^{k} (-a_1 + a_i - 1) \prod_{i=1}^{k} (-a_1 - b_i)v \neq 0.
\]
Example 3.12 Consider the tensor product of evaluation modules

The proof above) to distinguish notations from $Y$.

Remark 3.11 We remark that to determine if a $Y$ is isomorphic and one can identify

Proof of Theorem 3.3 It follows from Corollary 3.9 and Proposition 3.10.

We conclude this section by an example which we mentioned in the introduction.

Example 3.12 Consider the tensor product of evaluation modules $M := L(3, 0) \otimes L(-1, 0)$. Its highest $\ell$-weight is $(\frac{(a+3)(u-1)}{q^2}, 1)$. The corresponding polynomial $\varphi(u)$ is $(u + 3)(a - 1)$ which has no multiple roots. Therefore, the module $M$ is thin. Moreover, $M \cong L(2, 0) \oplus L(1, 1)$ as $gl_{1|1}$-modules. However, $M$ is not isomorphic to any tensor products of skew representations as the highest $\ell$-weights $\xi = (\xi_1, \xi_2)$ of skew representations satisfy $a_i + b_i \geq 0$ for all $i$ after rearranging $b_i$, where $a_i$ and $b_i$ are defined in (3.9) and (3.10).

4 Quantum affine superalgebra

Throughout this section, we shall assume $q \in \mathbb{C}^\times$ is generic. Recall $s_i$ from Sect. 2.1 and set $q_i = q^{s_i}$. For $k \in \mathbb{Z}$, we also set

$$[k] := \frac{q^k - q^{-k}}{q - q^{-1}}.$$
4.1 Quantum superalgebra $U_q(\mathfrak{g}[m|n])$

The quantum superalgebra $U_q(\mathfrak{g}[m|n])$ is a superalgebra with generators $e_i^±$ and $t_j^±$ for $i \in I$ and $j \in \bar{I}$ where $|e_i^±| = \bar{1}$ and $|t_j^±| = |e_i^±| = \bar{0}$ for $i \in \bar{I}$ and $j \in I \setminus \{m\}$. The relations of $U_q(\mathfrak{g}[m|n])$ [27, Proposition 10.4.1] are given by

\[ t_i t_j^{-1} = t_j^{-1} t_i = 1, \quad t_i e_j^± t_i^{-1} = q_i^{(e_i, \alpha_j)} e_j^±, \]

\[ [e_i^+, e_k^-] = \delta_{ik} \frac{t_j^{s_j} - t_j^{s_j+1}}{q_j - q_j^{-1}}, \]

\[ [e_i^±, e_j^±] = (e_i^±)^2 = 0, \quad [e_i^±, [e_j^±, e_j^±+1]_{q^{-1}}] = [e_i^±, [e_j^±, e_j^{-1}]_{q^{-1}}] = 0, \quad \text{if } j \in I \setminus \{m\}, \]

\[ [[[e_{m-1}^±, e_m^±]_q, e_{m+1}^±]_{q^{-1}}, e_m^±] = 0, \quad \text{when } m, n > 1, \]

where $[a, b]_q = ab - (-1)^{|a||b|}qba$ for homogeneous $a, b$ and $q \in \mathbb{C}$, and $[\cdot, \cdot] = [\cdot, \cdot]_1$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m+n})$ be a tuple of complex numbers. Denote $\mathcal{L}(\lambda)$ the irreducible module generated by a nonzero vector $v$ satisfying the conditions

\[ t_i v = q^{\lambda_i} v, \quad e_j v = 0, \]

for $i \in \bar{I}$ and $j \in I$. We call $\mathcal{L}(\lambda)$ the irreducible $U_q(\mathfrak{g}[m|n])$-module of highest weight $\lambda$.

Let $\mathcal{V} := \mathbb{C}^{m|n}$ be the vector superspace with a basis $v_i, i \in \bar{I}$, such that $|v_i| = |i|$. Let $E_{ij} \in \text{End}(\mathcal{V})$ be the linear operators such that $E_{ij} v_k = \delta_{jk} v_i$. The map $\rho_\mathcal{V} : U_q(\mathfrak{g}[m|n]) \rightarrow \text{End}(\mathcal{V})$,

\[ \rho_\mathcal{V}(t_i) = \sum_{k \in \bar{I}} q^{\delta_{ik}} E_{kk}, \quad \rho_\mathcal{V}(e_j^+) = E_{j,j+1}, \quad \rho_\mathcal{V}(e_j^-) = E_{j+1,j}, \quad i \in \bar{I}, \ j \in I, \]

defines a $U_q(\mathfrak{g}[m|n])$-module structure on $\mathcal{V}$. We call it the vector representation of $U_q(\mathfrak{g}[m|n])$. The highest weight of $\mathcal{V}$ is the tuple $(1, 0, \ldots, 0)$.

The following theorem was shown for essentially typical $\mathfrak{g}[m|n]$-weight $\lambda$ in [23, Theorem 4]. However, the same proof also works for covariant $\lambda$, cf. [26, Theorem 7] and [6].

**Theorem 4.1** ([23, Theorem 4]) Let $\lambda$ be an $(m|n)$-covariant weight, then $\mathcal{L}(\lambda)$ admits a basis $\xi_\lambda$ parameterized by all $\lambda$-admissible GT tableaux $\Lambda$. The actions of the generators of $U_q(\mathfrak{g}[m|n])$ are given by the formulas

\[ u_k \xi_\lambda = q^{\sum_{i=1}^k \lambda_{k+i} - \sum_{j=1}^{k-1} \lambda_{k-j-1}} \xi_\lambda, \quad 1 \leq k \leq m+n; \]

\[ e_k^+ \xi_\lambda = -\sum_{i=1}^k \frac{\prod_{j=i}^{k+1} [l_{k+1,j} - l_{ki}]}{\prod_{j \neq i, j=1}^k [l_{kj} - l_{ki}]} \xi_{\lambda+\delta_k}, \quad 1 \leq k \leq m-1; \]
\[ e_k^+ \xi_\Lambda = \sum_{i=1}^k \frac{\prod_{j=1}^{k-1} [l_{k-1,j} - l_{ki}]}{\prod_{j \neq i, j=1}^k [l_{kj} - l_{ki}]} \xi_{\Lambda - \delta_{ki}}, \quad 1 \leq k \leq m - 1; \]

\[ e_m^+ \xi_\Lambda = \sum_{i=1}^m \theta_{mi} (-1)^{i-1} (-1)^{\theta_{m1} + \cdots + \theta_{mi,1}} \]
\[ \times \frac{\prod_{1 \leq j < i \leq m} [l_{mj} - l_{mi} - 1]}{\prod_{1 \leq j < m} [l_{mj} - l_{mi}]} \xi_{\Lambda + \delta_{mi}}, \]

\[ e_m^- \xi_\Lambda = \sum_{i=1}^m (1 - \theta_{mi}) (-1)^{i-1} (-1)^{\theta_{m1} + \cdots + \theta_{mi,1}} \]
\[ \times \frac{[l_{mi} - l_{m+1,m+1}] \prod_{1 \leq j < i \leq m} [l_{mj} - l_{mi} + 1] \prod_{j = 1}^{m-1} [l_{m+1,j} - l_{mi} - 1]}{\prod_{1 \leq j < m} [l_{mj} - l_{mi}]} \xi_{\Lambda - \delta_{mi}}, \]

and for \( m + 1 \leq k \leq m + n - 1 \)

\[ e_k^- \xi_\Lambda = \sum_{i=1}^k \theta_{ki} (-1)^{\theta_{ki}} \xi_{\Lambda - \delta_{ki}} \]
\[ - \sum_{i=m+1}^k \prod_{j=1}^m \left( \frac{[l_{kj} - l_{ki}][l_{kj} - l_{ki} + 1]}{[l_{kj} - l_{ki}][l_{kj} - l_{ki} - 1]} \right) \]
\[ \times \frac{\prod_{j=1}^{k-1} [l_{k+1,j} - l_{ki}]}{\prod_{j \neq i, j=1}^k [l_{kj} - l_{ki}]} \xi_{\Lambda + \delta_{ki}}, \]

\[ e_k^+ \xi_\Lambda = \sum_{i=1}^m \theta_{k-1,i} (-1)^{\theta_{k-1,i}} \xi_{\Lambda - \delta_{ki}} \]
\[ \times \prod_{j \neq i, j=1}^m \left( \frac{[l_{kj} - l_{ki} + 1]}{[l_{kj} - l_{ki}]} \right) \]
\[ \times \frac{\prod_{j=1}^{k-1} [l_{k+1,j} - l_{ki}]}{\prod_{j \neq i, j=m+1}^k [l_{kj} - l_{ki}]} \xi_{\Lambda - \delta_{ki}}. \]

Here, we use the same notations and conventions as in Theorem 2.1 with \( r = 0 \). Note that \( k' = k \) for all \( k \in \mathbb{Z} \). \( \square \)

### 4.2 Quantum affine superalgebra \( U_q(\widehat{\mathfrak{g}_m|n}) \)

Let \( \mathcal{R}(u) \in \text{End}(\mathcal{V}^\otimes 2)[u] \) be the trigonometric R-matrix, see, e.g., [24],

\begin{equation}
\mathcal{R}(u) = \sum_{i \in \tilde{I}} (uq_i - q_i^{-1}) E_{ii} \otimes E_{ii} + (u - 1) \sum_{i \neq j} E_{ii} \otimes E_{jj} \\
+ u \sum_{i < j} (q_i - q_i^{-1}) E_{ji} \otimes E_{ij} + \sum_{i < j} (q_j - q_j^{-1}) E_{ij} \otimes E_{ji}. \tag{4.1}
\end{equation}

The quantum affine superalgebra \( U_q(\widehat{\mathfrak{g}_m|n}) \) is the unital associative superalgebra generated by \( t^{(a, \pm)}_{ij} \) for \( i, j \in \tilde{I} \) and \( a \in \mathbb{Z}_{\geq 0} \) with parity \( |i| + |j| \). For \( k = 1, 2 \), set

\[ T_k^\pm(u) = \sum_{i, j \in \tilde{I}} t^{\pm}_{ij}(u) \otimes E^{(k)}_{ij} \in U_q(\widehat{\mathfrak{g}_m|n}) \otimes \text{End}(\mathcal{V}^\otimes 2)[[u^\pm]], \]
where $E_{ij}^{(k)}$ are defined similarly as in (2.4) and

$$t_{ij}^{\pm}(u) = \sum_{a \geq 0} t_{ij}^{(a,\pm)} u^{\pm a} \in U_q(\widehat{gl}_{m|n})[[u^{\pm 1}]].$$

The relations of $U_q(\widehat{gl}_{m|n})$ are given by

\[
\begin{align*}
\mathcal{R}(u_1/u_2)T_1^{\pm}(u_1)T_2^{\pm}(u_2) &= T_2^{\pm}(u_2)T_1^{\pm}(u_1)\mathcal{R}(u_1/u_2), \\
\mathcal{R}(u_1/u_2)T_1^{\mp}(u_1)T_2^{\mp}(u_2) &= T_2^{\mp}(u_2)T_1^{\mp}(u_1)\mathcal{R}(u_1/u_2), \\
t_{ij}^{(0,-)} = t_{ij}^{(0,+)} &= 0, \quad \text{for } 1 \leq i < j \leq m + n, \\
t_{ii}^{(0,+)} t_{ii}^{(0,-)} = t_{ii}^{(0,-)} t_{ii}^{(0,+)} &= 1, \quad \text{for } i \in \tilde{I}.
\end{align*}
\]  

(4.2)

The Gauss decomposition of $U_q(\widehat{gl}_{m|n})$ gives generating series

$$e_{ij}^{\pm}(u) = \sum_{a \geq 0} e_{ij}^{(a,\pm)} u^{\pm a}, \quad f_{ji}(u) = \sum_{a \geq 0} f_{ji}^{(a,\pm)} u^{\pm a}, \quad d_{k}^{\pm}(u) = \sum_{a \geq 0} d_{k,a}^{\pm} u^{\pm a},$$

where $1 \leq i < j \leq m + n$ and $k \in \tilde{I}$, such that

$$T^{\pm}(u) = \left( \sum_{i < j} f_{ji}^{+}(u) \otimes E_{ji} + 1 \otimes \mathcal{I}_V \right) \left( \sum_{k} d_{k}^{+}(u) \otimes E_{kk} \right) \left( \sum_{i < j} e_{ij}^{+}(u) \otimes E_{ij} + 1 \otimes \mathcal{I}_V \right).$$

(4.3)

For $i \in I$, define

$$x_{i}^{+}(u) = \sum_{a \in \mathbb{Z}} x_{i,a}^{+} u^{a} := e_{i,i+1}^{+}(u) - e_{i,i+1}^{-}(u),$$

$$x_{i}^{-}(u) = \sum_{a \in \mathbb{Z}} x_{i,a}^{-} u^{a} := f_{i+1,i}^{-}(u) - f_{i+1,i}^{+}(u).$$

The following relations are known from [32] or [29, Theorem 3.5],

\[
\begin{align*}
[d_{i}^{*}(u), d_{k}^{\pm}(w)] &= 0, \quad \text{(4.4a)} \\
[d_{j}^{*}(u), x_{i}^{\pm}(w)] &= [x_{i}^{\pm}(u), x_{j}^{\pm}(w)] = 0, \quad \text{for } |i - j| \geq 2, \ |i - l| \geq 2, \quad \text{(4.4b)} \\
d_{i}^{*}(u)x_{i}^{\pm}(w) &= \left( q_i u - q_i^{-1} w \right) \mp 1 x_{i}^{\pm}(u) d_{i}^{*}(u), \quad \text{(4.4c)} \\
d_{i+1}^{*}(u)x_{i}^{\pm}(w) &= \left( q_i^{-1} u - q_i w \right) \mp 1 x_{i}^{\pm}(u) d_{i+1}^{*}(u), \quad \text{(4.4d)} \\
[x_{i}^{+}(u), x_{j}^{-}(w)] &= \delta_{il} (q_i - q_i^{-1}) \delta(u/w) (d_{i+1}^{+}(u) d_{i}^{\pm}(u)^{-1} - d_{i+1}^{+}(u) d_{i}^{-}(w)^{-1}), \quad \text{(4.4e)}
\end{align*}
\]

where $i, l \in I, j, k \in \tilde{I}, \delta(u) = \sum_{n \in \mathbb{Z}} u^n \in \mathbb{C}[[u^{\pm 1}]],$ and $\star$ is either $+$ or $-$. \[\Box\] Springer
Lemma 4.2 We have
\[ x_{i,0}^+ = (t_{ii}^{(0,+)}(0,+) - t_{i,i+1}^{(0,+)} - 1), \quad x_{i,0}^- = t_{i+1,i}^{(0,-)}(0,+) - 1. \]

**Proof** Since \( t_{ij}^{(0,-)} = i_{ij}^{(0,+)} = 0 \) for \( 1 \leq i < j \leq m + n \), it is not hard to show from Gauss decomposition (4.3) that \( f_i^+(u) \in uU_q(\mathfrak{gl}_{m|n})[[u]] \) and \( e_i^-(u) \in u^{-1}U_q(\mathfrak{gl}_{m|n})[[u^{-1}]] \). The statement follows that by comparing the constant terms in (4.3).

Lemma 4.3 ([4]) The quantum affine superalgebra \( U_q(\mathfrak{gl}_{m|n}) \) is generated by the coefficients of \( x_i^\pm(u) \) and \( d_j^\pm(u) \) for \( i \in I \) and \( j \in \tilde{I}. \)

**Proof** The lemma follows from Lemma 4.3 and relations (4.4).

Define \( t_{ij}^{(a,\pm)} \in U_q(\mathfrak{gl}_{m|n}) \) for \( i,j \in \tilde{I} \) and \( a \in \mathbb{Z}_{\geq 0} \) by
\[
(T^\pm(u))^{-1} = \sum_{i,j \in \tilde{I}} \sum_{a \geq 0} t_{ij}^{(a,\pm)}(u) \otimes E_{ij} \in U_q(\mathfrak{gl}_{m|n}) \otimes \text{End}(V)[[u^\pm]].
\]

Clearly, \( t_{ij}^{(a,\pm)} \) generate \( U_q(\mathfrak{gl}_{m|n}). \)

Lemma 4.5 The assignment
\[
t_{ij}^{(a,\pm)} \mapsto t_{i'j'}^{(a,\pm)}, \quad i,j \in \tilde{I}, \ a \in \mathbb{Z}_{\geq 0},
\]

extends uniquely to a superalgebra morphism \( \phi_r : U_q(\mathfrak{gl}_{m|n}) \rightarrow U_q(\mathfrak{gl}_{m'|n}). \) Moreover, we have
\[
\phi_r : x_i^\pm(u) \mapsto x_i^\pm(u), \quad d_j^\pm(u) \mapsto d_j^\pm(u), \quad \text{for } i \in I, \ j \in \tilde{I}.
\]

**Proof** The statement follows from the same strategy of [4, (5.39–5.41)]. More precisely, the first statement follows from the defining relations (4.2) for \( U_q(\mathfrak{gl}_{m|n}) \) by taking inverse to \( T_i^\pm(u), \ T_2^\pm(u) \) and restricting to \( E_{i'j'} \otimes E_{k'l'} \) for \( i,j,k,l \in \tilde{I} \). The second statement then follows from the Gauss decomposition.

4.3 Evaluation morphisms and \( \ell \)-weights

The quantum superalgebra \( U_q(\mathfrak{gl}_{m|n}) \) has another presentation as follows, see [29, Section 3.1.2]. Let \( \mathcal{R} = \mathcal{R}(1) \), see (4.1).
Let \( \mathcal{U}_q(\mathfrak{gl}_{m|n}) \) be the unital associative superalgebra generated by \( t_{ij}^\pm \), for \( 1 \leq i < j \leq m + n \), with parity \( |i| + |j| \) and with the relations in \( \mathcal{U}_q(\mathfrak{gl}_{m|n}) \otimes \text{End}(\mathcal{V} \otimes 2) \)

\[
R T_{i}^{\pm} T_{j}^{\mp} = T_{i}^{\pm} T_{j}^{\mp} R , \quad R T_{i}^{+} T_{j}^{-} = T_{i}^{+} T_{j}^{-} R , \quad t_{ij}^+ t_{ji}^- = t_{ji}^- t_{ij}^+ = 1 ,
\]

where \( T^+ = \sum_{i \leq j} t_{ij}^+ \otimes E_{ij} \) and \( T^- = \sum_{i \leq j} t_{ji}^- \otimes E_{ij} \in \mathcal{U}_q(\mathfrak{gl}_{m|n}) \otimes \text{End}(\mathcal{V}) \). The superalgebras \( \mathcal{U}_q(\mathfrak{gl}_{m|n}) \) and \( \mathcal{U}_q(\mathfrak{gl}_{m|n}) \) are isomorphic via the isomorphism extended by the assignments

\[
e_i^+ \mapsto \frac{(t_{ii}^+)^{-1} t_{i+1,i}^+}{1 - q_i^{-2}}, \quad e_i^- \mapsto \frac{t_{i+1,i}^- (t_{ii}^-)^{-1}}{1 - q_i^2}, \quad t_{ij}^\pm \mapsto (t_{jj}^\pm)^{-1}.
\]

The assignment

\[
t_{ij}^\pm (u) \mapsto \frac{t_{ij}^\pm - u^{\pm 1} t_{ij}^\mp}{1 - u^{\pm 1}}
\]

uniquely extends to a superalgebra homomorphism \( \sigma_{m|n} : \mathcal{U}_q(\hat{\mathfrak{gl}}_{m|n}) \to \mathcal{U}_q(\mathfrak{gl}_{m|n}) \cong \mathcal{U}_q(\mathfrak{gl}_{m|n}) \). We call \( \sigma_{m|n} \) the evaluation morphism. Let \( M \) be a \( \mathcal{U}_q(\mathfrak{gl}_{m|n}) \)-module. We call the module obtained by pulling back \( M \) through the evaluation morphism an evaluation module and denote it again by \( M \).

The assignment \( t_{ij}^\pm \mapsto t_{ij}^{(0,\pm)} \) defines a superalgebra morphism \( \iota : \mathcal{U}_q(\mathfrak{gl}_{m|n}) \to \mathcal{U}_q(\hat{\mathfrak{gl}}_{m|n}) \). Moreover, we clearly have \( \sigma_{m|n} \circ \iota = \text{Id}_{\mathcal{U}_q(\mathfrak{gl}_{m|n})} \). Hence, \( \mathcal{U}_q(\mathfrak{gl}_{m|n}) \cong \mathcal{U}_q(\hat{\mathfrak{gl}}_{m|n}) \) is identified as a subalgebra of \( \mathcal{U}_q(\hat{\mathfrak{gl}}_{m|n}) \) which is invariant under the evaluation morphism.

**Lemma 4.6** Identifying \( \mathcal{U}_q(\mathfrak{gl}_{m|n}) \) as a subalgebra of \( \mathcal{U}_q(\hat{\mathfrak{gl}}_{m|n}) \), we have \( x_{i,0}^\pm = (1 - q_i^{\pm 2}) e_i^\pm \).

**Proof** The lemma follows from Lemma 4.2 and the observations above. \( \square \)

Set \( \mathcal{B} := \mathbb{C}[[u^{-1}]]^\times \) and \( \mathfrak{B} := \mathcal{B}^{\bar{I}} \). We call an element \( \zeta \in \mathfrak{B} \) an \( \ell \)-weight. We write \( \ell \)-weights in the form \( \zeta = (\zeta_i(u))_{i \in \bar{I}} \), where \( \zeta_i(u) \in \mathcal{B} \) for all \( i \in \bar{I} \).

Let \( M \) be a \( \mathcal{U}_q(\hat{\mathfrak{gl}}_{m|n}) \)-module. We say that a nonzero vector \( v \in M \) is of \( \ell \)-weight \( \zeta \) if \( d_i^+(u)v = \zeta_i(u)v \) for \( i \in \bar{I} \). We say that a vector \( v \in M \) is a highest \( \ell \)-weight vector of \( \ell \)-weight \( \zeta \) if \( v \) is of \( \ell \)-weight \( \zeta \) and \( x_i^+(u)v = 0 \) for all \( i \in \bar{I} \).

Let \( M \) be a finite-dimensional \( \mathcal{U}_q(\hat{\mathfrak{gl}}_{m|n}) \)-module and \( \zeta \in \mathfrak{B} \) an \( \ell \)-weight. Let

\[
\zeta_i(u) = \sum_{j \geq 0} \zeta_{i,j} u^{-j}, \quad \zeta_{i,0} \in \mathbb{C}^\times, \quad \zeta_{i,j} \in \mathbb{C}.
\]

Denote by \( M_\zeta \) the generalized \( \ell \)-weight space corresponding to the \( \ell \)-weight \( \zeta \),

\[
M_\zeta := \{ v \in M \mid (d_i^+ - \zeta_{i,j}) \text{dim} M v = 0 \text{ for all } i \in \bar{I}, \; j \in \mathbb{Z}_{\geq 0} \}.
\]
We call $M$ thin if $\dim(M_\xi) \leq 1$ for all $\xi \in \mathcal{B}$. In particular, if $M$ is thin, then the actions of $d^\pm_i(u)$ are simultaneously diagonalizable.

**Example 4.7** Let $\lambda$ be a $\mathfrak{gl}_{m|n}$-weight. Then, the evaluation module $\mathcal{L}(\lambda)$ has the highest $\ell$-weight $\xi$,

$$
\xi_i(u) = \frac{q^{\lambda_i} - uq^{-\lambda_i}}{1 - u}, \quad \xi_j(u) = \frac{q^{-\lambda_j} - uq^{\lambda_j}}{1 - u}, \quad 1 \leq i < m, \quad m + 1 \leq j \leq m + n.
$$

\[\square\]

### 4.4 Skew representations

Recall that we have a fixed $r \in \mathbb{Z}_{\geq 0}$ and $k' := r + k$ for all $k \in \mathbb{Z}$. We identify $U_q(\mathfrak{gl}_r)$ as a subalgebra of $U_q(\mathfrak{gl}_{m'|n})$ via the natural embedding $e_i^\pm \mapsto e_i^\pm$ and $t_j^\pm1 \mapsto t_j^\pm1$. The quantum superalgebra $U_q(\mathfrak{gl}_{m'|n})$ is identified as a subalgebra of $U_q(\mathfrak{gl}_{m'|n})$ via the embedding $e_i^\pm \rightarrow e_i^\pm$ and $t_j^\pm1 \mapsto t_j^\pm1$. Clearly, the two subalgebras $U_q(\mathfrak{gl}_r)$ and $U_q(\mathfrak{gl}_{m'|n})$ commute with each other.

Let $\lambda = (\lambda_1, \ldots, \lambda_{m'+n})$ be an $(m'|n)$-covariant weight and $\mu = (\mu_1, \ldots, \mu_r)$ a $(r|0)$-covariant weight. Let $\mathcal{L}(\lambda/\mu)$ be the subspace of $\mathcal{L}(\lambda)$ given by

$$
\mathcal{L}(\lambda/\mu) := \{ v \in \mathcal{L}(\lambda) \mid t_i v = q^{\mu_i} v, e_j^+ v = 0, \quad 1 \leq i \leq r, \quad 1 \leq j < r \}.
$$

It is clear that $\mathcal{L}(\lambda/\mu)$ is a $U_q(\mathfrak{gl}_{m'|n})$-module and a $U_q(\mathfrak{gl}_{m'|n})_{U_q(\mathfrak{gl}_r)}$-module. It follows from Theorem 4.1 that $\mathcal{L}(\lambda/\mu)$ admits a basis $\xi_{\lambda}$ parameterized by all $\lambda/\mu$-admissible GT tableaux $\Lambda$. Moreover,

$$
e_k^{\pm} \xi_{\lambda} = \sum_{i=1}^{k'} [\mathcal{E}_{\lambda,ki}^\pm] \xi_{\lambda \pm \delta_i}, \quad (4.5)
$$

where $[\mathcal{E}_{\lambda,ki}^\pm]$ are obtained from $\mathcal{E}_{\lambda,ki}^\pm$ by replacing all numbers appearing in $\mathcal{E}_{\lambda,ki}^\pm$ from Theorem 2.1 to their associated $q$-numbers.

The quantum affine superalgebra $U_q(\widehat{\mathfrak{gl}}_r)$ is identified as a subalgebra of $U_q(\widehat{\mathfrak{gl}}_{m'|n})$ via the natural embedding $t_i^{(a,\pm)} \rightarrow t_i^{(a,\pm)}$. Recall $\phi_r$ from Lemma 4.5.

**Lemma 4.8** The image of $U_q(\widehat{\mathfrak{gl}}_{m'|n})$ under $\phi_r$ in $U_q(\widehat{\mathfrak{gl}}_{m'|n})$ supercommutes with the subalgebra $U_q(\widehat{\mathfrak{gl}}_r)$.

**Proof** The subalgebra $U_q(\widehat{\mathfrak{gl}}_r)$ of $U_q(\widehat{\mathfrak{gl}}_{m'|n})$ is generated by coefficients of $x_i^\pm(u)$ and $d_j^\pm(u)$ for $1 \leq i \leq r - 1$ and $1 \leq j \leq r$. The image of $U_q(\widehat{\mathfrak{gl}}_{m'|n})$ under $\phi_r$ is generated by coefficients of $x_i^\pm(u)$ and $d_j^\pm(u)$ for $i \in I$ and $j \in \tilde{I}$. Now, the lemma follows from (4.4).

\[\square\]

**Remark 4.9** Using Lemma 4.5, the lemma reduces to show that $t_i^{\pm}(u)$ supercommutes with $t_k^i(u)$ for $1 \leq i, j \leq r$ and $1' \leq k, l \leq m' + n$ in $U_q(\widehat{\mathfrak{gl}}_{m'|n})$. This can be shown directly from (4.2), see, e.g., [9, equation (5)].

\[\square\] Springer
It follows from Lemma 4.8 that the image of the homomorphism
\[ \varpi_{m'|n} \circ \phi_r : U_q(\widehat{gl}_{m'|n}) \to U_q(\widehat{gl}_{m'|n}) \]
supercommutes with the subalgebra \( U_q(\widehat{gl}_r) \) in \( U_q(\widehat{gl}_{m'|n}) \). This implies that the subspace \( \mathcal{L}(\lambda/\mu) \) is invariant under the action of the image of \( \varpi_{m'|n} \circ \phi_r \). Therefore, \( \mathcal{L}(\lambda/\mu) \) is a \( U_q(\widehat{gl}_{m'|n}) \)-module. We call \( \mathcal{L}(\lambda/\mu) \) a skew representation of \( U_q(\widehat{gl}_{m'|n}) \).

**Remark 4.10** Note that skew representations \( \mathcal{L}(\lambda/\mu) \) can also be defined using a Lie superalgebra \( gl_{r_1|r_2} \) instead of a Lie algebra \( gl_r \), see [12, Section 3]. However, the associated skew representation essentially depends on the shape of the skew Young diagram \( \Gamma_{\lambda/\mu} \), see [12, Remark 3.7]. Here, we only treat the case of \( gl_r \) for simplicity.

Define the series \( C_k(u) \), for \( 0 \leq k \leq m+n \), in \( U_q(\widehat{gl}_{m'|n}) \),
\[
C_k(u) := \prod_{i=1}^r d_i^+ (uq^{-2(r-i+1)}) \prod_{j=1}^k (d_j^+ (uq^{2\gamma_j}))^{s_j}
\]

For a \( \lambda/\mu \)-admissible GT tableau \( \Lambda \), define rational functions \( y_{\Lambda,k} \), for \( 0 \leq k \leq m+n \), by
\[
y_{\Lambda,k}(u) = \prod_{i=1}^r \left( q^{\lambda'_{ki}} \frac{1-uq^{-2l_{ki}}}{1-uq^{-2(r-i+1)}} \right) \prod_{j=1}^k \left( q^{\lambda'_{kj}} \frac{1-uq^{-2l_{kj}}}{1-uq^{2\gamma_j}} \right)^{s_j}.
\]

Define \( \zeta_{\Lambda,k}(u) \) by
\[
\zeta_{\Lambda,k}(u) = \left( \frac{y_{\Lambda,k}(uq^{-2\gamma_k})}{y_{\Lambda,k-1}(uq^{-2\gamma_k})} \right)^{s_k}.
\]

and set \( \zeta_{\Lambda} = (\zeta_{\Lambda,1}(u), \ldots, \zeta_{\Lambda,m+n}(u)) \).

**Lemma 4.11** We have \( C_k(u) \zeta_{\Lambda} = y_{\Lambda,k}(u) \zeta_{\Lambda} \) for \( 0 \leq k \leq m+n \). Moreover, the vector \( \zeta_{\Lambda} \in \mathcal{L}(\lambda/\mu) \) is of \( \ell \)-weight \( \zeta_{\Lambda} \). In particular, the skew representation \( \mathcal{L}(\lambda/\mu) \) is thin.

**Proof** The proofs of statements are similar to those of Lemmas 2.4, 2.5, and Corollary 2.8.

For each \( i \in I \) and \( a \in \mathbb{C} \), define the simple \( \ell \)-root \( A_{i,a} \in \mathfrak{B} \) by
\[
(A_{i,a})_j(u) = \frac{u-a}{q^{(\alpha_i,\epsilon_j)}u - q^{-(\alpha_i,\epsilon_j)}a}, \quad j \in \bar{I}.
\]

The following proposition will be used in the proof of one of our main results.
**Proposition 4.12** ([28, Proposition 3.1]) Let $V$ be a finite-dimensional $U_q(\mathfrak{gl}_m|n)$-module. Pick and fix any $i \in I$. Let $(\mu, \nu)$ be a pair of $\ell$-weights of $V$ such that $x_{i,j}^\pm (V_\mu) \cap V_\nu \neq \{0\}$ for some $j \geq 1$. Then:

1. $\nu = \mu A_{i,a}^{\pm 1}$ for some $a \in \mathbb{C}^\times$,
2. there exist bases $(v_k)_{1 \leq k \leq \dim(V_\mu)}$ of $V_\mu$ and $(w_l)_{1 \leq l \leq \dim(V_\nu)}$ of $V_\nu$, and complex polynomials $P_{k,l}(z)$ of degree $\leq k + l - 2$ such that

$$
(x_i^\pm(z)v_k)_\nu = \sum_{l=1}^{\dim(V_\nu)} w_l P_{k,l}(\partial z) \delta(a/z).
$$

4.5 Main results

Now, we are ready to formulate and prove the main results of this section.

**Theorem 4.13** We have

$$
d_k^\pm(u)\xi_\Lambda = \zeta_{\Lambda,k}(u)\xi_\Lambda,
$$

$$
x_k^\pm(u)\xi_\Lambda = (1 - q_i^{\pm 2}) \sum_{l=1}^{k'} [E^\pm_{\Lambda,ki}] \delta(q^{2(l_{ki} + s_{ki} + \gamma_k)/u}) \xi_{\Lambda \pm \delta_{ki}},
$$

where $l_{ki}, s_{ki}, [E^\pm_{\Lambda,ki}]$ and $\zeta_{\Lambda,k}(u)$ are defined in (2.2), (3.1), (4.5) and (4.6), respectively.

**Proof** The proof is similar to that of Theorem 3.1. We briefly describe the strategy of proof. By Lemma 4.11 and Proposition 4.12, it suffices to find the number $a$ appearing in Proposition 4.12 for each pair $\Lambda$ and $\Lambda \pm \delta_{ki}$ and the action of $x_i^\pm, 0$ on $\xi_\Lambda$. The number $a$ is computed by Lemma 4.11, while the action of $x_i^\pm, 0$ on $\xi_\Lambda$ is obtained from Theorem 4.1 and Lemma 4.6.

**Theorem 4.14** Every skew representation of $U_q(\mathfrak{gl}_m|n)$ is irreducible.

**Proof** The proof is similar to that of Theorem 3.2 using Theorem 4.13.

An analogue of Theorem 3.3 holds for $U_q(\mathfrak{gl}_{1|1})$ as well which we do not repeat.

**Acknowledgements** The author thanks E. Mukhin and V. Tarasov for stimulating discussion.

**References**

1. Brito, M., Mukhin, E.: Representations of quantum affine algebras of type $B_N$. Trans. Am. Math. Soc. **369**, 2775–2806 (2017)
2. Berele, A., Regev, A.: Hook young diagrams with applications to combinatorics and to representations of Lie superalgebras. Adv. Math. **62**(2), 118–175 (1987)
3. Cherednik, I.: A new interpretation of Gelfand–Zetlin bases. Duke Math. J. **54**, 563–577 (1987)
4. Ding, J., Frenkel, I.: Isomorphism of two realizations of quantum affine algebra $U_q(\hat{gl}(n))$. Commun. Math. Phys. **156**, 277–293 (1993)
5. Frenkel, E., Mukhin, E.: The Hopf algebra $\text{Rep}(U_q(\hat{gl}_\infty))$. Sel. Math. (N.S.) **8**, 537–635 (2002)
6. Futorny, V., Serganova, V., Zhang, J.: Gelfand–Tsetlin modules for $gl(m|n)$, to appear in Math. Res. Lett
7. Gautam, S., Wendlandt, C.: Poles of finite-dimensional representations of Yangians, pp. 1–61. arXiv:math.RT/2009.06427
8. Gow, L.: On the Yangian $Y(\mathfrak{gl}_{m|n})$ and its quantum Berezinian. Czech J. Phys. **55**, 1415–1420 (2005)
9. Gow, L.: Gauss Decomposition of the Yangian $Y(\mathfrak{gl}_{m|n})$. Commun. Math. Phys. **276**(3), 799–825 (2007)
10. Huang, C.-L., Lu, K., Mukhin, E.: Solutions of $(\mathfrak{gl}_{m|n})$ XXX Bethe ansatz equation and rational difference operators. J. Phys. A: Math. Theor. **52**, (37), 375204 (2019)
11. Lu, K., Mukhin, E.: On the supersymmetric XXX spin chains associated to $\mathfrak{gl}_1|1$. Commun. Math. Phys. **386**(2), 711–747 (2021)
12. Lu, K., Mukhin, E.: Jacobi–Trudi identity and Drinfeld functor for super Yangian. Int. Math. Res. Not. IMRN 2021(21), 16749–16808 (2021)
13. Molev, A.: Gelfand–Tsetlin bases for representations of Yangians. Lett. Math. Phys. **30**, 53–60 (1994)
14. Molev, A.: Combinatorial bases for covariant representations of the Lie superalgebra $\mathfrak{gl}_{m|n}$. Bull. Inst. Math. Acad. Sin. (N.S.) **6**(4), 415–462 (2011)
15. Molev, A., Olshanski, G.: Centralizer construction for twisted Yangians. Sel. Math. (N.S.) **6**, 269–317 (2000)
16. Nazarov, M.: Quantum Berezinian and the classical capelli identity. Lett. Math. Phys. **21**, 123–131 (1991)
17. Nazarov, M.: Representations of twisted Yangians associated with skew Young diagrams. Sel. Math. (N.S.) **10**, 71–129 (2004)
18. Nazarov, M.: Yangian of the General Linear Lie Superalgebra, SIGMA Symmetry Integrability Geom. Methods Appl. **16**, Paper No. 112 (2020)
19. Nazarov, M., Tarasov, V.: Yangians and Gelfand–Zetlin bases. Publ. RIMS **30**, 459–478 (1994)
20. Nazarov, M., Tarasov, V.: Representations of Yangians with Gelfand–Zetlin bases. J. Reine Angew. Math. **496**, 181–212 (1998)
21. Nazarov, M., Tarasov, V.: On irreducibility of tensor products of Yangian modules. Int. Math. Res. Not. IMRN **1998**(3), 125–150 (1998)
22. Nazarov, M., Tarasov, V.: On irreducibility of tensor products of Yangian modules associated with skew Young diagrams. Duke Math. J. **112**(2), 343–378 (2002)
23. Palev, T.D., Stoilova, N.I., Van der Jeugt, J.: Finite-dimensional representations of the quantum super-algebra $U_q[\mathfrak{gl}(n|m)]$ and related $q$-identities. Commun. Math. Phys. **166**, 367–378 (1994)
24. Perk, J., Schultz, C.: New families of commuting transfer matrices in q-state vertex models. Phys. Lett. **84** A, 407–410 (1981)
25. Sergeev, A.N.: The tensor algebra of the identity representation as a module over the Lie superalgebras $\mathfrak{gl}(n,m)$ and $\mathfrak{q}(n)$. Math. USSR Sbornik **51**(2), 419–427 (1985)
26. Stoilova, N.I., Van der Jeugt, J.: Gelfand–Zetlin basis and Clebsch-Gordan coefficients for covariant representations of the Lie superalgebra $\mathfrak{gl}(m|n)$. J. Math. Phys. **51**(9), 1–15 (2010)
27. Yamane, H.: Quantized enveloping algebras associated with simple Lie superalgebras and their universal R-matrices. Publ. RIMS. Kyoto Univ. **30**, 15–87 (1994)
28. Young, C.: Quantum loop algebras and r-root operators. Transform. Groups **20**(4), 1195–1226 (2015)
29. Zhang, H.-F.: RTT realization of quantum affine superalgebras and tensor products. Int. Math. Res. Not. IMRN **2016**(4), 1126–1157 (2016)
30. Zhang, R.-B.: Representations of super Yangian. J. Math. Phys. **36**, 38–54 (1995)
31. Zhang, R.-B.: The $\mathfrak{gl}(M|N)$ super Yangian and its finite dimensional-representations. Lett. Math. Phys. **37**(4), 419–434 (1996)
32. Zhang, Y.: Comments on the Drinfeld realization of quantum affine superalgebra $U_q(\mathfrak{gl}_{m|n})^{(1)}$ and its Hopf algebra structure. J. Phys. A **30**, 8325–8335 (1997)