LAGRANGIAN FORMALISM FOR TENSOR FIELDS

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Abstract

The Lagrangian formalism for tensor fields over differentiable manifolds with different (not only by sign) contravariant and covariant affine connections and a metric [(L_n, g)-spaces] is considered. The functional variation and the Lie variation of a Lagrangian density, depending on components of tensor fields (with finite rank) and their first and second covariant derivatives are established. A variation operator is determined and the corollaries of its commutation relations with the covariant and the Lie differential operators are found. The canonical (common) method of Lagrangians with partial derivatives (MLPD) and the method of Lagrangians with covariant derivatives (MLCD) are outlined. They differ from each other by the commutation relations the variation operator has to obey with the covariant and the Lie differential operator. The canonical and covariant Euler-Lagrange equations are found as well as their corresponding [(L_n, g)-spaces]. The energy-momentum tensors are considered on the basis of the Lie variation and the covariant Noether identities.

1 Introduction

The recent development of the models for description of the gravitational interaction (Hecht and Hehl 1991), (Hehl et al 1995) give rise to a tendency in using more general differentiable manifolds than the (pseudo) Riemannian spaces (Eddington 1925), (Schrödinger 1950).

The fact that affine connection, which in a point or over a curve in a Riemannian space can vanish [leading to the principle of equivalence in the Einstein theory of gravitation (ETG)], can also vanish under a special choice of the basic system in a space with affine connection and metric (von der Heyde 1975),

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reve (Iliev 1992), (Hartley 1995) shows that the equivalence principle in the ETG is only a corollary of the mathematical apparatus used in this theory. Therefore, every differentiable manifold with affine connection and metric can be used as a model for space-time in which the equivalence principle holds. But, if the manifold has two different (not only by sign) connections for tangent and co-tangent vector fields, the situation changes and is worth being investigated. The main characteristics of differentiable manifolds with different (not only by sign) contravariant and covariant affine connections and a metric \((\mathcal{T}_n,g)\)-spaces have been outlined in (Manoff 1995) [or in (Manoff 1996)].

There are countless numbers of papers concerning the applications of the Lagrangian formalism in classical field theories over (pseudo) Euclidean and (pseudo) Riemannian spaces. Since the crucial works of Noether about the role of the symmetries of a Lagrangian density for finding out local conserved quantities the most authors consider a Lagrangian formalism using invariant with respect to the co-ordinate action of a physical system, constructed by means of a Lagrangian density with certain symmetry properties [s. for example (Lovelock and Rund 1975)]. On this basis they have found the Euler-Lagrange equations and the corresponding conserved quantities. Other authors have used the invariance of a Lagrangian density with respect to functional variations [s. for example (Schmutzer 1968)]. Both types of investigations have found their applications in classical field theories over differentiable manifolds with an affine connection and a metric. But both types of considerations are using the common method of Lagrangians with partial derivatives.

The task of this paper is to show that a Lagrangian theory of tensor fields can be considered over \((\mathcal{T}_n,g)\)-spaces by means of the common (canonical) method of Lagrangians with partial derivatives (MLPD) or by means of the method of Lagrangians with covariant derivatives (MLCD). Both methods differ to each other. The second one is an entirely covariant, related to tensor fields over a differentiable manifold method, leading to independent of the affine connections (and therefore, to independent of the transport of the tensor fields, gauge invariant) field theoretical structures.

A Lagrangian formalism for tensor fields over differentiable manifolds with affine connections and a metric has three essential structures:

(a) The Lagrangian density.

(b) The Euler-Lagrange equations.

(c) The energy-momentum tensors.

In Sec. 2. a Lagrangian density for tensor fields and its properties are considered as well as its functional and Lie variations. A variation operator is introduced related to the functional variation and the corollaries of its commutation relations with the Lie differential operator and with the covariant differential operator are summarised.

In Sec. 3. the Euler-Lagrange equations are obtained as a result of the functional variation of the Lagrangian density.

In Sec. 4. the energy-momentum tensors are found by the use of the Lie variation of the Lagrangian density. The main objects under the consideration could be given schematically as follows
2 Lagrangian density $L$. Functional and Lie variations. Variation operator

A Lagrangian density $L$ is by definition a tensor density (relative tensor field) of rank 0 with the weight $q = \frac{1}{2}$, depending on tensor fields’ components and their first and second covariant derivatives. In accordance to the two different considerations of a Lagrangian density $L$ over a differentiable manifold $M$ ($\dim M = n$) with affine connections and a metric $([L_n,g]$-space), one can introduce the following definitions:

(a) Lagrangian density of type 1. Tensor density $L$ with weight $q = \frac{1}{2}$ and rank 0, considered as depending on components of tensor fields (with finite rank) and their first (and second) partial derivatives with respect to the co-ordinates $x^k$ as well as on components of affine connections and their partial derivatives,

$$L = \sqrt{-g} \cdot L(g_{ij}, g_{ij,k}, g_{ij,k,l}, V_A^B, V_A^{B,i}, V_A^{B,i,j}) ,$$

where $L(x^k) = L'(x^k)$ is a Lagrangian invariant, $g_{ij}$ are the components of the covariant metric tensor field $g = g_{ij} dx^i dx^j = g_{\alpha\beta} e^\alpha \otimes e^\beta$, $dx^i dx^j = \frac{1}{2} (dx^i \otimes dx^j + dx^j \otimes dx^i)$, $g_{ij} = g_{ji}$, $V_A^B$ are components of tensor fields $V$ or components of an affine connection $\Gamma$ (with components $\Gamma^i_{jk}$), or $P$ (with components $P^i_{jk}$),...
\[ d_\mu = \det(g_{ij}) < 0, \]

\[ V^A_{B;i} = \frac{\partial V^A_{B}}{\partial x^i}, \quad V^A_{B;i,j} = \frac{\partial^2 V^A_{B}}{\partial x^j \partial x^i}. \]

(b) Lagrangian density of type 2. Tensor density \( \mathbf{L} \) with weight \( q = \frac{1}{2} \) and rank 0, considered as depending on components of tensor fields (with finite rank) and their first (and second) covariant derivatives with respect to basic vector fields and to the corresponding affine connections,

\[ \mathbf{L} = \sqrt{-g} L(g_{ij}, g_{ij;k}, g_{ij;k;l}, V^A_{B}, V^A_{B;i}, V^A_{B;i,j}), \quad (2) \]

where \( L(x^k) = L'(x^{k'}) \) is a Lagrangian invariant, \( g_{ij} \) are the components of the covariant metric tensor field \( g \), \( V^A_{B} \) are components of tensor fields \( V = V^A_{B} \varepsilon_A \otimes e^B = V^A_{B} \partial_A \otimes dx^B \) with finite rank, \( A = i_1...i_k, B = j_1...j_l, k, l \in \mathbb{N} \). \( V^A_{B;i} \) is the covariant derivative of the components \( V^A_{B} \) along a contravariant basic vector field \( e_i \) (or \( \partial_i \)) (or along a coordinate \( x^i \)).

The Lagrangian density \( \mathbf{L} \) is (by definition) a tensor density of rank 0 and weight \( \frac{1}{2} \) with respect to the parameters \( x^m \). On the other side, \( \mathbf{L} \) can be written in the form

\[ \mathbf{L} = \mathbf{L}(y(x), w(x), z(x)) = \mathbf{L}(r(x)), \quad (3) \]

where \( y \sim (r^1, ..., r^s), \quad w \sim (r^{s+1}, ..., r^{s+p}), \quad z \sim (r^{s+p+1}, ..., r^{s+p+q}), \quad r \sim (r^1, ..., r^s, r^{s+1}, ..., r^{s+p}, r^{s+p+1}, ..., r^{s+p+q}), \quad s, p, q \in \mathbb{N}. \quad (4) \)

\( y \) is the set of the functions \( K^A_{B} \) (more precisely the image in \( R \) of the set of the functions over \( R^n \)), \( w \) is the set of the functions \( K^A_{B;i} \) (or \( K^A_{B;i,j} \)), \( z \) is the set of the functions \( K^A_{B;j} \) (or \( K^A_{B;i,j} \)). Every of the sets of these finite numbers of functions defines a vector space with the corresponding finite dimension over \( M \) (\( \dim M = n \)). The whole set \( Y \sim (y, w, z) \) builds a finite vector space as a direct product of the single vector spaces \( y, w \) and \( z \)

\[ Y = y \times w \times z, \quad Y \sim R^3, \quad y = R^s, \quad w = R^{p=n.s}, \quad z = R^{q=n^2.s}, \quad R = R^1 = R^s \times R^p \times R^q, \quad r \in Y. \quad (5) \]

Remark. In accordance with the number of the tensor fields, \( y, w \) and \( z \) could have some linear subspaces.

In this way, the Lagrangian density \( \mathbf{L} \) appears as a real function over \( R^3 \)

\[ \mathbf{L} : U(R^3) \to V(R), \quad U(R^3) \subset R^3, \quad V(R) \subset R \equiv R. \quad (6) \]

\( \mathbf{L} \) could have all properties of a real valued function of a finite number of variables over a manifold \( M \) (\( \dim M = n \)) and therefore, it could be considered as an object of the multidimensional analysis. The vector spaces \( y, w \) and \( z \) appear as linear subspaces of \( Y \). Under these premises the rules and theorems of the multi-dimensional analysis are also valid for \( \mathbf{L} \).
2.1 The Lagrangian density $L$ as a differentiable function over $Y = R^j$

Let we now consider a real function $L : U \to R$, where $U \subset Y$.

**Definition 1.** (Zoritch 1981, p. 435) The function $L : U \to R$, defined over the set $U \subset R^j$ (or over a neighbourhood of $r \in U$) is called differentiable at a point $r \in U$ ($r$ is a limit point for $U$) if

$$L(r + h) - L(r) = l(r).h + \alpha(r; h), \quad h \in R^j,$$

(7)

where $l(r) : h \to R$ is a linear function with respect to $h$ and $\alpha(r; h) \to o(h)$ for $h \to 0$ ($\lim_{h \to 0} [\alpha(r; h)] = 0$), $r + h \in U$.

**Remark.** A limit point for $U$ is a point $r$ for which $r \to r$, $r \in U$. The basis in $U$ is a set of neighbourhoods of $r$ in the set $U$. The elements of the basis are denoted as $\tilde{U}(r) = U \cap \tilde{U}(r)$.

The vectors

$$\triangle r(h) = (r + h) - r = h, \quad \triangle L(r; h) = L(r + h) - L(r)$$

(8)

are called correspondingly increment of the argument $r$ and increment of the function $L$. These vectors are usually denoted by the symbols of the functions of $h$: $\triangle r$ and $\triangle L(r)$

$$\triangle r : h \to \triangle r(h) = h, \quad \triangle L(r) : h \to \triangle L(r; h) = L(r + h) - L(r), \quad h \in Y.$$  

(9)

The linear function $l(r) : R^j \to R$ is called differential (tangential map, derivative map) of the function $L : U \to R$ at the point $r \in U$.

The differential of the function $L : U \to R$ is usually marked by means of the symbol $dL(r)$.

In accordance with the last symbol, $\square$ can be written in the form

$$\triangle L(r; h) = L(r + h) - L(r) = dL(r).h + \alpha(r; h).$$

(10)

One could see that the differential is in its nature defined over the increment $h$ of the considered point $r \in R^j$

$$dL(r).h \in R, \quad dL(r) : h \to dL(r).h, \quad dL(r) : \tilde{U} \subset R^j \to R.$$  

(11)

$$dL(r) : T_r(R^j = Y) \to T_{L(r)}(R).$$

(12)

In full accordance with the one-dimensional case, the function of several variables $L$ is differentiable at a point $r$, if its increment $\triangle L(r; h)$ at this point appears as a linear function of the increment $h$ of the argument $r$ within an accuracy of a term $\alpha(r; h)$ which is infinitesimally small for $h \to 0$, compared with the increment $h$ of the argument $r$. 

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2.2 Differential and partial derivatives of the Lagrangian density \(L\)

Let we now consider the real function \(L : U \to R\), defined over the set \(U \subset Y = R^j\) and differentiable at the point \(r \in U\). In the most cases \(U\) is a range of \(Y\). If \(r\) is an internal point of the set \(U\), then the point \(r + h\) will (after a sufficient small increment \(h\) of \(r\)) also belong to \(U\) and therefore, \(r + h \in U\) will lie in the domain of \(L : U \to R\). If we express \(r, h\) and \(l(r).h\) in a co-ordinate basis as

\[
\begin{align*}
  r & \sim (r^i) \sim (y^i, w^k, z^l), \\
  h & \sim (h^i) \sim (h^i, h^k, h^l), \\
  l(r).h & = \sum_j \pi_j(r).h^j = a_i(r).h^i + b_k(r).h^k + c_l(r).h^l,
\end{align*}
\]

(13)

then the relation \(L(r + h) - L(r) = l(r).h + \alpha(r; h)\) can be written in the form

\[
\begin{align*}
  & L(y^i + h^i, w^k + h^k, z^l + h^l) - L(y^i, w^k, z^l) = \\
  & = a_i(r).h^i + b_k(r).h^k + c_l(r).h^l + o(h) \quad \text{(for } h \to 0). \tag{14}
\end{align*}
\]

\(a_i(r), b_k(r)\) and \(c_l(r)\) are connected to the point \(r\) real numbers.

The norm in \(R^j\) can be defined in the usual form (Zoritch 1981, p.432), as

\[
\| r \| = \sqrt{\sum_i (y^i)^2 + \sum_k (w^k)^2 + \sum_l (z^l)^2}. \tag{15}
\]

Let we now determine the real numbers \(a_i(r), b_k(r)\) and \(c_l(r)\). For finding a solution of this problem we will concentrate our attention not on an arbitrary increment \(h\) of \(r\) but on a special increment \(h^m \in R^s\) colinear to the vector \(e_m\) in \(R^s\). For \(h = h^m\) it is obvious that \(\| h \| = | h^m |\) and we obtain from \([14]\)

\[
L(y^i, ..., y^m + h^m, ..., y^s, w^k, z^l) - L(y^i, ..., y^m, ..., y^s) = a_m(r).h^m + o(h)
\]

for \(h^m \to 0\). \(\tag{16}\)

The last equality means that if all variables of the function \(L(y^i, w^k, z^l)\) with an exception of \(y^m\) are fixed, then the constructed in this way function of the variable \(y^m\) is differentiable at the point \(y^m\). It follows from \([14]\) that

\[
a_m(r) = \lim_{h^m \to 0} \frac{1}{h^m} [L(y^i, ..., y^m + h^m, ..., y^s, w^k, z^l) - L(y^i, ..., y^m, ..., y^s, w^k, z^l)]. \tag{17}\n\]

Definition 2. The limit \([17]\) is called partial derivative of the function \(L(r)\) at the point \(r \sim (y^i, w^k, z^l)\) with respect to the variable \(y^m\). This derivative is marked by the symbol

\[
\frac{\partial L}{\partial y^m}(r).
\]

Proposition 1. (Zoritch 1981, p. 437-438). If the function \(L : U \to R\) defined over a set \(U \subset Y = R^j\) is differentiable at an internal point \(r \in U\), then the function \(L\) has at this point partial derivatives with respect to every one of
the variables and the differential of the function \( \mathbf{L} \) is uniquely determined by these partial derivatives in the form

\[
d \mathbf{L}(r).h = \frac{\partial \mathbf{L}}{\partial r^j}(r).j^i + \frac{\partial \mathbf{L}}{\partial w^k}(r).h^k + \frac{\partial \mathbf{L}}{\partial z^l}(r).h^l.
\]

(18)

Example. For the function \( \pi^i : x \to x^i, x^i \in R, x \in R^n \), i.e. \( \pi^i \) is a mapping of \( x = (x^1, \ldots, x^n) \) to the corresponding co-ordinate \( x^i \), one has

\[
\triangle \pi^i(x, \widetilde{h}) = (x^i + \widetilde{h}^i) - x^i = \widetilde{h}^i,
\]

because of \( \triangle f(x, \widetilde{h}) = f(x + \widetilde{h}) - f(x) \), where \( f = \pi^i \). The increment \( \triangle \pi^i \) of the function \( \pi^i \) is only a linear to \( \mathbf{h} \) function: \( \pi^i(\mathbf{h}) = \mathbf{h} = \triangle \pi^i(\mathbf{h}) \). Therefore, \( \triangle \pi^i(x, \mathbf{h}) = \delta \pi^i(x, \mathbf{h}) = \triangle \pi^i(\mathbf{h}) \) and \( d\pi^i(x) = d\pi^i \) is independent of \( x \in R^n \). If we write \( x^i(x) \) instead of \( \pi^i(x) \), then it follows that \( dx^i(x)\mathbf{h} = dx^i \cdot \mathbf{h} = \mathbf{h} \) [\( dx^i \cdot \mathbf{h} = dx^i(h) \)].

By means of the last relations and on the grounds of the expression [18], one can write the differential of an arbitrary function in the form of a linear combination of the differentials of the co-ordinates of its argument \( x \in R^n \)

\[
df(x).h = \partial_i f(x).dx^i(h) = \partial_i f(x).dx^i , \; \forall h \in R^n ,
\]

(19)

Therefore,

\[
d \mathbf{L}(r) = \frac{\partial \mathbf{L}}{\partial y^i}(r).dy^i + \frac{\partial \mathbf{L}}{\partial w^k}(r).dw^k + \frac{\partial \mathbf{L}}{\partial z^l}(r).dz^l , \; h^i = dr^i(h) , \; h = dr(h) ,
\]

(20)

\[
d \mathbf{L}(r).h = \frac{\partial \mathbf{L}}{\partial r^j}(r).h^j = \frac{\partial \mathbf{L}}{\partial r^j}(r).dr^j(h) , \; d \mathbf{L}(r) = \frac{\partial \mathbf{L}}{\partial r^j}.dr^j .
\]

(21)

2.3 Functional variation of the Lagrangian density \( \mathbf{L} \)

The differential \( d \mathbf{L}(r) \) of the function \( \mathbf{L}(r) \) maps an element \( h \in \mathbb{Y} = R^j \) into an element \( d \mathbf{L}(r).h \in R \)

\[
d \mathbf{L}(r).h : h \to d \mathbf{L}(r).h \in R , \; r, h \in R^j .
\]

\( d \mathbf{L}(r).h \) can be written in the form [18]

\[
d \mathbf{L}(r).h = \frac{\partial \mathbf{L}}{\partial r^j}(r).h^j = \frac{\partial \mathbf{L}}{\partial y^i}(r).h^i + \frac{\partial \mathbf{L}}{\partial w^k}(r).h^k + \frac{\partial \mathbf{L}}{\partial z^l}(r).h^l .
\]

Let \( r \) and \( r + h \) be elements of one and the same subset \( U \subset \mathbb{Y} = R^j \): \( r, r + h \in U_1 \subset U \subset \mathbb{Y} \). \( h \) is interpreted as the increment of \( r \). Let we now assume that \( h = \varepsilon.\delta r \) is constructed by means of the image \( \delta r \) of \( r \) under a certain linear mapping (operator) \( \delta : r \to \delta r \) with \( r, \delta r \in U_1 \), \( r, h = \varepsilon.\delta r \in U_1 \subset U \), \( \varepsilon \in [0, 1] \subset R \). Then,

\[
r + h = r + \varepsilon.\delta r , \; \; r^j + h^j = r^j + \varepsilon.\delta r^j ,
\]

(22)
depending on the parameters \( x \) and \( \varepsilon \) with \( \lim_{\varepsilon \to 0} \varepsilon.\delta \).

The operator (mapping) \( \delta \) is called the variation operator. In this case the expression \( \delta \) would have the form

\[ L(r + \varepsilon.\delta r) - L(r) = \varepsilon.\delta L(r) + \alpha(r; \varepsilon.\delta r), \quad \lim_{\varepsilon.\delta r \to 0} \alpha(r; \varepsilon.\delta r) = 0. \tag{24} \]

The last expression can also be written in the form

\[ L(r + \varepsilon.\delta r) = \varepsilon.\delta L(r) + \alpha(r; \varepsilon.\delta r), \]

where

\[ \delta L(r) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [L(r + \varepsilon.\delta r) - L(r)], \tag{25} \]

or

\[ \delta L(r) = dL(r).\delta r = \frac{\partial L}{\partial r^j}(r).\delta r^j = \frac{\partial L}{\partial y^i}(r).\delta y^i + \frac{\partial L}{\partial w^k}(r).\delta w^k + \frac{\partial L}{\partial z^l}(r).\delta z^l. \tag{26} \]

\( \delta L(r) \) could be considered as the image of \( L(r) \) under the action of the variation operator \( \delta \). This action of \( \delta \) on \( L(r) \) is determined by the definition of \( \delta L(r) \)

\[ \delta : L(r) \to \delta L(r) = dL(r).\delta r, \quad r, \delta r \in Y = R^j, \quad dL(r).\delta r \in R, \quad \delta : R \to R. \tag{27} \]

Let us now consider the integral (Lovelock and Rund 1975, pp. 188-189)

\[ S(\varepsilon) = \int_{V_n} L(r(x) + \varepsilon.\delta r(x)).d^{(n)}x, \tag{28} \]

where \( V_n \) is the volume of a limited region of \( R^m \); \( r = r(x) \), \( x \sim (x^1, \ldots, x^n) \); \( r(x), \delta r(x) \in Y = R^j \). The increment \( h = \varepsilon.\delta r(x) \) (respectively \( \delta r \) ) are also depending on the parameters \( x^m \). \( L(r(x)) \) is now considered as a function of the parameters \( x^m \) and the integral \( S(\varepsilon) \) is considered as a function of the parameter \( \varepsilon [\delta r(x) \) are considered to be arbitrary finite elements of \( Y \).]

The integral \( S(\varepsilon) \) can be developed with respect to the parameter \( \varepsilon \) by the use of the representation \([24] \) of \( L(r(x)) \) as a differentiable function of \( r(x) \)

\[ S(\varepsilon) = \int_{V_n} L(r(x) + \varepsilon.\delta r(x)).d^{(n)}x = \]

\[ = \int_{V_n} L(r(x)).d^{(n)}x + \varepsilon.\int_{V_n} \delta L(r(x)).d^{(n)}x + \int_{V_n} \alpha(r(x); \varepsilon.\delta r(x)).d^{(n)}x \tag{30} \]

with \( \lim_{\varepsilon \to 0} \alpha(r(x); \varepsilon.\delta r(x)) = 0. \) \( S(\varepsilon) \) will have the form

\[ S(\varepsilon) = S(0) + \varepsilon.\delta S + S(\alpha(\varepsilon)), \tag{31} \]
where
\[ S(0) = \int_{V_n} L(r(x)).d^{(n)}x , \quad \delta S = \int_{V_n} \delta L(r(x)).d^{(n)}x , \]
\[ S(\alpha(\varepsilon)) = \int_{V_n} \alpha(r(x); \varepsilon.\delta r(x)).d^{(n)}x . \]

The integral \( S(\alpha(\varepsilon)) \) is considered (as the other integrals in (31)) over a limited region \( U_0(x) \subset R^n \) with a finite volume \( V_n \). In this region \( U_0 \) the function \( \alpha(r(x); \varepsilon.\delta r(x)) \) is a limited function of \( r(x) \) and \( \delta r(x) \) [because of the differentiability of \( L(r(x)) \) at every \( r(x) \in U(U_0) \subset Y = R^j \) with \( \lim_{\varepsilon \to 0}[\alpha(r(x); \varepsilon.\delta r(x))] = 0 \). Therefore,
\[ \lim_{\varepsilon \to 0} \int_{V_n} \alpha(r(x); \varepsilon.\delta r(x)).d^{(n)}x = \int_{V_n} \lim_{\varepsilon \to 0} \alpha(r(x); \varepsilon.\delta r(x)).d^{(n)}x = 0 , \]
\[ \int_{V_n} o(\varepsilon.\delta r(x)).d^{(n)}x \leq o(\varepsilon).M , \quad M < \infty , \varepsilon \to 0 . \]

The value of the integral \( S(\alpha(\varepsilon)) \) aims at \( \varepsilon = 0 \) faster then the integral \( \delta S \)
\[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{V_n} \varepsilon.\delta L(r(x)).d^{(n)}x = \int_{V_n} \delta L(r(x)).d^{(n)}x = \delta S . \]

The function \( S(\varepsilon) \) is differentiable at \( \varepsilon = 0 \), where \( \delta S \) is the differential of \( S(\varepsilon) \) at the point \( \varepsilon = 0 \). \( \varepsilon.\delta S \) is uniquely determined because of the relation (Zoritch 1981, p.187)
\[ \lim_{\varepsilon \to 0} \frac{S(\varepsilon) - S(0)}{\varepsilon} = \lim_{\varepsilon \to 0} (\delta S + \frac{S(\alpha(\varepsilon))}{\varepsilon}) = \delta S . \]
\( \delta S \) appears as the derivative of \( S(\varepsilon) \) at \( \varepsilon = 0 \).
\[ \delta S = \left( \frac{dS(\varepsilon)}{d\varepsilon} \right)_{\varepsilon=0} = 0 \]
is the necessary condition for the existence of an extremum of \( S \) at the point \( \varepsilon = 0 \). Therefore, the condition
\[ \delta S = \int_{V_n} \delta L(r(x)).d^{(n)}x = 0 \]
appears as a necessary condition for the existence of an extremum of the function \( S(\varepsilon) \) at the point \( \varepsilon = 0 \), to which the point \( r(x) \in Y = R^j \) corresponds:
\( r(x) = \lim_{\varepsilon \to 0}[r(x) + \varepsilon.\delta r(x)] \).
\( \delta S = 0 \) can also be written in the form
\[ \delta S = \int_{V_n} dL(r(x)).\delta r.d^{(n)}x = 0 . \]
\( \delta r(x) \in U \subset Y = R^j \) can be considered as an arbitrary chosen element of the subset \( U \in Y \). It could have fixed values on the shell (the boundary) \( (V_n) \) of the volume \( V_n \) and especially, it can vanish on \( (V_n) \)

\[ \delta r|_{(V_n)} = 0. \tag{38} \]

The problem of finding out the conditions for \( L(r(x)) \) appearing as local necessary conditions for \( \delta S = 0 \) with \( \delta r|_{(V_n)} = 0 \) is related to the simple Lagrangian problem for the case of tensor fields with finite rank. As a solution of this problem the Euler-Lagrange equations follow for the variables \( r \) as functions of the co-ordinates \( x^m \) of a region of the manifold \( M \).

**Remark.** For simplicity, we identify every point \( x \) of the differentiable manifold \( M \) with its co-ordinates \( x \sim (x^i, \ i = 1, ..., n) \in R^n \) in a given map, where \( x : p = x \in M \rightarrow x(p = x) = x \in R^n \) \( (x \in M \simeq x \in R^n) \). All functions over \( x \in M \) are considered as functions over \( x \in R^n \).

\[ \delta S \] is denoted as the variation of the action \( S \), where the action \( S \) is defined as

\[ S = S(0) = \int_{V_n} L(r(x)) d^n x. \tag{39} \]

If we interpret \( L \) as a Lagrangian density depending on field variables and \( r \) as field variables, then we obtain the well known relations of the variational problem for the Lagrangian density \( L \).

(a) The Lagrangian density \( L \) depends on components (in a given basis) of tensor fields (with finite rank) and their first and second partial derivatives with respect to the co-ordinates as well as on components of the affine connections over \( M \) and their partial derivatives

\[ L = L(K^A, K^A_B, i, K^A_B, j), \tag{40} \]

where \( K^A \sim (g_{ij}, V^A_B, \Gamma^i_{jk}, P^j_{ik}) \).

\( g_{ij} \) are the components of the covariant metric tensor \( g \) in a given basis, \( V^A_B \) are components of non-metric tensor fields \( V \), \( \Gamma^i_{jk} \) are the components of the contravariant affine connection \( \Gamma \) and \( P^j_{ik} \) are the components of the covariant affine connection \( P \).

As a tensor density of rank 0 and weight \( \frac{1}{2} \) the Lagrangian density \( L \) can be constructed by means of the metric tensor \( g \) in the form \( L = \sqrt{-d_g} L \), \( d_g = \det(g_{ij}) < 0 \). \( L \) is the Lagrangian invariant with respect to the co-ordinate transformations (with respect to the diffeomorphisms of the manifold \( M \))

\[ L'(x^k) = L(x^k). \]
Therefore, the Lagrangian density \( \mathbf{L} \) will have the form

\[
\mathbf{L} = \sqrt{-g} L(K^A_B, K^A_{B;i}, K^A_{B;i,j})
\]

(41)

and the variation \( \delta \mathbf{L} \) will be written as

\[
\delta \mathbf{L} = \frac{\partial \mathbf{L}}{\partial K^A_B} \delta K^A_B + \frac{\partial \mathbf{L}}{\partial K^A_{B;i}} \delta (K^A_{B;i}) + \frac{\partial \mathbf{L}}{\partial K^A_{B;i,j}} \delta (K^A_{B;i,j})
\]

(42)

where

\[
\frac{\partial \mathbf{L}}{\partial K^A_B} = \frac{\partial \mathbf{L}}{\partial (K^A_B)}, \quad \frac{\partial \mathbf{L}}{\partial K^A_{B;i}} = \frac{\partial \mathbf{L}}{\partial (K^A_{B;i})}, \quad \frac{\partial \mathbf{L}}{\partial K^A_{B;i,j}} = \frac{\partial \mathbf{L}}{\partial (K^A_{B;i,j})}.
\]

The validity of commutation relations of the type

\[
\delta (K^A_{B;i}) = (\delta K^A_B)_i, \quad \delta (K^A_{B;i,j}) = (\delta K^A_B)_{i,j}
\]

(43)

is a problem connected with the validity of commutation relations between the variation operator \( \delta \) and the partial derivatives. It requires additional investigation. Usually it is assumed that \( 43 \) is a priori fulfilled (Schmutzer 1968) or follows on the basis of the condition \( (\partial / \partial x^i)(\delta r) = 0 \) and the differentiability conditions for the function \( \mathbf{L} \).

(b) The Lagrangian density \( \mathbf{L} \) depends on components of tensor fields (with finite rank) and their first and second covariant derivatives. In an analogous way as in the case (a), \( \mathbf{L} \) can be written in the form

\[
\mathbf{L} = \sqrt{-g} L(K^A_B, K^A_{B;i}, K^A_{B;i;j})
\]

(44)

where

\[
K^A_B \sim (g_{ij}, V^A_B).
\]

The variation \( \delta \mathbf{L} \) will have the form

\[
\delta \mathbf{L} = \frac{\partial \mathbf{L}}{\partial K^A_B} \delta K^A_B + \frac{\partial \mathbf{L}}{\partial K^A_{B;i}} \delta (K^A_{B;i}) + \frac{\partial \mathbf{L}}{\partial K^A_{B;i;j}} \delta (K^A_{B;i;j})
\]

(45)

where

\[
\frac{\partial \mathbf{L}}{\partial K^A_B} = \frac{\partial \mathbf{L}}{\partial (K^A_B)}, \quad \frac{\partial \mathbf{L}}{\partial K^A_{B;i}} = \frac{\partial \mathbf{L}}{\partial (K^A_{B;i})}, \quad \frac{\partial \mathbf{L}}{\partial K^A_{B;i;j}} = \frac{\partial \mathbf{L}}{\partial (K^A_{B;i;j})}.
\]

The validity of commutation relations of the type

\[
\delta (K^A_{B;i}) = (\delta K^A_B)_i, \quad \delta (K^A_{B;i;j}) = (\delta K^A_B)_{i;j}
\]

(46)

is a problem connected with the commutation relations between the variation operator \( \delta \) and the covariant derivatives. It requires additional investigation.

The Euler-Lagrange equations can be obtained by means of the functional variations of a Lagrangian density and of these of its field variables considered as dynamic field variables (in contrast to the non-varied field variables considered as fixed and non-dynamic field variables).
2.4 Lie variation of the Lagrangian density $\mathbf{L}$

The free choice of $h \in Y = R^j$ (under the prerequisite for $r + h \in U$, $r \in U$) in the expressions 7 and 8 allows us to consider another type of variation than the functional variation.

Let we suppose that the increment $h$ is constructed by means of the action of the Lie differential operator $\mathcal{L}_\xi$ on $r$, considered as tensor fields components, $\mathcal{L}_\xi : r \to \mathcal{L}_\xi r \in Y$, as

$$
h = \varepsilon.\mathcal{L}_\xi r, \quad \forall \xi \in T(M), \varepsilon \in [0,1] \subset R.
$$

(47)

$\mathcal{L}_\xi r$ are the (well defined) Lie derivatives of the components of the tensor fields and their covariant derivatives. In this case the increment $h$ appears as an infinitesimal dragging-along an arbitrary given contravariant vector field $\xi$ over $M$. Then

$$
r + h = r + \varepsilon.\mathcal{L}_\xi r, \quad r^i + h^i = r^i + \varepsilon.\mathcal{L}_\xi r^i,
$$

(48)

$$
d\mathbf{L}(r).h = \varepsilon.d\mathbf{L}(r).\mathcal{L}_\xi r = \varepsilon.[\frac{\partial \mathbf{L}(r)}{\partial y^j}(r).\mathcal{L}_\xi r^j] =$$

$$
= \varepsilon.[\frac{\partial \mathbf{L}}{\partial y^j}(r).\mathcal{L}_\xi y^j + \frac{\partial \mathbf{L}}{\partial w^k}(r).\mathcal{L}_\xi w^k + \frac{\partial \mathbf{L}}{\partial z^l}(r).\mathcal{L}_\xi z^l].
$$

(49)

d$\mathbf{L}(r).\mathcal{L}_\xi r$ is called Lie variation of the Lagrangian density $\mathbf{L}$. It follows from 7 and 8 that

$$
\mathbf{L}(r + \varepsilon.\mathcal{L}_\xi r) - \mathbf{L}(r) = \varepsilon.d\mathbf{L}(r).\mathcal{L}_\xi r + \alpha(r;\varepsilon.\mathcal{L}_\xi r),
$$

(50)

where

$$
\lim_{\varepsilon.\mathcal{L}_\xi r \to 0} \frac{[\alpha(r;\varepsilon.\mathcal{L}_\xi r)]}{\varepsilon} = 0, \text{ or } \lim_{\varepsilon \to 0} [\alpha(r;\varepsilon.\mathcal{L}_\xi r)] = 0, \text{ if } \mathcal{L}_\xi r |_{U \subset Y} \leq M < \infty.
$$

The Lagrangian density $\mathbf{L}$ considered as a function of the parameter $\varepsilon$ is differentiable at the point $\varepsilon = 0$

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [\mathbf{L}(r + \varepsilon.\mathcal{L}_\xi r) - \mathbf{L}(r)] = d\mathbf{L}(r).\mathcal{L}_\xi r.
$$

(51)

To the point $\varepsilon = 0 \in R$ corresponds the point $r \in Y = R^j$.

Let we now consider the integral

$$
\overline{S}(\varepsilon) = \int_{V_n} \mathbf{L}(r(x) + \varepsilon.\mathcal{L}_\xi r(x)).d^n x, \quad r = r(x), \mathcal{L}_\xi r = \mathcal{L}_\xi r(x),
$$

(52)

and develop this integral with respect to $\varepsilon$ by the use of the representation of $\mathbf{L}(r(x))$ as a differentiable function of $r(x)$

$$
\overline{S}(\varepsilon) = \int_{V_n} \mathbf{L}(r(x)).d^n x + \varepsilon. \int_{V_n} d\mathbf{L}(r(x)).\mathcal{L}_\xi r(x).d^n x +$$

$$
+ \int_{V_n} \alpha(r(x);\varepsilon.\mathcal{L}_\xi r(x)).d^n x = \overline{S}(0) + \varepsilon.\overline{S}(\varepsilon) + \overline{S}(\alpha(\varepsilon)),
$$

(53)

12
where
\[ S(0) = \int_{V_n} L(r(x))d^{(n)}x = S(0), \quad \mathcal{S}(\xi) = \int_{V_n} dL(r(x)).\mathcal{L}_\xi r(x).d^{(n)}x, \]
\[ \mathcal{S}(\alpha(\varepsilon)) = \int_{V_n} \alpha(r(x); \varepsilon).\mathcal{L}_\xi r(x).d^{(n)}x. \]

We define now the Lie derivative of the integral \( \mathcal{S}(\varepsilon) \) at the point \( \varepsilon = 0 \) along an arbitrary given contravariant vector field \( \xi \in T(M) \) as
\[ \mathcal{L}_\xi[\mathcal{S}(0)] = \lim_{\varepsilon \to 0} \frac{\mathcal{S}(\varepsilon) - \mathcal{S}(0)}{\varepsilon} = \mathcal{L}_\xi \mathcal{S}. \quad (54) \]

Since \( \mathcal{S}(\varepsilon) - \mathcal{S}(0) = \varepsilon.\mathcal{S}(\xi) + \mathcal{S}(\varepsilon(\varepsilon)) \), we obtain from \( \text{(54)} \)
\[ \mathcal{L}_\xi[\mathcal{S}(0)] = \lim_{\varepsilon \to 0} \frac{\mathcal{S}(\varepsilon) - \mathcal{S}(0)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\varepsilon.\mathcal{S}(\xi) + \mathcal{S}(\varepsilon(\varepsilon))] = \mathcal{S}(\xi), \quad (55) \]

Therefore,
\[ \mathcal{L}_\xi[\mathcal{S}(0)] = \mathcal{S}(\xi) = \int_{V_n} dL(r(x)).\mathcal{L}_\xi r(x).d^{(n)}x. \quad (56) \]

On the other side, the Lie differential operator \( \mathcal{L}_\xi \) (Yano 1957), (Manoff 1996) acting on the integral \( \mathcal{S}(0) \) has to commute with the integral, i.e., it has to obey the relation \( \mathcal{L}_\xi \circ \int = \int \circ \mathcal{L}_\xi \), and therefore,
\[ \mathcal{L}_\xi[\mathcal{S}(0)] = \mathcal{L}_\xi \int_{V_n} L(r(x))d^{(n)}x = \int_{V_n} \mathcal{L}_\xi[L(r(x))d^{(n)}x] = \]
\[ = \mathcal{L}_\xi[\mathcal{S}(0)] = \mathcal{S}(\xi) = \int_{V_n} dL(r(x)).\mathcal{L}_\xi r(x).d^{(n)}x. \quad (57) \]

Since
\[ \int_{V_n} \mathcal{L}_\xi[L(r(x))d^{(n)}x] = \int_{V_n} \mathcal{L}_\xi L(r(x))d^{(n)}x, \quad (58) \]

where \( \mathcal{L}_\xi L(r(x)) \) is the Lie derivative of the tensor density \( L \) and \( \mathcal{L}_\xi \) is the form-invariant Lie differential operator acting on covariant tensor densities as \( L \). From the relations \( \mathcal{L}_\xi \circ \int = \int \circ \mathcal{L}_\xi \) it follows that
\[ \mathcal{L}_\xi L(r(x)) \equiv dL(r(x)).\mathcal{L}_\xi r(x), \quad (59) \]
or
\[ \mathcal{L}_\xi L(r(x)) \equiv \frac{\partial L}{\partial r^j}(r).\mathcal{L}_\xi r^j = \frac{\partial L}{\partial y^i}(r).\mathcal{L}_\xi y^i + \frac{\partial L}{\partial w^k}(r).\mathcal{L}_\xi w^k + \frac{\partial L}{\partial z^l}(r).\mathcal{L}_\xi z^l. \quad (60) \]
Remark. From the expressions $\frac{54}{56}$, it follows that

$$\int_{V_n} dL(r(x)) \cdot L_\xi r(x) \cdot d^{(n)}x = \int_{V_n} \mathcal{L}_\xi L(r(x)) \cdot d^{(n)}x , \quad (61)$$

where $\mathcal{L}_\xi L(r(x)) = dL(r(x)) \cdot L_\xi r(x)$ is defined as the Lie variation of the Lagrangian density $L$. From $\xi$ and $\xi$, it follows that the Lie variation of the Lagrangian density $L$ is equal to the Lie derivative of the Lagrangian density $L$ considered as a tensor density on which the form-invariant Lie differential operator $\mathcal{L}_\xi$ acts.

If $L$ and $r$ are expressed in their explicit forms, then we obtain for $\forall \xi \in T(M)$ the s. c. Lie identity for the Lagrangian density $L$

$$\mathcal{L}_\xi L = \frac{\partial L}{\partial K^A_B} L_\xi K^A_B + \frac{\partial L}{\partial K^A_{B;i}} L_\xi (K^A_{B;i}) + \frac{\partial L}{\partial K^A_{B;ij}} L_\xi (K^A_{B;ij}), \quad (62)$$

where

$$K^A_B \sim (g_{ij}, V^A_B).$$

The next problem is the explicit representation of the identity $\xi$ by means of the explicit form of the Lie derivatives of the components and their first and second covariant derivatives of the metric tensor field $g$ and the non-metric tensor fields $V$. As a result we obtain the covariant Noether identities for the Lagrangian density $L$ and the corresponding energy-momentum tensors. This problem has been already solved for a Lagrangian theory of tensor fields (with finite rank) over differentiable manifolds with an affine connection and a metric $([L_n, g]-spaces)$ (Manoff 1991).

2.5 Variation operator

Definition 3. Variation (variational) operator. Operator $\delta$, acting on the components of tensor fields in a given basis and preserving the type of these tensor fields. The result of its action is identified with the result of the action of an operator (denoted with the same symbol $\delta$) on tensor fields with the following properties:

1. Action on a tensor field $K$

$$\delta : K \rightarrow \delta K,$$

$$K, \quad \delta K \in \otimes^k_l(M),$$

$$\delta K = \delta K^{A}_{B} e_A \otimes e^B = \delta K^{C}_{D} \partial_C \otimes dx^D,$$

$$K = K^{A}_{B} e_A \otimes e^B = K^{C}_{D} \partial_C \otimes dx^D,$$

$$K^A_B \in C^r(M), \quad \delta K^A_B \in C^r(M), \quad x \in M.$$

(1.1) For $k = l = 0$ this property leads to the action of $\delta$ on an invariant function $f \in \otimes^0_0(M)$:

$$\delta : f \rightarrow \delta f, \quad f, \quad \delta f \in C^r(M), \quad f, \quad \delta f \in \otimes^0_0(M).$$
Remark. A function $f$ over $M$ could be an invariant function $[f \in \otimes^0_0(M)$, $f \in C^r(M)]$ or a noninvariant function $f \in C^r(M)$, $f \notin \otimes^0_0(M)$ [changing its form and value under the change of the map (co-ordinates) in $M$].

2. Action on a function $f$

\[ \delta : f \rightarrow \delta f, \quad f, \delta f \in C^r(M), \]

3. Linear operator with respect to tensor fields

\[ \delta(\alpha.K_1 + \beta.K_2) = \alpha.\delta K_1 + \beta.\delta K_2, \]
\[ \alpha, \beta \in R (or C), \quad K_i \in \otimes^k i(M), \quad i = 1, 2, \]

4. Differential operator acting on tensor fields and obeying the Leibniz rule

\[ \delta(f.g) = \delta f.g + f.\delta g, \quad f, g \in C^r(M), \quad f, g \in \otimes^0_0(M), \]
\[ \delta(Q \otimes S) = \delta Q \otimes S + Q \otimes \delta S, \quad Q \in \otimes^k i(M), \quad S \in \otimes^m r(M), \]

5a. Commutation relations (as additional conditions) with the Lie differential operator

\[ \delta \circ \partial_j = \partial_j \circ \delta, \quad \delta \circ \partial_a = \partial_a \circ \delta, \]
\[ \delta \circ \xi - \xi \circ \delta = \xi \circ \delta, \]

5b. Commutation relations (as additional conditions) with the covariant differential operator

\[ \delta \circ \nabla_j = \nabla_j \circ \delta, \quad \delta \circ \nabla_a = \nabla_a \circ \delta, \]
\[ \delta \circ \nabla_\xi - \nabla_\xi \circ \delta = \nabla_\xi \circ \delta, \]

5c. Commutation relations (as additional conditions) with the contraction operator $S$

\[ \delta \circ S = S \circ \delta. \]

From the properties 2. and 4. it follows that $\delta 1 = 0, \quad 1 \in N \subset C^r(M)$.

Proof: $\delta(1.1) = (\delta 1).1 + 1.(\delta 1) = 2.(\delta 1) = \delta 1 : \delta 1 = 0$. From the properties 2., 3. and 4. it follows that $\delta \alpha = 0, \alpha = const. \in R (or C) \subset C^r(M)$.

Proof: $\delta(\alpha.g) = \alpha.\delta g = \delta \alpha.g + \alpha.\delta g : \delta \alpha = 0, \forall g \in C^r(M)$.

The commutation relations of the variational operator with the covariant differential operator, the Lie-differential operator and the contraction operator $S$ could be represented in the following scheme. The commutation relations of type A appear as sufficient conditions for the commutation of $\delta$ with the partial derivatives of the field variables. The commutation relation of type B appear as sufficient conditions for the commutation of $\delta$ with the covariant derivative of the field variables.
Commutation relations of type A
\[ \delta \circ L_{\partial_j} = L_{\partial_j} \circ \delta , \quad \delta \circ L_{e_\alpha} = L_{e_\alpha} \circ \delta , \]
\[ \delta \circ L_{\xi} = L_{\xi} \circ \delta \]
\[ \delta (f_{i,j}) = (\delta f)_{i,j} , \quad \delta (e_\alpha f) = e_\alpha (\delta f) \]
\[ \delta C_{\alpha \beta \gamma} = 0 , \quad \delta T^{\alpha \gamma}_{\beta \alpha} = \delta T^{\gamma}_{\alpha \beta} \]
\[ \delta (P^i_{jk}) + \Gamma^{i}_{jk} = 0 \]
\[ \delta (P^\alpha_{\beta j}) + \Gamma^\alpha_{\beta j} = 0 \]
\[ \delta (P^\alpha_{\beta j}) = e_\alpha (\delta f) \]

Commutation relations of type B
\[ \delta \circ \nabla_{\partial_j} = \nabla_{\partial_j} \circ \delta , \quad \delta \circ \nabla_{e_\alpha} = \nabla_{e_\alpha} \circ \delta \]
\[ \delta \circ \nabla_\xi = \nabla_\xi \circ \delta \]
\[ \delta (f_{i,j}) = (\delta f)_{i,j} , \quad \delta (e_\alpha f) = e_\alpha (\delta f) \]
\[ \delta T^{\alpha \gamma}_{\beta \alpha} = - \delta C_{\alpha \beta \gamma} \]
\[ \delta (K^A_{\beta j}) = (\delta K^A_{\beta j})_j \]
\[ \delta (e_\alpha K^A_{\beta j}) = e_\alpha (\delta K^A_{\beta j}) \]
\[ \delta (\Gamma^i_{jk}) = (\delta \Gamma^i_{jk})_j \]
\[ \delta (e_\alpha \Gamma^i_{\beta j}) = e_\alpha (\delta \Gamma^i_{\beta j}) \]
\[ \delta (P^i_{jk}) = (\delta P^i_{jk})_j \]
\[ \delta (e_\alpha P^\alpha_{\beta j}) = e_\alpha (\delta P^\alpha_{\beta j}) \]

Commutation relations of type C
\[ \delta (\xi^i_{j,k}) = (\delta \xi^i_{j,k})_j \]
\[ \delta (f^i_{j,k}) = 0 , \quad \delta (f^\alpha_{\sigma j}) = 0 \]
\[ \delta f^i_{j} = f^i_{j} - \delta f^i_{j} , \quad \delta f^\alpha_{j} = \delta f^\alpha_{j} \]
\[ \delta P^i_{jk} = - \delta P^i_{jk} \]
\[ \delta (\xi^i_{j,k}) = (\delta \xi^i_{j,k})_j , \quad \delta (e_\beta \xi^\alpha) = e_\beta (\delta \xi^\alpha) \]
\[ \delta (f^i_{j,k}) = (\delta f^i_{j,k})_k = 0 \]

The method using commutation relations of type A is the method of Lagrangians with partial derivatives (MLPD). The method using commutation relations of type B is the method of Lagrangians with covariant derivatives (MLCD). In this case the affine connections appear as non-dynamic fields variables and the variation commutes simultaneously with the partial and the covariant derivatives of the tensor fields components. The commutation relations of type C could be used when the contraction tensor field \( Sr = f^i_{j,k} \partial_j \otimes dx^j = f^\alpha_{\beta} e_\alpha \otimes e^\beta \) is considered as a (fixed) non-dynamical tensor field or when \( Sr = Kr = g^i_{\alpha} \partial_i \otimes dx^j = g^\alpha_{\beta} e_\alpha \otimes e^\beta \), i.e. when the contraction tensor field \( Sr \) is equal to the Kronecker tensor field \( Kr \). In both cases \( \delta \circ S = S \circ \delta \) appears as a sufficient condition for \( \delta f^i_{j,k} = 0 \).

On the basis of the two types 1. and 2. of the Lagrangian density \( L \) and the commutation relations \( A \) and \( B \) two types of methods determining a Lagrangian formalism can be developed:
(a) Method of Lagrangians with partial derivatives (MLPD).
(b) Method of Lagrangians with covariant derivatives (MLCD).
2.6 Method of Lagrangians with partial derivatives (MLPD)

The method of Lagrangians with partial derivatives (MLCD) is a Lagrangian formalism for tensor fields based on:

(a) A Lagrangian density \( L \) of type 1.

(b) The action \( S \) of a Lagrangian system described by means of the Lagrangian density \( L \)

\[
S = \int_{V_n} L \, d^{(n)}x = \int_{V_n} L \, d\omega,
\]

where \( V_n \) is a volume in the manifold \( M \) with \( \text{dim } M = n \); \( d\omega = \sqrt{-d_g \, d^{(n)}x} \) is the invariant volume element.

(c) The functional variation \( \delta S \) of the action \( S \) with the condition \( (\delta S = 0) \) for the existence of an extremum

\[
\delta S = \delta \int_{V_n} L \, d^{(n)}x = \int_{V_n} \delta L \, d^{(n)}x = 0 . \quad (63)
\]

(d) The functional variation of the Lagrangian density \( L \) in the form

\[
\delta L = \frac{\partial L}{\partial g_{ij}} \delta g_{ij} + \frac{\partial L}{\partial g_{ij,k}} \delta (g_{ij,k}) + \frac{\partial L}{\partial g_{ij,k,l}} \delta (g_{ij,k,l}) + \frac{\partial L}{\partial V_{A B}} \delta V_{A B} + \frac{\partial L}{\partial V_{A B,i}} \delta (V_{A B,i}) + \frac{\partial L}{\partial V_{A B,i,j}} \delta (V_{A B,i,j}) . \quad (64)
\]

(e) The variational operator \( \delta \) obeying the commutation relations with the Lie differential operator

\[
\delta \circ \xi = \xi \circ \delta + \mathcal{L}_\xi \,,
\]

\[
\delta \circ \ell_{\partial_j} = \xi_{\partial_j} \circ \delta \quad (\text{in a co-ordinate basis}),
\]

\[
\delta \circ \ell_{e_\alpha} = \xi_{e_\alpha} \circ \delta \quad (\text{in a non-co-ordinate basis}),
\]

leading to commutation of \( \delta \) with the partial derivatives:

\[
\delta (g_{ij,k}) = (\delta g_{ij})_{,k} , \quad \delta (g_{ij,k,l}) = (\delta g_{ij})_{,k,l} , \quad \delta (V_{A B,i}) = (\delta V_{A B})_{,i} , \quad \delta (V_{A B,i,j}) = (\delta V_{A B})_{,i,j} . \quad (66)
\]

(f) The Lie variation \( \mathcal{L}_\xi S \) of the action \( S \)

\[
\mathcal{L}_\xi S = \mathcal{L}_\xi \int_{V_n} L \, d^{(n)}x = \int_{V_n} \mathcal{L}_\xi L \, d^{(n)}x . \quad (67)
\]

(g) The Lie variation of the Lagrangian density \( L \) in the form

\[
\mathcal{L}_\xi L = \frac{\partial L}{\partial g_{ij}} \mathcal{L}_\xi g_{ij} + \frac{\partial L}{\partial g_{ij,k}} \mathcal{L}_\xi (g_{ij,k}) + \frac{\partial L}{\partial g_{ij,k,l}} \mathcal{L}_\xi (g_{ij,k,l}) + \ldots
\]
\[
\begin{align*}
\frac{\partial L}{\partial V_{A B}} \xi V_{A B} + \frac{\partial L}{\partial V_{A B,i}} \xi (V_{A B,i}) + \frac{\partial L}{\partial V_{A B,i,j}} \xi (V_{A B,i,j}) .
\end{align*}
\]

Remark. In the last expression, the Lie derivatives of the partial derivatives of the components of tensor fields have to be determined in an appropriate manner (Schmutzer 1968). There is no covariant definition of the Lie derivative of partial derivatives of the components of tensor fields.

2.7 Method of Lagrangians with covariant derivatives (MLCD)

The method of Lagrangians with covariant derivatives (MLCD) is a Lagrangian formalism for tensor fields based on:

(a) A Lagrangian density \( L \) of type 2.

(b) The action \( S \) of a Lagrangian system described by means of the Lagrangian density
\[
S = \int_{V_n} L \, d^{(n)} x = \int_{V_n} L \, d \omega,
\]
where \( V_n \) is a volume in the manifold \( M \) with \( \text{dim} M = n \); \( d \omega = \sqrt{-g} \, d^{(n)} x \) is the invariant volume element.

(c) The functional variation \( \delta S \) of the action \( S \) with the condition for the existence of an extremum
\[
\delta S = \delta \int_{V_n} L \, d^{(n)} x = \int_{V_n} \delta L \, d^{(n)} x = 0 .
\]

(d) The functional variation of the Lagrangian density \( L \) in the form
\[
\delta L = \frac{\partial L}{\partial g_{ij}} \delta g_{ij} + \frac{\partial L}{\partial g_{ij;k}} \delta (g_{ij;k}) + \frac{\partial L}{\partial g_{ij;k;l}} \delta (g_{ij;k;l}) +
\]
\[
+ \frac{\partial L}{\partial V_{A B}} \delta V_{A B} + \frac{\partial L}{\partial V_{A B;i}} \delta (V_{A B;i}) + \frac{\partial L}{\partial V_{A B;i,j}} \delta (V_{A B;i,j}) .
\]

(e) The variational operator \( \delta \) obeying the commutation relations with the covariant differential operator
\[
\delta \circ \nabla = \nabla \circ \delta + \nabla \delta ,
\]
\[
\delta \circ \nabla B = \nabla B \circ \delta \quad \text{(in a co-ordinate basis)},
\]
\[
\delta \circ \nabla e_a = \nabla e_a \circ \delta \quad \text{(in a non-co-ordinate basis)},
\]
leading to commutation of \( \delta \) with the covariant derivatives:
\[
\delta (g_{ij;k}) = (\delta g_{ij});k , \quad \delta (g_{ij;k;l}) = (\delta g_{ij});k;l ,
\]
\[
\delta (V_{A B;i}) = (\delta V_{A B});i , \quad \delta (V_{A B;i,j}) = (\delta V_{A B});i,j .
\]

(f) The Lie variation \( \mathcal{L}_\xi S \) of the action \( S \)
\[
\mathcal{L}_\xi S = \mathcal{L}_\xi \int_{V_n} L \, d^{(n)} x = \int_{V_n} \mathcal{L}_\xi L \, d^{(n)} x .
\]
(g) The Lie variation of the Lagrangian density $L$ in the form

$$\mathcal{L}_\xi L = \partial_L \mathcal{L}_\xi g_{ij} + \partial_L \mathcal{L}_\xi (g_{ij;}^k) + \partial_L \mathcal{L}_\xi (g_{ij;k;}^l) + \partial_L \mathcal{L}_\xi (g_{ij;k;l}^m) + \partial_L \mathcal{L}_\xi (g_{ij;k;l}^m).$$

In the MLCD [because of the commutation relations (e)] the affine connections $\Gamma$ and $P$, and their corresponding curvature tensors are considered as non-dynamic field variables ($\delta \Gamma_{ijk}^i = 0, \delta P_{ijk}^i = 0, \delta R_{ijkl}^i = 0, \delta P_{ijkl}^i = 0$). Therefore, in the MLCD variations of the components and their covariant derivatives of the covariant metric tensor $g$ and the non-metric tensor fields $V$ are considered for given (fixed) affine connections and for fixed and determined by them types of transports of the tensor fields. Of course, the question arises how the affine connections can be found if not by means of a Lagrangian formalism.

The first simple answer is: the affine connections (or the equations for them as functions of the co-ordinates in $M$) can be found on the grounds of the MLPD and then the components of the tensor fields (as functions of the co-ordinates in $M$) can be determined by means of MLCD. This answer could induce another question: why two methods have to be applied when one is enough for finding out all equations for all dynamic field variables. There are at least two possible answers to this question: 1. The MLCD ensure the finding out equations for tensor fields (as dynamic field variables). These equations are (a) covariant (tensorial) equations and (b) form-invariant (gauge invariant) equations with respect to the affine connections. The affine connections could be determined on the grounds of additional conditions and not exactly by means of a variational principle. 2. The MLPD can ensure the consideration of the affine connections as dynamic field variables and the finding out their field equations. It cannot give direct answer for the type (tensorial or non-tensorial) of the equations obtained for the tensor field variables and their corresponding energy-momentum quantities. The tensorial character of quantities and relations concerning the tensor fields variables has to be proved (which, in general, could be a matter of some difficulty) (Lovelock and Rund 1975).

In the development of the MLCD problems arise connected with the different conditions for obtaining Euler-Lagrange’s equations.

### 3 Euler-Lagrange’s equations

Let a Lagrangian density of the type 2

$$L = \sqrt{-d_g L(g_{ij}, g_{ij;k}, g_{ij;k;l}, V^A_B, V^A_{B;i}, V^A_{B;i;j})}$$

be given. $g_{ij}$ are the components of the metric tensor field (metric) $g$, $V^A_B$ are components of non-metric tensor fields $V$ with finite rank ($A = i_1...i_k$, $B = j_1...j_l$), $A$, $B$ are multi-indices, $d_g = \det(g_{ij}) < 0$, $;k$ is the covariant derivative (constructed by means of the affine connection $\Gamma$ and / or $P$) with
respect to a basic vector field $e_k$ (or $\partial_k$), $L$ is the corresponding to $L$ Lagrangian invariant.

The functional variation of $L$ can be considered under the following conditions:

1. The metric and the affine connection are in general independent characteristics of a $(\mathcal{T}_n, g)$-space.

2. As sufficient condition for the commutation of the functional variation $\delta$ with the covariant derivative $\nabla_{\partial_k}$, i.e., as sufficient condition for

$$\delta(g_{ij;k}) = (\delta g_{ij})_{;k}, \delta(V^A_{B;k}) = (\delta V^A_{B})_{;k},$$

appears the condition for the operators $\delta$ and $\nabla_{\partial_k}$, acting on tensor fields over $M$,

$$\delta \circ \nabla_{\partial_k} = \nabla_{\partial_k} \circ \delta,$$

and leading also to the conditions

$$\delta \Gamma^i_{jk} = 0, \delta P^i_{jk} = 0, \delta \circ e_k = e_k \circ \delta, \delta \circ \partial_k = \partial_k \circ \delta.$$  

(77)

Therefore, the contravariant affine connection $\Gamma$ with the components $\Gamma^i_{jk}$ and the covariant connection $P$ with the components $P^i_{jk}$ appear under these conditions as fixed, non-dynamic field variables.

3. $g_{ij}$ and $V^A_{B}$ are considered as dynamic field variables. The functional variation of $L$ will be found with respect to these dynamic tensor fields.

4. The functional variation $\delta \omega^A$ of the covariant basis $\omega^A$ determining the invariant volume element $d\omega$

$$d\omega = \frac{1}{n!} \sqrt{-d_g} \varepsilon_A \omega^A = \sqrt{-d_g} d^{(n)} x,$$

$$\omega^A = dx^1 \wedge ... \wedge dx^n = e^1 \wedge ... \wedge e^n, \quad d^{(n)} x = dx^1 \wedge ... \wedge dx^n,$$

$$\varepsilon_A = \varepsilon_{i_1 ... i_n} = \varepsilon_{e_1 ... e_n}, \quad \delta \varepsilon_{i_1 ... i_n} = 0,$$

is equal to zero

$$\delta \omega^A = 0.$$  

(78)

Remark. The last condition 4. follows from the properties of the functional operator $\delta$.

The functional variation of $L$ under the conditions 1÷4 can be considered in two different ways for obtaining the Euler-Lagrange equations for $g_{ij}$ and $V^A_{B}$. These two ways are determined by the different types of terms, separated during the variation of $L$, and different conditions for the affine connections $\Gamma$ and $P$ for obtaining a common divergency term, necessary for the application of the Stokes theorem.

3.1 Covariant Euler-Lagrange’s equations

The functional variation of $L$ with respect to $g_{ij}$ and $V^A_{B}$ as components of the dynamic tensor variables $g$ and $V$ can be written in the forms

$$\delta L = \sqrt{-d_g} (\delta_g L + \delta_v L),$$
\[ \delta \mathbf{L} = \sqrt{-g} \left[ \frac{\delta g^{i} L}{\delta g_{kl}} + \frac{1}{2} L \frac{\delta g_{kl}}{\delta g^{i}} \right] \delta g_{kl} + \frac{\delta \mathbf{L}}{\delta V^{A} B} \delta V^{A} B + j^{m}, \quad (79) \]

where
\[ j^{m} = g_j^{m} + v_j^{m}, \quad g_j^{m} = \left( \frac{\delta g^{i}}{\delta g_{kl}} + \frac{1}{2} L \frac{\delta g_{kl}}{\delta g^{i}} \right) \delta g_{kl} + g_j^{m}, \quad (80) \]

The functional variation \( \delta g (L, d\omega) \) with respect to \( g_{ij} \) under the conditions \( 1 \div 4 \) can be written in the form
\[ \delta g (L, d\omega) = (\delta g L). d\omega, \quad (81) \]

If one has to apply the variational principle to the action
\[ S = \int_{V_n} \sqrt{-g} L. d^{(n)} x = \int_{V_n} L. d^{(n)} x = \int_{V_n} L. d\omega, \quad (82) \]

with \( \delta S = \int_{V_n} \delta \mathbf{L}. d^{(n)} x = 0 \) and \( d^{(n)} x \) as a non-invariant volume element, and
\[ d\omega = \sqrt{-g} d^{(n)} x \]

as the corresponding invariant volume element, then the covariant Euler-Lagrange equations for \( g_{ij} \), respectively for \( V^{A} B \),
\[ \frac{\delta g_{ij}}{\delta g_{ij}} = 0 \iff \frac{\delta g_{ij}}{\delta \mathbf{L}} = - \frac{1}{2} L g^{ij} - P^{ij}, \quad \frac{\delta \mathbf{L}}{\delta V^{A} B} = - P_{A} B, \quad (83) \]

can be found. This can be done after rewriting the scalar density \( \sqrt{-g} j^{m}, \) as a common divergence \( \sqrt{-g} j^{m} \) of a vector field density \( j^{m} = \sqrt{-g} j^{m} \) and using Stokes’ theorem with the boundary conditions
\[ \delta g_{kl} \big|_{(V_n)} = 0, \quad \delta V^{A} B \big|_{(V_n)} = 0, \]

\[ (\delta g_{kl})_{;i} \big|_{(V_n)} = 0 \iff (\delta g_{kl})_{;i} \big|_{(V_n)} = 0, \quad (\sim \text{because of } \delta g_{kl} \big|_{(V_n)} = 0 \quad \text{and } \delta P_{ij}^{k} = 0), \]

\[ (\delta V^{A} B)_{;i} \big|_{(V_n)} = 0 \iff (\delta V^{A} B)_{;i} \big|_{(V_n)} = 0, \quad (\sim \text{because of } \delta V^{A} B \big|_{(V_n)} = 0 \quad \text{and } \delta P_{ij}^{k} = 0), \quad (84) \]

for the arbitrary variations \( \delta g_{kl}, \delta V^{A} B \) and their (first) covariant (or partial) derivatives on the shell \( (V_n) \) of the volume \( V_n \).

For every contravariant vector field \( j^{i} \) the necessary and sufficient conditions for the existence of the relation
\[ \sqrt{-g} j^{i} ;_{;i} = j^{i} ;_{;i} = (\sqrt{-g} j^{i})_{;i} \quad (85) \]

are the conditions for the components \( \Gamma_{jk}^{i} \) of the affine connection \( \Gamma \)
\[ q_{i} = \Gamma_{ik}^{j} - (\log \sqrt{-g})_{;i} = 0, \quad \text{where } \Gamma_{ik}^{j} = g_{ik}^{j} \Gamma_{ik}^{l}. \quad (86) \]
or the equivalent conditions

\[ q_i = T^k_{ji} - C_{ik} - \frac{1}{2}g^{jk}g_{kl;i} + g^l_kg^k_{li} = 0 \]

(in a non-coordinate basis \( \{e_i\} \) with \( [e_i, e_k] = C_{ik} e_l \) ),

where \( T^i_{jk} = \Gamma^i_{kj} - \Gamma^i_{jk} - C_{jk}^i, T^k_{ji} = g^l_kT^l_{ji}, C_{ik}^j = g^l_kC_{ik}^l \),

\[ g^k_{li} = \Gamma^k_{li} + P^k_{li}, \quad P^k_{li} - P^k_{lj} - C_{ik}^l = U_{li}. \]

The necessary and sufficient conditions follow from the relations

\[ \sqrt{-d_g} j^i = \sqrt{-d_g} j^i + (T^k_{ji} - C_{ik} - \frac{1}{2}g^{jk}g_{kl;i} + g^l_kg^k_{li}).(\sqrt{-d_g} j^i) = q^i + q^i_j, \quad j^i = \sqrt{-d_g} j^i, \quad q_i = T^k_{ji} - C_{ik} - \frac{1}{2}g^{jk}g_{kl;i} + g^l_kg^k_{li} \]

(93)

(94)

The necessary and sufficient conditions can also be written in the form \( q_i j^i = 0 \). By the use of the explicit form of \( j^i \) (see below) and the arbitrariness of \( \delta g_{ij} \) and \( \delta V^A_B \) and their covariant derivatives \( (\delta g_{ij})_{;ij} \) and \( (\delta V^A_B)_{;ij} \), there follow the covariant Euler-Lagrange equations for \( g \) and \( V \) in the form:

\[ \frac{\delta_s L}{\delta g_{kl}} + \frac{1}{2} L g^{kl} + P^{kl} = 0, \quad \frac{\delta_s L}{\delta V^A_B} + P^A_B = 0, \]  

(95)

where

\[ \frac{\delta_s L}{\delta g_{kl}} = \frac{\partial L}{\partial g_{kl}} - (\frac{\partial L}{\partial g_{kl;m}})_{;m} + (\frac{\partial L}{\partial g_{kl;m;n}})_{;m;n}, \]

(96)

\[ P^{kl} = q_i[\frac{\partial L}{\partial g_{kl;i}} - (\frac{\partial L}{\partial g_{kl;i;j}})_{;j} + (q_i q_j - q_{ij} q_i) \frac{\partial L}{\partial g_{kl;i}^j}], \]

(97)

\[ \frac{\delta_s L}{\delta V^A_B} = \frac{\partial L}{\partial V^A_B} - (\frac{\partial L}{\partial V^A_B;j})_{;j} + (q_i q_j - q_{ij} q_i) \frac{\partial L}{\partial V^A_B;j;i}, \]  

(98)

\[ P^A_B = q_i[\frac{\partial L}{\partial V^A_B;i} - (\frac{\partial L}{\partial V^A_B;i;j})_{;j} + (q_i q_j - q_{ij} q_i) \frac{\partial L}{\partial V^A_B;i;j}], \]

(99)

\[ \delta_s L = \sqrt{-d_g}(\frac{\delta_s L}{\delta g_{kl}} + \frac{1}{2} L g^{kl} \delta g_{kl} + \sqrt{-d_g} j^i,i = \]

\[ = \sqrt{-d_g}(\frac{\delta_s L}{\delta g_{kl}} + \frac{1}{2} L g^{kl} + P^{kl}) \delta g_{kl} + (\sqrt{-d_g} j^i)_i = \]

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Explicit form of the covariant Euler-Lagrange equations for $g_{ij}$ under the conditions $q_i = 0$ for the affine connections.

For $q_i = 0 : P^{kl} = 0$ and we have:

$$
\delta g L = \sqrt{-d_g} \partial_{g_{ij}} L = \sqrt{-d_g} \partial_{g_{kl}} \frac{\delta \bar{g} L}{\delta g_{kl}} + g^m_{,m} \delta g_{ij} ,
$$

(96)

$$
\frac{\delta \bar{g} L}{\delta g_{kl}} = \frac{\partial L}{\partial g_{ij}} + \frac{\partial L}{\partial g_{B;i;j}} \delta V^A_B + \frac{\partial L}{\partial g_{B;i;m}} \delta V^A_B ; n .
$$

(97)

From the Euler-Lagrange equations for $g_{ij}$, it follows that

$$
\frac{\delta \bar{g} L}{\delta g_{ij}} g_{ij} = - \frac{n}{2} L , \text{ (dim } M = n). 
$$

(98)

For the special case, when $L = L(g_{ij}, V^A_B, V^A_{B;i}, V^A_{B;i;j})$

$$
\frac{\delta \bar{g} L}{\delta g_{ij}} \partial_{g_{ij}} g_{ij} = \frac{\partial L}{\partial g_{ij}} , \frac{\partial L}{\partial g_{ij}} g_{ij} = - \frac{n}{2} L ,
$$

(99)

$L$ appears as a homogeneous function of $g_{ij}$ with a degree of homogeneity $m = - \frac{n}{2}$ (Euler’s theorem).

Explicit form of the covariant Euler-Lagrange equations for $V^A_B$ under the conditions $q_i = 0$ for the affine connections.

For $q_i = 0 : P^B = 0$ and we have:

$$
\delta_v L = \sqrt{-d_g} \delta_v L = \sqrt{-d_g} \frac{\delta \bar{V}^A_B}{\delta V^A_B} \delta V^A_B + v^m \delta g_{ij} ,
$$

(100)

$$
\frac{\delta \bar{V}^A_B}{\delta V^A_B} = \frac{\partial L}{\partial V^A_B ; i} - \frac{\partial L}{\partial V^A_B ; i ; j} \delta V^A_B ; j : i = 0 .
$$

(101)

23
3.2 Canonical Euler-Lagrange’s equations

The functional variation of the Lagrangian density \(L\) with respect to \(g_{ij}\) and \(V^{A\,B}\) can be presented in the form

\[
\delta L = \delta_g L + \delta_v L = \delta_g L + \frac{\delta_v L}{\delta V^{A\,B}} \delta V^{A\,B} + j^i_{\,;i},
\]

where

\[
j^i = g^i_j + v^i_j.
\]

The Stokes theorem can be applied for the common divergency \(j^i_{\,;i}\) in \((\mathcal{T}_n, g)\)-space (dim \(M = n\)). After introducing the boundary conditions

\[
\delta g_{kl}|_{(V_n)} = 0, \quad \delta V^{A\,B}|_{(V_n)} = 0,
\]

\[
(\delta g_{kl})_{;i}|_{(V_n)} = 0 \cong (\delta g_{kl})_{,i}|_{(V_n)} = 0, \quad (\cong \text{because of } \delta g_{kl}|_{(V_n)} = 0 \text{ and } \delta \Gamma^i_{jk} = 0),
\]

\[
(\delta V^{A\,B})_{;i}|_{(V_n)} = 0 \cong (\delta V^{A\,B})_{,i}|_{(V_n)} = 0, \quad (\cong \text{because of } \delta V^{A\,B}|_{(V_n)} = 0 \text{ and } \delta \Gamma^i_{jk} = 0),
\]

for the arbitrary \(\delta g_{kl}\) and \(\delta V^{A\,B}\) and their (first) covariant (or partial) derivatives on the shell \((V_n)\) of the volume \(V_n\), the canonical Euler-Lagrange equations for \(g_{kl}\) and \(V^{A\,B}\) can be found in the form identical to the form of the covariant Euler-Lagrange equations \(^{11}\). The result does not depend on the form-invariant covariant differential operators \(\nabla_u\) and \(\tilde{\nabla}_u\) acting on \(L\) (the definitions of \(\nabla_u\) and \(\tilde{\nabla}_u\) are given below).

If additional conditions are imposed on the affine connections for obtaining a common divergency term, then the canonical Euler-Lagrange equations can be found in the forms respectively:

\[
\frac{\delta_j L}{\delta g_{kl}} = 0, \quad \frac{\delta_v L}{\delta V^{A\,B}} = 0.
\]

where

\[
\frac{\delta_j L}{\delta g_{kl}} = \frac{\partial L}{\partial g_{kl}} - \left(\frac{\partial L}{\partial g_{kl;m}}\right)_{m} - \left(\frac{\partial L}{\partial g_{kl;m:n}}\right)_{n:m} = 0,
\]

\[
\frac{\delta_v L}{\delta V^{A\,B}} = \frac{\partial L}{\partial V^{A\,B}} - \left(\frac{\partial L}{\partial V^{A\,B;k}}\right)_{k} + \left(\frac{\partial L}{\partial V^{A\,B;k;l}}\right)_{l;k} = 0.
\]

The representation of the covariant divergency \(j^i_{\,;i}\) as a common divergency term \(j^i_{\,;i}\) and additional terms depends on the form-invariant covariant differential operator by the use of which the covariant divergency \(j^i_{\,;i}\) has been represented. There exist two different form-invariant covariant differential operators \(\nabla_u\) and \(\tilde{\nabla}_u\) \([u \in T(M)]\) (see the subsection below about these operators) acting on covariant tensor densities \(\tilde{Q} = (\delta_o)^\omega Q\) with \(\omega = q \in R\) and \(Q \in \otimes^k (M)\) of the type of \(j^i = \sqrt{-g} \cdot j^i\) (here \(Q = j = j^i, \partial_\ell = j^\alpha, e_\alpha, \omega = q = \frac{1}{2}\)). The covariant divergency of \(j^i\) will have the forms

(a) under the action of \(\nabla_u\):
\[ j^i;_i = j^i;_i + (P^k_{ik} + \Gamma^k_{ik})j^i = j^i;_i + (T^k_{ik} - C^k_{ik} + g^k_ig^k_j)j^i \]

(in a non-coordinate basis),

\[ j^i;_i = j^i;_i + (P^k_{ik} + \Gamma^k_{ik})j^i = j^i;_i + (T^k_{ik} + g^k_ig^k_j)j^i \]

(in a coordinate basis),

\[ \tilde{Q} = (d_K)_{\omega}Q, \]

where \( d_K = \det(K) = \det(K^\alpha^\beta) \) (in a coordinate basis), \( d_K = \det(K^\alpha^\beta) \) (in a non-coordinate basis), \( d_K \neq 0, Q \in \otimes^k \mathfrak{L}(M), \det(K^\alpha^\beta) = J^2, \det(K^\alpha^\beta'), \det(K^\alpha^\beta) = J^{-2}, \det(K^\alpha^\beta), J = \det(A_{\alpha^\alpha}) = \det(\partial x^i/\partial x^i) \) (in a coordinate basis), \( J^{-1} = \det(A_{\alpha^\alpha}), J^{-1} = J^{-1}, J = 1. \)

\[ \tilde{Q} = (d_K)_{\omega}Q, \]

where \( d_K = \det(K) = \det(K_{ij}) \) (in a coordinate basis), \( d_K = \det(K_{ij}) \) (in a non-coordinate basis), \( d_K \neq 0, Q \in \otimes^k \mathfrak{L}(M), \det(K_{ij}) = J^{-2}, \det(K_{ij}), \det(K_{ij}) = J^{-2}, \det(K_{ij}). \)

The transformation properties of tensor densities are determined by the transformation properties of their constructing quantities:

\[ \tilde{Q}' = (d_K')_{\omega}Q' = J^{-2\omega}(d_K)_{\omega}Q = J^{-2\omega}\tilde{Q}, \]

\[ \tilde{Q}' = (d_K')_{\omega}Q' = J^{2\omega}(d_K)_{\omega}Q = J^{2\omega}\tilde{Q}, \]

3.3 Tensor densities (relative tensor fields) over \((\mathcal{T}_n, g)\)-spaces. Form-invariant covariant differential operators for tensor densities

\[ m_{ij} = n_{ij} \quad (\text{the number of the matrix elements is equal to the square of the dimension of the manifold.}) \]

It is assumed that these tensor fields of second rank are symmetric or anti-symmetric tensor fields.

2. Contravariant and covariant tensor densities. For the construction of a contravariant or covariant tensor densities one can use the determinant (and its powers) of the components of contravariant or covariant tensor fields of second rank. It is assumed that these tensor fields of second rank are symmetric or anti-symmetric tensor fields.

**Definition 4.** Covariant tensor density \( \tilde{Q} \) with weight \( \omega \) is the product of the tensor field \( Q \) with the determinant \( d_K \) to the power \( \omega \) \((\omega \in R)\),

\[ \tilde{Q} = (d_K)_{\omega}Q, \]

where \( d_K = \det(K) = \det(K^\alpha^\beta) \) (in a coordinate basis), \( d_K = \det(K^\alpha^\beta) \) (in a non-coordinate basis), \( d_K \neq 0, Q \in \otimes^k \mathfrak{L}(M), \det(K^\alpha^\beta) = J^{-2}, \det(K^\alpha^\beta'), \det(K^\alpha^\beta) = J^{-2}, \det(K^\alpha^\beta), J = \det(A_{\alpha^\alpha}) = \det(\partial x^i/\partial x^i) \) (in a coordinate basis), \( J^{-1} = \det(A_{\alpha^\alpha}), J^{-1} = J^{-1}, J = 1. \)

**Definition 5.** Covariant tensor density \( \tilde{Q} \) with weight \( \omega \) is the product of the tensor field \( Q \) with the determinant \( d_K \) to the power \( \omega \) \((\omega \in R)\),

\[ \tilde{Q} = (d_K)_{\omega}Q, \]

where \( d_K = \det(K) = \det(K_{ij}) \) (in a coordinate basis), \( d_K = \det(K_{ij}) \) (in a non-coordinate basis), \( d_K \neq 0, Q \in \otimes^k \mathfrak{L}(M), \det(K_{ij}) = J^{-2}, \det(K_{ij}), \det(K_{ij}) = J^{-2}, \det(K_{ij}). \)

The transformation properties of tensor densities are determined by the transformation properties of their constructing quantities:
1. for covariant tensor densities

\[ \tilde{\nabla} = J^{-2\omega} \nabla = J^{-2\omega}.(d\Omega)^{\omega}.Q \] and \( \tilde{\nabla}' = J^{2\omega}.\tilde{\nabla} = J^{2\omega}.(d\Omega)^{\omega}.Q, \)

\[ Q = Q_{B}.e_{A} \otimes e^{B}, \]

it follows that \( \nabla_{u}\tilde{\nabla}' = -2\omega.[u(\log J)].\tilde{\nabla}' + J^{-2\omega}.\nabla_{u}\tilde{\nabla}, \) \( \nabla_{u}\tilde{\nabla}' = 2\omega.[u(\log J)].\tilde{\nabla}' + J^{2\omega}.\nabla_{u}\tilde{\nabla}, \) and that \( \nabla_{u}\tilde{\nabla} \) and \( \nabla_{u}\tilde{\nabla}' \) do not transform as tensor densities. Therefore, the result of the action of the covariant differential operator on a tensor density is not a tensor density with respect to its transformation properties. New covariant operators have to be constructed mapping a tensor density in another tensor density of the same kind. The determination of such form-invariant covariant differential operators acting on a tensor density and preserving its type and weight, i.e., mapping a density in a tensor density of the same type, is possible but not unique.

**Definition 6.** Form-invariant covariant differential operator of type 1. for contravariant tensor densities \( \tilde{\nabla} \). Covariant differential operator \( \nabla_{u} : \nabla_{u} = \nabla_{u} + 2\omega.\Gamma_{\beta\gamma}^{\alpha}.u^{\gamma} \) (in a non-co-ordinate basis), \( \nabla_{u} = \nabla_{u} + 2\omega.\Gamma_{ik}^{l}.u^{k} \) (in a co-ordinate basis): \( \nabla_{u}\tilde{\nabla} = \tilde{\nabla}_{A B} \Gamma^{A}_{B k} .u^{k} .\partial_{A} \otimes dx^{B} \), \( \tilde{\nabla} = \tilde{\nabla}^{A}_{B k} .\partial_{A} \otimes dx^{B} \),

\[ \tilde{\nabla}^{A}_{B k} = (d\Omega)^{\omega}.Q^{A}_{B k} + [(d\Omega)^{\omega}]_{k} .Q^{A}_{B} = \tilde{\nabla}^{A}_{B k} .\Gamma^{A}_{C k} .Q^{C}_{B} + P^{D}_{B k} .\tilde{\nabla}^{A}_{D} + 2\omega.\Gamma_{ik}^{l} .Q^{A}_{B} \]

\( \tilde{\nabla}^{A}_{B k} \) are called components of the covariant derivative of type 1. of the tensor density \( \tilde{\nabla} \) in a co-ordinate basis.

**Definition 7.** Form-invariant covariant differential operator of type 2. for contravariant tensor densities \( \tilde{\nabla} \). Covariant differential operator \( \nabla_{u} : \nabla_{u} = \nabla_{u} + 2\omega.\Gamma_{\beta\gamma}^{\alpha} + C_{\gamma}^{\beta \delta}.u^{\gamma} \) (in a non-co-ordinate basis), \( \nabla_{u} = \nabla_{u} + 2\omega.\Gamma_{ik}^{l}.u^{k} \) (in a co-ordinate basis):

\[ \nabla_{u}\tilde{\nabla} = \tilde{\nabla}^{A}_{B k} .\partial_{A} \otimes dx^{B} \], \( \tilde{\nabla} = \tilde{\nabla}^{A}_{B k} .\partial_{A} \otimes dx^{B} \),

\[ \tilde{\nabla}^{A}_{B k} = (d\Omega)^{\omega}.Q^{A}_{B k} + [(d\Omega)^{\omega}]_{k} .Q^{A}_{B} = \tilde{\nabla}^{A}_{B k} .\Gamma^{A}_{C k} .Q^{C}_{B} + P^{D}_{B k} .\tilde{\nabla}^{A}_{D} + 2\omega.\Gamma_{ik}^{l} .Q^{A}_{B} \]

\( \tilde{\nabla}^{A}_{B k} \) are called components of the covariant derivative of type 2. of the tensor density \( \tilde{\nabla} \) in a co-ordinate basis.

**Definition 8.** The form-invariant covariant differential operator of type 1. for covariant tensor densities \( \tilde{\nabla} \). Covariant differential operator \( \nabla_{u} : \nabla_{u} = \nabla_{u} + 2\omega.\Gamma_{\beta\gamma}^{\alpha}.u^{k} \) (in a co-ordinate basis), \( \nabla_{u} = \nabla_{u} + 2\omega.\Gamma_{ik}^{l}.u^{k} \) (in a co-ordinate basis):

\[ \nabla_{u}\tilde{\nabla} = \tilde{\nabla}^{A}_{B k} .u^{k} .\partial_{A} \otimes dx^{B} \], \( \tilde{\nabla} = \tilde{\nabla}^{A}_{B k} .\partial_{A} \otimes dx^{B} \),

\[ \tilde{\nabla}^{A}_{B k} = (d\Omega)^{\omega}.Q^{A}_{B k} + [(d\Omega)^{\omega}]_{k} .Q^{A}_{B} = \tilde{\nabla}^{A}_{B k} .\Gamma^{A}_{C k} .Q^{C}_{B} + P^{D}_{B k} .\tilde{\nabla}^{A}_{D} + 2\omega.\Gamma_{ik}^{l} .Q^{A}_{B} \]

\( \tilde{\nabla}^{A}_{B k} \) are called components of the covariant derivative of type 1. of the tensor density \( \tilde{\nabla} \) in a co-ordinate basis.

**Definition 9.** The form-invariant covariant differential operator of type 2. for covariant tensor densities of the type of \( \tilde{\nabla} \) is the covariant differential
operator $\nabla_u : \nabla_u = \nabla_u + 2\omega(P^{\gamma}_{\gamma \beta} - C^\gamma_{\gamma \beta})u^\gamma$ (in a non-co-ordinate basis), $\nabla_u = \nabla_u + 2\omega.P^k_{kl}u^k$ (in a co-ordinate basis); $\bar{\nabla}_u \bar{Q} = \bar{Q}^{A}_{B;k}.\partial_A \otimes dx^B$, $\bar{Q} = \bar{Q}^{A}_{B;k}.\partial_A \otimes dx^B$.

$$\bar{Q}^{A}_{B;k} = \sum (d^k)^{\omega}.Q^A_{B;k} + [(d^k)^{\omega}]_{;k}.Q^A_{B} + 2\omega.U^l_{kl}.Q^A_{B} = \bar{Q}^{A}_{B;k} + \Gamma^A_{ck}.\bar{Q}^C_{B} + P^D_{Bk}.\bar{Q}^A_{D} + 2\omega.P^l_{kl}.Q^A_{B}.$$  

$\bar{Q}^{A}_{B;k}$ are the components of the covariant derivative of type 2. of the covariant tensor density $\bar{Q}$ in a co-ordinate basis.

The properties of the form-invariant covariant differential operators are determined by the properties of the covariant differential operator and their construction.

### 3.4 Necessary and sufficient conditions for the application of the Stokes theorem as conditions on the affine connections

By means of the different form-invariant covariant differential operators different necessary and sufficient conditions can be found for the application of the theorem of Stokes allowing the use of the boundary conditions on the shell $(V_n)$ of the volume $V_n$.

(a) Under the action of $\nabla_u$ the necessary and sufficient conditions for the existence of the relation $j^i_{;i} = j^i_{;i}$ for $\forall j^i$ are the conditions

$$\bar{t}_i = P^k_{ki} + \Gamma^k_{ik} = T^k_{ki} - C^k_{ik} + g^k_{ik}g^l_{ki} = 0 \text{ (in a non-co-ordinate basis)},$$

$$\bar{q}_i = P^k_{ki} + \Gamma^k_{ik} = T^k_{ki} + g^k_{ik}g^l_{ki} = 0 \text{ (in a co-ordinate basis).}$$

(b) Under the action of $\nabla_u$ the necessary and sufficient conditions for the existence of the relation $j^i_{;i} = j^i_{;i}$ for $\forall j^i$ are the conditions

$$\bar{q}_i = P^k_{ki} + \Gamma^k_{ik} - C^k_{ik} = 0 \text{ (in a non-co-ordinate basis)},$$

$$\bar{q}_i = P^k_{ki} + \Gamma^k_{ik} = 0 \text{ (in a co-ordinate basis).}$$

Under the conditions in a co-ordinate basis the Stokes theorem can be applied for the common divergency $j^i_{;i}$ in $(\mathbb{L}_n, g)$-space $(\dim M = n)$.

**Explicit form of the canonical Euler-Lagrange equations for $g_{ij}$**

$$\delta_g L = \frac{\delta L}{\delta g_{kl}}.\delta g_{kl} + g_{ij} \partial_{g_{ij}} + g_{ij},$$  

$$\delta_{g_{ij}} L = 0 ,$$

$$g_{ij} = \frac{\partial L}{\partial g_{kl;m;n}} + \left(\frac{\partial L}{\partial g_{kl;m;n}}\right).\delta g_{kl} + \left(\frac{\partial L}{\partial g_{kl;m;n}}\right).\delta g_{kl;m;n}.$$
Explicit form of the canonical Euler-Lagrange equations for $V^A_B$

$$\delta_v L = \frac{\delta_v L}{\delta V^A_B} \delta V^A_B + v^i_A ;_i , \quad (115)$$

$$\frac{\delta_v L}{\delta V^A_B} = 0 , \quad (116)$$

$$v^i_A = \left[ \frac{\partial L}{\partial V^A_B ;_i} - \left( \frac{\partial L}{\partial V^A_B ;_i;k} + \frac{\partial L}{\partial V^A_B ;_i;k} \right) \right] \delta V^A_B \delta V^A_B + \left( \frac{\partial L}{\partial V^A_B ;_i;k} \right) \delta V^A_B , \quad (117)$$

Remark. For $(L_n, g)$-spaces and $\tilde{\nabla}_u = \nabla_u - 2q \cdot \Gamma_{ki}^i u^k$ in a co-ordinate basis the relation $j^i_A ;_i = j^i_A$ is fulfilled and no additional conditions for $\Gamma = -P$ are needed (Lovelock and Rund 1975).

3.5 Non-equivalence between the covariant and the canonical Euler-Lagrange equations under the different conditions for the affine connections

The canonical Euler-Lagrange equations for $g_{ij}$ and $V^A_B$ can be expressed by the use of the covariant Euler-Lagrange equations under the different conditions for the affine connections. There exist two different types of relations between both types of E-L equations, depending on the different form-invariant covariant differential operators $\nabla_u$ and $\tilde{\nabla}_u$. Both operators determine different conditions on the affine connections for the application of the theorem of Stokes and along with it the use of the boundary conditions.

1. If the form-invariant covariant operator $\nabla_u$ is used, then the canonical E-L equations can be written in the forms respectively

$$\frac{\delta_v L}{\delta g_{ij}} = 0 \Rightarrow \frac{\delta_v L}{\delta g_{ij}} = -\frac{1}{2} L \cdot g^{ij} + g^{kl} , \quad (118)$$

$$\frac{\delta_v L}{\delta V^A_B} = 0 \Rightarrow \frac{\delta_v L}{\delta V^A_B} = v^A_B , \quad (119)$$

with the following relations:

(a) Relations between the covariant and canonical Euler-Lagrange equations for $g_{ij}$

$$\frac{\delta_v L}{\delta g_{kl}} = \sqrt{-d_g} \cdot \left( \frac{\delta g}{\delta g_{kl}} - g^{kl} \right) = \sqrt{-d_g} \cdot \left( \frac{\delta g}{\delta g_{kl}} + \frac{1}{2} L \cdot g^{kl} - g^{kl} \right) , \quad (120)$$

$$g^{kl} = \frac{1}{2} \left[ \frac{\partial L}{\partial g_{kl;m;n}} + \frac{\partial L}{\partial g_{kl;m;n}} \right] \left[ Q_m + \frac{1}{2} Q_n - \frac{1}{2} Q_n \cdot Q_m \right] , \quad (121)$$

$$Q_i = \sqrt{-d_g} \cdot g_{ki}^{ik} = \left[ \log(-d_g) \right] ;_i , \quad g^{kl} = g^{lk} . \quad (122)$$

The relation between $g^i_A$ and $g^i_A$ can be found in the form

$$g^i_A = -d_g \cdot g^i_A + g^{ih} . \quad (123)$$
$$g\overline{D}^{i} = \frac{1}{2} \frac{\partial L}{\partial g_{kl;i;m}} Q_{m} \delta g_{kl} = g^{kl} \cdot \delta g_{kl}, \quad (124)$$

$$g\overline{D}^{kl} = \frac{1}{2} \frac{\partial L}{\partial g_{kl;i;m}} Q_{m}, \quad (125)$$

and the identity for $g\overline{D}^{k}$

$$g\overline{D}^{k} = \frac{1}{2} Q_{k} \cdot (g^{j} - g\overline{D}^{j}) - g\overline{D}^{kl} \cdot \delta g_{kl} \quad (126)$$

is valid.

(b) Relation between the covariant and canonical Euler-Lagrange equations for $V^{A}_{B}$

$$\frac{\delta v^{L}}{\delta V^{A}_{B}} = \sqrt{-d_{g}} (\frac{\delta v^{L}}{\delta V^{A}_{B}} - v\overline{D}^{A}_{B}), \quad (127)$$

$$v\overline{D}^{A}_{B} = \frac{1}{2} \left[ \frac{\partial L}{\partial V^{A}_{B;i;k}} Q_{i} - \frac{\partial L}{\partial V^{A}_{B;i;k}} (Q_{k;} + \frac{1}{2} Q_{k} Q_{i}) - Q_{i} (\frac{\partial L}{\partial V^{A}_{B;k;i}} + \frac{\partial L}{\partial V^{A}_{B;k;i}}) \right]. \quad (128)$$

The relation between $v^{j}_{i}$ and $v^{j}_{i}$ can be expressed in the form

$$v^{j}_{i} = \sqrt{-d_{g}} (v^{j}_{i} - v\overline{D}^{i}), \quad (129)$$

$$v\overline{D}^{i} = \frac{1}{2} Q_{k} \cdot \frac{\partial L}{\partial V^{A}_{B;i;k}} \delta V^{A}_{B}, \quad (130)$$

and the identity

$$v\overline{D}^{k} : k = \frac{1}{2} Q_{k} (v^{j}_{k} - v\overline{D}^{j}) - v\overline{D}^{A}_{B} \cdot \delta V^{A}_{B} \quad (131)$$

is valid.

2. If the form-invariant covariant operator $\overline{\nabla}_{a}$ is used, then the canonical E-L equations can be written in the forms respectively

$$\frac{\delta v^{L}}{\delta g_{ij}} = \left( \frac{\delta v^{L}}{\delta g_{ij}} - g\overline{D}^{ij} \right) = \sqrt{-d_{g}} \left( \frac{\delta v^{L}}{\delta g_{ij}} - \frac{1}{2} L \cdot g^{ij} - g\overline{D}^{ij} \right), \quad (132)$$

$$\frac{\delta v^{L}}{\delta V^{A}_{B}} = \left( \frac{\delta v^{L}}{\delta V^{A}_{B}} - v\overline{D}^{A}_{B} \right) = \frac{1}{2} Q_{k} \cdot \frac{\partial L}{\partial V^{A}_{B;i;k}} \delta V^{A}_{B}, \quad (133)$$

with the following relations:

(a) Relations between the covariant and canonical Euler-Lagrange equations for $g_{ij}$

$$\frac{\delta v^{L}}{\delta g_{kl}} = \sqrt{-d_{g}} \left( \frac{\delta v^{L}}{\delta g_{kl}} - g\overline{D}^{kl} \right) = \sqrt{-d_{g}} \left( \frac{\delta v^{L}}{\delta g_{kl}} + \frac{1}{2} L \cdot g^{kl} - g\overline{D}^{kl} \right), \quad (134)$$

$$g\overline{D}^{kl} = \frac{1}{2} Q_{m} - U_{rn} \right) \cdot \frac{\partial L}{\partial g_{kl;m}} - \frac{1}{2} Q_{n} - U_{rn} \right) \cdot \frac{\partial L}{\partial g_{kl;m;n}} - \frac{\partial L}{\partial g_{kl;m;n}} \quad (135)$$
\[-\frac{1}{2}Q_m - U_{rm}^r = \frac{1}{2}
abla \cdot (Q_n - U_{m}^n \nabla L) + \frac{\partial L}{\partial g_{kl;m;n}} + \frac{\partial L}{\partial g_{kl;m;n}}\nabla L = g_{ik}^{ik},\]
\[Q_i = g^{ik}g_{kl;i} = \log(-d_g)_{ij}, U_{ik}^k = P_{ik}^k - P_{il}^k \quad (-C_{ik}^k), U_{rm}^r = g_{ik}^k U_{rm}^k.\]

The relation between \(g_j^i\) and \(g_j^i\) can be found in the form
\[g_j^i = \sqrt{-d_g}(g_j^i - g_{ik}^k),\]
\[\bar{D}_j^i = \frac{1}{2}Q_m - U_{rm}^r \frac{\partial L}{\partial g_{kl;i}^k} \delta g_{kl},\]
\[\bar{D}_j^k \equiv \frac{1}{2}Q_m - U_{mk}^m \frac{\partial L}{\partial g_{kl;i}^k} \delta g_{kl}(140),\] is valid.

(b) Relation between the covariant and canonical Euler-Lagrange equations for \(V^A_B\)
\[\frac{\delta \mathcal{L}}{\delta V^A_B} = \sqrt{-d_g}(\frac{\delta \mathcal{L}}{\delta V^A_B} - v_{\bar{D}_A^B}),\]
\[v_{\bar{D}_A^B} = \frac{1}{2}Q_i - U_{ri}^i \frac{\partial L}{\partial V^A_B} - (\frac{1}{2}Q_i - U_{ik}^i)\nabla D^i = \frac{1}{2}Q_i - U_{mk}^m \frac{\partial L}{\partial V^A_B} \delta V^A_B + \frac{\partial L}{\partial V^A_B} \delta V^A_B,\]
\[-(\frac{1}{2}Q_i - U_{ri}^i)\nabla D^i = \frac{1}{2}Q_i - U_{mk}^m \frac{\partial L}{\partial V^A_B} \delta V^A_B,\] is valid.

It is obvious (compare for instance 108 and 109 with 83 for the case of \(\nabla\)) that in \((L_n, g)\)-spaces the covariant Euler-Lagrange equations are different from the canonical Euler-Lagrange equations if the explicit form of \(j^i\) and \(\bar{j}^i\) are not taken into account and only general conditions for every \(j^i\) or for every \(\bar{j}^i\) are imposed. The difference is a corollary of using different terms of \(\delta L\) and different conditions for \(\Gamma\) and \(P\) in both ways for obtaining common divergency.
term necessary for applying the Stokes theorem. Both type of equations are identical for $V_n$ and $U_n$-spaces, where $g_{ij;k} = 0$, $(Q_i = g^{kl} g_{kl;i} = 0)$ and therefore $g^{ij} D_j = 0$ and $\nu D_A B = 0$. In $(L_n, g)$-spaces they would be equivalent in a coordinate basis, if $g^{ij} g_{kl} ; i = 0$. On the other side, it is obviously that, because of the different conditions for $\Gamma$ and $P$, in a $(L_n, g)$-space in general only one type of Euler-Lagrange’s equations (canonical or covariant) could be used. Therefore, the unique way for avoiding ambiguities is the use of the covariant Euler-Lagrange equations without using additional conditions on the affine connections.

Remark. The dilemma of choosing the right type of the Euler-Lagrange equations can be easily solved in the case of $(L_n, g)$-spaces by using only the canonical E-L equations and the definition of $\tilde{\nabla} u$ (which in this case requires no additional conditions for the affine connection $\Gamma = -P$).

4 Energy-momentum tensors

The energy-momentum tensors can be determined using the method of Lagrangians with covariant derivatives which leads to the covariant Noether identities.

4.1 Lie variation of the Lagrangian density $L$

Let a Lagrangian density of the type

$$L = \sqrt{-g} \left[ L(g_{ij}, g_{ij;k}, V^A_B, V^A_B; V_{A B}; i, j) \right]$$

be given. The action of the system is defined as

$$S = \int_{V_n} L \, d^{(n)} x = \int_{V_n} L \, d\omega ,$$

(146)

where

$$\omega^A = \varepsilon_{i_1 \ldots i_n} \omega^i = dx^i_1 \wedge \ldots \wedge dx^i_n , \,
L \, d\omega = \int_{V_n} L \, d^{(n)} x = \int_{V_n} L \, d\omega = \int_{V_n} L \, d^{(n)} x = \int_{V_n} L \, d\omega = \int_{V_n} L \, d^{(n)} x = \int_{V_n} L \, d\omega = \int_{V_n} L \, d^{(n)} x = \int_{V_n} L \, d^{(n)} x = \int_{V_n} L \, d\omega.$$
4.2 Form-invariant Lie differential operators for tensor densities

The determination of Lie differential operators acting on a tensor density and preserving its type and weight, i.e., mapping a density in a tensor density of the same type, is possible but not unique.

**Lie derivatives of tensor densities.** The Lie differential operator acts on contravariant and covariant tensor densities in accordance with its action on functions and tensor fields over a manifold $M$:

$$L_\xi \overline{Q} = (d\overline{\chi})^\omega \cdot L_\xi Q + \omega [\nabla_\xi (\log d\overline{\chi})] \overline{Q} - 2\omega \Gamma^i_{jk} \xi^k \overline{Q},$$

$$L_\xi \tilde{Q} = (d\overline{\chi})^\omega \cdot L_\xi Q + \omega [\nabla_\xi (\log d\overline{\chi})] \tilde{Q} - 2\omega \beta_{\beta'} \xi^\beta \tilde{Q}.$$  

The transformation properties of $L_\xi \overline{Q}$ and $L_\xi \tilde{Q}$ are determined by the construction of $\overline{Q}$ and $\tilde{Q}$, their transformation properties and the transformation properties of the derivatives of the Jacobian $J$. From $\overline{Q'} = J^{-2\omega} \overline{Q} = J^{-2\omega} (d\overline{\chi})^\omega \cdot Q$, $\omega = q \neq 0$, it follows that $L_\xi \overline{Q} = -2\omega [\xi (\log J)] \overline{Q} + J^{-2\omega} \cdot L_\xi \overline{Q}$. It is obvious that $L_\xi \overline{Q}$ does not transform as a tensor density, i.e., the condition $L_\xi \overline{Q} = J^{-2\omega} \cdot L_\xi \overline{Q}$ is not fulfilled in general. From $\tilde{Q'} = J^{2\omega} \tilde{Q} = J^{2\omega} (d\overline{\chi})^\omega \cdot Q = \tilde{Q} A^{B'} B' e_A A' \otimes e_B'$, it follows that $L_\xi \tilde{Q}' = 2\omega [\xi (\log J)] \tilde{Q}' + J^{2\omega} \cdot L_\xi \tilde{Q}$, and that $L_\xi \tilde{Q}$ does not transform as a covariant tensor density. It is necessary a new Lie differential operator to be constructed which action on a tensor density results again in a tensor density with the corresponding transformation properties.

**Definition 10.** Form-invariant Lie differential operator of type 1, for contravariant tensor densities of the type of $\overline{Q}$. Lie differential operator $\overline{L}_\xi :$

$$\overline{L}_\xi = L_\xi + 2\omega \Gamma^i_{jk} \xi^k - 2\omega (\xi^i_{~~;i} - T^i_{jk} \xi^k)$$ (in a co-ordinate basis).

The result of the action of the form-invariant Lie differential operator on a tensor density of the type of $\overline{Q}$ is called Lie derivative of type 1, of the tensor density $\overline{Q}$ along the contravariant vector field $\xi : \overline{L}_\xi \overline{Q} = \overline{L}_\xi [(d\overline{\chi})^\omega \cdot Q] = [L_\xi (d\overline{\chi})^\omega] \cdot Q + (d\overline{\chi})^\omega \cdot L_\xi Q = \overline{\chi} \overline{Q} A \cdot B \cdot D_A \otimes dx$,

$$\overline{L}_\xi \overline{Q} A \cdot B = \overline{Q} A \cdot B k \cdot k + (S(C_{i j} A) \overline{Q} C \cdot B - S(D_{i j} A) \overline{Q} D \cdot A) \cdot (\xi^i_{~~;i} - T^i_{kj} \xi^k) - 2\omega (\xi^i_{~~;i} - T^i_{kj} \xi^k) \overline{Q} A \cdot B.$$

$\overline{L}_\xi \overline{Q} A \cdot B$ are called components of the Lie derivative of type 1, of the tensor density $\overline{Q}$ in a co-ordinate basis.

**Definition 11.** Form-invariant Lie differential operator of type 2, for contravariant tensor densities of the type of $\overline{Q}$. Lie differential operator $\overline{L}_\xi :$

$$\overline{L}_\xi = L_\xi + 2\omega \Gamma^i_{jk} \xi^k - 2\omega \xi^i_{~~;i}$$ (in a co-ordinate basis).

The result of the action of the form-invariant Lie differential operator $\overline{L}_\xi$ on a tensor density $\overline{Q}$ is called Lie derivative of type 2, of the contravariant tensor density $\overline{Q}$ along the contravariant vector field $\xi : \overline{L}_\xi \overline{Q} = \overline{L}_\xi [(d\overline{\chi})^\omega \cdot Q] =$
\[(d\bar{K})^\omega \cdot \xi Q + [\xi (d\bar{K})^\omega]Q = 2\omega \cdot T^k_{jk} \xi^k \bar{Q} + \bar{\xi} Q\, A^\omega_B^A B_B^A \otimes dx^B,\]

\[\bar{\xi} Q\, A^\omega_B^A B_B^A \xi^k + (S_{Ci}^A, Q^C_B^C B - S_{Bi}^A D_t, Q^A_B^A D_t^A) (\xi^i_{;i} - T^i_{jk} \xi^k) - 2\omega \, \xi^i_{;i} \bar{Q}^A_B^A B_B^A.\]

\[\bar{\xi} Q\, A^\omega_B^A B_B^A \text{ are called components of the Lie derivative of type 2. of the tensor density } \bar{Q} \text{ in a co-ordinate basis.}\]

**Definition 12.** Form-invariant Lie differential operator of type 1. for covariant tensor densities of the type of \(Q\). Lie differential operator \(\bar{\xi} : \bar{\xi} = \xi + 2\omega \cdot P^i_{ik} \xi^k + 2\omega \cdot (\xi^i_{;i} - T^i_{jk} \xi^k)\) (in a co-ordinate basis).

The result of the action of the form-invariant differential operator \(\bar{\xi}\) on a tensor density \(\bar{Q}\) is called Lie derivative of type 1. of the covariant tensor density \(\bar{Q}\) along a contravariant vector field \(\bar{\xi}\) in a co-ordinate basis \([\text{compare with } (\text{Lovelock and Rund 1975 - p.124})]\).

\[\bar{\xi} Q\, A^\omega_B^A B_B^A = \bar{Q}^A_B^A B_B^A \xi^k + (S_{Ci}^A, \bar{Q}^C_B^C B - S_{Bi}^A D_t, \bar{Q}^A_B^A D_t^A) (\xi^i_{;i} - T^i_{jk} \xi^k) + 2\omega \, (\xi^i_{;i} - T^i_{jk} \xi^k). Q^A_B^A B_B^A.\]

\[\bar{\xi} Q\, A^\omega_B^A B_B^A \text{ are called components of the Lie derivative of type 1. of the tensor density } \bar{Q} \text{ in a co-ordinate basis.}\]

**Definition 13.** Form-invariant Lie differential operator of type 2. for covariant tensor densities of the type of \(Q\). Lie differential operator \(\bar{\xi} : \bar{\xi} = \xi + 2\omega \cdot P^i_{ik} \xi^k + 2\omega \cdot (\xi^i_{;i} \xi^k + (\xi^k_{;i} - T^i_{jk} \xi^k)\) (in a co-ordinate basis).

The result of the action of the form-invariant differential operator \(\bar{\xi}\) on a tensor density \(\bar{Q}\) is called Lie derivative of type 2. of the covariant tensor density \(\bar{Q}\) along a contravariant vector field \(\bar{\xi}\) in a co-ordinate basis \([\text{compare with } (\text{Lovelock and Rund 1975 - p.124})]\).

\[\bar{\xi} Q\, A^\omega_B^A B_B^A = \bar{Q}^A_B^A B_B^A \xi^k + (S_{Ci}^A, \bar{Q}^C_B^C B - S_{Bi}^A D_t, \bar{Q}^A_B^A D_t^A) (\xi^i_{;i} - T^i_{jk} \xi^k) + 2\omega \, (\xi^i_{;i} - T^i_{jk} \xi^k). Q^A_B^A B_B^A.\]

\[\bar{\xi} Q\, A^\omega_B^A B_B^A \text{ are called components of the Lie derivative of type 2. of the tensor density } \bar{Q} \text{ in a co-ordinate basis.}\]

It follows from \([14]\) that the Lie variation of a Lagrangian density \(L\) is related to the Lie derivative of type 1. of the tensor density \(\bar{Q}\) of the type of \(\bar{Q}\) (rank \(Q = 0, K = g, \omega = q = \frac{1}{2}\)).

**4.3 Covariant Noether’s identities**

By means of the Lie variation of \(L\)

\[\bar{\xi} L = \sqrt{-g} (L_{,i} \xi^i + \frac{1}{2} L \cdot g^k \bar{\xi} g_{ik}) \equiv \]

33
with an arbitrary contravariant vector field \( \xi = \xi^i \partial_i = \xi^k e_k \) and the explicit form of the Lie derivatives of the metric and non-metric tensor field’s components and their covariant derivatives (s. Appendix) the Lie identity can be written in the form

\[
\mathcal{P} = (\mathcal{P}_i + \mathcal{P}_k \, T^k_{ij}) \xi^i = \mathcal{P}_i, \xi^i \equiv 0 \quad \text{for} \quad \forall \xi^i, (149)
\]

or as the two identities \( \mathcal{P}_i = \mathcal{P}_i + \vartheta_i \, \xi^i : j \equiv 0 \) and \( \mathcal{P}_i \, \xi^i = 0 \).

The covariant Noether identities can be found in (an analogous to the case of \( U_n \)- and \( L_n - g \))-spaces (Manoff 1987, 1991) form

\[
\mathcal{P}_i = \mathcal{P}_i = \mathcal{P}_i + \vartheta_i \, \xi^i \equiv 0 \quad \text{(first covariant Noether’s identity), (150)}
\]

\[
\mathcal{Q}_i \equiv \mathcal{Q}_i \equiv \mathcal{Q}_i \equiv 0 \quad \text{second covariant Noether’s identity), (151)}
\]

where

\[
\begin{align*}
\mathcal{P}_i &= v \mathcal{P}_i + g \mathcal{P}_i, \\
\vartheta_i \, \xi^i &= v \vartheta_i \, \xi^i + g \vartheta_i \, \xi^i, \\
\mathcal{T}_i \, \xi^i &= v \mathcal{T}_i \, \xi^i + g \mathcal{T}_i \, \xi^i, \\
\mathcal{Q}_i \, \xi^i &= v \mathcal{Q}_i \, \xi^i + g \mathcal{Q}_i \, \xi^i.
\end{align*}
\]

The quantities \( v, \mathcal{P}_i, \mathcal{T}_i \, \xi^i, \mathcal{Q}_i \) are the corresponding quantities to the components \( V^A_B \) of the non-metric tensors \( V \). The quantities \( g \mathcal{P}_i \) are the corresponding quantities to the components \( g_{ij} \) of the metric tensor field \( g \).

### 4.4 Canonical, generalized canonical, symmetric (of Belinfante) and variational (of Euler-Lagrange) energy-momentum tensors

\( \mathcal{P}_i \) is the generalized canonical energy-momentum tensor (GC-EMT) constructed as a sum of the GC-EMT \( \vartheta_i \) of the non-metric tensor fields \( V \) and the GC-EMT \( g \mathcal{P}_i \) of the metric tensor \( g \).

\[
\begin{align*}
v \vartheta_i \, \xi^i &= v \vartheta_i \, \xi^i - v K_i \, j - v W_j \, k, \\
g \vartheta_i \, \xi^i &= g \vartheta_i \, \xi^i - g K_i \, j - g W_j \, k.
\end{align*}
\]

\( v \vartheta_i \) and \( g \vartheta_i \) are the canonical energy-momentum tensors (C-EMT-s) of \( V \) and \( g \) correspondingly. When \( L \) is depending only on \( V^A_B, g_{ij} \) and their first partial derivatives, \( \mathcal{T}_i \, \xi^i \) has been symmetrized by means of the a priori introduced s.c. Belinfante terms to the symmetric EMT of Belinfante (Schmutzer 1968). Here \( v W_j \, k \) and \( g W_j \, k \) consist of the generalized Belinfante terms for the non-metric tensor fields \( V \) and for the metric \( g \) respectively. They appear
in the construction of $\bar{\theta}^i_j$ in a very natural way as a result of a symmetrization procedure.

$sT^i_j$ is the symmetric energy-momentum tensor of Belinfante (S-EMT-B) constructed as a sum of the S-EMT-B $v_sT^i_j$ for the non-metric tensor fields $V$ and the S-EMT-B $g_sT^i_j$ for the metric $g$, where (s. Appendix)

$$v_sT^i_j = v^i_j - g^i_jL, \quad g_sT^i_j = g^i_j - g^i_jL.$$ (154)

$\overline{Q}^i_j$ is the variational energy-momentum tensor of Euler-Lagrange (V-EMT-EL) constructed as a sum of the V-EMT-EL $vQ^i_j$ for the tensor fields $V$ and the V-EMT-EL $gQ^i_j$ for the metric $g$.

If the covariant Euler-Lagrange equations (E-L equations) for $V^A^B$ are of the type

$$\frac{\delta_vL}{\delta V^A^B} = 0, \quad \text{and} \quad \frac{\delta_gL}{\delta g_{ij}} = 0 \quad \text{for} \quad g_{ij},$$ (155)

then $\overline{Q}^i_j = 0$ and

$$\overline{\theta}^i_j = sT^i_j.$$ (156)

But one has to take into account that the covariant E-L equations for the metric $g$ and the non-metric tensor fields $V$ in $(\mathcal{L}_n, g)$-spaces are not the equations in (155) but the equations

$$\frac{\delta_gL}{\delta g_{ij}} = -\frac{1}{2}L_{g}^{ij} - P_{ij}, \quad \frac{\delta_vL}{\delta V^A^B} = -P^A^B.$$ (157)

Remark. The explicit forms of the EMT-s and their corresponding quantities are given in the Appendix. The explicit form of $\overline{Q}^i_j$ shows the role of the covariant Euler-Lagrange equations in its structure.

4.5 Symmetric energy-momentum tensor of Hilbert

The symmetric energy-momentum tensor of Hilbert (S-EMT-H) for the components of non-metric tensor fields $V$ is determined in $V_n$-spaces (where $S = C$ and $f^i_j = g^i_j$, $\gamma^i_{jk} = g_{ik}$, $g_{jk} = g_{jk}$, $vshT^i_j = vshT^{kj}$) as (Schmutzer 1968)

$$vshT^i_j = -\frac{2}{\sqrt{-d_g}}\frac{\delta_gL}{\delta g_{jk}}g_{ik}^{\gamma} = g_{ik}^{\gamma}vshT^{kj}.$$ (158)

The same definition could be accepted for $vshT^i_j$ in $(\mathcal{L}_n, g)$- and $(\mathcal{T}_n, g)$-spaces as a generalizations of that one in $V_n$-spaces.

By means of the explicit form of $\delta_gL/\delta g_{jk}$ and the form-invariant covariant differential operator $\nabla_u$, $vshT^i_j$ can also be written in the forms

$$vshT^i_j = g\overline{Q}^i_j - g^i_jL + 2gD^j_k\gamma^i_{jk} = g_sT^i_j - g^i_jL + 2gD^j_k\gamma^i_{jk} = v_sT^i_j - v^i_jL + 2gD^j_k\gamma^i_{jk}.$$ (159)
The last two relations are obtained using the connection between \( g_i^j L \) and \( v_{sT_i}^j \) or \( g_{sT_i}^j \) in [54].

**Non-equivalence between the symmetric energy-momentum tensor of Belinfante and the symmetric energy-momentum tensor of Hilbert**

It is obvious from [59] that there is in general no equivalence between the S-EMT-H and the S-EMT-B. The connection between \( v_{sT_i}^j \) and \( v_{shT_i}^j \) is based only on the common for both tensors term \( g_i^j L \). The same is valid to the connection between \( g_{sT_i}^j \) and \( v_{shT_i}^j \). By means of [59] one can easily prove the following proposition:

**Proposition 2.** The necessary and sufficient conditions for the equivalence between the S-EMT-B and the S-EMT-H \( \left( v_{sT_i}^j = v_{shT_i}^j \right) \) are the conditions

\[
 v_{sT_i}^j = g_{Q_i^j} + 2 g_{D_i^j} g_{k^j}.
\]

Therefore, \( v_{sT_i}^j \) is not equal in general to \( v_{shT_i}^j \). This fact has to be taken into account when energy-momentum problems are considered in Lagrangian theories for tensor fields in \((\mathcal{T}_n, g)\)-spaces.

**Non-equivalence between the generalized canonical energy–momentum tensor and the symmetric energy-momentum tensor of Hilbert**

The second covariant Noether identity \( \bar{\theta_i}^j = \theta_i^j \equiv g_{Q_i^j} \), written in the form

\[
 v_{\bar{\theta_i}^j} = g_{\theta_i^j} + (v_{sT_i}^j + g_{sT_i}^j) = v_{Q_i^j} + g_{Q_i^j},
\]

can be expressed by means of the relation

\[
 g_{Q_i^j} = v_{shT_i}^j + g_i^j L - 2 g_{D_i^j} g_{k^j}.
\]

It is obvious that there is in general no equivalence between the GC-EMT and the S-EMT-H. From the last equality it follows immediately the proposition:

**Proposition 3.** The necessary and sufficient conditions for the equivalence between the GC-EMT and the S-EMT-H \( \left( v_{\bar{\theta_i}^j} = v_{shT_i}^j \right) \) are the conditions

\[
 v_{\bar{\theta_i}^j} = g_{\theta_i^j} + g_{sT_i}^j + 2 g_{D_i^j} g_{k^j}.
\]

These conditions are not fulfilled in general.
5 Conclusions

It has been shown that the corresponding to a Lagrangian density covariant Noether identities contain in their structures the structure of the covariant Euler-Lagrange equations. The Euler-Lagrange equations, the covariant Noether identities and their corresponding energy-momentum tensors built a full theoretical scheme of a Lagrangian theory of tensor fields with compatible structure’s elements.

The covariant Noether identities induce the determination of three types of energy-momentum tensors: generalized canonical, symmetric (of Belinfante) and variational (of Euler-Lagrange). The last type of EMT-s for non-metric tensor fields \( V \) vanishes in \( V_n \)-spaces [but not in \( (\mathcal{T}_n, g) \)-spaces] if the covariant Euler-Lagrange equations for the corresponding dynamical fields are fulfilled. It does not vanish for the covariant metric tensor field \( g \). The different types of EMT-s appear for every field variable (tensor field) in a Lagrangian density regardless of the role of these field variables as dynamical or non-dynamical tensor fields. The symmetric energy-momentum tensor of Hilbert introduced in the Einstein field equations appears as an irrelevant element in the whole scheme of a covariant Lagrangian theory of tensor fields although some connections between it and the other EMT-s could be established. It has been shown that the Euler-Lagrange equations can be found in an independent of the affine connections way and therefore, in an independent of the type of transport of the tensor field manner.

A Explicit form of the energy-momentum tensors

A.1 Lie derivatives of components and their covariant derivatives of tensor fields

1. Lie derivatives of the components \( V^A_B \) of non-metric tensor fields \( V \)

\[
\mathcal{L}_\xi(V^A_B) = \mathcal{L}_\xi^A^B = [V^A_B;B] + \left( S_{\alpha B}^A + D_k^A V^A_B \right) T_{\alpha k}^B \xi^B + \left( S_{\alpha B}^C + D_k^C V^C_B \right) T_{\alpha k}^A \xi^A \equiv T_{\alpha k} \xi^B + \left( S_{\alpha B}^C + D_k^C V^C_B \right) \xi^A \equiv T_{\alpha k} \xi^A .
\]

(165)

2. Lie derivatives of the covariant derivatives of the components \( V^A_B \) of non-metric tensor fields \( V \)

\[
\mathcal{L}_\xi(V^A_B;B) = [V^A_B;B;i] + \left( S_{\alpha B}^{A k} V^C_B \right) T_{\alpha k}^B \xi^B + \left( S_{\alpha B}^{A k} V^C_B \right) \xi^B + \left( S_{\alpha B}^{A k} V^C_B \right) \xi^B + \left( S_{\alpha B}^{A k} V^C_B \right) \xi^B + \left( S_{\alpha B}^{A k} V^C_B \right) \xi^B + \left( S_{\alpha B}^{A k} V^C_B \right) \xi^B + \left( S_{\alpha B}^{A k} V^C_B \right) \xi^B + \left( S_{\alpha B}^{A k} V^C_B \right) \xi^B + \left( S_{\alpha B}^{A k} V^C_B \right) \xi^B .
\]

(166)
3. Lie derivatives of the components $g_{ij}$ of the metric tensor field $g$

$$
\mathcal{L}_\xi g_{ij} = (g_{ij;k} + g_{ij} T^k_{ij} + g_{il} T^i_{lj}) \xi^k + g_{ij} (\xi^l \nabla_l - g_{lj} \xi^l) .
$$

(168)

$$
-S_{B\tau} D_{ij} g_D = g^j_k g_{ia} + g^j_k g_{ak} , \quad B = kl , \quad D = mn .
$$

(169)

A.2 Energy-momentum tensors for the non-metric tensor fields $V^A_B$

1. Energy-momentum tensor of Belinfante $s_v T^i_j$ for $V^A_B$

$$
{s_v T}^i_j = v T^i_j - g^i_l L ,
$$

(170)

$$
{s_v T}^i_j = v T^i_j + g^i_l v T^j_l ,
$$

(171)

$$
 \xi^k \nabla_v r s v T^i_j = g_{im} (v \nabla_v r - \nabla_v T^i_j)^m - v T^i_j = q_{vk} T^k_i ,
$$

(172)

$$
\nabla_v T^i_j = v \nabla_v T^i_j + v T^i_j = \frac{1}{2} (v \nabla_v T^i_j + v \nabla_v r T^i_j) ,
$$

(173)

$$
\nabla_v T^i_j = \xi^k \nabla_v r s v T^i_j = q_v r s v T^i_j - q_v r s v T^i_j = \xi^k \nabla_v r s v T^i_j .
$$

(174)

$$
\nabla_v T^i_j = q_v r s v T^i_j = q_v r s v T^i_j .
$$

(175)

$$
{\nabla_v T}^i_j = S_{Cr} A_j [ - \left( \frac{\partial L}{\partial V^A_{B;k}} \right) V^C_B + (\nabla^L_{B;k:m} + \nabla^L_{B;m:k}) V^B_{B;} -
$$

$$
- (\frac{\partial L}{\partial V^A_{B;j:k}} V^A_{B;} + \nabla^L_{B;j:k}) V^A_{B;} ) ,
$$

(176)

$$
S_{Cr} A_j [ - \left( \frac{\partial L}{\partial V^A_{B;k}} V^C_B + (\nabla^L_{B;k:m} + \nabla^L_{B;m:k}) V^B_{B;} -
$$

$$
- (\frac{\partial L}{\partial V^A_{B;j:k}} V^A_{B;} + \nabla^L_{B;j:k}) V^A_{B;} ) ,
$$

(177)

$$
S_{Cr} A_j = - \sum_{k=1}^l g^j_k g^i_k g^j_{k-1} \cdots g^i_{k-1} g^j_{k+1} \cdots g^i_{k+1} ,
$$

(178)

$$
V^A_{B;} = e_i V^A_{B;} + \Gamma^A_{C;i}, V^C_B + P^D_{B;i}, V^A_D = V^A_{B;} + \Gamma^A_{C;i}, V^C_B + P^D_{B;i}, V^A_D ,
$$

(179)

$$
\Gamma^A_{C;i} = - S_{Cr} A_j, \Gamma^A_{j;k} , \quad P^A_{C;i} = - S_{Cr} A_j, P^A_{j;k} .
$$

(180)

2. Generalized canonical energy-momentum tensor $s_v T^i_j$ for $V^A_B$. $s_v T^i_j$ is the canonical energy-momentum tensor for $V^A_B$.

$$
{s_v T}^i_j = v T^i_j - v K^i_j - v W^i_j ,
$$

(181)

$$
{s_v T}^i_j = [ - \left( \frac{\partial L}{\partial V^A_{B;j}} + \nabla^L_{B;j:k} \right) ] V^A_{B;} + \nabla^L_{B;j:k} V^A_{B;} - g^j_l L ,
$$

(182)
\[ vK_{ij} = S_{Cm} A_n V^C B \frac{\partial L}{\partial V^A_{B;kj}} R^m_{nik} + S_{Bm} D_n V^A D \frac{\partial L}{\partial V^A_{B;kj}} P^m_{nik} + \]
\[ + \frac{\partial L}{\partial V^A_{B;kj}} V^A B_\ell T^l_{ik}, \] (183)
\[ T^l_{ij} = -T^l_{ik} = \Gamma^l_{ji} - \Gamma^l_{ij} - C_{il}^k (in \text{ a non-co-ordinate basis}), \] (184)
\[ T^l_{ij} = \Gamma^l_{ji} - \Gamma^l_{ij} \] (in a co-ordinate basis),
\[ vW^l_{ijk} = g_{kl} vW^j_{k} - g_{kl} vW^l_{j} + g_{kl} vW^m_{jk} l, \] (185)
\[ vW^l_{ijk} l = v_s V^n m l g^{mk} - v_s V^n k m l g^{lj} - v_a V^n j k l g^{ml} = -vW^m_{k j l}. \] (186)

3. Variational energy-momentum tensor of Euler-Lagrange \( vQ_{ij} \) for \( V^A_B \)
\[ vQ_{ij} = (S_{B\ell} D^l_{V^A D} - S_{Ci} A_j V^C B) \frac{\delta vL}{\delta V^A_B} \cdot \] (187)
\[ vT_i = \frac{\delta vL}{\delta V^A_B} V^A B : i + vW_i, \quad vW_i = vS_i - vS_k \cdot T^k_{ij} \] (188)
\[ vS_i = vW^l_{ijk} k_j + vV^l_{ijk} l - vQ^l_{ijk} k_j (g_{lj;k} - g_{lj;i} + g_{lj;m} T^m_{ik}) = \]
\[ = vW^l_{ijk} k_j - vW^l_{ijk} k_j - \frac{1}{2} [vW^l_{ijk} (T_{ljk} + vW^l_{ijk}) + vW^l_{ijk} (T_{jlk} + T_{jlk} T_{lk} - T_{lk} - T_{lk} T_{jlk} + T_{lk} T_{lk} - T_{lk} T_{lk} - T_{lk} T_{lk} - T_{lk}) + \]
\[ + vT_{mjk} m (T_{mj} m + T_{lj} l - T_{lj} l) + vW^l_{ijk} (T_{mjk} m - T_{lj} l - T_{lj} l) \]
\[ T_i = g_{li} T_{ij} l = T_{il} l, \quad T_{jk} = g_{jm} T_{ij} m \]
\[ vQ_{ij} = vW^l_{ijk} k_j - vF^l_{ijk}. \] (190)
\[ vQ^l_{ijk} k_j = S_{B\ell} D^l_{V^A D} + \frac{\partial L}{\partial V^A_{B;kj}} + \frac{\partial L}{\partial V^A_{B;kj}}, V^A_{D,m} - \frac{\partial L}{\partial V^A_{D;l;kj}}, V^A_{D,m} \] (192)
\[ vT_{ij} = S_{C\ell} A_j \frac{\partial L}{\partial V^A_{B;kj}} V^C B + \frac{\partial L}{\partial V^A_{B;kj}} + \frac{\partial L}{\partial V^A_{B;kj}}, V^C B,m \]
\[ - \frac{\partial L}{\partial V^A_{B;kj}}, V^C B,m \] (193)
\[ vV^l_{ijk} = \frac{1}{2} (vV^l_{ijk} + vV^l_{ijk}), \quad vV^l_{ijk} = \frac{1}{2} (vV^l_{ijk} - vV^l_{ijk}) \] (194)
\[ vV^l_{ijk} = vW^l_{ijk} + vT^l_{ij} k_j, vW^l_{ijk} = g_{ld} vW^l_{ijk} = -vW^l_{ijk}, \]
\[ vW^l_{ijk} = vW^l_{ijk} + vW^l_{ijk} + vW^l_{ijk} \] (195)
\[ vT_{ij}^l = vW^l_{ijk} + vW^l_{ijk} + vW^l_{ijk} - vW^l_{ijk} \] (196)
\[ vS_{ij} = vT^l_{ij} - vK_{ij} + g_{ij} L. \] (197)
A.3 Energy-momentum tensors for the metric tensor field $g_{ij}$

1. Symmetric energy-momentum tensor of Belinfante $s_glT_i^j$ for $g_{ij}$

\[ g_{sl}T_i^j = gT_i^j - g_i^jL , \]  

\[ gT_i^j = gT_i^{jk}k = g_{il}gT_i^{jk}k , \]

\[ gT_i^{jk}k = g_{il}(gsV_m^{kj}k - g_i^{m}m ) , \]

\[ gaV_i^{jk}l = \frac{1}{2}(g_aV_i^{kj}l + g_aV_i^{kj}l) , \]

\[ gT_i^{jk}l = \frac{1}{2}(gT_i^{jk}l - gT_i^{jk}l) , \]

\[ gT_i^{jk}l = gT_i^{jk}l + gT_i^{jk}l = gT_i^{jk}l + gaV_i^{jk}l . \]

\[ g_{s}V_i^{jk}l = s^D_p[\frac{\partial L}{\partial g_{B;k}}gD + (\frac{\partial L}{\partial g_{B;km;k}})gD_m - (\frac{\partial L}{\partial g_{B;km;m}})gD_m] , \]

\[ gaV_i^{jk}l = \frac{1}{2}(g_{s}V_i^{kj}l + g_{s}V_i^{kj}l) , \]

\[ g_{s}V_i^{jk}l = gaV_i^{jk}l - gT_i^{jk}l , \]

\[ g_{s}V_i^{jk}l = g_{s}V_i^{jk}l - g_{s}V_i^{jk}l , \]

\[ g_{s}V_i^{jk}l = gaV_i^{jk}l + gaV_i^{jk}l . \]

2. Generalized canonical energy-momentum tensor $g\overline{T}_i^j$ for $g_{ij}$, $g\overline{T}_i^j$ is the canonical energy-momentum tensor for $g_{ij}$

\[ g\overline{T}_i^j = g\overline{T}_i^j - gK_i^j - g\overline{W}_i^{jk}k , \]

\[ g\overline{T}_i^j = [\frac{\partial L}{\partial gkl;j} - (\frac{\partial L}{\partial gkl;j;m})gkl;i + (\frac{\partial L}{\partial gkl;m;j})gkl;i - g_i^jL] , \]

\[ gK_i^j = gK_i^j + g_{s}V_m^{jk}k - g_{s}V_m^{jk}k + g_{s}V_m^{jk}k - (g_{s}V_m^{jk}k - g_{s}V_m^{jk}k) , \]

\[ g\overline{W}_i^{jk}k = g\overline{W}_i^{jk}k - g\overline{W}_i^{jk}k , \]

\[ g\overline{W}_i^{jk}k = g\overline{W}_i^{jk}k - g\overline{W}_i^{jk}k , \]

\[ g\overline{W}_i^{jk}k = g\overline{W}_i^{jk}k - g\overline{W}_i^{jk}k , \]

\[ g\overline{W}_i^{jk}k = g\overline{W}_i^{jk}k - g\overline{W}_i^{jk}k . \]

3. Energy-momentum of Euler-Lagrange $g\overline{Q}_i^j$ for $g_{ij}$

\[ g\overline{Q}_i^j = -\frac{\delta L}{\delta g_{j;k}}g_{j;k} , \]

\[ g\overline{Q}_i^j = \frac{\delta L}{\delta g_{kl}}g_{kl;i} + g_{W}i , \]

\[ g\overline{Q}_i^j = \frac{\delta L}{\delta g_{kl;j;m}}g_{kl;i} + \frac{\delta L}{\delta g_{kl;m;j}}g_{kl;m;i} = g\overline{Q}_i^j - g\overline{Q}_i^j + g\overline{Q}_i^jL . \]

\[ \overline{Q}_i^j = -\frac{\delta L}{\delta g_{j;k}}g_{j;k} , \]
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