Polarization Effects in Lepton Flavor Violated Decays Induced by Axion-Like Particles

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Abstract

Lepton flavor violated process are strongly suppressed in the Standard Model due to very small neutrino mass, but can be sizable in some extended models. The current experimental bounds on decay modes $\ell' \pm \rightarrow a \ell \pm$ are much weaker than other flavor violated processes because of the huge irreducible backgrounds $\ell' \rightarrow \ell \bar{\nu} \nu \ell$. In this paper, we give the full helicity density matrix of both the signal and backgrounds, and then study polarization effects. Particularly, we treat inclusively the two missing neutrinos in the background, and we find that both longitudinal and transverse polarization effects survives even the relative kinematical degrees of freedom of the two neutrinos are integrated out. Furthermore, we have show that signal and backgrounds have distinctive polarization effects which can be measured by using energy fractions of the charged decay products. This is particularly useful because kinematical reconstruction is not required. In case of that the decaying lepton is not polarized, we show that correlations between polarizations of lepton pair generated at colliders are still useful to search for the signals. More interestingly, polarization correlations depends on product of scalar and pseudo-scalar ALP couplings, and hence are sensitive to their relative sign. We demonstrate that how the polarization correlation effects can be used to investigate flavor violating decays $\tau^\pm \rightarrow a \ell^\pm$ at the BelleII experiment.

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I. INTRODUCTION

It is generally accepted that the Standard Model (SM) need to be improved to account for new physics (NP) phenomenologies, for examples neutrino oscillations and Dark Matter etc. Usually, new particles are assumed to be heavy, and hence high energy colliders are expected to be sensitive to signals. However, so far no convincing evidence of beyond the SM has been observed, for instance, at the Large Hadron Collider (LHC). On the other hand, light or ultra light particles can appear naturally as pseudo-Nambu-Goldstone-Bosons of an underlying broken symmetry. A promising instance is the Axion [1–3], which is originally introduced to solve the strong CP problem, but is extended effectively to be correlated with other fundamental open questions in particle physics such as the origin of dark matter (DM) [4–7], as well the hierarchy [8, 9] and flavor problems [10–14]. Such kind of extensions are generally referred to as axion-like-particles (ALP) [15–17]. In view of effective field theory, the ALP mass and couplings to the SM fermions and gauge bosons are free parameters to be determined experimentally [18]. Stringent constrains on ALP couplings arise from cosmological and astrophysical observations [15–19]. The LEP collaboration studied ALP,
and obtained significant bounds for masses ranging from the MeV scale up to 90 GeV [20]. Searches for ALPs at the LHC and future colliders has also been extensively explored [20–30]. Furthermore, decays of mesons at flavor factories can provide rich possibilities to probe ALP couplings to both leptons and quarks [31–33].

Most interestingly, while lepton flavor changing neutral currents are strongly suppressed by the neutrino mass-squared difference in the SM [34, 35], the lepton flavor number in neutral currents can be effectively violated at tree-level by the ALPs [36–42]. Recent studies show that large parameter space of flavor violated ALPs can offers qualitatively new ways to explain the anomalies related to the magnetic and electric dipole moments of the muon and electron [43–46]. However, it was shown that in case of $m_a > m_\tau$, a simultaneous explanation of both anomalies in terms of flavor-violating ALP couplings to $\tau$-lepton is not possible due to relatively large contribution to $\mu \rightarrow e\gamma$, unless there is a large hierarchy between couplings of electron and muon to the $\tau$-lepton [43]. Similar situation happens for the case $m_\mu < m_a < m_\tau$. In case of that $2m_e < m_a < m_\mu$, flavor diagonal contributions have to be dominant in order to account for these two anomalies simultaneously, and the ALP couplings to electron should be larger than the ALP couplings to muon with opposite sign [43]. If the ALP is ultra light, flavor-violating ALP couplings can never give correct contributions to the anomalies, and hence can put strong limits on the corresponding couplings [47–54]. In this case, flavor violated decay modes $\tau \rightarrow \ell a$ can provide direct measurement on the relevant flavor violation parameters. However, It is challenging to distinguishing signal events because more than one neutrino appear in the final state of the irreducible backgrounds $\tau^\pm \rightarrow \ell^\pm \bar{\nu}_\ell \nu_\tau$. The ARGUS collaboration studied these channels, and an upper limit $\sim 2\%$ are obtained by tagging one side of the $\tau$-lepton pair decaying in 3-prong mode [55]. The 3 $-$ 1 prong searching method is also used by the Belle II experiment [56–59]. However, number of events is significantly reduced because of small branching fraction of the 3-prong decay channel. Some new methods are proposed [60–62], but still rely on double-side tagging of the $\tau$-lepton.

Measuring total cross section and/or decay rates are ineffective due to the huge irreducible background $\tau^\pm \rightarrow \ell^\pm \bar{\nu}_\ell \nu_\tau$. This is also the reason that experimental bounds on the decay modes $\ell^\pm \rightarrow a \ell^\pm$ are much weaker than other flavor violated processes [63].

Alternatively, polarization observables which can be inclusive with respect to the neutrinos, are efficient probes to the relevant flavor violation parameters. It is well-known that, because parity conservation is violated maximally via the $V - A$ electroweak current in
\(\tau\)-decays, its longitudinal polarization can be efficiently measured by using its decay products [64]. Particularly important is the hadronic decay mode \(\tau \to \pi \nu_\tau\), in which polar angle distribution of the \(\pi\)-meson in the rest frame of the \(\tau\)-lepton (or equivalently energy distribution of the \(\pi\)-meson in the flying frame of the \(\tau\)-lepton) has maximum sensitivity to the longitudinal polarization of the parent \(\tau\)-lepton [64, 65]. Sensitivities of the other hadronic decay modes, \(\tau \to \rho \nu_\tau\) and \(\tau \to a_1 \nu_\tau\), are relatively lower due to deconstructive contributions from different helicity amplitudes, but can be significantly improved by further selections on the helicity components via subsequent decays of the \(\rho\)-meson [66–68] and \(a_1\)-meson [68], respectively. It was shown that energy-energy correlation, as a consequence of longitudinal-longitudinal spin correlation, can be used to measure ratios between the electroweak axial and vector coupling constants [69–71]. Incorporating spin vectors of \(\tau\)-lepton pairs, CP-odd tensor observables [72] and \(T\)-odd observable [73] can be constructed, and a general analysis was done in Ref. [74]. Spin correlations were also analyzed based on decay helicity amplitudes of the \(\tau\)-lepton, and was employed to probing interaction between the \(\tau\)-lepton and a neutral scalar particle [75, 76].

For the leptonic decay modes, since more than one neutrino appear in the final state, polarization effects are weaker than the other modes, and accurate reconstruction of the \(\tau\)-lepton rest frame is rather hard. Because of this, the leptonic decay modes are usually not considered as useful channels for studying full spin correlations. In this paper, we study the full decay density matrix of the leptonic decay modes in the rest frame of the decaying \(\tau\)-lepton. We will show that polarization effects of the leptonic decay modes \(\tau^{\pm} \to \ell^{\pm} \bar{\nu}_\ell \nu_\tau\) survives when the invariant mass of the two neutrinos is inclusively measured. Furthermore, signal and backgrounds have distinctive polarization effects, and hence signals can be selected by studying polarizations of the decaying leptons. Even through \(\tau\)-leptons generated at low energy collider are usually unpolarized, and hence there is no net polarization effect, we will show that correlations between the polarizations of the \(\tau\)-lepton pair are still useful to search for the signals. We demonstrate that how the polarization correlation effects can be used to investigate flavor violating decays \(\tau^{\prime \pm} \to a \ell^{\pm}\) at the BelleII experiment.
II. LEPTON FLAVOR VIOLATED ALP

In this subsection, we study a lepton flavor violated model with a new scalar or pseudo scalar mediator. Up to dimension five, the relevant low-energy effective interaction Lagrangian of an Axion-like particle couples to charged leptons and photons are described by [18, 43, 63],

$$L_{\text{Eff}} = \frac{1}{2} \left( \left( \partial_\mu a \right)^2 - m_a^2 a^2 \right) + L_{a\ell\ell'} + L_{a\gamma\gamma'},$$

$$L_{a\ell\ell'} = - \frac{f_a}{1} \left( \partial_\mu a \right) \left( \bar{\ell}_L U_L \gamma^\mu \ell_L + \bar{\ell}_R U_R \gamma^\mu \ell_R \right),$$

$$L_{a\gamma\gamma'} = \frac{\alpha E}{4\pi} \frac{f_a}{1} c_{\gamma\gamma'} a F^{\mu\nu} \tilde{F}_{\mu\nu},$$

where $\ell = (e, \mu, \tau)^T$, the hermitian matrices $U_L$ and $U_R$ are defined in the mass eigenstates of the leptons, and $f_a$ is the decay constant of the Axion-like particle. Since we are considering an ALP as a pseudo-Nambu-Goldstone boson of a spontaneously broken global symmetry, the decay constant $f_a$ characterizing the broken scale is naturally much larger than the low energy scale parameter, $m_a$, mass of the ALP. In case of that the ALP couples to on-shell leptons, then by using the equation of motion, the corresponding interaction Lagrangian can be rewritten as,

$$L_{a\ell\ell'}^{\text{Eff}} = - a \left( \bar{\ell}_i U_{Vij} (m_i - m_j) \ell_j + \bar{\ell}_i U_{Aij} (m_i + m_j) \ell_j \gamma^5 \ell_i \right),$$

where $U_V = (U_R + U_L)/2$ and $U_A = (U_R - U_L)/2$, and $m_i$ are masses of the lepton $\ell_i$. For flavor-diagonal ALP couplings, one can easily find that only the pseudo-scalar Lorentz structure survives. Since we will mainly study the spin correlation effects at a collider, the Lagrangian (4) is hence the relevant part studied in this paper (even through we will also consider decays of the $\tau$-lepton, narrow width approximation is particularly good because of its very small width, $\sim 2 \times 10^{-12}$GeV [77]). Because of this we introduce dimensionless scalar couplings, and the Lagrangian (4) is rewritten as,

$$L_{a\ell\ell'}^{\text{Eff}} = - a \left( S_{ij} \ell_i + S_{ij}^{\gamma 5} \ell_i \gamma^5 \ell_i \right),$$

where the scalar and pseudo-scalar coupling matrices are given as,

$$S_{ij}^{ij} = \frac{i}{f_a} U_{Vij} (m_j - m_i), \quad S_{ij}^{ij} = - \frac{i}{f_a} U_{Aij} (m_i + m_j).$$
Here we have used different parameterizations such that the diagonal elements have the same sign conventions with the standard Yukawa couplings. These two kinds of conventions can have (sign) differences in some observables, particularly when interference effects between SM channels and Axion induced channels. Sometimes we also use notations with explicit charility

\[ S_{ij}^L = S_{ij}^S - S_{ij}^P, \quad S_{ij}^R = S_{ij}^S + S_{ij}^P. \]  

(7)

Phenomenologies details of the above ALP model strongly depends on its mass. In case of that the ALP is ultralight, star evolution can be significantly affected as a result of stellar cooling mechanism [78], and hence can place strong constraints on the ALP couplings with astrophysical observations [79–83]. A very strong bound on the pseudo-scalar coupling of electron, \( S_{e\mu}^S < 2.1 \times 10^{-13} \) at 90% C.L., was obtained in Ref. [80]. Similar constraint for \( \mu \)-lepton, \( S_{\mu\mu}^S < 2.1 \times 10^{-10} \), was reported in Ref. [82]. For the corresponding scalar couplings, \( S_{\ell\ell}^S \), similar upper bounds applies as argued in Ref. [44]. Here we further assume that those constrains are also valid as long as \( m_a < 2m_e \). It turns out that a light ALP can couple to leptons only through flavor violated couplings. For \( \mu - e \) flavor violated couplings, a conservative bound \( S_{S/P}^{e\mu} < 5.3 \times 10^{-11} \) was obtained [80, 84, 85]. Substantially improvement on this coupling will be reached by the Mu3e experiment in near future [80, 86]. For \( \tau \)-lepton, the currently best experimental limits were set by the ARGUS collaboration [55], \( S_{S/P}^{\tau\ell} \lesssim 10^{-7} \), which are relatively weak compared to the couplings \( S_{S/P}^{e\mu} \).

III. POLARIZED DECAY DENSITY MATRIX OF LEPTONS

A. Polarized Decay \( \ell' \to \ell a \)

In this subsection we give angular distributions of the following polarized decay process

\[ \ell^- (\vec{p}_{\ell'}, \lambda_{\ell'}) \longrightarrow a (\vec{p}_a) + \ell^- (\vec{p}_{\ell}, \lambda_{\ell}) . \]  

(8)

Here \( \lambda_{\ell'} \) and \( \lambda_{\ell} \) are helicities of the decaying and outgoing leptons, \( \ell' \) and \( \ell \), respectively. In principal, the ALP can undergo further transitions, \( a^* \to \ell'^\pm \ell'^\mp \), with either on-shell or off-shell \( a^* \). Within our scenario the only possible channel is \( \ell' \to \ell (a^* \to \mu^\pm e^\mp) \). However, because decay width of the ALP is very small, a fraction of on-shell \( a \) may not decay inside of the detector, and hence effectively an invisible particle. The typical decay length of \( a \),
\( l_a \), depends on its energy, and therefore dedicated experimental study is necessary to have a correct assessment of the lepton flavor violation process \( \ell' \to \ell (a^* \to \mu^\mp e^\mp) \) in the resonant region \([87, 88]\). In our analysis for the invisible decay at BelleII, we will naively require that the decay length is larger than 1m such that the ALP is effectively invisible. In contrast, if \( l_a < 1 \text{m} \), the ALP is considered to be decaying promptly, and hence the full process \( \ell' \to \ell (a^* \to \mu^\mp e^\mp) \) has to be considered. We will study this process in Sec. III B. For this moment let us focus on the (effectively) invisible decay \( \ell' \to \ell a \).

The helicity amplitudes as well as the corresponding helicity density matrix elements are calculated in the rest frame of the lepton \( \ell' \), in which the \( z \)-axis is defined as along the flying direction of the lepton \( \ell' \), and for completeness we assume that the \( x \)-axis can be defined unambiguously such that the helicity density matrix elements are also non-trivial functions of the azimuthal angle (we will give its definition when the production dynamics of \( \ell' \) is given).

Within above conventions, the momentum of leptons and ALP are parameterized as follows,

\[
p_{\ell'}^\mu = m_{\ell'} (1, \ 0, \ 0, \ 0),
\]

\[
p_a^\mu = \frac{1}{2} m_{\ell'} \alpha_a^+ (1, \ \beta_a \sin \theta_a \cos \phi_a, \ \beta_a \sin \theta_a \sin \phi_a, \ \beta_a \cos \theta_a), \tag{10}
\]

\[
p_{\ell}^- = \frac{1}{2} m_{\ell'} \alpha_a^- (1, -\beta_a \sin \theta_a \cos \phi_a, -\beta_a \sin \theta_a \sin \phi_a, -\beta_a \cos \theta_a), \tag{11}
\]

where \( \alpha_a^\pm = 1 \pm (m_a^2 - m_{\ell'}^2)/m_{\ell'}^2 \), and \( \beta_a = \beta_{\ell' a} / \alpha_a^+ \) and \( \beta_\ell = \beta_{\ell' \ell a} / \alpha_a^- \) with \( \beta_{\ell' \ell a} = \sqrt{1 - x} \).

According to the Lagrangian (5), the helicity amplitudes are given as,

\[
D_{\lambda, \lambda'} = \frac{u(\vec{p}_\ell, \lambda)}{u(\vec{p}_{\ell'}, \lambda')} \left( S_{\ell'\ell} + S_{\ell' P} \gamma_5 \right) \frac{u(\vec{p}_{\ell'}, \lambda')}{u(\vec{p}_{\ell}, \lambda)}.
\]

With our parameterizations the momentum then we have,

\[
D_{\lambda, \lambda'}^+ = \frac{\lambda_{\ell'} m_{\ell'}}{2} \sqrt{\alpha_a^+ \alpha_a^-} \left( f_+ S_{\ell' P}^\ell - f_- S_{\ell S}^\ell \right) e^{-i\phi_+ / 2} e^{i\lambda_{\ell'} \phi_+ / 2} \sqrt{1 - \lambda_{\ell'} \cos \theta_a}, \tag{13}
\]

\[
D_{\lambda, \lambda'}^- = \frac{\lambda_{\ell'} m_{\ell'}}{2} \sqrt{\alpha_a^+ \alpha_a^-} \left( f_+ S_{\ell S}^\ell + f_- S_{\ell' P}^\ell \right) e^{i\phi_- / 2} e^{i\lambda_{\ell'} \phi_- / 2} \sqrt{1 + \lambda_{\ell'} \cos \theta_a}, \tag{14}
\]

where \( f_\pm = (\sqrt{1 + \beta_{\ell'}} \pm \sqrt{1 - \beta_{\ell'}}) / \sqrt{2} \), and in massless limit of the lepton \( \ell \) (which generally works well for studying polarization effect), \( f_\pm = 1 \). For completeness, we will keep the mass dependence in the density matrix elements. The diagonal and off-diagonal elements can be
In terms of the energy fraction \(z\), the \(\ell\) fraction of the particle a more sensitive than the cross section measurement to the ALP coupling constants.\(\xi\) components, then there is nontrivial polarization effects. Most importantly, the polarizer we can clearly see that as long as \(\xi_{\ell'\ell} \neq 0\), i.e., both \(S_{S}^{\ell\ell'}\) and \(S_{P}^{\ell\ell'}\) have non-zero real components, then there is nontrivial polarization effects. Most importantly, the polarizer \(\xi_{\ell'\ell}\) is sensitive to the relative sign between \(S_{S}^{\ell\ell'}\) and \(S_{P}^{\ell\ell'}\). Furthermore, we will see that once the irreducible background process \(\ell' \rightarrow \ell\nu_{\ell}\nu_{\ell}\) are considered, polarization measurement is more sensitive than the cross section measurement to the ALP coupling constants.

Angular distributions of the polarized decay width can be obtained by including the corresponding 2-body phase space factor. Since the azimuthal angle distribution is trivial for the longitudinal polarizations, then in terms of polar angle of the charged lepton \(\ell\) \((\theta_{\ell} = \pi - \theta_{a})\) we have,

\[
\frac{1}{m_{\ell'}} \frac{d\Gamma_{\lambda_{\ell'}}}{d\cos\theta_{\ell}} = \frac{\overline{\beta}_{\ell'\ell} \alpha_{a\ell} \overline{\alpha}_{a\ell}}{64\pi} S_{\ell'\ell}^{S} \left[ 1 + \gamma_{\ell}^{-1} \kappa_{\ell'\ell} + \lambda_{\ell'} \xi_{\ell'\ell}^{+} \beta_{\ell} \cos\theta_{\ell} \right].
\]

We can clearly see that as long as \(\xi_{\ell'\ell} \neq 0\), i.e., both \(S_{S}^{\ell\ell'}\) and \(S_{P}^{\ell\ell'}\) have non-zero real components, then there is nontrivial polarization effects. Most importantly, the polarizer \(\xi_{\ell'\ell}\) is sensitive to the relative sign between \(S_{S}^{\ell\ell'}\) and \(S_{P}^{\ell\ell'}\). Furthermore, we will see that once the irreducible background process \(\ell' \rightarrow \ell\nu_{\ell}\nu_{\ell}\) are considered, polarization measurement is more sensitive than the cross section measurement to the ALP coupling constants.

In practice, kinematics of the final state usually can not know precisely due to the invisible particle a, and hence it is challenging to use the above polarization effects. However, energy fraction of the \(\ell\)-lepton, \(z_{\ell} = E_{\ell}/E_{e}\), which is correlated to the polar angle distribution of the \(\ell\)-lepton, can be measured with small uncertainty at \(e^{-}e^{+}\) collider. For a mother lepton \(\ell'\) generated with a boost factor \(\beta_{e}\) in the Lab. frame, the corresponding energy fraction is given as,

\[
z_{\ell} = \frac{1}{2} \beta_{e} \alpha_{a\ell} \left( 1 + \beta_{e} \beta_{\ell} \cos\theta_{\ell} \right).
\]

In terms of the energy fraction \(z_{\ell}\) the differential cross section can be written as,

\[
\frac{1}{m_{\ell'}} \frac{d\Gamma_{\lambda_{\ell'}}}{dz_{\ell}} = \frac{\overline{\beta}_{\ell'\ell} \alpha_{a\ell}}{16\pi \beta_{e} \beta_{\ell}} S_{\ell'\ell}^{S} \left[ \frac{1}{2} \left( 1 + \gamma_{\ell}^{-1} \kappa_{\ell'\ell} + \lambda_{\ell'} \xi_{\ell'\ell}^{+} \beta_{\ell} \alpha_{a\ell} \right) - \lambda_{\ell'} \beta_{\ell} \xi_{\ell'\ell}^{+} z_{\ell} \right].
\]
FIG. 1. Fig. 1(a) and Fig. 1(b) show the energy fraction distributions of electron and muon in decays $\tau^{-} \rightarrow \ell^{-} a$ with $\xi_{e\tau}^+ = 1$ and with $\xi_{\mu\tau}^+ = -1$, respectively. The ALP has been assumed to be massless, and the $\tau$-leptons are assumed to be generated with a fixed energy $E_{\tau} = 5.29$ GeV. $3 \times 10^4$ events (total events with $\lambda_\tau = \pm 1$) are used to obtain the normalized histograms.
Therefore, for positively polarized $\ell'$-lepton and $\xi_{\ell\ell'}^+ > 0$, the decay events are dominated in the region with $\cos \theta_\ell < 0$, and hence have small energy fractions. In contrast, most of the $\ell$-leptons have larger energy fractions if the $\ell'$-lepton is negatively polarized. The polarization effects are shown in Fig. 1(a) and Fig. 1(b) for $\tau^- \to e^-a$ with $\xi_{\ell\ell'}^+ = 1$ and $\tau^- \to \mu^-a$ with $\xi_{\ell\ell'}^+ = -1$, respectively. In both case, the ALP has been assumed to be massless, and the $\tau$-lepton are boosted to have an energy $E_\tau = 5.29\text{GeV}$ which corresponds to the case of $\tau$-lepton production in pairs at BelleII. We can clearly see sensitivities on the sign of $\xi_{\ell\ell'}^+$, and influence of the mass of the $\ell$-lepton is negligible. One more important property that we can learn form (20) is that, for relatively heavy ALP the $\gamma$-factor of $\ell$-lepton can be moderate, then there is a cancellation between the first two terms in (20) when the ALP interactions are dominated by the pseudo-scalar coupling ($S_{\ell\ell'}^S \ll S_{\ell\ell'}^P$). In this case the decay width is rapidly decreased and hence giving smaller branching ratio. However, this happens in very narrow region because of the large mass hierarchy in the lepton sector. For instance the decay process $\tau^- \to \mu^-a$, we can see in Fig. 2, the three configurations of $(S_{\mu\tau}^S, S_{\mu\tau}^P)$ are nearly impossible to distinguish near the boundary $m_a = m_\tau - m_\mu$.

![Mass dependence of the decay width $\Gamma(\tau^- \to \mu^-a)$ for three configurations of the ALP coupling constants: $(S_{\mu\tau}^S, S_{\mu\tau}^P) = (1, 0), (0, 1), (1, 1)/\sqrt{2}$](image_url)
B. Polarized Decay $\ell'^{\mp} \to \ell^{\mp} \ell \pm j$  

In this subsection we consider the case of that the ALP undergoes further decays. Our results will be given with either an on-shell or off-shell ALP, and we will see that the off-shell contribution is completely negligible. Within our senario, the decaying particle has to be the $\tau$-lepton ($i.e.$, $\ell = \tau$), and for the decay products either $\ell = \nu$ or $\ell = j$. In this work we restrict ourselves to the case $\ell = \nu$ such that interference effects due to identical particles are removed. Therefore, we will focus on the decay processes: $\ell'^{\mp} \to \ell^{\mp} \ell \pm j$.  

While at tree level those decay channels are initiated via intermediate $a$ as $\ell'^{\mp} \to \ell^{\mp} a \to \ell^{\mp} (\ell \pm j)$, the loop induced channels $\ell'^{\mp} \to j^{\mp} \gamma^{*} \to j^{\mp} (\ell^{\mp} \ell \pm)$ can also contribute. For on-shell decay of the ALP, the relative importance of these two decay channels are characterized by the ratio $m_{a}^{2}/(m_{a}^{2} \Gamma_{a}^{2})$. As long as the ALP’s width is small enough, $m_{a}^{-1} \Gamma_{a} < m_{\ell}^{2}/m_{a}^{2}$, the decay channels $\ell'^{\mp} \to \ell^{\mp} a \to \ell^{\mp} (\ell \pm j)$ are dominant. Furthermore, contributions of the channels $\ell'^{\mp} \to j^{\mp} \gamma^{*} \to j^{\mp} (\ell^{\mp} \ell \pm)$ can be reduced significantly by putting a cut on the invariant mass of $\ell^{\mp} \ell \pm$. Therefore in this work we will consider only the decay channel $\ell'^{\mp} \to \ell^{\mp} a \to \ell^{\mp} (\ell \pm j)$. Without loss of generality, we give the helicity amplitudes and density matrix elements for the decay process $\ell'^{\mp} \to \ell^{\mp} a \to \ell^{\mp} (\ell^{\mp} \ell \pm)$, where $\ell^{\mp} (\ell^{\mp} \ell \pm)$ is the ALP’s width is small enough, $m_{a}^{-1} \Gamma_{a} < m_{\ell}^{2}/m_{a}^{2}$, and the decay width $\Gamma_{a \to \ell^{\mp} j}$ is simply given as,  

$$\Gamma_{a \to \ell^{\mp} j} = \frac{1}{2 \pi} \left( \epsilon_{+} |S_{a}^{\uparrow}|^{2} + \epsilon_{-} |S_{a}^{\downarrow}|^{2} \right),$$  

where $\epsilon_{\pm} = 1 - (m_{j} \pm m_{\ell})^{2}/m_{X}^{2}$ and $\bar{\beta}_{a} = \sqrt{1 + (m_{j} - m_{\ell})^{2}/m_{X}^{2} - 2m_{j}^{2}/m_{X}^{2} - 2m_{\ell}^{2}/m_{X}^{2}}$.  

Because of the integrand in (23) is always positive, it is then clear then that spin correlations of the decay process $\ell'^{\mp} \to \ell^{\mp} \ell j$ described by $\mathcal{K}_{a \nu}^{\lambda}$ is the same as the process $\ell'^{\mp} \to \ell^{\mp} a$. This is also true even when the decay $a \to \ell^{\mp} j$ is measured inclusively (this can natruely
happens in practice when decay width of the ALP is sufficiently small, and hence become invisible inside of detector). Within our conventions $S_{S}^{ef} = (S_{S}^{ef})^*$ and $S_{P}^{ef} = -(S_{P}^{ef})^*$, hence the total decay width $\Gamma_a = 2\Gamma_{a \rightarrow \mu^+e^-}(m_a^2)$.

The ALP can be either on-shell and off-shell. Let us study the case of on-shell resonance decay. Since large mass hierarchy between $m_j$ and $m_\ell$, the kinematical factor $\epsilon_+ \approx \epsilon = 1 - m_\mu^2/m_X^2$, and $\beta_a \approx \sqrt{1 - m_\mu^2/m_X^2} = \sqrt{\epsilon}$. In this limit the decay width is simply given as, $m_X^{-1}\Gamma_{a \rightarrow \ell^+\ell^-}(m_X^2) = (8\pi)^{-1} e^{3/2} S_{\ell\ell}$, which depends only on sum of the squared ALP coupling constants. The most important thing is that the decay width is rather small, $m_a^{-1}\Gamma_{a \rightarrow \mu^+e^-}(m_a^2) \leq 4 \times 10^{-2} S_{e\mu} \sim 10^{-10}$ for a typical ALP couplings at TeV scale. Therefore, narrow width approximation which works excellent, and angular distributions of the polarized decay width of the process $\ell^-' \rightarrow \ell^-\ell^+\ell^-$ can be written as,

$$\frac{1}{m_\ell'} \frac{d\Gamma_{\lambda_{\ell'}}}{d\cos \theta_\ell} = \frac{\beta_{\ell\ell'}}{512\pi^2} \alpha_{\ell\ell'}^+ \epsilon_\ell \epsilon_{\ell'} \frac{e^{3/2}(m_a^2)}{S_{\ell\ell'}} S_{\ell\ell'} \left[1 + \gamma_\ell^{-1} \kappa_{\ell\ell'} - \lambda_\ell^+ \epsilon_{\ell\ell'} \beta_\ell \cos \theta_\ell \right].$$ (25)

All polarization effects discussed in last section survives completely. For off-shell ALP, the dominant contribution is given by $m_X^2 \sim m_\mu^2$. The kinematical factors $\epsilon_+ \sim 2$ and $\epsilon_- \sim 0$, hence off-shell conversion is sensitive to only the scalar coupling constant $S_{S}^{ef}$. The polarized decay angular distribution can be written as

$$\frac{1}{m_\ell'} \frac{d\Gamma_{\lambda_{\ell'}}}{d\cos \theta_\ell} \approx \frac{1}{64\pi} \alpha_{\ell\ell'}^+ S_{\ell\ell'} S_{\ell\ell'} \left[1 + \gamma_\ell^{-1} \kappa_{\ell\ell'} - \lambda_\ell^+ \epsilon_{\ell\ell'} \beta_\ell \cos \theta_\ell \right],$$ (26)

where $\alpha_{\ell\ell'}$, $\gamma_{\ell\ell'}^{-1}$ and $\beta_{\ell\ell'}$ are following integrations of the corresponding kinematical factors in (20)

$$\bar{k} = \int_{(m_\mu + m_\ell)^2}^{(m_\ell - m_\mu)^2} \frac{m_X^2 m_\ell^2 \, dm_X^2}{8\pi^2 (m_X^2 - m_\mu^2)^2} k e^{3/2}(m_X^2).$$ (27)

Since the kinematical factors are at an order $O(1)$ for light ALP, the integral is dominantly enhanced by the logarithm factor $\ln m_\ell^2/m_\mu^2$ which is moderate compared to the suppression factor $S_{\ell\ell'}$. In consideration of this the off-shell conversion can be completely neglected.

C. Polarized Decay $\ell' \rightarrow \ell \bar{\nu}_\ell \nu_\ell$

The leptonic decay modes $\ell' \rightarrow \ell \bar{\nu}_\ell \nu_\ell$ are usually irreducible backgrounds of new physics signals, and it is very hard to distinguish signals because of more than one neutrino appear in the decays. For instance, both $\tau \rightarrow \mu \gamma$ and $\tau \rightarrow \ell \alpha$ suffer from heavy contamination of
the SM process $\tau \to \ell \nu \bar{\nu}$ [57–59]. The decay channel $\tau \to \ell \alpha$ with $\alpha$ being a scalar particle was studied by the ARGUS collaboration [55], and an upper limits $\sim 2\%$ are obtained by tagging one side of the $\tau$-lepton pair decaying in 3-prong mode. The 3 $\pm$ 1 prong searching method is also used by the Belle II experiment [56–59]. However, number of events is significantly reduced because of small branching fraction of the 3-prong decay channel. Some new methods are proposed [60–62], but still rely on double-side tagging of the $\tau$-lepton.

In this section, we study the full decay density matrix of the decay $\ell' \to \ell \nu \bar{\nu}$ in the rest frame of the decaying $\ell'$-lepton. Although this frame cannot be accurately reconstructed in real experiments on account of the missing neutrinos, we will show that even when the relative kinematical degree of freedoms between the two neutrinos are integrated out, both longitudinal and transverse polarization effects survives. Those polarization effects can be either used to characterize backgrounds or to study signals. This is particularly important for the $\tau$-lepton, because its large leptonic decay branching ratio. Since large mass hierarchy in the lepton sector and also we are primarily concerned with decay of fast moving $\tau$-leptons, it is sufficient to study the distributions in mass less limit of the outgoing leptons. Without loss of generality, all the formula are given for decay of the $\ell^-$-lepton, density matrix for decay of its anti-particle can be obtained in a similar way. The kinematical variables are defined as follows (see Fig. 3 for details),

$$\ell^- (p_{\ell'}, \lambda_{\ell'}) \longrightarrow \nu_{\ell^'} (p_{\nu_{\ell^'}}) + \bar{\nu}_{\ell} (p_{\bar{\nu}_{\ell}}) + \ell^- (p_{\ell}, \lambda_{\ell}),$$ (28)

helicities of the $\ell'$-lepton and $\ell$-lepton take values of $\lambda_{\ell^'}/2 = \pm1/2$ and $\lambda_{\ell}/2 = \pm1/2$, respectively. Masses of the $\ell$-lepton and neutrinos will be neglected in all the calculations.

For the above leptonic decay mode, the final state posses a three body phase space. However, different from the usual decomposition, momentum of the two neutrino will be combined by introducing a virtual momenta $p_{\nu}^X = p_{\nu_{\ell'}} + p_{\nu_{\ell}}$ (see Fig. 3 for definitions of the kinematical variables). The lower limit of the $p_{\nu}^X$ can be calculated directly as $p_{\nu}^X \geq 0$ in the limit of massless neutrinos. Its upper bound can be obtained by calculating $p_{\nu}^X = (p_{\nu} - p_{\ell})^2$, which equals $m_{\nu_{\ell'}}^2 + m_{\nu_{\ell}}^2 - 2m_{\nu}E_{\ell}$ with the minimum energy $min\{E_{\ell}\} = m_{\ell}$ giving the upper bound $max \{m_{\nu}^2\} = (m_{\nu_{\ell'}} - m_{\ell})^2 \approx m_{\nu_{\ell'}}^2$. According to our parameterization the decay phase space can be written as,
FIG. 3. Definitions of the kinematical variables, particularly $p_X = p_{\nu\ell} + p_{\bar{\nu}}$, and $m_X^2 = p_X^2$. The angular variables $(\theta_X, \phi_X)$ and $(\theta_{\nu\ell}, \phi_{\nu\ell})$ of the momentum $p_X$ and $p_{\nu\ell}$ are defined in the rest frames of $p_{\nu\ell}$ and $p_X$, respectively.

\[
d\Phi_{\nu\ell} = \frac{1}{2\pi} dm_X^2 d\Pi_{\nu\ell}(\theta_X, \phi_X) d\Pi_X(\theta_{\nu\ell}, \phi_{\nu\ell}), \tag{29}
\]

\[
d\Pi_{\nu\ell} = \frac{d^3 p_{\nu\ell}}{(2\pi)^3 2E_{\nu\ell}} \frac{d^3 p_X}{(2\pi)^3 2E_X} (2\pi)^4 \delta^4(p_{\nu\ell} - p_{\ell} - p_X) = \frac{\bar{\beta}_X d\cos\theta_X d\phi_X}{8\pi} \frac{2}{2\pi}, \tag{30}
\]

\[
d\Pi_X = \frac{d^3 p_{\nu\ell}}{(2\pi)^3 2E_{\nu\ell}} \frac{d^3 p_X}{(2\pi)^3 2E_X} (2\pi)^4 \delta^4(p_X - p_{\nu\ell} - p_{\bar{\nu}}) = \frac{1}{8\pi} \frac{d\cos\theta_{\nu\ell} d\phi_{\nu\ell}}{2\pi}. \tag{31}
\]

Here the angular variables $\theta_X$ and $\phi_X$ are defined in the rest frame of the parent $\ell'$-lepton, and the corresponding momentum are parameterized as follows,

\[
p_{\nu\ell} = m_{\nu\ell} (1, 0, 0, 0), \tag{32}
\]

\[
p_X = \frac{1}{2} m_{\nu\ell} (\alpha_X, \bar{\beta}_X \sin\theta_X \cos\phi_X, \bar{\beta}_X \sin\theta_X \sin\phi_X, \bar{\beta}_X \cos\theta_X), \tag{33}
\]

\[
p_{\bar{\nu}} = \frac{1}{2} m_{\nu\ell} (\alpha_{\ell}, -\bar{\beta}_X \sin\theta_X \cos\phi_X, -\bar{\beta}_X \sin\theta_X \sin\phi_X, -\bar{\beta}_X \cos\theta_X), \tag{34}
\]

where the factors $\alpha_X = 1 + (m_X^2 - m_{\nu\ell}^2)/m_{\nu\ell}^2 \approx 1 + m_X^2/m_{\nu\ell}^2$, $\alpha_{\ell} = 1 - (m_X^2 - m_{\ell}^2)/m_{\ell}^2 \approx 1 - m_X^2/m_{\ell}^2$ and $\bar{\beta}_X = \sqrt{1 + (m_X^2 - m_{\bar{\nu}}^2)^2/m_{\bar{\nu}}^2 - 2(m_X^2 + m_{\bar{\nu}}^2)/m_{\bar{\nu}}^2} \approx 1 - m_X^2/m_{\bar{\nu}}^2$. The the angular variables $\theta_{\nu\ell}$ and $\phi_{\nu\ell}$ are defined in the rest frame of the momentum $p_X^\mu$, and the momentum of the two neutrinos are parameterized as follows (we have used massless limit
for the neutrino masses),

\[ p_X^\mu = m_X \left( 1, 0, 0, 0 \right), \quad (35) \]

\[ p_{\nu_{\nu'}}^\mu = \frac{1}{2} m_X \left( 1, \sin \theta_{\nu_{\nu'}}, \sin \theta_{\nu_{\nu'}}, \cos \theta_{\nu_{\nu'}} \right), \quad (36) \]

\[ p_{\nu_{\bar{\nu}}}^\mu = \frac{1}{2} m_X \left( 1, -\sin \theta_{\nu_{\nu'}}, -\sin \theta_{\nu_{\nu'}}, -\cos \theta_{\nu_{\nu'}} \right). \quad (37) \]

The complete decay density matrix can be written as

\[ \rho_{\lambda_{\nu'}}^{\lambda_{\nu}} = \mathcal{M}_{\lambda_{\nu'}} \mathcal{M}_{\lambda_{\nu}}^\dagger, \quad (38) \]

where \( \mathcal{M}_{\lambda_{\nu'}} \) is the decay helicity amplitude. Assuming that decay of the \( \ell' \)-lepton is completely described by the standard model (SM), then the decay helicity amplitude is given by contract of the charged currents of the \( \ell' - \nu_{\nu'} \) and \( \ell - \nu_{\nu} \) systems. In this form, the two neutrinos couple separately to the two leptons, and hence it is relatively hard to obtain the inclusive decay density matrix. In order to integrate out the relative kinematical variables of these two neutrinos, we apply Fierz transformation for the decay helicity amplitude which in turn is given by a contract of two neutral currents of the \( \ell' - \ell' \) and \( \nu_{\nu'} - \nu_{\nu} \) systems,

\[ \mathcal{M}_{\lambda_{\nu'}} = \frac{g_W^2}{2D_W} N_{\lambda_{\nu'}}^{\nu_{\nu}} (\ell', \ell) N_{\mu} (\nu_{\nu'}, \bar{\nu}_{\ell}), \quad (39) \]

\[ N_{\lambda_{\nu'}}^{\nu_{\nu}} (\ell', \ell) = \bar{u}_{\ell'}(\bar{p}_{\nu_{\nu'}}) \gamma_{\mu} \gamma_{L} u_{\nu_{\nu'}}(\bar{p}_{\nu_{\nu}}, \lambda_{\nu}), \quad (40) \]

\[ \mathcal{N}_{\nu_{\nu'}} (\nu_{\nu'}, \bar{\nu}_{\ell}) = \bar{u}_{\nu_{\nu'}}(\bar{p}_{\nu_{\nu'}}) \gamma_{\mu} \gamma_{L} v_{\nu_{\nu}}(\bar{p}_{\nu_{\nu}}). \quad (41) \]

where \( D_W^{-1} \) is propagator of the charged weak bosons. Accordingly, the decay spin density matrix can be rewritten as,

\[ \rho_{\lambda_{\nu'}}^{\lambda_{\nu}} = \frac{g_W^4}{4D_W^2} N_{\lambda_{\nu'}}^{\nu_{\nu} \nu_{\nu}} (\ell', \ell) N_{\mu \nu} (\nu_{\nu'}, \bar{\nu}_{\ell}), \quad (42) \]

where the two tensors are defined as

\[ N_{\lambda_{\nu'}}^{\nu_{\nu} \nu_{\nu}} (\ell', \ell) = N_{\lambda_{\nu'}}^{\nu_{\nu}} (\ell', \ell) N_{\lambda_{\nu}}^{\nu_{\nu}} (\ell', \ell), \quad (43) \]

\[ N_{\mu \nu} (\nu_{\nu'}, \bar{\nu}_{\ell}) = N_{\mu} (\nu_{\nu'}, \bar{\nu}_{\ell}) N_{\nu}^\dagger (\nu_{\nu'}, \bar{\nu}_{\ell}). \quad (44) \]

In this expression, the relative kinematical freedoms of the two neutrinos can be integrated out formally. Within our parameterizations of kinematical variables, the factor \( D_W^{-2} \) in (42)
can also depend on the angular variable $\theta_{\nu'\ell'}$ in general. Then, without loss of generality, the relative kinematical freedom of the two neutrinos can be integrated out by defining

$$\mathcal{N}^{\mu\nu}(m_X^2) = \frac{1}{4} g_W^4 \int d\Pi_X D_W^{-2} \mathcal{N}^{\mu\nu}(\nu_e, \bar{\nu}_e),$$

which is just a function of the invariant mass $m_X^2$. Correspondingly, the inclusive decay spin density matrix is given as

$$\rho^{\lambda'\ell'}_{\lambda\ell'} = N^{\lambda'\ell'}_{\lambda\ell'}(m_X^2),$$

(46)

Therefore, the tensor $\mathcal{N}^{\mu\nu}(m_X^2)$ is the most essential quantity for studying spin correlation effects in the decay process. The tensor $\mathcal{N}^{\mu\nu}(m_X^2)$ in the rest frame of the $\ell'$-lepton can be obtained by calculating its expression in the rest frame of the momentum $p_X$, and then applying rotations and boosts. With the excellent approximation, $D_W \approx -m_W^2 + im_W \Gamma_W$,

$$\mathcal{N}^{\mu\nu}(m_X^2) = f_\ell \left( p_\ell^\mu p^\nu_X - m^2_X g^{\mu\nu} \right),$$

(47)

where the coefficient $f_\ell = g_W^4 \left[ \frac{96\pi m_W^2 (m_W^2 + \Gamma_W^2)}{98 m_X^2} \right]^{-1}$.

The neutral current $\mathcal{N}^\mu_{\lambda'\ell'}(\ell', \ell)$ can be calculated directly in the rest frame of the $\ell'$-lepton, and is given as,

$$\mathcal{N}^\mu_{\lambda'\ell'} = \sqrt{\beta_X} \left( \begin{array}{c} e^{i\lambda' \phi_X} \sqrt{\frac{1 + \lambda' \cos \theta_X}{2 \beta_X}} \\ -e^{-i\lambda' \phi_X} \sqrt{\frac{1 - \lambda' \cos \theta_X}{2 \beta_X}} \\ -i\lambda' e^{-i\lambda' \phi_X} \sqrt{\frac{1 - \lambda' \cos \theta_X}{2 \beta_X}} \\ -\lambda' e^{-i\lambda' \phi_X} \sqrt{\frac{1 + \lambda' \cos \theta_X}{2 \beta_X}} \end{array} \right).$$

(48)

By using following relations,

$$p_X \cdot \mathcal{N}_{\lambda'\ell'} = \frac{m^2_{\ell'}}{\sqrt{2}} e^{i\lambda' \phi_X} \sqrt{\frac{1 + \lambda' \cos \theta_X}{\beta_X}},$$

(49a)

$$\mathcal{N}_{\lambda'\ell'} \cdot \mathcal{N}^\dagger_{\lambda'\ell'} = -m^2_X \beta_X \left( 1 - \lambda' \cos \theta_X \right),$$

(49b)

$$\mathcal{N}_{\lambda'\ell'} \cdot \mathcal{N}^\dagger_{\lambda'\ell'} = m^2_{\ell'} \beta_X \sin \theta_X e^{i\lambda' \phi_X}.$$

(49c)
Then we can obtain the density matrix,
\[
\rho_{\lambda \ell}^{\lambda' \ell'} = f_{\ell m}^4 \beta^{-1}_X \chi(z_X) \left( 1 + \lambda \ell' \mathcal{P}_\ell(z_X) \cos \theta_X \right),
\]
\[
\rho_{-\lambda \ell}^{\lambda' \ell'} = f_{\ell m}^4 \beta^{-1}_X \chi(z_X) \mathcal{P}_\ell(z_X) \sin \theta_X e^{i \lambda' \phi_X},
\]
(50)
where \( z_X = m_X^2 / m_\nu^2 \), and the coefficient are given as
\[
\chi(z_X) = (1 - z_X)^2 (1 + 2z_X),
\]
(52)
\[
\mathcal{P}_\ell(z_X) = \frac{1 - 2z_X}{1 + 2z_X}.
\]
(53)
Fig. 4(a) shows the functions \( \chi(z_X) \) and \( \mathcal{P}_\ell(z_X) \). We can see that the polarizor \( \mathcal{P}_\ell(z_X) \)
decrease quickly with respect to \( z_X \), and change its sign at \( z_X \approx 0.5 \). This is due to
that contributions of the spin-0 and the longitudinal component of the spin-1 states have
different sign (see Eq. (47)), and hence the \( m_X \)-dependent and \( p_X \)-dependent terms interfere
deconstructively (see Eq. (49)). Fig. 4(b) shows the normalized differential distribution of
the decay width \( \Gamma_{\ell}^{-1} d\Gamma_\ell / d\sqrt{z_X} \). Firstly, form the invariant mass distribution we can see that

![Fig. 4](image-url)
the hard invariant mass $m_X$ has relatively larger fraction. Since the correlation between $m_X$ and the energy of the lepton, leptons are expected to have a relatively soft energy spectrum in laboratory frame. This can reduce experimental efficiency due to cuts on momentum, but it is not too worse compared to the decay mode $\tau \to \pi \nu_{\tau}$ since usually harder cuts are used for the hadronic decay mode.

The $2D$ differential polarized decay width is given by

$$
\frac{d\Gamma_{\lambda'}}{dz_X d\cos \theta_\ell} = \frac{G_F^2 m_\ell^5}{192 \pi^3} \chi(z_X) \left( 1 - \lambda' P_\ell(z_X) \cos \theta_\ell \right),
$$

where $G_F = g_W^2/(4\sqrt{2} m_W^2)$ is the Fermi constant (we have used approximation $\Gamma_W \approx 0$), and we have used the relation $\theta_X = \pi - \theta_\ell$ such that the distributions are expressed in term of the polar angle of the charged lepton. Integrating over the invariant mass $m_X$, the polarized differential decay width is given as

$$
\frac{d\Gamma_{\lambda'}}{d\cos \theta_\ell} = \frac{G_F^2 m_\ell^5}{384 \pi^3} \left( 1 - \lambda' P_\ell \cos \theta_\ell \right),
$$

where $P_{\ell^-} = 1/3$. Compared to the hadronic decay mode $\tau \to \pi \nu_{\tau}$, $P_{\ell^-}$ is smaller, and hence the leptonic decay mode has lower sensitivity to the polarization state of the parent $\tau$-lepton. On the other hand, transverse spin correlation, i.e., the azimuthal angle distribution is also proportional to $P_{\ell^-}$, and hence is also reduced. However, because of its much larger branching ratio, the leptonic decay mode can also be important in practice. Furthermore, the exact transverse spin correlation also exist in the hadronic decay mode $\tau \to \pi \nu_{\tau}$. This means the same azimuthal angle distribution can be found in decay modes, $Y \to \tau_{\pi/\ell} \tau_{\pi/\ell}$, of a heavy particle $Y$. We will investigate elsewhere its possible application in searching $CP$-violation effect in decay of the Higgs boson $h(125)$ [89, 90] to $\tau$-lepton pair. Here we study only the longitudinal polarization effects.

In terms of energy fraction we have

$$
\frac{d\Gamma_{\lambda'}}{dz_X d\ell} = \frac{G_F^2 m_\ell^5}{48 \pi^3} \beta_\ell^{-1} \beta_X^{-1} \chi(z_X) \left[ \frac{1}{2} \left( 1 + \lambda' \beta_\ell^{-1} \beta_X^{-1} P_\ell(z_X) \right) - \lambda' \beta_\ell^{-1} \beta_X^{-1} P_\ell(z_X) z_\ell \right].
$$

Here we should note that upper and lower integration limits of the variable $z_\ell$ depend on $z_X$, and are given as $\text{Max}\{z_\ell\} = \alpha_\ell (1 + \beta_\ell \beta_X)/2$ and $\text{Min}\{z_\ell\} = \alpha_\ell (1 - \beta_\ell \beta_X)/2$. In the limit $\beta_\ell \approx 1$, $\text{Max}\{z_\ell\} \approx 1 + z_X$ and $\text{Min}\{z_\ell\} \approx 0$. Within this limit the above differential equation can be approximated as,

$$
\frac{d\Gamma_{\lambda'}}{dz_X d\ell} = \frac{G_F^2 m_\ell^5}{48 \pi^3} \beta_X^{-1} \chi(z_X) \left[ \frac{1}{2} \left( 1 + \lambda' \beta_\ell^{-1} P_\ell(z_X) \right) - \lambda' \beta_X^{-1} P_\ell(z_X) z_\ell \right] \Theta(1 - z_X - z_\ell),
$$

18
where $\Theta(1 - z_X - z_\ell)$ is the usual step-function. Then we can perform the integration with respect to the variable $z_X \in [0, 1 - z_\ell]$, and we obtain,

$$d\Gamma_{\lambda_\tau} = \frac{G_F^2 m_\nu^5}{192\pi^3} \left[ I(z_\ell) + \lambda_\nu K(z_\ell) \right],$$

(58)

where the two functions are given as

$$I(z_\ell) = \frac{1}{3} \left( 4z_\ell^3 - 9z_\ell^2 + 5 \right),$$

(59)

$$K(z_\ell) = \frac{1}{3} \left( 8z_\ell^3 - 9z_\ell^2 + 1 \right).$$

(60)

And we have normalized these two functions such that integration of $I(z_\ell)$ with respect to $z_\ell \in [0, 1]$ gives 1, and integration of $K(z_\ell)$ in the same integration is zero as it should be. Fig. 5(a) and Fig. 5(b) show energy fraction distributions of $\mu^-$ (a) and $e^+$ (b) in decays $\tau^- \to \mu^- \bar{\nu}_\mu \nu_\tau$ and $\tau^+ \to e^+ \nu_e \bar{\nu}_\tau$ with different polarizations $\lambda_\tau$ and $\bar{\lambda}_\tau$ of the $\tau$-leptons, respectively. The $\tau$-leptons are assumed to be generated with a fixed energy $E_\tau = 5.29\text{GeV}$. Our predictions given by (58) (in massless limit of the leptons) are shown by the curves. The filled squares and circles are simulating results by using MadGraph with $3 \times 10^4$ events in total (sum of $\lambda_\tau = \pm 1$).

FIG. 5. Energy fraction distributions of $\mu^-$ (a) and $e^+$ (b) in decays $\tau^- \to \mu^- \bar{\nu}_\mu \nu_\tau$ and $\tau^+ \to e^+ \nu_e \bar{\nu}_\tau$ with different polarizations $\lambda_\tau$ and $\bar{\lambda}_\tau$ of the $\tau$-leptons, respectively. The $\tau$-leptons are assumed to be generated with a fixed energy $E_\tau = 5.29\text{GeV}$. Our predictions given by (58) (in massless limit of the leptons) are shown by the curves. The filled squares and circles are simulating results by using MadGraph with $3 \times 10^4$ events in total (sum of $\lambda_\tau = \pm 1$).
respectively. Our predictions given by (58) (in massless limit of the leptons) are shown by the curves. We can see that there are some differences in the small $z_\pm$ region. This is because of we have neglected masses of the leptons, and have used the approximation $\beta_\tau \sim 1$.

**IV. POLARIZATION CORRELATION EFFECTS AT THE BELLEII**

In this section we study the spin correlation effects in decays of $\tau$-lepton pair which are produced at a $e^-e^+$ collider. Particularly we focus on the BelleII experiment at where the $\tau$-lepton pair is generated with energy $\sqrt{s} = 10.58\text{GeV}$ at the center of mass frame. Since the major irreducible backgrounds of the signal $\tau^\pm \rightarrow \ell^\pm a$ are the pure leptonic decay modes $\tau^\pm \rightarrow \ell^\pm \bar{\nu}_\tau \nu_\tau$ which has more than one neutrino, hence it is impossible to reconstruct the full kinematics of the final state. This is also true even when the relative degree of freedom of the two neutrinos in each side are integrated out, as we have down in Sec. III C. The ARGUS collaboration studied the signal processes, by tagging one side of the $\tau$-lepton pair decaying in 3-prong mode, and an upper limits $\sim 2\%$ was obtained [55]. Such $3-1$ prong searching method is also used by the Belle II experiment [56–59]. However, number of events is significantly reduced because of small branching fraction of the 3-prong decay channel.

At BelleII, the $\tau$-lepton pair is produced essentially by via a virtual photon, and hence are not polarized. Nevertheless, polarizations of the two $\tau$-leptons are correlated because (in massless limit of the beam particles) helicities of the $\tau$-lepton pair are required to satisfy the relation $\lambda_\tau = -\lambda_\tau^*$ thanks to spin conservation. As we have shown that polarization effect of the leptonic decay modes $\tau^\pm \rightarrow \ell^\pm \bar{\nu}_\tau \nu_\tau$ survives when the invariant mass of the two neutrinos is inclusively measured. Moreover, signal and backgrounds have distinctive polarization effects, and hence signals can be investigated by studying polarization correlations of these $\tau$-leptons. In this work we investigate following processes:

$$e^-e^+ \rightarrow \tau^-_\alpha \tau^+_\beta,$$

(61)

where $\alpha, \beta = \ell, \pi, a\ell$ with $\tau^\pm_\ell = \tau^\pm \rightarrow \ell^\pm \bar{\nu}_\tau \nu_\tau$, $\tau^\pm_\pi = \tau^\pm \rightarrow \pi^\pm \nu_\tau$ and $\tau^\pm_\ell^* = \tau^\pm \rightarrow \ell^\pm a$. The other hadronic decay channels of the $\tau$-lepton can also be used, but we will study them elsewhere.

For the $e^-e^+ \rightarrow \tau^-_\ell \tau^+_\ell^*$ channel, the normalized cross section can be written as,

$$\frac{1}{\sigma_{\tau^-_\ell \tau^+_\ell^*}} \frac{d\sigma}{dz^-_\ell dz^+_\ell^*} = I_\ell(z^-_\ell)I_{\ell^*}(z^+_\ell^*) + K_\ell(z^-_\ell)K_{\ell^*}(z^+_\ell^*).$$

(62)
In massless limits of the leptons $\ell$ and $\ell'$, we have $\mathcal{I}_\ell = \mathcal{I}'_{\ell'} = \mathcal{I}$ and $\mathcal{K}_\ell = \mathcal{K}_{\ell'} = \mathcal{K}$. In this case, we have a relatively soft positive correlations between $z_\ell^-$ and $z_{\ell'}^+$. Fig. 6(a) and Fig. 6(b) show contour plots of the above differential cross section in $z_\ell^- - z_{\ell'}^+$ plane for the channels $e^- e^+ \rightarrow \tau^- \tau^+$ and $e^- e^+ \rightarrow \tau^- \tau^+_\mu$ at $\sqrt{s} = 10.58$GeV, respectively. We can see clear positive correlations in both channels. There is a small difference in the left-bottom corner due to mass differences between $m_\mu$ and $m_e$. As we have mentioned, we will neglect this differences in this work.

For the corresponding signal processes, $e^- e^+ \rightarrow \tau^- \tau^+_\mu$ and $e^- e^+ \rightarrow \tau^- \tau^+_e$, the differential cross section can be written as (without loss of generality we discuss only the channel $e^- e^+ \rightarrow \tau^- \tau^+_e$, completely the same distributions can be found in the charge conjugated channels),

$$\frac{1}{\sigma_{\tau^- \tau^+_e}} \frac{d\sigma_{\tau^- \tau^+_e}}{dz^- dz^+} = \mathcal{I}(z^-) - \xi\mathcal{K}(m_a)\xi\mathcal{K}(z^+),$$

where $\xi\mathcal{K}(m_a)$ is a function of the mass of the ALP, and $\xi\mathcal{K}(0) = \xi\mathcal{K}$. Fig. 7(a) and Fig. 7(b) show contour plots of the differential cross section for the signal process $e^- e^+ \rightarrow \tau^- \tau^+_e$ with
$m_a = 0\text{GeV}$ and $\xi_{e\tau} = \pm 1$. We can clearly see that the correlation properties are completely different from the ones of backgrounds.

![Contour plots](image)

**FIG. 7.** Contour plots of the normalized differential cross sections in the $z_-$ – $z_+$ plane for the channels $e^- e^+ \rightarrow \tau^- \tau^+_{ae}$ at $\sqrt{s} = 10.58\text{GeV}$ with $m_a = 0\text{GeV}$ and $\xi_{e\tau} = \pm 1$. The distributions are obtained by using $10^5$ events for every channels.

Sophisticated analysis on the spin correlation effects can be done by using general fitting method (for instance the likelihood method) in the $z_-$ – $z_+$ plane. Here we analyze the spin correlation effects by introducing asymmetry observables in specified regions of the $z_-$ – $z_+$ plane. In consideration of that $z_-$ and $z_+$ of the backgrounds are positively correlated along $z_- = z_+$, we introduce following asymmetry observable,

$$A_{\ell\ell'} = \frac{N_{\ell\ell'}(z_- > z_+) - N_{\ell\ell'}(z_- < z_+)}{N_{\ell\ell'}(z_- > z_+) + N_{\ell\ell'}(z_- < z_+)}.$$ \hfill (64)

In massless limit of the leptons $\ell$ and $\ell'$, we have $\mathcal{I}_\ell = \mathcal{I}_{\ell'} = \mathcal{I}$ and $\mathcal{K}_\ell = \mathcal{K}_{\ell'} = \mathcal{K}$, hence the SM predictions are simply zero,

$$A_{\ell\ell'}^{\text{SM}} = 0.$$ \hfill (65)

However, for the ALP model, the asymmetry observable is proportional to the parameter $\xi_{e\tau}(m_a)$, and can be written as,

$$A_{\ell\ell'}^{\text{ALP}}(m_a) = \epsilon_{\ell\ell'} \xi_{e\tau}(m_a),$$ \hfill (66)
where $\epsilon_{\ell\ell'}$ is an efficiency factor. In case of that $m_a = 0$, we obtain

$$A_{\ell\ell'}^{ALP}(0) = -0.289 \, \epsilon_{\ell\ell'} \, \xi_{\ell\ell'}.$$  \hfill (67)

Combing the backgrounds and signals the total asymmetry can be written as

$$A_{\ell\ell'}(m_a) = \frac{\epsilon_{\ell\ell'}^{Exp} R_{\ell\ell'}^a}{(1 + \epsilon_{\ell\ell'}^{Exp} R_{\ell\ell'}^a)} A_{\ell\ell'}^{ALP}(m_a),$$  \hfill (68)

where $R_{\ell\ell'}^a = N_{\ell\ell'}^{ALP}/N_{\ell\ell'}^{SM}$ are branching ratios of the corresponding signal process, and $\epsilon_{\ell\ell'}^{Exp}$ are ratios of efficiencies due to experimental reconstructions and kinematical cuts whose efficiencies can be different for signals and backgrounds. Assuming that statistical uncertainties of the number of events in each region follows the standard Gaussian uncertainty, i.e.,

$$\delta N_{\ell\ell'}(z_- > z_+) = \sqrt{N_{\ell\ell'}(z_- > z_+)}$$

and

$$\delta N_{\ell\ell'}(z_- < z_+) = \sqrt{N_{\ell\ell'}(z_- < z_+)},$$

then statistical uncertainties of the asymmetry observables $A_{\ell\ell'}(m_a)$ are given as,

$$A_{\ell\ell'}(m_a) = \frac{1}{\delta N_{\ell\ell'}} \sqrt{1 - (A_{\ell\ell'}(m_a))^2}. \hfill (69)$$

The total cross section of $e^- e^+ \rightarrow \tau^- \tau^+$ at $\sqrt{s} = 10.58 \text{GeV}$ is about $8.25 \times 10^2 \text{pb}$ (at the leading order). Given the branching ratios $B_{\tau^-} = 17.41\%$ and $B_{\tau^+} = 17.83\%$, the total cross sections of $e^- e^+ \rightarrow \tau^- \tau^+$ is about $1.02 \times 10^2 \text{pb}$. For a projected luminosity $L_{\text{ab}}^{-1}$, we have approximately $L \times 10^8 \tau^- \tau^+$ events in total. The BelleII experimental aims to accumulate a luminosity $L = 50$. Hence by just using the leptonic decay modes (in both sides) the asymmetries can be probed at a level of $\sim 10^{-4}$, and can be even smaller according to (69) if $A_{\ell\ell'}(m_a)$ is relatively large.

Experimental searches for the lepton flavor violation in decays of the $\tau$-lepton can be further improved by including hadronic decay modes of the $\tau$-lepton. Fig. 8 shows polarization correlations in the process $e^- e^+ \rightarrow \tau^- \tau^+$. We can see that the correlation behaviours are completely different from the ones of signal processes, $e^- e^+ \rightarrow \tau^- \tau^+$, which are shown in Fig. 9(a) and Fig. 9(b) with $\xi_{\mu\tau} = \pm 1$, respectively. However, in contrast to the simple positive correlation in the leptonic decay channels $e^- e^+ \rightarrow \tau^- \tau^+$, energy fractions of the charge particles in the present channels are correlated nontrivially. From Fig. 1(a) and Fig. 5(a) we can see that in both $\tau^+ \tau^- \ell^-$ and $\tau^+ \tau^+ \ell^-$ modes the critical value of energy fraction $z_+ \approx 0.5$ is good to separate the event with different polarizations. Therefore, for the channel $e^- e^+ \rightarrow \tau^- \tau^+$, we introduce following two asymmetry observables,

$$A_{\pi\ell}^{\pm} = \frac{N_{\pi\ell}(z_+ > 0.5) - N_{\pi\ell}(z_+ < 0.5)}{N_{\pi\ell}(z_+ > 0.5) + N_{\pi\ell}(z_+ < 0.5)},$$  \hfill (70)
Where $A_{\ell\pi}^\pm$ corresponds to $z_- > 0.5$ and $z_- < 0.5$. Similarly for the charge conjugate channels $e^-e^+ \rightarrow \tau_-^\pm \pi_\ell^\mp$ we can define,

$$A_{\ell\pi}^\pm = \frac{N_{\ell\pi}(z_- > 0.5) - N_{\ell\pi}(z_- < 0.5)}{N_{\ell\pi}(z_- > 0.5) + N_{\ell\pi}(z_- < 0.5)},$$

(71)

where $A_{\ell\pi}^\pm$ corresponds to $z_+ > 0.5$ and $z_+ < 0.5$. In the massless limit of $\ell$-lepton, our results give for the process $e^-e^+ \rightarrow \tau_-^\pm \pi_\ell^\mp$, 

$$(A^{SM}_{\ell\pi})_{\pi\ell}^+ \approx -0.542, \quad \text{and} \quad (A^{SM}_{\ell\pi})_{\pi\ell}^- \approx -0.375.$$  

(72)

Again for the signal processes, the asymmetries are functions of the mass of the ALP,

$$(A^{ALP})_{\pi\ell}(m_a) = \epsilon_{\pi\ell}^\pm \xi_{\ell\tau}(m_a).$$

(73)

where $\epsilon_{\pi\ell}$ are efficiency factors. In the case of $m_a = 0$, we have,

$$(A^{ALP})_{\pi\ell}^\pm \approx \pm 0.25 \epsilon_{\pi\ell}^\pm \xi_{\ell\tau},$$

(74)

Given that $B_{\tau\pi} = 10.83\%$ and a projected luminosity $L \times 10^8$ events for the processes $e^-e^+ \rightarrow \tau_-^\pm \pi_\ell^\pm$, which is at the same order compared to the
leptonic decay modes. Since the asymmetries are relatively large, hence experimental uncertainties of these observables are expected to be at the same level. $\delta A_{\pi\ell}^{\pm} \approx 10^{-4}$ for luminosity $L = 1 \text{ ab}^{-1}$. It is also expected that the other hadronic decay channels, $\tau^{\pm} \rightarrow \rho^{\pm}\nu_{\tau}$ and $\tau^{\pm} \rightarrow a_{1}^{\mp}\nu_{\tau}$ can reach the same level.

V. SUMMARY

In summary we have studied polarization effects of the lepton flavor violated decays induced by an ALP. Experimental searches on these rare decay channels are challenging because of the huge irreducible backgrounds $\ell' \rightarrow \ell\ell'\nu_{\ell}\nu_{\ell'}$, and hence the current experimental bounds on the decay modes $\ell'^{\pm} \rightarrow a\ell^{\pm}$ are much weaker than other flavor violated processes. In this paper, we treat inclusively the two missing neutrinos in the SM leptonic decay modes, and their relative kinematical degrees of freedom are integrated. After that we find that both longitudinal and transverse polarization effects survives. Observing transverse polarization effects requires reconstructions of the mother leptons, and hence is challenging particularly
for $\tau$-leptons produced at low energy colliders, we will study this elsewhere (as well as its applications in other problems). In contrast, the energy fractions of its charged decay products, which can be defined unambiguously and measured precisely at $e^-e^+$ collider, are correlated with the longitudinal polarizations of the mother leptons. Furthermore, we have show that signal and backgrounds have distinctive polarization effects, and hence signals of lepton flavor violation can be searched for by studying longitudinal polarizations of the decaying leptons. On the other hand, leptons generated at low energy collider are usually unpolarized and hence there is no net polarization effect. However, we have show that correlations between polarizations of the $\tau$-lepton pair are still useful to search for the signals. More interestingly, polarization correlations depends on product of scalar and pseudo-scalar ALP couplings, and hence are sensitive to their relative sign. This is a distinctive property compared to the usually observable (cross section or decay branching ratio). By introducing asymmetry observables, we demonstrate that how the polarization correlation effects can be used to investigate flavor violating decays $\tau^\pm \rightarrow a\ell^\pm$ at the BelleII experiment.

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