Sangaku Problems About Ellipses: Why Primary Sources Matter

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In the literature of Japanese mathematics of the Edo period, ellipses are generally described either as oblique sections of right circular cylinders or as affine images of circles. Although it has been claimed that Seki Takakazu (1642–1708), sometimes thought of as the “father” of wasan, treated ellipses as oblique conic sections [14, p. 198], earlier studies deny this [7, p. 172], and Seki himself derived the formula for the area $A$ of an ellipse by considering the shear of a right circular cylinder of diameter $v$ and altitude $h$, as shown in Figure 1.

Under this transformation, one obtains an oblique elliptical cylinder with axes $u$ and altitude $k$. By similar triangles, $h/u = k/v$. Since the volumes of the cylinder and its image are the same (Cavalieri’s principle), it follows that

$$\pi v^2 b = Ak = \frac{dv}{u},$$

and so $A = \pi uv$ [9, pp. 116–118].

This proof shows more generally that for Seki, the cylindrical-section and affine-image definitions of an ellipse were equivalent. The near total absence of wasan problems involving hyperbolas or parabolas, in contrast to the great variety involving ellipses, also suggests that ellipses were not usually thought of as conic sections.

An ellipse may also be defined as the locus of a point in a plane whose distances to two points in the plane—the foci—sum to a given constant. The physical implementation of this definition is the familiar pins-and-string construction method, which the Japanese knew but hardly ever mentioned (see, e.g., [1]; see also [5, p. 192]). In analytic geometry, it is usual to derive the Cartesian equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ of the canonical ellipse from the locus definition. The wasanka knew this in the form $v^2x^2 + u^2y^2 - u^2v^2 = 0$, where $x > y$ are the sides of a rectangle circumscribed by an ellipse with axes $u > v$ parallel to the sides (e.g., [16, Proposition 92]). But if the proof of Aida Yasuaki (in [1]) is any indication, they arrived at it straight from the affine-image definition: by the crossed chords theorem, if a circle of diameter $u$ has half-chord $h$ perpendicular to it at some fixed diameter at distance $s/u$ from one of its endpoints, then $s(u - s) = b^2$. Now transform this circle into an ellipse by scaling the perpendicular distances of its points from the fixed diameter by $v/u$ for some $v, v < u$. The corresponding half-chord $k$ of the ellipse is

$$k^2 = \frac{v^2}{u^2} s(u - s) = \frac{v^2s}{u} - \frac{v^2s^2}{u^2};$$

hence $uv^2s - v^2s^2 - u^2k^2 = 0$. If a rectangle circumscribed in this ellipse has sides $y = 2k$ parallel to its minor axis, then $s = \frac{u}{2} - \frac{v}{2}$, so

1Premodern Japanese mathematics is retroactively called wasan, in contrast to yōsan, “Western mathematics.” Its practitioners, or wasanka, frequently displayed problems that they had devised or solved on plaques called sangaku in temples or shrines, so outside Japan, sangaku is often used metonymically for all geometry problems in the premodern Japanese tradition. Manuscripts and woodblock-printed books documenting these problems proliferated in the eighteenth and nineteenth centuries.
or $v^2x^2 + u^2y^2 = u^2v^2 = 0$. The greater difficulty of proving that the locus and conic section definitions of ellipses are equivalent is evident from the late date of what is now given in textbooks as the standard proof [2].

Given these differences between the Japanese and Western approaches to ellipses, particularly the absence of the Cartesian plane and the indifference to the algebraic significance of the foci in the Japanese approach, many modern solutions of sangaku ellipse problems for which the original solutions are missing cannot be what wasanka had in mind despite their seeming simplicity. To understand what the wasanka were likely to be thinking requires looking whenever possible at the primary sources they left behind.

The Magical Equation

For example, modern solutions of many sangaku problems (e.g. [4, Problems 6.1.5, 6.1.6, 6.1.7, and 7.1.1]) seem to cry out for a lemma based on the premise that each tangent has a slope $m$ with respect to one of the axes of the ellipse. Such a lemma is our Proposition 1.

**Proposition 1.** If the lengths of the major and minor axes of the ellipse are $2u$ and $2v$, then parallel tangents of finite slope $m$ with respect to the major axis meet the minor axis at a distance $\pm\sqrt{m^2u^2 + v^2}$ from the center of the ellipse.

**Proof** Assume a Cartesian coordinate system aligned with the axes, and assume that the line $y = mx + c$ is tangent to the ellipse. Substituting $mx + c$ for $y$ in the standard equation yields $(m^2u^2 + v^2)x^2 + 2cmu^2x + c(u^2 - v^2) = 0$. This is a quadratic in $x$; its discriminant must be zero, because a tangent meets an ellipse in a single point. Hence

$$\left(2cmu^2\right)^2 - 4(m^2u^2 + v^2)u^2(c^2 - v^2) = 0,$$

or $c = \pm\sqrt{m^2u^2 + v^2}$, with one value of $c$ for each of the parallel tangents with slope $m$.

The equation $y = mx \pm \sqrt{m^2u^2 + v^2}$ was called the magical equation to the tangent in nineteenth-century Western mathematics, because it represents the tangents “magically” without referring to the coordinates of the points of tangency (e.g., [15]). The “magical equation to the normal” is $y = nx \pm \sqrt{n^2u^2 + v^2}$, where $mn = -1$. The tangent and normal meet at the same point on the ellipse for $m > 0$ if the $\pm$ signs are the same and for $m < 0$ if the $\pm$ signs are opposite. For $m = 0$, the tangents are $y = \pm v$ (as Proposition 1 asserts), and the two normals coincide with the $y$-axis ($n = \infty$). For $n = 0$, the two normals coincide with the $x$-axis (as the foregoing equation asserts), and the tangents are the lines $x = \pm u (m = \infty)$, which have no $y$-intercepts.

Affine Transformation

Because the Japanese thought of ellipses as sections of cylinders, they naturally understood the affine relationship between the diameters of cylinders and the axes of an ellipse based on it, and they relied heavily on this knowledge to solve problems. The following four discussions can be thought of as short case studies that illustrate the difference between the Japanese approach and more modern ones.

The Director Theorem

Consider, for example, the following theorem and its modern proof.

**Proposition 2.** The locus of the points at which two orthogonal tangents to an ellipse with semiaxes $u, v$ intersect is the concentric circle of radius $u^2 + v^2$, the director circle of the ellipse.

**Proof** By Proposition 1, if a tangent passes through $P(b, k)$, then $k = mb \pm \sqrt{m^2u^2 + v^2}$, that is, $(k - mb)^2 = m^2u^2 + v^2$, or $(b^2 - u^2)m^2 - 2bkm + (k^2 - v^2) = 0$. The roots of this quadratic are the slopes of the two tangents through $P$. They are orthogonal if and only if their product is $-1$. This product is the ratio of the constant term to the leading coefficient; i.e., $\frac{2bkm}{m^2u^2 + v^2} = -1$, or $b^2 + k^2 = u^2 + v^2$. Therefore, orthogonal tangents meet at $P$ if and only if $P$ lies on the circle $x^2 + y^2 = u^2 + v^2$.

This is very nearly Proposition 89 in [16], which asserts, without proof, that if an ellipse is inscribed in a rectangle with sides $a, b$, then $a^2 + b^2 = u^2 + v^2$, where $u, v$ are the whole axes. In this instance, we have an explicit proof in a primary source [6], which I freely translate as follows:

Figure 1. Seki's sheared cylinder.
PROOF (FUKUDA) Scale the figure by $v/u$ in the direction of the major axis (see Figure 2). The images of segments orthogonal to the major axis, like $g$ and $b$, do not change length. In the preimage, $p + q$ is the diagonal of the larger rectangle, and

$$c^2 - b^2 = a^2, \quad b^2 + d^2 = (p + q)^2, \quad \frac{bd}{p+q} = b,$$

$$\frac{b^2}{p+q} = b.$$

In the image, we have $\frac{b^2}{(p+q)^2} = q$, so

$$b^2 = b^2 + q^2 = \frac{b^2 d^2}{(p+q)^2} + \frac{b^4 v^2}{(p+q)^2 u^2}.$$

Hence $b^2 v^2$, the square of the area of the rhombus in the image, is

$$\frac{b^2 d^2 v^2}{(p+q)^2} + \frac{b^4 v^2}{(p+q)^2 u^2}.$$

But in terms of the preimage, the same area is $\frac{a^2 b^2 v^2}{u^2}$, or

$$\frac{(c^2 - b^2) b^2 v^2}{u^2} = \frac{b^2 c^2 v^2}{u^2} - \frac{b^4 v^2}{u^2}.$$

Hence

$$\frac{b^2 d^2 v^2}{(p+q)^2} + \frac{b^4 v^2}{(p+q)^2 u^2} = \frac{b^2 c^2 v^2}{u^2} - \frac{b^4 v^2}{u^2}.$$

Simplifying yields

$$d^2 u^2 + b^2 v^2 - c^2 (p + q)^2 + b^2 (p + q)^2 = 0.$$

On substituting $b^2 + d^2$ for $(p + q)^2$, we obtain

$$d^2 u^2 + b^2 v^2 - b^2 c^2 - c^2 d^2 + b^4 + b^2 d^2 = 0. \quad [*]$$

Now, $\frac{d^4 v^2}{(p+q)^2 u^2} + \frac{b^2 d^2}{(p+q)^2} = d^2$.

In the same way as before, we equate two expressions for the area of the parallelogram in the image, which gives us $d^4 v = \frac{bd}{p+q}$. Hence $bd = d^4 u$, or on squaring, $b^2 d^2 = d^2 u^2$. Therefore,

$$\frac{d^4 v^2}{(p+q)^2} + \frac{b^2 d^2 u^2}{(p+q)^2} = b^2 d^2,$$

or on substituting $b^2 + d^2$ for $(p + q)^2$ and simplifying,

$$b^2 u^2 + d^2 v^2 - b^4 - b^2 d^2 = 0. \quad [**]$$

Adding [*] and [**] yields

$$u^2 (b^2 + d^2) + v^2 (b^2 + d^2) - c^2 (b^2 + d^2) = 0,$$

which reduces to $u^2 + v^2 = c^2$. \hfill \square

Working Around Proposition 1

Because the proof in [6] of Proposition 2 does not use Proposition 1, it is more complicated than the modern proof given previously. Though it seems natural to invoke Proposition 1 to solve many ellipse problems, other theorems, such as Proposition 82 in [16], can often be used instead; it states that if an ellipse with axes $u > v$ is inscribed in an isosceles trapezoid with one axis parallel to the bases $a > b$, then the length of that axis is the geometric mean of the bases. This is easily proved from the fact that if an isosceles trapezoid has an incircle of diameter $b$, then, since the tangents to a circle from the same point are equal, we must have $(\frac{x}{2} + \frac{y}{2})^2 = b^2 + (\frac{x}{2} - \frac{y}{2})^2$, or $ab = b^2$. Now consider an ellipse with axes $u > v$ inscribed in an isosceles trapezoid of height $v$ with bases $a > b$ parallel to the major axis of length $u$. By scaling all distances in that direction by $\frac{v}{b}$ one obtains an
isosceles trapezoid with bases $a' > b'$ about a circle of diameter $v$ (distances orthogonal to the direction of scaling are unaffected). Since $a'b' = v^2$, the inverse transformation gives $\frac{a'}{b'} = \frac{b'}{v^2}$, or $ab = u^2$.2

Of course, $u$ and $v$ can be interchanged. The proposition is particularly useful when one has three tangents, two parallel to an axis of the ellipse and the third a transversal; one uses half the lengths of the bases and parallel axis. We postpone an example of this to later.

Ellipses Inscribed in Parallelograms

A study of a second, more challenging, proposition, mentioned (without proof and written incorrectly without the denominator $b^2$) in [3, p. 335], suggests another way the Japanese may have solved problems without the aid of Proposition 1. Here is the proposition and my own proof of it. (I write $(E)u, v$ for an ellipse with center $E$ and semiaxes $u, v$ by analogy to the notation $(O)r$ for a circle with center $O$ and radius $r$.)

**Proposition 3.** If an ellipse $(E)u, v$ is inscribed in a parallelogram with sides $2a > 2b$ and distance $2b$ between the longer sides, then

$$\left( a^2 - b^2 - u^2 + v^2 \right)^2 = 4(b^2 - v^2)(b^2u^2 - a^2b^2)$$

**Proof** If the slopes of the sides are $p, q$, then according to Proposition 1, $y = px \pm \sqrt{u^2p^2 + v^2}$ and $y = qx \pm \sqrt{u^2q^2 + v^2}$. Assume first that $p > 0$ and $q < 0$. On computing any three of the four points where these lines intersect and the distances between them, one finds that half the sides of the parallelogram are $a = \sqrt{(q^2 + 1)(a^2p^2 + q^2v^2)}$ and $b = \sqrt{(p^2 + 1)(u^2p^2 + q^2v^2)}/p$. On the other hand, the distance along the line $y = -\frac{a}{b}$ from the origin to the side $y = qx + \sqrt{u^2q^2 + v^2}$ is $b = \sqrt{\frac{u^2q^2 + v^2}{q}}$. Whether one substitutes the foregoing expressions for $a, b$, and $b$ into $(a^2 - b^2 - u^2 + v^2)^2$ or into $\left( \frac{(b^2 - v^2)(b^2u^2 - a^2b^2)}{b^2} \right)$, the result is $a^2b^2 = a^2b^2$, so the two expressions are equal.

If the longer sides are parallel to the major axis of the ellipse, then $b = v > 0$, and the equation simplifies to $a^2 - b^2 = u^2 - v^2$. (This is a sangaku result in its own right; see [3, p. 334].) If the parallelogram is a rectangle, then $b = v > 0$, and the equation simplifies to $(a^2 - b^2 - u^2 + v^2)^2 = 4(b^2 - v^2)(b^2u^2 - a^2b^2)$, or $a^2 + b^2 = u^2 + v^2$. (This is Proposition 2, the director circle theorem.) Finally, if $b = v = b$, then the rectangle is aligned with the axes of the ellipse, and of course, $a = u$ and $b = v$.

I have not been able to locate a primary source for Proposition 3, but it is obvious that despite the power of Proposition 1, the foregoing proof is rather deficient in geometric motivation. So how would the wasanka have proved it? Proposition 3 is easily proved for bounding parallelograms, so perhaps that was the starting point.

A parallelogram circumscribed about an ellipse is called bounding if the lines joining opposing touch points are the bimedians of the parallelogram. (The lines are then called conjugate diameters of the ellipse.) It is easy to prove Proposition 3 for bounding parallelograms because of the algebraic identity

$$(a^2 - b^2 - u^2 + v^2)^2 = 4(u^2 - a^2)(b^2 - v^2).$$

For the one hand, $a^2 + b^2 - u^2 - v^2 = 0$ only if $a$ and $b$ are conjugate semidiameters of an ellipse with semiaxes $u, v$ (first theorem of Apollonius—Proposition 2 again). And on the other, the areas of all bounding parallelograms are $4uv$; hence $uv = ab$ (second theorem of Apollonius), which is equivalent to the solution of [4, Problem 6.1.1]. Setting the second term on the left equal to zero yields $(a^2 - b^2 - u^2 + v^2)^2 = 4(u^2 - a^2)(b^2 - v^2)$; and expanding the right side of this equation gives us

$$4u^2(b^2 - u^2 - v^2) = \frac{4u^2(b^2 - u^2 - v^2)(b^2u^2 - a^2b^2)}{u^2v^2}$$

Thus, if an ellipse is inscribed in a bounding parallelogram of sides $2a > 2b$, the proposition is true. For an ellipse in a nonbounding parallelogram, one can construct a bounding parallelogram also of altitude $2b$, as shown in Figure 3. (There are generally two ways to do this.)

Geometrically, this amounts to applying an invertible affine transformation (a composition of a shear and a scaling) that maps the nonbounding parallelogram onto the bounding parallelogram. Did some wasanka argue from this fact to a generalization? The Japanese often applied affine transformations that take ellipses to circles and vice versa, as in Fukuda’s proof of Proposition 2. Another example is explained further below. In the absence of a primary source, we cannot be sure how the wasanka proved Proposition 3 for nonbounding parallelograms, but

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2It is easy to prove, in similar fashion, Proposition 83 in [16], which states that in the case of a right trapezoid, the parallel axis is the harmonic mean of the bases.

Hasegawa Hiroshi (1782–1839) used various transformations to simplify specific sangaku problems, but not in a systematic way. His method, called kyokkei jutsu 枢形術 [13, pp. 225–228], seems to have included shears. Even accomplished wasanka such as Uchida Iisumi (or Gokan, 1805–1882) went astray when they used Hasegawa’s kyokkei jutsu. For a famous and well-documented case, see [3, pp. 290–330], [4, pp. 64, 156].
it seems unlikely that they availed themselves of Proposition 1.

**Promised Example**

Here is a simpler and clearer case of working around Proposition 1. [4, Problem 7.2.4].

Figure 4. Fukagawa and Pedoe Problem 7.2.4.

**A solution without Proposition 1 goes as follows:** Label the various points as in Figure 4. All the right triangles are similar, with legs in the ratio 2 : 1, since $MA = CN = I\sqrt{5}$, we have $FO = I\sqrt{5} / 4$, $EO = I\sqrt{5}$, and $EF = \frac{5}{2}I$, since $MF = FC = \frac{1}{2}I$, we have $EM : EF : EC : a : b : l$. Hence $MH = \frac{1}{2}FO = \frac{l\sqrt{5}}{2}$. Likewise, $CK = \frac{2}{3}FO = \frac{l\sqrt{5}}{3}$. Applying Proposition 82 in [16] to $HMCK$, $u^2 = MH \cdot CK = \frac{4}{3}I^2$, we have $5u^2 = 6I^2$, and $u = l\sqrt{6/5}$.

Nowadays, a solver might be inclined to say that $FO$ and $EO$ lie, respectively, on the $x$- and $y$-axes of a Cartesian coordinate system with origin $O$. Then line $CE$ has the equation $y = 2x + l\sqrt{5}$, and so by Proposition 1, $4u^2 + v^2 = 5I^2$. By the first corollary of Proposition 3, $w^2 - v^2 = \frac{2I^2}{5} - \frac{I^2}{5} = I^2$. Adding equations yields $5u^2 = 6I^2$, so $u = l\sqrt{6/5}$.

Insofar as the more modern solution requires two hardly obvious supporting propositions, it is arguably no improvement over the first.

**Invariants Under Affine Transformations**

Although some *wasanka* evidently understood that certain properties of figures remain invariant under affine transformations, as Seki’s derivation of the area of an ellipse shows, one may wonder how fully developed a theory they had about these facts. Consider, for example, three different ways of proving Proposition 95 in [16]:

**Proposition 4.** If an ellipse is inscribed in a rectangle as shown (Figure 5, left), then $BE \cdot CF = EC \cdot FD$.

This could more descriptively be stated as follows: an ellipse inscribed in a rectangle touches adjacent sides at points that divide them in the same ratio.

**Proof** [11, p. 116]. Transform the ellipse and rectangle as shown in Figure 5. The image of the rectangle is a rhombus, so $B'C' = C'D'$. But $E'C' = C'F'$ (equal tangents to a circle from the same point). Therefore, $B'E' = F'D'$. Noting that $a^2 + b^2 = c^2 + d^2$ in the image and that the pairs of red-legged right triangles with hypotenuses on $B'C'$ and $D'C'$ are similar, we see that $B'E' = F'D'$ implies that the same constant of proportionality $t$ applies to both pairs. Hence $a^2t^2 + b^2t^2 = c^2t^2 + d^2t^2$. Reverting to the preimage yields

$BE = \sqrt{a^2 + \left(\frac{bu}{v}\right)^2} = \frac{\sqrt{a^2v^2 + b^2u^2}}{v}$,

$EC = \sqrt{a^2t^2 + \left(\frac{bu}{v}\right)^2} = \frac{t\sqrt{a^2v^2 + b^2u^2}}{v}$.

Likewise,

$CF = \frac{t\sqrt{c^2v^2 + d^2u^2}}{v}$, $FD = \frac{\sqrt{c^2v^2 + d^2u^2}}{v}$.

Therefore, $BE \cdot CF = EC \cdot FD$. □

Nakayama, a modern author, says in his preface that he avoids “clever proofs” [11, preface p. 2] for the sake of novice readers. Lacking a primary source, we do not know whether Japanese of the Edo period proved Proposition 4 in the same manner as he. Those who knew that the ratios of segments defined by collinear points are not altered by affine transformations could have taken a shortcut and deduced $\frac{BE}{EC} = \frac{CF}{FD}$ directly from $\frac{B'E'}{C'E'} = \frac{F'D'}{C'D'}$. In fact, one can do even better, not only shortening the proof further but also generalizing Proposition 4 to parallelograms.

**Proof** (the author). The polar of a point external to an ellipse meets it in the touch points of the two tangents to the ellipse through the point. The polars of all the points on a line through the center of the ellipse are parallel. In the projective plane, if and only if the pole is the point at infinity does the polar pass through the center of the ellipse. In Figure 5, left, $AC$ is the polar of the point at infinity on $BD$, and vice versa. Therefore, $EF \parallel BD$ and $\Delta ECF \sim \Delta ABC$:

$\frac{EC}{BC} = \frac{CF}{FD} \Rightarrow \frac{EC + CF}{BC} = \frac{EC - CF}{BC} \Rightarrow \frac{EC}{BF} = \frac{CF}{BE}$. □

The *wasanka* would certainly not have proceeded in this way, but they did know that Proposition 4 applies to all parallelograms, not just to rectangles [3, p. 334]. As with

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For an introduction to this concept, see, for instance, the Wikipedia article “Pole and Polar.”
Reconstructing a Solution

Interestingly, the first two sentences of the last proof are true for hyperbolas as well as for ellipses, provided “a point external to a hyperbola” is understood to mean that one can connect the point to the center without crossing either branch of the hyperbola. But as remarked at the outset, the Japanese treated ellipses as sections of cylinders, not cones, so their understanding of the relationship among ellipses, parabolas, and hyperbolas must have been limited. This is illustrated by the lone example of a hyperbola in a sangaku problem documented in the literature [4, Example 5.2]. The problem makes a fitting conclusion for our discussion, because it also highlights a modern solution that is almost surely not the original one.

One branch of a hyperbola appears in the section of a cone cut by a plane parallel to the cone’s axis, as shown in Figure 6. Two planes orthogonal to the axis section the cone in circles of radii $R > r$, the bases of a frustum, separated by height $h$ along the axis.

According to [4], the problem is to express, in terms of $r, R, b$, the radius of the circle $(C)m$ in the plane $x = r$, which touches the hyperbola and larger base of the frustum, and the semiaxes of the ellipse $(E)u, v$ in the same plane, which touches $(C)$ and the hyperbola at its vertex. A remark in [4] states that Fukagawa found this problem, which had been posted on a sangaku, “now lost,” in Aichi prefecture in 1841; that he published it in 1976; and that no Edo period solution is known.2

As [4] explains, if the height of the cone is $k$, then $\frac{b}{k} = \frac{k - \sqrt{b^2 + c^2}}{k}$. For every point $(x, y, z)$ on the cone, we have $\frac{\sqrt{x^2 + y^2}}{k} = \frac{k - \sqrt{z^2}}{k}$. Setting $x = r$ yields $\frac{\sqrt{r^2 + y^2}}{k} = \frac{k - \sqrt{z^2}}{k}$. This leads to $z = \frac{b(k - \sqrt{r^2 + y^2})}{k}$, which is true for every point where the plane $x = r$ meets the cone, and is therefore the equation of the hyperbola. Notice that this equation makes no mention of the lengths of the transverse and conjugate axes of the hyperbola, as the standard Cartesian equation must. From this, [4] proceeds by replacing $y^2$ in $z = \frac{b(k - \sqrt{r^2 + y^2})}{k - r}$ with $m^2 - (z - m)^2$, setting the discriminant of the resulting quadratic in $z$ to zero (because $z$ has the same value at both of the points at which $(C)$ touches the hyperbola), and solving to get $m = \frac{bR - k\sqrt{r^2 + y^2}}{k - r}\sqrt{\frac{b^2 + (r - k)^2}{(k - r)^2}}$.

Up to this point, there is nothing in the discussion that would have puzzled the wasanka. The basic geometry condensed in the equation of the circle $y^2 + (z - m)^2 = m^2$ is easily deduced from the Pythagorean theorem, whereas the modern standard equation of an ellipse is based on its locus definition, which the Japanese did not exploit. Indeed, when we say that $z = \frac{b(k - \sqrt{r^2 + y^2})}{k - r}$ represents one branch of a hyperbola, we are implicitly ascribing a meaning to the equation of which the Japanese author of the problem was probably unaware.

By contrast, turning to the ellipse $(E)$, [4] begins with the locus-definition equation $\frac{x^2}{b^2} + \frac{(z - b)^2}{m^2} = 1$; substitutes

2Lost or not, this problem is listed in [12, p. 71].
\[\frac{k^2}{b^2}(z - k)^2 - r^2\] for \(y^2\); sets the discriminant of the resulting quadratic in \(z\) to zero (because the ellipse touches the hyperbola at just one point); simplifies the equation, making use of \(k = \frac{bR}{k_r}\) along the way; and finally obtains \(v^2 = \frac{r(R - r)u}{b}\). I doubt a wasankaka would have done things that way. Furthermore, the conditions of the problem as stated in [4] do not preclude ellipses with smaller \(v\): the problem should properly demand the greatest possible \(v\).

Using calculus, one can show that the radius of curvature of the hyperbola at its vertex is \(\frac{r(R - r)}{b}\) and that the radius of curvature of the corresponding vertex of the ellipse is \(\frac{r}{b}\); therefore, \(v = \sqrt{\frac{r(R - r)u}{b}}\) is, in fact, maximal.

THE JAPANESE FOUND THAT THE RADIUS OF CURVATURE AT THE
ENDPOINTS OF THE MAJOR AXIS OF AN ELLIPSE IS \(\frac{1}{b}\) WITHOUT CALCULUS
([4, p. 140], quoting [10]; see also [5, p. 220]).

But [4] fails to note that the same kind of calculation is possible for the hyperbolic locus in the foregoing problem: one simply replaces \(y^2\) in \(z = \frac{k(R - \sqrt{r^2 + y^2})}{k_r}\) with \(m^2 - (z - b + m)^2\), sets the discriminant of the resulting quadratic in \(z\) to zero, and obtains \(m = \frac{r(R - r)}{b}\). Since an ellipse touching the hyperbola at its vertex also meets it in two other points unless \(\frac{v}{b} \leq \frac{r(R - r)}{b}\), it follows, without resort to the standard equation for the ellipse or the lengthy algebra in [4], that \(v = \sqrt{\frac{r(R - r)u}{b}}\) is maximal.

**Conclusion:**

Clearly, there are at least two distinct ways in which one can approach sangaku problems. On the one hand, from the viewpoint of recreational mathematics or mathematics pedagogy, they are challenging exercises, often hard to solve even with modern mathematical tools such as analytic geometry. If, on the other hand, one wants to understand what the Japanese of the time probably had in mind when they composed or solved sangaku problems, just getting a correct numerical solution or verifying a proof with modern methods is not enough. One must examine primary sources and see how they fit into the chronology of relevant documentary evidence. For linguistic reasons, not many non-Japanese are likely to pursue this line of research—even many modern Japanese would find the old-fashioned language of the texts daunting. Still, mathematical history is more than a catalogue of true theorems, as the case of ellipses in Edo-period Japan shows.

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**Appendix: Conjectured Japanese Proof of Proposition 3**

We have already proven Prop. 3 for an ellipse with axes \(2a, 2b, a > b\), and a tangential bounding parallelogram. For a tangential non-bounding parallelogram, we must seek alternative expressions for \(m^2 + n^2 - u^2 - v^2 (\neq 0)\) and \(bm (\neq uv)\). To that end, we construct a bounding parallelogram of identical altitude in three steps: by scaling by \(\frac{u}{u}\) in the direction of the major axis, thereby creating a circle and tangential rhombus; then constructing a tangential square about the circle with a side through an endpoint of the image of the altitude; and finally scaling the square by \(\frac{u}{u}\). Say that the sides of the resulting bounding parallelogram have length \(2a, 2b, a > b\), and, of the rhombus, \(2r, r > v\). The inverse scaling also takes the right triangle with sides \(r\) and \(v\) (Fig. 7, right) to the triangle with sides \(b\) and \(n\) and altitude \(b\) (Fig. 7, left). Its base has length \(\frac{u}{u}\). Applying the Law of Cosines as the Japanese knew it [9, pp. 101–104],

\[n^2 + \left(\frac{a}{u} \sqrt{r^2 - v^2}\right)^2 - b^2 = 2a \frac{u}{u} \sqrt{r^2 - v^2} \sqrt{n^2 - b^2}.
\]

Since \(\frac{u}{u} = \frac{u}{u}\) (in the bounding parallelogram) and \(\frac{u}{u} = \frac{u}{u}\)

(scaling is an affine transformation), we have \(bm = ru\), and so, in turn,

\[n^2 + m^2 - a^2 - b^2 = 2b \frac{u}{u} \sqrt{r^2 - v^2} \sqrt{n^2 - b^2},
\]

\[(m^2 + n^2 - u^2 - v^2)^2 = 4b^2 \left(n^2 - b^2\right)\]

\[b^2 (m^2 + n^2 - u^2 - v^2)^2 = 4b^2 (m^2 + n^2 - u^2 - v^2)^2 (n^2 - b^2).
\]

Therefore, the identity \(b^2 (m^2 - n^2 - u^2 + v^2)^2 = 4b^2 (m^2 + n^2 - u^2 - v^2)^2 (n^2 - b^2)\) (which we used to prove Prop. 3 for bounding parallelograms) becomes

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\(^{6}\)The crucial lemma gives the distance between the centers of an ellipse and a circle inscribed in it in terms of the ellipse’s semiaxes and the circle’s radius. It is based on similar triangles produced when a cylinder and a sphere around which it is wrapped are both sectioned by a plane oblique to the axis of the cylinder. The triangles lie in the plane through the axis of the cylinder orthogonal to the sectioning plane. The proof (found in [4] and [5]) is not hard; one cannot help recalling that Desargues’s theorem too is more easily proved in three dimensions than in two.

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\[ h^2 (m^2 - n^2 - u^2 + v^2) = 4h^2 (u^2 - m^2)(n^2 - v^2) + 4 (h^2 m^2 - u^2 v^2) (n^2 - h^2). \]

Simplifying the right side yields \( 4(n^2 u^2 - b^2 m^2)(b^2 - v^2) \), and thus proves Prop. 3 for the non-bounding parallelogram.

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