Entanglement between distant atoms mediated by a hybrid quantum system consisting of superconducting flux qubit and resonators

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Abstract
A hybrid quantum system consisting of spatially separated two-level atoms is studied. Two atoms do not interact directly, but they are coupled via an intermediate system which consists of a superconducting flux qubit interacting with a mechanical and an electrical resonator which are coupled to one of the atoms. Moreover, the superconducting flux qubit is driven by a classical microwave field. Applying adiabatic elimination, an effective Hamiltonian for the atomic subsystem is obtained. Our results demonstrate that entanglement degradation decay as well as fidelity decay in the dispersive regime are faster. Moreover, the driven field amplitude possesses an important role in entanglement and fidelity evolution.

Keywords: quantum entanglement, fidelity, circuit electrodynamics

(Some figures may appear in colour only in the online journal)

1. Introduction

Coupling between two distant qubits plays an important role in the implementation of quantum information protocols. Mediated features between distant qubits due to long-range indirect interaction provide long coherence times and can also be considered as promising candidates for quantum state transfer control. Two far coupled atoms can be entangled and controlled by the photon numbers of thermal field. One of the two atoms would interact with the thermal field inside the cavity, while the other would move outside freely [1]. In semiclassical consideration, two distant quantum dipole emitters in the proximity of a metal nanoparticle are entangled by exhibition of the localized surface plasmons. The steady-state degree of entanglement only depends on the ratio of distances between the metal nanoparticle and the quantum dipole emitters [2]. The quantum entanglement of two quantum dot heavy-hole spins separated at a long distance is studied using single-photon interference. Moreover, heavy-hole spins demonstrate a long coherence time [3]. Long-distance coupling between spin-qubits can be achieved by applying a long-range interaction such as qubit coupling to a ferromagnet [4] or an electromagnetic field with significant photon modes [5]. Two long-distance resonant exchange qubits are coupled by means of the electromagnetic field in a microwave cavity. The energy levels of qubits are matched by the resonant frequency of the cavity [6]. The long-range interaction of spin–orbit coupling is used to prepare the entangled spin-qubits in quantum dots in order to propose a quantum computer architecture [7]. Charge transport behavior through triple quantum dot arrays exhibits long-distance coherent tunnel coupling between the outward dots [8]. In a hybrid solid structure, the transfer of quantum information through the nitrogen-vacancy ensemble acts as the...
long-distance memory ingredients which are coupled with the LC circuit as the transmitter [9]. In all of the mentioned research, the distant entangled qubits are resources of quantum information. Moreover, the state transfer between the distant qubits is an important step in quantum information protocols. Therefore, the introduction and realization of physical systems consisting of the coupling between distant qubits are important steps for quantum information processing.

To investigate novel nonclassical quantum information processing, circuit quantum electrodynamics (circuit QED) and nanomechanical resonators have recently attracted a great deal of attention. Circuit quantum electrodynamics (cQED) acts as an on-chip analogy of cavity quantum electrodynamics (CQED) at microwave frequencies. The circuit QED consists of a superconducting qubit as an artificial atom coupled with microwave resonators. Superconducting qubits with dissipationless nonlinearity of the Josephson junction when connected to the microwave resonators provide strong coupling in order to transport single photons only with classical microwave fields.

Superconducting devices provide long coherence time without dissipation and play a significant role in these circuits. Superconducting qubits which interact with a superconducting microwave resonator were introduced as the circuit QED [10]. In the following, the circuit QED has progressed considerably in quantum computation [11–13]. For instance, the state transfer between an electromagnetic resonator and a mechanical resonator was investigated in plenty of studies [14–17].

A system consisting of two superconducting qubits which are coupled with two or more resonators provides a platform for quantum systems in the GHz range [18–21]. Two superconducting resonators in interaction with a superconducting qubit constitute a quantum switch [22]. This system is used to connect two nonlinear microwave resonators [23]. The realization of coupling between a superconducting flux qubit and two electrical and mechanical resonators has been demonstrated [24]. The entangled superconducting qubits in a multi-cavity system have been studied recently [25]. Therefore, superconducting qubits and resonators are one of important systems in the quantum regime.

In CQED, coupled qubit-cavity systems and the coupled qubit-spin system have been extensively studied experimentally and theoretically. A single quantum system of an atom or ion with discrete energy levels is coupled to the quantised radiation field in a cavity in order to develop the single-photon emission [26]. A coupled atom-cavity system with strong damping which cannot follow strong coupling is considered to investigate the phase response of the system in the regime of high atomic phase-shift experimentally [27]. In two-coupled qubits mediated with a resonator, the energy of a single photon excites two qubits simultaneously. This QED system containing longitudinal couplings operates around the resonant regime [28]. The dynamics of a qubit-cavity coupled system is studied for quantum state protection, storage and engineering. In this system which is inhomogeneously broadened spin, the states are weakly coupled to the light [29].

Coupled-spin qubit systems with localized electron spins demonstrate weak inter-qubit couplings which cause low fidelity and reproducibility [30,31]. To realize the behavior of coupled-spin qubit system dynamics, inter-qubit coupling and environmental noise has been studied theoretically [32]. In the coupling of qubits with optical cavities, the Zeeman splitting of QDs has a small magnitude compared with the linewidths of a cavity which is important in spin-cavity interactions [33,34]. In order to enhance the small ground-state splitting, the coupling of the QD molecule to a cavity is used to achieve large spin splitting [35].

In comparison of the cQED and CQED systems, the advantage of hybrid quantum systems is that the artificial atoms have larger transition dipoles than the natural atoms which can enhance the coupling strength magnitude [36]. In addition, one-dimensional superconducting microwave resonators contain a larger strength of coupling than the ordinary three-dimensional one. These elements can induce strong coupling to the coupled subsystems through the circuit [37]. These properties of CQED systems explore the quantum optics studies and quantum information research from the microscopic regime of atoms to the macroscopic of artificial atoms on a chip. Additionally, the coupling strength of systems in cQED can be tuned easily by manipulating the circuit parameters. In other words, the hybrid systems of CQED show significantly more tunability, scalability and large coupling than the microscopic systems of trapped atom and spin [38].

In the present contribution, we introduce a physical system which provides two distant interacting atoms. Our system is based on the circuit QED setup which is composed of an intermediate superconducting flux qubit interacting with an electrical resonator as well as a mechanical resonator. Moreover, each of these resonators are coupled to a separate two-level atom. The three-level superconducting flux qubit is driven by a classical microwave field which is considered as a tuning factor. The two qubits (two atoms) which are coupled indirectly, provide a platform to achieve entanglement. Since this system gives two interaction regimes, resonant and dispersive, the dynamics of entanglement and state transfer are investigated in both interaction regimes. To quantify the entanglement, a specific measure is required for each quantum system. The appropriate measure of the entanglement corresponding to the two-qubit systems is concurrence [39,40]. In order to study the state transfer, we would investigate the time evolution of fidelity. This quantity is defined to know how well the entanglement preserves between the initial entangled state and the desired state. This concept would characterize the maximal overlap of a desired state with a maximally entangled state [41–45].

This paper is structured as follows. In section 2, first we introduce the Hamiltonian of the system. In the last part of this section, we derive the Markovian master equation for each case. Next, the evolution of entanglement is studied in section 3 and the dynamics of fidelity is investigated in
Figure 1. The physical system under consideration. Two atoms $A$ and $B$ are coupled indirectly via two resonators $a$ and $b$ which interact with a three-level superconducting flux qubit. The coupling coefficient between the second excited flux qubit level with resonator $a$ is selected as $G_a$ and the coupling strength between the ground state level with resonator $b$ is chosen as $G_b$. The energy difference between flux qubit levels are $\omega_{g2,e} = \omega_2 - \omega_b$ and $\omega_{g1,e} = \omega_1 - \omega_b$. The system is driven by a classical driven field with Rabi frequency $\Omega$.

section 4 for both resonant and dispersive regimes. In section 4, we obtain the effective Hamiltonians for both resonant and dispersive regimes as an appendix.

2. Physical model

The physical system under study is shown in figure 1 schematically. The proposed system consists of a superconducting flux qubit which interacts with two resonators, a mechanical resonator $a$ and an electrical resonator $b$. The flux qubit is driven by an external classical field with Rabi frequency $\Omega$. Moreover, each resonator is coupled to a two-level atom, $A$ or $B$. Our purpose is to provide an effective indirect interaction between the atomic subsystems in which the driven field, superconducting flux qubit and resonators play the role of a transducer.

Usually the coupling strength in the circuit electromechanics is in the range of the microwave. Therefore, circuit electromechanics is known as the microwave counterpart of cavity optomechanics [46, 47]. In order to increase the coupling strength of these systems, it is shown that the replacement of the coupling capacitor by a superconducting qubit substantially enhances the coupling strength [48]. Superconducting qubits are composed of Josephson junctions which provide the quantum mechanics platform in macroscopic circuits at low temperatures [49].

The nonlinear nature of the Josephson junction leads to a nonuniform separation of energy levels [50]. Taking into account the two lowest energy levels, this system would be supposed as a qubit. Superconducting qubits are divided into three categories: charge, flux and phase qubit. In the flux qubit, the Josephson coupling energy is greater than the Coulomb energy which can be controlled by an external microwave field. This applied microwave field would be used to manipulate the flux qubit [51].

Resonators are considered as devices which are able to carry the electromagnetic field and to exchange energy between two ultimate destinations, such as atoms, through interaction with them. Resonators with a high quality factor would work in the GHz frequency regime and would be divided into superconducting and nanomechanical resonators [10, 52]. One kind of superconducting resonator is the LC resonator which utilizes a tunable electrical element. This type of resonator is known as an electrical resonator.

2.1. Hamiltonian

The Hamiltonian of the whole system may be written as $H = H_0 + H_{int}$. In this relation, $H_0$ corresponds to the free Hamiltonian and is composed of the following terms:

$$H_0 = H_A + H_R + H_q.$$  

In this equation, $H_A$ describes the free Hamiltonian of atoms $A$ and $B$:

$$H_A = \hbar \omega_A \sigma_A^+ + \hbar \omega_B \sigma_B^+.$$  

The atomic energy transitions are given by $\omega_A$ and $\omega_B$ and also $\sigma_i^+ = |e_i\rangle \langle e_i| - |g_i\rangle \langle g_i|$ in which $i = A$ or $B$. We have assumed that $|e_i\rangle$ is the excited (ground) state of the $i$th atom. Moreover, in equation (1) $H_R$ is the free Hamiltonian of the electrical and mechanical resonators and is defined as

$$H_R = \hbar \omega_a a^+ a + \hbar \omega_b b^+ b.$$  

In the present system, we have supposed that every resonator possesses a single mode with frequency $\omega_a$ and $\omega_b$. In this equation, $a$ and $b$ are the annihilation operators associated with the resonators. The third term of equation (1) corresponds to the free Hamiltonian of the flux qubit. This flux qubit consists of three Josephson junctions in a loop configuration. For this qubit, the two lowest energy states are localized while the third one is delocalized. Biasing the loop by a magnetic flux, this system may serve as a $V$ type artificial atom with a cyclic transition configuration [24]. If the three lowest energy levels of this loop are labeled as $|k\rangle$ with $k \in \{g, e_1, e_2\}$, the qubit Hamiltonian could be written as

$$H_q = \sum_k \hbar \omega_k \sigma_k^z, \quad k = g, e_1, e_2.$$  

The interaction Hamiltonian between different subsystems can be written as

$$H_{int} = H_{A,a} + H_{B,b} + H_{A,q} + H_{B,q} + H_{Driv}.$$  

(5)
In this equation, \( H_{A,a} \) and \( H_{B,b} \) describe the interaction between the resonators, \( a \) and \( b \), with the two-level atoms, \( A \) and \( B \). In the present contribution, we are going to study two different resonant and dispersive interaction regimes. The interaction between atoms and resonators would be described with the following Hamiltonians [56]:

\[
H_{A,a} = G_A(a^\dagger \sigma_A^a + a \sigma_A^a), \quad H_{B,b} = G_B(b^\dagger \sigma_B^b + b \sigma_B^b).
\]

(6)

In this equation, \( \sigma_A^a = |g_A\rangle \langle e_A| \) and \( \sigma_B^b = |g_B\rangle \langle e_B| \) are the lowering operators of atomic systems. In these relations, \( G_A \) and \( G_B \) are the coupling strengths between the resonators and two-level atoms. Therefore, the transition frequency of the atom \( A \) (qubit \( A \)), \( \omega_A \), should be close to the frequency of electrical resonator \( a \), \( \omega_a \), and also \( \omega_B \), the transition frequency of atom \( B \) (qubit \( B \)) must be near to \( \omega_b \), the frequency of mechanical resonator \( b \).

The energy scale of each physical system describes the physical regime of the system. In addition, the energy scale determines which physical systems could be coupled to each other. Therefore, an important point about the hybrid systems corresponds to their energy scales. For a typical qubit loop, the energy difference between qubit eigenstates may be within the range of \( 2\pi \times [0, 10] \) GHz [53]. Furthermore, the electrical resonator can be assumed as an \( LC \) part of a superconducting transmission line forming a one-dimensional cavity with frequency regime \( \sim 2\pi \times [1, 10] \) GHz [54]. Moreover, the recent progress of experimental techniques reveals that the mechanical resonators would be characterized in GHz frequency regimes [55].

In the same manner, the \( H_{A,a} \) and \( H_{B,b} \) terms in Hamiltonian (5) describe the interaction between the resonators and artificial atom (qubit subsystem). In the qubit eigen basis, these coupling are described by

\[
H_{A,a} = G_a(a^\dagger \sigma_{e_2,g} + a \sigma_{e_2,g}), \quad H_{B,b} = G_b(b^\dagger \sigma_{e_1,g}^+ + b \sigma_{e_1,g}^+).
\]

(7)

In these relations, \( G_a \) and \( G_b \) are the coupling strengths between resonators and qubit systems. Also, \( \sigma_{e_2,g} = |e_2\rangle \langle g| \) is a lowering operator. As the resonator \( a \) is coupled with \( e_2 \leftrightarrow g \) transition of the superconducting qubit, their frequencies need to be close to each other similarly to the resonator \( b \) and \( e_1 \leftrightarrow g \) transition of the superconducting qubit. In a recent experiment, the coupling strengths between the electrical resonator and flux qubit as well as the mechanical resonator are obtained of the order of \( O(1) \) MHz [55–57]. Therefore, we choose the coupling strength of all coupled subsystems as the same \( G_a = G_b = G_A = G_B = 40 \) MHz.

In Hamiltonian (5), \( H_{D_{\text{Bow}}} \) is the driven interaction of the flux qubit. In order to manipulate the system, the flux qubit should be inductively driven by an external microwave field. We have supposed that this driven field dispersively couples with the transition of \( |e_1\rangle \leftrightarrow |e_2\rangle \). This driven interaction would be modeled as

\[
H_{D_{\text{Bow}}} = \Omega (\sigma_{e_1,e_2}^- + \sigma_{e_1,e_2}^+).
\]

(8)

Here, \( \Omega \) is the coupling coefficient of the microwave driven field and is related to the amplitude of the driven field. This quantity possesses an important role in the dynamics of the present system. The system under consideration is composed of different subsystems and its evolution provides a complex dynamics.

3. Dynamics of system

The main physical system is a hybrid system which is composed of two atoms, the resonators and one flux qubit which is driven by a classical field. The atomic decay rate is of the order of GHz. In comparison to this decay rate, the electrical resonator decay rate is of several kHz, whereas the mechanical resonators possess the usual decay rates of several MHz. Furthermore, an interesting feature of superconducting qubits is their long decoherence times [58].

To study the dynamics of the present system, we start from the Liouville-von Neumann equation for the complete system in the interaction picture [59]. The whole system consists of a multilevel structure involving two indirectly coupled qubits \( A \) and \( B \) mediated by an electrical resonator \( a \) and a mechanical resonator \( b \) connecting to a superconducting flux qubit. To simplify the computational process and obtain the effective Hamiltonian, the procedure of adiabatic elimination is applied [60–62]. Superconducting flux qubit with three-level lambda type system is reduced to a two-level one with the adiabatic elimination strategy [63–65]. Additionally, for single-photon transport, a superconducting transmission line resonator (TLR) array coupled with a Cooper pair box (CPB) was used in order to connect two TLRs. In this research, an effective interaction between the TLRs was obtained by adiabatically eliminating the variables of the CPB [66]. By the adiabatic elimination of atomic and photonic states, the coupling of atomic qubits at a quantum network with some cavities coupled with optical fibers is performed, leading to qubit–qubit interactions [67]. In another study, by adiabatic elimination of atomic and photonic states, atomic qubits at a quantum network with some cavities which are coupled to optical fibers lead to qubit–qubit interactions [68]. Therefore, we can define an effective Hamiltonian for our system by the adiabatic elimination of superconducting flux qubit with three-level to a two-level one firstly and then by adiabatically eliminating of mediated resonators. The effective Hamiltonian is calculated for dispersive and resonant regimes in the Appendix. Our effective open system of two distant qubits is coupled to the common fermionic reservoir. To obtain the time evolution of the system, we trace out the bath degrees of freedom which defines the lead correlation function of master equation. Then under the Born-Markov approximations, the quantum master equation (QME) for the reduced density matrix which is obtained:

\[
\dot{\rho}(t) = -\frac{i}{\hbar} [H, \rho(t)] + \mathcal{L}_A \rho + \mathcal{L}_B \rho,
\]

(9)

where \( \rho \) denotes the reduced density matrix of system. The first term shows the lamb shift and \( \mathcal{L}_i \rho = \frac{i}{2} [2\sigma_i^a \rho \rho \sigma_i^a - \sigma_i^a \sigma_i^a \rho - \rho \sigma_i^a \sigma_i^a], \ i, j = A, B \) is the Lindblad operator which describes the dissipation in the system. It is
worth noting that the introduced hybrid system leads to an indirect interaction between two remote atoms. Thus, we have supposed that the effective physical system is composed of two interacting atoms. So, the master equation for the present system is achieved:

\[
\dot{\rho} = -iG[\sigma^A_+\rho\sigma^-_B + \sigma^A_-\rho\sigma^+_B - \rho\sigma^+_A\sigma^-_B - \rho\sigma^-_A\sigma^+_B] + \frac{\Gamma}{2} [2\sigma^A_+\rho\sigma^-_A - \sigma^+_A\sigma^-_A\rho - \rho\sigma^+_A\sigma^-_A] + \frac{\Gamma}{2} [2\sigma^-_B\rho\sigma^+_B - \sigma^-_B\sigma^+_B\rho - \rho\sigma^-_B\sigma^+_B] + \frac{\Gamma_{AB}}{2} [2\sigma^+_A\sigma^-_B + \sigma^+_A\sigma^-_B\rho - \rho\sigma^+_A\sigma^-_B] + \frac{\Gamma_{BA}}{2} [2\sigma^-_B\sigma^+_A + \sigma^-_B\sigma^+_A\rho - \rho\sigma^-_B\sigma^+_A].
\]

In this equation, the dissipation coefficient \(\Gamma_j\) (\(i, j = A, B\)) denotes the effective atomic decay rates without loss of generality relating to the bath correlation function which is defined as:

\[
\Gamma_j(\omega) = 2\pi \sum_{i\sigma} \int_{0}^{\infty} dt e^{i\omega t} \int_{\nu} d\nu |(c_i^\dagger c_i)|^2 |\rho_{\nu}\rangle\langle \rho_{\nu}|,
\]

in which \(c_i^\dagger\) indicates the annihilation (creation) fermionic operator of reservoir, \(\nu\) shows the wave vector and \(\sigma\) denotes the spin of the central system. Here, we assume \(\Gamma_A = \Gamma_B = 2\pi \times 0.1\) MHz. In addition, as the qubits do not interact with each other directly, the induced indirect interaction is absent, so we have \(\Gamma_{AB} = \Gamma_{BA} = 0\).

In the present contribution, we study the system in the computational basis, that is \([1] = |ee\rangle, [2] = |eg\rangle, [3] = |ge\rangle\) and \([4] = |gg\rangle\). Here, \([ij] = [i]_A [j]_B\) are the excited or ground states of atomic systems. As indicated above, the introduced hybrid system provides an effective interaction between two atoms. Therefore, in this system we encountered with two-coupled qubits. An interesting physical quantity would be the entanglement between qubits (atomic subsystems). To quantify the entanglement in the two-qubit systems, concurrence is an appropriate measure for both pure and mixed states [39, 40]. Accordingly, we will obtain concurrence in the present system.

![Resonant regime](image_url)

**Figure 2.** Left panel: concurrence for resonant regime, thick line \(\Omega_r = 2\pi \times 0.01\) MHz, thin line \(\Omega_r = 2\pi \times 0.015\) MHz. Right panel: concurrence for dispersive regime, thick line \(\Omega_d = 2\pi \times 50\) MHz, thin line \(\Omega_d = 2\pi \times 75\) MHz.

3.1. Concurrence

Concurrence is defined as \(C(\rho) = \text{Max} [0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4]\), where \(\lambda_i\) (\(i = 1, 2, 3, 4\)) are non-negative eigenvalues of a matrix \(R\) with decreasing order \(\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4\). The matrix \(R\) is defined as \(R = \sqrt{\rho} \sqrt{\rho}^T\), where \(\rho\) refers to the density matrix of the system and \(\rho^T = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)\). Here, \(\sigma_y\) represents the \(y\) component of the Pauli matrices and \(\rho^*\) is the complex conjugate of the density matrix. The time-dependence of the concurrence in both interaction regimes, resonant and dispersive, is shown in figure 2. In this figure each panel corresponds to the specific interaction regime, whereas in each case different plots correspond to a given driven field amplitude, \(\Omega\). The driven field is an experimental parameter would be employed to control the dynamics of the system. The relevant deserved magnitude for a given field depends on the validity of adiabatic elimination. This validity, which leads to the effective Hamiltonian by elimination of irrelevant states, is that the driven field Rabi frequency should be less than the detuning considerably [64].
The initial state of atoms is considered as
\[
\rho[0] = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0.5 & -0.5i & 0 \\
0 & 0.5i & 0.5 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]
which possesses the highest degree of entanglement. The interesting point about the present system is that during evolution, the qubits cannot be entangled for the initial separable state. The main reason for this event is that when the qubits from the first are in interaction with each other and then they experience an additional interaction which is different for each of them, the qubits would be entangled even they started from separable initial states. The figures show the entanglement degradation of the system as time elapses. Additionally, entanglement degradation is governed by the driven field. Therefore, the classical microwave driven field may be used as a control parameter for the entanglement between remote atoms. A comparison between the panels of figure 2 demonstrates that the entanglement between atoms follows an oscillatory time evolution. In the resonant regime, the period is smaller than the dispersive regime. Moreover, the entanglement degradation in the dispersive regime is faster than resonant one. This comparison between two interaction regimes is illustrated in figure 3. This figure exhibits long-lasting entanglement in the resonant regime. That is, the entanglement degradation is faster in the dispersive regime. As a consequence, the interaction between atoms and resonators provides a platform for the entanglement in the present system. Additionally, the driven field is another important parameter in the entanglement time evolution.

3.2. Fidelity

The physical system under study can be used for state transfer between atoms. In this situation, fidelity is an important parameter which characterizes the overlap of a desired state with a maximally entangled one. In the present contribution, to quantify the fidelity we have chosen the Bell state which is used for the concurrence as the reference state. The fidelity of a mixed state is defined as
\[
F(\rho_1, \rho_2) = \max[\langle \Phi_1 | \Phi_2 \rangle^2] \quad [41],
\]
which is known as Uhlmann formula [69]. In this relation, $|\Phi_1\rangle$ and $|\Phi_2\rangle$ are the purifications of $\rho_1$ and $\rho_2$, respectively. This expression of fidelity can be written in an equivalent relation $F(\rho_1, \rho_2) = Tr[\sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}}]$. In this definition, $\rho_1$ is the initial state, or target state, and $\rho_2$ is the elapsed state. Then, this quantity determines to what extent the evolved state of the system is close to the target state $\rho_1$. The fidelity time evolution in the system under study is illustrated in figure 4 for both the resonant and dispersive regimes. Different plots in each panel correspond to the given driven field. Here, our target state is a Bell state and we choose the Bell state which is given in equation (12) as the initial state of the system. For that, the fidelity started at the maximum value and as time elapses it is decayed to zero. That is, the initial maximally entangled state evolves into a complete separable final state. In other words, fidelity follows a similar scenario to entanglement. Moreover, a comparison between the time evolution of fidelity in two different resonant and dispersive regimes is illustrated in figure 5. This plot shows that the fidelity decay rate is faster in the dispersive interaction regime, similarly to entanglement evolution. Thus, the

Figure 4. Left panel: fidelity for resonant regime, solid line $\Omega_R = 2\pi \times 0.01 \text{MHz}$ and dashed line $\Omega_R = 2\pi \times 0.015 \text{MHz}$. Right panel: fidelity for dispersive regime, solid line $\Omega_D = 2\pi \times 50 \text{MHz}$, dashed line $\Omega_D = 2\pi \times 75 \text{MHz}$.

Figure 5. Comparison of Fidelity between resonant and dispersive regimes. Solid line: resonant regime, $\Omega_R = 2\pi \times 0.01 \text{MHz}$; dashed line: dispersive regime, $\Omega_D = 2\pi \times 50 \text{MHz}$.
resonant interaction between atoms and resonators keeps the atomic subsystem state closer to the Bell state.

4. Conclusion

We have proposed an analysis on an electromechanical circuit consisting of a superconducting flux qubit which interacts with electrical and mechanical resonators. In turn, each resonator is in interaction with a single two-level atomic system. The flux qubit is driven by a microwave field. We have shown that in the specific situation, one could achieve an effective interaction between two atoms in the system. Therefore, the introduced hybrid system reduces to a coupled-qubits system. Next, the time evolution of quantum entanglement and fidelity were studied. The interaction between atoms and resonators is considered in two different resonant and dispersive regimes. Our results have illustrated that the entanglement degradation in the dispersive interaction is faster than the resonant interaction regime. Similarly, the decay rate of fidelity in the dispersive regime is faster than the resonant interaction regime. That is, entanglement and fidelity follow a similar time evolution in the two interaction regimes. Also, the increase in driven field amplitude leads to slower entanglement degradation as well as fidelity decay. As a result, in the introduced system, the resonant interaction regime and the strong driven field lead to more robust entanglement. Moreover, the dispersive interaction between the atoms and resonators provides an appropriate platform for state transfer.

Appendix. Calculation of dispersive and resonant regime

To obtain the effective Hamiltonian in the dispersive and resonant regimes, all parts of the total Hamiltonian should be written in the interaction picture:

\[
\hat{H}_f(t) = G_A \hat{a} \hat{\sigma}_A^+ e^{i\Delta_A t} + \hat{a}^+ \hat{\sigma}_A e^{-i\Delta_A t} \\
+ G_B (\hat{b} \hat{\sigma}_B^+ e^{i\Delta_B t} + \hat{b}^+ \hat{\sigma}_B e^{-i\Delta_B t}) \\
+ G_A (\hat{b} \hat{\sigma}_A^+ e^{i\Delta_B t} + \hat{b}^+ \hat{\sigma}_A e^{-i\Delta_B t}) \\
+ \Omega (\hat{\sigma}_{c_{12}} e^{i\Delta_{c_{12}} t} + \hat{\sigma}_{c_{12}}^+ e^{-i\Delta_{c_{12}} t}),
\]

Here, \( G_A, G_B, G_A \) and \( G_B \) denote the coupling strength of respectively and \( \Omega \) indicates the frequency of the external driven field. Additionally, we describe the detuning parameters as: \( \Delta_A = [\omega_a - \omega_b], \Delta_B = [\omega_a - \omega_b], \Delta_B = [\omega_b - \omega_b], \Delta_B = [\omega_b - \omega_b], \Delta_B = [\omega_b - \omega_b] \) and \( \Delta_{c_{12}} = \omega_{c_{12}} - \omega_{c_{12}} \). The electrical and mechanical resonators work in the order of GHz frequency, we typically consider their frequencies as \( \omega_a = 2\pi \times 5.12 \) GHz and \( \omega_b = 2\pi \times 1.4 \) GHz respectively [24]. Resonators are coupled to the qubits \( A \) and \( B \) in one side and superconducting flux qubit in other side, so the frequency of each coupled qubit should be in the range of the mutual resonator. The level of qubits can be tuned by means of the gate voltage biasing, the external applied microwave or magnetic flux which let us to set the frequency of the subsystems for each regime.

In the following, we calculate the effective Hamiltonian for the dispersive and resonant regimes.

A.1. Dispersive regime

In the dispersive regime, the coupled resonator-atom subsystems are far detuned compared with their coupling strength. This limitation for the present system can be expressed as \( \Delta_{a} \gg \Delta_{b} \gg \Delta_{a} \gg \Delta_{a} \gg \Delta_{a} \gg \Delta_{a} \). In addition, we suppose that \( \Delta_{a} \gg \Delta_{b} \gg \Omega^2 \). However, in this regime, the interaction does not lead to energy exchange and the interaction effects would be followed through frequency shift and other physical phenomena as well [70].

According to the constant magnitude of coupling strength for each part and also the conditions of the dispersive regime, we arrange the detuning parameters \( \Delta_{a} = \Delta_{b} = \Delta_{a} = \Delta_{b} = 2\pi \times 120 \) MHz. As the resonators work in the GHz frequency regime, we take the frequency of the electrical and mechanical resonators as \( \omega_a = 2\pi \times 5.12 \) GHz and \( \omega_b = 2\pi \times 1.4 \) GHz respectively [24]. With respect to the coupling of the qubit \( A \) and the \( e_2, g \) level of superconducting qubit with resonator \( a \) from both sides, the relevant frequencies \( \omega_{a,2} = 2\pi \times 5.24 \) GHz and \( \omega_{c_{12},g} = 2\pi \times 5 \) GHz are chosen respectively. Additionally, we select \( \omega_{a,2} = 2\pi \times 1.52 \) GHz and \( \omega_{b} = 2\pi \times 1.28 \) GHz for \( e_1, g \) level of superconducting qubit and qubit \( B \) which are connected to resonator \( b \) respectively.

According to adiabatic elimination, the Hamiltonian for the present regime is transformed as \( H^D_{ef} = e^{i\theta}H^D_e \). The new operator \( \lambda^D \) is defined as:

\[
\lambda^D = \frac{G_A}{\Delta_{a}} (\hat{\sigma}_A^+ \hat{\sigma}_A - \hat{\sigma}_A^+ \hat{\sigma}_A) + \frac{G_B}{\Delta_{a}} (\hat{\sigma}_B^+ \hat{\sigma}_B - \hat{\sigma}_B^+ \hat{\sigma}_B) \\
+ \frac{\Omega}{\Delta_{c_{12}}} (\hat{\sigma}_{c_{12}} - \hat{\sigma}_{c_{12}}^+) \\
+ \frac{G_B}{\Delta_{b}} (\hat{\sigma}_B^+ \hat{\sigma}_B - \hat{\sigma}_B^+ \hat{\sigma}_B) + \frac{G_B}{\Delta_{b}} (\hat{\sigma}_B^+ \hat{\sigma}_B - \hat{\sigma}_B^+ \hat{\sigma}_B). \quad (14)
\]

Using the Barker-Campbell-Hausdorff relation, the transformed Hamiltonian could be expanded as:

\[
H^D_{ef} = H + [\lambda^D, H] + \frac{1}{2} [\lambda^D, [\lambda^D, H]] + \cdots \quad (15)
\]

Following this expansion up to fifth order, the effective Hamiltonian describing the remote interaction between atomic subsystems is obtained as:

\[
H^D_{ef} = G_D (\hat{\sigma}_A^+ \hat{\sigma}_A^+ + \hat{\sigma}_B^+ \hat{\sigma}_B^+ ). \quad (16)
\]

Here \( G_D \) is an effective coupling strength in the dispersive regime and is given as

\[
G_D = \frac{1}{20} \frac{\Omega G_A G_B G_A G_B}{\Delta_{c_{12}} \Delta_{a} \Delta_{b} \Delta_{a} \Delta_{b}} \cdot \Delta_{a} \\
- \Delta_{b} + 4 (\Delta_{a} - \Delta_{b}) + 6 \Delta_{c_{12}}. \quad (17)
\]
A.2. Resonant regime

The resonant regime is defined when the coupled subsystems (qubit-resonator or atom-resonator) are in resonant or near-resonant limit. The limitation of this regime in the physical realization is $\Delta_{aq} \ll G_a$, $\Delta_{bq} \ll G_b$, $\Delta_{au} \ll G_a$ and $\Delta_{bu} \ll G_B$.

Therefore in these circumstances, the frequency of qubits should be so close to resonators that we choose them in this way: $\omega_a = 2\pi \times 5.135$ GHz and $\omega_{e,G_a} = 2\pi \times 5.09$ GHz which are coupled with resonator $a$, moreover $\omega_1 = 2\pi \times 1.43$ GHz and $\omega_2 = 2\pi \times 1.385$ GHz which are connected to the resonator $b$. However the frequency of resonators are considered the same as before.

These considerations provide us with the same detuning parameters of $\Delta_{aq} = \Delta_{bq} = 2\pi \times 30$ MHz and $\Delta_{au} = \Delta_{bu} = 2\pi \times 15$ MHz. In addition, we suppose that $\Delta_{aq} \Delta_{bq} \gg \Omega_a^2$. To obtain an effective Hamiltonian for our distant qubits system in resonant regime, we apply the time evolution operator [60]:

$$u(t, t_0 = 0) \approx 1 - \frac{i}{\hbar} \int_0^t dt' H_{\text{int}}(t') - \frac{1}{2\hbar^2} \int_0^t dt_1 \int_0^t dt_2 \text{tr} H_{\text{int}}(t_1) \text{tr} H_{\text{int}}(t_2) + \ldots \quad \text{(18)}$$

After calculation, the unitary transformation becomes $u(t, t_0 = 0) \approx 1 - \frac{i}{\hbar} H_{\text{eff}} t$ which can be written as $u(t, t_0 = 0) \approx \exp[-i\frac{t}{\hbar} H_{\text{eff}}]$. For our system, we calculate the unitary equation (18) up to fifth order and average the unitary relation for the ground state $|0\rangle$ which becomes:

$$u(t, t_0 = 0) \approx 1 - \frac{i}{\hbar} \frac{2G_a G_b G_A G_B}{\Delta_{au} \Delta_{bu} (\Delta_{aq} - \Delta_{au})(\Delta_{bq} - \Delta_{bu})} H_{\text{eff}} t \quad \text{(19)}$$

Here, the effective Hamiltonian for resonant regime is achieved:

$$H_{\text{eff}} = G_R (\hat{\sigma}_3^a \hat{\sigma}_3^u + \hat{\sigma}_3^a \hat{\sigma}_3^b), \quad \text{(20)}$$

in which the coupling coefficient for resonant regime is given as:

$$G_R = 2G_a G_b G_A G_B \times \frac{1}{\Delta_{au} \Delta_{bu} (\Delta_{aq} - \Delta_{au})(\Delta_{bq} - \Delta_{bu})} \quad \text{(21)}$$

Here, we use this detuning condition $\Delta_{bu} - \Delta_{bq} - \Delta_{e_1 e_2} + \Delta_{aq} - \Delta_{au} = 0$ in the resonant regime.

Drawing a comparison between Hamiltonians in equation (16) and equation (20) reflects this fact that both effective Hamiltonians describe an indirect interaction between atomic subsystems. In this case, the significant point is concerned about the difference order of magnitude between the coupling strengths of these interaction regimes which is $G_a / G_0 \ll 10$. Therefore, the introduced system may be supposed as a coupler between distanced atomic systems.

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