RAZUMIKHIN-TYPE THEOREMS ON POLYNOMIAL STABILITY OF HYBRID STOCHASTIC SYSTEMS WITH PANTOGRAPH DELAY

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Abstract. The main aim of this paper is to investigate the polynomial stability of hybrid stochastic systems with pantograph delay (HSSwPD). By using the Razumikhin technique and Lyapunov functions, we establish several Razumikhin-type theorems on the $p$th moment polynomial stability and almost sure polynomial stability for HSSwPD. For linear HSSwPD, sufficient conditions for polynomial stability are presented.

1. Introduction. Systems in many branches of science and industry do not only depend on the present state but also the past ones. Stochastic differential delay equations (SDDEs) have been widely used to model such systems. On the other hand, these systems may often experience abrupt changes in their structure and parameters and continuous time Markov chains have been used to model these abrupt changes. Hence, SDDEs with Markovian switching, known also as hybrid SDDEs, have appeared frequently in practice. The analysis and control of these systems involve investigating their stability, which is often regarded as one of the important issues of dynamical systems studied. There have been much work on the stability of hybrid SDDEs, we here mention Fei et al. [3], Hu et al. [8], Ji and Chizeck [11], Mao et al. [14, 16, 21, 32], Shaikhet [25], Wu et al. [30] and Zhang et al. [35] among others.

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It is well known that the classical and powerful techniques applied in the study of stability are based on the Lyapunov direct method. However, the greatest disadvantage of this approach is that it is difficult to construct the Lyapunov functionals effectively. In this case, the Razumikhin technique has been proposed and was used to study the stability of time delay systems. Razumikhin [24] was the first to develop this technique to study the stability of deterministic systems with delay. Mao [17, 18] extended this technique to stochastic systems with delay and employed the Razumikhin approach to study both pth moment and almost sure exponential stability for stochastic functional differential equations (SFDEs) and neutral SFDEs. After that, the Razumikhin technique has become very popular and there have appeared many results based on it to discuss the stability of stochastic systems with time delay, see for example, [9, 10, 15, 22, 23, 26, 28, 29, 33] and the many references therein.

On the other hand, it is noted that the literature cited above mainly focuses on pth moment and almost sure exponential stability. However, in many cases we may find that the equation is not exponentially stable, but the solution does tend to zero asymptotically. Consequently, it appears necessary to study other stability, for instance, polynomial stability. Mao [19, 20] considered the polynomial stability of stochastic system, which shows that solution tends to zero polynomially. Liu and Chen [12] studied the polynomial stability for stochastic differential equations (SDEs) with bounded delay. In recent years, Appleby and Buckwar [1] investigated the polynomial stability of SFDEs with unbounded delay or pantograph delay. Milosevic [13] has established the sufficient conditions of almost sure polynomial stability of the solution for nonlinear SDEs with pantograph delay. Guo and Li [1] has established the sufficient conditions of almost sure polynomial stability of the nonlinear SDEs with pantograph delay by using the Razumikhin-type technique.

Motivated by [19, 18, 12, 1, 13, 7], in this paper, we are concerned with the polynomial stability of hybrid stochastic system with pantograph delay

\[ dx(t) = f(x(t), x(qt), t, r(t))dt + g(x(t), x(qt), t, r(t))dw(t), \quad (1) \]

where \( 0 < q < 1 \), the coefficients \( f : \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^n \) and \( g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^{n \times m} \) are Borel-measurable, \( S = \{1, 2, \cdots, N\} \), and \( r(t) \) is a continuous-time Markov chain taking value in \( S \), \( w(t) \) is an \( m \)-dimensional Brownian motion. In fact, as a class of special stochastic delay systems, stochastic system with pantograph delay has been investigated by many scholars, we can refer to Baker and Buckwar [2], Fan et al. [5, 4], Xiao et al. [31], Guo and Li [6], Zhou and Xue [34], Shen et al. [27].

To the best of our knowledge, there are no results based on the Razumikhin technique referring to the polynomial stability of hybrid stochastic system with pantograph delay (1). The main aim of the present paper is to close this gap by extending the Razumikhin approach to the study of both the pth moment polynomial stability and almost sure polynomial stability for (1). Moreover, in [7], the authors only studied the polynomial stability in mean square, but in this paper, we shall also study the almost sure polynomial stability. On the other hand, since hybrid stochastic system (1) we are studied have continuous components as well as discrete components, their asymptotic behaviour is completely different from a single system of pantograph equation. Actually, (1) can be regarded as the result of the following \( N \) equations

\[ dx(t) = f(x(t), x(qt), t, i)dt + g(x(t), x(qt), t, i)dw(t), \quad 1 \leq i \leq N \quad (2) \]
switching from one to the others according to the movement of the Markov chain.
Due to the presence of the Markov chains, we will show that even if some subsystems
(2) are not polynomially stable, the overall hybrid system (1) may still be polynomially stable.
In this paper, some preliminaries and notations on hybrid stochastic system
with pantograph delay will be introduced in section 2. In sections 3, we will show
the main results of our paper, where several useful criteria will be established on
the polynomial stability in mean square as well as the almost sure polynomial
stability for (1.1). In section 4, the general results established in sections 3 will be
applied to deal with the polynomial stability of linear hybrid stochastic system with
pantograph delay.

2. Preliminaries. Let \((\Omega, \mathcal{F}, P)\) be a complete probability space with a filtration
\(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual conditions. Let \(w(t)\) be an \(m\)-dimensional Brownian
motion defined on the probability space \((\Omega, \mathcal{F}, P)\). Let \(t \geq t_0 \geq 0\) and \(C([q(t), t]; \mathbb{R}^n)\)
denote the family of the continuous functions \(\varphi\) from \([q(t), t] \to \mathbb{R}^n\) with the norm
\(\|\varphi\| = \sup_{q(t) \leq \theta \leq t} |\varphi(\theta)|\), where \(|\cdot|\) is the Euclidean norm in \(\mathbb{R}^n\). If \(A\) is a vector or
matrix, its transpose is denoted by \(A^T\). If \(A\) is a matrix, its norm \(\|A\|\) is defined by
\(\|A\| = \sup\{|Ax| : |x| = 1\}\). For \(p > 0\), \(L^p_p([q(t), t]; \mathbb{R}^n)\) denote the family of
all \((\mathcal{F}_t)\)-measurable, \(C([q(t), t], \mathbb{R}^n)\)-valued random variables \(\varphi = \{\varphi(\theta) : q(t) \leq \theta \leq t\}\)
such that \(E\|\varphi\|^p < \infty\).

Let \(r(t), t \geq 0\) be a right-continuous Markov chain on the probability space
\((\Omega, \mathcal{F}, P)\) taking values in a finite state space \(S = \{1, 2, \ldots, N\}\) with generator \(\Gamma = (\gamma_{ij})_{N \times N}\) given by:

\[
P(r(t + \Delta) = j | r(t) = i) = \begin{cases} 
\gamma_{ij} \Delta + o(\Delta), & \text{if } i \neq j, \\
1 + \gamma_{ij} \Delta + o(\Delta), & \text{if } i = j.
\end{cases}
\]

where \(\Delta > 0\). Here \(\gamma_{ij} \geq 0\) is the transition rate from \(i\) to \(j\), \(i \neq j\), while
\(\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}\). We assume that the Markov chain \(r(.)\) is independent of the
Brownian motion \(w(.)\). Let us consider the nonlinear hybrid stochastic systems
with pantograph delay

\[
dx(t) = f(x(t), x(q(t)), t, r(t))dt + g(x(t), x(q(t)), t, r(t))dw(t), \quad t \geq t_0
\]

with initial data \(\{x(t) : q(t_0) \leq t \leq t_0\} = \xi \in L^2_{\mathcal{F}_{t_0}}([q(t_0), t_0]; \mathbb{R}^n)\). Here
\(f : \mathbb{R}^n \times \mathbb{R}^n \times [t_0, \infty) \times S \to \mathbb{R}^n\), and \(g : \mathbb{R}^n \times \mathbb{R}^n \times [t_0, \infty) \times S \to \mathbb{R}^{n \times m}\).

In this paper, the following hypothesis are imposed on the coefficients \(f\) and \(g\).

**Assumption 2.1.** For each integer \(d \geq 1\), there exists a positive constant \(k_d\) such that

\[
|f(x, y, t, i) - f(\bar{x}, \bar{y}, t, i)|^2 + |g(x, y, t, i) - g(\bar{x}, \bar{y}, t, i)|^2 \leq k_d(|x - \bar{x}|^2 + |y - \bar{y}|^2),
\]

for those \(x, y, \bar{x}, \bar{y} \in \mathbb{R}^n\) with \(|x| \lor |y| \lor |\bar{x}| \lor |\bar{y}| \leq d\) and \((t, i) \in [t_0, \infty) \times S\).

It is known that Assumption 2.1 only guarantees that (3) has a unique maximal solution, which may explode to infinity at a finite time. To avoid such a possible explosion, we need to impose an additional condition in terms of Lyapunov functions.

Let \(C(\mathbb{R}^n \times [q(t_0), \infty) \times S; \mathbb{R}^+)\) denote the family of continuous functions from
\(\mathbb{R}^n \times (q(t_0), \infty) \times S \to \mathbb{R}^+\). Also denote by \(C^{2,1} = C^{2,1}(\mathbb{R}^n \times (q(t_0), \infty) \times S; \mathbb{R}^+)\) the
family of all continuous non-negative functions $V(x, t, i)$ defined on $\mathbb{R}^n \times [qt_0, \infty) \times S$ such that they are continuously differentiable twice in $x$ and once in $t$. Given $V \in C^{2,1}$, we define the function $LV : \mathbb{R}^n \times \mathbb{R}^n \times [qt_0, \infty) \times S \to \mathbb{R}$ by

$$LV(x, y, t, i) = V_t(x, t, i) + V_x(x, t, i)f(x, y, t, i) + \frac{1}{2}\text{trace}[g(y, t, i)V_{xx}(x, t, i)g(x, y, t, i)] + \sum_{j=1}^{N} \gamma_{ij}V(x, t, j),$$

where $V_t(x, t, i) = \frac{\partial V(x, t, i)}{\partial t}$, $V_x(x, t, i) = \left(\frac{\partial V(x, t, i)}{\partial x_1}, \cdots, \frac{\partial V(x, t, i)}{\partial x_n}\right)$, and $V_{xx}(x, t, i) = \left(\frac{\partial^2 V(x, t, i)}{\partial x_i \partial x_j}\right)_{n \times n}$.

**Assumption 2.2.** There exist a Lyapunov function $V \in C^{1,2}$ and some positive constants $c_1, c_2, \alpha_1$ and $\alpha_2$ such that for any $x, y \in \mathbb{R}^n$, $t \geq qt_0$, and $i \in S$,

$$c_1|x|^p \leq V(x, t, i) \leq c_2|x|^p,$$  \hfill (5)

and

$$LV(x, y, t, i) \leq -\alpha_1|x|^p + \alpha_2|y|^p.$$  \hfill (6)

Now, we present the definitions of the $p$th moment polynomial stability and almost surely polynomial stability of hybrid stochastic systems with pantograph delay (3).

**Definition 2.3.** The solution of (3) is said to be $p$th moment polynomially stable if there exists a constant $\gamma$ such that

$$\limsup_{t \to \infty} \frac{\log \mathbb{E}|x(t)|^p}{\log (1+t)} < -\gamma.$$  \hfill (7)

**Definition 2.4.** The solution of (3) is said to be almost surely polynomially stable if there exists a constant $\tilde{\gamma}$ such that

$$\limsup_{t \to \infty} \frac{\log |x(t)|}{\log (1+t)} < -\tilde{\gamma} \quad \text{a.s.}$$  \hfill (8)

3. **Main results.** In this section, we shall investigate the $p$th moment polynomial stability and almost surely polynomial stability of (3) by using the Razumikhin-type technique. Before analyzing the stability, we firstly show that (3) has a global solution.

**Theorem 3.1.** Let Assumptions 2.1 and 2.2 hold. Then for any given initial data $\xi$, there is a unique global solution $x(t)$ to (3) on $t \in [qt_0, \infty)$. Moreover, the solution has the property that

$$\mathbb{E}|x(t)|^p < \infty$$  \hfill (9)

for any $t \geq qt_0$.

**Proof.** Since the coefficients of (2.1) are locally Lipschitz continuous, for any given initial data $\xi$, there is a maximal local solution $x(t)$ on $t \in [t_0, \sigma_\infty)$, where $\sigma_\infty$ is the explosion time. Let $k_0 > 0$ be sufficiently large for $\|\xi\| < k_0$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \in [t_0, \sigma_\infty) : |x(t)| \geq k\}.$$
Clearly, $\tau_k$ is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k \to \infty} \tau_k$, whence $\tau_\infty \leq \sigma_\infty$ a.s. Note if we can show that $\tau_\infty = \infty$ a.s., then $\sigma_\infty = \infty$ a.s. So we just need to show that $\tau_\infty = \infty$ a.s. We shall first show that $\tau_\infty \geq \frac{t_0}{q}$ a.s. By the Itô formula (see e.g. [14]) and condition (6), we can show that, for any $k \geq k_0$ and $t_1 \geq t_0$,
\[
EV(x(\tau_k \wedge t_1), \tau_k \wedge t_1, r(\tau_k \wedge t_1)) \leq EV(x(t_0), t_0, r(t_0)) + E \int_{t_0}^{\tau_k \wedge t_1} \left( -\alpha_1|x(t)|^p + \alpha_2 q|x(qt)|^p \right) dt.
\]
(10)
Let us now restrict $t_1 \in [t_0, \frac{t_0}{q}]$. By condition (2.3), we then get
\[
c_1E|x(\tau_k \wedge t_1)|^p \leq H_1 - \alpha_1E \int_{t_0}^{\tau_k \wedge t_1} |x(t)|^p dt,
\]
where
\[
H_1 = c_2E|x(t_0)|^p + \alpha_2 qE \int_{t_0}^{\frac{t_0}{q}} |x(qt)|^p dt
\]
\[
\leq c_2E||\xi||^p + \alpha_2 E \int_{t_0}^{\frac{t_0}{q}} |x(t)|^p dt < \infty.
\]
It then follows that
\[
E|x(\tau_k \wedge t_1)|^p \leq \frac{H_1}{c_1}, \quad t_0 \leq t_1 \leq \frac{t_0}{q}
\]
(12)
for any $k \geq k_0$. In particular, $E|x(\tau_k \wedge \frac{t_0}{q})|^p \leq \frac{H_1}{c_1}, \forall k \geq k_0$. This implies $k^pP(\tau_k \leq \frac{t_0}{q}) \leq \frac{H_1}{c_1}$. Letting $k \to \infty$, we hence obtain that $P(\tau_\infty \leq \frac{t_0}{q}) = 0$, namely
\[
P(\tau_\infty > \frac{t_0}{q}) = 1.
\]
(13)
Letting $k \to \infty$ in (12) yields
\[
E|x(t_1)|^p \leq \frac{H_1}{c_1}, \quad t_0 \leq t_1 \leq \frac{t_0}{q}.
\]
(14)
Let us now proceed to prove $\tau_\infty > \frac{t_0}{q}$ a.s. given that we have shown (13)-(14). For any $k \geq k_0$ and $t_1 \in [t_0, \frac{t_0}{q}]$, it follows from (10) that
\[
c_1E|x(\tau_k \wedge t_1)|^p \leq H_2 - \alpha_1E \int_{t_0}^{\tau_k \wedge t_1} |x(t)|^p dt,
\]
(15)
where
\[
H_2 = c_2E|x(t_0)|^p + \alpha_2 qE \int_{t_0}^{\frac{t_0}{q}} |x(qt)|^p dt
\]
\[
\leq H_1 + \alpha_2 qE \int_{t_0}^{\frac{t_0}{q}} |x(qt)|^p dt = H_1 + \alpha_2 qE \int_{t_0}^{\frac{t_0}{q}} |x(t)|^p dt < \infty.
\]
Consequently
\[
E|x(\tau_k \wedge t_1)|^p \leq \frac{H_2}{c_1}, \quad t_0 \leq t_1 \leq \frac{t_0}{q^2}.
\]
(16)
In particular, \( E|\eta(k \wedge t_0/\eta)|^p \leq \frac{H_2}{c_1}, \forall k \geq k_0 \). This implies \( k^p P(\tau_k \leq t_0/\eta) \leq \frac{H_2}{c_1} \).

Letting \( k \to \infty \), we then obtain that \( P(\tau_\infty \leq t_0/\eta) = 0 \), namely \( P(\tau_\infty > t_0/\eta) = 1 \).

Letting \( k \to \infty \) in (16) yields
\[
E|x(t_1)|^p \leq \frac{H_2}{c_1}, \quad t_0 \leq t_1 \leq \frac{t_0}{\eta^2}.
\]

Repeating this procedure, we can show that, for any integer \( i \geq 1 \), \( \tau_\infty > \frac{t_0}{\eta^i} \) a.s.,
\[
E|x(t_1)|^p \leq \frac{H_i}{c_1}, \quad t_0 \leq t \leq \frac{t_0}{\eta^i},
\]
where
\[
H_i = c_2 E|x(t_0)|^p + \alpha_2 E \int_{t_0}^{t} |x(qt)|^p dt < \infty.
\]

We must therefore have \( \tau_\infty = \infty \) a.s. and the required assertion (9) holds as well. The proof is therefore complete.

**Remark 3.2.** In this theorem, we removed an additional condition \( \alpha_1 > \alpha_2 \) which has played an important role in the proof of the existence results \([6, 7, 13, 27, 34]\). Hence, we improve and generalize the corresponding existence results of \([6, 7, 13, 27, 34]\).

**Theorem 3.3.** Let \( p > 0 \) and let Assumptions 2.1, 2.2 hold. Assume that there exists a constant \( \lambda > 0 \) such that
\[
E \left[ \max_{1 \leq t \leq N} LV(x(t), x(qt), t, i) \right] \leq -\lambda E \left[ \max_{1 \leq t \leq N} V(x(t), t, i) \right]
\]
for all \( t \geq t_0 \) and those \( V \) satisfying
\[
E \left[ \min_{1 \leq t \leq N} V(x(qt), qt, i) \right] \leq q^{-\lambda} E \left[ \max_{1 \leq t \leq N} V(x(t), t, i) \right].
\]

Then (3) is \( p \)th moment polynomially stable, that is, for any initial data \( \xi \in L_{\mathcal{F}_{t_0}}^p ([q t_0, t_0]; R^n) \),
\[
E|x(t)|^p \leq \frac{c_2}{c_1} (1 + t_0)^\Lambda E\|\xi\|^p (1 + t)^{-\Lambda}, \quad \text{on } t \geq t_0.
\]

**Proof.** Let \( \eta \in (0, \lambda) \) be arbitrary. Define
\[
U(t) = (1 + t)^\eta EV(x(t), t, r(t)).
\]

Since for any \( t \geq t_0 \), \( E|x(t)|^p < \infty \) and by (5), we have
\[
U(t) \leq c_2 (1 + t)^\eta E|x(t)|^p < \infty.
\]

Hence, \( U(t) \) is well defined and is right continuous on \( t \geq t_0 \). For any \( \xi \in L_{\mathcal{F}_{t_0}}^p ([q t_0, t_0]; R^n) \), if we prove that
\[
U(t) \leq c_2 (1 + t_0)^\eta E\|\xi\|^p =: \beta,
\]
then (19) follows straightforwardly. It is obvious that
\[
U(t_0) = (1 + t_0)^\eta EV(x(t_0), t_0, r(t_0)) \leq c_2 (1 + t_0)^\eta E|x(t_0)|^p \leq c_2 (1 + t_0)^\eta E\|\xi\|^p = \beta.
\]

We claim that (21) holds for all \( t > t_0 \). Otherwise, by the right continuity of \( U(t) \), there exists a smallest \( \bar{t} \geq 0 \) such that for any \( t \in [t_0, t_0 + \bar{t}] \), \( U(t) \leq \beta \) and
$U(t_0 + \bar{t}) = \beta$ as well as $U(t_0 + \bar{t} + \delta) > U(t_0 + \bar{t})$ for all sufficiently small $\delta$. Then, if $q(t_0 + \bar{t}) < t_0$, by condition (5),
\[
EV(x(q(t_0 + \bar{t})), q(t_0 + \bar{t}), r(q(t_0 + \bar{t}))) \\
\leq c_2E|q(t_0 + \bar{t})|^p \leq c_2E|\xi|^p = (1 + t_0)^{-\eta}\beta = (1 + t_0)^{-\eta}U(t_0 + \bar{t}) \\
\leq [1 + q(t_0 + \bar{t})]^{-\eta}[1 + (t_0 + \bar{t})]^nEV(x(t_0 + \bar{t}), t_0 + \bar{t}, r(t_0 + \bar{t})) \\
\leq q^{-\eta}EV(x(t_0 + \bar{t}), t_0 + \bar{t}, r(t_0 + \bar{t})) \\
\leq q^{-\lambda}EV(x(t_0 + \bar{t}), t_0 + \bar{t}, r(t_0 + \bar{t})).
\]
If $q(t_0 + \bar{t}) \geq t_0$, we have
\[
EV(x(q(t_0 + \bar{t})), q(t_0 + \bar{t}), r(q(t_0 + \bar{t}))) \\
= [1 + q(t_0 + \bar{t})]^{-\eta}U(q(t_0 + \bar{t})) \\
\leq [1 + q(t_0 + \bar{t})]^{-\eta}U(t_0 + \bar{t}) \\
\leq [1 + q(t_0 + \bar{t})]^{-\eta}[1 + (t_0 + \bar{t})]^nEV(x(t_0 + \bar{t}), t_0 + \bar{t}, r(t_0 + \bar{t})) \\
\leq q^{-\eta}EV(x(t_0 + \bar{t}), t_0 + \bar{t}, r(t_0 + \bar{t})) \\
\leq q^{-\lambda}EV(x(t_0 + \bar{t}), t_0 + \bar{t}, r(t_0 + \bar{t})).
\]
Consequently,
\[
E\left[ \min_{1 \leq i \leq N} V(x(q(t_0 + \bar{t})), q(t_0 + \bar{t}), i) \right] \leq q^{-\lambda}E\left[ \max_{1 \leq i \leq N} V(x(t_0 + \bar{t}), t_0 + \bar{t}, i) \right].
\]
Thus, by condition (17), we have
\[
E\left[ \max_{1 \leq i \leq N} LV(x(t_0 + \bar{t}), x(q(t_0 + \bar{t})), t_0 + \bar{t}, i) \right] \\
\leq -\lambda E\left[ \max_{1 \leq i \leq N} V(x(t_0 + \bar{t}), t_0 + \bar{t}, i) \right] \\
< -\eta E\left[ \max_{1 \leq i \leq N} V(x(t_0 + \bar{t}), t_0 + \bar{t}, i) \right],
\]
(22)
which implies
\[
ELV(x(t_0 + \bar{t}), x(q(t_0 + \bar{t})), t_0 + \bar{t}, r(t_0 + \bar{t})) < -\eta EV(x(t_0 + \bar{t}), t_0 + \bar{t}, r(t_0 + \bar{t})).
\]
By the right continuity of $V$, $t \in [t_0 + \bar{t}, t_0 + \bar{t} + \delta]$ for sufficiently small $\delta$
\[
ELV(x(t), x(qtt), t, r(t)) \leq -\eta EV(x(t), t, r(t)).
\]
Applying the Itô formula to $U(t)$ yields
\[
U(t_0 + \bar{t} + \delta) = U(t_0 + \bar{t}) + \int_{t_0 + \bar{t}}^{t_0 + \bar{t} + \delta} (1 + t)^{\eta} \left( ELV(x(t), x(qt), t, r(t)) + \frac{\eta}{1 + t} EV(x(t), t, r(t)) \right) dt \leq U(t_0 + \bar{t}).
\]
However, this contradicts $U(t_0 + \bar{t} + \delta) > U(t_0 + \bar{t})$, so (21) must hold. The proof is therefore complete.

Now, we use this theorem to establish a useful result on $p$th moment polynomial stability.

**Theorem 3.4.** Let $p > 0$, $\alpha_1 > \alpha_2 \geq 0$ and let Assumptions 2.1 and 2.2 hold. Assume that, for all $x, y \in \mathbb{R}^n$, $i \in S$ and $t \geq 0$,
\[
\max_{1 \leq i \leq N} LV(x, y, t, i) \leq -\alpha_1 \max_{1 \leq i \leq N} V(x, t, i) + \alpha_2 \min_{1 \leq i \leq N} V(y, qt, i).
\]
(23)
Then (3) is \( p \)-th moment polynomially stable, that is, for any initial data \( \xi \in \mathbb{L}^p_{\mathcal{F}}(0, t_0; \mathbb{R}^n) \),

\[
E|x(t)|^p \leq \frac{c_2}{c_1} (1 + t_0)^{\lambda} E||\xi||^p (1 + t)^{-\lambda}
\]  

(24)

where \( \lambda \) is the unique root to the following equation

\[
\lambda = \alpha_1 - \alpha_2 q^{-\lambda}.
\]

Proof. If the solution \( x(t) \) of (3) satisfies that for all \( t \geq t_0 \),

\[
E\left[ \min_{1 \leq i \leq N} V(x(qt), qt, i) \right] \leq q^{-\lambda} E\left[ \max_{1 \leq i \leq N} V(x(t), t, i) \right],
\]

(25)

then by (23) and (25), we have

\[
E\left[ \max_{1 \leq i \leq N} LV(x(t), x(qt), t, i) \right] 
\leq -\alpha_1 E\left[ \max_{1 \leq i \leq N} V(x(t), t, i) \right] + \alpha_2 E\left[ \min_{1 \leq i \leq N} V(x(qt), qt, i) \right] 
\leq -\alpha_1 E\left[ \max_{1 \leq i \leq N} V(x(t), t, i) \right].
\]

This shows that condition (17) is satisfied and (24) follows from Theorem 3.3.

The following theorem gives the sufficient conditions for the almost sure polynomial stability of (3).

**Theorem 3.5.** Let \( p \geq 2, \eta \in (1, \lambda) \) and let all the conditions of Theorem 3.4 hold. Assume that there exists a constant \( L > 0 \) such that

\[
|g(x, y, t, i)| \leq L(|x| + |y|).
\]

Then (24) implies

\[
\limsup_{t \to \infty} \frac{\log|x(t)|}{\log(1 + t)} < -\frac{\lambda - \eta}{p}, \quad \text{a.s.}
\]

(27)

In other words, the \( p \)-th moment polynomial stability implies almost sure polynomial stability.

Proof. For any integer \( n \geq 0 \). Applying the Itô formula to \( V(x, t, i) = |x|^p \), we can obtain that

\[
E\left( \sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} |x(t)|^p \right) 
= E|x(q^{-n}t_0)|^p + E\left( \sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} \int_{q^{-n}t_0}^t LV(x(s), x(qs), s, r(s)) ds \right) 
+ E\left( \sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} \int_{q^{-n}t_0}^t V_x(x(s), s, r(s))g(x(s), x(qs), s, r(s)) dw(s) \right).
\]

(28)

By condition (6), we have

\[
E\left( \sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} \int_{q^{-n}t_0}^t LV(x(s), x(qs), s, r(s)) ds \right) 
\leq \alpha_2 qE\left( \sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} \int_{q^{-n}t_0}^t |x(qs)|^p ds \right) 
\leq \alpha_2 qE \int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(qt)|^p dt = \alpha_2 E \int_{q^{-(n+1)}t_0}^{q^{-n}t_0} |x(t)|^p dt.
\]

(29)
By the Burkholder-Davis-Gundy inequality, condition (26) and the basic inequality
\[ |a + b|^\frac{p}{2} \leq |a|^\frac{p}{2} + |b|^\frac{p}{2}, \]
we have
\[
E\left( \sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} \int_{q^{-n}t_0}^{t} V_x(x(s), s, r(s))g(x(s), x(qs), s, r(s))dw(s) \right) 
\]
\[
\leq C_p E\left( \int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |V_x(x(t), t, r(t))g(x(t), x(qt), t, r(t))|^2 dt \right)^{\frac{1}{2}} 
\]
\[
\leq pC_p LE\left( \int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(t)|^{2p-2}(|x(t)| + |x(qt)|)^2 dt \right)^{\frac{1}{2}} 
\]
\[
\leq \sqrt{2}pC_p LE\left( \int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(t)|^{2p} dt \right)^{\frac{1}{2}} 
\]
\[
+ \sqrt{2}pC_p LE\left( \int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(t)|^{2p-2} |x(qt)|^2 dt \right)^{\frac{1}{2}} 
\]
\[
\leq \sqrt{2}pC_p LE\left( \sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} |x(t)|^p \int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(t)|^p dt \right)^{\frac{1}{2}} 
\]
\[
+ \sqrt{2}pC_p LE\left( \sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} |x(t)|^{p-2} |x(qt)|^2 dt \right)^{\frac{1}{2}} 
\]
\[
\leq \sqrt{2}pC_p L E\left( \sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} |x(t)|^p \right) + \frac{\sqrt{2}pC_p L E}{2\epsilon} \int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(t)|^p dt 
\]
\[
+ \frac{\sqrt{2}pC_p L E}{2\epsilon} \int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(t)|^{p-2} |x(qt)|^2 dt. \tag{30} 
\]

Letting \( \epsilon = \frac{1}{2\sqrt{2}pC_p L} \), it follows from (30) that
\[
E\left( \sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} \int_{q^{-n}t_0}^{t} V_x(x(s), s, r(s))g(x(s), x(qs), s, r(s))dw(s) \right) 
\]
\[
\leq \frac{1}{2} E\left( \sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} |x(t)|^p \right) + 2p^2 C_p^2 L^2 E \int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(t)|^p dt 
\]
\[
+ 2p^2 C_p^2 L^2 E \int_{q^{-n}t_0}^{q^{-(n+1)}t_0} (|x(t)|^p + |x(qt)|^p) dt 
\]
\[
\leq \frac{1}{2} E\left( \sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} |x(t)|^p \right) + 4p^2 C_p^2 L^2 E \int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(t)|^p dt 
\]
In general, the solution do not imply each other. In Remark 3.6, the proof is therefore completed. Therefore, in view of the Borel-Cantelli lemma, there exists an \( \Omega \) with\[
E \left( \sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} |x(t)|^p \right) \leq 2E|x(q^{-n}t_0)|^p + 8p^2C_p^2L^2 \int_{q^{-n}t_0}^{q^{-(n+1)}t_0} E|x(t)|^p dt + (\alpha_2 + 2p^2C_p^2L^2) \int_{q^{-n-1}t_0}^{q^{-n}t_0} E|x(t)|^p dt \leq K_1(1 + q^{-n}t_0)^{-\lambda} + K_2 \int_{q^{-n}t_0}^{q^{-(n+1)}t_0} (1 + t)^{-\lambda} dt, \]
for any \( q \in \mathbb{R} \), where \( K_1 = 2\alpha_2 (1 + t_0)^{\lambda} E[|\xi|^p], K_2 = 12p^2C_p^2L^2 \alpha_2 (1 + t_0)^{\lambda} E[|\xi|^p], K_3 = 2(\alpha_2 + 2p^2C_p^2L^2) \alpha_2 (1 + t_0)^{\lambda} E[|\xi|^p]. \) Now, by the Chebyshev inequality, we have
\[
P \left\{ \sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} |x(t)|^p \geq (1 + q^{-n}t_0)^{-(\lambda - \eta)} \right\} \leq (1 + q^{-n}t_0)^{-(\lambda - \eta)} E \left( \sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} |x(t)|^p \right) \leq K_1(1 + q^{-n}t_0)^{-(\lambda - \eta)} + K_2(1 + q^{-n}t_0)^{-(\lambda - \eta)}(q^{-(n+1)}t_0 - q^{-n}t_0) + K_3(1 + q^{-n}t_0)^{-(\lambda - \eta)}(1 + q^{-(n-1)}t_0)^{-\lambda} (q^{-n}t_0 - q^{-(n-1)}t_0) \leq K_1(1 + q^{-n}t_0)^{-(\lambda - \eta)} + K_2(1 + q^{-n}t_0)^{-(\lambda - \eta)}(q^{-(n+1)}t_0 - q^{-n}t_0) + K_3q^{-\lambda}(1 + q^{-n}t_0)^{-(\lambda - \eta)}(q^{-n}t_0 - q^{-(n-1)}t_0).
\]
Clearly, when \( \eta > 1, \)
\[
K_1 \sum_{n=0}^{\infty} (1 + q^{-n}t_0)^{-(\lambda - \eta)} + K_2 \sum_{n=0}^{\infty} (1 + q^{-n}t_0)^{-(\lambda - \eta)}(q^{-(n+1)}t_0 - q^{-n}t_0) + K_3q^{-\lambda} \sum_{n=0}^{\infty} (1 + q^{-n}t_0)^{-(\lambda - \eta)}(q^{-n}t_0 - q^{-(n-1)}t_0) \leq [K_1 + K_2(\lambda - 1) - 1]t_0 + K_3q^{-\lambda}(1 - q^{-n}t_0) \sum_{n=0}^{\infty} (1 + q^{-n}t_0)^{-\eta/\alpha} < \infty.
\]
In view of the Borel-Cantelli lemma, there exists an \( \Omega_0 \subseteq \Omega \) with \( P(\Omega_0) = 1 \) such that for any \( w \in \Omega_0 \), there exists an integer \( n_0 = n_0(w) > 0 \), when \( n \geq n_0 \) and \( q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0, \)
\[
|x(t)|^p \leq (1 + q^{-n}t_0)^{-(\lambda - \eta)}.
\]
Therefore,
\[
\limsup_{t \to \infty} \frac{\log |x(t)|}{\log (1 + t)} < -\frac{\lambda - \eta}{p}, \quad \text{a.s.}
\]
The proof is therefore completed.

**Remark 3.6.** In general, the \( p \)-th moment stability and almost sure stability of the solution do not imply each other. In [17, 18, 26, 10, 23], the authors proved that the
pth moment stability imply the almost sure stability when the diffusion term and the drift term of stochastic systems obey the linear growth condition. It should be mentioned that the drift term of stochastic systems (3) does not satisfy the linear growth condition in this paper. Therefore, the proposed method in [17, 18, 26, 10, 23] can not be used here.

**Example 3.7.** Let \( w(t) \) is a scalar Brownian motion. Let \( r(t) \) be a right-continuous Markov chain taking values in \( S = \{1, 2\} \) with the generator
\[
\Gamma = \begin{pmatrix}
-8 & 8 \\
\gamma & -\gamma
\end{pmatrix}.
\]

Of course, \( w(t) \) and \( r(t) \) are assumed to be independent. Consider the following scalar hybrid stochastic systems with pantograph delay
\[
dx(t) = f(x(t), t, r(t))dt + g(x(0.5t), t, r(t))dw(t), \quad t \geq 1,
\]
with \( q = 0.5 \), the initial data \( \xi(t) = x_0 \) \((0.5 \leq t \leq 1)\) and \( r(1) = 1 \). Moreover, for \((x, y, t, i) \in R \times R \times (0.5, \infty) \times S, \)
\[
f(x, t, 1) = x, \quad g(y, t, 1) = 0.5y,
\]
\[
f(x, t, 2) = -5x - x^3, \quad g(y, t, 2) = \sqrt{2} y.
\]

We note that (33) can be regarded as the result of the two equations
\[
dx(t) = x(t)dt + 0.5x(0.5t)dw(t)
\]
and
\[
dx(t) = (-5x(t) - x^3(t))dt + \frac{\sqrt{2}}{2} x(0.5t)dw(t),
\]
switching among each other according to the movement of the Markov chain \( r(t) \). It is easy to see that (35) is polynomially stable but (34) is unstable. However, we shall see that due to the Markovian switching, the overall system (33) will be polynomially stable. In fact, the coefficients \( f \) and \( g \) satisfy the local Lipschitz condition but the coefficient \( f \) do not satisfy the linear growth condition. To find out whether hybrid stochastic systems with pantograph delay (33) is polynomial stability in mean square, we use the Lyapunov function
\[
V(x, t, i) = \begin{cases}
x^2, & \text{if } i = 1, \\
0.4x^2, & \text{if } i = 2.
\end{cases}
\]

It is easy to see that the operator \( LV \) from \( R \times R \times (0.5, \infty) \times S \) to \( R \) has the form
\[
LV(x, y, t, 1) \leq -2.8x^2 + 0.25y^2,
\]
and
\[
LV(x, y, t, 2) \leq -(4 - 0.6\gamma)x^2 + 0.2y^2.
\]

According to (23), we require \((4 - 0.6\gamma) \land 2.8 > 0.625.\)

**Case I.** If \( 4 - 0.6\gamma \geq 2.8 \), namely \( 0 < \gamma \leq 2 \), then
\[
\max_{i=1, 2} LV(x, y, t, i) \leq -2.8x^2 + 0.25y^2
\]
\[
\leq -2.8 \max_{i=1, 2} V(x, t, i) + 0.625 \min_{i=1, 2} V(y, qt, i).
\]

By Theorem 3.4, we can conclude that if the transition rate \( \gamma \in (0, 2] \), then hybrid stochastic systems with pantograph delay (33) will be mean square polynomially stable and the 2th moment Lyapunov exponent is not greater than the unique root.
If \( 0.625 < 4 - 0.6\gamma < 2.8 \), namely \( 2 < \gamma < 5.625 \), then

\[
\max_{i=1,2} LV(x, y, t, i) \leq -(4 - 0.6\gamma) \max_{i=1,2} V(x, t, i) + 0.625 \min_{i=1,2} V(y, qt, i).
\]

By Theorem 3.4, we can conclude that if the transition rate \( \gamma \in (2, 5.625) \), then for any initial data \( x_0 \), (33) will be mean square polynomially stable. On the other hand, by Theorem 3.5, we can conclude that if the transition rate \( \gamma \in (2, 2.917) \), (33) will be almost surely polynomially stable. But, if the transition rate \( \gamma \in (2.917, 5.625) \), (33) will not be almost surely polynomially stable. For example, if we choosing \( \gamma = 3 \), we can conclude that (33) will be mean square polynomially stable but not be almost surely polynomially stable. In fact, the root \( \lambda \) of the following equation \( \lambda = 2.2 - 0.625 \times 0.5^{-\lambda} \) is not great than 1 which contradicts the fact \( \lambda > 1 \).

4. Linear Hybrid Stochastic Systems with Pantograph delay. As an application of Theorem 3.4 and 3.5, we consider the linear hybrid stochastic systems with pantograph delay of the form

\[
dx(t) = [A(r(t))x(t) + B(r(t))x(qt)]dt + [C(r(t))x(t) + D(r(t))x(qt)]dw(t),
\]

on \( t \geq t_0 \). Here \( A, B, C, D : S \to \mathbb{R}^{n \times n} \) and we shall write \( A(i) = A_i, B(i) = B_i, C(i) = C_i, D(i) = D_i \).

**Theorem 4.1.** Assume that there exist positive constants \( \theta_i \), \( 1 \leq i \leq N \) and \( \lambda > 1 \), such that \( \lambda_3 > 0 \) and \( q^{-\lambda} < \frac{\lambda_3}{A_2 \lambda_1} \), where \( \lambda_1 = \min_{1 \leq i \leq N} \theta_i, \lambda_2 = \max_{1 \leq i \leq N} \theta_i, \)

\[
\lambda_3 = - \max_{1 \leq i \leq N} \lambda_{\max} \left( |A_i + A_i^\top + 2C_i^\top C_i| \theta_i + (\theta_i + \sum_{j=1}^{N} \gamma_{ij} \theta_j) I \right),
\]

and

\[
\lambda_4 = \max_{1 \leq i \leq N} \| (B_i^\top B_i + 2D_i^\top D_i) \theta_i \|.
\]

Then (36) is mean square polynomially stable and almost surely polynomially stable.

**Proof.** Let \( V(x(t), t, i) = \theta_i |x(t)|^2 \), where \( \theta_i > 0 \) for all \( i \in S \). Clearly,

\[
\lambda_1 |x|^2 \leq V(x, t, i) \leq \lambda_2 |x|^2.
\]

The operator \( LV \) has the form

\[
LV(x(t), x(qt), t, i) = 2\theta_i x^\top(t) [A_i x(t) + B_i x(qt)] \\
+ \theta_i [C_i x(t) + D_i x(qt)]^\top [C_i x(t) + D_i x(qt)] \\
+ \sum_{j=1}^{N} \gamma_{ij} \theta_j x^\top(t) x(t).
\]

Note that

\[
2\theta_i x^\top(t) A_i x(t) = \theta_i x^\top(t) (A_i + A_i^\top) x(t),
\]

\[
2\theta_i x^\top(t) B_i x(qt) \leq \theta_i x^\top(t) x(t) + \theta_i x^\top(qt) B_i^\top B_i x(qt)
\]

and

\[
\theta_i [C_i x(t) + D_i x(qt)]^\top [C_i x(t) + D_i x(qt)] \leq 2\theta_i x^\top(t) C_i^\top C_i x(t) \\
+ 2\theta_i x^\top(qt) D_i^\top D_i x(qt).
\]
It follows that
\[ LV(x(t), x(qt), t, i) \leq \theta_i x^\top(t) \left( A_i + A_i^\top + 2C_i^\top C_i + \left(1 + \frac{1}{\theta_i} \sum_{j=1}^N \tau_{ij}\theta_j\right)I\right) x(t) \]
\[ + \theta_i x^\top(qt) \left( B_i^\top B_i + 2D_i^\top D_i\right) x(qt) \]
\[ \leq -\lambda_3|x(t)|^2 + \lambda_4|x(qt)|^2. \]  
\( (40) \)

For any \( t \geq t_0 \) and \( V \) satisfying
\[ E[\min_{1 \leq i \leq N} V(x(qt), qt, i)] \leq q^{-\lambda}E[\max_{1 \leq i \leq N} V(x(t), t, i)], \]
we have
\[ \lambda_1 E|x(qt)|^2 \leq q^{-\lambda}\lambda_2 E|x(t)|^2 \text{ on } t \geq t_0. \]
Hence, by (40), we obtain
\[ E[\max_{1 \leq i \leq N} LV(x(t), x(qt), t, i)] \leq -\lambda_3 E|x(t)|^2 + \lambda_4 E|x(qt)|^2 \]
\[ \leq -\left(\lambda_3 - q^{-\lambda}\frac{\lambda_2 \lambda_4}{\lambda_1}\right) E|x(t)|^2. \]

Recalling \( q^{-\lambda} < \frac{\lambda_1}{\lambda_2 \lambda_4} \), we see all assumptions of Theorem 3.3 are satisfied with \( p = 2 \), therefore, linear hybrid stochastic systems with pantograph delay (36) is mean square polynomially stable. On the other hand, for any \( i \in S \), we have
\[ |C(i)x(t) + D(i)x(qt)| \leq ||C(i)|||x(t)|| + ||D(i)|||x(qt)|| \]
\[ \leq L(|x(t)| + |x(qt)|), \]
where \( L = \max_{1 \leq i \leq N} ||C(i)|| + ||D(i)||. \) Hence, by Theorem 3.5, (36) is also almost surely polynomially stable.

**Example 4.2.** Let \( w(t) \) be a scalar Brownian motion. Let \( r(t) \) be a right-continuous Markov chain taking values in \( S = \{1, 2\} \) with the generator
\[ \Gamma = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}. \]

Assume that \( w(t) \) and \( r(t) \) are independent. Consider a two-dimensional hybrid stochastic systems with pantograph delay
\[ dx(t) = A(r(t))x(t)dt + B(r(t))x(0.4t)dt + D(r(t))x(0.4t)dw(t), \quad t \geq 1, \]
\( \quad (41) \)
with \( q = 0.4 \), the initial data \( \xi(t) = x_0 \) \((0.4 \leq t \leq 1)\) and \( r(1) = 1 \). Here
\[ A_1 = \begin{pmatrix} -8 & 1 \\ 2 & -4 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0.3 & 0.2 \\ -0.1 & 0.4 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0.5 & 0.2 \\ -0.5 & 0.1 \end{pmatrix} \]
and
\[ A_2 = \begin{pmatrix} -5 & -1 \\ 1 & -6 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0.2 \\ 0.1 & 0.4 \end{pmatrix}, \quad D_2 = \begin{pmatrix} -0.5 & 0.3 \\ 0.1 & 0.1 \end{pmatrix}. \]

Define the Lyapunov function \( V(x, t, 1) = 0.4|x|^2 \) and \( V(x, t, 2) = 0.3|x|^2 \). Though a straight computation, it is not difficult to obtain \( \lambda_1 = 0.3 \), \( \lambda_2 = 0.4 \), \( \lambda_3 = 2.3 \), \( \lambda_4 = 0.4886 \). Obviously, \( \lambda_3 = 2.3 > 0 \). If \( 0.4^{-\lambda} < \frac{\lambda_1}{\lambda_2 \lambda_4} = 3.5305 \), i.e., \( \lambda \in (1, 2.4694) \), then by Theorem 4.1, we obtain that hybrid stochastic systems with pantograph delay (41) is polynomially stable in mean square and is also almost surely polynomially stable.
Conclusion. This paper is devoted to the $p$th moment polynomial stability and almost sure polynomial stability of hybrid stochastic systems with pantograph delay. The Razumikhin technique and Lyapunov functions is used to derive sufficient conditions for stabilities of nonlinear hybrid stochastic systems with pantograph delay, which further helps to derive easily verifiable conditions for linear hybrid stochastic systems with pantograph delay. Finally, two examples are provided to verify the effectiveness of the main results.

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