Spin electrodynamics in a mesoscopic ring

Dann S Olesen and Ole Keller

Abstract

The minimal-coupled Pauli equation is used to calculate the linear microscopic spin conductivity tensor of a circular mesoscopic free-electron ring possessing complete (single-level) transverse electron confinement. A detailed comparison of the structures (and magnitudes) of the spin and space conductivity tensors is made. When the incident electric field is polarized perpendicular to the plane of the ring only the spin conductivity is nonvanishing. The polarization selection rules for the spin conductivity always involve a transfer of orbital angular momentum between neighbouring running ring states. Numerical spectral results are presented for a nano-sized Al ring excited perpendicular to the ring plane for different temperatures. The frequency spectra of the relevant spin conductivity component are qualitatively different for rings holding even and uneven half number of electrons due to different state occupancy probabilities near the Fermi edge. Far-field electromagnetic scattering spectra related to the spin dynamics are calculated in the limit where the quantum ring behaves as an electric-dipole absorber and radiator.

1. Introduction

In the linearized electrodynamics of mesoscopic media the microscopic conductivity tensor, \( \sigma_{\text{space}}(r, r'; \omega) \), relates the selfconsistent microscopic electric field \( E(r'; \omega) \) at space point \( r' \) to the prevailing microscopic current density, \( J(r; \omega) \), at the space point \( r \). \( \omega \) is the angular frequency. The current density response of the dynamic magnetic field, \( B(r; \omega) = \nabla \times E(r; \omega)/(i\omega) \), is included as a part of \( \sigma_{\text{space}}(r, r'; \omega) \) for \( r \neq r' \). The conductivity tensor is derived in a perturbative scheme from the Schrödinger Hamiltonian. Beginning from the field-coupled Dirac equation a unitary Foldy-Wouthuysen transformation \( \mathcal{H} \rightarrow \mathcal{H}_\text{mass} \) is included as a part of \( \mathcal{H}_\text{field-coupled Dirac equation} \).

In order to include dynamic spin effects in the conductivity one must start from a weakly relativistic Hamiltonian. Beginning from the field-coupled Dirac equation a unitary Foldy-Wouthuysen transformation gives in a rigorous manner all the weakly relativistic corrections (used here in lowest order) to the Hamiltonian of the Schrödinger equation. Two of the corrections relate to the electron spin, \( S = (\hbar/2)\sigma \), viz., the spin-orbit term, \( [\hbar/(4m_0 c^2)] \nabla V(r) \times (p + eA) \cdot \sigma \), and the Pauli term, \( [\hbar/(2m_0)] \sigma \cdot \nabla \times A \). The dynamic parts of these, proportional to \( A(r; \omega) \) and \( \nabla \times A(r; \omega) \), respectively, give rise to an additive contribution, \( \sigma_{\text{spin}}(r, r'; \omega) \), to the spatial conductivity tensor. Since the vector potential, \( A(r; \omega) \), is assumed to have no static part the Zeeman effect is absent.

In this work we shall limit ourselves to a theoretical study of the dynamic Pauli contribution to \( \sigma_{\text{spin}}(r, r'; \omega) \). In a forthcoming paper we shall analyze the dynamic spin-orbit term in mesoscopic media, paying particular attention to the fact that this term has a significant contribution from the surface of a mesoscopic system where the gradient of the potential energy, \( V(r) \), varies rapidly in space.

Although the general expression for the Pauli contribution to \( \sigma_{\text{spin}}(r, r'; \omega) \) has been derived on the basis of many-body response theory more than two decades ago, quantitative calculations for specific mesoscopic media are needed to enlighten important physical aspects and details. The angular momentum photon-drag...
current in a mesoscopic cylinder shell and a quantum ring has been studied in [9]. The photon drag is a nonlinear, spatially nonlocal effect. The photon momentum drag involving spin-flip transitions in nonsymmetric quantum wells has been studied both experimentally [10] and theoretically [11] in InSb and GaAs. The angular-momentum photon drag’s relation to persistent currents [12, 13] has been studied in [14]. Comprehensive theoretical studies of spin and diamagnetism in linear and nonlinear optics also have been carried out [15–17]. In the present paper we calculate and discuss the Pauli spin contribution to \( \sigma^{\text{spin}}(r, r'; \omega) \) for a mesoscopic (nano-sized) circular ring in the limit where it behaves like a circular quantum wire. Despite the fact that the spin dynamics usually only gives a small contribution to the total conductivity tensor, an investigation of the differential electromagnetic far-field scattering of a properly polarized incident field allows one to pick out a part of the dynamic spin contribution to the linear conductivity tensor.

Our motivation for carrying out the present analysis comes from recent microscopic studies of the diffraction of electromagnetic radiation from mesoscopic holes in metallic screens, in particular Al screens. The Fermi surface of Al is close to a free-electron surface (band structure effects negligible at infrared frequencies and below). Thus, a model without spin–orbit coupling appears sufficient. From a microscopic point of view the scattering form a circular hole originates from a ring-shaped region in the vicinity of the hole where the electron density various in space (Friedel-like oscillations). In previous works, electron spin effects have been neglected, and our idea is to include these via the present model (for the ring-shaped surface region). In cases where the incident field excites the electron system perpendicular to the plane of the screen, and the screen is so thin that the in-plane electron dynamics dominates, the spin contribution is believed to dominate the diffraction, cf the analyses in sections 6–8. A schematic illustration of the mesoscopic ring, and the spin currents induced by (i) in-plane electromagnetic fields of opposite spin, and (ii) fields polarized perpendicular to the plane of the ring (figure 1).

In sections 2 and 3, we summarize the general results for the linear spin- and space conductivity tensors. In section 5, the spin- and space conductivities of a mesoscopic circular ring is calculated under the assumptions that (i) the incident field acting on the ring does not vary across the ring domain [electric dipole (ED) limit] and (ii) the cross sectional dimensions of the ring are so small that the ring electrons essentially are confined to a 'one-dimensional' motion around the ring (quantum wire approach). Qualitatively, the confinement assumption requires that the field quantum energy, \( h\omega \), is much smaller than the electronic level spacings related to the transverse coordinates. In section 6, we discuss the physical structure of the quantum wire’s conductivity.
tensor, emphasizing the fact that orbital angular momentum between neighboring running electron states always are involved. In sections 4 and 7, we analyze the electromagnetic scattering from the spins and calculate the scattered Poynting vector in the far field in a configuration where only the spin system contributes to the scattering process. In section 8, numerical results are presented for the frequency dependence of the relevant component of the spin conductivity tensor, and the related far-field Poynting vector. Spectra are calculated for nano-sized Al wires containing an even number (N) of free electrons, viz., \(10^3\) and \(10^5\). Results are presented for three temperatures (10 K, 300 K, 500 K). The two choices for \(N\) illustrate the theoretical prediction that the spectra for \(N/2\) even and odd are qualitatively different due to the different electron occupancy probabilities near the Fermi edge.

2. Spin conductivity tensor

In a solid state plasma, treated in the framework of the electron gas (jellium) approximation, spin–orbit couplings are absent. In consequence, the linear conductivity tensor at frequency \(\omega\), \(\sigma(\mathbf{r}, \mathbf{r}'; \omega)\), is made up of additive contributions related to the space- and spin-electrodynamics of the jellium (see the appendix), i.e.,

\[
\sigma(\mathbf{r}, \mathbf{r}'; \omega) = \sigma^{\text{space}}(\mathbf{r}, \mathbf{r}'; \omega) + \sigma^{\text{spin}}(\mathbf{r}, \mathbf{r}'; \omega).
\]

(1)

The microscopic conductivity correlates the electron dynamics at two different space points, \(\mathbf{r}\) and \(\mathbf{r}'\).

In the weakly relativistic (Pauli) limit each electron gives a spin contribution

\[
\hat{v}^{\text{spin}} = -\frac{g}{2m_e} \hat{\sigma} \cdot \mathbf{B}
\]

(2)

to the Hamilton operator, and adds a Hermitian spin current density operator

\[
\hat{j}^{\text{spin}} = \frac{\hbar}{2m_e} \left[ \hat{\sigma} \cdot (\mathbf{r} - \mathbf{r}_e) \times \mathbf{p} \right] = \hat{\sigma} \cdot \mathbf{p} \delta(\mathbf{r} - \mathbf{r}_e)
\]

(3)

to the spatial part of the current density operator. In equation (2), \(g \approx 2\), \(q \approx e[\epsilon > 0]\), \(m_0\), \(\hat{\sigma}\), and \(\mathbf{B} = \mathbf{B}(\mathbf{r}; \omega)\) denote the gyromagnetic factor, the electron charge and mass, the (dimensionless) Pauli spin operator, and the local magnetic field. In equation (3) \(\mathbf{r}_e = \mathbf{r}_e\) and \(\mathbf{p} = (\hbar/i) \nabla\) are the electron position and momentum operators in the direct-space representation.

An iterative solution of the Liouville equation for the spin density matrix operator, \(\hat{P}^{\text{spin}}\), allows one to determine the spin contribution to the conductivity tensor. A many-body calculation of \(\sigma^{\text{spin}}(\mathbf{r}, \mathbf{r}'; \omega)\), carried out with \(\hat{V}^{\text{spin}}\) and \(\hat{j}^{\text{spin}}\) in their second-quantized form, can be found in [6]. In the random-phase-approximation (RPA), where the eigenstates of the many-electron Hamiltonian consist of direct product of single-particle-like states one obtains

\[
\sigma^{\text{spin}}(\mathbf{r}, \mathbf{r}'; \omega) = \frac{g^2 \hbar}{2} \sum_{ij} \frac{\varepsilon_j - \varepsilon_i}{\hbar \omega} \left( \frac{1}{\varepsilon_j - \varepsilon_i} \frac{f_j - f_i}{\hbar \omega} \right) \times R_{i \\ j}^{\text{spin}}(\mathbf{r}, \mathbf{r}').
\]

(4)

The single-particle states are characterized by the spatial quantum number \(i, j, \cdots\). Each number is a triplet set. \(\varepsilon_i(\varepsilon_j)\) is the eigenenergy of state \(|i\rangle\) and the related occupation (Fermi–Dirac) factor is \(f_i(f_j)\). The same factor, \(\varepsilon_i - \varepsilon_j\), appears in the nominator and denominator of equation (4). For a comparison to the expression for the space conductivity given in equation (7) it is convenient to keep the form in equation (4). The frequency independent dyadic tensor

\[
R_{i \\ j}^{\text{spin}}(\mathbf{r}, \mathbf{r}') = j_{i \\ j}^{\text{spin}}(\mathbf{r}') \times \mathbf{U} \times j_{j \\ i}^{\text{spin}}(\mathbf{r}') = j_{i \\ j}^{\text{spin}}(\mathbf{r}') j_{j \\ i}^{\text{spin}}(\mathbf{r}) - U_{j \\ i}^{\text{spin}}(\mathbf{r}) \cdot j_{i \\ j}^{\text{spin}}(\mathbf{r}')
\]

(5)

has a structure build from the spin transition current density between the states \(i\) and \(j\), viz.,

\[
\hat{j}^{\text{spin}}(\mathbf{r}) = \frac{\hbar}{2m_0} \left[ \nabla \psi_i(\mathbf{r}) \psi_j(\mathbf{r}) + \nabla \psi_j(\mathbf{r}) \psi_i(\mathbf{r}) \right] = \frac{\hbar}{2m_0} \nabla [\psi_i(\mathbf{r}) \psi_j^*(\mathbf{r})]
\]

(6)

and an analogous expression for \(j_{i \\ j}^{\text{spin}}(\mathbf{r}')\). The unit tensor \(\mathbf{U}\) can be written in an infinite variety of dyadic forms. In section 5 a form containing helicity unit vectors is used.

3. Space conductivity tensor

In general, the electrodynamic properties of the electron gas will be dominated by the field-induced spatial motion of the particles. The related linear space conductivity tensor, calculated in the framework of RPA, is given by [6]
\[ \sigma^{pace}(r', r'; \omega) = \frac{2\hbar}{i} \sum_{ij} \frac{f_j - f_i}{\varepsilon_j - \varepsilon_i} \frac{1}{\hbar \omega + \varepsilon_j - \varepsilon_i} \times R^{pace}_{j-i}(r', r'), \]  
(7)

where

\[ R^{pace}_{j-i}(r', r') = j^{pace}_{j-i}(r) j^{pace}_{j-i}(r'). \]  
(8)

The frequency independent tensor in equation (8) is a dyadic product of two transition current densities, \( j_{j-i}(r) \) and \( j_{j-i}(r') \). In explicit form

\[ j^{pace}_{j-i}(r) = \frac{q\hbar}{2m_0 i} (\psi^* s_j(r) \nabla \psi_j(r) - \psi_j(r) \nabla \psi^* s_j(r)), \]  
(9)

and an analogous formula for \( j^{pace}_{j-i}(r') \).

A comparison of equations (4) and (7) shows three important qualitative differences between \( \sigma^{pace}(r', r'; \omega) \) and \( \sigma^{spin}(r', r'; \omega) \). It is important to appreciate these differences in order to select possible cases in which the spin electrodynamics does not make an insignificant contribution to the jellium dynamics, and hence to electromagnetic scattering processes. The three differences, which we shall quantify in our study of a circular quantum wire (sections 5 and 6), are: (i) The extra prefactor \((\varepsilon_j - \varepsilon_i)/(\hbar \omega)\) on each \((i,j)\) terms in \( \sigma^{spin}(r', r'; \omega) \). In consequence, one expects that the spin electrodynamics tends to play a more important role at frequencies far below all Bohr \((B)\) transition frequencies \(\omega_B^0 = |\varepsilon_j - \varepsilon_i|/\hbar\). (ii) The different symmetries of \( R^{pace}_{j-i}(r, r') \) and \( R^{spin}_{j-i}(r, r') \) which enables one to select electromagnetic scattering configurations which favour the spin contribution; see section 7. (iii) The sign difference between the two terms in the transition current densities \( j^{pace}_{j-i}(r') \) \(= j^{pace}_{j-i}(r)^* \) and \( j^{spin}_{j-i}(r') \) \(= j^{spin}_{j-i}(r)^* \) has the consequence that the relative importance of the spin dynamics increases if the linear dimensions of the solid state plasma are reduced from macroscopic to mesoscopic sizes. We shall exemplify points (i)–(iii) in our quantum wire study.

4. Electromagnetic scattering from a mesoscopic object

Let us assume that an incident electromagnetic field, with an electric part denoted by \( E^0(r; \omega) \) in the space-frequency domain, interacts linearly with a solid state plasma. Inside the plasma a yet unknown local electric field \( E(r; \omega) \) prevails. In the RPA description a microscopic electron current density

\[ J(r; \omega) = \int_{-\infty}^{\infty} \sigma(r, r'; \omega) \cdot E(r'; \omega) \, dr' \]  
(10)

is induced by the local field. In order to determine the local field equation (10) must be supplemented by the scattering integral equation

\[ E(r; \omega) = E^0(r; \omega) + i\mu_0 \omega \int_{-\infty}^{\infty} G(r, r'; \omega) \cdot J(r'; \omega) \, dr'. \]  
(11)

This integral equation is established from the microscopic Maxwell-Lorentz equations [6]. The dyadic quantity \( G(r, r'; \omega) \) is the standard electromagnetic Green function. Note that its properties (form) inside matter and in the near-field zone of matter are quite complicated and first derived in macroscopic electrodynamics by Yaghjian [18], and in microscopic physics in [6] (see also [19]). Once the local electric field inside the plasma has been determined from the selfconsistent integral equation obtained by inserting equation (10) into equation (11), the scattered field

\[ E^{scatt}(r; \omega) = E(r; \omega) - E^0(r; \omega) \]  
(12)

can be determined everywhere in space by direct integration over the scattering medium. A selfconsistent calculation of the local field poses a difficult problem even in mesoscopic systems. Integral equations of the type in equation (11), with equation (10) inserted, often are used in near-field electrodynamics [20, 21] and in macroscopic scattering analyses [22]. In recent years these difficulties have been illustrated, e.g., in a series of papers dealing with the electromagnetic (light) scattering from a nano-sized hole in a flat metallic (or semiconductor) quantum-well screen [23].

In our study of the scattering from a quantum wire it will be assumed that the wire cross section is so small that the electron motion in the directions perpendicular to the wire is frozen. Microscopically, this corresponds to a situation where only one bound quantum level is associated to each of the two transverse coordinates. In this case the difference between the local and external fields inside the quantum wire is expected to be small, and the total field outside the scattering medium therefore is determined by

\[ E(r; \omega) = E^0(r; \omega) + i\mu_0 \omega \int_{-\infty}^{\infty} G(r, r'; \omega) \cdot \sigma(r', r''; \omega) \cdot E^0(r''; \omega) \, dr'' \, dr'. \]  
(13)
In a somewhat different perspective, equation (13) appears as scattering calculated in a first-order Born approximation [20, 22].

Before turning our attention towards the circular quantum ring, we shall assume that the dimensions of the mesoscopic medium are so small that the spatial variations of the dyadic Green function and the incoming field across the scattering object are negligible. Physically, this corresponds to a situation where the mesoscopic object can be considered both as an electric dipole (ED) absorber and radiator [23]. In this double ED approximation equation (13) is reduced to the form

\[
\mathbf{E}(\mathbf{r}; \omega) = \mathbf{E}^0(\mathbf{r}; \omega) + i \mu_0 \omega G(\mathbf{r}, \mathbf{r}'; 0; \omega) \cdot \sigma_{\text{ED}}(\omega) \cdot \mathbf{E}^0(0; \omega),
\]

where

\[
\sigma_{\text{ED}}(\omega) = \int_{\mathbb{R}^2} \boldsymbol{\sigma}(\mathbf{r}, \mathbf{r}'; \omega) \, d^3\mathbf{r} \, d^3\mathbf{r}'.
\]

In equation (14) we have assumed that the origo of our coordinate is placed inside the mesoscopic object.

5. Spin and space conductivities (ED) of a mesoscopic circular-cylindrical ring

5.1. Quantum states

Let us consider a circular cylindrical shell having its center axis coincident with the z-axis in a cylindrical \((r, \phi, z)\)-coordinate system. The height of the cylinder is \(L\) and its shell thickness is \(d\). Inside the shell, i.e., for \(r_0 \leq r \leq r_0 + d\) and \(-L/2 \leq z \leq L/2\), we assume that the electron potential energy is constant \(V(\mathbf{r}) = 0\). In the general case, the stationary single-electron product states can be described by wave functions with three indices, \(\psi_{n, l, m}\), where \(n, l,\) and \(m\) are associated to the radial states \((n)\) and the states along \((l)\) and around \((m)\) the cylinder axis. It is well known that the radial parts of the wave functions also depend on \(l\) and \(m\). When \(L\) and \(d\) become sufficiently small (ring case) one is left with only one bound eigenstate for the \(r\) - and \(z\)-parts of the eigenfunctions:

\[
\psi_{n, l, m} \Rightarrow R(r) Z(z) \Phi_{nl}(\phi).
\]

In the remaining part of the paper Greek letter indices \((\alpha, \beta, \cdots)\) are used to classify the various \(m\)-states. For what follows it is sufficient to assume complete spatial electron confinement in the \(r\) and \(z\) coordinates; cf the delta-function potential method often used in quantum-well systems.

The separable normalized probability densities for the mesoscopic circular ring (quantum wire) we thus approximate by

\[
|R(r)|^2 = \frac{\delta(r - r_0)}{r_0},
\]

and

\[
|Z(z)| = \delta(z),
\]

where \(\delta\) is the Dirac delta function. For the stationary free-electron states \(\Phi_{nl}(\phi)\) we use the set

\[
\Phi_{nl}(\phi) = \frac{1}{\sqrt{2\pi}} e^{im_\phi}, \quad m_\phi = 0, \pm 1, \pm 2, \cdots.
\]

The associated eigenenergies are

\[
\varepsilon_m = \frac{\hbar^2}{2m_0\hbar^2} m^2.
\]

The use of left- and right-hand running eigenmodes instead of standing modes is convenient in the study of spin electrodynamics, as we shall see in section 6. To calculate the spin (equation (6))—and space (equation (9))—transition current densities for a circular ring, assuming complete transverse electron confinement, one must make the reduction

\[
\nabla \Rightarrow \hat{\phi} \frac{1}{r_0} \frac{\partial}{\partial \phi},
\]

where

\[
\hat{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi
\]

is the relevant local unit vector, decomposed in its Cartesian \(x\) and \(y\) components.
5.2. Integrated spin and space transition current densities

On the basis of equations (17)–(21) it is a straightforward matter to calculate the spin and space transition current densities for the circular quantum wire. Hence,

\[
J_{p \alpha \rightarrow \beta}(r) = -\frac{q\hbar}{4\pi m_0 r_0^2} \delta(r - r_0) \delta(z)(m_\alpha - m_\beta) e^{i(m_\alpha - m_\beta)\phi} \tag{23}
\]

and

\[
J_{spac e \alpha \rightarrow \beta}(r) = -\frac{q\hbar}{4\pi m_0 r_0^2} \delta(r - r_0) \delta(z)(m_\alpha + m_\beta) e^{i(m_\alpha - m_\beta)\phi}. \tag{24}
\]

It appears from equations (23) and (24) that

\[
J_{spac e \alpha \rightarrow \beta}(r) = \frac{m_\alpha - m_\beta}{m_\alpha + m_\beta} J_{p \alpha \rightarrow \beta}(r). \tag{25}
\]

The two different combinations \(m_\alpha \pm m_\beta\), appearing in equations (23) and (24), can be traced back to the sign difference between the terms inside the square brackets of equations (6)[first member] and (9).

The integrated transition current densities, given in general by

\[
I_{\alpha \rightarrow \beta}^{\text{spin}} = \int_{-\infty}^{\infty} J_{p \alpha \rightarrow \beta}^{\text{spin}}(r) dr,
\]

and

\[
I_{\alpha \rightarrow \beta}^{\text{space}} = \int_{-\infty}^{\infty} J_{p \alpha \rightarrow \beta}^{\text{space}}(r) dr,
\]

are easily calculated for the quantum wire. Hence,

\[
I_{\alpha \rightarrow \beta}^{\text{spin}} = \int_{-\infty}^{\infty} \frac{q\hbar}{2m_0 r_0} (m_\alpha - m_\beta) \frac{1}{2\pi} \int_0^{2\pi} \phi e^{i(m_\alpha - m_\beta)\phi} d\phi
\]

\[
= \frac{q\hbar}{2\sqrt{2} m_0 r_0} \delta_{m_\alpha m_\beta} (\hat{e}_+ + \hat{e}_-),
\]

\[
(28)
\]

and by use of equation (25)

\[
I_{\alpha \rightarrow \beta}^{\text{space}} = \frac{q\hbar}{2\sqrt{2} m_0 r_0} \big[(2m_\alpha - 1) \delta_{m_\alpha m_\beta} \hat{e}_+ - (2m_\alpha + 1) \delta_{m_\alpha m_\beta} \hat{e}_-\big].
\]

(29)

In equations (28) and (29) we have introduced the two (+, −) helicity unit vectors

\[
\hat{e}_\pm = \frac{1}{\sqrt{2}} (\hat{e}_x \pm i \hat{e}_y),
\]

(30)

descriving oppositely rotating circular states in the \(xy\)-plane, in a manner to be discussed in section 6 C. The quantity \(\delta_{m,m'}\) is the Kronecker delta, here relating only neighbouring states, \(m_\beta = m_\alpha = \pm 1\).

For later use, we note that

\[
\hat{e}_+ \cdot \hat{e}_\pm = 0, \quad \hat{e}_- \cdot \hat{e}_\pm = 1, \quad \hat{e}_\pm \cdot \hat{z} = 0;
\]

\[
\hat{e}_\pm \times \hat{e}_+ = \mp i \hat{z}, \quad \hat{e}_\pm \times \hat{z} = \pm i \hat{e}_\pm,
\]

(31)

(32)

and that the unit tensor has a dyadic decomposition

\[
U = \hat{e}_+ \hat{e}_+ + \hat{e}_- \hat{e}_- + 2 \hat{z} \hat{z}
\]

(33)

5.3. Integrated spin and space dyads

In the double electric-dipole approximation only the integrals of the dyads \(R_{\alpha \rightarrow \beta}(r, r')\) (equation (5)) and \(R_{\alpha \rightarrow \beta}^{\text{space}}(r, r')\) (equation (8)) over \(r\) and \(r'\) are needed for a determination of the spin and space parts of the ED conductivity tensor (equation (15)). By means of the formulas

\[
\delta_{m_\alpha m_\beta} \delta_{m_\alpha m_\beta} = 0, \tag{34}
\]

\[
\delta_{m_\alpha m_\beta} \delta_{m_\alpha m_\beta} = \delta_{m_\alpha m_\alpha}, \tag{35}
\]

the unit vector decomposition (equation (33)), and the vector product relations in equation (32), one obtains in the quantum wire case after a somewhat tedious calculation.
\[ \Gamma_{\alpha \rightarrow \beta}^{\text{pin}} \times U \times \Gamma_{\beta \rightarrow \alpha}^{\text{pin}} = -\left( \frac{\hbar}{2 \sqrt{2} m_0 \omega} \right)^2 \times \left[ (\hat{\mathbf{E}}_x \cdot \hat{\mathbf{e}} - \hat{\mathbf{E}}_z \cdot \hat{\mathbf{e}}) \delta_{m_\alpha, m_{\beta - 1}} + (\hat{\mathbf{E}}_z \cdot \hat{\mathbf{e}}) \delta_{m_\beta, m_{\alpha + 1}} \right]. \]  

(36)

The corresponding space part is much easier to calculate; only equations (34) and (35) are needed. Hence,

\[ \Gamma_{\alpha \rightarrow \beta}^{\text{space}} \times \Gamma_{\beta \rightarrow \alpha}^{\text{space}} = \left( \frac{\hbar}{2 \sqrt{2} m_0 \omega} \right)^2 \left[ (2m_\alpha - 1)^2 \hat{\mathbf{e}}_x \cdot \hat{\mathbf{e}} - (2m_\alpha + 1)^2 \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}} \right] \delta_{m_\beta, m_{\alpha + 1}} \].

(37)

5.4. Summation over states

It appears from the analysis in section 5.1–5.3, and equation (4) that the frequency dependent ED spin conductivity of the circular quantum wire is given by

\[ \sigma_{\text{ED}}^{\text{pin}}(\omega) = \sum_{\alpha, \beta} \frac{f_\beta - f_\alpha}{\hbar \omega + \varepsilon_\beta - \varepsilon_\alpha} \Gamma_{\alpha \rightarrow \beta}^{\text{pin}} \times U \times \Gamma_{\beta \rightarrow \alpha}^{\text{pin}}, \]

(38)

with \( f_\alpha \equiv f_{m_\alpha} \left( f_\beta \equiv f_{m_\beta} \right) \). The Kronecker delta’s in equation (36) reduce the double sum in equation (38) to a single sum, giving

\[ \sigma_{\text{ED}}^{\text{pin}}(\omega) = i \frac{\tilde{\omega}}{\omega} \left( \frac{\hbar}{2 \sqrt{2} m_0 \omega} \right)^2 \sum_\alpha \left[ \frac{f_{m_\alpha - 1} - f_{m_\alpha}}{\hbar \omega + \varepsilon_{m_\alpha - 1} - \varepsilon_{m_\alpha}} (\hat{\mathbf{E}}_x \cdot \hat{\mathbf{e}}) \delta_{m_\alpha, m_{\beta - 1}} + \frac{f_{m_\alpha + 1} - f_{m_\alpha}}{\hbar \omega + \varepsilon_{m_\alpha + 1} - \varepsilon_{m_\alpha}} (\hat{\mathbf{E}}_z \cdot \hat{\mathbf{e}}) \right]. \]

(39)

Our final expression for the spin conductivity tensor is obtained using the expression for the eigenenergies (equation (20)), and trivial relabelings \( \sum_\alpha \Rightarrow \sum_{m_\alpha} \). Thus,

\[ \sigma_{\text{ED}}^{\text{pin}}(\omega) = i \frac{\tilde{\omega}}{\omega} \left( \frac{\hbar}{2 \sqrt{2} m_0 \omega} \right)^2 \sum_\alpha \left[ \frac{f_{m_\alpha - 1} - f_{m_\alpha}}{\hbar \omega + \frac{\hbar^2}{2 m_0 c^2} (1 - 2m_\alpha)} (\hat{\mathbf{E}}_x \cdot \hat{\mathbf{e}}) + \frac{f_{m_\alpha + 1} - f_{m_\alpha}}{\hbar \omega + \frac{\hbar^2}{2 m_0 c^2} (1 + 2m_\alpha)} (\hat{\mathbf{E}}_z \cdot \hat{\mathbf{e}}) \right]. \]

(40)

Following the analogous procedure to the one above, it follows that the space conductivity tensor in the ED approximation is given by

\[ \sigma_{\text{ED}}^{\text{space}}(\omega) = \frac{\hbar}{2 \omega} \sum_m \left[ (1 - 2m_m) (f_{m_{m - 1}} - f_{m_m}) \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}} + \frac{(1 + 2m_m) (f_{m_{m + 1}} - f_{m_m})}{\hbar \omega + \frac{\hbar^2}{2 m_0 c^2} (1 + 2m_\alpha)} \hat{\mathbf{e}}_x \cdot \hat{\mathbf{e}} \right]. \]

(41)

From the outset it has been assumed that the electron gas in the absence of the incident electromagnetic field is spin degenerated (no time independent magnetic field present, no spin–orbit coupling, etc). This implies that the Fermi–Dirac distribution in the one-electron approach shall be normalized to half the number of electrons in the given closed system. Hence, for our quantum wire it is required that

\[ \frac{N}{2} = \sum_m \exp \left( \frac{\varepsilon_m - \mu(T)}{kT} \right) + 1 \]

(42)

where \( N \) is the number of jellium electrons, and \( \mu(T) \) is the chemical potential at the absolute temperature \( T \). If the number of electrons carried in the wire and the temperature are known, \( \mu(T) \) can be determined from equation (42).

6. Structure of spin conductivity tensor

In this section we discuss the physical structure of the key result obtained for the spin conductivity tensor in the electric–dipole approximation (equation (40)). By a comparison with the corresponding result found for the ED space conductivity tensor (equation (41)), it is realized that the relative importance of the spin dynamics can be significantly increased in a scattering process by a proper choice of the external parameters \( \omega, \gamma_0 \), and \( \mathbf{E}(\mathbf{0}; \omega) \).

6.1. Net occupations

If one rewrites the difference \( f_{m_{m - 1}} - f_{m_m} \) in the form

\[ f_{m_{m - 1}} - f_{m_m} = f_{m_{m - 1}} - f_{m_{m - 1}} + f_{m_m}(1 - f_{m_{m - 1}}), \]

(43)

it appears that the first part behind the summation sign in equation (40) describes the net transition probability (normalized to \( N \)) from state \( m_{m - 1} \) to \( m_m \). The upward transition \( (m_{m - 1} \rightarrow m_m) \) is proportional to the product of the probabilities that the lower state is occupied \( (f_{m_{m - 1}}) \) and the higher state is empty \( (1 - f_{m_m}) \). The downward transition rate \( (m_m \rightarrow m_{m - 1}) \), given by the second part of equation (43) is proportional to the product of the
probabilities for the $m$ and $m-1$ states being occupied $(f_m)$ and empty $(1-f_{m-1})$, respectively. The difference between the first and second parts gives the net excitation rate $(f_{m-1} - f_m)$ from $m-1$ to $m$.

A rewriting

$$f_{m+1} - f_m = f_{m+1}(1 - f_m) - f_m(1 - f_{m+1})$$

makes it obvious that the second part behind the summation sign in equation (40) relates to the net deexcitation rate form state $m+1$ to $m$. A schematic illustration of the cycle related to equation (43) discussed above is shown in figure 2.

6.2. Energy conservation

The two denominators appearing in the individual $m$-terms of equation (40) become zero when the Bohr conditions ($\omega = \omega_B$)

$$\hbar \omega_B = \varepsilon_m - \varepsilon_{m-1} = \frac{\hbar^2}{2m_0 r_0^2} (2m-1),$$

and

$$\hbar \omega_B = \varepsilon_{m+1} - \varepsilon_m = \frac{\hbar^2}{2m_0 r_0^2} (2m+1),$$

are satisfied, respectively. The (unphysical) divergence of $\sigma^\text{spin}_{\text{EL}}(\omega)$ at these frequencies relates to the fact that irreversible damping processes were neglected in the derivation of equation (4) [6]. If these inevitably present processes are included in the framework of a simple (frequency-independent) relaxation-time ($\tau$) approach one just needs to make the replacement $\omega \rightarrow \omega + i/\tau$ in the various denominators of equation (40). In the $\omega^{-1}$-factor appearing in front of the $m$-summation sign this replacement must not be made, due to the fact that this overall $\omega^{-1}$-factor originates in the relation $E_T = i\omega A_T$ between the divergence-free parts of the electric field ($E_T$) and vector potential ($A_T$). Despite of the presence of the $\omega^{-1}$-factor the spin current density is not singular in the limit $\omega \rightarrow 0$ because the transverse (divergence-free) part of the retarded incident field vanishes at least as fast as $\omega$ in the limit $\omega \rightarrow 0$; see [7]. Inclusion of damping mechanisms hence implies that the various frequency resonances in $\sigma^\text{spin}_{\text{EL}}(\omega)$ all have finite magnitudes.

6.3. Orbital angular momenta transfer. Polarization selection rules

For our circular quantum ring with complete confinement in the $z$ and $r$ coordinates the orbital angular momentum operator ($\hat{L}$) takes a particularly simple form, viz.,

$$\hat{L} = \hat{r} \times \hat{p} = r_0 \hat{r} \times \left( \frac{\hbar}{i} \frac{1}{r_0} \frac{\partial}{\partial \phi} \right) = \hat{z} \frac{\hbar}{i} \frac{\partial}{\partial \phi},$$

since $\hat{r} \times \hat{\phi} = \hat{z}$. The free-electron state $\Phi_n(\phi)$ (equation (19)) are the local unit vectors in the plane of the ring. The free-electron state $\Phi_n(\phi)$ (equation (19)) is an eigenstate of the angular momentum operator with
Table 1. 1: Electron transition. 2: Net transition probability. 3: Angular momentum transfer (to electron). 4: Electric field (E\textsubscript{0}) helicity unit vector. 5: Force (-eE\textsubscript{0}) helicity unit vector. 6: Normalized resonance condition (h\omega_{0}/\epsilon = R). 7: Induced current density helicity vectors.

|   |   |   |
|---|---|---|
| 1 | m - 1 → m | m + 1 → m |
| 2 | f_{m-1} - f_{m} | f_{m+1} - f_{m} |
| 3 | h\hat{z} | h\hat{z} |
| 4 | \hat{e}_{+} | \hat{e}_{-} |
| 5 | \hat{e}_{-} | \hat{e}_{+} |
| 6 | R = (2m - 1) | R = (2m + 1) |
| 7 | \hat{e}_{i} | \hat{e}_{i} |

Eigenvalue mh\hat{z}:

\[ \hat{z} \frac{h}{i} \frac{\partial}{\partial \phi} \phi_{m}(\phi) = mh\hat{z} \phi_{m}(\phi). \]  

(48)

In the transitions m - 1 → m and m + 1 → m, appearing in the expression for \( \sigma_{ED}^{spin}(\omega) \) (equation (40)), amounts of orbital angular momenta \( [mh - (m - 1)h] \hat{z} = h\hat{z} \) and \( [mh - (m + 1)h] \hat{z} = -h\hat{z} \) are transferred to the ring.

To understand how this relates the ED spin current density, \( J_{z}^{ED}(r = 0; \omega) \equiv J_{z}^{ED}(\omega) \), induced by the incoming electric field, viz.,

\[ J_{z}^{ED}(\omega) = \sigma_{ED}^{spin}(\omega) \cdot E_{0}(0; \omega) = \sigma_{ED}^{spin}(\omega) \cdot [\hat{e}_{+} \hat{e}_{+} + \hat{e}_{-} \hat{e}_{-}] \cdot E_{0}(0; \omega), \]  

(49)

let us temporarily neglect the z-component of the incident field. [For a plane wave propagating in a direction perpendicular to the plane of the ring \( \hat{z} \cdot E_{0} = 0 \), exactly]. The vectorial form of the spin current density contribution from the group of \( m - 1 \rightarrow m \) transitions then satisfies the proportionality \( \propto \)

\[ J_{z}^{ED}(\omega; m - 1 \rightarrow m) \propto (\hat{z} \hat{z} + \hat{e}_{+} \hat{e}_{-}) \cdot (\hat{e}_{+} \hat{e}_{+} + \hat{e}_{-} \hat{e}_{-}) \cdot E_{0}(0; \omega) = \hat{e}_{+} \cdot E_{0}(0; \omega). \]  

(50)

The orbital angular momentum \( \text{added} \) to the ring hence is subtracted from the positive helicity part of the prescribed incident field. The group of \( m + 1 \rightarrow m \) transitions gives

\[ J_{z}^{ED}(\omega; m + 1 \rightarrow m) \propto \hat{e}_{-} \cdot E_{0}(0; \omega). \]  

(51)

The orbital angular momentum \( \text{subtracted} \) from the ring therefore is given to the negative helicity part of the incoming field. An overview of the above processes appears in schematic form in table 1.

Because the spin current density operator (equation (3)) contains the electron momentum operator \( \hat{p} \) (via \( \hat{p} \times \hat{\sigma} \)) it is not surprising that orbital angular momentum transfer enters the spin conductivity dynamics. The form of the space current density operator [3, 6]

\[ j^{space} = \frac{q}{2m_{0}} [\hat{p} \delta (r - r') + \delta (r - r') \hat{p}] \]  

(52)

clearly implies that orbital angular momentum transfer processes also underly the expression for \( \sigma_{ED}^{space}(\omega) \) (equation (41)). Since the spin operator \( \hat{\sigma}, co ipsi, does not appear in spatial dynamics, and since the complete electron confinement does not allow electron oscillations perpendicular to the ring plane, it is obvious that only the dyadic forms \( \hat{e}_{+} \hat{e}_{-} \) and \( \hat{e}_{-} \hat{e}_{+} \), enter \( \sigma_{ED}^{space}(\omega) \) (equation (41)).

6.4. Spin current density perpendicular to the quantum wire plane

Let us now discuss the important zz-part of the ED spin conductivity tensor, namely,

\[ \sigma_{ED,zz}^{spin}(\omega) = i\frac{g}{\omega} \left( \frac{q\hbar}{2\sqrt{2} m_{0} \delta} \right)^{2} \sum_{m} \left[ \frac{f_{m-1} - f_{m}}{h\omega + \frac{k_{B}^{2}}{2m_{0}\delta}(1 - 2m)} + \frac{f_{m+1} - f_{m}}{h\omega + \frac{k_{B}^{2}}{2m_{0}\delta}(1 + 2m)} \right]. \]  

(53)

First of all, why is it possible to obtain a spin current density in the z-direction, taking into account the complete electron confinement in this direction? To give the answer, let us return to equation (4), and calculate \( \nabla \cdot \sigma_{z}^{spin}(r, r'; \omega) \). The calculation requires a determination of

\[ \nabla \cdot R_{z}^{spin}(r, r') = \nabla \cdot [ j_{z}^{spin}(r) \times U \times j_{z}^{spin}(r')]. \]  

(54)
Since only $j_{i-j}(r)$ depends on $r$, simple tensor calculations give
\[ \nabla \cdot R_{i-j}^{\text{spin}}(r, r') = (\nabla \times j_{i-j}^{\text{spin}}(r)) \cdot U \times j_{i-j}^{\text{spin}}(r'). \] (55)
Utilizing equation (6) one obtains
\[ \nabla \times j_{i-j}^{\text{spin}}(r) = \frac{q\hbar}{2m_0} \nabla \times \nabla [\psi_i(r) \psi_j(r)] = 0, \] (56)
and therefore
\[ \nabla \cdot R_{i-j}^{\text{spin}}(r, r') = 0, \quad \forall i, j. \] (57)
For the divergence of the spin current density, $j_{i-j}^{\text{spin}}(r; \omega)$ one then gets
\[ \nabla \cdot j_{i-j}^{\text{spin}}(r; \omega) = \int_{-\infty}^{\infty} [\nabla \cdot \sigma_{i-j}^{\text{spin}}(r, r'; \omega)] \cdot E(r'; \omega) d^3r' = 0. \] (58)
From the equation of continuity for the spin dynamics, i.e.,
\[ \nabla \cdot j_{i-j}^{\text{spin}}(r; \omega) = i\omega \rho_{i-j}^{\text{spin}}(r; \omega), \] (59)
it then follows that the spin charge density, $\rho_{i-j}^{\text{spin}}(r; \omega) = 0$. Hence, the excitation/deexcitation of the spin dynamics in the $z$-direction is not accompanied by a spin–charge current density perpendicular to the plane of the wire. This result is not so surprising since (i) the spin is an internal degree of freedom for the electron, and (ii) spin–orbit couplings are absent in the electron gas model used.

A significant simplification of the expression in equation (53) is possible because $m$ runs over all integers. In the last part of equation (53) [containing the factor $\int_{m+1}^{m} f_i \mathrm{d} m$] we then make the replacement $m \to m - 1$. After the replacement the two terms in the square bracket are added. Upon a number of algebraic manipulations one obtains
\[ \sigma_{i-j}^{\text{spin}}(\omega, z) = \frac{ig^2\hbar^4}{\omega 8m_0^3 r_0^4} \sum_{m} (2m - 1)(f_m - f_{m-1}) \frac{\epsilon_m^2}{(\epsilon_m^2)^2 - \left(\frac{\hbar^2}{2m_0^2 r_0^2}\right)^2}, \] (60)
The result of various numerical calculations of $\sigma_{i-j}^{\text{spin}}(\omega, z)$ is shown and discussed in section 8.

6.5 Qualitative comparison of spin and space ED-conductivity magnitudes
In most cases the dominating part of the ED current density, $j_{\text{ED}}(\omega) = j_{\text{ED}}^{\text{space}}(\omega) + j_{\text{ED}}^{\text{spin}}(\omega)$ comes from $j_{\text{ED}}^{\text{space}}(\omega)$ ($j_{\text{ED}}^{\text{spin}}(\omega)$ is totally neglected in many electrodynamics studies). A qualitative comparison of equations (40) and (41) shows possibilities for enhancing the relative importance of the spin dynamics.

The ratio $(R)$ between the magnitudes of the spin and space factors appearing in front of the summation signs in equations (40) and (41) are with $g = 2$ given by
\[ R = \frac{2m_0^2 r_0^2}{\hbar \omega}. \] (61)
The nominator in equation (61) equals the smallest energy difference between the electron states in the ring, viz., $\epsilon_{\pm 1} - \epsilon_0 = \epsilon_{\pm 1}$. As one might have anticipated interaction with photons of energy, $\hbar \omega$, well below $\epsilon_{\pm 1}$ will favour the spin part of the electron-field coupling.

Behind the summation sign in equation (41) appears for the space conductivity factors $1 \mp 2m$ not present for the spin conductivity (equation (40)). For increasing ring radius the spatial conductivity thus tends to dominate since large $m$-values contribute. At low temperatures this tendency is clear since states near the (one-dimensional) Fermi level gives the main contributions to both conductivity parts. By decreasing $r_0$: (i) the ratio SPIN/SPACE (equation (61)) increases and (ii) the effective maximum value of $|1 \mp 2m|$ becomes smaller. Both (i) and (ii) makes the spin conductivity more important.

7. Far-field scattering from the spins
It appears from equations (40) and (41) that an incident electric field polarized perpendicular to the plane of the quantum wire, i.e.
\[ E^0(0; \omega) = E^0(0; \omega) \hat{z}, \] (62)
with $E^0(\omega) \equiv E^0(0; \omega)$, only gives rise to an induced spin current density. The scattered electric ($E_{\text{scatt}}^{\text{spin}}(r; \omega)$) and magnetic ($B_{\text{scatt}}^{\text{spin}}(r; \omega)$) fields from the induced spin current density in turn are given by
\[
\mathbf{E}^{\text{scatter}}(\mathbf{r}; \omega) = i\mu_0 \omega \mathbf{G}(\mathbf{r}, 0; \omega) \cdot \hat{z} \sigma^{\text{spin}}_{ED,zz}(\omega) \mathbf{E}^0(\omega),
\]

and
\[
\mathbf{B}^{\text{scatter}}(\mathbf{r}; \omega) = \frac{i\mu_0 \omega}{c} \mathbf{G}^M(\mathbf{r}, 0; \omega) \cdot \hat{z} \sigma^{\text{spin}}_{ED,zz}(\omega) \mathbf{E}^0(\omega),
\]

where \(\sigma^{\text{spin}}_{ED,zz}(\omega) = \hat{z} \cdot \sigma^{\text{spin}}_{ED}(\omega) \cdot \hat{z}\). Explicit expressions for the electric and magnetic (superscript M) Green functions, \(\mathbf{G}\) and \(\mathbf{G}^M\), can be found, e.g., in [19].

In a scattering experiment usually one is interested mainly in electromagnetic field quantities in the far field (FF) of the scatterer. Except for scattering in the forward direction of the incoming beam (or wave packet), asymptotically there will be no interference between \(\mathbf{E}(\mathbf{r}; \omega)\) and \(\mathbf{E}^{\text{scatter}}(\mathbf{r}; \omega)\). In the far field the Green functions are given by the well known dyadic expressions
\[
\mathbf{G}_{FF}(\mathbf{r}; \omega) = \frac{e^{iq_0r}}{4\pi r}(\mathbf{U} - \hat{\mathbf{r}}),
\]

\[
\mathbf{G}^M_{FF}(\mathbf{r}; \omega) = \frac{e^{iq_0r}}{4\pi r} \mathbf{U} \times \hat{\mathbf{r}},
\]

where \(q_0 = \omega/c\) and \(\hat{\mathbf{r}} = \mathbf{r}/r\).

A schematic illustration of the scattering configuration of relevance for pure spin scattering is presented in figure 3.

To obtain the essentials of the far–field scattering from the spins it is sufficient to limit ourselves to monochromatic (z-polarized) incident fields for which
\[
\mathbf{E}^0(0, t) = \frac{1}{2} \{ \mathbf{E}^0 e^{-i\omega t} + (\mathbf{E}^0)^* e^{i\omega t} \} \hat{z},
\]

\(\mathbf{E}^0\) being a constant. The cycle-average \(\langle \cdots \rangle\) of the energy density flow, the Poynting vector \(\mathbf{S}(\mathbf{r}, t)\), given in general by
\[
\langle \mathbf{S}(\mathbf{r}, t) \rangle = c^2 \langle \mathbf{g}(\mathbf{r}, t) \rangle = \frac{1}{4\mu_0} \{ \mathbf{E}(\mathbf{r}; \omega) \times \mathbf{B}^*(\mathbf{r}; \omega) + \text{c.c.} \},
\]

where \(\mathbf{g}(\mathbf{r}, t)\) is the momentum density of the electromagnetic field, can be calculated in the far field by combining equations (63)–(68). Thus, one obtains for the scattered flow
\[
\langle \mathbf{S}^{\text{scatter}}(\mathbf{r}, t) \rangle_{FF} = \frac{2}{4\mu_0} \left\{ \frac{(\mu_0 \omega)^2}{c} \sigma^{\text{spin}}_{ED,zz}(\omega) \right\} \left\{ \frac{1}{4\pi r^2} \times \{ [\mathbf{U} - \hat{\mathbf{r}}] \cdot \hat{z} ] \times [ \mathbf{U} \times \hat{\mathbf{r}} \cdot \hat{z} ] \right\}.
\]

By expressing the unit tensor in the dyadic form given by the local unit vectors in polar coordinates, i.e.,
\[
\mathbf{U} = \hat{\mathbf{r}} + \hat{\theta} \hat{\phi} + \hat{\phi} \hat{\theta},
\]

one readily gets

\[\text{Figure 3. Electromagnetic spin scattering configuration in the case where the incoming electric field (E; red arrow) of a plane wave (wave-vector q; thick black arrow) is polarized perpendicular to the plane of the circular mesoscopic ring (in blue). } (\mathbf{S}^{\text{scatter}})_{FF} \text{ is the cycle-averaged far-field Poynting vector (red arrow).}\]
\begin{align}
[(\mathbf{U} - \hat{r} \mathbf{r}) \times \hat{z}] \times [\mathbf{U} \times \hat{r} \cdot \hat{z}] &= \mathbf{r} \sin^2 \theta,
\end{align}
where \( \theta \) is the polar angle. Finally,
\begin{align}
\langle S_{\text{corr}}(r, t) \rangle_{\text{FF}} &= \frac{\mu_0 (E^0)^2}{32 \pi^2 c} \left( \frac{\sin \theta}{r} \right)^2 \omega^2 |\sigma_{\text{ED,zz}}^{\text{pin}}(\omega)|^2 \mathbf{r}.
\end{align}
As always in the far field, the energy flow is in the radial direction, and varies as \( r^{-2} \) (necessary for energy conservation in the vacuum field). The flow is independent of the azimuth angle (\( \phi \)), as required by symmetry: \( E^0 \) points in the direction perpendicular to the plane of the ring, which physical properties are invariant against arbitrary rotations around \( \hat{z} \).

8. Numerical results

8.1. Spin conductivity spectra

In what follows we shall assume that the number (N) of jellium electrons is even. We shall realize that the spin conductivity (\( \sigma_{\text{ED,zz}}^{\text{pin}}(\omega) \)) spectrum at low temperatures has a single resonance when \( N/2 \) is even \([N/2 = 2M + 1; M = 0, 1, 2, \ldots]\), and two resonances for even \( N/2 [N/2 = 2M; M = 0, 1, 2, \ldots] \). Since the wire radius (\( r_0 \)) relates to \( N \) via
\begin{align}
\eta_0 &= \frac{Na}{2\pi^2},
\end{align}
where \( a \) and \( \nu \) are the interatomic distance and the number of free electrons per atom, respectively, a nano-sized ring has \( N \gg 1 \). For an Al wire with \( a \approx 3.0 \times 10^{-10} \text{m} \) and \( \nu = 3 \), one has \( r_0 \approx 1.6 \times 10^{-8} \text{m} \) for \( N = 10^3 \). An electronic relaxation time \( \tau = 5.0 \times 10^{-11} \text{s} \) is used. In all subsequent spectra the Al data are used, and only numerical spin spectra for \( N/2 = 500 (M = 250) \) and \( N/2 = 501 (M = 250) \) are shown in the figures.

In the low-temperature limit (\( T \to 0 \)) all states up to (and including) \( m_{\text{max}}(>0) \) are occupied (with one or two electrons). The relation between \( N \) and \( m_{\text{max}} \) is
\begin{align}
\frac{N}{2} &= 2m_{\text{max}} + \Delta,
\end{align}
where
\begin{align}
\Delta &= \begin{cases} 
0, & \text{for } N/2 \text{ even}, \\
1, & \text{for } N/2 \text{ odd}.
\end{cases}
\end{align}
For \( \Delta = 0 \), the state at \( m_{\text{max}} \) holds one electron (of spin up or down); [the state \(-m_{\text{max}} \) also holds just one electron]. For \( \Delta = 1 \), the state \(-m_{\text{max}} \) holds two electrons of opposite spin [two electrons present also in the state \( m_{\text{max}} \)]. As illustrated in figure 4, only a single transition, \( m_{\text{max}} \to m_{\text{max}} + 1 \), contributes to the spectra for \( \Delta = 1 \), whereas two transitions, \( m_{\text{max}} - 1 \to m_{\text{max}} \) and \( m_{\text{max}} \to m_{\text{max}} + 1 \), contribute for \( \Delta = 0 \).

The \( \sigma_{\text{ED,zz}}^{\text{pin}}(\omega) \) spectra associated to \( \Delta = 0,1 \) are shown in figures 5 and 6 for \( N = 1000, 1002 \) for three different temperatures (10 K, 300 K, 500 K). In these figures the real and imaginary parts of the spin conductivity are plotted as a function \( \hbar \omega/\varepsilon_1 \) [the monochromatic photon energy normalized to the lowest (non-zero)
electron eigenenergy]. The main (strongest) resonances are located at $+m_{\text{max}}$ ($\Delta = 1$) and $-m_{\text{max}}$ ($\Delta = 0$) according to equations (45) and (46). Since $m_{\text{max}} = 250$ for $N = 1000, 1002$, the resonances are at $\hbar \omega / \varepsilon_1 = 501$ ($\Delta = 1$) and $499,501$ ($\Delta = 0$); $\hbar \omega / \varepsilon_1 = 500$ corresponds $\hbar \omega = 75.3$ meV.

As the temperature is increased additional weaker resonances appear in the spin conductivity spectra. The strength of the various resonances are determined by the factor $-m_{\text{ff}}$. The resonances are present because the factor $-f_{m-1}^{\text{fm}}$ in equation (60) is associated to a net transition rate; see equation (43). All resonances are located at uneven integer values for $\omega / \varepsilon_1$.

8.2. Poynting vector spectra

It appears from equation (71) that the far-field Poynting vector related to the spin conductivity tensor $\sigma_{\text{ED},zz}^{\text{spin}}(\omega) \hat{z} \hat{z}$ has simple azimuth angle an radial dependencies viz., $\sin^2 \theta$ and $r^{-2}$. Therefore it is sufficient to present numerical results for the frequency dependence of the scalar quantity

$$F(\omega) = |\omega \sigma_{\text{ED},zz}^{\text{spin}}(\omega)|^2.$$  

For $N = 1000, 1002$ we have plotted $F(\hbar \omega / \varepsilon_1)$ for three temperatures ($T = 10$ K, 300 K, 500 K) in figures 7 and 8. The physical interpretation of the Poynting vector spectra follows directly from the discussion given for $\sigma_{\text{zz}}^{\text{spin}}(\omega)$ in section 8.1.
9. Outlook

Although the integrated spin conductivity tensor

\[
\sigma_{\text{spin}} = \int_{-\infty}^{\infty} \sigma_{\text{spin}}(r', r; \omega) \, d^3r' \, d^3r
\]

relates the induced spin current density to the incoming electric field in a spatially local manner, it must not be forgotten that the underlying spin dynamics, given by \( \sigma_{\text{spin}}(r', r; \omega) \), is nonlocal. This follows from the fact that the Pauli interaction \( -\mathbf{\sigma} \cdot \mathbf{B} = (i\omega)^{-1} \mathbf{\sigma} \cdot \nabla \times \mathbf{E} \) is nonlocal; it relates in a nonlocal fashion to the electric field in the neighbourhood of the point where the current density is considered. If we allow the ring cross section to be finite (having at least two quantum-well levels) one will realize that the ring spin dynamics attains magnetic dipole plus electric quadrupole properties.

The remarks above makes it interesting to study the dynamic spin contribution in microscopic small-hole diffraction, where it has been shown that the radiation from the effective aperture (hole) besides an ED term contains magnetic dipole (MD) and electric quadrupole (EQ) contributions [23]. Whether the spin dynamics contributes to the MD and EQ scattering in a significant manner is an open question. The selvedge spin response may perhaps be modelled as a quantum ring (with finite cross section). Furthermore, the Young interference pattern arising from two circular spin selvedges may show resemblance to the diffraction from two magnetic scatterer. To a certain extent the spin scattering from the rings can be compared to magnetic scattering from two ‘flat’ atoms oscillating harmonically. The Young diffraction pattern from two identical two-level atoms has been

Figure 6. Spin conductivity spectra as in figure 5, just with \( N = 1002 \). Since \( N/2 \) is odd, a single-peak structure with sidebands appears.
analyzed in [24] [see also [25]]. If the rings are excited by a single (or few) photon field an interesting interplay between quantum electrodynamics (QED) and particle spin dynamics may result.

In a recent theoretical study of the electrodynamics of a Möbius quantum wire it was indicated that electron spin dynamics perhaps might be of significant importance in some cases [26].

It will be interesting also to study the case where the ring is superconducting. In the jellium-like Al case the electron conductivity tensor can be calculated on the basis of the simplest BCS-pairing model. Below the superconducting transition temperature $\sigma_{\text{space}}$ composed of para(par)- and dia(dia)magnetic parts. $\sigma_{\text{space}} = \sigma_{\text{par}} + \sigma_{\text{dia}}$ (equation (1)), will be replaced by a new $\sigma_{\text{dia}}$ described in the Cooper pairing approximation. The spin conductivity tensor now must be calculated from the pairing states. The calculation for Al in the superconducting state is beyond the scope of the present work.

In this work the study of the field-induced spin electrodynamics of a mesoscopic jellium ring has been limited to the linear regime. However, as in other branches of electrodynamics it certainly would be of interest to extend the investigations to the nonlinear regime. In lowest order of nonlinearities two important phenomena appear for a monochromatic incident field: (i) Second-harmonic generation, and (ii) angular-momentum electromagnetic (photon) drag. For neither of these phenomena the role of the spin electrodynamics has been investigated. Guided by the circumstance that the overall ratio of the spin/space conductivities is known to increase up to two orders of magnitude in quantum-well systems, it seems of central importance to seek to calculate the nonlinear spin dynamics in mesoscopic rings. In the nonlinear regime also a dynamic Zeeman effect may appear. To estimate the magnitude of this requires a calculation beyond the scope of this work, even if spin dynamics is neglected.

**Figure 7.** Scaled Poynting vector spectra $F \equiv 10^4 F(\omega)$: equation (75)) for three different temperatures as a function of the normalized photon energy $(\hbar \omega/\varepsilon_1)$ for $N = 1000$. Structure: Double-peaked with sidebands. SI-units are used for $F$. 
Appendix: Space-spin crosscoupling in linear electrodynamics

A.1. Density matrix approach

It is well-known that the current density, \( \mathbf{J}(\mathbf{r}, t) \), induced in the electron system by an incident electromagnetic field can be studied (calculated) in an elegant manner using a density matrix operator approach [see e.g. \([19]\), and references therein]. Thus,

\[
\mathbf{J}(\mathbf{r}, t) = \text{Tr}\{\mathbf{\hat{J}}\},
\]

where \( \text{Tr}\{\cdots\} \) means the trace of the operator inside \{\cdots\}, and \( \mathbf{\hat{J}} \) is the electron current density operator. The density matrix operator, \( \hat{\rho} \), satisfies the Liouville equation

\[
\frac{i\hbar}{\partial t} \hat{\rho} = [\hat{H}, \hat{\rho}],
\]

where \( \hat{H} \) is the total Hamilton operator of the coupled particle-field system, and \([\cdots, \cdots]\) stands for commutator.

After having been transferred to the space-frequency domain, the Liouville equation is adequately solved by expanding the density matrix operator in a power series of the interaction Hamiltonian. Within the framework of linear response theory, one obtains [6]

\[
\mathbf{J}(\mathbf{r}, t) = \mathbf{J}^{\text{pace}}(\mathbf{r}; \omega) + \mathbf{J}^{\text{pin}}(\mathbf{r}; \omega) + \mathbf{J}^{\text{pace-pin}}(\mathbf{r}; \omega).
\]

Each of the three parts in equation (79) relates the relevant induced microscopic current density in a spatially nonlocal fashion to the prevailing electric field, and the connections are the linear conductivity tensors \( \sigma^{\text{pace}}(\mathbf{r}, \mathbf{r}'; \omega) \), \( \sigma^{\text{pin}}(\mathbf{r}, \mathbf{r}'; \omega) \) and \( \sigma^{\text{pace-pin}}(\mathbf{r}, \mathbf{r}'; \omega) \). The explicit expressions obtained for \( \sigma^{\text{pace}}(\mathbf{r}, \mathbf{r}'; \omega) \) and \( \sigma^{\text{pace}}(\mathbf{r}, \mathbf{r}'; \omega) \) are those given in equations (4) and (7), [calculational details are given in [6]].

\[\text{Figure 8. Normalized Poynting vector spectra as in figure 7, just with } N = 1002. \text{ Structure: Single-peaked with sidebands.}\]
A.2. Cross-coupling in the absence of spin–orbit coupling

In the present work the electrodynamics in the mesoscopic ring has been studied under the assumption that the spin–orbit coupling is negligible. In this case it turns out that the cross-coupling term, $J_{\text{space–spin}}(r; \omega)$, vanishes, as we prove below.

The cross-coupled current density is calculated from

$$J_{\text{space–spin}}(r; \omega) = Tr \{ \hat{\rho}_1 \hat{J}_F^\sigma \} + Tr \{ \hat{\rho}_F^\sigma \hat{J}_1 \},$$  \hspace{1cm} (80)

where $\hat{\rho}_1$ and $\hat{\rho}_F^\sigma$ are the space ($\hat{\rho}_1$) and spin ($\hat{\rho}_F^\sigma$; upper index $\sigma$) parts of the density matrix. These are linearly related to the spatial ($\hat{H}_1$) and spin ($\hat{H}_F^\sigma$) parts of the interaction Hamiltonian. The spin part $\hat{H}_F^\sigma$ (eq $\hat{H}^\text{spin}$) is the one given in equation (2). The free (F) part of the spin current density operator, $\hat{J}_F^\sigma$, is given in equation (3).

Let us now consider the first part of equation (80), viz.,

$$Tr \{ \hat{\rho}_1 \hat{J}_F^\sigma \} = \sum_{i,j} \langle i| \hat{\rho}_1 |j \rangle \langle j| \hat{J}_F^\sigma |i \rangle.$$  \hspace{1cm} (81)

To demonstrate that this part is zero, we use a complete set of single-particle states

$$|i\rangle = |i, s\rangle = |i\rangle \otimes |s\rangle,$$  \hspace{1cm} (82)

where $\psi_i(r) = \langle r|\rangle$ and $\psi_j(r) = \langle r|j \rangle$ are the electron wave functions (in direct space) of states $i$ and $j$ [see sections 2 and 3]. The eigenket in equation (82) is the tensor product ($\otimes$) of the kets belonging to the wave function ($\langle i\rangle$) and spin-state ($|s\rangle$) spaces. Utilizing the relation [6]

$$\langle i| \hat{\rho}_1 |j \rangle = \frac{f_j - f_i}{\hbar \omega + \varepsilon_j - \varepsilon_i} \langle i| \hat{H}_1 |j \rangle,$$  \hspace{1cm} (83)

one obtains

$$Tr \{ \hat{\rho}_1 \hat{J}_F^\sigma \} = \sum_{i,j,i',j'} \frac{f_j - f_i}{\hbar \omega + \varepsilon_j - \varepsilon_i} \times \langle i, s'| \hat{H}_1 |j, s \rangle \langle j, s| \hat{J}_F^\sigma |i, s' \rangle.$$  \hspace{1cm} (84)

To show that $Tr \{ \hat{\rho}_1 \hat{J}_F^\sigma \} = 0$ it is sufficient to calculate the spin summation in equation (84). Since $\hat{H}_1$ is spin-independent $\langle \psi'| \hat{H}_1 |\psi \rangle \propto \delta_{i',i}$, and since $\hat{J}_F^\sigma$ is linear in $\sigma$, the type of matrix element entering the spin summation is $\langle \psi|\sigma|\psi' \rangle$. The spin summation thus gives a typical factor

$$S = \sum_{s,s'} \delta_{s,s'} \langle \psi|\sigma|\psi' \rangle = \sum_{s=-1}^{+1} \langle \psi|\sigma|\psi' \rangle = 0,$$  \hspace{1cm} (85)

and hence it has been proven that $Tr \{ \hat{\rho}_1 \hat{J}_F^\sigma \} = 0$. The second part of equation (80) also vanishes since

$$\langle \psi'| \hat{\rho}_F^\sigma |\psi \rangle \sim \langle \psi'| \hat{H}_F^\sigma |\psi \rangle$$

with replacements $\hat{\rho}_1 \rightarrow \hat{\rho}_F^\sigma$, $\hat{H}_1 \rightarrow \hat{H}_F^\sigma$, and $\langle \psi| \hat{J}_F^\sigma |\psi' \rangle \propto \delta_{i',i}$. The same $S$-factor as in equation (85) hence appears in the calculation of $Tr \{ \hat{\rho}_F^\sigma \hat{J}_1 \}$.

In the absence of spin–orbit coupling, we have come to the conclusion that

$$J_{\text{space–spin}}(r; \omega) = 0,$$  \hspace{1cm} (86)

and therefore the linear conductivity tensor has no cross-coupling part.

A.3. Spin-orbit cross-coupling in conductivity tensor

The expression given for the spin-orbit (s-o) correction to the Schrödinger Hamiltonian in the INTRODUCTION shows that the density matrix operator in the presence of spin–orbit coupling has five terms, i.e.,

$$\hat{\rho} = \hat{\rho}_F + \hat{\rho}_F^{s-o} + \hat{\rho}_1 + \hat{\rho}_1^{s-o} + \hat{\rho}_F^{s-o}.$$  \hspace{1cm} (87)

The addition $\hat{\rho}_F^{s-o}$ to the free density matrix operator changes the field-unperturbed eigenstates, and the Fermi factors. The first-order term $\hat{\rho}_1^{s-o}$ leads to a space-spin current density contribution of the form

$$J_{\text{space–spin}}(r; \omega) = Tr \{ \hat{\rho}_1^{s-o} \hat{J}_F \}$$  \hspace{1cm} (88)

in linear response theory. Although the spin-orbit contribution to the Hamiltonian is proportional to $\sigma$, one cannot conclude that the cross-coupling part in equation (88) is zero. This follows from the fact that field unperturbed eigenstates depend on the electron spin [via the s-o Hamiltonian part $(\hbar/4m_e c^2) \nabla V(r) \times \vec{p} \cdot \vec{\sigma}$].

ORCID iDs

Dann S Olesen  @ https://orcid.org/0000-0001-8762-6038
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