Zeno paradox in classical stochastic processes with memory

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We investigate Zeno paradox in classical stochastic processes and attribute the observed time dependence of survival probability to the noise with memory.

Introduction. In many stochastic processes the survival probability $S(t)$ attracts considerable attention. The standard exponential decay law $S(t) = e^{-t/\tau}$ is never exact. The decay law is typically quadratic for short times, $S(t) \simeq 1 - t^2/\tau^2$. As a consequence, for frequently repeated measurements at very small time intervals, the Zeno effect, that is the freezing of state in its unstable initial configuration, takes place. For measurements repeated for slightly larger time intervals, also the inverse Zeno effect, that is an increased decay rate by measurements, is possible. Finally, at large times a power law sets in, $S(t) \simeq t^{-\alpha}$, where $\alpha > 0$ depends on the details of the interaction that governs that particular decay. In this Letter we discuss classical Zeno paradox (formulated in terms of probability) for processes influenced by noise with memory.

Quantum Zeno paradox. Simple derivation of the quantum Zeno effect is possible by considering time behaviour of the state vector. Let the state vector at time $t$ be $e^{-iHt}|\Psi\rangle$, where $|\Psi\rangle$ is the state vector at time $t = 0$ and $H$ is the Hamiltonian in units where the reduced Planck’s constant is set to unity, $\hbar = 1$. For $t$ small enough, it is possible to make a power series expansion: $e^{-iHt} \simeq 1 - iHt - H^2t^2/2 + \ldots$. The survival probability is

$$S(t) = \langle \Psi | e^{-iHt} | \Psi \rangle^2 \simeq 1 - (\Delta H)^2 t^2$$

where

$$(\Delta H)^2 = \langle \Psi | H^2 | \Psi \rangle - \langle \Psi | H | \Psi \rangle^2$$

is the variance of Hamiltonian.

Interrupted the interval $[0, t]$ by $n$ measurements at times $t/n$, the survival probability is

$$S(t) \simeq \left[ 1 - (\Delta H)^2 \left( \frac{t}{n} \right)^2 \right]^n$$

and approaches 1 for $n \to \infty$. It illustrate the situation that an unstable particle, if observed continuously, will never decay.

Non-local master equation with memory. In classical case we consider the rate of change of the probability $P(t)$ at time $t$ and dependence on its history (starting at $t = 0$). The non-local master equation is

$$\frac{dP(t)}{dt} = - \int_0^t K(t-s)P(s) \, ds + \eta(t)$$

where $\eta$ is the stochastic noise. The memory effects are taken into account through the introduction of the memory kernel

$$K(t, s) = \langle \eta(t) \eta(s) \rangle = \gamma f(t - s)$$

with $\gamma = (\Delta \eta)^2 = \langle \eta^2 \rangle - \langle \eta \rangle^2$ being variance of $\eta$ (notice that $\langle \eta \rangle = 0$). We assume that $\eta$ is a Lorentzian noise with spectrum

$$G(\omega) = \langle \eta^2 \rangle \frac{\tau}{1 + \omega^2 \tau^2}$$

with $\tau$ being the memory time since it sets the time scale over which time correlations of noise decay.

Averaging probability over ensemble, for survival probability $S(t) = \langle P(t) \rangle$ we have

$$\frac{dS(t)}{dt} = - \int_0^t \gamma f(t-s) S(s) \, ds$$

Differentiating both side of Eq. (8) we have

$$\frac{d^2S(t)}{dt^2} + \frac{1}{\tau} \frac{dS(t)}{dt} + \gamma S(t) = 0$$

which allow simply to solve the problem analytically. Survival probability is given by

$$S(t) = \frac{e^{-t(1+A)/(2\tau)}(1 + A)}{2A} + \frac{e^{-t(1-A)/(2\tau)}(1 - A)}{2A}$$

2 The Langevin equation in approach given by Eq. (4) was considering in description of particles motion

3 $\langle \eta(0) \eta(t) \rangle = \int_0^{\infty} G(\omega)e^{i\omega t} d\omega = (\eta^2)e^{i|t|/\tau}$

4 For $f(t-s) = \delta(t-s)$ we have standard exponential decay law, $S(t) = e^{-t/\tau}$. 

$\quad$
with $A = \sqrt{1 - 4\gamma^2}$. Time dependence of the survival probability $S(t)$ for different parameters $A$ is shown in Fig. 1. Only for large $t$ we observe exponential behaviour, $S(t) \sim \exp[-t(1-A)/(2\tau)]$. For $t$ small enough we have

$$S(t) \simeq 1 - \frac{1}{2} \gamma t^2 = 1 - \frac{1}{2} (\Delta \eta)^2 t^2$$

with the same behaviour as $S(t)$ given by Eq. (1). The variance of noise, characterizing by $(\Delta \eta)^2/2$ term, correspond with $(\Delta H)^2$, and survival probability $S(t)$ the same exhibit Zeno effect [14].

Some remarks. Usually, the memory kernel $K(t-s)$ can lead to non-Markovian dynamics depending on the structure and time scale of the processes. Also a Markovian stochastic processes (where the time correlation $\langle \eta(t)\eta(s) \rangle \sim e^{-|t-s|/\tau}$ as is given by Eq. (3)) can lead to non-Markovian dynamics of the system is coupled to. In general, Markovianity or non-Markovianity are not a feature of just the noise but of the dynamics of the system coupled to the noise [14].

Survival probability given by Eq. (10) has the formal form

$$S(t) = \omega_a \exp(-t/\tau_a) - \omega_b \exp(-t/\tau_b)$$

(12)

For $t$ small enough we have

$$S(t) \simeq (\omega_a - \omega_b) + \frac{\omega_b - \omega_a}{\tau_b - \tau_a} t + \frac{\omega_a}{2\tau_a^2} - \frac{\omega_b}{2\tau_b^2} t^2$$

From normalization, $S(t=0) = 1$, we have $\omega_a - \omega_b = 1$. For $\omega_a/\tau_a = \omega_b/\tau_b$ and $\tau_a > \tau_b$ we observe Zeno paradox (as given by Eq. (11)). However, for $\omega_a/\tau_a = \omega_b/\tau_b$ and $\tau_a < \tau_b$ we have "normal" behaviour with $S(t) \simeq 1 - (\omega_b/\tau_b - \omega_a/\tau_a)t$.

Memory destroy independence. Exponential decay law, being memory-less, obeys the relation $S(t = t_1 + t_2 | t_1) = S(t_2)$. Due to independence we have $S(t_1 + t_2) = S(t_1)S(t_2)$. This is not valid for dependent variables, and (as in discussed above survival probabilities and its power law expansions given by Eq. (11) and Eq. (11))

$$S(t = t_1 + t_2) \neq S(t_1)S(t_2)$$

(14)

For dependent variables (due to noise correlation) we have rather use the joint survival probability, in the form (in analogy to the power expansion of multivariate normal distribution)

$$S(t_1, t_2, ..., t_n) = 1 - \sum_{i=1}^{n} \sum_{j=1}^{n} t_it_j C_{i,j}$$

(15)

where $C_{i,j} = \langle \eta(t_i)\eta(t_j) \rangle - \langle \eta(t_i) \rangle \langle \eta(t_j) \rangle$ and for noise $\eta(t)$ we can write $C_{i,j} = C$. Even for non-correlated noise, $C_{i,j} = 0$ for $i \neq j$, we have $S(t_1, t_2, ..., t_n) = 1 - \sum_{i=1}^{n} t_i^2 C_{i,i} \neq S(t_1^2)$.

Interrupted the interval $[0, t]$ by $n$ measurements at times $\delta = t/n$, the survival probability is

$$S(\delta_1, \delta_2, ..., \delta_n) = 1 - C \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{t}{n} \right)^2 = 1 - Ct^2$$

(16)

and the Zeno paradox not exist.

Summary. The dynamical evolution of statistical system is always influenced by its environment, which exhibiting time-correlated random fluctuations can lead to non-Markovian dynamics [15, 16]. By studying time-correlated noise we have shown how survival probability depends on the time scale of the noise correlations. The time correlations in the noise field determine how fast the survival probability converges to its exponential law behaviour. Approximations, considering only one variable survival probability $S(t)$ do not take into account all memory effects. The assumption that $S(\sum_{i=1}^{n} t_i) = \prod_{i=1}^{n} S(t_i)$, leading to the conclusion expressed by Eq. (3), seems very doubtful (for discussion of the conditional survival probability $S(t_i | t_{i-1})$ see Appendix A). Multivariate joint probability $S(t_1, t_2, ..., t_n)$ is needed for proper description of frequently repeated measurements at very small time intervals.

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Appendix A: Conditional $S(t + t' | t)$. Consider the survival probability $S(t) \simeq 1 - at^2$, as given by Eqs. (11) and (11), where the time interval $[0, t]$ is interrupted by $n$
measurements at times $t/n, 2t/n, \ldots, t$. The width of bins is $\delta = t/n$. Conditional probability for $i-th$ bin is

$$S_i = S(t_i = i\delta|t_{i-1} = (i-1)\delta) = \frac{S(i\delta \cap (i-1)\delta)}{S((i-1)\delta)} = \frac{1 - a\delta^2i^2}{1 - a\delta^2(i-1)^2}$$

(A-1)

and is not the same as $S(\delta) = 1 - a\delta^2$, contrary to the memory-less exponential $S(t)$ for which $S_i = S(\delta) = e^{-\delta}$.

For $\delta \rightarrow 0$, the survival probability in each one of bins $S_i \rightarrow 1$. Nevertheless, the survival probability for $n$ measurements

$$S(t = n\delta) = \prod_{i=1}^{n} \frac{1 - a\delta^2i^2}{1 - a\delta^2(i-1)^2} = 1 - at^2$$

(A-2)

and is not dependent on the number of measurements (in particular, $S(n\delta)$ is the same when $n \rightarrow \infty$). Contrary to arguments justifying the Zeno effect, observed continuously an unstable particle decay "normally".

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