Consistent Evolution with Different Time-Slicings in Quantum Gravity

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Abstract

Rovelli’s “quantum mechanics without time” motivates an intrinsically time-slicing independent picture of reduced phase space quantum gravity, which may be described as “quantization after evolution”. Sufficient criteria for carrying out quantization after evolution are developed in terms of a general concept of the classical limit of quantum mechanics. If these criteria are satisfied then it is possible to have consistent unitary evolution of operators, with respect to an infinite parameter family of time-slicings (and probably all time-slicings), with the correct classical limit. The criteria are particularly amenable to study in (2+1)-dimensional gravity, where the reduced phase space is finite dimensional.
Part I

A Time-Slicing Independent Picture of Reduced Phase Space Quantum Gravity

1 Introduction

In reduced phase space quantum gravity a time-slicing is fixed, and evolution considered with respect to it. If a different time-slicing connecting the same initial and final slices is used instead, it is natural to ask whether the two evolutions will agree. While in classical geometrodynamics the answer is yes—after all, the evolution is induced by a single spacetime metric—in quantum geometrodynamics the answer is in general no. Because of operator ordering ambiguities, an operator valued spacetime metric does not induce unique operator valued metrics on the slices of a time-slicing. A better question is whether evolution along different time-slicings can be made to agree by judicious choice of the operator orderings of the Hamiltonians.

Actually, this general problem involves three distinct problems. In the terminology of Kuchař [1] and Isham [2], these are the global time problem, the multiple choice problem, and the functional evolution problem.

The global time problem concerns whether there exists a canonical transformation

\[(g_{ab}, K^{ab}) \rightarrow (X^A, P_A, \phi^r, p_r)\]

from the intrinsic geometry \(g_{ab}\) and extrinsic curvature \(K^{ab}\) of a spacelike hypersurface \(\Sigma \hookrightarrow M\), to “internal spacetime coordinates” \(X^A\) which determine the embedding \(\Sigma \hookrightarrow M\), their canonical conjugates \(P_A\), and the coordinates on the reduced phase space \((\phi^r, p_r)\). We assume that the canonical transformation \((1.1)\) exists. Here the term “reduced phase space” means the space obtained by solving the constraints on the hypersurface \(\Sigma \hookrightarrow M\) and modding out by the transformations generated by the momentum constraints. Fixing the embedding \(\Sigma \hookrightarrow M\) may be interpreted as gauge fixing the Hamiltonian constraint.

The multiple choice problem concerns the fact that the canonical transformation \((1.1)\) may not be unique. Fixing the internal coordinates \(X^A\) determines an embedding \(\Sigma \hookrightarrow M\) only after the data \((\phi^r, p_r)\) on \(\Sigma\) have been specified and the equations of geometrodynamics solved. That is, fixing the internal coordinate functions \(X^A\) determines a spacetime metric dependent embedding \(\Sigma \hookrightarrow M\), which we call a slice. If \((1.1)\) is not unique, then given a spacetime metric we can use internal coordinates \(X'^A\) obtained from a different canonical transformation \((1.1)\) to fix the same embedding. However, the metric dependence of the embedding may be different. That is the slice may be different. It seems possible that using two different canonical transformations \((1.1)\) to fix time-slicings, that is spacetime metric dependent foliations with spacelike leaves, may result in two different time-slicings which share the same end slices \([1]\). The question of whether or not evolution of the wave function
will agree for the different time-slicings can be raised here. This is not the problem we are interested in. We assume that a particular choice of canonical transformation (1.1) has been made.

The problem of functional evolution remains even after the canonical transformation (1.1) has been fixed. The embedding variables $X^A$ can be used to specify various time-slicings. For each time-slicing gravity can, in theory, be reduced to a Hamiltonian system by solving the constraints on each slice and modding out by the transformations generated by the momentum constraints. The reduced phase space $(\phi^r, p_r)$ plays the role of an ordinary phase space of an unconstrained Hamiltonian system, with time a parameter labeling slices. An example of this reduction has been carried out by Moncrief [3], and by Hosoya and Nakao [4], for (2+1)-dimensional gravity. Hence, we can again arrive at the situation of having the same end slices connected by many different time-slicings. We come now back to our question, can evolution with respect to all the different time-slicings be made to agree by judicious choice of the operator orderings of the Hamiltonians?

Rovelli’s quantum mechanics without time [5] suggests a way to think about this problem. In time reparametrization-invariant systems time evolution takes the form of a gauge transformation. In this context, Rovelli advocates regarding observables not as objects which take different values at different times (gauges), but rather as objects whose values are independent of time (gauge). The traditional observables are broken up into a sequence of observables, one for each time. The position of a particle at 1:00 PM is independent of time. Rovelli proposes regarding such objects as observables.

In this work Rovelli’s observables, sometimes called evolving constants of the motion, are interpreted in terms of covariant canonical quantization [1, 4]. A traditional observable is a function on phase space—the same function for all times. The “value” of a traditional observable changes with time because the point in phase space changes. Rovelli’s observables may be interpreted as functions on the space of solutions of the classical equations of motion. Of course a solution does not change with time; it is the whole path in phase space. To get evolution the function on solution space must change with time. For each traditional observable, that is each function on phase space, there correspond many functions on solution space, one for each time. For example, if the traditional observable is the position of a particle, one of the functions on solution space, when evaluated at a particular solution, yields the position at 1:00 PM, another yields the position at 2:00 PM etc.

For a particular traditional observable in gravity, that is a particular function on the reduced phase space, we obtain a family of functions on solution space—one for each slice. Together, the functions on solution space constitute a consistent evolution system. Traditionally in quantization, the function on phase space is raised to operator status. Here we propose to lift the whole family of functions on solution space, slice by slice, to operators, thus obtaining a consistent operator evolution system. We call this procedure quantization after evolution because the family of functions on solution space represent the complete already evolved classical system. The procedure is motivated by Carlip’s [8] (see also [9]) comparison of the Witten [10], and Moncrief-Hosoya-Nakao [3, 4] quantizations of (2+1)-gravity. Kuchař [11] also mentions this procedure in his discussion of Rovelli’s work.

Quantization after evolution will be defined herein in terms of a general concept of the
classical limit of quantum mechanics \cite{11,12}. It gives a Heisenberg picture of quantum mechanics, in the sense that operators evolve while the state remains fixed. However, it is not completely clear that it is equivalent to evolution by the Heisenberg equations of motion. The main issue concerns the classical limit of Hamiltonians, which are not properly defined on individual slices. To find the Hamiltonian on a slice, the surrounding time-slicing must be known. Hence, although quantization after evolution yields completely consistent evolution of operators defined on slices with the right classical limit, we are unable to prove that the quantum Hamiltonian has the right classical limit. One could claim that this is not enough to show that the functional evolution problem may be solved, because it does not actually show that the Hamiltonians may be “operator ordered” (implying that they have the right classical limit) to give consistent evolution. The issue of whether or not Hamiltonians should be considered as observables in geometrodynamics arises.

After defining quantization after evolution, the problem of constructing a quantization after evolution will be formulated as the problem of finding a lifting of canonical transformations to unitary transformations. This formulation will be used to show that if certain criteria are satisfied, then for manifolds of the form $M = \Sigma \times \mathbb{R}^+$ quantization after evolution may be implemented on an infinite parameter family of time-slicings, and probably on all time-slicings. The main criterion which must be satisfied is that it must be locally possible to continuously lift real valued functions on phase space to hermitian operators with the right classical limit. The results of this paper mean that if the criteria are satisfied, then consistent evolution of operators defined on slices with the right classical limit may be achieved for at least an infinite parameter family of time-slicings. In addition it is shown that any particular time-slicing, with evolution defined by any operator ordering of the associated Hamiltonian, may be included in such a consistent infinite parameter family.

The article is organized into three broad parts. Part I gives a conceptual introduction to the intrinsically time-slicing independent picture of reduced phase space quantum gravity mentioned above. Part II makes the ideas of part I precise by defining quantization after evolution and formulating the question of existence. Part III discusses the question of existence, deriving the result on consistent evolution.

2 Motivation

Consider the solution of a classical evolution problem on a two dimensional phase space by means of Hamilton-Jacobi theory. Find a (time-dependent) canonical transformation from the variables $(q, p)$ to variables $(\alpha, \beta)$, for which the Hamiltonian is zero. Starting with initial data $(q(t_0), p(t_0))$, implement the canonical transformation at time $t_0$ to find

\[
(\alpha(t_0) = \alpha_{t_0}(q(t_0), p(t_0)), \quad \beta(t_0) = \beta_{t_0}(q(t_0), p(t_0))). \tag{2.1}
\]

Evolve $(\alpha, \beta)$ (they don’t change since $H=0$). Then implement the inverse canonical transformation at time $t$ to find

\[
(q(t) = q_{t}(\alpha(t), \beta(t)), \quad p(t) = p_{t}(\alpha(t), \beta(t))). \tag{2.2}
\]
A natural question seems to be: Can we use the above method to solve the evolution problem in Heisenberg picture quantum mechanics? To this end, quantize the classical mechanical system in the variables $(\alpha, \beta)$: $\alpha \mapsto \hat{\alpha}$, $\beta \mapsto \hat{\beta}$, $0 = H(t) \mapsto \hat{H}(t)$. If ever there were a natural operator ordering for $\hat{H}(t)$ there is here, $\hat{H}(t) = 0$. So $(\hat{\alpha}, \hat{\beta})$ are constant in time, just like their classical counterparts.

Therefore, starting with the initial operators $(\hat{q}(t_0), \hat{p}(t_0))$, use the canonical transformation (2.1), elevated to an operator equation, to find
\[
\left(\hat{\alpha}(t_0) = \alpha_{st_0}(\hat{q}(t_0), \hat{p}(t_0)), \hat{\beta}(t_0) = \beta_{st_0}(\hat{q}(t_0), \hat{p}(t_0))\right).
\]
(2.3)

Evolve $(\hat{\alpha}, \hat{\beta})$ (they don’t change since $\hat{H} = 0$). Then implement the inverse canonical transformation at time $t$ (2.2), elevated to an operator equation, to find
\[
\left(\hat{q}(t) = q_{st}(\hat{\alpha}(t), \hat{\beta}(t)), \hat{p}(t) = p_{st}(\hat{\alpha}(t), \hat{\beta}(t))\right).
\]
(2.4)

Unfortunately, equations (2.3) and (2.4) are not uniquely defined by equations (2.1) and (2.2). A choice of operator ordering must be made. However, an interesting question has been spawned. To what extent is the operator ordering ambiguity associated with (2.3) and (2.4) equivalent to the operator ordering ambiguity associated with making $H(q, p, t)$ into an operator $\hat{H}(t)$ and solving the Heisenberg equations of motion to find $(\hat{q}(t), \hat{p}(t))$ in terms of $(\hat{q}(t_0), \hat{p}(t_0))$? (Clearly they are not equivalent unless the operator orderings in (2.3) and (2.4) are restricted to give unitary evolution.)

The above discussion motivates a picture of evolution in quantum mechanics which I call quantization after evolution. In the following two sections this discussion is generalized and a rough preliminary definition of quantization after evolution given.

### 3 Already Evolved Classical System

With the right notation and definitions we can take the evolution out of evolution. The symbols $p$ and $q$ are most properly regarded as representing coordinate functions on phase space $PS$, that is mappings
\[
q : PS \rightarrow \mathbb{R},
\]
\[
p : PS \rightarrow \mathbb{R}.
\]

What is meant by $q(t)$ and $p(t)$ in the previous example? We like to think of evolution as a path in the phase space $\xi : \mathbb{R} \rightarrow PS$. The symbols $q(t)$ and $p(t)$ are short for $q(\xi(t))$ and $p(\xi(t))$, which are real numbers (for each fixed $t$). Similarly $\alpha(t)$ and $\beta(t)$ must be real numbers; but how should we interpret them?

The solution space $SS$ is the space of parametrized (by time) paths in phase space which satisfy the equations of motion. (Here we are speaking of an unconstrained Hamiltonian system.) That is, an element $\xi$ of $SS$ is a mapping
\[
\xi : \mathbb{R} \rightarrow PS.
\]
(3.1)
Given a solution $\xi \in SS$, real numbers $\alpha(t)$ and $\beta(t)$ are determined which are independent of time. Hence $\alpha(t)$ and $\beta(t)$ are really just bad notation for $\alpha(\xi)$ and $\beta(\xi)$. That is, the symbols $\alpha$ and $\beta$ are most properly regarded as representing coordinate functions on solution space, that is mappings

$$\alpha : SS \rightarrow \mathbb{R},$$

$$\beta : SS \rightarrow \mathbb{R}.$$

Now that we have straightened things out a bit, define the following pair of functions on solution space by substituting $\alpha$ for $\alpha(t)$ and $\beta$ for $\beta(t)$ in equations (2.2):

$$(q_t = q_{st}(\alpha, \beta), \quad p_t = p_{st}(\alpha, \beta)).$$

That is:

$$q_t : SS \rightarrow \mathbb{R}$$

$$\xi \mapsto q_{st}(\alpha(\xi), \beta(\xi))$$

$$p_t : SS \rightarrow \mathbb{R}$$

$$\xi \mapsto p_{st}(\alpha(\xi), \beta(\xi)).$$

(3.3)

For the mapping $q$, $q_t(\xi)$ is the image of the point in phase space $\xi(t)$:

$$q_t(\xi) = q(\xi(t)).$$

(3.4)

The mapping $p$ and $p_t(\xi)$ have the same relation. In sloppy language we may say: $q_t(\xi)$ is the “value of $q$ at time $t$” when the solution is $\xi$.

Let us now generalize the concept of the functions on solution space $q_t$ and $p_t$ to any phase space and any observable. We still only deal in unconstrained Hamiltonian systems. A function on phase space

$$O : PS \rightarrow \mathbb{R}$$

will be called a phase space observable. $q$ and $p$ are special examples. The pair $(q_t, t)$ will be called the solution space observable corresponding to the phase space observable $q$ at time $t$, in the example. Generally, we define the solution space observable corresponding to the phase space observable $O$ at time $t$ to be the pair $(O_t, t)$ where $O_t$ is defined by analogy with (3.4). Given the mapping $O : PS \rightarrow \mathbb{R}$ and the time $t$, $O_t$ is the mapping $O_t : SS \rightarrow \mathbb{R}$ defined by

$$O_t(\xi) = O(\xi(t)),$$

(3.6)

where $\xi(t)$ is the point in phase space corresponding to the solution $\xi$ and the time $t$. The mappings $O_t$ are essentially equivalent to Rovelli’s evolving constants of the motion [5]. However, the definition given here is more closely related to the covariant canonical quantization point of view [6, 7].
Figure 1 The Already Evolved Classical System

The reason for calling the pair \((\mathcal{O}_t, t)\) the solution space observable instead of \(\mathcal{O}_t\) is that \((\mathcal{O}_t, t)\) is distinguished from \((\mathcal{O}_{t'}, t')\) when \(t \neq t'\) even if \(\mathcal{O}_t = \mathcal{O}_{t'}\). Now we define the already evolved classical system \((AES)\) as the set of all such pairs:

\[
AES = \{ (\mathcal{O}_t, t) \mid t \in \mathbb{R}, \mathcal{O}_t \in SS^* \},
\]

\((SS^* \text{ denotes the dual space of } SS)\). \((3.7)\)

The already evolved classical system is represented schematically in Figure 1. For brevity, Figure 1 represents the phase space observables \(q\) and \(p\) from the motivating example. Note, however, that the definitions of \(\mathcal{O}_t\) and the \(AES\) do not depend at all on this example.

The name “already evolved classical system” is meant to be descriptive. Input a solution, and out pops the whole already evolved system. The AES does not evolve; however it has a causal structure. If we let \(\{\mathcal{O}\}\) be the set of phase space observables, then \((3.8)\) gives a natural bijection mapping \(\{\mathcal{O}\} \times \mathbb{R}\) to the AES:

\[
B : \{\mathcal{O}\} \times \mathbb{R} \to AES,
\]

\[
B(\mathcal{O}, t) = (\mathcal{O}_t, t).
\] \((3.8)\)

\(B\) induces a Poisson bracket on the space of functions on solution space \(\{\mathcal{O}_t\}\) from the Poisson bracket on the space of functions on phase space \(\{\mathcal{O}\}\). Specifically, we fix \(t = t_0\) and define

\[
\{(\mathcal{O}_{t_0}, \mathcal{O}_{2t_0}), t_0\} = B \left( \{\mathcal{O}_1, \mathcal{O}_2\}, t_0 \right).
\] \((3.9)\)

The definition is independent of \(t_0\) because time evolution is a canonical transformation. Of course we can also look at this as the solution space acquiring a symplectic structure from the phase space.
An explicit construction of solution space observables for the constant mean curvature time-slicing of (2+1)-gravity is given by Carlip \[8\]. In this construction, Witten’s holonomy variables \[10\] can be viewed as coordinates on the solution space. Equivalently, Witten’s holonomy variables can be viewed as the Hamilton-Jacobi variables.

4 Quantization After Evolution I

Quantization after evolution is quantizing the already evolved classical system. For simplicity, let us return to our two dimensional example. One way to perform quantization after evolution is to quantize the solution space in the canonical variables \((\alpha, \beta)\):

\[
(\alpha, \beta) \mapsto (\hat{\alpha}, \hat{\beta}),
\]

\[
\{\alpha, \beta\}_{pb} \mapsto \frac{1}{\imath \hbar} [\hat{\alpha}, \hat{\beta}],
\]

then substitute the operators \((\hat{\alpha}, \hat{\beta})\) for \((\alpha, \beta)\) in the functions on solution space which comprise the AES. This is represented schematically in Figure 2, which is Figure 1 with hats added. In doing this we should choose operator orderings which respect the canonical structure of \((q, p)\), and the causal structure of the AES. Specifically we require

\[
\frac{1}{\imath \hbar} [\hat{q}_t, \hat{p}_t] = 1 \quad \text{and},
\]

\[
\hat{q} \equiv U \hat{q}_t U^\dagger, \quad \hat{p} \equiv U \hat{p}_t U^\dagger, \quad \text{where} \quad U U^\dagger = 1.
\]
Due to results such as Van Hove’s theorem [13, 14], the success of this venture will depend on the parametrization \((\alpha, \beta)\) of the solution space. Because of this problem, a slightly different and more general definition of quantization after evolution will be given in Section 10, and used for the analysis portion of this paper.

An explicit construction of a quantization after evolution has been carried out by Carlip [8] for the example mentioned above, in the context of comparing Moncrief’s and Witten’s quantizations of \((2+1)\)-gravity. The relation here is that Witten’s quantization can be viewed as a quantization of solution space \(((\alpha, \beta) \mapsto (\hat{\alpha}, \hat{\beta}))\), while Moncrief’s quantization can be viewed as a quantization of the (reduced) phase space and subsequent evolution under the Heisenberg equations of motion. Although he does not name his construction, Carlip essentially constructs a quantization after evolution, with Witten’s holonomy variables parametrizing the solution space, and finds an operator ordering of Moncrief’s Hamiltonian such that the Heisenberg equation evolution agrees with the quantization after evolution.

This example further motivates the question: is quantization after evolution equivalent, in general, to quantizing the phase space and evolving by means of the Heisenberg equations of motion. This question will be considered after more formal definitions have been given (Sections 9–11). The next section begins discussion of systems which don’t have a preferred time parameter.

5 Systems Without Preferred Time Parameter

For the purposes of this paper, the most important feature of quantization after evolution is that it applies to systems without a preferred time parameter and leads to intrinsically consistent evolution. So far we have studied Hamiltonian systems. For a Hamiltonian system the space of time is \(\mathbb{R}\). A solution \(\xi\) in \(SS\) is a mapping

\[\xi : \mathbb{R} \rightarrow PS.\] (5.1)

Gravity can be made to fit this mold by choosing a time-slicing. On each slice in a one parameter (\(\mathbb{R}\)) family of slices a spatial metric and extrinsic curvature tensor (point in \(PS\)) are induced by a spacetime metric (point in \(SS\)). (This statement is meant for illustration only; the term “slice” requires explanation. A more careful discussion involving the reduced phase space is given in the next section.) However this reduction is unnatural. Which time-slicing should we choose? More naturally, gravity fits a generalization of this mold. On each slice in the space of all slices \((S)\) a spatial metric and extrinsic curvature tensor (point in \(PS\)) are induced by a spacetime metric (point in \(SS\)). That is for gravity, a solution \(\xi\) in \(SS\) is a mapping

\[\xi : S \rightarrow PS.\] (5.2)

Much of what follows will be valid either in the situation (5.1) or in the situation (5.2). In fact it will be valid for any “system with global time \(\Gamma\)”:

**Definition 1** A system with global time \(\Gamma\) is a triple \((PS, SS, \Gamma)\) where \(PS, SS, \) and \(\Gamma\) are topological spaces with differential structure such that:
1. An element $\xi$ of $SS$ is a differentiable mapping
   $$\xi : \Gamma \rightarrow PS.$$ 

2. $PS$ is a symplectic space. For all $\tau$ and $\tau'$ in $\Gamma$, $\xi(\tau)$ and $\xi(\tau')$ are related by a canonical transformation $u(\tau, \tau')$, which does not depend on $\xi$.

3. For every $\tau$ in $\Gamma$ and $\eta$ in $PS$ there exists $\xi$ in $SS$ with $\xi(\tau) = \eta$.

If $\Gamma$ is not $\mathbb{R}$ we will also call this system a “system without preferred time parameter”. Of course we will call $PS$ the phase space, $SS$ the solution space, and $\Gamma$ the space of global time. This definition is introduced so that we may carry on our discussion in general terms, and then when appropriate set $\Gamma$ equal to $\mathbb{R}$, $S$, or a subspace of $S$. It will also be used in the next section to state an important assumption regarding the nature of time in gravity.

It is useful to describe the consistent evolution problem in the general setting of a system with global time $\Gamma$. A Hamiltonian system may be obtained from a system with global time $\Gamma$ by choosing a “preferred time”, that is by choosing a differentiable embedding $\mathbb{R} \hookrightarrow \Gamma$. For gravity, a choice of time-slicing is a special case of a choice of preferred time. Of course a Hamiltonian system can be quantized in the usual Heisenberg picture. If this is done for more then one choice of preferred time, specifically for two embeddings $\mathbb{R} \overset{e_1}{\hookrightarrow} \Gamma$ and $\mathbb{R} \overset{e_2}{\hookrightarrow} \Gamma$ which coincide at two points, the operator evolution between the two points will not agree in general. The question naturally arises: can we operator order the two Hamiltonians so that the operator evolution agrees? More generally: can we operator order the infinitude of Hamiltonians corresponding to all possible embeddings $\mathbb{R} \hookrightarrow \Gamma$ so that the evolution is completely consistent? This question will be referred to as the question of achieving consistent evolution in the Heisenberg picture for a quantum theory without preferred time parameter. Of course our emphasis is on gravity. It is simply convenient to speak in general terms.

It is interesting to note that most of the systems studied in physics have (probable) interpretations as systems of global time $\Gamma$ for some $\Gamma$. For example, special relativistic systems have a (probable) interpretation as systems of global time $\Gamma$ with $\Gamma$ the space of flat spacelike planes in Minkowski space. In an upcoming paper it will be shown explicitly that the relativistic free particle system in (1+1)-dimensions is a system with global time $\mathbb{R} \times \mathbb{R}$. The system is found to have two Hamiltonians associated with two time variables which parametrize the straight spacelike lines. A consistent quantization (in the sense just described) is carried out explicitly and the result appears to be a viable single particle interpretation of relativistic quantum mechanics in (1+1)-dimensions.

The definitions given in Section 3 are applicable to any system with global time $\Gamma$. Simply substitute $\Gamma$ for $\mathbb{R}$ and the words “global time” for “time”. For definiteness, let us write them here as formal definitions.

**Definition 2** Given the mapping $O : PS \rightarrow \mathbb{R}$ and the global time $\tau \in \Gamma$, $O_{\tau}$ is the mapping $O_{\tau} : SS \rightarrow \mathbb{R}$ defined by

$$O_{\tau}(\xi) = O(\xi(\tau)), \quad (5.3)$$

where $\xi(\tau)$ is the point in phase space corresponding to the solution $\xi$ and the global time $\tau$. 


Definition 3

\[ AES = \{ (O_\tau, \tau) \mid \tau \in \Gamma, \, O_\tau \in SS^* \}, \]

\[(SS^* \text{ denotes the dual space of } SS).\]

The bijection \( B \) becomes

\[ B : \{ O \} \times \Gamma \rightarrow AES, \]

\[ B(O_\tau, \tau) = (O_\tau, \tau). \quad (5.4) \]

It still induces Poisson brackets on the space of functions on solution space.

Quantization after evolution is also applicable to any system with global time \( \Gamma \). An already evolved classical system without preferred time is represented schematically in Figure 3, which should be compared with Figure 1. Quantization after evolution is shown schematically in Figure 4, which should be compared with Figure 2, and proceeds precisely as in Section 4 except that \( t \) and \( t' \) in equation (4.2) should be replaced by \( \tau \) and \( \tau' \), which are elements of \( \Gamma \).

Of course, the above doesn’t prove that it is possible to find “operator orderings” which satisfy the conditions (4.2), that is to carry out quantization after evolution. But if we can, the resulting evolution is intrinsically consistent. Classical functions on solution space are assigned operator status on an instant by instant basis (instant of global time of course).
6 Gravity as a System Without Preferred Time Parameter

Missing from the previous section is a clear description of what is meant by a “slice”. This is a non-trivial question involving the global time and the multiple choice problems. In this section we will first motivate and then state an assumption concerning the nature of time in gravity. Along the way, the meaning of the term “slice” will be explained.

The definition of the “reduced phase space” is the constraint surface modulo the transformations generated by the first class constraints. In many cases finding the reduced phase space reduces a theory to an unconstrained Hamiltonian system with the reduced phase space playing the role of an ordinary phase space. In gravity, however, the symmetries are the diffeomorphisms of the spacetime manifold. After finding the fully reduced phase space, no equations of motion remain (at least if the spacetime is spatially closed). Evolution amounts to a spacetime diffeomorphism, and hence has been modded out of the fully reduced phase space. The fully reduced phase space is thus naturally isomorphic to the solution space.

The complete reduction has been carried out for (2+1)-dimensional gravity by Witten [10]. If an unconstrained Hamiltonian system with phase space coordinates \((q, p)\) and Lagrangian

\[
L = \int (p \frac{dq}{dT} - H) dT
\]

*When equations of motion are present, the reduced phase space is isomorphic to the solution space, but not naturally isomorphic. A different isomorphism is obtained for each time \(t_0\) by mapping a point in phase space to the solution for which it is the initial data at time \(t_0\).
is parametrized, then it acquires an equivalent description as a constrained system with phase space coordinates \((q, p, T, P_T, K, P_K)\) and Lagrangian

\[
L' = \int \left( p \frac{dq}{dt} + P_T \frac{dT}{dt} - (H + P_T)K \right) dt.
\]

If the fully reduced phase space is found for this constrained system, the result is not the original phase space \((q, p)\). The constraint \(H + P_T = 0\) generates evolution of \(q\) and \(p\), so that the reduced phase space is \textit{naturally} isomorphic to the solution space just as for gravity. To regain the original Hamiltonian system, what must be done is to fix the gauge

\[
\frac{dT}{dt} = K = 1,
\]

not mod out by the transformations generated by the constraint.

In gravity, the analogous gauge condition is the fixing of a time-slicing. However, after a time-slicing has been fixed, a constrained system still remains. (The gauge has not been completely fixed.) If at this point the reduced phase space is found, the result is an unconstrained Hamiltonian system with time a parameter labeling slices. This “partially reduced phase space” is to be distinguished from the fully reduced phase space mentioned earlier. This (partial) reduction has been carried out for the constant mean curvature time-slicing (York time) in \((2+1)\)-dimensional gravity by Moncrief [3], and by Hosoya and Nakao [4].

The global time problem concerns whether this reduction can be carried out in general, for any foliation of the manifold with spacelike leaves. The general reduction amounts to finding a canonical transformation

\[
(g_{ab}, K^{ab}) \rightarrow (X^A, P_A, \phi^r, p_r)
\]

from the intrinsic geometry \(g_{ab}\) and extrinsic curvature \(K^{ab}\) of a spacelike hypersurface \(\Sigma \hookrightarrow M\) to “internal spacetime coordinates” \(X^A\), their canonical conjugates \(P_A\), and the coordinates on the partially reduced phase space \((\phi^r, p_r)\). Using the term “gauge transformation” to denote a transformation generated by the momentum constraints, one might hope for the following: (1) for arbitrary \((X^A, \phi^r, p_r)\) there exists unique \(P_A\) such that the constraints are satisfied; (2) the embedding \(\Sigma \hookrightarrow M\) is determined by the functions \(X^A\), which are otherwise pure gauge; and (3) the functions \((\phi^r, p_r)\) may be regarded as coordinates on a gauge invariant space, the partially reduced phase space. If this may be achieved then given a spacetime (a manifold with Lorentzian metric satisfying Einstein’s equations), and a canonical transformation \((6.1)\), any foliation into spacelike leaves may be determined by fixing a one parameter family of functions \(X^A(t)\). Such a one parameter family \(X^A(t)\) will be referred to as a time-slicing. For each fixed \(t\), \(X^A(t)|_{t \text{ fixed}}\) will be referred to as a slice.

The space of time-slicings is homeomorphic to the space of foliations with spacelike leaves. However the two should not be identified; if the spacetime metric is changed the foliation determined by a time-slicing \(X^A(t)\) will change. For example in the Moncrief-Hosoya-Nakao reduction, the time coordinate is the (constant) mean curvature. Given a spacetime, there is a unique embedding \(\Sigma \hookrightarrow M\) such that the mean curvature \(K\) has a particular value, for
example $K = 3$. However if the metric is changed, the embedding for which $K = 3$ will change. So in gravity a particular time is a slice, which is closely related to, but not quite the same as, an embedding $\Sigma \hookrightarrow M$.

Classically, evolution from slice to slice does not depend on the intermediate time-slicing. Therefore, given a solution, a point in the partially reduced phase space is uniquely specified for each slice. This phenomenon is usually called “many-fingered time”. In the language of Section 3, gravity is a system with global time $S$, where $S$ is the space of slices. As noted in this and the previous section, a Hamiltonian system does not emerge unless a time-slicing (preferred time) is specified. For this reason, given a slice, a Hamiltonian is not identified. We must be given the slice and the local time-slicing to identify the Hamiltonian. This is why it is not clear if Hamiltonians should be considered as observables in gravity.

It must be noted that if the canonical transformation (6.1) is not unique, then viewing gravity as a system with global time $S$ is unnatural. A decomposition into time and physical degrees of freedom must still be chosen. The quantum theories based on different choices may be inequivalent. This is known as the multiple choice problem [4,5], a serious problem which is not addressed in this article. In this article it is assumed that a particular canonical transformation (6.1) has been chosen, so that the problem known as the functional evolution problem [4,5] is addressed.

As described, the above conclusions depend on a positive resolution of the global time problem, that is the existence of the canonical transformation (6.1). However these are complex issues, and the above is meant to be only a cursory introduction, a motivation for viewing gravity as a system without preferred time parameter as defined in Section 3. So let us finish this section by stating clearly what we will assume in the remainder of this work.

Assumption 1 Gravity is a system with global time $S$ as defined in Definition 3 (perhaps not canonically). For a manifold of the form $M = \Sigma \times \mathbb{R}^+$, $S$ is homeomorphic to the space of spacelike embeddings $\Sigma \hookrightarrow M$ for a particular spacetime $(M, g)$.

The remainder of this paper will refer to the partially reduced phase space simply as the phase space, with the understanding that only systems with global time $\Gamma$ (as defined in Definition 1) are considered.

7 Progress Report

The achievement so far has been to replace the question,

Can we achieve consistent evolution in the Heisenberg picture for a quantum theory without preferred time parameter (for example gravity)?

with the questions:

1. Is quantization after evolution equivalent to Heisenberg equation evolution when a preferred time has been chosen?

2. Can we carry out quantization after evolution for the theory in question?
If question number 2 is answered affirmatively, but question number 1 unanswered, then we may achieve consistent evolution of observables defined at specific global times with the correct classical limit. However, the evolution may not be generated by a Hamiltonian which has the correct classical limit. Here it becomes important to know if the Hamiltonian should be considered an observable.

Part II
Quantization After Evolution

In part II, an assumption concerning the existence of a “classical limit” of an algebra of $\hbar$ dependent operators on Hilbert space will be spelled out and used to define quantization after evolution, as well as ordinary Heisenberg picture quantum mechanics. The question of equivalence between the two will be discussed. Finally, quantization after evolution will be recast in the light of quantizing canonical transformations, with a view toward the question of existence to be discussed in part III.

8 The Classical Limit

It is generally assumed that quantum mechanics reduces to classical mechanics in some appropriate limit, usually termed “$\hbar$ goes to zero”. This is called the correspondence principle. Operators which arise from quantization should depend on $\hbar$ in an appropriate way, and there should be a definition of a limiting process, call it $\lim_{\hbar \to 0}$, which tests this appropriateness. The following assumption is motivated by the work of Werner and Berezin [11, 12].

Assumption 2 There exists a classical limit, $\lim_{\hbar \to 0}$, which satisfies the following:

1. There is an algebra $\hat{A}$ of $\hbar$-sequences, that is $\hbar$-dependent operators on an appropriate Hilbert space, which converges to an abelian algebra whose elements are identified with functions on phase space:

   - **Convergent Set** For $\hat{a} \in \hat{A}$, $\lim_{\hbar \to 0}(\hat{a}) = \text{complex valued function on phase space}$.
   - **Product Law** For $\hat{a}_1, \hat{a}_2 \in \hat{A}$, $\lim_{\hbar \to 0}(\hat{a}_1 \hat{a}_2) = (\lim_{\hbar \to 0}(\hat{a}_1))(\lim_{\hbar \to 0}(\hat{a}_2))$.

2. The subset $\hat{A}_{\text{diff}} \subset \hat{A}$ of “differentiable-$\hbar$-sequences”, that is $\hbar$-sequences which converge to differentiable functions, satisfies the following:

   - **Lie Bracket Convergence** For $\hat{a}_1, \hat{a}_2 \in \hat{A}_{\text{diff}}$, $\lim_{\hbar \to 0}(\frac{1}{\hbar}[\hat{a}_1, \hat{a}_2]) = \{\lim_{\hbar \to 0}\hat{a}_1, \lim_{\hbar \to 0}\hat{a}_2\}_{pb}$.

3. $\lim_{\hbar \to 0}$ induces a locally-sectionable linear mapping of the space of Hermitian differentiable-$\hbar$-sequences, to the space of differentiable real valued functions on phase space:

$$\rho : \hat{R}_{\text{diff}} \to R_{\text{diff}}, \quad \rho = \lim_{\hbar \to 0}|\hat{R}_{\text{diff}}|$$
Locally-sectionable means that there is an open covering \( \{ U_i \} \) of \( R_{\text{diff}} \), such that for each \( U_i \) there is a continuous mapping \( s_i : U_i \to \hat{R}_{\text{diff}} \) with \( \rho \circ s_i = I \). That is, it means that locally it is possible to continuously lift real valued differentiable functions on phase space to Hermitian operators with the correct classical limit.

If an operator is not in \( \hat{A} \) we shall say that it does not converge, or is not a convergent \( \hbar \)-sequence, etc. Most notably, the unitary evolution operators of quantum mechanics cannot converge. However, in Section 12, we shall define a sense in which unitary transformations converge to canonical transformations.

At present, there is no generally accepted implementation of \( \lim_{\hbar \to 0} \). However, in the finite dimensional case, there are mappings known as the Wigner-Weyl quantization and dequantization maps \([11]\). Taking the Weyl symbol \([12]\) of an operator provides a one-to-one mapping of a certain class of operators \( \hat{A}' \) to functions on phase space. For example, if the phase space is two dimensional the mapping is

\[
\hat{O} \mapsto \frac{1}{2} \left( \langle q | \hat{O} | p \rangle + \frac{1}{\langle q | p \rangle} \right),
\]

and we find

\[
\hat{q} \mapsto q, \quad \hat{p} \mapsto p, \quad \hat{q} \hat{p} \mapsto qp + \frac{i\hbar}{2}, \quad \hat{p} \hat{q} \mapsto qp - \frac{i\hbar}{2} \quad \text{etc.}
\]

By taking the ordinary limit as \( \hbar \) goes to zero of the Weyl symbols, we get a mapping which satisfies the product law and Lie bracket convergence for operators in \( \hat{A}' \). This operation is generally regarded as giving the right classical limit for operators in \( \hat{A}' \). Because the Weyl symbol-operator correspondence is one-to-one and continuous, with real valued functions corresponding to Hermitian operators, we see that item 3 of Assumption 2 is satisfied at least for the subset \( \hat{A}' \cap \hat{R}_{\text{diff}} \subset \hat{R}_{\text{diff}} \) and its Weyl symbols. Hence, if all the functions we cared about were Weyl symbols of operators in \( \hat{A}' \), we could proclaim Assumption 2 satisfied and \( \hat{A} = \hat{A}' \).

Actually the preceding paragraph is not rigorous. What has been achieved, as I understand it (see \([12]\)), is to set up a one to one correspondence between the functions \( f \in C^\infty(\mathbb{R}^n) \) for which there exist fixed real numbers \( C \) and \( m \) such that

\[
|\partial^\alpha_q \partial^\beta_p f(q, p)| \leq C (1 + |q| + |p|)^m
\]

for all \( \alpha \) and \( \beta \), and a class of operators that have a dense domain in \( L^2(\mathbb{R}^n) \). Equation (8.1) is at least valid for polynomial functions.

Recently Werner \([11]\) has proposed a definition for \( \lim_{\hbar \to 0} \) in the finite dimensional case which purports to agree with the accepted folklore and be valid on the set of bounded operators. Werner shows that his construction satisfies item 1 and item 2 of Assumption 4. Item 3 is not explored. It is quite possible that in the case of (2+1)-dimensional gravity, where the reduced phase space is finite dimensional, Werner’s work will provide the hoped for definition of the classical limit, and allow all the assumptions (except Assumption 4) of this paper to be tested.

The following lemma brings out the importance of item 3 of Assumption 4, and is required in part III. It is also tacitly used in the definitions that follow, although they could be modified to do without it.
Lemma 1 If the topology on the space of differentiable real valued functions on phase space is paracompact, and if Assumption 2 holds, then the differentiable real valued functions may be continuously lifted to hermitian operators on Hilbert space with the correct classical limit.

Proof: Let $R_{\text{diff}}$ be the space of real valued differentiable functions with a paracompact topology. Let $\hat{R}_{\text{diff}}$ be the space of hermitian differentiable-$\hbar$-sequences. We need to show that the bundle over $R_{\text{diff}}$,

$$\rho : \hat{R}_{\text{diff}} \to R_{\text{diff}}, \quad \rho = \lim_{\hbar \to 0} |_{\hat{R}_{\text{diff}}},$$

has a section.

Let $\hat{r}$ and $\hat{r}'$ be in $\rho^{-1}(r)$ where $r$ is in $R_{\text{diff}}$, then

$$\rho(\lambda \hat{r} + (1 - \lambda)\hat{r}') = r.$$

Therefore $\rho^{-1}(r)$ is a convex domain.

If $R_{\text{diff}}$ is paracompact, then a locally-sectionable bundle over $R_{\text{diff}}$ with convex fibers has a section. A paracompact space has a partition of unity subordinate to any locally finite covering. Therefore choose a sufficiently fine covering $\{U_i\}$ of $R_{\text{diff}}$, and a partition of unity $\{f_i\}$ subordinate to it. Because the bundle over $R_{\text{diff}}$ is locally sectionable we can construct mappings

$$\sigma_i : U_i \to \hat{R}_{\text{diff}}, \quad \rho \circ \sigma_i = I|_{U_i}.$$ 

Then, because the fibers are convex,

$$\sum_i f_i \sigma_i : R_{\text{diff}} \to \hat{R}_{\text{diff}}$$

is a section.

QED

From a physics point of view, the paracompactness requirement is innocuous. Any topology which may be defined by a metric (metrizable topology) is paracompact. If for some reason one wanted to drop the paracompactness requirement, the word “locally” would have to be removed from item 3 of Assumption 2 in what follows.

9 Heisenberg Picture Quantum Mechanics

The general concept of $\lim_{\hbar \to 0}$ will now be used to make various definitions. The first one is Heisenberg picture quantum mechanics.

Definition 4 Let $O_{H(t)}$ be the phase space observable corresponding to the Hamiltonian at time $t$. Heisenberg picture quantum mechanics is a mapping,

$$HQ : \{O\} \sqcup \{(O_{H(t)}, t) \mid t \in \mathbb{R}\} \to \{\text{Hermitian operators}\}$$

$$O \mapsto \hat{O}(t_0), \quad (O_{H(t)}, t) \mapsto \hat{H}(t),$$

such that:
1. \( \lim_{\hbar \to 0} (\hat{O}(t_0)) = \mathcal{O} \), \( \lim_{\hbar \to 0} (\hat{H}(t)) = \mathcal{O}_{H(t)} \).

2. For a certain “preferred” Lie subalgebra of phase space observables \( \{\alpha\} \subset \{O\} \), \( HQ \) is a Lie algebra isomorphism:

\[
HQ(\{\beta, \alpha\}_pb) = \frac{1}{i\hbar} [\hat{\beta}(t_0), \hat{\alpha}(t_0)], \quad \beta, \alpha \in \{\alpha\},
\]

together with the Heisenberg evolution equation,

\[
\frac{d\hat{O}}{dt}(t) = \frac{1}{i\hbar} [\hat{O}(t), \hat{H}(t)],
\]

which determines \( \hat{O}(t) \) for \( t > t_0 \).

Note the use of the disjoint union to adjoin the set of Hamiltonians to the set of phase space observables. A time-dependent Hamiltonian is not properly a function on phase space. It only gives rise to a phase space observable \( \mathcal{O}_{H(t)} \) (function on phase space) when evaluated at a particular time. As mentioned earlier (Section 3), in geometrodynamics the situation is even worse. The Hamiltonian is not defined at a time (that is on a slice), but only for a local time-slicing. However, it is still true that given a time and a local time-slicing, a function on phase space is given, the phase space observable corresponding to the Hamiltonian for the time and local time-slicing. By considering the set of Hamiltonians as separate from the set of phase space observables, we are allowing the Hamiltonian to lift to an operator which differs from the operator for the related phase space observable by a term with zero classical limit. Also, because we are considering the pairs \( (\mathcal{O}_{H(t)}, t) \), Hamiltonians at different times, but which correspond to the same phase space observable may lift to operators that differ by a term with zero classical limit. Hence some of the structure of the classical theory, specifically the relationship between observables and generators of evolution, is lost in the quantum theory. In run-of-the-mill quantum mechanics, the set of Hamiltonians is regarded as a subset of the space of phase space observables.

In determining whether or not Definition 4 is reasonable, one should consider that we always give up structure when quantizing. For example, Van Hove’s theorem implies that we can never completely preserve the Lie algebra given by the Poisson bracket. Only a very small subalgebra can be preserved [14]. Actually, when quantizing a theory, we generally consider only a few phase space observables. So to evaluate the relation of Definition 4 to experimentally tested aspects of quantum mechanics, let us restrict our attention to a discrete subset of phase space observables, \( \{q, p\} \subset \{O\} \), which parametrize phase space. \( HQ \) is replaced by the restricted mapping \( HQ' \):

\[
HQ' : \{q, p\} \sqcup \{(\mathcal{O}_{H(t)}, t) \mid t \in \mathbb{R}\} \rightarrow \{\text{Hermitian operators}\}
\]

\[
q \mapsto \hat{q}(t_0), \ p \mapsto \hat{p}(t_0), \ (\mathcal{O}_{H(t)}, t) \mapsto \hat{H}(t).
\]

It is rare that the function on phase space corresponding to the Hamiltonian is \( q \) or \( p \). So the concern that a phase space observable and a Hamiltonian which corresponds to the
same function on phase space may map to different operators is almost moot. The only remaining distinction between Definition 4 and run-of-the-mill quantum mechanics is that even if the classical Hamiltonian is not time-dependent, Definition 4 allows the operator Hamiltonian to be time-dependent. So the phrase to remember is that Definition 4 allows time dependent operator orderings for Hamiltonians. In a theory where a time-independent classical Hamiltonian is the exception, not the rule, time-dependent operator ordering does not seem a big liberty.

10 Quantization After Evolution

The mapping \( \lim_{\hbar \to 0} \) can be used to give an equivalent formulation of the classical limit in terms of the solution space observables. We choose an isomorphism between the phase space \( PS \) and the solution space \( SS \) by choosing an “initial” global time, \( \tau_0 \), and mapping a point \( \eta \) in phase space to the solution which has the “initial data” \( \eta \) at time \( \tau_0 \) (note that Assumption 2 guarantees that this mapping is an isomorphism):

\[
I_{\tau_0} : PS \to SS.
\]

\( I_{\tau_0} \) induces the obvious isomorphism \( I_{\tau_0}^* \) between functions on phase space and functions on solution space:

\[
I_{\tau_0}^* : \{ \text{functions on } PS \} \to \{ \text{functions on } SS \}
\]

\[
O \mapsto O_{\tau_0}, \quad O(\eta) = O_{\tau_0}(I_{\tau_0}\eta).
\]

(10.2)

Of course \( I_{\tau_0}^* \) is just the mapping of Definition 2 (Section 5) for \( \tau = \tau_0 \); that is \( I_{\tau_0}\eta(\tau_0) = \eta \). Finally define the limit \( \lim_{\hbar \to 0} \) to be the composition of \( \lim_{\hbar \to 0} \) with \( I_{\tau_0}^* \):

\[
\tilde{\lim}_{\hbar \to 0} : \{ \text{convergent operators} \} \to \{ \text{functions on } SS \}
\]

\[
\tilde{\lim}_{\hbar \to 0} = I_{\tau_0}^* \circ \lim_{\hbar \to 0}.
\]

(10.3)

Now we are in a position to define quantization after evolution:

**Definition 5** A quantization after evolution is a mapping,

\[
QAE : AES \to \{ \text{Hermitian operators} \}, \quad (O_\tau, \tau) \mapsto \hat{O}(\tau),
\]

differentiable in \( \tau \), such that:

1. It has the correct classical limit, that is

\[
\tilde{\lim}_{\hbar \to 0} (\hat{O}(\tau)) = O_\tau.
\]
2. For a certain “preferred” Lie subalgebra of solution space observables at a fixed time \( \tau_0 \),

\( O \subset \{(\mathcal{O}_\tau, \tau) \mid \tau = \tau_0\} \subset AES, \)

\( QAE \) is a Lie algebra isomorphism:

\[
QAE((\{\alpha_{\tau_0}, \beta_{\tau_0}\}, \tau_0)) = \frac{1}{i\hbar}[\hat{\alpha}(\tau_0), \hat{\beta}(\tau_0)], \quad (\alpha_{\tau_0}, \tau_0), (\beta_{\tau_0}, \tau_0) \in O.
\]

3. \( QAE \) respects the causal structure \( AES \simeq \{O\} \times \Gamma \) provided by the bijection \( B \) (5.4).

The natural (and known from quantum mechanics) meaning of this is:

For all \( \tau_1, \tau_2 \in \Gamma \), there exists a unitary operator \( \hat{U}(\tau_1, \tau_2) \) such that:

\[
\hat{O}(\tau_2) = \hat{U}(\tau_1, \tau_2)\hat{O}(\tau_1)\hat{U}^\dagger(\tau_1, \tau_2).
\]

Definition 3 depends on \( \tau_0 \), because equation (10.3) depends on \( \tau_0 \). However, the dependence is trivial (and is therefore not reflected in the notation, \( \tilde{\lim}_{\hbar \to 0} \)). The relation between phase space and solution space is simply shifted by a canonical transformation of solution space (or a canonical transformation of phase space).

To understand the relationship between Definition 3 and the brief description given in Section 4, consider the following diagram:

\[
\begin{align*}
\text{quantized} \quad \text{solution space} & \xrightarrow{D} \quad \text{quantized} \quad \text{AES} \\
C \uparrow \quad \{\text{solution space}\} & \xrightarrow{A} \quad \{\text{AES}\}
\end{align*}
\]

The single arrow \( C \) denotes quantizing the solution space. This involves choosing a preferred Lie subalgebra of solution space observables to be preserved. The double arrow \( D \) denotes quantization after evolution by substituting the operators obtained from \( C \) into the functions of the \( AES \) (see Figures 1 and 2) and attempting to find operator orderings such that items 2 and 3 of Definition 3 are satisfied. Due to results such as Van Hove’s theorem this may not be possible. For example, preservation of the preferred Lie subalgebra of the process \( C \) may be incompatible with preservation of the preferred Lie subalgebra of the quantization after evolution. Hence the path \( D \circ C \), which is the path described in Section 4, is not passable for all quantizations of solution space \( C \). In reference [8], Carlip had to follow this path, using Witten’s parametrization of solution space, to show that Witten’s quantization of \((2+1)\)-gravity was equivalent to the Moncrief-Hosoya-Nakao quantization.

The double arrow \( A \) denotes construction of the already evolved classical system. This involves defining functions on solution space. The single arrow \( B \) denotes quantization after evolution any way possible. For example if the space of global time is \( \mathbb{R} \), we can always use evolution under the Heisenberg equations of motion (except see Assumption 3). The path \( B \circ A \) is the path described in this section. If the space of global time is \( \mathbb{R} \), it is passable whenever the corresponding Heisenberg picture quantization (Definition 4) is possible. It was briefly described by Kuchař in reference [4].
11 Comparison of Quantization After Evolution and Heisenberg Picture Quantum Mechanics

In the case $\Gamma = \mathbb{R}$, or when a preferred time has been chosen, we can compare quantization after evolution with Heisenberg picture quantum mechanics. This is done by using the bijection $B : \{O\} \times \Gamma \to AES$ (5.4), to map Heisenberg picture quantum mechanics into a quantization of the already evolved classical system ($AES$). The result is the following equivalent definition of Heisenberg picture quantum mechanics:

**Definition 6 (equivalent to Definition 4)** Let $(H_t, t)$ be the solution space observable corresponding to the Hamiltonian at time $t$, that is $(H_t, t) = B(O_H(t), t)$. Heisenberg picture quantum mechanics is a mapping $HQA : AES \sqcup \{(H_t, t) \mid t \in \mathbb{R}\} \to \{\text{Hermitian operators}\}$, defined by

$(O_t, t) \mapsto \hat{O}(t), \ (H_t, t) \mapsto \hat{H}(t)$,

such that:

1. $\lim_{\hbar \to 0} \hat{H}(t) = H_t$, and for a fixed time $t_0$ $\lim_{\hbar \to 0} \hat{O}(t_0) = O_{t_0}$.

2. $\frac{d\hat{O}}{dt}(t) = \frac{1}{i\hbar} [\hat{O}(t), \hat{H}(t)]$.

3. For a certain “preferred” Lie subalgebra of the solution space observables at time $t_0$,

$O \subset \{(O_t, t) \mid t = t_0\} \subset AES$,

$HQA$ is a lie algebra isomorphism:

$HQA((\{\alpha_{t_0}, \beta_{t_0}\}_{pb}, t_0)) = \frac{1}{i\hbar} [\hat{\alpha}(t_0), \hat{\beta}(t_0)], \ (\alpha_{t_0}, t_0), \ (\beta_{t_0}, t_0) \in O$.

To see that Definition 6 is equivalent to Definition 4, we must check that the operators of Definition 6 have the right classical limit under $\lim_{\hbar \to 0}$. Specifically we require

$$\lim_{\hbar \to 0} \hat{O}(t_0) = O \quad \text{and},$$

$$\lim_{\hbar \to 0} \hat{H}(t) = O_{H(t)}.$$

Applying the identity in the form $I_{t_0*}I_{t_0*}$ we find

$$I_{t_0*}^{-1}I_{t_0*} \lim_{\hbar \to 0} \hat{O}(t_0) = I_{t_0*}^{-1} \lim_{\hbar \to 0} \hat{O}(t_0) = I_{t_0*}^{-1}O_{t_0} = O \quad \text{and},$$

$$I_{t_0*}^{-1}I_{t_0*} \lim_{\hbar \to 0} \hat{H}(t) = I_{t_0*}^{-1} \lim_{\hbar \to 0} \hat{H}(t) = I_{t_0*}^{-1}H_t. \quad (11.1)$$

The last equality of Equation (11.1) follows because $I_{t_0*}$ is the mapping of Definition 4 for $t = t_0$ (see section 10).
It remains to show that \( I_{t_0}^{-1} H_t = \mathcal{O}_{H(t)} \). Again from the definition of \( I_{t_0} \) and Definition 2 (Section 5) we have

\[
I_{t_0} \mathcal{O}_{H(t)}(\xi) = \mathcal{O}_{H(t)}(\xi(t_0)) \quad \text{and,}
H_t(\xi) = \mathcal{O}_{H(t)}(\xi(t)).
\] (11.3)

Evolution of \( \mathcal{O}_{H(t)}(\xi(t)) \) is given by the Hamiltonian equations of motion:

\[
\frac{d\mathcal{O}_{H(t)}}{dt} - \frac{\partial \mathcal{O}_{H(t)}}{\partial t} = \{\mathcal{O}_{H(t)}, \mathcal{O}_{H(t)}\}_{pb} = 0,
\]
so
\[
\frac{d\mathcal{O}_{H(t)}}{dt} = \frac{\partial \mathcal{O}_{H(t)}}{\partial t}.
\]

Therefore \( \mathcal{O}_{H(t)}(\xi(t_0)) = \mathcal{O}_{H(t)}(\xi(t)) \) and from (11.3) we have

\[ H_t = I_{t_0} \mathcal{O}_{H(t)}, \]

which completes the demonstration that Definition 6 and Definition 4 are equivalent.

Comparing Definition 5 with Definition 6 we see that Heisenberg picture quantum mechanics is equivalent to quantization after evolution (at least if we restrict our attention to differentiable functions) if the following assumptions hold:

**Assumption 3** Evolution under the Heisenberg equations of motion implies that the correct classical limit is obtained for \( t > t_0 \). That is \( \lim_{\hbar \to 0} (\hat{\mathcal{O}}(t)) = \mathcal{O}_t \) for all \( t > t_0 \).

**Assumption 4** Differentiable unitary evolution with the correct classical limit implies that

\[ \frac{d\hat{\mathcal{O}}}{dt}(t) = \frac{1}{i\hbar} [\hat{\mathcal{O}}(t), \hat{H}(t)] \]

for some \( \hat{H}(t) \) with the correct classical limit.

If Assumption 3 holds, then Heisenberg picture quantum mechanics is contained in quantization after evolution. If Assumption 4 holds, then quantization after evolution is contained in Heisenberg picture quantum mechanics.

Assumptions 3 and 4 can in principle be checked once we have a physically accepted definition for \( \lim_{\hbar \to 0} \). Using his definition, Werner has shown that Assumption 3 holds for time-independent Hamiltonians. It may be somewhat over cautious to refer to Assumption 3 as an assumption. It is, after all, a general tenet of quantum mechanics. Never the less we will continue to do so.

If Assumption 2 (Section 8) holds, then Assumption 4 holds if the Hamiltonian is in \( \hat{A}_{\text{diff}} \) and the classical limit commutes with the time derivative. The only thing we can say here is that if the classical limit commutes with the time derivative then it seems probable that the
Hamiltonian is constrained to be in $\hat{A}_{\text{diff}}$. To understand this statement consider application of the classical limit to the operator evolution equation:

$$\lim_{\hbar \to 0} \left( \frac{d\hat{\mathcal{O}}}{dt} \right)(t) = \lim_{\hbar \to 0} \left( \frac{1}{i\hbar} [\hat{\mathcal{O}}(t), \hat{H}(t)] \right), \quad \hat{H}(t) = -i\hbar \hat{U}^{-1} \frac{d\hat{U}}{dt}.$$

If we assume that the classical limit commutes with the time derivative we find

$$\{\lim_{\hbar \to 0} (\hat{\mathcal{O}}), \mathcal{O}_{H(t)}\}_{pb} = \lim_{\hbar \to 0} \left( \frac{1}{i\hbar} [\hat{\mathcal{O}}, \hat{H}(t)] \right). \quad (11.4)$$

Now write $\hat{H}(t) = \hat{H}'(t) + \hat{h}(t)$ where $\lim_{\hbar \to 0}(\hat{H}'(t)) = \mathcal{O}_{H(t)}$. If Assumption \[ holds then $\lim_{\hbar \to 0}(\frac{1}{i\hbar}[\hat{\mathcal{O}}(t), \hat{H}'(t)]) = \{\lim_{\hbar \to 0}(\hat{\mathcal{O}}(t)), \mathcal{O}_{H(t)}\}_{pb}$ and we have

$$\lim_{\hbar \to 0} \left( \frac{1}{i\hbar} [\hat{\mathcal{O}}(t), \hat{h}(t)] \right) = 0. \quad (11.5)$$

If the Hamiltonian is not in $\hat{A}_{\text{diff}}$, then $\lim_{\hbar \to 0}(\hat{h}(t))$ is not a differentiable function. In contradiction to (11.3), it seems likely that if $\lim_{\hbar \to 0}(\hat{h}(t))$ is not a differentiable function, and $\lim_{\hbar \to 0}(\frac{1}{i\hbar}[\hat{f}, \hat{g}]) = \{\lim_{\hbar \to 0}(\hat{f}), \lim_{\hbar \to 0}(\hat{g})\}_{pb}$ when $\lim_{\hbar \to 0}(\hat{f})$ and $\lim_{\hbar \to 0}(\hat{g})$ are differentiable (Assumption \[), then $\lim_{\hbar \to 0}(\frac{1}{i\hbar}[\hat{\mathcal{O}}(t), \hat{h}(t)])$ would be something like the Poisson bracket with a non-differentiable function, or worse.

### 12 Unitary Transformations Over Canonical Transformations

There is another way to look at quantization after evolution which focuses less on observables and more on unitary transformations. In this respect it is better tied to the original motivation of Section \[. To explain this picture, we first define a bundle of unitary transformations over canonical transformations. By a “bundle” is meant a triple, $(Y, X, \rho)$, consisting of two two topological spaces $Y$ and $X$ and a continuous surjective mapping $\rho : Y \to X$ \[.

Let $U$ be the connected component of the identity of the space of canonical transformations of solution space. Let $\hat{U}$ be the space of unitary transformations of Hilbert space such that for $\hat{u}$ in $\hat{U}$ there exists $u$ in $U$ with $\lim_{\hbar \to 0}(\hat{u}(\hat{f})) = u(\lim_{\hbar \to 0}(\hat{f}))$ for all differentiable-\hbar-sequences $\hat{f}$.

We wish to define topologies and differential structures on $U$ and $\hat{U}$ consistent with the topologies and differential structures assumed to exist on the space of differentiable functions on solution space $A_{\text{diff}}$ and the space of differentiable-\hbar-sequences on Hilbert space $\hat{A}_{\text{diff}}$, respectively. Let $V$ and $\hat{V}$ be arbitrary open sets in $A_{\text{diff}}$ and $\hat{A}_{\text{diff}}$, respectively. Let $\hat{f}$ and $f$ be arbitrary elements of $\hat{A}_{\text{diff}}$ and $A_{\text{diff}}$, respectively. Define an open set in $U$ labeled by $f$ and $V$ to be

$$U_{f,V} = \{ u \in U | (u(f) \in V) \}. \quad (12.1)$$

Define an open set in $\hat{U}$ labeled by $\hat{f}$ and $\hat{V}$ to be

$$\hat{U}_{\hat{f},\hat{V}} = \{ \hat{u} \in \hat{U} | (\hat{u}(\hat{f}) \in \hat{V}) \}. \quad (12.2)$$
For all \( f, \hat{f}, V, \hat{V} \) these open sets generate topologies on \( U \) and \( \hat{U} \). These topologies ensure that:

1. Given any \( \hat{f} \), a continuous path in \( \hat{U} \) maps to a continuous path in \( \hat{R}_{\text{diff}} \) by its action on \( \hat{f} \).

2. Given any \( f \), a continuous path in \( U \) maps to a continuous path in \( R_{\text{diff}} \) by its action on \( f \).

Define the differentiable structures on \( \hat{U} \) and \( U \) similarly. That is \( C^1 \) paths map to \( C^1 \) paths.

To complete the definition of the bundle, \( (\hat{U}, U, \rho) \), define the obvious mapping

\[
\rho : \hat{U} \to U \text{ by } \hat{u} \mapsto u \text{ if } \lim_{h \to 0} \hat{u}(\hat{f}) = u(\lim_{h \to 0} \hat{f})).
\tag{12.3}
\]

The inverse images \( \rho^{-1}(u) \) for \( u \) in \( U \) are diffeomorphic to each other, since they are related by unitary transformations. I shall call them fibers. Points within a fiber are related by unitary transformations which lie in the fiber over the identity. These are the only unitary transformations enacted by unitary operators which can converge under \( \lim_{h \to 0} \) because if \( \hat{U}_n \) is a convergent unitary operator then by the product law of Assumption 2 (Section 8),

\[
\lim_{h \to 0} (\hat{U}_n \hat{f} \hat{U}_n^+) = \lim_{h \to 0} (\hat{f}) \lim_{h \to 0} (\hat{U}_n \hat{U}_n^+) = \hat{f}.
\]

### 13 Quantizing Canonical Transformations

The alternative picture of quantization after evolution we have in mind may be described as quantizing canonical transformations [17]. If the bundle, \( (\hat{U}, U, \rho) \), has a section, then the symmetries of the classical theory can be reproduced exactly in the quantum theory. That is the canonical transformations can be lifted continuously to unitary transformations with the correct classical limit. It is tempting to conclude from Van Hove’s theorem that this cannot be done. However this is not clear, we are not requiring that unitary transformations be generated by observables (see Section 4). Regardless, in quantization after evolution we are only interested in the canonical transformations which generate time evolution.

In order to limit consideration to canonical transformations which generate time evolution, define a mapping of the space of global time \( \Gamma \) into the base space \( U \) of canonical transformations. Fix an initial time \( \tau_0 \), and map it to the identity transformation. Then associated to every other time \( \tau \) is a canonical transformation defined by classical evolution.
from \( \tau_0 \) to \( \tau \). This defines the mapping \( T_\Gamma \) and gives rise to the following diagram:

\[
\begin{array}{ccc}
\hat{U} & \downarrow \rho \\
\Gamma & \rightarrow & U \\
\end{array}
\] (13.1)

Together with Definition \( \| \) (Section \( \| \)), the topology and differential structure on \( U \) ensure that \( T_\Gamma \) is differentiable.

The pullback bundle over \( \Gamma \), \((P, \Gamma, \lambda)\), is defined as

\[ P = \{ (\tau, \hat{u}) \in \Gamma \times \hat{U} \mid T_\Gamma(\tau) = \rho(\hat{u}) \} \]
\[ \lambda: P \rightarrow \Gamma, \]
\[ (\tau, \hat{u}) \mapsto \tau. \] (13.2)

A section in the pullback bundle over \( \Gamma \) is a continuous mapping of the space of global time to unitary transformations which has the correct classical limit in the sense of Section \( 12 \). That is, it is a commutative diagram

\[
\begin{array}{ccc}
\hat{U} & \downarrow \rho \\
\Gamma & \rightarrow & U \\
\end{array}
\] (13.3)

Now assume that a mapping of the solution space observables corresponding to time \( \tau_0 \), to operators, which satisfies conditions 1 and 2 in the definition of quantization after evolution (Definition \( 5 \), Section \( 10 \)) has been chosen. By Lemma \( 6 \) (Section \( 8 \)) this may always be done. The remaining freedom in choice of a quantization after evolution is the choice of a \( C^1 \) section in the pullback bundle over \( \Gamma \). The existence of a \( C^1 \) section is equivalent to the existence of a quantization after evolution.

**Part III**

**Consistent Evolution With Different Time-Slicings**

It will serve as a good introduction to sketch the most obvious approach to showing that consistent evolution may be achieved, although in the end we will not be able follow through

\[1\] In gravity, simply take any time-slicing connecting the “initial” slice \( \sigma_0 \) and the slice \( \sigma \). (In gravity the space of global time is the space of slices. The label \( \sigma \) is used to distinguish this particular case from the general case.) Since the classical theory is independent of time-slicing, the resulting canonical transformation is too. If \( \sigma_0 \) and \( \sigma \) are overlapping slices, evolve from \( \sigma_0 \) to \( \sigma' \), which does not overlap either, and then from \( \sigma' \) to \( \sigma_0 \). Actually this can be viewed as a piecewise differentiable choice of preferred time which is not a time-slicing. Clearly there is also a differentiable choice of preferred time connecting \( \sigma_0 \) and \( \sigma \).
with this approach, and will settle for a slightly weaker result. Consider a spacetime manifold of the form $M = \Sigma \times \mathbb{R}^+$, and take $\sigma_0 = \Sigma \times 0$. Assume we can show that each slice $\sigma$ can be connected to $\sigma_0$ by a particular time-slicing, such that the variation of the time-slicings with $\sigma$ is continuous. By Assumption 2 and Lemma 1 (both in Section 8) the classical Hamiltonians for these time-slicings can be continuously lifted to operator Hamiltonians with the right classical limit. Hence each slice $\sigma$ can be assigned a unitary transformation by evolving from $\sigma_0$ (which is assigned the identity transformation) to $\sigma$ using the Heisenberg equations of motion. Because the time-slicings and Hamiltonians vary continuously with $\sigma$, the variation of the unitary transformations with $\sigma$ will be continuous. Because the unitary transformations are derived from the Heisenberg equations of motion, they will have the right classical limit in the sense of Section 12. Hence we will have shown that a section of the pullback bundle over $\Gamma$ exists when $\Gamma$ is the space of slices (Diagram (13.3) with $\Gamma = S =$ \textit{space of slices}). If we further assume that the existence of a $C^0$ section implies the existence of a $C^1$ section, then we will have shown that a quantization after evolution of gravity exists.

The criterion that each slice $\sigma$ can be connected to $\sigma_0$ by a particular time-slicing, such that the variation of the time-slicings with $\sigma$ is continuous, will be called the time-slicing criterion. It is difficult to prove that the time-slicing criterion holds in general. Hence, I will not be able to show existence of consistent evolution in complete generality. As a substitute I will first offer a non-rigorous physical argument for the time-slicing criterion in (2+1)-dimensions. Then, feeling that this is not enough, I will derive a sufficient condition for the pullback bundle over $\Gamma$ (Diagram (13.3)) (when $\Gamma$ is an arbitrary space of global time) to have a section (13.3), and show that the condition is satisfied when $\Gamma$ is any subspace of the space of slices parametrized by a (finite or infinite) CW-complex. This result will essentially mean that if Assumption 2 holds, then we can have consistent evolution for at least an infinite parameter family of time-slicings. Finally I will show that any particular time-slicing, with evolution defined by any operator ordering of the associated Hamiltonian, may be included in such an infinite parameter family.

14 Physical Argument

Although we cannot prove that the time-slicing criterion holds—that is that each slice $\sigma$ can be connected to $\sigma_0$ by a particular time-slicing, such that the variation of the time-slicings with $\sigma$ is continuous—we can see the likelihood of this result from a simple physical model. Consider (2+1)-dimensional gravity with $\Sigma = \Sigma_g$, a genus $g$ surface. The question is mostly one of topology, not geometry. So instead of the $M^{2+1} = \Sigma_g \times \mathbb{R}^+$ spacetime, consider a thickened genus $g$ surface made of foam rubber. Further imagine that the piece of foam rubber has been constructed by gluing together thin genus $g$ shells of foam rubber (a genus $g$ onion). This foam rubber object is a “physical representation” of the topology of the region of $M^{2+1}$ between $\sigma$ and $\sigma_0$, with preferred time-slicing.

*The choice of such continuous family of time-slicings may be quite arbitrary.

†The last assumption can be described as an assumption of reasonable compatibility between the topology and differential structure of the space of operators on Hilbert space.
Note that we have exchanged the topology on the space of slices $S$ (which is related to the topology on $U$ via $T_{\Gamma}|_{\Gamma=S}$) for the topology placed on a layered piece of foam rubber by our intuition. Clearly this is not rigorous. However, the topology on $S$ is one in which the spatial metrics induced on the slices (points of $S$) by the spacetime metric change continuously. Locally the spacetime manifold has the topology of $\mathbb{R}^3$ so that the spacetime metric is locally a continuous function on $\mathbb{R}^3$. Since $\mathbb{R}^3$ is surely the topology of our intuition, our intuition should be applicable to judging what is meant by a continuous change in slice. At least it seems worthwhile to continue.

To see that the time-slicing changes continuously with $\sigma$, simply poke and prod at the outside of the foam rubber (that is deform $\sigma$) and imagine the deformation of the interior layered foam rubber structure (that is deformation of the time-slicing). Clearly for each poke and prod we get a unique distribution of the interior foam rubber, and when we remove our fingers, it springs back to its original distribution. This suggests that the time-slicing condition is satisfied. Unfortunately, we have little experience with 4-dimensional foam rubber.

15 Generalized Condition

We will now derive a sufficient condition for the pullback bundle over an arbitrary space of global time $\Gamma$ to have a section.

**Condition 1** Let $E$ be the space of pairs, of points $\tau$ in $\Gamma$, and paths $p$ in $U$ connecting $T_{\Gamma}(\tau)$ to $T_{\Gamma}(\tau_0) = I$ (see (13.1)):

$$E = \{ (\tau, p) \mid \tau \in \Gamma, \ p \text{ a } C^1 \text{ path in } U \text{ connecting } T_{\Gamma}(\tau) \text{ to } I \}.$$  

We will say Condition 1 is satisfied if the bundle, $(E, \Gamma, \sigma)$, defined by

$$\sigma : E \to \Gamma,$$

$$(\tau, p) \mapsto \tau,$$

has a section.

If the time-slicing criterion of Section 14 is satisfied, then Condition 1 is satisfied when $\Gamma$ is the space of slices $S$, because a time-slicing is a path in $S$ which is mapped to a path in $U$ by $T_{\Gamma}$.

**Claim 1** If the space of functions on solution space is paracompact, and if Condition 1, Assumption 2 (Section 8), and Assumption 3 (Section 11) hold, then the pullback bundle over $\Gamma$ has a section.

The argument for Claim 1 is analogous to the argument for the time-slicing criterion sketched in the introduction to Part III. In view of Lemma 1 (Section 8), fix a particular
continuous lifting of the differentiable real valued functions on solution space to Hermitian operators:

\[ L : \mathcal{R}_{diff} \to \hat{\mathcal{R}}_{diff}, \quad \lim_{\hbar \to 0} L = I. \quad (15.1) \]

Then consider a \( C^1 \) path in \( U \) starting at \( T_\Gamma(\tau_0) = I \) and ending at the image of another point in \( \Gamma \), \( T_\Gamma(\tau) \):

\[ p : [0, 1] \to U, \quad p(0) = I, \quad p(1) = T_\Gamma(\tau), \quad \tau \in \Gamma. \quad (15.2) \]

Together, the mappings \( L \) and \( p \) define a lifting of \( \tau \) in the pullback bundle over \( \Gamma \) as follows.

To each point \( t \in [0, 1] \) on the path in \( U \) is associated a Hamiltonian \( H_{pt} \). Use the lifting \( L \) to map it to a Hermitian operator:

\[ \hat{H}_p(t) = L(H_{pt}), \quad \lim_{\hbar \to 0} \hat{H}_p(t) = H_{pt}. \quad (15.3) \]

The lifting of \( \tau \) is then defined by evolution under the Heisenberg equations of motion. (Here Assumption 3 is used.) That is \( \tau \) is lifted to the unitary transformation defined by conjugation with the unitary operator

\[ \hat{U}_p = Te^{\frac{i}{\hbar} \int_0^1 \hat{H}_p(t')dt'}, \quad (15.4) \]

where \( T \) denotes the time ordered product.

In general this construction is not enough to define a section. However, it is enough if to each point \( \tau \) in \( \Gamma \) we can associate a particular path in \( U \) connecting \( T_\Gamma(\tau) \) to \( T_\Gamma(\tau_0) = I \), such that the paths vary continuously with \( \tau \). In this case we obtain a section by assigning to each point \( \tau \) in \( \Gamma \) the unitary transformation corresponding to the particular path \( p_\tau \) which ends at \( T_\Gamma(\tau) \). (Presumably \( p_{\tau_0} \) will be the constant path so that \( \tau_0 \) lifts to the identity.):

\[ s : \Gamma \to \{ (\tau, \hat{u}) \in \Gamma \times \hat{U} \mid T_\Gamma(\tau) = \rho(\hat{u}) \}
\]

\[ \tau \mapsto (\tau, \hat{u}_\tau), \]

\[ \hat{u}_\tau = \text{unitary transformation implemented by conjugation with} \]

\[ \hat{U}_{p_\tau} = Te^{\frac{i}{\hbar} \int_0^1 \hat{H}_{p_\tau(t')}dt'}. \quad (15.5) \]

Because the paths vary continuously with \( \tau \), \( \hat{H}_{p_\tau}(t) \) varies continuously with \( \tau \), and therefore \( \hat{U}_{p_\tau} \) varies continuously with \( \tau \). Hence (15.3) defines a section. The condition that to each point \( \tau \) in the base space \( \Gamma \), we can associate a particular \( C^1 \) path in \( U \) connecting \( T_\Gamma(\tau) \) to \( I \), such that the paths vary continuously with \( \tau \) is Condition 1.

We now make the assumption that the existence of a \( C^0 \) section implies the existence of a \( C^1 \) section. That is we assume some reasonable compatibility between the topology and differential structure of the space of operators on Hilbert space. Then, if the provisions of Claim 3 hold, a quantization after evolution can be carried out for a theory with global time \( \Gamma \). In the next section we will show that Condition 1 holds when \( \Gamma \) is a subset of the space of slices parametrized by a CW-complex, and the spacetime manifold is of the form \( M = \Sigma \times \mathbb{R}^+ \). The result will be interpreted in Section 17 and extended in Section 18.
16 Consistent Evolution on a CW-Complex of Global Times

To obtain definite results, consideration will now be limited to a subspace of the space of slices parametrized by a CW-complex. Consider a spacetime manifold of the form \( M = \Sigma \times \mathbb{R}^+ \), and take \( \sigma_0 = \Sigma \times 0 \). Let \( T_2 \) be the mapping of the space of slices \( S \) into \( U \) determined as in Section 13 with \( \sigma_0 \) mapped to the identity. Let \( T_1 \) be a mapping of a CW-complex \( \Gamma_C \) to \( S \), which is a homeomorphism to its image, such that there is a point \( \tau_0 \) which maps to \( \sigma_0 \). The diagram,

\[
\begin{array}{ccc}
\hat{U} & \downarrow \rho \\
\Gamma_C & \rightarrow & S \\
T_1 & \rightarrow & T_2 \\
U \\
\end{array}
\]

leads to the pullback bundle over \( \Gamma_C \), \((P_C, \Gamma_C, \lambda_C)\), defined by

\[
P_C = \{(\gamma, \hat{u}) \in \Gamma_C \times \hat{U} \mid T_2 \circ T_1(\Gamma_C) = \rho(\hat{u})\}
\]

\[
\lambda_C: P_C \rightarrow \Gamma_C
\]

\[
(\gamma, \hat{u}) \mapsto \gamma.
\] (16.1)

Of course the bundle (16.1) is the bundle (13.2) with \( \Gamma \) set equal to \( \Gamma_C \), a subspace of \( S \) parametrized by a CW-complex. A section of (16.1) provides a quantization after evolution on the space of global time \( \Gamma_C \). We will show that Condition 1 (Section 15) holds for (16.1). Therefore, by Claim 1, a section of (16.1) exists if Assumption 2, etc., is correct.

Claim 2 Let \( E \) be the space of pairs of points \( \tau \) in the CW-complex \( \Gamma_C \), and paths in \( U \) connecting \( T_2 \circ T_1(\tau) \) to \( T_2 \circ T_1(\tau_0) = I 
\):

\[
E = \{ (\tau, p) \mid \tau \in \Gamma_C, \ p \ a \ C^1 \ path \ in \ U \ connecting \ T_2 \circ T_1(\tau) \ to \ I \}.
\]

Then, if Assumption 2 (Section 14) holds, the bundle \((E, \Gamma_C, \sigma)\) defined by

\[
\sigma: E \rightarrow \Gamma_C,
\]

\[
(\tau, p) \mapsto \tau,
\]

has a section.

To argue for Claim 2 consider the smaller bundle obtained by restricting \( E \) to paths that are the images of time-slicings (A time-slicing is a path in \( S \), but not all paths in \( S \) are time-slicings.):

\[
E' = \{ (\tau, p) \mid \tau \in \Gamma_C, \ p \ a \ time \ slicing \ connecting \ T_1(\tau) \ to \ T_1(\tau_0) \},
\]

\[
\sigma': E' \rightarrow \Gamma_C,
\]

\[
(\tau, p) \mapsto \tau.
\] (16.2)
Of course if \((E', \Gamma_C, \sigma')\) has a section then so does \((E, \Gamma_C, \sigma)\). We will use a theorem due to Gromov \cite{18} to show that the fiber \(\sigma'^{-1}(\gamma), \gamma \in \Gamma_C\), is weak homotopy equivalent to a point. In particular, all the homotopy groups are trivial. Using arguments from obstruction theory \cite{19}, this fact, together with the fact that \(\Gamma_C\) is a CW-complex, can be used to show that a section exists.

Gromov’s theorem is quite general. The basic result we will need is that the space of spacelike \((n-1)\)-plane fields on an \(n\)-dimensional non-closed Lorentzian manifold is weak homotopy equivalent to the space of integrable spacelike \((n-1)\)-plane fields. A \(p\)-plane field is an assignment of a \(p\)-dimensional subspace of the tangent space to each point in a manifold. It can also be described as a section in the fiber bundle associated with the tangent bundle, with fibers the space of \(p\)-dimensional subspaces of the tangent space. An integrable spacelike \((n-1)\)-dimensional plane field on an \(n\)-dimensional Lorentzian manifold is a time-slicing\footnote{Actually it is a foliation with spacelike leaves. By Assumption \[\text{there is a homeomorphism between the space of such foliations and the space of time-slicings. Here we fix a spacetime and use this homeomorphism to to obtain topological statements about time-slicings.}\].}.

If we take as the non-closed manifold the piece of the spacetime manifold between \(T_1(\tau_0)\) and \(T_1(\tau)\) together with \(T_1(\tau_0)\) and \(T_1(\tau)\) (a manifold with boundary), call it \(X\), then a time-slicing necessarily includes \(T_1(\tau_0)\) and \(T_1(\tau)\) as slices. The space of time-slicings of \(X\) is the fiber of \((E', \Gamma_C, \sigma')\) over \(\tau\). Therefore, if we can show that the space of spacelike \((n-1)\)-plane fields on \(X\) is contractible, then by Gromov’s theorem the homotopy groups of the fibers of \((E', \Gamma_C, \sigma')\) are all trivial.

Let \((Y, X, \alpha)\) be the fiber bundle associated with the tangent bundle of \(X\) with fibers the space of spacelike \((n-1)\)-dimensional subspaces of the tangent space. As stated earlier, the space of sections of \((Y, X, \alpha)\) is the space of spacelike \((n-1)\)-plane fields. First we will show that if the fibers of \((Y, X, \alpha)\) are convex domains, then the space of sections of \((Y, X, \alpha)\) is a convex domain. Let \(s_1 : X \rightarrow Y\) and \(s_2 : X \rightarrow Y\) be two sections of \((Y, X, \alpha)\). Then for \(x\) in

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fibers}
\caption{The fibers of \((Y, X, \alpha)\) are convex domains}
\end{figure}
$X$, $s_1(x)$ and $s_2(x)$ are in $\alpha^{-1}(x)$, and

$$(\lambda s_1 + (1 - \lambda)s_2)(x) = \lambda s_1(x) + (1 - \lambda)s_2(x)$$

is in $\alpha^{-1}(x)$ if $\alpha^{-1}(x)$ is convex. Therefore $\lambda s_1 + (1 - \lambda)s_2$ is a section and the space of sections is convex if the fibers of $(Y, X, \alpha)$ are convex.

We must show that each fiber of $(Y, X, \alpha)$—that is the space of $(n - 1)$-dimensional spacelike subspaces of the tangent space—is a convex domain. Each tangent space of $X$ acquires a Minkowski metric from the Lorentzian metric on $X$. By choosing a coordinate system in which the metric is diag $(-1, 1, 1, \ldots)$, the spacelike subspaces are put in one to one correspondence with the time-like vectors of the form $(1, \vec{x})$ (see Figure 5). The time-like vectors of the form $(1, \vec{x})$ are in one to one correspondence with the vectors in $\mathbb{R}^{n-1}$ of length less then one. (The former are a graph over the latter.) The space of vectors in $\mathbb{R}^{n-1}$ of length less then one is clearly a convex domain. Hence each fiber of $(Y, X, \alpha)$ acquires the structure of a convex domain.

Following the above line of reasoning, we conclude that the homotopy groups of the fibers of $(E', \Gamma_C, \sigma')$ are all trivial. This means that for all $m$, any mapping of the sphere $S^m$ into a fiber of $(E', \Gamma_C, \sigma')$ is extendible to a mapping of the ball $D^{m+1}$ into the fiber. Begin constructing a section of $(E', \Gamma_C, \sigma')$ by choosing arbitrary liftings of the zero cells of $\Gamma_C$ to $E'$ (see Figure 6). The pullback bundle over a single cell is necessarily a trivial fiber bundle, because a cell is contractible and all the fibers are homeomorphic. Therefore, because any mapping of $S^0$ (two points) to a fiber may be extended to a mapping of $D^1$ to the fiber, the lifting of the zero cells may be extended to a lifting of the 1-cells. The argument repeats for the 2-cells, 3-cells, \ldots, on up to $\infty$. There is not even a requirement that the number of cells be countable.

We have shown that a section of $(E', \Gamma_C, \sigma')$, and hence of $(E, \Gamma_C, \sigma)$, exists for an arbitrary CW-complex $\Gamma_C$. Therefore, Condition $\mathbb{H}$ is satisfied. Provided Assumption $\mathbb{E}$ holds, etc., we may have consistent evolution on any subspace of the space of global time parametrized by a (finite or infinite) CW-complex.
17 Interpretation of a CW-Complex of Global Times

How much of $S$ may be parametrized by a CW-complex? That is, how much does this result say about the general problem of consistent evolution? Let us consider the question: given a particular time-slicing connecting $\sigma_0$ to $\sigma$, how many other time-slicings connecting $\sigma_0$ and $\sigma$ can define the same evolution?

A particular time-slicing connecting $\sigma_0$ to $\sigma$ is a family of slices (global times) parametrized by the closed interval $[0, 1]$. A family of deformations of the time-slicing, parametrized by another closed interval, is a family of slices parametrized by a closed disc. A family of deformations of the time-slicing parametrized by $[0, 1] \times [0, 1]$, is a family of slices parametrized by a closed 3-ball, etc. Clearly we may have an infinite parameter family of deformations of the time-slicing with slices parametrized by a CW-complex. If there is any particular time-slicing we wish to include, we may always include it§. By Claims 1 and 2 all these time-slicings may be brought into agreement (if Assumption 2, etc., holds). Furthermore, if the time-slicings include common intermediate slices, they may be brought into agreement on those slices as well.

The situation is probably something like trying to cover the real number line with a closed interval. You can always add in any additional points you like. But you can never cover the whole real number line. If we could show that the space of slices $S$ was homotopic to a CW-complex, then we could show that Condition 1 (Section 15) holds for $S$. There is a theorem due to John Milnor [20] which states roughly: “The space of mappings of a compact topological space to a CW-complex is homotopic to a CW-complex.” Well, the space $S$ is the space of spacelike mappings of $\Sigma$ to $M = \Sigma \times \mathbb{R}^+$. It seems likely that a proof of the general result may proceed along these lines.

18 Quantization by Preferred Time-Slicing

Arguably the most practical approach to making a physical prediction using quantum gravity is to reduce gravity to a Hamiltonian system using a convenient time-slicing connecting the initial and final slices. Assuming that Assumption 4 (Section 11) is correct, this reduces the problem to quantizing a Hamiltonian system with the attendant operator ordering ambiguities. Let us avoid the many conceptual issues by assuming that this at least would be sensible and correct if we used the right time-slicing and the right operator ordering. Then assuming that physics requires consistent evolution with respect to all time-slicings, there must be an operator ordering which is right for our “most convenient” time-slicing. Lacking experimental data, we will have to search for this ordering by invoking some esthetic principle or new physical theory. An interesting question is: is there any ordering choice which would preclude the possibility of consistent evolution with respect to some other time-slicing, and hence be excluded on this basis? At least for an infinite parameter family of time-slicings—more precisely a subspace of $S$ parametrized by a CW-complex—if the assumptions of this paper are correct the answer is no.

§The space of time-slicings is connected, because its 0-th homotopy group is trivial. Therefore we may reach any time-slicing by deforming the original time-slicing.
To see this let $\Gamma_C$ be as in Section 16, and let the initial and final slices be $\sigma_i$ and $\sigma_f$ respectively (they should be in the image of $T_1$). If $\sigma_i$ and $\sigma_f$ do not correspond to zero cells, further subdivide $\Gamma_C$ so that they do. There exists a time-slicing $T$ containing $\sigma_0$, $\sigma_i$, and $\sigma_f$ which coincides with the “most convenient” time-slicing between $\sigma_i$ and $\sigma_f$. Now construct a quantization after evolution of the system with global time $\Gamma_C$ by choosing a section of (16.2) and a lifting $L$ of the real valued functions on phase space to Hermitian operators. Following the argument for Claim 2 (Section 16) we find that the liftings of the zero cells of $\Gamma_C$ are arbitrary. Hence in choosing the section of (16.2), we can lift the zero cell corresponding to $\sigma_f$ to $T$, and the zero cell corresponding to $\sigma_i$ to the portion of $T$ between $\sigma_0$ and $\sigma_i$. If $H_T(t)$ ($t \in [0,1]$) is the classical Hamiltonian for the time-slicing $T$, then we now have a quantization after evolution where evolution from $\sigma_i$ to $\sigma_f$ is consistent with evolution along the most convenient time-slicing with the operator Hamiltonian $L(H_T(t))$. Given any operator valued continuous function $\hat{f}$ on $S$, with $\lim_{h \to 0} \hat{f}(\sigma) = 0$ for all $\sigma$, we can always redefine $L$ by

$$L(H(t)) \to L'(H(t)) = L(H(t)) + \hat{f}(\sigma(t)).$$

Hence, $L(H_T(t))$ can have any operator ordering whatsoever.

19 Conclusion

The relation of this work to gravity is based on Assumption 1 (Section 6), essentially that gravity may be reduced to an unconstrained system with the “space of slices” playing the role of time. If this assumption fails, then this work may be applicable to some other theory, but not gravity. If Assumption 2 (Section 8) fails, that is if it is not locally possible to continuously lift the real valued functions on phase space to Hermitian operators, then the result of this work is only to place further doubt on the possibility for consistent evolution. There are also a few seemingly innocuous assumptions: paracompactness, deformability of continuous mappings to smooth ones, and Assumption 3 (Section 11). To avoid repeating them over and over again, let us assume that all the above assumptions are correct for the remainder of this discussion.

This said, to interpret our results, it must be known if quantization after evolution really is equivalent to Heisenberg equation evolution. (Here we mean Assumption 4 in Section 11.) If it is, then the interpretation is that it is possible to operator order the myriad of Hamiltonians, corresponding to an infinite parameter family of time-slicings (at least), so that evolution is consistent. However, time dependent operator orderings of the Hamiltonians may be required. In addition any particular time-slicing, with evolution defined by any operator ordering of the associated Hamiltonian, may be included in such a consistent infinite parameter family.

If, however, quantization after evolution is not equivalent to Heisenberg equation evolution, that is if there is some exotic form of differentiable unitary evolution with the correct

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If $\hat{f}$ is not constant then $L$ will become slice-dependent; but this causes no problem.
classical limit, but which is generated by a Hamiltonian with the wrong or possibly no classical limit, then the interpretation depends on whether the Hamiltonians should be considered observables. This, I suppose, depends on whether or not the Hamiltonians can be measured. Observables which are defined on slices are fine. They evolve consistently and have the right classical limit. But the Hamiltonians are defined not on slices, but on local time-slicings. If this makes them unmeasurable, then quantization after evolution is still a reasonable physical theory. However, if Hamiltonians in geometrodynamics are measurable, then quantization after evolution must be equivalent to Heisenberg equation evolution if it is to have the right classical limit.

Of course one should at least ask whether or not evolution along different time-slicings should agree. In the past we have found that the existence of a classical symmetry, that is the fact that the classical theory transformed under a trivial representation of some transformation group, did not always mean that the quantum theory transformed under a trivial representation. The classic example is the electron and its transformation under rotations. Electron spin is a purely quantum mechanical effect and possibly some phenomena involving different time-slicings also exists. However, this would mean that the unified picture of space and time, so treasured by relativists, would not hold in the quantum theory. What this work has attempted to show is that such a result is not forced on us.

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References

[1] K. Kuchař, “Time and Interpretations of Quantum Gravity,” in 4th Canadian Conference on General Relativity and Relativistic Astrophysics (World Scientific, Singapore, 1992)

[2] C.J. Isham, “Canonical Quantum Gravity and the Problem of Time” in “Integrable Systems, Quantum Groups, and Quantum Field Theories”, pp157–288, edited by L. A. Ibort and M.A. Rodriguez (Kluwer Academic Publishers, London, 1993).

[3] V. Moncrief, J. Math. Phys. 30 2907 (1989).

[4] A. Hosoya and K. Nakao, Class. Quant. Grav. 7 163 (1990).

[5] C. Rovelli, Phys. Rev. D42 2638 (1990); Phys. Rev. D43 442 (1991).

[6] A. Ashtekar and A. Magnon, Gen. Relativ. Gravit. 12, 205 (1980).

[7] C. Crnkovic and E. Witten, in Three Hundred Years of Gravitation, edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1987).
[8] S. Carlip, *Phys. Rev.* **D42** 2647 (1990).

[9] S. Carlip and J.E. Nelson, *Phys. Rev.* **D51** 5643 (1995).

[10] E. Witten, *Nucl. Phys.* **B311**, 46 (1988); **B323**, 113 (1989).

[11] R. F. Werner, Osnabruck University preprint: RFW-95-CLS, e-Print Archive: quant-ph/9504016.

[12] F. A. Berezin and M.A. Shubin, “The Schrödinger Equation”, Mathematics and its Applications, (Kluwer Academic Publishers, 1991).

[13] P. R. Chernoff, Hadronic Journal **4**, 879 (1981).

[14] Guillemin and Sternberg, “Symplectic Techniques in Physics,” (Cambridge University Press, Cambridge, 1984).

[15] Y. Choquet-Bruhat and C. Dewitt-Morette with M. Dillard-Bleick, “Analysis Manifolds and Physics Part I,” (North-Holland, 1982).

[16] For an introduction to homotopy theory see, for example, I. M. James, “General Topology and Homotopy Theory” (Springer-Verlag, New York, 1984); or R. A. Piccinini, “Lectures on Homotopy Theory”, Mathematics Studies **171**, (North-Holland, 1992).

[17] For a related approach see, A. Anderson, *Ann. Phys.* **232** 292 (1994).

[18] M. L. Gromov, Math. USSR-Izvestija, Vol.3, No.4, 671 (1969); and M. L. Gromov, “Partial Differential Relations”, (Springer-Verlag, Berlin, 1986).

[19] For an introduction to obstruction theory see for example E. H. Spanier, “Algebraic Topology” (McGraw-Hill, New York, 1966).

[20] J. Milnor, Trans. Amer. Math. Soc. **90** 272 (1959).