QUANTITATIVE ILLUMINATION OF CONVEX BODIES AND VERTEX DEGREES OF GEOMETRIC STEINER MINIMAL TREES

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ABSTRACT. In this note we prove two results on the quantitative illumination parameter \( f(d) \) of the unit ball of a \( d \)-dimensional normed space introduced by K. Bezdek (1992). The first is that \( f(d) = O(2^d d^2 \log d) \). The second involves Steiner minimal trees. Let \( v(d) \) be the maximum degree of a vertex, and \( s(d) \) of a Steiner point, in a Steiner minimal tree in a \( d \)-dimensional normed space, where both maxima are over all norms. F. Morgan (1992) conjectured that \( s(d) \leq 2^d \), and D. Cieslik (1990) conjectured \( v(d) \leq 2(2^d - 1) \). We prove that \( s(d) \leq v(d) \leq f(d) \) which, combined with the above estimate of \( f(d) \), improves the previously best known upper bound \( v(d) < 3^d \).

1. INTRODUCTION

Let \( K \) denote a convex body in the \( d \)-dimensional real vector space \( \mathbb{R}^d \). Denote its volume by \( \mu(K) \) and its translatively covering density by \( \theta(K) \). A (positive) homothet with ratio \( \lambda > 0 \) of \( K \) is any set of the form \( \lambda K + t \), with \( t \in \mathbb{R}^d \). The difference body of \( K \) is \( K - K \). According to the Rogers-Shephard inequality [RS57], \( \mu(K - K) / \mu(K) \leq (2^d/d) \). If \( K \) is centred (that is, \( K = -K \)), then of course \( \mu(K - K) / \mu(K) = 2^d \), and \( K \) defines a norm

\[
\|x\|_K := \inf\{\lambda > 0 : \lambda^{-1}x \in K\},
\]

which turns \( \mathbb{R}^d \) into a normed space. Let \( \mathcal{K}^d \) denote the class of all \( d \)-dimensional convex bodies, and \( \mathcal{K}_o^d \) the class of all centred \( d \)-dimensional convex bodies.

1.1. Quantitative illumination and covering. A point \( p \notin K \) illuminates a point \( q \) on the boundary of \( K \) if the ray

\[
\{\lambda p + (1 - \lambda)q : \lambda < 0\}
\]

intersects the interior of \( K \). A set of points \( P \subseteq \mathbb{R}^d \setminus K \) illuminates \( K \) if each boundary point of \( K \) is illuminated by some point in \( P \). Let \( L(K) \) be the smallest size of a set that illuminates \( K \). Also let \( L(d) := \max\{L(K) : K \in \mathcal{K}^d\} \), and \( L_o(d) := \max\{L(K) : K \in \mathcal{K}_o^d\} \). Since \( L(K) = 2^d \) if \( K \) is a cube, \( L(d) \geq L_o(d) \geq 2^d \). The well-known illumination problem is to show that \( L(d) = 2^d \). For large \( d \) the best known upper bounds are \( L(d) \leq \binom{2^d}{d} d(\log d + \log \log d + 5) \) and \( L_o(d) \leq 2^d d(\log d + \log \log d + 5) \), due to Rogers [Grü63, p. 284]; see also [RZ97].

There are other equivalent formulations of this illumination problem. For example, let \( L'(K) \) be the smallest number of positive homothets of
K, with each homothety ratio less than 1, whose union contains K. Then \( L(K) = L'(K) \). See [MS99] for a survey on this problem and its history.

We consider quantitative versions of the above two formulations of the illumination problem. The first was introduced by K. Bezdek [Bez92]. For \( K \in \mathcal{K}_o^d \) let
\[
B(K) := \inf \left\{ \sum_i \|p_i\|_K : \{p_i\} \text{ illuminates } K \right\}.
\]
This ensures that far-away light sources are penalised. Let
\[
B(d) := \sup \{ B(K) : K \in \mathcal{K}_o^d \}.
\]
Bezdek asked for the value of \( B(d) \), and in particular, if \( B(d) \) is finite for \( d \geq 3 \). He showed that \( B(2) = 6 \); the regular hexagon giving equality. Note that \( B(K) \geq L(K) \), hence \( B(d) \geq L_0(d) \geq 2^d \). It is also easily seen that \( B(K) = 2^d \) if \( K \) is a \( d \)-cube, and \( B(K) = 2d \) if \( K \) is a \( d \)-cross polytope.

We introduce the following quantitative covering parameter for \( K \in \mathcal{K}_o^d \):
\[
C(K) := \inf \left\{ \sum_i (1 - \lambda_i)^{-1} : K \subseteq \bigcup_i (\lambda_i K + t_i), 0 < \lambda_i < 1, t_i \in \mathbb{R}^d \right\}.
\]
In this way homothets almost as large as \( K \) are penalised.

**Proposition 1.** For any \( K \in \mathcal{K}_o^d \) we have \( B(K) \leq 2C(K) \).

Let
\[
C(d) := \sup \{ C(K) : K \in \mathcal{K}_o^d \},
\]
and
\[
C_o(d) := \sup \{ C(K) : K \in \mathcal{K}_o^d \}.
\]
Hence \( C(d) \geq C_o(d) \geq B(d)/2 \). It is easy to see that \( C(K) = 2^{d+1} \) if \( K \) is a \( d \)-cube, hence \( C(d) \geq C_o(d) \geq 2^{d+1} \). As before, it is not clear whether \( C(d) \) is finite. Levi [Lev54] showed that any planar convex body can be covered with 7 homothets, each with homothety ratio 1/2; hence \( C(2) \leq 14 \). Lassak’s result [Las86] that any planar convex body can be covered with 4 homothets, each with ratio 1/\( \sqrt{2} \), improves this to \( C(2) \leq 8 + 4\sqrt{2} \). Lassak [Las93] also showed that any convex body in \( \mathbb{R}^3 \) can be covered with 28 homothets, each with ratio 7/8; hence \( C(3) \leq 224 \). We show that a result of Rogers and Zong [RZ97] implies the following upper bound.

**Theorem 1.** For any \( d \)-dimensional convex body \( K \) we have
\[
C(K) < e(d + 1) \frac{\mu(K - K)}{\mu(K)} \theta(K).
\]
Using Rogers’ estimate [Rog57] \( \theta(K) \leq d(\log d + \log \log d + 5) \) for \( d \geq 2 \) and the Rogers-Shephard inequality one finds
\[
C(d) < \binom{2d}{d} e(d + 1)d(\log d + \log \log d + 5) = O(4^d d^{3/2} \log d),
\]
and
\[
B(d) \leq 2C_o(d) < 2^{d+1} e(d + 1)d(\log d + \log \log d + 5) = O(2^d d^2 \log d).
\]
Perhaps \( C(d) = O(2^d) \).
1.2. Steiner minimal trees. Given a finite set of points $V$ in $\mathbb{R}^d$, a Steiner tree $T$ of $V$ is any tree in $\mathbb{R}^d$ whose vertex set contains $V$, and whose edges are straight-line segments in $\mathbb{R}^d$. The vertices of $T$ not in $V$ are called Steiner points. (Usually Steiner points are required to have degree at least 3, but this is unnecessary here.) The $K$-length of a Steiner tree is the total length in $\|\cdot\|_K$ of the edges of the tree, where $K$ is a centred convex body. It is easily seen \cite{Coc67} that any given point set has a Steiner tree of smallest $K$-length, called a $K$-Steiner minimal tree ($K$-SMT).

Steiner minimal trees have been studied mostly in the Euclidean plane and the rectilinear plane ($K$ a parallelogram) \cite{HKW92}. Other normed planes have also been considered; see \cite{Bra01} \S 3.1 for further references. Steiner minimal trees in normed spaces of higher dimension have been investigated by Cieslik \cite{Cie98} and Morgan \cite{Mor92} among others.

Let $v(K)$ be the maximum possible degree of a vertex in a $K$-SMT, and $s(K)$ the maximum possible degree of a Steiner point in a $K$-SMT. Clearly $s(K) \leq v(K)$. The following table gives some examples of known values of $s(K)$ and $v(K)$. See \cite{Swa99, Swa00, BTW00} for further examples.

| $K$               | $s(K)$ | $v(K)$ |
|-------------------|--------|--------|
| Euclidean $d$-ball| 3      | 3      |
| $d$-cube         | $2^d$  | $2^d$  |
| $d$-cross polytope| $2d$   | $2d$   |
| regular hexagon  | 4      | 6      |

Let $s(d) := \max\{s(K) : K \in \mathbb{K}_d^d\}$, and $v(d) := \max\{v(K) : K \in \mathbb{K}_d^d\}$. Then $2^d \leq s(d) \leq v(d)$. The following two conjectures have been made:

Conjecture 1 \cite{Cie90, Cie98 ch. 4]. $v(d) \leq 2(2^d - 1)$ for all $d \geq 2$.

Conjecture 2 \cite{Mor92, Mor98 ch. 10]. $s(d) \leq 2^d$ for all $d \geq 2$.

Cieslik \cite{Cie90} has shown that $v(K) \leq H(K)$ where $H(K)$ is the transative kissing number of $K$. See \cite{Zor98} for a survey and for references to the following upper bounds on $H(K)$. Since $H(K) \leq 3^d - 1$ with equality only for (affine images of) the $d$-cube, it follows that $v(d) \leq 3^d - 2$. Since for planar $K$ we have $H(K) \leq 6$ if $K$ is not a parallelogram, we obtain $v(2) = 6$ \cite{Cie90}; thus Conjecture 1 is true for $d = 2$. Conjecture 2 is also true for $d = 2$ \cite{Swa00}. The two-dimensional methods are very special and offer no hope for generalisation to higher dimensions. We find upper bounds within a factor of $O(d^2 \log d)$ from the conjectured values, using the following relationship with Bezdek’s illumination parameter.

Theorem 2. For any $K \in \mathbb{K}_d^d$ we have $v(K) \leq B(K)$.

Note that equality holds, for example, if $K$ is a regular hexagon, a $d$-cube or a $d$-cross polytope, but not if $K$ is a $d$-ball.

Corollary 1. For any $K \in \mathbb{K}_d^d$ we have $s(K) \leq v(K) < 2^{d+1}v(d + 1)\theta(K)$.

Corollary 2. $s(d) \leq v(d) = O(2^dd^2 \log d)$. 
2. PROOFS

**Proof of Proposition**[1] Let \( \{ \lambda_i K + t_i \} \) be a finite covering of \( K \), with \( 0 < \lambda_i < 1 \) for all \( i \). Let \( \varepsilon > 0 \) be sufficiently small such that all \( \lambda_i + \varepsilon < 1 \). If a boundary point \( q \) of \( K \) is covered by \( \lambda_i K + t_i \), then \( 1 - \lambda_i \leq \| t_i \|_K \leq 1 + \lambda_i < 2 \), and the centre of the homothety mapping \( K \) to \( (\lambda_i + \varepsilon) K + t_i \), namely \( p_i := (1 - \lambda_i - \varepsilon)^{-1} t_i \), is outside \( K \) and illuminates \( q \). Therefore, the set \( \{ p_i \} \) illuminates \( K \), and \( \sum \| p_i \|_K < \sum 2/(1 - \lambda_i - \varepsilon) \). Since \( \varepsilon > 0 \) can be made arbitrarily small, \( \sum \| p_i \|_K \leq 2 \sum (1 - \lambda_i)^{-1} \). □

**Proof of Theorem**[1] It is known [RZ97] that for any \( 0 < \lambda < 1 \) there exists a covering of \( K \) by homothets \( \{ \lambda K + t_i : i = 1, \ldots, N \} \), with

\[
N \leq \frac{\mu(K - \lambda K)}{\mu(K)} \theta(K) < \lambda^{-d} \frac{\mu(K - K)}{\mu(K)} \theta(K).
\]

Choosing \( \lambda = d/(d + 1) \) we obtain

\[
\sum_{i=1}^{N} (1 - \lambda)^{-1} < (d + 1) \left( 1 + \frac{1}{d} \right)^d \frac{\mu(K - K)}{\mu(K)} \theta(K) < (d + 1) e \frac{\mu(K - K)}{\mu(K)} \theta(K).
\]

□

**Lemma 1.** If \( p \) illuminates the boundary point \( u \) of \( K \in K^d_0 \), then for all sufficiently small \( \varepsilon > 0 \), \( \| u - \varepsilon p \|_K < 1 - \varepsilon \).

**Proof.** The lemma is trivial if \( p = \lambda u \) for some \( \lambda \). Therefore, assume that \( p \) and \( u \) are linearly independent and consider the two-dimensional subspace spanned by them (Figure 1). Since \( p \) illuminates \( u \), we may choose \( \varepsilon_0 > 0 \) such that the line through \( o \) and \( u - \varepsilon_0 p \) intersects the line \( \ell \) through \( u \) and \( p \) in the interior of \( K \). Then clearly for all \( \varepsilon > 0 \) with \( \varepsilon < \varepsilon_0 \) the line through \( o \) and \( u - \varepsilon p \) still intersects \( \ell \) in the interior of \( K \). Let \( v = (\| u - \varepsilon p \|_K)^{-1} (u - \varepsilon p) \). Then the lines \( ou \) and \( op \) intersect in \( p' \), say, with \( \| p' \|_K < \| p \|_K \). Using similar triangles, \( \| u - \varepsilon p \|_K = 1 - \| \varepsilon p \|_K / \| p' \|_K < 1 - \varepsilon \). □

**Proof of Theorem**[2] Consider a vertex of a \( K \)-SMT of degree \( v(K) \). By translating we may assume that the vertex is the origin \( o \). By scaling we may also assume that each edge emanating from \( o \) has \( K \)-length at least 1. Let these edges be \( ov_i \), with \( \| v_i \|_K \geq 1 \). Let \( u_i = \| v_i \|_K^{-1} v_i \). Then the star \( T \) joining \( o \) to each \( u_i \) is a \( K \)-SMT of \( \{ o, u_1, u_2, \ldots, u_{v(K)} \} \) (otherwise we would be able to shorten the original tree).

Let \( \{ p_1, \ldots, p_k \} \) illuminate \( K \). For each \( j = 1, \ldots, k \), let

\[
U_j = \{ u_i : p_j \text{ illuminates } u_i \}.
\]
Then \( \{u_i\} = \bigcup_j U_j \). We estimate the number of points \( |U_j| \) in each \( U_j \). By Lemma 4, we may find \( \varepsilon > 0 \) such that \( \|u_i - \varepsilon p_j\|_K < 1 - \varepsilon \) for all \( i \). Consider the tree \( T' \) obtained from the star \( T \) by replacing, for each \( u_i \in U_j \), the edge from \( o \) to \( u_i \) by the edge from \( \varepsilon p_j \) to \( u_i \), and joining the Steiner point \( \varepsilon p_j \) to \( o \). Then \( T' \) is not shorter than \( T \). This implies that
\[
|U_j| = \sum_{u_i \in U_j} \|u_i\|_K \leq \|\varepsilon p_j\|_K + \sum_{u_i \in U_j} \|u_i - \varepsilon p_j\|_K
\]
\[
< \varepsilon \|p_j\|_K + (1 - \varepsilon)|U_j|,
\]
and \( |U_j| < \|p_j\|_K \). Hence \( v(K) \leq \sum_{j=1}^k |U_j| < \sum_{j=1}^k \|p_j\|_K \). Taking the infimum over all sets \( \{p_i\} \) that illuminate \( K \), we obtain that \( v(K) \leq B(K) \).  

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