BLOW-UP FOR PERIODIC NON-GAUGE INVARIANT NLS WITH NONCONSTANT INITIAL DATA

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Abstract. We study 1D NLS with non-gauge-invariant quadratic nonlinearity on the torus. The Cauchy problem admits trivial global solutions which are constant with respect to space. Moreover, the existence of blowup solutions has been studied by focusing on the behavior of the Fourier 0 mode of solutions. In this paper, the precise blow up criteria for the Cauchy problem is shown in a hat Lebesgue space by studying the interaction between the Fourier 0 mode and oscillation of solutions. Namely, solutions in the hat Lebesgue space are shown to blow up if they are different from the trivial ones.

1. Introduction

It is well known that nonlinear dispersive equations may have different qualitative behaviour as a result of the competition between non linear source terms and the dispersion of the solution. If \( u(t, x) \) is the complex valued solution of the nonlinear Schrödinger problem

\[
i\partial_t u + \Delta_x u = f(u),
\]

where \( t \geq 0 \) and \( x \) denote time and space variables, respectively, then typical nonlinear source terms are of type

\[
f(u) = |u|^p \quad (1.1)
\]

\[
f(u) = \pm |u|^{p-1}u. \quad (1.2)
\]

The strength of the dispersion is measured by the quantity

\[
\sup_{t > 1} t^\sigma \| u(t, x) \|_{L^\infty_x} < \infty, \sigma \geq 0. \quad (1.3)
\]

If the space variables are in \( \mathbb{R}^n \), then the linear Schrödinger equation has dispersion parameter \( \sigma = n/2 > 0 \).

If the space variables are in a compact manifold, then \( \sigma = 0 \) and there is a lack of dispersion of type \((1.3)\) with \( \sigma > 0 \).

In this work, we consider the simplest cases of \( x \in \mathbb{T} \) and nonlinearity \( f(u) = |u|^2 \) and therefore we study the Cauchy problem,

\[
\begin{align*}
  i\partial_t u + \Delta u &= |u|^2, \quad t \in [0, T), \ x \in \mathbb{T}, \\
  u(0, x) &= \phi(x), \quad x \in \mathbb{T}
\end{align*}
\]

(1.4)

for \( T > 0 \). Our goal is to provide a sharp condition of initial data for the non-existence of global solutions.
The Cauchy problem
\begin{equation}
\begin{cases}
    i \partial_t u + \Delta u = |u|^p, & t \in [0, T), \ x \in \mathbb{T}^n, \\
    u(0, x) = \phi(x), & x \in \mathbb{T}^n
\end{cases}
\end{equation}

with \( n \geq 1 \) and \( p > 1 \) is known to be locally well-posed with sufficiently smooth initial data and suitable power of nonlinear term. For example, for \( s > n/2 \) and \( p \in 2\mathbb{N} \), the local well-posedness of \((1.5)\) is shown in the \( H^s(\mathbb{T}^n) \) framework. Here the Sobolev space of order \( s \) on the torus \( \mathbb{T}^n \) is defined by
\[
H^s(\mathbb{T}^n) = \left\{ u \in L^2(\mathbb{T}^n); \sum_{k \in \mathbb{Z}^n} (1 + |k|^2)^s |\hat{u}(k)|^2 < \infty \right\},
\]
where \( \hat{u}(k) = (2\pi)^{-n} \int_{\mathbb{T}^n} e^{-ik \cdot x} u(x) dx \). There is also a large literature on the detail of local well-posedness of \((1.4)\). We refer the reader to \([1, 2, 3, 7]\) and references therein, for instance.

The non-existence of global solutions to \((1.5)\) has been studied under some condition of \( \hat{\phi}(0) \), the Fourier 0 mode of \( \phi \) or integral of \( \phi \). Especially, the positivity of the nonlinearity has played an important role. In \([12]\), for any \( p > 1 \), Oh showed the non-existence of the global weak solutions to \((1.5)\) if initial data \( \phi \) satisfies
\[
\text{Im} \int_{\mathbb{T}^n} \phi(x) dx < 0.
\]
Here we say \( u \in L^p_{\text{loc}}(0, \infty; L^p(\mathbb{T}^n)) \) is a global weak solution to \((1.5)\) if for any \( T > 0 \), \( u \) satisfies the weak form
\[
\int_0^T \int_{\mathbb{T}^n} u(t, x)(-i \partial_t \psi(t, x) + \Delta \psi(t, x)) dx \, dt
\]
\[
= i \int_{\mathbb{T}^n} \phi(x) \psi(0, x) dx + \int_0^T \int_{\mathbb{T}^n} |u(t, x)|^p \psi(t, x) dx \, dt
\]
for any \( \psi \in C([0, T] \times \mathbb{T}^n) \cap C^\infty([0, T] \times \mathbb{T}^n) \) with \( \psi(T, x) = 0 \) for any \( x \in \mathbb{T}^n \). The non-existence was shown by a so-called test function method, namely, it is shown that if \( \phi \) satisfies \((1.6)\), then the weak form \((1.7)\) cannot be satisfied with some \( T > 0 \) and \( \psi \). We remark that the test function method is a general method to show the non-existence of weak solutions with the nonlinearity of the type \( |u|^p \).

For the test function method and related topics, we refer the reader to \([8, 11, 12, 13]\) and references therein. Moreover, in \([3]\), the first author and Ozawa generalized the condition of Oh for smooth initial data.

**Proposition 1.1** ([3] Proposition 1.3). For any \( n \geq 1 \) and \( p > 1 \), if initial data \( \phi \in H^2(\mathbb{T}^n) \) satisfies
\[
\text{Im} \int_{\mathbb{T}^n} \phi(x) dx < 0 \quad \text{or} \quad \text{Re} \int_{\mathbb{T}^n} \phi(x) dx \neq 0,
\]
then there is no \( C^1([0, \infty); L^2(\mathbb{T}^n)) \cap C([0, \infty); (H^2 \cap L^p)(\mathbb{T}^n)) \) function satisfying \((1.5)\) on \([0, \infty)\) in the \( L^2 \) framework.

Proposition 1.1 may be shown by an ODE argument. Indeed, when \((1.8)\) holds, there is a complex number \( \alpha \) such that \( \text{Re}(\alpha) > 0 \) and
\[
M(t) := -(2\pi)^n \text{Im}(\alpha \hat{u}(t, 0)) = -\text{Im} \left( \alpha \int_{\mathbb{T}^n} u(t, x) dx \right)
\]
satisfies the estimate
\[ M(0) = -2\pi^n \text{Im}(\alpha \hat{\phi}(0)) = -\text{Im} \left( \alpha \int_{\mathbb{T}^n} \phi(x)dx \right) > 0 \]
and the ordinary differential inequality
\[ \dot{M}(t) \leq C \text{Re}(\alpha)M(t)^p \]  
with some \( C > 0 \). Then (1.9) implies that \( M \) is not bounded. By a similar argument, it is also seen that if \( \phi \) satisfies
\[ \int_{\mathbb{T}^n} \phi(x)dx = 0 \quad \text{and} \quad \phi \neq 0, \]  
there is no global solution to (1.5). For the completeness, this approach is revisited in the last section.

For the non-existence of global solutions, (LS) is the sharp condition of the \( \hat{\phi}(0) \). Indeed, it is easy to see that if \( \phi(x) \equiv i\mu_0 \) with \( \mu_0 \in \mathbb{R} \),
\[ u(t, x) = \begin{cases} i(t + \mu_0^{-1})^{-1} & \text{if } \mu_0 > 0, \\ 0 & \text{if } \mu_0 = 0, \\ -i(|\mu_0^{-1}| - t)^{-1} & \text{if } \mu_0 < 0 \quad \text{and} \quad t < |\mu_0|^{-1} \end{cases} \]
satisfies (1.14) for \( t > 0 \) both weakly and in the sense of \( L^2 \). We note that if \( \mu_0 \geq 0 \), then \( \phi = i\mu_0 \) does not satisfy the condition (LS) and \( u \) blows up at \( t = |\mu_0|^{-1} \).

Even though the sharp condition of \( \hat{\phi}(0) \) is known, as far as the authors know, the necessary and sufficient condition of \( \phi \) for the non-existence of global solutions was not known. Especially, in earlier works, only the behavior of the Fourier 0 mode has been focused but the contribution of the oscillation of solutions has not been considered sufficiently. It is because the Fourier 0 mode of solutions is simply controlled by the positivity of the nonlinearity. We note that this is similar in other problems: the Schrödinger equations on \( \mathbb{R}^n \), Damped wave equations \( \mathbb{R}^n \), for example. We refer the reader [9, 10] and the references therein. However, it is insufficient to focus only on the Fourier 0 mode to study the sharp criteria of blowup phenomena. For example, the initial data of (1.10) does not possess global solutions nor satisfy (LS). Therefore, we need to see the contribution of the oscillation of solutions in order to clarify the condition of initial data for blowup.

The aim of this research is to consider the sharp condition of initial data for the non-existence of global solutions. For this purpose, we consider (LS) in the \( \ell^1 \) framework, where for norm space \( X \) for sequences, \( \hat{X} \) is the collection of \( f \in L^1(\mathbb{T}) \) satisfying \( \hat{f} \in X \). For the detail of hat-Lebesgue space in the Euclidian case, we refer the reader to [9, 10] and the references there. We remark that we have the following embedding relations
\[ L^1(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T}) \hookrightarrow \ell^1 \hookrightarrow H^2(\mathbb{T}). \]
We say that \( u \) is a \( \ell^1 \) global solution to (LS) if \( u \in C^1([0, \infty); (1 - \Delta)^{-1}\ell^1) \cap C([0, \infty); \ell^1) \) satisfies (LS) for any \( t \geq 0 \) in the sense of \( (1 - \Delta)^{-1}\ell^1 \). Our main statement is the following:

**Proposition 1.2.** Let \( \phi \in \ell^1 \). Then there exist \( \ell^1 \) global solutions if and only if \( \phi(x) = i\mu_0 \) with \( \mu_0 \geq 0 \). More precisely, If \( u \) is a \( \ell^1 \) global solution, then \( u \) is constant with respect to \( x \) and satisfies \( \text{Im} u(t, 0) > 0 \) for any \( t \geq 0 \).
We remark that local solutions to \((1.4)\) in the \(\hat{\ell}^1\) framework are easily constructed by the standard contraction argument. Therefore, the blowup alternative holds, namely if solutions does not exists globally,
\[
\limsup_{t \to T} \|u\|_{\hat{\ell}^1} = \infty
\]
holds with some \(T > 0\).

We may show Proposition 1.2 by a contradiction argument. We remark that when initial data does not satisfy \((1.8)\), unlike earlier works, it seems no longer possible to show the blowup phenomena by ignoring the effect of oscillation. Therefore, in this paper, we may focus on the nonlinear interaction between the Fourier 0 mode and oscillation of solutions. Namely, we show that, thanks to the oscillation of solutions \(u, \tilde{u}(t, 0) \leq 0\) and \(u(t) \neq 0\) hold at some \(t > 0\) nevertheless \(\tilde{u}(0, 0)\) may be positive. Then by taking \(\tilde{u}(t, 0) \leq 0\) as initial data, Proposition (1.1) implies that \(u\) blows up at a finite time.

In the next section, we prepare the proof of Proposition 1.2 by introducing a decomposition of solutions and a priori controls of \(u\). Proposition 1.1 is shown in the last section.

### 2. Preparation

In order to show Proposition 1.2 it is sufficient to show the case where
\[
\text{Im} \int_T^\infty \phi(x)dx > 0 \quad \text{and} \quad \text{Re} \int_T^\infty \phi(x)dx = 0. \tag{2.1}
\]
We may rewrite our solution \(u\) by
\[
u(t, x) = \sum_{k \in \mathbb{Z}} u_k(t)e^{ikx - ik^2t}.
\]
We remark that \(u_k(t) = \tilde{u}(t, k)e^{ik^2t}\) and \(e^{ikx - ik^2t}\) is a solution of linear Schrödinger equation for any \(k \in \mathbb{Z}\). Then the nonlinearity is rewritten by
\[
|u(t, x)|^2 = \sum_m \sum_{\ell \neq 0} \overline{u_{\ell}(t)}u_{\ell}(t)e^{i(m-l)x - i(m^2 - \ell^2)t}
\neq \sum_k \left( \sum_{\ell \neq 0} \overline{u_{\ell}(t)}u_{\ell+k}(t)e^{-2ikt} \right)e^{ikx - ik^2t}. \tag{2.2}
\]
We note that \((1.4)\) implies that \(u_0(t) = \tilde{u}(t, 0)\) is purely imaginary when \(\phi\) satisfies \((2.1)\), so we denote \(\text{Im} u_0(t) = \mu(t)\) for simplicity. Then \((1.4)\) is rewritten by the system,
\[
\dot{\mu}(t) = -\mu(t)^2 - \sum_{\ell \neq 0} |u_{\ell}(t)|^2, \tag{2.3}
\]
\[
\dot{u}_k(t) = -\mu(t)u_k(t) - i \sum_{\ell \neq 0} \overline{u_{\ell}(t)}u_{\ell+k}(t)e^{-2ikt} \tag{2.4}
\]
for \(k \neq 0\). Here, we put
\[
\nu(t) = \sum_{\ell \neq 0} |u_{\ell}(t)|^2,
\]
where
\[ R(t) = \sum_{k \neq 0} \sum_{\ell \neq 0} \bar{u}_k(t) \bar{u}_\ell(t) u_{\ell+k}(t)e^{-2ikt}. \]

Now we show an estimate of \( R \).

**Lemma 2.1.** We assume that \( \mu_0 > 0 \) and \( \nu_0 > 0 \). We also assume that we have a solution to (1.4) with \( \mu(t) \geq 0 \) for \( t \in [0, T_0) \) with some \( T_0 > 0 \). Then there exist positive constants \( A \) and \( B \) such that the estimate
\[
\left| \int_{0}^{t} R(\tau)d\tau \right| \leq A(\mu(t)\nu(t) + \nu^{3/2}(t) + \mu_0\nu_0 + \nu_0^{3/2})
+ B \int_{0}^{t} \mu(\tau)\nu^{3/2}(\tau) + \nu^2(\tau)d\tau. \tag{2.6}
\]
holds for any \( t \in (0, T_0) \).

**Proof.** We, at first, decompose \( R \) by
\[
R(t) = \sum_{k \neq 0} \sum_{\ell \neq 0} R_{k,\ell}(t),
\]
where
\[ R_{k,\ell}(t) = \bar{u}_k(t) \bar{u}_\ell(t) u_{\ell+k}(t)e^{-2ikt}. \]

When \( \ell = -k \), by integration by parts, (2.3) and (2.4) imply that we have
\[
i \int_{0}^{t} R_{k,-k}(\tau)d\tau
= \frac{1}{2k^2} \left( \bar{u}_k(t) u_{-k}(t) \mu(t)e^{2ikt} - \bar{u}_k(0) u_{-k}(0) \mu(0) \right)
+ \frac{1}{2k^2} \int_{0}^{t} \bar{u}_k(\tau) u_{-k}(\tau) (\mu_0^2(\tau) + \nu^2(\tau))e^{2ikt}d\tau
+ i \int_{0}^{t} (\bar{u}_k(\tau) u_{-k}(\tau)e^{2ikt} - i|u_k(\tau)|^2 - i|u_{-k}(\tau)|^2) \mu_0^2(\tau)d\tau
+ i \int_{0}^{t} \bar{u}_k(\tau) \mu_0(\tau) \sum_{m \neq 0, -k} u_m(\tau) u_{m-k}(\tau)e^{2ik(m+k)\tau}d\tau
+ i \int_{0}^{t} \bar{u}_k(\tau) \mu_0(\tau) \sum_{m \neq 0, k} u_m(\tau) u_{m-k}(\tau)e^{2ik(k-m)\tau}d\tau. \tag{2.7}
\]
When \( \ell \neq 0, -k \), the identity
\[
i \int_{0}^{t} R_{k,\ell}(\tau)d\tau
= \frac{i}{2k\ell} \left( \bar{u}_k(t) \bar{u}_\ell(t) u_{k+\ell}(t)e^{-2ikt} - \bar{u}_k(0) \bar{u}_\ell(0) u_{k+\ell}(0) \right)
= \frac{i}{2k\ell} (I_{k,\ell} + I_{\ell,k} + J_{k,\ell}) \tag{2.8}
\]
follows similarly, where

\[ I_{k, \ell} = - \int_0^t \overline{\mu_k(\tau)\overline{u_k(\tau)}} u_{k+\ell}(\tau) e^{-2ikt\tau} d\tau \]

\[ = \int_0^t \mu(\tau)\overline{\mu_k(\tau)\overline{u_k(\tau)}} u_{k+\ell}(\tau) e^{-2ikt\tau} d\tau + i \sum_{m \neq 0, -k} \int_0^t u_m(t)\overline{u_{m+k}(t)} u_{k+\ell}(t) e^{-2ikt(m-\ell)} dt \]

(2.9)

and

\[ J_{k, \ell} = - \int_0^t \overline{\mu_k(\tau)} \mu(\tau) u_{k+\ell}(\tau) e^{-2ikt\tau} d\tau \]

\[ = \int_0^t \mu(\tau)\overline{\mu_k(\tau)} u_{k+\ell}(\tau) e^{-2ikt\tau} d\tau + i \int_0^t \mu(\tau)\overline{\mu_k(\tau)} \mu(\tau) e^{-2ikt(k+\ell)^2} \tau \]

\[ + i \sum_{m \neq 0, -k} \int_0^t \overline{u_m(\tau)} \mu_k(\tau) u_{m+k+\ell}(\tau) e^{-2ikt(km+m\ell)} d\tau. \]

(2.10)

We recall that for positive sequences \( a, b, c, d \), the estimates

\[ \sum_{k \neq 0, \ell \neq 0, -k} \sum_{\ell \neq 0, -k} \frac{1}{k \ell} a_k b_k c_k e^{k+\ell} \leq \| k^{-1} a_k \|_{\ell^1_{k \neq 0}} \sum_{\ell \neq 0, -k} b_{k+\ell} \| c_k \|_{\ell^2_{k \neq 0}} \]

\[ \leq \frac{\pi}{\sqrt{3}} \| a_k \|_{\ell^2_{k \neq 0}} \| b_k \|_{\ell^2_{k \neq 0}} \| c_k \|_{\ell^2_{k \neq 0}}, \]

and

\[ \sum_{k \neq 0, \ell \neq 0, -k} \sum_{m \neq 0, -k} \frac{1}{k \ell} a_m b_{m+k} c_{\ell+k} \]

\[ \leq \| k \| \sum_{m \neq 0, -k} a_m b_{m+k} \| c_{\ell+k} \|_{\ell^2_{\ell \neq 0}} \| k^{-1} \ell_{\ell \neq 0} \|_{\ell^2_{\ell \neq 0}} \]

\[ \leq \pi^2 \| a_k \|_{\ell^2_{k \neq 0}} \| b_k \|_{\ell^2_{k \neq 0}} \| c_k \|_{\ell^2_{k \neq 0}} \| d_k \|_{\ell^2_{k \neq 0}}, \]

\[ \sum_{k \neq 0, \ell \neq 0} \sum_{m \neq 0, -k} \sum_{\ell \neq 0, -k} \frac{1}{k \ell} a_k b_{k+\ell} c_{m+k+\ell} \]

\[ \leq \| k^{-1} a_k \|_{\ell^1_{k \neq 0}} \| k^{-1} b_k \|_{\ell^1_{k \neq 0}} \| c_k \|_{\ell^2_{k \neq 0}} \| d_k \|_{\ell^2_{k \neq 0}} \]

\[ \leq \frac{\pi^2}{3} \| a_k \|_{\ell^2_{k \neq 0}} \| b_k \|_{\ell^2_{k \neq 0}} \| c_k \|_{\ell^2_{k \neq 0}} \| d_k \|_{\ell^2_{k \neq 0}}, \]

where we have used the fact that

\[ \sum_{k=1}^{\infty} k^{-2} = \frac{\pi^2}{6}. \]

These estimates above and (2.7), (2.8), and (2.9) imply (2.6). \( \square \)

Thanks to Lemma 2.1, we have the following a priori controls:
Lemma 2.2. Under the assumption of Lemma 2.1, if \( \mu_0 \) and \( \nu_0 \) are small enough to satisfy the estimate

\[
4A\mu_0 + 4A(2\nu_0)^{1/2} + 3B\mu_0 < \frac{1}{2}
\]

with A and B of Lemma 2.1, then the following estimate holds for \( t \in [0, T_0) \):

\[
\nu(t) \leq 2\nu_0. \tag{2.11}
\]

Proof. The positivity of \( \mu \) and \( \nu \), and (2.5) imply that \( \nu \) satisfies

\[
\nu(t) = \nu_0 - 2\int_0^t \mu(\tau)\nu(\tau) + R(\tau)d\tau
\]

\[
\leq \nu_0 + 2\left| \int_0^t R(\tau)d\tau \right|. \tag{2.12}
\]

Let \( M(t) = \sup_{\tau \in [0, t]} \nu(\tau) \). Since \( \mu \) is non-negative and decreasing on \([0, T_0)\), Lemma 2.1 and (2.12) imply that the estimate

\[
M(t) \leq \nu_0 + 4A(\mu_0 M(t) + M^{3/2}(t)) + 2B\int_0^t \mu(\tau)\nu(\tau)^{3/2} + \nu(\tau)^2d\tau \tag{2.13}
\]

holds. Here we have used the positivity of the integrand of the RHS of (2.13). Thanks to the monotonicity of \( \mu \), we can change variable of the integral in the right hand side of (2.13) by

\[
t \in [0, T_0) \to s = \mu_0 - \mu(t) \in [0, \mu_0 - \mu(T_0)).
\]

We put \( V(s) = \nu(t) \) and

\[
T_1 = \sup \{ t \in (0, T_0) \mid M(t) \leq 2\nu_0 \}.
\]

Then for \( t \in [0, T_1] \), we have

\[
M(t) \leq \nu_0 + 4A(\mu_0 M(t) + M(t)^{3/2}) + 2B\int_0^{\mu_0 - \mu(t)} \frac{(\mu_0 - \sigma)V(\sigma)^{3/2} + V(\sigma)^2}{(\mu_0 - \sigma)^2 + V(\sigma)}d\sigma
\]

\[
\leq \nu_0 + (4A\mu_0 + 4AM(t)^{1/2} + 3B\mu_0)M(t)
\]

\[
< \nu_0 + \frac{1}{2}M(t),
\]

where we have used the Young inequality,

\[
2(\mu_0 - \sigma)V(s)^{1/2} \leq (\mu_0 - \sigma)^2 + V(s).
\]

Therefore \( T_1 = T_0 \) and (2.11) is shown. \( \square \)

The next statement is a direct consequence of Lemma 2.1.

Corollary 2.3. Under the assumption of Lemma 2.1, if \( \mu_0 \) and \( \nu_0 \) are sufficiently small, then the estimate

\[
\delta := \inf_{t \in [0, T_0)} \left[ \nu_0 - 2A(\mu(t)\nu(t) + \nu(t)^{3/2} + \mu_0\nu_0 + \nu_0^{3/2}) \right] > 0
\]

holds.
3. Proof of Proposition 1.2

When \( \mu_0 < 0 \), \( 2.3 \) implies a priori that \( \mu \) goes to \(-\infty\) at a finite time. Moreover, when \( \mu_0 = 0 \) and \( \nu_0 > 0 \), \( 2.3 \) and the continuity argument implies that for \( t_0 > 0 \), \( \mu(t_0) < 0 \). Therefore, \( \mu \) again goes to \(-\infty\) at a finite time.

Now we consider the case where \( \mu_0 > 0 \) and \( \nu > 0 \) and assume \( u \) exists for any positive time. It is sufficient to show that \( \mu(T_0) = 0 \) and \( \nu(T_0) > 0 \) holds at a time \( T_0 > 0 \). Indeed, the equation

\[
\dot{\nu} = -\mu^2 - \nu^2
\]

implies that there exists \( T_0 \in (0, \infty) \) such that \( \mu(T_0) = 0 \). We remark that such \( T_0 \) is finite because \( \mu \) does not converge as long as \( \nu \) is positive. We also note that there does not exist \( t > 0 \) such that \( \mu(t) = \nu(t) = 0 \) because if such \( t \) exists, \( 2.3 \) and \( 2.5 \) imply that \( \nu = \nu \equiv 0 \). We show the assertion by a contraction argument, namely we assume \( \lim_{t\to\infty} \mu(t) = \lim_{t\to\infty} \nu(t) = 0 \) holds but this assumption implies \( \inf_{t>0} \nu(t) > 0 \).

Without loss of generality, we assume that \( \mu_0 \) and \( \nu_0 \) are so small that Corollary 2.3 is applicable. \( 2.6 \) and Lemma 2.1 Corollary 2.3 imply that \( \nu \) enjoys the estimate,

\[
\nu(t) \geq \delta - 2 \int_0^t \mu(\tau) \nu(\tau) + B\mu(\tau) \nu(\tau)^{3/2} + B\nu(\tau)^2 d\tau
\]

with some \( \delta > 0 \) for any \( t \in [0, T_0] \). Then we change \( t \) into \( s = \mu_0 - \mu(t) \) and put \( \nu(t) = V(s) \) as the proof of Corollary 2.3 Then the estimate above implies that we have

\[
V(s) \geq \delta - 2 \int_0^s \frac{(\mu_0 - \sigma)V(\sigma) + B(\mu_0 - \sigma)V(\sigma)^{3/2} + BV(\sigma)^2}{(\mu_0 - \sigma)^2 + V(\sigma)} d\sigma,
\]

\[
\geq \delta - \int_0^s 2 \frac{(\mu_0 - \sigma)V(\sigma)}{(\mu_0 - \sigma)^2 + V(\sigma)} + 3BV(\sigma) d\sigma, \tag{3.1}
\]

where we have used the Young inequality,

\[
2(\mu_0 - \sigma)V(\sigma)^{1/2} \leq (\mu_0 - \sigma)^2 + V(\sigma).
\]

Now we claim that there exists \( g \in C([0, \mu_0]) \cap C^1((0, \mu_0)) \) satisfying the estimates

\[
g(s) \leq -2 \frac{(\mu_0 - s)g(s)}{(\mu_0 - s)^2 + g(s)} - 3Bg(s),
\]

\( g(0) \leq \delta \), and \( g(\mu_0) > 0 \). Then by the comparison argument, we have

\[

\nu(T_0) = V(\mu_0) \geq g(\mu_0) > 0
\]

and this shows the contradiction.

In order to find \( g \) above, for \( s \in [0, \mu_0) \), we put

\[
f(s) = \frac{(\mu_0 - s)^2}{W(C_1(\mu_0 - s)^2)}, \quad C_1 = \frac{e^{\mu_0^2/f_0}}{f_0}
\]

with \( f_0 > 0 \), where \( W \) is the Lambert \( W \) function given by

\[
W(\sigma)e^{W(\sigma)} = \sigma
\]

for \( \sigma \geq 0 \). For the detail of the Lambert \( W \) function, we refer [3], for example. We note that the identity \( f(0) = f_0 \) holds because the definition implies that

\[
W(\sigma e^{\sigma}) = \sigma
\]
Therefore, if \( f \) holds for \( \sigma \geq 0 \). We also note that the identity

\[
\lim_{s \to s_0} f(s) = f_0 e^{-\mu_0^2 f_0} > 0
\]

follows from \([4, (3.1)]\). Now we claim that \( f \) satisfies the identity

\[
\dot{f}(s) = -\frac{2(\mu_0 - s)f(s)}{(\mu_0 - s)^2 + f(s)}
\]

for \( s \in (0, \mu_0) \). Indeed, since \( W \) enjoys the identity

\[
\dot{W}(\sigma) = \frac{W(\sigma)}{\sigma(1 + W(\sigma))}
\]

for \( \sigma > 0 \) (See \([4, (3.2)]\)), we compute

\[
\dot{f}(s) = -2\frac{\mu_0 - s}{W(C_1(\mu_0 - s)^2)} + 2C_1\frac{(\mu_0 - s)^3}{W(C_1(\mu_0 - s)^2)} W(C_1(\mu_0 - s)^2)
\]

\[
= -2\frac{\mu_0 - s}{W(C_1(\mu_0 - s)^2)} + 2\frac{\mu_0 - s}{W(C_1(\mu_0 - s)^2)(1 + W(C_1(\mu_0 - s)^2))}
\]

\[
= -2\frac{\mu_0 - s}{W(C_1(\mu_0 - s)^2)}
\]

\[
= -2\frac{\mu_0 - s}{(\mu_0 - s)^2 + f(s)}
\]

Then we put \( g(s) = e^{3B(\mu_0 - s)} f(s) \). Since \( g(s) \geq f(s) \) for \( s \in [0, \mu_0) \), \( \dot{g} \) is estimated by

\[
\dot{g}(s) = -\frac{2(\mu_0 - s)g(s)}{(\mu_0 - s)^2 + f(s)} - 3Bg(s)
\]

\[
\leq -\frac{2(\mu_0 - s)g(s)}{(\mu_0 - s)^2 + g(s)} - 3Bg(s)
\]

Therefore, if \( f_0 < e^{-3B\mu_0} \), then the desired \( g \) is obtained.

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