Homology Groups of Cubical Sets with Connections

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Abstract
Toward defining commutative cubes in all dimensions, Brown and Spencer introduced the notion of “connection” as a new kind of degeneracy. In this paper, for a cubical set with connections, we show that the connections generate an acyclic subcomplex of the chain complex of the cubical set. In particular, our results show that the homology groups of a cubical set with connections are independent of whether we normalize by the connections or we do not, that is, connections do not contribute to any nontrivial cycle in the homology groups of the cubical set.

Keywords Cubical homology · Connection · Cubical set

Mathematics Subject Classification 55N35 · 55U15

1 Introduction
Cubical sets stemmed naturally from the development of homology theory of various spaces. Instead of simplices, cubes were, for the first time, used by Serre to develop (co)homology
theory for fiber spaces [19], and Eilenberg and MacLane [13] developed the singular, cubical homology theory of topological spaces. Massey’s classical book [18] presents a comprehensive treatment of singular homology using the cubical approach.

Kan introduced and studied abstract cubical sets for the purpose of developing a general homotopy theory, see [16]. Cubical sets come with a singular homology theory [12, Section 14.7] and a geometric realization [12, Definition 11.1.11]. Federer [14, Theorem 3.9.12] showed that the singular homology groups of a cubical set and that of its geometric realization are isomorphic.

In order to develop applications of higher groupoids to work by J.H.C. Whitehead in his paper Combinatorial Homotopy II [23], Brown and Higgins in [9] found it necessary to introduce the notion of connection in cubical sets, see also [1]. The recent paper [8] explains the origin of the notion of connection as well as the need for it. Cubical sets with connections have been shown to have many desirable properties [10], and have characteristics similar to those of simplicial sets [15]. For example, cubical abelian groups with connections are equivalent to chain complexes [11], and cubical groups with connections are Kan fibrant [21], a property shared with simplicial sets. Recently, in [17], it was shown that cubical sets with connections form a strict test category. In particular, the geometric realization of the product of cubical sets with connections has the “right” homotopy type; a property that cubical set (without connections) do not have in general.

In this note we study the singular homology groups of cubical sets with connections. We were originally motivated by computational considerations encountered in [5,6]. Since the chain groups are very large, we explored cutting down the size of the the chain complex by dropping connection cubes. Toward this end, we investigate the contribution of connections to the nontrivial cycles in the homology groups. A simple relation (Lemma 4) between the singular cubical differential, the face maps, the degeneracy maps and the connections maps immediately implies (Corollary 6) that connections generate a chain subcomplex of the singular chain complex of the cubical set. We prove (Theorem 10) that the homology groups of this subcomplex are trivial, by defining natural filtrations in each dimension and showing inductively that each of the corresponding subcomplexes has trivial homology.

Several variants of the definition of a cubical set with connections appear in the literature. We have adopted the most inclusive definition, employed in [1], which yields the richest structure. In fact our proof of Theorem 10 shows that the principal result holds for the other variants as well (see Remark 11).

As another byproduct of our main result, it follows that the quotient of the singular chain complex of the cubical set by the subcomplex generated by the connection cubes computes the same homology as the singular chain complex itself.

In an appendix we provide the arguments showing that this quotient complex indeed is the cellular chain complex of the canonical CW-structure on the geometric realization of a cubical set with connections (see Theorem 19). In particular, for a cubical set with connections, we state in Corollary 20 that the singular homology groups of the geometric realizations with and without connection identifications coincide.

The latter result would also be a consequence of a result by Antolini [2], who states (using a more restricted definition of connections) that the two realizations are homotopy equivalent. Since we consider Antolini’s arguments hard to penetrate in any case, we see value in the purely algebraic derivation given here.
2 Background and Notations

In this section we recall the definition of a cubical set with connections and the homology theory of cubical sets. Then we give two examples of such sets to demonstrate the motivation for this study.

Throughout the paper, \( R \) denotes a commutative ring with unit which shall be the ring of coefficients. For any positive integer \( n \), let \([n] := \{1, \ldots, n\} \).

**Definition 1** [16] A cubical set \( K \) is a collection of sets \( \{K_n\}_{n \geq 0} \) together with, for each \( n \geq 1 \) and each \( i \in [n] \),

1. two maps \( f_j^K, f_j^- : K_n \longrightarrow K_{n-1} \), which are called face maps, and
2. a map \( \varepsilon_i : K_{n-1} \longrightarrow K_n \), which is called a degeneracy map,

satisfying the following relations for \( \alpha, \beta \in \{+, -\} \):

(i) \( f^\alpha_i f^\beta_j = f^\beta_j f^\alpha_i \) if \( i < j \).

(ii) \( \varepsilon_i \varepsilon_j = \varepsilon_{j+1} \varepsilon_i \) if \( i \leq j \).

(iii) \( f^\alpha_i \varepsilon_j = \begin{cases} 
\varepsilon_{j-1} f^\alpha_i & \text{if } i < j; \\
\varepsilon_j f^\alpha_i & \text{if } i > j; \\
id & \text{if } i = j.
\end{cases} \)

In a cubical set \( K \), an element \( \sigma \in K_n \) is called a singular \( n \)-cube. A singular \( n \)-cube \( \sigma \) is said to be degenerate if \( \sigma = \varepsilon_i f_i^+ \sigma \) for some \( i \in [n] \). Otherwise, \( \sigma \) is called non-degenerate.

**Definition 2** [1] A cubical set with connections is a cubical set \( K \) together with, for \( n \geq 1 \) and each \( i \in [n] \), two additional maps (called connections)

\( \Gamma_i^+, \Gamma_i^- : K_n \longrightarrow K_{n+1} \)

such that, for \( \alpha, \beta \in \{+, -\} \) and \( i, j \in [n] \), the following relations are satisfied:

(i) \( \Gamma_i^\alpha \Gamma_j^\beta = \Gamma_{j+1}^\beta \Gamma_i^\alpha \) if \( i \leq j \).

(ii) \( \Gamma_i^\alpha \varepsilon_j = \begin{cases} 
\varepsilon_{j+1} \Gamma_i^\alpha & \text{if } i < j; \\
\varepsilon_i \Gamma_{j-1}^\alpha & \text{if } i > j; \\
\varepsilon_i & \text{if } i = j.
\end{cases} \)

(iii) \( f_i^\alpha \Gamma_j^\beta = \begin{cases} 
\Gamma_j^\beta f_i^\alpha & \text{if } i > j + 1; \\
id & \text{if } i = j, j + 1, \alpha = \beta; \\
\varepsilon_i f_i^\alpha & \text{if } i = j, j + 1, \alpha \neq \beta.
\end{cases} \)

2.1 Homology Groups of Cubical Sets

Let \( K \) be a cubical set and let \( R \) be the ring of coefficients. For each \( n \geq 0 \), let \( \mathcal{L}_n(K) \) be the free \( R \)-module generated by the singular \( n \)-cubes with coefficients from \( R \), that is,

\( \mathcal{L}_n(K) := \left\{ \sum_{\sigma \in S} r_\sigma \sigma : S \text{ a finite subset of } K_n \text{ and } r_\sigma \in R \right\} \).

For \( n > 0 \), define the map \( \partial_n : \mathcal{L}_n(K) \longrightarrow \mathcal{L}_{n-1}(K) \) such that, for each singular \( n \)-cube \( \sigma \),

\( \partial_n(\sigma) = \sum_{i=1}^{n} (-1)^i (f_i^- \sigma - f_i^+ \sigma) \).
and extend linearly to all elements of $L_n(K)$. Furthermore, define the map $\partial_0 : L_0(K) \rightarrow L_{-1}(K)(= \{0\})$ to be the zero map, that is $\partial_0(\sigma) = 0$ for all $\sigma \in L_0$.

For each $n \geq 1$, let $D_n(K)$ be the $R$-submodule of $L_n(K)$ that is generated by all degenerate singular $n$-cubes, and let $C_n(K)$ be the free $R$-module $L_n(K)/D_n(K)$, whose elements are called $n$-chains. Clearly, the cosets of non-degenerate singular $n$-cubes freely generate $C_n(K)$.

Using Definition 1(iii), it is easy to check that $\partial_n[D_n(K)] \subseteq D_{n-1}(K)$ and, for $n \geq 1$, $\partial_{n-1}\partial_n = 0$, see [4,18]. Hence, $\partial_n : C_n(K) \rightarrow C_{n-1}(K)$ is a boundary operator, and $C(K) = (C_*(K), \partial_*)$ is a chain complex of free $R$-modules. We call $C(K)$ the non-degenerate chain complex of the cubical set $K$.

The homology groups of $K$ are defined to be the homology groups of the chain complex $C(K)$, that is, $\mathcal{H}_n(K) := \text{Ker}(\partial_n)/\text{Im}(\partial_{n+1})$, see [16]. For more information about the homology and homotopy of cubical sets see [12, Sections 14.7 and 13.1].

2.2 Cubical Sets of Topological Spaces

Let $X$ be a topological space, and, for $n \geq 0$, let $I^n$ be the geometric $n$-dimensional cube, that is, $I^n := \{(x_1, \ldots, x_n) : x_i \in [0, 1], i \in [n]\}$ with the standard topology. Define $KX_n$ to be the set of all continuous maps $\sigma : I^n \rightarrow X$. For each $i \in [n]$ and $\sigma \in KX_n$, define face maps $f_i^+\sigma, f_i^-\sigma \in KX_{n-1}$ such that, for $(a_1, \ldots, a_{n-1}) \in I^{n-1}$,

$$(f_i^+\sigma)(a_1, \ldots, a_{n-1}) := \sigma(a_1, \ldots, a_{i-1}, 1, a_i, \ldots, a_{n-1}),$$

$$(f_i^-\sigma)(a_1, \ldots, a_{n-1}) := \sigma(a_1, \ldots, a_{i-1}, 0, a_i, \ldots, a_{n-1}).$$

Also, define $\varepsilon_i\sigma \in KX_{n+1}$ such that, for $(a_1, \ldots, a_{n+1}) \in I^{n+1}$,

$$(\varepsilon_i\sigma)(a_1, \ldots, a_{n+1}) := \sigma(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n+1}).$$

It is easy to check that $KX := \{KX_n\}_{n \geq 0}$ along with the face maps $f_i^\pm$ and degeneracy maps $\varepsilon_i$ is a cubical set.

Furthermore, $KX$ is a cubical set with connections defined as follows. For each $i \in [n]$, set

$$f_i^\beta \sigma (a_1, \ldots, a_{n+1}) := \sigma(a_1, \ldots, a_{i-1}, m_\beta(a_i, a_{i+1}), a_{i+2}, \ldots, a_{n+1})$$

where

$$m_\beta(x, y) = \begin{cases} \min(x, y) & \text{if } \beta = +; \\ \max(x, y) & \text{if } \beta = -. \end{cases}$$

The set $KX$ was initially constructed by Eilenberg and Mac Lane [13] and was used to define the cubical singular homology groups of $X$, which turned out to be the same as the (classical) singular homology groups of $X$, that is, $H_n(X) = H_n(KX)$ for all $n$, see [18, Section 2, Chapter II]. Furthermore, the geometric realization $|KX|$ of $KX$ and $X$ are weakly homotopy equivalent [12, Proposition 11.1.16], in particular $H_n(|KX|)$ and $H_n(X)$ are isomorphic for all $n$, see [20, Theorem 7.6.25].

2.3 Discrete Cubical Sets of Graphs

Another cubical set with connections arises from the development of a discrete homology theory for metric spaces [3,4]. For a given metric space $X$, the singular $(n, r)$-cubes are defined to be the $r$-Lipschitz maps from the $n$-dimensional Hamming cube to the metric
space $X$, and the (discrete) homology groups of the metric space $X$ are defined to be the singular homology groups of the resulting singular chain complex.

In a recent paper [5] we study the theory from [4] in the combinatorially interesting case where the singular $n$-cubes are the graph homomorphisms from the $n$-dimensional Hamming cube to a given undirected, simple graph $G$. This results in a cubical set $KG$ which is used to define a (discrete) cubical homology of the graph $G$.

For $n \geq 0$, let $Q_n$ be the Hamming $n$-dimensional cube, that is, $Q_n := \{(x_1, \ldots, x_n) : x_i \in \{0, 1\}, i \in [n]\}$. Define $KG_n$ to be the set of all graph homomorphisms $\sigma : Q_n \to G$. For each $i \in [n]$ and $\sigma \in KG_n$, define face maps $f_i^+ \sigma, f_i^- \sigma \in KG_{n-1}$ such that, for $(a_1, \ldots, a_{n-1}) \in Q_{n-1}$,

$$\begin{align*}
(f_i^+ \sigma) (a_1, \ldots, a_{n-1}) &:= \sigma (a_1, \ldots, a_{i-1}, 1, a_i, \ldots, a_{n-1}) , \\
(f_i^- \sigma) (a_1, \ldots, a_{n-1}) &:= \sigma (a_1, \ldots, a_{i-1}, 0, a_i, \ldots, a_{n-1}) .
\end{align*}$$

Also, define $\varepsilon_i \sigma \in KG_{n+1}$ such that, for $(a_1, \ldots, a_{n+1}) \in Q_{n+1}$,

$$(\varepsilon_i \sigma) (a_1, \ldots, a_{n+1}) := \sigma (a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{n+1}) .$$

Furthermore, for each $i \in [n]$, define connection maps $\Gamma_i^+ \sigma, \Gamma_i^- \sigma \in KG_{n+1}$ such that

$$\Gamma_i^\beta \sigma (a_1, \ldots, a_{n+1}) := \sigma (a_1, \ldots, a_{i-1}, m_\beta (a_i, a_{i+1}), a_{i+2}, \ldots, a_{n+1}) ,$$

where

$$m_\beta (x, y) = \begin{cases} 
\min (x, y) & \text{if } \beta = + ; \\
\max (x, y) & \text{if } \beta = - .
\end{cases}$$

The proof of the following lemma is straightforward and is similar to that of $KX$ being a cubical set with connections.

**Lemma 3** The collection $KG := \{KG_n\}_{n \geq 0}$ along with the face maps $f_i^\pm$, degeneracy maps $\varepsilon_i$ and connections $\Gamma_i^\pm$ is a cubical set with connections.

Even though we were able to compute the homology groups of many classes of graphs [5, Sections 4 and 7], in general such computations are not feasible and, once again, the need for better understanding of the cubical set itself is evident. Investigating the role of the connections in the nontrivial cycles in the homology groups of $KG$ is a natural and useful step.

### 3 Homology of the Connection Chain Subcomplex

Throughout this section, let $K$ be a cubical set with connections and let $C(K)$ be its non-degenerate chain complex. It is easy to see that the set of connections of $K$ does not form a cubical subset of $K$ as not all faces of a connection are necessarily connections. However, we will show that the connections generate a subcomplex $\text{Con}(K) \subseteq C(K)$. Our main result in this section is that the homology groups of $\text{Con}(K)$ are trivial. The proof also establishes this result for several related subcomplexes of $\text{Con}(K)$.

**Lemma 4** If $\theta \in K_n$ is a singular $n$-cube and $\beta \in \{+, -\}$,

(i) $\partial_{n+1} \Gamma_1^\beta (\theta) = - \Gamma_1^\beta \sum_{i=2}^{n} (-1)^i (f_i^- f_i^+) (\theta)$.

(ii) $\partial_{n+1} \Gamma_n^\beta (\theta) = \Gamma_{n-1}^\beta \sum_{i=1}^{n-1} (-1)^i (f_i^- f_i^+) (\theta)$.
(iii) For any $1 < t < n$,
\[
\partial_{n+1} \Gamma_i^\beta (\theta) = \Gamma_{t-1}^\beta \sum_{i=1}^{t-1} (-1)^i (f_i^+ - f_i^-) (\theta) - \Gamma_t^\beta \sum_{i=t+1}^{n} (-1)^i (f_i^+ - f_i^-) (\theta).
\]

**Proof** Let $\theta \in K_n$ be a singular $n$-cube and $\beta \in \{+, -\}$. Then, for $t \in [n]$,
\[
\partial_{n+1} \Gamma_i^\beta (\theta) = \sum_{i=1}^{n+1} (-1)^i (f_i^- - f_i^+) \left( \Gamma_i^\beta (\theta) \right).
\]

By Definition 2(iii), $f_i^\alpha \Gamma_i^\beta (\theta) = f_i^\alpha \Gamma_i^\beta (\theta)$ and $f_i^\alpha \Gamma_i^\beta = \begin{cases} \Gamma_i^\beta f_i^\alpha & \text{if } i < t; \\ \Gamma_i^\beta & \text{if } i > t + 1. \end{cases}$ Setting $t = 1$ and $t = n$ give (i) and (ii) immediately. For $1 < t < n$, the following computation implies (iii):
\[
\partial_{n+1} \Gamma_i^\beta (\theta) = \Gamma_{t-1}^\beta \sum_{i=1}^{t-1} (-1)^i (f_i^- - f_i^+) (\theta) + \Gamma_t^\beta \sum_{i=t+2}^{n+1} (-1)^i (f_i^- - f_i^+) (\theta)
\]
\[
= \Gamma_{t-1}^\beta \sum_{i=1}^{t-1} (-1)^i (f_i^- - f_i^+) (\theta) - \Gamma_t^\beta \sum_{i=t+1}^{n} (-1)^i (f_i^- - f_i^+) (\theta).
\]

We are now in a position to define our principal object of study.

**Definition 5** For $n \geq 0$, let $\text{Con}_n (K)$ be the $R$-submodule of $C_n (K)$ that is generated by the cosets of $\Gamma_i^\beta (\tau)$ where $\tau \in K_{n-1}$, $i \in [n-1]$, and $\beta \in \{+, -\}$.

**Corollary 6** If $\theta \in \text{Con}_n (K)$, $n > 0$, then $\partial_n (\theta) \in \text{Con}_{n-1} (K)$. In particular, $\text{Con} (K) = (\text{Con}_*, \partial_*)$ is a chain subcomplex of the chain complex $C(K)$.

**Proof** This is an immediate consequence of Lemma 4.

We call $\text{Con} (K)$ the **connection chain complex** of $K$. Clearly, $\text{Con}_n (K)$ is generated by the cosets of $\Gamma_i^\beta (\tau)$ where $\tau$ is a non-degenerate singular $(n-1)$-cube. In particular, $\text{Con}_1 (K) = (0)$. For all $\beta$, $i$, $n$ we will extend by linearity and regard $\Gamma_i^\beta$ as a linear operator on $\text{Con}_n (K)$.

When $\theta \in \text{Con}_n (K)$, the identities in Lemma 4 can be sharpened considerably.

**Corollary 7** Let $\theta = \Gamma_i^\beta (\tau)$ where $\tau \in K_{n-1}$, and $\beta \in \{+, -\}$. Then:

(i) If $t = 1$, that is $\theta = \Gamma_1^\beta (\tau)$, then
\[
\partial_{n+1} \Gamma_1^\beta (\theta) + \Gamma_1^\beta \partial_n (\theta) = \beta \theta.
\]

(ii) If $1 < t < n$, then
\[
\partial_{n+1} \Gamma_i^\beta (\theta) + \Gamma_i^\beta \partial_n (\theta) = (-1)^{t+1} \beta \theta + 2 \Gamma_{t-1}^\beta \sum_{i=1}^{t-1} (-1)^i (f_i^- - f_i^+) (\tau).
\]
Proof Adding the terms $\Gamma_i^\beta \partial_n(\theta)$ and $\Gamma_i^\beta \partial_n(\theta)$ to both sides of Lemma 4(i) and (iii), respectively, we obtain

$$\partial_{n+1} \Gamma_i^\beta(\theta) + \Gamma_i^\beta \partial_n(\theta) = \Gamma_i^\beta \sum_{i=1}^{t-1} (-1)^i \left( f_i^- - f_i^+ \right)(\theta) + \Gamma_i^\beta \sum_{i=1}^{t} (-1)^i \left( f_i^- - f_i^+ \right)(\theta).$$

By Definition 2(iii), the coset $\Gamma_i^\beta \left[ (-1)^i \left( f_i^- - f_i^+ \right)(\theta) \right] = (-1)^i + 1 \beta \theta$ and, for $i < t$, $(f_i^- - f_i^+)(\theta) = (f_i^- - f_i^+)(\Gamma_i^\beta(\tau)) = \Gamma_i^\beta(f_i^- - f_i^+)(\tau)$. Thus

$$\partial_{n+1} \Gamma_i^\beta(\theta) + \Gamma_i^\beta \partial_n(\theta) = (-1)^i + 1 \beta \theta + \left( \Gamma_i^\beta \sum_{i=1}^{t-1} (-1)^i \left( f_i^- - f_i^+ \right)(\tau) \right)$$

$$= (-1)^i + 1 \beta \theta + 2 \Gamma_i^\beta \sum_{i=1}^{t-1} (-1)^i \left( f_i^- - f_i^+ \right)(\tau),$$

since $\Gamma_i^\beta \sum_{i=1}^{t-1} (-1)^i \left( f_i^- - f_i^+ \right)(\tau) = \Gamma_i^\beta \Gamma_i^\beta$. \hfill \Box

Definition 8 For $1 \leq p, q < \infty$, define $\text{Con}_{p,q}^n(K)$ to be the submodule of $\text{Con}_n(K)$ spanned by connections of the form $\Gamma_i^+(\tau)$ with $1 \leq i \leq p$ and $\Gamma_i^-(\tau)$ with $1 \leq i \leq q$, where $\tau \in \text{Con}_{n-1}(K)$.

Note that

$$\text{Con}_{p,q}^n(K) = \text{Con}_{n-1,q}^n(K) \text{ for } p \geq n - 1, \text{ and}$$

$$\text{Con}_{p,q}^n(K) = \text{Con}_{p,n-1}^n(K) \text{ for } q \geq n - 1.$$ 

We will sometimes abbreviate $\text{Con}_{p,q}^n(K) = \text{Con}_{p,q}^n = \text{C}_{p,q}^n$, $\text{Con}_{n-1,q}^n(K) = \text{Con}_{n-1,q}^n = \text{C}_{n-1,q}^n$, and $\text{Con}_{p,n-1}^n(K) = \text{Con}_{p,n-1}^n = \text{C}_{p,n-1}^n$.

Lemma 9 For any fixed $p, q$, $\text{Con}_{p,q}^n(K) = (\text{Con}_{p,q}^n(K), \partial_n)$ is a chain complex.

Proof This follows from Lemma 4. \hfill \Box

Theorem 10 (Main Theorem) For any fixed $p, q$, the complexes

$$\text{Con}_{p,0}^n(K), \text{Con}_{p,\infty}^n(K), \text{Con}^0_{p,q}(K), \text{Con}^{p,\infty}(K)$$

have trivial homology in all dimensions $n > 0$, as does the complex $\text{Con}(K)$ itself.

Proof It will suffice to prove the first two cases, i.e.

$$\mathcal{H}_n(\text{Con}_{p,0}^n) = (0) \text{ and } \mathcal{H}_n(\text{Con}_{p,\infty}^n) = (0) \quad (1)$$

for all $n, p, q > 0$. The third and fourth cases are proved analogously and the result for $\text{Con}(K)$ follows easily once the other cases have been proved. The statements in (1) will be proved by induction on $p$ and $q$, respectively, using the filtration

$$\text{C}_{n,0}^1 \subseteq \text{C}_{n,0}^2 \subseteq \cdots \subseteq \text{C}_{n,0}^{n-1} \subseteq \text{C}_{n,1}^{n-1} \subseteq \text{C}_{n,2}^{n-1} \subseteq \cdots \subseteq \text{C}_{n,1}^{n-1,n-1}$$

as a guide for each $n$. The induction argument will consist of four separate steps.

Base Case For $p$. We will prove that $\mathcal{H}_n(\text{Con}_{p,0}^n) = (0)$ for all $n > 0$. Suppose that $\theta \in \text{Con}_{n,0}^{1,0}$ and $\partial(\theta) = 0$. Then

$$\theta = \sum_{\alpha} c_{\alpha} \theta_{\alpha},$$
where each $\theta_\alpha = \Gamma_1^+(\tau_\alpha)$, with $\tau_\alpha \in \text{Con}_{n-1}(K)$. For each such $\theta_\alpha$ we have

$$\partial_{n+1} \Gamma_1^+(\theta_\alpha) + \Gamma_1^+ \partial_n (\theta_\alpha) = \theta_\alpha,$$

by Corollary 7(i), and by linearity $\partial_{n+1} \Gamma_1^+(\theta) + \Gamma_1^+ \partial_n (\theta) = \theta$. Since $\partial_n (\theta) = 0$, it follows that

$$\partial_{n+1} \Gamma_1^+(\theta) = \theta,$$

proving that $\theta$ is the boundary of an element $\Gamma_1^+(\theta) \in \text{Con}_{n-1}^{1,0}$, as desired.

**Induction Step for $p$.** We will prove that if $\mathcal{H}_n(\text{Con}_{i-1,0}) = (0)$ for all $n > 0$, then $\mathcal{H}_n(\text{Con}_{i,0}) = (0)$ for all $n > 0$. By elementary homological algebra (e.g., [22][Ex.1.3.1]), this will follow if we can show that $\mathcal{H}_n(\text{Con}_{i,0}/\text{Con}_{i-1,0}) = (0)$ for all $n > 0$.

If $[\theta] \in \text{Con}_{n,0}/\text{Con}_{n-1,0}$, then

$$[\theta] = \sum_{\alpha} \epsilon_\alpha [\theta_\alpha], \quad (2)$$

where each $\theta_\alpha$ is a singular $n$-cube in $\text{Con}_{n,0}$ (i.e., a generator), with $\theta_\alpha = \Gamma_p^+(\tau_\alpha)$ for some $\tau_\alpha \in \text{Con}_{n-1}(K)$. By Corollary 7(ii) we have

$$\partial_{n+1} \Gamma_p^+(\theta_\alpha) + \Gamma_p^+ \partial_n (\theta_\alpha) = (-1)^{p+1} \theta_\alpha + 2 \Gamma_p^+ \sum_{i=1}^{p-1} (-1)^i (f_i^+ - f_i^-) (\tau_\alpha). \quad (3)$$

We will sum (3) over $\alpha$, using (2), and evaluate the result in $\text{Con}_{n,0}/\text{Con}_{n-1,0}$. The second term on the right side of (3) is in $\text{Con}_{n-1,0}$, and hence can be ignored. The remaining terms combine to yield

$$\left[\partial_{n+1} \Gamma_p^+(\theta)\right] + \left[\Gamma_p^+ \partial_n (\theta)\right] = (-1)^{p+1}[\theta] \quad (4)$$

in $\text{Con}_{n,0}/\text{Con}_{n-1,0}$.

Now suppose that $[\theta]$ is a cycle in $\text{Con}_{n,0}/\text{Con}_{n-1,0}$. Since $[\partial_n (\theta)] = [0]$, we can write

$$\partial_n (\theta) = \sum_{\gamma} c_\gamma \psi_\gamma, \quad (5)$$

where each $\psi_\gamma$ is a generator of $\text{Con}_{n-1,0}$, i.e. $\psi_\gamma = \Gamma_i^+(\phi_\gamma)$ with $i \leq p - 1$. We claim that this implies $[\Gamma_p^+ \partial_n (\theta)] = [0]$. Some care must be taken in justifying this statement, because $\Gamma_p^+$ does not operate on either $\text{Con}_{n-1,0}$ or $\text{Con}_{n-1,0}$. From Definition 2(i), we have

$$\Gamma_p^+(\psi_\gamma) = \Gamma_p^+ \Gamma_i^+(\phi_\gamma) = \Gamma_i^+ \Gamma_{p-1}^+(\phi_\gamma) \in \text{Con}_{n-1}^{p-1,0}$$

for each $\gamma$, and hence

$$\Gamma_p^+ \partial(\theta) = \sum_{\gamma} c_\gamma \Gamma_p^+(\psi_\gamma) \in \text{Con}_{n-1}^{p-1,0}.$$

This shows that $[\Gamma_p^+ \partial_n (\theta)] = [0]$ as claimed, and now (4) reduces to

$$\left[\partial_{n+1} \Gamma_p^+(\theta)\right] = (-1)^{p+1}[\theta]. \quad (6)$$

We conclude that if $[\theta]$ is a cycle in $\text{Con}_{n,0}/\text{Con}_{n-1,0}$, then it is a boundary, and the argument is complete.
BASE CASE FOR $q$. We will prove that $\mathcal{H}_n(\Con_{\infty,1}) = (0)$, having already proved that $\mathcal{H}_n(\Con_{p,1}) = (0)$ for all $p, n > 0$. By the reasoning used in the previous step, it will be sufficient to prove that $\mathcal{H}_n(\Con^{n-1,1}/\Con^{n-1,0}) = (0)$.

Suppose that $[\theta] \in \mathcal{H}_n(\Con^{n-1,1}/\Con^{n-1,0})$, with $\theta = \sum_n c_n \theta_n$, where each $\theta_n$ is a singular $n$-cube such that $\theta_n = \Gamma^+_n(\tau_n) \in \Con^{n-1,1}$ for some $\tau_n$. For each such $\theta_n$ we have
\[
\partial_{n+1} \Gamma^+_n(\theta_n) + \Gamma^+_n \partial_n(\theta_n) = -\theta_n,
\]
by Corollary 7(ii), as before. It follows that
\[
\partial_{n+1} \Gamma^+_n(\theta) + \Gamma^+_n \partial_n(\theta) = -\theta,
\]
in $\Con^{n-1,1}$, by linearity. If $[\theta]$ is a cycle in $\mathcal{H}_n(\Con^{n-1,1}/\Con^{n-1,0})$, then $\partial_n(\theta) = \sum_n \sum_n c_n \psi_n$ with $\psi_n \in \Con^{n-1,0}$, i.e., each $\psi_n$ is a linear combination of terms of the form $\Gamma^+_i(\phi_n)$ with $1 \leq i \leq n - 1$. By Definition 2(i), for each $\phi_n$ we have
\[
\Gamma^+_1 \Gamma^+_i(\phi_n) = \Gamma^+_i \Gamma^+_1(\phi_n) \in \Con^{n-1,0},
\]
which implies $\Gamma^+_1 \psi_n \in \Con^{n-1,0}$. Hence $\Gamma^+_1 \partial_n(\theta) \in \Con^{n-1,0}$, i.e., $[\Gamma^+_1 \partial_n(\theta)] = [0]$, and from (7) we now conclude that $[\partial_{n+1} \Gamma^+_1(\theta)] = [\theta]$. Thus every cycle $[\theta] \in \mathcal{H}_n(\Con^{n-1,1}/\Con^{n-1,0})$ is a boundary, as desired.

INDUCTION STEP FOR $q$. For $q > 1$, we will prove that if $\mathcal{H}_n(\Con^{n-1,q-1}/\Con^{n-1,q-1}) = (0)$ for all $n > 0$, then $\mathcal{H}_n(\Con^{n-1,q}) = (0)$ for all $n > 0$, by showing that $\mathcal{H}_n(\Con^{n-1,q}/\Con^{n-1,q-1}) = (0)$ for all $n > 0$. Consider an element $[\theta] \in \mathcal{H}_n(\Con^{n-1,q}/\Con^{n-1,q-1})$, where $\theta = \sum_n c_n \theta_n$ with each $\theta_n \in \Con^{n-1,q}$ and $\theta_n = \Gamma^+_n(\tau_n)$ for some $\tau_n$.

From Corollary 7(ii) we have
\[
\partial_{n+1} \Gamma^+_n(\theta_n) + \Gamma^+_n \partial_n(\theta_n) = (-1)^{q+1} \theta_n + 2 \Gamma^+_n \Gamma^+_n \sum_{i=1}^{q-1} (-1)^i (f_i^+ - f_i^-) (\tau_n).
\]
If $[\theta]$ is a cycle in $\mathcal{H}_n(\Con^{n-1,q}/\Con^{n-1,q-1})$, then $\partial_n(\theta) = \sum_n \sum_n c_n \psi_n$ with each $\psi_n \in \Con^{n-1,q-1}$, i.e., $\psi_n = \Gamma^+_i(\phi_n)$ where either $\beta = +$ and $1 \leq i \leq n - 1$, or $\beta = -$ and $1 \leq i \leq q - 1$. We claim that the second terms on both the left and right side of (8) vanish when we replace $\theta_n$ by $\theta$ and evaluate the result in $\Con^{n-1,q}/\Con^{n-1,q-1}$.

For the second term on the left, we need to compute $\Gamma^+_q \Gamma^+_i(\phi_n)$ in each possible $\phi_n$. Applying the commutation axiom in Definition 2(ii) we get
\[
\Gamma^+_q \Gamma^+_i(\phi_n) = \left\{ \begin{array}{ll}
\Gamma^+_q \Gamma^+_i(\phi_n) & \text{if } q \leq i \leq n - 1 \\
\Gamma^+_q \Gamma^+_i(\phi_n) & \text{if } 1 \leq i < q,
\end{array} \right.
\]
and
\[
\Gamma^+_q \Gamma^+_i(\phi_n) = \Gamma^+_i \Gamma^+_q(\phi_n) \quad \text{if } 1 \leq i \leq n - 1.
\]
It follows that $[\Gamma^+_q \partial_n(\theta)] = [0]$, as claimed.

The second term on the right is trivially in $\Con^{n-1,q-1}$, and hence for each $\alpha$ it vanishes in $\Con^{n-1,q}/\Con^{n-1,q-1}$. Consequently, (8) reduces to
\[
[\partial_n \Gamma^+_q(\theta)] = (-1)^{q+1}[\theta].
\]
in $\Con^{n-1,q}/\Con^{n-1,q-1}$. Again we have proved that every cycle is a boundary, completing the induction argument, and Theorem 10 is proved. □
**Remark 11** In some of the literature, cubical sets with connections have often been defined using only one of the two connection operators, $I_i^+$ and $I_i^-$, introduced in this paper. This convention appears in the earliest treatments \([9,10]\) as well as several later works \([2,7,12,17]\). We have used the alternate definition appearing in \([1]\) and also \([8]\), because it affords direct access to a richer and more interesting algebraic and topological structure. Since the subcomplex $\text{Con}^{p,0}(K) \subseteq \text{Con}(K)$ is the connection complex corresponding to definition with a single connection operator, our main result Theorem 10 applies to this version as well.

**Corollary 12** The short exact sequence of chain complexes

$$0 \rightarrow \text{Con}_n(K) \rightarrow C_{n+1}(K) \rightarrow C_{n+1}(K)/\text{Con}_n(K) \rightarrow 0$$

induces a long exact sequence of homology groups, and since $\mathcal{H}_n(\text{Con}(K))$ is trivial, we have $\mathcal{H}_n(C(K)) \cong \mathcal{H}_n(C(K)/\text{Con}(K))$.

It is well-known that, over a suitable category, the category of chain complexes and the category of crossed complexes are equivalent \([11]\). It would be interesting to see whether the results in this paper can be properly stated and extended to the context of crossed complexes.

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**Appendix: Homology of Cubical Sets and Homology of Their Geometric Realization**

Recall that $I^n$ is the geometric $n$-dimensional cube $[0,1]^n$. Let $(f_i^\alpha)^*: I^{n-1} \rightarrow I^n$ be the map sending $(x_1, \ldots, x_{n-1}) \in I^{n-1}$ to $(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_{n-1})$ where $y = 0$ if $\alpha = -$ and $y = 1$ if $\alpha = +$. Let further $(\varepsilon_i)^*: I^n \rightarrow I^{n-1}$ be the map sending $(x_1, \ldots, x_n)$ to $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. The geometric realization $|K|$ of a cubical set is the quotient space of the disjoint union $\bigsqcup I^n \times K_n$ by the equivalence relation $\sim$, which is generated by the following elementary equivalences: For $(x_1, \ldots, x_n) \in I^n$ and $\sigma \in K_{n-1}$ we set

$$((x_1, \ldots, x_n), \varepsilon_i(\sigma)) \sim ((\varepsilon_i)^*(x_1, \ldots, x_n), \sigma)$$

and, for $(x_1, \ldots, x_{n-1}) \in I^{n-1}$ and $\sigma \in K_n$, we set

$$((x_1, \ldots, x_{n-1}), f_i^\alpha(\sigma)) \sim ((f_i^\alpha)^*(x_1, \ldots, x_{n-1}), \sigma).$$

Then $|K|$ can be given the structure of a CW-complex whose (open) $n$-cells are the images $e^{(n)}_\sigma$ of the cells $I^n \times \{\sigma\}$ in $|K|$ for $\sigma \in K_n^{nd}$. Here $K_n^{nd}$ denotes the set of non-degenerate $n$-cubes in $K$, see \([12, \text{Remark 11.1.14}]\). Let $S(K)$ be the cellular chain complex of $|K|$. By the definition of $S(K)$ the cells $e^{(n)}_\sigma$ for $\sigma \in K_n^{nd}$ form a basis of its $n$th chain group $S_n(K)$. It is well known (see \([14, \text{Corollary 3.9.11}]\)) that identifying $\sigma \in K_n^{nd}$ with $e^{(n)}_\sigma$ yields the following isomorphism of chain complexes.

**Lemma 13** (Corollary 3.9.11 [14]) $C(K) \cong S(K)$.

If the cubical set $K$ is a cubical set with connections then there is an associated geometric realization $|K’|$ which is the quotient of the disjoint union $\bigsqcup I^n \times K_n$ by the equivalence.
relation $\sim'$, which is generated by (9), (10) and the relation
\begin{equation}
\left( \Gamma^{\alpha}_i \right)^* (x_1, \ldots, x_n), \sigma \sim' \left( (x_1, \ldots, x_n), \Gamma^{\alpha}_i (\sigma) \right)
\end{equation}
for $\sigma \in K_{n-1}$ and $(x_1, \ldots, x_n) \in I^n$. Here $(\Gamma^\alpha_i)^* : I^n \rightarrow I^{n-1}$ is defined by
\begin{equation}
(\Gamma^\alpha_i)^* (x_1, \ldots, x_n) = \begin{cases} (x_1, \ldots, x_{i-1}, \max (x_i, x_{i+1}), x_{i+2}, \ldots, x_n) & \text{if } \alpha = - \\ (x_1, \ldots, x_{i-1}, \min (x_i, x_{i+1}), x_{i+2}, \ldots, x_n) & \text{if } \alpha = + \end{cases}
\end{equation}
In particular, $\sim'$ is coarser than $\sim$ and hence $|K'|$ can be seen as a quotient of $|K|$ by the additional identifications implied by (11). Let $K^{ndc}_n$ be the set of $n$-cubes in $K$ that are neither degenerate nor connections.

In order to understand the relation between $|K|$ and $|K'|$ we need to understand the face structure of cubes in $K^{ndc}_n$. For that we consider for any cube $\sigma \in K_n$ the set of all of its faces $\tau$; i.e. all cubes $\tau$ such that $\tau = f^\alpha_i (\cdots (f^\alpha_j (\cdots (\sigma) \cdots ) ) )$ for a choice of $i_1, \ldots, i_r$ and $\alpha_1, \ldots, \alpha_r$. For $\sigma \in K$ we denote by $F_\sigma$ the set of its faces. We order the cubes from $K$ by saying that $\tau$ is smaller than $\sigma$ if $\tau$ is a face of $\sigma$. With this notation we are in position to formulate the following structural result on the role of non-degenerate and non-connection cubes in the face structure.

**Lemma 14** For any $\tau \in K_n$ there is a unique face $\rho$ of $\tau$ that is maximal with the property that it is neither degenerate nor a connection. Moreover, if $\tau = \varepsilon_i (\sigma)$ or $\tau = \Gamma^{\alpha}_i (\sigma)$ then $\rho$ is a subface of $\sigma$ and $\tau = g_k \cdots g_1 (\rho)$ for suitably chosen connection and degeneracy maps $g_1, \ldots, g_k$ for some $k \geq 0$.

**Proof** We prove the assertion by induction on the dimension $n$.

If $n = 0$ then $\tau$ is non-degenerate and non-connection. Hence $\tau$ itself is the maximal face we are looking for.

Let $n > 0$. If $\tau$ is neither degenerate nor a connection then again $\tau$ itself is the unique maximal face.

Let $\tau$ be degenerate, say $\tau = \varepsilon_i (\sigma)$ for some $i \in [n]$ and some $(n-1)$-cube $\sigma$. Then, by (iii) of Definition 1, $f^\alpha_i (\tau) = \sigma$ if $i = j$, and $f^\alpha_i (\tau) = \varepsilon_{i-1} (f^\alpha_j (\sigma))$ if $j < i$ and $= \varepsilon_i (f^\alpha_{i-1} (\sigma))$ if $j > i$. By induction, we know that there is an unique maximal non-degenerate and non-connection face $\rho$ of $\sigma$. We claim that $\rho$ is the unique maximal non-degenerate and non-connection face of $\tau$. By induction we know that each $\varepsilon_i (f^\beta_j (\sigma))$ has a unique maximal non-degenerate, non-connection face which is a subface of $f^\beta_j (\sigma)$ and hence of $\sigma$. In particular, they must be subfaces of $\rho$. If follows by induction that $\sigma = g_k \cdots g_1 (\rho)$ for a sequence of degeneracy and connection maps $g_1, \ldots, g_k$ and $k \geq 0$. Then $\tau = \varepsilon_i g_k \cdots g_1 (\rho)$.

Finally, consider the case that $\tau$ is a connection. Say $\tau = \Gamma^{\alpha}_i (\sigma)$ for some $i \in [n]$ and some $(n-1)$-cube $\sigma$. Notice that, by (iii) of Definition 2, every $(n-1)$-face of $\tau$ other than $\sigma$ is either $\Gamma^{\beta}_j (f^\alpha_i (\sigma))$ or $\varepsilon_j (f^\alpha_i (\sigma))$ for some $j \in [n]$, and $\alpha \in \{+, -, \}$. By induction $\sigma$ and any $\Gamma^{\beta}_j (f^\alpha_i (\sigma))$ have an unique maximal non-degenerate, non-connection face. Again by induction the latter are subfaces of $\sigma$. In particular, they must be subfaces of the unique maximal non-degenerate, non-connection face $\rho$ of $\sigma$. From the induction hypothesis it follows $\sigma = g_k \cdots g_1 (\rho)$ for a sequence of degeneracy and connection maps $g_1, \ldots, g_k$ and $k \geq 0$. Then $\tau = \Gamma^{\alpha}_i g_k \cdots g_1 (\rho)$. \qed

Note that along the same lines one can show that for any cube there is a unique maximal non-degenerate face.

The relations among the degeneracy and connection maps allow the following strengthening of Lemma 14.
Lemma 15 For any $\tau \in K_n$ there is a unique face $\rho$ of $\tau$ that is maximal with the property that it is neither degenerate nor a connection. Moreover, if $\tau$ is non-degenerate then $\tau = g_k \cdots g_1(\rho)$ for suitably chosen connection maps $g_1, \ldots, g_k$ and some $k \geq 0$.

Proof From Lemma 14 it follows that there is a unique maximal face $\rho$ of $\tau$ that is neither degenerate nor a connection. It also follows from that lemma that $\tau = g_k \cdots g_1(\rho)$, for degeneracy and connection maps $g_1, \ldots, g_k$. If all $g_i$ are connection maps we are done. Assume there is an $i$ such that $g_i$ is a degeneracy map. We claim that then $\tau$ is degenerate. We prove the claim by downward induction on the maximal $i$ such that $g_i$ is a degeneracy map. If $i = k$ then $\tau$ is degenerate, contradicting the assumptions. If $i < k$ then by Definition 2(iii) there is a connection or degeneracy map $g_i'$ and a degeneracy map $g_{i+1}$ such that

$$\tau = g_k \cdots g_{i+1} g_i' g_{i-1} \cdots g_1(\rho).$$

By induction this implies that $\tau$ is degenerate. □

Now we apply the results on the face structure in order to understand the attachment of cells in $|K|$ and $|K'|$. We assume without stating the proofs the following fact:

- Let $(x, \sigma), (y, \sigma) \in I_{\dim \sigma}^\sigma \times \{\sigma\}$. Then $(x, \sigma), (y, \sigma)$ are identified through the equivalence relation generated by $9, 10$ (resp. 9, 10 and 11) on $\bigsqcup_{\tau \in K} I_{\dim \tau}^{\dim \tau} \times \{\tau\}$ if and only if they are identified by the equivalence relation generated by $9, 10$ (resp. 9, 10 and 11) on $\bigsqcup_{\tau \in F_{\sigma}} I_{\dim \tau}^{\dim \tau} \times \{\tau\}$.

This fact allows us to consider the identifications by the equivalence relations we consider as local identifications among points in the cells corresponding to the faces of a given cell.

Lemma 16 Let $\tau \in K_n$ be such that $\tau = g_k \cdots g_1(\rho)$ for some cube $\rho$ and connection maps $g_1, \ldots, g_k$. Let $\sim_{\tau}$ be the restriction of the equivalence relation generated by (9), (10), (11) to $M_{\tau} = \bigsqcup_{\sigma \in F_{\tau}} I_{\dim \sigma}^{\dim \sigma} \times \{\sigma\}$ and define $\sim_{\rho}$ analogously. Then there is a retraction $p_{\tau} : M_{\tau} / \sim_{\tau} \to M_{\rho} / \sim_{\rho}$.

Proof We construct the retraction by induction on $k$. For $k = 0$ the identity is the desired retraction.

Let $k \geq 1$ and assume that for $\tau' = g_{k-1} \cdots g_1(\rho)$ there is such a retraction $p_{\tau'} : M_{\tau'} / \sim_{\tau'} \to M_{\rho} / \sim_{\rho}$. Then $\tau = g_k \tau'$. The equivalence relation on $I_{\dim \tau}^{\dim \tau} \times \{\tau\}$ induced by the connection map $g_k = I_i^{\beta}$ has equivalence classes being sets with fixed maximum or minimum of the $i$th and $(i + 1)$th coordinate depending on $\beta$ being $+$ or $-$. Each equivalence class has exactly two points that via the face maps $f_i^\beta$ and $f_i^{\beta+1}$ are identified with points in $I_{\dim \tau'}^{\dim \tau'} \times \{\tau'\}$, indeed both points are identified with the same point. The map that sends each equivalence class to the image of this point in $M_{\tau'} / \sim_{\tau'}$ provides a retraction from $M_{\tau} / \sim_{\tau}$ to $M_{\tau'} / \sim_{\tau'}$. Composing this retraction with the retraction from $p_{\tau'}$ provides the asserted retraction. This concludes the induction step. □

We now introduce the concept of pushing cells for a general CW-complex which we will then match with the process of passing from $|K|$ to $|K'|$ in our case. Let $X$ be a CW-complex where, for $n \geq 0$, $X_n = (e^{(n)}_\sigma)_{\sigma \in J_n}$ is the set of open $n$-cells in $X$ for some indexing set $J_n$. For each $\sigma \in J_n$ let $g_\sigma : \partial e^{(n)}_\sigma \to X^{(n-1)}$ be the attaching map. For some fixed $N \geq 0$, let $J_N \subseteq J_N$ be a subset of the index set of the cells in dimension $N$ such that, for each $\sigma \in J_N$,

- there is a $\tau \in J_\ell$ for some $\ell < N$ such that $\exists g_\sigma \subseteq e^{(\ell)}_\tau$, and
- for this $\tau$ there is a retraction $p_\sigma : e^{(N)}_\sigma \to e^{(\ell)}_\tau$.

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Now let $X^{\text{push}}$ be the CW-complex with $X^{\text{push}}_n = (c^{(n)}_{e_{\sigma}})_{\sigma \in J'_n}$ the open $n$-cells in $X^{\text{push}}$ where $J'_n = J_n$ for $n \neq N$ and $J'_N = J_N \setminus \bar{J}_N$ and attaching maps $g^{(N)}_{\sigma}(x) = g_{\sigma}(x)$ if $g_{\sigma}(x) \notin e^{(N)}_{\sigma}$ for some $\sigma \in \bar{J}_N$ and $g_{\sigma}(x) = p_{\sigma}(g_{\sigma}(x))$ otherwise. In this situation we say that $X^{\text{push}}$ arises from $X$ by pushing the cells $e^{(N)}_{\sigma}$ for $\sigma \in \bar{J}_N$.

Next we show that $|K|$ and $|K'|$ are examples of CW-complexes that arise from each other by pushing cells.

**Lemma 17** The geometric realization $|K'|$ is a CW-complex that arises from the CW-complex of the geometric realization $|K|$ by pushing the cells corresponding to connections successively by dimension in increasing order. In particular, $|K'|$ can be given the structure of a CW-complex with $n$-cells indexed by the $K^\text{ndc}_n$.

**Proof** Since the first connection cells (that are not already degenerate) arise in dimension 2, we can assume the following situation. For some $n \geq 2$ we have constructed a complex $X$ such that

(a) $X$ arises from $|K|$ by pushing all cells that correspond to connections of dimensions $< n$ where $n \geq 2$.

(b) $|K'| \sim_n X$ where $\sim_n$ is the equivalence relation which has singleton equivalence classes outside the closure of the cells of dimension $< n$ and equals (11) when applied to the union of the closures of all other cells.

Now let $\sigma \in K_n$ be a connection that is non-degenerate. Then by Lemma 14 there is a unique maximal face $\tau \in K_\ell$ of $\sigma$ which is non-degenerate and non-connection. Since all proper connection faces of $\sigma$ have been pushed the attaching map $g_{\sigma}$ of the $N$-cell $I^N$ corresponding to $\sigma$ has as its image the $\ell$-cell corresponding to $\tau$. Furthermore, by Lemma 15 the conditions of Lemma 16 are satisfied and there is a retraction $p_{\sigma}$ from then closure of the $N$-cell corresponding to $\sigma$ to the closure of the $\ell$-cell corresponding to $\sigma$. Moreover, by Lemma 16 the map $\sigma$ identifies the exactly those elements which lie in the same equivalence class of $\sim_n$.

Hence the conditions for a pushing to the cells corresponding to non-degenerate connections $\sigma$ are satisfied. It follows that (a) and (b) are satisfied for $\sim_n$.

Finally, we need to understand the impact of pushing cells on the cellular chain complex of a CW-complex.

**Lemma 18** Let $X$ be a CW-complex with cells $X_n = (e^{(n)}_{e_{\sigma}})_{\sigma \in J_n}$, $n \geq 0$. Assume that there is a dimension $N$ such that $X^{\text{push}}$ arises from $X$ by pushing the cells $e^{(N)}_{\sigma}$ for $\sigma \in \bar{J}_N \subseteq J_N$. Let

$$\partial e^{(n)}_{\sigma} = \sum_{\sigma' \in J_{n-1}} d_{\sigma,\sigma'} e^{(n-1)}_{\sigma'}$$

be the differential of the cellular chain complex associated to $X$. Then for $\sigma \in J_n \setminus J_N$, $\sigma' \in J_{n-1} \setminus J_N$ the coefficient $d^{\text{push}}_{\sigma,\sigma'}$ in the differential of the cellular chain complex of $X^{\text{push}}$ we have $d^{\text{push}}_{\sigma,\sigma'} = d_{\sigma,\sigma'}$.

**Proof** The coefficient $d_{\sigma,\sigma'}$ is given as the degree of the composition

$$S^{n-1} \cong \partial e^{(n)}_{\sigma} \otimes_{\mathbb{Z}} X^{(n-1)} \to X^{(n-1)} / \left( X^{(n-1)} \setminus e^{(n-1)}_{\sigma'} \right) \cong S^{n-1}.$$
The composition depends on the attaching maps $g_\sigma$ of the cells corresponding to $\sigma$ only. Now consider the same sequence in $X^{\text{push}}$, which in particular implies $\sigma, \sigma' \neq \tau$. Let $g'_{\sigma}$ be the corresponding attaching maps. If $g_\sigma(x) \notin \epsilon_{\tau}^{(N)}$ for some $\tau \in \tilde{J}_N$ then $g_\sigma(x) = g'_{\sigma}(x)$. If $g_\sigma(x) \in \epsilon_{\tau}^{(N)}$ for some $\tau \in \tilde{J}_N$ then $g'_{\sigma}(x) = p_\sigma(g_\sigma(x))$ for a retraction $p_\sigma$. But in the latter case $g_\sigma(x)$ and $g'_{\sigma}(x)$ lie in the complement of any $(n-1)$ cell different from $\epsilon_{\tau}^{(N)}$. In that situation the composition is again determined by $g_\sigma$. It follows that $d_\sigma,\sigma' = d_{\sigma,\sigma'}^{\text{push}}$. □

By definition $C_n(K)/\text{Con}_n(K)$ has a basis indexed by $K_n^{\text{ndc}}$. The differential of the complex $C_n(K)/\text{Con}_n(K)$ are arises from the differential in $C(K)$ in the following way. Let $\sigma\alpha$ is the differential of $\alpha \in K_n^{\text{ndc}}$ in $C_n(K)$ then we set all coefficients of element from $K_{n-1}^{\text{ndc}} \setminus K_{n-1}^{\text{ndc}}$ to 0. Now the following theorem is an immediate consequence of Lemma 18 and Lemma 17.

**Theorem 19** The cellular chain complex $S'(K)$ of $|K|$ is isomorphic to the quotient complex $C(K)/\text{Con}(K)$. In particular,

$$H_i(|K|) \cong H_i(S'(K)) \cong H_i(C(K)/\text{Con}(K)).$$

**Proof** The assertion follows immediately from Lemma 17 and Lemma 18. □

The theorem together with Corollary 12 implies the following.

**Corollary 20** Let $K$ be a cubical set with connections. Then

$$H_i(|K|) \cong H_i(S'(K)) \cong H_i(C(K)/\text{Con}(K)) \cong H_i(C(K)) \cong H_i(|K|).$$

This fact provides another motivation for the study of connections.

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