Self-intersections of the Riemann zeta function on the critical line

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Abstract

We show that the Riemann zeta function $\zeta$ has only countably many self-intersections on the critical line, i.e., for all but countably many $z \in \mathbb{C}$ the equation $\zeta(\frac{1}{2} + it) = z$ has at most one solution $t \in \mathbb{R}$. More generally, we prove that if $F$ is analytic in a complex neighborhood of $\mathbb{R}$ and locally injective on $\mathbb{R}$, then either the set

$$\{(a, b) \in \mathbb{R}^2 : a \neq b \text{ and } F(a) = F(b)\}$$

is countable, or the image $F(\mathbb{R})$ is a loop in $\mathbb{C}$.

1 Introduction

In the half-plane $\{s \in \mathbb{C} : \sigma > 1\}$ the Riemann zeta function is defined by the equivalent expressions

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}.$$  

In the extraordinary memoir of Riemann [7] it is shown that $\zeta$ extends to a meromorphic function on the entire complex plane with its only singularity being a simple pole at $s = 1$, and it satisfies the functional equation relating its values at $s$ and $1 - s$. The Riemann hypothesis asserts that every non-real zero of $\zeta$ lies on the critical line

$$\mathcal{L} := \{s \in \mathbb{C} : \sigma = \frac{1}{2}\}.$$  

By a self-intersection of $\zeta$ on the critical line we mean an element of the set

$$\{(s_1, s_2) \in \mathcal{L}^2 : s_1 \neq s_2 \text{ and } \zeta(s_1) = \zeta(s_2)\}.$$  

Our aim in the present paper is to prove that this set is countable.

Theorem 1. The Riemann zeta function has only countably many self-intersections on the critical line.
In other words, the equation $\zeta(\frac{1}{2} + it) = z$ has at most one solution $t \in \mathbb{R}$ for all but countably many $z \in \mathbb{C}$. This complements the fact that $\zeta(\frac{1}{2} + it) = z$ has at least two solutions $t \in \mathbb{R}$ for infinitely many $z \in \mathbb{C}$, which follows from a recent result of Banks and Kang [1, Theorem 1.2]. Moreover, it has been conjectured in [1] that $\zeta(\frac{1}{2} + it) = z$ has no more than two solutions $t \in \mathbb{R}$ for every nonzero $z \in \mathbb{C}$; our Theorem 1 makes it clear that there are at most countably many counterexamples to this conjecture.

There are two main ingredients in the proof of Theorem 1. The first is that the curve $f_{p, t, q} : \zeta(\frac{1}{2} + it)$ is locally injective on $\mathbb{R}$, i.e., for every $t \in \mathbb{R}$ there is an open real interval containing $t$ on which $f$ is one-to-one; this is Proposition 3 of §2. A result of Levinson and Montgomery [2] guarantees the local injectivity of $f$ around points $t$ for which $f_{p, t, q} \neq 0$, and in Proposition 3 the local injectivity of $f$ is also established around points $t$ with $f(t) = 0$.

The second ingredient in the proof of Theorem 1 is the following general result about self-intersections of locally injective analytic curves.

**Theorem 2.** Let $F$ be a function which is analytic in a complex neighborhood of the real line $\mathbb{R}$, and suppose that $F$ is locally injective on $\mathbb{R}$. If the set of self-intersections

$$\{(a, b) \in \mathbb{R}^2 : a \neq b \text{ and } F(a) = F(b)\}$$

is uncountable, then $F(\mathbb{R})$ is a loop in $\mathbb{C}$.

By a loop in $\mathbb{C}$ we mean the image of a continuous map $L : [\gamma, \delta] \to \mathbb{C}$ such that $L(\gamma) = L(\delta)$. Since every loop is compact, Theorem 2 applied with $F := f$ immediately implies Theorem 1 in view of the fact that $\zeta$ is unbounded on the critical line (see, for example, [8, Theorem 8.12]).

The proof of Theorem 2 in [3] relies on intersection properties of analytic curves that were first discovered by Markushevich [3] and were later extended by Mohon’ko [4, 5]; see Proposition 4 of [3] and the remarks that follow. We believe that the statement of Theorem 2 is new and may be of independent interest.

### 2 Local injectivity

**Proposition 3.** The curve

$$f(t) := \zeta(\frac{1}{2} + it) \quad (t \in \mathbb{R}) \quad (1)$$

is locally injective on $\mathbb{R}$.
Proof. For every \( a \in \mathbb{R} \), let \( \Sigma_a \) denote the collection of open intervals \( \mathcal{I} \) in \( \mathbb{R} \) that contain \( a \). For any fixed \( a \) we must show that \( f \) is one-to-one on an interval \( \mathcal{I} \in \Sigma_a \).

In the case that \( f(a) \neq 0 \) we use a result of Levinson and Montgomery \[2\] which states that \( \zeta(s) = 0 \) whenever \( \zeta'(s) = 0 \) and \( \sigma = \frac{1}{2} \). As \( f(a) \neq 0 \) we have \( f'(a) \neq 0 \), hence \( f \) is locally invertible in a complex neighborhood of \( a \); in particular, \( f \) is one-to-one on some interval \( \mathcal{I} \in \Sigma_a \).

Let \( \mathcal{Z} \) denote the set of all zeros of \( f \). If \( t \notin \mathcal{Z} \), we define

\[
\vartheta(t) := \arg f(t) = \Im \log \zeta\left(\frac{1}{2} + it\right)
\]

by continuous variation of the argument from 2 to \( 2 + it \) to \( \frac{1}{2} + it \), starting with the value 0, and we denote by \( N(t) \) the number of zeros \( \rho = \beta + i\gamma \) of \( \zeta(s) \) in the rectangle \( 0 < \beta < 1, 0 < \gamma < t \). Then

\[
\vartheta(t) = \pi \cdot (N(t) - 1) + g(t) \quad (t \notin \mathcal{Z}),
\]

where

\[
g(t) := -\arg \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) + \frac{t}{2} \log \pi \quad (t \in \mathbb{R});
\]

see, for example, Montgomery and Vaughan \[6\] Theorem 14.1. Then

\[
g'(t) = -\frac{1}{2} \Re \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{it}{2}\right) + \frac{1}{2} \log \pi \quad (t \in \mathbb{R}),
\]

and from the well known relation \( (\Gamma'/\Gamma)'(s) = \sum_{n=0}^{\infty} (n+s)^{-2} \) we see that

\[
g''(t) = -16t \sum_{n=0}^{\infty} \frac{4n + 1}{(4n + 1)^2 + 4t^2} \quad (t \in \mathbb{R}).
\]

Thus, if \( \Theta := 6.2898 \cdots \) is the unique positive root of the function defined by the right side of (2), it is easy to see that \( g \) is strictly decreasing at any \( t \in \mathbb{R} \) with \( |t| > \Theta \).

We now consider the case that \( f(a) = 0 \), i.e., \( a \in \mathcal{Z} \). Since \( \mathcal{Z} \) is a discrete subset of \( \mathbb{R} \), there exists an interval \( \mathcal{I} \in \Sigma_a \) for which \( \mathcal{I} \cap \mathcal{Z} = \{a\} \). Using a shorter interval \( \mathcal{I} \), if necessary, and noting that \( |a| \geq 14.1347 \cdots > \Theta \), we can also ensure that \( g \) is strictly decreasing on \( \mathcal{I} \), and that \( |g(t) - g(a)| < 1 \) for all \( t \in \mathcal{I} \). In particular, \( |g(t_1) - g(t_2)| < 2 \) for all \( t_1, t_2 \in \mathcal{I} \).
Now suppose that \( f(t_1) = f(t_2) \) with \( t_1, t_2 \in \mathcal{I} \). If \( f(t_1) = f(t_2) = 0 \), then \( t_1 = t_2 (= a) \) since \( \mathcal{I} \cap \mathcal{Z} = \{a\} \). If \( f(t_1) = f(t_2) \neq 0 \) we have
\[
0 = \vartheta(t_1) - \vartheta(t_2) = \pi(N(t_1) - N(t_2)) + g(t_1) - g(t_2),
\]
and therefore \( g(t_1) - g(t_2) \in (-2, 2) \cap \pi\mathcal{Z} = \{0\} \), i.e., \( g(t_1) = g(t_2) \). Since \( g \) is strictly decreasing on \( \mathcal{I} \), we again have \( t_1 = t_2 \). This shows that \( f \) is one-to-one on \( \mathcal{I} \), and the proof is complete. \(
\)

3 Analytic functions sharing a curve

As in \( \S 2 \), for every \( a \in \mathbb{R} \) we use \( \Sigma_a \) to denote the collection of open real intervals that contain \( a \).

For each \( \mathcal{I} \in \Sigma_0 \) we put \( \mathcal{I}^+ := \mathcal{I} \cap [0, \infty) \) and \( \mathcal{I}^- := \mathcal{I} \cap (-\infty, 0] \). Given a function \( F \) defined in a complex neighborhood of zero, we say that
\[
\begin{itemize}
  \item the curve of \( F \) bounces back at zero if there are arbitrarily short intervals \( \mathcal{I} \in \Sigma_0 \) such that \( F(\mathcal{I}^+) = F(\mathcal{I}^-) \).
\end{itemize}
\]

Next, given two functions \( F, G \) defined in a complex neighborhood of zero, we say that
\[
\begin{itemize}
  \item \( F, G \) share a curve around zero if there are arbitrarily short intervals \( \mathcal{I}, \mathcal{J} \in \Sigma_0 \) for which \( F(\mathcal{I}) = G(\mathcal{J}) \);
  \item \( F, G \) roughly agree near zero if there are two sequences of nonzero real numbers, \( (u_k)_{k=1}^{\infty} \) and \( (v_k)_{k=1}^{\infty} \), such that \( u_k \to 0 \) and \( v_k \to 0 \) as \( k \to \infty \), and \( F(u_k) = G(v_k) \) for all \( k \geq 1 \);
  \item \( F, G \) roughly agree to the right of zero if there are two sequences of positive real numbers, \( (u_k)_{k=1}^{\infty} \) and \( (v_k)_{k=1}^{\infty} \), such that \( u_k \to 0^+ \) and \( v_k \to 0^+ \) as \( k \to \infty \), and \( F(u_k) = G(v_k) \) for all \( k \geq 1 \).
\end{itemize}
\]

Proposition 4. Suppose that \( F, G \) are non-constant and analytic in some neighborhood of zero, and that neither curve bounces back at zero. Then the following statements are equivalent:
\[
\begin{itemize}
  \item[(i)] \( F, G \) roughly agree near zero;
  \item[(ii)] \( F, G \) share a curve around zero.
\end{itemize}
\]

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Remarks: With a few modifications to the proof given below, one can establish the more general statement: If $F, G$ are non-constant and analytic in a neighborhood of zero and roughly agree to the right (or left) of zero, then either $F, G$ share a curve around zero, or at least one of the two curves bounces back at zero. After proving Proposition 4 we discovered that the result is essentially contained in the (untranslated) work of Mohon’ko [4, 5], who extended earlier results of Markushevich [3, Vol. III, Theorem 7.20] from regular analytic curves to arbitrary analytic curves. Here, we present our own proof for the sake of completeness and for the convenience of the reader.

We begin with the observation that Proposition 4 is a consequence of the following statement.

Lemma 5. If $F, G$ satisfy the hypotheses of Proposition 4 and roughly agree to the right of zero, then they share a curve around zero.

Indeed, suppose the lemma has already been established. The implication $(ii) \Rightarrow (i)$ of Proposition 4 being clear, we need only show that $(i) \Rightarrow (ii)$. To this end, let $\mathcal{U} := (u_k)_{k=1}^\infty$ and $\mathcal{V} := (v_k)_{k=1}^\infty$ be sequences of nonzero real numbers such that $u_k \to 0$ and $v_k \to 0$ as $k \to \infty$, and $F(u_k) = G(v_k)$ for all $k \geq 1$. Replacing $\mathcal{U}$ with a subsequence, if necessary, we can assume that all of the numbers in $\mathcal{U}$ are of the same sign, and likewise for $\mathcal{V}$; that is, for some $\varepsilon_\mathcal{U}, \varepsilon_\mathcal{V} \in \{\pm 1\}$ we have $u_k = \varepsilon_\mathcal{U}|u_k|$ and $v_k = \varepsilon_\mathcal{V}|v_k|$ for all $k \geq 1$. Now, the functions $F_1, G_1$ defined by

$$F_1(z) := F(\varepsilon_\mathcal{U}z) \quad \text{and} \quad G_1(z) := G(\varepsilon_\mathcal{V}z)$$

are non-constant and analytic in a neighborhood of zero, and neither the curve of $F_1$ nor that of $G_1$ bounces back at zero. Also, $F_1, G_1$ roughly agree to the right of zero since $|u_k| \to 0^+$ and $|v_k| \to 0^+$ as $k \to \infty$, and

$$F_1(|u_k|) = F(u_k) = G(v_k) = G_1(|v_k|) \quad (k \geq 1).$$

By Lemma 5 it follows that $F_1, G_1$ share a curve around zero. Hence, there are arbitrarily short intervals $\mathcal{I}, \mathcal{J} \in \Sigma_0$ such that

$$F(\varepsilon_\mathcal{U}\mathcal{I}) = F_1(\mathcal{I}) = G_1(\mathcal{J}) = G(\varepsilon_\mathcal{V}\mathcal{J}).$$

Since the intervals $\varepsilon_\mathcal{U}\mathcal{I}, \varepsilon_\mathcal{V}\mathcal{J}$ lie in $\Sigma_0$ and are arbitrarily short, it follows that $F, G$ share a curve around zero. Thus, $(i) \Rightarrow (ii)$ as required.
Proof of Lemma 5. Let \((u_k)_{k=1}^\infty\) and \((v_k)_{k=1}^\infty\) be two sequences of positive real numbers tending to zero such that \(F(u_k) = G(v_k)\) for all \(k \geq 1\). Clearly, \(F(0) = G(0)\), and we can assume \(F(0) = G(0) = 0\) without loss of generality. Since the functions \(F, G\) are non-constant and analytic, we can write
\[
F(z) = a_m z^m + a_{m+1} z^{m+1} + \cdots \quad \text{and} \quad G(z) = b_n z^n + b_{n+1} z^{n+1} + \cdots
\]
for all \(z\) in a neighborhood of zero, where \(m, n \geq 1\) and \(a_m b_n \neq 0\). As neither \(F\) nor \(G\) is identically zero, we have \(F(u_k) = G(v_k) \neq 0\) for all large \(k\).

Noting that \(v_k^m/u_k^m > 0\) and
\[
\frac{b_n v_k^m}{a_m u_k^m} = \frac{F(u_k)}{a_m u_k^m} \frac{b_n v_k^m}{G(v_k)} = \frac{1 + O(u_k)}{1 + O(v_k)} \to 1 \quad (k \to \infty),
\]
it follows that \(b_n/a_m > 0\); hence, replacing \(F, G\) with \(a_m^{-1} F, a_m^{-1} G\) we can assume without loss of generality that \(a_m = 1\) and \(b_n > 0\).

Now put
\[
\ell := \text{lcm}[m, n], \quad m_1 := \frac{\ell}{m}, \quad n_1 := \frac{\ell}{n}, \tag{3}
\]
and write
\[
F_1(z) := F(z^{m_1}) = z^\ell + \cdots + z^\ell(1 + F_2(z)),
\]
\[
G_1(z) := G(z^{n_1}) = b_n z^\ell + \cdots + (c_n z)^\ell(1 + G_2(z)),
\]
where \(F_2, G_2\) are analytic in a neighborhood of zero, with \(F_2(0) = G_2(0) = 0\), and \(c_n = b_n^{1/\ell} > 0\). The functions
\[
F_3(z) := z \cdot \exp\big(\ell^{-1} \log(1 + F_2(z))\big),
\]
\[
G_3(z) := c_n z \cdot \exp\big(\ell^{-1} \log(1 + G_2(z))\big),
\]
are well-defined and analytic near zero, where \(\log\) denotes the principal branch of the logarithm, and since \(F_3'(0) = 1 \neq 0\) and \(G_3'(0) = c_n \neq 0\) these functions are invertible near zero. Therefore, we can define \(H = F_3^{-1} \circ G_3\) as an (invertible) analytic function in some neighborhood of zero. Now, since \(F_3^\ell = F_1\) and \(G_3^\ell = G_1\), for all large \(k\) we have
\[
F_3(u_k^{1/m_1})^\ell = F_1(u_k^{1/m_1}) = F(u_k) = G(v_k) = G_1(v_k^{1/n_1}) = G_3(v_k^{1/n_1})^\ell,
\]
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and thus $F_3(u_k^{1/m_1}) = \xi_k \cdot G_3(v_k^{1/m_1})$ holds with some $\ell$-th root of unity $\xi_k$. On the other hand, for all large $k$ the relation

$$\frac{F_3(u_k^{1/m_1})}{G_3(v_k^{1/m_1})} = \frac{u_k^{1/m_1}}{c_n v_k^{1/m_1}} \exp \left( \ell^{-1} \log \left( \frac{1 + F_2(u_k^{1/m_1})}{1 + G_2(v_k^{1/m_1})} \right) \right)$$

implies that

$$\arg \xi_k = (i \ell)^{-1} \log \left( \frac{1 + F_2(u_k^{1/m_1})}{1 + G_2(v_k^{1/m_1})} \right) \to 0 \quad (k \to \infty);$$

thus, for all sufficiently large $k$ we must have $\xi_k = 1$, $F_3(u_k^{1/m_1}) = G_3(v_k^{1/m_1})$, and $H(v_k^{1/m_1}) = u_k^{1/m_1}$, and it follows that $H$ defines a real-valued invertible real analytic function on some interval $K \in \Sigma_0$. 

From the relation

$$G_3(t) = F_3(H(t)) \quad (t \in K),$$

we see that

$$G(t^{n_1}) = G_1(t) = F_1(H(t)) = F(H(t)^{m_1}) \quad (t \in K). \quad (4)$$

We claim that $n_1$ is odd. Indeed, suppose on the contrary that $n_1$ is even. From the definition it follows that $\gcd(m_1, n_1) = 1$, hence $m_1$ is odd in this case. Since $H$ is real-valued and invertible, $H(0) = 0$, and $H$ maps $v_k^{1/m_1}$ to $u_k^{1/m_1} > 0$ if $k$ is large, we have $H(K^+) \subseteq [0, \infty)$ and $H(K^-) \subseteq (-\infty, 0]$. Thus, if $\varepsilon > 0$ is small enough so that the interval $L := (H(-\varepsilon)^{m_1}, H(\varepsilon)^{m_1}) \subseteq \Sigma_0$ is contained in $K$, then $F(L^+) = F(L^-)$ since by (4):

$$F(H(t)^{m_1}) = G(t^{n_1}) = G((-t)^{n_1}) = F(H(-t)^{m_1}) \quad (t \in [0, \varepsilon]).$$

But this means that the curve of $F$ bounces back at zero, which contradicts our original hypothesis on $F$. This contradiction establishes our claim that $n_1$ is odd. A similar argument shows that $m_1$ is also odd.

Since $m_1$ and $n_1$ are both odd, the intervals $I := H(K)^{m_1}$, $J := K^{n_1}$, are both open and contain zero, thus $I, J \in \Sigma_0$. By (4) we have $F(I) = G(J)$. Replacing $K$ by an arbitrarily short interval in $\Sigma_0$, the intervals $I, J$ become arbitrarily short as well, and therefore $F, G$ share a curve around zero. 

\[ \text{To see why $H$ is real-valued on some in $\Sigma_0$, observe that $h(z) := H(z) - H(z)$ is analytic near zero, $h(v_k^{1/m_1}) = 0$ for all large $k$ since $u_k, v_k \in \mathbb{R}$, and $v_k^{1/m_1} \to 0$ as $k \to \infty$, hence by the principle of analytic continuation $h$ must vanish identically near zero.} \]
The ideas presented above can be adapted as follows. Given \( a, b \in \mathbb{R} \) and a function \( F \) defined in some complex neighborhoods of \( a \) and \( b \), we say that

- \( F \) shares curves around \( a, b \) if there exist arbitrarily short intervals \( I \in \Sigma_a \) and \( J \in \Sigma_b \) such that \( F(I) = F(J) \).

This is equivalent to the statement that \( F_a, F_b \) share a curve around zero, where for each \( a \in \mathbb{R} \) we denote by \( F_a \) the function given by \( F(z) = F(z + a) \).

When viewed as a function of a real variable, if \( F \) is one-to-one locally at \( a \) and \( b \), then the curves of \( F_a, F_b \) cannot bounce back at zero; hence, the next statement is an easy consequence of Proposition 4.

**Lemma 6.** Fix \( a, b \in \mathbb{R} \). Suppose that \( F \) is analytic in some neighborhoods of \( a \) and \( b \), and that there are intervals in both \( \Sigma_a \) and \( \Sigma_b \) on which \( F \) is one-to-one. Then the following statements are equivalent:

(i) \( F \) shares curves around \( a, b \);

(ii) there are sequences \( (u_k)_{k=1}^\infty \) and \( (v_k)_{k=1}^\infty \) such that
   - \( u_k \neq a \) and \( v_k \neq b \) for all \( k \geq 1 \);
   - \( u_k \to a \) and \( v_k \to b \) as \( k \to \infty \);
   - \( F(u_k) = F(v_k) \) for all \( k \geq 1 \).

For real numbers \( \alpha < \beta \) we now assume that \( F \) is analytic in a complex neighborhood of the interval \( [\alpha, \beta] \), and that \( F \) is locally injective at every point of \( [\alpha, \beta] \), i.e., for any \( a \in [\alpha, \beta] \) there is an interval in \( \Sigma_a \) on which \( F \) is one-to-one. By a self-intersection of \( F \) on \( [\alpha, \beta] \) we mean an element of

\[ S_{\alpha, \beta} := \{(a, b) \in [\alpha, \beta]^2 : a \neq b \text{ and } F(a) = F(b)\} \]

Our goal is to understand the structure of this set.

**Lemma 7.** The set \( S_{\alpha, \beta} \) is compact.

**Proof.** Since \( [\alpha, \beta]^2 \) is compact, it suffices to show that \( S_{\alpha, \beta} \) is closed.

Let \((a, b) \in [\alpha, \beta]^2\) and suppose that \((u_k, v_k) \to (a, b)\) as \( k \to \infty \), where \((u_k, v_k) \in S_{\alpha, \beta}\) for all \( k \geq 1 \). Since \( F(u_k) = F(v_k) \) for each \( k \), the continuity of \( F \) implies that \( F(a) = F(b) \).

Assume that \( a = b \). Since \( F \) is locally injective at \( a \), there is an interval \( I \in \Sigma_a \) on which \( F \) is one-to-one. On the other hand, for all large \( k \) we have \( u_k, v_k \in I \), \( u_k \neq v_k \), and \( F(u_k) = F(v_k) \), which shows that \( F \) is not one-to-one on \( I \). The contradiction implies that \( a \neq b \), hence \((a, b) \in S_{\alpha, \beta} \).
We now express $S_{\alpha,\beta}$ as a disjoint union $S^*_{\alpha,\beta} \cup S^\circ_{\alpha,\beta}$, where $S^*_{\alpha,\beta}$ is the set of limit points of $S_{\alpha,\beta}$, and $S^\circ_{\alpha,\beta}$ is the set of isolated points in $S_{\alpha,\beta}$. The next lemma provides a useful characterization of the set $S^*_{\alpha,\beta}$.

**Lemma 8.** We have

$$S^*_{\alpha,\beta} = \{(a, b) \in [\alpha, \beta]^2 : a \neq b \text{ and } F \text{ shares curves around } a, b\}.$$  \hspace{1cm} (5)

**Proof.** Let $T$ denote the set on the right side of (5). In view of Lemma 6, the inclusion $T \subseteq S^*_{\alpha,\beta}$ is clear. On the other hand, if $(a, b)$ is a limit point of $S_{\alpha,\beta}$ that does not lie in $T$, then one of the following statements is true:

(i) there is a sequence $(u_k)_{k=1}^\infty$ such that $u_k \neq a$ for all $k \geq 1$, $u_k \to a$ as $k \to \infty$, and $F(u_k) = F(b)$ for all $k \geq 1$;

(ii) there is a sequence $(v_k)_{k=1}^\infty$ such that $v_k \neq b$ for all $k \geq 1$, $v_k \to b$ as $k \to \infty$, and $F(a) = F(v_k)$ for all $k \geq 1$.

However, since $F$ is analytic in a neighborhood of $[\alpha, \beta]$, either statement implies that $F$ is constant on the same neighborhood, contradicting the local injectivity of $F$ on $[\alpha, \beta]$. This shows that $S^*_{\alpha,\beta} \setminus T = \emptyset$, and (5) follows. \hfill \square

**Lemma 9.** For every $(a, b) \in S^*_{\alpha,\beta}$ there are intervals $I \in \Sigma_a$, $J \in \Sigma_b$, and a continuous bijection $\phi_{a,b} : I \to J$ such that $\phi_{a,b}(a) = b$, and $(t, \phi_{a,b}(t)) \in S^*_{a,\beta}$ for all $t \in I$. Moreover, $\phi_{a,b}$ is a strictly increasing function of $t$.

**Proof.** For any $(a, b) \in S^*_{\alpha,\beta}$, there are arbitrarily short open intervals $I \in \Sigma_a$ and $J \in \Sigma_b$ such that $F(I) = F(J)$. Since $a \neq b$ we can assume $I \cap J = \emptyset$. Since $F$ is locally injective, we can further assume $F$ is one-to-one on $I$ and on $J$. Then $\phi_{a,b}$ is the map obtained via the composition

$$I \xrightarrow{F|_I} F(I) \xrightarrow{(F|_J)^{-1}} J.$$

Clearly, $(t, \phi_{a,b}(t)) \in S_{\alpha,\beta}$ for all $t \in I$, and as $I$ is open and $\phi_{a,b}$ is continuous, every such $(t, \phi_{a,b}(t))$ is a limit point of $S_{\alpha,\beta}$. This proves the first statement.

Observe that, since $F$ is one-to-one on $I$ and $\phi_{a,b}$ is continuous, it follows that $\phi_{a,b}$ is either strictly increasing or strictly decreasing on its interval of definition. Thus, to finish the proof it suffices to show that the set

$$S'_{\alpha,\beta} := \{(a, b) \in S^*_{\alpha,\beta} : \phi_{a,b} \text{ is strictly decreasing on } I\}$$
is empty. First, we claim that $S'_{\alpha,\beta}$ is closed, hence compact by Lemma 7. Indeed, if $(a, b)$ is any limit point of $S'_{\alpha,\beta}$, then $(a, b)$ lies in the closed set $S^*_{\alpha,\beta}$. Let $I, J, \phi_{a,b}$ be defined as before. Since $(a, b)$ is a limit point, we can find $(u, v) \in S'_{\alpha,\beta}$ for which $u \in I$, $v \in J$. As $\phi_{u,v}$ is strictly decreasing near $u$, the same is true of $\phi_{a,b}$, hence $\phi_{a,b}$ is strictly decreasing on $I$. This implies that $(a, b) \in S'_{\alpha,\beta}$, and the claim follows.

Assume that $S'_{\alpha,\beta} \neq \emptyset$. Since $S'_{\alpha,\beta}$ is compact, there exists $(a, b) \in S'_{\alpha,\beta}$ for which

$$|a - b| = \min_{(u,v) \in S'_{\alpha,\beta}} |u - v| \neq 0. \quad (6)$$

Interchanging $a$ and $b$, if necessary, we can assume that $a < b$. Let $I, J, \phi_{a,b}$ be defined as before. Since $\phi_{a,b}$ is strictly decreasing on $I$, using the first statement of the lemma it is easy to see that $(t, \phi_{a,b}(t))$ lies in $S'_{\alpha,\beta}$ for all $t \in I$; in particular, if $u \in I$, $u > a$, and $v := \phi_{a,b}(u)$, then $(u, v) \in S'_{\alpha,\beta}$ with $v < b$. But then we have $|u - v| < |a - b|$, which contradicts $|a - b|$. Hence, it must be the case that $S'_{\alpha,\beta} = \emptyset$, and we are done.

Given $a, b \in [\alpha, \beta]$, we write $a \equiv b$ and say that $a$ is $F$-equivalent to $b$ if $a = b$ or $(a, b) \in S^*_{\alpha,\beta}$; otherwise, we say that $a$ and $b$ are $F$-inequivalent. Using Lemma 6 one verifies that this defines an equivalence relation on $[\alpha, \beta]$. We also write $a \prec b$ and say that $a$ is the immediate $F$-predecessor of $b$ if $a < b$, $a \equiv b$, and there is no number $c$ such that $a < c < b$ and $c \equiv a$.

It is important to note that if $a \equiv b$ and $a < b$, then $b$ has an immediate $F$-predecessor $c$ that lies in the interval $[a, b)$; this follows from the fact that the numbers in a fixed $F$-equivalence class are isolated since $F$ is analytic and locally injective.

**Proposition 10.** Suppose that $\gamma, \delta$ are $F$-equivalent numbers in $[\alpha, \beta]$, and that $\gamma$ is the immediate $F$-predecessor of $\delta$. Then

(i) every number in $[\alpha, \beta]$ is $F$-equivalent to some number in $[\gamma, \delta]$;

(ii) distinct numbers in $[\gamma, \delta]$ are $F$-inequivalent.

**Proof.** To prove (i), suppose on the contrary that there exists $\xi \in [\alpha, \beta]$ which is $F$-inequivalent to every number in $[\gamma, \delta]$. To achieve the desired contradiction, we examine three cases separately.

**Case 1:** $\xi \in [\gamma, \delta]$. Since $\gamma \equiv \delta$, we have $\xi \in (\gamma, \delta)$, so this case is not possible as $\xi$ is $F$-equivalent to itself.
Case 2: $\xi \in [\alpha, \gamma)$. Under this assumption, the supremum

\[ \eta := \sup \{ \tau \in [\alpha, \gamma) : \tau \text{ is } F\text{-inequivalent to every number in } [\gamma, \delta) \} \]

exists, and we have $\eta \in [\alpha, \gamma)$. First, we claim that $\eta \neq \gamma$. Assume on the contrary that $\eta \equiv \gamma \equiv \delta$. We cannot have $\eta = \xi$ since $\xi$ is $F$-inequivalent to every number in $[\gamma, \delta)$; therefore, $\eta > \xi \geq \alpha$. By Lemma 9, $t \equiv \phi_{\eta, \delta}(t)$ for all $t$ in some open interval $I \in \Sigma_{\eta}$. Since $\phi_{\eta, \delta}$ is strictly increasing on $I$ and $\phi_{\eta, \delta}(\eta) = \delta$, we see that every $\tau \in [\alpha, \eta)$ which is sufficiently close to $\eta$ is $F$-equivalent to a number in $[\gamma, \delta)$, namely $\phi_{\eta, \delta}(\tau)$. This contradicts the definition of $\eta$ and thereby establishes the claim.

Now let $(u_k)_{k=1}^{\infty}$ be a strictly decreasing sequence in the interval $(\eta, \gamma)$, with $u_k \to \eta^+$ as $k \to \infty$. From the definition of $\eta$, each $u_k$ is $F$-equivalent to some $v_k \in [\gamma, \delta) \subseteq [\gamma, \delta]$. Since $S_{\alpha, \beta}^{*}$ and $[\gamma, \delta]$ are compact, choosing a subsequence (if necessary) we can assume that $(u_k, v_k) \to (\eta, \lambda) \in S_{\alpha, \beta}^{*}$ with some $\lambda \in [\gamma, \delta]$. Moreover, such $\lambda$ lies in the open interval $(\gamma, \delta)$ since $\eta \neq \gamma$ as shown above. However, using Lemma 9 again, it follows that any $t$ that is sufficiently close to $\eta$ is $F$-equivalent to a number in $[\gamma, \delta)$, namely $\phi_{\eta, \lambda}(t)$, and this again contradicts the definition of $\eta$. Therefore, the case $\xi \in [\alpha, \gamma)$ is not possible.

Case 3: $\xi \in (\delta, \beta]$. This is also impossible by an argument similar to that given in Case 2, using instead the definition

\[ \eta := \inf \{ \tau \in (\delta, \beta] : \tau \text{ is } F\text{-inequivalent to every number in } [\gamma, \delta) \} \]

We omit the details. This completes our proof of (i).

Turning to the proof of (ii), suppose on the contrary that $\mu \equiv \theta$ and $\gamma \leq \mu < \theta < \delta$; without loss of generality, we can assume that $\mu \nless \theta$. Since $\gamma \nless \delta$, the interval $(\gamma, \delta)$ contains no number that is $F$-equivalent to $\gamma$; therefore, $\theta \neq \gamma$, so $\mu \neq \gamma$ and $\gamma < \mu$.

Now, fix an arbitrary number $\lambda \in (\theta, \delta)$. Applying the previous result (i) with the sets $[\mu, \theta)$ and $[\gamma, \lambda)$ in place of $[\gamma, \delta]$ and $[\alpha, \beta]$, respectively, we see that every number in $[\gamma, \lambda)$ is $F$-equivalent to some number in $[\mu, \theta)$; in particular, $\gamma$ is $F$-equivalent to a number $\eta$ for which $\gamma < \mu \leq \eta < \theta < \delta$. But this is impossible since $\gamma \nless \delta$, and the contradiction completes the proof of (ii).

Proof of Theorem 2. Suppose that $S$ is uncountable, and put

\[ S^* := \{(a, b) \in \mathbb{R}^2 : a \neq b \text{ and } F \text{ shares curves around } a, b \} \]
We claim that $S^* \neq \emptyset$. Indeed, assuming that $S^* = \emptyset$, we have $S_{-n,n}^* = \emptyset$ for every natural number $n$, and therefore $S_{-n,n} = S_{-n,n}^*$ is finite as it is a closed and isolated subset of the compact set $[-n,n]^2$. But then $S = \bigcup_{n \geq 1} S_{-n,n}$ is a countable union of finite sets, hence countable. This contradiction yields the claim.

Now, since $S^* \neq \emptyset$, we can find distinct real numbers $\gamma, \delta$ such that $\gamma \simeq \delta$, and we can assume that $\gamma \not\implies \delta$. As $[\gamma, \delta] \subseteq [-n,n]$ for all large $n$, and $F$ is constant on every $F$-equivalence class, Proposition 10 implies that $F([-n,n]) = F([\gamma, \delta])$ for all large $n$. Letting $n \to \infty$, we see that $C = F([\gamma, \delta])$, which is a loop in $\mathbb{C}$ since $F(\gamma) = F(\delta)$.  

\section{Concluding remarks}

It would be interesting to determine whether the appropriate analogue of Theorem 1 can be established for arbitrary automorphic $L$-functions. For many functions in the Selberg class, including all Dirichlet $L$-functions, an analogue of Voronin’s Universality Theorem is available and thus one can prove that those $L$-functions are unbounded on the critical line. However, establishing the local injectivity of such functions may be difficult.

With a little extra work, Theorem 2 can be generalized and applied to arbitrary analytic curves on $\mathbb{R}$ (or on an interval in $\mathbb{R}$). For such functions $F$, if the set of self-intersections is uncountable, then the curve continues to wind in the same direction around a loop (when $F$ is locally injective), or else it bounces back at least once along a regular curve, which need not be a loop nor even bounded.

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\section*{References}

[1] W. Banks and S. Kang, ‘On repeated values of the Riemann zeta function on the critical line,’ Experiment. Math. 12 (2012), no. 2, 132–140.
[2] N. Levinson and H. L. Montgomery, ‘Zeros of the derivatives of the Riemann zeta function,’ Acta Math. 133 (1974), 49–65.

[3] A. I. Markushevich, Theory of functions of a complex variable. Vol. I, II, III. Chelsea Publishing Co., New York, 1977.

[4] A. Z. Mohon’ko, ‘Certain properties of analytic curves. I,’ (Russian) Teor. Funkciĭ Funkcional. Anal. i Priložen 21 (1974), 3–13, 126.

[5] A. Z. Mohon’ko, ‘Certain properties of analytic curves. II,’ (Russian) Teor. Funkciĭ Funkcional. Anal. i Priložen 21 (1975), 111–118, 162.

[6] H. L. Montgomery and R. C. Vaughan, Multiplicative number theory I. Classical theory. Cambridge Studies in Advanced Mathematics, 97. Cambridge University Press, Cambridge, 2007.

[7] B. Riemann, ‘Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse,’ Monatsberichte der Berliner Akademie, 1859.

[8] E. C. Titchmarsh, The theory of the Riemann zeta-function. Second edition. The Clarendon Press, Oxford University Press, New York, 1986.