Inductive limits of compact quantum groups and their unitary representations

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Abstract
We introduce inductive limits of compact quantum groups and their unitary representation theories. Those give a more explicit representation-theoretic meaning to our previous study of quantization of characters and central probability measures in the asymptotic representation theory. We also study tensor product representations of the infinite-dimensional quantum unitary group. As a by-product, we give an explicit representation-theoretic interpretation to certain transformations that play an important role in analyzing $q$-central probability measures on the paths in the $q$-Gelfand–Tsetlin graph.

Keywords Asymptotic representation theory · Quantum groups · Infinite-dimensional quantum groups · Operator algebras · Spherical representations · Spherical functions.

Mathematics Subject Classification 20C15, 17B37, 46L65

1 Introduction
The asymptotic representation theory is the unitary representation theory of inductive limit groups of compact groups (see [6,11,17], etc.). One of the successful operator algebraic approaches is based on Stratila–Voiculescu AF-algebras (see [23]). Actually, the characters of inductive limit groups correspond to certain traces of the associated Stratila–Voiculescu AF-algebras. In our previous works [20,21], we have initiated extending the asymptotic representation theory to quantum groups. We particularly introduced a character theory of inductive systems of compact quantum groups based on Stratila–Voiculescu AF-algebras. Then, we gave a representation-theoretic
interpretation to Gorin’s asymptotic analysis [9] of the \(q\)-Gelfand–Tsetlin graph in terms of \(q\)-deformed quantum groups. However, we have not yet constructed any explicit “inductive limit quantum groups” of compact quantum groups. We need a new framework of operator algebraic quantum groups because we need a quantization of not locally compact groups. Actually, the inductive limit topology of the infinite-dimensional unitary group \(U(\infty) = \lim_{N \to \infty} U(N)\) is not locally compact.

The purpose of this paper is to give a natural inductive limit of compact quantum groups. When \(G = (G_N)_{N=0}^{\infty}\) is an inductive system of compact quantum groups, then compact quantum group \(W^*\)-algebras \(W^*(G_N)\) (in the sense of Yamagami [32]) form an inductive system of \(W^*\)-algebras and unital normal \(*\)-homomorphisms. Our first result is that Takeda’s inductive limit (see [26]), denoted by \(W^*(G_{\infty})\) here, has a natural quantum group structure similar to Woronowicz algebra of Masuda–Nakagami [14]. Namely, \(W^*(G_{\infty})\) has a comultiplication \(\hat{\delta}_{\infty}\), a unitary antipode \(\hat{R}_{\infty}\), and a deformation automorphism group \(\hat{\tau}_{\infty}\). Moreover, Takeda’s construction admits that the Stratila–Voiculescu AF-algebra \(\mathfrak{A}(G)\) of \(G\) sits in \(W^*(G_{\infty})\) as a \(\sigma\)-weakly dense \(*\)-subalgebra. More importantly, all the normal \(\hat{\tau}_{\infty}\)-KMS states on \(W^*(G_{\infty})\) completely correspond to our previous quantized characters defined on \(\mathfrak{A}(G)\) as restriction or extension. Therefore, we can reformulate our previous character theory of \(G\) in terms of \(G_{\infty} = (W^*(G_{\infty}), \hat{\delta}_{\infty}, \hat{R}_{\infty}, \hat{\tau}_{\infty})\) except for the “inductive limit topology” on the set of quantized characters of \(G_{\infty}\). In fact, \(W^*(G_{\infty})\) is too large and needs a smaller subalgebra like \(\mathfrak{A}(G)\) to introduce a natural topology on the quantized characters. See Remark 3.2.

The introduction of this explicit inductive limit \(G_{\infty}\) of \(G\) allows us to give a natural unitary representation theory of \(G_{\infty}\) rather than \(G\). We can consequently formulate challenging problems about the fusion rules of \(G_{\infty}\). For instance, it is a natural problem to describe tensor products of two unitary representations associated with quantized characters. In this paper, we investigate this problem when \(G_N = U_q(N)\). Any quantized characters of \(U_q(\infty)\), which is the inductive limit of the \(U_q(N)\), correspond to Gorin’s \(q^2\)-central probability measures on the paths in \(q^2\)-Gelfand–Tsetlin graph (see [20]). Thus, we can study our problem by Gorin’s analysis in [9]. Our second result in this paper is explicitly describing the tensor products of two unitary representations of \(U_q(\infty)\) associated with quantized characters. See Theorem 4.1. Certain concrete transformations on the set of \(q\)-central probability measures describe such the tensor product unitary representations when one of the characters is simple like a character of \(U(\infty)\) given as the powers of determinant. Here, we would like to emphasize that those transformations often appear in the analysis of \(q\)-central probability measures on the paths in the \(q\)-Gelfand–Tsetlin graph (see [7,9]), and hence, the present work gives, as a by-product, an explicit representation-theoretic interpretation to such transformations in the analysis of \(q\)-central probability measures.

One of the other challenging problems is to develop a quantum group version of the theory of spherical representations and spherical functions in the spirits of Olshanski. They also plays an important role in the asymptotic representation theory. See [17,18]. We give some basic formulations and results in the appendix of this paper. We can also refer to Ueda’s paper [30], which is devoted to such spherical representation theory for \(C^*\)-algebras and flows.
In Sect. 2, we review necessary facts on compact quantum groups. Section 3 is the main part of this paper, where we will introduce the notion of inductive limits of compact quantum groups. In Sect. 4, we investigate tensor products of unitary representations of the infinite-dimensional quantum unitary group $U_q(\infty)$. As an appendix, we give a possible formulation of spherical representations and spherical functions in our setting of quantum group $W^\ast$-algebras following Olshanski’s idea, e.g., [5,17,18]. As a further development of our character theory, we can generalize an algebraic construction of Markov dynamics in [1,13] to quantum groups. Namely, we can construct quantum Markov semigroups using the comultiplication and quantized characters of a compact quantum group. Then, they induce dynamics on the unitary dual of the compact quantum group. See [22].

2 Preliminaries

We review necessary facts on compact quantum groups and compact quantum group $W^\ast$-algebras.

Let $G = (C(G), \delta_G)$ be a compact quantum group; that is, $C(G)$ is a unital $C^\ast$-algebra and $\delta_G : C(G) \to C(G) \otimes C(G)$ is a unital $\ast$-homomorphism satisfying

- $(\delta_G \otimes \text{id}) \circ \delta_G = (\text{id} \otimes \delta_G) \circ \delta_G$ as $\ast$-homomorphisms from $C(G)$ to $C(G) \otimes C(G) \otimes C(G)$,
- $(C(G) \otimes 1)\delta_G(C(G)), (1 \otimes C(G))\delta_G(C(G)) \subset C(G) \otimes C(G)$ are dense,

where the symbol $\otimes$ denotes the operation of minimal tensor products of $C^\ast$-algebras. Let $h_G$ be the Haar state of $G$, which is a quantum analog of integration by a Haar probability measure. Moreover, $\{f_z^G\}_{z \in \mathbb{C}}$ denotes the so-called Woronowicz characters of $G$. See [15] for more details.

In this paper, we always assume that $\hat{G}$ is countable, where $\hat{G}$ is the set of all unitary equivalence classes of irreducible unitary representations of $G$. For each $\alpha \in \hat{G}$, we fix a representative $U_\alpha \in B(\mathcal{H}_{U_\alpha}) \otimes C(G)$, where the representation space $\mathcal{H}_{U_\alpha}$ must be finite dimensional. Then, the matrix $F_{U_\alpha} := (\text{id} \otimes f_1^G)(U_\alpha) \in B(\mathcal{H}_{U_\alpha})$ is positive and invertible with $\text{Tr}(F_{U_\alpha}) = \text{Tr}(F_{U_\alpha}^{-1})$, where this trace is called the quantum dimension of $U_\alpha$ (and of $\alpha$), denoted by $\text{dim}_q(\alpha)$.

We denote by $\mathcal{A}(G)$ the linear subspace of $C(G)$ generated by all matrix coefficients of finite-dimensional representations of $G$. Then, $\mathcal{A}(G)$ becomes a $\ast$-subalgebra of $C(G)$. In this paper, we always assume that $C(G)$ is the universal $C^\ast$-algebra generated by $\mathcal{A}(G)$. The linear dual $\mathcal{A}(G)^\ast$ also becomes a $\ast$-algebra and $\mathcal{A}(G)^\ast \cong \prod_{\alpha \in \hat{G}} B(\mathcal{H}_{U_\alpha})$ naturally as $\ast$-algebras. Then, we define the three $\ast$-subalgebras $\mathbb{C}[G], C^\ast(G)$ and $W^\ast(G)$ of $\mathcal{A}(G)^\ast$ that are $\ast$-isomorphic to

$$\text{alg-} \bigoplus_{\alpha \in \hat{G}} B(\mathcal{H}_{U_\alpha}) := \left\{(x_\alpha)_{\alpha \in \hat{G}} \in \prod_{\alpha \in \hat{G}} B(\mathcal{H}_{U_\alpha}) \mid x_\alpha = 0 \text{ without finitely many } \alpha \in \hat{G}\right\}.$$
by the above *-isomorphism, respectively. Then, \( C^* (G) \) becomes a \( C^* \)-algebra called the group \( C^* \)-algebra of \( G \), and \( W^* (G) \) becomes a von Neumann algebra called the group von Neumann algebra of \( G \). In what follows, we identify three *-algebras \( \mathbb{C} [G] \subset C^* (G) \subset W^* (G) \) with

\[
\text{alg-} \bigoplus_{\alpha \in \hat{G}} B(\mathcal{H}_{U_{\alpha}}) \subset \text{c}_0- \bigoplus_{\alpha \in \hat{G}} B(\mathcal{H}_{U_{\alpha}}) \subset \ell^\infty- \bigoplus_{\alpha \in \hat{G}} B(\mathcal{H}_{U_{\alpha}}).
\]

Let \( \{ \tau^G_t \}_{t \in \mathbb{R}} \) be the scaling group of \( G \), which is a one-parameter automorphism group on \( A (G) \) (see [15, Sect. 1.7]). It is known that their dual maps \( \{ \tau^G_t \}_{t \in \mathbb{R}} \) (defined by \( \tau^G_t (f) := f \circ \tau^G_t \) for \( f \in A (G)^* \)) are given as \( ( \prod_{\alpha \in \hat{G}} \text{Ad} F_{U_{\alpha}}^H )_{t \in \mathbb{R}} \). Thus, \( \{ \tau^G_t \}_{t \in \mathbb{R}} \) preserves \( \mathbb{C} [G] \), \( C^* (G) \) and \( W^* (G) \), respectively, and forms a one-parameter automorphism group. In what follows, we denote by the same symbol \( \{ \tau^G_t \}_{t \in \mathbb{R}} \) the restrictions to \( W^* (G) \) and \( C^* (G) \).

Let \( (\pi_G, L^2 (G), \xi_G) \) be the GNS-triple associated with the Haar state \( \h_G \). Then, using the orthogonality relations of matrix coefficients of irreducible representations (see [15, Theorem 1.4.3]), we can show that \( L^2 (G) \) is isometrically isomorphic to \( \ell^2- \bigoplus_{\alpha \in \hat{G}} B(\mathcal{H}_{U_{\alpha}}) \), where each \( B(\mathcal{H}_{U_{\alpha}}) \) has the inner product given by \( \langle X, Y \rangle := \text{Tr}_{\mathcal{H}_{U_{\alpha}}} (F_{U_{\alpha}}^H Y^* X) / \dim_q (\alpha) \). Thus, we can regard operators in \( C^* (G) \) and \( W^* (G) \), which act on \( \ell^2- \bigoplus_{\alpha \in \hat{G}} B(\mathcal{H}_{U_{\alpha}}) \) by the left multiplication, as bounded operators on \( L^2 (G) \). Moreover, we always assume that \( \h_G \) is faithful. Thus, \( C (G) \) is faithfully embedded into \( B (L^2 (G)) \).

We define \( U^G := \bigoplus_{\alpha \in \hat{G}} U_{\alpha} \in \bigoplus_{\alpha \in \hat{G}} W^* (G) \otimes B(\mathcal{H}_{\alpha}) = W^* (G) \otimes B (L^2 (G)) \). By [19, Theorem 3.1] and [29, Sect. 2.2], there exists a unique unital *-homomorphism \( \delta_G : W^* (G) \to W^* (G) \otimes W^* (G) \) such that

\[
(\hat{\delta}_G \otimes \text{id}) \circ \hat{\delta}_G = (\text{id} \otimes \hat{\delta}_G) \circ \hat{\delta}_G, \quad (\hat{\delta}_G \otimes \text{id})(U^G) = U^G_{23} U^G_{13}, \quad (2.1)
\]

\[
\hat{\delta}_G \circ \tau^G_t = (\tau^G_t \otimes \tau^G_t) \circ \hat{\delta}_G. \quad (2.2)
\]

We denote by \( \epsilon_G \) and \( \hat{\epsilon}_G \) the counit and the unitary antipode of \( W^* (G) \), respectively. See [15, Sect. 1.6] for definitions. The counit \( \epsilon_G : W^* (G) \to \mathbb{C} \) is a normal *-homomorphism, and the unitary antipode \( \hat{\epsilon}_G : W^* (G) \to W^* (G) \) is an involutive normal *-anti-automorphism. It is known that

\[
\tau^G_t \circ \hat{\epsilon}_G = \hat{\epsilon}_G \circ \tau^G_t, \quad \hat{\delta}_G \circ \hat{\epsilon}_G = (\hat{\epsilon}_G \otimes \hat{\epsilon}_G) \circ \hat{\delta}_G^\text{op}, \quad (2.3)
\]
where $\hat{\delta}^{\text{op}} := \sigma \circ \hat{\delta}$ and $\sigma : W^*(G) \hat{\otimes} W^*(G) \to W^*(G) \hat{\otimes} W^*(G)$ is the flip map. Remark that $\hat{R}_G \circ \hat{\tau}_{-i/2}^G = \hat{\tau}_{-i/2}^G \circ \hat{R}_G$ becomes an antipode, where $\hat{\tau}_{-i/2}$ is the analytic continuation of $\{ \hat{\tau}_t \}_{t \in \mathbb{R}}$. More precisely, for any $x \in \mathbb{C}[G]$

$$m \circ (\text{id} \otimes (\hat{R}_G \circ \hat{\tau}_{-i/2}^G))(\hat{\delta}_G(x)) = \hat{\epsilon}_G(x)1 = m \circ ((\hat{R}_G \circ \hat{\tau}_{-i/2}^G) \otimes \text{id})(\hat{\delta}_G(x)),$$

where $m : W^*(G) \circ W^*(G) \to W^*(G)$ is the multiplication. Then, following Yamagami [32], we call the $(W^*(G), \hat{\delta}_G, \hat{\epsilon}_G, \hat{R}_G, \{ \hat{\tau}_t \}_{t \in \mathbb{R}})$ a compact quantum group $W^*$-algebra. In what follows, we use the same symbol $\hat{G}$ to denote a compact quantum group $W^*$-algebra, i.e., $G = (W^*(G), \hat{\delta}_G, \hat{\epsilon}_G, \hat{R}_G, \{ \hat{\tau}_t \}_{t \in \mathbb{R}})$.

**Definition 2.1** A normal $\hat{\tau}_G$-KMS state on $W^*(G)$ with the inverse temperature $-1$ is called a quantized character of $G$. We denote by $\text{Char}(G)$ the set of quantized characters of $G$. For any $\alpha \in \hat{G}$, the $\chi^\alpha \in \text{Char}(G)$ defined by

$$\chi^\alpha((x_{\alpha'})_{\alpha' \in \hat{G}}) := \frac{\text{Tr}(FU_{\alpha}x_{\alpha})}{\dim_q(\alpha)}$$

is called the indecomposable quantized character associated with $\alpha \in \hat{G}$.

**Remark 2.1** Our previous definition of quantized characters ([20, Definition 2.1]) is equivalent to the above definition of quantized characters. See Proposition 2.1. If $\{ \hat{\tau}_G \}$ is trivial (i.e., $G$ is of Kac type), quantized characters are nothing less than normal tracial states on $W^*(G)$. If $G$ is a compact group, then each normal tracial state on $W^*(G)$ corresponds to each characters of $G$, where a character of $G$ means a positive-definite continuous function on $G$ satisfying that $f(gh) = f(hg)$ for any $g, h \in G$ and $f(e) = 1$.

We equip $\text{Char}(G)$ with the weak* topology induced from $W^*(G)$. Let KMS($C^*(G)$) be the set of $\hat{\tau}_G$-KMS state with the inverse temperature $-1$ on $C^*(G)$, equipped with the weak* topology induced from $C^*(G)$. Remark that $\text{Char}(G)$ and KMS($C^*(G)$) are $w^*$-metrizable since $W^*(G)$ and $C^*(G)$ are separable.

**Proposition 2.1** The mapping $\chi \in \text{Char}(G) \mapsto \chi|_{C^*(G)} \in \text{KMS}(C^*(G))$ is an affine homeomorphism. In particular, a sequence $\chi_n \in \text{Char}(G)$ converges to $\chi \in \text{Char}(G)$ as $n \to \infty$ if and only if $\chi_n(x) \to \chi(x)$ as $n \to \infty$ for any $x \in C^*(G)$.

**Proof** By the similar way as [20, Lemma 2.2], we can show that any $\chi \in \text{Char}(G)$ can be uniquely decomposed as $\chi = \sum_{\alpha \in \hat{G}} c_{\alpha} \chi^\alpha$, where the $c_{\alpha}$ are nonnegative coefficients satisfying $\sum_{\alpha \in \hat{G}} c_{\alpha} = 1$. Thus, there exists an affine bijection between $\text{Char}(G)$ and KMS($C^*(G)$). In the rest of proof, we show that a sequence $\chi_n \in \text{Char}(G)$ converges to $\chi \in \text{Char}(G)$ if $\chi_n(x) \to \chi(x)$ as $n \to \infty$ for any $x \in C^*(G)$. We denote by $\chi_n = \sum_{\alpha \in \hat{G}} c_{\alpha}^{(n)} \chi^\alpha$ and $\chi = \sum_{\alpha \in \hat{G}} c_{\alpha} \chi^\alpha$ the unique decompositions of $\chi_n, \chi$, respectively. By the assumption, $c_{\alpha}^{(n)} \to c_{\alpha}$ as $n \to \infty$ for any $\alpha \in \hat{G}$. Thus, by the dominated convergence theorem with respect to the counting measure on $\hat{G}$, the sequence $\chi_n(x)$ converges to $\chi(x)$ for any $x \in W^*(G)$. \(\square\)
Definition 2.2 The tensor product representation of two normal $\ast$-representation $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ of $W^*(G)$ is defined as $(\pi_1 \otimes \pi_2, \mathcal{H}_1 \otimes \mathcal{H}_2)$, where $\pi_1 \otimes \pi_2 := (\pi_1 \otimes \pi_2) \circ \hat{\delta}_G$. The conjugate representation of $(\pi, \mathcal{H})$ is defined as $(\pi^\ast, \overline{\mathcal{H}})$, where $\overline{\mathcal{H}}$ is the conjugate Hilbert space of $\mathcal{H}$ and $\pi^\ast(x)\xi := \pi(\hat{R}_G(x^*))\xi$ for any $x \in W^*(G)$ and $\xi \in \mathcal{H}$.

We introduce the notion of tensor products of products quantized characters.

Lemma 2.1 For given two quantized characters $\chi_1, \chi_2$, the $\chi_1 \otimes \chi_2 := (\chi_1 \otimes \chi_2) \circ \hat{\delta}_G$ becomes a quantized character.

Proof It easy to see that $\chi_1 \otimes \chi_2$ is a $\hat{\tau}^G \otimes \hat{\tau}^G$-KMS state on $W^*(G) \otimes W^*(G)$. Thus, for any $x, y \in W^*(G)$ there exists a bounded continuous complex function $F$ on the zonal region $\{ z \in \mathbb{C} \mid -1 \leq \text{Im}(z) \leq 0 \}$, which is analytic in $\{ z \in \mathbb{C} \mid -1 < \text{Im}(z) < 0 \}$, such that

$$F(t) = \chi_1 \otimes \chi_2(\delta(x)(\hat{\tau}^G \otimes \hat{\tau}^G)_t(\delta(y))),$$

$$F(t - i) = \chi_1 \otimes \chi_2((\hat{\tau}^G \otimes \hat{\tau}^G)_t(\delta(y)))\delta(x))$$

for any $t \in \mathbb{R}$. By Eq. (2.2), we have $F(t) = \chi_1 \otimes \chi_2(x^\ast_t \hat{\tau}^G(y))$ and $F(t - i) = \chi_1 \otimes \chi_2(\hat{\tau}^G_t(y)x)$ for any $t \in \mathbb{R}$. Therefore, $\chi_1 \otimes \chi_2$ is a $\hat{\tau}^G$-KMS state on $W^*(G)$. □

3 Quantum group $W^*$-algebras and their unitary representation theory

3.1 Quantum group $W^*$-algebras of inductive limits of compact quantum groups

We will introduce the notion of inductive limits of compact quantum groups as inductive limits of quantum group $W^*$-algebras.

Let $\mathcal{G} = (G_N)_{N=0}^\infty$ be a sequence of compact quantum groups such that $G_0 = (\mathbb{C}, \text{id}_\mathbb{C})$ and $G_N$ is a quantum subgroup of $G_{N+1}$, i.e., there is a surjective $\ast$-homomorphism $\theta_N : A(G_{N+1}) \to A(G_N)$ satisfying $\delta_{G_N} \circ \theta_N = (\theta_N \otimes \theta_N) \circ \delta_{G_{N+1}}$, called the restriction map. Note that $\theta_N|_{A(G_{N+1})}$ is also surjective onto $A(G_N)$ (see [29, Lemma 2.8]). By [29, Lemma 2.10], there exists a faithful normal unital $\ast$-homomorphism $\Theta_N : W^*(G_N) \to W^*(G_{N+1})$ satisfying $(\Theta_N \otimes \text{id})(U_{G_N}^\ast) = (\text{id} \otimes \theta_N)(U_{G_{N+1}}^\ast)$, where $\Theta_N$ is the dual map of $\theta_N$. By Eq. (2.1), we can prove that

$$\hat{\delta}_{G_{N+1}} \circ \Theta_N = (\Theta_N \otimes \Theta_N) \circ \hat{\delta}_{G_N}. \quad \text{(3.1)}$$

Let $\{ \hat{\tau}_t^N \}_{t \in \mathbb{R}}$ be the dual maps of $\{ \tau_t^N \}_{t \in \mathbb{R}}$ and $\hat{R}_{G_N}$ the unitary antipode of $W^*(G_N)$. Then, by [29, Lemma 2.9], we have

$$\Theta_N \circ \hat{\tau}_t^N = \hat{\tau}_{t+1}^N \circ \Theta_N, \quad \Theta_N \circ \hat{R}_{G_N} = \hat{R}_{G_{N+1}} \circ \Theta_N. \quad \text{(3.2)}$$
With those observations, by [26, Theorem 7], there exists a von Neumann algebra \( W^*(G_\infty) \), where \( G_\infty \) is just a symbol, and we have normal unital \( * \)-homomorphisms \( \Theta_N^\infty : W^*(G_N) \to W^*(G_\infty) \) satisfying \( \Theta_{N+1}^\infty \circ \Theta_N = \Theta_N^\infty \) and the following property: If a von Neumann algebra \( M \) and normal unital \( * \)-homomorphisms \( \Phi_N : W^*(G_N) \to \) \( M \) satisfy that \( \Phi_{N+1} \circ \Theta_N = \Phi_N \) for every \( N \geq 0 \), then there exists a unique normal \( * \)-homomorphism \( \Phi : W^*(G_\infty) \to M \) satisfying \( \Phi \circ \Theta_N^\infty = \Phi_N \) for every \( N \geq 0 \). This \( W^*(G_\infty) \) is the \( W^* \)-inductive limit of the inductive system \( (W^*(G_N), \Theta_N^\infty)_{N=1}^\infty \) in the sense of Takeda [26]. The Stratila–Voiculescu AF-algebra \( \mathcal{A}(\mathbb{G}) \) of the inductive system \( \mathbb{G} \) can be defined as the \( C^* \)-subalgebra of \( W^*(G_\infty) \) generated by \( \bigcup_{N=0}^\infty \Theta_N^\infty(C^*(G_N)) \).

**Remark 3.1** Let \( \mathcal{M}(\mathbb{G}) := \lim_{\to} (W^*(G_N), \Theta_N) \) be the inductive limit in the category of \( C^* \)-algebras and \( * \)-homomorphisms. Remark that the both \( C^*(G_N) \subset W^*(G_N) \) are faithfully embedded into \( \mathcal{M}(\mathbb{G}) \) since \( \Theta_N \) is injective for every \( N \geq 0 \). A state \( \varphi \) on \( \mathcal{M}(\mathbb{G}) \) is said to be locally normal (see, e.g., [27]) if \( \varphi|_{W^*(G_N)} \) is normal for any \( N \geq 0 \). Let \( \mathcal{N} \) be the set of all locally normal states on \( \mathcal{M}(\mathbb{G}) \). We define \( (\pi, \mathcal{H}) := \bigoplus_{\varphi \in \mathcal{N}} (\pi_{\varphi}, \mathcal{H}_{\varphi}) \), where \( (\pi_{\varphi}, \mathcal{H}_{\varphi}) \) is the GNS-representation of \( \mathcal{M}(\mathbb{G}) \) associated with \( \varphi \in \mathcal{N} \). Then, a \( W^* \)-inductive limit of the inductive system \( (W^*(G_N), \Theta_N^\infty)_{N=0}^\infty \) is obtained as \( \pi(\mathcal{M}(\mathbb{G}))^\prime \prime \). Indeed, the following facts are known:

1. \( \pi|_{W^*(G_N)} \) is unital and normal for each \( N \geq 0 \).
2. By [26, Lemma 2], \( \mathcal{N} \) is dense in the set of states on \( \mathcal{M}(\mathbb{G}) \) with respect to the weak* topology. Thus, \( \pi \) is an injective \( * \)-homomorphism.
3. By [24, Theorem 1] and [26, Lemma 2], \( \mathcal{N} = \{ \tilde{\varphi} \circ \pi \mid \tilde{\varphi} \) is normal state on \( \pi(\mathcal{M}(\mathbb{G}))^\prime \prime \} \).

Let \( M \) be a von Neumann algebra and \( \Phi_N : W^*(G_N) \to M \) normal \( * \)-homomorphisms for any \( N \geq 0 \). If \( \Phi_{N+1} \circ \Theta_N = \Phi_N \) for any \( N \geq 0 \), then, by the universality of \( \mathcal{M}(\mathbb{G}) \), we have a \( * \)-homomorphism \( \Phi^0 : \mathcal{M}(\mathbb{G}) \to M \) satisfying \( \Phi^0|_{W^*(G_N)} = \Phi_N \) for any \( N \geq 0 \). Then, we have

\[
\{ \tilde{\varphi} \circ \Phi^0 \mid \tilde{\varphi} \) is normal state on \( M \} \subseteq \mathcal{N} = \{ \tilde{\varphi} \circ \pi \mid \tilde{\varphi} \) is normal state on \( \pi(\mathcal{M}(\mathbb{G}))^\prime \prime \} \).

Thus, by [25, Theorem 2], there is a unique normal \( * \)-homomorphism \( \Phi : \pi(\mathcal{M}(\mathbb{G}))^\prime \prime \to M \) satisfying \( \Phi \circ \Theta_N^\infty = \Phi_N \) for any \( N \geq 0 \).

Here, we discuss a Hopf algebra structure of \( W^*(G_\infty) \) with a distinguished flow. By [26, Theorem 10], we have

\[
W^*(G_\infty) \hat{\otimes} W^*(G_\infty) = \bigcup_{N \geq 0} \Theta_N^\infty(W^*(G_N)) \hat{\otimes} \Theta_N^\infty(W^*(G_N))^{SOT},
\]

\[
W^*(G_\infty) \hat{\otimes} W^*(G_\infty) \hat{\otimes} W^*(G_\infty) = \bigcup_{N \geq 0} \Theta_N^\infty(W^*(G_N)) \hat{\otimes} \Theta_N^\infty(W^*(G_N)) \hat{\otimes} \Theta_N^\infty(W^*(G_N))^{SOT},
\]

where the each right-hand side is the closure with respect to the strong operator topology. By Eq. (3.1), we obtain the unital normal \( * \)-homomorphism \( \delta_\infty : W^*(G_\infty) \to \)
$W^*(G_{\infty}) \hat{\otimes} W^*(G_{\infty})$ satisfying $\hat{\delta}_{\infty} \circ \Theta_N^{\infty} = (\Theta_N^{\infty} \otimes \Theta_N^{\infty}) \circ \hat{\delta}_N$ for any $N \geq 0$. Then, by Eq. (2.1), we have $(\text{id} \otimes \hat{\delta}_{\infty}) \circ \hat{\delta}_G = (\delta_{\infty} \otimes \text{id}) \circ \hat{\delta}_{\infty}$. By Eq. (3.2), we also obtain a one-parameter automorphism group $\{\hat{\iota}_t^{\infty}\}_{t \in \mathbb{R}}$ on $W^*(G_{\infty})$ and an involutive normal $*$-anti-automorphism $\hat{R}_{\infty} : W^*(G_{\infty}) \to W^*(G_{\infty})$ satisfying $\hat{\iota}_t^{\infty} \circ \Theta_N^{\infty} = \Theta_N^{\infty} \circ \hat{\iota}_t^{G_N}$ and $\hat{R}_{\infty} \circ \Theta_N^{\infty} = \hat{R}_{G_N}$ for every $N \geq 0$. Remark that $\Theta_N^{\infty} \circ \hat{R}_{G_N}$ is normal $*$-homomorphism from $W^*(G_N)$ to the opposite algebra of $W^*(G_{\infty})$. Thus, we can apply the universality of $W^*(G_{\infty})$ to the family of normal $*$-anti-homomorphisms $\Theta_N^{\infty} \circ \hat{R}_{G_N}$. Moreover, by Eqs. (2.2), (2.3), for any $t \in \mathbb{R}$ we have

$$
\hat{\delta}_{\infty} \circ \hat{\iota}_t^{\infty} = (\hat{\iota}_t^{\infty} \otimes \hat{\iota}_t^{\infty}) \circ \hat{\delta}_G, \quad \hat{\iota}_t^{\infty} \circ \hat{R}_{\infty} = \hat{R}_{\infty} \circ \hat{\iota}_t^{\infty},
\hat{\delta}_{\infty} \circ \hat{R}_{\infty} = (\hat{R}_G \otimes \hat{R}_G) \circ \hat{\delta}_{\infty}^{\text{op}},
$$

(3.3)

where $\hat{\delta}_{\infty}^{\text{op}} := \hat{\delta}_{\infty} \circ \sigma$ and $\sigma : W^*(G_{\infty}) \hat{\otimes} W^*(G_{\infty}) \to W^*(G_{\infty}) \hat{\otimes} W^*(G_{\infty})$ is the flip map.

Following Yamagami’s work [32] and Masuda and Nakagami’s work [14], we introduce a suitable notion of inductive limits of compact quantum groups and more generally a notion of not necessarily locally compact quantum group.

**Definition 3.1** We call the above $G_{\infty} := (W^*(G_{\infty}), \mathfrak{A}(\mathbb{G}), \hat{\delta}_{\infty}, \hat{R}_{\infty}, \{\hat{\iota}_t^{\infty}\}_{t \in \mathbb{R}})$ the inductive limit quantum group $W^*$-algebra of $\mathbb{G}$. More generally, we call $G = (M, \mathfrak{A}, \delta, R, \{\tau_t\}_{t \in \mathbb{R}})$ a quantum group $W^*$-algebra if

- $M$ is a von Neumann algebra,
- $\mathfrak{A}$ is a $\sigma$-weakly dense $C^*$-subalgebra of $M$,
- $\delta : M \to M \hat{\otimes} M$ is a comultiplication, i.e., unital normal $*$-homomorphism satisfying $(\text{id} \otimes \delta) \circ \delta = (\delta \otimes \text{id}) \circ \delta$,
- $R : M \to M$ is an involutive normal $*$-anti-homomorphism called the unitary antipode, and $\{\tau_t\}_{t \in \mathbb{R}}$ is a one-parameter automorphism group on $M$ preserving $\mathfrak{A}$ called the deformation automorphism group. Moreover, $R$ and $\{\tau_t\}_{t \in \mathbb{R}}$ satisfy Eq. (3.3).

The necessity of $\mathfrak{A}$ might be unclear in the above definition. However, $\mathfrak{A}(\mathbb{G})$ is indeed necessary to give a natural topology on the convex set of quantized characters of $G_{\infty}$. See Theorem 3.1 and Remark 3.2. If we assume the existence of Haar weight $h$ of $G$, then the $(M, \delta, R, \{\tau_t\}_{t \in \mathbb{R}}, h)$ is a Woronowicz algebra in the sense of Masuda and Nakagami [14], which is an approach to locally compact quantum groups based on von Neumann algebras. On the other hand, if $M$ is a direct sum of finite-dimensional von Neumann algebras and $G$ has a counit $\epsilon$, then $(M, \delta, \epsilon, R, \{\tau_t\}_{t \in \mathbb{R}})$ is nothing less than a compact quantum group $W^*$-algebra. Here, we do not assume anything about the algebraic structures of $M$. Instead, we impose $\mathfrak{A}$, which plays a role of the “topology on $G$,” on our formulation of $G$. 

\[ \mathfrak{A} \]

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3.2 Unitary representations and quantized characters of quantum group $W^*$-algebras

Here, we introduce the concepts of unitary representations and quantized characters of quantum group $W^*$-algebras. When a quantum group $W^*$-algebra comes from an inductive system of compact quantum groups, we compare the concepts of quantized characters in this paper and in our previous paper [20]. In the next two definitions and two lemmas, let $G = (M, \mathfrak{A}, \delta, R, \{\tau_t\}_{t \in \mathbb{R}})$ be a general quantum group $W^*$-algebra.

**Definition 3.2** A normal $\tau$-KMS state on $M$ with inverse temperature $-1$ is called a quantized character of a quantum group $W^*$-algebra $G$. We denote by $\text{Char}(G)$ the convex set of quantized characters of $G$ and equip $\text{Char}(G)$ with the topology of pointwise convergence on $\mathfrak{A}$.

**Lemma 3.1** A quantized character $\chi \in \text{Char}(G)$ is extreme if and only if $\chi$ is factorial.

**Proof** By [3, Theorem 2.3.19], we can show that extreme normal $\tau$-KMS states are also extreme points in the simplex of not necessarily normal $\tau$-KMS states. Thus, the claim immediately follows from [4, Theorem 5.3.30].

**Definition 3.3** A normal $*$-representation $(\pi, \mathcal{H})$ of $M$ is called a unitary representation of $G$ if there exists a one-parameter automorphism group $\{\gamma_t\}_{t \in \mathbb{R}}$ on $\pi(M)$ satisfying $\gamma_t(\pi(x)) = \pi(\tau_t(x))$ for any $x \in M$ and $t \in \mathbb{R}$. A unitary representation $(\pi, \mathcal{H})$ of $G$ is called a factor representation if $\pi(M)$ is a factor. A unitary representation $(\pi, \mathcal{H})$ of $G$ is called a quantum finite (resp. semifinite) representation if $\{\gamma_t\}_{t \in \mathbb{R}}$ is the modular automorphism group of some faithful normal state (resp. semifinite weight) on $\pi(M)$.

It is easy to see that the GNS-representation of any quantized characters of $G$ is a quantum finite representation of $G$.

**Definition 3.4** The tensor product representation of two representation $(\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2)$ of $G$ is defined as $(\pi_1 \otimes \pi_2, \mathcal{H}_1 \otimes \mathcal{H}_2)$, where $\pi_1 \otimes \pi_2 := (\pi_1 \otimes \pi_2) \circ \delta$. The contragredient representation of $(\pi, \mathcal{H})$ is defined as $(\pi^\vee, \overline{\mathcal{H}})$, where $\pi^\vee(x)\overline{\xi} := \overline{\pi(R(x^*))\xi}$ for any $x \in M$ and $\overline{\xi} \in \overline{\mathcal{H}}$.

It is easy to show that tensor product representations and contragredient representations of unitary representations are also unitary. However, it is unclear to us whether the contragredient representation of a quantum finite representation is again quantum finite. By the following lemma, the tensor product of any given two quantized characters is also a quantized character.

**Lemma 3.2** For two quantized characters $\chi_1, \chi_2$, the $\chi_1 \otimes \chi_2 := (\chi_1 \otimes \chi_2) \circ \delta$ becomes a quantized character of $G$.

**Proof** Remark that $\chi_1 \otimes \chi_2$ is a $\tau \otimes \tau$-KMS state on $M \otimes M$. Thus, for any pair $x, y \in M$ there exists a bounded continuous complex function $F$ on $\{z \in \mathbb{C} \mid -1 \leq \text{Im}(z) \leq 0\}$, which is analytic in $\{z \in \mathbb{C} \mid -1 < \text{Im}(z) < 0\}$, such that for any $t \in \mathbb{R}$

$$F(t) = \chi_1 \otimes \chi_2(\delta(x)(\tau \otimes \tau)_t(\delta(y))),$$

$$F(t - i) = \chi_1 \otimes \chi_2((\tau \otimes \tau)_t(\delta(y))\delta(x)).$$
By Eq. (3.3), we have $F(t) = \chi_1 \odot \chi_2(x \tau(y))$ and $F(t - i) = \chi_1 \odot \chi_2(\tau(y)x)$ for any $t \in \mathbb{R}$. Hence, $\chi_1 \odot \chi_2$ is $\tau$-KMS state.

In the rest of this section, let $G_{\infty} := (W^*(G_{\infty}), \mathcal{A}(G), \delta_{\infty}, \hat{R}_{\infty}, \{t_i^{\infty}\}_{i \in \mathbb{R}})$ be the inductive limit quantum group $W^*$-algebra of inductive system $G = (G_N)_{N=0}^\infty$. We will compare the above definition of quantized character of $G_{\infty}$ with our previous one [20, Definition 2.2]. Recall that $\{t_i^{\infty}\}_{i \in \mathbb{R}}$ preserves $\mathcal{A}(G_{\infty})$ globally. We denote by the same symbol $t_i^{\infty}$ its restriction to $\mathcal{A}(G_{\infty})$. Let $(e_{N,i})_{i=0}^\infty$ be an approximate unit for $C^*(G_N)$ for every $N \geq 1$.

**Lemma 3.3** Let $(\pi, \mathcal{H})$ be a non-degenerate $\ast$-representation of $\mathcal{A}(G)$. Then, there is a unique normal extension $\tilde{\pi} : W^*(G_{\infty}) \to B(\mathcal{H})$ of $\pi$ if and only if $\pi(e_{N,i})$ converges to $\text{id}_{\mathcal{H}}$ in the strong operator topology for every $N \geq 1$.

**Proof** If there is a normal extension $\tilde{\pi} : W^*(G_{\infty}) \to B(\mathcal{H})$, then $\pi(e_{N,i})$ converges to $\pi(1) = \text{id}_{\mathcal{H}}$ in the strong operator topology for every $N \geq 1$. Conversely, we assume that $\pi(e_{N,i})$ converges to $\text{id}_{\mathcal{H}}$ in the strong operator topology for every $N \geq 1$. Then, the restriction of $(\pi, \mathcal{H})$ to $C^*(G_N)$ is also non-degenerate for every $N \geq 1$, and we can prove that there exists a unique extension of $(\pi|C^*(G_N), \mathcal{H})$ to a normal $\ast$-homomorphism $(\tilde{\pi}_N, \mathcal{H})$ of $W^*(G_N)$. By the uniqueness of normal extensions, we have $\tilde{\pi}_{N+1} \circ \Theta_N = \tilde{\pi}_N$. Therefore, by the universality of $W^*(G_{\infty})$, we have a unique normal $\ast$-representation $(\tilde{\pi}, \mathcal{H})$ satisfying that $\tilde{\pi}|\mathcal{A}(G) = \pi$. □

**Lemma 3.4** Let $\chi$ be a $\tau^{\infty}$-KMS state on $\mathcal{A}(G)$ and $(\pi_\chi, \mathcal{H}_X, \xi_\chi)$ the associated GNS-triple. Then, $\|\chi|C^*(G_N)\| = 1$ for every $N \geq 1$ if and only if the $\ast$-representation $(\pi_\chi, \mathcal{H}_X)$ satisfies the equivalent conditions in Lemma 3.3.

**Proof** Assume that $\|\chi|C^*(G_N)\| = 1$ for every $N \geq 1$. Let $\mathcal{A}(G)_{\tau^{\infty}}$ be the set of $\tau^{\infty}$-analytic elements. For any $x \in \mathcal{A}(G)_{\tau^{\infty}}$, we have

$$
\|\pi_\chi(x)\xi_\chi - \pi_\chi(e_{N,i})\pi_\chi(x)\xi_\chi\|^2 = \chi(\tilde{\tau}^{\infty}(x)x^*(1 - e_{N,i})^2) \leq \|x^{\tilde{\tau}^{\infty}}(x^*)\| \|\pi_\chi(1 - e_{N,i})^2\xi_\chi\|.
$$

By $\|\chi|C^*(G_N)\| = 1$, we have $\lim_{i \to \infty} \chi(e_{N,i}) = \lim_{i \to \infty} \chi(e_{N,i})^2 = 1$, see [3, Proposition 2.3.11(2)]. Namely, $\lim_{i \to \infty} \|\pi_\chi(1 - e_{N,i})^2\xi_\chi\| = 0$ since $\|\pi_\chi(1 - e_{N,i})\|^2 \leq 2\|\pi_\chi(1 - e_{N,i})\|$. Therefore, $\pi_\chi(e_{N,i})$ converges to $\text{id}_{\mathcal{H}_X}$ in the strong operator topology, since $\mathcal{A}(G)_{\tau^{\infty}}$ is norm dense in $\mathcal{A}(G)$ and hence $\pi_\chi|_{\mathcal{A}(G)_{\tau^{\infty}}} : \mathcal{A}(G)_{\tau^{\infty}} \to \mathcal{H}_X$ is dense in $\mathcal{H}_X$. Conversely, if $\pi_\chi(e_{N,i})$ converges to $\text{id}_{\mathcal{H}_X}$ in the strong operator topology for every $N \geq 1$, then we have

$$
1 \geq \|\chi|C^*(G_N)\| \geq \limsup_{i \to \infty} \chi(e_{N,i}) = \lim_{i \to \infty} \langle \pi_\chi(e_{N,i})\xi_\chi, \xi_\chi \rangle = 1.
$$

□

Let $\text{KMS}(\mathcal{A}(G))^0$ be the set of $\tau^{\infty}$-KMS states $\varphi$ on $\mathcal{A}(G)$ satisfying $\|\varphi|C^*(G_N)\| = 1$ for every $N \geq 1$. We equip $\text{KMS}(\mathcal{A}(G))^0$ with the weak$^*$ topology induced from $\mathcal{A}(G)$. By the following theorem, the above definition of quantized characters of $G_{\infty}$ and our previous one [20, Definition 2.2] agree with each other.
Theorem 3.1 For any $\chi \in \text{Char}(G_\infty)$, its restriction $\chi|_{\mathfrak{A}(G)}$ falls into $\text{KMS}(\mathfrak{A}(G))^0$. Moreover, this correspondence is affine and homeomorphic.

Proof By Lemma 3.3, 3.4, $\chi|_{\mathfrak{A}(G)}$ falls into $\text{KMS}(\mathfrak{A}(G))^0$ for any $\chi \in \text{Char}(G_\infty)$ and this correspondence is affine and bijective. Moreover, by the definition of topology on $\text{Char}(G_\infty)$, this correspondence is also homeomorphic.

Corollary 3.1 $\text{Char}(G_\infty)$ is a Choquet simplex.

Proof If follows from the above theorem and [20, Proposition 2.4].

Remark 3.2 By Proposition 2.1, it is easy to show that $\text{Char}(G_\infty)$ with the topology of pointwise convergence on $\mathfrak{A}(G)$ is homeomorphic to $\text{Char}(G_\infty)$ with the topology of pointwise convergence on $\bigcup_{N \geq 0} W^*(G_N)$. For a while, we assume that $G_\infty$ is not just a symbol, but an inductive limit of usual compact group, i.e., $G_\infty = \lim_{\to} G_N$ (equipped with the inductive limit topology). Then, the topology on the set of characters of $G_\infty$ by the uniform convergence on compact subsets is equivalent to the weakest topology such that the mappings $\chi \mapsto \int_{G_N} \chi(g)f(g)d\mu_N(g)$ are continuous for any $f \in L^1(G_N, \mu_N)$ and any $N \geq 1$, where $\mu_N$ is the Haar probability measure of $G_N$ (see [10]). Moreover, the set of characters of $G_\infty$ is homeomorphic to the set of tracial states $\varphi$ on $\mathfrak{A}(G_\infty)$ satisfying $\|\varphi|_{C^*(G_N)}\| = 1$ for every $N \geq 1$ (see [8, Sect. 2.1] and [20, Sect. 2.4]). By Theorem 3.1, they are also homeomorphic to the set of normal tracial states on $W^*(G_\infty)$ with the topology of pointwise convergence on $\mathfrak{A}(G_\infty)$, which is nothing less than the set of quantized characters since $\{\hat{\tau}_f^\infty\}_{f \in \mathbb{R}}$ is trivial in this case.

Remark 3.3 We can regard $W^*(G_\infty)$ as the usual group von Neumann algebra of $G_\infty$. Indeed, if $G_\infty$ is an inductive limit of usual compact groups, then we have a bijective correspondence between the set of all unitary representations of $G_\infty$ and the set of all non-degenerate normal $*$-representations of $W^*(G_\infty)$. Let $(\pi, \mathcal{H})$ be a unitary representation of $G_\infty$. Then, each restriction $\pi|_{G_N}$ induces a unique non-degenerate representation $(\tilde{\pi}_N, \mathcal{H}_N)$ of $W^*(G_N)$, and we have $\tilde{\pi}_{N+1} \circ \Theta_N = \tilde{\pi}_N$ for every $N$. Thus, we obtain a non-degenerate representation $(\tilde{\pi}, \mathcal{H})$ of $W^*(G_\infty)$ satisfying $\tilde{\pi} \circ \Theta_N = \tilde{\pi}_N$ for every $N$. We can also obtain the reverse correspondence in the same way. Moreover, we have $\tilde{\pi}(W^*(G_\infty))'' = \pi(G_\infty)''$. Thus, $W^*(G_\infty)$ and $G_\infty$ must have the same representation theory.

4 The inductive limit $U_q(\infty)$ and their tensor product representations

4.1 Basics of quantum unitary groups $U_q(N)$

Let $q \in (0, 1)$ and $U_q(N)$ be the quantum unitary group of rank $N$. See, e.g., [12,16] for its definition. We can regard $U_q(N)$ as a quantum subgroup $U_q(N+1)$. Thus, their inductive system $\mathbb{U}_q$ is well defined. It is known that the branching rule of irreducible representations of $U_q(N)$ does not depend on the quantization parameter $q$; that is, the following facts hold (see [16,33]):

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• All the equivalence classes of irreducible representations of $U_q(N)$ and $U(N)$ are labeled with the set of *signatures* given as $\text{Sign}_N := \{ \lambda = (\lambda_p)_{p=1}^N \in \mathbb{Z}^N \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \}$, and we set $\text{Sign}_0 := \{ \ast \}$.

• The restriction of the irreducible representation labeled with $\nu \in \text{Sign}_{N+1}$ contains the irreducible one labeled with $\lambda \in \text{Sign}_N$ if and only if $\nu_1 \geq \lambda_1 \geq \nu_2 \geq \cdots \geq \lambda_N \geq \nu_{N+1}$. We write $\lambda \prec \nu$ in this case. Moreover, we assume that $\ast \prec \lambda$ for every $\lambda \in \text{Sign}_1$.

**Remark 4.1** Moreover, $U_q(N)$ and $U(N)$ have the same fusion ring and the dimension of each irreducible representation does not depend on $q$. Therefore, by [15, Theorem 2.7.10, Proposition 2.7.12], the Haar state of $U_q(N)$ is also faithful.

**Remark 4.2** Noumi, Yamada and Mimachi ([16, Theorem 2.5]) gave a concrete realization of irreducible representations of $U_q(N)$. In particular, the irreducible representation of $U_q(N)$ corresponding to $(k, \ldots, k)$ for any $k \in \mathbb{Z}$ is given as $\det_q^k(N) \in \mathbb{C} \otimes A(U_q(N))$, where $\det_q(N)$ is the quantum determinant.

Let $T(N) = (C(T^N), \delta_{T(N)})$ be the compact quantum group of $N$-dimensional torus $\mathbb{T}^N$ (see [15, Example 1.2.2]). Then, we can regard that $T(N)$ is a quantum subgroup of $U_q(N)$. See [12], for example. We denote by $\pi_N$ the restriction map.

### 4.2 $q$-Coherent systems and $q$-Schur generating functions

Here, we briefly summarize $q$-coherent systems and $q$-Schur generating functions. We define $w_q(\lambda, \nu) := q^{(N+1)|\lambda| - N|\nu|}$ for any pair $\lambda \in \text{Sign}_N$, $\nu \in \text{Sign}_{N+1}$ with $\lambda \prec \nu$. See [20] for the representation-theoretic meaning of this definition. A sequence of probability measures $P_{\lambda}$ on $\text{Sign}_N$ is called a $q$-coherent system if

$$
\frac{P_{\lambda}(\lambda)}{\dim_q(\lambda)} = \sum_{\nu \in \text{Sign}_{N+1} : \lambda \prec \nu} w_q(\lambda, \nu) \frac{P_{\lambda+1}(\nu)}{\dim_q(\nu)}
$$

holds for every $\lambda \in \text{Sign}_N$ and $N \geq 1$, where $\dim_q(\lambda)$ is the quantum dimension of $\lambda \in \text{Sign}_N = U_q(N)$.

For a probability measure $P_{\lambda}$ on $\text{Sign}_N$, its $q$-Schur generating function $S(z_1, \ldots, z_N; P_{\lambda})$ is defined as

$$
S(z_1, \ldots, z_N; P_{\lambda}) := \sum_{\lambda \in \text{Sign}_N} P_{\lambda}(\lambda) \frac{s_{\lambda}(x_1, \ldots, x_N)}{s_{\lambda}(1, q^{-2}, \ldots, q^{-2(N-1)})},
$$

where $s_{\lambda}(x_1, \ldots, x_N)$ is the Schur (Laurent) polynomial with label $\lambda \in \text{Sign}_N$.

Remark that $S(x_1, \ldots, x_N; P_{\lambda})$ absolutely converges on $T_N := \{(x_1, \ldots, x_N) \in \mathbb{C}^N \mid |x_i| = q^{-2(i-1)}\}$ for any probability measure $P_{\lambda}$ on $\text{Sign}_N$. See [9, Proposition 4.5].

In what follows, we use the same symbol $U_q(N)$ to denote the associated compact quantum group $W^*$-algebra. For any quantized character $\chi$ of $U_q(N)$, there is a unique
probability measure $P_N$ on $\text{Sign}_N$ satisfying that
\[(\chi \otimes \pi_N)(U_{q}^{(N)}(z_1, \ldots, z_N)) = S(z_1, q^{-2}z_2, \ldots, q_{-1}^{2(N-1)}z_N; P_N) \tag{4.1}\]
on $(z_1, \ldots, z_N) \in \mathbb{T}^N$. See Sect. 2 for the notation $U_{q}^{(N)}$. In particular, we have
\[(\chi^\lambda \otimes \pi_N)(U_{q}^{(N)}(z_1, \ldots, z_N)) = \frac{s_\lambda(z_1, q^{-2}z_2, \ldots, q_{-1}^{2(N-1)}z_N)}{s_\lambda(1, q^{-2}, \ldots, q_{-1}^{2(N-1)})}, \tag{4.2}\]
where $\chi^\lambda$ is the indecomposable quantized character associated with $\lambda \in \text{Sign}_N$.

We denote by $U_q(\infty)$ the inductive limit quantum group $W^*$-algebra of $U_q$, called the infinite-dimensional quantum unitary group. By Theorem 3.1 and [20, Sect. 3], two simplexes of $q$-coherent systems and quantized characters of $U_q(\infty)$ are affine homeomorphic. Indeed, Eq. (4.1) gives an affine homeomorphism between them. Moreover, their extreme points are completely parametrized by $\mathcal{N} := \{((\theta_i)_{i=1}^\infty \in \mathbb{Z}^\infty | \theta_1 \leq \theta_2 \leq \cdots \}$. See [20], [9] for more details. In particular, if $\chi$ is the quantized character of $U_q(\infty)$ corresponding to a $q$-coherent system $(P_N)_{\mathcal{N}=1}^\infty$, then we have
\[(\chi|_{W^*(U_q(\infty))} \otimes \pi_N)(U_{q}^{(N)}(z_1, \ldots, z_N)) = S(z_1, q^{-2}z_2, \ldots, q_{-1}^{2(N-1)}z_N; P_N). \tag{4.3}\]

### 4.3 Tensor product representation of $U_q(\infty)$ and $q$-Schur generating functions

In this section, we study tensor product representations of $U_q(\infty)$ and give a representation-theoretic interpretation of the transformations $A_k$ on $\mathcal{N}$ defined as $A_k((\theta_i)_{i=1}^\infty) := (\theta_i + k)_{i=1}^\infty$, which often appear in the analysis of $q$-central probability measures (see [7,9]).

**Proposition 4.1** Let $\chi, \chi_1, \chi_2$ be quantized characters of $U_q(N)$ and $P, P_1, P_2$ their corresponding probability measures on $\text{Sign}_N$, respectively. Then, $\chi = \chi_1 \circ \chi_2$ if and only if
\[S(x_1, \ldots, x_N; P) = S(x_1, \ldots, x_N; P_1)S(x_1, \ldots, x_N; P_2) \tag{4.4}\]
holds for every $(x_1, \ldots, x_N) \in \mathbb{T}_N$.

**Proof** Let $U_N := U_{q}^{(N)}$. By Eq. (2.1), we have
\[
[(\chi_1 \circ \chi_2) \otimes \pi_N](U_N) = (\chi_1 \otimes \chi_2 \otimes \pi_N)(U_{N23}U_{N13}) \\
= (\chi_1 \otimes \chi_2 \otimes \text{id})(\text{id} \otimes \pi_N)(U_{N23}(\text{id} \otimes \pi_N)(U_{N13}) \\
= (\chi_1 \otimes \pi_N)(U_N)_{13}(\chi_2 \otimes \pi_N)(U_N)_{23}
\]
Thus, by Eq. (4.1), $\chi = \chi_1 \circ \chi_2$ if and only if Eq. (4.4) holds. \qed

This type of claim holds even in the infinite-dimensional case.
Theorem 4.1 Let \( \chi, \chi_1, \chi_2 \) be quantized characters of \( U_q(\infty) \) and \( (P_N)_N \), \( (P_{1,N})_N \), \( (P_{2,N})_N \) their corresponding \( q \)-coherent systems, respectively. Then, \( \chi = \chi_1 \oplus \chi_2 \) if and only if Eq. (4.4) holds for every \( N \geq 1 \).

Proof Since \( \bigcup_{N \geq 0} W^*(U_q(N)) \) is \( \sigma \)-weakly dense in \( W^*(U_q(\infty)) \), we obtain that \( \chi = \chi_1 \oplus \chi_2 \) if and only if \( \chi | W^*(U_q(N)) = (\chi_1 \oplus \chi_2)| W^*(U_q(N)) \) for every \( N \geq 1 \). Thus, this theorem immediately follows from Proposition 4.1. \( \square \)

Corollary 4.1 Let \( \chi_{\theta} \) be the extreme quantized character corresponding to \( \theta \in \mathcal{N} \). Then, \( \chi_{\theta} \oplus \chi(k,k,\ldots) = \chi_{A_k(\theta)} \) for every \( k \in \mathbb{Z} \).

Proof By Proposition 4.1, it suffices to show that for every \( N \geq 1 \)

\[
S(x_1, \ldots, x_N; P_N^{A_k(\theta)}) = S(x_1, \ldots, x_N; P_N^\theta)S(x_1, \ldots, x_N; P_N^{(k,k,\ldots)}),
\]

where \( (P_N^\theta)_{N=1}^\infty \) is the extreme \( q \)-coherent system corresponding to \( \theta \in \mathcal{N} \). We define \( A_k(\lambda) := (\lambda_1 + k, \ldots, \lambda_N + k) \) for any \( \lambda \in \text{Sign}_N \). By [9, Proposition 5.13], we have \( P_N^{A_k(\theta)} = P_N^\theta \circ A_{-k} \). Thus, we have

\[
S(x_1, \ldots, x_N; P_N^{A_k(\theta)}) = x_1^k(q^2x_2)^k \cdots (q^{2(N-1)}x_N)^k S(x_1, \ldots, x_N; P_N^\theta)
\]

since \( S_{A_k(\lambda)}(x_1, \ldots, x_N) = x_1^k \cdots x_N^k S_\lambda(x_1, \ldots, x_N) \). By [9, Theorem 5.1, Proposition 4.10],

\[
S(x_1, \ldots, x_N; P_N^{(k,k,\ldots)}) = \lim_{L \to \infty, N \leq L} \frac{S(k,\ldots,k)(x_1, \ldots, x_N, q^{-2N}, \ldots, q^{-2(L-1)})}{S(k,\ldots,k)(1, q^{-2} \ldots, q^{-2(L-1)})} = x_1^k(q^2x_2)^k \cdots (q^{2(N-1)}x_N)^k.
\]

Therefore, we obtain \( S(x_1, \ldots, x_N; P_N^{A_k(\theta)}) = S(x_1, \ldots, x_N; P_N^\theta)S(x_1, \ldots, x_N; P_N^{(k,k,\ldots)}) \). \( \square \)

Remark 4.3 Recall that the set of extreme quantized characters of \( U_q(\infty) \) is completely parametrized by \( \mathcal{N} = \{ (\theta_i)_{i=1}^\infty \in \mathbb{Z}^\infty | \theta_1 \leq \theta_2 \leq \cdots \} \). Since the GNS-representation associated with any extreme KMS state is factorial, we obtain a factor (a von Neumann algebra with trivial center) corresponding to any parameter of \( \mathcal{N} \). In our previous paper [21], the Murray–von Neumann–Connes type of the resulting factor is explicitly determined in terms of \( \mathcal{N} \). Combining with the above corollary, we obtain the following rule on tensor products of factor representations associated with extreme quantized characters of \( U_q(\infty) \):

\[
\text{type } X \otimes \text{type } I_1 = \text{type } X,
\]

where \( X = I_1, I_1 \) or \( \text{III}_q, \).

\( \square \) Springer
The description of tensor product representations is of importance in representation theory. The determinant \( \det(U) \) of \( U \in U(N) \subset U(\infty) \) gives one of the simplest extreme characters, and the product of the determinant and an arbitrary extreme character (i.e., the tensor product of the determinant and the finite factor representation associated with each extreme character). In the case of \( U_q(\infty) \), Voiculescu functions are replaced with \( q \)-Schur generating functions. Moreover, using \( q \)-Schur functions (see [31]) of extreme characters of \( U(\infty) \), we can explicitly describe such a multiplication of the determinant and an extreme character (i.e., the tensor product of the determinant and the finite factor representation associated with each extreme character). In the case of \( U_q(\infty) \), Voiculescu functions are replaced with \( q \)-Schur generating functions. In fact, we have described the tensor product representations associated with any pair of quantized characters in terms of \( q \)-Schur generating functions in Theorem 4.1. By Remark 4.2, the quantum analog of \( \det^k (k \in \mathbb{Z}) \) is given as the extreme quantized character corresponding to \( (k, k, \ldots) \in \mathcal{N} \). Similarly to the case of \( U(\infty) \), we have proved that the tensor product of extreme quantized character corresponding to \( (k, k, \ldots) \in \mathcal{N} \) and an arbitrary parameter must be extreme. Moreover, the resulting new parameter is given by the transformation \( A_k \) on \( \mathcal{N} \). Therefore, we have been able to give an explicit representation theoretic interpretation to \( A_k \). Finally, we describe tensor products of two unitary representations associated with quantized characters of \( U_q(\infty) \) when one of them is extreme with the parameter \( (k, k, \ldots) \). Recall that the set of extreme quantized characters of \( U_q(\infty) \) is homeomorphic to \( \mathcal{N} \) endowed with the component-wise convergence topology. See [20]. Since \( \text{Char}(U_q(\infty)) \) is a simplex, for any \( \chi \in \text{Char}(U_q(\infty)) \) there exists a unique probability measure \( P \) such that \( \chi = \int_{\mathcal{N}} \chi_{\theta} dP(\theta) \). We now extend \( A_k \) to the transformation on \( \text{Char}(U_q(\infty)) \) by \( A_k(\chi) := \int_{\mathcal{N}} \chi_{A_k(\theta)} dP(\theta) \). Then, the following is a consequence of Corollary 4.1:

**Corollary 4.2** For any \( k \in \mathbb{Z} \), we have \( \chi \odot \chi_{(k,k)} = A_k(\chi) \).

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**Appendix A. Spherical representations and spherical functions**

This appendix is based on what Yoshimichi Ueda explained to us in a rather general framework. See [30]. The goal of this appendix is to develop a minimal foundation of Olshanski’s spherical representation theory in the quantum setting in order to provide a basis for subsequent investigations of unitary representation theory for \( U_q(\infty) \). It may be regarded as an answer to a question to the first version of this paper asked by Grigori Olshanski. In what follows, we will freely use standard facts in modular theory, see, e.g., [3, Sect. 2.5], [28, Chapter IV–IX]. Actually, the theory of standard forms based on modular theory lies behind the materials here. We also use the standard notation \( (a \cdot \varphi \cdot b)(x) = \varphi(bxa) \) with a state \( \varphi \) on a \( C^* \)-algebra \( A \) and \( a, b, x \in A \).

Let \( G \) be a topological group. A triple \((T, \mathcal{H}, \xi)\) is called a spherical representation if \((T, \mathcal{H})\) is a unitary representation of \( G \times G \) and \( \xi \in \mathcal{H} \) is a cyclic \( G \)-invariant unit vector, where we remark that \( G \) can be identified a subgroup of
$G \times G$ by $g \in G \mapsto (g, g) \in G \times G$. Then, the function $\varphi : G \times G \to \mathbb{C}$ given as $\varphi(g, h) := \langle T(g, h)\xi, \xi \rangle$ is called a spherical function. Namely, the spherical function $\varphi$ is a positive-definite continuous function satisfying that $\varphi(e) = 1$ and $\varphi$ is $G$-biinvariant, that is, $\varphi(kgk', khk') = \varphi(g, h)$ for any $g, h, k, k' \in G$. It is known that there exists an affine bijective correspondence between the characters of $G$ and the spherical functions. In this way, there also exists a correspondence between finite factor representations of $G$ and irreducible spherical representations. See [17] for more details.

Let $G = (M, \mathfrak{A}, \delta, R, \{\tau_t\}_{t \in \mathbb{R}})$ be a quantum group $W^*$-algebra (see Definition 3.1). We remark that the presence of dense $C^*$-algebra $\mathfrak{A}$ and comultiplication $\delta$ is not necessary in what follows. We denote by $M \otimes_{\text{bin}} M$ the binormal tensor product of $M$, see [2, Sect. 4.3]. For any map $f$ on $M \otimes_{\text{bin}} M$, we define $f_L$ and $f_R$ on $M$ by $f_L(x) := f(x \otimes 1)$, $f_R(x) := f(1 \otimes x)$. If $f_L$ and $f_R$ are normal, then $f$ is said to be binormal. Note that the binormal tensor product $M \otimes_{\text{bin}} M$ enjoys the universality with respect to two commuting normal $\ast$-representations of $M$. We denote by $M_\tau$ the $\sigma$-weakly dense $\ast$-subalgebra of $\tau$-analytic elements.

**Definition A.1** A triple $(T, \mathcal{H}, \xi)$ is called a spherical representation if $(T, \mathcal{H})$ is a binormal $\ast$-representation of $M \otimes_{\text{bin}} M$ and $\xi \in \mathcal{H}$ is a cyclic unit vector satisfying that

$$T(1 \otimes x)\xi = T(\kappa(x) \otimes 1)\xi$$

for any $x \in M_\tau$, where $\kappa := R \circ \tau_{-i/2} = \tau_{-i/2} \circ R$. Then, $\xi$ is called a spherical vector. Two spherical representations $(T_i, \mathcal{H}_i, \xi_i)$ ($i = 1, 2$) are unitarily equivalent if there exists a unitary intertwiner $U$ from $(T_1, \mathcal{H}_1)$ to $(T_2, \mathcal{H}_2)$ satisfying that $U\xi_1 = \xi_2$.

Recall that $\kappa$ becomes the canonical antipode when $G$ comes from a usual quantum group like $SU_q(2)$, etc., see [14, Remark 1.3(a)]. In this viewpoint, Eq. (A.1) can be regarded as an analog of $T(e, g)\xi = T(g^{-1}, e)\xi$ for every $g \in G$ with a unitary representation $(T, \mathcal{H})$ of an ordinary group $G \times G$ and $\xi \in \mathcal{H}$. This is clearly equivalent to that $T(g, g)\xi = \xi$ for every $g \in G$. Therefore, our definition is a natural generalization of Olshansky’s one for ordinary groups.

Let $\chi$ be a quantized character of $G$ and $(\pi_\chi, \mathcal{H}_\chi, \xi_\chi)$ the associated GNS-triple. Since $\chi$ is a $\tau$-KMS state, $\xi_\chi$ is separating and cyclic for $\pi_\chi(M)$. This guarantees that $\chi$ “extends” to a faithful normal state $\hat{\chi}$ on $\pi_\chi(M)$, i.e., $\hat{\chi}(\pi_\chi(x)) = \chi(x)$ for any $x \in M$. See [4, Corollary 5.3.8] and around there. Moreover, the modular automorphism group $\{\sigma_t^\hat{\chi}\}_{t \in \mathbb{R}}$ associated with $\hat{\chi}$ satisfies that $\sigma_t^\hat{\chi}(\pi_\chi(x)) = \pi_\chi(\sigma_t(x))$ for every $x \in M$ and $t \in \mathbb{R}$. Hence, the modular conjugation $J_{\hat{\chi}} : \mathcal{H}_\chi \to \mathcal{H}_{\hat{\chi}}$ is known to be given by the formula: $J_{\hat{\chi}}\pi_\chi(x)\xi_\chi = \pi_\chi(\tau_{-i/2}(x^*))\xi_\chi$ for every $x \in M_\tau$. Then, it is well known that $J_{\hat{\chi}}\pi_\chi(M)J_{\hat{\chi}}$ coincides with the commutant $\pi_\chi(M)'$. By universality of binormal tensor product, there is a unique binormal $\ast$-representation $(T_{\hat{\chi}}, \mathcal{H}_{\hat{\chi}})$ of $M \otimes_{\text{bin}} M$ such that $T_{\hat{\chi}}(x \otimes y) = \pi_\chi(x)J_{\hat{\chi}}(\pi_\chi(R(y)^*))J_{\hat{\chi}}$ for any simple tensor $x \otimes y$.

**Lemma A.1** The following three assertions hold true:

1. $(T_{\hat{\chi}}, \mathcal{H}_{\hat{\chi}}, \xi_{\hat{\chi}})$ is a spherical representation.
2. $(T_{\hat{\chi}}, \mathcal{H}_{\hat{\chi}})$ is irreducible if and only if $\chi$ is extreme; that is, $(\pi_\chi, \mathcal{H}_\chi)$ is factorial.
Lemma A.2
For any spherical function of spherical representation theory. x for every T have satisfies Eq. (A.2). for any x ∈ M.

Proof The first and the second assertions are easy to prove. We leave them to the reader and prove only the third assertion. Let (T, H, ξ) be a spherical representation satisfying (T_L(x)ξ, ξ) = χ(x) for any x ∈ M. By the uniqueness of GNS-triple associated with χ, it suffices to show that ξ is cyclic for T_L(M). By Eq. (A.1), we have T(x ⊗ y)ξ = T_L(xκ(y))ξ for any x ∈ M and y ∈ M_T. Thus, T(M ⊗_bin M)ξ ⊂ T_L(M)∥·∥ since M_T is σ-weakly dense in M and T is binormal. Namely, ξ is also cyclic for T_L(M).

For a spherical representation (T, H, ξ), we define a state ϕ := ϕ_{(T, H, ξ)} on M ⊗_bin M by ϕ(X) := (T(X)ξ, ξ) for any X ∈ M ⊗_bin M, which we call the spherical function associated with (T, H, ξ). Recall that original spherical functions are defined as certain positive-definite continuous functions on G × G with an ordinary group G in Olshanski’s theory. Our naive idea here is to translate positive-definite continuous functions on groups into positive linear functionals on corresponding C*-algebra.

It is fairly elementary to check that the above ϕ is binormal and satisfies that

\[(\kappa(x) \otimes 1) \cdot \varphi = (1 \otimes x) \cdot \varphi\] (A.2)

for every x ∈ M_T. Here is an abstract definition of spherical functions:

Definition A.2 A binormal state ϕ on M ⊗_bin M is called a spherical function if ϕ satisfies Eq. (A.2).

The following lemma gives naturality of quantized characters from the viewpoint of spherical representation theory.

Lemma A.2 For any spherical function ϕ, the state ϕ_L on M becomes a quantized character of G. Moreover, (T_L, H, ξ) gives the GNS-triple associated with ϕ_L if ϕ is associated with a spherical representation (T, H, ξ).

Proof Since ϕ_L is normal, it suffices to show that ϕ_L is a τ-KMS state. Since τ_{-i} = κ^2 and τ_{i/2}(y^*) = τ_{-i/2}(y^*), we have

\[
ϕ_L(xτ_{-i}(y)) = [(κ^2(y) \otimes 1) \cdot ϕ](x \otimes 1) \\
= [(1 \otimes κ(y)) \cdot ϕ](x \otimes 1) \\
= [((1 \otimes κ^{-1}(y^*))) \cdot ϕ](x^* \otimes 1) \\
= [(y^* \otimes 1) \cdot ϕ](x^* \otimes 1) \\
= ϕ_L(yx)
\]

for any x ∈ M and y ∈ M_T. If ϕ is associated with a spherical representation (T, H, ξ), then it is clear that ϕ_L(x) = ⟨T_L(x)ξ, ξ⟩ for any x ∈ M. Moreover, ξ is cyclic for T_L(M) (see the proof of Lemma A.1). Thus, (T_L, H, ξ) is the GNS-triple associated with ϕ.
In a similar way to Lemma A.2, we can prove that $\varphi_R(x\kappa^{-2}(y)) = \varphi_R(yx)$ for any $x \in M$ and $y \in M_\tau$. Since $\kappa^{-2} = \tau$, we conclude that $\varphi_R$ is $\tau$-KMS state on $M$ with the inverse temperature 1.

We are in a position to give a precise relationship between spherical functions and quantized characters. The next theorem can be understood as a quantum analog of one of the key facts in Olshanski’s spherical representation theory (see [13, Sect. 24]).

**Theorem A.1** The following four assertions hold true:

1. The correspondence $(T, \mathcal{H}, \xi) \mapsto \varphi(T, \mathcal{H}, \xi)$ gives a bijection from the unitarily equivalent classes of spherical representations to the spherical functions.
2. The correspondence $\chi \mapsto (T_\chi, \mathcal{H}_\chi, \xi_\chi)$ gives a bijection from the spherical functions to the unitarily equivalent classes of spherical representations.
3. By (1), (2), the correspondence $\chi \mapsto \varphi(T_\chi, \mathcal{H}_\chi, \xi_\chi)$ gives a bijection from the spherical functions to the quantized characters. Moreover, the inverse correspondence is given as $\varphi \mapsto \varphi_L$.
4. Under the bijection in (2) (resp. in (1)), extreme quantized characters (resp. extreme spherical functions) correspond to irreducible spherical representations.

**Proof** To prove (1), (2) and (3), it suffices to show the following three claims:

(a) Any spherical representation $(T, \mathcal{H}, \xi)$ is unitarily equivalent to $(T_\chi, \mathcal{H}_\chi, \xi_\chi)$, where $\chi := \varphi_L$ and $\varphi := \varphi(T, \mathcal{H}, \xi)$.
(b) Any spherical function $\varphi$ is equal to $\varphi(T, \mathcal{H}, \xi)$, where $(T, \mathcal{H}, \xi) := (T_\chi, \mathcal{H}_\chi, \xi_\chi)$ and $\chi := \varphi_L$.
(c) Any quantized character $\chi$ of $G$ is equal to $\varphi_L$, where $\varphi := \varphi(T, \mathcal{H}, \xi)$ and $(T, \mathcal{H}, \xi) := (T_\chi, \mathcal{H}_\chi, \xi_\chi)$.

Claims (a) and (c) clearly follow from Lemma A.1(1), (3) and Lemma A.2. We prove Claim (b). Let $\varphi$ be a spherical function. By Lemma A.2 and Lemma A.1(1), we obtain a quantized character $\chi := \varphi_L$ of $G$ and the spherical representation $(T_\chi, \mathcal{H}_\chi, \xi_\chi)$. Then, we have $(T_\chi(x \otimes y)\xi_\chi, \xi_\chi) = (x\kappa(y) = \varphi(x\kappa(y) \otimes 1) = \varphi(x \otimes y)$ for any $x \in M$ and $y \in M_\tau$. Since $M_\tau$ is $\sigma$-weakly dense in $M$ and $T_\chi$ and $\varphi$ are binormal, we obtain $\varphi(T_\chi, \mathcal{H}_\chi, \xi_\chi) = \varphi$.

The fourth assertion follows from Lemma A.1(2). Therefore, we can formulate the following proposition, which can be regarded as quantum analogs of other key facts in Olshanski’s spherical representation theory. For a $*$-representation $(T, \mathcal{H})$ of $M \otimes_{\text{bin}} M$, we define

$$\mathcal{H}^0 := \{ \eta \in \mathcal{H} \mid T(1 \otimes x)\eta = T(\kappa(x) \otimes 1)\eta \text{ for any } x \in M_\tau \}.$$  

The first one is rather easy to prove. Hence, we leave it to the reader.

**Proposition A.1** Let $(T, \mathcal{H}, \xi)$ be a spherical representation. Then, $(T, \mathcal{H})$ is irreducible if $\dim \mathcal{H}^0 = 1$.

The usual proof of the following fact in the original setting uses the notion of Gelfand pairs crucially, but our proof below uses the Connes Radon–Nikodym theorem instead. It seems an interesting question to formulate a quantum analog of Gelfand pairs (in this context).
Proposition A.2 Let \((T, \mathcal{H})\) be a ∗-representation of \(M \otimes_{\text{bin}} M\) such that \(T_L\) and \(T_R\) are normal. Then, \(\dim \mathcal{H}^0 \leq 1\) if \((T, \mathcal{H})\) is irreducible.

Proof Assume that \((T, \mathcal{H})\) is irreducible and \(\mathcal{H}^0 \neq \{0\}\). Let \(\xi_i \in \mathcal{H}^0 (i = 1, 2)\) be unit vectors. It suffices to show that \(\xi_2\) is proportional to \(\xi_1\). Since \((T, \mathcal{H})\) is irreducible, \(\xi_i\) is cyclic for \(T (M \otimes_{\text{bin}} M)\). Thus, \((T, \mathcal{H}, \xi_i)\) is a spherical representation, and hence, by Lemma A.2, we obtain a quantized character \(\chi_i\) of \(G\) and its GNS-triple \((T_L, \mathcal{H}, \xi_i)\).

By Lemma A.1(2), (3), the representation \((T_L, \mathcal{H})\) is factorial. Recall that the state \(\hat{\chi}_i\) on \(T_L(M)\) given by \(\hat{\chi}_i(T_L(x)) = \chi_i(x)\) for any \(x \in M\) is faithful normal and its modular automorphism group \(\{\sigma_t^{\hat{\chi}_i}\}_{t \in \mathbb{R}}\) is given by \(\sigma_t^{\hat{\chi}_i}(T_L(x)) = T_L(\tau_t(x))\) for any \(x \in M\) and \(t \in \mathbb{R}\). Namely, \(\sigma_t^{\hat{\chi}_1} = \sigma_t^{\hat{\chi}_2}\) for any \(t \in \mathbb{R}\). Then, by [28, Theorem VIII.3.3(d)], the Connes Radon–Nikodym cocycle \(\{(D\hat{\chi}_1 : D\hat{\chi}_2)\}_{t \in \mathbb{R}}\) of \(\hat{\chi}_1\) with respect to \(\hat{\chi}_2\) must fall into the center of \(T_L(M)\). Since \((T_L, \mathcal{H})\) is a factor representation, the center of \(T_L(M)\) must be trivial, and thus, \(\{(D\hat{\chi}_1 : D\hat{\chi}_2)\}_{t \in \mathbb{R}}\) is a one-parameter group of scalar unitary operators, i.e., \((D\omega_t : D\omega_s) = \lambda^{it}\) for some \(\lambda > 0\). Using the well-known uniqueness result for Connes Radon–Nikodym cocycle, we can prove that \(\hat{\chi}_1 = \lambda \hat{\chi}_2\). Evaluating this equation at 1, we obtain that \(\lambda = 1\), that is, \(\hat{\chi}_1 = \hat{\chi}_2\) and hence \(\chi_1 = \chi_2\). By Theorem A.1, \((T, \mathcal{H}, \xi_1)\) and \((T, \mathcal{H}, \xi_2)\) are unitarily equivalent. Since \((T, \mathcal{H})\) is irreducible, any unitary intertwiner from \((T, \mathcal{H}, \xi_1)\) to \((T, \mathcal{H}, \xi_2)\) must be a scalar operator; that is, \(\xi_2\) is proportional to \(\xi_1\). \(\square\)

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