HEAT KERNEL RECURRENCE ON SPACE FORMS
AND APPLICATIONS

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Abstract. In this paper, we first give a direct proof for two recurrence relations of the heat kernels for hyperbolic spaces in [5]. Then, by similar computation, we give two similar recurrence relations of the heat kernels for spheres. Finally, as an application, we compute the diagonal of heat kernels for odd dimensional hyperbolic spaces and the heat trace asymptotic expansions for odd dimensional spheres.

1. Introduction

Let $K_n(t, r(x, y))$ be the heat kernel of the hyperbolic space $\mathbb{H}^n$. In [5], the authors obtained the following two recurrence relations:

\begin{equation}
K_{n+2} = -\frac{e^{-nt}}{2\pi \sinh r} \partial_r K_n,
\end{equation}

and

\begin{equation}
K_n(t, r) = \sqrt{2} e^{\frac{(2n-1)t}{4}} \int_r^\infty \frac{K_{n+1}(t, \rho) \sinh \rho}{(\cosh \rho - \cosh r)\frac{2}{\pi}} d\rho,
\end{equation}

by the expression of $K_n$ computed by using Selberg’s transform. Another method of obtaining the expressions of heat kernels on hyperbolic spaces using wave kernels can be found in [7]. The recurrence relations (1.1) and (1.2) are useful in obtaining heat kernel estimates on hyperbolic spaces. For examples, in [5], the authors obtained sharp upper and lower bound of the heat kernels for hyperbolic spaces, and in [18], we obtained optimal Li-Yau gradient estimate for hyperbolic spaces by using (1.1) and (1.2).

Let $H_n(t, r(x, y))$ be the heat kernel of $\mathbb{R}^n$. By the expression

\begin{equation}
H_n(t, r) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{r^2}{4t}}
\end{equation}

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of $H_n$, it is not hard to see that two similar recurrence relations:

(1.4) \[ H_{n+2} = -\frac{1}{2\pi r} \partial_r H_n, \]

and

(1.5) \[ H_n(t, r) = 2 \int_{r}^{\infty} \frac{H_{n+1}(t, \rho) \rho}{(\rho^2 - r^2)^{\frac{1}{2}}} d\rho \]

are also true on $\mathbb{R}^n$. According to this, a natural question is: are there any similar recurrence relations on spheres? The same question was asked in [13] where the author only obtained asymptotic recurrence relations of the heat kernels on spheres for short distance. In this paper, we first give a direct proof of the recurrence relations (1.1) and (1.2) on hyperbolic spaces without using the expressions of the heat kernels. Then, by similar computation, we give an affirmative answer to the question. More precisely, we obtain the following two recurrence relations of heat kernels on spheres.

**Theorem 1.1.** Let $\kappa_n(t, r(x, y))$ be the heat kernel of $S^n$ for $n = 1, 2, \ldots$. Then

(1) \[ \kappa_{n+2} = -\frac{e^{nt}}{2\pi \sin r} \partial_r \kappa_n \]

for $n = 1, 2, \ldots$

(2) \[ \kappa_n = \sqrt{2} e^{-\frac{n-1}{4} t} \int_{r}^{\pi} \frac{\kappa_{n+1}(t, \rho) \sin \rho}{(\cos r - \cos \rho)^{\frac{1}{2}}} d\rho \]

\[ + (n - 1) 2^{n-1} \omega_{n-2} \int_{0}^{t} e^{-\frac{n}{4} s} \kappa_{n+1}(s, \pi) \int_{0}^{\pi} \sin^{n-1} \left( \frac{\rho}{2} \right) \cos^{n-2} \left( \frac{\rho}{2} \right) \]

\[ \int_{0}^{\pi} \kappa_n(t - s, \arccos(\cos r \cos \rho + \sin r \sin \rho \cos \theta)) \sin^{n-2} \theta d\theta d\rho ds \]

for $n = 2, 3, \ldots$.

The reason that (1.7) is not true for $n = 1$ is mainly that removability of singularity for heat equations does not hold for $n = 1$ (see [9, 10, 14, 16]).

From the recurrence relation (1.6), one can immediately obtain explicit expressions of heat kernels for odd dimensional spheres:

(1.8) \[ \kappa_{2m+1} = \frac{e^{nt}}{(2\pi)^m} \left( -\frac{1}{\sin r} \partial_r \right)^m \kappa_1 \]
for $m = 1, 2, \cdots$, where

$$(1.9) \quad \kappa_1(t, r) = \frac{1}{\sqrt{4\pi t}} \sum_{k=-\infty}^{\infty} e^{-\frac{(r+2k\pi)^2}{4t}}.$$  

This expression was already known in literature, see for examples [4, 8, 17]. Moreover, by letting $r = 0, \pi$ in (1.7), we have the following recurrence relations:

$$(1.10) \quad \kappa_n(t, 0) = 2e^{-\frac{2n-1}{4}t} \int_0^\pi \kappa_{n+1}(t, \rho) \cos \left(\frac{\rho}{2}\right) d\rho + (n - 1)2^{n-1}\omega_{n-1} \times \int_0^t e^{-\frac{2n-1}{4}s} \kappa_{n+1}(s, \pi) \int_0^\pi \kappa_n(t - s, \rho) \sin^{n-1} \left(\frac{\rho}{2}\right) \cos^{n-2} \left(\frac{\rho}{2}\right) d\rho ds.$$  

and

$$(1.11) \quad \kappa_n(t, \pi) = (n - 1)2^{n-1}\omega_{n-1} \int_0^t e^{-\frac{2n-1}{4}s} \kappa_{n+1}(s, \pi) \int_0^\pi \kappa_n(t - s, \rho) \sin^{n-2} \left(\frac{\rho}{2}\right) \cos^{n-1} \left(\frac{\rho}{2}\right) d\rho ds.$$  

As an application of the recurrence relations (1.1) and (1.6), we have the following recurrence relations of the diagonal of heat kernels and $\kappa_n(t, \pi)$.

**Corollary 1.1.** Let the notations be the same as before. Then, we have the following recurrence relations:

1. $K_{n+2}(t, 0) = -e^{-nt} \frac{\partial_t K_n(t, 0)}{2n\pi}.$

2. $\kappa_{n+2}(t, 0) = -e^{nt} \frac{\partial_t \kappa_n(t, 0)}{2n\pi}.$

3. $\kappa_{n+2}(t, \pi) = e^{nt} \frac{\partial_t \kappa_n(t, \pi)}{2n\pi}.$

The recurrence relations are useful in the computation of the diagonal of heat kernels for hyperbolic spaces and spheres. As an application, we use them to compute the diagonal of heat kernels of odd dimensional hyperbolic spaces and the heat trace asymptotic expansions of odd dimensional spheres. Although heat trace coefficients of spheres have been obtained in [2, 3, 12, 15, 17], the recurrence relation in (2) of Corollary 1.1 provides a different and simpler approach.
**Theorem 1.2.** Let the notations be the same as before. Then,

(1) \[ K_{2m+1}(t,0) = (4\pi t)^{-\frac{2m+1}{2}} e^{-t^2} \sum_{k=0}^{m-1} \frac{\Gamma(m-k+\frac{1}{2}) c_{m,k}}{\Gamma(m+\frac{1}{2})} t^k \]

for \( m = 1, 2, \ldots \).

(2) \[ \kappa_{2m+1}(t,0) \sim (4\pi t)^{-\frac{2m+1}{2}} e^{-t^2} \sum_{k=0}^{m-1} (-1)^k \frac{\Gamma(m-k+\frac{1}{2}) c_{m,k}}{\Gamma(m+\frac{1}{2})} t^k \]

as \( t \to 0^+ \), where

\[ a_{2m+1,k} = \sum_{l=0}^{k} (-1)^l \frac{m^{2k-2l} \Gamma(m-\frac{l}{2}) c_{m,l}}{(2m)! (k-l)!}, \quad k = 0, 1, 2, \ldots \]

are the heat trace coefficients of \( S^{2m+1} \).

Here

\[ c_{m,k} = \begin{cases} 
1 & k = 0 \\
\sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq m-1} i_1^2 i_2^2 \cdots i_k^2 & 1 \leq k \leq m-1 \\
0 & k \geq m 
\end{cases} \]

for \( m = 1, 2, \ldots \).

One should note that (1.14) has been obtained in [15, Theorem 1.3.1]. We would also like to mention that explicit expressions of heat kernels of some symmetric spaces were presented in [13, 3], and in [6], an expression of the heat kernel of \( S^2 \) in series was presented.

The organization of the rest of this paper is as follows: In Section 2, we give a direct proof to (1.1) and (1.2) and prove (1) of Corollary 1.1.

In Section 3, we prove Theorem 1.1 and (2) and (3) of Corollary 1.1.

Finally, in Section 4, we prove Theorem 1.2.

### 2. A Direct Proof of Heat Kernel Recurrence on Hyperbolic Spaces

In this section, we give proofs of the recurrence relations (1.1) and (1.2) for hyperbolic spaces by direct computation.

Let \( h(t,r) \) be a smooth function. To check that \( h(t,r(x,y)) \) is the heat kernel of the \( n \)-dimensional hyperbolic space \( \mathbb{H}^n \), we only need to
check that
\begin{equation}
(2.1) \quad h_t - \Delta_n h = h_t - h_{rr} - (n - 1) \coth(r) h_r = 0
\end{equation}
and
\begin{equation}
(2.2) \quad \int_0^\infty h(t, r) f(r) \omega_{n-1} \sinh^{n-1} r dr \to f(0)
\end{equation}
as \( t \to 0^+ \) for any smooth function \( f \) with compact support. Here and
throughout this section, \( \Delta_n \) is the Laplacian operator on \( \mathbb{H}^n \) and
\[ \omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \]
is the volume of the \( n - 1 \) dimensional sphere. Moreover, setting \( \sigma = \cosh r \), then (2.1) is equivalent to
\begin{equation}
(2.3) \quad \partial_t h - [(\sigma^2 - 1)\partial^2_\sigma h + n\sigma \partial_\sigma h] = 0.
\end{equation}

We now give a direct proof to (1.1).

**Theorem 2.1.** Let \( K_n(t, r(x, y)) \) be the heat kernel of \( \mathbb{H}^n \). Then,
\begin{equation}
(2.4) \quad K_{n+2} = \frac{e^{-nt}}{2\pi \sinh r} \partial_r K_n.
\end{equation}

**Proof.** Since \( K_n(t, r(x, y)) \) is a smooth function, it is not hard to see
that \( \partial_r K_n(t, 0) = 0 \) and \( \frac{1}{\sinh r} \partial_r K_n(t, r(x, y)) \) is a smooth function (see
[11, Proposition 2.7]).

Moreover, note that
\begin{equation}
(2.5) \quad \partial_t K_n - [(\sigma^2 - 1)\partial^2_\sigma K_n + n\sigma \partial_\sigma K_n] = 0.
\end{equation}
Taking derivative on the last equality with respect to \( \sigma \) gives us
\begin{equation}
(2.6) \quad \partial_t \partial_\sigma K_n - [(\sigma^2 - 1)\partial^2_\sigma \partial_\sigma K_n + (n + 2)\sigma \partial_\sigma \partial_\sigma K_n] - n\partial_\sigma K_n = 0
\end{equation}
So
\begin{equation}
(2.7) \quad (\partial_t - \Delta_{n+2})[e^{-nt} \partial_\sigma K_n] = 0.
\end{equation}
Furthermore, for any smooth function \( f(r) \) with compact support,

\[
- \int_{\mathbb{H}^{n+2}} \frac{e^{-nt}}{2 \pi \sinh r} \partial_r K_n(t, r) f(r) \, dV \\
= - \omega_{n+1} \int_0^\infty \frac{e^{-nt}}{2 \pi} \partial_r K_n(t, r) f(r) \sinh^n r \, dr \\
= \frac{e^{-nt} \omega_{n+1}}{2 \pi} \int_0^\infty K_n(t, r) (f(r) \sinh^n r) \, dr \\
= \frac{e^{-nt} \omega_{n+1}}{2 \pi \omega_{n-1}} \int_0^\infty K_n(f_r \sinh r + n f(r) \cosh r) \partial_r K_n(r, t) \sinh^{n-1} r \, dr \\
\to f(0)
\]

as \( t \to 0^+ \), by noting that \( K_n \) is the heat kernel of \( \mathbb{H}^n \). Then, by noting that \( \partial_r K_n = \frac{1}{\sinh r} \partial_r K_n \), we get the conclusion. \( \square \)

Next, we come to prove (1) of Corollary 1.1.

**Corollary 2.1.** Let \( K_n(t, r(x, y)) \) be the heat kernel of \( \mathbb{H}^n \). Then,

\[
K_{n+2}(t, 0) = \frac{e^{-nt}}{\omega_{n+1}} \partial_t K_n(t, 0).
\]

**Proof.** Setting \( r = 0 \) in

\[
\partial_t K_n - [\partial^2_r K_n + (n - 1) \coth r \partial_r K_n] = 0,
\]

we have

\[
\partial_t K_n(t, 0) = n \partial^2_r K_n(t, 0).
\]

Then, by (1.1),

\[
K_{n+2}(t, 0) = -\frac{e^{-nt}}{2n \pi} \partial^2_r K_n(t, 0) = -\frac{e^{-nt}}{2n \pi} \partial_t K_n(t, 0).
\]

\( \square \)

We next come to give a direct proof to (1.2).

**Theorem 2.2.** Let \( K_n(t, r(x, y)) \) be the heat kernel of \( \mathbb{H}^n \). Then,

\[
K_n(t, r) = \sqrt{2} e^{\frac{(2n-1)t}{4}} \int_r^\infty \frac{K_{n+1}(t, \rho) \sinh \rho}{(\cosh \rho - \cosh r)^{\frac{n+1}{2}}} \, d\rho.
\]

**Proof.** Let \( \sigma = \cosh r, s = \cosh \rho \) and \( \xi = s - \sigma \). Then

\[
\int_r^\infty \frac{K_{n+1}(t, \rho) \sinh \rho}{(\cosh \rho - \cosh r)^{\frac{n+1}{2}}} \, d\rho = \int_0^\infty \frac{K_{n+1}}{\sqrt{s - \sigma}} \, ds = \int_0^\infty \frac{K_{n+1}(t, \xi + \sigma)}{\sqrt{\xi}} \, d\xi.
\]
Hence,

\[
(\partial_t - \Delta_n) \int_0^\infty K_{n+1}(t, \rho) \frac{\sinh \rho}{(\cosh \rho - \cosh r)^{\frac{1}{2}}} d\rho \\
= \int_0^\infty \left( \partial_t - \left( (\sigma^2 - 1) \partial^2_{\sigma} + n \sigma \partial_{\sigma} \right) \right) K_{n+1}(t, \xi + \sigma) d\xi \\
= \int_0^\infty \frac{\left[ ((\sigma + \xi)^2 - 1) \partial^2_{\sigma} + (n + 1)(\sigma + \xi) \partial_{\sigma} - ((\sigma^2 - 1) \partial^2_{\sigma} + n \sigma \partial_{\sigma}) \right] K_{n+1}(t, \xi + \sigma)}{\sqrt{\xi}} d\xi \\
= \int_0^\infty \xi^{-\frac{1}{2}} \left[ (2\sigma \xi + \xi^2) \partial^2_{\xi} + ((n + 1) \xi + \sigma) \partial_{\xi} \right] K_{n+1}(t, \sigma + \xi) d\xi \\
= - \int_0^\infty \left( \sigma^2 - 1 \frac{3}{2} \xi^2 \right) \partial_{\xi} K_{n+1}(t, \sigma + \xi) d\xi + \int_0^\infty \xi^{-\frac{1}{2}} \left[ ((n + 1) \xi + \sigma) \partial_{\xi} \right] K_{n+1}(t, \sigma + \xi) d\xi \\
= \left( n - \frac{1}{2} \right) \int_0^\infty \xi^2 \partial_{\xi} K_{n+1}(t, \sigma + \xi) d\xi \\
= - \left( \frac{n}{2} - \frac{1}{4} \right) \int_0^\infty \xi^{-\frac{3}{2}} K_{n+1}(t, \sigma + \xi) d\xi \\
= - \left( \frac{n}{2} - \frac{1}{4} \right) \int_r^\infty K_{n+1}(t, \rho) \sinh \rho \frac{1}{(\cosh \rho - \cosh r)^{\frac{1}{2}}} d\rho.
\]

So,

\[
(\partial_t - \Delta_n) \left[ e^{\frac{2n-1}{4} t} \int_0^\infty K_{n+1}(t, \rho) \frac{\sinh \rho}{(\cosh \rho - \cosh r)^{\frac{1}{2}}} d\rho \right] = 0.
\]

Moreover,

\[
(2.15) \quad \sqrt{2\omega_{n-1}} e^{\frac{(2n-1)t}{4}} \int_0^\infty \int_0^\infty K_{n+1}(t, \rho) \frac{\sinh \rho}{(\cosh \rho - \cosh r)^{\frac{1}{2}}} d\rho f(r) \sinh^{n-1} r dr \\
= \sqrt{2\omega_{n-1}} e^{\frac{(2n-1)t}{4}} \int_0^\infty K_{n+1}(t, \rho) \sinh \rho \int_0^\rho \frac{f(r) \sinh^{n-1} r}{(\cosh \rho - \cosh r)^{\frac{1}{2}}} dr d\rho \\
\xrightarrow{t \to 0^+} \frac{\sqrt{2\omega_{n-1}}}{\omega_n} \lim_{\rho \to 0} \sinh^{1-n} \rho \int_0^\rho \frac{f(r) \sinh^{n-1} r}{(\cosh \rho - \cosh r)^{\frac{1}{2}}} dr \\
= f(0)
\]

as \( t \to 0^+ \). The last equality can be computed as follows.
Let $x = \cosh r - 1$, $y = \cosh \rho - 1$ and $z = \frac{x}{y}$. Then,

$$\sinh^{1-n} \rho \int_0^\rho \frac{\sinh^{n-1} r}{(\cosh \rho - \cosh r)^2} dr$$

$$= [(1 + y)^2 - 1]^{\frac{1-n}{2}} \int_0^y \frac{[(1 + x)^2 - 1]^{\frac{1}{2}}}{(y - x)^{\frac{1}{2}}} dx$$

$$= (2 + y)^{\frac{1-n}{2}} \int_0^1 z^{\frac{n-2}{2}} (1 - z)^{-\frac{1}{2}} (2 + y) z^{\frac{n-2}{2}} dz$$

$$\rightarrow \frac{1}{\sqrt{2}} B \left( \frac{1}{2}, \frac{n}{2} \right)$$

as $\rho \to 0^+$. This completes the proof of theorem. □

3. Heat kernel recurrence on spheres

In this section, we come prove to the recurrence relations (1.6) and (1.7) for heat kernels on spheres. Similarly as before, to check that $h(t, r(x, y))$ is the heat kernel of $S^n$, we only need to check that

(3.1) \[ h_t - \Delta_n h = h_t - h_{rr} - (n - 1) \cot(r) h_r = 0 \]

and

(3.2) \[ \int_0^\pi h(t, r \omega) \omega^{n-1} r dr \rightarrow f(0) \]

as $t \to 0^+$ for any smooth function $f(r)$. Here and throughout this section, $\Delta_n$ is the Laplacian operator on $S^n$. Moreover, by setting $\sigma = \cos r$, the equation (3.1) is equivalent to

(3.3) \[ \partial_t h - [(1 - \sigma^2) \partial^2_\sigma h - n \sigma \partial_\sigma h] = 0. \]

We first come to proof (1.6).

**Theorem 3.1.** Let $\kappa_n(t, r(x, y))$ be the heat kernel of $S^n$. Then

(3.4) \[ \kappa_{n+2} = -\frac{e^{nt}}{2\pi \sin r} \partial_r \kappa_n. \]

**Proof.** Since $\kappa_n(t, r(x, y))$ is smooth on $S^n$, one can see that $\partial_r \kappa(t, 0) = \partial_r \kappa(t, \pi) = 0$ and $\frac{1}{\sin r} \partial_r \kappa_n(t, r(x, y))$ is a smooth function (see [III Proposition 2.7]).

Note that

(3.5) \[ \partial_t \kappa_n - [(1 - \sigma^2) \partial^2_\sigma \kappa_n - n \sigma \partial_\sigma \kappa_n] = 0. \]
Taking derivative on the last equality with respect to $\sigma$, we have

$$\partial_t \partial_\sigma \kappa_n - [(1 - \sigma^2) \partial^2_\sigma \partial_\sigma \kappa_n - (n + 2) \sigma \partial_\sigma \partial_\sigma \kappa_n] + n \partial_\sigma \kappa_n = 0.$$ 

Thus,

$$\partial_t [e^{nt} \partial_\sigma \kappa_n] = 0.$$ 

Moreover,

$$- \int_{S^{n+2}} \frac{e^{nt}}{2\pi \sin r} \partial_r \kappa_n (t, r) f(r) dV = - \omega_{n+1} \int_0^\pi \frac{e^{nt}}{2\pi} \partial_r \kappa_n (t, r) f(r) \sin^n r dr$$

$$= \frac{e^{nt} \omega_{n+1}}{2\pi} \int_0^\pi \kappa_n (t, r) (f(r) \sin^n r) dr$$

$$= \frac{e^{nt} \omega_{n+1}}{2\pi \omega_{n-1}} \int_0^\pi \kappa_n (f_r \sin r + nf(r) \cos r) \omega_{n-1} \sin^{n-1} r dr$$

$$\to f(0)$$

as $t \to 0^+$, where we have used that $\kappa_n$ is the heat kernel on $S^n$. Noting that $\partial_\sigma \kappa_n = - \frac{1}{\sin r} \partial_r \kappa_n$, we get the conclusion. \qed

Next, we come to prove (2) and (3) of Corollary 1.1

**Corollary 3.1.** Let $\kappa_n (t, r(x, y))$ be the heat kernel of $S^n$. Then,

1. $\kappa_{n+2} (t, 0) = - \frac{e^{nt}}{2\pi} \partial_t \kappa_n (t, 0)$.
2. $\kappa_{n+2} (t, \pi) = e^{nt} \partial_t \kappa_n (t, \pi)$.

**Proof.** Similarly as in the proof of Corollary 2.1 setting $r = 0, \pi$ in

$$\partial_t \kappa_n - [\partial^2_\sigma \kappa_n + (n - 1) \cot r \partial_r \kappa_n] = 0$$

and (1.6) will give us the conclusion. \qed

For the proof of (1.7), we need the following lemmas.

**Lemma 3.1.** Let $\kappa_n (t, r(x, y))$ be the heat kernel of $S^n$. Then,

$$\left( \partial_t - \Delta_n \right) \int_r^\pi \frac{\kappa_{n+1} (t, \rho) \sin \rho}{(\cos r - \cos \rho)^\frac{1}{2}} d\rho$$

$$= \frac{2n - 1}{4} \int_r^\pi \frac{\kappa_{n+1} (t, \rho) \sin \rho}{(\cos r - \cos \rho)^\frac{1}{2}} d\rho - (n - 1)(1 + \cos r)^{-\frac{1}{2}} \kappa_{n+1} (t, \pi).$$

**Proof.** (1) Let $\sigma = \cos r$, $s = \cos \rho$ and $\xi = \sigma - s$. Then,

$$\int_r^\pi \frac{\kappa_{n+1} (t, \rho) \sin \rho}{(\cos r - \cos \rho)^\frac{1}{2}} d\rho = \int_{-1}^\sigma \frac{\kappa_{n+1} (t, \sigma)}{(\sigma - s)^\frac{1}{2}} ds = \int_0^{1+\sigma} \frac{\kappa_{n+1} (t, \sigma - \xi)}{\xi^\frac{1}{2}} d\xi.$$
Hence,

\[
(\partial_t - \Delta_n) \int_r^\pi \frac{\kappa_{n+1}(t, \rho) \sin \rho}{(\cos r - \cos \rho)^{\frac{1}{2}}} d\rho
\]

\[= \{\partial_t - [(1 - \sigma^2)\partial^2_\sigma - n\sigma \partial_\sigma]\} \int_0^{1+\sigma} \frac{\kappa_{n+1}(t, \sigma - \xi)}{\xi^{\frac{1}{2}}} d\xi \]

\[= \int_0^{1+\sigma} \frac{\partial_\xi \kappa_{n+1}(t, \sigma - \xi)}{\xi^{\frac{1}{2}}} d\xi + \left( \frac{1}{2} + \left( n - \frac{1}{2} \right) \sigma \right) (1 + \sigma)^{-\frac{1}{2}} \kappa_{n+1}(t, \pi) - (1 - \sigma)(1 + \sigma)^{\frac{1}{2}} \partial_\sigma \kappa_{n+1}(t, \pi)
\]

\[= \int_0^{1+\sigma} \big[ (1 - (\sigma - \xi)^2) \partial^2_\sigma - (n + 1)(\sigma - \xi) \partial_\sigma \big] \kappa_{n+1}(t, \sigma - \xi) d\xi
\]

\[= \int_0^{1+\sigma} [(2\sigma \xi - \xi^2) \partial^2_\sigma + ((n + 1)\xi - \sigma) \partial_\sigma] \kappa_{n+1}(t, \sigma - \xi) d\xi
\]

\[= \int_0^{1+\sigma} \left( -\sigma \xi^{-\frac{1}{2}} + \frac{3}{2} \xi^{\frac{1}{2}} \right) \partial_\xi \kappa_{n+1}(t, \sigma - \xi) d\xi - \int_0^{1+\sigma} \xi^{-\frac{1}{2}} ((n + 1)\xi - \sigma) \partial_\xi \kappa_{n+1}(t, \sigma - \xi) d\xi
\]

\[= \left( \frac{1}{2} + \left( n - \frac{1}{2} \right) \sigma \right) (1 + \sigma)^{-\frac{1}{2}} \kappa_{n+1}(t, \pi)
\]

\[= - (n - \frac{1}{2}) \int_0^{1+\sigma} \xi^{\frac{1}{2}} \partial_\xi \kappa_{n+1}(t, \sigma - \xi) d\xi + \left( \frac{1}{2} + \left( n - \frac{1}{2} \right) \sigma \right) (1 + \sigma)^{-\frac{1}{2}} \kappa_{n+1}(t, \pi)
\]

\[= \frac{2n - 1}{4} \int_r^\pi \frac{\kappa_{n+1}(t, \rho) \sin \rho}{(\cos r - \cos \rho)^{\frac{1}{2}}} d\rho - (n - 1)(1 + \sigma)^{-\frac{1}{2}} \kappa_{n+1}(t, \pi).
\]

\[
\square
\]

**Lemma 3.2.** Let \( \kappa_n(t, r(x, y)) \) be the heat kernel of \( \mathbb{S}^n \). Then,

\[\sqrt{2} \omega_{n-1} \int_0^\pi \int_r^\pi \frac{\kappa_{n+1}(t, \rho) \sin \rho}{(\cos r - \cos \rho)^{\frac{1}{2}}} d\rho f(r) \sin^{n-1} r dr \rightarrow f(0)\]

as \( t \rightarrow 0^+ \) for any smooth function \( f \).
Proof. Note that

$$\omega_{n-1} \int_0^\pi \int_0^\pi \frac{\kappa_{n+1}(t, \rho) \sin \rho}{(\cos r - \cos \rho)^{\frac{3}{2}}} d\rho f(r) \sin^{n-1} r dr = \omega_{n-1} \int_0^\pi \frac{\kappa_{n+1}(t, \rho) \sin \rho}{(\cos r - \cos \rho)^{\frac{3}{2}}} \int_0^\rho f(r) \sin^{n-1} r dr d\rho$$

$$= \frac{\omega_{n-1}}{\omega_n} \lim_{\rho \to 0} \sin^{-n} \rho \int_0^\rho \frac{f(r) \sin^{n-1} r}{(\cos r - \cos \rho)^{\frac{3}{2}}} dr.$$

Moreover, let $$\sigma = 1 - \cos \rho$$ and $$s = 1 - \cos r$$, $$\xi = \frac{s}{\sigma}$$, we have,

$$\sin^{-n} \rho \int_0^\rho \frac{\sin^{n-1} r}{(\cos r - \cos \rho)^{\frac{3}{2}}} dr = \left[ 1 - (1 - \sigma)^2 \right]^{\frac{n}{2}} \int_0^\sigma \frac{[1 - (1 - s)^2]^{\frac{n-2}{2}}}{(\sigma - s)^{\frac{3}{2}}} ds$$

$$= \left[ \sigma(2 - \sigma) \right]^{\frac{1-n}{2}} \int_0^\sigma \frac{s(2 - s)\frac{n-2}{2}}{(\sigma - s)^{\frac{3}{2}}} ds$$

$$= (2 - \sigma) \frac{1-n}{2} \int_0^{1} \frac{\xi^{\frac{n-2}{2}}(2 - \sigma \xi)^{\frac{n-2}{2}}}{(1 - \xi)^{\frac{3}{2}}} d\xi$$

$$\to \frac{1}{\sqrt{2}} B(1/2, n/2)$$

$$= \frac{\omega_n}{\sqrt{2}\omega_{n-1}}.$$

This gives us the conclusion. \(\square\)

We are now ready to prove (1.7).

**Theorem 3.2.** Let $$k_n(t, r(x, y))$$ be the heat kernel of $$\mathbb{S}^n$$. Then

$$\kappa_n = \sqrt{2} e^{-\frac{2n-1}{4} t} \int_0^\pi \kappa_{n+1}(t, \rho) \sin \rho \frac{d\rho}{(\cos r - \cos \rho)^{\frac{3}{2}}}$$

$$+ (n - 1)2^{n-1}\omega_{n-2} \int_0^t e^{-\frac{2n-1}{4} s} \kappa_{n+1}(s, \pi) \int_0^\pi \sin^{n-1} \left( \frac{\rho}{2} \right) \cos^{n-2} \left( \frac{\rho}{2} \right)$$

$$\int_0^\pi \kappa_n(t - s, \arccos(\cos r \cos \rho + \sin r \sin \rho \cos \theta)) \sin^{n-2} \theta d\theta dp ds$$

for $$n = 2, 3, \ldots$$
Proof. Let \( v \) be the solution of the Cauchy problem:

\[
\begin{cases}
v_t - \Delta_n v = (n - 1)e^{-\frac{2n - 1}{4}t}\kappa_{n+1}(t, \pi) \sec \left( \frac{r(o,x)}{2} \right) \\
v(0, x) = 0
\end{cases}
\]

on \( S^n \). Here \( o \) is a fixed point in \( S^n \). By Duhamel’s principle,

\[
v(x) = (n - 1) \int_0^t e^{-\frac{2n - 1}{4}s}\kappa_{n+1}(s, \pi) \int_{S^n} \kappa_n(t - s, r(x, y)) \sec \left( \frac{r(o,y)}{2} \right) dV(y)ds.
\]

It is clear that \( v \) is rotationally symmetric with respect to \( o \). Moreover, by the spherical law of cosines, one has

\[
v(x) = (n - 1)\omega_{n-2} \int_0^t e^{-\frac{2n - 1}{4}s}\kappa_{n+1}(s, \pi) \int_0^\pi \sec \left( \frac{\rho}{2} \right) \sin^{n-1} \rho \\
\quad \int_0^\pi \kappa_n(t - s, \arccos(r \cos \rho + s \sin \rho \cos \theta)) \sin^{n-2} \theta d\rho d\theta ds
\]

\[
= (n - 1)2^{n-1}\omega_{n-2} \int_0^t e^{-\frac{2n - 1}{4}s}\kappa_{n+1}(s, \pi) \int_0^\pi \sin^{n-1} \left( \frac{\rho}{2} \right) \cos^{n-2} \left( \frac{\rho}{2} \right) \\
\quad \int_0^\pi \kappa_n(t - s, \arccos(r \cos \rho + s \sin \rho \cos \theta)) \sin^{n-2} \theta d\rho d\theta ds
\]

where \( r = r(o, x) \). Moreover, by Lemma 3.1, the function

\[
u(t, x) = \sqrt{2}e^{-\frac{2n - 1}{4}t} \int_r^\pi \frac{\kappa_{n+1}(t, \rho) \sin \rho}{(\cos r - \cos \rho)^{\frac{n}{2}}} d\rho + v(x)
\]

satisfies the heat equation on \( S^n \setminus \{ o' \} \) where \( o' \) is the antipodal point of \( o \). Note that \( u \) is continuous, by removability of singularity of the heat equation (see [9, 10, 14, 16]), we know that \( u \) is smooth. Moreover, by Lemma 3.2, \( u(t, x) \to \delta_o \) as \( t \to 0^+ \). This completes the proof of the theorem.

4. Heat trace of odd dimensional hyperbolic spaces and spheres

In this section, by using Corollary 1.1, we prove Theorem 1.2.

We first prove (1) of Theorem 1.2.

Proof of (1) of Theorem 1.2. Suppose that

\[
K_{2m+1}(t, 0) = (4\pi t)^{-\frac{m+1}{2}}e^{-m^2t}P_m(t).
\]
Then, by (1) of Corollary 1.1,

\begin{equation}
(4.2)\quad P_m = \left(1 + \frac{2(m - 1)^2}{2m - 1}t\right) P_{m-1} - \frac{2t}{2m - 1} P'_{m-1}
\end{equation}

with \(P_0 = 1\). Let

\begin{equation}
(4.3)\quad Q_m = \Gamma \left( m + \frac{1}{2} \right) P_m.
\end{equation}

Then

\begin{equation}
(4.4)\quad Q_m = \left( m - \frac{1}{2} + (m - 1)^2 t \right) Q_{m-1} - t Q'_{m-1}.
\end{equation}

Let

\begin{equation}
(4.5)\quad F_m(z) = \prod_{k=0}^{m-1} (1 + k^2 z) = \sum_{k=0}^{m-1} c_{m,k} z^k.
\end{equation}

We claim that

\begin{equation}
(4.6)\quad Q_m = \sum_{k=0}^{m-1} \Gamma(m - k + \frac{1}{2}) t^k \int_C \frac{F_m(z)}{z^{k+1}} \, dz = \sum_{k=0}^{m-1} \Gamma \left( m - k + \frac{1}{2} \right) c_{m,k} t^k,
\end{equation}

where \(C\) is the unit circle. We will show this by induction. It is clearly true for \(m = 1\) and suppose it is true for \(m - 1\), by (4.4),

\begin{equation}
(4.7)\quad Q_m = \left( m - \frac{1}{2} + (m - 1)^2 t \right) Q_{m-1} - t Q'_{m-1}
\end{equation}

\begin{align*}
&= \left( m - \frac{1}{2} + (m - 1)^2 t \right) \sum_{k=0}^{m-2} \frac{\Gamma(m - k - \frac{1}{2}) t^k}{2\pi \sqrt{-1}} \int_C \frac{F_{m-1}(z)}{z^{k+1}} \, dz \\
&\quad - t \sum_{k=0}^{m-2} \frac{\Gamma(m - k - \frac{1}{2}) t^{k-1}}{2\pi \sqrt{-1}} \int_C \frac{F_{m-1}(z)}{z^{k+1}} \, dz \\
&= \sum_{k=0}^{m-1} \frac{t^k}{2\pi \sqrt{-1}} \int_C \left( \frac{(m - k - \frac{1}{2}) \Gamma(m - k - \frac{1}{2})}{z^{k+1}} + \frac{(m - 1)^2 \Gamma(m - k + \frac{1}{2})}{z^k} \right) F_{m-1}(z) \, dz \\
&= \sum_{k=0}^{m-1} \frac{\Gamma(m - k + \frac{1}{2}) t^k}{2\pi \sqrt{-1}} \int_C \frac{F_m(z)}{z^{k+1}} \, dz.
\end{align*}

So,

\begin{equation}
(4.8)\quad P_m = \sum_{k=0}^{m-1} \frac{\Gamma(m - k + \frac{1}{2})}{\Gamma(m + \frac{1}{2})} c_{m,k} t^k.
\end{equation}
This completes the proof. □

We next come to prove (2) of Theorem 1.2. Note that

\[ \kappa_1(t, 0) = (4\pi t)^{-\frac{1}{2}} \left( 1 + 2 \sum_{k=1}^{\infty} e^{-\frac{k^2}{t}} \right). \]  

Because \( t^m e^{-\frac{k^2}{t}} \) tends to 0 exponentially as \( t \to 0^+ \) for any constant \( m \) and any positive constant \( k \), by (2) of Corollary 1.1, the terms \( e^{-\frac{k^2}{t}} \) with \( k = 1, 2, \ldots \) in (4.9) have no contribution to the heat trace asymptotic as \( t \to 0^+ \) for odd dimensional sphere. So, to compute the heat trace asymptotic for odd dimensional sphere, we only need to take care of \((4\pi t)^{-\frac{1}{2}}\) in the expression (4.9) of \( \kappa_1(t, 0) \).

**Proof of (2) of Theorem 1.2.** Let \( \bar{\kappa}_1(t) = (4\pi t)^{-\frac{1}{2}} \) and

\[ \bar{\kappa}_{n+2} = -\frac{e^{nt}}{2n\pi} \partial_t \bar{\kappa}_n. \]

Then, \( \bar{\kappa}_{2m+1} \) is the heat trace asymptotic for \( S^{2m+1} \). Suppose that

\[ \bar{\kappa}_{2m+1}(t) = (4\pi t)^{-\frac{2m+1}{2}} e^{m^2t} p_m(t). \]

Then, by (4.10), we know that

\[ p_m = \left( 1 - \frac{2(m - 1)^2}{2m - 1} t \right) p_{m-1} - \frac{2t}{2m-1} p'_{m-1} \]

with \( p_0 = 1 \). Let \( q_m = \Gamma(m + \frac{1}{2}) p_m \). Then,

\[ q_m = \left( m - \frac{1}{2} - (m - 1)^2 t \right) q_{m-1} - t q'_{m-1} \]

with \( q_0 = \Gamma(\frac{1}{2}) \). Let

\[ \tilde{F}_m(z) = \prod_{k=0}^{m-1} (1 - k^2 z) = \sum_{k=0}^{m-1} (-1)^k c_{m,k} z^k. \]

We claim that

\[ q_m = \sum_{k=0}^{m-1} \frac{\Gamma(m - k + \frac{1}{2}) t^k}{2\pi \sqrt{-1}} \int_C \tilde{F}_m(z) \frac{dz}{z^{k+1}} = \sum_{k=0}^{m-1} (-1)^k \Gamma \left( m - k + \frac{1}{2} \right) c_{m,k} t^k, \]
where \( C \) is the unit circle. We will show this by induction. It clearly true for \( m = 1 \) and suppose it is true for \( m - 1 \), by (4.13),

\[
(4.16)
\]

\[
q_m = \left( m - \frac{1}{2} - (m - 1)^2t \right) q_{m-1} - t\bar{q}'_{m-1}
\]

\[
= \left( m - \frac{1}{2} - (m - 1)^2t \right) \sum_{k=0}^{m-2} \frac{\Gamma(m - k - \frac{1}{2})t^k}{2\pi \sqrt{-1}} \int_C \frac{\tilde{F}_{m-1}(z)}{z^{k+1}} dz
\]

\[
- t \sum_{k=0}^{m-2} \frac{\Gamma(m - k - \frac{1}{2})kt^{k-1}}{2\pi \sqrt{-1}} \int_C \frac{\tilde{F}_{m-1}(z)}{z^{k+1}} dz
\]

\[
= \sum_{k=0}^{m-1} \frac{t^k}{2\pi \sqrt{-1}} \int_C \left( \frac{(m - k - \frac{1}{2})\Gamma(m - k - \frac{1}{2})}{z^{k+1}} - \frac{(m - 1)^2\Gamma(m - k + \frac{1}{2})}{z^{k}} \right) \tilde{F}_{m-1}(z) dz
\]

\[
= \sum_{k=0}^{m-1} \frac{\Gamma(m - k + \frac{1}{2})t^k}{2\pi \sqrt{-1}} \int_C \tilde{F}_m(z) dz.
\]

So,

\[
(4.17)
\]

\[
p_m = \sum_{k=0}^{m-1} (-1)^k \frac{\Gamma(m - k + \frac{1}{2})}{\Gamma(m + \frac{1}{2})} c_{m,k} t^k.
\]

and

\[
\bar{\kappa}_{2m+1} = (4\pi t)^{\frac{2m+1}{2}} e^{m^2t} p_m
\]

\[
= (4\pi t)^{\frac{2m+1}{2}} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{m^{2k-l}}{k!} \frac{\Gamma(m - k + \frac{1}{2})}{\Gamma(m + \frac{1}{2})} \frac{1}{(m + \frac{1}{2})(k-l)!} c_{m,l} t^k
\]

\[
= (4\pi t)^{\frac{2m+1}{2}} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{(-1)^l m^{2k-2l} \Gamma(m - l + \frac{1}{2})}{\Gamma(m + \frac{1}{2})(k-l)!} c_{m,l} t^k
\]

So the heat coefficients of \( S^{2m+1} \) are

\[
(4.19)
\]

\[
a_k = \omega_{2m+1}(4\pi) \frac{\frac{2m+1}{2}}{2m+1} \sum_{l=0}^{k} (-1)^l \frac{m^{2k-2l} \Gamma(m - l + \frac{1}{2})}{\Gamma(m + \frac{1}{2})(k-l)!} c_{m,l}
\]

\[
= \sum_{l=0}^{k} (-1)^l \frac{m^{2k-2l} \Gamma(m - l + \frac{1}{2})}{(2m)!(k-l)!} c_{m,l}
\]

for \( k = 0, 1, \ldots \). This completes the proof. □
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