RESEARCH PAPER

Darboux and Analytic First Integrals of Kingni–Jafari System with Only One Stable Equilibrium Point.

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A B S T R A C T:
In this paper, we illustrate by an evidence that the Kingni–Jafari differential system $\dot{u} = -w$, $\dot{v} = -u - w$, $\dot{w} = 3u - av + u^2 - w^2 - v w + b$, where $a$ and $b$ are real parameters has no Darboux and rational first integrals for any value of $a, b$. Furthermore, we show that this system has no global $C^1$ first integrals for $a \in (0,3), b > 0$ and $3b - ab > a^2$. Also, an analytic first integral for some generic condition is studied for this system at the neighborhood of the equilibrium point $\left(0, \frac{b}{a}, 0\right)$.

KEY WORDS: Invariant Algebraic Surfaces, Darboux First Integral, Exponential Factor, Analytic First Integral.
DOI: http://dx.doi.org/10.21271/ZJPAS.32.3.1
ZJPAS (2020), 32(3):1-9.

1.INTRODUCTION:

Kingni and Jafari in (Kingni et al., 2014) proposed the simplest electronic circuit design. This electronic circuit consists of a resistors, AD633 multiplier, capacitors and operational amplifiers. This circuit can be considered by the following three-dimensional chaotic differential system

\[
\begin{align*}
\dot{u} &= -w, \\
\dot{v} &= -u - w, \\
\dot{w} &= 3u - av + u^2 - w^2 - v w + b.
\end{align*}
\]

has a rare equilibrium point $\left(0, \frac{b}{a}, 0\right)$ for $a \neq 0$.

The study of chaotic System (1) is significant in physics and engineering applications, especially in circuit, control and communications. In (Wei et al., 2016), the authors proved that this system is the chaotic system with invisible attractors and that the stable equilibrium point can coexist with a strange attractor for specific parameters. Dynamics of the Kingni and Jafari system have explained via numerical simulations such as phase portraits, bifurcation diagrams and new cost function for parameter estimation. Wei et al (2016) have learned complex dynamical behaviors and topological structure of the system such as the dynamics of this system at infinity, periodic solutions, Hopf bifurcation and zero Hopf bifurcation. System (1) has been studied in the papers (Kingni et al., 2014 and Wei et al., 2016) but none of those papers mentions the integrability or non-integrability. In this paper, we investigate...
first integrals of a Darboux and an analytic type of system (1).
Solutions of a differential system can be compared with the existing behavior of a system to make if the theoretical detailing of the system is accurate. This is interesting topic in the sciences. A Darboux integrability is a method to find a solution of a differential system, for more details see (Ollagnier, 1997, Christopher et al., 2007, Llibre and Valls, 2011a, Llibre and Valls, 2011b and Hussien and Amen, 2018).

2. PRELIMINARY RESULTS.
This section is started with a short overview of the integrability problem, the Darboux method and the auxiliary results which are given (Llibre and Zhang, 2002, Llibre and Zhang, 2010, Llibre and Valls, 2011b and Llibre and Zhang, 2012). To prove our important results, firstly we give some basic definitions and theorems as a background to this study.

The associated vector field to system (1) is defined by

\[
\chi = -w \frac{\partial}{\partial u} + (-u - w) \frac{\partial}{\partial v} + (3u - a v + u^2 - w^2 - v w + b) \frac{\partial}{\partial w}.
\]
(2)

Let \( D \) be an open subset of \( \mathbb{C}^3 \), a non-constant function \( H: D \rightarrow \mathbb{C} \) is a first integral of the polynomial vector field \( \chi \) on \( D \) if it is a constant on all orbits \( (u(t), v(t), w(t)) \) of \( \chi \) contained in \( D \). Obviously, that \( H \) is called a first integral of \( \chi \) on \( D \) if and only if

\[
\chi(H) = -w \frac{\partial H}{\partial u} + (-u - w) \frac{\partial H}{\partial v} + (3u - a v + u^2 - w^2 - v w + b) \frac{\partial H}{\partial w} = 0.
\]
(3)

A local (global) first integral \( H \) is a first integral whose domain of definition is a neighborhood of an equilibrium point (whose domain of definition is \( \mathbb{R}^3 \)) of system (1). We recall that \( H \) is an analytic (rational) first integral if it is an analytic (rational) function.

An equilibrium point \((u_0, v_0, w_0)\) of system (1) is said to be an attractor if all eigenvalues \( \lambda_i \) of the Jacobian matrix of (1) at \((u_0, v_0, w_0)\) have negative real parts.

**Theorem 2.1.** Routh-Hurwitz Criterion. The zero of \( \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0 \) have negative real parts if and only if \( a_1 > 0, a_3 > 0 \) and \( a_1 a_2 - a_3 > 0 \).

We present the following results concerning with the non-existence of first integral, that we use later on, this is due to (Llibre et al., 2015).

**Theorem 2.2.** If system (1) has an equilibrium point \((u_0, v_0, w_0)\) which is either repeller or attractor, then system (1) has no \( C^1 \) first integrals defined in a neighborhood at \((u_0, v_0, w_0)\).

A Darboux theory of integrability has a best method to determine that systems have a first integral or not. Now, we will describe its some basic nations, for more deep information look at (Christopher and Llibre, 2000 and Llibre and Valls, 2011b). Suppose that \( f = f(u, v, w) \in \mathbb{C}[u, v, w] \), then \( f = 0 \) is said to be an invariant algebraic surface or it is called a Darboux polynomial of \( \chi \) if there exist a polynomial \( K_f \in \mathbb{C}[u, v, w] \) such that

\[
\chi(f) = -w \frac{\partial f}{\partial u} + (-u - w) \frac{\partial f}{\partial v} + (3u - a v + u^2 - w^2 - v w + b) \frac{\partial f}{\partial w} = f K_f,
\]
(4)

we recall \( K_f \) is the cofactor of \( f \) and the degree of \( K_f \) here is at most 1.

**Proposition 2.3.** System (1) has a rational first integral if it has two different Darboux polynomials with the match cofactors.

We denote an exponential factor of system (1) by \( E \) which defined by a non-constant function of the form \( E = e^{\varphi} \) with greatest common divisor between \( g \) and \( f \) is equal to one. That means \((g, f) = 1\), where \( g, f \in \mathbb{C}[u, v, w] \) and it is satisfied

\[
\chi(E) = -w \frac{\partial E}{\partial u} + (-u - w) \frac{\partial E}{\partial v} + (3u - a v + u^2 - w^2 - v w + b) \frac{\partial E}{\partial w} = E L,
\]
(5)

for some polynomial \( L = L(u, v, w) \in \mathbb{C}[u, v, w] \) of degree at most 1 which is called the cofactor of \( E \).
Proposition 2.4. i) The function \( E = e^{qf} \) is an exponential factor of polynomial differential system (1) and \( f \) is a non-constant polynomial, then \( f = 0 \) is an invariant algebraic surface.

ii) Finally \( e^g \) can be an exponential factor, getting from the multiplicity of the infinity invariant plane.

Theorem 2.5. Darboux Theorem (Christopher and Llibre, 2000). Suppose that a polynomial vector field \( \chi \) of degree \( d \) in \( \mathbb{C}^3 \) have \( p \) irreducible invariant algebraic surfaces \( f_i = 0 \) such that the \( f_i \) are pairwise relatively prime with cofactors \( K_i \) for \( i = 1, ..., p \) and \( q \) exponential factors \( e^{\theta_j} \) together cofactors \( L_j \) for \( j = 1, ..., q \). There exist \( \lambda_i, \mu_j \in \mathbb{C} \) not all zero such that

\[
\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0, \tag{6}
\]

if and only if the function

\[
f_1^{\lambda_1} \cdots f_p^{\lambda_p} \left( e^{\theta_1} \right)^{\mu_1} \cdots \left( e^{\theta_q} \right)^{\mu_q}, \tag{7}
\]

is the first integral of system (1).

The form (7) is called a Darboux first integral. The following proposition is essential to prove the existence of an analytic first integral of system (1) which is due to (Llibre and Valls, 2008).

Proposition 2.6. (Llibre and Valls, 2008). The 3-dimensional linear differential system

\[
\dot{U} = PU, \text{ where } U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad \dot{U} = \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix},
\]

has two independent first integrals which are given in the following cases

1. \( F_1 = \frac{w^{\lambda_1}}{u^{\lambda_2}} \) and \( F_2 = \frac{w^{\lambda_3}}{u^{\lambda_1}} \) if \( P = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \), with \( \lambda_i \in \mathbb{R}\{0\}, \quad i = 1,2,3.

2. \( F_1 = \frac{w^{\lambda_1}}{u^{\lambda_2}} \) and \( F_2 = w \exp \left( -\frac{\lambda_2 v}{u} \right) \) if

\[
P = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \quad \text{with } \lambda_i \in \mathbb{R}\{0\}, \quad i = 1,2.
\]

3. \( F_1 = \frac{u^2}{2u w - v^2} \) and \( F_2 = u \exp \left( -\frac{\lambda_1 w}{u} \right) \) if

\[
P = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_1 & 0 \\ 0 & 1 & \lambda_1 \end{pmatrix}, \quad \text{with } \lambda_1 \in \mathbb{R}\{0\}.
\]

4. \( F_1 = \frac{(u^2 + v^2)^\lambda}{w^2 \alpha} \) and

\[
F_2 = \exp \left( -2\alpha \arctan \left( \frac{v}{u} \right) \right) \left( u^2 + v^2 \right)^\beta \text{ if } P = \begin{pmatrix} \alpha & -\beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \text{with } \lambda, \alpha, \beta \in \mathbb{R}\{0\}.
\]

5. \( F_1 = (u^2 + v^2) \) and

\[
F_2 = \exp \left( -\lambda \arctan \left( \frac{v}{u} \right) \right) \left( u^2 + v^2 \right)^\beta \text{ if } P = \begin{pmatrix} 0 & -\beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \text{with } \lambda, \beta \in \mathbb{R}\{0\}.
\]

3. MAIN RESULTS AND THEIR PROVING

In this part, the existence of rational first integrals (see Theorem 3.3), Darboux first integrals (see Theorem 3.5) and an analytic first integral (Theorem 3.8) are the main results of system (1) are described. Moreover, some other results relative to this topic are studied in this work such as a polynomial first integral, invariant algebraic surfaces, exponential factors and \( C^1 \) first integrals of system (1).

The following proposition is the first result in this work.

Proposition 3.1. System (1) has no polynomial first integrals.

Proof. Let \( H = \sum_{i=1}^{n} H_i(u, v, w) \) be a polynomial first integral of system (1), where each \( H_i \) is a
homogeneous polynomial in its variables of degree \( i \). By definition of first integral, we have

\[
-w \frac{\partial}{\partial u} H + (-u - w) \frac{\partial}{\partial v} H + (3 u - a v + u^2 - w^2 - v w + b) \frac{\partial}{\partial w} H = 0.
\]

Computing the terms of degree \( n + 1 \), we obtain

\[
(u^2 - w^2 - v) \frac{\partial}{\partial w} H_n(u, v, w) = 0,
\]

that is

\[
H_n(u, v, w) = F_1(u, v),
\]

where \( F_1 \) is a polynomial of variables \( u \) and \( v \) of degree \( n \). Also, computing the terms of degree \( n \) in equation (8), we have

\[
-w \frac{\partial}{\partial u} F_1(u, v) + (-u - w) \frac{\partial}{\partial v} F_1(u, v) + (3 u - a v) \frac{\partial}{\partial w} F_1(u, v) + (u^2 - w^2 - v) \frac{\partial}{\partial w} H_n-1(u, v, w) = 0,
\]

this gives

\[
H_{n-1}(u, v, w) = \frac{1}{\sqrt{u^2 + v^2}} \left( \left( 2 u - v \left( \frac{\partial}{\partial v} F_1(u, v) \right) \right) v \left( \frac{\partial}{\partial u} F_1(u, v) \right) \right) \arctanh \left( \frac{v + 2 w}{\sqrt{4u^2 + v^2}} \right) + \left( \left( -\frac{1}{2} \frac{\partial}{\partial u} F_1(u, v) - \frac{1}{2} \frac{\partial}{\partial v} F_1(u, v) \right) \ln(-u^2 + w^2 + v w) + F_2(u, v) \right) \sqrt{4u^2 + v^2}.
\]

Since \( H_{n-1}(u, v, w) \) is a polynomial of degree \( n - 1 \), then we have

\[
\frac{\partial}{\partial u} F_1(u, v) + \frac{\partial}{\partial v} F_1(u, v) = 0,
\]

and

\[
(2u - v) \left( \frac{\partial}{\partial v} F_1(u, v) \right) - v \left( \frac{\partial}{\partial u} F_1(u, v) \right) = 0,
\]

It is clear that the solution of equation (9) is

\[
F_1(u, v) = F_3(v - u),
\]

where \( F_3 \) is a polynomial of variables \( u \) and \( v \). Since, \( F_1 \) is the polynomial of degree \( n \) then it must be in the form

\[
F_1(u, v) = c (v - u)^n,
\]

where \( c \) is an arbitrary constant. Putting (11) in (10), we obtain

\[
c n (v - u)^n u = 0,
\]

This gives \( c n = 0 \), then \( c = 0 \) or \( n = 0 \). If \( c = 0 \), this implies that \( F_1 = 0 \), then \( H_n(u, v, w) = 0 \), in this case system (1) has no polynomial first integrals. If \( n = 0 \) then \( H \) is a constant function, this is trivial. This means that there is no a polynomial first integral of system (1).

**Proposition 3.2.** System (1) does not have invariant algebraic surfaces with non-zero cofactors.

Proof. Suppose that \( f = \sum_{i=1}^{n} f_i(u, v, w) \) is an invariant algebraic surfaces of system (1) with the cofactor \( K = k_0 + k_1 u + k_2 v + k_3 w \), where \( k_i \in \mathbb{C} \) for \( i = 0, \ldots, 3 \), and each \( f_i \) is a homogeneous polynomial in its variables of degree \( i \). Assume that \( f_n \neq 0 \) for \( n > 1 \), then by definition of invariant algebraic surface, we obtain

\[
-w \frac{\partial}{\partial u} f + (-u - w) \frac{\partial}{\partial v} f + (3 u - a v + u^2 - w^2 - v w + b) \frac{\partial}{\partial w} f = K f.
\]

We first compute the terms of degree \( n + 1 \) to obtain

\[
\frac{u^2 - w^2 - v w}{v w} \frac{\partial}{\partial w} F_n(u, v, w) = (k_1 u + k_2 v + k_3 w) f_n(u, v, w).
\]

(13)

This gives

\[
f_n(u, v, w) = G_1(u, v) (-u^2 + w^2 + v) \quad \text{where} \quad \frac{2w + v}{\sqrt{4u^2 + v^2}} \quad \text{and} \quad k_3 \quad \text{are polynomial of variables} \quad u \quad \text{and} \quad v \quad \text{of degree} \quad n - 2m.
\]

Also, calculating the terms of degree \( n \) in equation (12), we take out

\[
-w \left( \left( \frac{\partial}{\partial u} G_1(u, v) \right) (-u^2 + w^2 + v w) - 2m u G_1(u, v) (-u^2 + w^2 + v w)^m \right) + (-u - w) \left( \left( \frac{\partial}{\partial v} G_1(u, v) \right) (-u^2 + w^2 + v w)^m + \right.
\]

\[
\left. -2m u G_1(u, v) (-u^2 + w^2 + v w)^m \right)
\]

since \( f_n(u, v, w) \) is a polynomial function, this implies that \( k_1 = 0, k_2 = -m \) and \( k_3 = -2m \) where \( m \in \mathbb{N} \). Then equation (14) becomes

\[
f_n(u, v, w) = G_1(u, v) (-u^2 + w^2 + v w)^m,
\]

where \( G_1 \) is a polynomial of variables \( u \) and \( v \) of degree \( n - 2m \). Also, calculating the terms of degree \( n \) in equation (12), we take out

\[
-w \left( \left( \frac{\partial}{\partial u} G_1(u, v) \right) (-u^2 + w^2 + v w) - 2m u G_1(u, v) (-u^2 + w^2 + v w)^m \right) + (-u - w) \left( \left( \frac{\partial}{\partial v} G_1(u, v) \right) (-u^2 + w^2 + v w)^m + \right.
\]

\[
\left. -2m u G_1(u, v) (-u^2 + w^2 + v w)^m \right)
\]
From equation (17), if $G_1 = 0$ then $f_n(u, v, w) = 0$, this gives that system (1) has no invariant algebraic surfaces. Or, if $m = 0$, this gives $k_2 = k_3 = 0$. Then equation (12) becomes

$$-w \frac{\partial}{\partial u} f(u, v, w) + (-u - w) \frac{\partial}{\partial v} f(u, v, w) + (3 u - av + u^2 - \frac{1}{2} w^2 - v w + b) \frac{\partial}{\partial w} f(u, v, w) = k_0 f(u, v, w).$$

(18)

It is not easy to discover a solution of equation (18). So, the weight change of variables is used as described in (Libre & Pessoa, 2009) in order to find an invariant algebraic surfaces of system (1).

Let $u = \mu U, v = V, w = W$ and $t = \mu T$, with $\mu \in \mathbb{C} \setminus \{0\}$. Then, system (1) turn into

$$\dot{U} = -W,$n  
$$\dot{V} = -\mu^2 U - \mu W,$n  
$$\dot{W} = \mu^3 U^2 + 3 \mu^2 U - a \mu V - \mu W^2 - \mu V W + b \mu,$n  

(19)

where the dots denote the derivative of the variables $U, V$ and $W$ with respect to $T$. Set $F(U, V, W) = \mu^n f(\mu U, V, W) = \sum_{j=0}^{n} \mu^j F_j(U, V, W)$, where $F_j$ is the weight homogeneous part with weight degree $n - j$ of $F$, and $n$ is the weight degree of $F$ with weight exponent $s = (1,0,0)$. And $K(U, V, W) = k_0$.

Then, by invariant algebraic surfaces, we have

$$-W \sum_{j=0}^{n} \mu^j \frac{\partial}{\partial U} F_j(U, V, W) + (-\mu^2 U - \mu W) \sum_{j=0}^{n} \mu^j \frac{\partial}{\partial V} F_j(U, V, W) + (\mu^3 U^2 + 3 \mu^2 U - a \mu V - \mu W^2 - \mu V W + b \mu) \sum_{j=0}^{n} \mu^j \frac{\partial}{\partial W} F_j(U, V, W) = k_0 \sum_{j=0}^{n} F_j(U, V, W).$$

(20)

We calculate the terms which contain $\mu^0$ to obtain

$$-\frac{\partial}{\partial V} F_0(U, V, W) Z = k_0 F_0(U, V, W) = 0,$n  

that is

$$F_0(U, V, W) = G_0(V, W) e^{-\frac{\mu^0}{W}},$$

where $G_0$ is a polynomial function of variables $V$ and $W$. Since, $F_0(U, V, W)$ is a polynomial function. Thus, we obtain $k_0 = 0$. This implies that system (1) has no invariant algebraic surfaces with non-zero cofactors.

**Theorem 3.3.** System (1) has no rational first integrals.
Proof. From Proposition 3.2, system (1) has no Darboux polynomials. Then by Proposition 2.3, system (1) has no proper rational first integral.

We proved that in Proposition 3.2, system (1) does not have invariant algebraic surfaces. So, by Proposition 2.4, an exponential function must be in the following

\[ E = e^{g(u,v,w)}, \]

for more details see (Libre & Valls, 2012).

**Proposition 3.4.** System (1) has only two exponential factors \( e^u \) and \( e^v \) with cofactors \(-w\) and \(-u - w\), respectively.

Proof. Let \( E = e^{g(u,v,w)} \),

\[ g(u, v, w) = \sum_{k=0}^{n} g_k(u, v, w) \]

be an exponential factor with non-zero cofactor \( L = L_0 + L_1 u + L_2 v + L_3 w \), where each \( g_k \) is a homogeneous polynomial in its variables of degree \( k \). Then, we have

\[ -w \frac{\partial}{\partial u} e^{g(u,v,w)} + (-u - w) \frac{\partial}{\partial v} e^{g(u,v,w)} + (3 u - a v + u^2 - w^2 - v w + b) \frac{\partial}{\partial w} e^{g(u,v,w)} = L e^{g(u,v,w)}. \]

Simplifying

\[ -w \frac{\partial}{\partial u} g(u, v, w) + (-u - w) \frac{\partial}{\partial v} g(u, v, w) + (3 u - a v + u^2 - w^2 - v w + b) \frac{\partial}{\partial w} g(u, v, w) = L. \]

Firstly, we assume that \( n > 1 \). Calculating the terms of degree \( n + 1 \) in equation (22), we take out

\[ (u^2 - w^2 - v w) \frac{\partial}{\partial w} g_n(u, v, w) = 0, \]

that is

\[ g_n(u, v, w) = F_1(u, v), \]

where \( F_1 \) is a polynomial of degree \( n \). Also, computing the terms of degree \( n \) in equation (22), we obtain

\[ -w \frac{\partial}{\partial u} F_1(u, v) + (-u - w) \frac{\partial}{\partial v} F_1(u, v) + (3 u - a v + u^2 - w^2 - v w + b) \frac{\partial}{\partial w} F_1(u, v) + (u^2 - w^2 - v w) \frac{\partial}{\partial w} g_{n-1}(u, v, w) = 0, \]

this gives

\[ \begin{align*}
    g_{n-1}(u, v, w) &= \\
    &= \frac{1}{\sqrt{4u^2 + v^2}} \left( (2u - v) \left( \frac{\partial}{\partial v} F_1(u, v) \right) - \right. \\
    &\left. v \left( \frac{\partial}{\partial u} F_1(u, v) \right) \right) \arctan \left( \frac{v + 2w}{\sqrt{4u^2 + v^2}} \right) + \\
    &\left( \left( -\frac{1}{2} \frac{\partial}{\partial u} F_1(u, v) - \frac{1}{2} \frac{\partial}{\partial v} F_1(u, v) \right) \ln(-u^2 + \\
    &w^2 + v w) + F_2(u, v) \right) \sqrt{4u^2 + v^2} \\
\end{align*} \]

Since \( g_{n-1}(u, v, w) \) is a polynomial of degree \( n - 1 \), then we have

\[ \frac{\partial}{\partial u} F_1(u, v) + \frac{1}{2} \frac{\partial}{\partial v} F_1(u, v) = 0 \quad (23) \]

and

\[ (2u - v) \left( \frac{\partial}{\partial v} F_1(u, v) \right) - v \left( \frac{\partial}{\partial u} F_1(u, v) \right) = 0, \quad (24) \]

it is clearly that the solution of equation (23) is

\[ F_1(u, v) = F_3(v - u), \]

where \( F_3 \) is a polynomial of the variables \( u \) and \( v \). Since, \( F_1(u, v) \) is a polynomial of degree \( n \), then it must be in the formula

\[ F_1(u, v) = c (v - u)^n, \quad (25) \]

where \( c \) is arbitrary constant. Putting (25) in (24) we take out

\[ c n (v - u)^n = 0. \]

Since, \( n > 1 \) then \( c = 0 \), this implies that \( F_1 = 0 \), this gives \( g_n = 0 \). Thus \( g = 0 \), for \( n > 1 \). Now, we assume that \( g(u, v, w) \) is a polynomial of degree \( n = 1 \).

Letting \( g(u, v, w) = c_0 + c_1 u + c_2 v + c_3 w \).

Then, by equation (22), we have

\[ -w c_1 + (-u - w) c_2 + (3 u - a v + u^2 - w^2 - v w + b) c_3 = L_0 + L_1 u + L_2 v + L_3 w. \]

Comparing the coefficient, we obtain \( c_3 = L_2 \), \( c_1 = L_1 \) and \( c_2 = -L_3 \). That is

\[ g(u, v, w) = (L_1 - L_3) (u - L_1 v). \]

This implies that \( e^{(L_1 - L_3) (u - L_1 v)} \) is the exponential factor with cofactor \( L_3 u + L_3 w \). Hence, the only two independent exponential factors of system (1) are \( e^u \) and \( e^v \) with cofactors \(-w\) and \(-u - w\), respectively.

Now, having a Darboux first integral is illustrated in the following theorems.
Theorem 3.5. System (1) has no Darboux first integrals.  
Proof. Since, $e^u$ and $e^v$ are the unique exponential factors with cofactors $-w$ and $-u-w$, respectively. Then by Darboux Theorem 2.5, we have  
$$
\mu_1(-w) + \mu_2(-u-w) = 0, \quad (26)
$$
with non-zero constants $\mu_1, \mu_2 \in \mathbb{C}$. The above equation has no non-trivial solution. Then, system (1) does not have a Darboux first integral.

The condition $ab \neq 0$ that has assumed in system (1) is an essential condition to prove the existence of $C^1$ and an analytic first integral.

Theorem 3.6. If $a \in (0,3), b > 0$ and $3b - ab > a^2$ then system (1) has no a global $C^1$ first integral.  
Proof. Since $s_0 = \left(0, \frac{b}{a}, 0\right)$ is the equilibrium point of system (1), then the Jacobian matrix at $s_0$ of system (1) is  
$$
J = \begin{bmatrix}
0 & 0 & -1 \\
-1 & 0 & -1/b \\
3 & -a & -b/a 
\end{bmatrix}.
$$

A characteristic equation of the above matrix is  
$$
P(\lambda) = \lambda^3 + \frac{b}{a} \lambda^2 + (3-a) \lambda + a = 0.
$$

Then by Theorem 2.1 the eigenvalues have non-zero negative real parts if and only if $a \in (0,3), b > 0$ and $3b - ab > a^2$. Then, by Theorem 2.2 system (1) has no global $C^1$ first integrals in the neighborhood of $s_0$.

Proposition 3.7. The linear part of system (1) has no polynomial first integrals at the equilibrium point $s_0 = \left(0, \frac{b}{a}, 0\right)$, where $a$ and $b$ satisfy  
$$
27a^4 - b^3 = 0, 
$$
$$
3a^3 - 9a^2 + b^2 = 0. \quad (27)
$$
Proof. Firstly, we use the linear transformation  
$$
(u, v, w) \rightarrow \left(u, v + \frac{b}{a}, w\right)
$$
to move $s_0$ into the origin by then system (1) becomes  
$$
\begin{align*}
\dot{u} &= -w, \\
\dot{v} &= -u-w, \\
\dot{w} &= 3u - a v - \frac{b}{a} w - v w + u^2 - w^2. \quad (28)
\end{align*}
$$

The linear part can be written of the above system as  
$$
\begin{bmatrix}
\dot{u} \\
\dot{v} \\
\dot{w}
\end{bmatrix} =  
\begin{bmatrix}
0 & 0 & -1 \\
-1 & 0 & -1/b \\
3 & -a & -b/a 
\end{bmatrix} 
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}. \quad (29)
$$

The characteristic equation at (0,0,0) is  
$$
u^3 + \frac{b}{a} u^2 + (3-a) \ u + a = 0. \quad (30)
$$

Simply, we can see that equation (30) has a triple real root say $\lambda$ if and only if it could be written as  
$$(u - \lambda)^3 = u^3 - 3\lambda u^2 + 3\lambda^2 u - \lambda^3.$$

That is, $\lambda = \frac{-b}{3a}$, with  
$$L_1(a,b) = \lambda^3 + a, \quad L_2(a,b) = 3a - 3\lambda^2.$$

Putting the value of $\lambda = \frac{-b}{3a}$ in $L_1$ and $L_2$. We compute the value of $a$ and $b$ such that $L_1 = L_2 = 0$, to obtain  
$$
27a^4 - b^3 = 0, 
$$
$$
3a^3 - 9a^2 + b^2 = 0.
$$
This implies that $\lambda_1 = \lambda_2 = \lambda_3 = -\frac{b}{3a}$, then from cases 1-3 in Proposition 2.6, we obtain the linear part of system (1) has no a polynomial first integrals.

**Theorem 3.8.** If $a$ and $b$ satisfy the condition (27), then system (1) does not have analytic first integrals at $s_0 = \left(0, \frac{b}{a}, 0\right)$.

Proof. Firstly, we move the equilibrium point $\left(0, \frac{b}{a}, 0\right)$ into the (0,0,0). Then, by the linear change of coordinates $(u, v, w) \rightarrow \left(u, v + \frac{b}{a}, w\right)$, system (1) can be transformed into system (28).

Suppose that $H = \sum_{i \geq 1} H_i(x, y, z)$ is analytic first integral of system (1), where $H_i$ is a homogeneous polynomial of degree $i$ for all $i \geq 1$.

We will illustrate by induction that $H_i = 0$ for all $i \geq 1$.

Since, $H$ is a first integral of system (28), then by definition of first integral, we have

$$-w \frac{\partial H}{\partial u} + (-u - w) \frac{\partial H}{\partial v} + \left(3u - a v - \frac{b}{a} w - v w + u^2 - w^2\right) \frac{\partial H}{\partial w} = 0. \quad (31)$$

Calculating the terms of degree 1 in equation (31), we take out

$$-w \frac{\partial}{\partial u} H_1(u, v, w) + (-u - w) \frac{\partial}{\partial v} H_1(u, v, w) + \left(3u - a v - \frac{b}{a} w\right) \frac{\partial}{\partial w} H_1(u, v, w) = 0. \quad (32)$$

Then $H_1$ could be a zero polynomial or it could be a polynomial first integral of first degree. Since, $a$ and $b$ satisfy the condition (27), then, by Proposition 3.7, the linear part of system (28) has no polynomial first integral. This gives $H_1 = 0$, which proves $H_i = 0$ for $i = 1$.

Now, assume that $H_i = 0$ for $i = 1, ..., m - 1$ with $m \geq 2$, and it will be proved for $i = m$.

Using induction supposition, calculating the terms of degree $m$ in equation (31), we obtain

$$-w \frac{\partial}{\partial u} H_m(u, v, w) + (-u - w) \frac{\partial}{\partial v} H_m(u, v, w) + \left(3u - a v - \frac{b}{a} w\right) \frac{\partial}{\partial w} H_m(u, v, w) = 0. \quad (33)$$

Then, $H_m$ could be a zero polynomial or it could be a polynomial first integral of $m$ degree. Since, $a$ and $b$ satisfy the condition (27), we proceed as the case $H_i$ for $i = 1$, and using by Proposition 3.7, we obtain that $H_m = 0$. This proves that $H_i = 0$ for all $i \geq 1$. Thus, system (28) has no analytic first integral. Going back under the change of coordinates $(u, v, w) \rightarrow \left(u, v + \frac{b}{a}, w\right)$, gives that system (1) does not have an analytic first integral at $s_0$.

**4. CONCLUSION**

In this paper, we proved that the Kingni–Jafari system has no Darboux first integrals. Also, this system has no analytic first integrals at the neighborhood of the equilibrium point and we obtained that the system has no global $C^1$ first integrals for $a \in (0,3), b > 0$ and $3b - ab > a^2$.

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