SARNAK’S CONJECTURE FOR NILSEQUENCES ON ARBITRARY NUMBER FIELDS AND APPLICATIONS

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Abstract. We formulate the generalized Sarnak’s M"{o}bius disjointness conjecture for an arbitrary number field $K$, and prove a quantitative disjointness result between polynomial nilsequences $(\Phi(g(n)\Gamma))_{n \in \mathbb{Z}^D}$ and aperiodic multiplicative functions on $O_K$, the ring of integers of $K$. Here $D = [K: \mathbb{Q}]$, $X = \mathbb{G}/\Gamma$ is a nilmanifold, $g : \mathbb{Z}^D \rightarrow \mathbb{G}$ is a polynomial sequence, and $\Phi : X \rightarrow \mathbb{C}$ is a Lipschitz function. The proof uses tools from multi-dimensional higher order Fourier analysis, multi-linear analysis, orbit properties on nilmanifold, and an orthogonality criterion of Kátai in $O_K$.

We also use variations of this result to derive applications in number theory and combinatorics: (1) we prove a structure theorem for multiplicative functions on $K$, saying that every bounded multiplicative function can be decomposed into the sum of an almost periodic function (the structural part) and a function with small Gowers uniformity norm of any degree (the uniform part); (2) we give a necessary and sufficient condition for the Gowers norms of a bounded multiplicative function in $O_K$ to be zero; (3) we provide partition regularity results over $K$ for a large class of homogeneous equations in three variables. For example, for $a, b \in \mathbb{Z}\{0\}$, we show that for every partition of $O_K$ into finitely many cells, where $K = \mathbb{Q}(\sqrt{a}, \sqrt{b}, \sqrt{a+b})$, there exist distinct and non-zero $x, y$ belonging to the same cell and $z \in O_K$ such that $ax^2 + by^2 = z^2$.

1. Introduction

1.1. Sarnak’s Conjecture on number fields. Let $\mu : \mathbb{Z} \rightarrow \{-1, 0, 1\}$ be the M"{o}bius function, which is defined by $\mu(n) = (-1)^k$ if $n$ is the product of $k$ distinct prime numbers in $\mathbb{N}$, and $\mu(n) = 0$ otherwise. It is widely believed the function $\mu$ satisfies the “M"{o}bius randomness law” (see Section 13.1 of [31]), in the sense that $\mu$ is not correlated with any sequence of complex numbers of “low complexity”. This vague principle turns out to often provide heuristic asymptotics for various averages along primes (see [47] for examples). In [42], a precise conjecture was formulated by Sarnak:

Conjecture 1.1 (Sarnak’s Conjecture for integers). Let $(X, T)$ be a topological system with zero topological entropy. Then for all $\Phi \in C(X)$ and $x \in X$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n)\Phi(T^n x) = 0.$$ 

Many instances of Sarnak’s Conjecture have been proven. We give a few examples but stress that this is an incomplete list: [1, 3, 8, 9, 10, 15, 16, 29, 30, 35, 37, 39, 40, 42, 52]. It is natural to ask whether Sarnak’s Conjecture holds with $\mu$ replaced by other functions which are interesting in

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1The definition of the M"{o}bius function is usually stated for $\mathbb{N}$, but for the convenience of this paper we state it for $\mathbb{Z}$. 

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analytic number theory. We say that a function \( \chi : \mathbb{Z} \rightarrow \mathbb{C} \) is multiplicative (written as \( \chi \in \mathcal{M}_\mathbb{Q} \)) if \( \chi(mn) = \chi(m)\chi(n) \) for all \( (m, n) = 1 \). Let \( \mathcal{M}_\mathbb{Q}^a \) denote the set of all multiplicative functions \( \chi \) of modulus at most 1 which is aperiodic, meaning that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi(an + b) = 0
\]

for all \( a, b \in \mathbb{Z}, a \neq 0 \). It is a classical result that the Möbius function \( \mu \) is aperiodic. One can ask the following question:

**Question 1.2** (Generalized Sarnak’s Conjecture for integers). For which \( \chi \in \mathcal{M}_\mathbb{Q}^a \) does the following hold: for every topological system \((X, T)\) with zero topological entropy, every \( \Phi \in C(X) \), and every \( x \in X \), we have that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \chi(n)\Phi(T^n x) = 0.
\]

It is not hard to see that the answer to Question 1.2 is false when \( \chi \) is not aperiodic. Motivated by the results in [36], a natural conjecture is that the answer to Question 1.2 is affirmative if \( \chi \) is “strongly aperiodic” (which in particular includes all real-valued aperiodic multiplicative functions, see [36] for the definition).

In this paper, we enhance the scope of Sarnak’s Conjecture (and related topics) to multiplicative functions on general number fields, and seek applications in this broader setting. Let \( K \) be an algebraic number field and \( O_K \) be its ring of integers (see Section 2 for definitions). Denote \( D := [K: \mathbb{Q}] \) and let \( \mathcal{B} = \{b_1, \ldots, b_D\} \) be an integral basis of \( O_K \). For convenience we call \( K = (K, O_K, D, \mathcal{B} = \{b_1, \ldots, b_D\}) \) an integral tuple. In analog to the case \( K = \mathbb{Q} \), for a general number field \( K \), one can define the Möbius function \( \mu_K \), the set of bounded multiplicative functions \( \mathcal{M}_K^b \), and that of bounded aperiodic multiplicative functions \( \mathcal{M}_K^a \) in a natural way (see Section 1.2 for the precise definitions).

However, not all the Möbius functions \( \mu_K \) enjoy the same properties as the Möbius function \( \mu \) on \( \mathbb{Z} \). As we shall see in Section 1.2 for certain number fields \( K \), the Möbius function \( \mu_K \) does not even take the value -1. So we turn to the generalized Sarnak’s Conjecture instead.

**Question 1.3** (Generalized Sarnak’s Conjecture for algebraic number fields). Let \( K = (K, O_K, D, \mathcal{B} = \{b_1, \ldots, b_D\}) \) be an integral tuple. For which \( \chi \in \mathcal{M}_K^a \) does the following hold: for every topological system \((X, T_1, \ldots, T_d)\) with commuting transformations \( T_1, \ldots, T_d \) with zero topological entropy, every \( \Phi \in C(X) \), and every \( x \in X \), we have that

\[
\lim_{N \to \infty} \frac{1}{N^D} \sum_{1 \leq n_1, \ldots, n_D \leq N} 1_p(n_1, \ldots, n_D)\chi(n_1 b_1 + \cdots + n_D b_D)\Phi(T_1^{n_1} \cdots T_D^{n_D} x) = 0.
\]

The main result of this paper is to provide an affirmative answer to Question 1.3 for nilsystems (see Section 4 for definitions) with respect to all aperiodic functions, in a more general sense that one can replace \( T_1^{n_1} \cdots T_D^{n_D} x \) by any polynomial sequence (see Section 4 for definitions), and taking the average along any arithmetic progression:

**Theorem 1.4** (Generalized Sarnak’s Conjecture along nilsequences). Let \( K = (K, O_K, D, \mathcal{B} = \{b_1, \ldots, b_D\}) \) be an integral tuple. Let \( X = G/\Gamma \) be a nilmanifold and \( g : \mathbb{Z}^D \rightarrow \mathbb{C} \) be a polynomial sequence. Then for every \( \chi \in \mathcal{M}_K^a \), every \( \Phi \in C(X) \), and every \( D \)-dimensional arithmetic
progression \( P \) we have that
\[
\lim_{N \to \infty} \frac{1}{(2N + 1)^D} \sum_{-N \leq n_1, \ldots, n_D \leq N} 1_P(n_1, \ldots, n_D) \chi(n_1 b_1 + \cdots + n_D b_D) \Phi(g(n_1, \ldots, n_D) \cdot e_X) = 0.
\]

In particular, Theorem 1.4 implies that Conjecture 1.3 holds for every integral tuple \( K \) and every nilmanifold \( X \) with \( T_1, \ldots, T_d \) being translations on \( X \) (not necessarily commuting with each other).

If the sequence \( (g(n_1, \ldots, n_D) \cdot e_X)_{(n_1, \ldots, n_D) \in \mathbb{Z}^D} \) in Theorem 1.4 is equidistributed on \( X \), meaning that
\[
\lim_{N \to \infty} \frac{1}{(2N + 1)^D} \sum_{-N \leq n_1, \ldots, n_D \leq N} \Phi(g(n_1, \ldots, n_D) \cdot e_X) = 0
\]
for every \( \Phi \in C(X) \) such that \( \int_X \Phi \, dm_X = 0 \) (where \( m_X \) is the Haar measure on \( X \)), then one can deduce a generalization of a result of Daboussi, which can be viewed as a variation of Theorem 1.4.

**Theorem 1.5 (Generalized Daboussi’s Theorem).** Let \( K = (K, O_K, D, \mathcal{B} = \{b_1, \ldots, b_D\}) \) be an integral tuple. Let \( X = G/T \) be a nilmanifold and \( g : \mathbb{Z}^D \to C \) be a polynomial sequence such that \( (g(n_1, \ldots, n_D) \cdot e_X)_{(n_1, \ldots, n_D) \in \mathbb{Z}^D} \) is equidistributed on \( X \). Then for every \( \Phi \in C(X) \) such that \( \int_X \Phi \, dm_X = 0 \) and every \( D \)-dimensional arithmetic progression \( P \), we have that
\[
\lim_{N \to \infty} \sup_{x \in M_K} \left| \frac{1}{(2N + 1)^D} \sum_{-N \leq n_1, \ldots, n_D \leq N} 1_P(n_1, \ldots, n_D) \chi(n_1 b_1 + \cdots + n_D b_D) \Phi(g(n_1, \ldots, n_D) \cdot e_X) \right| = 0.
\]

For the special case \( O_K = \mathbb{Z} \) of Theorem 1.5 when \( X = \mathbb{T} \) and \( g \) is a linear polynomial, this is also known as Daboussi’s theorem ([12, 13, 14]). when \( X = \mathbb{T} \) and \( g \) is a general polynomial, this was essentially proved by Kátai [32]. The general case for \( O_K = \mathbb{Z} \) was proved by Frantzikinakis and Host (Theorem 2.2 of [18]), and the case when \( O_K = \mathbb{Z}[i] \) and \( X \) is at most 2-step was proved in Propositions 7.8 and 7.9 of [43].

In addition to Theorems 1.4 and 1.5, we also provide quantitative versions of them, which are Theorems 8.1 and 7.1 respectively.

**Remark 1.6.** The proofs of the quantitative Theorems 7.1 and 8.1 being much more complicated than the qualitative Theorems 1.4 and 1.5 occupy the bulk of this paper. The paper could be largely shorten if one is satisfied with the qualitative results only. However, in order for these results to be useful for applications, it is important to have quantitative versions of them.

### 1.2. Applications.

It turns out that there are many applications of Theorems 1.4 and 1.5 (and their quantitative versions), which we explain in this section.

#### 1.2.1. Structure theorem for multiplicative functions.

We start with the precise definition for multiplicative functions on \( O_K \) (see Section 2 for terminologies arising from algebraic number theory):

**Definition 1.7 (Multiplicative functions on \( O_K \)).** Let \( K \) be a number field and \( O_K \) be its ring of integers. We say that a function \( \chi : O_K \to \mathbb{C} \) is multiplicative on \( O_K \) if for all \( m, n \in O_K \) such that the \( K \)-norm \( N_K(m) \) of \( m \) is coprime with \( N_K(n) \) in \( \mathbb{Z} \), we have that \( \chi(mn) = \chi(m)\chi(n) \).

\(^2\)A set \( P \subseteq \mathbb{Z}^D \) is a \( D \)-dimensional arithmetic progression if \( P = \{n_0 + \sum_{i=1}^D M_i n_i : n_i \in \{0, \ldots, N_i - 1\}, 1 \leq i \leq D\} \) for some \( M_i, N_i \in \mathbb{N}, \) and \( n_0 \in \mathbb{Z}^D \) is called the length and \((M_1, \ldots, M_D)\) the step of \( P \).

\(^3\)The additional assumption \( \int_X \Phi \, dm_X = 0 \) is necessary, as otherwise the theorem fails for \( \Phi \equiv 1 \) and \( \chi \equiv 1 \).
Let \( \mathcal{M}_K \) denote the collection of all multiplicative functions on \( \mathcal{O}_K \) with modulus at most 1, and \( \mathcal{M}_K^a \) denote the collection of all aperiodic functions \( \chi \) in \( \mathcal{M}_K \), meaning that

\[
\lim_{N \to \infty} \sup_P \left| \frac{1}{(2N+1)^D} \sum_{-N \leq a_1, \ldots, a_D \leq N} 1_P(n_1b_1 + \cdots + n_Db_D)\chi(n_1b_1 + \cdots + n_Db_D) \right| = 0,
\]

where \( \sup_P \) is taken over all \( D \)-dimensional arithmetic progressions \( P \).

We show in Appendix A that the definitions in (1) and (3) coincide when \( K = \mathbb{Q} \). In particular, one can define the Möbius function \( \mu_K : K \to \{-1, 0, 1\} \) on \( \mathcal{O}_K \) by letting \( \mu_K(n) = (-1)^k \) if the ideal \( (n) \) is the product of \( k \) distinct prime ideals, and \( \mu_K(n) = 0 \) otherwise.\(^4\) Then \( \mu_K \in \mathcal{M}_K \). However, we caution the readers that \( \mu_K \) does not always belong to \( \mathcal{M}_K^a \). For example, it is not hard to show that \( \mu_K \) does not take the value -1 when \( K = \mathbb{Q}(\sqrt{2}, \sqrt{3}) \), as for every prime integer \( p \in \mathbb{N} \), \( (p) \) is the product of 2 or 4 prime ideals in \( \mathcal{O}_K \).

One application of the Sarnak’s Conjecture along nilsequences is to provide structure theorems for multiplicative functions \( \chi \) in \( \mathcal{M}_K \) on an arbitrary number field \( K \). Roughly speaking, our structure theorem says that \( \chi \) can be written as the sum of two functions \( \chi_s \) (the “structural part”) and \( \chi_u \) (the “uniform part”), where \( \chi_s \) is approximately periodic and \( \chi_u \) behaves randomly enough to have a negligible contribution for the applications we are interested in. The uniformity of a function is measured by the Gowers norms.

**Definition 1.8** (Gowers uniformity norms on \( \mathbb{Z}_N^D \)). For \( d, D, N \in \mathbb{N}_+ \), we define the \( d \)-th Gowers uniformity norm of \( f \) on \( \mathbb{Z}_N^D \) inductively by

\[
\|f\|_{U^d(\mathbb{Z}_N^D)} := \left| \frac{1}{N^D} \sum_{n \in \mathbb{Z}_N^D} f(n) \right|
\]

and

\[
\|f\|_{U^d+1(\mathbb{Z}_N^D)} := \left( \frac{1}{N^D} \sum_{m \in \mathbb{Z}_N^D} \|f_{m} \cdot \overline{f}\|_{U^d(\mathbb{Z}_N^D)}^2 \right)^{1/2^{d+1}}
\]

for \( d \geq 1 \), where \( \overline{f} \) denotes the conjugate of \( f \) and \( f_m(n) := f(m + n) \) for all \( n \in \mathbb{Z}_N^D \).

Gowers\(^5\) showed that this defines a norm on functions on \( \mathbb{Z}_N \) for \( d > 1 \). These norms were later used by Green, Tao, Ziegler and others in studying the primes (see, for example,\(^2\) \( {22, 49, 27} \)). Analogous semi-norms were defined in the ergodic setting by Host and Kra\(^2\)\(^8\).

**Convention 1.9.** For an integral tuple \( K = (K, \mathcal{O}_K, D, \mathcal{B} = \{b_1, \ldots, b_D\}) \), let \( \iota_B : \mathbb{Z}^D \to \mathcal{O}_K \) denote the bijection given by

\[
\iota_B(n_1, \ldots, n_D) = n_1b_1 + \cdots + n_Db_D.
\]

Let \( \chi : \mathcal{O}_K \to \mathbb{C} \) be a function and \( N, \tilde{N} \in \mathbb{N} \) be such that \( N < \tilde{N} \). We use \( \chi_{N, \tilde{N}} : \mathbb{Z}_{\tilde{N}}^D \to \mathbb{C} \) to denote the truncated function given by \( \chi_{N, \tilde{N}}(n) = \chi \circ \iota_B(n) \) for all \( n \in \{1, \ldots, N\}^D \) and \( \chi_{N, \tilde{N}}(n) = 0 \) otherwise. Through this paper, we write \( \chi_N := \chi_{N, \tilde{N}} \) to simplify the notations of truncated functions. The quantity \( \tilde{N} \) will be always clear from the context.

\(^4\)In fact, multiplicative functions (and in particular the Möbius function) can be defined on all the ideals of \( \mathcal{O}_K \) rather than just on the principle ideals \( (n) \) in a natural way. In this paper, we restrict the domain of the functions to the principle ideals only since such functions are already good enough for applications.

\(^5\)\( \mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z} \).
We assume that the set $M_K$ is endowed with the topology of pointwise convergence and thus is a compact metric space. The main structure theorem we have is the following, which generalizes Theorem 1.1 of [18] and answers Problem 1 of [18]:

**Theorem 1.10** ($U^d$ structure theorem for multiplicative functions). Let $\Omega \in \mathbb{N}$. For all $N \in \mathbb{N}$, let $\tilde{N}$ denote the smallest prime integer greater than $\Omega N$. Let $K = (K, O_K, D, B = \{b_1, \ldots, b_D\})$ be an integral tuple and $\nu$ be a probability measure on the group $M_K$. For every $\epsilon > 0$ and $d \geq 2$, there exist $Q := Q(\nu, D, \epsilon, \Omega)$, $R := R(\nu, D, \epsilon, \Omega)$ and $N_0 := N_0(\nu, D, \epsilon, \Omega) \in \mathbb{N}$ such that for every $N \geq N_0$ and $\chi \in M_K$, the truncated function $\chi_N : \mathbb{Z}_N^d \rightarrow \mathbb{C}$ can be written as

$$
\chi_N(n) = \chi_{N,s}(n) + \chi_{N,a}(n)
$$

for all $n \in \mathbb{Z}_N^d$ such that the following holds:

1. $|\chi_{N,s}(n)| \leq 1$ and $|\chi_{N,s}(n + Qe) - \chi_{N,s}(n)| \leq \frac{\epsilon}{N}$ for every $n \in \mathbb{Z}_N^d$ and $1 \leq i \leq D$.
2. $\|\chi_{N,a}\|_{U^d(\mathbb{Z}_N^d)} \leq \epsilon$.

We remark that by Bertrand’s postulate, $\Omega N < \tilde{N} < 2\Omega N$. The reason that we work on $\mathbb{Z}_N^d$ rather than $\mathbb{Z}_N^d$ is that for all $a \in O_K$ such that $0 < |N_K(a)| < \tilde{N}$, the map $n \mapsto na_{A_8}(a) \mod \mathbb{Z}_N^d$ (see Section 2 for the definition of $a_{A_8}(a)$) is a bijection from $\mathbb{Z}_N^d$ to itself (see also the discussion after Proposition 10.10 for the reason).

We prove Theorem 1.10 (and its stronger version Theorem 9.2) in Section 9.

1.2.2. A criteria for aperiodic multiplicative functions. Another application of the main results of the paper is to provide a criteria for aperiodic multiplicative functions using Gowers norms. Denote $[N] := \{1, \ldots, N\}$. The Gowers norms can be extended to functions taking values in $\mathbb{Z}^d$ (as was done in [18] [22]).

**Definition 1.11** (Gowers uniformity norms on intervals). Let $d \geq 2$ and $D, N \in \mathbb{N}$. For every functions $f : [N]^D \rightarrow \mathbb{C}$, by Lemma A.2 of Appendix A of [18], the quantity

$$
\|f\|_{U^d([N]^D)} := \frac{1}{\|1_{[N]^D}\|_{U^d(\mathbb{Z}_N^d)}} \cdot \|1_{[N]^D} \cdot f\|_{U^d(\mathbb{Z}_N^d)}
$$

is independent of $N^*$ if $N^* > 2N$, which is called the $U^d([N]^D)$-norm of $f$.

If the $U^d([N]^D)$-norm of a function goes to 0 as $N \rightarrow \infty$, then it is an aperiodic function. However, the converse is not always the case. But for multiplicative functions, these two conditions are equivalent:

**Theorem 1.12** (Structure theorem for aperiodic multiplicative functions). Let $K = (K, O_K, D, B = \{b_1, \ldots, b_D\})$ be an integral tuple and $\chi \in M_K$. Then $\chi$ is aperiodic if and only if $\lim_{N \rightarrow \infty} \|\chi \circ \iota_B\|_{U^d([N]^D)} = 0$ for all $d \geq 2$.

Theorem 1.12 generalizes Theorem 2.5 of [18], and we prove it in Section 8.
1.2.3. Partition regularity for homogeneous equations. An important question in Ramsey theory is to determine which algebraic equations, or systems of equations, are partition regular. As there are various formulations of the partition regular questions, we start with a technical definition in order to cover as many cases as possible:

**Definition 1.13** (Partition regularity). Let \( K = (K, O_K, D, \mathcal{B} = \{b_1, \ldots, b_D\}) \) be an integral tuple, \( s \in \mathbb{N}_+, r \in \mathbb{N}, \) and \( p \in \mathbb{C}[x_1, \ldots, x_i; z_1, \ldots, z_r] \) be a polynomial. We say that \( p \) is **partition regular** over \( O_K \) with respect to \( x_1, \ldots, x_i \) if for every finite partition \( O_K = \bigcup_{m=1}^{\infty} U_i \) of \( O_K \), there exist \( 1 \leq i \leq m, x_1, \ldots, x_i \in U_j \) non-zero and pairwise distinct such that \( p(x_1, \ldots, x_i; z_1, \ldots, z_r) = 0 \) for some \( z_1, \ldots, z_r \in O_K \).

It was proved by Rado \cite{rado} that for \( a, b, c \in \mathbb{Z}\setminus\{0\} \), the linear polynomial \( p(x_1, x_2, x_3) = ax_1 + bx_2 + cx_3 \) is partition regular over \( \mathbb{Z} \) with respect to \( x_1, x_2 \) and \( x_3 \) (in this case \( r = 0 \)) if and only if one of \( a + b, a + c, b + c, a + b + c \) is \( 0 \). The situation is much less clear for second or higher degree equations \( p \), or for integer rings \( O_K \) other than \( \mathbb{Z} \), unless we allow some of the variables in \( p \) to take values freely (namely \( r > 0 \)). It is a classical result of Furstenberg \cite{furstenberg} and Sarközy \cite{sarkozy} that the equation \( p(x_1, x_2; z_1) = x_1 - x_2 - z_1^2 \) is partition regular over \( \mathbb{Z} \) with respect to \( x_1 \) and \( x_2 \). Bergelson and Leibman \cite{bergelson} provided other examples of translation invariant equations by proving a polynomial version of the van der Waerden Theorem. A result of Khalfalah and Szemerédi \cite{khalfalah} showed that the equation \( p(x_1, x_2; z_1) = x_1 + x_2 - z_1^2 \) is partition regular over \( \mathbb{Z} \) with respect to \( x_1 \) and \( x_2 \).

In the work of Frantzikinakis and Host \cite{frantzikinakis}, by using the structure theorem for multiplicative functions on \( \mathbb{Z} \), they proved that certain class of quadratic equations with two restricted variables \( x_1 \) and \( x_2 \) are partition regularity (as well as scattered examples of higher degree equations). For example, they showed that the equation

\[
(4) \quad p(x_1, x_2; z_1) = ax_1^2 + bx_2^2 - z_1^2,
\]

is partition regular over \( \mathbb{Z} \) with respect to \( x_1 \) and \( x_2 \) if \( a, b, a + b \) are non-zero square integers (for example, \( a = 16, b = 9 \)).

Note that not all equations of the form (4) are partition regular over \( \mathbb{Z} \). For example, the equation \( x_1^2 + 3x_2^2 - z_1^2 = 0 \) has even no non-trivial integer solutions. So it is natural to consider the partition regularity problems over a larger ring of integers. In \cite{khalfalah}, by using a partial structure theorem for multiplicative functions on \( \mathbb{Z}[i] \), the author proved that (4) is partition regular over \( \mathbb{Z}[i] \) with respect to \( x_1 \) and \( x_2 \) if \( \sqrt{a}, \sqrt{b}, \sqrt{a+b} \in \mathbb{Z}[i] \) (for example, \( a = 1, b = -1 \)). We remark that the question whether \( x_1^2 - x_2^2 - z_1^2 \) is partition regular over \( \mathbb{Z} \) with respect to \( x_1 \) and \( x_2 \) remains an open question.

In this paper, we provide partition regularity results for a larger family of polynomials over certain ring of integers. Our main result is Theorem 10.2. We postponed its precise statement to Section 10 but provide a few sample applications of Theorem 10.2 in the introduction.

The first is an example for quadratic equations:

**Theorem 1.14.** Let \( p(x_1, x_2; z_1) = ax_1^2 + bx_2^2 - z_1^2 \) for some \( a, b \in \mathbb{Z}\setminus\{0\} \), then \( p \) is partition regular over the ring of integers of \( \mathbb{Q}(\sqrt{a}, \sqrt{b}, \sqrt{a+b}) \) with respect to \( x_1 \) and \( x_2 \).

For example, \( p(x_1, x_2; z_1) = 9x_1^2 + 16x_2^2 - z_1^2 \) is partition regular over \( \mathbb{Z} \) with respect to \( x_1 \) and \( x_2 \), which reproves a result in \cite{frantzikinakis}: \( p(x_1, x_2; z_1) = x_1^2 - x_2^2 - z_1^2 \) is partition regular over \( \mathbb{Z}[i] \) with

\footnote{The original result of \cite{rado} was stated for \( \mathbb{N} \) but a similar argument holds for \( \mathbb{Z} \).}
Sarnak’s Conjecture for Nilsequences and Applications

1.3. Methods and Organizations. The first part of this paper is devoted to the proof of Theorem 7.1 (the quantitative version of Theorem 1.5), which is the central result in this paper. In Sections 2, 3, and 4, we provide background materials used in this paper. In Section 2, we provide all the results we need from algebraic number theory. In particular, we provide the Kátai’s Lemma on algebraic number fields (Lemma 2.2.1), which is a useful tool for the study of Sarnak’s Conjecture on an arbitrary number field. In Sections 3 and 4, we provide basic properties on nilmanifolds and equidistribution properties for polynomial sequences, respectively. The materials in these two sections is a mixture of classical knowledge and original results.

Sections 5, 6, and 7 are the main novelties of this paper, where we use the tools from previous sections to prove Theorem 7.1. In Section 7, we use the materials in Sections 2, 3, and 4 to reduce Theorem 7.1 to two questions (which are also the two main innovations of this paper): (i) the description of a special subnilmanifold of the product space \( X \times X \), which is carried out in Section 5 (Theorem 5.5); and (ii) a problem in multi-linear algebra, which is answered in Section 6 (Theorem 6.1).

The second part of this paper is to use Theorem 7.1 to prove all other results. In Section 8, we prove some immediate consequences of Theorem 7.1, including Theorem 1.5, the generalized Daboussi’s Theorem, Theorem 8.1 (the quantitative version of Theorem 1.4), the Sarnak’s Conjecture for nilsequences, and Theorem 1.12, the criteria for aperiodic multiplicative functions. In Section 9, we prove the structure theorem for multiplicative functions, namely Theorem 9.2 (which is a stronger version of Theorem 1.10). In Section 10, we prove the partition regularity results (i.e., Theorem 1.14 in its full generality) by using the structure theorem.

Remark 1.16 (Overlapping with literatures). Due to the unavoidable formalism in the proofs of the results, many parts of this paper have overlapping with [18, 26, 43]. To be more precise, Sections 3, 4, 7, 8, 9, and 10 partially overlap with [18, 43] (Sections 3 and 4 also partially overlap with [26]); Sections 2, 5, and 6 are completely new and have no counterparts in [18, 26, 43].

1.4. Open Questions. For multiplicative functions on number fields, there are many natural questions in addition to Sarnak’s Conjecture (Conjecture 1.3). For example, one can ask the logarithm generalization of Sarnak’s Conjecture:

**Conjecture 1.17** (Generalized logarithm Sarnak’s Conjecture). Let \( K = (K, O_K, \mathcal{D}, \mathcal{B} = (b_1, \ldots, b_D)) \) be an integral tuple. For which \( \chi \in \mathcal{M}_K^{\mathbb{Z}} \) does the following hold: for every topological system \( (X, T_1, \ldots, T_d) \) with commuting transformations \( T_1, \ldots, T_d \) with zero topological entropy, every \( \Phi \in C(X) \) and every \( x \in X \), we have that

\[
\lim_{N \to \infty} \frac{1}{(\log N)^D} \sum_{1 \leq n_1, \ldots, n_D \leq N} \frac{\chi(n_1b_1 + \cdots + n_Db_D)\Phi(T_{1}^{n_1} \cdots T_{D}^{n_D}x)}{n_1n_2 \cdots n_D} = 0.
\]
It is worth noting that for the case $K = \mathbb{Q}$, Question 1.17 (and Question 1.18 below) is not true for all aperiodic multiplicative functions (to see this, one can use the example in Theorem B.1 of [36]). For the case $K = \mathbb{Q}$, under certain ergodicity assumption of the system, Conjecture 1.17 was proved by Frantzikinakis and Host in [19] when $\chi$ is the Möbius function, and then in [17] when $\chi$ is strongly aperiodic.

It is also natural to ask the analog of Chowla’s Conjecture:

**Question 1.18** (Generalized Chowla’s (and logarithm Chowla’s) Conjecture). Let $K = (K, O_K, D, \mathcal{B} = \{b_1, \ldots, b_D\})$ be an integral tuple. For which $\chi \in M_K^d$ does the following hold: for every $k \in \mathbb{N}_+$ and $m_1, \ldots, m_k \in O_K$ which are pairwise distinct, we have that

$$
\lim_{N \to \infty} \frac{1}{N^D} \sum_{n = n_1 b_1 + \cdots + n_D b_D, 1 \leq n_1, \ldots, n_D \leq N} \chi(n + m_1)\chi(n + m_2)\cdots\chi(n + m_k) = 0
$$

(or

$$
\lim_{N \to \infty} \frac{1}{(\log N)^D} \sum_{n = n_1 b_1 + \cdots + n_D b_D, 1 \leq n_1, \ldots, n_D \leq N} \chi(n + m_1)\chi(n + m_2)\cdots\chi(n + m_k) = 0.
$$

for the logarithm version).

For $K = \mathbb{Q}$, it is not hard to show that Chowla’s Conjecture for $\chi = \mu$ implies Sarnak’s Conjecture (see for example [48]). Moreover, if $\chi$ is the Möbius function on $\mathbb{Z}$, then logarithm Chowla’s Conjecture is equivalent to the logarithm Sarnak’s Conjecture [50], both of which are known to be true when $k = 2$ [49] and when $k$ is an odd number [51].

Another natural question is whether Sarnak’s Conjecture holds in the measure theoretic setting:

**Conjecture 1.19** (Generalized measurable Sarnak’s Conjecture). Let $K = (K, O_K, D, \mathcal{B} = \{b_1, \ldots, b_D\})$ be an integral tuple and $\chi \in M_K^d$. Then for every measure preserving system $(X, \mu, T_1, \ldots, T_d)$ with commuting transformations $T_1, \ldots, T_d$ [19] every $\Phi \in L^\infty(\mu)$ and $\mu$-a.e. $x \in X$,

$$
\lim_{N \to \infty} \frac{1}{N^D} \sum_{1 \leq n_1, \ldots, n_D \leq N} \chi(n_1 b_1 + \cdots + n_D b_D)\Phi(T_{n_1}^{b_1} \cdot \cdots \cdot T_{n_D}^{b_D} x) = 0.
$$

One possible approach to prove Conjecture 1.19 for the case $K = \mathbb{Q}$ is to combine Kátai’s Lemma (see for example Lemma 2.21), Bourgain’s double pointwise convergence theorem [7], the Host-Kra structure theorem (Theorem 10.1 of [26]), and the orthogonality of multiplicative functions and nilsequences (see Theorem 2.5 of [18], or Theorem 1.4 in this paper). In Theorem 3.1 of [2], by using a result from Green and Tao [24], another proof was given for Conjecture 1.19 for the case when $K = \mathbb{Q}$ and $\chi$ is the Möbius function. The case when $K$ is a number field other than $\mathbb{Q}$ remains open.

Due to the importance of the Möbius function in number theory and additive combinatorics, the following question is also interesting to ask:

**Question 1.20.** On which integral tuple $K = (K, O_K, D, \mathcal{B} = \{b_1, \ldots, b_D\})$ is the Möbius function $\mu_K$ aperiodic? For such $K$, are the answers to Questions 1.13, 1.17 and 1.18 affirmative for $\chi = \mu_K$? Does Conjecture 1.19 hold for the special case $\chi = \mu_K$?

\(^{10}\)Note that there is no assumption on entropy in this conjecture.
SARNAK’S CONJECTURE FOR NILSEQUENCES AND APPLICATIONS

1.5. Notations. We introduce the notations we use in this paper.

- \( M_K \) and \( M_K^a \) are the sets of multiplicative and aperiodic multiplicative functions on \( O_K \) with modulus at most 1, respectively.
- For \( N \in \mathbb{N}_+ \), denote \([N] := \{1, \ldots, N\} \) and \( \mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z} \).
- For \( D \in \mathbb{N}_+ \) and \( N \geq 0 \), denote \( R_{N,D} := \{(n_1, \ldots, n_D) \in \mathbb{Z}^D : |n_i| \leq N, 1 \leq i \leq D\} \).
- Let \( a : \mathbb{C} \to \mathbb{C} \) be a map with \( V \) being a finite set, denote
  \[ \mathbb{E}_{x \in V} a(x) := \frac{1}{|V|} \sum_{x \in V} a(x). \]
- Let \( a, b : \mathbb{N} \to \mathbb{C} \) and \( I \) be a collection of parameters. The notion \( C := C(I) \) means that \( C \) is a quantity depending only on the parameters in \( I \). We write \( a \ll_I b \) if there exist \( C := C(I) > 0, N_0 := N_0(I) > 0 \) such that \( a(N) \leq Cb(N) \) for all \( N \geq N_0 \).
- For \( x \in \mathbb{R} \), \( \lfloor x \rfloor \) is the largest integer which is not smaller than \( x \), and \( \lceil x \rceil \) is the smallest integer which is not larger than \( x \).
- Let \( e : \mathbb{R} \to \mathbb{C} \) denote the function \( e(x) := e^{2\pi i x} \).
- For \( i \in \mathbb{N}_+ \), \( e_i \) denotes the vector whose \( i \)-th coordinate is 1 and all other coordinates are 0 (the dimension of \( e_i \) will be clear from the context).
- For a vector \( v = (v_1, \ldots, v_m) \in \mathbb{C}^m \) for some \( m \in \mathbb{N}_+ \), denote \( |v| := |v_1| + \cdots + |v_m| \).
- For \( v = (v_1, \ldots, v_m) \in \mathbb{R}^m \), let
  \[ ||v||_{T_m} := \inf_{n = (n_1, \ldots, n_m) \in \mathbb{Z}} |v - n|. \]
- For \( w = u + vi \in \mathbb{C}^m \) for some \( u, v \in \mathbb{R}^m \), let \( ||w||_{T_m} := ||u||_{T_m} + ||v||_{T_m} \).
- For \( m, n \in \mathbb{Z} \) and \( R \) be a ring. Then \( M_{m \times n}(R) \) denote all the \( m \times n \) matrices whose entries take values from \( R \). Let \( O_{m \times n} \) denote the \( m \times n \) matrix whose all entries are 0.
- Throughout this paper, for \( A \in M_{s \times s}(\mathbb{R}) \), we use \( A : \mathbb{R}^s \to \mathbb{R}^s \) to denote the map given by \( A(x) := x \cdot A, x \in \mathbb{R}^s \), i.e. the right multiplication of \( A \).

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2. Ingredients from algebraic number theory

2.1. Algebraic number field and minimal polynomials.

Definition 2.1 (Integral tuple). An (algebraic) number field \( K \) is a finite degree and (hence algebraic) field extension of the field of rational numbers \( \mathbb{Q} \). The ring of integers \( \mathcal{O}_K \) of a number field \( K \) is the ring of all integral elements contained in \( K \).[1] Denote \( D = [K : \mathbb{Q}] \). It is classical that there exists an integral basis \( \mathcal{B} = \{b_1, \ldots, b_D\} \) of \( \mathcal{O}_K \), i.e. a basis \( b_1, \ldots, b_D \in \mathcal{O}_K \) of the \( \mathbb{Q} \)-vector space \( K \) such that each element \( x \in \mathcal{O}_K \) can be uniquely represented as \( x = \sum_{i=1}^{D} a_ib_i \) for some \( a_i \in \mathbb{Z} \). We call \( K = (K, \mathcal{O}_K, D, \mathcal{B} = \{b_1, \ldots, b_D\}) \) an integral tuple.

[1] An integral element is a root of polynomial with integer coefficients and leading coefficient 1.

[2] If \( K \) is a field extension of \( L \), then \([K : L]\) denotes the degree of this extension.
Let \( K = (K, \mathcal{O}_K, D, \mathcal{B} = \{b_1, \ldots, b_D\}) \) be an integral tuple. Recall that \( \iota_B : \mathbb{Z}^D \rightarrow \mathcal{O}_K \) is the bijection given by \( \iota_B(n_1, \ldots, n_D) = n_1b_1 + \cdots + n_Db_D \). For \( x \in K \), let \( A_B(x) \in M_{D \times D}(\mathbb{Q}) \) be the unique matrix such that
\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_D 
\end{bmatrix}
= A_B(x)
\begin{bmatrix}
    b_1 \\
    b_2 \\
    \vdots \\
    b_D 
\end{bmatrix}
\].
This implies that for all \( x, y \in K \),
\[
(5) \quad \iota_B^{-1}(xy) = \iota_B^{-1}(y)A_B(x).
\]
We remark that \( A_B(x) \in M_{D \times D}(\mathbb{Z}) \) if \( x \in \mathcal{O}_K \).

**Definition 2.2 (K-norm).** The \( K \)-norm of \( x \in K \) is \( N_K(x) := \det(A_B(x)) \).

Note that \( N_K(x) \) is independent of the choice of the basis \( \mathcal{B} \).

We say that a polynomial \( f \in \mathbb{Q}[x] \) is **monic** if the leading coefficient of \( f \) is 1. We say that \( f \in \mathbb{Q}[x] \) is **irreducible** if \( f = gh \), \( g, h \in \mathbb{Q}[x] \) implies that one of \( g \) and \( h \) is a constant. We say that \( f \in \mathbb{Q}[x] \) is the **minimal polynomial** of an algebraic number \( x \) (or a matrix \( A \in M_{s \times s}(\mathbb{Q}) \)) if \( f \) is a monic polynomial of the smallest possible positive degree such that \( f(x) = 0 \) (or \( f(A) = O_{s \times s} \)).

The following lemma is classical and we omit the proof:

**Lemma 2.3 (Properties on the K-norm).** Let \( K = (K, \mathcal{O}_K, D, \mathcal{B} = \{b_1, \ldots, b_D\}) \) be an integral tuple. Then
1. For all \( x, y \in K \) and \( q \in \mathbb{Q} \), we have that \( A_B(x)A_B(y) = A_B(y)A_B(x) = A_B(xy) \), \( A_B(x) + A_B(y) = A_B(x + y) \) and \( A_B(qx) = qA_B(x) \). In particular, \( N_K(xy) = N_K(x)N_K(y) \);
2. If \( K \) is algebraically closed and \( f \) is the minimal polynomial of some \( x \in K \setminus \{0\} \), then \( N_K(x) = (-1)^{|K: \mathbb{Q}|}f(0)^{|K: \mathbb{Q}|/\deg(f)} \).

The following are some basic properties about minimal polynomials:

**Lemma 2.4 (Minimal polynomials).** Let \( K = (K, \mathcal{O}_K, D, \mathcal{B} = \{b_1, \ldots, b_D\}) \) be an integral tuple and \( \overline{K} \) be the algebraic closure of \( K \). Let \( x \in K \setminus \{0\} \) and \( f \) denote the minimal polynomial of the matrix \( A_\mathcal{B}(x) \). Then
1. \( f \) is also the minimal polynomial of \( x \);
2. \( f \) is irreducible, and has no repeated roots;
3. \( y \in \overline{K} \) is a root of \( f \) if and only if \( y \) is an eigenvalue of \( A_\mathcal{B}(x) \). In particular, by (ii), all the eigenvalues of \( A_\mathcal{B}(x) \) are distinct;
4. All the roots of \( f \) have the same \( \overline{K} \)-norm as \( x \).

**Proof.**
1. Let \( g \in \mathbb{Q}[x] \) be any polynomial. By Lemma 2.3(i), \( A_\mathcal{B}(g(x)) = g(A_\mathcal{B}(x)) \). Then
\[
g(x) = 0 \Leftrightarrow A_\mathcal{B}(g(x)) = O_{D \times D} \Leftrightarrow g(A_\mathcal{B}(x)) = O_{D \times D}.
\]
So the minimal polynomial \( f \) of \( A_\mathcal{B}(x) \) is also the minimal polynomial of \( x \).
2. Since the minimal polynomial of \( x \) is irreducible, by (i), \( f \) is irreducible. Since \( f' \neq 0 \), \( f \) is coprime with \( f' \) and so \( f \) has no repeated roots.
3. See [53].
4. By Lemma 2.3(ii), the \( \overline{K} \)-norm of all roots of \( f \) equal to \( (-1)^{|\overline{K}: \mathbb{Q}|}f(0)^{|\overline{K}: \mathbb{Q}|/\deg(f)} \), which are the same. \( \Box \)

The next is a characterization for diagonalizable matrices, which is used in later sections.
Lemma 2.5 (A characterization for diagonalizable matrices). Let \( f \in \mathbb{C}[x] \) be a non-constant polynomial with no repeated roots (in \( \mathbb{C} \)). Let \( s \in \mathbb{N}_+ \) and \( B \) be an \( s \times s \) matrix such that \( f(B) = O_{s \times s} \) for some \( s \in \mathbb{N}_+ \). Then there exist an \( s \times s \) invertible matrix \( S \) and a diagonal matrix
\[
J = \begin{bmatrix}
\mu_1 & \mu_2 & \cdots & \mu_s \\
\end{bmatrix}
\]
with \( f(\mu_1) = \cdots = f(\mu_s) = 0 \) such that \( B = JSJ^{-1} \).

Proof: The case when \( s = 1 \) is straightforward. So we assume that \( s \geq 2 \).

Let \( E_i \) denote the \( s \times s \) matrix whose \((k, k+i)\)-th entry is 1 for all \( 1 \leq k \leq s-i \) and all the other entries are 0. Converting \( B \) to the Jordan normal form, it suffices to show that if \( f(B) = O_{s \times s} \) and \( B = \mu I_s \) or \( B = \mu I_s + E_1 \) for some \( \mu \in \mathbb{C} \), then \( B = \mu I_s \) and \( f(\mu) = 0 \).

In fact, if \( B = \mu I_s \), then \( f(B) = f(\mu)I_s \). So \( f(\mu) = 0 \). If \( B = \mu I_s + E_1 \), note that \( E_i E_j = E_{i+j} \) for all \( 1 \leq i, j \leq s-1 \) (for convenience denote \( E_i = O_{s \times s} \) if \( i \geq s \)). Writing \( f(x) = \sum_{m=0}^{M} a_m x^m \) for some \( M \geq 1 \), we have that
\[
f(B) = \sum_{n=0}^{M} a_n \left( \sum_{i=0}^{n} \mu^{n-i} \binom{n}{i} \right) E_i = \sum_{i=0}^{M} E_i \left( \sum_{n=i}^{M} \binom{n}{i} a_n \mu^{n-i} \right) = 0,
\]
where \( E_0 = I_s \). This implies that \( \sum_{n=i}^{M} \binom{n}{i} a_n \mu^{n-i} = 0 \) for all \( 0 \leq i \leq \min(M, s-1) \). By assumption, \( \min(M, s-1) \geq 1 \). Setting \( i = 0 \), \( f(\mu) = \sum_{n=0}^{M} a_n \mu^n = 0 \). Setting \( i = 1 \), \( f'(\mu) = \sum_{n=1}^{M} n a_n \mu^{n-1} = 0 \).

A contradiction to the fact that \( f \) has no repeated roots. This finishes the proof.

\[ \square \]

2.2 Ideals and unique factorization.

Definition 2.6 (Units). We say that \( e \) is a unit of \( \mathcal{O}_K \) if there exists \( e' \in \mathcal{O}_K \) such that \( ee' = 1 \). Since \( N_K(x) \in \mathbb{Z} \) for all \( x \in \mathcal{O}_K \) and \( N_K(1) = 1 \), it is not hard to see that the \( K \)-norm of a unit is \( \pm 1 \).

Let \( K \) be a number field and \( \mathcal{O}_K \) be its ring of integers. A subset \( I \subseteq \mathcal{O}_K \) is an ideal of \( \mathcal{O}_K \) if for all \( x, y \in I, z \in \mathcal{O}_K \), we have that \( x - y, xz \in I \). An ideal \( I \) of \( \mathcal{O}_K \) is principle if there exists \( c \in \mathcal{O}_K \) such that \( I = \{ cx : x \in \mathcal{O}_K \} \). If \( I \) is generated by \( a_1, \ldots, a_k \in \mathcal{O}_K \), meaning that \( I = \{ c_1 a_1 + \cdots + c_k a_k : c_1, \ldots, c_k \in \mathbb{Z} \} \), then we write \( I = (a_1, \ldots, a_k) \) for short. Since \( \mathcal{O}_K \) is Noetherian, by Theorem 5.2.3 of [38], every ideal of \( \mathcal{O}_K \) is finitely generated, and so we may always write \( I \) as \( I = (a_1, \ldots, a_k) \) for some \( k \in \mathbb{N}_+ \) and \( a_1, \ldots, a_k \in \mathcal{O}_K \).

There are many different concepts of primes in a number field, which we clarify as follows:

Definition 2.7 (Primes). Let \( K = (K, \mathcal{O}_K, D, \mathcal{B} = \{b_1, \ldots, b_D\}) \) be an integral tuple.

- \( x \in \mathbb{N}_+ \) is a prime integer if \( x \) is a prime in the field \( \mathbb{Q} \).
- An ideal \( I \) of \( \mathcal{O}_K \) is a prime ideal if for all \( x, y \in \mathcal{O}_K \) such that \( xy \in I \), either \( x \in I \) or \( y \in I \).
- \( p \in \mathcal{O}_K \) is a prime element if \( (p) \) is a principle and prime ideal.
- For \( p, q \in \mathcal{O}_K \), we say that \( p \) divides \( q \) (written as \( q | p \)) if \( pq^{-1} \in \mathcal{O}_K \).

Let \( I \subseteq \mathcal{O}_K \) be an ideal of \( \mathcal{O}_K \). Then \( \mathcal{O}_K/I \) is a finite set (see for example Exercise 4.4.3 of [38]), and the cardinality of this set is called the index of \( I \) in \( \mathcal{O}_K \), or the \( K \)-norm of the ideal \( I \) (denoted as \( N(I) \)).

The following lemma is standard (see for example Exercise 5.3.15 of [38]):

Lemma 2.8 (Norms of ideals). Let \( K = (K, \mathcal{O}_K, D, \mathcal{B} = \{b_1, \ldots, b_D\}) \) be an integral tuple and \( I = (a) \) be a principle ideal of \( \mathcal{O}_K \) for some \( a \in \mathcal{O}_K \). The \( K \)-norm \( N(I) \) of \( I \) coincides with the absolute value of the \( K \)-norm \( |N_K(a)| \) of \( a \).
It is well known that every ideal $I$ of $\mathcal{O}_K$ can be factorized into the form 

\[ I = \prod_{i=1}^{m} I_i^{e_i} \]

for some $m \in \mathbb{N}_+$, $e_i \in \mathbb{N}_+$, prime ideal $I_i$ for all $1 \leq i \leq m$ in a unique way (modulo the order of the ideals $I_i$).

**Lemma 2.9 (Properties of prime ideals).** For every prime integer $p \in \mathbb{N}$, let $\mathcal{J}_p$ denote the collection (possibly an empty collection) of prime ideals of $\mathcal{O}_K$ which contains $(p)$. Then

(i) for all $J \in \mathcal{J}_p$, $N(J) = p^f$ for some $1 \leq f \leq D$.

(ii) $\mathcal{J}_p$ is a finite set of cardinality at most $D$.

(iii) every principle prime ideal of $\mathcal{O}_K$ belongs to some $\mathcal{J}_p$. In particular, the $K$-norm of every principle prime ideal is a power of a prime integer, and the $K$-norms of principle prime ideals from different $\mathcal{J}_p$ are coprime (in $\mathbb{Z}$).

**Proof.** (i) and (ii). Suppose that

\[ (p) = \prod_{i=1}^{m} I_i^{f_i} \]

for some $m \in \mathbb{N}_+$, $e_i \in \mathbb{N}_+$, prime ideal $I_i$ for all $1 \leq i \leq m$. By the unique factorization of $(p)$, $\mathcal{J}_p = \{I_1, \ldots, I_m\}$ and $|\mathcal{J}_p| = m$. Since $p \in \mathbb{N}$,

\[ p^D = |N_K(p)| = \prod_{i=1}^{m} N(I_i)^{e_i}. \]

So $N(I_i) = p^{f_i}$ for some $f_i \in \mathbb{N}_+$ for all $1 \leq i \leq m$. Therefore,

\[ m \leq \sum_{i=1}^{m} e_i f_i = D. \]

(iii) Let $J = (a)$ be a principle prime ideal for some $a \in \mathcal{O}_K$. We first claim that there exists $n \in \mathbb{N}_+$ such that $a|n$. Let $f(x) = \sum_{n=0}^{M} a_n x^n \in \mathbb{Q}[x]$ be the minimal polynomial of $a$. Pick $C \in \mathbb{N}_+$ such that $Cf \in \mathbb{Z}[x]$. Then

\[ -a(\sum_{n=1}^{M} C a_n a^{n-1}) = C f(0) \in \mathbb{Z}. \]

Since $C a_n, C f(0)$ and $a \in \mathcal{O}_K$, we have that $a y \in \mathbb{Z}\setminus\{0\}$ for $y := \sum_{n=1}^{M} C a_n a^{n-1} \in \mathcal{O}_K$. This implies that $a|n$ and finishes the proof of the claim.

Since $J$ is a prime ideal, by the unique factorization of $(n)$, there exists a prime integer $p$ (dividing $n$) such that $J$ contains $(p)$. So $J \in \mathcal{J}_p$. \qed

**2.3. Regularization of algebraic numbers.** Let $K = (K, \mathcal{O}_K, D, \mathcal{B} = \{b_1, \ldots, b_D\})$ be an integral tuple. For $N \in \mathbb{N}_+$ and $a \in \mathcal{O}_K$, denote

\[ R_{N,\mathcal{B}} := \{z \in \mathcal{B} : \mathcal{O}_K^{-1}(z) \in \mathcal{R}_{N,D}\} \]

and

\[ a^{-1} R_{N,\mathcal{B}} := \{z \in \mathcal{O}_K : az \in R_{N,\mathcal{B}}\} \]
throughout this section. We caution the reader that the set $a^{-1}R_{N,B}$ is a subset of $O_K$, and is NOT the set of $z \in K$ such that $az \in R_{N,B}$.

We need to use the following estimate of the density of ideals frequently in this section:

**Lemma 2.10 (Density of ideals).** Let $K = (K, O_K, D, B = \{b_1, \ldots, b_D\})$ be an integral tuple and $I$ be an ideal of $O_K$. We have that

$$\lim_{N \to \infty} \frac{|I \cap R_{N,B}|}{(2N + 1)^D} = \frac{1}{N(I)}$$

**Proof.** There exist $d_1, \ldots, d_{N(I)} \in O_K$ with $d_1 = 0$ such that $d_i + I$ are disjoint subsets of $O_K$ and their union is $O_K$. Suppose that $I = (a_1, \ldots, a_k)$ for some $k \in \mathbb{N}_+$ and $a_1, \ldots, a_k \in O_K$. Let $M$ be a positive integer such that $a_i^M \in O_K$ for all $1 \leq i \leq k$. Then there exists a constant $C > 0$ such that for all $x \in O_K$, the cardinality of the set $(x + I) \cap R_{M,B}$ equals to $(2M + 1)^D C$. For $N \in \mathbb{N}_+$, by partitioning $R_{N,B}$ into smaller cubes of the form $R_{M,B}$, it is easy to see that the limit exists and equals to $C$ for all $1 \leq i \leq N(I)$. Since the union of $d_i + I, 1 \leq i \leq N(I)$ is $O_K$, we have that $C = \frac{1}{N(I)}$. \(\square\)

If $a \in \mathbb{Z}\setminus\{0\}$, then clearly $a^{-1}R_{N,B}$ is a subset of $R_{N/|a|B}$, which is a cube with $|N_K(a)| = a^{-D}$ of the size of $R_{N,B}$. However, this is not the case when $a \notin \mathbb{Z}$.

**Example 2.11.** Consider the integral tuple $K = (K, O_K, D, B = \{b_1, \ldots, b_D\}) = (\mathbb{Q}(\sqrt{2}), \mathbb{Z}[\sqrt{2}], 2, \{1, \sqrt{2}\})$.

For all $m, n \in \mathbb{Z}$, $A_B(m + n \sqrt{2}) = \begin{bmatrix} m & n \\ 2n & m \end{bmatrix}$ and it has two real eigenvalues $m \pm n \sqrt{2}$. Let $a = (2 + \sqrt{2})^4$. Then $\det(A_B(a)) = 16$, and the two eigenvalues of $A_B(a)$ are respectively $(2 + \sqrt{2})^4 \approx 135.882$ and $(2 - \sqrt{2})^4 \approx 0.118$. Although the “volume” of $a^{-1}R_{N,B}$ is approximately $1/16$ of that of $R_{N,B}$, $a^{-1}R_{N,B}$ is not contained in $R_{N,B}$ (it is only contained in a much larger rectangle $R_{136N,B}$).

On the other hand, if we multiple a with the unit $e = (-1 + \sqrt{2})^4$ and denote $a' := ae = 4$, then $a'^{-1}R_{N,B} \subseteq R_{N/4,B}$, which is $|N_K(a)| = 1/16$ the size of $R_{N,B}$.

We prove the following theorem in this section, which generalizes the phenomena appeared in the previous example.

**Definition 2.12 (C-regular number).** Let $K = (K, O_K, D, B = \{b_1, \ldots, b_D\})$ be an integral tuple and $C > 0$. We say that $a \in O_K$ is **C-regular** if for all $N \in \mathbb{N}$, we have that $a^{-1}R_{N,B} \subseteq R_{C|N_K(a)|^{-1/2}N,B}$.

**Theorem 2.13 (Regularization of algebraic numbers).** Let $K = (K, O_K, D, B = \{b_1, \ldots, b_D\})$ be an integral tuple. There exists a constant $C_B > 0$ depending only on $B$ such that for every $a \in K$, there exists a unit $e$ of $O_K$ such that $ea$ is $C_B$-regular.

We start with a structure theorem of the eigenspaces of $A_B(x)$.

**Lemma 2.14 (Structures for the eigenspaces of $A_B(x)$).** Let $K = (K, O_K, D, B = \{b_1, \ldots, b_D\})$ be an integral tuple. Then there exist $r_1, r_2 \in \mathbb{N}$ with $r_1 + 2r_2 = D$, and a decomposition of $\mathbb{C}^D$ into $1$ dimensional subspaces

$$\mathbb{C}^D = \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_{r_1} \oplus (\mathbb{C}v_{r_1+1} \oplus \mathbb{C}v_{r_1+2}) \oplus \cdots \oplus (\mathbb{C}v_{r_1+r_2} \oplus \mathbb{C}v_{r_1+r_2})$$

\(^{13}\)We use the notation $R_{N,D}$ in all other sections, but use $R_{N,B}$ in this section as it is more convenient.
for some \( v_1, \ldots, v_r \in \mathbb{R}^D \) and \( v_{r+1}, \ldots, v_{r+2} \in \mathbb{C}^D \) such that for all \( x \in K \), there exist \( \lambda_1(x), \ldots, \lambda_{r+2}(x) \in \mathbb{R} \) such that \( v_i A_{\mathcal{B}}(x) = \lambda_i(x)v_i \) for all \( 1 \leq i \leq r+2 \). Moreover, \( \lambda_i(x_1)\lambda_i(x_2) = \lambda_i(x_1x_2) \) for all \( x_1, x_2 \in K \) and \( 1 \leq i \leq r+2 \).

**Proof.** Let \( y \in K \) be any number the degree of whose minimal polynomial is \( D \) (the existence of such \( y \) is guaranteed by the Theorem of the Primitive Element, see for example Theorem 3.3.2 of [38]). By Lemma 2.14, \( \mathcal{B} \) has \( D \) distinct eigenvalues. So we may decompose \( \mathbb{C}^D \) into 1 dimensional subspaces

\[
\mathbb{C}^D = \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_D
\]

for some \( v_1, \ldots, v_D \in \mathbb{C}^D \) such that \( v_i A_{\mathcal{B}}(y) = \lambda_i(y)v_i \) for some \( \lambda_i(y) \in \mathbb{C} \) for all \( 1 \leq i \leq D \). Since \( A_{\mathcal{B}}(x) \) is a matrix with real coefficients, complex eigenvalues and eigenvectors come in pairs, and so we may assume that there exist \( r_1, r_2 \in \mathbb{N} \) with \( r_1 + 2r_2 = D \), such that

\[
\mathbb{C}^D = \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_{r_1} \oplus (\mathbb{C}v_{r_1+1} \oplus \mathbb{C}v_{r_1+1}^\perp) \oplus \cdots \oplus (\mathbb{C}v_{r_1+r_2} \oplus \mathbb{C}v_{r_1+r_2}^\perp)
\]

for some \( v_1, \ldots, v_{r_1} \in \mathbb{R}^D \) and \( v_{r_1+1}, \ldots, v_{r_1+r_2} \in \mathbb{C}^D \), \( \lambda_1(y), \ldots, \lambda_{r_1}(y) \in \mathbb{R} \) and \( \lambda_{r_1+1}(y), \ldots, \lambda_{r_1+r_2}(y) \in \mathbb{C} \) such that \( v_i A_{\mathcal{B}}(y) = \lambda_i(y)v_i \) for all \( 1 \leq i \leq r_1 + r_2 \).

Let \( x \in K \). By Lemma 2.14(i), for all \( 1 \leq i \leq r_1 + r_2 \),

\[
(v_i A_{\mathcal{B}}(x)) A_{\mathcal{B}}(y) = (v_i A_{\mathcal{B}}(y)) A_{\mathcal{B}}(x) = \lambda_i(y)(v_i A_{\mathcal{B}}(x)).
\]

So both \( v_i \) and \( v_i A_{\mathcal{B}}(x) \) are eigenvectors of eigenvalue \( \lambda_i(y) \) for the matrix \( A_{\mathcal{B}}(y) \). Since \( \lambda_1(y), \ldots, \lambda_{r_1}(y) \) and \( \lambda_{r_1+1}(y), \ldots, \lambda_{r_1+r_2}(y) \) are distinct and the eigenspace of every eigenvalue of \( A_{\mathcal{B}}(y) \) is 1-dimensional, we have that \( v_i A_{\mathcal{B}}(x) \) for \( 1 \leq i \leq r_1 \) and \( v_i A_{\mathcal{B}}(x) \) for \( r_1 \leq i \leq r_1 + r_2 \). So for all \( 1 \leq i \leq r_1 + r_2 \), there exist \( \lambda_1(x), \ldots, \lambda_{r_1}(x) \in \mathbb{R} \) and \( \lambda_{r_1+1}(x), \ldots, \lambda_{r_1+r_2}(x) \in \mathbb{C} \) such that \( v_i A_{\mathcal{B}}(x) = \lambda_i(x)v_i \).

For \( x_1, x_2 \in K \) and \( 1 \leq i \leq r_1 + r_2 \), note that

\[
\lambda_i(x_1x_2)v_i = v_i A_{\mathcal{B}}(x_1x_2) = v_i A_{\mathcal{B}}(x_1) A_{\mathcal{B}}(x_2) = \lambda_i(x_1)v_i A_{\mathcal{B}}(x_2) = \lambda_i(x_1) \lambda_i(x_2)v_i.
\]

So \( \lambda_i(x_1) \lambda_i(x_2) = \lambda_i(x_1x_2) \). \qed

Let the notations be as in Lemma 2.14 and denote \( r = r_1 + r_2 - 1 \). Let \( W : K \rightarrow \mathbb{R}^r \) be the map given by

\[
W(x) = (\log |\lambda_1(x)|, \ldots, \log |\lambda_r(x)|).
\]

Then

\[
(6) \quad \prod_{i=1}^{r_1} \lambda_i(x) \cdot \prod_{i = r_1+1}^{r_1+r_2} |\lambda_i(x)|^2 = \det (A_{\mathcal{B}}(x)) = N_K(x),
\]

and so the value of \( \log |\lambda_{r_1+1}(x)| \) is uniquely determined by \( N_K(x) \) and \( W(x) \). The following result is essentially proved in Theorem 8.1.6 of [38]:

**Proposition 2.15.** Let \( U_K \) denote the group of units in \( O_K \). Then \( W(U_K) \) is a lattice of \( \mathbb{R}^r \), i.e. there exist \( \epsilon_1, \ldots, \epsilon_r \in U_K \) such that the \( \mathbb{R} \)-span of \( W(\epsilon_1), \ldots, W(\epsilon_r) \) is \( \mathbb{R}^r \).

**Proof of Theorem 2.17** Let \( \epsilon_1, \ldots, \epsilon_r \in U_K \) be such that the \( \mathbb{R} \)-span of \( W(\epsilon_1), \ldots, W(\epsilon_r) \) is \( \mathbb{R}^r \). By Proposition 2.15 there exist a constant \( C_1 := C_1(K) > 0 \) and \( x_1, \ldots, x_r \in \mathbb{Z} \) such that denoting

\[
\epsilon = \epsilon_1^{x_1} \ldots \epsilon_r^{x_r},
\]

we have that

\[
\log |\lambda_i(\epsilon a)| = \log |\lambda_i(a)| + \sum_{j=1}^{r} x_j \log |\lambda_j(\epsilon_j)| \in \left[ \frac{1}{D} \log |N_K(a)|, \frac{1}{D} \log |N_K(a)| + C_1 \right]
\]
for all $1 \leq i \leq r$. Note that $\varepsilon$ is a unit of $\mathcal{O}_K$ and so $|N_K(\varepsilon)| = 1$. By (6),
\[
\log |\lambda_{r+1}(\varepsilon)| = \frac{1}{\alpha_{r+1}}(\log |N_K(\varepsilon)| - \sum_{i=1}^{r} \alpha_{i} \log |\lambda_{i}(\varepsilon)|) = \frac{1}{\alpha_{r+1}}(\log |N_K(a)| - \sum_{i=1}^{r} \alpha_{i} \log |\lambda_{i}(\varepsilon)|),
\]
where $\alpha_{i} = 1$ if $1 \leq i \leq r_1$ and $\alpha_{i} = 2$ if $r_1 + 1 \leq i \leq r + 1$. So
\[
\log |\lambda_{r+1}(\varepsilon)| \in \left[ \frac{1}{D} \log |N_K(a)| - rC_1, \frac{1}{D} \log |N_K(a)| \right].
\]

Now let $m_1 b_1 + \cdots + m_D b_D \in (ea)^{-1} R_{N,\mathcal{B}}$ for some $m_1, \ldots, m_D \in \mathbb{Z}$. By definition, there exists $n_1 b_1 + \cdots + n_D b_D \in R_{N,\mathcal{B}}$ for some $n_1, \ldots, n_D \in \mathbb{Z}$ such that
\[
n_1 b_1 + \cdots + n_D b_D = (ea)(m_1 b_1 + \cdots + m_D b_D) = (m_1, \ldots, m_D)A_{\mathcal{B}}(ea) \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_D \end{bmatrix}.
\]
In other words, $(m_1, \ldots, m_D) = (n_1, \ldots, n_D)A_{\mathcal{B}}^{-1}(ea)$. Suppose that
\[
(n_1, \ldots, n_D) = \sum_{i=1}^{r_1} c_i v_i + \sum_{i=r_1+1}^{r+1} (c_i v_i + c_i' \nu_i)
\]
for some $c_i, c_i' \in \mathbb{C}$, where $v_i$ is defined in Lemma 2.14. Then
\[
(m_1, \ldots, m_D) = \sum_{i=1}^{r_1} c_i \lambda_i^{-1}(ea)v_i + \sum_{i=r_1+1}^{r+1} (c_i \lambda_i^{-1}(ea)v_i + c_i' (\lambda_i) \nu_i).
\]
Since $-N \leq n_1, \ldots, n_D \leq N$ and the basis $v_1, \ldots, v_{r+1}, \nu_{r+1}, \ldots, \nu_{r_1}$ depends only on $\mathcal{B}$, there exists $C_2 := C_2(\mathcal{B}) > 0$ such that $|c_i|, |c_i'| \leq C_2 N$ for all $1 \leq i \leq r + 1$. Then all of $|c_i \lambda_i^{-1}(ea)|$, $|c_i' (\lambda_i) \nu_i)$ are at most $C_2 N e^{2rC_1} |N_K(a)|^{-\frac{1}{r}}$. Again there exists $C_3 := C_3(\mathcal{B}) > 0$ such that $|m_i| \leq C_2 C_3 e^{2rC_1} |N_K(a)|^{-\frac{1}{r}} N$ for all $1 \leq i \leq D$. Setting $C_{\mathcal{B}} := C_2 C_3 e^{2rC_1}$, we have that $(m_1, \ldots, m_D) \in R_{C_{\mathcal{B}}|N_K(a)|^{-\frac{1}{r}}, N, \mathcal{B}}$, and so $(ea)^{-1} R_{N,\mathcal{B}} \subseteq R_{C_{\mathcal{B}}|N_K(a)|^{-\frac{1}{r}}, N, \mathcal{B}}$. \hfill \Box

**Remark 2.16.** The dimension $r$ of the $\mathbb{R}$-span of $W(U_K)$ equals to 0 if and only if $r_1 = 1, r_2 = 0$ or $r_1 = 0, r_2 = 1$, which implies that $K = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-d})$ for some square-free positive integer $d$. In this case, there exists $C_{\mathcal{B}} > 0$ such that every $a \in \mathcal{O}_K$ is $C_{\mathcal{B}}$-regular.

The following is another property of $C$-regular numbers:

**Lemma 2.17.** Let $K = (K, \mathcal{O}_K, D, \mathcal{B} = (b_1, \ldots, b_D))$ be an integral tuple and $C, N \geq 0$. Then there exists $C' := C'(\mathcal{B}, C) > 0$ such that for every prime element $a \in \mathcal{O}_K$ which is $C$-regular, we have that
\[
\left| \left\{ z \in a^{-1} R_{N,\mathcal{B}} : N_K(z) \text{ is not coprime with } N_K(z) \in \mathbb{Z} \right\} \right| \leq C' \cdot \frac{N_D}{|N_K(a)|^{1+\frac{1}{r}}} + o_{\mathcal{B}}(N^D).
\]

**Proof.** Let $\mathcal{J}_p$ be defined as in Lemma 2.9. Since $a$ is a prime element, $J := (a)$ is a prime ideal. By Lemma 2.9, there exists a prime integer $p \in \mathbb{N}$ such that $N(J) = p^i$ for some $1 \leq i \leq D$ and $J \in \mathcal{J}_p$. Again by Lemma 2.9 for all $J' \in \mathcal{J}_p$,
\[
N(J') \geq p \geq |N_K(a)|^{\frac{1}{r}}.
\]
Let \( z \in O_K \) be such that \(|N_K(a)| = N(J) = p^j \) is not coprime with \( N_K(z) \) in \( \mathbb{Z} \). Then \( N_K(z) \) is divisible by \( p \). By the unique factorization of \( (z) \), there exists a prime ideal \( J' \) of \( O_K \) such that \( z \in J' \) and \( N(J') \) divides \( p \). By Lemma 2.19, \( J' \in \mathcal{J}_p \). Then

\[
\left| \left\{ z \in a^{-1}R_{N,B} : N_K(a) \text{ is not coprime with } N_K(z) \text{ in } \mathbb{Z} \right\} \right| \\
\leq \left| \left\{ z \in R_{C[|N_K(a)|^{1/2}N,B]} : N_K(a) \text{ is not coprime with } N_K(z) \text{ in } \mathbb{Z} \right\} \right| \quad \text{(since } a \text{ is } C\text{-regular)} \\
\leq \sum_{J' \in \mathcal{J}_p} \left| R_{C[|N_K(a)|^{1/2}N,B]} \cap J' \right| \quad \text{(by the discussion above)} \\
\leq \sum_{J' \in \mathcal{J}_p} \frac{(2C|N_K(a)|^{1/2}N + 1)^D}{|N_K(a)|^{1/2}N} + o_{B,d}(N^D) \quad \text{(by Lemma 2.10)} \\
\leq D \cdot \frac{(2C|N_K(a)|^{1/2}N + 1)^D}{|N_K(a)|^{1/2}N} + o_{B,d}(N^D) \quad \text{(by Lemma 2.9(ii))} \\
\leq D2^{D-1}((2C)^D(N)^D|N_K(a)|^{1/2}N + 1) + o_{B,d}(N^D).
\]

This finishes the proof. \( \square \)

2.4. Kátaï’s Lemma on algebraic number fields. Kátaï’s Lemma is an important tool in the study of correlations between a multiplicative function and an arbitrary sequence in the integer ring \( \mathbb{Z} \). It was first proved by Kátaï [32] for \( K = \mathbb{Q} \) and generalized to \( K = \mathbb{Q}(\sqrt{-d}) \) for all positive square-free integer \( d \) by Frantzikinakis and Host [18]. In this section, we introduce a version of Kátaï’s Lemma for general number fields.

The proof of the following theorem can be found on pages 148–149 of [38].

**Theorem 2.18.** Let \( K \) be a number field and \( O_K \) be its ring of integers. Then

\[
\sum_{p \in O_K \text{ is a prime element}} \frac{1}{|N_K(p)|} = \infty \quad \text{and} \quad \sum_{p \in O_K \text{ is a prime element}} \frac{1}{|N_K(p)|^{1+c}} < \infty \quad \text{for all } c > 0.
\]

Let \( \mathcal{P} \) be a finite subset of \( O_K \) and \( z \in O_K \). Denote

\[
\mathcal{A}_p = \sum_{p \in \mathcal{P}} \frac{1}{|N_K(p)|} \quad \text{and} \quad \omega_p(z) = \sum_{p \in \mathcal{P} : p|z} 1.
\]

**Lemma 2.19** (Turán-Kubilius’ Lemma). Let \( K = (K, O_K, D, \mathcal{B} = \{b_1, \ldots, b_D\}) \) be an integral tuple. For every \( N \in \mathbb{N} \) and every finite subset of prime elements \( \mathcal{P} \) of \( O_K \) whose \( K\)-norms are pairwise coprime (in \( \mathbb{Z} \)), we have that

\[
\sum_{z \in \mathbb{Z}_{N,B}} |\omega_p(z) - \mathcal{A}_p| \ll_{\mathcal{B}} \sqrt{\mathcal{A}_p + 1} \cdot N^D + o_{B,D}(N^D).
\]

**Proof.** By the Cauchy-Schwartz inequality, it suffices to show that

\[
\sum_{z \in \mathbb{Z}_{N,B}} (\omega_p(z) - \mathcal{A}_p)^2 \ll_{\mathcal{B}} (\mathcal{A}_p + 1) \cdot N^D + o_{B,D}(N^D).
\]

Note that

\[
\sum_{z \in \mathbb{Z}_{N,B}} \mathcal{A}_p^2 = \mathcal{A}_p^2 \cdot (2N^2 + 1)^D.
\]
By Lemma 2.10,
\[
\sum_{z \in R_{K,B}} 2\mathcal{A}_p \cdot \omega_p(z) = 2\mathcal{A}_p \sum_{z \in R_{K,B}} \sum_{p \in \mathcal{P} : \ p \mid z} 1 = 2\mathcal{A}_p \sum_{p \in \mathcal{P}} |p^{-1}R_{K,B}|
\]
(10)
\[
= 2\mathcal{A}_p \sum_{p \in \mathcal{P}} \left( \frac{(2N + 1)^D}{|N_K(p)|} + o_B,p(N^D) \right) = 2\mathcal{A}_p^2 \cdot (2N + 1)^D + o_B,p(N^D).
\]

We claim that if \( z \in O_K, \ p \mid z \) and \( q \mid z \) for some \( p, q \in \mathcal{P}, p \neq q \), then \( pq \mid z \). It suffices to show that there is no prime ideal \( I \) such that \( I \supseteq (p) \) and \( I \supseteq (q) \). If not, by the unique factorization of \( (p) \) and \( (q) \), both \( N_K(p) \) and \( N_K(q) \) are divisible by \( N_I \). Since \( N_K(p) \) is coprime with \( N_K(q) \) in \( \mathbb{Z} \) by assumption, we have that \( |N_I| = 1 \), a contradiction. This proves the claim.

By Lemma 2.10 and the claim,
\[
\sum_{z \in R_{K,B}} \omega_p^2(z) = \sum_{z \in R_{K,B}} \left( \sum_{p,q \in \mathcal{P}, p \neq q} \sum_{\mathcal{P}} 1 + \sum_{p \in \mathcal{P}} 1 \right) = \sum_{p,q \in \mathcal{P}, p \neq q} \sum_{\mathcal{P}} \frac{(2N + 1)^D}{|N_K(pq)|} + o_B,p(N^D) + \sum_{p \in \mathcal{P}} \frac{(2N + 1)^D}{|N_K(p)|} + o_B,p(N^D)
\]
(11)
\[
\leq \mathcal{A}_p^2 \cdot (2N + 1)^D + (\mathcal{A}_p + C) \cdot (2N + 1)^D + o_B,p(N^D),
\]
where \( C := \sum_{p \in \mathcal{O}_K} \) a prime \( \frac{1}{|N_K(p)|} < \infty \) by Theorem 2.18. Then (9), (10) and (11) implies (8).

We are now ready to state Kátai’s Lemma on arbitrary algebraic number fields. An important difference between Kátai’s Lemma for \( K = \mathbb{Q} \) or \( \mathbb{Q}(\sqrt{-d}), d \in \mathbb{N} \) and that for arbitrary number field is that we require some regularity condition in the latter case (whereas the regularity condition always holds in the former case as it is mentioned in Remark 2.16).

Lemma 2.20 (Kátai’s Lemma on algebraic number fields (multiplicative version) ). Let \( K = (K, \mathcal{O}_K, D, \mathcal{B} = (b_1, \ldots, b_D)) \) be an integral tuple and \( C > 0 \). Let \( \chi \in \mathcal{M}_K \), and \( h : \mathcal{O}_K \to \mathbb{C} \) be a function with modulus at most 1. Let \( \mathcal{P} \) be a finite collection of \( C \)-regular prime elements of \( \mathcal{O}_K \) whose \( K \)-norms are pairwise coprime in \( \mathbb{Z} \). For \( N \in \mathbb{N}_+ \), let
\[
S(N) := \sum_{z \in R_{K,B}} \chi(z)h(z)
\]
and
\[
C_{\mathcal{P}}(N) := \sum_{p,q \in \mathcal{P}, p \neq q} \sum_{\mathcal{P}} h(pz) \overline{h(qz)}.
\]
Then
\[
\left| \frac{S(N)}{N^D} \right|^2 \leq C_{\mathcal{B}} \frac{1}{\mathcal{A}_p^2} \frac{C_{\mathcal{P}}(N)}{N^D} + \left( \frac{1}{\mathcal{A}_p} + \frac{1}{\mathcal{A}_p^2} \right) + o_{C,\mathcal{B},p}(1)
\]

Proof. Let
\[
S'(N) := \sum_{z \in R_{K,B}} \chi(z)h(z)\omega_p(z).
\]
By Lemma 2.19 \( |S'(N) - \mathcal{A}_pS(N)| \leq \sqrt{\mathcal{A}_p + 1} \cdot N^D + o_B,p(N^D) \). We may rewrite \( S'(N) \) as
\[
S'(N) = \sum_{z \in R_{K,B}} \sum_{p \in \mathcal{P}, p \mid z} \chi(z)h(z) = \sum_{p \in \mathcal{P}} \sum_{p \mid z} \chi(pz)h(pz).
\]
In this sum, the term $\chi(pz)h(pz)$ is equal to $\chi(p)\chi(z)h(pz)$ unless $N_K(p)$ is not coprime with $N_K(z)$ in $\mathbb{Z}$. By the $C$-regularity of $p$ and Lemma 2.21, if we set

$$S''(N) := \sum_{p \in \mathcal{P}} \sum_{z \in \mathbb{Z}^2 \cdot \mathcal{P}^{-1} R_{K, B}} \chi(p)\chi(z)h(pz),$$

then there exists $C_1 := C_1(C, B) > C$ such that

$$|S'(N) - S''(N)| \leq 2 \sum_{p \in \mathcal{P}} \left| \{z \in \mathbb{Z}^2 \cdot \mathcal{P}^{-1} R_{K, B} : N_K(p) \text{ is not coprime with } N_K(z) \text{ in } \mathbb{Z} \} \right| \leq 2C_1 \frac{N^D}{|N_K(p)|^{1 + \frac{1}{D}}} + o_{\mathbb{Z}, \mathcal{P}}(N^D) = 2C_1 N^D + o_{\mathbb{Z}, \mathcal{P}}(N^D),$$

where

$$C_2 = \sum_{a \in O_K \text{ is a prime element}} \frac{1}{|N_K(a)|^{1 + \frac{1}{D}}} < \infty$$

is a constant depending only on $K$. Let $R'_{K, B} := \bigcup_{p \in \mathcal{P}} p^{-1} R_{K, B}$. By the $C$-regularity of $p \in \mathcal{P}$, $R'_{K, B} \subseteq R_{K, B}$. By Cauchy-Swartz inequality,

$$|S''(N)|^2 = \left| \sum_{z \in \mathbb{Z}^2 \cdot \mathcal{P}^{-1} R_{K, B}} \chi(z) \sum_{p \in \mathcal{P}} \chi(p)h(pz) \right|^2 \leq (2CN + 1)^D \sum_{z \in \mathbb{Z}^2 \cdot \mathcal{P}^{-1} R_{K, B}} \left| \sum_{p \in \mathcal{P}} \chi(p)h(pz) \right|^2 \leq (2CN + 1)^D \sum_{p, q \in \mathcal{P}} \sum_{z \in \mathbb{Z}^2 \cdot \mathcal{P}^{-1} R_{K, B}} \chi(p)h(pz)\overline{\chi(q)h(qz)} \leq (2CN + 1)^D \sum_{p, q \in \mathcal{P}} \sum_{z \in \mathbb{Z}^2 \cdot \mathcal{P}^{-1} R_{K, B}} h(pz)\overline{h(qz)}.$$

Again by the $C$-regularity of $p \in \mathcal{P}$,

$$(2CN + 1)^D \sum_{p \in \mathcal{P}} \left| \sum_{z \in \mathbb{Z}^2 \cdot \mathcal{P}^{-1} R_{K, B}} h(pz)\overline{h(pz)} \right| \leq (2CN + 1)^D \sum_{p \in \mathcal{P}} \left| R_{C|N_K(p)|^{1 + \frac{1}{D}}, R_{K, B}} \right| = (2CN + 1)^D \sum_{p \in \mathcal{P}} (2C|N_K(p)|^{-\frac{1}{D}}N + 1)^D \ll_{C, \mathcal{B}} \mathcal{A}_p \cdot N^{2D} + |\mathcal{P}| \cdot N^D. $$

Combining all the previous estimates, we have that

$$|\mathcal{A}_p S(N)|^2 \leq |S'(N) - \mathcal{A}_p S(N)|^2 + |S''(N) - S'(N)|^2 + |S''(N)|^2 \ll_{C, \mathcal{B}} \mathcal{A}_p \cdot N^{2D} + o_{\mathbb{Z}, \mathcal{P}}(N^{2D}) + N^{2D} + |\mathcal{P}|^2 + N^D \cdot (C_F(N) + |\mathcal{P}|).$$

This finishes the proof by dividing both sides by $(\mathcal{A}_p \cdot N^D)^2$. \hfill \square

By using (5), we have the following additive version of Lemma 2.21:

**Lemma 2.21 (Kátai’s Lemma on algebraic number fields (additive version)).** Let $K = (K, O_K, D, B = \{b_1, \ldots, b_D\})$ be an integral tuple and $C > 0$. Let $\chi \in M_K$, and $h : \mathbb{Z}^D \to \mathbb{C}$ be a function with
modulus at most 1. Let $\mathcal{P}$ be a finite collection of $C$-regular prime elements of $O_K$ whose $K$-norms are pairwise coprime in $\mathbb{Z}$. For $N \in \mathbb{N}$, let

$$S(N) := \sum_{n \in \mathcal{R}_{N,D}} \chi(\iota_G(n)) h(n)$$

and

$$C_\mathcal{P}(N) := \sum_{p,q \in \mathcal{P}, p \neq q} \sum_{n \in \mathbb{Z}^D, n \mathcal{A}_G(p), n \mathcal{A}_G(q) \in \mathcal{R}_{N,D}} \lvert h(n \mathcal{A}_G(p)) \overline{\eta}(n \mathcal{A}_G(q)) \rvert.$$ Then

$$\left| \frac{S(N)}{(2N + 1)^D} \right|^2 \leq C_B \frac{1}{\mathcal{A}_p^2} \frac{C_\mathcal{P}(N)}{(2N + 1)^D} + \left( \frac{1}{\mathcal{A}_p} + \frac{1}{\mathcal{A}_q} \right) + o_{C,B,\mathcal{P}}(1).$$

3. NILMANIFOLDS

We provide the background material and the notations we use for nilmanifolds in this section. Some of the notions we use follow from [18, 26, 43].

3.1. Nilmanifolds and nil-structures. Let $G$ be a connected and simply connected Lie group with the identity element $e_G$. For $a, b \in G$, denote $[a, b] := aba^{-1}b^{-1}$. For subgroups $H_1$ and $H_2$ of $G$, let $[H_1, H_2]$ denote the smallest subgroup of $G$ generated by $[a, b], a \in H_1, b \in H_2$.

**Definition 3.1** (Nilpotent groups and Filtrations). Let $G$ be a connected and simply connected Lie group with the identity element $e_G$. The *natural filtration* (or the lower central series) $G_{\bullet, \star} := (G_i)_{i \in \mathbb{N}}$ is the sequence of subgroups of $G$ defined by $G_0 := G_1 := G, G_{i+1} := [G, G_i]$ for all $i \in \mathbb{N}_+$. We say that $G$ is nilpotent if there exists $d \in \mathbb{N}_+$ such that $G_{d+1} = \{e_G\}$. The smallest such $d \in \mathbb{N}_+$ is called the *natural step* of $G$.

A *pre-filtration* $G_{\bullet} := (G(i))_{0 \leq i \leq k+1}$ of a nilpotent Lie group $G$ is a sequence of subgroups $G(i)$ of $G$ and some $k \in \mathbb{N}$ such that

$$G = G(0) \supseteq G(1) \supseteq G(2) \supseteq \cdots \supseteq G(k+1) = \{e_G\}$$

and $[G(i), G(j)] \subseteq G(i+j)$ for all $i, j \in \mathbb{N}$, where we denote $G(i) = \{e_G\}$ for all $i \geq d + 1$ for convenience. We say that $G_{\bullet}$ is a filtration if in addition $G(i) \subseteq [G_1, \ldots, G_i, G_{d+1} = \{e_G\}]$ for all $i \in \mathbb{N}$. The smallest $k \in \mathbb{N}$ such that $G^{(k+1)} = \{e_G\}$ is called the *degree* of $G_{\bullet}$. It is easy to see that $k \geq d$.

**Remark 3.2.** Note that what we define as a “pre-filtration” is called a “filtration” in literature. In this paper, we only work with filtrations instead of the more general pre-filtrations, since the Mal’cev basis adapted to a filtration (see Definition 3.5) is compatible with the natural filtration.

**Definition 3.3** (Nilmanifold). Let $G$ be a connected and simply connected nilpotent Lie group and $\Gamma$ be a discrete, cocompact subgroup of $G$. Denote $X = G/\Gamma$, and let $\mathcal{B}$ and $m_X$ be the Borel $\sigma$-algebra and Haar measure of $X$, respectively. The probability space $(X, \mathcal{B}, m_X)$ is called a *nilmanifold*. When there is no confusion, we also say that $(X, m_X)$ or simply $X$ is a nilmanifold.

---

\[14\text{In this paper, we only concern connected and simply connected Lie groups as we will eventually reduce all the results to this special case.}\]
Convention 3.4. For convenience, in this paper, when we say that “\(X = G/\Gamma\) is a nilmanifold”, we implicitly assume that \(G\) is a nilpotent connected and simply connected Lie group, and \(\Gamma\) is a discrete and cocompact subgroup of \(G\).

If \(X = G/\Gamma\) is a nilmanifold, then we use \(m_X\) to denote the Haar measure on \(X\), and \(e_X := e_G\Gamma = \Gamma\) the identity element in \(X\).

Let \(X = G/\Gamma\) be a nilmanifold and \(G'\) be a subgroup of \(G\). We say that \(G'\) is rational for \(\Gamma\) if \(G'\) is connected, simply connected, closed, and \(\Gamma' := G' \cap \Gamma\) is cocompact in \(G'\). We say that a filtration \(G_\bullet := (G^{(i)})_{0 \leq i \leq k+1}\) of \(G\) is rational for \(\Gamma\) if \(G^{(i)}\) is rational for \(\Gamma\) for all \(0 \leq i \leq k + 1\). It was shown in [11] that the natural filtration of \(G\) is rational for \(\Gamma\).

We say that \(X'\) is a sub nilmanifold of \(X = G/\Gamma\) if \(X' = G'/\Gamma' := G'/(G' \cap \Gamma)\) for some \(G' < G\) rational for \(\Gamma\).

Every nilmanifold has an explicit algebraic description by using the Mal’cev basis:

**Definition 3.5** (Mal’cev basis). Let \(X = G/\Gamma\) be a nilmanifold and \(G_\bullet := (G^{(i)})_{0 \leq i \leq k+1}\) be a filtration of \(G\) for some \(k \in \mathbb{N}\). Let \(\dim(G) = m\) and \(\dim(G^{(i)}) = m_i\) for all \(0 \leq i \leq k + 1\). A basis \(X := \{\xi_1, \ldots, \xi_m\}\) for the Lie algebra \(\mathfrak{g}\) of \(G\) (over \(\mathbb{R}\)) is a Mal’cev basis for \(X\) adapted to the filtration \(G_\bullet\) if

- for all \(0 \leq j \leq m - 1\), \(b_j := \text{Span}_{\mathbb{R}}(\xi_{j+1}, \ldots, \xi_m)\) is a Lie algebra ideal of \(\mathfrak{g}\) and so \(H_j := \text{Span}_{\mathbb{R}}(\xi_1, \ldots, \xi_j)\) is a normal Lie subgroup of \(G\);
- \(G^{(i)} = H_{m-m_i}\) for all \(0 \leq i \leq k\);
- the map \(\psi^{-1} : \mathbb{R}^m \to G\) give by
  \[
  \psi^{-1}(t_1, \ldots, t_m) = \exp(t_1\xi_1) \cdots \exp(t_m\xi_m)
  \]
  is a bijection;
- \(\Gamma = \psi^{-1}(\mathbb{Z}^m)\).

We call \(\psi\) the *Mal’cev coordinate map* with respect to the Mal’cev basis \(X\). If \(g = \psi^{-1}(t_1, \ldots, t_m)\), we say that \((t_1, \ldots, t_m)\) are the Mal’cev coordinates of \(g\) with respect to \(X\).

It is known that for every filtration \(G_\bullet\) which is rational for \(\Gamma\), there exists a Mal’cev basis adapted to it. See for example the discussion on pages 11–12 of [26].

Let \(\mathfrak{g}\) be endowed with an Euclidean structure such that the Mal’cev basis \(X\) is an orthogonal basis. This induces a Riemann structure on \(G\) which is invariant under the right translations. We use \(d_G\) to denote the distance on the group \(G\) endowed with the corresponding geodesic distance (which is again invariant under the right translations).

Let \(X = G/\Gamma\) be a nilmanifold and \(p : G \to X\) be the projection. Let \(d_X\) denote the metric on \(X\) given by

\[
  d_X(x, y) := \inf_{g, h \in G} \{d_G(g, h) : p(g) = x, p(h) = y\}.
\]

By the right invariance of \(d_G\), it is not hard to show that \(d_X\) is indeed a metric on \(X\). Note that the infimum in the definition of \(d_X\) can always be obtained since \(\Gamma\) is discrete. We say that \(d_X\) and \(d_G\) are metrics induced by \(G_\bullet\) (or \(X\)).

In order to simplify the notations of all the structures imposed above on a nilmanifold, we introduce the following notation:

\(\exp : \mathfrak{g} \to G\) is the exponential map.
Definition 3.6 (Nil-structure). Let \( X = G / \Gamma \) be a nilmanifold. If \( G_* \) is a filtration of \( X \), \( X \) is a Mal’cev basis adapted to \( G_* \). \( \psi : \mathbb{R}^n \to G \) is the Mal’cev coordinate map with respect to \( X \), and \( d_G, d_X \) are the metrics induced by \( G_* \). Then we say that the tuple \( \bar{x} = (G_*, X, \psi, d_G, d_X) \) is a nil-structure of \( X \). We say that \( X \) is a \( k \)-step nilmanifold with respect to \( \bar{x} \) if the degree of \( G_* \) is \( k \).

We say that \( \bar{x} = (G_*, X, \psi, d_G, d_X) \) is a natural nil-structure of \( X \) if \( G_* = G_{e_*} \) is the natural filtration of \( G \).

We define some special nil-structures which are used in later sections:

Definition 3.7 (Variations of nil-structures). Let \( X = G / \Gamma \) be a nilmanifold with a nil-structure \( \bar{x} = (G_* = (G^{(i)})_{0 \leq i \leq k+1}, X, \psi, d_G, d_X) \) and suppose that \( \dim(G) = m \).

Quotient nilmanifold. Let \( G' \) be a subgroup of \( G \) rational for \( \Gamma \). Let \( \pi : G \to G' \) be the quotient map. Denote \( G_* := G / G' \) and \( \Gamma_* := \Gamma / (G' \cap \Gamma) \). Then \( \bar{x}_\pi := G_* / \Gamma_* \) is a nilmanifold. Then we use \( G_{\pi_*} := (G_*(\pi))_{0 \leq i \leq n} \) to denote the filtration of \( G_* \) given by \( G_*(\pi)_i := G^{(i)} / G' , i \in \mathbb{N} \). We say that any nil-structure of \( X' \) of the form \( \bar{x}' = (G'_*, X'_*, \psi', d_{G'_*}, d_{X'_*}) \) (i.e. the filtration of \( X' \) is \( G'_* \)) is a nil-structure induced by \( \bar{x} \) (or by \( \bar{x}' \)).

Conjugated sub nilmanifold. Let \( \bar{x}' := G' / (G' \cap \Gamma) \) be a sub nilmanifold of \( X \) with a nil-structure \( X' = (G'_*, X'_*, \psi', d_{G'_*}, d_{X'_*}) \) induced by \( \bar{x} \) and suppose that \( \dim(G') = m' \). Let \( \eta \in G \) be rational for \( \Gamma \), meaning that \( \eta^{(i)} \in \Gamma \) for some \( m \in \mathbb{Z} \setminus \{0\} \). Denote \( G'_a := a^{-1}G'a, X'_a := G'_a / (G'_a \cap \Gamma) \) and let \( G_{a_*} := (G^{(i)}_{a_*})_{0 \leq i \leq n} \) be the filtration of \( G'_a \). Let \( G_{a_*} := a^{-1}G^{(i)}a, i \in \mathbb{N} \). We say that any nil-structure of \( X'_a \) of the form \( \bar{x}'_a = (G_{a_*}, X'_a, \psi'_a, d_{G_{a_*}}, d_{X'_a}) \) (i.e. the filtration of \( X'_a \) is \( G_{a_*} \)) is a nil-structure induced by \( \bar{x} \) (or by \( \bar{x}'_a \)).

Product nilmanifold. Let \( X \times X := G \times G / (\Gamma \times \Gamma) \) be the product nilmanifold of \( X \). Then we use \( \bar{x} \times \bar{x} := ((G \times G)_{(i)})_{0 \leq i \leq d+1} \) to denote the filtration of \( G \times G \) given by \( G \times G^{(i)} := G^{(i)} \times G^{(i)} , i \in \mathbb{N} \), \( \psi \times \psi : G \times G \to \mathbb{R}^{2d} \) the Mal’cev coordinate map such that for \( \psi(g) = (x_1, \ldots, x_m) \) and \( \psi(g') = (x'_1, \ldots, x'_m) \), \( \psi \times \psi(g, g') = (x_1, \ldots, x_m, x'_1, \ldots, x'_m) \) and \( \psi \times \psi : G \times G \to \mathbb{R}^{2d} \). Let \( d_{G \times G} := d_G \times d_G \) and \( d_{X \times X} := d_X \times d_X \). We use \( \bar{x} \times \bar{x} \) to denote the nil-structure \((G \times G)_*, \psi \times \psi, d_{G \times G}, d_{X \times X})\).
Lemma 3.8. Let $X = G/\Gamma$ be a $k$-step nilmanifold with a nil-structure $X = (G_\ast, X, \psi, d_G, dx)$ for some $k \in \mathbb{N}_+$. Let $X' = G'/\Gamma'$ be a sub nilmanifold of $X$ and $X' = (G_\ast', X', \psi', d_{G'}, dx')$ be a nil-structure of $X'$ induced by $X$. Then

(i) For every bounded subset $F$ of $G$, there exists $C > 0$ such that for all $g, h, h' \in F$, $d_G(gh, gh') \leq C d_G(h, h')$;
(ii) For every bounded subset $F$ of $G$, there exists $C > 0$ such that for all $x, y \in X$ and $g \in F$, $d_X(g \cdot x, g \cdot y) \leq C d_X(x, y)$;
(iii) For every bounded subset $F$ of $G$, there exists $C_s > 0$ for every $s \in \mathbb{N}$ such that for every $f \in C^s(X)$ and $g \in F$, writing $f_g(x) := f(g \cdot x)$, we have that $\| f_g \|_{C^s(X)} \leq C_s \| f \|_{C^s(X)}$;
(iv) There exists $\delta > 0$ such that for all $1 \leq i \leq k$, $\gamma \in \Gamma$ and $g \in G^{(i)}$, $d_G(\gamma, g) < \delta$ implies that $\gamma \in G^{(i)}$.
(v) There exists $C \geq 1$ such that for all $x, y \in X'$, $C^{-1} d_X(x, y) \leq d_X'(x, y) \leq C d_X(x, y)$.

3.2. Properties on the Lie bracket.

Definition 3.9 (Iterated Lie bracket). Let $G$ be a Lie group. For $d \in \mathbb{N}_+$ and $g_1, \ldots, g_d \in G$, denote
\[
[g_1, \ldots, g_d]_d := [[\ldots [[g_1, g_2], g_3], \ldots], g_d].
\]
When $d = 1$, we denote $\{g_1\}_1 := g_1$. When $d = 2$, we have that $\{g_1, g_2\}_2 = [g_1, g_2].$

We provide a lemma regarding to the map $[\cdot, \ldots, \cdot]_d$ for later uses.

Lemma 3.10. Let $d \in \mathbb{N}_+$ and $G$ be a nilpotent Lie group of natural step $d$ with the natural filtration $(G_i)_{0 \leq i \leq d+1}$

(i) Let $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{N}_+$. For all $g_i \in G_{a_i}, 1 \leq i \leq n$, $\{g_1, \ldots, g_i\}_i \in G_{a_1 + \cdots + a_n}$.
(ii) For all $g_1', g_1, \ldots, g_d \in G$,
\[
\{g_1, \ldots, g_d\}_d - \{g_1', \ldots, g_d\}_d = \{g_1 \cdot g_1', \ldots, g_d\}_d.\]
(iii) For all $g_1, \ldots, g_d \in G$,
\[
\{g_1, \ldots, g_d\}_d^{-1} = \{g_1^{-1}, \ldots, g_d\}_d;\]

Proof: (i) is straightforward by induction. (iii) is a corollary of (ii) by setting $g_1' = g_1^{-1}$.

We now prove (ii). By (i), $\{g_2, \ldots, g_d\}_d-1 \in G_{d-1}$. So it suffices to show that for all $a, b, c \in G$, we have that
\[
\{ab, c\} = [a, [b, c]] \cdot [b, c] \cdot [a, c],
\]
which can be verified by a direct computation. \[\square\]

3.3. Special factors of a nilmanifold. We introduce three special factors of a nilmanifold in this section. The first one is the horizontal torus, which plays an important role in equidistribution properties:

Definition 3.11 (Horizontal torus and characters). Let $X = G/\Gamma$ be a nilmanifold with a natural nil-structure $X = (G_c, X, \psi_c, d_{c,G}, d_{c,X})$ with $d$ being the natural step of $X$. Suppose that $\dim(G) = m$ and $\dim(G_2) = m_2$. Then $\psi_c : G \to \mathbb{R}^m$ induces an isometric identification between the horizontal torus $G/\left\langle G_2 \right\rangle$ (endowed with the quotient metric) and $\mathbb{T}^{m-m_2}$ (endowed with the canonical metric).

A horizontal character is a continuous group homomorphism $\eta : G \to \mathbb{T}$ such that $\eta(\Gamma) = \{0\}$.

\[\text{Property (iv) is stated for the natural filtration in [138], but its proof applies easily to any filtration rational for } \Gamma \text{ (i.e. the filtration } G_i).\]

\[\text{This lemma also holds for any filtration } G_\ast \text{ of } G. \text{ But we do not need it.}\]
Then every horizontal character \( \eta \) vanishes on \( G_2 \) and induces a continuous group homomorphism between \( G/G_2 \) and \( \mathbb{R}^{m-m_2} \).

Let \( X = (G, X, \psi, d_G, d_X) \) be any nil-structure of \( X \). Under the Mal’cev basis \( X \), we can write
\[
\eta \circ \psi^{-1}(x_1, \ldots, x_m) = \ell_1 x_1 + \cdots + \ell_m x_m \mod \mathbb{Z}
\]
for some \( \ell_1, \ldots, \ell_m \in \mathbb{Z} \) (called the coordinates of \( \eta \) with respect to \( X \)) for all \((x_1, \ldots, x_m) \in \mathbb{R}^m\) in a unique way. Denote the \( \mathcal{X} \)-norm of \( \eta \) by
\[
||\eta||_X := |\ell_1| + \cdots + |\ell_{m-m_2}|.
\]

The second special factor is a sub torus of the horizontal torus \( G/G_2 \Gamma \) which we call the faithful horizontal torus. This concept is uncommon from literature, but is essential in understanding certain sub nilmanifolds of the product space \( \mathbb{R} \times X \).

**Definition 3.12** (Faithful horizontal torus). Let \( X = G/\Gamma \) be a nilmanifold of natural step \( d \in \mathbb{N}_+ \) with the natural filtration \( G = (G_i)_{0 \leq i \leq d+1} \). Let \( G_{\text{ker}} \) be the collection of all \( g \in G \) such that for all \( g_1, \ldots, g_d \in G, [g_1, g_2, \ldots, g_d] = e_G \). By Lemma 3.10 it is easy to see that \( G_{\text{ker}} \) is a subgroup of \( G \) and contains \( G_2 \). We say that \( G/(G_{\text{ker}} \Gamma) \) is the faithful horizontal torus of \( X \) (endowed with the quotient metric).

The faithful horizontal torus \( G/G_{\text{ker}} \Gamma \) is a sub torus of the horizontal torus \( G/G_2 \Gamma \), but the converse may not be true.

**Example 3.13.** Let \( H = \mathbb{R}^3 \) be endowed with a group structure given by
\[
(x, y; z) \cdot (x', y'; z') := (x + x', y + y'; z + z' + xy')
\]
for all \((x, y; z), (x', y'; z') \in \mathbb{R}^3\). It is easy to see that \((H, \cdot)\) is a group, and \( H_2 = \{0\} \times \{0\} \times \mathbb{R} \), \( H_3 = \{(0, 0, 0)\} \). This group is called the Heisenberg group.

Let \( G = \mathbb{R} \times H, \Gamma = \mathbb{Z}^2 \) and \( X = G/\Gamma \). Then \( G_2 = \{0\} \times \{0\} \times \mathbb{R} \) and the horizontal torus \( G/G_2 \Gamma \) is \( \mathbb{T}^3 \times \{0\} \). On the other hand, \( G_{\text{ker}} = \mathbb{R} \times \{0\} \times \{0\} \times \mathbb{R} \), and so the faithful horizontal torus \( G/G_{\text{ker}} \Gamma \) is \( \{0\} \times \mathbb{T}^2 \times \{0\} \).

We postpone further properties of the faithful horizontal torus to Section 5. Given a filtration \( G_\bullet \) of a nilmanifold \( X = G/\Gamma \) of natural step \( d \), it is convenient for us to work on a Mal’cev adapted to \( G_\bullet \) where the subgroups \( G_{\text{ker}} \) and \( G_d \) of \( G \) can be expressed in a nice way.

**Definition 3.14** (Standard Mal’cev basis). Let \( X = G/\Gamma \) be a nilmanifold of natural step \( d \) for some \( d \in \mathbb{N}_+ \) with \( G = (G_i)_{0 \leq i \leq d+1} \) being its natural filtration. Let \( G_\bullet = (G^{(i)})_{0 \leq i \leq k+1} \) be another filtration of \( G \) for some \( k \in \mathbb{N}_+ \). Suppose that \( \dim(G) = m, \dim(G_{\text{ker}}) = m'_2 \) and \( \dim(G_d) = r \). Let \( X = (\xi_1, \ldots, \xi_m) \) be a Mal’cev basis for \( X \) adapted to the filtration \( G_\bullet \) with \( \psi: G \to \mathbb{R}^m \) the Mal’cev coordinate map. We say that \( X \) is standard if \( G_{\text{ker}} = \psi^{-1}((0)^{m-m'_2} \times \mathbb{R}^{m'_2}) \) and \( G_d = \psi^{-1}((0)^{m-r} \times \mathbb{R}^r) \).

We say that a nil-structure \( X = (G, X, \psi, d_G, d_X) \) is standard if \( X \) is standard.

It is easy to see that every filtration \( G_\bullet \) admits one (but not necessarily unique) standard Mal’cev basis, as \( G^{(i)} = [G_1, \ldots, G_d, G_{d+1} = \{e_G\}] \) for all \( i \in \mathbb{N} \).

The last special factor is the vertical torus, a concept which allows us to conduct Fourier analysis on nilmanifolds.

**Definition 3.15** (Vertical torus and nilcharacters (or vertical characters)). Let \( X = G/\Gamma \) be a nilmanifold of natural step \( d \) for some \( d \in \mathbb{N}_+ \) with \( G = (G_i)_{0 \leq i \leq d+1} \) being its natural filtration.
Suppose that \( \dim(G_d) = r \). Then \( G_d \) lies in the center of \( G \). We call \( G_d/(G_d \cap \Gamma) \) the \textit{vertical torus} on \( X \). For a standard nil-structure \( \Phi = (G_\Phi, \chi, \psi, d_G, d_\chi) \) of \( X \), we say that \( \Phi : X \to \mathbb{C} \) is a \textit{nilcharacter} (or \textit{vertical character}) with frequency \((h_1, \ldots, h_r) \in \mathbb{Z}^r\) with respect to \( \Phi \) if
\[
\Phi(g \cdot x) = e(h_1 \gamma_1 + \cdots + h_r \gamma_r) \Phi(x)
\]
for all \( g = \psi^{-1}(0, \ldots, 0; y_1, \ldots, y_r) \in G_d \) and \( x \in X \).

The following are some basic properties of nilcharacters, which will be used in later sections:

**Lemma 3.16** (Translation invariance of nilcharacters). Let \( X = G/\Gamma \) be a nilmanifold of natural step \( d \) for some \( d \in \mathbb{N}_+ \) with \( G_{c,\bullet} = (G_i)_{0 \leq i \leq d+1} \) being its natural filtration. Let \( \Phi = (G_\Phi, \chi, \psi, d_G, d_\chi) \) be a standard nil-structure of \( X \) and \( \Phi \) be a nilcharacter of \( X \) with respect to \( \Phi \). For \( g_0 \in G \), let \( \Phi_{g_0}(x) := \Phi(g_0 \cdot x) \) for all \( x \in X \). Then \( \Phi_{g_0} \) is also a nilcharacter of \( X \) with the same frequency as \( \Phi \) with respect to \( \Phi \).

**Proof.** Suppose that \( \dim(G_d) = r \) and \( \Phi \) is with frequency \((h_1, \ldots, h_r) \in \mathbb{Z}^r\) with respect to \( \Phi \). Since \( \chi \) is standard,
\[
\Phi(g \cdot x) = e(h_1 \gamma_1 + \cdots + h_r \gamma_r) \Phi(x)
\]
for all \( g = \psi^{-1}(0, \ldots, 0; y_1, \ldots, y_r) \in G_d \) and \( x \in X \). Since \( g \in G_d \) is in the center of \( G \),
\[
\Phi_{g_0}(g \cdot x) = \Phi(g_0g \cdot x) = (h_1 \gamma_1 + \cdots + h_r \gamma_r) \Phi_{g_0}(x) = (h_1 \gamma_1 + \cdots + h_r \gamma_r) \Phi_{g_0}(x).
\]
This implies that \( \Phi_{g_0} \) is also a nilcharacter of \( X \) with frequency \((h_1, \ldots, h_r) \) with respect to \( \Phi \).

---

**Lemma 3.17** (Nilcharacters on \( X \times X \)). Let \( X = G/\Gamma \) be a nilmanifold of natural step \( d \) for some \( d \in \mathbb{N}_+ \) with \( G_{c,\bullet} = (G_i)_{0 \leq i \leq d+1} \) being its natural filtration. Suppose that \( \dim(G_d) = 1 \). Let \( \Phi = (G_\Phi, \chi, \psi, d_G, d_\chi) \) be a standard nil-structure of \( X \) and \( \Phi \) be a nilcharacter of \( X \) with frequency \( \ell \) with respect to \( \Phi \) for some \( \ell \in \mathbb{Z} \setminus \{0\} \). Let \( H \) be a subgroup of \( G \times G \) rational for \( \Gamma \times \Gamma \) and \( Y := H/(H \cap (\Gamma \times \Gamma)) \) be a sub nilmanifold of \( X \times X \) with a standard nil-structure \( \mathfrak{Y} \) induced by \( \chi \times \chi \). Then

(i) \( \Phi \otimes \mathfrak{Y} \) is a nilcharacter of \( X \times X \) with frequency \((\ell, -\ell)\) with respect to \( \chi \times \chi \).

(ii) If \( \dim(H_d) = 2 \) (i.e. \( H_d = G_d \times G_d \)), then \( \Phi \otimes \mathfrak{Y}_{|Y} \) is a nilcharacter of \( Y \) with frequency \((\ell, -\ell)\) with respect to \( \mathfrak{Y} \).

(iii) If \( \dim(H_d) = 1 \), and suppose that
\[
H_d = \{(\psi^{-1}(0, \ldots, 0; \ell_1 t), \psi^{-1}(0, \ldots, 0; \ell_2 t)) \in G_d \times G_d : t \in \mathbb{R} \}
\]
for some \( \ell_1, \ell_2 \in \mathbb{Z} \) not all equal to 0, then \( \Phi \otimes \mathfrak{Y}_{|Y} \) is a nilcharacter of \( Y \) with respect to \( \mathfrak{Y} \). Moreover, its frequency is non-zero if and only if \( \ell_1 = \ell_2 \).

**Proof.** By assumption, \( \Phi(g \cdot x) = e(\ell y) \Phi(x) \) for all \( g = \psi^{-1}(0, \ldots, 0; y) \in G_d \) and \( x \in X \).

(i) For all \( g = \psi^{-1}(0, \ldots, 0; y), g' = \psi^{-1}(0, \ldots, 0; y') \in G_d \) and \((x, x') \in X \times X\),
\[
(12) \quad \Phi \otimes \mathfrak{Y}((g, g') \cdot (x, x')) = \Phi(gx) \mathfrak{Y}(g'x') = e(\ell y - \ell y')\Phi(x) \mathfrak{Y}(x') = e((\ell, -\ell) \cdot (y, y')) \Phi \otimes \mathfrak{Y}(x, x').
\]
So \( \Phi \otimes \mathfrak{Y} \) is a nilcharacter of \( X \times X \) with frequency \((\ell, -\ell)\) with respect to \( \chi \times \chi \).

(ii) If \( H_d = G_d \times G_d \), then \( (12) \) holds for all \((g, g') \in H_d \). So \( \Phi \otimes \mathfrak{Y}_{|Y} \) is a nilcharacter of \( Y \) with frequency \((\ell, -\ell)\) with respect to \( \mathfrak{Y} \).

\[g \in G_d \text{ because } \chi \text{ is standard.}\]
(iii) Let \( h = (\psi^{-1}(0, \ldots, 0; \ell_1 t), (\psi^{-1}(0, \ldots, 0; \ell_2 t)) \in H_d \) for some \( t \in \mathbb{R} \). Then for all \((x, x') \in X \times X\), by (12),

\[
\Phi \otimes \overline{\Phi}(h \cdot (x, x')) = e(\ell(\ell_1 - \ell_2) t) \Phi \otimes \overline{\Phi}(x, x').
\]

So \( \Phi \otimes \overline{\Phi}_{\gamma} \) is a nilcharacter of \( Y \) with respect to \( \gamma \), and its frequency is zero if and only if \( \ell_1 - \ell_2 = 0 \) (since \( \ell \neq 0 \)). \( \square \)

4. Equidistribution properties for polynomial sequences on nilmanifolds

In this section, we collect all the equidistribution results we need in this paper.

4.1. Polynomial sequences and smooth norms. We start with the definition of polynomial sequences.

**Definition 4.1** (Polynomial sequences). Let \( G \) be a group endowed with a pre-filtration \( G_\bullet = (G^{(i)})_{0 \leq i \leq k+1} \) for some \( k \in \mathbb{N} \). Let \( D \in \mathbb{N}_+ \) and \( g: \mathbb{Z}^D \to G \) be a map. For \( h \in \mathbb{Z}^D \), define \( \partial_h g: \mathbb{Z}^D \to G \) by \( \partial_h g(n) := g(n + h)^{-1} g(n) \) for all \( n \in \mathbb{Z}^D \). Let \( \text{poly}_D(G_\bullet) \) denote the collection of all \( g: \mathbb{Z}^D \to G \) such that for all \( i \in \mathbb{N} \), and \( n, h_1, \ldots, h_i \in \mathbb{Z}^D \), we have that \( \partial_{h_1} \ldots \partial_{h_i} g(n) \in G^{(i)} \).

We call functions in \( \text{poly}_D(G_\bullet) \) polynomial sequences with respect to \( G_\bullet \).

We say that \( g: \mathbb{Z}^D \to G \) is a polynomial sequence on \( G \) (written as \( g \in \text{poly}_D(G) \)) without specifying the pre-filtration if \( g \in \text{poly}_D(G_\bullet) \) for some pre-filtration \( G_\bullet \) of \( G \). The degree of \( g \) is the smallest degree of all the filtrations \( G_\bullet \) of \( G \) such that \( g \in \text{poly}_D(G_\bullet) \).

**Remark 4.2.** Clearly, if \( g \in \text{poly}_D(G_\bullet) \) for some pre-filtration \( G_\bullet \), then \( g \in \text{poly}_D(G_\bullet') \) for some filtration \( G_\bullet' \). So the definition of polynomial sequences in this paper coincides with the one used in [18] and [26].

Note that there is an implicitly upper bound for the “degree” of every polynomial sequence in \( \text{poly}_D(G_\bullet) \), namely the degree of the pre-filtration \( G_\bullet \).

**Remark 4.3.** As we shall see later in this paper, in many theorems, we endow two filtrations (and two nil-structures adapted to them) on a nilmanifold simultaneously: a natural filtration \( G_\bullet \) through which the horizontal, faithful horizontal and vertical toruses are defined, and a filtration \( G_\bullet' \) through which the polynomial sequence is defined.

For \( D \in \mathbb{N}_+ \), \( n = (n_1, \ldots, n_D) \in \mathbb{Z}^D \), and \( j = (j_1, \ldots, j_D) \in \mathbb{N}^D \), recall that \( |j| := j_1 + \cdots + j_D \).

Denote \( n^j := n_1^j \cdots n_D^j \) and

\[
\binom{n}{j} := \prod_{i=1}^D \binom{n_i}{j_i}
\]

The following description of polynomial sequences is Lemma 6.7 of [26] (or Section 4 of [34]):

**Lemma 4.4** (Polynomials in Mal’cev basis). Let \( X = G/\Gamma \) be a nilmanifold with a nil-structure \( \mathcal{X} = (G_\bullet = (G^{(i)})_{0 \leq i \leq k+1}, X, \psi, d_G, d_X) \). Suppose that \( \dim(G) = m \) and \( \dim(G^{(i)}) = m_i \) for all \( 0 \leq i \leq k + 1 \). Then \( g \in \text{poly}_D(G_\bullet) \) if and only if

\[
\psi \circ g(n) = \sum_{j \in \mathbb{N}^D} \alpha_j(n^j)
\]

for some \( \alpha_j = (\alpha_{j,1}, \ldots, \alpha_{j,m}) \in \mathbb{R}^m \) for all \( j \in \mathbb{N}^D \) such that \( \alpha_{j,i} = 0 \) for all \( i \leq m - mj \).
Let $D, k, r \in \mathbb{N}_+$ and $g \in \text{poly}_D(\mathbb{R}^*_*)$, where the filtration $\mathbb{R}^*_* := (\mathbb{R}^*)^{(i)}_{0 \leq i \leq k}$ of $\mathbb{R}$ is given by $(\mathbb{R}^*)^{(0)} := \mathbb{R}$ for all $0 \leq i \leq k$ and $(\mathbb{R}^*)^{(k+1)} := \{0\}$. By Lemma 4.4, $g$ can be expressed alternatively in two different ways (in each way there is a unique expression):

$$g(n) = \sum_{j \in \mathbb{N}^D} \alpha_j n^j$$

for some $\alpha_j, \alpha'_j \in \mathbb{R}$ for all $j \in \mathbb{N}^D$ such that $\alpha_j = \alpha'_j = 0$ whenever $|j| > k$.

**Definition 4.5** (Smooth norms). Let the notations be as above. For all $N \in \mathbb{N}$, we define the smooth norms of $g \in \text{poly}_D(\mathbb{R}^*_*)$ as

$$\|g\|_{C^N(R_N,D)} := \max_{j \neq 0} (2N+1)\|\alpha_j\|_{\mathbb{Z}^r}$$

and

$$\|g\|_{C^N(R_N,D)}' := \max_{j \neq 0} (2N+1)\|\alpha'_j\|_{\mathbb{Z}^r}.$$

It is easy to check that there exists $C := C(k,D) > 1$ such that

$$C^{-1}\|g\|_{C^N(R_N,D)} \leq \|g\|_{C^N(R_N,D)}' \leq C\|g\|_{C^N(R_N,D)}$$

for all $r \in \mathbb{N}_+$ and $g \in \text{poly}_D(\mathbb{R}^*_*)$. So we can use both norms alternatively without affecting our proofs. Roughly speaking, it was shown in [18, 26] that the smallness of the smooth norms of $g$ indicates that $g$ is a slow-varying function.

Obviously, the smallness of the $\| \cdot \|_{\mathbb{Z}^r}$-norms of the coefficients $\alpha_j$ (or $\alpha'_j$) implies the smallness of the smooth norm of $g$. Conversely, we have the following lemma:

**Lemma 4.6.** Let $D, m \in \mathbb{N}_+$ and $g: \mathbb{Z}^D \to \mathbb{R}$ be a homogeneous polynomial of the form

$$g(n) = \sum_{|j|=m} a'_j n^j$$

for some $a'_j \in \mathbb{R}$ for all $n \in \mathbb{Z}^D$. There exist $C := C(D,m) > 0$ and $Q := Q(D,m) \in \mathbb{N}_+$ such that if $\|g(n)\|_{\mathbb{Z}} \leq C_0$ for all $n \in \mathbb{Z}^D, |n| \leq m$, then $\|Qa'_j\|_{\mathbb{Z}} \leq C_0 Q$ for all $|j| = m$.

**Proof.** Recall that $\partial_{n^j} g(n) := g(n+m) - g(n)$. Let $j = (j_1, \ldots, j_D) \in \mathbb{N}^D$ be any vector with $|j| = m$. Then it is easy to check that

$$\partial_{n^1}^{j_1} \cdots \partial_{n^D}^{j_D} g(0) = (j_1! \cdots j_D!) a'_j.$$

Since $\|g(n)\|_{\mathbb{Z}} \leq C_0$ for all $|n| \leq m$,

$$\|(j_1! \cdots j_D!) a'_j\|_{\mathbb{Z}} \leq 2^D C_0.$$

Let $Q = (m!)^D$, which divides $j_1! \cdots j_D!$. We have that

$$\|Qa'_j\|_{\mathbb{Z}} \leq 2^D C_0 Q / (j_1! \cdots j_D!) \leq 2^D C_0 Q$$

for all $|j| = m$. This finishes the proof by setting $C = 2^D Q$. \hfill \Box

### 4.2. Smooth norms on the faithful horizontal torus

Let $X = G/\Gamma$ be a nilmanifold with a standard nil-structure $X = (G_*, X, \psi, d_G, d_X)$. Suppose that $\dim(G) = m$, $\dim(G_{\ker}) = m'$ and let $s' = m - m'$. For convenience, we use the same notation $\pi_{\ker}$ to denote the following two different maps, the meaning of which will always be clear from the context: (i) $\pi_{\ker}: \mathbb{R}^m \to \mathbb{R}^{s'}$, the projection from $\mathbb{R}^m$ to its first $s'$ coordinates; (ii) $\pi_{\ker}: G \to G/G_{\ker}$, the quotient map of $G$ by $G_{\ker}$.
Clearly, the Mal’cev coordinate map \( \psi \) induces an isometric identification \( \psi_{ker} : G/G_{ker} \rightarrow \mathbb{R}^{s'} \) between \( G/G_{ker} \) and \( \mathbb{R}^{s'} \) such that \( \psi_{ker} \circ \pi_{ker} = \pi_{ker} \circ \psi : G \rightarrow \mathbb{R}^{s'} \). \( \psi_{ker} \) also induces an isometric identification between the faithful horizontal torus \( G/(G_{ker} \Gamma) \) and \( \mathbb{T}^{s'} \) (endowed with the canonical metric). We define the smooth norm on the faithful horizontal torus as follows, which will be used in later sections.

**Definition 4.7** (Smooth norm on the faithful horizontal torus). Let \( X = G/\Gamma \) be a nilmanifold with a standard nil-structure \( \mathcal{X} = (G_\ast, X, \psi, d_\ast, d_\mathcal{X}) \). Suppose that \( \dim(G) = m, \dim(G_{ker}) = m_2' \) and let \( s' = m - m_2' \). Let \( D, N \in \mathbb{N}_+ \) and \( g \in \text{poly}_D(G_\ast) \). Then \( \pi_{ker} \circ \psi \circ g : \mathbb{Z}^D \rightarrow \mathbb{R}^{s'} \) can be written as

\[
\pi_{ker} \circ \psi \circ g(n) = \sum_{j \in \mathbb{N}^D, \|j\| \leq k} \alpha_j n^j
\]

for some \( d \in \mathbb{N}, \alpha_j, \alpha'_j \in \mathbb{R}^s \) for all \( j \in \mathbb{N}^D, \|j\| \leq k \). We define the smooth norm of \( g \) on the faithful horizontal torus by

\[
\|g\|_{C_{ker,X}^s(R_{N,D})} := \|\pi_{ker} \circ \psi \circ g\|_{C^s(R_{N,D})} \quad \text{and} \quad \|g\|'_{C_{ker,X}^s(R_{N,D})} := \|\pi_{ker} \circ \psi \circ g\|'_{C^s(R_{N,D})},
\]

where \( \| \cdot \|_{C^s(R_{N,D})} \) and \( \| \cdot \|'_{C^s(R_{N,D})} \) are the norms defined in Definition 4.5.

**4.3. Leibman’s Theorem and total equidistribution.** By the quantitative nature of the results in this paper, we need to use the concept of total \( \epsilon \)-equidistribution first introduced in [26], which can be viewed as a quantitative version of [2].

**Definition 4.8** (Total \( \epsilon \)-equidistribution). Let \( (X = G/\Gamma, m_\mathcal{X}) \) be a nilmanifold with a nil-structure \( \mathcal{X} \). Let \( D, N \in \mathbb{N}_+, \epsilon > 0 \) and \( g : \mathbb{Z}^D \rightarrow G \). We say that the sequence \( (g(n) \cdot e_X)_{n \in R_{N,D}} \) is **totally \( \epsilon \)-equidistributed** on \( X \) with respect to \( \mathcal{X} \) if for every \( D \)-dimensional arithmetic progression \( P \), every function \( f \) on \( X \) with \( \|f\|_{\text{Lip}(X)} \leq 1 \) and \( \int_X f \, dm_\mathcal{X} = 0 \), we have that

\[
\left| \mathbb{E}_{n \in R_{N,D}} 1_P(n) f(g(n) \cdot e_X) \right| \leq \epsilon.
\]

The next result is a variation of Theorem 8.6 in [26], which provides a convenient criteria for establishing equidistribution properties of polynomial sequences on nilmanifolds (see also Theorem 7.3 of [43]):

**Theorem 4.9** (A variation of the quantitative Leibman’s Theorem). Let \( \epsilon > 0, D \in \mathbb{N}_+ \) and \( X = G/\Gamma \) be a nilmanifold with a nil-structure \( \mathcal{X} = (G_\ast, X, \psi, d_\ast, d_\mathcal{X}) \). There exists \( C := C(\mathcal{X}, \epsilon, D) > 0 \) such that for every \( N \in \mathbb{N} \) and polynomial sequence \( g \in \text{poly}_D(G_\ast) \), if \( (g(n) \cdot e_X)_{n \in R_{N,D}} \) is not totally \( \epsilon \)-equidistributed on \( X \) with respect to \( \mathcal{X} \), then there exists a horizontal character \( \eta \) such that

\[
0 < \|\eta\|_X \leq C \quad \text{and} \quad \|\eta \circ g\|_{C^s(R_{N,D})} \leq C.
\]

This theorem is stated in [23] and [26] under the stronger hypothesis that the sequence is not “\( \epsilon \)-equidistributed on \( X \)”, meaning that (13) fails for \( P = R_{N,D} \). The stronger result Theorem 4.9 can be obtained by using Theorem 5.2 of [18] combined with a similar argument in Lemma 3.1 in [23]. We omit the proof.

The following is a partial converse of the above result (see also Lemma 5.3 of [18] and Theorem 7.5 of [43]):

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23If a quantity depends on \( X \) (such as \( C \) ), then it also implicitly depends on the nilmanifold \( X \).
Theorem 4.10 (Inverse Leibman’s Theorem). Let $D \in \mathbb{N}_+$, $C_0 > 0$ and $X = G/\Gamma$ be a nilmanifold with a nil-structure $\bar{x} = (G_\ast, X, \psi, d_G, d_X)$. There exist $C := C(\bar{x}, D), N_0 := N_0(\bar{x}, C_0, D) > 0$ such that for every $C_0 > 0$, every $N \geq N_0$, and every polynomial sequence $g \in \text{poly}_D(G_\ast)$, if there exists a non-trivial horizontal character $\eta$ of $X$ with $\|\eta\|_X \leq C_0$ and $\|\eta \circ g\|_{C^0(\mathbb{R}_0, D)} \leq C_0$, then the sequence $(g(n) \cdot e_X)_{n \in \mathbb{R}_0, D}$ is not totally $C_0(\mathbb{R}_0, D)$-equidistributed on $X$ with respect to $\bar{x}$.

Proof. Since $\|\eta \circ g\|_{C^0(\mathbb{R}_0, D)} \leq C_0$, we have that

$$\eta \circ g(n) = \sum_{j \in \mathbb{Z}^D} a_j(n),$$

where $\|a_j\|_T \leq \frac{C_0}{(2N+1)^D}$ for all $0 < |j| \leq k$ with $k$ being the degree of $G_\ast$ which depends only on $\bar{x}$. Thus $|e(\eta \circ g(n)) - e(\eta \circ g(0))| \leq 1/2$ for all $n \in R_{C_1, N}$ for some $C_1 := C_1(k, D) > 0$. Then for all $N \in \mathbb{N}$,

$$\left| E_{n \in R_{C_1, N}} e(\eta \circ g(n)) \right| \geq \frac{1}{2},$$

which implies that

$$E_{n \in R_{C_1, N}}(n) e(\eta \circ g(n)) \geq \frac{C_1^D}{2C_0^D} = \frac{C_2}{C_0(N)}$$

for some $C_2 := C_2(\bar{x}, D) > 0$. So if $N > 4C_2C_0/C_1^D$, then the left hand side of (14) is at least $\frac{C_2}{4C_0}$.

Since $\|\eta\|_X \leq C_0$, the function $x \to e(\eta(x))$ defined on $X$ is Lipschitz with respect to $\bar{x}$ with Lipschitz constant at most $C_0C_3$ for some $C_3 := C_3(\bar{x}, D) > 0$, and has integral 0 since $\eta$ is non-trivial. Therefore, the sequence $(g(n) \cdot e_X)_{n \in \mathbb{R}_0, D}$ is not totally $C_4C_0^{D+1}$-equidistributed with $C_4 := C_1^D/4C_3$ for all $N > 4C_2C_0/C_1^D$.

We also need the following alternative description of total equidistribution:

Proposition 4.11 (Total equidistribution on general subsets). Let $X = G/\Gamma$ be a nilmanifold with a nil-structure $\bar{x} = (G_\ast, X, \psi, d_G, d_X)$. Let $D \in \mathbb{N}_+$, $\epsilon > 0$. There exist $\delta := \delta(\bar{x}, \epsilon) > 0$ and $N_0 := N_0(\bar{x}, \epsilon) > 0$ such that for every $g \in \text{poly}_D(G_\ast)$, if there exist $N \in \mathbb{N}, N \geq N_0$, a set $P \subseteq \mathbb{R}_0, D$ such that for any line $\ell \subseteq \mathbb{R}_0, D$, and every $\Phi: X \to \mathbb{C}$ with $\int_X \Phi d\mu = 0, \|\Phi\|_{L^1(\mathbb{R}_0, D)} \leq 1$ such that

$$\left| E_{n \in \mathbb{R}_0, D} 1_P(n) \Phi(g(n) \cdot e_X) \right| > \epsilon,$$

then $(g(n) \cdot e_X)_{n \in \mathbb{R}_0, D}$ is not totally $\delta$-equidistributed on $X$ with respect to $\bar{x}$.

To prove this proposition, we need the following technical lemma, whose proof is the argument on pages 6–9 of [26].

Lemma 4.12. Let $D, N \in \mathbb{N}_+$ and $\epsilon > 0$. Let $X = G/\Gamma$ be a nilmanifold with a nil-structure $\bar{x} = (G_\ast, X, \psi, d_G, d_X)$ and $g \in \text{poly}_D(G_\ast)$ be a polynomial sequence. Let $N, L > 0$ be such that $N > L^2$ and $L > C/\epsilon$ for some $C$ sufficiently large depending only on $\bar{x}, D$ and $\epsilon$. Suppose that for all $v \in [L]^D$, there exist $J_v \subseteq \mathbb{R}_0, D$ with $|J_v| > \frac{1}{2} e^{N^D}$ such that $(g(n + nv) \cdot e_X)_{n \in \mathbb{R}_0, D}$
Remark 4.14. Let \( C := C(\mathfrak{x}, e) > 0 \) be sufficiently large to be chosen latter. Let \( N > (Ce^{-1})^2 \) and pick \( C e^{-1} < L < N^2 \). Since \( \|\Phi\|_{L^p(\mathfrak{x})} \leq 1 \), for all \( v \in [L]^D \), we have that
\[
\mathbb{E}_{n \in [L]^D} 1_{P(n)} \Phi(g(n) \cdot e \mathfrak{x}) = \mathbb{E}_{n \in [L]^D} \mathbb{E}_{-N/L^2 \leq n \leq N/L^2} 1_{P(m + nv)} \Phi(g(m + nv) \cdot e \mathfrak{x}) + O(\frac{1}{L}).
\]

So if \( Ce^{-1} > 4 \), then \((16)\) and \((16)\) imply that there exists a set \( J_v \subseteq R_{N,D} \) with \( |J_v| > \epsilon(2N + 1)^D / 4 \) such that for all \( m \in J_v \),
\[
\left| \mathbb{E}_{-N/L^2 \leq n \leq N/L^2} 1_{P(m + nv)} \Phi(g(m + nv) \cdot e \mathfrak{x}) \right| > \epsilon / 2.
\]

By assumption, the set \( \{ n \in \mathbb{Z} : m + nv \in P \} \) is a 1-dimensional arithmetic progression. So \((17)\) implies that the sequence
\[
(g_{m,v}(n) \cdot e \mathfrak{x})_{n \in [-N/L^2 \leq n \leq N/L^2]} := (g(m + nv) \cdot e \mathfrak{x})_{n \in [-N/L^2 \leq n \leq N/L^2]}
\]
is not totally \( \epsilon / 2 \)-equidistributed on \( X \) with respect to \( \mathfrak{x} \) for all \( v \in [L]^D \) and \( m \in J_v \). We may then use Lemma 4.12 to conclude that there is a nilmanifold \( G / \Gamma \) such that
\[
0 < ||\eta||_{\mathfrak{x}} \leq W \text{ and } ||\eta \circ g||_{C^0(\mathfrak{x}, \mathcal{D})} \leq W.
\]
By Theorem 4.10, \((g(n) \cdot e \mathfrak{x})_{m,v} \) is not totally \( \delta \)-equidistributed on \( X \) for some \( \delta := \delta(\mathfrak{x}, e) > 0 \) and \( N \) sufficiently large depending only on \( \mathfrak{x}, e \).

Let \( X = G / \Gamma \) be a nilmanifold. Recall that \( a \in G \) is rational for \( \Gamma \) if \( a^m \in \Gamma \) for some \( m \in \mathbb{N}_+ \). The following is an application of Theorems 4.9 and 4.10 which is used in later sections.

Corollary 4.13 (Changing the base point). Let \( X = G / \Gamma \) be a nilmanifold with a nil-structure \( \mathfrak{x} = (G_\bullet, \mathfrak{x}, \psi, d_G, d_X) \). Let \( G' \) be a subgroup of \( G \) rational for \( \Gamma \) and \( X' = G' / (G' \cap \Gamma) \) be a nilmanifold with a nil-structure \( \mathfrak{x}' = (G'_\bullet, \mathfrak{x}', \psi', d_{G'}, d_{X'}) \) induced by \( \mathfrak{x} \). Let \( a \in G \) be rational for \( \Gamma \) and denote \( G'_a := a^{-1}G'a. \) Let \( X'_\mathfrak{x} := G'_\mathfrak{x} / (G'_\mathfrak{x} \cap \Gamma) \) be a nil-structure \( \mathfrak{x}' = (G'_\bullet, \mathfrak{x}', \psi', d_{G'}, d_{X'}) \) induced by \( a \)-conjugate from \( \mathfrak{x}' \). Let \( D \in \mathbb{N}_+ \). Then there exist a function \( \rho := \rho_{X', \mathfrak{x}', \psi', \mathfrak{x}'_D} : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( \lim_{m \to 0^+} \rho(t) = 0 \) and \( N_0 := N_0(\mathfrak{x}, \mathfrak{x}', \mathfrak{x}'_D) \in \mathbb{N} \) such that for all \( g \in \text{poly}_D(G'_\bullet) \) and \( N \geq N_0 \), if \( (g(n) \cdot e \mathfrak{x})_{m,v} \) is totally \( t \)-equidistributed on \( X' \) with respect to \( \mathfrak{x}' \), then \( (a^{-1}g(n)a) \cdot e \mathfrak{x}' \) is totally \( \rho(t) \)-equidistributed on \( X'_\mathfrak{x} \) with respect to \( \mathfrak{x}'_D \).

Remark 4.14. Let the notations be as in Corollary 4.13. It was proved in Lemma B.4 of [13] that \( G'_a \) is a subgroup of \( G \) rational for \( \Gamma \). So \( G'_\bullet \cap \Gamma \) is cocompact in \( G'_a \) and \( X'_\mathfrak{x} = G'_a \cdot e \mathfrak{x} \). Moreover, writing \( g_a(n) := a^{-1}g(n)a \) and \( (G'_\bullet)_a := a^{-1}G'_a \cdot e \mathfrak{x} \), we have that \( g_a \in \text{poly}_D(G'_\bullet)_a \).

Corollary 4.13 was proved in Corollary 5.5 of [13] for the case \( D = 1 \), but the general case can be proved by a similar method (by using Theorems 4.9 and 4.10, higher dimensional versions of Leibman’s theorems), and so we omit the proofs.

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\(^{24}\) The conclusion of Lemma 4.12 is exactly the same as that of Lemma 3.1 in [23].
4.4. Factorization theorem.

**Definition 4.15** (Smooth and rational sequences). Let $D, N \in \mathbb{N}_+$, $M \in \mathbb{N}$ and $X = G/\Gamma$ be a nilmanifold with a nil-structure $\hat{x} = (G, X, \psi, dG, dx)$. Suppose that $\dim(G) = m$.

- A sequence $\epsilon \in \text{poly}_D(G_* \bullet)$ is $(M, N)$-*smooth* with respect to $\hat{x}$ if $\|\psi \circ \epsilon\|_{C^0_m(R_{N,D})} \leq M$, $dG(e(n), e_G) \leq M$, and $dG(e(n), e(n + \epsilon)) \leq M/N$ for all $n \in R_{N,D}$ and $1 \leq i \leq D$.
- $g \in G$ is $M$-*rational* for $\Gamma$ if $g^m \in \Gamma$ for some $m \in \mathbb{N}$, $1 \leq m \leq M$. A sequence $\gamma : R_{N,D} \to G$ is $M$-*rational* for $\Gamma$ if $\gamma(n)$ is $M$-rational for $\Gamma$ for all $n \in R_{N,D}$.

The following result says that every polynomial sequence is concentrated near a finite collection of sub nilmanifolds:

**Theorem 4.16** (Factorization theorem). Let $D, M \in \mathbb{N}_+$ and $X = G/\Gamma$ be a nilmanifold with a nil-structure $\hat{x} = (G_*, X, \psi, dG, dx)$. There exists a finite family $\mathcal{F}(M) \subseteq \mathcal{F}_{X,d}(M)$ of subnilmanifolds of $X$, which increases with $M$, each of the form $X' = G'/\Gamma'$ for some subgroup $G'$ of $G$ rational for $\Gamma$ and $\Gamma' := G' \cap \Gamma'$, endowed with a nil-structure $\hat{x}'$ induced by $\hat{x}$, such that the following holds: for every function $\omega : \mathbb{N} \to \mathbb{R}_+$, there exists $M_1 := M_1(\hat{x}, \omega, D) \in \mathbb{N}_+$, and for every $N \in \mathbb{N}_+$ and $g \in \text{poly}_D(G_*)$, there exist $M \in \mathbb{N}$ with $M \leq M_1$, a sub nilmanifold $X' \subseteq \mathcal{F}(M)$, and a factorization $g(n) = g(n)g'(n)\gamma(n), n \in R_{N,D}$ with $\epsilon, g', \gamma \in \text{poly}_D(G_*)$ such that

(i) $\epsilon : R_{N,D} \to G$ is $(M, N)$-smooth with respect to $\hat{x}$;
(ii) $g' \in \text{poly}_D(G_*)$ and $(g'(n) \cdot e_n)_{n \in R_{N,D}}$ is totally $\omega(M)$-equidistributed on $X'$ with respect to $\hat{x}'$;
(iii) $\gamma : R_{N,D} \to G$ is $M$-rational for $\Gamma$, and $\gamma(n) \cdot e_n = \gamma(n + Me_i) \cdot e_{n'}$ for all $1 \leq i \leq D$, $n, n + Me_i \in R_{N,D}$.

**Remark 4.17.** The proof of Theorem 4.16 is essentially the same as Theorem 10.2 of [26]. So we omit its proof, but only pointing out the differences:

- The definition of $(M, N)$-smoothness in this paper is stronger than the one used by Green and Tao [26] as we require that $\|\psi \circ \epsilon\|_{C^0_m(R_{N,D})} \leq M$ in addition. This stronger conclusion was in fact proved implicitly by using the construction of $\epsilon$ on pages 49–50 in the proof of Proposition 9.2 of [26], the proof of Theorem 10.2 of [26], and the fact that

$$\|\psi \circ \epsilon_1 \epsilon_2\|_{C^0_m(R_{N,D})} \leq C(\|\psi \circ \epsilon_1\|_{C^0 m(R_{N,D})} + \|\psi \circ \epsilon_2\|_{C^0 m(R_{N,D})})$$

for some $C > 0$ depending only on $X$ and $D$.
- The part “$\gamma(n) \cdot e_n = \gamma(n + Me_i) \cdot e_{n'}$ for all $1 \leq i \leq D$, $n, n + Me_i \in R_{N,D}$” is not mentioned in Theorem 10.2 of [26], but it follows immediately from Lemma A.12 of [26].
- Theorem 10.2 of [26] is stated only for $\omega(M) = M^A$ for some $A > 0$, but the same method can be used to prove it for a general function $\omega$ (see also the remark on page 29 of [18]).
- Theorem 10.2 of [26] provides a more explicit description of the family $\mathcal{F}(M)$, but we do not need it in this paper (see also the remark on page 29 of [18]).

5. Description of certain sub nilmanifold of $X \times X$

The purpose of this section is to study a special type of sub nilmanifolds of $X \times X$ for some nilmanifold $X$. Though short in length, Section 5 is the most important piece of ingredient in the proof of Theorem 1.5. The main result in this section is Theorem 5.5, but we need some definitions before stating it.
5.1. $d$-automorphisms on nilmanifolds. Let $X = G/\Gamma$ be a nilmanifold of natural step $d$. For all $g_1, \ldots, g_d, h_1, \ldots, h_d \in G$ such that $g_i h_i^{-1} \in G_{\ker}$, $1 \leq i \leq d$, we have that $[g_1, \ldots, g_d]_d = [h_1, \ldots, h_d]_d$. So the map $[\cdot, \ldots, \cdot]_d \colon G^d \to G_d$ factors through $G_d^2$, and it induces a map
\[
[\cdot, \ldots, \cdot]_d \colon (G/G_{\ker})^d \to G_d
\]
in the natural way (which is still denoted as $[\cdot, \ldots, \cdot]_d$ for convenience).

Let $X = G/\Gamma$ be a nilmanifold. We say that a map $\sigma \colon G \to G$ is an automorphism of $X$ if $\sigma$ is a continuous bijection such that $\sigma(\Gamma) \subseteq \Gamma$ and $\sigma(gh) = \sigma(g)\sigma(h)$ for all $g, h \in G$. In this paper, we need to study a special type of automorphisms.

**Definition 5.1** ($d$-automorphisms). Let $X = G/\Gamma$ be a nilmanifold and $d \in \mathbb{N}_+$ be its natural step. Let $\pi_{\ker} \colon G \to G/G_{\ker}$ be the quotient map. We say that a map $\sigma \colon G/G_{\ker} \to G/G_{\ker}$ is a $d$-automorphism of $X$ if $\sigma$ is an automorphism of $G/G_{\ker}\Gamma$, and for all $g_1, \ldots, g_d \in G/G_{\ker}$, we have that
\[
[g_1, \ldots, g_d]_d = [\sigma(g_1), \ldots, \sigma(g_d)]_d.
\]
Let $\text{Aut}_d(X)$ denote the collection of all $d$-automorphisms of $X$.

**Remark 5.2.** In the degenerate case $d = 1$, a 1-automorphism of $X$ is just an automorphism of $G/G_{\ker}\Gamma$.

Let $d \in \mathbb{N}_+$ and $G$ be a nilpotent group with a standard nil-structure $\mathfrak{x} = (G_\bullet, X, \psi, d_G, d_X)$. Suppose that $\dim(G) = m$, $\dim(G_2) = m_2$, $s' = m - m_2$. Recall that $\psi_{\ker} \colon G/G_{\ker} \to \mathbb{R}^{s'}$ is the isometric identification between $G/G_{\ker}$ and $\mathbb{R}^{s'}$ induced by $\psi$. Then $\sigma \colon G/G_{\ker} \to G/G_{\ker}$ is an automorphism of $G/G_{\ker}\Gamma$ if and only if there exists $A \in M_{s' \times s'}(\mathbb{Z})$ such that
\[
\sigma(g) = \psi_{\ker}^{-1} \circ A \circ \psi_{\ker}(g) := \psi_{\ker}^{-1}(\psi_{\ker}(g) \cdot A)
\]
for all $g \in G/G_{\ker}$ (recall that $A \colon \mathbb{R}^{s'} \to \mathbb{R}^{s'}$ denotes the map given by $A(x) := x \cdot A, x \in \mathbb{R}^{s'}$, i.e. the right multiplication of $A$). For convenience we denote $\sigma$ by $\sigma_{A}$, and write
\[
\text{Aut}_{A,d}(X) := \{ A \in M_{s' \times s'}(\mathbb{Z}) : \sigma_{A} \in \text{Aut}_d(X) \}.
\]

**Convention 5.3.** In order to lighten the notation, we make the following convention. Let $\mathbf{x}$ be a vector in $\mathbb{R}^{s'}$ (then $\psi_{\ker}^{-1}(\mathbf{x}) \in G/G_{\ker}$). We use $\psi^{-1}(\mathbf{x})$ to denote any element in $G$ of the form $\psi^{-1}(\mathbf{x'})$ for some $\mathbf{x'} \in \mathbb{R}^m$ whose first $s$ coordinates is the vector $\mathbf{x}$ (i.e. $\pi_{\ker}(\mathbf{x'}) = \mathbf{x}$). For every $g \in G$ and every $\sigma \in \text{Aut}_d(X)$, we use $\sigma(g)$ to denote any element in $G$ whose projection $\pi_{\ker}(\sigma(g))$ on $G/G_{\ker}$ is $\sigma(\pi_{\ker}(g))$.

Although $\psi^{-1}(\mathbf{x})$ and $\sigma(g)$ are not uniquely defined as elements of $G$, they are well defined modulo $G_{\ker}$. Since the map $[\cdot, \ldots, \cdot]_d \colon G^d \to G_d$ factors through $G_{\ker}^d$, expressions such as $[\psi^{-1}(\mathbf{x}_1), \ldots, \psi^{-1}(\mathbf{x}_d)]_d$ and $[\sigma(g_1), \ldots, \sigma(g_d)]_d$ are well defined even though $\psi^{-1}(\mathbf{x}_i)$ and $\sigma(g_i)$ are not (and we will use this convention in such expressions only).

Under this convention, we have that $\sigma \in \text{Aut}_d(G)$ if for all $g_1, \ldots, g_d \in G,$
\[
[\sigma(g_1), \ldots, \sigma(g_d)]_d = [g_1, \ldots, g_d]_d.
\]

5.2. **Main result of this section.** We need some quantitative definitions before stating the main result.
**Definition 5.4** (Height). The *height* of a rational number $\frac{p}{q}$, $p, q \in \mathbb{Z}$, $(p, q) = 1$ is $\max(|p|, |q|)$. We denote the height of an irrational number by $\infty$.

The *height* of a matrix is the maximum of the heights of entries of this matrix. The *height* of a vector is the maximum of the heights of coordinates of this vector.

For a subspace $A$ of $\mathbb{R}^n$ with dimension $r$, the *height* of $A$ is the minimum of the heights of matrices $B \in M_{r \times s}(\mathbb{Z})$ such that $A = \{xB \in \mathbb{R}^n : x \in \mathbb{R}^r \}$ (denote the height of $A$ to be $\infty$ if such a $B$ does not exist). Since natural numbers are well-ordered, the height of $A$ is always well-defined.

Let $X = G/\Gamma$ be a nilmanifold with a nil-structure $\mathfrak{X} = (G, X, \psi, d_G, d_X)$. The height of $\sigma \in \text{Aut}_d(G)$ with respect to $\mathfrak{X}$ is the smallest height of the matrix $A$ such that $\sigma$ can be written as $\sigma = \sigma_{\psi, A}$.

Let $X = G/\Gamma$ be a nilmanifold with a standard nil-structure $\mathfrak{X} = (G, X, \psi, d_G, d_X)$. Suppose that $\dim(G) = m$, $\dim(G_{\ker}) = m_2'$, and $s' = m - m_2'$. Recall that $\psi$ induces an identification $\psi_{\ker} : G/G_{\ker} \to \mathbb{R}^{s'}$. For every subgroup $H$ of $G$ rational for $\Gamma$, $H/(H \cap G_{\ker})$ is a subgroup of $G/G_{\ker}$ rational for $G_{\ker} \cap \Gamma$. Assume that $\dim(H/(H \cap G_{\ker})) = r$. The *height* of $H$ with respect to $\mathfrak{X}$ is the minimum of the heights of matrices $\sigma \in M_{r \times s}(\mathbb{Z})$ such that

$$\psi_{\ker}(H/(H \cap G_{\ker})) = \{x \in \mathbb{R}^{s'} : x \in \mathbb{R}^r \}.$$ 

If $Y = H/(H \cap \Gamma)$ is a sub nilmanifold of $X$, then the height of $Y$ with respect to $\mathfrak{X}$ is that of $H$ with respect to $\mathfrak{X}$.

We are now ready to state the main result of this section, which is the heart of this paper:

**Theorem 5.5** (Description of a special sub nilmanifold of $X \times X$). Let $d \in \mathbb{N}_+$ and $X = G/\Gamma$ be a nilmanifold with a standard nil-structure $\mathfrak{X} = (G, X, \psi, d_G, d_X)$ and the natural filtration $G_{\epsilon, \psi} = (G_0)_{0 \leq i \leq d+1}$ of natural step $d$. Suppose that $\dim(G_{\epsilon}) = 1$. Then for all $C > 0$, there exists $C' := C'(X, C) > C$ such that for every sub nilmanifold $Y = H/(H \cap (\Gamma \times \Gamma))$ of $X \times X$ of height at most $C$ with respect to $\mathfrak{X}$, (where $H$ is a subgroup of $G \times G$ rational for $\Gamma \times \Gamma$) satisfying

- the projection of $Y$ to both coordinates equals to $X$;
- $H_d$ can be written as $H_d = \{(\psi^{-1}(0, \ldots, 0; t), \psi^{-1}(0, \ldots, 0; t)) \in G^{(d)} \times G^{(d)} : t \in \mathbb{R} \}$;

there exists $\sigma \in \text{Aut}_d(G)$ of height at most $C'$ with respect to $\mathfrak{X}$ such that $h_1 = \sigma(h_2) \mod G_{\ker}$ for all $(h_1, h_2) \in H$.

**Proof.** Suppose that $\dim(G) = m$, $\dim(G_{\ker}) = m_2'$ and $s' = m - m_2'$. Denote $\varphi := \psi \times \psi$. Recall that $\mathfrak{X}$ induces the product nil-structure $\mathfrak{X} \times \mathfrak{X}$ on $X \times X$. Since $\mathfrak{X}$ is a standard nil-structure on $X$, $\mathfrak{X} \times \mathfrak{X}$ is a standard nil-structure on $X \times X$. In other words, $\varphi(G_{\ker} \times G_{\ker}) = \mathbb{Z}^{s'} \times \mathbb{R}^{m_2'} \times \mathbb{Z}^{s'} \times \mathbb{R}^{m_2'}$. This naturally induces a Mal’cev coordinate map $\varphi_{\ker} := \psi_{\ker} \times \psi_{\ker}$ from the abelian group $(G/G_{\ker}) \times (G/G_{\ker})$ to $\mathbb{R}^{2s'}$.

Since $H/(H \cap (G_{\ker} \times G_{\ker}))$ is a subgroup of $(G/G_{\ker}) \times (G/G_{\ker})$ rational for $(G_{\ker} \cap \Gamma) \times (G_{\ker} \cap \Gamma)$ of height at most $C$ with respect to $\mathfrak{X}$, denoting $r := \dim(H/(G_{\ker} \times G_{\ker})) \leq 2s'$, there exists $A = (A_1, A_2) \in M_{r \times 2s'}(\mathbb{Z})$ with rank$(A) = r$ and height at most $C$ such that

$$\varphi_{\ker}(H/(H \cap (G_{\ker} \times G_{\ker}))) = \{x \in \mathbb{R}^{2s'} : x = (x_1, \ldots, x_r) \in \mathbb{R}^r \}.$$

2\textsuperscript{25}Note that there are two different filtrations in the statement of this theorem. See also Remark 4.3.

2\textsuperscript{26}The constant $C'$ and thus the whole theorem is independent of the choice of the nil-structure $\mathfrak{X}$. But we do not need it in this paper.

2\textsuperscript{27}Meaning that $h_1 \sigma^{-1}(h_2) \in G_{\ker}$, or equivalently, $\pi_{\ker}(h_1) = \pi_{\ker} \circ \sigma(h_2)$. 

}\end{document}
By the description of $H_d$, we have that $[g_{1,1}, \ldots, g_{1,d}]_d = [g_{2,1}, \ldots, g_{2,d}]_d$ for all $(g_{1,i}, g_{2,i}) \in H, 1 \leq i \leq d$. So
\begin{equation}
(18) \quad [\psi^{-1}_{\ker}(x_1A_1), \ldots, \psi^{-1}_{\ker}(x_dA_1)]_d = [\psi^{-1}_{\ker}(x_1A_2), \ldots, \psi^{-1}_{\ker}(x_dA_2)]_d
\end{equation}
for all $x_1, \ldots, x_d \in \mathbb{R}^r$.

Since the projection of $Y$ to both coordinates equals to $X$, we have that $r \geq s'$ and $\text{rank}(A_1) = \text{rank}(A_2) = s'$. Suppose that $A_2 = Y \begin{bmatrix} I_{s' \times s'} & 0_{(s'-r) \times s'} \end{bmatrix}$ for some invertible $r \times r$ matrix $Y$ of height at most $C$. Denote $A_1 = Y \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, where $B_1$ and $B_2$ are $s' \times s'$ and $(r-s') \times s'$ matrices of heights at most $C' := C'(s', C) = s' C^2$, respectively. Setting $x_i = (y_i, z_i) Y^{-1}, y_i \in \mathbb{R}^{s'}, z_i \in \mathbb{R}^{r-s'}, 1 \leq i \leq d$ in (18), we have that
\begin{equation}
(19) \quad [\psi^{-1}_{\ker}(y_1B_1 + z_1B_2), \ldots, \psi^{-1}_{\ker}(y_dB_1 + z_dB_2)]_d = [\psi^{-1}_{\ker}(y_1), \ldots, \psi^{-1}_{\ker}(y_d)]_d
\end{equation}
for all $y_i \in \mathbb{R}^{s'}, z_i \in \mathbb{R}^{r-s'}, 1 \leq i \leq d$.

Let $\sigma = \psi^{-1}_{\ker} \circ B_1 \circ \psi_{\ker}$. Then $\sigma$ is of height at most $C'$ with respect to $\times \times$. By (19), we have that
\begin{equation*}
[\sigma(g_1), \ldots, \sigma(g_d)]_d = [\psi^{-1}_{\ker} \circ B_1 \circ \psi_{\ker}(g_1), \ldots, \psi^{-1}_{\ker} \circ B_1 \circ \psi_{\ker}(g_d)]_d
\end{equation*}
\begin{equation*}
= [\psi^{-1}_{\ker}(\psi_{\ker}(g_1)B_1), \ldots, \psi^{-1}_{\ker}(\psi_{\ker}(g_d)B_1)]_d = [\psi^{-1}_{\ker}(\psi_{\ker}(g_1)), \ldots, \psi^{-1}_{\ker}(\psi_{\ker}(g_d))]_d = [g_1, \ldots, g_d]_d
\end{equation*}
for all $g_1, \ldots, g_d \in G$. So $\sigma \in \text{Aut}_d(G)$.

Pick any point $(h_1, h_2) \in H$. Since $Y$ is invertible, there exists $x = (y, z) \in \mathbb{R}^{s'}, z \in \mathbb{R}^{r-s'}$ such that $(h_1, h_2) = \varphi^{-1}_{\ker}(xA_1, xA_2) = \varphi^{-1}_{\ker}(yB_1 + zB_2, y)$. Then $h_1 = \psi^{-1}_{\ker}(yB_1 + zB_2), h_2 = \psi^{-1}_{\ker}(y)$, and $\sigma(h_2) = \psi^{-1}_{\ker}(yB_1)$. By (19), for all $y_i \in \mathbb{R}^{s'}, z_i \in \mathbb{R}^{r-s'}, 2 \leq i \leq d$,
\begin{equation*}
[h_1, \psi^{-1}_{\ker}(y_2B_1 + z_2B_2), \ldots, \psi^{-1}_{\ker}(y_dB_1 + z_dB_2)]_d
\end{equation*}
\begin{equation*}
= [\psi^{-1}_{\ker}(yB_1 + zB_2), \psi^{-1}_{\ker}(y_2B_1 + z_2B_2), \ldots, \psi^{-1}_{\ker}(y_dB_1 + z_dB_2)]_d
\end{equation*}
\begin{equation*}
= [\psi^{-1}_{\ker}(y), \psi^{-1}_{\ker}(y_2), \ldots, \psi^{-1}_{\ker}(y_d)]_d
\end{equation*}
\begin{equation*}
= [\sigma(h_2), \psi^{-1}_{\ker}(y_2B_1 + z_2B_2), \ldots, \psi^{-1}_{\ker}(y_dB_1 + z_dB_2)]_d.
\end{equation*}

Since $\text{rank}(A_1) = s'$, we have that
\begin{equation*}
[h_1, g_2, \ldots, g_d]_d = [\sigma(h_2), g_2, \ldots, g_d]_d
\end{equation*}
for all $g_2, \ldots, g_d \in G$. By Lemma 3.10, $[\sigma(h_2)h_1^{-1}, g_2, \ldots, g_d]_d = e_G$ for all $g_2, \ldots, g_d \in G$. By the definition of $G_{\ker}$, we have that $h_1 = \sigma(h_2) \mod G_{\ker}$.

6. Multi-linear analysis along polynomial sequences

In this section, we prove the following theorem, which is another important ingredient for the proofs of our main results:

**Theorem 6.1.** Let $K = (K, O_K, D, \mathcal{B} = \{b_1, \ldots, b_p\})$ be an integral tuple and $p, q \in O_K \setminus \{0\}$ with $|N_K(p)| \neq |N_K(q)|$. Let $d \in \mathbb{N}_+$ and $X = G/\Gamma$ be a nilmanifold with a standard nil-structure $\mathfrak{X} = (G, \mathcal{X}, \psi, d\mathcal{G}, d\mathcal{X})$ and the natural filtration $G_{\mathfrak{c}, \mathfrak{s}} = (G_{\mathfrak{s}})_{0 \leq \mathfrak{s} \leq d+1}$ of natural step $d$. Suppose that $\dim(G_d) = 1$. For all $C > 0$, there exist $\delta := \delta(X, \mathcal{B}, p, q, C) > 0$ and $N_0 := N_0(X, \mathcal{B}, p, q, C) \in \mathbb{N}$
such that for every $N \geq N_0$, every $g \in \text{poly}_D(G_\bullet)$ and every $\sigma \in \text{Aut}_d(X)$ of height at most $C$ with respect to $X$, denoting

$$ h(n) := g(nA_G(p)) \cdot (\sigma \circ g(nA_G(q)))^{-1}, n \in \mathbb{Z}^D, $$

if $||h||_{C}\ker(X) \leq C$, then $(g(n) \cdot e_X)_{c \in \mathbb{R}X}$ is not totally $\delta$-equidistributed on $X$ with respect to $X$.

**Remark 6.2.** Since $\sigma$ is not well defined as a map on $G$, neither is $h$. However, since $h$ is well defined modulo $G\ker$, the norm $||h||_{C}\ker(X) \leq C$ is well-defined.

Theorem 6.1 is a main technical innovation of this paper, whose argument is rather different from the ones used in [18]. Before giving the proof of Theorem 6.1 we provide some examples to illustrate the ideas.

### 6.1 Some examples

Let $H = \mathbb{R}^3$ be the Heisenberg group endowed with a group structure given by

$$(x, y; z) \cdot (x', y'; z') := (x + x', y + y' + z' + xy')$$

for all $(x, y; z), (x', y'; z') \in \mathbb{R}^3$. Recall that $H_2 = H\ker = \{0\} \times \{0\} \times \mathbb{R}$ and $H_3 = \{(0, 0; 0)\}$. Let $\Gamma = \mathbb{Z}^3$ and $X_{\text{Hei}} = H/\Gamma$. Our first example is a special case of Theorem 6.1 for $K = \mathbb{Q}$ (translated into the qualitative version for the convenience of explanations):

**Proposition 6.3.** (Example for $K = \mathbb{Q}$, qualitative version). Let $g: \mathbb{Z} \to H$ be given by $g(n) = a^n$ for some $a = (x_0, y_0, z_0) \in H$ for all $n \in \mathbb{Z}$. Suppose that there exist $\sigma \in \text{Aut}_2(X_{\text{Hei}})$ and $p, q \in \mathbb{Z}\setminus\{0\}, [p] \neq [q]$ such that

$$ g(pn) = \sigma \circ g(qn) \mod H\ker $$

for all $n \in \mathbb{Z}$, then $(g(n) \cdot e_{X_{\text{Hei}}})_{n \in \mathbb{Z}}$ is not equidistributed on $X_{\text{Hei}}$ (recall [2] for the definition).

**Proof.** Suppose that on the contrary $(g(n) \cdot e_{X_{\text{Hei}}})_{n \in \mathbb{Z}}$ is equidistributed on $X_{\text{Hei}}$. Then $x_0, y_0, 1$ are linear independent over $\mathbb{Q}$. Suppose that $\sigma(x, y) = (x, y)A$ for some $A \in M_{2 \times 2}(\mathbb{Z})$ for all $(x, y) \in \mathbb{R}^2 = H/H\ker$. By assumption, we have that $p(x_0, y_0) = q(x_0, y_0)A$. Since $x_0, y_0, 1$ are linear independent over $\mathbb{Q}$, we have that $A = \begin{bmatrix} p/q & 0 \\ 0 & p/q \end{bmatrix}$. In other words, $\sigma(x, y) = \frac{p}{q}(x, y)$ for all $(x, y) \in \mathbb{R}^2$. However, by the multi-linearity of $[\cdot, \cdot]$, for all $h_1, h_2 \in H/H\ker$,

$$ [h_1, h_2] = [\sigma(h_1), \sigma(h_2)] = \frac{p}{q} h_1, \frac{p}{q} h_2 = \frac{p}{q} [h_1, h_2], $$

where we used Convention 5.3. Since $[p] \neq [q], [h_1, h_2] = (0, 0; 0)$ for all $h_1, h_2 \in H/H\ker$, a contradiction.

**Remark 6.4.** In fact, a similar argument applies to the case where $H$ is replaced by a $d$-step nilpotent group. In this case, one can deduce that

$$ [h_1, \ldots, h_d]_d = \left(\frac{p}{q}\right)^d [h_1, \ldots, h_d] $$

to get a contradiction. This idea provides an alternative approach to prove Theorem 6.1 of [18].

We provide another example for the case $K = \mathbb{Q}[i]$. In this case $O_K = \mathbb{Z}[i]$ and we can choose $B = \{1, i\}$ as the integral basis. Then for every $p = p_1 + p_2 i \in \mathbb{Q}[i], p_1, p_2 \in \mathbb{Q}$, we have that

$$ A_B(p) = \begin{bmatrix} p_1 & p_2 \\ -p_2 & p_1 \end{bmatrix}. $$
Proposition 6.5 (Example for $K = \mathbb{Q}[i]$, qualitative version). Let $g: \mathbb{Z}^2 \to H$ be given by $g(n_1, n_2) = a_1 n_1^d n_2^{d_2}$ for some $a_1 = (x_1, y_1; z_1), a_2 = (x_2, y_2; z_2) \in H$ for all $(n_1, n_2) \in \mathbb{Z}^2$. Suppose that there exist $\sigma \in \text{Aut}_2(X_{Hel})$ and $p, q \in \mathbb{Z}[i]\{0\}, |N_{\mathbb{Q}[i]}(p)| \neq |N_{\mathbb{Q}[i]}(q)|$ such that

$$g(nA_B(p)) = \sigma \circ g(nA_B(q)) \mod H_{ker}$$

for all $n \in \mathbb{Z}^2$, where $B = \{1, i\}$. Then $(g(n) \cdot e_{hel})_{n \in \mathbb{Z}^2}$ is not equidistributed on $X_{Hel}$.

**Proof.** The idea of the proof is similar to Proposition 7.9 of [43]. Suppose that $\sigma(x, y) = (x, y)A$ for some $A \in M_{2 \times 2}(\mathbb{Z})$ for all $(x, y) \in \mathbb{R}^2 = H/H_{ker}$. Let $g': \mathbb{Z} \to \mathbb{R}^2 = H/H_{ker}$, $g'(n_1, n_2) := n_1(x_1, y_1) + n_2(x_2, y_2)$ be the projection of $g$ onto $H/H_{ker}$. By assumption, we have that

$$g'(nA_B(p/q)) = \sigma \circ g'(n) = g'(n)A$$

for all $n \in \mathbb{Z}^2$. Let $f \in \mathbb{Q}[x]$ be the minimal polynomial of $A_B(p/q)$. By the linearity of $g'$,

$$g'(n(f(A_B(p/q)))) = g'(n)f(A)$$

for all $n \in \mathbb{Z}^2$.

Suppose that on the contrary $(g(n) \cdot e_{hel})_{n \in \mathbb{Z}^2}$ is equidistributed on $X_{Hel}$. Then $(g'(n) \mod \mathbb{Z}^2)_{n \in \mathbb{Z}^2}$ is equidistributed on $\mathbb{T}^2$. Therefore, (20) implies that $f(A) = O_{2 \times 2}$. By Lemma 2.3 there exist a $2 \times 2$ invertible matrix $S$ and a diagonal matrix $J = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$ with $f(\mu_1) = f(\mu_2) = 0$ such that

$$A = SJS^{-1}.$$  

By Lemma 2.4 $N_{\mathbb{Q}[i]}(\mu_1) = N_{\mathbb{Q}[i]}(\mu_2) = \det(A_B(p/q))$.

Since $\sigma \in \text{Aut}_2(X_{Hel})$, for all $i, j \in \{1, 2\},$

$$[e_iS^{-1}, e_jS^{-1}] = [\sigma(e_iS^{-1}), \sigma(e_jS^{-1})] = [e_iJS^{-1}, e_jJS^{-1}] = [\mu_i\prime \mu_j, e_iS^{-1}, e_jS^{-1}],$$

where $e_1 := (1, 0)$ and $e_2 := (0, 1)$. Since $|N_{\mathbb{Q}[i]}(\mu_1\mu_2)| = |\det(A_B(p/q))|^2 \neq 1$, we have that $[e_iS^{-1}, e_jS^{-1}] = (0, 0; 0)$ for all $i, j \in \{1, 2\}$. This implies that $[h_1, h_2] = (0, 0; 0)$ for all $h_1, h_2 \in H/H_{ker}$ since $S$ is invertible, which is impossible. This finishes the proof. \(\square\)

6.2. $m$-symmetric and $m$-diagonal forms. As we shall see later in this section, Theorem 6.1 is related to a problem on certain multi-linear functions. So we start with a generalization of the quadratic form to higher order cases:

**Definition 6.6 ($m$-symmetric and $m$-diagonal forms).** Let $D, m, s \in \mathbb{N}_+$. We say that a map $L: (\mathbb{Z}^D)^m \to \mathbb{R}^s$ is a ($D$-dimensional) $m$-symmetric form if for all $n_i = (n_{i,1}, \ldots, n_{i,D}) \in \mathbb{Z}^D, 1 \leq i \leq m$, we have that

$$L(n_1, \ldots, n_m) = \sum_{i_1, \ldots, i_m=1}^{D} u_{i_1,\ldots,i_m} \prod_{j=1}^{m} n_{j,i_j}$$

for some $u_{i_1,\ldots,i_m} \in \mathbb{R}^s$ such that for any permutation $\tau: \{1, \ldots, m\} \to \{1, \ldots, m\}$, $u_{i_1,\ldots,i_m} = u_{\tau(i_1),\ldots,\tau(i_m)}$.

We say that a map $R: \mathbb{Z}^D \to \mathbb{R}^s$ is a ($D$-dimensional) $m$-diagonal form if

$$R(n) = \sum_{j \in \mathbb{Z}^D, |j|=m} v_j n^j$$

for some $v_j \in \mathbb{R}^s$. 

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28 We clarify that $A_B(p/q)$ is a $2 \times 2$ matrix because $[\mathbb{Q}[i] : \mathbb{Q}] = 2$, while $A$ is a $2 \times 2$ matrix because dim$(H/H_{ker}) = 2$. 

Conceotion 6.7. In the rest of this section, the dimension $D$ is considered as fixed, and we simply say that a function is an $m$-symmetric or $m$-diagonal form for short.

For example, a 1-symmetric form is of the form $L: \mathbb{Z}^D \to \mathbb{R}^s$, $L(n) = v \cdot n$ for some $v \in \mathbb{R}^D$ for all $n \in \mathbb{Z}^D$, which is just a linear function. A 2-symmetric form is of the form $L: \mathbb{Z}^2 \to \mathbb{R}^s$, $L(m, n) = mAn^T$ for some $A \in M_{D \times D}(\mathbb{R})$ such that $A^T = A$ for all $m, n \in \mathbb{Z}^D$, which is a quadratic form.

The following lemma says that there exists a canonical bijection between $m$-symmetric and $m$-diagonal forms:

Lemma 6.8 (Identification between $m$-symmetric and $m$-diagonal forms). Let $D, m, s \in \mathbb{N}_+$. For every $m$-symmetric form $L: (\mathbb{Z}^D)^m \to \mathbb{R}^s$, there exists a unique $m$-diagonal form $R: \mathbb{Z}^D \to \mathbb{R}^s$ such that $L(n, \ldots, n) = R(n)$ for all $n \in \mathbb{Z}^D$, and vice versa.

Proof. Suppose that $L: (\mathbb{Z}^D)^m \to \mathbb{R}^s$ is an $m$-symmetric form given by (21), and $R: \mathbb{Z}^D \to \mathbb{R}^s$ is an $m$-diagonal form given by (22). Then $L(n, \ldots, n) = R(n)$ for all $n \in \mathbb{Z}^D$ if and only if for all $j = (j_1, \ldots, j_D) \in \mathbb{N}^D$,

$$\sum_{i_1, \ldots, i_m} u_{i_1, \ldots, i_m} = 1, \quad \forall j \in U(j),$$

where the set $U(j)$ consists of all $(i_1, \ldots, i_m) \in \{1, \ldots, D\}^m$ such that the set $\{1 \leq k \leq m: i_k = i\}$ is of cardinality $j_i$ for all $1 \leq i \leq D$. Since $L$ is an $m$-symmetric form, for all $(i_1, \ldots, i_m) \in U(j),$

$$u_{i_1, \ldots, i_m} = \frac{1}{|U(j)|} v_j = \frac{1}{m} v_j,$$

where $\binom{m}{j} := \binom{m}{j_1} \binom{m-j_1}{j_2} \cdots \binom{m-j_1-j_2-\cdots-j_{D-1}}{j_D}$. (23) implies that $L$ uniquely determines $R$, and (24) implies that $R$ uniquely determines $L$. \hfill $\Box$

If $L(n, \ldots, n) = R(n)$ for all $n \in \mathbb{Z}^D$ for some $m$-symmetric form $L: (\mathbb{Z}^D)^m \to \mathbb{R}^s$ and $m$-diagonal form $R: \mathbb{Z}^D \to \mathbb{R}^s$, then we denote $R = \hat{L}$ and $L = \hat{R}$. Clearly, $\hat{L} = L$ and $\hat{R} = R$.

From (23) and (24), the following lemma is straightforward:

Lemma 6.9 (Vanishing property). Let $D, m, s \in \mathbb{N}_+$ and $L: (\mathbb{Z}^D)^m \to \mathbb{R}^s$ be an $m$-symmetric form. Then $L \equiv 0$ if and only if $\hat{L} \equiv 0$.

Similar to the quadratic forms, the $m$-symmetric forms enjoy many invariance properties:

Lemma 6.10 (Invariance properties). Let $D, m, s \in \mathbb{N}_+$ and $L: (\mathbb{Z}^D)^m \to \mathbb{R}^s$ be an $m$-symmetric form. Then

(i) for all $m$-symmetric form $L': (\mathbb{Z}^D)^m \to \mathbb{R}^s$, $L + L'$ is an $m$-symmetric form;

(ii) for all $c \in \mathbb{R}$, $cL$ is an $m$-symmetric form;

(iii) for all $A \in M_{s \times s}(\mathbb{Z})$, denoting $A \circ L(n_1, \ldots, n_m) := L(n_1, \ldots, n_m) \cdot A$, then $A \circ L$ is an $m$-symmetric form;

(iv) for all $A \in M_{n \times n}(\mathbb{Z})$, denoting $L \circ A(n_1, \ldots, n_m) := L(n_1A, \ldots, n_mA)$, then $L \circ A$ is an $m$-symmetric form.

Proof. (i), (ii) and (iii) are straightforward by definition, and so we only prove (iv).

Denote $n_i = (n_{i1}, \ldots, n_{iD}) \in \mathbb{Z}^D$ for $1 \leq i \leq m$. Suppose that

$$L(n_1, \ldots, n_m) = \sum_{i_1, \ldots, i_m} u_{i_1, \ldots, i_m} \prod_{j=1}^m n_{ij}. $$

Therefore, for all $A \in M_{n \times n}(\mathbb{Z}),$

$$L(A(n_1, \ldots, n_m)) = \sum_{i_1, \ldots, i_m} u_{i_1, \ldots, i_m} \prod_{j=1}^m A(n_{ij}).$$
Let $\tau: \{1, \ldots, m\} \to \{1, \ldots, m\}$ be a permutation. Then $u_{i_1, \ldots, i_m} = u_{i_{\tau(1)}, \ldots, i_{\tau(m)}}$ for all $1 \leq i_1, \ldots, i_m \leq D$.

Suppose that $A = (a_{k,i})_{1 \leq j, k \leq D}$. Then $(\mathbf{n}_j A)_i = \sum_{k=1}^{D} n_{ji} a_{k,i}$ for all $1 \leq j \leq m, 1 \leq i \leq D$. So

$$L \circ A(\mathbf{n}_1, \ldots, \mathbf{n}_m) = \sum_{i_1, \ldots, i_m = 1}^{D} u_{i_1, \ldots, i_m} \prod_{j=1}^{m} \left( \sum_{k=1}^{D} n_{ji} a_{k,i} \right) = \sum_{i_1, \ldots, i_m = 1}^{D} u'_{i_1, \ldots, i_m} \prod_{j=1}^{m} n_{ji},$$

where

$$u'_{i_1, \ldots, i_m} = \sum_{i_1', \ldots, i_m' = 1}^{D} u_{i_1', \ldots, i_m'} a_{i_1', i_1} \cdots a_{i_m', i_m}.$$}

So

$$u'_{i_1, \ldots, i_m} = \sum_{j_1, \ldots, j_m = 1}^{D} u_{j_1, \ldots, j_m} a_{i_1, j_1} \cdots a_{i_m, j_m} = \sum_{j_1, \ldots, j_m = 1}^{D} u_{j_1, \ldots, j_m} a_{i_1, j_1} \cdots a_{i_m, j_m} = u'_{i_1, \ldots, i_m}$$

for all $1 \leq i_1, \ldots, i_m \leq D$. This implies that $L \circ A$ is an $m$-symmetric form. \hfill \Box

### 6.3. Proof of Theorem 6.1

We are now ready to prove Theorem 6.1 in this section.

**Step 1:** converting $g(\mathbf{n})$ into $m$-symmetric forms. Let $s' = \dim(G) - \dim(G_{\ker})$. Recall that $\pi_{\ker}: G \to G/G_{\ker}$ is the quotient map and $\psi_{\ker}: G / G_{\ker} \to \mathbb{R}^{s'}$ is the map induced by $\psi$. Suppose that $g_{\ker} := \pi_{\ker} \circ \psi \circ g: \mathbb{Z}^{D} \to \mathbb{R}^{s'}$ is given by

$$g_{\ker}(\mathbf{n}) := \sum_{|j| \leq k} a'_j \mathbf{n}^j$$

for some $k \in \mathbb{N}, a'_j \in \mathbb{R}^{s'}$ for all $|j| \leq k$ (where $k$ depends only on $\chi$). Suppose that $\sigma = \sigma_{\psi, A}$ for some $A \in M_{s' \times s'}(\mathbb{Z})$ of height at most $C$. Then letting $h_{\ker} := \pi_{\ker} \circ \psi \circ h: \mathbb{Z}^{D} \to \mathbb{R}^{s'}$, we have that

$$h_{\ker}(\mathbf{n}) = \sum_{|j| \leq k} \left( a'_j (\mathbf{n} A G_{\psi} (p))^j - (a'_j \cdot \mathbf{n} A G_{\psi} (q))^j \right).$$

Since $||h||^{(R_{\ker}, (R_{\psi})^\chi)}_{(R_{\psi})^\chi} \leq C$, by definition, $||h_{\ker}||^\chi_{(R_{\psi})^\chi} \leq C$. We may assume without loss of generality that $|N_{K}(p)| > |N_{K}(q)|$. Denote $R: \mathbb{Z}^{D} \to \mathbb{R}^{s'}$ by

$$R(\mathbf{n}) := h_{\ker}(\mathbf{n} A G_{\psi} (q)^{-1}) = \sum_{|j| \leq k} \left( a'_j (\mathbf{n} A G_{\psi} (p/q))^j - (a'_j \cdot \mathbf{n})^j \right).$$

For $0 \leq m \leq k$, let $R_m$ be the $m$-diagonal form given by

$$R_m(\mathbf{n}) := \sum_{|j| = m} \left( a'_j (\mathbf{n} A G_{\psi} (p/q))^j - (a'_j \cdot \mathbf{n})^j \right) := \sum_{|j| = m} \epsilon_{m,j} \mathbf{n}^j$$

for some $\epsilon_{m,j} \in \mathbb{R}^{s'}$ for all $|j| = m$. Then $R(\mathbf{n}) = \sum_{m=0}^{k} R_m(\mathbf{n})$. Since $||h_{\ker}||^\chi_{(R_{\psi})^\chi} \leq C$, by (25) and (26), $||R_m||^\chi_{(R_{\psi})^\chi} \leq C D^m H(q^{-1})^m$ for all $1 \leq m \leq k$, where $H(q^{-1})$ is the height of $A G_{\psi} (q^{-1})$ which is finite. So

$$||\epsilon_{m,j}||_{T'} \leq C_1 m / N^m$$
for some $C_{1,m} = C_{1,m}(\mathcal{X}, p, q, \mathcal{B}, C) > 0$.

Fix $1 \leq m \leq k$, and let $R_m' : \mathcal{Z}^D \to \mathbb{R}^{r'}$ be the $m$-diagonal form given by

$$R_m'(n) := \sum_{j \in \mathcal{D}} a_j' n_j.$$

Then

$$L_m := R_m$$

are $m$-symmetric forms. By Lemma 6.10

(30)

$$L_m^\prime : = L_m' \circ A_{\mathcal{B}}(p/q) - A \circ L_m' - L_m$$

is also an $m$-symmetric form (where one should consider $L_m$ as the “error term”). By (27), (29) and (30),

$$\tilde{L}_m(n) = L_m''(n, \ldots, n) = L_m'(nA_{\mathcal{B}}(p/q), \ldots, nA_{\mathcal{B}}(p/q)) - A \circ L_m'(n, \ldots, n) - L_m(n, \ldots, n)$$

$$= R_m'(nA_{\mathcal{B}}(p/q)) - A \circ R_m'(n) - R_m(n) = 0$$

for all $n \in \mathcal{Z}^D$. By Lemma 6.9 for all $1 \leq m \leq k$.

(31)

$$L_m'' \equiv 0.$$

**Step 2: using eigenvectors to express $L_m'$ and $L_m$.** For all $x_1, \ldots, x_d \in \mathbb{R}^{r'}$, $[\psi^{-1} x_1, \ldots, \psi^{-1} x_d]_d \in G_d$ (where we use Convention 5.3 to define $\psi^{-1} x$) and so $\psi([\psi^{-1} x_1, \ldots, \psi^{-1} x_d]_d) = (0, \ldots, 0; t)$ for some $t \in \mathbb{R}$. Denote $F(x_1, \ldots, x_d) := t$. Then $F : (\mathbb{R}^{r'})^d \to \mathbb{R}$ is a multi-linear function on $(\mathbb{R}^{r'})^d$. So $F$ can be extended to a multi-linear function from $(\mathbb{C}^{r'})^d$ to $\mathbb{C}$ in the natural way, which for convenience is still denoted by $F$. Since $\sigma = \sigma_{\psi, \mathcal{A}} \in \text{Aut}_d(\mathcal{X})$, we have that

$$[\psi^{-1}(x_1A), \ldots, \psi^{-1}(x_dA)]_d = [\psi^{-1} x_1, \ldots, \psi^{-1} x_d]_d$$

for all $x_1, \ldots, x_d \in \mathbb{R}^{r'}$. So

(32)

$$F(x_1A, \ldots, x_dA) = F(x_1, \ldots, x_d)$$

for all $x_1, \ldots, x_d \in \mathbb{R}^{r'}$ and so for all $x_1, \ldots, x_d \in \mathbb{C}^{r'}$.

Since $L_m, L_m', L_m'' : (\mathbb{Z}^D)^m \to \mathbb{R}^{r'}$ are multi-linear functions, they can also be extend to multi-linear functions from $(\mathbb{C}^D)^m$ to $\mathbb{C}^{r'}$ in the natural way, which for convenience are still denoted by $L_m, L_m', L_m''$, respectively. Since $L_m''(n_1, \ldots, n_m) = 0$ for all $n_1, \ldots, n_m \in \mathcal{Z}^D$ by (31), we have that for all $u_1, \ldots, u_m \in \mathcal{C}^D, L_m''(u_1, \ldots, u_m) \equiv 0$. So by (30),

(33)

$$L_m'(u_1A_{\mathcal{B}}(p/q), \ldots, u_mA_{\mathcal{B}}(p/q)) = A \circ L_m'(u_1, \ldots, u_m) + L_m(u_1, \ldots, u_m).$$

By Lemma 2.14 there exist a basis $v_1, \ldots, v_D \in \mathbb{C}^D$ of $\mathbb{C}^D$ (over $\mathbb{C}$) depending only on $\mathcal{B}$, and $\lambda_i \in \mathbb{C}, 1 \leq i \leq D$ depending on $\mathcal{B}, p$ and $q$, such that $v_i A_{\mathcal{B}}(p/q) = \lambda_i v_i$ for all $1 \leq i \leq D$. Since $|N_K(p)| > |N_K(q)|$, by Lemma 2.4 $|N_K(\lambda_1)|, \ldots, |N_K(\lambda_D)| > 1$. Denote

$$\kappa := \kappa(p, q, \mathcal{B}) = \min_{1 \leq i \leq D} \frac{1}{|N_K(\lambda_i)|} > 1.$$

By (33), for all $1 \leq i_1, \ldots, i_m \leq D$, we have that

$$\prod_{j=1}^m \lambda_{i_j} \cdot L_m'(v_{i_1}, \ldots, v_{i_m}) = L_m'(v_{i_1} A_{\mathcal{B}}(p/q), \ldots, v_{i_m} A_{\mathcal{B}}(p/q))$$

$$= A \circ L_m'(v_{i_1}, \ldots, v_{i_m}) + L_m(v_{i_1}, \ldots, v_{i_m}).$$
Denote
\[ V_m = \{ (v_{i_1}, \ldots, v_{i_m}) \in (\mathbb{C}^D)^m : 1 \leq i_1, \ldots, i_m \leq D \} . \]
Clearly, \( \text{span}_C V_m = (\mathbb{C}^D)^m \). For \( \tilde{v}_m = (v_{i_1}, \ldots, v_{i_m}) \in V_m \), let \( \lambda_{\tilde{v}_m} = \prod_{j=1}^m \lambda_{i_j} \). Then (34) implies that
\[ \lambda_{\tilde{v}_m} \cdot L_m'(\tilde{v}_m) = A \circ L_m'(\tilde{v}_m) + L_m(\tilde{v}_m) \]
for all \( \tilde{v}_m \in V_m \) and \( 1 \leq m \leq k \).

**Step 3:** iterating (35) with polynomials. By induction, it is not hard to show from (35) that for all \( n \in \mathbb{N}_+ \),
\[ \lambda_{\tilde{v}_m}^n \cdot L_m'(\tilde{v}_m) = A^n \circ L_m'(\tilde{v}_m) + B_{\tilde{v}_m,A,n} \circ L_m(\tilde{v}_m), \]
where
\[ B_{\tilde{v}_m,A,n} := \sum_{i=0}^{n-1} \lambda_{\tilde{v}_m}^{n-1-i} A^i \in M_{s' \times s'}(\mathbb{C})^\otimes n \]
So for all \( f(x) = \sum_{i=0}^r a_i x^i \in \mathbb{C}[x] \), we have that
\[ f(\lambda_{\tilde{v}_m}) \cdot L_m'(\tilde{v}_m) = f(A) \circ L_m'(\tilde{v}_m) + B_{\tilde{v}_m,A,f} \circ L_m(\tilde{v}_m), \]
where
\[ B_{\tilde{v}_m,A,f} := \sum_{i=0}^r a_i B_{\tilde{v}_m,A,i}. \]
Let \( f_0 \in \mathbb{Q}[x] \) denote the monic polynomial of the smallest possible positive degree such that
\( f_0(\lambda_{\tilde{v}_m}) = 0 \) for all \( 1 \leq m \leq k \) and \( \lambda_{\tilde{v}_m} \in V_m \). Then
\[ f_0(A) \circ L_m'(\tilde{v}_m) = -B_{\tilde{v}_m,A,f_0} \circ L_m(\tilde{v}_m) \]
for all \( 1 \leq m \leq k \) and \( \lambda_{\tilde{v}_m} \in V_m \), where the heights of \( f_0(A) \) and \( B_{\tilde{v}_m,A,f_0} \), \( \tilde{v}_m \in V_m, 1 \leq m \leq k \) are bounded above by some constant \( C_2 := C_2(\bar{\chi}, D, p, q, C) > 0 \).

Let \( \overline{K} \) denote the algebraic closure of \( K \).

**Claim 1:** \( f_0 \) has no repeated roots, and the absolute value of the \( \overline{K} \)-norm of all the roots of \( f_0 \) are at least \( \kappa \).

Let \( f_{\tilde{v}_m} \) denote the minimal polynomial of \( \lambda_{\tilde{v}_m} \) for all \( 1 \leq m \leq k \) and \( \tilde{v}_m \in V_m \). Then for all \( \tilde{v}_m \in V_m \) and \( \tilde{v}_m' \in V_m', 1 \leq m, m' \leq k \), either \( f_{\tilde{v}_m} = f_{\tilde{v}_m'} \), or \( f_{\tilde{v}_m} \) and \( f_{\tilde{v}_m'} \) have no common roots.

So \( f_0 \) is a constant multiple of the products of all the different polynomials appearing in the set \( \{ f_{\tilde{v}_m} : 1 \leq m \leq k, \tilde{v}_m \in V_m \} \). Since each \( f_{\tilde{v}_m} \) has no repeated roots by Lemma 2.4, so does \( f_0 \).

On the other hand, by Lemma 2.4, all the roots of \( f_{\tilde{v}_m} \) have the same absolute value of the \( \overline{K} \)-norm as \( |N_{\overline{K}}(\lambda_{\tilde{v}_m})| = |N_K(\lambda_{\tilde{v}_m})| |\overline{K} : K| \), which is at least \( \kappa^m |\overline{K} : K| \geq \kappa \). So the absolute value of the \( \overline{K} \)-norm of all the roots of \( f_0 \) are at least \( \kappa \). This finishes the proof of the claim.

**Claim 2:** \( f_0(A) \neq O_{s' \times s'} \).

Suppose that \( f_0(A) = O_{s' \times s'} \). By Claim 1, \( f_0 \) has no repeated roots. By Lemma 2.5 there exist an \( s' \times s' \) invertible matrix \( S \) and a diagonal matrix \( J = \begin{bmatrix} \mu_1 & & \\ & \mu_2 & \\ & & \cdots \\ & & \mu_{s'} \end{bmatrix} \) with \( f_0(\mu_1) = \cdots = \)

\footnote{Here \( \lambda_{\tilde{v}_m}^k := 1 \) and \( A^k := I_{s' \times s'} \) is the \( s' \times s' \) identity matrix.}
\( f_0(\mu') = 0 \) such that \( A = JS^{-1} \). Again by Claim 1, we have that \( |N_X(\mu_1)|, \ldots, |N_X(\mu_d)| > \kappa \). By (32), for all \( x_1, \ldots, x_d \in \mathbb{C}' \),

\[
F(x_1S^{-1}, \ldots, x_dS^{-1}) = F(x_1S^{-1}A, \ldots, x_dS^{-1}A) = F(x_1JS^{-1}, \ldots, x_dJS^{-1}).
\]

Recall that \( e_i \in \mathbb{C}' \) denotes the vector whose \( i \)-th coordinate is 1 and all other coordinates are 0. By the definition of \( G_\ker \), \( F \) is not constant 0. So by the multi-linearity of \( F \) and invertibility of \( S \), there exist \( 1 \leq i_1, \ldots, i_d \leq s' \) such that \( F(e_{i_1}S^{-1}, \ldots, e_{i_d}S^{-1}) \neq 0 \). Then

\[
F(e_{i_1}S^{-1}, \ldots, e_{i_d}S^{-1}) = F(e_{i_1}JS^{-1}, \ldots, e_{i_d}JS^{-1}) = (\prod_{j=1}^d \mu_{i_j}) \cdot F(e_{i_1}S^{-1}, \ldots, e_{i_d}S^{-1}).
\]

Since \( |N_X(\prod_{j=1}^d \mu_{i_j})| \geq x^d > 1 \), this is impossible. This contradiction implies that \( f_0(A) \neq O_{s' \times s'} \).

**Step 4: finishing the proof.** By Claim 2, there exists a row \( c = (c_1, \ldots, c_s) \in \mathbb{Q}' \) of the matrix \( f_0(A) \) which is non-zero. Moreover, the height of \( c \) is at most \( C_2 \). By (36), for all \( 1 \leq m \leq k \) and \( \tilde{v}_m \in V_m \),

\[
c \cdot L_m'(\tilde{v}_m) = c_{m'} \cdot L_m(\tilde{v}_m),
\]

where \( c_{m'} \in \mathbb{C}' \) is a row of the matrix \( -B_{m,A,F} \). By (28),

\[
\|c \cdot L_m'(\tilde{v}_m)\|_T \leq C_{3,m}\frac{m}{N_m}
\]

for all \( 1 \leq m \leq k \), \( \tilde{v}_m \in V_m \) for some \( C_{3,m} := C_{3,m}(X, D, p, q, C) > 0 \). Since \( \text{span}_{\mathbb{C}} V_m = (\mathbb{C}D)^m \), by the multi-linearity of \( L_m' \), we have that for all \( n_1, \ldots, n_m \in \mathbb{Z}D \),

\[
\|c \cdot L_m'(n_1, \ldots, n_m)\|_T \leq \frac{C_{4,m}}{N_m} \prod_{i=1}^m |n_i|
\]

for some \( C_{4,m} := C_{4,m}(X, D, p, q, C) > 0 \). So

\[
\|c \cdot R_m'(n)\|_T \leq \frac{C_{4,m}|n|^m}{N_m}
\]

for all \( n \in \mathbb{Z} \) and \( 1 \leq m \leq k \). By Lemma 4.6

\[
\|Q_m c \cdot a'_j\|_T \leq \frac{C_{5,m}}{N_m}
\]

for all \( 1 \leq m \leq k \), \( |j| = m \) for some \( C_{5,m} := C_{5,m}(X, D, p, q, C) > 0 \) and \( Q_m := Q_m(X, D, p, q, C) \in \mathbb{N}_+ \). Letting \( Q = \prod_{m=1}^k Q_m \), we have that

\[
\|Qc \cdot a'_j\|_T \leq \max_{1 \leq m \leq k} \frac{C_{5,m}}{N_m} \cdot \frac{Q}{Q_m}
\]

for all \( 1 \leq |j| \leq k \). Note that \( Qc \cdot a'_j \in \mathbb{R} \) and \( c \) is independent of the choice of \( 1 \leq m \leq k \). Since \( \mathfrak{X} \) is a standard nil-structure, the map \( \eta: \mathcal{G} \to \mathbb{T} \) defined by

\[
\eta(g_0) := (\mathcal{Q}c, 0, \ldots, 0) \cdot \psi(g_0) \mod \mathbb{Z}, g_0 \in \mathcal{G}
\]

is a horizontal character of \( X \) with \( 0 < \|\eta\|_X \leq \text{dim}(G)C_2Q \). Since \( g_{\ker} = \sum_{m=1}^k R'_m \), we have that

\[
\|\eta \circ g\|_{C^r (\mathbb{R}_{X,D})} \leq \max_{1 \leq m \leq k} \frac{C_5,mQ}{Q_m}.
\]

By Theorem 6.1 there exist \( \delta := \delta(\mathfrak{X}, D, p, q, C) > 0 \) and \( N_0 := N_0(\mathfrak{X}, D, p, q, C) \in \mathbb{N} \) such that for all \( N \geq N_0 \), \( (g(n) - \mathfrak{X})_{n \in \mathbb{R}_{X,D}} \) is not totally \( \delta \)-equidistributed on \( X \) with respect to \( \mathfrak{X} \). This finishes the proof of Theorem 6.1.

\footnote{Recall that for \( z = a + bi \in \mathbb{C} \) for some \( a, b \in \mathbb{R} \), \( \|z\|_T \) denotes the quantity \( \|a\|_T + \|b\|_T \).}
7. Orthogonality of Multiplicative Functions and Nilsequences

In this section, we prove the following central quantitative correlation result of this paper.

**Theorem 7.1** (Main quantitative correlation result). Let $K = (K, O_K, D, \mathcal{B} = \{b_1, \ldots, b_D\})$ be an integral tuple and $X = G/\Gamma$ be a nilmanifold with a nil-structure $\hat{x}$. For all $w, \epsilon > 0$, there exist $\delta := \delta(\hat{x}, w, \mathcal{B}, \epsilon) > 0$ and $N_0 := N_0(\hat{x}, w, \mathcal{B}, \epsilon) \in \mathbb{N}$ such that for all $N \geq N_0$, the following holds: if there exist $g \in \text{poly}_D(G)$ of degree at most $w$, $m \in \mathbb{Z}^D, \chi \in \mathcal{M}_K, \Phi: X \to \mathbb{C}$ such that $\|\Phi\|_{\text{Lip}(X)} \leq 1$ and $\int_X \Phi \, dm_X = 0$, and a $D$-dimensional arithmetic progression $P$ such that

$$\tag{37} \left| \mathbb{E}_{n \in \mathbb{Z}^D} 1_P(n) \chi(t^g(n)) \Phi(g(n + m) \cdot e_X) \right| \geq \epsilon,$$

then the sequence $(g(n) \cdot e_X)_{n \in \mathbb{Z}^D}$ is not totally $\delta$-equidistributed on $X$ with respect to $\chi$.

**7.1 Preliminary reductions.** Suppose that $X = G/\Gamma$ is of natural step $d$ for some $d \in \mathbb{N}_+$. By induction, we may assume the following.

**Assumption 1:** either (i) $d = 1$; or (ii) $d \geq 2$ and the conclusion holds for $d - 1$.

Assume that $g \in \text{poly}_D(G_\ast)$ for some filtration $G_\ast$ of $G$ (which depends only on the degree $w$ of $g$), and let $X' \subseteq X$ be a standard nil-structure of $X$ adapted to $G_\ast$. Since the metrics generated by all nil-structures of $X$ generate the same topology of $X$, all such metrics are equivalent. So there exists $C_0 := C_0(\hat{x}, w) > 1$ such that for all $\Phi: X \to \mathbb{C}$,

$$\tag{38} C_0^{-1} \|\Phi\|_{\text{Lip}(X)} \leq \|\Phi\|_{\text{Lip}(X')} \leq C_0 \|\Phi\|_{\text{Lip}(X)}.$$

Therefore, we can make the following assumption:

**Assumption 2:** $X = (G_\ast, X, \psi, d_G, d_X)$ is a standard nil-structure of $X$, and $g \in \text{poly}_D(G_\ast)$.

We need some further reductions similar to the ones used in Theorem 6.1 of [18], Lemma 3.7 of [26] and Proposition 7.9 of [43].

Denote $m = \dim(G), m_\ast = \dim(G_\ast), s' = m - m_\ast$ and $r = \dim(G_\ast)$. By approximating $\Phi$ with a smooth function, there exist $C_1 := C_1(\hat{x}, e) > 0$ and $\Phi': X \to \mathbb{C}$ such that

$$\|\Phi - \Phi'\|_{L^\infty(m_X)} \leq \epsilon/2, \int_X \Phi' \, dm_X = 0, \text{ and } \|\Phi'\|_{L^\infty(m_X)} \leq C_1.$$

So (37) Implies that

$$\tag{39} \left| \mathbb{E}_{n \in \mathbb{Z}^D} 1_P(n) \chi(t^g(n)) \Phi'(g(n + m) \cdot e_X) \right| \geq \epsilon/2.$$

Recall that $\psi: G \to \mathbb{R}^m$ is the Mal’cev coordinate map with respect to $G_\ast$. Define $\tilde{\psi}: G_d \to \mathbb{T}^r$ by

$$\tilde{\psi}(\psi^{-1}(0, \ldots, 0; y_1, \ldots, y_r)) := (y_1, \ldots, y_r) \mod \mathbb{T}^r$$

for all $(y_1, \ldots, y_r) \in \mathbb{R}^r$. Since $\tilde{\psi}$ factors through $\Gamma$, $\hat{\psi}$ induces an identification between $G_d/(G_d \cap \Gamma)$ with $\mathbb{T}^r$, as well as an identification between the dual group of $G_d/(G_d \cap \Gamma)$ with $\mathbb{Z}^r$. For $v \in \mathbb{Z}^r$, let $\Phi'_\ast(\cdot), \Phi''_\ast: X \to \mathbb{C}$ be the functions

$$\Phi'_\ast(x) := \int_{\mathbb{T}^r} e(-y \cdot v) \Phi'(\tilde{\psi}^{-1}(v) \cdot x) \, dm_{\mathbb{T}^r}(v) \text{ and } \Phi''_\ast(x) := \Phi'_\ast(x)/\|\Phi'_\ast\|_{L^\infty(m_X)}.$$

Denote $\Phi''_\ast := 0$ if $\|\Phi'_\ast\|_{L^\infty(m_X)} = 0$. 
for all \( x \in X \), where \( \tilde{\psi}^{-1}(v) \) is viewed as an arbitrary pre-image of \( v \) in \( G_d \). Then for all \( y \in \mathbb{Z}^r \), 
\[ \| \Phi'_y \|_{lip(X)} \leq C_1. \]
\( \int_X \Phi'_y dm_X = 0 \), and \( \Phi'_y \) is a nilcharacter of \( X \) with frequency \( y \) with respect to \( \tilde{x} \). Since \( \| \Phi'_y \|_{L^\infty(mx)} \leq C_1 \), using integration by parts, we have that \( \| \Phi'_y \|_{L^\infty(mx)} \leq C_2(1 + |y|)^{-2r} \) for some \( C_2 := C_2(\tilde{x}, \epsilon, e) > 0 \). Since for all \( x \in X \),
\[ \Phi(x) = \sum_{y \in \mathbb{Z}^r} \Phi'_y(x) = \sum_{y \in \mathbb{Z}^r} \| \Phi'_y \|_{L^\infty(mx)} \cdot \Phi''_y(x), \]
by (39), there exist \( \epsilon_1 := \epsilon_1(\tilde{x}, \epsilon, e) > 0 \), \( C_3 := C_3(\tilde{x}, \epsilon, e) > 0 \) and \( y \in \mathbb{Z}^r \) such that \( |y| \leq C_3 \) and
\[ \sum_{y \in \mathbb{Z}^r} \| \Phi'_y \|_{L^\infty(mx)} \cdot \Phi''_y(g(n + m) \cdot e_X) \geq \epsilon_1. \]

For \( y \in \mathbb{Z}^r \), let 
\[ G_{d,Y} := \{ g \in G_d : y \cdot \tilde{\psi}(g) = 0 \}. \]
Then \( G_{d,Y} \) is a subgroup of \( G_d \) rational for \( G_d \cap \Gamma \). Let \( G_Y := G/G_{d,Y} \) and \( \Gamma_Y := \Gamma/(G_{d,Y} \cap \Gamma) \). Then \( X_Y := G_Y/\Gamma_Y \) is a nilmanifold. Let \( \pi_Y : X \rightarrow X_Y \) be the quotient map and \( x_{n_y} = (G_{d,Y}, \pi_Y, d_{G_{d,Y}}, d_{X_Y}) \) be any standard nil-structure of \( X_Y \) induced by the quotient map \( \pi_Y \). Then \( \| f|_{lip(X_Y)} \|_{lip(X)} \leq C_d \| f \|_{lip(X)} \) for some \( C_d := C_d(\tilde{x}, \epsilon, e) > 0 \) for all \( |y| \leq C_3 \) and all \( f : X \rightarrow \mathbb{C} \).

We first assume that (40) holds for \( y = 0 \). If \( d = 1 \), then \( G = G_d \) and so \( \Phi''_y \) is a constant. Since \( \int_X \Phi''_y dm_X = 0 \), we have that \( \Phi''_y = 0 \), a contradiction to (40).

Now suppose that \( d \geq 2 \). Then \( G_{d,0} = G_0, G_0 = G/G_d, \Gamma_0 = G_{d,Y} \cap \Gamma \). So \( \tilde{x}_0 = \tilde{x}_0/\Gamma_0 \) is of natural step \( d - 1 \). The function \( \Phi''_y \) factors through \( G_{d,Y} \) and so can be written as \( \Phi''_y = \Phi \circ \pi_0 \) for some function \( \tilde{\Phi} : X_Y \rightarrow \mathbb{C} \). It is easy to see that \( \int_{X_0} \tilde{\Phi} dm_{x_0} = \int_X \Phi''_y dm_X = 0 \). By (40), we have that
\[ \left| \mathbb{E}_{n \in R_{N,D}} 1_p(n) \chi_0(tg(n)) \Phi_0 \circ g(n + m) \cdot e_X \right| \geq \epsilon_1. \]

Since \( g \in \text{poly}_d(G_\epsilon) \), \( \pi_0 \circ g \in \text{poly}_d(G_\epsilon) \). Since \( G_0 \) is of natural step \( d - 1 \), by induction hypothesis, if \( N \geq N_0(\tilde{x}_n, \epsilon_1, D) \), then the sequence \( (\pi_0 \circ g(n) \cdot e_X)_{n \in R_{N,D}} \) is not totally \( \delta := \delta(\tilde{x}, \tilde{x}_n, \epsilon_1C_4^{-1}, D) \)-equidistributed on \( X_0 \) with respect to \( \tilde{x}_n \), which implies that \( (g(n) \cdot e_X)_{n \in R_{N,D}} \) is not totally \( C_4^{-1} \)-equidistributed on \( X \) with respect to \( \tilde{x} \). This finishes the proof.

Now assume that \( y \neq 0 \) and suppose that Theorem 7.1 holds when \( \text{dim}(G_d) = 1 \). Note that \( (G_{d,Y})_d = G_d/G_{d,Y} \) is of dimension 1. Since \( w \cdot y = 0 \) for all \( g = \tilde{\psi}^{-1}(w) \in G_{d,Y} \). We have that
\[ \Phi''_y(gx) = \int_{\mathbb{T}^r} e(-y \cdot \tilde{\Phi}^r(\tilde{\psi}^{-1}(v) \cdot g x) dm_{\mathbb{T}^r}(v) = \int_{\mathbb{T}^r} e(-y \cdot \tilde{\Phi}^r(\tilde{\psi}^{-1}(v + w) \cdot x) dm_{\mathbb{T}^r}(v) \]
\[ = \int_{\mathbb{T}^r} e(-y \cdot (v - w)) dm_{\mathbb{T}^r}(v) = \int_{\mathbb{T}^r} e(-y \cdot (v - w)) dm_{\mathbb{T}^r}(v) = \int_{\mathbb{T}^r} e(-y \cdot v) dm_{\mathbb{T}^r}(v) \]
for all \( x \in X \). So there exists \( \Phi_Y : X_Y \rightarrow \mathbb{C} \) such that \( \Phi''_y = \Phi_Y \circ \pi_Y \). It is easy to see that \( \int_{X_Y} \Phi_Y dm_{X_Y} = \int_X \Phi''_y dm_X = 0 \) and \( \| \Phi_Y \|_{lip(X_Y)} \leq C_4 \). By (40), we have that
\[ \left| \mathbb{E}_{n \in R_{N,D}} 1_p(n) \chi_0(tg(n)) \Phi_Y \circ g(n + m) \cdot e_X \right| \geq \epsilon_1. \]

Since \( g \in \text{poly}_d(G_\epsilon) \), \( \pi_0 \circ g \in \text{poly}_d(G_\epsilon) \), by assumption, if \( N \geq \max_{|y| \leq C_3} N_0(\tilde{x}_n, \epsilon_1, D) \) (which depends only on \( x \), \( \epsilon \) and \( D \) ), then the sequence \( (\pi_0 \circ g(n + m) \cdot e_X)_{n \in R_{N,D}} \) is not totally \( \max_{|y| \leq C_3} \delta(\tilde{x}_n, C_4^{-1} \epsilon_1, D) \)-equidistributed on \( X_Y \) with respect to \( \tilde{x}_n \) for some \( |y| \leq C_3 \), which implies that \( (g(n + m) \cdot e_X)_{n \in R_{N,D}} \) is not totally \( \delta \)-equidistributed on \( X \) with respect to \( \tilde{x} \) for some \( \delta := \delta(\tilde{x}, \epsilon, D, C_4) > 0 \). This finishes the proof.
Note that $\Phi_n'$ is a nilcharacter of $X$ with non-zero frequency. In conclusion, it now suffices to prove Theorem 7.1 under the following assumption:

**Assumption 3:** $\dim(G_d)=1$, and $\Phi$ is a nilcharacter of $X$ with frequency $\ell \in \mathbb{Z}\setminus\{0\}$ with respect to $\chi$.

By using Theorems 4.9 and 4.10 we may further assume that:

**Assumption 4:** $m=0$, and $g(0) = e_G$.

The justification of Assumption 4 is identical to the argument in Section 7.3 of [18], and so we omit the proof.

### 7.2. Using Katai’s Lemma

We now use Katai’s Lemma (Lemma 2.21) to get rid of the multiplicative function $\chi$ in the expression of (37).

Let $G > 0$ be defined as in Lemma 2.13 and $J_p$ be defined as in Lemma 2.9. We construct a set $\mathcal{P} \subseteq \mathbb{N}$ as follows: for every prime integer $p \in \mathbb{N}_+$, if $J_p$ consists of principle prime ideals, let $(a)$ be one of them with the smallest $K$-norm $N(a) = |N_K(a)|$ for some $a \in O_K$. By Lemma 2.13 we may pick some $a' \in O_K$ which is $C_{G}$-regular such that $(a') = (a)$. We put such an element $a'$ into the set $\mathcal{P}$. Then all the elements in $\mathcal{P}$ are $C_{G}$-regular, and have pairwise coprime $K$-norms in $\mathbb{Z}$ by Lemma 2.9. For $W \in \mathbb{N}_+$, let $\mathcal{P}_W$ denote the first $W$ elements in $\mathcal{P}$ (in an arbitrary order).

By Lemma 2.9 the cardinality of each $\mathcal{P}_p$ is at most $D$. By the minimality of $N((a'))$ and Theorem 2.18 we have that

$$\lim_{W \to \infty} \mathcal{A}_{\mathcal{P}_W} \geq \frac{1}{D} \sum_{a \in O_K \text{ is a prime element}} \frac{1}{|N_K(a)|} = \infty,$$

where $\mathcal{A}_{\mathcal{P}_W}$ is defined in (7). So by (37), the assumption that $m=0$, and Lemma 2.21 there exist $N_0 := N_0(e, G, W := W(e, G) > 0, p, q \in \mathcal{P}_W$ with $|N_K(p)| \neq |N_K(q)|$, and $\epsilon_2 := \epsilon_2(e, G) > 0$ such that for all $N \geq N_0$,

$$\left| \mathbb{E}_{n \in \mathbb{N}_+, d} 1_{p,p,q}(n)\Phi(g(nA_{G}(p)) \cdot e_X) \cdot \overline{\Phi}(g(nA_{G}(q)) \cdot e_X) \right| \geq \epsilon_2,$$

where

$$P(p,q) := \{n \in \mathbb{Z}^D : nA_{G}(p), nA_{G}(q) \in \mathcal{P}\}.$$

In order to simplify the notations, from now on, we assume implicitly that all the quantities are dependent on $p, q$ and so on $W$. Since there are only finitely many pairs of such $p, q$, from now on we may consider $p, q$ as fixed.

### 7.3. Factorizing the polynomial sequence

Let

$$h_1(n) := g(nA_{G}(p)), h_2(n) := g(nA_{G}(q)) \text{ and } h(n) := (h_1(n), h_2(n)) \text{ for all } n \in \mathbb{Z}^D.$$  

Then $h \in \text{poly}_D((G \times G)_*)$. We now use Theorem 4.16 to convert $h(n)$ into a sequence which is totally equidistributed on a sub nilmanifold of $X \times X$. This step is again similar to the ones used in Theorem 6.1 of [18] and Proposition 7.9 of [43].

Let $\omega : \mathbb{N} \to \mathbb{R}_+$ be a function to be defined later. By Theorem 4.16 there exists a finite family $\mathcal{F}(M) := \mathcal{F}(X, D, M)$ of sub nilmanifolds of $X \times X$, which increases with $M$ and independent of $\omega$, a constant $M_1 := M_1(\chi, \omega, D) \in \mathbb{N}_+$, an integer $M^* \in \mathbb{N}$ with $M^* \leq M_1$, a closed subgroup $H$ of $G \times G$ rational for $\Gamma \times \Gamma$, a nilmanifold $Y := H/(H \cap (\Gamma \times \Gamma))$ belonging to $\mathcal{F}(M^*)$ with a nil-structure
\( \mathcal{Y} = (H_\bullet, \mathcal{Y}, \psi_Y, d_H, d_Y) \) induced by \( \mathcal{X} \times \mathcal{X} \), and a factorization \( h(n) = \epsilon(n)h'(n)\gamma(n), n \in R_{N,D} \) with \( \epsilon, g', \gamma \in \text{poly}_D(G_\bullet) \) such that
\begin{itemize}
  \item \( \epsilon: R_{N,D} \rightarrow G \times G \) is \( (M^*, N) \)-smooth;
  \item \( h' \in \text{poly}_D(H_\bullet) \) and \( (h'(n) \cdot e_Y)_{n \in R_{N,D}} \) is totally \( \omega(M^*) \)-equidistributed on \( Y \) with respect to \( \mathcal{Y} \);
  \item \( \gamma: R_{N,D} \rightarrow G \times G \) is \( M^* \)-rational for \( \Gamma \times \Gamma \) and \( \gamma(n) \cdot e_Y = \gamma(n + M^* \mathbf{e}_i) \cdot e_Y \) for all \( 1 \leq i \leq D \) and \( n, n + M^* \mathbf{e}_i \in R_{N,D} \).
\end{itemize}

We may rewrite (41) as

\[
\left| \sum_{n \in R_{N,D}} \Phi(n) \right| \geq \varepsilon_2.
\]

Our goal is to remove \( \epsilon(n) \) and \( \gamma(n) \) on the left hand side of (42). By Corollary B.3 of \cite{18}, there exists a finite subset \( \Sigma(M^*) \) of \( G \times G \), which consists of elements \( M^* \)-rational for \( \Gamma \times \Gamma \) such that every element in \( G \times G \) which is \( M^* \)-rational for \( \Gamma \times \Gamma \) can be written as \( a \gamma_0 \) for some \( a \in \Sigma(M^*) \) and \( \gamma_0 \in \Gamma \times \Gamma \). We may also assume that \( e_{G, G} \in \Sigma(M^*) \). For all \( a \in \Sigma(M^*) \), let \( H_a := a^{-1} H_a \), \( \Gamma_a := H_a \cap (\Gamma \times \Gamma) \) and \( Y_a := H_a \Gamma_a \). Lemma B.4 of \cite{18} implies that \( H_a \) is a subgroup of \( G \times G \) rational for \( \Gamma \times \Gamma \), and so \( Y_a \) is a sub-nilmanifold of \( X \times X \). Let \( \mathcal{Y}_a := ((H_a), \mathcal{Y}_a, \psi_Y, d_H, d_Y) \) be a nil-structure of \( Y_a \) induced by the \( a \)-conjugate from \( Y \). Then \( (H_a) \) is the filtration of \( H_a \) given by \( H_a^{(j)} = H_a \cap H_a^{(j)} \) for all \( j \in \mathbb{N} \). Since \( G^{(j)} \) is a normal subgroup of \( G \), we have that \( H_a^{(j)} = a^{-1} H_a^{(j)} \).

Let
\[
\mathcal{F}'(M^*) := \{ Y_a : Y \in \mathcal{F}(M^*), a \in \Sigma(M^*) \}.
\]

By Lemma \cite{3,8} and Corollary \cite{4,13} there exists a function \( C_1 := C_{1, \lambda} : \mathbb{N} \rightarrow \mathbb{R}_+ \) such that the following properties hold:

\begin{enumerate}
  \item [(P1)] For all \( a \in \Sigma(M^*) \) and \( g \in G \times G \) with \( d_G(g, e_{G, G}) \leq M^* \), we have \( d_G(a^{-1} g, g, e_{G, G}) \leq C_1(M^*)d_G(g, e_{G, G}) \);
  \item [(P2)] For all \( a \in \Sigma(M^*) \), \( g \in G \times G \) with \( d_G(g, e_{G, G}) \leq M^* \), and \( x, y \in X \times X \), we have that \( d_G(g, x, ga \cdot y) \leq C_1(M^*)d_G(x, y) \);
  \item [(P3)] As a result, for all \( a \in \Sigma(M^*) \), \( g \in G \times G \) with \( d_G(g, e_{G, G}) \leq M^* \), and \( f \in \text{Lip}(\mathcal{X} \times \mathcal{X}) \), denoting \( f_g(x) := f(g \cdot x) \) for all \( x \in X \times X \), we have that \( \| f_g \|_{\text{Lip}(\mathcal{X} \times \mathcal{X})} \leq C_1(M^*)\| f \|_{\text{Lip}(\mathcal{X} \times \mathcal{X})} \);
  \item [(P4)] For all \( Y_a \in \mathcal{F}'(M^*) \) and \( x, x' \in Y_a \), \( C_1(M^*)^{-1}d_{X \times X}(x, y) \leq d_{Y_a}(x, y) \leq C_1(M^*)d_{X \times X}(x, y) \);
  \item [(P5)] There exist a function \( \rho : \mathcal{X} \times \mathcal{Y}_a : \mathbb{N} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) with \( \lim_{t \rightarrow 0^+} \rho(M, t) = 0 \) for all \( M \in \mathbb{N} \) and \( N_1 := N_{1, \lambda} : \mathbb{N} \rightarrow \mathbb{N} \) such that for every \( Y = H/ (H \cap (\Gamma \times \Gamma)) \in \mathcal{F}(M^*) \), \( a \in \Sigma(M^*) \), \( t > 0 \), \( N \in \mathbb{N} \) with \( N \geq N_1(M^*) \) and \( f \in \text{poly}_D(H_\bullet) \), if \( (f(n) \cdot e_Y)_{n \in R_{N,D}} \) is totally \( t \)-equidistributed on \( Y \) with respect to \( \mathcal{Y} \), then \( a^{-1} f a \in \text{poly}_D((H_a) \bullet) \) and \( (a^{-1} f(n) a - e_{Y_a})_{n \in R_{N,D}} \) is totally \( \rho(M^*, t) \)-equidistributed on \( Y_a \) with respect to \( \mathcal{Y}_a \).
\end{enumerate}

We now return to (42). For convenience, for every subset \( R \subseteq \mathbb{Z}^D \), denote
\[
I(R) := \{ n \in \mathbb{Z}^D : nA_{\mathbb{Z}}(p), nA_{\mathbb{Z}}(q) \in R \}.
\]
Then \( P(p, q) = I(P) \). Set

\[
L := \left\lfloor \frac{\varepsilon_2 N}{20D^3(M^*)|K(pq)(M^*)^2} \right\rfloor \quad \text{and} \quad N_2(M^*) = N_2(\chi, \omega, K, \varepsilon, M^*) := \frac{20D_1^3(M^*)|K(pq)(M^*)^2}{\varepsilon_2}.
\]
From now on we assume that \( N \geq N_0 + N_1(M^*) + N_2(M^*) \). Then \( L \geq 1 \) and
\[
\frac{e_2 N}{40DC_1^2(M^*)|N_K(pq)|(M^*)^2} \leq L \leq \frac{e_2 N}{20DC_1^2(M^*)|N_K(pq)|(M^*)^2}.
\]
Since \( e_2 \leq 1 \) and \( C_1(M^*) \geq 1 \), \( M^*L \leq N \).

Let \( P_0 \) be a \( D \)-dimensional arithmetic progression in \( R_{N,D} \) of step \((|N_K(pq)| \cdot M^*, \ldots, |N_K(pq)| \cdot M^*) \) and length \((L_1, \ldots, L_D)\) for some \( L \leq L_i < 2L, 1 \leq i \leq D \). Then for all \( n, n' \in I(P_0) \), \( n - n' \in M^* \cdot Z^D \). So there exist \( a \in \Sigma(M^*) \) and \( n_0 \in \Gamma \times \Gamma \) such that for all \( n \in I(P_0) \), \( \gamma(n) \cdot e_{X \times X} = a \cdot \epsilon_{X \times X} \). Denote \( h'_n(n) := a^{-1}h(n)a, n \in Z^D \) and \((\Phi \otimes \overline{\Phi})_\alpha(x) := \Phi \otimes \overline{\Phi}((n_0)a \cdot x), x \in X \times X \) for some fixed \( n_0 \in I(P_0) \). For all \( n \in P_0 \), we have
\[
\Phi \otimes \overline{\Phi}(h(n) \cdot e_{X \times X}) = (\Phi \otimes \overline{\Phi})_\alpha(a^{-1}(n_0))^{-1}e(n)a h'_n(n) \cdot e_{X \times X}.
\]
Since \( \epsilon \) is \((M^*, N)\)-smooth,
\[
d_{G \times G}(e(n_0)^{-1}e(n), e_{G \times G}) \leq (2DL|N_K(pq)|M^*) \cdot \frac{M^*}{N} = 2DL|N_K(pq)|(M^*)^2 / N.
\]
By \( (P1) \),
\[
d_{G \times G}(a^{-1}e(n_0)^{-1}e(n)a, e_{G \times G}) \leq 2C_1(M^*)DL|N_K(pq)|(M^*)^2 / N.
\]
Since \( ||\Phi||_{Lip(X)} \leq 1 \), we have that \( ||\Phi \otimes \overline{\Phi}||_{Lip(X \times X)} \leq 2 \). By \( (P3) \), \( ||\Phi \otimes \overline{\Phi}||_{Lip(X \times X)} \leq 2C_1(M^*) \).

By \( (P4) \),
\[
||(\Phi \otimes \overline{\Phi})_\alpha||_{Lip(X)} \leq 2C_1^2(M^*).
\]

Since \( P_0 \) is of length at most \( 2L \), \( I(P_0) \) is of cardinality at most \( (2L)^D \). So
\[
\left| \mathbb{E}_{n \in R_{N,D}} 1_{R(P_0)}(n) 1_{R(P_1)}(n) \Phi \otimes \overline{\Phi}(h(n) \cdot e_{X \times X}) - \mathbb{E}_{n \in R_{N,D}} 1_{R(P_0)}(n) 1_{R(P_1)}(n) (\Phi \otimes \overline{\Phi})_\alpha(h'_n(n) \cdot e_{Y_a}) \right| \leq \frac{(2L)^D}{(2N + 1)^D} \cdot 2C_1^2(M^*) \cdot \frac{2C_1(M^*)DL|N_K(pq)|(M^*)^2}{N} \leq 4DC_1^2(M^*)|N_K(pq)|(M^*)^2 \frac{L}{N}^{D+1} \leq e_2 \frac{L}{N}^D.
\]

Since \( N \geq N_2(M^*) \), \( L \geq 1 \) and \( (44) \) holds. Since \( M^*L \leq N \), we may partition \( R_{N,D} \) into \( \mathcal{D} \)-dimensional arithmetic progressions \( R_{N,D} = \bigcup_i P'_i \) of step \((|N_K(pq)| \cdot M^*, \ldots, |N_K(pq)| \cdot M^*) \) and length between \( L \) and \( 2L \) in each of the \( \mathcal{D} \) directions. The number of these progressions is bounded above by \((N/L^D)\). Note that
\[
1_{R(P)}(n) = 1_{R(P)}(n) 1_{R_{N,D}}(n) = \sum_i 1_{R(P)}(n) 1_{R(P'_i)}(n) = 1_{R(P'_i \cap P)}(n).
\]

It follows from \( (42) \) that there exist one of them \( P'_i \) such that
\[
\left| \mathbb{E}_{n \in R_{N,D}} 1_{R(P'_i \cap P)}(n) \Phi \otimes \overline{\Phi}(h(n) \cdot e_{X \times X}) \right| \geq e_2 \frac{L}{N}^D.
\]

We deduce from \( (46) \) that for some \( a \in \Sigma(M^*) \),
\[
\left| \mathbb{E}_{n \in R_{N,D}} 1_{R(P'_i \cap P)}(n) (\Phi \otimes \overline{\Phi})_\alpha(h'_n(n) \cdot e_{Y_a}) \right| \geq e_2 \frac{L}{N}^D - e_2 \frac{L}{N}^D \leq e_2 \frac{L}{N}^D \geq e_3(M^*) \geq e_3(M^*) \geq e_3(M^*) = \frac{\epsilon_3}{40DC_1^2(M^*)|N_K(pq)|(M^*)^2} \frac{e_2}{2} \frac{L}{N}^D \text{ with the last inequality coming from } (44).
\]
It is easy to see that for every line $\ell \subseteq \mathbb{R}^D$, the set $I(P_1 \cap P) \cap \ell$ is a 1-dimensional arithmetic progression. By (45) and (47) and Proposition 4.11 there exist $\epsilon_4(M^*) := \epsilon_{4, \ell}(M^*)$ and $N_3(\omega, M^*) := N_3,\ell,\chi(\omega, M^*) > N_0 + N_1(M^*) + N_2(M^*)$ such that for all $N > N_3(\omega, M^*)$, \(\int_{Y_a} \Phi \otimes \overline{\Phi}|_{Y_a} \ d\mu_{Y_a} = 0\) implies that
\[(48) \quad (h'_a(n) \cdot e_{Y_a})_{n \in \mathbb{N}} \text{ is not totally } \epsilon_4(M^*)\text{-equidistributed on } Y_a \text{ with respect to } \mathcal{U}_a.\]

Moreover, $\epsilon_4 : \mathbb{N} \to \mathbb{R}_+$ as a function of $M^*$ is independent of the choice of the function $\omega$.

On the other hand, $\epsilon_4 : \mathbb{N} \to \mathbb{R}_+$ as a function of $M^*$ is independent of the choice of the function $\omega$.

By (45) and (47) and Proposition 4.11, there exist $\epsilon_4(M^*) := \epsilon_{4, \ell}(M^*)$ and $N_3(\omega, M^*) := N_3,\ell,\chi(\omega, M^*) > N_0 + N_1(M^*) + N_2(M^*)$ such that for all $N > N_3(\omega, M^*)$, \(\int_{Y_a} \Phi \otimes \overline{\Phi}|_{Y_a} \ d\mu_{Y_a} = 0\) implies that
\[(48) \quad (h'_a(n) \cdot e_{Y_a})_{n \in \mathbb{N}} \text{ is not totally } \epsilon_4(M^*)\text{-equidistributed on } Y_a \text{ with respect to } \mathcal{U}_a.\]

Moreover, $\epsilon_4 : \mathbb{N} \to \mathbb{R}_+$ as a function of $M^*$ is independent of the choice of the function $\omega$.

On the other hand, $\epsilon_4 : \mathbb{N} \to \mathbb{R}_+$ as a function of $M^*$ is independent of the choice of the function $\omega$.

By (45) and (47) and Proposition 4.11, there exist $\epsilon_4(M^*) := \epsilon_{4, \ell}(M^*)$ and $N_3(\omega, M^*) := N_3,\ell,\chi(\omega, M^*) > N_0 + N_1(M^*) + N_2(M^*)$ such that for all $N > N_3(\omega, M^*)$, \(\int_{Y_a} \Phi \otimes \overline{\Phi}|_{Y_a} \ d\mu_{Y_a} = 0\) implies that
\[(48) \quad (h'_a(n) \cdot e_{Y_a})_{n \in \mathbb{N}} \text{ is not totally } \epsilon_4(M^*)\text{-equidistributed on } Y_a \text{ with respect to } \mathcal{U}_a.\]

We are now ready to state the restriction of the function $\omega$: we pick $\omega$ to be any function such that (49) holds for $\zeta(M^*) = \epsilon_4(M^*)$ (recall that $\epsilon_4 : \mathbb{N} \to \mathbb{R}_+$ as a function of $M^*$ is independent of the choice of $\omega$). Then for every $a \in \Sigma(M^*)$ and $N \geq N_3(\omega, M^*)$,
\[(50) \quad (h'_a(n) \cdot e_{Y_a})_{n \in \mathbb{N}} \text{ is totally } \epsilon_4(M^*)\text{-equidistributed on } Y_a \text{ with respect to } \mathcal{U}_a.\]

Combining (48) and (50), we have that
\[(51) \quad \int_{Y_a} \Phi \otimes \overline{\Phi}|_{Y_a} \ d\mu_{Y_a} \neq 0.\]

### 7.4. Invoking the key ingredients

Denote $h'_a(n) = (h'_{a,1}(n), h'_{a,2}(n))$, $h = (h_1(n), h_2(n))$, $\epsilon(n) = (\epsilon_1(n), \epsilon_2(n))$ and $\gamma(n) = (\gamma_1(n), \gamma_2(n))$. We are now ready to use the results from Sections 5 and 6 to finish the proof of Theorem 7.1.

Recall that $\Phi$ is a nilcharacter of $X$ with frequency $\ell \neq 0$ with respect to $\mathfrak{x}$ by Assumption 2. By Lemma 4.16 $\Phi \otimes \overline{\Phi}$ is a nilcharacter of $X \times X$ with frequency $(\ell, -\ell)$ with respect to $\mathfrak{x} \times \mathfrak{x}$, and so is $(\Phi \otimes \overline{\Phi})_{\mathfrak{y}}$.

Since $H_a < G \times G$, $(H_a)_i < (G \times G)_i = G_i \times G_i$ for all $i \in \mathbb{N}_+$. So $(H_a)_{\mathfrak{g}} < G_{\mathfrak{g}} \times G_{\mathfrak{g}}$. Since $\dim(G_{\mathfrak{g}}) = 1$, $\dim((H_a)_{\mathfrak{g}}) = 0, 1$ or 2.

**Case that** $\dim((H_a)_{\mathfrak{g}}) = 0$. Then the projection of $Y_a$ to the first coordinate is not $X$. By the choice of $Y_a$, there exist $C_4(M^*) := C_{4,\mathfrak{x},\mathfrak{g}}(M^*) > 0$ and horizontal character $\eta$ of $X$ such that $0 < \|\eta\|_{L^2} \leq C_4(M^*)$ and $\eta \circ h'_{a,1} = \eta \circ h'_a \equiv 0$. Since $\gamma$ takes value in the finite set $\Sigma(M^*)$, there exists $Q := Q(M^*)$ such that $\eta \circ \gamma \equiv 0$.

Since $\epsilon_1(n)$ is $(M^*, N)$-smooth, $\epsilon_1(Qn)$ is $(Q^D M^*, N)$-smooth. By definition,\[\|\eta^O \circ (\epsilon_1 h'_{a,1})\|_{C^\gamma(R_{\mathfrak{y}, D})} \leq C_5(M^*)\]
for some $C_5(M^*) := C_{5,\mathfrak{x},\mathfrak{g}}(M^*) > 0$ for all $N \geq N_4(M^*) := N_{4,\mathfrak{x},\mathfrak{g}}(M^*) > N_5(M^*)$. So we have that \[\|\eta^O \circ g(nA_{\mathfrak{g}}(p))\|_{C^\gamma(R_{\mathfrak{y}, D})} = \|\eta^O \circ (\epsilon_1 h'_{a,1} \gamma_1)\|_{C^\gamma(R_{\mathfrak{y}, D})} \leq C_5(M^*).\]

By Theorem 4.10 there exist $C_6(M^*) := C_{6,\mathfrak{x},\mathfrak{g}}(M^*)$ and $N_5(\omega, M^*) := N_{5,\mathfrak{x},\mathfrak{g}}(M^*) > N_4(M^*)$, such that for all $N \geq N_5(M^*)$, $(g(nA_{\mathfrak{g}}(p)) \cdot e_{Y_a})_{n \in \mathbb{N}}$ is not totally $C_6(M^*)$-equidistributed on $X$ with respect to $\mathfrak{x}$. By Proposition 4.11 there exist $\delta := \max_{M^* \leq M_1} \delta_{C_6(M^*)} > 0$ and $N_6 := \max_{M^* \leq M_2} N_6(M^*)$ such that for all $N \geq N_6(M^*)$, $(g(nA_{\mathfrak{g}}(p)) \cdot e_{Y_a})_{n \in \mathbb{N}}$ is the totally $C_6(M^*)$-equidistributed on $X$ with respect to $\mathfrak{x}$. By Proposition 4.11 there exist $\delta := \max_{M^* \leq M_1} \delta_{C_6(M^*)} > 0$ and $N_6 := \max_{M^* \leq M_2} N_6(M^*)$ such that for all $N \geq N_6(M^*)$, $(g(nA_{\mathfrak{g}}(p)) \cdot e_{Y_a})_{n \in \mathbb{N}}$ is the totally $C_6(M^*)$-equidistributed on $X$ with respect to $\mathfrak{x}$.
max_{M \leq M_1} N_{6, C_8}(M') > N_5(M')$, such that for all $N \geq N_6$, $(g(n) \cdot e_X)_{n \in R_{N,D}}$ is not totally $\delta$-equidistributed on $X$ with respect to $\mathfrak{x}$. This finishes the proof.

**Case that** $\dim((H_d)_{a}) = 2$. In this case, $(H_d)_{a} = G_d \times G_d$. Since $\Phi \otimes \overline{\Phi}$ is with frequency $(\ell, -\ell)$ on $X \times X$ with respect to $\mathfrak{x} \times \mathfrak{x}$, by Lemma 3.17, $(\Phi \otimes \overline{\Phi})_{(a)}$ is also with frequency $(\ell, -\ell)$ on $Y$ with respect to $\mathfrak{y}$. So $\int_{Y_{a}} \Phi \otimes \overline{\Phi}_{Y_{a}} d\mu_{Y_{a}} = 0$, a contradiction to (51).

**Case that** $\dim((H_d)_{a}) = 1$. Since $X$ is standard, in this case

$$(H_d)_{a} = [(\psi^{-1}(0, \ldots, 0; t), \psi^{-1}(0, \ldots, 0; t_2)) \in G^{(d)} \times G^{(d)} : t \in \mathbb{R}]$$

for some $t_1, t_2 \in \mathbb{Z}$ not all equal to 0. If $t_1 \neq t_2$, then by Lemma 3.17, $(\Phi \otimes \overline{\Phi})_{(a)}$ is also with frequency $(\ell, -\ell)$ on $Y$ with respect to $\mathfrak{y}$. So $\int_{Y_{a}} \Phi \otimes \overline{\Phi}_{Y_{a}} d\mu_{Y_{a}} = 0$, a contradiction to (51).

So we must have that

$$\int_{Y_{a}} \Phi \otimes \overline{\Phi}_{Y_{a}} d\mu_{Y_{a}} = 0$$

for some $t \in \mathbb{R}$. This finish the proof.

If the projection of $Y_{a}$ to one of the two coordinates is not $X$, we are done by the same argument as in the case that $\dim((H_d)_{a}) = 0$. So we may assume that the projection of $Y_{a}$ to both coordinates are $X$. Since $\mathcal{F}^{\mathcal{F}}(M')$ is a finite set, by Theorem 5.5, there exists $\sigma \in \text{Aut}(G)$ of height at most $C_7(M') := C_{8, X, D, 1}(M') > 0$ such that $h_1 = \sigma(h_2) \mod \ker$ for all $(h_1, h_2) \in H$. Then $h'_{a}(n) = \sigma \circ h'_{a}(2) \mod \ker$ and so $h'_{a}(n) = \sigma \circ h'_{a}(n) \mod \ker$ for all $n \in R_{N,D}$.

Since $\gamma$ takes value in the finite set $\Sigma(M')$, there exists $Q(M') \in \mathbb{N}_+$ such that

$$\gamma(Q(M') \cdot n) \in \Gamma \times \Gamma$$

for all $n \in \mathbb{Z}^D$. Let $Q = \prod_{M' \leq M_1} Q_{M'}$. Since $e_1(n)$ and $e_2(n)$ are $(M', N)$-smooth and $\sigma$ is of height at most $C_7(M')$ with respect to $X$, $e_1(Qn)$ and $\sigma \circ e_2(Qn)$ are $(C_8(M'), N)$-smooth for some $C_8(M') := C_{8, X, D, 1, 2}(M') > 0$. By definition, for all $N \geq N_7(M') := N_{7, X, D, C_7, Q}(M') > N_3(M')$,

$$||e_1(Qn)||_{C_{\ker}^{(R_{N,D})}} \leq ||e_1(Qn)||_{C_{\ker}^{(R_{N,D})}} \leq C_9(M')$$

for some $C_9(M') := C_{9, X, D, \epsilon}(M') > 0$. Since $h'_{a} = \sigma \circ h'_{a} \mod \ker$,

$$||g(QnA_2(p)) \cdot (\sigma \circ g(QnA_2(q)))^{-1}||_{C_{\ker}^{(R_{N,D})}} = ||e_1(h'_{a}(Qn) \cdot (\sigma \circ (e_2(h'_{a}(Qn)))^{-1})||_{C_{\ker}^{(R_{N,D})}}$$

and

$$||e_1(h'_{a}(Qn) \cdot (\sigma \circ (e_2(h'_{a}(Qn)))^{-1}||_{C_{\ker}^{(R_{N,D})}} = ||e_1(Qn) \cdot (\sigma \circ (e_2(Qn)))^{-1}||_{C_{\ker}^{(R_{N,D})}} \leq 2C_9(M').$$

Since $|N_{K}(p) \neq |N_{K}(q)|$ and $M' \leq M_1$, by Theorem 6.1, there exist

$$\delta' := \delta'(X, D, \epsilon) := \max_{M' \leq M_1} \delta_{X, D, C_8}(M') > 0$$

and

$$N'_{8} := N'_{8}(X, D, \epsilon) := \max_{M' \leq M_1} N_{8, X, D, C_8}(M') > \max_{M' \leq M_1} N_{7}(M')$$

such that for every $N \geq N'_{8}$, $(g(Qn) \cdot e_{X})_{n \in R_{N,D}}$ and thus $(g(n) \cdot e_{X})_{n \in R_{N,D}}$ is not totally $\delta'$-equidistributed on $X$ with respect to $\mathfrak{x}$. Since $X$ depends only on $X_0$ and $w$, and one can verify that all other quantities in the proof depend eventually only on $X_0$, $w$, and $D$. This (finally!) finishes the proof.

8. **Consequences of Theorem 7.1**

In this section, we deduce Theorems 1.4, 1.5 and 1.12 by using Theorem 7.1.
8.1. Proofs of Theorems [1.4] and [1.5]

Proof of Theorem [7.3] Denote \( g(n_1, \ldots, n_d) := T_{n_1}^{a_1} \cdots T_{n_d}^{a_d} \) and let \( \bar{x} = (G_\bullet, X, \psi, dG, dX) \) be a nil-structure on \( X \). If the conclusion of Theorem [1.5] does not hold, then there exist a function \( \Phi \in C(X) \) with \( \int_X \Phi \, dm_X = 0 \), a \( D \)-dimensional arithmetic progression \( P, \epsilon > 0 \) and an infinite set \( J \subseteq \mathbb{N} \) such that for all \( N \in J \), there exists \( X \in \mathcal{M}_K \) such that

\[
\left| \mathbb{E}_{n \in \mathbb{N}, d, \Phi} \mathbf{l}_P(n) \chi(t_g(n)) \Phi(g(n) \cdot e_X) \right| \geq \epsilon.
\]

By Theorem [7.1], there exist \( \delta > 0 \) and \( N_0 \in \mathbb{N} \) such that the sequence \( \langle g(n) \cdot e_X \rangle_{n \in \mathbb{N}} \) is not \( \delta \)-equidistributed on \( X \) with respect to \( \bar{x} \) for all \( N > N_0, N \in J \). By Theorem [4.9], there exist \( C > 0 \) independent of \( N \) and a horizontal character \( \eta \) such that

\[
0 < ||\eta||_X \leq C \quad \text{and} \quad ||\eta \circ g||_{C^\infty(X)} \leq C.
\]

Since there are only finitely many \( \eta \) with \( 0 < ||\eta||_X \leq C \), there exist an infinite set \( J' \subseteq J \) and a horizontal character \( \eta_0 \) such that for all \( N \in J', ||\eta_0 \circ g||_{C^\infty(X)} \leq C \). Letting \( N \to \infty \), we have that \( \eta_0 \circ g \equiv 0 \). So

\[
\lim_{N \to \infty} \mathbb{E}_{n \in \mathbb{N}, d} e(\eta_0(g(n) \cdot e_X)) = 1 \neq 0.
\]

Since \( ||\eta_0||_X > 0 \), we have that \( \int_X e(\eta_0) \, dm = 0 \). So \( \langle g(n) \cdot e_X \rangle_{n \in \mathbb{N}} \) is not equidistributed on \( X \), a contradiction. This finishes the proof. \( \Box \)

For Theorem [1.4], we prove the following stronger version:

**Theorem 8.1** (Quantitative version of Theorem [1.4]). Let \( K = (K, O_K, D, B = \{b_1, \ldots, b_D\}) \) be an integral tuple and \( X = G/\Gamma \) be a nil-manifold with a nil-structure \( \bar{x} \). For every \( w, \epsilon > 0 \), there exist \( \delta' := \delta'(\bar{x}, w, B, \epsilon) > 0 \) and \( N_0 := N_0(\bar{x}, w, B, \epsilon) \in \mathbb{N} \) such that for every \( N \geq N_0 \), the following holds: if there exist \( g \in \text{poly}_D(G) \) of degree at most \( m \), \( m \in \mathbb{Z}^D, \chi \in \mathcal{M}_K \), \( \Phi: X \to \mathbb{C} \) such that \( ||\Phi|| \leq 1, ||\Phi||_{\text{Lip}(X)} \leq 1 \), and a \( D \)-dimensional arithmetic progression \( P \) such that

\[
\left| \mathbb{E}_{n \in \mathbb{N}, d} \mathbf{l}_P(n) \chi \circ t_g(n) \Phi(g(n + m) \cdot e_X) \right| \geq \epsilon,
\]

then there exists a \( D \)-dimensional arithmetic progression \( P' \) such that

\[
\left| \mathbb{E}_{n \in \mathbb{N}, d} \mathbf{l}_{P'}(n) \chi \circ t_g(n) \right| \geq \delta'.
\]

**Proof:** We assume implicitly that all the quantities in the proof depend on \( \bar{x}, B \) and \( \epsilon \). Similar to the deduction of Assumption 2 in Section [7.1], we may assume that \( \bar{x} = (G_\bullet, X, \psi, dG, dX) \) and \( g \in \text{poly}_D(G_\bullet) \).

Suppose that (52) holds for some choice of the parameters. By using the factorization theorem (Theorem [3.16]), we may deduce from (52) that there exist

- A function \( \lambda := \lambda_{\bar{x}, B}: \mathbb{N} \to \mathbb{R}_+ \);
- For all \( M \in \mathbb{N} \) a finite subset \( \Sigma(M) \subseteq G \) and a finite collection \( \mathcal{F}(M) \) of sub nilmanifolds \( X' = G'/\Gamma' \) of \( X \) with nil-structures \( X' \) induced by \( \bar{x} \);
- For each \( a \in \Sigma(M) \), a sub nilmanifold \( X'_a = G'_a/\Gamma'_a \) of \( X \) with a nil-structure \( X'_a \) adapted to the filtration \( G'_a \) induced by the \( a \)-conjugate of \( \bar{x} \), a polynomial sequence \( g'_a \in \text{poly}_D(G'_a) \) and a function \( \Phi'_a: X'_a \to \mathbb{C} \) with \( ||\Phi'_a|| \leq 1 \) and \( ||\Phi'_a||_{\text{Lip}(X'_a)} = 1 \),
such that for every function \( \zeta : \mathbb{N} \to \mathbb{R}_+ \), there exist a function \( N_0 : \mathbb{N} \to \mathbb{R}_+ \) and \( M_1 \in \mathbb{N}_+ \) such that the following holds:

**Property 1.** There exist \( M^* \in \mathbb{N} \) with \( M^* \leq M_1 \), such that if (52) holds for some \( N \geq N_0(M^*) \), then exist a \( D \)-dimensional arithmetic progression \( P' \) of \( \mathbb{Z}^D \) an element \( a \in \Sigma(M^*) \) such that

\[
\left| \mathbb{E}_{n \in \mathbb{N}_+} \mathbf{1}_{P'}(n) \chi \circ \iota_B(n) \cdot \Phi'(g'_a(n) \cdot e_X) \right| > 2\lambda(M^*).
\]

**Property 2.** For all \( M^* \in \mathbb{N} \), \( a \in \Sigma(M^*) \) and \( N \geq N_0(M^*) \),

\[(g'_a(n) \cdot e_X)'_{a \in \mathbb{N}_+} \text{ is totally } \zeta(M^*)\text{-equidistributed on } X'_a \text{ with respect to } X'_a.\]

We left the choice of \( \zeta \) to the end of the proof (note that the choice of \( \zeta \) must be dependent only on \( \lambda, \epsilon, w, x, B \)). The method we deduce Properties 1 and 2 from (52) is similar to the one we used to deduce (49) and (47) from (42), or the one used in [18] to deduce (8.11) and (8.12) from (8.9), or the one used in [45] to deduce (28) and the equidistribution condition right after (28) from the last inequality at the end of page 101 (or at the end page of 41 for the arXiv version). So for conciseness, we omit the proof of Properties 1 and 2 and leave them to the interested readers.

Assume that \( N \geq \max_{M \leq M_1} N_0(M) \). Let \( z := \mathbb{E}_{X'_a} \Phi'_a \cdot dm_{X'_a} \) and \( \Phi'_a := \Phi_a - z \). Then \( \int_{X'_a} \Phi'_a \cdot dm_{X'_a} = 0 \). By Theorem 7.1 and Property 2, if we choose \( \zeta \) to be the function \( \zeta(M) := \delta(\lambda, w, B, \lambda(M)) \), where \( \delta(\lambda, w, B, \lambda(M)) \) is defined in Theorem 7.1, then

\[
\left| \mathbb{E}_{n \in \mathbb{N}_+} \mathbf{1}_{P'}(n) \chi \circ \iota_B(n) \cdot \Phi'(g'_a(n) \cdot e_X) \right| < \lambda(M^*).
\]

Since \( |z| \leq 1 \), by Property 1,

\[
\left| \mathbb{E}_{n \in \mathbb{N}_+} \mathbf{1}_{P'}(n) \chi \circ \iota_B(n) \right| \geq \left| \mathbb{E}_{n \in \mathbb{N}_+} \mathbf{1}_{P'}(n) \chi \circ \iota_B(n) \cdot z \right| \geq \lambda(M^*),
\]

which finishes the proof by setting \( \epsilon' := \min_{M' \leq M_1} \lambda(M^*) \). \( \square \)

### 8.2. Properties about the Gowers norms.

We introduce some basic properties about the Gowers norms before proving Theorem 1.12. We start with the definitions of the convolution product and the Fourier transformation on \( \mathbb{Z}_N^D \).

**Definition 8.2** (Convolution product). Let \( N \in \mathbb{N}_+ \). The convolution product of two functions \( f, g : \mathbb{Z}_N^D \to \mathbb{C} \) is defined by

\[
f \ast g(n) := \mathbb{E}_{m \in \mathbb{Z}_N^D} f(n - m) g(m).
\]

**Definition 8.3** (Fourier transformation). For every \( n = (n_1, \ldots, n_D) \) and \( m = (m_1, \ldots, m_D) \in \mathbb{Z}_N^D \), write

\[n \circ_N m := \frac{1}{N}(n \cdot m) = \frac{1}{N}(n_1 m_1 + \cdots + n_D m_D).\]

For every function \( f : \mathbb{Z}_N^D \to \mathbb{C} \), let \( \hat{f} : \mathbb{Z}_N^D \to \mathbb{C} \) denote the Fourier transformation of \( f \) given by

\[\hat{f}(\xi) := \mathbb{E}_{n \in \mathbb{Z}_N^D} f(n) \cdot e(-n \circ_N \xi)\]

for all \( \xi \in \mathbb{Z}_N^D \) (recall that \( e(x) := \exp(2\pi i x) \) for all \( x \in \mathbb{R} \)).

A direct computation shows that for any function \( f \) on \( \mathbb{Z}_N^D \), we have

\[
\|f\|_{L^2(\mathbb{Z}_N^D)}^2 = \sum_{\xi \in \mathbb{Z}_N^D} |\hat{f}(\xi)|^4.
\]
Lemma A.4 of [18]:

If \( \xi \) then there exist a function that generalizes Lemma A.6 in [18]:

\[
\left| E_{n \in \mathbb{Z}_N^D} [f(n)] \right| \leq C \| f \|_{U^2(\mathbb{Z}_N^D)}
\]

Proof. Since \( N \) is a prime integer, the norm \( \| f \|_{U^2(\mathbb{Z}_N^D)} \) is invariant under any change of variables of the form \( n = (n_1, \ldots, n_D) \rightarrow (c_1n_1 + m_1, \ldots, c_Dn_D + m_D) \) for any \( m_i, c_i \in \mathbb{Z} \) such that \( (c_i, N) = 1 \) for all \( 1 \leq i \leq D \). So we may assume without loss of generality that \( P = [d_1] \times \ldots \times [d_D] \) for some \( 1 \leq d_1, \ldots, d_D \leq N \). A direct computation shows that

\[
\left| \hat{f}(\xi_1, \ldots, \xi_D) \right| \leq \frac{2^D}{ND} \prod_{i=1}^D \left| \frac{\xi_i}{N} \right| = \frac{2^D}{\prod_{i=1}^D \min[\xi_i, N - \xi_i]}
\]

for all \( (\xi_1, \ldots, \xi_D) \in \mathbb{Z}_N^D \). Thus

\[
\left\| \hat{f}(\xi_1, \ldots, \xi_D) \right\|_{e^2(\mathbb{Z}_N^D)} \leq C
\]

for some \( C := C(D) > 0 \). Then by Parseval’s identity, Hölder’s inequality, and identity (53), we deduce that

\[
\left| E_{n \in \mathbb{Z}_N^D} \hat{f}(n) \right| = \left| \sum_{\xi \in \mathbb{Z}_N^D} \hat{f}(\xi) \right| \leq C \left( \sum_{\xi \in \mathbb{Z}_N^D} |\hat{f}(\xi)|^4 \right)^{1/4} \leq C \| f \|_{U^2(\mathbb{Z}_N^D)}.
\]

The following inverse theorem can be deduced from Theorem 11 of [45] (or from [46]) and Lemma A.4 of [18]:

Theorem 8.5 (The inverse theorem for \( \mathbb{Z}^D \) actions). For every \( \epsilon > 0, d \geq 2 \) and \( D \in \mathbb{N}_+ \), there exist \( \delta := \delta(d, D, \epsilon) > 0, N_0 := N_0(d, D, \epsilon) \in \mathbb{N} \) and a nilmanifold \( X := X(d, D, \epsilon) \) with a nil-structure \( \hat{X} := \hat{X}(d, D, \epsilon) = (G_\ast, X, \psi, d_G, d_X) \) such that for every \( N \geq N_0 \) and every \( f : \mathbb{Z}_N^D \rightarrow \mathbb{C} \) with \( |f| \leq 1 \), if

\[
\| f \|_{U^d(\mathbb{Z}_N^D)} \geq \epsilon,
\]

then there exist a function \( \Phi : X \rightarrow \mathbb{C} \) with \( \| \Phi \|_{\text{Lip}(X)} \leq 1 \) and a polynomial sequence \( g \in \text{poly}_D(G_\ast) \) such that

\[
\left| E_{n \in \mathbb{N}^P} f(n) \cdot \Phi(g(n) \cdot e_X) \right| \geq \delta,
\]

where we regard \( f \) as a function from \( \mathbb{Z}^D \) to \( \mathbb{C} \) supported on \([N]^D \) in the obvious way.

We are now ready to prove Theorem 1.12.

Proof of Theorem 1.12. Suppose first that \( \lim_{N \rightarrow \infty} \| \hat{X} \circ t_B \|_{U^d([N]^P)} \neq 0 \) for some \( d \geq 2 \). Then by definition, there exist \( \epsilon > 0 \) and an infinite set \( J \subseteq \mathbb{N} \) such that \( \| \hat{1}_{[N]^P} \circ \hat{X} \circ t_B \|_{U^d([N]^P)} > \epsilon \) for all \( N \in J \). By Theorems 8.5 there exist \( \epsilon' > 0, N_0 \in \mathbb{N} \) and a nilmanifold \( X = G/\Gamma \) such that for every \( N \geq N_0, N \in J \), there exist a function \( \Phi : X \rightarrow \mathbb{C} \) with \( \| \Phi \|_{\text{Lip}(X)} \leq 1 \) and a polynomial sequence \( g \in \text{poly}_D(G) \) such that

\[
\left| E_{n \in \mathbb{N}^P} [f(n) \circ t_B(n) \cdot \Phi(g(n) \cdot e_X)] \right| \geq \epsilon'.
\]
By Theorem 8.1, there exist $\delta > 0, N_1 \geq N_0$ and a $D$-dimensional arithmetic progression $P'$ such that for all $N \geq N_1, N \in J$,
\[
\left| \sum_{n \in R_{N,D}} 1_{P'}(n) \chi \circ \nu_B(n) \right| \geq \delta.
\]
By definition, $\chi$ is not aperiodic.

Conversely, suppose that $\lim_{N \to \infty} \|\chi \circ \nu_B(\mathbb{Z}^D_N)\| = 0$. Let $N^*$ denote the smallest prime number greater than $2N$. Then $N^* \leq 4N$. Similar to Lemma A.3 of [18], $\lim \inf_{N \to \infty} \|1_{[N]}\|_{U^2(\mathbb{Z}^D_{N^*})}$ is bounded below by a positive constant depending only on $D$. This implies that
\[
\lim_{N \to \infty} \|1_{[N]} \circ \chi \circ \nu_B\|_{U^2(\mathbb{Z}^D_{N^*})} = 0.
\]

By Lemma 8.1, we have
\[
\lim \sup \sup_{N \to \infty} \sum_{n \in R_{N,D}} 1_{P}(n) \chi \circ \nu_B(n) \leq 4^D \lim \sup \sup_{N \to \infty} \sum_{n \in R_{N,D}} 1_{P}(n) 1_{[N]}(n) \chi \circ \nu_B(n) \leq 4^D \lim \sup_{N \to \infty} \|1_{[N]} \circ \chi \circ \nu_B\|_{U^2(\mathbb{Z}^D_{N^*})} = 0,
\]
which implies that $\chi$ is aperiodic. \qed

9. Structure theorem for multiplicative functions

We prove Theorem 1.10 and its stronger form Theorem 9.2 in this section. The approach we use is similar to the ones used in [18, 43].

9.1. Strong $U^d$ structure theorem.

**Definition 9.1 (Kernel).** A function $\phi: \mathbb{Z}_N^D \to \mathbb{C}$ is a kernel of $\mathbb{Z}_N^D$ if it is non-negative and $\mathbb{E}_{n \in \mathbb{Z}_N^D} \phi(n) = 1$. The set $\{\xi \in \mathbb{Z}_N^D: \hat{\phi}(\xi) \neq 0\}$ is called the spectrum of $\phi$.

In order to show Theorem 1.10, it suffices to show the following stronger theorem, which generalizes the main structure theorems in [18, 43].

**Theorem 9.2 (Strong $U^d$ structure theorem for multiplicative functions).** Let $\Omega \in \mathbb{N}$. For $N \in \mathbb{N}$, let $\hat{N}$ denote the smallest prime integer greater than $\Omega N$. Let $K = (K, \mathcal{O}_K, D, B = \{b_1, \ldots, b_D\})$ be an integral tuple and $\nu$ be a probability measure on the group $M_K$. Let $F: \mathbb{N} \times \mathbb{N} \times \mathbb{R}^+ \to \mathbb{R}^+$ be a function. For every $\varepsilon > 0$ and $d \geq 2$, there exist $Q = Q(D, d, F, \varepsilon, \Omega), R = R(D, d, F, \varepsilon, \Omega), N_0 = N_0(D, d, F, \varepsilon, \Omega) \in \mathbb{N}_+$ such that for every $N \geq N_0$ and $\chi \in M_K$, the truncated function $\chi_N: \mathbb{Z}_N^D \to \mathbb{C}$ can be written as
\[
\chi_N(n) = \chi_{N,s}(n) + \chi_{N,u}(n) + \chi_{N,e}(n)
\]
for all $n \in \mathbb{Z}_N^D$ such that the following holds:

(i) $|\chi_{N,s}| \leq 1$, $\chi_{N,s} = \chi_N \ast \phi_{N,1}$ and $\chi_{N,s} + \chi_{N,e} = \chi_N \ast \phi_{N,2}$, where $\phi_{N,1}$ and $\phi_{N,2}$ are kernels of $\mathbb{Z}_N^D$ that are independent of $\chi$, and the convolution product is defined on $\mathbb{Z}_N^D$;

(ii) $|\chi_{N,s}(n + Qe_i) - \chi_{N,s}(n)| \leq \frac{\varepsilon}{2}$ for every $n \in \mathbb{Z}_N^D$ and $1 \leq i \leq D$;

(iii) For every $\xi = (\xi_1, \ldots, \xi_D) \in \mathbb{Z}_N^D$ such that $\hat{\chi}(\xi) \neq 0$ and every $1 \leq i \leq D$, there exists $p_i \in \{0, \ldots, Q - 1\}$ such that $|\xi_N - \xi_N^i| \leq \frac{\varepsilon}{2}$;

(iv) $\|\chi_{N,u}\|_{U^d(\mathbb{Z}_N^D)} \leq \frac{1}{f(Q, R, \varepsilon)}$;

(v) $\mathbb{E}_{n \in \mathbb{Z}_N^D} |\chi_{N,e}(n)| d\nu(\chi) \leq \varepsilon$. 

9.2. Weak $U^2$ structure theorem. Our first step is to prove a weak $U^2$ structure theorem:

**Theorem 9.3** (Weak $U^2$ structure theorem for multiplicative functions). Let $\Omega \in \mathbb{N}$. For $N \in \mathbb{N}$, let $\hat{N}$ denote the smallest prime integer greater than $\Omega N$. Let $K = (K, O_K, D, \mathcal{B} = \{b_1, \ldots, b_D\})$ be an integral tuple. For every $\epsilon > 0$, there exist $Q := Q(\epsilon, \mathcal{B}, \Omega), R := R(\epsilon, \mathcal{B}, \Omega), N_0 := N_0(\epsilon, \mathcal{B}, \Omega) \in \mathbb{N}_+$ such that for every $N \geq N_0$ and $\chi \in \mathcal{M}_K$, the truncated function $\chi_N : \mathbb{Z}_N^D \rightarrow \mathbb{C}$ can be written as

$$
\chi_N(n) = \chi_{N,s}(n) + \chi_{N,u}(n)
$$

for all $n \in \mathbb{Z}_N^D$ such that the following holds:

1. $|\chi_{N,s}| \leq 1$ and $\chi_{N,s} = \chi_N * \phi_{N,\epsilon}$ for some kernel $\phi_{N,\epsilon}$ of $\mathbb{Z}_N^D$ which is independent of $\chi$, where the convolution product is defined on $\mathbb{Z}_N^D$;
2. $|\chi_{N,s}(n) + Qe_i - \chi_{N,s}(n)| \leq \frac{1}{\epsilon}$ for every $n \in \mathbb{Z}_N^D$ and $1 \leq i \leq D$;
3. For every $\xi = (\xi_1, \ldots, \xi_D) \in \mathbb{Z}_N^D$ such that $f_{N,s}(\xi) \neq 0$ and every $1 \leq i \leq D$, there exists $p_i \in \{0, \ldots, Q - 1\}$ such that $|\xi_i - \frac{p_i}{Q}| \leq \frac{1}{\epsilon}$;
4. $\|\chi_{N,u}\|_{U^2(\mathbb{Z}_N^D)} \leq \epsilon$;
5. For every $0 < \epsilon' \leq \epsilon$, $N \geq \max\{N_0(\epsilon, \mathcal{B}), N_0(\epsilon', \mathcal{B})\}$ and $\xi \in \mathbb{Z}_N^D$, we have that $\hat{\phi}_{N,\epsilon}(\xi) \geq \frac{1}{\epsilon}$.

In the rest of Section 9.2, we consider $\Omega$ as fixed, and all the quantities depend implicitly on $\Omega$. Moreover, $\hat{N}$ always denotes the smallest prime integer greater than $\Omega N$.

We first explain what happens when the Fourier coefficient of $\chi$ is away from 0.

**Corollary 9.4** (A consequence of Theorem 7.1). Let $K = (K, O_K, D, \mathcal{B} = \{b_1, \ldots, b_D\})$ be an integral tuple. For every $\epsilon > 0$, there exist $Q := Q(\epsilon, \mathcal{B}), V := V(\epsilon, \mathcal{B}), N_0 := N_0(\epsilon, \mathcal{B}) \in \mathbb{N}_+$ such that for every $N \geq N_0$, every $\chi \in \mathcal{M}_K$ and every $\xi = (\xi_1, \ldots, \xi_D) \in \mathbb{Z}_N^D$, if $|\hat{\chi}_N(\xi)| \geq \epsilon$, then

$$
\sum_{i=1}^{D} \left| \frac{Q \xi_i}{N} \right| \leq \frac{QV}{N}.
$$

One way to prove Corollary 9.4 is to follow the method used in Corollary 5.2 of [43]. Here we provide a different proof by using Theorem 7.1 as a black box:

**Proof:** Let $G = \mathbb{R}, \Gamma = \mathbb{Z}$ and $X = G/\Gamma = \mathbb{T}$. Let $g_{N,\xi} \in \text{poly}_D(\mathbb{R})$ be the function given by $g_{N,\xi}(n) := -n \circ_K \xi$ for all $n \in \mathbb{Z}^D$. Since $|R_{N,D}| \leq (3N)^D$, $|\hat{\chi}_N(\xi)| \geq \epsilon$ implies that

$$
\sum_{n \in \mathbb{R}} 1_{|\mathcal{N}|} 1 \chi(t_g(n)) e(g_{N,\xi}(n) \cdot e_X) \geq 3^D \epsilon.
$$

By Theorem 7.1 there exist $\delta := \delta(\epsilon, \mathcal{B}) > 0$ and $N_0 := N_0(\epsilon, \mathcal{B}) \in \mathbb{N}$ such that if $N \geq N_0$, then $(g_{N,\xi}(n) \cdot e_X)_{n \in \mathbb{R}}$ is not totally $\delta$-equidistributed. Since every horizontal character $\eta : \mathbb{T} \rightarrow \mathbb{T}$ on $\mathbb{T}$ can be written as $\eta(t) = qt \mod Z$ for some $q \in \mathbb{Z}$, by Theorem 4.9 there exist $V := V(\delta, D) = V(\epsilon, \mathcal{B}) > 0$ and $0 < \xi \leq V$ such that

$$
\|\eta_{\xi} \circ g_{N,\xi}|C^\infty_{\mathbb{R}}| = \sum_{i=1}^{D} \left| \frac{Q \xi_i}{N} \right| \leq \frac{V}{N}.
$$
where \( \eta_q(t) := q \xi t \mod \mathbb{Z} \) for all \( t \in T \). Let \( Q := V! \). Then \( Q \) depends only on \( \epsilon \) and \( B \), and for all \( \xi \in \mathbb{Z}_N^D \\

\sum_{i=1}^D \left\| \frac{O \cdot \xi_i}{N} \right\|_T = \sum_{i=1}^D \left\| \frac{O \cdot \xi_i}{N} \right\|_T \leq \frac{Q}{q} \sum_{i=1}^D \left\| \frac{q \xi_i}{N} \right\|_T \leq \frac{QV}{qN} \leq \frac{QV}{N}.

This finishes the proof. \( \square \)

Let the integral tuple \( K = (K, O_K, D, B = \{b_1, \ldots, b_D\}) \) be fixed, and we assume that all the quantities depend implicitly on \( B \) in the rest of this section. For \( \epsilon > 0, N \in \mathbb{N}, q \in \mathbb{Z}_N^D \), define

\[
\mathcal{A}(N, \epsilon) := \left\{ \xi \in \mathbb{Z}_N^D : \sup_{\chi \in \mathcal{M}_K} |\widehat{\chi N}(\xi)| \geq \epsilon^2 \right\};
\]

\[
W(N, q, \epsilon) := \max_{\xi=(\xi_1, \ldots, \xi_D) \in \mathcal{A}(N, \epsilon)} N \left\| \frac{q \xi_i}{N} \right\|_2;
\]

\[
Q(\epsilon) := \min_{k \in \mathbb{N}} \left( k! \limsup_{N \to \infty} W(N, k!, \epsilon) < \infty \right);
\]

\[
V(\epsilon) := 1 + \left[ \frac{1}{Q(\epsilon)} \limsup_{N \to \infty} W(N, Q(\epsilon), \epsilon) \right].
\]

It follows from Corollary 9.4 that \( Q(\epsilon) \) is well defined. Notice that for all \( 0 < \epsilon' \leq \epsilon, Q(\epsilon') \geq Q(\epsilon) \) and \( Q(\epsilon') \) is a multiple of \( Q(\epsilon) \). Thus \( V(\epsilon') \geq V(\epsilon) \) (it is easy to verify that \( V(\epsilon) \) increases as \( \epsilon \) decreases). By definition, there exists \( N_1(\epsilon) \in \mathbb{N} \) such that for all \( N \geq N_1(\epsilon), \chi \in \mathcal{M}_K \) and \( \xi \in \mathbb{Z}_N^D \),

\[
|\widehat{\chi N}(\xi)| \geq \epsilon^2 \Rightarrow \sum_{i=1}^D \left\| \frac{Q(\epsilon) \xi_i}{N} \right\|_2 \leq \frac{Q(\epsilon) V(\epsilon)}{N}.
\]

For every \( m \geq 1, N > 2m \), we define the function \( f_{N,m} : \mathbb{Z}_N^D \to \mathbb{C} \) by

\[
f_{N,m}(n) := \sum_{-m \leq \xi_1, \ldots, \xi_D \leq m} \left( \prod_{i=1}^D \left( 1 - \frac{|\xi_i|}{m} \right) \right) \cdot e(n \circ_{\hat{N}} (\xi_1, \ldots, \xi_D)).
\]

It is easy to verify that \( f_{N,m} \) is a kernel of \( \mathbb{Z}_N^D \) whose spectrum is \( \{-(m-1), \ldots, -m-1\}^D \). Let \( Q_N(\epsilon)^* \) be the unique integer in \( \{1, \ldots, N-1\} \) such that \( Q(\epsilon) Q_N(\epsilon)^* \equiv 1 \mod N \). Let

\[
N_0(\epsilon) := \max\{N_1(\epsilon), 2DQ(\epsilon) V(\epsilon) |\epsilon^{-1}| \}.
\]

For \( N > N_0 \), we define \( \phi_{N,\epsilon} : \mathbb{Z}_N^D \to \mathbb{C} \) by

\[
\phi_{N,\epsilon}(\xi) := f_{N,DQ(\epsilon) V(\epsilon) |\epsilon^{-1}|}(Q_N(\epsilon)^* \xi).
\]

In other words, \( f_{N,DQ(\epsilon) V(\epsilon) |\epsilon^{-1}|}(\xi) = \phi_{N,\epsilon}(Q(\epsilon) \xi) \). Then \( \phi_{N,\epsilon} \) is also a kernel of \( \mathbb{Z}_N^D \), and the spectrum of \( \phi_{N,\epsilon} \) is the set

\[
\Xi_{N,\epsilon} := \left\{ \xi = (\xi_1, \ldots, \xi_D) \in \mathbb{Z}_N^D : \left\| \frac{Q(\epsilon) \xi_i}{N} \right\|_2 < \frac{DQ(\epsilon) V(\epsilon) |\epsilon^{-1}|}{N}, 1 \leq i \leq D \right\}.
\]
Moreover,
\[
\widehat{\phi_{N,e}}(\xi) = \prod_{i=1}^{D} \left(1 - \left\| \frac{Q(e)\xi_i}{N} \right\|_{\mathbb{T}} \right) \cdot \frac{\hat{N}}{DQ(e)V(e)\epsilon^{-4}}
\]
if \(\xi \in \Xi_{N,e}\) and \(\widehat{\phi_{N,e}}(\xi) = 0\) otherwise.

**Proof of Theorem 9.3.** In this proof, we assume implicitly that every constant depends on \(\mathcal{B}\) and \(\Omega\). Let the notations be defined as above. Fix \(\epsilon > 0\), and let \(Q(e)\) and \(N_0(e)\) be defined as in (54) and (55), respectively. Let \(R(e)\) be sufficiently large to be chosen later. For all \(\chi \in \mathcal{M}_K\), let \(\chi_{N,s} := \chi_N \ast \phi_{N,e}\) and \(\chi_{N,s} = \chi_N - \chi_{N,s}\), where \(\phi_{N,e}\) is defined in (56). We show that \(\chi_{N,s}\) and \(\chi_{N,u}\) satisfy all the requirements.

We now fix \(\chi \in \mathcal{M}_K\) and \(N \geq N_0(e)\). Since \(|\chi| \leq 1\), by definition, \(|\chi|_{\mathcal{N}_e} \leq 1\). So Property (i) holds.

Using Fourier inversion formula and the estimate \(|e(x) - 1| \leq 2\pi|x|_T\), for all \(1 \leq i \leq D\), we have that
\[
|\chi_{N,s}(n + Q(e)n_i) - \chi_{N,s}(n)| \leq \sum_{\xi = (\xi_1, \ldots, \xi_D) \in \mathbb{Z}_N^D} |\phi_{N,e}| |2\pi| \frac{Q(e)\xi_i}{N} |_T \leq |\Xi_{N,e}| \frac{2\pi DQ(e)V(e)\epsilon^{-4}}{N}.
\]
Since \(|\Xi_{N,e}|\) is finite and depends only on \(\epsilon\), Property (ii) follows by taking \(R(e)\) sufficiently large depending only on \(\epsilon\), \(\mathcal{B}\) and \(\Omega\).

Let \(\xi \in \mathbb{Z}_N^D\) be such that \(\widehat{\chi_{N,s}}(\xi) \neq 0\). Then \(\phi_{N,e}(\xi) \neq 0\) and so \(\xi \in \Xi_{N,e}\). By definition, there exist \(p_1, \ldots, p_D \in \{0, \ldots, Q(e) - 1\}\) such that \(\xi^T - p_\xi = \xi^D\). So Property (iii) holds by taking \(R(e)\) sufficiently large depending only on \(\epsilon\) and \(\mathcal{B}\).

For every \(\chi \in \mathcal{M}_K\) and \(\xi = (\xi_1, \ldots, \xi_D) \in \mathbb{Z}_N^D\) if \(|\widehat{\chi_N}(\xi)| \geq \epsilon^2\), then by the definition of \(Q(e)\) and \(N_1(e)\),
\[
\left\| \frac{Q(e)\xi_i}{N} \right\|_T \leq \frac{Q(e)V(e)}{N}
\]
for all \(1 \leq i \leq D\). Thus \(\phi_{N,e}(\xi) \geq (1 - \epsilon^4/D)^D \geq 1 - \epsilon^4\). So
\[
(57) \quad |\widehat{\chi_N}(\xi) - \phi_{N,e}(\widehat{\chi_N}(\xi))| = |\widehat{\chi_N}(\xi) - \phi_{N,e}(\widehat{\chi_N}(\xi))| \leq \epsilon^4 \leq \epsilon^2.
\]
Note that (57) also holds if \(|\widehat{\chi_N}(\xi)| \leq \epsilon^2\). Thus by identity (53) and Parseval’s identity, we have
\[
|\chi_{N,s}|_{U_1^{\mathbb{Z}_N^D}}^2 = \sum_{\xi \in \mathbb{Z}_N^D} |\widehat{\chi_N}(\xi) - \phi_{N,e}(\widehat{\chi_N}(\xi))|^2 \leq \epsilon^4 \sum_{\xi \in \mathbb{Z}_N^D} |\chi_N(\xi) - \phi_{N,e}(\chi_N(\xi))|^2 \leq \sum_{\xi \in \mathbb{Z}_N^D} |\chi_N(\xi)|^2 \leq \epsilon^4.
\]
This proves Property (iv).

Suppose that \(0 < e' \leq \epsilon\). Since \(Q(e') \geq Q(e), V(e') \geq V(e)\) and \(Q(e')\) is a multiple of \(Q(e)\), we have \(\Xi_{N,e} \subseteq \Xi_{N,e'}\) and \(\phi_{N,e'}(\xi) \geq \phi_{N,e}(\xi)\) for every \(\xi \in \mathbb{Z}_N^D\). This proves Property (v), which finishes the proof of the whole theorem. \(\square\)

**9.3. Weak \(U^d\) structure theorem.** Our section step is to prove a weak \(U^d\) structure theorem:

**Theorem 9.5 (Weak \(U^d\) structure theorem for multiplicative functions).** Let \(\Omega \in \mathbb{N}\). For \(N \in \mathbb{N}\), let \(\hat{N}\) denote the smallest prime integer greater than \(\Omega N\). Let \(K = (K, O_K, D, \mathcal{B} = \{b_1, \ldots, b_D\})\) be an integral tuple. For every \(d \in \mathbb{N}, d \geq 3\) and \(\epsilon > 0\), there exists \(\theta_0 := \theta_0(d, \epsilon, \mathcal{B}, \Omega) > 0\) such that for
all $0 < \theta < \theta_0$, there exist $Q := Q(d, \epsilon, \theta, \mathcal{B}, \Omega), R := R(d, \epsilon, \theta, \mathcal{B}, \Omega), N_0 := N_0(d, \epsilon, \theta, \mathcal{B}, \Omega) \in \mathbb{N}_+$ such that for every $N \geq N_0$ and $\chi \in M_K$, the truncated function $\chi_N : \mathbb{Z}_N^D \to \mathbb{C}$ can be written as
\[ \chi_N(n) = \chi_{N,s}(n) + \chi_{N,\epsilon}(n) \]
for all $n \in \mathbb{Z}_N^D$ such that the following holds:

(i) $|\chi_{N,s}| \leq 1$ and $\chi_{N,s} = \chi_N * \phi_{N,0}$, where $\phi_{N,0}$ is the kernel of $\mathbb{Z}_N^D$ defined in (56) which is independent of $\chi$, and the convolution product is defined on $\mathbb{Z}_N^D$;

(ii) $|\chi_{N,s}(n + Qe_i) - \chi_{N,s}(n)| \leq \frac{\epsilon}{N}$ for every $n \in \mathbb{Z}_N^D$ and $1 \leq i \leq D$;

(iii) For every $\xi = (\xi_1, \ldots, \xi_D) \in \mathbb{Z}_N^D$ such that $f_{N,s}(\xi) \neq 0$ and every $1 \leq i \leq D$, there exists $p_i \in \{0, \ldots, Q - 1\}$ such that $|\xi_i - p_i| \leq \frac{\epsilon}{N}$;

(iv) $\|\chi_{N,\epsilon}\|_{U^d(\mathbb{Z}_N^D)} \leq \epsilon$.

The proof of Theorem 9.5 is similar to Theorem 8.1 of [18] and Theorem 8.2 of [43]. We provide the details for completeness. Again, in the rest of Section 9.3 we consider $\Omega$ as fixed, and all the quantities depend implicitly on $\Omega$. Moreover, $\bar{N}$ always denotes the smallest prime integer greater than $\Omega N$.

Let $K, d, \epsilon$ be as in the statement of Theorem 9.5 and let $\phi_{N,\epsilon}$ be defined as in (56). For all $\epsilon > 0$, $N \in \mathbb{N}$ and $\chi \in M_K$, denote
\begin{equation}
\chi_{N,\epsilon,\theta} := \chi_N * \phi_{N,\epsilon} \quad \text{and} \quad \chi_{N,\epsilon,\theta} := \chi_N - \chi_{N,s,\epsilon}.
\end{equation}

By Theorem 9.3, for all $\theta > 0$, there exist $Q(\delta, \mathcal{B}), R(\delta, \mathcal{B}), N_0(\delta, \mathcal{B}) \in \mathbb{N}_+$ such that for every $N \geq N_0(\delta, \mathcal{B})$ and $\chi \in M_K$, Properties (i)-(v) of Theorem 9.3 holds with $\epsilon$ replaced with $\theta$.

Comparing Theorem 9.5 with Theorem 9.3 it is easy to see that we only need to show that for every $\epsilon > 0$ and $d \geq 3$, there exists $\theta_0 := \theta_0(d, \epsilon, \mathcal{B}) > 0$ such that for all $0 < \theta < \theta_0$, there exists $N_0 := N_0(d, \epsilon, \theta, \mathcal{B}) \in \mathbb{N}$ such that for all $N \geq N_0(d, \epsilon, \theta, \mathcal{B})$ and $\chi \in M_K$, we have that
\[ \|\chi_{N,\epsilon,\theta}\|_{U^d(\mathbb{Z}_N^D)} \leq \epsilon. \]

By Properties (iv) of Theorem 9.3, $|\chi_{N,\epsilon,\theta}|_{U^d(\mathbb{Z}_N^D)} \leq \theta$ for all $\theta > 0$. Our strategy is to show that for multiplicative functions, the smallness of the $U^d$ norm implies the smallness of the $U^d$ norm for $d \geq 3$. By Theorem 8.5 (the inverse theorem for Gowers norms), in order to prove Theorem 9.5 it suffices to show the following:

**Proposition 9.6.** Let $K = (K, O_K, D, \mathcal{B} = \{b_1, \ldots, b_D\})$ be an integral tuple, $\delta > 0$ and $d \in \mathbb{N}_+$. Let $X = G/\Gamma$ be a nilmanifold with a nil-structure $\tilde{X} = (G_\bullet, \chi, \psi, d_G, d_\chi)$. There exists $\theta_0 := \theta_0(\chi, d, \delta, \mathcal{B}) > 0$ such that for all $0 < \theta < \theta_0$, there exists $N_0 := N_0(\chi, d, \delta, \theta, \mathcal{B}) \in \mathbb{N}$ such that for every $N \geq N_0$, every $\chi \in M_K$, every $g \in \text{poly}_D(G_\bullet)$ and every $\Phi : X \to \mathbb{C}$ with modulus at most 1 and $\|\Phi\|_{\text{lip}(\chi)} \leq 1$, we have that
\begin{equation}
|\mathbb{E}_{n \in [X]^{D}} \chi_{N,\epsilon,\theta}(n) \cdot \Phi(g(n) \cdot e_X)| \leq \delta,
\end{equation}
where $\chi_{N,\epsilon,\theta}$ is defined in (58).

**Proof.** We may assume without loss of generality that $\tilde{X}$ is standard. To simplify the notations, in the proof, $\delta, d, X, \mathcal{B}$ and $\Omega$ are fixed and all the quantities depend implicitly on them.

Suppose on the contrary that there exist arbitrarily small $\theta > 0$, arbitrarily large $N \in \mathbb{N}$, function $\chi \in M_K$, polynomial sequence $g \in \text{poly}_D(G_\bullet)$, and $\Phi : X \to \mathbb{C}$ with modulus at most 1 and
\[\|\Phi\|_{\text{Lip}(X)} \leq 1, \text{ such that}\]

\[\mathbb{E}_{n \in [N]^D} \Phi(g(n) \cdot e_X) > \delta.\]  

By using the factorization theorem (Theorem 4.16), we may deduce from (60) that there exist

- A function \(\lambda := \lambda(\delta, \epsilon, \Omega) : \mathbb{N}_+ \to \mathbb{R}_+;\)
- For all \(M \in \mathbb{N}\) a finite subset \(\Sigma(M) \subseteq G\) and a finite collection \(\mathcal{F}(M)\) of sub nilmanifolds \(X' = G'/\Gamma'\) of \(X\) with nil-structures \(X'_a\) induced by \(X\);
- For each \(a \in \Sigma(M),\) a sub nilmanifold \(X'_a = G'_a/\Gamma'_a\) of \(X\) with a nil-structure \(X'_a\) adapted to the filtration \(G'_a\) induced by the \(a\)-conjugate of \(X',\) a polynomial sequence \(g'_a \in \text{poly}_D(G'_a)\) and a function \(\Phi'_a : X'_a \to \mathbb{C}\) with \(\|\Phi'_a\| \leq 1\) and \(\|\Phi'_a\|_{\text{Lip}(X'_a)} = 1,\)

such that for every function \(\zeta : \mathbb{N} \to \mathbb{R}_+\), there exist a function \(N_0 : \mathbb{N} \to \mathbb{R}_+\) and \(M_1 \in \mathbb{N}_+\) such that the following holds:

**Property 1.** There exist \(M^* \in \mathbb{N}\) with \(M^* \leq M_1\), such that if (52) holds for some \(N \geq N_0(M^*)\), then exist a \(D\)-dimensional arithmetic progression \(P\) of \(\mathbb{Z}^D\) an element \(a \in \Sigma(M^*)\) such that

\[\mathbb{E}_{n \in [N]^D} \Phi(g'_a(n) \cdot e_X) > 3 \cdot 2^D C \lambda(M^*),\]

where \(C := C(D) > 0\) is the constant defined in Lemma 8.4.

**Property 2.** For all \(M^* \in \mathbb{N}\), \(a \in \Sigma(M^*)\) and \(N \geq N_0(M^*),\)

\((g'_a(n) \cdot e_{X_a})_{n \in \mathbb{N}_+}\) is totally \(\zeta(M^*)\)-equidistributed on \(X'_a\) with respect to \(X'_a\).

We left the choice of \(\zeta\) to the end of the proof (note that the choice of \(\zeta\) must be dependent only on \(\delta, \epsilon, \mathcal{B}\)). The method we deduce Properties 1 and 2 from (60) is again similar to the one we used to deduce (47) and (49) from (42), as well as the one to deduce Properties 1 and 2 from (52) in the proof of Theorem 8.1. So we omit it.

Set \(\theta_0 := \min_{M^* \leq M_1} \lambda(M^*).\) We may assume that (60) holds for some \(\theta < \theta_0\) and \(N > \max_{M^* \leq M_1} N_0(M^*).\) Let \(z := \int_{X'_a} d\Phi'_a\) and \(\Phi'_0 := \Phi'_a - z.\) Then \(\int_{X'_a} \Phi'_0 \cdot e_{X_a} = 0.\) Applying Lemma 8.4 Property (iv) of Theorem 9.4 and the definition of \(\theta_0\) consecutively, we have that

\[\mathbb{E}_{n \in [N]^D} \Phi(g'_a(n) \cdot e_X) \leq C \lambda(M^*),\]

By Property 1,

\[\mathbb{E}_{n \in [N]^D} \Phi(g'_a(n) \cdot e_X) \leq 2 \cdot 2^D C \lambda(M^*).\]

By (58), we may write \(X_{N,a,0} = X_{N,a,0} \times X_{N,a,0}'\) where \(\psi_{N,a} : \mathbb{Z}_N^D \to \mathbb{R}\) is the function given by \(\psi_{N,a}(0) := \tilde{N}^D - \Phi_{N,a}(0)\) and \(\psi_{N,a}(n) := -\Phi_{N,a}(n)\) for all \(n \in \mathbb{Z}_N^D \setminus \{0\}\). Since \(\mathbb{E}_{n \in \mathbb{Z}_N^D} \Phi_{N,a} = 1\), we have that \(\mathbb{E}_{n \in \mathbb{Z}_N^D} \psi_{N,a} \leq 2.\) Therefore, by (61), there exists \(m = (m_1, \ldots, m_D) \in \mathbb{Z}_N^D\) such that

\[\mathbb{E}_{n \in [N]^D} \Phi(g'_a(n) \cdot e_X) \leq 2 \cdot 2^D C \lambda(M^*),\]

where the residue class \(n + m \mod \tilde{N}^D\) is taken in \([\tilde{N}]^D = \{1, \ldots, \tilde{N}\}^D\) instead of the more conventional \([0, \ldots, \tilde{N} - 1]^D\). Therefore, there exist \(J = J_1 \times \cdots \times J_{D} \subseteq [\tilde{N}]^D\) and \(m' = (m_1, \ldots, m_D) \in [\tilde{N}]^D\), such that for all \(1 \leq i \leq D\), either \(J_i = \{1, \ldots, \tilde{N} - m_i\}\) and \(m_i = m_i\) or \(J_i = \{\tilde{N} - m_i + 1, \ldots, \tilde{N}\}\) and \(m_i = m_i - \tilde{N}\), and that

\[\mathbb{E}_{n \in [N]^D} \Phi(g'_a(n + m') \cdot e_X) > \lambda(M^*).\]
Note that $1_P(n + m')1_f(n)1_{M'}(n) = 1_P(n)$ for some $D$-dimensional arithmetic progression $P' \subseteq [N]^D$. Denoting $g'_{a,m}(n) := g'_a(n + m') \in \text{poly}_D(G'_{a,m})$, we deduce from (62) that
\begin{equation}
\left| \mathbb{E}_{n \in [N]^P} 1_P(n) \chi(n) \cdot \Phi'_a(g'_{a,m}(n) \cdot eX'_a) \right| > C \lambda(M').
\end{equation}
Since $\int_{X'_a} \Phi'_{0,a} dm_{X'_a} = 0$ and $\| \Phi'_{0,a} \|_{\text{Lip}(X'_a)} = \| \Phi'_{a} \|_{\text{Lip}(X'_a)} = 1$, by Theorem 7.1 there exists a function $\zeta : [N] \to \mathbb{N}^+$ such that if $N$ is sufficiently large, then $(g'_a(n) \cdot eX'_a)_{n \in [N]^P}$ is not totally $\zeta(M')$-equidistributed on $X'_a$ with respect to $X'_a$. By choosing $\zeta$ to be the function defined above (which is a function of $M'$ depending only on $\delta, \delta, B$, and $\Omega$), we get a contradiction to Property 2. This finishes the proof. \hfill \square

9.4. **Deducing the strong $U^d$ structure theorem from the weak one.** Our last step is to finish the proof of Theorem 9.2, i.e. the strong $U^d$ structure theorem. By using an iterative argument of energy increment, we can deduce that the weak $U^d$ structure theorem (Theorem 9.5) implies Theorem 9.2. As the method is identical to Section 8.10 in [18], we omit the proof.

10. **Partition regularity properties**

In this section, we explain how Theorem 9.2 can be applied to deduce partition regularity properties.

10.1. **Statement of the main result on partition regularity problems.** We start with a technical definition which captures the algebraic structure behind partition regularity problems.

**Definition 10.1** (Types of polynomials). Let $K = (K, O_K, D, B = \{b_1, \ldots, b_D\})$ be an integral tuple, $r \in \mathbb{N}_+$, and $p \in \mathbb{C}[x, y; z_1, \ldots, z_r]$ be a homogeneous polynomial, meaning that
\[ p(tx, ty; tz_1, \ldots, tz_r) = t^k p(x, y; z_1, \ldots, z_r) \]
for all $t \in \mathbb{C}$. We say that $p$ is a $K$-type polynomial if there exist $d \in \mathbb{N}_+$ and $a_1, \ldots, a_d, a'_1, \ldots, a'_d \in O_K$ satisfying (i) $a_i \neq a'_i$ and $a'_i \neq a'_j$ for all $1 \leq i, j \leq d, i \neq j$; and (ii) $\{a_1, \ldots, a_d\} \neq \{a'_1, \ldots, a'_d\}$, such that for all $m, n, k \in K$, there exist $z_1, \ldots, z_r \in K$ such that
\[ p(k \prod_{i=1}^d (m + a_in), k \prod_{i=1}^d (m + a'_in); z_1, \ldots, z_r) = 0. \]

Our main result is the following:

**Theorem 10.2** (Partition regularity result in full generality). Let $K = (K, O_K, D, B = \{b_1, \ldots, b_D\})$ be an integral tuple, $r \in \mathbb{N}_+$, and $p \in \mathbb{C}[x, y; z_1, \ldots, z_r]$ be a $K$-type polynomial. Then $p$ is partition regular over $O_K$ with respect to $x$ and $y$.

**Remark 10.3.** Although Theorem 10.2 already covers many classes of equations, there are three important restrictions. The first is that the number of variables taking values in $U_i$ (i.e. $x$ and $y$) equals to 2. The second is that the polynomial $p$ is homogeneous. The third is that we require $p(x, y; z_1, \ldots, z_r) = 0$ to have a parametrized solution of the form $x = k \prod_{i=1}^d (m + a_in)$ and $y = k \prod_{i=1}^d (m + a'_in)$.

We start with explaining the applications of Theorem 10.2 and differ its proof to the end of the section.
10.2. Applications of Theorem 10.2 to partition regularity problems. We first provide a criteria for partition regularity properties for quadratic equations.

**Proposition 10.4** (Partition regularity for quadratic equations). Let \( p \) be a quadratic equation of the form

\[
p(x, y; z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz
\]

for some \( a, b, c, d, e, f \in \mathbb{Z} \). Denote

\[
\Delta_1(p) := \sqrt{e^2 - 4ac}, \Delta_2(p) := \sqrt{f^2 - 4bc}, \Delta_3(p) := \sqrt{(e + f)^2 - 4c(a + b + d)}.
\]

Suppose that \( c, \Delta_1^2(p), \Delta_2^2(p) \neq 0 \), and at least one of \( \Delta_1^2(p) \) and \( \Delta_1^2(p) - \Delta_2^2(p) \) is non-zero, then \( p \) is a \( K \)-type polynomial for \( K := \mathbb{Q}(\Delta_1(p), \Delta_2(p), \Delta_3(p)) \). In particular, \( p \) is partition regular over \( \mathcal{O}_K \) with respect to \( x \) and \( y \) by Theorem 10.2.

It is not hard to see that Proposition 10.4 implies Proposition 1.14.

**Remark 10.5.** The quadratic equations which are not covered by Proposition 10.4 are the following: (i) \( c = 0 \); (ii) \( c \neq 0 \), one of \( \Delta_1^2(p), \Delta_2^2(p), \Delta_3^2(p) \) equals to 0 and the other two are equal; (iii) \( c \neq 0 \), one of \( \Delta_1^2(p) \) and \( \Delta_2^2(p) \) equals to 0, and the other one is not equal to \( \Delta_3^2(p) \).

The equations \( p \) in these three cases are “degenerate” in one way or another, which are completely different from the case discussed in Proposition 10.4. For Case (ii), it is not hard to show that for every algebraic number field \( K, p \) is partition regular over \( \mathcal{O}_K \) with respect to \( x \) and \( y \) if and only if \( \mathbb{Q}(\Delta_1(p), \Delta_2(p), \Delta_3(p)) \subseteq K \). It is an interesting question to ask whether in Cases (i) and (iii) \( p \) is partition regular over \( \mathbb{Z} \) with respect to \( x \) and \( y \). But this is beyond the theme of this paper.

**Proof of Theorem 10.4.** Let

\[
p'(x, y; z) := p(2cx, 2cy, z - ex - fy) = -c(\Delta_1^2(p)x^2 + \Delta_2^2(p)y^2 - z^2 + (\Delta_3^2(p) - \Delta_1^2(p) - \Delta_2^2(p))xy).
\]

(i) Suppose first that \( \Delta_2^2(p) = 0 \) and \( \Delta_1^2(p) \neq \Delta_3^2(p) \). By a direct computation, for all \( k, m, n \in K \), we have that \( p'(x', y', z') = 0 \) for

\[
x' := k(m - \Delta_3(p)n)(m + \Delta_2(p)n);
\]

\[
y' := k(m - \Delta_1(p)n)(m + \Delta_1(p)n);
\]

\[
z' := \pm k(\Delta_1^2(p) - \Delta_2^2(p))mn.
\]

Let \( a_1 = -\Delta_2(p), a_2 = \Delta_2(p), a_1' = -\Delta_1(p) \) and \( a_2' = \Delta_1(p) \). We have that there exists \( z \in K \) such that

\[
p(2kc(m + a_1n)(m + a_2n), 2kc(m + a_1'n)(m + a_2'n); z) = 0.
\]

Since \( \Delta_1^2(p), \Delta_2^2(p) \neq 0 \), we have that \( a_1 \neq a_2 \) and \( a_1' \neq a_2' \). If \( \{a_1, a_2\} = \{a_1', a_2'\} \), then \( \Delta_2^2(p) = \Delta_3^2(p) \), a contradiction. This implies that \( \{a_1, a_2\} \neq \{a_1', a_2'\} \), and so \( p \) is a \( K \)-type polynomial.

(ii) We now assume that \( \Delta_3^2(p) \neq 0 \). This case is similar to Appendix C of [18]. By a direct computation, for all \( k, m, n \in K \), we have that \( p'(x', y', z') = 0 \) for

\[
x' := k(m + c(\Delta_2^2(p) + \Delta_2(p)\Delta_3(p))n)(m + c(\Delta_2^2(p) - \Delta_2(p)\Delta_3(p))n);
\]

\[
y' := k(m + c(\Delta_3^2(p) - \Delta_3^2(p) + \Delta_1(p)\Delta_3(p))n)(m + c(\Delta_3^2(p) - \Delta_3^2(p) - \Delta_1(p)\Delta_3(p))n);
\]

\[
z' := \pm k(\Delta_2^2(p) - \Delta_3^2(p))mn.
\]
\[ z' := \pm k\Delta_3(p)\left(m^2 + c\Delta_1^2(p) + \Delta_2^2(p) - \Delta_3^2(p)\right)mn + c^2\Delta_1^3(p)\Delta_2^3(p)n^3. \]

Let \( a_1 = c(\Delta_1^2(p) + \Delta_2(p)\Delta_3(p)), a_2 = c(\Delta_2^2(p) - \Delta_2(p)\Delta_3(p)), a'_1 = c(\Delta_1^2(p) - \Delta_2(p)\Delta_3(p)) + \Delta_1(p)\Delta_3(p) \)
and \( a'_2 = c(\Delta_2^2(p) - \Delta_2(p)\Delta_3(p)) - \Delta_1(p)\Delta_3(p). \)
Since \( \Delta_1(p), \Delta_2(p), \Delta_3(p) \in K \), we have that \( a_1, a_2, a'_1, a'_2 \in K \) and \( x', y', z' \in K \). So for all \( k, m, n \in K \), there exists \( z \in K \) such that
\[ p(2kc(m + a_1)n(m + a_2)n, 2kc(m + a'_1)n(m + a'_2)n; z) = 0. \]

Since \( \Delta_1^2(p), \Delta_2^2(p), \Delta_3^2(p), c \neq 0 \), we have that \( a_1 \neq a_2 \) and \( a'_1 \neq a'_2 \). If \( \{a_1, a_2\} = \{a'_1, a'_2\} \), then \( a_1 + a_2 = a'_1 + a'_2 \) and so \( \Delta_2^2(p) = 0 \), a contradiction. This implies that \( \{a_1, a_2\} \neq \{a'_1, a'_2\} \), and so \( p \) is a \( K \)-type polynomial.

Another application of Theorem \([10,2]\) is to prove Corollary \([1,15]\).

**Proof of Corollary \([1,15]\)** By using an identity of Gérardin:
\[ (m^2 - n^2)^4 + (2mn + m^2)^4 + (2mn + n^2)^4 = 2(m^2 + mn + n^2)^4, \]
we have that the polynomial \( p(x_1, x_2; z_1, z_2) := x_1^2 - 2x_2^2 + z_1^2 - z_2^2 \) is of \( \mathbb{Q}(\sqrt{-3}) \)-type (by setting \( x_1 = k(m + n)(m - n) \) and \( x_2 = k(m + \frac{1 + \sqrt{-3}}{2}n)(m + \frac{1 - \sqrt{-3}}{2}n) \)). Since \( \mathbb{Z}[\frac{1 + \sqrt{-3}}{2}] \) is the ring of integers of \( \mathbb{Q}(\sqrt{-3}) \), Theorem \([10,2]\) implies that \( p \) is partition regular over \( \mathbb{Z}[\frac{1 + \sqrt{-3}}{2}] \) with respect to \( x_1 \) and \( x_2 \). \( \square \)

### 10.3. Multiplicative measure preserving systems

To study Theorem \([10,2]\) we introduce the multiplicative density of a subset of a number field. Let \( K = (K, O_K, D, B = \{b_1, \ldots, b_D\}) \) be an integral tuple and \( I_1, I_2, \ldots \) be an enumeration of prime ideals in \( O_K \) (the number of primes ideals are countable) such that \( N(I_1) \leq N(I_2) \leq \ldots \) Let \( \Phi_N = (\Phi_N)_{N \in \mathbb{N}} \) be the sequence of finite subsets of \( O_K \) defined by
\[ \Phi_N := \{n \in O_K : (n) \supseteq (I_1I_2 \ldots I_N)^N\}. \]

Then \( (\Phi_N)_{N \in \mathbb{N}} \) is a multiplicative Følner sequence on \( O_K \), meaning that for all \( a \in O_K^* \)
\[ \lim_{N \to \infty} \frac{|a^{-1}\Phi_N \Delta \Phi_N|}{|\Phi_N|} = 0, \]
where \( a^{-1}\Phi_N := \{a^{-1}x \in O_K : x \in \Phi_N\} = \{y \in O_K : ay \in \Phi_N\}. \)

**Definition 10.6** (Multiplicative density). Let \( K = (K, O_K, D, B = \{b_1, \ldots, b_D\}) \) be an integral tuple. The (upper) multiplicative density of a subset \( E \) of \( O_K \) (with respect to the multiplicative Følner sequence \( \Phi_N \)) is defined to be
\[ d_{\text{mult}, K}(E) := \limsup_{N \to \infty} \frac{|E \cap \Phi_N|}{|\Phi_N|}. \]

When there is no confusion, we write \( d_{\text{mult}}(E) := d_{\text{mult}, K}(E) \) for short. Since \( (\Phi_N)_{N \in \mathbb{N}} \) is a multiplicative Følner sequence, for all \( E \subseteq O_K \) and \( a \in O_K^* \),
\[ d_{\text{mult}}(E) = d_{\text{mult}}(a^{-1}E). \]

To show Theorem \([10,2]\), our strategy is to convert the question to a recurrence problem on a special type of dynamical systems:
\[ 32O_K := O_K \setminus \{0\}. \]
Definition 10.7 (Action by dilation). Let $K = (K, O_K, D, \mathcal{B} = \{b_1, \ldots, b_D\})$ be an integral tuple. An action by dilation over $K$ on a probability space $(X, \mathcal{D}, \mu)$ is a family $(T_n)_{n \in O_K^*}$ of invertible measure preserving transformations of $(X, \mathcal{D}, \mu)$ that satisfy $T_1 = id$ and $T_m \circ T_n = T_{mn}$ for all $m, n \in O_K^*$.

Note that an action by dilation $(T_n)_{n \in O_K^*}$ can be extended to a measure preserving action $(T_n)_{n \in K^*}$ by defining $T_{m/n} = T_m \circ T_n^{-1}$ for all $m, n \in O_K^*$. We remark that $(T_n)_{n \in K^*}$ is well defined even though $O_K$ may not be a unique factorization domain. In fact, let $m/n = m'/n'$ for some $m, m', n' \in O_K^*$. Then

$$T_m \circ T_{n'} = T_m \circ T_{m' n} = T_{m' n} = T_{m'} \circ T_n,$$

and so $T_m \circ T_n^{-1} = T_{m'} \circ T_n^{-1}$. Since $O_K^*$ with multiplication is a discrete amenable semi-group, we have the Furstenberg correspondence principle (see for example Theorem 2.1 of [6] and Theorem 6.4.17 of [4]):

Theorem 10.8 (Furstenberg correspondence principle). Let $K = (K, O_K, D, \mathcal{B} = \{b_1, \ldots, b_D\})$ be an integral tuple and $E$ be a subset of $O_K$. Then there exist an action by dilation $(T_n)_{n \in O_K^*}$ on a probability space $(X, \mathcal{D}, \mu)$ and a set $A \in \mathcal{D}$ with $\mu(A) = d_{mult}(E)$ such that for every $k \in \mathbb{N}_+$ and $n_1, \ldots, n_k \in O_K$, we have

$$d_{mult}(n_1^{-1} E \cap \cdots \cap n_k^{-1} E) \geq \mu(T_{n_1}^{-1} A \cap \cdots \cap T_{n_k}^{-1} A).$$

Let $\mathcal{M}_K^c$ denote the collection all completely multiplicative functions $\chi : O_K \rightarrow \mathbb{C}$ with modulus equals to 1, meaning that $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in O_K^*$, and that $|\chi| \equiv 1$. Every $\chi \in \mathcal{M}_K^c$ can be extended to a multiplicative function on $K^*$ by setting

$$\chi(m/n) := \chi(m)\overline{\chi(n)}$$

for all $m, n \in O_K^*$. $\chi(m/n)$ is well defined by a reason similar to (64). Endowing $\mathcal{M}_K^c$ with the pointwise multiplication and the topology of pointwise convergence, $\mathcal{M}_K^c$ is a compact Abelian group with the constant function 1 being the unit element. Moreover, $\mathcal{M}_K^c$ is the dual group of $K^*$.

Let $(X, \mathcal{D}, \mu)$ be a probability space with an action by dilation $(T_n)_{n \in O_K^*}$. For every $f \in L^2(\mu)$, by the spectral theorem, there exists a positive finite measure $\nu$ (called the spectral measure of $f$) on the dual group $\mathcal{M}_K^c$ of $K^*$ such that for all $m, n \in O_K^*$,

$$\int_X f \cdot T_n \overline{f} \, d\mu = \int_X f \cdot T_{m/n} \overline{f} \, d\mu = \int_{\mathcal{M}_K^c} \chi(m/n) \, d\nu(\chi) = \int_{\mathcal{M}_K^c} \chi(m)\overline{\chi(n)} \, d\nu(\chi).$$

The following lemma can be deduced by the same argument on pages 64–65 of [18]:

Lemma 10.9 (Positivity properties for spectrum measures). Let $K = (K, O_K, D, \mathcal{B} = \{b_1, \ldots, b_D\})$ be an integral tuple and $(X, \mathcal{D}, \mu)$ be a probability space with an action by dilation $(T_n)_{n \in O_K^*}$. Let $A \in \mathcal{D}$ with $\mu(A) > 0$ and $\nu$ be the spectral measure of the function $1_A$. Then

$$\nu(\{1\}) > 0 \text{ and } \int_{\mathcal{M}_K^c} \chi(m)\overline{\chi(n)} \, d\nu(\chi) \geq 0 \text{ for all } m, n \in O_K^*.$$

In order to prove Theorem 10.2, it suffices to show the following multiple recurrence property for multiplicative functions:

Proposition 10.10 (Multiple recurrence property for multiplicative functions). Let $K = (K, O_K, D, \mathcal{B} = \{b_1, \ldots, b_D\})$ be an integral tuple. Let $d \in \mathbb{N}_+$ and $a_1, \ldots, a_d, a'_1, \ldots, a'_d \in O_K$ be such that (i)
\[ a_i \neq a_j \text{ and } a'_i \neq a'_j \text{ for all } 1 \leq i, j \leq d; \text{ and (ii) } \{a_1, \ldots, a_d\} \neq \{a'_1, \ldots, a'_d\}. \]

Let \( \nu \) be a probability measure on \( \mathcal{M}_K^c \) satisfying (66). Then there exist \( m, n \in \mathcal{O}_K \) such that \( \prod_{i=1}^d (m + a_i n) \) and \( \prod_{i=1}^d (m + a'_i n) \) are distinct and nonzero, and that

\[
\int_{\mathcal{M}_K^c} \prod_{i=1}^d \chi(m + a_i n) \prod_{i=1}^d \bar{\chi}(m + a'_i n) d\nu(\chi) > 0.
\]

We postpone the proof of Proposition 10.10 to the next section, but explain first how to derive Theorem 10.2 from Proposition 10.10.

**Proof of Theorem 10.2 assuming Proposition 10.10.** Let \( d \in \mathbb{N}_+ \) and \( a_1, \ldots, a_d, a'_1, \ldots, a'_d \in \mathcal{O}_K \) be as in Definition 10.11 for the \( K \)-type polynomial \( p(x, y; z_1, \ldots, z_r) \). By the sub-additivity of \( d_{\text{mult}} \), in order to show the partition regularity of \( p \), it suffices to show that for all \( E \subseteq \mathcal{O}_K \) with \( d_{\text{mult}}(E) > 0 \), there exist \( m, n, k \in \mathcal{O}_K \) such that \( x := k \prod_{i=1}^d (m + a_i n) \) and \( y := k \prod_{i=1}^d (m + a'_i n) \) are distinct and nonzero elements in \( E \). It suffices to show that there exist \( m, n \in \mathcal{O}_K \) such that

\[
(67) \quad \prod_{i=1}^d (m + a_i n) \text{ and } \prod_{i=1}^d (m + a'_i n) \text{ are distinct and non-zero,}
\]

and that

\[
d_{\text{mult}}\left( \prod_{i=1}^d (m + a_i n)^{-1} E \cap \prod_{i=1}^d (m + a'_i n)^{-1} E \right) > 0.
\]

Let the probability space \((X, \mathcal{D}, \mu)\), the action by dilation \( (T_n)_{n \in \mathcal{O}_K} \) and the set \( A \in \mathcal{D} \) with \( \mu(A) = d_{\text{mult}}(E) > 0 \) be as in Theorem 10.8. By Theorem 10.8 it suffices to show that there exist \( m, n \in \mathcal{O}_K \) such that (67) holds and that

\[
\mu(T_{\prod_{i=1}^d (m + a_i n)}^{-1} A \cap T_{\prod_{i=1}^d (m + a'_i n)}^{-1} A) = \int_{\mathcal{M}_K^c} \prod_{i=1}^d \chi(m + a_i n) \prod_{i=1}^d \bar{\chi}(m + a'_i n) d\nu(\chi) > 0,
\]

where \( \nu \) is the spectrum measure of \( 1_A \). By Lemma 10.9 and Proposition 10.10 we are done. \( \square \)

### 10.4. A sketch of the proof of Proposition 10.10

Since

\[
\lim_{N \to \infty} \frac{1}{N^{2D}} \left| \left\{ (m, n) \in \mathfrak{G}(\mathbb{N})^D \times \mathfrak{G}(\mathbb{N})^D : \text{ two of } \prod_{i=1}^d (m + a_i n), \prod_{i=1}^d (m + a'_i n) \text{ and } 0 \text{ are equal} \right\} \right| = 0,
\]

in order to finish the proof of Proposition 10.10 it suffices to show the following:

**Proposition 10.11.** (Multiple averages for multiplicative functions) Let \( \mathbf{K} = (\mathcal{K}, \mathcal{O}_K, D, \mathcal{B} = \{b_1, \ldots, b_D\}) \) be an integral tuple. Let \( d \in \mathbb{N}_+ \) and \( a_1, \ldots, a_d, a'_1, \ldots, a'_d \in \mathcal{O}_K \) be such that (i) \( a_i \neq a_j \) and \( a'_i \neq a'_j \) for all \( 1 \leq i, j \leq d \); and (ii) \( \{a_1, \ldots, a_d\} \neq \{a'_1, \ldots, a'_d\} \). Let \( \nu \) be a probability measure on \( \mathcal{M}_K^c \) satisfying (66). Then

\[
\lim_{N \to \infty} \mathbb{E}_{m, n \in \mathfrak{G}(\mathbb{N})^D} \int_{\mathcal{M}_K^c} \prod_{i=1}^d \chi(m + a_i n) \prod_{i=1}^d \bar{\chi}(m + a'_i n) d\nu(\chi)
\]

\[
= \lim_{N \to \infty} \mathbb{E}_{m, n \in \mathfrak{G}(\mathbb{N})^D} \int_{\mathcal{M}_K^c} \prod_{i=1}^d \chi \circ \tau_{b_i}(\mathfrak{G}(m + nA_{\mathcal{B}}(a_i))) \prod_{i=1}^d \bar{\chi} \circ \tau_{b_i}(\mathfrak{G}(m + nA_{\mathcal{B}}(a'_i))) d\nu(\chi) > 0.
\]
The proof of Proposition [10.11] is similar to Proposition 10.4 of [18] and Proposition 3.3 of [43]. We omit the proof but stress the differences.

Suppose first that \( a_i = a'_i \) for some \( 1 \leq i, j \leq d \). We assume without loss of generality that \( a_d = a'_d \). Since \( \chi \) is of modulus 1,

\[
\prod_{i=1}^{d} \chi(m + a_i n) = \prod_{i=1}^{d} \chi(m + a'_i n).
\]

This implies that we may remove the terms \( m + a_i n \) and \( m + a'_i n \) simultaneously from the statement of Proposition [10.11] and replace \( d \) with \( d - 1 \). Since \( \{a_1, \ldots, a_d\} \neq \{a'_1, \ldots, a'_d\} \), we can not remove all of \( a_1, \ldots, a_d, a'_1, \ldots, a'_d \) by using this induction. In conclusion, it suffices to prove Proposition [10.11] under the additional assumption that \( d \geq 1 \) and all of \( a_1, \ldots, a_d, a'_1, \ldots, a'_d \) are distinct. By a change of variables, we may further assume that one (and only one) of them is 0.

For \( A \in M_{D \times D}(\mathbb{Z}) \), let \( H(A) \) denote the height of \( A \). Let

\[
\ell := \sum_{i=1}^{d} H(A_g(a_i)) + \sum_{i=1}^{d} H(A_g(a'_i)) + 10
\]

and \( \tilde{N} \) be the smallest prime number (in \( \mathbb{N} \)) greater than \( 10D/\ell \). Let \( \chi_N : \mathbb{Z}_\tilde{N}^D \to \mathbb{C} \) denote the truncated function given by \( \chi_N(n) = \chi \circ \tau_{g}(n) \) for all \( n \in \{1, \ldots, N\}^D \) and \( \chi_N(n) = 0 \) otherwise. In order to show (68), it suffices to show that

\[
\liminf_{N \to \infty} \frac{1}{N} \sum_{m \in \mathbb{Z}_N^D} \left| \prod_{i=1}^{d} \chi_N(m + nA_g(a_i)) \right| \prod_{i=1}^{d} \chi_N(m + nA_g(a'_i)) dv(\chi) > 0
\]

(in fact the left hand side of (68) equals to a constant multiple of (69) for a reason similar to (10.13) of [18]).

Applying Theorem [9.2] for \( \Omega = 10D/\ell \) and with \( d \) replaced by \( 2d - 1 \), we may decompose the truncated function \( \chi_{N, \tilde{N}} := \chi_N \) into the sum \( \chi_{N} = \chi_{N,s} + \chi_{N,u} + \chi_{N,e} \) satisfying the statements in Theorem [9.2] and expand the left hand side of (69) into \( 3^{2d} \) terms. Let \( \epsilon > 0 \) be a sufficiently small error term. By a similar argument as in the proof of Proposition 10.5 in [18] (the estimation of the \( A_3(N) \) term on pages 71–72), we have that

\[
\liminf_{N \to \infty} \frac{1}{N} \sum_{m \in \mathbb{Z}_N^D} \left| \prod_{i=1}^{d} \chi_{N,s}(m + nA_g(a_i)) \right| \prod_{i=1}^{d} \chi_{N,u}(m + nA_g(a'_i)) dv(\chi)
\]

is bounded below by a positive constant which is independent of \( \epsilon \) (to obtain such an estimate, one needs to invoke the property (66) of the measure \( \nu \) and use an immediate generalization of Lemma 10.6 of [18]).

Now it suffices to show that all other terms are negligible. A term is obviously \( O(\epsilon) \) if it contains the expression \( \chi_{N,e} \). Since \( \|\chi_{N,u}\|_{L^{2d-1}(\mathbb{Z}_N^D)} \ll \epsilon \), it suffices to show that all terms containing the expression \( \chi_{N,u} \) are negligible, which holds immediately if one can show that

\[
\left| \frac{1}{N} \sum_{m \in \mathbb{Z}_N^D} \left| f_i(m + nA_g(a_i)) \right| \right|_{L^{2d-1}(\mathbb{Z}_N^D)} \leq C \min_{1 \leq i \leq 2d} \|f_i\|_{L^{2d-1}(\mathbb{Z}_N^D)} + \frac{10}{N}
\]

for all functions \( f_1, \ldots, f_{2d} : \mathbb{Z}_N^D \to \mathbb{C} \) with modulus at most 1 for some \( C := C(a_1, \ldots, a_{2d}) > 0 \). The proof of (70) is a straightforward generalization of Lemma 10.7 in [18], and so we are done.
It is worth noting that in the proof of (70), we need to use the fact that for all \( a \in O_K \) such that \( |N_m(a)| < N \), the map \( x \rightarrow xA_m(a) \mod \mathbb{Z}_N^D \) is a bijection from \( \mathbb{Z}_N^D \) to itself.

**Appendix A. Equivalent definitions for aperiodic functions**

In this appendix, we show that the two definitions of aperiodic functions (1) and (3) are equivalent.

**Lemma A.1.** Let \( f: \mathbb{Z} \rightarrow \mathbb{C} \) be a function with modulus at most 1. Then

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(an + b) = 0
\]

for all \( a, b \in \mathbb{Z}, a \neq 0 \) if and only if

\[
\lim_{N \to \infty} \sup_{L \in \mathbb{N}, a, b \in \mathbb{Z}, a \neq 0} \left| \frac{1}{2N+1} \sum_{n=-N}^{N} 1_{P_{a,b,L}}(n) \cdot f(n) \right| = 0,
\]

where \( P_{a,b,L} := \{ am + b \in \mathbb{Z} : 0 \leq m \leq L - 1 \} \).

**Proof.** (72) obviously implies (71). Now suppose that (72) fails for some function \( f \). Then there exist \( \epsilon > 0, N_i, L_i \in \mathbb{N}_+ \) and \( a_i, b_i \in \mathbb{Z}, a_i \neq 0 \) for \( i \in \mathbb{N} \) such that \( N_{i+1} > N_i \) and that

\[
\left| \frac{1}{2N_i+1} \sum_{n=-N_i}^{N_i} 1_{P_{a_i,b_i,L_i}}(n) \cdot f(n) \right| > \epsilon
\]

for all \( i \in \mathbb{N} \). Since \( |f| \leq 1 \), we get from (73) that

\[
\epsilon < \frac{1}{2N_i+1} \left| P_{a_i,b_i,L_i} \right| \leq \frac{2N_i+1}{2N_i} + 1.
\]

So if \( i \) is sufficiently large, then \( |a_i| \leq \frac{2}{\epsilon} \). Since \( a_i \) only take finitely many values, there exist infinitely many \( i \) such that these \( a_i \) take the same value \( a_0 \), and \( b_i \equiv b_0 \mod a_0 \) for some \( 0 \leq b_0 < |a_0| \). In conclusion, there exist an infinitely sequence of integers \( N_i \in \mathbb{N} \) (which is still denoted by \( N_i \)), and \( M_i, M_i' \in \mathbb{Z} \) with \( -N_i \leq a_0M_i + b_0 \leq a_0M_i' + b_0 \leq N_i \) such that

\[
\left| \frac{1}{2N_i+1} \sum_{n=M_i}^{M_i'} f(a_0n + b_0) \right| > \epsilon.
\]

We may assume without loss of generality that \( a_0 > 0 \) as the other case is similar. Then \( M_i - M_i' \leq 2N_i \).

Assume that (71) holds for all \( a, b \in \mathbb{Z}, a \neq 0 \). Then there exists \( M_0 := M_0(\epsilon) > 0 \) such that for all \( M > M_0 \), we have that

\[
\left| \sum_{n=0}^{M-1} f(a_0n + b_0) \right| \left| \sum_{n=-M+1}^{0} f(a_0n + b_0) \right| < M\epsilon/2.
\]

In other words, for all \( M \in \mathbb{N}_+ \),

\[
\left| \sum_{n=0}^{M-1} f(a_0n + b_0) \right| \left| \sum_{n=-M+1}^{0} f(a_0n + b_0) \right| < \max\{M\epsilon/2, M_0\}.
\]
So we may deduce from (74) that
\[
(2N_i + 1)\varepsilon < (M'_i - M_i)\varepsilon/2 + 2M_0 \leq (2N_i + 1)\varepsilon/2 + 2M_0,
\]
which is a contradiction if \(i\) is sufficiently large. This contradiction implies that (71) implies (72). □

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