VALUATIONS WITH INFINITE LIMIT-DEPTH

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ABSTRACT. For a certain field $K$, we construct a valuation-algebraic valuation on the polynomial ring $K[x]$, whose underlying Maclane–Vaquié chain consists of an infinite (countable) number of limit augmentations.

INTRODUCTION

Let $(K, v)$ be a valued field. In a pioneering work, Maclane studied the extensions of the valuation $v$ to the polynomial ring $K[x]$ in the case $v$ discrete of rank one [10]. He proved that all extensions of $v$ to $K[x]$ can be obtained as a kind of limit of chains of augmented valuations:

\[
\mu_0 \xrightarrow{\phi_1, \gamma_1} \mu_1 \xrightarrow{\phi_2, \gamma_2} \cdots \xrightarrow{\phi_n, \gamma_n} \mu_n \xrightarrow{\phi_{n+1}, \gamma_{n+1}} \mu_{n+1} \xrightarrow{\phi_{n+2}, \gamma_{n+2}} \cdots \xrightarrow{\phi_m, \gamma_m} \mu_m
\]

involving the choice of certain key polynomials $\phi_n \in K[x]$ and elements $\gamma_n$ belonging to some extension of the value group of $v$.

These chains of valuations contain relevant information on $\mu$ and play a crucial role in the resolution of many arithmetic-geometric tasks in number fields and function fields of curves [2, 3].

For valued fields of arbitrary rank, several approaches to this problem were developed by Alexandru-Popescu-Zaharescu [1], Kuhlmann [8], Herrera-Mahboub-Olalla-Spivakovsky [11, 12] and Vaquié [15, 17].

In this general context, limit augmentations and the corresponding limit key polynomials appear as a new feature. In the henselian case, limit augmentations are linked with the existence of defect in the extension $\mu/v$ [16]. Thus, they are an obstacle for local uniformization in positive characteristic.

A chain as in (1) is said to be a MacLane–Vaquié chain if it is constructed as a mixture of ordinary and limit augmentations, and satisfies certain additional technical condition (see Section 1.5). In this case, the intermediate valuations $\mu_n$ are essentially unique and contain intrinsic information about the valuation $\mu$ [12, Thm. 4.7].

In particular, the number of limit augmentations of any MacLane–Vaquié chain of $\mu$ is an intrinsic datum of $\mu$, which is called the limit-depth of $\mu$.

In this paper, we exhibit an example of a valuation with an infinite limit-depth, inspired in a construction by Kuhlmann of infinite towers of Artin-Schreier extensions with defect [9].
1. Maclane–Vaquié chains of valuations on \( K[x] \)

In this section we recall some well-known results on valuations on a polynomial ring, mainly extracted from the surveys \([11]\) and \([12]\).

Let \((K, v)\) be a valued field, with valuation ring \(\mathcal{O}_v\) and residue class field \(k\). Let \(\Gamma = v(K^*)\) be the value group and denote by \(\Gamma_Q = \Gamma \otimes \mathbb{Q}\) the divisible hull of \(\Gamma\). In the sequel, we write \(\Gamma_Q \infty\) instead of \(\Gamma_Q \cup \{\infty\}\).

Consider the set \(T\) of all \(\Gamma_Q\)-valued extensions of \(v\) to the field \(K(x)\) of rational functions in one indeterminate. That is, an element \(\mu \in T\) is a valuation on \(K[x]\),\[ \mu : K[x] \rightarrow \Gamma_Q \infty, \]
such that \(\mu|_K = v\) and \(\mu^{-1}(\infty) = \{0\}\). Let \(\Gamma_\mu = \mu(K(x)^*)\) be the value group and \(k_\mu\) the residue field.

This set \(T\) admits a partial ordering. For \(\mu, \nu \in T\) we say that \(\mu \leq \nu\) if \[ \mu(f) \leq \nu(f), \quad \forall f \in K[x], \]
As usual, we write \(\mu < \nu\) to indicate that \(\mu \leq \nu\) and \(\mu \neq \nu\).

This poset \(T\) has the structure of a tree; that is, all intervals \((-\infty, \mu] := \{\rho \in T | \rho \leq \mu\}\) are totally ordered \([12, \text{Thm. 2.4}]\).

A node \(\mu \in T\) is a leaf if it is a maximal element with respect to the ordering \(\leq\). Otherwise, we say that \(\mu\) is an inner node.

The leaves of \(T\) are the valuation-algebraic valuations in Kuhlmann’s terminology \([8]\). The inner nodes are the residually transcendental valuations, characterized by the fact that the extension \(k_\mu/k\) is transcendental. In this case, its transcendence degree is necessarily equal to one \([8]\).

1.1. Graded algebra and key polynomials. Take any \(\mu \in T\). For all \(\alpha \in \Gamma_\mu\), consider the \(O_v\)-modules:
\[ P_\alpha = \{g \in K[x] | \mu(g) \geq \alpha\} \supset \mathcal{P}_\alpha^+ = \{g \in K[x] | \mu(g) > \alpha\}. \]
The graded algebra of \(\mu\) is the integral domain:
\[ G_\mu = \bigoplus_{\alpha \in \Gamma_\mu} \mathcal{P}_\alpha/\mathcal{P}_\alpha^+. \]

There is an initial term mapping \(\text{in}_\mu : K[x] \rightarrow G_\mu\), given by \(\text{in}_\mu 0 = 0\) and \(\text{in}_\mu g = g + \mathcal{P}_\mu^+\) for all nonzero \(g \in K[x]\).

The following definitions translate properties of the action of \(\mu\) on \(K[x]\) into algebraic relationships in the graded algebra \(G_\mu\).

**Definition.** Let \(g, h \in K[x]\).

We say that \(g\) is \(\mu\)-divisible by \(h\), and we write \(h |_\mu g\), if \(\text{in}_\mu h | \text{in}_\mu g\) in \(G_\mu\).

We say that \(g\) is \(\mu\)-irreducible if \(\text{in}_\mu g\) is a prime element; that is, the homogeneous principal ideal of \(G_\mu\) generated by \(\text{in}_\mu g\) is a prime ideal.

We say that \(g\) is \(\mu\)-minimal if \(g \nmid_\mu f\) for all nonzero \(f \in K[x]\) with \(\deg(f) < \deg(g)\).

Let us recall a well-known characterization of \(\mu\)-minimality \([11, \text{Prop. 2.3}]\).
Lemma 1.1. A polynomial \( g \in K[x] \setminus K \) is \( \mu \)-minimal if and only if \( \mu \) acts as follows on \( g \)-expansions:
\[
f = \sum_{0 \leq n} a_n g^n, \quad \deg(a_n) < \deg(g) \quad \implies \quad \mu(f) = \min_{0 \leq n} \{\mu(a_n g^n)\}.
\]

Definition. A (Maclane-Vaquié) key polynomial for \( \mu \) is a monic polynomial in \( K[x] \) which is simultaneously \( \mu \)-minimal and \( \mu \)-irreducible. The set of key polynomials for \( \mu \) is denoted \( KP(\mu) \).

All \( \phi \in KP(\mu) \) are irreducible in \( K[x] \). For all \( \phi \in KP(\mu) \) let \([\phi]_\mu \subset KP(\mu)\) be the subset of all key polynomials \( \varphi \in KP(\mu) \) such that \( \in_\mu \varphi = \in_\mu \phi \).

Lemma 1.2. [15] Thm. 1.15] Let \( \mu < \nu \) be two nodes in \( \mathcal{T} \). Let \( t(\mu, \nu) \) be the set of monic polynomials \( \phi \in K[x] \) of minimal degree satisfying \( \mu(\phi) < \nu(\phi) \). Then, \( t(\mu, \nu) \subset KP(\mu) \) and \( t(\mu, \nu) = [\phi]_\mu \) for all \( \phi \in t(\mu, \nu) \).

Moreover, for all \( f \in K[x] \), the equality \( \mu(f) = \nu(f) \) holds if and only if \( \phi \nmid_\mu f \).

The existence of key polynomials characterizes the inner nodes of \( \mathcal{T} \).

Theorem 1.3. A node \( \mu \in \mathcal{T} \) is a leaf if and only if \( KP(\mu) = \emptyset \).

Definition. The degree \( \deg(\mu) \) of an inner node \( \mu \in \mathcal{T} \) is defined as the minimal degree of a key polynomial for \( \mu \).

1.2. Depth zero valuations. For all \( a \in K, \gamma \in \Gamma_Q \), consider the depth-zero valuation
\[
\mu = \omega_{a,\delta} = [v; x-a, \gamma] \in \mathcal{T},
\]
defined in terms of \((x-a)\)-expansions as
\[
f = \sum_{0 \leq n} a_n(x-a)^n \quad \implies \quad \mu(f) = \min\{v(a_n) + n\gamma \mid 0 \leq n\}.
\]

Note that \( \mu(x-a) = \gamma \). Clearly, \( x-a \) is a key polynomial for \( \mu \) of minimal degree and \( \Gamma_\mu = \langle \Gamma, \gamma \rangle \). In particular, \( \mu \) is an inner node of \( \mathcal{T} \) with \( \deg(\mu) = 1 \).

One checks easily that
\[
\omega_{a,\delta} \leq \omega_{b,\epsilon} \iff v(a-b) \geq \delta \text{ and } \epsilon \geq \delta.
\]

1.3. Ordinary augmentation of valuations. Let \( \mu \in \mathcal{T} \) be an inner node. For all \( \phi \in KP(\mu) \) and all \( \gamma \in \Gamma_Q \) such that \( \mu(\phi) < \gamma \), we may construct the ordinary augmented valuation
\[
\mu' = [\mu; \phi, \gamma] \in \mathcal{T},
\]
defined in terms of \( \phi \)-expansions as
\[
f = \sum_{0 \leq n} a_n\phi^n, \quad \deg(a_n) < \deg(\phi) \quad \implies \quad \mu'(f) = \min\{\mu(a_n) + n\gamma \mid 0 \leq n\},
\]

Note that \( \mu'(\phi) = \gamma, \mu < \mu' \) and \( t(\mu, \mu') = [\phi]_\mu \).

By [11] Cor. 7.3], \( \phi \) is a key polynomial for \( \mu' \) of minimal degree. In particular, \( \mu' \) is an inner node of \( \mathcal{T} \) too, with \( \deg(\mu') = \deg(\phi) \).
1.4. Limit augmentation of valuations. Consider a totally ordered family of inner nodes of $T$, not containing a maximal element:

$$\mathcal{C} = (\rho_i)_{i \in A} \subset T.$$ 

We assume that $\mathcal{C}$ is parametrized by a totally ordered set $A$ of indices such that the mapping $A \to \mathcal{C}$ determined by $i \mapsto \rho_i$ is an isomorphism of totally ordered sets.

If $\deg(\rho_i)$ is stable for all sufficiently large $i \in A$, we say that $\mathcal{C}$ has stable degree, and we denote this stable degree by $\deg(\mathcal{C})$.

We say that $f \in K[x]$ is $\mathcal{C}$-stable if, for some index $i \in A$, it satisfies

$$\rho_i(f) = \rho_j(f), \quad \text{for all } j > i.$$

Lemma 1.4. A nonzero $f \in K[x]$ is $\mathcal{C}$-stable if and only if $\text{in}_{\rho_i} f$ is a unit in $\mathcal{G}_{\rho_i}$ for some $i \in A$.

Proof. Suppose that $\text{in}_{\rho_i} f$ is a unit in $\mathcal{G}_{\rho_i}$ for some $i \in A$. Take any $j > i$ in $A$, and let $t(\rho_i, \rho_j) = [\varphi]_{\rho_i}$. By Lemma 1.2, $\varphi \in \text{KP}(\rho_i)$, so that $\text{in}_{\rho_i} \varphi$ is a prime element. Hence, $\varphi \mid_{\rho_i} f$, and this implies $\rho_i(f) = \rho_j(f)$, again by Lemma 1.2.

Conversely, if $f$ is $\mathcal{C}$-stable, there exists $i_0 \in A$ such that $\rho_{i_0}(f) = \rho_i(f)$ for all $i > i_0$. Hence, $\text{in}_{\rho_i} f$ is the image of $\text{in}_{\rho_{i_0}} f$ under the canonical homomorphism $\mathcal{G}_{\rho_{i_0}} \to \mathcal{G}_{\rho_i}$. By [12 Cor. 2.6], $\text{in}_{\rho_i} f$ is a unit in $\mathcal{G}_{\rho_i}$.

We obtain a stability function $\rho_{\mathcal{C}}$, defined on the set of all $\mathcal{C}$-stable polynomials by

$$\rho_{\mathcal{C}}(f) = \max\{\rho_i(f) \mid i \in A\}.$$ 

Definition. We say that $\mathcal{C}$ has a stable limit if all polynomials in $K[x]$ are $\mathcal{C}$-stable. In this case, $\rho_{\mathcal{C}}$ is a valuation in $T$ and we say that

$$\rho_{\mathcal{C}} = \lim_{i \in A} \rho_i.$$

Suppose that $\mathcal{C}$ has no stable limit. Let $\text{KP}_\infty(\mathcal{C})$ be the set of all monic $\mathcal{C}$-unstable polynomials of minimal degree. The elements in $\text{KP}_\infty(\mathcal{C})$ are said to be limit key polynomials for $\mathcal{C}$. Since the product of stable polynomials is stable, all limit key polynomials are irreducible in $K[x]$.

Definition. We say that $\mathcal{C}$ is an essential continuous family of valuations if it has stable degree and it admits limit key polynomials whose degree is greater than $\deg(\mathcal{C})$.

For all limit key polynomials $\phi \in \text{KP}_\infty(\mathcal{C})$, and all $\gamma \in \Gamma_Q$ such that $\rho_i(\phi) < \gamma$ for all $i \in A$, we may construct the limit augmented valuation

$$\mu = [\mathcal{C}; \phi, \gamma] \in T$$

defined in terms of $\phi$-expansions as:

$$f = \sum_{0 \leq n} a_n \phi^n, \quad \deg(a_n) < \deg(\phi) \implies \mu(f) = \min\{\rho_{\mathcal{C}}(a_n) + n\gamma \mid 0 \leq n\}.$$ 

Since $\deg(a_n) < \deg(\phi)$, all coefficients $a_n$ are $\mathcal{C}$-stable. Note that $\mu(\phi) = \gamma$ and $\rho_i < \mu$ for all $i \in A$. By [11 Cor. 7.13], $\phi$ is a key polynomial for $\mu$ of minimal degree, so that $\mu$ is an inner node of $T$ with $\deg(\mu) = \deg(\phi)$. 
1.5. Maclane–Vaquié chains. Consider a countable chain of valuations in $\mathcal{T}$:

\[(3) \quad v \xrightarrow{\phi_0, \gamma_0} \mu_0 \xrightarrow{\phi_1, \gamma_1} \mu_1 \xrightarrow{\phi_2, \gamma_2} \cdots \xrightarrow{\phi_n, \gamma_n} \mu_n \longrightarrow \cdots\]

in which $\phi_0 \in K[x]$ is a monic polynomial of degree one, $\mu_0 = [v; \phi_0, \gamma_0]$ is a depth-zero valuation, and each other node is an augmentation of the previous node, of one of the two types:

- **Ordinary augmentation:** $\mu_{n+1} = [\mu_n; \phi_{n+1}, \gamma_{n+1}]$, for some $\phi_{n+1} \in \text{KP}(\mu_n)$.

- **Limit augmentation:** $\mu_{n+1} = [\mathcal{C}_n; \phi_{n+1}, \gamma_{n+1}]$, for some $\phi_{n+1} \in \text{KP}_\infty(\mathcal{C}_n)$, where $\mathcal{C}_n$ is an essential continuous family whose first valuation is $\mu_n$.

Therefore, $\phi_n$ is a key polynomial for $\mu_n$ of minimal degree and $\deg(\mu_n) = \deg(\phi_n)$, for all $n \geq 0$.

**Definition.** A chain of mixed augmentations as in (3) is said to be a Maclane–Vaquié (MLV) chain if every augmentation step satisfies:

- If $\mu_n \rightarrow \mu_{n+1}$ is ordinary, then $\deg(\mu_n) < \deg(\mu_{n+1})$.
- If $\mu_n \rightarrow \mu_{n+1}$ is limit, then $\deg(\mu_n) = \deg(\mathcal{C}_n)$ and $\phi_n \not\in \mathfrak{t}(\mu_n, \mu_{n+1})$.

In this case, we have $\phi_n \not\in \mathfrak{t}(\mu_n, \mu_{n+1})$ for all $n$. As shown in [12, Sec. 4.1], this implies that $\mu(\phi_n) = \gamma_n$ and $\Gamma_{\mu_n} = \langle \Gamma_{\mu_{n-1}}, \gamma_n \rangle$ for all $n$.

The following theorem is due to Maclane, for the discrete rank-one case [10], and Vaquié for the general case [15]. Another proof may be found in [12, Thm. 4.3].

**Theorem 1.5.** Every node $\mu \in \mathcal{T}$ falls in one, and only one, of the following cases.

(a) It is the last valuation of a finite MLV chain.

\[
\mu_0 \xrightarrow{\phi_1, \gamma_1} \mu_1 \xrightarrow{\phi_2, \gamma_2} \cdots \xrightarrow{\phi_{r-1}, \gamma_{r-1}} \mu_r = \mu.
\]

(b) It is the stable limit of an essential continuous family, $\mathcal{C} = (\rho_i)_{i \in A}$, whose first valuation $\mu_r$ falls in case (a):

\[
\mu_0 \xrightarrow{\phi_1, \gamma_1} \mu_1 \xrightarrow{\phi_2, \gamma_2} \cdots \xrightarrow{\phi_{r-1}, \gamma_{r-1}} \mu_{r-1} \xrightarrow{\phi_r, \gamma_r} \mu_r = \mu.
\]

Moreover, we may assume that $\deg(\mu_r) = \deg(\mathcal{C})$ and $\phi_r \not\in \mathfrak{t}(\mu_r, \mu)$.

(c) It is the stable limit, $\mu = \lim_{n \in \mathbb{N}} \mu_n$, of an infinite MLV chain.

The main advantage of MLV chains is that their nodes are essentially unique, so that we may read in them several data intrinsically associated to the valuation $\mu$.

For instance, the sequence $(\deg(\mu_n))_{n \geq 0}$ and the character “ordinary” or “limit” of each augmentation step $\mu_n \rightarrow \mu_{n+1}$, are intrinsic features of $\mu$ [12, Sec. 4.3].

Thus, we may define order preserving functions

\[
\text{depth, lim-depth}: \mathcal{T} \rightarrow \mathbb{N}_\infty,
\]

where depth($\mu$) is the length of the MLV chain underlying $\mu$, and lim-depth($\mu$) counts the number of limit augmentations in this MLV chain.

It is easy to construct examples of valuations on $K[x]$ of infinite depth. In the next section, we show the existence of valuations with infinite limit-depth too. Their construction is much more involved.
2. A valuation with an infinite limit-depth

In this section, we exhibit an example of a valuation with an infinite limit-depth, based on explicit constructions by Kuhlmann, of infinite towers of field extensions with defect [9].

For a prime number $p$, let $\mathbb{F}$ be an algebraic closure of the prime field $\mathbb{F}_p$. For an indeterminate $t$, consider the fields of Laurent series, Newton-Puiseux series and Hahn series in $t$, respectively:

$$\mathbb{F}((t)) \subset K = \bigcup_{N \in \mathbb{N}} \mathbb{F}((t^{1/N})) \subset H = \mathbb{F}((t^\mathbb{Q})).$$

For a generalized power series $s = \sum_{q \in \mathbb{Q}} a_q t^q$, its support is a subset of $\mathbb{Q}$:

$$\text{supp}(s) = \{ q \in \mathbb{Q} \mid a_q \neq 0 \}.$$ 

The Hahn field $H$ consists of all generalized power series with well-ordered support. The Newton-Puiseux field $K$ contains all series whose support is included in $\frac{1}{N}\mathbb{Z} \geq m$ for some $N \in \mathbb{N}$, $m \in \mathbb{Z}$.

From now on, we denote by $\text{Irr}_K(b)$ the minimal polynomial over $K$ of any $b \in \overline{K}$. On these three fields we may consider the valuation $v$ defined as

$$v(s) = \min(\text{supp}(s)),$$

which clearly satisfies,

$$v(\mathbb{F}((t))^*) = \mathbb{Z}, \quad v(K^*) = v(H^*) = \mathbb{Q}.$$ 

The valued field $(\mathbb{F}((t)), v)$ is henselian, because it is the completion of the discrete rank-one valued field $(\mathbb{F}(t), v)$. Since the extension $\mathbb{F}((t)) \subset K$ is algebraic, the valued field $(K, v)$ is henselian too.

The Hahn field $H$ is algebraically closed. Thus, it contains an algebraic closure $\overline{K}$ of $K$. The algebraic generalized power series have been described by Kedlaya [6, 7]. Let us recall [8, Lem. 3], which is essential for our purposes.

**Lemma 2.1.** If $s \in H$ is algebraic over $K$, then it is contained in a tower of Artin-Schreier extensions of $K$. In particular, $s$ is separable over $K$ and $\deg_K s$ is a power of $p$.

Any $s \in H$ determines a valuation on $H[x]$ extending $v$:

$$v_s : H[x] \rightarrow \mathbb{Q}_\infty, \quad g \mapsto v_s(g) = v(g(s)).$$

We are interested in the valuation on $K[x]$ obtained by restriction of $v_s$, which we still denote by the same symbol $v_s$. If $s$ is algebraic over $K$ and $f = \text{Irr}_K(s) \in K[x]$, we have $v_s(f) = \infty$. Hence, $v_s$ cannot be extended to a valuation on $K(x)$.

On the other hand, suppose that $s = \sum_{q \in \mathbb{Q}} a_q t^q \in H$ is transcendental over $K$ and all its truncations

$$s_r = \sum_{q \in \mathbb{Q}} a_q t^q, \quad r \in \mathbb{R},$$

are algebraic over $K$ and have a bounded degree over $K$. Then, it is an easy exercise to check that $v_s$ falls in case (b) of Theorem 1.5.

Therefore, our example of a valuation with infinite limit-depth must be given by a transcendental $s \in H$, all whose truncations are algebraic over $K$ and have unbounded
degree over $K$. In this case, $v_s$ will necessarily fall in case (c) of Theorem 1.5. We want to find an example such that, moreover, all steps in the MLV chain of $v_s$ are limit augmentations.

By Lemma 2.1, the truncations of $s$ must belong to some tower of Artin-Schreier extensions of $K$. Let us use a concrete tower constructed by Kuhlmann [9, Ex. 3.14].

2.1. A tower of Artin-Schreier extensions of $K$. Let $\text{AS}(g) = g^p - g$ be the Artin-Schreier operator on $K[x]$. It is $\mathbb{F}_p$-linear and has kernel $\mathbb{F}_p$.

Let us start with the classical Abhyankar’s example

$$s_0 = \sum_{i \geq 1} t^{-1/p^i} \in H,$$

which is a root of the polynomial $\phi_0 = \text{AS}(x) - t^{-1} \in K[x]$. Since the denominators of the support of $s_0$ are unbounded, we have $s_0 \notin K$. Since the roots of $\phi_0$ are $s_0 + \ell$, for $\ell$ running on $\mathbb{F}_p$, the polynomial $\phi_0$ has no roots in $K$. Hence, $\phi_0$ is irreducible in $K[x]$, because all irreducible polynomials in $K[x]$ have degree a power of $p$.

Now, we iterate this construction to obtain a tower of Artin-Schreier extensions

$$K \subset K(s_0) \subset K(s_1) \subset \cdots \subset K(s_n) \subset \cdots$$

where $s_n \in H$ is taken to be a root of $\phi_n = \text{AS}(x) - s_{n-1}$. The above argument shows that $\phi_n$ is irreducible in $K(s_{n-1})$ as long as $s_n \notin K(s_{n-1})$, which is easy to check.

From the algebraic relationship $\text{AS}(s_n) = s_{n-1}$ we may deduce a concrete choice for all $s_n$:

$$s_n = \sum_{j \geq n} \binom{j}{n} t^{1/p^{j+1}}, \quad \text{for all } n \geq 0,$$

which follows from the well-known identity

$$\binom{j+1}{n+1} = \binom{j}{n+1} + \binom{j}{n}, \quad \text{for all } j \geq n.$$

In particular,

$$\deg_K s_n = p^{n+1}, \quad v(s_n) = -1/p^{n+1}, \quad \text{for all } n \geq 0.$$

For all $n \geq 0$, we have $\text{Irr}_K(s_n) = \text{AS}^{n+1}(x) - t^{-1}$, and the set of roots of this polynomial is

$$(4) \quad Z(\text{Irr}_K(s_n)) = s_n + \text{Ker}(\text{AS}^{n+1}) \subset s_n + \mathbb{F}.$$ 

In particular, the support of all these conjugates of $s_n$ is contained in $(-1,0]$, and Krasner’s constant of $s_n$ is zero:

$$(5) \quad \Delta(s_n) = \max \{v(s_n - \sigma(s_n)) \mid \sigma \in \text{Gal}(\overline{K}/K), \sigma(s_n) \neq s_n\} = 0.$$

We are ready to define our transcendental $s \in H$ as:

$$s = \sum_{n \geq 0} t^n s_n.$$

Let us introduce some useful notation to deal with the support of $s$ and its truncations. Consider the well-ordered set

$$S = \left\{(n,i) \in \mathbb{Z}^2_p \mid 0 \leq n \leq i, \quad p \nmid \binom{i}{n}\right\}.$$
The support of $s$ is the image of the following order-preserving embedding

$$
\delta: S \hookrightarrow \mathbb{Q}, \quad (n, i) \mapsto \delta(n, i) = n - \frac{1}{p^{i+1}}.
$$

The limit elements in $S$ are $(n, n)$ for $n \geq 0$. These elements have no immediate predecessor in $S$. On the other hand, all elements in $S$ have an immediate successor:

$$(n, i) \rightsquigarrow (n, i + m),$$

where $m$ is the least natural number such that $p \nmid \binom{i + m}{n}$.

For all $(n, i) \in S$ we consider the truncations of $s$ determined by the rational numbers $\delta(n, i)$:

$$s_{n,i} := s_{\delta(n,i)} = \sum_{m=0}^{n-1} t^m s_{m} + t^n \sum_{j=n}^{i-1} \binom{j}{n} t^{-1/p^{j+1}}.$$

For the limit indices $(n, n) \in S$ the truncations are:

$$s_{n,n} = \sum_{m=0}^{n-1} t^m s_{m}.$$

Since $(0, 0) = \min(S)$, the truncation $s_{0,0} = 0$ is defined by an empty sum.

All truncations of $s$ are algebraic over $K$. Their degree is

$$\deg_K s_{n,i} = p^n, \quad \text{for all } (n, i) \in S,$$

because $s_{n-1}$ has degree $p^n$, and all other summands have strictly smaller degree. For instance, the “tail” $t^n \sum_{j=n}^{i-1} \binom{j}{n} t^{-1/p^{j+1}}$ belongs to $K$.

The unboundedness of the degrees of the truncations of $s$ is not sufficient to guarantee that $s$ is transcendental over $K$. To this end, we must analyze some more properties of these truncations.

For any pair $(a, \delta) \in \overline{K} \times \mathbb{Q}$, consider the ultrametric ball

$$B = B_\delta(a) = \{ b \in \overline{K} \mid v(b - a) \geq \delta \}.$$

We define the degree of such a ball over $K$ as

$$\deg_K B = \min\{ \deg_K b \mid b \in B \}.$$

**Lemma 2.2.** For all $n \geq 1$, we have $\deg_K B_{n-1} (s_{n,n}) = p^n$.

**Proof.** Denote $B = B_{n-1} (s_{n,n})$. From the computation in [5], we deduce that Krasner’s constant of $s_{n,n}$ is $\Delta(s_{n,n}) = n - 1$. Any $u \in B$ may be written as

$$u = s_{n,n} + \ell t^{n-1} + b, \quad \ell \in \mathbb{F}, \quad b \in \overline{K}, \quad v(b) > n - 1.$$

Let $z = s_{n,n} + \ell t^{n-1}$. Since $\ell t^{n-1}$ belongs to $K$, we have

$$\deg_K z = p^n, \quad \Delta(z) = n - 1.$$ 

Since $v(u - z) > \Delta(z)$, we have $K(z) \subset K(u)$ by Krasner’s lemma. Hence, $\deg_K u \geq p^n$. Since $B$ contains elements of degree $p^n$, we conclude that $\deg_K B = p^n$. \hfill \square

**Corollary 2.3.** The element $s \in H$ is transcendental over $K$. 
Lemma 2.4. For all \( v \) let us show that all polynomials \( x \) valuation on \( K \) coincides with \( v \) of \( n, i \) this result to \( K \). Indeed, this follows from (2) because \( v \) of \( n, i \) \( \omega_{s_n,i,\delta(n,i)} \) be the depth zero valuation on \( \overline{K}[x] \) associated to the pair \((s_n,i,\delta(n,i)) \in K \times \mathbb{Q} \); that is, \[
v_{n,i} \left( \sum_{0 \leq \ell} a_{\ell} (x - s_n,i)^{\ell} \right) = \min_{0 \leq \ell} \left\{ v_s \left( a_{\ell} (x - s_n,i)^{\ell} \right) \right\} = \min_{0 \leq \ell} \{ v(a_{\ell}) + \ell \delta(n,i) \}.
\]

Lemma 2.4. For all \((n,i),(m,j) \in S \) we have \( v_{n,i}(x - s_{m,j}) = \min\{\delta(n,i),\delta(m,j)\} \). In particular, \( v_{n,i} < v_s \) for all \((n,i) \in S \).

Proof. The computation of \( v_{n,i}(x - s_{m,j}) \) follows immediately form the definition of \( v_{n,i} \). The inequality \( v_{n,i} \leq v_s \) follows from the comparison of the action of both valuation on \((x - s_{n,i})\)-expansions. Finally, if we take \( \delta(n,i) < \delta(m,j) \), we get
\[
v_{n,i}(x - s_{m,j}) = \delta(n,i) < \delta(m,j) = v_s(x - s_{m,j}).
\]

This shows that \( v_{n,i} < v_s \).

Lemma 2.5. The family \( C = (v_{n,i})_{(n,i) \in S} \) is a totally ordered family of valuations on \( \overline{K}[x] \) of stable degree one, admitting \( v_s \) as its stable limit.

Proof. Let us see that \( C \) is a totally ordered family of valuations. More precisely,
\[
(n,i) < (m,j) \implies \delta(n,i) < \delta(m,j) \implies v_{n,i} < v_{m,j} < v_s.
\]

Indeed, this follows from (2) because \( v(s_{n,i} - s_{m,j}) = v(s_{n,i} - s) = \delta(n,i) \).

Clearly, \( C \) contains no maximal element, and all valuations in \( C \) have degree one. Let us show that all polynomials \( x - a \in \overline{K}[x] \) are \( C \)-stable, and the stable value coincides with \( v_s(x - a) = v(s - a) \).

Since \( s \) is transcendental over \( K \), we have \( s \neq a \) and \( q = v(s - a) \) belongs to \( \mathbb{Q} \).

For all \((n,i) \in S \) such that \( \delta(n,i) > q \) we have
\[
v_{n,i}(x-a) = \min\{v(a - s_{n,i}),\delta(n,i)\} = \min\{q,\delta(n,i)\} = q = v_s(x-a).
\]

This ends the proof of the lemma.

Therefore, \( v_s \) falls in case (b) of Theorem 1.5 as a valuation on \( \overline{K}[x] \). A MLV chain of \( v_s \) is, for instance,
\[
v_{0,0} \xrightarrow{C} v_s = \lim(C).
\]

In order to obtain a MLV chain of \( v_s \) as a valuation of \( K[x] \), we need to “descend” this result to \( K[x] \). In this regard, we borrow some ideas of [17].
2.3. A MLV chain of $v_s$ as a valuation on $K[x]$. We say that $(a, \delta) \in K \times \mathbb{Q}$ is a minimal pair if $\deg_K B_{\delta}(a) = \deg_K a$. This concept was introduced in [1]. By equation (2), for all $b \in K$ we have

$$\omega_{a, \delta} = \omega_{b, \delta} \iff b \in B_{\delta}(a).$$

However, only the minimal pairs $(a, \delta)$ of this ball contain all essential information about the valuation on $K[x]$ that we obtain by restriction of $\omega_{a, \delta}$.

**Lemma 2.6.** [17] Prop. 3.3] For $(a, \delta) \in K \times \mathbb{Q}$, let $\mu$ be the valuation on $K[x]$ obtained by restriction of the valuation $\omega = \omega_{a, \delta}$ on $K[x]$. Then, for all $g \in K[x]$, $\text{in}_\mu g$ is a unit in $\mathcal{G}_\mu$ if and only if $\text{in}_\omega g$ is a unit in $\mathcal{G}_\omega$.

The following result was originally proved in [13]; another proof can be found in [13] Thm. 1.1.

**Lemma 2.7.** For a minimal pair $(a, \delta) \in K \times \mathbb{Q}$, let $\mu$ be the valuation on $K[x]$ obtained by restriction of the valuation $\omega_{a, \delta}$ on $K[x]$. Then, $\text{Irr}_K(a)$ is a key polynomial for $\mu$, of minimal degree.

We need a last auxiliary result.

**Lemma 2.8.** For all $(n, i) \in S$ the pair $(s_{n, i}, \delta(n, i))$ is minimal.

**Proof.** All $(s_{0, i}, \delta(0, i))$ are minimal pairs, because $\deg_K s_{0, i} = 1$. For $n > 0$, denote $B_{n, i} = B_{\delta(n, i)}(s_{n, i})$. Since $B_{n, i} \subset B_{n-1}(s_{n, n})$, Lemma 2.2 shows that

$$\deg_K B_{n, i} \geq \deg_K B_{n-1}(s_{n, n}) = p^n.$$

Since the center $s_{n, i}$ of the ball $B_{n, i}$ has $\deg_K s_{n, i} = p^n$, we deduce $\deg_K B_{n, i} = p^n$. Thus, $(s_{n, i}, \delta(n, i))$ is a minimal pair. \qed

**Notation.** Let us denote the restriction of $v_{n, i}$ to $K[x]$ by

$$\rho_{n, i} = (v_{n, i})_{|K[x]}.$$

Moreover, for the limit indices $(n, n)$, $n \geq 0$, we denote:

$$\mu_n = \rho_{n, n}, \quad \phi_n = \text{Irr}_K(s_{n, n}), \quad \gamma_n = v_s(\phi_n).$$

By Lemmas 2.4 and 2.5, the set of all valuations $(\rho_{n, i})_{(n, i) \in S}$ is totally ordered, and $\rho_{n, i} < v_s$ for all $(n, i)$.

**Proposition 2.9.** For all $n \geq 0$, the set $C_n = (\rho_{n, i})_{(n, i) \in S}$ is an essential continuous family of stable degree $p^n$. Moreover, the polynomial $\phi_{n+1}$ belongs to $\text{KP}_\infty(C_n)$ and $\mu_{n+1} = [C_n; \phi_{n+1}, \gamma_{n+1}]$.

**Proof.** Let us fix some $n \geq 0$. By Lemmas 2.4 and 2.8, all valuations in $C_n$ have degree $p^n$. Hence, $C_n$ is a totally ordered family of stable degree $p^n$.

Let us show that all monic $g \in K[x]$ with $\deg(g) \leq p^n$ are $C_n$-stable. Let $u \in K$ be a root of $g$. By Lemma 2.2, $u \not\in B_n(s_{n+1, n+1})$, so that $v(s_{n+1, n+1} - u) < n$. Since $v(s - s_{n+1, n+1}) = \delta(n + 1, n + 1) > n$, we deduce that $v(s - u) < n$.

Therefore, we may find $j \geq n$ such that

$$v(s - u) < n - \frac{1}{p^{j+1}}.$$
for all roots \( u \) of \( g \). As we showed along the proof of Lemma 2.3, this implies

\[ v_{n,i}(x - u) = v_s(x - u) \quad \text{for all } (n, i) \geq (n, j) \]

simultaneously for all roots \( u \) of \( g \). Therefore, \( \rho_{n,i}(g) = v_s(g) \) for all \( (n, i) \geq (n, j) \) and \( g \) is \( C_n \)-stable.

Now, let us show that \( \phi_{n+1} \) is \( C_n \)-unstable. For all \( i \geq n \), we have

\[ v(s_{n+1,n+1} - s_{n,i}) = \delta(n, i) = v_n(i - s_{n,i}). \]

By 11 Prop. 6.3, \( x - s_{n+1,n+1} \) is a key polynomial for \( v_{n,i} \); thus, \( v_{n,i}(x - s_{n+1,n+1}) \)

is not a unit in the graded algebra \( G_{v_{n,i}} \). Hence, \( v_{n,i}(x - s_{n+1,n+1}) \)

is not a unit in \( G_{v_{n,i}} \) and Lemma 2.6 shows that \( v_{n,i}(x - s_{n+1,n+1}) \)

is not a unit in \( G_{\rho_{n,i}} \). Since this holds for all \( i \), Lemma 1.4 shows that \( \phi_{n+1} \) is \( C_n \)-unstable.

Since the irreducible polynomials in \( K[x] \) have degree a power of \( p \) (Lemma 2.1), \( \phi_{n+1} \) is an \( C_n \)-unstable polynomial of minimal degree. Therefore, \( C_n \) is an essential continuous family and \( \phi_{n+1} \in K\phi_\infty(C_n) \).

Since \( \phi_{n+1} \) is \( C_n \)-unstable, \( \rho_{n,i}(\phi_{n+1}) < v_s(\phi_{n+1}) = \gamma_{n+1} \) for all \( i \). Thus, it makes sense to consider the limit augmentation \( \mu = [C_n; \phi_{n+1}, \gamma_{n+1}] \). Let us show that \( \mu = \mu_{n+1} \) by comparing their action on \( \phi_{n+1} \)-expansions. For all \( g = \sum_{0 \leq \ell} a_\ell \ell \phi_{n+1} \),

\[
\mu_{n+1}(g) = \min_{0 \leq \ell} \left\{ \mu_{n+1}(a_\ell \phi_{n+1}^\ell) \right\}, \quad \mu(g) = \min_{0 \leq \ell} \left\{ \mu(a_\ell \phi_{n+1}^\ell) \right\}.
\]

Since \( \deg(a_\ell) < p^{n+1} = \deg(\phi_{n+1}) \), all these coefficients \( a_\ell \) are \( C_n \)-stable. Hence, \( \rho_{n,i}(a_\ell) = v_s(a_\ell) \) for all \( (n, i) \) sufficiently large. Since \( \rho_{n,i} < \mu_{n+1} < v_s \), we deduce

\[
\mu(a_\ell) = \rho_{\infty}(a_\ell) = \rho_{n,i}(a_\ell) = \mu_{n+1}(a_\ell) = v_s(a_\ell).
\]

Finally, for all \( i \geq n + 1 \), we have \( v(s_{n+1,i} - s_{n+1,n+1}) = \delta(n + 1, n + 1) \), so that

\[
v_{n+1,n+1}(x - s_{n+1,n+1}) = \delta(n + 1, n + 1) = v_{n+1,i}(x - s_{n+1,n+1}).
\]

By 4, for all the other roots \( u \) of \( \phi_{n+1} \), the support of \( u \) is contained in \((-1, n] \). Thus, for all \( i \geq n + 1 \) we get

\[
v_{n+1,n+1}(x - u) = v(s_{n+1,n+1} - u) = v(s_{n+1,i} - u) = v_{n+1,i}(x - u).
\]

Since \( \mu_{n+1} = \rho_{n+1,i,n+1} < \rho_{n+1,i} < v_s \), 12 Cor. 2.5 implies

\[
\mu_{n+1}(\phi_{n+1}) = \rho_{n+1,i}(\phi_{n+1}) = v_s(\phi_{n+1}) = \gamma_{n+1} = \mu(\phi_{n+1}).
\]

By 6, we deduce that \( \mu = \mu_{n+1} \).

Therefore, we get a countable chain of limit augmentations

\[
\begin{array}{cccccc}
\phi_1 \xrightarrow{\gamma_1} & \mu_1 \xrightarrow{\phi_2, \gamma_2} & \cdots & \xrightarrow{\phi_{n-1}, \gamma_{n-1}} & \mu_{n-1} \xrightarrow{\phi_n, \gamma_n} & \mu_n \rightarrow \\
\end{array}
\]

which is an MLV chain. Indeed, the MLV condition amounts to

\[
\phi_n \not\in t(\mu_n, \mu_{n+1}) \quad \text{for all } n \geq 0.
\]

This means \( \mu_n(\phi_n) = \mu_{n+1}(\phi_n) \) for all \( n \). Since \( \mu_n < \mu_{n+1} < v_s \), the desired equality follows from \( \mu_n(\phi_n) = \gamma_n = v_s(\phi_n) \).

Finally, the family \( (\mu_n)_{n \in \mathbb{N}} \) has stable limit \( v_s \). Indeed, for all nonzero \( f \in K[x] \), there exists \( n \in \mathbb{N} \) such that \( \deg(f) < p^n = \deg(\mu_n) \). Let \( t(\mu_n, v_s) = [\phi]_{\mu_n} \). Since \( \deg(f) < \deg(\phi) \), we have \( \phi \not\in t(\mu_n, f) \) and this implies \( \mu_n(f) = v_s(f) \) by Lemma 1.2.

As a consequence, \( v_s \) has infinite limit-depth.
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