Abstract

The electric losses in a bulk or film superconductor exposed to a parallel radio-frequency magnetic field may have three origins: In homogeneous vortex-free superconductors losses proportional to the frequency squared originate from the oscillating normal-conducting component of the charge carriers which is always present at temperatures $T > 0$. With increasing field amplitude the induced supercurrents approach the depairing current at which superconductivity breaks down. And finally, if magnetic vortices can penetrate the superconductor they typically cause large losses since they move driven by the AC supercurrent.

INTRODUCTION

The phenomenon of superconductivity was discovered in 1911 by Heike Kamerlingh-Onnes in Leiden. After he had achieved to liquify helium at the temperature of $T = 4.2K$ he observed that the resistivity of Hg became unmeasurably small below some “critical temperature” $T_c = 4.15$ K. A sensitive method to measure the residual resistivity in this “superconducting state” is to observe the temporal decay of the persistent “supercurrents” in a ring, say of Pb ($T_c = 7.2$ K), Sn ($T_c = 3.72$ K), or Nb ($T_c = 7.2$ K) by monitoring the magnetic field generated by the circulating current. It turned out [1] that the supercurrent does not decay measurably, even after several years. Ideally loss-free superconducting wires may thus be used to build coils which keep their magnetic field for years, after the windings have been loaded with current and then are cut short by a superconducting switch.

Thus, DC currents in a superconductor can flow practically loss-free if they are not too large. However, it turned out that alternating currents (AC) in superconductors are not completely loss-free, in particular at high frequencies ($RF =$ radio frequencies, $MW =$ microwave frequencies). There are essentially three effects which cause energy dissipation during current flow in superconductors:

(a) Even in ideally homogeneous bulk superconductors an electric field $E \propto \omega$ (with frequency $\omega/2\pi$) is required to accelerate the “superconducting electrons”, the Cooper pairs of the microscopic BCS theory [2]. This electric field also moves the “normalconducting” electrons that are always present at finite temperatures $T>0$. The dissipated power of this effect is $\propto E^2 \times \omega^2$.

(b) When the current density inside the superconductor reaches the depairing current density $j_{dp}$, the superconducting order parameter is suppressed to zero at the place where $j = j_{dp}$. This means superconductivity disappears and electric losses appear. This nucleation of the normal state typically occurs at the specimen surface or in the center of Abrikosov vortices. In particular, when an increasing magnetic field $H_a$ is applied along a superconducting half space $x>0$ one initially has $j(x) = (H_a/\lambda) \exp(-x/\lambda)$ where $j < j_{dp}$, but when $H_a$ reaches the thermodynamic critical field $H_{c1} = \Phi_0/(\sqrt{8\pi} \lambda \xi_\mu_0)$ one has $j \approx j_{dp}$ near the surface, thus one has $j_{dp} \approx H_a/\lambda$. Here $\Phi_0 = h/2e = 2.07 \cdot 10^{-15}$ Tm$^2$ is the quantum of magnetic flux, $\lambda$ is the magnetic penetration depth, and $\xi$ is the superconducting coherence length. Within Ginzburg-Landau (GL) theory, valid near $T = T_c$, the lengths $\lambda(T) = \kappa \xi(T) \propto (T_c - T)^{-1/2}$ diverge as $T \to T_c$, but the GL parameter $\kappa = \lambda/\xi$ is independent of temperature $T$.

(c) Large dissipation may be caused by vortices inside the superconductor. These move under the action of the induced AC current, which exerts a Lorentz force on the vortices and causes them to oscillate and dissipate energy. At high frequencies the amplitude of this oscillation is smaller then the range of possible pinning forces caused by material inhomogeneities, e.g., precipitates or defects in the crystal lattice. This vortex dissipation then cannot be suppressed by introducing pins.

For a flat bulk type-II superconductor (defined by $\kappa > 1/\sqrt{2}$) in thermodynamic equilibrium it is favorable that part of the magnetic flux penetrates in form of Abrikosov vortices when the applied field $H_a$ equals or exceeds the lower critical field $H_{c1} \approx (\Phi_0/4\pi \lambda^2 \mu_0)(\ln \kappa + 0.5)$ (for $\kappa > 1.5$) and has not yet reached the upper critical field $H_{c2} \approx \Phi_0/(2\pi \xi^2 \mu_0)$ where superconductivity vanishes. The penetration of vortices at an ideally flat surface may be delayed by a surface barrier leading to a higher penetration field $H_p \approx H_c \geq H_{c1}$ (overheating). On the other hand, with superconductors of finite size, demagnetization effects may allow the vortices to penetrate already at much lower fields. In particular, for a large film of width $w$ and thickness $d \ll w$, the penetration field is strongly reduced, $H_p/H_{c1} \approx d/w \cdots \sqrt{d/w} \ll 1$ depending on the edge profile, see below. An infinitely large thin film will thus be penetrated by any perpendicular magnetic field component, even if very small.

AC RESPONSE OF VORTEX-FREE SUPERCONDUCTORS

The puzzling fact that superconductors may carry loss-free DC current but AC currents exhibit electric losses, was explained by the two-fluid model of Gorter and Casimir in 1934 [1]. Later, the microscopic BCS Theory [2] essen-
tially confirmed the two-fluid picture, giving for its phe-
nomenological parameters a microscopic interpretation and explicit expressions.

Two-Fluid Model

The phenomenological two-fluid model assumes that the total electron density is composed of the density of superconducting electrons \( n_s \) and that of normal electrons \( n_n \), which have different relaxation times \( \tau_n \) and \( \tau_s \). Historically, Gorter and Casimir assumed \( n_n \propto t^4 \) (\( t = T/T_c \)) and \( n_s \propto 1 - t^4 \). As in the Drude model [1], the drift velocity \( v \) of each of these two fluids should obey a Newton law,

\[
m \frac{dv}{dt} = eE - m v/\tau
\]

with \( m \) and \( e \) the mass and charge of the electron. The total current density \( J = J_s + J_n \) is the sum of the supercurrent \( J_s = en_s v_s \) and the normal current \( J_n = en_n v_n \). (In all other Sections of this paper the current density is denoted by \( J \).) If one assumes \( \tau_s = \infty \), one obtains from Eq. (1) the first London equation

\[
dJ_s/dt = (n_v^2/m) E = E/(\mu_0 \lambda^2)
\]

with \( \lambda = (m/n_s e^2 \mu_0)^{1/2} \) the London depth. In the London gauge where \( E = -dA/dt \) (induction law) Eq. (2) may be written in form of the second London equation \( J_s = (\mu_0 \lambda^2)^{-1} A \). For the normal electrons one may assume \( \tau_n \ll 1/\omega \) with periodic electric field \( E \propto \exp(i\omega t) \). This gives for the normal current

\[
J_n = (n_v^2 \tau_n/m) E.
\]

Defining the complex conductivity \( \sigma(\omega) = \sigma_1(\omega) - i\sigma_2(\omega) \) by \( J = \sigma(\omega)E \propto \exp(i\omega t) \), one obtains

\[
\sigma_1(\omega) = \left( \frac{n_v e^2/m \omega} {n_v e^2 \tau_n/m} \right) \omega \propto n_v e^2 \tau_n/m,
\]

\[
\sigma_2(\omega) = \frac{n_v e^2/m \omega}{(\mu_0 \lambda^2)^{-1}} \propto 1.
\]

For a normal conductor this yields \( \sigma_1 = \sigma_n, \sigma_2 = 0 \), and the skin depth \( \delta_{\text{skin}} = (2/\mu_0 \sigma_n \omega)^{1/2} \). For superconductors at \( \omega = 0 \) the \( \delta \)-function in \( \sigma_1 \) reflects the ideal DC conductivity, while at finite frequencies the inductive part dominates, \( \sigma_2 \gg \sigma_1 \). However, the small dissipative part \( \sigma_1 = n_v e^2 \tau_n/m \) is important since it causes the AC losses. In the situation with an AC magnetic field parallel to the superconductor surface, the current is forced (a current bias as opposed to a voltage bias) and one has a dissipation per unit volume \( \rho J^2 = \text{Re}\{1/\sigma\}J^2 \approx (\sigma_1/\sigma_2^2)J^2 \approx \sigma_1 E^2 \) since \( \sigma_1 \ll \sigma_2 \). The dissipation is thus proportional to \( n_v \omega^2 \).

The sum \( \sigma = \sigma_1 - i\sigma_2 \) is analogous to a circuit of a resistive channel \( 1/R \propto \sigma_1 \) in parallel to an inductive channel of admittance \( 1/i\omega L \propto \sigma_2 \). Below a frequency \( \omega_0 = R/L \) this circuit is mainly inductive and above mainly resistive. The ratio of the currents in the two channels is \( J_s/J_n = n_s/(n_n \omega_0 T) \). This defines the crossover frequency \( \omega \approx (n_s/n_n)(1/\tau_n) \approx (n_s/n_n) \cdot 10^{11} \text{ Hz} \) [1].

When the superconductor forms the inner wall of a microwave cavity with incident parallel magnetic field of amplitude \( H_{\text{inc}} \), the wave is almost ideally reflected by the wall since a surface screening current \( J_s = 2H_{\text{inc}} \) is induced. The small dissipated power per unit area is then

\[
P_{\text{abs}} = J_s^2 R_s,
\]

where

\[
R_s = \delta^{-1} \text{Re}\{1/\sigma\} = \delta^{-1} \sigma_1/\sigma^2 \approx \delta^{-1} \sigma_1/\sigma_2^2
\]

is the surface resistance (e.g. in units \( \Omega \)). Here \( \delta = [2/\mu_0(|\sigma| + \sigma_2 \omega)]^{1/2} \) is the general skin depth reducing to the superconducting penetration depth \( \lambda \) or to the skin depth \( \delta_{\text{skin}} \) in the super or normal conducting limits. For the superconducting wall one has \( R_s \approx \sigma_1 \mu_0^2 \lambda^2 \omega^2/2 \) and the absorbed versus incident power of this wall is [1]

\[
\frac{P_{\text{abs}}}{P_{\text{inc}}} = J_s^2 R_s = \frac{4R_s}{\mu_0^2} \approx \frac{1}{Q}.
\]

The quality factor \( Q \) of the superconducting cavity is thus inversely proportional to \( R_s \propto Q^{-1} \propto n_v \omega^2 \).

Microscopic Theory

After the BCS theory [2] had given the microscopic explanation of superconductivity, the complex AC conductivity was calculated within this weak-coupling theory [3, 4]. In the extreme local limit (\( \lambda \ll \xi_0 = h v_F/\pi \Delta \) with \( v_F = \text{Fermi velocity} \) and \( \Delta = \text{energy gap} \); this assumption actually is not satisfied for type-II superconductors with GL parameter \( \kappa > 0.7 \), in the impure limit (electron mean free path \( l = v_F T \ll \xi_0 \)), and for frequencies below the energy-gap frequency \( (h\nu < 2\Delta, \nu = \omega/2\pi) \) the resulting AC conductivity may be expressed as two integrals over an energy variable,

\[
\frac{\sigma_1}{\sigma_n} = \int f_1(\epsilon, \Delta, T, \omega) \, d\epsilon,
\]

where \( \sigma_n = n e^2 \tau_n/m \approx n e^2/\hbar v_F (p_F = mv_F = \text{Fermi momentum, } n = \text{electron density}) \) is the Drude conductivity in the normal state and \( f_1 \) and \( f_2 \) are some functions. Evaluating these integrals for the case \( \omega \ll T \ll \Delta \) (in units \( h = k_B = 1 \)) one obtains [5]

\[
\frac{\sigma_1}{\sigma_n} = \frac{2\Delta}{T} \exp\left(-\frac{\Delta}{T}\right) \ln \frac{9T}{4\omega},
\]

\[
\frac{\sigma_2}{\sigma_n} = \frac{\pi \Delta}{\omega}.
\]

The dissipative part \( \sigma_1 \) and inductive part \( \sigma_2 \) may be written in the form of the two-fluid model:

\[
\sigma_1 \approx n_{qp} e^2 l/p_F, \quad \sigma_2 \approx n_s e^2/m \omega,
\]

where \( n_{qp} \) is the quasiparticle density (replacing the normal electron density \( n_n \) of the two-fluid model) and \( n_s \) the superconducting electron density,

\[
n_{qp} = n \frac{\Delta}{T} \exp\left(-\frac{\Delta}{T}\right) 2 \ln \frac{9T}{4\omega},
\]

\[
n_s = n l / \xi_0.
\]
The quality factor $Q$ of the resonator is now
\[ Q^{-1} \propto R_s \approx \frac{1}{2} \mu_0 \lambda \sigma_1 \omega^2 \propto n_{qp} \omega^2. \tag{11} \]
Since the quasiparticle density $n_{qp} \sim \exp(-\Delta/T)$ strongly decreases at low temperatures $T$, $Q$ should increase drastically. Note that with increasing purity (increasing $\sigma_1$) increases but the penetration depth $\lambda \approx \lambda_{\text{pure}} \sqrt{1 + \xi_0 / l}$ decreases in Eq. (11). Thus, maximum $Q$ is reached at some intermediate, not too high purity of the superconductor.

High-Purity Niobium

For the high-purity Nb used in the TESLA cavities, the frequency dependent surface resistance has been computed by Kurt Scharnberg within the Eliashberg model that extends the BCS model to strong coupling superconductors [6]. Strong (electron–phonon) coupling effects change the amplitude and the temperature dependence of the gap parameter, they lead to a renormalization (enhancement) of the quasiparticle mass, which in turn affects the London penetration depth, and they result in temperature and energy dependent quasiparticle lifetimes. The electron-phonon interaction enters in form of the Eliashberg function $\alpha^2 F(\omega)$ which was taken from tunneling experiments. A Coulomb pseudopotential $\mu^* = 0.17$ and a Coulomb cut-off $\omega_c = 240$ meV were used. At sufficiently low $T$ and low $\omega$ of the incident radiation, inelastic scattering is negligible and only disorder induced elastic scattering is important, which is parameterized by the normal state scattering rate $\Gamma_N = 1/2\tau$. This is fit to the surface resistance $R_s$ measured at $\nu = 1.3$ GHz, yielding $\Gamma_N \approx 1$ meV and $\tau \approx 3 \cdot 10^{-13}$ sec. Nonlocal effects (wave vector $q > 0$) were disregarded, which is partly corrected for by using the lifetime $\tau$ fitted at 1.3 GHz.

With these assumptions the surface resistance $R_s \approx \frac{1}{2} \sigma_1 \mu(\lambda^2 \omega^2)$ of high-purity Nb was computed at $T = 2$ K. Note that $R_s$ is related to the reflectivity $r$ of the metal by $R_s = (Z_0/4)(1 - r)$ where $Z_0 = (\epsilon_0 c)^{-1} = 377 \Omega$ is the impedance of the vacuum. Starting from $R_s \approx 20$ m$\Omega$ at 1.3 GHz the resistance rises to a few $\mu$W at 600 GHz and then exhibits a large step at 750 GHz to a value of 15 m$\Omega$. Above this energy-gap frequency $\nu = 2\Delta / h$ one has nearly constant $R_s$ till at least 2000 GHz.

VORTICES IN SUPERCONDUCTORS

Ginzburg-Landau and London Theories

Before the microscopic explanation of superconductivity was given in 1957 by BCS [2] there were very powerful phenomenological theories that were able to describe the thermodynamic and electrolytic behavior of superconductors. In 1935 Fritz and Heinz London established the London theory, see Eq.(2) above, and in 1952 Vitalii Ginzburg and Lev Landau conceived the Ginzburg-Landau (GL) theory. The GL theory may be written as a variational problem that minimizes the spatially averaged GL free energy density,
\[ \langle - |\psi|^2 | + \frac{1}{2} |\psi|^4 | + \left( |i\xi \nabla + A|^2 + (\lambda \nabla \times A)^2 \right) \rangle = \text{Minimum} \tag{12} \]
Here $\psi(r)$ is the complex GL-function, or order parameter, and $A(r)$ the vector potential of the magnetic induction $B = \nabla \times A$. The two lengths are the magnetic penetration depth $\lambda$ (usually taken as unit length) and the GL coherence length $\xi$; both lengths diverge as the temperature $T$ approaches the critical temperature $T_c$, figure 1, 2.
λ \propto \xi \propto (T_c - T)^{-1/2}. Their ratio, the GL parameter \( \kappa = \lambda / \xi \) within GL theory (valid near \( T_c \)) is independent of \( T \).

The GL theory can be derived from the microscopic BCS theory (L. P. Gor’kov 1959) in the limit \( T_c - T \ll T_c \), yielding for the GL function \( \psi(\mathbf{r}) = \Delta(\mathbf{r})/\Delta_{\text{BCS}} \) where \( \Delta \) is the energy gap function. The London theory follows from GL theory in the cases when the magnitude of the order parameter is nearly constant, \( |\psi| \approx 1 \). This condition is fulfilled when \( \xi \) is small as compared to the specimen extension and to \( \lambda \), requiring \( \kappa \gg 1 \). An arrangement of straight or arbitrarily curved vortex lines positioned at \( \mathbf{r}_\nu(z) = [x_\nu(z), y_\nu(z), z] \) (\( \nu = 1, 2, 3 \ldots \)) then has a magnetic field that obeys the London equation modified by adding \( \delta \) functions centered at the vortex cores,

\[
(1 - \lambda^2 \nabla^2) \mathbf{B}(\mathbf{r}) = \Phi_0 \sum_\nu \int \delta_3(\mathbf{r} - \mathbf{r}_\nu) \, d\mathbf{r}_\nu. \tag{13}
\]

**Ideal Vortex Lattice**

In 1957 Alexei Abrikosov, a thesis student of Lev Landau in Moscow, obtained a periodic solution of the Ginzburg-Landau equations and recognized that this corresponds to a lattice of vortices of supercurrent, circumla-
Losses by Moving Vortices

When a supercurrent flows in a superconductor, either applied by contacts or caused by a gradient or curvature of the local magnetic field, this current density $\mathbf{j}$ exerts a Lorentz force $\mathbf{f} = \mathbf{j} \times \hat{\mathbf{B}}$ on a vortex. The Lorentz force density on a vortex lattice is $\mathbf{F} = \mathbf{j} \times \bar{\mathbf{B}}$. Neglecting a small Hall effect, the vortices move along this force with velocity $\mathbf{v}$.
moving vortex core induces electric field

\[ E = \eta^{-1} \mathbf{F} \] where \( \eta \) is a drag coefficient or viscosity. The vortex motion induces an average electric field

\[
E = \mathbf{B} \times \mathbf{v} = \eta^{-1} \mathbf{B} \times (\mathbf{j} \times \mathbf{B}) = \rho \eta \mathbf{j}, \tag{15}
\]

\[
\rho \eta \approx (\mathbf{B}/B_{c2}) \rho_n. \tag{16}
\]

Here \( \rho \eta \) is the flux-flow velocity, which at large average inductions \( B \) is comparable to the normal resistivity of the superconductor at that temperature (measurable by applying a large field \( B_{a} > B_{c2} \)). However, when only a few vortices have penetrated (\( B \ll B_{c2} \)) one has much smaller resistivity \( \rho \eta \ll \rho_n \). But even then the vortex-caused dissipation at low \( T \) is typically much larger than the dissipation caused by the normal excitations.

Where does this resistive dissipation come from? There are two effects of comparable size, see Fig. 9. First, as pointed out by Bardeen and Stephen [1, 13], the motion of the magnetic field induces a dipolar electric field that drives current through the superconductor and through the vortex core dissipates energy that leads to the \( \rho \eta \) of Eq. (16). Second, as stated by Tinkham [1, 14], the moving vortex core moves that at a given position the order parameter \( |\psi|^2 \) goes down and up again when the core passes. If one assumes a delay of the recovery of \( |\psi|^2 \) by a relaxation time \( \tau \approx h/\Delta \) one obtains an additional dissipation of the order of Eq. (16). These two sources of losses are nice for physical understanding. In the exact calculation of the dissipation of a moving vortex lattice from time-dependent GL theory [15] these two sources cannot be separated but the approximate Eq. (16) is essentially confirmed [16], also by microscopic theory [17]. The numerical and also the measured flux-flow resistivity in the middle between the exact values 0 and \( \rho_n \) is somewhat smaller than the Eq. (16), i.e., for constant current source the real dissipation is lower.

**Pinning of Vortices**

When the material is inhomogeneous on the microscopic length scale of the vortex core \( \xi \), then the vortices are pinned and cannot move as long as the Lorentz force does not exceed the pinning force, or the current density \( j \) is smaller than the critical current density \( j_c \), see the reviews [18, 19, 20]. In this way the electric losses caused by flux flow can be avoided, and completely loss-free conductors of DC current can be tailored by introducing appropriate pinning centers into the material, e.g., precipitates and crystal lattice defects. For AC currents small losses remain, however. One source of AC dissipation is due to the (albeit small) concentration of normal carriers or excitations and can be understood from the two-fluid model as discussed above. The other source is the oscillation of vortices in the pinning wells. At small displacements \( u \) from their equilibrium position one may assume linear elastic binding of the vortices to the pins, with a force density \( -ku \). Adding to this the viscose drag force \(-\eta \dot{u}\) and the Lorentz force one obtains the force balance equation in an AC current

\[
\mathbf{j}_{ac} \times \mathbf{B} = -ku - i\omega n \mathbf{u}. \tag{17}
\]

One can see that at frequencies above \( k/\eta \), of order \( \omega/2\pi > 10^7 \) Hz, the viscose force dominates [21]. Pinning thus cannot prevent vortex oscillations at high fre-
Figure 11: Penetration of vortex lines into pin-free cylinders with radius \(a\) and height \(2b\). Top: \(b/a = 2\). Bottom: \(b/a = 0.3\). Forced by the applied field \(H_a\), the vortices enter from the corners, but only when the applied field has reached some threshold field do they jump to the middle leaving a vortex-free zone near the surface. With further increasing \(H_a\) the vortices eventually fill the cylinder uniformly from the middle. This delayed penetration without pinning is called geometrical barrier. Such a barrier is absent only for ellipsoid-shaped specimens.

Figure 12: Bean model with constant critical current density \(j_c\) for a superconducting bar with rectangular cross section \(2a \times 2b\) \((b/a = 0.35)\) put into a perpendicular magnetic field \(H_a\) that first increases from 0 to 0.5 (left column) and then decreases again (right column). The parameter 0.01, 0.1, 0.3, 0.5, 0.3, 0.1, 0, -0.1 is \(H_a/(\alpha j_c)\). Shown are the magnetic field lines (solid lines) and the penetrating fronts (dashed lines) where the current density \(j\) (flowing along the bar) jumps from \(\pm j_c\) to 0 (in the field-free and current-free core) or from \(j_c\) to \(-j_c\) (after full penetration of flux).

Interesting theoretical problems are the statistical summation of random pinning forces to obtain the critical current density \(j_c\), and the problem of thermally activated depinning [18, 19, 20]. The latter leads to vortex motion even at small currents densities \(j < j_c\) due to finite temperature. This flux creep may be described by a highly nonlinear resistivity. In particular, a logarithmic dispersive activation energy for depinning, \(U(j) = U_0 \ln(j_c/j)\), leads to an often observed power-law current–voltage curve,

\[
E(j) = E_0 \exp[-U(j)/k_B T] = E_0 (j/j_c)^n
\]

with a creep exponent \(n = U_0/k_B T\). For \(n = 1\) one has Ohmic behavior [free flux flow, Eq. (15)], for \(n \gg 1\) one has flux creep, and in the limit \(n \to \infty\) this power law yields the Bean model, in which \(j\) is either 0 or \(j_c\): When at some position one has \(j > j_c\), the vortices rearrange immediately such that \(j\) is reduced to \(j_c\) again. This concept is useful for DC currents and at not too high frequencies where the pinning forces exceed the viscous drag force.

Geometry Effects

The electromagnetic properties of a superconductor (and of any conductor or isolator) depend not only on the material but also on the geometry of the problem, i.e., on the shape of the specimen and on the way a magnetic or electric field is applied. For example, the reversible magnetization curves of a pin-free superconductor in Fig. 5 apply to the unrealistic case of very long slabs or cylinders in exactly parallel field, where demagnetization effects are absent. For the still unrealistic situation of a perfect ellipsoidal shape one may calculate from these ideal curves the reversible magnetization curves of any ellipsoid by using the concept of the demagnetization factor. But when
the specimen shape is not an ellipsoid, then even for a pin-free superconductor the magnetization curves have to be computed numerically, since now the induction (or vortex density) inside the specimen is no longer spatially constant.

It turns out that even without pinning such magnetization curves in general show a hysteresis, i.e., they are irreversible and depend on the magnetic history, see Fig. 10. This irreversibility is due to a geometric barrier [22, 23] for the penetration of vortices as illustrated in Fig. 11 for cylinders (or long bars) with rectangular cross section: When the applied uniform field $H_a$ is increased, vortex lines enter at the corners, pulled by the screening currents (Meissner currents) that flow at the surface, and held back by their line tension (like a rubber band). With increasing $H_a$ the vortices penetrate deeper and become longer. When the vortices from two corners meet at the equator, they connect and form one long vortex line that contracts and immediately jumps to the specimen center. During this rapid jump all their elastic energy is dissipated by the viscous drag force $F = \eta \dot{v}$, see text above Eq. (15). With further increasing $H_a$ more vortices jump to the center, crossing the flux-free zone near the surface, and eventually the entire specimen is filled with vortices coming from the growing central zone.

Flux penetration thus occurs with a threshold, over a “geometrical barrier”. The sudden onset of flux penetration to the center leads to the sharp maximum in the small (inner, pin-free) hysteresis loop of $M(H_a)$ in Fig. 10. When $H_a$ is decreased again, the vortices leave the specimen essentially without barrier, and at $H_a = 0$ all vortices have left, i.e., one has $B = 0$ and also $M = 0$ (since no screening currents flow anymore). The perpendicular field at which the first vortices enter at the corners of a pin-free long strip and a circular disk, both with rectangular cross section $2a \times 2b$, was computed in [23]:

$$H_{pen}^{strip} \approx H_{c1} \tanh \frac{0.36 b}{a},$$
$$H_{pen}^{disk} \approx H_{c1} \tanh \frac{0.67 b}{a}. \quad (19)$$

In the presence of pinning the hysteresis loops of $M(H_a)$ in Fig. 10 become larger. The area of such loops is the energy dissipated during one cycle due to depinning of vortices. When $H_{c1}$ is negligibly small as compared to $H_a$, the hysteresis curves and the vortex density and currents in a superconductor with pinning may be computed by treating it as a nonlinear conductor, Eq. (18). Figure 12 shows how the magnetic field lines (and vortices) penetrate and exit a thick disk with pinning when an axial $H_a$ is first increased beyond the field of full penetration, and then is decreased again [24, 25]. The chosen large creep exponent $n = 50$ practically reproduces the Bean model.

Figures 10, 11, and 12 were computed by time-integration of an equation for the (scalar) current density $j$ inside the superconductor; this method implicitly accounts for the infinitely extended magnetic stray field outside the specimen, without need to compute it and to cut it off. From the resulting current density the magnetic field lines are then easily calculated by the Biot-Savart law.

A completely different geometry is shown in Fig. 13, namely, the current stream lines and the contours of the magnetic field $B_z(x, y)$ in a thin film or platelet of rectangular shape with pinning and same large creep exponent $n = 50$ corresponding to the Bean model like in Fig. 12. An increasing magnetic field $H_a$ is applied perpendicular to the film. Initially, when $H_a \ll J_c = dj_c$ is small, no magnetic flux penetrates the film, i.e., the circulating screening currents generate a magnetic field that in the film area is constant (of size $-H_a$) and exactly compensates the applied field $H_a$. With increasing $H_a$, magnetic flux penetrates mainly from the middle of the sides of the rectangle (not from the corners), leaving still a flux-free zone in the middle. At and beyond some field of full penetration the current stream lines are concentric rectangles of constant distance, since the magnitude of the sheet current has saturated to the constant value $J_c = dj_c$. The magnetic field has then penetrated to the center, and the contour lines of $B_z(x, y)$ do not change anymore with further increasing $H_a$.

**Penetration of First Vortex**

An important question for RF superconductivity is under what circumstances and at which applied magnetic field $H_a$ the first vortex enters the superconductor, since the presence of even a few vortices can cause large losses. First
I summarize the expressions for the three critical fields which for type-II superconductors (with $\kappa \geq 1/\sqrt{2}$) obey $B_{c1} \leq B_c \leq B_{c2}$:

$$B_{c1} \approx \frac{\Phi_0}{4\pi \lambda^2} \left( \ln \kappa + \alpha \right),$$

$$B_c = \frac{\Phi_0}{\sqrt{8\pi \lambda \xi}} = \frac{\sqrt{2\kappa}}{\ln \kappa + \alpha} B_{c1},$$

$$\alpha(\kappa) = \frac{1}{2} + \frac{1 + \ln 2}{2\kappa - \sqrt{2} + 2} = \begin{cases} 1.35, & \kappa \approx 0.71 \\ 0.50, & \kappa \gg 1 \end{cases}.$$
Magnetic field $H$ at the equator of:

- cylinder
- sphere
- ellipsoid
- rectangle

first predicted by Bean and Livingston (BL) [27] for a superconductor with planar surface in a parallel applied field $H_a$. The Gibbs free energy $G(x)$ for this case reads

$$G(x) = \Phi_0 \left( H_a e^{-x/\lambda} - \frac{1}{2} H_o (2x) + (H_{c1} - H_a) \right). \quad (21)$$

In it the first term is the interaction of the vortex with the applied field $H_a$ or with its screening currents $(H_a/\lambda) e^{-x/\lambda}$, the second term is the interaction with the image vortex (at position $-x$, of opposite orientation), and the third term is an integration constant. Using the fact that for not too small $\kappa$ one has $B_1(0) \approx 2B_{c1}$ (see Fig. 1) one has with Eq. (14), $B_{c1} \approx (\Phi_0/4\pi \lambda^2)K_0(r_c/\lambda)$ yielding with $B_{c1}$ (20) a core radius $r_c \approx \xi \exp[-\alpha(\kappa)]$. With this we may write $G(x)$ (21) in the dimensionless form

$$\frac{G(x)}{\Phi_0 H_{c1}} \approx \frac{H_a}{H_{c1}} (e^{-x/\lambda} - 1) + 1 - \frac{K_0(\sqrt{4r_c^2 + \gamma_c^2}/\lambda)}{K_0(r_c/\lambda)} \quad (22)$$

that is plotted in Fig. 14 for $\kappa \approx 1.3$. Of course, this $G(x)$ is a only approximate, in particular at small $\kappa$, for which vortex penetration has to be computed numerically. Anyway, Fig. 14 shows that vortex penetration becomes favorable at $H_a = H_c$ and that the Bean-Livingston barrier vanishes at $H_a \approx H_c > H_{c1}$.

The assumption of BL that the entering vortex is long, straight, and exactly parallel to a planar surface is not very realistic. Alternatively, one may assume that the first vortex nucleates and penetrates in form of a small loop, say a half circle of radius $R$, see Fig. 15 top. The self-energy of this half circle is approximately $U_{self} = \pi R (\Phi_0^2/4\pi \lambda^2 \mu_0)$, putting the outer cut-off radius $\approx R$ instead of $\Lambda \gg R$. In the logarithm $\ln(\lambda/\xi) \rightarrow \ln(R/\xi) \approx 1$ when $R$ is of order of $\xi$. The interaction of this vortex loop with the surface screening current of density $j_s$ is $U_{js} \approx (\pi R^2/2) \Phi_0 j_s$ (flux quantum times loop area times $j_s$). For a planar surface one has $j_s = H_a/\lambda$ directly at the surface. The criterion that $U_{js} \geq U_{self}$ at $H_a \geq H_p$ yields then

$$H_p \approx \frac{\Phi_0/\mu_0}{2\pi \lambda R} = \frac{\sqrt{2\kappa}}{R} H_c \approx H_c, \quad (23)$$

which is just the BL result. Thus, the assumption of a penetrating vortex loop does not change much the penetration field of a planar surface.

However, when the surface has roughness with characteristic length $\geq \xi$, then vortices will penetrate at sharp points or cusps, see Fig. 15. At a corner with angle $\alpha = 90^\circ$, the screening current directly at the surface is strongly enhanced at this corner; Fig. 16 shows this for a superconducting bar with square cross section $2a \times 2a$ and penetration depth $\lambda = 0.025a$, to which a uniform transverse $H_a$ is applied. A rough estimate gives for this geometry an enhancement of the screening current at this corner, $j_s = C H_a/\lambda$, by a factor $C \approx 4$. The field of first vortex penetration $H_p$ is then reduced from Eq. (23) by just this factor, $H_p \approx H_c/C \approx H_c/4$.

For sharper corners the enhancement of $j_s$ and reduction of $H_p$ are even larger. As shown in the textbook of Landau-Lifshitz (Electrodynamics of Continua) for an ideal diamagnetic material at a corner with angle $\alpha$ (Fig. 17) the magnetic field diverges as $H \propto 1/r^3$ with exponent $\beta = (\pi - \alpha)/(2\pi - \alpha)$, where $r$ is the distance to the point of the corner. This gives $H \propto 1/r^{3/2}$ for $\alpha = \pi/2$ and $H \propto 1/r^{1/2}$ for $\alpha \rightarrow 0$.

Similarly, an axially applied magnetic field flowing around a ideal diamagnetic cylinder, sphere, or disks with elliptical or rectangular cross section or disks of aspect ratio $b/a \ll 1$, is enhanced at its equator by factors 2, 3/2, $a/b$, or $\approx (a/b)^{3/2}$, respectively, due to the strong curvature of the field lines at this line, see Fig. 18.

**Vortices in thin films**

One has to distinguish two quite different types of vortices in thin film superconductors: vortices perpendicular or parallel to the film plane. In wide films with width $w = 2a \gg$ thickness $d = 2b$, the vortices will nearly always run perpendicular across the film thickness, even in tilted applied field $H_a$, because of the large demagnetization factor of this film. This means the circulating currents prefer to flow in the film plane. Only when $H_a$ is exactly parallel to the film surface, or when the film is coating a bulk superconductor that screens any perpendicular field component, then vortices parallel to the film plane may occur.

When the film is of finite size, one may use Eqs. (19) to estimate at which applied perpendicular field component $H_{az}$ the first vortices penetrate, namely already at a very small field, smaller than $H_{c1}\sqrt{d/w}$. When the film edges are wedge-shaped or sharp, the penetration field is even smaller, cf. Fig. 17 and Fig. 18 (elliptical edge). Into infinitely extended or closed films (e.g., a Nb layer covering the inner surface of a Cu cavity) any perpendicular field
will penetrate since the field lines cannot flow around the film. Only when this film has holes or slits can some magnetic flux cross the film via these holes, but the magnetic field in the holes will be larger than $H_{c2}$ by at least the ratio of film area over the total area of all holes. However, the field in the holes will penetrate into the film when it is of the order of $H_{c1}$ times the square root of film thickness over hole distance. Thus, even such a perforated pin-free film will be penetrated by a perpendicular field that is very much smaller than $H_{c1}$. Numerical investigation of this problem is under way.

Pinning of vortices will not appreciably enhance all these penetration fields at high radio frequencies, where the (elastic) pinning forces are smaller than the viscous drag force. If the small applied perpendicular magnetic field is a DC field (e.g., some stray field or the earth magnetic field) then the additional RF field will even favor the penetration of the DC field in form of vortices, since it “shakes” the vortices. As shown in [28, 29], shaking of vortices by an AC field oriented perpendicular to the vortices leads to the relaxation of irreversible currents if the AC amplitude exceeds some threshold value. This vortex creep means that even in very small $H_{c2}$, perpendicular vortices will penetrate under the action of a large-amplitude RF field, and then these vortices oscillate and dissipate energy.

The properties of parallel vortex lines in a thin film with $d \ll \lambda$ was solved by Alexei Abrikosov (1964), Vadim Shmidt (1969), and in an elegant way by Alex Gurevich [31]. The lower critical field is enhanced in thin films as compared to bulk superconductors,

$$B_{c1} = \frac{2\Phi_0}{\pi d} \left( \ln \frac{\lambda}{d} - 0.07 \right),$$

and the field at which the surface barrier for vortex penetration disappears is also enhanced,

$$B_{p} = \frac{\Phi_0}{2\pi d \xi}. \quad (25)$$

For example, a NbN film with $\xi = 5$ nm, $d = 20$ nm has $B_{c1} = 4.2$ T and $B_{p} = 6.37$ T, much better than the penetration field $B_{p} \approx B_{c} = 0.18$ T for Nb at low $T$.

To enhance the operating RF amplitude in microwave cavities for accelerators and reduce the losses, Gurevich [31] suggests to use solid Nb or Pb with multilayer coating on its inner surface by alternating superconducting and insulating layers with $d < \lambda$. This will prevent penetration of vortices into the bulk superconductor when the vortex penetration field $B_{p}$ is large; e.g., for NbN films with $d = 20$ nm the RF field can be as high as 4.2 T, see Gurevich’s contribution to this conference.

Considering the elastic and viscous forces on a parallel vortex in a thin film, Gurevich estimates its characteristic relaxation time as

$$\tau \approx 2d \mu_0 \lambda^2 / (\xi \rho_n). \quad (26)$$

For a 30 nm Nb$_3$Sn film this $\tau \approx 10^{-12}$ s is much shorter than the RF period of $10^{-9}$ s. The maximum amplitude of the RF field at which the surface barrier of a single thin film coating disappears is of the order of the bulk $H_{c}$ of the film material, e.g., 0.54 T for Nb$_3$Sn. Thus, Nb$_3$Sn coating more than doubles the vortex penetration field for Nb, $B_{p} \approx B_{c} = 0.18$ T at low $T$. Thus, it appears that Nb cavities coated with a Nb$_3$Sn layer or with NbN multilayers allow for much higher RF amplitudes than uncoated Nb, or Cu coated by a Nb film, if this can be achieved technically.

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