Mathematical modeling of power law and Herschel - Buckley non-Newtonian fluid of blood flow through a stenosed artery with permeable wall: Effects of slip velocity

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Abstract. A mathematical model of non-Newtonian blood flow through a stenosed artery is considered. The steady non-Newtonian model is chosen characterized by the generalized power-law model and Herschel-Bulkley model incorporating the effect of slip velocity due to stenosed artery with permeable wall. The effects of slip velocity for non-Newtonian nature of blood on velocity, flow rate and wall shear stress of the stenosed artery with permeable wall are solved analytically. The effects of various parameters such as slip parameter ($\lambda$), power index (m) and different thickness of the stenosis (h) on velocity, volumetric flow rate and wall shear stress are discussed through graphs.

1. Introduction

Now a days many people are suffering for cardiovascular disease such as stenosis (atherosclerosis). Arteriosclerosis literally means “hardening of the arteries”; it is a generic term reflecting arterial wall thickening and loss of elasticity. Three distinct types are recognized, each with different clinical and pathologic consequences. A stenosis is defined as a partial occlusion of the blood vessels due to the accumulation of cholesterol and fats and the abnormal growth of tissue. Pulsatile flow of blood through catheterized artery by modeling blood as Herschel-Bulkley fluid and artery as rigid coaxial circular cylinders is studied by [5]. Mathematical modeling of pulsatile flow of non-Newtonian fluid through an elastic artery with effects of elasticity discussed by [2]. Study of blood flow using power law and Herschel-Bulkley non-Newtonian fluid model through elastic artery gave idea [1]. The pulsatile flow of blood through mild stenosed artery treating the blood as Herschel-Bulkley fluid is studied by [3]. [4] is presented a mathematical model for blood flow through stenosed arteries with axially variable peripheral layer thickness and variable slip at the wall. The objective of in this paper is to study the steady non-Newtonian blood flow of power law and Herschel-Bulkley fluid model through a stenosed artery with permeable wall. The effects of slip velocity of both power law and Herschel-Bulkley fluid model on velocity, flow rate and wall stress solved analytically and discussed through graphs.
2. Formulation of the problem:

A non-Newtonian Stenosis artery with permeable wall geometry as shown in the Fig. 1.

![Fig. 1 Flow geometry of a stenosed artery with permeable wall](image)

\[ \frac{R(z)}{R_0} = \left\{ \begin{array}{ll} 1 - \frac{\delta}{R_0} \left[ 1 + \cos \frac{2\pi}{l_0} (z - \frac{d}{2}) \right]; & \text{when } d \leq z \leq d + l_0 \\ 1 & ; \text{otherwise} \end{array} \right. \]  

(1)

[Where \( R(z) \) - radius of the stenosed artery, \( R_0 \) - radius of the artery with out stenosed artery, \( \delta \) - height of stenosed artery, \( L_0 \) - length of the stenosis, \( L \) - length of the artery, \( d \) - length of non stenosis region].

An governing equation of non-Newtonian fluid in cylindrical co-ordinate system \((r', \theta', z')\) of the is given by,

\[ \mu \left[ \frac{1}{r'} \frac{d}{dr'} \left( r' \frac{dv'}{dr'} \right) \right] = \frac{dp'}{dz'} \quad \text{at} \quad 0 \leq r' \leq R(z) \]  

(2)

[where \( r' \) and \( z' \) denote the radial and axial coordinates respectively and \( \theta' \) is the angle, \( \mu \) is the viscosity, \( v' \) is the axial velocity, \( p' \) the pressure and \( R_0 \) the radius of the tube].

Boundary conditions for the problem stated above may be listed as:

**Slip condition**: \( \dot{v}' = \hat{A} \frac{\partial v'}{\partial r'} \) at \( r' = R(z) \)

**Symmetry condition**: \( \frac{\partial v'}{\partial r'} = 0 \) at \( r' = 0 \)  

where \( \hat{A} \) is slip parameter.

3. Mathematical Solution

It is convenient to write equation (2) and equation (3) in dimensionless form by using dimensionless quantities

\[ r = \frac{r'}{R_0}, v = \frac{v'}{v_0}, \tau = \frac{\tau'}{\tau_0} \quad \text{where} \quad \nu_0 = \frac{R_0^2}{2\mu_0} \frac{dp'}{dz'} \quad \text{and} \quad \tau_0 = \frac{\mu_0}{R_0} v_0 \]
3.1 Calculation for Power-law model:
Using above non-dimensionless quantities in (2) the power law model of stenosed artery with permeable wall becomes (Surendrakumar 2015)

\[
\left( -\frac{dv}{dr} \right) = \left( \frac{\mu_o}{\tau_0} \right)^{\frac{m-1}{m}} \cdot (\tau)^{\frac{1}{m}}
\]

(4)
m is the power index, \(\mu_o\) is the apparent viscosity of the fluid.

**Slip condition**: \(v = \lambda \frac{\partial v}{\partial r} \) at \( r = \frac{R(z)}{R_0} \)

**Symmetry condition**: \(\frac{\partial v}{\partial r} = 0\) at \( r = 0 \)

The equation (4) is solved by using boundary condition (5) we get the Velocity profile of power law model in stenosed artery with permeable wall is,

\[
v = A_i \left[ \frac{m}{m+1} r \frac{R(z)}{R_0} + \left( \lambda - \frac{m}{m+1} \frac{R(z)}{R_0} \right) \right]
\]

(6)

Volumetric flow rate for power law model of stenosed artery with permeable wall

\[
Q = 2\pi \int_0^{R(z)} rvdr
\]

\[
Q = 2\pi A_i \left[ \frac{m^2}{m+1} \frac{R(z)}{R_0 (3n+1)} + \frac{R(z)}{2R_0} \left( \lambda - \frac{m}{m+1} \frac{R(z)}{R_0} \right) \right]
\]

(7)

Wall shear stress for power law model of stenosed artery with permeable wall

\[
\tau = \left[ -\frac{dv}{dr} \right]_{r=R(z)} \quad \tau = A_i \left[ \frac{R(z)^{\frac{1}{m}}}{R_0} \right]
\]

(8)

3.2 Calculation for Herschel-Bulkley model:
Using non-dimensionless in (2) the Herschel-Bulkley fluid model of stenosed artery with permeable wall is (Surendrakumar 2015)

\[
-\frac{dv}{dr} = \left( \frac{\mu_o}{\tau_0} \right)^{\frac{m-1}{m}} \left( \tau - \dot{\gamma} \right)^{\frac{1}{m}}, \quad \text{where} \quad \tau' = \mu_o \left( \gamma \right)^m + \tau_y, \quad \tau' \geq \tau_y, \quad \gamma = 0, \tau' \leq \tau_y
\]

(9)

Solving the above equation with boundary condition (5) we get the velocity profile of Herschel-Bulkley fluid is,
$$v = \frac{a_i}{Eh} \left(-1 + \frac{a_i r}{Eh}\right)^{\frac{1}{m}} \left[\frac{1}{a_i} - \frac{r}{Eh}\right]\left(\frac{ma_i}{1 + m}\right),$$

$$\frac{a_i a_m R(z)}{(1 + m) Eh} \left(-1 + \frac{a_i R(z)}{EhR_0}\right)^{\frac{1}{m}} \left[\frac{1}{a_i} - \frac{R(z)}{R_0 Eh}\right] - \left[-1 + \frac{a_i R(z)}{R_0 Eh}\right]^{\frac{1}{m}} \left[\frac{1}{Eh}\right] - \left[-1 + \frac{a_i R(z)}{R_0 Eh}\right]^\frac{1}{m} \left[\frac{1}{Eh}\right] \right\}$$

(10)

where $\delta$ yields stress and Eh is elasticity.

Volumetric flow rate for Herschel-Bulkley model of stenosed artery with permeable wall

$$Q = 2\pi \int_0^{R(z)} rv dr$$

$$Q = \frac{\pi a_i R^2(z)}{EhR_0^2} \left(-1 + \frac{a_i R(z)}{EhR_0}\right)^{\frac{1}{m}} \left[\frac{1}{a_i} - \frac{R(z)}{R_0 Eh}\right]\left(\frac{ma_i}{1 + m}\right),$$

$$\frac{\pi a_i a_m R^2(z)}{(1 + m) R_0^2 (Eh)} \left(-1 + \frac{a_i R(z)}{R_0 Eh}\right)^{\frac{1}{m}} \left[\frac{1}{a_i} - \frac{R(z)}{R_0 Eh}\right] - \left[-1 + \frac{a_i R(z)}{R_0 Eh}\right]^{\frac{1}{m}} \left[\frac{1}{Eh}\right] - \left[-1 + \frac{a_i R(z)}{R_0 Eh}\right]^\frac{1}{m} \left[\frac{1}{Eh}\right] \right\}$$

(11)

Wall shear stress for Herschel-Bulkley fluid model of stenosed artery with permeable wall

$$\tau = \left[-\mu \frac{dv}{dr}\right]_{r=R(z)}$$

$$\tau = \left(\frac{a_i}{Eh}\right) \left(\frac{ma_i}{1 + m}\right) \left[-1 + \frac{a_i R(z)}{EhR_0}\right]^{\frac{1}{m}} \left(-\frac{1}{Eh}\right) + \left(\frac{1}{a_i}\right) \left(a_i R(z)\right)^{\frac{1}{m}} \left(\frac{1}{Eh}\right) - \left[-1 + \frac{a_i R(z)}{EhR_0}\right]^{\frac{1}{m}} \left(\frac{1}{Eh}\right) \right\}$$

(12)

4. Results and discussion

In order to have an estimate of the quantitative effects of parameters involved in the of power law and Herschel-Bulkley model of blood through a stenosed artery with permeable wall in the flow analysis, computer codes are developed to evaluate the analytical results obtained for wall shear stress $\tau$, volumetric flow rate $Q$ and velocity $v$ in the stenotic region on slip parameter ($\lambda$), power index ($m$) and stenotic region ($\delta$) are computed graphically.
Fig. 2 variation of power law fluid of velocity $v$ with radial distance $r$ for different number of slip parameter $\lambda$ and $m=1$.

Fig. 3 variation of power law fluid of volumetric flow rate $Q$ with $z/L$ for different number of thickness of stenosis $\delta$, $\lambda=0.1$, $m=1$.

Fig. 4 variation of power law fluid of wall shear stress $\tau$ with $z/L$ for different number of power index (m), $\delta=0.2$, $\lambda=0.1$.

Fig. 5 variation of Herschel-Bulkley fluid of velocity $v$ with radial distance $r$ for different number of slip parameter $\lambda$ and $m=1$. 
Fig. 6 variation of Herschel-Bulkley fluid of volumetric flow rate $Q$ with $z/L$ for different number of thickness of stenosis $\delta$ and $m=1, \lambda=0.1$.

Fig. 7 variation of Herschel-Bulkley fluid of wall shear stress $\tau$ with $z/L$ for different number of power index ($m$), $\delta=0.2, \lambda=0.1$.

In figure 2 shows that the power law fluid of velocity decreases as the increasing slip parameter $\lambda$. Figure 3 shows that the power law fluid of flow rate increases as the increasing thickness of stenosis $\delta$. In figure 4 power law fluid shows that wall shear stress decreases with the increasing power index ($m$). Figure 5 shows that Herschel-Bulkley fluid of velocity increases as the increasing slip parameter $\lambda$. In figure 6 shows that the Herschel-Bulkley fluid of volumetric flow rate increases with the increasing thickness of stenosis $\delta$. In figure 7 shows that Herschel-Bulkley fluid of wall shear stress decreases as the increasing power index ($m$).

5. Conclusion

In this analysis a laminar, incompressible steady flow of blood through a stenosed artery is studied. The artery is considered as mild stenosis with permeable wall, blood is under power-law and Herschel-Bulkley model. The comparison in flow profiles for power-law model and Herschel-Bulkley model of blood flow are demonstrated in the figures. We see that the velocity profiles for power law model is advanced compare to the Herschel-Bulkley model for fixed value of slip parameter $\lambda$. It is noticed that for fixed value thickness of stenosis, the volumetric flow rate increases with the increase in the value thickness of stenosis $\delta$ which leads to more and more deposition of LDL molecules along the walls of artery and ultimately forms the arteriosclerotic plaques and retards the blood flow. The slip condition plays an important role in shear skin, spurt and hysteresis effects. The fluids that exhibit boundary slip have important technological applications such as in polishing valves of artificial heart and internal cavities.
References

[1] Surendrakumar 2015, Study of blood flow using power law and Herschel-Bulkley non-Newtonian fluid model through elastic artery, *Proceedings of ICFM*, International Conference of Frontiers in Mathematics.

[2] Chitra et al. 2017, “Mathematical Modeling of Pulsatile flow of non-Newtonian fluid through an elastic artery: Effects of elasticity”, *International Journal of Pure and Applied Mathematical Sciences*, 10, Number 1, pp. 29-40.

[3] Sankar, D.S. and K. Hemalatha 2007, *Pulsatile flow of Herschel-bulkley fluid through catheterized arteries- a mathematical model*, Applied Mathematical Modelling, 31, 1497-1517.

[4] Islam M. Eldesoky 2012, Slip Effects on the Unsteady MHD Pulsatile Blood Flow through Porous Medium in an Artery under the Effect of Body Acceleration, *International Journal of Mathematical Sciences*, Article Id 860239, 26 pages.

[5] Sankar D. S. and U., Lee, Mathematical modelling of pulsatile flow of non-Newtonian fluid in stenosed arteries, Commun. in Nonlinear Sci. Numer. Simul. 14, 2971 -2981.