A SURVEY OF SOME RECENT APPLICATIONS OF OPTIMAL TRANSPORT METHODS TO ECONOMETRICS

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Abstract. This paper surveys recent applications of methods from the theory of optimal transport to econometric problems.

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1. Introduction

Optimal transport, popularized by Villani’s texts Villani, 2003 and Villani, 2009, is currently a very active research area of mathematics, and it has found applications in many sciences. Economics is no exception. However, up to a recent period, the appearance of optimal transport in economics was only in connection with two-sided models of matching (see e.g. Becker, 1973, Shapley and Shubik, 1972, Gretsky et al., 1992): indeed, as shown by Chiappori et al. Chiappori et al., 2010, equilibrium outcomes in two-sided matching problems with transferable utility coincide with the solutions of an optimal transport problem. More recently however, methods from optimal transport theory have been used as a tool in a number of problems in econometrics, microeconomic theory, and finance. These methods are exposed in a comprehensive way in my recent monograph, Optimal Transport Methods in Economics Galichon, 2016, aimed at an audience of economists. The goal of the present paper, which partly follows the presentation there, is to provide a short introduction to the

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use of optimal transport methods in econometrics. The reader will hopefully forgive me for a bias toward my own research in selecting applications.

The paper is organized as follows. Section 2 provides a brief overview of the theory, and states its main result, the Monge-Kantorovich theorem. Section 3 discusses three particular cases where, because of additional restrictions, the analysis of solutions of the Monge-Kantorovich problem can be pushed further, which is most helpful in applications. Section 4 reviews a number of econometric applications involving optimal transport methods.

2. Monge-Kantorovich theory in a nutshell

2.1. Optimal coupling. We start by describing the optimal transport problem at an intermediate level of generality, which will suffice for our applications. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two closed subsets of \( \mathbb{R}^d \) and \( \mathbb{R}^d' \), respectively, and consider two (Borel) probability distributions \( P \) and \( Q \) of respective supports \( \mathcal{X} \) and \( \mathcal{Y} \). A coupling of probabilities \( P \) and \( Q \) is a joint probability distribution \( \pi \) on \( \mathcal{X} \times \mathcal{Y} \) with marginal distributions \( P \) and \( Q \), which means that if \( (X,Y) \) is a random vector with probability distribution \( \pi \), then its projections \( X \) and \( Y \) on \( \mathcal{X} \) and \( \mathcal{Y} \) should be random vectors with respective probability distributions \( P \) and \( Q \). The set of such couplings will be denoted as

\[
\mathcal{M}(P,Q) = \{ \pi : (X,Y) \sim \pi \text{ implies } X \sim P \text{ and } Y \sim Q \}.
\]

Hence, \( \pi \in \mathcal{M}(P,Q) \) encodes the “missing information” needed in order to build a random vector \( (X,Y) \) when \( X \sim P \) and \( Y \sim Q \). When \( \mathcal{X} = \mathcal{Y} = [0,1] \), and \( P = Q = U([0,1]) \), \( \mathcal{M}(P,Q) \) coincides with the set of copulas, which are well-known objects in applied probability whose purpose is also to build bivariate distributions based on univariate ones. Therefore, couplings can be seen as a generalization of copulas beyond the univariate case.

Following Monge, 1781 and Kantorovich, 1939, 1948, we shall consider the Monge-Kantorovich problem of finding the “optimal” coupling of probability distributions \( P \) and \( Q \). By optimal, we mean here that it should maximize the expectation of
some surplus function $\Phi : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, that is,

$$\sup_{\pi \in \mathcal{M}(P,Q)} \mathbb{E}_\pi [\Phi (X,Y)].$$

(2.1)

We can interpret this problem as a problem of worker-firm assignment: a central planner needs to assign on a one-to-one basis a population of workers, whose skills are distributed on set $\mathcal{X}$ with distribution $P$, to a population of firms whose characteristics are distributed according to $Q$ on set $\mathcal{Y}$. The economic value created by worker $x$ if employed by firm $y$ is $\Phi (x,y)$. An assignment of workers to firms is defined as $\pi \in \mathcal{M}(P,Q)$, which measures the distribution of the matches. The total economic value created by an assignment $\pi$ is therefore $\mathbb{E}_\pi [\Phi (X,Y)]$, and hence, the maximum economic value that the central planner can hope to achieve is (2.1).

2.2. Duality and the Monge-Kantorovich theorem. Problem (2.1) is an infinite-dimensional linear programming problem. Indeed, the objective function is linear because it is an expectation with respect to measure $\pi$, which is the optimization variable, and the constraint $\pi \in \mathcal{M}(P,Q)$ can be expressed as $\mathbb{E}_\pi [\varphi (X) + \psi (Y)] = \mathbb{E}_P [\varphi (X)] + \mathbb{E}_Q [\psi (Y)]$ for all integrable test functions $\varphi : \mathcal{X} \to \mathbb{R}$ and $\psi : \mathcal{Y} \to \mathbb{R}$, and these constraints are linear with respect to $\pi$.

As a result, problem (2.1) has a dual formulation. The dual formulation can be worked out by hand, by noticing that if $\varphi$ and $\psi$ are test functions such that $\Phi (x,y) \leq \varphi (x) + \psi (y)$ for all $x$ and $y$ in their domains, then by integration with respect to any $\pi \in \mathcal{M}(P,Q)$ and by using the fact that $\pi$ has margins $P$ and $Q$, it follows that

$$\mathbb{E}_\pi [\Phi (X,Y)] \leq \mathbb{E}_P [\varphi (X)] + \mathbb{E}_Q [\psi (Y)].$$

This being true for any $\pi \in \mathcal{M}(P,Q)$ and any test functions $\varphi$ and $\psi$ such that $\Phi (x,y) \leq \varphi (x) + \psi (y)$, it follows by taking the maximum on the left and the minimum on the right hand side that the value of problem (2.1) is less or equal to the value of the dual problem

$$\inf_{\varphi(x)+\psi(y)\geq\Phi(x,y)} \mathbb{E}_P [\varphi (X)] + \mathbb{E}_Q [\psi (Y)]$$

(2.2)

where the minimum is over all $P$- and $Q$- integrable functions $\varphi$ and $\psi$ such that $\varphi (x) + \psi (y) \geq \Phi (x,y)$. Under a few additional assumptions on $P$, $Q$ and $\Phi$, the Monge-Kantorovich
Theorem asserts that this inequality actually holds as an equality, and the optimal coupling $\pi$ exists. Under an additional assumption, pairs of solution potentials $(\varphi, \psi)$ to the dual problem (2.2) also exist. This is made precise in the following result:

**Theorem 1** (Monge-Kantorovich duality theorem). Assume $\Phi : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \cup \{-\infty\}$ is an upper semicontinuous surplus function bounded from above by $\underline{a}(x) + \underline{b}(y)$ where $\underline{a}$ and $\underline{b}$ are respectively integrable with respect to $P$ and $Q$. Then:

(i) Strong duality holds, that is:

$$
\sup_{\pi \in \mathcal{M}(P,Q)} \mathbb{E}_\pi [\Phi (X,Y)] = \inf_{\varphi(x)+\psi(y) \geq \Phi(x,y)} \mathbb{E}_P [\varphi (X)] + \mathbb{E}_Q [\psi (Y)],
$$

(2.3)

where the infimum on the right hand-side is taken over measurable and integrable functions $\varphi$ and $\psi$, and the inequality constraint should be satisfied for $P$—almost every $x$ and $Q$—almost every $y$.

(ii) An optimal solution $\pi$ to the primal problem on the left hand-side exists.

(iii) Assume further $\Phi$ is bounded from below by $\underline{a}(x) + \underline{b}(y)$ where $\underline{a}$ and $\underline{b}$ are respectively integrable with respect to $P$ and $Q$. Then the dual problem on the right hand-side also has solutions.

For a proof of this result, refer to Villani, 2009, Theorem 5.10.

Let us now explore a criterion for jointly checking that a coupling $\pi$ and a pair of potential functions $(\varphi, \psi)$ are simultaneously optimal for the primal and the dual respectively. Take $\pi \in \mathcal{M}(P,Q)$ and take $\varphi$ and $\psi$ such that $\varphi(x) + \psi(y) \geq \Phi(x,y)$. Then $\pi$ and $(\varphi, \psi)$ are respectively solutions to (2.1) and (2.2) if and only if the equality $\mathbb{E}_\pi [\varphi (X) + \psi (Y)] = \mathbb{E}_\pi [\Phi (X,Y)]$ holds. This is equivalent to the fact that

$$
\text{Supp} (\pi) \subseteq \{(x,y) \in \mathcal{X} \times \mathcal{Y} : \varphi (x) + \psi (y) = \Phi (x,y)\}.
$$

(2.4)

In the worker-firm interpretation alluded to above, this condition has an interpretation in terms of pairwise stability. Indeed, it implies that $\varphi(x)$ can be interpreted as the payoff of a worker of type $x$, and $\psi(y)$ can be interpreted as the payoff of a firm of type $y$. If $x$ and $y$ are matched at equilibrium, then $(x,y) \in \text{Supp} (\pi)$, thus $\varphi(x) + \psi(y) = \Phi(x,y)$,
that is, the output $\Phi(x, y)$ created by the $(x, y)$ pair should be divided into the payoff of the worker $\varphi(x)$, and the payoff of the firm $\psi(y)$. If $x$ and $y$ are not matched, then $\varphi(x) + \psi(y) \geq \Phi(x, y)$, which expresses the fact that $x$ and $y$ do not have an incentive to leave their existing partners to form a blocking pair. Thus $(\varphi, \psi)$ is a pair of “stable” payoffs.

2.3. Some remarks. Let us note a few immediate properties of this problem.

First, note that if one replaces $\Phi(x, y)$ by $\Phi(x, y) + a(x) + b(y)$, then the set of optimal couplings $\pi$ for problem (2.1) remains unchanged: indeed, the value of $\mathbb{E}_\pi[a(X) + b(Y)]$ does not depend on the choice of $\pi \in \mathcal{M}(P, Q)$. Also note that if $(\varphi, \psi)$ is a solution to problem (2.2), then for any real number $c$, $(\varphi - c, \psi + c)$ is also a solution.

Next, note that if a minimizer $(\varphi, \psi)$ of the dual problem (2.2) exists, then one has necessarily

\begin{align}
\varphi(x) &= \max_{y \in \mathcal{Y}} \{ \Phi(x, y) - \psi(y) \} \\
\psi(y) &= \max_{x \in \mathcal{X}} \{ \Phi(x, y) - \varphi(x) \},
\end{align}

which has another interpretation, this time in terms of Walrasian equilibrium: $\varphi(x)$ can be seen as the equilibrium wage of a worker of type $x$, while $\psi(y)$ can be interpreted as the equilibrium surplus of a firm of type $y$. In particular, formula (2.6) expresses the problem of a firm $y \in \mathcal{Y}$ looking for a worker $x \in \mathcal{X}$ yielding the highest surplus $\Phi(x, y) - \varphi(x)$.

Formulas (2.5) and (2.6) allow to rewrite problem (2.2) using a number of alternative reformulations:

- Using the expression of $\varphi$ given in (2.5), problem (2.2) rewrites as

\begin{equation}
\inf_{\psi} \left\{ \mathbb{E}_P \left[ \max_{y \in \mathcal{Y}} \{ \Phi(X, y) - \psi(y) \} \right] + \mathbb{E}_Q [\psi(Y)] \right\}
\end{equation}

- Using the fact that a constant can be freely added to or subtracted from $\psi$, one may look at the solutions of problem (2.2) among those such that $\mathbb{E}_Q [\psi(Y)] = 0$, that
is, problem (2.2) yet rewrites as
\[ \inf_{\psi} \mathbb{E}_P \left[ \max_{y \in Y} \{ \Phi (X, y) - \psi (y) \} \right] \]
\[ \text{s.t.} \quad \mathbb{E}_Q [\psi (Y)] = 0. \]

These alternative formulations will turn out to be useful below.

3. THREE CASES OF INTEREST

3.1. Discrete optimal assignment problem. One first case of interest is found when \( P \) and \( Q \) are discrete distributions with finite support: \( P = \sum_{i=1}^{n} p_i \delta_{x_i} \) and \( Q = \sum_{j=1}^{m} q_j \delta_{y_j} \). In which case, we denote \( \Phi_{ij} = \Phi (x_i, y_j) \), and duality (2.3) rewrites as
\[
\max_{\pi \geq 0} \sum_{ij} \pi_{ij} \Phi_{ij} = \min_{\varphi, \psi} \sum_i p_i \varphi_i + \sum_j q_j \psi_j \]
\[ \text{s.t.} \quad \sum_i \pi_{ij} = p_i \quad \text{s.t.} \quad \varphi_i + \psi_j \geq \Phi_{ij} \]

(3.1)

The complementary slackness condition states that if \( \pi_{ij} \), which is the Lagrange multiplier associated to the dual constraint \( \varphi_i + \psi_j \geq \Phi_{ij} \), is strictly positive, then the corresponding constraint is saturated. Thus \( \pi_{ij} > 0 \) implies \( \varphi_i + \psi_j \geq \Phi_{ij} \), which recovers the general condition (2.4) viewed above.

Formulated as a linear programming problem as in (3.1), the optimal transport problem can be solved using standard linear programming toolboxes; see Galichon, 2016, section 3.4 how to perform computations efficiently using the sparse structure of the constraint matrix.

3.2. Continuous-to-discrete case. One second case of interest arises when \( P \) is a continuous distribution, but when \( Q \) is a discrete distribution with finite support: \( Q = \sum_{j=1}^{m} q_j \delta_{y_j} \). In this case, the reformulation of the Monge-Kantorovich problem using expression (2.7) is particularly useful. Indeed, following Aurenhammer, 1987, one may then expresses problem (2.2) as
\[
\min_{\psi \in \mathbb{R}^m} \mathbb{E}_P \left[ \max_{j \in \{1, \ldots, m\}} \{ \Phi (X, y_j) - \psi_j \} \right] + \sum_{j=1}^{m} q_j \psi_j, \]

(3.2)

which is simply the problem of minimizing a convex function in \( \mathbb{R}^m \). See Galichon, 2016, section 5.3 for a discussion on the implementation of this method using gradient descent.
3.3. **Scalar product surplus.** A third case of interest is the case when \( P \) and \( Q \) are continuous distributions on \( \mathcal{X} = \mathcal{Y} = \mathbb{R}^d \) (\( d = d' \)), and when \( \Phi (x, y) = x^\top y \) is the scalar product between \( x \) and \( y \). In this case, under the assumption that \( P \) and \( Q \) have finite second moments, a solution \((\varphi, \psi)\) to (2.2) exists, and relations (2.5) and (2.6) become

\[
\begin{align*}
\varphi (x) &= \max_{y \in \mathbb{R}^d} \{ x^\top y - \psi (y) \} \\
\psi (y) &= \max_{x \in \mathbb{R}^d} \{ x^\top y - \varphi (x) \}
\end{align*}
\]

hence \( \varphi \) and \( \psi \) are related by Legendre-Fenchel conjugation, which is classically denoted \( \psi = \varphi^* \) and \( \varphi = \psi^* \). A short tutorial on convex analysis from the point of view of optimal transport is provided in Galichon, 2016, section 6.1, where the basic notions used in the sequel, such as Legendre transform and subdifferential, are recapitulated.

By (2.4), if \( \pi \in \mathcal{M}(P, Q) \) is an optimal coupling and if \((\varphi, \psi)\) is a dual solution, then \( \pi \) and \((\varphi, \psi)\) are both optimal if and only if the support of \( \pi \) is included in the set of \((x, y)\) such that \( \varphi (x) + \varphi^* (y) = x^\top y \). But this relation is equivalently reexpressed in convex analysis by the fact that \( y \in \partial \varphi (x) \), where \( \partial \varphi \) denotes the subgradient of \( \varphi \) at \( x \). Hence, \( \pi \) and \((\varphi, \psi)\) are both optimal if and only if

\[
\text{Supp} (\pi) \subseteq \left\{ (x, y) \in \mathbb{R}^d \times \mathbb{R}^d : y \in \partial \varphi (x) \right\}.
\]

However, it is a well-known fact in convex analysis\(^1\) that if \( \varphi \) is convex, then the set of points where \( \varphi \) is not differentiable is of zero Lebesgue measure. As a result, if \( P \) is absolutely continuous with respect to the Lebesgue measure, then \( \partial \varphi (x) \) \( P \)-almost surely coincides with \( \{ \nabla \varphi (x) \} \). Therefore, in this case, an optimal coupling \((X, Y) \sim \pi \) can be represented by \((X, \nabla \varphi (X))\), where \( \varphi \) is convex and is such that \( \nabla \varphi (X) \sim Q \). This is denoted as

\[
\nabla \varphi \# P = Q \quad (3.3)
\]

and one says that \( \nabla \varphi \) "pushes forward \( P \) onto \( Q \)." The existence and uniqueness (up to a constant) of a convex function \( \varphi \) satisfying (3.3) are the object of Brenier’s theorem Brenier, 1987 (see also Knott and Smith, 1984 and Rüschendorf and Rachev, 1990). This theorem

\(^1\)It follows from Rademacher’s theorem; see Villani, 2009, theorem 10.8.
was improved by McCann (McCann, 1995), who obtained the existence of $\varphi$ in (3.3) without requiring $P$ and $Q$ to have second moments.

4. Econometric applications

4.1. Demand inversion in discrete choice models. Our first application is the problem of demand inversion in discrete choice, or random utility models. Consider a discrete choice model, where agents face alternatives $y \in \mathcal{Y}$. The systematic utility associated to alternative $y$ is a real number $\delta_y$, and the unobserved heterogeneity associated to it is $\varepsilon_y$. It is assumed that $\varepsilon \sim P$, where $P$ is a probability distribution on $\mathbb{R}^\mathcal{Y}$.

Let $q$ be a probability vector on $\mathcal{Y}$. One says that $q$ is a vector of market shares induced by systematic utility vector $\delta$ if $q$ is the probability distribution of a random variable $Y$ such that $Y \in \arg\max_{y \in \mathcal{Y}} \{\delta_y + \varepsilon_y\}$. The problem of demand inversion in discrete choice models, as defined for instance in Berry, 1994, is as follows: given a vector of market shares $q$, what is the set of systematic utility vectors $D(q)$ whose elements $\delta$ induce the choice probability $q$. Formally,

$$D(q) = \left\{ \delta \in \mathbb{R}^\mathcal{Y} : \exists Y \sim q, Y \in \arg\max_{y \in \mathcal{Y}} \{\delta_y + \varepsilon_y\} \right\}.$$

The main observation here, due to Galichon and Salanié, 2014, is that $\delta \in D(q)$ if and only if $\psi = -\delta$ appears in the solution of the dual Monge-Kantorovich problem (2.2). Indeed, define

$$G(\delta) = \mathbb{E}_P \left[ \max_{y \in \mathcal{Y}} \{\delta_y + \varepsilon_y\} \right]$$

then the envelope theorem shows that $\delta \in D(q)$ if and only if $q \in \partial G(\delta)$. But according to a basic result in convex analysis (e.g. Galichon, 2016, section 6.1), $q \in \partial G^*(\delta)$ if and only if $\delta \in \partial G^*(q)$; thus $D(q) = \partial G^*(q)$, and

$$D(q) = \arg\min_{\delta} \left\{ G(\delta) - \sum_{y \in \mathcal{Y}} q_y \delta_y \right\}.$$
Thus, $\delta \in D(q)$ if and only if $\psi = -\delta$ minimizes $G(-\psi) + \sum_{y \in Y} q_y \psi_y$, that is, if and only if it solves

$$\min_{\psi} \mathbb{E}_\mathbf{P} \left[ \max_{y \in Y} \{ \varepsilon_y - \psi_y \} \right] + \sum_{y \in Y} q_y \psi_y$$

which is exactly problem (3.2) with $\Phi(\varepsilon, y) = \varepsilon_y$. Therefore the problem of demand inversion in discrete choice models is equivalent to a optimal transport problem, and numerical methods for solving the latter can be used for the former. See the papers Galichon and Salanié, 2014, Chiong et al., 2015, and an overview in Galichon, 2016, section 9.2.

4.2. Multivariate quantiles. Quantiles play a fundamental role in econometrics and applied statistics. They are useful for comparing distributional outcomes, measuring risk and inequality, identifying willingness-to-pay, etc. One of the most stringent limitations of quantiles is the fact that they are fundamentally univariate objects: indeed, the quantile function of real-valued random variable $Y$ is defined as the inverse map of its cumulative distribution function, which can only be inverted when $Y$ is univariate.

There is a considerable literature aiming at providing various generalizations of the notion of quantile to the multivariate case, which we will not review here. Our point here is to show that optimal transport provides a sensible such generalization, see Galichon, 2016, section 6.3.

One possible way to define the (univariate) quantile map is as the monotone map which pushes forward the uniform distribution on $[0, 1]$ to a distribution of interest; in other words, the quantile map associated to random variable $Y \sim \nu$ is the map $T$ such that (i) $T$ is nondecreasing and such that if $U \sim \mathcal{U}([0, 1])$, then $T(U)$ has the same distribution $\nu$ as $Y$. This point of view will be the basis of our multivariate generalization of the notion of quantiles to the case of a random vector of dimension $d$: letting $\mu$ be a probability distribution of reference over $\mathbb{R}^d$ (when $d = 1$, a natural choice is the uniform distribution), the $\mu$-quantile map associated to $\nu$ is the map $T : \mathbb{R}^d \to \mathbb{R}^d$ such that:

- $T \# \mu = \nu$, which means that $T$ pushes forward the distribution $\mu$ to $\nu$; and such that
• $T$ is “monotone” in the sense that it is the gradient of a convex function: $T = \nabla \varphi$, where $\varphi : \mathbb{R}^d \to \mathbb{R}$.

The shape restriction that $T$ should be the gradient of a convex function is a natural generalization as, in dimension one, monotone maps are the gradients/derivatives of convex functions. That a unique solution to this requirement exists and is unique is the object of McCann’s theorem referred above.

Let us now discuss how the $\mu$-quantile associated to an empirical distribution is constructed. Let $\{y_1, ..., y_n\}$ be a sample, and let $\nu_n = n^{-1} \sum_{i=1}^n \delta_{y_i}$ be the associated empirical distribution. Because the quantile map is constructed as the gradient of a convex function, it is easy to see that in the case $\nu$ has a finite support, the quantile map should be the gradient of a piecewise affine and convex function – indeed, the gradient of such a map will take a finite number of possible values. In this case, the quantile map $T_n$ associated to $\nu_n$ will be defined as

$$T_n(u) = \arg \max_{y_i \in \{y_1, ..., y_n\}} \left\{ u^\top y_i - \psi_i \right\}$$

where the weights $\psi_i$ form a solution to problem (3.2) with the surplus $\Phi$ chosen as the scalar product. Hence, the empirical quantile map can be obtained as a finite-dimensional optimization problem. Consistency of $T_n$, which expresses that the $\mu$-quantile of $\nu_n$ converges to the $\mu$-quantile of $\nu$ has been shown in Chernozhukov et al., Forthcoming.

This definition of $\mu$-quantiles, which was originally introduced in Ekeland et al., 2011, has a number of applications, including:

• A notion of multivariate comonotonicity, which allows to construct multivariate measures of financial exposures. This was the original motivation in Ekeland et al., 2011, see also Bosc and Galichon, 2014. The definition is that $Y_1$ and $Y_2$ are $\mu$-comonotone if the $\mu$-quantile of $Y_1 + Y_2$ is the sum of the $\mu$-quantiles of $Y_1$ and $Y_2$. 

• Multivariate counterparts for a number of stochastic orders, which are known to rely on the notion of quantiles in the univariate case, such as first-order stochastic dominance. See Charpentier et al., 2016.

• An extension of the theory of rank-dependent expected utility to multivariate risky prospects. Rank-dependent utility functions were built as a response to paradoxes in expected utility theory such as Allais’ paradox. A multivariate extension of Yaari’s utility function was proposed in Galichon and Henry, 2012.

• A characterization of Pareto efficient risk-sharing arrangements in the multivariate case, as was done in Carlier et al., 2012, extending a result by Landsberger and Meilijson, 1994 of the efficient risk-sharing arrangements as the comonotone ones.

• An extension of Matzkin’s quantile-based identification results in hedonic models (see e.g. Matzkin, 2003 and Heckman et al., 2010) to the case with more than one attribute, as is done in Chernozhukov et al., 2015.

• A multivariate version of quantile regression based on a semiparametric extension of the Monge-Kantorovich case proposed in Carlier et al., 2016.

4.3. Partial identification. Optimal transport can also be a useful tool to handle partial identification in incomplete models. Consider an economic model with parameter $\theta \in \Theta$ which predicts that the population’s income $X$ will have distribution $P_\theta$. Assume that the income $X$ is not perfectly observed, but that only the tax bracket in which the income belongs is observed. If $y$ is the mid-point of the bracket, let $\Gamma(y) = [l(y), u(y)]$ be the corresponding bracket. The brackets are indexed by their mid-point $Y$, whose distribution $Y \sim Q$ is assumed to be observed. The identified set $\Theta_I$ is therefore the set of parameters $\theta \in \Theta$ such that the distribution $X \sim P_\theta$ predicted by the model is compatible with the observed distribution of the brackets $Y \in Q$. (We abstract away from any sample uncertainty here). More precisely, $\Theta_I$ is the set of $\theta$ such that there is a joint probability $\pi \in \mathcal{M}(P_\theta, Q)$ such that $\pi(X \in \Gamma(Y)) = 1$.

This compatibility problem can be formulated as an optimal transport problem. Indeed, if

$$
\Phi(x, y) = 1 \{x \in \Gamma(y)\}
$$
then \( \theta \in \Theta_I \) if and only if

\[
\max_{\pi \in \mathcal{M}(P_\theta, Q)} E_\pi \left[ 1 \{ X \in \Gamma (Y) \} \right] = 1.
\] (4.1)

Problem (4.1) is an optimal transport problem, and thus the numerical determination of \( \Theta_I \) boils down to the computation of the value of such a problem. This equivalence has been put to use in Galichon and Henry, 2011 and Ekeland et al., 2010; see a synthetic presentation in Galichon, 2016, section 9.1.

4.4. **Revealed preference.** There is an interesting connection between optimal transport and Afriat’s theorem on revealed preference inequalities Afriat, 1967. Indeed, the problem of revealed preference under its most basic form consists in the following: given the observation of \( n \) bundles \( x_1, \ldots, x_n \) in \( \mathbb{R}^d \), and given corresponding price vectors \( p_1, \ldots, p_n \) in \( \mathbb{R}^d \), one would like to recover nontrivial utility functions \( u \) such that

\[
x_i = \arg \max_{x_k : p_i^\top x_k \leq p_i^\top x_i} u(x_j).
\]

In the affirmative, one shall say that observations \( \{(x_i, p_i)\} \) are rationalizable. As shown by Afriat, an affirmative answer to this problem is equivalent to the existence of \( \lambda \in \mathbb{R}^n_+ \), \( \lambda \neq 0 \) and \( u \in \mathbb{R}^n \) such that

\[
u_i - u_j \geq \lambda_i p_i^\top (x_i - x_j).
\] (4.2)

As it was pointed out in Ekeland and Galichon, 2013, see also Kolesnikova et al., 2013, this problem can be reformulated using an optimal transport problem. Let \( p \) be the vector of uniform probability over \( \{1, \ldots, n\} \), i.e. \( p_i = 1/n \) for \( i = 1, \ldots, n \). Let \( \Delta \) be the set of \( \lambda \in \mathbb{R}^n_+ \) such that \( \sum_{i=1}^n \lambda_i = 1 \). For \( \lambda \in \Delta \), set

\[
W(\lambda) = \max_{\pi \in \mathcal{M}(p,p)} \sum_{1 \leq i, j \leq n} \pi_{ij} \Phi_{ij}^\lambda
\] (4.3)

where \( \Phi_{ij}^\lambda = \lambda_i p_i^\top (x_i - x_j) \). By Monge-Kantorovich duality, we have

\[
W(\lambda) = \frac{1}{n} \min \left\{ \sum_{i=1}^n u_i + \sum_{j=1}^n v_j : u_i + v_j \geq \Phi_{ij}^\lambda \right\}.
\]
Note that one has $W(\lambda) \geq 0$ as can be seen by taking $\pi^{*}_{ij} = 1 \{i = j\}/n$ in expression (4.3). Assume that $W(\lambda) = 0$. Then it means that $\pi^{*}_{ij}$ is optimal. Take a pair $(\phi, \psi)$ which is optimal for the dual problem. Thus by complementary slackness, $u_i + v_i = \Phi^\lambda_{ii} = 0$, hence $u_i = -u_i$. Thus the dual feasibility condition then implies Afriat’s inequalities (4.2). Conversely, it is quite easy to show that if Afriat’s inequalities are satisfied, then $W(\lambda) = 0$. This provides a particularly simple criterion: observations $\{(x_i, p_i)\}$ are rationalizable if and only if

$$\min_{\lambda \in \Delta} W(\lambda) = 0$$

which is a convex minimization problem. Further, the subgradient of $W(\lambda)$ is the set of $Z(\pi) \in \mathbb{R}^d$ such that $Z_i(\pi) = p_i^\top \left( x_i/n - \sum_{j=1}^n \pi_{ij} x_j \right)$, for any $\pi$ solution of (4.3).

5. Conclusion

This short article has hopefully convinced the reader of the growing importance of optimal transport methods as part of the standard econometrician’s toolbox. Because it is so intrinsically connected with notions such as linear programming, convex analysis, duality, quantiles, copulas, clustering, graphs, and numerical methods, investing some time in the study of this theory can only be fruitful. While this article has kept a focus on econometrics, a number of other economic applications of optimal transport exist, notably in mechanism design, labor economics, family economics, and asset pricing. Some of these applications are reviewed in *Optimal Transport Methods in Economics* Galichon, 2016.
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