Research article

On $\theta_\omega$-continuity

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**A R T I C L E   I N F O**

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**A B S T R A C T**

We use the closure and the theta omega closure operators to introduce $\theta_\omega$-continuous, $\omega$-$\theta$-continuous, weakly $\theta_\omega$-continuous and faintly $\theta_\omega$-continuous as new four classes of functions. We obtain several properties, relationships, examples and counter-examples related to them.

1. Introduction and preliminaries

Let $A$ be any subset of a topological space $(X, \tau)$. The set $\{x \in X : \forall O \in \tau \text{ with } x \in O, O \cap A \text{ is uncountable}\}$ is called the set of condensation points of $A$ and is denoted by $\text{Cond}(A)$. $A$ is called an $\omega$-closed set [1] if $\text{Cond}(A) \subseteq A$ and $A$ is called an $\omega$-open set if $X - A$ is $\omega$-closed. The family of all $\omega$-open sets of $(X, \tau)$ forms a topology on $X$ which is finer than $\tau$ and this topology is denoted by $\tau_\omega$. $\omega$-open sets played a vital role in general topology research see, [2, 3, 4, 5, 6, 7, 8, 9]. The closure of $A$ in $(X, \tau)$ (resp. $(X, \tau_\omega)$) is called by $\overline{A}$ (resp. $\overline{A}_\omega$). $\text{Cl}_\omega(A) = \{x \in X : \forall O \in \tau \text{ with } x \in O, \overline{O} \cap A \neq \emptyset\}$ [10] and $\text{Cl}_\omega(A) = \{x \in X : \forall O \in \tau_\omega \text{ with } x \in O, \overline{O}_\omega \cap A \neq \emptyset\}$ [11] are the theta-closure and the theta-$\omega$-closure operators. $A$ is $\theta$-closed [10] (resp. $\theta_\omega$-closed [11]) if $\text{Cl}_\omega(A) = A$ (resp. $\text{Cl}_\omega(A) = A$). $A$ is $\theta$-open [10] (resp. $\theta_\omega$-open [11]) if $X - A$ is $\theta$-closed (resp. $\theta_\omega$-closed). The family of $\theta$-open (resp. $\theta_\omega$-open) sets in $(X, \tau)$ are denoted by $\tau_\theta$ (resp. $\tau_{\theta_\omega}$). It is proved in [10] and [11] that $(X, \tau_\theta)$ and $(X, \tau_{\theta_\omega})$ are topological spaces, $\tau_\theta$ coarser than $\tau_{\theta_\omega}$ and $\tau_{\theta_\omega}$ coarser than $\tau$. Authors in [11] defined and investigated $\theta_{\omega}$-open sets. And they used them to characterize some separation axioms. Studying continuities in topological spaces is still a hot area of research see, [12, 13, 14, 15, 16]. This paper is devoted to introduce and investigate four new classes of functions, namely: $\theta_\omega$-continuous, $\omega$-$\theta$-continuous, weakly $\theta_\omega$-continuous and faintly $\theta_\omega$-continuous.

In this paper, for any nonempty set $X$, $\tau_{\text{ind}}, \tau_{\text{cof}}, \tau_{\text{cov}}$ will denote respectively the indiscrete topology, the cofinite topology, and the countable topology on $X$.

The following sequence of definitions and propositions will be used in the sequel:

**Definition 1.1.** [17] A topological space $(X, \tau)$ is called anti-locally countable if each $U \in \tau - \{\emptyset\}$ is uncountable.

**Proposition 1.2.** [18] If $(X, \tau)$ is an anti-locally countable topological space, then for all $A \in \tau_{\omega}, A = \overline{A}$.

**Definition 1.3.** [18] A topological space $(X, \tau)$ is called $\omega$-regular if for each closed set $F$ in $(X, \tau)$ and $x \in X - F$, there exist $U \in \tau$ and $V \in \tau_{\omega}$ such that $x \in U, F \subseteq V$ and $U \cap V = \emptyset$.

**Proposition 1.4.** [18] A topological space $(X, \tau)$ is $\omega$-regular if and only if for each $U \in \tau$ and each $x \in U$ there is $V \in \tau$ such that $x \in V \subseteq \overline{V} \subseteq U$.

Recall that a topological space $(X, \tau)$ is locally indiscrete if every open set in $(X, \tau)$ is closed and $(X, \tau)$ is locally countable if for each $x \in X$, there is $U \in \tau$ such that $x \in U$ and $U$ is countable.

**Definition 1.5.** [11] A topological space $(X, \tau)$ is said to be $\omega$-locally indiscrete if every open set in $(X, \tau)$ is $\omega$-closed.

**Proposition 1.6.** [11] a. Every locally indiscrete topological space is $\omega$-locally indiscrete.

- Every locally countable topological space is $\omega$-locally indiscrete.
2. Continuity

Definition 2.1. [19] A function $f : (X, τ) \rightarrow (Y, σ)$ is said to be $θ$-continuous if for every $x \in X$ and every open subset $V$ in $Y$ containing $f(x)$, there exists an open subset $U$ in $X$ containing $x$ such that $f(U) \subseteq V$.

Definition 2.2. A function $f : (X, τ) \rightarrow (Y, σ)$ is said to be $θ_{α}$-continuous if for every $x \in X$ and every open subset $V$ in $Y$ containing $f(x)$, there exists an open subset $U$ in $X$ containing $x$ such that $f(U) \subseteq V^α$.

Theorem 2.3. Every $θ_{α}$-continuous function is $θ$-continuous.

Proof. Let $f : (X, τ) \rightarrow (Y, σ)$ be $θ_{α}$-continuous. Let $x \in X$ and let $V$ be any open set in $Y$ containing $f(x)$. Since $f$ is $θ_{α}$-continuous, there exists an open subset $U$ in $X$ containing $x$ such that $f(U) \subseteq V^α$. It follows that $f$ is $θ$-continuous.

The converse of Theorem 2.3 is not true in general as the following example clarifies:

Example 2.4. Consider the function $f : (ℕ, τ_{ind}) \rightarrow (ℕ, τ_{cof})$ defined as $f(x) = x$. Then
- $f$ is $θ$-continuous.
- $f$ is not $θ_{α}$-continuous.

Proof. a. Let $x \in ℕ$ and let $V \in τ_{cof}$ such that $f(x) = x \in V$. Take $U = ℕ$. Then $x \in U \in τ_{ind}$ and $f(U) \subseteq V = ℕ$. It follows that $f$ is $θ$-continuous.

b. Let $x = 1$ and let $V = ℕ - \{2\}$. Then $V \in τ_{cof}$ with $f(\{1\}) = 1 \in V$. If there exists $U \in τ_{ind}$ such that $1 \in U \in τ_{ind}$ and $f(U) \subseteq V^α$, then $U = ℕ$ and $f(ℕ) = ℕ$. But $V^α = ℕ - \{2\}$. It follows that $f$ is not $θ_{α}$-continuous.

Theorem 2.5. If $f : (X, τ) \rightarrow (Y, σ)$ is a $θ$-continuous function and $(Y, σ)$ is an anti-locally countable topological space, then $f : (X, τ) \rightarrow (Y, σ)$ is $θ_{α}$-continuous.

Proof. Let $x \in X$ and let $V$ be any open subset in $Y$ containing $f(x)$. Since $f$ is $θ$-continuous, there exists an open subset $U$ in $X$ containing $x$ such that $f(U) \subseteq V$. Since $(Y, σ)$ is anti-locally countable, then by Proposition 1.2 we have $V = V^α$ and thus $f(U) \subseteq V^α$. It follows that $f$ is $θ_{α}$-continuous.

Theorem 2.6. [19] Every continuous function is $θ$-continuous but not conversely.

The following two examples show that continuity and $θ_{α}$-continuity are independent:

Example 2.7. Consider the function $f : (ℝ, τ_{ind}) \rightarrow (ℝ, τ_{cof})$ defined as $f(x) = x$. Clearly that $f$ is discontinuous. Let $x \in ℝ$ and let $V \in τ_{cof}$ such that $f(x) = x \in V$. Then $V^α = ℝ$. Take $U = ℝ$. Then $x \in U \in τ_{ind}$ and $f(U) = ℝ \subseteq ℝ = V$. It follows that $f$ is $θ_{α}$-continuous.

Example 2.8. Consider the function $f : (ℕ, τ) \rightarrow (ℕ, σ)$ where $τ = σ = \{∅, ℕ, \{1\}\}$ and $f(x) = x$. Clearly that $f$ is continuous. Suppose that $f$ is $θ_{α}$-continuous. Take $x = 1$ and $V = \{1\}$. Then $V \in σ$ with $f(x) = x \in V$ and $V^α = V$. On the other hand, if $U \in τ$ with $1 \in U$, then either $U = \{1\}$ or $U = ℕ$. In both cases, $V = ℕ$ and $f(U) = ℕ \subseteq V^α = \{1\}$ which is impossible. It follows that $f$ is not $θ_{α}$-continuous.

In the following result we give a sufficient condition for a $θ_{α}$-continuous function to be continuous:

Theorem 2.9. If $f : (X, τ) \rightarrow (Y, σ)$ is a $θ_{α}$-continuous function with $(Y, σ)$ is $ω$-regular, then $f$ is continuous.

Proof. Let $x \in X$ and let $V$ be any open set in $Y$ containing $f(x)$. Since $(Y, σ)$ is $ω$-regular, then by Proposition 1.4 there exists an open set $H$ in $Y$ such that $f(x) \in H \subseteq \overline{H} \subseteq V$. Since $f$ is $θ_{α}$-continuous, there exists an open set $U$ in $X$ containing $x$ such that $f(U) \subseteq \overline{H}$. Thus we have $f(U) \subseteq f(U) \subseteq \overline{H} \subseteq V$.

It follows that $f$ is continuous.

Definition 2.10. [20] A function $f : (X, τ) \rightarrow (Y, σ)$ is said to be weakly continuous if for every $x \in X$ and every open set $V$ in $Y$ containing $f(x)$, there exists an open subset $U$ in $X$ containing $x$ such that $f(U) \subseteq \overline{V}$.

Definition 2.11. A function $f : (X, τ) \rightarrow (Y, σ)$ is said to be $ωθ$-continuous if for every $x \in X$ and every open set $V$ in $Y$ containing $f(x)$, there exists an open subset $U$ in $X$ containing $x$ such that $f(U) \subseteq \overline{V}$.

Theorem 2.12. Every $ωθ$-continuous function is weakly continuous.

Proof. Let $f : (X, τ) \rightarrow (Y, σ)$ be $ωθ$-continuous. Let $x \in X$ and let $V$ be any open set in $Y$ containing $f(x)$. Since $f$ is $ωθ$-continuous, there exists an open set $U$ in $X$ containing $x$ such that $f(U) \subseteq \overline{V}$. Thus $f(U) \subseteq f(U) \subseteq \overline{V}$. It follows that $f$ is weakly continuous.

Theorem 2.13. If $f : (X, τ) \rightarrow (Y, σ)$ is weakly continuous such that $(X, τ)$ is $ω$-locally indiscrete, then $f$ is $ωθ$-continuous.

Proof. Let $x \in X$ and let $V$ be any open set in $Y$ containing $f(x)$. Since $f$ is weakly continuous, there exists an open set $U$ in $X$ containing $x$ such that $f(U) \subseteq \overline{V}$. Since $(X, τ)$ is $ω$-locally indiscrete, then $U$ is $ω$-closed and $U^ω = U$. Thus $f(U^ω) = f(U) \subseteq \overline{V}$. It follows that $f$ is $ωθ$-continuous.

Corollary 2.14. If $f : (X, τ) \rightarrow (Y, σ)$ is weakly continuous such that $(X, τ)$ is locally indiscrete, then $f$ is $ωθ$-continuous.

Proof. Proposition 1.6 (b) and Theorem 2.13.

Corollary 2.15. If $f : (X, τ) \rightarrow (Y, σ)$ is weakly continuous such that $(X, τ)$ is locally countable, then $f$ is $ωθ$-continuous.

Proof. Proposition 1.6 (b) and Theorem 2.13.

Theorem 2.16. If $f : (X, τ) \rightarrow (Y, σ)$ is weakly continuous such that $(X, τ)$ is $ω$-regular, then $f$ is $ωθ$-continuous.

Proof. Let $x \in X$ and let $V$ be any open set in $Y$ containing $f(x)$. Since $f$ is weakly continuous, there exists an open set $H$ in $X$ containing $x$ such that $f(H) \subseteq \overline{V}$. Since $(X, τ)$ is $ω$-regular, then there is an open set $U$ in $X$ containing $x$ such that $U^ω \subseteq H$. Thus $f(U^ω) = f(H) \subseteq \overline{V}$. It follows that $f$ is $ωθ$-continuous.

Theorem 2.17. Every $θ$-continuous function is $ωθ$-continuous.

Proof. Let $f : (X, τ) \rightarrow (Y, σ)$ be $θ$-continuous. Let $x \in X$ and let $V$ be any open set in $Y$ containing $f(x)$. Since $f$ is $θ$-continuous, there exists an open set $U$ in $X$ containing $x$ such that $f(U) \subseteq \overline{V}$. Thus $f(U^ω) \subseteq f(U) \subseteq \overline{V}$. It follows that $f$ is $ωθ$-continuous.
Proposition 1.4, \( f \) is \( \omega \)-\( \theta \)-continuous if and only if \( f \) is \( \theta \)-continuous.\]

**Remark 2.19.** The function in Example 3.3 of [21] is weakly continuous but not \( \theta \)-continuous, moreover, its domain is anti-locally countable. So by Theorem 2.18, this function is not \( \omega \)-\( \theta \)-continuous. Therefore, the converse of Theorem 2.12 is not true in general.

The following example shows that the converse of Theorem 2.17 is not true in general:

**Example 2.20.** We utilize Example 3.2 of [21]. Let \( X = Y = \{ a, b, c, d \} \) and \( \tau = \sigma = \{ \emptyset, X, \{ a \}, \{ b \}, \{ a, b \}, \{ a, b, c \}, \{ a, b, c, d \} \} \). Define \( f : (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = c, f(b) = d, f(c) = b, f(d) = a \). As in [21] \( f \) is weakly continuous and by Corollary 2.15 \( f \) is \( \omega \)-\( \theta \)-continuous. On the other hand, it is proved in [21] that \( f \) is not \( \theta \)-continuous.

**Definition 2.21.** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be weakly \( \theta \)-continuous if for every \( x \in X \) and every open set \( V \) in \( Y \) containing \( f(x) \), there exists an open subset \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq V \).

**Theorem 2.22.** Every weakly \( \theta \)-continuous function is weakly continuous.

**Proof.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be weakly \( \theta \)-continuous. Let \( x \in X \) and let \( V \) be any open set in \( Y \) containing \( f(x) \). Since \( f \) is weakly \( \theta \)-continuous, there exists an open set \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq V \). Thus \( f(U) \subseteq V \). It follows that \( f \) is weakly continuous.

The following example shows that the converse of Theorem 2.22 is not true in general:

**Example 2.23.** Consider the identity function \( f : (\mathbb{N}, \tau) \rightarrow (\mathbb{N}, \sigma) \) where \( \tau = \{ \emptyset, \mathbb{N} \} \) and \( \sigma = \{ \emptyset, \{ 1 \} \} \). It is not difficult to check that \( \{ 1 \} = \mathbb{N} \) and \( \mathbb{N} = \{ 1 \} \). To see that \( f \) is weakly \( \theta \)-continuous, let \( x \in \mathbb{N} \) and \( V \in \sigma \) such that \( f(x) = x \in V \). Then \( V = \mathbb{N} \) or \( V = \{ 1 \} \) and in both cases \( f(V) = \mathbb{N} \). Choose \( U = \mathbb{N} \). Then \( x \in U \) and \( f(U) = \mathbb{N} \subseteq \mathbb{N} = V \). To see that \( f \) is not weakly \( \theta \)-continuous, suppose to the contrary that \( f \) is weakly \( \theta \)-continuous. Let \( x = 1 \) and take \( V = \{ 1 \} \). Then there is \( U \in \tau \) such that \( 1 \in U \) and \( f(U) \subseteq \{ 1 \} \). Then \( f(U) = f(\mathbb{N}) = \mathbb{N} \subseteq \{ 1 \} \) which is a contradiction.

**Theorem 2.24.** If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is weakly continuous such that \( (Y, \sigma) \) is anti-locally countable, then \( f \) is weakly \( \theta \)-continuous.

**Proof.** Let \( x \in X \) and let \( V \) be any open set in \( Y \) containing \( f(x) \). Since \( f \) is weakly continuous, there exists an open set \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq V \). Since \( (Y, \sigma) \) is anti-locally countable, then by Proposition 1.4, \( \overline{V} = \overline{V} \), and so \( f(U) \subseteq \overline{V} = \overline{V} \). It follows that \( f \) is weakly \( \theta \)-continuous.

**Theorem 2.25.** Every continuous function is weakly \( \theta \)-continuous.

**Proof.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be continuous. Let \( x \in X \) and let \( V \) be any open set in \( Y \) containing \( f(x) \). Since \( f \) is continuous, there exists an open set \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq V \). Thus \( f(U) \subseteq \overline{V} \). It follows that \( f \) is weakly \( \theta \)-continuous.
Theorem 2.36. Let \( f : (X,\tau) \rightarrow (Y,\sigma) \) be a function. Then the following are equivalent:

a. \( f : (X,\tau) \rightarrow (Y,\sigma) \) is faintly \( \theta_{\omega} \)-continuous.

b. \( f : (X,\tau) \rightarrow \{Y,\sigma_{\omega}\} \) is continuous.

c. The inverse image of each \( \theta_{\omega} \)-open set in \( Y \) is open in \( X \).

d. The inverse image of each \( \theta_{\omega} \)-closed set in \( Y \) is closed in \( X \).

Theorem 2.37. Every weakly \( \theta_{\omega} \)-continuous function is faintly \( \theta_{\omega} \)-continuous.

Proof. \( f : (X,\tau) \rightarrow (Y,\sigma) \) be weakly \( \theta_{\omega} \)-continuous. Let \( x \in X \) and \( Y \) be \( \theta_{\omega} \)-open set containing \( f(x) \). There is \( B \in \theta_{\omega} \) such that \( f(x) \subseteq B \subseteq Y \). Since \( f \) is weakly \( \theta_{\omega} \)-continuous, then there exists \( U \in \tau \) containing \( x \) such that \( f(U) \subseteq B \subseteq Y \). It follows that \( f \) is faintly \( \theta_{\omega} \)-continuous.

The following example shows that the implication in Theorem 2.37 is not reversible in general:

Example 2.38. Consider the identity function \( f : (\mathbb{R},\sigma) \rightarrow (\mathbb{R},\tau) \) where \( \sigma = \{\emptyset,\mathbb{R},\mathbb{N}\} \) and \( \tau \) as in Example 2.26. Then:

a. \( f \) is faintly \( \theta_{\omega} \)-continuous.

b. \( f \) is not weakly continuous.

Proof. (a) Since by Example 2.26 \( \tau_{\omega} = \{\emptyset,\mathbb{R},\mathbb{N}\} = \sigma \), then \( f : (\mathbb{R},\sigma) \rightarrow (\mathbb{R},\tau) \) is continuous and by Theorem 2.36 \( f : (\mathbb{R},\sigma) \rightarrow (\mathbb{R},\tau) \) is faintly \( \theta_{\omega} \)-continuous.

(b) Suppose to the contrary that \( f \) is weakly continuous. Since \( f(\sqrt{2}) = \sqrt{2} \in \mathbb{Q} \subseteq \tau \), there is \( U \subseteq \sigma \) such that \( \sqrt{2} \in U \) and \( f(U) \subseteq \mathbb{Q} = \mathbb{R} - \mathbb{N} \). Since \( \sqrt{2} \in U \subseteq \sigma \), then \( U = \mathbb{R} \) and \( f(U) = \mathbb{R} \), a contradiction.

Theorem 2.39. Every faintly \( \theta_{\omega} \)-continuous function is faintly \( \theta_{\omega} \)-continuous.

Proof. Theorems 2.18, 2.35 and 2.36.

The implication in Theorem 2.39 is not reversible as it can be seen from the following example:

Example 2.40. Consider the identity function \( f : (\mathbb{R},\tau) \rightarrow (\mathbb{R},\sigma) \) where \( \tau = \{\emptyset,\mathbb{R}\} \) and \( \sigma = \{\emptyset,\mathbb{R},\mathbb{Q}\} \). It is not difficult to check that \( \sigma_{\theta} = \{\emptyset,\mathbb{R}\} \) and \( \sigma_{\omega} = \sigma \). Therefore, by Theorems 2.35 and 2.36, it follows that \( f \) is faintly continuous but not faintly \( \theta_{\omega} \)-continuous.

We can summarize the results and examples above by means of the following diagram:

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Declarations

Author contribution statement

S. Al Ghour, B. Irshidat: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

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Additional information

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