0. Introduction

The theory has its origin in the work of Grothendieck [Grk] who introduced the following notation of properties of abelian categories:

Ab3. An abelian category with coproducts or equivalently, a cocomplete abelian category.

Ab5. Ab3-category, in which for any directed family \( \{ A_i \}_{i \in I} \) of subobjects of an arbitrary object \( X \) and for any subobject \( B \) of \( X \) the following relation holds:

\[
\left( \sum_{i \in I} A_i \right) \cap B = \sum_{i \in I} (A_i \cap B)
\]
Ab5-categories possessing a family of generators are called Grothendieck categories. They constitute a natural extension of the class of module categories, with which they share a great number of important properties.

Although our interest lies exclusively with the category Mod \( A \) with \( A \) a ring (associative, with identity), here by a module category Mod \( A \) we mean a Grothendieck category possessing a family of finitely generated projective generators \( \mathcal{A} = \{ P_i \}_{i \in I} \). We also refer to the family \( \mathcal{A} \) as a ring. It is equivalent to the category \( (A^{op}, \text{Ab}) \) of additive functors from the category \( A^{op} \) to the category of abelian groups Ab. Conversely, every functor category \( (\mathcal{B}, \text{Ab}) \) with \( \mathcal{B} \) a preadditive category is a module category Mod \( A \) with a ring of representable functors \( \mathcal{A} = \{ h_B = (B, -) \}_{B \in \mathcal{B}} \). When \( A = \{ A \} \) with \( A \) a ring we write Mod \( \mathcal{A} = \text{Mod} A \).

In the fundamental work of Gabriel \([G]\) it was presented perhaps the basic tool for studying Grothendieck categories: localization theory. Such concepts as a localizing subcategory, a quotient category (and arising in this case the respective concepts of a torsion functor and a localizing functor) play a key role in our analysis.

Recall that the full subcategory \( \mathcal{S} \) of a Grothendieck category \( \mathcal{C} \) is a Serre subcategory if for every short exact sequence
\[
0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0
\]
in \( \mathcal{C} \) the object \( C \in \mathcal{S} \) if and only if \( C', C'' \in \mathcal{S} \). A Serre subcategory \( \mathcal{S} \) of \( \mathcal{C} \) is localizing if it is closed under coproducts.

Gabriel observed that there is a bijection on the class of all idempotent topologizing sets of right ideals of the ring \( A \) ("The Gabriel topologies"), and the class of all localizing subcategories of Mod \( A \). Here we define a Gabriel topology on an arbitrary ring \( A = \{ P_i \}_{i \in I} \) as follows. Let \( \mathfrak{F} = \{ \mathfrak{F}^i \}_{i \in I} \) be some family of subobjects of \( P_i \in \mathcal{A} \). Then the family \( \mathfrak{F}^i \) is a Gabriel topology on \( \mathcal{A} \) if it satisfies the following axioms:

\begin{enumerate}
  \item \( P_i \in \mathfrak{F}^i \) for each \( i \in I \).
  \item If \( a \in \mathfrak{F}^i \) and \( \mu \in \text{Hom}_A(P_j, P_i) \), \( P_j \in \mathcal{A} \), then \( \{ a : \mu \} = \mu^{-1}(a) \) belongs to \( \mathfrak{F}^j \).
  \item If \( a \) and \( b \) are subobjects of \( P_i \) such that \( a \in \mathfrak{F}^i \) and \( \{ b : \mu \} \in \mathfrak{F}^j \) for any \( \mu \in \text{Hom}_A(P_j, P_i) \) with \( \text{Im}(\mu) \subseteq a \), \( P_j \in \mathcal{A} \), then \( b \in \mathfrak{F}^j \).
\end{enumerate}

**Theorem** (Gabriel). The map
\[
\mathcal{S} \mapsto \mathfrak{F}(\mathcal{S}) = \{ a \subseteq P_i \mid i \in I, P_i/a \in \mathcal{S} \}
\]
establishes a bijection between Gabriel topologies on \( \mathcal{A} \) and the class of localizing subcategories of Mod \( \mathcal{A} \).

Similar to the category of modules Mod \( A \) the respective localizing functor
\[
(-)_\mathcal{S} : \text{Mod} \mathcal{A} \longrightarrow \text{Mod} \mathcal{A}/\mathcal{S}
\]
is constructed in the following way:
\[
M_\mathcal{S}(P_i) = \lim_{a \in \mathfrak{F}^i} \text{Hom}_A(a, M/t_\mathcal{S}(M)).
\]
Here \( t_\mathcal{S} \) denotes an \( \mathcal{S} \)-torsion functor that takes each right \( \mathcal{A} \)-module \( M \in \text{Mod} \mathcal{A} \) to the largest subobject \( t_\mathcal{S}(M) \) of \( M \) belonging to \( \mathcal{S} \).

The famous Popescu-Gabriel Theorem \([P,G]\) reduces (at least in principle) the Grothendieck categories theory to a study of quotient categories of the module category.
Mod $A$. It asserts that every Grothendieck category $\mathcal{C}$ with a family of generators $\mathcal{U} = \{U_i\}_{i \in I}$ is equivalent to the quotient category $\text{Mod}A/\mathcal{S}$ with $A$ the endomorphism ring $\text{End}U$ of a generator $U = \oplus_{i \in I}U_i$ of $\mathcal{C}$ and $\mathcal{S}$ some localizing subcategory of $\text{Mod}A$.

However, the main inconvenience of working with Popescu-Gabriel’s Theorem is that, as a rule, the endomorphism ring $\text{End}U$ has enough complicated form. It often arises a necessity to study Grothendieck categories locally, with the help of “visible” subcategories of $\mathcal{C}$. In this vein, in [GGT] it has been shown that if $P$ is a finitely generated projective object of $\mathcal{C}$ (if such an object exists), then the full subcategory generated by the object $P$

$$\mathcal{S} = \{C \in \mathcal{C} \mid \text{Hom}_\mathcal{C}(P,C) = 0\}$$

is localizing in $\mathcal{C}$ and the quotient category $\mathcal{C}/\mathcal{S}$ of $\mathcal{C}$ with respect to $\mathcal{S}$ is equivalent to the category of modules $\text{Mod}A$ with $A = \text{End}P$. In particular, every module category $\text{Mod}\mathcal{A}$ with $\mathcal{A} = \{P_i\}_{i \in I}$ can be partially covered with the categories $\text{Mod}A_i$, where $A_i = \text{End}P_i$, i.e. the following relation holds:

$$\bigcup_{i \in I} \text{Mod}A_i \subseteq \text{Mod}\mathcal{A}.$$ We extend this result to arbitrary projective families of objects. We say that a family $\mathcal{U}$ of the Grothendieck category $\mathcal{C}$ generates a full subcategory $\mathcal{B}$ of $\mathcal{C}$ if for every object $C$ of $\mathcal{B}$ there is an exact sequence

$$\oplus_{i \in I}U_i \rightarrow \oplus_{j \in J}U_j \rightarrow C \rightarrow 0$$

with $U_i \in \mathcal{U}$.

**Theorem.** Let $\mathcal{C}$ be a Grothendieck category and $\mathcal{U} = \{P_i\}_{i \in I}$ some family of projective objects of $\mathcal{C}$. Then the subcategory $\mathcal{S} = \{C \in \mathcal{C} \mid c(P,C) = 0 \text{ for all } P \in \mathcal{U}\}$ is localizing in $\mathcal{C}$ and $\mathcal{C}/\mathcal{S}$ is equivalent to the quotient category $\text{Mod}\mathcal{A}/\mathcal{P}$, where $\mathcal{A} = \{h_P = (-,P)\}_{P \in \mathcal{U}}, \mathcal{P}$ is some localizing subcategory of $\text{Mod}\mathcal{A}$. Moreover, $\mathcal{C}/\mathcal{S}$ is equivalent to a subcategory of $\mathcal{C}$ generated by $\mathcal{U}$. If, in addition, each $P \in \mathcal{U}$ is finitely generated, then $\mathcal{C}/\mathcal{S}$ is equivalent to the module category $\text{Mod}\mathcal{A}$.

The Popescu-Gabriel Theorem is generalized as follows.

**Theorem** (Popescu and Gabriel). Let $\mathcal{C}$ be a Grothendieck category with a family of generators $\mathcal{U} = \{U_i\}_{i \in I}$ and $T = (-,?) : \mathcal{C} \rightarrow \text{Mod}\mathcal{A}$ be the representation functor that takes each $X \in \mathcal{C}$ to $(-,X)$, where $\mathcal{A} = \{h_{U_i} = (-,U_i)\}_{i \in I}$. Then:

1. $T$ is full and faithful.
2. $T$ induces an equivalence between $\mathcal{C}$ and the quotient category $\text{Mod}\mathcal{A}/\mathcal{S}$, where $\mathcal{S}$ denotes the largest localizing subcategory in $\text{Mod}\mathcal{A}$ for which all modules $TX = (-,X)$ are $\mathcal{S}$-closed.

The advantage of the Theorem is that we can freely choose a family of generators $\mathcal{U}$ of $\mathcal{C}$. To be precise, if $\mathcal{M}$ is an arbitrary family of objects of $\mathcal{C}$, then the family $\overline{\mathcal{U}} = \mathcal{U} \cup \mathcal{M}$ is also a family of generators.

We say that an object $C$ of $\mathcal{C}$ is $\mathcal{U}$-finitely generated ($\mathcal{U}$-finitely presented) if there is an epimorphism $\eta : \oplus_{i \in I}U_i \rightarrow C$ (if there is an exact sequence $\oplus_{i=1}^mU_i \rightarrow \oplus_{j=1}^nU_i \rightarrow C$) where $U_i \in \mathcal{U}$. The full subcategory of $\mathcal{U}$-finitely generated ($\mathcal{U}$-finitely presented) objects of $\mathcal{C}$ is denoted by $\text{fg}_\mathcal{U}\mathcal{C}$ ($\text{fp}_\mathcal{U}\mathcal{C}$). When every $U_i \in \mathcal{U}$ is finitely generated
(finitely presented), that is the functor \((U_i, -)\) preserves direct unions (limits), we write \(fg_{U_i} \mathcal{C} = fg \mathcal{C} (fp_{U_i} \mathcal{C} = fp \mathcal{C})\). Then every Grothendieck category is locally \(U\)-finitely generated (locally \(U\)-finitely presented) that means every object \(C\) of \(\mathcal{C}\) is a direct union (limit) \(C = \sum_{i \in I} C_i\) \((C = \lim_{i \in I} C_i)\) of \(U\)-finitely generated (\(U\)-finitely presented) objects \(C_i\).

Recall also that a localizing subcategory \(S\) of \(\mathcal{C}\) is of prefinite (finite) type provided that the inclusion functor \(\mathcal{C}/S \rightarrow \mathcal{C}\) commutes with direct unions (limits). So the following assertion holds.

**Theorem** (Breitsprecher). Let \(\mathcal{C}\) be a Grothendieck category with a family of generators \(U = \{U_i\}_{i \in I}\). Then the representation functor \(T = (-, ?) : \mathcal{C} \rightarrow ((fp_{U_i} \mathcal{C})^{op}, Ab)\) defines an equivalence between \(\mathcal{C}\) and \(((fp_{U_i} \mathcal{C})^{op}, Ab)/S\), where \(S\) is some localizing subcategory of \(((fp_{U_i} \mathcal{C})^{op}, Ab)\). Moreover, \(S\) is of finite type if and only if \(fp_{U_i} \mathcal{C} = fp \mathcal{C}\). In this case, \(\mathcal{C}\) is equivalent to the category \(\text{Lex}((fp \mathcal{C})^{op}, Ab)\) of contravariant left exact functors from \(fp \mathcal{C}\) to \(Ab\).

The Gabriel topology stated above is taken up again in expanded and more general form, to be used for the characterization of finiteness conditions on localizing subcategories. Let \(U = \{U_i\}_{i \in I}\) be a family of generators of the Grothendieck category \(\mathcal{C}\) and \(S\) a localizing subcategory. By Gabriel topology \(\mathfrak{G}\) on \(U\) we mean a collection of the following sets:

\[
\mathfrak{G}^i = \{a \subseteq U_i \mid U_i/a \in S\}
\]

where \(U_i \in U\). Then \(\mathfrak{G}\) satisfies axioms \(T1 - T3\) (see above) and similar to the module category the localizing functor \((-)_S : \mathcal{C} \rightarrow \mathcal{C}/S\) is defined by the rule

\[
X_S(U_i) = \lim_{a \in \mathfrak{G}^i} (a, X/t_S(X))
\]

where \(U_i \in U\) and \(t_S\) is an \(S\)-torsion functor.

At present in modern theory of rings and modules and also in theory of abelian categories there is a number of fundamental concepts having come from model theory of modules. The model theory has brought essentially new principles and statements of questions that touch upon purely algebraic objects. For this reason, it has stipulated a number of investigations, which are, on the one hand, a translation of model-theoretic idioms into algebraic language and, on the other hand, methods obtained so prove, in a certain context, a very convenient tool for studying the category of modules.

It turned out that the most important model-theoretic conceptions are realized in the category

\[
\mathcal{C}_A = (\text{mod} A^{op}, Ab)
\]

whose objects are additive covariant functors from the category \(\text{mod} A^{op}\) of finitely presented left \(A\)-modules into the category of abelian groups \(Ab\). We refer to the category \(\mathcal{C}_A\) as the category of *generalized right \(A\)-modules* on account of the right exact, fully faithful functor \(M_A \mapsto M \otimes_A -\) from the category \(\text{Mod} A\) of right \(A\)-modules to \(\mathcal{C}_A\). This functor identifies pure-injective right \(A\)-modules with injective objects of \(\mathcal{C}_A\). Auslander [Aus] observed that the full subcategory \(fp \mathcal{C}_A\) of finitely presented functors of \(\mathcal{C}_A\) is abelian. Equivalently, \(\mathcal{C}_A\) is a locally coherent Grothendieck category.
Note separately that the category \( C \) plays an important role in modern representation theory of finite dimensional algebras (see e.g. \([\text{Aus2, Kr2}]\)).

One of the fundamental model-theoretic concepts is a Ziegler spectrum of a ring introduced by Ziegler \([\text{Zgr}]\) in model-theoretic terms. Recently Herzog \([\text{Hrz}]\) and Krause \([\text{Kr1}]\) have defined (algebraically) the Ziegler spectrum of an arbitrary locally coherent Grothendieck category. An extension of the Ziegler spectrum to arbitrary locally coherent categories is especially constructive in the study of purity in arbitrary locally finitely presented (not only Grothendieck) categories. Generally, a locally coherent Grothendieck category plus the couple \((t_S, (-)_S)\) of torsion/localizing functors turns out a useful tool every time in model theory it arises something new about the category of modules! This is, if you will, a good heuristic method in the indicated context.

So then the Ziegler spectrum \( \text{Zg} C \) of a locally coherent Grothendieck category \( C \) is a topological space whose points are the isomorphism types of the indecomposable injective objects of \( C \) and an open basis of \( \text{Zg} C \) is given by the collection of subsets

\[
\mathcal{O}(C) = \{ E \in \text{Zg} C \mid \text{Hom}_C(C, E) \neq 0 \}
\]

as \( C \in \text{coh} C \) ranges over the coherent objects of \( C \). If \( C = C_A \), this space is called a Ziegler spectrum of a ring.

The second half of the paper, for the most part, is devoted to a generalization of results of \([\text{GG1, GG2}]\). At first, given a ring \( A = \{ P_i \}_{i \in I} \) of finitely generated projective generators of the module category \( \text{Mod} A = (A^{\text{op}}, \text{Ab}) \), we define, similar to the category \( C_A \), the category of generalized \( A \)-modules

\[
C_A = (\text{mod} A^{\text{op}}, \text{Ab}).
\]

Here \( \text{mod} A^{\text{op}} = \text{fp}(A, \text{Ab}) \) denotes a full subcategory of finitely presented left \( A \)-modules. The most important notions of the category \( C_A \) are easily carried over to the same notions for the category \( C_A \).

The following Theorem is the main result for this part.

**Theorem.** Let \( C \) be a Grothendieck category with a family of generators \( U = \{ U_i \}_{i \in I} \) and \( A = \{ h_{U_i} \}_{i \in I} \) a ring generated by \( U \). Then \( C \) is equivalent to the quotient category of \( C_A \) with respect to some localizing subcategory \( S \) of \( C_A \). In particular, every module category \( \text{Mod} A \) with \( A = \{ P_i \}_{i \in I} \) a ring is equivalent to the quotient category of \( C_A \) with respect to the localizing subcategory \( \mathcal{P} A = \{ F \in C_A \mid F(P) = 0 \text{ for all } P \in A \} \).

The preceding Theorem leads to a description of different classes of rings and modules in terms of torsion/localizing functors in the category \( C_A \). The aim of such a description is to give a criterion of a duality for categories of finitely presented left and right \( A \)-modules.

The paper is organized as follows. The first section is preliminary, collecting the necessary category-theoretic background. In the second section we discuss Gabriel topologies and localization in module categories. The following section is of purely technical interest. The results of this section will be needed in proving Popescu-Gabriel’s Theorem. The principle section of the paper is fourth, in which we prove the Popescu-Gabriel Theorem and also discuss projective generating sets. In the fifth section we show how various finiteness conditions of the localizing subcategory \( S \) of
\( \mathcal{C} \) are reflected by properties of the family of generators \( \mathcal{U} \) of \( \mathcal{C} \). We also prove here Breitsprecher’s Theorem. The basic facts about categories of generalized \( \mathcal{A} \)-modules, including Auslander-Gruson-Jensen’s Duality and Theorem of Herzog is presented in the sixth section. In the remaining sections we present Grothendieck categories as quotient categories of \( \mathcal{C}_\mathcal{A} \) and illustrate how localizing subcategories of \( \mathcal{C}_\mathcal{A} \) are used to study rings and modules.

When there is no doubt about the ring \( A \) or the category \( \mathcal{B} \), we usually abbreviate \( \text{Hom}_A(M,N) \) or \( \text{Hom}_B(M,N) \) as \( (M,N) \). We shall freely invoke the fact that every object \( X \in \mathcal{C} \) of a Grothendieck category \( \mathcal{C} \) has an injective envelope \( E(X) \in \mathcal{C} \) (for details see [Fr, Chapter VI]). This fact is also discussed in section 4. If \( \mathcal{B} \) is a category, then by a subcategory \( \mathcal{A} \) of \( \mathcal{B} \) we shall always mean a full subcategory of \( \mathcal{B} \). For concepts such as subobject, epimorphism, injectivity, etc. we shall use the prefix \( \mathcal{A} \)-subobject or \( \mathcal{B} \)-subobject to indicate the context. This prefix can be omitted if the concept in question is absolute with respect to the inclusion \( \mathcal{A} \subseteq \mathcal{B} \). To indicate the context of an operation, for example \( \text{Ker} \mu, E(X) \) or \( \varinjlim X_i \), we shall use a subscript, for example, \( \text{Ker}_\mathcal{A} \mu, E_\mathcal{A}(X) \), etc. which may also be omitted in case of absoluteness.

Acknowledgements. I should like to thank A. I. Generalov and M. Prest for helpful discussions. Many thanks also to Vladimir Kalinin for his help in preparing the manuscript and constant interest in the work.

1. Preliminaries

In this preliminary section, we collect the basic facts about Grothendieck categories. Some constructions can be defined for more general (than Grothendieck) categories. For details and proofs we refer the reader to [Gbl, Fa].

1.1. Ab-conditions

Recall that an abelian category \( \mathcal{C} \) is cocomplete or \( \text{Ab3-category} \) if it has arbitrary direct sums. The cocomplete abelian category \( \mathcal{C} \) is said to be \( \text{Ab5-category} \) if for any directed family \( \{A_i\}_{i \in I} \) of subobjects of \( A \) and for any subobject \( B \) of \( A \), the relation

\[
\left( \sum_{i \in I} A_i \right) \cap B = \sum_{i \in I} (A_i \cap B)
\]

holds.

The condition \( \text{Ab3} \) is equivalent to the existence of arbitrary direct limits. Also, \( \text{Ab5} \) is equivalent to the fact that there exist inductive limits and the inductive limits over directed families of indices are exact, i.e. if \( I \) is a directed set and

\[
0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0
\]

is an exact sequence for any \( i \in I \), then

\[
0 \rightarrow \varinjlim A_i \rightarrow \varinjlim B_i \rightarrow \varinjlim C_i \rightarrow 0
\]

is an exact sequence.

Let \( \mathcal{C} \) be a category and \( \mathcal{U} = \{U_i\}_{i \in I} \) a family of objects of \( \mathcal{C} \). The family \( \mathcal{U} \) is said to be a \textit{family of generators} of the category \( \mathcal{C} \) if for any object \( A \) of \( \mathcal{C} \) and any subobject
of $A$ distinct from $A$ there exists at least an index $i \in I$ and a morphism $u : U_i \to A$ which cannot be factorized through the canonical injection $i : B \to A$ of $B$ into $A$. An object $U$ of $\mathcal{C}$ is said to be a generator of the category $\mathcal{C}$ provided that the family $\{U\}$ is a family of generators of the category $\mathcal{C}$.

Let $\mathcal{C}$ be a cocomplete abelian category; then $\mathcal{U} = \{U_i\}_{i \in I}$ is a family of generators for $\mathcal{C}$ if and only if the object $\bigoplus_{i \in I} U_i$ is a generator of $\mathcal{C}$ [BD, Proposition 5.33]. According to [BD, Proposition 5.35] the cocomplete abelian category $\mathcal{C}$ which possesses a family of generators $\mathcal{U}$ is locally small and similar to [BD, Proposition 5.34] one can be proved that any object of $\mathcal{C}$ is isomorphic to a quotient of an object $\bigoplus_{j \in J} U_j$, where $J$ is some set of indices, $U_j \in \mathcal{U}$ for any $j \in J$.

An abelian category which satisfies the condition Ab5 and which possesses a family of generators is called a Grothendieck category.

Examples. (1) The category of left (or right) $A$-modules, where $A$ is a ring, and the category of (pre-)sheafs of $A$-modules on an arbitrary topological space are Grothendieck categories.

(2) Let $\mathcal{B}$ be a preadditive small category. We denote by $(\mathcal{B}, \text{Ab})$ the category whose objects are the additive functors $F : \mathcal{B} \to \text{Ab}$ from $\mathcal{B}$ to the category of abelian groups $\text{Ab}$ and whose morphisms are the natural transformations between functors. That it is Grothendieck follows from [Stm, Example V.2.2]. Besides, the family of representable functors $\{h^B = (B, -)\}_{B \in \mathcal{B}}$ is a family of projective generators for $(\mathcal{B}, \text{Ab})$ [Stm, Corollary IV.7.5].

Let $\mathcal{C}$ be an abelian category and $\mathcal{U} = \{U_i\}_{i \in I}$ some set of objects of $\mathcal{C}$. Consider $\mathcal{U}$ as a small preadditive category and let $(\mathcal{U}^{\text{op}}, \text{Ab})$ be the category of contravariant functors from $\mathcal{U}$ to $\text{Ab}$. By $T : \mathcal{C} \to (\mathcal{U}^{\text{op}}, \text{Ab})$ we denote a functor defined as follows.

$$TX = c(-, X), \quad Tf = ( -, f)$$

where $X \in \mathcal{C}$ and $f$ is a morphism of $\mathcal{C}$.

**Proposition 1.1.** The functor $T : \mathcal{C} \to (\mathcal{U}^{\text{op}}, \text{Ab})$ defined above is faithful if and only if $\mathcal{U}$ is a family of generators of $\mathcal{C}$.

**Proof.** Assume $T$ is faithful; if $i : X' \to X$ is a monomorphism of $\mathcal{C}$ which is not an isomorphism, and let $j : X \to X/X'$ is the cokernel of $i$, then, if $Ti$ is an isomorphism, it follows that $Tj = 0$ since $T$ is left exact. Therefore $j = 0$, which is a contradiction.

Conversely, assume $\mathcal{U}$ is a family of generators of $\mathcal{C}$ and let $f : X \to Y$ be a morphism in $\mathcal{C}$ such that $Tf = 0$. If $f = pj$ is the canonical decomposition of $f$ with $p$ monomorphism and $j$ epimorphism, then $Tj = 0$ and if $i : \text{Ker} f \to X$ is the kernel of $j$, we get that $Ti$ is an isomorphism. Since $\mathcal{U}$ is a family of generators, $i$ is an isomorphism and therefore $j = 0$, hence $f = 0$.

1.2. Localization in Grothendieck categories

A subcategory $\mathcal{S}$ of the Grothendieck category $\mathcal{C}$ is closed under extensions if for any short exact sequence

$$0 \to X' \to X \to X'' \to 0$$
in \( C \) in which \( X', X'' \) belong to \( S \) the object \( X \) belongs to \( S \). The subcategory \( S \) is a **Serre subcategory** provided that it is closed under extensions, subobjects, and quotient objects. The corresponding **quotient category** \( C/S \) is constructed as follows. The objects of \( C/S \) are those of \( C \) and

\[
c/S(X,Y) = \lim \inf_c (X', Y/Y')
\]

with \( X' \subseteq X \), \( Y' \subseteq Y \) and \( X/X', Y' \in S \). The set of such pairs is a partially ordered directed set with respect to the relation \((X', Y/Y') \leq (X'', Y/Y'')\), which holds if and only if \( X'' \subseteq X' \) and \( Y' \subseteq Y'' \). The direct limit is indexed by this partial order. Again \( C/S \) is abelian and there is canonically defined the **quotient functor** \( q : C \to C/S \) such that \( q(X) = X \); it is exact with \( \text{Ker} q = S \) (see [GB1, FA]). Here the kernel \( \text{Ker} f \) of a functor \( f : C \to D \) is, by definition, the subcategory of all objects \( X \) such that \( f(X) = 0 \).

A Serre subcategory \( S \) of \( C \) is called **localizing** provided that the corresponding quotient functor admits a right adjoint \( s : C/S \to C \). Note that \( S \) is localizing if and only if \( S \) is closed under taking coproducts [FA, Theorem 15.11]. In this case, \( S \) and \( C/S \) are again Grothendieck categories [GB1, Proposition III.4.9]. Besides, the inclusion functor \( S \to C \) admits a right adjoint \( t = t_S : C \to S \) which assigns to \( X \in C \) the largest subobject \( t(X) \) of \( X \) belonging to \( S \) [Kr1, Lemma 2.1]. The functor \( t_S \) is called a **torsion functor** and an object \( X \) is called **\( S \)-periodic** or simply **periodic** provided that \( t_S(X) = X \). Furthermore, for any object \( X \in C \) there is a natural morphism \( \lambda_X : X \to sq(X) \) such that \( \text{Ker} \lambda_X \), \( \text{Coker} \lambda_X \in S \) and \( \text{Ker} \lambda_X = t_S(X) \) (see [FA, Chapter XV]).

An object \( X \in C \) is said to be **\( S \)-closed** (respectively **\( S \)-torsionfree**) provided that \( \lambda_X \) is an isomorphism (respectively a monomorphism). Thus the **section functor** \( s \) induces an equivalence between \( C/S \) and the subcategory of \( S \)-closed objects in \( C \) [FA, Proposition 15.19B]. Moreover, the quotient category \( C/S \) is equivalent to the **perpendicular category** \( S^\perp \) consisting of all objects \( X \in C \) satisfying \( c(S,X) = 0 \) and \( \text{Ext}^1_{C}(S,X) = 0 \) for any \( S \in S \) [Kr1, Lemma 2.2]. Henceforth, the object \( sq(X) \) is denoted by \( X_S \) and the morphism \( sq(\alpha) \) is denoted by \( \alpha_S \) for every \( X \in C \) and \( \alpha \in \text{Mor} C \); the morphism \( \lambda \) we shall call an **\( S \)-envelope** of the object \( X \). Thus for any object \( X \) of \( C \) there is an exact sequence

\[
0 \to A' \to X \xrightarrow{\lambda_X} X_S \xrightarrow{} A'' \to 0
\]

with \( A', A'' \in S \) and \( \lambda_X \) the \( S \)-envelope of \( X \). Note that any two \( S \)-envelopes \( \lambda^i_X : X \to X_S \), \( i = 1, 2 \), of \( X \in C \) are isomorphic and \( X_S \approx (X_S)_S \). Also, note that \( X_S = 0 \) if and only if the object \( X \) belongs to \( S \).

**Proposition 1.2.** [FA, Proposition 15.19C] Let \( X \) be an \( S \)-torsionfree object; then a monomorphism \( \mu : X \to Y \) is an \( S \)-envelope if and only if \( Y \) is \( S \)-closed and \( X/Y \in S \). In this case, the following properties hold:

1. \( \mu \) is an essential monomorphism.
2. If \( E \) is an essential extension of \( Y \), then both \( E \) and \( E/Y \) are \( S \)-torsionfree.

Conversely, if (1) and (2) hold and \( Y/X \in S \), then \( \mu \) is an \( S \)-envelope. Moreover, if \( E(X) \) is an injective envelope of \( X \) and \( X \) is \( S \)-torsionfree, then its \( S \)-envelope is the
Lemma 1.3. Let \( S \subseteq C \) be the subcategory of \( C \)-closed objects. Consider the \textit{localizing functor} \((-\)\(_S): C \to S \subseteq C\), \((-\)\(_S) = sq\); then the inclusion functor \( i: S \subseteq C \to C \), by definition, is fully faithful and the localizing functor \((-\)\(_S\)) is exact since \( q \) is exact and the section functor \( s \), as we have already noticed above, induces an equivalence of \( S \subseteq C \) and \( C/S \otimes \alpha \). Let \( X, Y \) be objects of \( C \) and \( \alpha \in C(X,Y) \), \((-\)\(_S)\(\alpha) = \alpha_S = (\lambda_Y \alpha)_S \) with \( \lambda_Y \) the \( S \)-ensvelope of \( Y \). Clearly that \( \alpha_S = 0 \) if and only if \( \text{Im} \alpha \subseteq \text{t}_S(Y) \) whence it easily follows that for \( X \in C, Y \in S \subseteq C \) there is an isomorphism \( \alpha(X,Y) \approx S \subseteq \lambda_Y(X_S,Y) \), that is \( i \) is right adjoint to the localizing functor \((-\)\(_S\)). On the other hand, if \( C \) and \( D \) are Grothendieck categories, \( q': C \to D \) is an exact functor and the functor \( s': D \to C \) is fully faithful and right adjoint to \( q' \), then \( \text{Ker} q' \) is a localizing subcategory and there exists an equivalence \( C/\text{Ker} q' \cong D \) such that \( Hq' = q \) with \( q \) the canonical functor [Fa, Proposition 15.18].

Later on, the quotient category \( C/S \) always means the subcategory of \( S \)-closed objects \( S \subseteq C \) with the pair of functors \((i,(-)\(_S\))\), where \( i: C/S \to C \) is an inclusion functor, \((-\)\(_S): C \to C/S \) is a localizer.

The following Lemma characterizes \( S \)-closed injective objects.

**Lemma 1.3.** (1) An object \( E \in C/S \) is \( C/S \)-injective if and only if it is \( C \)-injective.

(2) An \( S \)-torsionfree and \( C \)-injective object \( E \) is \( S \)-closed.

**Proof.** (1). The inclusion functor \( i: C/S \to C \) preserves injectivity since it is right adjoint to the exact functor \((-\)\(_S\)). If \( E \in C/S \) is \( C \)-injective, then any \( C/S \)-monomorphism \( \mu: E \to X \) is also a \( C \)-monomorphism, and so splits.

(2). It follows from Proposition 1.2. \( \square \)

As for \( C/S \)-morphisms, it holds the following.

**Lemma 1.4.** Let \( \alpha: X \to Y \) be a morphism in \( C/S \). Then:

(1) The \( C \)-kernel of \( \alpha \) is \( S \)-closed.

(2) \( \alpha \) is a \( C/S \)-epimorphism if and only if \( Y/\text{Im}_C \alpha \in S \).

**Proof.** (1). It suffices to notice that the inclusion functor \( i: C/S \to C \), being right adjoint to the localizing functor \((-\)\(_S\)), is left exact.

(2). Localizing the exact sequence

\[
X \xrightarrow{\alpha} Y \xrightarrow{\beta} Y/\text{Im}_C \alpha \to 0
\]

with \( \beta = \text{Coker} \alpha \), we get that \( \beta_S = 0 \) that implies \( \text{Y/Im}_C \alpha_S = 0 \). \( \square \)

In particular, a \( C/S \)-morphism is a monomorphism if and only if it is a \( C \)-monomorphism. We shall refer to this as the \textit{absoluteness} of monomorphism. So for \( A, B \in C/S \) the relation \( A \leq B \) holds in \( C/S \) if and only if it holds in \( C \). Also, it is easily shown that for a \( C \)-morphism \( \alpha: X \to Y \) the \( C/S \)-morphism \( \alpha_S \) is a \( C/S \)-monomorphism if and only if \( \text{Ker} \alpha \in S \) and \( \alpha_S \) is a \( C/S \)-epimorphism if and only if \( Y/\text{Im} \alpha \in S \). Finally \( \alpha_S \) is a \( C/S \)-isomorphism if and only if \( \text{Ker} \alpha \in S \) and \( Y/\text{Im} \alpha \in S \).
1.3. Lattices of localizing subcategories

The results of this paragraph are of purely technical interest, but they will be needed later. Let \( C \) be a Grothendieck category with a family of generators \( U = \{ U_i \}_{i \in I} \). Denote by \( L(C) \) a lattice consisting of localizing subcategories of \( C \) ordered by inclusion.

Recall that \( X \in C \) is \( U \)-finitely generated provided that there is an epimorphism \( \bigoplus_{i=1}^{n} U_i \to X \) with \( U_i \in U \). A subcategory consisting of \( U \)-finitely generated objects is denoted by \( \text{fg}_U C \). The fact that \( L(C) \) is a set follows from that any localizing subcategory \( S \) is generated by its intersection \( \text{fg}_U S = S \cap \text{fg}_U C \) with \( \text{fg}_U C \). This means that every object \( X \in S \) can be written as a direct union \( \sum X_i \) of objects from \( \text{fg}_U S \). Because the category \( \text{fg}_U C \) is skeletally small, \( L(C) \) is indeed a set.

**Proposition 1.5.** [GG] Proposition 2.7) Let \( P \) and \( S \) be localizing subcategories of \( C \); then \( P \subseteq S \) if and only if \( C/P \) is a quotient category of \( C/S \) with respect to the localizing in \( C/P \) subcategory \( S/P = \{ X \in C/P \mid X_S = 0 \} \).

**Proposition 1.6.** Let \( P \) be a localizing subcategory of \( C \) and \( A \) a localizing subcategory of \( C/P \); then there is a localizing subcategory \( S \) of \( C \) containing \( P \) such that \( S/P = A \).

**Proof.** Suppose the pair

\[
i_P : C/P \to C, \quad (\_)_P : C \to C/P
\]

defines \( C/P \) as a quotient category of \( C \). Also, suppose

\[
i_A : (C/P)/A \to C/P, \quad (\_)_A : C/P \to (C/P)/A
\]

defines \( (C/P)/A \) as a quotient category of \( C/P \).

Denote by

\[
Q = (\_)_A \circ (\_)_P : C \to (C/P)/A
\]

\[
I = i_P \circ i_A : (C/P)/A \to C.
\]

Then \( Q \), being a composition of exact functors, is an exact functor. Similarly, \( I \), being a composition of fully faithful functors, is a fully faithful functor. Furthermore, given \( X \in C \) and \( Y \in (C/P)/A \), we have

\[
c(X, Y) \approx (C/P)(X_P, Y) \approx (C/P)/A(Q(X), Y).
\]

Hence \( Q \) is a left adjoint functor to \( I \). Thus the pair \( (I, Q) \) defines \( (C/P)/A \) as a quotient category of \( C \) with respect to the localizing subcategory \( S = \text{Ker} \ Q \). By construction of \( S \) it is easily seen that \( P \subseteq S \) and \( S/P = A \). \( \square \)

Given a localizing subcategory \( P \) of \( C \), consider the following sublattice of \( L(C) \):

\[
L_P(C) = \{ S \in L(C) \mid S \supseteq P \}.
\]

**Corollary 1.7.** If \( P \) is a localizing subcategory of \( C \), then the map

\[
L : L_P(C) \to L(C/P), \ S \mapsto S/P
\]

is a lattice isomorphism.
Note also that for any $S \in \mathcal{L}_P(C)$

$$S/P = S_P = \{ S_P \mid S \in S \}.$$ 

Indeed, clearly that $S/P \subset S_P$. In turn, for $S \in S$ consider exact sequence (1.1)

$$0 \to A' \to S \to S_P \to A'' \to 0$$

with $A', A'' \in \mathcal{P}$. Because $\mathcal{P} \subseteq S$, it follows that $A', A'' \in S$. Hence $S_P \in S$ and since $S_P$ is $\mathcal{P}$-closed, one gets $S_P \in S/P$.

1.4. Locally finitely presented Grothendieck categories

Throughout this paragraph we fix a Grothendieck category $\mathcal{C}$. We define here the most important subcategories of $\mathcal{C}$, essentially used further. Namely we describe the subcategories consisting of finitely generated, finitely presented and coherent objects respectively. These categories are ordered by inclusion as follows:

$$\mathcal{C} \supseteq \text{fg}\, \mathcal{C} \supseteq \text{fp}\, \mathcal{C} \supseteq \text{coh}\, \mathcal{C}.$$ 

Recall an object $A \in \mathcal{C}$ is finitely generated if whenever there are subobjects $A_i \subseteq A$ for $i \in I$ satisfying $A = \sum_{i \in I} A_i$, then there is already a finite subset $J \subseteq I$ such that $A = \sum_{i \in J} A_i$. The category of finitely generated subobjects of $\mathcal{C}$ is denoted by $\text{fg}\, \mathcal{C}$. The category is locally finitely generated provided that every object $X \in \mathcal{C}$ is a directed sum $X = \sum_{i \in I} X_i$ of finitely generated subobjects $X_i$ or equivalently, $\mathcal{C}$ possesses a family of finitely generated generators.

**Theorem 1.8.** [Stm, Proposition V.3.2] An object $C \in \mathcal{C}$ is finitely generated if and only if the canonical homomorphism $\Phi : \lim_{\to} c(C, D_i) \to c(C, \sum D_i)$ is an isomorphism for every object $D \in \mathcal{C}$ and directed family $\{D_i\}_I$ of subobjects of $D$.

A finitely generated object $B \in \mathcal{C}$ is finitely presented provided that every epimorphism $\eta : A \to B$ with $A$ finitely generated has a finitely generated kernel $\text{Ker}\, \eta$. The subcategory of finitely presented objects of $\mathcal{C}$ is denoted by $\text{fp}\, \mathcal{C}$. The respective categories of finitely presented left and right $A$-modules over the ring $A$ are denoted by $\text{mod}\, A = \text{fp}(\text{Mod}\, A)$ and $\text{mod}\, A = \text{fp}(\text{Mod}\, A)$. Notice that the subcategory $\text{fp}\, \mathcal{C}$ of $\mathcal{C}$ is closed under extensions. Besides, if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence in $\mathcal{C}$ with $B$ finitely presented, then $C$ is finitely presented if and only if $A$ is finitely generated.

The most obvious example of a finitely presented object of $\mathcal{C}$ is a finitely generated projective object $P$. We say that $\mathcal{C}$ has enough finitely generated projectives provided that every finitely generated object $A \in \mathcal{C}$ admits an epimorphism $\eta : P \to A$ with $P$ a finitely generated projective object. If $\mathcal{C}$ has enough finitely generated projectives, then by the remarks above, every finitely presented object $B \in \mathcal{C}$ is isomorphic to the cokernel of a morphism between finitely generated projective objects. This is expressed by an exact sequence

$$P_1 \to P_0 \to B \to 0$$

called a projective presentation of $B$. 


Examples. The category $\text{Mod } A^{\text{op}}$ of left $A$-modules has enough finitely generated projectives.

Another example of a category having enough finitely generated projectives is the category of functors $(\mathcal{B}, \text{Ab})$ from the small preadditive category $\mathcal{B}$ to the category of abelian groups $\text{Ab}$. In this category every finitely generated projective object is a coproduct factor of a finite coproduct of representable objects $\bigoplus_{i=1}^{n} (B_i, -)$ (see [Hrz, §1.2]). In addition, if $\mathcal{B}$ is an additive category, that is $\mathcal{B}$ is preadditive, has finite products/coproducts and idempotents split in $\mathcal{B}$, then every finitely generated projective object in $(\mathcal{B}, \text{Ab})$ is representable [Hrz, Proposition 2.1].

The category $\mathcal{C}$ is locally finitely presented provided that every object $B \in \mathcal{C}$ is a direct limit $B = \lim_{\longrightarrow} B_i$ of finitely presented objects $B_i$ or equivalently, $\mathcal{C}$ possesses a family of finitely presented generators. As an example, any locally finitely generated Grothendieck category having enough finitely generated projectives $\{P_i\}_{i \in I}$ is locally finitely presented [Laz, Appendix]. In this case, $\{P_i\}_{i \in I}$ are generators for $\mathcal{C}$. For instance, the set of representable functors $\{h^B\}_{B \in \mathcal{B}}$ of the functor category $(\mathcal{B}, \text{Ab})$ with $\mathcal{B}$ a small preadditive category form a family of finitely generated projective generators for $(\mathcal{B}, \text{Ab})$. Therefore $(\mathcal{B}, \text{Ab})$ is a locally finitely presented Grothendieck category (see [Hrz, Proposition 1.3]).

Theorem 1.9. [Stm, Proposition V.3.4] Let $\mathcal{C}$ be locally finitely generated. An object $B \in \mathcal{C}$ is finitely presented if and only if the functor $\mathcal{C}(B, -) : \mathcal{C} \rightarrow \text{Ab}$ commutes with direct limits.

A finitely presented object $C \in \mathcal{C}$ is coherent provided that every finitely generated subobject $B \subseteq C$ is finitely presented. Evidently, a finitely generated subobject of a coherent object is also coherent. The subcategory of coherent objects of $\mathcal{C}$ is denoted by $\text{coh } \mathcal{C}$. The category $\mathcal{C}$ is locally coherent provided that every object of $\mathcal{C}$ is a direct limit of coherent objects. Equivalently, $\text{fp } \mathcal{C}$ is abelian [Rs, §2] or $\mathcal{C}$ possesses a family of coherent generators. For example, a category of left $A$-modules is locally coherent if and only if the ring $A$ is left coherent.

In order to characterize the fact that $(\mathcal{B}, \text{Ab})$ with $\mathcal{B}$ an additive category is locally coherent, that is $\text{fp } \mathcal{C} = \text{coh } \mathcal{C}$ [Rs, §2], recall that a morphism $\psi : Y \rightarrow Z$ is a pseudo-cokernel for $\varphi : X \rightarrow Y$ in $\mathcal{B}$ if the sequence $h^Z (\psi, -) h^Y (\varphi, -) h^X$ is exact, i.e. every morphism $\delta : Y \rightarrow Z'$ with $\delta \varphi = 0$ factors through $\psi$.

Lemma 1.10. [Kr2, Lemma C.3] The following are equivalent for $\mathcal{C}$:

(1) $\text{fp } \mathcal{C}$ is abelian.

(2) Every morphism in $\mathcal{B}$ has a pseudo-cokernel.

The classical example of a locally coherent Grothendieck category is the category of right (left) generalized $A$-modules $\mathcal{C}_A = (\text{mod } A^{\text{op}}, \text{Ab})$ ($\mathcal{A}_C = (\text{mod } A, \text{Ab})$) consisting of covariant additive functors from the category $\text{mod } A^{\text{op}} (\text{mod } A)$ of finitely presented left (right) $A$-modules to $\text{Ab}$. By the preceding Lemma the category $\text{fp } \mathcal{C}_A$, henceforth the category of coherent functors, is abelian. As we have already said, the finitely generated projective objects of $\mathcal{C}_A$ are the representable functors $(M, -) = \text{Hom}_A (A M, -)$ for some $M \in \text{mod } A^{\text{op}}$ and they are generators for $\mathcal{C}_A$. 


There is a natural right exact and fully faithful functor

\[ ? \otimes A : \text{Mod} A \longrightarrow C_A \quad (1.2) \]

which takes each module \( M_A \) to the tensor functor \( M \otimes_A \). Recall that a short exact sequence

\[ 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0 \]

of right \( A \)-modules is pure if for any \( M \in \text{mod} A \) the sequence of abelian groups

\[ 0 \longrightarrow \text{Hom}_A(M, X) \longrightarrow \text{Hom}_A(M, Y) \longrightarrow \text{Hom}_A(M, Z) \longrightarrow 0 \]

is exact. Equivalently, the \( C_A \)-sequence

\[ 0 \longrightarrow X \otimes A \longrightarrow Y \otimes A \longrightarrow Z \otimes A \longrightarrow 0 \]

is exact. The module \( Q \in \text{Mod} A \) is pure-injective if the functor \( \text{Hom}_A(-, Q) \) takes pure-monomorphisms to epimorphisms.

Functor (1.2) identifies pure-injective \( A \)-modules with injective objects of \( C_A \) \([\text{GJ}, \text{Proposition 1.2}]\) (see also \([\text{HRZ}, \text{Proposition 4.1}]\)). Furthermore, the functor \( M \otimes_A - \in \text{coh} C_A \) if and only if \( M \in \text{mod} A \) \([\text{Aus1}](\text{see also [HRZ]}\).

The category \( \text{coh} C_A \) has enough injectives and they are precisely objects of the form \( M \otimes_A - \) with \( M \in \text{mod} A \) \([\text{HRZ}, \text{Proposition 5.2}]\). Thus every coherent object \( C \in \text{coh} C_A \) has both a projective presentation in \( C_A \)

\[ (K, -) \longrightarrow (L, -) \longrightarrow C \longrightarrow 0 \]

and an injective presentation in \( \text{coh} C_A \)

\[ 0 \longrightarrow C \longrightarrow M \otimes A \longrightarrow N \otimes A \rightarrow \]

Here \( K, L \in \text{mod} A^{\text{op}} \) and \( M, N \in \text{mod} A \).

It should be remarked that most important for the applications in representation theory of finite dimensional algebras is the concept of purity because the pure-injective modules play a prominent role among non-finitely generated modules. It is therefore that many concepts and problems of the theory are naturally formulated and solved in the category \( C_A \). For this subject we recommend the reader Auslander’s work \([\text{Aus2}]\) and Krause’s thesis \([\text{Kr2}]\).

Another important application came from model theory of modules since the main its conceptions are realized in \( C_A \) \(\text{see [JL, HRZ]}. \) One of such concepts ("The Ziegler spectrum") will be discussed in section 5.

2. Grothendieck categories possessing finitely generated projective generators

The following terminology is inspired from the classical theory for categories of modules \( \text{Mod} A \) where \( A \) is a ring. Similar to \( \text{Mod} A \), Grothendieck categories \( C \) possessing finitely generated projective generators \( \mathcal{A} = \{P_i\}_{i \in I} \) are denoted by \( \text{Mod} \mathcal{A} \) and \( \mathcal{A} \) we call a ring of projective generators \( \{P_i\}_{i \in I} \) or simply a ring. The category \( \text{Mod} \mathcal{A} \) is called a category of right \( \mathcal{A} \)-modules. Finally any submodule \( \mathfrak{a} \) of \( P_i \in \mathcal{A} \) is called an ideal of the ring \( \mathcal{A} \) corresponding to the object \( P_i \).
2.1. The Gabriel topology

Let us consider a family of ideals $\mathcal{F} = \{\mathcal{F}^i\}_{i \in I}$, where $\mathcal{F}^i$ is some family of ideals of $A$ corresponding to the object $P_i$. Then $\mathcal{F}$ is a Gabriel topology on $A$ if it satisfies the following axioms:

$T1$. $P_i \in \mathcal{F}$ for each $i \in I$.

$T2$. If $a \in \mathcal{F}^i$ and $\mu \in \text{Hom}_A(P_j, P_i)$, $P_j \in \mathcal{A}$, then $\{a : \mu\} = \mu^{-1}(a)$ belongs to $\mathcal{F}^j$.

$T3$. If $a$ and $b$ are ideals of $\mathcal{A}$ corresponding to $P_i$ such that $a \in \mathcal{F}^i$ and $\{b : \mu\} \in \mathcal{F}^j$ for any $\mu \in \text{Hom}_A(P_j, P_i)$ with $\text{Im}\mu \subseteq a$, $P_j \in \mathcal{A}$, then $b \in \mathcal{F}^i$.

If $\mathcal{A} = \{A\}$ is a ring and $a$ a right ideal of $A$, for an arbitrary endomorphism $\mu : A \to A$ of the module $A_A$

$$
\mu^{-1}(a) = \{a : \mu(1)\} = \{a \in A \mid \mu(1)a \in a\}.
$$

On the other hand, given an element $x \in A$, one has $\{a : x\} = \mu^{-1}(a)$ with $\mu \in \text{End} A$ such that $\mu(1) = x$.

Remark. We shall need the following properties of Gabriel topologies $\mathcal{F} = \{\mathcal{F}^i\}_{i \in I}$ on the ring $\mathcal{A} = \{P_i\}_{i \in I}$:

1. If $a \in \mathcal{F}^i$ and $b$ is an ideal of $\mathcal{A}$ corresponding to $P_i \in \mathcal{A}$ containing $a$, then $b \in \mathcal{F}^i$. Indeed, if $\mu \in \{P_j, P_i\}$ such that $\text{Im}\mu \subseteq a$, then $\{b : \mu\} = P_j \in \mathcal{F}^j$.

2. If $a, b \in \mathcal{F}^i$, then $a \cap b \in \mathcal{F}^i$. Indeed, since $\{a \cap b : \mu\} = \{a : \mu\} \cap \{b : \mu\}$ for any $\mu \in \{P_j, P_i\}$, it follows that $\{a \cap b : \mu\} = \{a : \mu\} \in \mathcal{F}^i$ for $\mu \in \{P_j, P_i\}$ such that $\text{Im}\mu \subseteq b$.

Thus every $\mathcal{F}^i$, $i \in I$, is a downwards directed system of ideals.

Theorem 2.1 (Gabriel). There is a bijective correspondence between Gabriel topologies $\mathcal{F} = \{\mathcal{F}^i\}_{i \in I}$ on the ring $\mathcal{A} = \{P_i\}_{i \in I}$ and localizing subcategories in $\text{Mod} \mathcal{A}$.

Proof. Suppose $\mathcal{F} = \{\mathcal{F}^i\}_{i \in I}$ is a Gabriel topology on $\mathcal{A}$; then by $\mathcal{S}$ denote the following subcategory in $\text{Mod} \mathcal{A}$: $A \in \mathcal{S}$ if and only if for any $\delta : P_i \to A$ the kernel $\ker \delta \in \mathcal{F}^i$. We claim that $\mathcal{S}$ is a localizing subcategory. Indeed, let $A \in \mathcal{S}$ and $i : A' \to A$ be a monomorphism, and $\delta : P_i \to A'$; then $\ker \delta = \ker(i\delta) \in \mathcal{F}^i$. Suppose now $p : A \to A''$ is an epimorphism, $\delta : P_i \to A''$. Since $P_i$ is projective, there is $\gamma : P_i \to A$ such that $p\gamma = \delta$ whence $\ker \gamma \subseteq \ker \delta$, and hence $\ker \delta \in \mathcal{F}^i$.

Let us show now that $\mathcal{S}$ is closed under extensions. To see this, consider a short exact sequence

$$
0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0
$$

with $A', A'' \in \mathcal{S}$. Let $\delta : P_i \to A$; then $a = \ker(p\delta) \in \mathcal{F}^i$. Consider $\gamma : P_j \to P_i$ such that $\text{Im}\gamma \subseteq a$. As $p\delta\gamma = 0$, there exists $\alpha : P_j \to A'$ such that $\delta\gamma = i\alpha$. Therefore $\{\ker \delta : \gamma\} = \gamma^{-1}(\ker \delta) = \ker(\delta\gamma) = \ker(\alpha) \in \mathcal{F}^j$. From $T3$ it follows that $\ker \delta \in \mathcal{F}^i$. Thus $\mathcal{S}$ is a Serre subcategory. If $\delta_i : P_i \to \oplus A_k, A_k \in \mathcal{S}$, then there is $k_1, \ldots, k_n$ such that $\text{Im}\delta \subseteq \oplus_{i=1}^n A_k$ (since $P_i$ is finitely generated). Because $\mathcal{S}$ is a Serre subcategory, it follows that any finite direct sum of objects from $\mathcal{S}$ belongs to $\mathcal{S}$, and so $\ker \delta \in \mathcal{F}^i$, hence $\mathcal{S}$ is a localizing subcategory.

Conversely, assume that $\mathcal{S}$ is a localizing subcategory in $\text{Mod} \mathcal{A}$. Let $\mathcal{F}^i = \{a \subseteq P_i \mid P_a \cap a \in \mathcal{S}\}$. Obviously that $P_i \in \mathcal{F}^i$. If $a \in \mathcal{F}^i$ and $\delta : P_j \to P_i$ is a morphism, then
\{a : \delta \} = \text{Ker}(p\delta) \text{ with } p : P_i \to P_i/a \text{ the canonical epimorphism. Since } P_i/a \in \mathcal{S}, \text{ it follows that } \{a : \delta \} \in \mathfrak{S}^i. \text{ It remains to check } T3. \text{ Assume that } a \in \mathfrak{S}^i \text{ and let } b \subset P_i \text{ be such that for any } \mu : P_j \to P_i \text{ with } \text{Im } \mu \subset a \text{ the ideal } \mu^{-1}(b) \in \mathfrak{S}^j. \text{ Let us consider an exact sequence}

\[ 0 \to a + b/\mathfrak{S} \to P_i/\mathfrak{S} \to P_i/a + b \to 0.\]

Since \( a \subset a + b \) one has \( a + b \in \mathfrak{S}^i. \) Let \( p : P_i \to P_i/\mathfrak{S} \) be the canonical epimorphism, \( \gamma_\mu = p\mu \text{ for } \mu \in (P_j, P_i). \) Because \( \mu^{-1}(b) = \text{Ker } \gamma_\mu, \) it follows that \( P_j/\mu^{-1}(b) = \text{Im } \gamma_\mu = p(\mu(P_j) + b). \) In particular, if \( \text{Im } \mu \subset a, \) then \( p(\mu(P_j) + b) = P_j/\mu^{-1}(b) \in \mathcal{S}, \) and hence we obtain then that

\[ a + b/\mathfrak{S} = \sum_{\mu \in (P_j, P_i) : \text{Im } \mu \subset a} p(\mu(P_j) + b) \]

belongs to \( \mathcal{S}. \) Since \( \mathcal{S} \) is closed under extensions, we conclude that \( P_i/\mathfrak{S} \in \mathcal{S}. \)

The following result was obtained by Freyd [Fr, p. 120] for cocomplete abelian categories possessing a family of finitely generated projective generators (see also [Pop, Corollary III.6.4]).

**Proposition 2.2.** Let Mod\( \mathcal{A} \) be the category of right \( \mathcal{A} \)-modules with \( \mathcal{A} = \{P_i\}_{i \in I} \) a ring of finitely generated projective generators, \( T : \text{Mod } \mathcal{A} \to (\mathcal{A}^{\text{op}}, \text{Ab}) \) a functor which assigns to the right \( \mathcal{A} \)-module \( M \) the functor \( \text{Hom}_{\mathcal{A}}(-, M) \). Then \( T \) is an equivalence.

**Proof.** By definition the functor \( T \) is exact, faithful by Proposition [], and by Theorem [] it preserves direct limits. Clearly that \( T \) also preserves finitely presented modules. \( \hat{\gamma} \) From Yoneda Lemma it easily follows that

\[ \text{Hom}_{\mathcal{A}}(\bigoplus_{i=1}^{n} P_i, \bigoplus_{j=1}^{m} P_j) \approx (T(\bigoplus_{i=1}^{n} P_i), T(\bigoplus_{j=1}^{m} P_j)). \]

Let us show that for any finitely presented \( \mathcal{A} \)-modules \( M, N \in \text{mod } \mathcal{A} \) there is an isomorphism

\[ \text{Hom}_{\mathcal{A}}(M, N) \approx (T(M), T(N)). \]

To see this, consider the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
\bigoplus_{i=1}^{n} h_{P_i} & \xrightarrow{a} & \bigoplus_{j=1}^{m} h_{P_j} & \xrightarrow{p} & TM & \to & 0 \\
\bigoplus_{l=1}^{l} h_{P_l} & \xrightarrow{q'} & \bigoplus_{k=1}^{l} h_{P_k} & \xrightarrow{p'} & TN & \to & 0.
\end{array}
\]

Since \( \bigoplus h_{P_i} \) is projective, there exists \( g \) such that \( p'g = fp. \) In turn, because \( p'gq = fpg = 0, \) it follows that \( \text{Im } (gq) \subset \text{Ker } p' = \text{Im } q'. \) Since \( \bigoplus h_{P_i} \) is projective, there exists \( h \) such that \( q'h = gq. \) Thus there results a commutative diagram

\[
\begin{array}{ccc}
\bigoplus_{i} P_i & \xrightarrow{h'} & M & \to & 0 \\
\bigoplus_{i} P_i & \xrightarrow{g'} & N & \to & 0.
\end{array}
\]

where \( T(g') = g, T(h') = h. \) We get then that there exists \( f' : M \to N \) such that \( T(f') = f. \)
Consider now right \( \mathcal{A} \)-modules \( M \) and \( N \). Write them as direct limits \( M = \varinjlim_j M_j \) and \( N = \varprojlim_j N_j \) of finitely presented \( \mathcal{A} \)-modules \( M_j, N_j \). One has

\[
\text{Hom}_\mathcal{A}(M, N) \approx \varinjlim_j \lim_j \text{Hom}_\mathcal{A}(M_j, N_j) \approx \varinjlim_j \lim_j (\text{TM}_i, \text{TN}_i)
\]

that is \( T \) is a fully faithful functor. It thus remains to check that any functor \( F \in (\mathcal{A}^{\text{op}}, \text{Ab}) \) is isomorphic to \( \text{TM} \) for some \( M \in \text{Mod} \mathcal{A} \). To see this, choose for \( F \) a projective presentation

\[
\oplus_I h_{P_i} = T(\oplus_I P_i) \xrightarrow{\alpha} \oplus_J h_{P_j} = T(\oplus_J P_j) \rightarrow F \rightarrow 0
\]

with \( I, J \) some sets of indices. Because \( T \) is fully faithful, there exists \( \beta : \oplus_I P_i \rightarrow \oplus_J P_j \) such that \( T(\beta) = \alpha \). Define \( M \) by the exact sequence

\[
\oplus_I P_i \xrightarrow{\beta} \oplus_J P_j \rightarrow M \rightarrow 0.
\]

We result in the following commutative diagram

\[
\begin{array}{ccc}
\oplus_I h_{P_i} & \xrightarrow{\alpha} & \oplus_J h_{P_j} \\
\text{T}(\oplus_I P_i) & \xrightarrow{T(\beta)} & \text{T}(\oplus_J P_j) \\
\end{array}
\]

Therefore we obtain that \( F \approx \text{TM} \), i.e. \( T \) is an equivalence.

In fact, the technique of the proof of the preceding Proposition allows to establish more strong result used later. Before formulating it, we make some notation. Let us consider an arbitrary Grothendieck category \( \mathcal{C} \) and suppose \( \mathcal{A} = \{ P_i \}_{i \in I} \) is some family of finitely generated projective objects of \( \mathcal{C} \) (if such a family exists). For clearness we also denote by \( \text{Mod} \mathcal{A} \) the following subcategory of \( \mathcal{C} \): an object \( M \in \text{Mod} \mathcal{A} \) if and only if there exists an exact sequence

\[
\oplus_{k \in K} P_k \rightarrow \oplus_{j \in J} P_j \rightarrow M \rightarrow 0
\]

with \( J, K \) some sets of indices, \( P_i, P_k \in \mathcal{A} \). As usual, let us consider \( \mathcal{A} \) as a preadditive category and let \( T : \text{Mod} \mathcal{A} \rightarrow (\mathcal{A}^{\text{op}}, \text{Ab}) \) be the functor taking \( M \) to \((- , M)\) and \( f \) to \((- , f)\).

**Proposition 2.3.** The functor \( T \) establishes an equivalence of \( \text{Mod} \mathcal{A} \) and \((\mathcal{A}^{\text{op}}, \text{Ab})\). In particular, if \( \mathcal{A} = \{ P \} \), where \( P \) is some finitely generated projective object of \( \mathcal{C} \), then \( \text{Mod} \mathcal{A} \) is equivalent to the category of right \( A \)-modules with \( A = \text{End} P \) the endomorphism ring of \( P \).

**Proof.** By the slight modification the proof of the first part repeats the proof of Proposition 2.2. Furthermore, if \( \mathcal{A} = \{ P \} \), then \( h_P \) is a finitely generated projective generator for \((\mathcal{A}^{\text{op}}, \text{Ab})\) whence it easily follows that \((\mathcal{A}^{\text{op}}, \text{Ab})\), in view of the Mitchell Theorem, is equivalent to the category of right \( A \)-modules \( \text{Mod} A \) with \( A = \text{End} P \).

It will be shown in section 4 that the category \( \text{Mod} \mathcal{A} \) is in fact equivalent to a quotient category \( \mathcal{C}/\mathcal{S} \), where \( \mathcal{S} = \{ C \in \mathcal{C} \mid c(P_i, C) = 0 \text{ for all } P_i \in \mathcal{A} \} \).
From Proposition 2.2 it also follows that in order to define a right $\mathcal{A}$-module $M$ (respectively an $\mathcal{A}$-homomorphism), it suffices to define $M$ as a functor from $\mathcal{A}^{\text{op}}$ to $\text{Ab}$ (respectively as a natural transformation between functors). And conversely, any functor $F : \mathcal{A}^{\text{op}} \to \text{Ab}$ (respectively a natural transformation between functors from $(\mathcal{A}^{\text{op}}, \text{Ab})$) can be considered as a right $\mathcal{A}$-module (respectively an $\mathcal{A}$-homomorphism).

Later on, we shall not distinguish $\mathcal{A}$-modules and functors of $(\mathcal{A}^{\text{op}}, \text{Ab})$ and shall freely make use of this fact without additional reserves.

### 2.2. Localization in module categories

Fix a localizing subcategory $\mathcal{S}$ of $\text{Mod} \mathcal{A}$, $\mathcal{A} = \{ P_i \}_{i \in I}$. Let $\mathfrak{S} = \{ \mathfrak{S}^i \}_{i \in I}$ be the respective Gabriel topology on $\mathcal{A}$. As we have already noticed on p. 14, $\mathfrak{S}^i$ is a downwards directed system. Let $X$ be an arbitrary right $\mathcal{A}$-module and $t = ts$ the $\mathcal{S}$-torsion functor. For every pair $a, b \in \mathfrak{S}^i$ such that $b \subset a$ there is a homomorphism

$$\text{Hom}_\mathcal{A}(a, X/t(X)) \to \text{Hom}_\mathcal{A}(b, X/t(X))$$

induced by inclusion of $b$ into $a$. Clearly that the abelian groups $\text{Hom}_\mathcal{A}(a, X/t(X))$ with these homomorphisms form an inductive system over $\mathfrak{S}^i$.

Consider a functor $H : \text{Mod} \mathcal{A} \to \text{Mod} \mathcal{A}$ defined as follows. For every $M \in \text{Mod} \mathcal{A}$ and every $P_i \in \mathcal{A}$ we put

$$H(M)(P_i) = \lim_{a \in \mathfrak{S}^i} \text{Hom}_\mathcal{A}(a, M/t(M))$$  \hspace{1cm} (2.1)

Let us show that abelian groups (2.1) define $H(M)$ as a functor from $\mathcal{A}^{\text{op}}$ to $\text{Ab}$. To see this, we consider a morphism $\mu : P_j \to P_i$ and an element $m$ of $H(M)(P_i)$. Let $u : a \to M/t(M)$ be a morphism representing the element $m$ of the direct limit. We then define $H(M)(\mu)(m) \in H(M)(P_j)$ to be represented by the composed map

$$\mu^{-1}(a) \xrightarrow{\mu} a \xrightarrow{u} M/t(M)$$  \hspace{1cm} (2.2)

It is easy to see that $H(M)(\mu)$ is well defined, i.e. is independent of the choice of the representing morphism $u$. Thus $H(M)$ becomes a right $\mathcal{A}$-module.

Let now $f : M \to N$ be a morphism in $\text{Mod} \mathcal{A}$. It is obvious that $f(t(M))$ is contained in $t(N)$. Thus $f$ induces a unique morphism $f' : M/t(M) \to N/t(N)$. In turn, $f'$ induces a unique morphism $H(f) : H(M) \to H(N)$ which is a homomorphism of right $\mathcal{A}$-modules. This concludes the construction of the functor $H$.

The Gabriel topology $\mathfrak{S}$ ordered by inclusion is a directed set (see remark on p. 14). From the fact that $\text{Ab}$ satisfies Ab5-condition and from the construction of $H$ we deduce that $H$ is a left exact functor. Moreover, if $M \in \mathcal{S}$, then $H(M) = 0$ since $t(M) = M$.

Let $\zeta_i$ be the canonical morphism from $\text{Hom}_\mathcal{A}(P_i, M/t(M))$ into the direct limit $H(M)(P_i)$, $p : M \to M/t(M)$ be the canonical epimorphism. Consider a map of $\mathcal{A}$-modules $\Phi_M : M \to H(M)$ defined as follows. For $P_i \in \mathcal{A}$, $\alpha \in \text{Hom}_\mathcal{A}(P_i, M)$ we put $\Phi_M(P_i)(\alpha) = \zeta_i p \alpha$. Let $\mu : P_j \to P_i$ be a morphism. From construction of the map
$H(M)(\mu)$ it easily follows that the diagram

$$
\begin{array}{ccl}
\text{Hom}_A(P_i, M) & \xrightarrow{\Phi_M(P_i)(\mu)} & H(M)(P_i) \\
\downarrow \langle \mu, M \rangle & & \downarrow H(M)(\mu) \\
\text{Hom}_A(P_j, M) & \xrightarrow{\Phi_M(P_j)(\mu)} & H(M)(P_j)
\end{array}
$$

is commutative, and so $\Phi_M$ is an $A$-homomorphism. It is directly verified that $\Phi_M$ is functorial in $M$. Thus we obtain a functorial morphism $\Phi : 1_{\text{Mod}_A} \to H$.

Concerning the functorial morphism $\Phi$ we prove the following.

**Proposition 2.4.** Ker $\Phi_M$ and Coker $\Phi_M$ belong to $S$ for any right $A$-module $M$.

**Proof.** As above, we can construct a morphism $\Psi_M : M/t(M) \to H(M)$, $\Psi_M(P_i)(\mu) = \xi_i \mu$ with $\mu \in \text{Hom}_A(P_i, M/t(M))$. One analogously verifies that $\Psi_M$ is an $A$-homomorphism. Then from definitions of the morphisms $\Phi_M$ and $\Psi_M$ it easily follows that the diagram with exact rows

$$
\begin{array}{ccl}
0 & \longrightarrow & t(M) & \longrightarrow & M & \longrightarrow & M/t(M) \\
& & \downarrow \Psi_M & & \downarrow \Phi_M & & \downarrow H(M) \\
0 & \longrightarrow & \text{Ker} \Phi_M & \longrightarrow & M & \longrightarrow & H(M)
\end{array}
$$

is commutative whence one gets $t(M) \subset \text{Ker} \Phi_M$. Let us show that $\Psi_M$ is a monomorphism or equivalently, $\Phi_M$ is a monomorphism for any $S$-torsionfree object $M$. Indeed, assume that $M$ is $S$-torsionfree and let $\mu : P_i \to M$ be such that $\Phi_M(P_i)(\mu) = 0$. Then there exists an element $a \in \mathfrak{A}$ such that the restriction $\mu|_a = 0$. But this implies that $a$ is contained in $\text{Ker} \mu$ whence $\text{Ker} \mu \in \mathfrak{A}$ (see remark on p. 4), and so $\text{Im} \mu \in S$. Because $M$ is $S$-torsionfree, it follows that $\text{Im} \mu = 0$, that is $\mu = 0$. Thus $\text{Ker} \Phi_M = t(M)$. Therefore, if $H(M) = 0$, then $M \in S$. Indeed, we obtain then that $\text{Ker} \Phi_M = M$, hence $M \in S$. Therefore $H(M) = 0$ if and only if $M \in S$.

It remains to check that Coker $\Phi_M \in S$. Let $\mu : P_i \to H(M)$. We suffice to show that the ideal $\mu^{-1}(\text{Im} \Phi_M)$ is an element of $\mathfrak{A}$. Indeed, if $p : P_i \to \text{Coker} \Phi_M$, then there exists $\mu : P_i \to H(M)$ such that $p = \text{Coker} \Phi_M \circ \mu$ since $P_i$ is projective. Further, because the sequence

$$
0 \longrightarrow \mu^{-1}(\text{Im} \Phi_M) \longrightarrow P_i \xrightarrow{p} \text{Coker} \Phi_M
$$

is exact and $\mu^{-1}(\text{Im} \Phi_M)$, by assumption, belongs to $\mathfrak{A}$, it will follow then that Coker $\Phi_M$ belongs to $S$. Without loss a generality we can assume that $\Phi_M$ is a monomorphism and identify $M$ with $\text{Im} \Phi_M$. We put $b = \mu^{-1}(M)$.

Let $u : a \to M$ be an $A$-homomorphism representing $\mu$ in the direct limit $H(M)(P_i)$ and $\xi : P_j \to P_i$ be such that $\text{Im} \xi \subseteq a$. Let us consider the following commutative diagram

$$
\begin{array}{ccl}
\xi^{-1}(b) & \longrightarrow & b & \longrightarrow & M \\
\downarrow \iota & & \downarrow & & \downarrow \Phi_M \\
P_j & \xrightarrow{\xi} & P_i & \xrightarrow{\mu} & H(M).
\end{array}
$$
Recall that the element $\mu \xi$ is represented by the composed morphism $u \bar{\xi} : \{a : \xi\} \to M$ (see sequence (2.2)). In view of that $\operatorname{Im} \xi \subseteq a$, we have $\{a : \xi\} = P_j$, and therefore $\Phi_M(u \bar{\xi}) = \mu \xi$. Since both squares of the diagram are pullback, it follows that the outer square is pullback, and so there exists a morphism $\varpi : P_j \to \xi^{-1}(b)$ such that $\nu \varpi = 1_{P_j}$ whence $\xi^{-1}(b) = P_j \in \mathfrak{F}^i$. By T3 we deduce that $b \in \mathfrak{F}^i$.

**Theorem 2.5.** For any right $\mathcal{A}$-module $M$ the module $H(M)$ is $\mathcal{S}$-closed. Moreover, the homomorphism $\Phi_M$ is an $\mathcal{S}$-envelope of $M$.

**Proof.** To begin with, we shall show that $H(M)$ is $\mathcal{S}$-torsionfree. Let $S$ be a subobject of $H(M)$ belonging to $\mathcal{S}$, $\mu : P_i \to S$ and let $\mu$ be represented by $u : a \to M/t(M)$ in the direct limit $H(M)(P_i)$. Suppose also $\xi : P_j \to P_i$ is such that $\operatorname{Im} \xi \subseteq \operatorname{Ker} \mu$; then the equality $\mu \xi = 0$ implies that the image of the composed map $u \bar{\xi} : \{a : \xi\} \to M/t(M)$ in $H(M)(P_i)$ equals to zero, that is (using the properties of direct limits) there is an ideal $b \in \mathfrak{F}^i$ such that the restriction $u \bar{\xi}$ representing $\mu \xi$ to $b$ equals to zero, so that $b \subseteq \operatorname{Ker}(u \bar{\xi})$. Then $\operatorname{Ker}(u \bar{\xi}) \in \mathfrak{F}^i$, i.e. $\operatorname{Im}(u \bar{\xi}) \subseteq S$. But $M/t(M)$ is $\mathcal{S}$-torsionfree whence $\operatorname{Im}(u \bar{\xi}) = 0$, hence $u \bar{\xi} = 0$. Because this holds for every $\xi : P_j \to P_i$ such that $\operatorname{Im} \xi \subseteq \operatorname{Ker} \mu$, we infer that $\operatorname{Ker} \mu \subseteq \operatorname{Ker} u$ and since $\operatorname{Ker} \mu \in \mathfrak{F}^i$, it follows that $\operatorname{Ker} u$ is also an element of $\mathfrak{F}^i$. But in that case, $\mu$, being an image of zero homomorphism $u \circ \operatorname{Ker} u$ from $\operatorname{Hom}_\mathcal{A}(\operatorname{Ker} u, M/t(M))$, equals to zero. Since it holds for any $\mu \in \operatorname{Hom}_\mathcal{A}(P_i, S)$, we deduce that $S = 0$.

Let us prove that a module $M$ is $\mathcal{S}$-closed if and only if $\Phi_M$ is an isomorphism. Indeed, if $M$ is $\mathcal{S}$-closed, then, in view of the preceding Proposition, $\operatorname{Ker} \Phi_M = t(M) = 0$, that is $\Phi_M$ is a monomorphism. Since $\operatorname{Ext}^1(\operatorname{Coker} \Phi_M, M) = 0$, there exists a morphism $\alpha : H(M) \to M$ such that $\Phi_M \alpha = 1_M$, i.e. $\operatorname{Coker} \Phi_M$ is a direct summand of $H(M)$ and since $H(M)$ is $\mathcal{S}$-torsionfree, we conclude that $\operatorname{Coker} \Phi_M = 0$ that imply $\Phi_M$ is an isomorphism.

On the other hand, if for $M$ the morphism $\Phi_M$ is an isomorphism, then $t(M) = \operatorname{Ker} \Phi_M = 0$, that is $M$ is $\mathcal{S}$-torsionfree. Now if we showed that every short exact sequence

$$0 \longrightarrow M \overset{i}{\longrightarrow} N \longrightarrow S \longrightarrow 0$$

with $S \in \mathcal{S}$ splits, it would follow then that $\operatorname{Ext}^1(S, M) = 0$ for any $S \in \mathcal{S}$ that implies $M$ would be $\mathcal{S}$-closed. To see this, consider a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & M \\
\downarrow{\Phi_M} & & \downarrow{\Phi_N} \\
0 & \longrightarrow & H(M) \\
& & \downarrow{H(i)} \\
& & H(N) \\
& & \longrightarrow H(S)
\end{array}
$$

where the bottom row is exact and $H(S) = 0$. We deduce that $H(i)$ is an isomorphism. Hence $\Phi^{-1}_M H(i)^{-1} \Phi_N i = 1_M$ that implies $i$ is a split monomorphism.

Thus to see that $H(M)$ is $\mathcal{S}$-closed for any module $M$, it suffices to show that $\Phi_{H(M)} : H(M) \to H(H(M))$ is an isomorphism. To begin, let us prove that $H(\Phi_M)$ is an isomorphism. From construction of $H(M)$ it follows that $H(M) = H(M/t(M))$. Let $p : M \to M/t(M)$ be the canonical epimorphism. If we apply the functor $H$ to the
Commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{p} & M/t(M) \\
\phi_M & & \downarrow \phi_{M/t(M)} \\
H(M) & \xrightarrow{=} & H(M/t(M)),
\end{array}
\]

we obtain \(H(\phi_M) = H(\phi_{M/t(M)})\). Since \(H(\text{Coker } \phi_{M/t(M)}) = 0\), it follows that \(H(\phi_{M/t(M)})\) is an isomorphism, and so \(H(\phi_M)\) is an isomorphism.

Further, since \(\phi\) is a functorial morphism, the following relations

\[
\phi_{H(M)}\phi_M = H(\phi_M)\phi_M \tag{2.3}
\]

and

\[
\phi_{H^2(M)}\phi_{H(M)} = H(\phi_{H(M)})\phi_{H(M)} \\
\phi_{H^2(M)}H(\phi_M) = H^2(\phi_M)\phi_{H(M)} \tag{2.4}
\]

hold. Applying the functor \(H\) to (2.3), one gets \(H(\phi_{H(M)}H(\phi_M)) = H^2(\phi_M)H(\phi_M)\) and since \(H(\phi_M)\) is an isomorphism, one has \(H(\phi_{H(M)}) = H^2(\phi_M)\). Then from equalities (2.4) it follows that

\[
\phi_{H^2(M)}\phi_{H(M)} = \phi_{H^2(M)}H(\phi_M). \tag{2.5}
\]

Because \(H^2(M)\) is \(S\)-torsionfree, according to the first part of the proof of the preceding Proposition \(\phi_{H^2(M)}\) is a monomorphism, and so from (2.5) it follows that \(\phi_{H(M)} = H(\phi_M)\), that is \(\phi_{H(M)}\) is an isomorphism, hence \(H(M)\) is \(S\)-closed.

In particular, if we consider an exact sequence

\[
0 \longrightarrow \text{Ker } \phi_M = t(M) \longrightarrow M \xrightarrow{\phi_M} H(M) \longrightarrow \text{Coker } \phi_M \longrightarrow 0
\]

and apply the exact localizing functor \((-)_S\), one obtains

\[
M_S \approx (H(M))_S \approx H(M)
\]

whence it immediately follows that \(\phi_M\) is an \(S\)-envelope of \(M\).

Suppose now \(i : \text{Mod } \mathcal{A}/S \rightarrow \text{Mod } \mathcal{A}\) is an inclusion functor. The functor \(H\) is left adjoint to \(i\). An adjunction morphism of \(i\) with \(H\) is furnished by the functorial morphism \(\phi : 1_{\text{Mod } \mathcal{A}} \rightarrow H = i \circ H\) constructed above and a quasi-inverse morphism to \(\phi\) is a functorial morphism \(\psi : H \circ i \rightarrow 1_{\text{Mod } \mathcal{A}/S}\) defined by the equality \(\psi_M = (\phi_M)^{-1}\) for all \(M \in \text{Mod } \mathcal{A}/S\). The fact that \(\psi_{H(M)}H(\phi_M) = 1_{H(M)}\) is deduced from the equality \(\phi_{H(M)} = H(\phi_M)\) proved in the preceding Theorem. The fact that \(i(\psi_M)\phi_{i(M)} = 1_{i(M)}\) is trivial, so that \(H\) is indeed left adjoint to \(i\). But the localizing functor \((-)_S\) is also left adjoint to \(i\), and therefore the functors \((-)_S\) and \(H\) are equivalent.

3. The structure of localizing subcategories

The results of this section are of purely technical interest, but they will be needed in proving the Popescu-Gabriel Theorem.

**Proposition 3.1.** Suppose that \(\mathcal{C}\) is an abelian category, \(\mathcal{M}\) is some class of objects of \(\mathcal{C}\), \(\mathcal{S} = \{C \in \mathcal{C} \mid c(C, M) = 0, \text{Ext}^1_C(C, M) = 0 \text{ for all } M \in \mathcal{M}\}\). Then the following assertions hold:
(1) $S$ is closed under extensions and $S \in S$ if and only if for any $M \in \mathcal{M}$, $X \in \mathcal{C}$, and epimorphism $f : X \to S$ the canonical homomorphism $(X,M) \to (\ker f,M)$ is an isomorphism.

In addition, if $\mathcal{C}$ is cocomplete, then $S$ is closed under taking coproducts.

(2) For $S$ the following conditions are equivalent:
(a) $S$ is a Serre subcategory.
(b) $S$ is closed under subobjects.
(c) $S \in S$ if and only if for any $M \in \mathcal{M}$, $X \in \mathcal{C}$, and morphism $f : X \to S$ the canonical homomorphism $(X,M) \to (\ker f,M)$ is an isomorphism.

In addition, if $\mathcal{C}$ is a Grothendieck category, then $S$ is a localizing subcategory.

Proof. (1). Let us consider a short exact sequence in $\mathcal{C}$

$$0 \to S' \xrightarrow{i} S \xrightarrow{j} S'' \to 0,$$

which induces an exact sequence

$$0 \to (S'',M) \to (S,M) \to (S',M) \to \text{Ext}^1(S'',M) \to \text{Ext}^1(S,M) \to \text{Ext}^1(S',M) \quad (3.2)$$

If $M \in \mathcal{M}$ and $S',S'' \in S$, then evidently $S \in S$, so $S$ is closed under extensions.

Let $S \in S$, $M \in \mathcal{M}$, $X \in \mathcal{C}$, $f : X \to S$ be an epimorphism. If we consider an exact sequence

$$0 \to (S,M) = 0 \to (X,M) \to (\ker f,M) \to \text{Ext}^1(S,M) = 0 \quad (3.3)$$

one obtains $(X,M) \cong (\ker f,M)$.

Conversely, consider the identity morphism $1_S : S \to S$ of the object $S$. Then $0 = (\ker 1_S,M) \cong (S,M)$. It remains to check that $\text{Ext}^1(S,M) = 0$. It suffices to show that any short exact sequence

$$0 \to M \xrightarrow{h} P \xrightarrow{f} S \to 0$$

splits. By assumption for $1_M : M \to M$ there is $g : P \to M$ such that $gh = 1_M$. So $h$ splits.

In turn, if $\mathcal{C}$ is cocomplete, then the fact that $S$ is closed under under taking coproducts follows from the functor $\text{Ext}^1(\cdot,M)$ commutes with direct sums.

(2). (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c): Suppose that $S \in S$, $f : X \to S$ is an arbitrary morphism; then $\text{Im} f$, by hypothesis, belongs to $S$. Substituting $S$ for $\text{Im} f$ in (3.3), we get that the canonical morphism $(X,M) \to (\ker f,M)$ is an isomorphism for all $M \in \mathcal{M}$. The converse is proved similar to (1).

(c) $\Rightarrow$ (a): From the first assertion it follows that $S$ is closed under extensions. Suppose now that in exact sequence (3.1) the object $S \in S$ and $f : X \to S'$, $X \in \mathcal{C}$. Then $\ker f = \ker(if)$ whence it easily follows that $S' \in S$. Further, if we consider exact sequence (3.2) with $S',S \in S$, we obtain that also $S'' \in S$.

If $\mathcal{C}$ is a Grothendieck category, then $S$ is closed under taking coproducts, and so it is a localizing subcategory [EJ, Theorem 15.11].
3.1. Negligible objects and covering morphisms
Throughout this paragraph \( \mathcal{C} \) is assumed to be a Grothendieck category. Let \( \mathcal{M} \) be some class of objects of the category \( \mathcal{C} \); then an object \( S \in \mathcal{C} \) is \( \mathcal{M} \)-negligible provided that for any \( M \in \mathcal{M} \), \( X \in \mathcal{C} \), and \( f : X \to S \) the canonical homomorphism \((X, M) \to (\text{Ker } f, M)\) is an isomorphism.

**Example.** Any localizing subcategory \( \mathcal{S} \) of \( \mathcal{C} \) is a subcategory consisting of \( \mathcal{C}/\mathcal{S} \)-negligible objects, where \( \mathcal{C}/\mathcal{S} \) is the quotient category of \( \mathcal{C} \) relative to \( \mathcal{S} \), since \( \mathcal{S} = (\mathcal{C}/\mathcal{S})^\perp \).

**Lemma 3.2.** The subcategory \( \mathcal{S} \) of \( \mathcal{C} \) consisting of \( \mathcal{M} \)-negligible objects is the largest localizing subcategory such that all \( M \in \mathcal{M} \) are \( \mathcal{S} \)-closed.

**Proof.** Indeed, if \( \mathcal{P} \) is a localizing subcategory such that any object \( M \in \mathcal{M} \) is \( \mathcal{P} \)-closed, then for any \( f : X \to S \), \( S \in \mathcal{P} \), the object \( \text{Im } f \in \mathcal{P} \), and hence, if we consider exact sequence (3.3) with \( S = \text{Im } f \), we get that the homomorphism \((X, M) \to (\text{Ker } f, M)\) is an isomorphism. Therefore, in view of Proposition 3.1, \( S \in \mathcal{S} \), that is \( \mathcal{P} \subset \mathcal{S} \).

It is useful to have available the following characterization of \( \mathcal{M} \)-negligible objects in terms of generators \( \mathcal{U} = \{U_i\}_{i \in I} \) of the category \( \mathcal{C} \).

**Proposition 3.3.** An object \( S \) is \( \mathcal{M} \)-negligible if and only if for an arbitrary finite set \( J \) and morphism \( f : U_J = \bigoplus_{j \in J} U_j \to S \), \( U_j \in \mathcal{U} \), the canonical homomorphism \((U_J, M) \to (\text{Ker } f, M)\) is an isomorphism for every \( M \in \mathcal{M} \).

**Proof.** The necessary condition is straightforward. Assume the converse. Let us consider the commutative diagram

\[
\begin{array}{ccc}
K_i = \text{Ker}(fp\psi_i) & \xrightarrow{\gamma_i} & U_i \\
\downarrow{\delta_i} & & \downarrow{p} \\
0 & \xrightarrow{f} & X \xrightarrow{\varphi} S \\
\end{array}
\]

where the bottom row is exact, the couple \((K_i, \gamma_i)\) is the kernel of \( fp\psi_i \), \( p \) is an epimorphism, \( U_i \in \mathcal{U} \) for any \( i \in I \) (here \( I = \bigcup_{U \in \mathcal{U}} (U, X) \)), \( \psi_i \) are the canonical monomorphisms, and \( \delta_i \) is a unique morphism that makes the diagram commute. Let \( M \in \mathcal{M} \) and \( u : X \to M \) a morphism such that \( u\varphi = 0 \). Then \( u\varphi\delta_i = 0 \) for any \( i \in I \). By assumption it follows that \( up\psi_i = 0 \) for any \( i \), and therefore \( up = 0 \) that implies \( u = 0 \) since \( p \) is an epimorphism. Thus the canonical homomorphism induced by \( \varphi \)

\[
(X, M) \longrightarrow (\text{Ker } f, M)
\]

is a monomorphism.

Consider now a morphism \( u : \text{Ker } f \to M \). Then \( u\delta_i : K_i \to M \). Let \( \lambda_i : U_i \to M \) be such that \( \lambda_i\gamma_i = u\delta_i \). The morphisms \( \lambda_i \) induce the morphism \( \lambda : U_I \to M \) such that \( \lambda\psi_i = \lambda_i \).

Suppose now that \( J \) is an arbitrary finite subset of \( I \), \( \psi_J \) is the canonical monomorphism \( U_J \to U_I \), the couple \((K_J, \gamma_J)\) is the kernel of \( fp\psi_J \), \( \delta_J : K_J \to \text{Ker } f \) is a unique
morphism such that \( p\psi_J \gamma_J = \varphi \delta_J \), \( \lambda_J : U_J \to M \) is a unique morphism such that \( \lambda_J \gamma_J = u \delta_J \), and \( \psi'_i \) is the canonical map \( U_i \to U_J \), so that \( \psi_i = \psi_J \psi'_i \). As the diagram

\[
\begin{array}{ccc}
K_J & \xrightarrow{\gamma_J} & U_J \\
\delta_J \downarrow & & \downarrow p \psi_J \\
\text{Ker } f & \xrightarrow{\varphi} & M
\end{array}
\]

is pullback and \( p\psi_J \psi'_i \gamma_i = p\psi_i \gamma_i = \varphi \delta_i \), there exists a morphism \( \delta'_i : K_i \to K_J \) such that \( \gamma_J \delta'_i = \psi'_i \gamma_i \) and \( \delta_J \delta'_i = \delta_i \). Then we have \( u \delta_i = u \delta_J \delta'_i = \lambda_J \gamma_J \delta'_i = \lambda_J \psi'_i \gamma_i \). In turn, \( u \delta_i = \lambda_i \gamma_i \) whence, using the hypothesis, we conclude that \( \lambda_J \psi'_i = \lambda_i \) for all \( i \in J \). But also \( \lambda \psi_J \psi'_i = \lambda \psi_i = \lambda_i \) for all \( i \in J \). Consequently, \( \lambda_J = \lambda \psi_J \). On the other hand, since \( p\psi_J \gamma_J = \varphi \delta_J \) and the square from the diagram

\[
\begin{array}{ccc}
\text{Ker } p & \xrightarrow{\chi} & K = \text{Ker}(fp) \xrightarrow{\gamma} U_J \\
\downarrow & & \downarrow p \\
0 & \xrightarrow{\varphi} & X \xrightarrow{f} S
\end{array}
\] (3.4)

is pullback, there exists \( \delta'_J : K_J \to K \) such that \( \gamma_J \delta'_J = \psi_J \gamma_J \) and \( \delta_J \delta'_J = \delta_J \). Then \( \lambda \gamma \delta'_J = \lambda \psi_J \gamma_J = \lambda_J \gamma_J = u \delta_J = u \delta \delta'_J \), i.e. (if we identify \( U_J \) with his image in \( U_I \)) \( \lambda \gamma \) coincides with \( u \delta \) on \( K \cap U_J \) for all finite \( J \subset I \). Because \( U_I = \sum U_J \), we conclude that \( \lambda \gamma = u \delta \). From the diagram (3.4) we have then \( \lambda \chi = \lambda \gamma \chi' = u \delta \chi' \). But since \( \varphi \delta \chi' = p \chi = 0 \) and \( \varphi \) is a monomorphism, it follows that \( \delta \chi' = 0 \). Hence \( \lambda \chi = 0 \), and therefore there is a morphism \( v : X \to M \) such that \( vp = \lambda \). But then \( v \varphi \delta = vp \gamma = \lambda \gamma = u \delta \) and since \( \delta \) is an epimorphism, it follows that \( u = v \varphi \) as was to be proved.

Let \( S \) be a localizing subcategory of the category \( \mathcal{C} \). We say that the object \( M \in \mathcal{C} \) cogenerates \( S \) if \( S = \{ C \in \mathcal{C} \mid \mathcal{C}(C, M) = 0 \} \).

**Lemma 3.4.** The localizing subcategory \( S \) consisting of \( \mathcal{M} \)-negligible objects is cogenerated by the objects \( E(M) \oplus E(E(M)/M) \) where \( M \in \mathcal{M} \), \( E(M) \) (respectively \( E(E(M)/M) \)) is an injective envelope of \( M \) (respectively \( E(M)/M) \).

**Proof.** Denote by \( \mathcal{P} \) the localizing subcategory cogenerated by \( E(M) \oplus E(E(M)/M) \). We need to show that \( S = \mathcal{P} \).

Let \( M \in \mathcal{M} \). Consider a short exact sequence

\[
0 \longrightarrow M \longrightarrow E(M) \longrightarrow E(M)/M \longrightarrow 0.
\]

It induces an exact sequence

\[
0 \longrightarrow (S, M) \longrightarrow (S, E(M)) \longrightarrow (S, E(M)/M) \longrightarrow \text{Ext}^1(S, M) \longrightarrow 0.
\]
for any $S \in C$. If $S \in \mathcal{S}$, then $\text{Ext}^1(S, M) = 0$ and also $(S, E(M)) = 0$ since $M$ is an $S$-torsionfree object. Therefore $(S, E(M)/M) = 0$, and hence $(S, E(E(M)/M)) = 0$ that implies $\mathcal{S} \subseteq \mathcal{P}$.

On the other hand, suppose $S \in \mathcal{P}$; then $(S, E(M)/M) = 0$ since $E(E(M)/M)$ is a $\mathcal{P}$-torsionfree object. Hence we obtain that $(S, M) = 0$ and $\text{Ext}^1(S, M) = 0$ that means $M$ is $\mathcal{P}$-closed. But $\mathcal{S}$, by hypothesis, is the largest localizing subcategory such that any $M$ is $\mathcal{S}$-closed, hence $\mathcal{P} \subseteq \mathcal{S}$. \hfill \Box

In the proof we made use of the fact that an object $M$ of $C$ is $\mathcal{S}$-torsionfree if and only if the injective envelope $E(M)$ of $M$ is $\mathcal{S}$-torsionfree.

We shall say that an object $C$ of the category $C$ is cyclic if there is an epimorphism $f : U_i \rightarrow C$ for some $U_i \in \mathcal{U}$.

**Lemma 3.5.** Suppose $C, D$ are objects of $C$; then $c(C, E(D)) = 0$ if and only if $c(C', D) = 0$ for any cyclic subobject $C'$ of $C$.

**Proof.** Easy. \hfill \Box

**Proposition 3.6.** Let $\text{Mod} \mathcal{A}$ be the category of right $\mathcal{A}$-modules with $\mathcal{A}$ the ring of finitely generated projective generators $\{P_i\}_{i \in I}$ and let $\mathcal{M}$ be some class of $\mathcal{A}$-modules. Then an object $S$ is $\mathcal{M}$-negligible if and only if for an arbitrary morphism $f : P_i \rightarrow S$, $P_i \in \mathcal{A}$, the canonical homomorphism $(P_i, M) \rightarrow (\text{Ker} f, M)$ with $M \in \mathcal{M}$ is an isomorphism.

**Proof.** The necessary condition is straightforward. Let $S'$ be an arbitrary cyclic subobject of $S$; then $(S', M) = 0$. Indeed, there is an epimorphism $f : P_i \rightarrow S'$ for some $P_i \in \mathcal{A}$ and if we consider a short exact sequence

$$0 \rightarrow \text{Ker} f \xrightarrow{i} P_i \xrightarrow{f} S' \rightarrow 0$$

it will follow then $(S', M) = 0$ since the homomorphism $(i, M)$, by hypothesis, is an isomorphism. Therefore, in view of Lemma 3.3, $(S, E(M)) = 0$.

Consider now a short exact sequence

$$0 \rightarrow M \rightarrow E(M) \rightarrow E(M)/M \rightarrow 0.$$

It induces an exact sequence

$$(S', E(M)) \rightarrow (S', E(M)/M) \rightarrow \text{Ext}^1(S', M) \rightarrow 0.$$

But $(S', E(M)) = 0$ (since $(S, E(M)) = 0$), and hence $(S', E(M)/M) \cong \text{Ext}^1(S', M)$.

Our proof will be finished if we show that any short exact sequence

$$0 \rightarrow M \xrightarrow{g} N \xrightarrow{h} S' \rightarrow 0$$
splits and then $\text{Ext}^1(S', M) = 0$. To see this, let us consider the following commutative diagram

$$
\begin{array}{ccccccccc}
\text{Ker } f & \text{Ker } f \\
\downarrow i' & \downarrow i \\
0 & \longrightarrow M & \longrightarrow N \Pi_{S'} P_i & \longrightarrow P_i \\
\text{Ker } f & \text{Ker } f & \text{Ker } f & \text{Ker } f \\
0 & \longrightarrow M & \longrightarrow N & \longrightarrow S' & \text{Coker } u & \longrightarrow 0
\end{array}
$$

where the lower right square is pullback, so that the morphisms $f'$ and $\pi$ are epimorphisms since $f$ and $h$ are epimorphisms. As $P_i$ is a projective $A$-module, it follows that $\nu$ is a split monomorphism. Let $\pi'$ be the canonical projection onto $M$, that is $\pi' \nu = 1_M$. By assumption there exists a morphism $\beta : P_i \rightarrow M$ such that $\pi' \nu = q f' \nu = q g$, that is $g$ is a split monomorphism that finishes the proof.

Let $M$ be a class of objects of the Grothendieck category $C$. We shall say that a morphism $u : X \rightarrow Y$ is $M$-covering provided that $\text{Coker } u$ is an $M$-negligible object.

**Lemma 3.7.** Consider a category of right $A$-modules $\text{Mod } A$ with $A = \{P_i\}_{i \in I}$ a ring. Then a morphism $u : X \rightarrow Y$ is $M$-covering if and only if for any $P_i \in A$, $M \in M$, and morphism $f : P_i \rightarrow Y$ the sequence

$$
0 \rightarrow \text{Hom}_A(P_i, M) \xrightarrow{\varphi} \text{Hom}_A(X \Pi_Y P_i, M) \xrightarrow{\psi} \text{Hom}_A(\text{Ker } u, M)
$$

is induced by $u$ and $f$ is exact.

**Proof.** Consider the commutative diagram

$$
\begin{array}{ccccccccc}
\text{Ker}(pf) & \text{Ker}(pf) \\
\downarrow h \\
0 & \longrightarrow \text{Ker } u & \longrightarrow X \Pi_Y P_i & \longrightarrow P_i \\
\text{Ker}(pf) & \text{Ker}(pf) & \text{Ker}(pf) & \text{Ker}(pf) \\
0 & \longrightarrow \text{Ker } u & \longrightarrow X & \longrightarrow Y & \longrightarrow \text{Coker } u & \longrightarrow 0
\end{array}
$$

where the rows are exact, the couple $(\text{Ker}(pf), h)$ is the kernel of $pf$. Because $pf u' = pu f' = 0$, there exists a unique morphism $q : X \Pi_Y P_i \rightarrow \text{Ker}(pf)$ such that $u' = h q$. We claim that $q$ is an epimorphism. Indeed, let us consider the following diagram

$$
\begin{array}{ccccccccc}
X \Pi_Y P_i & \xrightarrow{q} & \text{Ker}(pf) & \xrightarrow{h} & P_i \\
f' \downarrow & & \downarrow & & \downarrow f \\
X & \xrightarrow{n} & \text{Ker } p & \xrightarrow{k} & Y & \xrightarrow{p} & \text{Coker } u
\end{array}
$$
where \((\text{Ker}p,k)\) is the kernel of \(p\), \(kn = u\), and the middle arrow means a unique morphism that makes the diagram commute. As the outer and right squares are pullback, it follows that also the left square is pullback. But then the epimorphism \(n : M \to \text{Im}u = \text{Ker} p\) implies \(q\) is an epimorphism.

Suppose that \(u\) is \(\mathcal{M}\)-covering and \(\alpha : P_i \to M\) a morphism with \(\alpha u' = 0\). Then \(\alpha h q = 0\), and so \(\alpha h = 0\), hence \(\alpha = 0\) since \(\text{Coker} u\) is \(\mathcal{M}\)-negligible. Thus the homomorphism \(\varphi\) induced by \(u'\) is a monomorphism. Suppose now that a morphism \(\alpha : X\Pi Y P_i \to M\) satisfies \(\alpha i' = 0\). Since \((\text{Ker}(pf),q)\) is the cokernel of \(i'\), we deduce that there exists a morphism \(t : \text{Ker}(pf) \to M\) such that \(tq = \alpha\). Since \(\text{Coker} u\) is \(\mathcal{M}\)-negligible, there exists \(l : P_i \to M\) such that \(lh = t\). But then \(lu' = lh q = t q = \alpha\).

Conversely, suppose the sequence of Lemma is exact and \(g\) is an arbitrary morphism \(P_i \to \text{Coker} u\). Since \(P_i\) is projective, there exists a morphism \(f : P_i \to Y\) such that \(pf = g\), so that \((\text{Ker}(pf),h)\) is the kernel of \(g\). If a morphism \(\alpha : P_i \to M\) satisfies \(\alpha h = 0\), then \(\alpha u' = \alpha h q = 0\) whence \(\alpha = 0\) since \(\varphi\) is a monomorphism. Let now \(\beta \in \text{Hom}_A(\text{Ker}g,M)\). Because \(\beta q i' = 0\) and \(\text{Ker} \psi = \text{Im} \varphi\), it follows that there exists \(\alpha : P_i \to M\) such that \(\beta q = \alpha u'\), that is \(\beta q = \alpha h q\), and therefore \(\beta = \alpha h\). Thus the canonical homomorphism \(\text{Hom}_A(P_i,M) \to \text{Hom}_A(\text{Ker} g,M)\) is an isomorphism. From Proposition 3.6 it follows that \(\text{Coker} u\) is \(\mathcal{M}\)-negligible.

In proving the preceding Lemma we have used the fact that any right \(\mathcal{A}\)-module \(M\) has an injective envelope \(E(M)\). However there is another way (which does not use the existence of injective envelopes) of describing \(\mathcal{M}\)-covering morphisms. Namely it holds the following.

**Lemma 3.8.** An \(\mathcal{A}\)-homomorphism \(u : X \to Y\) of right \(\mathcal{A}\)-modules \(X\), \(Y\) is \(\mathcal{M}\)-covering if and only if for any finite set of indices \(J\), \(M \in \mathcal{M}\), and morphism \(f : P_J = \oplus_{j \in J} P_j \to Y\) with \(P_j \in \mathcal{A}\) the sequence

\[
0 \to \text{Hom}_A(P_j,M) \xrightarrow{\varphi} \text{Hom}_A(X \Pi Y P_j,M) \xrightarrow{\psi} \text{Hom}_A(\text{Ker} u, M)
\]

induced by \(u\) and \(f\) is exact.

**Proof.** Our assertion is proved by the slight modification of the proof of the preceding Lemma. Namely it is necessary to substitute \(P_i\) for \(P_j\) in the respective places and then apply Proposition 3.3. \(\square\)

4. Representation of Grothendieck categories

Consider a Grothendieck category \(\mathcal{C}\) and fix a family of generators \(\mathcal{U} = \{U_i\}_{i \in I}\) of \(\mathcal{C}\). In subsection 3.1 we have defined a functor \(T : \mathcal{C} \to \text{Mod} \mathcal{A} = (\mathcal{U}^{\text{op}}, \text{Ab})\) which assigns to \(X \in \mathcal{C}\) the object \((-,-)\) (here \(\mathcal{A}\) denotes a ring of finitely generated projective generators \(h_{U_i} = (-,U_i)\), \(i \in I\), of the functor category \((\mathcal{U}^{\text{op}}, \text{Ab})\). The following result was also obtained by Prest [Pr1].

**Theorem 4.1** (Popescu and Gabriel). Let \(\mathcal{C}\) be a Grothendieck category with a family of generators \(\mathcal{U} = \{U_i\}_{i \in I}\), \(\mathcal{A} = \{h_{U_i} = (-,U_i)\}_{i \in I}\), and \(T\) be the functor defined above. Then:

1. \(T\) is full and faithful.
(2) \( T \) induces an equivalence between \( \mathcal{C} \) and the quotient category \( \text{Mod} \, \mathcal{A}/\mathcal{S} \), where \( \mathcal{S} \) denotes the largest localizing subcategory in \( \text{Mod} \, \mathcal{A} \) for which all modules \( TX = (-, X) \) are \( \mathcal{S} \)-closed.

Proof. (1): From Lemma 3.1 it follows that \( T \) is a faithful functor. To see that it is full, we must show that if \( X \) and \( Y \) are objects in \( \mathcal{C} \) and \( \Phi : (-, X) \to (-, Y) \) is a functor, then \( \Phi \) is of the form \( \Phi(f) = \varphi f \) for some \( \varphi : X \to Y \). Denote by \( A_i = \text{Hom}_{\mathcal{A}}(U_i, X) \) the set of all morphisms \( U_i \to X \) and put \( A = \cup_{i \in I} A_i \). For each \( \alpha \in A, \text{we let } i_\alpha : U_i \to U_A = \oplus_{i \in A} U_i \text{ denote the corresponding injection. There exists a unique morphism } \lambda : U_A \to X \text{ such that } \lambda i_\alpha = \alpha \text{ for each } \alpha \in A, \text{ and } \lambda \text{ is an epimorphism } \) since \( \mathcal{U} \) is a family of generators. Similarly there exists a unique morphism \( \mu : U_A \to Y \) such that \( \mu i_\alpha = \Phi(\alpha) \) for each \( \alpha \in A \). Let \( \kappa : K \to U_A \) be the kernel of \( \lambda \). We can show that \( \mu \kappa = 0 \), then \( \mu \) factors as \( \mu = \varphi \lambda \) for some \( \varphi : X \to Y \) and for each \( i \in I, \alpha : U_i \to X \) we get \( \Phi(\alpha) = \mu i_\alpha = \varphi \lambda i_\alpha = \varphi \alpha \), and our assertion would be proved.

So we need to check that \( \mu \kappa = 0 \). For each finite subset \( J \) of \( A \) and each \( \alpha \in J \) there are canonical morphisms \( \pi'_\alpha : U_J = \oplus_{i \in J} U_i \to U_\alpha, i'_\alpha : U_\alpha \to U_J \) and \( i_J : U_J \to U_A \). Let \( \kappa_J : K_J \to U_J \) be the kernel of the composed morphism \( \lambda i_J : U_J \to X \). Since \( K \) is the direct limit of the kernels \( K_J \) for all finite subsets \( J \) of \( A \), it suffices to show that \( \mu i_J \kappa_J = 0 \). Now for each \( U_i \in \mathcal{U}, \beta : U_i \to K_J \) we have, using the fact that \( \Phi \) is a functor, that

\[
\mu i_J \kappa_J \beta = \mu i_J \left( \sum_{\alpha \in J} i'_\alpha \beta \pi'_\alpha \right) \kappa_J \beta = \sum_{\alpha \in J} \mu i_\alpha \pi'_\alpha \kappa_J \beta = \sum_{\alpha \in J} \Phi(\alpha) \pi'_\alpha \kappa_J \beta
= \sum_{\alpha \in J} \Phi(\alpha \pi'_\alpha \kappa_J \beta) = \Phi(\sum_{\alpha \in J} \lambda i_J i'_\alpha \pi'_\alpha \kappa_J \beta) = \Phi(\lambda i_J \kappa_J \beta) = 0
\]

since \( \lambda i_J \kappa_J = 0 \). Since this holds for arbitrary \( \beta : U_i \to K_J \), it follows that \( \mu i_J \kappa_J = 0 \).

(2): Let \( \mathcal{S} \) be the largest localizing subcategory in \( \text{Mod} \, \mathcal{A} \) for which all modules of the form \( TX = (-, X) \) are \( \mathcal{S} \)-closed. This subcategory exists by Lemmas 3.2 and 3.4 and it is cogenerated by the class of injective modules of the form \( E(TX) \oplus E(TX)/TX \). Let \( \mathcal{T} = \{TX\}_{X \in \mathcal{C}} \); then the respective \( \mathcal{T} \)-negligible objects and \( \mathcal{T} \)-covering morphisms will be referred to as negligible and covering respectively, omitting the prefix \( \mathcal{T} \). Since every module \( TX \) is \( \mathcal{S} \)-closed, there is a functor \( \mathcal{T}' : \mathcal{C} \to \text{Mod} \, \mathcal{A}/\mathcal{S} \) such that \( T = i \mathcal{T}' \) with \( i : \text{Mod} \, \mathcal{A}/\mathcal{S} \to \text{Mod} \, \mathcal{A} \) an inclusion functor. We must show that \( \mathcal{T}' \) is an equivalence. Since \( i \mathcal{T}' = T \) is full and faithful by (1), also \( \mathcal{T}' \) is full and faithful, and thus it suffices to show that every \( \mathcal{S} \)-closed module is isomorphic to a module of the form \( TX \).

To see this, for each \( \mathcal{A} \)-module \( M \) we choose an exact sequence

\[
\oplus_{i \in I} h_{U_i} \xrightarrow{\alpha} \oplus_{j \in J} h_{U_j} \longrightarrow M \longrightarrow 0
\]

with \( U_i \in \mathcal{U}, I, J \) some sets of indices. Then \( \alpha \) induces in a natural way a morphism \( \beta : \oplus_{i \in I} U_i \to \oplus_{j \in J} U_j \) in \( \mathcal{C} \). To be precise, for each \( X \in \mathcal{C} \) we have a functorial
isomorphism

\[ \psi_0(X) : \text{Hom}_A(\oplus_{i \in I} h_{U_i}, TX) \approx \prod_{i \in I} \text{Hom}_A(h_{U_i}, TX) \]

\[ \approx \prod_{i \in I} c(U_i, X) \approx c(\oplus_{i \in I} U_i, X) \]

and analogously, one has a functorial isomorphism

\[ \psi_1(X) : \text{Hom}_A(\oplus_{j \in J} h_{U_j}, TX) \approx \prod_{j \in J} \text{Hom}_A(h_{U_j}, TX) \]

\[ \approx \prod_{j \in J} c(U_j, X) \approx c(\oplus_{j \in J} U_j, X). \]

The morphism \( \alpha \) induces the functorial morphism

\[ \zeta = \psi_0 \circ (\alpha, -) \circ \psi_1^{-1} : c(\oplus_{j \in J} U_j, -) \longrightarrow c(\oplus_{i \in I} U_i, -). \]

According to [BD, Corollary 1.7], there exists a unique morphism \( \beta : \oplus_{i \in I} U_i \rightarrow \oplus_{j \in J} U_j \) such that \((\beta, -) = \zeta\).

Now let us define \( \tilde{M} \) by the exact sequence

\[ \oplus_{i \in I} U_i \overset{\beta}{\longrightarrow} \oplus_{j \in J} U_j \longrightarrow \tilde{M} \longrightarrow 0. \] (4.1)

We now apply the functor \( T' \) to (4.1). If we knew that \( T' \) preserves colimits, we would have then the following isomorphisms

\[ (\oplus_{i \in I} h_{U_i}, \oplus_{j \in J} h_{U_j}) \approx (\oplus_{i \in I} T'U_i, \oplus_{j \in J} T'U_j) \]

\[ \approx (T'(\oplus_{i \in I} U_i), T'(\oplus_{j \in J} U_j)) \approx c(\oplus_{i \in I} U_i, \oplus_{j \in J} U_j) \]

and obtain the following commutative diagram in \( \text{Mod } A/S \)

\[
\begin{array}{ccc}
\oplus_{i \in I} h_{U_i} & \xrightarrow{\alpha_S} & \oplus_{j \in J} h_{U_j} & \longrightarrow & M_S & \longrightarrow & 0 \\
\| & & & & \| & & \\
\oplus_{i \in I} h_{U_i} & \xrightarrow{T', \beta = \alpha_S} & \oplus_{j \in J} h_{U_j} & \longrightarrow & T'(\tilde{M}) & \longrightarrow & 0 \\
\end{array}
\]

with exact in \( \text{Mod } A/S \) rows, and that \( M_S \) is \( S \)-closed would imply \( M_S = T'(\tilde{M}) \). To conclude the proof it thus remains to show:

**Lemma 4.2.** The functor \( T' : \mathcal{C} \rightarrow \text{Mod } A/S \) is exact and preserves direct sums.

**Proof.** First we prove the exactness of \( T' \). The functor \( T' = (-)_S T \), where \( (-)_S \) is the respective localizing functor, is obviously left exact, and so it suffices to prove that \( T' \) preserves epimorphisms. This means that if \( u : X \rightarrow Y \) is an epimorphism in \( \mathcal{C} \), then the morphism \( Tu \) of \( \text{Mod } A \) is covering, i.e., in view of Lemma 3.7, that for any object \( Z \) of \( \mathcal{C} \) and \( U_i \in \mathcal{U} \) we have the exact sequence

\[ 0 \rightarrow \text{Hom}_A(h_{U_i}, TZ) \rightarrow \text{Hom}_A(h_{U_i}, \Pi_T Y X, TZ) \rightarrow \text{Hom}_A(\text{Ker } Tu, TZ) \] (4.2)
induced by any $A$-homomorphism $f : hU_i \to TY$. Since $T$ is full and faithful and $hU_i = TU_i$, we deduce that there exists a morphism $g : U_i \to Y$ such that $f = Tg$. Therefore we have the commutative diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \text{Ker } u & \longrightarrow & U_i \Pi_Y X & \longrightarrow & U_i & \longrightarrow & 0 \\
& & \downarrow & & \downarrow g & & \downarrow u & & \\
& & X & \longrightarrow & Y & \longrightarrow & 0
\end{array}
\]

with exact rows. The short exact sequence

\[
0 \longrightarrow \text{Ker } u \longrightarrow U_i \Pi_Y X \longrightarrow U_i \longrightarrow 0
\]

induces the exact sequence

\[
0 \longrightarrow c(U_i, Z) \longrightarrow c(U_i \Pi_Y X, Z) \longrightarrow c(\text{Ker } u, Z)
\]

and so, since $T$ is fully faithful, we have the exact sequence

\[
0 \to \text{Hom}_A(hU_i, TZ) \to \text{Hom}_A(T(U_i \Pi_Y X), TZ) \to \text{Hom}_A(T(\text{Ker } u), TZ).
\]

But since $T$ is left exact, $T(\text{Ker } u)$ is isomorphic to $\text{Ker}(Tu)$, $T(U_i \Pi_Y X)$ is obviously isomorphic to $hU_i \Pi_T TX$ and these isomorphisms are functorial. Thus sequences (1.2) are exact for all $Z \in C$, $U_i \in U$ and now it suffices to apply Lemma 3.7.

It remains to prove that $T'$ preserves direct sums. Actually we prove a little more, namely that it preserves direct unions. Suppose $\{X_\gamma\}_{\gamma \in \Gamma}$ is a directed family of sub-objects of $X \in C$ such that $X = \sum_{\gamma \in \Gamma} X_\gamma$ with $\Gamma$ some set of indices. We need to show that the canonical monomorphism

\[
u : \sum_{\gamma \in \Gamma} TX_\gamma \longrightarrow T\left(\sum_{\gamma \in \Gamma} X_\gamma\right).
\]

is covering. Let $g$ be an arbitrary morphism $TU_i \to \text{Coker } u$. Since $TU_i = hU_i$ is a projective object of $\text{Mod } A$, there exists a morphism $g : TU_i \to TX$ such that $g = vq$ where $v = \text{Coker } u$. Because $T$ is fully faithful, there exists a morphism $f : U_i \to X$ such that $g = Tf$. Let $X_\gamma \Pi_X U_i$ be the fibered product associated to the scheme

\[
\begin{array}{ccc}
U_i & \longrightarrow & \text{Ker } u \\
\downarrow f & & \downarrow g \\
X_\gamma & \longrightarrow & X
\end{array}
\]

with $u_\gamma$ the canonical monomorphism. Since $T$ is left exact, it follows that $T(X_\gamma \Pi_X U_i)$ is the fibered product associated to the scheme

\[
\begin{array}{ccc}
TU_i & \longrightarrow & \text{Ker } u \\
\downarrow Tf & & \downarrow g \\
TX_\gamma & \longrightarrow & TX
\end{array}
\]
We obtain thus that $\sum_{\gamma \in \Gamma} T(X_\gamma \Pi_X U_i)$ is the fibered product associated to the scheme

\[
\begin{array}{c}
TU_i \\
\downarrow \tau_f \\
\sum_{\gamma \in \Gamma} TX_\gamma \longrightarrow TX
\end{array}
\]

that implies $\sum_{\gamma \in \Gamma} T(X_\gamma \Pi_X U_i) \approx (\sum_{\gamma \in \Gamma} TX_\gamma) \Pi_X TU_i$. Since

\[
\sum_{\gamma \in \Gamma} c(X_\gamma \Pi_X U_i) \approx (\sum_{\gamma \in \Gamma} X_\gamma) \Pi_X U_i = X \Pi_X U_i \approx U_i
\]

and $T$ is fully faithful, one obtains

\[
\text{Hom}_A(\sum_{\gamma \in \Gamma} T(X_\gamma \Pi_X U_i), TZ) \approx \lim \rightarrow \text{Hom}_A(T(X_\gamma \Pi_X U_i), TZ)
\]

\[
\approx \lim \rightarrow c((X_\gamma \Pi_X U_i), Z) \approx c((\sum_{\gamma \in \Gamma} X_\gamma \Pi_X U_i), Z) \approx c(U_i, Z) \approx \text{Hom}_A(h_{U_i}, TZ)
\]

for all $Z \in \mathcal{C}$. Hence we get, in view of Lemma 3.7, that Coker $u$ is negligible.

Thus the functor $T'$ is an equivalence. This concludes the proof of the Popescu-Gabriel Theorem.

**Corollary 4.3.** Let $\mathcal{C}$ be a Grothendieck category with a family of generators $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{M} = \{M_j\}_{j \in J}$ an arbitrary family of objects of $\mathcal{C}$. Put $\mathcal{U}' = \mathcal{U} \cup \mathcal{M}$ and $\mathcal{A} = \{h_{U} = (-, U)\}_{U \in \mathcal{U}'}$. Then the functor $T : \mathcal{C} \to \text{Mod} \mathcal{A}$, $TX = (-, X)$, defines an equivalence between $\mathcal{C}$ and the quotient category $\text{Mod} \mathcal{A}/S$, where $S$ denotes the largest localizing subcategory in $\text{Mod} \mathcal{A}$ for which all modules $TX$ are $S$-closed.

**Proof.** It suffices to note that $\mathcal{U}'$ is again a family of generators for $\mathcal{C}$ (that directly follows from definition).

**Corollary 4.4** (Popescu and Gabriel [PG]). If $\mathcal{C}$ is a Grothendieck category, $U$ is a generator of $\mathcal{C}$, then the functor $c(U, -)$ establishes an equivalence between categories $\mathcal{C}$ and $\text{Mod} \mathcal{A}/S$ with $\mathcal{A} = \text{End} U$ the endomorphism ring of $U$, $S$ some localizing subcategory of $\text{Mod} \mathcal{A}$.

**Proof.** By Theorem 4.1 the category $\mathcal{C}$ is equivalent to the quotient category of $(\mathcal{U}^{\text{op}}, \text{Ab})$ with $\mathcal{U} = \{U\}$ the family of generators. Since the representable functor $h_U$ is a finitely generated projective generator for $(\mathcal{U}^{\text{op}}, \text{Ab})$, it follows that $(\mathcal{U}^{\text{op}}, \text{Ab})$, in view of the Mitchell Theorem, is equivalent to the category of right $\mathcal{A}$-modules $\text{Mod} \mathcal{A}$ whence our assertion easily follows.

**Remark.** In the proof of Lemma 4.2 we essentially used Lemma 3.7, which uses, in turn, the fact that any $\mathcal{A}$-module has an injective envelope. There are different ways of (independent of Popescu-Gabriel Theorem [PG]) proving this fact (e.g. see [Fr, Chapter VI] or [Pop, §III.3.10]). Nevertheless, Lemma 4.2 can be easily deduced from Lemma 3.8, which, as we have noticed on p. 29, does not use injective envelopes. To
be precise, if we substitute \( P_i \) for \( P_j \) in the respective places of the proof of Lemma \( \text{(L.2)} \) with \( J \) an arbitrary finite set of indices, our proof will literally repeat given one and then the fact that any object \( X \) of a Grothendieck category has an injective envelope is proved similar to [BD, Corollary 6.32].

Let \( \mathcal{P} \) be a localizing subcategory of \( \mathcal{C} \). We shall identify \( \mathcal{C} \), via the functor \( T : \mathcal{C} \rightarrow \text{Mod} \mathcal{A} \), \( \mathcal{A} = \{ h_{U_i} \}_{i \in I} \), with the quotient category \( \text{Mod} \mathcal{A}/\mathcal{S} \). According to Proposition \( \text{(L.3)} \) there is a localizing subcategory \( \mathcal{T} \) of \( \text{Mod} \mathcal{A} \) such that \( \mathcal{T} \supseteq \mathcal{S} \) and \( \mathcal{T}/\mathcal{S} = \mathcal{P} \). Moreover, the quotient category \( \mathcal{C}/\mathcal{P} \) is equivalent to the quotient category \( \text{Mod} \mathcal{A}/\mathcal{T} \).

Let \( \mathfrak{G} \) be the Gabriel topology on \( \mathcal{A} \) corresponding to \( \mathcal{T} \). Similar to the category of modules, the family \( \mathfrak{G} = \{ \mathfrak{G}^i \}_{i \in I} \)

\[
\mathfrak{G}^i = \{ X \mid X \subseteq U_i, U_i/X \in \mathcal{P} \}
\]

we call a Gabriel topology on \( \mathcal{U} \). It is easily verified that \( \mathfrak{G} \subseteq \mathfrak{F} \) and

\[
\mathfrak{G} = \mathfrak{F} \mathcal{S} = \{ a_{\mathcal{S}} \mid a \in \mathfrak{F} \}.
\]

**Proposition 4.5.** For a Gabriel topology \( \mathfrak{G} \) on \( \mathcal{U} \) the following assertions hold:

\( T1' \). \( U_i \in \mathfrak{G} \) for each \( i \in I \).

\( T2' \). If \( a \in \mathfrak{G}^i \) and \( \mu \in \mathcal{C}(U_j, U_i) \), \( U_j \in \mathcal{U} \), then \( \{ a : \mu \} = \mu^{-1}(a) \) belongs to \( \mathfrak{G}^j \).

\( T3' \). If \( a \) and \( b \) are subobjects of \( U_i \) such that \( a \in \mathfrak{G}^i \) and \( \{ b : \mu \} \in \mathfrak{G}^j \) for any \( \mu \in \mathcal{C}(U_j, U_i) \) with \( \text{Im} \mu \subseteq a \), \( U_j \in \mathcal{U} \), then \( b \in \mathfrak{G}^i \).

**Proof.** \( T1' \): Straightforward.

\( T2' \): Because the left square of the commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mu^{-1}(a) & \longrightarrow & U_j & \longrightarrow & U_j/\mu^{-1}(a) & \longrightarrow & 0 \\
\downarrow & & \downarrow \mu & & \downarrow \delta & & \\
0 & \longrightarrow & a & \longrightarrow & U_i & \longrightarrow & U_i/a & \longrightarrow & 0
\end{array}
\]

is pullback, it follows that \( \delta \) is a monomorphism. Hence \( U_j/\mu^{-1}(a) \in \mathcal{P} \) since \( \mathcal{P} \) is closed under subobjects.

\( T3' \): Suppose \( a \in \mathfrak{G}^i \) and \( \mu : h_{U_j} \rightarrow h_{U_i} \) is an \( \mathcal{A} \)-homomorphism such that \( \text{Im} \mu \subseteq a \); then \( a \in \mathfrak{F} \). Because \( (-)_T = (-)_P(-)_S \), it follows that

\[
(h_{U_j}/\mu^{-1}(b))_T = (U_j/\mu_S^{-1}(b))_P = 0
\]

since \( U_j/\mu_S^{-1}(b) \in \mathcal{P} \). Therefore \( \mu^{-1}(b) \in \mathfrak{F} \) and by \( T3 \) we deduce that \( b \in \mathfrak{F} \). Since \( b \) is \( \mathcal{S} \)-closed, we get that \( b \in \mathfrak{G} \).

Further, the \( \mathcal{P} \)-envelope \( X_P \) of an arbitrary object \( X \) of \( \mathcal{C} \) is constructed similar to \( \text{Mod} \mathcal{A} \). Namely, since the localizing functor \( (-)_T \) factors as \( (-)_T = (-)_P(-)_S \), one has

\[
X_P(U_i) = X_T(U_i) = \lim_{a \in \mathfrak{G}}(a, X/t_T(X)) = \lim_{a \in \mathfrak{F}}(a, X/t_P(X)) \approx \lim_{a_{\mathcal{S}} \in \mathfrak{G}(\mathcal{S})}(a_{\mathcal{S}}, X/t_{\mathcal{P}}(X))
\]

and this isomorphism is functorial in both arguments. Here we made use of the fact that \( t_T(X) = t_P(X) \) for any \( X \in \mathcal{C} \).
4.1. Projective generating sets

Let \( U = \{U_i\}_{i \in I} \) be some family of objects of \( C \). By \( C_U \) we shall denote a subcategory of \( C \) consisting of objects generated by \( U \). To be precise, \( C \in C_U \) if and only if there is a presentation
\[
\bigoplus_I U_i \twoheadrightarrow \bigoplus_J U_j \rightarrow C \rightarrow 0
\]
of \( C \) by objects from \( U \) and say that \( U \) is a generating set for \( C_U \). When \( U = \{U\} \) we write \( C_U = C_{U_U} \). Clearly \( C_U = C \) if and only if \( U \) is a family of generators of \( C \).

To begin, we study categories \( C_U \) with \( U \) a generating set consisting of projective objects. So suppose \( P \) is a projective object of \( C \). For any \( M \in C_P \) there is a projective presentation
\[
\bigoplus_I P_i \twoheadrightarrow \bigoplus_J P_j \rightarrow M \rightarrow 0 \tag{4.3}
\]
with \( I, J \) some sets of indices, \( P_i = P_j = P \) for all \( i, j \).

**Theorem 4.6.** Let \( C \) be a Grothendieck category, \( P \) some projective object of \( C \). Then the subcategory \( S = \{C \in C \mid c(P, C) = 0\} \) is localizing and the localized object \( P_S \) is \( C/S \)-projective and a \( C/S \)-generator. Moreover, the localizing functor \( (-)_S \) induces an equivalence of \( C/S \) and \( C_P \).

**Proof.** To begin, we shall show that \( S \) is localizing. Indeed, let
\[
0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0
\]
be a short exact sequence in \( C \). If we apply the exact functor \( c(P, -) \), we shall get a short exact sequence of abelian groups
\[
0 \rightarrow c(P, A') \rightarrow c(P, A) \rightarrow c(P, A'') \rightarrow 0
\]
whence it easily follows that \( c(P, A) = 0 \) if and only if \( c(P, A') = 0 \) and \( c(P, A'') = 0 \) that implies \( S \) is a Serre subcategory. Furthermore, if we consider the map
\[
c(P, \bigoplus A_i) \overset{\varphi}{\rightarrow} c(P, \prod A_i) \approx \prod c(P, A_i) = 0
\]
with \( A_i \in S \) and \( \varphi \) a monomorphism induced by the canonical monomorphism \( \bigoplus A_i \rightarrow \prod A_i \), it will be follow that \( S \) is closed under taking coproducts, and therefore \( S \) is a localizing subcategory.

In the rest of the proof we shall show that \( P_S \) is a projective generator of \( C/S \). First let us consider a short exact sequence
\[
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0
\]
in \( C/S \). It induces an exact sequence in \( C \)
\[
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow S \rightarrow 0
\]
with \( S \in S \). If we apply the exact functor \( c(P, -) \), one gets the following commutative diagram of abelian groups:
\[
\begin{array}{cccccc}
0 & \rightarrow & c(P, A) & \rightarrow & c(P, B) & \rightarrow & c(P, C) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & c/S(P_S, A) & \rightarrow & c/S(P_S, B) & \rightarrow & c/S(P_S, C) & \rightarrow & 0
\end{array}
\]
where vertical arrows are isomorphisms. Therefore $P_S$ is $C/S$-projective.

It remains to check that $P_S$ is a generator of $C/S$. Let $A$ be an arbitrary object of $C/S$, $I = c(P, A)$; then there exists a morphism $u : \oplus_{i \in I} P_i \to A$ with $P_i = P$, $u_i = i$ for all $i \in I$. We have an exact sequence in $C$

$$\oplus P_i \xrightarrow{u} A \to \text{Coker } u \to 0.$$ 

Let $w : P \to A$ be an arbitrary morphism. By construction of $u$ we have that $\text{Im } w \subseteq \text{Im } u$, and so there exists a morphism $p : P \to \oplus P_i$ such that $w = up$ (since $P$ is projective) that evidently implies $c(P, \text{Coker } u) = 0$, i.e. $\text{Coker } u \in S$.

But then,

$$u_s : (\oplus P_i)_S \approx \oplus (P_i)_S \to A_S$$

is a $C/S$-epimorphism. Thus any object $A \in C/S$ is a quotient object of $\oplus (P_i)_S$ whence it is easily follows that $P_S$ is a generator of $C/S$.

Now let us show that the restriction of $(-)_S$ to $C_P$ defines an equivalence of $C/S$ and $C_P$. For $M \in C$ there is an exact sequence of the form

$$0 \to A' \to M \xrightarrow{\lambda} M_S \to A'' \to 0$$

with $A', A'' \in S$. Hence $\lambda$ induces an isomorphism:

$$c(P, M) \approx c(P, M_S) \approx c_{/S}(P_S, M_S).$$

But then we have that

$$c(\oplus P_i, M) \approx \prod c(P_i, M) \approx c(\oplus P_i, M_S) \approx c_{/S}(\oplus (P_i)_S, M_S)$$

with $P_i = P$ for any $i \in I$. Now, if we consider projective presentation (4.3) for $M \in C_P$, we obtain a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & c(M, N) & \to & c(\oplus P_i, N) & \to & c(\oplus P_i, N) \\
& & \downarrow & & \downarrow & & \\
0 & \to & c(M, N_S) & \to & c(\oplus P_i, N_S) & \to & c(\oplus P_i, N_S) \\
\end{array}
\]

with exact rows and vertical arrows isomorphisms. Thus,

$$c(M, N) \approx c(M, N_S) \approx c_{/S}(M_S, N_S)$$

that implies $(-)_S|_{C_P}$ is a fully faithful functor. Finally let $N \in C/S$. Consider a $C/S$-projective presentation of $N$

$$\oplus (P_i)_S \xrightarrow{\alpha} \oplus (P_j)_S \to N \to 0$$

where $P_i = P_j = P$. Then there exists $\beta : \oplus P_i \to \oplus P_j$ such that $\alpha = \beta_S$, hence $N \approx (\text{Coker } \beta)_S$, as was to be proved.

**Corollary 4.7.** Under the conditions of Theorem 4.6 the quotient category $C/S$ is equivalent to the quotient category $\text{Mod } A/P$, where $A = \text{End } P$ is the endomorphism ring of $P$, $P$ is some localizing subcategory of $\text{Mod } A$.

**Proof.** It suffices to note that isomorphisms (4.4) induce a ring isomorphism

$$A = c(P, P) \approx c(P, P_S) \approx c_{/S}(P_S, P_S).$$

Now our assertion follows from Corollary 4.4 since $P_S$ is a generator for $C/S$. \qed
Corollary 4.8. Let $\mathcal{C}$ be a Grothendieck category and $\mathcal{U} = \{P_i\}_{i \in I}$ some family of projective objects of $\mathcal{C}$. Then the subcategory $\mathcal{S} = \{C \in \mathcal{C} \mid \mathcal{C}(P, C) = 0 \text{ for all } P \in \mathcal{U}\}$ is localizing and $\mathcal{C}/\mathcal{S}$ is equivalent to the quotient category $\text{Mod} \mathcal{A}/\mathcal{P}$, where $\mathcal{A} = \{h_P\}_{P \in \mathcal{U}}$, $\mathcal{P}$ is some localizing subcategory of $\text{Mod} \mathcal{A}$. Moreover, the localizing functor $(-)_{\mathcal{S}}$ induces an equivalence of $\mathcal{C}/\mathcal{S}$ and $\mathcal{C}_\mathcal{U}$.

Proof. Denote by $Q = \bigoplus_{P \in \mathcal{U}} P$; then $\mathcal{S} = \{C \in \mathcal{C} \mid \mathcal{C}(Q, C) = 0\}$. From the preceding Theorem it follows that $\mathcal{S}$ is localizing and that $Q_\mathcal{S} = \bigoplus_{P \in \mathcal{U}} P_\mathcal{S}$ is a $\mathcal{C}/\mathcal{S}$-projective generator that implies $\mathcal{U}_\mathcal{S} = \{P_\mathcal{S} \mid P \in \mathcal{U}\}$ is a family of projective generators of $\mathcal{C}/\mathcal{S}$.

In view of isomorphisms (4.4), there is an equivalence of $\mathcal{U}$ and $\mathcal{U}_\mathcal{S}$. Now our assertion follows from Theorem 4.3. \hfill \Box

Remark. Under the conditions of Corollary 4.8 the localizing functor $(-)_{\mathcal{S}}$ factors through $\mathcal{C}_\mathcal{U}$

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{(-)_{\mathcal{S}}} & \mathcal{C}/\mathcal{S} \\
F & \downarrow & \downarrow G \\
\mathcal{C}_\mathcal{U} & & \\
\end{array}$$

where $G$ is a restriction of $(-)_{\mathcal{S}}$ to $\mathcal{C}_\mathcal{U}$ and $F$ is constructed as follows. Let $C \in \mathcal{C}$ and $I = \bigcup_{P \in \mathcal{U}} I_P$ with $I_P = \mathcal{C}(P, C)$; then there is a morphism $\varphi : \bigoplus_{\mu \in I} P_\mu \to C$, $\varphi_\mu = \mu$. Similarly, there is a morphism $\psi : \bigoplus_{\nu \in J} P_\nu \to \text{Ker} \varphi$ with $J = \bigcup_{P \in \mathcal{U}} J_P$, $J_P = \mathcal{C}(P, \text{Ker} \varphi)$, so that we have the commutative diagram:

$$\begin{array}{ccc}
\bigoplus_{\nu \in J} P_\nu & \xrightarrow{\zeta} & \bigoplus_{\mu \in I} P_\mu & \xrightarrow{i} & C \\
\downarrow \psi & & \downarrow & & \downarrow \varphi \\
\text{Ker} \varphi & & & & \\
\end{array}$$

with $i = \text{Ker} \varphi$. By definition we put $F(C)$ to be equal to $\text{Coker} \zeta$.

Following notation of section 2, given $\mathcal{A} = \{P_i\}_{i \in I}$ a family of finitely generated projective objects of $\mathcal{C}$, the category $\mathcal{C}_\mathcal{A}$ is denoted by $\text{Mod} \mathcal{A}$. By Proposition 2.3 $\text{Mod} \mathcal{A} \approx (\mathcal{A}^{op}, \text{Ab})$.

Theorem 4.9. Suppose $\mathcal{C}$ is a Grothendieck category and $\mathcal{A} = \{P_i\}_{i \in I}$ is some family of finitely generated projective objects of $\mathcal{C}$. Then the subcategory $\mathcal{S} = \{C \in \mathcal{C} \mid \mathcal{C}(P_i, C) = 0 \text{ for all } P_i \in \mathcal{A}\}$ is localizing and $\mathcal{A}_\mathcal{S} = \{(P_i)_\mathcal{S}\}_{i \in I}$ is a family of finitely generated projective generators of the quotient category $\mathcal{C}/\mathcal{S}$. Moreover, the localizing functor $(-)_{\mathcal{S}}$ induces an equivalence of $\mathcal{C}/\mathcal{S}$ and $\text{Mod} \mathcal{A}$.

Proof. By Corollary 4.8 $\mathcal{A}_\mathcal{S}$ is a family of projective generators for $\mathcal{C}/\mathcal{S}$. It thus remains to show that every $(P_i)_\mathcal{S} \in \mathcal{A}_\mathcal{S}$ is $\mathcal{C}/\mathcal{S}$-finitely generated.

To see this, we consider an object $X$ of $\mathcal{C}/\mathcal{S}$. Let $\{X_j\}_{j \in J}$ be a directed family of $\mathcal{C}/\mathcal{S}$-subobjects of $X$ such that $X = \sum_{\mathcal{C}/\mathcal{S}} X_j$. Because $X_j$ are also $\mathcal{C}$-subobjects, it follows that the $\mathcal{C}$-direct union $\sum_{\mathcal{C}} X_j$ is a subobject of $X$ and the quotient object
$A = X/\sum C X_j$ belongs to $S$. Indeed, if we apply the exact localizing functor $(-)_S$ commutes with direct limits to the short exact sequence

$$0 \to \sum C X_j \to X \to A \to 0$$

we shall obtain then the short exact sequence

$$0 \to (\sum C X_j)_S = \sum_{C/S} X_j \to X \to A_S \to 0.$$

Whence $A_S = 0$, that is $A \in S$. For any $P_i \in A$ one has then

$$c(P_i; \sum C X_j) \approx c(P_i, X) \approx c(S(P_i)_S, \sum_{C/S} X_j).$$

Thus,

$$c(S(P_i)_S, \sum_{C/S} X_j) \approx c(P_i, \sum C X_j) \approx \lim c(P_i, X_j) \approx \lim c(S(P_i)_S, X_j).$$

By Theorem 1.8 $\{P_i\}_S \in \text{fg} C/S$.

The fact that the localizing functor $(-)_S$ induces an equivalence between $C/S$ and $\text{Mod}_A$ is proved similar to Theorem 1.6.

Corollary 4.10. (CGT, Theorem 2.1) Let $C$ be a Grothendieck category and $P$ be some finitely generated projective object of $C$. Then the subcategory $\mathcal{S} = \{C \in C \mid c(P, C) = 0\}$ is localizing and the functor $C/S(P, -)$ gives an equivalence of categories $C/S$ and $\text{Mod}_A$ with $A = \text{End} P$ the endomorphism ring of the object $P$.

Proof. According to the preceding Theorem the localized object $P_S$ is a finitely generated projective generator, and so, in view of the Mitchell Theorem and Corollary 1.7, the quotient category $C/S$ is equivalent to the category of modules over the ring $A = \text{End}_C P$. 

Example. Consider the category of generalized right $A$-modules $\mathcal{C}_A = (\text{mod} A^{op}, \text{Ab})$. Let $M \in \text{mod} A^{op}$, $R = \text{End}_A M$ and $S_M = \{F \in \mathcal{C}_A \mid F(M) = 0\}$. By Corollary 1.10 there exists an equivalence of categories $\mathcal{C}_A/S_M \overset{h}{\to} \text{Mod} R$ where $h(F_{SM}) = F_{SM}(M) = F(M)$ for any $F \in \mathcal{C}_A$. A quasi-inverse functor to $h$ is constructed as follows: $g : \text{Mod} R \to \mathcal{C}_A/S_M$, $E \mapsto ((M, E) \otimes_A -)_{SM}$. In particular, given $F \in \mathcal{C}_A$, there is an isomorphism $F_{SM} \approx ((M, F(M)) \otimes_A -)_{SM}$.

Suppose $\text{Mod}_A$, $A = \{P_i\}_{i \in I}$, is a category of right $A$-modules and for any $P_i \in A$ we put

$$\mathcal{S}_i = \{M \in \text{Mod}_A \mid \text{Hom}_A(P_i, M) = 0\}.$$ 

By Corollary 1.10 $\text{Mod} A/\mathcal{S}_i \approx \text{Mod} A_i$ with $A_i = \text{End} P_i$. We consider this equivalence as identification. Then the following inclusion holds:

$$\bigcup_{P_i \in A} \text{Mod} A_i \subseteq \text{Mod}_A.$$

Generally speaking, as it shows the next example, categories of $A_i$-modules $\text{Mod} A_i$ do not cover $\text{Mod}_A$ entirely.
Example. Consider the category of generalized abelian groups $\mathbb{Z}C = (\text{mod} \mathbb{Z}, \text{Ab})$. It has been said in section 1 that the functor $M \mapsto - \otimes M$ identifies pure injective abelian groups and injective objects of $\mathbb{Z}C$. By Kaplansky’s Theorem [Kap], indecomposable pure injective abelian groups $\mathbb{Z}g\mathbb{Z}C$ are precisely abelian groups of the form:

1. The injective modules $\mathbb{Q}$ and, for every prime $p$, the Prüfer groups $\mathbb{Z}_{p^\infty}$.
2. Every cyclic group $\mathbb{Z}_{p^n}$ of order a prime power.
3. For every prime $p$, the $p$-adic completion $\mathbb{Z}_p = \varprojlim \mathbb{Z}_{p^n}$ of the integers.

Ziegler $\mathbb{Zgr}$ has shown that a similar argument holds for Dedekind domains.

Let $S_M = \{C \in \mathbb{Z}C \mid C(M) = 0\}$ with $M \in \text{mod} \mathbb{Z}$. As any finitely generated abelian group is isomorphic to $\mathbb{Z}^{\oplus m} \oplus \mathbb{Z}^{\oplus n}_{p^i}$ with $p_1 \ldots p_n$ prime, it suffices to show that the relation

$$\text{Mod} \mathbb{Z} \bigcup \left( \bigcup_{p \text{ is prime}} \text{Mod} \mathbb{Z}_{p^n} \right) \subseteq \mathbb{Z}C$$

holds. According to [GG], Corollary 2.4 $- \otimes Q \in \mathbb{Z}g\mathbb{Z}C$ belongs to $\mathbb{Z}C/S_M \approx \text{Mod} A_M$ with $A_M = \text{End} M$ if and only if $Q \approx \text{Hom}_{A_M}(M, E)$, where $E$ is an indecomposable injective $A_M$-module and in this case $E \approx M \otimes Q$. Clearly that $Q, \mathbb{Z}_{p^n} \in \mathbb{Z}C/S_{\mathbb{Z}}$ and $\mathbb{Z}_{p^n} \in \mathbb{Z}C/S_{\mathbb{Z}_{p^n}}$. However $\mathbb{Z}_p$ does not satisfy the indicated condition. Indeed, it is not an injective abelian group, and therefore $- \otimes \mathbb{Z}_p \notin \mathbb{Z}C/S_{\mathbb{Z}}$. Because for prime $p \neq q$ the object $(\mathbb{Z}_p, -) \in S_{\mathbb{Z}_{p^n}}$ and $\mathbb{Z}_p \otimes \mathbb{Z}_p \approx \mathbb{Z}_p \neq 0$, it follows that $\text{t}_{S_{\mathbb{Z}_{p^n}}}(\mathbb{Z}_p) \neq 0$ where $\text{t}_{S_{\mathbb{Z}_{p^n}}}$ is an $S_{\mathbb{Z}_{p^n}}$-torsion functor. Hence, in view of Lemma [3], $- \otimes \mathbb{Z}_p \notin \mathbb{Z}C/S_{\mathbb{Z}_{p^n}}$. Finally $- \otimes \mathbb{Z}_p \notin \mathbb{Z}C/S_{\mathbb{Z}_{p^n}}$ since $(\mathbb{Z}_{p^n}, \mathbb{Z}_{p^n} \otimes \mathbb{Z}_p) \approx \mathbb{Z}_{p^n} \neq \mathbb{Z}_p$.

5. Finiteness conditions for localizing subcategories

Let $S$ be a localizing subcategory of the Grothendieck category $C$. In this section, we shall investigate how various finiteness properties of $S$ are reflected by properties of the family of generators $U = \{U_i\}_{i \in I}$ of $C$. We study the precise conditions on $U$ which make the quotient category $C/S$ into e.g. a locally finitely generated category.

Let us consider an inclusion functor $i : C/S \to C$. We say that $S$ is of prefinite type (respectively of finite type) provided that $i$ commutes with direct unions (respectively with direct limits), i.e. for any $S$-closed object $C$ and any directed family $\{C_i\}_I$ of $S$-closed subobjects of $C$ the relation

$$\sum_{C/S} C_i = \sum_C C_i$$

holds (respectively $\varinjlim_{C/S} C_i = \varinjlim_C C_i$). It should be remarked that if $S$ is of prefinite type, then in particular every direct sum of $S$-closed objects is $S$-closed. Recall also that a subcategory $B \subseteq A$ of an abelian category $A$ is exact provided that it is abelian and the inclusion functor of $B$ into $A$ is exact.

Proposition 5.1. [Fr] Theorem 3.41] A subcategory $B$ of $A$ is an exact subcategory if and only if the following two conditions hold:

1. If $B_1, B_2 \in B$ then the coproduct $B_1 \oplus B_2$ is an object of $B$.
2. If $\beta : B_1 \to B_2$ is a morphism in $B$, then both the $A$-kernel and $A$-cokernel of $\beta$ are objects of $B$. 
Finally a subcategory $\mathcal{B}$ of an abelian category $\mathcal{A}$ is called \textit{coexact} provided that the perpendicular subcategory
\[
\mathcal{B}^\perp = \{ A \in \mathcal{A} \mid (B, A) = 0, \text{Ext}^1(B, A) = 0 \text{ for all } B \in \mathcal{B} \}
\]
is exact. For example, if $\mathcal{C}$ is a Grothendieck category, $\mathcal{S} \subseteq \mathcal{C}$ is localizing, then $\mathcal{S}$ is coexact if and only if the quotient category $\mathcal{C}/\mathcal{S}$ is exact because $\mathcal{C}/\mathcal{S} = \mathcal{S}^\perp$. Note also that any quotient category $\mathcal{C}/\mathcal{S}$ is coexact since $\mathcal{S} = (\mathcal{C}/\mathcal{S})^\perp$ is an exact subcategory.

**Proposition 5.2.** If $\mathcal{S} \subseteq \mathcal{C}$ is a coexact localizing subcategory, then for any projective object $P$ of $\mathcal{C}$ the object $P_\mathcal{S}$ is $\mathcal{C}/\mathcal{S}$-projective. If, in addition, $\mathcal{C}$ has a family of projective generators $\mathcal{A} = \{ P_i \}_{i \in I}$ and any $(P_i)_\mathcal{S}$ is $\mathcal{C}/\mathcal{S}$-projective, then $\mathcal{S}$ is coexact.

**Proof.** If $\mathcal{S}$ is coexact, then any short exact sequence
\[
0 \longrightarrow A \longrightarrow B \xrightarrow{\beta} C \longrightarrow 0 \tag{5.1}
\]
in $\mathcal{C}/\mathcal{S}$ is also exact in $\mathcal{C}$. Suppose $P \in \mathcal{C}$ is projective; then we have the following commutative diagram
\[
\begin{array}{cccccc}
0 & \longrightarrow & c(P, A) & \longrightarrow & c(P, B) & \xrightarrow{(P, \beta)} & c(P, C) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & c/\mathcal{S}(P_\mathcal{S}, A) & \longrightarrow & c/\mathcal{S}(P_\mathcal{S}, B) & \xrightarrow{(P_\mathcal{S}, \beta)} & c/\mathcal{S}(P_\mathcal{S}, C) & \longrightarrow & 0
\end{array}
\]
with exact rows and vertical arrows being isomorphisms. Hence $P_\mathcal{S}$ is $\mathcal{C}/\mathcal{S}$-projective.

Conversely, we need to show that any $\mathcal{C}/\mathcal{S}$-epimorphism is $\mathcal{C}$-epimorphism. Consider exact sequence \([\mathcal{S}, \beta]\). By Lemma [\ref{Lem2} \cite{St} $C/\text{Im}_\mathcal{C} \beta \in \mathcal{S}$. By assumption the morphism $(P_\mathcal{S}, \beta)$ is an epimorphism, where $P \in \mathcal{A}$, hence $(P, \beta)$ is an epimorphism. Therefore $(P, C/\text{Im}_\mathcal{C} \beta) = 0$ for every $P \in \mathcal{A}$ that implies $C/\text{Im}_\mathcal{C} \beta = 0$ since $\mathcal{A}$ is a family of generators.

The following statement characterizes coexact localizing subcategories of (pre)finite type.

**Proposition 5.3.** For a coexact localizing subcategory $\mathcal{S}$ of a Grothendieck category $\mathcal{C}$ the following assertions are equivalent:

1. $\mathcal{S}$ is of finite type.
2. $\mathcal{S}$ is of prefinite type.
3. The inclusion functor $i : \mathcal{C}/\mathcal{S} \to \mathcal{C}$ commutes with coproducts.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3): Straightforward.

(3) $\Rightarrow$ (1): Let $\lim_{\mathcal{C}/\mathcal{S}} C_i$ be a $\mathcal{C}/\mathcal{S}$-direct limit of $C_i \in \mathcal{C}/\mathcal{S}$, $i \in I$. Denote by $A$ the subset of $I \times I$ consisting of pairs $(i, j)$ with $i \leq j$ and for any $\lambda \in A$ we put $s(\lambda) = i$, $t(\lambda) = j$. By \cite{St}, Proposition IV.8.4]
\[
\lim_{\mathcal{C}/\mathcal{S}} C_i = \text{Coker}_{\mathcal{C}/\mathcal{S}} \left[ \bigoplus_{\lambda \in A} C_{s(\lambda)} \xrightarrow{\varphi} \bigoplus_{i \in I} C_i \right]
\]
with $\varphi$ induced by $\varphi_\lambda = u_j \gamma_{ij} - u_i : C_{s(\lambda)} \to \bigoplus_{i \in I} C_i$, $\lambda = (i, j)$ and $\gamma_{ij} : C_{s(\lambda)} \to C_{t(\lambda)}$ the canonical morphism. By hypothesis the inclusion functor $\mathcal{C}/\mathcal{S} \to \mathcal{C}$ is exact and
commutes with coproducts, and so we obtain
\[
\lim_{\to} c_i = \text{Coker}_c \left[ \bigoplus_{\lambda \in A} C_s(\lambda) \to \bigoplus_{i \in I} C_i \right] = \lim_{\to} c_i
\]
as was to be proved. \(\square\)

**Lemma 5.4.** Let \(C\) be a Grothendieck category and \(S\) a localizing subcategory of \(C\). Then the \(S\)-torsion functor \(t = t_S\) commutes with direct limits if and only if any \(C\)-direct limit of \(S\)-closed objects is \(S\)-torsionfree.

**Proof.** Suppose that \(t\) commutes with direct limits. Let us consider an exact sequence
\[
0 \to t(\lim_{\to} C_i) \to \lim_{\to} C_i \xrightarrow{\lambda} \lim_{\to} c_i S C_i
\]
with \(C_i \in C/S\), \(\lambda\) an \(S\)-envelope of \(\lim_{\to} C_i\). Applying the left exact functor \(t\), we have
\[
t(t(\lim_{\to} C_i)) = t(\lim_{\to} C_i) = \lim_{\to} t(C_i) = 0
\]
since \(t(C_i) = 0\). So \(\lim_{\to} c_i\) is \(S\)-torsionfree.

Conversely, let \(\{C_i\}_I\) be a direct system of objects from \(C\). As above, there is an exact sequence
\[
0 \to t(C_i) \to C_i \xrightarrow{\lambda} (C_i)_S.
\]
Because the direct limit functor is exact, one gets an exact sequence
\[
0 \to \lim_{\to} t(C_i) \to \lim_{\to} C_i \to \lim_{\to} (C_i)_S. \tag{5.2}
\]
Since \(S\) is closed under taking direct limits, it follows that \(\lim_{\to} t(C_i) \in S\). Now, if we apply \(t\) to sequence (5.2), one obtains
\[
\lim_{\to} t(C_i) = t(\lim_{\to} t(C_i)) = t(\lim_{\to} C_i)
\]
since \(t(\lim_{\to} (C_i)_S) = 0\). \(\square\)

Let \(\mathcal{G}\) be a Gabriel topology on \(U\) corresponding to \(S\). By a basis for the topology \(\mathcal{G}\) we mean a subset \(\mathcal{B}\) of \(\mathcal{G}\) such that every object in \(\mathcal{G}\) contains some \(b \in \mathcal{B}\).

**Theorem 5.5.** Let \(C\) be a locally finitely generated Grothendieck category with the family of finitely generated generators \(U \subseteq \text{fg} C\) and suppose that \(S\) is a localizing subcategory of \(C\). Then the following conditions are equivalent:

1. \(S\) is of prefinite type.
2. For any \(U \in U\) the natural morphism \(\lim_{\to} c(U,C_i) \to c(U, \sum_{C_i} c_i)\) induced by the \(S\)-envelope \(\lambda\) of \(\sum_{C_i} c_i\) is an isomorphism.
3. \(U_S = \{U_S\}_{U \in U}\) is a family of \(C/S\)-finitely generated generators for \(C/S\).
4. If \(C\) is \(C\)-finitely generated, then \(C_S\) is \(C/S\)-finitely generated.
5. The torsion functor \(t\) commutes with direct limits.
6. Any \(C\)-direct limit of \(S\)-closed object is \(S\)-torsionfree.
7. \(\mathcal{G}\) has a basis of finitely generated objects.

Thus \(C/S\) is a locally finitely generated Grothendieck category and in this case, any \(C/S\)-finitely generated object \(D\) is a localization \(C_S\) of some \(C \in \text{fg} C\).
Proof. Equivalence (5) ⇔ (6) follows from Lemma 5.4.

(1) ⇒ (2): By definition, \( \sum_{C/S} C_i = \sum C_i \). Now our assertion follows from Theorem 1.8.

(2) ⇒ (3): Let \( \lambda : \sum C_i \rightarrow \sum_{C/S} C_i \) be an \( S \)-envelope of \( \sum C_i \). Then the composed map

\[
\lim_{C/S}(U, C_i) \rightarrow c(U, \sum_{C/S} C_i)
\]

with \( \Phi \) the canonical morphism is, by hypothesis, an isomorphism. Hence,

\[
\lim_{C/S}(U, C_i) \approx \lim_{C/S}(U, C_i) \approx c(U, \sum_{C/S} C_i) \approx c(U, \sum_{C/S} C_i).
\]

By Theorem 1.8 \( U_S \in \text{fg} C/S \).

(3) ⇒ (4): If \( C \in \text{fg} C \) there is an epimorphism \( \eta : \bigoplus_{i=1}^{n} U_i \rightarrow C \) for some \( U_1 \ldots U_n \in U \). Since \( \bigoplus_{i=1}^{n} (U_i)_S \in \text{fg} C/S \), it follows that \( C_S \in C/S \).

(4) ⇒ (7): Suppose \( a \in S \), that is \( a_{S} = U_S \) for some \( U \in U \). Write \( a = \sum C a_i \) as a directed sum of \( C \)-finitely generated subobjects \( a_i \). Then \( U_S = \sum_{C/S}(a_i)_S \). By assumption there is \( i_0 \) such that \( U_S = (a_{i_0})_S \), and hence \( a_{i_0} \in S \).

(7) ⇒ (1): First let us show that \( U_S \in \text{fg} C/S \) where \( U \in U \). Indeed, suppose \( U_S = \sum_{C/S} a_i \); then \( a = \lambda_U^{-1}(\sum C a_i) = \sum \lambda_U^{-1}(a_i) \in S \) with \( \lambda_U \) an \( S \)-envelope of \( U \). By assumption there is a finitely generated subobj \( b \) of \( a \) such that \( b \in S \). Then there is \( i_0 \) such that \( b \subseteq \lambda_U^{-1}(a_{i_0}) \). One has

\[
U_S = b_S \subseteq (\lambda_U^{-1}(a_{i_0}))_S \subseteq a_{i_0} \subseteq U_S.
\]

So \( a_{i_0} = U_S \) that implies \( U_S \in \text{fg} C/S \).

Further, the isomorphism

\[
c(U, \sum_{C/S} C_i) \approx c(S, \sum_{C/S} C_i) \approx \lim_{C/S}(U, C_i) \approx c(U, \sum_{C/S} C_i)
\]

is functorial in \( U \), and so \( \sum_{C/S} C_i = \sum C_i \).

(1) ⇒ (6): The direct limit \( \lim_{C/S} C_i \) may be described as a quotient object of a coproduct \( \bigoplus_i C_i \). To be precise, let \( R \) be the subset of \( I \times I \) consisting of pairs \((i, j)\) with \( i \leq j \) and for each \( S \subseteq R \) we put

\[
C_S = \bigoplus_{(i, j) \in S} \text{Im}(u_i - u_j \gamma_{ij}) \subseteq \bigoplus C_i,
\]

where \( u_i : C_i \rightarrow \bigoplus C_i \) is the canonical monomorphism for \( i \in I \), \( \gamma_{ij} : C_i \rightarrow C_j \) is the canonical morphism for \( i \leq j \). Then \( \lim_{C/S} C_i = \bigoplus C_i/C_R = \bigoplus C_i/\sum C_S \), where \( S \) runs over all finite subsets of \( R \). By assumption both \( \bigoplus C_i \) and \( \sum C_i \) are \( S \)-closed, and therefore \( \lim_{C/S} C_i \), as a quotient object of \( S \)-closed objects, is \( S \)-torsionfree.

(5) ⇒ (1): Suppose that \( X \) is \( S \)-closed. Write \( X = \sum_{C/S} X_i \) as a direct union of \( S \)-closed subobjects. Then \( X/\sum C X_i = t(X/\sum C X_i) = \lim t(X/\sum C X_i) = 0 \) since \( t(X/X_i) = 0 \). Thus \( \sum_{C/S} X_i = \sum C X_i \).

In turn, let \( D \in \text{fg} C/S \). Write \( D = \sum D_i \) as a directed sum of \( D_i \in \text{fg} C \). Then \( D = D_S = \sum_{C/S}(D_i)_S \) whence \( D = (D_{i_0})_S \) for some \( i_0 \).
Proposition 5.6. Let \( \mathcal{C} \) be a locally finitely presented Grothendieck category with the family of finitely presented generators \( \mathcal{U} \subseteq \text{fp}\mathcal{C} \). Then the following conditions are equivalent:

1. \( \mathcal{S} \) is of finite type.
2. For any \( U \in \mathcal{U} \) the natural morphism \( \varinjlim \mathcal{C}(U, C_i) \to \mathcal{C}(U, \varinjlim \mathcal{C}/S C_i) \) induced by the \( \mathcal{S} \)-envelope \( \lambda \) of \( \varinjlim \mathcal{C}_i \) is an isomorphism.
3. \( \mathcal{U}_S = \{ U_S \}_{U \in \mathcal{U}} \) is a family of \( \mathcal{C}/\mathcal{S} \)-finitely presented generators for \( \mathcal{C}/\mathcal{S} \).
4. If \( \mathcal{C} \) is \( \mathcal{C} \)-finitely presented, then \( \mathcal{C}_S \) is \( \mathcal{C}/\mathcal{S} \)-finitely presented.

Thus \( \mathcal{C}/\mathcal{S} \) is a locally finitely presented Grothendieck category and in this case, any \( \mathcal{C}/\mathcal{S} \)-finitely presented object \( D \) is a localization \( \mathcal{C}_S \) of some \( C \in \text{fp}\mathcal{C} \).

Proof. It suffices to note that for each \( C \in \text{fp}\mathcal{C} \) the representable functor \( (C, -) \) commutes with direct limits (see Theorem [1.3]) and there is a presentation
\[
\bigoplus_{i=1}^n U_i \to \bigoplus_{j=1}^m U_j \to C \to 0
\]
of \( C \) by objects from \( \mathcal{U} \); and then our proof literally repeats Theorem [5.3].

In turn, if \( D \in \text{fp}\mathcal{C}/\mathcal{S} \) there is an epimorphism \( \eta : B_S \to D \) with \( B \in \text{fp}\mathcal{C} \); then \( \text{Ker} \eta \in \text{fg}\mathcal{C}/\mathcal{S} \). According to [Hrz, Lemma 2.13] we can choose \( C \subseteq B \) such that \( C \in \text{fg}\mathcal{C} \) and \( \mathcal{C}_S = \text{Ker} \eta \). Hence \( D = (B/\mathcal{C})_S \). \( \square \)

Now let us consider a localizing subcategory \( \mathcal{S} \) of the module category \( \text{Mod}\mathcal{A} \) with \( \mathcal{A} = \{ P_I \}_{I \subseteq I} \) a ring. Then the family \( \mathcal{A}_S = \{ P_S \}_{P \in \mathcal{A}} \) generates \( \text{Mod}\mathcal{A}/\mathcal{S} \) and we call it the ring of quotients of \( \mathcal{A} \) with respect to \( \mathcal{S} \).

There is a naturally defined functor \( j : \text{Mod}\mathcal{A}/\mathcal{S} \to \text{Mod}\mathcal{A}_S = (\mathcal{A}_S^{\text{op}}, \text{Ab}) : \)
\[
\text{Hom}_{\mathcal{A}_S}(P_S, j(M)) := \text{Hom}_{\mathcal{A}}(P, M)
\]
for every \( P \in \mathcal{A} \) and \( M \in \text{Mod}\mathcal{A}/\mathcal{S} \). The next assertion is an analog of the Walker and Walker Theorem (see [Stm, Proposition XI.3.4]).

Proposition 5.7. The functor \( j : \text{Mod}\mathcal{A}/\mathcal{S} \to \text{Mod}\mathcal{A}_S \) is an equivalence if and only if the localizing subcategory \( \mathcal{S} \) is of finite type and coexact.

Proof. Suppose \( j \) is an equivalence. Then every \( P_S \) finitely generated and projective in \( \text{Mod}\mathcal{A}/\mathcal{S} \). By Proposition [5.2] \( \mathcal{S} \) is coexact and by Proposition [5.6] it is of finite type.

Conversely, let \( \mathcal{S} \) be of finite type and coexact. By Proposition [5.2] every \( P_S \) is projective in \( \text{Mod}\mathcal{A}/\mathcal{S} \) and by Proposition [5.6] it is finitely generated in \( \text{Mod}\mathcal{A}/\mathcal{S} \). Now our assertion follows from Proposition [2.2]. \( \square \)

5.1. Left exact functors

Let \( \mathcal{C} \) be a locally finitely presented Grothendieck category. By Theorem [1.1] the functor \( T : \mathcal{C} \to \text{Mod}\mathcal{A} \) with \( \mathcal{A} = \{ h_X \}_{X \in \text{fp}\mathcal{C}} \) induces an equivalence of \( \mathcal{C} \) and \( \text{Mod}\mathcal{A}/\mathcal{S} \) where \( \mathcal{S} \) is some localizing subcategory of \( \text{Mod}\mathcal{A} \). Let \( \mathfrak{S} = \{ \mathfrak{S}^X \}_{X \in \text{fp}\mathcal{C}} \) be the respective Gabriel topology on \( \mathcal{A} \).

Denote by \( \mathcal{L} \) the subcategory of \( \text{Mod}\mathcal{A} \) consisting of \( L \in \text{Mod}\mathcal{A} \) such that for any \( x \in L(X), X \in \text{fp}\mathcal{C} \), there exists an epimorphism \( f : Y \to X \) such that \( L(f)(x) = 0 \). It is directly verified that \( \mathcal{L} \) is a localizing subcategory.
Lemma 5.8. If \( X' \xrightarrow{p} X \xrightarrow{f} X'' \to 0 \) is an exact sequence in \( \text{fp} \mathcal{C} \), then \( \text{Coker}(Tf) \in \mathcal{L} \).

Proof. Let \( Y \in \text{fp} \mathcal{C} \) and \( y \in \text{Coker}(Tf)(Y) \); then there is a morphism \( g \in \mathcal{C}(Y, X'') \) such that \( u_y = r \circ Tg \), where \( r : h_{X''} \to \text{Coker}(Tf) \) is the canonical epimorphism and \( u_y, y(1_Y) = y \). Consider in \( \mathcal{C} \) the commutative diagram

\[
\begin{array}{ccc}
X \Pi_{X''} Y & \xrightarrow{f'} & Y \\
\downarrow g' & & \downarrow g \\
X & \xrightarrow{f} & X''
\end{array}
\]

in which \( f' \) is an epimorphism. Because \( Y \in \text{fp} \mathcal{C} \), there is a finitely generated subobject \( Z' \) of \( X \Pi_{X''} Y \) such that \( f'(Z') = Y \). There exists an epimorphism \( h : Z \to Z' \) with \( Z \in \text{fp} \mathcal{C} \). So \( f'h \) is an epimorphism. It is easily seen that \( \text{Coker}(Tf)(f'h)(y) = 0 \), hence \( \text{Coker}(Tf) \in \mathcal{L} \).

Recall that the functor \( M \in \text{Mod} \mathcal{A}, \mathcal{A} = \{ h_X \}_{X \in \text{fp} \mathcal{C}} \), is left exact if for any exact sequence

\[
X' \to X \to X'' \to 0
\]

in \( \text{fp} \mathcal{C} \) the sequence of abelian groups

\[
0 \to M(X'') \to M(X) \to M(X')
\]

is exact.

Proposition 5.9. Every left exact functor \( M \in \text{Mod} \mathcal{A} \) is \( \mathcal{L} \)-closed.

Proof. Let \( M : \text{fp} \mathcal{C} \to \text{Ab} \) be a contravariant left exact functor, \( X \in \text{fp} \mathcal{C} \) and \( x \in M(X) \) such that \( \text{Ker} \ u_x \in \mathcal{F}^X \). Here \( u_x \) denotes a unique morphism such that \( u_x, X(1_X) = x \). Then there exists an epimorphism \( f : Y \to X \) in \( \text{fp} \mathcal{C} \) such that \( M(f)(x) = 0 \). But \( M(f) \) is a monomorphism of abelian groups and thus \( x = 0 \). So \( M \) is \( \mathcal{L} \)-torsionfree.

Now let \( a \in \mathcal{F}^X \) and \( g : a \to M \) a morphism in \( \text{Mod} \mathcal{A} \). There exists an epimorphism \( f : Y \to X \) such that \( \text{Im}(Tf) \subseteq a \). By Lemma 5.8, \( \text{Im}(Tf) \in \mathcal{F}^X \). One has a morphism \( g \circ Tf = t : h_Y \to M \). Since \( \text{Ker} f \in \text{fg} \mathcal{C} \), there exists an epimorphism \( Z \to \text{Ker} f \) and thus one gets an exact sequence

\[
Z \xrightarrow{p} Y \xrightarrow{f} X \to 0.
\]

We have the following commutative diagram of abelian groups:

\[
\begin{array}{c}
0 \to h_Y(X) \xrightarrow{h_Y(f)} h_Y(Y) \xrightarrow{h_Y(p)} h_Y(Z) \\
\downarrow t_x \downarrow t_Y \downarrow t_z \\
0 \to M(X) \xrightarrow{M(f)} M(Y) \xrightarrow{M(p)} M(Z)
\end{array}
\]

So,

\[
(M(p)t_Y)(1_Y) = (t_z h_Y(p))(1_Y) = (g_z(Tf)z h_Y(p))(1_Y) = g_z(f p) = 0.
\]

Then there exists an element \( x \in M(X) \) such that \( M(f)(x) = t_Y(1_Y) = (gTf)_Y(1_Y) = g_Y(f) \). Thus \( u_x |_{\text{Im}(Tf)} = g |_{\text{Im}(Tf)} \). Since \( \text{Im}(Tf) \in \mathcal{F}^X \) and \( M \) is \( \mathcal{L} \)-torsionfree, it follows
that $u_x|_a = g$. Thus $u_x : h_X \to M$ is a morphism prolonging $g$. The uniqueness of $u_x$ follows from the fact that $M$ is $L$-torsionfree. So $M$ is $L$-closed. 

Define $\text{Lex}((\text{fp}\mathcal{C})^{\text{op}}, \text{Ab})$ to be the category of contravariant left exact functors from $\text{fp}\mathcal{C}$ to $\text{Ab}$. So, we are now in a position to prove the following result.

**Theorem 5.10** (Breitsprecher [Br]). Let $\mathcal{C}$ be a locally finitely presented Grothendieck category. The representation functor $T = c(-,?): \mathcal{C} \to ((\text{fp}\mathcal{C})^{\text{op}}, \text{Ab})$ defined by $X \mapsto (-, X)$ is a natural equivalence between $\mathcal{C}$ and $\text{Lex}((\text{fp}\mathcal{C})^{\text{op}}, \text{Ab})$.

**Proof.** Our assertion would be proved, if we showed that $\mathcal{S} = \mathcal{L}$. Because for every $X \in \mathcal{C}$ the functor $T(X)$ is left exact, in view of Proposition 5.9, it follows that $T(X)$ is $L$-closed, and so $\mathcal{L} \subseteq \mathcal{S}$. Conversely, let $M \in \mathcal{S}$. We can choose a projective presentation of $M$

$$\oplus h_{Y_j} \xrightarrow{g} \oplus h_{X_i} \to M \to 0.$$

By Proposition 5.6 $\mathcal{S}$ is of finite type, and so any coproduct of $\mathcal{S}$-closed objects is an $\mathcal{S}$-closed object. Therefore there exists a morphism $f \in \text{Mor}\mathcal{C}$ such that $Tf = g$. Furthermore, $f$ is an epimorphism in $\mathcal{C}$ since $M_S = 0$. Thus, without loss of generality, we can assume that for every $M \in \mathcal{S}$ there is an exact sequence

$$TY \xrightarrow{Tf} TX \to M \to 0,$$

where $f$ is a $\mathcal{C}$-epimorphism.

According to [Kr3, Lemma 5.7] $f$ is a direct limit $f_\alpha : Y_\alpha \to X_\alpha$ of epimorphisms $f_\alpha$ in $\text{fp}\mathcal{C}$, and so $M \approx \lim_{C} \text{Coker}(Tf_\alpha)$. By Lemma 5.3 every $\text{Coker}(Tf_\alpha) \in \mathcal{L}$, hence $M \in \mathcal{L}$. 

The next result describes localizing subcategories of finite type of a locally coherent Grothendieck category.

**Theorem 5.11.** Let $\mathcal{C}$ be a locally coherent Grothendieck category with the family of coherent generators $\mathcal{U} \subseteq \text{coh}\mathcal{C}$. Then the following conditions are equivalent:

1. $\mathcal{S}$ is of finite type.
2. $\mathcal{S}$ is of prefinite type.
3. The torsion functor $t$ commutes with direct limits.
4. Any $\mathcal{C}$-direct limit of $\mathcal{S}$-closed objects is $\mathcal{S}$-torsionfree.
5. For any $U \in \mathcal{U}$ the natural morphism $\lim_{C} c(U, C_i) \to c(U, \lim_{C/S} C_i)$ induced by the $\mathcal{S}$-envelope $\lambda$ of $\lim_{C/S} C_i$ is an isomorphism.
6. $\mathcal{U}_S = \{ U_S \}_{U \in \mathcal{U}}$ is a family of $\mathcal{C}/\mathcal{S}$-coherent generators for $\mathcal{C}/\mathcal{S}$.
7. If $\mathcal{C}$ is $\mathcal{C}$-coherent, then $\mathcal{C}_S$ is $\mathcal{C}/\mathcal{S}$-coherent.
8. $\mathcal{S}$ has a basis of coherent objects.

Thus $\mathcal{C}/\mathcal{S}$ is a locally coherent Grothendieck category and in this case, any $\mathcal{C}/\mathcal{S}$-coherent object $D$ is a localization $C_S$ of some $C \in \text{coh}\mathcal{C}$.

**Proof.** (1) $\Rightarrow$ (5): It follows from Proposition 5.6.

(5) $\Rightarrow$ (6): By Proposition 5.3 $\mathcal{U}_S \subseteq \text{fp}\mathcal{C}/\mathcal{S}$. Let us show that for any $U \in \mathcal{U}$ the object $U_S$ is coherent. Let $C \subseteq U_S$ be $\mathcal{C}/\mathcal{S}$-finitely generated subobject of $U_S$. From [HrZ, Lemma 2.13] it follows that there is a $\mathcal{C}$-finitely generated subobject $A \subseteq U$
of $U$ such that $A_S = C$. Since $U$ is coherent, the object $A \in \text{coh} \mathcal{C}$. By Proposition 5.6, $A_S \in \text{fp} \mathcal{C}/S$.

(6) $\Rightarrow$ (7): Easy.

(7) $\Rightarrow$ (1): $S$ is of finite type by Proposition 5.6.

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4): It follows from Theorem 5.3.

(2) $\Leftrightarrow$ (8): Since any finitely generated subobject of a coherent object is coherent, our assertion follows from Theorem 5.3.

(1) $\Rightarrow$ (2): Straightforward.

(3) $\Rightarrow$ (1): It follows from [Kr1, Lemma 2.4].

As in [Hrz] for an arbitrary subcategory $\mathcal{X}$ of $\mathcal{C}$, denote by $\vec{\mathcal{X}}$ the subcategory of $\mathcal{C}$ consisting of direct limits of objects in $\mathcal{X}$. Herzog [Hrz] and Krause [Kr1] observed that localizing subcategories of finite type in a locally coherent category are determined by Serre subcategories of the abelian subcategory $\text{coh} \mathcal{C}$. Namely it holds the following.

**Theorem 5.12** (Herzog and Krause). Let $\mathcal{C}$ be a locally coherent Grothendieck category. There is a bijective correspondence between Serre subcategories $\mathcal{P}$ of $\text{coh} \mathcal{C}$ and localizing subcategories $\mathcal{S}$ of $\mathcal{C}$ of finite type. This correspondence is given by the functions

$$
\mathcal{P} \mapsto \vec{\mathcal{P}}
$$

$$
\mathcal{S} \mapsto \text{coh} \mathcal{S} = \mathcal{S} \cap \text{coh} \mathcal{C}
$$

which are mutual inverses.

Later on, given a Serre subcategory $\mathcal{P}$ of $\text{coh} \mathcal{C}$, the $\vec{\mathcal{P}}$-torsion functor will be denoted by $t_\mathcal{P}$.

Recall that an object $C \in \mathcal{C}$ of a Grothendieck category $\mathcal{C}$ is *noetherian* if any subobject of $C$ is finitely generated. $\mathcal{C}$ is called *locally noetherian* if it has a family of noetherian generators. In that case, the relations

$$\text{fg} \mathcal{C} = \text{fp} \mathcal{C} = \text{coh} \mathcal{C}$$

hold. If in addition $\mathcal{C}$ is a locally finitely generated, it is locally coherent and any localizing subcategory $\mathcal{S}$ of $\mathcal{C}$ is of finite type. Any $X \in \mathcal{S}$ is a direct union $\sum_{i \in I} X_i$ of objects $X_i \in \text{fg} \mathcal{C} \cap \mathcal{S} = \text{coh} \mathcal{C} \cap \mathcal{S} = \text{coh} \mathcal{S}$. Furthermore, any quotient category $\mathcal{C}/\mathcal{S}$ is locally noetherian with a family of noetherian generators $\{C_S\}_{C \in \text{coh} \mathcal{C}}$.

### 5.2. The Ziegler topology

The study of pure-injective (= algebraically compact) modules over different classes of rings plays an important role in the theory of rings and modules. It goes back to Cohn and has been further developed by various mathematicians. Because pure-injective modules can be defined, using some condition of solvability (in this module) of linear equations systems, many problems of (algebraic!) structure of pure-injective modules admit reformulations using concepts of the model theory. It is such an approach that led Ziegler [Zgr] to construction of the topological space (“The Ziegler spectrum”) whose points are indecomposable pure-injective modules. Recently Herzog [Hrz] and Krause [Kr1] have proposed an algebraic definition of the Ziegler spectrum.
Let \( \mathcal{C} \) be a Grothendieck category with the family of generators \( \mathcal{U} \). We denote by \( \text{Zg} \mathcal{C} \) the set of isomorphism classes of indecomposable injective objects in \( \mathcal{C} \) and call \( \text{Zg} \mathcal{C} \) the **Ziegler spectrum** of \( \mathcal{C} \). The fact that \( \text{Zg} \mathcal{C} \) forms a set follows from that any indecomposable injective object in \( \mathcal{C} \) occurs as the injective envelope of some \( \mathcal{U} \)-finitely generated object \( X \in \text{fg} \mathcal{C} \) and \( \text{fg} \mathcal{C} \) is skeletally small. It will be convenient to identify each isomorphism class with a representative belonging to it. If \( S \) is a localizing subcategory in \( \mathcal{C} \), the assignment \( X \mapsto E(X) \) induces injective maps \( \text{Zg} S \to \text{Zg} \mathcal{C} \) and \( \text{Zg} \mathcal{C} / S \to \text{Zg} \mathcal{C} \). We consider both maps as identifications. They satisfy \( \text{Zg} S \cup \text{Zg} \mathcal{C} / S = \text{Zg} \mathcal{C} \) and \( \text{Zg} S \cap \text{Zg} \mathcal{C} / S = \emptyset \) [Gbl, Corollaire III.3.2].

Now let \( \mathcal{C} \) be a locally coherent category, i.e. \( \mathcal{U} \subseteq \text{coh} \mathcal{C} \). To an arbitrary subcategory \( \mathcal{X} \subseteq \text{coh} \mathcal{C} \), we associate the subset of \( \text{Zg} \mathcal{C} \)

\[
\mathcal{O}(\mathcal{X}) = \{ E \in \text{Zg} \mathcal{C} \mid c(C, E) \neq 0 \text{ for some } C \in \text{coh} \mathcal{C} \}.
\]

If \( \mathcal{X} = \{ C \} \) is singleton, we abbreviate \( \mathcal{O}(\mathcal{X}) \) to \( \mathcal{O}(C) \); thus \( \mathcal{O}(\mathcal{X}) = \cup_{C \in \mathcal{X}} \mathcal{O}(C) \).

We restrict the discussion to subcategories of the form \( \mathcal{O}(S) \) where \( S \subseteq \text{coh} \mathcal{C} \) is a Serre subcategory. In that case,

\[
\mathcal{O}(S) = \{ E \in \text{Zg} \mathcal{C} \mid t_S(E) \neq 0 \}.
\]

**Theorem 5.13** (Herzog [Hrz] and Krause [Kr1]). For a locally coherent Grothendieck category \( \mathcal{C} \) the following assertions hold:

1. The collection of subsets of \( \text{Zg} \mathcal{C} \)

\[\{ \mathcal{O}(S) \mid S \text{ is a Serre subcategory} \}\]

satisfies the axioms for the open sets of a topological space on \( \text{Zg} \mathcal{C} \). This topological space we call the **Ziegler spectrum** of \( \mathcal{C} \) too.

2. There is a bijective inclusion preserving correspondence between the Serre subcategories \( S \) of \( \text{coh} \mathcal{C} \) and the open subsets \( \mathcal{O} \) of \( \text{Zg} \mathcal{C} \). This correspondence is given by the functions

\[
S \mapsto \mathcal{O}(S) \quad \mathcal{O} \mapsto S_{\mathcal{O}} = \{ C \in \text{coh} \mathcal{C} \mid \mathcal{O}(C) \subseteq \mathcal{O} \}
\]

which are mutual inverses.

Recall that a topological space \( \mathcal{X} \) is **quasi-compact** provided that for every family \( \{ \mathcal{O}_i \}_{i \in I} \) of open subsets \( \mathcal{X} = \cup_{i \in I} \mathcal{O}_i \) implies \( \mathcal{X} = \cup_{i \in J} \mathcal{O}_i \) for some finite subset \( J \) of \( I \). A subset of \( \mathcal{X} \) is quasi-compact if it is quasi-compact with respect to the induced topology.

By [Hrz, Corollary 3.9] and [Kr1, Corollary 4.6] an open subset \( \mathcal{O} \) of \( \text{Zg} \mathcal{C} \) is quasi-compact if and only if it is one of the basic open subsets \( \mathcal{O}(C) \) with \( C \in \text{coh} \mathcal{C} \).

Serre subcategories \( S \) of \( \text{coh} \mathcal{C} \) arise in the following natural way. An object \( M \in \mathcal{C} \) is **coh-injective** if \( \text{Ext}^1_{\mathcal{C}}(C, M) = 0 \) for each \( C \in \text{coh} \mathcal{C} \). Then the subcategory

\[ S_M = \{ C \in \text{coh} \mathcal{C} \mid c(C, M) = 0 \}\]

generated by \( M \) is Serre. Moreover, every Serre subcategory of \( \text{coh} \mathcal{C} \) arises in this fashion [Hrz, Corollary 3.11].
Examples. Here are some examples of the Ziegler-closed subsets:

1) Let \( C \) be a locally coherent Grothendieck category; then the functor category \((\mathrm{coh} C)^\mathrm{op}, \mathrm{Ab})\) is locally coherent. By Theorem 5.10 the category \( C \) is equivalent to \( \mathrm{Lex}(\mathrm{coh} C)^\mathrm{op}, \mathrm{Ab}) = (\mathrm{coh} C)^\mathrm{op}, \mathrm{Ab})/\vec{L} \), where \( L \) is a Serre subcategory of \( \mathrm{coh}((\mathrm{coh} C)^\mathrm{op}, \mathrm{Ab})\) consisting of objects isomorphic to \( \mathrm{Coker}(\mu, -) \) for some epimorphism \( \mu : A \to B \) in \( \mathrm{coh} C \). Thus \( Zg C \) is closed in \( Zg((\mathrm{coh} C)^\mathrm{op}, \mathrm{Ab}) \).

2) If \( C = C_A \), the Ziegler spectrum of \( C_A \)
\[ Zg C_A = \{ Q \otimes_A - | Q_A \text{ is an indecomposable pure-injective module} \} \]
We notice that \( Zg C_A \) is quasi-compact since \( Zg C_A = \mathcal{O}(A \otimes_A -) \). Prest, Rothmaler and Ziegler have shown \([\text{PRZ}, \text{Corollary} \ 4.4]\) (see also \([\text{GG1}, \text{Theorem} \ 2.5]\)) that the ring \( A \) is right coherent if and only if the set \( I_{\text{inj}} = \{ Q \otimes_A - \in Zg C_A | Q \text{ is an injective module} \} \) is closed in \( Zg C_A \).

3) If \( A \) is left coherent, from \([\text{GG2}, \text{Theorem} \ 2.4]\) it follows that \( I_{\text{flat}} = \{ Q \otimes_A - \in Zg C_A | Q \text{ is a flat module} \} \) is closed in \( Zg C_A \).

Recollect that the ring \( A \) is weakly Quasi-Frobenius if the functor \( \mathrm{Hom}_A(-, A) \) gives a duality of categories \( \mathrm{mod} A^\mathrm{op} \) and \( \mathrm{mod} A \). Weakly Quasi-Frobenius rings are described as (two-sided) coherent absolutely pure rings \([\text{GG2}, \text{Theorem} \ 2.11]\). Over weakly Quasi-Frobenius rings \( I_{\text{inj}} = I_{\text{flat}} \). And backwards, if \( A \) is a (two-sided) coherent ring and \( I_{\text{inj}} = I_{\text{flat}} \), then \( A \) is a weakly Quasi-Frobenius ring (see \([\text{GG2}, \text{Corollary} \ 2.12]\)).

4) Let \( \rho : A \to B \) be a ring homomorphism; then \( \rho \) induces the exact functor \( \rho^* : \mathrm{coh} C_A \to \mathrm{coh} C_B \). If \( \rho \) is an epimorphism of rings, then the map \( M_B \mapsto M_A \) induces a homeomorphism \( Zg C_B \to Zg C_A \setminus \mathcal{O}(S) \) with \( S = \mathrm{Ker} \rho^* \). Thus \( Zg C_B \) is closed in \( Zg C_A \) \([\text{Pr2}, \text{Corollary} \ 9]\).

6. Categories of generalized modules

The most important notions and properties of the category of generalized \( A \)-modules \( C_A \) can be easily extended to an arbitrary functor category \( \mathrm{Mod} \mathcal{A} = (\mathcal{A}^\mathrm{op}, \mathrm{Ab}) \). The model-theoretic background can be found in \([\text{Bur}, \text{Pr3}]\). In this section, for the most part, we adhere to the reference \([\text{Hrz}]\).

6.1. Tensor products

Let \( \mathrm{Mod} \mathcal{A} \) be the category of right \( \mathcal{A} \)-modules with the family of finitely generated projective generators \( \mathcal{A} = \{ P_i \}_{i \in I} \). According to Proposition 2.2 \( \mathrm{Mod} \mathcal{A} \approx (\mathcal{A}^\mathrm{op}, \mathrm{Ab}) \). We refer to the functor category \( (\mathcal{A}, \mathrm{Ab}) \) as a category of left \( \mathcal{A} \)-modules and denote it by \( \mathrm{Mod} \mathcal{A}^\mathrm{op} \). It is a locally finitely presented Grothendieck category with the family of finitely generated projective generators \( \mathcal{A}^\mathrm{op} = \{ h_P = \mathrm{Hom}_A(P, -) \}_{P \in \mathcal{A}} \).

**Proposition 6.1.** \([\text{Pop}, \text{Theorem III.6.3}]\) Let \( M : \mathcal{A}^\mathrm{op} \to \mathrm{Ab} \) (\( N : \mathcal{A} \to \mathrm{Ab} \)) be a right (left) \( \mathcal{A} \)-module. Then, a unique functor \( M \otimes_{\mathcal{A}} - : \mathrm{Mod} \mathcal{A}^\mathrm{op} \to \mathrm{Ab} \) (\( - \otimes_{\mathcal{A}} N : \mathrm{Mod} \mathcal{A} \to \mathrm{Ab} \)) exists such that:
(1) There are functorial isomorphisms $M \otimes_A h^P \approx M(P)$ and $h_P \otimes_A N \approx N(P)$ for $P \in \mathcal{A}$.

(2) $M \otimes_A -$ and $- \otimes_A N$ have right adjoints.

Note also that the tensor product functor $M \otimes_A -$ is right exact and commutes with direct limits.

So let $\text{Mod} \, \mathcal{A} = (\mathcal{A}^{\text{op}}, \text{Ab})$ (Mod $\mathcal{A}^{\text{op}} = (\mathcal{A}, \text{Ab})$) be the category of right (left) $\mathcal{A}$-modules with $\mathcal{A} = \{P_i\}_{i \in I}$, $\text{mod} \, \mathcal{A}^{\text{op}}$ (mod $\mathcal{A}$) be the category of finitely presented left (right) $\mathcal{A}$-modules. By definition every $M \in \text{mod} \, \mathcal{A}$ has a projective presentation

$$\bigoplus_{j=1}^n h_{P_j} \rightarrow \bigoplus_{k=1}^m h_{P_k} \rightarrow M \rightarrow 0 \tag{6.1}$$

in $\text{Mod} \, \mathcal{A}$. Similarly, every $M \in \text{mod} \, \mathcal{A}^{\text{op}}$ has a presentation

$$\bigoplus_{j=1}^n h_{P_j} \rightarrow \bigoplus_{k=1}^m h_{P_k} \rightarrow M \rightarrow 0$$

in $\text{Mod} \, \mathcal{A}^{\text{op}}$.

Denote by $\mathcal{C}_A = (\text{mod} \, \mathcal{A}^{\text{op}}, \text{Ab})$ ($\mathcal{A} = (\text{mod} \, \mathcal{A}, \text{Ab})$) the category of additive covariant functors defined on $\text{mod} \, \mathcal{A}^{\text{op}}$ (mod $\mathcal{A}$) and call $\mathcal{C}_A$ ($\mathcal{A}$) the category of generalized right (left) $\mathcal{A}$-modules. They naturally extend the respective categories of generalized $\mathcal{A}$-modules for which $\mathcal{A} = \{A\}$ is a ring. Because $\text{mod} \, \mathcal{A}^{\text{op}}$ is closed under cokernels, $\mathcal{C}_A$ is a locally coherent Grothendieck category. Finitely generated projective objects of $\mathcal{C}_A$ are precisely of the form $\{(M, -)\}_{M \in \text{mod} \, \mathcal{A}^{\text{op}}}$ and this family generates $\mathcal{C}_A$. The tensor product functor $\otimes \mathcal{A} - : \text{Mod} \, \mathcal{A} \rightarrow \mathcal{C}_A$ defined by the rule $M_A \mapsto M \otimes \mathcal{A} -$ is fully faithful and right exact.

Each finitely presented (= coherent) generalized module $C \in \mathcal{C}_A$ has a projective presentation

$$(K, -) \xrightarrow{f} (L, -) \rightarrow C \rightarrow 0$$

with $K, L \in \text{mod} \, \mathcal{A}^{\text{op}}$. In particular, if $M \in \text{mod} \, \mathcal{A}$, then $M \otimes \mathcal{A} - \in \text{coh} \mathcal{C}_A$. Indeed, consider projective presentation (6.1) of $M$. As tensoring is right exact, this gives an exact sequence in $\mathcal{C}_A$

$$\bigoplus_{j=1}^n h_{P_j} \otimes \mathcal{A} - \rightarrow \bigoplus_{k=1}^m h_{P_k} \otimes \mathcal{A} - \rightarrow M \rightarrow 0$$

which is a presentation of $M \otimes \mathcal{A} -$ in $\mathcal{C}_A$ by finitely generated projective objects since $\bigoplus h_{P_j} \otimes \mathcal{A} - \approx (\bigoplus P_j, -)$.

**Lemma 6.2.** An object $E \in \mathcal{C}_A$ is coh$\mathcal{C}_A$-injective if and only if it is isomorphic to one of the functors $M \otimes \mathcal{A} -$ where $M$ is a right $\mathcal{A}$-module.

**Proof.** Herzog [Hrz. Proposition 2.2] has shown that $E$ is coh$\mathcal{C}_A$-injective if and only if it is right exact. Therefore the functor $M \otimes \mathcal{A} -$ is coh-injective.

Let $E$ be coh-injective. Define $M_A$ by putting $\text{Hom}_\mathcal{A}(P, M) = c_A((h_P, -), E) = E(h_P)$. Now our assertion is proved similar to [Stm. Proposition IV.10.1].

Thus the category $\text{Mod} \, \mathcal{A}$ of right $\mathcal{A}$-modules can be considered as the subcategory of coh$\mathcal{C}_A$-injective objects of the category $\mathcal{C}_A$.

In order to describe the points of the Ziegler spectrum $Zg \mathcal{C}_A$ of $\mathcal{C}_A$, recall that a short exact sequence

$$0 \rightarrow L \overset{\mu}{\rightarrow} M \overset{\delta}{\rightarrow} N \rightarrow 0$$

of an arbitrary locally finitely presented Grothendieck category \( C \) is pure-exact provided that the sequence
\[
0 \longrightarrow c(X, L) \longrightarrow c(X, M) \longrightarrow c(X, N) \longrightarrow 0
\]
is exact for all \( X \in \text{fp} C \). In this case, \( \mu \) is called a pure-monomorphism. An object \( Q \in C \) is said to be pure-injective if every pure-exact sequence with first term \( Q \) splits.

**Proposition 6.3.** For a short exact sequence \( \varepsilon : 0 \to L \to M \to N \to 0 \) in \( C \), the following are equivalent:

1. \( \varepsilon \) is pure-exact in \( C \).
2. \( \varepsilon \) is a direct limit of split exact sequences \( 0 \to L_i \to M_i \to N_i \to 0 \) in \( C \).

**Proof.** Write \( N = \lim_{\longrightarrow} N_i \) as a direct limit of finitely presented objects \( N_i \). For every \( i \in I \) consider the following commutative diagram
\[
\begin{array}{ccc}
\varepsilon_i : 0 & \longrightarrow & L_i \\
\downarrow & & \downarrow \\
0 & \longrightarrow & L \\
\end{array}
\]
in which the right square is pullback and \( \varphi_i \) the canonical morphism. Since for \( i \leq j \) the relations
\[
\delta \psi_i = \varphi_i \delta_i = \varphi_j (\varphi_i \delta_i)
\]
hold, there exists a unique \( \psi_{ij} : M_i \to M_j \) such that \( \psi_i = \psi_j \psi_{ij} \). Clearly that the system \( \{M_i, \psi_{ij}\}_I \) is direct and \( \lim_{\longrightarrow} \varepsilon_i = \varepsilon \). By hypothesis there exists a morphism \( f : N_i \to M \) such that \( \delta f = \varphi_i \), and hence there exists \( g : N_i \to M_i \) such that \( \delta_i g = 1_{M_i} \), i.e. each sequence \( \varepsilon_i \) splits.

Conversely, if each \( \varepsilon_i \) splits, then the sequence
\[
0 \longrightarrow \text{Hom}_A(X, L) \longrightarrow \text{Hom}_A(X, M) \longrightarrow \text{Hom}_A(X, N) \longrightarrow 0
\]
is exact. The fact that \( \text{Hom}_A(X, -) \), \( X \in \text{fp} C \), commutes with direct limits implies (1).

**Corollary 6.4.** The sequence \( \varepsilon : 0 \to L \to M \to N \to 0 \) of right \( \mathcal{A} \)-modules is pure-exact if and only if the \( \mathcal{C}_A \)-sequence \( \varepsilon \otimes_A - : 0 \to L \otimes_A - \to M \otimes_A - \to N \otimes_A - \to 0 \) is exact.

**Proof.** As tensoring commutes with direct limits, the necessary condition follows from the preceding Proposition. Conversely, if \( \varepsilon \otimes_A - \) is exact, then for any \( X \in \text{mod} \mathcal{A} \) one has the following exact sequence
\[
0 \longrightarrow (X \otimes_A -, L \otimes_A -) \longrightarrow (X \otimes_A -, M \otimes_A -) \longrightarrow (X \otimes_A -, N \otimes_A -) \longrightarrow \text{Ext}^1(X \otimes_A -, L \otimes_A -).
\]
Because \( X \otimes_A - \in \text{coh} C_A \) and the object \( L \otimes_A - \) is \( \text{coh} C_A \)-injective, it follows that \( \text{Ext}^1(X \otimes_A -, L \otimes_A -) = 0 \). \( \square \)
Notice that as tensoring commutes with direct limits, a monomorphism $\mu : L \to M$ is pure if and only if $\mu \otimes_A X$ is a monomorphism for any right $A$-module $X$.

The next result is proved similar to [Hrz, Proposition 4.1].

**Proposition 6.5.** An object $E \in \mathcal{C}_A$ is an injective object if and only if it is isomorphic to one of the functors $Q \otimes_A -$ where $Q$ is a pure-injective right $A$-module.

Thus the points of the Ziegler spectrum $Zg \mathcal{C}_A$ of $\mathcal{C}_A$ are represented by the pure-injective indecomposable right $A$-modules. Note that every indecomposable injective right $A$-module $E_A$ is pure-injective and hence $E \otimes_A -$ is a point of $Zg \mathcal{C}_A$.

It is easy to see that $Zg \mathcal{C}_A = \cup_{P \in A} O(P \otimes_A -$). Prest has shown [Pr3, Example 1.5] that $Zg \mathcal{C}_A$ need not be compact, in contrast to the case $A = \{ A \}$ where $A$ is a ring, the whole space need not be basic open.

6.2. Auslander-Gruson-Jensen Duality

Let $A$ be an arbitrary ring. Gruson and Jensen [GJ] and Auslander [Aus3] proved that there is a duality $D : (\text{coh}_A \mathcal{C})^{\text{op}} \approx \text{coh} \mathcal{C}_A$ between the respective subcategories of the coherent objects of $\mathcal{A} \mathcal{C}$ and $\mathcal{C}_A$. It is not hard to show that the same holds for an arbitrary ring $A = \{ P_i \}_{i \in I}$.

Namely let the functor $D : (\text{coh}_A \mathcal{C})^{\text{op}} \to \mathcal{C}_A$ be given by

$$(DC)(A N) = \mathcal{A}C(C, - \otimes_A N)$$

where $C \in \text{coh}_A \mathcal{C}$ and $N \in \text{mod} A^{\text{op}}$. If $\delta : B \to C$ is a morphism in $\text{coh}_A \mathcal{C}$, then

$$D(\delta)_N : (DC)(A N) \to (DB)(A N)$$

is put to be $\mathcal{A}C(\delta, - \otimes_A N)$.

We claim that $D$ is exact. Indeed, if

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is a short exact sequence in $\text{coh}_A \mathcal{C}$, then because $- \otimes_A N$ is $\text{coh}_A \mathcal{C}$-injective, the sequence

$$0 \to (C, - \otimes_A N) \xrightarrow{(\beta, - \otimes_A N)} (B, - \otimes_A N) \xrightarrow{(\alpha, - \otimes_A N)} (A, - \otimes_A N) \to 0$$

is exact for each $N \in \text{mod} A^{\text{op}}$. Therefore the sequence

$$0 \to DC \xrightarrow{D\beta} DB \xrightarrow{Da} DA \to 0$$

is exact.

By construction of $D$, $D(M_A, -)(A X) = ((M_A, -), - \otimes_A X) \approx M \otimes_A X$.

Thus $D(M_A, -) \approx M \otimes_A -. Given C \in \text{coh}_A \mathcal{C}$, consider a projective presentation

$$(N_A, -) \looparrowright (M_A, -) \to C \to 0$$

of $C$. Applying the exact functor $D$, we obtain an exact sequence

$$0 \to DC \to M \otimes_A - \to N \otimes_A -.$$
Thus $DC$ is a coherent object of $\mathcal{C}_A$ and therefore the functor $D$ has its image in $\text{coh} \mathcal{C}_A$.

**Theorem 6.6** (Auslander, Gruson and Jensen). The functor $D : (\text{coh} \mathcal{A} \mathcal{C})^{\text{op}} \to \text{coh} \mathcal{C}_A$ defined above constitutes a duality between the categories $\text{coh} \mathcal{A} \mathcal{C}$ and $\text{coh} \mathcal{C}_A$. Furthermore, for $M_A \in \text{mod} \mathcal{A}$ and $A_N \in \text{mod} \mathcal{A}^{\text{op}}$ we have that

$$D(M_A, -) \approx M \otimes_A - \quad \text{and} \quad D(- \otimes_A N) \approx (A_N, -).$$

**Proof.** Since $D(- \otimes_A N)(A_M) \approx (- \otimes_A N, - \otimes_A M) \approx \text{Hom}_A(A_N, A_M)$, it follows that $D(- \otimes_A N) \approx (A_N, -)$. Similarly, we can define the functor $D' : (\text{coh} \mathcal{A}_C)^{\text{op}} \to \text{coh} \mathcal{A}_C$ in the other direction. Both of the compositions $DD'$ and $D'D$ are exact functors that are equivalences on the respective categories of finitely generated projective objects. Therefore they are both natural equivalences.

Because the category $\text{coh} \mathcal{A} \mathcal{C}$ has enough projectives, the duality gives the following.

**Proposition 6.7** (Auslander). The category $\text{coh} \mathcal{C}_A$ has enough injectives and they are precisely the objects of the form $M \otimes_A -$ where $M_A \in \text{mod} \mathcal{A}$.

Thus every coherent object $C \in \text{coh} \mathcal{C}_A$ has both a projective presentation in $\mathcal{C}_A$

$$(A_K, -) \rightarrow (A_L, -) \rightarrow C \rightarrow 0 \quad (6.2)$$

and an injective presentation in $\text{coh} \mathcal{C}_A$

$$0 \rightarrow C \rightarrow M \otimes_A - \rightarrow N \otimes_A - \quad (6.3)$$

where $K, L \in \text{mod} \mathcal{A}^{\text{op}}$ and $M, N \in \text{mod} \mathcal{A}$.

We conclude the section by Herzog’s Theorem. Let $\mathcal{S} \subseteq \mathcal{A} \mathcal{C}$; then the subcategory

$$DS = \{DC \mid C \in \mathcal{S}\}$$

is Serre in $\text{coh} \mathcal{C}_A$ and the restriction to $\mathcal{S}$ of the duality $D$ gives a duality $D : \mathcal{S}^{\text{op}} \to DS$. By Theorem 5.13 the map $\mathcal{O}(\mathcal{S}) \leftrightarrow \mathcal{O}(DS)$ induced on the open subsets of the Ziegler spectrum is an inclusion-preserving bijection. Similar to [Hrz], it is easily shown that the functor $D$ induces an isomorphism of abelian groups

$$\mathcal{A} / \mathcal{S} (A, B) \approx \mathcal{C}_A / DS (DB, DA)$$

where $A$ and $B \in \text{coh} \mathcal{A} \mathcal{C}$ and the assignment given by $A_S \mapsto (DA)_{DS}$ is functorial. Thus we have the following.

**Theorem 6.8** (Herzog). Let $\mathcal{A}$ be a ring. There is an inclusion-preserving bijective correspondence between Serre subcategories of $\text{coh} \mathcal{A} \mathcal{C}$ and those of $\text{coh} \mathcal{C}_A$ given by

$$\mathcal{S} \mapsto DS.$$
abelian categories:

\[
\begin{array}{cccccc}
0 & \longrightarrow & S & \longrightarrow & \text{coh}_A \mathcal{C} & \longrightarrow & \text{coh}_A \mathcal{C}/S & \longrightarrow & 0 \\
D & \downarrow & D & \downarrow & D & & D & \downarrow & \\
0 & \longrightarrow & DS & \longrightarrow & \text{coh} \mathcal{C}_A & \longrightarrow & \text{coh} \mathcal{C}_A/DS & \longrightarrow & 0
\end{array}
\]

7. Grothendieck categories as quotient categories of \((\text{mod} \ A^{op}, \text{Ab})\)

In this section we give another representation of the Grothendieck category \(\mathcal{C}\) as a quotient category of \(\mathcal{C}_A\). To begin, let us prove the following.

Proposition 7.1. Let \(\text{Mod} \ A\) be the category of right \(A\)-modules with \(A = \{P_i\}_{i \in I}\) a ring and let \(\mathcal{C}_A\) be the respective category of generalized right \(A\)-modules. Then the category \(\text{Mod} \ A\) is equivalent to the quotient category of \(\mathcal{C}_A\) with respect to the localizing subcategory \(\mathcal{P}^A = \{F \in \mathcal{C}_A \mid F(P) = 0 \text{ for all } P \in A\}\).

Proof. For an arbitrary functor \(F \in \mathcal{C}_A\) by \(F(\mathcal{A})\) denote a right \(A\)-module defined as follows. If \(P \in A\), then we put \(F(\mathcal{A})(P) = F(\mathcal{A}P)\). It is directly checked that \(F(\mathcal{A}) \in \text{Mod} \ A\). From Theorem 4.3 it follows that the functor \(\Phi : \mathcal{C}_A \rightarrow \text{Mod} \ A, F \mapsto F(\mathcal{A})\), defines an equivalence of categories \(\text{Mod} \ A\) and \(\mathcal{C}_A/\ker \Phi\). Clearly, \(\mathcal{P}^A = \ker \Phi\). \(\square\)

Because there is a natural equivalence of functors \(P \otimes_A - \approx (\_P, -)\), from Theorem 4.3 it also follows that the quotient category \(\mathcal{C}_A/\mathcal{P}^A\) is equivalent to the subcategory \(\text{Mod} \ \overline{A} = \{M \otimes_A - \mid M \in \text{Mod} \ A\}\).

Remark. It easy to see that the subcategory \(\mathcal{P}^A = \{\ker(\mu \otimes_A -) \mid \mu \text{ is a monomorphism in } \text{Mod} \ A\}\).

It should also be remarked that \(\mathcal{C}_A/\mathcal{P}^A\)-injective objects, in view of Lemma 1.3, are precisely objects \(E \otimes_A -\) where \(E \mathcal{A}\) is an injective right \(A\)-module.

Now suppose that \(\mathcal{C}\) is a Grothendieck category with the family of generators \(U\). As usual, let \(\text{Mod} \ A\) be the category of right \(A\)-modules with \(A = \{h_U\}_{U \in \mathcal{U}}\). We are now in a position to prove the following.

Theorem 7.2. Every Grothendieck category \(\mathcal{C}\) with the family of generators \(U\) is equivalent to the quotient category of \(\mathcal{C}_A\) with respect to some localizing subcategory \(S\) of \(\mathcal{C}_A\).

Proof. By Theorem 4.3 there is a pair of functors \((s, q)\), where \(s : \mathcal{C} \rightarrow \text{Mod} \ A\) and \(q : \text{Mod} \ A \rightarrow \mathcal{C}\), defining \(\mathcal{C}\) as a quotient category of \(\text{Mod} \ A\). In turn, by Proposition 7.1 there is a pair of functors \((g, h)\), where \(g : \text{Mod} \ A \rightarrow \mathcal{C}_A\) and \(h : \mathcal{C}_A \rightarrow \text{Mod} \ A\), defining \(\text{Mod} \ A\) as a quotient category of \(\mathcal{C}_A\). It thus suffices to show that \(gs\) is a fully faithful functor, the functor \(qh\) is exact and left adjoint to \(gs\). Indeed, the composition \(gs\) of fully faithful functors \(g\) and \(s\) is again a fully faithful functor and the composition \(qh\) of exact functors \(q\) and \(h\) is an exact functor. The fact that \(gs\) is right adjoint to \(qh\) follows from the following isomorphisms:

\[c_A(F, gs(C)) \approx \text{Hom}_A(h(F), s(C)) \approx c(qh(F), C)\]

Hence \(\mathcal{C}\) is equivalent to the quotient category of \(\mathcal{C}_A\) with respect to the localizing subcategory \(S = \ker(qh)\). \(\square\)
Corollary 7.3. [GG1] Theorem 2.3 Every Grothendieck category \( C \) with a generator \( U \) is equivalent to the quotient category of \( C_A, A = \text{End}_C U \), with respect to some localizing subcategory \( S \) of \( C_A \).

**Proof.** It follows from Corollary 4.4 and Theorem 7.2.

The ring \( A \) of the module category \( \text{Mod} A \) is said to be right coherent if each object \( P \in A \) is coherent. Suppose now \( C \) is a locally coherent, i.e. \( U \subseteq \text{coh} C \) and \( \text{Mod} A \) the respective module category with \( A = \{ h_U \}_{U \in U} \). One easily verifies that \( A \) is right coherent.

**Theorem 7.4.** Let \( C \) be a Grothendieck category with the family of generators \( U \). Consider the following conditions:

1. \( C \) is locally coherent, i.e. \( U \subseteq \text{coh} C \).
2. The localizing subcategory \( S \) from the preceding Theorem is of finite type.
3. \( S \) is of prefinite type.
4. \( \text{Zg} C = \left\{ E \otimes_A - \mid E \text{ is } C\text{-injective} \right\} \) is closed in \( \text{Zg} C_A \).

Then conditions (1), (2) and (3) are equivalent and (1), (2), (3) imply (4). If \( C \) is a locally finitely generated category, then also (4) implies (1), (2), (3).

**Proof.** Equivalence (2) \( \Leftrightarrow \) (3) follows from Theorem 5.11.

(1) \( \Rightarrow \) (4). By assumption the ring \( A \) is right coherent. Therefore the category of right \( A \)-modules \( \text{Mod} A \) is locally coherent. By Theorem 5.11 \( \text{Zg} C \) is closed in \( \text{Zg} (\text{Mod} A) \). Our assertion would be proved if we showed that \( \text{Zg} (\text{Mod} A) \) is a closed subset of \( \text{Zg} C_A \). In view of Theorem 5.13 and Theorem 5.12 this is equivalent to the localizing subcategory \( \mathcal{P}^A \) of \( C_A \) to be of finite type. Thus we must show that \( \mathcal{P}^A = \overline{\mathcal{S}}^A \) where \( \mathcal{S}^A = \mathcal{P}^A \cap \text{coh} C_A \) is a Serre subcategory of \( \text{coh} C_A \). Clearly that \( \overline{\mathcal{S}}^A \subseteq \mathcal{P}^A \). Let us show the inverse inclusion.

Let \( F \in \mathcal{P}^A \); then there exists an exact sequence

\[
0 \rightarrow F \xrightarrow{\iota} M \otimes_A - \xrightarrow{\mu \otimes -} N \otimes_A - \xrightarrow{\nu \otimes -} L \otimes_A - \rightarrow 0,
\]

where \( M \otimes_A - = E(F), N \otimes_A - = E(\text{coker } \iota), \mu : M \rightarrow N \) is a monomorphism. By hypothesis \( A \) is right coherent, and so the exact sequence \( 0 \rightarrow M \xrightarrow{\mu} N \xrightarrow{\nu} L \rightarrow 0 \) is a direct limit of exact sequences \( 0 \rightarrow M_i \xrightarrow{\mu_i} N_i \xrightarrow{\nu_i} L_i \rightarrow 0 \) with \( M_i, N_i, L_i \in \text{mod} A \) [Kr3, Lemma 5.7]. If \( C_i = \text{Ker}(\mu_i \otimes -) \), then \( C_i \in \mathcal{S}^A \). Consider a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & C_i \\
\downarrow & & \downarrow \\
0 & \rightarrow & F \\
\end{array}
\]

where \( M \otimes_A - = E(F), N \otimes_A - = E(\text{coker } \iota), \mu : M \rightarrow N \) is a monomorphism. By hypothesis \( A \) is right coherent, and so the exact sequence \( 0 \rightarrow M \xrightarrow{\mu} N \xrightarrow{\nu} L \rightarrow 0 \) is a direct limit of exact sequences \( 0 \rightarrow M_i \xrightarrow{\mu_i} N_i \xrightarrow{\nu_i} L_i \rightarrow 0 \) with \( M_i, N_i, L_i \in \text{mod} A \) [Kr3, Lemma 5.7]. If \( C_i = \text{Ker}(\mu_i \otimes -) \), then \( C_i \in \mathcal{S}^A \). Consider a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & C_i \\
\downarrow & & \downarrow \\
0 & \rightarrow & F \\
\end{array}
\]

with \( C_i \in \mathcal{S}^A \). Consider a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & C_i \\
\downarrow & & \downarrow \\
0 & \rightarrow & C_j \\
\end{array}
\]

with \( C_i \in \mathcal{S}^A \). Consider a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & C_i \\
\downarrow & & \downarrow \\
0 & \rightarrow & C_j \\
\end{array}
\]
It is directly verified that the family \( \{ C_i, \gamma_{ij} \} \) is direct. Now, taking a direct limit on the upper row in diagram (7.1), one obtains \( F = \lim_{\to} C_i \). Thus \( F \in S^A \).

(2) \( \Rightarrow \) (1). By Theorem 5.11 the \( S \)-localization \( (h_U \otimes_A -)_S \approx U \) of the \( C \)-coherent object \( h_U \otimes_A - \) is \( C \)-coherent.

(4) \( \Rightarrow \) (2). Suppose \( C \) is a locally finitely generated category and \( Zg C = Zg C_A / S \) is a closed subset of \( Zg C_A \). By Theorem 5.13 there is a Serre subcategory \( P \) of \( coh C_A \) such that \( Zg C = Zg C_A / P \). From [GG1, Proposition 2.7] it follows that \( S = P \). Now our assertion follows from Theorem 5.12.

Corollary 7.5. Let \( Mod A \) be the module category with \( A = \{ P_i \}_{i \in I} \). Then the following assertions are equivalent:

(1) The ring \( A \) is right coherent.
(2) The localizing subcategory \( P^A \) is of finite type.
(3) The localizing subcategory \( P^A \) is of prefinite type.
(4) \( Zg(\text{Mod} A) = \{ E \otimes_A - \mid E_A \text{ is an injective right } A \text{-module} \} \) is closed in \( Zg C_A \).

It is useful to have available the following criterion of coherence of a ring.

Proposition 7.6. For a ring \( A = \{ P_i \}_{i \in I} \) the following are equivalent:

(1) \( A \) is right coherent.
(2) For any finitely presented left \( A \)-module \( M \) the right \( A \)-module \( M^* = \text{Hom}_A(M, A) \) is finitely presented.
(3) For any finitely presented left \( A \)-module \( M \) the right \( A \)-module \( M^* = \text{Hom}_A(M, A) \) is finitely generated.
(4) For any coherent object \( C \in coh C_A \) the right \( A \)-module \( C(A) \) is finitely presented.
(5) For any coherent object \( C \in coh C_A \) the right \( A \)-module \( C(A) \) is finitely generated.

Proof. By Proposition 7.1 the functor \( C_A \to Mod A, F \mapsto F(A) \), defines an equivalence of categories \( C_A / P^A \) and \( Mod A \). By the preceding Corollary \( A \) is right coherent if and only if \( P^A \) is of finite type (= of prefinite type). Because the family \( \{ (A M, -) \}_{M \in \text{mod } A^{\text{op}}} \) is a family of generators of \( C_A \), our assertion immediately follows from Theorem 5.5 and Proposition 5.6.

Corollary 7.7. For a ring \( A = \{ P_i \}_{i \in I} \) the following are equivalent:

(1) \( P^A \) is coexact.
(2) For any finitely presented left \( A \)-module \( M \) the right \( A \)-module \( M^* = \text{Hom}_A(M, A) \) is projective.

In particular, \( P^A \) is of finite type and coexact if and only if for any \( M \in \text{mod } A^{\text{op}} \) the module \( M^* = \text{Hom}_A(M, A) \) is a finitely generated projective right \( A \)-module. In that case, the ring \( A \) is right coherent.

Proof. It follows from Proposition 5.2. The second part follows from Proposition 5.6 and Proposition 7.6.

Example. Let \( C_Z \) be the category of generalized abelian groups. Then the localizing subcategory \( P^Z \) is evidently of finite type and coexact.
Now we intend to describe localizing subcategories of prefinite type in locally finitely generated Grothendieck categories in terms of localizing subcategories of finite type of the category $C_A$. Let $S^A$ be a Serre subcategory $P^A \cap \text{coh} C_A$ of $C_A$.

**Proposition 7.8.** For a localizing subcategory $S$ of $\text{Mod} A$, $A = \{P_i\}_{i \in I}$, the following assertions are equivalent:

1. $S$ is of prefinite type.
2. There is a Serre subcategory $T$ of $\text{coh} C_A$ such that $T \supseteq S^A$ and $\bar{T}(A) = S$.

**Proof.** Denote by $P$ a localizing subcategory of $C_A$ such that $P \supseteq P^A$ and $P(A) = S$ (see Proposition 1.6). Let $T = P \cap \text{coh} C_A$; then $T$ is a Serre subcategory of $\text{coh} C_A$. Clearly, $T \supseteq S^A$. Obviously that $\bar{T} \subseteq P$. Hence $\bar{T}(A) \subseteq S$. Let us show the inverse inclusion.

Let $\mathfrak{S} = \{S^P\}_{P \in A}$ be a Gabriel topology corresponding to $S$. Our assertion would be proved, if we showed that $\mathfrak{S}$ has a basis consisting of those ideals $a$ of $P$ such that $a = b(A)$, where $b$ is a coherent subobject of $P \otimes_A -$ such that $(P \otimes_A -)/b \in T$.

So let $a \in \mathfrak{S}^P$. Consider the following exact sequence

$$0 \longrightarrow \text{Ker}(\alpha \otimes -) \longrightarrow a \otimes_A - \xrightarrow{\alpha \otimes -} P \otimes_A -.$$

Because $\alpha$ is a monomorphism, $\text{Ker}(\alpha \otimes -) \in P^A$. Let $\tilde{a} = \text{Im}(\alpha \otimes -)$; then $\tilde{a}(A) = a$ and $(P \otimes_A -)/\tilde{a} \in P$. Write $\tilde{a} = \sum_{i \in I} a_i$ as a direct union of finitely generated subobjects $a_i$ of $\tilde{a}$. Because each $a_i$ is a subobject of the $C_A$-coherent object $P \otimes_A -$, it follows that $a_i$ is coherent. One has

$$P_S = a_S = (\tilde{a}(A))_S = \sum_{i \in I} (a_i(A))_S.$$

By Theorem 5.5 the object $P_S \in \text{fg}(\text{Mod} A/S)$, and so there is a finite subset $J$ of $I$ such that $P_S = \sum_{i \in J} (a_i(A))_S$. Let $b = \sum_{i \in J} a_i$; then $(P \otimes_A -)/b \in P$ and since $(P \otimes_A -)/b$ is a coherent object, one has $(P \otimes_A -)/b \in T$. Hence $b(A) = a$ as was to be proved. \hfill \Box

Now let us consider a locally finitely generated Grothendieck category $C$. By Theorem 7.8 there is a localizing subcategory $S$ of $C_A$ such that $C$ is equivalent to $C_A/S$ with $A = \{h_U\}_{U \in \mathcal{U}}$. We consider this equivalence as identification. If $Q$ and $P$ are localizing subcategories of $C_A$, by $Q_P$ denote a subcategory of $C_A/P$ which consist of $\{Q_P \mid Q \in Q\}$. Denote by $\mathcal{L}$ a Serre subcategory $S \cap \text{coh} C_A$ of $\text{coh} C_A$.

**Proposition 7.9.** Let $Q$ be a localizing subcategory of a locally finitely generated Grothendieck category $C$ with $\mathcal{U}$ a family of generators of $C$. Let $A = \{h_U\}_{U \in \mathcal{U}}$ be a ring generated by $\mathcal{U}$. Then the following assertions are equivalent:

1. $Q$ is of prefinite type.
2. There is a Serre subcategory $T$ of $\text{coh} C_A$ such that $T \supseteq \mathcal{L}$ and $\bar{T}(A) = Q$.

**Proof.** By Theorem 4.3 there is a localizing subcategory $P$ of $\text{Mod} A$ such that $C$ is equivalent to $\text{Mod} A/P$ and by Proposition 1.6 there is a localizing subcategory $V$ of $\text{Mod} A$ such that $V \supseteq P$ and $V/P = Q$. Since both $P$ and $Q$ are of prefinite type, it follows that $V$ is of prefinite type. From the preceding Proposition it follows that $V = \bar{T}(A)$ for some localizing subcategory of finite type $\bar{T}$ of $C_A$. Then $Q = V_P = (\bar{T}(A))_P = \bar{T}_S$. \hfill \Box
**Question.** Is it true that if \( \overline{\mathcal{T}} \) is a localizing subcategory of finite type in \( \mathcal{C}_A \) containing the subcategory \( \overline{\mathcal{S}}^A \), then the subcategory \( \overline{\mathcal{T}}(\mathcal{A}) \) is localizing and of prefinite type in \( \text{Mod} \mathcal{A} \)? If this was true, we could define the Ziegler topology of an arbitrary locally finitely generated Grothendieck category.

8. Absolutely pure and flat modules

In this section we sketch how classes of absolutely pure, flat \( \mathcal{A} \)-modules may be studied with the help of some torsion/localization functors in the category \( \mathcal{C}_A \).

**Definitions.** (1) Let \( \mathcal{C} \) be a locally finitely presented Grothendieck category. An object \( C \in \mathcal{C} \) is said to be **absolutely pure** (or \( \text{FP} \)-injective) if \( \text{Ext}^1_C(X, C) = 0 \) for any \( X \in \text{fp} \mathcal{C} \). An object \( C \in \mathcal{C} \) is called \( \text{fp} \)-injective if for any monomorphism \( \mu : X \to Y \) in \( \text{fp} \mathcal{C} \) the morphism \( C(\mu, C) \) is an epimorphism. Evidently that every absolutely pure object is \( \text{fp} \)-injective and every \( \text{fp} \)-injective finitely presented object is absolutely pure. The ring \( \mathcal{A} = \{P_i\}_{i \in I} \) is **right absolutely pure** if every right \( \mathcal{A} \)-module \( P \in \mathcal{A} \) is absolutely pure.

(2) Let \( \text{Mod} \mathcal{A}, \mathcal{A} = \{P_i\}_{i \in I} \), be the category of right \( \mathcal{A} \)-modules. A module \( M \in \text{Mod} \mathcal{A} \) is **flat** if the tensor functor \( M \otimes - \) is exact. \( M \) is called \( \text{fp} \)-flat if for any monomorphism \( \mu : \mathcal{A}K \to \mathcal{A}L \) in \( \text{mod} \mathcal{A}^{\text{op}} \) the morphism \( M \otimes \mathcal{A} \mu \) is a monomorphism. Evidently that every flat module is \( \text{fp} \)-flat.

**Remark.** Some authors refer to absolutely pure objects as \( \text{fp} \)-injective. Therefore, to make not terminological displacements, every time we refer to somebody the reader should make more precise the terminology.

One easily verifies:

**Lemma 8.1.** Let \( \mathcal{C} \) be a locally finitely presented Grothendieck category. The following statements are equivalent for an object \( C \in \mathcal{C} \):

1. \( C \) is absolutely pure.
2. Every exact sequence \( 0 \to C \to C' \to C'' \to 0 \) is pure.
3. There exists a pure-exact sequence \( 0 \to C \to C' \to C'' \to 0 \) with \( C' \) absolutely pure.

We shall adhere to the following notation:

\[
\mathcal{P}^A = \{F \in \mathcal{C}_A \mid F(\mathcal{A}) = 0\}
\]

\[
\mathcal{S}^A = \{C \in \text{coh} \mathcal{C}_A \mid C(\mathcal{A}) = 0\}
\]

\[
\mathcal{S}_A = \{C \in \text{coh} \mathcal{C}_A \mid (C, P \otimes \mathcal{A} -) = 0 \text{ for all } P \in \mathcal{A}\}.
\]

Subcategories \( \mathcal{A} \mathcal{S} \) and \( \mathcal{A}_\mathcal{S} \) of \( \text{coh} \mathcal{A} \mathcal{C} \) are defined similar to subcategories \( \mathcal{S}^A \) and \( \mathcal{S}_A \) respectively. By Theorem 5.12 \( \overline{\mathcal{S}}^A \) and \( \overline{\mathcal{S}}_A \) are localizing subcategories of finite type. By Corollary 7.5 \( \mathcal{P}^A = \overline{\mathcal{S}}^A \) if and only if the ring \( \mathcal{A} \) is right coherent. If we consider presentations \( \{6,3\} \), it is easily seen that

\[
\mathcal{S}^A = \{\text{Ker}(\mu \otimes -) \mid \mu : M \to N \text{ is a monomorphism in } \text{mod} \mathcal{A}\}.
\]
Similarly,

\[ S_A = \{ \text{Coker}(\mu, -) \mid \mu : L \to K \text{ is a monomorphism in mod } \mathcal{A}^{\text{op}} \}. \]

**Proposition 8.2.** Let \( K \in \text{Mod } \mathcal{A} \); the following assertions hold:

1. \( K \) is absolutely pure if and only if the functor \( K \otimes_{\mathcal{A}} - \) is \( \mathcal{P}^A \)-torsionfree.
2. \( K \) is \( \text{fp-injective} \) if and only if the functor \( K \otimes_{\mathcal{A}} - \) is \( \mathcal{S}^A \)-torsionfree.
3. \( K \) is \( \text{fp-flat} \) if and only if the functor \( K \otimes_{\mathcal{A}} - \) is \( \mathcal{S}^A \)-torsionfree.

**Proof.** Adapt the proof for modules over a ring \( \mathcal{A} = \{ A \} \) [CG2, Proposition 2.2]. \( \Box \)

**Corollary 8.3.** The set of indecomposable pure-injective \( \text{fp-injective} \) (\( \text{fp-flat} \)) modules is closed in \( Zg \mathcal{C}_A \).

It is well-known (see e.g. [Stm]) that for a ring \( A \) a right \( A \)-module \( M \) is finitely presented (finitely generated) if and only if the natural map \( M \otimes_{\mathcal{A}} (\prod_{i \in I} N_i) \to \prod_{i \in I}(M \otimes_{\mathcal{A}} N_i) \) is an isomorphism (epimorphism) for every family \( \{ N_i \}_{i \in I} \) of right \( A \)-modules. This generalises to arbitrary module categories \( \text{Mod } \mathcal{A} \) as follows.

**Lemma 8.4.** [Kr3, Lemma 7.1] Let \( \mathcal{A} \) be a ring. For \( M \in \text{Mod } \mathcal{A} \) the following are equivalent:

1. \( M \) is finitely presented (finitely generated).
2. The natural morphism \( M \otimes_{\mathcal{A}} (\prod_{i \in I} N_i) \to \prod_{i \in I}(M \otimes_{\mathcal{A}} N_i) \) is an isomorphism (epimorphism) for every family \( \{ N_i \}_{i \in I} \) in \( \text{Mod } \mathcal{A} \).
3. The natural morphism \( M \otimes_{\mathcal{A}} (\prod_{i \in I} P_i) \to \prod_{i \in I}(M \otimes_{\mathcal{A}} P_i) = \prod_{i \in I} M(P_i) \) is an isomorphism (epimorphism) for every family \( \{ P_i \}_{i \in I} \) in \( \mathcal{A} \).

Now the next result may be proved similar to [CG2, Proposition 2.3].

**Proposition 8.5.** Let \( \mathcal{A} = \{ P_i \}_{i \in I} \) be a ring; then:

1. For every family of right \( \mathcal{A} \)-modules \( \{ M_i \}_{i \in I} \) the module \( \prod_I M_i \) is absolutely pure (respectively \( \text{fp-injective} \), \( \text{fp-flat} \)) if and only if every \( M_i \) is absolutely pure (respectively \( \text{fp-injective} \), \( \text{fp-flat} \)).
2. The direct limit \( \lim_{\to} M_i \) of \( \text{fp-injective} \) (respectively \( \text{fp-flat} \)) right \( \mathcal{A} \)-modules \( M_i \) is an \( \text{fp-injective} \) (respectively \( \text{fp-flat} \)) module.

Let us consider now a locally finitely presented Grothendieck category \( \mathcal{C} \) with the family of generators \( \mathcal{U} \subseteq \text{fp } \mathcal{C} \). As usual, consider the category of modules \( \text{Mod } \mathcal{A} \) with \( \mathcal{A} = \{ h_U \}_{U \in \mathcal{U}} \). By Theorem [1.1] \( \mathcal{C} \) is equivalent to the quotient category \( \text{Mod } \mathcal{A}/\mathcal{S} \). Furthermore, by Proposition 5.6 \( \mathcal{S} \) is of finite type. By Theorem 7.2 there is a localizing subcategory \( \mathcal{P} \) of \( \mathcal{C}_A \) such that \( \mathcal{C} \) is equivalent to the quotient category \( \mathcal{C}_A/\mathcal{P} \). Similar to the category of modules, absolutely pure/\( \text{fp-injective} \) objects of \( \mathcal{C} \) can be described in terms of torsion functors in \( \mathcal{C}_A \). To begin, let us prove the following.

**Proposition 8.6.** For an object \( C \in \mathcal{C} \) the following assertions hold:

1. \( C \) is absolutely pure if and only if it is absolutely pure as a right \( \mathcal{A} \)-module.
2. \( C \) is \( \text{fp-injective} \) if and only if it is \( \text{fp-injective} \) as a right \( \mathcal{A} \)-module.
Proof. (1). Let $C$ be a $C$-absolutely pure object and $M \in \text{mod} \mathcal{A}$. We must show that $\text{Ext}^1_{\mathcal{A}}(M, C) = 0$. Equivalently, any short exact sequence

$$0 \longrightarrow C \stackrel{\alpha}{\longrightarrow} X \longrightarrow M \longrightarrow 0$$

of right $\mathcal{A}$-modules splits. By Proposition 5.6 $M_S \in \text{fp} C$. By assumption, the morphism $\alpha_S$ splits, i.e. there exists $\beta : X_S \rightarrow C$ such that $\beta \alpha_S = 1_C$. Then $(\beta \lambda_X)\alpha = \beta \alpha_S = 1_C$ where $\lambda_X$ is an $S$-envelope of $X$. So $\alpha$ splits.

Conversely, let $C$ be an absolutely pure right $\mathcal{A}$-module and let

$$\varepsilon : 0 \longrightarrow C \stackrel{\alpha}{\longrightarrow} E \stackrel{\beta}{\longrightarrow} X \longrightarrow M \longrightarrow 0$$

be a $C$-exact sequence with $E = E(C)$ and $X = E/C$. By assumption, the short exact sequence

$$\bar{\varepsilon} : 0 \longrightarrow C \stackrel{\alpha}{\longrightarrow} E \stackrel{\beta}{\longrightarrow} \text{Im} \beta \longrightarrow 0$$

is pure-exact in $\text{Mod} \mathcal{A}$. Clearly that $\bar{\varepsilon}_S = \varepsilon$. From Proposition 6.3 it follows that $\bar{\varepsilon}$ is a direct limit of split exact sequences

$$\bar{\varepsilon}_i : 0 \longrightarrow C_i \longrightarrow E_i \longrightarrow M_i \longrightarrow 0$$

in $\text{Mod} \mathcal{A}$. Then $\varepsilon$ is a direct limit of split exact sequences $\varepsilon_i = (\bar{\varepsilon}_i)_S$. Thus $C$ is $C$-absolutely pure.

(2). Suppose $C$ is an $fp$-injective object in $\mathcal{C}$ and $\mu : M \rightarrow N$ is a monomorphism in $\text{mod} \mathcal{A}$. Since $S$ is of finite type, the morphism $\mu_S$ is a monomorphism in $\text{fp} \mathcal{C}$. Consider the commutative diagram

$$
\begin{array}{ccc}
\text{Hom}_\mathcal{A}(N, C) & \stackrel{(\mu, C)}{\longrightarrow} & \text{Hom}_\mathcal{A}(M, C) \\
\downarrow & & \downarrow \\
\text{c}(N_S, C) & \stackrel{(\mu_S, C)}{\longrightarrow} & \text{c}(M_S, C)
\end{array}
$$

where vertical arrows are isomorphisms. Because $(\mu_S, C)$ is an epimorphism, it follows that $(\mu, C)$ is an epimorphism.

Conversely, suppose $\mu : X \rightarrow Y$ is a monomorphism in $\text{fp} \mathcal{C}$. Then there is a monomorphism $\gamma : M \rightarrow N$ in $\text{mod} \mathcal{A}$ such that $\gamma_S = \mu$. Indeed, we can embed $\mu$ into the commutative diagram in $\mathcal{C}$ with exact rows:

$$
\begin{array}{ccc}
(-, \oplus_{i=1}^n U_i) & \stackrel{\psi}{\longrightarrow} & (-, \oplus_{j=1}^m U_j) \longrightarrow X \longrightarrow 0 \\
\downarrow & & \downarrow \\
(-, \oplus_{k=1}^s U_k) & \stackrel{\varphi}{\longrightarrow} & (-, \oplus_{l=1}^t U_l) \longrightarrow Y \longrightarrow 0.
\end{array}
$$

Because each $\oplus U_i$ is $S$-closed and finitely generated projective in $\text{Mod} \mathcal{A}$, both $\text{Coker} \psi$ and $\text{Coker} \varphi$ are finitely presented right $\mathcal{A}$-modules. We put $M = \text{Coker} \psi$ and $N = \text{Coker} \varphi$. There is a unique morphism $\gamma : M \rightarrow N$. Since $M_S = X$ and $N_S = Y$, it
follows that $\gamma_S = \mu$. Consider the commutative diagram

$$
\begin{array}{ccc}
\text{Hom}_A(N, C) & \xrightarrow{(\gamma, C)} & \text{Hom}_A(M, C) \\
\downarrow & & \downarrow \\
c(Y, C) & \xrightarrow{(\mu, C)} & c(X, C)
\end{array}
$$

where vertical arrows are isomorphisms. Because $(\gamma, C)$ is an epimorphism, it follows that $(\mu, C)$ is an epimorphism. So $C$ is $fp$-injective in $\mathcal{C}$.

**Corollary 8.7.** The ring $\mathcal{A} = \{h_U\}_{U \in \mathcal{U}}$ is right absolutely pure if and only if each $U \in \mathcal{U}$ is an absolutely pure object in $\mathcal{C}$.

**Proof.** It suffices to note that each $h_U$ is $S$-closed (see Theorem 4.1) and then apply the preceding Proposition.

Denote by $T = \text{coh} \mathcal{C} \cap \mathcal{P}$ and let $t_\mathcal{P}$ and $t_\mathcal{T}$ be the torsion functors corresponding to the localizing subcategories $\mathcal{P}$ and $\mathcal{T}$ of $\mathcal{C}_\mathcal{A}$.

**Proposition 8.8.** Let $C \in \mathcal{C}$; the following assertions hold:

1. $C$ is an absolutely pure object of $\mathcal{C}$ if and only if $t_\mathcal{P}(C \otimes \mathcal{A} -) = 0$.
2. $C$ is an $fp$-injective object of $\mathcal{C}$ if and only if $t_\mathcal{T}(C \otimes \mathcal{A} -) = 0$.

**Proof.** (1). Let $C$ be absolutely pure. By the preceding Proposition it is an absolutely pure right $\mathcal{A}$-module. Now let $E$ be an injective envelope of $C$. Then $C \otimes \mathcal{A} -$ is a subobject of $E \otimes \mathcal{A} -$. Because $E \otimes \mathcal{A} -$ is $\mathcal{P}$-torsionfree, it follows that $C \otimes \mathcal{A} -$ is $\mathcal{P}$-torsionfree. Conversely, since $\mathcal{P} \supseteq \mathcal{P}^\mathcal{A}$, our assertion follows from Propositions 8.2 and 8.6.

(2). Let $C$ be $fp$-injective and $T \in \mathcal{T}$. Consider exact sequence (6.3)

$$0 \to T \to M \otimes \mathcal{A} \xrightarrow{\mu \otimes -} N \otimes \mathcal{A} -$$

where $M, N \in \text{mod} \mathcal{A}$. Because $0 = T_\mathcal{P} = T(\mathcal{A})_S$, it follows that the morphism $\mu_S$ is a monomorphism in $fp \mathcal{C}$. Consequently, the morphism $(\mu_S, C)$ is an epimorphism, and so the morphism $(\mu, C)$ is an epimorphism too. As $C \otimes \mathcal{A} -$ is a $\text{coh} \mathcal{C}_\mathcal{A}$-injective object, one has an exact sequence

$$(N \otimes \mathcal{A} -, C \otimes \mathcal{A} -) \xrightarrow{(\mu \otimes -, C \otimes \mathcal{A} -)} (M \otimes \mathcal{A} -, C \otimes \mathcal{A} -) \to (T, C \otimes \mathcal{A} -) \to 0.$$

But $(\mu, C)$ is an epimorphism, hence $(T, C \otimes \mathcal{A} -) = 0$. So $t_\mathcal{T}(C \otimes \mathcal{A} -) = 0$. Because $T \supseteq \mathcal{S}^\mathcal{A}$, the converse follows from the preceding Proposition and Proposition 8.2.

For a ring $A$ the Chase Theorem asserts that $A$ is left coherent if and only if any direct product $\prod M_i$ of flat right $A$-modules $M_i$ is flat. This generalizes to arbitrary module category as follows.

**Proposition 8.9 (Chase).** Let $\mathcal{A} = \{P_i\}_{i \in I}$ be a ring. Then the following are equivalent:

1. $\mathcal{A}$ is left coherent.
2. Every product of flat right $\mathcal{A}$-modules is flat.
3. Every product $\prod_{j \in J} P_j$ of $P_j \in \mathcal{A}$ is a flat right $\mathcal{A}$-module for every set $J$. 
Proof. (1) ⇒ (2): Let \(\{M_j\}_{j \in J}\) be the family of flat right \(A\)-modules. By Proposition 8.3 the module \(\prod_{j \in J} M_j\) is \(fp\)-flat. Let \(\varphi : K \to L\) be a monomorphism in \(\text{Mod}\, A^{\text{op}}\). As \(A\) is left coherent, it follows that \(\varphi = \lim \varphi_i\) is a direct limit of monomorphisms \(\varphi_i\) in \(\text{mod}\, A^{\text{op}}\) [Kr3, Lemma 5.9]. Then the morphism \(\varphi \otimes \prod M_j = \lim (\varphi_i \otimes \prod M_j)\) is a direct limit of monomorphisms \(\varphi_i \otimes \prod M_j\). So it is a monomorphism too.

(2) ⇒ (3) is trivial.

(3) ⇒ (1): Let \(\_A K\) be a finitely generated submodule of finitely presented module \(\_A L\). For each index set \(J\) we have a commutative diagram

\[
\begin{array}{ccc}
K \otimes_A \prod P_j & \longrightarrow & L \otimes_A \prod P_j \\
\varphi_K \downarrow & & \varphi_L \downarrow \\
\prod K(P_j) & \longrightarrow & \prod L(P_j)
\end{array}
\]

where the horizontal arrows are monomorphisms. Since \(\varphi_L\) is a monomorphism by Lemma 8.4, also \(\varphi_K\) is a monomorphism. Thus \(K\) is finitely presented by Lemma 8.4.

In contrast to absolutely pure right \(A\)-modules a class of flat right \(A\)-modules is realized in \(C_A\) as a class of those \(M \otimes_A -\) for which \(t_S(M \otimes_A -) = 0\) for some localizing subcategory \(S\) of \(C_A\) if and only if \(A\) is left coherent.

**Theorem 8.10.** For a ring \(A\) the following assertions are equivalent:

1. \(A\) is left coherent.
2. In \(C_A\) there is a localizing subcategory \(S\) such that any right \(A\)-module \(M\) is flat if and only if the functor \(M \otimes_A -\) is \(S\)-torsionfree.
3. Every left \(fp\)-injective \(A\)-module is absolutely pure.
4. Every right \(fp\)-flat \(A\)-module is flat.
5. A direct limit of absolutely pure left \(A\)-modules is absolutely pure.

Proof. (1) ⇔ (5): It follows from [Kr3, Lemma 9.3]. The rest is proved similar to [GG2, Theorem 2.4].

**Theorem 8.11.** For a ring \(A\) the following conditions are equivalent:

1. \(A\) is right absolutely pure.
2. \(S^A \subseteq S_A\).
3. \(_A S \subseteq _A S\).
4. Every \(fp\)-flat right \(A\)-module is \(fp\)-injective.
5. Every indecomposable pure-injective \(fp\)-flat right \(A\)-module is \(fp\)-injective.
6. Every pure-injective \(fp\)-flat right \(A\)-module is \(fp\)-injective.
7. Every \(fp\)-injective left \(A\)-module is \(fp\)-flat.
8. Every indecomposable pure-injective \(fp\)-injective left \(A\)-module is \(fp\)-flat.
9. Every pure-injective \(fp\)-injective left \(A\)-module is \(fp\)-flat.

Proof. Adapt the proof for modules over a ring (see [GG2, Theorem 2.5]).

**Example.** Let \(A = \{A\}\) be a ring and \(C_A\) the category of generalized right \(A\)-modules. Then the ring \(B = \{(M, -)\}_{M \in \text{Mod}\, A^{\text{op}}}\) is right absolutely pure if and only if \(A\) is (von Neumann) regular.
Indeed, let $\mathcal{B}$ be right absolutely pure; then each $(K, -)$ with $K \in \text{mod } A^{\text{op}}$ is\text{coh} $\mathcal{C}_A$-injective. Therefore $(K, -)$ is isomorphic to the object $K^* \otimes_A -$ where $K^* = \text{Hom}_{A}(K, A).$ Since every coherent object $C \in \text{coh} \mathcal{C}_A$ is $\text{coh} \mathcal{C}_A$-injective, and hence $A$ is a regular ring [Hrz, Theorem 4.4].

The converse follows from [Hrz, Theorem 4.4].

To conclude, we shall give a criterion of a duality for categories of finitely presented left and right $A$-modules. The ring $A$ over which the functor $\text{Hom}_A(-, A)$ constitutes a duality of indicated categories one calls \textit{weakly Quasi-Frobenius}. For the case $A = \{A\}$ with $A$ a ring, we refer the reader to [GG2].

**Theorem 8.12.** For a ring $A = \{P\}_{i \in I}$ the following assertions are equivalent:

1. $A$ is \textit{weakly Quasi-Frobenius}.
2. $A$ is (left and right) absolutely pure and (left and right) coherent.
3. Classes of flat right $A$-modules and absolutely pure right $A$-modules coincide.
4. $A$ is left absolutely pure and left coherent and any flat right $A$-module is absolutely pure.
5. $A$ is right absolutely pure and right coherent, and any absolutely pure right $A$-module is flat.

And also assertions $(3') - (5')$, obtained from $(3) - (5)$ respectively by substituting the word “right” for “left” and vice versa.

**Proof.** (1) $\Rightarrow$ (2): By assumption, given a finitely presented left $A$-module $M$, the right $A$-module $M^* = \text{Hom}_A(M, A)$ is finitely presented. From Proposition 7.6 it follows that $A$ is right coherent. Symmetrically, $A$ is left coherent. Because the functor $\text{Hom}_A(-, P)$ where $P \in A$ is exact both on $\text{mod } A^{\text{op}}$ and on $\text{mod } A$, it follows that $P$ is an absolutely pure both left and right $A$-module. So $A$ is an (two-sided) absolutely pure ring.

(2) $\Rightarrow$ (1): Since $A$ is a (two-sided) coherent ring, from Proposition 7.5 it follows that $\mathcal{P}^A = \mathcal{S}^A$, and hence there is an equivalence of categories $\text{mod } A$ and $\text{coh} \mathcal{C}_A/\mathcal{S}^A$. Similarly, there is an equivalence of categories $\text{mod } A^{\text{op}}$ and $\text{coh} \mathcal{C}/\mathcal{A} \mathcal{S}$. In view of Theorem 8.11 we have the following relations:

$$\mathcal{S}^A = D(\mathcal{A}S) = D(\mathcal{A}S)$$

Now our assertion follows from Theorem 8.8.

(2) $\Rightarrow$ (3), (2) $\Rightarrow$ (4): Apply Theorems 8.10 and 8.11.

(3) $\Rightarrow$ (5): It suffices to show that $A$ is right coherent. To see this, consider a direct system of absolutely pure right $A$-modules $\{M_i\}_{i \in I}$. Since each $M_i$, by assumption, is flat, it follows that the module $\lim \rightarrow M_i$ is flat, and so it is absolutely pure. Therefore $A$ is right coherent by Theorem 8.10.

(4) $\Rightarrow$ (3): By Theorem 8.11 any absolutely pure right $A$-module is $fp$-flat, and hence flat by Theorem 8.10.
(5) ⇒ (2): Since the ring \( A \) is right absolutely pure, the module \( \prod_j P_j \), where \( P_j \in A \) and \( J \) is some set of indices, is also absolutely pure, and therefore it is flat. In view of Proposition 8.9, \( A \) is left coherent. By Theorem 8.10 any \( fp \)-injective right \( R \)-module is absolutely pure, and hence flat. From Theorem 8.11 it follows that \( A \) is left absolutely pure. □

References

[Aus1] M. Auslander, ‘Coherent functors’, in Proc. Conf. on Categorical Algebra (La Jolla, 1965), Springer, 1966, 189-231.

[Aus2] M. Auslander, ‘A functorial approach to representation theory’, Lect. Notes Math. 944 (1982), 105-179.

[Aus3] M. Auslander, ‘Isolated singularities and almost split sequences’, Lect. Notes Math. 1178 (1986), 194-242.

[Br] S. Breitsprecher, ‘Lokal endlich präsentierbare Grothendieck-Kategorien’, Mitt. Math. Sem. Giessen 85 (1970), 1-25.

[BD] I. Bucur, A. Deleanu, Introduction to the theory of categories and functors, Wiley, London, 1968.

[Bur] K. Burke, ‘Some Model-Theoretic Properties of Functor Categories for Modules’, Doctoral Thesis, University of Manchester, 1994.

[Fa] C. Faith, Algebra: rings, modules and categories, vol. 1, Mir, Moscow, 1977 (in Russian).

[Fr] P. Freyd, Abelian categories, Harper and Row, New-York, 1964.

[Gbl] P. Gabriel, ‘Des catégories abéliennes’, Bull. Soc. Math. France 90 (1962), 323-448.

[GG1] G. A. Garkusha, A. I. Generalov, ‘Grothendieck categories as quotient categories of \( (R \text{-mod, Ab}) \)’, Fund. Prikl. Mat., to appear (in Russian).

[GG2] G. A. Garkusha, A. I. Generalov, ‘Duality for categories of finitely presented modules’, Algebra i Analiz, to appear (in Russian).

[Grk] A. Grothendieck, ‘Sur quelques points d’algèbre homologique’, Tohoku Math. J. 9 (1957), 119-221.

[GJ] L. Gruson, C. U. Jensen, ‘Dimensions cohomologiques reliées aux foncteurs \( \lim^{(i)} \)’, Lect. Notes Math. 867 (1981), 234-294.

[Her] I. Herzog, ‘The Ziegler spectrum of a locally coherent Grothendieck category’, Proc. London Math. Soc. 47 (1997), 503-558.

[JL] C. U. Jensen, M. Lenzing, Model theoretic algebra, Logic and its Applications 2, Gordon and Breach, New York, 1989.

[Kap] I. Kaplansky, Infinite abelian groups, Ann Arbor, University of Michigan Press, 1969.

[Kr1] H. Krause, ‘The spectrum of a locally coherent category’, J. Pure Appl. Algebra 114 (1997), 259-271.

[Kr2] H. Krause, ‘The spectrum of a module category’, Habilitationsschrift, Universität Bielefeld, 1998.

[Kr3] H. Krause, ‘Functors on locally finitely presented additive categories’, Colloq. Math. 75 (1998), 105-132.

[Laz] D. Lazard, ‘Autour de la platitude’, Bull. Soc. Math. France 97 (1969), 81-128.

[PG] N. Popescu, P. Gabriel, ‘Caractérisation des catégories abéliennes avec générateurs et limites inductives exactes’, C. R. Acad. Sc. Paris 258 (1964), 4188-4190.

[Pop] N. Popescu, Abelian categories with applications to rings and modules, Academic Press, London and New-York, 1973.

[Pr1] M. Prest, ‘Elementary torsion theories and locally finitely presented Abelian categories’, J. Pure Appl. Algebra 18 (1980), 205-212.

[Pr2] M. Prest, ‘Epimorphisms of rings, interpretations of modules and strictly wild algebras’, Comm. Algebra 24 (1996), 517-531.
[Pr3] M. Prest, ‘The Zariski spectrum of the category of finitely presented modules’, *preprint, University of Manchester*, 1998.

[PRZ] M. Prest, Ph. Rothmaler, M. Ziegler, ‘Absolutely pure and flat modules and “indiscrete” rings’, *J. Algebra* 174 (1995), 349-372.

[Rs] J.-E. Roos, ‘Locally noetherian categories’, *Lect. Notes Math.* 92 (1969), 197-277.

[Stm] B. Stenström, *Rings of quotients*, Springer-Verlag, New York and Heidelberg, 1975.

[Zgr] M. Ziegler, ‘Model theory of modules’, *Ann. Pure Appl. Logic* 26 (1984), 149-213.

Saint-Petersburg State University, Higher Algebra and Number Theory Department, Faculty of Mathematics and Mechanics, Bibliotechnaya Sq. 2, 198904, Russia

*E-mail address: ggarkusha@hotmail.com*