ON A THEOREM BY JUHÁSZ AND SZENTMIKLÓSSY

PAOLO LIPPARINI

Abstract. We extend a theorem by Juhász and Szentmiklóssy to notions related to pseudocompactness. We also allow the case when one of the cardinals under consideration is singular.

We give an application to the study of decomposable ultrafilters: if \( \kappa \) is singular, \( D \) is a uniform ultrafilter over \( \kappa^+ \), and \( D' \) is a uniform ultrafilter over cf \( \kappa \), then \( D' \times D \) is \( \kappa \)-decomposable.

1. Introduction

Juhász and Szentmiklóssy [JS] introduced the principle \( \Phi(\mu, \kappa, \lambda) \), and used it to get some “interpolation” theorems for \( \kappa \)-compactness of topological spaces; in more detail, they proved that if \( \Phi(\mu, \kappa, \lambda) \) holds and a space is both \( \mu \)-compact and \( \lambda \)-compact, then it is also \( \kappa \)-compact. They also obtained many further interesting consequences.

In what follows \( \mu, \kappa \) and \( \lambda \) shall always assumed to be infinite cardinals; as usual, \( [S]^{<\nu} = \{ Z \subseteq S \mid |Z| < \nu \} \); similarly, \( [S]^\nu = \{ Z \subseteq S \mid |Z| = \nu \} \). The principle \( \Phi(\mu, \kappa, \lambda) \) introduced by Juhász and Szentmiklóssy is the assertion that there exists a family \( \{ S_\xi \mid \xi < \lambda \} \subseteq [\kappa]^{\mu} \) such that \( |\{ \xi < \lambda \mid |A \cap S_\xi| = \mu \}| < \lambda \), for every \( A \in [\kappa]^{<\kappa} \). Notice that Juhász and Szentmiklóssy assumed that \( \lambda \) is regular in the above definition; we shall not need this assumption.

In Section 2 we show, by a small modification of Juhász and Szentmiklóssy argument, that, when \( \lambda \) is not assumed to be regular, the above mentioned result still holds under the additional assumption of cf \( \lambda \)-compactness. Perhaps more significantly, we present a version which deals with properties connected with pseudocompactness.

In Section 3 we show that if \( \Phi(\mu, \kappa, \lambda) \) holds, \( D' \) is a uniform ultrafilter over \( \mu \), and \( D \) is a uniform ultrafilter over \( \lambda \), then \( D \times D' \) is \( \kappa \)-decomposable, together with some related results. We expect to

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be able to present even deeper connections among Juhász and Szentmiklóssy principle, and regularity and decomposability of ultrafilters in the nearest future.

2. $\kappa$-COMPACTNESS RELATIVE TO SOME FAMILY $\mathcal{F}$

First, some definitions are needed. We shall consider a topological space $X$ together with a family $\mathcal{F}$ of nonempty subsets of $X$. This is done in order to treat simultaneously the following two main cases: $\mathcal{F} = \mathcal{S}$, the set of all singletons of $X$, in which case we get notions and theorems related to compactness, $\kappa$-compactness, . . . , and the case when $\mathcal{F} = \mathcal{O}$, the set of all nonempty open subsets of $X$, in which case we get notions related to pseudocompactness. At first reading, the reader is advised to consider only the above examples (or the example he or she is more interested in).

If $X$ is a topological space, and $\mathcal{Y}$ is an infinite subset of $\mathcal{P}(X)$ we say that $x \in X$ is a complete accumulation point of $\mathcal{Y}$ if $|\{Y \in \mathcal{Y} \mid Y \cap U \neq \emptyset\}| = |\mathcal{Y}|$, for every neighborhood $U$ of $x$ in $X$. When all the members of $\mathcal{Y}$ are singletons, we get the usual notion of a complete accumulation point of a subset $Y$ of $X$ (where, in this case $Y = \bigcup Y$). If $\mathcal{F} \subseteq \mathcal{P}(X)$, we say that $X$ is $\mathcal{F}$-$\lambda$-compact if every subfamily $Y \subseteq \mathcal{F}$ of cardinality $\lambda$ has a complete accumulation point. When $\mathcal{F} = \mathcal{S}$, this is called $\lambda$-compactness, and means that every subset of $X$ of cardinality $\lambda$ has a complete accumulation point. The other interesting case is when $\mathcal{F} = \mathcal{O}$. Just to mention an example, for Tychonoff spaces, $\mathcal{O}$-$\omega$-compactness in the above sense is equivalent to pseudocompactness. We refer to [L], in particular, Section 3, for more details about $\mathcal{F}$-$\lambda$-compactness (called there $\mathcal{F}$-CAP$_\lambda$), equivalent formulations, related notions, references to the literature as well as references to alternative terminology used in the literature. Throughout the present section, we shall assume that $X$ is some fixed topological space, and $\mathcal{F}$ is a fixed family such that $\emptyset \neq \mathcal{F} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$.

**Theorem 2.1.** Suppose that $\Phi(\mu, \kappa, \lambda)$ holds, that $X$ is $\mathcal{F}$-$\mu$-compact and $\lambda$-compact and, in case $\lambda$ is singular, suppose further that $X$ is cf $\lambda$-compact. Then $X$ is $\mathcal{F}$-$\kappa$-compact.

**Proof.** The proof follows essentially the lines of the proof of [JS, Theorem 2] with a small variation in the case $\lambda$ singular. Let $\mathcal{Y} \subseteq \mathcal{F}$ be a family of cardinality $\kappa$ and enumerate it as $\{Y_\alpha \mid \alpha \in \kappa\}$ with all the $Y_\alpha$'s distinct. Let $\{S_\xi \mid \xi < \lambda\} \subseteq [\kappa]^\mu$ be given by $\Phi(\mu, \kappa, \lambda)$. By $\mathcal{F}$-$\mu$-compactness, for every $\xi < \lambda$, the family $\{Y_\alpha \mid \alpha \in S_\xi\}$ (of cardinality $\mu$) has a complete accumulation point $p_\xi$, that is, $|\{\alpha \in S_\xi \mid Y_\alpha \cap U \neq \emptyset\}| = \mu$, for every neighborhood $U$ of $p_\xi$. 
If $\lambda$ is regular, there are two cases: (1) there is $p \in X$ such that $|\{\xi < \lambda \mid p_\xi = p\}| = \lambda$; (2) $|\{p_\xi \mid \xi < \lambda\}| = \lambda$. In this latter case, choose some complete accumulation point $p$ of $\{p_\xi \mid \xi < \lambda\}$; the existence of such a $p$ is guaranteed by $\lambda$-compactness. If $\lambda$ is singular, a third case can occur: (3) for every $\lambda' < \lambda$, there is $p(\lambda') \in X$ such that $\lambda' \leq |\{\xi < \lambda \mid p_\xi = p(\lambda')\}| < \lambda$. In this case, fix an increasing sequence $(\lambda_\gamma)_{\gamma \in \text{cf}\lambda}$ cofinal in $\lambda$, and, for every $\gamma \in \text{cf}\lambda$, choose some $p_\gamma = p(\lambda_\gamma)$ as above. Then necessarily $|\{p_\gamma \mid \gamma \in \text{cf}\lambda\}| = \text{cf}\lambda$, and, by $\text{cf}\lambda$-compactness, the set $\{p_\gamma \mid \gamma \in \text{cf}\lambda\}$ has a complete accumulation point $p$.

In each of the above cases, for every neighborhood $U$ of $p$, we have that $|\{\xi < \lambda \mid p_\xi \in U\}| = \lambda$, hence $|\{\xi < \lambda \mid |A^U \cap S_\xi| = \mu\}| = \lambda$, where we have put $A^U = \{\alpha \in \kappa \mid Y_\alpha \cap U \neq \emptyset\}$. Here we have used the fact that if $p_\xi \in U$, $U$ a neighborhood of $p$, then $U$ is also a neighborhood of $p_\xi$. We show that $p$ is a complete accumulation point of $\{Y_\alpha \mid \alpha \in \kappa\}$. Suppose not. Then, for some neighborhood $U$ of $p$, $|A^U| < \kappa$. Then $\Phi(\mu, \kappa, \lambda)$ implies that $|\{\xi < \lambda \mid |A^U \cap S_\xi| = \mu\}| < \lambda$, a contradiction. □

Notice that if in the above theorem $\lambda$ is a regular cardinal, and $F$ is taken to be equal to $S$, then we get exactly the statement of [JS, Theorem 2]. The arguments in the proof of Theorem 2.1 actually give the following stronger "local" version.

**Corollary 2.2.** Suppose that $\Phi(\mu, \kappa, \lambda)$ holds, that $X$ is $\lambda$-compact and, in case $\lambda$ is singular, suppose further that $X$ is $\text{cf}\lambda$-compact.

If $\mathcal{Y} \subseteq \mathcal{P}(X)$, $|\mathcal{Y}| = \kappa$, and every subset of $\mathcal{Y}$ of cardinality $\mu$ has a complete accumulation point, then $\mathcal{Y}$ has a complete accumulation point. In particular, if $Y \subseteq X$, $|Y| = \kappa$, and every subset of $Y$ of cardinality $\mu$ has a complete accumulation point, then $Y$ has a complete accumulation point.

Using the theorems and arguments in [JS], and by Theorem 2.1, we get the following corollary (notice that if $\emptyset \not\in F$, then $\kappa$-compactness trivially implies $F$-$\kappa$-compactness).

**Corollary 2.3.** If $X$ is linearly Lindelöf and $F$-$\kappa_\omega$-compact, then $X$ is $F$-$\kappa$-compact, for every uncountable cardinal $\kappa$.

**Remark 2.4.** Let us remark that in the present section we have used very little topology. In all the above arguments (and in a large part of the paper by Juhász and Szentmiklóssy) the only needed assumption is that to every point $x$ of $X$ there is associated a family of "neighborhoods" with the only properties that (1) each "neighborhood" of
x contains \{x\}, and (2) if \(U\) is a “neighborhood” of \(x\), then every \(y \in U\) has some “neighborhood” contained in \(U\). In particular, we have never used the topological property that the intersection of two neighborhoods of \(x\) is still a neighborhood of \(x\).

3. Decomposability of products of ultrafilters

In this section we use \(\Phi(\mu, \kappa, \lambda)\) to prove theorems about decomposability of ultrafilters. In fact, the proofs work without changes for arbitrary families of subsets of some set, not even necessarily being filters; so we state the results in such a generality. Of course, the reader may always assume that \(D, D'\) below are ultrafilters, and it might happen this is the only interesting case.

We say that \(D \subseteq \mathcal{P}(I)\) is uniform over \(I\) if every member of \(D\) has cardinality \(|I|\). Here \(\mathcal{P}(I)\) is the set of all subsets of \(I\). We say that \(D\) is \(\mu\)-decomposable if there exists a function \(f : I \to \mu\) such that \(f^{-1}(Z) \notin D\), whenever \(Z \in [\mu]^{<\mu}\). Such an \(f\) is called a \(\mu\)-decomposition for \(D\). Clearly, \(f\) is a \(\mu\)-decomposition for \(D\) if and only if \(f(D)\) is uniform over \(\mu\), where we put \(f(D) = \{Z \subseteq \mu \mid f^{-1}(Z) \in D\}\).

In the above definitions, \(\mu\) can be equivalently replaced by any set of cardinality \(\mu\). Throughout the paper, \(D, D'\) are assumed to be subsets of \(\mathcal{P}(I), \mathcal{P}(I')\), respectively.

The next theorem exploits the connection between \(\Phi(\mu, \kappa, \lambda)\) and decomposability of families of sets. As in the preceding section, we are not necessarily assuming that \(\lambda\) is regular, though we do not know how much this is an actual gain in generality. We define the product \(D \times D'\) to be the subset of \(\mathcal{P}(I \times I')\) defined by: \(X \in D \times D'\) if and only if \(\{i \in I \mid \{i' \in I' \mid (i, i') \in X\} \in D'\} \in D\). Of course, for ultrafilters, this coincides with a classical definition. See \([K]\) and further references there.

**Theorem 3.1.** If \(\Phi(\mu, \kappa, \lambda)\) holds, \(D'\) is \(\mu\)-decomposable, and \(D\) is \(\lambda\)-decomposable, then \(D \times D'\) is \(\kappa\)-decomposable.

**Proof.** We first prove the theorem in the particular case in which \(D'\) is uniform over \(\mu\) and \(D\) is uniform over \(\lambda\). Let \(\{S_\xi \mid \xi < \lambda\}\) be given by \(\Phi(\mu, \kappa, \lambda)\) and, for every \(\xi < \lambda\), choose a bijection \(f_\xi : \mu \to S_\xi\). Since \(D'\) is uniform over \(\mu\), \(f_\xi^{-1}(C) \notin D'\), for every \(\xi < \lambda\) and \(C \in [S_\xi]^{<\mu}\). Define \(f : \lambda \times \mu \to \kappa\) by \(f(\xi, \eta) = f_\xi(\eta)\). We claim that \(f\) witnesses that \(D \times D'\) is \(\kappa\)-decomposable. Indeed, let \(A \in [\kappa]^{<\kappa}\). By \(\Phi(\mu, \kappa, \lambda)\), \(|\{\xi < \lambda \mid |A \cap S_\xi| = \mu\}| < \lambda\), thus, from the above remark, \(|\{\xi < \lambda \mid f_\xi^{-1}(A) \in D'\}| < \lambda\), hence, since \(D\) is uniform, \(|\{\xi < \lambda \mid f_\xi^{-1}(A) \in D'\} \notin D\), that is, \(|\{\xi < \lambda \mid \{\eta \mid f(\xi, \eta) \in A\} \in D'\} \notin D\), and this means exactly \(f^{-1}(A) \notin D \times D'\).
Now consider the general case in which \( D' \) is \( \mu \)-decomposable, and \( D \) is \( \lambda \)-decomposable. Then there are \( g : I \to \lambda \) and \( g' : I' \to \mu \) such that \( g(D) \) is uniform over \( \lambda \) and \( g'(D') \) is uniform over \( \mu \). Applying the above particular case to \( g(D) \) and \( g'(D') \), we get some \( f : \lambda \times \mu \to \kappa \) witnessing that \( g(D) \times g'(D') \) is \( \kappa \)-decomposable. Then clearly the function \( h : I \times I' \to \kappa \) defined by \( h(i, i') = f(g(i), g'(i')) \) witnesses that \( D \times D' \) is \( \kappa \)-decomposable.

The same proof as above provides a slightly more general result. If \( D \subseteq \mathcal{P}(I) \), and, for every \( i \in I \), \( D_i \subseteq \mathcal{P}(I_i) \), the \( D \)-sum \( \sum_D D_i \) of the \( D_i \)'s modulo \( D \) is the subset of \( \mathcal{P}(\{(i, j)| i \in I_i, j \in I_j\}) \) defined by \( X \in \sum_D D_i \) if and only if \( \{i \in I| \{j \in I_i|(i, j) \in X\} \in D_i\} \in D \).

**Theorem 3.2.** If \( \Phi(\mu, \kappa, \lambda) \) holds, \( D \subseteq \mathcal{P}(I) \) is \( \lambda \)-decomposable and, for every \( i \in I \), \( D_i \) is \( \mu \)-decomposable (or just \( \{i \in I| D_i \) is \( \mu \)-decomposable\} \in D), then \( \sum_D D_i \) is \( \kappa \)-decomposable.

**Corollary 3.3.** If \( \kappa \) is singular, \( D' \) is \( \text{cf} \kappa \)-decomposable, and \( D \) is \( \kappa^+ \)-decomposable, then \( D \times D' \) is \( \kappa \)-decomposable.

**Proof.** Immediate from Theorem 3.1 and and Juhász and Szentmiklóssy Theorem 4 in [JS], asserting that \( \Phi(\text{cf} \kappa, \kappa, \kappa^+) \) holds.

Notice that Juhász and Szentmiklóssy also proved that, for example, \( \Phi(\mu, \kappa, \kappa^+) \) holds whenever \( \text{cf} \kappa = \text{cf} \mu < \mu < \kappa \). See [JS] Theorem 5. However this adds nothing to the theorems of the present section, since \( \mu \)-decomposability implies \( \text{cf} \mu \)-decomposability.

### 4. Further Remarks

We add a very simple observation, which nevertheless might be of some interest. It elaborates on a classical argument, which dates back at least to Katětov [K]. See the last lines on p. 173 therein.

If \( D \subseteq \mathcal{P}(I) \), \( X \) is a topological space, and \( (F_i)_{i \in I} \) is a sequence of subsets of \( X \), then a point \( p \in X \) is said to be a \( D \)-limit point (or a \( D \)-accumulation point) of \( (F_i)_{i \in I} \) if \( \{i \in I| U \cap F_i \neq \emptyset\} \in D \), for every neighborhood \( U \) of \( p \). In case each \( F_i \) is a singleton \( \{x_i\} \), we shall simply say that \( p \) is a \( D \)-limit point of \( (x_i)_{i \in I} \). In this situation, it is sometimes said that \( (x_i)_{i \in I} \) \( D \)-converges to \( p \).

If \( F \subseteq \mathcal{P}(X) \), we say that \( X \) is \( F \)-\( D \)-compact (or that \( X \) satisfies the \( F \)-\( D \)-accumulation property) if, for every sequence \( (F_i)_{i \in I} \) of (not necessarily distinct) members of \( F \), there is \( p \in X \) which is a \( D \)-limit point of \( (F_i)_{i \in I} \). When \( F = \mathcal{S} \) this is simply called \( D \)-compactness; when \( F = O \) this is called \( D \)-pseudocompactness (sometimes, \( D \)-feeble compactness). Again, see [L] for further comments and references. The
above definitions are usually given only in the particular case when $D$ is an ultrafilter, but here, as in Section 3, we shall need no particular assumption on $D$.  

**Lemma 4.1.** Suppose that $D \subseteq \mathcal{P}(I)$, that $D_i \subseteq \mathcal{P}(I_i)$, for each $i \in I$, and that $(F_{i,j})_{i \in I, j \in I_i}$ is a sequence of subsets of $X$. If, for each $i \in I$, the subsequence $(F_{i,j})_{j \in I_i}$ has some $D_i$-limit point $p_i$, and if the sequence $(p_i)_{i \in I}$ has a $D$-limit point $p$, then $p$ is also a $\sum D_i$-limit point of $(F_{i,j})_{j \in I_i}$.

**Proof.** Applying the definitions, we have that $\{ j \in I_i \mid U \cap F_{i,j} \neq \emptyset \} \in D_i$, for every $i \in I$ and every neighborhood $U_i$ of $p_i$; moreover, $\{ i \in I \mid p_i \in U \} \in D$, for every neighborhood $U$ of $p$. The above statements, together with the fact that if $p_i \in U$, then $U$ is also a neighborhood of $p_i$, give $\{ (i, j) \mid i \in I, j \in I_i, U \cap F_{i,j} \neq \emptyset \} \in \sum D_i$. □

**Proposition 4.2.** If $X$ is both $D$-compact and $\mathcal{F}$-$D_i$-compact, for every $i \in I$, then $X$ is $\mathcal{F}$-$\sum D_i$-compact.

In particular, if $X$ is $D'$-compact and $\mathcal{F}$-$D_i'$-compact, then $X$ is $\mathcal{F}$-$D \times D'$-compact.

**Proof.** The second statement is a particular case of the first one, so let us prove the first statement. Suppose that $(F_{i,j})_{i \in I, j \in I_i}$ is a sequence of elements of $\mathcal{F}$. By $\mathcal{F}$-$D_i$-compactness, for every $i \in I$, the subsequence $(F_{i,j})_{j \in I_i}$ has some $D_i$-limit point $p_i$. By $D$-compactness, the sequence $(p_i)_{i \in I}$, has a $D$-limit point, and we are done by the previous lemma. □

Notice that Remark 2.4 applies to Lemma 4.1 and Proposition 4.2, too.

**Remark 4.3.** Proposition 4.2 can be used to appreciate the connections between Theorem 2.1 and Theorem 3.1. Indeed, suppose that $\lambda$ is a regular cardinal. Then it is easy to see that $\lambda$-compactness is equivalent to $D$-compactness, for $D = [\lambda]$. Similarly, if also $\mu$ is regular, then $\mu$-compactness is equivalent to $D'$-compactness, for $D' = [\mu]$. From Theorem 3.1 we get that if $\Phi(\mu, \kappa, \lambda)$ holds, then $D \times D'$ is $\kappa$-decomposable, that is, $D'' = f(D \times D')$ is uniform over $\kappa$, for some appropriate function $f$. If, under the above assumptions, $X$ is both $\lambda$-compact and $\mu$-compact, then, by Proposition 4.2, $X$ is also $D \times D'$-compact, and it is trivial to see that then $X$ is also $D''$-compact, and this implies that $X$ is $\kappa$-compact.

More generally, for $\mu$ regular, $\mathcal{F}$-$\mu$-compactness is equivalent to $\mathcal{F}$-$D'$-compactness, for the same $D'$ as above, and all the above arguments work for arbitrary $\mathcal{F}$. Hence, in the case when $\mu$ is regular, Theorem 2.1 is in fact a corollary of Theorem 3.1 and of Proposition 4.2. We
do not need the assumption that $\lambda$ is regular, since, in the case when $\lambda$ is singular, $D$-compactness (where, as above, $D = [\lambda]^1$) is equivalent to the conjunction of both $\lambda$-compactness and of $\text{cf} \lambda$-compactness (see [L, Proposition 3.3]).

On the other hand, some technical differences arise when $\mu$ is singular. Indeed, notice that Juhász and Szentmiklóssy [JS] give significant applications of $\Phi(\mu, \kappa, \lambda)$ also in the case when $\mu$ is singular, $\mu$-compactness holds, but $\text{cf} \mu$-compactness does not. Compare also Corollary 2.3 in the present paper.

References

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