A Fire Fighter’s Problem

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Abstract. Suppose that a circular fire spreads in the plane at unit speed. A fire fighter can build a barrier at speed \(v > 1\). How large must \(v\) be to ensure that the fire can be contained, and how should the fire fighter proceed? We provide two results. First, we analyze the natural strategy where the fighter keeps building a barrier along the frontier of the expanding fire. We prove that this approach contains the fire if \(v > v_c = 2.6144\ldots\) holds. Second, we show that any “spiralling” strategy must have speed \(v > 1.618\) in order to succeed.

Keywords: Motion Planning, Dynamic Environments, Spiralling strategies, Lower and upper bounds

1 Introduction

Fighting wildfires and epidemics has become a serious issue in the last decades. Professional fire fighters need models and simulation tools on which strategic decisions can be based. Thus, a good understanding of the theoretical foundations seems necessary.

Substantial work has been done on the fire fighting problem in graphs; see, e.g., the survey article [3]. Here, initially one vertex is on fire. Then a firefighter can be placed at one of the other vertices. Next, the fire spreads to each adjacent vertex that is not defended by a fighter, and so on. The game continues until the fire cannot spread anymore. The objective, to save a maximum number of vertices from the fire, is NP-hard to achieve, even for trees.

A more geometric setting has recently been studied in [6]. Suppose that inside a simple polygon \(P\) a candidate set of barrier cuts has been defined. If a fire starts at some point inside \(P\) one wants to build a subset of these barriers in order to save a maximum area from the fire. But each point on a barrier must be built before the fire arrives there. This maximization problem is also NP-hard, even if the candidate barriers are the diagonals of a convex polygon, but there exists an 11.65 approximation algorithm.

In this paper we are studying a purely geometric version of the fire fighter problem. Suppose there is a circular fire of initial radius \(A\) in the plane, centered at the origin. The fire spreads at unit speed. Initially, the plane is empty, except for a single fire fighter who is placed on the boundary of the fire. (One could imagine that the fire fighter is woken up by the approaching fire.) The fighter can move at speed \(v\), and build a barrier along his path. The fire cannot cross
this barrier, and the fighter cannot move into the fire. Will the fighter be able to contain the fire, and how should he proceed to achieve this?

Clearly, the answer depends on speed $v$. For $v = 1$ the fighter can barely save himself by moving along a straight line away from the fire. Given a sufficiently large speed ($v > 2\pi + 1$ would suffice), he can move some distance to the right and complete a circle around the fire before the fire can reach it. What happens in between?

*In this paper we show that a speed $v > 2.6144$ is sufficient to contain a fire, and that a speed $v > 1.618$ is necessary, at least for a reasonably large class of strategies.*

The first bound is established in the following way. We consider a conscientious fire fighter who tries to contain the fire by building a barrier along its ever expanding frontier, at her maximum speed $v$. Building firebreaks near the fire is a technique fire departments employ [5]. Let us denote this strategy by FF (short for Follow Fire). A spiralling barrier curve results. While the fighter keeps building the barrier, the fire is coming after her along the outside of the barrier, as shown in Figure 1. The fighter can only win this race, and contain the fire, if the last coil of the spiral hits the previous one. In the hand-drawn example shown in Figure 1 this happens in the second round if $v = 5.759$; but for smaller values of $v$, more rounds may be necessary. We have the following result.

![Fig. 1: The race between the fire and the fighter for speed $v = 3.738$. The firebreak was constructed from $p_0$ to $p_2$ whereas the fire expands along the outer side of the barrier up to point $q$. Can the fire figther finally catch the fire?](image)
Fig. 2: At speed $v = 5.759$ the fire will be fully contained by the fire fighter’s barrier in the second round.

Theorem 1.

(i) Strategy FF contains the fire if $v > v_c \approx 2.6144$ holds.

(ii) But as $v$ decreases to $v_c$, the number of rounds to containment tends to infinity.

Although strategy FF is rather simple, the proof of Theorem 1 is not. First, we establish a recursive system of linear differential equations associated with each round. They can be solved easily by standard methods, but the resulting recursions are complicated. Therefore, we apply techniques from analytic combinatorics. We look at the generating function $F(Z)$ that arises from these recursions, and find a presentation of $F(Z)$ as a ratio of analytic functions. The denominator equals

$$e^w Z - s Z = 0,$$

where $w = \frac{2\pi + \alpha}{\sin \alpha}$ and $s = e^{(2\pi + \alpha) \cot \alpha}$ are functions of a real variable $\alpha$ which equals $\cos^{-1}(1/v)$ in our setting. Our target are the coefficients of $F(Z)$; they are linked to the zeroes of equation (1).

Let $\alpha_c \approx 1.1783$ be the smallest positive solution of $s = e^w$, corresponding to $v_c \approx 2.6144$. For this value of $\alpha$, equation (1) has a real zero $Z = 1/w$, as direct substitution shows. For $\alpha > \alpha_c$, corresponding to $v > v_c$, this real zero splits
into a complex zero \( z_0 = \rho (\cos \phi + \sin \phi i) \) and its conjugate, where \( \phi \in (0, \pi) \), and no real zeroes of equation \( \frac{1}{4} \) remain.

At this point, part (i) of Theorem 1 follows from a Theorem of Pringsheim’s in complex function theory; see Section 6. To find out how many rounds it takes to contain the fire, we apply Cauchy’s residue theorem and find that their number is \( \approx \frac{\pi}{\phi} \). Since \( \phi \), the angle of the complex root \( z_0 \), tends to zero as \( z_0 \) becomes real for \( \alpha \to \alpha_c \), part (ii) of Theorem 1 also follows. How \( j \), the number of rounds, depends on \( v \) is shown in Figure 3. For speeds \( v \geq 3 \) strategy FF needs at most 4 rounds to contain the fire.

Fig. 3: The approximate number of rounds needed by strategy FF, as a function of speed \( v \).

While we conjecture that speeds \( \leq v_c \approx 2.6144 \) are generally insufficient, we can currently prove only a somewhat lower bound. We restrict ourselves to the class of “spiralling” strategies that visit the four coordinate axes in cyclic order, and at increasing distances from the origin. (Note that strategy FF is spiralling even though the fighter’s distance to the origin may be decreasing: the barrier’s intersection points with any ray from 0 are in increasing order since the curve does not self-intersect.) Here we have the following.

**Theorem 2.** In order to enclose the fire, a spiralling strategy must be of speed

\[
v > \frac{1 + \sqrt{5}}{2} \approx 1.618,
\]

the golden ratio.
The proof of Theorem 2 is given in Section 7. An (almost) complete proof of Theorem 1 (i) is given in the main text; only for some details we refer to the Appendix. Proving part (ii) of Theorem 1 requires considerably more work; we sketch only the essential ideas in the main text. A complete proof of (i) and (ii), which can be read independently of the main text, is given in the Appendix.

2 The barrier curve generated by strategy FF

We would like to show how the barrier curve shown in Figure 2 has been developed. A more detailed view of the starting situation of Figure 2 from $p_0$ to $p_2$ is depicted in Figure 4.

![Figure 4](image)

Fig. 4: The first part of the barrier curve constructed by FF consists of two different logarithmic spirals of eccentricity $\alpha$ where $\alpha = \cos^{-1}(1/v)$ holds. Namely, a logarithmic spiral around the origin 0 from $p_0$ to $p_1$ and a logarithmic spiral around $p_0$ from $p_1$ to $p_2$. At $p_2$ the fire figther’s curve starts wrapping around the constructed barrier as show in Figure 2.

Consider some point $p$ in the first round between $p_0$ and $p_1$ as shown in Figure 4. If $\alpha$ denotes the angle between the fighter’s velocity vector at $p$ and the ray from 0 through $p$, the fighter advances at speed $v \cos \alpha$ away from the fire.
This implies $v \cos \alpha = 1$ because the fire expands at unit speed and the fighter stays on its frontier, by definition of strategy FF. Consequently, the barrier curve between $p_0$ and $p_1$ is part of a logarithmic spiral centered at 0, whose tangents include the angle $\alpha = \cos^{-1}(1/v)$ with the extensions of the rays from 0 through $p$.

In polar coordinates a logarithmic spiral (with excentricity $\alpha$) is defined by $(\varphi, A \cdot e^{\varphi \cot \alpha})$ and the barrier curve from $p_0$ to $p_1$ is represented by the interval $\varphi \in [0, 2\pi]$. The curve length of the logarithmic spiral of excentricity $\alpha$ around origin $O$ between two points $C$ and $D$ appearing on the spiral in this order is given by $\frac{1}{\cos \alpha} (|DO| - |CO|)$, where $|CO|$ and $|DO|$ denote the distances from $D$ and $C$ to the origin 0, respectively. Thus, for example the curve length from $p_0$ to $p_1$ is given by $l_1 = \frac{A}{\cos(\alpha)} \cdot (e^{2\pi \cot(\alpha)} - 1)$.

From point $p_1$ on, the geodesic shortest paths $\pi(p)$ from 0 to $p$, along which the fire spreads, start with segment $0p_0$, followed by segment $p_0p$, until the fighter reaches the point $p_2$ on the barrier’s tangent to $p_0$; see Figure 4. Thus, by the previous argument, between $p_1$ and $p_2$ the barrier curve constructed by FF is part of a logarithmic spiral of excentricity $\alpha$ now centered at $p_0$. This spiral starts at $p_1$ with distance $A' = A(e^{2\pi \cot(\alpha)} - 1)$ from its origin $p_0$, and the curve length from $p_1$ to $p_2$ is given by $l'_2 = \frac{A'}{\cos(\alpha)}(e^{\alpha \cot(\alpha)} - 1) = \frac{A}{\cos(\alpha)} \cdot (e^{2\pi \cot(\alpha)} - 1)(e^{\alpha \cot(\alpha)} - 1)$. This means that the overall curve length from $p_0$ to $p_2$ is given by $l_1 + l'_2 = l_2 = \frac{A}{\cos(\alpha)} \cdot (e^{2\pi \cot(\alpha)} - 1)e^{\alpha \cot(\alpha)}$.

How does the curve constructed by FF develop from $p_2$ on? We turn over to Figure 2 From $p_2$ on, the geodesic shortest path $\pi(p)$ from 0 to fighter’s current position $p$ starts wrapping around the existing spiral part of the curve, beginning at $p_0$. The last edge of $\pi(p)$ ending at $p$ will be called the free string in the sequel. The fire will be contained if and only if the free string ever attains length 0.

Thus, after the first round the curve progresses like an involute, “drawn” by endpoint $p$ of the free string. But unlike a regular involute, the string is not normal to the outer layer. Rather, its extension beyond $p$ includes the angle $\alpha$ with the barrier’s tangent at $p$. This causes the string to grow in length by $\cos \alpha$ for each unit drawn. At the same time, part of the string gets wrapped around the inner layer. It is this interplay between growing and shrinking we will investigate below.

As the fighter is building the barrier at speed $1/\cos \alpha$, the fire is coming after her at unit speed along the outside of the barrier, as indicated in Figure 1. Thus, each barrier point $p$ is caught by fire twice, once from the inside, when the fighter passes through $p$, and a second time from the outside, if the fire is not stopped before.
3 Linkages

That the innermost part of the curve consists of two different spiral segments, around 0 and around \( p_0 \), carries over to subsequent layers. The structure of the curve can be described as follows. Let

\[
\begin{align*}
    l_1 &= \frac{A}{\cos(\alpha)} \cdot (e^{2\pi \cot(\alpha)} - 1) \\
    l_2 &= \frac{A}{\cos(\alpha)} \cdot (e^{2\pi \cot(\alpha)} - 1) e^{\alpha \cot(\alpha)}
\end{align*}
\]

denote the curve lengths from \( p_0 \) to \( p_1 \) and \( p_2 \), respectively, as derived before in Section 2. For \( l \in [0, l_1] \) let \( F_0(l) \) denote the segment connecting 0 to the point of curve length \( l \); see the sketch given in Figure 5.

Fig. 5: A sketch of the general situation. Two types of linkages defining subsegments of the curve.

At the endpoint of \( F_0(l) \) we construct the tangent and extend it until it hits the next layer of the curve, creating a segment \( F_1(l) \), and so on. This construction gives rise to a “linkage” connecting adjacent layers of the curve. Each edge of the linkage is turned counterclockwise by \( \alpha \) with respect to its predecessor. The outermost edge of a linkage is the free string mentioned above. As parameter \( l \)
increases from 0 to \(t_1\), edge \(F_0(l)\), and the whole linkage, rotate counterclockwise. While \(F_0(0)\) equals the line segment from the center to \(p_0\), edge \(F_0(t_1)\) equals segment \(0p_1\).

Analogously, let \(l \in [t_1, t_2]\), and let \(\phi_0(l)\) denote the segment from \(p_0\) to the point at curve length \(l\) from \(p_1\). This segment can be extended into a linkage in the same way. We observe that

\[
F_{j+1}(t_1) = \phi_{j+1}(t_1) \quad (2)
\]

\[
F_{j+1}(0) = \phi_j(t_2) \quad (3)
\]

hold. But initially, we have \(F_0(l) = A + \cos(\alpha)l\) and \(\phi_0(l) = \cos(\alpha)l\), so that \(F_0(t_1) \neq \phi_0(t_1)\). Clearly, each point on the curve can be reached by a linkage, as tangents can be constructed backwards.

## 4 Analysis

A detailed proof of the following general facts is given in the Appendix in Lemma 6 and 7. We present the intuitive ideas here.

As the endpoint of a taught string of length \(F\), tangent to a smooth curve \(C\) at some point \(p\), is moved in direction \(\alpha\), as shown in Figure 6 (i), the length \(l\) of the wrapped string grows at rate \(r \sin \alpha / F\), where \(r\) denotes the curve’s radius of curvature at \(p\). (Intuitively, the larger the motion \(\sin \alpha\) perpendicular to the string, and the larger the osculating circle, the more of the string gets wrapped; but the larger \(F\), the smaller is the effect of the perpendicular motion.) The center of the osculating circle at \(p\) is known to be the limit of the intersections of the normals of all points near \(p\) with the normal at \(p\). If, instead of the normals, we consider the lines turned by the angle \(\pi/2 - \alpha\), their limit intersection point has distance \(r \sin \alpha\) from \(p\); an example is shown in Figure 6 (ii) for the case where curve \(C\) itself is a circle.

For the barrier curve, the limit intersection point of the turned normals near \(p\) is just the tangent point from \(p\) to the previous layer of the curve. If we denote by \(L_i\) the length of the barrier curve from \(p_0\) to the outer endpoint of the \(i\)th edge of an \(F\)-linkage, the above observations imply the following for \(L_{j-1}, F_j\) and \(F_{j-1}\) as functions of \(L_j\):

\[
\frac{L'_{j-1}}{L'_{j}} = \frac{L'_{j-1}}{1} = \frac{r \sin \alpha}{F_j} = \frac{F_{j-1}}{F_j}.
\]

As we change to the parameter \(l \in [0, t_1]\) introduced in Section 3, inner derivatives cancel out, and we obtain

**Lemma 1.**

\[
\frac{L'_{j-1}(l)}{L'_{j}(l)} = \frac{F_{j-1}(l)}{F_{j}(l)}.
\]
Fig. 6: In (i), the wrapped string grows at a rate of $r \sin \alpha / F$. In (ii), the turned normals meet at a point $r \sin \alpha$ away from $p$.

By multiplication, Lemma 1 generalizes to non-consecutive edges. Thus,

$$
\frac{F_j(l)}{F_0(l)} = \frac{L'_j(l)}{L'_0(l)} = L'_j(l)
$$

(4)

holds.

On the other hand, a point $p$ on the $j$th layer of the barrier curve has geodesic distance $L_{j-1}(l) + F_j(l)$ from the initial fire of radius $A$, and the fire arrives at $p$ (from the inside) simultaneously with the fighter, who has then completed a barrier of length $L_j(l)$ at speed $1 / \cos \alpha$. This yields, $F_j(l) + L_{j-1}(l) = \cos \alpha L_j(l)$ and after taking derivatives,

$$
F'_j(l) + L'_{j-1}(l) = \cos \alpha L'_j(l).
$$

(5)

From 5 and 4 we obtain a linear differential equation for $F_j(l)$,

$$
F'_j(l) - \frac{\cos(\alpha)}{F_0(l)} F_j(l) = - \frac{F_{j-1}(l)}{F_0(l) l}.
$$

The textbook solution for $y'(x) + f(x)y(x) = g(x)$ is

$$
y(x) = \exp(-a(x)) \left( \int g(t) \exp(a(t)) dt + \kappa \right),
$$

where $a = \int f$ and $\kappa$ denotes a constant that can be chosen arbitrarily. In our case,

$$
a(l) = \int -\frac{\cos(\alpha)}{A + \cos(\alpha)} dt = -\ln(F_0(l))
$$
because of \( F_0(l) = A + \cos(\alpha) l \), and we obtain

\[
F_j(l) = F_0(l) \left( \kappa_j - \int \frac{F_{j-1}(t)}{F_0^2(t)} \, dt \right). \tag{6}
\]

Next, we consider a linkage of \( \phi \)-type, for parameters \( l \in [l_1, l_2] \), and obtain analogously

\[
\phi_j(l) = \phi_0(l) \left( \lambda_j - \int \frac{\phi_{j-1}(t)}{\phi_0^2(t)} \, dt \right). \tag{7}
\]

Now we determine the constants \( \kappa_j, \lambda_j \) such that the solutions 6 and 7 describe a contiguous curve. To this end, we must satisfy conditions 2 and 3.

We define \( \kappa_0 := 1 \) and

\[
\kappa_{j+1} := \frac{\phi_j(l_2)}{\phi_0(l_1)} + \int \frac{\phi_j(t)}{\phi_0^2(t)} \, dt |_{l=0}
\]

so that 6 becomes

\[
F_{j+1}(l) = F_0(l) \left( \frac{\phi_j(l_2)}{\phi_0(l_1)} - \int_0^l \frac{\phi_j(t)}{\phi_0^2(t)} \, dt \right),
\]

which, for \( l = 0 \), yields \( F_{j+1}(0) = \phi_j(l_2) \) (condition 3). Similarly, we set \( \lambda_0 := 1 \) and

\[
\lambda_{j+1} := \frac{F_{j+1}(l_1)}{\phi_0(l_1)} + \int \frac{\phi_j(t)}{\phi_0^2(t)} \, dt |_{l=l_1}
\]

so that 7 becomes

\[
\phi_{j+1}(l) = \phi_0(l) \left( \frac{F_{j+1}(l_1)}{\phi_0(l_1)} - \int_{l_1}^l \frac{\phi_j(t)}{\phi_0^2(t)} \, dt \right),
\]

and for \( l = l_1 \) we get \( F_{j+1}(l_1) = \phi_{j+1}(l_1) \) (condition 2).

For simplicity, let us write

\[ G_j(l) := \frac{F_j(l)}{F_0(l)} \text{ and } \chi_j(l) := \frac{\phi_j(l)}{\phi_0(l)}, \tag{8} \]

which leads to

\[
G_{j+1}(l) = \frac{\phi_j(l_2)}{\phi_0(l_1)} \chi_j(l_2) - \int_0^l \frac{G_j(t)}{F_0(t)} \, dt \tag{9}
\]

\[
\chi_{j+1}(l) = \frac{F_0(l_1)}{\phi_0(l_1)} G_{j+1}(l_1) - \int_{l_1}^l \frac{\chi_j(t)}{\phi_0(t)} \, dt. \tag{10}
\]

In order to find out if the fire fighter is successful we only need to check the values of \( F_j(l) \) at the end of each round, as the following lemma shows.
Lemma 2. The curve encloses the fire if and only if there exists an index \( j \) such that \( F_j(l_1) \leq 0 \) holds.

Proof. The free string shrinks to zero if and only if there exist an index \( j \) and argument \( l \) such that \( F_j(l) \leq 0 \) or \( \phi_j(l) \leq 0 \). Clearly, \( G_j \) and \( F_j \) have identical signs, as well as \( \chi_j \) and \( \phi_j \) do. Suppose that \( G_j > 0 \) and \( G_{j+1}(l) = 0 \), for some \( j \) and some \( l \in [0, l_1] \). By 9 function \( G_{j+1} \) is decreasing, therefore \( G_{j+1}(l_1) \leq 0 \).

Now assume that \( G_i > 0 \) holds for all \( i \), and that we have \( \chi_{j-1} > 0 \) and \( \chi_j(l) = 0 \) for some \( j \) and some \( l \in [l_1, l_2] \). By 10 this implies \( \chi_j(l_2) \leq 0 \), and from 25 we conclude \( G_{j+1} \leq 0 \), in particular \( G_{j+1}(l_1) \leq 0 \).

5 Recursions

The integrals in 9 and 10 disappear by iterated substitution. This process is not entirely trivial, and the calculations can be found in Section C in the Appendix.

After plugging in values, one obtains cross-wise recursions

\[
F_j(l_1) = \frac{F_0(l_1)}{F_0(0)} \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!} \left( \frac{2\pi}{\sin \alpha} \right)^\nu \phi_{j-1-\nu}(l_2) \tag{11}
\]

\[
\phi_j(l_2) = \frac{\phi_0(l_2)}{\phi_0(l_1)} \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!} \left( \frac{\alpha}{\sin \alpha} \right)^\nu \hat{F}_{j-\nu}(l_1) \tag{12}
\]

where \( \phi_{-1}(l_2) := F_0(0), \hat{F}_0(l_1) := \phi_0(l_1) \), and \( \hat{F}_{i+1}(l_1) := F_{i+1}(l_1) \).

In order to solve the cross-wise recursions 11 and 12 for the numbers \( F_j(l_1) \) we define the formal power series

\[
F(X) := \sum_{j=0}^{\infty} F_j X^j \quad \text{and} \quad \phi(X) := \sum_{j=0}^{\infty} \phi_j X^j
\]

where \( F_j := F_j(l_1) \) and \( \phi_j := \phi_j(l_2) \), for short. From 11 we obtain

\[
F(X) = \frac{F_0}{F_0(0)} e^{-\frac{2\pi}{\sin \alpha} X} (X \phi(X) + F_0(0)), \tag{13}
\]

and from 12

\[
\phi(X) = \frac{\phi_0}{\phi_0(l_1)} e^{-\frac{\alpha}{\sin \alpha} X} (X F(X) - F_0 + \phi_0(l_1)); \tag{14}
\]

both equalities can be easily verified by computing the products and comparing coefficients. Now we substitute 14 into 13, solve for \( F(X) \), divide both sides by \( F_0 \) and expand by \( e^{\frac{2\pi}{\sin \alpha} X} \) to obtain

\[
\frac{F(X)}{F_0} = \frac{e^{vX} - rX}{e^{uX} - sX}, \tag{15}
\]
where \( v, r, w, s \) are the following functions of \( \alpha \):

\[
\begin{align*}
v &= \frac{\alpha}{\sin \alpha} \quad \text{and} \quad r = e^{\alpha \cot \alpha} \\
w &= \frac{2\pi + \alpha}{\sin \alpha} \quad \text{and} \quad s = e^{(2\pi + \alpha) \cot \alpha}.
\end{align*}
\]

(16)

It is possible to expand the inverse of the denominator in 15 into a power series. This leads to interesting expressions for the \( F_j \); but how to derive their signs seems not obvious.

6 Singularities and Residues

For this reason, we consider the right hand side of (15) as a function

\[
f(z) := \frac{e^{vz} - rz}{e^{wz} - sz},
\]

of a complex variable, \( z \). Both numerator and denominator of \( f \) are analytic on the complex plane. Thus, singularities of \( f \) can only arise from zeroes of the denominator \( e^{wz} - sz \). This equation has received some attention in the area of delay differential equations [2]. As in the Introduction, let \( \alpha_c \approx 1.1783 \) be the unique solution of \( s = ew \) in \((0, \pi/2] \), corresponding to speed \( v_c = 1/\cos \alpha_c \approx 2.6144 \).

**Lemma 3.** For \( \alpha = \alpha_c \), equation \( e^{wz} - sz \) has a real root \( 1/w \approx 0.1238 \). For \( \alpha > \alpha_c \) (corresponding to speed \( v > v_c \)), this root splits into a complex conjugate pair \( z_0 \) and \( \overline{z_0} \), whose absolute values are < 0.31. All other zeroes of numerator and denominator in (15) are strictly complex, and of absolute values \( \geq 1 \). Function \( f(z) \) in (17) has only poles as singularities.

For a proof of Lemma 3, see Lemma 10 to 13 in the Appendix.

From now on we assume that \( \alpha > \alpha_c \) holds. Now we would like to make use of a general Theorem concerning the sign of coefficients of power series within their convergence radius, in order to prove the first part of Theorem 1.

**Pringsheim’s Theorem** (see for example [4] p. 240): Let \( h(z) = \sum_{n=0}^{\infty} a_n z^n \) be a power series with finite convergence radius \( R \). If \( h(z) \) has non-negative coefficients, \( a_j \), then point \( z = R \) is a singularity of \( h(z) \).

**Proof.** (of Theorem 1 (i)) Let \( \alpha > \alpha_c \). Because of the singularities \( z_0 \) and \( \overline{z_0} \), the power series expansion of \( f(z) \) in (17) has a finite radius, \( R \), of convergence. If all coefficients \( F_i \) were \( \geq 0 \) then, by Pringsheim’s Theorem function \( f(z) \) would have a singularity at \( R \). But, by Lemma 3, there can be only complex singularities. Thus, there must be coefficients \( F_j < 0 \), proving that the fire fighter succeeds.
Now we sketch the proof of Theorem 1(ii). A complete version can be found in the Appendix Sections E and F. This will also lead to another, and constructive, proof of part (i) of Theorem 1.

We are using a technique described in [4], p. 258 ff. Let \( \Gamma \) denote the circle of radius 0.9 around the origin. By Cauchy’s Residue Theorem,

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{f(u)}{u^{j+1}} \, du = \sum_{z \text{ inside } \Gamma} \text{res}(z)
\]

holds, where the sum is over all residues of the poles of \( f(z) \) encircled by \( \Gamma \). By Lemma 3, these poles are \( z_0, \bar{z}_0, \) and 0, which has residue \( F_j/F_0 \). Computing the residues of \( z_0, \bar{z}_0 \) yields

\[
\frac{F_j}{F_0} = \sin(j\phi + p) \frac{|z_0|^{-j}}{|z_0 - x_0|} \Theta(1) + \frac{1}{2\pi i} \int_{\Gamma} \frac{f(u)}{u^{j+1}} \, du,
\]

where \( z_0 = \rho (\cos \phi + \sin \phi i) \), with \( 0 < \phi < \pi \), and \( x_0 = (1/w, 0) \) is the limit of \( z_0 \) as \( \alpha \) tends to \( \alpha \). The rightmost term’s absolute value is upper bounded by the maximum of \( |f(z)| \) on \( \Gamma \), times 0.9\(^{-j}\); its influence turns out to be negligible.

The oscillation \( \sin(t\phi + p) \) has wavelength \( 2\pi/\phi \). For \( j \) near its negative minimum, the value of \( 18 \) becomes negative. This proves that the fire fighter will succeed in containing the fire in round \( j \), for some \( j \leq c \cdot 2\pi/\phi \) (in fact, one can choose \( c = 1 \)). As \( \alpha \) decreases towards \( \alpha_c \), both \( \phi \) and phase \( p \) tend to zero, but

\[
\lim_{\alpha \to \alpha_c} \frac{p}{\phi} \approx 1.315
\]

holds. This value denotes how much the graph of \( \sin(t\phi + p) \) is shifted to the left, as compared to \( \sin t \). We see that \( j \) must increase through almost the whole positive halfwave of \( \sin(t\phi + p) \) before negative values can occur. Since wavelength \( 2\pi/\phi \) goes to infinity, so does the number of rounds the fire fighter needs. This completes the proof of Theorem 1. All details are given in the Appendix.

7 Lower bound

Let us call a barrier building strategy \( S \) spiralling if it starts on the boundary of a fire of radius \( A \), and visits the four coordinate axes in counterclockwise order and at increasing distances from the origin.

Now let \( S \) be a spiralling strategy of maximum speed \( v \leq (1 + \sqrt{5})/2 \approx 1.618 \), the golden ratio. We can assume that \( S \) proceeds at constant speed \( v \). Let \( p_0, p_1, p_2, \ldots \) denote the points on the coordinate axes visited, in this order, by \( S \). The following lemma shows that \( S \) cannot succeed because there is still fire burning outside the barrier on the axis previously visited.

**Lemma 4.** When \( S \) visits point \( p_{i+1} \), the interval \( [p_i, p_i + \text{sign}(p_i)A] \) on the axis visited before is on fire.
Proof. The proof is by induction on $i$. Suppose strategy $S$ builds a barrier of length $x$ between $p_0$ and $p_1$, as shown in Figure 7(i). During this time the fire advances $x/v$ along the positive $X$-axis, so that $A + x/v \leq p_1 \leq x$ must hold, or

$$\frac{x}{v} \geq \frac{1}{v-1} A > A;$$

the last inequality follows from $v < 2$. Thus, the fire has time enough to move a distance of $A$ from $p_0$ downwards along the negative $Y$-axis.

Now let us assume that strategy $S$ builds a barrier of length $y$ between $p_i$ and $p_i+1$, as shown in Figure 7(ii). By induction, the interval of length $A$ below $p_i-1$ is on fire. Also, when the fighter moves on from $p_i$, there must be a burning interval of length at least $A + x/v$ on the positive $Y$-axis which is not bounded by a barrier from above. This is clear if $p_i+1$ is the first point visited on the positive $Y$-axis, and it follows by induction, otherwise. Thus, we must have $A + x/v + y/v \leq p_{i+1} \leq y$, hence

$$\frac{y}{v} \geq \frac{1}{v-1} A + \frac{1}{v(v-1)} x > A + x,$$

since the assumption on $v$ implies $v^2 \leq v + 1$. This shows that the fire can crawl along the barrier from $p_{i-1}$ to $p_i$, and a distance $A$ to the right, as the fighter moves to $p_{i+1}$, completing the proof of Theorem 2.

8 Conclusions

A number of interesting questions arise. Are there strategies that can contain the fire at a speed $v < v_c$? How about starting points away from the fire? Given a feasible speed $v$, how can the fighter minimize the time to completion, or the
area burned? Is it possible to generalize to fires of more realistic shapes, as they result under the influence of wind? These problems seem to define a nice area in the field of path planning in dynamic environments, where obstacle shapes depend on the agent’s actions.

For practical purposes, one would wish for a strategy that contains the fire in a single closed round. Also, starting points away from the fire could be allowed. If the fighter is free to pick her starting point she can contain the fire in a single closed round if, and only if, her speed is at least $v \geq 3.7788\ldots$. In this case the shortest possible (i.e., completion time optimal) solution consists of a line segment $q_0q_1$ followed by a segment of a logarithmic spiral of eccentricity $\alpha$, where $v = \frac{1}{\cos(\alpha)}$. See Figure 8 for an example of the time optimal single closed loop for $\alpha = 1.41$ and $v \approx 6.25$.

A single closed loop solution only exists for

$$\alpha > \arctan \left( \frac{\frac{3}{2}\pi}{W \left( \frac{4}{2}\pi \right)} \right) \approx 74.66^\circ$$
in which \( W \) denotes Lamberts W function \(^1\) defined by the functional equation \( W(x) e^{W(x)} = x \). This gives \( \alpha \geq 1.3029 \ldots \) or \( v \geq 3.7788 \ldots \)

References

1. R. M. Corless and G. H. Gonnet and D. E. G. Hare and D. J. Jeffrey. Lambert’s W function in Maple. The Maple Technical Newsletter, Issue 9, pp. 12–22, 1993.
2. C.E. Falbo. Analytic and Numerical Solutions to the Delay Differential Equation \( y'(t) = \alpha y(t-\delta) \). Joint Meeting of the Northern and Southern California Sections of the MAA, San Luis Obispo, CA, 1995. Revised version at [www.mathfile.net](http://www.mathfile.net)
3. S. Finbow and G. MacGillivray. The Firefighter Problem: a survey of results, directions and questions. Australasian J. Comb, 43, pp. 57-78, 2009.
4. P. Flajolet and R. Sedgewick. Analytic Combinatorics. Cambridge, 2009.
5. Food and Agriculture Organization of the United Nations (FAO). International Handbook on Forest Fire Protection. [www.fao.org/forestry/27221-06293a5348df137bc8b14e24472df64810.pdf](http://www.fao.org/forestry/27221-06293a5348df137bc8b14e24472df64810.pdf)
6. R. Klein, Ch. Levcopoulos, and A. Lingas. Approximation algorithms for the geometric firefighter and budget fence problems. in A. Pardo and A. Viola (eds.) LATIN 2014, Montevideo, LNCS 8392, pp. 261–272.
A Appendix

Note: This Appendix contains the proof of Theorem 4. To save the reader from jumping forth and back from the main text, we include here a contiguous text that can be read independently. The text starts with the description of linkages.

That the innermost part of the curve consists of two different spiral segments, around 0 and around \( p_0 \), carries over to subsequent layers. The structure of the curve can be described as follows. Let

\[
\begin{align*}
  l_1 &= \frac{A}{\cos(\alpha)} \cdot (e^{2\pi\cot(\alpha)} - 1) \\
  l_2 &= \frac{A}{\cos(\alpha)} \cdot (e^{2\pi\cot(\alpha)} - 1)e^{\alpha\cot(\alpha)}
\end{align*}
\]

denote the curve lengths from \( p_0 \) to \( p_1 \) and \( p_2 \), respectively. For \( l \in [0, l_1] \) let \( F_0(l) \) denote the segment connecting 0 to the point of curve length \( l \); see the sketch given in Figure 9.

![Fig. 9: Two types of linkages defining subsegments of the curve.](image)

At the endpoint of \( F_0(l) \) we construct the tangent and extend it until it hits the next layer of the curve, creating a segment \( F_1(l) \), and so on. This construction
gives rise to a “linkage” connecting adjacent layers of the curve. Each edge of the linkage is turned counterclockwise by $\alpha$ with respect to its predecessor. The outermost edge of a linkage is the free string mentioned above. As parameter $l$ increases from 0 to $l_1$, edge $F_0(l)$, and the whole linkage, rotate counterclockwise. While $F_0(0)$ equals the line segment from the center to $p_0$, edge $F_0(l_1)$ equals segment $0p_1$.

Analogously, let $l \in [l_1, l_2]$, and let $\phi_0(l)$ denote the segment from $p_0$ to the point at curve length $l$ from $p_1$. This segment can be extended into a linkage in the same way. We observe that

$$F_{j+1}(l_1) = \phi_{j+1}(l_1) \quad (19)$$
$$F_{j+1}(0) = \phi_j(l_2) \quad (20)$$

hold. But initially, we have $F_0(l) = A + \cos(\alpha) l$ and $\phi_0(l) = \cos(\alpha) l$, so that $F_0(l_1) \neq \phi_0(l_1)$. Clearly, each point on the curve can be reached by a linkage, as tangents can be constructed backwards.

In what follows both types of linkages will be considered separately. Since the arguments are almost identical, we focus on the $F$-type. The following fact is quite useful. We identify a link with its length, and a point on the curve with its curve length from $p_0$; see Figure 10.

$$t + F = \cos(\alpha) b.$$  

Fig. 10: Here $t + F = \cos(\alpha) b$.

**Lemma 5.** Let $F$ be a link from either linkage type with endpoints $t < b$ (so that $F$ is not an $F_0$). Then, $t + F = \cos(\alpha) b$. 

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Proof. The total string length from 0 to \( b \) equals \( A + t + F \). On the other hand, as the fighter arrives at the outer point \( b \), his walk along the curve has increased the initial string length \( A \) by \( \cos(\alpha) \cdot b \).

Next, we analyze how the free end of a string shrinks by wrapping.

![Diagram of a string wrapping around a curve.](image)

**Fig. 11:** A string wrapping around a curve.

**Lemma 6.** Suppose a string of length \( F \) is tangent to a point \( t \) on some smooth curve \( C \). Now the end of the string moves a distance of \( \epsilon \) in the direction of \( \alpha \), as shown in Figure 6. Then for the curve length \( C^t_{\epsilon} \) between \( t \) and the new tangent point, \( t_{\epsilon} \), we have

\[
\lim_{\epsilon \to 0} \frac{C^t_{\epsilon}}{\epsilon} = \frac{\sin(\alpha) \cdot r}{F}
\]

where \( r \) denotes the radius of the osculating circle at \( a \).

Intuitively, the claim of Lemma 6 is clear. The larger the radius of curvature, and the larger the advance perpendicular to the tangent at \( t \), the more string gets wrapped. On the other hand, the larger \( F \) is, the smaller becomes the effect of an advance of \( \sin(\alpha) \epsilon \).
Proof. From \( r \sin(\phi/2) = s = a \cos(\phi/2) \) we obtain \( a = r \tan(\phi/2) \). For short, let \( c := C_t^{t_s} \). By l'Hospital’s rule

\[
\frac{c}{2a} = \frac{r \phi}{2r \tan(\phi/2)} \approx \cos^2(\phi/2) \to 1
\]
as \( \epsilon \), hence \( \phi \), go to 0. Thus, \( 2a \) is a good approximation of \( c = C_t^{t_s} \). By the law of sines,

\[
\frac{\epsilon \sin(\alpha)}{\sin(\phi)} = \frac{F \epsilon + a}{\sin(\pi/2)},
\]
hence

\[
\frac{\sin(\phi)}{\epsilon} = \frac{\sin(\alpha)}{F \epsilon + a} \to \frac{\sin(\alpha)}{F}.
\]
This implies \( \sin(\phi/2)/\epsilon \to \sin(\alpha)/(2F) \), and we conclude

\[
\frac{C_t^{t_s}}{\epsilon} = \frac{c}{2a} \frac{\epsilon}{\epsilon} \approx \frac{2r \tan(\phi/2)}{\epsilon} = \frac{2r \sin(\phi/2)}{\epsilon \cos(\phi/2)} \to \frac{r \sin(\alpha)}{F}.
\]

The center of the osculating circle at \( t \) is known to be the limit point of the intersections of the normals to \( C \) near \( t \). This fact can be generalized to turned normals as follows.

**Lemma 7.** Let \( t \) be a point on a smooth curve \( C \), whose osculating circle at \( t \) is of radius \( r \). Consider the lines \( L_s \) resulting from turning the normal at points \( s \) by an angle of \( \pi/2 - \alpha \). Then their limit intersection point with \( L_t \) has distance \( \sin(\alpha)r \) to \( t \).

It is easy to verify Lemma 7 in case \( C \) is a circle.

Proof. Let us assume that \( C \) is locally parameterized by \( Y = f(X) \) and that \( t = (x_0, f(x_0)) \). Then the tangent in \( t \) is

\[
Y = f'(x_0)X - f'(x_0)x_0 + f(x_0),
\]
and line \( L_t \), the tangent turned counterclockwise by \( \alpha \), is given by

\[
Y = \tan(\arctan(f'(x_0)) + \alpha)X - \tan(\arctan(f'(x_0)) + \alpha)x_0 + f(x_0).
\]
Now let \((v, w)\) denote the point of intersection of \( L_t \) and \( L_s \), where \( s = (x_0 + \epsilon, f(x_0 + \epsilon)) \). Equating the two line equations we obtain

\[
(h(x_0 + \epsilon) - h(x_0))v = g(x_0 + \epsilon) - g(x_0) + f(x_0) - f(x_0 + \epsilon)
\]
where

\[
h(x) := \tan(\arctan(f'(x)) + \alpha) \quad \text{and} \quad g(x) := h(x)x
\]
After dividing by \( \epsilon \) and taking limits, we have

\[
h'(x_0)v_0 = g'(x_0) - f'(x_0) = h'(x_0)x_0 + h(x_0) - f'(x_0),
\]

This implies \( \sin(\phi/2)/\epsilon \to \sin(\alpha)/(2F) \), and we conclude
which results in

\[ v_0 = x_0 + \frac{h(x_0) - f'(x_0)}{h'(x_0)} \]

\[ w_0 = h(x_0) v_0 - g(x_0) + f(x_0) = f(x_0) + \frac{h^2(x_0) - h(x_0)f'(x_0)}{h'(x_0)}. \]

In other words,

\[ (v_0, w_0) - (x_0, f(x_0)) = \frac{h(x_0) - f'(x_0)}{h'(x_0)} (1, h(x_0)) \]

\[ |(v_0, w_0) - (x_0, f(x_0))| = \left| \frac{h(x_0) - f'(x_0)}{h'(x_0)} \right| \sqrt{1 + h^2(x_0)} \]

Using the addition formula for \( \tan \),

\[ h(x) = \tan(\arctan(f'(x)) + \alpha) = \frac{f'(x) + \tan(\alpha)}{1 - f'(x) \tan(\alpha)} \]

we obtain

\[ h(x_0) - f'(x_0) = \frac{1 + (f'(x_0))^2 + \tan(\alpha)}{1 - f'(x_0) \tan(\alpha)}. \]

and

\[ 1 + h^2(x_0) = \frac{(1 + (f'(x_0))^2)(1 + \tan^2(\alpha))}{(1 - f'(x_0) \tan(\alpha))^2}. \]

Moreover,

\[ h'(x_0) = \frac{f''(x)(1 + \tan^2(\alpha))}{(1 - f'(x_0) \tan(\alpha))^2}. \]

Putting expressions together we obtain

\[ |(v_0, w_0) - (x_0, f(x_0))| = \left| \frac{1 + (f'(x_0))^2}{f''(x_0)} \right| \frac{\tan(\alpha)}{\sqrt{1 + \tan^2(\alpha)}}. \]

The first term equals the radius of the osculating circle, \( r \), and the second is \( \sin(\alpha) \).

For our special curves, the limit intersection point considered in Lemma 7 is well-known: it is the tangent point from \( t \) to the previous layer of the curve. Thus, we obtain the following from Lemma 5 and Lemma 7.

**Lemma 8.** Let \( F_j, F_k \) denote edges of a linkage, and let \( v_j, v_k \) denote the velocities of their respective outer endpoints, as the linkage rotates. Then,

\[ \frac{v_j}{v_k} = \frac{F_j}{F_k}. \]

**Proof.** Lemma 6 proves the claim for \( k = j + 1, F = F_{j+1}, \) and \( v_{j+1} = 1 \). We observe that, under a change of parameters, inner derivatives cancel out in the velocity quotient. Also, the claim carries over to non-consecutive edges in a linkage, by multiplication.
B Differential equations

First, we consider a linkage of $F$-type, parameterized by $l \in [0, l_1]$, as described in the Introduction. Let $L_j(l)$ denote the curve length from $p_0$ to the outer endpoint of edge $F_j(l)$; clearly, $L_0(l) = l$.

By Lemma 5, we have $F_j(l) + L_{j-1}(l) = \cos(\alpha) L_j(l)$, which implies

$$F'_j(l) + L'_{j-1}(l) = \cos(\alpha) L'_j(l).$$

From Lemma 8, we know

$$F_j(l) = F_0(l) \left( \frac{L'_j(l)}{F'_0(l)} \right).$$

Together, these equations imply the linear differential equation

$$F'_j(l) - \frac{\cos(\alpha)}{F_0(l)} F_j(l) = - \frac{F_{j-1}(l)}{F_0(l)}. \quad (21)$$

The textbook solution for $y'(x) + f(x)y(x) = g(x)$ is

$$y(x) = \exp(-a(x)) \left( \int g(t) \exp(a(t)) \, dt + \kappa \right),$$

where $a = \int f$ and $\kappa$ denotes a constant that can be chosen arbitrarily. In our case,

$$a(l) = \int - \frac{\cos(\alpha)}{A + \cos(\alpha)} \, dt = - \ln(F_0(l))$$

because of $F_0(l) = A + \cos(\alpha) l$, and we obtain

$$F_j(l) = F_0(l) \left( \kappa_j - \int \frac{F_{j-1}(t)}{F_0^2(t)} \, dt \right). \quad (22)$$

Next, we consider a linkage of $\phi$-type, for parameters $l \in [l_1, l_2]$, and obtain analogously

$$\phi_j(l) = \phi_0(l) \left( \lambda_j - \int \frac{\phi_{j-1}(t)}{\phi_0^2(t)} \, dt \right). \quad (23)$$

Now we determine the constants $\kappa_j, \lambda_j$ such that the solutions 6 and 7 describe a contiguous curve. To this end, we must satisfy conditions 19 and 20.

We define $\kappa_0 := 1$ and

$$\kappa_{j+1} := \frac{\phi_j(l_2)}{F_0(l_0)} + \int \frac{F_j(t)}{F_0^2(t)} \, dt|_{l=0}$$

so that 22 becomes

$$F_{j+1}(l) = F_0(l) \left( \frac{\phi_j(l_2)}{F_0(l_0)} - \int_0^l \frac{F_j(t)}{F_0^2(t)} \, dt \right), \quad (22)$$
which, for \( l = 0 \), yields \( F_{j+1}(0) = \phi_j(l_2) \) (condition \( 20 \)).

Similarly, we set \( \lambda_0 := 1 \) and

\[
\lambda_{j+1} := \frac{F_{j+1}(l_1)}{\phi_0(l_1)} + \int_{l_1}^l \frac{\phi_j(t)}{\phi_0(t)} \, dt \big|_{l=l_1}
\]

so that \( 23 \) becomes

\[
\phi_{j+1}(l) = \phi_0(l) \left( \frac{F_{j+1}(l_1)}{\phi_0(l_1)} - \int_{l_1}^l \frac{\phi_j(t)}{\phi_0(t)} \, dt \right),
\]

and for \( l = l_1 \) we get \( F_{j+1}(l_1) = \phi_{j+1}(l_1) \) (condition \( 19 \)).

For simplicity, let us write

\[
G_j(l) := \frac{F_j(l)}{F_0(l)} \quad \text{and} \quad \chi_j(l) := \frac{\phi_j(l)}{\phi_0(l)},
\]

which leads to

\[
\begin{align*}
G_{j+1}(l) & = \frac{\phi_0(l_2)}{F_0(0)} \chi_j(l_2) - \int_0^l \frac{\chi_j(t)}{F_0(t)} \, dt, \\
\chi_{j+1}(l) & = \frac{F_0(l_1)}{\phi_0(l_1)} G_{j+1}(l_1) - \int_{l_1}^l \frac{\chi_j(t)}{\phi_0(t)} \, dt.
\end{align*}
\]

In order to find out if the fire fighter is successful we only need to check the values of \( F_j(l) \) at the end of each round, as the following lemma shows.

**Lemma 9.** The curve encloses the fire if and only if there exists an index \( j \) such that \( F_j(l_1) \leq 0 \) holds.

**Proof.** The free string shrinks to zero if and only if there exist an index \( j \) and argument \( l \) such that \( F_j(l) \leq 0 \) or \( \phi_j(l) \leq 0 \). Clearly, \( G_j \) and \( F_j \) have identical signs, as well as \( \chi_j \) and \( \phi_j \) do. Suppose that \( G_j > 0 \) and \( G_{j+1}(l) = 0 \), for some \( j \) and some \( l \in [0,l_1] \). By \( 23 \) function \( G_{j+1} \) is decreasing, therefore \( G_{j+1}(l_1) \leq 0 \). Now assume that \( G_i > 0 \) holds for all \( i \), and that we have \( \chi_{j-1} > 0 \) and \( \chi_j(l) = 0 \) for some \( j \) and some \( l \in [l_1,l_2] \). By \( 26 \) this implies \( \chi_j(l_2) \leq 0 \), and from \( 25 \) we conclude \( G_{j+1} \leq 0 \), in particular \( G_{j+1}(l_1) \leq 0 \).

**C Recursions**

The recursion given in equation \( 25 \) can be solved by iterated substitution. One obtains

\[
G_{j+1}(l) = \frac{\phi_0(l_2)}{F_0(0)} \sum_{\nu=0}^j (-1)^\nu I_\nu(l) \chi_{j-\nu}(l_2) + (-1)^{j+1} I_{j+1}(l) \quad (27)
\]
where

\[ I_n(x_n) = \int_0^{x_n} \frac{1}{F_0(x_{n-1})} \int_0^{x_{n-1}} \frac{1}{F_0(x_{n-2})} \cdots \int_0^{x_1} \frac{1}{F_0(x_0)} \, dx_0 \cdots dx_{n-1}. \]

By induction on \( n \) we derive

\[ I_n(x_n) = \frac{1}{n!} \frac{1}{\cos^n \alpha} \left( \ln \left( \frac{A + \cos(\alpha) x_n}{x_n} \right) \right)^n \]

since \( F_0(x) = A + \cos(\alpha) x \). By definition of \( l_1 \), we have \( \ln \left( \frac{A + \cos(\alpha) l_1}{l_1} \right) = 2\pi \cot \alpha \), so that setting \( l = l_1 \) in formula (27) leads to

\[ G_j(l_1) = \frac{\phi_0(l_2)}{F_0(0)} \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!} \left( \frac{2\pi}{\sin \alpha} \right)^\nu \chi_{j-\nu}(l_2) \]  

(28)

where, for convenience, \( \chi_{-1}(l_2) := \frac{\phi_0(l_2)}{F_0(0)} \). We observe that this formula is also true for \( j = 0 \). Multiplying both sides by \( F_0(l_1) \), and re-substituting (24), results in

\[ F_j(l_1) = \frac{F_0(l_1)}{F_0(0)} \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!} \left( \frac{2\pi}{\sin \alpha} \right)^\nu \phi_{j-\nu}(l_2) \]  

(29)

where \( \phi_{-1}(l_2) = F_0(0) \).

In a similar way we solve the recursion in (26) using

\[ \int_0^{x_n} \frac{1}{\phi_0(x_{n-1})} \int_0^{x_{n-1}} \frac{1}{\phi_0(x_{n-2})} \cdots \int_0^{x_1} \frac{1}{\phi_0(x_0)} \, dx_0 \cdots dx_{n-1} = \frac{1}{n!} \frac{1}{\cos^n \alpha} \left( \ln \left( \frac{x_n}{l_1} \right) \right)^n \]

and \( \ln \left( \frac{x_n}{l_1} \right) = \alpha \cot \alpha \). One obtains, after substituting \( l = l_2 \),

\[ \phi_j(l_2) = \frac{\phi_0(l_2)}{\phi_0(l_1)} \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!} \left( \frac{\alpha}{\sin \alpha} \right)^\nu \hat{F}_{j-\nu}(l_1) \]  

(30)

where \( \hat{F}_0(l_1) := \phi_0(l_1) \) and \( \hat{F}_{i+1}(l_1) := F_{i+1}(l_1) \).

In order to solve the cross-wise recursions (29) and (30) for the numbers \( F_j(l_1) \) we define the formal power series

\[ F(X) := \sum_{j=0}^{\infty} F_j X^j \quad \text{and} \quad \phi(X) := \sum_{j=0}^{\infty} \phi_j X^j \]

where \( F_j := F_j(l_1) \) and \( \phi_j := \phi_j(l_2) \), for short. From (29) we obtain

\[ F(X) = \frac{F_0}{F_0(0)} e^{-\frac{2\pi}{\sin \alpha} X} (X \phi(X) + F_0(0)), \]  

(31)
and from \(30\)

\[
\phi(X) = \frac{\phi_0}{\phi_0(l)} e^{-\frac{2\pi}{\sin \alpha} X} \left( X F(X) - F_0 + \phi_0(l) \right); \tag{32}
\]

both equalities can be easily verified by computing the products and comparing coefficients. Now we substitute \(32\) into \(31\), solve for \(F(X)\), divide both sides by \(F_0\) and expand by \(e^{\frac{2\pi}{\sin \alpha} X}\) to obtain

\[
\frac{F(X)}{F_0} = \frac{e^{vX} - rX}{e^{wX} - sX}, \tag{33}
\]

where \(v, r, w, s\) are the following functions of \(\alpha\):

\[
v = \frac{\alpha}{\sin \alpha} \quad \text{and} \quad r = e^{\alpha \cot \alpha},
\]

\[
w = \frac{2\pi + \alpha}{\sin \alpha} \quad \text{and} \quad s = e^{(2\pi + \alpha) \cot \alpha}. \tag{34}
\]

**Remark.** One can show that

\[
s^{-j} \frac{F_j}{F_0} = T_j(j + 1 - \frac{w}{w}) - \frac{r}{s} T_j(j)
\]

holds, where

\[
T_j(Y) := \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!} \left( \frac{w}{s} \right)^\nu (Y - \nu)^\nu,
\]

but it seems not obvious how to derive the signs of the \(F_j\) from this representation.

**D Singularity and residues**

Therefore, we resort to convex analysis and consider the right hand side of \(33\)

\[
f(z) := \frac{e^{vz} - rz}{e^{wz} - sz}, \tag{35}
\]

as a function of a complex variable, \(z\). Both numerator and denominator are analytic on the complex plane. Thus, singularities of \(f\) can only arise from zeroes of the denominator.

The equation \(e^{wZ} - sZ = 0\) has received some attention in the field of delay differential equations, see, e.g., Falbo \[2\]. With the following lemma, our main interest will be in case (i) and its transition into case (ii).

**Lemma 10.**

(i) If \(\frac{s}{w} < \epsilon\) then the equation \(e^{wZ} - sZ = 0\) has an infinite number of non-real, discrete conjugate pairs of complex roots.

(ii) As \(\frac{s}{w}\) increases to \(\epsilon\), the pair of complex roots \(z_0\) and \(\bar{z_0}\) of minimum imaginary part converge to the real zero \(x_0 = 1/w\).

(iii) For \(\frac{s}{w} > \epsilon\), the real zero \(x_0\) splits into two different real zeros.
Proof. Let \( z = a + i b \) be a complex zero of \( e^{wZ} - sZ = 0 \), for real parameters \( w, s \neq 0 \), that is,

\[
e^{wa} \left( \cos(wb) + i \sin(wb) \right) = sa + i sb.
\]

(36)

If the imaginary part \( b \) of \( z \) equals zero then \( e^{wa} = sa \), hence

\[
e^{wa} = \frac{s}{w}.
\]

This implies \( \frac{s}{w} \geq e \); see Figure 12. Now suppose that \( b \neq 0 \) holds. Then

\[
e^{wa} \cos(wb) = sa
\]

\[
e^{wa} \sin(wb) = sb,
\]

implies

\[
e^{wa} \cos(wb) = sa
\]

\[
e^{wa} \sin(wb) = sb,
\]

\[
\cot(wb) = \frac{a}{b} \text{ and }
\]

\[
e^{wb} \cot(wb) \sin(wb) = \frac{s}{w} wb.
\]

The graph of the real function \( h(X) := e^{X \cot X} \sin X \) intersects the line \( qX \) in an infinite number of discrete points; see Figure 13. Each intersection point \( p \) with abscissa \( x \) corresponds to a zero \( \cot(x) = \frac{a}{b} \) of \( e^{wZ} - sZ = 0 \), of absolute value \( \frac{x^2}{\sin^2 x \frac{w}{m}} \). Function \( h(X) \) has poles at the integer multiples of \( \pi \). As shown
in Figure 14, the first intersection point to the right of 0 has abscissa \( x_0 < \pi \), the following ones, \( x_k > 2k\pi \).

As slope \( q \) of the line \( qX \) increases beyond \( e \), its two innermost intersections \( \neq 0 \) with the graph of \( h(X) \) disappear. Thus, the imaginary parts of \( z_0, \bar{z}_0 \) become zero, causing a “double” real zero at \( 1/w \). As \( q \) grows beyond \( e \), this zero splits into two simple zeroes \( a/w \) and \( a'/w \); compare Figure 12.

For later use we note the following. While slope \( q \) is less than \( e \) we can write the zero \( z_0 \) of positive imaginary part associated with \( p_0 \) as

\[
z_0 = a + b i = \rho (\cos(\phi) + \sin(\phi)) i.
\]

This representation yields \( \cot(\phi) = a/b \). Since we have also derived \( \cot(wb) = a/b \) it follows that angle \( \phi \) of \( z_0 \) and \( wb \) are congruent modulo \( \pi \). Since \( wb = x_0 < \pi \) and \( \phi < \pi \) because of \( b > 0 \) we conclude that

\[
\phi = wb = w\rho \sin(\phi)
\]

is the smallest positive solution of \( e^X \cot X \sin X = qX \).

First we show that only poles can arise from these zeroes.

**Lemma 11.** Each zero \( u = a + b i \) (except \( 1/w \) in case (ii) of Lemma 10) is a pole of order one of the function \( g(z) = (e^{uz} - s z)^{-1} \), with residue \( \mu = ((wu - 1)s)^{-1} \).

**Proof.** We have

\[
\frac{z - u}{e^{uz} - sz} = \frac{z - u}{e^{u}e^{zs} - e^{uw} + su - sz} = \frac{1}{\frac{e^{uz} - e^{uw}}{z-u} - s}.
\]
Fig. 14: The first intersection points of $h(X) := e^X \cot X \sin X$ with line $qX$ to the right of 0. Here, $q = s/w$ is a function of angle $\alpha$. The numbers $\text{abs}_i$ denote the absolute values of the zeroes of $e^{wZ} - sZ = 0$ that correspond to the intersection points $p_i$, for $i = 0, \ldots, 3$. The value of $\text{abs}0$ is decreasing towards $0.30563$, as $\alpha$ tends to $\pi/2$. We have $\text{abs}1 = 1$ because of the zero in Lemma 12 (i). All other zeroes have absolute values substantially larger than 1.
As \( z \) tends to \( u \), the differential quotient in the denominator tends to the finite number \((e^w)'(u) = we^wu\). Hence, \( u \) is a pole of order 1, and \( g(z) \) has a local expansion

\[
g(z) = \frac{\mu}{z-u} + \sum_{i=0}^{\infty} w_i (z-u)^i.
\]

One can show that, in case (ii), zero \( 1/w \) gives rise to a pole of order two of \( g(z) \). Thus, function \( g(Z) \), and therefore \( f(Z) \), are meromorphic.

Next, we consider which of the poles of \( g(z) \) cancel out in the numerator of \( f(z) \). From now on, the parameters \( v, r, w, s \) are no longer considered independent but functions of the angle \( \alpha \), as introduced in 33.

**Lemma 12.** Numerator and denominator of function \( f(z) \) in 35 have the following zeroes in common:

(i) \( \cos(\alpha) + \sin(\alpha) i \)

(ii) \( \cos(\alpha) + (q+1) \sin(\alpha) i \) for each integer \( q \) satisfying \( \alpha = \frac{2p}{q} \pi \), for some integer \( p \).

**Proof.** Let \( z = a + bi \) be a common zero of \( e^{vZ} - rZ \) and \( e^{wZ} - sZ \). As in the proof of Lemma 10 we have

\[
e^{wb \cot(wb)} \sin(wb) = sb \quad (39)
\]

\[
e^{wb \cot(wb)} \cos(wb) = sa \quad (40)
\]

and, analogously,

\[
e^{vb \cot(vb)} \sin(vb) = rb \quad (41)
\]

\[
e^{vb \cot(vb)} \cos(vb) = ra. \quad (42)
\]

This implies \( \cot(wb) = a/b = \cot(vb) \), hence \( wb = vb + k\pi \) for some integer \( k \). Because \( s \) and \( r \) are positive for all \( \alpha \), the expressions in 40 and 42 must have the same sign, and we conclude that \( k = 2h \) is even. This implies \( \sin(wb) = \sin(vb) \) and from

\[
\frac{2\pi}{\sin \alpha} b = (w-v)b = 2h\pi
\]

follows \( b = h \sin \alpha \). Moreover, we have

\[
e^{\frac{2\pi}{\sin \alpha} a} = e^{(w-v)a} = \frac{e^{ua} \sin(vb)}{e^{va} \sin(vb)} = \frac{sb}{rb} = \frac{s}{r} = e^{2\pi \cot \alpha},
\]

and we obtain \( a = \cos \alpha \). This yields

\[
r \cos(vb) = e^{v \cos \alpha} \cos(vb) = e^{va} \cos(vb) = ra = r \cos \alpha,
\]

hence \( vb = \alpha + 2p\pi \) for some integer \( p \), and from \( h\alpha = hv \sin \alpha = vb \) follows

\[
\alpha = \frac{2p}{h-1} \pi.
\]
From now on we consider only case (i) of Lemma 10. Since $\frac{s}{w}$ is a strictly decreasing function in $\alpha$, and we have

$$\frac{e^{(2\pi+\alpha)\cot\alpha}}{2\pi+\alpha \sin \alpha} = \frac{s}{w} < e$$

$$\iff \alpha > \alpha_c := 1.17830 \ldots$$

The critical angle $\alpha_c$ corresponds to a speed $v_c = \frac{1}{\cos(\alpha_c)} = 2.61440 \ldots$. For $\alpha \in (\alpha_c, \pi/2)$ we can summarize our findings as follows.

**Lemma 13.** For $\alpha \in (\alpha_c, \pi/2)$, function $f(z)$ in (33) has only non-real, first-order poles as singularities. A conjugate pair $z_0, \overline{z_0}$ is situated at distance $< 0.31$ from the origin. All other poles are of absolute value $> 1$. For $\alpha \to \alpha_c$ both $z_0, \overline{z_0}$ converge to the real pole $(1/w, 0)$ where $1/w \approx 0.12383$.

**Proof.** By Lemma 11 and by Lemma 10 (i), function $f(z)$ has only poles of order one for singularities, none of which are real. Zero $\cos \alpha + \sin \alpha i$ of the denominator is of absolute value 1, but it is not a pole of $f(z)$, by Lemma 12 (i). All other zeroes of the denominator canceling out must be of absolute value $> 1$, by Lemma 12 (ii). Hence, $z_0$ and $\overline{z_0}$ are in fact poles. The bounds on the absolute values can be obtained by numerical evaluation; see Figure 14.

Now we are ready to state the first result.

**Theorem 3.** If the fire fighter’s speed exceeds $v_c \approx 2.6144$ then he will contain the fire.

**Proof.** We know that power series expansion at zero of $f(z)$ in (33) has a finite radius of convergence, $R$, because of the poles $z_0, \overline{z_0}$. Suppose that all coefficients $F_i$ of $F(X)$ were positive. By a Theorem of Pringsheim’s (cf. Flajolet and Sedgewick [4], p. 240) this would imply that $z = R$ is a singularity of $f(z)$. But by Lemma 13 $f(z)$ has only poles as singularities, and none of them is real because of $\alpha > \alpha_c$. Thus, there must be negative coefficients $F_i$, i.e., fire fighter’s curve encloses the fire.

### E An upper bound for strategy FF

In order to find out how many rounds are necessary before the fire is contained we use complex analysis to relate the size of the coefficients $F_i$ of $F(X)$ in (33) to the poles of $f(z)$ (cf. [4], p. 258 ff). Again, we are assuming that $\alpha > \alpha_c$ holds. Let $\Gamma$ denote a circle of radius $\gamma$ centered at the origin, counterclockwise oriented. We choose $\gamma := 0.9 \in (0.31, 1)$; Lemma 13 ensures that $\Gamma$ encircles the poles $z_0, \overline{z_0}$ but contains no other singularity of $f(z)$ in (33) either inside or on its boundary.

By Cauchy’s Residue Theorem,

$$\frac{1}{2\pi i} \int_{\Gamma} f(u) \frac{du}{u^{j+1}} = \sum_{z \text{ inside } \Gamma} \text{res}(z),$$

(45)
where the sum is over all residues of the poles $z$ of $\frac{f(z)}{z^{j+1}}$ encircled by $\Gamma$. These poles are

$$z_0 \text{ and } \overline{z_0} \text{ with residues } \mu = \frac{e^{vz_0} - rz_0}{(wz_0 - 1)s z_0^{j+1}} \text{ and } \overline{\mu} = \frac{e^{v\overline{z_0}} - r\overline{z_0}}{(w\overline{z_0} - 1)s \overline{z_0}^{j+1}}$$

$0$ with residue $\frac{F_j}{F_0}$;

this follows from Lemma 11 and from

$$f(z) \frac{z}{z^{j+1}} = \frac{F_0}{z} + \frac{F_1}{z} + \frac{F_2}{zz_0^{j+1}} + \ldots + \frac{F_j}{z} + \sum_{i=0}^{\infty} \frac{F_{j+i+1}}{z^i}.$$

Thus, equation 45 implies

$$\frac{F_j}{F_0} = -(\mu + \overline{\mu}) + \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z^{j+1}} \, dz. \quad (46)$$

The integral can be upper bounded by

$$\left| \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z^{j+1}} \, dz \right| \leq \frac{1}{2\pi} D \int_{\Gamma} \frac{1}{|z|^{j+1}} \, dz = \frac{1}{2\pi} D \left( \frac{1}{\gamma} \right)^{j+1} 2\pi \gamma \quad (47)$$

$$= D \gamma^{-j} \quad (48)$$

because all $z$ on $\Gamma$ have absolute value $\gamma$. Here, $D$ denotes the maximum value function $|f(z)|$ attain on the compact set $\Gamma \times [\alpha_c, \pi/2]$.

Now let

$$z_0 = \rho (\cos \phi + \sin \phi i) = a + bi \quad (49)$$

be the pole different from zero whose imaginary part, $b$, is positive. Let us recall from 38 in the proof of Lemma 10 that $\phi$, the angle of $z_0$, is the smallest positive real number solving $e^x \cot(x) \sin(x) = \frac{s}{w} x$, with $s, w$ as defined in 34. We also have shown that

$$\phi = wb = w\rho \sin \phi \quad (50)$$

holds. Furthermore, $\rho = |z_0|$. Let $x_0 = (1/w, 0)$ denote the real pole to which $z_0$ and $\overline{z_0}$ converge as $\alpha$ decreases to $\alpha_c$; compare Lemma 13. We obtain the following identities.

$$w |z_0 - x_0| = \sqrt{w^2 \rho^2 - 2w \rho \cos(\phi) + 1} \quad (51)$$

$$= \sqrt{\frac{\phi^2}{\sin^2 \phi} - 2\phi \frac{\cos \phi}{\sin \phi} + 1} \quad (52)$$

$$= \sqrt{(wa - 1)^2 + w^2 b^2} \quad (53)$$

The sum of residues can be written as follows.
Lemma 14. We have

\[- (\mu + \overline{\mu}) \begin{array}{l}
= \left( \frac{e^{va}}{\rho^2} \cos((j + 1)\phi - vb) - \frac{e^{va} w}{\rho} \cos((j + 2)\phi - vb) \right) \end{array} \]

(54)

\[- \left( \frac{r}{\rho} \cos(j\phi) - rw \cos((j + 1)\phi) \right) \end{array} \}

(55)

\[
\frac{2}{s((wa - 1)^2 + w^2 b^2)} \cdot \frac{1}{\rho j - 1}
\]

(56)

Proof. Using $z_0 = a + bi$ we obtain

\[
\mu + \overline{\mu} = \frac{e^{va} - r z_0}{s(wz_0 - 1) z_0^{j+1}} + \frac{e^{va} w}{s(wz_0 - 1) z_0^{j+1}}
\]

\[
= \frac{1}{s} \frac{we^{va} z_0^{j+2} - e^{va} z_0^{j+1} - rw(a^2 + b^2) z_0^{j+1} + r(a^2 + b^2) z_0^j}{(wa - 1)^2 + w^2 b^2} \frac{(a^2 + b^2) z_0^j}{2}
\]

\[
+ \frac{1}{s} \frac{we^{va} z_0^{j+2} - e^{va} z_0^{j+1} - rw(a^2 + b^2) z_0^{j+1} + r(a^2 + b^2) z_0^j}{(wa - 1)^2 + w^2 b^2} \frac{(a^2 + b^2) z_0^j}{2}
\]

With $e^{va} = e^{va} (\cos(vb) + \sin(vb) i)$ and $z_0^j = \rho^j (\cos(j\phi) + \sin(j\phi) i)$ one gets

\[
\text{Re}(e^{va} z_0^{j+2}) = \text{Re}(e^{va} (\cos(vb) + \sin(vb) i) \rho^{j+2} (\cos((j + 2)\phi) - \sin((j + 2)\phi) i))
\]

\[
= e^{va} \rho^{j+2} (\cos(vb) \cos((j + 2)\phi) + \sin(vb) \sin((j + 2)\phi))
\]

\[
= e^{va} \rho^{j+2} \cos(vb - (j + 2)\phi),
\]

and substituting $a^2 + b^2 = \rho^2$ and $z + \overline{z} = 2\text{Re}(z)$ shows that $\mu + \overline{\mu}$ equals

\[
\frac{2}{s} \frac{we^{va} \rho^{j+2} \cos(vb - (j + 2)\phi) - e^{va} \rho^{j+1} \cos(vb - (j + 1)\phi) - rw \rho^{j+3} \cos((j + 1)\phi) + r \rho^{j+2} \cos(j\phi)}{(wa - 1)^2 + w^2 b^2} \rho^{2j+2}
\]

The sign of $-(\mu + \overline{\mu})$ in Lemma 14 is determined by the four cosine terms. If we substitute $j$ with a real “time” variable $t$, we can consider them as sine waves of the same frequency, $\phi$, but different amplitudes and phases. A finite sum of such waves is again a sine wave of frequency $\phi$, so that

\[
\left( \frac{e^{va}}{\rho^2} \cos((t + 1)\phi - vb) - \frac{e^{va} w}{\rho} \cos((t + 2)\phi - vb) \right) \end{array} \]

(57)

\[- \left( \frac{r}{\rho} \cos(t\phi) - rw \cos((t + 1)\phi) \right) \end{array} \}

(58)

\[
= L \sin(t\phi + p)
\]

(59)

holds, with amplitude $L$ and phase $p$. In this section we need only a lower bound to amplitude $L$.

Lemma 15. We have

\[
L \geq L_0 := \sqrt{w^2 \rho^2 - 2 w \rho \cos(\phi) + 1} \left( \frac{e^{va}}{\rho^2} - \frac{r}{\rho} \right).
\]

(60)
Proof. In general, one has

\[ a_1 \sin(t\phi + p_1) + a_2 \sin(t\phi + p_2) = \sqrt{a_1^2 + a_2^2 + 2a_1a_2 \cos(p_1 - p_2)} \sin(t\phi + p), \]

where phase \( p \) depends on \( a_1, a_2, p_1, p_2 \). These formulae for the sum of two waves of identical frequency can be found in textbooks or, for example, in Bronstein et al., Taschenbuch der Mathematik, 1993. This yields

\[
\frac{e^{va}}{\rho} \cos((t + 1)\phi - vb) - \frac{e^{wa}}{\rho} \cos((t + 2)\phi - vb)
\]

\[=\frac{e^{va}}{\rho^2} \sin(t\phi + \phi - vb + \frac{\pi}{2}) + \frac{e^{wa}}{\rho} \sin(t\phi + 2\phi - vb + \frac{3\pi}{2}) \] \hspace{1cm} (61)

\[=\sqrt{\frac{e^{2va}}{\rho^4} + \frac{e^{2wa}w^2}{\rho^2}} + 2\frac{e^{va}w}{\rho^2} \cos(-\phi - \pi) \sin(t\phi + p) \] \hspace{1cm} (62)

\[=\frac{e^{va}}{\rho^2} \sqrt{w^2\rho^2 - 2w\rho \cos(\phi) + 1} \sin(t\phi + p). \] \hspace{1cm} (63)

Similarly,

\[- \left( \frac{r}{\rho} \cos(t\phi) - \frac{rw}{\rho} \cos((t + 1)\phi) \right) \]

\[=\frac{r}{\rho} \sqrt{w^2\rho^2 - 2w\rho \cos(\phi) + 1} \sin(t\phi + q). \] \hspace{1cm} (64)

Thus, the sum of these two sine waves has amplitude

\[ \sqrt{w^2\rho^2 - 2w\rho \cos(\phi) + 1} \left( \frac{e^{va}}{\rho^2} - \frac{r}{\rho} \right). \] \hspace{1cm} (65)

\[ \geq \sqrt{w^2\rho^2 - 2w\rho \cos(\phi) + 1} \left( \frac{e^{va}}{\rho^2} - \frac{r}{\rho} \right). \] \hspace{1cm} (66)

Now we can prove a quantitative version of Theorem 3.

**Theorem 4.** Let \( \alpha > \alpha_c \). Then there is an index \( j \in O(\frac{2\pi}{\phi}) \) such that \( F_j < 0 \) holds.

Proof. The function \( h(t) := L \sin(t\phi + p) \) of 59 attains its minima \(-L\) at arguments \( t^* \) where \( t^* \phi + p = \frac{3\pi}{2} \) mod 2\( \pi \). For an integer \( j \) at most 1/2 away from \( t^* \) we have

\[ h(j) \leq h(t^* + \frac{1}{2}) = L \sin(\frac{3\pi + \phi}{2}) = -L \cos(\frac{\phi}{2}) \]

these terms are negative because of \( \phi < 2.09 < \pi \). This implies

\[ \frac{F_j}{F_0} < -\frac{2}{s} \left( \frac{e^{va}}{|z_0|} - r \right) \frac{1}{w} \frac{1}{|z_0 - x_0|} \cos(\frac{\phi}{2}) |z_0|^{-j} + D \cdot 0.9^{-j}, \] \hspace{1cm} (67)
We observe that such integers \( j \) occur (at least) once in every period of length \( 2\pi/\phi \) of function \( h(t) \).

Since \(|z_0| < 0.31 < 0.9\), the powers \( |z_0|^{-j} \) grow in \( j \) much faster than \( 0.9^{-j} \) does. All coefficients in \( 69 \) are positive, and lower bounded by independent constants on \([\alpha_c, \pi/2]\). Indeed, we have \( \frac{2s}{\pi} \geq 0.091 \) with a minimum at \( \alpha = \pi/2 \), and \( w |z_0 - x_0| \leq 0.33 \) with a maximum at \( \alpha = \pi/2 \).

Hence, after a constant number of periods the value of \( 69 \) becomes negative. This completes the proof of Theorem 4.

Numerical inspection shows that a suitable integer \( j \) can already be found in the first period of function \( h \), so that \( j \leq \frac{2\pi}{\phi} + 1 \).

F  A lower bound for strategy FF

Now we let \( \alpha \) decrease to the critical value \( \alpha_c \), and prove that the first index \( j \), for which \( F_j \) becomes negative, grows with \( \pi/\phi \) to infinity. To this end we must show that the sine wave in \([59]\) starts, at zero, near the beginning of its positive half-cycle, so that it takes half a period before negative values can occur.

The graph of \( \sin(t\phi + p) \) is shifted, along the \( t \)-axis, by \( p/\phi \) to the left, as compared to the graph of \( \sin(t) \). As \( \alpha \) tends to \( \alpha_c \), frequency \( \phi \) goes to zero, and so does phase \( p \). But, surprisingly, their ratio rapidly converges to a small constant.

**Lemma 16.** We have

\[
\sigma := \lim_{\alpha \to \alpha_c} \frac{p}{\phi} = \left( \frac{r}{\pi w} - r \right) \left( 1 - \frac{v}{w} \right) + \frac{1}{3} \approx 1.351.
\]

Figure 15 shows \(-(\mu + \pi)p^{j-1}\) as a function of time parameter \( j = t \); see \([54]\) to \([56]\). Crucial in the proof is the following geometric fact.

**Lemma 17.** Consider the triangle shown in Figure [16] which has a base of length 1, a base angle of \( \phi \), and height \( \phi \). As \( \phi \) goes to zero, the ratio \( \frac{\tau}{\phi} \) tends to \( 1/3 \), and \( \gamma \) converges to \( \pi/2 \).

**Proof.** From \( C \cos \tau = \phi \) and \( C \sin \tau = 1 - \phi \cot \phi \) we obtain

\[
\tan \tau = \frac{\sin \phi - \phi \cos \phi}{\phi \sin \phi},
\]

and because \( \tau \) must go to zero as \( \phi \) does, we have

\[
\frac{\tau}{\phi} = \cos \tau \frac{\tau}{\sin \tau} \tan \tau \approx \frac{\sin \phi - \phi \cos \phi}{\phi^2 \sin \phi}.
\]

A twofold application of l’Hospital’s rule shows that the last term has the same limit as

\[
\frac{\sin \phi}{2 \sin \phi \sin \phi} \sim \frac{\cos \phi}{3 \cos \phi - \phi \sin \phi}.
\]

34
Fig. 15: The shift to the left is almost constant, as $\alpha$ tends to $\alpha_c$ and period $2\pi/\phi$ goes to infinity.

Fig. 16: Ratio $\tau/\phi$ tends to $1/3$, as $\phi$ goes to zero.
which converges to $1/3$. Moreover, we have

$$\sin \gamma = \sin(\pi/2 - \phi) = \cos(\phi)$$

which tends to 1, so that $\gamma$ converges to $\pi/2$.

Now we give the proof of Lemma 16.

**Proof.** As in the proof of Lemma 15 one generally has

$$a_1 \sin(t\phi + p_1) + a_2 \sin(t\phi + p_2) = a_3 \sin(t\phi + p_3),$$

where the new amplitude, $a_3$, is given by

$$a_3 = \sqrt{a_1^2 + a_2^2 + 2a_1a_2 \cos(p_1 - p_2)},$$

and the new phase, $p_3$, fulfills

$$p_3 = \arcsin\left(\frac{a_2 \sin(p_2 - p_1)}{a_3}\right) + p_1.$$

First, we are applying this formula to

$$\frac{e^{va}}{\rho^2} \cos((t+1)\phi - vb) - \frac{e^{va}}{\rho} \cos((t+2)\phi - vb)$$

$$= \frac{e^{va}}{\rho^2} \sin(t\phi + \phi - vb + \pi/2) + \frac{e^{va}}{\rho} \sin(t\phi + 2\phi - vb + 3\pi/2)$$

and obtain, as in

$$\frac{e^{va}}{\rho^2} \sqrt{\sin^2 \phi - 2w \cos(\phi) + 1},$$

and, for the new phase,

$$\frac{e^{va}}{\rho^2} \sqrt{\sin^2 \phi - 2w \cos(\phi) + 1}$$

using [59] and the triangle in Figure 16. We conclude that the value of arcsin goes to $-\pi/2$, so that $p_3$ converges to zero. For the resulting shift we obtain

$$\frac{p_3}{\phi} = \frac{\tau}{\phi} + 1 - \frac{v}{w} \rightarrow \frac{1}{3} + 1 - \frac{v}{w}$$
by Lemma 17

Next, we consider 58

\[ r w \cos((t + 1)\phi) - \frac{r}{\rho} \cos(t\phi) = r w \sin(t\phi + \phi + \pi/2) + \frac{r}{\rho} \sin(t\phi + 3\pi/2) \]

As in 66

\[ a_{58} = \frac{r}{\rho} \sqrt{w^2\rho^2 - 2w\rho \cos(\phi) + 1} \]

and for the phase,

\[ p_{58} = \arcsin\left(\frac{\sin(\phi)}{\sqrt{w^2\rho^2 - 2w\rho \cos(\phi) + 1}}\right) + \phi + \pi/2 = \pi/2 + \gamma + \tau + \phi, \]

observing that the argument of arcsin equals

\[ \frac{\sin(\phi)}{C} = \frac{\sin(\gamma + \tau)}{1}, \]

applying the law of sines to the triangle shown in Figure 16. We see that \( p_{58} \) goes to \( \pi \); in this case, the shift does not converge.

Now we consider the sum of 57 and 58. We know from 67 the final amplitude,

\[ a = \sqrt{w^2\rho^2 - 2w\rho \cos(\phi) + 1} \sqrt{\frac{e^{2\nu a}}{\rho^2} + \frac{2}{\rho^2} + 2 \frac{e^{2\nu a}}{\rho^2} \cos(p_{58} - p_{57})}, \]

and obtain for the phase

\[ p = \arcsin\left(\frac{r}{\rho} \frac{\sin(p_{58} - p_{57})}{\sqrt{\frac{2}{\rho^2} + 2 \frac{e^{2\nu a}}{\rho^2} \cos(p_{58} - p_{57})}}\right) + p_{57} \]

For short, let \( \overline{p} \) denote the arcsin term, and let \( R \) be the square root in the denominator. Since \( p_{58} - p_{57} \) tends to \( \pi \) we conclude that \( \overline{p} \) goes to zero. Thus, we obtain

\[
\frac{p}{\phi} = \frac{\overline{p}}{\sin \overline{p}} \frac{\sin \overline{p}}{\phi} + \frac{p_{57}}{\phi} = \frac{\overline{p}}{\sin \overline{p}} \frac{r}{\rho} \frac{1}{R} \frac{\sin(p_{58} - p_{57})}{\pi - (p_{58} - p_{57})} - \frac{\pi - (p_{58} - p_{57})}{\phi} + \frac{p_{57}}{\phi} \sim \frac{r}{\rho} \frac{1}{R} \frac{\phi - \gamma - \nu b}{\phi} + \frac{p_{57}}{\phi} = \frac{r}{\rho} \frac{1}{R} \frac{\phi - \nu b}{\phi} + \frac{p_{57}}{\phi} \rightarrow \frac{r w}{e^{\nu/w} w^2 - rw} \left(1 - \frac{v}{w}\right) + 1/3 + 1 - \frac{v}{w}. \]

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This concludes the proof of Lemma 16.

Now let \( j \) be an integer satisfying
\[
j \leq \frac{\pi}{\phi} - 2\sigma.
\]
Lemma 16 implies that the sign of \( \sin(j\phi + p) \) in 59 is positive, and, even more, that
\[
\sin(j\phi + p) > \sin(p) > \sin(1.35\phi) \tag{70}
\]
holds. For such integers \( j \) we obtain, similarly to 69 in Section E,
\[
\frac{F_j}{F_0} > \frac{2}{s} \left( e^{v\alpha} - r \right) \frac{1}{w} \frac{1}{|z_0 - x_0|} \sin(1.35\phi) |z_0|^{-j} - 0.1 \cdot 0.9^{-j} \tag{71}
\]
Here we have used the following estimate for \( D \) in 48.

**Lemma 18.** With the radius of \( \Gamma \) equal to \( \gamma = 0.9 \), we have \( |f(z)| \leq 0.1 \) for all \( z \) on \( \Gamma \), if \( \alpha \) is close enough to \( \alpha_c \).

**Proof.** Let \( z = \gamma e^{i\psi} \) be a parameterization of circle \( \Gamma \) for \( \psi \in [0 \ldots 2\pi] \). By multiplication with complex conjugates,
\[
|f(z)|_{z \in \Gamma} = \left| \frac{e^{v\gamma e^{i\psi}} - r \gamma e^{i\psi}}{e^{w\gamma e^{i\psi}} - s \gamma e^{i\psi}} \right|
\]
\[
= \left( \frac{e^{2v\gamma \cos \psi} - 2e^{v\gamma \cos \psi} r\gamma \cos(v\gamma \sin \psi - \psi) + r^2\gamma^2}{e^{2w\gamma \cos \psi} - 2e^{w\gamma \cos \psi} s\gamma \cos(w\gamma \sin \psi - \psi) + s^2\gamma^2} \right) \sqrt{e^{2v\gamma \sin^2 \phi} - 2e^{v\gamma \sin \phi} \gamma r \sin \phi + 1}
\]
The maximum is attained at \( \psi = \pi \), and it grows monotonically from 0.09... for \( \alpha = \alpha_c \) to 1.269... for \( \alpha = \pi/2 \).

Now we can state the lower bound.

**Theorem 5.** As angle \( \alpha \) decreases to the critical value \( \alpha_c \), the number \( j \) of rounds necessary to contain the fire is at least \( j > \frac{\pi}{\phi} - 2.71 \). This lower bound grows to infinity.

**Proof.** By the preceding discussion, estimate 71 holds for each \( j \) that stays below this bound. As \( \phi \) tends to 0 we get
\[
\sin(1.35\phi) = \frac{\sin(1.35\phi)}{\sqrt{1 - 2w\rho \cos(\phi) + 1}}
\]
\[
= \frac{\sin(1.35\phi)}{\sqrt{\frac{2\phi}{\sin \phi} \frac{\cos \phi}{\sin \phi} + 1}}
\]
\[
\sim 1.35 \frac{\sin \phi}{\sqrt{\frac{\phi^2}{\sin \phi^2} - 2\phi \frac{\cos \phi}{\sin \phi} + 1}}
\]
\[
= 1.35 \sin(\gamma + \tau)
\]
\[
\sim 1.35.
\]
Here, the first equality follows from 53 and the second, from 50. Then we have applied l’Hospital’s rule, and the next line follows from Lemma 17. Indeed, the square root is equal to $c$ in Figure 16, and we can apply the law of sines together with the fact that $\gamma$ goes to $\pi/2$, and $\tau$ to zero.

Substituting in 71 the other limit values (none of which is critical) we find

$$\frac{F_j}{F_0} > 0.091 \cdot 7.82 \cdot 1.35 \cdot |z_0|^{-j} - 0.1 \cdot 0.9^{-j}$$
$$\geq 0.96 \cdot 0.1239^{-j} - 0.1 \cdot 0.9^{-j}$$
$$> 0.$$

Here 7.82 is the limit of $\frac{e^{\alpha}}{|z_0|-r}$ as $\alpha$ tends to $\alpha_c$. This completes the proof of Theorem 5.