A NOTE ON THREE TYPES OF QUASISYMMETRIC FUNCTIONS

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Abstract. In the context of generating functions for $P$-partitions, we revisit three flavors of quasisymmetric functions: Gessel’s quasisymmetric functions, Chow’s type B quasisymmetric functions, and Poirier’s signed quasisymmetric functions. In each case we use the inner coproduct to give a combinatorial description (counting pairs of permutations) to the multiplication in: Solomon’s type A descent algebra, Solomon’s type B descent algebra, and the Mantaci-Reutenauer algebra, respectively. The presentation is brief and elementary, our main results coming as consequences of $P$-partition theorems already in the literature.

1. Quasisymmetric functions and Solomon’s descent algebra

The ring of quasisymmetric functions is well-known (see [8], ch. 7.19). Recall that a quasisymmetric function is a formal series

$$Q(x_1, x_2, \ldots) \in \mathbb{Z}[[x_1, x_2, \ldots]]$$

of bounded degree such that the coefficient of $x_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_k}^{\alpha_k}$ is the same for all $i_1 < i_2 < \cdots < i_k$ and all compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. Recall that a composition of $n$, written $\alpha \vdash n$, is an ordered tuple of positive integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ such that $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k = n$. In this case we say that $\alpha$ has $k$ parts, or $\# \alpha = k$. We can put a partial order on the set of all compositions of $n$ by reverse refinement. The covering relations are of the form

$$(\alpha_1, \ldots, \alpha_i + \alpha_{i+1}, \ldots, \alpha_k) \prec (\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_k).$$

Let $Q_{sym}^n$ denote the set of all quasisymmetric functions homogeneous of degree $n$. The ring of quasisymmetric functions can be defined as $Q_{sym} := \bigoplus_{n \geq 0} Q_{sym}^n$, but our focus will be on the quasisymmetric functions of degree $n$, rather than the ring as a whole.
The most obvious basis for \( \mathcal{Q} \text{sym}_n \) is the set of monomial quasisymmetric functions, defined for any composition \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \vdash n \),
\[
M_\alpha := \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.
\]

We can form another natural basis with the fundamental quasisymmetric functions, also indexed by compositions,
\[
F_\alpha := \sum_{\alpha \preceq \beta} M_\beta,
\]
since, by inclusion-exclusion we can express the \( M_\alpha \) in terms of the \( F_\alpha \):
\[
M_\alpha = \sum_{\alpha \preceq \beta} (-1)^{\#\beta - \#\alpha} F_\beta.
\]

As an example,
\[
F_{(2,1)} = M_{(2,1)} + M_{(1,1,1)} = \sum_{i < j} x_i^2 x_j + \sum_{i < j < k} x_i x_j x_k = \sum_{i \leq j < k} x_i x_j x_k.
\]

Compositions can be used to encode descent classes of permutations in the following way. Recall that a descent of a permutation \( \pi \in \mathfrak{S}_n \) is a position \( i \) such that \( \pi_i > \pi_{i+1} \), and that an increasing run of a permutation \( \pi \) is a maximal subword of consecutive letters \( \pi_{i+1} \pi_{i+2} \cdots \pi_{i+r} \) such that \( \pi_{i+1} < \pi_{i+2} < \cdots < \pi_{i+r} \). By maximality, we have that if \( \pi_{i+1} \pi_{i+2} \cdots \pi_{i+r} \) is an increasing run, then \( i \) is a descent of \( \pi \) (if \( i \neq 0 \)), and \( i + r \) is a descent of \( \pi \) (if \( i + r \neq n \)). For any permutation \( \pi \in \mathfrak{S}_n \), define the descent composition, \( C(\pi) \), to be the ordered tuple listing (from left to right) the lengths of the increasing runs of \( \pi \). If \( C(\pi) = (\alpha_1, \alpha_2, \ldots, \alpha_k) \), we can recover the descent set of \( \pi \):
\[
\text{Des}(\pi) = \{ \alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_k \}.
\]

Since \( C(\pi) \) and \( \text{Des}(\pi) \) have the same information, we will use them interchangeably. For example, the permutation \( \pi = (3, 4, 5, 2, 6, 1) \) has \( C(\pi) = (3, 2, 1) \) and \( \text{Des}(\pi) = \{3, 5\} \).

Recall ([7], ch. 4.5) that a \( P \)-partition is an order-preserving map from a poset \( P \) to some (countable) totally ordered set. To be precise, let \( P \) be any labeled partially ordered set (with partial order \( <_P \)) and let \( S \) be any totally ordered countable set. Then \( f : P \to S \) is a \( P \)-partition if it satisfies the following conditions:

(1) \( f(i) \leq f(j) \) if \( i <_P j \)
(2) \( f(i) < f(j) \) if \( i <_P j \) and \( i > j \) (as labels)
We let $A(P)$ (or $A(P; S)$ if we want to emphasize the image set) denote the set of all $P$-partitions, and encode this set in the generating function
\[ \Gamma(P) := \sum_{f \in A(P)} x_{f(1)} x_{f(2)} \cdots x_{f(n)}, \]
where $n$ is the number of elements in $P$ (we will only consider finite posets). If we take $S$ to be the set of positive integers, then it should be clear that $\Gamma(P)$ is always going to be a quasisymmetric function of degree $n$. As an easy example, let $P$ be the poset defined by $3 >_P 2 <_P 1$. In this case we have
\[ \Gamma(P) = \sum_{f(3) \geq f(2) < f(1)} x_{f(1)} x_{f(2)} x_{f(3)}. \]

We can consider permutations to be labeled posets with total order $\pi_1 <_\pi \pi_2 <_\pi \cdots <_\pi \pi_n$. With this convention, we have
\[ A(\pi) = \{ f : [n] \to S \mid f(\pi_1) \leq f(\pi_2) \leq \cdots \leq f(\pi_n) \}
\]
and $k \in \text{Des}(\pi) \Rightarrow f(\pi_k) < f(\pi_{k+1})$,

and
\[ \Gamma(\pi) = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n \atop k \in \text{Des}(\pi) \Rightarrow i_k < i_{k+1}} x_{i_1} x_{i_2} \cdots x_{i_n}. \]

It is not hard to verify that in fact we have
\[ \Gamma(\pi) = F_{C(\pi)}, \]
so that generating functions for the $P$-partitions of permutations $\pi \in S_n$ form a basis for $\mathbb{Q} \text{sym}_n$.

We have the following theorem related to $P$-partitions of permutations, due to Gessel [3].

**Theorem 1.** As sets, we have the bijection
\[ A(\pi; ST) \leftrightarrow \bigoplus_{\sigma \tau = \pi} A(\tau; S) \oplus A(\sigma; T), \]
where $ST$ is the cartesian product of the sets $S$ and $T$ with the lexicographic ordering and $\bigoplus$ denotes the disjoint union.

Let $X = \{ x_1, x_2, \ldots \}$ and $Y = \{ y_1, y_2, \ldots \}$ be two two sets of commuting indeterminates. Then we define the bipartite generating function,
\[ \Gamma(\pi)(XY) = \sum_{(i_1, j_1) \leq (i_2, j_2) \leq \cdots \leq (i_n, j_n) \atop k \in \text{Des}(\pi) \Rightarrow (i_k, j_k) < (i_{k+1}, j_{k+1})} x_{i_1} \cdots x_{i_n} y_{j_1} \cdots y_{j_n}. \]

We will apply Theorem 1 with $S = T = \mathbb{P}$, the positive integers.
Corollary 1. We have
\[ F_{C(\pi)}(XY) = \sum_{\sigma \tau = \pi} F_{C(\tau)}(X) F_{C(\sigma)}(Y). \]

Following [3], we can define a coalgebra \( Q_{sym}^* \) in the following way. If \( \pi \) is any permutation with \( C(\pi) = \gamma \), let \( a_{\alpha,\beta}^\gamma \) denote the number of pairs of permutations \( (\sigma, \tau) \in S_n \times S_n \) with \( C(\sigma) = \alpha \), \( C(\tau) = \beta \), and \( \sigma \tau = \pi \). Then Corollary [4] defines a coproduct \( Q_{sym}^* \rightarrow Q_{sym}^* \otimes Q_{sym}^* \):  
\[ F_\gamma \mapsto \sum_{\alpha,\beta=\gamma} a_{\alpha,\beta}^\gamma F_\beta \otimes F_\alpha. \]

The dual space to \( Q_{sym}^* \) is then \( Q_{sym}^\ast \) equipped with multiplication  
\[ F_\beta \ast F_\alpha = \sum_\gamma a_{\alpha,\beta}^\gamma F_\gamma. \]

Let \( \mathbb{Z}S_n \) denote the group algebra of the symmetric group. We can define its dual coalgebra \( \mathbb{Z}S_n^* \) with comultiplication  
\[ \pi \mapsto \sum_{\sigma \tau = \pi} \tau \otimes \sigma. \]

Then we have a surjective homomorphism of coalgebras \( \varphi^*: \mathbb{Z}S_n^* \rightarrow Q_{sym}^* \) given by  
\[ \varphi^*(\pi) = F_{C(\pi)}. \]

The dualization of this map is then an injective homomorphism \( \varphi : Q_{sym} \rightarrow \mathbb{Z}S_n \) with  
\[ \varphi(F_\alpha) = \sum_{C(\pi) = \alpha} \pi. \]

The image of \( \varphi \) is then a subalgebra of the group algebra, with basis  
\[ u_\alpha := \sum_{C(\pi) = \alpha} \pi. \]

This subalgebra is well-known as Solomon’s descent algebra, denoted \( \text{Sol}(A_{n-1}) \). Corollary [4] has then given a combinatorial description to multiplication in \( \text{Sol}(A_{n-1}) \):  
\[ u_\beta u_\alpha = \sum_{\gamma=\gamma} a_{\alpha,\beta}^\gamma u_\gamma. \]

The above arguments are due to Gessel [3]. We give them here in full detail for comparison with later sections, when we will outline a similar relationship between type Chow’s B quasisymmetric functions and \( \text{Sol}(B_n) \), and between Poirier’s signed quasisymmetric functions and the Mantaci-Reutenauer algebra.
2. Type B quasisymmetric functions and Solomon’s descent algebra

The type B quasisymmetric functions can be viewed as the natural objects related to type B $P$-partitions (see [2]). Define the type B posets (with $2n + 1$ elements) to be posets labeled distinctly by $\{-n, \ldots, -1, 0, 1, \ldots, n\}$ with the property that if $i <_{P} j$, then $-j <_{P} -i$. For example, $-2 <_{P} 1 <_{P} 0 <_{P} -1 >_{P} 2$ is a type B poset.

Let $P$ be any type B poset, and let $S = \{s_0, s_1, \ldots\}$ be any countable totally ordered set with a minimal element $s_0$. Then a type B $P$-partition is any map $f : P \to \pm S$ such that

1. $f(i) \leq f(j)$ if $i <_{P} j$
2. $f(i) < f(j)$ if $i <_{P} j$ and $i > j$ (as labels)
3. $f(-i) = -f(i)$

where $\pm S$ is the totally ordered set $\cdots < -s_2 < -s_1 < s_0 < s_1 < s_2 < \cdots$

If $S$ is the nonnegative integers, then $\pm S$ is the set of all integers.

The third property of type B $P$-partitions means that $f(0) = 0$ and the set $\{f(i) \mid i = 1, 2, \ldots, n\}$ determines the map $f$. We let $A_B(P) = A_B(P; \pm S)$ denote the set of all type B $P$-partitions, and define the generating function for type B $P$-partitions as

$$\Gamma_B(P) := \sum_{f \in A_B(P)} x^{f(1)} x^{f(2)} \cdots x^{f(n)}.$$

Signed permutations $\pi \in \mathfrak{S}_n$ are type B posets with total order

$$-\pi_n < \cdots < -\pi_1 < 0 < \pi_1 < \cdots < \pi_n.$$

We then have

$$A_B(\pi) = \{f : \pm [n] \to \pm S \mid f(-i) = -f(i),$$

$$0 \leq f(\pi_1) \leq f(\pi_2) \leq \cdots \leq f(\pi_n),$$

$$\text{and } k \in \text{Des}_B(\pi) \Rightarrow f(\pi_k) < f(\pi_{k+1})\},$$

and

$$\Gamma_B(\pi) = \sum_{0 \leq \ell_1 \leq \ell_2 \leq \cdots \leq \ell_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Here, the type B descent set, $\text{Des}_B(\pi)$, keeps track of the ordinary descents as well as a descent in position 0 if $\pi_1 < 0$. Notice that if $\pi_1 < 0$, then $f(\pi_1) > 0$, and $\Gamma_B(\pi)$ has no $x_0$ terms, as in

$$\Gamma_B((-3, 2, -1)) = \sum_{0 < i < j < k} x_i x_j x_k.$$
The possible presence of a descent in position zero is the crucial difference between type A and type B descent sets. Define a pseudo-composition of $n$ to be an ordered tuple $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_1 \geq 0$, and $\alpha_i > 0$ for $i > 1$, such that $\alpha_1 + \cdots + \alpha_k = n$. We write $\alpha \vdash n$ to mean $\alpha$ is a pseudo-composition of $n$. Define the descent pseudo-composition $C(\pi)$ of a signed permutation $\pi$ be the lengths of its increasing runs as before, but now we have $\alpha_1 = 0$ if $\pi_1 < 0$. As with ordinary compositions, the partial order on pseudo-compositions of $n$ is given by reverse refinement. We can move back and forth between descent pseudo-compositions and descent sets in exactly the same way as for type A. If $C(\pi) = (\alpha_1, \ldots, \alpha_k)$, then we have

$$\text{Des}_B(\pi) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}.$$

We will use pseudo-compositions of $n$ to index the type B quasisymmetric functions. Define $\mathcal{BQ}_{\text{sym}}_n$ as the vector space of functions spanned by the type B monomial quasisymmetric functions:

$$M_{B,\alpha} := \sum_{0 < i_2 < \cdots < i_k} x_0^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k},$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is any pseudo-composition of $n$, or equivalently by the type B fundamental quasisymmetric functions:

$$F_{B,\alpha} := \sum_{\alpha \leq \beta} M_{B,\beta}.$$

The space of all type B quasisymmetric functions is defined as the direct sum $\mathcal{BQ}_{\text{sym}} := \bigoplus_{n \geq 0} \mathcal{BQ}_{\text{sym}}_n$. By design, we have

$$\Gamma_B(\pi) = F_{B,C(\pi)}.$$

From Chow [2] we have the following theorem and corollary.

**Theorem 2.** As sets, we have the bijection

$$\mathcal{A}_B(\pi; ST) \leftrightarrow \prod_{\sigma \tau = \pi} \mathcal{A}_B(\tau; S) \oplus \mathcal{A}_B(\sigma; T).$$

We take $S = T = \mathbb{Z}$ and we have the following.

**Corollary 2.** We have

$$F_{B,C(\pi)}(XY) = \sum_{\sigma \tau = \pi} F_{B,C(\tau)}(X) F_{B,C(\sigma)}(Y).$$

The coalgebra structure on $\mathcal{BQ}_{\text{sym}}_n$ works just the same as the type A case so we will omit some details. Corollary 2 gives us the coproduct

$$F_{B,\gamma} \mapsto \sum_{\alpha, \beta \vdash n} b_{\alpha, \beta} F_{B,\beta} \otimes F_{B,\alpha},$$
where for any \( \pi \) such that \( C(\pi) = \gamma \), \( b_{\alpha,\beta}^{\gamma} \) is the number of pairs of signed permutations \((\sigma, \tau)\) such that \( C(\sigma) = \alpha \), \( C(\tau) = \beta \), and \( \sigma\tau = \pi \). The dual algebra is isomorphic to \( \text{Sol}(B_n) \), where if \( u_\alpha \) is the sum of all signed permutations with descent pseudo-composition \( \alpha \), the multiplication given by

\[
 u_\beta u_\alpha = \sum_{\gamma \vdash n} b_{\alpha,\beta}^{\gamma} u_\gamma.
\]

3. Signed quasisymmetric functions and the Mantaci-Reutenauer algebra

One thing to have noticed about the generating function for type B \( P \)-partitions is that we are losing a certain amount of information when we take absolute values on the subscripts. We can think of signed quasisymmetric functions as arising naturally by dropping this restriction.

For a type B poset \( P \), define the signed generating function for type B \( P \)-partitions to be

\[
 \Gamma(P) := \sum_{f \in A_B(P)} x_{f(1)}x_{f(2)} \cdots x_{f(n)},
\]

where we will write

\[
x_i = \begin{cases} u_i & \text{if } i < 0, \\ v_i & \text{if } i \geq 0. \end{cases}
\]

In the case where \( P \) is a signed permutation, we have

\[
 \Gamma(\pi) = \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1}x_{i_2} \cdots x_{i_n},
\]

so that now we are keeping track of the set of minus signs of our signed permutation along with the descents. For example,

\[
 \Gamma((-3, 2, -1)) = \sum_{0 < i < j < k} u_iv_ju_k.
\]

To keep track of both the set of signs and the set of descents, we introduce the signed compositions as used in \( \Pi \). A signed composition \( \alpha \) of \( n \), denoted \( \alpha \vdash n \), is a tuple of nonzero integers \((\alpha_1, \ldots, \alpha_k)\) such that \( |\alpha_1| + \cdots + |\alpha_k| = n \). For any signed permutation \( \pi \) we will associate a signed composition \( sC(\pi) \) by simply recording the length of increasing runs with constant sign, and then recording that sign. For example, if \( \pi = (-3, 4, 5, -6, -2, -7, 1) \), then \( sC(\pi) = (-1, 2, -2, -1, 1) \). The signed composition keeps track of both the set of signs and the set of descents of the permutation, as we demonstrate with an example.
If \( sC(\pi) = (-3, 2, 1, -2, 1) \), then we know that \( \pi \) is a permutation in \( \mathfrak{S}_9 \) such that \( \pi_4, \pi_5, \pi_6, \) and \( \pi_9 \) are positive, whereas the rest are all negative. The descents of \( \pi \) are in positions 5 and 6. Note that for any ordinary composition of \( n \) with \( k \) parts, there are \( 2^k \) signed compositions, leading us to conclude that there are

\[
\sum_{k=1}^{n} \binom{n-1}{k-1} 2^k = 2 \cdot 3^{n-1}
\]
signed compositions of \( n \).

We will use signed compositions to index the signed quasisymmetric functions (see [6]). For any signed composition \( \alpha \), define the monomial signed quasisymmetric function

\[
\overline{M}_\alpha := \sum_{\substack{1 < i_1 < \cdots < i_k \\ \alpha_r < 0 \Rightarrow x_{i_r} = u_{ir} \\ \alpha_r > 0 \Rightarrow x_{i_r} = v_{ir}}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k},
\]

and the fundamental signed quasisymmetric function

\[
\overline{F}_\alpha := \sum_{\alpha \leq \beta} \overline{M}_\beta.
\]

By construction, we have

\[
\overline{\Gamma}(\pi) = \overline{F}_{sC(\pi)}.
\]

Notice that if we set \( u = v \), then our signed quasisymmetric functions become type B quasisymmetric functions.

Let \( \mathcal{SQ}_{\text{sym}}n \) denote the span of the \( \overline{M}_\alpha \) (or \( \overline{F}_\alpha \)), taken over all \( \alpha \| n \). The space of all signed quasisymmetric functions, \( \mathcal{SQ}_{\text{sym}} := \bigoplus_{n \geq 0} \mathcal{SQ}_{\text{sym}}n \), is a graded ring whose \( n \)-th graded component has rank \( 2 \cdot 3^{n-1} \).

Theorem 2 is a statement about splitting apart bipartite \( P \)-partitions, independent of how we choose to encode the information. So while Corollary 2 is one such way of encoding the information of Theorem 2, the following is another.

**Corollary 3.** We have

\[
\overline{F}_{sC(\pi)}(XY) = \sum_{\sigma \tau = \pi} \overline{F}_{sC(\tau)}(X) \overline{F}_{sC(\sigma)}(Y).
\]

We define the coalgebra \( \mathcal{SQ}_{\text{sym}}^*n \) as we did in the earlier cases. Let \( \pi \in \mathfrak{B}_n \) be any signed permutation with \( sC(\pi) = \gamma \), and let \( c_{n,\gamma}^\alpha \) be the number of pairs of permutations \( (\sigma, \tau) \in \mathfrak{B}_n \times \mathfrak{B}_n \) with \( sC(\sigma) = \alpha \),
$sC(\tau) = \beta$, and $\sigma \tau = \pi$. Corollary 3 gives a coproduct $\mathcal{SQ} sym_n^* \rightarrow \mathcal{SQ} sym_n^* \otimes \mathcal{SQ} sym_n^*$:

$$\mathcal{F}_\gamma \mapsto \sum_{\alpha, \beta \vdash n} c_{\alpha, \beta}^\gamma \mathcal{F}_\beta \otimes \mathcal{F}_\alpha.$$ 

Multiplication in the dual algebra $\mathcal{SQ} sym_n$, the signed quasisymmetric functions of degree $n$, is given by

$$\mathcal{F}_\beta \ast \mathcal{F}_\alpha = \sum_{\gamma \vdash n} c_{\alpha, \beta}^\gamma \mathcal{F}_\gamma.$$ 

The group algebra of the hyperoctahedral group, $\mathbb{Z} \mathfrak{B}_n$, has a dual coalgebra $\mathbb{Z} \mathfrak{B}_n^*$ with comultiplication given by the map

$$\pi \mapsto \sum_{\sigma \tau = \pi} \tau \otimes \sigma.$$ 

The following is a surjective homomorphism of coalgebras $\psi^* : \mathbb{Z} \mathfrak{B}_n^* \rightarrow \mathcal{SQ} sym_n^*$ given by

$$\psi^*(\pi) = \mathcal{F}_{sC(\pi)}.$$ 

The dualization of this map is an injective homomorphism $\psi : \mathcal{SQ} sym_n \rightarrow \mathbb{Z} \mathfrak{B}_n$ with

$$\psi(\mathcal{F}_\alpha) = \sum_{sC(\pi) = \alpha} \pi.$$ 

The image of $\psi$ is then a subalgebra of $\mathbb{Z} \mathfrak{B}_n$ of dimension $2 \cdot 3^{n-1}$, with basis

$$v_\alpha := \sum_{sC(\pi) = \alpha} \pi.$$ 

This subalgebra is called the Mantaci-Reutenauer algebra, with multiplication given explicitly by

$$v_\beta v_\alpha = \sum_{\gamma \vdash n} c_{\alpha, \beta}^\gamma v_\gamma.$$ 

In closing, we remark that this same method is seen in [5], where Stembridge’s enriched $P$-partitions [9] are generalized and put to use to study peak algebras. Variations on the theme can also be found in [4].

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