AN INTRODUCTION TO WEINSTEIN HANDLEBODIES FOR COMPLEMENTS OF SMOOTHED TORIC DIVISORS

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ABSTRACT. In this article, we provide an introduction to an algorithm for constructing Weinstein handlebodies for complements of smoothed toric divisors using explicit coordinates and a simple example. This article also serves to welcome newcomers to Weinstein handlebody diagrams and Weinstein Kirby calculus. Finally, we include one complicated example at the end of the article to showcase the algorithm and the types of Weinstein Kirby diagrams it produces.

1. INTRODUCTION

A key way to study closed symplectic manifolds, is to break them down into two more easily understood parts: a neighborhood of a divisor and a complementary Weinstein domain. A divisor is a symplectic submanifold of co-dimension 2. One can allow this submanifold to have certain controlled singularities, such as normal crossing singularities or more general singularities modeled on complex hypersurfaces. Donaldson proved that every symplectic manifold has a divisor [Don96] and Giroux proved that this divisor can be chosen such that the complement of a regular neighborhood admits the structure of a Weinstein domain [Gir02, Gir17]. A Weinstein domain is a symplectic manifold with convex contact type boundary which can be broken down into symplectic handles modeled and glued as described by Weinstein [Wei91]. The symplectic topology of a Weinstein manifold is encoded in the attaching spheres of the handles which are isotropic and Legendrian submanifolds of the sphere which can be drawn using front projections. This Weinstein handlebody diagram gives a combinatorial/diagrammatic method to encode a symplectic manifold. There is a calculus of moves which relates different diagrams for equivalent Weinstein manifolds [Gom98, DG09].

Recently, there has been increased study of symplectic divisors in symplectic manifolds, particularly in the case when the complement is Weinstein. Some of the motivation comes from homological mirror symmetry, where generalizing the link between coherent sheaves and Fukaya categories to larger classes of manifolds has required one to consider a mirror pair that includes not only a space, but also a divisor [Aur07]. One way to associate a Fukaya categories for a divisor pair, is to look at the wrapped Fukaya category of the complement of the divisor. The Weinstein handle decomposition is key to understanding the wrapped Fukaya category, due to recent results that the co-cores of the handles generate the category [CDRGG17, GPS19, GPS18]. The Floer homology of these co-cores is intrinsically tied to the Legendrian DGA of the Weinstein handlebody diagram [BEE12, Ekh19] which is combinatorially calculated by Ekholm-Ng [EN15].

An important class of symplectic manifolds are toric manifolds. These have been studied extensively as they form a large class of examples of integrable system because of the symmetry provided by the Hamiltonian action of a torus on such manifold. According to the famous Delzant classification, all compact symplectic toric 2n-manifolds are uniquely (up to equivariant symplectomorphism) determined by convex n-dimensional polytopes, which correspond to the orbit space of the action. Much of the sympletic information can be encoded in the combinatorics of these polytopes. Moreover, toric manifolds have their origin in algebraic geometry, and they come by definition with a fibration by tori.

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given by the action, so that they have been among the first cases of interest for homological mirror symmetry, especially in view of the SYZ philosophy (see for instance [Abo09]). Every compact toric symplectic manifold is naturally equipped with a toric divisor. This is precisely the set of all points with non trivial stabilizer and the fixed points of the toric action are normal crossing singularities of the divisor. The complement of a neighborhood of the divisor is symplectomorphic to a Weinstein domain whose completion is $T^n$. Hypersurfaces with normal crossing singularities can naturally be deformed to become less singular at the expense of increasing the topological complexity of the divisor and its complement. A toric manifold, together with its toric divisor or any smoothing of the divisor is a Log Calabi-Yau pair which is a convenient setting for studying mirror symmetry of a space with a divisor [GHK15].

A manifold of dimension 4 will have symplectic surface divisors. Normal crossing singularities in this dimension are just positive transverse intersections of two smooth branches, or nodes. A deformation of this node smooths out the surface, trading the node for an annular tube which thus joins two different components or increases the genus of the surface. For a toric 4-manifold, the complement of the (fully singular) toric divisor looks like $T^*T^2$, which has a natural Weinstein structure described by a diagram discovered by Gompf [Gom98]. When the nodes of the toric divisor are smoothed out, the complement changes in a prescribed way which we investigate in detail.

In this article, we explain an algorithm to produce a Weinstein handlebody diagram for the complement of any divisor obtained by smoothing any number of nodes of a toric divisor in a toric 4-manifold. Fully detailed proofs backing up our algorithm are included in our upcoming paper [ACSG+]. That article also will explain how toric moment data determines the input to our algorithm, include many more examples, and analyze the corresponding Legendrian Chekanov-Eliashberg DGA. This article focuses on the most accessible example, as well as showcasing one fun case.

**Main Results.** There exists an algorithm to produce a Weinstein handlebody diagram for the complement of a toric divisor smoothed at some chosen collection of nodes.

1. Applying this algorithm to $\mathbb{CP}^2$ smoothed in one node yields the self-plumbing of $T^*S^2$ as illustrated in Figure 1. Moreover, the same output is obtained for the complement of a toric divisor in any toric 4-manifold smoothed in one node.

2. Applying the algorithm to $\mathbb{CP}^2 \# 5\mathbb{CP}^2$ smoothed at all eight nodes yields a 7-component link of Legendrian unknots with maximal $tb = -1$ as illustrated in Figure 2.
We would like to note that the handlebody diagram in the first example above has already been observed by Casals-Murphy in [CM19], viewed as a Weinstein handlebody for the complement of an affine smooth conic in $\mathbb{C}^2$ (which is the same as the complement of a smooth conic together with a generic line in $\mathbb{CP}^2$, which is the one smoothed toric divisor in $\mathbb{CP}^2$). We obtain this diagram from a completely different method and provide a systematic recipe which applies much more generally. This first example provides an accessible way to explain the steps of our more general algorithm.

This paper is organized as follows. In Section 2, we give definitions and discuss the relevant preliminary background on Weinstein Kirby calculus. In Section 3, we study a fundamental ingredient of our handle attachment in the Weinstein Kirby diagram by describing the core and co-core of the handle in the smoothing local model. The remainder of the paper is dedicated to the algorithm for producing the desired handlebody diagrams. We present the algorithm in two sections. In Section 4, we produce a Weinstein handlebody diagram for the complement of a toric divisor smoothed in one node and apply sequences of Kirby calculus moves to simplify the diagrams. Finally, we present a more complicated example, coming from $\mathbb{CP}^2 \# 5 \mathbb{CP}^2$, with the toric divisor smoothed at all eight nodes, to showcase the scope of applications and the corresponding Weinstein Kirby diagrams.

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2. Weinstein Handlebodies and Kirby Calculus

2.1. Weinstein Handle Structure. A Liouville vector field $V$ for a symplectic manifold $(W, \omega)$ is a vector field satisfying $L_V \omega = \omega$. By Cartan’s formula for the Lie derivative and the fact that the symplectic form is closed, this is equivalent to saying $d(\iota_V \omega) = \omega$. In particular, when there exists a Liouville vector field, the symplectic structure is exact. The 1-form $\lambda = \iota_V \omega$ which satisfies $d\lambda = \omega$ is called the Liouville form. The primary use of Liouville vector fields is to glue symplectic manifolds along contact type boundaries. When the Liouville vector field is transverse to the boundary, it defines a contact structure on the boundary and can be used to identify a collared neighborhood of the boundary with a piece of the symplectization of that contact boundary.

When Weinstein defined a model of a handle decomposition for symplectic manifolds, he equipped the handle with a Liouville vector field so that the gluing of the handle attachment could be performed using only contact information on the boundary. More specifically, the handle attachment is completely specified by a Legendrian attaching sphere, or an isotropic attaching sphere together with data on its normal bundle. The limitation is that the index of the handle is required to be less than or equal to $n$ in a $2n$ dimensional manifold. In particular, Weinstein 4-manifolds must be built entirely from handles of index 0, 1, and 2.

The model Weinstein handle of index $k$ in dimension $2n$ for $k \leq n$ is a subset of $\mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \cdots, x_n, y_n)$, with the standard symplectic structure $\omega = \sum_j dx_j \wedge dy_j$ and Liouville vector field

$$V_k = \sum_{j=1}^{k} (-x_j \partial_{x_j} + 2y_j \partial_{y_j}) + \sum_{j=k+1}^{n} \left( \frac{1}{2} x_j \partial_{x_j} + \frac{1}{2} y_j \partial_{y_j} \right).$$
As with smooth handle theory, the handles are in one to one correspondence with the critical points of a Morse function. The Liouville vector field agrees with the gradient of such a Morse function (for some choice of metric), in other words, the Liouville vector field is gradient-like. In the model index $k$ handle, the Liouville vector field is the gradient (with the standard Euclidean metric) of the function

$$\phi_k = \sum_{j=1}^{k-1} \left(-\frac{1}{2} x_j^2 + y_j^2\right) + \sum_{j=k+1}^{n} \left(\frac{1}{4} x_j^2 + \frac{1}{4} y_j^2\right).$$

The handle can be considered to be the subset of $\mathbb{R}^{2n}$ given by $D^k \times D^{2n-k}$ where the first factor corresponds to the coordinates $(x_1, \ldots, x_k)$ and the second corresponds to the remaining coordinates $(x_{k+1}, \ldots, x_n, y_1, \ldots, y_n)$. The key terminology for important parts of the handle is as follows.

- **The core** of the handle is $D^k \times \{0\}$ where $x_{k+1} = \cdots = x_n = y_1 = \cdots = y_n = 0$. This is the stable manifold of flow-lines of $V_k$ which limit positively towards the zero at the origin.

- **The co-core** of the handle is $\{0\} \times D^{2n-k}$ where $x_1 = \cdots = x_k = 0$. This is the unstable manifold of flow-lines of $V_k$ which limit negatively towards the zero at the origin.

- **The attaching sphere** is the boundary of the core, $S^{k-1} \times \{0\}$. This will be identified with an isotropic sphere in the boundary of the existing manifold to which the handle is attached.

- **The attaching region** is a neighborhood of the attaching sphere $S^{k-1} \times D^{2n-k}$. This is the entire part of the handle which will be glued on to a piece of the boundary of the existing manifold when the handle is attached. Therefore the Liouville vector field $V_k$ points inward into the handle along this part of the boundary (it is concave).

- **The belt sphere** is the boundary of the co-core $\{0\} \times S^{2n-k-1}$. It is a Legendrian sphere in the boundary of the manifold obtained after attaching the handle.

In general we can piece together the Liouville vector fields on the handles, and put together adjusted versions of the locally defined Morse functions to get a global Morse function on the manifold. A Weinstein structure is often encoded analytically as a quadruple $(W, \omega, V, \phi)$ where $W$ is a smooth manifold, $\omega$ is a symplectic structure on $W$, $V$ is a Liouville vector field for $\omega$ on $W$, and $\phi$ is a Morse function such that $V$ is the gradient-like for $\phi$.

**Remark 1.** When a manifold with a Weinstein structure has contact type boundary, it is called a Weinstein domain. Such a domain can be extended by a cylindrical end to make the Liouville vector field complete to give a non-compact infinite volume Weinstein manifold.
2.2. Weinstein Kirby calculus. The data needed to encode the Weinstein domain are the attaching maps. In dimension 4, the attaching map of a handle is completely determined by the Legendrian or isotropic attaching sphere. The attaching sphere of a 1-handle is a pair of points. Diagrammatically we draw a pair of 3-balls implicitly identified by a reflection, representing the attaching region $S^0 \times D^1$. The attaching sphere of a 2-handle is a Legendrian embedded circle (a Legendrian knot). The 4-dimensional 2-handle attachment is determined by the knot together with a framing, but in the Weinstein case, the framing is determined by the contact structure. More specifically, the contact planes along a Legendrian knot determine a framing by taking a vector field transverse to the contact planes. The contactomorphism gluing the attaching region of the 2-handle to the neighborhood of the Legendrian identifies the product framing in the 2-handle with the $tb - 1$ framing. Here $tb$ denotes the contact framing (Thurston-Bennequin number), which is identified with an integer by looking at the difference between the contact framing and the Seifert framing (this must be appropriately interpreted when the diagram contains 1-handles—see [Gom98]).

The diagram we draw should specify the Legendrian attaching knots in $S^3$ along with the pairs of 3-balls indicating the attachments of the 1-handles. By removing a point away from these attachments, we reduce the picture in $S^3$ to a picture in $\mathbb{R}^3$. After a contactomorphism, the contact structure on $\mathbb{R}^3$ is $\ker(dx - ydz)$ in coordinates $(x, y, z)$. The front projection is the map $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ with $\pi(x, y, z) = (x, z)$. A Legendrian curve in this contact structure is tangent to the contact planes, which happens precisely when the $y$-coordinate is equal to the slope $\frac{dz}{dx}$ of the front projection. Therefore, Legendrian knots can be recovered from their front projections with the requirement that the diagram has no vertical tangencies (it will have cusp singularities where the knot is tangent to the fibers of the projection) and the crossings are always resolved so that the over-strand is the strand with the more negative slope (we orient the $y$-axis into the page to maintain the standard orientation convention for $\mathbb{R}^3$ so the over-strand is the strand with a more negative $y$-coordinate). In these front projections, the contact framing $tb$ can be computed combinatorially in terms of the oriented crossings and cusps of the diagram, when the diagram is placed in a standard form when the pairs of 3-balls giving the attaching regions of 1-handles related by a reflection across a vertical axis. Namely, $tb$ of a legendrian knot is the difference of the writhe of the knot and half the number of cusps in the front projection.

The set of moves that relate Weinstein handlebody diagrams in Gompf standard form for equivalent Weinstein domains includes Legendrian Reidemeister moves (including how they interact with the 1-handles) listed in [Gom98], see Figures 4 and 5, as well as handle slides, and handle pair cancellations and additions. Given two $k$-handles, $h_1$ and $h_2$, a handle slide of $h_1$ over $h_2$ is given by isotoping the attaching sphere of $h_1$, and pushing it through the belt sphere of $h_2$. We depict a 1-handle slide (along with intermediate Reidemeister and Gompf moves) in Figure 6 and a 2-handle slide in Figure 7. A 1-handle $h_1$ and a 2-handle $h_2$ can be cancelled, provided that the attaching sphere of $h_2$ intersects the belt sphere of $h_1$ transversely in a single point. We call this a handle cancellation and the pair of handles a cancelling pair. Likewise a cancelling pair can be added to a Weinstein handlebody diagram, as depicted in Figure 8. When multiple 2-handles intersect a single 1-handle, the simplification in Figure 9 can be performed to reduce the overall complexity of a Weinstein diagram.

Before approaching our goal of presenting an algorithm to construct Weinstein–Kirby diagrams for complements of smoothed toric divisors, we will start with the unsmoothed case, where the complement has a Liouville completion, $T^*T^2$. The Legendrian handlebody we present, was originally found by Gompf [Gom98]. It follows from that article that this handlebody gives a Stein/Weinstein structure on the smooth manifold $D^*T^2$ (which is the trivial bundle $D^2 \times T^2$). More generally, Stein handlebody diagrams are given on the smooth manifolds $D^*\Sigma$ for any surface in [Gom98]. In [ACSG⁺], we show that the Weinstein structures induced on these diagrams are Weinstein homotopic to the canonical co-tangent Weinstein structure on $D^*\Sigma$.

For $T^*T^2$ specifically, it is known that there is a unique Weinstein fillable contact structure on the boundary $T^3$ and one can then deduce that the Gompf handlebody agrees with the canonical Weinstein structure by Wendl’s result that $S^*T^2$ has a unique Stein/Weinstein filling up to deformation.
FIGURE 4. The Legendrian Reidemeister moves, up to 180 degree rotation about each axis, where the top, middle, and bottom moves are called Reidemeister I, Reidemeister II, and Reidemeister III, respectively.

FIGURE 5. Gompf’s three additional isotopic moves, up to 180 degree rotation about each axis. The top, middle, and bottom moves are called Gompf move 4, Gompf move 5, and Gompf move 6, respectively.

FIGURE 6. An example of a 1-handle slide on $T^*T^2$. 
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Figure 7. An example of a 2-handle slide of the black unknot over the red unknot.

Figure 8. An example of a 1-handle cancelling with a 2-handle.

Figure 9. An example of handle slides and cancellations when multiple 2-handles pass through a 1-handle. Red and blue 2-handles are slid over the central green 2-handle. The green 2-handle is then cancelled with the 1-handle.

[Wen10]. To see that the diagram in Figure 11 represents $D^*T^2 \cong D^2 \times T^2$ smoothly, we can start with a handle decomposition for $T^2$ with one 0-handle, two 1-handles and a single 2-handle. Thickening this diagram to a 4-dimensional handlebody yields a disk bundle over $T^2$ with Euler number $e$, agreeing with the framing coefficient of the 2-handle attachment. One then needs to put the diagram into Gompf standard form as seen in Figure 10 by sliding the uppermost attaching ball below the attaching ball on the right so that both 1-handles are related by a reflection across the same vertical axis. Then we must realize the knot as a Legendrian knot by replacing vertical tangencies by cusps and making sure the crossings always have the over-strand corresponding to the more negative slope. The most obvious way to do this yields a Legendrian knot whose Thurston-Bennequin framing 0 (see the diagram on the right of Figure 10), so this would correspond to a $D^2$ bundle over $T^2$ with Euler number $-1$. By wrapping one strand around the lower left attaching ball as in Figure 11, we obtain a smoothly isotopic picture where the new Legendrian has $tb = 1$, so the Euler number is $1 - 1 = 0$ as needed for $D^*T^2$. Since this is one Weinstein filling of $S^*T^2$, and we know that such fillings are
unique up to deformation, it must agree up to Weinstein homotopy with the canonical co-tangent Weinstein structure on $D^*T^2$.

![Diagram](image1.png)

**Figure 10.** Diagram depicting how to move the usual picture of $T^*T^2$ into standard form after inserting the necessary Legendrian data, i.e. replacing vertical tangencies by cusps.

![Diagram](image2.png)

**Figure 11.** Stein structure on $D^2$ bundle over $T^2$ with framing coefficient $e(T^*T^2) \leq 0$.

### 3. The Local Model for Our Handle Attachment

In this section we study the local model for the smoothing of a normal crossing singularity of a symplectic divisor in dimension 4 and describe the local handle attachment information (core and co-core) when we understand how this smoothing corresponds to a 2-handle attachment in the complement.

A local model in a 4-dimensional manifold $M$ for the normal crossing of two symplectic divisors can be given by a Darboux chart $(\mathbb{C}^2, \omega_{\text{std}})$ at the intersection point, where the two divisors are mapped to the two axes $\Sigma_1 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_2 = 0\}$ and $\Sigma_2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 = 0\}$. Smoothing this normal crossing means that locally one substitutes the union of these divisors by the smooth surface

$$\Sigma = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \cdot z_2 = \varepsilon^2\}$$

for some $\varepsilon > 0$. Topologically, the complement of the smoothed surface differs from the complement of the normal crossing divisors by one 2-handle attachment and one can show that this also holds when considering Weinstein structures, namely that the Weinstein domain in the complement of the smoothing corresponds to a Weinstein handle attachment on the complement of the normal crossing divisors. See [ACSG+2] for a detailed proof.

In order to encode this handle attachment in the Weinstein Kirby diagram, we need to identify the corresponding Legendrian attaching sphere. For this purpose, one can describe the co-core of the handle in the smoothing local model, determine the core and deduce the corresponding attaching sphere.
One can find the co-core by looking for a Lagrangian disk with boundary on the boundary of the handle, that is in our case the smoothed hypersurface \( \Sigma \). In the local model, one can consider the disk \( D_1 \) defined as the image of the map

\[
\phi : [0, 1] \times [0, 2\pi] \rightarrow \mathbb{C}^2 \\
(r, \theta) \mapsto \begin{pmatrix} re^{i\theta} \\ r e^{-i\theta} \end{pmatrix}
\]

This disk is Lagrangian as for \( r \neq 0 \), the derivative of \( \phi \)

\[
d_{(r, \theta)}\phi = \begin{pmatrix} e^{i\theta} & ire^{i\theta} \\ e^{-i\theta} & -ire^{-i\theta} \end{pmatrix}
\]
is an isomorphism so that the tangent space to \( D_1 \) at the point \( \phi(r, \theta) \) is spanned by the vectors

\[
u_1 = \begin{pmatrix} e^{i\theta} \\ e^{-i\theta} \end{pmatrix} \quad \text{and} \quad \nu_2 = \begin{pmatrix} ire^{i\theta} \\ -ire^{-i\theta} \end{pmatrix}
\]

and one can check that

\[
\omega_{std}(\nu_1, \nu_2) = \Im m(\langle \nu_1, \nu_2 \rangle) = 0
\]

where \( \langle \cdot, \cdot \rangle \) is the standard Hermitian product in \( \mathbb{C}^2 \) (equivalently one can check that \( \phi^*(\frac{1}{2}(dz_1 \wedge \bar{dz}_1 + dz_2 \wedge \bar{dz}_2)) = 0 \)). The point for \( r = 0 \) corresponds to the origin of \( \mathbb{C}^2 \). At this point, the two following curves \( c_1 \) and \( c_2 \) in \( D_1 \) parametrized for \( t \in (-1, 1) \) by:

\[
c_1(t) = \begin{pmatrix} t\varepsilon \\ t\varepsilon \end{pmatrix} \quad \text{and} \quad c_2(t) = \begin{pmatrix} it\varepsilon \\ -it\varepsilon \end{pmatrix}
\]
give the two independent vectors in the tangent space at the origin of the disk \( D_1 \):

\[
u_1 = \begin{pmatrix} \varepsilon \\ \varepsilon \end{pmatrix} \quad \text{and} \quad \nu_2 = \begin{pmatrix} i\varepsilon \\ -i\varepsilon \end{pmatrix}
\]

One can note that we have again \( \omega_{std}(\nu_1, \nu_2) = 0 \).

The boundary of this disk is the image by \( \phi \) of \( \{1\} \times [0, 2\pi] \) that is the circle

\[
B = \left\{ \begin{pmatrix} e^{i\theta} \\ e^{-i\theta} \end{pmatrix} \bigg| \theta \in [0, 2\pi] \right\}.
\]

The circle \( B \) lies on the smoothed \( \Sigma \) as \( z_1 z_2 = e^{i\theta} e^{-i\theta} = \varepsilon^2 \) and goes to the origin when \( \varepsilon \) goes to 0, so that \( B \) is the belt curve of the handle and \( D_1 \) is indeed its co-core.

To find the core, one can look for a Lagrangian disk transversal to the co-core at one point and which avoids the boundary of the Weinstein manifold, so in our model, it should avoid the smoothed \( \Sigma \).

Let \( D_2 \) be the disk defined as the image of the map

\[
\psi : [0, 1] \times [0, 2\pi] \rightarrow \mathbb{C}^2 \\
(r, \theta) \mapsto \begin{pmatrix} re^{i\theta} \\ r e^{-i(\theta+\pi)} \end{pmatrix} = \begin{pmatrix} re^{i\theta} \\ -r e^{-i\theta} \end{pmatrix}
\]

This disk is also Lagrangian as, similarly as before, the tangent space to \( D_2 \) at the point \( \psi(r, \theta) \) for \( r \neq 0 \) is spanned by the vectors

\[
u_3 = \begin{pmatrix} e^{i\theta} \\ -e^{-i\theta} \end{pmatrix} \quad \text{and} \quad \nu_4 = \begin{pmatrix} ire^{i\theta} \\ ire^{-i\theta} \end{pmatrix}
\]

and one can check that

\[
\omega_{std}(\nu_3, \nu_4) = \Im m(\langle \nu_3, \nu_4 \rangle) = 0.
\]

Similarly, at the origin, the two following curves \( c_3 \) and \( c_4 \) in \( D_2 \) parametrized for \( t \in (-1, 1) \) by:

\[
c_3(t) = \begin{pmatrix} t\varepsilon \\ -t\varepsilon \end{pmatrix} \quad \text{and} \quad c_4(t) = \begin{pmatrix} it\varepsilon \\ it\varepsilon \end{pmatrix}
\]
give the two independent vectors in the tangent space at the origin of the disk $D_2$:

$$u_3 = \begin{pmatrix} \varepsilon \\ -\varepsilon \end{pmatrix} \quad \text{and} \quad u_4 = \begin{pmatrix} i\varepsilon \\ i\varepsilon \end{pmatrix}$$

Note again that $\omega_{std}(u_3, u_4) = 0$. This disk does not intersect the smoothed $\Sigma$ as

$$z_1z_2 = r\varepsilon e^{i\theta}(-r\varepsilon e^{-i\theta}) = -r^2\varepsilon^2 \neq \varepsilon^2.$$

Moreover, $D_1$ and $D_2$ intersect at the origin and this intersection is transverse as one can check that the family $(u_1, u_2, u_3, u_4)$ spans $\mathbb{C}^2$ as a real vector space. This shows that $D_2$ is the core of the handle and the attaching sphere is the image by $\psi$ of $\{1\} \times [0, 2\pi]$, that is,

$$A = \left\{ \begin{pmatrix} \varepsilon e^{i\theta} \\ \varepsilon e^{-i(\theta+\pi)} \end{pmatrix} \middle| \theta \in [0, 2\pi] \right\}.$$

In the toric description of the toric manifolds and divisors we consider, the Hamiltonian torus action in the local Darboux model corresponds to the torus action on $\mathbb{C}^2$ given in coordinates: $(e^{i\theta_1}, e^{i\theta_2}) \mapsto (z_1, z_2) = (e^{i\theta_1}z_1, e^{i\theta_2}z_2)$. In particular, through the symplectomorphism between the complement of the normal crossing divisor we consider and $T^*T^2$, the orbit of a point corresponds to the torus $T^2$ and the quotient space under the Hamiltonian action corresponds to a cotangent fiber, so that the attaching sphere corresponds in this symplectic identification to a lift of a circle of slope $(1, -1)$ in the base $T^2$ to the cotangent bundle $T^*T^2$ (lift corresponding to the point $(\varepsilon^2, \varepsilon^4)$ in the quotient space $(\mathbb{R}_{>0})^2$). For a more detailed description of the toric point of view, see [ACSG].

4. The Algorithm through an Example

Here we will show how to obtain a Weinstein handle diagram for the complement of a toric divisor with exactly one node smoothed. For example, we could start with the toric 4-manifold $\mathbb{CP}^2$ whose toric divisor is a collection of three $\mathbb{CP}^1$’s intersecting at three points, and smooth this divisor at one of the intersection points. More explicitly in homogeneous coordinates $[z_0 : z_1 : z_2]$ on $\mathbb{CP}^2$, the toric divisor is given by the union of the three lines $L_0 = \{z_0 = 0\}$, $L_1 = \{z_1 = 0\}$ and $L_2 = \{z_2 = 0\}$. Let us smooth the intersection of $L_1$ with $L_2$ at $[1 : 0 : 0]$. In the affine coordinate chart where $z_0 = 1$, with coordinates $(z_1, z_2)$, this aligns exactly with our local model in section 3. After smoothing, the lines $L_1$ and $L_2$ are joined to form a conic $Q$, which intersects the remaining line $L_0$ at two points. By our model in section 3, the complement of this smoothed divisor is obtained by attaching a single 2-handle to $D^*T^2$ along the Legendrian lift of a curve in $T^2$ of slope $(1, -1)$.

![Figure 12](image-url) - Left: The Legendrian unknot $\mathcal{K}$ in the boundary of the 4-dimensional 0-handle of the Gompf diagram, indicating the intersection of this boundary with the Lagrangian torus (0-section of $D^*T^2$). The black portions coincide with segments of the attaching circle of the 2-handle, and the blue portions give the attaching arcs of the 2-dimensional 1-handles of the torus. Right: The corresponding decomposition of the Lagrangian torus.
In order to translate Legendrian attaching circles in $S^*T^2$ described as the co-normal lift of a curve in $T^2$ into Legendrian curves drawn in the Gompf diagram (Figure 11), we need to understand how these two pictures get identified. As mentioned in section 2.2, the Gompf handle diagram is obtained by starting with a smooth handle decomposition of $T^2$ with a single 0-handle, two 1-handles, and one 2-handle. This diagram is thickened by two dimensions to obtain $T^2 \times D^2$, and then the attaching curve of the 2-handle is isotoped around until it agrees with a Legendrian front diagram with induced framing $tb - 1 = 0$. On the other hand, the co-normal lift construction is more compatible with the canonical (Morse-Bott) Weinstein structure on $D^*T^2$ which has critical locus along the 0-section. In [ACSG+], we prove that these two structures are Weinstein homotopic and identify the image of the Lagrangian torus giving the zero-section of $D^*T^2$ in the Gompf diagram. The handle decomposition on the 4-manifold induces the corresponding handle decomposition on the Lagrangian torus by intersection. In particular, we see a Legendrian (un)knot $K$ in the boundary of the 4-dimensional 0-handle, which partially coincides with the attaching sphere of the 2-handle, and partially corresponds with attaching arcs for the 1-handles of the Legendrian torus. See Figure 12.

Now consider the Legendrian co-normal lift to $S^*T^2$ of the circle in $T^2$ which is the boundary of the 2-dimensional 0-handle, with the inward co-orientation. This is a Legendrian push-off of $K$ in $\partial B^4 = \partial D^*D^2$, because a small positive Reeb flow applied to $K = \partial D^2$ yields the co-normal lift of a concentric circle close to $\partial D^2$. We will perform an isotopy to the curves in our torus corresponding to attaching spheres of the additional 2-handles so that these curves agree with parallel copies of such circles except where they enter the 1-handles. Circles which are further inward will be pushed off more in the positive Reeb direction. Note that every Legendrian circle has a standard neighborhood which is contactomorphic to a neighborhood of the zero section in $J^1(S^1)$ with the contact form $dz - ydx$ where $x$ is the coordinate on $S^1$. We will translate the diagram on the torus to a front projection diagram of $J^1(S^1)$. Since the Reeb direction is the positive $\partial_z$ direction in $J^1(S^1)$, circles which are further inwards in the torus (pushed further by the Reeb flow) will correspond to curves which are pushed upwards more in the $J^1(S^1)$ diagram. See an example of this procedure in Figures
Figure 14. The circle with \((1,0)\)-slope on \(T^2\) with the dotted arrow indicating the co-orientation of the curve, and an isotoped version corresponding to a positive Reeb push-off of the boundary of the square.

Figure 15. The curve in \(J^1(S^1)\) which is identified with the curve \((1,0)\) on \(T^2\) as in Figure 14.

14 and 15. Finally, once we have our diagram in the 1-jet space, we can satellite the diagram onto the image of \(S^1\) in the Gompf diagram. The images of the co-normal lifts of curves in various regions of the torus is illustrated in Figure 13.

Initially, let us apply this procedure in the simple example where we are attaching a 2-handle along the co-normal lift of a circle in the torus with slope \((1,0)\). The cotangent projection of this model is presented on the left in Figure 14. Push the curve \((1,0)\) to the upper side of the square, so it lies close to the boundary, and then cut the rectangle at the bottom left vertex to map to \(J^1(S^1)\), obtaining Figure 15. Satelliting this onto the Legendrian unknot in Figure 12, we obtain Figure 16.

Figure 16. The Legendrian handle diagram of the complement of the toric divisor smoothed in one node. That is, \(T^*T^2 \cup \Lambda_{(1,0)}\).

In general, when satellitting, we need to be somewhat careful with the behavior of the curves near the 1-handle. If the curves pass above the attaching region of a 1-handle without entering the 1-handle, they will follow an upward Reeb push-off of the cusps that appear inside the 1-handle attaching balls in Figure 12. Note that we will typically push these cusps out of the attaching regions of the 1-handles by a Legendrian isotopy. If the curves pass through the 1-handle in the torus, they
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Figure 17. Mapping the red, blue and purple curves from $J^1(S^1)$ to $T^*T^2$. The order of the set of curves is preserved.

will pass through the corresponding 1-handle in the 4-dimensional handlebody. See Figure 17 for the conventions in a more complicated example.

Figure 18. In columns from left to right, a series of Reidemeister and Gompf moves and 1-handle slides that take the Legendrian lift of a $(1, -1)$ curve to the Legendrian lift of a $(1, 0)$ curve. (1) Reidemeister III, (2) Reidemeister II and I, (3) Reidemeister II, (4) Reidemeister II, (5) slide green 1-handle over orange 1-handle, (6) Gompf move 5, (7) Gompf move 4, and (8) Reidemeister II.

Although, our initial explicit example asked us to attach along a curve of slope $(1, -1)$, we can see that in fact the resulting Weinstein manifold is equivalent to using the $(1, 0)$ slope. Figure 18
FIGURE 19. A series of Reidemeister moves and handle slides simplifying the Weinstein handle diagram of the complement of the toric divisor smoothed in one node. (1) Reidemeister III, (2) Reidemeister II and I, (3) 2-handle slide, (4) Handle cancellation, (5) Reidemeister III, (6) Reidemeister I and II, (7) Reidemeister III, I and II, (8) Gompf move 6, (9) Reidemeister II.
shows the series of 1 handle slides, Reidemeister and Gompf moves that take the Legendrian lift of the resulting diagram obtained by attaching along a \((1, -1)\) curve to the diagram corresponding to attaching along a \((1, 0)\) curve.

![Diagram](image)

**Figure 20.** The diagram of the complement of the toric divisor smoothed in one node after simplifications, in particular after applying a single Reidemeister I to the last diagram in Figure 19.

In fact, if we start with any toric 4-manifold, smooth a single node of the toric divisor, and study the complement, the result will be an equivalent Weinstein manifold. This is because attaching of a single 2-handle to \(T^*T^2\) along the Legendrian curve given by lifting a slope \((a, b)\) curve in the torus does not depend on the choice of the slope. We can see that \(T^*T^2 \cup \Lambda_{(a,b)}\) is symplectomorphic to \(T^*T^2 \cup \Lambda_{(1,0)}\), by performing 1-handle slides on \(T^*T^2 \cup \Lambda_{(a,b)}\) similar to Figure 18, to take a \(\Lambda_{(a,b)}\) curve to \(\Lambda_{(a,b \pm a)}\) or to \(\Lambda_{(a \pm b, b)}\). Using the Euclidean algorithm, with an appropriate choice of 1-handle slides, one can start with \(T^*T^2 \cup \Lambda_{(a,b)}\), for any pair \(a, b \in \mathbb{Z}\) that are relatively prime, and end with \(T^*T^2 \cup \Lambda_{(1,0)}\).

This fact can also be proved using toric arguments (see [ACSG+] for more).

The diagram where we attach along the \((1, 0)\) curve is preferable to the one where we attach along slope \((1, -1)\) (or a more complicated \((a, b)\) curve) because it is easier to simplify. Starting with Figure 16, we can perform Reidemeister moves, Gompf moves, handle cancellations and handle slides. We choose such a simplifying sequence in Figure 19 to obtain the diagram illustrated in Figure 20.

5. A MORE COMPLICATED EXAMPLE: SMOOTHING TORIC DIVISOR IN \(\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}\)

Consider \(\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}\), whose Delzant polytope is illustrated in Figure 21. We will smooth all eight singularities that map under the moment map to the vertices of the octagon. By [ACSG+], the complement of the smoothed divisor \(\Sigma \subset \mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}\) is given by attaching 2-handles to \(T^*T^2\) along Legendrian attaching spheres:

\[
\Lambda_{(1,0)}, \Lambda_{(0,-1)}, \Lambda_{(0,-1)}, \Lambda_{(-1,0)}, \Lambda_{(-1,0)}, \Lambda_{(0,1)}, \Lambda_{(0,1)}, \Lambda_{(1,0)}.
\]

![Diagram](image)

**Figure 21.** The Delzant polytope of \(\mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}\).
Figure 22 shows the curves $\Gamma = (1,0), (1,0), (0,-1), (0,-1), (-1,0), (-1,0), (0,1), (0,1)$ on $T^2$. As in the previous section, we perturb the curves by a Legendrian isotopy in order to identify them with parallel curves in $J^1(S^1)$. We choose to perturb each curve in the direction is given by its counterclockwise normal. Since there are multiple curves, we choose a perturbation which minimizes the number of crossings. Once these curves agree with parallel copies of suitable concentric circles which are positive Reeb push-offs of boundary of the square (except where they enter the $1$-handles), we identify a neighborhood of boundary of the square (containing all our isotoped curves) with a neighborhood of the zero-section of $J^1(S^1)$. This identifies all our curves in $T^2$ with curves in $J^1(S^1)$ as in Figure 23. We then satellite the image of the curves in $J^1(S^1)$ onto the image of $S^1$ in the Gompf diagram of $T^*T^2$, using the conventions described in the previous section and illustrated in Figures 13, and 17 to maintain the relative positions of the curves. The result is Figure 24.

**Figure 22.** The curves $(1,0), (1,0), (0,-1), (0,-1), (-1,0), (-1,0), (0,1), (0,1)$ on $T^2$, and their resulting isotopies which are push-offs of the boundary of the square in the positive Reeb direction.

**Figure 23.** The curves in $J^1(S^1)$ which are identified with the curves $(1,0), (1,0), (0,-1), (0,-1), (-1,0), (-1,0), (0,1), (0,1)$ on $T^2$ as in Figure 22.

As in the previous example we perform a series of Reidemeister moves, Gompf moves, handle cancellations and handle slides, and obtain the leftmost diagram illustrated in Figure 25. Note that both 1-handles were canceled, and that if we slide the black trefoil under the red Legendrian unknot we obtain a 7-component link of Legendrian unknots with maximal Thurston-Bennequin number equal to $-1$. The homology of $\mathbb{CP}^2 \# 5 \mathbb{CP}^2 \setminus \nu(\tilde{\Sigma})$, where $\tilde{\Sigma}$ is the smoothed toric divisor and $\nu(\tilde{\Sigma})$ is the neighborhood of $\tilde{\Sigma}$, is easily computed from the Weinstein handle decomposition. In particular,

$$H_0(\mathbb{CP}^2 \# 5 \mathbb{CP}^2 \setminus \nu(\tilde{\Sigma}); \mathbb{Z}) = \mathbb{Z},$$

$$H_2(\mathbb{CP}^2 \# 5 \mathbb{CP}^2 \setminus \nu(\tilde{\Sigma}); \mathbb{Z}) = \mathbb{Z}^7,$$

$$H_i(\mathbb{CP}^2 \# 5 \mathbb{CP}^2 \setminus \nu(\tilde{\Sigma}); \mathbb{Z}) = 0 \text{ for } i = 1, 3, 4.$$
Note that the second homology is generated by exact Lagrangian spheres built from the Lagrangian core of the 2-handles and the Lagrangian disk fillings of the attaching Legendrian spheres.

**Figure 24.** The Weinstein handle diagram of the complement of the toric divisor of $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$ smoothed in 8 nodes.

**Figure 25.** The leftmost diagram is obtained via a simplification of Figure 24. In the middle diagram, we perform an additional 2-handle slide of the black trefoil under the red unknot to obtain the link of Legendrian unknots in the rightmost picture.
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