Statistical Tracing of Magnetic Fields: Comparing and Improving the Techniques

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Abstract

Magnetohydrodynamic turbulence displays velocity anisotropies that reflect the direction of the magnetic field. This anisotropy has led to the development of a number of statistical techniques for studying magnetic fields in the interstellar medium. In this paper, we review and compare three techniques that use radio position–position–velocity data to determine magnetic field strength and morphology: the correlation function anisotropy (CFA), principal component analysis of anisotropies (PCAA), and the more recent velocity gradient technique (VGT). We compare these three techniques and suggest improvements to the CFA and PCAA techniques to increase their accuracy and versatility. In particular, we suggest and successfully implement a much faster way to calculate nonperiodic correlation functions for the CFA. We discuss possible improvements to the current implementation of the PCAA. We show the advantages of the VGT in terms of magnetic field tracing and stress the complementary nature with the other two techniques.

Key words: ISM: general – ISM: magnetic fields – ISM: structure – magnetohydrodynamics (MHD) – methods: numerical

1. Introduction

Turbulence is a ubiquitous phenomenon in astrophysics (see Draine 2011), and it has been detected in the interstellar medium ranges from kiloparsecs to sub-astronomical-unit scales (Armstrong et al. 1995; Elmegreen & Scalo 2004) and is responsible for the nonthermal broadening of line emission (Kainulainen & Tan 2013; Correia et al. 2014). It is a well-established fact that the interstellar medium (ISM) is turbulent and magnetized (Padoan et al. 2004; Burkhart et al. 2010; Federrath et al. 2011; Vázquez-Semadeni et al. 2011; see Mac Low & Klessen 2004; McKee & Ostriker 2007 for reviews). Magnetic turbulence controls a number of key astrophysical processes, like cosmic-ray propagation (see Schleicher et al. 2010), heat transfer (Narayan & Medvedev 2001; Lazarian 2006) and transfer of polarized radio emission (Draine 2005; Havercorn et al. 2006). Moreover, ISM turbulence and magnetic fields are key components of the star formation paradigm (Padoan et al. 2004; McKee & Ostriker 2007; Bialy et al. 2017b; Burkhart 2018). Scientists have known for decades, for example, that magnetic fields can control the collapse of molecular clouds (Mestel & Spitzer 1956; Spitzer 1978; Shu 1983; Mouschovias 1991) and remove angular momentum from accretion disks (see Krasnopolsky et al. 2012). More recently, magnetic turbulence has been identified as a driver of magnetic field diffusion from collapsing clouds and accretion disks via the process termed “reconnection diffusion” (see Lazarian et al. 2012; Mocz et al. 2017).

The importance of magnetic fields and turbulence has resulted in the development of a number of techniques for studying these phenomena in observations. In general, observational techniques (e.g., polarization or Zeeman studies) exist in the radio to optical wavelengths to study interstellar magnetic fields (see Crutcher 2012 for a review). Techniques based on the statistical imprints of turbulence or gravity have also been suggested and employed (Heyer et al. 2008; Lazarian 2009; Burkhart et al. 2017; see the thesis Burkhart 2014 for a review). Suggested by the aforementioned statistical works, observational studies should use the feature of turbulence anisotropy when tracing magnetic field directions, for example, the turbulence anisotropy measured in M51 using the method of correlation functions from polarized synchrotron data (Houde et al. 2013). This opens up a radical new way to study magnetic fields compared to the traditional polarization or Zeeman studies.

Magnetohydrodynamic (MHD) turbulence has been explored both theoretically and numerically (see Shebalin et al. 1983; Higdon 1984; Montgomery & Matthaeus 1995). In the present day, the theoretical foundations of magnetic field-tracing techniques through turbulence statistics are based on the well-known theory proposed by Goldreich & Sridhar (1995, GS95). They developed a theory for strong, incompressible MHD turbulence that provides definitive predictions of the energy spectrum and anisotropy of velocity fields. GS95 propose that there is a critical balance between nonlinear interactions and wave propagation, such that the timescales to transfer energy along the two directions are comparable. For an energy-conserving cascade, GS95 imply

\[ L_1 \propto L_2^{2/3}. \]

The above relation is not available in the global system of reference. Therefore, one should not expect the anisotropic relation to be observed in the global system of reference. The theory of turbulent reconnection (Lazarian & Vishniac 1999) has demonstrated the deep relationship and interdependence between MHD turbulence and magnetic field dynamics. The
framework of Lazarian & Vishniac (1999) allows one to understand why, unlike the original GS95 treatment, the anisotropy of turbulence reflects not the mean magnetic field direction, but the direction of the magnetic field that percolates turbulent eddies. Indeed, the turbulent reconnection theory predicts that magnetic field lines reconnect so fast that eddies are not constrained by magnetic fields if they perpendicularly mix in the direction of the magnetic field of the eddy. This result was confirmed by numerical simulations (Cho & Vishniac 2000; Maron & Goldreich 2001; Cho & Lazarian 2003) and suggested that the study of anisotropy not only can define the mean magnetic field directions, but can also trace local variations of magnetic field direction.

The first suggestion to study magnetic fields statistically using the theoretical understanding of GS95 was the correlation function analysis (CFA) of the velocities (Lazarian et al. 2002; Esquivel & Lazarian 2011; Burkhart et al. 2014). In the aforementioned papers, the CFA analysis was applied to velocity channel maps obtained from MHD simulations. It was also demonstrated that the velocity anisotropies can indeed provide the direction of the mean magnetic field. In Esquivel & Lazarian (2005) and Esquivel et al. (2007), the CFA was quantified and elaborated. The technique was further explored as a way not only to find magnetic field direction but also to determine magnetization (Esquivel & Lazarian 2011; Esquivel et al. 2015), as well as to determine the contribution of the fast, slow, and Alfvén modes in observed turbulence (Kandel et al. 2016, 2017a, 2017b).

The principal component analysis of anisotropies (PCAA) provides another way to trace magnetic fields using the turbulence anisotropy (Heyer et al. 2008). The PCAA was successfully applied to the observations and shown to correspond to the polarimetry data (Heyer et al. 2008), as well as the directions that were obtained with the technique.

Finally, the latest statistical technique for magnetic field studies, the velocity gradient technique (VGT), was demonstrated as a tool for tracing magnetic fields in the interstellar medium and molecular clouds. The first work (González-Casanova & Lazarian 2017) on VGT used velocity centroid gradients (VCGs) to trace magnetic fields. Only approximate tracing was available, and the accuracy of the technique was resolution dependent. A radical improvement of the VGT was achieved in Yuen & Lazarian (2017a), who used the procedure of block averaging to provide reliable magnetic field tracing. The further development of the VGT for centroids was done in Yuen & Lazarian (2017b), suggesting that removing some wave modes can improve the accuracy of magnetic field tracing. Another branch of the VGT, namely, velocity channel gradients (VChGs), was developed in Lazarian & Yuen (2018). The gradients of intensity fluctuations in thin channel maps were used to represent velocity fluctuations (see Lazarian & Pogosyan 2000). In the same paper, Lazarian & Yuen (2018) suggested using the galactic rotation curve in order to obtain the 3D distribution of magnetic fields. With the numerical studies of velocity gradients in self-absorbing media (González-Casanova et al. 2017) and application of the VCGs and VChGs to observed neutral hydrogen (H I) and molecular tracer maps (Yuen & Lazarian 2017a, 2017b; Lazarian & Yuen 2018), the VGT was identified as a powerful new approach to tracing magnetic fields.

While all three techniques appeal to GS95 as their foundation, it is not yet clear whether their predictions of magnetic field directions are in agreement with each other. Common questions in comparing these three field-tracing techniques would be, (1) what are the constraints of the techniques?, (2) how precise can we trace the B-field?, and (3) are the methods self-consistent? In short, a benchmark study of all three methods in the same framework has yet to be performed. Yuen & Lazarian (2017a) first showed that VGT is superior to the CFA technique in tracing magnetic fields in observational data. This result has also been verified in a parallel work using the gradients of synchrotron intensities (Lazarian et al. 2017). They point out that, compared to VGT, CFA requires a larger area to perform the ensemble average in calculating the correlation function. The empirical nature of PCAA also brings up questions related to its applicability; that is, there is no self-consistent check for whether PCAA is working in a certain region. VGT shows that having 202 samples is sufficient to satisfy the Gaussian condition shown in Yuen & Lazarian (2017a, 2017b).

This paper aims to compare the three techniques and quantify their ability to trace magnetic fields (equivalently, detect anisotropy) using synthetic maps generated from numerical simulations. As PCAA is not applicable to studying anisotropy in individual channels, we do not show the results of the VChG analysis, although this technique provides the best tracing of magnetic fields among the different versions of the VGT. We investigate the advantages, limitations, and constraints of these methods. We organize the paper as follows. In Section 2 we describe the details and properties of the simulations. In Section 3 we introduce the three methods of tracing magnetic fields in detail. In Section 4 we show the results of the comparison, and in Section 5 we present a discussion of the results. We conclude our paper in Section 6.

2. Synthetic Data from MHD Turbulence Simulations

The numerical data that we analyzed in this work are obtained by 3D MHD simulations using the single-fluid, operator-split, staggered-grid MHD Eulerian code ZEUS-MP/3D (Hayes et al. 2006) to set up a 3D, uniform, and isothermal turbulent medium. Periodic boundary conditions are applied to emulate a part of the interstellar cloud. Solenoidal turbulence injections are employed. Our simulations employ various Alfvénic Mach numbers $M_A = V_A / V_A$ and sonic Mach numbers $M_S = V_S / V_A$, where $V_A$ represents the injection velocity, $V_A$ the Alfvén velocities, and $V_S$ the sonic velocity. All of the cubes related to this work are listed in Table 1. The ranges of $M_S$, $M_A$, and $\beta = 2M_A^2/M_S^2$ are specifically selected so that they cover different possible scenarios of astrophysical turbulence from subsonic to supersonic cases. However, limited by the turbulence scaling (see LV99), we devote most of our research to the sub-Alfvénic and trans-Alfvénic cases in this study.

To reduce the complexity of comparing the three methods, we only consider the optically thin case and synthesize observational maps using the following treatment. We first compute the PPV cubes from 3D numerical simulations. A

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8 Fluctuations of the intensity in so-called “thin velocity channel maps” are mostly influenced by velocity fluctuations; the meaning of “thin channel” is quantified in Lazarian & Pogosyan (2000).

9 We use PCAA to distinguish this analysis from the earlier studies in Brunt & Heyer (2002b), who used the principal component analysis to get the spectral indices of turbulence.
position–position–velocity (PPV) cube corresponds to a 3D array with size \(n_x \times n_y \times n_v\), where \(n_x\) and \(n_y\) represent the sizes along the \(x\) and \(y\) axes, and \(n_v\) the number of velocity channels along the spectral line direction \(v\) (line of sight, LOS). The number of velocity channels is an adjustable parameter, and in our simulation, we choose \(n_v = 400\) for our simulation and observation. The velocity centroid map \(C(x, y)\) is a map weighted by velocity channel speed and has the size \(n_x \times n_y\). It is obtained by multiplying each velocity channel by its velocity, and then integrating along the velocity direction and dividing by the total emission in the direction of integration:

\[
C(x, y) = I^{-1}(x, y) \int dv \, \rho(x, y, v),
\]

where \(I\) represents the integrated intensity of the spectroscopic cube, and \(\rho\) is the density of the PPV cube:

\[
I(x, y) = \int dv \, \rho(x, y, v).
\]

In our implementations for the three methods below, most of our calculations are based on either the velocity centroid \(C(x, y)\)\(^{10}\) or the \(\rho(x, y, v)\) (henceforth \(\rho\) when the meaning is clear in the context).

The orientations of anisotropy/gradients from the three methods are compared with synthetic polarization assuming a constant-emissivity dust grain alignment process. In other words, the Stokes parameters \(Q(x, y)\) and \(U(x, y)\) can be expressed in terms of the angle \(\theta\) between the \(y\)- and \(z\)-direction magnetic fields by \(\tan \theta(x, y, z) = B_z(x, y, z)/B_y(x, y, z)\):

\[
Q(x, y) \propto \int dz \rho(x, y, z) \cos(2\theta(x, y, z))
\]

\[
U(x, y) \propto \int dz \rho(x, y, z) \sin(2\theta(x, y, z)).
\]

The dust polarized intensity \(I_P = \sqrt{Q^2 + U^2}\) and angle \(\Phi = 0.5 \alpha \tan^2(U/Q)\) are then defined correspondingly. The alignment between the prediction of magnetic field (CFA, PCAA, VG) and projected magnetic field orientations (polarization angles) is quantified by the alignment measure (AM), introduced in analogy with the grain alignment studies,

\[
AM = \langle 2 \cos^2 \phi - 1 \rangle
\]

(see González-Casanova & Lazarian 2017; Yuen & Lazarian 2017a), where \(\phi\) is the relative angle between the vectors representing respectively the magnetic field orientation predicted by the three methods and polarization. AM ranges between \(-1\) and \(1\). When AM \(\sim 1\), it means that the two vectors are statistically perfectly aligned; when AM \(\sim -1\), it means that the two vectors are essentially perpendicular to each other.

### 3. Improving the CFA and PCAA

In this section, we propose improvements to the CFA and PCAA techniques. In particular, for the CFA technique, we propose and test a fast method for calculating the correlation function in nonperiodic regions. We also demonstrate a new way of contour tracing to significantly improve the accuracy of the CFA technique.

We also modify PCAA in order to trace magnetic fields with higher accuracy. To do this, we borrow the block-averaging approach that was successfully employed earlier in Yuen & Lazarian (2017a) with velocity gradients. We implement the improvements we suggested for CFA to the correlation functions of centroids. Our approaches to improving the calculation of correlation functions are also applicable to the analysis of the velocity channel maps.

#### 3.1. Correlation Functions from Velocity Centroids

The (second-order) correlation function of a velocity centroid map is defined as

\[
\overline{C(r)} = \langle C(r) C(r + R) \rangle.
\]

The direction of the major axis determines the orientation of the averaged magnetic field in a sampled region (Esquivel & Lazarian 2011; Burkhart et al. 2014). For the case of periodic boundary condition simulations, a special form of correlation function and the periodicity of the centroid map allow us to obtain the correlation function through the cross-correlation theorem and the fast Fourier transform (FFT):

\[
\overline{C(r)} = \overline{\mathbf{\gamma}^{-1} [||C||^2]},
\]

where \(\mathbf{\overline{\gamma}}\) is the Fourier transform operator. Figure 1 shows an example of how the correlation function should behave in terms of the contour plot. One can observe that in scales with the major axis \(r < 40\), the contours are elongated along the mean field direction, and the major axes of the smallest contours are aligned with the parallel direction. Notice that the larger contours are slightly misaligned from the parallel

| Model | \(M_s\) | \(M_a\) | \(\beta = 2M_s^2/M_a^2\) | Resolution |
|-------|--------|--------|-----------------|------------|
| Ms0.4Ma0.04 | 0.41 | 0.04 | 0.02 | 480\(^3\) |
| Ms0.8Ma0.08 | 0.92 | 0.09 | 0.02 | 480\(^3\) |
| Ms1.6Ma0.16 | 1.95 | 0.18 | 0.02 | 480\(^3\) |
| Ms3.2Ma0.32 | 3.88 | 0.35 | 0.02 | 480\(^3\) |
| Ms6.4Ma0.64 | 7.14 | 0.66 | 0.02 | 480\(^3\) |
| Ms0.4Ma0.132 | 0.47 | 0.15 | 0.2178 | 480\(^3\) |
| Ms0.8Ma0.264 | 0.98 | 0.32 | 0.2178 | 480\(^3\) |
| Ms1.6Ma0.528 | 1.92 | 0.59 | 0.2178 | 480\(^3\) |
| Ms0.04Ma0.4 | 0.05 | 0.52 | 200 | 480\(^3\) |
| Ms0.08Ma0.8 | 0.10 | 1.08 | 200 | 480\(^3\) |

Note. \(M_s\) and \(M_a\) are the instantaneous values at each of the snapshots taken.
direction (see Section 5 in our explanation of why this is the case).

To compare the CFA with the VGT, we also implement the strategy of block averaging first suggested in Yuen & Lazarian (2017a) to CFA. In the framework of VGT, block averaging reveals the statistically most probable direction of the magnetic field in the region of consideration. Considering the subblock statistics, computation of the correlation function anisotropy should also reveal the direction of the magnetic field, similar to the block averaging in VGT.

The implementation of block statistics in CFA requires computing Equation (8) for nonperiodic maps. However, Equation (8) is limited to maps that are periodic in both boundaries. For nonperiodic regions, one needs to compute the correlation function through the direct computation of Equation (7), which requires an amount of computation in proportion to the square of the number of pixels. In particular, to utilize CFA inside blocks, which naturally are not periodic, a faster calculation on par with Equation (8) should be developed. To calculate the correlation function of a nonperiodic centroid map, we adopt the Hockney method (Hockney 1968) to solve the open-boundary convolution problem as shown in Figure 2 formally with Equation (8) (see Section A for a formal discussion),11 which decreases the time complexity of the computation process. We pad the centroid map C with size \( n_x \times n_y \) (in Figure 2 \( n_x = n_y = 3 \)) into two \((2n_x + 1) \times (2n_y + 1)\) blocks \(X, Y\) as shown in Figure 2, where

\[
X(i, j) = \begin{cases} 
C(\text{mod}(i - 1, n_x) + 1, 1 \leq i, j \leq 2n \\
\text{mod}(j - 1, n_y) + 1), \\
0, \quad \text{otherwise}
\end{cases}
\]

\[11\]

Figure 1. Illustration of how to obtain the correlation function from the velocity centroid (i.e., Equation (8)). The correlation function here is calculated from cube Ms3.2Ma0.32.

\[12\]

Figure 2. Illustration of how to implement Hockney’s approach in our method, that is, obtain the correlation function from the velocity centroid (see Appendix A for the technical explanation).

\[
Y(i, j) = \begin{cases} 
C(i, j), & 1 \leq i, j \leq n_x, \\
0, & \text{otherwise}
\end{cases}
\]

where mod is the modulo operation. The open-boundary correlation function is therefore

\[
CF_{C(R)}[i, j] = \mathcal{F}^{-1}\{\mathcal{F}\{X}\mathcal{F}\{Y\}^*\}, \quad 1 \leq i, j \leq n_x.
\]

This implementation enables one to compute the correlation function and structure function efficiently in nonperiodic cases, which are particularly useful for our comparison between CFA and VGT.\(^{12}\) Figure 3 shows how to locate the direction of anisotropy given a specific correlation map or structure function map. Concretely, the algorithm plots the contour lines of the map and detects the orientation of the elongated major axes and minor axes of each (elliptical) contour line. Then the map is rotated such that the major axis of a contour with a particular radius (in our case, the searching radius is 10 pixels) is parallel to the horizontal direction. The direction of anisotropy is then determined by the direction of the major axis of the contour.

There are additional difficulties in using CFA when the line of sight makes a different angle to the mean magnetic field. Burkhart et al. (2014) suggested that the detected degree of anisotropy will drop when the angle between the line of sight and the mean magnetic field decreases. We also see the same effect for our numerical cubes as in Figure 4. For VGT, the respective investigation has been done in Yuen & Lazarian (2017b) and having a similar drop of AM when the angle between the line of sight and the mean magnetic field decreases. However, observers should be aware of the fact that, while the degree of anisotropy is decreasing with respect to the decrease in angles between the line of sight and the mean magnetic field, the predicted orientations are still evident for both VGT (Yuen & Lazarian 2017b) and CFA (Figure 4).

3.2. Analysis with Principal Component Analysis

3.2.1. Finding Anisotropies with PCA

The PCA is widely used in image processing and image compression. In terms of astrophysical applications, the PCA analysis was used in Brunt & Heyer (2002a, 2002b) to obtain the turbulence spectrum from observations. In Heyer et al. (2008), the PCA was employed to study turbulence anisotropies. The physical meaning of the eigenvalues from the PCA

11 Assuming one has \(N\) data for a centroid, then the traditional method (Equation (7)) requires a complexity of \(O(N^2)\); with the FFT method (Equation (8)), the complexity is reduced to \(O(N \log N)\).

12 The Big-O factor for Hockney’s method, is \(O((2N + 1)\log(2N + 1))\), compared to the traditional method with \(O(N^2)\), where \(N\) is the number of discrete elements in an array.
analysis is closely related to the value of the turbulence velocity dispersion $v^2$. In particular, those larger eigenvalues correspond to the largest-scale contributions of turbulence eddies along the line of sight, assuming GS95 scaling applies.

To study anisotropy, Heyer et al. (2008) applied the PCA to the spectroscopic data as a tool for tracing anisotropy, similar to what was done earlier in the statistical analysis of channel maps and centroids (Lazarian et al. 2002; Esquivel & Lazarian 2005). Similar to the latter techniques, the directional PCAA demonstrated its ability to identify the direction of the mean magnetic field. To help the reader understand the essence of the technique, we provide a simple version of its mathematical formalism as well as an illustration in Figure 5.

Assuming a proper normalization is used,13 we can treat the PPV cube $\rho(x, y, v)$ as the probability density function of three random variables $x$, $y$, $v$. The covariance matrix is14

$$S(v_1, v_2) \propto \int dx dy \rho(x, y, v) \rho(x, y, v_2).$$

In the later treatment of anisotropy tracing, Heyer et al. (2008) split the PPV cube into vertical and horizontal position–velocity tires (PV tires), where every PV tire is a vertical or horizontal slice from the PPV map $\rho(x, y, v)$ averaged over the $x$ direction (y direction):

$$W(y, v) \propto \int dx \rho(x, y, v)$$

$$W(x, v) \propto \int dy \rho(x, y, v).$$

The covariance matrices ($S_x$ and $S_y$) for the PV tires ($W$) are

$$S_x(v_1, v_2) \propto \int dx W(x, v_1) W(x, v_2)$$

$$S_y(v_1, v_2) \propto \int dy W(y, v_1) W(y, v_2),$$

so the eigenvalue equations for these covariance matrices are

$$S_{x} u_x = \lambda_x u_x$$

$$S_{y} u_y = \lambda_y u_y,$$

where $\lambda_{x,y,i}$ are the eigenvalues associated with the eigenvectors $u_{x,y,i}$ with $i = 1, 2, \ldots, n_v$. The eigenvectors contain the information of velocity variations along this particular PV tire. To get information on the spatial variance, one must project each eigenvector into the PV tires. These eigen-projections $P_{x,i}$, $P_{y,i}$ are

$$P_{x,i}(x) = \int dv W(x, v) u_{x,i}(v)$$

$$P_{y,i}(y) = \int dv W(y, v) u_{y,i}(v).$$

With the sets of eigenvectors and eigen-projections on the $x$ and $y$ directions in hand, one can apply the method of

\begin{enumerate}
\item In principle, one can use the normalized PPV cube $\rho' = \rho / \int \rho$. However, for the treatment of PCAA, the difference of a constant does not alter the result. Therefore we stay with using $\rho$ for simplicity.
\item The correct definition of a covariance matrix should be $S(v_1, v_2) = E(\rho(v_1)\rho(v_2))-E(\rho(v_1))E(\rho(v_2))$, where $E$ is the expectation operator. However, the second part was not included in Heyer et al. (2008).
\end{enumerate}
autocorrelation functions (ACFs) $\text{ACF}\{X\} = \text{CF}\{X\} / \text{Var}\{X\}$ to these sets of data in order to obtain the characteristic velocity and scale. Each characteristic velocity (scale) is calculated when the ACF for one eigenvector (eigen-projection) drops by one e-fold. Due to resolution limitations, the characteristic velocities (scales) are interpolated between the points nearest to $1/e$:

$$\frac{\text{ACF}\{u\}(\delta v)}{\text{ACF}\{u\}(0)} = e^{-1}$$ \hspace{1cm} (21)

$$\frac{\text{ACF}\{P\}(L)}{\text{ACF}\{P\}(0)} = e^{-1}. \hspace{1cm} (22)$$

We obtain at least 10 pairs (see right panel in Figure 16) of characteristic velocity $\delta v_{x,y}$, and we scale $L_{x,y}$ from the ACFs of the corresponding eigenvectors and eigen-projections.

If, as is the case in numerical data cubes, the magnetic field is oriented along either the $x$ or $y$ axes, one can expect the ACFs for the $x$ and $y$ directions to be different. When using observational data, Heyer et al. (2008) attempted to find the direction of the magnetic field by calculating the ACFs while rotating the directions of the $x$ and $y$ axes. This by itself can provide the magnetic field direction. However, Heyer et al. (2008) were studying the scaling of the ACFs while changing the orientation of the coordinate axes. The anisotropy was determined by the variations of the exponent $\alpha$ in

$$\delta v_x = v_{0,x} L_x^{\alpha_x},$$

$$\delta v_y = v_{0,y} L_y^{\alpha_y}. \hspace{1cm} (24)$$

An example can be found in Figure 6. There are some challenges associated with this procedure. Indeed, according to both turbulence theory and MHD turbulence simulations (GS95, LV99, Cho & Lazarian 2003), the differences of indices should not be observed in the global system of reference related to the mean magnetic field. In Appendix B, we show that the observed differences between the indices are the result of both the limited inertial range of numerical simulations and the isotropic driving of turbulence at the injection scale.

While we find that the approach in Heyer et al. (2008) has problems, for the sake of comparison, we use the formalism on the velocity and length-scale determinations (Equation (21)) as it is presented in their original work. In particular, we find the differences between the exponents $\alpha_x$ and $\alpha_y$. Compared to the actual observational study in which the direction of the magnetic field is not known a priori, a rotation of the coordinate system is required to guess the direction of the magnetic field before applying PCA.\(^{15}\)

3.2.2. Testing PCAA

We first prepare the PPV cubes with constant $n$ and PPV density $\rho$ by the distribution function of the line-of-sight projection.

\(^{15}\) The corresponding procedure is not elaborated in detail in Heyer et al. (2008). We feel that this procedure of rotating the coordinate system is not straightforward in terms of its practical implementation. For instance, the calculation of the covariance matrix $S$, for the PV tires (Equation (15)) requires an addition along the $y$ axis in the rotated coordinate. For both synthetic and observation maps, the information of the map is usually stored in a rectangular coordinate. Any kind of addition after a rotation, as the PCA method had, will result in a distortion in the covariance matrix $S_{x,y}$.
velocity $f(v; z)$:

$$n(x, y, v) = \int dz f(v; z)$$

$$\rho(x, y, v) = \int dz \rho(x, y, z)f(v; z).$$

We then apply the method of PCAA as illustrated. Figure 6 shows how the density scaling in PPV cubes would change the anisotropies found by PCAA on the same numerical cube but with different weighting of density.

We also test our implementation of PCAA on our simulations as listed in Table 1. Figure 7 shows how the sonic Mach number $M_s$ and Alfvénic Mach number $M_A$ could possibly change the anisotropy. In this work, we adopt an isotropy index that can be obtained directly from the PCAA exponents so we can compare to the velocity centroid isotropy index:

$$\Upsilon = 1 - \frac{|\alpha_1 - \alpha_2|}{\sqrt{\alpha_1 \alpha_2}}.$$ (27)

For isotropic velocity fields, $\Upsilon \sim 1$. Note that $\Upsilon$ can be negative for highly anisotropic clouds.

In Figure 7, we do not see a clear relationship between the isotropy index and $M_s$, but a slightly positive relationship between isotropy index and $M_A$ is found. This is expected as in the PPV formulation using PCAA. Only the largest variance contributions are extracted, and it is well known that the variance of density is a function of sonic Mach number $\sigma^2 \propto \log(1 + b^2 M_s^2)$ for some $b \sim 1/3$–1/2 (Federrath et al. 2011; Burkhart & Lazarian 2012). If as mentioned in the previous section only the largest eigenvalue is extracted, only the density clumps with the highest dispersion will, therefore, be analyzed.

We explained earlier that, in the global system of reference, there should not be a difference in the spectral indices based on the theory of MHD turbulence. To compare with VGT, which is a local measure of anisotropy, we have to improve the method of PCAA on the local scale instead of a global direction. In Section 4.2 we shall show our method of improving PCAA and compare with VGT.

3.3. The Velocity Gradient Technique

3.3.1. Block Averaging

The VGT is a recently developed technique for tracing magnetic field directions based on the anisotropic turbulence
scaling (GL17a, YL17a). In terms of the GS95 scaling, turbulent eddies are elongated along the local magnetic field directions. As a result, the gradients of velocity are perpendicular to the local magnetic field directions. As a result, a simple 90° rotation of gradients traces the direction of magnetic field locally.

We adopt the subblock averaging and the respective error-estimation method as suggested in YL17ab. While the gradients are good probes of magnetic field directions as suggested by our series of papers (Yuen & Lazarian 2017a, 2017b; Lazarian & Yuen 2018), knowing the errors of individual gradient vectors is always beneficial when applying to observations. We have to emphasize on the basis of GS95 turbulence that the statistical nature of gradients acted similarly to the techniques based on turbulence anisotropy and principal component analysis (Esquivel & Lazarian 2005; Heyer et al. 2008; Burkhart et al. 2014). An insufficient number of pixels within the block will result in a significant error of magnetic field direction estimation. The recipe we proposed in YL17ab allows us to acquire the statistical gradient orientation average within a block from the peak value of the Gaussian fitting function

\[ N(\theta; p_1, p_2, p_3) = p_1 \exp\left(-\theta - \frac{p_2^2}{4p_3^2}\right) \]

The standard error of the Gaussian peak, \( \sigma_{x,y} \), which is one of the free parameters of the Gaussian function for fitting, will tell us how well the gradient orientation distribution follows the Gaussian distribution, and how accurately the peak can represent the averaged direction of gradients inside a particular block of a certain size.

### 3.3.2. Recent Improvements for the VGT

Recently, Hu et al. (2018) demonstrated that the method of PCA is capable of extracting the anisotropic velocity modes along the line of sight. Different from the method of PCAA, Hu et al. (2018) construct the eigen-intensity maps \( I_{\text{eigen}} \) and eigen-centroid maps \( C_{\text{eigen}} \) using PCA as

\[
C_{\text{eigen}}(x, y) = \int dv \rho(x, y, v) \cdot v \cdot \lambda(v) / I_{\text{eigen}}(x, y)
\]

\[
I_{\text{eigen}}(x, y) = \int dv \rho(x, y, v) \cdot \lambda(v),
\]

where \( \lambda \) are the eigenvalues associated with the eigenvectors \( \mathbf{u} \).

In both synthetic and observational maps, the extraction of eigen-centroids can effectively probe the direction of magnetic field with very high accuracy. As a result, for studies of the projected magnetic field, the improved technique can provide a higher accuracy of magnetic field tracing.

### 4. Comparison between the Three Techniques

The common goal of the three methods (CFA, PCAA, VGT; see Table 3 for the abbreviations) is to trace magnetic field orientation independently from polarimetry measurements. For VGT, the accuracy of determining the magnetic field direction can be obtained through block averaging (Yuen & Lazarian 2017a; Lazarian & Yuen 2018).\(^{16}\) The other two techniques have their limitations. For instance, the CFA technique depends strongly on the viewing angle chosen (Burkhart et al. 2014). We have not developed yet a self-consistent procedure for estimating the accuracy of the magnetic field orientation with CFA like the Gaussian fitting criterion in VGT subblock averaging (Yuen & Lazarian 2017a). The PCAA technique in its present incarnation seems to have even more problems. The determination of the anisotropy angle using PCAA requires tedious checking of the anisotropy index in every possible angle that the map can rotate. Moreover, the projection of PV tires after rotation will result in distortion of the PV tire statistics. As a result, the ACF may not provide the anisotropy direction correctly. Moreover, the perpendicular and parallel velocity scaling indices \( \alpha \) are not expected to change for turbulence with extensive inertial range.

Nevertheless, the synergy of the techniques should be utilized. We note that an advantage of CFA is that the related anisotropies are analytically described in Kandel et al. (2017a) and are related to the contributions from slow, fast, and Alfvén modes that constitute the MHD turbulence cascade (Cho & Lazarian 2003). Below we provide a more quantitative comparison of the three techniques.

#### 4.1. VGT versus CFA

Figure 9 shows a visual comparison of applying both VGT and CFA to the same centroid data from the Ms1.6Ma5.28 simulation by selecting a block size of 30 pixels. We compare the performance of CFA with the older VGT recipe (from Yuen & Lazarian 2017a); the latter carries only one user-defined variable, the block size. For the B-field orientation probed by CFA, we use the orientation of the major axis as a prediction, following the treatment in Section 3.1 (see Figure 3). The algorithm suggested in Section 3.1 is more adaptive in dealing with irregular anisotropic shapes compared to our previous treatment in Yuen & Lazarian (2017a) using a highly simplified layout.

\[^{16}\] In our forthcoming paper, we are comparing the VGT with the tracing of filaments in channel maps that is suggested in Clark et al. (2014, 2015). We find that the filaments in the latter papers are the result of velocity crowding, the same effect that makes thin channel maps sensitive to velocity fluctuations (Lazarian & Pogosyan 2000).
gradient at origin for the correlation function method.\textsuperscript{17} We select the pixel distance of 10 pixels for anisotropy contour detection.

One can see a significant advantage of VGT compared to CFA in Figure 9 in terms of the AM. Figure 10 shows a scatter plot of $\Delta M_{\text{VGT}} - \Delta M_{\text{CFA}}$ with respect to $M_A$ using the gradient recipe of Yuen & Lazarian (2017a) for a block size of 30 pixels. The mean AM for VGT is $\sim0.43$, while that for CFA is about $\sim0.31$. While we do not see a clear trend of $\Delta M = M_{\text{VGT}} - M_{\text{CFA}}$ versus $M_A$, there is a general $\Delta M = 0.2$ advantage for VGT over CFA.

We do expect that the tracing power of VGT will increase appreciably after the improvements suggested by Lazarian & Yuen (2018). But whether we can use the same improvement technique for the CFA-probed B-field is not clear yet. One can see from Figure 9 that the magnetic field estimations from CFA are more likely to be bimodal; that is, the vectors are likely to be either parallel or perpendicular to the real field. This might because the small-scale statistical shape studies are not well studied (see Figure 15 for an illustration).

\textsuperscript{17} In principle, the shorter axis direction for the anisotropy corresponds to larger gradients. Therefore one can try to detect the anisotropy by taking gradients at the origin of correlation functions and rotate $90^\circ$ for magnetic field directions. This, however, cannot tackle complex structures like what we see in Figure 15.

\textbf{4.2. VGT versus PCAA}

To compare PCAA with VGT, we update the implementations of PCAA to make it comparable to the subblock-averaged method in VGT. The general guideline would be performing \textit{PCAA on subblocked PPV cubes}. We extract the partial PPV (pPPV) cubes $\rho_{ij}$ that cover partial spatial regions:

$$\rho_{ij}(x, y, v) = \rho((i - 1)n + x, (j - 1)n + y, v).$$

The sequence of PPV cubes $\rho_{ij}$ contains in total $n_i n_j / n^2$ elements. For each pPPV cube, we assume they are independent and processed using the steps from Section 3.2. One can refer to Figure 5 for the simplified, pictorial work flow for PCAA. The product from the pipeline would be a six-element array $(\delta v_x, \delta v_y, L_x, L_y, \alpha_x, \alpha_y)$ for each pPPV cube, which should have an empirical scaling as shown in Equations (23) and (24).

One can try to convert the anisotropic direction predicted by PCAA in each PPV cube\textsuperscript{18} to some magnetic field orientation prediction similar to VGT. The trick is to use the fact that the six-element array provides a measure of velocity gradients, and the maximal-perpendicular properties of velocity gradients allow us to predict the direction of the magnetic field by rotating the PCA-backed velocity gradient by $90^\circ$. We start with the \textit{statistical average} maximal PCAA gradient orientation inside the block $(i, j)$ assuming both $L_x, L_y$ are small and not aligned with the $\parallel, \perp$ coordinate:

$$\nabla v(L_x, L_y) \sim \left[ \begin{array}{c} \delta v_x \\ \delta v_y \\ L_x \\ L_y \end{array} \right].$$

The orientation of the PCAA gradient is then given by

$$\tan \theta(L_x, L_y) = \frac{\delta v_x}{\delta v_y} = \frac{L_y^{\alpha_x - 1}}{L_x^{\alpha_y - 1}}.$$  \hspace{1cm} (32)

For $L$ smaller than some turbulence scales depending on the Alfvén Mach number (see LV99 for a complete discussion), that is, if $L$ is sufficiently small, we can then obtain an expression from Equation (32) for approximating the PCAA angle:

$$\tan \theta = \frac{1 - \alpha_x}{1 - \alpha_y}.$$  \hspace{1cm} (33)

Figure 11 shows how the magnetic field orientation predicted by $\theta + \pi/2$ from Equation (33) is compared to polarization measurements and VGT. In the following, we discuss the two separate cases regarding properties of PPV cubes.

\textbf{4.2.1. Constant-density Case}

We first investigate how the block averaging would behave in the constant-density PPV cube (Burkhart et al. 2013). In other words, we create PPV cubes with a uniform density field and a turbulent velocity field. Thus all fluctuations in such a cube are entirely due to velocity caustics. The left part of Figure 11 shows how the performance of PCAA is compared to VCG visually in a centroid map from super-Alfvénic simulation Ms3.2Ma0.32. For some part of the region, PCAA is able

\textsuperscript{18} This is, however, not a very accurate statement, as PCAA has a preassumed anisotropy direction. Heyer et al. (2008) illustrate how to obtain the direction of anisotropy by rotating the Taurus map and seeing which orientation can give the largest anisotropy difference.
also true for observations where bulk motions exist and scale contributions are considered to be meaningful. This is considered in terms of correlation and structure functions, only the small-scale part is often limited to only several pixels and highly depends on the quality of the data.19 If the resolution of the map is small (in observation) or the dissipation process is strong (in numerical studies), the small-scale anisotropy determination under the assumption of elliptical elongation would fail (see Section 3.1 for the method building). The change in anisotropy is even more severe when the number of samples for the statistical studies is not enough (e.g., Section 4.1). To what extent the correlation function anisotropy can provide a correct answer given a map with a certain resolution is uncertain.

We therefore want to test the dependencies of resolution on the anisotropy method using multiple resolutions. We prepared some higher resolution cubes (Table 2) and compare with what we have (Table 1) for both the anisotropy axis ratio and its orientation.

5. Additional Effects

5.1. Fluctuation of Anisotropy Scale and Directions in CFA

The method of CFA has its limitations in both resolution and quality of the data, for both simulations and observations. In principle, numerical simulations have limited inertial ranges. In terms of correlation and structure functions, only the small-scale contributions are considered to be meaningful. This is also true for observations where bulk motions exist (e.g., galactic motions and shear, outflow) aside from turbulence. As a result, the large-scale part of the correlation function may not be so meaningful in determining the anisotropy. However, this immediately brings a paradox for the anisotropy direction, as the small-scale part is often limited to only several pixels and highly depends on the quality of the data.19 If the resolution of the method of CFA is held constant. In this figure, we use the model Ms3.2Ma0.32. Block size = 30.

4.2.2. Real-density Case

We also explore how the use of the real-density PPV cube would change the AM of VGT and PCAA. In theory, the involvement of real density will make the velocity channel map contain both density and velocity contributions, in which the proportion between the two contributions is determined by the channel thickness (LP00) and the sonic Mach number. When we sum up the channels, density is expected to dominate over the velocity contribution, and the AM is expected to drop compared to the constant-density case.

The right side of Figure 11 shows how the performance of PCA is compared to VCG visually in a centroid map from super-Alfvénic simulation Ms3.2Ma0.32. Compared with the constant-density case, the alignment of PCA is improved closer to zero. However, VCG still provides an excellent performance in tracing the magnetic field, even though we do see a drop of AM from the constant-density case.

Figure 11. Comparison between the magnetic field predictions from VGT and PCAA when the constant-density condition (first and second panels) and turbulent real-density condition (third and fourth panels) are applied, respectively. In the first two panels, the fluctuations are entirely due to velocity fluctuations, since the density is held constant. In this figure, we use the model Ms3.2Ma0.32. Block size = 30.

5.1. Fluctuation of Anisotropy Scale and Directions in CFA

Table 2

| Model | $M_s$ | $M_A$ | $\beta = 2M_s^2/M_A^2$ | Resolution |
|-------|-------|-------|------------------------|------------|
| H0    | 7.36  | 0.22  | 0.0017                 | 792$^3$    |
| H1    | 6.41  | 0.41  | 0.0083                 | 792$^3$    |
| H2    | 6.47  | 0.61  | 0.0176                 | 792$^3$    |
| H3    | 6.47  | 0.80  | 0.0309                 | 792$^3$    |
| H4    | 6.15  | 1.00  | 0.0531                 | 792$^3$    |

Note. $M_s$ and $M_A$ are the instantaneous values at each of the snapshots taken.

5.1. Distortion of Anisotropy over Scales

We first illustrate the effect of scale-dependent anisotropy in our numerical cube with lower resolution (480$^3$). Figure 12 shows how the shape and orientation of the correlation function anisotropy in Ms0.4Ma0.04 are changing with respect to length. One can directly see that while the numerical cube is somewhat anisotropic in all scales visually, both the axis ratio and the orientation are changing when one steps away from the center of the ellipses. The left panel of Figure 13 shows a clearer effect with a scatter plot from three numerical simulations Ms0.4Ma0.04, Ms0.8Ma0.08, and Ms1.6Ma0.16, illustrating the pixel distance (defined as the distance from the common center for the anisotropic ellipses) to the relative angle that the smallest anisotropy elongates to.

### Other contextual information:

19 Readers might challenge whether the large-scale shearing/rotation motion or the small-scale outflow motions may alter the result of VGT. In principle, with proper scale filtering (Yuen & Lazarian 2017b), one can remove the contribution from large-scale structures, which is also true for the method of CFA. For the small-scale outflow motion, using a large-enough block size can average out the contribution of non-turbulent motions.
What if we increase the resolution and simulate cubes with appropriate $M_A$ so that its scale $L_{\text{inj}}M_A^2$ is within the inertial range? Figure 14 shows how the cubes from the same code with higher resolution would behave. One can see from the trend for $a/b$ axis ratio and orientation oscillations closer to theoretical expectations (Esquivel & Lazarian 2005) that the $a/b$ ratio is decreasing with respect to $M_A$, and the orientation is more stable. This illustrates that the resolution of the map is critical for the CFA study.

5.1.2. Disappearance of Elliptical Anisotropy

The resolution problem can not only distort the shape of the anisotropies in different scales but also destroy the prominent elliptical shape, especially when performing the subblock CFA analysis (Equation (11)). Figure 15 shows how the shape of anisotropy is changed when one selects a different size of a partial region from the same cube $M_{s0.4}M_{a0.04}$. While the anisotropy is changed when one selects a different size of a region, both the direction and the shape of the contours change significantly. One possibility is that the measured region only contains small-scale strip-like structures, and, intuitively, the anisotropy is considered indeterminable in the region of interest. What if we increase the resolution and simulate cubes with higher resolution would behave. One can see from the trend for $a/b$ axis ratio and orientation oscillations closer to theoretical expectations (Esquivel & Lazarian 2005) that the $a/b$ ratio is decreasing with respect to $M_A$, and the orientation is more stable. This illustrates that the resolution of the map is critical for the CFA study.

5.2. Dependence of Channel Resolution in PCA

While PCAA is a very powerful tool in extracting spectrum properties through a relatively simple statistical pipeline, there are concerns about its consistency and arbitrariness when applied to both synthetic and real observation. Two very important questions would be, (1) what is the minimal velocity channel number for PCAA?, and (2) what is the optimal number that one can pick for the ACF analysis? They are both crucial because the importance of a particular velocity spectral line eigenvector is dependent on both the channel resolution and the number of biggest eigenvalues that are picked when fitting the $\alpha$ values.

We first illustrate how the channel resolution would change the answer of PCAA. The left panel of Figure 16 shows a relation of isotropy index to the channel resolution for both constant- and real-density PPV cubes (see Equation (25)) on the cube $M_{s1.6}M_{a0.528}$ using 10 eigenvalues. One can see that the isotropy index fluctuates dramatically when the channel number is less than 200 pixels and stays constant afterward. This indicates the method of PCAA has a more significant error if the velocity channel resolution is not high enough. In particular, our test shows an approximately 20% error for the constant-density case and 12% for the real-density case in terms of the isotropy index. If one converts the isotropy index back to the angle, a larger error is expected. In this test, we did not include the noise produced by the instruments (which is very common in observational spectroscopic cubes). However, due to the nature of the PCAA pipeline, the noise only contributes to the velocity spectra modes with small eigenvalues.

We also test whether there is a way to search for the optimal number of principal components for PCAA analysis. In previous literature, only a handful of modes are used in PCAA (e.g., Brunt & Heyer 2002b used eight modes, and Correia et al. 2016 used 3–12). We use an $n_v = 100$ cube to test how the change in the number of eigenvalues picked for PCAA would change the anisotropy measurement. Figure 16 (right panel) shows the variation of isotropy index with the number of principal components we used for PCAA analysis, for both the actual turbulent density field (i.e., denoted as “real”) and a constant-density field. One can see a significantly larger variation when the number of principal components is smaller than 20. For instance, the isotropy index changes from 0.05 to 0.8 when the number of principal components changes from 5 to 15 for the real-density case. Relatively, the constant-density PPV cube is more robust when the number of principal components is changing, but the variation is still large up to 20 components.

One may argue whether the use of an intensity threshold for the higher eigen-projections may provide more stability to the isotropy index. However, the extra dependency on the intensity threshold will also increase the difficulty for observers in finding the best combination of channel number, number of components, and threshold value to analyze the result with PCAA.

6. Summary

In this paper, we studied three different techniques that have been suggested in the literature for tracing magnetic fields. We improved these techniques and compared them with each other. In particular, for the CFA technique, we suggested and successfully tested a way to find the direction of the magnetic
field more accurately, as well as a way to calculate correlation functions quickly. We also suggested the subblock-averaging technique for the PCAA. Our findings can be summarized as follows:

1. The VGT technique is superior to CFA and PCAA in tracing magnetic fields.
2. Correlation function anisotropy faces several issues when the block size is small. In particular, the anisotropies may

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**Figure 13.** Two main properties of CFA to difference $M_a$ under resolution $480^3$. Panel (a) shows the axis ratio to difference radius from the center. Panel (b) shows the variation degree difference to difference radius from the center.

**Figure 14.** Two main properties of CFA to difference $M_a$ under resolution $792^3$. Panel (a) shows the axis ratio to difference radius from the center. Panel (b) shows the variation degree difference to difference radius from the center.

**Figure 15.** Variation of correlation function anisotropy shapes with respect to block size for the cube $M_s0.4M_a0.04$. We draw contours to specify the isocontours.
be distorted or multicentered or the contours are not closed. That significantly affects the determination of the direction of anisotropy, and thus the magnetic field. Poor resolution may also hinder the CFA technique from correctly determining the Alfvén Mach number.

3. Principal component analysis provides a method for extracting the most important velocity components in a PPV map. However, the detection of anisotropy strongly depends on the quality of spectroscopic cubes and the number of components that are being analyzed. We report only a weak dependence on $M_A$ and no dependence on $M_S$. With the block-averaging technique applied, we show that VGT has a significant advantage compared to PCA for finding magnetic field detections.

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Software: Julia, Matlab, Python.

Appendix A

The FFT Open-boundary Cross-correlation Method

Computationally, the cross-correlation function is pretty expensive with its computer complexity of $O(N^2)$. This limits scientists from studying the statistical behavior, often forcing the calculation of anisotropy to be truncated to small scales only. This is because the usual Fourier transform method (Equation 8) is not valid in the case when we select a partial region for CFA analysis. In the following, based on the formulation from Hockney & Eastwood, we explain how one can compute the open-boundary problem with a treatment similar to that in Equation (7), which facilitates the method of CFA for observational maps and also subblock studies in parallel to VGT.

In the following treatment, we shall interchange integrals $\int dx$ and summation signs $\sum_x$ freely to address the feature that the numerical data (both synthetic and observational) are discrete and have finite resolution.

Assume we have a piece of complex numerical data $C(r)$ and we would like to obtain its correlation function:

$$CF(r) = \int dr' C^*(r') C(r + r'),$$

where the sign $*$ means complex conjugate. Using the definition of Fourier transform and assuming the functions are all $C_2$ converging, we have

$$CF(r) = \frac{1}{L^2} \int dr' dk_1 dk_2 C^*(k_1) e^{-ik_1(r'+r)} C(k_2) e^{ik_2 r'},$$

$$= \frac{1}{L} \int dk C^*(k) C(k) e^{-ikr},$$

where $L$ is the normalization constant. This formula is essentially Equation (8).

Hockney proposed a novel way to tackle the common convolution problem in computational physics, especially to tackle Newtonian gravity using the following implementation, which has been tested in our code in accelerating the open-space discrete Poisson calculation from $O(N^2)$ to $O(N \log N)$. Here we follow their idea and calculate the cross-correlation counterpart following the appendix of Ryne (2011) (arXiv:1111.4971). Suppose one is interested in the discrete cross-correlation

$$f_i = \sum_{j=0}^{n-1} g_j^* h_{i+j}$$

Figure 16. Plot showing how the isotropy index (y axis) varies with respect to channel resolution (left panel, x axis) and the number of principal components remaining (right panel, x axis).
where $i = 0, 1, 2, \ldots, m - 1$. Define the zero-pad sequence of $g$, which has the length of $N > m + n$:

$$G_j = \begin{cases} g_j, & \text{if } j = 0, 1, 2, \ldots, n - 1 \\ 0, & \text{if } j = n + 1, \ldots, N. \end{cases}$$

(38)

Similarly, we can define the periodic cross-term

$$H_k = \begin{cases} h_k, & \text{if } j = 0, 1, 2, \ldots, m + n - 1 \\ 0, & \text{if } j = m + n, \ldots, N \\ h_{\text{mod}(k,N)}, & \text{otherwise.} \end{cases}$$

(39)

The above is the basis for the padding strategy shown in Figure 2.

The analogous summation formula for cross-correlation is similar to Equation (37) of Ryne (2011), assuming $W = e^{-2\pi i/N}$:

$$F_j = \frac{1}{N} \sum_{k=0}^{N-1} W^{-jk} \left( \sum_{l=0}^{N-1} G_l^* W^{-lk} \right) \left( \sum_{l=0}^{N-1} H_l W^{lk} \right). \quad 1 \leq j \leq N. \quad (40)$$

From Equations (35) and (36) and the consideration of the zero-padding in Equation (38), we arrive:

$$F_i = \sum_{j=0}^{N-1} G_i^* H_j = \sum_{i=0}^{m+n-1} g_i^* h_j. \quad (41)$$

For the specific case that we are interested in, $g = h = C$ and having the same size. The minimal number that satisfies the condition $N > m + n \sim 2n$ is $N = 2n + 1$. Noticing that the multidimensional FFT in the rectangular case is orthogonal, we therefore arrive at the pictorial description in Figure 2.

One might question whether the appearance of bad pixels might alter the result we showed in the main text. We therefore perform a simple “hole-punching” test on our existing data. We select a centroid map $C$ from the cube $H0$ (see Table 2) and randomly set a certain percentage of the pixels to be NaN. We then directly use the open-boundary FFT method on the centroid map with different block size to see whether the detected CFA orientation and axis ratio are changed after we zero the NaN pixels. Figure 17 shows the change of axis ratio and orientation when we set a certain percentage of the data to be NaN and then perform CFA using the open-boundary FFT method. We see that even as 40% of the data are punched out, we still have approximately the same predictions on the axis ratio or major-axis orientation, which suggest that the current open-boundary FFT method would still be robust to real data, which will usually carry a number of empty pixels.

**Appendix B**

**Anisotropy**

A correct anisotropy can be revealed in a local frame of reference, which is defined with respect to the local mean magnetic field. However, it is possible to observe a (fake) scale-dependent anisotropy in the global frame of reference, which is aligned with the mean magnetic field. For the sake of simplicity, we assume the driving is isotropic throughout this appendix.

If the turbulence is sub-Alfvénic, it is easy to understand why we observe a (fake) scale-dependent anisotropy in the global frame of reference. First, we note that large-scale structures are isotropic because driving is isotropic. Second, we note that structures measured in the global frame of reference are anisotropic on very small scales due to field-line wandering on large scales. The anisotropy in the global frame of reference is scale-independent on very small scales and of order $l_1/l_\perp \sim B_0/b_1 \sim 1/M_A$, where $l_1$ and $l_\perp$ are the parallel and perpendicular size of eddies, respectively, and $M_A$ is the Alfvén Mach number. Third, since large-scale structures are isotropic and very small-scale ones are anisotropic, there should be transition scales on which anisotropy is scale-dependent. Note that the scale-dependent anisotropy on the transition scales in the global frame of reference is different from the true anisotropy that can be revealed in a local frame of reference.

Even if $b_L/B_0 \sim M_A \sim 1$, we can have a (fake) scale-dependent anisotropy near the energy injection scale $L$ in the
global frame of reference. Suppose that we try to reveal anisotropy using the second-order structure function

$$SF_2(r_{\perp}, r_{\parallel}) = \langle |A(x + r) - A(x)|^2 \rangle_{\text{avg, over } x},$$

where $A$ can be either the velocity or magnetic field, and $r_{\perp}$ and $r_{\parallel}$ are components of the separation vector $r$ perpendicular and parallel to the mean magnetic field, respectively. If contours of $SF_2(r_{\perp}, r_{\parallel})$ are isotropic, we can say structures are isotropic (see Cho & Vishniac 2000). If $b_{\perp}/B_0 \sim 1$, we expect that small-scale structures are isotropic and $SF_2(r_{\perp}, 0) = SF_2(0, r_{\parallel})$. Note, however, that roughly speaking the second-order structure function represents power near the scale of interest. For example, $SF_2(r_{\perp}, 0)$ represents power near the (perpendicular) scale $r_{\perp}$, which is approximately equal to the power in the shaded area in Figure 18(a). Similarly, $SF_2(r_{\parallel}, 0)$ is approximately equal to the power in the shaded area in Figure 18(b). Although we will not show it rigorously, Figure 18 clearly tells us that $SF_2(r_{\perp}, 0) > SF_2(0, r_{\parallel})$, which means that structures look anisotropic. The fact that $SF_2(r_{\perp}, 0) > SF_2(0, r_{\parallel})$ implies that contours of $SF_2(r_{\perp}, r_{\parallel})$ are elongated along the direction parallel to the mean magnetic field. Note that this kind of anisotropy appears on sufficiently small scales. Now, the situation is similar to that of sub-Alfvénic turbulence: large scales are isotropic due to isotropic driving, and small scales are anisotropic as we have shown above. Therefore, there should be transition scales on which we observe a (fake) scale-dependent anisotropy.

### Appendix C

#### Term and Abbreviations

| Abbreviation | Term |
|--------------|------|
| AM           | Alignment measure |
| ACF          | Autocorrelation function |
| CFA          | Correlation function anisotropy |
| VGT          | Velocity gradient technique |
| PCA          | Principal component analysis |
| PCAA         | Principal component analysis of anisotropies |
| PPV          | Position–position–velocity |
| pPPV         | Partial position–position–velocity |
| pSIG         | Polarized synchrotron intensity gradient |
| VGT          | Velocity gradient technique |
| VCG          | Velocity centroid gradient |
| MHD          | Magnetohydrodynamics |

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Figure 18. Structure functions for the directions perpendicular (SF$_2$(r$_{\perp}$, 0); left panel) and parallel (SF$_2$(0, r$_{\parallel}$); right panel) to the mean magnetic field. Here r$_{\perp}$ $\propto$ 1/k$_{\parallel}$ and r$_{\parallel}$ $\propto$ 1/k$_{\parallel}$.
