ASYMPTOTIC PRESERVING DISCONTINUOUS GALERKIN METHODS FOR A LINEAR BOLTZMANN SEMICONDUCTOR MODEL

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Abstract. A key property of the linear Boltzmann semiconductor model is that as the collision frequency tends to infinity, the phase space density \( f = f(x, v, t) \) converges to an isotropic function \( M(v)\rho(x, t) \), called the drift-diffusion limit, where \( M \) is a Maxwellian and the physical density \( \rho \) satisfies a second-order parabolic PDE known as the drift-diffusion equation. Numerical approximations that mirror this property are said to be asymptotic preserving. In this paper we build two discontinuous Galerkin methods to the semiconductor model: one with the standard upwinding flux and the other with an \( \epsilon \)-scaled Lax-Friedrichs flux, where \( 1/\epsilon \) is the scale of the collision frequency. We show that these schemes are uniformly stable in \( \epsilon \) and are asymptotic preserving. In particular, we discuss what properties the discrete Maxwellian must satisfy in order for the schemes to converge in \( \epsilon \) to an accurate \( h \)-approximation of the drift diffusion limit. Discrete versions of the drift-diffusion equation and error estimates in several norms with respect to \( \epsilon \) and the spatial resolution are also included.

Key words. drift-diffusion, asymptotic preserving, discontinuous Galerkin, semiconductor models

AMS subject classifications. 65M08, 65M12, 65M15, 65M60

1. Introduction. Kinetic equations are an established tool for modeling charged-particle transport in semiconductors, particularly in non-equilibrium settings [18, 26, 29]. However, numerical simulations of such equations are known to be challenging, due to the size of the space on which they are defined (in general three position, three momentum variables, plus time) and the multiscale nature of the equations. With regards to the latter, it is well-known that for large collision frequencies and long-time scales, the kinetic solution is well-approximated by a drift-diffusion equation which depends on space and time only. Under reasonable conditions, this limit was established rigorously for the case of an applied electric field in [28]. The case of a self-consistent field was later treated in [1, 27].

Because of the drift-diffusion approximation, solving a kinetic model of charge transport in collisional regimes may be unnecessarily expensive; to ameliorate this cost, methods which leverage the drift-diffusion approximation, either via domain decomposition [20] or acceleration [10, 22] are sometimes used. At a minimum, it

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is important that a discretization of the kinetic equation recover a stable and consistent discretization of the drift-diffusion limit as the collision frequency becomes infinitely large; this is the so-called asymptotic preserving (AP) property [15, 16]. While standard finite volume or finite-difference methods that rely on upwinding to discretize advection terms are not asymptotic preserving, there are specialized spatial discretizations [31] and operator splitting techniques [16, 17, 21] that are.

A different approach for capturing the numerical drift-diffusion limit is to use discontinuous Galerkin (DG) methods. These methods have been developed both for kinetic semiconductor equations [6–9, 25] and for the drift-diffusion equations [5, 24]. While not yet rigorously established in the literature, it is reasonable to assume that DG methods will recover the numerical drift-diffusion limit. Such a conjecture rests on a similar body of work for kinetic equations of radiation transport. In that setting, collisional dynamics over long time scales lead to a standard diffusion equation [3,14,23]. The asymptotic preserving properties of DG methods for transport equations were first established in [23] for one-dimensional (slab) geometries and later extended to the general multi-dimensional setting in [2]. In [13], the work in [2] was re-established using a rigorous functional analysis framework. The work presented here follows in the spirit of that framework.

In the current paper, we rigorously prove the numerical drift-diffusion limit for a DG method applied to a linear kinetic semiconductor equation. In particular, the collision operator approximates very complicated material interactions with a simple relaxation model and the electric field is not self-consistent, but rather assumed to be given. The DG method relies on a reformulation of the kinetic equation in terms of a weighted distribution function. While such a reformulation is likely not necessary, partially due to the numerical results in [22], it does make some stability results easier to prove. Such results are challenging because, unlike the radiation transport case, the advection operators and collision operator of the kinetic semiconductor equation are stable in $L^2$ spaces with two different weightings. Even at the analytical level, this mismatch poses significant challenges [27]. Even so, we expect that the analysis presented here can be leveraged for “more standard” implementations.

Beyond linearity, there are several other assumptions made in the analysis. Some of these are technical, but others are quite important. Among these, the most important is a zero-inflow boundary condition which precludes the development of a boundary layer. We also assume that the initial data is well-prepared in the sense that it is consistent with the state of local thermal equilibrium. Removing these three assumptions—linearity, zero inflow, and well-prepared initial data—will be important steps in future work. In addition, uniform error estimates independent of the collision frequency, along the lines of [32] for the radiation transport case, should be considered. However, the analysis here is already fairly involved and requires more work than the radiation transport case. The main novelty of this work is the rigorous analysis of the numerical diffusion limit of a kinetic equation. In contrast to [13] whose work and novelty closely resembles and inspires the work here, the kinetic equation in the current setting is time-dependent, involves advection in both the physical and velocity variables, is defined over an unbounded velocity domain, and has a collision operator with a kernel (the local thermal equilibrium) that is not contained in a standard finite element space. Additionally, this work provides several lemmas concerning stability and control of projecting discontinuous Galerkin finite element functions onto a continuous Galerkin finite element space. These technical results will aid in the current and future numerical analysis of AP discontinuous Galerkin schemes.

The remainder of the paper is organized as follows. In Section 2, we introduce
the relevant equations, preliminary notation, assumptions used to construct a discrete Maxwellian, and the numerical method for solving the kinetic semiconductor model given in (2.1) below. We characterize the collision frequency by an asymptotic parameter $\varepsilon > 0$ which is inversely proportional to the mean-free-path between collisions, and in Section 3, we develop stability and pre-compactness estimates that allow us to take the $\varepsilon$-limit to 0. Additionally, we give several technical results which will aid in the general analysis of discrete drift-diffusion limits. In Section 4, we show the numerical density $\rho_h^\varepsilon$ of the kinetic model converges to the solution of a discretized version $\rho_0^\varepsilon$ of the drift diffusion system, given in (2.7) below. In Section 5, we show error estimates for $\|\rho_h^\varepsilon - \rho_0^\varepsilon\|$ in $\varepsilon$ and $h$ as well as error estimates for $\|\rho_0^\varepsilon - \rho_0\|$ in $\varepsilon$. This allows us to build estimates for $\|\rho_h^\varepsilon - \rho_0\|$ in $\varepsilon$ and $h$.

2. Background, Preliminaries, and Assumptions. Given $\varepsilon > 0$, a Lipschitz spatial domain $\Omega_x \subset \mathbb{R}^3$, and data $f_0$ prescibed on $\Omega_x$, let $f_\varepsilon(x,v,t)$ be the solution of the following kinetic semiconductor model

\begin{align}
(2.1a) & \quad \varepsilon \frac{\partial f_\varepsilon}{\partial t} + v \cdot \nabla_x f_\varepsilon + E(x,t) \cdot \nabla_v f_\varepsilon - \frac{1}{\varepsilon} Q(f_\varepsilon) = 0, \quad (x,v) \in \Omega_x \times \mathbb{R}^3, t > 0; \\
(2.1b) & \quad f_\varepsilon(x,v,t) = f_-(x,v,t), \quad (x,v) \in \partial \Omega_-, t > 0; \\
(2.1c) & \quad f_\varepsilon(x,v,0) = f_0(x,v), \quad (x,v) \in \Omega_x \times \mathbb{R}^3,
\end{align}

where $E \in W^{1,\infty}([0,\infty);L^\infty(\Omega_x))$ is a given electric field, $f_-$ is the inflow data, and

\begin{equation}
\partial \Omega_- = \{(x,v) \in \partial \Omega_x \times \mathbb{R}^3 : v \cdot n_x(x) < 0\},
\end{equation}

with $n_x(x_0)$ being the normal to $\Omega_x$ at the point $x_0$, is the inflow component of the boundary. Additionally the collision operator $Q$ is defined by

\begin{equation}
Q(f_\varepsilon) = \omega(M \rho_\varepsilon - f_\varepsilon)
\end{equation}

where

\begin{equation}
\rho_\varepsilon(x,t) = \int_{\mathbb{R}^3} f_\varepsilon(x,v,t) \, dv; \quad M(v) = (2\pi\theta)^{-3/2} e^{-|v|^2/2\theta}
\end{equation}

with $\theta > 0$ the (lattice) temperature; and $\omega \in L^{\infty}(\Omega_x)$ with $0 < \omega_{\text{min}} \leq \omega$ on $\Omega_x$ is the (scaled) collision frequency.

Definition 2.1. The function $\rho_\varepsilon$ defined in (2.4) is the number density and

\begin{equation}
J_\varepsilon = \frac{1}{\varepsilon} \int_{\mathbb{R}^3} v f_\varepsilon(x,v,t) \, dv
\end{equation}

is the current density.

It has been shown in [28] that if the inflow and initial data are isotropic, that is, $f_-(x,v,t) = m(x)M(v)$ and $f_0(x,v) = \rho_0(x,t)M(v)$, then as $\varepsilon \to 0$, $f_\varepsilon$ converges to $M(v)\rho^0(x)$, where $\rho^0$ solves the drift-diffusion equation:

\begin{align}
(2.6a) & \quad \frac{\partial \rho_0}{\partial t} + \nabla_x \cdot \left( \frac{1}{\omega} \left( -\theta \nabla_x \rho_0 + E \rho_0 \right) \right) = 0, \quad x \in \Omega_x, t > 0; \\
(2.6b) & \quad \rho_0(x,t) = m(x), \quad x \in \partial \Omega_x, t > 0; \\
(2.6c) & \quad \rho_0(x,0) = \rho_0(x), \quad x \in \Omega_x.
\end{align}
Let \( J^0 = \frac{1}{2}(-\theta \nabla_x \rho^0 + E \rho^0) \). Then the pair \( \{\rho^0, J^0\} \) solves the equivalent first-order system:

\[
\begin{align*}
\frac{\partial \rho^0}{\partial t} + \nabla_x \cdot J^0 &= 0, & x \in \Omega_x, t > 0; \\
\omega J^0 + \theta \nabla_x \rho^0 - E \rho^0 &= 0, & x \in \Omega_x, t > 0; \\
\rho^0(x,t) &= m(x), & x \in \partial \Omega_x, t > 0; \\
\rho^0(x,0) &= \rho_0(x), & x \in \Omega_x.
\end{align*}
\]

### 2.1. Notation

Given a measurable open set \( D \subset \mathbb{R}^3 \), let \( L^2(D) \) and \( W^{k,p}(D) \) be the standard Lebesgue and Sobolev spaces of functions on \( D \) and let \( H^k(D) := W^{k,2}(D) \). When \( D \) is a volume (a three-dimensional manifold) in \( \mathbb{R}^3 \), we use \( (\cdot, \cdot)_D \) to denote the standard \( L^2 \) inner product with respect to the Lebesgue measure \( dx \). If \( D \) is a surface (a two-dimensional manifold) in \( \mathbb{R}^3 \), we use \( (\cdot, \cdot)_{\Sigma} \) to denote the \( L^2 \) inner product with respect to the Lebesgue measure on the surface. These inner products can be extended to vector valued functions in a natural way by use of the Euclidean inner product.

To discretize (2.1), we first restrict the domain in \( v \). Given \( L > 0 \), let \( \Omega_v = [-L, L]^3 \) and define \( \Omega = \Omega_x \times \Omega_v \). Given a mesh parameters \( h_x > 0 \), let \( \mathcal{T}_{x,h} := \mathcal{T}_{x,h_x} \) be a mesh on \( \Omega_x \) constructed from open polyhedral cells \( K \) of maximum diameter \( h_x \), and let \( \mathcal{E}_{x,h}^i \) be the interior skeleton of \( \mathcal{T}_{x,h} \), i.e., the set of edges \( e \subseteq \partial K \not\subseteq \partial \Omega_x \). Similarly, given \( h_v > 0 \), let \( \mathcal{T}_{v,h} := \mathcal{T}_{v,h_v} \) and \( \mathcal{E}_{v,h}^j \) be a mesh and interior skeleton for \( \Omega_v \), respectively. We assume that \( \mathcal{T}_{x,h} \) is quasi-uniform and shape regular. The conditions of \( \mathcal{T}_{v,h} \) are given in Subsection 2.3.

Given an edge \( e \in \mathcal{E}_{x,h}^i \), let \( e \in \partial K^+ \cap \partial K^- \) for some \( K^+, K^- \in \mathcal{T}_{x,h} \), let \( z \in L^2(\Omega_x) \) and \( \tau \in [L^2(\Omega_x)]^3 \) be scalar and vector-valued functions, respectively, each with well-defined traces on \( K^+ \) and \( K^- \). For such functions, we define the average and jump methods

\[
\begin{align*}
\{z\} &= \frac{1}{2} \left( [z]_{K^+} + [z]_{K^-} \right), & [z] = [z]_{K^+} n_x^+ + [z]_{K^-} n_x^-; \\
\{\tau\} &= \frac{1}{2} \left( [\tau]_{K^+} + [\tau]_{K^-} \right), & [\tau] = [\tau]_{K^+} n_x^+ + [\tau]_{K^-} n_x^-,
\end{align*}
\]

where \( n_x^\pm \) are the unit normal vectors pointing outward from \( K^\pm \), respectively. These definitions can be modified to average and jumps in the \( v \)-direction in a natural way with normal unit vector \( n_v \).

For ease of presentation we will use \( a \lesssim b \) to denote \( a \leq Cb \) where \( C > 0 \) is a constant independent of \( h_x \) and \( \varepsilon \). The constant additionally depends on the data \( \omega \) and \( E \) (see Assumption 2.9), the final time \( T, \Omega_x, L, h_x \)-independent mesh parameters of \( \mathcal{T}_{x,h} \), and the discrete Maxwellian discussed in Subsection 2.3 which depends on \( h_v, L, \) and \( \theta \).

Given integers \( k_x \geq 0 \) and \( k_v \geq 0 \), let

\[
\begin{align*}
V_{x,h} &= \{ z \in L^2(\Omega_x) : z|_T \in \mathbb{Q}_{k_x}(T) \ \forall K \in \mathcal{T}_{x,h} \} \\
V_{v,h} &= \{ z \in L^2(\Omega_v) : z|_T \in \mathbb{Q}_{k_v}(T) \ \forall K \in \mathcal{T}_{v,h} \},
\end{align*}
\]

where \( \mathbb{Q}_k(T) \) is the set of all polynomials on \( K \) with \( k \) being the maximum degree in any variable, and let

\[
V_h = V_{x,h} \otimes V_{v,h}
\]
be the tensor DG discrete space. For purposes of this paper, we assume \( k_x \geq 0 \) and \( k_v \geq 1.\) For any function \( z_h \in V_h, \) let \( \nabla x z_h \subset V_h \) and \( \nabla v z_h \subset V_h \) denote piece-wise gradients defined on \( K \) for all \( T \in T_h.\)

For the discretization in \( x, \) some additional notation is needed. Let

\[
V_{x,h}^0 = \{ q_h \in V_{x,h} : q_h \big|_{\partial \Omega_x} = 0 \}
\]

be the space of DG functions with vanishing trace, and let

\[
S_{x,h} = V_{x,h} \cap C^0(\overline{\Omega_x}) \quad \text{and} \quad S_{x,h}^0 = V_{x,h}^0 \cap C^0(\overline{\Omega_x})
\]

be the continuous finite element analogues to the DG spaces \( V_{x,h} \) and \( V_{x,h}^0, \) respectively.

Below we consider two discretizations parameterized by an integer \( \beta \in \{0,1\} \) that determines the type of numerical flux used and, consequently, the space that the discrete drift-diffusion limit will live; see Remark 2.4. Let \( S_h^\beta \) be an \( L^2 \)-orthogonal projection operator from \( L^2(\Omega_x) \) onto \( V_{x,h}^0 \) if \( \beta = 1 \) and from \( L^2(\Omega_x) \) onto \( S_{x,h}^0 \) if \( \beta = 0. \) Moreover, let \( (S_{x,h}^0)^* \) and \( (V_{x,h}^0)^* \) denote the topological dual of \( S_{x,h}^0 \) and \( V_{x,h}^0, \) respectively and let \( \rightharpoonup \) represent convergence in the weak topology. Additionally, for \( \beta \in \{0,1\} \) define the discrete dual-norm \( H_{\kappa,\beta}^{-1}(\Omega_x) \) by

\[
\| z_h \|_{H_{\kappa,\beta}^{-1}(\Omega_x)} = \sup_{q_h \in S_{x,h}^0, q_h \neq 0} \frac{\langle z_h, q_h \rangle_{\Omega_x}}{\| \nabla x q_h \|_{L^2(\Omega_x)}} \quad \text{or} \quad \| z_h \|_{H_{\kappa,\beta}^{-1}(\Omega_x)} = \sup_{q_h \in V_{x,h}^0, q_h \neq 0} \frac{\langle z_h, q_h \rangle_{\Omega_x}}{\| q_h \|_{H_{\kappa}^{-1}(\Omega_x)}}
\]

where \( \| \cdot \|_{H_{\kappa}^{-1}(\Omega_x)} \) is a discrete \( H^1 \) norm:

\[
\| q_h \|_{H_{\kappa}^{-1}(\Omega_x)}^2 = \| \nabla x q_h \|_{L^2(\Omega_x)}^2 + \frac{1}{h_x} \| q_h \|_{L^2(\partial \Omega_x)}^2 + \frac{1}{h_x} \| q_h \|_{L^2(\partial \Omega_x)}^2.
\]

A discrete Poincaré-Friedrichs inequality \cite[Theorem 10.6.12]{4} yields

\[
\| q_h \|_{L^2(\Omega_x)} \lesssim \| q_h \|_{H_{\kappa}^{-1}(\Omega_x)}.
\]

Given a Banach space \( X, 1 \leq p \leq \infty, \) and the final time \( T, \) we let \( L^p_T(X) := L^p([0,T]; X), \) \( C^0([0,T]; X), \) and \( H^1([0,T]; X) \) be the standard \( L^p/C^0/H^1 \) spaces of Banach-space valued functions with Bochner integration.

Finally we will often write \( \frac{\partial}{\partial t} \) as \( \partial_t \) in order to keep the spacing consistent in longer estimates. Both will be used interchangeably.

### 2.2. Alternate form of the PDE

It is easy to show that the collision operator \( Q, \) defined in (2.3), is semi-coercive in the weighted norm \( \| M^{-\frac{1}{2}}(\cdot) \|_{L^2(\Omega)}. \) Indeed, testing \( Q \) by \( M^{-\frac{1}{2}} f^\varepsilon \) gives

\[
-(M^{-1} f^\varepsilon, Q(f^\varepsilon))_{\Omega} = \| \omega_{\varepsilon} M^{-\frac{1}{2}} (f^\varepsilon - M \rho^\varepsilon) \|_{L^2(\Omega)}^2.
\]

This structure is critical to achieving the drift-diffusion limit. However since standard discretizations of (2.1) do not allow test functions with an \( M^{-1} \) weight, we instead

---

1. The assumption on \( k_v \) is to enable the construction of the discrete Maxwellian; see Subsection 2.3.
2. The discrete gradient ignores the jumps in \( z_h \) across the boundary, but agrees with the standard definition of gradient for continuous functions.
rewrite (2.1) in terms of the weighted distribution $g^\varepsilon = M^{-\frac{1}{2}}f^\varepsilon$:

\begin{align}
(2.18a) \\
\varepsilon \frac{\partial g^\varepsilon}{\partial t} + v \cdot \nabla x g^\varepsilon + E(x,t) \cdot \nabla v g^\varepsilon - \frac{\omega}{\varepsilon} \left( M^{\frac{1}{2}} \rho^\varepsilon - g^\varepsilon \right) = \frac{1}{2\theta} E(x,t) \cdot v g^\varepsilon, \quad (x,v) \in \Omega_v \times \mathbb{R}^3, \quad t > 0;
\end{align}

\begin{align}
(2.18b) \\
g^\varepsilon(x,v,t) = f_-(x,v,t)/M^{\frac{1}{2}}(v), \quad (x,v) \in \partial \Omega_-, \quad t > 0;
\end{align}

\begin{align}
(2.18c) \\
g^\varepsilon(x,v,0) = f_0(x,v)/M^{\frac{1}{2}}(v), \quad (x,v) \in \Omega_x \times \mathbb{R}^3; \quad \blacksquare
\end{align}

where, in terms of $g^\varepsilon$, $\rho^\varepsilon = (M^{\frac{1}{2}}, g^\varepsilon)_{\mathbb{R}^3}$. Since $\|g^\varepsilon\|_{L^2(\Omega)} = \|M^{-\frac{1}{2}}f^\varepsilon\|_{L^2(\Omega)}$, the weighted collision operator

\begin{align}
M^{-\frac{1}{2}} Q(M^{\frac{1}{2}} g^\varepsilon) = \omega \left( M^{\frac{1}{2}} \rho^\varepsilon - g^\varepsilon \right)
\end{align}

will be $L^2$-coercive and symmetric as a function of $g^\varepsilon$. We refer to the function $M^{\frac{1}{2}} g^\varepsilon$ as the weighted equilibrium. The cost of this additional structure is the electric field term on the right-hand side of (2.18a).

**2.3. Construction of Discrete Maxwellian.** In order to recover the proper drift-diffusion limit, we need to construct a suitable discrete Maxwellian on the bounded domain $\Omega_v$. This is done via an approximation of the square root of the one-dimensional Maxwellian. Assume that $\mathcal{T}_{v,h}$ is a tensor product mesh, i.e., $\mathcal{T}_{v,h} = \mathcal{T}_{v,h}^1 \otimes \cdots \otimes \mathcal{T}_{v,h}^3$, and let $M^{\frac{1}{2}}_{h,i}$ be a continuous, strictly positive, piecewise-polynomial approximation of the one-dimensional root-Maxwellian over $\mathcal{T}_{v,h}^i$:

\begin{align}
M^{\frac{1}{2}}_{h,i}(v_i) \approx M^{\frac{1}{2}}_{i}(v_i) := \left( \frac{1}{\sqrt{2\pi} \theta} e^{-\frac{v_i^2}{2\theta}} \right)^{1/2}, \quad i = 1, \ldots, 3,
\end{align}

with the following properties:

**Assumption 2.2.** For each $i = 1, \ldots, 3$, the function $M^{\frac{1}{2}}_{h,i}$ satisfies the following properties:

a. $(M^{\frac{1}{2}}_{h,i}, M^{\frac{1}{2}}_{h,i})_{[-L,L]} = 1$, c. $(\partial v M^{\frac{1}{2}}_{h,i}, \partial v M^{\frac{1}{2}}_{h,i})_{[-L,L]} = \frac{1}{\theta^2}$,

b. $M^{\frac{1}{2}}_{h,i}(L) = M^{\frac{1}{2}}_{h,i}(-L)$, d. $(\partial v M^{\frac{1}{2}}_{h,i}, M^{\frac{1}{2}}_{h,i})_{[-L,L]} = 0$.

**Definition 2.3.** The discrete root-Maxwellian $M^{\frac{1}{2}}_h \in V_{v,h} \cap C^0(\overline{\Omega_v})$ is

\begin{align}
M^{\frac{1}{2}}_h(v) = \prod_{i=1}^{3} M^{\frac{1}{2}}_{h,i}(v_i),
\end{align}

and the discrete velocity $v_h$ is

\begin{align}
v_h = -2\theta \frac{\nabla v M^{\frac{1}{2}}_h}{M^{\frac{1}{2}}_h} = -2\theta \nabla v \log(M^{\frac{1}{2}}_h).
\end{align}

Since $M^{\frac{1}{2}}_h > 0$ on $\overline{\Omega_v}$, it follows that $v_h \in L^\infty(\Omega_v)$. Moreover $v_h M^{\frac{1}{2}}_h \in V_{v,h}$, even though $v_h \notin V_{v,h}$.

**Remark 2.4.** In defining the discrete root-Maxwellian:
1. The continuity requirement on $M_{h,i}^\frac{1}{2}$ is the reason for assumption $k_v \geq 1$ in Subsection 2.1.

2. Assumption 2.2.d is not independent, but rather is implied by Assumption 2.2.b.

3. If Assumption 2.2.c is not satisfied, then the numerical discretization below will still converge in the $\varepsilon$-limit to a discretization of a drift-diffusion system, but $\theta$ from (2.7) will be $\theta_{v,\ast}$, see (2.23), instead of the proper temperature. This can be seen by substituting $\theta_{v,\ast}$ for $\theta$ in Sections 3 and 4.

**Remark 2.5.** The existence of such a discrete Maxwellian satisfying every assumption given is not discussed. Rather, to create a discrete Maxwellian that satisfies every assumption but Assumption 2.2.c, take $M_{h,i}^\frac{1}{2}$ to be the Lagrange piecewise linear nodal interpolant of $M_i^\frac{1}{2}$ and scale it to have an $L^2$ norm of 1. With mild restrictions on $L$ and $h_v$ based on $\theta$, we show in Lemma B.1, given in the appendix, that $M_{h,i}^1$ is an $O(h_v^2)$ approximation to $M_{h,i}^1$ in the $L^2$-norm and an $O(h_v)$ approximation in the $H^1$-norm.\[^3\] While the discrete temperature

\[
\theta_{h,v} := \frac{1}{4}(\partial_v M_{h,i}^\frac{1}{2}, \partial_v M_{h,i}^\frac{1}{2})^{-1}[-L,L]
\]
is not exactly $\theta$, it is readily seen from Lemma B.1 that $\theta_{h,v}$ is an $O(h_v)$ approximation to $\theta$.

**2.4. The Numerical Method.** We now give our numerical method for (2.18).

**Problem 2.6.** Find $g_h^\varepsilon \in H^1([0,T]; V_h)$ such that

\[
\varepsilon \frac{\partial g_h^\varepsilon}{\partial t} + A(g_h^\varepsilon, z_h) + B(g_h^\varepsilon, z_h) + \mathcal{D}(g_h^\varepsilon, z_h) - \frac{1}{\varepsilon} \mathcal{Q}(g_h^\varepsilon, z_h) = \mathcal{C}(g_h^\varepsilon, z_h) + \mathcal{R}(z_h),
\]

for all $z_h \in V_h$ and a.e $0 < t \leq T$ where

\[
A(w_h, z_h) = -(w_h w_h, \nabla_x z_h)_\Omega + \left( w_h \| w_h \| + \varepsilon \beta \frac{|v_h \cdot n_x|}{2} \| w_h \|, [z_h] \right)_{\varepsilon \Omega \times \partial \Omega_v},
\]

\[
B(w_h, z_h) = - (E w_h, \nabla_v z_h)_\Omega + \left( E \| w_h \| + \frac{|E \cdot n_v|}{2} \| w_h \|, [z_h] \right)_{\Omega \times \varepsilon \partial \Omega_v},
\]

\[
\mathcal{D}(w_h, z_h) = \left( E M_h^\frac{1}{2} P(w_h), n_v z_h \right)_{\Omega \times \varepsilon \partial \Omega_v},
\]

\[
P(w_h) = (M_h^\frac{1}{2}, w_h)_\Omega,
\]

\[
\mathcal{Q}(w_h, z_h) = \left( \omega (M_h^\frac{1}{2} P(w_h) - w_h), z_h \right)_\Omega,
\]

\[
\mathcal{C}(w_h, z_h) = \frac{1}{20} (E \cdot v_h w_h, z_h)_\Omega,
\]

\[
\mathcal{R}(z_h) = - (v_h g_{-h}, n_x z_h)_{\partial \Omega_-},
\]

\[^3\]We are neglecting the errors due to the finite velocity domain. See Lemma B.1 for the full estimate.
and the functions $g_{-h} \in V_h$ and $g_{0,h} \in V_h$ are the discrete inflow and initial data respectively.

**Definition 2.7.** The discrete number density $\rho^\varepsilon_h$ and current density $J^\varepsilon_h$ are given by

\begin{align}
\rho^\varepsilon_h &= P(g^\varepsilon_h) = (M^\frac{2}{3}_h, g^\varepsilon_h)_{\Omega_+}, \\
J^\varepsilon_h &= \frac{1}{\varepsilon} P(v_h g^\varepsilon_h) = \frac{1}{\varepsilon} (M^\frac{2}{3}_h v_h, g^\varepsilon_h)_{\Omega_+}.
\end{align}

**Remark 2.8.** In Problem 2.6,

1. The bilinear form $A$ and functional $R$ are the result of the discretizations of the operator $v \cdot \nabla_x$ with $v$ replaced by $v_h$. The parameter $\beta$ is a switch between a standard upwinding and a scaled upwinding flux. If $\beta = 0$, the flux $\hat{v}_h g$ is the standard upwinding flux, namely

$$
\hat{v}_h g = \begin{cases}
   v_h \|g\| + \frac{|v_h \cdot n_x|}{2} [g] & \text{on } \mathcal{E}_{x,h} \times \Omega_v \\
   v_h g & \text{on } \partial\Omega_+ \\
   v_h g_{-h} & \text{on } \partial\Omega_-
\end{cases}
$$

In this case, $\rho^\varepsilon_h$ will converge to a continuous finite element function as $\varepsilon \to 0$; see Section 4. As a result, locking will occur when $k_x = 0$, i.e. when $V_{x,h}$ is comprised of piecewise constant functions in $x$. If $\beta = 1$, the flux $\hat{v}_h g$ instead contains an $\varepsilon$-scaled Lax-Friedrichs penalty in the jump. This modification yields a LDG-like discretization of the drift-diffusion equations; see Section 4.

2. The bilinear forms $B$ and $D$ are constructed using standard upwind fluxes for the Vlasov operator $E \cdot \nabla_v$ but, due to the velocity boundary domain restriction, we weakly impose the boundary condition $g^\varepsilon_h = M^\frac{2}{3}_h \rho^\varepsilon_h$ on $\Omega_x \times \partial\Omega_v$. This provides two benefits. First, the density $g^\varepsilon_h$ will not lose or gain mass out of the velocity boundary. Second, this boundary condition keeps the restriction of $v$ to the bounded domain $\Omega_v$ from polluting the discrete drift-diffusion limit.

3. The bilinear form $Q$ is a standard discretization of the collision operator.

4. The bilinear form $C$ is a standard discretization of the term $E \cdot v g$, with $v$ replaced by $v_h$.

2.5. Other Assumptions. Here we collect any other assumptions used in the analysis of Problem 2.6.

**Assumption 2.9.** The collision frequency $\omega$ and the electric field $E$ in (2.1) are specified as $\omega \in L^{\infty}(\Omega_x)$ with $0 < \omega_{\text{min}} \leq \omega$ on $\Omega_x$ and $E \in W^{1,\infty}([0,T]; L^{\infty}(\Omega_x))$.

**Assumption 2.10.** The inflow data in (2.1) is isotropic, that is, $f_0(x,v) = \rho_0(x)(v)$ Additionally we require $g_{0,h}$ in Problem 2.6 to be discretely isotropic, that is, $g_{0,h} = \rho_{0,h} M^\frac{2}{3}_h$ where $\rho_{0,h} \in V_{x,h}$ is defined by

$$(\rho_{0,h}, q_h)_{\Omega_x} = (\rho_{0}, q_h)_{\Omega_x} \quad \forall q_h \in V_{x,h}.$$
It follows from Assumption 2.10, Assumption 2.2.a, and Assumption 2.2.d that

\begin{align}
(2.28) \quad \rho_h \big|_{t=0} &= (M^2, g_0, h)_{\Omega_v} = (M^2, M^2)_{\Omega_v} \rho_0 = \rho_0,
\end{align}

\begin{align}
(2.29) \quad J_h \big|_{t=0} &= \frac{1}{\varepsilon} (v_h M^2, g_0, h)_{\Omega_v} = \frac{1}{\varepsilon} (v_h M^2, M^2)_{\Omega_v} \rho_0 = 0.
\end{align}

Assumption 2.11. The continuous incoming data \( f_- \) in (2.1b), the discrete incoming data \( g_{-, h} \) in Problem 2.6 and, consequently, the functional \( R \) in Problem 2.6 are identically zero.

Assumptions like Assumption 2.11 are commonly made in the analysis of diffusion limits to avoid handling complications due to boundary layers. While it is not expected that the equilibrium boundary condition will induce a boundary layer, the case of non-zero incoming data will be treated elsewhere.

Assumption 2.12. The inverse Laplacian \( S : L^2(\Omega_x) \rightarrow H^1_0(\Omega_x) \) defined by

\begin{align}
(2.30) \quad (\nabla S q, \nabla w)_{\Omega_x} = (q, w)_{\Omega_x} \quad \forall w \in H^1_0(\Omega),
\end{align}

is a bounded linear operator from \( L^2(\Omega_x) \rightarrow H^2(\Omega_x) \cap H^1_0(\Omega_x) \), that is,

\begin{align}
(2.31) \quad \|S q\|_{H^2(\Omega_x)} \lesssim \|q\|_{L^2(\Omega_x)}
\end{align}

for any \( q \in L^2(\Omega_x) \).

Assumption 2.12 is needed for the proof of stability of the \( L^2 \) projection \( S_h^\beta \) in the \( H^1_0(\Omega_x) \)-norm (see Lemma 3.5). If \( \Omega_x \) is convex, then Assumption 2.12 is automatically satisfied [12, Section 3.2].

Assumption 2.13. There is a constant \( C > 0 \) such that \( \frac{\varepsilon}{h_x} < C \), that is, \( h_x \) cannot go to zero faster than \( \varepsilon \).

Assumption 2.13 is to improve the readability of the paper. There are several estimates that include bounds of \( 1 + \frac{\varepsilon}{h_x} \) and these bounds will not improve any rates of convergence regardless of the choice of \( \varepsilon \) and \( h_x \). Thus with Assumption 2.13 we have the bounds \( 1 + \frac{\varepsilon}{h_x} \lesssim 1 \) and \( 1 + \frac{\sqrt{\varepsilon}}{h_x} \lesssim 1 \).

3. A Priori Estimates. In this section we develop space-time stability estimates for \( g_h \), the number density \( \rho_h \), and the current density \( J_h \) in Problem 2.6.

3.1. Preliminary Estimates and Identities. In this subsection we list an inverse and trace inequality, derived from the standard estimates in [30], technical interpolation and projections estimates, and a useful integration by parts identity.

Lemma 3.1 (Trace Inequality). For any \( z_h \in V_h \) and \( q_h \in V_{x,h} \) we have

\begin{align}
(3.1) \quad \|\| z_h \|_{L^2(\Omega_x \times \Omega_v)} + \| z_h \|_{L^2(\Omega_v \times \Omega_v)} \leq \frac{C}{h_x} \| z_h \|_{L^2(\Omega)},
\end{align}

\begin{align}
(3.2) \quad \|\| q_h \|_{L^2(\Omega_x \times \Omega_v)} + \| q_h \|_{L^2(\Omega_v \times \Omega_v)} \leq \frac{C}{h_x} \| q_h \|_{L^2(\Omega_v)},
\end{align}

\begin{align}
(3.3) \quad \|\| z_h \|_{L^2(\Omega_v \times \Omega_v)} + \| z_h \|_{L^2(\Omega_x \times \Omega_v)} \leq \frac{C}{h_v} \| z_h \|_{L^2(\Omega)}.
\end{align}

Here \( C > 0 \) is an \( \varepsilon, h_x \), and \( h_v \)-independent constant. \( C \) depends on the polynomial degree of \( V_h \) and other \( h_x \) and \( h_v \)-independent mesh parameters of \( T_{x,h} \) and \( T_{v,h} \).
Lemma 3.2 (Inverse Inequality). For any $q_h \in V_{x,h}$ we have
\begin{equation}
\|\nabla q_h\|_{L^2(\Omega_x)} \leq C h_x^{-1} \|q_h\|_{L^2(\Omega_x)},
\end{equation}
where $C > 0$ is some $\varepsilon$ and $h_x$-independent constant that depends on the polynomial degree of $V_{x,h}$.

Lemma 3.3 (Integration by Parts). For any $q_h \in V_{x,h}$ and $\tau_h \in [V_{x,h}]^3$, there holds
\begin{equation}
(q_h, \nabla \cdot \tau_h)_{\Omega_x} = -(\nabla q_h, \tau_h)_{\Omega_x} + \langle \{q_h\}, \{\tau_h\}\rangle_{E_{x,h}^0} + \langle \{q_h\} \cdot \{n_x\}, \tau_h\rangle_{\partial \Omega_x}.
\end{equation}

3.2. Technical Estimates for Drift-Diffusion Analysis. In this subsection, we present several technical results which are useful both in the analysis in the drift-diffusion limit to Problem 2.6, and for the future analysis of similar problems.

The first result is an error estimate of an interpolant from $V_{x,h} \rightarrow S^0_{x,h}$ if $\beta = 0$ and $V_{x,h} \rightarrow V^0_{x,h}$ if $\beta = 1$. The result allows us to control a DG function’s distance to $S^0_{x,h}$, in the $H^1_{\Omega}$-norm, by the function’s interior jumps and boundary data. The interpolant onto the conforming finite element space, $I^0_{x,h}$, is the KP interpolant from [19, Theorem 2.2]. The construction of $I^0_{x,h}$ is achieved in a similar manner and proved below.

Lemma 3.4 (Conforming Interpolant). There is an interpolant $I^0_{x,h} : V_{x,h} \rightarrow S^0_{x,h}$ if $\beta = 0$ and $I^1_{x,h} : V_{x,h} \rightarrow V^0_{x,h}$ if $\beta = 1$ such that for any $q_h \in V_{x,h}$ we have
\begin{align}
\|q_h - I^0_{x,h} q_h\|_{H^1_{\Omega_x}} &\lesssim \frac{1}{h_x} \|q_h\|_{L^2(\Omega_x)} + \frac{1}{h_x} \|[q_h]\|_{L^2(E_{x,h}^0)}, \\
\|q_h - I^1_{x,h} q_h\|_{H^1_{\Omega_x}} &\lesssim \frac{1}{h_x} \|q_h\|_{L^2(\Omega_x)}. 
\end{align}

Proof. The existence of $I^0_{x,h}$ and (3.6a) is given in [19, Theorem 2.2]. The construction of $I^1_{x,h}$ and proof of (3.6b) uses ingredients and notation from Theorem 2.2 of [19]. For $K \in T_{x,h}$, let $N_K := \{x_K^i, i = 1 \ldots, m\}$ be the Lagrange nodes of $K$ with $\{\varphi^i_K, i = 1 \ldots, m\}$ be the associated local Lagrange basis functions with $\varphi^i_K(x_K^j) = \delta_{ij}$. Let $N \subseteq \cup_{K \in T_{x,h}} N_K$. We partition $N = N_i \cup N_b$ into the interior nodes $N_i$ and the boundary nodes $N_b$ with $N_i \cap N_b = \emptyset$. Given the representation of $q_h \in V_{x,h}$ in the basis $\varphi^i_K$, define $I^1_{x,h} q_h \in V^0_{x,h}$ by
\begin{equation}
I^1_{x,h} q_h = \sum_{K \in T_{x,h}} \sum_{i=1}^m \begin{cases}
\alpha^i_K \varphi^i_K & \text{if } x_K^i \in N_i \\
0 & \text{if } x_K^i \in N_b
\end{cases}
\end{equation}
where $q_h = \sum_{K \in T_{x,h}} \sum_{i=1}^m \alpha^i_K \varphi^i_K$.

From (3.7) it is easy to see that $q_h - I^1_{x,h} q_h$ is zero at the interior nodes. Via scaling arguments ($\Omega_x \subset \mathbb{R}^3$), we have
\begin{align}
\|\nabla \varphi^i_K\|_{L^2(K)}^2 &\lesssim h_x, \\
\frac{1}{h_x} \sum_{e \in E_{x,h}} \|\varphi^i_K\|_{L^2(e)}^2 &\lesssim h_x, \\
\frac{1}{h_x} \|\varphi^i_K\|_{L^2(\partial \Omega_x \cap \partial K)}^2 &\lesssim h_x.
\end{align}
Thus by (3.8) we have

\begin{equation}
\|q_h - I_h q_h\|_{H^1_0(\Omega_x)}^2 \lesssim h_x \sum_{K \in T_{x,h}} \sum_{i : x^*_K \in N_h} \|\alpha_i K\|^2 = h_x \sum_{\nu \in N_h} \sum_{x^*_K = \nu} \|\alpha_i K\|^2.
\end{equation}

where the last equality is just re-indexing of the double sum. From (2.21) of [19] and the quasi-uniformity of $T_{x,h}$, we have for all $\nu \in N_h$:

\begin{equation}
\sum_{x^*_K = \nu} \|\alpha_i K\|^2 \lesssim h_x^{-1} \|q_h\|^2_{L^2(\partial \Omega \cap \partial K)}.
\end{equation}

Therefore (3.6b) follows from (3.9) and (3.10). The proof is complete.

Additionally, we give an $H^1_\beta$ stability estimate for the $L^2$ projection from $V_{x,h}$ to the spaces $S^0_{x,h}$ and $V^0_{x,h}$. This interpolant, $S^\beta_h$ is vital to the analysis of evolution problems and this estimate is needed to extend the results of the interpolant $I_h^\beta$ to $S^\beta_h$; see (3.76).

**Lemma 3.5.** For $\beta \in \{0, 1\}$, the projection $S^\beta_h$ is stable on $V_{x,h}$ with respect to the $H^1_\beta(\Omega_x)$-norm, that is,

\begin{equation}
\|S^\beta_h q_h\|_{H^1_\beta(\Omega_x)} \lesssim \|q_h\|_{H^1_\beta(\Omega_x)} \quad \forall q_h \in V_{x,h}.
\end{equation}

We will prove Lemma 3.5 by first showing an equivalent result. Given $\gamma > 0$, define the DG discrete Laplacian energy on $V_{x,h}$ via the symmetric bilinear form

\begin{equation}
(q_h, z_h)_E := (\nabla q_h, \nabla z_h)_{\Omega_x} - \langle q_h, \nabla^\dagger z_h \rangle_{E^\dagger_{x,h}} - \langle \nabla q_h, \nabla z_h \rangle_{E_{x,h}}
- \langle q_h, \nabla z_h \rangle_{\partial \Omega_{x,h}} - \gamma h_x \langle q_h, \nabla z_h \rangle_{\partial \Omega_{x,h}} + \gamma h_x \langle q_h, \nabla q_h \rangle_{E_{x,h}} + \gamma h_x \langle q_h, z_h \rangle_{\partial \Omega_{x,h}}.
\end{equation}

Standard DG elliptic theory shows that there exists a $\gamma_* > 0$, independent of $h$, such that $(\cdot, \cdot)_E$ is an inner product on $V_{x,h}$ for all $\gamma > \gamma_*$ [30]. We fix some $\gamma > \gamma_*$ and therefore $(\cdot, \cdot)_E$ induces a norm $\|\cdot\|_E$ on $V_{x,h}$. Moreover, by use of the trace inequality, Lemma 3.1, we have

\begin{equation}
\|q_h\|_{H^1_\beta(\Omega_x)} \lesssim \|q_h\|_E \lesssim \|q_h\|_{H^1_\beta(\Omega_x)}
\end{equation}

for all $q_h \in V_{x,h}$. From (3.13), the following lemma immediately implies Lemma 3.5.

**Lemma 3.6.** The $L^2$-projection $S^\beta_h$ is stable on $V_{x,h}$ with respect to $\|\cdot\|_E$, that is, there is a constant $C > 0$, independent of $h_x$, such that

\begin{equation}
\|S^\beta_h q_h\|_E \leq C \|q_h\|_E \quad \forall q_h \in V_{x,h}.
\end{equation}

**Proof.** We will focus on the case $\beta = 0$. The proof is written such that the $\beta = 1$ case is shown by substituting $S^0_{x,h}$ with $V^0_{x,h}$. Let $\{\psi_i\}_{i=1}^M \subset V_{x,h}$ be an orthonormal eigenbasis in $L^2(\Omega_x)$ with associated eigenvalues $\lambda_i > 0$, in increasing order, for the following eigenvalue problem: find $\psi \in V_{x,h}$ and $\lambda \in \mathbb{R}$ such that

\begin{equation}
(\psi, q_h)_E = \lambda (\psi, q_h)_{\Omega_x} \quad \forall q_h \in V_{x,h}.
\end{equation}
Similarly, let \( \{ \varphi_j \}_{j=1}^N \subset S^0_{x,h} \) be an orthonormal eigenbasis in \( L^2(\Omega_z) \) with associated eigenvalues \( \mu_j > 0 \), in increasing order, for the following eigenvalue problem on \( S^0_{x,h} \). Find \( \varphi \in S^0_{x,h} \) and \( \mu \in \mathbb{R} \) such that

\[
\langle \varphi, w_h \rangle_E = \mu \langle \varphi, w_h \rangle_{\Omega_z} \quad \forall w_h \in S^0_{x,h}.
\]

(3.16)

We also define the solution operators \( T_h : V_{x,h} \to V_{x,h} \), and \( T_h^0 : S^0_{x,h} \to S^0_{x,h} \) by

\[
\langle T_h q_h, z_h \rangle_E = \langle q_h, z_h \rangle_{\Omega_z} \quad \forall z_h \in V_{x,h},
\]

(3.17)

\[
\langle T_h^0 w_h, s_h \rangle_E = \langle w_h, s_h \rangle_{\Omega_z} \quad \forall s_h \in S^0_{x,h}.
\]

(3.18)

We note that while written using the inner product \( \langle \cdot, \cdot \rangle_E \), (3.18) is the standard continuous Galerkin finite element method for the Poisson problem and (3.16) is its respective eigenvalue problem. Note \( \psi_i \) and \( \varphi_j \) are eigenvectors of \( T_h \) and \( T_h^0 \) with associated eigenvalues \( \lambda_i^{-1} \) and \( \mu_j^{-1} \) respectively. We also recall the inverse Laplacian \( S: L^2(\Omega_z) \to H^2(\Omega_z) \cap H_0^1(\Omega_z) \) which is given in Assumption 2.12. Standard continuous and discontinuous Galerkin theory \([4, 30]\) and Assumption 2.12 yield the following estimates:

\[
\| T_h q_h - S q_h \|_E \lesssim h \| S q_h \|_{H^2(\Omega)} \lesssim h \| q_h \|_{L^2(\Omega_z)} \quad \forall q_h \in V_{x,h},
\]

(3.19)

\[
\| T_h^0 w_h - S w_h \|_E \lesssim h \| S w_h \|_{H^2(\Omega)} \lesssim h \| w_h \|_{L^2(\Omega_z)} \quad \forall w_h \in S^0_{x,h}.
\]

(3.20)

Therefore by (3.19)-(3.20) we have

\[
\| T_h w_h - T_h^0 w_h \|_E \lesssim h \| w_h \|_{L^2(\Omega_z)} \quad \forall w_h \in S^0_{x,h}.
\]

Given \( q_h \in V_{x,h} \), let \( \alpha \in \mathbb{R}^M \) be the coefficients of \( q_h \) w.r.t the basis \( \{ \psi_i \} \) given by \( \alpha_i = \langle q_h, \psi_i \rangle_{\Omega_z} \). Similarly, given \( w_h \in S^0_{x,h} \), let \( \xi \in \mathbb{R}^N \) be the coefficients of \( w_h \) w.r.t the basis \( \{ \varphi_j \} \) given by \( \xi_j = \langle w_h, \varphi_j \rangle_{\Omega_z} \). Due to the eigenbasis decomposition we have

\[
q_h = \sum_{i=1}^M \alpha_i \psi_i, \quad w_h = \sum_{j=1}^N \xi_j \varphi_j,
\]

(3.22a)

\[
\| q_h \|_{L^2(\Omega_z)}^2 = \sum_{i=1}^M \alpha_i^2 = |\alpha|^2, \quad \| w_h \|_{L^2(\Omega_z)}^2 = \sum_{j=1}^N \xi_j^2 = |\xi|^2.
\]

(3.22b)

\[
\| q_h \|_E^2 = \sum_{i=1}^M \lambda_i \alpha_i^2 =: |\alpha|_E^2, \quad \| w_h \|_E^2 = \sum_{j=1}^N \mu_j \xi_j^2 =: |\xi|_E^2.
\]

(3.22c)

Through the decompositions in (3.22a), we also have

\[
T_h q_h = \sum_{i=1}^M \alpha_i \psi_i, \quad T_h^0 w_h = \sum_{j=1}^N \xi_j \varphi_j.
\]

(3.23)

Additionally, we define the operator \( A_h^0 : S^0_{x,h} \to S^0_{x,h} \) by

\[
A_h^0 w_h = \sum_{j=1}^N \xi_j \mu_j^{1/2} \varphi_j.
\]

(3.24)
The fact that \( \{ \varphi_j \} \) is an orthonormal set in \( L^2(\Omega_x) \) and (3.22c) yield
\[
(3.25) \quad \| A^0_h w_h \|_{L^2(\Omega_x)} = \| w_h \|_E.
\]
Also, \( \varphi_j \) is an eigenvector of \( A^0_h \) with associated eigenvalue \( \mu_j^{1/2} \).

Note that (3.14) is equivalent to uniformly bounding
\[
(3.26) \quad \sup_{q_h \in V_{x,h} \setminus \{0\}} \frac{\| S^\beta_h q_h \|_E}{\| q_h \|_E} = \sup_{q_h \in V_{x,h} \setminus \{0\}} \frac{(S^\beta_h q_h, w_h)_E}{\| q_h \|_E \| w_h \|_E}
\]
in \( h_x \); thus we seek to bound \( (S^\beta_h q_h, w_h)_E \). Using (3.16), (3.22a), (3.22c), and (3.25) we have
\[
(S^\beta_h q_h, w_h)_E = \sum_{ij} \alpha_i(S^\beta_h \psi_i, \varphi_j)_E \xi_j
\]
\[
= \sum_{ij} \alpha_i(S^\beta_h \psi_i, \varphi_j)_{\Omega_x} \mu_j \xi_j = \sum_{ij} \alpha_i(\psi_i, \varphi_j)_{\Omega_x} \mu_j \xi_j
\]
\[
= \sum_{ij} \alpha_i \lambda_i^{1/2} (\lambda_i^{-1/2} \psi_i, \mu_j^{1/2} \varphi_j)_{\Omega_x} \xi_j \xi_j^{1/2}
\]
\[
= \sum_{ij} \alpha_i \lambda_i^{1/2} (\lambda_i^{-1/2} \psi_i, A^0_{h,\varphi_j})_{\Omega_x} \xi_j \xi_j^{1/2}
\]
\[
\leq |\alpha|_E \| \overline{C} \|_2 |\xi|_E = \| \overline{C} \|_2 \| q_h \|_E \| w_h \|_E.
\]
Here \( \overline{C} \in \mathbb{R}^{M \times N} \) with \( \overline{C}_{ij} = (\lambda_i^{-1/2} \psi_i, A^0_{h,\varphi_j})_{\Omega_x} \), and
\[
(3.28) \quad \| \overline{C} \|_2 = \max_{\xi \in \mathbb{R}^N \setminus \{0\}} \frac{\xi^T \overline{C}^T \overline{C} \xi}{|\xi|^2}.
\]
To bound \( \| \overline{C} \|_2 \), we use the decomposition of (3.22a) along with (3.23) and (3.17) to compute the following:
\[
\xi^T \overline{C}^T \overline{C} \xi = \sum_{j=1}^N \sum_{i=1}^M \xi_j(A^0_{h,\varphi_j}, \lambda_i^{-1/2} \psi_i)_{\Omega_x} (\lambda_i^{-1/2} \psi_i, A^0_{h,\varphi_i})_{\Omega_x} \xi_i
\]
\[
= \sum_{j=1}^N \xi_j \left( A^0_{h,\varphi_j} \sum_{i=1}^M \frac{(A^0_{h,\varphi_i,\psi_i})_{\Omega_x} \psi_i}{\lambda_i} \right) \xi_i
\]
\[
(3.29) \quad = \sum_{j=1}^N \xi_j (A^0_{h,\varphi_j}, T_h A^0_{h,\varphi_j})_{\Omega_x} \xi_i = \sum_{j=1}^N \xi_j (T_h A^0_{h,\varphi_j}, T_h A^0_{h,\varphi_j})_E \xi_i
\]
\[
= \left( T_h A^0_{h,\varphi_j} \sum_{j=1}^N \xi_j \varphi_j, T_h A^0_{h,\varphi_j} \sum_{j=1}^N \xi_j \varphi_j \right)_E
\]
\[
(3.30) \quad \| \overline{C} \|_2 = \max_{w_h \in S^0_{x,h} \setminus \{0\}} \frac{\| T_h A^0_{h,\varphi_j} \|^2}{\| w_h \|_{L^2(\Omega_x)}}
\]
Therefore the desired estimate (3.14) holds provided we can show

\[ \| T_h A_h^0 w_h \|_E \lesssim \| w_h \|_{L^2(\Omega_x)} \]

for all \( w_h \in S_{x,h}^0 \).

Let \( w_h \in S_{x,h}^0 \) with decomposition given in (3.22a). To show (3.31), we will add and subtract \( T_h^0 A_h^0 w_h \) where \( T_h^0 \) is given in (3.18) and apply the triangle inequality to obtain

\[ \| T_h A_h^0 w_h \|_E \leq \| (T_h - T_h^0) A_h^0 w_h \|_E + \| T_h^0 A_h^0 w_h \|_E. \]

To bound \( \| (T_h - T_h^0) A_h^0 w_h \|_E \), we use (3.21), (3.25), (3.13), and the inverse inequality (4.4) which gives

\[ \| (T_h - T_h^0) A_h^0 w_h \|_E \lesssim h_x \| A_h^0 w_h \|_{L^2(\Omega_x)} = h_x \| w_h \|_E \lesssim h_x \| w_h \|_{H^1_h(\Omega)} \lesssim \| w_h \|_{L^2(\Omega_x)}. \]

Direct computation of \( T_h^0 A_h^0 w_h \) gives us

\[ T_h^0 A_h^0 w_h = \sum_{j=1}^N \xi_j T_h^0 A_h^0 \varphi_j = \sum_{j=1}^N \xi_j \mu_j^{1/2} T_h^0 \varphi_j = \sum_{j=1}^N \xi_j \mu_j^{-1/2} \varphi_j. \]

Additionally, since \( \varphi_j \) is an eigenvector to the problem given in (3.16), then

\[ (\varphi_j, \varphi_l) = \delta_{jl} \mu_j, \quad \text{and} \quad \| \varphi_j \|_E^2 = \mu_j, \]

where \( \delta_{jl} \) is the Kronecker delta. Then using (3.34), (3.35), and (3.22b) we obtain

\[ \| T_h A_h^0 w_h \|_E^2 = \sum_{j=1}^N \frac{\xi_j^2}{\mu_j} \| \varphi_j \|_E^2 = \sum_{j=1}^N \xi_j^2 \| w_h \|_{L^2(\Omega_x)}^2. \]

Therefore (3.32), (3.33), and (3.36) imply (3.31). The proof is complete.

\[ \square \]

**3.3. Initial Estimates.** We first focus on estimates for \( g_h^0 \) that will lead to estimates for \( \rho_h^0 \) and \( J_h^0 \).

**Lemma 3.7.** The bilinear forms defined in Problem 2.6 satisfy the following bounds:

\[ -\mathcal{Q}(g_h^0, g_h^0) \geq \omega_{\min} \| g_h^0 - M_h^\frac{1}{2} \rho_h^0 \|_{L^2(\Omega)}^2, \]

\[ \mathcal{C}(g_h^0, g_h^0) \leq \frac{1}{2} \left( 2C_1 + \frac{\omega_{\min}}{\varepsilon} \right) \| g_h^0 - M_h^\frac{1}{2} \rho_h^0 \|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2\omega_{\min}} C_3^2 \| g_h^0 \|_{L^2(\Omega)}^2, \]

\[ \mathcal{A}(g_h^0, g_h^0) = \left\langle \frac{|v_h \cdot n_x|}{2} [g_h^0], [g_h^0] \right\rangle_{\partial \Omega_x \times \Omega_v} + \varepsilon \left\langle \frac{|v_h \cdot n_x|}{2} [g_h^0], [g_h^0] \right\rangle_{E_{x,h}^l \times \Omega_v}, \]

\[ \mathcal{B}(g_h^0, g_h^0) + \mathcal{D}(g_h^0, g_h^0) \geq \left\langle \frac{|E \cdot n_v|}{2} [g_h^0], [g_h^0] \right\rangle_{\Omega_x \times E_{x,h}^l} - \frac{C_2}{2h_v} \| g_h^0 - M_h^\frac{1}{2} \rho_h^0 \|_{L^2(\Omega)}^2, \]

\[ \square \]
where
\begin{align}
(3.41a) \quad C_1 & := \frac{\|E \cdot v_h\|_{L^\infty(L^\infty(\Omega))}}{2\theta}, \\
(3.41b) \quad C_2 & := C_T \|E\|_{L^\infty(L^\infty(\Omega_x))}, \\
(3.41c) \quad C_3 & := \frac{3}{2\theta} \|E\|_{L^\infty(L^\infty(\Omega_x))},
\end{align}
are constants independent of \( \varepsilon, h_x, \) and \( h_v; \) and \( C_T > 0 \) is a constant from the trace inequality (3.2).

Proof. For (3.37), the definition of \( \rho^*_h, \) along with Assumption 2.2.a, implies that
\begin{equation}
(3.42) \quad (M^\frac{1}{2}_h \rho^*_h, M^\frac{1}{2}_h \rho^*_h)_{\Omega_x} = |\rho^*_h|^2 = (M^\frac{1}{2}_h \rho^*_h, g_h)_{\Omega_x}.
\end{equation}
Therefore by expansion we see
\begin{equation}
(3.43) \quad \|g_h - M^\frac{1}{2}_h \rho^*_h\|^2_{L^2(\Omega_x)} = (g_h, g_h - M^\frac{1}{2}_h \rho^*_h)_{\Omega_x}.
\end{equation}
Using (3.43) gives
\begin{align*}
- \mathcal{Q}(g_h, g_h) = (\omega, g_h - M^\frac{1}{2}_h \rho^*_h)_{\Omega_x} &= (\omega, \|g_h - M^\frac{1}{2}_h \rho^*_h\|^2_{L^2(\Omega_x)})_{\Omega_x} \\
&\geq \omega_{\text{min}} \|g_h - M^\frac{1}{2}_h \rho^*_h\|^2_{L^2(\Omega_x)},
\end{align*}
which is (3.37).

For (3.38), by Assumption 2.2.d,
\begin{equation}
(3.44) \quad C(M^\frac{1}{2}_h \rho^*_h, M^\frac{1}{2}_h \rho^*_h) = \frac{1}{2\theta} (E \rho^*_h, \rho^*_h)_{\Omega_x} \cdot (v_h M^\frac{1}{2}_h, M^\frac{1}{2}_h)_{\Omega_x}
\end{equation}
by (3.44),
\begin{align}
(3.45) \quad C(g_h - M^\frac{1}{2}_h \rho^*_h, g_h - M^\frac{1}{2}_h \rho^*_h) &= C(g_h, g_h) - 2C(g_h, M^\frac{1}{2}_h \rho^*_h) + C(M^\frac{1}{2}_h \rho^*_h, M^\frac{1}{2}_h \rho^*_h) \\
&= C(g_h, g_h) - 2C(g_h, M^\frac{1}{2}_h \rho^*_h) + 2C(M^\frac{1}{2}_h \rho^*_h, M^\frac{1}{2}_h \rho^*_h) \\
&= C(g_h, g_h) - 2C(g_h, M^\frac{1}{2}_h \rho^*_h, M^\frac{1}{2}_h \rho^*_h).
\end{align}
Rewriting (3.45) yields
\begin{equation}
(3.46) \quad C(g_h, g_h) = C(g_h - M^\frac{1}{2}_h \rho^*_h, g_h - M^\frac{1}{2}_h \rho^*_h) + 2C(g_h - M^\frac{1}{2}_h \rho^*_h, M^\frac{1}{2}_h \rho^*_h) \\
:= I_1 + I_2.
\end{equation}
Applying Hölder’s inequality to (3.46) gives
\begin{equation}
(3.47) \quad |I_1| \leq \frac{\|E \cdot v_h\|_{L^\infty(\Omega)}}{2\theta} \|g_h - M^\frac{1}{2}_h \rho^*_h\|^2_{L^2(\Omega)}
\end{equation}
Using Hölder’s inequality, (2.22), and Assumption 2.2.c, we bound \( I_2 \) as
\begin{align}
|I_2| &\leq \frac{1}{\theta} \|E \cdot v_h M^\frac{1}{2}_h \rho^*_h\|_{L^2(\Omega)} \|g_h - M^\frac{1}{2}_h \rho^*_h\|_{L^2(\Omega)} \\
&\leq 2 \|E\|_{L^\infty(\Omega_x)} \|\rho^*_h\|_{L^2(\Omega_x)} \|v_h M^\frac{1}{2}_h\|_{L^2(\Omega_x)} \|g_h - M^\frac{1}{2}_h \rho^*_h\|_{L^2(\Omega)} \\
&\leq 2 \|E\|_{L^\infty(\Omega_x)} \|\rho^*_h\|_{L^2(\Omega_x)} \|\nabla v M^\frac{1}{2}_h\|_{L^2(\Omega_x)} \|g_h - M^\frac{1}{2}_h \rho^*_h\|_{L^2(\Omega)} \\
&\leq \frac{3}{2\theta} \|E\|_{L^\infty(\Omega_x)} \|\rho^*_h\|_{L^2(\Omega_x)} \|g_h - M^\frac{1}{2}_h \rho^*_h\|_{L^2(\Omega)}.
\end{align}
Meanwhile, integrating (3.42) over \( \Omega_x \) and applying the Cauchy-Schwarz inequality gives

\[
\| \rho_h \|_{L^2(\Omega_x)} = \| M_h^\frac{1}{2} \rho_h \|_{L^2(\Omega)} \leq \| g_h \|_{L^2(\Omega)}.
\]

Substituting (3.47)-(3.49) into (3.46) and invoking Young’s inequality gives

\[
C(g_h, g_h) \leq C_1 \| g_h - M_h^\frac{1}{2} \rho_h \|_{L^2(\Omega)}^2 + C_3 \| g_h \|_{L^2(\Omega)} \| g_h - M_h^\frac{1}{2} \rho_h \|_{L^2(\Omega)}
\]

\[
\leq \left( C_1 + \frac{\omega_{\text{min}}}{2\varepsilon} \right) \| g_h - M_h^\frac{1}{2} \rho_h \|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2\omega_{\text{min}}} C_3^2 \| g_h \|_{L^2(\Omega)}^2.
\]

which yields (3.38).

For (3.39), it follows from Lemma 3.3 (setting \( q_h = g_h \) and \( \tau_h = v_h g_h \)) that

\[
(v_h g_h, \nabla_{x} g_h)_\Omega = \left( v_h \| g_h \|, [g_h] \right)_{E_{x,h} \times \Omega_v} + \frac{1}{2} \left( v_h g_h, n_x g_h \right)_{\partial \Omega_x \times \Omega_v}.
\]

Direct substitution of this formula into the definition of \( A \) gives

\[
A(g_h, g_h) = \varepsilon a \left( \left| \frac{v_h \cdot n_x}{2} \right| g_h^2, g_h^2 \right)_{E_{x,h} \times \Omega_v} + \left( \left| \frac{v_h \cdot n_x}{2} \right| g_h, g_h \right)_{\partial \Omega_x \times \Omega_v},
\]

which is (3.39).

For (3.40), a formula similar to (3.51) gives

\[
B(g_h, g_h) = \left( \left| \frac{E \cdot n_v}{2} \right| g_h^2, g_h^2 \right)_{\Omega_v \times E_{x,h}} - \frac{1}{2} \left( E g_h, n_v g_h \right)_{\Omega_v \times \partial \Omega_v}.
\]

Meanwhile, invoking the divergence theorem and Assumption 2.2.d yields

\[
\left( E \cdot n_v, (M_h^\frac{1}{2})^2 \right)_{\partial \Omega_v} = 2E \cdot (\nabla_v M_h^\frac{1}{2}, M_h^\frac{1}{2})_{\Omega_v} = 0.
\]

Therefore applying the polarization identity \( ab = \frac{1}{2}(a^2 + b^2 - (a - b)^2) \) with \( a := g_h \)
and \( b := M_h^\frac{1}{2} \rho_h \) and (3.54) gives the following bound on \( D \):

\[
D(g_h, g_h) = \left( E \cdot n_v, M_h^\frac{1}{2} \rho_h g_h \right)_{\Omega_v \times \partial \Omega_v}
\]

\[
= \frac{1}{2} \left( E \cdot n_v, (M_h^\frac{1}{2} \rho_h)^2 \right)_{\Omega_v \times \partial \Omega_v}
\]

\[
= \frac{1}{2} \left( E \cdot n_v, (g_h - M_h^\frac{1}{2} \rho_h)^2 \right)_{\Omega_v \times \partial \Omega_v}
\]

\[
\geq \frac{1}{2} \left( E g_h, n_v g_h \right)_{\Omega_v \times \partial \Omega_v} - \frac{\| E \|_{L^\infty(\Omega_v)}}{2h_v} \| g_h - M_h^\frac{1}{2} \rho_h \|_{L^2(\Omega_v \times \partial \Omega_v)}.
\]

Applying the discrete trace estimate (3.3) to (3.55) gives

\[
D(g_h, g_h) \geq \frac{1}{2} \left( E g_h, n_v g_h \right)_{\Omega_v \times \partial \Omega_v} - \frac{C_T \| E \|_{L^\infty(\Omega_v)}}{2h_v} \| g_h - M_h^\frac{1}{2} \rho_h \|_{L^2(\Omega_v)}.
\]
where the constant $C_T > 0$ is independent of $\varepsilon, h_x,$ and $h_v$. Adding (3.53) and (3.56) yields

$$
B(g_h^\varepsilon, g_h^\varepsilon) + D(g_h^\varepsilon, g_h^\varepsilon) \geq \left\langle \frac{|E \cdot n_v|}{2} [g_h^\varepsilon], [g_h^\varepsilon] \right\rangle_{\Omega_x \times E_{x,h}^t} \frac{C_T \| |E| \|_{L^\infty(\Omega_x)}}{2h_v} \|g_h^\varepsilon - M_h^\varepsilon \rho_h^\varepsilon\|_{L^2(\Omega)}^2
$$

(3.57)

which is (3.40). The proof is complete. □

Using Lemma 3.7, we derive a space-time energy estimate for $g_h^\varepsilon$ and $M_h^\varepsilon \rho_h^\varepsilon - g_h^\varepsilon$.

Define

$$
\varepsilon_{h_v} := \frac{\omega \min h_v}{4C_1 h_v + 2C_2}.
$$

LEMMA 3.8. Given $h_v > 0$, $h_x > 0$, and $\varepsilon \leq \varepsilon_{h_v}$,

$$
\|g_h^\varepsilon(T)\|_{L^2(\Omega)}^2 + \frac{\omega \min}{2\varepsilon^2} \|g_h^\varepsilon - M_h^\varepsilon \rho_h^\varepsilon\|_{L^2(\Omega)}^2 \leq \|g_{0,h}^\varepsilon\|_{L^2(\Omega)}^2 \exp \left( \frac{C_3^2}{\omega \min} T \right).
$$

Proof. Letting $z_h = \frac{2}{\varepsilon} g_h^\varepsilon$ in (2.24a) yields the energy equation

$$
\frac{d}{dt} \|g_h^\varepsilon\|_{L^2(\Omega)}^2 + \frac{2}{\varepsilon} A(g_h^\varepsilon, g_h^\varepsilon) + 2 \varepsilon B(g_h^\varepsilon, g_h^\varepsilon) + 2 \varepsilon D(g_h^\varepsilon, g_h^\varepsilon) - \frac{2}{\varepsilon^2} Q(g_h^\varepsilon, g_h^\varepsilon) = 2 \varepsilon C(g_h^\varepsilon, g_h^\varepsilon)
$$

Substituting the estimates in Lemma 3.7 yields

$$
\frac{d}{dt} \|g_h^\varepsilon\|_{L^2(\Omega)}^2 + \frac{\omega \min}{\varepsilon^2} \|g_h^\varepsilon - M_h^\varepsilon \rho_h^\varepsilon\|_{L^2(\Omega)}^2 \leq \frac{1}{\omega \min} C_3^2 \|g_h^\varepsilon\|_{L^2(\Omega)}^2 + \left( \frac{2C_3^2}{\varepsilon^2} + \frac{\omega \min}{\varepsilon^2} \right) \|g_h^\varepsilon - M_h^\varepsilon \rho_h^\varepsilon\|_{L^2(\Omega)}^2
$$

(3.61)

Dropping the positive contribution $\frac{1}{\varepsilon} \langle |E \cdot n_v| [g_h^\varepsilon], [g_h^\varepsilon] \rangle_{\Omega_x \times E_{x,h}^t}$ and collecting like terms gives

$$
\frac{d}{dt} \|g_h^\varepsilon\|_{L^2(\Omega)}^2 + \frac{\omega \min}{\varepsilon^2} \|g_h^\varepsilon - M_h^\varepsilon \rho_h^\varepsilon\|_{L^2(\Omega)}^2 \leq \frac{C_3^2}{\omega \min} \|g_h^\varepsilon\|_{L^2(\Omega)}^2.
$$

(3.62)
Since \( \varepsilon_{h_v} = \frac{\omega_{\min h_v}}{4C_{11} h_v + 2C_2} \) (see (3.58)), it follows that for any \( \varepsilon \leq \varepsilon_{h_v} \),
\[
\omega_{\min} - 2\varepsilon C_1 - \frac{C_2}{h_v} \geq \frac{\omega_{\min}}{2}.
\]

Therefore
\[
\frac{d}{dt} \|g_h^\epsilon\|_{L^2(\Omega)}^2 + \varepsilon^{-1} \|v_h \cdot n_x \|_\Omega + \frac{1}{\varepsilon} \|v_h \cdot n_x \|_\Omega + \frac{\omega_{\min}}{2 \varepsilon^2} \|M_h^\frac{1}{2} \rho_h^\epsilon - g_h^\epsilon\|_{L^2(\Omega)}^2 \leq \frac{C_3^2}{\omega_{\min}} \|g_h^\epsilon\|_{L^2(\Omega)}^2.
\]

Applying Grönwall’s inequality to (3.64) yields (3.59). The proof is complete. \( \Box \)

With Lemma 3.8 in hand, we can obtain stability estimates for \( \rho_h^\epsilon \) and \( J_h^\epsilon \) as well as some projection estimates which will be useful in the next section. We first list a technical lemma whose proof is provided in the appendix.

**Lemma 3.9.** Let
\[
\gamma_I(x) := \left( \frac{|v_h \cdot n_x(x)|}{2} M_h^\frac{1}{2}, M_h^\frac{1}{2} \right)_{\Omega_x} \quad \text{and} \quad \gamma_B(x) := \left( v_h M_h^\frac{3}{2}, M_h^\frac{3}{2} \right)_{\{v_h \cdot n_x > 0\}}
\]
for \( x \in \mathcal{E}_x^I \) and \( x \in \partial \Omega \) respectively. Then there exists \( \gamma_I = \gamma_I(h_v) > 0 \) such that \( \gamma_I > \gamma_B \) on \( \mathcal{E}_x^I \) and \( \gamma_B \cdot n > \gamma_B \) on \( \partial \Omega_x \) for all \( h_v \) and \( \varepsilon \).

**Lemma 3.10.** Recall the definitions of \( \rho_h^\epsilon \) and \( J_h^\epsilon \) from (2.26) and (2.27), respectively. For all \( h_v > 0 \) and every \( \varepsilon \leq \varepsilon_{h_v} \), where \( \varepsilon_{h_v} \) is defined in (3.58), the following space-time stability estimates hold:
\[
\|\rho_h^\epsilon\|_{L^\infty_t(L^2(\Omega_x^I))} \leq \|\rho_h^\epsilon\|_{L^\infty_t(L^2(\Omega))} \lesssim \|g_0,h\|_{L^2(\Omega)},
\]
\[
\frac{1}{\varepsilon} \|M_h^\frac{1}{2} \rho_h^\epsilon - g_h^\epsilon\|_{L^2_t(L^2(\Omega))} \lesssim \|g_0,h\|_{L^2(\Omega)},
\]
\[
\|J_h^\epsilon\|_{L^\infty_t(L^2(\Omega_x^I))} \lesssim \|g_0,h\|_{L^2(\Omega)}.
\]

Moreover,
\[
\frac{\sqrt{\gamma}}{\sqrt{1 - \beta}} \|\rho_h^\epsilon\|_{L^2_t(L^2(\mathcal{E}_x^I))} + \frac{\sqrt{\gamma}}{\sqrt{\varepsilon}} \|\rho_h^\epsilon\|_{L^2_t(L^2(\Omega_x^I))} \lesssim \left( \frac{\varepsilon}{h_v} + 1 \right) \|g_0,h\|_{L^2(\Omega)}
\]
\[
\sqrt{\frac{h_v}{\varepsilon}} \|\rho_h^\epsilon - S_h^\beta \rho_h^\epsilon\|_{L^2_t(L^2(\Omega_x^I))} + \sqrt{\frac{h_v}{\varepsilon}} \|\rho_h^\epsilon - S_h^\beta \rho_h^\epsilon\|_{L^2_t(L^2(\Omega_x^I))} \lesssim \left( \frac{\varepsilon}{h_v} + 1 \right) \|g_0,h\|_{L^2(\Omega)}
\]
where \( \gamma \) is defined in Lemma 3.9.

**Proof.** Estimates (3.66) and (3.67) follow from (3.59). For (3.68), the definition of \( J_h^\epsilon \) and Assumption 2.2.d give
\[
J_h^\epsilon = \frac{1}{\varepsilon} \left( v_h M_h^\frac{3}{2}, g_h^\epsilon \right)_{\Omega_x} = \frac{1}{\varepsilon} \left( v_h M_h^\frac{3}{2}, M_h^\frac{1}{2} \rho_h^\epsilon \right)_{\Omega_x} + \frac{1}{\varepsilon} \left( v_h M_h^\frac{3}{2}, M_h^\frac{1}{2} \right)_{\Omega_x} \rho_h^\epsilon
\]
\[
= -\frac{2\theta}{\varepsilon} \left( \nabla v, M_h^\frac{3}{2}, g_h^\epsilon - M_h^\frac{1}{2} \rho_h^\epsilon \right)_{\Omega_x}.
\]
Together (3.67) and (3.71) imply that

\[(3.72) \quad \| J_h^\varepsilon \|_{L^2_h(L^2(\Omega_x))} \leq \frac{2\theta}{\varepsilon} \| M_h^\varepsilon \|_{H^1(\Omega_x)} \| M_h^\varepsilon \rho_h^\varepsilon - g_h^\varepsilon \|_{L^2_h(L^2(\Omega_x))} \lesssim |g_{0,h}| L^2(\Omega_x). \]

We now focus on (3.69). We will only prove the bound on the first term of (3.69) as the bound on the second term is similar. Using the definition of \(\gamma_I\); adding and subtracting \(g_h^\varepsilon\); and using the trace inequality (3.1), we obtain

\[(3.73) \quad \sqrt{\gamma_I} \| \rho_h^\varepsilon \|_{L^2(\varepsilon_x \times h^\varepsilon)} \leq \| \sqrt{\gamma_I} \rho_h^\varepsilon \|_{L^2(\varepsilon_x \times h^\varepsilon)} = \left\| \sqrt{\frac{|v_h|}{2}} [M_h^\varepsilon \rho_h^\varepsilon - g_h^\varepsilon] \right\|_{L^2(\varepsilon_x \times h^\varepsilon, \Omega_x)} + \left\| \sqrt{\frac{|v_h|}{2}} [g_h^\varepsilon] \right\|_{L^2(\varepsilon_x \times h^\varepsilon, \Omega_x)} \lesssim \frac{1}{\sqrt{h_x}} \| M_h^\varepsilon \rho_h^\varepsilon - g_h^\varepsilon \|_{L^2(\Omega_x)} + \left( \sqrt{\frac{|v_h|}{2}} [g_h^\varepsilon] \right)_{L^2(\varepsilon_x \times h^\varepsilon, \Omega_x)} . \]

Integrating (3.73) from 0 to \(T\) and using both (3.59) and (3.67) yields

\[(3.74) \quad \sqrt{\gamma_I} \| \rho_h^\varepsilon \|_{L^2_h(L^2(\varepsilon_x \times h^\varepsilon))} \lesssim \left( \frac{\varepsilon}{\sqrt{h_x}} + \sqrt{\varepsilon^{1-\beta}} \right) |g_{0,h}| L^2(\Omega_x). \]

If \(\beta = 1\), the proof is complete by noting that \(\varepsilon \lesssim \sqrt{\varepsilon}\). If \(\beta = 0\), then we can divide (3.74) by \(\sqrt{\varepsilon}\) and arrive at (3.69).

For (3.70), we first focus on \(\beta = 0\). Recall \(I_h^\varepsilon\) from Lemma 3.4. From (3.6a) and (3.69) we have

\[(3.75) \quad \sqrt{\frac{h_x}{\varepsilon}} \| (\rho_h^\varepsilon - I_h^\varepsilon \rho_h^\varepsilon) \|_{L^2_h(L^2(\varepsilon_x \times h^\varepsilon))} \lesssim \frac{1}{\sqrt{\varepsilon}} \| \rho_h^\varepsilon \|_{L^2_h(L^2(\varepsilon_x \times h^\varepsilon))} + \frac{1}{\sqrt{\varepsilon}} \| \rho_h^\varepsilon \|_{L^2_h(L^2(\varepsilon_x \times h^\varepsilon))} \lesssim \left( \frac{\varepsilon}{\sqrt{h_x}} + 1 \right) |g_{0,h}| L^2(\Omega_x). \]

We now extend (3.75) to \(\rho_h^\varepsilon - S_h^\varepsilon \rho_h^\varepsilon\) using the stability of \(S_h^\varepsilon\). By Lemma 3.5 and the triangle inequality we have

\[(3.76) \quad \| \rho_h^\varepsilon - S_h^\varepsilon \rho_h^\varepsilon \|_{H^1(\Omega_x)} \leq \| \rho_h^\varepsilon - I_h^\varepsilon \rho_h^\varepsilon \|_{H^1(\Omega_x)} + \| I_h^\varepsilon \rho_h^\varepsilon - S_h^\varepsilon \rho_h^\varepsilon \|_{H^1(\Omega_x)} \]

\[\leq \| \rho_h^\varepsilon - I_h^\varepsilon \rho_h^\varepsilon \|_{H^1(\Omega_x)} + \| S_h^\varepsilon I_h^\varepsilon \rho_h^\varepsilon - S_h^\varepsilon \rho_h^\varepsilon \|_{H^1(\Omega_x)} \]

\[\leq \| \rho_h^\varepsilon - I_h^\varepsilon \rho_h^\varepsilon \|_{H^1(\Omega_x)} + \| S_h^\varepsilon (I_h^\varepsilon \rho_h^\varepsilon - \rho_h^\varepsilon) \|_{H^1(\Omega_x)} \]

\[\lesssim \| \rho_h^\varepsilon - I_h^\varepsilon \rho_h^\varepsilon \|_{H^1(\Omega_x)}. \]

Applying (3.75) to (3.76) yields

\[(3.77) \quad \sqrt{\frac{h_x}{\varepsilon}} \| (\rho_h^\varepsilon - S_h^\varepsilon \rho_h^\varepsilon) \|_{L^2_h(H^1(\Omega_x))} \lesssim \left( \frac{\varepsilon}{\sqrt{h_x}} + 1 \right) |g_{0,h}| L^2(\Omega_x). \]

We can extend (3.77) to the \(L^2_h(L^2(\Omega_x))\) norm by (2.16) and arrive at (3.70). The case \(\beta = 1\) is similar. The proof is complete.
3.4. Time Derivative Estimates. In this subsection, we construct temporal estimates for $\partial_t \rho_h^e$ and $\partial_t J_h^e$ by determining the evolution equations for $\rho_h^e$ and $J_h^e$. The evolution equations (see Lemma 3.11 and Lemma 3.13) are formed by choosing a particular type of test function in Problem 2.6. By adding and subtracting the discrete weighted equilibrium $M_h^4 \rho_h^e$, we can write the evolution equations (3.78) and (3.99) into the terms that will build the discretization of (2.7) and the remainder terms $\Theta_i$, where $\Theta_i$ is uniformly bounded in $\varepsilon$ when integrated over time.

We begin with the evolution equation for $\rho_h^e$:

**Lemma 3.11.** For any $\varepsilon > 0$, $\rho_h^e$ and $J_h^e$ satisfy

$$
\begin{align*}
\left( \frac{\partial}{\partial t} \rho_h^e, q_h \right)_{\Omega_x} &- (J_h^e, \nabla_x q_h)_{\Omega_x} + \left( \|J_h^e\|, [q_h] \right)_{E_{x,h}} + \varepsilon^{\beta-1} \left( \|\gamma I\|, \|q_h\| \right)_{E_{x,h}} \\
&+ \frac{1}{\varepsilon} \left( \gamma B \rho_h^e, n_x q_h \right)_{\partial \Omega_x} = \varepsilon^\beta \Theta_1 (\tilde{g}_h^e, q_h) + \Theta_2 (\tilde{g}_h^e, q_h),
\end{align*}
$$

(3.78)

for all $q_h \in V_{x,h}$, where $\tilde{g}_h^e = g_h - M_h^4 \rho_h^e$ and

$$
\begin{align*}
\Theta_1 (\tilde{g}_h^e, q_h) &= -\frac{1}{\varepsilon} \left( \frac{|v_h \cdot n_x|}{2} \|g_h - M_h^4 \rho_h^e\|, [M_h^4 q_h] \right)_{E_{x,h} \times \Omega_x}, \\
\Theta_2 (\tilde{g}_h^e, q_h) &= -\frac{1}{\varepsilon} \left( v_h (g_h - M_h^4 \rho_h^e), n_x M_h^4 q_h \right)_{\partial \Omega_x}.
\end{align*}
$$

(3.79) (3.80)

Additionally, we have the following bounds:

$$
\begin{align*}
|\Theta_1 (\tilde{g}_h^e, q_h)| &\leq \frac{1}{\varepsilon \sqrt{h_x}} \|g_h - M_h^4 \rho_h^e\|_{L^2(\Omega)} \|q_h\|_{L^2(E_{x,h})}, \\
|\Theta_2 (\tilde{g}_h^e, q_h)| &\leq \frac{1}{\varepsilon \sqrt{h_x}} \|g_h - M_h^4 \rho_h^e\|_{L^2(\Omega)} \|q_h\|_{L^2(\partial \Omega_x)}.
\end{align*}
$$

(3.81) (3.82)

**Proof.** To show (3.78), we let $q_h \in V_{x,h}$ and choose $z_h = M_h^4 q_h \in V_h$ into (2.24a) and evaluate term by term. First, the time derivative term reduces to

$$
\begin{align*}
\varepsilon \left( \frac{\partial}{\partial t} \rho_h^e, M_h^4 q_h \right)_{\Omega_x} &\varepsilon \left( \frac{\partial}{\partial t} \rho_h^e, M_h^4 q_h \right)_{\Omega_x} = \varepsilon \left( \frac{\partial}{\partial t} \rho_h^e, q_h \right)_{\Omega_x}.
\end{align*}
$$

(3.83)

Next, using the definition of $J_h^e$ in (2.27), we compute

$$
\begin{align*}
\mathcal{A}(g_h^e, M_h^4 q_h) &= -\left( (v_h g_h^e, M_h^4 q_h)_{\Omega_x}, \nabla_x q_h \right) + \left( \|v_h g_h^e, M_h^4 q_h\|, [q_h] \right)_{E_{x,h}} \\
&+ \varepsilon^\beta \left( \frac{1}{2}, v_h \cdot n_x g_h^e, [M_h^4 q_h] \right)_{E_{x,h} \times \Omega_x} + \left( v_h g_h^e, n_x M_h^4 q_h \right)_{\partial \Omega_x} \\
&= -\varepsilon(J_h^e, \nabla q_h)_{\Omega_x} + \varepsilon \left( \|J_h^e\|, [q_h] \right)_{E_{x,h}} \\
&+ \varepsilon^\beta \left( \frac{1}{2}, v_h \cdot n_x g_h^e, [M_h^4 q_h] \right)_{E_{x,h} \times \Omega_x} + \left( v_h g_h^e, n_x M_h^4 q_h \right)_{\partial \Omega_x}.
\end{align*}
$$

(3.84)

Adding and subtracting $M_h^4 \rho_h^e$ from the last two terms of (3.84) and using the defi-
nitions of $\Theta_1$ and $\Theta_2$ gives

\begin{equation}
(3.85)
A(g_h^x, M_h^2 q_h) = -\varepsilon (J_h^x, \nabla q_h)_{\Omega_x} + \varepsilon \left( \left( \int J_h^x \right), [q_h] \right)_{\mathcal{E}_h^{l,h}} + \varepsilon \left( \left( \int J_h^x \right), [q_h] \right)_{\mathcal{E}_h^{l,h} \times \Omega_v} + \varepsilon \left( M_h^2 \rho_h^x, n_x M_h^2 q_h \right)_{\partial \Omega_x} + \varepsilon \left( M_h^2 \rho_h^x, n_x M_h^2 q_h \right)_{\partial \Omega_x} = -\varepsilon (J_h^x, \nabla q_h)_{\Omega_x} + \varepsilon \left( \left( \int J_h^x \right), [q_h] \right)_{\mathcal{E}_h^{l,h}} + \varepsilon \left( \gamma \left( \int [\rho_h^x], q_h \right) \right)_{\mathcal{E}_h^{l,h}} + \varepsilon \left( \gamma B \rho_h^x, n_x q_h \right)_{\partial \Omega_x} \nonumber
\end{equation}

\begin{equation}
-\varepsilon \varepsilon^{\beta+1} \varepsilon (\tilde{g}_h^x, q_h) - \varepsilon \Theta_2 (\tilde{g}_h^x, q_h).
\end{equation}

After division by $\varepsilon$, (3.83) and (3.85) recover (3.78). Thus it remains to show that

\begin{equation}
(3.86) \quad B(g_h^x, M_h^2 q_h) + D(g_h^x, M_h^2 q_h) + Q(g_h^x, M_h^2 q_h) = C(g_h^x, M_h^2 q_h).
\end{equation}

For $\mathcal{B}$, any edge integral in (2.25b) vanishes because $M_h^2 q_h$ is continuous in $v$. Thus by the definition of the discrete velocity $v_h$ in (2.22),

\begin{equation}
(3.87) \quad B(g_h^x, M_h^2 q_h) = -(E g_h^x, q_h \nabla_v M_h^2)_{\Omega_x} + \frac{1}{2\theta} (E \cdot v_h g_h^x, M_h^2 q_h)_{\Omega_x} = C(g_h^x, M_h^2 q_h).
\end{equation}

For $\mathcal{D}$, Assumption 2.2.b implies that $D(g_h^x, M_h^2 q_h) = 0$. For $Q$, because $M_h^2 q_h$ is isotropic, $Q(g_h^x, M_h^2 q_h) = 0$ as well. Thus (3.86) holds and consequently so does (3.78).

We now prove the bounds on $\Theta_1$ and $\Theta_2$. For $\Theta_1$, Hölder’s inequality and the trace inequality (3.1) give

\begin{equation}
(3.88) \quad |\Theta_1 (\tilde{g}_h^x, q_h) | \leq \frac{1}{\varepsilon} \| v_h M_h^2 \|_{L^\infty(\Omega_x)} \| g_h^x - M_h^2 \rho_h^x \|_{L^2(\mathcal{E}_h^{l,h})} \| q_h \|_{L^2(\mathcal{E}_h^{l,h})} \leq \frac{1}{\varepsilon \sqrt{h_x}} \| v_h M_h^2 \|_{L^\infty(\Omega_x)} \| g_h^x - M_h^2 \rho_h^x \|_{L^2(\Omega_x)} \| q_h \|_{L^2(\mathcal{E}_h^{l,h})}
\end{equation}

which is (3.81). A similar argument for $\Theta_2$ recovers (3.82). The proof is complete.

** Lemma 3.12.** For any $\varepsilon \leq \varepsilon_{h_x}$, where $\varepsilon_{h_x}$ is defined in (3.58), and all $h_x > 0$,

\begin{equation}
(3.89) \quad \| \partial_1 \rho_h^x \|_{L^2 (H^{-1}_h(\Omega_x))} \lesssim \| g_{0,h} \|_{L^2(\Omega_x)},
\end{equation}

\begin{equation}
(3.90) \quad \| \partial_1 S_{h}^\beta \rho_h^x \|_{L^2 (L^2(\Omega_x))} \lesssim \frac{1}{h_x} \| g_{0,h} \|_{L^2(\Omega_x)},
\end{equation}

where the $H^{-1}_h(\Omega_x)$ norm is defined in (2.14).

**Proof.** We first focus on (3.89) and $\beta = 1$. Let $q_h \in V_{x,h}^0$ in (3.78), with $q_h \neq 0$. For such $q_h$, $\Theta_2 (\tilde{g}_h^x, q_h) = 0$; therefore

\begin{equation}
(3.91) \quad (\partial_1 \rho_h^x, q_h) = (J_h^x, \nabla q_h)_{\Omega_x} - \left( \left( \int J_h^x \right), [q_h] \right)_{\mathcal{E}_h^{l,h}} - \left( \gamma \int (\rho_h^x, q_h) \right)_{\mathcal{E}_h^{l,h}} + \Theta_1 (\tilde{g}_h^x, q_h).
\end{equation}
We now bound the remaining terms on the right-hand side of (3.91). For the first term, the Cauchy-Schwarz inequality implies that,
\[
(J_h^\varepsilon, \nabla_x q_h)_{\Omega_x} \lesssim \| \nabla_x q_h \|_{L^2(\Omega_x)} \| J_h^\varepsilon \|_{L^2(\Omega_x)}.
\]
For the second and third terms, the Cauchy-Schwarz inequality and the trace inequality (3.2) imply that
\[
\langle \| J_h^\varepsilon \|, [q_h] \rangle_{E^I_{x,h}} \lesssim \frac{1}{\sqrt{h_x}} \| [q_h] \|_{L^2(E^I_{x,h})} \| J_h^\varepsilon \|_{L^2(\Omega)}
\]
and
\[
\langle \gamma_l [\rho_h^\varepsilon], [q_h] \rangle_{E^I_{x,h}} \lesssim \frac{1}{\sqrt{h_x}} \| [q_h] \|_{L^2(E^I_{x,h})} \| \rho_h^\varepsilon \|_{L^2(\Omega)}.
\]
Substituting (3.92) through (3.94) and the bound on $\Theta_1$ in (3.88) into (3.91) gives
\[
(\partial_t \rho_h^\varepsilon, q_h)_{\Omega_x} \lesssim \left( \| J_h^\varepsilon \|_{L^2(\Omega_x)} + \| \rho_h^\varepsilon \|_{L^2(\Omega_x)} + \frac{1}{\varepsilon} \| g_h - M_h^\varepsilon \rho_h^\varepsilon \|_{L^2(\Omega_x)} \right) \| q_h \|_{H^1(\Omega_x)}.
\]
After dividing (3.95) by $\| q_h \|_{H^1(\Omega_x)}$ and taking the supremum over all $q_h \in V_{x,h}$, the result is
\[
\| \partial_t \rho_h^\varepsilon \|_{H^{-1}_0(\Omega_x)} \lesssim \| J_h^\varepsilon \|_{L^2(\Omega_x)} + \| \rho_h^\varepsilon \|_{L^2(\Omega_x)} + \frac{1}{\varepsilon} \| g_h - M_h^\varepsilon \rho_h^\varepsilon \|_{L^2(\Omega_x)}.
\]
Integrating (3.96) over $t \in [0, T]$ and applying (3.66)–(3.68) yields (3.89).

When $\beta = 0$, the bound on (3.89) is simpler to show. Indeed, if $q_h \in S^0_{x,h}$ is continuous, only the first term on the right-hand side of (3.91) remains. This term is bounded using (3.92) so that the argument follows when replacing $\| q_h \|_{H^1(\Omega_x)}$ by $\| \nabla_x q_h \|_{L^2(\Omega_x)}$.

To show (3.90), we first use the trace inequality (3.2) and inverse inequality (3.4) to obtain
\[
\| q_h \|_{H^1(\Omega_x)} \lesssim \frac{1}{h_x} \| q_h \|_{L^2(\Omega_x)}
\]
Choosing $q_h = \partial_t S_h^\beta \rho_h^\varepsilon$ in (3.95) and using (3.97) and the identity
\[
(\partial_t \rho_h^\varepsilon, \partial_t S_h^\beta \rho_h^\varepsilon)_{\Omega_x} = (\partial_t S_h^\beta \rho_h^\varepsilon, \partial_t S_h^\beta \rho_h^\varepsilon)_{\Omega_x}
\]
we see that the desired estimate holds. The proof is complete.

We next turn to the evolution equation for $J_h^\varepsilon$.

**Lemma 3.13.** For any $\varepsilon > 0$, $\rho_h^\varepsilon$ and $J_h^\varepsilon$ satisfy
\[
\varepsilon^2 \left( \frac{\partial}{\partial t} J_h^\varepsilon, \tau_h \right)_{\Omega_x} + (\omega J_h^\varepsilon, \tau_h)_{\Omega_x} + \theta (\nabla_x \rho_h^\varepsilon, \tau_h)_{\Omega_x} - \theta \left( \| J_h^\varepsilon \|, \| \tau_h \| \right)_{E^I_{x,h}}
\]
\[- (E \rho_h^\varepsilon, \tau_h)_{\Omega_x} = \varepsilon \Theta_3 (g_h^\varepsilon, \tau_h) + \varepsilon^{\beta+1} \Theta_4 (\rho_h^\varepsilon, \tau_h) + \sqrt{\varepsilon} \Theta_5 (\rho_h^\varepsilon, \tau_h),
\]
for all $\tau_h \in [\mathcal{V}_{x,h}]^3$. Here $\Theta_3$ is a remainder that includes several terms that depend on $\tilde{g}_h = g_h^\varepsilon - M_h^2 \rho_h^\varepsilon$; it satisfies the bound

$$|\Theta_3(\tilde{g}_h, \tau_h)| \lesssim \frac{1}{\varepsilon h_x} ||g_h^\varepsilon - M_h^2 \rho_h^\varepsilon||_{L^2(\Omega)} ||\tau_h||_{L^2(\Omega_x)}.$$  

(3.100)

The terms

$$\Theta_4(\rho_h^\varepsilon, \tau_h) = -\frac{1}{\sqrt{\varepsilon^{1-\beta} h_x}} \left( v_h \cdot n_x M_h^2 \rho_h^\varepsilon, [M_h^2 \rho_h^\varepsilon, \varepsilon^2 |M_h^2 \rho_h^\varepsilon|] \right)_{\varepsilon^{1-\beta} \varepsilon^{1-\beta} h_x},$$

(3.101)

$$\Theta_5(\rho_h^\varepsilon, \tau_h) = \frac{1}{\sqrt{\varepsilon^2}} \left( v_h \cdot n_x M_h^2 \rho_h^\varepsilon, [M_h^2 \rho_h^\varepsilon, \varepsilon^2 |M_h^2 \rho_h^\varepsilon|] \right)_{\varepsilon^{1-\beta} \varepsilon^{1-\beta} h_x},$$

(3.102)

are also remainder terms satisfying the bounds

$$|\Theta_4(\rho_h^\varepsilon, \tau_h)| \lesssim \frac{1}{\varepsilon^{3/2} h_x} ||\rho_h^\varepsilon||_{L^2(\varepsilon^{1-\beta} \varepsilon^{1-\beta} h_x)} ||\tau_h||_{L^2(\varepsilon^{1-\beta} \varepsilon^{1-\beta} h_x)},$$

(3.103)

$$|\Theta_5(\rho_h^\varepsilon, \tau_h)| \lesssim \frac{1}{\varepsilon^{3/2} h_x} ||\rho_h^\varepsilon||_{L^2(\varepsilon^{1-\beta} \varepsilon^{1-\beta} h_x)} ||\tau_h||_{L^2(\varepsilon^{1-\beta} \varepsilon^{1-\beta} h_x)},$$

(3.104)

Proof. Let $v_h^i = v_h \cdot e_i$, where $e_i$ is the standard unit basis vector in the $i$-th coordinate direction. From the definition of $v_h$ in (2.22), it follows that $v_h^i M_h^2 \in \mathcal{V}_{x,h}$. To derive (3.99), we let $z_h = v_h^i M_h^2 \tau_h \in V_h$ in (2.24a), where $\tau_h \in \mathcal{V}_{x,h}$ is arbitrary, and evaluate the result term by term. For brevity, we will not gather all of the terms of $\Theta_3$ together, but rather identify each piece as we go and show that it satisfies the bound in (3.100).

For the time derivative, the definition of $J_h^\varepsilon$ in (2.27) gives

$$\varepsilon \left( \frac{\partial}{\partial t} g_h^\varepsilon, v_h^i M_h^2 \tau_h \right)_\Omega = \varepsilon \left( \frac{\partial}{\partial t} (g_h^\varepsilon, v_h M_h^2)_{\Omega_x}, \tau_h e_i \right)_\Omega = \varepsilon^2 \left( \frac{\partial}{\partial t} J_h^\varepsilon, \tau_h e_i \right)_\Omega.$$  

(3.105)

To evaluate $A$, we add and subtract $M_h^2 \rho_h^\varepsilon$ from the first argument and write

$$A(g_h^\varepsilon, v_h^i M_h^2 \tau_h) = A(M_h^2 \rho_h^\varepsilon, v_h^i M_h^2 \tau_h) - \varepsilon \left\{ \frac{1}{\varepsilon} A(g_h^\varepsilon - M_h^2 \rho_h^\varepsilon, v_h^i M_h^2 \tau_h) \right\} =: I_1 - \varepsilon \{I_2\}.$$  

(3.106)

The term $I_2$ belongs to $\Theta_3$. Since $I_2$ contains $g_h^\varepsilon - M_h^2 \rho_h^\varepsilon$, following a similar treatment to $\Theta_1$ in Lemma 3.11, we can show $I_2$ satisfies (3.100). For $I_1$, the definition of $A$ in (2.25a) implies that,

$$I_1 = I_3 - \sqrt{\varepsilon^{1+1}} \{I_4\} - \sqrt{\varepsilon \{I_5\}},$$  

(3.107)
where

\begin{align}
I_3 &= -\langle \rho_h^e, \nabla_x \tau_h \rangle_{\Omega_v} \cdot (v_h M_h^{1/2}, v_h M_h^{1/2})_{\Omega_v} + \left( \langle \rho_h^e \rangle_{\Omega_v} \right)_{E_{\tau,h}} \cdot (v_h M_h^{1/2}, v_h M_h^{1/2})_{\Omega_v} \\
&\quad + \langle \rho_h, n_x \tau_h \rangle_{\partial \Omega_v} \cdot (v_h M_h^{1/2}, v_h M_h^{1/2})_{\Omega_v} \\
I_4 &= -\frac{1}{\sqrt{\beta}} \left\langle \nabla_x \cdot \left[ \frac{1}{2} M_h^{1/2} \left[ \rho_h^e \right]_{\Omega_v} \cdot M_h^{1/2} v_h \cdot e_i \tau_h \right] \right\rangle_{E_{\tau,h} \times \Omega_v} \\
I_5 &= \frac{1}{\sqrt{\varepsilon}} \left\langle v_h \cdot n_x M_h^{1/2} \rho_h^e, M_h^{1/2} v_h \cdot e_i \tau_h \right\rangle_{\partial \Omega_v}.
\end{align}

The definitions of $M_h^{1/2}$ and $v_h$ in Definition 2.3, combined with Assumptions 2.2.d and 2.2.c, imply that

\begin{equation}
(v_h M_h^{1/2}, v_h M_h^{1/2})_{\Omega_v} = (-2\theta \nabla_v M_h^{1/2}, -2\partial_v, M_h^{1/2})_{\Omega_v} = \theta e_i.
\end{equation}

Substituting (3.109) into (3.108a) and then applying the discrete integration-by-parts formula from (3.5) gives

\begin{align}
I_3 &= -\theta \langle \rho_h^e e_i, \nabla_x \tau_h \rangle_{\Omega_v} + \theta \left( \left\langle \rho_h^e e_i \right\rangle_{\Omega_v} \right)_{E_{\tau,h}} + \theta \langle \rho_h^e e_i, n_x \tau_h \rangle_{\partial \Omega_v} \\
&= \theta \left( \nabla_x \rho_h^e, \tau_h e_i \right) - \theta \left( \left\langle \rho_h^e \rangle_{\Omega_v} \right\rangle_{E_{\tau,h}} \right)_{\partial \Omega_v}.
\end{align}

Meanwhile $I_4$ is the only component of $\Theta_4$ and can be bounded using the trace inequality (3.2) to obtain

\begin{equation}
|I_4| \lesssim \frac{1}{h \varepsilon^{1/2}} \|v_h\|_{L^\infty(\Omega_v)} \|M_h^{1/2}\|_{L^2(\Omega_v)} \|\rho_h^e\|_{L^2(E_{\tau,h})} \|\tau_h\|_{L^2(\Omega_v)}.
\end{equation}

Likewise, $I_5$ is the only component of $\Theta_5$ and can be bounded in a similar fashion.

To evaluate $B$, we add and subtract $M_h^{1/2} \rho_h^e$ from the first argument and write

\begin{align}
B(g_h^e, v_h M_h^{1/2} \tau_h) &= B(M_h^{1/2} \rho_h^e, v_h M_h^{1/2} \tau_h) - \varepsilon \left\{ \frac{1}{\varepsilon} B(g_h^e - M_h^{1/2} \rho_h^e, v_h M_h^{1/2} \tau_h) \right\} \\
&= I_5 + \varepsilon \{I_6\}.
\end{align}

Here $I_6$ is a remainder term that belongs to $\Theta_3$ and satisfies the bound in (3.100) due to the trace estimate (3.3) and inverse estimate (3.4). Meanwhile, the upwind penalty term in $B(M_h^{1/2} \rho_h^e, v_h M_h^{1/2} \tau_h)$ vanishes because the first argument is continuous in $v$; this leaves

\begin{equation}
I_5 = -\langle E \rho_h^e, \tau_h \rangle_{\Omega_v} \cdot \left( M_h^{1/2}, \nabla_x (v_h M_h^{1/2}) \right)_{\Omega_v} - \left( \nabla_x (v_h M_h^{1/2}), [v_h M_h^{1/2}]_v \right)_{E_{\tau,h}}.
\end{equation}

From the integration-by-parts identity (3.5) (applied to functions in $V_v,h$) and continuity of $M_h^{1/2}$,

\begin{align}
(M_h^{1/2}, \nabla_x (v_h M_h^{1/2}))_{\Omega_v} - \left( M_h^{1/2}, [v_h M_h^{1/2}]_v \right)_{E_{\tau,h}} &= - \frac{1}{\varepsilon} \left( \nabla_x M_h^{1/2}, v_h M_h^{1/2} \right)_{\Omega_v} + \left( M_h^{1/2}, n_x v_h M_h^{1/2} \right)_{\partial \Omega_v}.
\end{align}
The definition of \(v_h\) in (2.22), along with (3.109), implies that for the first term above,
\[
(3.115) \quad - \left( \nabla_x M_h^\frac{1}{2}, v_h^i M_h^\frac{1}{2} \right)_{\Omega_x} = \frac{1}{2\theta} \left( v_h M_h^\frac{1}{2}, v_h^i M_h^\frac{1}{2} \right)_{\Omega_x} = \frac{1}{2} \epsilon_i.
\]
Substituting (3.114) with (3.115) into (3.113) and recalling the definition of \(D\) from (2.25c) gives
\[
(3.116) \quad I_5 = -\frac{1}{2} (E \rho_h^*, \tau_h e_i)_{\Omega_x} + \frac{1}{2} (E \rho_h^*, \tau_h e_i)_{\Omega_x} \left( M_h^\frac{1}{2}, n_x v_h^i M_h^\frac{1}{2} \right)_{\partial \Omega_x} = \frac{1}{2} (E \rho_h^*, \tau_h e_i)_{\Omega_x} - D(g_h^*, v_h^i M_h^\frac{1}{2} \tau_h)
\]
To evaluate \(Q\), we use Assumption 2.2.d:
\[
(3.117) \quad \frac{1}{\epsilon} Q(g_h^*, v_h^i M_h^\frac{1}{2} \tau_h) = -\frac{1}{\epsilon} (\omega(M_h^\frac{1}{2}, \rho_h - g_h^*) v_h^i M_h^\frac{1}{2} \tau_h)_{\Omega_x}
\]
\[
= -\frac{1}{\epsilon} (\omega \rho_h^*, \tau_h)_{\Omega_x} (M_h^\frac{1}{2}, v_h^i M_h^\frac{1}{2})_{\Omega_x} + \left( \frac{\omega}{\epsilon} (g_h^*, v_h M_h^\frac{1}{2})_{\Omega_x}, \tau_h e_i \right)_{\Omega_x}
\]
\[
= (\omega J_5^*, \tau_h e_i)_{\Omega_x}.
\]
Lastly, to evaluate \(C\), we add and subtract \(M_h^\frac{1}{2} \rho_h^*\) from the first argument and write
\[
(3.118) \quad C(g_h^*, v_h^i M_h^\frac{1}{2} \tau_h) = \frac{1}{2\theta} (E \cdot v_h M_h^\frac{1}{2}, \rho_h^* v_h^i M_h^\frac{1}{2} \tau_h)_{\Omega_x} + \epsilon \left( \frac{1}{2\theta \epsilon} (E \cdot v_h (g_h^* - M_h^\frac{1}{2} \rho_h), v_h^i M_h^\frac{1}{2} \tau_h)_{\Omega_x} \right)
\]
\[
= I_7 + \epsilon \{I_8\}.
\]
Here \(I_8\) is a remainder term of \(\Theta_3\) which satisfies the bound in (3.100) and, because of (3.109),
\[
(3.119) \quad I_7 = \frac{1}{2\theta} (E \rho_h^*, \tau_h)_{\Omega_x}, (v_h M_h^\frac{1}{2}, v_h^i M_h^\frac{1}{2} \tau_h)_{\Omega_x} = \frac{1}{2} (E \rho_h^*, \tau_h e_i)_{\Omega_x}.
\]
We have shown (3.99) for all \(\tau_h e_i\) where \(\tau_h \in V_{x,h}\), and thus (3.99) holds for all \(\tau_h \in [v_{x,h}]^3\). The proof is complete.

We now use (3.99) to get a space-time bound on \(\partial_t J_h^\tau\). Recall the definition of \(\varepsilon_{h_x}\) in (3.58).

**Lemma 3.14.** Assume \(\varepsilon \leq \varepsilon_{h_x} \lesssim h_x \leq 1\). Then
\[
(3.120) \quad \| J_h^\tau \|_{L^\infty(L^2(\Omega_x))} \lesssim \frac{1}{h_x} \| g_{0,h} \|_{L^2(\Omega)},
\]
\[
(3.121) \quad \varepsilon^{3/2} \| \partial_t J_h^\tau \|_{L^2(L^2(\Omega_x))} \lesssim \frac{1}{\sqrt{h_x}} \| g_{0,h} \|_{L^2(\Omega)}.
\]
*Proof.* We will first prove (3.120) which is an estimate needed to obtain (3.121). Setting \(\tau_h = J_h^\tau\) in (3.99) gives
\[
(3.122) \quad \frac{\varepsilon^2}{2} \frac{d}{dt} \| J_h^\tau \|_{L^2(\Omega_x)}^2 + \omega_{\mathrm{min}} \| J_h^\tau \|_{L^2(\Omega_x)}^2 \leq -\theta (\nabla_x \rho_h^*, J_h^\tau)_{\Omega_x} + \theta \{ \| \rho_h \|, \| J_h^\tau \| \}_{L^2(\Omega_x)}
\]
\[
+ (E \rho_h^*, J_h^\tau)_{\Omega_x} + \varepsilon \Theta_3(g_h^*, J_h^\tau) + \varepsilon \Theta_4(\rho_h^*, J_h^\tau) + \sqrt{\varepsilon} \Theta_5(\rho_h^*, J_h^\tau).
\]
The first three terms on the right-hand side of (3.122) can be bounded using Cauchy-Schwarz together with the inverse inequality (3.4) for the first, a trace inequality (3.2) for the second, and the $L^\infty$ bound on $E$ for the third. After absorbing $\varepsilon$-independent constants,

\begin{equation}
- \theta(\nabla_x \rho_h^\varepsilon, J_h^\varepsilon)_{\Omega_x} + \theta \left( \| \rho_h^\varepsilon \|, \| J_h^\varepsilon \| \right)_{\mathcal{E}_{\varepsilon,h}} + (E \rho_h^\varepsilon, J_h^\varepsilon)_{\Omega_x} \lesssim \frac{1}{h_x} \| \rho_h^\varepsilon \|_{L^2(\Omega_x)} \| J_h^\varepsilon \|_{L^2(\Omega_x)}
\end{equation}

Meanwhile, the bounds on $\Theta_3$, $\Theta_4$, and $\Theta_5$ from Lemma 3.11 and Lemma 3.13 imply that

\begin{equation}
\varepsilon \Theta_3(\tilde{g}_h, J_h) + \sqrt{\varepsilon^{\beta+1}} \Theta_4(\rho_h, J_h) + \sqrt{\varepsilon} \Theta_5(\rho_h, J_h) \lesssim \frac{1}{h_x} \| \rho_h^\varepsilon \|_{L^2(\Omega_x)}
\end{equation}

\begin{equation}
+ \| g_h - M_h^\varepsilon \rho_h^\varepsilon \|_{L^2(\Omega)} \| J_h^\varepsilon \|_{L^2(\Omega_x)}.
\end{equation}

Substituting these bounds into (3.122) and dividing by $\varepsilon^2 \| J_h^\varepsilon \|_{L^2(\Omega_x)}$ yields

\begin{equation}
\frac{d}{dt} \| J_h^\varepsilon \|_{L^2(\Omega_x)} + \frac{\omega_{\min}}{\varepsilon^2} \| J_h^\varepsilon \|_{L^2(\Omega_x)} \lesssim \frac{1}{\varepsilon^2 h_x} \left( \| \rho_h^\varepsilon \|_{L^2(\Omega_x)} + \| g_h - M_h^\varepsilon \rho_h^\varepsilon \|_{L^2(\Omega)} \right).
\end{equation}

According to (2.29), $J_h^\varepsilon|_{t=0} = 0$. Thus Grönwall’s Lemma applied to (3.125), along with the $L^\infty$ in time bound in (3.66), recovers (3.120).

We now prove (3.121). As the proof below is quite technical, we first briefly summarize the process. The idea is to pass the time derivative from $J_h^\varepsilon$ to $\rho_h^\varepsilon$ for the terms that are not sufficiently small in $\varepsilon$ to bound with the usual techniques. However, since $\partial_t \rho_h^\varepsilon$ is not uniformly bounded w.r.t $\varepsilon$ in $L^2_T(L^2(\Omega_x))$, we will first add and subtract its projection $S_h^\beta \rho_h^\varepsilon$, whose time derivative is uniformly bounded in $\varepsilon$ by Lemma 3.12, before passing the time derivative over. Having an explicit bound on the size of $\rho_h^\varepsilon - S_h^\beta \rho_h^\varepsilon$ by (3.70), we can obtain the $\varepsilon^{3/2}$ scale in (3.121).

Setting $\tau_h = \varepsilon \partial_t J_h^\varepsilon$ in (3.99) gives

\begin{equation}
\varepsilon^3 \| \partial_t J_h^\varepsilon \|_{L^2(\Omega_x)}^2 + \frac{\varepsilon}{2} \frac{d}{dt} \| \sqrt{\omega} J_h^\varepsilon \|_{L^2(\Omega_x)}^2 = -\varepsilon \theta(\nabla_x \rho_h^\varepsilon, \partial_t J_h^\varepsilon)_{\Omega_x} + \varepsilon \theta \left( \| \rho_h^\varepsilon \|, \| \partial_t J_h^\varepsilon \| \right)_{\mathcal{E}_{\varepsilon,h}}
\end{equation}

\begin{equation}
+ (E \rho_h^\varepsilon, \partial_t J_h^\varepsilon)_{\Omega_x} + \varepsilon^2 \Theta_3(\tilde{g}_h, \partial_t J_h^\varepsilon) + \varepsilon^{\beta+1} \Theta_4(\rho_h, \partial_t J_h^\varepsilon) + \varepsilon^{3/2} \Theta_5(\rho_h, \partial_t J_h^\varepsilon).
\end{equation}

We then integrate (3.126) over $t \in [0,T]$, use the zero initial condition in (2.29), and drop the positive term $\| \sqrt{\omega} J_h^\varepsilon \|_{L^2(\Omega_x)}^2|_{t=T}$. This gives

\begin{equation}
\int_0^T \left[ -\varepsilon \theta(\nabla_x \rho_h^\varepsilon, \partial_t J_h^\varepsilon)_{\Omega_x} + \varepsilon \theta \left( \| \rho_h^\varepsilon \|, \| \partial_t J_h^\varepsilon \| \right)_{\mathcal{E}_{\varepsilon,h}}
\end{equation}

\begin{equation}
+ (E \rho_h^\varepsilon, \partial_t J_h^\varepsilon)_{\Omega_x} + \varepsilon^2 \Theta_3(\tilde{g}_h, \partial_t J_h^\varepsilon) + \varepsilon^{(\beta+3)/2} \Theta_4(\rho_h, \partial_t J_h^\varepsilon) + \varepsilon^{3/2} \Theta_5(\rho_h, \partial_t J_h^\varepsilon) \right] dt.
\end{equation}

We now add and subtract $S_h^\beta \rho_h^\varepsilon$ (recall $S_h^\beta$ is an $L^2$ projection) to several terms of
We first bound \( K(3.130) \) inequality (3.2), as well as the bound on \( \partial_t (3.132) \)
\( t \) treatment as for \( \nu > (3.129) \). We will bound
\[ \varepsilon^3 \| \partial_t J_h^s \|_{L^2_x((L^2(\Omega_z))} \leq \int_0^T \left\{ - \varepsilon \theta (\nabla_x (\rho_h^s - S_h^2 \rho_h^s), \partial_t J_h^s)_{\Omega_z} \right. \\
+ \varepsilon \theta \left( [\rho_h^s - S_h^2 \rho_h^s], [\partial_t J_h^s] \right)_{\mathcal{E}_{x,h}} + \varepsilon (E(\rho_h^s - S_h^2 \rho_h^s), \partial_t J_h^s)_{\Omega_z} \right. \\
+ \left\{ - \varepsilon (\nabla_x S_h^2 \rho_h^s, \partial_t J_h^s)_{\Omega_z} + \varepsilon \theta \left( [S_h^2 \rho_h^s], [\partial_t J_h^s] \right)_{\mathcal{E}_{x,h}} \right. \\
+ \left\{ \varepsilon (\partial_t S_h^2 \rho_h^s - S_h^2 \rho_h^s, \partial_t J_h^s)_{\Omega_z} \right. \\
+ \left\{ \nu \varepsilon \theta (\partial_t S_h^2 \rho_h^s, \partial_t J_h^s)_{\Omega_z} + \varepsilon (\partial_t S_h^2 \rho_h^s, \partial_t J_h^s)_{\Omega_z} \right. \\
\leq \int_0^T \left[ \{ I_1 \} + \{ I_2 \} + \{ I_3 \} + \{ I_4 \} \right] dt. \\
\] (3.129)
\[ \int_0^T I_1 dt \lesssim \varepsilon \left( \| \rho_h^s - S_h^2 \rho_h^s \|_{L^2_x((L^2(\Omega_z)))} + \| \rho_h^s - S_h^2 \rho_h^s \|_{L^2((H^1(\Omega_z)))} \right) \| \partial_t J_h^s \|_{L^2_x((L^2(\Omega_z)))} \]
\[ \lesssim \frac{\varepsilon^{3/2}}{h_x} \left( \sqrt{\frac{\varepsilon}{h_x}} + 1 \right) \| g_{0,h} \|_{L^2(\Omega)} \| \partial_t J_h^s \|_{L^2_x((L^2(\Omega_z)))} \]
\[ \lesssim \frac{1}{h_x \nu} \| g_{0,h} \|^2_{L^2(\Omega)} + \nu \varepsilon^3 \| \partial_t J_h^s \|^2_{L^2_x((L^2(\Omega_z)))}. \]
for all \( \nu > 0 \).

For \( I_2 \), we integrate by parts in time to obtain
\[ \int_0^T I_2 dt = \left\{ \varepsilon^2 \theta (\nabla_x \partial_t S_h^2 \rho_h^s, J_h^s)_{\Omega_z} - \theta \left( [\partial_t S_h^2 \rho_h^s], [J_h^s] \right)_{\mathcal{E}_{x,h}} \right\} \\
+ \left\{ \varepsilon \theta \left( [S_h^2 \rho_h^s], [J_h^s] \right)_{\mathcal{E}_{x,h}} \right\} \\
= \{ K_1 \} + \{ K_2 \}. \]
(3.130)
We first bound \( K_1 \). Using Cauchy-Schwarz, the inverse inequality (3.4), and the trace
inequality (3.2), as well as the bound on \( \partial_t S_h^2 \rho_h^s \) in (3.90) and the bound on \( J_h^s \) in
(3.68), we have
\[ K_1 \lesssim \frac{\varepsilon}{h_x} \| \partial_t S_h^2 \rho_h^s \|_{L^2((L^2(\Omega_z)))} \| J_h^s \|_{L^2((L^2(\Omega_z)))} \lesssim \frac{\varepsilon}{h_x^2} \| g_{0,h} \|^2_{L^2(\Omega)}. \]
For \( K_2 \), terms evaluated at \( t = 0 \) vanish due to Assumption 2.10. Following a similar
treatment as for \( K_1 \), but instead using the \( L^2 \) estimates (3.68) and (3.66), we obtain
\[ K_2 \lesssim \frac{\varepsilon}{h_x} \| S_h^2 \rho_h^s \|_{L^\infty((L^2(\Omega_z)))} \| J_h^s \|_{L^\infty((L^2(\Omega_z)))} \lesssim \frac{\varepsilon}{h_x^2} \| g_{0,h} \|^2_{L^2(\Omega)}. \]
For $I_3$, the treatment is similar to that of $I_2$. Integrating by parts in time and applying bounds similar to those used for $K_1$ and $K_2$, we find

\( (3.133) \)

\[
\int_0^T I_3 \, dt = -\varepsilon \int_0^T \left( \partial_t (E S_k^\beta \rho_h^\varepsilon), J_h^\varepsilon \right)_\Omega \, dt + \varepsilon \left( E S_k^\beta \rho_h^\varepsilon, J_h^\varepsilon \right)_\Omega \bigg|_0^T \\
= -\varepsilon \int_0^T \left( S_k^\beta \rho_h^\varepsilon \partial_t E, J_h^\varepsilon \right)_\Omega + \left( E \partial_t S_k^\beta \rho_h^\varepsilon, J_h^\varepsilon \right)_\Omega \bigg|_0^T \\
\lesssim \varepsilon \| \rho_h^\varepsilon \|_{L^2(\Omega)} \| J_h^\varepsilon \|_{L^2(\Omega)} + \varepsilon \| \partial_t S_k^\beta \rho_h^\varepsilon \|_{L^2(\Omega)} \| J_h^\varepsilon \|_{L^2(\Omega)} \\
+ \varepsilon \| \rho_h^\varepsilon \|_{L^\infty(\Omega)} \| J_h^\varepsilon \|_{L^\infty(\Omega)} \\
\lesssim \frac{\varepsilon}{\beta \| \nu \|_{L^2(\Omega)}} \| g_{0,h} \|_{L^2(\Omega)}^2.
\]

We now focus on each term of $I_4$. To bound $\Theta_3$, we use (3.100), (3.67), and Young’s inequality:

\( (3.134) \)

\[
\int_0^T \varepsilon^2 \Theta_3 (g_h^\varepsilon, \partial_t J_h^\varepsilon) \, dt \lesssim \sqrt{\frac{\varepsilon}{h_x}} \left( \frac{1}{\varepsilon} \| g_h^\varepsilon - M_h^\varepsilon \rho_h^\varepsilon \|_{L^2(\Omega)} \right) \left( \varepsilon^{3/2} \| \partial_t J_h^\varepsilon \|_{L^2(\Omega)} \right) \\
\lesssim \frac{\varepsilon}{\nu h_x} \| g_{0,h} \|_{L^2(\Omega)}^2 + \varepsilon \| \partial_t J_h^\varepsilon \|^2_{L^2(\Omega)}
\]

for any $\nu > 0$. We treat $\Theta_4$ in a similar manner. Using (3.103) and (3.69) with Assumption 2.13, we have

\( (3.135) \)

\[
\int_0^T \varepsilon^{(\beta+3)/2} \Theta_4 (\rho_h^\varepsilon, \partial_t J_h^\varepsilon) \, dt \lesssim \frac{\varepsilon^{\beta/2}}{\sqrt{h_x}} \left( \frac{1}{\varepsilon (1-\beta)/2} \| \rho_h^\varepsilon \|_{L^2(\Omega)} \right) \left( \varepsilon^{3/2} \| \partial_t J_h^\varepsilon \|_{L^2(\Omega)} \right) \\
\lesssim \frac{\varepsilon^\beta}{\nu h_x} \left( \frac{1}{\varepsilon (1-\beta)} \| \rho_h^\varepsilon \|^2_{L^2(\Omega)} \right) + \varepsilon \| \partial_t J_h^\varepsilon \|^2_{L^2(\Omega)} \\
\lesssim \frac{\varepsilon^\beta}{\nu h_x} \| g_{0,h} \|_{L^2(\Omega)}^2 + \varepsilon \| \partial_t J_h^\varepsilon \|^2_{L^2(\Omega)}
\]

for any $\nu > 0$. We treat $\Theta_5$ similar to the $\beta = 0$ case of $\Theta_4$; cf. (3.135):

\( (3.136) \)

\[
\int_0^T \varepsilon^{3/2} \Theta_5 (\rho_h^\varepsilon, \partial_t J_h^\varepsilon) \, dt \lesssim \frac{1}{\nu h_x} \| g_{0,h} \|_{L^2(\Omega)}^2 + \varepsilon \| \partial_t J_h^\varepsilon \|^2_{L^2(\Omega)}
\]

for any $\nu > 0$. Combining (3.134)-(3.136) yields the following bound for $I_4$:

\( (3.137) \)

\[
\int_0^T I_4 \, dt \lesssim \frac{1}{\nu} \left( \frac{1}{h_x} + \frac{\varepsilon}{h_x^2} \right) \| g_{0,h} \|_{L^2(\Omega)}^2 + \varepsilon \| \partial_t J_h^\varepsilon \|^2_{L^2(\Omega)}
\]

for any $\nu > 0$. Combining (3.128)-(3.132) and (3.137) we obtain

\( (3.138) \)

\[
\varepsilon^3 \| \partial_t J_h^\varepsilon \|^2_{L^2(\Omega)} \lesssim \frac{1}{\nu} \left( \frac{1}{h_x} + \frac{\varepsilon}{h_x^2} \right) \| g_{0,h} \|_{L^2(\Omega)}^2 + \frac{\varepsilon}{h_x^2} \| g_{0,h} \|_{L^2(\Omega)}^2 + \varepsilon \| \partial_t J_h^\varepsilon \|^2_{L^2(\Omega)}
\]
Choosing \( \nu \), independent of \( \varepsilon \) and \( h_x \), sufficiently small to move \( \varepsilon^3 \| \partial_t J_h^\varepsilon \|_{L^2_\nu(L^2(\Omega_\varepsilon))} \) from the right-hand side of (3.138) and applying Assumption 2.13 to the first two terms on the right-hand side of (3.138) gives us (3.121). The proof is complete.

4. The Drift Diffusion Limit. The bounds in Section 3 allow us to take the limit of \( \rho_h^\varepsilon \) and \( J_h^\varepsilon \) as \( \varepsilon \to 0 \). In this section, we show these limits satisfy (4.2), a discrete version of the drift-diffusion equations (2.7). Recall the definition of \( \varepsilon_{h\nu} \) from (3.58).

**Theorem 4.1.** Let \( h_x, h_\nu > 0 \) be fixed. Then for all \( \varepsilon \leq \varepsilon_{h\nu} \), we have \( \rho_h^\varepsilon \in L^2_\nu(L^2(\Omega_\varepsilon)) \), \( \partial_t \rho_h^\varepsilon \in L^2_\nu(H^{-1}_\nu(\Omega_\varepsilon)) \) and \( J_h^\varepsilon \in L^2_\nu(L^2(\Omega_\varepsilon)) \) with bound

\[
(\partial \rho_h^\varepsilon, q_h)_{\Omega_\varepsilon} + (J_h^\varepsilon, q_h)_{\Omega_\varepsilon} + (\| J_h^\varepsilon \|, [q_h])_{\mathcal{E}_x, h} + (\gamma \| \rho_h^\varepsilon \|, [q_h])_{\mathcal{E}_x, h} = 0, 
\]

\[
(\omega J_h^0, \tau_h)_{\Omega_\varepsilon} + \theta (\nabla x \rho_h^0, \tau_h)_{\Omega_\varepsilon} - \theta (\| \rho_h^0 \|, [\tau_h])_{\mathcal{E}_x, h} - (E \rho_h^0, \tau_h)_{\Omega_\varepsilon} = 0, 
\]

\[
(\rho_h^0(0), q_h)_{\Omega_\varepsilon} = (\rho_{0,h}, q_h)_{\Omega_\varepsilon},
\]

for all \( \tau_h \in [V_{x,h}, 3] \) and \( q_h \in V_{x,h}^0 \) if \( \beta = 1 \) and for all \( \tau_h \in [V_{x,h}, 3] \) and \( q_h \in S_{x,h}^0 \) if \( \beta = 0 \).

**Remark 4.2.** When \( \beta = 0 \) all of the interior edge terms in (4.2) vanish due to the continuity of the \( \rho_h^\varepsilon \) and \( q_h \).

**Proof.** The proof proceeds in two steps.

**Step 1: Existence of Limits.** It follows from (3.66), (3.68), and (3.89) that \( \| \rho_h^\varepsilon \|_{L^2_\nu(L^2(\Omega_\varepsilon))} \) and \( \| J_h^\varepsilon \|_{L^2_\nu(L^2(\Omega_\varepsilon))} \) are uniformly bounded in \( \varepsilon \). Since each of these spaces are Hilbert spaces, we can extract a subsequence \( \rho_h^\varepsilon \) and \( J_h^\varepsilon \), not relabeled, and limiting functions \( \rho_h^0 \in L^2_\nu(V_{x,h}) \) and \( J_h^0 \in L^2_\nu(\Omega_{x,h}) \) such that \( \rho_h^\varepsilon \to \rho_h^0 \) in \( L^2_\nu(L^2(\Omega_\varepsilon)) \) and \( J_h^\varepsilon \to J_h^0 \) in \( L^2_\nu(L^2(\Omega_\varepsilon)) \). We now show \( \rho_h^0 \in L^2_\nu(V_{x,h}) \). By (3.69),

\[
\| \rho_h^0 \|_{L^2_\nu(L^2(\Omega_{x,h}))} \lesssim \| g_{0,h} \|_{L^2_\nu(\Omega_{x,h})}. 
\]

Since \( V_h \) is finite dimensional, then \( \rho_h^\varepsilon \to \rho_h^0 \) in \( L^2_\nu(\partial \Omega_\varepsilon) \). Because the norm is weakly lower semi-continuous, (4.3) implies that

\[
\| \rho_h^0 \|_{L^2_\nu(L^2(\partial \Omega_\varepsilon))} = 0.
\]

Therefore \( \rho_h^0 \in L^2_\nu(V_{x,h}) \). If \( \beta = 0 \), then (3.69) additionally gives us

\[
\| \rho_h^0 \|_{L^2_\nu(L^2(\mathcal{E}_{x,h}))} \lesssim \| g_{0,h} \|_{L^2_\nu(\Omega_{x,h})},
\]

and thus passing the limit as \( \varepsilon \to 0 \) in (4.4) we obtains

\[
\| \rho_h^0 \|_{L^2_\nu(L^2(\mathcal{E}_{x,h}))} = 0.
\]
Hence $\rho^0_h(t)$ is continuous in $x$ for a.e. time $t$ and $\rho^0_h \in L^2_T(S^0_{x,h})$ if $\beta = 0$.

For this paragraph we will consider the case $\beta = 0$ and put the respective $\beta = 1$ result in parentheses. By (3.89), $\partial_t \rho^0_h$ is uniformly bounded in $\varepsilon$ in $L^2_T((S^0_{x,h})^*)$ (resp $L^2_T((V^0_{x,h})^*)$) where $(S^0_{x,h})^*$ (resp $(V^0_{x,h})^*$) is the dual space of $S^0_{x,h}$ (resp $V^0_{x,h}$). Hence there is a subsequence of $\partial_t \rho^0_h$ such that $\partial_t \rho^0_h \rightharpoonup \zeta$ for some $\zeta \in L^2_T((S^0_{x,h})^*)$ (resp $L^2_T((V^0_{x,h})^*)$). Since $S^0_{x,h}$ (resp. $V^0_{x,h}$) is finite-dimensional and thus a Hilbert space with respect to the $L^2$ inner product on $\Omega_x$, we can apply the Riesz representation theorem to show there is $\zeta_h \in L^2_T(S^0_{x,h})$ (resp $L^2_T(V^0_{x,h})$) such that 

$$
\zeta(t; q_h) = (\zeta_h(t), q_h(t))_{\Omega_x}
$$

for all $q_h \in S^0_{x,h}$ (resp $V^0_{x,h}$) and a.e. $t$ where $\zeta(t; \cdot) \in (S^0_{x,h})^*$ (resp $(V^0_{x,h})^*$). By a standard density argument (see [11, Chapter 7, Problem 5]), we have $\zeta_h = \partial_t \rho^0_h$. Therefore $\rho^0_h \in H^1(0,T; S^0_{x,h}) \rightarrow C^0([0,T]; S^0_{x,h})$ (resp $H^1(0,T; V^0_{x,h}) \rightarrow C^0([0,T]; V^0_{x,h})$).

**Step 2: The Limiting System.** We first recover (4.2a). We choose $q_h \in L^2_T(S^0_{x,h})$ if $\beta = 0$ and $q_h \in L^2_T(V^0_{x,h})$ if $\beta = 1$ in (3.78) and integrate in time to obtain

$$
(4.5) \int_0^T [(\partial_t \rho^0_h, q_h)_{\Omega_x} - (J^0_h, \nabla x q_h)_{\Omega_x} + \langle \|J^0_h\|, [q_h] \rangle_{\varepsilon^1, h} + (\gamma I [\|q^0_h\|], [q_h])_{\varepsilon^1, h}] dt = \int_0^T \varepsilon^3 \Theta_1(\tilde{g}^\varepsilon_h, q_h) dt.
$$

Note that $\int_0^T \Theta_2(\tilde{g}^\varepsilon_h, q_h) dt = 0$ since for a.e. $t$, $q_h(t) = 0$ on $\partial\Omega_x$. If $\beta = 0$, then $\int_0^T \Theta_1(\tilde{g}^\varepsilon_h, q_h) dt = 0$ since $q_h(t)$ is continuous in $x$ for a.e. $t$; additionally, the interior penalty term is also zero so we drop its $\varepsilon^{\beta-1}$ multiplier found in (3.78). If $\beta = 1$, then $\int_0^T \Theta_1(\tilde{g}^\varepsilon_h, q_h) dt$ is uniformly bounded with respect to $\varepsilon$ by (3.81) and (3.67); thus the right hand side will vanish as $\varepsilon \to 0$. Therefore we can pass the weak limit as $\varepsilon \to 0$ in (4.5) and obtain

$$
\int_0^T \left( \frac{\partial}{\partial t} \rho^0_h, q_h \right)_{\Omega_x} - (J^0_h, \nabla x q_h)_{\Omega_x} + \langle \|J^0_h\|, [q_h] \rangle_{\varepsilon^1, h} + (\gamma I [\|q^0_h\|], [q_h])_{\varepsilon^1, h} dt = 0,
$$

which implies

$$
\left( \frac{\partial}{\partial t} \rho^0_h, q_h \right)_{\Omega_x} - (J^0_h, \nabla x q_h)_{\Omega_x} + \langle \|J^0_h\|, [q_h] \rangle_{\varepsilon^1, h} + (\gamma I [\|q^0_h\|], [q_h])_{\varepsilon^1, h} = 0.
$$

for all $q_h \in V^0_{x,h}$ if $\beta = 1$ and all $q_h \in S^0_{x,h}$ if $\beta = 0$ and a.e. $0 < t \leq T$. Thus we arrive at (4.2a).

For (4.2b), choose $\tau_h \in L^2_T([V^0_{x,h}]^3)$ in (3.99) and integrate in time to obtain

$$
(4.6) \int_0^T (\omega J^\varepsilon_h, \tau_h)_{\Omega_x} + \theta (\nabla x \rho^\varepsilon_h, \tau_h)_{\Omega_x} - \theta \langle \|\rho^\varepsilon_h\|, \|\tau_h\| \rangle_{\varepsilon^1, h} - (E \rho^\varepsilon_h, \tau_h)_{\Omega_x} dt = \int_0^T \varepsilon \Theta_3(\tilde{g}^\varepsilon_h, \tau_h) + \sqrt{\varepsilon^{\beta+1}} \Theta_4(\rho^\varepsilon_h, \tau_h) + \sqrt{\varepsilon} \Theta_5(\rho^\varepsilon_h, \tau_h) - \varepsilon^2 (\partial_t J^\varepsilon_h, \tau_h)_{\Omega_x} dt.
$$

Since $\varepsilon^{3/2} \|\partial_t J^\varepsilon_h\|_{L^2([L^2(\Omega_x)])}$ is bounded in $\varepsilon$ by (3.121), the time derivative term in (4.6) vanishes as $\varepsilon \to 0$. Additionally since $\int_0^T \Theta_3(\tilde{g}^\varepsilon_h, \tau_h) dt$, $\int_0^T \Theta_4(\rho^\varepsilon_h, \tau_h) dt$, and
\[ \int_0^T \Theta_\varepsilon(\rho_h^\varepsilon, \tau_h) \, dt \text{ are bounded in } \varepsilon \text{ by (3.100), (3.103), and (3.104) respectively, then the entire right hand side of (4.6) vanishes as } \varepsilon \text{ vanishes. Therefore the limiting equation of (4.6) as } \varepsilon \to 0 \text{ is} \]

\[ \int_0^T (\omega J_H^0, \tau_h)_{\Omega_x} + \theta(\nabla_x \rho_H^0, \tau_h)_{\Omega_x} - \theta \langle [\rho_H^0]_x, [\tau_h]_x \rangle_{\varepsilon, h} - (E \rho_H^0, \tau_h)_{\Omega_x} \, dt = 0, \]

which implies

\[ (\omega J_H^0, \tau_h)_{\Omega_x} + \theta(\nabla_x \rho_H^0, \tau_h)_{\Omega_x} - \theta \langle [\rho_H^0]_x, [\tau_h]_x \rangle_{\varepsilon, h} - (E \rho_H^0, \tau_h)_{\Omega_x} = 0 \]

for all \( \tau_h \in [V_{x,h}]^3 \) and a.e. \( 0 < t \leq T \). Therefore we recover (4.2b).

We now derive the projected initial condition (4.2c) which is similarly shown in [11, Page 379] but is listed here for completeness. Let \( \beta = 1 \) and choose \( q_h \in H^1([0, T], V_{x,h}^0) \) with \( q_h(T) = 0 \) in (4.5). Integration by parts on the first term of (4.5) recovers

\[ (4.7) \]

\[ - \int_0^T \left( \rho_h^0, \frac{\partial}{\partial t} q_h \right)_{\Omega_x} - (J_H^0, \nabla_x q_h)_{\Omega_x} + \langle [J_H^0], [q_h] \rangle_{\varepsilon, h} + \langle \gamma_I [\rho_h^0], [q_h] \rangle_{\varepsilon, h} \, dt = \int_0^T \varepsilon \Theta_1(g^\varepsilon_h, q_h) \, dt + (\rho_{0,h}, q_h(0))_{\Omega_x}. \]

Sending \( \varepsilon \to 0 \) in (4.7) yields

\[ (4.8) \]

\[ - \int_0^T \left( \rho_h^0, \frac{\partial}{\partial t} q_h \right)_{\Omega_x} - (J_H^0, \nabla_x q_h)_{\Omega_x} + \langle [J_H^0], [q_h] \rangle_{\varepsilon, h} + \langle \gamma_I [\rho_h^0], [q_h] \rangle_{\varepsilon, h} \, dt = (\rho_{0,h}, q_h(0))_{\Omega_x}. \]

Choosing the same test function in (4.2a) and integrating by parts in time also yields

\[ (4.9) \]

\[ - \int_0^T \left( \rho_h^0, \frac{\partial}{\partial t} q_h \right)_{\Omega_x} - (J_H^0, \nabla_x q_h)_{\Omega_x} + \langle [J_H^0], [q_h] \rangle_{\varepsilon, h} + \langle \gamma_I [\rho_h^0], [q_h] \rangle_{\varepsilon, h} \, dt = (\rho_h^0(0), q_h(0))_{\Omega_x}. \]

Subtracting (4.8) from (4.9) implies (4.2c). The case for \( \beta = 0 \) is similar. The proof is complete.

Now we show that the whole sequence \( \{ \rho_h^\varepsilon \} \) and \( \{ J_h^\varepsilon \} \) must converge to \( \rho_h^0 \) and \( J_h^0 \) respectively. We do this by showing uniqueness of solutions to the drift-diffusion system (4.2) with the following lemma.

**Lemma 4.3.** Suppose \( \rho_h \) and \( J_h \) satisfy (4.2), then we have the following bound for any \( \rho_{x} > 0 \):

\[ (4.10) \]

\[ \| \rho_h \|^2_{L^\infty([0,T] \times \Omega_x)} + \frac{\omega_{\min}}{\theta} \| J_h \|^2_{L^2([0,T] \times \Omega_x)} \leq \exp \left( \frac{\| E \|_{L^\infty([0,T] \times \Omega_x)}}{\theta \omega_{\min}} T \right) \| \rho_{0,h} \|^2_{L^2(\Omega_x)} \]

Moreover, the solution pair \( \{ \rho_h, J_h \} \) to (4.2) is unique.
Proof. We first focus on (4.10). Choose \( q_h = \theta \rho_h \) and \( \tau_h = J_h \) in (4.2). Adding both equations in (4.2) gives us

\[
\frac{\theta}{2} \frac{d}{dt} \| \rho_h \|_{L^2(\Omega_x)}^2 + (\omega J_h, J_h)_{\Omega_x} + \theta \langle \gamma_l \| \rho_h \|, \| \rho_h \| \rangle_{E^T_{x,h}} = (E \rho_h, J_h)_{\Omega_x}
\]

for each system. We note if \( \beta = 0 \), then the interior penalty term is zero since \( \rho_h \) is continuous. Thus (4.11) is true independent of \( \beta \). Dropping the interior penalty term, bounding the right hand side of (4.11) by Hölder’s and Young’s inequality, and dividing by \( \theta/2 \) we arrive at

\[
\frac{d}{dt} \| \rho_h \|_{L^2(\Omega_x)}^2 + \frac{\omega_{\min}}{\theta} \| J_h \|_{L^2(\Omega_x)}^2 \leq \frac{\| E \|_{L^\infty([0,T]\times\Omega_x)}}{\theta \omega_{\min}} \| \rho_h \|_{L^2(\Omega_x)}^2.
\]

Applying Grönwall’s to (4.12) gives us (4.10).

Due to the uniqueness result from Lemma 4.3, we attach the additional corollary.

**Corollary 4.4.** The full sequences \( \{ \rho_h^\varepsilon \} \) and \( \{ J_h^\varepsilon \} \) weakly converge to \( \rho_h^0 \) and \( J_h^0 \) respectively in the topologies given in Theorem 4.1.

**5. Error Estimates.** In this section we develop error estimates for \( \rho_h^\varepsilon \) against the true drift-diffusion limit \( \rho^0 \) which solves (2.6). The error estimates are created by comparing both against the discrete drift-diffusion \( \rho_h^\varepsilon \) which solves (4.2). This is summarized in the following theorem whose proof we delay until the end of the section.

**Theorem 5.1.** Suppose \( \rho^0 \in L^s_x(H^s(\Omega)) \) and \( J^0 \in L^s_x([L^2(\Omega_x)]^3) \) satisfy (2.7) for some \( s \geq 2 \), \( \omega \in W^{r,\infty}(\Omega_x) \), and \( E \in L^r_T(W^{r,\infty}(\Omega_x)) \) for some \( r \geq s - 1 \). Define

\[
C_{\omega,r} = \| \omega \|_{W^{r,\infty}(\Omega_x)} \| E \|_{L^r_T(W^{r,\infty}(\Omega_x))}.
\]

Then for any \( \varepsilon \leq \varepsilon_{h_0} \), where \( \varepsilon_{h_0} \) is defined in (3.58) we have the following error estimate:

\[
\| \rho_h^\varepsilon - \rho^0 \|_{L^2_x(L^2(\Omega_x))} \leq \sqrt{C_{h_0}} \| g_{0,h} \|_{L^2(\Omega)} + h_x^{\min(k+1,s)-1} \| \rho^0 \|_{L^2_x(H^s(\Omega_x))} + C_{\omega,r} h_x^{\min(k+1,s)-1} \| \rho^0 \|_{L^2_x(H^s(\Omega_x))}.
\]

**5.1. Error Estimates in \( \varepsilon \).** Here we build estimates comparing \( \rho_h^\varepsilon \) to \( \rho_h^0 \). Define \( e^\varepsilon = \rho_h^\varepsilon - \rho_h^0 \) and \( e^\varepsilon = J_h^\varepsilon - J_h^0 \). Subtracting (4.2) from the system (3.78) and (3.99) gives us the following error equations:

\[
(\partial_t e^\varepsilon, q_h)_{\Omega_x} + (e^\varepsilon, \nabla_x q_h)_{\Omega_x} + \langle [e^\varepsilon^T], [q_h] \rangle_{E^T_{x,h}} + \varepsilon e^\beta \langle \gamma_l [e^\varepsilon^T], [q_h] \rangle_{E^T_{x,h}}
\]

\[
= \varepsilon \Theta_1(\tilde{g}_h^\varepsilon, q_h)
\]

\[
(\omega e^\varepsilon, \tau_h)_{\Omega_x} + \gamma_l (\| \nabla_x e^\varepsilon \|, \| \tau_h \|)_{E^T_{x,h}} - (E e^\varepsilon, \tau_h)_{\Omega_x}
\]

\[
= \varepsilon \Theta_3(\tilde{g}_h^\varepsilon, \tau_h) + \varepsilon^2 \Theta_4(\tilde{g}_h^\varepsilon, \tau_h) + \sqrt{\varepsilon} \Theta_5(\tilde{g}_h^\varepsilon, \tau_h) + \sqrt{\varepsilon} \left( \varepsilon^{3/2} \Theta_5(J_h^\varepsilon, \tau_h) \right)_{\Omega_x}
\]

for all \( \tau_h \in [V_{x,h}]^3 \) and \( q_h \in V^0_{x,h} \) if \( \beta = 1 \) and \( q_h \in S^0_{x,h} \) if \( \beta = 0 \).
In order to bound the error of $e^\varepsilon_\rho$, we decompose the $\rho$ error as $e^\varepsilon_\rho = \eta^\varepsilon_\rho - \xi^\varepsilon_\rho := (\rho^\varepsilon_h - S^\varepsilon_h \rho^h) - (\rho^0_h - S^\varepsilon_h \rho^0_h)$. Thus $\eta^\varepsilon_\rho \in V_{x,h}$ and $\xi^\varepsilon_\rho \in V^0_{x,h}$ if $\beta = 1$ and $\xi^\varepsilon_\rho \in S^0_{x,h}$ if $\beta = 0$.

**Lemma 5.2.** For any $h_x > 0$, $h_v > 0$, and $\varepsilon \leq \varepsilon_{h_v}$, where $\varepsilon_{h_v}$ is defined in Lemma 3.8, $e^\varepsilon_\rho$ and $e^\varepsilon_\gamma$ satisfy the following error bound:

$$
\|\xi^\varepsilon_\rho(T)\|_{L^2_\rho}^2 + \frac{\omega_{\min}}{2} \|e^\varepsilon_\gamma\|_{L^2_\rho}^2 + \frac{\theta\gamma_3}{2} \|\xi^\varepsilon_\rho\|_{L^2_\rho}^2 \leq \varepsilon \rho_{0,h}\|g_{0,h}\|_{L^2_\rho}.
$$

(5.4)

**Proof.** Choose $q_h = -\theta \xi^\varepsilon_\rho$ and $\tau_h = e^\varepsilon_\gamma$ in (5.3). Adding the two equations in (5.3) we arrive at

$$
\theta \left( \partial_t \xi^\varepsilon_\rho, \xi^\varepsilon_\rho \right)_{\Omega_x} + (\omega e^\gamma_\rho, e^\gamma_\rho)_{\Omega_x} + \theta \left( \gamma_1 \left( \xi^\varepsilon_\rho, \xi^\varepsilon_\rho \right)_{\xi^\varepsilon_\rho} + (\nu_{\min}) \right)_{\xi^\varepsilon_\rho} + (\theta \eta^\varepsilon_\rho, \eta^\varepsilon_\rho + \xi^\varepsilon_\rho)
$$

$$
- (E \xi^\varepsilon_\rho, e^\varepsilon_\gamma)_{\Omega_x} - \varepsilon \theta \Theta_1(\tilde{g}^\varepsilon_\rho, \xi^\varepsilon_\rho) + \varepsilon \Theta_3(\tilde{g}^\varepsilon_\rho, e^\gamma_\rho) + \sqrt{\varepsilon^{\beta+1}} \Theta_4(\rho^\varepsilon_\rho, e^\varepsilon_\gamma)
$$

(5.5)

$$
+ \sqrt{\varepsilon} \Theta_5(\rho^\varepsilon_\rho, e^\gamma_\rho) + \frac{1}{\sqrt{\varepsilon}} \left( \varepsilon^{3/2} \partial_t J^\varepsilon_\gamma, e^\gamma_\rho \right)_{\Omega_x} + (\partial_t \eta^\varepsilon_\rho, \eta^\varepsilon_\rho)_{\Omega_x}.
$$

We seek to bound each of the terms on the right hand side of (5.5). Let

$$
I_1 := -\theta (\rho_{\min}) \eta^\varepsilon_\rho \left( \rho_{\min} \right) \eta^\varepsilon_\rho + \theta \left( \rho_{\min} \right) \eta^\varepsilon_\rho \left( \rho_{\min} \right) \eta^\varepsilon_\rho + (\nu_{\min}) \eta^\varepsilon_\rho \left( \rho_{\min} \right) \eta^\varepsilon_\rho.
$$

(5.6)

By use of inverse inequalities, trace inequalities, and Young’s inequality, we can bound $I_1$ for any $\nu > 0$ by

$$
|I_1| \lesssim \frac{1}{\nu} \|\eta^\varepsilon_\rho\|_{L^2_\rho}^2 + \frac{1}{\nu} \|\eta^\varepsilon_\rho\|_{H^1_\rho}^2 + \nu \|e^\varepsilon_\gamma\|_{L^2_\rho}^2.
$$

(5.7)

We can bound $(E \xi^\varepsilon_\rho, e^\varepsilon_\gamma)_{\Omega_x}$ with Young’s inequality to obtain

$$
|E \xi^\varepsilon_\rho, e^\varepsilon_\gamma| \lesssim \frac{1}{\nu} \|\xi^\varepsilon_\rho\|_{L^2_\rho}^2 + \nu \|e^\varepsilon_\gamma\|_{L^2_\rho}^2
$$

(5.8)

for any $\nu > 0$. For $\Theta_1$, we note if $\beta = 0$, then $\xi^\varepsilon_\rho$ is continuous in $x$, so $\Theta_1(\tilde{g}^\varepsilon_\rho, \xi^\varepsilon_\rho) = 0$. Thus we focus on the case $\beta = 1$ for which we use (3.81) and Young’s inequality to bound $\Theta_1$ as

$$
|\varepsilon \Theta_1(\tilde{g}^\varepsilon_\rho, \xi^\varepsilon_\rho)| \lesssim \frac{1}{\nu h_x} \|M^\varepsilon_\rho \rho^\varepsilon_\rho - g^\varepsilon_\rho\|_{L^2_\rho}^2 + \nu \|\xi^\varepsilon_\rho\|_{L^2_\rho}^2
$$

(5.9)

for any $\nu > 0$. We use (3.100) and Young’s inequality to obtain

$$
|\varepsilon \Theta_3(\tilde{g}^\varepsilon_\rho, \xi^\varepsilon_\rho)| \lesssim \frac{1}{\nu h_x} \|M^\varepsilon_\rho \rho^\varepsilon_\rho - g^\varepsilon_\rho\|_{L^2_\rho}^2 + \nu \|e^\varepsilon_\gamma\|_{L^2_\rho}^2
$$

(5.10)

for any $\nu > 0$. Similarly, we can bound $\Theta_4$ and $\Theta_5$ by (3.103), (3.104), and Young’s inequality to find

$$
|\sqrt{\varepsilon^{\beta+1}} \Theta_4(\rho^\varepsilon_\rho, e^\varepsilon_\gamma)| \lesssim \frac{\varepsilon^{\beta+1}}{\nu h_x} \|\rho^\varepsilon_\rho\|_{L^2_\rho}^2 + \frac{1}{\nu h_x} \|\rho^\varepsilon_\rho\|_{L^2_\rho}^2 + \nu \|e^\varepsilon_\gamma\|_{L^2_\rho}^2.
$$

(5.11)
for any $\nu > 0$. We bound the $\partial_t J^\varepsilon_h$ term as

$$
(5.12) \quad \frac{1}{2} \frac{d}{dt} \frac{\varepsilon^3}{\nu} \partial_t J^\varepsilon_h \| \varepsilon^3 \partial_t J^\varepsilon_h \|_{L^2(\Omega_\varepsilon)}^2 + \nu \| \varepsilon^3 \partial_t J^\varepsilon_h \|_{L^2(\Omega_\varepsilon)}^2
$$

for any $\nu > 0$. The term $\left( \partial_t \eta^\varepsilon_h, \varepsilon^j \right)_{\Omega_\varepsilon}$ is zero since by definition $\eta^\varepsilon_h$ is orthogonal to $\xi^\varepsilon_h$ in $L^2(\Omega_\varepsilon)$. Injecting (5.7) through (5.12) into (5.5) gives us

$$
(5.13) \quad \frac{1}{2} \frac{d}{dt} \frac{\varepsilon^3}{\nu} \| \varepsilon^3 \partial_t J^\varepsilon_h \|_{L^2(\Omega_\varepsilon)}^2 + \nu \| \varepsilon^3 \partial_t J^\varepsilon_h \|_{L^2(\Omega_\varepsilon)}^2
$$

for all $\nu > 0$. Choosing $\nu$, independent of $\varepsilon$ and $h_x$, sufficiently small we can move the last two terms on the right of (5.13) over to the left. Since $\xi^\varepsilon_h(0) = 0$, we can then apply Grönwall’s to obtain (assuming $h_x \lesssim 1$)

$$
(5.14) \quad \frac{1}{2} \frac{d}{dt} \frac{\varepsilon^3}{\nu} \| \varepsilon^3 \partial_t J^\varepsilon_h \|_{L^2(\Omega_\varepsilon)}^2 + \nu \| \varepsilon^3 \partial_t J^\varepsilon_h \|_{L^2(\Omega_\varepsilon)}^2
$$

Note that the appropriate norms of $\eta^\varepsilon_h = \rho^\varepsilon_h - S_\varepsilon^\varepsilon_h \rho^\varepsilon_h$, $\rho^\varepsilon_h$, and $M^\varepsilon_h \rho^\varepsilon_h - g^\varepsilon_h$ can all be bounded from Lemma 3.10 while $\frac{d}{dt} J^\varepsilon_h$ can be bounded in $L^2$ by Lemma 3.14. Applying these bounds to (5.14), recalling Assumption 2.13, and noticing $\varepsilon^2 \lesssim \varepsilon$ gives us (5.4). The proof is complete.

We can now show the $\varepsilon$-error estimate.

**Theorem 5.3.** Let $\{ \rho^0_h, J^0_h \}$ satisfy (4.2). Then for any $\varepsilon \leq \varepsilon_{h, \varepsilon}$, where $\varepsilon_{h, \varepsilon}$ is defined in (3.58), we have the following error estimate:

$$
(5.15) \quad \| \rho^\varepsilon_h - \rho^0_h \|_{L^2(\Omega_\varepsilon)} + \| J^\varepsilon_h - J^0_h \|_{L^2(\Omega_\varepsilon)} \lesssim \sqrt{\frac{\varepsilon}{h_x}} \| g_{0, h} \|_{L^2(\Omega)}.
$$

**Proof.** Using the triangle inequality and Hölder’s inequality, we have

$$
\| \rho^\varepsilon_h - \rho^0_h \|_{L^2(\Omega_\varepsilon)} \leq \| \rho^\varepsilon_h - \rho^\varepsilon_h \|_{L^2(\Omega_\varepsilon)} + \| \rho^\varepsilon_h - \rho^\varepsilon_h \|_{L^2(\Omega_\varepsilon)} \lesssim \| \rho^\varepsilon_h \|_{L^2(\Omega_\varepsilon)} + \| \rho^\varepsilon_h \|_{L^2(\Omega_\varepsilon)}
$$

which we can bounded by (3.70) and (5.4). The $J^\varepsilon_h$ error estimate follows from (5.4). The proof is complete.
5.2. Error Estimate in $h$. We now focus on the error estimates of the limiting drift diffusion system (4.2). Here we guarantee a positive rate of convergence for $k_x \geq 1$. The polynomial degree restriction is not surprising as traditionally an interior penalty term on $J^0_h$ is required in order to obtain a positive rate of convergence for piecewise-constant polynomial approximations (see [30, Table 2.6]). We note that DG discretizations of (2.7) have been studied for the one-dimensional case in [24]; however, their discretization defines the auxiliary variable in the system as a scalar multiple of $\nabla \cdot \rho^0$ rather than $\nabla \cdot \rho^0_h$. Additionally, their error estimates rely on $\rho^0_h$ being discontinuous in space and the fluxes for $\rho^0_h$ and $J^0_h$ to be alternating so that the Gauss-Radau projection can be utilized. Since both of these properties do not hold for (4.2), we include our own error estimates in $h$.

**Lemma 5.4.** Suppose $\rho^0 \in L^2_s(\Omega)$ and $J^0 \in L^2_s([L^2(\Omega_x)]^3)$ satisfy (2.7) for some $s \geq 2$, $\omega \in W^{r,\infty}(\Omega_x)$, and $E \in L^\infty(T^hW^{r,\infty}(\Omega_x))$ for some $r \geq 1$. Define $\mu = \min\{r, s - 1\}$ and recall $C_{\omega, r}$ from Theorem 5.1. Then

\[
\| \rho^0_h - \rho^0 \|^2_{L^2(\Omega_x)} + \| J^0_h - J^0 \|^2_{L^2(\Omega_x)} \lesssim \max(k, s)^{-1} \| \rho^0 \|^2_{L^\infty(H^\ast(\Omega_x))} + C_{\omega, r} \max(k, s)^{-\beta/2} \| \rho^0 \|^2_{L^\infty(H^\ast(\Omega_x))}.
\]

**Proof.** We decompose $e_{h, \rho} := \rho^0_h - \rho_0$ and $e_{h, J} := J^0_h - J^0$ as

\[
e_{h, \rho} = \xi_{h, \rho} - \eta_{h, \rho} := (\rho^0_h - S_h \rho^0) - (\rho^0 - S_h \rho^0),
\]

\[
e_{h, J} = \xi_{h, J} - \eta_{h, J} := (J^0_h - P_h J^0) - (J^0 - P_h J^0),
\]

where $P_h$ is the $L^2$-orthogonal projection onto $V_h$. Because $\rho^0, J^0$ also solve (4.2), the differences $e_{h, \rho}$ and $e_{h, J}$ satisfy

\[
(\partial t e_{h, \rho}, q_h)_{\Omega_x} - (e_{h, J}, \nabla_x q_h)_{\Omega_x} + (\langle e_{h, \rho} \rangle, [q_h])_{E^t_{x, h}} + (\gamma_I [e_{h, \rho}], [q_h])_{E_I_{x, h}} = 0,
\]

\[
(\omega e_{h, J}, \tau_h)_{\Omega_x} + \theta(\nabla_x e_{h, \rho}, \tau_h)_{\Omega_x} - \theta ([e_{h, \rho}], \tau_h)_{E_I_{x, h}} - (E e_{h, \rho}, \tau_h)_{\Omega_x} = 0.
\]

for all $\tau_h \in [V_{x, h}]^3$ and $q_h \in S^h_{x, h}$ if $\beta = 0$ and $q_h \in V_{x, h}$ if $\beta = 1$. Choosing $q_h = \theta \xi_{h, \rho}$ and $\tau_h = \xi_{h, J}$ in (5.18) gives

\[
\theta (\partial_t \xi_{h, \rho}, \xi_{h, J})_{\Omega_x} - \theta (\xi_{h, J}, \nabla_x \xi_{h, \rho})_{\Omega_x} + \theta \langle \xi_{h, J} \rangle, [\xi_{h, \rho}]_{E^t_{x, h}} + \theta \langle \gamma_I [\xi_{h, \rho}] \rangle, [\xi_{h, \rho}]_{E_I_{x, h}} = 0.
\]

\[
(\omega \xi_{h, J}, \xi_{h, J})_{\Omega_x} + \theta(\nabla_x \xi_{h, \rho}, \xi_{h, J})_{\Omega_x} - \theta \langle \xi_{h, \rho} \rangle, [\xi_{h, J}]_{E^t_{x, h}} - (E \xi_{h, \rho}, \xi_{h, J})_{\Omega_x} = 0.
\]

By properties of the $L^2$-projection. $(\partial_t \eta_{h, \rho}, \xi_{h, J})_{\Omega_x} = 0$ and $(\eta_{h, J}, \nabla_x \xi_{h, \rho})_{\Omega_x} = 0$. Adding (5.19) and (5.20) gives

\[
\frac{d}{dt} \| \xi_{h, \rho} \|^2_{L^2(\Omega_x)} + \omega \min \| \xi_{h, J} \|^2_{L^2(\Omega_x)} + \gamma_s \theta \| [\xi_{h, \rho}] \|^2_{L^2(\Omega_x)} \leq \theta \langle \xi_{h, \rho} \rangle, [\xi_{h, J}]_{E^t_{x, h}} + \theta \langle \gamma_I [\xi_{h, \rho}] \rangle, [\xi_{h, J}]_{E_I_{x, h}} + (E \xi_{h, \rho}, \xi_{h, J})_{\Omega_x} + (\omega \eta_{h, J}, \xi_{h, J})_{\Omega_x} + \theta(\nabla_x \eta_{h, \rho}, \xi_{h, J})_{\Omega_x} - \theta \langle \eta_{h, \rho} \rangle, [\xi_{h, J}]_{E_I_{x, h}} - (E \eta_{h, \rho}, \xi_{h, J})_{\Omega_x}.
\]
We now bound the terms on the right hand side of (5.18) using a combination of Cauchy-Schwarz, trace (3.2), inverse (3.4), and Young’s inequalities. Note any integral on \( E_{x,h}^I \) vanishes if \( \beta = 0 \) so we will multiply the edge contributions by \( \beta \) to compensate. Doing this gives us

\[
\frac{4}{\beta} ||\xi_{h,\rho}||_{L^2(\Omega_{x,h})}^2 + ||\xi_{h,J}||_{L^2(\Omega_{x,h})}^2 + \beta ||\xi_{h,\rho}||_{L^2(\Omega_{x,h})}^2 \lesssim ||\xi_{h,\rho}||_{L^2(\Omega_{x,h})}^2
\]

Since \( \omega, E(t) \in W^{r,\infty}(\Omega) \), then \( J^0(t) \in H^\mu(\Omega_x) \) where \( \mu = \min\{r, s-1\} \). From standard finite element interpolation theory (cf \([4]\)) we have the projection estimates

\[
||\eta_{h,\rho}||_{L^2_0(\Omega_{x,h})} \lesssim h_x^{min\{k+1,s\}} ||\rho^0||_{L^2_0(\Omega_x)}^2,
\]

\[
||\eta_{h,J}||_{L^2_0(\Omega_{x,h})} \lesssim C_{\omega,r} h_x^{min\{k+1,\mu\}} ||\rho^0||_{L^2_0(\Omega_x)}^2.
\]

Integrating (5.22) from 0 to \( T \), applying the bounds (5.23), and invoking Grönwall’s lemma we have

\[
||\xi_{h,\rho}||_{L^\infty(\Omega_{x,h})} + ||\xi_{h,J}||_{L^2_0(\Omega_{x,h})} \lesssim h_x^{2min\{k+1,s\}-2} ||\rho^0||_{L^2_0(\Omega_x)}^2
\]

\[
+ C\omega_r h_x^{2min\{k+1,\mu\}-\beta} ||\rho^0||_{L^2_0(\Omega_x)}^2.
\]

Thus by (5.24) and a triangle inequality we have (5.16). The proof is complete. \( \square \)

We can now prove Theorem 5.1 as a consequence of Theorem 5.3 and Lemma 5.4.

**Proof of Theorem 5.1.** Using a triangle inequality and applying Theorem 5.3 and Lemma 5.4 immediately implies the result. The proof is complete. \( \square \)

**Remark 5.5.** Consider an \( H^2 \) solution \( \rho^0 \), \( k_x = 1 \), and full upwinding, that is, \( \beta = 0 \). Then the error in (5.2) is \( O(\sqrt{\varepsilon/h_x + h_x}) \) which is optimal if we set \( h_x = \varepsilon^{1/3} \).

**6. Conclusion.** We have developed two stable discontinuous Galerkin methods for the a linear Boltzmann semiconductor problem, rigorously showed that they are asymptotically preserving, and explicitly showed their limiting discrete drift-diffusion systems as \( \varepsilon \to 0 \).

Future work includes extending the results presented in this paper to a self-consistent electric field \( E \), non-homogeneous inflow boundary data \( f_\ast \), and non-isotropic initial data.

**Appendix A. Technical Lemmas.**

**A.1. Proof of Lemma 3.9.** Lemma 3.9 is a result of the following lemma:

**Lemma A.1.** There exists \( \gamma_\ast > 0 \) independent of \( \varepsilon \) and \( h_x \) such that

(A.1) \[ \inf_{\xi \in \mathbb{R}^d \atop ||\xi||_2 = 1} \left( v_h M_h \frac{\xi}{|h_x^\frac{1}{2}} \right)_{\{v: v_h (v) > 0\}} > \gamma_\ast, \]

(A.2) \[ \inf_{\xi \in \mathbb{R}^d \atop ||\xi||_2 = 1} \left( \frac{|v_h \cdot \xi|}{2} M_h^\frac{1}{2} M_h^\frac{1}{2} \right)_{\Omega_v} > \gamma_\ast. \]

**Proof.** We first focus on (A.1). We will show the function \( \gamma : \mathbb{R}^d \to \mathbb{R} \) defined by

\[ \gamma(\xi) = \left( v_h M_h^\frac{1}{2} \frac{\xi}{|h_x^\frac{1}{2}} \right)_{\{v: v_h (v) > 0\}} \]
is lower semi-continuous. Let $\xi_n \to \xi$. Define $\gamma_n : \Omega_v \to \mathbb{R}$ by

$$\gamma_n(v) = v_h : \xi_n M_h^\frac{1}{2} M_h^\frac{1}{2} \chi_{\{v_h(v) : \xi > 0\}}$$

where $\chi_A$ is the indicator function for the set $A$. By Fatou’s Lemma we have

$$\liminf_{n \to \infty} \gamma(\xi_n) = \liminf_{n \to \infty} \int_{\Omega_v} \gamma_n(v) \, dv \geq \int_{\Omega_v} \liminf_{n \to \infty} \gamma_n(v) \, dv.$$  
We claim

$$\lim_{n \to \infty} \gamma_n(v) = v_h : \xi M_h^\frac{1}{2} M_h^\frac{1}{2} \chi_{\{v_h(v) : \xi > 0\}}$$

for all a.e. $v \in \Omega_v$. Let $v \in \Omega_v$ with $v_h(v) : \xi < 0$. Then eventually we have $v_h(v) : \xi_k < 0$ for all $k$ sufficiently large. Thus the indicator function evaluates to zero and $\gamma_k(v) = 0$; thus (A.4) holds. Let $v \in \Omega_v$ with $v_h(v) : \xi > 0$. Then similarly $v_h(v) : \xi_k > 0$ for all $k$ sufficiently large. Hence the indication function evaluates to 1 and we can pass the limit to show (A.4) holds. Since the set $\{v : v_h(v) : \xi = 0\}$ is a set of measure zero, (A.4) holds for all a.e. $v \in \Omega_v$. Using (A.4) we continue (A.3) to obtain

$$\liminf_{n \to \infty} \gamma(\xi_n) \geq \int_{\Omega_v} \liminf_{n \to \infty} \gamma_n(v) \, dv = \int_{\Omega_v} v_h : \xi M_h^\frac{1}{2} M_h^\frac{1}{2} \chi_{\{v_h(v) : \xi > 0\}} \, dv = \gamma(\xi).$$

Therefore $\gamma$ is lower semi-continuous. Since $\gamma > 0$ on the compact unit sphere, it obtains a positive minimum. Thus the first equality of (A.1) holds.

For (A.2), we note that the function $\xi \to \langle \frac{\varepsilon}{h} \xi, \xi M_h^\frac{1}{2} M_h^\frac{1}{2} \rangle_{\Omega_v}$ is Lipschitz continuous. Since it is also positive on the compact unit sphere, it obtains a positive minimum. The proof is complete.

**Appendix B. Maxwellian Approximation.**

Lemma B.1 gives precise bounds for the discrete 1D root Maxwellian constructed in Remark 2.5.

**Lemma B.1.** Let $L > 0$ with

$$L \geq \sqrt{\theta},$$
and suppose $\Omega_v = [-L, L]$. Furthermore, assume

$$h_v^2 \leq \frac{4}{\sqrt{3}} \theta.$$  

Let $Q_{k_v} : C^0(\overline{\Omega_v}) \to S_{v,k_v}$ with $k_v = 1$ be the piecewise linear nodal Lagrange interpolant. Define

$$\tilde{Q}_{k_v} := \frac{Q_{k_v} u}{\|Q_{k_v} u\|_{L^2(\Omega_v)}}.$$  

For $i = 1, \ldots, 3$, let $M_i^\frac{1}{2} = \tilde{Q}_h(M_i^\frac{1}{2})$ where $M_i^\frac{1}{2}(v_i)$ is defined in (2.20). Then $M_i^\frac{1}{2}$ is positive, continuous, and satisfies Assumption 2.2.a, Assumption 2.2.b, Assumption 2.2.d. Moreover, we have the following approximation results:

$$\|M_i^\frac{1}{2} - \tilde{Q}_{k_v} M_i^\frac{1}{2}\|_{L^2(\Omega_v)} \leq \frac{5}{2} \left(1 - \text{erf} \left(\frac{L}{\sqrt{29}}\right)^{1/2}\right) + \frac{5}{2} h_v^2 \frac{\sqrt{3}}{8\theta},$$

$$\|\partial_\nu(M_i^\frac{1}{2} - \tilde{Q}_{k_v} M_i^\frac{1}{2})\|_{L^2(\Omega_v)} \leq \frac{5}{2} \left(1 - \text{erf} \left(\frac{L}{\sqrt{29}}\right)^{1/2}\right) + \frac{5}{2} h_v^2 \frac{\sqrt{3}}{16\theta^{3/2}} + \frac{5 \sqrt{3}}{2} \frac{1}{\sqrt{40}} h_v.$$
Proof. For ease of notation set $\mathcal{M}(v) = M^4(v)$. Equation (B.1) gives us the estimate
\begin{equation}
\frac{16}{25} \leq \mathrm{erf}\left(\frac{1}{\sqrt{2}}\right) \leq \mathrm{erf}\left(\frac{L}{\sqrt{29}}\right) < 1.
\end{equation}
Direction calculation and (B.1) yields
\begin{align}
\|M\|_{L^2(\Omega_v)}^2 & \leq \text{erf}\left(\frac{L}{\sqrt{29}}\right) \leq 1, \tag{B.7a} \\
\|\partial_v M\|_{L^2(\Omega_v)}^2 & \leq \frac{1}{4\theta} \text{erf}\left(\frac{L}{\sqrt{29}}\right) - \frac{L}{2\sqrt{2\pi\theta^3}} \exp\left(-\frac{L^2}{2\theta}\right) \leq \frac{1}{4\theta} \text{erf}\left(\frac{L}{\sqrt{29}}\right), \tag{B.7b} \\
\|\partial_v^2 M\|_{L^2(\Omega_v)}^2 & \leq \frac{3}{16\theta^2} \text{erf}\left(\frac{L}{\sqrt{29}}\right) - \frac{L(\theta - L^2)}{8\sqrt{2\pi\theta^3}} \exp\left(-\frac{L^2}{2\theta}\right) \leq \frac{3}{16\theta^2} \text{erf}\left(\frac{L}{\sqrt{29}}\right). \tag{B.7c}
\end{align}
We can give precise bounds on the interpolation error $\|M - \tilde{Q}_{h_v} M\|$ from the proof in [4, Theorem (0.4.5)], (B.7c), and (B.7a):
\begin{align}
\|M - Q_{h_v} M\|_{L^2(\Omega_v)} & + \frac{1}{\sqrt{2h_v}} \|\partial_v (M - Q_{h_v} M)\|_{L^2(\Omega_v)} \\
& \leq \frac{1}{2} h_v \|\partial_v^2 M\|_{L^2(\Omega_v)} \leq \frac{\sqrt{3}}{8\theta} h_v^2 \|M\|_{L^2(\Omega_v)} \leq \frac{\sqrt{3}}{8\theta} h_v^2. \tag{B.8}
\end{align}
Using the reverse triangle inequality, (B.8), and (B.6), we obtain
\begin{equation}
\|Q_{h_v} M\|_{L^2(\Omega_v)} \geq \left(1 - \frac{\sqrt{3}h_v^2}{8\theta}\right) \|M\|_{L^2(\Omega_v)} \geq \frac{4}{5} \left(1 - \frac{h_v^2 \sqrt{3}}{8\theta}\right) \tag{B.9}
\end{equation}
From (B.2), we have
\begin{equation}
1 - \frac{\sqrt{3}}{8\theta} h_v^2 \geq \frac{1}{2}, \tag{B.10}
\end{equation}
and hence
\begin{equation}
\|Q_{h_v} M\|_{L^2(\Omega_v)} \geq \frac{2}{5}, \quad \text{and} \quad \frac{1}{\|Q_{h_v} M\|_{L^2(\Omega_v)}} \leq \frac{5}{2}. \tag{B.11}
\end{equation}
We now show (B.4). Define
\begin{equation}
\alpha_M = \|Q_{h_v} M\|_{L^2(\Omega_v)}. \tag{B.12}
\end{equation}
Using in definition of $\tilde{Q}_{h_v}$ we obtain
\begin{equation}
\|M - \tilde{Q}_{h_v} M\|_{L^2(\Omega_v)} = \frac{1}{\alpha_M} \|\alpha_M M - Q_{h_v} M\|_{L^2(\Omega_v)}. \tag{B.13}
\end{equation}
Adding and subtracting key quantities and several uses of the standard and reverse triangle inequalities both yield
\begin{align}
\|\alpha_M M - Q_{h_v} M\|_{L^2(\Omega_v)} & \leq |1 - \alpha_M| \|M\|_{L^2(\Omega_v)} + \|M - Q_{h_v} M\|_{L^2(\Omega_v)} \\
& \leq \left|1 - \|M\|_{L^2(\Omega_v)}\right| \left|\|\alpha_M M\|_{L^2(\Omega_v)} - 1\right| + \|M - Q_{h_v} M\|_{L^2(\Omega_v)} \\
& \leq |1 - \|M\|_{L^2(\Omega_v)}| \|M\|_{L^2(\Omega_v)} \\
& + \|M - Q_{h_v} M\|_{L^2(\Omega_v)} \|\alpha_M M\|_{L^2(\Omega_v)} \\
& + \|M - Q_{h_v} M\|_{L^2(\Omega_v)}. \tag{B.14}
\end{align}
The terms on the right hand side of (B.14) can be bounded using (B.7) and (B.8). These estimates along with (B.13) and (B.11) yield (B.4). For (B.5), a similar $H^1$ estimate to (B.14) can be formed, namely:

\[
\| \partial_v (\alpha M M - Q_h v M) \|_{L^2(\Omega_v)} \leq |1 - \| M \|_{L^2(\Omega_v)}| \| \partial_v M \|_{L^2(\Omega_v)} \\
+ \| M - Q_h v M \|_{L^2(\Omega_v)} \| \partial_v M \|_{L^2(\Omega_v)} \\
+ \| \partial_v (M - Q_h v M) \|_{L^2(\Omega_v)}.
\]

(B.15)

Estimate (B.15) along with the estimates above yield (B.5). The proof is complete.\]

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