Group theory

A new canonical induction formula for \( p \)-permutation modules

Une nouvelle formule d’induction canonique pour modules de \( p \)-permutation

Laurence Barker, Hatice Mutlu

Department of Mathematics, Bilkent University, 06800 Bilkent, Ankara, Turkey

A R T I C L E   I N F O
Article history:
Received 1 October 2018
Accepted after revision 9 April 2019
Available online 24 April 2019
Presented by the Editorial Board

A B S T R A C T
Applying Robert Boltje’s theory of canonical induction, we give a restriction-preserving formula expressing any \( p \)-permutation module as a \( \mathbb{Z}[1/p] \)-linear combination of modules induced and inflated from projective modules associated with subquotient groups. The underlying constructions include, for any given finite group, a ring with a \( \mathbb{Z} \)-basis indexed by conjugacy classes of triples \((U, K, E)\) where \( U \) is a subgroup, \( K \) is a \( p' \)-residue-free normal subgroup of \( U \), and \( E \) is an indecomposable projective module of the group algebra of \( U/K \).

© 2019 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

R É S U M É
En application de la théorie de l’induction canonique de Robert Boltje, nous présentons une formule stable par restriction au moyen de laquelle tout module de \( p \)-permutation est exprimé sous forme de combinaison \( \mathbb{Z}[1/p] \)-linéaire des inductions des inflations des modules projectifs associés à des groupes de sous-quotients. Les constructions concernées comprennent, pour tout groupe fini, un anneau qui a une \( \mathbb{Z} \)-base indexée par les classes de conjugaison des triplets \((U, K, E)\) avec \( U \) un sous-groupe, \( O^p(K) = K \triangleleft U \) et \( E \) un module projectif indécomposable de l’algèbre de groupe de \( U/K \).

© 2019 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

We shall be applying Boltje’s theory of canonical induction \cite{Boltje} to the ring of \( p \)-permutation modules. Of course, \( p \) is a prime. We shall be considering \( p \)-permutation modules for finite groups over an algebraically closed field \( \mathbb{F} \) of characteristic \( p \). A review of the theory of \( p \)-permutation modules can be found in Bouc–Thévenaz \cite[Section 2]{Bouc-Thévenaz}.

E-mail addresses: barker@fen.bilkent.edu.tr (L. Barker), hatice.mutlu@bilkent.edu.tr (H. Mutlu).

https://doi.org/10.1016/j.crma.2019.04.004
1631-073X © 2019 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
A canonical induction formula for $p$-permutation modules was given by Boltje [3, Section 4] and shown to be $\mathbb{Z}$-integral. It expresses any $p$-permutation module, up to isomorphism, as a $\mathbb{Z}$-linear combination of modules induced from a special kind of $p$-permutation module, namely, the 1-dimensional modules.

We shall be inducing from another special kind of $p$-permutation module. Let $G$ be a finite group. We understand all $\mathbb{F}G$-modules to be finite-dimensional. An indecomposable $\mathbb{F}G$-module $M$ is said to be exprojective provided the following equivalent conditions hold up to isomorphism: there exists a normal subgroup $K \leq G$ such that $M$ is inflated from a projective $\mathbb{F}G/K$-module; there exists $K \leq G$ such that $M$ is a direct summand of the permutation $\mathbb{F}G$-module $G/K$; every vertex of $M$ acts trivially on $M$; some vertex of $M$ acts trivially on $M$. Generally, an $\mathbb{F}G$-module $X$ is called exprojective provided every indecomposable direct summand of $X$ is exprojective.

The exprojective modules do already play a special role in the theory of $p$-permutation modules. Indeed, the parametrization of the indecomposable $p$-permutation modules, recalled in Section 2, characterizes any indecomposable $p$-permutation module as a particular direct summand of a module induced from an exprojective module.

We shall give a $\mathbb{Z}[1/p]$-integral canonical induction formula, expressing any $p$-permutation $\mathbb{F}G$-module, up to isomorphism, as a $\mathbb{Z}[1/p]$-linear combination of modules induced from exprojective modules. More precisely, we shall be working with the Grothendieck ring for $p$-permutation modules $T(G)$ and we shall be introducing another commutative ring $\mathbb{K}T(G)$ which, roughly speaking, has a free $\mathbb{Z}$-basis consisting of lifts of induced modules of indecomposable exprojective modules. We shall consider a ring epimorphism $\text{lin}_G : T(G) \to \mathbb{K}T(G)$ and its $\mathbb{Q}$-linear extension $\text{lin}_G : \mathbb{Q}T(G) \to \mathbb{Q}\mathbb{K}T(G)$. The latter is split by a $\mathbb{Q}$-linear map $\text{can}_G : \mathbb{Q}T(G) \to \mathbb{Q}\mathbb{K}T(G)$ which, as we shall show, restricts to a $\mathbb{Z}[1/p]$-linear map $\text{can}_G : \mathbb{Z}[1/p]\mathbb{K}T(G) \to \mathbb{Z}[1/p]\mathbb{K}T(G)$.

Let $\mathbb{K}$ be a field of characteristic zero that is sufficiently large for our purposes. To motivate further study of the algebras $\mathbb{Z}[1/p]\mathbb{K}T(G)$ and $\mathbb{K}T(G)$, we mention that, notwithstanding the formulas for the primitive idempotents of $\mathbb{K}T(G)$ in Boltje [4, 36], Bouc–Thévenaz [6, 4.12] and [1], the relationship between those idempotents and the basis $\{[M^E_G : (P, E) \in \mathcal{P}(E)]\}$ remains mysterious. In Section 4, we shall prove that $\mathbb{K}T(G)$ is $\mathbb{K}$-semisimple as well as commutative, in other words, the primitive idempotents of $\mathbb{K}T(G)$ comprise a basis for $\mathbb{K}T(G)$. We shall also describe how, via $\text{lin}_G$, each primitive idempotent of $\mathbb{K}T(G)$ lifts to a primitive idempotent of $\mathbb{K}T(G)$.

2. Exprojective modules

We shall establish some general properties of exprojective modules.

Given $H \leq G$, we write $\mathcal{I}\text{Ind}_H$ and $\mathcal{I}\text{Res}_G$ to denote the induction and restriction functors between $\mathbb{F}G$-modules and $\mathbb{F}H$-modules. When $H \leq G$, we write $\mathcal{I}\text{Ind}_{G/H}$ to denote the inflation functor to $\mathbb{F}G$-modules from $\mathbb{F}G/H$-modules. Given a finite group $L$ and an understood isomorphism $L \to G$, we write $\mathcal{I}\text{Iso}_G$ to denote the isogation functor from $\mathbb{F}G$-modules to $\mathbb{F}L$-modules, we mean to say, $\mathcal{I}\text{Iso}_G(X)$ is the $\mathbb{F}L$-module obtained from an $\mathbb{F}G$-module $X$ by transport of structure via the understood isomorphism.

Let us classify the exprojective $\mathbb{F}G$-modules up to isomorphism. We say that $G$ is $p'$-residue-free provided $G = O^{p'}(G)$, equivalently, $G$ is generated by the Sylow $p'$-subgroups of $G$. Let $\mathcal{Q}(G)$ denote the set of pairs $(K, F)$, where $K$ is a $p'$-residue-free normal subgroup of $G$ and $F$ is an indecomposable projective $\mathbb{F}G/K$-module, two such pairs $(K, F)$ and $(K', F')$ being deemed the same provided $K = K'$ and $F \cong F'$. We define an indecomposable exprojective $\mathbb{F}G$-module $M^E_G = \mathcal{I}\text{Ind}_{G/K}(F)$. By considering vertices, we obtain the following result.

**Proposition 2.1.** The condition $M \cong M^E_G$ characterizes a bijective correspondence between:
(a) the isomorphism classes of indecomposable exprojective $\mathbb{F}G$-modules $M$,
(b) the elements $(K, F)$ of $\mathcal{Q}(G)$.

In particular, for a $p'$-subgroup $P$ of $G$, the condition $E \cong N_G(P)\mathcal{I}\text{Ind}_{N_G(P)/P}(E)$ characterizes a bijective correspondence between, up to isomorphism, the indecomposable exprojective $\mathbb{F}N_G(P)$-modules $E$ with vertex $P$ and the indecomposable projective $\mathbb{F}N_G(P)/P$-modules $\mathcal{E}$. It follows that the well-known classification of the isomorphism classes of indecomposable $p$-permutation $\mathbb{F}G$-modules, as in Bouc–Thévenaz [6, 2.9] for instance, can be expressed as in the next result. Let $\mathcal{P}(G)$ denote the set of pairs $(P, E)$ where $P$ is a $p$-subgroup of $G$ and $E$ is an exprojective $\mathbb{F}N_G(P)$-module with vertex $P$, two such pairs $(P, E)$ and $(P', E')$ being deemed the same provided $P = P'$ and $E \cong E'$. We make $\mathcal{P}(G)$ become a $G$-set via the actions on the coordinates. We define $M^E_{P, E}$ to be the indecomposable $p$-permutation $\mathbb{F}G$-module with vertex $P$ in Green correspondence with $E$.

**Theorem 2.2.** The condition $M \cong M^E_{P, E}$ characterizes a bijective correspondence between:
(a) the isomorphism classes of indecomposable $p$-permutation $\mathbb{F}G$-modules $M$,
(b) the $G$-conjugacy classes of elements $(P, E) \in \mathcal{P}(G)$.

We now give a necessary and sufficient condition for $M^E_{P, E}$ to be exprojective.
Proposition 2.3. Let \((P, E) \in \mathcal{P}(G)\). Let \(K\) be the normal closure of \(P\) in \(G\). Then \(M_{P,E}^G\) is exprojective if and only if \(N_K(P)\) acts trivially on \(E\). In that case, \(K\) is \(p^i\)-residue-free, \(P\) is a Sylow \(p\)-subgroup of \(K\), we have \(G = N_G(P)K\), the inclusion \(N_G(P) \hookrightarrow G\) induces an isomorphism \(N_G(P)/N_K(P) \cong G/K\), and \(M_{P,E}^G \cong M_{K,F}^G\), where \(F\) is the indecomposable projective \(FG/K\)-module determined, up to isomorphism, by the condition \(E \cong N_G(P)/N_K(P)\). \(\text{ISO}_G(K)\). \(P\).

Proof. Write \(M = M_{P,E}^G\). If \(M\) is exprojective then \(K\) acts trivially on \(M\) and, perforce, \(N_K(P)\) acts trivially on \(E\).

Conversely, suppose \(N_K(P)\) acts trivially on \(E\). Then \(P\), being a vertex of \(E\), must be a Sylow \(p\)-subgroup of \(N_K(P)\). Hence, \(P\) is a Sylow \(p\)-subgroup of \(K\). By a Frattini argument, \(G = N_K(P)\) and we have an isomorphism \(N_G(P)/N_K(P) \cong G/K\) as specified. Let \(X = c\text{Ind}_{N_K(P)}(E)\). The assumption on \(E\) implies that \(X\) has well-defined \(FG\)-submodules

\[ Y = \left\{ \sum_k k \otimes N_G(P) x : x \in E \right\}, \quad Y' = \left\{ \sum_k k \otimes N_K(P) x_k : x_k \in E, \sum_k x_k = 0 \right\} \]

summed over a left transversal \(kN_K(P) \subseteq K\). Making use of the well-definedness, an easy manipulation shows that the action of \(N_G(P)\) on \(X\) stabilizes \(Y\) and \(Y'\). Similarly, \(K\) stabilizes \(Y\) and \(Y'\). So \(Y\) and \(Y'\) are \(FG\)-submodules of \(X\). Since \(|K : N_K(P)|\) is coprime to \(p\), we have \(Y \cap Y' = 0\). Since \(|K : N_K(P)| = |G : N_G(P)|\), a consideration of dimensions yields \(X = Y \oplus Y'\).

Fix a left transversal \(L\) for \(N_K(P)\) in \(K\). For \(g \in N_G(P)\) and \(\ell \in L\), we can write \(g\ell = \ell h_g\) with \(\ell h_g \in L\) and \(h_g \in N_K(P)\). By the assumption on \(E\) again, \(h_g x = x\) for all \(x \in E\). So

\[ g \sum_{\ell \in L} \ell \otimes x = \sum_{\ell \in L} g \ell \otimes gx = \sum_{\ell \in L} \ell \otimes gx \]

summed over \(\ell \in L\). We have shown that \(n_{G(P)} \text{Res}_G(Y) \equiv E\). A similar argument involving a sum over \(L\) shows that \(K\) acts trivially on \(Y\). Therefore, \(Y \cong M_{K,F}^G\). On the other hand, \(Y\) is indecomposable with vertex \(P\) and, by the Green correspondence, \(Y \equiv M_{P,E}^G\). \(\square\)

We shall be making use of the following closure property.

Proposition 2.4. Given exprojective \(FG\)-modules \(X\) and \(Y\), then the \(FG\)-module \(X \otimes_F Y\) is exprojective.

Proof. We may assume that \(X\) and \(Y\) are indecomposable. Then \(X\) and \(Y\) are, respectively, direct summands of permutation \(FG\)-modules having the form \(FG/K\) and \(FG/L\) where \(K \leq G \leq L\). By Mackey decomposition and the Krull–Schmidt Theorem, every indecomposable direct summand of \(X \otimes Y\) is a direct summand of \(FG/(K \cap L)\). \(\square\)

3. A canonical induction formula

Throughout, we let \(R\) be a class of finite groups that is closed under taking subgroups. We shall understand that \(G \in R\).

We shall abuse notation, neglecting to use distinct expressions to distinguish between a linear map and its extension to a larger coefficient ring.

Specializing some general theory in Boltje [2], we shall introduce a commutative ring \(T(G)\) and a ring epimorphism \(\text{lin}_G : T(G) \rightarrow T(G)\). We shall show that the \(\mathbb{Z}[1/p]\)-linear extension \(\text{lin}_G : \mathbb{Z}[1/p]T(G) \rightarrow \mathbb{Z}[1/p]T(G)\) has a splitting \(\text{can}_G : \mathbb{Z}[1/p]T(G) \rightarrow \mathbb{Z}[1/p]T(G)\). As we shall see, \(\text{can}_G\) is the unique splitting that commutes with restriction and isogation.

To be clear about the definition of \(T(G)\), the Grothendieck ring of the category of \(p\)-permutation \(FG\)-modules, we use the split short exact sequences are the distinguished sequences determining the relations on \(T(G)\). The multiplication on \(T(G)\) is given by tensor product over \(F\). Given a \(p\)-permutation \(FG\)-module \(X\), we write \([X]\) to denote the isomorphism class of \(X\). We understand that \([X] \in T(G)\). By Theorem 2.2,

\[ T(G) = \bigoplus_{(P,E) \in \mathcal{P}(G)} \mathbb{Z}[M_{P,E}^G] \]

as a direct sum of regular \(Z\)-modules, the notation indicating that the index runs over representatives of \(G\)-orbits. Let \(T^{\text{ex}}(G)\) denote the \(Z\)-submodule of \(T(G)\) spanned by the isomorphism classes of exprojective \(FG\)-modules. By Proposition 2.4, \(T^{\text{ex}}(G)\) is a subring of \(T(G)\). By Proposition 2.1,

\[ T^{\text{ex}}(G) = \bigoplus_{(K,F) \in \mathcal{Q}(G)} \mathbb{Z}[M_{G,F}^K] \].

For \(H \leq G\), the induction and restriction functors \(\text{ind}_H\) and \(\text{Res}_G\) give rise to induction and restriction maps \(\text{ind}_H\) and \(\text{Res}_G\) between \(T(H)\) and \(T(G)\). Similarly, given \(L \in R\) and an isomorphism \(\theta : L \rightarrow G\), we have an evident isogation map \(\text{iso}_G^L : T(L) \rightarrow T(G)\). In particular, given \(g \in G\), we have an evident conjugation map \(\sigma_g\). Boltje noted that, when \(\mathcal{R}\) is the set of subgroups of a given fixed finite group, \(T\) is a Green functor in the sense of [2, 1.1c]. For arbitrary \(\mathcal{R}\), a class of admitted isogations must be understood, and the isogations and inclusions between groups in \(\mathcal{R}\) must satisfy the
axioms of a category. Granted that, then $T$ is still a Green functor in an evident sense whereby the conjugations replaced by isogations.

Following a construction in [2, 2.2], adaptation to the case of arbitrary $\mathcal{R}$ being straightforward, we form the $G$-cofixed quotient $\mathcal{Z}$-module

$$\mathcal{T}(G) = \left( \bigoplus_{U \leq G} T^{ex}(U) \right)_G$$

where $G$ acts on the direct sum via the conjugation maps $\xi \mapsto \xi U \cdot \xi$. Harnessing the Green functor structure of $T$, the restriction functor structure of $T^{ex}$ and noting that $T^{ex}(G)$ is a subring of $T(G)$, we make $\mathcal{T}$ become a Green functor much as in [2, 2.2], with the evident isogation maps. In particular, $\mathcal{T}(G)$ becomes a ring, commutative because $T(G)$ is commutative. Given $x_U \in T^{ex}(U)$, we write $[U, x_U]_G$ to denote the image of $x_U$ in $\mathcal{T}(G)$. Any $x \in \mathcal{T}(G)$ can be expressed in the form

$$x = \sum_{U \leq G} [U, x_U]_G$$

where the notation indicates that the index runs over representatives of the $G$-conjugacy classes of subgroups of $G$. Note that $x$ determines $[U, x_U]$ and $x_G$ but not, in general, $x_U$. Let $\mathcal{R}(G)$ be the $G$-set of pairs $(U, K, F)$ where $U \leq G$ and $(K, F) \in \mathcal{Q}(U)$. We have

$$\mathcal{T}(G) = \bigoplus_{U \leq G, (K, F) \in \mathcal{Q}(U)} \mathcal{Z}[U, [M^K_U, F]] = \bigoplus_{(U, K, F) \in \mathcal{R}(G)} \mathcal{Z}[U, [M^K_U, F]].$$

We define a $\mathcal{Z}$-linear map $\text{lin}_G : \mathcal{T}(G) \to \mathcal{T}(G)$ such that $\text{lin}_G[U, x_U] = c \text{ind}_U(x_U)$. As noted in [2, 3.1], the family $(\text{lin}_G : G \in \mathcal{R})$ is a morphism of Green functors $\text{lin} : \mathcal{T} \to \mathcal{T}$. In particular, the map $\text{lin}_G : \mathcal{T}(G) \to \mathcal{T}(G)$ is a ring homomorphism. Extending to coefficients in $\mathcal{Q}$, we obtain an algebra map

$$\text{lin}_G : \mathcal{Q} \mathcal{T}(G) \to \mathcal{Q} \mathcal{T}(G).$$

Let $\pi_G : \mathcal{T}(G) \to T^{ex}(G)$ be the $\mathcal{Z}$-linear epimorphism such that $\pi_G$ acts as the identity on $T^{ex}(G)$ and $\pi_G$ annihilates the isomorphism class of every indecomposable non-exprojective $p$-permutation $\mathcal{F}G$-module. By $\mathcal{Q}$-linear extension again, we obtain a $\mathcal{Q}$-linear epimorphism $\pi_G : \mathcal{T}(G) \to \mathcal{Q} T^{ex}(G)$. After [2, 5.3a, 6.1a], we define a $\mathcal{Q}$-linear map

$$\text{can}_G : \mathcal{Q} \mathcal{T}(G) \to \mathcal{Q} \mathcal{T}(G), \xi \mapsto \frac{1}{|G|} \sum_{U, V \leq G} |U| \text{mőb}(U, V) [U, \text{res}_V(\pi_V(\text{res}_G(\xi)))].$$

where $\text{mőb}()$ denotes the Möbius function on the poset of subgroups of $G$.

**Theorem 3.1.** Consider the $\mathcal{Q}$-linear map $\text{can}_G$.

1. We have $\text{lin}_G \circ \text{can}_G = \text{id}_{\mathcal{Q} \mathcal{T}(G)}$.
2. For all $H \leq G$, we have $\text{res}_H \circ \text{can}_G = \text{can}_H \circ \text{res}_G$.
3. For all $L \in \mathcal{R}$ and isomorphisms $\theta : L \to G$, we have $\text{iso}_L \circ \text{can}_G = \text{can}_L \circ \text{iso}_G$.
4. $\text{can}_G[X] = [X]$ for all exprojective $\mathcal{F}G$-modules $X$.

Those four properties, taken together for all $G \in \mathcal{R}$, determine the maps $\text{can}_G$.

**Proof.** By [2, 6.4], part (1) will follow when we have checked that, for every indecomposable non-exprojective $p$-permutation $\mathcal{F}G$-module $M$, we have $[M] \in \sum_{K \leq G} c \text{ind}_K([\mathcal{T}(K)]).$ By [3, 2.1, 4.7], we may assume that $G$ is $p$-hypoelementary. By [3, 1.3(b)], $M$ is induced from $N_G(P)$ where $P$ is a vertex of $M$. But $M$ is non-exprojective, so $P$ is not normal in $G$. The check is complete. Parts (2), (3), (4) follow from the proof of [2, 5.3a]. □

Parts (2) and (3) of the theorem can be interpreted as saying that $\text{can}_G : T \to T$ is a morphism of restriction functors. It is not hard to check that, when $\mathcal{R}$ is closed under the taking of quotient groups, the functors $T$, $T^{ex}$, $\mathcal{T}$ can be equipped with inflation maps, and the morphisms $\text{lin}_G$ and $\text{can}_G$ are compatible with inflation.

The latest theorem immediately yields the following corollary.

**Corollary.** Given a $p$-permutation $\mathcal{F}G$-module $X$, then

$$[X] = \frac{1}{|G|} \sum_{U, V \leq G} |U| \text{mőb}(U, V) c \text{ind}_U \text{res}_V(\pi_V(\text{res}_G[X])).$$
Given $p$-permutation $\mathbb{F}G$-modules $M$ and $X$, with $M$ indecomposable, we write $m_G(M, X)$ to denote the multiplicity of $M$ as a direct summand of $X$. We write $\pi_G(X)$ to denote the direct summand of $X$, well-defined up to isomorphism, such that $[\pi_G(X)] = \pi_G[X]$.

**Lemma 3.3.** Let $p$ be a set of primes. Suppose that, for all $V \in \mathfrak{A}$, all $p$-permutation $\mathbb{F}V$-modules $Y$, all $U \triangleleft V$ such that $V/U$ is a cyclic $p$-group, and all $V$-fixed elements $(K, F) \in \mathcal{Q}(U)$, we have

$$m_U(M^K_U, \pi_U(\mathcal{R}(Y))) = \sum_{(J,E) \in \mathcal{Q}(V)} m_U(M^K_U, \mathcal{R}(V)) m_V(M^J_V, \pi_Y(V)).$$

Then, for all $G \in \mathfrak{A}$, we have $|G|p'$ can$_C[Y] \in \mathcal{T}(G)$, where $|G|p'$ denotes the $p'$-part of $|G|$.

**Proof.** This is a special case of [2, 9.4]. □

We can now prove the $\mathbb{Z}[1/p]$-integrality of can$_G$.

**Theorem 3.4.** The $\mathbb{Q}$-linear map can$_C$ restricts to a $\mathbb{Z}[1/p]$-linear map $\mathbb{Z}[1/p]\mathcal{T}(G) \to \mathbb{Z}[1/p]\mathcal{T}(G)$.

**Proof.** Let $p$ be the set of primes distinct from $p$. Let $V, Y, U, K, F$ be as in the latest lemma. We must obtain the equality in the lemma. We may assume that $Y$ is indecomposable. If $Y$ is exprojective, then $\pi_U(\mathcal{R}(Y)) \cong Y$, and $\pi_Y(Y) \cong X$, whence the required equality is clear. So we may assume that $Y$ is non-exprojective. Then $\pi_Y(Y)$ is the zero module. It suffices to show that $M^K_U$ is not a direct summand of $\mathcal{R}(Y)$. For a contradiction, suppose otherwise. The hypothesis on $\mathcal{R}(V)$ implies that $U$ contains the vertices of $Y$. So $Y \mid \mathcal{R}(X)$ for some indecomposable $p$-permutation $\mathbb{F}U$-module $X$. Bearing in mind that $(K, F)$ is $V$-stable, a Mackey decomposition argument shows that $M^K_U \cong X$. The $V$-stability of $(K, F)$ also implies that $K \triangleleft V$. So

$$Y \mid \mathcal{R}(X) \cong Y \mathcal{R}(X) \cong Y \mathcal{R}(X) \mathcal{R}(Y).$$

We deduce that $Y$ is exprojective. This is a contradiction, as required. □

**Proposition 3.5.** The $\mathbb{Z}$-linear map $\text{lin}_C : \mathcal{T}(G) \to \mathcal{T}(G)$ is surjective. However, the $\mathbb{Z}[1/p]$-linear map $\text{can}_C : \mathbb{Z}[1/p]\mathcal{T}(G) \to \mathbb{Z}[1/p]\mathcal{T}(G)$ need not restrict to a $\mathbb{Z}$-linear map $\mathcal{T}(G) \to \mathcal{T}(G)$. Indeed, putting $p = 3$ and $G = SL_2(3)$, letting $Y$ be the isomorphically unique indecomposable non-simple non-projective $p$-permutation $\mathbb{F}G$-module and $X$ the isomorphically unique 2-dimensional simple $\mathbb{F}Q_8$-module, then the coefficient of the standard basis element $[Q_8, X]_C$ in $\text{can}_C([Y])$ is equal to $2/3$.

**Proof.** Since every 1-dimensional $\mathbb{F}G$-module is exprojective, the surjectivity of the $\mathbb{Z}$-linear map $\text{lin}_C$ follows from Boltje [3, 4.7]. Routine techniques confirm the counter-example. □

4. The $\mathbb{K}$-semisimplicity of the commutative algebra $\mathbb{K}\mathcal{T}(G)$

Let $\mathcal{I}(G)$ be the $G$-set of pairs $(P, s)$ where $P$ is a $p$-subgroup of $G$ and $s$ is a $p'$-element of $N_G(P)/P$. Let $\mathbb{K}$ be a field of characteristic zero such that $\mathbb{K}$ has roots of unity whose order is the $p'$-part of the exponent of $G$. Choosing and fixing an arbitrary isomorphism between a suitable torsion subgroup of $\mathbb{K} - \{0\}$ and a suitable torsion subgroup of $\mathbb{F} - \{0\}$, we can understand Brauer characters of $\mathbb{F}G$-modules to have values in $\mathbb{K}$. For a $p'$-element $s \in G$, we define a species $\epsilon_{P,s}^G$ of $\mathbb{K}\mathcal{T}(G)$, we mean, an algebra map $\mathbb{K}\mathcal{T}(G) \to \mathbb{K}$, such that $\epsilon_{P,s}^G[M]$ is the value, at $s$, of the Brauer character of a $p$-permutation $\mathbb{F}G$-module $M$. Generally, for $(P, s) \in \mathcal{I}(G)$, we define a species $\epsilon_{P,s}^G$ of $\mathbb{K}\mathcal{T}(G)$ such that $\epsilon_{P,s}^G[M] = \epsilon_{s1,s}^N(C/P)$ $[M(P)]$, where $M(P)$ denotes the $P$-relative Brauer quotient of $M^P$. The next result, well-known, can be found in Bouc–Thévenaz [6, 2.18, 2.19].

**Theorem 4.1.** Given $(P, s), (P', s') \in \mathcal{I}(G)$, then $\epsilon_{P,s}^G = \epsilon_{P',s'}^G$ if and only if we have $G$-conjugacy $(P, s) =_G (P', s')$. The set $\{\epsilon_{P,s}^G : (P, s) \in \mathcal{I}(G)\}$ is the set of species of $\mathbb{K}\mathcal{T}(G)$ and it is also a basis for the dual space of $\mathbb{K}\mathcal{T}(G)$. The dual basis $\{\epsilon_{P,s}^G : (P, s) \in \mathcal{I}(G)\}$ is the set of primitive idempotents of $\mathbb{K}\mathcal{T}(G)$. As a direct sum of trivial algebras over $\mathbb{K}$, we have

$$\mathbb{K}\mathcal{T}(G) = \bigoplus_{(P, s) \in \mathcal{I}(G)} \mathbb{K}\epsilon_{P,s}^G.$$

Let $\mathcal{J}(G)$ be the $G$-set of pairs $(L, t)$ where $L$ is a $p'$-residue-free normal subgroup of $G$ and $t$ is a $p'$-element of $G/L$. We define a species $\epsilon_{L,t}^G$ of $\mathbb{K}T^e(G)$ such that, given an indecomposable exprojective $\mathbb{F}G$-module $M$, then $\epsilon_{L,t}^G[M] = 0$ unless $M$
is the inflation of an $FG/L$-module $M$, in which case, $e_G^{L,G}$ is the value, at $t$, of the Brauer character of $M$. It is easy to show that, given a $p$-subgroup $P \leq G$ and a $p'$-element $s \in N_G(P)/P$, then $e_{P,s}^G[M] = e_G^{L,G}[M]$ for all exprojective $FG$-modules $M$ if and only if $L$ is the normal closure of $P$ in $G$ and $t$ is conjugate to the image of $s$ in $G/L$. Hence, via the latest theorem, we obtain the following lemma.

**Lemma 4.2.** Given $(L, t), (L', t') \in J(G)$, then $e_G^{L,G} = e_{L'}^{L',t'}$ if and only if $L = L'$ and $t = G/L t'$, in other words, $(L, t) = G (L', t')$. The set \( \{ e_G^{L,G} : (L, t) \in G \} \) is the set of species of $K_{T^n(G)}$ and it is also a basis for the dual space of $K_{T^n(G)}$.

Let $K(G)$ be the $G$-set of triples $(V, L, t)$ where $V \leq G$ and $(L, t) \in J(V)$. Given $(L, t) \in J(G)$, we define a species $e_G^{V, L, t}$ of $K_T(G)$ such that, for $x \in T(G)$ expressed as a sum as in Section 3,

$$e_G^{V, L, t}(x) = e_G^{L, t}(x_G).$$

Generally, for $(V, L, t) \in K(G)$, we define a species $e_G^{V, L, t}$ of $K_T(G)$ such that

$$e_G^{V, L, t}(x) = e_{V, L, t}(v \res_G(x)).$$

Using Lemma 4.2, a straightforward adaptation of the argument in [6, 2.18] gives the next result. This result also follows from Boltje–Raggi–Cárdenas–Valero-Elizondo [5, 7.5].

**Theorem 4.3.** Given $(V, L, t), (V', L', t') \in K(G)$, then $e_G^{V, L, t} = e_G^{V', L', t'}$ if and only if $(V, L, t) = G (V', L', t')$. The set \( \{ e_G^{V, L, t} : (V, L, t) \in K(G) \} \) is the set of species of $K_T(G)$ and it is also a basis for the dual space of $K_T(G)$. The dual basis \( \{ e_G^{V, L, t} : (V, L, t) \in K(G) \} \) is the set of primitive idempotents of $K_T(G)$. As a direct sum of trivial algebras over $K$, we have

$$K_T(G) = \bigoplus_{(V, L, t) \in K(G)} K e_G^{V, L, t}. $$

We have the following easy corollary on lifts of the primitive idempotents $e_G^{P,s}$.

**Corollary 4.4.** Given $(P, s) \in I(G)$, then $e_G^{P,s}$ is the unique primitive idempotent $e$ of $K_T(G)$ such that $\lin_G(e) = e_G^{P,s}$.

**References**

[1] L. Barker, An inversion formula for the primitive idempotents of the trivial source algebra, J. Pure Appl. Math. (2019). https://doi.org/10.1016/j.jpaa.2019.04.008, in press.
[2] R. Boltje, A general theory of canonical induction formulae, J. Algebra 206 (1998) 293–343.
[3] R. Boltje, Linear source modules and trivial source modules, Proc. Symp. Pure Math. 63 (1998) 7–30.
[4] R. Boltje, Representation rings of finite groups, their species and idempotent formulae, preprint.
[5] R. Boltje, G. Raggi-Cárdenas, L. Valero-Elizondo, The $-^s$ and $-^{s'}$ constructions for biset functors, J. Algebra 523 (2019) 241–273.
[6] S. Bouc, J. Thévenaz, The primitive idempotents of the $p$-permutation ring, J. Algebra 323 (2010) 2905–2915.