Tilting objects in singularity categories of toric Gorenstein varieties

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Abstract

We study certain toric Gorenstein varieties with isolated singularities which are the quotient spaces of generic unimodular representations by the one-dimensional torus, or by the product of the one-dimensional torus with a finite abelian group. Based on the works of Špenko and Van den Bergh [Invent. Math. 210 (2017), no. 1, 3-67] and Mori and Ueyama [Adv. Math. 297 (2016), 54-92], we show that the singularity categories of these varieties admit tilting objects, and hence are triangle equivalent to the perfect categories of some finite dimensional algebras.

Keywords: singularity category, tilting object, non-commutative crepant resolution

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1 Introduction

For a Noetherian graded Gorenstein algebra $S$ with singularities, its singularity category $\mathcal{D}_{sg}(S) := D^b(\text{grmod}(S)) / \text{Perf}(S)$ is a triangulated category, which reflects many properties of the singularities of $S$. For example, Buchweitz shows in [2] that $\mathcal{D}_{sg}(S)$ is triangle equivalent to the stable category $\text{CM}^Z(S)$, which is the quotient category of the category of graded maximal Cohen-Macaulay modules over $S$ by the full subcategory of graded projective modules.

In a series of papers [14, 15, 16], Orlov shows that the graded singularity categories are determined by the local properties of the singularities, and have a deep relationship with the Homological Mirror Symmetry conjecture, where the category of graded D-branes of type B with a homogeneous superpotential is equivalent to the singularity category $\mathcal{D}_{sg}(A)$ for some graded commutative algebra $A$. If $S$ is a Gorenstein isolated singularity and we forget the gradings of the $S$-modules, then $\mathcal{D}_{sg}(S) \cong \text{CM}(S)$ is a Calabi-Yau triangulated category, which has fruitful homological properties and applications, and has been extensively studied in recent years.

For an algebraic triangulated category, the existence of a tilting object is very important since it replaces the abstract triangulated category by the concrete perfect category of modules over some algebra; see, for example, [3, 6] and references therein for more details and applications. For the case of singularities categories, finding the tilting objects is also an interesting topic, which turns out to have deep relationships with representation theory and algebraic geometry.

Recall that every affine toric Gorenstein variety is in the form $\text{Spec}(\text{Sym}(W)^G)$, where $W$ is a generic unimodular representation of an abelian reductive group $G$, and $G$ is the product of a torus and a finite abelian group (c.f. [18]). When $G$ is a finite group, Iyama and Takahashi show in [8] that the corresponding singularity category has a tilting object. Later in [12], Mori and Ueyama generalize the above result to Noetherian Koszul Artin-Schelter (AS-) regular algebras.

However, for the case that $G$ contains a torus, the singularity categories are considered very rarely in the literature. The difficulty lies in two folds. On the one hand, there may be infinitely many non-isomorphic irreducible maximal Cohen-Macaulay modules over $\text{Sym}(W)^G$, and we therefore cannot apply the McKay-type correspondence, such as in [8], to construct the corresponding quiver and hence the tilting object. On the other hand, the non-commutative crepant resolution of the quotient singularity is not a Koszul algebra in general, and thus the method of [12] cannot be used to this case directly, either. Nevertheless, we prove in this paper the following.

**Theorem 1.1.** Let $k$ be an algebraically closed field of characteristic zero. Let $G$ be

(1) the one-dimensional torus $T$, or

(2) the product of $T$ with a nontrivial abelian finite group.

Suppose $k^n$ is a generic unimodular representation of $G$, and the categorical quotient $S$ is an isolated affine toric Gorenstein singularity. Then the singularity category $\mathcal{D}_{sg}(S)$ admits a tilting object, where the grading of $S$ is canonically induced from that of $k[x_1, \cdots, x_n]$.
The main idea of the proof of the above theorem is as follows. First, we use the method of Špenko and Van den Bergh \cite{17} Theorem 1.6.2 to construct a non-commutative crepant resolution $\Lambda$ of $R^G$. By the results of Iyama and Reiten \cite{7} Lemma 3.6 & Theorem 3.7 and Wemyss \cite{22} Theorem 4.6.3, $\Lambda$ is an $(n-1)$-Calabi-Yau algebra, and hence is an AS-regular algebra of dimension $n$ with Gorenstein parameter $n$ (see Proposition 2.13 below). Therefore by Orlov \cite{16} Theorem 16, $D^b(\text{tail } \Lambda)$ has a tilting object which is $\bigoplus_{i=-n+1}^{0} \Lambda(i)$, where, for a graded algebra $A$, $\text{tail } A$ := grmod $A$/tors $A$ with tors $A$ being the full subcategory of grmod $A$ consisting of graded torsion $A$-modules.

Second, we have the following diagram which is constructed in \cite{12}:

$$
\begin{array}{c}
D^b(\text{grmod } \Lambda) \\
\downarrow \pi \\
D^b(\text{tail } \Lambda)
\end{array}
\begin{array}{c}
\xrightarrow{(-)e} \\
\xrightarrow{\pi}
\end{array}
\begin{array}{c}
D^b(\text{grmod } R^G) \\
\downarrow \pi \\
D^b(\text{tail } R^G)
\end{array}
\begin{array}{c}
\xrightarrow{e} \\
\xrightarrow{\Phi (\text{Orlov embedding})}
\end{array}
\begin{array}{c}
D^b(\text{tail } R^G) \\
\downarrow \mu \\
D^b(\text{gr} \text{sg} (R^G))
\end{array}
\begin{array}{c}
\xrightarrow{(1.1)} \\
\xrightarrow{\Phi (\text{Orlov embedding})}
\end{array}

$$

where $e$ is an idempotent of $\Lambda$ (see Notation 3.4 below), $\pi$ is the natural projection, $\mu$ is the projection from $D^b(\text{tail } R^G)$ to its right admissible subcategory $D^b_{\text{gr} \text{sg}}(R^G)$, and $\Phi$ and $\nu$ are the Orlov embedding and Verdier localization respectively. Moreover, there is an identity of functors

$$( - )e \circ \pi \cong \pi \circ ( - )e.$$ 

We shall show that $R^G$ is a graded Gorenstein algebra (see Proposition 2.12 below). By Mori and Ueyama \cite{13} Lemma 2.7, the algebra $\Lambda/\Lambda e \Lambda$ is finite dimensional if and only if the functor

$$(-)e : D^b(\text{tail } \Lambda) \rightarrow D^b(\text{tail } R^G)$$

is an equivalence. The isolatedness of the singularity ensures that the tilting objects in $D^b(\text{tail } \Lambda)$ can be transferred to $D^b(\text{tail } R^G)$ by the functor $(-)e$ (see Lemma 3.5 below).

Now to prove Theorem 1.1, we improve the method of Mori and Ueyama in \cite{12} Lemma 4.5. Since $\Lambda$ is an AS-regular algebra of dimension $n - 1$ with Gorenstein parameter $n$, we add a summand $(1 - e)\Lambda e$ to the direct sum $\bigoplus_{i=1}^{n-1} (\Omega_{\Lambda}^i (1 - e)(i)) e$ obtained by using syzygies in loc. cit.. Then taking the minimal left approximation (see Definition 3.9 below) of

$$\nu(\bigoplus_{i=1}^{n-1} (\Omega_{\Lambda}^i (1 - e)(i)) e \oplus (1 - e)\Lambda e)$$

with respect to $e\Lambda e$, we get the tilting object

$$\nu(\bigoplus_{i=1}^{n-1} (\Omega_{\Lambda}^i (1 - e)(i)) e \oplus (1 - e)\Lambda e).$$

The rest of the paper is devoted to giving a detailed proof of the above theorem. It is organized as follows: In \S 2 we introduce some often used notations and recall some properties of AS-regular algebras, graded Gorenstein algebras and non-commutative crepant resolutions. In \S 3 we construct a tilting object in $D^b_{\text{gr} \text{sg}}(R^G)$. In \S 4 we prove Theorem 1.1 and also study an example with the tilting object explicitly given.
Notations. Throughout the paper, $k$ is an algebraically closed field of characteristic zero. All modules are right modules. The dimensions of all commutative rings and varieties are more than one unless otherwise stated.

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2 Non-commutative crepant resolutions

In this section, we go over the definition and some properties of the non-commutative crepant resolution (NCCR) of singularities, which is introduced by Van den Bergh in [20, 21]. Some materials are also taken from [17, 18, 19].

2.1 Basics of NCCR

Definition 2.1 (Van den Bergh). Let $R$ be a Gorenstein normal domain. A non-commutative crepant resolution (NCCR) of $R$ is an algebra of the form $\text{End}_R(M)$ for some reflexive $R$-module $M$ such that

1. $\text{End}_R(M)$ is a Cohen-Macaulay $R$-module, and
2. the global dimension of $\text{End}_R(M)$ is finite.

Later Wemyss in [22] replaces Condition (2) in the above definition by that the global dimension of $\text{End}_R(M)$ is equal to the Krull dimension of $R$. However, if $R$ is an equicodimensional Gorenstein normal domain, then both definitions coincide.

Definition 2.2. Let $G$ be a reductive group, $T$ be a maximal torus of $G$ and $V$ be a finite dimensional representation of $G$. The representation is said to be quasi-symmetric if for every line $\ell \subset X(T) \otimes \mathbb{Z} \mathbb{R}$ through the origin, we have

$$\sum_{\alpha_i \in \ell} \alpha_i = 0,$$

where $X(T) := \text{Hom}(T, k^\times)$ is the character space of $T$, and $\alpha_i$ runs through the weights of $T$ on $V$.

Definition 2.3. Let $G$ be an algebraic group and $X$ be a smooth affine variety with an action of $G$. We say $G$ acts generically on $X$ if the action satisfies that:

1. $X$ contains a closed point with closed orbit and its stabilizer is trivial, and
2. if $X^s \subseteq X$ is the set of points satisfying (1), then $\text{codim}(X \setminus X^s) \geq 2$.

Let $V$ be an $n$-dimensional representation of $T$, where $T$ now is $k^\times$. Then $T$ acts on the graded polynomial ring $R := k[x_1, x_2, \cdots, x_n]$ by $t(x_i) := t^{\chi_i}x_i$ for $i = 1, \cdots, n$, where $\deg(x_i) = 1$. The weights of the action are denoted by

$$\alpha_T = (\chi_1, \chi_2, \cdots, \chi_n) \in \mathbb{Z}^n.$$
Definition 2.4. We say $\chi := (\chi_1, \chi_2, \cdots, \chi_n) \in \mathbb{Z}^n$ is effective if

1. there are at least two positive terms and two negative terms in $\chi$,
2. $\sum_{i=1}^{n} \chi_i = 0$, and
3. $\gcd(\chi_i, \chi_j) = 1$, for any $1 \leq i, j \leq n$ such that $\chi_i \chi_j < 0$.

Lemma 2.5. If the weights of the action of $T$ on $R := k[V]$ is effective in the sense of Definition 2.4, then $R^T$ is a Gorenstein algebra.

Proof. In [17, §1.6] and [21, Theorem 8.9], Špenko and Van den Bergh show that if the action of $T$ on $\text{Spec}(R)$ satisfies that:

1. $T$ acts generically on $\text{Spec}(R)$, and
2. $V$ is a unimodular representation of $T$,

then $R^T$ is a Gorenstein algebra. Thus, to prove the lemma, it suffices to show the action of $T$ on $R$ satisfies these two conditions.

First, choose a closed point $x \in \text{Spec}(R)$ such that there is at most one coordinate is zero; it is direct to check that the orbit of $x$ is closed and, by Definition 2.4(3), the stabilizer of $x$ is trivial. Thus, we have that $\text{codim}(\text{Spec}(R) \setminus \text{Spec}(R)^*) \geq 2$, and hence the action of $T$ on $R$ satisfies the first condition.

Second, the triviality of the action of $T$ on the canonical bundle follows from Definition 2.4(2). Therefore, the action of $T$ on $R$ satisfies the second condition.

Now, let $\Delta$ be a bounded closed interval on $\mathbb{R}$, and $\Delta_\epsilon$ be obtained from $\Delta$ by removing the left endpoint.

Let $G$ be a reductive group that acts on $V$ as above and on a finite dimensional vector space $W$. Denote by $M^G_R(W)$ the $R^G$-module $(W \otimes R)^G$, where $R := k[x_1, x_2, \cdots, x_n]$ is the coordinate ring of $V$. Now, let $(\beta_i)_{i=1}^{n}$ be the $T$-weights of $W$, where $T$ is a maximal torus of $G$. Set

$$\Sigma := \left\{ \sum_i a_i \beta_i | a_i \in (-1, 0) \right\} \subset X(T)_R := X(T) \otimes_{\mathbb{Z}} \mathbb{R}.$$ 

The following is proved in [17, Theorem 1.19] and [21, Theorem 8.9]:

Proposition 2.6. Let $G = T$ be the one-dimensional torus $k^\times$ and $V$ be a finite dimensional representation of $G$, which is quasi-symmetric and generic. Let

$$\mathcal{L} = X(T) \cap (1/2)\Sigma, \quad U = \bigoplus_{\chi \in \mathcal{L}} V_\chi,$$

where $V_\chi$ is the representation of $G$ with weight $\chi$. Then $\text{End}_{R^G}(M)$ is an NCCR of $R^G$, where $M := M^G_R(U)$ and $R = k[V]$.

Notation 2.7. In what follows, we use $\Lambda$ to denote the NCCR of $R^G$ constructed above.
The following theorem is obtained in [7, Theorem 3.7] and is explicitly stated in [22, Theorem 4.6.3], which says that $D^b(\Lambda)$ is a Calabi-Yau triangulated category:

**Theorem 2.8.** If $R$ is an equidimensional Gorenstein normal domain over $k$, and $\Lambda$ is an NCCR of $R$, then

$$\text{Hom}_{D^b(\Lambda)}(X,Y) \cong D\text{Hom}_{D^b(\Lambda)}(Y,X)$$

for all $X \in D^b(\Lambda)$ and $Y \in D^b(\Lambda)$, where $D^b(\Lambda)$ is the full subcategory of $D^b(\Lambda)$ consisting of objects whose homologies are finite dimensional, and $D(-) := \text{Hom}_k(-,k)$.

Going back to Proposition 2.6, $k \oplus L$ is equipped with a $\Lambda$-module structure given by the algebra homomorphism $p$ such that

$$\bigoplus_{\chi \in \Lambda} \text{End}_R(M(V_\chi)) \xrightarrow{\bigoplus_{\chi \in \Lambda} \phi_\chi} \bigoplus_{\chi \in \Lambda} \text{End}_R(M(U)) \xrightarrow{p} k \oplus L$$

is commutative, where

$$\phi_\chi : \text{End}_R(M(U)) \longrightarrow \text{End}_R(M(V_\chi))$$

is the canonical projection from $\text{End}_R(M(U))$ to $\text{End}_R(M(V_\chi))$, and

$$\eta_\chi : \text{End}_R(M(V_\chi)) \rightarrow k$$

is the canonical projection from $\text{End}_R(M(V_\chi))$ to $k$. That is,

$$p = \left( \bigoplus_{\chi \in \Lambda} \eta_\chi \right) \circ \left( \bigoplus_{\chi \in \Lambda} \phi_\chi \right).$$

Plugging $X = k \oplus L$ and $Y = \Lambda = \text{End}_R(M(U))$ in Theorem 2.8, we have the following.

**Corollary 2.9.** Under the conditions of Theorem 2.8, we have

$$\text{Hom}_{D^b(\Lambda)}(k \oplus L, \Lambda) \cong D\text{Hom}_{D^b(\Lambda)}(\Lambda, k \oplus L) \cong k \oplus L.$$

### 2.2 AS-regular algebras

NCCRs have a close relationship with Artin-Schelter regular algebras, whose definition we now recall.

Suppose $A$ is a graded associative $k$-algebra. For $M \in \text{grmod } A$ a finitely generated graded $A$-module, we write $M = \bigoplus_i M_i$, where $M_i$ is the degree $i$ component of $M$. Let $A(j)$ be the graded $A$-module such that $A(j)_i = A_{i+j}$. For any $M, N \in \text{grmod } A$, we denote

$$\text{Ext}_A^i(M, N) := \bigoplus_j \text{Ext}_{\text{grmod } A}^i(M, N(j)),$$

for $i \in \mathbb{Z}$. 
Definition 2.10. Suppose $A$ is a non-negatively graded algebra with $A_0$ semi-simple over $k$. We say $A$ is an Artin-Schelter (AS-) regular algebra of dimension $d$ with Gorenstein parameter $a$ if the following two conditions hold:

1. the global dimension $\text{gldim}(A) = d$, and

2. there is an isomorphism

$$\text{Ext}^i_A(A_0, A) \cong \begin{cases} D(A_0)(a), & i = d, \\ 0, & i \neq d \end{cases}$$

in $\text{grmod} A_0$, where $D(-) := \text{Hom}_k(-, k)$ as before.

If an algebra $A$ satisfies (2) in above definition and has finite injective dimension in both $\text{grmod} A$ and $\text{grmod} A^{op}$, then we call it a graded Gorenstein algebra of dimension $d$ with Gorenstein parameter $a$.

Suppose $A$ is a Noetherian Gorenstein algebra of global dimension $d$. If we endow $A$ with a $\mathbb{Z}$-grading such that $A_0$ is finite dimensional, one may ask whether $A$ is an AS-regular algebra of dimension $d$ with Gorenstein parameter $a$, for some $a \in \mathbb{Z}$. We have the following.

Proposition 2.11. Let $A$ be a unital Noetherian algebra of global dimension $d$ over a finite dimensional $k$-algebra $K$ with an augmentation map $\varepsilon : A \to K$ such that $A$ satisfies the Gorenstein condition

$$\text{Ext}^i_A(K, A) \cong \begin{cases} K, & i = d, \\ 0, & i \neq d. \end{cases} \quad (2.1)$$

If we endow $A$ with a $\mathbb{Z}$-grading such that $A_0 = K$ is semi-simple and $A_j = 0$ for $j < 0$, then there is a decomposition

$$A_0 \cong \bigoplus_j A_{0,j}$$

in $\text{grmod} A_0$ such that $A_{0,j} \cong k$ as vector spaces and a sequence of numbers $a_1, a_2, \cdots, a_r$ in $\mathbb{Z}$ such that

$$\text{Ext}^i_A(A_{0,j}, A) \cong \begin{cases} A_{0,j}(a_j), & i = d, \\ 0, & i \neq d \end{cases}$$

in $\text{grmod} A_0$.

Proof. Let $P^\bullet$ be a projective resolution of $A_0$ in $\text{grmod} A$. Then $P^\bullet$ is bounded with length at most $d$ by [5, Theorem 12]. Now we view $P^\bullet$ as a finitely generated projective resolution of $A_0$ in $\text{mod} A$, which is denote by $\overline{P}^\bullet$. Then we have

$$\text{RHom}_{D^b(A)}^*(A_0, A) \cong \text{RHom}_{D^b(A)}^*(\overline{P}^\bullet, A)$$

$$\cong \text{Ext}_A^*(P^\bullet, A)$$

$$\cong \bigoplus_i \text{Ext}_{\text{grmod} A}^*(P^\bullet, A(i))$$

$$\cong \bigoplus_i \text{Ext}_{\text{grmod} A}^*(A_0, A(i)),$$
in grmod $A_0$. Since $A$ satisfies the Gorenstein condition (2.1) with $K = A_0$, we have

$$A_0 \cong \text{Ext}^d_{A_0}(A, A) \cong \bigoplus_b \text{Ext}^d_{\text{grmod} A}(A_0, A(b)),$$

where $b \in \mathbb{Z}$ such that $\text{Ext}^d_{\text{grmod} A}(A_0, A(b)) \neq 0$.

Therefore we get a sequence numbers $b_1, b_2, \cdots, b_r$ in $\mathbb{Z}$ which are given as above such that

$$A_0 \cong \bigoplus_{l=1}^r \bigoplus_{s=1}^{m_l} A_{0, l_s}$$

(2.2)

and

$$\bigoplus_{s=1}^{m_l} A_{0, l_s} \cong \text{Ext}^d_{\text{grmod} A}(A_0, A(b_l)) \cong \text{Ext}^d_{\text{grmod} A} \left( \bigoplus_{l=1}^r \bigoplus_{s=1}^{m_l} A_{0, l_s}, A(b_l) \right)$$

(2.3)

in grmod $A_0$, for any $1 \leq l \leq r$, where each $A_{0, l_s} \cong k$. Since both isomorphisms (2.2) and (2.3) hold in grmod $A_0$, we have

$$\text{Ext}^d_{\text{grmod} A}(A_{0, l_s}, A(b_l)) \cong A_{0, l_s}$$

and

$$\text{Ext}^d_{\text{grmod} A}(A_{0, l_s}, A(b_{l'})) \cong 0$$

for any $l \neq l'$. Moreover, from the decomposition (2.2) we get an injection

$$\text{Ext}^i_A(A_{0, l_s}, A) \hookrightarrow \text{Ext}^i_A(A_0, A),$$

from which we obtain

$$\text{Ext}^i_A(A_{0, l_s}, A) = 0$$

for any $i \neq d$. Thus, we get

$$\text{Ext}^i_A(A_{0, l_s}, A) \cong \begin{cases} A_{0, l_s}(-b_l), & i = d, \\ 0, & i \neq d \end{cases}$$

in grmod $A_0$. \qed

**Proposition 2.12.** With the setting of Proposition 2.6, we have that $R^T$ is a Noetherian graded Gorenstein algebra of dimension $n - 1$ with Gorenstein parameter $n$.

**Proof.** First, it is obvious that

$$\dim(R^T) = \dim(\text{Spec}(R^T)) = \dim(\text{Spec}(R)) - \dim(T) = n - 1.$$

By Orlov [16] Corollary 25 & Proposition 28, which says that the Gorenstein parameter of $R^T$ is equal to the number $r \in \mathbb{N}$ such that $O_X(-r) \cong \omega_X$, where $X = (\text{Spec}(R^T) \setminus \{0\})/k^\times$ and the action of $k^\times$ on Spec($R^T$) is given by the grading on $R^T$. Thus to prove the proposition, it suffices to prove that $O_X(-n) \cong \omega_X$, which is equivalent to showing that the degree of the divisor of the anti-canonical line bundle on $X$ is $n$. 

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Observe that $X$ is isomorphic to the projective variety $(((\text{Spec}(R) \setminus \{0\})/k^\times) \setminus H)/T$, where the action of $T$ is induced by its action on $\text{Spec}(R)$, and $H = H_1 \cup H_2$, where

$$H_1 := \{x = (x_1 : x_2 : \cdots : x_n) \in \mathbb{P}^{n-1} | x_i = 0 \text{ for } \chi_i < 0\},$$

and

$$H_2 := \{x = (x_1 : x_2 : \cdots : x_n) \in \mathbb{P}^{n-1} | x_i = 0 \text{ for } \chi_i > 0\}.$$

Denote $((\text{Spec}(R) \setminus \{0\})/k^\times) \setminus H$ by $Y$; then

$$\omega^{-1}_Y = i^* \omega^{-1}_{P^{n-1}} \otimes \omega_Y,$$

where $i : Y \to \mathbb{P}^{n-1}$ is the embedding. Hence the degree of the divisor of the anti-canonical line bundle on $Y$ is $n$.

Moreover, the divisor of the anti-canonical line bundle on $Y$ is given by the section $s_Y = x_1 x_2 \cdots x_n$ of $\omega^{-1}_Y$.

By [10] Lemma 4.7, which says that

$$\omega_X \cong \omega_Y / T \cong t \otimes \omega_Y$$

in the category $\text{Coh}(Y, G)$ of $G$-equivariant coherent sheaves, where $t$ is the Lie algebra of $T$, we have

$$\omega^{-1}_X \cong \omega^{-1}_Y / T \cong t^* \otimes \omega^{-1}_Y,$$

and therefore, $\omega^{-1}_X = \rho_* \omega^{-1}_Y$, where $t^*$ is the $k$-linear dual of $t$ and $\rho : Y \to X$ is the quotient morphism given by $T$. Thus, there is a lift of $\rho$ to the total space of the anti-canonical line bundle, which is also a quotient morphism given by $T$, such that

$$\tilde{\rho} : \omega^{-1}_Y \to \omega^{-1}_X.$$

Since the action of $T$ on vector space $k \cdot x_1 x_2 \cdots x_n$ is trivial, we have that $\tilde{\rho} \circ s_Y$ is a $T$-equivariant morphism, and therefore there is a section $s_X$ of $\omega^{-1}_X$ such that the following diagram

$$\begin{array}{ccc}
Y & \overset{s_Y}{\longrightarrow} & \omega^{-1}_Y \\
\rho \downarrow & & \downarrow \tilde{\rho} \\
X & \overset{s_X}{\longrightarrow} & \omega^{-1}_X
\end{array}$$

commutes. Thus the degree of the section $s_X$ of $\omega^{-1}_X$ is $n$ and therefore the degree of the divisor of the anti-canonical line bundle on $X$ is $n$.

Now, recall that $\Lambda$ is the NCCR of $R^G$ by Proposition 2.6, we have the following.

**Proposition 2.13.** $\Lambda$ is an AS-regular algebra of dimension $n - 1$ with Gorenstein parameter $n$.

**Proof.** By Theorem 2.8 and Propositions 2.11 and 2.12 we only need to prove

$$a_1 = a_2 = \cdots = a_r = n,$$

where $\{a_1, a_2, \cdots, a_r\}$ is given in Proposition 2.11.

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To this end, let $\Lambda$ be the graded algebra $A$ in Proposition 2.11 and $P^*_i$ be the projective resolution of $\Lambda_0,i \cong e_i\Lambda_0$ of graded $\Lambda$-modules, where $e_i$ is the idempotent corresponding to $\Lambda_0,i$. From the proof of Proposition 2.11 we know that

$$\text{Ext}^n_\Lambda(A_0,i, \Lambda) \cong A_0,i(a_i).$$

Therefore we have an isomorphism

$$k(a_i) \cong \text{Ext}^n_\Lambda(A_0,i, \Lambda)$$

$$\cong \text{Hom}^n_{C^\Lambda(\text{grmod } \Lambda)}(P^*_i, \Lambda)$$

$$\cong \text{Hom}^n_{C^\Lambda(\text{grmod } \Lambda)}(P^*_i, e_i\Lambda)$$

$$\cong \text{Hom}^n_{C^\Lambda(\text{grmod } \Lambda)}(P^*_i, \text{Hom}_{e_i\Lambda e_i}(\Lambda e_i, e_i\Lambda e_i))$$

$$\cong \text{Hom}^n_{C^\Lambda(\text{grmod } e_i\Lambda e_i)}(P^*_i \otimes_\Lambda \Lambda e_i, e_i\Lambda e_i)$$

Moreover, since $(P^*_i)^j \in \text{add}(A)$ for any $j$, $(P^*_i)^j e_i$ is a Cohen-Macaulay $R^T$-module. Observe that when $R^T$ is a Gorenstein algebra, $M$ is a Cohen-Macaulay $R^T$-module if and only if

$$\text{Hom}^i_{D^b(S)}(M, S) = 0,$$

(2.4)

for any $i \neq 0$. Thus from the fact that $R^T$ is a Noetherian graded Gorenstein algebra of dimension $n - 1$ with Gorenstein parameter $n$, we get an isomorphism:

$$\text{Hom}^n_{C^\Lambda(\text{grmod } e_i\Lambda e_i)}(P^*_i e_i, e_i\Lambda e_i) \cong \text{Hom}^n_{D^b(\text{grmod } e_i\Lambda e_i)}(P^*_i e_i, e_i\Lambda e_i)$$

$$\cong \text{Hom}^n_{D^b(\text{grmod } R^T)}(k, e_i\Lambda e_i)$$

$$\cong k(n)$$

in $\text{grmod } k$, where the second isomorphism holds since we have

$$P^*_i e_i \cong P^*_i \otimes^L_\Lambda \Lambda e_i \cong e_i\Lambda_0 \otimes^L_\Lambda \Lambda e_i \cong e_i\Lambda_0 \cong k$$

in $D^b(\text{grmod } e_i\Lambda e_i)$. From these two isomorphisms we get that $a_i = n$ for $i = 1, \cdots, r$. \hfill \Box

We next state several facts that will be used in next section.

First, consider the two categories $\text{mod}(R^G)$ and $\text{mod}(R, G)$, where $\text{mod}(R, G)$ is the category of $G$-equivariant $R$-modules. In general, $\text{mod}(R^G)$ and $\text{mod}(R, G)$ are not equivalent. However, in the case of reflexive modules, we have the following lemma.

Lemma 2.14 (II Lemma 3.3). Under the conditions of Theorem 2.8 assume moreover that $G$ acts generically on $k^n$. Let $\text{ref}(G, R)$ be the category of $G$-equivariant $R$-modules which are reflexive as $R$-modules and $\text{ref}(R^G)$ be the category of reflexive $R^G$-modules. Then the functors

$$\text{ref}(G, R) \to \text{ref}(R^G) : M \mapsto M^G$$

and

$$\text{ref}(R^G) \to \text{ref}(G, R) : N \mapsto (R \otimes_R N)^{**}$$

are mutually inverse equivalences between the two symmetric monoidal categories.
Thus, by the above lemma, we obtain that
\[
\text{Hom}_{\text{mod}(R^G)}(M^G_R(V_1), M^G_R(V_2)) \cong \text{Hom}_{\text{mod}(R,G)}(R \otimes V_1, R \otimes V_2) \\
\cong M^G_R(\text{Hom}_k(V_1, V_2))
\]
in mod($R^G$).

Second, observe that
\[
M^G_R(V) = (V \otimes R)^G = ((V^*)^* \otimes R)^G \cong \text{Hom}_{\text{Rep}(G)}(V^*, R)
\]
for any $V \in \text{Rep}(G)$, where Rep($G$) is the category of representations of $G$. If $V$ is an irreducible representation of $G$, then the elements of $M^G_R(V)$ are in one to one correspondence with the irreducible sub-representations of $G$ in $R$ which are isomorphic to $V^*$ in Rep($G$).

3 Generators of singularity categories

In this section, we study a generator for the singularity category of $R^T$, where $T$ is the one-dimensional torus. For $R^G$ where $G$ is the product of $T$ with a finite abelian group, the argument is similar and will be studied in §4.2. The main results of this section are Propositions 3.10 and 3.17.

3.1 Existence of the generator

Let us start with the following lemma, which is due to Mori and Ueyama:

Lemma 3.1 ([12, Lemma 3.17], [13, Lemma 2.7]). Let $A$ be a Noetherian graded algebra and $e$ be an idempotent of $A$ such that $eAe$ is also a Noetherian graded algebra, $Ae \in \text{grmod } eAe$ and $eA$ is a finitely generated left $eAe$-module. Then $A/(e) \in \text{tors } A$ if and only if
\[
- \otimes^L A e : \text{tail } A \rightarrow \text{tail } eAe
\]
is an equivalence functor.

The following result also appears in [13], which we state as a lemma:

Lemma 3.2. Let $A$ be a Noetherian AS-regular algebra, and $e$ be an idempotent of $A$ such that $eAe$ is a Noetherian graded algebra. If $(-) : \text{tail } A \rightarrow \text{tail } eAe$ is an equivalence functor, then so is $(-) : D^b(\text{tail } A) \rightarrow D^b(\text{tail } eAe)$.

Proof. Since $(-) : \text{tail } A \rightarrow \text{tail } eAe$ is an equivalence functor of abelian categories, we have that
\[
(-) : C^b(\text{tail } A) \rightarrow C^b(\text{tail } eAe)
\]
is also an equivalence functor, where $C^b(\text{tail } A)$ and $C^b(\text{tail } eAe)$ are the bounded chain complex categories of tail $A$ and tail $eAe$ respectively. Thus to prove the lemma it suffices to prove that $M^* \in C^b(\text{tail } A)$ is exact if and only if $M^* e \in C^b(\text{tail } eAe)$ is exact.
If \( M^* \in C^b(\text{tail} \ A) \) is exact, then \( M^* \otimes_A eA \in C^b(\text{tail} \ eA) \) is also exact since \((-)e \cong (-) \otimes_A eA \) is an exact functor on \( \text{grmod} A \).

Conversely, without loss of generality, we assume that \( N_* \) is a short exact sequence

\[
0 \longrightarrow N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} N_3 \longrightarrow 0.
\]

After tensoring \( N_* \) with \( eA \), we obtain an exact sequence

\[
0 \longrightarrow \ker \left( f_1 \otimes eA eA \right) \longrightarrow N_1 \otimes eA eA \longrightarrow N_2 \otimes eA eA \longrightarrow N_3 \otimes eA eA \longrightarrow 0
\]

since the functor \((-) \otimes eA eA \) is right exact on \( \text{grmod}(eA) \). Since

\[
\left( N_* \otimes eA eA \right) \otimes_A eA \cong N_*
\]

in \( C(\text{tail} eA) \), we have

\[
\left( \ker \left( f \otimes eA eA \right) \right) \otimes_A eA \cong 0
\]

in \( \text{tail} eA \). Therefore we obtain that

\[
\ker \left( f \otimes eA eA \right) \cong 0
\]

in \( \text{tail} A \) since \((-)e \) is an equivalent functor in \( \text{tail} A \). Thus

\[
0 \longrightarrow N_1 \otimes eA eA \xrightarrow{f_1 \otimes eA eA} N_2 \otimes eA eA \xrightarrow{f_2 \otimes eA eA} N_3 \otimes eA eA \longrightarrow 0
\]

is an exact sequence in \( C(\text{tail} A) \). \( \square \)

**Definition 3.3** ([13] Definition 1.1). A right Noetherian graded algebra \( A \) is called a graded isolated singularity if \( \text{gldim}(\text{tail} A) < \infty \), where

\[
\text{gldim}(\text{tail} A) := \sup \left\{ i \in \mathbb{Z} \mid \text{Hom}_{D^b(\text{tail} A)}^i(M, N) \neq 0, M, N \in \text{tail} A \right\}.
\]

Next we recall a result of Mori and Ueyama obtained in [13, Theorem 3.10], which is stated as follows: suppose a finite group \( G \) acts on an AS-regular algebra \( S \) of dimension \( d \geq 2 \) and suppose the action has homological determinant one; then \((S \ast G)/(e)\) is finite dimensional over \( k \), where \( e = \frac{1}{|G|} \sum_{g \in G} (1, g) \), if and only if \( S^G \) is a graded isolated singularity and

\[
\psi : S \ast G \rightarrow \text{End}_G(S); \quad (s, g) \mapsto \{ t \mapsto sg(t) \}
\]

is an isomorphism of graded algebras.

**Notation 3.4.** From now on, let \( e \) denote the idempotent of \( \Lambda \) which corresponds to the summand \( M^*_R(V_l) \), where \( l \) is the minimal number in \( L \) given in Proposition 2.6.

The following lemma generalizes the above result of Mori and Ueyama to the case where \( G = T \) is the one-dimensional torus.

**Lemma 3.5.** Let \( R = k[x_1, x_2, \cdots, x_n] \) and \( T \) be the one-dimensional torus acting on \( R \) with weights \( \chi = (\chi_1, \chi_2, \cdots, \chi_n) \). If \( R^T \) satisfies the first and second conditions of Definition 2.4, and is a graded Gorenstein algebra with the canonical grading, then the following are equivalent:
(1) $\chi$ is effective;

(2) the functor $(-)_e : \text{tail} \Lambda \to \text{tail} R^T$ is an equivalence functor with inverse $(-) \otimes_{e\Lambda} e\Lambda$;

(3) $R^T$ is graded isolated singularity;

(4) Spec$(R^T)$ has a unique isolated singularity at the origin.

Proof. From (2) to (1): If $R^T$ does not satisfy the third condition in Definition 2.3, without loss of generality we may assume that gcd$(\chi_1, \chi_2) = h > 1$. From the construction of $\Lambda = \text{End}_{R^T}(\bigoplus_{\lambda \in \mathcal{L}} M_{R^T}(V_{\lambda}))$ in Proposition 2.6, we may choose an indecomposable element $e' \in \Lambda_0$ associated to the summand $M_{R^T}(V_{l+1})$, where $l$ is the minimal number in $\mathcal{L}$.

Assume that $\chi_1 > 0 > \chi_2$. Since $x_1^{\chi_2}x_2^{\chi_1} \in R^T \subset \Lambda$, we have that $e'(x_1^{\chi_2}x_2^{\chi_1})^N \in e'R^T \subset \Lambda$, for any $N > 0$.

Now, fix $N > 0$. Since $e'(x_1^{\chi_2}x_2^{\chi_1})_e = e'(x_1^{\chi_2}x_2^{\chi_1})e'$, we have that $e'(x_1^{\chi_2}x_2^{\chi_1})^N \in \Lambda e\Lambda$ if and only if there are two elements $f_1 \in e'\Lambda e \cong (R \otimes V_{-1})^T$ and $f_2 \in e\Lambda e' \cong (R \otimes V_1)^T$ such that

$$f_1f_2 = e'(x_1^{\chi_2}x_2^{\chi_1})^N.$$ 

On the other hand

$$f_1f_2 = e'(x_1^{\chi_2}x_2^{\chi_1})^N$$

if and only if there are two numbers $\alpha, \beta \in \mathbb{Z}$ such that $-\chi_2N \geq \alpha \geq 0$, $\chi_1N \geq \beta \geq 0$ and $f_2 = e\Lambda e' \cong (R \otimes V_1)^T$ such that

$$\chi_1\alpha + \chi_2\beta = -1$$

since $k(f_2) \cong V_{-1}$. However, we have assumed that gcd$(\chi_1, \chi_2) = h > 1$, which is a contradiction.

Thus we can choose infinity many $N$ making

$$e'(x_1^{\chi_2}x_2^{\chi_1})^N \notin \Lambda e\Lambda.$$ 

It contradicts to the fact that $\Lambda/\Lambda e\Lambda$ is a finite dimensional vector space.

From (3) to (2): Since $R^T$ is a Noetherian graded Gorenstein algebra, the result follows from [13] Theorem 2.5.

From (4) to (3): Suppose Spec$(R^T)$ has a unique isolated singularity at the origin. Then Spec$(R^T) \setminus \{0\}$ is smooth, and so is $(\text{Spec}(R^T) \setminus \{0\}) / k^\times$ by Luna’s slice theorem in [9] (see also the proof of [17], Lemma 3.8), which says that for any closed point $p \neq \{0\} \in \text{Spec}(R^T)$, suppose it is the image of $\bar{p} \in \text{Spec}(R)^e$, then there are two étale morphisms

$$S//T_{\bar{p}} \to N_{\bar{p}}//T_{\bar{p}},$$

and

$$S//G_{\bar{p}} \to \text{Spec}(R)//T \cong \text{Spec}(R^T),$$

where $T_{\bar{p}}$ is the stabilizer of $\bar{p}$, $N_{\bar{p}}$ is a $T_{\bar{p}}$-invariant complement to the inclusion of $T_{\bar{p}}$-representations $T_{\bar{p}}(T)/T_{\bar{p}}(T_{\bar{p}}) \subseteq T_{\bar{p}}(\text{Spec}(R))$, and $S$ is the affine $T_{\bar{p}}$-invariant slice to the $T$-orbit of $\bar{p}$.  

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Thus by the fact that the category of coherent sheaves \( \text{Coh}(\text{Spec}(R^T) \setminus \{0\})/k^\times \) \( \cong \text{tail } R^T \), we have \( \text{gldim}(\text{tail } R^T) < \infty \).

From (1) to (4): Again by Luna’s slice theorem in [9] and the proof of [17] Lemma 3.8 as cited above, \( \tilde{p} \) is not singular in \( \text{Spec}(R^T) \) if and only if the origin is not singular in \( N_{\tilde{p}}/T_{\tilde{p}} \), which is equivalent to that \( T_{\tilde{p}} \) is trivial. Since \( \tilde{p} = (x_1,x_2,\cdots,x_n) \in \text{Spec}(R)^n \), then there are \( x_i, x_j \in \{x_1,x_2,\cdots,x_n\} \) such that \( \chi_i\chi_j < 0 \). Therefore \( \gcd(\chi_i, \chi_j) = 1 \) by effectiveness. Thus for any \( t \in T_{\tilde{p}} \), we have \( t^{\chi_i} = 1 \) and \( t^{\chi_j} = 1 \), from which we obtain that \( t = 1 \in T \). Therefore \( T_{\tilde{p}} \) is trivial.

Suppose \( \chi \) is effective as above. Since \( \Lambda \) is an AS-regular algebra and a Cohen-Macaulay \( R^T \)-module, \( (1-e) \) has a bounded graded projective \( \Lambda \)-modules resolution, denoted by \( P^* \), with length \( n \) such that \( P^{i-n} = (1-e)\Lambda(-n) \), \( P^0 = (1-e)\Lambda \), \( P^i = \text{add}_{j \in [-n+1,-1]}(\bigoplus_j e_i\Lambda(j)) \) for \( -n+1 \leq i \leq -1 \). In fact, \( P^i \) is given by the minimal right \( \text{add}(\Lambda) \)-approximation of \( \ker d_{i+1} : P^{i+1} \rightarrow P^{i+2} \) over \( \Lambda \) in the sense of Definition 3.9 below.

**Notation 3.6.** Let \( Q \) be the following set of objects in \( D^gtr_s(R^T) \):

\[
Q := \{ \nu((1-e)\Lambda e), \nu(\Omega^1_\Lambda(1-e)e(1)), \nu(\Omega^2_\Lambda(1-e)e(2)), \cdots , \nu(\Omega^{n-1}_\Lambda(1-e)e(n-1)) \}
\]

and denote by \( E_Q \) the direct sum of the objects in \( Q \).

The following two lemmas are due to Mori and Orlov respectively, which will be used soon.

**Lemma 3.7** ([10] Lemma 2.9]). Let \( A \) be a right Noetherian graded algebra with \( A_0 \) semi-simple over \( k \). Then the natural morphism

\[
\text{Hom}_{\text{grmod}}(M,N) \rightarrow \text{Hom}_{\text{tail}}(M,N)
\]

is an isomorphism of vector spaces for any \( N,M \in \text{grmod } A \) with \( \text{depth}(N) \geq 2 \).

**Lemma 3.8** ([16] Theorem 16]). Suppose \( A \) is a Gorenstein algebra of dimension \( d \) with positive Gorenstein parameter \( a \). If \( A \) is Noetherian, then there is a fully faithful functor \( \Phi : D^gtr_s(A) \rightarrow D^b(\text{tail } A) \), and a semi-orthogonal decomposition

\[
D^b(\text{tail } A) = \langle \pi A(-a+1), \pi A(-a+2), \cdots , \pi A, \Phi D^gtr_s(A) \rangle.
\]

Note that the functor \( \Phi \) in above lemma is \( \Phi_0 \) in [16].

Now let us remind that an action of a group, say \( G \), on a vector space \( V \) is said to be unimodular if all elements of \( G \) have determinant one. In other words, we may view \( G \) as a subgroup of \( \text{SL}(V) \); in particular, an effective torus action on \( V \) in the sense of Definition 2.4 is always unimodular.

**Definition 3.9** (Minimal approximation). Let \( A \) be a graded algebra, \( C \) be a full subcategory of the category of graded \( A \)-modules which is closed under isomorphism and \( M \) be a graded \( A \)-module. A **minimal left \( C \)-approximation** of \( M \) is a morphism of graded \( A \)-modules

\[
f : M \rightarrow X
\]

such that...
(1) for any $N \in \text{Ob}(\mathcal{C})$, $\text{Hom}_{\text{grmod}\, A}(N, f) : \text{Hom}_{\text{grmod}\, A}(X, N) \to \text{Hom}_{\text{grmod}\, A}(M, N)$ is surjective;

(2) if $g \in \text{End}_{\text{grmod}\, A}(X)$ satisfies $f = g \circ f$, then $g$ is an automorphism.

The minimal right $C$-approximation is defined dually.

In the above definition, let $A$ be the graded commutative ring $R^T \cong e\Lambda$, and $\mathcal{C}$ be the category of graded projective $e\Lambda e$-modules. Since $e\Lambda$ is a graded locally finite-dimensional algebra with $(e\Lambda)_0 = k$, we know that such an $f$ exists and is unique up to isomorphism for any graded $e\Lambda e$-module $M$ [4 Definition 5.1]. Now, we denote by $L_{e\Lambda}M$ the cone of the minimal left $C$-approximation of $M$. For convenience, we call $L_{e\Lambda}M$ the minimal left approximation of $M$ by $e\Lambda e$.

**Proposition 3.10.** With the setting of Theorem 1.1 in the first case, i.e., $G = T$, we have

$$D_{sg}^r(R^T) = \text{thick}(E_Q),$$

where $E_Q$ is given in Notation 3.7. In other words, $E_Q$ as an object in $D_{sg}^r(R^T)$ generates $D_{sg}^r(R^T)$ itself.

**Proof.** By Lemma 3.8 and Proposition 2.13 we have

$$D^b(\text{tail } \Lambda) = \text{thick} \left( \bigoplus_{i=-n+1}^{0} \Lambda(i) \right).$$

Applying $(-)e$ in Lemma 3.3 on both sides of the above identity, we get

$$D^b(\text{tail } R^T) = \text{thick} \left( \bigoplus_{i=-n+1}^{0} \Lambda e(i) \right).$$

Now by Lemma 3.8 applying $\mu$ on both sides of the above identity we further obtain

$$D_{sg}^r(R^T) = \text{thick} \left( \bigoplus_{i=-n+1}^{0} \mu \circ \pi((1 - e)\Lambda e(i)) \right).$$

Thus to prove the proposition it suffices to prove that

$$\mu \circ \pi((1 - e)\Lambda e(j)) \in \text{thick}(E_Q),$$

for any $-n + 1 \leq j \leq 0$. We next prove (3.1), which then finishes the proof of the proposition. \hfill $\square$

We start with the following lemma:

**Lemma 3.11.** We have that

$$\Phi(E_Q) \cong \Phi \circ \nu \left( L_{e\Lambda}(\bigoplus_{i=1}^{n-1} (\Omega^1_{\Lambda}(1 - e)(i)) e \oplus (1 - e)\Lambda e) \right)$$

$$\cong \pi \left( L_{e\Lambda}(\bigoplus_{i=1}^{n-1} (\Omega^1_{\Lambda}(1 - e)(i)) e \oplus (1 - e)\Lambda e) \right).$$
Admit this lemma for a moment. Since \( \mu \circ \Phi \) is the identify functor on \( D_{sg}^R(R^T) \), we have

\[
\nu \left( L_{e\Lambda e} \left( \bigoplus_{i=1}^{n-1} (\Omega^i_{\Lambda}(1-e)(i))e \oplus (1-e)\Lambda e \right) \right)
\]

\[\cong \mu \circ \Phi \circ \nu \left( L_{e\Lambda e} \left( \bigoplus_{i=1}^{n-1} (\Omega^i_{\Lambda}(1-e)(i))e \oplus (1-e)\Lambda e \right) \right)\]

\[\cong \mu \circ \pi \left( L_{e\Lambda e} \left( \bigoplus_{i=1}^{n-1} (\Omega^i_{\Lambda}(1-e)(i))e \oplus (1-e)\Lambda e \right) \right),\]

where the last equality follows from Lemma 3.11. Hence, we have that

\[ E_Q = \nu \left( L_{e\Lambda e} \left( \bigoplus_{i=1}^{n-1} (\Omega^i_{\Lambda}(1-e)(i))e \oplus (1-e)\Lambda e \right) \right) \]

\[\cong \mu \circ \pi \left( L_{e\Lambda e} \left( \bigoplus_{i=1}^{n-1} (\Omega^i_{\Lambda}(1-e)(i))e \oplus (1-e)\Lambda e \right) \right). \tag{3.2} \]

**Lemma 3.12.** We have that

\[ \pi((1-e)\Lambda e(j)) \in \text{thick} \left( \pi \left( \bigoplus_{i=1}^{n-1} (\Omega^i_{\Lambda}(1-e)(i))e \oplus (1-e)\Lambda e \right), \pi \left( \bigoplus_{r=0}^{-n+1} e\Lambda e(r) \right) \right) \tag{3.3} \]

for any \(-n+1 \leq j \leq 0\).

**Proof of 3.12.** Applying \( \mu \) to both sides of (3.3), we get

\[ \mu \circ \pi((1-e)\Lambda e(j)) \in \text{thick} \left( \mu \circ \pi \left( \bigoplus_{i=1}^{n-1} (\Omega^i_{\Lambda}(1-e)(i))e \oplus (1-e)\Lambda e \right) \right), \]

where the right hand side is equivalent to \( \text{thick}(E_Q) \) by (3.2). \( \square \)

**Proof of Lemma 3.12** We show the proof by induction. First, observe that

\[ \pi((1-e)\Lambda e(j)) \in \text{thick} \left( \pi \left( \bigoplus_{i=1}^{n-1} (\Omega^i_{\Lambda}(1-e)(i))e \oplus (1-e)\Lambda e \right), \pi \left( \bigoplus_{r=0}^{-n+1} e\Lambda e(r) \right) \right) \]

for \( j = 0, -1 \), since \( \pi\Omega_{\Lambda}^{n-1}(1-e)(n-1)e \cong \pi(1-e)\Lambda e(-1) \).

Next, we assume the claim holds for \( l \leq j \leq 0 \); that is,

\[ \pi((1-e)\Lambda e(j)) \in \text{thick} \left( \pi \left( \bigoplus_{i=1}^{n-1} (\Omega^i_{\Lambda}(1-e)(i))e \oplus (1-e)\Lambda e \right), \pi \left( \bigoplus_{r=0}^{-n+1} e\Lambda e(r) \right) \right) \]

for \( l \leq j \leq 0 \) and some \(-n+1 \leq l \leq -1\). Consider the following exact sequence of graded \( \Lambda \)-modules

\[ 0 \rightarrow (1-e)\Lambda(l-1) \cong P^{1-n}(n+l-1) \rightarrow P^{2-n}(n+l-1) \rightarrow \cdots \rightarrow \Omega_{\Lambda}^{l+n-1}(1-e)(n+l-1) \rightarrow 0, \]
which implies that
\[ 0 \to \pi((1 - e)\Lambda e)(l - 1) \to \pi(P^{2-n}e(n + l - 1)) \to \cdots \to \pi(\Omega^{i+n-1}_{\Lambda}(1 - e)e(n + l - 1)) \to 0 \]
is equal to 0 in \( D^b(\text{tail } R^T) \). Now observe that \( \pi(P^{2-n}e(n + l - 1)) \) is the direct sum of some summands of \( \pi(\Lambda e(l)) \) and \( \pi(\Lambda e(l + 1)) \). Indeed, in the resolution \( P^* \), for any \(-i\) and any summand \( e'\Lambda(r') \) of \( P^{-i} \) and summand \( e''\Lambda(r'') \) of \( P^{-i+1} \), the composition
\[ e'\Lambda(r') \hookrightarrow P^{-i} \xrightarrow{d_{-i}} P^{-i+1} \to e''\Lambda(r'') \]
is zero unless \( r' < r'' \), since \( P^{-i} \) is constructed by the minimal right \( \text{Add}(\Lambda) \)-approximation of \( \text{Ker}(d_{-i+1}) \). Moreover, since \( \Lambda \) is an AS-regular algebra of dimension \( n - 1 \) with Gorenstein parameter \( n \), we have that the length of \( P^* \) is \( n - 1 \) and \( P^{1-n} \) is the direct sum of some summands of \( \Lambda(-n) \). Hence, we obtain that for any \(-i\), \( P^{-i} \) is the direct sum of some summands of \( \Lambda(-i - 1) \) and \( \Lambda(-i) \).

In the same way, we have that \( \pi(P^{l-n+1}e(n+l-1)) \) is the direct sum of some summands of \( \pi(\Lambda e(-1)) \) and \( \pi(\Lambda e) \). Combining these two statements, we have
\[ \pi(P^r e(n + l - 1)) \in \text{thick}\left( \pi\left( \bigoplus_{i=l}^{0}(1 - e)\Lambda e(i) \right), \pi\left( \bigoplus_{r=-n+1}^{0} e\Lambda e(r) \right) \right) \]
for any \( 2 - n \leq r \leq l - n - 1 \). Hence from
\[ \text{thick}\left( \pi\left( \bigoplus_{i=l}^{0}(1 - e)\Lambda e(i) \right) \right) \subseteq \text{thick}\left( \pi\left( \bigoplus_{i=1}^{n-1} \Omega^{i}_{\Lambda}(1 - e)(i)e \oplus (1 - e)\Lambda e \right), \pi\left( \bigoplus_{r=-n+1}^{0} e\Lambda e(r) \right) \right), \]
we have
\[ \pi(P^r e(n + l - 1)) \in \text{thick}\left( \pi\left( \bigoplus_{i=1}^{n-1} \Omega^{i}_{\Lambda}(1 - e)(i)e \oplus (1 - e)\Lambda e \right), \pi\left( \bigoplus_{r=-n+1}^{0} e\Lambda e(r) \right) \right) \]
for any \( 2 - n \leq r \leq l - n - 1 \). Therefore we obtain that
\[ \pi((1 - e)\Lambda e)(l - 1) \in \text{thick}\left( \pi\left( \bigoplus_{i=1}^{n-1} \Omega^{i}_{\Lambda}(1 - e)(i)e \oplus (1 - e)\Lambda e \right), \pi\left( \bigoplus_{r=-n+1}^{0} e\Lambda e(r) \right) \right), \]
and the lemma follows.

Now we prove Lemma 3.13. To this end, denote by \( \text{grmod}(\Lambda)_{\geq 0} \) the full subcategory of \( \text{grmod} \) \( \Lambda \) whose objects consist of \( N \in \text{grmod} \Lambda \) such that \( N_i = 0 \) for \( i < 0 \). Moreover, denote by \( D^b(\text{grmod}(\Lambda)_{\geq 0}) \) the full subcategory of \( D^b(\text{grmod} \Lambda) \) whose objects consist of \( M^* \in D^b(\text{grmod} \Lambda) \) such that \( M^j \in \text{Ob}(\text{grmod}(\Lambda)_{\geq 0}) \) for \( j \in \mathbb{Z} \). The following lemma is due to Amiot [1].

**Lemma 3.13** ([1, Theorem 4.3]). Let \( \Lambda \) be a Noetherian graded Gorenstein algebra and \( M^* \in D^b(\text{grmod} \Lambda) \). If moreover
(1) \( M^* \in D^b(\text{grmod}(A)_{\geq 0}) \) and

(2) \( \text{Hom}_{D^b(\text{grmod}(A))}^*(M^*, A(i)) = 0 \) for any \( i \leq 0 \),

then

\[
\pi(M^*) = \Phi \circ \nu(M^*).
\]

**Proof of Lemma 3.11.** The first equality holds since

\[
E_Q = \nu\left( L_{e\Lambda}(\bigoplus_{i=1}^{n-1}(\Omega^i_{\Lambda}(1-e)(i))e \oplus (1-e)\Lambda e) \right)
\]

in \( D^b_{sg}(R^T) \).

To prove the second equality, we only need to show

\[
L_{e\Lambda}(\bigoplus_{i=1}^{n-1}(\Omega^i_{\Lambda}(1-e)(i))e \oplus (1-e)\Lambda e)
\]

satisfies the two conditions of Lemma 3.13; that is, we need to show the following:

**Claim 3.14.** \( L_{e\Lambda}(\bigoplus_{i=1}^{n-1}(\Omega^i_{\Lambda}(1-e)(i))e \oplus (1-e)\Lambda e) \in D^b(\text{grmod}(\Lambda)_{\geq 0}) \).

**Claim 3.15.** \( \text{Hom}_{D^b(\text{grmod}(A))}^*(L_{e\Lambda}(\bigoplus_{i=1}^{n-1}(\Omega^i_{\Lambda}(1-e)(i))e \oplus (1-e)\Lambda e), \Lambda(j)) = 0 \) for any \( j \leq 0 \).

Once these two claims are proved, Lemma 3.11 follows immediately.

**Proof of Claim 3.14.** First, since \( \Lambda \) is an AS-regular algebra of dimension \( n-1 \) with Gorenstein parameter \( n \), in the \((-i)\)-th and \((-i+1)\)-th positions of the resolution \( P^*(i) \) after shifting \( i \) times, which are

\[
P^{-i}(i) \twoheadrightarrow \Omega^i_{\Lambda}(1-e)(i) \hookrightarrow P^{-i+1}(i),
\]

we have that \( P^{-i}(i) \) is the direct sum of some summands of \( \Lambda(-1) \) and \( \Lambda(0) \), and \( P^{-i+1}(i) \) is the direct sum of some summands of \( \Lambda(0) \) and \( \Lambda(1) \). Since \( \Omega^i_{\Lambda}(1-e)(i) \) is a submodule of \( P^{-i+1}(i) \), if \( \Omega^i_{\Lambda}(1-e)(i) \) is not in \( \text{grmod}(\Lambda)_{\geq 0} \), then the image of \( \Omega^i_{\Lambda}(1-e)(i) \) in \( P^{-i+1}(i) \) contains a summand of \( \Lambda_0(1) \). But the image of

\[
d_i(i) : P^{-i}(i) \rightarrow P^{-i+1}(i)
\]

does not contain any summand of \( \Lambda_0(1) \) from the construction of \( P^* \). Thus

\[
\Omega^i_{\Lambda}(1-e)(i) \in \text{grmod}(\Lambda)_{\geq 0}
\]

for any \( n-1 \geq i \geq 1 \).

Moreover, we know that \( 1-e\Lambda \in \text{grmod}(\Lambda)_{\geq 0} \). Thus tensoring with \( (-) \otimes_{\Lambda} \Lambda e \), we have

\[
\bigoplus_{i=1}^{n-1}(\Omega^i_{\Lambda}(1-e)(i))e \oplus (1-e)\Lambda e \in \text{grmod}(R^T)_{\geq 0}.
\]
Applying the minimal left approximation of the above $R^T$-module by $e\Lambda e \in \text{grmod}(R^T)_{\geq 0}$, we also have
\[
L_{e\Lambda e} \left( \bigoplus_{i=1}^{n-1} \left( \Omega^*_A(1-e)(i) \right) e \oplus (1-e)\Lambda e \right) \in \text{grmod}(R^T)_{\geq 0}. 
\]
\[\square\]

**Proof of Claim 3.14** Since

\[
\text{Hom}^l_{D^b(\text{grmod} R^T)}((1-e)\Lambda e, e\Lambda e(j)) = 0
\]
for any $j \leq 0$ and $l \in \mathbb{Z}$, and
\[
\text{Hom}^l_{D^b(\text{grmod} R^T)}(L_{e\Lambda e}((\Omega^*_A(1-e)(i))e), e\Lambda e(j)) = 0
\]
for any $l \leq -2$, $1 \leq i \leq n-1$ and $j \leq 0$, we have to show that
\[
\text{Hom}^l_{D^b(\text{grmod} R^T)}(L_{e\Lambda e}((\Omega^*_A(1-e)(i))e), e\Lambda e(j)) = 0
\]
for any $l \geq -1$, $1 \leq i \leq n-1$ and $j \leq 0$.

In the meantime, from the argument in the proof of Lemma 3.12, we know that $P^{-i}(i)$ is the direct sum of some summands of $\Lambda(-1)$ and $\Lambda$. Thus, we have
\[
\text{Hom}^l_{D^b(\text{grmod} R^T)}(L_{e\Lambda e}((\Omega^*_A(1-e)(i))e), e\Lambda e(j)) = 0
\]
for any $l \geq -1$, $1 \leq i \leq n-1$ and $j \leq -2$.

Thus, combining the above two cases, to show the Claim, we only need to show that
\[
\text{Hom}^l_{D^b(\text{grmod} R^T)}(L_{e\Lambda e}((\Omega^*_A(1-e)(i))e), e\Lambda e(j)) = 0
\]
for any $l \geq -1$, $1 \leq i \leq n-1$ and $-1 \leq j \leq 0$. We divide the proof into several cases.

**Case 1:** $l \geq 1$, $1 \leq i \leq n-1$ and $j = -1$ or 0. We give $\Omega^*_A(1-e)(i)$ the following resolution
\[
0 \to (1-e)\Lambda(i-n) \cong P^{-n+1}(i) \to \cdots \to P^{-i-1}(i) \to P^{-i}(i) \to \Omega^*_A(1-e)(i)) \to 0
\]
from $P^*$, which is denoted by $P^*(i)$. Since $\Lambda e$ is a Cohen-Macaulay $R^T$-module, we have
\[
\text{Hom}^l_{D^b(\text{grmod} R^T)}(\Omega^*_A(1-e)(i)e, e\Lambda e(j)) = \text{Hom}^l_{C^*(\text{grmod} R^T)}(P^*_i(i)e, e\Lambda e(j)).
\]

Moreover, since $\Lambda$ is an AS-regular algebra, we have
\[
\text{Hom}^l_{C^*(\text{grmod} R^T)}(P^*_i(i)e, e\Lambda e(j)) 
\cong \text{Hom}^l_{C^*(\text{grmod} R^T)}(P^*_i(i)e, e\Lambda e(j)) 
\cong \text{Hom}^l_{D^b(\text{grmod} R^T)}(P^*_i(i)e, e\Lambda e(j))) 
\cong \text{Hom}^l_{D^b(\text{grmod} R^T)}((1-e) \otimes \Lambda e(i), e\Lambda e(j)) 
\cong \text{Hom}^l_{D^b(\text{grmod} R^T)}((1-e) \otimes \Lambda e(i), e\Lambda e(j)) 
\cong 0
\]
for $l \geq 1$. Thus we have
\[
\text{Hom}^l_{D^b(\text{grmod } R^T)}(\Omega^i \Lambda (1 - e)(i) e, e\Lambda e(j)) = 0
\]
for $l \geq 1$. Applying the minimal left approximation of $\Omega^i \Lambda (1 - e)(i) e$ by $e\Lambda e$, we have
\[
\text{Hom}^l_{D^b(\text{grmod } R^T)}(L_e \Lambda e(\Omega^i \Lambda (1 - e)(i) e), e\Lambda e(j)) = 0
\]
for $l \geq 1$.

**Case 2:** $l = 0$ or $-1$, $1 \leq i \leq n - 1$ and $j = 0$. From the first condition in the definition of the minimal left approximation, we have
\[
\text{Hom}^l_{D^b(\text{grmod } R^T)}(L_e \Lambda e(\Omega^i \Lambda (1 - e)(i) e), e\Lambda e) = 0,
\]
for any $i = 1, \ldots, n - 1$ and $l = 0$ or $-1$.

**Case 3:** $l = -1$, $1 \leq i \leq n - 1$ and $j = -1$. Since $P^{-i} e(i)$ is a direct sum of some summands of $\Lambda e(-1)$ and $\Lambda e(0)$,
\[
\text{Hom}^{-1}_{D^b(\text{grmod } R^T)}(L_e \Lambda e(\Omega^i \Lambda (1 - e)(i) e), e\Lambda e(-1)) = 0.
\]

**Case 4:** $l = 0$, $i = 1$ and $j = -1$. Similarly to the above case,
\[
\text{Hom}^0_{D^b(\text{grmod } R^T)}(\Omega^1 \Lambda (1 - e)(1) e, e\Lambda e(-1)) = 0.
\]

**Case 5:** $l = 0$, $2 \leq i \leq n - 1$ and $j = -1$. We have to show that, for $i \geq 2$,
\[
\text{Hom}^0_{D^b(\text{grmod } R^T)}(L_e \Lambda e(\Omega^i \Lambda (1 - e)(i) e), e\Lambda e(-1)) \cong \text{Hom}^0_{D^b(\text{grmod } R^T)}(\Omega^i \Lambda (1 - e)(i) e, e\Lambda e(-1)) = 0.
\]

We show the equality by contradiction. If there is a morphism
\[
0 \neq s \in \text{Hom}^0_{D^b(\text{grmod } R^T)}(\Omega^i \Lambda (1 - e)(i) e, e\Lambda e(-1)),
\]
then there is a morphism
\[
0 \neq \tilde{s} \in \text{Hom}^0_{D^b(\text{grmod } R^T)}(P^{-i} e(i) e, e\Lambda e(-1)),
\]
where $\tilde{s}$ is the composition of $s$ with the morphism $P^{-i} e(i) e \to \Omega^i \Lambda (1 - e)(i) e$ in $P^{-i}$. Since $P^{-i} e(i) e$ is the direct sum of some summands of $\Lambda e(-1)$ and $\Lambda e(0)$, there is a summand $e\Lambda e(-1)$ of $P^{-i} e(i) e$ such that the embedding
\[
eq 0 \Lambda e(-1) \hookrightarrow \Lambda e(-1) \hookrightarrow P^{-i} e(i) e
\]
composed with $\tilde{s}$ is the identity. Now fix the above summand $e\Lambda e(-1)$ of $P^{-i}(i)$. By Lemmas 3.5 and 3.7 we have
\[
\text{Hom}_{\text{grmod } R^T}(\Omega^i \Lambda (1 - e)(i) e, e\Lambda e(-1))
\]
\[ \cong \text{Hom}_{\text{tail}} R T (\Omega^1(1 - e)(i), e\Lambda e(-1)) \]
\[ \cong \text{Hom}_{\text{tail}} \Lambda (\Omega(1 - e)(i), e\Lambda(-1)) \]
\[ \cong \text{Hom}_{\text{grmod}} \Lambda (\Omega(1 - e)(i), e\Lambda(-1)). \]

Denote the above isomorphism by
\[ \kappa : \text{Hom}_{\text{grmod}} \Lambda (\Omega(1 - e)(i), e\Lambda(-1)) \cong \text{Hom}_{\text{grmod}} R T (e\Lambda e(-1), e\Lambda e(\Omega(1 - e)(i))). \]

By the same method, we also have the following two isomorphisms:
\[ \kappa' : \text{Hom}_{\text{grmod}} \Lambda \left( e\Lambda(i), e\Lambda(-1) \right) \cong \text{Hom}_{\text{grmod}} R T \left( e\Lambda e(-1), e\Lambda e(i) \right), \]
and
\[ \kappa'' : \text{Hom}_{\text{grmod}} \Lambda \left( e\Lambda(-1), e\Lambda(1 - e)(i) \right) \cong \text{Hom}_{\text{grmod}} R T \left( e\Lambda e(\Omega(1 - e)(i)), e\Lambda e(-1) \right). \]

Denote by \( p_i \) the composition of morphisms \( e\Lambda(i) \to P^i(i) \to \Omega(1 - e)(i) \) in \( \text{grmod}(\Lambda) \).

Then we have
\[ \kappa' (s \circ p_i) = \kappa (s) \circ \kappa'' (p_i) = \text{id}_{e\Lambda e(-1)}. \]

Since \( s \circ p_i \in \text{Hom}_{\text{grmod}\Lambda}(e\Lambda(1 - e)(i), e\Lambda(-1)) = e\Lambda e \), we have that \( s \circ p_i \) is equal to \( \text{id}_{\Lambda} \) up to a scalar. Thus \( e\Lambda(-1) \) is a summand of \( \Omega(1 - e)(i) \) and \( p_i \) is injective. However, this contradicts to the construction of the resolution \( P^* \) and the fact that
\[ \text{Hom}_{\mathcal{D}(\text{grmod}\Lambda)}(P^i, e\Lambda(i)) = 0 \]
for any \( i, j \in \mathbb{Z} \). Therefore we have that
\[ \text{Hom}_{\mathcal{D}(\text{grmod} R T)}(L_{e\Lambda e}(\Omega(1 - e)(i)), e\Lambda e(-1)) = 0. \]

The proof of Proposition 3.10 is now complete.

### 3.2 Ext group of the generator

In this subsection we study the Ext-groups of the generator discussed in the previous subsection. We start with the following lemma due to Minamoto and Mori.

**Lemma 3.16 (\[11, Proposition 4.4\]).** Let \( \Lambda \) be an AS-regular algebra of dimension \( d \geq 1 \) with Gorenstein parameter \( a \). Then we have
\[ \text{Hom}_{\mathcal{D}(\text{grmod})}^q(\Lambda(i), \Lambda(j + ma)) = 0 \]
for \( q \neq 0, 0 \leq i, j \leq a - 1 \) and \( m \geq 1 \).

With this lemma, we are ready to show the main result of this subsection.

**Proposition 3.17.** Let \( E_Q \) be given in Notation 3.6. With the setting of Proposition 3.10, we have
\[ \text{Hom}_{\mathcal{D}(\text{grmod} R T)^r}^r(E_Q, E_Q) = 0 \]
for any \( r \neq 0 \).
Proof. We divide the proof into three cases, namely when \( r \leq -2 \), \( r \geq 1 \) and \( r = -1 \) respectively.

**Case 1: \( r \leq -2 \).** By Lemmas 3.5 and 3.11 we have

\[
\text{Hom}^r_{D^b(R^r)}(E_Q, E_Q) \\
\cong \text{Hom}^r_{D^b(\text{tail } R^r)}(\Phi E_Q, \Phi E_Q) \\
\cong \text{End}^r_{D^b(\text{tail } R^r)}\left( \pi(L_{e\Lambda}(\bigoplus_{i=1}^{n-1} \Omega_i^1(1-e)(i) e \oplus (1-e)\Lambda e)) \right) \\
\cong \text{End}^r_{D^b(\text{tail } \Lambda)}\left( \pi(L_{e\Lambda}(\bigoplus_{i=1}^{n-1} \Omega_i^1(1-e)(i) e \oplus (1-e)\Lambda)) \right),
\]

where \( L_{e\Lambda}(\Omega_i^1(1-e)(i)) \) is the cone of the minimal left \( \text{add}(e\Lambda) \)-approximation of graded \( \Lambda \)-module \( \Omega_i^1(1-e)(i) \), and \( \text{add}(e\Lambda) \) is the subcategory of \( \text{grmod } \Lambda \) consisting of direct summands of finite direct sums of \( e\Lambda \). In fact, the minimal left \( \text{add}(e\Lambda) \)-approximation exists and is unique up to isomorphism for any graded \( \Lambda \)-module, since \( \Lambda \) is a graded locally finite-dimensional algebra with semi-simple \( A_0 \). Since

\[
\text{Hom}^j_{D^b(\text{grmod } \Lambda)}(P^*_{-i}(i), e\Lambda) = \text{Hom}^{j+i}_{D^b(\text{grmod } \Lambda)}(P^*(i), e\Lambda) = 0
\]

for any \( j \geq 1 \), there is a morphism \( q_{-i} : \Omega_i^1(1-e)(i) \to e\Lambda \oplus m_i \) for some \( m_i \in \mathbb{N} \) such that \( L_{e\Lambda}(\Omega_i^1(1-e)(i)) \) is equal to

\[
0 \to (1-e)\Lambda(-n) \to P^{1-n}(-n+1) \to \cdots \to \Omega_i^1(1-e) \xrightarrow{q_{-i}} e\Lambda \oplus m_i \to 0
\]

in \( D^b(\text{grmod } \Lambda) \), which we denote by \( \widetilde{P^*_{-i}} \).

By Lemmas 3.7 and 3.16 we have that

\[
\text{End}^r_{D^b(\text{tail } \Lambda)}\left( \pi(\widetilde{P^*_{-i}(i)}) \right) = \text{End}^r_{D^b(\text{grmod } \Lambda)}(\widetilde{P^*_{-i}(i)})
\]

for \( n-1 \geq i \geq 2 \) and therefore

\[
\text{End}^r_{D^b(\text{tail } \Lambda)}\left( \pi(\bigoplus_{i=2}^{n-1} \widetilde{P^*_{-i}(i)} \oplus \bigoplus_{i=0}^{1}(1-e)\Lambda(i)) \right) \\
= \text{End}^r_{D^b(\text{grmod } \Lambda)}\left( \bigoplus_{i=2}^{n-1} \widetilde{P^*_{-i}(i)} \oplus \bigoplus_{i=0}^{1}(1-e)\Lambda(i) \right).
\]

This equality implies that

\[
\text{End}^r_{D^b(\text{tail } \Lambda)}\left( \pi(L_{e\Lambda}(\bigoplus_{i=1}^{n-1} \Omega_i^1(1-e)(i) e \oplus (1-e)\Lambda)) \right) \\
= \text{End}^r_{D^b(\text{grmod } \Lambda)}\left( \bigoplus_{i=2}^{n-1} \widetilde{P^*_{-i}(i)} \oplus \bigoplus_{i=0}^{1}(1-e)\Lambda(i) \right)
\]

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since $\Omega^i_\Lambda(1 - e)(1) \cong (1 - e)\Lambda(1)$ in tail $\Lambda$. Thus we get

$$\text{End}_{D^b(\text{grmod } \Lambda)}\left( \bigoplus_{i=0}^{n-1} P^*_{-i}(i) \oplus \bigoplus_{i=0}^{1} (1 - e)\Lambda(i) \right) = 0$$

for $r \leq -2$ since the homologies of the complex $\bigoplus_{i=2}^{n-1} P^*_{-i}(i) \oplus \bigoplus_{i=0}^{1} (1 - e)\Lambda(i)$ are concentrated in degree 0 and 1.

**Case 2:** $r \geq 1$. We consider the following vector spaces

$$\text{Hom}_{D^b(\text{grmod } \Lambda)}\left( \bigoplus_{i=2}^{n-1} P^*_{-i}(i) \oplus \bigoplus_{i=0}^{1} (1 - e)\Lambda(i), \bigoplus_{i=2}^{n-1} P^*_{-i}(i) \oplus \bigoplus_{i=0}^{1} (1 - e)\Lambda(i)[r] \right).$$

By the fact that

$$\text{Hom}_{D^b(\text{grmod } \Lambda)}(1 - e, \Lambda(j)) = 0$$

for either $r \neq n - 1$ and $j \in \mathbb{Z}$, or $r = n - 1$ and $j \neq -n$, and by the first condition in the definition of minimal right add(\Lambda)-approximation over $\Lambda$, we have that these vector spaces

$$\text{Hom}_{D^b(\text{grmod } \Lambda)}\left( \bigoplus_{i=2}^{n-1} P^*_{-i}(i) \oplus \bigoplus_{i=0}^{1} (1 - e)\Lambda(i), \bigoplus_{i=2}^{n-1} P^*_{-i}(i) \oplus \bigoplus_{i=0}^{1} (1 - e)\Lambda(i)[r] \right)$$

are all trivial.

**Case 3:** $r = -1$. We need to show

$$\text{Hom}_{D^b(\text{grmod } \Lambda)}\left( \bigoplus_{i=2}^{n-1} P^*_{-i}(i) \oplus \bigoplus_{i=0}^{1} (1 - e)\Lambda(i), \bigoplus_{i=2}^{n-1} P^*_{-i}(i) \oplus \bigoplus_{i=0}^{1} (1 - e)\Lambda(i)[r] \right)$$

are all trivial.

First, since the homologies of the complex $\bigoplus_{i=2}^{n-1} P^*_{-i}(i) \oplus \bigoplus_{i=0}^{1} (1 - e)\Lambda(i)$ are concentrated in degree 0 and 1, we have

$$\text{Hom}_{D^b(\text{grmod } \Lambda)}\left( \bigoplus_{i=0}^{1} (1 - e)\Lambda(i), \bigoplus_{i=2}^{n-1} P^*_{-i}(i) \oplus \bigoplus_{i=0}^{1} (1 - e)\Lambda(i) \right) = 0.$$.

Second, consider the vector spaces

$$\text{Hom}_{D^b(\text{grmod } \Lambda)}(P^*_{-i}(i), P^*_{-j}(j)).$$

Observe that they are isomorphic to the vector spaces of the following morphisms of chain complexes

```
0 \rightarrow \Omega^i_\Lambda(1 - e)(i) \xrightarrow{q_i} e\Lambda \xrightarrow{e} 0 \\
\downarrow \quad \downarrow \quad \downarrow f \quad \downarrow \\
0 \rightarrow 0 \rightarrow \Omega^j_\Lambda(1 - e)(j) \xrightarrow{q_j} e\Lambda
```

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in \(C^b(\text{grmod } \Lambda)\), for any \(i, j = 2, \cdots, n - 1\). Furthermore, the above diagram is contained in the following commutative diagram of chain complexes

\[
\begin{array}{cccccc}
0 & \xrightarrow{P^{-i}(i)} & P^{-j+1}(j) & \xrightarrow{0} \\
0 & \xrightarrow{0} & 0 & \xrightarrow{e\Lambda^{\oplus m_i}} \xrightarrow{\Omega^1_i} 0 \\
\end{array}
\]

in \(C^b(\text{grmod } \Lambda)\), for any \(i, j = 2, \cdots, n - 1\), where \(\Omega^1_i\) is given by composing \(q_{-i}\) with

\[
P^{-i}(i) \rightarrow \Omega^1_i(1 - e)(i),
\]

and \(g\) is given by composing \(f\) with

\[
\Omega^1_i(1 - e)(j) \rightarrow P^{-j+1}(j).
\]

By definition of \(\Lambda\), the idempotent element \(e\) corresponds to the \(R^T\)-module \(M^0_{\varnothing}(V_i)\), where \(l\) is the minimal integer in \(\mathcal{L}\) in Proposition 2.14.

Without loss of generality, we choose two indecomposable summands \(e'\Lambda(-1) \in P^{-i}(i)\) and \(e''\Lambda(1) \in P^{-j+1}(j)\) respectively. By the argument after Lemma 2.14, we know that any morphism from \(e\Lambda^{\oplus m_i}\) to \(e''\Lambda(1)\) is given by some linear combination of \(\{x_{j_1}, x_{j_2}, \cdots, x_{j_s}\}\) which is the subset of \(\{x_1, x_2, \cdots, x_n\}\) consisting of elements whose weights are all negative. Similarly, any morphism from \(e'\Lambda(-1)\) to \(e\Lambda^{\oplus m_i}\) is given by some linear combinations of \(\{x_{i_1}, x_{i_2}, \cdots, x_{i_t}\}\) which is the subset of \(\{x_1, x_2, \cdots, x_n\}\) consisting of elements whose weights are all positive.

Thus we can write the morphism

\[
e\Lambda^{\oplus m_i} \xrightarrow{g} P^{-j+1}(j) \xrightarrow{e''\Lambda(1)}
\]
as \(\sum_{r=1}^s a^r_{j_r}x_{j_r}, \cdots, \sum_{r=1}^s a^{m_i}_r x_{j_r}\), where \(x_{j_r} \in \{x_{j_1}, x_{j_2}, \cdots, x_{j_s}\}\) and \(a^1_r, \cdots, a^{m_i}_r \in k\). In the same way, we can write the morphism

\[
e''\Lambda(-1) \xrightarrow{P^{-i}(i)} \xrightarrow{\hat{q}_{-i}} e\Lambda^{\oplus m_i}
\]
as \(\sum_{r=1}^t b^r_{i_r}x_{i_r}, \cdots, \sum_{r=1}^t b^{m_i}_r x_{i_r}\), where \(x_{i_r} \in \{x_{i_1}, x_{i_2}, \cdots, x_{i_t}\}\) and \(b^1_r, \cdots, b^{m_i}_r \in k\). Therefore we have

\[
\sum_{u=1}^{m_i} \left( \sum_{r=1}^s a^r_{j_r}x_{j_r} \right) \left( \sum_{r=1}^t b^u_r x_{i_r} \right) = 0
\]
in \(R\). Now the above equality can be written in the form

\[
\sum_{r=1}^{j_j} x_{j_r} f_{j_r} = 0,
\]

where \(f_{j_r}\) is a linear combination of \(\{x_{j_1}, x_{j_2}, \cdots, x_{j_t}\}\) for any \(j_r\). However, since \(x_{j_1} \notin \sum_{r=2}^{j_s} x_{j_r} f_{j_r}\), the above equality is impossible unless all \(f_{j_r} = 0\). In fact, we have

\[
x_{j_1} \notin x_{j_2}, \cdots, x_{j_s}, x_{i_1}, x_{i_2}, \cdots, x_{i_t}.
\]
Thus we obtain that
\[ \text{Hom}_{D^b(\text{grmod } \Lambda)}^{-1}(\widetilde{P}_{-i}^*(i), \widetilde{P}_{-j}^*(j)) = 0. \]

In the completely analogous way, we have that
\[ \text{Hom}_{D^b(\text{grmod } \Lambda)}^{-1}(\bigoplus_{i=2}^{n-1} \widetilde{P}_{-i}^*(i), \bigoplus_{i=0}^{1} (1 - e)\Lambda(i)) = 0, \]
and therefore we have
\[ \text{Hom}_{D^b(\text{grmod } \Lambda)}^{-1}(\bigoplus_{i=2}^{n-1} \widetilde{P}_{-i}^*(i) \oplus \bigoplus_{i=0}^{1} (1 - e)\Lambda(i)) = 0. \]

4 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. As stated in the theorem, the proof is divided into two cases: the first case is when \( G \) is the one dimensional torus \( T \), and the second case is when \( G \) is the product of \( T \) with a finite abelian group. We start with the following definition:

Definition 4.1 (Tilting object). Let \( \mathcal{C} \) be triangulated category. An object \( X \in \text{Ob}(\mathcal{C}) \) is said to be tilting if

(1) \( \mathcal{C} = \text{thick}(X) \), and

(2) \( \text{Hom}_\mathcal{C}(X, X[i]) = 0 \) for any \( i \neq 0 \).

An important property of tilting theory is the following (see [6] for more details). If \( X \) is a tilting object of an algebraic Krull-Schmidt triangulated \( \mathcal{C} \), then the following functor
\[ \mathcal{C} \to K^b(\text{proj}(\text{End}_\mathcal{C}(X))) : M \mapsto \text{Hom}_\mathcal{C}(X, M) \]
is an equivalence of triangulated categories.

4.1 Proof of Theorem 1.1: the first case

Proof of Theorem 1.1 (when \( G = T \)). By Propositions 3.10 and 3.17 \( E_Q \) is a tilting object in \( D^b_{gr}(R^T) \), from which the theorem follows. \( \square \)

Example 4.2. Let \( R = k[x_1, x_2, x_3, x_4] \). Assume the weights of the action of one dimensional tours \( T \) on \( R \) are \((1, 1, -1, -1)\). By Proposition 2.6 \( \mathcal{C} = \{0, 1\} \) if we choose \( e = \frac{1}{7} \), and then \( M = R^T \oplus M^L_k(V_1) \), and \( \Lambda = \text{End}_{R^T}(M) \). Moreover, \( \Lambda \) can be written as a quiver algebra \( kQ_\Lambda/I \), where \( Q_\Lambda \) is given as follows:

\[ \begin{array}{c}
\bullet 1 \\
\downarrow \quad \downarrow \\
\bullet 2 \\
\end{array} \quad \begin{array}{c}
\bullet x_4 \\
\uparrow \quad \uparrow \\
\bullet x_3 \\
\end{array} \quad \begin{array}{c}
\bullet x_1 \\
\downarrow \quad \downarrow \\
\bullet x_2 \\
\end{array} \]
Thus, from above projective resolution, we obtain the tilting object
\begin{equation}
x_3, x_4 \in \text{Hom}_{R^T}(R^T, M_R^T(V_1)) \cong \text{Hom}_{R,k}(V_0, R \otimes V_1) \subseteq R
\end{equation}
respectively, the arrows \( \bar{x}_1, \bar{x}_2 \) correspond to the morphisms
\begin{equation}
x_1, x_2 \in \text{Hom}_{R^T}(M_R^T(V_1), R^T) \cong \text{Hom}_{R,k}(V_1, R \otimes V_0) \subseteq R
\end{equation}
respectively, and \( I \) is generated by elements
\begin{equation}
(\bar{x}_1 \bar{x}_3 \bar{x}_2 - \bar{x}_2 \bar{x}_3 \bar{x}_1, \bar{x}_1 \bar{x}_4 \bar{x}_2 - \bar{x}_2 \bar{x}_4 \bar{x}_1, \bar{x}_3 \bar{x}_1 \bar{x}_4 - \bar{x}_4 \bar{x}_1 \bar{x}_3, \bar{x}_3 \bar{x}_2 \bar{x}_4 - \bar{x}_4 \bar{x}_2 \bar{x}_3).
\end{equation}
Now, we have a projective resolution of \( \Lambda_0 \cong k^2 \) as a graded \( \Lambda \)-module as follows:
\begin{equation}
\begin{array}{c}
0 \to \Lambda(-4) \xrightarrow{\varphi_3} \Lambda^{\oplus 4}(-3) \xrightarrow{\varphi_2} \Lambda^{\oplus 4}(-1) \xrightarrow{\varphi_1} \Lambda \xrightarrow{\varphi_0} \Lambda_0 \to 0
\end{array}
\end{equation}
where
(1) \( \varphi_0 \) is given by the canonical projection \( \Lambda = \bigoplus_i \Lambda_i \to \Lambda_0 \);
(2) \( \varphi_1 \) is given by
\begin{equation}
\varphi_1(a, b, c, d) = (\bar{x}_1 a + \bar{x}_2 b + \bar{x}_3 c + \bar{x}_4 d),
\end{equation}
for any \( (a, b, c, d) \in \Lambda^{\oplus 4}(-1) \);
(3) \( \varphi_2 \) is given by
\begin{equation}
\varphi_2(a, b, c, d) = (\bar{x}_4 \bar{x}_2 a - \bar{x}_3 \bar{x}_2 b, -\bar{x}_4 \bar{x}_1 a + \bar{x}_3 \bar{x}_1 b, \bar{x}_4 \bar{x}_2 c - \bar{x}_3 \bar{x}_2 d, -\bar{x}_4 \bar{x}_3 c + \bar{x}_2 \bar{x}_3 d),
\end{equation}
for any \( (a, b, c, d) \in \Lambda^{\oplus 4}(-3) \);
(4) \( \varphi_3 \) is given by
\begin{equation}
\varphi_3(1_\Lambda(-4)) = (\bar{x}_3(-4), \bar{x}_4(-4), \bar{x}_2(-4), \bar{x}_1(-4)) \in \Lambda^{\oplus 4}(-3),
\end{equation}
and \( 1_\Lambda \) is the identity element of \( \Lambda \).

Thus, from above projective resolution, we obtain the tilting object \( E_Q \) of \( D^b_{sg}(R^T) \) to be the direct sum of the following objects:
(1) \( (1 - e)\Lambda e = M_R^T(V_1) \);
(2) \( L_{e\Lambda e}(\Omega^1_\Lambda(1 - e) e(1)) = (1 - e)\ker(\varphi_1) e(1) = \bigoplus_{i \geq 1} (1 - e) \Lambda_i e \), where the first equality holds since we have
\begin{equation}
\text{Hom}_{D^b(\text{grmod}R^T)}(\Omega^1_\Lambda(1 - e) e(1), e\Lambda e) = 0,
\end{equation}
and \( \Omega^1_\Lambda(1 - e) e = (1 - e)\ker(\varphi_1) e \) in \( D^b(\text{tail}R^T) \);
(3) \( L_{e\Lambda e}(\Omega^2_{\Lambda}(1-e)e(2)) \), which is equal to the following complex:

\[(e\Lambda e)^{\mathbb{P}2}(1) \xrightarrow{\varphi_1} (1-e)\Lambda e(2) \xrightarrow{\varphi_0} k(2); \]

(4) \( L_{e\Lambda e}(\Omega^3_{\Lambda}(1-e)e(3)) \), which is equal to the following complex

\[(1-e)\Lambda e(-1) \xrightarrow{\varphi_3} e\Lambda e^{\mathbb{P}2}. \]

We therefore have

\[ \text{End}_{Dgr\operatorname{gr}(R_T)}(E_Q) = \operatorname{diag}(k,k,k,k) \sim k^{\oplus 4}. \]

4.2 Proof of Theorem 1.1: the second case

By [17, Lemma 3.15], an NCCR can be constructed for the case where \( G \) is not a connected group if an NCCR exists for the connected component \( G_0 \) containing the identity element. Based on this lemma, we construct an NCCR of \( R_G \) as follows. Let

\[ \Lambda := \text{End}_{R_T}\left( \bigoplus_{\chi \in \mathcal{L}} M^T_R(V_{\chi}) \right) \cong \text{End}_{(R,G)}\left( \bigoplus_{\chi \in \mathcal{L}} R \otimes V_{\chi} \right) \]

be the NCCR of \( R^T \) constructed in \$3\$, where \( T \subset G \) is the maximal one-dimensional torus containing the identity element in \( G \) and \( H := G/T \) is a finite abelian group in \( \text{SL}(N,k) \). Set

\[ \Lambda' := \text{End}_{R^G}\left( \bigoplus_{\chi \in \mathcal{L}} M^G_R(U_{\chi}) \right) \cong \text{End}_{(R,G)}\left( \bigoplus_{\chi \in \mathcal{L}} R \otimes U_{\chi} \right), \]

which is an NCCR of \( R^G \), where \( U_{\chi} := \text{Ind}^G_T(V_{\chi}) \). We have

\[ \Lambda' \cong M^G_R\left( \text{End}_{(k,G)}\left( \text{Ind}^G_T\left( \bigoplus_{\chi \in \mathcal{L}} V_{\chi} \right) \right) \right). \]

For any \( \chi, \chi' \in \mathcal{L} \), we have that

\[ \text{Hom}_{k,G}(U_{\chi}, U_{\chi'}) \cong \text{Hom}_{(k,G)}\left( \text{Ind}^G_T(V_{\chi}), \text{Ind}^G_T(V_{\chi'}) \right) \cong \text{Hom}_{(k,T)}(\text{Ind}^T_T(V_{\chi}), V_{\chi'}), \]

where \( \text{Ind}^G_T(V_{\chi}) \) is considered as a \( T \) representation in the last term of above equality. At the same time, the \( T \) representation \( \text{Ind}^G_T(V_{\chi}) \) is isomorphic to \( kH \otimes V_{\chi} \), whose \( T \)-action is induced from \( V_{\chi} \). Thus we have that

\[ \Lambda' \cong kH \otimes \Lambda, \]

and its multiplicative structure is given by those of \( kH \) and \( \Lambda \).

Next, we endow \( \Lambda' \) with a canonical grading from \( R \) such that \( \deg(h \otimes 1) = 0 \) for any \( h \in H \). Then we obtain the following two propositions and a lemma, whose proofs are completely analogous to the case when \( G \) is the one-dimensional torus, and hence are omitted:

**Proposition 4.3** (Compare with Proposition 2.12). \( R^G \) is a Noetherian graded Gorenstein algebra of dimension \( n - 1 \) with Gorenstein parameter \( n \).
Proposition 4.4 (Compare with Proposition 2.13). \( \Lambda' \) is an AS-regular algebra of dimension \( n - 1 \) with Gorenstein parameter \( n \).

Lemma 4.5 (Compare with Lemma 3.5). Let \( R = k[x_1, x_2, \ldots , x_n] \), and \( T \) be the maximal one-dimensional torus in \( G \) such that \( T \) contains the identity element of \( G \), which acts on \( R \) with weights \( \chi = (\chi_1, \chi_2, \ldots , \chi_n) \). If \( R^G \) satisfies the first and second conditions in Definition 2.4, and is a graded Gorenstein algebra with grading endowed canonically, then the following are equivalent:

1. \( \chi \) is effective;
2. the functor \( (-)^{e'} : \text{tail} \Lambda \to \text{tail} R^G \) is an equivalence functor with inverse \( (-) \otimes e' \Lambda', e' \Lambda', \) where \( e' := (\frac{1}{|H|} \sum_{h \in H} h) \otimes e; \)
3. \( R^G \) is graded isolated singularity;
4. \( \text{Spec}(R^G) \) has unique isolated singularity at the origin.

By Proposition 4.4, we have an object 
\[
E_Q' = \nu \left( L_{e'} \Lambda' \left( \bigoplus_{i=1}^{n-1} \Omega_{\Lambda'}^{1 - e'}(i) e' \otimes (1 - e') \Lambda' e' \right) \right)
\]
in \( D_{gr}^{sg}(R^G) \).

By Propositions 4.3 and 4.4 and Lemma 4.5 we obtain the following two propositions, whose proofs are again analogous and hence are omitted:

Proposition 4.6 (Compare with Proposition 3.10). With the setting of Theorem 1.1 in the second case, we have 
\[
D_{gr}^{sg}(R^G) = \text{thick}(E_Q').
\]
In other words, \( E_Q' \) as an object of \( D_{gr}^{sg}(R^G) \) generates \( D_{gr}^{sg}(R^G) \) itself.

Proposition 4.7 (Compare with Proposition 3.17). With the setting of Proposition 4.6, we have 
\[
\text{Hom}_{D_{gr}^{sg}(R^G)}^{r}(E_Q', E_Q') = 0
\]
for any \( r \neq 0 \).

Proof of Theorem 1.1 continued (when \( G \) is the product of \( T \) with a finite abelian group). The proof follows from the combination of Propositions 4.6 and 4.7. \( \square \)

Remark 4.8. The above method of constructing the tilting object does not work for the case when \( G \) contains a two or higher dimensional torus.

In fact, our method uses \cite{17} Lemma 1.19], which assumes that the action of the connected subgroup \( G_0 \) of \( G \) on \( R \) is quasi-symmetric. Now suppose \( R^{G_0} \) is an isolated singularity. Without loss of generality, we assume that \( G_0 \) is an \( m \)-dimensional torus and the weights of its action on \( R \) are \( v_1, v_2, \ldots , v_m \), each of which is in \( \mathbb{Z}^m \). Similarly to Lemma 3.3, the
isolatedness of the singularity of $R_{G_0}$ implies that, if there is a sequence of natural numbers $c_1, c_2, \ldots, c_r$ and a subset $\{u_1, u_2, \ldots, u_r\} \subseteq \{v_1, v_2, \ldots, v_n\} \subset X(T)$ such that
\[ \sum_i c_i u_i = 0, \]
then $\{u_1, u_2, \ldots, u_r\}$ spans $\mathbb{Z}^m$ as a $\mathbb{Z}$-module.

Since the action of $G_0$ on $R$ is quasi-symmetric, there is a line $\ell \subset \mathbb{Z}^m$ through the origin such that
\[ \{\beta_1, \beta_2, \ldots, \beta_t\} := \{v_1, v_2, \ldots, v_n\} \cap \ell \neq \emptyset \]
and $\sum_{\beta_i \in \ell} \beta_i = 0$. Thus, $\{\beta_1, \beta_2, \ldots, \beta_t\}$ spans $\mathbb{Z}^m$ as a $\mathbb{Z}$-module. But this is impossible since $\{\beta_1, \beta_2, \ldots, \beta_t\}$ is contained in a line which goes through the origin.

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