On Possibility of Extending the Optimal Replacement time of Series and Parallel Systems

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Abstract

Among all systems, the series system has the lowest optimal replacement time, while the parallel system has the highest optimal replacement time. This paper is comparing the standard age replacement strategy (SARS) with some proposed replacement strategies (strategy A and strategy B) for two multi-unit systems. Two numerical examples are provided for a simple illustration of the proposed replacement cost models under SARS, strategies A and B. The results obtained showed that strategy B can extend the optimal replacement time of a series system.

Keywords: Failure, Optimal, Strategy, System, Unit.

1. Introduction

The failure of an operating unit or system might sometimes be costly, dangerous, negatively affect revenue, the production of defective items, or causes a delay in customer services. It is an important problem to determine when best to preventively replace or maintain an operating unit or system before failure. Under the age replacement policy, the series system has the lowest optimum replacement time, while the parallel system has the highest optimum replacement time. As the series system is the having the lowest optimum replacement time, this may lead to the early replacement of a series system.

There is extensive literature on age replacement models with minimal repair. Cha and Finkelstein [1] introduced a new type of minimal repair to be called conditional statistical minimal repair, and their approach goes further and deals with the corresponding minimal repair processes for systems operating in a random environment. Chang [2] considered a system that suffers one of two types of failures based on a specific random mechanism. Chang and Chen [3] discussed that effective replacement policies should be collaborative once gathering data from the time of operations, mission durations, minimal repairs, and maintenance triggering approaches. Coria et al. [4] proposed an analytical optimization method for preventive maintenance replacement cost rate. Fallahnejhad and Najafian [5] studied the best time for performing preventive maintenance operations for many systems. Gheisary and Goli [6] investigated an efficient method to compute the exact reliability for a multi-state system by using the distribution of bivariate order statistics. Based on the continuous-time Markov theory, Huang and Wang [7] construct a time-replacement policy for multistate systems with aging multistate components so as determine the optimal time to replace the entire system. Jain and Gupta [8] studied optimal replacement policy for a repairable system with multiple vacations and imperfect coverage. Enogwe et al. [9] applied the knowledge of probability distribution of failure times and proposed a replacement model for items that fail suddenly. Lim et al. [10] presented some age replacement policies in which a system is replaced by a new one at the planned age and when a failure occurs before the planned replacement age, it can be either perfectly repaired with random probability $p$ or minimally repaired with random probability $1-p$. Liu et al. [11] established uncertain reliability mathematical models of simple repairable series systems, simple repairable parallel systems, simple repairable series-parallel systems, and simple repairable parallel-series systems, respectively. Malki et al. [12] presented some age replacement policies for a parallel system with
stochastic dependence. Mirjalili and Kazempoor [13] investigated three replacement policies including cold standby and minimal repair policies for a system consisting of independent components with increasing failure rate functions. Murthy and Hwang [14] discussed that, in a probabilistic sense, failures can be reduced through effective maintenance actions, and such maintenance actions can occur either at discrete time instants or continuously over time. Nakagawa [15] presented a modified standard age replacement (SAR) model to a discrete-time age replacement model. Nakagawa et al. [16] presented the advantages of some proposed replacement policies. In an approach for analyzing the behavior of an industrial system under the cost-free warranty policy, Niwas and Garg [17] developed a mathematical model of a system based on the Markov process, they also derived various parameters such as reliability, mean time to system failure, availability and expected profit for the system. Rebaiaia and Ait-kadi [18] presented the problem of selecting the best among three maintenance strategies for conducting maintenance planning that is the most economical. Safaei et al. [19] investigated the optimal period for preventive maintenance and the best decision for repair or replacement in terms of some measures. Safaei et al. [20] used the copula framework and present two optimal age replacement policies based on the minimum expected cost function and maximum availability function for series or parallel systems with dependent components. Sanoubar et al. [21] considered an age-replacement policy (without minimal repair) under which the system is replaced at failure or at a prescribed replacement time, whichever occurs first, where it is assumed that replacement costs are non-decreasing in system age. Sheu et al. [22] presented preventive replacement models for a system subjected to shocks that arrive according to a non-homogeneous Poisson process, such that when a shock takes place, the system is either replaced by a new one (type 2 failure) or minimally repaired (type 1 failure). Sudheesh et al. [23] considered the discrete age-replacement model, and then studied the properties of mean time to failure of a system. Tsoukalas and Agrafiotis [24] introduced a new replacement policy for a system with correlated failure and usage time. Waziri et al. [25] explored some characteristics of an age replacement model with minimal repair for a series-parallel system with six units, such that the six units are having non-uniform failure rates. Waziri [26] presented a discrete scheduled replacement model with the discounting rate for a unit that is subjected to three categories of failures. Furthermore, Waziri and Yusuf [27] presented a discrete scheduled replacement cost model for a multi-component system that is subjected to two categories of failures. Wu et al. [28] proposed a new replacement policy and established corresponding replacement models for a deteriorating repairable system with multiple vacations of one repairman. Xie et al. [29] analyzed the impacts of cascading failures on the reliability of series-parallel systems, where they studied the effects of safety barriers on preventing occurring failures. Yaun and Xu [30] studies a cold standby repairable system with two different components and one repairman taking multiple vacations. Yusuf and Ali [31] constructed an age replacement cost model for a parallel system with units under some assumptions. Zhao et al. [32] investigated the problem of which replacement is better between continuous and discrete scheduled replacement times. Zhao et al. [33] developed some analytic replacement cost rates under two proposed policies considering random mission durations time, to avoid preventive replacement during the mission period.

The literature review presented in this paper did not capture a method or strategy for extending the optimal replacement time of a multi-component system. This paper will come up with some replacement cost models under some proposed strategies, to see the possibility of extending the optimum replacement time of series and parallel systems, and this will be achieved through the following objectives:

1. By constructing an age replacement cost model for series and parallel systems under the standard age replacement strategy (SARS).
2. By constructing age replacement cost models for series and parallel systems under two proposed strategies.
3. By providing some two numerical examples for a simple illustration of the constructed replacement cost models.

2. Notations and systems description

2.1 Notations

- \( r_i(t) \) Type I failure rate of unit \( A_i \) for \( i = 1, 2, 3, 4, 5, 6 \)
- \( r'_i(t) \) Type II failure rate of unit \( A_i \) for \( i = 1, 2, 3, 4, 5, 6 \)
- \( R^*_i(t) \) Reliability function of Type II failure of unit \( A_i \) for \( i = 1, 2, 3, 4, 5, 6 \)
- \( SARS \) Standard age replacement strategy
- \( CS_i(T) \) Cost rate of system \( S_i \) under SARS, for \( i = 1, 2 \)
- \( CY_i(T) \) Cost rate of system \( S_i \) under strategy \( A_i \), for \( i = 1, 2 \)
- \( CZ_i(T) \) Cost rate of system \( S_i \) under strategy \( B_i \), for \( i = 1, 2 \)
- \( X^*_i \) Optimal replacement time of system \( S_i \) under SARS, for \( i = 1, 2 \)
- \( Y^*_i \) Optimal replacement time of system \( S_i \) under strategy \( A_i \), for \( i = 1, 2 \)
- \( Z^*_i \) Optimal replacement time of system \( S_i \) under strategy \( B_i \), for \( i = 1, 2 \)
- \( C_{ir} \) Cost of minimal repair of failed unit \( A_i \) due to Type II failure, for \( i = 1, 2, 3, 4, 5, 6 \)
- \( C_{im} \) Cost of minimal repair of failed unit \( A_i \) due to Type II failure, for \( i = 1, 2, 3, 4, 5, 6 \).
Consider six units $A_1, A_2, A_3, A_4, A_5,$ and $A_6,$ arranged to form two different systems, series system ($S_1$) and parallel system ($S_2$). It is assumed that all the six units are subjected to Type I and Type II failures, such that, Type I failure is a repairable one, while Type II failure is a non-repairable failure. Now, since all the six units are subjected to Type I and Type II failures, then we can say that all the two systems are also subjected to Type I and Type II failures. See the Figures 1 and 2 for the diagram of the two systems ($S_1$ and $S_2$).

![Figure 1. Reliability block diagram of system $S_1$](Image)

![Figure 2. Reliability block diagram of system $S_2$](Image)

### 3. Formulation of cost model under SARS

#### 3.1 Some assumptions under SARS

1. If a system fails due to Type I failure, then the system is minimally repaired.
2. If a system fails due to Type II failure, then the whole system is replaced completely with a new one.
3. Both the two types of failures for the six units arrive according to a non-homogeneous Poisson process.
4. The rate of Type II failure follows the order: $r_4^*(t) \geq r_5^*(t) \geq r_6^*(t)$.
5. The rate of Type I failure follows the order: $r_1(t) \geq r_2(t) \geq r_3(t) \geq r_5(t) \geq r_6(t)$.
6. A system is replaced at a planned time $T(T > 0)$ after its installation or at Type II failure, whichever arrives first.
7. The cost of the planned replacement of a system is less than the cost of un-planned replacement.
8. The cost of repair of a failed unit is less than the cost of un-planned replacement.
9. All costs are positive numbers.

The probability that system $S_1$ will be replaced at planned replacement time $T$, before Type II failure occurs, is

$$R_{T_1}^*(T) = R_1^*(T)R_2^*(T)R_3^*(T)R_4^*(T)R_5^*(T)R_6^*(T).$$

(1)

The probability that system $S_2$ will be replaced at planned replacement time $T$, before Type II failure occurs, is

$$R_{T_2}^*(T) = 1 - \int_{T}^{\infty} (1 - R_{T_2}^*(T)) dt.$$  

(2)

In the meantime systems of $S_1$ and $S_2$ under SARS, is

$$\text{Mean time} = \int_{0}^{T} R_{T_2}^*(t) dt, \text{ for } i = 1, 2.$$  

(3)

The cost of un-planned replacement (failure due to Type II failure) of $S_1$ and $S_2$ in one replacement cycle, is

$$\text{Cost of unplanned replacement} = C_{sr}(1 - R_{T_i}^*(T)), \text{ for } i = 1, 2.$$  

(4)

The cost of planned replacement at time $T$ of $S_1$ and $S_2$ in one replacement cycle, is

$$\text{Cost of planned replacement} = C_{sp}R_{T_i}^*(T), \text{ for } i = 1, 2.$$  

(5)

The cost of minimal repair of units $A_1, A_2, A_3, A_4, A_5,$ and $A_6$ due to Type I failure in one replacement cycle, is

$$\text{Cost of repair} = \int_{0}^{T} C_{1m}r_1(t)R_{T_1}^*(t)dt + \int_{0}^{T} C_{2m}r_2(t)R_{T_1}^*(t)dt + \int_{0}^{T} C_{3m}r_3(t)R_{T_1}^*(t)dt + \int_{0}^{T} C_{4m}r_4(t)R_{T_1}^*(t)dt + \int_{0}^{T} C_{5m}r_5(t)R_{T_1}^*(t)dt + \int_{0}^{T} C_{6m}r_6(t)R_{T_1}^*(t)dt.$$  

(6)

The replacement cost rate of $S_1$ and $S_2$ under SARS, is

$$C_{sr}(T) = \frac{C_{sr}(1 - R_{T_i}^*(T)) + C_{sp}R_{T_i}^*(T) + \int_{0}^{T} f(t)R_{T_i}^*(t)dt}{\int_{0}^{T} R_{T_i}^*(t)dt}, \text{ for } i = 1, 2.$$  

(7)

where

$$f(t) = C_{1m}r_1(t) + C_{2m}r_2(t) + C_{3m}r_3(t) + C_{4m}r_4(t) + C_{5m}r_5(t) + C_{6m}r_6(t).$$  

(8)

### 4. Formulation of cost model under strategy A

Strategy A is a preventive maintenance strategy, in which the un-planned replacement of a whole system depends on the failure of units $A_1, A_3,$ and $A_5$ due to Type II. Noting that, the reliability function of a system due to strategy A, depends on the location of units.
A₁, A₃ and A₅ in a system. But when any of the units A₂, A₄ or A₆ fails due to Type II failure, the failed unit is replaced completely with a new one and allows the system to continue operating from where it stopped.

Under strategy A, we have the following reliability functions:

1. System S₁: the system is replaced completely with a new one when at least one of the units A₁, A₃ or A₅ fails due to Type II failure. Now, the probability that system S₁ will be replaced at planned replacement time T, before Type II failure occurs under strategy A, is

\[ R_{S₁}^{A}(T) = R_1(T)R_2^A(T)R_3^A(T) \] (9)

2. System S₂: the system is replaced completely with a new one when all the three units A₁, A₃ and A₅ fails due to Type II failure. Now, the probability that system S₂ will be replaced at planned replacement time T, before Type II failure occurs under strategy A, is

\[ R_{S₂}^{A}(T) = 1 - (1 - R_1(T))(1 - R_2^A(T))(1 - R_3^A(T)) \] (10)

In the meantime systems of S₁ and S₂ in one replacement cycle under strategy A, is

Mean time = \[ \int \frac{R_{S₁}^{A}(t)}{R_{S₁}^{A}(t)} dt \], for \( i = 1, 2 \). (11)

The cost of un-planned replacement (failure due to Type II failure) of S₁ and S₂ in one replacement cycle, is

Cost of unplanned replacement = \[ C_{nr}(1 - R_{S₁}^{A}(T)) \], for \( i = 1, 2 \). (12)

The cost of planned replacement at time T of S₁ and S₂ in one replacement cycle, is

Cost of planned replacement = \[ C_{pr}R_{S₁}^{A}(T) \], for \( i = 1, 2 \). (13)

The cost of minimal repair of units A₁, A₂, A₃, A₄, A₅ and A₆ due to Type I failure in one replacement cycle, is

Cost of repair = \[ - \int_0^T \left[ (C_{1m}r_1(t)R_{S₁}^A)^n(a*n) + (C_{2m}r_2(t)R_{S₁}^A)^p + (C_{3m}r_3(t)R_{S₁}^A)^q \right] dt \] + \[ \int_0^T C_{2m}r_2(t)R_{S₁}^{A}(t)dt + \int_0^T C_{3m}r_3(t)R_{S₁}^{A}(t)dt \] + \[ \int_0^T C_{4m}r_4(t)R_{S₁}^{A}(t)dt + \int_0^T C_{5m}r_5(t)R_{S₁}^{A}(t)dt \] + \[ \int_0^T C_{6m}r_6(t)R_{S₁}^{A}(t)dt \] (14)

The cost of replacement of units A₂, A₄ and A₆ due to Type II failure in one replacement cycle, is

Cost of replacement = \[ \int_0^T C_{2r_2^A}(t)R_{S₁}^{A}(t)dt + \int_0^T C_{4r_4^A}(t)R_{S₁}^{A}(t)dt + \int_0^T C_{6r_6^A}(t)R_{S₁}^{A}(t)dt \] (15)

The replacement cost rate of S₁ and S₂ under strategy A, is

\[ \text{Cost}(T) = \text{Cost}(T) + \text{Cost}(T) + \int_0^T \text{Cost}(t)dt \] (16)

for \( i = 1, 2 \), where

\( K(t) = C_{1m}r_1(t) + C_{2m}r_2(t) + C_{3m}r_3(t) + C_{4m}r_4(t) + C_{5m}r_5(t) + C_{6m}r_6(t) \),

and

\( L(t) = C_{2r_2^A}(t) + C_{4r_4^A}(t) + C_{6r_6^A}(t) \). (18)

5. Formulation of cost model under strategy B

Strategy B is a preventive maintenance strategy, in which the un-planned replacement of a whole system depends on the failure of units A₂, A₄ and A₆ due to Type II. Noting that, the reliability function of a system due to strategy B, depends on the location of units A₂, A₄ and A₆ in a system. But when any of the units A₂, A₃ or A₅ fails due to Type II failure, the failed units are replaced completely with new ones and allow the system to continue operating from where it stopped. Under strategy B, we have the following reliability functions:

1. System S₁: the system is replaced completely with a new one when at least one of the units A₂, A₄ or A₆ fails due to Type II failure. Now, the probability that system S₁ will be replaced at planned replacement time T, before Type II failure occurs under strategy B, is

\[ R_{S₁}^{B}(T) = R_2(T)R_4^B(T)R_6^B(T) \] (19)

2. System S₂: the system is replaced completely with a new one when all the three units A₂, A₄ or A₆ fails due to Type II failure. Now, the probability that system S₂ will be replaced at planned replacement time T, before Type II failure occurs under strategy B, is

\[ R_{S₂}^{B}(T) = 1 - (1 - R_2(T))(1 - R_4^B(T))(1 - R_6^B(T)) \] (20)

The mean time of systems of S₁ and S₂ in one replacement cycle under strategy B, is

Mean time = \[ \int \frac{R_{S₁}^{B}(t)}{R_{S₁}^{B}(t)} dt \], for \( i = 1, 2 \). (21)

The cost of un-planned replacement (failure due to Type II failure) of S₁ and S₂ in one replacement cycle, is

Cost of unplanned replacement = \[ C_{nr}(1 - R_{S₁}^{B}(T)) \], for \( i = 1, 2 \). (22)

The cost of planned replacement at time T of S₁ and S₂ in one replacement cycle, is

Cost of planned replacement = \[ C_{pr}R_{S₁}^{B}(T) \], for \( i = 1, 2 \). (23)

The cost of minimal repair of units A₁, A₂, A₃, A₄, A₅ and A₆ due to Type I failure in one replacement cycle, is

Cost of repair = \[ \int_0^T C_{1m}r_1(t)R_{S₁}^{B}(t)dt \] + \[ \int_0^T C_{2m}r_2(t)R_{S₁}^{B}(t)dt + \int_0^T C_{3m}r_3(t)R_{S₁}^{B}(t)dt \] + \[ \int_0^T C_{4m}r_4(t)R_{S₁}^{B}(t)dt + \int_0^T C_{5m}r_5(t)R_{S₁}^{B}(t)dt \] + \[ \int_0^T C_{6m}r_6(t)R_{S₁}^{B}(t)dt \] (24)

The cost of replacement of units A₂, A₄ and A₆ due to Type II failure in one replacement cycle, is

Cost of replacement = \[ \int_0^T C_{2r_2^B}(t)R_{S₁}^{B}(t)dt \] + \[ \int_0^T C_{4r_4^B}(t)R_{S₁}^{B}(t)dt \] + \[ \int_0^T C_{6r_6^B}(t)R_{S₁}^{B}(t)dt \] (25)

The replacement cost rate of systems S₁ and S₂ under strategy B, is

\[ \text{Cost}(T) = \text{Cost}(T) + \text{Cost}(T) + \int_0^T \text{Cost}(t)dt \] (26)

for \( i = 1, 2 \).
where
\[ M(t) = C_{1m}r_1^*(t) + C_{2m}r_2^*(t) + C_{3m}r_3^*(t) + C_{4m}r_4^*(t) + C_{5m}r_5^*(t) + C_{6m}r_6^*(t). \] (27)

and
\[ N(t) = C_{1r}r_1^r(t) + C_{3r}r_3^r(t) + C_{5r}r_5^r(t) \] (28)

5.1 Numerical examples

In this section, two numerical examples were provided to illustrate the characteristics of the proposed replacement cost models constructed above.

Example 1

Let the failure time of Type I failure for the six units follows the Weibull distribution:
\[ r_i^*(t) = \lambda_i \alpha_i t^{\alpha_i - 1}, \quad \text{for } i = 1, 2, 3, 4, 5, 6, \] (29)

where \( \alpha_i > 1 \) and \( t \geq 0 \).

Also, let the failure time of Type II failure for the six units follows the Weibull distribution:
\[ r_i^r(t) = \lambda_i^r \alpha_i^r t^{\alpha_i^r - 1}, \quad \text{for } i = 1, 2, 3, 4, 5, 6, \] (30)

where \( \alpha_i^r > 1 \) and \( t \geq 0 \).

Let the set of parameters and cost of repair/replacement be used throughout this particular example:

1. \( \alpha_1 = 4, \alpha_2 = 3, \alpha_3 = 3, \alpha_4 = 3, \alpha_5 = 4 \) and \( \alpha_6 = 2 \).
2. \( \lambda_1 = 0.03, \lambda_2 = 0.002, \lambda_3 = 0.03, \lambda_4 = 0.001 \) \( \lambda_2 = 0.001 \) and \( \lambda_6 = 0.001 \).
3. \( \alpha_1^r = 4, \alpha_2^r = 3.5, \alpha_3^r = 4, \alpha_4^r = 3.5, \alpha_5^r = 4, \) and \( \alpha_6^r = 3.5 \).
4. \( \lambda_1^r = 0.00033, \lambda_2^r = 0.00025, \lambda_3^r = 0.00030, \lambda_4^r = 0.00023, \lambda_5^r = 0.00025 \) \( \lambda_6^r = 0.0002 \).
5. \( C_{sr} = 70, C_{sp} = 45 \) and \( C_{im} = 0.4 \), for \( i = 1, 2, 3, 4, 5, 6 \).

By substituting the parameters of equations (29) and (30), the rates of Type I and Type II failures are obtained as follows:

\[ r_1(t) = 0.12t^3. \] (31)
\[ r_2(t) = 0.06t. \] (32)
\[ r_3(t) = 0.09t^2. \] (33)
\[ r_4(t) = 0.003t^3. \] (34)
\[ r_5(t) = 0.004t^3. \] (35)
\[ r_6(t) = 0.002t. \] (36)
\[ r_1^r(t) = 0.00132t^3. \] (37)
\[ r_2^r(t) = 0.00875t^{2.5}. \] (38)
\[ r_3^r(t) = 0.00012t^3. \] (39)
\[ r_4^r(t) = 0.000805t^{2.5}. \] (40)
\[ r_5^r(t) = 0.001t^4. \] (41)
\[ r_6^r(t) = 0.0007t^{2.5}. \] (42)

The tables below in this example are obtained by substituting all the rates of the two failures (Type I and Type II failures), and costs of replacement and repair in equations (7), (16), and (26).

Table 1. Results obtained from evaluating the replacement cost rates of systems \( S_1 \) and \( S_2 \) under SARS.

| \( T \) | \( CS_1(T) \) | \( CS_2(T) \) |
|---|---|---|
| 1 | 240.42 | 240.04 |
| 2 | 122.77 | 120.16 |
| 3 | 88.58 | 80.36 |
| 4 | 79.17 | 60.69 |
| 5 | 82.73 | 49.16 |
| 6 | 91.47 | 41.80 |
| 7 | 94.78 | 36.93 |
| 8 | 95.87 | 33.87 |
| 9 | 97.93 | 32.50 |
| 10 | 99.00 | 32.92 |
| 11 | 99.52 | 34.91 |
| 12 | 100.22 | 37.87 |

Table 2. Results obtained from evaluating the replacement cost rates of systems \( S_1 \) and \( S_2 \) under strategy A.

| \( T \) | \( CYS_1(T) \) | \( CYS_2(T) \) |
|---|---|---|
| 1 | 240.78 | 240.57 |
| 2 | 122.87 | 121.18 |
| 3 | 87.58 | 81.96 |
| 4 | 76.01 | 62.89 |
| 5 | 76.62 | 52.13 |
| 6 | 83.71 | 46.27 |
| 7 | 89.44 | 44.88 |
| 8 | 90.24 | 48.34 |
| 9 | 92.73 | 55.49 |
| 10 | 93.99 | 60.79 |
| 11 | 95.45 | 62.77 |
| 12 | 98.00 | 64.57 |

Table 3. Results obtained from evaluating the replacement cost rates of systems \( S_1 \) and \( S_2 \) under strategy B.

| \( T \) | \( CZS_1(T) \) | \( CZS_2(T) \) |
|---|---|---|
| 1 | 240.80 | 240.64 |
| 2 | 122.54 | 121.62 |
| 3 | 85.61 | 83.11 |
| 4 | 70.10 | 65.11 |
| 5 | 63.97 | 55.61 |
| 6 | 62.95 | 50.64 |
| 7 | 64.67 | 49.55 |
| 8 | 67.21 | 48.55 |
| 9 | 67.67 | 50.20 |
| 10 | 68.74 | 53.05 |
| 11 | 70.46 | 56.43 |
| 12 | 72.97 | 59.34 |

Table 4. The optimal replacement times of systems \( S_1 \) and \( S_2 \) under SARS, strategies A and B from Tables 1, 2 and 3.

| System | SARS | Strategy A | Strategy B |
|---|---|---|---|
| \( S_1 \) | \( X_{S_1}^{\star} = 4.00 \) | \( Y_{S_1}^{\star} = 4.00 \) | \( Z_{S_1}^{\star} = 6.00 \) |
| \( S_2 \) | \( X_{S_2}^{\star} = 9.00 \) | \( Y_{S_2}^{\star} = 7.00 \) | \( Z_{S_2}^{\star} = 8.00 \) |
Some observations of results obtained from example 1 are as follows:

1. From Table 4, observe that, the optimal replacement time of the system $S_1$ obtained under strategy B is higher than that of SARS and strategy A, while the optimal replacement time of system $S_2$ obtained under SARS is higher than that of strategies A and B.

2. From Figure 3, observe that, the cost rate of the system $S_1$ obtained under strategy B is lower than that of SARS and strategy A.

3. From Figure 4, observe that, the cost rate of the system $S_2$ obtained under SARS is lower than that of strategies A and B.

**Example 2**

Let the failure time of Type I failure for the six units follows Power law distribution:

$$ r_i(t) = \lambda_i \propto (\lambda_i t)^{\alpha_i}, \text{for } i = 1, 2, 3, 4, 5, 6, \quad (43) $$

where $\alpha_i > 1$ and $t \geq 0$.

Also, let the failure time of Type II failure for the six units follows Power law distribution:

$$ r'_i(t) = \lambda'_i \propto (\lambda'_i t)^{\alpha'_i}, \text{for } i = 1, 2, 3, 4, 5, 6, \quad (44) $$

where $\alpha'_i > 1$ and $t \geq 0$.

The cost and the parameters of both Type I and Type II failures in example 1 were adopted. Similarly, the tables below in this example are obtained by substituting all the failure rates of Type I, Type II, and costs of replacement and repair in equations (7), (16), and (26).

**Table 5.** Results obtained from evaluating the replacement cost rates of systems $S_1$ and $S_2$ under SARS.

| T   | $C_{S_1}(T)$ | $C_{S_2}(T)$ |
|-----|--------------|--------------|
| 10  | 224.67       | 224.29       |
| 20  | 107.02       | 104.41       |
| 30  | 72.83        | 64.61        |
| 40  | 63.42        | 44.94        |
| 50  | 66.98        | 33.41        |
| 60  | 75.72        | 26.05        |
| 70  | 79.03        | 21.18        |
| 80  | 80.12        | 18.12        |
| 90  | 82.18        | 16.75        |
| 100 | 83.25        | 17.17        |
| 110 | 83.77        | 19.16        |
| 120 | 84.47        | 22.12        |

**Table 6.** Results obtained from evaluating the replacement cost rates of systems $S_1$ and $S_2$ under strategy A.

| T   | $C_{S_1}(T)$ | $C_{S_2}(T)$ |
|-----|--------------|--------------|
| 10  | 225.03       | 224.82       |
| 20  | 107.12       | 105.43       |
| 30  | 71.83        | 66.21        |
| 40  | 60.26        | 47.14        |
| 50  | 60.87        | 36.38        |
| 60  | 67.96        | 30.52        |
| 70  | 73.69        | 29.13        |
| 80  | 74.49        | 32.59        |
| 90  | 76.98        | 39.74        |
| 100 | 78.24        | 46.04        |
| 110 | 79.7         | 47.02        |
| 120 | 82.25        | 48.82        |

**Table 7.** Results obtained from evaluating the replacement cost rates of systems $S_1$ and $S_2$ under strategy B.

| T   | $C_{S_1}(T)$ | $C_{S_2}(T)$ |
|-----|--------------|--------------|
| 10  | 225.05       | 240.64       |
| 20  | 106.79       | 121.62       |
| 30  | 69.86        | 83.11        |
| 40  | 54.35        | 65.11        |
| 50  | 48.22        | 55.61        |
| 60  | 47.20        | 50.64        |
| 70  | 48.92        | 48.55        |
| 80  | 51.46        | 48.00        |
| 90  | 51.92        | 50.20        |
| 100 | 52.99        | 53.05        |
| 110 | 54.71        | 56.43        |
| 120 | 57.22        | 59.34        |

**Table 8.** The optimal replacement times of systems $S_1$ and $S_2$ under SARS, strategy A and strategy B from tables 5, 6 and 7.

| System | SARS | Strategy A | Strategy B |
|--------|------|------------|------------|
| $S_1$  | $X_{S_1} = 40.00$ | $Y_{S_1} = 40.00$ | $Z_{S_1} = 60.00$ |
| $S_2$  | $X_{S_2} = 90.00$ | $Y_{S_2} = 70.00$ | $Z_{S_2} = 80.00$ |
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6. General observation of results

From the results obtained from examples 1 and 2, one can clearly see that strategy B can extend the optimal replacement time of the series system, while it cannot extend the optimal replacement time of the parallel system. In terms of the cost rate, the results showed that the cost rate of the series system is lower than that of SARS and strategy A, while the cost rate of the parallel system under SARS is lower than that of the strategies A and B.

7. Significance of results

From the results obtained in this research, one can clearly see that the strategy B is a good preventive maintenance plan for maintaining series multi-unit systems because strategy B has the following advantages over SARS and strategy A:

1. The optimal replacement time of the series system obtained under strategy B has a higher optimal replacement time than that of SARS and strategy A. Thus, this will reduce the chances of early replacement of the series systems at an early stage.
2. The cost of maintenance of the series system under strategy B is lower than that of SARS and strategy A.

Also, from the other way round of the findings, maintenance managers and plant management are advised to adopt SARS as a good preventive maintenance strategy for maintaining the parallel multi-unit system, because preventive maintenance under SARS, has the following advantages over strategies A and B:

1. The optimal replacement time of a parallel system obtained under SARS has a higher optimal replacement time than that of strategies A and B.
2. The cost of maintenance of the parallel system under SARS is lower than that of strategies A and B.

One can relate the findings of these results obtained to real life, one can use the results to select the best strategy for maintaining the following:

1. Series and parallel configurations of a combined heat and power (CHP) plant coupled to thermal networks.
2. Subsystems of industrial plants.
3. Subsystems of air crafts

8. Summary and conclusion

This research covered the age replacement policy with the concept of repair at failure. In trying to explore some possible ways of extending the optimal replacement time of some multi-component systems, this paper presented some proposed age replacement cost models under standard age replacement strategy (SARS), strategy A and strategy B for series and parallel systems. It is assumed that the two systems are subjected to Type I and Type II failures. Below are the tables that compare the three proposed strategies.
The following cases:

1. By applying the proposed strategy A and strategy B, to series-parallel and parallel-series systems, to see the possible extension of their optimal replacement time.

2. By considering periodic replacement time $NT (N = 1, 2, 3, ...)$ for a fixed T, to see the possible extension of the optimal replacement time of multi-component systems.

3. By in-cooperating a warranty period in the proposed replacement models under strategies A and B.

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Table 9. Comparing SARS, Strategy A and Strategy B for system $S_1$.

| SARS | If all the six units fails due Type II failure, then replace the whole system |
|------|--------------------------------------------------------------------------|
|      | If all the six units fails due Type I failure, then repair the failed units, minimally. |
|      | If at least one of $A_1, A_2$ and $A_3$ fails due Type II failure, then replace the whole system |
|      | If at least one of $A_4, A_5$, and $A_6$ fails due Type II failure, then replace the failed unit(s) |
|      | If all the six units fails due Type I failure, then repair the failed unit, minimally. |

| Strategy A | If all the six units fails due Type I failure, then repair the failed units, minimally. |
|------------|--------------------------------------------------------------------------------------|
| If all the six units fails due Type I failure, then repair the failed units, minimally. |
| If at least one of $A_1, A_2$ and $A_3$ fails due Type II failure, then replace the whole system |
| If at least one of $A_4, A_5$, and $A_6$ fails due Type II failure, then replace the failed unit(s) |
| If all the six units fails due Type I failure, then repair the failed unit, minimally. |

| Strategy B | If all the six units fails due Type II failure, then replace the whole system |
|------------|--------------------------------------------------------------------------------|
| If all the six units fails due Type II failure, then replace the whole system |
| If all the six units fails due Type II failure, then replace the failed unit(s) |
| If all the six units fails due Type I failure, then repair the failed units, minimally. |

Table 10. Comparing SARS, Strategy A and Strategy B for system $S_2$.

| SARS | If all the six units fails due Type II failure, then replace the whole system |
|------|--------------------------------------------------------------------------------|
|      | If all the six units fails due Type I failure, then repair the failed units, minimally. |
|      | If all the six units fails due Type II failure, then replace the whole system |
|      | If all the six units fails due Type I failure, then repair the failed unit, minimally. |

| Strategy A | If all the six units fails due Type I failure, then repair the failed units, minimally. |
|------------|--------------------------------------------------------------------------------------|
| If all the three units $A_1, A_2$, and $A_3$ fails due Type II failure, then replace the whole system |
| If all the three units $A_4, A_5$, and $A_6$ fails due Type II failure, then replace the failed unit |
| If all the six units fails due Type I failure, then repair the failed units, minimally. |

| Strategy B | If all the three units $A_1, A_2$, and $A_3$ fails due Type II failure, then replace the whole system |
|------------|--------------------------------------------------------------------------------------|
| If all the three units $A_4, A_5$, and $A_6$ fails due Type II failure, then replace the failed unit(s) |
| If all the six units fails due Type I failure, then repair the failed units, minimally. |
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