1. Introduction

Hilbert schemes of points have a rich literature in algebraic geometry, commutative algebra, combinatorics, representation theory, and approximation theory. Various aspects of them have been studied in many contexts. In this paper we study the local equations and the singularities of $\text{Hilb}^n(\mathbb{C}^d)$. For a general introduction to the field, see [17, Chapter 18].

In [11], Haiman proved the remarkable result that the isospectral Hilbert scheme of points in the plane is normal, Cohen-Macaulay and Gorenstein. He also showed that this implies the $n!$ conjecture and the positivity conjecture for the Kostka-Macdonald coefficients. In addition, he conjectured that the isospectral Hilbert scheme over the principal component of $\text{Hilb}^n(\mathbb{C}^d)$ is Cohen-Macaulay for any $d, n \geq 1$. In particular, his conjecture implies that the principal component of $\text{Hilb}^n(\mathbb{C}^d)$ is Cohen-Macaulay (see [11, Section 5.2] and [17, Conjecture 18.38]).

We provide a counterexample to the conjecture. The idea is to look at the local neighborhood near $m_2$ on the principal component of $\text{Hilb}^9(\mathbb{C}^8)$, which is an affine cone over a certain projective variety. We will see that its local equations contain generators of high degree. Then the geometry of the projective variety implies that its affine cone is not Cohen-Macaulay. Our main result is the following:

**Theorem A.** The principal component of $\text{Hilb}^9(\mathbb{C}^8)$ is not locally Cohen-Macaulay at $m_2$.

Vakil showed that a number of important moduli spaces satisfy Murphy’s law, and many others studied badly-behaved moduli spaces of positive-dimensional objects (see [21] and the references therein). However very little is known about how bad the singularities of the Hilbert scheme of points on a smooth variety of dimension $> 2$ can be. On the other hand, Haiman [12, Proposition 2.6 and Remark.(2) in p.213] showed that a certain blow-up of $\text{Sym}^n(\mathbb{C}^d)$ is the principal component of $\text{Hilb}^n(\mathbb{C}^d)$, and Ekedahl and Skjelnes [7] generalized it to the case of quasi-projective schemes. If $d = 2$ then the blow-up is a resolution of singularities, but Theorem A implies that if $d, n \gg 0$ then the blow-up destroys the Cohen-Macaulayness of $\text{Sym}^n(\mathbb{C}^d)$.

Turning to a more detailed description, we consider the Hilbert scheme $\text{Hilb}^{d+1}(\mathbb{C}^d)$ of $(d+1)$ points in affine $d$-space $\mathbb{C}^d$, because it contains the squares $m^2$ of maximal ideals. It parameterizes the ideals $I$ of colength $(d+1)$ in $\mathbb{C}[x] = \mathbb{C}[x_1, ..., x_d]$. 

Let \( V_d \subset \text{Hilb}^{d+1}(\mathbb{C}^d) \) denote the affine open subscheme consisting of all ideals \( I \in \text{Hilb}^{d+1}(\mathbb{C}^d) \) such that \( \{1, x_1, \ldots, x_d\} \) is a \( \mathbb{C} \)-basis of \( \mathbb{C}[x]/I \). We will call \( V_d \) the symmetric affine subscheme. We note that the square of any maximal ideal in \( \mathbb{C}[x] \) belongs to the symmetric affine subscheme. One may think of \( V_d \) as a deformation space of \( m^2 \).

The following proposition is probably well-known to experts [10, Section 6], [13].

**Proposition 1.** Let \( d \geq 2 \). Let \( V_d \) be the symmetric affine open subscheme of \( \text{Hilb}^{d+1}(\mathbb{C}^d) \). Then \( V_d \) is isomorphic to

\[
\mathbb{C}^d \times \text{Spec}(R_d/I_d),
\]

where \( R_d \) is a \( d((d+1)/2) - 1 \)-dimensional polynomial ring and \( I_d \) is a homogeneous ideal generated by certain quadratic polynomials. (When \( d = 2 \), \( I_2 \) is the zero ideal \((0)\).)

More precisely, since \( V_d \) admits a natural action of \( GL(d) \), we can describe the quotient ring in terms of Schur functors.

**Theorem 2.** Let \( d \geq 3 \). Then \( V_d \) is isomorphic to

\[
\mathbb{C}^d \times \text{Spec} \left( \text{Sym}^*(S_{(3,1,1,\ldots,1,0)} W) \right)_{<S_{(4,3,2,\ldots,2,1)} W>},
\]

where \( W \) is a \( d \)-dimensional \( \mathbb{C} \)-vector space, \((3,1,1,\ldots,1,0)\) is a partition of \((d+1)\) and \((4,3,2,\ldots,2,1)\) is of \((2d+2)\).

If \( d \leq 6 \) then \( V_d \) is irreducible [8], [18], [3]. However if \( d \geq 7 \) then \( V_d \) is reducible, and there is a distinguished component called a principal component. For any \( d \), let \( P_d \) denote the principal component of \( V_d \). Here we regard it as its reduced structure. The general elements in \( P_d \) are radical ideals defining \((d+1)\) distinct points whose linear span is non-degenerate, i.e. there is no hyperplane passing through them in \( \mathbb{C}^d \). The most special elements in \( P_d \) are \( m^2 \). It is clear that the dimension of the principal component is \( d(d+1) \).

Let \( J_d \) denote the defining ideal of \( P_d \), in other words,

\[ P_d \cong \mathbb{C}^d \times \text{Spec}(R_d/J_d), \]

where \( J_d \) is a reduced homogeneous ideal.

There has been some interest in trying to find the equations \( P_d \) satisfy (e.g. [17, Problem 18.40], [20, Remark 3.4]). But up to now they have not been known to satisfy any other equations, besides the quadratic Plücker relations. We present some new equations and obtain the following result.

**Theorem 3.** Let \( d = 8 \) and let \( P_8 \) be the principal component of \( V_8 \). Then \( P_8 \) is isomorphic to

\[ \mathbb{C}^8 \times \text{Spec}(R_8/J_8), \]

where \( R_8 \) is a \( 8((8)/2)-1 \)-dimensional polynomial ring and a set of the minimal homogeneous generators of \( J_8 \) contains certain polynomials of degree 90. In particular, the Castelnuovo-Mumford regularity of \( J_8 \) is \( \geq 90 \), while the dimension of \( \text{Proj}(R_8/J_8) \) is 63.

\[ ^1 \text{It is not known whether } \text{Hilb}^n(\mathbb{C}^d) \text{ is reduced or not, for } d \geq 3. \]
Again more precisely,

**Proposition 4.** The principal component \( P_8 \) is isomorphic to
\[
\mathbb{C}^8 \times \text{Spec} \frac{\text{Sym}^\cdot(S_{(3,1,1,\ldots,1,0)}W)}{J_8},
\]
where the vector space of the minimal homogeneous generators of \( J_8 \) contains
\[
S_{(133,130,126,122,60,60,60)}W.
\]

**Proof of Theorem 3.** Together with the following lemma and proposition, Theorem 3 implies that the principal component \( P_8 \) is not Cohen-Macaulay. More concretely, if \( P_8 \) were Cohen-Macaulay, then \( \text{Proj}(R_8/J_8) \) would be arithmetically Cohen-Macaulay, but then Lemma 5 and Proposition 6 would imply \( \text{reg}(\text{Proj}(R_8/J_8)) \leq 64 \), which would contradict Theorem 3.

**Lemma 5.** Let \( S \subset \mathbb{P}^N \) be a projective arithmetically Cohen-Macaulay variety of dimension \( n \). Suppose that there is a smooth open set \( \tilde{U} \subset S \) such that

\* \( \text{codim}_S(S \setminus \tilde{U}) \geq 2 \), and

\* \( \tilde{U} \) is covered by rational proper curves, i.e., for any point \( x \in \tilde{U} \), there is a smooth irreducible rational proper curve on \( \tilde{U} \) passing through \( x \).

Then \( \text{reg}(S) \leq n + 1 \).

**Proposition 6.** Let \( X = \text{Proj}(R_d/J_d) \) for \( 2 \leq d \leq 8 \). Then there is a smooth open set \( \tilde{U}_d \subset X \) such that

\* \( \text{codim}_X(X \setminus \tilde{U}_d) = 2 \), and

\* \( \tilde{U}_d \) is covered by rational proper curves.

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2. **Local equations of the Hilbert scheme of points**

In this section we prove Theorem 2. In fact the defining ideal of \( V_d \) will be obtained by very concrete computations.

Before we begin the proof, let us explain the notation more precisely. By Lemma 14 there is an injective homomorphism
\[
j : S_{(4,3,2,\ldots,2,1)}^W \hookrightarrow \text{Sym}^2(S_{(3,1,1,\ldots,1,0)}^W)
\]
of Schur modules. Then \( j \) induces natural maps
\[
S_{(4,3,2,\ldots,2,1)}^W \otimes \text{Sym}^{r-2}(S_{(3,1,1,\ldots,1,0)}^W) \hookrightarrow \text{Sym}^r(S_{(3,1,1,\ldots,1,1,0)}^W) \otimes \text{Sym}^{r-2}(S_{(3,1,1,\ldots,1,0)}^W) \to \text{Sym}^r(S_{(3,1,1,\ldots,1,0)}^W), \quad r \geq 2,
\]
which define the quotient ring $\frac{\text{Sym}^* (\mathbb{C}[x_1, \ldots, x_d])}{\text{Sym}^* (\mathbb{C}[x_1, \ldots, x_d])}$.

To ease notations and references, we introduce the notion of ideal projectors (cf. [1], [5], [6], [20]).

**Definition 7.** (cf. [1]) A linear idempotent map $P$ on $\mathbb{C}[x]$ is called an **ideal projector** if $\ker P$ is an ideal in $\mathbb{C}[x]$.

We will use de Boor’s formula:

**Theorem 8.** ([5], de Boor) A linear mapping $P : \mathbb{C}[x] \to \mathbb{C}[x]$ is an ideal projector if and only if the equality

\begin{equation}
(2.1) \quad P(gh) = P(gP(h))
\end{equation}

holds for all $g, h \in \mathbb{C}[x]$.

Let $\mathcal{P}$ be the space of ideal projectors onto span $\{1, x_1, \ldots, x_d\}$, in other words,

$$
\mathcal{P} := \{ P : \text{ideal projector} \mid \ker P \in V_d \}.
$$

The space $\mathcal{P}$ is isomorphic to the symmetric affine subscheme $V_d$ [19, p3]. For the sake of simplicity, we prefer to work on $\mathcal{P}$ in place of $V_d$.

First we consider the natural embedding of $\mathcal{P}$. Gustavsen, Laksov and Skjelnes [10] gave more general description of open affine coverings of Hilbert schemes of points.

**Lemma 9.** The space $\mathcal{P}$ can be embedded into $\mathbb{C}^{(d+1)(\frac{d+1}{2})}$.

**Sketch of proof.** For each ideal projector $P \in \mathcal{P}$ and each pair $(i, j)$, $1 \leq i, j \leq d$, there is a collection $p_{0,ij}, p_{1,ij}, \ldots, p_{d,ij}$ of complex numbers such that

\begin{equation}
(2.2) \quad P(x_ix_j) = p_{0,ij} + \sum_{m=1}^{d} p_{m,ij}x_m.
\end{equation}

As $(i, j)$ varies over $1 \leq i, j \leq d$, each ideal projector $P \in \mathcal{P}$ gives rise to a collection $p_{0,ij}, p_{r,st}$ ($1 \leq i, j, r, s, t \leq d$) of complex numbers. Of course $p_{0,ij} = p_{0,ji}$ and $p_{r,st} = p_{r,ts}$. So we have a map $f : \mathcal{P} \to \mathbb{C}^{(d+1)(\frac{d+1}{2})}$.

Here we only show that $f$ is one-to-one. It is proved in [10] that $f$ is in fact a scheme-theoretic embedding.

We will show that if $P_1, P_2 \in \mathcal{P}$ and if $f(P_1) = f(P_2)$, i.e. $P_1(x_ix_j) = P_2(x_ix_j)$ for every $(i, j), 1 \leq i, j \leq d$, then $P_1 = P_2$. Since $P_1$ and $P_2$ are linear maps, it is enough to check that $P_1(x_{i_1} \ldots x_{i_r}) = P_2(x_{i_1} \ldots x_{i_r})$ for any monomial $x_{i_1} \ldots x_{i_r}$. This follows from de Boor’s formula (2.1):

$$
P_1(x_{i_1} \ldots x_{i_r}) = P_1(x_{i_1} P_1(x_{i_2} \ldots P_1(x_{i_{r-1}, i_r}) \cdots))
= P_2(x_{i_1} P_2(x_{i_2} \ldots P_2(x_{i_{r-1}, i_r}) \cdots)) = P_2(x_{i_1} \ldots x_{i_r}),
$$

where we have used the property that $P(g)$ is a linear combination of $1, x_1, \ldots, x_d$ for any $g \in \mathbb{C}[x]$. $\square$
Next we describe the ideal defining $\mathcal{P}$ in
\[
\mathbb{C}[p_{0,ij}, p_{r,st}|_{1 \leq i,j,r,s,t \leq d}] / \langle p_{0,ij} - p_{0,ji}, p_{r,st} - p_{r,ts} \rangle =: R,
\]
where we keep the notations in the above proof. Let $I_{\mathcal{P}}$ denote the ideal.

**Lemma 10.** Let $C(a; j, (i,k)) \in R$ denote the coefficient of $x_a$ in
\[
P(x_k P(x_i x_j)) - P(x_i P(x_k x_j)) \in R[x_1, \ldots, x_d].
\]
Then $I_{\mathcal{P}}$ is generated by $C(a; j, (i,k))$’s $(0 \leq a \leq d, 1 \leq i,j,k \leq d)$. (We regard an element in $R[x_1, \ldots, x_d]_0 \cong R$ as a coefficient of $x_0$.)

For example, if $a \neq j, i, k$ then
\[
(2.3) \quad C(a; j, (i,k)) = \sum_{m=1}^{d} (p_{m,ij}p_{a,km} - p_{m,kj}p_{a,im}).
\]
If $a = k$ then
\[
(2.4) \quad C(k; j, (i,k)) = p_{0,ij} + \sum_{m=1}^{d} (p_{m,ij}p_{k,km} - p_{m,kj}p_{k,im}).
\]

**Proof of Lemma 10.** The de Boor’s formula (2.1) implies that $I_{\mathcal{P}}$ is generated by coefficients of $x_a$’s $(1 \leq a \leq d)$ in $P(gP(h)) - P(hP(g))$ (all $g, h \in \mathbb{C}[x]$). But any $P(gP(h)) - P(hP(g))$ can be generated by $P(x_k P(x_i x_j)) - P(x_i P(x_k x_j))$’s.

□

We note that $C(a; j, (i,k)) + C(a; j, (k,i)) = 0$ so from now on we identify $C(a; j, (i,k))$ with $-C(a; j, (k,i))$.

**Lemma 11.** In fact, $I_{\mathcal{P}}$ is generated by $C(a; j, (i,k))$’s $(1 \leq a \leq d, 1 \leq i,j,k \leq d)$.

**Proof.** It is enough to prove that for any $1 \leq i,j,k \leq d$, the polynomial $C(0; j, (i,k))$ is generated by $C(a; b, (e,f))$’s $(1 \leq a,b,e,f \leq d)$. Fix any $u, 1 \leq u \leq d$. Then we have
\[
C(0; j, (i,k)) = \sum_{m=1}^{d} (p_{m,ij}p_{0,km} - p_{m,kj}p_{0,im})
\]
\[
= - \sum_{m=1}^{d} \left( p_{m,ij} \sum_{t=1}^{d} (p_{t,km}p_{u,tu} - p_{t,ku}p_{u,tm}) - p_{m,kj} \sum_{t=1}^{d} (p_{t,im}p_{u,tu} - p_{t,nu}p_{u,tm}) \right)
\]
\[
+ \sum_{m=1}^{d} \left( p_{m,ij}C(u; k, (m, u)) - p_{m,kj}C(u; i, (m, u)) \right)
\]
\[
= - \sum_{t=1}^{d} \left( p_{u,tu} \sum_{m=1}^{d} (p_{m,ij}p_{t,km} - p_{m,kj}p_{t,im}) \right. \\
\left. - p_{t,ku} \sum_{m=1}^{d} (p_{m,ij}p_{u,tm} - p_{m,ij}p_{u,tm}) + p_{t,iu} \sum_{m=1}^{d} (p_{m,km}p_{u,im} - p_{m,km}p_{u,im}) \right) \\
+ \sum_{m=1}^{d} p_{u,jm} \sum_{t=1}^{d} (p_{t,ku}p_{m,it} - p_{t,iu}p_{m,kt}) \\
+ \sum_{m=1}^{d} \left( p_{m,ij}C(u; k, (m, u)) - p_{m,kj}C(u; i, (m, u)) \right) \\
= - \sum_{t=1}^{d} \left( p_{u,ta}C(t; j, (i, k)) - p_{t,ku}C(u; i, (j, k)) + p_{t,ku}C(u; k, (j, t)) \right) \\
+ \sum_{m=1}^{d} p_{u,jm}C(m; u, (k, i)) \\
+ \sum_{m=1}^{d} \left( p_{m,ij}C(u; k, (m, u)) - p_{m,kj}C(u; i, (m, u)) \right).
\]

So the set of generators of \( I_P \) is
\[
\{ C(a; j, (i, k)) \mid 1 \leq a, i, j, k \leq d \}.
\]

We associate to this a representation of \( GL(W) \), where \( W \) is a \( d \)-dimensional vector space.

**Proposition 12.** The \( \mathbb{C} \)-vector space \( Y \) of generators
\[
\begin{align*}
\langle C(a; j, (i, k)) \mid 1 \leq a, i, j, k \leq d \rangle & > \\
C(a; j, (i, k)) + C(a; j, (k, i))
\end{align*}
\]

is canonically isomorphic to
\[
S_{(3,2,1,\ldots,1,0)}W \bigoplus S_{(3,1,1,\ldots,1,1)}W
\]
as \( \mathbb{C} \)-vector spaces, where \( W \) is a \( d \)-dimensional vector space and \( S_{(3,2,1,\ldots,1,0)} \) (resp. \( S_{(3,1,1,\ldots,1,1)} \)) is the Schur functor corresponding to the partition \( (3, 2, 1, \ldots, 1, 0) \) (resp. \( (3, 1, 1, \ldots, 1, 1) \)) of \( d + 2 \).

**Proof.** Let \( W = \bigoplus_{i=1}^{d} \mathbb{C}v_i \). Define
\[
\varphi : Y \rightarrow \bigwedge^{d-1} W \otimes W \otimes \bigwedge^2 W
\]
by
\[
\varphi : C(a; j, (i, k)) \mapsto (-1)^a(v_1 \wedge \cdots \wedge \hat{v}_a \wedge \cdots \wedge v_d) \otimes v_j \otimes (v_i \wedge v_k).
\]
Then it is clear that \( \varphi \) is injective.
By Littlewood-Richardson rule, we have

\[ d - 1 \bigwedge W \otimes W \otimes \bigwedge^2 W \]

\[ \cong S_{(1,1,1,\ldots,1,0)} W \otimes W \otimes S_{(1,1,0,\ldots,0,0)} W \]

\[ \cong S_{(3,2,1,\ldots,1,1)} W \bigoplus (S_{(2,2,1,\ldots,1,1,1)} W) \bigoplus S_{(2,2,2,1,\ldots,1,0)} W \]

\[ \cong S_{(3,2,1,\ldots,1,1,1)} W \bigoplus \bigwedge^d W \otimes \bigwedge^2 W \bigoplus \bigwedge^{d-1} W \otimes \bigwedge^3 W, \]

where each partition is of \((d + 2)\). We will show that the image of any element of \(Y\) under \(\varphi\) lies neither on \(\bigwedge^d W \otimes \bigwedge^2 W\) nor \(\bigwedge^{d-1} W \otimes \bigwedge^3 W\).

Since

\[ \sum_{j=1}^{d} (-1)^j (v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_d) \otimes v_j \otimes (v_i \wedge v_k), \quad 1 \leq i < k \leq d, \]

generate \(\bigwedge^d W \otimes \bigwedge^2 W\), we need to show that

\[ (2.5) \quad \sum_{j=1}^{d} C(j; j, (i, k)) = 0. \]

But this is elementary because

\[ \sum_{j=1}^{d} C(j; j, (i, k)) = \sum_{j=1}^{d} \sum_{m=1}^{d} (p_m,ijp_j,km - p_m,kjp_j,im) = 0. \]

Since

\[ (v_1 \wedge \cdots \wedge \hat{v}_a \wedge \cdots \wedge v_d) \otimes v_j \otimes (v_i \wedge v_k) \]

\[ + (v_1 \wedge \cdots \wedge \hat{v}_a \wedge \cdots \wedge v_d) \otimes v_k \otimes (v_j \wedge v_i) \]

\[ + (v_1 \wedge \cdots \wedge \hat{v}_a \wedge \cdots \wedge v_d) \otimes v_i \otimes (v_k \wedge v_j), \quad 1 \leq a \leq d, \quad 1 \leq j < i < k \leq d, \]

generate \(\bigwedge^{d-1} W \otimes \bigwedge^3 W\), we need to show that

\[ (2.6) \quad C(a; j, (i, k)) + C(a; k, (j, i)) + C(a; i, (k, j)) = 0. \]

But this is again elementary because

\[ \sum_{m=1}^{d} (p_m,ijp_a,km - p_m,kjp_a,im) \]

\[ + \sum_{m=1}^{d} (p_m,jkp_a,im - p_m,ikp_a,jm) \]

\[ + \sum_{m=1}^{d} (p_m,kip_a,jm - p_m,jip_a,km) = 0. \]
Therefore $\varphi(Y) \subset S_{(3,2,1,\cdots,1,0)} W \bigoplus S_{(3,1,1,\cdots,1,1)} W$, in other words,

$$\varphi : Y \rightarrow S_{(3,2,1,\cdots,1,0)} W \bigoplus S_{(3,1,1,\cdots,1,1)} W$$

is injective.

The next lemma completes the proof.

**Lemma 13.** $\varphi : Y \rightarrow S_{(3,2,1,\cdots,1,0)} W \bigoplus S_{(3,1,1,\cdots,1,1)} W$ is surjective.

*Proof.* It is enough to show that there are no nontrivial $\mathbb{C}$-linear relations among $C(a; j, (i, k))$’s other than $\mathbb{C}$-linear combinations of (2.5) and (2.6).

Suppose

$$C(a; j, (i, k)) + \sum_{u, b, e, f} c_{u,b,e,f} C(u; b, (e, f)) = 0, \quad c_{u,b,e,f} \in \mathbb{C}. \quad (2.7)$$

If $a \neq i, j, k$ then $C(a; j, (i, k))$ contains a term $p_{m,ij}p_{a,km}$ and a term $p_{m,kj}p_{a,im}$. The term $p_{m,ij}p_{a,km}$ appears only in $C(a; j, (i, k))$ and $C(a; i, (k, j))$ among all $C(u; b, (e, f))$, $1 \leq u, b, e, f \leq d$. Similarly the term $p_{m,kj}p_{a,im}$ appears only in $C(a; j, (i, k))$ and $C(a; k, (j, i))$. So the left hand side of (2.7) must be a nontrivial linear combination of (2.5) and (2.6). Repeating the argument, (2.7) becomes a linear combination of (2.5) and (2.6).

Similarly even if $a = i, j, k$, each term in $C(a; j, (i, k))$ appears only in the ones involved in (2.5) or (2.6). To get cancelation among these, the left hand side of (2.7) must contain (2.5) or (2.6). Repeating the argument, (2.7) becomes a linear combination of (2.5) and (2.6). \hfill \square

The following decomposition of Schur functors will be used later.

**Lemma 14.** We have

$$\bigwedge^{d-1} W \otimes \text{Sym}^2 W \cong S_{(2,1,1,\cdots,1,1)} W \bigoplus S_{(3,1,1,\cdots,1,0)} W,$$

and

$$\text{Sym}^2 (S_{(3,1,1,\cdots,1,0)} W) \cong S_{(6,2,2,\cdots,2,0)} W \bigoplus S_{(5,3,2,\cdots,1,1)} W \bigoplus S_{(5,2,2,\cdots,2,1)} W \bigoplus S_{(4,4,2,\cdots,2,0)} W \bigoplus S_{(4,3,2,\cdots,2,1)} W \bigoplus S_{(4,2,2,\cdots,2,2)} W.$$

(If $d = 3$ then $S_{(5,3,2,\cdots,1,1)} W$ does not appear.)

*Proof.* The first isomorphism follows from the Littlewood-Richardson rule. The second isomorphism can be calculated by [4, pp.124–128]. \hfill \square

**Lemma 15.** There is an injective homomorphism

$$j : S_{(4,3,2,\cdots,2,1)} W \hookrightarrow \text{Sym}^2 \left( \bigwedge^{d-1} W \otimes \text{Sym}^2 W \right)$$

such that $\mathcal{P}$ (hence the symmetric affine open subscheme $V_{\mathcal{P}}$) is isomorphic to

$$\text{Spec} \frac{\text{Sym}^* (\bigwedge^{d-1} W \otimes \text{Sym}^2 W)}{S_{(4,3,2,\cdots,2,1)} W},$$
where \((4, 3, 2, \cdots, 2, 1)\) is a partition of \((2d+2)\).

**Proof.** Consider a diagram

\[
\begin{array}{ccc}
\mathbb{C}[p'_{0,ij} \mid 1 \leq i, j, r, s, t \leq d] / (p_{0,ij} - p_{0,j1}, p_{r,st} - p_{r,ts}) & \xrightarrow{f} & \mathbb{C}[p'_{0,ij} \mid 1 \leq i, j, r, s, t \leq d] / (p_{0,ij} - p_{0,j1}, p_{r,st} - p_{r,ts}) =: R \\
g \downarrow & & \downarrow \\
T := \mathbb{C}[p'_{r,st} \mid 1 \leq r, s, t \leq d] / (p_{r,st} - p_{r,ts}) & & \\
\end{array}
\]

where \(g\) is the natural projection and \(f^{-1}\) is defined by

\[
p'_{0,ij} \mapsto C(i+1; j, (i, i+1)), \quad 1 \leq i, j \leq d,
\]

\[(\text{if } i = d \text{ then } i+1 := 1)\]

\[
p'_{r,st} \mapsto p_{r,st}, \quad 1 \leq r, s, t \leq d.
\]

In fact \(f\) is an isomorphism because \(p_{0,ij}\) is a linear term in

\[
C(i+1; j, (i, i+1)) = p_{0,ij} + \sum_{m=1}^{d} (p_{m,ij}p(i+1),(i+1)m - p_{m,(i+1)j}p(i+1),im).
\]

Since \(C(i+1; j, (i, i+1)) \in I_P\), we have an induced isomorphism

\[
(2.8) \quad \frac{R}{I_P} \cong \frac{T}{I_PT},
\]

where \(I_PT\) is the expansion of \(I_P\) to \(T\). We note that in this construction \(C(i+1; j, (i, i+1))\) can be replaced by any \(C(k; j, (i, k))\) or \(C(k; i, (j, k))\) \((k \neq i, j)\), because the resulting \(I_PT\) does not depend on the choice \(C(k; j, (i, k))\) or \(C(k; i, (j, k))\). In fact this construction is natural in the sense that we eliminate all the linear terms appearing in \(C(a; j, (i, k))\) so that the ideal \(I_PT\) is generated by quadratic equations.

Since \(p'_{0,ij}\) are eliminated under passing \(g\), the direct summand \(S_{(3,1,1,\cdots,1,1)}W (\cong \text{Sym}^2W)\) in \(W\) is eliminated. Then, by Proposition 12 the vector space of generators of \(I_PT\) is canonically isomorphic to \(S_{(3,2,1,\cdots,1,0)}W\) hence to

\[
\bigwedge^d W \otimes S_{(3,2,1,\cdots,1,0)}W \cong S_{(4,3,2,\cdots,2,1)}W \subset \text{Sym}^2 \left(\bigwedge^{d-1} W \otimes \text{Sym}^2W\right),
\]

where the last containment follows from Lemma 14.

The isomorphism of rings

\[
T = \mathbb{C}[p'_{r,st} \mid 1 \leq r, s, t \leq d] / (p'_{r,st} - p'_{r,ts}) \cong \text{Sym}^\bullet \left(\bigwedge^{d-1} W \otimes \text{Sym}^2W\right)
\]

naturally induces the isomorphism of quotient rings

\[
(2.9) \quad \frac{T}{I_PT} \cong \frac{\text{Sym}^\bullet (\bigwedge^{d-1} W \otimes \text{Sym}^2W)}{< S_{(4,3,2,\cdots,2,1)}W >}.
\]

Combining this with (2.8) gives the desired result. \qed
Theorem 16. \( \mathcal{P} \) (hence the symmetric affine open subscheme \( V_d \)) is isomorphic to
\[
\mathbb{C}^d \times \text{Spec} \left( \text{Sym}^* (S(3,1,1, \cdots ,1,0)W) \right) \left/ \langle S_{(1,3,2, \cdots ,2,1)}W \rangle \right.,
\]
where \((3, 1, 1, \cdots , 1, 0)\) is a partition of \((d+1)\) and \((4, 3, 2, \cdots ,2, 1)\) is of \((2d+2)\).

Sketch of Proof. Define an isomorphism of rings
\[
T = \mathbb{C}[p_{r,s,t}]_{1 \leq r,s,t \leq d} \cong \mathbb{C}[q_{r,s,t}]_{1 \leq r,s,t \leq d} \Rightarrow Q
\]
by
\[
p'_{r,s,t} \mapsto \begin{cases} 
q_{r,s} + q_{s,ss}, & \text{if } r = t \\
q_{r,s}, & \text{if } r \neq s, t.
\end{cases}
\]

As a matter of fact this is a natural isomorphism, because the square of any maximal ideal in \( \mathbb{C}[x] \) satisfies \( p'_{r,s} - \frac{1}{2}p'_{s,s} = 0 \) \((r \neq s)\), i.e. \( q_{r,s} = 0 \). It is straightforward to check that no element in minimal generators of \( I_{\mathcal{P}Q} \) contains terms involving \( q_{s,ss}, 1 \leq s \leq d \). For example, if \( a, i, j, k \) are distinct, then
\[
C(a; j, (i, k)) = \sum_{m=1}^{d} (p_{m,ij}p_{a,km} - p_{m,kj}p_{a,im})
\]
becomes
\[
\sum_{m \neq a, i, k} (q_{m,ij}q_{a,km} - q_{m,kj}q_{a,im}) + \sum_{m \neq a, i, k} \left( (q_{j,ij} + q_{i,ii})q_{a,kj} - (q_{j,kj} + q_{k,kk})q_{a,ij} \right) + \left( q_{a,ij}(q_{a,ka} + q_{k,kk}) - q_{a,kj}(q_{a,ia} + q_{i,ii}) \right) + \sum_{m \neq a, i, k} \left( (q_{i,ij} + q_{j,ij})q_{a,ki} - q_{i,kj}q_{a,ii} \right) + \left( q_{a,ij}(q_{a,ka} + q_{k,kk}) - q_{a,kj}(q_{a,ia} + q_{i,ii}) \right) + \sum_{m \neq a, i, k} \left( q_{k,ij}q_{a,km} - q_{k,kj}q_{a,im} \right)
\]
in which no term involves \( q_{s,ss}, 1 \leq s \leq d \).

Therefore we get
\[
\frac{T}{I_{\mathcal{P}T}} \cong \frac{Q}{I_{\mathcal{P}Q}} \cong \mathbb{C}[q_{s,ss}]_{1 \leq s \leq d} \otimes_{\mathbb{C}} \frac{\mathbb{C}[q_{r,s,t}]_{1 \leq r,s,t \leq d}}{(q_{r,t} - q_{r,s})/I_{\mathcal{P}Q}}.
\]
We may identify the basis of $S_{(2,1,1,\ldots,1,1)} W$ with $\{q_{s,s} | 1 \leq s \leq d\}$. So, by (2.9), we have

$$T_{\mathcal{P} \mathcal{Q}} \cong \mathbb{C}[q_{s,s}]_{1 \leq s \leq d} \otimes \mathbb{C}[q_{r,s,t}]_{1 \leq r,s,t \leq d, \ r \neq s \ or \ t \neq s} / \mathcal{P} \mathcal{Q}$$

$$\cong \text{Sym}^* \left( S_{(2,1,1,\ldots,1,1)} W \right) \otimes \text{Sym}^* \left( S_{(3,1,1,\ldots,1,0)} W \right).$$

Combining this with (2.8) gives the desired result. \hfill \Box

**Example 17.** It is well known (15) that if $d = 3$ then $V_d$ is isomorphic to a cone over the Plücker embedding of the Grassmannian $G(2, 6)$ with a three-dimensional vertex. Let $W$ be a 3-dimensional vector space and $W'$ a 6-dimensional vector space. Then

$$\text{Sym}^* \left( S_{(3,1,0)} W \right)_{< S_{(4,3,1)} W >} \cong \text{Sym}^* \left( \Lambda^2 \left( S_{(2,0,0)} W \right) \right)_{< \Lambda^4 \left( S_{(2,0,0)} W \right) >} \cong \text{Sym}^* \left( \Lambda^2 W' \right)_{< \Lambda^4 W' >}.$$

\hfill \Box

3. Local equations of the principal component of the Hilbert scheme of points

In this section, we prove Proposition 4. We start by showing that $J_d$ has a representation-theoretic expression.

**Lemma 18.** The $\mathbb{C}$-vector space of the minimal generators of $J_d$ is the direct sum of some irreducible Schur functors.

**Proof.** We prove a more general statement: the vector space $(J_d)_{\leq n} := \bigoplus_{i=0}^{n} (J_d)_i$ is the direct sum of some irreducible Schur functors for every $n$. It is enough to show that there is a group homomorphism from $GL(d)$ to $GL((J_d)_{\leq n})$ which is comparable with the natural action of the symmetric group $S_d$.

First, there is a natural way of defining $g \cdot p_{r,s,t}$ for $g \in GL(d)$. If $p_{r,s,t}$ is given by $I_{\{p_1,\ldots,p_d\}} \in P_d$ as in (2.2), then $g \cdot p_{r,s,t}$ is given by $I_{\{g \cdot p_1,\ldots,g \cdot p_d\}}$.

Next, we define a homomorphism $\rho$ from $GL(d)$ to $GL((J_d)_{\leq n})$ as follows. For every $f(p_{r,s,t})_{1 \leq r,s,t \leq d} \in (J_d)_{\leq n}$, define

$$g \cdot f(p_{r,s,t})_{1 \leq r,s,t \leq d}$$

by

$$g \cdot f(p_{r,s,t})_{1 \leq r,s,t \leq d} := f(g \cdot p_{r,s,t})_{1 \leq r,s,t \leq d}.$$ 

Since any point in $P_d$ satisfies $f = 0$, we have $f(g \cdot p_{r,s,t}) \in (J_d)_{\leq n}$. It is easy to check that $\rho$ is a homomorphism from $GL(d)$ to $GL((J_d)_{\leq n})$. It is obvious that this is comparable with the natural action of $S_d$. \hfill \Box
Lemma 19. The vector space of the minimal homogeneous generators of the ideal \( J_8 \subset \text{Sym}^* (S_{(3,1,\ldots,1,0)} W) \) contains
\[
S_{(133,130,126,122,119,60,60,60)} W.
\]

Sketch of proof. The idea is to observe that there are relations among
\[
\{ p_{r,st} \}_{1 \leq r \leq 3, 4 \leq s,t \leq 8}.
\]
This is suggested by the fact that a general ideal having the Hilbert function of the type \((1,5,3)\) is not contained in the principal component \( P_8 \) \[8, 18, 3\]. In particular, if \( \{ p_{r,st} \}_{1 \leq r \leq 3, 4 \leq s,t \leq 8} \) are general complex numbers and if the other coordinates are 0, then the colength 9 ideal determined by those coordinates does not belong to \( P_8 \).

By the algorithm in \[4, \text{pp.124–128}\], one can check that \( S_{(133,130,126,122,119,60,60,60)} W \) appears in the decomposition of \( \text{Sym}^{90} (S_{(3,1,\ldots,1,0)} W) \). We find elements in \( S_{(133,130,126,122,119,60,60,60)} W \) in a very explicit way.

Recall from (2.3) that if \( a \neq j, i, k \) then
\[
C(a; j, (i, k)) = \sum_{m=1}^{d} (p_{m,ij}p_{a,km} - p_{m,kj}p_{a,im})
\]
in the polynomial ring \( \frac{C[p_{r,st}]}{(p_{r,st} - p_{r,133})} \). The key fact is that any term in any \( C(a; j, (i, k)) \) with \( 1 \leq a \leq 3, 4 \leq i, j, k \leq d \) is a product of two coordinates, one of which is in \( \{ p_{r,st} \}_{1 \leq r \leq 3, 4 \leq s,t \leq 8} \) and the other is not. We consider the following 90 \( \times \) 1 matrix each of whose entry is a polynomial of degree 2.

\[
\begin{pmatrix}
C(1; 4, (5, 6)) \\
\vdots \\
C(a; j, (i, k)) \\
\vdots \\
C(3; 6, (5, 6))
\end{pmatrix},
\]

where \( 1 \leq a \leq 3 \), and \( 4 \leq j < i < k \leq 8 \) or \( 4 \leq k < j < i \leq 8 \) or \( 4 = i = j < k \leq 8 \) or \( 4 = i < j = k \leq 8 \) or \( 5 = i = j < k \leq 6 \) or \( 5 = i < j = k \leq 6 \).

Then we can observe that there is a 90 \( \times \) 115 matrix \( M \) such that each entry of \( M \) is one of the elements in
\[
\{0, \pm p_{r,st} \}_{1 \leq r \leq 3, 4 \leq s,t \leq 8},
\]
and \( M \) fits into the following matrix factorization:
\[
\begin{pmatrix}
C(1; 4, (5, 6)) \\
\vdots \\
C(a; j, (i, k)) \\
\vdots \\
C(3; 6, (5, 6))
\end{pmatrix} = M \cdot \begin{pmatrix}
p_{r',s',t'} - \delta_{r',t'} \frac{p_{r',s',t'}}{2} - \delta_{r',s'} \frac{p_{r',s',t'}}{2} \\
\vdots \\
\end{pmatrix},
\]
where \( \delta \) denotes the Kronecker delta, and \( 1 \leq r', s' \leq 3 < t' \leq 8 \) or \( 4 \leq r' \leq 8 \), \( 4 \leq s' \leq t' \leq 8 \) but \( r', s', t' \) are not all equal.
Then exhaustive computations show that the determinant of any $90 \times 90$ minor of $M$ lies in $S_{133,130,126,122,119,60,60,60}W$ and in $J_8$, and that the determinant of some $90 \times 90$ minor is nonzero.\footnote{One can check whether or not a given polynomial belongs to $J_8$, since $P_8$ admits an explicit rational parametrization. (For example, see \cite[Theorem 3.3]{20} or \cite[Proposition 2.6]{12}.)} Then, thanks to Lemma 18, $J_8$ contains $S_{133,130,126,122,119,60,60,60}W$.

It remains to show that $S_{133,130,126,122,119,60,60,60}W$ is contained in the minimal generators of $J_8$. If $S_{133,...}W$ were not minimal, there would be a partition $\lambda$ of $89 \cdot 9$ such that

$$S_\lambda W \subset J_8 \cap \text{Sym}^{89}(S_{(3,1,...,1,0)}W),$$

and $S_\lambda W$ generates $S_{133,...}W$. But we have to check that there is no such $\lambda$.

To this end, we consider all the partitions $\lambda$ such that

$$S_\lambda W \subset (S_{(3,1,...,1,0)}W)^{\otimes 89}$$

and

$$S_\lambda W \otimes S_{(3,1,...,1,0)}W \supset S_{133,130,126,122,119,60,60,60}W.$$ 

There are 15 such partitions, and I checked that none of their embeddings into $\text{Sym}^{89}(S_{(3,1,...,1,0)}W)$ are in $J_8$. We remark that each $90 \times 90$ minor of $M$ belongs to one of such $S_\lambda W$. \hfill \Box

**Remark 20.** We note that the generator $S_{(4,3,2,\ldots,2,1)}W$ of $I_8$ does not generate $S_{133,...}W$, in other words,

$$S_{133,130,126,122,119,60,60,60}W \notin S_{(4,3,2,\ldots,2,1)}W.$$  

It is an elementary consequence of the combinatorial Littlewood-Richardson rule (for example, see \cite[p456]{9}). In fact any $S_\lambda W (\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_8))$ appearing in the decomposition of $S_{(4,3,2,\ldots,2,1)}W \otimes (S_{(3,1,1,...,1,0)}W)^{\otimes (r-2)}$ satisfies $\lambda_8 + \cdots + \lambda_8 \geq rk + 1$, for any $r \geq 2$ and any $k = 0,...,7$.

Concretely speaking, the ideal generated by $C(a,j,(i,k))$ does not contain any nonzero determinants of $90 \times 90$ minors of $M$. It is easy to prove this without using Schur functors, because for any term $\prod_i p_{r_i,s_{i,t_i}}$ in any determinant of $90 \times 90$ minors of $M$, we have $r_i \neq s_j, t_j$ for all $i,j$.

### 4. Proof of Lemma 5

Lemma 5 can be considered as a standard fact. We do not claim any novelty for its proof.

**Proof of Lemma 5.** Recall that the regularity index of $S$, $r(S)$, is the minimum degree in which the Hilbert function of $S$ agrees with the Hilbert polynomial (see \cite{2} for more details). If $S \subset \mathbb{P}^N$ is an aCM scheme of dimension $n$, then $r(S) = \text{reg}(S) - n - 1$ (this follows from \cite[Theorem 4.4.3 (b)]{2}). Hence it is enough to show that $r(S) \leq 0$.

The Hilbert function of $S$ is $H(S,t) = h^0(S,\mathcal{O}_S(t))$ and its Hilbert polynomial is $\chi(S,t) = h^0(S,\mathcal{O}_S(t)) + (-1)^nh^n(S,\mathcal{O}_S(t)) = H(S,t) + (-1)^nh^0(S,\omega_S(-t))$ where $\omega_S$ denotes the dualizing sheaf.
The second condition on $\tilde{U}$ implies $H^0(\tilde{U}, \omega_{\tilde{U}}) = 0$ (see [16, Chapter 4]). Since $S$ is Cohen-Macaulay and $\text{codim } S \setminus \tilde{U} \geq 2$, we have $H^0(S, \omega_S) = 0$. So we have $h^0(S, \omega_S(-t)) = 0$ for all $t \geq 0$ and this establishes the lemma.

5. Proof of Proposition 6

In this section we prove Proposition 6. We first construct an open subset $U_d$ of $\text{Spec}(R_d/J_d)$, where $\tilde{U}_d$ will be the projective counterpart of $U_d$ in $\text{Proj}(R_d/J_d)$.

Let $U_d$ be the open subset of $P_d$ consisting of all ideals $I \in P_d$ such that the radical $\text{Rad}(I)$ of $I$ defines at least $d$ distinct points. Then $U_d$ is smooth and $\text{codim}_{P_d}(P_d \setminus U_d) = 2$.

We consider the Hilbert–Chow morphism on $U_d$,

$$\rho : U_d \longrightarrow \text{Sym}^{d+1}(\mathbb{C}^d),$$

and the averaging map

$$\pi : \text{Sym}^{d+1}(\mathbb{C}^d) \longrightarrow \mathbb{C}^d$$

given by

$$\pi(\{(x_{1,1}, \ldots, x_{1,d}), \ldots, (x_{d+1,1}, \ldots, x_{d+1,d})\}) = \left( \frac{x_{1,1} + \ldots + x_{d+1,1}}{d+1}, \ldots, \frac{x_{1,d} + \ldots + x_{d+1,d}}{d+1} \right).$$

Let $j$ be the natural morphism

$$j : U_d \hookrightarrow P_d \cong \mathbb{C}^d \times \text{Spec}(R_d/J_d),$$

and let $pr_1$ be the projection

$$pr_1 : \mathbb{C}^d \times \text{Spec}(R_d/J_d) \longrightarrow \mathbb{C}^d.$$

Lemma 21. $pr_1|_{j(U_d)}$ agrees with $\pi \circ \rho \circ j^{-1}|_{j(U_d)}$, in other words, the following diagram

$$\begin{array}{ccc}
U_d & \xrightarrow{\rho} & \text{Sym}^{d+1}(\mathbb{C}^d) \\
\uparrow^{j^{-1}} & & \downarrow^{\pi} \\
\mathbb{C}^d \times \text{Spec}(R_d/J_d) \supset j(U_d) & \xrightarrow{pr_1} & \mathbb{C}^d
\end{array}$$

is commutative (up to automorphisms of $\mathbb{C}^d \times \text{Spec}(R_d/J_d)$).

Proof. For each element $I$ in $U_d$, there is a corresponding ideal projector $P_I$, which gives rise to $pr_{s,t}, 1 \leq r, s \leq d$ as in [2.2]. From the proof of Theorem 16 we may define

$$pr_1 \circ j : U_d \longrightarrow \mathbb{C}^d$$

by

$$I \mapsto \left( \frac{\sum_{r=1}^{d} P_{r,1} r}{d+1}, \ldots, \frac{\sum_{r=1}^{d} P_{r,d} r}{d+1} \right).$$

It is elementary to check that this map is the same as $\pi \circ \rho$. $\square$

---

\(^3\)For $2 \leq d \leq 8$, it can be checked by the computer algebra system Macaulay 2.
We identify the fiber \((pr_1)^{-1}(O)\) over the origin \(O = (0, \ldots, 0) \in \mathbb{C}^d\), with the affine cone over \(\text{Proj}(R_d/J_d)\). By construction, \(j(U_d) \cap (pr_1)^{-1}(O)\) parameterizes the ideals defining \(d\) distinct points and one more (possibly infinitely near) point, whose average (=center of mass) is the origin \(O\). Hence scaling distances from \(O\) by any nonzero constant preserves membership in \(j(U_d) \cap (pr_1)^{-1}(O)\). In other words, if an ideal defining \((d + 1)\) points, say \(p_1, \ldots, p_{d+1}\), belongs to \(j(U_d) \cap (pr_1)^{-1}(O)\), then so does the ideal defining \(\lambda \cdot p_1, \ldots, \lambda \cdot p_{d+1}\) for any \(\lambda \neq 0 \in \mathbb{C}\). In fact we have

\[
\text{Proj}(R_d/J_d) = (\text{Spec}(R_d/J_d) \setminus O) / \sim,
\]

where the equivalence relation is given by \(I_{\{p_1, \ldots, p_{d+1}\}} \sim I_{\{\lambda p_1, \ldots, \lambda p_{d+1}\}}, \lambda \neq 0\).

Therefore all told, \(j(U_d) \cap (pr_1)^{-1}(O)\) is the affine cone over a certain open subset of \(\text{Proj}(R_d/J_d)\), with the vertex of the cone removed. We denote the open subset by \(\tilde{U}_d\). So we get

\[
\tilde{U}_d = \left( j(U_d) \cap (pr_1)^{-1}(O) \right) / \sim.
\]

Of course, for any point \(q \in \mathbb{C}^d\), we have

\[
\tilde{U}_d \cong \left( j(U_d) \cap (pr_1)^{-1}(q) \right) / \sim.
\]

**Lemma 22.** For any \(d \geq 2\), \(\tilde{U}_d\) is covered by rational proper curves, i.e., for any point \(x \in \tilde{U}_d\), there is a smooth irreducible rational proper curve on \(\tilde{U}_d\) passing through \(x\).

**Proof.** The idea of the proof is to find a smooth irreducible rational proper curve on \(\tilde{U}_d\), and to apply the \(GL(d)\)-action to the curve.

First we find a \(\mathbb{P}^1\) on \(\tilde{U}_d\) as follows. Consider the following \((d + 1)\) points on \(\mathbb{C}^d : x_i = (x_{i,1}, \ldots, x_{i,d})\), \(1 \leq i \leq d + 1\), where

\[
x_{i,j} = \begin{cases} -d, & \text{if } i = j \\ 1, & \text{if } i \neq j. \end{cases}
\]

Note that the image of the ideal defining \(x_1, \ldots, x_{d+1}\) under \((pr_1 \circ j)\) is the origin \(O\).

We fix \(d\) distinct points \([a_j : b_j]\) \((1 \leq j \leq d)\) on \(\mathbb{P}^1 \setminus [1 : 1]\), and define a morphism

\[
\varphi : \mathbb{P}^1 \longrightarrow \tilde{U}_d
\]

by

\[
[\alpha : \beta] \mapsto \left[ \text{the ideal vanishing along } \left( \frac{b_1 \alpha - a_1 \beta}{b_1 - a_1} x_{i,1}, \ldots, \frac{b_d \alpha - a_d \beta}{b_d - a_d} x_{i,d} \right), 1 \leq i \leq d + 1 \right],
\]

where \([\text{ideal}]\) denotes the equivalence class of the ideal. For each \(j\), we define \(\varphi([a_j : b_j])\) by the equivalence class of the non-radical ideal as a limit of radical ideals, where two points collide. In other words,

\[
\varphi([a_j : b_j]) = \lim_{[\alpha : \beta] \rightarrow [a_j : b_j]} \varphi([\alpha : \beta]).
\]

It is straightforward to check that \(\varphi\) is well-defined and that \(\varphi(\mathbb{P}^1)\) is smooth and irreducible.

Now we can prove the lemma. For any element \([I] \in \tilde{U}_d\), there is \(g \in GL(d)\) such that

\[
[I] \in g \cdot (\varphi(\mathbb{P}^1)),
\]
where $\cdot$ is the natural action of $GL(d)$. □

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