PENTAGON EQUATION AND COMPACT QUANTUM SEMIGROUPS

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Abstract. The generalization of multiplicative unitary notion from compact quantum groups to compact quantum semigroups is considered. We show why the same construction doesn’t work in this case by giving examples of C*-algebras with non-trivial comultiplication which do not admit multiplicative unitaries. By the use of the pentagon equation we suggest a notion of an operator which gives comultiplication on any C*-algebra. The multiplicative unitary turns out to be its special case. We prove for some compact quantum semigroups that the comultiplication is given by such operator.

1. Introduction

The theory of multiplicative unitaries for compact quantum groups is a well-developed area of the theory of operator algebras. Given the importance and the success of that theory, it is natural to attempt to extend it to a more general situation by, for example, developing a theory of multiplicative unitaries for compact quantum semigroups. However, in this work we show that there are significant differences and explain in details why the theory can not be repeated for compact quantum semigroups. The analogues of the classical results turn out to be false.

Baaj and Skandalis proved in [4] that there exists a multiplicative unitary $u \in B(H \otimes H)$ for every compact quantum group $(\mathcal{A}, \Delta)$ such that

$$\Delta(a) = u(a \otimes 1)u^* \text{ for any } a \in \mathcal{A}. \tag{1.1}$$

The way they construct this operator is based on the GNS-representation of $\mathcal{A}$ corresponding to the Haar functional $h$ of $(\mathcal{A}, \Delta)$, which existence was proved in [3]. Woronowicz proved that a unique faithful Haar functional corresponds to each compact quantum group. In the proof of this theorem the density conditions are used essentially. Hence, this statement may fail to be true for compact quantum semigroups. So, the first difference with the case of compact quantum semigroups is the fact that $h$ may not exist.

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One can suggest that the theory should be developed for compact quantum semigroups with Haar functional. But the second difference is that even if $h$ exists, it may not be faithful. This is the case for the compact quantum semigroup on the Toeplitz algebra $\mathbb{T}$ and the algebra of continuous functions on a compact semigroup, which we describe in Section 3. In the first example, despite that $h$ is not faithful, the corresponding GNS-representation of this algebra is still isomorphic to itself. But, in Section 3 we prove that there does not exist multiplicative unitary satisfying (1.1). Nevertheless, there exists a unitary $u$, which satisfies (1.1), but is not a multiplicative unitary. It follows that multiplicative unitary does not correspond to every compact quantum semigroup.

The second example shows us another situation. The Haar functional for this compact quantum semigroup exists, and even there is a multiplicative unitary. But, due to the fact that $h$ is not faithful, the corresponding GNS-representation is one-dimensional. This implies that the comultiplication defined by the multiplicative unitary does not induce the comultiplication on the algebra as in the classical case.

All these reasons bring us to the need for a new notion of operator, which could generalize the notion of a multiplicative unitary. This new operator must

1. not depend on the Haar functional;
2. induce the comultiplication on the algebra;
3. be based on the pentagon equation;
4. give a multiplicative unitary in the case of compact quantum group;
5. exist for all examples mentioned above.

In Section 4 we suggest a notion of an operator which possesses all these properties. Furthermore, we show the way it is defined for the examples from Section 3. We also prove that this operator exists for compact quantum semigroups with some additional properties.

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2. Preliminaries

A unital $C^*$-algebra $A$ with unital $^*$-homomorphism $\Delta: A \to A \otimes A$ is called a compact quantum semigroup (2) if $\Delta$ satisfies coassociativity condition:

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta.$$ 

$\Delta$ is called a comultiplication. If the linear subspaces

$$\{\Delta(b)(a \otimes I); \ a, b \in A\}, \quad (2.1)$$

$$\{\Delta(b)(I \otimes a); \ a, b \in A\}, \quad (2.2)$$
are dense in $A \otimes A$, then $(A, \Delta)$ is called a compact quantum group \[3\]. A $*$-homomorphism $\epsilon : A \to \mathbb{C}$ is called a counit if for any $a \in A$

$$(\epsilon \otimes \text{id})\Delta(a) = a, \quad (\text{id} \otimes \epsilon)\Delta(a) = a.$$  

State $h \in A^*$ is called a Haar functional in $A^*$ if the following

conditions hold for any $\rho \in A^*$:

$$h^* \rho = \rho^* h = \lambda_\rho h,$$

where $\lambda_\rho \in \mathbb{C}$ depends on $\rho$. The operation $*$ is a multiplication in $A^*$ induced by comultiplication $\Delta$ on $A$:

$$(\rho^* \varphi)(a) = (\rho \otimes \varphi)\Delta(a).$$

**Remark 1.** $h$ is Haar functional iff the following relations hold for any $a \in A$

$$(h \otimes \text{id}) \Delta(a) = h(a) I,$$

$$(\text{id} \otimes h) \Delta(a) = h(a) I.$$  

Further we recall the definition of multiplicative unitary and give a new definition of multiplicative isometry.

**Definition 1.** Let $H$ be a Hilbert space, $\sigma : H \otimes H \to H \otimes H$ a flip operator: $\sigma(x \otimes y) = y \otimes x$, where $x, y \in H$. For $a \in B(H \otimes H)$ we denote

$$a_{12} = a \otimes I, a_{23} = I \otimes a, a_{13} = \sigma_{12}a_{23}\sigma_{12} = \sigma_{23}a_{12}\sigma_{23}.\quad (2.6)$$

Operator $u \in B(H \otimes H)$ is called multiplicative isometry, if it is isometric and satisfies pentagon equation:

$$u_{12}u_{13}u_{23} = u_{23}u_{12}.\quad (2.7)$$

If in the above definition operator $u$ is unitary it is called a multiplicative unitary \[4\]. Recall the construction of multiplicative unitary for a compact quantum group. Consider the GNS-representation $(H_\varphi, \pi_\varphi)$ of $C^*$-algebra $A$, corresponding to a state $\varphi$. Denote by $N_\varphi$ a left ideal $\{a \in A | \varphi(a^*a) = 0\}$. For any $a \in A$ denote by $\overline{a}$ the corresponding equivalence class in the quotient space $A/N_\varphi$.

The main result concerning multiplicative unitary for compact quantum groups, proved in \[4\], is the following.

**Theorem 1.** Let $(A, \Delta)$ be a compact quantum group with Haar functional $h$. Consider the GNS-representation $(H_h, \pi_h)$ corresponding to $h$. Then there exists multiplicative unitary $u \in B(H_h \otimes H_h)$ such that

$$(\pi_h \otimes \pi_h)\Delta(a) = u(\pi_h(a) \otimes I)u^*.$$  

For any $a, b \in A$ operator $u$ here is defined in the following way

$$u(\overline{a} \otimes \overline{b}) = \overline{\Delta(a)(I \otimes b)}.$$  

3. Multiplicative unitary and compact quantum semigroups

In this section we attempt to repeat the construction of multiplicative unitary for compact quantum semigroups. Firstly we give an analogue of Theorem 1.

**Theorem 2.** Let \((A, \Delta)\) be a compact quantum semigroup with Haar functional \(h\) and \((H_h, \pi_h)\) the GNS-representation corresponding to \(h\). Then there exists multiplicative isometry \(u \in B(H_h \otimes H_h)\) such that

\[
u^*(\pi_h \otimes \pi_h)(\Delta(a))u = \pi_h(a) \otimes I. \tag{3.1}\]

**Proof.** Take operator \(u\) defined by (2.9). Since \(h\) is Haar functional, using (2.4) we obtain

\[
\langle u (\pi \otimes \bar{b}), u (\pi \otimes \bar{b}) \rangle = (h \otimes h)((I \otimes b^*) \Delta (a^*) \Delta (a) (I \otimes b)) =
\]

\[
= h ((h \otimes id)((I \otimes b^*) \Delta (a^*) (I \otimes b))) = h (b^* (h \otimes id) \Delta (a^*) b) =
\]

\[
= h (b^* b) h (a^* a) = \langle \pi \otimes \bar{b}, \pi \otimes \bar{b} \rangle
\]

Consequently, \(u\) is isometric.

The following calculations show that \(u\) is a multiplicative isometry.

\[
u_{12}u_{13}u_{23} (\pi \otimes \bar{b} \otimes \bar{c}) = u_{12}u_{13} (\pi \otimes u (\bar{b} \otimes \bar{c})) =
\]

\[
= u_{12}u_{13} \left( \pi \otimes \Delta (b) (I \otimes c) \right) = u_{12}u_{13}(a \otimes \Delta (b)) (I \otimes I \otimes c) =
\]

\[
= (\Delta \otimes id) (\Delta (a)) (I \otimes b) (I \otimes I \otimes c) =
\]

\[
= (id \otimes \Delta) (\Delta (a)) (I \otimes b) (I \otimes I \otimes c) = u_{23}u_{12} (\pi \otimes \bar{b} \otimes \bar{c})\]

It is easy to check (3.1). \(\Box\)

Since the density conditions may not hold for an arbitrary compact quantum semigroup, operator \(u\) from theorem 2 may not be unitary. Hence, we cannot go further and get the same result as in theorem 1. Next we give examples to describe the situation in details.

**Example 1.** Toeplitz algebra.

Consider the Toeplitz algebra \(T\) – the minimal \(C^*\)-algebra generated by an isometric right-shift operator \(T\) and \(T^*\) on a Hilbert space \(H\) with orthonormal basis \(\{e_n\}_{n=0}^\infty\). It was shown in [1] that this algebra admits a comultiplication \(\Delta\) and the corresponding Haar functional \(h\), defined by:

\[
\Delta (T) = T \otimes T, h (I) = 1, h (T_{n,m}) = 0, T_{n,m} = T^n T^{m*} \tag{3.2}
\]

for all \((m,n) \in (Z_+ \times Z_+)\). Here we use notation introduced in [1]. Obviously, functional \(h\) is not faithful.
Proposition 1. The GNS-representation of algebra $\mathcal{T}$ corresponding to the Haar functional $h$ is the same Toeplitz algebra $T$, i.e. we have the isomorphism $\mathcal{T} \cong \pi_h(\mathcal{T})$ with $\pi_h(T)$ being right-shift operator on $H_h$.

Proof. One can easily see that $N_h$ is a linear space generated by $\{T_{n,m}\}_{m \neq 0}$. Here $H$ is a Hilbert space with basis $\{e_k\}_{k=0}^{\infty}$, where $e_k = [T^k] = T^k + N_h$. Then the corresponding GNS-representation $\pi_h$ acts in the following way:

$$\pi_h(T_n)e_k = \pi_h(T_n)[T^k] = [T^{n+k}] = e_{n+k}.$$ 

Thus, the algebra $\pi_h(T)$ is a $C^*$-algebra generated by right-shift isometric $\pi_h(T)$.

Theorem 3. There doesn’t exist any multiplicative unitary $u$ for $(\mathcal{T}, \Delta)$ satisfying (2.8). But there exists a unitary operator $u$ satisfying (2.8) which doesn’t satisfy pentagon equation.

Proof. By virtue of proposition 1 we may identify $\pi_h(T)$ with $T$ and omit notation $\pi_h$. Assume that $u \in B(H \otimes H)$ is a unitary operator satisfying (2.8) for all $a \in \mathcal{A}$. Particularly, we have $T \otimes T = u(T \otimes 1) u^*$. Hence, such operator is equivalent to one defined by following relations:

$$u(e_0 \otimes e_{2k}) = e_0 \otimes e_k, \quad k = 0, 1, 2, \ldots,$$

$$u(e_0 \otimes e_{2k-1}) = e_k \otimes e_0, \quad k = 1, 2, \ldots.$$

It is sufficient to show that pentagon equation does not hold for this operator. To this end take vector $e_0 \otimes e_1 \otimes e_0$ and calculate first the left-hand side of pentagon equation.

$$u_{12}u_{13}u_{23}(e_0 \otimes e_1 \otimes e_0) = u_{12}u_{13}(e_0 \otimes u(e_1 \otimes e_0)) = u_{12}u_{13}(e_0 \otimes u(T \otimes 1)(e_0 \otimes e_0)) =$$

$$= u_{12}u_{13}(e_0 \otimes (T \otimes T)u(e_0 \otimes e_0)) = u_{12}u_{13}(e_0 \otimes e_1 \otimes e_1) = u_{12}(e_1 \otimes e_1 \otimes e_0) =$$

$$= (u(T \otimes 1)(e_0 \otimes e_1) \otimes e_0) = ((T \otimes T)u(e_0 \otimes e_1) \otimes e_0) = e_2 \otimes e_1 \otimes e_0.$$ 

And the right-hand side of pentagon equation on the same vector.

$$u_{23}u_{12}(e_0 \otimes e_1 \otimes e_0) = u_{23}(e_1 \otimes e_0 \otimes e_0) = e_1 \otimes e_0 \otimes e_0.$$ 

Consequently, this operator $u$ is not a multiplicative unitary.

Thus, there are compact quantum semigroups without multiplicative unitary. Next example describes another kind of situation.

Example 2. Compact semigroup algebra
Let $S$ be a compact semigroup with zero element. Consider the algebra of continuous functions on $S$, $C(S)$. Define the natural comultiplication on $C(S)$.

\[(\Delta (f))(x, y) = f(xy). \quad (3.3)\]

One can easily verify that $(C(S), \Delta)$ is a compact quantum semigroup. The Haar functional on $(C(S), \Delta)$ is the next one:

\[h(f) = f(0)\]

Obviously, $h$ is not faithful. Since $h$ is a pure state, the corresponding GNS-representation of $C(S)$ is one-dimensional. Therefore, there is no interest in considering multiplicative unitary in this example.

In the next section we give the definition of operator based on pentagon equation. This operator generalizes the notion of multiplicative unitary and is sufficient for these two examples.

4. NEW OPERATOR

The notion of multiplicative unitary is based on the GNS-representation corresponding to Haar functional $h$. Since the representation $\pi_h$ is faithful for a compact quantum group $(A, \Delta)$, the comultiplication $\Delta$ induces comultiplication $\Delta'$ on $\pi_h(A)$, defined by the following commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\Delta} & A \otimes A \\
\pi_h \downarrow & & \downarrow \pi_h \otimes \pi_h \\
\pi_h(A) & \xrightarrow{\Delta'} & \pi_h(A) \otimes \pi_h(A)
\end{array}
\]

In the case of compact quantum group we identify $A$ with $\pi_h(A)$. The multiplicative unitary from theorem 1 then satisfies the next condition

\[\Delta'(a) = u^* (a \otimes 1) u \quad (4.1)\]

for any $a \in A$.

The right-hand side is the unitary operator $W$ defined as multiplication by $u^*$ from the left, and $u$ from the right, calculated on $a \otimes 1$, the element of $\pi_h(A) \otimes \pi_h(A)$, which we identify with $A$. This shows that $W : A \otimes A \to A \otimes A$ is a linear unitary operator. The pentagon equation for operator $u$ is encoded in the similar equation for operator $W$. We can rewrite (4.1) in the following way:

\[\Delta'(a) = W(a \otimes 1).\]

We have already shown that there may not exist (see ex. 1) a multiplicative unitary, or it may give no interest since the GNS-representation may not be faithful (see ex. 2). Nevertheless, the operator $W$ described above may still exist and may give the comultiplication, without being
unitar. This leads us to an idea of an operator $W : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ which satisfies the pentagon equation on algebra.

Let $\mathcal{L}(\mathcal{A})$ be the algebra of all linear continuous operators on $C^*$-algebra $\mathcal{A}$.

Operator $\Sigma \in \mathcal{L}(\mathcal{A} \otimes \mathcal{A})$ defined as follows $\Sigma (a \otimes b) = b \otimes a$, for $a, b \in \mathcal{A}$, is called a flip.

Suppose that $V \in \mathcal{L}(\mathcal{A} \otimes \mathcal{A})$ and $a \in \mathcal{A} \otimes \mathcal{A}$, then we denote $V_{12} = V \otimes \text{id}$, $V_{23} = \text{id} \otimes V$, $V_{13} = \Sigma_{12} V_{23} \Sigma_{12} = \Sigma_{23} V_{12} \Sigma_{23}$.

$a_{12} = a \otimes 1$, $a_{23} = 1 \otimes a$, $a_{13} = (\Sigma \otimes \text{id}) (a_{23}) = \Sigma_{12} a_{23}$.

**Definition 2.** Let $\mathcal{A}$ be a $C^*$-algebra. We say that the linear operator $W : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ satisfies pentagon equation if the following condition holds

$$W_{12} W_{13} W_{23} = W_{23} W_{12}.$$  \hspace{1cm} (4.2)

Given a unital $C^*$-algebra $\mathcal{A}$ we may define two trivial comultiplications on it:

$$\Delta^L(a) = a \otimes 1$$

$$\Delta^R(a) = 1 \otimes a$$

Then $(\mathcal{A}, \Delta^L)$ and $(\mathcal{A}, \Delta^R)$ are compact quantum semigroups.

**Theorem 4.** Let $\mathcal{A}$ be a unital $C^*$-algebra and $W \in \mathcal{L}(\mathcal{A} \otimes \mathcal{A})$ be a unital $C^*$-homomorphism, which satisfies pentagon equation. Then operators $\Delta(a) = W \Delta(a)$ and $\hat{\Delta}(a) = W^* \Delta^R(a)$ define comultiplications on $\mathcal{A}$ and $(\mathcal{A}, \Delta)$, $(\mathcal{A}, \hat{\Delta})$ are compact quantum semigroups.

**Proof.** It is enough to check the coassociativity of maps $\Delta, \hat{\Delta}$. Take $a \in \mathcal{A} \otimes \mathcal{A}$. Then we have

$$(\Delta \otimes \text{id}) a = (W \Delta^L \otimes \text{id}) a =$$

$$= (W \otimes \text{id}) (\Delta^L \otimes \text{id}) a = (W \otimes \text{id}) a_{13} = W_{12} (a_{13})$$

Using (4.2) and the above relation we get

$$(\Delta \otimes \text{id}) \Delta (a) = W_{12} (\Delta (a))_{13} = W_{12} (W (a \otimes 1))_{13} =$$

$$W_{12} W_{13} (a \otimes 1)_{13} =$$

$$= W_{12} W_{13} (a \otimes 1 \otimes 1) = W_{12} W_{13} (id \otimes W) (a \otimes 1 \otimes 1) =$$

$$= W_{12} W_{13} W_{23} (a \otimes 1 \otimes 1) = W_{23} W_{12} (a \otimes 1 \otimes 1) =$$

$$W_{23} (W (a \otimes 1))_{12} =$$

$$W_{23} (\Delta (a))_{12} = (id \otimes \Delta) \Delta (a).$$

By the similar calculations we obtain the same for $\hat{\Delta}$. \hfill $\square$

**Theorem 5.** For any compact quantum semigroup $(\mathcal{A}, \Delta)$ with counit $\epsilon$, there exist $C^*$-homomorphisms $W^L, W^R \in \mathcal{L}(\mathcal{A} \otimes \mathcal{A})$ satisfying the pentagon equation, such that

$$\Delta = W^L \Delta^L = W^R \Delta^R.$$
Remark 2. Under conditions of Theorem 5, \( W_L, W_R \) are projections and satisfy the following conditions
\[
W_L W_R = W_R W_L = W_L, W_R W_L = W_L.
\]
Moreover, there exist \( C^* \)-homomorphisms \( W'_L, W'_R \in \mathcal{L}(\mathcal{A} \otimes \mathcal{A}), \)
\[
W'_L = (id \otimes \epsilon (\cdot) I), W'_R = (\epsilon (\cdot) I \otimes id)
\]
which are also projections such that
\[
W'_L \Delta = \Delta L, W'_R \Delta = \Delta R, W_L W'_L = W_L, W_R W'_R = W_R.
\]

Consider two operators \( V \in \mathcal{L}(\mathcal{A} \otimes \mathcal{A}), W \in \mathcal{L}(\mathcal{B} \otimes \mathcal{B}) \) satisfying pentagon equation. Define linear operator
\[
V \Box W = (id \otimes \Sigma \otimes id)(V \otimes W)(id \otimes \Sigma \otimes id).
\]
Clearly, \( V \Box W \) satisfies pentagon equation on \( A \otimes B \).

Theorem 6. Let \( (A, \Delta_A), (B, \Delta_B) \) be compact quantum semigroups
and \( W_A \in \mathcal{L}(\mathcal{A} \otimes \mathcal{A}), W_B \in \mathcal{L}(\mathcal{B} \otimes \mathcal{B}) \) be operators satisfying pentagon equation, such that
\[
\Delta_A = W_A \Delta_1, \quad \Delta_B = W_B \Delta_1.
\]
Then \( (A \otimes B, \Delta) \) is a compact quantum semigroup with \( \Delta \) given by:
\[
\Delta = (W_A \Box W_B) \Delta_1.
\]

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