Riemann–Hilbert analysis for Laguerre polynomials with large negative parameter

A.B.J. Kuijlaars\textsuperscript{1} and K.T-R McLaughlin\textsuperscript{2}

Abstract. We study the asymptotic behavior of Laguerre polynomials $L_n^{(\alpha_n)}(nz)$ as $n \to \infty$, where $\alpha_n$ is a sequence of negative parameters such that $-\alpha_n/n$ tends to a limit $A > 1$ as $n \to \infty$. These polynomials satisfy a non-hermitian orthogonality on certain contours in the complex plane. This fact allows the formulation of a Riemann–Hilbert problem whose solution is given in terms of these Laguerre polynomials. The asymptotic analysis of the Riemann–Hilbert problem is carried out by the steepest descent method of Deift and Zhou, in the same spirit as done by Deift et al. for the case of orthogonal polynomials on the real line. A main feature of the present paper is the choice of the correct contour.

Keywords: Riemann–Hilbert problems, generalized Laguerre polynomials, strong asymptotics, steepest descent method.

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1 Introduction

1.1 Generalized Laguerre polynomials

The classical Laguerre polynomials $L_n^{(\alpha)}$ are orthogonal on the interval $[0, \infty)$ with respect to the weight $x^{\alpha}e^{-x}$, that is

$$\int_0^\infty L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)x^{\alpha}e^{-x} dt = 0, \quad \text{if } n \neq m.$$  \hfill (1.1)

The integral in (1.1) converges only if $\alpha > -1$. The Laguerre polynomials are given by the explicit formula

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n + \alpha}{n - k} \frac{(-x)^k}{k!}$$  \hfill (1.2)

and by the Rodrigues formula

$$L_n^{(\alpha)}(x) = \frac{1}{n!}x^{-\alpha}e^x \left( \frac{d}{dx} \right)^n [x^{\alpha+n}e^{-x}],$$  \hfill (1.3)

see e.g. [3]. Both (1.2) and (1.3) make sense for arbitrary $\alpha$ (even complex) and they define the generalized non-classical Laguerre polynomials. The recurrence relation

$$-xL_n^{(\alpha)}(x) = (n + 1)L_{n+1}^{(\alpha)}(x) - (2n + \alpha + 1)L_n^{(\alpha)}(x) + (n + \alpha)L_{n-1}^{(\alpha)}(x),$$  \hfill (1.4)

with $L_0^{(\alpha)} \equiv 1$, $L_{-1}^{(\alpha)} \equiv 0$, and the second order differential equation

$$xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0, \quad y(x) = L_n^{(\alpha)}(x),$$  \hfill (1.5)

continue to hold for arbitrary $\alpha$. We consider in this paper only real and negative $\alpha$, although extensions to complex $\alpha$ are possible.

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1.2 Relations to other polynomials

Laguerre polynomials with negative parameters appear in the literature in a number of forms. First we note the special cases $\alpha = -n$ and $\alpha = -n - 1$, where we have

$$L_n^{(-n)}(z) = \frac{(-1)^n}{n!} z^n,$$

and

$$L_n^{(-n-1)}(z) = (-1)^n \sum_{k=0}^{n} \frac{z^k}{k!},$$

respectively. Thus for $\alpha = -n - 1$, the generalized Laguerre polynomial agrees (up to a sign if $n$ is odd) with the partial sum of the exponential series. More generally, we have that for $\alpha = -n - m$ with $n, m \in \mathbb{N}$, the generalized Laguerre polynomials appear as the numerator and denominator polynomials in the rational Padé approximant for the exponential function. To be precise, if

$$p_{n,m}(x) = (-1)^n \binom{n+m}{n}^{-1} L_n^{(-n-m-1)}(x)$$

and

$$q_{n,m}(x) = (-1)^m \binom{n+m}{m}^{-1} L_m^{(-n-m-1)}(x)$$

then $p_{n,m}(0) = q_{n,m}(0) = 1$, and

$$p_{n,m}(x) - q_{n,m}(x) e^x = O \left(x^{n+m+1}\right) \quad \text{as} \quad x \to 0,$$

see e.g. [23], [25].

Laguerre polynomials with negative parameters are also related to the so-called generalized Bessel polynomials

$$y_n(z; a) = \sum_{k=0}^{n} \binom{n}{k} (n + a - 1)k \left(\frac{z}{2}\right)^k,$$

since

$$y_n(z; a) = (-1)^n n! \left(\frac{z}{2}\right)^n L_n^{(-2n-a+1)} \left(\frac{2}{z}\right).$$

The usual Bessel polynomials correspond to $a = 2$ in (1.11). See Grosswald [17] for a comprehensive account of these polynomials.

1.3 Earlier work on asymptotics

For $\alpha > -1$, the Laguerre polynomials satisfy the orthogonality (1.1) on the positive real axis, and therefore they have only positive real zeros. This property is lost for $\alpha < -1$. Indeed, the generalized Laguerre polynomials may have many non-real zeros. In Figure 1 we have plotted the zeros of $L_n^{(-40A)}(40z)$ for a number of values of $A > 0$. Similar plots are shown in the paper [20].
Figure 1: Zeros of generalized Laguerre polynomials $L_{n}^{(-A\rho)}(nx)$ for $n = 40$ and $A = 0.81$ (left), $A = 1.01$ (middle), and $A = 2$ (right).

In Figure 1 we see that the zeros cluster along certain curves in the complex plane. Martín et al. [20] identified these curves as trajectories of a quadratic differential, depending on a parameter $A$. For $A > 1$, the curve is a simple arc, which as $A$ decreases to 1 closes itself to form for $A = 1$ the well-known Szegő curve [29], [24]. For $0 < A < 1$, the curve consists of a closed loop together with an interval on the positive real axis.

A number of rigorous results on the asymptotic behavior of the zeros of generalized Laguerre polynomials $L_{n}^{(\alpha)}(nx)$ such that

$$\lim_{n \to \infty} \frac{\alpha_{n}}{n} = -A$$

are known from the literature. The first result is due to Szegő [29] who studied the partial sum of the exponential series, that is $\alpha_{n} = -n - 1$ see (1.7). Szegő showed that the normalized zeros tend to the curve which now bears his name. Olver [21] considered the zeros of Hankel functions, which includes the Bessel polynomials as a special case. In terms of the generalized Laguerre polynomials, this is the case $\alpha_{n} = -2n - 1$. Saff and Varga [26] studied the zeros and poles of Padé approximants to the exponential function, see (1.8)–(1.9). Their main result says that for integers $\alpha_{n} < -n$ such that (1.13) holds, all zeros of $L_{n}^{(\alpha_{n})}(nx)$ tend to a well-defined curve as $n \to \infty$. The curve depends on $A \geq 1$ only, and coincides with the curve described in [20]. Saff and Varga also obtained the weak limit of the zero counting measures. The proofs in [26] can be extended without any difficulty to non-integer $\alpha_{n} < -n$.

Stated in terms of generalized Bessel polynomials (1.11)–(1.12) asymptotic results on zeros are due to De Bruin et al. [4], Carpenter [5], and Wong and Zhang [31]. These papers deal with the limit (1.13) with $A = 2$. The latter paper also presents uniform asymptotic expansions of the generalized Bessel polynomials. In very recent work, Dunster [14] establishes uniform asymptotic expansions in the complex plane for the case of general $A$, with the exception of $A = 0$ and $A = 1$. The results on zeros in [14] are restricted to the case $A > 1$.

### 1.4 Asymptotics from Riemann–Hilbert problems

Most papers cited above use some form of the steepest descent technique for integrals, see especially [26] and [21]. Martín et al. [20] use an orthogonality relation in the complex plane satisfied by generalized Laguerre polynomials. The approach of Dunster [14] starts from the differential equation (1.5) and is based on techniques developed by Olver [22].

In this paper we derive asymptotics of generalized Laguerre polynomials using the non-linear steepest descent / stationary phase method for Riemann–Hilbert problems introduced by Deift and Zhou in [12], and further developed in [13] and [11]. In later developments,
the method was applied successfully to problems in random matrix theory [3], [9], and in orthogonal polynomials [3], [4], [11], [14], and combinatorics [2]. For review of some of these developments, and a pedagogic introduction to some of the material of random matrix theory, orthogonal polynomials, and Riemann–Hilbert problems, see [1].

The Riemann–Hilbert approach to the asymptotics of generalized Laguerre polynomials starts from the observation that these polynomials satisfy orthogonality relations in the complex plane. The orthogonality is on a contour \( \Sigma \) going around the positive real axis, but otherwise being quite arbitrary. The orthogonality property allows the formulation of a Riemann–Hilbert problem, due to Fokas, Its, and Kitaev [13], whose solution is given in terms of \( L_n^{(\alpha)} \). The Riemann–Hilbert problem is analyzed in the large \( n \) limit with the steepest descent method as done in [9] and [10] for orthogonal polynomials on the real line.

A novel feature for the problem at hand is that the arbitrary contour \( \Sigma \) has to be chosen in a correct way in order to arrive at a Riemann–Hilbert problem which is amenable to subsequent asymptotic analysis. The correct contour was described in [20]. It is a curve with the S-property of Stahl [27] and Gonchar and Rakhmanov [16]. The structure of the curve depends on the value of \( A \) as already explained before. In this paper we analyze the case of an open contour, that is, the case \( A > 1 \). In subsequent work we consider the case of a closed loop plus an interval \((0 < A < 1)\) and the case of a single closed contour \((A = 1)\). We note that the steepest descent / stationary phase method for Riemann–Hilbert problems was augmented to handle cases in which the contour selection involves determining a set of nontrivial curves in the plane in [18], by Kamvissis, McLaughlin, and Miller, in the context of the semi-classical limit of the focusing nonlinear Schrödinger equation.

We emphasize that the main interest in the present paper lies in the method we use and not in the results obtained for the Laguerre polynomials. In particular, we do not improve upon the asymptotic expansions of Dunster [14]. The steepest descent method for Riemann–Hilbert problems is a very powerful new method, and its use in the study of classical special functions is new. In future work we consider generalized Laguerre polynomials for the cases \( 0 < A < 1 \) and \( A = 1 \), and the Riemann–Hilbert approach will lead to new results for these cases.

2 Complex orthogonality and the formulation of the Riemann–Hilbert problem

2.1 Orthogonality

For \( \alpha < -1 \), the generalized Laguerre polynomial \( L_n^{(\alpha)} \) is not orthogonal on the positive real axis, but instead satisfies a non-hermitian orthogonality in the complex plane.

Let \( \mathcal{F} \) be the collection of all simple Jordan curves \( \Sigma \) in \( \mathbb{C} \setminus [0, \infty) \) that are symmetric with respect to the real axis, and such that there is \( M > 0 \) such that for all \( x \geq M \), there is \( y(x) > 0 \), such that the intersection of \( \Sigma \) with \( \text{Re} \, z = x \) consists of the two points \( x \pm iy(x) \), and \( \lim_{x \to \infty} y(x) = L \) exists and is finite (possibly 0). Any curve \( \Sigma \in \mathcal{F} \) divides the complex plane into two domains, \( \Omega_+ \) and \( \Omega_- \), where \( \Omega_- \) contains the positive real axis. We choose the orientation of \( \Sigma \) such that \( \Omega_+ \) is on the +–side (i.e., on the left) while traversing \( \Sigma \) and \( \Omega_- \) is on the −–side. So \( \Sigma \) is oriented clockwise as in Figure 2.

In what follows we define \( x^\alpha \) with a branch cut along the positive real axis. Thus \( x^\alpha = |x|^\alpha e^{i\alpha \arg x} \) with \( \arg x \in [0, 2\pi) \).
Lemma 2.1 Let $\Sigma \in \mathcal{F}$, $n \in \mathbb{N}$, and $\alpha \in \mathbb{R}$. Then

$$\int_{\Sigma} L_n^{(\alpha)}(x)x^kx^\alpha e^{-x} \, dx = 0, \quad \text{for } k = 0, 1, \ldots, n-1. \quad (2.1)$$

If in addition $\alpha + n + 1 \not\in \mathbb{N}$, then

$$\int_{\Sigma} L_n^{(\alpha)}(x)x^kx^\alpha e^{-x} \, dx \neq 0, \quad \text{for } k = n. \quad (2.2)$$

**Proof.** The orthogonality (2.1) follows from the Rodrigues formula (1.3) by repeated integration by parts. In the same way, we also get

$$\int_{\Sigma} L_n^{(\alpha)}(x)x^n x^\alpha e^{-x} \, dx = \frac{1}{n!} \int_{\Sigma} \left( \frac{d}{dx} \right)^n (x^{\alpha+n} e^{-x}) x^n \, dx = (-1)^n \int_{\Sigma} x^{\alpha+n} e^{-x} \, dx$$

For $\alpha + n > -1$, we deform $\Sigma$ to the positive real axis to obtain

$$\int_{\Sigma} L_n^{(\alpha)}(x)x^n x^\alpha e^{-x} \, dx = (-1)^n \left( 1 - e^{2\pi i \alpha} \right) \int_{0}^{\infty} x^{\alpha+n} e^{-x} \, dx$$

$$= (-1)^{n+1} 2i e^{\pi i \alpha} \sin(\pi \alpha) \Gamma(\alpha + n + 1), \quad (2.3)$$

where $\Gamma$ denotes the Gamma function. By analytic continuation the integral in (2.2) is equal to (2.3) for every $\alpha$, and (2.2) follows.

The formula (2.1) expresses orthogonality with respect to the complex measure $x^\alpha e^{-x} \, dx$ on $\Sigma$.

### 2.2 Riemann–Hilbert problem

Let $\alpha \in \mathbb{R}$. We consider the monic polynomials

$$P_n(z) = \frac{(-1)^n n!}{n^n} L_n^{(\alpha)}(nz), \quad n = 0, 1, \ldots, \quad (2.4)$$
Introducing a change of variables \( x = nz \) in (2.1) and (2.2), we see that
\[
\int_\Sigma P_n(z)z^kz^\alpha e^{-nz} \, dz \begin{cases} = 0, & \text{for } k = 0, 1, \ldots, n - 1, \\ \neq 0, & \text{for } k = n, \end{cases}
\]  
(2.5)
for every contour \( \Sigma \in \mathcal{F} \), provided that \( \alpha + n + 1 \not\in \mathbb{N} \).

The polynomial \( P_n \) is characterized through a Riemann–Hilbert problem due to Fokas, Its, and Kitaev [15].

**Riemann–Hilbert problem for \( Y \):**

Let \( \Sigma \) be a contour from the class \( \mathcal{F} \), that divides the complex plane into two parts \( \Omega_+ \) and \( \Omega_- \), as above. The problem is to determine a \( 2 \times 2 \) matrix valued function \( Y : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{2 \times 2} \) such that the following hold.

(a) \( Y(z) \) is analytic for \( z \in \mathbb{C} \setminus \Sigma \),

(b) \( Y(z) \) possesses continuous boundary values for \( z \in \Sigma \), denoted by \( Y_+(z) \) and \( Y_-(z) \), where \( Y_+(z) \) and \( Y_-(z) \) denote the limiting values of \( Y(z') \) as \( z' \) approaches \( z \in \Sigma \) from \( \Omega_+ \) and \( \Omega_- \), respectively, and

\[
Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^\alpha e^{-nz} \\ 0 & 1 \end{pmatrix}, \quad \text{for } z \in \Sigma,
\]  
(2.6)

(c) \( Y(z) \) has the following behavior as \( z \to \infty \):

\[
Y(z) = \left( I + O \left( \frac{1}{z} \right) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \quad \text{as } z \to \infty, \; z \in \mathbb{C} \setminus \Sigma.
\]  
(2.7)

**Proposition 2.2** Let \( \alpha \in \mathbb{R} \) with \( \alpha + n \not\in \mathbb{N} \). Then the unique solution of the Riemann–Hilbert problem for \( Y \) is given by

\[
Y(z) = \begin{pmatrix} P_n(z) & 1 \frac{1}{2\pi i} \int_\Sigma P_n(\zeta)z^\alpha e^{-nz} \, d\zeta \\ Q_{n-1}(z) & 1 \frac{1}{2\pi i} \int_\Sigma Q_{n-1}(\zeta)z^\alpha e^{-nz} \, d\zeta \end{pmatrix}
\]  
(2.8)

where \( P_n(z) \) is the monic generalized Laguerre polynomial (2.4) and

\[
Q_{n-1}(z) = \frac{(-1)^n n^{n+\alpha} \pi e^{-\pi i \alpha}}{\sin(\pi \alpha) \Gamma(\alpha + n)} L^{(\alpha)}_{n-1}(nz).
\]  
(2.9)

**Proof.** The proof is as in [3, Section 3.2]. Here we will only give the proof for the second row, since that is where the condition on \( \alpha \) plays a role.

From (2.4) it follows that \( Y_{21} \) is an entire function, which by (2.7) satisfies \( Y_{21}(z) = O(z^{n-1}) \) as \( z \to \infty \). Therefore \( Y_{21}(z) = Q_{n-1}(z) \) for some polynomial \( Q_{n-1} \) of degree at most \( n - 1 \). The (2,2) entry of the condition (2.6) is \( (Y_{22})_+(z) = (Y_{22})_-(z) + Q_{n-1}(z)z^\alpha e^{-nz} \), which by the Sokhotskii-Plemelj formula yields

\[
Y_{22}(z) = \frac{1}{2\pi i} \int_\Sigma Q_{n-1}(\zeta)z^\alpha e^{-nz} \, d\zeta.
\]  
(2.10)
From the (2,2) entry of (2.7) it follows that \( Y_{22}(z) = z^{-n} + O(z^{-n-1}) \) as \( z \to \infty \). Because of (2.10) this gives the conditions
\[
\int_{\Sigma} Q_{n-1}(\zeta) \zeta^\alpha e^{-n\zeta} \zeta^k d\zeta = 0 \quad \text{for } k = 0, \ldots, n-2, \tag{2.11}
\]
and
\[
\int_{\Sigma} Q_{n-1}(\zeta) \zeta^\alpha e^{-n\zeta} \zeta^{-1} d\zeta = -2\pi i. \tag{2.12}
\]
The orthogonality conditions (2.11) are satisfied if \( Q_{n-1}(z) = d_n L^{(\alpha)}_{n-1}(nz) \), and the constant \( d_n \) must be chosen so that (2.12) holds as well. Because of Lemma 2.1 this can be done if \( \alpha + n \not\in \mathbb{N} \), and the result is given by (2.9). \( \square \)

3 Selection of the right contour

3.1 The right contour \( \Sigma \)

In Section 3–6, we assume \( n \in \mathbb{N} \) and \( \alpha < -n \) are fixed. We write \( A = -\alpha/n \), so that \( A > 1 \). [Later, when we let \( n \to \infty \), \( \alpha \) and \( A \), as well as other notions introduced in these sections, will depend on \( n \).]

A major step in the analysis of the Riemann–Hilbert problem for \( Y \) is the selection of the right contour. In order that the subsequent analysis works, the contour cannot be arbitrary but has to chosen in a precise way. The contour depends on \( A \). We define
\[
\beta = 2 - A + 2i\sqrt{A - 1} \tag{3.1}
\]
Martinez et al. [20] showed that the values of \( z \) for which
\[
\frac{1}{2\pi i} \int_{\bar{\beta}}^{\beta} \frac{(s - \beta)^{1/2}}{s} \frac{(s - \bar{\beta})^{1/2}}{s} ds \text{ is real,}
\]
where the branch of the square roots is chosen so that they are analytic and single valued on the path of integration from \( \bar{\beta} \) to \( \beta \), form a system of curves as shown in Figure 3 for a number of values of \( A \). In geometric function theory these curves are known as trajectories of the quadratic differential \( \frac{(s - \beta)(s - \bar{\beta})}{s^2} d^2s \), see [28]. We see four smooth (in fact analytic)

Figure 3: Curves where \( \frac{1}{2\pi i} \int_{\bar{\beta}}^{\beta} \frac{R(s)}{s} ds \) is real, for the values \( A = 1.01 \) (left) and \( A = 2 \) (right). The two points of intersection are \( \bar{\beta} \) and \( \beta \).

curves. Two curves are connecting \( \bar{\beta} \) with \( \beta \), one of them crosses the negative real axis, and the other one crosses the positive real axis.
**Definition 3.1** We define $\Gamma$ as the trajectory of the quadratic differential $\frac{(s-\beta)(s-\bar{\beta})}{s^2} \, ds$ from $\bar{\beta}$ to $\beta$ which crosses the negative real axis. $\Gamma$ is oriented from $\bar{\beta}$ to $\beta$.

We put

$$R(z) = (z - \beta)^{1/2}(z - \bar{\beta})^{1/2}, \quad z \in \mathbb{C} \setminus \Gamma,$$

where the branch is chosen which is defined and analytic on $\mathbb{C} \setminus \Gamma$, and which is such that $R(z) \sim z$ as $z \to \infty$. For $s \in \Gamma$, we use $R_+(s)$ and $R_-(s)$ to denote the limits from the $+$-sides and $-$-sides, respectively. As usual, the $+$-side of an oriented curve lies to the left, and the $-$-side lies to the right, if one traverses the curve.

Then by definition of $\Gamma$, we have

$$\frac{1}{2\pi i} \int_{\bar{\beta}}^{z} R_+(s) \, ds \quad \text{is real for every } z \in \Gamma,$$

with integration along the $+$-side of $\Gamma$.

It is also of interest to know where $\frac{1}{2\pi i} \int_{\bar{\beta}}^{z} R(s) \, ds$ or $\frac{1}{2\pi i} \int_{\beta}^{z} R(s) \, ds$ is purely imaginary. These are the dotted curves shown in Figure 4. The dotted curves are analytic extensions of the solid ones.

![Figure 4](image)

Figure 4: Curves where $\frac{1}{2\pi i} \int_{\bar{\beta}}^{z} R(s) \, ds$ is real (solid lines) and curves where $\frac{1}{2\pi i} \int_{\bar{\beta}}^{z} R(s) \, ds$ or $\frac{1}{2\pi i} \int_{\beta}^{z} R(s) \, ds$ is purely imaginary (dotted lines), for the values $A = 1.01$ (left) and $A = 2$ (right).

We can now state which contour $\Sigma$ to choose.

**Definition 3.2** We let $\Sigma$ be the contour in $\mathcal{F}$ consisting of $\Gamma$ together with the two dotted curves that form the analytic extension of $\Gamma$.

We denote the part of $\Sigma \setminus \Gamma$ in the lower half-plane by $\Sigma_1$ and its mirror image in the upper half-plane by $\Sigma_2$.

So we have a disjoint union $\Sigma = \Gamma \cup \Sigma_1 \cup \Sigma_2$. Figure 8 shows the curve $\Sigma$ for two values of $A$, together with the zeros of the corresponding Laguerre polynomial $L_{40}^{(-40A)}(40x)$ of degree 40. The figure shows that the zeros are close to $\Gamma$, and that they are in the domain $\Omega_-$. These findings will be confirmed by our final result, Corollary 7.2 below.

**Remark 3.3** We have chosen $\Sigma_2$ so that $\int_{\bar{\beta}}^{z} \frac{R(s)}{s} \, ds$ is real and positive on $\Sigma_2$. This is not essential. What is important for the subsequent analysis is that it has positive real part on $\Sigma_2$. This means that we have the freedom to deform $\Sigma_2$, as long as we take care that the real part of $\int_{\bar{\beta}}^{z} \frac{R(s)}{s} \, ds$ is positive on $\Sigma_2$. However, it will be convenient to do this deformation only away from $\beta$, so that in a neighborhood of $\beta$, we have $\Sigma_2$ exactly as we defined it.

Similar remarks apply to $\int_{\beta}^{z} \frac{R(s)}{s} \, ds$ and $\Sigma_1$. 

8
3.2 A probability measure on $\Gamma$

The following proposition gives one of the crucial properties of $\Gamma$.

**Proposition 3.4** *The (complex) measure $\frac{1}{2\pi i} \frac{R_+(s)}{s} ds$ is a probability measure on $\Gamma$.***

**Proof.** We show first that

$$\frac{1}{2\pi i} \int_\Gamma \frac{R_+(s)}{s} ds = 1. \quad (3.4)$$

Let $I = \int_\Gamma \frac{R_+(s)}{s} ds$. Since $R_-(-s) = -R_+(s)$ for $s \in \Gamma$, we have

$$2I = \int_\Gamma \left( \frac{R_+(s)}{s} - \frac{R_-(s)}{s} \right) ds = \oint_\gamma \frac{R(s)}{s} ds,$$

where $\gamma$ is a closed contour encircling the curve $\Gamma$ once in the clockwise direction and not encircling $z = 0$.

After contour deformation, we pick up residues at $z = 0$ and at $z = \infty$, namely

$$2I = 2\pi i \text{Res}_{z=0} \frac{R(z)}{z} - 2\pi i \text{Res}_{z=\infty} \frac{R(z)}{z}. \quad (3.5)$$

The residue at $z = 0$ is

$$\text{Res}_{z=0} \frac{R(z)}{z} = R(0) = |\beta| = \sqrt{(2 - A)^2 + 4(A - 1)} = A \quad (3.6)$$

and the residue at $z = \infty$ is the coefficient of $z^{-1}$ in the Laurent expansion of $R(z)/z$

$$\frac{R(z)}{z} = (1 - \beta/z)^{1/2}(1 - \bar{\beta}/z)^{1/2}$$

$$= \left(1 - \frac{\beta}{2z} + O(z^{-2}) \right) \left(1 - \frac{\bar{\beta}}{2z} + O(z^{-2}) \right)$$

$$= 1 - \frac{\beta + \bar{\beta}}{2z} + O(z^{-2}).$$

Thus

$$\text{Res}_{z=\infty} \frac{R(z)}{z} = -\frac{\beta + \bar{\beta}}{2} = -\text{Re} \beta = -(2 - A). \quad (3.7)$$
Hence by (3.3)–(3.7) we have $2I = 2\pi i(A + (2 - A)) = 4\pi i$, so that (3.4) follows.

Having (3.4) we can now prove the proposition. Let $t \mapsto z(t)$ for $t \in [0, t_0]$, be the arc length parametrization of $\Gamma$ starting at $\beta$. Thus $z(0) = \beta$ and $z(t_0) = \beta$. Then we have for $z = z(t)$ with $t \in (0, t_0),$

$$
\frac{1}{2\pi i} \int_{\beta}^{z} \frac{R_+(s)}{s} ds = \frac{1}{2\pi i} \int_{0}^{t} \frac{R_+(z(t))}{z(t)} z'(t) dt,
$$

and this is real for every $t$ by construction of $\Gamma$. It has the value 0 for $t = 0$ and the value 1 for $t = t_0$ (due to (3.4)). The derivative of (3.8) with respect to $t$ is $\frac{1}{2\pi i} \frac{R_+(z(t))}{z(t)} z'(t)$ which is not zero for $t \in (0, t_0)$. Thus (3.8) can only increase from 0 to 1 as $t$ increases from 0 to $t_0$. Hence $\frac{1}{2\pi i} \frac{R_+(s)}{s} ds$ is a positive measure on $\Gamma$. It is a probability measure because of (3.4). □

### 3.3 Auxiliary functions

With the measure $d\mu(s) = \frac{1}{2\pi i} \frac{R_+(s)}{s} ds$ on $\Gamma$ we define the so-called $g$-function as follows.

**Definition 3.5** The $g$-function is the complex logarithmic potential of $\mu$, that is,

$$
g(z) = \int_{\Gamma} \log(z - s) d\mu(s), \quad z \in \mathbb{C} \setminus (\Gamma \cup \Sigma_1),
$$

where for each $s$ we view $\log(z - s)$ as an analytic function of the variable $z$, with branch cut emanating from $z = s$. The cut is taken along $\Gamma \cup \Sigma_1$.

We need two more functions.

**Definition 3.6** The $\phi$-function is defined as

$$
\phi(z) = \frac{1}{2} \int_{\beta}^{z} \frac{R(s)}{s} ds, \quad z \in \mathbb{C} \setminus (\Gamma \cup \Sigma_1 \cup [0, \infty)),
$$

where the path of integration from $\beta$ to $z$ lies entirely in $\mathbb{C} \setminus (\Gamma \cup \Sigma_1 \cup [0, \infty))$, except for the initial point $\beta$.

The $\tilde{\phi}$-function is defined as

$$
\tilde{\phi}(z) = \frac{1}{2} \int_{\beta}^{z} \frac{R(s)}{s} ds, \quad z \in \mathbb{C} \setminus (\Gamma \cup \Sigma_2 \cup [0, \infty)),
$$

where the path of integration from $\tilde{\beta}$ to $z$ lies entirely in $\mathbb{C} \setminus (\Gamma \cup \Sigma_2 \cup [0, \infty))$, except for the initial point $\tilde{\beta}$.

It is immediate from (3.3) that $\tilde{\phi}_+(z)$ is purely imaginary for $z \in \Gamma$. By Proposition 3.4 its imaginary part increases from 0 to $\pi$ as $z$ traverses the curve $\Gamma$ from $\tilde{\beta}$ to $\beta$. In particular we have $\tilde{\phi}_+(\beta) = \pi i$. Similarly $\tilde{\phi}_-(\beta) = -\pi i$. Thus we have

$$
\tilde{\phi}(z) = \begin{cases} 
\phi(z) + \pi i & \text{for } z \in \Omega_+, \\
\phi(z) - \pi i & \text{for } z \in \Omega_-. 
\end{cases}
$$

**Proposition 3.7** There is a constant $\ell$ such that

$$
g(z) = \frac{1}{2} (A \log z + z + \ell) - \phi(z), \quad z \in \mathbb{C} \setminus (\Gamma \cup \Sigma_1 \cup [0, \infty)),
$$

where $\log z$ is defined with a branch cut along $[0, \infty)$.
Proof. We note that
\[ g'(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{R(s)}{s} ds = \frac{1}{2\pi i} \oint_{\gamma} \frac{R(s)}{s} ds \]
where \( \gamma \) is a closed contour in \( \mathbb{C} \setminus \Gamma \), that encircles \( \Gamma \) once in the clockwise direction, but does not encircle \( z \) and 0. Then exactly as in the proof of Proposition 3.4, we deform the contour, and now pick up residues at \( z, 0 \) and \( \infty \). We find for \( z \in \mathbb{C} \setminus \Gamma \),
\[ g'(z) = \frac{1}{2} \left[ \text{Res}_{s=z} \left( \frac{1}{z-s} \frac{R(s)}{s} \right) + \text{Res}_{s=0} \left( \frac{1}{z-s} \frac{R(s)}{s} \right) - \text{Res}_{s=\infty} \left( \frac{1}{z-s} \frac{R(s)}{s} \right) \right] \]
\[ = \frac{1}{2} \left[ -\frac{R(z)}{z} + \frac{R(0)}{z} + 1 \right] \]
\[ = \frac{1}{2} \left[ -\frac{R(z)}{z} + \frac{A}{z} + 1 \right]. \quad (3.14) \]

Let \( z \in \mathbb{C} \setminus (\Gamma \cup \Sigma_1 \cup [0, \infty)) \). Integrating (3.14) from \( \beta \) to \( z \) along a curve in \( \mathbb{C} \setminus (\Gamma \cup \Sigma_1 \cup [0, \infty)) \), we find
\[ g(z) = g(\beta) + \frac{1}{2} (A \log z + z) - \frac{1}{2} (A \log \beta + \beta) - \phi(z). \quad (3.15) \]
Thus (3.13) holds with \( \ell = 2g(\beta) - (A \log \beta + \beta) \). \( \square \)

3.4 Jump properties of \( g \)

From Proposition 3.7 we obtain the following jump relations for \( g \) across the contour \( \Sigma \). These jumps are crucial for the subsequent analysis.

Proposition 3.8  (a) We have
\[ g_+(z) - g_-(z) = 2\pi i \quad \text{for } z \in \Sigma_1, \quad (3.16) \]
and
\[ g_+(z) - g_-(z) = -\phi_+(z) + \phi_-(z) = -2\phi_+(z) = 2\phi_-(z) \quad \text{for } z \in \Gamma. \quad (3.17) \]

(b) We have, with the same constant \( \ell \) as in Proposition 3.7,
\[ g_+(z) + g_-(z) = A \log z + z + \ell \quad \text{for } z \in \Gamma, \quad (3.18) \]
\[ g_+(z) + g_-(z) = A \log z + z + \ell - 2\tilde{\phi}(z) \quad \text{for } z \in \Sigma_1, \quad (3.19) \]
and
\[ g_+(z) + g_-(z) = A \log z + z + \ell - 2\phi(z) \quad \text{for } z \in \Sigma_2. \quad (3.20) \]

Proof. In (3.13) we let \( z \) approach \( \Sigma \), from the + and - sides, respectively, to obtain
\[ g_\pm(z) = \frac{1}{2} (A \log z + z) + \frac{1}{2} \ell - \phi_\pm(z), \quad \text{for } z \in \Sigma. \quad (3.21) \]
Since \( \phi \) changes sign across \( \Gamma \), (3.17) and (3.18) immediately follow from (3.21). For \( z \in \Sigma_1 \), we have by (3.12) and (3.21)

\[
g_+(z) - g_-(z) = -\phi_+(z) + \phi_-(z) = -\tilde{\phi}_+(z) - \pi i + (\tilde{\phi}_-(z) + \pi i)
\]

\[
= 2\pi i - \tilde{\phi}_+(z) + \tilde{\phi}_-(z),
\]

which is (3.16) as \( \tilde{\phi} \) is analytic across \( \Sigma_1 \). In addition, we have

\[
g_+(z) + g_-(z) = A \log z + z + \ell - \phi_+(z) - \phi_-(z)
\]

\[
= A \log z + z + \ell - \tilde{\phi}_+(z) - \tilde{\phi}_-(z) \quad \text{for } z \in \Sigma_1.
\]

which yields (3.19). Similarly, (3.20) follows. \( \square \)

4 First two transformations: \( Y \mapsto U \mapsto T \)

4.1 First transformation \( Y \mapsto U \)

With the \( g \)-function and the constant \( \ell \) from Proposition 3.7, we perform the first transformation of the Riemann–Hilbert problem.

**Definition 4.1** We define for \( z \in \mathbb{C} \setminus \Sigma \),

\[
U(z) = e^{-n(\ell/2)\sigma_3}Y(z)e^{-ng(z)\sigma_3}e^{n(\ell/2)\sigma_3}.
\]  

(4.1)

Here, and in what follows, \( \sigma_3 \) denotes the Pauli matrix \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), so that for example

\[
e^{-ng(z)\sigma_3} = \begin{pmatrix} e^{-ng(z)} & 0 \\ 0 & e^{ng(z)} \end{pmatrix}.
\]

From the Riemann–Hilbert problem for \( Y \) it follows by a straightforward calculation that \( U \) is the unique solution of the following Riemann–Hilbert problem.

**Riemann–Hilbert problem for \( U \):**

The problem is to determine a \( 2 \times 2 \) matrix valued function \( U : \mathbb{C} \setminus \Sigma \rightarrow \mathbb{C}^{2 \times 2} \) such that

(a) \( U(z) \) is analytic for \( z \in \mathbb{C} \setminus \Sigma \),

(b) \( U(z) \) possesses continuous boundary values for \( z \in \Sigma \), denoted by \( U_+(z) \) and \( U_-(z) \), and

\[
U_+(z) = U_-(z) \begin{pmatrix} e^{-n(g_+(z)-g_-(z))} & z^{-An}e^{-nz}e^{n(g_+(z)+g_-(z)-\ell)} \\ 0 & e^{n(g_+(z)-g_-(z))} \end{pmatrix}  
\]

(4.2)

for \( z \in \Sigma \),

(c) \( U(z) \) behaves like the identity at infinity:

\[
U(z) = I + O\left(\frac{1}{z}\right) \quad \text{as } z \rightarrow \infty, \quad z \in \mathbb{C} \setminus \Sigma.
\]  

(4.3)
The jump relation (4.2) for \( U \) has a different form on the three parts \( \Gamma, \Sigma_1 \) and \( \Sigma_2 \). On \( \Gamma \) we have that \( g_+ - g_- = -2\phi_+ = 2\phi_- \) by (3.17), and \( g_+ + g_- = A \log z + \ell \) by (3.18) so that

\[
U_+(z) = U_-(z) \begin{pmatrix} e^{2n\phi_+(z)} & 1 \\ 0 & e^{2n\phi_-(z)} \end{pmatrix} \quad \text{for } z \in \Gamma. \tag{4.4}
\]

On \( \Sigma_1 \) we use (3.16) and (3.19) to obtain

\[
U_+(z) = U_-(z) \begin{pmatrix} 1 & e^{-2n\tilde{\phi}(z)} \\ 0 & 1 \end{pmatrix} \quad \text{for } z \in \Sigma_1, \tag{4.5}
\]

and similarly it follows that

\[
U_+(z) = U_-(z) \begin{pmatrix} 1 & e^{-2n\phi_+(z)} \\ 0 & 1 \end{pmatrix} \quad \text{for } z \in \Sigma_2. \tag{4.6}
\]

The transformation \( Y \mapsto U \) has the effect of normalizing the Riemann–Hilbert problem at infinity. In addition, by the construction of \( \Sigma \), we have that \( \phi \) is real and positive on \( \Sigma_2 \), and \( \tilde{\phi} \) is real and positive on \( \Sigma_1 \). So the jump matrices for \( U \) in (4.4) and (4.6) are close to the identity if \( n \) is large. Since \( \phi \) has purely imaginary boundary values on both sides of \( \Gamma \), the jump matrix for \( U \) on \( \Gamma \) in (4.4) has oscillatory diagonal entries.

### 4.2 Second transformation \( U \mapsto T \)

The jump matrix for \( U \) on \( \Gamma \), see (4.4), factors as

\[
\begin{pmatrix} e^{2n\phi_+(z)} & 1 \\ 0 & e^{2n\phi_-(z)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{2n\phi_-(z)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-2n\phi_+(z)} & 1 \end{pmatrix}. \tag{4.7}
\]

Observe that the first matrix in the right-hand side of (4.7) can be analytically continued to the \(-\) side of the contour \( \Gamma \), and in doing so, the \((1,2)\) entry becomes exponentially decaying in \( n \). Similarly, the third matrix in the right-hand side of (4.7) can be analytically continued to the \(+\) side of the contour \( \Gamma \), and in doing so, the \((1,2)\) entry also becomes exponentially decaying in \( n \). We are thus led to introduce the following “contour augmentation” step, as part of the steepest descent / stationary phase method for Riemann–Hilbert problems developed by Deift and Zhou. The oriented contour \( \Sigma^T \) consists of \( \Sigma \) plus two simple curves \( \Sigma_3 \) and \( \Sigma_4 \) from \( \bar{\beta} \) to \( \beta \), contained in \( \Omega_+ \) and \( \Omega_- \), respectively, as shown in Figure 3. We choose \( \Sigma_3 \) and \( \Sigma_4 \) such that \( \text{Re} \phi(z) < 0 \) on \( \Sigma_3 \) and \( \Sigma_4 \).

It is possible to choose such curves. Indeed, \( \phi \) is positive on \( \Sigma_2 \), and its real part vanishes on \( \Gamma \) and on the other solid lines shown in Figure 4. So \( \text{Re} \phi > 0 \) in the full region on the right. In the two other regions, bounded by the solid lines, we then have that \( \text{Re} \phi < 0 \). Note that \( \text{Re} \phi \) does not change sign across \( \Gamma \).

Note that by (3.12) we also have \( \text{Re} \Phi(z) < 0 \) on \( \Sigma_3 \) and \( \Sigma_4 \).

Then \( \mathbb{C} \setminus \Sigma^T \) has four connected components, denoted by \( \Omega_1, \Omega_2, \Omega_3, \) and \( \Omega_4 \) as indicated in Figure 4.

**Definition 4.2** We define \( T : \mathbb{C} \setminus \Sigma^T \to \mathbb{C}^{2\times 2} \) by

\[
T(z) = U(z) \quad \text{for } z \in \Omega_1 \cup \Omega_4. \tag{4.8}
\]
Figure 6: Contour $\Sigma^T = \Gamma \cup \bigcup_j \Sigma_j$ and the domains $\Omega_j$, $j = 1, \ldots, 4$, for the Riemann–Hilbert problem for $T$, for the value $A = 1.01$.

$$T(z) = U(z) \begin{pmatrix} 1 & 0 \\ -e^{2n\phi(z)} & 1 \end{pmatrix} \quad \text{for } z \in \Omega_2,$$

$$T(z) = U(z) \begin{pmatrix} 1 & 0 \\ e^{2n\phi(z)} & 1 \end{pmatrix} \quad \text{for } z \in \Omega_3,$$

Then from the Riemann–Hilbert problem for $U$ and the factorization (4.7) we obtain that $T$ is the unique solution of the following Riemann–Hilbert problem.

**Riemann–Hilbert problem for $T$:**

The problem is to determine a $2 \times 2$ matrix valued function $T : \mathbb{C} \setminus \Sigma^T \to \mathbb{C}^{2 \times 2}$ such that the following hold:

(a) $T(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma^T$, 

(b) $T(z)$ possesses continuous boundary values for $z \in \Sigma^T$, denoted by $T_+(z)$ and $T_-(z)$, and

$$T_+(z) = T_-(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for } z \in \Gamma,$$

$$T_+(z) = T_-(z) \begin{pmatrix} 1 & 0 \\ e^{2n\phi(z)} & 1 \end{pmatrix} \quad \text{for } z \in \Sigma_3 \cup \Sigma_4,$$

$$T_+(z) = T_-(z) \begin{pmatrix} 1 & e^{-2n\tilde{\phi}(z)} \\ 0 & 1 \end{pmatrix} \quad \text{for } z \in \Sigma_1,$$

and

$$T_+(z) = T_-(z) \begin{pmatrix} 1 & e^{-2n\phi(z)} \\ 0 & 1 \end{pmatrix} \quad \text{for } z \in \Sigma_2,$$

(c) $T(z)$ behaves like the identity at infinity:

$$T(z) = I + O\left(\frac{1}{z}\right) \quad \text{as } z \to \infty, \quad z \in \mathbb{C} \setminus \Sigma^T.$$
5 Construction of the parametrix for $T$

5.1 Parametrix away from $\beta$ and $\bar{\beta}$

As remarked following (4.7), the jump matrices appearing in (4.12) for $z \in \Sigma_3 \cup \Sigma_4$ are exponentially close to the identity matrix away from $\beta$ and $\bar{\beta}$. Similarly, the jump matrices appearing in (4.13) and (4.14) are also exponentially close to the identity matrix away from $\beta$ and $\bar{\beta}$. This hints that these portions of the contour on which the Riemann–Hilbert problem for $T$ is posed should be somehow negligible. Thus we expect that the leading order asymptotics is determined by the solution $N$ of the following model Riemann–Hilbert problem:

Riemann–Hilbert problem for $N$:

The problem is to determine $N : \mathbb{C} \setminus \Gamma \to \mathbb{C}^{2 \times 2}$ such that the following hold.

(a) $N(z)$ is analytic for $z \in \mathbb{C} \setminus \Gamma$,

(b) $N(z)$ possesses continuous boundary values for $z \in \Gamma \setminus \{\beta, \bar{\beta}\}$, denoted by $N_+(z)$ and $N_-(z)$, and

$$N_+(z) = N_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{for } z \in \Gamma \setminus \{\beta, \bar{\beta}\},$$  \hspace{1cm} (5.1)

(c) $N(z) = I + O \left(\frac{1}{z}\right)$ for $z \to \infty$.

This Riemann–Hilbert problem for $N$ is solved explicitly by, see [10, p.1520] or [6, p.200],

$$N(z) = \begin{pmatrix} \frac{a(z) + a(z)^{-1}}{2} & \frac{a(z) - a(z)^{-1}}{2i} \\ \frac{a(z) - a(z)^{-1}}{2i} & \frac{a(z) + a(z)^{-1}}{2} \end{pmatrix},$$  \hspace{1cm} (5.2)

where

$$a(z) = \frac{(z - \beta)^{1/4}}{(z - \bar{\beta})^{1/4}}.$$  \hspace{1cm} (5.3)

The branches of the roots in (5.3) are chosen such that $a(z)$ is analytic on $\mathbb{C} \setminus \Gamma$ and $\lim_{z \to \infty} a(z) = 1$.

**Remark 5.1** The solution (5.2) is not the only solution to the Riemann–Hilbert problem for $N$. It is the unique solution that satisfies, in addition to (a), (b), and (c), the condition

(d) Near the endpoints $\beta$ and $\bar{\beta}$, we have

$$N(z) = O(|z - \beta|^{-1/4}) \quad \text{as } z \to \beta,$$

$$N(z) = O(|z - \bar{\beta}|^{-1/4}) \quad \text{as } z \to \bar{\beta},$$

with the $O$-term being taken entry-wise.
Remark 5.2 For explicit calculations, it is useful to have an alternative expression for $N$. For $z = x$ real, it is clear from (5.3) that $a(x)$ has modulus one. If $\arg(a(x)) = \theta(x)$, then (5.2) shows

$$N(x) = \begin{pmatrix} \cos \theta(x) & \sin \theta(x) \\ -\sin \theta(x) & \cos \theta(x) \end{pmatrix}.$$  

Since $\tan(2\theta(x)) = -\frac{2\sqrt{A-1}}{x-2+A}$, we then find

$$N(z) = \begin{pmatrix} \cos \left( \frac{1}{2} \arctan \left( \frac{2\sqrt{A-1}}{x-2+A} \right) \right) & \sin \left( \frac{1}{2} \arctan \left( \frac{2\sqrt{A-1}}{x-2+A} \right) \right) \\ \sin \left( \frac{1}{2} \arctan \left( \frac{2\sqrt{A-1}}{x-2+A} \right) \right) & \cos \left( \frac{1}{2} \arctan \left( \frac{2\sqrt{A-1}}{x-2+A} \right) \right) \end{pmatrix}$$

first for $z = x$ real, but then also for arbitrary $z \in \mathbb{C} \setminus \Gamma$ by analytic continuation. We have to take the appropriate branch of the multivalued arctan function in (5.4). Using trigonometric identities, one may then check that (5.4) reduces to

$$N(z) = \begin{pmatrix} \left( \frac{1+R'(z)}{2} \right)^{1/2} - \left( \frac{1-R'(z)}{2} \right)^{1/2} \\ \left( \frac{1-R'(z)}{2} \right)^{1/2} + \left( \frac{1+R'(z)}{2} \right)^{1/2} \end{pmatrix}$$

for $z \in \mathbb{C} \setminus \Gamma$, (5.5)

where as usual we have $R(z) = (z - \beta)^{1/2}(z - \bar{\beta})^{1/2}$. We can also verify directly that (5.5) solves the Riemann–Hilbert problem for $N$.

5.2 Parametrix near $\beta$

The next step is a local analysis around the points $\beta$ and $\bar{\beta}$. We need to construct a local parametrix $P$ in a neighborhood $U_\delta = \{z \in \mathbb{C} \mid |z - \beta| < \delta\}$ of $\beta$ such that

- $P$ satisfies the jumps for $T$ exactly in $U_\delta$,
- $P$ matches $N$ on the boundary of $U_\delta$ up to order $1/n$.

See Figure 7 for the contours $\Sigma^T \cap U_\delta$.

More precisely, we have

Riemann–Hilbert problem for $P$:

The problem is to determine, for a given $\delta > 0$ sufficiently small, a $2 \times 2$ matrix valued function $P : \overline{U}_\delta \setminus \Sigma^T \to \mathbb{C}^{2 \times 2}$ such that

(a) $P(z)$ is analytic for $z \in U_\delta \setminus \Sigma^T$, and continuous on $\overline{U}_\delta \setminus \Sigma^T$,

(b) $P(z)$ possesses continuous boundary values for $z \in \Sigma^T \cap U_\delta$, denoted by $P_+(z)$ and $P_-(z)$, and

$$P_+(z) = P_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{for } z \in \Gamma \cap U_\delta,$$  

$$P_+(z) = P_-(z) \begin{pmatrix} 1 & e^{2n\phi(z)} \\ e^{-2n\phi(z)} & 0 \end{pmatrix} \quad \text{for } z \in (\Sigma_3 \cup \Sigma_4) \cap U_\delta,$$  

and

$$P_+(z) = P_-(z) \begin{pmatrix} 1 & e^{-2n\phi(z)} \\ 0 & 1 \end{pmatrix} \quad \text{for } z \in \Sigma_2 \cap U_\delta,$$  

where $\phi(z) = \arg(z - \beta)$.
Figure 7: Neighborhood $U_\delta$ of $\beta$ and the parts of the contours $\Gamma$, $\Sigma_2$, $\Sigma_3$, and $\Sigma_4$ that are within $U_\delta$. The value of $A$ is 1.01.

(c) There exists a constant $C > 0$ such that for every $z \in \partial U_\delta \setminus \Sigma^T$,

$$\|P(z)N^{-1}(z) - I\| \leq \frac{C}{n},$$

where $\| \cdot \|$ is any matrix norm.

The construction of $P$ follows along the same lines as given by Deift et al. From its definition (3.10) it is easy to see that the $\phi$-function has a convergent expansion

$$\phi(z) = (z - \beta)^{3/2} \sum_{k=0}^{\infty} c_k (z - \beta)^k, \quad c_0 \neq 0,$$

in a neighborhood of $\beta$. The factor $(z - \beta)^{3/2}$ is defined with a cut along $\Gamma \cup \Sigma_1$. Then

$$f(z) = \left[\frac{3}{2} \phi(z)\right]^{2/3}$$

is defined and analytic in a neighborhood of $\beta$. We choose the $2/3$-root with a cut along $\Gamma$ and such that $f(z) > 0$ for $z \in \Sigma_2$. Recall $\phi > 0$ on $\Sigma_2$.

Then $f(\beta) = 0$ and $f'(\beta) \neq 0$. Therefore we can and do choose $\delta$ so small that $\zeta = f(z)$ is a one-to-one mapping from $U_\delta$ onto a convex neighborhood $f(U_\delta)$ of $\zeta = 0$. Under the mapping $\zeta = f(z)$, we then have that $\Sigma_2 \cap U_\delta$ corresponds to $(0, \infty) \cap f(U_\delta)$ and that $\Gamma \cap U_\delta$ corresponds to $(-\infty, 0] \cap f(U_\delta)$.

Now we specify how to choose $\Sigma^T$ near $\beta$. For an arbitrary, but fixed $\sigma \in (\pi/3, \pi)$, we choose $\Sigma_3$ and $\Sigma_4$ such that $f$ maps $\Sigma_3 \cap U_\delta$ and $\Sigma_4 \cap U_\delta$ onto $\{\zeta \in f(U_\delta) \mid \arg \zeta = \sigma\}$ and $\{\zeta \in f(U_\delta) \mid \arg \zeta = -\sigma\}$, respectively.

**Proposition 5.3** The Riemann–Hilbert problem for $P$ is solved by

$$P(z) = E(z)\Psi^\sigma(n^{2/3}f(z))e^{n\phi(z)\sigma_3}$$

(5.12)
where
\[ E(z) = \sqrt{\pi} e^{\frac{z^2}{4}} \left( \begin{array}{cc} 1 & -1 \\ -i & -i \end{array} \right) \left( \frac{n^{1/6} f(z)^{1/4}}{a(z)} \right)^{\sigma_3} \] (5.13)

and \( \Psi^\sigma \) is an explicit matrix valued function built out of the Airy function \( \text{Ai} \) and its derivative \( \text{Ai}' \) as follows

\[
\Psi^\sigma(\zeta) = \left\{ \begin{array}{ll}
\text{Ai}(\zeta) & \text{Ai}(\omega^2 \zeta) \\
\text{Ai}'(\zeta) & \omega^2 \text{Ai}'(\omega^2 \zeta) \\
\text{Ai}(\zeta) & \omega^2 \text{Ai}(\omega^2 \zeta) \\
\text{Ai}'(\zeta) & -\text{Ai}(\omega \zeta) \\
\text{Ai}(\zeta) & -\text{Ai}'(\omega \zeta) \\
\text{Ai}'(\zeta) & -\text{Ai}'(\omega \zeta) \\
\end{array} \right.
\]

\[
e^{-\frac{z \zeta}{\sigma_3}} \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right)
\]

for \( 0 < \arg \zeta < \sigma \), \( \sigma < \arg \zeta < \pi \), \( -\pi < \arg \zeta < -\sigma \), \( -\sigma < \arg \zeta < 0 \),

with \( \omega = e^{2\pi i/3} \).

**Proof.** The proof is similar to [10, p.1523–1525]. \( \square \)

**Remark 5.4** An analysis of the proof in [10] shows that the constant \( C \) in (5.9) can be taken uniformly for \( \sigma \) in a compact subset \( K_\sigma \) of \((\pi/3, \pi)\), for \( A \) in a compact subset \( K_A \) of \((1, \infty)\), and for \( \delta \) uniformly in some interval \((\delta_0, \delta_1)\) depending on \( K_\sigma \) and \( K_A \).

A similar remark applies to the parametrix \( \tilde{P} \) to be constructed below.

### 5.3 Parametrix near \( \tilde{\beta} \)

A similar construction yields a parametrix \( \tilde{P} \) in a neighborhood \( \tilde{U}_\delta = \{ z \mid |z - \tilde{\beta}| < \delta \} \) that satisfies the following Riemann–Hilbert problem.

**Riemann–Hilbert problem for \( \tilde{P} \):**

The problem is to determine, for a given \( \delta > 0 \) sufficiently small, a \( 2 \times 2 \) matrix valued function \( \tilde{P} : \tilde{U}_\delta \setminus \Sigma^T \to \mathbb{C}^{2 \times 2} \) such that

(a) \( \tilde{P}(z) \) is analytic for \( z \in U_\delta \setminus \Sigma^T \), and continuous on \( \overline{U_\delta} \setminus \Sigma^T \),

(b) \( \tilde{P}(z) \) possesses continuous boundary values for \( z \in \Sigma^T \cap \overline{U_\delta} \), denoted by \( \tilde{P}_+(z) \) and \( \tilde{P}_-(z) \), and

\[
\tilde{P}_+(z) = \tilde{P}_-(z) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \text{ for } z \in \Gamma \cap \tilde{U}_\delta, \quad (5.15)
\]

\[
\tilde{P}_+(z) = \tilde{P}_-(z) \left( \begin{array}{cc} 1 & 0 \\ e^{2n \tilde{\phi}(z)} & 1 \end{array} \right) \text{ for } z \in (\Sigma_3 \cup \Sigma_4) \cap \tilde{U}_\delta, \quad (5.16)
\]

and

\[
\tilde{P}_+(z) = \tilde{P}_-(z) \left( \begin{array}{cc} 1 & e^{-2n \tilde{\phi}(z)} \\ 0 & 1 \end{array} \right) \text{ for } z \in \Sigma_1 \cap \tilde{U}_\delta. \quad (5.17)
\]

Recall that the \( \tilde{\phi} \)-function is defined in (3.11).
There exists a constant \( C > 0 \) such that for every \( z \in \partial \tilde{U}_\delta \setminus \Sigma^T \),
\[
\| \tilde{P}(z)N^{-1}(z) - I \| \leq \frac{C}{n}.
\] (5.18)

There is a one-to-one analytic mapping \( \zeta = \tilde{f}(z) \) such that
\[
\frac{2}{3}(-\tilde{f}(z))^{3/2} = \tilde{\phi}(z).
\] (5.19)

Then \( \tilde{f} \) maps \( \tilde{U}_\delta \) onto a convex neighborhood of \( \zeta = 0 \).

**Proposition 5.5** The Riemann–Hilbert problem for \( \tilde{P} \) is solved by
\[
\tilde{P}(z) = \tilde{E}(z)\tilde{\Psi}(n^{2/3}\tilde{f}(z))e^{n\tilde{\phi}(z)\sigma_3}
\] (5.20)

with
\[
\tilde{E}(z) = \sqrt{\pi}e^{\pi i/6} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \left( n^{1/6}(-\tilde{f}(z))^{1/4}a(z) \right)^{\sigma_3},
\] (5.21)

and
\[
\tilde{\Psi}(\zeta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tilde{\Psi}(\zeta) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\] (5.22)

**Proof.** This follows as in [10, p.1527]. \( \square \)

**Remark 5.6** As in [10] there is a full asymptotic expansion of \( P_{N-1} \) and \( \tilde{P}_{N-1} \) in inverse powers of \( n \). This expansion leads to uniform asymptotic expansions for the generalized Laguerre polynomials. In the present paper we will compute asymptotics for the polynomials to first order, but we will not carry out the further computations to determine the coefficients of a complete asymptotic expansion.

### 6 Final transformation \( T \mapsto S \)

Using \( N, P, \) and \( \tilde{P} \), we define for every \( n \in \mathbb{N} \),
\[
S(z) = T(z)N(z)^{-1} \quad \text{for} \quad z \in \mathbb{C} \setminus (\Sigma^T \cup \overline{U}_\delta \cup \overline{\tilde{U}}_\delta),
\] (6.1)
\[
S(z) = T(z)P(z)^{-1} \quad \text{for} \quad z \in U_\delta \setminus \Sigma^T,
\] (6.2)
\[
S(z) = T(z)\tilde{P}(z)^{-1} \quad \text{for} \quad z \in \tilde{U}_\delta \setminus \Sigma^T.
\] (6.3)

Then \( S \) is defined and analytic on \( \mathbb{C} \setminus \left( \Sigma^T \cup \partial U_\delta \cup \partial \tilde{U}_\delta \right) \). However it follows from the construction that \( S \) has no jumps on \( \Gamma \) and on \( \Sigma^T \cap (U_\delta \cup \tilde{U}_\delta) \). Therefore \( S \) has an analytic continuation to \( \mathbb{C} \setminus \Sigma^S \) (also denoted by \( S \)), where \( \Sigma^S \) is the contour indicated in Figure 8 for \( A = 1.01 \).

Formally, the contours \( \Sigma^S \) are given by \( \Sigma^S_j = \Sigma_j \setminus (\overline{U}_\delta \cup \overline{\tilde{U}}_\delta) \) for \( j = 1, 2, 3, 4 \), and the domains \( \Omega^S \) are given by \( \Omega^S_1 = \Omega_1 \setminus (\overline{U}_\delta \cup \overline{\tilde{U}}_\delta), \Omega^S_2 = (\Omega_2 \cup \Gamma \cup \Omega_3) \setminus (\overline{U}_\delta \cup \overline{\tilde{U}}_\delta), \) and \( \Omega^S_3 = \Omega_4 \setminus (\overline{U}_\delta \cup \overline{\tilde{U}}_\delta) \).

Then \( S \) satisfies the following Riemann–Hilbert problem.
Figure 8: Contour $\Sigma^S = \partial U_\delta \cup \partial U_\delta \cup \bigcup_j \Sigma^S_j$ and the domains $U_\delta$, $\tilde{U}_\delta$, and $\Omega^S_j$, $j = 1, 2, 3$, for the Riemann–Hilbert problem for $S$, for the value $A = 1.01$.

**Riemann–Hilbert problem for $S$:**

The problem is to determine $S : \mathbb{C} \setminus \Sigma^S \to \mathbb{C}^{2 \times 2}$ such that the following hold.

(a) $S(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma^S$,

(b) $S(z)$ possesses continuous boundary values for $z \in \Sigma^S$, denoted by $S_+(z)$ and $S_-(z)$, and

$$S_+(z) = S_-(z) P(z) N(z)^{-1} \quad \text{for} \quad z \in \partial U_\delta,$$

$$S_+(z) = S_-(z) \tilde{P}(z) N(z)^{-1} \quad \text{for} \quad z \in \partial \tilde{U}_\delta,$$

$$S_+(z) = S_-(z) N(z) \begin{pmatrix} 1 & 0 \\ e^{2n\phi(z)} & 1 \end{pmatrix} N(z)^{-1} \quad \text{for} \quad z \in \Sigma^S_3 \cup \Sigma^S_4,$$

$$S_+(z) = S_-(z) N(z) \begin{pmatrix} 1 & e^{-2n\tilde{\phi}(z)} \\ 0 & 1 \end{pmatrix} N(z)^{-1} \quad \text{for} \quad z \in \Sigma^S_1,$$

$$S_+(z) = S_-(z) N(z) \begin{pmatrix} 1 & e^{-2n\phi(z)} \\ 0 & 1 \end{pmatrix} N(z)^{-1} \quad \text{for} \quad z \in \Sigma^S_2,$$

(c) $S(z) = I + O \left( \frac{1}{z^2} \right)$ for $z \to \infty$.

The jump matrices for $S$ are close to the identity matrix if $n$ is large. Indeed, by (5.9) and (5.18) we have

$$\|P(z) N^{-1}(z) - I\| \leq \frac{C}{n} \quad \text{for} \quad z \in \partial U_\delta,$$
and
\[ \| \tilde{P}(z)N^{-1}(z) - I \| \leq \frac{C}{n} \quad \text{for } z \in \partial U_\delta, \]
with a constant $C$ that is independent of $z$. The constant can also be chosen independently of the value of $A$ for $A$ in a compact subset of $(1, \infty)$, see Remark 5.4.

The jump matrices in (6.6)–(6.8) are exponentially close to the identity matrix.

7 Asymptotic for polynomials $L_n^{(\alpha_n)}(nz)$

7.1 Asymptotics for $S$

In Sections 3–6 the values of $n$ and $\alpha$ were assumed to be fixed. In order to study the asymptotics, we now let $\alpha = \alpha_n$ depend on $n$ and we write $A_n = -\frac{\alpha_n}{n}$. We assume that $A_n > 1$ for every $n$, and
\[ \lim_{n \to \infty} -\frac{\alpha_n}{n} = \lim_{n \to \infty} A_n = A > 1. \] (7.1)

Because of the $n$-dependence of $A_n$, all of the notions and results introduced in Sections 3–6 are $n$-dependent. For example, we have that the curves $\Gamma, \Sigma, \Sigma^T, \text{and } \Sigma^S$ are all varying with $n$, and so we denote them by $\Gamma_n, \Sigma_n, \Sigma^T_n, \text{and } \Sigma^S_n$. Likewise, we have that the functions $R, g, \phi, \text{and } \tilde{\phi}$, as well as all matrix-valued functions are $n$-dependent, and we also use a subscript $n$ to denote their dependence on $n$. If now we use $\Gamma, \Sigma, R, g, \text{etc.},$ without subscript $n$, then this refers to the limiting case. Due to (7.1) we have that the curves $\Gamma_n$ tend to the limiting curve $\Gamma$, etc.

At the end of the previous section, we observed that the jump matrix for $S_n$ is $I + O(1/n)$ uniformly on $\Sigma^S_n$ as $n \to \infty$. In addition, the jump matrix converges to the identity matrix as $z \to \infty$ along the unbounded components of $\Sigma^S_n$ sufficiently fast, so that the jump matrix is also close to $I$ in the $L^2$-sense. Since the contours $\Sigma^S_n$ are only slightly varying with $n$, we may follow arguments as those given in [2] and [3], to conclude that
\[ S_n(z) = I + O\left(\frac{1}{n}\right) \quad \text{as } n \to \infty \] (7.2)
uniformly for $z \in \mathbb{C} \setminus \Sigma^S_n$. Briefly, the Riemann–Hilbert problem for $S_n$ is equivalent to a system of singular integral equations. This system is of the form $\gamma(\zeta) - F(\gamma)(\zeta) = g(\zeta)$, where $\gamma$ is a matrix valued function defined on the contour $\Sigma^S_n$, and where the singular integral operator $F(\gamma)(\zeta)$ has operator norm $O(\frac{1}{n})$. The system of singular integral equations can therefore be solved by Neumann series in powers of $\frac{1}{n}$, and this leads to a series expansion for the solution of the Riemann–Hilbert problem.

Using more precise information on the jump matrix for $S_n$ as indicated in Remark 5.6, and with additional assumptions on the limit (7.1), one is able to obtain a full asymptotic expansion for $S_n(z)$ in powers of $1/n$. This in return would give a full asymptotic expansion for the generalized Laguerre polynomials.

7.2 Strong asymptotics for generalized Laguerre polynomials

Unraveling the steps $Y_n \mapsto U_n \mapsto T_n \mapsto S_n$ and using (7.2), we obtain strong asymptotics for $Y_n$ in all regions of the complex plane. In particular we are interested in the $(1,1)$ entry of $Y_n$, since this is the monic generalized Laguerre polynomial.
Theorem 7.1  Suppose for each \( n \in \mathbb{N} \), we have a parameter \( \alpha_n < -n \) such that \((7.1)\) holds where \( A_n = -\frac{\alpha_n}{n} \). Then we have the following asymptotic results for the generalized Laguerre polynomials \( L_n^{(\alpha_n)}(nz) \) as \( n \to \infty \).

(a) (asymptotics away from \( \Gamma \))
Uniformly for \( z \) in compact subsets of \( \mathbb{C} \setminus \Gamma \), we have as \( n \to \infty \),
\[
L_n^{(\alpha_n)}(nz) = \frac{(-n)^n}{n!} e^{ng_n(z)} \left( \frac{1 + R'_n(z)}{2} \right)^{1/2} \left( 1 + O \left( \frac{1}{n} \right) \right). \tag{7.3}
\]

(b) (asymptotics on + -side of \( \Gamma_n \), away from endpoints)
Uniformly for \( z \) on the + -side of \( \Gamma_n \) away from \( \beta \) and \( \bar{\beta} \), we have as \( n \to \infty \),
\[
L_n^{(\alpha_n)}(nz) = \frac{(-n)^n}{n!} e^{ng_n(z)} \left( \frac{1 + R'_n(z)}{2} \right)^{1/2} \times \left[ 1 - \left( \frac{1 - R'_n(z)}{1 + R'_n(z)} \right)^{1/2} e^{2n\phi_n(z)} + O \left( \frac{1}{n} \right) \right]. \tag{7.4}
\]

(c) (asymptotics on --side of \( \Gamma_n \), away from endpoints)
Uniformly for \( z \) on the --side of \( \Gamma_n \) away from \( \beta \) and \( \bar{\beta} \), we have as \( n \to \infty \),
\[
L_n^{(\alpha_n)}(nz) = \frac{(-n)^n}{n!} e^{ng_n(z)} \left( \frac{1 + R'_n(z)}{2} \right)^{1/2} \times \left[ 1 + \left( \frac{1 - R'_n(z)}{1 + R'_n(z)} \right)^{1/2} e^{2n\phi_n(z)} + O \left( \frac{1}{n} \right) \right]. \tag{7.5}
\]

(d) (asymptotics near \( \beta \))
Uniformly for \( z \) in a (small) neighborhood of \( \beta \), we have as \( n \to \infty \),
\[
L_n^{(\alpha_n)}(nz) = \frac{(-n)^n}{n!} \exp \left( \frac{n}{2} (A_n \log z + z + \ell) \right) \sqrt{\pi} \times \left[ \left( \frac{z - \bar{\beta}_n}{z - \beta_n} \right)^{1/4} \left( n^{2/3} f_n(z) \right)^{1/4} \text{Ai}(n^{2/3} f_n(z)) \left( 1 + O \left( \frac{1}{n} \right) \right) \right. \\
\left. - \left( \frac{z - \beta_n}{z - \bar{\beta}_n} \right)^{1/4} \left( n^{2/3} f_n(z) \right)^{-1/4} \text{Ai}^\prime(n^{2/3} f_n(z)) \left( 1 + O \left( \frac{1}{n} \right) \right) \right]. \tag{7.6}
\]

Proof. (a) Let \( K \) be a compact subset of \( \mathbb{C} \setminus \Gamma \). Then, for \( n \) large enough, say \( n \geq n_0 \), we have \( K \subset \mathbb{C} \setminus \Gamma_n \). Let \( n \geq n_0 \). While choosing the contours \( (\Sigma_3)_n \) and \( (\Sigma_4)_n \) in Section 4, we then take care that \( (\Omega_2)_n \) and \( (\Omega_3)_n \) do not meet the compact \( K \), see Figure 4. We also choose \( \delta \) so that the disks of radius \( \delta \) around \( \beta_n \) and \( \bar{\beta}_n \) are disjoint from \( K \). Then \( K \subset (\Omega_1^\ell)_n \cup (\Omega_3^\ell)_n \), see Figure 4. For \( z \in K \), we then have by \((4.1), (4.8), \) and \((6.1)\)
\[
Y_n(z) = e^{n(\ell_n/2)\sigma_3} U_n(z) e^{ng_n(z)\sigma_3} e^{-n(\ell_n/2)\sigma_3} \\
= e^{n(\ell_n/2)\sigma_3} T_n(z) e^{ng_n(z)\sigma_3} e^{-n(\ell_n/2)\sigma_3} \\
= e^{n(\ell_n/2)\sigma_3} S_n(z) N_n(z) e^{ng_n(z)\sigma_3} e^{-n(\ell_n/2)\sigma_3}. \tag{7.7}
\]
Using (2.8), (3.3), and (7.2), and the fact that \(|(N_n)_{11}(z)|\) is bounded away from zero on \(K\), we obtain (7.3) from (7.7).

(b) For \(z \in (\Omega_2)_n \setminus ((U_{\delta})_n \cup (\bar{U}_{\delta})_n)\), we have by (4.1) and (4.3)
\[(Y_n)_{11}(z) = (U_{11}(z)e^{n\phi_n(z)} = [(T_{n11}(z) + (T_{n12}(z)e^{2n\phi_n(z)})] e^{n\phi_n(z)} \quad (7.8)\]
Since \(T_n = S_n/N_n\) by (6.1), \(S_n = I + O(1/n)\), and the entries of \(N_n\) are uniformly bounded away from zero in the region under consideration, we then get
\[(Y_n)_{11}(z) = [(N_{11})(z) + (N_{12}(z)e^{2n\phi_n(z)} + O(1/n)] e^{n\phi_n(z)}, \quad (7.9)\]
uniformly for \(z \in (\Omega_2)_n \setminus ((U_{\delta})_n \cup (\bar{U}_{\delta})_n)\). This proves (7.4).

(c) This is proved similarly as part (b). The only difference is that for \(z \in (\Omega_3)_n \setminus ((U_{\delta})_n \cup (\bar{U}_{\delta})_n)\) we use (4.10) instead of (4.3). This leads to
\[(Y_n)_{11}(z) = [(N_{11}(z)(P_{n11}(z) + (N_{12}(z)(P_{n21}(z)) e^{n\phi_n(z)} \quad (7.10)\]
By Proposition 5.3 we have
\[P_n(z) = E_n(z)\Psi^\sigma(\zeta)e^{n\phi_n(z)}\]
where \(\zeta = n^{2/3}f_n(z)\). For \(z \in (\Omega_1)_n\), we have that \(\zeta\) belongs to the sector \(0 < \arg \zeta < \sigma\), so that the first formula for \(\Psi^\sigma(\zeta)\) in (5.14) applies. It follows that
\[
\left(\begin{array}{c}
(P_{n11}(z) \\
(P_{n12}(z)
\end{array}\right) = E_n(z) \left(\begin{array}{c}
Ai(\zeta) \\
Ai'(\zeta)
\end{array}\right) e^{-\frac{n\phi}{\sigma}e^{n\phi_n(z)}. \quad (7.11)\]
Using the definition (5.13) of \(E_n(z)\), we obtain from (7.11)
\[
\left(\begin{array}{c}
(P_{n11}(z) \\
(P_{n12}(z)
\end{array}\right) = \sqrt{n} \left(\begin{array}{cc}
1 & -1 \\
- \frac{1}{2} & -1
\end{array}\right) \left(\begin{array}{c}
\zeta^{1/4} \\
\frac{\zeta^{1/4}}{a_n(z)}
\end{array}\right) \left(\begin{array}{c}
Ai(\zeta) \\
Ai'(\zeta)
\end{array}\right) e^{n\phi_n(z)}. \quad (7.12)\]
Then
\[(S_{n11}(z)(P_{n11}(z) + (S_{n12}(z)(P_{n21}(z)) \]
\[
= \sqrt{n} \left(\begin{array}{cc}
1 + O(\frac{1}{n}) & -1 + O(\frac{1}{n})
\end{array}\right) \left(\begin{array}{c}
\frac{\zeta^{1/4}}{a_n(z)} \\
\frac{\zeta^{1/4}}{a_n(z)}
\end{array}\right) \left(\begin{array}{c}
Ai(\zeta) \\
Ai'(\zeta)
\end{array}\right) e^{n\phi_n(z)}. (7.12)\]
Combining (7.10) and (7.12) with \(\zeta = n^{2/3}f_n(z)\), we obtain an expression for \((Y_{n11}(z)\), which leads to (7.6) in view of (2.4) and (3.13).

Next, for \(z\) in the other regions \((\Omega_j)_n, j = 2, 3, 4\), similar calculations lead to the same expression for \((Y_{n11}(z)\). Hence (7.6) holds uniformly for \(z \in U_\epsilon\).

This completes the proof of Theorem 7.1. \(\square\)

Remark 7.2 Since \(\alpha_n\) is real, we have \(L_n^{(\alpha_n)}(nz) = L_n^{(\alpha_n)}(nz)\), and so (7.8) also describes the asymptotic behavior near \(\bar{\beta}\).
7.3 Asymptotics for zeros

From Theorem 7.1 we deduce the following results concerning the zeros of the generalized Laguerre polynomials \( L^{(\alpha n)}_{n}(nz) \).

**Corollary 7.3** Suppose we are in the same situation as in Theorem 7.1.

(a) **(All zeros tend to \( \Gamma \))** For every neighborhood \( \Omega \) of \( \Gamma \), there is \( n_0 \) such that for every \( n \geq n_0 \), all zeros of \( L^{(\alpha n)}_{n}(nz) \) are in \( \Omega \).

(b) **(Zeros are on the \(-\)-side)** For every \( \delta > 0 \), there is \( n_0 \) such that for every \( n \geq n_0 \), there are no zeros of \( L^{(\alpha n)}_{n}(nz) \) in the region \( (\Omega_2)_n \setminus ((U_\delta) \cup (\bar{U}_\delta)) \).

**Proof.** (a) This is immediate from the asymptotic formula (7.3) since \( 1 + R'_n(z) \neq 0 \).

(b) Suppose \( z \) is a zero of \( L^{(\alpha n)}_{n}(nz) \) lying in \( (\Omega_2)_n \setminus ((U_\delta) \cup (\bar{U}_\delta)) \), that is, on the +\(-\)side of \( \Gamma_n \). Then we have by (7.4)

\[
\left( \frac{1 - R'_n(z)}{1 + R'_n(z)} \right)^{1/2} e^{2n\phi_n(z)} = 1 + O \left( \frac{1}{n} \right).
\]

We know that \( \text{Re} \phi_n(z) \leq 0 \). In order to obtain a contradiction it is thus enough to prove that

\[
|1 - R'_n(z)| < |1 + R'_n(z)|.
\]

Equality holds in (7.14) if and only if \( R'_n(z) \) is purely imaginary, so that \( (R'_n)^2(z) \) is real and negative. From the definition (7.2) of \( R_n \), we get

\[
(R'_n)^2(z) = \frac{(z - \beta_n + \bar{\beta}_n)^2}{(z - \beta_n)(z - \bar{\beta}_n)} = \frac{(z - \text{Re} \beta_n)^2}{(z - \text{Re} \beta_n)^2 + (\text{Im} \beta_n)^2}.
\]

This is real and negative if and only if \( z \) belongs to the vertical segment connecting \( \bar{\beta}_n \) and \( \beta_n \). Consequently, this is the set where \( |1 - R'_n(z)| = |1 + R'_n(z)| \). The vertical segment and \( \Gamma \) form a closed contour, and it may be checked that (7.14) holds for \( z \) outside of this contour, which includes the region \( (\Omega_2)_n \setminus ((U_\delta) \cup (\bar{U}_\delta)) \).

So we have a contradiction, and it follows that there are no zeros in \( (\Omega_2)_n \setminus ((U_\delta) \cup (\bar{U}_\delta)) \) for \( n \) large enough. \( \square \)

**Remark 7.4** From the uniform asymptotics (7.9) in a neighborhood of \( \beta \) it is possible to derive asymptotics for the extreme zeros of \( L^{(\alpha n)}_{n}(nz) \). For example, it follows that for fixed \( k \in \mathbb{N} \) and \( n \geq k \), there is a zero \( z_{k,n} \) of \( L^{(\alpha n)}_{n}(nz) \) such that

\[
z_{k,n} = \beta_n - \frac{\epsilon_k}{f'_n(\beta_n)n^{2/3}} + O \left( \frac{1}{n} \right),
\]

where \(-\epsilon_k\) is the \( k \)th largest (negative) zero of the Airy function \( \text{Ai}(x) \).
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A.B.J. Kuijlaars arno@wis.kuleuven.ac.be
Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200 B, 3001 Leuven, Belgium

K.T-R McLaughlin mcl@amath.unc.edu
Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599, USA

and mcl@math.arizona.edu
Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA