A CHARACTERIZATION OF KOISO’S TYPED SOLITONS

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Abstract. By extending Koiso’s examples to the non-compact case, we construct complete gradient Kähler-Ricci solitons of various types on certain holomorphic line bundles over compact Kähler-Einstein manifolds. Moreover, a uniformization result on steady gradient Kähler-Ricci solitons with non-negative Ricci curvature is obtained under additional assumptions.

1. Introduction

A Kähler metric $g$ is called a Kähler-Ricci soliton on a complex manifold $M$ if there is a holomorphic vector field $V$ and a real number $\rho$ such that the equality $\text{Ric} + \rho g - LVg = 0$ holds on $M$. It is called steady when $\rho = 0$, expanding when $\rho > 0$ and shrinking when $\rho < 0$. In addition, when $V$ is the gradient vector field of a real function $f$ on $M$ we call it a gradient Kähler-Ricci soliton.

Like Ricci solitons in the real case, Kähler-Ricci solitons naturally arise as one takes limits of dilations of singularities in Kähler-Ricci flow (cf. [H2] and [C2]). Due to their importance in the study of Kähler-Ricci flow, it is interesting to learn more examples of Kähler-Ricci solitons. One typical method of constructing Kähler-Ricci solitons is to impose certain symmetry conditions and reduce the soliton equation to ODEs which are more tractable (for instance, in [K], [C1] for the compact case and [H1], [C1], [C2], [F-I-K] for the non-compact case.). In the other direction, Wang and Zhu [W-Z] constructed Kähler-Ricci solitons on toric Kähler manifolds with positive first Chern class by solving equations of Monge–Ampère type on toric Fano manifolds.

To the author’s knowledge, all known examples in the non-compact case rely on the $U(n)$-symmetry assumption of the soliton metric. However, it is easy to note that Koiso’s methods in [K], which allows less restriction on the symmetry assumption, may work similarly in the non-compact case. Since there seems to be no literature containing this explicitly, in this note we carry out this construction in details. We prove:

Theorem 1.1. Let $M$ be a compact $(n-1)$-dim Kähler-Einstein manifold satisfying $\text{Ric}(g_0) = g_0$ and $L$ be a holomorphic line bundle over $M$. Assume there exists a Hermitian metric on $L$ such that the eigenvalues $\lambda_i$ of the Ricci form of $L$ with respect to $g_0$ are constant on $M$. Then We have:

1) If $-1 < \lambda_i < 0$ for $1 \leq i \leq n-1$, there exists a complete shrinking gradient Kähler-Ricci soliton on $L$.

2) If $\lambda_i < -1$ for $1 \leq i \leq n-1$, there exists a family of complete expanding gradient Kähler-Ricci solitons on $L$.

3) If $L$ is the canonical line bundle, there exists a family of complete steady gradient Kähler-Ricci solitons on $L$. 

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Rotationally symmetric Kähler-Ricci solitons on holomorphic line bundles over \( \mathbb{CP}^{n-1} \) were constructed in [C1] and [F-I-K]. It can be checked that Theorem 1.1 includes their examples as a special case. Moreover, in the steady and expanding case, Theorem 1.1 provides a family of Kähler-Ricci solitons over certain holomorphic line bundles of compact Kähler-Einstein manifolds. Those line bundles can be chosen to be canonical line bundles or their tensor products.

One can check that the steady gradient Kähler-Ricci solitons on canonical line bundles over Kähler-Einstein manifolds in Theorem 1.1 have nonnegative Ricci curvature on the zero section and positive Ricci curvature away from it. This motivates us to prove the following uniformization result as a characterization of Koiso’s typed soliton in the steady case.

**Theorem 1.2.** Let \( M \) be a non-compact steady gradient Kähler-Ricci soliton with non-negative Ricci curvature. Assume its scalar curvature attains a positive maximum along a compact complex submanifold \( K \) with codimension 1 and the Ricci curvature is positive away from \( K \). Then \( M \) is biholomorphic to a holomorphic line bundle over \( K \).

It was shown in [B] and [C-T] independently that a non-compact complex manifold is biholomorphic to \( \mathbb{C}^n \) if it admits a steady gradient Kähler-Ricci soliton metric with positive Ricci curvature and its scalar curvature attaining a maximum at some point. In [B] Bryant proved a crucial lemma on the existence of a nice local coordinate near singular points of the gradient holomorphic vector field associated to the soliton and obtained his uniformization result. Here we apply Bryant’s lemma in [B] in our case to prove Theorem 1.2.

It is also interesting to note that some results concerning the geometric classification of shrinking Ricci solitons under curvature assumptions was shown in [N] and the recent preprints [N-W], [P-W] and [Na].

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## 2. Example of Kähler-Ricci solitons

### 1 Background and calculations

In this part, we quickly review some facts on Koiso’s construction of a Kähler metric on a \( \mathbb{C}^* \)–bundle over a compact Kähler manifold.

Given a holomorphic line bundle \( L \rightarrow M \) on a complex manifold \( M \) where \( \pi \) is the natural projection, assume its local trivialization is given by \( \varphi_\alpha : U_\alpha \times \mathbb{C} \rightarrow \pi^{-1}(U_\alpha) \). We define \( \mathbb{C}^* \)–action on \( L^* = L \setminus \{0\text{-section}\} \) by \( \varphi_\alpha(p, \lambda) \circ g = \varphi_\alpha(p, \lambda g) \) (\( \forall g \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}, p \in M, \lambda \in \mathbb{C} \)). It can be checked that this definition is independent of the choice of local trivialization and this \( \mathbb{C}^* = \mathbb{R}^+ \times S^1 \) action is free. We denote the two holomorphic vector fields generated by \( \mathbb{R}^+ \) and \( S^1 \) action \( H \) and \( S \).

Let \( L \) be a Hermitian Line bundle over a compact Kähler manifold \( M \). Denote \( \bar{J} \) the complex structure on \( L \) and \( \rho_L \) the Ricci form of \( L \). Assume \( t \) is a function on \( L \) depending only on the norm and increasing on the norm. We consider a hermitian metric on \( L^* \) of the form

\[
\bar{g} = \pi^* g_t + dt^2 + (dt \circ \bar{J})^2,
\]
where \( g_t \) is a family of Riemannian metrics on \( M \). Denote \( u(t)^2 = \tilde{g}(H, H) \). It can
checked that \( u \) depends only on \( t \).

The following facts can be found in [K-S].

**Fact 2.1.** \( \tilde{g} \) is Kähler on \( L^* \) if and only if each \( g_t \) is Kähler on \( M \) and \( g_t = g_0 - UB \)
where \( B(JX, Y) = \rho_L(X, Y) \) \( U = \int_0^t u(t) dt \).

We further assume the eigenvalues of \( B \) with respect to \( g \)
are constant on \( M \). Let \( z^1 \cdots z^n \) be local coordinates of \( M \) and denote \( z^0 \cdots z^n \) be local coordinates of
\( L_0 \) such that \( \frac{\partial}{\partial z^0} = H - \sqrt{-1} S \).

**Fact 2.2.** \( \tilde{g}_{00} = 2u^2, \tilde{g}_{\alpha 0} = 2u\partial_\alpha t, \tilde{g}_{\alpha \beta} = g_{\alpha \beta} + 2\partial_\alpha t\partial_\beta t \). Define \( p = \det(g_0^{-1} \cdot g_t) \),
then \( \det(\tilde{g}) = 2u^2 \cdot p \cdot \det(g_0) \).

**Fact 2.3.** If we assume that \( \partial_\alpha t = \partial_\beta t = 0 \) \( (1 \leq \alpha \leq n-1) \) on a fiber. If a function
\( f \) is defined on \( L^* \) and only depending on \( t \), then \( \partial_\alpha \partial_\beta f = u \frac{\partial^2 f}{\partial t^2} + \frac{1}{2} u \cdot \frac{\partial}{\partial t} (\log(u^2 p)) \cdot B_{\alpha \beta} \).

**Fact 2.4.** Under the same assumption of Fact 2.3, the Ricci curvature of \( \tilde{g} \) becomes:
\( \tilde{R}_{00} = -u \cdot \frac{\partial}{\partial t} (u \cdot \frac{\partial}{\partial t} (\log(u^2 p))) \), \( \tilde{R}_{\alpha 0} = 0 \), \( \tilde{R}_{\alpha \beta} = R_{\alpha \beta} + \frac{1}{2} u \cdot \frac{\partial}{\partial t} (\log(u^2 p)) \cdot B_{\alpha \beta} \).

Reset \( \phi(U) = u(t)^2, Q(U) = p \) and one can compute the following:

**Fact 2.5.** Set \( V = -\frac{d}{dt} H \), under the same assumption of Fact 2.3, one can compute:
\( \tilde{R}_{00} - \tilde{g}_{00} - L_\nu \tilde{g}_{00} = -H(P \circ \phi - E \phi), \tilde{R}_{\alpha 0} = \tilde{g}_{\alpha 0} = L_\nu \tilde{g}_{00} = 0, \tilde{R}_{\alpha \beta} - \tilde{g}_{\alpha \beta} - L_\nu \tilde{g}_{1,\alpha \beta} = \frac{1}{2} (P \circ \phi - E \phi) B_{\alpha \beta} + (R_{\alpha \beta} - g_{\alpha \beta}), \) where \( P \circ \phi = \frac{dp}{dt} + \frac{d\phi}{dt} Q + 2U \).

If the initial metric \( g_0 \) on \( M \) is Kähler-Einstein satisfying \( \text{Ric}(g_0) = g_0 \), then
from the above one can reduce the shrinking soliton equation \( \text{Ric} - \tilde{g} - L_\nu \tilde{g} = 0 \)
to a one order ODE and get the formal solution
\( \phi(U) = -\frac{2e^{EU}}{Q(U)} \int_{U_{\min}}^U xe^{-Ex}Q(x) \ dx, \)
where we denote \([U_{\min}, U_{\max}]\) and \([t_{\min}, t_{\max}]\) to be the range of of the function
\( U \) and \( t \) respectively. We assume that \( U_{\min} \not= -\infty \) and \( t_{\min} \not= -\infty \).

We state the following lemma concerning the condition on growth of \( \phi(U) \) in order to get a well-defined Kähler metric on \( L^* \).

**Lemma 2.1.** If \( \phi(U) > 0 \) and \( g_0 - UB \) remains positive on \([U_{\min}, U_{\max}]\), in addition, \( \int_{U_{\min}}^U \frac{dt}{\phi(U)} = +\infty, \int_{U_{\max}}^U \frac{dt}{\phi(U)} = +\infty \) and \( \int_{U_{\min}}^U \frac{dU}{\phi(U)} \) is finite for all
\( U \in (U_{\min}, U_{\max}) \), then we can get a unique expression of \( t \) w.r.t. the Hermitian
metric \( r \) on \( L \) with given initial value \( t_{\min} \), this results a Kähler metric on \( L^* \)
which satisfies soliton equation on \( L^* \).

**Proof.** According to definition we know
\( \frac{dU}{\sqrt{\phi(U)}} = dt \)
\( \sqrt{\phi(U)} = u(t) = r \frac{dt}{dr} \)
this implies:
\( \int_{U_{\min}}^U \frac{dU}{\phi(U)} = \int_0^r \frac{dr}{r} \)
\[ \int_{U_{\min}}^{U} \frac{dU}{\sqrt{\phi(U)}} = \int_{t_{\min}}^{t} dt \]

Clearly when the assumption in the lemma holds, we can solve the ODEs to get the expression of \( t \) in terms of \( r \) from the above formula and \( r \) varies from 0 to \( +\infty \). \( \square \)

We remark that by solving \( \frac{df}{dt} = u \) one can also make the holomorphic vector \( V \) be given by a gradient vector field of a real-valued function \( f \) which depends only on \( t \).

2 The shrinking case

For convenience we introduce the following assumption.

**Assumption 2.1.** \( \pi : L \to M \) is a Hermitian holomorphic line bundle over a compact \((n-1)\)-dim Kähler-Einstein manifold \( M \) with \( \text{Ric}(g_0) = g_0 \), where the eigenvalues of the Ricci form of \( L \) with respect to \( g_0 \) are constant on \( M \) and satisfying \(-1 < \lambda_i < 0 \) for \( 1 \leq i \leq n-1 \).

Solving \( \tilde{\text{Ric}} - \tilde{g} - L\nabla \tilde{g} = 0 \), we get the formal expression:

\[ \phi(U) = -\frac{2e^{EU}}{Q(U)} \int_{U_{\min}}^{U} xe^{-Ex} Q(x) \, dx \]

on \( L^* \). According to Assumption 2.1 the eigenvalue of \( B \) with respect to \( g_0 \) are \(-1 < \lambda_1 \cdots \lambda_{n-1} < 0 \), then \( Q(U) = \prod_{i=1}^{n-1} (1 - U\lambda_i) \), now the above expression of \( \phi(U) \) can be computed explicitly:

\[ \phi(U) = \frac{2\eta(U, E)}{Q(U)} - \frac{2e^{E(U-U_{\min})}}{Q(U)} \eta(U_{\min}, E), \]

where \( \eta(U, E) \) is a degree-\( n \) polynomial with respect to \( U \) with the principal term \( \frac{1}{E} U^n \).

Our goal is to discuss the possibility to make it complete. If it is incomplete along the zero section we only consider adding \( M \) to complete it, and if it is still incomplete along infinity, we also consider adding \( M \) along infinity hence compactify \( L \) to a projective bundle (see [F-I-K] for other ways to complete the metric.).

If we require \( U_{\min} = -1 \), following the computation on the first Chern class on the zero section in Example 2.1 in [F-I-K], we can check that this is a necessary condition to extend the metric to zero section. Calculating the metric \( \tilde{g} \) in the local coordinates one can also check that this suffices to extend the metric non-degenerate along the zero section. A similar analysis shows that \( U_{\max} = 1 \) will suffice to extend metric \( \tilde{g} \) non-degenerate at infinity if we want to compactify it to a projective bundle.

In order to get a complete non-compact metric we have to learn more about the behavior of \( \tilde{g} \) along infinity. One can also check that if the metric is in the form of \( \tilde{g} = \pi^* g_t + dt^2 + (dt \circ \tilde{J})^2 \) the geodesic starting in the fibre direction moves along the holomorphic vector field \( \nabla f \) away from zero section. In order to make the soliton metric complete along infinity, one only need to check whether those geodesic tends to infinity. This further means the growth of those geodesics is reflected in the growth of function \( t \). (i.e. \( t - t_{\min} \) measures the length of those geodesics).
It turns out that value of $E$ determines the behavior of this metric along infinity. First we introduce two values of $E$ which are critical in our analysis. Define $E_0$ be the solution of an algebraic equation $\eta(-1, E) = 0$ and $E_1$ be the solution to

$$\phi(1) = \frac{2\eta(1, E)}{Q(1)} - \frac{2e^{2E}}{Q(1)} \eta(-1, E) = 0$$

We have the following lemma to ensure the existence and uniqueness of $E_1$ and $E_0$:

**Lemma 2.2.** For any $-1 < \lambda_1 \cdots \lambda_{n-1} < 0$, $E_0$ and $E_1$ exist uniquely. And $0 < E_1 < E_0 < +\infty$.

**Proof.** This can be shown by writing those polynomial equations explicitly and analyzing the signs of coefficients carefully. \hfill $\square$

We now begin to analyze how $E$ affects the asymptotic behavior of the metric in details.

(Case 1): When $E = E_0$, $\phi(U)$ satisfies the assumptions in Lemma 2.1, and one can check that when $r$ changes from 0 to $+\infty$, $t$ changes from $t_{min}$ to $+\infty$, $U$ from $U_{min} = -1$ to $+\infty$ and $\phi(U)$ from 0 to $+\infty$. The resulting metric is complete on the total space of $L$.

(Case 2): When $E > E_0$, $\phi(U)$ does not satisfy the assumptions in Lemma 2.1. When $U$ changes from $U_{min} = -1$ to $+\infty$, we have $t$ changes from $t_{min}$ to $t_{max}$ with finite value and $\phi(U)$ from 0 to $+\infty$, however $r$ changes from 0 to a finite value. This results a metric which is not well defined on $L^*$. 

(Case 3): When $E = E_1$, when $r$ changes from 0 to $+\infty$, $t$ changes from $t_{min}$ to $t_{max}$ with finite value, $U$ from $U_{min} = -1$ to $U_{max} = 1$ and $\phi(U)$ from 0 to 0. One can complete the metric by compactifying $L$ to a projective bundle and this results a compact gradient Kähler-Ricci soliton on $P(L)$. This has been obtained in [K] and [C1]. It is interesting to note that $E_1$ is related to the holomorphic invariant defined in [T-Z].

(Case 4): For all other $E$, $\phi(U)$ satisfies the assumptions in Lemma 2.1. When $r$ changes from 0 to $+\infty$, $t$ changes from $t_{min}$ to $t_{max}$ with finite value, $U$ from $U_{min} = -1$ to a finite $U_{max} \neq 1$ and $\phi(U)$ from 0 to 0. The resulting metric is incomplete along infinity. Moreover, it can not be completed by adding a $M$ along infinity since $U_{min} \neq 1$.

To sum up the above discussion, the shrinking case in Theorem 1.1 is restated as follows.

**Theorem 2.1.** Under the Assumption 2.1, then there exists a complete shrinking Kähler-Ricci soliton on $L$ and the projectified line bundle $P(L)$ respectively satisfying $\tilde{\text{Ric}} - \tilde{g} - L_V\tilde{g} = 0$ such that $\tilde{g}$ is in the form $\tilde{g} = \pi^* g_t + dt^2 + (dt \circ J)^2$, here $V$ can be uniquely determined by the natural holomorphic $\mathbb{R}^+$ action.

3 The expanding case and steady case

**Assumption 2.2.** The only difference with Assumption 2.1 is the eigenvalues of the Ricci form of $L$ with respect to $g_0$ are constant on $M$ and satisfying $\lambda_i < -1$ for $1 \leq i \leq n - 1$. 

Solving $\tilde{\text{Ric}} + \tilde{\nabla} \tilde{g} - L_V \tilde{g} = 0$, we get the formal expression:
\[
\phi(U) = \frac{2e^{EU}}{Q(U)} \int_{U_{\min}}^{U} xe^{-Ex}Q(x) \, dx
\]
on $L^*$.  

The analysis of the asymptotic behavior of the metric is similar to shrinking case, we only list the result:

(1) if we require $U_{\min} = 1$, we can show that this will suffice to extend the metric to the zero section non-degenerate.

(2) Compared with the shrinking case, the restriction on $E$ is moderate in the expanding case.

(Case 1): for any $E < 0$, it can be show that the resulting metric is complete along infinity.

(Case 2): for any $E > 0$, when $U$ from $U_{\min} = -1$ to $+\infty$, we have $t$ changes from $t_{\min}$ to $t_{\max}$ with finite value and $\phi(U)$ from 0 to $+\infty$, however $r$ changes from 0 to a finite value. This results a metric which is not well defined on $L^*$.

We now turn to steady case.

**Assumption 2.3.** $L$ is the canonical line bundle a compact $(n-1)$-dim Kähler – Einstein manifold $M$ with $\text{Ric}(g_0) = g_0$.

Solving $\tilde{\text{Ric}} - L_V \tilde{g} = 0$ one can get the formal expression is:
\[
\phi(U) = \frac{2e^{EU}}{Q(U)} \int_{U_{\min}}^{U} xe^{-Ex}Q(x) \, dx
\]
on $L^*$.

The analysis of the asymptotic behavior of the metric shows that the following holds:

(1) we only need to require $U_{\min} > -1$. one can show that this will suffice to extend the metric to the zero section non-degenerate.

(2) the result on $E$ is similar to the expanding case.

(Case 1): for all $E < 0$, it can be show that the resulting metric is complete along infinity. What is interesting is that in this case the resulting metric has positive Ricci curvature away from the zero section and nonnegative on the zero section. Unfortunately this metric can not have nonnegative bisectional curvature everywhere, the author thanks Prof. Albert Chau and Prof. Fangyang Zheng for providing this information. In fact, since the zero section is totally geodesic one can verify that at any point on the zero section the holomorphic bisectional curvature of the plane by the fiber direction and the tangent direction is always negative using the curvature formula in submanifold geometry.

(Case 2): if $E > 0$, then this results a metric which is not well defined even on $L^*$.

We now restate Theorem 1 in steady and expanding case as follows:

**Theorem 2.2.** Under Assumption 2.2 (or Assumption 2.3), then we can find a family of complete expanding Kahler-Ricci solitons (or complete steady Kahler-Ricci solitons) on $L$ where $\tilde{g}$ is in the form of $\tilde{g} = \pi^* g_t + dt^2 + (dt \circ \tilde{J})^2$. Moreover, in the steady case the soliton metrics have positive Ricci curvature away from the zero section.
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3. A UNIFORMIZATION THEOREM

In this section, we want to prove Theorem 1.2. First we state the following lemma due to Bryant in [B].

**Lemma 3.1.** Let $Z$ be the holomorphic vector field associated to a gradient Kähler-Ricci soliton and $p$ be one of singular point of $Z$, there exists a $p$-centered holomorphic coordinates $w^1 \cdots w^n$ on a neighborhood $U_p$ on which $Z = \text{Ric}(p, \frac{\partial}{\partial w^i})w^i \frac{\partial}{\partial w^i}$ and $\frac{\partial}{\partial w^1} \cdots \frac{\partial}{\partial w^n}$ are orthogonal at $p$.

**Proof of Theorem 1.2.** First we show that $\nabla f$ vanishes on $K$, hence $K$ is a totally geodesic complex submanifold.

This can be proved by the similar method in [H2]. One only need to rule out the possibility that the integral curve generated by $\nabla f$ from any point $p$ on $K$ may always stay in $K$ if $\nabla f$ does not vanish at $p$. This is impossible because:

$$\frac{df(\phi_t(p))}{dt} = -|\nabla f|^2,$$

$$R + |\nabla f|^2 = \text{const},$$

where $\phi_t$ is the holomorphic flow generated by $-\nabla f$. Then $f$ decays to infinity on the compact set $K$ if $\phi_t(p)$ always stay in $K$. This contradiction shows this integral curve must leave $K$ after a finite time, then the remaining argument is similar in [H2].

We further show that for any point in $M$, it converges to $K$ under the holomorphic flow $\phi_t$ generated by $-\nabla f$. Since $\nabla f$ vanishes only on $K$ since $f$ is strictly convex outside $K$ and weakly convex on $M$, we know that $\nabla f$ vanishes only on $K$.

Since

$$\frac{dR(\phi_t(x))}{dt} = R_{ij} \nabla_i f \nabla_j f,$$

and $\text{dis}(\phi_t(x), K)$ is non-increasing when $t$ increases. We conclude that $\text{dist}(\phi_t(x), K)$ converges to zero as $t$ goes to infinity.

We applied Lemma 3.1 under the assumption of Theorem 3.1. For an arbitrary point $p$ on $K$, the Ricci curvature of $M$ vanishes when restricted to $T_pK$. From the original proof of Lemma 3.1 in [B], we can also pick the coordinates $w^1 \cdots w^n$ on a neighborhood $U_p$ such that $\frac{\partial}{\partial w^i} \cdots \frac{\partial}{\partial w^n}$ at $p$ lie in $T_pK$. Now we find a local holomorphic coordinates $w$ on $U_p$ which satisfies

$$w^i(\exp_t \nabla f(q)) = w^i(q)$$

when $1 \leq i \leq n - 1$ and

$$w^n(\exp_t \nabla f(q)) = \exp(ht)w^n(q)$$

where $q$ is any point in $U_p$ and $h = \text{Ric}(p, \frac{\partial}{\partial w^n})$ is a real constant.

Define

$$W_p = \{ x \mid \text{dist}(\phi_t(x), p) \to 0 \quad \text{as} \quad t \to +\infty \},$$

then $M$ can be written as $\bigcup_{p \in K} W_p$. We now show that the above coordinates $w^1 \cdots w^n$ on $U_p$ satisfy $w^n(q) = 0$ for any $q \in U_p \cap K$. In fact, this easily follows by using $w^n(\exp_t \nabla f(z)) = \exp(ht)w^n(z)$ to a sequence picked from $W_q \cap U_p$.

As in [B], one can define a global holomorphic map from $W_p$ to $C$ by extending the local parametrization above. For any point $q$ distinct from $p$ in $W_p$ we can find
a point \( q_1 \) in \( W_p \cap U_p \) such that \( q = \exp_{t_1} \nabla f(q_1) \) for some \( t_1 \). Define

\[
z^i(q) = w^i(q_1)
\]

for \( 1 \leq i \leq n - 1 \) and

\[
z^n(q) = \exp(ht_1)w^n(q_1).
\]

One can easily check that this definition does not depend on the choice of \( q_1 \). It can also be checked that the holomorphic map \( z \) gives a biholomorphic map from \( W_p \) to \( C \).

The holomorphic line bundle structure of \( M \) can be derived from the above global holomorphic parametrization of \( W_p \), thus the theorem is proved.

Before ending this note, we add two remarks here.

**Remark 3.1.** Due to Bryant [B], the singular locus of the holomorphic vector field associated to a gradient Kähler-Ricci soliton is a disjoint union of nonsingular complex manifolds, each of which is totally geodesic. In view of this the assumption of Theorem 3.1 is natural in some sense.

**Remark 3.2.** From Theorem 1.1, we can construct steady gradient Kähler-Ricci solitons on the canonical line bundle over a compact Kähler-Einstein manifold which satisfy all assumptions in the above theorem Theorem 3.1. It will be interesting to investigate whether this is the only example.

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