GROUPS OF ASYMPTOTIC DIFFEOMORPHISMS

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ABSTRACT. We consider classes of diffeomorphisms of Euclidean space with partial asymptotic expansions at infinity; the remainder term lies in a weighted Sobolev space whose properties at infinity fit with the desired application. We show that two such classes of asymptotic diffeomorphisms form topological groups under composition. As such, they can be used in the study of fluid dynamics according to the method of V. Arnold [1]. Specific applications have been obtained for the Camassa-Holm equation [14] and the Euler equations [15].

Keywords. Groups of diffeomorphisms, asymptotic expansions, weighted Sobolev spaces, Camassa-Holm equation, Euler equation.

0. Introduction

A modern development in fluid dynamics is to view the motion of an incompressible fluid as a geodesic flow on a group of diffeomorphisms of the underlying physical space. This approach was initiated by V. Arnold [1] and further developed by Ebin & Marsden [9] and Bourguignon and Brezis [5] to obtain well-posedness of initial-value problems associated with the Euler and Navier-Stokes equations. In these papers, the underlying physical space was compact (a compact manifold with or without boundary). Subsequently, Cantor [7] used this approach to study the Euler equations on $\mathbb{R}^d$ by considering diffeomorphisms $\phi : \mathbb{R}^d \to \mathbb{R}^d$ of the form

\[ \phi = \text{Id} + f, \]

where $\text{Id}$ is the identity map and the function $f$ is in a weighted Sobolev space that requires $f$ to decay rapidly at infinity. However, one would like to consider diffeomorphisms of the form \[ \phi = \text{Id} + u, \]

where $u$ is bounded but not required to decay rapidly. Moreover, if the initial condition has asymptotics at infinity, one would like to know that the solution has similar asymptotics at infinity (with coefficients depending on $t$). To make these improvements, we require additional structure.

In this paper, we study groups of diffeomorphism on $\mathbb{R}^d$ of the form

\[ \phi = \text{Id} + u, \]

where $u$ is taken from a function space that we call an asymptotic space: these consist of bounded maps on $\mathbb{R}^d$ having a partial asymptotic expansion at infinity of the form

\[ u(x) = a_0(\theta) + \frac{a_1(\theta)}{r} + \cdots + \frac{a_N(\theta)}{r^N} + f_N(x) \quad \text{for} \quad r = |x| > R, \]

where $\theta = x/|x|$, the functions $a_0, \ldots, a_N$ lie in certain Sobolev spaces on the unit sphere $S^{d-1}$, and the remainder function $f_N$ belongs to a function space $R_N$ which ensures that

\[ |f_N(x)| = o(|x|^{-N}) \quad \text{as} \quad |x| \to \infty. \]

The remainder space $R_N$ will be a weighted Sobolev space, but there are different possibilities: the choice will depend upon the application, since it must be compatible with the equations being

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studied. In this paper we shall consider as remainder space two different classes of weighted Sobolev spaces. In one class, that we shall denote by \( H_N^{m,p}(\mathbb{R}^d) \), the functions have derivatives up to order \( m \) that are in \( L^p_{\text{loc}}(\mathbb{R}^d) \) := \( \{ f \in L^p(\mathbb{R}^d) : (1 + |x|)^N f(x) \in L^p(\mathbb{R}^d) \} \). In the other class of weighted Sobolev spaces, that we shall denote by \( W_N^{m,p}(\mathbb{R}^d) \), the derivatives of functions satisfy \( D^m f \in L^p_{N+|\alpha|-d/p}(\mathbb{R}^d) \) for all \( |\alpha| \leq m \). Neither of these classes of weighted Sobolev spaces is new to the literature, but for convenience we shall define them and summarize their properties in Section 1; our exposition is self-contained, with proofs provided in the Appendix.

In Section 2 we give the formal definition of our asymptotic spaces on \( \mathbb{R}^d \). If we use as \( \mathcal{R}_N \) the weighted Sobolev space \( H_N^{m,p}(\mathbb{R}^d) \) with \( m \geq 1 \) and \( 1 < p < \infty \), then the functions satisfying (1) define an asymptotic space that we denote by \( \mathcal{A}_N^{m,p}(\mathbb{R}^d) \). On the other hand, if we let \( \mathcal{R}_N = W_N^{m+1/p}(\mathbb{R}^d) \), then we denote the associated asymptotic space by \( \mathcal{A}_N^{m,p}(\mathbb{R}^d) \). At times it is useful to consider functions with \( a_0 = \cdots = a_{n-1} = 0 \) for an integer \( n \leq N \); we denote the corresponding classes of weighted Sobolev spaces by \( \mathcal{A}_{n,N}^{m,p}(\mathbb{R}^d) \) and \( \mathcal{A}_{0,N}^{m,p}(\mathbb{R}^d) \), and identify \( \mathcal{A}_{0,N}^{m,p}(\mathbb{R}^d) = \mathcal{A}_N^{m,p}(\mathbb{R}^d) \). Our primary interest in these asymptotic spaces is to control the behavior of diffeomorphisms at infinity. However, in Section 3, we consider an application of the asymptotic spaces to the Helmholtz decomposition of vector fields on \( \mathbb{R}^d \); this requires an analysis of the inverse of the Laplacian, which is important in many applications, including our study \[15\] of Euler’s equation on \( \mathbb{R}^d \).

In Section 4, we introduce and study the associated spaces of diffeomorphisms \( \mathcal{A}D_{m,N}^{m,p} \) and \( \mathcal{A}D_{n,N}^{m,p} \), i.e. diffeomorphisms \( \phi : \mathbb{R}^d \to \mathbb{R}^d \) that are of the form (2) where the components of \( u \) are in \( \mathcal{A}_{n,N}^{m,p} \) or \( \mathcal{A}_{m,N}^{m,p} \) respectively. The main result of this paper is Theorem \[1.1\] which states that, provided \( m > 2 + d/p \), both \( \mathcal{A}D_{m,N}^{m,p} \) and \( \mathcal{A}D_{n,N}^{m,p} \) are topological groups when composition is used as the group operation. (For compact domains and manifolds, the regularity assumption \( m > 1 + d/p \) is typically sufficient to show that the associated Sobolev spaces form topological groups under composition; cf. \[9\] and \[3\]. Our assumption of additional regularity stems from the fixed point argument that we use in Section 6 to prove the existence of inverses within the group; this argument avoids some of the technical difficulties caused by the asymptotics.)

The fact that \( \mathcal{A}D_{m,N}^{m,p} \) and \( \mathcal{A}D_{n,N}^{m,p} \) are topological groups allows us to use them to study the asymptotics of various fluid flows on \( \mathbb{R}^d \). With \( d = 1 \), for example, the Camassa-Holm equation \[6\] is a completely integrable equation that has attracted considerable attention recently. From the differential geometric point of view, Misiolek \[17\] showed that the equation can be realized as the geodesic flow for a certain metric on the Bott-Virasoro group. Moreover, Constantin \[3\] studied initial-value problems for Camassa-Holm on \( \mathbb{R}^1 \) by using a group of diffeomorphisms of the form (4) that not only do not contain asymptotics but require decay \( o(|x|^{-3/2}) \) as \( |x| \to \infty \). In our companion paper \[14\], we used groups of asymptotic diffeomorphisms on \( \mathbb{R}^1 \) to show that the initial-value problem for the Camassa-Holm equation is well-posed with respect to asymptotic spaces. For example, we showed that if the initial condition \( u_0 \) is in \( \mathcal{A}_{n,N}^{m,2} \) for \( m \geq 3 \) and \( N \geq n \geq 0 \), then there is a unique solution \( u \) of Camassa-Holm in \( C^0([0,T], \mathcal{A}_{n,N}^{m,2}) \cap C^1([0,T], \mathcal{A}_{n,N}^{m-1,2}) \). With \( n = 0 \) this allows \( u_0 = O(1) \) as \( |x| \to \pm \infty \), a great improvement over previous results.

With \( d \geq 2 \), we have also used our groups of asymptotic diffeomorphisms \( \mathcal{A}D_{n,N}^{m,p} \) to study Euler’s equation for a velocity field \( u \) and pressure \( p \) of an incompressible fluid on \( \mathbb{R}^d \) with external forcing \( f \). In fact, in \[15\] we show that if \( m > 2 + d/p \), \( 1 \leq N \leq d - 1 \), and \( f \in C([0,T], \mathcal{A}_{n,N}^{m+1,p}) \), then, for any \( u_0 \in \mathcal{A}_{n,N}^{m,p} \) with \( \text{div} u_0 = 0 \), there exists \( \tau \in (0,T] \) and functions \( u \in C^0([0,\tau], \mathcal{A}_{1,N}^{m,p}) \cap C^1([0,\tau], \mathcal{A}_{1,N}^{m-1,p}) \), and \( p \in C([0,\tau], H_{\text{loc}}^{m+1,p}) \) satisfying Euler’s equation; the function \( u \) is unique. A crucial step in this analysis is the use of the Euler projector as constructed in Section 3 of the present paper.
There is strong evidence that these spaces of asymptotic diffeomorphisms will be equally useful in the study of other fluid equations on $\mathbb{R}^d$. With $d = 1$, for example, we note that there are important equations such as KdV and mKdV that have been studied on $\mathbb{R}^1$ with asymptotic conditions at infinity. For example, Menikoff [13] proved the existence of unbounded solutions for KdV on $\mathbb{R}^1$ which are $O(|x|)$ as $|x| \to \infty$; this was subsequently refined and generalized by Bondareva & Shubin [3, 4] and by Kenig, Ponce, and Vega [12]. Building on these ideas, Kappeler, Perry, Shubin, & Topalov [11] proved the existence and uniqueness of unbounded solutions for mKdV on $\mathbb{R}^1$ with asymptotic expansions at infinity. We believe that spaces of asymptotic diffeomorphisms will be useful in the study of these and other fluid equations on $\mathbb{R}^d$.

1. Weighted Sobolev Spaces on $\mathbb{R}^d$

Let $\langle x \rangle = \sqrt{|x|^2 + 1}$. For $1 < p < \infty$, $\delta \in \mathbb{R}$, and a nonnegative integer $m$, we define the Banach spaces $H_\delta^{m,p}(\mathbb{R}^d)$ and $W_\delta^{m,p}(\mathbb{R}^d)$ to be the closures of $C_0^\infty(\mathbb{R}^d)$ in the respective norms:

\[ \|f\|_{H_\delta^{m,p}} = \sum_{|\alpha| \leq m} \|\langle x \rangle^\delta D^\alpha f\|_{L^p}, \]

\[ \|f\|_{W_\delta^{m,p}} = \sum_{|\alpha| \leq m} \|\langle x \rangle^{d+|\alpha|} D^\alpha f\|_{L^p}. \]

Notice that $H_\delta^{0,p}(\mathbb{R}^d) = W_\delta^{0,p}(\mathbb{R}^d)$ is just a weighted $L^p$-space that may be denoted by $L^p_\delta(\mathbb{R}^d)$. If we let $H_{\text{loc}}^{m,p}(\mathbb{R}^d)$ be functions whose first $m$ derivatives are $L^p$-integrable over compact subsets of $\mathbb{R}^d$, then we can use mollifiers to show that $H_{\text{loc}}^{m,p}(\mathbb{R}^d) = \{ f \in H_{\text{loc}}^{m,p}(\mathbb{R}^d) : \|f\|_{H_{\text{loc}}^{m,p}} < \infty \}$ and $W^{m,p}_\delta(\mathbb{R}^d) = \{ f \in H_{\text{loc}}^{m,p}(\mathbb{R}^d) : \|f\|_{W^{m,p}_\delta} < \infty \}$. Note also that $W^{m,p}_\delta(\mathbb{R}^d) \subset H_{\text{loc}}^{m,p}(\mathbb{R}^d)$.

These weighted Sobolev spaces enjoy the following properties:

**Lemma 1.1.** Let $1 < p < \infty$, $\delta \in \mathbb{R}$, and $m$ be a nonnegative integer.

(a) For $m \geq 1$, $f \mapsto \frac{\partial f}{\partial x}$ defines a continuous map $H^{m,p}_\delta(\mathbb{R}^d) \to H^{m-1,p}_\delta(\mathbb{R}^d)$.

(b) For $m \geq 0$ and $\gamma \in \mathbb{R}$, $f \mapsto \langle x \rangle^{-\gamma} f$ defines a continuous map $H^{m,p}_\delta(\mathbb{R}^d) \to H^{m,p}_{\delta+\gamma}(\mathbb{R}^d)$.

(c) For $mp < d$, if $f \in H^{m,p}_\delta(\mathbb{R}^d)$ then $f \in L^q_\delta(\mathbb{R}^d)$ for all $q \in [p, dp/(d - mp)]$, and

\[ \|f\|_{L^q_\delta} \leq C \|f\|_{H^{m,p}_\delta}, \quad \text{where } C = C(d, m, p, q, \delta). \]

For $mp = d$, the same conclusions hold for all $q \in [p, \infty)$.

(d) For $mp > d$, if $f \in H^{m,p}_\delta(\mathbb{R}^d)$ then $f \in C^k(\mathbb{R}^d)$ for all $k < m - (d/p)$, and

\[ \sup_{x \in \mathbb{R}^d} \langle x \rangle^\delta |D^\alpha f(x)| \leq C \|f\|_{H^{m,p}_\delta} \quad \text{for all } 0 \leq |\alpha| \leq k, \quad \text{where } C = C(d, m, p, k, \delta). \]

In fact, for all $0 \leq |\alpha| \leq k$, we have

\[ |x|^\delta |D^\alpha f(x)| \to 0 \quad \text{as } |x| \to \infty. \]

**Lemma 1.2.** Let $1 < p < \infty$, $\delta \in \mathbb{R}$, and $m$ be a nonnegative integer.

(a) For $m \geq 1$, $f \mapsto \frac{\partial f}{\partial x}$ defines a continuous map $W^{m,p}_\delta(\mathbb{R}^d) \to W^{m-1,p}_{\delta+1}(\mathbb{R}^d)$.

(b) For $m \geq 0$ and $\gamma \in \mathbb{R}$, $f \mapsto \langle x \rangle^{-\gamma} f$ defines a continuous map $W^{m,p}_\delta(\mathbb{R}^d) \to W^{m,p}_{\delta+\gamma}(\mathbb{R}^d)$.

(c) For $mp < d$, if $f \in W^{m,p}_\delta(\mathbb{R}^d)$ then $f \in L^q_{\delta-\frac{d}{p}}(\mathbb{R}^d)$ for all $q \in [p, dp/(d - mp)]$, and

\[ \|f\|_{L^q_{\delta-\frac{d}{p}}} \leq C \|f\|_{W^{m,p}_\delta}, \quad \text{where } C = C(d, m, p, q). \]

For $mp = d$, the same conclusions hold for all $q \in [p, \infty)$.
(d) For \( mp > d \), if \( f \in W^{m,p}_{\frac{d}{d-m}}(\mathbb{R}^d) \) then \( f \in C^k(\mathbb{R}^d) \) for all \( k < m - (d/p) \), and

\[
\sup_{x \in \mathbb{R}^d} \left( |x|^{k+|\alpha|} |D^\alpha f(x)| \right) \leq C \|f\|_{W^{m,p}_{\frac{d}{d-m}}} \quad \text{for all } 0 \leq |\alpha| \leq k, \quad \text{where } C = C(m, p, k, \delta).
\]

In fact, for all \( 0 \leq |\alpha| \leq k \), we have

\[
|x|^{\delta + |\alpha|} |D^\alpha f(x)| \to 0 \quad \text{as } |x| \to \infty.
\]

For both lemmas, the properties (a) and (b) are obvious; properties (c) and (d) are proved in the Appendix.

Using these lemmas, we can prove the following results about pointwise multiplication:

**Proposition 1.1.** For \((m + \ell - k)p > d\) where \(0 \leq k \leq \ell \leq m\), pointwise multiplication

\[
(f, g) \mapsto fg
\]

defines a continuous map \( H^{m,p}_{\delta_1}(\mathbb{R}^d) \times H^{\ell,p}_{\delta_2}(\mathbb{R}^d) \to H^{k,p}_{\delta_1+\delta_2}(\mathbb{R}^d)\). In fact, there is a constant \( C = C(d, m, \ell, k, p, \delta_1, \delta_2) \) such that

\[
\|fg\|_{H^{k,p}_{\delta_1+\delta_2}} \leq C \|f\|_{H^{m,p}_{\delta_1}} \|g\|_{H^{\ell,p}_{\delta_2}} \quad \text{for all } f \in H^{m,p}_{\delta_1} \text{ and } g \in H^{\ell,p}_{\delta_2}.
\]

**Proof.** To begin with, it is easy to check that the following weighted Hölder inequality holds:

\[
\|fg\|_{L^{p/\delta}_x} \leq \|f\|_{L^{q_1}_{x_1}} \|g\|_{L^{q_2}_{x_2}}, \quad \text{if } \frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} \text{ and } \delta = \delta_1 + \delta_2.
\]

Consequently, we will know that \((f, g) \mapsto fg\) defines a continuous map \( H^{m,p}_{\delta_1}(\mathbb{R}^d) \times H^{\ell,p}_{\delta_2}(\mathbb{R}^d) \to L^{p}_{\delta_1+\delta_2}(\mathbb{R}^d)\), i.e. the proposition holds for \(k = 0\), provided we can find \(1 \leq q_1, q_2 \leq \infty\) so that

\[
H^{m,p}_{\delta_1}(\mathbb{R}^d) \subset L^{q_1}_{\delta_1}(\mathbb{R}^d), \quad H^{\ell,p}_{\delta_2}(\mathbb{R}^d) \subset L^{q_2}_{\delta_2}(\mathbb{R}^d), \quad \frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2}.
\]

To obtain (8), let us first assume that \( mp < d \), so we also have \( \ell p < d \). According to Lemma 1.1 (c), we have

\[
H^{m,p}_{\delta_1}(\mathbb{R}^d) \subset L^{q_1}_{\delta_1}(\mathbb{R}^d) \quad \text{provided } p \leq q_1 \leq \frac{dp}{d-mp}
\]

\[
H^{\ell,p}_{\delta_2}(\mathbb{R}^d) \subset L^{q_2}_{\delta_2}(\mathbb{R}^d) \quad \text{provided } p \leq q_2 \leq \frac{dp}{d-\ell p}.
\]

Now it is clear that the function \( f(q_1, q_2) = q_1^{-1} + q_2^{-1} \) takes on all values between \(2/p\) and

\[
\frac{d-mp}{dp} + \frac{d-\ell p}{dp} = \frac{2}{p} - \frac{m + \ell}{d},
\]

so whether \( f(q_1, q_2) \) ever equals \( p^{-1} \) is determined by whether

\[
\frac{2}{p} - \frac{m + \ell}{d} \leq \frac{1}{p} \leq \frac{2}{p}.
\]

The second inequality is trivial but the first holds precisely when \((m + \ell)p \geq d\).

How does this result change when \( mp \geq d \)? For \( mp = d \) we need to require \( q_1 < \infty \), which translates into a strict inequality in (8), so we require \((m + \ell)p > d\). For \( mp > d \), then by Lemma 1(d) we can take \( q_1 = \infty \) and \( q_2 = p \). Thus we always have (8) under the assumption \((m + \ell)p > d\). This proves the proposition for \(k = 0\).

Now, to prove the proposition for all \(0 \leq k \leq \ell \), we must show that \( D^\alpha(fg) \in L^{p}_{\delta_1+\delta_2} \) for all \( |\alpha| \leq k \). But if we use the Leibniz rule to write

\[
D^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (D^\beta f)(D^{\alpha-\beta} g),
\]
and we observe that $D^\beta f \in H^{m-|\beta|}_{\delta_1}(\mathbb{R}^d)$ and $D^{\alpha-\beta} g \in H^{d-|\alpha-\beta|}_{\delta_2}(\mathbb{R}^d)$, then we can use provided $m - |\beta| + \ell - |\alpha - \beta| > d/p$. But this is guaranteed since $|\beta| + |\alpha - \beta| = |\alpha| \leq k$. □

**Proposition 1.2.** For $(m + \ell - k)p > d$ where $0 \leq k \leq \ell \leq m$, pointwise multiplication 

$$(f, g) \mapsto fg$$ defines a continuous map $W^{m,p}_{\delta_1}(\mathbb{R}^d) \times W^{\ell,p}_{\delta_2}(\mathbb{R}^d) \to W^{k,p}_{\delta_1+\delta_2}(\mathbb{R}^d)$. In fact, there is a constant $C = C(d, m, \ell, k, p)$ such that 

$$\|fg\|_{W^{k,p}_{\delta_1+\delta_2}} \leq C \|f\|_{W^{m,p}_{\delta_1}} \|g\|_{W^{\ell,p}_{\delta_2}}$$ for all $f \in W^{m,p}_{\delta_1}$ and $g \in W^{\ell,p}_{\delta_2}$.

**Proof.** As a special case of 7 we have 

$$\|fg\|_{L^p_{\delta_1+\delta_2}} \leq \|f\|_{L^{q_1}_{\delta_1}} \|g\|_{L^{q_2}_{\delta_2}}$$, if $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2}$ and $\delta = \delta_1 + \delta_2$.

Using this, we will know that $(f, g) \mapsto fg$ defines a continuous map $W^{m,p}_{\delta_1}(\mathbb{R}^d) \times W^{\ell,p}_{\delta_2}(\mathbb{R}^d) \to L^p_{\delta_1+\delta_2}(\mathbb{R}^d)$ provided we can find $1 \leq q_1, q_2 \leq \infty$ so that 

$$(10) \quad W^{m,p}_{\delta_1}(\mathbb{R}^d) \subset L^{q_1}_{\delta_1}(\mathbb{R}^d), \quad W^{\ell,p}_{\delta_2}(\mathbb{R}^d) \subset L^{q_2}_{\delta_2}(\mathbb{R}^d), \quad \frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2}.$$ 

Let us assume first that $mp < d$, so we also have $\ell p < d$. According to Lemma 1.2(c), we have 

$$W^{m,p}_{\delta_1}(\mathbb{R}^d) \subset L^{q_1}_{\delta_1}(\mathbb{R}^d) \quad \text{provided } p \leq q_1 \leq \frac{dp}{d-mp}$$ 

and 

$$W^{\ell,p}_{\delta_2}(\mathbb{R}^d) \subset L^{q_2}_{\delta_2}(\mathbb{R}^d) \quad \text{provided } p \leq q_2 \leq \frac{dp}{d-\ell p}.$$ 

For the same reasons as in the proof of Proposition 1.1, this is possible when $(m + \ell)p \geq d$. The case $mp \geq d$ also follows as in the proof of Proposition 1.1.

Now, to prove the proposition, we must show that $(x)^{|\alpha|} D^\alpha(fg) \in L^p_{\delta_1+\delta_2}$ for all $|\alpha| \leq k$. But if we write 

$$(x)^{|\alpha|} D^\alpha(fg) = (x)^{|\alpha|} \sum_{\beta \leq |\alpha|} \binom{|\alpha|}{|\beta|} (D^\beta f)(D^{\alpha-\beta} g)$$

$$= \sum_{\beta \leq |\alpha|} \binom{|\alpha|}{|\beta|} (x)^{|\beta|} D^\beta f \left( (x)^{|\alpha-\beta|} D^{\alpha-\beta} g \right),$$

and we observe that $(x)^{|\beta|} D^\beta f \in W^{m-|\beta|}_{\delta_1}(\mathbb{R}^d)$ and $(x)^{|\alpha-\beta|} D^{\alpha-\beta} g \in W^{d-|\alpha-\beta|}_{\delta_2}(\mathbb{R}^d)$, then we can use provided $m - |\beta| + \ell - |\alpha - \beta| > d/p$. But this is guaranteed since $|\beta| + |\alpha - \beta| = |\alpha| \leq k$. □

In the next section, we shall also need to consider Sobolev spaces $H^{m,p}(S^{d-1})$ on the unit sphere $S^{d-1}$ in $\mathbb{R}^d$. The boundedness of multiplication on these spaces can be found in the literature or easily derived using the Hölder inequality and Sobolev embedding on $S^{d-1}$ as in the proofs above. We record here the result.

**Proposition 1.3.** For $(m + \ell - k)p > d - 1$ where $0 \leq k \leq \ell \leq m$, pointwise multiplication 

$$(f, g) \mapsto fg$$ defines a continuous map $H^{m,p}(S^{d-1}) \times H^{\ell,p}(S^{d-1}) \to H^{k,p}(S^{d-1})$. In fact, there is a constant $C = C(d, m, \ell, k, p)$ such that 

$$\|fg\|_{H^{k,p}} \leq C \|f\|_{H^{m,p}} \|g\|_{H^{\ell,p}}$$ for all $f \in H^{m,p}$ and $g \in H^{\ell,p}$.
2. Asymptotic Spaces of Functions on $\mathbb{R}^d$

We want to consider functions $u \in H^{m,p}_{loc}(\mathbb{R}^d)$ which are bounded on $\mathbb{R}^n$ and admit a partial asymptotic expansion as $|x| \to \infty$. To describe this partial asymptotic expansion, let $\chi(t)$ be a smooth function satisfying $\chi(t) = 0$ for $t \leq 1$, $\chi(t) = 1$ for $t \geq 2$, and $|\chi^{(k)}(t)| \leq M$ for $0 \leq k \leq m$ and all $t$. For a nonnegative integer $N$, the functions that we consider are of the following form:

\begin{equation}
(11a) \quad u(x) = a(x) + f(x), \quad \text{where}
\end{equation}

\begin{equation}
(11b) \quad a(x) = \chi(r) \left( a_0(\theta) + \frac{a_1(\theta)}{r} + \cdots + \frac{a_N(\theta)}{r^N} \right) \text{ with } a_k \in H^{m+1-N-k,p}(S^{d-1}), \quad \text{and}
\end{equation}

\begin{equation}
(11c) \quad f(x) = o \left( |x|^{-N} \right) \quad \text{as } |x| \to \infty.
\end{equation}

In (11b) and throughout this paper, we use $r = |x|$ and $\theta = x/|x| \in S^{d-1}$. We refer to $a$ in (11b) as the asymptotic function, the function $a_k$ on $S^{d-1}$ as the asymptotic of order $k$, and $f$ as the remainder function for $u$. We want to achieve (11c) by requiring the remainder function $f$ to belong to one of the weighted Sobolev spaces discussed in the previous section. Let us begin with $W^{m,p}_d$. From Lemma 1.2(d) we see that $f \in W^{m,p}_d(\mathbb{R}^d)$ satisfies (11c) provided $mp > d$ and $\delta + \frac{d}{p} \geq N$. However, for reasons that will become clear in the next section, we want to avoid values of $\delta$ for which $\delta + \frac{d}{p}$ is an integer. Consequently, let us define

\begin{equation}
(12) \quad \gamma_N = N + \gamma_0, \quad \text{where } \gamma_0 \text{ has been chosen to satisfy } 0 < \gamma_0 + \frac{d}{p} < 1,
\end{equation}

so that $f \in W^{m,p}_{\gamma_N}(\mathbb{R}^d)$ with $mp > d$ satisfies $f(x) = o(|x|^{-N-\varepsilon})$ where $\varepsilon \in (0,1)$. Now we define

\begin{equation}
(13) \quad A^{m,p}_N(\mathbb{R}^d) = \{ u \text{ is in the form } (11) \text{ where } f \in W^{m,p}_{\gamma_N}(\mathbb{R}^d) \}.
\end{equation}

When the domain $\mathbb{R}^d$ is understood, we simply write $A^{m,p}_N$ instead of $A^{m,p}_N(\mathbb{R}^d)$; when $N$ is fixed or understood, we may simply write $\gamma$ instead of $\gamma_N$. The norm on $A^{m,p}_N$ is given by

\begin{equation}
(14) \quad \| u \|_{A^{m,p}_N} = \| a_0 \|_{H^{m+1-N,p}(S^{d-1})} + \cdots + \| a_N \|_{H^{m+1-N,p}(S^{d-1})} + \| f \|_{W^{m,p}_d(\mathbb{R}^d)}.
\end{equation}

This norm is complete, so $A^{m,p}_N$ is a Banach space. For an integer $n$ with $0 \leq n \leq N$, we define closed subspaces

\begin{equation}
(15) \quad A^{m,p}_{n,N} = A^{m,p}_{n,N}(\mathbb{R}^d) = \{ u \in A^{m,p}_N : a_k = 0 \text{ for } k < n \}.
\end{equation}

**Remark 2.1.** That the regularity of the asymptotic $a_k$ depends on $k$, i.e. $a_k \in H^{m+1-N-k,p}(S^{d-1})$, is an important feature of (11); it will prove essential many times in the analysis below.

**Remark 2.2.** In the definition (11), the specification that $\chi(r) \equiv 1$ for $r > 2$ is somewhat arbitrary. In fact, if we introduce $\chi_R(t) = \chi(R^{-1}t)$, then $\chi_R(r) \equiv 1$ for $r > 2R$ and we can write

$$
 u = \chi \sum_{k=0}^{N} \frac{a_k(\theta)}{r^k} + f = \chi_R \sum_{k=0}^{N} \frac{a_k(\theta)}{r^k} + \tilde{f},
$$

where $\tilde{f}$ differs from $f$ by a function with compact support:

$$
 \tilde{f} = f + (\chi - \chi_R) \sum_{k=0}^{N} \frac{a_k(\theta)}{r^k}.
$$

But we can estimate

$$
 \left\| (\chi - \chi_R) \sum_{k=0}^{N} \frac{a_k(\theta)}{r^k} \right\|_{W^{m,p}_{\gamma_N}} \leq C \sum_{k=0}^{N} \| a_k \|_{H^{m,p}} \leq C \sum_{k=0}^{N} \| a_k \|_{H^{m+1-N-k,p}},
$$
where $C$ depends on $R$, $\chi$, $m$, $p$, $d$, and $N$, to conclude $\|f\|_{W^{m,p}_\gamma(B)} \leq C\|u\|_{A^{m,p}_N}$. Similarly, we can estimate $f$ in terms of the $a_k$ and $\tilde{f}$, so if we use $\chi_R$ in place of $\chi$ in (11), we will get a norm on the Banach space $A^{m,p}_N$ that is equivalent to (14). This will be important in subsequent sections. In fact, it is sometimes convenient to consider the restriction of $u = a + f$ to the exterior domain $B^*_R = \mathbb{R}^d \setminus B_R$. This generates a Banach space $A^{m,p}_N(B^*_R)$ with norm

$$\|u\|_{A^{m,p}_N(B^*_R)} := \sum_{k=0}^{N} |a_k|_{H^{m+1-N+k,p}(S^{d-1})} + \|f\|_{W^{m,p}_\gamma(B^*_R)}.$$  

Notice that $\|u\|_{A^{m,p}_N(\mathbb{R}^d)}$ is equivalent to $\|u\|_{A^{m,p}_N(B^*_R)} + \|f\|_{H^{m,p}(B_R)}$.

Now let us use $H^{m,p}_{\delta}$ as the remainder space. From Lemma 11(d) we see that $f \in H^{m,p}_{\delta}(\mathbb{R}^d)$ satisfies (11e) provided $mp > d$ and $\delta \geq N$. This suggests that we use $H^{m,p}_N$ as the remainder space. However, if we use (11b) then we would want $|r|^{-N-1} \in H^{m,p}_{\delta}(B^*_R)$, but this is only true if $d < p$. Consequently, for given nonnegative integer $N$, let $N^*$ be the positive integer satisfying

$$N^* = \frac{d}{p} \quad \text{if} \quad N = \frac{d}{p} \quad \text{and} \quad N^* = \frac{d}{p} + 1 \quad \text{if} \quad N > \frac{d}{p},$$

and replace (11b) with

$$a(x) = \chi(|x|) \sum_{k=0}^{N^*} \frac{a_k(\theta)}{r^k}, \quad \text{with} \; a_k \in H^{m+1-N^*+k,p}(S^{d-1}).$$

We always have $N^* \geq N$, but we have $N^* = N$ when $d = 1$, or more generally if $d < p$. In any case, let us define

$$A^{m,p}_N(\mathbb{R}^d) = \left\{ u \in H^{m,p}(\mathbb{R}^d) : u = a_0 + f, \; a_0 \in H^{m,p}_N(\mathbb{R}^d), \; f \in H^{m,p}_N(\mathbb{R}^d) \right\}.$$  

When the domain $\mathbb{R}^d$ is understood, we simply write $A^{m,p}_N$. We replace (14) by

$$\|u\|_{A^{m,p}_N} = \sum_{k=0}^{N^*} |a_k|_{H^{m+1-N^*+k,p}(S^{d-1})} + \|f\|_{H^{m,p}(\mathbb{R}^d)}.$$  

Under this norm, $A^{m,p}_N$ is a Banach space, and we define closed subspaces $A^{m,p}_{N_1}$ by requiring $a_k = 0$ for $k < n$. Of course, Remarks 2.1 and 2.2 apply as well to (17) and (18).

We next investigate some of the properties of these asymptotic spaces. We begin with an elementary result.

**Lemma 2.1.** If $a \in H^{m,p}(S^{d-1})$ then $a(\theta) r^{-k} \in W^{m,p}_\delta(B^*_R) \subset H^{m,p}_{\delta}(B^*_R)$ for all $\delta < k - d/p$.

The lemma is easy to prove using integration in spherical coordinates and the simple computation:

$$a \in H^{m,p}(S^{d-1}), \quad 0 \leq |\beta| \leq m \Rightarrow D^\beta(a(\theta) r^{-k}) = b_\beta(\theta) r^{-k-|\beta|} \quad \text{for} \quad |x| = r > 1,$$

where $b_\beta \in H^{m-|\beta|,p}(S^{d-1})$.

We will use Lemma 2.1 in confirming that our asymptotic spaces have the following properties:

**Proposition 2.1.**

(a) If $n_1 \geq n$ and $N_1 \geq N$, then $A^{m,p}_{n_1,N_1} \subset A^{m,p}_{n,N}$ and $A^{m,p}_{n_1,N_1} \subset A^{m,p}_{n,N}$.

(b) If $m \geq 1$, then $u \mapsto \partial u / \partial x_j$ is continuous $A^{m,p}_{n,N} \rightarrow A^{m-1,p}_{n+1,N+1}$ and $A^{m,p}_{n,N} \rightarrow A^{m-1,p}_{n+1,N+1}$.

(c) Multiplication by $\chi(r) r^{-k}$ is bounded $A^{m,p}_{n,N} \rightarrow A^{m,k,\beta}_{n,N+k}$ and $A^{m,p}_{n,N} \rightarrow A^{m,k,\beta}_{n+N,N+k}$.

(d) Assume $m > d/p$. If $u \in A^{m,p}_{n,N}$, then

$$\sup_{x \in \mathbb{R}^d} (x)^{n+|\alpha|} |D^\alpha u(x)| \leq C \|u\|_{A^{m,p}_{n,N}} \quad \text{for all} \quad |\alpha| < m - d/p.$$
If \( u \in \mathcal{A}_{m,N_1}^{m,p} \), then

\[
\sup_{x \in \mathbb{R}^d} |x|^n |D^\alpha u(x)| \leq C \|u\|_{\mathcal{A}_{m,N_1}^{m,p}} \quad \text{for all } |\alpha| < m - d/p.
\]

We note that the constants \( C \) in (d), and throughout the rest of this paper, may depend on \( d \) and parameters associated with the functions spaces but not on the functions themselves.

**Proof.** (a) Write \( u \in \mathcal{A}_{n_1,N_1}^{m,p} \) as

\[
u(x) = \chi \left( \frac{a_{n_1}(\theta)}{n_1} + \cdots + \frac{a_{N_1}(\theta)}{N_1} \right) + f_{N_1} \quad \text{with } a_k \in H^{m+1+N_1-k}(S^{d-1}), \ f_{N_1} \in W_{\gamma_{N_1}}^m.
\]

\[
u(x) = \chi \left( \frac{a_n(\theta)}{n} + \cdots + \frac{a_N(\theta)}{N} \right) + g_N \quad \text{where } a_k = 0 \quad \text{for } n \leq k < n_1 \text{ and}
\]

\[
g_N = \begin{cases} 
0 & \text{if } N_1 = N, \\
\chi \left( \frac{a_{N+1}}{N+1} + \cdots + \frac{a_{N_1}}{N_1} \right) + f_{N_1} & \text{if } N_1 > N.
\end{cases}
\]

We clearly have \( a_k \in H^{m+1+N-k}(S^{d-1}) \) and (using Lemma 2.1) \( g_N \in W_{\gamma_{N_1}}^m, \) so \( u \in \mathcal{A}_{n,N}^{m,p} \).

Similarly for \( \mathcal{A}_{n_1,N_1}^{m,p} \subset \mathcal{A}_{m,p}^{m,p} \).

To prove (b) let us first consider \( u \in \mathcal{A}_{n_1,N_1}^{m,p} \) with asymptotics \( a_k \in H^{m+1+N-k}(S^{d-1}) \) for \( k = n_1, \ldots, N \). We use (14) with \( |\beta| = 1 \) to compute

\[
\frac{\partial}{\partial x_j} \left( \chi(r) \frac{a_k(\theta)}{r^k} \right) = \chi(r) \frac{b_{r,j}(\theta)}{r^{k+1}} + \theta^j a_k(\theta) \frac{r^{k}}{r^k},
\]

where \( b_{r,j} \in H^{m+N-k}(S^{d-1}) \). The term \( b_{r,j} r^{-k} \) for \( k = n_1, \ldots, N \) is of the form of an asymptotic function in \( \mathcal{A}_{n_1,N_1}^{m-1,p} \) while \( \chi'(r) \theta_j a_k(\theta) r^{-k} \) has compact support so certainly belongs to the remainder space \( W_{\gamma_{N_1}}^{m-1,p} \). Since we also know that \( \partial_j : W_{\gamma_{N_1}}^{m,p} \to W_{\gamma_{N_1}+1,p}^{m-1,p} \), we have \( \partial_j : \mathcal{A}_{n,N}^{m,p} \to \mathcal{A}_{n_1,N_1}^{m-1,p} \). We consider \( u \in \mathcal{A}_{n_1,N_1}^{m,p} \), with asymptotics \( a_k \in H^{m+1+N-k}(S^{d-1}) \) for \( k = n_1, \ldots, N \).

Using (22) where \( b_{r,j} \in H^{m+N-k}(S^{d-1}) \subset H^{m+N-k-1}(S^{d-1}) \) for \( k = n_1, \ldots, N \), with \( \chi(r) \theta_j a_k(\theta) r^{-k} \) has support in the annulus \( \mathcal{A}_{n_1,N_1}^{m,p} \) is bounded.

The proof of (c) is immediate.

To prove (d), first consider \( u = \chi(r) a_k(\theta) r^{-k} \) with \( n \leq k \leq N \) and \( a_k \in H^{m+1+N-k}(S^{d-1}) \). We generalize (22) to conclude

\[
D^\alpha \left( \chi(r) \frac{a_k(\theta)}{r^k} \right) = \chi(r) D^\alpha \left( \frac{a_k(\theta)}{r^k} \right) + g(x) = \chi(r) \frac{b_{r,\alpha}(\theta)}{r^{k+1}} + g(x),
\]

where \( b_{r,\alpha} \in H^{m+1+N-k-|\alpha|}(S^{d-1}) \) with

\[
\|b_{r,\alpha}\|_{H^{m+1+N-k-|\alpha|}(S^{d-1})} \leq c \|a_k\|_{H^{m+1+N-k}(S^{d-1})},
\]

and \( g \in H^{m+2+N-k-|\alpha|}(S^{d-1}) \) has support in the annulus \( \mathcal{A} = \{x : 1 < |x| < 2\} \) with

\[
\|g\|_{H^{m+2+N-k-|\alpha|}(A)} \leq c \|a_k\|_{H^{m+1+N-k}(S^{d-1})},
\]

Now \( |\alpha| < m - d/p \) certainly implies \( m + 1 + N - k - |\alpha| > (d - 1)/p \), so we can use the Sobolev embedding theorem on \( S^{d-1} \) to conclude

\[
\sup_{\theta \in S^{d-1}} \|b_{r,\alpha}(\theta)\| \leq C \|b_{r,\alpha}\|_{H^{m+1+N-k-|\alpha|}(S^{d-1})}.
\]
and we can apply the Sobolev embedding theorem on $A$ to conclude

\[ \sup_{1 < |x| < 2} |g(x)| \leq C \|g\|_{H^{m+2N-k-|\alpha|}(A)}. \]

(24b)

Combining these inequalities, we have

\[ \sup_{|x| > 1} |x|^{n+|\alpha|} |D^\alpha \left( \chi(x) \frac{a_k(\theta)}{r^k} \right)| \leq C \|a_k\|_{H^{m+1+N-k}(S^{d-1})}. \]

Thus, for an asymptotic function $a$ as in (11b), we have

\[ \sup_{|x| > 1} |x|^{n+|\alpha|} |D^\alpha a(x)| \leq C \left( \|a_0\|_{H^{m+1+N-\alpha}(S^{d-1})} + \cdots + \|a_N\|_{H^{m+1}(S^{d-1})} \right), \]

and the same holds for the asymptotic function in (16b) provided we replace $N$ by $N^*$. Now let us consider the remainder function $f$. If $f \in W^{m,p}_\gamma(\mathbb{R}^d)$, then we use Lemma 1.2 (d) with $\delta = N$ to conclude

\[ \sup_{x \in \mathbb{R}^d} \langle x \rangle^{N+|\alpha|} |D^\alpha f(x)| \leq C \|f\|_{W^{m,p}_\gamma} \quad \text{for all } |\alpha| < m - d/p. \]

On the other hand, if $f \in H^{m,p}_\gamma(\mathbb{R}^d)$, then we use Lemma 1.4 (d) with $\delta = N$ to conclude

\[ \sup_{x \in \mathbb{R}^d} \langle x \rangle^{N} |D^\alpha f(x)| \leq C \|f\|_{H^{m,p}_\gamma} \leq C \|f\|_{H^{m,p}} \quad \text{for all } |\alpha| < m - d/p. \]

Since $n \leq N$, these estimates imply (20) and (21).

What about multiplication? The product of two partial asymptotic expansions involves a number of terms. The product of the remainder functions is covered by Propositions 1.1 and 1.2 for convenience, we record here the following special case of those results:

**Lemma 2.2.** Assume $m > d/p$ and $k = 0, \ldots, m$.

(a) $\|fg\|_{H^{m,p}_{\alpha_1+\beta_1, \gamma_1}} \leq \|f\|_{H^{m,p}_{\beta_1, \gamma_1}} \|g\|_{H^{m,p}_{\alpha_2, \gamma_2}}$ for $f \in H^{m,p}_{\beta_1, \gamma_1}$ and $g \in H^{m,p}_{\alpha_2, \gamma_2}$.

(b) $\|fg\|_{W^{m,p}_{\alpha_1+\beta_1-d/p, \gamma_1}} \leq \|f\|_{W^{m,p}_{\beta_1-d/p, \gamma_1}} \|g\|_{W^{m,p}_{\alpha_2-d/p, \gamma_2}}$ for $f \in W^{m,p}_{\beta_1-d/p, \gamma_1}$ and $g \in W^{m,p}_{\alpha_2-d/p, \gamma_2}$.

The product of an asymptotic term like $a_k(\theta)/|x|^k$ and a remainder function is covered by the following (in which we use Lemma 2.1(c) to assume $k = 0$).

**Lemma 2.3.** Assume $a \in H^{s,p}(S^{d-1})$ for an integer $s > (d-1)/p$, $m$ is an integer $0 \leq m \leq s$, and $\delta \in \mathbb{R}$.

(a) $f \in W^{m,p}_\delta(B^\circ_1) \Rightarrow af \in W^{m,p}_\delta(B^\circ_1)$ and $\|af\|_{W^{m,p}_\delta(B^\circ_1)} \leq C \|a\|_{H^{s,p}(S^{d-1})} \|f\|_{W^{m-p}(B^\circ_1)}$.

(b) $f \in H^{m,p}_\delta(B^\circ_1) \Rightarrow af \in H^{m,p}_\delta(B^\circ_1)$ and $\|af\|_{H^{m-p}_\delta(B^\circ_1)} \leq C \|a\|_{H^{s,p}(S^{d-1})} \|f\|_{H^{m-p}_\delta(B^\circ_1)}$.

**Proof.** For a nonnegative integer $\ell$, we simply denote by $D^\ell f$ a partial derivative of $f$ of order $\ell$. To show (a), we want to estimate $\int_{|x| > 1} |x|^{(\delta+\ell)p} |D^\ell (af)|^p dx$ for $\ell = 0, \ldots, m$. But $D^\ell (af)$ is a sum of products of the form $D^i a D^j f$ where $i + j = \ell$. For $i = 0$, we have

\[ \int_{|x| > 1} |x|^{(\delta+\ell)p} |D^\ell f|^p dx \leq \sup_{\theta \in S^{d-1}} \|a(\theta)|^p \int_{|x| > 1} |x|^{(\delta+\ell)p} |D^\ell f|^p dx \leq C \|a\|_{H^{s,p}(S^{d-1})} \|f\|_{W^{m,p}_\delta(B^\circ_1)}^p. \]

For $i > 0$, $D^i a$ is a sum of products of the form $r^{-i} c_k(\theta) D^k_\delta a$ where $c_k$ is a polynomial in $\theta$ and $k = 1, \ldots, i$. Thus we want to estimate

\[ \int_{|x| > 1} |x|^{(\delta+j)p} |D^k_\delta a|^p |D^i f|^p dx \quad \text{for } k = 1, \ldots, i; \ i + j = \ell. \]
For fixed $r > 1$, let us denote by $f_r$ the function on $S^{d-1}$ defined by $f_r(\theta) = f(r\theta)$. Now let us use Proposition 2.2 (and $s > (d - 1)/p$) to estimate

$$\int_{S^{d-1}} |D^r_\theta a(\theta)|^p |D^j f(\theta)|^p d\theta = \|D^r_\theta a(D^j f)(r\theta)\|_{L^p(S^{d-1})}^p \leq C \|D^r_\theta a\|_{H^{s-1-k,p}(S^{d-1})}^p \|(D^j f)(r\theta)\|_{H^{s,p}(S^{d-1})}^p \leq C \|a\|_{H^{s,p}(S^{d-1})} \|f\|_{H^{s,p}(S^{d-1})}^p.$$ 

By trace theory, $D^j f \in W^{m-j,p}(B_1)$ implies $(D^j f)(r) \in H^k(S^{d-1})$ provided $k \leq m - j - \frac{1}{p}$; this last condition only fails when $j = m$, which does not occur since we have assumed $i > 0$. Moreover, for a function $g(x)$ we can use $\partial g/\partial \theta_i = r \partial g/\partial x_i$ to estimate any derivative $D^r_\theta g$ on $S^{d-1}$ by $|D^r_\theta g(r\theta)| \leq r|\nabla g(x)| + \cdots + r^k|\nabla^k g(x)|$, where $|\nabla^k g(x)|$ denotes the sum of the absolute values of all $x$ derivatives of $g$ of order $k$. Applying this to $g = D^j f$, we obtain

$$\int_{|x| > 1} |x|^{(d+j)p} |D^j f||_{H^{s,p}(S^{d-1})}^p dx \leq C \|f\|_{W^{s,p}(B_1)}^p \text{ for } j + k \leq \ell.$$ 

Thus we have shown for $i + j = \ell$ and $k = 1, \ldots, i$ that

$$\int_{|x| > 1} |x|^{(d+j)p} |D^r_\theta a_p| |D^j f|^p dx \leq C \|a\|_{H^{s,p}(S^{d-1})}^p \|f\|_{W^{s,p}(B_1)}^p \leq C \|a\|_{H^{s,p}(S^{d-1})} \|f\|_{W^{s,p}(B_1)}^p.$$ 

Combining the cases $i = 0$ and $i > 0$, we have shown (a).

The proof of (b) follows the same outline as for (a). Again we write $D^\ell(af)$ as a sum of products $D^\ell a^r f$ and treat the case $i = 0$ by

$$\int_{|x| > 1} |x|^{(d-j)p} |D^r_\theta a|^p |D^j f|^p dx \leq \left( \sup_{\theta \in S^{d-1}} |a(\theta)|^p \right) \int_{|x| > 1} |x|^{(d-j)p} |D^j f|^p dx \leq C \|a\|_{H^{s,p}(S^{d-1})} \|f\|_{H^{s,p}(B_1)}^p.$$ 

For $i > 0$, we want to estimate

$$\int_{|x| > 1} |x|^{(d-i)p} |D^r_\theta a|^p |D^j f|^p dx \text{ for } k = 1, \ldots, i; \ i + j = \ell.$$ 

But arguing as above and using $(\delta - i)p < \delta p$, we can conclude

$$\int_{|x| > 1} |x|^{(d-i)p} |D^\ell(af)|^p dx \leq C \|a\|_{H^{s,p}(S^{d-1})} \|f\|_{H^{s,p}(B_1)}^p \text{ for } \ell \leq m. \quad \square$$

Combining Lemmas 2.2 and 2.3 we obtain the following

**Corollary 2.1.** If $m > d/p$, $0 \leq n \leq N$, and $u \in A^{m,p}_{n,N}$, then for any $f \in W^{k,p}_\delta$ where $0 \leq k \leq m$ and $\delta \in \mathbb{R}$ we have

$$\|fu\|_{W^{k,p}_\delta} \leq C \|f\|_{W^{k,p}_\delta} \|u\|_{A^{m,p}_{n,N}}.$$ 

The analogous statement with $W$ replaced by $H$ and $A$ replaced by $A^p$ is also true.

We are now able to prove the following result on products for our asymptotic spaces:

**Proposition 2.2.** For $m > d/p$ and $0 \leq n_i \leq N_i$ for $i = 1, 2$, let $\bar{n} = n_1 + n_2$ and $\bar{N} = \min(N_1 + n_2, N_2 + n_1)$. Then

$$(27a) \quad \|uv\|_{A^{m,p}_{\bar{n},\bar{N}}} \leq C \|u\|_{A^{m,p}_{n_1,N_1}} \|v\|_{A^{m,p}_{n_2,N_2}} \quad \text{for } u \in A^{m,p}_{n_1,N_1}, \ v \in A^{m,p}_{n_2,N_2},$$

$$(27b) \quad \|uv\|_{A^{m,p}_{\bar{n},\bar{N}}} \leq C \|u\|_{A^{m,p}_{n_1,N_1}} \|v\|_{A^{m,p}_{n_2,N_2}} \quad \text{for } u \in A^{m,p}_{n_1,N_1}, \ v \in A^{m,p}_{n_2,N_2}.$$
Proof. We shall prove (27b); the proof of (27a) is analogous. Since \( p \) is fixed, we shall drop that notation, but let us introduce \( \gamma_i = \gamma_{N_i} \) for \( i = 1, 2 \) and \( \bar{\gamma} = \gamma_{\bar{N}} \). For \( u \in A_{m_1, N_1}^m \), \( v \in A_{m_2, N_2}^m \), let us write

\[
 u = \chi \left( \frac{a_{n_1}}{r^{n_1}} + \cdots + \frac{a_{N_1}}{r^{N_1}} \right) + f \quad \text{and} \quad v = \chi \left( \frac{b_{n_2}}{r^{n_2}} + \cdots + \frac{b_{N_2}}{r^{N_2}} \right) + g
\]

where \( a_k \in H^{m+1+N_1-k}(S^{d_1}) \), \( b_k \in H^{m+1+N_2-k}(S^{d_2}) \), \( f \in W_{\gamma_1}^m \), and \( g \in W_{\gamma_2}^m \). Taking the product, we can write

\[
u v = \chi^2 \sum_{i=n_1}^{N_1} \sum_{j=n_2}^{N_2} \frac{a_i b_j}{r^{i+j}} + \chi \left( \frac{a_{n_1}}{r^{n_1}} + \cdots + \frac{a_{N_1}}{r^{N_1}} \right) g + \chi \left( \frac{b_{n_2}}{r^{n_2}} + \cdots + \frac{b_{N_2}}{r^{N_2}} \right) f + fg
\]

(28)

\[
 = \chi^2 \sum_{k=n_1+n_2}^{N_1+N_2} \sum_{i+j=k} c_k \quad \text{where} \quad c_k = \frac{a_i b_j}{r^k} \quad \text{and} \quad h \quad \text{is all terms involving} \quad f \quad \text{or} \quad g.
\]

In order to show that \( u v \in A_{h, N}^m \) we need to show (i) \( c_k \in H^{m+1+N_1-k}(S^{d_1}) \) and (ii) \( h \in W_{\bar{\gamma}}^m \).

Of course, we also need to show that we can replace \( \chi^2 \) in (28) by \( \chi \). But \((\chi^2 - \chi)\) is supported in \( 1 < \gamma < 2 \), so

\[
 \left\| (\chi^2 - \chi) \sum_{k=n}^{N} \frac{c_k}{r^k} \right\|_{W_{\bar{\gamma}}^m} \leq \sum_{k=n}^{N} \|c_k\|_{H^m(S^{d_1})}
\]

\[
 \leq C \sum_{i=n_1}^{N_1} \|a_i\|_{H^m(S^{d_1})} \sum_{j=n_2}^{N_2} \|b_j\|_{H^m(S^{d_2})} \leq C \|u\|_{A_{m_{1}, N_{1}}} \|v\|_{A_{m_{2}, N_{2}}}.
\]

To prove \( c_k \in H^{m+1+N_1-k}(S^{d_1}) \), we can use Proposition 1.3 to conclude

\[
\|a_i b_j\|_{H^{m+1+N_1-k}} \leq C \|a_i\|_{H^m(S^{d_1})} \|b_j\|_{H^m(S^{d_2})} \quad \text{for} \quad k = i + j,
\]

since the condition

\[
(m + 1 + N_1 - i) + (m + 1 + N_2 - j) - (m + 1 + \tilde{N} - k) > (d - 1)/p
\]

reduces to just \( m + 1 > (d - 1)/p \), which is guaranteed by our assumption \( m > d/p \). This also shows the desired estimate for (27b).

To show \( h \in W_{\bar{\gamma}}^m \) we have several terms to consider. Let us first consider \( f g \). But \( f \in W_{\delta_1, \delta_2 - \frac{d}{p}}^{m, p} \) and \( g \in W_{\delta_1, \delta_2 + \frac{d}{p}}^{m, p} \) for \( N_i < \delta_1 = \gamma_i + \frac{d}{p} < N_i + 1 \), so we can apply Proposition 1.2 (using \( m > d/p \)) to conclude \( f g \in W_{\delta_1, \delta_2 - \frac{d}{p}}^{m, p} \). But \( \bar{\gamma} \leq N_1 + N_2 < \delta_1 + \delta_2 \), so we have \( W_{\delta_1, \delta_2 - \frac{d}{p}}^{m, p} \subset W_{\bar{\gamma}}^m \), i.e.

\[
\|f g\|_{W_{\bar{\gamma}}^m} \leq C \|f\|_{W_{\gamma_1}^m} \|g\|_{W_{\gamma_2}^m}
\]

As for the other terms, we can use \( \bar{\gamma} \leq n_2 + \gamma_1, n_1 + \gamma_2 \) and Lemmas 1.2 and 2.3 to conclude

\[
\sum_{k=n_1}^{N_1} \left\| \frac{a_k}{r^k} g \right\|_{W_{\gamma}^m(B_{\gamma_1}^\infty)} \leq C \sum_{k=n_1}^{N_1} \left\| \frac{a_k}{r^k} g \right\|_{W_{n_1+\gamma_2}^m(B_{\gamma_1}^\infty)} \leq C \left( \sum_{k=n_1}^{N_1} \left\| a_k \right\|_{H^m(S^{d_1})} \right) \|g\|_{W_{\gamma_1}^m}
\]

and

\[
\sum_{k=n_2}^{N_2} \left\| \frac{b_k}{r^k} f \right\|_{W_{\gamma}^m(B_{\gamma_2}^\infty)} \leq C \sum_{k=n_2}^{N_2} \left\| b_k f \right\|_{W_{n_2+\gamma_2}^m(B_{\gamma_2}^\infty)} \leq C \left( \sum_{k=n_2}^{N_2} \left\| b_k \right\|_{H^m(S^{d_2})} \right) \|f\|_{W_{\gamma_2}^m}.
\]
Finally we use $\bar{\gamma} = \bar{N} + \gamma_0$ where $\gamma_0 < 1 - \frac{d}{p}$ to show the term $\chi^2 \sum_{k=N+1}^{N_1+N_2} r^{-k} c_k$ is in the
remainder space:

$$\left\| \chi^2 \sum_{k=N+1}^{N_1+N_2} \frac{c_k}{r^k} \right\|_{W^m(B_1)} \leq \sum_{k=N+1}^{N_1+N_2} \|c_k\|_{H^{m(S^{d-1})}} \|\chi^2\|_{W^m_{\gamma_0-1}} \leq C \|u\|_{A_{m,N_1}^1} \|v\|_{A_{m,N_2}^1}. \quad \square$$

As a special case of Proposition 2.2 we obtain

\textbf{Corollary 2.2.} If $m > d/p$, and $0 \leq n \leq N$, then $A_{m,N}^{m,p}$ and $A_{n,N}^{m,p}$ are Banach algebras.

Since this paper is mostly concerned with diffeomorphisms of $\mathbb{R}^d$, we need to consider asymptotic spaces of vector-valued functions. Here we use bold-face $u$ for a vector-valued function and denote its components by $u^j$. Let us define the Banach spaces

$$\text{(29a)} \quad \mathcal{A}_{m,N}^{m,p}(\mathbb{R}^d, \mathbb{R}^d) = \left\{ u : \mathbb{R}^d \to \mathbb{R}^d \mid u^j \in A_{m,N}^{m,p}(\mathbb{R}^d) \right\}, \quad \|u\|_{\mathcal{A}_{m,N}^{m,p}} = \sum_{j=1}^{d} \|u^j\|_{A_{m,N}^{m,p}}.$$

and

$$\text{(29b)} \quad A_{m,N}^{m,p}(\mathbb{R}^d, \mathbb{R}^d) = \left\{ u : \mathbb{R}^d \to \mathbb{R}^d \mid u^j \in A_{m,N}^{m,p}(\mathbb{R}^d) \right\}, \quad \|u\|_{A_{m,N}^{m,p}} = \sum_{j=1}^{d} \|u^j\|_{A_{m,N}^{m,p}}.$$

As in the scalar-valued case, we will abbreviate $A_{0,N}^{m,p}$ simply as $A_{N}^{m,p}$ and suppress the notation $(\mathbb{R}^d, \mathbb{R}^d)$ when it is clear from the context that we are considering vector fields on $\mathbb{R}^d$.

3. Application to the Laplacian and Helmholtz Decompositions

The asymptotic spaces $A_{m,N}^{m,p}$ are generally preferable to $A_{n,N}^{m,p}$ in applications involving the Laplacian $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$. In this section we will illustrate this by discussing the mapping properties of $\Delta$ and an application to the Helmholtz decomposition of vector fields.

To begin with, consider the mapping

$$\text{(30)} \quad \Delta : W_{\delta}^{m+1,p}(\mathbb{R}^d) \to W_{\delta+2}^{m-1,p}(\mathbb{R}^d) \quad \text{for} \quad m \geq 1.$$ 

Clearly, (30) is continuous for all $\delta \in \mathbb{R}$, and in \textbf{[3]} it was shown that (30) is injective for $\delta > -d/p$ and an isomorphism (in particular invertible) for $0 < \delta + d/p < d - 2$ (when $d \geq 3$). For $N < \delta + d/p < N + 1$, where $N$ is an integer $\geq d - 2$, it was also shown in \textbf{[3]} that (30) is Fredholm with explicitly specified cokernel. We now observe that for arbitrary $g \in W_{\delta+2}^{m-1,p}(\mathbb{R}^d)$ for $N < \delta + d/p < N + 1$, we can find $u \in A_{\delta-2,N}^{m+1,p}$ such that $\Delta u = g$; here we have used $\gamma_N = \delta$ in defining $A_{\delta-2,N}^{m+1,p}$, so our hypothesis on $g$ can instead be written $g \in W^{m-1,p}_{\gamma_N+2}(\mathbb{R}^d)$.

\textbf{Lemma 3.1.} Suppose $d \geq 2$ and $m \geq 1$.

(a) For $d \geq 3$, there is a bounded operator

$$\text{(31a)} \quad K : W^{m-1,p}_{\gamma_N+2}(\mathbb{R}^d) \to A_{\delta-2,N}^{m+1,p}(\mathbb{R}^d),$$

such that $\Delta K g = g$. In other words, $u = Kg$ is of the form

$$\text{(31b)} \quad u(x) = \chi(|x|) \left( \frac{a_{d-2}}{r^{d-2}} + \cdots + \frac{a_N(\theta)}{r^N} \right) + f(x)$$

where each $a_k(\theta)/r^k$ is harmonic for $x \neq 0$ (so $a_k \in C^\infty(S^{d-1})$) and $f \in W^{m,p}_{\gamma_N}(\mathbb{R}^d)$. 

(b) For $d = 2$, the result also holds, except $1/r^{2-d}$ in (31b) is replaced by $\log r$. Of course, this means that the asymptotic space $A_{0,N}^{m+1,p}$ in (31a) must be replaced by

$$\mathcal{A}_{0,N}^{m+1,p}(\mathbb{R}^d) = \left\{ u = x \left( a_0^* \log r + a_0(\theta) + \cdots + \frac{a_N(\theta)}{r^N} \right) + f : a_0^* = \text{const.}, a_k \in H^{m+2+N-k}(S^{d-1}), f \in W_{\gamma N}^{m+1,p}(\mathbb{R}^d) \right\}.$$  

Proof. Let $\Gamma(|x|)$ denote the fundamental solution for the Laplace operator in $\mathbb{R}^d$ and $K = \Gamma^*$ denote the convolution operator. As shown in [13], $K : W_{\delta+2}^{-m-1,p}(\mathbb{R}^d) \rightarrow W_{\delta+2}^{m+1,p}(\mathbb{R}^d)$ is an isomorphism for $0 < \delta + d/p < d - 2$ (when $d \geq 3$); since we know $\gamma_N$ satisfies $N < \gamma_N + d/p < N + 1$, we conclude that (31a) is an isomorphism for $0 < \delta + d/p < d - 2$ and (31b) is no longer bounded, and we either need to restrict the domain space or expand the range space. Let us first describe what happens for $N = d - 2$ and then consider the general case.

For $d - 2 < \delta + d/p < d - 1$, it was shown in [13] that (30) is injective with constants as cokernel: if we let $\tilde{W}_{\delta+2}^{-m-1,p}(\mathbb{R}^d) = \{ g \in W_{\delta+2}^{-m-1,p}(\mathbb{R}^d) : \int_{\mathbb{R}^d} g(x)dx = 0 \}$, then $K : \tilde{W}_{\delta+2}^{-m-1,p}(\mathbb{R}^d) \rightarrow W_{\delta}^{m+1,p}(\mathbb{R}^d)$ is bounded. Taking $\gamma_N = \delta$, we have $K : \tilde{W}_{\gamma N+2}^{-m-1,p}(\mathbb{R}^d) \rightarrow W_{\gamma N+2}^{m+1,p}(\mathbb{R}^d)$ is bounded. To extend $K$ to general $g \in W_{\gamma N+1}^{-m-1,p}(\mathbb{R}^d)$, let us observe that $\Delta (\chi(r) \Gamma(r))$ has compact support and we use Green’s first identity to calculate

$$\int_{\mathbb{R}^d} \Delta (\chi(r) \Gamma(r)) \, dx = \int_{|x|<2} \Delta (\chi(r) \Gamma(r)) \, dx = \int_{|x|=2} \frac{\partial}{\partial r} \Gamma(r) \, ds = 1.$$  

Now we define

$$\tilde{g}(x) = g(x) - c_0 \Delta (\chi(r) \Gamma(r)) \quad \text{where} \quad c_0 = \int_{\mathbb{R}^d} g \, dx.$$  

Notice that $c_0$ is finite; in fact, using $N = d - 2$ and Hölder’s inequality, we easily confirm that

$$|c_0| \leq \|g\|_{L^1} \leq C \|g\|_{L_{\gamma N+2}^p}.$$  

Then $\int \tilde{g} \, dx = 0$, so $\tilde{g} \in \tilde{W}_{\gamma N+2}^{-m-1,p}(\mathbb{R}^d)$ and we can let $f = Kg$ to find $f \in W_{\gamma N+2}^{m+1,p}(\mathbb{R}^d)$. Finally, we define $Kg$ by

$$Kg = f + c_0 \chi(r) \Gamma(r).$$  

For $d \geq 3$, $u = Kg$ is of the form (31b) for $N = d - 2$ and satisfies $\Delta u = g$ as well as the estimate

$$\|u\|_{A_{N-2}^{m+1,p}^p} = |c_0| + \|f\|_{W_{\gamma N+2}^{m+1,p}} \leq C \|g\|_{W_{\gamma N+2}^{m+1,p}}.$$  

For $d = 2$ we have $u = c_0 \chi(r) \log r + f$, so it is clear how to treat this case as well. This proves the result for $N = d - 2$.

More generally, for $k + d - 2 < \delta + d/p < k + d - 1$ where $k = N - d + 2 > 0$, it was shown in [13] that (20) is injective with cokernel equal to the harmonic polynomials of degree less than or equal to $k$. If we let $\mathcal{H}_k$ denote the spherical harmonics of degree $k$, let $N(k) = \dim \mathcal{H}_k$, and choose an orthonormal basis $\{ \phi_{k,j} : j = 1, \ldots, N(k) \}$ for $\mathcal{H}_k$, then a basis for the space of harmonic polynomials that are homogeneous of degree $k$ is $\{ \phi_{k,j}(\theta) r^k : j = 1, \ldots, N(k) \}$. Consequently, if we define

$$\tilde{W}_{\delta+2}^{-m-1,p}(\mathbb{R}^d) = \left\{ g \in W_{\delta+2}^{-m-1,p}(\mathbb{R}^d) : \int_{\mathbb{R}^d} g(x)\phi_{\ell,j}(\theta) r^\ell \, dx = 0, \; j = 1, \ldots, N(\ell), \; \ell = 0, \ldots, k \right\},$$
then showed that $K: \tilde{W}_{\delta+k+2}^{m-1,p}(\mathbb{R}^d) \to W_{\delta+k+1}^{m+1,p}(\mathbb{R}^d)$ is bounded. Taking $\gamma_N = \delta$ and considering a general $g \in W_{\gamma_N+k+2}^{m-1,p}(\mathbb{R}^d)$, we define

$$c_{\ell,j} = \int_{\mathbb{R}^d} g(x) \phi_{\ell,j}(\theta) r^{q} dx \quad \text{for } j = 1, \ldots, N(k), \ell = 1, \ldots, k.$$  

Using Hölder’s inequality, we can confirm

$$|c_{\ell,j}| \leq C \|g\|_{L_p^{\gamma_N+2}},$$

and in particular that $c_{\ell,j}$ is finite. Recall that

$$\frac{\phi_{\ell,j}(\theta)}{r^{d-2+q}}$$

is harmonic for $r > 0$, so we can use Green’s second identity to calculate

$$\int_{\mathbb{R}^d} \phi_{\ell',j'}(\theta) r^{q} \Delta \left( \frac{\phi_{\ell,j}(\theta)}{r^{d-2+q}} \right) dx = \int_{|x| \leq 2} \phi_{\ell',j'}(\theta) r^{q} \Delta \left( \frac{\phi_{\ell,j}(\theta)}{r^{d-2+q}} \right) dx$$

$$= \int_{|x| = 2} \left( \phi_{\ell',j'}(\theta) r^{q} \frac{\partial \phi_{\ell,j}(\theta)}{\partial r} - \phi_{\ell,j}(\theta) \frac{\partial \phi_{\ell',j'}(\theta)}{\partial r} r^{q} \right) ds$$

$$= \begin{cases} 2 - d - 2q & \text{if } \ell' = \ell \text{ and } j' = j, \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$\tilde{g}(x) = g(x) - c_0 \Delta(\chi(r) \Gamma(r)) - \sum_{\ell=1}^{k} \sum_{j=1}^{N(\ell)} \frac{c_{\ell,j}}{2 - d - 2q} \Delta \left( \chi(r) \frac{\phi_{\ell,j}(\theta)}{r^{d-2+q}} \right).$$

Then $\tilde{g} \in \tilde{W}_{\gamma_N+k+2}^{m-1,p}(\mathbb{R}^d)$ and we can let $f = K\tilde{g} \in W_{\gamma_N+k+2}^{m+1,p}$. Finally, we define

$$Kg = f + \chi(r) \left[ c_0 \Gamma(r) + \sum_{\ell=1}^{k} \sum_{j=1}^{N(\ell)} \frac{c_{\ell,j}}{2 - d - 2q} \frac{\phi_{\ell,j}}{r^{d-2+q}} \right].$$

We see that $u = Kg$ is of the form $\Box$ and satisfies $\Delta u = g$ as well as the estimate

$$\|u\|_{A_{\gamma_N+k+2}^{m+1,p}} \leq C \left( |c_0| + \sum_{\ell=1}^{k} \sum_{j=1}^{N(\ell)} |c_{\ell,j}| \right) + \|f\|_{W_{\gamma_N+k+2}^{m+1,p}} \leq C \|g\|_{W_{\gamma_N+k+2}^{m+1,p}},$$

where $C$ depends on the Sobolev norms of $\phi_{\ell,j}$ on $S^{d-1}$, but not on $g$. This completes the proof.

Notice that $\Box$ generalizes to

$$\Delta: A_{\gamma_N+k+2}^{m+1,p}(\mathbb{R}^d) \to A_{\gamma_N+k+2}^{m-1,p}(\mathbb{R}^d),$$

and we want to consider its invertibility. For $v \in A_{\gamma_N+k+2}^{m-1,p}(\mathbb{R}^d)$, we write $v = b + g$ where $b = \chi(r^{-2}b_2 + \cdots + r^{-N-2}b_{N+1})$ with $b_{k+2} \in H^{m+N-k}(S^{d-1})$ and $g \in W_{\gamma_N+k+2}^{m-1}$. To define $\Delta^{-1}v$, we first try to find an asymptotic function $a = \chi(a_0 + \cdots + r^{-N}a_N) \in A_N^{m+1,p}(\mathbb{R}^d)$ satisfying

$$\Delta \left( \frac{a_k(\theta)}{r^q} \right) = \frac{b_{k+2}(\theta)}{r^{k+2}} \quad \text{for } k = 0, \ldots, N.$$  

Then we will use Lemma $\Box$ to find a remainder function $f$ so that $u = a + f \in A_{\gamma_N+k+2}^{m+1,p}(\mathbb{R}^d)$ is an exact solution of $\Delta u = v$. To solve $\Box$, we can use separation of variables. In fact, using
\[ \Delta = \partial_r^2 + (d - 1) r^{-1} \partial_r + r^{-2} \Delta_h, \] where \( h \) is the induced metric on \( S^{d-1} \), we find that \( a_k \) must satisfy

\[ \Delta h a_k - k(d - 2 - k) a_k = b_{k+2} \text{ on } S^{d-1}. \] (39)

If \( k(d - 2 - k) > 0 \), then we can uniquely solve (39) to find \( a_k \). However, for \( k = 0 \) or \( k = d - 2 \), we have a simple solvability condition, namely \( \int_{S^{d-1}} b_{k+2} \, ds = 0 \), and the solution \( a_k \) is only unique up to an additive constant; this is expected since \( c_0 \) and \( c_{d-2} r^{d-2} \) are harmonic for \( r > 0 \).

Let us consider two closed subspaces of \( A^{m-1,p}_{2,N+2}(\mathbb{R}^d) \):

\[ \mathcal{A}^{m-1,p}_{2,N+2}(\mathbb{R}^d) = \left\{ u = \chi \left( \frac{b_2(\theta)}{r^2} + \cdots + \frac{b_{N+2}(\theta)}{r^{N+2}} \right) + f \in A^{m-1,p}_{2,N+2}(\mathbb{R}^d) : \int_{S^{d-1}} b_d(\theta) \, ds = 0 \right\}. \] (40)

\[ \mathcal{A}^{m-1,p}_{2,N+2}(\mathbb{R}^d) = \left\{ u = \chi \left( \frac{b_2(\theta)}{r^2} + \cdots + \frac{b_{N+2}(\theta)}{r^{N+2}} \right) + f \in A^{m-1,p}_{0,0}(\mathbb{R}^d) : \int_{S^{d-1}} b_d(\theta) \, ds = 0 \right\}. \] (41)

Of course, if \( d > N + 2 \), then the solvability condition \( \int b_d \, ds = 0 \) is vacuous.

**Proposition 3.1.**

(a) For \( d \geq 3, m \geq 1, \) and \( 0 \leq N \leq d - 2 \), there is a bounded operator

\[ K : A^{m-1,p}_{2,N+2}(\mathbb{R}^d) \to A^{m+1,p}_{0,0}(\mathbb{R}^d) \]

satisfying \( \Delta K v = v \) for all \( v \in A^{m-1,p}_{2,N+2}(\mathbb{R}^d) \). This operator is also bounded.

(b) For \( d = 2 \) and \( m \geq 1 \) the operator is bounded

\[ K : A^{m-1,p}_{2,2}(\mathbb{R}^2) = A^{m-1,p}_{0,0}(\mathbb{R}^2) \to A^{m+1,p}_{0,0}(\mathbb{R}^2). \] (42b)

**Proof.**

As indicated above, for \( v = b + g = \chi(r^{-2} b_2 + \cdots + r^{-N-2} b_{N+2}) + g \in A^{m-1,p}_{2,N+2} \) with \( b_{k+2} \in H^{m+N-k,p}(S^{d-1}) \), we have the necessary solvability conditions so that we can find \( a_k \) in \( H^{m+2-N-k,p}(S^{d-1}) \) solving (39) with \( \| a_k \|_{H^{m+N-k,p}(S^{d-1})} \leq C \| b_{k+2} \|_{H^{m+N-k,p}(S^{d-1})} \) for \( k = 0, \ldots, N \); in fact, the \( a_k \) are unique except for \( k = 0, d - 2 \). Of course, the same analysis applies for \( d = 2 \). However, let us assume \( d > 2 \) and \( v \in A^{m-1,p}_{2,N+2}(\mathbb{R}^d) \) with \( \int b_2(\theta) \, ds \neq 0 \). Then the necessary solvability condition does not hold in order to be able to solve (39) for \( k = 0 \). Instead, let us replace \( a_0(\theta) \) by \( a_0^* \log r + a_0(\theta) \) and instead of (38) try to solve

\[ \Delta (a_0^* \log r + a_0(\theta)) = \frac{b_2(\theta)}{r^2}, \]

with \( a_0^* \) being a constant. In place of (39) we have \( \Delta_h a_0 + (d - 2)a_0^* = b_2 \). If we choose

\[ a_0^* = \frac{1}{d - 2} \int_{S^{d-1}} b_2(\theta) \, ds, \]

then \( |a_0| \leq C \| b_2 \|_{L^p(S^{d-1})} \leq C \| b_2 \|_{H^{m+N,p}(S^{d-1})} \) and we can find \( a_0 \in H^{m+2-N,p}(S^{d-1}) \) with \( \| a_0 \|_{H^{m+2-N,p}(S^{d-1})} \leq C \| b_2 \|_{H^{m+N,p}(S^{d-1})} \). To summarize, we have defined \( a_0^*, a_0(\omega), \ldots, a_N(\omega) \) (where \( a_0^* = 0 \) unless \( d > 2 \) and \( \int b_2 \, ds \neq 0 \)) so that

\[ \Delta (a_0^* \log r + a_0(\theta) + \cdots + r^{-N} a_N(\theta)) = r^{-2} b_2(\theta) + \cdots + r^{-N-2} b_{N+2}(\theta). \]

Now let \( u = a + f \) where

\[ a = \chi (a_0^* \log r + a_0(\theta) + \cdots + r^{-N} a_N(\theta)). \]
We need to use Lemma 3.1 to find the remainder function $f$ so that $\Delta u = v$. We compute

$$\Delta a = \chi b + \Delta \chi (a_0^* \log r + a_0 + \cdots + r^{-N} a_N) + \nabla \chi \cdot \nabla (a_0^* \log r + a_0 + \cdots + r^{-N} a_N).$$

So we want $f$ to satisfy

$$\Delta f = h := g - \Delta \chi (a_0^* \log r + a_0 + \cdots + r^{-N} a_N) - \nabla \chi \cdot \nabla (a_0^* \log r + a_0 + \cdots + r^{-N} a_N).$$

But $h$ and $g$ differ by a function in $H^{m+1,p}(\mathbb{R}^d)$ with compact support, and $g \in W^{m-1,p}_{\gamma+2}$, so $h \in W^{m-1,p}_{\gamma+2}$. Consequently, we can apply Lemma 3.1 to find $f = Kh \in A^{m+1,p}_{d-2,N}$ (or $u \in A^{m+1,p}_{d-2,N}$ if $d = 2$) satisfying $\Delta u = h$. We see that $u = a + f$ satisfies $\Delta u = v$ and the mapping $K : v \mapsto u$ is continuous between the appropriate spaces.

**Remark 3.1.** The problem with extending this result to $N > d - 2$ is that $\log r$ terms arise in the solution of (38) for large values of $k$. Cf. Example 3.1 below.

Now we turn to the application to Helmholtz decompositions. It is well-known that a $C^1$-vector field $u$ in $\mathbb{R}^3$ satisfying $D^k u = O(|x|^{-1-k-\varepsilon})$ as $|x| \to \infty$ for $k = 0, 1$ and some $\varepsilon > 0$ can be decomposed into the sum of a unique divergence-free vector field with the same decay property and a gradient field:

\begin{equation}
(43) \quad u = v + \nabla w, \quad \text{div } v = 0.
\end{equation}

Moreover, $v$ and $\nabla w$ are orthogonal in that $\int v \cdot \nabla w \, dx = 0$. This is called the Helmholtz decomposition in $\mathbb{R}^3$. We now show that (43) can be achieved when $d \geq 2$ and $u \in A^{m,p}_{1,N}$; this allows some vector fields $u$ satisfying $O(|x|^{-1})$ as $|x| \to \infty$ instead of requiring $O(|x|^{-1-\varepsilon})$. (While $v, \nabla w \in A^{m,p}_{1,N}$, the orthogonality $\int v \cdot \nabla w \, dx = 0$ need not hold for $d \geq 2$ and $u \in A^{m,p}_{1,N}$.)

**Theorem 3.1.** If $d \geq 2$, $m \geq 1$, and $1 \leq N \leq d - 1$, then every vector field $u \in A^{m,p}_{1,N}$ can be written in the form (43) where $v \in A^{m,p}_{1,N}$ is divergence-free and $w \in H^{m+1,p}_{\infty}$ with $|w(x)| = O(|x|)$ as $|x| \to \infty$; in fact, $v$ is uniquely determined and $w$ is unique up to an additive constant. The map $P_0 : u \mapsto v$ defines a bounded linear map $A^{m,p}_{1,N} \to A^{m,p}_{1,N}$ that is a projection: $P_0^2 u = P_0 u$.

The operator $P_0$ is called the Euler projector. We can reformulate Theorem 3.1 as a statement about closed subspaces.

**Corollary 3.1.** Under the hypotheses of Theorem 3.1, $A^{m,p}_{1,N}$ can be decomposed into a direct sum of closed subspaces $A^{m,p}_{1,N} = \mathcal{A} \oplus \mathcal{G}$, where $\mathcal{A} = \{u \in A^{m,p}_{1,N} : \text{div } u = 0\}$ and $\mathcal{G}$ is the nullspace of the Euler projector $P_0$.

**Proof of Theorem 3.1** When the 1st-order derivatives of $u$ are $O(|x|^{-2-\varepsilon})$ as $|x| \to \infty$, the decomposition (43) can be found by letting $K \text{div } u$, where $K$ is defined by convolution with the fundamental solution. However, for a vector field $u \in A^{m,p}_{1,N}$, we will replace $K$ by the operator discussed in Proposition 3.1. However, we first need to use separation of variables to study $\text{div } u$.

Let $\omega^1, \ldots, \omega^{d-1}$ be local coordinates on $S^{d-1}$, considered as functions of Euclidean coordinates which, in this proof, we index by superscripts, i.e. $x^1, \ldots, x^d$. Let $h = h_{\alpha\beta} \, dx^\alpha \, dx^\beta$ denote the Riemannian metric on $S^{d-1}$ induced by the Euclidean metric $g = dx^2 = dr^2 + r^2 \, h$. Recall that the divergence of a vector field $v$ may be computed in general coordinates $\bar{x}^1, \ldots, \bar{x}^d$ by

$$\text{div } v = \frac{1}{\sqrt{\det g_{ij}}} \frac{\partial}{\partial \bar{x}^j} \left( \sqrt{\det g_{ij}} \, \bar{v}^i \right),$$

where $g = \bar{g}_{ij} \, d\bar{x}^i \, d\bar{x}^j$ and $\bar{v}^i = \frac{\partial \bar{v}^j}{\partial \bar{x}^i} \, v^j$. To compute the divergence of $v$ in the coordinates $(\bar{x}^0, \ldots, \bar{x}^{d-1}) = (r, \omega^1, \ldots, \omega^{d-1})$, we first compute its components in these coordinates by

$$\bar{v}^0 = \frac{\partial r}{\partial x^j} \, v^j = \theta_j \, v^j,$$

where $\theta_j = \frac{x^j}{r} \in S^{d-1},$
In fact, let $x^v = \frac{\partial \omega^v}{\partial x^v} = \omega^v r^{-1} v^j$, where $\omega^v_j = r \frac{\partial \omega^v}{\partial x^v} \in S^{d-1}$, and then compute the divergence to find

$$
\text{div } v = \frac{1}{r^{d-1} \sqrt{\det h_{\alpha \beta}}} \frac{\partial}{\partial x^\alpha} \left( r^{d-1} \sqrt{\det h_{\alpha \beta}} v^j \right)
$$

(44)

$$
= \frac{1}{r^{d-1} \sqrt{\det h_{\alpha \beta}}} \frac{\partial}{\partial x^r} \left( r^{d-1} \sqrt{\det h_{\alpha \beta}} \theta^j v^j \right) + \frac{1}{r} \frac{\partial}{\partial x^\alpha} \left( \sqrt{\det h_{\alpha \beta}} \omega^v_j v^j \right).
$$

We can use (44) to compute the divergence of a vector field of the form $v = r^{-k} a_k$ where $k \geq 1$ and $a_k$ is a vector field with components $a_k^j(\omega) \in H^{m-N+1-k,p}(S^{d-1})$ for $j = 1, \ldots, d$. We conclude

$$
\text{div}(r^{-k} a_k) = \frac{d-1-k}{r^{k+1}} \theta^j a_k^j(\omega) + \frac{1}{r^{k+1} \sqrt{h}} \frac{\partial}{\partial \omega^\alpha} \left( \sqrt{h} \omega^v_j a_k^j(\omega) \right),
$$

(45)

where we have used the abbreviation $\sqrt{h}$ for $\sqrt{\det h_{\alpha \beta}}$.

Now we consider $u \in A^{m,p}_{1,N}$ and claim that $\text{div } u \in \widetilde{A}^{m-1,p}_{2,N+1}$. In fact, using (45) we see that $\text{div } u = \chi(r^{-2}c_2 + \cdots + r^{-N-1}c_{N+1}) + g$ where $c_d$ is given by

$$
c_d(\omega) = \frac{1}{\sqrt{h}} \frac{\partial}{\partial \omega^\alpha} \left( \sqrt{h} \omega^v_j a_d^j(\omega) \right).
$$

Since $c_d$ is a divergence on $S^{d-1}$, we conclude $\int_{S^{d-1}} b_d \, ds = 0$ and so $\text{div } u \in \widetilde{A}^{m-1,p}_{2,N+1}$. By Proposition 3.1 we have $w = K(\text{div } u) \in A^{m+1,p}_{1,N-1}$, and hence $\nabla w \in A^{m,p}_{1,N}$. In fact, the map $u \mapsto \nabla K(\text{div } u)$ is bounded $A^{m,p}_{1,N} \rightarrow A^{m,p}_{1,N}$.

Finally, we let $v = u - \nabla w$. Then $v \in A^{m,p}_{1,N}$ and if we compute the divergence, we get

$$
\text{div } v = \text{div } u - \Delta w = 0.
$$

Thus $u = v + \nabla w$ satisfies (45). Now let us confirm uniqueness. If we had

$$
u = v_1 + \nabla w_1 = v_2 + \nabla w_2, \quad \nabla v_1 = \nabla v_2 = 0,
$$

then $\nabla (w_1 - w_2) = v_2 - v_1$, and we take divergence to conclude $\Delta (w_1 - w_2) = 0$. But our assumption $(w_1 - w_2)(x) = o(|x|)$ as $|x| \to \infty$ then implies $w_1 - w_2 = \text{const}$, and we see that $v_2 - v_1 = 0$, i.e. $v$ is unique. Thus $P_0 : u \mapsto v$ is well-defined and bounded $A^{m,p}_{1,N} \rightarrow A^{m,p}_{1,N}$. If $\text{div } u = 0$ then $w = \Delta^{-1} \text{div } u = 0$, so $P_0 u = u$. In particular, $P_0^2 u = P_0 u$, so $P_0$ is indeed a projection.

The restriction $N \leq d - 1$ is necessary to avoid log $r$ terms in the Helmholtz decomposition. To see this, let us consider an example.

**Example 3.1.** Let us consider $d = N = 2$ and try to obtain the Helmholtz decomposition (43). In fact, let $x = r \cos \phi, y = r \sin \phi$, and simply consider

$$
u = \chi(r) \frac{a_2(\phi)}{r}, \quad \phi \in S^1.
$$

If we try to find $b_2(\phi)$ satisfying

$$
\Delta \left( \frac{b_2(\phi)}{r} \right) = \text{div} \left( \frac{a_2(\phi)}{r} \right),
$$

a computation shows that $b_2$ must satisfy

$$
\frac{\partial^2 b_2}{\partial \phi^2} + b_2 = -\sin \phi \frac{\partial a_1}{\partial \phi} + \cos \phi \frac{\partial a_2}{\partial \phi} - 2 \cos \phi a_1^2 - 2 \sin \phi a_2^2.
$$
Proposition 4.2. For integers $m > 1 + d/p$ and $N \geq n \geq 0$, define
\[ \mathbb{A}^m_{n,N}(\mathbb{R}^d, \mathbb{R}^d) := \{ \phi \in \text{Diff}^1_+(\mathbb{R}^d, \mathbb{R}^d) \mid \phi(x) = x + u(x), \ u \in \mathbb{A}^m_{n,N}(\mathbb{R}^d, \mathbb{R}^d) \} \]
and
\[ \mathbb{A}^m_{n,N}(\mathbb{R}^d, \mathbb{R}^d) := \{ \phi \in \text{Diff}^1_+(\mathbb{R}^d, \mathbb{R}^d) \mid \phi(x) = x + u(x), \ u \in \mathbb{A}^m_{n,N}(\mathbb{R}^d, \mathbb{R}^d) \} \cdot \]
Similar to Section 2, we will abbreviate these collections by $\mathbb{A}^m_{n,N}$ and $\mathbb{A}^m_{n,N}$ when it is clear that we are considering diffeomorphisms of $\mathbb{R}^d$; we also let $\mathbb{A}^{m+1}_{n,N}$ and $\mathbb{A}^{m+1}_{n,N}$ denote $\mathbb{A}^{m+1}_{0,N}$ and $\mathbb{A}^{m+1}_{0,N}$ respectively.

Now we list some important properties of these spaces of asymptotic diffeomorphisms; as stated before, proofs will be given in the next section. First, we want to show that $\mathbb{A}^m_{n,N}$ and $\mathbb{A}^m_{n,N}$ are topological groups under composition of functions. Since $\phi = Id + u$ means $\phi(\psi) = \psi + u(\psi)$, we see that continuity of $(\phi, \psi) \rightarrow \phi(\psi)$ in $\phi$ reduces to continuity of $(u, \psi) \rightarrow u(\psi)$. Consequently, we need the following:

**Proposition 4.1.** For integers $m > 1 + d/p$ and $N \geq n \geq 0$, composition $(u, \psi) \mapsto u \circ \psi$ defines continuous mappings
\[ \mathbb{A}^m_{n,N} \times \mathbb{A}^m_{n,N} \rightarrow \mathbb{A}^m_{n,N} \] and $C^1$-mappings
\[ \mathbb{A}^{m+1}_{n,N} \times \mathbb{A}^{m+1}_{n,N} \rightarrow \mathbb{A}^{m+1}_{n,N} \] and $\mathbb{A}^{m+1}_{n,N} \times \mathbb{A}^{m+1}_{n,N} \rightarrow \mathbb{A}^{m+1}_{n,N}$.\]

Next we need to know that inverses of asymptotic diffeomorphisms are asymptotic diffeomorphisms. Due to the complexity of the asymptotics, this is simplest to prove for one degree of regularity greater than that required for the continuity of composition.

**Proposition 4.2.** For integers $m > 1 + d/p$ and $N \geq n \geq 0$, if $\phi \in \mathbb{A}^{m+1}_{n,N}$ then $\phi^{-1} \in \mathbb{A}^{m+1}_{n,N}$, and $\phi \rightarrow \phi^{-1}$ defines a $C^1$-mapping $\mathbb{A}^{m+1}_{n,N} \rightarrow \mathbb{A}^{m+1}_{n,N}$. The same result holds when $\mathbb{A}$ is replaced by $\mathbb{A}$.\]
These two propositions together suggest that $\mathbb{AD}^{m,p}_{n,N}$ is a topological group for $m > 2 + d/p$, but we have not shown that $\phi \to \phi^{-1}$ is continuous $\mathbb{AD}^{m,p}_{n,N} \to \mathbb{AD}^{m,p}_{n,N}$. However, since the topology in $\mathbb{AD}^{m,p}_N$ is just a translation of the Banach space topology of $\mathbb{A}^{m,p}_N$, this follows from the result of Montgomery [13]. Analogous statements can be made about $\mathbb{AD}^{m,p}_N$. Consequently, once we have proved the two propositions above, we will have shown:

**Theorem 4.1.** For integers $m > 2 + d/p$ and $N \geq n \geq 0$, $\mathbb{AD}^{m,p}_{n,N}$ and $\mathbb{AD}^{m,p}_{n,N}$ are both topological groups under composition.

5. **Proof of the Continuity of Composition (Proposition 4.1)**

Our first result concerns scalar functions and is useful in taking the composition of partial asymptotic expansions. Recall from Remark 2.2 the asymptotic space $\mathcal{A}^{m,p}_{n,N}$ in the exterior of the ball $B_R$.

**Lemma 5.1.** Suppose $m > d/p$, $N \geq 1$, and $\alpha > 0$. If $u \in \mathcal{A}^{m,p}_{1,N}(B_R^c)$ satisfies $1 + u(x) \geq \varepsilon$ for some $\varepsilon > 0$ and all $|x| > R$, then $(1 + u)^{-\alpha} - 1 \in \mathcal{A}^{m,p}_{1,N}(B_R^c)$ and

$$\|(1 + u)^{-\alpha} - 1\|_{\mathcal{A}^{m,p}_{1,N}} \leq C_\alpha \|u\|_{\mathcal{A}^{m,p}_{1,N}}.$$

The same result holds with $A$ replaced by $\mathbb{A}$.

**Proof.** The hypotheses imply that $u$ is continuous and bounded on $B_R^c$, so we may assume $-1 + \varepsilon \leq u(x) \leq M$ for $|x| > R$. Now $(1 + t)^{-\alpha}$ is a smooth function for $-1 + \varepsilon \leq t \leq M$, so by Taylor’s theorem with remainder, we have

$$(47) \quad f_\alpha(t) := (1 + t)^{-\alpha} = 1 - \alpha t + \cdots + (-1)^\ell [\alpha(\alpha + 1) \cdots (\alpha + \ell - 1)] t^\ell + R_\ell(t),$$

where $R_\ell$ is a smooth function of $t \in [-1 + \varepsilon, M]$ satisfying

$$(48) \quad \left|R^{(j)}_\ell(t)\right| \leq C \|t\|^\ell + j \quad \text{for} \quad j = 0, 1, \ldots, \ell + 1.$$

(The standard statement of Taylor’s theorem has $j = 0$ in (48); but for $j = 1, \ldots, \ell + 1$ we can first differentiate (47) with respect to $t$ and then use the Taylor estimate for $f^{(j)}_\alpha$.) Hence we can write

$$(49) \quad (1 + u(x))^{-\alpha} = 1 - \alpha u(x) + \cdots + (-1)^\ell \alpha(\alpha + 1) \cdots (\alpha + \ell - 1) \frac{u(x)}{\ell!} + R_\ell(u(x)).$$

Now we have assumed $u \in \mathcal{A}^{m,p}_{1,N}(B_R^c)$, so by Proposition 2.2 we know that $u^2, \ldots, u^\ell \in \mathcal{A}^{m,p}_{1,N}(B_R^c)$. Consequently, we will have completed our proof provided we can show

$$(50) \quad \|R_N(u)\|_{W_{n,p}^{m,p}(B_R^c)} \leq C \|u\|_{\mathcal{A}^{m,p}_{1,N}}.$$

To prove (50), we need to consider derivatives of $R_\ell(u)$ up to order $m$; for notational simplicity, at this point let us assume $d = 1$. If we calculate the first few derivatives

$$D_x R_\ell(u) = R'_\ell(u)u'$$
$$D_x^2 R_\ell(u) = R''_\ell(u)(u')^2 + R'_\ell(u)u''$$
$$D_x^3 R_\ell(u) = R'''_\ell(u)(u')^3 + 3R''_\ell(u)u'u'' + R'_\ell(u)u'''$$

we see that, for each $k = 1, 2, \ldots, m$ we have

$$D_x^k R_\ell(u) = \sum_{j=1}^k R^{(j)}_\ell(u) P_j^k(u', u'', \ldots, u^{(k)}),$$

where $P_j^k(t_1, \ldots, t_k)$ is a homogeneous polynomial of degree $j$ and the total number of derivatives is $k$. 

$$D_x^k R_\ell(u) = \sum_{j=1}^k R^{(j)}_\ell(u) P_j^k(u', u'', \ldots, u^{(k)}),$$

where $P_j^k(t_1, \ldots, t_k)$ is a homogeneous polynomial of degree $j$ and the total number of derivatives is $k$. 

$$D_x^k R_\ell(u) = \sum_{j=1}^k R^{(j)}_\ell(u) P_j^k(u', u'', \ldots, u^{(k)}),$$

where $P_j^k(t_1, \ldots, t_k)$ is a homogeneous polynomial of degree $j$ and the total number of derivatives is $k$. 

$$D_x^k R_\ell(u) = \sum_{j=1}^k R^{(j)}_\ell(u) P_j^k(u', u'', \ldots, u^{(k)}),$$

where $P_j^k(t_1, \ldots, t_k)$ is a homogeneous polynomial of degree $j$ and the total number of derivatives is $k$.
In fact, we can easily prove (51) by induction. It is certainly true for $k = 1$ (in which case $P^1_k(u') = u'$). Now assume that (51) is true for $k$. To prove (51) for $k + 1$, we calculate

$$D_x^{k+1} (R_\ell (u)) = \sum_{j=1}^{k} D_x \left[ R^{(j)}_\ell (u) P_j^{k}(u', \ldots, u^{(k)}) \right]$$

$$= \sum_{j=1}^{k} R^{(j+1)}_\ell (u) u' P_j^{k}(u', \ldots, u^{(k)}) + R^{(j)}_\ell (u) D_x \left[ P_j^{k}(u', \ldots, u^{(k)}) \right].$$

But $u' P_j^{k}(u', \ldots, u^{(k)})$ is a homogeneous polynomial of degree $j + 1$ with total number of derivatives $k + 1$, and $R^{(j)}_\ell (u) D_x \left[ P_j^{k}(u', \ldots, u^{(k)}) \right]$ is a homogeneous polynomial of degree $j$ with total number of derivatives $k + 1$. Relabeling, we have (51) for $k + 1$, completing the induction step.

Now we want to use the representation (51) to estimate $D_x^k (R_\ell (u))$. Using $u \in A_{1,N}^{m,p}$, we have $|u(x)| \leq C \langle x \rangle^{-1}$, and so (56) implies

$$|R^{(j)}_\ell (u)| \leq C \langle x \rangle^{\ell - j} \leq C \langle x \rangle^{-j+1}$$

We also have $|u'(x)| \leq C \langle x \rangle^{-2}$, so to estimate $P_j^{k}(u', \ldots, u^{(k)})$, every occurrence of $u$ contributes $\langle x \rangle^{-1}$ and each derivative of $u$ contributes an additional $\langle x \rangle^{-1}$, so we obtain

$$|P_j^{k}(u', \ldots, u^{(k)})| \leq C \langle x \rangle^{-j-k}.$$ 

Combining (52) and (53) with (51), we obtain the estimate

$$|D_x^k (R_\ell (u))| \leq C \langle x \rangle^{-\ell-k} \quad \text{for } k = 0, \ldots, m.$$ 

Thus $\langle x \rangle^{N-k} |D_x^k (R_\ell (u))| \leq C \langle x \rangle^{N-\ell-1}$, which is in $L^p$ for $\ell \geq N$ and $k = 0, \ldots, m$, showing that $R_\ell (u) \in W_{N, \ell}^{m,p} (B_R^c)$. Since $W_{N, \ell}^{m,p}(\mathbb{R}) \subset W_{N, \ell}^{m,p} (B_R^c)$, this proves (50) and hence the Lemma when $u \in A_{1,N}^{m,p}(B_R^c)$.

If we instead assume $u \in A_{1,N}^{m,p}(B_R^c)$ and perform the same calculations, then $|u^{(k)}(x)| \leq C \langle x \rangle^{-1}$ for $k = 0, \ldots, m - 1$, so (52) still holds, but in place of (53) we obtain

$$|P_j^{k}(u', \ldots, u^{(k)})| \leq C \langle x \rangle^{-j},$$

and in place of (54) we have

$$|D_x^k (R_\ell (u))| \leq C \langle x \rangle^{-\ell-1} \quad \text{for } k = 0, \ldots, m.$$ 

Then $\langle x \rangle^{N-\ell} |D_x^k (R_\ell (u))| \leq C \langle x \rangle^{N-\ell-1}$, which is in $L^p$ for $\ell \geq N$, showing that in place of (50) we have

$$R_\ell (u) \in H_{N, \ell}^{m,p}(\mathbb{R}) \quad \text{for } \ell \geq N, m.$$ 

But this shows $(1 + u)^{-\alpha} \in A_{1,N}^{m,p}(B_R^c)$ and proves the Lemma when $u \in A_{1,N}^{m,p}(B_R^c)$.

We will need upper and lower bounds on $\langle \phi(x) \rangle$ when $\phi = Id + u$ is an asymptotic diffeomorphism. But these estimates do not require that $\phi$ be a diffeomorphism, so we formulate them simply in terms of the vector function $u$.

**Lemma 5.2.** If $u \in A_{0}^{m,p}$ where $m > d/p$, then there exist positive constants $c_1$ and $c_2$ so that

$$c_1 \langle x \rangle \leq \langle x + u(x) \rangle \leq c_2 \langle x \rangle \quad \text{for all } x \in \mathbb{R}^d,$$

holds uniformly for bounded $\|u\|_{A_{0}^{m,p}}$. The same result holds with $A$ replaced by $\mathbb{A}$.
Proof. By Lemma 1.2 (d), we have \( \|u\|_\infty \leq C \|u\|_{\mathcal{A}^{m,p}_0} =: M \). Then we can use \( 2x \cdot u \geq -\frac{1}{2} |x|^2 - 2 |u|^2 \) to conclude

\[
|x| \geq 2M \Rightarrow \frac{1 + |x + u|^2}{1 + |x|^2} \geq \frac{1 + |u|^2}{1 + |x|^2} \geq 1 + \frac{M^2}{1 + |x|^2} \geq \frac{1}{4},
\]

and

\[
|x| \leq 2M \Rightarrow \frac{1 + |x + u|^2}{1 + |x|^2} \geq \frac{1 + |u|^2}{1 + |x|^2} \geq 1 + \frac{M^2}{1 + |x|^2} \geq \frac{1}{4M^2}.
\]

Similarly, we use \( 2x \cdot u \leq |x|^2 + |u|^2 \) to conclude

\[
\frac{1 + |x + u|^2}{1 + |x|^2} \leq \frac{1 + 2|x|^2 + 2|u|^2}{1 + |x|^2} \leq \frac{2 + 2|x|^2 + 2M^2}{1 + |x|^2} \leq c(M).
\]

These three estimates complete the proof for \( u \in \mathcal{A}^{m,p}_0 \). Since Lemma 1.2 (d) also implies \( \|u\|_\infty \leq C \|u\|_{\mathcal{A}^{m,p}_0} \), the same proof establishes (58) for \( u \in \mathcal{A}^{m,p}_0 \).

Our next estimates concern the Jacobian matrix \( D\phi \) of an asymptotic diffeomorphism \( \phi \). (We adopt the convention that the \( i \)-th row and \( j \)-th column element of \( D\phi \) is \( (D\phi)_{ij} = \partial \phi^j / \partial x^i \), so the chain rule may be written \( D(\phi \circ \psi) = (D\phi) \circ (D\psi) \).) Let \( |D\phi| \) denote the sum of the absolute values of all elements of \( D\phi \). For \( \phi \in AD^{m,p}_0 \) (or \( \phi \in AD^{m,p}_0 \)) we want to show that the estimate

\[
|D\phi(x)| \leq C \quad \text{for all } x \in \mathbb{R}^d
\]

holds uniformly for bounded \( \|u\|_{\mathcal{A}^{m,p}_0} \). Moreover, since we assumed that our diffeomorphisms are orientation-preserving, we know that \( \det(D\phi(x)) > 0 \), but we need to confirm a lower bound at infinity, so that we have

\[
0 < \varepsilon \leq \det(D\phi(x)) \quad \text{for all } x \in \mathbb{R}^d.
\]

In fact, we want to show that (59) holds locally uniformly for \( u \in \mathcal{A}^{m,p}_0 \): this means that for fixed \( \phi^* = Id + u^* \in AD^{m,p}_0 \) we can choose \( \varepsilon \) so that (60) holds for all \( \phi = Id + u \in AD^{m,p}_0 \) with \( \|u - u^*\|_{\mathcal{A}^{m,p}} \leq \delta \) and \( \delta \) sufficiently small.

Lemma 5.3. Suppose \( m > 1 + d/p \). If \( \phi \in AD^{m,p}_0 \), then (59) holds uniformly for bounded \( \|u\|_{\mathcal{A}^{m,p}} \) and (60) holds locally uniformly for \( u \in \mathcal{A}^{m,p}_0 \). The analogous statement holds for \( \phi \in AD^{m,p}_0 \).

Proof. Write \( \phi(x) = x + u(x) \) where \( u \in \mathcal{A}^{m,p}_0 \). Then \( D\phi(x) = I + D(u(x)) \) where \( D(u) \leq C \|D(u)\|_{\mathcal{A}^{m-1,p}} \) using \( m - 1 > d/p \) and Proposition 2.1. Thus (59) holds uniformly for bounded \( \|u\|_{\mathcal{A}^{m,p}} \). But \( D(u) \) with \( m - 1 > d/p \) also implies \( |D(u(x))| \to 0 \) as \( |x| \to \infty \), so \( \phi \) satisfies (60). However, we need to show that (60) holds locally uniformly.

Now suppose we fix \( \phi^* = Id + u^* \in AD^{m,p}_0 \) satisfying (60), i.e. \( \det(D\phi^*(x)) \geq \varepsilon > 0 \) for all \( x \in \mathbb{R} \). If we choose \( \tilde{u} \in \mathcal{A}^{m,p}_0 \) with \( \|\tilde{u}\|_{\mathcal{A}^{m,p}} \) sufficiently small, we can make \( |D\tilde{u}|_{\mathcal{A}^{m,p}} \) as small as we like. But \( \det : \mathbb{R}^{2d} \to \mathbb{R} \) is continuous, so we can arrange that \( \phi = \phi^* + \tilde{u} \) satisfies

\[
\det(D\phi(x)) = \det(D\phi^*(x) + D\tilde{u}(x)) \geq \frac{\varepsilon}{2}
\]

uniformly for small \( \|\tilde{u}\|_{\mathcal{A}^{m,p}} \). Thus we can choose \( \varepsilon > 0 \) and \( \delta > 0 \) so that (60) holds not only for \( \phi^* \), but for all \( \phi = \phi^* + \tilde{u} \) with \( \|\tilde{u}\|_{\mathcal{A}^{m,p}} \leq \delta \); i.e. (60) holds locally uniformly.

The proof for \( \phi \in AD^{m,p}_0 \) leads to \( D(u) \in H^{m-1,p} \) but otherwise is analogous.

Given an asymptotic diffeomorphism \( \phi \) in either \( AD^{m,p}_0 \) or \( AD^{m,p}_0 \) with \( m > 1 + d/p \), the inverse function \( \phi^{-1} \) is a diffeomorphism, but we want to obtain estimates at infinity. Letting \( x = \phi^{-1}(y) \) in (58) yields in particular that

\[
C_1 \langle y \rangle \leq \langle \phi^{-1}(y) \rangle \leq C_2 \langle y \rangle \quad \text{for all } y \in \mathbb{R}^d.
\]
where \( C_1 = 1/c_2 \) and \( C_2 = 1/c_1 \). But since \( c_1 \) and \( c_2 \) were uniform for bounded \( \|u\|_{A_0^m,p(\mathbb{R}^d)} \), we find the same is true of \( C \).

Similarly, for \( \phi = \text{Id} + u \), we want to show
\[
(D\phi^{-1})(x) \leq C_3 \quad \text{for all } x \in \mathbb{R}^d,
\]
and
\[
0 < \varepsilon \leq \det(D\phi^{-1})(x) \quad \text{for all } x \in \mathbb{R}^d,
\]
both holding locally uniformly for bounded \( \|u\|_{A_0^m,p} \) (and similarly for \( \hat{A}_0^m,p \)).

**Lemma 5.4.** Let \( \phi = \text{Id} + u \in \mathcal{AD}_0^{m,p} \) where \( m > 1 + d/p \). Then \( (62) \) and \( (63) \) both hold locally uniformly for \( u \in \mathcal{A}^{m,p}_0 \). The same is true with \( \mathcal{A} \) replaced by \( \hat{\mathcal{A}} \).

**Proof.** Let us fix \( \phi^* = \text{Id} + u \) and consider \( \phi = \text{Id} + u = \phi^* + \tilde{u} \) with \( \|\tilde{u}\|_{A_0^{m,p}} \) small. Let \( \psi = \phi^{-1} \) and let \( y = \phi(x) \). Since \( D\phi(x) = I + Du(x) \), we can use Lemma 5.3 to conclude
\[
|I + Du \circ \psi(y)| \leq C \quad \text{for all } y \in \mathbb{R}^d,
\]
and
\[
0 < \varepsilon \leq \det(I + Du \circ \psi(y)) \quad \text{for all } y \in \mathbb{R}^d,
\]
where \( C \) and \( \varepsilon \) are uniform for \( \|\tilde{u}\|_{A_0^{m,p}} \leq \delta \). If we differentiate \( \phi(y) = \psi(y) + u(\psi(y)) = y \), we obtain \( D\psi + (Du \circ \psi)D\psi = I \), or \( (I + Du \circ \psi)D\psi = I \). However, \( (65) \) shows that \( I + Du \circ \psi \) is invertible, so we can write
\[
D\psi = (I + Du \circ \psi)^{-1}.
\]
Consequently we have
\[
\det D\psi = \det[(I + Du \circ \psi)^{-1}] = [\det(I + Du \circ \psi)]^{-1}.
\]
Using \( (64) \) and \( (65) \), we conclude
\[
0 < \varepsilon_1 \leq \det D\psi(y) \leq C_1 \quad \text{for all } y \in \mathbb{R}^d,
\]
where \( \varepsilon_1 = 1/C_2 \) and \( C_1 = 1/\varepsilon \) are uniform for \( \|\tilde{u}\|_{A_0^{m,p}} \leq \delta \). In particular, this confirms \( (63) \). To prove \( (62) \) we want to bound \( |D\psi| \) uniformly for small \( \|\tilde{u}\|_{A_0^{m,p}} \). But if we use the adjoint formula for the inverse of a matrix,
\[
(I + Du \circ \psi)^{-1} = \frac{1}{\det(I + Du \circ \psi)} \text{Adj}(I + Du \circ \psi),
\]
we see that \( (62) \) follows from \( (61), (63) \), and \( (64) \).

The proof for the result with \( \hat{\mathcal{A}} \) replaced by \( \hat{\mathcal{A}} \) is strictly analogous. \( \square \)

We now consider properties of the composition \( f \circ \phi \) when \( f \) is in the remainder space and \( \phi \) is an asymptotic diffeomorphism. In our first result, we allow \( f \) to be less regular than \( \phi \) since this will be useful for later application. We may assume that \( f \) is scalar-valued, and we denote its gradient by \( \nabla f \).

**Lemma 5.5.** Suppose \( m > 1 + d/p \) and \( \delta \in \mathbb{R} \). For every \( \phi \in \mathcal{AD}_0^{m,p} \) and every \( 0 \leq k \leq m \), we have
\[
\|f \circ \phi\|_{W^{k,p}_\delta} \leq C \|f\|_{W^{k,p}_\delta} \quad \text{for all } f \in W^{k,p}_\delta,
\]
where \( C \) may be taken locally uniformly in \( \phi \in \mathcal{A}^{m,p}_0 \). The analogous result with \( \hat{\mathcal{A}} \) replacing \( \mathcal{A} \) (and \( H_{\delta} \) replacing \( W_{\delta} \)) is also true.
Proof. We prove (67) by induction. For $k = 0$, we simply use the change of variables $x = \psi(y) = \phi^{-1}(y)$:

$$
\|f \circ \phi\|_{W^{k,p}_\delta}^p = \int \left( (\langle x \rangle^d |f \circ \phi(x)|)^p \right) dx \\
= \int \left( (\psi(y))^d |f(y)|^p \det(D\psi(y)) \right) dy \\
\leq C \int \left( (|y|^d |f(y)|)^p \right) dy = C \|f\|_{L^p}^p,
$$

where $C$ can be taken locally uniformly by Lemma 5.3.

Now we assume (67) holds for $k < m$ and prove it for $k + 1$. It suffices to assume $f \in W^{k+1,p}_\delta$ and show

$$
\|\nabla (f \circ \phi)\|_{W^{k+1,p}_{\delta+1}}^p \leq C \|f\|_{W^{k+1,p}_\delta}^p.
$$

But $\nabla (f \circ \phi) = (\nabla f \circ \phi) \cdot D\phi$ where $\nabla f \in W^{k,p}_{\delta+1}$ and $D\phi = I + Du$ with $Du \in A_{m-1,p}$, so we can use Corollary 2.1 concerning products (since $m - 1 > d/p$) to conclude

$$
\|\nabla (f \circ \phi)\|_{W^{k+1,p}_{\delta+1}}^p = \|(f \circ \phi) \cdot (I + Du)\|_{W^{k,p}_{\delta+1}}^p \\
\leq C \|I + Du\|_{A_{m-1,p}} \|\nabla f \circ \phi\|_{W^{k,p}_{\delta+1}}^p \leq C \|\nabla f \circ \phi\|_{W^{k,p}_{\delta+1}}^p,
$$

where $C$ can be chosen uniformly for $\|u\|_{A_{m,p}} \leq M$. Now we can apply (67) to $\nabla f$ (with $\delta + 1$ in place of $\delta$) to conclude

$$
\|\nabla f \circ \phi\|_{W^{k+1,p}_{\delta+1}}^p \leq C \|\nabla f\|_{W^{k,p}_{\delta+1}}^p \leq C \|f\|_{W^{k+1,p}_\delta}^p,
$$

where $C$ may be taken locally uniformly for $\phi \in AD_{0,m,0}^m$. Putting these two inequalities together yields (68), where $C$ may be taken locally uniformly for $\phi \in AD_{0,m,0}^m$.

The proof for $f \in H^{m,p}_{\delta}$ and $\phi \in AD_{0,m,0}^m$ leads to $\nabla f \in H^{m-1,p}_{\delta}$ and $Du \in H^{m-1,p}_0$ but is otherwise analogous. 

The following result will play an important role in proving the continuity of $f \circ \phi$ with respect to $\phi$.

**Lemma 5.6.** Assume $m > 1 + d/p$, $\delta \in \mathbb{R}$, and $f \in C^\infty_0(\mathbb{R}^d)$. If $\phi_k, \phi \in AD_{0,m,0}^m$ with $\phi_k \to \phi$ in $AD_{0,m,0}^m$ as $k \to \infty$, then $f \circ \phi_k \to f \circ \phi$ in $W^{m,p}_\delta$. The same is true with $A$ replaced by $A$ (and $W^{m,p}_\delta$ replaced by $H^{m,p}_\delta$).

**Proof.** Since $m > d/p$, we know that $\phi_k$ and $\phi$ are continuous functions with $\phi_k \to \phi$ uniformly on compact sets in $\mathbb{R}^d$. Moreover, since $\phi_k(x) = x + u_k(x)$ and $\phi(x) = x + u(x)$ where $u_k$ and $u$ are bounded functions while $f$ has compact support, there is a compact set $K$ such that

$$
f(\phi_k(x)) = 0 = f(\phi(x)) \quad \text{for all} \ x \in K^c.
$$

Now we show $f \circ \phi_k \to f \circ \phi$ in $W^{\ell,p}_\delta$ for $0 \leq \ell \leq m$ by induction. For $\ell = 0$, we use the estimate

$$
|f(\phi_k(x)) - f(\phi(x))| \leq \left( \max_{y \in \mathbb{R}^d} |Df(y)| \right) |\phi_k(x) - \phi(x)|
$$

along with (69) and the fact that $\phi_k \to \phi$ uniformly on $K$ to conclude that

$$
\int_{\mathbb{R}^d} (\langle x \rangle^d)^p |f \circ \phi_k(x) - f \circ \phi(x)|^p \, dx \leq C \int_K |\phi_k(x) - \phi(x)|^p \, dx \to 0.
$$
Next we assume the result for $0 \leq \ell < m$ and prove it for $\ell + 1$. Since we may assume $f$ is scalar-valued, this means showing $\nabla(f \circ \phi_k) \to \nabla(f \circ \phi)$ in $W^{\ell+1,p}_0$. But we may compute

$$\nabla(f \circ \phi_k) = (\nabla f) \circ \phi_k \circ D\phi_k$$

and

$$\nabla(f \circ \phi) = (\nabla f) \circ \phi \circ D\phi.$$

Moreover, we know $D\phi_k \to D\phi$ in $\mathcal{A}D^{m-1,p}_0$ and (by the induction hypothesis) $(\nabla f) \circ \phi_k \to (\nabla f) \circ \phi$ in $W^{\ell,p}_0$. Hence, by Corollary 5.1 concerning products, we find that $\nabla(f \circ \phi_k) \to \nabla(f \circ \phi)$ in $W^{\ell+1,p}_0$, as desired.

The proof for $\mathcal{A}D^m_0$ and $H^m_\delta$ is strictly analogous.

The previous two lemmas may be used to obtain the following continuity result.

**Corollary 5.1.** Assume $m > 1 + d/p$ and any $\delta \in \mathbb{R}$. Then composition $(f, \phi) \mapsto f \circ \phi$ is continuous as a map: a) $H^m_\delta \times \mathcal{A}D^m_0 \to H^m_\delta$, and b) $W^m_\delta \times \mathcal{A}D^m_0 \to W^m_\delta$.

**Proof.** a) Fix $(f^*, \phi^*) \in H^m_\delta \times \mathcal{A}D^m_0$ and consider a sequence $(f_j, \phi_j) \to (f^*, \phi^*)$ in $H^m_\delta \times \mathcal{A}D^m_0$. By the triangle inequality

$$\|f_j \circ \phi_j - f^* \circ \phi^*\|_{H^m_\delta} \leq \|f_j \circ \phi_j - f^* \circ \phi_j\|_{H^m_\delta} + \|f^* \circ \phi_j - f^* \circ \phi^*\|_{H^m_\delta}. $$

Using (67), we have

$$\|f_j \circ \phi_j - f^* \circ \phi_j\|_{H^m_\delta} = \|(f_j - f^*) \circ \phi_j\|_{H^m_\delta} \leq C_1 \|f_j - f^*\|_{H^m_\delta}$$

for sufficiently large $j$. Now let us use the density of $C^\infty_0(\mathbb{R}^d)$ in $H^m_\delta$ to find $\tilde{f}^*$ such that $\|\tilde{f}^* - f^*\|_{H^m_\delta}$ is small. Now we can write

$$\|f^* \circ \phi_j - f^* \circ \phi^*\|_{H^m_\delta} \leq \|(f^* - \tilde{f}^*) \circ \phi_j\|_{H^m_\delta} + \|\tilde{f}^* \circ \phi_j - \tilde{f}^* \circ \phi^*\|_{H^m_\delta} + \|\tilde{f}^* - f^*\|_{H^m_\delta} \circ \phi^*\|_{H^m_\delta}. $$

Again we can use (67) to make

$$\|(f^* - \tilde{f}^*) \circ \phi_j\|_{H^m_\delta} + \|\tilde{f}^* \circ \phi_j - \tilde{f}^* \circ \phi^*\|_{H^m_\delta} \leq C_2 \|\tilde{f}^* - f^*\|_{H^m_\delta}$$

for sufficiently large $j$. Now, given $\varepsilon > 0$, we first pick $\tilde{f}^*$ so that $C_2 \|\tilde{f}^* - f^*\|_{H^m_\delta} < \varepsilon/2$. Then we pick $J$ sufficiently large that both $C_1 \|f - f^*\|_{H^m_\delta} < \varepsilon/4$ and (by Lemma 5.5)

$$\|\tilde{f}^* \circ \phi_j - \tilde{f}^* \circ \phi^*\|_{H^m_\delta} < \varepsilon/4$$

for $j \geq J$. This shows $\|f_j \circ \phi_j - f^* \circ \phi^*\|_{H^m_\delta} < \varepsilon$ for $j \geq J$, i.e. $H^m_\delta \times \mathcal{A}D^m_0 \to H^m_\delta$ is continuous.

b) The proof is exactly the same as for a).

The following estimates provide a stronger description of the continuity of $f \circ \phi$ when $f$ has an additional degree of regularity.

**Lemma 5.7.** Assume $m > 1 + d/p$ and $\delta \in \mathbb{R}$.

a) Fix $\phi_\ast \in \mathcal{A}D^m_0$. For $f \in H^{m+1}_\delta$ and $\phi \in \mathcal{A}D^m_0$ sufficiently close to $\phi_\ast$ we have

$$\|f \circ \phi - f \circ \phi_\ast\|_{H^m_\delta} \leq C \|f\|_{H^{m+1}_\delta} \|\phi - \phi_\ast\|_{H^m_\delta}.$$  

b) Fix $\phi_\ast \in \mathcal{A}D^m_0$. For $f \in W^{m+1}_\delta$ and $\phi \in \mathcal{A}D^m_0$ sufficiently close to $\phi_\ast$ we have

$$\|f \circ \phi - f \circ \phi_\ast\|_{W^{m+1}_\delta} \leq C \|f\|_{W^{m+1}_\delta} \|\phi - \phi_\ast\|_{\mathcal{A}D^m_0}.$$  

In both (70a) and (70b), $C$ can be taken uniformly for all $\phi$ in a fixed neighborhood of $\phi_\ast$. 
Proof. For \( \phi = Id + u \in \mathcal{A}^m_{\mathcal{D}_0} \) sufficiently close to \( \phi_* = Id + u_* \), let \( \tilde{u} = \phi - \phi_* = u - u_* \) so that \( \phi_* + t\tilde{u} \in \mathcal{A}^m_{\mathcal{D}_0} \) for \( 0 \leq t \leq 1 \). Now \( m > 1 + d/p \) implies \( f \in C^1 \) (in fact \( C^2 \)), so we can write

\[
f \circ \phi - f \circ \phi_* = \int_0^1 \frac{d}{dt} f(\phi_* + t\tilde{u}) \, dt = \int_0^1 (\nabla f)(\phi_* + t\tilde{u}) \cdot \tilde{u} \, dt.
\]

By Corollary 5.1 we know that \( t \mapsto \nabla f(\phi_* + t\tilde{u}) \) is continuous \([0, 1] \to H^m_{d,p}(\mathbb{R}^d)\), so this mapping is Riemann integrable. Thus we can conclude that

\[
\|f \circ \phi - f \circ \phi_*\|_{H^m_{d,p}} \leq \int_0^1 \| (\nabla f)(\phi_* + t\tilde{u}) \cdot \tilde{u} \|_{H^m_{d,p}} \, dt
\]

where we have also used Corollary 2.1. But now we can apply Lemma 5.3 to \( \nabla f \in H^m_{d,p}(\mathbb{R}^d)\):

\[
\| (\nabla f)(\phi_* + t\tilde{u}) \|_{H^m_{d,p}} \leq C \| \nabla f \|_{H^m_{d,p}} \leq C \| f \|_{H^m_{d+1,p}},
\]

where \( C \) can be taken uniform in a neighborhood of \( \phi_* \). Putting this together yields (70a).

The proof of (70b) is analogous, except now \( \nabla f \in W^{m,p}_{d+1} \) so the estimates become

\[
\|f \circ \phi - f \circ \phi_*\|_{W^{m,p}_{d+1}} \leq \int_0^1 \| (\nabla f)(\phi_* + t\tilde{u}) \|_{W^{m,p}_{d+1}} \, dt \| \tilde{u} \|_{A^m_{d+1,p}}
\]

where \( C \) can be taken uniform in a neighborhood of \( \phi_* \).

Before considering the continuity of \( a \circ \phi \) when \( a \) is an asymptotic function and \( \phi \) is an asymptotic diffeomorphism, we need a refinement of Lemma 5.1. For \( u \in \mathcal{A}^m_{\mathcal{D}_1}(B^c_1) \) with \( m > d/p \), let us introduce the \textit{scalar-valued} function \( \rho(u) \) defined by

\[
\rho(u)(x) := \frac{2x \cdot u(x) + |u(x)|^2}{|x|^2}.
\]

Using Propositions 2.1 and 2.2 we see that \( \rho : \mathcal{A}^m_{\mathcal{D}_1}(B^c_1) \to \mathcal{A}^m_{\mathcal{D}_1}(B^c_1) \) is continuous and we can calculate the asymptotics of \( \rho(u) \) in terms of the asymptotics of \( u \). In fact, since we also know by Proposition 2.1 that \( u \) is bounded on \( \mathbb{R}^d \), we see that \( \rho(u(x)) \to 0 \) as \( |x| \to \infty \), and hence we have \( 1 + \rho(u) \geq \varepsilon > 0 \) for \( |x| > R \) with \( R \) sufficiently large. Note that \( R \) depends on \( u \), but we can take it uniformly on a bounded neighborhood \( \mathcal{U} \) of a fixed \( u^* \in \mathcal{A}^m_{\mathcal{D}_1}(B^c_1) \). Using Lemma 5.1 the \textit{scalar-valued} function \( \sigma(u) \) defined by

\[
\sigma(u)(x) := (1 + \rho(u(x)))^{-1/2} - 1
\]

is in \( \mathcal{A}^m_{\mathcal{D}_1}(B^c_R) \) and we can calculate its asymptotics in terms of the asymptotics of \( u \), so we have \( \sigma : \mathcal{U} \to \mathcal{A}^m_{\mathcal{D}_1}(B^c_R) \). We now want to show that we can choose \( \mathcal{U} \) so that \( \sigma : \mathcal{U} \to \mathcal{A}^m_{\mathcal{D}_1}(B^c_R) \) is real-analytic; in particular, this map is continuous.

**Lemma 5.8.** If \( u^* \in \mathcal{A}^m_{\mathcal{D}_1}(B^c_1) \) for \( m > d/p \) and \( N \geq 0 \), then there is a neighborhood \( \mathcal{U} \) of \( u^* \) and \( R \) sufficiently large that \( \sigma : \mathcal{U} \to \mathcal{A}^m_{\mathcal{D}_1}(B^c_R) \) is real analytic. The same is true with \( \mathcal{A} \) replaced by \( \mathbb{H} \).

**Proof.** Let us fix \( u^* \in \mathcal{A}^m_{\mathcal{D}_1}(B^c_1) \) and consider \( u = u^* + \bar{u} \). As observed above, we know \( \rho(u) \in \mathcal{A}^m_{\mathcal{D}_1}(B^c_R) \) and there is a neighborhood \( \mathcal{U} \) of \( u^* \) such that, for \( R \) sufficiently large, we
Lemma 5.9. For some degree) to \( \mathbb{R} \) of homogeneity does not matter since we will only be using the behavior of \( \mathbb{R} \) and extend it to \( \phi \) function and \( \theta \) function and \( \sigma \) have

If we let \( \mathcal{U} = \{ u = u + \bar{u} : \| \bar{u} \|_{A_{m,p}^m(B_1^c)} < \eta \} \) and use the power series \((1 + t)^{-1/2} = 1 - \frac{1}{2} t + \cdots \)
we find that

Consequently, the same is true of \((1 + \rho(u))^{-1/2} \), from which the result follows.

The proof for the result with \( \mathcal{A} \) replaced by \( \mathcal{A} \) is strictly analogous.

Another lemma will be useful in controlling the remainder term in \( a \circ \phi \) when \( a \) is an asymptotic function and \( \phi \) is an asymptotic diffeomorphism. In this lemma we consider a function \( b \) on \( S^{d-1} \) and extend it to \( \mathbb{R}^d \setminus \{ 0 \} \) as a function of some degree of homogeneity; however, the specific degree of homogeneity does not matter since we will only be using the behavior of \( b \) near \( |x| = 1 \).

Lemma 5.9. For \( m > 1 + d/p \) and \( N > 0 \), suppose \( b \in H^{m,p}(S^{d-1}) \) is extended (by homogeneity of some degree) to \( \mathbb{R}^d \setminus \{ 0 \} \), and \( v \in A_0^{m,p}(\mathbb{R}^d, \mathbb{R}^d) \). Then for \( R \) sufficiently large we have

\[
\frac{b(\frac{x+v}{|x|^{N+1}})}{|x|^{N+1}} \leq C \| b \|_{H^{m,p}(S^{d-1})},
\]

where \( C \) is locally uniform in \( v \in A_0^{m,p}. \) The analogous estimate for \( v \in A_0^{m,p}(\mathbb{R}^d, \mathbb{R}^d) \) is

\[
\frac{b(\frac{x+v}{|x|^{N+1}})}{|x|^{N+1}} \leq C \| b \|_{H^{m,p}(S^{d-1})}.
\]

Proof. Writing \( \gamma = \gamma_N \), we shall prove by induction that

\[
b \in H^{\ell,p}(S^{d-1}) \implies \frac{b(\frac{x+v}{|x|^{N+1}})}{|x|^{N+1}} \leq C \| b \|_{H^{\ell,p}(S^{d-1})} \text{ for } \ell = 0, \ldots, m.
\]

For \( \ell = 0 \) we easily obtain

\[
\frac{b(\frac{x+v}{|x|^{N+1}})^p}{|x|^{N+1}} \leq C \int_{B_1^c} I(r) \, dr
\]

where

\[
I(r) = \int_{S^{d-1}} \left| b \left( \theta + \frac{v(r\theta)}{r} \right) \right|^p \, ds.
\]

We first want to show that, for \( R \) sufficiently large depending locally uniformly on \( v \), \( I(r) \) can be estimated by \( C \| b \|_{L^p(S^{d-1})}^p \). To do this we consider the surface \( \Xi \) in \( \mathbb{R}^d \setminus \{ 0 \} \) parameterized by \( \theta \in S^{d-1} \):

\[
\xi(\theta) = \theta + \frac{v(r\theta)}{r} = \frac{x + v(x)}{|x|}.
\]
We compute the Jacobian:
\[
\frac{\partial x^i}{\partial \theta^j} = \delta_{ij} + \sum_{k=1}^d \frac{\partial u^i}{\partial x^k} \delta_{jk} = \delta_{ij} + \frac{\partial u^i}{\partial x^j}.
\]
But \( \nabla v \in A_{n+1}^{m-1,p} \) with \( m > 1 + d/p \) implies by Proposition 2.1(d) that \( |\nabla v(x)| \leq C |v|_{A_{n+1}^{m-1,p}}/|x| \). So, for \( R \) sufficiently large, we conclude that for \( r > R \) the Jacobian is nonsingular and we have
\[
I(r) = \int_{S^{d-1}} \left| b \left( \theta + \frac{v(r\theta)}{r} \right) \right|^p ds \leq C \int_{S^{d-1}} |b(\theta)|^p ds,
\]
where \( C \) is locally uniform in \( v \in A_0^{m,p} \). Finally, using \( \gamma - N - 1 < -d/p \), we conclude
\[
\int_R^{\infty} r^{(\gamma - N - 1)p + d - 1} I(r) dr \leq C \|b\|_{L_p(S^{d-1})}^p,
\]
which gives us (76) for \( \ell = 0 \).

Now we assume (76) for \( \ell = m - 1 \) and prove it for \( \ell = m \). It suffices to show
\[
\left\| \nabla \left( \frac{b(x + v)}{|x|^{N+1}} \right) \right\|_{W_{\gamma+1}^{m-1,p}(B_R^c)} \leq C \|b\|_{H^{m,p}(S^{d-1})}.
\]
But
\[
\frac{\partial}{\partial x^j} \left( \frac{b(x + v)}{|x|^{N+1}} \right) = (\partial_i b) \left( \frac{x + v}{|x|} \right) \left( \frac{\delta_{ij} - \theta_i \theta_j + \partial_j v^i}{|x|} \right) |x|^{-N-1} - (N + 1) b \left( \frac{x + v}{|x|} \right) |x|^{-N} \theta_j.
\]
We can use Lemma 2.3 to estimate
\[
\left\| (\partial_i b) \left( \frac{x + v}{|x|} \right) |x|^{-N-2} \right\|_{W_{\gamma+1}^{m-1,p}(B_R^c)} \leq C \left\| (\partial_i b) \left( \frac{x + v}{|x|} \right) |x|^{-N-1} \right\|_{W_{\gamma+1}^{m-1,p}(B_R^c)} \leq C \|\partial_i b\|_{H^{m-1,p}(S^{d-1})} \leq C \|b\|_{H^{m,p}(S^{d-1})},
\]
where we have used the induction hypothesis for \( \ell = m - 1 \) applied to \( \partial_j b \in H^{m-1,p}(S^{d-1}) \). We can also apply the induction hypothesis to estimate
\[
\left\| b \left( \frac{x + v}{|x|} \right) |x|^{-N-2} \right\|_{W_{\gamma+1}^{m-1,p}} = \left\| b \left( \frac{x + v}{|x|} \right) |x|^{-N-1} \right\|_{W_{\gamma+1}^{m-1,p}} \leq C \|b\|_{H^{m-1}(S^{d-1})} \leq C \|b\|_{H^{m,p}(S^{d-1})}.
\]
Putting these together proves (76), which completes the induction.

For \( v \in A_0^{m,p} \) we follow the same outline, using \( N - N^* - 1 < -d/p \) to conclude convergence of the radial integral. \( \square \)

We now consider compositions \( u \circ \phi \) when \( u = a + f \) as in (110). We may assume that \( u \) is scalar-valued but the diffeomorphism \( \phi = Id + u \) is necessarily vector-valued. We start with generalizing Lemma 5.5.

**Lemma 5.10.** Suppose \( m > 1 + d/p \) and \( N \geq n \geq 0 \). For any \( \phi \in AD_{n,N}^{m,p} \) we have
\[
\|u \circ \phi\|_{A_{n,N}^{m,p}} \leq C \|u\|_{A_{n,N}^{m,p}} \text{ for all } u \in A_{n,N}^{m,p},
\]
where \( C \) may be taken locally uniformly in \( \phi \in AD_{N}^{m,p} \). The analogous result with \( \wedge \) replacing \( A \) is also true.
Proof. To simplify notation, we assume $n = 0$. Using the form (14) of the $\mathcal{A}$-norm and Lemma 5.5, it suffices to consider

$$u(x) = a(x) = \chi((x) a_k(\theta)/|x|^k \quad \text{where } a_k \in H^{m+1+N-k,p}(S^{d-1}) \text{ and } 0 \leq k \leq N.$$

Moreover, since $\phi = \text{Id} + v$ where $v \in A_N^{m,p} \subset C_B(\mathbb{R}^d)$, we may assume that $|v(x)| \leq M$. Since $\phi(x) = x + v(x)$, for $|x| > 2M$ we have $M \leq |\phi(x)| \leq |x| + M \leq 3|x|/2$. Let us assume $M \geq 1$ and let $R = 2M$; for $x \in B_R^c$ we have $\chi(|x|) = 1 = \chi(|\phi(x)|)$, so it suffices to estimate $a \circ \phi$ in $|x| > R$ in terms of $a_k$ on $S^{d-1}$. So we want to show

$$\|a \circ \phi\|_{A_N^{m,p}(B_R^c)} \leq C \|a_k\|_{H^{m+1+N-k,p}(S^{d-1})} = C \|a\|_{A_N^{m,p}},$$

where $C$ is locally uniform on $v \in A_N^{m,p}$. But to estimate $\|a \circ \phi\|_{A_N^{m,p}(B_R^c)}$, we need a partial asymptotic expansion for $a \circ \phi$.

Let us consider $a_k(x)$ as a homogeneous of degree 0 function on $\mathbb{R}^d \setminus \{0\}$. In particular, $a_k \in H^{m+1+N-k,p}(\mathbb{R}^d \setminus \{0\}) \subset C^{N-k+1}(\mathbb{R}^d \setminus \{0\})$ since $m > d/p$. So, by Taylor’s theorem with remainder at a point $y^*$ in $\mathbb{R}^d \setminus \{0\}$, we can write

$$(80a) \quad a_k(y) = \sum_{|\alpha| \leq N-k} D^\alpha a_k(y^*) \frac{(y - y^*)^\alpha}{\alpha!} + R_{N,k}(y, y^*),$$

where the remainder $R_{N,k}(y, y^*)$ can be expressed in integral form as

$$(80b) \quad R_{N,k}(y, y^*) = \sum_{|\alpha| = N-k+1} \frac{N-k+1}{\alpha!} \int_0^1 (1-t)^{N-k} D^\alpha a_k(y^* + t(y - y^*)) \, dt \, (y - y^*)^\alpha.$$  

This approximation holds for $y$ in a neighborhood of $y^*$, and more generally provided $0 \notin \{y^* + t(y - y^*) : 0 \leq t \leq 1\}$. But we now want to take both $y$ and $y^*$ on $S^{d-1}$. In fact, we shall replace $y$ by $\phi(x)/|\phi(x)|$ and $y^*$ by $\theta = x/|x|:

$$(81) \quad a_k \left( \frac{\phi(x)}{|\phi(x)|} \right) = \sum_{|\alpha| \leq N-k} \frac{D^\alpha a_k(\theta)}{\alpha!} \left( \frac{\phi(x)}{|\phi(x)|} - \frac{x}{|x|} \right)^\alpha + R_{N,k} \left( \frac{\phi(x)}{|\phi(x)|}, \frac{x}{|x|} \right).$$

Notice that $\phi(x) = x + v(x)$, where $v$ is bounded, means that $\phi(x)/|\phi(x)| \to x/|x|$ as $|x| \to \infty$, so for $|x| > R$ with $R$ sufficiently large we can arrange

$$0 \notin \left\{ \frac{x}{|x|} + t \left( \frac{\phi(x)}{|\phi(x)|} - \frac{x}{|x|} \right) : 0 \leq t \leq 1 \right\}.$$

But we need to investigate the difference $\phi(x)/|\phi(x)| - x/|x|$ in more detail.

Notice that we can write

$$|\phi(x)|^{-1} = |x + v|^{-1} = |x|^{-1}(1 + \rho(v)(x))^{-1/2} = |x|^{-1}(1 + \sigma(v)(x)),$$

where $\rho(v)$ and $\sigma(v)$ are defined in (72) and (73) respectively. But by Lemma 5.8 we know that $\|\sigma(v)\|_{A_N^{m,p}(B_R^c)}$ is bounded locally uniformly for $v$ in $A_N^{m,p}(B_R^c)$. It is easy to confirm that

$$\frac{\phi(x)}{|\phi(x)|} - \frac{x}{|x|} = \frac{w(x)}{|x|} \quad \text{where } w(x) := \sigma(v)(x) [x + v(x)] + v(x).$$

Note that $w \in A_N^{m,p}(B_R^c)$ and that its asymptotics can be computed in terms of $v$; in particular, $\|w\|_{A_N^{m,p}(B_R^c)}$ is bounded locally uniformly in $v \in A_N^{m,p}(B_R^c)$.  


We plug (83) and (82) into (81) to conclude

\[
a \circ \phi(x) = \chi(|\phi(x)|) a_k \left( \frac{\phi(x)}{|\phi(x)|} \right) |\phi(x)|^{-k}
\]

(84)

\[
= \chi(|\phi(x)|) \left[ \sum_{|\alpha| \leq N-k} \frac{D^\alpha a_k(\theta)}{\alpha!} \frac{w^\alpha}{|x|^{k+|\alpha|}} + R_{N,k} \left( \frac{\phi(x)}{|\phi(x)|} \frac{x}{|x|} \right) \frac{1}{|x|^k} \right] (1 + \sigma(v))^k.
\]

Although this is not quite the partial asymptotic expansion for \(a \circ \phi\), it can be used to estimate \(\|a \circ \phi\|_{A^m_{N,p}}\). In fact, using Proposition 2.2 and Lemma 5.2, we know \(\|a(1 + \sigma(v))\|_{A^m_{N,p}(B_R^c)} \leq C \|a\|_{A^m_{N,p}(B_R^c)}\), where \(C\) is locally uniform in \(v \in A^m_{N,p}(B_1^c)\); consequently, to estimate \(\|a \circ \phi\|_{A^m_{N,p}(B_R^c)}\) we need only estimate the \(A^m_{N,p}(B_R^c)\)-norm of the two terms in the brackets in (84).

First of all, we claim

\[
\sum_{|\alpha| \leq N-k} \frac{D^\alpha a_k(\theta)}{\alpha!} \frac{w^\alpha}{|x|^{k+|\alpha|}} \leq C \|a\|_{A^m_{N,p}(B_R^c)},
\]

where \(C\) is locally uniform in \(v \in A^m_{N,p}(B_1^c)\). To see this, we use the algebra property to estimate

\[
\left\| \sum_{|\alpha| \leq N-k} \frac{D^\alpha a_k(\theta)}{\alpha!} \frac{w^\alpha}{|x|^{k+|\alpha|}} \right\|_{A^m_{N,p}(B_R^c)} \leq \sum_{|\alpha| \leq N-k} \left\| \frac{D^\alpha a_k(\theta)}{\alpha!} \frac{w^\alpha}{|x|^{k+|\alpha|}} \right\|_{A^m_{N,p}(B_R^c)} \leq C \|a\|_{A^m_{N,p}(B_R^c)},
\]

where \(C\) is locally uniform in \(w \in A^m_{N,p}(B_1)\). If we recall that \(v \mapsto w\) is continuous as a map \(A^m_{N,p}(B_1^c) \to A^m_{N,p}(B_R^c)\), then we can consider \(C\) in (85) as being locally uniform in \(v\).

Secondly, we claim

\[
\left\| R_{N,k}(\phi/|\phi|, x/|x|) \right\|_{W^{m,p}_{N+1}(B_R^c)} \leq C \|a_k\|_{H^{m+1+N-k,p}(S^{d-1})},
\]

where \(C\) may be taken locally uniform in \(v \in A^m_{N,p}(B_1^c)\). To see this, notice from (80b) that

\[
R_{N,k}(\phi/|\phi|, x/|x|) = \sum_{|\alpha| = N-k+1} \frac{N-k+1}{\alpha!} \int_0^1 (1-t)^{N-k} D^\alpha a_k \left( \frac{x}{|x|} + t \frac{w(x)}{|x|} \right) dt \frac{w^\alpha}{|x|^{N+1}}.
\]

We can apply Lemma 5.9 with \(b = D^\alpha a_k \in H^{m,p}(S^{d-1})\) to conclude

\[
\left\| \frac{D^\alpha a_k(x/|x|)}{|x|^{N+1}} \right\|_{W^{m,p}_{N+1}(B_R^c)} \leq C \|D^\alpha a_k\|_{H^{m,p}(S^{d-1})} \leq C \|a_k\|_{H^{m+1+N-k,p}(S^{d-1})}.
\]

Using Corollary 2.1 and the above remarks regarding \(v \mapsto w\), we obtain (80). Putting this together with (81) and (83), we obtain (79), as desired.

To prove the corresponding result for \(A\), we replace \(N\) by \(N^*\) in (80a) and (80b) and replace \(83\) by

\[
\left\| R_{N^*,k}(\phi/|\phi|, x/|x|) \right\|_{H^{m,p}_{N^*+1}(B_R^c)} \leq C \|a_k\|_{H^{m+1+N^*-k,p}(S^{d-1})}.
\]

The details are straight-forward.
Lemma 5.11. Suppose \( m > 1 + d/p \) and \( N \geq 0 \). Let \( u(x) = \chi(|x|)a_k(\theta)/|x|^k \) where \( n \leq k \leq N \) and \( a_k \in C^\infty(S^{d-1}) \). If \( \phi, \phi_j \in AD_N^{m,p} \) with \( \phi_j \to \phi \) in \( AD_N^{m,p} \), then \( u \circ \phi_j \to u \circ \phi \) in \( A_n^{m,p} \). The same is true if \( A \) is replaced by \( \mathbb{A} \).

Proof. As in the proof of Lemma 5.10 we assume \( n = 0 \). Since we can write \( \phi_j(x) = x + v_j(x) \) and \( \phi(x) = x + v(x) \) where \( v_j \) and \( v \) are uniformly bounded functions, we can take \( R \) large enough that \( \chi(|\phi_j(x)|) = \chi(|\phi(x)|) = 1 \) for all \( |x| > R \), so we want to estimate in \( A_n^{m,p}(B_R^n) \) the difference

\[
\frac{a_k\left(\frac{\phi(x)}{|\phi(x)|}\right)}{|\phi(x)|^k} - \frac{a_k\left(\frac{\phi_j(x)}{|\phi_j(x)|}\right)}{|\phi_j(x)|^k}.
\]

Using the scalar function \( \sigma \) defined in (73), let us introduce (as we did in (83)) the vector functions

\[
w_j := \sigma(v_j)[I + v_j] + v_j \quad \text{and} \quad w := \sigma(v)[I + v] + v.
\]

Since \( v_j \to v \) in \( A_n^{m,p} \), we see by Lemma 5.8 that \( w_j \to w \) in \( A_n^{m,p} \). Now if we apply (84) to both \( u \circ \phi_j \) and \( u \circ \phi \), we find for \( |x| > R \) that

\[
\begin{align*}
u \circ \phi_j - u \circ \phi &= \sum_{|\alpha| \leq N-k} D^\alpha a_k(\theta) \frac{[w_j^\alpha(1 + \sigma(v_j))^k - w^\alpha(1 + \sigma(v))^k]}{\alpha! |x|^{k+|\alpha|}} \\
&\quad + \frac{1}{|x|^k} \sum_{|\alpha| \leq N-k} \frac{D^\alpha a_k(\theta)}{\alpha! |x|^{k+|\alpha|}} \left( \sum_{\alpha} \sigma(v_j) \frac{\phi_j(x)}{|\phi_j(x)|} - \sum_{\alpha} \sigma(v) \frac{\phi(x)}{|\phi(x)|} \right).
\end{align*}
\]

(88)

Using Lemma 5.8 again, for each fixed \( \alpha \) we know

\[
\|w_j^\alpha(1 + \sigma(v_j))^k - w^\alpha(1 + \sigma(v))^k\|_{A_n^{m,p}} \to 0.
\]

As observed in the proof of Lemma 5.10 \(|x|^{k+|\alpha|} D^\alpha a_k(\theta) \in A_n^{m,p}(B_R^n) \), and so

\[
\sum_{|\alpha| \leq N-k} \frac{D^\alpha a_k(\theta) [w_j^\alpha(1 + \sigma(v_j))^k - w^\alpha(1 + \sigma(v))^k]}{\alpha! |x|^{k+|\alpha|}} \to 0.
\]

To handle the remainder terms in (88), it suffices to show

\[
\frac{1}{|x|^k} \left[ R_N,k \left( \frac{\phi_j(x)}{|\phi_j(x)|} - \frac{\phi(x)}{|\phi(x)|} \right) \right] \to 0 \quad \text{in} \quad A_n^{m,p}(B_R^n).
\]

But, using (57), this quantity is given by

\[
\sum_{|\alpha| = N-k+1} \frac{N-k+1}{\alpha!} \int_0^1 (1-t)^{N-k} t^{N+1} \left[ D^\alpha a_k \left( \frac{x}{|x|} + t \frac{w_j(x)}{|x|} \right) w_j^\alpha - D^\alpha a_k \left( \frac{x}{|x|} + t \frac{w(x)}{|x|} \right) w^\alpha \right] dt.
\]

However, \( w_j \to w \) in \( A_n^{m,p} \) implies \( w_j^\alpha \to w^\alpha \) in \( A_n^{m,p} \) and also \( D^\alpha a_k((x + tw_j(x))/|x|) \to D^\alpha a_k((x + tw(x))/|x|) \) in \( A_n^{m,p}(B_R^n) \), so the remainder term in (88) also tends to zero in \( A_n^{m,p}(B_R^n) \) as \( j \to \infty \), which is what we needed to show. The proof for \( \mathbb{A} \) is identical.

Similar to Corollary 5.1, we can use Lemmas 5.10 and 5.11 to show that composition on our asymptotic spaces is continuous; we shall not repeat the argument.

Corollary 5.2. Suppose \( m > 1 + d/p \) and \( N \geq n \geq 0 \). Then composition \( (u, \phi) \mapsto u \circ \phi \) is continuous as a map: a) \( A_n^{m,p} \times AD_n^{m,p} \to A_n^{m,p} \), and b) \( A_n^{m,p} \times AD_n^{m,p} \to A_n^{m,p} \).

Now we extend Lemma 5.7 to general asymptotic functions \( u \). Again we may assume that \( u \) is scalar-valued.
Lemma 5.12. Suppose $m > 1 + d/p$ and $N \geq 0$.

a) For $u \in A_{n,N}^{m+1,p}$, $\phi_{*} \in AD_{N}^{m,p}$, and all $\phi \in AD_{n,N}^{m,p}$ sufficiently close to $\phi_{*}$ we have

$$
\|u \circ \phi - u \circ \phi_{*}\|_{A_{m,p}^{N}} \leq C\|u\|_{A_{m+1,p}^{N}}\|\phi - \phi_{*}\|_{A_{N}^{m,p}}.
$$

(89a)

b) For $u \in A_{N-1}^{m+1,p}$, $\phi_{*} \in AD_{n,N}^{m,p}$, and all $\phi \in AD_{n,N}^{m,p}$ sufficiently close to $\phi_{*}$ we have

$$
\|u \circ \phi - u \circ \phi_{*}\|_{A_{m,p}^{N}} \leq C\|u\|_{A_{m+1,p}^{N-1}}\|\phi - \phi_{*}\|_{A_{N}^{m,p}}.
$$

(89b)

In both cases, the constant $C$ is uniformly local in $\phi$.

Proof. As in the proof of Lemma 5.7 let $\phi = Id + v$ and $\phi_{*} = Id + v_{*}$, and let $\tilde{v} = \phi - \phi_{*} = v - v_{*}$.

We first prove (89a). Assuming $u \in A_{N}^{m+1,p}$, we know $\nabla u \in A_{N}^{m,p}$. By Corollary 5.2 the function $F : t \mapsto (\nabla u)(\phi_{*} + t\tilde{v}) \cdot \tilde{v}$ is continuous as a map $F : [0,1] \rightarrow A_{N}^{m,p}$, hence Riemann integrable. Consequently, we can apply the algebra property and then Lemma 5.10 to (90) to obtain the desired estimate:

$$
\|u \circ \phi - u \circ \phi_{*}\|_{A_{m,p}^{N}} \leq \sup_{0 \leq t \leq 1} \|\nabla u(\phi_{*} + t\tilde{v})\|_{A_{N}^{m,p}}\|\tilde{v}\|_{A_{N}^{m,p}}
$$

$$
\leq C\|\nabla u\|_{A_{N}^{m,p}}\|\tilde{v}\|_{A_{N}^{m,p}} \leq C\|u\|_{A_{m+1,p}^{N}}\|\tilde{v}\|_{A_{N}^{m,p}}.
$$

Now we consider (89b). Assuming $u \in A_{N-1}^{m+1,p}$, we know $\nabla u \in A_{N}^{m,p}$, so the above steps show

$$
\|u \circ \phi - u \circ \phi_{*}\|_{A_{m,p}^{N}} \leq C\|\nabla u\|_{A_{N}^{m,p}}\|\tilde{v}\|_{A_{N}^{m,p}} \leq C\|u\|_{A_{m+1,p}^{N-1}}\|\tilde{v}\|_{A_{N}^{m,p}}.
$$

(90)

Proof of Proposition 4.1. The continuity of $\{n_{N}^{m,p}\}$ is contained in Corollary 5.2. For the proof that $\{n_{N}^{m,p}\}$ is $C^{1}$, we shall abbreviate our notation. Let $X^{m}$ represent $A_{n,N}^{m,p}$ (or $A_{n,N}^{m,p}$) and $XD^{m}$ represent $AD_{n,N}^{m,p}$ (or $AD_{n,N}^{m,p}$); the norm $X^{m}$ will be denoted simply by $\|\cdot\|_{m}$. We can also assume that $u$ is scalar-valued (although we will not distinguish notation between the $m$-norms of vector and scalar-valued functions).

We fix $u \in X^{m+1}$ and $\phi \in XD^{m}$, and consider nearby $u + \delta u \in X^{m+1}$ and $\phi + \delta \phi \in XD^{m}$; note that $\delta u \in X^{m+1}$ and $\delta \phi \in X^{m}$. We want to show that

$$
(u + \delta u) \circ (\phi + \delta \phi) = u \circ \phi + L_{n,\phi}(\delta u, \delta \phi) + R_{n,\phi}(\delta u, \delta \phi),
$$

where $L_{n,\phi} : X^{m+1} \times X^{m} \rightarrow X^{m}$ is a bounded linear map, and $\|L_{n,\phi}(\delta u, \delta \phi)\|_{m} = o(\|\delta u\|_{m+1} + \|\delta \phi\|_{m})$ as $\|\delta u\|_{m+1} + \|\delta \phi\|_{m} \rightarrow 0$.

We shall repeatedly use the following simple identity for $u \in C^{1}(\mathbb{R}^{d})$:

$$
u + d\nu = u + \nabla u(y) \cdot d\nu + R(u, y, d\nu),
$$

(92a)

where

$$
R(u, y, d\nu) = \int_{0}^{1} (\nabla u(y + t\delta y) - \nabla u(y)) \cdot d\nu dt.
$$

(92b)

Applying this identity with $y$ replaced by $\phi$, we find

$$
u \circ (\phi + d\phi) = u \circ \phi + (\nabla u \circ \phi) \cdot d\phi + R_{1}(u, \phi, d\phi),
$$

where

$$
R_{1}(u, \phi, d\phi) = \int_{0}^{1} (\nabla u \circ (\phi + t\delta \phi) - \nabla u \circ \phi) \cdot d\phi dt.
$$

(92b)
Then, replacing $u$ by $\delta u$, we find

$$
\delta u \circ (\phi + \delta \phi) = \delta u \circ \phi + (\nabla (\delta u) \circ \phi) \cdot \delta \phi + R_2(u, \delta u, \phi, \delta \phi),
$$

where

$$
R_2(u, \delta u, \phi, \delta \phi) = \int_0^1 (\nabla (\delta u) \circ (\phi + t \delta \phi) - \nabla (\delta u) \circ \phi) \cdot \delta \phi \, dt.
$$

Putting these together, we obtain \((91)\) where

$$
L_{u, \phi}(\delta u, \delta \phi) = \delta u \circ \phi + (\nabla u \circ \phi) \cdot \delta \phi,
$$

and

$$
R_{u, \phi}(\delta u, \delta \phi) = (\nabla (\delta u) \circ \phi) \cdot \delta \phi + R_1(u, \phi, \delta \phi) + R_2(u, \phi, \delta u, \delta \phi).
$$

Clearly $L_{u, \phi}$ is linear in $\delta u$ and $\delta \phi$ and bounded as desired, so we need to show $\| R_{u, \phi}(\delta u, \delta \phi) \|_m = o(\| \delta u \|_{m+1} + \| \delta \phi \|_m)$ as $\| \delta u \|_{m+1} + \| \delta \phi \|_m \to 0$. But, applying Lemma \ref{lemma:fixed_point_argument} and the algebra property, we can estimate the first term in $R_{u, \phi}$:

$$
\| \nabla (\delta u) \circ \phi \cdot \delta \phi \|_m \leq C \| \nabla (\delta u) \|_m \| \delta \phi \|_m \leq C \| \delta u \|_{m+1} \| \delta \phi \|_m.
$$

We can also use Lemma \ref{lemma:fixed_point_argument} and the algebra property to estimate the third term in $R_{u, \phi}$, namely $R_2$:

$$
\| R_2(u, \phi, \delta u, \delta \phi) \|_m \leq \int_0^1 \| (\nabla (\delta u) \circ (\phi + t \delta \phi) - \nabla (\delta u) \circ \phi) \|_m \, dt \| \delta \phi \|_m
\leq 2 \sup_{0 < t < 1} \| \nabla (\delta u) \circ (\phi + t \delta \phi) \|_m \| \delta \phi \|_m \leq C \| \delta u \|_{m+1} \| \delta \phi \|_m.
$$

Using $\| \delta u \|_{m+1} \| \delta \phi \|_m \leq \| \delta u \|_{m+1} + \| \delta \phi \|_m$, in both estimates above, we see that the $X^m$-norms of the first and third terms in $R_{u, \phi}$ are actually $O(\| \delta u \|_{m+1} + \| \delta \phi \|_m)$ as $\| \delta u \|_{m+1} + \| \delta \phi \|_m \to 0$.

To estimate $R_1$ in the $X^m$-norm as $\| \delta \phi \|_m \to 0$, we use the continuity of $\phi \to \nabla u \circ \phi$ in $X^m$ from the first part of Proposition \ref{prop:invertibility} to conclude that $\| \nabla u \circ (\phi + t \delta \phi) - \nabla u \circ \phi \|_m = o(1)$ uniformly for $0 < t < 1$ as $\| \delta \phi \|_m \to 0$. Using this in the definition of $R_1(u, \phi, \delta \phi)$ we find that

$$
\| R_1(u, \phi, \delta \phi) \|_m = o(\| \delta \phi \|_m) \quad \text{as} \quad \| \delta \phi \|_m \to 0.
$$

This completes the proof. \hfill \square

6. Proof of Invertibility (Proposition \ref{prop:invertibility})

Before we begin the proof of Proposition \ref{prop:invertibility} we prove several lemmas that will be useful. The first shows invertibility near the identity; but we require one additional order of differentiability.

**Lemma 6.1.** For $m > 1 + d/p$ and any $\phi \in \mathcal{K}D_N^{m+1,p}$ with $\| \phi - Id \|_{\mathcal{K}D_N^{m+1,p}} < \varepsilon$ sufficiently small, then $\phi^{-1} \in \mathcal{K}D_N^{m+1,p}$. The same result holds when $\mathcal{K}D$ is replaced by $\mathcal{A}D$.

**Proof.** We proceed in two steps: we first use a fixed point argument to show $\phi^{-1} \in \mathcal{K}D_N^{m+1,p}$ and then show in fact that $\phi^{-1} \in \mathcal{K}D_N^{m+1,p}$. Having fixed $p$ and $N$, let us denote the norm in $\mathcal{K}D_N^{m,p}$ simply by $\| \cdot \|_m$.

To formulate the fixed point argument, let $X = \{ v \in \mathcal{K}D_N^{m,p} : \| v \|_m < \eta \}$, where we have chosen $\eta$ small enough that $Id + v \in \mathcal{K}D_N^{m,p}$ for all $v \in X$. Now let us write $\phi = Id + u$ with $\| u \|_{m+1} < \varepsilon$, where $\varepsilon > 0$ will be specified below, and let $\psi := \phi^{-1}$. Then $\psi$ is a diffeomorphism and $\phi \circ \psi = Id$ implies $\psi = Id - u \circ \psi$. But we know that $u \in \mathcal{K}D_N^{m+1,p}$ is bounded, so we can write $\psi = Id + v$, where $v := -u \circ \psi$ is a bounded function; we want to show that $v \in \mathcal{K}D_N^{m,p}$. However, we know that $v$ satisfies $Id + v = Id - u \circ (Id + v)$. Consequently, let us define

$$
F_u(w) := -u \circ (Id + w).
$$

(95)
By continuity, we see that det ($\phi$) satisfies (16a) and $f$ we know that $A$ to one whose difference from the identity has compact support. Consequently, $A$ are in $\phi$ Lemma 5.1, we know (det($A$) is comprised of products of these elements, it too is a matrix with elements in $\phi$ $m,p$ $\phi$ $A$ $\phi$ $m, p$ $\phi$ $m, p$ $\phi$ $m, p$ $\phi$ $m, p$ $\phi$. Now sup $m > 1 + a/p$ and any $\phi \in \mathcal{A}^m_{m,p}$, there exists a continuous map $\phi_1 : [0,1] \to \mathcal{A}^m_{m,p}$ such that $\phi_1 = \phi$ and $\phi_0 - Id$ has compact support. The same result holds when $\mathcal{A}\mathcal{D}$ is replaced by $\mathcal{A}\mathcal{D}$.

**Proof.** Write $\phi = Id + u$ where $u \in \mathcal{A}^m_{m,p}$. Since $\phi$ is an orientation-preserving diffeomorphism, we know that

$$\det(D\phi) = \det \begin{pmatrix} 1 + \partial_1 u^1 & \partial_2 u^1 & \cdots & \partial_d u^1 \\ \partial_1 u^2 & 1 + \partial_2 u^2 & \cdots & \partial_d u^2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 u^d & \partial_2 u^d & \cdots & 1 + \partial_d u^d \end{pmatrix} > 0 \text{ for all } x \in \mathbb{R}^d.$$  

By continuity, we see that $\det(D(\phi + v)) > 0$ provided $v$ is chosen so that $\sup |Dv| < \varepsilon$ with $\varepsilon > 0$ sufficiently small. But recall that we can write $u = \chi_R(r)(a_0(\theta) + \cdots r^{-N}a_N(\theta)) + f$ where $N^*$ satisfies [1] and $f \in H^m_{m,p}$, so we can choose $R$ sufficiently large that

$$\left| D \left[ \chi_R(r)(a_0(\theta) + \cdots r^{-N}a_N(\theta)) + \chi_R(r) f(x) \right] \right| < \varepsilon.$$
(This can be done since $|\nabla(\chi_R(r))| = R^{-1}|\chi'(R^{-1}r)| \leq M/R$ and $|Df| = o(|x|^{-N})$ as $|x| \to \infty$.)

Thus, provided $R$ is sufficiently large, we see that

$$\phi_t := Id + u - (1-t) \left[ \chi_R(a_0 + \cdots + a_N \chi_R) \right] + \chi_R f$$

is an asymptotic diffeomorphism for all $t \in [0,1]$. But $\phi_1 = \phi$ and $\phi_0 = Id + (1 - \chi_R)f$, where $(1 - \chi_R)f$ has compact support, so we have proved the lemma.

The next lemma concerns the differential of an asymptotic diffeomorphism $\phi \in \mathcal{AD}^{m+1,p}_N$ for $m > 1 + d/p$ as a $C^1$-map $\mathcal{AD}_N^{m,p} \to \mathcal{AD}_N^{m,p}$, defined by composition. Of course, we must take the differential of $\phi$ at a particular diffeomorphism $F$, which we take as the identity $\psi = Id$ and denote the resultant differential $d_\psi \phi$ by $d_0 \phi$; this will be a linear map $d_0 \phi : \mathcal{AD}^{m,p}_N \to \mathcal{AD}^{m,p}_N$. If we write $\phi = Id + u$, then we have $d_0 \phi = I + Du$, since for $v \in \mathcal{AD}^{m,p}_N$ we can calculate pointwise

$$\lim_{t \to 0} \phi(Id + tv) = v + Du \cdot v = (I + Du) \cdot v.$$ 

We now show that this linear map $d_0 \phi : \mathcal{AD}^{m,p}_N \to \mathcal{AD}^{m,p}_N$ is invertible.

**Lemma 6.3.** For $m > 1 + d/p$ and any $\phi \in \mathcal{AD}^{m+1,p}_N$, the linear map $d_0 \phi : \mathcal{AD}^{m,p}_N \to \mathcal{AD}^{m,p}_N$ is invertible. If $\mathcal{AD}$ is replaced by $\mathcal{AD}$ the conclusion becomes $d_0 \phi : \mathcal{AD}^{m,p}_{N+1} \to \mathcal{AD}^{m,p}_{N+1}$ is invertible.

**Proof.** Write $\phi = Id + u$, where $u \in \mathcal{AD}^{m+1,p}_N$, so $d_0 \phi = I + Du$. But since $\phi$ is a diffeomorphism, considered as a matrix, $I + Du$ is invertible and its inverse is given by the adjoint formula (97).

Now $Du$ is a matrix, all of whose elements are in $\mathcal{A}^{m,p}_{1,N}$, since $\text{Adj}(I + Du)$ is comprised of products of these elements, it too is a matrix with elements in $\mathcal{A}^{m,p}_N$. Since $m > d/p$, $\mathcal{A}^{m,p}_N$ is an algebra and $\text{Adj}(I + Du)$ maps $\mathcal{A}^{m,p}_N \to \mathcal{A}^{m,p}_N$. Also, $\det(I + Du)$ is a product of elements of $\mathcal{A}^{m,p}_N$, so it too is in $\mathcal{A}^{m,p}_N$. Since $\det(I + Du)(x) > 0$ for all $x \in \mathbb{R}^d$ and $Du(x) \to 0$ as $|x| \to \infty$, we have $\det(I + Du)(x) \geq \varepsilon > 0$; therefore, we can use Lemma 5.1 to conclude that $\det(I + Du)^{-1} \in \mathcal{A}^{m,p}_N$. Consequently, $(I + Du)^{-1}$ is a matrix with all elements in $\mathcal{A}^{m,p}_N$, so it is bounded as a map $\mathcal{A}^{m,p}_N \to \mathcal{A}^{m,p}_N$, completing the proof.

The proof for $\mathcal{AD}$ is strictly analogous.

Next we want to show that left-translation by a fixed asymptotic diffeomorphism is an open map in a neighborhood of the identity; the next lemma shows that this is true provided we have one additional order of differentiability in the fixed diffeomorphism.

**Lemma 6.4.** For $m > 1 + d/p$ and any fixed $\phi_* \in \mathcal{AD}^{m+1,p}_N$, there is an open neighborhood $U$ of $Id$ in $\mathcal{AD}^{m+1,p}_N$ such that $\phi_*(U)$ is an open neighborhood of $\phi_*$ in $\mathcal{AD}^{m+1}_N$. The same result holds when $\mathcal{AD}$ is replaced by $\mathcal{AD}$.

**Proof.** Let $\phi_* = Id + u_*$ and for $\bar{\phi} = Id + \bar{u}$ near $Id$ in $\mathcal{AD}^{m,p}_N$, we can write

$$\phi_* \circ \bar{\phi} = \phi_* + F_*(\bar{u}), \text{ where } F_*(\bar{u}) = u_* \circ (Id + \bar{u}) - u_* + \bar{u}.$$ 

Hence there is an open neighborhood $U_0$ of $0$ in $\mathcal{A}^{m,p}_N$ such that $F_* : U_0 \to \mathcal{A}^{m,p}_N$ and $F_*(0) = 0$; in fact, since $u_* \in \mathcal{A}^{m+1,p}_N$, by Proposition 4.1 $F_* : U_0 \to \mathcal{A}^{m,p}_N$ is $C^1$. If we compute the differential of $F_*$ at 0, which we also denote by $d_0 F_*$, we find

$$d_0 F_*(v) = Du_* \cdot v + v = (I + Du_*) \cdot v = d_0 \phi_*(v) \text{ for any } v \in \mathcal{A}^{m,p}_N.$$ 

But, using Lemma 6.3 (with $m$ in place of $m - 1$), we know that $d_0 \phi_* = d_0 F_*$ is invertible. By the inverse function theorem, we conclude that $F_*$ admits a continuous inverse near 0 in $\mathcal{A}^{m,p}_N$, which translates as the desired conclusion for $\phi_*$ and $U = Id + U_0$.

Finally, we are ready to prove Proposition 4.2.
Proof of Proposition 4.2. We give the proof for \( \mathcal{A}D \), the case of \( \mathcal{A}D \) being analogous. Using Lemma \( 6.2 \) there exists a continuous map \( \phi_s : [0, 1] \to \mathcal{A}D^{m+1,p}_N \) such that \( \phi_1 = \phi \) and \( \phi_0 - I_d \) has compact support. But then \( \phi_0^{-1} \) is the identity outside a compact set, so trivially we have \( \phi_0^{-1} \in \mathcal{A}D^{m+1,p}_N \). We want to use the continuity method to show that this property can gradually be extended to \( \phi_1 = \phi \).

Let us denote by \( U^{m+1} \) the neighborhood of \( I_d \) in \( \mathcal{A}D^{m+1,p}_N \) which Lemma \( 6.1 \) guarantees consists of diffeomorphisms that are invertible in \( \mathcal{A}D^{m+1,p}_N \). Now let \( U^{m+1}_1 = \phi_t(U^{m+1}) = \{ \phi_t \circ \psi : \psi \in U^{m+1} \} \). We first want to show that every \( \phi_s \) in \( U^{m+1}_0 \) is invertible in \( \mathcal{A}D^{m+1,p}_N \). But this is trivial since \( \phi_s = \phi_0 \circ \psi \) implies \( \phi_s^{-1} = \psi^{-1} \circ \phi_0^{-1} \in \mathcal{A}D^{m+1,p}_N \). Now by compactness we can cover the path \( \phi_t \) for \( 0 \leq t \leq 1 \) by a finite number of these translated neighborhoods, i.e. \( U^{m+1}_t, U^{m+1}_{t_1}, \ldots, U^{m+1}_{t_k}, U^{m+1}_1 \). Next we want to show every \( \phi_s \) in \( U^{m+1}_t \) is invertible in \( \mathcal{A}D^{m+1,p}_N \); it suffices to show \( \phi_t^{-1} \in \mathcal{A}D^{m+1,p}_N \). But we can pick \( \hat{\phi} \in U^{m+1}_0 \cap U^{m+1}_t \), which is both invertible in \( \mathcal{A}D^{m+1,p}_N \) and of the form \( \hat{\phi} = \phi_t \circ \psi \) for some \( \psi \in U^{m+1} \). However, we can compose with \( \psi^{-1} \) to conclude \( \hat{\phi}_t = \hat{\phi} \circ \psi^{-1} \) and hence \( \hat{\phi}_t^{-1} = \psi \circ \phi_t^{-1} \in \mathcal{A}D^{m+1,p}_N \). Clearly this process can be continued to show every \( \phi_s \) in \( U^{m+1}_t \) is invertible in \( \mathcal{A}D^{m+1,p}_N \). In particular, \( \phi_1 = \phi \) is invertible in \( \mathcal{A}D^{m+1,p}_N \), as desired.

Finally, we want to show that \( \phi \to \phi^{-1} \) is \( C^1 \) as a map \( \mathcal{A}D^{m+1,p}_N \to \mathcal{A}D^{m,p}_N \). We shall do this using the implicit function theorem; this is valid since a neighborhood of a fixed \( \phi_s \) in \( \mathcal{A}D^{m+1,p}_N \) may be parameterized by a neighborhood of \( 0 \) in the Banach space \( \mathcal{A}^N \). In fact, let \( F : \mathcal{A}D^{m+1,p}_N \times \mathcal{A}D^{m,p}_N \to \mathcal{A}D^{m,p}_N \) represent composition, i.e. \( F(\psi, \phi) = \psi \circ \phi \), which we know from Proposition 4.1 is \( C^1 \). Let us fix \( \phi_s \in \mathcal{A}D^{m+1,p}_N \), and consider the differential of the map \( \psi \to F(\phi, \psi) \) at the point \( \psi = \phi_s^{-1} \in \mathcal{A}D^{m,p}_N \), which is given by

\[
T(h) = d\phi_s \circ \phi_s^{-1} \cdot h \quad \text{for } h \in \mathcal{A}D^{m,p}_N.
\]

Since \( d\phi_s \in \mathcal{A}^m \), by Proposition 2.2 we know that \( d\phi_s \circ \phi_s^{-1} \in \mathcal{A}^m \), and hence (using Proposition 2.2) the linear operator \( T \) is bounded on \( \mathcal{A}^m \); since its inverse is just \( T^{-1}(h) = d(\phi_s)^{-1} \circ \phi_s \cdot h \), which is also bounded on \( \mathcal{A}^m \). Consequently, the implicit function theorem implies that there is a neighborhood \( U \) of \( \phi_s \) in \( \mathcal{A}D^{m+1,p}_N \) and a unique \( C^1 \) map \( G : U \to \mathcal{A}D^{m,p}_N \) such that \( F(\phi, G(\phi)) = I_d \) holds for all \( \phi \) near \( \phi_s \). But uniqueness of the inverse of \( \phi \) shows \( G(\phi) = \phi^{-1} \), and hence \( \phi \to \phi^{-1} \) is \( C^1 \).

\[\square\]

Appendix A. Proofs of Lemmas 1.1 and 1.2

Proof of (c) & (d) in Lemma 1.1 Let \( Q \) be a d-box of side length 1. First suppose \( d > mp \). For \( g \in H^{m,p}(Q) \) and \( p \leq q \leq pd/(d - mp) \), the Sobolev inequality states

\[
\|g\|_{L^q(Q)} \leq C(d, m, p, q) \|g\|_{H^{m,p}(Q)}.
\]

Apply this to \( g = \langle x \rangle^\delta f \) and for \( |\alpha| \leq m \) use

\[
\|\mathcal{D}\alpha\langle x \rangle^\delta f\|_{L^p(Q)} = \| \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \mathcal{D}\beta\langle x \rangle^\delta \mathcal{D}^{\alpha-\beta} f \|_{L^p(Q)}
\]

\[
\leq C(|\alpha|, \delta) \sum_{\beta \leq \alpha} \| \langle x \rangle^{\delta - |\beta|} \mathcal{D}^{\alpha-\beta} f \|_{L^p(Q)}
\]

\[
\leq C(|\alpha|, \delta) \sum_{|\gamma| \leq |\alpha|} \| \langle x \rangle^{\delta} \mathcal{D}\gamma f \|_{L^p(Q)}
\]
to conclude
\[
\|x\|^\delta f\|_{L^q(Q)} \leq C(d, m, p, q, \delta) \sum_{|\alpha| \leq m} \|x\|^\delta D^\alpha f\|_{L^p(Q)}.
\]

Now let \(Q_0\) denote the \(d\)-box of side length 1 centered at the origin in \(\mathbb{R}^d\), and \(Q_\ell\) for \(\ell = 0, 1, \ldots\) be an enumeration of all \(d\)-boxes of side length 1 and centers at integral coordinates, so
\[
\mathbb{R}^d = \bigcup_{\ell=0}^\infty Q_\ell.
\]

Then we use the inequality \eqref{eq:1} and then the elementary inequality \((\sum a_j)^{1/q} \leq (\sum q_j)^{1/p}\) to estimate
\[
\|f\|_{L^q_1(\mathbb{R}^d)} = \|x\|^\delta f\|_{L^q(\mathbb{R}^d)} = \left(\sum_{\ell=0}^\infty \|\langle x\rangle^\delta D^\alpha f\|_{L^q(Q_\ell)}^p\right)^{1/p} = C \left(\sum_{|\alpha| \leq m} \|\langle x\rangle^\delta D^\alpha f\|_{L^p(Q_\ell)}^p\right)^{1/p},
\]
where \(C = C(d, m, p, q, \delta)\). But the last term is equivalent to \(C\|f\|_{H^{m,p}_0}\), which establishes the inequality in (c). The same argument works for \(d = mp\), provided we assume \(p \leq q < \infty\).

Now suppose \(d < mp\) and \(k < m - (d/p)\). For \(g \in H^{m,k,p}(Q)\), Morrey’s inequality implies \(g \in C(\mathbb{R}^d)\) and
\[
\sup_{x \in Q_\ell} |g(x)| \leq C(m, p, k) \|g\|_{H^{m,k,p}(Q)}.
\]
Apply this on \(Q_\ell\) (as above) to \(g = \langle x\rangle^\delta D^\alpha f\) for any \(\alpha\) satisfying \(0 \leq |\alpha| \leq k\):
\[
\sup_{x \in Q_\ell} \langle x\rangle^\delta |D^\alpha f(x)| \leq C(m, p, k) \|\langle x\rangle^\delta D^\alpha f\|_{H^{m-k,p}(Q_\ell)} = C(m, p, k, \delta) \sum_{|\beta| \leq m-k} \|D^\beta \langle x\rangle^\delta D^\alpha f\|_{L^p(Q_\ell)} \leq C(m, p, k, \delta) \sum_{|\alpha| \leq m} \|\langle x\rangle^\delta D^\alpha f\|_{L^p(Q_\ell)} \leq C(m, p, k, \delta) \|f\|_{H^{m,p}_0(\mathbb{R}^d)}.
\]

Now letting \(\ell\) vary on the left, we obtain the desired inequality. Moreover, since the series \(\sum_\ell \|\langle x\rangle^\delta D^\alpha f\|_{H^{m,p}(Q_\ell)}\) converges, we must have \(\|\langle x\rangle^\delta D^\alpha f\|_{H^{m,p}(Q_\ell)} \to 0\) as \(\ell \to \infty\). Consequently,
\[
\sup_{x \in Q_\ell} \langle x\rangle^\delta |D^\alpha f(x)| \to 0 \quad \text{as} \ \ell \to \infty,
\]
which implies that \(x|\delta|D^\alpha f(x)| \to 0\) as \(|x| \to \infty\). \(\square\)

**Proof of (c) & (d) in Lemma 1.2** We shall use similar arguments as in \cite{2}. Let us introduce the Sobolev norm with homogeneous weight function:
\[
\|f\|^p_{m,p,\delta} = \sum_{j=0}^m \int_{\mathbb{R}^d \setminus \{0\}} |x|^{\delta+j} D^j f(x)|^p \ dx.
\]
For $R > 0$ define $f_R(x) = f(Rx)$; it is easy to compute that
\begin{equation}
\|f_R\|_{m,p,\delta} = R^{-\delta - \frac{d}{p}} \|f\|_{m,p,\delta}.
\end{equation}
Letting $A_R = B_{2R} \setminus B_R$ where $B_R = \{x : |x| < R\}$, we can integrate over annuli instead of all of $\mathbb{R} \setminus \{0\}$ (and adjust notation in the obvious way) to obtain the localized version of (100):
\begin{equation}
\|f_R\|_{m,p,\delta; A_1} = R^{-\delta - \frac{d}{p}} \|f\|_{m,p,\delta; A_R}.
\end{equation}
But the weighting factor $|x|^\delta$ is bounded above and below by constants on $A_1$ and by $c R^\delta$ on $A_R$. We may conclude the following equivalence:
\begin{equation}
\|f_R\|_{m,p; A_1} \approx R^{-\delta} \|f\|_{m,p,\delta; A_R}.
\end{equation}
To prove (c), we apply the Sobolev inequality to $f_R$ on $A_1$ to conclude:
\[\|f_R\|_{q,A_1} \leq C \|f_R\|_{m,p; A_1}, \quad C = C(d, m, p, q).\]
Then apply (102) to both sides of this to conclude
\begin{equation}
\|f\|_{q,\delta - \frac{d}{p}; A_R} \leq C \|f\|_{m,p,\delta - \frac{d}{p}; A_R}, \quad C = C(d, m, p, q).
\end{equation}
Now let us write
\[\mathbb{R}^d = B_0 \cup A_1 \cup A_2 \cup \cdots \quad \text{where} \quad A_j = A_{2j-1} = B_{2^j} \setminus B_{2^j-1}.\]
Then we can use the Sobolev inequality on $B_0$ and (103) on each $A_j$ to obtain
\[\|f\|_{L_q; B_0} \leq C \left( \|f\|_{L_q; B_0}^{\frac{1}{p}} + \|f\|_{L_q; \frac{d}{p}; A_1}^{\frac{1}{p}} + \cdots + \|f\|_{L_q; \frac{d}{p}; A_k}^{\frac{1}{p}} \right)^{1/p},\]
where in the last step we used $p \leq q$ and the elementary inequality $(\sum a_j^p)^{1/p} \leq (\sum a_j^q)^{1/q}$. But this last term is equivalent to $\|f\|_{W^p_{\delta - \frac{d}{p}}}$, so we have proved (c).
To prove (d), when $mp > d$ the scaling argument yields in place of (103)
\begin{equation}
\sup_{x \in A_R} |x|^{\delta + |\alpha|} |D^\alpha f(x)| \leq C \|f\|_{m,p,\delta - \frac{d}{p}; A_R}.
\end{equation}
Now, we can replace the right hand side of (104) by $C \|f\|_{W^{m,p}_{\delta - \frac{d}{p}}}$ and then allow $R$ on the left hand side to range freely to conclude
\[\sup_{x \in \mathbb{R}^d} (x)^{\delta + |\alpha|} |D^\alpha f(x)| \leq C \|f\|_{W^{m,p}_{\delta - \frac{d}{p}}}.\]
But since the series
\[\|f\|_{m,p,\delta - \frac{d}{p}; A_1} + \|f\|_{m,p,\delta - \frac{d}{p}; A_2} + \cdots \]
converges, we see that $\|f\|_{m,p,\delta - \frac{d}{p}; A_j} \to 0$ as $j \to \infty$, so from (104) we conclude that
\[|x|^{\delta + |\alpha|} |D^\alpha f(x)| \to 0 \quad \text{as} \quad |x| \to \infty. \quad \square\]
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