General relativistic polytropes in anisotropic stars

A. A. Isayev

Kharkov Institute of Physics and Technology,
Academicheskaya Street 1, Kharkov, 61108, Ukraine

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Abstract

Spherically symmetric relativistic stars with the polytropic equation of state (EoS), which possess the local pressure anisotropy, are considered within the framework of general relativity. The generalized Lane-Emden equations are derived for the arbitrary anisotropy parameter $\Delta = p_t - p_r$ ($p_t$ and $p_r$ being the transverse and radial pressure, respectively). They are then applied to some special ansatz for the anisotropy parameter in the form of the differential relation between the anisotropy parameter $\Delta$ and the metric function $\nu$. The analytical solutions of the obtained equations are found for incompressible fluid stars and then used for getting their mass-radius relation, gravitational and binding energy. Also, following the Chandrasekhar variational approach, the dynamical stability of incompressible anisotropic fluid stars with the polytropic EoS against radial oscillations is studied. It is shown that the local pressure anisotropy with $p_t > p_r$ can make the incompressible fluid stars unstable with respect to radial oscillations, in contrast to incompressible isotropic fluid stars with the polytropic EoS which are dynamically stable.

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I. INTRODUCTION

It was argued in Ref. [1] that, despite the spherically symmetric distribution of matter inside a compact stellar object, it can be characterized by the local pressure anisotropy. The analysis of the generalized equations of hydrostatic equilibrium, allowing for the pressure anisotropy, shows that anisotropy may have the substantial effect on the maximum equilibrium mass and gravitational surface redshift [1–3]. At not too high densities, the impact of anisotropy can be studied within Newtonian gravity [4, 5]. At higher densities ($\rho \gtrsim 10^{15} \text{g/cm}^3$), both the relativistic effects and the effects of general relativity become important [6–12]. In order to study the influence of the local pressure anisotropy on the specific basis, it is necessary to know the concrete physical reasons responsible for its appearance, such as, e. g., the existence of a solid core [6, 7], occurrence of spontaneous deformation of Fermi surfaces [13, 14], availability of superfluid states with the finite orbital momentum of Cooper pairs [15–19] or finite superfluid momentum [20, 21], or the presence of strong magnetic fields inside a star [22–34].

With account of the pressure anisotropy, the equation of state (EoS) of the system will be also necessarily anisotropic. The EoS is the essential ingredient in solving the equations of the hydrostatic equilibrium and its importance was underlined in Ref. [35]. In the given work, we choose the polytropic EoS, which is widely used in many astrophysical applications [36–42]. For the polytropic index $n = 2$, the exact solutions of Einstein equations with the pressure anisotropy were obtained in Ref. [43]. Anisotropic spheres with the uniform energy density in general relativity were studied in Refs. [12, 44], and with the variable energy density in Ref. [45]. Note also that the exact solutions of Einstein equations for spherical anisotropic stars with the linear EoS were obtained in Ref. [46] for some particular mass distribution.

In the present work, we will study spherically symmetric relativistic anisotropic stars with the polytropic EoS, aiming to obtain the generalized Lane-Emden equations for the special ansatz for the anisotropy parameter in the form of the differential relation between the anisotropy parameter and the metric function $\nu$. This approach is different from that suggested in Refs. [1, 43] which consists in setting the anisotropy parameter, or some metric function, in the specific functional form. In general case, the obtained Lane-Emden equations can be integrated only numerically, but the analytic solutions can be found for incompressible fluid stars which are then used to get their mass-radius relation, gravitational and binding
energy. Also, we apply the Chandrasekhar variational procedure \cite{47} to study the dynamical stability of incompressible anisotropic fluid stars with respect to radial oscillations.

II. BASIC EQUATIONS

For spherically symmetric stars, the line element is written in the form

\[ ds^2 = e^{\nu(r,t)} dt^2 - e^{\lambda(r,t)} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \]  

where we use the system of units with \( c = 1 \). While the matter distribution inside a star is spherically symmetric, we allow the existence of the local pressure anisotropy in its interior with the different values of the radial \( p_r \) and transverse \( p_\theta = p_\varphi \equiv p_t \) pressures. The anisotropic energy-momentum tensor for the static configuration reads

\[ T^k_i = \text{diag}(\varepsilon, -p_r, -p_t, -p_t), \]  

where \( \varepsilon \) is the energy density of the system. The space-time geometry and matter distribution are related by Einstein equations:

\[ R^k_i - \frac{1}{2} R \delta^k_i = 8\pi GT^k_i. \]  

With account of Eqs. (1), (2), Einstein equations read

\[ e^{-\lambda} \left( \frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = 8\pi G \varepsilon, \]  

\[ e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = 8\pi G p_r, \]  

\[ \frac{1}{2} e^{-\lambda} \left( \nu'' - \frac{1}{2} \nu' \lambda' + \frac{1}{2} \nu'^2 + \frac{\nu'}{r} \right) = 8\pi G p_t, \]  

where prime denotes the differentiation with respect to \( r \). From Eqs. (4)–(6), or, equivalently, from the vanishing divergence of the energy-momentum tensor \( T^k_{i;k} = 0 \), one can get the equation for the hydrostatic equilibrium in the presence of the pressure anisotropy in the form

\[ p'_r = -\frac{\nu'}{2}(\varepsilon + p_r) + \frac{2\Delta}{r}, \quad \Delta \equiv p_t - p_r, \]  

where \( \Delta \) is the anisotropy parameter. The interior metric function \( \lambda \) can be found from Eq. (4):

\[ e^{-\lambda(r)} = 1 - \frac{2G}{r}m(r), \quad r < R, \]  

\[ \frac{1}{2} \]
where $R$ is the radial coordinate at the surface of a star and $m(r)$ is the mass enclosed in the sphere of radius $r$:

$$m(r) = 4\pi \int_0^r \varepsilon r^2 dr.$$  \hspace{1cm} (9)

From Eq. (5), with account of Eq. (8), one can find

$$\nu' = 2G\frac{m(r) + 4\pi p_r r^3}{r(r - 2Gm(r))}.$$  \hspace{1cm} (10)

Hence, the equation of hydrostatic equilibrium for a spherically symmetric anisotropic star takes the form

$$p_r' = -G\frac{(\varepsilon + p_r)(m(r) + 4\pi p_r r^3)}{r(r - 2Gm(r))} + \frac{2\Delta}{r},$$  \hspace{1cm} (11)

which is the generalization of the Tolman-Oppenheimer-Volkoff (TOV) equation with account of the local pressure anisotropy. In order to solve Eq. (11), considered together with Eq. (2), it is necessary to set the EoS of the system, $\varepsilon = \varepsilon(p_r)$, and to use a specific model-dependent expression for $\Delta$, or to utilize some approximation for it. Further we will assume that any physical quantity or metric function in the interior of a star is free of singularities. As follows from Eq. (11), the gradient $p_r'$ will be finite at $r = 0$, if, at least, $\Delta \propto r$ at $r \to 0$. As the boundary condition to Eq. (11), we will set the radial pressure at the center of a star $p_r(0) = p_{r0}$, and will determine the radial coordinate $R$ at the surface from the condition $p_r(R) = 0$. The total mass then can be determined as $M = m(R)$, assuming that $m(0) = 0$. After finding the radial pressure distribution $p_r(r)$ (together with the mass distribution $m(r)$), the metric functions $\lambda(r), \nu(r)$ can be determined from Eqs. (8),(10). At the boundary $r = R$, the metric functions are matchable to the exterior vacuum Schwarzschild metric:

$$\lambda(R) = -\nu(R) = -\ln \left(1 - \frac{2GM}{R}\right).$$  \hspace{1cm} (12)

III. GENERALIZED LANE–EMDEN EQUATIONS

Further, as the EoS of the system, we choose the polytropic EoS in the form [37]:

$$p_r = K \varrho^{\gamma} \equiv K \varrho^{1+\frac{1}{\gamma}},$$  \hspace{1cm} (13)
where \( \rho \) is the mass (baryon) density, \( K \) is some constant, which can be, in principle, temperature dependent, \( \gamma \) is the polytropic exponent, \( n \) is the polytropic index. Note that in some works [12, 36, 38, 41] the polytropic EoS is set in the form \( p_r = K \varepsilon^{\gamma} \), which will not be considered here. It is possible to show (see, e.g., Ref. [42]) that for the EoS (13) the energy density \( \varepsilon \) is related to the mass density \( \rho \) and the radial pressure \( p_r \) by the equation

\[
\varepsilon = \rho + \frac{p_r}{\gamma - 1}.
\] (14)

It is convenient to introduce the auxiliary dimensionless Lane-Emden function \( \theta \) according to the equations:

\[
p_r = p_r0 \theta^{n+1}, \quad \rho = \varrho_0 \theta^n,
\] (15)

where \( \varrho_0 \) is the central mass density. It follows from the boundary conditions for the radial pressure \( p_r \) that

\[
\theta(0) = 1, \quad \theta(R) = 0.
\] (16)

Then Eq. (7) of hydrostatic equilibrium can be rewritten as

\[
2q_0(n + 1)d\theta - \frac{4\Delta dr}{\varrho_0 r \theta^n} + (1 + (n + 1)q_0 \theta) d\nu = 0,
\] (17)

where \( q_0 \equiv \frac{p_r}{\varrho_0} \). This equation can be integrated to give

\[
\nu = \nu_0 - \ln \left( \frac{1 + (n + 1)q_0 \theta}{1 + (n + 1)q_0} \right)^2 + \frac{4}{q_0} \int_0^r \frac{\Delta dr}{r \theta^n (1 + (n + 1)q_0 \theta)}.
\]

\( \nu_0 \equiv \nu(0) \) being the integration constant. In order to find it, one can use the boundary condition (12). This gives

\[
\nu_0 = \ln \frac{1 - 2GM_R}{(1 + (n + 1)q_0)^2} - \frac{4}{q_0} \int_0^R \frac{\Delta dr}{r \theta^n (1 + (n + 1)q_0 \theta)}.
\]

Therefore, the metric function \( \nu(r) \) can be written as

\[
\nu(r) = \ln \frac{1 - 2GM_R}{(1 + (n + 1)q_0 \theta)^2} - \frac{4}{q_0} \int_r^R \frac{\Delta dr}{r \theta^n (1 + (n + 1)q_0 \theta)}.
\] (18)

Let us define the auxiliary function

\[
u(r) = \frac{m(r)}{M} = \frac{r}{2GM} \left( 1 - e^{-\lambda(r)} \right), \quad u(0) = 0, \quad u(R) = 1,
\] (19)
which, according to Eq. (9), satisfies the differential equation

$$Mu' = 4\pi\varepsilon r^2. \quad (20)$$

Then, substituting $e^{-\lambda}$ from Eq. (19), and the derivative $\nu'$ from Eq. (17) to Eq. (5), and using Eq. (20), one gets

$$q_0(n+1)\theta' r \left(1 - \frac{2GM}{r} u\right) + \frac{GMq_0\theta}{1 + nq_0\theta} u' + \frac{GM}{r} u - \frac{2\Delta}{q_0\theta^n(1 + (n+1)q_0\theta)} \left(1 - \frac{2GM}{r} u\right) = 0. \quad (21)$$

Defining the dimensionless variable $\xi$ and dimensionless function $\eta$ by the equations

$$r = \alpha \xi, \quad \eta = \frac{M}{4\pi q_0\alpha^3} u, \quad (22)$$

where $\alpha^2 = \frac{q_0(n+1)}{4\pi Gq_0}$, Eqs. (20) and (21) can be rewritten as

$$\frac{\xi - 2(n+1)q_0\eta}{1 + (n+1)q_0\theta} \left\{ \frac{d\theta}{d\xi} - \frac{2\Delta}{q_0\theta^n(1 + (n+1)q_0\theta)} \right\} + \eta + q_0\xi^3\theta^{n+1} = 0, \quad (23)$$

$$\frac{d\eta}{d\xi} = \xi^2\theta^n(1 + nq_0\theta). \quad (24)$$

As follows from Eqs. (16) and (22), the boundary conditions for the functions $\theta(\xi)$ and $\eta(\xi)$ read

$$\theta(0) = 1, \quad \theta(\xi_R) = 0, \quad (25)$$

$$\eta(0) = 0, \quad \eta(\xi_R) = \frac{M}{4\pi q_0\alpha^3}, \quad (26)$$

where $\xi_R = R/\alpha$. Eqs. (23) and (24) represent the generalized Lane-Emden equations for relativistic anisotropic polytropes with the EoS (13), after solving which one can find from Eqs. (15) and (22) the radial distribution of the radial pressure and mass in the interior of a spherically symmetric relativistic anisotropic star. At $\Delta = 0$, Eqs. (23) and (24) go over to the equations for the relativistic isotropic polytropes [37].

Note that for a given $q_0$, the radius and mass can be found as functions of the constant $K$ in EoS (13):

$$R = R^* q_0^{\frac{1-n}{2}}, \quad M = M^* q_0^{\frac{3-n}{2}} \eta(\xi_R), \quad (27)$$
where the dependence on $K$ goes through the quantities $R^*$ and $M^*$:

$$R^* = \sqrt{\frac{n+1}{4\pi G} K^\frac{2}{3}}, \quad M^* = \frac{1}{\sqrt{4\pi}} \left(\frac{n+1}{G}\right)^{\frac{3}{2}} K^\frac{2}{3}. \quad (28)$$

The mass-radius relation has the form

$$\frac{GM}{R} = (n + 1) q_0^\frac{n-3}{2} \eta(\xi_R). \quad (29)$$

The total energy of a star is

$$E = 4\pi \int_0^R \varepsilon r^2 dr = M^* q_0^\frac{n-3}{2} \eta(\xi_R). \quad (30)$$

The proper energy $E_0$ is the integral of the energy density over the proper spatial volume [36]:

$$E_0 = 4\pi \int_0^R \varepsilon e^{\frac{4}{3} r^2} dr = 4\pi q_0 a^3 \int_0^{\xi_R} (1 + n q_0 \theta) \frac{\xi^2 \theta^n}{\sqrt{1 - 2 q_0 (n+1) \eta(\xi)}} d\xi. \quad (31)$$

The gravitational potential energy $\Omega$ is determined as

$$\Omega = E - E_0 = M^* q_0^\frac{n-3}{2} \eta(\xi_R) \left( 1 - \frac{1}{\eta(\xi_R)} \int_0^{\xi_R} (1 + n q_0 \theta) \frac{\xi^2 \theta^n}{\sqrt{1 - 2 q_0 (n+1) \eta(\xi)}} d\xi \right). \quad (32)$$

The binding energy is the difference between the energy of the particles scattered to infinity and the total energy of the system [36]:

$$E_B = E_{0g} - E, \quad E_{0g} = 4\pi \int_0^R \varrho e^{\frac{4}{3} r^2} dr, \quad (33)$$

which reads

$$E_B = M^* q_0^{\frac{n-3}{2}} \eta(\xi_R) \left( u_g(\xi_R) - 1 \right), \quad u_g(\xi) = \int_0^{\xi} \frac{\xi^2 \theta^n}{\sqrt{1 - 2 q_0 (n+1) \eta(\xi)}} d\xi. \quad (34)$$

### IV. ANALYTICAL SOLUTIONS AND NUMERICAL RESULTS FOR INCOMPRESSIBLE ANISOTROPIC FLUID STARS

In order to solve the generalized Lane–Emden equations (23), (24) for the functions $\theta, \eta$ with the boundary conditions (25), (26), one needs to specify the anisotropy parameter $\Delta$. This can be done within the given model framework with the concrete physical mechanism responsible for the appearance of the local pressure anisotropy. The other approach to study the effects of the pressure anisotropy is to set the anisotropy parameter $\Delta$ in a
phenomenological way, with the use of some phenomenological ansatz for the anisotropy parameter. This approach allows to study general properties of spherically symmetric relativistic anisotropic stars and will be followed in the given research. The conclusions obtained within this approach are rather of a general character and independent of the details of a specific physical mechanism.

In fact, we will follow the point of view, suggested in Refs. [5, 48], and which consists in setting some additional differential relation between the unknown functions. Namely, we will suppose that the presence of the anisotropy parameter \( \Delta \) doesn’t change the general form of Lane-Emden equations for relativistic isotropic stars [37], but can only change the coefficients in these equations. Specifically, we will assume that the parameter \( \Delta \) and the metric function \( \nu \) are related by the differential equation

\[
-\frac{4\Delta}{q_0r^n} + (1 + (n + 1)q_0\theta)\,d\nu = (1 + \beta q_0\theta)\,d\nu,
\]

(35)

where \( \beta \) is some real constant. Substituting Eq. (35) into Eq. (17) and integrating it, one can obtain the metric function \( \nu(r) \) in the form

\[
\nu(r) = \ln \frac{1 - \frac{2GM}{R}}{(1 + \beta q_0\theta)^{2(n+1)}}.
\]

(36)

Using Eqs. (8) and (36), and introducing the same dimensionless variable \( \xi \) and dimensionless function \( \eta \) by Eq. (22), the modified Lane-Emden equations, corresponding to the ansatz (35), take the form

\[
\frac{\xi - 2(n + 1)q_0\eta}{1 + \beta q_0\theta} \xi \frac{d\theta}{d\xi} + \eta + q_0\xi^3\theta^{n+1} = 0,
\]

(37)

\[
\frac{d\eta}{d\xi} = \xi^2\theta^n(1 + nq_0\theta)
\]

(38)

with the same boundary conditions (25) and (26). One can see that the obtained Lane-Emden equations formally look as in the isotropic case [37], but with that difference that the impact of the anisotropy parameter is reflected in the coefficient \( \beta \) (substituting the multiplier \( n + 1 \)).

In general case, the Lane-Emden equations (37) and (38) can be integrated only numerically, but the analytical solutions can be found for incompressible anisotropic fluid stars, characterized by the constant density \( \varrho = \text{const} \). At \( n = 0 \), the function \( \eta(\xi) \), with account of the boundary condition \( \eta(0) = 0 \), is given by

\[
\eta(\xi) = \frac{\xi^3}{3}.
\]

(39)
The solution for the function \( \theta(\xi) \) reads

\[
\frac{1 + 3q_0 \theta}{1 + \beta q_0 \theta} = \pm \frac{1}{1 + \beta q_0} \left( 1 - \frac{2q_0}{3} \xi^2 \right)^{\frac{3-\beta}{4}}.
\] (40)

In the last equation, only the branch corresponding to the upper plus sign, satisfies the boundary conditions (25), and the respective solution is given by

\[
\theta(\xi) = \frac{1}{q_0} \frac{(1 + 3q_0)(1 - \frac{2q_0}{3} \xi^2)^{\frac{3-\beta}{4}} - (1 + \beta q_0)}{3(1 + \beta q_0) - \beta (1 + 3q_0)(1 - \frac{2q_0}{3} \xi^2)^{\frac{3-\beta}{4}}}.
\] (41)

The positive root of the \( \theta(\xi) \) reads

\[
\xi_R = \sqrt{\frac{3}{2q_0} \left[ 1 - \left( \frac{1 + \beta q_0}{1 + 3q_0} \right)^{\frac{3-\beta}{3}} \right]}.
\] (42)

It is possible to check that the subradical function is nonnegative at any \( \beta \) satisfying the inequality \( 1 + \beta q_0 > 0 \).

Calculating the function \( u_\theta(\xi) \) in Eq. (34) at \( n = 0 \), the binding energy of incompressible anisotropic fluid stars can be written as

\[
E_B = M^* \left\{ -\frac{3}{4} \xi_R \sqrt{q_0 (1 - \frac{2q_0}{3} \xi_R^2)} + \frac{1}{2} \sqrt{\left( \frac{3}{2} \right)^3 \arcsin \left( \sqrt{\frac{2q_0}{3} \xi_R} - \frac{1}{3} \left( \sqrt{q_0} \xi_R \right)^3 \right)} \right\}.
\] (43)

It can be verified that at \( n = 0 \) the gravitational potential energy is just \( \Omega = -E_B \).

Using the obtained analytical results for incompressible anisotropic fluid stars, the basic quantities of interest can be represented also in the graphical form. From Eq. (35), one can find the anisotropy parameter \( \Delta \) at \( n = 0 \) in terms of the dimensionless variable \( \xi \):

\[
\Delta(\xi) = \frac{p_\nu \xi}{4} (1 - \beta) \theta(\xi) \frac{\partial \nu}{\partial \xi}.
\] (44)
behavior is demonstrated by the transverse pressure $p_t$. The important feature of the model ansatz (35) is that not only $p_r$, but also $p_t$ is positive in the interior of a star, and vanishes at its surface. The positiveness of the radial $p_r$ and transverse $p_t$ pressures in the interior of a star guarantees its mechanical stability. If some of these pressures becomes negative, like the radial pressure $p_r$ in ultrastrong magnetic fields, this leads to the appearance of the corresponding instability; in the case of strong magnetic fields, this is the longitudinal instability developed along the magnetic field direction [22–24, 26–29]. Vanishing of the transverse pressure $p_t$ at the surface of anisotropic fluid stars is also important for their stability [24, 25, 30, 31], although in some studies this is not a required feature [1, 11, 12].

Fig. 2 shows the dimensionless mass $M/M^*$ and dimensionless binding energy $E_B/M^*$ for incompressible anisotropic fluid stars, determined according to Eqs. (27) and (43), respectively, as functions of the parameter $q_0$ for the set of fixed values of the parameter $\beta$. It is seen that both quantities, first, rapidly increase with $q_0$ and then gradually approach their asymptotic values, dependent on the given value of $\beta$. The mass of a star decreases with increasing the parameter $\beta$ at the given $q_0$, and, hence, the pressure anisotropy with $\beta < 1$ ($p_t > p_r$) leads to the increase of the mass of a star compared to that in the isotropic case. The binding energy stays always positive, as required by the stability of a star (although not all stars with $E_B > 0$ are stable, as will be shown in the next section). The gravitational potential energy at $n = 0$ differs only by sign from the binding energy, $\Omega/M^* = -E_B/M^*$, and is always negative.

Fig. 3 shows the mass–radius relation at $n = 0$ for two values of the central mass density, $\varrho_0 = 10^{18}$ kg/m$^3$ and $\varrho_0 = 2 \times 10^{18}$ kg/m$^3$, obtained by varying the parameter $q_0$ at fixed values of the parameter $\beta$. For the specific central mass density, the masses and radii, corresponding to different fixed $q_0$ and $\beta$, are different. Nevertheless, when $q_0$ varies at the given $\beta$, the current point moves along almost the same curve for this specific central density independently of $\beta$, the difference being only in the limiting masses for different $\beta$. Note that the maximum mass for incompressible anisotropic fluid stars with the polytropic EoS cannot be reached at finite $q_0$, but only asymptotically at $q_0 \to \infty$. This is in contrast to the common behavior when a mass-radius curve reaches a maximum value and then decreases (cf. the mass-radius curves, for example, in Refs. [49, 50] for strange quark stars). This result does not rely on any numerical procedure, but just is the reflection in the graphical form of the mass-radius relation (29) with the analytically found solutions of the modified Lane-
Emden equations (37), (38) at \( n = 0 \), where the quantities \( \eta(\xi), \xi_R \) are given by Eqs. (39), (42).

The recent discovery of two heavy neutron stars PSR J1614-2230 [51] and PSR J0348+0432 [52] with the masses \( M \sim 2M_\odot \) (\( M_\odot \approx 1.989 \times 10^{30} \) kg being the solar mass) set the corresponding constraint, to which any reasonable EoS should satisfy. It is seen from Fig. 3 that, despite its simplicity, the polytropic EoS for incompressible anisotropic fluid stars can fulfill this constraint even without reaching the maximum masses.

V. DYNAMICAL STABILITY OF INCOMPRESSIBLE ANISOTROPIC FLUID STARS

Let us consider the stability of spherically symmetric anisotropic stars with respect to radial oscillations, assuming that they do not violate the spherical symmetry. In the spherically symmetric case, allowing for the motions in the radial direction, Einstein equations (3) read

\[
e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi GT_0^0, \tag{45}
\]
\[
-e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi GT_1^1, \tag{46}
\]
\[
-\frac{1}{2}e^{-\lambda}\left(\nu'' - \frac{1}{2}\nu'\lambda' + \frac{1}{2}\nu'^2 + \frac{\nu' - \lambda'}{r}\right) + \frac{1}{2}e^{-\nu}\left(\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})\right) = 8\pi GT_2^2 = 8\pi GT_3^3, \tag{47}
\]
\[
-\frac{e^{-\lambda}}{r}\dot{\lambda} = 8\pi GT_0^1, \tag{48}
\]

where dot means the differentiation with respect to \( t \). The radial component of the equation \( T_{i;k}^k = 0 \), expressing the vanishing of the divergence of the energy-momentum tensor, can be written as

\[
\dot{T}_0^0 + T_1^1 + \frac{1}{2}T_1^0(\dot{\nu} + \dot{\lambda}) + \frac{\nu'}{2}(T_1^1 - T_0^0) + \frac{2}{r}(T_1^1 - T_2^2) = 0. \tag{49}
\]

The energy–momentum tensor for a spherically symmetric anisotropic star reads

\[
T_i^k = (\varepsilon + p_t)u_iu_k - p_t\delta_i^k + (p_r - p_t)s_is_k, \tag{50}
\]

where \( u^i = \frac{dx^i}{ds} \) is the fluid four-velocity, and \( s^i \) is the unit spacelike vector with the properties

\[
s^iu_i = 0, \ s^is_i = -1. \tag{51}
\]
For the motions in the radial direction, the four-vectors \( u^i \) and \( s^i \) have the structure

\[
\begin{align*}
  u^i &= \left( \frac{e^{-\frac{\nu}{2}}}{\sqrt{1 - v^2 e^{\lambda - \nu}}}, \frac{ve^{-\frac{\nu}{2}}}{\sqrt{1 - v^2 e^{\lambda - \nu}}}, 0, 0 \right), \\
  s^i &= \left( \frac{ve^{\frac{\lambda}{2} - \nu}}{\sqrt{1 - v^2 e^{\lambda - \nu}}}, \frac{e^{-\frac{\lambda}{2}}}{\sqrt{1 - v^2 e^{\lambda - \nu}}}, 0, 0 \right),
\end{align*}
\]

where \( v = \frac{dv}{dt} \) is the velocity in the radial direction. In the static limit, Eq. (50) for the energy-momentum tensor goes over to Eq. (2).

Further we will study the small radial oscillations and will represent the unknown quantities as

\[
\begin{align*}
  \varepsilon &= \varepsilon^0 + \delta\varepsilon, \quad p_r = p^0_r + \delta p_r, \quad p_t = p^0_t + \delta p_t, \\
  \nu &= \nu^0 + \delta\nu, \quad \lambda = \lambda^0 + \delta\lambda,
\end{align*}
\]

where \( \delta\varepsilon, \delta p_r, \delta p_t, \delta\nu, \delta\lambda \) are small perturbations with respect to the corresponding values at the state of hydrostatic equilibrium, denoted by the upper index ”0”. Also, for the small radial oscillations we will consider that \( \nu \ll 1 \). In the linear approximation on the small perturbations, for the nonzero components of the energy–momentum tensor we have

\[
\begin{align*}
  T^0_0 &= \varepsilon, \quad T^1_1 = -p_r, \quad T^2_2 = T^3_3 = -p_t, \\
  T^1_0 &= (\varepsilon^0 + p^0_r)v, \quad T^0_1 = -e^{-\lambda^0 - \nu^0}T^1_0,
\end{align*}
\]

where quantities \( \varepsilon, p_r \) and \( p_t \) are given by Eq. (54). With account of Eq. (56), Eqs. (45) and (46) can be written in the linearized form:

\[
\begin{align*}
  \frac{\partial}{\partial r} \left( r e^{-\lambda^0} \delta\lambda \right) &= 8\pi G r^2 \delta\varepsilon, \\
  \frac{1}{r} e^{-\lambda^0} \left( \delta\nu' - \nu^0 \delta\lambda \right) &= \frac{1}{r^2} e^{-\lambda^0} \delta\lambda + 8\pi G \delta p_r.
\end{align*}
\]

The linearized form of Eq. (48) is

\[
-\frac{e^{-\lambda^0}}{r} \delta\lambda = 8\pi G (\varepsilon^0 + p^0_r) v.
\]

In the linear approximation, Eq. (49) reads

\[
- e^{\lambda^0 - \nu^0} \left( \varepsilon^0 + p^0_r \right) \dot{v} - \delta p'_r - \frac{1}{2} \left( \varepsilon^0 + p^0_r \right) \delta\nu' - \frac{1}{2} (\delta\varepsilon + \delta p_r) \nu^0 + \frac{2}{r} (\delta p_r + \delta p_t) = 0.
\]
Following Ref. [47], it is convenient to introduce a "Lagrange displacement" \( \psi \) by the equation \( v = \dot{\psi} \). Then integration of Eq. (59) gives

\[
\frac{1}{r} e^{-\lambda_0} \delta\lambda = -8\pi G (\varepsilon^0 + p_r^0) \psi. \tag{61}
\]

Note that in the state of hydrostatic equilibrium, as a consequence of Eqs. (4) and (5), the following relationship holds true

\[
\frac{1}{r} e^{-\lambda_0} (\nu^0 + \lambda^0) = 8\pi G (\varepsilon^0 + p_r^0). \tag{62}
\]

With account of the last equation, Eq. (61) becomes

\[
\delta\lambda = -\psi (\nu^0 + \lambda^0). \tag{63}
\]

Taking into account Eq. (61), Eq. (57) reads

\[
\delta\varepsilon = -\psi \varepsilon^0 - \psi p_r^0 - (\varepsilon^0 + p_r^0) \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \psi). \tag{64}
\]

In view of the condition of hydrostatic equilibrium (7), the last equation can be written as

\[
\delta\varepsilon = -\psi \varepsilon^0 - (\varepsilon^0 + p_r^0) \frac{\varepsilon^0}{r^2} \frac{\partial}{\partial r} (r^2 \psi) - \frac{2\psi}{r} (p_t - p_r^0). \tag{65}
\]

Next, considering Eq. (58), with account of Eqs. (61) and (62), we get

\[
(\varepsilon^0 + p_r^0) \delta\nu = \left( \delta p_r - (\varepsilon^0 + p_r^0) (\nu^0 + \frac{1}{r}) \psi \right) (\nu^0 + \lambda^0). \tag{66}
\]

Now let us assume that all perturbations contain the dependence on time only through the exponential factor \( e^{i\omega t} \), where \( \omega \) is the frequency of radial oscillations. Then, with account of Eq. (66), Eq. (60) can be written as

\[
\omega^2 e^{\lambda^0 - \varepsilon^0} (\varepsilon^0 + p_r^0) \psi = \delta p_r + \left( \nu^0 + \frac{\lambda^0}{2} \right) \delta p_r
\]

\[
+ \frac{\nu^0}{2} \delta\varepsilon - \frac{1}{2} (\varepsilon^0 + p_r^0) \left( \nu^0 + \frac{1}{r} \right) (\nu^0 + \lambda^0) \psi + \frac{2}{r} (\delta p_r - \delta p_t),
\]

where all variations now stand for the amplitudes of the corresponding quantities with the time dependence given by the exponent \( e^{i\omega t} \). Note that the variation \( \delta\varepsilon \) is expressed through the Lagrange displacement \( \psi \) by Eq. (64). In order to relate the perturbation of the radial pressure \( \delta p_r \) to \( \psi \), we will assume, following Ref. [47], the conservation of the total baryon number. The corresponding continuity equation in general relativity reads

\[
\frac{\partial}{\partial x^k} (n_b u^k) + n_b u^k \frac{\partial \ln \sqrt{-g}}{\partial x^k} = 0, \tag{68}
\]
where \( n_b \) is the baryon number density, \( g \) is the determinant of the metric tensor, \( g \equiv \det |g_{ik}| = -\exp(\nu + \lambda)r^4\sin^2 \theta \). Taking into account Eq. (52), the last equation in the linear approximation on the small perturbations takes the form

\[
e^{-\frac{\psi}{r}} \delta n_b + \frac{1}{2} \frac{\partial}{\partial r} \left( n_b^0 e^{-\frac{\psi}{r}} v r^2 \right) + \frac{1}{2} n_b^0 e^{-\frac{\psi}{r}} \delta \lambda + \frac{1}{2} n_b^0 e^{-\frac{\psi}{r}} v (\nu^0 + \lambda^0) = 0, \tag{69}\]

where \( n_b^0(r) \) is the baryon number density in the state of hydrostatic equilibrium. Since \( v = \dot{\psi} \), this equation can be integrated and one gets

\[
\delta n_b + e^{-\frac{\psi}{r}} \frac{\partial}{\partial r} \left( n_b^0 e^{-\frac{\psi}{r}} r^2 \dot{\psi} \right) + \frac{1}{2} n_b^0 (\delta \lambda + \psi (\nu^0 + \lambda^0)) = 0. \tag{70}\]

In view of Eq. (63), the last term in Eq. (70) vanishes and one obtains

\[
\delta n_b = -e^{-\frac{\psi}{r}} \frac{\partial}{\partial r} \left( n_b^0 e^{-\frac{\psi}{r}} r^2 \dot{\psi} \right). \tag{71}\]

Let the EoS of the system have the following general structure: \( n_b = n_b(\varepsilon, \rho_r) \). Then it follows from Eqs. (65) and (71) that in the linear approximation

\[
\delta \rho_r = -\rho_r^0 \dot{\psi} - \gamma \rho_r^0 e^{-\frac{\psi}{r}} \frac{\partial}{\partial r} \left( e^{-\frac{\psi}{r}} r^2 \dot{\psi} \right) - 2 \frac{\partial \rho_r}{\partial \varepsilon} (\rho_r^0 - \rho_r^0) \dot{\psi}, \tag{72}\]

where the adiabatic coefficient \( \gamma \) is determined by

\[
\gamma = \frac{1}{\rho_r \frac{\partial \rho_r}{\partial \varepsilon}} \left( n_b - (\varepsilon + \rho_r) \frac{\partial m_b}{\partial \varepsilon} \right), \tag{73}\]

and, analogously to Ref. [47], is considered to be a constant for the matter inside a star. For the polytropic EoS (14), \( \frac{\partial \rho_r}{\partial \varepsilon} = \gamma - 1 \). Substituting expressions (65) and (72) for \( \delta \varepsilon \) and \( \delta \rho_r \) to Eq. (64), and using the field equation (6) and equation (7) of hydrostatic equilibrium, in the case of the polytropic EoS one gets

\[
\omega^2 e^{\lambda^0 - \nu^0} (\varepsilon^0 + p_r^0) \psi = \frac{2\psi}{r} p_r^0 \omega - \frac{2\psi}{r} (\gamma (\nu^0 + \frac{\lambda^0}{2} + \frac{1}{2}) \rho_r^0 - \rho_r^0) \tag{74}\]

\[
+ 8\pi Ge^{\lambda^0} p_t^0 (\varepsilon^0 + p_r^0) \psi - \gamma \frac{d}{dr} \left( \frac{2}{\varepsilon^0 + p_t^0} (\rho_r^0 - \rho_t^0) \psi \right) - \frac{\psi}{\varepsilon^0 + p_t^0} \left( \frac{2}{\varepsilon^0 + p_r^0} (\rho_r^0 - \rho_t^0) \right)^2 \]

\[
- \gamma e^{-(\nu^0 + \frac{\lambda^0}{2})} \frac{d}{dr} \left( e^{-\frac{\psi}{r}} \frac{p_r^0}{r^2} \frac{d}{dr} (r^2 e^{-\frac{\psi}{r}} \dot{\psi}) \right) - \frac{2}{r} \left( \gamma p_r^0 \frac{e^{\frac{\psi}{r}}}{r^2} \frac{d}{dr} (r^2 e^{-\frac{\psi}{r}} \dot{\psi}) + \delta \rho_t \right). \]

Solutions of Eq. (74) for the frequencies of radial oscillations should be sought under the boundary conditions

\[
\psi(r = 0) = 0, \quad \delta \rho_t(r = R) = 0. \tag{75}\]
In order to get the variational basis for finding the frequencies ω, let us multiply both parts of Eq. (74) on \( r^2 \psi \exp \left( \frac{\lambda^2 + \lambda''}{2} \right) \) and integrate over the range of \( r \). We will write the corresponding equation already for incompressible fluid stars (\( n = 0 \)), when the polytropic exponent \( \gamma \to \infty \). Omitting the upper indices zero as no longer necessary, one gets

\[
\omega^2 \int_0^R e^{\frac{\lambda^2 + \lambda''}{2} (\varepsilon + p_r) r^2 \psi^2} \, dr = \gamma \int_0^R e^{\frac{\lambda^2 + \lambda''}{2} p_r r^2} \left( \frac{d}{dr} \left( r^2 e^{-\frac{\lambda}{2} \psi} \right) \right)^2 \, dr = \gamma \int_0^R e^{\frac{\lambda^2 + \lambda''}{2} p_r r^2} \left( \frac{d}{dr} \left( r^2 e^{-\frac{\lambda}{2} \psi} \right) \right)^2 \, dr 
\]

In the variational equation (76), the Lagrange displacement \( \psi \) should be chosen such that \( \omega^2 \) is minimized. If all frequencies of radial oscillations are real, a spherically symmetric anisotropic star is dynamically stable; if some frequency appears to be imaginary, the configuration is unstable. A sufficient condition for the occurrence of the dynamical instability is vanishing of the right–hand side of Eq. (76) for some trial form of the Lagrange displacement \( \psi \) satisfying the boundary conditions. Let us introduce, following Ref. [37], the auxiliary function \( \chi = e^{-\frac{\lambda}{2} \psi} \). After changing the integration variable in Eq. (76) according to Eq. (22), substituting \( p_r = q_0 \theta_0 \), \( \varepsilon = \theta_0 \), and using Eq. (44) for the anisotropy parameter \( \Delta = p_t - p_r \) and expressions for the metric functions at \( n = 0 \):

\[
e^{-\lambda} = 1 - \frac{2q_0 \eta(\xi)}{\xi} = 1 - \frac{2q_0 \xi^2}{3}, \quad e^\nu = \frac{1 - \frac{2GM}{R}}{(1 + \beta q_0 \theta)^\frac{\gamma}{2}}, \quad (77)
\]

Eq. (76) takes the form

\[
\frac{\omega^2}{\omega_0^2} \frac{1}{1 - \frac{2GM}{R}} \int_0^{\xi_R} \frac{(1 + q_0 \theta) \xi^2 \chi^2}{(1 - \frac{2q_0 \xi^2}{3})^\frac{\gamma}{2} (1 + \beta q_0 \theta)^\frac{\gamma}{2}} \frac{d\xi}{\xi^2} = \gamma \int_0^{\xi_R} \frac{\theta \left( \frac{d}{d\xi} (\xi^2 \chi) \right)^2}{\xi^2 (1 - \frac{2q_0 \xi^2}{3})^\frac{\gamma}{2} (1 + \beta q_0 \theta)^\frac{\gamma}{2}} \frac{d\xi}{\xi^2} - \frac{\gamma (1 - \beta)}{2} \int_0^{\xi_R} \frac{\xi^2 \chi^2 \theta (\nu' + \frac{\lambda'}{2} + \frac{2}{\xi})}{(1 - \frac{2q_0 \xi^2}{3})^\frac{\gamma}{2} (1 + \beta q_0 \theta)^\frac{\gamma}{2}} \frac{d\xi}{\xi^2} \chi_1 = e^{-\frac{\lambda}{2} \xi}, \quad \chi_2 = \sqrt{\xi}. \quad (79)
\]
Then for each given $\beta$ we will try to find such $q_{0c}$ at which the right-hand side of Eq. (78) vanishes, and, hence, the dynamical instability for an incompressible anisotropic fluid star occurs at $q_0 > q_{0c}$.

The results of calculations are presented in Table I. The most important conclusion is that there are solutions for $q_{0c}$ in the case of the trial function $\chi_1$ at $\beta < 1$, i.e., for $\Delta = p_t - p_r > 0$ (and there are no solutions at $\beta > 1$). This means that the local pressure anisotropy with $p_t > p_r$ can affect the dynamical stability of spherically symmetric incompressible fluid stars, the result which is in contrast to the conclusion for incompressible isotropic fluid stars with the polytropic EoS (13) in Ref. [37], which are stable against radial oscillations.

It is seen also that the choice of the trial function does matter: the use of $\chi_1$ allows us to find the critical value $q_{0c}$ for $\beta < 1$ while the right-hand side of Eq. (78) does not vanish for the trial function $\chi_2$ at any $\beta$. Fig. 4 shows the behavior of these trial functions at $\beta = 0.5, q_0 = 3$. While the derivative $\chi_2'(\xi)$ is always positive, the derivative $\chi_1'(\xi)$ changes sign in the interval $[0, \xi_R]$, and, hence, the subintegral functions containing $\chi_1'(\xi)$ contribute qualitatively differently to the respective integrals for $\chi_1(\xi)$ and $\chi_2(\xi)$.

If dynamical instability occurs at $\beta < 1$, the question naturally arises: is the mass of an incompressible anisotropic fluid star still compatible with the two–solar–mass constraint at the moment of the appearance of dynamical instability at the critical value $q_{0c}$? Table II shows the values of the mass of an incompressible anisotropic fluid star at the critical value $q_{0c}$ in the case of the trial function $\chi_1(\xi)$. It is seen that for both values of the central mass density $\rho_0$, used in the calculations, the two–solar–mass constraint is still satisfied at the moment of the appearance of dynamical instability.

In summary, we have considered spherically symmetric relativistic stars with the polytropic equation of state, which possess the local pressure anisotropy, within the framework of general relativity. The generalized Lane-Emden equations have been derived for the case of the arbitrary anisotropy parameter $\Delta = p_t - p_r$. In this research, in order to study the effects of the pressure anisotropy, we follow a phenomenological approach, in which the anisotropy parameter is set with the help of some phenomenological ansatz. The conclusions obtained within this approach are rather of a general character and independent of the details of a specific physical mechanism. Specifically, the generalized Lane-Emden equations are applied to the special ansatz (35) for the anisotropy parameter $\Delta$ in the form of the differential relation.
between $\Delta$ and the metric function $\nu$. The analytical solutions of the obtained equations have been found for incompressible fluid stars and then used for getting their mass-radius relation, gravitational and binding energy. It has been clarified that the pressure anisotropy with $p_t > p_r$ leads to the increase of the mass of a star compared to that in the isotropic case, and this factor can be helpful in explaining the observational data of heavy compact stars with the mass $M \sim 2M_\odot$. Also, following the Chandrasekhar variational approach [47], the dynamical stability of incompressible anisotropic fluid stars with the polytropic EoS against radial oscillations has been studied. It has been shown that the local pressure anisotropy with $p_t > p_r$ can make the incompressible fluid stars unstable with respect to radial oscillations, in contrast to incompressible isotropic fluid stars with the polytropic EoS which are dynamically stable [37]. As shown in the model calculations of the given work, if dynamical instability occurs, the mass of an incompressible anisotropic fluid star at the moment of the appearance of instability is still compatible with the two–solar–mass constraint.

Note that in the interior of magnetars – strongly magnetized neutron or quark stars, magnetic fields of about $10^{18}$ G, or even larger, can potentially occur. Such strong magnetic fields can produce the substantial pressure anisotropy with $p_t > p_r$ [22,34], and, hence, can cause the dynamical instability of a magnetized stellar object.

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FIG. 1. The anisotropy parameter $\Delta$ (upper row), the radial $p_r$ (middle row) and transverse $p_t$ (bottom row) pressures at $n = 0$, normalized to the central radial pressure $p_{r0}$, as functions of the dimensionless variable $\xi$ for a set of fixed values of $\beta$ and $q_0$. 
FIG. 2. The dimensionless mass $M/M^*$ and dimensionless binding energy $E_B/M^*$ at $n = 0$ as functions of the parameter $q_0$ for the same set of fixed values of the parameter $\beta$ as in Fig. 1.
FIG. 3. (Color online) The mass–radius relation at $n = 0$ for two values of the central mass density, $\rho_0 = 10^{18}$ kg/m$^3$ and $\rho_0 = 2 \times 10^{18}$ kg/m$^3$, with the variable parameter $q_0$ and fixed values of the parameter $\beta$: $\beta = 0.5$ (black curves), $\beta = 2$ (blue curves) and $\beta = 4$ (green curves). The limiting masses for each configuration are shown by full dots.
FIG. 4. The trial functions $\chi_1 = e^{-\frac{1}{2} \xi}$ and $\chi_2 = \sqrt{\xi}$ at $\beta = 0.5$, $q_0 = 3$. 
TABLE I. The critical values of the parameter $q_0$ for the appearance of the dynamical instability of an incompressible anisotropic fluid star at different values of the parameter $\beta$ and two types of the trial functions used in the calculations.

| $\beta$ | $q_{0c}$ evaluated with the trial function | $\chi_1 = e^{-\frac{2}{7}\xi}$ | $\chi_2 = \sqrt{\xi}$ |
|---------|-------------------------------------------|--------------------------------|------------------------|
| 0.1     | 1.391                                     | -                             |                        |
| 0.3     | 1.796                                     | -                             |                        |
| 0.5     | 2.526                                     | -                             |                        |
| 0.7     | 4.210                                     | -                             |                        |
| 0.9     | 11.646                                    | -                             |                        |
TABLE II. The mass $M_c$ of an incompressible anisotropic fluid star at the critical value $q_{0c}$ for the appearance of dynamical instability in the case of the trial function $\chi_1(\xi)$.

| $\rho_0$, kg/m$^3$ | $M_c$ evaluated at the critical value $q_{0c}$, kg |
|---------------------|-----------------------------------------------|
|                     | $\beta = 0.1$ | $\beta = 0.3$ | $\beta = 0.5$ | $\beta = 0.7$ | $\beta = 0.9$ |
| $10^{18}$           | $6.999 \times 10^{30}$ | $7.030 \times 10^{30}$ | $7.064 \times 10^{30}$ | $7.097 \times 10^{30}$ | $7.122 \times 10^{30}$ |
| $2 \times 10^{18}$  | $4.949 \times 10^{30}$ | $4.971 \times 10^{30}$ | $4.995 \times 10^{30}$ | $5.018 \times 10^{30}$ | $5.036 \times 10^{30}$ |