ERGODIC FREQUENCY MEASURES FOR RANDOM SUBSTITUTIONS

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Abstract. We construct a family of ergodic measures on random substitution subshifts (RS-subshifts) associated to a primitive random substitution. In particular, the word frequencies of every finite legal word exist for almost every element of the random substitution subshift with respect to these measures. As an application, we show that for a certain class of random substitutions the measures of maximal entropy are frequency measures.

1. Introduction

Deterministic substitutions and the corresponding dynamical systems are well-studied objects in symbolic dynamics, which give rise to applications in other branches of mathematics and science [2, 23, 24]. Early efforts to understand the properties of randomised versions of these substitutions were made in a pioneering work by Peyrière [22]. Later, the concept of random substitutions also evoke interest in the physics community [7, 11]. Over the decades, random substitutions have appeared in various disguises and contexts, including $0L$-systems in formal language theory [25] and expansion-modification systems [14] that where used to model long-term correlation decay in DNA sequences [28, 16]. However, a systematic study of random substitutions is still in its infancy. Some basic topological properties were studied in [27], a study of periodic points associated to random substitution subshifts was undertaken in [26]. When passing to the associated subshift, it is desirable to construct measures that keep the structure of the underlying stochastic process meaningful, stressing the point of view that it can be considered as some limiting object of the iterated random substitution. Recently, for a certain class of random substitutions, equilibrium measures have been constructed that are invariant under the substitution action [15]. In this work, we are concerned with measures which are shift-invariant, a property which is usually taken as a starting point for assigning objects like diffraction measures or the spectrum of a Schrödinger operator to the subshift. In diffraction theory, random substitutions are used to generalise results from the deterministic setting [4, 31]. For instance, it is a well-known fact that a point set which comes from certain primitive substitutions gives rise to a pure point diffraction measure, and, consequently, to a pure point dynamical spectrum [3]. In contrast, in the case of randomised versions of these substitutions, one may obtain a mixed diffraction spectrum and a mixed dynamical spectrum, consisting of a pure point part and an absolutely continuous part [18, 4]. The ergodicity of the associated measures plays a vital role to ensure that this diffraction spectrum is the same

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for almost every realisation. Similarly, the generic choice of a Schrödinger spectrum as an almost-surely constant object, relies on the existence of a generic measure \( \mu \). Therefore, ergodicity is a desirable property of measures that arise from random substitutions.

We are going to construct these measures in the following way. It is a well-known fact that, if \( \rho \) is a primitive substitution, its hull \( X_\rho \) is strictly ergodic under the \( \mathbb{Z} \)-action of the shift [2, Thm. 4.3]. Moreover, if \( x \in X_\rho \) and \( v \) is a subword of \( x \), then \( \mu([v]) \) is given by the word frequency of \( v \) in \( x \) (which exits uniformly due to Oxtoby’s theorem), where \( \mu \) is the unique shift invariant and ergodic measure on \( X_\rho \) and \( [v] \) is the cylinder set which corresponds to \( v \). We would like to transfer this idea to random substitutions. Here, we are no longer in the strictly ergodic setting. Instead, we choose a measure \( \mu \) that is compatible with the word frequencies that arise almost surely in the limit of large inflation words. We establish the ergodicity of \( \mu \) by showing that the word frequencies of all legal words exist and coincide for \( \mu \)-almost all possible realisations. The ergodicity of the frequency measures arising from a certain class of random substitutions was previously claimed in [19, Thm. 4.22]. The proof, however, contains a small gap, since it relies on the independence of certain word-valued random elements, which is only asymptotically true and thus requires some extra work.

The paper is organised as follows. In Section 2, we introduce random substitutions and describe them as Markov processes. Here, we follow the approach presented in [12], which was also implicitly used in [19]. The corresponding probability measure is used in Section 3 to establish well-defined word frequencies in expectation. In Section 4, we generalise this result to the almost-sure existence of word frequencies. From these, we construct a shift-invariant measure on the stochastic hull of the random substitution in Section 5, and show that it is ergodic, comprising the main result of this paper. In the last section, we compute the metric entropy of a class of frequency measures for an explicit example. Moreover, we show that for a certain class of random substitutions one of the frequency measures is indeed the measure of maximal entropy.

2. Random Substitutions

The definition of a random substitution, presented in this section, basically follows the notion of a substitution Markov chain, presented in [12] and going back to [22]. However, our notation is slightly different to parallel the one used for deterministic substitutions [2]. It is mainly in accordance with [19] and [27].

Given a finite alphabet \( \mathcal{A} \), a word in \( \mathcal{A} \) is any finite concatenation of symbols in \( \mathcal{A} \). The length of a word \( u \) is denoted by \( |u| \). For a given word \( u = u_1 \cdots u_n \) and two positions \( k, m \in \mathbb{N} \) with \( 1 \leq k \leq m \leq n \), we define \( u_{[k,m]} = u_k \cdots u_m \). The number of occurrences of a word \( v = v_1 \cdots v_k \) within a word \( u = u_1 \cdots u_n \) is given by \( |u|_v = \text{card}\{0 \leq i \leq n-k : u_{[i+1,i+k]} = v\} \), provided that \( k \leq n \) and 0 otherwise. If \( |u|_v \geq 1 \), we say that \( v \) is a subword of \( u \) and we write \( v \preceq u \) in that situation. Assuming any (possibly lexicographic) order on \( \mathcal{A} = \{a_1, \ldots, a_m\} \), the Abelianisation of a word \( u \) is the \( m \)-dimensional vector \( \Phi(u) \), defined via \( \Phi(u)_i = |u|_{a_i} \). We write \( \mathcal{A}^+ \) for the set of finite words in \( \mathcal{A} \) and equip \( \mathcal{A}^+ \) with the discrete topology. Thus, \( \mathcal{B} = \mathcal{P}(\mathcal{A}^+) \), with \( \mathcal{P}(\mathcal{A}^+) \) the power set of \( \mathcal{A}^+ \), gives the \( \sigma \)-algebra of Borel sets. Similarly, \( \mathcal{A}^\mathbb{Z} \)
denotes the set of all bi-infinite sequences in \( A \) and we set \( x_{[k,m]} = x_k \cdots x_m \) for all \( x \in A^\mathbb{Z} \) and \( k \leq m \in \mathbb{Z} \). The shift-action on \( A^\mathbb{Z} \) is defined via \( (Sx)_i = x_{i+1} \) for all \( x \in A^\mathbb{Z} \) and \( i \in \mathbb{Z} \).

**Definition 2.1.** Let \( A \) be a finite alphabet and \( \{(\Omega_a, \mathbb{P}_a)\}_{a \in A} \) a collection of finite probability spaces. Further, let \( X_a \) be the set of random elements \( \chi: \Omega_a \to A^+ \) and \( \chi = \bigcup_{a \in A} X_a \). A random substitution on the alphabet \( A \) is a map \( \vartheta: A \to X \), \( a \mapsto \vartheta(a) \in X_a \).

We can represent \( \vartheta \) in the more 'classical' way

\[
\vartheta: a_i \mapsto \begin{cases} 
  u_{i,1}, & \text{with probability } p_{i,1}, \\
  \vdots \\
  u_{i,k_i}, & \text{with probability } p_{i,k_i},
\end{cases}
\]

where \( p_i = (p_{i,1}, \ldots, p_{i,k_i}) \) are probability vectors. To avoid trivialities, we always assume that \( p_{i,j} > 0 \) for every \( i \) and \( j \). The connection to the probabilities in the above formalism is given by \( p_{i,j} = \mathbb{P}_a[\vartheta(a_i) = \omega_{ij}] \). In this case, we can choose \( \Omega_a = \{1, \ldots, k_i\} \), numbering the possible realisations of \( \vartheta \) on \( a_i \).

Next, we can extend the action of \( \vartheta \) to finite words in a similar spirit as for a deterministic substitution. For \( v \in A^+ \), \( v = v_1 \cdots v_n \) with \( v_i \in A \), we set \( \Omega_v = \times_{i=1}^{m} \Omega_{v_i} \) and \( \mathbb{P}_v = \bigotimes_{i=1}^{m} \mathbb{P}_{v_i} \), giving a new probability space \( (\Omega_v, \mathbb{P}_v) \). Then, \( \vartheta(v): \Omega_v \mapsto A^+ \) is a random element, defined by \( \vartheta(v)(\omega_1, \ldots, \omega_m) = \vartheta(v_1)(\omega_1) \cdots \vartheta(v_m)(\omega_m) \) with the usual concatenation of words. This ensures that \( \vartheta \) acts independently on every letter of \( v \).

It is worth mentioning that the independence assumption might be relaxed leading to a more general concept of a random substitution, coined \( M \)-system in [22].

**Definition 2.2.** Let \( \vartheta \) be a random substitution with an associated stationary transition kernel, defined by \( \mathbb{P}(v,u) = \mathbb{P}_v[\vartheta(v) = u] \), for every \( v, u \in A^+ \). Let \( \{X_n\}_{n \in \mathbb{N}} \) be the Markov process induced by this kernel, starting from some initial distribution \( X_0 \) of words. Then, we define the action of \( \vartheta^n \) on \( X_0 \) as \( \vartheta^n(X_0) = X_n \), for all \( n \in \mathbb{N} \). For any \( a \in A \), we call \( \vartheta^n(a) \) a (random) inflation word of level \( n \).

We may thus interpret \( \vartheta \) as a map between word-valued random elements that is determined by its transition kernel. Let us assume for a moment that the initial distribution is concentrated on some word \( u = u_1 \ldots u_k \in A^+ \). Note that \( \vartheta^{m+n}(u) \) and \( \vartheta^m(\vartheta^n(u)) \) are distributed alike for any \( n, m \in \mathbb{N} \) by the Markov property of the process. An underlying probability space \( (\Omega, \mathbb{P}) \) for the Markov process \( \{\vartheta^n(u)\}_{n \in \mathbb{N}} \) is taken for granted. Operations on words like taking the length or considering subwords extend to word-valued random elements with obvious meaning. For example, \( \vartheta(u)_1 \) denotes the \( A \)-valued random element defined via \( \vartheta(u)_1(\omega) = (\vartheta(u)(\omega))_1 \). Similarly, \( \Phi(\vartheta(u)) \) becomes a random variable as a concatenation of the Abelianisation \( \Phi \) with the random element \( \vartheta(u) \).

We assume that \( (\Omega, \mathbb{P}) \) is rich enough in structure that also random elements like \( \vartheta(u_m) \) or \( \vartheta(\vartheta(u)_1) \) are well-defined, which do not have an interpretation as functions on the Markov process but were explicitly used for its construction.
Example 2.3. The random Fibonacci substitution is defined via
\[ \vartheta : \quad b \mapsto a, \quad a \mapsto \begin{cases} ab, & \text{with probability } p_1, \\ ba, & \text{with probability } p_2. \end{cases} \]

The distribution of \( \vartheta^2(b) = \vartheta(\vartheta(b)) = \vartheta(a) \) is immediate. The probability that \( \vartheta^2(a) = aba \) can be determined to be
\[
\mathbb{P}[\vartheta^2(a) = aba] = \mathbb{P}_a[\vartheta(a) = ab] \mathbb{P}_{ab}[\vartheta(b) = aba] + \mathbb{P}_a[\vartheta(a) = ba] \mathbb{P}_{ba}[\vartheta(b) \vartheta(a) = aba]
= p_1^2 + p_2^2.
\]

Note that there are two possible paths \( a \mapsto ab \mapsto aba \) and \( a \mapsto ba \mapsto aba \), contributing to this event. The full distribution of \( \vartheta^2(a) \) is given by
\[
\vartheta^2(a) = \begin{cases} aab, & \text{with probability } p_2p_1, \\ aba, & \text{with probability } p_1^2 + p_2^2, \\ baa, & \text{with probability } p_1p_2. \end{cases}
\]

Example 2.4. Consider the random substitution
\[ \vartheta : \quad a \mapsto \begin{cases} b, & \text{with probability } p_1, \\ ba, & \text{with probability } p_2, \end{cases} \quad b \mapsto \begin{cases} b, & \text{with probability } q_1, \\ ab, & \text{with probability } q_2. \end{cases} \]

Then, the event \( \vartheta(ab) = bab \) can be realised either via \( \vartheta(a) = ba, \vartheta(b) = b \) or via \( \vartheta(a) = b, \vartheta(b) = ab \). Thus, although the path \( ab \mapsto bab \) is identical, there are two ways to realise it. For the transition kernel \( P \), we therefore find \( P(ab, bab) = p_2q_1 + p_1q_2 \).

In the same spirit as for deterministic substitutions, we can define a substitution matrix which captures some information about the Abelianisation of the random substitution process.

Definition 2.5. The mean substitution matrix \( M_{\vartheta} \) of a random substitution \( \vartheta \) is defined via
\[
(M_{\vartheta})_{ij} = \mathbb{E}[\vartheta(a_j)]_{a_i} = \mathbb{E}[\Phi(\vartheta(a_j))_{a_i}].
\]

A random substitution \( \vartheta \) is called primitive if \( M_{\vartheta} \) is a primitive matrix. In this case, it is called expanding if \( \mathbb{P}_a[|\vartheta(a)| > 1] > 0 \) for some \( a \in \mathcal{A} \). For a primitive random substitution, we denote by \( \lambda \) the Perron–Frobenius (PF) eigenvalue of \( M_{\vartheta} \) and by \( R \) and \( L \) the corresponding right and left eigenvectors, respectively. The normalisation of these vectors is chosen such that \( \|R\|_1 = 1 \) and \( L \cdot R = 1 \).

Remark 2.6. Unlike in the deterministic setting, a primitive random substitution need not be expanding. An easy counterexample is given by
\[ \vartheta : \quad a, b \mapsto \begin{cases} a, & \text{with probability } p_1, \\ b, & \text{with probability } p_2, \end{cases} \quad M_{\vartheta} = \begin{pmatrix} p_1 & p_1 \\ p_2 & p_2 \end{pmatrix}. \]

We can construct from a random substitution the set of legal words \( \mathcal{L}_{\vartheta} \) and the symbolic hull \( \mathbb{X}_{\vartheta} \), much as in the deterministic case, compare [19], [27].

Definition 2.7. The language of a random substitution \( \vartheta \) on \( \mathcal{A} \) is defined as
\[
\mathcal{L}_{\vartheta} = \{ u \in \mathcal{A}^+ : u \vartheta^n, \mathbb{P}[\vartheta^n(a) = v] > 0, \text{ for some } n \in \mathbb{N}, a \in \mathcal{A} \}.
\]
The set of (\( \vartheta \))-legal words of length \( \ell \) is given by \( \mathcal{L}_{\vartheta}^{\ell} = \{ u \in \mathcal{L}_{\vartheta} : |u| = \ell \} \).
The definition of $L_{\vartheta}$ is independent of the probability vectors (as long as none of their entries is 0), making it a purely combinatorial object. The same holds for the subshift that we can construct from this language. A probabilistic structure will be obtained by a convenient choice of a measure on this subshift in Section 5.

**Definition 2.8.** For a random substitution $\vartheta$ on $A$, we define the random substitution subshift (RS-subshift) or stochastic hull as

$$X_\vartheta = \{x \in A^\mathbb{Z} : x_{[k,m]} \in L_\vartheta \text{ for all } k \leq m\}.$$

We equip the RS-subshift with the usual topology generated by cylinder sets and denote the corresponding Borel $\sigma$-algebra by $\mathcal{B}_\vartheta$.

**Remark 2.9.** A general cylinder set is of the form $Z_{[k,m]}(v) = \{x \in A^\mathbb{Z} : x_{[k,m]} = v\}$, for some $k \leq m \in \mathbb{Z}$ and $v \in L_\vartheta^{m-k+1}$. It is worth noticing that $\mathcal{B}_\vartheta$ is already generated by the smaller class of those cylinder sets that specify the position 0:

$$\mathcal{Z}_0(X_\vartheta) = \{Z_{[k,m]}(v) : k \leq 0 \leq m, v \in L_\vartheta^{m-k+1} \} \cup \{X_\vartheta, \emptyset\}.$$

The class $\mathcal{Z}_0(X_\vartheta)$ has the convenient property of forming a semi-algebra on $X$ (it contains the full space, is stable under intersections, and complements can be written as finite disjoint unions).

### 3. Word Frequencies, Convergence in Expectation

Many of the results in this section essentially go back to [12]. However, we do not assume the existence of a letter $b \in A$ and a realisation $\omega \in \Omega_\vartheta$ such that $\vartheta(b)(\omega)$ starts with $b$, which is taken as a standing assumption in [12]. Also note that the terminology in [12] is slightly different in so far as properties that hold under taking expectation are often called ‘almost sure’.

An important tool for determining letter frequencies of a deterministic substitution is its corresponding induced substitution [24, Ch. 5.4.1]. Before we continue, we need to extend this notion to the case of expanding random substitutions. For an arbitrary $\ell \in \mathbb{N}$, let $A_\ell := L_\vartheta^\ell$ be the alphabet consisting of letters that are $\vartheta$-legal words of length $\ell$. Given $u = u_1 \cdots u_\ell \in A_\ell$, we define the random element $\vartheta_\ell^n(u) : \Omega \to A_\ell^+$ by the prescription

$$\vartheta_\ell^n(u)(\omega) := \left(\vartheta^n(u)(\omega)_{[k,k+\ell-1]}\right)_{1 \leq k \leq |\vartheta^n(u_1)(\omega)|}.$$

Note that $\vartheta_\ell^n(u)$ is a concatenation of letters in the alphabet $A_\ell$ that is consistent with their right collars. Therefore, we need to extend the action of $\vartheta_\ell$ only to those words $V \in A_\ell^+$ which satisfy $V = V_1 \cdots V_m = (u_{[k,k+\ell-1]})_{1 \leq k \leq m}$ for some $u \in L_\vartheta$ with $|u| \geq m + \ell - 1$. This defines a new space $D_\ell \subseteq A_\ell^+$ of consistent concatenations of right collared words of length $\ell$.

For $V \in D_\ell$, given as above, we define

$$\vartheta_\ell(V)(\omega) = \vartheta_\ell(V_1)(\omega) \cdots \vartheta_\ell(V_m)(\omega) = \left(\vartheta(u_{[1,\ell]})(\omega)_{[k,k+\ell-1]}\right)_{1 \leq k \leq |\vartheta(u_1)(\omega)|} \cdots \left(\vartheta(u_{[m,m+\ell-1]})(\omega)_{[k,k+\ell-1]}\right)_{1 \leq k \leq |\vartheta(u_m)(\omega)|}.$$

An important observation at this point is that the random elements \( \vartheta(V_1), \ldots, \vartheta(V_m) \) are not independent in contrast to the action of \( \vartheta \) on the concatenation of letters in \( \mathcal{A} \). That is because e.g. the words \( V_1 \) and \( V_2 \) have some overlap which, by force, is mapped to the same word under \( \vartheta \) for any realisation. In this sense, \( \vartheta \) does not act on \( \mathcal{A}^+ \) in the same way that \( \vartheta \) acts on \( \mathcal{A}^+ \) and thus, strictly speaking, \( \vartheta \) is not a random substitution as defined above. Nevertheless, it still defines a stationary transition kernel on \( (\mathcal{D}_\ell, \mathcal{P}(\mathcal{D}_\ell)) \) and thus a Markov chain with state space \( \mathcal{D}_\ell \) over the alphabet \( \mathcal{A}_\ell \) — in fact, it forms an M-system in the sense of Peyrière [22]. Therefore, the notion of the Abelianisation map \( \Phi \) and the mean substitution matrix \( M_{\vartheta \ell} \) carry over to this case. We emphasise that for \( V \in \mathcal{D}_\ell \), the vector \( \Phi(V) \) is indexed by the letters of the alphabet \( \mathcal{A}_\ell \) which comprise \emph{words} in \( \mathcal{A} \).

It is worth mentioning at this point that, for any letter \( \ell \) of \( \mathcal{A}_\ell \), the (marginal) distribution of \( \vartheta(V) \) is well-defined and independent of its position in \( V \), a property which is inherited from the underlying random substitution \( \vartheta \). This is why, for \( v \in \mathcal{L}_\alpha \) and \( V = V_1 \cdots V_m \in \mathcal{D}_\ell \), we can conclude from the pointwise identity

\[
|\vartheta(V)(\omega)|_v = \sum_{i=1}^m |\vartheta(V_i)(\omega)|_v
\]

the corresponding relation under taking expectations

\[
\mathbb{E}|\vartheta(V)|_v = \sum_{i=1}^m \mathbb{E}|\vartheta(V_i)|_v.
\]  

We can also check by induction that \( (\vartheta(V))^n(\omega) = \vartheta(V^n)(\omega) \) for all \( V \in \mathcal{D}_\ell \) and \( \omega \in \Omega \), using the identities above — compare [19, Lem. 4.3].

**Proposition 3.1.** Let \( \vartheta \) be an expanding random substitution on \( \mathcal{A} \). Let \( n_\ell = \text{card} \mathcal{L}_\alpha \) be the number of legal words of length \( \ell \) and \( \mathcal{X}_\ell \) the space of random elements \( X : \Omega \to \mathcal{D}_\ell \) such that \( \mathbb{P}[X = V] \neq 0 \) only for finitely many \( V \in \mathcal{D}_\ell \). Then, the following diagram commutes,

\[
\begin{array}{ccc}
\mathcal{X}_\ell & \xrightarrow{\vartheta_\ell} & \mathcal{X}_\ell \\
\mathbb{E} \circ \Phi & \downarrow & \mathbb{E} \circ \Phi \\
\mathbb{R}^{n_\ell} & \xrightarrow{M_{\vartheta \ell}} & \mathbb{R}^{n_\ell}
\end{array}
\]

In the case \( \ell = 1 \), the above assertion holds also for substitution rules that are not expanding.

**Proof.** Suppose \( X \in \mathcal{X}_\ell \) and set \( A_X = \{V \in \mathcal{D}_\ell : \mathbb{P}[X = V] \neq 0\} \). We have \( \mathbb{E}\Phi(X) = \sum_{V \in A_X} \mathbb{P}[X = V] \Phi(V) \), where \( \Phi(V) = (|V|_{v_1}, \ldots, |V|_{v_{n_\ell}}) \), with \( v_i \in \mathcal{L}_\alpha \) for \( i \in \{1, \ldots, n_\ell\} \).

Let \( V = V_1 \cdots V_m \in \mathcal{D}_\ell \), in particular \( V_j \in \mathcal{L}_\alpha \) for all \( j \in \{1, \ldots, m\} \). By Eq. (1), we find

\[
\mathbb{E}\Phi(\vartheta(V)) = \mathbb{E}|\vartheta(V)|_{v_i} = \sum_{k=1}^m \mathbb{E}\Phi(\vartheta(V_k))_{v_i} = \sum_{j=1}^{n_\ell} \mathbb{E}\Phi(\vartheta(v_j))_{v_i} |V|_{v_j} = (M_{\vartheta_\ell} \Phi(V))_i.
\]
We can write the distribution of the random element \( \vartheta_\ell(X) \) via the Chapman–Kolmogorov equation as \( \mathbb{P}[\vartheta_\ell(X) = W] = \sum_{V \in A_X} \mathbb{P}[X = V] \mathbb{P}[\vartheta_\ell(V) = W] \), for \( W \in D_\ell \), and thus

\[
\mathbb{E}\Phi(\vartheta_\ell(X)) = \sum_W \mathbb{P}[\vartheta_\ell(X) = W] \Phi(W) = \sum_{V \in A_X} \sum_W \mathbb{P}[X = V] \mathbb{P}[\vartheta_\ell(V) = W] \Phi(W)
\]

\[
= \sum_{V \in A_X} \mathbb{P}[X = V] \mathbb{E}\Phi(\vartheta_\ell(V)) = \sum_{V \in A_X} \mathbb{P}[X = V] M_{\vartheta_\ell} \Phi(V) = M_{\vartheta} \mathbb{E}\Phi(X).
\]

In the case that \( \ell = 1 \) and \( \vartheta \) is non-expanding, only one-letter words are legal. The above proof is then still literally true, with the additional constraint that \( m = 1 \).

Note that, for \( \ell = 1 \), the induced substitution \( \vartheta_1 \) coincides with the original substitution \( \vartheta \). This gives an alternative characterisation of primitive substitutions, compare [27, Rem. 6] and [19, Rem. 2.13].

**Corollary 3.2.** Let \( \vartheta \) be a random substitution rule on \( A \). Then, \( \vartheta \) is primitive if and only if there exists a number \( k \in \mathbb{N} \) such that, for all \( a_i, a_j \in A \), we have \( \mathbb{P}[a_i \vartheta(a_j)] > 0 \).

**Proof.** From Proposition 3.1, we know \( \mathbb{E}\Phi(\vartheta^k(a_j))_i = (M^k_\vartheta \Phi(a_j))_i = (M^k_\vartheta)_{ij} \). Therefore,

\[
(M^k_\vartheta)_{ij} > 0 \iff \mathbb{E}\Phi(\vartheta^k(a_j))_i > 0 \iff \mathbb{P}[\vartheta^k(a_j)_i > 0] > 0.
\]

\( \square \)

It was shown in [27, Prop. 26] that \( M_{\vartheta_\ell} \) is primitive if \( M_{\vartheta} \) is primitive and in [12, Lem. 2.4.9] that the PF eigenvalues of both matrices coincide. The following result is inspired by [12, Lem. 2.4.5, Lem. 2.4.6, Cor. 2.4.8]. Note that our results differs from the one given in the proof of [12, Lem. 2.4.5] by a factor of \( L_u^{(\ell)} \) as we fixed the norm of the right eigenvector to be 1.

**Lemma 3.3.** Let \( \vartheta \) be an expanding primitive random substitution on the finite alphabet \( A \). For any \( \ell \in \mathbb{N} \), let \( \vartheta_\ell \) be the induced substitution with mean substitution matrix \( M_{\vartheta_\ell} \), PF eigenvalue \( \lambda \) and right and left PF-eigenvector \( R^{(\ell)} \) and \( L_u^{(\ell)} \), normalised as above. Then, we have the following convergence results:

\[
\lim_{n \to \infty} \frac{\mathbb{E}\Phi(\vartheta_n^\ell(u))}{\lambda^n} = L_u^{(\ell)} R^{(\ell)}, \tag{2}
\]

\[
\lim_{n \to \infty} \frac{\mathbb{E}|\vartheta_{n+1}^\ell(u)|}{\mathbb{E}|\vartheta_n^\ell(u)|} = \lambda, \tag{3}
\]

\[
\lim_{n \to \infty} \frac{\mathbb{E}\Phi(\vartheta_n^\ell(u))}{\mathbb{E}|\vartheta_n^\ell(u)|} = R^{(\ell)}, \tag{4}
\]

for all \( u \in A_\ell \). The same statement is still true for non-expanding \( \vartheta \) if \( \ell = 1 \).

**Proof.** Note that \( \|\Phi(\vartheta_n^\ell(u))\|_1 = |\vartheta_n^\ell(u)| \) and the expectation value is a linear operator, so the second and the third relations follow immediately from the first, observing the identity \( \lim_{n \to \infty} \frac{1}{\lambda^n} \mathbb{E}|\vartheta_n^\ell(u)| \to L_u^{(\ell)} \neq 0 \). It thus remains to show Eq. (2).
Using Proposition 3.1, we find $E \Phi(\vartheta^n(u)) = M^n_\vartheta \Phi(u)$ and thus, by an application of the PF theorem,

$$E \Phi(\vartheta^n(u)) = \frac{1}{\lambda^n} M^n_\vartheta \Phi(u) \xrightarrow{n \to \infty} R^{(\ell)}(L^{(\ell)} \cdot e_u) = R^{(\ell)} L^{(\ell)}_u.$$ 

\[ \square \]

In the next result, we establish an exponentially decaying bound for the probability that the length of $\vartheta^n(a)$ grows sublinearly. This will be a convenient tool in later constructions.

**Proposition 3.4.** If $\vartheta$ is an expanding primitive substitution, there are constants $C, K > 0$ such that, for all $a \in A$ and large enough $n$,

$$\mathbb{P}[|\vartheta^n(a)| < Kn] \leq e^{-Cn}. $$

In particular,

$$\lim_{n \to \infty} |\vartheta^n(a)| = \infty,$$

almost surely for all $a \in A$.

**Proof.** The second claim is an immediate consequence of the first one by an application of the Borel–Cantelli lemma.

Let $a_j \in A$ and $u$ be a word such that $|u| \geq 2$ and $\vartheta(a_j)(\omega) = u$ for some $\omega$. From the primitivity of $\vartheta$, we deduce that there is $\tilde{N} \in \mathbb{N}$ such that, for all $a \in A$, we have $a_j < \vartheta^{\tilde{N}}(a)$, and consequently $u < \vartheta^{\tilde{N}+1}(a)$ with positive probability $\tilde{q}$. Set $N := \tilde{N} + 1$. It follows

$$\mathbb{P}[|\vartheta^N(a)| \geq 2] \geq \tilde{q},$$

hence,

$$\mathbb{P}[|\vartheta^N(a)| = 1] < (1 - \tilde{q}) =: q < 1. \quad (5)$$

Consider the finite sequence of positive random variables $(X_k)_{0 \leq k \leq n - 1}$ defined via $X_k = |\vartheta^N(\vartheta^{Nk}(a)_1)|$, where by convention $X_0 = |\vartheta^N(a)|$. Note that, for any $\omega \in \Omega$, the assumption $X_k(\omega) \geq 2$ implies

$$|\vartheta^{N(k+1)}(\omega)| \geq |\vartheta^{Nk}(\omega)| + 1,$$

because the action of $\vartheta^N$ does not decrease the length of any word. Thus, for any $n \in \mathbb{N}$ and $K > 0$, $|\vartheta^N(a)| \leq Kn$ implies that card$\{j : X_j \geq 2\} \leq Kn \leq [Kn]$, or equivalently

$$\text{card}\, J \geq n - [Kn], \quad J = \{j : X_j = 1\}.$$ 

For a fixed set $J_0 \subseteq \{0, \ldots, n - 1\}$, we find

$$\mathbb{P}[X_j = 1 \text{ for all } j \in J_0] \leq q^{\text{card}\, J_0},$$

by the fact that Eq. (5) holds irrespective of $a \in A$ and the Markov property. Simple combinatorial arguments yield

$$\mathbb{P}[|\vartheta^{Nn}(a)| \leq Kn] \leq \mathbb{P}[\text{card}\, J \geq n - [Kn]] = \sum_{k=n-[Kn]}^{n} \mathbb{P}[\text{card}\, J = k]$$

$$\leq \sum_{k=n-[Kn]}^{n} \binom{n}{k} q^k \leq ([Kn] + 1) \left(\frac{n}{[Kn]}\right)^n q^{n-[Kn]}.$$
Using Stirling’s formula, we find
\[
\liminf_{n \to \infty} \frac{1}{n} \log \binom{n}{K_n} = -K \log(K) - (1 - K) \log(1 - K)
\]
and thus
\[
\liminf_{n \to \infty} \frac{1}{n} \log \left( \mathbb{P} \left[ |\vartheta^n(a)| \leq K_n \right] \right) \leq (1 - K) \log(q) - K \log(K) - (1 - K) \log(1 - K)
\]
\[
\lim_{K \to 0} \frac{1}{n} \log(q) < 0.
\]

Thereby, we can find $\tilde{K}, \tilde{C} > 0$ such that
\[
\mathbb{P} \left[ |\vartheta^n(a)| \leq \tilde{K} n \right] \leq e^{-\tilde{C} n},
\]
for large enough $n$. Setting $C = \frac{\tilde{C}}{2n}$ and $K = \frac{\tilde{K}}{2n}$, the assertion follows by a straightforward interpolation argument. \qed

**Corollary 3.5.** A primitive random substitution $\vartheta$ is expanding if and only if the corresponding PF-eigenvalue $\lambda$ fulfils $\lambda > 1$.

**Proof.** We will use the assertion of Lemma 3.3 for the case $\ell = 1$, such that it is still true for non-expanding substitutions. Therefore, $\lambda > 1$ is equivalent to $\mathbb{E}[|\vartheta^0(a)|] \to \infty$ as $n \to \infty$. So, $\lambda > 1$ clearly implies that $\vartheta$ is an expanding substitution. On the other hand, if $\vartheta$ is expanding, $|\vartheta^n(a)(\omega)| \to \infty$, as $n \to \infty$, for all $\omega \in B$ with $\mathbb{P}(B) = 1$, by Proposition 3.4. Therefore,
\[
\liminf_{n \to \infty} \mathbb{E}[|\vartheta^n(a)|] = \liminf_{n \to \infty} \int_B |\vartheta^n(a)(\omega)| \, d\mathbb{P}(\omega) \geq \int_B \liminf_{n \to \infty} |\vartheta^n(a)(\omega)| \, d\mathbb{P}(\omega) = \infty,
\]
by Fatou’s Lemma. Thus, $\lim_{n \to \infty} \mathbb{E}[|\vartheta^n(a)|] =$ $\infty$ for all $a \in A$, implying $\lambda > 1$. \qed

We finally conclude with the following result — compare the proof of [12, Thm. 2.4.10].

**Proposition 3.6.** Let $\vartheta$ be an expanding, primitive random substitution on $A$. For any $\ell \in \mathbb{N}$, let $\vartheta_\ell$ be the induced substitution with mean substitution matrix $M_{\vartheta_\ell}$ and right normalised PF eigenvector $R^{(\ell)}$. Then, for any $v \in \mathcal{L}_0^\ell$ and $a \in A$,
\[
\lim_{n \to \infty} \mathbb{E}[|\vartheta^n(a)|] = R^{(\ell)}_v.
\]
Furthermore, for any $u \in \mathcal{L}_0^\ell$ with $u_1 = a$,
\[
L^{(\ell)}_u = \lim_{n \to \infty} \frac{\mathbb{E}[|\vartheta^n(u)|]}{\lambda^n} = \lim_{n \to \infty} \frac{\mathbb{E}[|\vartheta^n(a)|]}{\lambda^n} = L_a.
\]

**Proof.** Let $\ell \in \mathbb{N}$, $a \in A$ and $v \in \mathcal{L}_0^\ell$. Since $\vartheta$ is expanding and primitive, there are words of arbitrary length containing $a$ — compare Corollary 3.2. Choose any legal word of length $\ell$, $u = aa_2 \cdots a_\ell \in \mathcal{L}_0^\ell$, starting with the letter $a$. From the definition of the induced substitution, we find $|\vartheta^n(u)(\omega)| = |\vartheta^n(a)(\omega)|$, for all $\omega \in \Omega$. This also implies that $\mathbb{E}[|\vartheta^n(u)|] = \mathbb{E}[|\vartheta^n(a)|]$, for every $n \in \mathbb{N}$, yielding the second identity when combined with Eq. (2).

Furthermore, we have $|\vartheta^n(u)(\omega)|_v = |\vartheta^n(a)(\omega)|_v + O(\ell)$. This is because the last $\ell - 1$ ‘letters’ $(\in \mathcal{L}_0^\ell)$ of $\vartheta^n(u)(\omega)$ are not entirely included in $\vartheta^n(a)(\omega)$, whereas all others comprise
the subwords of $\vartheta^n(a)$. Thus, $E[|\vartheta^n(u)|_v] = E[|\vartheta^n(a)|_v] + O(\ell)$. Since $\vartheta$ is expanding, it is $\lambda > 1$ and so the error term $O(\ell)$, when divided by $E[|\vartheta^n(a)|]$, gets arbitrarily small, as $n \to \infty$. Consequently, writing $|\vartheta^n(u)|_v = \Phi(\vartheta^n(u))_v$, 

$$\lim_{n \to \infty} \frac{E[|\vartheta^n(a)|_v]}{E[|\vartheta^n(a)|]} = \lim_{n \to \infty} \frac{E[|\vartheta^n(u)|_v]}{E[|\vartheta^n(a)|]} = R_v(\ell),$$

by an application of the third identity in Lemma 3.3.

For the rest of this paper, we set $\vartheta$ to be an expanding primitive random substitution on some finite alphabet $\mathcal{A} = \{a_1, \ldots, a_m\}$ with $m$ letters. This allows us to use the powerful machinery derived from PF theory which we developed in this chapter.

4. Word Frequencies in Inflation Words

The word frequency of some $v \in \mathcal{L}_\vartheta^\ell$ within a (generally larger) word $w \in \mathcal{L}_\vartheta$ will be denoted by $\nu_v(w) = \frac{|w|}{|w|}$. In this section, we will show that

$$\nu_v(\vartheta^n(a)) = \frac{|\vartheta^n(a)|_v}{|\vartheta^n(a)|} \xrightarrow{n \to \infty} R_v(\ell)$$

holds almost surely, irrespective of the choice of $a \in \mathcal{A}$. To this end, we will split the word $\vartheta^n(a) = \vartheta^k(\vartheta^{n-k}(a))$ into a large number of inflation words with level smaller than $n$. One technical obstacle is given by the fact that the (random) word $\vartheta^{n-k}(a)$ might comprise more than one type of letter, such that the action of $\vartheta^k$ on these letters cannot be interpreted as i.i.d. random elements. This can be circumvented with the help of an elementary result. It basically establishes that if $\nu_v(\vartheta^n(u))$ deviates from $R_v(\ell)$ by a certain amount, then there is a letter $a \in \mathcal{A}$, comprising a positive fraction of $u$, such that a similar statement holds if we regard only the inflation words $\vartheta^n(u_i)$ for those letters $u_i \prec u$ that coincide with $a$. We set up a slightly more general form for this result to allow for its application also in a different context.

**Lemma 4.1.** Let $c = \sum_{i=1}^m c_i$ and $d = \sum_{i=1}^m d_i$ with $0 \leq c_i \leq d_i$ and $1 \leq d_i$ for all $i \in \{1, \ldots, m\}$. Further, suppose that

$$\left| \frac{c}{d} - K \right| > \delta,$$

for some $0 \leq \delta, K \leq 1$. Then, there exists some $j \in \{1, \ldots, m\}$ such that $d_j \geq \frac{\delta}{m} d$ and

$$\left| \frac{c_j}{d_j} - K \right| \geq \frac{\delta}{2}. \quad (6)$$

Further, for this choice of $j$ and any $f \in \mathbb{R}_+$,

$$\left| \frac{d_j}{f} - 1 \right| \geq \frac{\delta}{8} \quad \text{or} \quad \left| \frac{c_j}{f} - K \right| \geq \frac{\delta}{8}. \quad (7)$$

**Proof.** Suppose Eq. (6) does not hold and set

$$S_1 = \left\{ 1 \leq i \leq m : |c_i - Kd_i| \geq \frac{\delta}{2d_i} \right\}$$

Also, set $S_2 = \{1, \ldots, m\} \setminus S_1$. Then, $\frac{c}{d} = \frac{\sum_{i \in S_1} c_i}{\sum_{i \in S_1} d_i} + \frac{\sum_{i \in S_2} c_i}{\sum_{i \in S_2} d_i}$.

Since $d_i \geq \frac{\delta}{m} d$ for all $i \in S_1$, we get

$$\left| \frac{\sum_{i \in S_1} c_i}{\sum_{i \in S_1} d_i} - K \right| \geq \frac{\delta}{2m} \quad \text{and} \quad \left| \frac{\sum_{i \in S_2} c_i}{\sum_{i \in S_2} d_i} - K \right| \geq \frac{\delta}{2m}.$$
Proof. Let us start with some arbitrary positions of the letter a in contradiction to the assumption. This proves Eq. (6). As follows. Setting d = |φ(u)| and c = Σ |φ(u_j)|, one of the following two cases needs to hold,

\[ \frac{c}{d} - R_v^\ell > \frac{\epsilon}{2} \quad \text{or} \quad \ell n > \frac{\epsilon}{2} |φ(u)|. \]

Let us first treat case (I). We want to translate this to a similar statement that involves only one type of letter as a starting point for the random inflation φ; compare Figure 2 for

and S_2 = \{1, \ldots, m\} \setminus S_1. Then, i ∈ S_1 implies d_i < δ_d. Note that generally |c_i - Kd_i| ≤ d_i. Consequently,

\[ |c - Kd| \leq \sum_{i \in S_1} |c_i - Kd_i| + \sum_{i \in S_2} |c_i - Kd_i| < \sum d_i + \sum_{i \in S_1} \frac{\delta}{2}d_i < \sum_{i \in S_1} \frac{\delta}{2}d + \frac{\delta}{2}d \leq \delta d, \]

in contradiction to the assumption. This proves Eq. (6). Assuming |d_j/f - 1| ≤ \delta/8, we obtain,

\[ \frac{\delta}{4} \leq \frac{d_j\delta}{f^2} \leq \left| \frac{c_j}{f} - Kd_j \right| \leq \left| \frac{c_j}{f} - K \right| + K\left| 1 - \frac{d_j}{f} \right| \leq \left| \frac{c_j}{f} - K \right| + \frac{\delta}{8}, \]

and thus Eq. (7) holds.

Lemma 4.2. Let v ∈ L_0 and ε > 0. There is a k_0 ∈ N such that, for all k > k_0, there is a C > 0 with

\[ \mathbb{P}\left[ |\nu_\ell(φ(u)) - R_v^\ell| > \epsilon \right] \leq e^{-Cn}, \]

provided that u ∈ L_0^n and n is large enough.

Proof. Let us start with some arbitrary n ∈ N and u ∈ L_0^n. For any letter a ∈ A, denote the positions of the letter a within the word u by P_a = \{1 ≤ i ≤ n : u_i = a\}, possibly empty and n_a = card P_a = |u|_a. Assume card A = m and

\[ \left| \frac{|φ(u)|}{|φ(u)|} - R_v^\ell \right| > \epsilon. \]

Note that, in contrast to the total length |φ(u)| = \sum_{j=1}^n |φ(u_j)|, for the occurrences of v within φ(u) it is not enough to count how often v appears in any inflation word,

\[ |φ(u)|_v = \sum_{j=1}^n |φ(u_j)|_v + \text{Ovl}, \]

where the additional term Ovl (in general a random variable) gives the number of times that the position of v within φ(u) overlaps two (or more) neighbouring inflation words. A situation where such an overlap occurs is depicted in Figure 1. Since there are only n - 1 different boundaries of inflation words of the form φ(u_j), the bound 0 ≤ Ovl < nε follows immediately. Setting d = |φ(u)| and c = \sum_{j=1}^n |φ(u_j)|_v, one of the following two cases needs to hold,

(I) \[ \frac{c}{d} - R_v^\ell > \frac{\epsilon}{2} \quad \text{or} \quad \ell n > \frac{\epsilon}{2} |φ(u)|. \]
an illustrative example. With the identifications \( \delta = \xi, K = R^{(f)}_a, c_a = \sum_{j \in P_a} |\vartheta^k(u_j)|_v, d_a = \sum_{j \in P_a} |\vartheta^k(u_j)| \) we can apply Lemma 4.1. That is, for all realisations of the random variables that we consider, there is some letter \( a \in A \) such that \( \sum_{j \in P_a} |\vartheta^k(u_j)| \geq \frac{\varepsilon}{4m} |\vartheta^k(u)| \) and (choosing \( f = \lambda^k L_a n_a \), in the notation of Lemma 4.1),

\[
\frac{1}{n_a} \sum_{j \in P_a} |\vartheta^k(u_j)|_v - R^{(f)}_v \geq \frac{\varepsilon}{16} \quad \text{or} \quad \frac{1}{n_a} \sum_{j \in P_a} |\vartheta^k(u_j)| - 1 \geq \frac{\varepsilon}{16},
\]

which we denote by case (Ia) and (Ib), respectively. Setting \( \mu = \max_{a \in A} \max_{\omega \in \Omega} |\vartheta(a)(\omega)| \) as the maximal possible inflation factor, we have the crude estimate,

\[
\mu^k n_a \geq \sum_{j \in P_a} |\vartheta^k(u_j)| \geq \frac{\varepsilon}{4m} |\vartheta^k(u)| \geq \frac{\varepsilon}{4m} n,
\]

that is, there is some constant \( C_1(k, \varepsilon) = C_1 > 0 \) such that \( n_a \geq C_1 n \). Let us define the random variables

\[
X_{(j)}^k = |\vartheta^k(u_j)|_v \quad \text{and} \quad Y_{(j)}^k = |\vartheta^k(u_j)| - |\vartheta^k(u)|,
\]

and note that each of the corresponding families with \( j \in P_a \) comprise i.i.d. random variables with common distribution \( X^k \) and \( Y^k \), respectively. By Lemma 3.3 and Proposition 3.6,

\[
\mathbb{E} Y_{(j)}^k \xrightarrow{k \to \infty} 1 \quad \text{and} \quad \mathbb{E} X_{(j)}^k \xrightarrow{k \to \infty} R^{(f)}_v,
\]

for any \( j \in P_a \). We can thus find a \( k_0^{(1)} \in \mathbb{N} \) such that, for all \( k \geq k_0^{(1)} \),

\[
\frac{1}{n_a} \sum_{j \in P_a} X_{(j)}^k - \mathbb{E} X^k \geq \frac{\varepsilon}{32} \quad \text{or} \quad \frac{1}{n_a} \sum_{j \in P_a} Y_{(j)}^k - \mathbb{E} Y^k \geq \frac{\varepsilon}{32}.
\]

Let us fix such a \( k \). Then, the joint distribution of the family \( (X_{(j)}^k)_{j \in P_a} \) does not depend on the structure of \( u \) or the exact form of \( P_a \), but only on \( n_a = \text{card} P_a \). Also, since \( X^k \) can take only finitely many values, the bound \( \mathbb{E} e^{tX^k} < \infty \) is trivial for all \( t \in \mathbb{R} \). By Cramér’s theorem on large deviations [10, Thm. I.4], we therefore obtain,

\[
\lim_{n_a \to \infty} \frac{1}{n_a} \log \mathbb{P} \left[ \frac{1}{n_a} \sum_{j \in P_a} X_{(j)}^k - \mathbb{E} X^k \geq \frac{\varepsilon}{32} \right] < 0,
\]

Figure 2. Inflation procedure of \( \vartheta^k \) on \( u \). The occurrences of some \( a \in A \) in \( u \) are highlighted. These are mapped to i.i.d random elements, distributed like \( \vartheta^k(a) \), which appear as subwords of \( \vartheta^k(u) \).
and analogously for the family \((Y^k)_{j \in P_a}\). Thus, for large enough \(n\) and any \(u \in L_n\),
\[
P[(I)] \leq \sum_{a \in A} P[(Ia) \text{ holds for } a] + \sum_{a \in A} P[(Ib) \text{ holds for } a] \leq \sum_{a \in A} e^{-\tilde{C}n_a} \leq |A|e^{-\tilde{C}C_1n}.
\]
We now turn to case (II). This can be rephrased as
\[
\frac{\sum_{a \in A} |u|_a}{\sum_{a \in A} \sum_{j \in P_a} |\vartheta^k(u_j)|} > \frac{\varepsilon}{2\ell}
\]
such that we can again apply Lemma 4.1, this time with \(\delta = \frac{\varepsilon}{2\ell}, K = 0, c_a = |u_a|\) and \(d_a = \sum_{j \in P_a} |\vartheta^k(u_j)|\), yielding the existence of some \(a \in A\) with \(\sum_{j \in P_a} |\vartheta^k(u_j)| \geq \frac{\varepsilon}{4\ell} |\vartheta^k(u)|\)
and, recalling \(n_a = |u_a|,
\[
\frac{1}{n_a} \sum_{j \in P_a} |\vartheta^k(u_j)| < \frac{4\ell}{\varepsilon}.
\]
(8)
Note that all the random elements \(\vartheta^k(u_j)\) are distributed as \(\vartheta^k(a)\) and again i.i.d. for \(j \in P_a\).
At the same time,
\[
n_a > \frac{\varepsilon}{4\ell} \sum_{j \in P_a} |\vartheta^k(u_j)| \geq \frac{\varepsilon^2}{16\ell^2 m} |\vartheta^k(u)| \geq \frac{\varepsilon^2}{16\ell^2 m} n.
\]
Since \(E[|\vartheta^k(u_j)|] \xrightarrow{k \to \infty} \infty\), as \(k \to \infty\), we can find a \(k^{(2)}_0\) such that, for all \(k \geq k^{(2)}_0\), Eq. (8) implies
\[
\frac{1}{n_a} \sum_{j \in P_a} |\vartheta^k(u_j)| < \frac{1}{2} E[|\vartheta^k(a)|].
\]
Similarly to case (I), we can apply basic large deviation results to conclude that the probability for this to happen is bounded by \(e^{-C_2n}\), for some \(C_2 > 0\) and \(n \in \mathbb{N}\) large enough.
Finally, taking \(k_0 = \max\{k^{(1)}_0, k^{(2)}_0\}\) and adjusting the constant in the exponent, \(k \geq k_0\) implies
\[
P \left[ |\nu_v(\vartheta^k(u)) - R_v^{(\ell)} | > \varepsilon \right] \leq P[(I)] + P[(II)] \leq e^{-Cn}.
\]
for large enough \(n \in \mathbb{N}\). \(\square\)

**Proposition 4.3.** Let \(v \in L^\ell_0\) be a \(\vartheta\)-legal word of length \(\ell \in \mathbb{N}\). Then, in the limit of large inflation words, the word frequency of \(v\) exists and is the same for almost all realisations. It coincides with the corresponding entry \(R_v^{(\ell)}\) of the right PF eigenvector of the induced substitution matrix. More precisely, for any \(a \in A\),
\[
\lim_{n \to \infty} \nu_v(\vartheta^n(a)) = R_v^{(\ell)}
\]
holds almost surely.

**Proof.** Let \(\varepsilon > 0\) and \(a \in A\). We are going to establish that
\[
\sum_{n=1}^{\infty} P \left[ |\nu_v(\vartheta^n(a)) - R_v^{(\ell)} | > \varepsilon \right] < \infty.
\]
(9)
From this, the assertion follows by a standard application of the Borel–Cantelli lemma.
Due to Proposition 3.4, there are some \( \tilde{C}, K > 0 \) such that \( \mathbb{P}[|\vartheta^n(a)| < Kn] \leq e^{-\tilde{C}n} \) for large enough \( n \in \mathbb{N} \). Choose \( C > 0 \) and a fixed \( k \geq k_0 \) as in Lemma 4.2. Then, for large enough \( n \in \mathbb{N} \),

\[
\mathbb{P} \left[ |\nu_v(\vartheta^{k+n}(a)) - R_v^{(\ell)}| > \varepsilon \right] \leq e^{-\tilde{C}n} + \sum_{u \in \mathcal{L}, |u| > Kn} \mathbb{P} \left[ |\nu_v(\vartheta^k(u)) - R_v^{(\ell)}| > \varepsilon \right] \mathbb{P}[\vartheta^n(a) = u] \\
\leq e^{-\tilde{C}n} + e^{-CKn},
\]

by Lemma 4.2. This verifies Eq. (9).

\( \square \)

**Remark 4.4.** The statement of Proposition 4.3 for the special case \( \ell = 1 \) (frequency of letters) follows from a standard result on branching processes. More precisely, performing the Abelianisation, we obtain a vector-valued Markov process \( \{\Phi(\vartheta^n(a))\}_{n \in \mathbb{N}} \) which forms a multitype Galton–Watson (GW) process as described in [17] and [1]. From [1, Ch. V.6], we find that, for every \( a \in \mathcal{A} \),

\[
\lim_{n \to \infty} \frac{\Phi(\vartheta^n(a))}{\lambda^n} = RW_a
\]

holds almost surely, where \( W_a \) is a non-negative random variable with \( \mathbb{E}[W_a] = L_a \) and \( \mathbb{P}[W_a = 0] = 0 \). Note that \( \{\Phi(\vartheta^n_\ell(a))\}_{n \in \mathbb{N}} \) for \( \ell \geq 2 \) does not form a multitype GW process, because neighbouring words have some overlap which is necessarily mapped to the same word under \( \vartheta \). This violates the requirement for a GW process that each individual produces offspring independently. However, combining Proposition 4.3 with Eq. (10) we obtain that for any \( a \in \mathcal{A} \) and \( v \in \mathcal{L}^\ell_\vartheta \),

\[
\lim_{n \to \infty} \frac{|\vartheta^n(a)|_v}{\lambda^n} = \lim_{n \to \infty} \frac{|\vartheta^n(a)|_v}{|\vartheta^n(a)|} \frac{|\vartheta^n(a)|}{\lambda^n} = R_v^{(\ell)}W_a,
\]

holds almost surely. Note that, despite our preceding words of precaution, this result (and thereby Proposition 4.3) can also be obtained by generalising ideas from the theory of GW processes [22, Thm. 2].

5. **Ergodic frequency measures on the hull**

From the word frequencies in inflation words, as established in the last section, we want to define a shift-invariant measure on the hull \( X_\vartheta \) of the substitution in a consistent way.

**Definition 5.1.** A map \( \mu : \mathcal{G} \to [0, \infty] \) on a semi-algebra \( \mathcal{G} \) is called a measure on \( \mathcal{G} \), if the following conditions hold:

1. If \( \bigcup_{i=1}^\infty A_i \in \mathcal{G} \) is a disjoint union of elements \( A_i \in \mathcal{G} \), then \( \mu(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i) \),
2. \( \mu(\emptyset) = 0 \).

Often, this is also called a premeasure in the literature. We know from [21, Cor. 2.4.9] and [21, Prop. 2.5.1] that this can be extended uniquely to a measure on the \( \sigma \)-algebra generated by \( \mathcal{G} \) if \( \mu \) is \( \sigma \)-finite on \( \mathcal{G} \). Comparing this with Remark 2.9, we find that it is enough to specify a measure on \( \mathcal{V}_0(X_\vartheta) \) in order to uniquely define a measure on \( (X_\vartheta, B_\vartheta) \).
Proposition 5.2. Let \( \mu_\theta : \mathcal{Z}_0(X_\theta) \rightarrow [0, 1] \) be defined via \( \mu_\theta(\emptyset) = 0 \), \( \mu_\theta(X_\theta) = 1 \) and \( \mu_\theta(Z_{[k,m]}(v)) = R_v^{(\ell)} \), for any \( k \leq 0 \leq m \in \mathbb{Z} \) and \( v \in \mathcal{L}_\theta^\ell \) with \( |v| = m - k + 1 \). Here, \( R_v^{(\ell)} \) denotes the right PF-eigenvector of the induced substitution \( \theta_\ell \). Then, \( \mu_\theta \) defines a \( \sigma \)-finite measure on \( \mathcal{Z}_0(X_\theta) \) and thereby also on \( (X_\theta, \mathcal{B}_\theta) \).

As a first step, we establish a consistency relation for the word frequencies which ensures the finite additivity of \( \mu_\theta \).

Lemma 5.3. Suppose \( v = v_1 \cdots v_{\ell_0} \in \mathcal{L}_\theta \), and fix an arbitrary \( \ell \in \mathbb{N} \), with \( \ell \geq \ell_0 \) and a position \( k \in \{1, \ldots, \ell - \ell_0 + 1\} \). If \( \mathcal{G}_k^\ell(v) = \{ u \in \mathcal{L}_\theta^\ell : u_{[k,k+\ell_0-1]} = v \} \), then

\[
R_v^{(\ell_0)} = \sum_{u \in \mathcal{G}_k^\ell(v)} R_u^{(\ell)}.
\]

Proof. Fix \( \omega \in \Omega \) and \( a \in \mathcal{A} \) such that \( \nu_u(\vartheta^n(a)(\omega)) \) converges to \( R_u^{(|u|)} \) as \( n \rightarrow \infty \), for all \( u \in \mathcal{L}_\theta \). Now,

\[
|\vartheta^n(a)(\omega)|_v = \text{card} \left\{ j \in \{1, \ldots, |\vartheta^n(a)(\omega)| - \ell_0 + 1 \} : |\vartheta^n(a)(\omega)|_{j,j+\ell_0-1} = v \right\}.
\]

Note that, for \( j \geq k \), \( \vartheta^n(a)(\omega)|_{j,j+\ell_0-1} = v \) if and only if \( \vartheta^n(a)(\omega)|_{j-k+1,j-k+\ell} = u \) for some \( u \in \mathcal{G}_k^\ell(v) \), as long as \( j - k + \ell \leq |\vartheta^n(a)(\omega)| \). That is, if we count the cardinality of the corresponding set, we miss at most \( \ell - \ell_0 \) subwords of type \( v \) in \( \vartheta^n(a)(\omega) \). More precisely, with \( i = j - k \),

\[
|\vartheta^n(a)(\omega)|_v = \text{card} \bigcup_{u \in \mathcal{G}_k^\ell(v)} \{ i \in \{0, \ldots, |\vartheta^n(a)(\omega)| - \ell \} : \vartheta^n(a)(\omega)|_{i+1,i+\ell} = u \} + O(\ell - \ell_0).
\]

Dividing both sides by \( |\vartheta^n(a)(\omega)| \) and taking the limit \( n \rightarrow \infty \) yield the desired relation. \( \square \)

Proof of Proposition 5.2. First, note that every \( Z \in \mathcal{Z}_0(X_\theta) \) is compact as a closed subset of a compact space. Hence, every countable disjoint union \( \bigcup_{i=1}^\infty Z_i = Z \in \mathcal{Z}_0(X_\theta) \) is in fact finite, that is \( \bigcup_{i=1}^\infty Z_i = \bigcup_{j \in I} Z_j \) for some finite subset \( I \in \mathbb{N} \). It thus suffices to show that \( \mu_\theta \) is finitely additive on \( \mathcal{Z}_0(X_\theta) \).

We can rewrite any cylinder set \( Z_{[j,m]}(v) \) as a union of cylinder sets that specify a larger set of positions

\[
Z_{[j,m]}(v) = \bigcup_{u \in \mathcal{G}_k^{2n+1}(v)} Z_{[\ell-n,n]}(u), \tag{11}
\]

for any \( n \geq \max\{|j|, |m|\} \) and \( k = j + n + 1 \). Lemma 5.3 yields

\[
\mu_\theta(Z_{[j,m]}(v)) = R_v^{(m-j+1)} = \sum_{u \in \mathcal{G}_k^{2n+1}(v)} R_u^{(2n+1)} = \sum_{u \in \mathcal{G}_k^{2n+1}} \mu_\theta(Z_{[\ell-n,n]}(u)), \tag{12}
\]

providing the additivity of \( \mu_\theta \) on disjoint unions of the form in Eq. (11). The same kind of relation holds for \( Z = X_\theta \) if we replace \( \mathcal{G}_k^{2n+1}(v) \) by \( \mathcal{L}_\theta^{2n+1} \).
The finite additivity of $\mu_\vartheta$ on disjoint unions of cylinder sets of the form $\bigcup_{j \in I} Z_j = Z$ follows in a straightforward manner. Just consider a common refinement into cylinder sets of the form $Z_{[-n,n]}(u)$ for large enough $n \in \mathbb{N}$. Hence, $\mu_\vartheta$ is a ($\sigma$-)finite measure on $\mathcal{Z}_0(X_\vartheta)$. □

**Remark 5.4.** It is clear by definition that $\mu_\vartheta$ is shift-invariant on the cylinder sets of the semi-algebra $\mathcal{Z}_0(X_\vartheta)$. This also implies that $\mu_\vartheta$ is shift-invariant as a measure on $(X_\vartheta, \mathcal{B}_\vartheta)$ — compare [30, Thm. 1.1]. We are therefore left with a shift-invariant measure space $(X_\vartheta, \mathcal{B}_\vartheta, \mu_\vartheta)$.

As an important step towards ergodicity, we give a sufficient condition for the almost sure convergence of word frequencies with respect to the measure $\mu_\vartheta$. For the following, we fix an arbitrary averaging sequence of intervals $I_n = [a_n, b_n]$ with $a_n \leq b_n$ and $a_n, b_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$ such that the length of the intervals $d_n = b_n - a_n + 1$ is strictly monotonously increasing.

**Lemma 5.5.** Let $(X_\vartheta, \mathcal{B}_\vartheta, \mu_\vartheta)$ be the measure space introduced above. For a given word $v \in \mathcal{L}_\vartheta^+$ and $\varepsilon > 0$, let $C^n_v := \{ u \in \mathcal{L}_\vartheta^+ : |\nu_v(u) - R_v^{(\ell)}| > \varepsilon \}$. Suppose that, for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \sum_{u \in C^n_v} R_u^{(n)} < \infty. \tag{13}$$

Then, for almost every element $y \in X_\vartheta$, the frequency of $v$ in $y$ with respect to the averaging sequence $\{(a_n, b_n)\}_{n \in \mathbb{N}}$ is well-defined and given by $R_v^{(\ell)}$. That is, there exists a set of full measure $A^v \subset X_\vartheta$ such that, for any $y \in A^v$, we have

$$\lim_{n \to \infty} \nu_v(y_{[a_n, b_n]}) = R_v^{(\ell)}. \tag{14}$$

**Proof.** Consider the random variables

$$X_n : X_\vartheta \to \mathbb{R}, \quad y \mapsto \nu_v(y_{[a_n, b_n]}) = \frac{|y_{[a_n, b_n]}|_v}{d_n},$$

for all $n \in \mathbb{N}$, and the constant random variable $X : X_\vartheta \to \mathbb{R}, y \mapsto R_v^{(\ell)}$. It is a well-known fact (by an application of the Borel–Cantelli lemma) that $(X_n)_{n \in \mathbb{N}}$ converges $\mu_\vartheta$-almost surely to $X$ if

$$\sum_{n=1}^{\infty} \mu_\vartheta(\{ y \in X_\vartheta \mid |X_n(y) - X(y)| > \varepsilon \}) < \infty,$$

for every $\varepsilon > 0$. Therefore, our claim follows from

$$\mu_\vartheta(\{ y \in X_\vartheta \mid |X_n(y) - X(y)| > \varepsilon \}) = \mu_\vartheta(\{ y \in X_\vartheta \mid y_{[a_n, b_n]} \in C^n_v \}) = \mu_\vartheta(\{ y \in X_\vartheta \mid y \in \bigcup_{u \in C^n_v} Z_{[a_n, b_n]}(u) \}) = \mu_\vartheta(\bigcup_{u \in C^n_v} Z_{[a_n, b_n]}(u)) = \sum_{u \in C^n_v} R_u^{(d_n)} \leq \sum_{u \in C^n_v} R_u^{(n)},$$

where the last step is a consequence of the assumption that $(d_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$. □
Remark 5.6. The proof of the ergodicity result for a family of random noble means substitutions in [19] follows a somewhat different approach. There, for a given \( \ell \in \mathbb{N}_0 \) and \( k, m \in \mathbb{Z} \), the subwords \( X_{[k,k+\ell]} \) and \( X_{[m,m+\ell]} \) of a \( \mu_\theta \)-distributed random element \( X \) are interpreted as i.i.d. random words, provided that \( k \) and \( m \) are separated by a certain distance. The small gap in the proof, mentioned in the introduction, amounts to the fact that independence for the given, fixed distance does not hold and should be replaced by asymptotic independence as the distance approaches infinity. However, the varied correlation structure of different examples of random substitutions precludes using a similar approach in the general case. We therefore speak of i.i.d. random elements only in the context of the Markov measure \( \mathbb{P} \) and use the estimate in Lemma 4.2 to relate this to properties of \( \mu_\theta \).

Next, we want to show that Eq. (13) indeed holds for any expansive primitive random substitution. The intuition behind this is the following. For a word \( u \) to be in \( C^n_\varepsilon \), it must be exceptional regarding the occurrences of the subword \( v \). The sum of the frequencies of such words \( u \) can not be too large since otherwise it would contradict the well-defined frequency of \( v \) in the typical limit words of the random substitution Markov process. The idea is to exhaust exceptional words \( u \) with inflation words and to show that \( u \in C^n_\varepsilon \) essentially requires a positive fraction of these inflation words to have exceptional relative words frequencies as well. We can then employ the large deviation estimates in Lemma 4.2 to show that these events must have summable probabilities.

Proposition 5.7. Let \( v \in \mathcal{L}^\ell_\theta \) be a \( \vartheta \)-legal word of length \( \ell \) and \( \{I_n\}_{n \in \mathbb{N}} \) an averaging sequence of strictly increasing length. Then, for \( \mu_\theta \)-almost every element \( y \in \mathbb{X}^\vartheta \), the frequency of \( v \) in \( y \) with respect to \( \{I_n\}_{n \in \mathbb{N}} \) is well-defined and given by \( R_v^{(\ell)} \).

Proof. It suffices to show that Eq. (13) indeed holds for all \( \varepsilon > 0 \). Let \( v \in \mathcal{L}^\ell_\theta \) be fixed in the following. Because of Proposition 3.1, it is for \( x, u \in \mathcal{L}^n_\theta \) and \( k \in \mathbb{N} \): \((M^n_k)_{xu} = \mathbb{E}\Phi(\vartheta^n_k(u))_x \). Hence, we can rewrite the frequency of \( x \in \mathcal{C}^n_\varepsilon \) as

\[
R_x^{(n)} = \frac{1}{\lambda^k} \left( M^n_k R^n_u \right)_x = \frac{1}{\lambda^k} \sum_{u \in \mathcal{L}^n_\theta} \mathbb{E}\Phi(\vartheta^n_k(u))_x R_u^{(n)} = \sum_{u \in \mathcal{L}^n_\theta} R_u^{(n)} \frac{1}{\lambda^k} \sum_{V \in \mathcal{D}_n} \mathbb{P}[\vartheta^n_k(u) = V] |V|, \tag{15}
\]

for any \( k \in \mathbb{N} \). Thus, with \( Q^n_k(u, C^n_\varepsilon) := \sum_{x \in \mathcal{C}^n_\varepsilon} Q^n_k(u, x) \), it is

\[
\sum_{x \in \mathcal{C}^n_\varepsilon} R_x^{(n)} = \sum_{u \in \mathcal{L}^n_\theta} R_u^{(n)} Q^n_k(u, C^n_\varepsilon). \tag{15}
\]

Given a large enough \( k \), our strategy is to find a bound for \( Q^n_k(u, C^n_\varepsilon) \), uniform in \( u \), that is small enough to be summable over \( n \). Note that

\[
Q^n_k(u, C^n_\varepsilon) = \frac{1}{\lambda^k} \sum_{V \in \mathcal{D}_n} \sum_{j=1}^{|V|} \mathbb{P}[\vartheta^n_k(u) = V] \delta_j(C^n_\varepsilon) = \frac{1}{\lambda^k} \sum_{m=1}^\infty \sum_{j=1}^m \mathbb{P}[\vartheta^n_k(u)_j \in C^n_\varepsilon \cap |\vartheta^n_k(u)| = m].
\]
Figure 3. Illustration of how the random elements $x^k$, $N = N_k$ and $\tilde{x}^k$ are built from $\vartheta^k(u)$, given a fixed length $n \in \mathbb{N}$, a number of inflation steps $k \in \mathbb{N}$ and an initial position $1 \leq j \leq |\vartheta^k(u_1)|$.

This is still a finite sum, since $|\vartheta^k(u)|$ can take only finitely many values. We can exchange the order of summation and express this as

$$Q^k_n(u, C^n_\varepsilon) = \frac{1}{\lambda^k} \sum_{j=1}^{\infty} \mathbb{P}[\vartheta^k_n(u)_j \in C^n_\varepsilon \land |\vartheta^k_n(u)| \geq j].$$

Now, by definition, $|\vartheta^k_n(u)(\omega)| = |\vartheta^k(u_1)(\omega)|$ and $\vartheta^k_n(u)(\omega)_j = \vartheta^k(u)(\omega)_{[j,j+n-1]}$ for every realisation $\omega$. Thus,

$$Q^k_n(u, C^n_\varepsilon) = \frac{1}{\lambda^k} \sum_{j=1}^{\mu^k} \mathbb{P}[\vartheta^k(u)_{[j,j+n-1]} \in C^n_\varepsilon \land |\vartheta^k(u_1)| \geq j],$$

(16)

where $\mu = \max_{\omega \in \Omega} \max_{a \in A} |\vartheta(a)(\omega)|$ is an upper bound for the 1-step inflation factor. Note that, at this point, $k \in \mathbb{N}$ is still a free parameter.

Let us go back to Eq. (16). For the bulk of the remainder, we fix a position $j \in \mathbb{N}_0$ and consider the random variable given by $x^k = \vartheta^k(u)_{[j,j+n-1]}$, under the additional condition that $|\vartheta^k(u_1)| \geq j$. This ensures that $x^k$ overlaps the inflation word $\vartheta^k(u_0)$. Note that, in general, $x^k$ depends only on a prefix of $u$, given by $u_{[1,N_k+1]}$, where we take $N_k \in \mathbb{N}_0$ to be minimal with this property. Clearly, $N_k$ depends on the chosen realisation of $x^k$. It is thus also a random variable and we may regard it as a stopping time. By the minimality of $N_k$, we obtain that

$$\vartheta^k(u_{[2,N_k(\omega)]})(\omega) \prec x^k(\omega) \prec \vartheta^k(u_{[1,N_k(\omega)+1]})(\omega)$$

holds for every realisation $\omega$. Note that the ‘interior’ of $x^k$ on the left hand side is a concatenation of inflation words

$$\tilde{x}^k := \vartheta^k(u_{[2,N_k]}) = \vartheta^k(u_2) \cdots \vartheta^k(u_{N_k}),$$

suppressing the dependence on $\omega$ in our notation. Compare Figure 3 for a graphical representation of those quantities. Note that the ratio of $|\tilde{x}^k|$ and $n$ approaches 1 as $n \to \infty$ since
Consider the event $x^k \in C_v^k$ that is equivalent to
$$\left| \frac{|\varphi^k|_{v}}{n} - R_{v}^{(\ell)} \right| > \varepsilon. \quad (17)$$

We wish to express this in terms of the exact inflation word $\varphi^k$. We can split up the occurrences of $v$ in $x^k$ into those that are contained in $\varphi^k$, those contained in the overlap of $x^k$ with the boundary inflation words $\varphi^k(u_1)$ or $\varphi^k(u_{N_k+1})$ and finally those $v$ that overlap both $\varphi^k$ and one of the boundary inflation words. For an illustration of the three qualitatively different positions of $v$ within $x^k$, compare Figure 4. Thus,
$$|\varphi^k|_v \leq |x^k|_v \leq |\varphi^k|_v + |\varphi^k(u_1)|_v + |\varphi^k(u_{N_k+1})|_v + 2\ell.$$

The last term originates from the fact that there are two boundaries of $\varphi^k$ with boundary inflation words each of which can contribute at most $|v| = \ell$ occurrences of $v$ to the sum. For large enough $n$, we therefore conclude that
$$\left| \frac{|x^k|_v}{n} - \frac{|\varphi^k|_v}{|\varphi^k|} \right| \leq \left| \frac{|x^k|_v}{n} - \frac{|\varphi^k|_v}{|\varphi^k|} \right| + \left| \frac{|\varphi^k|_v}{n} - \frac{|\varphi^k|_v}{|\varphi^k|} \right| \leq \frac{\varepsilon}{2},$$

such that Eq. (17) implies
$$\left| \nu_v(\varphi^{k}(u_{[2,N_k]})) - R_{v}^{(\ell)} \right| = \left| \frac{|\varphi^k|_v}{|\varphi^k|} - R_{v}^{(\ell)} \right| > \frac{\varepsilon}{2}. \quad (18)$$

Corresponding to $\frac{\varepsilon}{2}$, let us fix some $k \geq k_0$ as in Lemma 4.2. Note that $N_k$ grows linearly with $n$ because
$$N_k \leq n \leq |\varphi^k(u_{[1,N_k+1]})| \leq \mu^k(N_k + 1),$$

that is, there is some $C_1 > 0$ such that $C_1n \leq N_k(\omega) \leq n$ for large enough $n$, uniformly for all realisations $\omega \in \Omega$. An application of Lemma 4.2 yields
$$\mathbb{P}[(17)] \leq \mathbb{P}[(18)] = \sum_{N=C_1n}^{n} \mathbb{P} \left[ \left| \nu_v(\varphi^{k}(u_{[2,N]})) - R_{v}^{(\ell)} \right| > \frac{\varepsilon}{2} \wedge N_k = N \right]$$
$$\leq \sum_{N=C_1n}^{n} e^{-CN} \leq ne^{-CC_1n} \leq e^{-\tilde{C}n},$$
for large enough \(n\) and appropriate choice of \(\tilde{C} > 0\). Since this bound holds irrespective of the choice of \(j\), we have for any \(j \in \{1, \ldots, \mu^k\}\),

\[
P\left|\vartheta(u)[j,j+n-1] \in C^m_\varepsilon \land |\vartheta(u_1)| \geq j\right| \leq P[(17)] \leq e^{-\tilde{C} n}.
\]

Comparing with Eq. (16), we find

\[
Q^k_n(u, C^m_\varepsilon) \leq \mu^k e^{-\tilde{C} n} \leq e^{-C'n},
\]

for \(n\) larger than some \(n_0\), by another adjustment of the constant. Since this is summable over \(n\), we find, using Eq. (15),

\[
\sum_{n=1}^{\infty} \sum_{x \in C^m_\varepsilon} R_x^{(n)} \leq \sum_{n=1}^{n_0} \sum_{x \in C^m_\varepsilon} R_x^{(n)} + \sum_{n=n_0+1}^{\infty} \sum_{u \in L^m_\vartheta} R_u^{(n)} e^{-C'n} < \infty.
\]

This finishes the proof. \(\square\)

Finally, we are ready to prove the main result of this paper.

**Theorem 5.8.** Let \(\vartheta\) be an expansive, primitive random substitution. Then, the measure-theoretic dynamical system \((X_\vartheta, B_\vartheta, \mu_\vartheta)\) is ergodic.

**Proof.** It suffices to show that the identity

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} f(S^j x) = \int_{X_\vartheta} f \, d\mu_\vartheta \tag{19}
\]

holds for every \(f \in L^1(X_\vartheta, B_\vartheta, \mu_\vartheta)\) and almost all \(x \in X_\vartheta\).

First, let \(Z \in \mathcal{F}_0(X_\vartheta)\) be a cylinder set and \(f = 1_Z\) be the indicator function on \(Z\). More precisely, let \(Z = Z_{[k,m]}(v)\), where \(k \leq m \in \mathbb{Z}\), and \(v \in L^{m-k+1}_\vartheta\). Now, by definition of \(\mu_\vartheta\) and Proposition 5.7, we obtain

\[
\int_{X_\vartheta} f \, d\mu_\vartheta = \mu_\vartheta(Z_{[k,m]}(v)) = R^{(m-k+1)}_v = \lim_{N \to \infty} \frac{|x[k,m+N-1]|_\vartheta}{N} = \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} f(S^j x),
\]

for almost all \(x\). Hence, Eq. (19) is true if \(f\) is the indicator function on a cylinder set.

Second, let \(\Gamma := \{ \sum_{Z \in S} a_Z 1_Z \mid S \subseteq \mathcal{F}_0(X_\vartheta)\) is finite and \(a_Z \in \mathbb{C} \}\) be the set of simple functions on \((X_\vartheta, B_\vartheta, \mu_\vartheta)\). By linearity, the validity of Eq. (19) for indicator functions on cylinder sets extends to arbitrary functions in \(\Gamma\).

Third, since \(\Gamma\) is dense in \(L^1(X_\vartheta, B_\vartheta, \mu_\vartheta)\), the claim follows. \(\square\)

### 6. Unique, strict and intrinsic ergodicity

The last part of this paper is devoted to studying further properties of the ergodic measures \(\mu_\vartheta\) that we constructed in the preceding section. It turns out that random substitutions are a general enough class of objects to allow for a variety of different behaviours. We illustrate this by a number of examples. First, let us show that, in general, the measures \(\mu_\vartheta\) are not uniquely ergodic.
Example 6.1. Consider the random period doubling substitution

$$
\sigma : a \mapsto \begin{cases} 
ab & \text{with probability } p \\
ba & \text{with probability } q := 1 - p 
\end{cases}, 
b \mapsto aa,
$$

for some $0 < p < 1$. A short computation gives

$$
M_\sigma = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad M_{\sigma^2} = \begin{pmatrix} pq & q & 1 + p & 2 \\
1 - pq & p & q & 0 \\
1 - pq & 1 & 0 & 0 \\
pq & 0 & 0 & 0 \end{pmatrix}.
$$

Hence, we obtain

$$
R^{(2)}(\mu) = \begin{pmatrix} 2(1 - p + p^2) & 2(1 - p + p^2) & p - p^2 & 3(1 - p + p^2) \\
3(p^2 - p + 2) & 3(p^2 - p + 2) & 3(p^2 - p + 2) & 3(p^2 - p + 2) \end{pmatrix}^T. \quad \text{Therefore,}
$$

$$
\int_{X_\sigma} 1_{Z_0(bb)} \, d\mu = \mu_\sigma(Z_0(bb)) = R^{(2)}_{bb} = \frac{p - p^2}{3(p^2 - p + 2)} > 0.
$$

On the other hand, we have

$$
\frac{1}{N} \sum_{k=0}^{N-1} 1_{Z_0(bb)}(S^kx) = 0
$$

for all $N \in \mathbb{N}$ if $x$ is an element of the hull of the deterministic period doubling substitution (which is a subset of the random hull). Consequently, Eq. (19) does not hold for all $x \in X_\sigma$ and thus $\mu_\sigma$ is not uniquely ergodic.

In fact, it is possible to characterise those random substitution subshifts which are uniquely ergodic — compare [27, Thm. 27(b) and Cor. 28] for a proof of the following result.

**Proposition 6.2.** Let $(X_\vartheta, S)$ be a primitive random substitution subshift.

(a) $(X_\vartheta, S)$ is uniquely ergodic if and only if the right PF eigenvectors $R^{(\ell)}$ are independent of the probability vectors $p_i$, for every $\ell$ and every $i$.

(b) If $(X_\vartheta, S)$ is uniquely ergodic, it is also strictly ergodic.

Most interesting examples of random substitutions however, give rise to highly non-minimal subshifts $X$; see [27]. In that case, we can find an uncountable family of ergodic frequency measures with full support on $X$. Let us expand a bit on this point.

Recall that apart from the set of possible 1-level inflation words the choice of a random substitution also fixes the probability vectors $p_i$ as distributions of $\vartheta(a_i), a_i \in A = \{a_1, \ldots, a_m\}$. We can slightly change our point of view regarding the set of possible inflation words $\text{Im}(\vartheta(a_i))$ as fixed and taking $\{p_i\}_{1 \leq i \leq m}$ to be free parameters. As long as none of the probability entries is 0, the subshift $X_\vartheta$ is independent of those parameters. Given any $\ell \in \mathbb{N}$, the entries of $M_{\vartheta_\ell}$ are polynomials in the probabilities $(p_i)_k$ by construction and thus $R^{(\ell)}$ depends continuously on those parameters, compare $R^{(2)}$ in Example 6.1. Assuming $R^{(\ell)}$ is not constant in all of the $p_i$, it thus takes a continuum of values giving rise to a continuum of different ergodic frequency measures on $X_\vartheta$. We summarise this observation as follows.

**Corollary 6.3.** Let $X$ be a random substitution subshift. For any pair $\vartheta, \vartheta'$ of expanding primitive random substitution generating $X = X_\vartheta = X_{\vartheta'}$, the frequency measures $\mu_\vartheta$ and $\mu_{\vartheta'}$
are either equal or mutually singular. If \((X, S)\) is not uniquely ergodic, there is an uncountable set of random substitutions \(\Theta\) such that \(\mu_\vartheta \perp \mu_{\vartheta'}\) for all \(\vartheta, \vartheta' \in \Theta\) with \(\vartheta \neq \vartheta'\). \(\square\)

Next, let us focus on the question whether or not a primitive random substitution subshift can be intrinsically ergodic, that is whether it allows a unique measure which maximises the entropy. We know that there is at least one measure of maximal entropy, see [6, Sec. (17.15), Cor. 2]. Consider the following two examples.

**Example 6.4.** Let \(X\) be a topologically transitive shift of finite type. It was shown in [9] that \(X\) can be realised as a primitive RS-subshift. Since every topologically transitive shift of finite type is intrinsically ergodic [20], there are intrinsically ergodic RS-subshifts. At present, it remains open whether the measure of maximal entropy on \(X\) can be realised as a frequency measure \(\mu_\vartheta\) arising from some random substitution \(\vartheta\).

**Example 6.5.** The Dyck shift is a coded shift, which is not sofic. Moreover, it has two measures of maximal entropy [13]. Hence, it is not intrinsically ergodic. The Dyck shift can be realised as a primitive RS-subshift via the random substitution

\[
(\rightarrow \{ ((), ([]) , ([]) \}, \quad [\rightarrow \{ ([()], [()], [()] \}, \quad ) \rightarrow \{ (0), ([()]), ([]) \}, \quad \] \rightarrow \{ ([)], (0), ([]) \},
\]

where the assignment of (non-degenerate) probability vectors is arbitrary. Therefore, not every RS-subshift is intrinsically ergodic.

To get a better feeling, let us compute the metric entropy for a specific example. To do so, we need the following result, which follows by an application of standard techniques from linear algebra; compare [29, Thm. 1.1, Cor. 1].

**Lemma 6.6.** Let \(M \in \text{Mat}(d, \mathbb{R})\) be a primitive matrix such that \(M_{ij} = m_i \in \mathbb{R}_{\geq 0}\) for all \(i, j \in \{1, \ldots, d\}\). Then, \(\lambda = \sum_{i=1}^{d} m_i\) is the PF eigenvalue of \(M\), and the corresponding (normalised) right PF eigenvector is given by

\[
R = \frac{1}{\lambda} (m_1, \ldots, m_d)^T.
\]

\(\square\)

**Proposition 6.7.** Consider the random substitution

\[
\zeta: a \mapsto \begin{cases} \text{ab, with probability } p \\ \text{ba, with probability } 1-p \end{cases}, \quad b \mapsto \begin{cases} \text{ab, with probability } p \\ \text{ba, with probability } 1-p \end{cases},
\]

for \(0 < p < 1\). If \(\mu\) denotes the corresponding frequency measure, the metric entropy \(h_\mu\) is given by

\[
h_\mu = -\frac{1}{2} \left( p \log(p) + (1-p) \log(1-p) \right).
\]
Proof. By definition of the metric entropy and the frequency measures, we obtain from Lemma 6.6

\[ h_\mu = \lim_{n \to \infty} -\frac{1}{n} \sum_{w \in L_\xi^n} \mu([w]) \cdot \log(\mu([w])) = \lim_{n \to \infty} -\frac{1}{n} \sum_{w \in L_\xi^n} R_w^{(n)} \cdot \log(R_w^{(n)}) \]

\[ = \lim_{n \to \infty} -\frac{1}{n} \sum_{i=1}^{\left|L_\xi^n\right|} \frac{1}{\lambda} m_i^{(n)} \cdot \log \left( \frac{1}{\lambda} m_i^{(n)} \right) = \frac{1}{\lambda} \lim_{n \to \infty} -\frac{1}{n} \sum_{i=1}^{\left|L_\xi^n\right|} m_i^{(n)} \cdot \log(m_i^{(n)}), \]

where \( m_i^{(n)} = \mathbb{E}[\zeta_n(u)|u_i]\), for \( u, u_i \in L_\xi^n \), is independent of \( u \). From now on, we consider the subsequence \((m_i^{(2n)})_{n \in \mathbb{N}}\). Let \( u = u_1 \ldots u_{2n} \in L_\xi^{2n} \). Then, for all \( \omega \in \Omega \), \( \zeta(u)(\omega) \) is of the form

\[ \zeta(u)(\omega) = \frac{2n}{\bigodot} v^{(k)}, \quad v^{(k)} \in \{ab, ba\}, \]

where \( \bigodot \) denotes concatenation of words from left to right. Therefore,

\[ \zeta_{2n}(u)(\omega) = (v^{(1)} \ldots v^{(n)}) (v_1^{(1)} v^{(2)} \ldots v^{(n)} v_1^{(n+1)}). \]

This implies

\[ \mathbb{E}[|\zeta_{2n}(u)|u_i] = \sum_{k=1}^{2n} \mathbb{E}[\delta_{u_i}(\zeta(u)|k,k+2n-1)] = \sum_{k=1}^{2n} \mathbb{P}[\zeta(u)|k,k+2n-1] = u_i. \]

(21)

Note that, for a specific choice of the \( v^{(k)} \), \( k \in \{1, \ldots, n\} \),

\[ \mathbb{P}\left[\zeta(u)|[1,2n] = \frac{n}{\bigodot} v^{(k)}\right] = p^{\{k|v(k)=ab\}} \cdot (1 - p)^{\{k|v(k)=ba\}}. \]

Since the suffix and prefix \( v_1^{(1)} \) and \( v_1^{(n+1)} \) determine the inflation words \( v^{(1)} \) and \( v^{(n+1)} \) uniquely, we also have that

\[ \mathbb{P}\left[\zeta(u)|[2,2n+1] = v_2^{(1)} \left(\frac{n}{\bigodot} v^{(k)}\right) v_1^{(n+1)}\right] = p^{\{k|v(k)=ab\}} \cdot (1 - p)^{\{k|v(k)=ba\}}, \]

where \( k \) ranges from 1 to \( n + 1 \) on the right hand side. For \( u_i \) to be legal, it must either be of the form \( u_i = v^{(1)} \ldots v^{(n)} \) or \( u_i = v_2^{(1)} v^{(2)} \ldots v^{(n)} v_1^{(n+1)} \), determining the words \( v^{(k)} \in \{ab, ba\} \) uniquely in either case. The only words that can be written in both forms are \( u_a = ab \ldots ab \) and \( u_b = baba \ldots ba \). Note that there are exactly \( \binom{n}{j} \) different words of the form \( u = v^{(1)} \ldots v^{(n)} \) with \( \{|k = 1, \ldots, n|v^{(k)} = ab\} = j \) for every \( j \in \{0, \ldots, n\} \) and \( \binom{n+1}{j} \) pairwise different words of the form \( u_i = v_2^{(1)} v^{(2)} \ldots v^{(n)} v_1^{(n+1)} \) with \( \{|k = 0, \ldots, n|v^{(k)} = ab\} = j \) for
all \( j \in \{0, \ldots, n+1\} \). Together with Eq. (21), we obtain

\[
\sum_{i=1}^{L(2^n)} m_i^{(2n)} \cdot \log(m_i^{(2n)}) = \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \log(p^j (1-p)^{n-j}) + \sum_{j=0}^{n+1} \binom{n+1}{j} p^j (1-p)^{n+1-j} \log(p^j (1-p)^{n+1-j})
\]

\[+ \text{Err}_{2n}
\]

\[= \log(p)(2n+1)p + \log(1-p)(2n+1)(1-p) + \text{Err}_{2n},
\]

where the error term \( \text{Err}_{2n} = \text{Err}_a^{2n} + \text{Err}_b^{2n} \), with

\[\text{Err}_a^{2n} = (p^n+(1-p)^{n+1}) \log(p^n+(1-p)^{n+1}) - p^n \log(p^n) - (1-p)^{n+1} \log((1-p)^{n+1})
\]

\[\text{Err}_b^{2n} = (p^{n+1}+(1-p)^n) \log(p^{n+1}+(1-p)^n) - p^{n+1} \log(p^{n+1}) - (1-p)^n \log((1-p)^n)
\]

accounts for the fact that there are two words \( u_a \) and \( u_b \) for which both summands in Eq. (21) are non-zero. Clearly, we have \( \frac{1}{2n} \text{Err}_{2n} \to 0 \) for \( n \to \infty \). Hence, we conclude

\[h_\mu = -\frac{1}{\lambda} \lim_{n \to \infty} \frac{1}{2n} \left( \log(p)(2n+1)p + \log(1-p)(2n+1)(1-p) + \text{Err}_{2n} \right)
\]

\[= -\frac{1}{2} \left( p \log(p) + (1-p) \log(1-p) \right),
\]

since \( \lambda = 2 \). \( \square \)

**Corollary 6.8.** Let \( \zeta \) be as in the previous proposition. Then, the RS-subshift \( X_\zeta \) is intrinsically ergodic, and the frequency measure which corresponds to \( p = \frac{1}{2} \) is the measure of maximal entropy.

**Proof.** It is not difficult to see that the subshift generated by the graph in Figure 5 coincides with \( X_\zeta \). Therefore, it is a sofic shift and thus intrinsically ergodic.

The topological entropy of \( X_\zeta \) is given by \( h(X_\zeta) = \frac{1}{2} \log(2) \) which follows from simple combinatorics on \( X_\zeta \); alternatively, see [8] for a criterion that allows to read off \( h(X_\zeta) \) immediately from the form of the random substitution. Now, if \( p = \frac{1}{2} \), the previous proposition implies that \( h_\mu = \frac{1}{2} \log(2) \), which finishes the proof. \( \square \)

Consequently, if \( p = (p, 1-p)^T \) is the uniform distribution, we obtain the measure of maximal entropy. This result also holds for a larger class of random substitutions. For this,
consider the random substitution
\[ \vartheta : a_i \mapsto \begin{cases} 
  w^{(1)}, & \text{with probability } p_1 \\
  \vdots & \\
  w^{(\ell)}, & \text{with probability } p_\ell
\end{cases} 
\]
for all \( a_i \in \mathcal{A} \),

where \( \ell \in \mathbb{N} \), \( \mathbf{p} = (p_1, \ldots, p_\ell)^T \) is a probability vector, and \( w^{(1)}, \ldots, w^{(\ell)} \) are legal words of length \( N \in \mathbb{N} \), such that \( \vartheta \) is primitive and \( |w^{(j_1)}|_{a_i} = |w^{(j_2)}|_{a_i} \) for all \( j_1, j_2 \in \{1, \ldots, \ell\} \) and for all \( a_i \in \mathcal{A} \).

**Proposition 6.9.** Let \( \vartheta \) be as above. Then, \( \mathcal{X}_\vartheta \) is intrinsically ergodic. Moreover, the frequency measure corresponding to the vector \( \mathbf{p} = (\frac{1}{\ell}, \ldots, \frac{1}{\ell})^T \) is the measure of maximal entropy.

**Proof.** The subshift \( \mathcal{X}_\vartheta \) is a sofic shift, and therefore it is intrinsically ergodic; compare the previous proposition.

By Eq. (20), the metric entropy can be computed via
\[ h_\mu = \lim_{n \to \infty} -\frac{1}{n} \sum_{i=1}^{\lfloor \mathcal{L}_\vartheta^N \rfloor} \frac{1}{\lambda_i} m_i^{(n)} \cdot \log \left( \frac{1}{\lambda_i} m_i^{(n)} \right) = \lim_{n \to \infty} -\frac{1}{n} \sum_{i=1}^{\lfloor \mathcal{L}_\vartheta^N \rfloor} 1 \cdot m_i^{(n)} \cdot \log \left( \frac{1}{N} m_i^{(n)} \right), \]
since \( \lambda = N \) is the PF eigenvalue of \( M_\vartheta \). Let us consider the subsequence \( (Nn)_{n \in \mathbb{N}} \) of \( (n)_{n \in \mathbb{N}} \). Let \( u = u_1 \ldots u_{Nn} \in \mathcal{L}_\vartheta^{Nn} \). Then, we have for all \( \omega \in \Omega \)
\[ \vartheta(u)(\omega) = \bigcirc_{j=1}^{Nn} v^{(j)}, \quad v^{(j)} \in \{w^{(1)}, \ldots, w^{(\ell)}\} \]
by the inflation-word structure and thus
\[ \vartheta_{Nn}(u)(\omega) = (v^{(1)}_1 \ldots v^{(1)}_{Nn}) (v^{(2)}_1 \ldots v^{(2)}_{Nn}) \ldots (v^{(n+1)}_1 \ldots v^{(n+1)}_{Nn}), \]
due to the fact that all \( v^{(j)} \) have the same length \( N \). Analogously to Eq. (21), we obtain
\[ m_i^{(Nn)} = \sum_{k=1}^{N} \mathbb{P} \left[ \vartheta(u)[k,k+Nn-1] = u_i \right]. \]

Since every inflation word \( w^{(j)} \), \( j = 1, \ldots, \ell \) appears with the same probability \( \frac{1}{\ell} \), we find that, for all \( 1 \leq k \leq N \) and \( u_i \in \mathcal{L}_\vartheta^N \), the term \( \mathbb{P} \left[ \vartheta(u)[k,k+Nn-1] = u_i \right] \) is either 0 or
\[ \ell^{-2} \ell^{-(n-1)} \leq \mathbb{P} \left[ \vartheta(u)[k,k+Nn-1] = u_i \right] \leq \ell^{-(n-1)} \]
Also, since \( u_i \) is legal, at least one of the terms needs to be non-zero. This implies
\[ \frac{1}{N} \ell^{-2} \ell^{-(n-1)} \leq \frac{1}{N} m_i^{(Nn)} \leq \ell^{-(n-1)} \]
and
\[ 1 = \sum_{i=1}^{\lfloor \mathcal{L}_\vartheta^{Nn} \rfloor} \frac{1}{N} m_i^{(Nn)} \in \left[ \frac{1}{N} \ell^{-2} \ell^{-(n-1)}, \| \mathcal{L}_\vartheta^{Nn} \| \ell^{-(n-1)} \right], \]
which leads to
\[ \frac{1}{N} \ell^{-2} \ell^{-(n-1)} \leq \frac{1}{\| \mathcal{L}_\vartheta^{Nn} \|} \leq \ell^{-(n-1)}. \]
This together with Eq. (22) gives
\[ c_1 \frac{1}{|L^{(N_n)}_\vartheta|} \leq \frac{1}{N} m_i^{(N_n)} \leq c_2 \frac{1}{|L^{(N_n)}_\vartheta|}, \]
where \( c_1 := \frac{1}{N\ell^2} \) and \( c_2 := N\ell^2 \). That is, the entries of the right eigenvector \( R^{(N_n)} \) can differ from the uniform distribution only by a non-zero factor, which can be chosen independent of \( n \). This is enough to conclude that the metric entropy coincides with the topological entropy. More precisely, we have
\[
\begin{align*}
  h_\mu &= \lim_{n \to \infty} -\frac{1}{Nn} \sum_{i=1}^{|L^{(N_n)}_\vartheta|} \frac{1}{N} m_i^{(N_n)} \log \left( \frac{1}{N} m_i^{(N_n)} \right) \\
  &\geq \lim_{n \to \infty} -\frac{1}{Nn} \sum_{i=1}^{|L^{(N_n)}_\vartheta|} \frac{1}{N} m_i^{(N_n)} \log \left( c_1 \frac{1}{|L^{(N_n)}_\vartheta|} \right) \\
  &= \lim_{n \to \infty} -\frac{1}{Nn} \log \left( \frac{1}{|L^{(N_n)}_\vartheta|} \right) = h(X_\vartheta),
\end{align*}
\]
i.e. \( h_\mu \geq h(X_\vartheta) \). Since one always has \( h_\mu \leq h(X_\vartheta) \), the claim follows. \( \square \)

Similarly as in Corollary 6.8, this allows us to conclude

**Corollary 6.10.** Let \( \vartheta \) be as above, and denote by \( \mu \) the measure of maximal entropy, i.e. the frequency measure that corresponds to the vector \( p = (\frac{1}{2}, \ldots, \frac{1}{2})^\top \). Then,
\[
  h_\mu = \frac{1}{N} \log(\ell).
\]

**Remark 6.11.** Proposition 6.9 gives an alternative characterisation of the measure of maximal entropy for a certain class of RS-subshifts, which are sofic shifts. Numerical computations suggest that this result can be extended to include non-sofic subshifts, such as the RS-subshift which arises from the random Fibonacci substitution
\[
\vartheta_{\text{Fib}} : a \mapsto \begin{cases} 
  ba, & \text{with probability } p \\
  ab, & \text{with probability } 1 - p
\end{cases}, \quad b \mapsto a.
\]
The difficulty in this situation is the computation of the normalised right PF eigenvectors \( R^{(n)} \) for large \( n \).

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