Dynamical systems with finite stopping times.
Part 2: Dissipative Oscillations and their semigroups

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Abstract
In this paper, we model, classify and investigate the solutions of (normalized) second order odes with nonconstant continuous coefficients. We introduce a generalized frequency function as the solution of a nonlinear integro-differential equation, show its existence and then derive a general representation formula for solutions of (normalized) second order odes with nonconstant continuous coefficients. We mainly focus on problems with a finite stopping time $T$. However, our results remain true if $T = \infty$. Subsequently, the energy behaviour of dissipative oscillations as well as various classes of oscillations are discussed. Finally, it is shown how semigroups of oscillations can be used to design and analyze dissipative waves. In the complementary paper [13], we investigated dynamical system with finite stopping time and their consequences for diffusion and wave dissipation via suitable control functions.

1 Introduction
This paper is concerned with modeling, classifying and investigating dissipative oscillations that satisfy (normalized) second order odes with nonconstant coefficients (cf. [8]). Here we are specially interested in oscillations that stop after a finite time period $T$, which we call the stopping time if $T \in (0, \infty)$. (For convenience, we define $[0, T]$ to be $[0, \infty)$ if $T = \infty$.) In contrast to odes with constant coefficients, such equations do not require
a control term to guarantee a finite stopping time (cf. complementary paper \cite{13}). We present and discuss several examples and show how our results can be applied to the field of dissipative wave and wave propagation (cf. \cite{17, 23, 24, 11, 26, 28, 7, 1, 27, 19, 15}).

1.1 Two approaches?

There are essentially two approaches to model or describe dissipative oscillations. First one could claim that (any) reasonable oscillation is of the form

\[ v(t) = \left[ C_1 \cos(w(t)) + C_2 \frac{\sin(w(t))}{\omega(0)} \right] \varrho(t), \]

where \( C_1, C_2 \) are appropriate constants and \( w \) and \( \varrho^{-1} \) are nonnegative increasing functions. Obviously, \( w \) and \( \varrho \) are generalizations of \( \omega_0 t \) and \( e^{-a_0 t/2} \), where \( \omega_0 \) is a frequency and \( a_0 \) is a damping constant. For this approach, one has a "formula" at disposal and it is frequently employed by engineers and physicists. On the other hand, one may generalize the usual oscillation equation to nonconstant coefficients and claim that this equation includes all reasonable oscillations (and maybe other processes). In this case, the oscillation is the solutions of

\[ v''(t) + a(t) v'(t) + b(t) v(t) = f_0 \quad \text{for} \quad t \in (0, T) \quad \text{with} \quad v(0) := v(0+) = \varphi \quad \text{and} \quad v'(0) := v'(0+) = \psi, \]

where \( a, b : \mathbb{R} \to \mathbb{R} \) are continuous functions, \( f_0 \) is continuous satisfying \( \text{supp}(f_0) \subset (0, \infty) \) and \( \varphi, \psi \in \mathbb{R} \). This second strategy is usually employed by mathematicians. If these two approaches are essentially the same, then one expects a relation between the functions \( w, \varrho \) in (1) and the functions \( a, b \) in (2). This motivates our first investigation.

1.2 The solution formula

Indeed, we show that there exists a differentiable frequency function \( \omega : [0, T) \to \mathbb{R} \) satisfying a nonlinear integro-differential equation such that the solution of (2) (with \( f_0 = 0 \)) reads as follows

\[ v(t) = \left[ \varphi \cos(\tilde{\omega}(t)) + (\psi + \alpha(0) \varphi) \frac{\sin(\tilde{\omega}(t))}{\omega(0)} \right] \varrho(t), \]

where

\[ \alpha := \frac{a}{2} + \frac{\omega'}{2 \omega}, \quad \varrho(t) := \exp(-\tilde{\alpha}(t)) \quad \text{and} \quad \tilde{g}(t) := \int_0^t g(s) \, ds \]
for $t \in [0, T)$; $g$ is integrable. It turns out, if $a$ and $b$ are continuous on $[0, T)$, then the existence of this frequency function is always guaranteed. For the special case, $\omega = \omega_0$ and $a = a_0$ on $[0, T)$, the frequency function $\omega$ and the relaxation function $g$ reduce to the classical form

$$\tilde{\omega}(t) = \omega_0 t \quad \text{and} \quad g = e^{-a_0 t/2} \quad \text{for} \quad t \in [0, T),$$

where $b$ is constant and equals to $\omega_0^2 + a_0^2/4$. We note that there also exists a formula for $f \neq 0$ (cf. Theorem 2 below). Employing the general representation formula enables us to specify and analyze the three main types of dissipative oscillations. We show especially that the generalized aperiodic limit case can always be solved without using a numerical solution method. In particular, the formula shows how to model dissipative oscillation with a finite stopping time and is very valuable for many control problems related to oscillations. Due to lake of space, we do not discuss this later issue.

### 1.3 The energy function

For dissipative oscillations satisfying a second order ode with constant coefficient, it is easy to show that its energy function is always decreasing. Unfortunately, this is not always true for nonconstant coefficients. Hence we investigate criteria as well a special examples for which the energy function is decreasing. In contrast to the simpler processes like relaxations (cf. [13]), this is much more complicated. Of course, we do not claim that the energy function of an oscillating medium must always be decreasing; this depends on the application respectively situation.

### 1.4 Application to dissipative waves

Finally, we show how our results can be used to design and investigate dissipative oscillations. If a spherical dissipative wave $G$ (defined on $\mathbb{R}^4$) is modelled, then it determines a dissipative wave operator $A$ by

$$A(G)(x, t) = \delta(t) \delta(x) \quad \text{for} \quad (x, t) \in \mathbb{R}^4 \quad \text{with} \quad G|_{t<0} = 0$$

and $G$ can be considered as the fundamental solution of the respective dissipative wave equation (cf. [14 13]). We show that $K_1 := \rho \cos(\tilde{\omega}_*)$ for reasonable relaxation function $\rho$ and frequency function $\omega_*$ is of the form $\hat{K}_1 = e^{-\alpha}$ with some dissipation law $\alpha$ and $\hat{K}_R := e^{-\alpha R}$ defines a semigroup $(K_R)_{R \geq 0}$, which we call the semigroup of dissipative oscillations. Here $\hat{K}_1$ denotes the Fourier transform of $K$ (cf. Appendix). Because of this, the
distribution

\[ G(x, t) := \frac{\mathcal{K}_{|x|}\left(t - \frac{|x|}{c_0}\right)}{4 \pi |x|} \quad (c_0 \text{ constant}) \]

corresponds to a spherical dissipative wave for which \( c_0 \) is an upper bound for the speed of the wave front. We see that \( \mathcal{K}_{|x|} \) for \( x \in \mathbb{R}^3 \) characterizes the oscillation at position \( x \) that is caused by the wave \( G \). We consider some basic examples for which the oscillations \( \{ \mathcal{K}_R \mid R > 0 \} \) have different qualitative behaviour for \( R \in (0, 1) \), \( R = 1 \) and \( R > 0 \). In particular, we see that there are uncountable many (different) dissipation models for which the allenuation law is approximately a \textit{power law} with exponent 2 (for small absolute frequencies). In other words, there exists a vast amount of models for dissipative tissue similar to water. For all these models we have the governing equation, the fundamental solution, the dissipative semigroup and the dissipation law at our disposal.

This paper is organized as follows: In Section 2 we introduce the concept of frequency function and show its existence under fairly weak assumptions. Then we tackle the representation formula and the classification of general dissipative oscillations. This section is concluded with a discussion of the energy behaviour of dissipative oscillations. An investigation of the various classes of oscillations is carried out in the subsequent section. In Section 4 we show how semigroups of oscillations can be used to design and analyze dissipative waves. For the convenience of the reader, an appendix with some facts about the Fourier transform is included. Finally, a short reflection about this paper is given in a concluding section.

2 Second order odes with time dependent coefficient

In this section, we introduce the concept of a \textit{frequency function} \( \omega : (0, T) \to \mathbb{R} \cup i\mathbb{R} \) \((T \in (0, \infty])\), which generalizes the classical frequency and which is a solution of an \textit{integro differential equation}. Then we show its existence and derive the general representation formula for dissipative oscillations that make use of the frequency function. Moreover, we give a classification of damped oscillations and shortly discuss the energy behaviour of dissipative oscillations.
From our point of view, the governing equation for dissipative oscillations is the second order ODE with time dependent coefficients \(2\). According to Chapter VIII in \(8\), this problem has always a twice differentiable solution if \(a\) and \(b\) are continuous on \((0, T)\). Here \(T \in (0, \infty]\) denotes the stopping time of the oscillation.

### 2.1 The frequency function and the solution formula

**Definition 1.** Let \(T \in (0, \infty]\) and \(a, b : [0, T) \rightarrow [0, \infty)\) be continuous. If there exists a differentiable function \(\omega : (0, T) \rightarrow \mathbb{R} \cup i\mathbb{R}\) satisfying

\[
\int_{0}^{t} \omega^2(s) \, ds = \int_{0}^{t} \left[ b(s) - \left( \frac{a(s)}{2} \right)^2 + \left( \frac{\omega'}{2\omega} \right)^2(s) \right] \, ds
- \frac{a(t)}{2} + \frac{a(0)}{2} - \frac{\omega'}{2\omega}(t) + \frac{\omega'}{2\omega}(0),
\]

then we call \(\omega\) the frequency function of the process \(v\) modeled by \(2\). We define the frequency at \(t = t_0 \in (0, T)\) to be zero, if

\[
\int_{0}^{t_0} \left[ b(s) - \left( \frac{a(s)}{2} \right)^2 \right] \, ds - \frac{a(t_0)}{2} + \frac{a(0)}{2} = 0
\]

holds. Moreover, we define \(\omega(0) := \omega(0+)\) and \(\omega(T) := \omega(T-)\).

If the coefficient \(a\) is differentiable, then the condition for the frequency simplifies to

\[
\omega^2 - \left( \frac{\omega'}{2\omega} \right)^2 + \left( \frac{\omega'}{2\omega} \right)' = b - \left( \frac{a}{2} \right)^2 - \frac{a'}{2},
\]

and for the special case \(\omega' = 0\), we arrive at

\[
\omega(t) = \sqrt{b(t) - \left( \frac{a(t)}{2} \right)^2} - \frac{a'(t)}{2} = \omega(0) \quad \text{for} \quad t \in (0, T).
\]

In the following Lemma and Theorem, we show that a frequency function exists if the coefficients \(a\) and \(b\) are continuous and prove the claimed representation formula for the solution of problem \(2\).

**Lemma 1.** Let \(\omega_0 > 0, \omega_1 \in \mathbb{R}, a, b : [0, T) \rightarrow \mathbb{R}\) be continuous,

- \(v_1\) denote the solution of \(2\) with \(\varphi := 0\) and \(\psi := 1\),
\[ \omega_2 \text{ denote the solution of (3) with } \varphi := \frac{1}{\omega_0} \text{ and } \psi := -\frac{1}{\omega_0} \left( \frac{a(0)}{2} + \frac{\omega_1}{2\omega_0} \right). \]

Then \( v_1 \) and \( v_2 \) exist and are twice continuously differentiable on \((0, T)\). If \( v_1^2(0) + v_2^2(0) > 0 \), then

\[ \omega(t) := \frac{v_1(t)v_2(t) - v_1(t)v_2'(t)}{v_1'(t) + v_2'(t)} \quad \text{for } t \in I_\omega \]

is a real valued frequency function on \( I_\omega := \{ t \in [0, T) \mid v_1^2(t) + v_2^2(t) \neq 0 \} \) and satisfies \( \omega(0) = \omega_0 > 0 \) and \( \omega'(0) = \omega_1 \). If \( v_2^2(0) - v_1^2(0) > 0 \), then

\[ i \epsilon(t) := \omega(t) := -\frac{v_1'(t)v_2(t) - v_1(t)v_2'(t)}{v_2'(t) - v_1'(t)} \quad \text{for } t \in I_\epsilon \]

is an imaginary valued frequency function on \( I_\epsilon := \{ t \in [0, T) \mid v_2^2(t) - v_1^2(t) \neq 0 \} \) and \( \epsilon(0) = \omega_0 > 0 \) and \( \epsilon'(0) = \omega_1 \).

**Proof.** Because of the continuity of \( a \) and \( b \), the solutions \( v_1 \) and \( v_2 \) exist and are twice continuously differentiable and real valued. In this proof, we denote the distributive derivative of a continuous function \( f \) by \( f' \). Then the proof is more instructive.

a) Let \( v_1^2(0) + v_2^2(0) > 0 \). We have to show that

\[ X_\omega := \omega^2 - \left( \frac{\omega'}{2\omega} \right)^2 - \left( \frac{\omega'}{2\omega} \right) - b + \left( \frac{a}{2} \right)^2 + \frac{a'}{2} = 0 \quad \text{on } I_\omega, \]

From (10) together with \( v_j' = -a v_j' - b v_j \) for \( j = 1, 2 \), we obtain

\[ \frac{\omega'}{2\omega} = -\frac{v_1 v_1' + v_2 v_2'}{v_1^2 + v_2^2} + \frac{a}{2}, \]

which implies

\[ \left( \frac{\omega'}{2\omega} \right)^2 = \frac{a^2}{4} + a \frac{v_1 v_1' + v_2 v_2'}{v_1^2 + v_2^2} + \frac{(v_1 v_1' + v_2 v_2')^2}{(v_1^2 + v_2^2)^2} \]

and

\[ \left( \frac{\omega'}{2\omega} \right)' = b - \omega^2 - \frac{a'}{2} + a \frac{v_1 v_1' + v_2 v_2'}{v_1^2 + v_2^2} + \frac{(v_1 v_1' + v_2 v_2')^2}{v_1^2 + v_2^2}. \]

Consequently,

\[ \left( \frac{\omega'}{2\omega} \right)' - \left( \frac{\omega'}{2\omega} \right)^2 = b - \omega^2 - \left( \frac{a}{2} \right)^2 - \frac{a'}{2}. \]

\(^1\text{Cf. Formula in Theorem 1 below.}\)
and therefore
\[(12) \quad X_\omega = 0 \quad \text{on} \quad I_\omega.\]

From the definition of \( v_1 \) and \( v_2 \), it follows that \( v_1(0+) = 0 \), \( v_1'(0+) = 1 \), \( v_2(0+) = \frac{1}{\omega_0} \neq 0 \) and \( v_2(0+) = -\frac{1}{\omega_0} \left( \frac{a(0)}{2} + \frac{\omega_1}{\omega_0} \right) \) and therefore
\[\omega(0+) = \omega_0 > 0 \quad \text{and} \quad \omega'(0+) = \omega_1.\]

This concludes item a).

b) Now let \( v_2^2(0) - v_1^2(0) > 0 \). We note that \( \epsilon \) in (11) can be obtained from (10) if \( v_1 \) is replaced by \( i v_1 \). Because \( v_1 \) satisfies equation \( w'' + a w' + b w = 0 \), \( i v_1 \) satisfies the same equation and thus identity (12) is also true for this case.

Similar as in Item a), the definition of \( v_1 \) and \( v_2 \) imply
\[i \epsilon(0+) = \omega(0+) = i \omega_0 > 0 \quad \text{and} \quad (i \epsilon)'(0+) = \omega'(0+) = i \omega_1,
\]
which concludes the proof.

\[\square\]

**Theorem 1.** Let \( \phi, \psi \in \mathbb{R} \), \( a, b : [0, T) \to \mathbb{R} \) be continuous and \( \omega \) be as in Lemma 1 for given \( \omega_0 > 0 \) and \( \omega_1 \in \mathbb{R} \). Then the frequency function \( \omega \) is well-defined on \([0, T)\) with \( \omega(0) = \omega_0 \) and \( \omega'(0) = \omega_1 \), and the solution of (2) is given by (3) with \( \alpha \) and \( \varrho \) defined by (4).

**Proof.** a) Existence. Let \( v_1, v_2, I_\omega \) and \( I_\epsilon \) be defined as in Lemma 1. The existence of the frequency function on \( I_\omega \) or \( I_\epsilon \), respectively, follows from Lemma 1. It is clear that each set \( I_\omega \) and \( I_\epsilon \) differs from \([0, T)\) by a countable set of real numbers. In item c) and d) below, we show with the help of the representation formula that \( v_1^2 + v_2^2 \) or \( v_2^2 - v_1^2 \) has no zeros on \([0, T)\) if \( v_1^2(0) + v_2^2(0) > 0 \) or \( v_2^2(0) - v_1^2(0) > 0 \) holds, respectively.

b) Solution formula. Let \( \lambda := \alpha \pm i \omega \) with \( \alpha := \frac{a}{2} + \frac{\omega_1}{\omega_0} \). Because \( \omega \) satisfies condition \((7)\), it follows that \( \lambda \) satisfies the (integrated) characteristic equation of \((2)\), i.e.
\[-\lambda(t) + \lambda(0) + \int_0^t \lambda^2(s) \, ds - \int_0^t a(s) \lambda(s) \, ds + \int_0^t b(s) \, ds = 0.\]

Thus
\[v_1(t) := \exp(-\tilde{\alpha}(t) - i \tilde{\omega}(t)) \quad \text{and} \quad v_2(t) := \exp(-\tilde{\alpha}(t) + i \tilde{\omega}(t))\]

are solutions of the equation in \((2)\). Because these solutions are linear independent for non-vanishing \( \omega \), the solution of \((2)\) reads as follows
\[v = [C_1 \exp(-i \tilde{\omega}) + C_2 \exp(i \tilde{\omega})] \varrho \quad \text{with} \quad \varrho := \exp(-\tilde{\alpha}).\]
From this together with \( v(0) = \varphi \) and \( v'(0) = \psi \), we get
\[
C_1 = \frac{\varphi}{2} + i \frac{\psi + \alpha(0) \varphi}{2\omega(0)} \quad \text{and} \quad C_2 = \frac{\varphi}{2} - i \frac{\psi + \alpha(0) \varphi}{2\omega(0)},
\]
which yields the claimed solution formula.

c) \( v_1^2 + v_2^2 \) has no zeros on \((0, T)\) if \( v_1^2(0) + v_2^2(0) > 0 \). From the solution formula and the definition of \( v_j \) for \( j = 1, 2 \), we get
\[
v_1(t) = \sin \left( \frac{\tilde{\omega}(t)}{\omega(0)} \right) \varrho(t) \quad \text{and} \quad v_2(t) = \cos \left( \frac{\tilde{\omega}(t)}{\omega(0)} \right) \varrho(t)
\]
and thus \( v_1^2(t) + v_2^2(t) \neq 0 \) for all \( t \in (0, T) \).

d) \( v_2^2 - v_1^2 \) has no zeros on \((0, T)\) if \( v_2^2(0) - v_1^2(0) > 0 \). Similarly as before, the solution formula and the definition of \( v_1 \) and \( v_2 \) imply
\[
v_1(t) = \frac{\sinh \left( \tilde{\epsilon}(t) \right)}{\omega_0} \varrho(t) \quad \text{and} \quad v_2(t) = \frac{\cos \left( \tilde{\epsilon}(t) \right)}{\omega_0} \varrho(t)
\]
and hence \( v_2^2(t) - v_1^2(t) \neq 0 \) for all \( t \in (0, T) \). This concludes the proof.

We now present an example of a purely frequency dependent damped oscillation.

**Example 1.** Let \( T \in (0, \infty], \omega_0 > 0, \omega_1 \in \mathbb{R} \) and \( w : (0, \infty) \to [0, \infty) \) be a differentiable function that is increasing and that satisfies \( w(0+) = \omega_0, w'(0+) = \omega_1 \) and \( \lim_{t \to T} w(t) = \infty \). Moreover, let
\[
a := \frac{w'}{w} \quad \text{and} \quad b := w^2 + a'.
\]

Then \( a, b \) and \( \omega := w \) satisfy condition \( (7) \) and \( \frac{\omega(t)}{w(t)} = \exp \left( - \int_0^t a(s) \, ds \right) = \varrho(t) \) for \( t \in (0, T) \) and consequently the solution of \( (2) \) is given by
\[
v(t) = \varphi \frac{\omega_0}{\omega(t)} \cos \left( \tilde{\omega}(t) \right) + \left( \psi + \frac{\omega_1}{\omega_0} \varphi \right) \frac{\sin \left( \tilde{\omega}(t) \right)}{\omega(t)}.
\]

Because of \( \lim_{t \to T} \omega(T- \omega(T-) = \infty \), it follows that \( \lim_{t \to T} v(t) = 0 \). This example shows that the relaxation function \( \varrho \) may depend only on the frequency.

Before we consider the second main theorem in this paper, we think that it is very instructive to the reader to check the following statement. The function
\[
v(t) := \int_0^t f_0(s) \exp \left( - \int_0^{t-s} \alpha(r + s) \, dr \right) \, ds \, H(t) \quad \text{for} \quad t \in \mathbb{R}
\]

8
for continuous $\alpha$ and $f$ with $\text{supp}(f) \subset (0, \infty)$ is the unique solution of
\[
v' + \alpha v = f \quad \text{on } \mathbb{R} \quad \text{with } v|_{t<0} = 0,
\]
due to $\frac{d}{dt} \left( \int_0^{t-s} \alpha(r+s) \, dr \right) = \alpha(t)$.

**Theorem 2.** Let $a, b, \omega$ be as in Lemma 1 and $\alpha, \varrho$ be as in Theorem 1. Then the solution of (2) with $\varphi = \psi = 0$ and continuous $f$ satisfying $\text{supp}(f) \subset (0, \infty)$ is given by
\[
v(t) = \int_0^t f_0(s) \frac{\sin(\tilde{\omega}_s(t-s))}{\omega_s(0)} \varrho_s(t-s) \, ds \, H(t) \quad \text{for } t \in \mathbb{R},
\]
where $\omega_s := \omega(\cdot + s)$, $\alpha_s := \alpha(\cdot + s)$ and $\varrho_s := \exp(-\tilde{\alpha}_s(\cdot - s))$.

**Proof.** It is clear that the solution of
\[
v'' + a v' + b v = f_0 \quad \text{on } \mathbb{R} \quad \text{with } v|_{t<0} = 0 \quad \text{and } f_0|_{t<0} = 0,
\]
is equal to $v(t) := \int_0^t v_s \, ds \, H(t)$ for $t \in \mathbb{R}$, where $v_s$ solves
\[
v''_s + a v'_s + b v_s = f_0(s) \delta(t-s) \quad \text{on } \mathbb{R} \quad \text{with } v_s|_{t<s} = 0.
\]
The claimed formula follows from this and $v_s(t) = f_0(s) \frac{\sin(\tilde{\omega}_s(t-s))}{\omega_s(0)} \varrho_s(t-s)$.

### 2.2 Classification of oscillations

The frequency of a standard oscillation, i.e. the solution of (2) with constant $a$ and $b$, is defined by $\omega = \sqrt{b - \frac{a^2}{4}}$. With this definition the weakly dissipative oscillation the aperiodic limit and creeping can be classified by $\omega > 0$, $\omega = 0$ and $\omega < 0$, respectively. This motivates:

**Definition 2.** Let $T$, $a$, $b$, $\omega$ be as in Definition 1, $I \subseteq [0, T)$ and $v$ be the solution of (2) on $(0, T)$.

1) We call $v$ a weakly dissipative oscillation on $I$, if $\omega^2(t) > 0$ for $t \in I$.
2) We call the solution $v$ an aperiodic limit on $I$ if $\omega^2(t) = 0$ for $t \in I$.
3) We call $v$ a creeping process on $I$, if $\omega^2(t) < 0$ for $t \in I$. 

9
4) We call $v$ a mixed oscillation, if none of the previous items are true on the whole time interval $(0, T)$.

The following example shows that a solution of $v'' + av' + bv = 0$ with $a > 0$ need not be damped.

**Example 2.** Let $T \in (0, \infty]$, $\omega : (0, T) \to (0, \infty)$ be differentiable with $\omega > 0$, $a := -\frac{\omega'}{\omega}$ and $b : (0, T) \to (0, \infty)$ be defined by $b = \int_0^T \omega^2(s) \, ds + \frac{\omega'}{\omega}(t) - \frac{\omega'}{\omega}(0)$. Then $\alpha$ vanishes and the solution of problem (2) reads as follows $v = \varphi \cos(\bar{\omega}) + \psi \sin(\bar{\omega})$. Here we assume that $\varphi$ and $\psi$ do not both vanish.

Although $v$ solve the "damped" oscillation equation with non-vanishing $a$ and $b$, it does not describe a damped oscillation. However, its frequency function is not constant.

For example, if $\omega(t) = \omega_0 e^{-\eta t}$ for $t \in [0, T)$, where $T \in (0, \infty]$ and $\eta, \eta'$ are positive monotonic increasing functions with $\lim_{t \to T} \eta(t) = \infty$, then $a = \eta$ and $b = \omega^2 + \eta'$. We see that $a$ is monotonic increasing with $\lim_{t \to T} a(t) = \infty$ and that $b$ is monotonic increasing on $(T_0, T)$ for some $T_0 > 0$ and $\lim_{t \to T} b(t) = \infty$. Moreover, the oscillation $v$ does not stop at $T$, i.e. not both numbers $v(T)$ and $v'(T)$ vanish.

In the next example, we show that the relaxation function $\varrho$ can be modeled independently of the frequency function $\omega$ for appropriately chosen coefficients $a$ and $b$.

**Example 3.** Let $T \in (0, \infty]$, $\omega : (0, T) \to (0, \infty)$ be twice differentiable with $\omega > 0$ and $\varrho = \exp \left( -\tilde{\alpha} t \right)$ with monotonic increasing $\alpha : (0, T) \to (0, \infty)$ satisfying $\lim_{t \to T} \alpha(t) = \infty$. Moreover, let $a := 2\alpha - \frac{\omega'}{\omega}$ and $b := \omega^2 + \alpha^2 - \alpha \frac{\omega'}{\omega} + \alpha'$. Then it follows from Theorem 1 that $v := \left[ \varphi \cos(\bar{\omega}) + (\psi + \alpha(0) \varphi) \frac{\sin(\bar{\omega})}{\omega(0)} \right] \varrho$ solves problem (2).

2.3 Short reflection about the energy

For many applications it is required that the coefficients $a$ and $b$ in (2) are such that the resulting oscillation has a decreasing energy function (that converges to zero as time approaches the stopping time). Hence we require some facts about the energy of oscillations. Of course, we do not exclude that there are materials that can absorbs or emits energy such that the oscillation energy is not always decreasing.

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2 If $\omega$ is twice differentiable, then $b = \omega^2 + \left( \frac{\omega'}{\omega} \right)'$. 

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10
Proposition 1. Let $T \in (0, \infty]$, $a : [0, T) \to \mathbb{R}$ be a non-negative, piecewise differentiable function satisfying $a(T^-) = \infty$, $b : [0, T) \to \mathbb{R}$ be a non-negative continuous function and $\varphi, \psi \in \mathbb{R}$. If

(i) $b$ is the zero function or constant or $b' \leq 0$, or

(ii) $b$ is piecewise differentiable and

$$b'(t) v^2(t) \leq 2a(t) (v')^2(t) \quad \text{for} \quad t \in (0, T),$$

then the energy $E$ of the oscillation described by (3) is decreasing and converges to zero for $t \to T^-$. 

Proof. From $E \propto \frac{1}{2} ((v')^2 + b v^2)$ and the ode in (2), we get $E' = -a (v')^2 + \frac{1}{2} b v^2$ and thus

$$E'(t) \leq 0 \iff b'(t) v^2(t) \leq 2a(t) (v')^2(t) \quad \text{for} \quad t \in (0, T),$$

which proves the claim for item (ii). The claim about item (i) follows at once from item (ii) with $b = 0$, $b' = 0$ and $b' \leq 0$, respectively, and the non-negativity of $a$. \hfill \Box

Remark 1. We now present a simple idea to model a dissipative oscillation for a given (non-constant) differentiable frequency function such that its energy is always decreasing. Let $a_0 > 0$ and $\omega$ be a given frequency function. First we choose the coefficient $b$ as the constant $b_0 := \left( \frac{a_0}{2} \right)^2 + \omega^2 - \left( \frac{\omega'}{\omega} \right)^2 + \left( \frac{\omega'}{\omega} \right)^2$. Second the function $a$ is determined by solving (7) with $a(0) = a_0$ and $a(0)' = 0$. That is to say, $a$ solves

$$\left( \frac{a}{2} \right)^2 + \frac{a'}{2} = R \quad \text{on} \quad (0, T) \quad \text{with} \quad a(0) = a_0 \quad \text{and} \quad a(0)' = 0,$$

where $R := b - \omega^2 + \left( \frac{\omega'}{\omega} \right)^2 - \left( \frac{\omega'}{\omega} \right)^2$. Then the full oscillation model is specified. Apart from the existence problem, the main issue is to solve the non-linear equation for $a$.

Remark 2. We note that an equivalence relation can be defined on the set of oscillations $O := \{ v \in C^2(0, T) \mid v \text{ solves (2) for given } a, b \in C(0, T) \}$ by

$$v \cong v_* \quad \iff \quad b_* - \frac{a_*^2}{4} - \frac{a_*'}{2} = b - \frac{a^2}{4} - \frac{a'}{2} \quad \text{on} \quad (0, T),$$

where $v$ solves solves (3) and $v_*$ solves (3) with $(a, b)$ replaced by $(a_*, b_*)$. Indeed, this is equivalent to $v$ and $v_*$ have the same frequency function, but different relaxation functions. Thus, if the pair $(a, b)$ is given, then a reasonable mathematical problem is to find a pair $(a_*, b_*)$ such that the energy function of the respective oscillation $v_*$ is always decreasing.
There is no general and simple strategy how to model reasonable oscillations or the respective odes such that the energy functions of the described processes are decreasing. However, in the following chapter, we give some examples with decreasing energy functions.

3 Investigation of the main types of oscillations

Now we start our investigation of the three main types of oscillations. In particular, we discuss some examples and compare them to the known ”classical” results. A view mixed examples are also considered.

3.1 Aperiodic limit case

In case of an oscillation with constant coefficient, the solution of the aperiodic limit case is a linear combination of \( \exp(-\lambda t) \) and \( t \exp(-\lambda t) \), where \( \lambda \) is the double solution of characteristic equation \( \lambda^2 + a \lambda + b = 0 \) with \( b := \frac{a^2}{4} \).

Now we present an example (toy problem) that shows that in case of time dependent coefficient, the aperiodic limit case is slightly different.

Example 4. We consider the oscillation equation

\[
(15) \quad v''(t) + 2 t v'(t) + (t^2 + 1) v = 0 \quad \text{for} \quad t \in (0, T)
\]

for some \( T \in (0, \infty) \) with \( a = 2 t \) and \( b = t^2 + 1 \). Because \( b = \frac{a^2}{4} + \frac{a'}{2} \) holds, this oscillation is aperiodic (cf. Definition 4). It is easy to see that \( \lambda_1(t) = t \) as well as \( \lambda_2(t) = t - \frac{\mu}{1+\mu t} \) for any \( \mu \in \mathbb{R} \setminus \{0\} \) are solutions of the characteristic equation \(-\lambda'(t) + \lambda^2(t) - 2 t \lambda(t) + t^2 + 1 = 0 \) for \( t \in (0, T) \) if \( \mu \geq 0 \) and for \( t \in (0, T) \setminus \{-1/\mu\} \) if \( \mu < 0 \). That is to say, for each \( \mu \in \mathbb{R} \)

\[
v_\mu = \exp \left( -\frac{1}{2} t^2 + \int_0^t \frac{\mu}{1+\mu s} \, ds \right) = |1 + \mu t| \exp \left( -\frac{1}{2} t^2 \right)
\]

is a solution of the above ode and thus the general solution reads as follows

\[v(t) = C_1 \exp \left( -\frac{1}{2} t^2 \right) + C_2 |1 + \mu t| \exp \left( -\frac{1}{2} t^2 \right),\]

where \( C_1 \) and \( C_2 \) are constants. Although, we have infinitely many different solutions of the characteristic equation, they lead ”only” to two linear independent solutions of the homogeneous ode \((15)\). Moreover, we see that there are two cases \( \mu > 0 \) and \( \mu < 0 \).

\(^3\)For this example, \( T \) is not the stopping time, i.e. \( v(T) \) and \( v'(T) \) are not both zero.
• If $\mu > 0$, then $t \mapsto \exp \left(-\frac{1}{2} t^2\right)$ and $t \mapsto t \exp \left(-\frac{1}{2} t^2\right)$ generate the solution, as for the case with constant coefficient.

• If $\mu < 0$, then $t \mapsto \exp \left(-\frac{1}{2} t^2\right)$ and $t \mapsto \exp \left(-\frac{1}{2} t^2\right) |1 + \mu t|$ are linear independent solutions of the homogeneous ode on the intervals $(0, -\frac{1}{\mu})$ and $(-\frac{1}{\mu}, \infty)$, respectively. This solution is not a classical one on $(0, T)$, i.e. it is not twice differentiable, due to the kink at $t = -\frac{1}{\mu}$.

In contrast to weak oscillation and creeping oscillation, the general aperiodic limit case can be completely analyzed. Indeed, we have

**Theorem 3.** Let $T \in (0, \infty]$, $a, b : [0, T) \to \mathbb{R}$ be positive and continuous satisfying

$$
\int_0^t \left[ b(s) - \frac{a^2(s)}{4} \right] \, ds - \left( \frac{a(t)}{2} - \frac{a(0)}{2} \right) = 0 \quad \text{for} \quad t \in (0, T),
$$

i.e. the frequency defined as in Definition 1 vanishes. Then the classical solution of (2) is given by

$$
v(t) = \left[ \varphi + \left( \psi + \frac{a(0)}{2} \varphi \right) t \right] \exp \left( -\frac{1}{2} \int_0^t a(s) \, ds \right).$$

**Proof.** First of all, because $a$ and $b$ are continuous on $(0, T)$ the solution of (2) exists and is unique on $(0, T)$. The claim follows from the solution formula (3) and the fact that $\alpha = \frac{a}{2}$ holds for the aperiodic limit. \(\square\)

If $b$ is constant for the aperiodic limit case, then we have for $a$ only three possibilities. None of them stops at a finite time. Indeed, this is shown by the following two Corollaries.

**Corollary 1.** Let $b_0 > 0$ be a constant and $v$ be an aperiodic oscillation with $b = b_0$ and $a$ non-negative. If $a(t) \neq 2 \sqrt{b_0}$ for all times, then the stopping time is infinite and

$$a(t) = a_+(t) := 2 \sqrt{b_0} \frac{\mu_0 e^{2 \sqrt{b_0} t} - 1}{\mu_0 e^{2 \sqrt{b_0} t} + 1} \quad \text{if} \quad a_0 < 2 \sqrt{b_0},$$

and

$$a(t) = a_-(t) = 2 \sqrt{b_0} \frac{\mu_0 e^{2 \sqrt{b_0} t} + 1}{\mu_0 e^{2 \sqrt{b_0} t} - 1} \quad \text{if} \quad a_0 > 2 \sqrt{b_0},$$

$$13$$
where \( a_0 := a(0) \) and \( \mu_0 := \frac{2\sqrt{b_0 + a_0}}{2\sqrt{b_0 - a_0}} \). In particular, if \( a_0 < 2\sqrt{b_0} \), then \( a \) is increasing and converges to \( 2\sqrt{b_0} \) for \( t \to \infty \) and if \( a_0 > 2\sqrt{b_0} \), then \( a \) is decreasing and converges to \( 2\sqrt{b_0} \) for \( t \to \infty \). Moreover, the energy function of this oscillation is always decreasing.

**Proof.** From the previous theorem, we see that an aperiodic limit oscillation with constant coefficient \( b = b_0 > 0 \) must satisfy \( b_0 = a^2(s) + \frac{a'(s)}{2} \) and thus \( a \) satisfies \( \frac{1}{a_0 - a} \, da = \frac{1}{2} \, dt \). Integration and simplification yields

\[
\left| \frac{2\sqrt{b_0} + a(t)}{2\sqrt{b_0} - a(t)} \right| = \left| \frac{2\sqrt{b_0} + a_0}{2\sqrt{b_0} - a_0} \right| \exp \left( 2\sqrt{b_0} \, t \right),
\]

which is equivalent to the claimed representation of \( a \). Moreover, the claimed growth behaviour follows at once from the representation formula of \( a \). Finally, because \( b \) is constant, the energy function of this oscillation is always decreasing. This concludes the proof. \( \square \)

**Corollary 2.** Let \( b_0 > 0 \) be a constant and \( v \) be an aperiodic oscillation with \( b = b_0 \) and a non-negative. Then the coefficient function \( a \) in (2) satisfies

- \( a(t) = 2\sqrt{b_0} \) for all \( t \in [0, T_0) \) with some \( T_0 \in [0, \infty) \) and
- either \( a(t) = a_+ (t - T_0) \) or \( a(t) = a_- (t - T_0) \) for all \( t \in [T_0, \infty) \)

with \( a_\pm \) defined as in Corollary 1. This is true for any \( T_0 \geq 0 \).

**Proof.** The claim follows from Corollary 1 and the fact that \( a := 2\sqrt{b_0} \) on \( (0, T_0) \) satisfies \( b_0 = a^2(s) + \frac{a'(s)}{2} \). \( \square \)

**Example 5.** Let \( T, a, b, \varphi, \psi \) and \( v \) be as in Theorem 3. For the coefficients \( a \) and \( b \) in (2), we choose

\[
a(t) := \frac{a_0 \, T^n}{(T - t)^n} \quad \text{and} \quad b(t) := \frac{a_0^2 \, T^{2n}}{4 \, (T - t)^{2n}} + \frac{n \, a_0 \, T^n}{2 \, (T - t)^{n+1}}
\]

for \( t \in (0, T) \) with \( n \in \mathbb{N} \) and \( a_0 > 0 \). According to Theorem 3, the solution of (3) is aperiodic oscillations and reads as follows

\[
v(t) = \left[ \varphi + \left( \psi + \frac{a_0}{2} \varphi \right) t \right] \left( 1 - \frac{t}{T} \right)^{\frac{a_0 \, T}{2}} \quad (n = 1)
\]

and

\[
v(t) = \left[ \varphi + \left( \psi + \frac{a_0}{2} \varphi \right) t \right] \exp \left( -\frac{a_0 \, T^n}{2 \, (n - 1)} \left[ \frac{1}{(T - t)^{n-1}} - \frac{1}{T^{n-1}} \right] \right) \quad (n \neq 1)
\]
for \( t \in (0, T) \), due to 

\[
-\frac{1}{2} \int_0^t a(s) \, ds = \log \left( \left| 1 - \frac{1}{T} \right|^{a_0 T} \right) \quad \text{for } n = 1 \quad \text{and} \quad
-\frac{1}{2} \int_0^t a(s) \, ds = -\frac{a_0 T}{2(n-1)} \left[ \left( \frac{1}{(T-t)^{n-1}} - \frac{1}{T^{n-1}} \right) \right] \quad \text{for } n \in \mathbb{N}\setminus\{1\}, \text{ respectively.}
\]

**Theorem 4.** Let \( T, a, b, \varphi, \psi \) and \( v \) be as in Theorem 3, \( a_0 := a(0) \) and \( T_0 := \frac{\varphi}{\psi + \frac{a_0}{2} \varphi} \). If \( a(T-) = \infty \) holds, then the energy \( E \) of the oscillation \( v \) converges to zero for \( t \to T- \). If \( a \) is twice differentiable, positive and 

\[
a'(t) + \frac{a''(t)}{a(t)} \leq \left[ a(t) - \frac{2}{T_0 + t} \right]^2 \quad \text{for } t \in (0, T) \quad \text{and} \quad \psi \neq -\frac{a_0}{2} \varphi
\]

or

\[
a'(t) + \frac{a''(t)}{a(t)} \leq a^2(t) \quad \text{for } t \in (0, T) \quad \text{and} \quad \psi = -\frac{a_0}{2} \varphi \neq 0,
\]

then the energy \( E \) is a decreasing function on \((0, T)\).

**Proof.** From \( a(T-) = \infty \) and the representation formula for \( v \) in Theorem 3, we infer \( v(T-) = 0 \) and \( v'(T-) = 0 \) and thus \( E(T-) = 0 \). Now let \( \psi - \frac{a_0}{2} \varphi \neq 0 \) be true. Then the energy is given by 

\[
E' \asymp \left[ v'' + b v \right] v' + \frac{1}{2} b' v^2 = -a (v')^2 + \frac{1}{2} b' v^2,
\]

due to the ode in (2). Hence \( E' \leq 0 \) if and only if \( \frac{b'}{2a} v^2 \leq (v')^2 \), which is equivalent to

\[
\frac{2b'}{a} \leq \left[ a(t) - \frac{2}{\psi + \frac{a_0}{2} \varphi} \right]^2 \quad \text{for } t \in (0, T).
\]

Employing \( 2b' = a a' + a'' \) and \( T_0 := \frac{\varphi}{\psi + \frac{a_0}{2} \varphi} \) to this inequality yields the claimed condition. Finally, if \( \psi + \frac{a_0}{2} \varphi = 0 \) is true with \( \varphi \neq 0 \), then \( v = \varphi \exp \left( -\frac{a}{2} \right) \) and thus

\[
\frac{b'}{2a} \leq (v')^2 \iff a' + \frac{a''}{a} \leq a^2,
\]

as was to be shown.

We now focus on a specific example with non-constant functions \( b \) and \( a \) and discuss its energy behaviour. A natural model for the coefficients
\( a, b : [0, T] \to \mathbb{R} \), which guarantees (a sufficiently large and finite) stopping time \( T > 1 \), is given by

\[
(16) \quad a(t) := \frac{T a_0}{T - t} - \frac{a_1}{T} t \quad \text{for} \quad t \in (0, T) \quad \text{and} \quad b := \frac{a^2}{4} + \frac{a'}{2},
\]

where \( a_0 > 0 \) and \( a_1 \geq 0 \) are constants. For simplicity, we consider the special case \( a_0 > 0 \) and \( a_1 = 0 \). The general case is handled in the same way, but is more tedious and not more instructive.

**Example 6.** Let \( a \) and \( b \) be defined by (16) with \( a_0 > 0 \) and \( a_1 = 0 \) and let \( T_0 := \frac{\varphi}{\psi + \varphi} \) satisfy \( 0 \leq T_0 \leq \infty \). Then the solution \( u \) of (2) is an aperiodic limit oscillation for any \( T > 0 \). For clarity, we split the proof in three parts depending on the values of \( T_0 \).

1) Let \( 0 < T_0 < \infty \) hold. Then condition in Theorem 4 reads as follows

\[
T a_0 + 2 \leq \left[ \frac{T a_0 - 2 (T - t)}{T_0 + t} \right]^2,
\]

where \( a(t) = \frac{T}{T - t}, \quad a'(t) = \frac{T}{(T - t)^2} \) and \( a''(t) = \frac{2 T}{(T - t)^3} \) have been employed. This condition is equivalent to \( P(t) := A(t) T^2 + B(t) T + C(t) \geq 0 \) with

\[
A(t) := \left( a_0 + \frac{2}{T_0 + t} \right)^2, \quad B(t) := -a_0 + \frac{4 a_0 t}{T_0 + t} - \frac{8 t}{(T_0 + t)^2}
\]

and

\[
C(t) := \frac{4 t^2}{(T_0 + t)^2} - 2.
\]

Because the functions \( A, B \) and \( C \) are bounded on \([0, T]\) and \( A \) does not vanish on \([0, T]\), it follows that \( P(t) \geq 0 \) for sufficiently large \( T \). Note that \( T \) is not completely independ of \( T_0 \).

2) Now let \( T_0 = 0 \). Then condition in Theorem 4 is equivalent to

\[
T^2 (2 + a_0 t)^2 + T (3 a_0 t - 8) t + 2 t^2 \geq 0,
\]

which is true if \( T \) is sufficiently large.

3) For \( T_0 = \infty \) condition in Theorem 4 reads as follows

\[
T^2 a_0^2 - 2 a_0 - 2 \geq 0, \quad \text{which is true if} \quad T \geq \frac{2}{a_0}.
\]

Hence, for given \( T_0 \in [0, \infty] \) and sufficiently large \( T \), the energy function of the considered oscillation is decreasing.
3.2 Creeping oscillation

From Theorem 1 together with \( \sin(i x) = i \sinh(x) \) and \( \cos(i x) = \cosh(x) \) for \( x \in \mathbb{R} \), we obtain the following solution formula for creeping oscillations

**Corollary 3.** Let \( \varphi, \psi \in \mathbb{R} \) and \( a, b : [0, T) \to [0, \infty) \) be continuous. If the frequency function defined as in (7) satisfies \( \omega^2 < 0 \) on \( [0, T) \), then there exists a function \( \epsilon \) satisfying \( \omega = i \epsilon \) with \( \epsilon(t) > 0 \) for \( t \in (0, T) \) and the solution of (2) exists, is unique and reads as follows

\[
v = \left[ \varphi \cosh(\tilde{\epsilon}) + (\psi + \alpha(0) \varphi) \frac{\sinh(\tilde{\epsilon})}{\epsilon(0)} \right] \varrho,
\]

where \( \tilde{\epsilon} \) and \( \varrho \) are defined as in (4) and \( \alpha := \frac{a}{2} + \frac{\epsilon'}{2\epsilon} \).

**Remark 3.** The classical problem

\[
v'' + a_0 v' = 0 \quad \text{on} \quad (0, \infty) \quad \text{with} \quad v(0) = \varphi \quad \text{and} \quad v'(0) = \psi
\]

has the solution and energy

\[
v(t) = \left( \varphi + \frac{\psi}{a_0} \right) - \frac{\psi}{a_0} \exp(-a_0 t) \quad \text{and} \quad E(t) = \frac{m}{2} \psi^2 \exp(-2a_0 t),
\]

respectively. It describes a creeping oscillation of a particle with mass \( m \). Because the coefficient \( b \) vanishes, its energy is always decreasing. For the case \( \varphi = 0 \) and \( \psi > 0 \), i.e. zero initial elongation and positive initial velocity the oscillation reads as follows

\[
v(t) = \frac{\psi}{a_0} - \frac{\psi}{a_0} \exp(-a_0 t) \quad \text{with} \quad \lim_{t \to \infty} v(t) = \frac{\psi}{a_0} > 0 = v(0).
\]

We see that the velocity is always directed away from the center of the string and its speed is decreasing. In this example, the string is not able to pull the particle back to zero. In the following example, we present an oscillation model that is similar to this one, but the final value is attained after the finite time period \( T \).

**Example 7.** Let \( 0 < T_1 < T < \infty \), \( a_0 > 0 \) and \( a, b : (0, T) \to \mathbb{R} \) be defined by

\[
a(t) := \begin{cases} 
a_0 & \text{if} \quad t \in (0, T_1) \\
a_0 \frac{T-t}{T-T_1} & \text{if} \quad t \in (T_1, T)
\end{cases} \quad \text{and} \quad b(t) := 0.
\]
The function $a$ is continuous on $(0, T)$ and differentiable on $(0, T)\setminus\{T_1\}$ and satisfies $\lim_{t\to T} a(t) = \infty$. Employing the ansatz $v_j := \exp(-\lambda_j t)$ to the underlying ode leads to the characteristic equation $-\lambda^2 + \lambda - a = 0$, which has the solutions $\lambda_1(t) = 0$ and $\lambda_2(t) = a(t) + \frac{a(t)}{a_0} \frac{\ddot{v}}{a(t)}$ for $t \in (0, T)$.

Hence two (linear independent) solutions of $v'' + a v' = 0$ are given by $v_1(t) = 1$ and $v_2(t) = \exp \left(-\int_0^t \lambda_2(s) \, ds\right)$. From $\frac{\dd v}{a} = \frac{a(t)}{a_0} \frac{\dd v}{a(t)}$ and $\int_{T_1}^T a(s) \, ds = -\log \left(\frac{T-t}{T-T_1}\right)^{a_0 T}$ for $t \in (T_1, T)$, we infer

$$v_1 = 1 \quad \text{and} \quad v_2(t) = \begin{cases} e^{-a_0 t} & \text{if } t \in (0, T_1) \\ e^{-a_0 T_1} \left(\frac{T-t}{T-T_1}\right)^{1+a_0 T} & \text{if } t \in (T_1, T) \end{cases}$$

and consequently the solution of $v'' + a v' = 0$ on $(0, T)$ reads as follows

$$v(t) = \left[ \varphi + \frac{\psi}{a_0} - \frac{\psi}{a_0} \exp(-a_0 t) \right] \chi_{[0, T_1]}(t)$$

$$+ \left[ \xi_0 - \frac{\psi}{a_0} \xi_1 \left(\frac{T-t}{T-T_1}\right)^{1+a_0 T} \right] \chi_{(T_1, T)}(t)$$

with $\xi_1 := \frac{a_0}{1+a_0 T} e^{-a_0 T_1}$ and $\xi_0 := \frac{\psi}{a_0} \xi_1 + v(T_1^-)$. This describes a creeping process for the whole time interval $(0, T)$, because the frequency function defined by $\omega = i \epsilon$ with $\epsilon := \frac{\alpha}{T} + \frac{\alpha'}{2 a}$ satisfies condition (7) and the above solution $v$ equals to the solution $\tilde{v}$ in Theorem 7. For this example, we have $\alpha = \epsilon = \frac{1}{2} \lambda_2$.

Because of $b = 0$ on $(0, T)$ and $v' = 0$ on $(T, \infty)$, the energy of this process is decreasing and reaches the valued zero at $t = T$. As in the standard situation, i.e. $a' = b' = 0$, we have two major cases (i) $\varphi = -\frac{\psi}{a_0}$ and (ii) $\varphi \neq -\frac{\psi}{a_0}$. In the first case, the elongation $v(t)$ reaches the value zero after the finite period $T$ and in the second case the elongation $v(t)$ reaches the value $\varphi + \frac{\psi}{a_0}$ after the finite period $T$. The difference between this model and the standard one is that the "final" values of the process are attained during a finite period $T$.

We now generalize the previous example from "exponent 1" to any "natural exponent".

**Example 8.** Let $T$, $a_0 > 0$, $n \in \mathbb{N} \setminus \{1\}$, $b = 0$ and $a : (0, T) \to \mathbb{R}$ be defined by

$$a(t) := \frac{a_0 T - n(T-t)^{n-1}}{(T-t)^n} \quad \text{for } \quad t \in (0, T)$$

\footnote{Note $\lambda_1 = \alpha + i \omega = \alpha - \epsilon$ and $\lambda_2 = \alpha - i \omega = \alpha + \epsilon$. Hence $\lambda_1 = 0$ implies $\alpha = \epsilon$ and $\lambda_2 = 2 \epsilon$.}
Then the solution of (2) is given by

\[ v(t) = \left( \varphi + \frac{T^{n-1}}{a_0} \psi \right) - \psi \frac{T^{n-1}}{a_0} \exp \left( -\frac{a_0 T}{n-1} \left[ \frac{1}{(T-t)^{n-1}} - \frac{1}{T^{n-1}} \right] \right). \]

Indeed, because \( a \) and \( b \) are continuous on \((0, T)\) the solution of the ode exists and is unique on \((0, T)\) (cf. Theorem 1), the characteristic eigenvalues of \(-\lambda' + \lambda^2 - a \lambda = 0\) are given by \( \lambda_1(t) = 0 \) and \( \lambda_2(t) = \frac{a_0 T}{(T-t)^n} \) for \( t \in (0, T) \) and thus two (linear independent) eigenfunction of the form \( v = \exp(-\lambda) \) reads as follows \( v_1(t) = 1 \) and \( v_2(t) = \exp \left( -\frac{a_0 T}{n-1} \left[ \frac{1}{(T-t)^n} - \frac{1}{T^{n-1}} \right] \right) \). From this and \( v = C_1 v_1 + C_2 v_2 \) with the initial condition \( v(0) = \varphi \) and \( v'(0) = \psi \), we infer

\[ C_1 = \varphi + \frac{T^n}{a_0 T} \psi \quad \text{and} \quad C_2 = -\frac{T^n}{a_0 T} \psi, \]

which proves the claim. From \( 0 = \lambda_1 = \alpha - \epsilon \) with \( \alpha \in \mathbb{R} \) and \( \epsilon \in \mathbb{R} \cup i \mathbb{R} \), we infer \( \alpha = \epsilon \in \mathbb{R} \). From this and \( \lambda_2 = \alpha + \epsilon \), we infer

\[ \alpha(t) = \epsilon(t) = \frac{\lambda_2(t)}{2} = \frac{a_0 T}{2 (T-t)^n} > 0 \quad \text{for} \quad t \in (0, T). \]

In particular, we see that this process is creeping.

### 3.3 Weak dissipative and mixed dissipative oscillations

In this section, we present examples of oscillations that are weakly dissipative. For appropriate initial frequency (\( \omega_0 = 0 \) or \( \omega_0 \in i \mathbb{R} \)), each of these examples reduces to an aperiodic limit or creeping process. But these cases we will not be further investigated. Moreover, we discuss an oscillation that is weak dissipative until the time instant \( T_1 \) and aperiodic from \( T_1 \) till the stopping time \( T \) (cf. Example 11).

**Example 9.** Let \( T, a_0, b_0 \) and \( \omega_0 \) be positive, \( a_1 \in (-\infty, a_0] \),

\[ a(t) := \frac{a_0 T}{T-t} - \frac{a_1}{T} t, \quad \omega(t) := \frac{\omega_0 T}{T-t} \quad \text{with} \quad \omega_0 := \sqrt{b_0 - \left( \frac{a_0}{2} + \frac{1}{2 T} \right)^2} \]

and

\[ b(t) := \frac{b_0 T^2}{(T-t)^2} + \frac{a_1^2}{4 T^2} t^2 - \frac{a_1 a_0}{2 (T-t)} t - \frac{a_1}{2 T}. \]
for \( t \in [0, T) \). Then \( a, b \) and \( \omega \) satisfy identity (7) and the solution of (2) describes an oscillation that is weak dissipative or creeping if \( \omega_0 > 0 \) or \( \omega_0 = i \epsilon_0 \) with \( \epsilon_0 > 0 \), respectively. For the special case \( a_1 = a_0 \), we have \( a'(0) = 0 \). Here we focus on the weak dissipative case. According to Theorem 7 the solution of (2) reads as follows

\[
v(t) = \left[ \varphi \cos \left( \omega_0 T \log \left( \frac{1 - t}{T} \right) \right) - \left( \psi + \left( \frac{a_0}{2} + \frac{1}{2T} \right) \varphi \right) \sin \left( \frac{\omega_0 T \log \left( \frac{1 - t}{T} \right)}{\omega_0} \right) \right] g(t)
\]

with

\[
g(t) = \sqrt{\left( 1 - \frac{t}{T} \right)^{1+\alpha_0} T} \exp \left( \frac{a_1}{4T} t^2 \right)
\]

for \( t \in [0, T] \). Here we have used that \( a(0) = a_1, \quad \frac{\omega'(0)}{2\omega(0)} = \frac{1}{2T}, \quad \omega(0) = \omega_0, \quad \int_0^t \frac{1}{T-s} \, ds = -\log \left( 1 - \frac{t}{T} \right) \) and \( \alpha(t) = \frac{1+\alpha_0}{2(T-t)} - \frac{a_1}{2T} t \). We note that

- \( \alpha' \) is strictly increasing if \( a_1 < a_0 + \frac{1}{T} \) (above we required \( a_1 < a_0 \)) and thus \( g \) is strictly decreasing.
- \( g \) is a linear (and decreasing) if \( a_0 = \frac{1}{T} \) and \( a_1 = 0 \);
- if ”\( T = \infty \)”, then \( a = a_0, b = b_0 \) and \( \omega_0 = \sqrt{b_0 - \frac{a^2_0}{4}} \), more precisely, for the limit \( T \to \infty \), we obtain the standard oscillation with constant coefficients.

Two numerical examples are presented in Fig. 1 and Fig. 2 for the setting (i) \( T = 10, a_0 T = 0.1, a_1 = 0, b_0 T = 20, \varphi = \psi = 1 \) and (ii) \( T = 10, a_0 T = 5, a_1 = 0, b_0 T = 1000, \varphi = \psi = 1 \), respectively.

The previous example can be generalized as follows.

**Example 10.** Let \( T, a_0 \) and \( b_0 \) be positive, \( n \in \mathbb{N}\setminus\{1\} \) and \( a_1 \in (-\infty, na_0] \). Then identity (7) is satisfied for \( a, b \) and \( \omega \) defined by

\[
a(t) := \frac{a_0 T^n}{(T-t)^n} - \frac{a_1}{T} t, \quad \omega(t) := \frac{\omega_0 T^n}{(T-t)^n} \quad \text{with} \quad \omega_0 := \sqrt{b_0 - \frac{a^2_0}{4}}
\]

and

\[
b(t) := \frac{b_0 T^{2n}}{(T-t)^{2n}} + \frac{n a_0 T^n}{2(T-t)^{n+1}} + \frac{n (2-n)}{4(T-t)^2} - \frac{a_1 a_0 T^{n-1}}{2(T-t)^n} t + \frac{a^2_1}{4T^2} t^2 - \frac{a_1}{2T}
\]
Figure 1: Visualization of the oscillation satisfying \( v'' + \frac{v'}{10(10-t)} + \frac{20v}{(10-t)^2} = 0 \) with \( v(0) = v'(0) = 1 \) (cf. Example 9). Here the function \( t \mapsto \tilde{\omega}(t) := \int_0^t \omega(s) \, ds \) is called phase function.

Figure 2: Visualization of the oscillation satisfying \( v'' + \frac{5v'}{10-t} + \frac{10^3v}{(10-t)^2} = 0 \) with \( v(0) = v'(0) = 1 \) (cf. Example 9).
for \( t \in [0, T) \). Again, the solution \( v \) of \((2)\) describes an oscillation that is weak dissipative or creeping if \( \omega_0 > 0 \) or \( \omega_0 = \pm \epsilon_0 \) with \( \epsilon_0 > 0 \), respectively. We focus on the weak dissipative case and obtain
\[
v(t) = \left[ \phi \cos \left( \frac{\omega_0 T^{2n-1}}{n-1} \left[ \left( 1 - \frac{t}{T} \right)^{1-n} - 1 \right] \right) \right. \\
+ \left( \psi + \left( \frac{a_0}{2} + \frac{n}{2T} \right) \phi \right) \frac{\sin \left( \frac{\omega_0 T^{2n-1}}{n-1} \left[ \left( 1 - \frac{t}{T} \right)^{1-n} - 1 \right] \right)}{\omega_0} \] \[
\left\{ \begin{array}{ll}
\varrho(t) = \left( 1 - \frac{t}{T} \right)^\frac{n}{2} \exp \left( \frac{a_1}{4T} t^2 - \frac{a_0 T^{2n-1}}{2(n-1)} \left( \left( 1 - \frac{t}{T} \right)^{1-n} - 1 \right) \right). 
\end{array} \right.
\]

If \( a_1 \leq n a_0 \) (as in the above assumption), then \( \alpha' \) is strictly increasing and therefore \( \varrho \) is strictly decreasing.

We now come to an example of a mixed dissipative oscillation.

**Example 11.** Let \( T > T_1 > 0 \) (with \( T \) very large), \( a_0, b_0 > 0 \) be such that \( \omega_0 := \sqrt{b_0 - \frac{a_0^2}{4}} > 0 \) and let the functions \( a, b : [0, T] \to \mathbb{R} \) be defined by
\[
a(t) := \frac{T a_0}{T - t} - \frac{a_0}{T} t \quad \text{for} \quad t \in (0, T)
\]
and
\[
b(t) := \left\{ \begin{array}{ll}
\frac{b_0}{4} + \frac{a'(t)}{2} & \text{for} \; t \in [0, T_1] \\
\frac{a^2(t)}{4} + \frac{a'(t)}{2} & \text{for} \; t \in (T_1, T) \end{array} \right.
\]
Moreover, let us assume that
\[
T_1 > 0 \quad \text{is the smallest number satisfying} \quad b_0 - \frac{a^2(T_1)}{4} - \frac{a'(T_1)}{2} = 0.
\]
as well as \( 0 < T_1 < T \). Note that \( B : t \mapsto \frac{a^2(t)}{4} + \frac{a'(t)}{2} \) is strictly increasing to infinite and \( b(0) > B(0) \) and therefore \( T_1 \) exists. For this setting, the solution \( v \) of \((3)\) is weakly dissipative on \((0, T_1)\), aperiodic on \((T_1, T)\) and stops at time \( T \). Because \( a \) and \( b \) are continuous on \((0, T)\), it follows that the oscillation \( v \) is twice differentiable on \((0, T)\).

**Remark 4.** It is easy to modify the setting in Example 11 such that \( a_{\text{new}}, b_{\text{new}} \) are constant on the interval \((0, T_0)\) and satisfy \( a_{\text{new}} = a(\cdot - T_0) \) and \( b_{\text{new}} = b(\cdot - T_0) \) on the interval \([T_0, T_{\text{new}})\) with \( T_{\text{new}} := T_0 + T \). If the product
If $a_0 T$ is sufficiently large, then the energy of this new oscillation is decreasing and stops at the time instant $T_{\text{new}}$. Obviously, if $T_0$ is sufficiently large, then this oscillation is very close to the standard weakly dissipative oscillation with coefficients $a_0$ and $b_0$ that has an infinite stopping time.

We conclude this section with a representation of $\omega$ which is for many theoretical considerations very convenient. It permits to consider all three cases of oscillations; the aperiodic limit case is obtained for $\eta \to 0$, where $\eta$ is a special help variable.

**Example 12.** Let $T \in (0, \infty]$, $\omega_0 > 0$, $a, \eta : [0, T] \to \mathbb{R}$ be non-negative continuous functions and

$$b := \omega_0^2 \exp(2\tilde{\eta}) + \frac{a^2 - \eta^2}{4} + \frac{a' + \eta'}{2} \quad \text{and} \quad \omega = \omega_0 \exp(\tilde{\eta}) .$$

It is easy to see that condition [7] is satisfied by these functions $a$, $b$ and $\omega$ for the three cases

- $\omega_0 > 0$ (weak dissipation)
- $\omega_0 = 0$ and $\eta$ is the zero function (aperiodic limit)
- $\omega_0 < 0$ (creeping)

For these cases we have $\omega_0 = \sqrt{b(0) - \frac{a^2(0) - \eta^2(0)}{4} - \frac{a'(0) + \eta'(0)}{2}}$. If $\eta$ and $a'$ are the zero function, then we obtain $\omega = \omega_0 := \sqrt{b(0) - \left(\frac{a(0)}{2}\right)^2}$, which is the classical expression for the frequency.

## 4 Dissipative waves and semigroups of dissipative oscillations

In this section, we show that a vast amount of *dissipative wave models* can be modelled and investigated via semigroups of oscillations. In particular, we discuss a family of examples for which the respective attenuation law is approximately a power law with exponent 2 (for small absolute frequencies). In other words, there exists a vast amount of models for dissipative tissue similar to water; not just one or a handful.

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[5] We note that relaxations are special cases of oscillations and thus we also considered them here.
In the following, we call a function $\alpha : \mathbb{R} \to \mathbb{C}$ a *dissipation law* if $\Re(\alpha)$ is nonnegative and even and $\Im(\alpha)$ is odd. ($\Re(\alpha)$ is usually called attenuation law.) Moreover, we call $(K_R)_{R \geq 0}$ a *semigroup of dissipative oscillations* if there exists a dissipation law $\alpha$ such that $\hat{K}_R = e^{-\alpha R}$ for $R \geq 0$ and $K_R(t) = 0$ for $t < 0$ and $R > 0$. Then we have (i) $K_{R_1+R_2} = K_{R_1} * K_{R_2}$ for $R_1, R_2 > 0$ and (ii) $\lim_{R \to 0} K_R = \delta$ in the distributive sense. As shown in [13], such a semigroup can be used to model a dissipative spherical wave [6]. Here $K_{|x|}$ for $x \in \mathbb{R}^3$ characterizes the oscillation at position $x$ that is caused by the dissipative spherical wave $G$ (cf. [3]). $G$ (or equivalently the pair $(\alpha, c_0)$) determines the dissipative wave operator $\mathcal{A}$, due to (5).

It is natural to model $K_1$ as a relaxation function $\varrho$ or as the product of a relaxation function with an appropriate oscillation such as $\cos(\omega_0 t)$ with some constant $\omega_0 \neq 0$. We now show that the frequency need not to be constant. For convenience, we define $[0, T]$ to be $[0, T]$ if $T = \infty$.

**Proposition 2.** Let $\omega_\ast$ be a real-valued and odd frequency function such that $h := e^{i\omega_\ast}$ satisfies $\int_{\mathbb{R}} |\hat{h}(s)| \, ds \leq 1$\(^{\text{6}}\). Moreover, let $\varrho \in L^1(\mathbb{R})$ satisfy $\text{supp}(\varrho) = [0, T]$ for some $T \in (0, \infty]$ and $\hat{\varrho} = e^{-\beta}$ for some dissipation law $\beta$\(^{7}\). Then $K_1 := \varrho \cos(\omega_\ast)$ induces a semigroup of dissipative oscillations by $K_R := K_1^{R}$ satisfying $\text{supp}(K_R) = [0, RT]$ for $R \geq 0$.

**Proof.** We have to show that there exists a dissipation law $\alpha$ satisfying $\hat{K}_1 = e^{-\alpha}$. Such an dissipation law $\alpha$ exists, if $\Re(\alpha) \equiv -\frac{1}{2} \log |\hat{K}_1|^2 \in [0, \infty)$, $\Re(\alpha)$ is even and $\Im(\alpha) \equiv \frac{\hat{\alpha}}{\hat{K}_1}$ is odd. We note that $|\hat{K}_1| \neq 0$ on $\mathbb{R}$, due to $\Re(\alpha) \neq \infty$. That $\Re(\alpha)$ is even and $\Im(\alpha)$ is odd follows at once from the fact that $K$ is real valued. Thus it remains to show that $|\hat{K}_1| \in [0, 1]$ is true. Let $h_1 := h(t) := e^{i\omega_0 t}$ and $h_2 := h(-t)$ for $t \in \mathbb{R}$. Because $\omega_\ast$ is real-valued, odd and $h$ is bounded, it follows that $\hat{h}_1$ and $\hat{h}_2$ are real-valued and tempered. From this and $\hat{h}_2(\omega) = \overline{\hat{h}_1(-\omega)} = \hat{h}_1(-\omega)$, we infer

$$\hat{K}_1(\omega) = \frac{1}{2} \int_{\mathbb{R}} \hat{\varrho}(\omega - s) (\hat{h}_1 + \hat{h}_2)(s) \, ds = \frac{1}{2} \int_{\mathbb{R}} [\hat{\varrho}(\omega - s) + \hat{\varrho}(\omega + s)] \hat{h}(s) \, ds,$$

and consequently

$$|\hat{K}_1(\omega)| \leq \int_{\mathbb{R}} |\hat{h}_1(s)| \, ds \leq 1, \text{due to } |\hat{\varrho}| = |e^{-\beta}| \leq 1.$$

If $\omega_\ast = \omega_0 = \text{const.}$, then $\hat{h}_1(\omega) = \delta(\omega - \omega_0)$ and the above conclusion is still valid. The last claim $\text{supp}(K_R) = [0, RT]$ for $R \geq 0$ follows at once from

---

\(^6\)For example, this is satisfied for the special case $\omega_\ast = \omega_0 = \text{const.}$, i.e. $h(t) = e^{i\omega_0 t}$.

\(^7\)Because of these assumptions, $\beta$ is entire.
Proposition 6 in the appendix. Note, $\alpha$ is entire, due to $\text{supp}(\mathcal{K}_1) = [0, T]$. This concludes the proof.

Remark 5. According to Proposition 6 in the appendix, there exists a dissipation law $\beta$ such that $\hat{\varrho} = e^{-\beta}$, if $\varrho \in L^1(\mathbb{R})$ and $\|\varrho\|_{L^1(\mathbb{R})} \leq \frac{1}{\sqrt{2}}$. This includes all relaxation functions (with sufficiently small initial value), i.e. nonnegative and monotonic decreasing functions that vanish on $(-\infty, 0)$ (and with sufficiently small $\varrho(0)$).

In the following two propositions, we present a family of natural dissipative semigroups. We start with the semigroup induced by the classical relaxation function, i.e. $\varrho(t) := e^{-t \tau H(t)}$ for $t \in \mathbb{R}$.

Proposition 3. Let $a_0 > 1$, $\tau := a_0^{-1}$ and $\mathcal{K}_R(t) = a_0 \chi^{R-1}_+(a_0 t) e^{-a_0 t}$ for $R > 0, t \in \mathbb{R}$, where (cf. Section 3.2 in [9])

$$\chi^{r}_+(t) := \begin{cases} \frac{t^r}{\Gamma(r+1)} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0 \end{cases} \quad \text{and} \quad r \in (-1, \infty).$$

Then $\mathcal{K}_R \in L^1(\mathbb{R})$ for $R > 0$ and $\hat{\mathcal{K}}_R(\omega) = \mathcal{F}(\mathcal{K}_R)(\omega) = \frac{1}{(1-12\pi^2 \omega)^R}$ for $\omega \in \mathbb{R}$ and $R > 0$, i.e. $(\mathcal{K}_R)_{R \geq 0}$ is a semigroup. If $R > \frac{1}{2}$, then $\mathcal{K}_R \in L^2(\mathbb{R})$.

Proof. That $\mathcal{K}_R \in L^1(\mathbb{R})$ for $R > 0$ (as well as $\mathcal{K}_{R_1+R_2} = \mathcal{K}_{R_1} \ast \mathcal{K}_{R_2}$ for $R_1, R_2 > 0$) follow from Exercise 6.3.18 (iii) in [22] with the substitution $t \rightarrow \frac{t}{\tau}$. For the second claim. We recall that the inverse Laplace transform of $f$ is defined by

$$\mathcal{L}^{-1}(f)(t) = \begin{cases} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{st} f(s) \, ds, & \text{for } t > 0, \\ \mathcal{L}^{-1}(h(s-a))(t) = e^{at} \mathcal{L}^{-1}(h(s))(t) & \text{for } a, t \in \mathbb{R} \\ \mathcal{L}^{-1}(s^{-r})(t) = \chi^{r-1}_+(t) & \text{for } t \in \mathbb{R}, r > 0. \end{cases}$$

Thus, if we use the Fourier transform in the form (different as in the appendix)

$$\mathcal{F}(f)(\omega) := \hat{f}(\omega) := \frac{1}{\sqrt{2 \pi}} \int_{\mathbb{R}} f(t) e^{i\omega t} \, dt \quad \text{for } \omega \in \mathbb{R},$$

25
then identity $\hat{K}_R(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{(1 + i\tau\omega)^r}$ follows from these properties and

$$
F^{-1}\left\{ \frac{1}{(1 - i\tau\omega)^r} \right\} = \frac{\sqrt{2\pi}}{\tau} \mathcal{L}\{ (1 + s)^{-r} \} \left( \frac{t}{\tau} \right).
$$

Because we use in this paper the Fourier transform \[^{[18]}\] (cf. Appendix), we have $\hat{K}_R(\omega) = \frac{1}{(1 - i\tau\omega)^r}$. ($\omega$ is replaced by $2\pi\omega$ and the factor $\frac{1}{\sqrt{2\pi}}$ is removed.) The last claim for $R > \frac{1}{2}$ follows from

$$
\|\hat{K}_R\|_{L^2}^2 = \int_0^\infty (1 + \tau^2\omega^2)^{-R} d\omega < C_1 + C_2 \int_1^\infty \omega^{-2R} d\omega < \infty \quad \text{if} \quad R > \frac{1}{2}
$$

for positive constant $C_1, C_2$. Hence $\|\hat{K}_R\|_{L^2}^2 = \|\hat{K}_R\|_{L^2}^2 < \infty$, which concludes the proof.

**Proposition 4.** Let $\tau \in (0, 1)$, $\nu > 0$ and $g_\nu(t) := \epsilon e^{-\left( \frac{t}{\nu} \right)^\nu} H(t)$ with $\epsilon$ such that $\|g_\nu\|_{L^1(\mathbb{R})} \leq \frac{1}{\sqrt{2}}$. Then $g_\nu \in L^1(\mathbb{R})$ for each $\nu > 0$.

**Proof.** Without loss of generality, we assume that $\tau = 1$. It is clear that $g_1 \in L^1(\mathbb{R})$ and as a consequence $g_\nu \in L^1(\mathbb{R})$ for each $\nu \geq 1$. Now let $\nu \in (0, 1)$. Then $\int_0^M e^{-t^\nu} dt < \infty$ for an $M > 0$ such that $s^{(1-\nu)/\nu} \leq e^{s/2}$ holds for all $s \geq M$. But then, it follows

$$
0 < \int_{M^\nu}^\infty e^{-t^\nu} dt = \frac{1}{\nu} \int_M^\infty s^{(1-\nu)/\nu} e^{-s} ds \leq \frac{1}{\nu} \int_M^\infty e^{-s/2} ds < \infty,
$$

where $s := t^\nu$ and $\nu \in (0, 1)$. But this shows $g_\nu \in L^1(\mathbb{R})$ for $\nu \in (0, 1)$ and concludes the proof.

We conclude this section with numerical examples.

**Example 13.** Let $K_R$ for $R > 0$ be as in Proposition 3 with $\tau := 10^{-9} s$ and $G$ be defined as in [\[^{[8]}\)]. In Fig. 3, we have visualized the qualitative behaviour of the “oscillations” $K_R$ for various values of $R$. We see

a) if $R \in (0, 1)$ then $K_R$ has a pole at $t = 0$ and is convex,

b) if $R = 1$ then $K_R(0^+)$ exists and $K_R$ is positive and convex, and

c) if $R > 1$ then $K_R(0^+) = 0$, $K_R$ is non-negative and has (exactly) one maximum at $t = (R - 1) \tau$.  

26
Figure 3: Visualisation of the oscillations $K_R$ from Example 13 for $R \in \{0.2, 0.4, 0.6, 0.8\}$ and $R \in \{1, 3, 5, 7\}$, respectively. As parameters, we used $\tau = 10^{-9}$ s and $c_\infty = 1481$ m/s. Moreover, we plotted $K_{T,R}$ for $R \in \{1, 3, 5, 7\}$, where $\hat{K}_{T,R} := \hat{\varrho}_T R$ with $\varrho_T$ defined as in Example 15 with $T := \frac{1}{3} 10^{-8}$ s. Note that supp($K_1$) = $[0, \infty)$ and supp($K_{T,1}$) = $[0, T]$.

Thus there are three different "types" of oscillations depending on the distance $R$ from the origin of the wave which obey the dissipation law $\alpha/R$.

**Example 14.** Let us shortly consider the semigroup defined by

$$K_R(t) := \frac{t^{R-1}}{\tau_0^R \Gamma(R)} \exp \left( -\frac{t}{\tau_0} \right) \cos(\omega_0 t) \quad \text{for} \quad R \geq 0,$$

where $\tau_0$ and $\omega_0$ are positive constants. For $\omega_0 = 0$ the oscillation $K_R$ reduces to the oscillation $K_R$ from the previous example. Via calculus, we see that the set of locations of extrem values of $K_R$ is given by $M_1 \cup M_2 \cup M_3$, where

$$M_1 = \{ t > 0 \mid \cos(\omega_0 t) = 0, \ R \neq 1 \},$$

$$M_2 = \{ t > 0 \mid t = \frac{(R - 1) \tau_0}{1 + \tau_0 \omega_0 \tan(\omega_0 t), \ R \neq 1} \}.$$
and

\[ M_3 = \{ t > 0 \mid \tau_0 \omega_0 \tan(\omega_0 t) = -1, \ R = 1 \} . \]

Hence in contrast to the previous examples, the oscillations \( K_R \) have (countable many) extrem values for each \( R \geq 0 \). Nevertheless, the qualitative behaviour between \( K_r \) (pole at \( t = 0 \)) and \( K_R \) (no pole) for \( r < 1 < R \) is different similarly as in Example 13.

**Example 15.** Let \( \rho_\infty \) denote the function \( \rho_1 \) from Example 13. Now we model a function \( M \) such that \( \varrho_T := M \ast_1 \varrho_\infty \) has support in \([0, T]\). It is easy to see that

\[ \rho_T(t) := \frac{\rho_\infty(t) - \sum_{m=0}^{2} \rho_\infty^{(m)}(T) \frac{(t-T)^m}{m!} \chi(0,T)}{1 - \sum_{m=0}^{2} \rho_\infty^{(m)}(0+)^{m} \frac{m!}{m!}} \]

is twice continuously differentiable on \((0, \infty)\) and has support in \([0, T]\). Therefore we define \( \hat{M} := \hat{\rho}_T \hat{\rho}_\infty^{-1} \). Moreover, because \( F^{-1}(\hat{\rho}_{\infty}^{-1}) = 1 + \delta' \), we infer

\[ \mathcal{M} = \rho_T + \frac{d\rho_T}{dt} \quad \text{and thus} \quad \mathcal{M} \text{ is an element of } L^1(\mathbb{R}). \]

In Fig. 3 we visualized the oscillations \( K_{T,R} \) defined by \( \hat{K}_{T,R} := \hat{\rho}_T^R \) for \( R > 0 \) and \( T := \frac{1}{3} \cdot 10^{-8} \) s.

We note that formulas for the dissipation laws of the above propositions and examples can be derived, but due to limit of space, we skip them. In particular, it can be shown that the dissipation law of the above models satisfy

\[ \beta(\omega) \approx a_0 + a_1 (\tau \omega)^2 - i a_2 \tau \omega \quad \text{if } |\omega| \text{ is sufficiently small} , \]

for approprate positive constants \( a_0(\nu, \tau, \epsilon), a_1(\nu) \) and \( a_2(\nu) \). Thus, although \( \varrho_\nu \) is strongly decreasing for large \( \nu \), the small frequency approximation of the attenuation law \( \Re(\beta) \) has exponents 2 and 0. (Usually the zero exponent is neglected, because of \( e^{a_0} \) in \( F^{-1} e^{-a_0-a_1 (\tau \omega)^2} = e^{a_0} F^{-1} e^{-a_1 (\tau \omega)^2} \) is just a ”normalization factor”. ) In the literatur of dissipative waves it is said that water is characterized by an exponent 2 for small frequencies. From the above examples, we see that there are uncountable many models that satisfy this criteria. The field of dissipative waves is much more various than expected.
5 Conclusion

The main goal of this paper was to analyse and discuss (normalized) second order odes with nonconstant coefficients and related them to (generalized) dissipative oscillations. In the first step, we showed that there exists a unique (nonnegative) “frequency function” solving an integro differential equation and in the second step, we proved a general representation formula for solutions of such problems. This solution formula (together with essential properties of the frequency function) is very useful and handy for (i) investigating (normalized) second order odes with nonconstant coefficients and (ii) modeling dissipative oscillations. In particular, it permits a classification of solutions of this type of problem in a similar manner as the classification of dissipative oscillations (satisfying a second order odes with constant coefficients). An advantage of odes with nonconstant coefficients (in contrast to odes with constant coefficients) is that special additional properties, like a finite stopping time, exists and thus can be incorporated into standard models. Indeed, in this paper, we mainly focused on modeling and investigating such types of models.

As an application, we shortly discussed and analysed semigroups with finite and infinite stopping times and dissipative waves that are linked them. Such a type of dissipative waves cause oscillations with either finite or infinite stopping times at each position in space. In the former case, we say that such a wave has a finite stopping time locally in space. Our examples demonstrate that the oscillation (cause by a dissipative wave) may have different qualitative behaviour at different distances from the wave origin. It is evident, that knowledge about this behaviour for various dissipation laws is important to understand dissipative waves.

We note that this work is complementary to our work \[13\]. We hope that this work inspires other scientists that works in control theory, pde theory and inverse problem theory.

6 Appendix: Some basic facts about the Fourier transform

For convenience of the reader, we list some basics about the Fourier transform (\[6, 9, 3\]), thereby fixing our notation. We also include a proposition that is useful for our investigation of dissipative waves.
If \( f \) is an \( L^1 \)–function, then its Fourier transform is defined by
\[
\mathcal{F}(f)(\omega) := \hat{f}(\omega) := \int_{\mathbb{R}} f(t) e^{i2\pi \omega t} \, dt \quad \text{for} \quad \omega \in \mathbb{R}.
\]

Then the Convolution Theorem for \( L^1 \)–functions reads as follows
\[
\mathcal{F}(f * t \, g) = \mathcal{F}(f) \, \mathcal{F}(g)
\]
and, if \( f \in L^1(\mathbb{R}) \) is differentiable, then
\[
\mathcal{F}(f')(\omega) = (-i \, 2 \pi \omega) \, \mathcal{F}(f)(\omega) \quad \text{for} \quad \omega \in \mathbb{R}.
\]
The inverse Fourier transform is denoted by \( \hat{f} \) as well as \( \mathcal{F}^{-1}(f) \).

The following Propositions specify which relaxation functions are of the form \( \mathcal{F}^{-1}(e^{-\alpha}) \) for some dissipation law \( \alpha \). They can be used to model dissipative spherical waves.

**Proposition 5.** Let \( \mathcal{K}_1 \in L^1(\mathbb{R}) \) be a positive, monotonic decreasing causal function (i.e. \( \mathcal{K}_1|_{t<0} = 0 \)) satisfying \( \| \mathcal{K}_1 \|_{L^1(\mathbb{R})} \leq \frac{1}{\sqrt{2}} \). Then there exists a dissipation law \( \alpha \) such that \( \hat{\mathcal{K}}_1 := e^{-\alpha} \) holds and \( \mathcal{K}_R := \mathcal{F}^{-1} \left( \hat{\mathcal{K}}_1^R \right) \) is well-defined for \( R \geq 0 \).

**Proof.** The proof is carried out in [13].

**Proposition 6.** Let \( \mathcal{K}_1, \alpha \) and \( \mathcal{K}_R (R \geq 0) \) be as in Proposition 5 with the additional assumption \( \text{supp} (\mathcal{K}_1) = [0, T] \) for some \( T \in (0, \infty) \). If \( \alpha \) is entire, then \( \text{supp}(\mathcal{K}_R) = [0, RT] \) for \( R \geq 0 \).

**Proof.** The proof is carried out in [13].

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