Composable Security of Generalized BB84 Protocols
Against General Attacks

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Abstract
Quantum key distribution (QKD) protocols make it possible for two parties to generate
a secret shared key. One of the most important QKD protocols, BB84, was suggested by
Bennett and Brassard in 1984. Various proofs of unconditional security for BB84 have
been suggested, but the first security proofs were not composable. Here we improve a
security proof of BB84 given by Biham, Boyer, Boykin, Mor, and Roychowdhury [1] to
be composable and match the state-of-the-art results for BB84, and we extend it to prove
unconditional security of several variants of the BB84 protocol. Our composable security
proof for BB84 and its variants is mostly self-contained, algebraic, and relatively simple,
and it gives tight finite-key bounds.

1 Introduction
Quantum key distribution (QKD) makes it possible for two legitimate parties, Alice and Bob, to
generate an information-theoretically secure key [2], that is secure against any possible attack
allowed by the laws of quantum physics. Alice and Bob use an insecure quantum channel and
an authenticated classical channel. The adversary Eve may interfere with the quantum channel
and is limited only by the laws of nature; however, she cannot modify the data sent in the
authenticated classical channel (she can only listen to it).

The main objective of analyzing a QKD protocol is proving its unconditional security:
namely, proving that even if the adversary Eve applies the strongest and most general attacks
allowed by the laws of nature (the “joint attacks”), Eve’s average information about the final
key is still negligible—that is, exponentially small in the number of qubits.

The first and most important QKD protocol was BB84 [2]; full definition of the BB84 pro-
tocol is available in Section 3. BB84 has many proofs of unconditional security (see, among
others, [3, 1, 4, 5, 6, 7, 8]), each of them having its own advantages and disadvantages. Ex-
tending the toolkit available for security proofs is still important, because many practical im-
plementations of theoretical protocols and many specifically designed practical protocols do
not yet have their full and unconditional security proved, and because existence of many proofs makes the security result more certain and less prone to errors.

In our work, we extend and generalize the security proof of BB84 presented by Biham, Boyer, Boykin, Mor, and Roychowdhury (BBBMR) [1]: first, we change it to prove the composable security of BB84—namely, to prove that the secret key remains secure even if Alice and Bob use it for cryptographic applications (see details in Subsection 1.1); second, we generalize it to apply to several variants of BB84 and not only to the original protocol; and third, we improve it to match the state-of-the-art results and security bounds for BB84.

Our paper gives composable security and tight finite-key bounds in a relatively simple way. This paper and the BBBMR paper [1] together are mostly self-contained. On the other hand, our paper leaves further possible generalizations for future research.

1.1 Unconditional Security Definitions of QKD Protocols

Originally, a QKD protocol was defined to be secure if the (classical) mutual information between Eve’s information and the final key, maximized over all the possible attack strategies and measurements by Eve, was exponentially small in the number of qubits, $N$. Examples of security proofs of BB84 that use this security definition are [3, 1, 4]: these security proofs used the observation that one cannot analyze the classical data held by Eve before privacy amplification, but must analyze the quantum state held by Eve [9]. In other words, they assumed that Eve could keep her quantum state until the end of the protocol, and only then choose the optimal measurement (based on all the data she observed) and perform the measurement.

Later, it was noticed that this security definition may not be “composable”. In other words, the final key is secure if Eve measures the quantum state she holds at the end of the QKD protocol, but the proof does not apply to cryptographic applications (e.g., encryption) of the final key: Eve might gain non-negligible information after the key is used, even though her information on the key itself was negligible. This means that the proof is not sufficient for practical purposes. In particular, these applications may be insecure if Eve keeps her quantum state until Alice and Bob use the key (thus giving Eve some new information) and only then measures.

Therefore, a new notion of “(composable) full security” was defined [10, 5, 11] using the trace distance, following universal composability definitions for non-quantum cryptography [12]. Intuitively, this notion requires that the final joint quantum state of Alice, Bob, and Eve at the end of the protocol is very close to their final state at the end of an ideal key distribution protocol, that distributes a completely random and secret final key to both Alice and Bob (namely, the trace distance between the real state and the ideal state is exponentially small in $N$). In other words, if a QKD protocol is secure, then except with an exponentially small probability, one of the two following events happens: the protocol is aborted, or the secret key generated by the protocol is the same as a perfect key that is uniformly distributed (i.e., each possible key having the same probability), is the same for both parties, and is independent of the adversary’s information. Details about the formal definition of composable security are available in Subsection 4.6.
1.2 Security Proofs of QKD

Composable security proofs for many QKD protocols, including BB84, have been presented [10, 5, 11]. The proofs of [5, 11] used different methods from the earlier (non-composable) results of [3, 1, 4].

The security proof of BB84 presented by Biham, Boyer, Boykin, Mor, and Roychowdhury (BBBMR) [1] (which follows previous works by some of its authors) used a connection between the information obtained by Eve and the disturbance she induces in the opposite (conjugate) basis. The proof bounded the trace distance between the density matrices held by Eve by using algebraical analysis, and then it used this bound for calculating the mutual information between Eve and the final key. BBBMR proved non-composable security of BB84 against the most general theoretical attacks.

The security proof of BBBMR has various advantages and disadvantages compared to other security proofs for QKD. On the one hand, the security proof of BBBMR is mostly self-contained, while other security approaches require many results from other areas of quantum information (such as various notions of entropy needed for the security proof of [5, 11], and entanglement purification and quantum error correction needed for the security proof of [4]); it gives tight finite-key bounds, unlike several other proof methods [4] and similarly to the most updated results (for the BB84 protocol) in some proof methods [6, 7, 8]; after our improvements, it matches state-of-the-art results (most notably, we proved its asymptotic error rate threshold to be 11%, as given by [4, 5], instead of the 7.56% threshold found by [3, 1]); and, at least in some sense, it is simpler than other proof techniques. On the other hand, it is currently limited to BB84-like protocols.

1.3 Our Contribution

In our work, we extend the security proof of BBBMR in the three following ways:

First, we prove composable security (instead of the non-composable security proved in [1]) against the most general attacks. Intuitively, this improvement is achieved by replacing the manipulations of classical mutual information in [1] by similar manipulations on the quantum trace distance.

Second, we prove security of four QKD protocols similar to BB84 (“generalized BB84”):

1. the “BB84-INFO-z” protocol (defined and analyzed by [13, 14] against collective attacks), in which the final key includes only bits sent in the $z$ basis \{\ket{0}_0, \ket{1}_0\};
2. the standard BB84 protocol;
3. the “efficient BB84” protocol described in [15], in which Alice and Bob choose the $z$ basis \{\ket{0}_0, \ket{1}_0\} or the $x$ basis \{\ket{0}_1 \triangleq \frac{\ket{0}_0 + \ket{1}_0}{\sqrt{2}}, \ket{1}_1 \triangleq \frac{\ket{0}_0 - \ket{1}_0}{\sqrt{2}}\} with non-uniform probabilities; and
4. the “modified efficient BB84” protocol (a simplified version of “efficient BB84”).

Intuitively, this improvement is achieved by using the correct statistical argument (namely, the correct application of Hoeffding’s theorem) for proving security of each protocol. Full details are given in Section 5.

Third, we improve the security results of BBBMR to match the state-of-the-art results achieved for BB84. Most importantly, we improve the error rate threshold in the “asymptotic” scenario, where an infinite number of qubits is transmitted (and the corresponding error
rate thresholds in the finite-key scenario, where only a finite number of qubits is transmitted, is decreased accordingly, depending on the specific number of qubits). In the very first security proofs of BB84 [3, 1], the asymptotic error rate threshold was found to be 7.56% (namely, above this error rate, the protocol must have been aborted); however, later security proofs such as [4, 5] succeeded to improve this error rate to 11%. While the original BBBMR proof only supported the original error rate of 7.56%, here we present the algebraic improvement required to make it support an error rate of 11%, as detailed in Subsection 4.4. (Intuitively, it is based on properties of error-correcting codes detailed in Subsection 2.3.) This improvement assures that BBBMR methods give security results compatible with other proof methods.

2 Preliminaries

2.1 The Notations of Quantum Information

In quantum information, information is represented by quantum states. A quantum pure state is denoted by \( |\psi \rangle \), and it is a normalized vector in a Hilbert space. The qubit Hilbert space is the 2-dimensional Hilbert space \( \mathcal{H}_2 \equiv \text{Span}\{ |0 \rangle_0, |1 \rangle_0 \} \), where the two states \( |0 \rangle_0, |1 \rangle_0 \) (denoted \( |0 \rangle, |1 \rangle \) in most papers) form an orthonormal basis of \( \mathcal{H}_2 \) named “the computational basis” or “the z basis”. The two states \( |0 \rangle_1 \equiv \frac{|0 \rangle_0 + |1 \rangle_0}{\sqrt{2}} \) and \( |1 \rangle_1 \equiv \frac{|0 \rangle_0 - |1 \rangle_0}{\sqrt{2}} \) (denoted \( |+\rangle, |-\rangle \) in most papers) form another orthonormal basis of \( \mathcal{H}_2 \), named “the Hadamard basis” or “the x basis”. These two bases are said to be conjugate bases.

A quantum mixed state is a probability distribution of several pure states, and it is represented by a density matrix: \( \rho = \sum_j q_j |\psi_j \rangle \langle \psi_j | \), where \( q_j \) is the probability that the system is in the pure state \( |\psi_j \rangle \) (this definition should not be confused with the probabilities of measurement results). For example, if the mixed state of a system is \( \rho_1 = \frac{1}{3} |0 \rangle_0 \langle 0 |_0 + \frac{2}{3} |0 \rangle_1 \langle 0 |_1 \), this means that the system is in the \( |0 \rangle_0 \) state with probability \( \frac{1}{3} \) and in the \( |0 \rangle_1 \) state with probability \( \frac{2}{3} \).

The most general operations allowed by quantum physics for the Hilbert space \( \mathcal{H} \) are: performing any unitary transformation \( U : \mathcal{H} \rightarrow \mathcal{H} \); adding an ancillary state inside another Hilbert space; measuring a state with respect to some orthonormal basis; and tracing out a quantum system (namely, ignoring and forgetting a quantum system).

See [16] for more background about quantum information.

2.2 The Notations of Bit Strings

In this paper, we denote bitstrings (of \( t \) bits, where \( t \geq 0 \) is some integer) by a bold letter (e.g., \( i = i_1 \ldots i_t \) with \( i_1, \ldots, i_t \in \{0, 1\} \)); and we refer to these bitstrings as elements of \( \mathbb{F}_2^t \)—namely, row vectors in a \( t \)-dimensional vector space over the field \( \mathbb{F}_2 = \{0, 1\} \), where addition of two vectors corresponds to a XOR operation between them. The number of 1-bits in a bitstring \( s \) is denoted by \( |s| \), and the Hamming distance between two strings \( s \) and \( s' \) is \( d_H(s, s') \equiv |s \oplus s'| \).

2.3 Linear Error-Correcting Codes

2.3.1 Basic Notions

Assume a sender and a receiver want to communicate over a noisy channel. They can use an error-correcting code: the sender adds redundancy to the input of the channel (“encoding”),
and the receiver can correct the errors induced by the noisy channel ("decoding"). In QKD, binary linear error-correcting codes can be used for correcting the errors caused during the quantum transmission.

An \([n,k]\) binary linear error-correcting code \(C\) includes \(2^k\) bitstrings named “codewords”, where each codeword is of length \(n\). Formally, \(C\) is a \(k\)-dimensional vector subspace of \(F_2^n\) (over the field \(F_2\)). Since \(C\) is linear, for all \(a, b \in C\) it holds that \(a \oplus b \in C\). In the standard use of the error-correcting code \(C\), the \(k\)-bit input string \(u\) is translated into an \(n\)-bit codeword \(a \in C\), which is then transmitted through the noisy channel.

The minimum distance \(d\) of a binary linear error-correcting code \(C\) is the smallest Hamming distance between any two codewords: namely, \(d \triangleq \min_{a \neq b \in C} d_H(a, b)\). Equivalently, it is the smallest Hamming weight of a non-zero codeword: \(d = \min_{a \in C \setminus \{0\}} |a|\).

The \(k \times n\) generator matrix \(G_C\) defines the code \(C\), as follows:

\[
C = \{uG_C \mid u \in F_2^k\}.
\] (1)

Namely, the code \(C\) is the row space of \(G_C\).

The \(r \times n\) parity-check matrix \(P_C\) (where \(r \triangleq n - k\)) similarly defines the code \(C\), as follows: for any \(a \in F_2^n\),

\[
a \in C \iff aP_C^T = 0.
\] (2)

Namely, the code \(C\) is the kernel of \(P_C\); we note that \(P_C\) must be of rank \(r\). The parity-check matrix \(P_C\) is also a generator matrix of the dual code \(C^\perp\), defined as:

\[
C^\perp = \{vP_C \mid v \in F_2^r\},
\] (3)

which means, in particular, that the inner product \(a \cdot b = 0\) for any \(a \in C, b \in C^\perp\). Therefore, the dual code \(C^\perp\) is the row space of \(P_C\) (and it can be shown to be the kernel of \(G_C\)).

Standard error correction for classical channels can be performed as follows: the sender sends a codeword \(a \in C\) through the noisy channel, and the receiver gets a noisy result \(b = a \oplus e\) (where \(e \in F_2^n\) is the error word). A standard decoder, which we use throughout this paper, is the nearest-codeword decoder: the word \(b\) is decoded into the codeword minimally distanced from it. The receiver knows the sender sent a codeword in \(C\); therefore, the receiver can check all possible codewords \(a' \in C\), compute the error words \(e' = b \oplus a'\) that would change \(a'\) into \(b\), and choose the lowest-weight error word \(e'\) (which thus corresponds to the nearest codeword, \(a'\)). If the minimum distance \(d\) equals \(2t + 1\), this decoding procedure can correct up to \(t\) errors with certainty; in Subsubsection 2.3.2 we show how to improve this performance.

Error correction for standard QKD protocols is a little different: the input word \(i_t\) (sent by Alice as a quantum state) is random and is not necessarily a codeword. Let us assume that Bob gets the output word \(j_t \triangleq i_t \oplus c_t\) (where \(c_t\) is the error word). In this case, errors can still be corrected in the following way: the sender Alice sends to Bob the syndrome \(\xi \in F_2^5\), defined as

\[
\xi \triangleq i_t(\xi)C.
\] (4)

Then, let us define the code coset \(C_\xi\) as follows:

\[
C_\xi \triangleq \{z \in F_2^5 \mid zP_C^T = \xi\}.
\] (5)

The set \(C_\xi\) is equal to \(\ell_\xi + C\), given any arbitrary word \(\ell_\xi \in F_2^5\) having the same syndrome \(\xi\) (namely, satisfying \(\ell_\xi P_C^T = \xi\)), which is why we name it “code coset”: this is true because
Equation (2) implies $C_0 = C$. Therefore, since Bob knows $\xi$, he also knows that the input word $i_f$ must be in the code coset $C_\xi = \ell_\xi + C$ (namely, there must exist $a \in C$ such that $i_f = \ell_\xi \oplus a$). Therefore, Bob can apply a similar method to standard error correction on the code coset $C_\xi$ (instead of applying it on the original code $C$)—namely, he can use the affine code $C_\xi$ instead of the linear code $C$. Specifically, Bob can check all possible words in the code coset $C_\xi$ (instead of applying it on the original code $C$)—namely, he can use the affine code $C_\xi$ instead of the linear code $C$. As explained in Subsubsection 2.3.1, for any two codewords $a, b \in C$, the bitstring $a \oplus e$ is correctly decoded into $a$ if and only if the codeword nearest to $a \oplus e$ is uniquely $a$. Therefore, both $x$ and $y$ are codewords nearest to $e$, and both $a \oplus x$ and $a \oplus y$ are codewords nearest to $a \oplus e$.

2.3.2 Finding Codes for Error Correction and Privacy Amplification

We now present several important results on linear error-correcting codes, that are essential both for fully proving the results discussed in Appendix E of [1] and for our improved results in Subsection 4.4.

For the nearest-codeword decoder used throughout this paper, we assume that whenever it gets an input $x \in \mathbb{F}_2^n$ such that two or more codewords are nearest to $x$ and have equal distances from $x$, the decoder fails (namely, its output may be arbitrary and is ignored). Namely, we assume the decoder never “breaks ties” between candidates to being nearest codewords, but simply fails if there is a tie and no unique decoding is possible.

First, we prove the success of error correction to be independent of the original word:

**Lemma 1.** For any $[n, k]$ binary linear error-correcting code $C$ and any error word $e \in \mathbb{F}_2^n$:

1. A nearest-codeword decoder successfully corrects the error $e$ if and only if the codeword nearest to $e$ is uniquely $0$ (namely, $0 \in C$ is strictly closer to $e$ than any non-zero codeword in $C$).

Formally, if the input to the nearest-codeword decoder is $a \oplus e$, where $a \in C$ is a codeword and $e \in \mathbb{F}_2^n$ is the error word, then $a \oplus e$ is correctly decoded into $a$ if and only if the codeword nearest to $e$ is uniquely $0$.

2. In particular, the success of correcting the error $e$ using a nearest-codeword decoder is independent of the codeword, and depends only on the error word.

Formally, for any two codewords $a, b \in C$, the bitstring $a \oplus e$ is correctly decoded into $a$ if and only if the bitstring $b \oplus e$ is correctly decoded into $b$.

**Proof.** As explained in Subsubsection 2.3.1, for $a \in C$, the bitstring $a \oplus e$ is decoded to its nearest codeword. Let us denote by $x \in C$ a codeword nearest to $e$, and denote by $y' \in C$ a codeword nearest to $a \oplus e$. We can easily denote $y' = a \oplus y$ (by denoting $y \triangleq y' \oplus a$; let us remember that $y', a \in C$, so $y \in C$). Then, because $x$ is a codeword nearest to $e$ and $a \oplus y$ is a codeword nearest to $a \oplus e$, we get:

$$d_H(x, e) = d_H(a \oplus x, a \oplus e) \geq d_H(a \oplus y, a \oplus e) = d_H(y, e) \geq d_H(x, e).$$  \hspace{1cm} (6)

Because the left-hand-side and right-hand-side are equal, it follows that all inequalities must be equalities, so $d_H(x, e) = d_H(y, e)$ and $d_H(a \oplus x, a \oplus e) = d_H(a \oplus y, a \oplus e)$. Therefore, both $x$ and $y$ are codewords nearest to $e$, and both $a \oplus x$ and $a \oplus y$ are codewords nearest to $a \oplus e$. 


This particularly means that the codeword nearest to $e$ is uniquely $x$ if and only if the codeword nearest to $a \oplus e$ is uniquely $a \oplus x$, in which case $x = y$.

Therefore, the nearest-codeword decoder correctly decodes $a \oplus e$ into $a$ if and only if the codeword nearest to $a \oplus e$ is uniquely $a$ \footnote{If the codeword nearest to $a \oplus e$ is \textit{not} unique, then according to our above assumption, the nearest-codeword decoder fails.}, which happens if and only if the codeword nearest to $e$ is uniquely $0$; this proves item 1.

For deducing item 2, we apply item 1 twice, proving that for any two codewords $a, b \in C$, the bitstring $a \oplus e$ is correctly decoded into $a$ if and only if the codeword nearest to $a \oplus e$ is uniquely $0$, which happens if and only if the bitstring $b \oplus e$ is correctly decoded into $b$. Therefore, the decoding’s correctness is independent of the codeword ($a$ or $b$), and depends only on the error word $e$.

Lemma 1 can be extended to QKD’s modified error correction process described in Subsubsection 2.3.1. For this, we use a \textit{modified nearest-codeword decoder} (which we call “nearest-word-in-code-coset decoder”), which gets an input word $x \in F_2^n$ and an input syndrome $\xi \in F_2^r$ and decodes $x$ into the nearest word in the code coset $C_\xi \triangleq \{ z \in F_2^n \mid zP_T^T = \xi \}$ defined in Equation (5). As before, we assume that if \textit{two or more words in the code coset} are nearest to $x$ and have equal distances from $x$, the decoder fails—namely, that the decoder never “breaks ties”. The generalization is now presented in the following Lemma:

\textbf{Lemma 2. For any $[n, k]$ binary linear error-correcting code C and any error word $e \in F_2^n$:}

1. A nearest-word-in-code-coset decoder successfully corrects the error $e$ if and only if the codeword (in $C$) nearest to $e$ is uniquely $0$ (namely, $0 \in C$ is strictly closer to $e$ than any non-zero codeword in $C$).

Formally, if the input word to the nearest-word-in-code-coset decoder is $i \oplus e$ and the input syndrome is $\xi$, where $i \in F_2^n$ is the original word, $e \in F_2^n$ is the error word, and $\xi = iP_T^T$ is the syndrome of the original word, then $i \oplus e$ is correctly decoded into $i$ if and only if the codeword (in $C$) nearest to $e$ is uniquely $0$.

2. In particular, the success of correcting the error $e$ using the nearest-word-in-code-coset decoder is independent of the original word or the input syndrome, and depends only on the error word.

Formally, for any two original words $i_1, i_2 \in F_2^n$, the bitstring $i \oplus e$ is correctly decoded into $i_2$ (given the input syndrome $\xi_2 \triangleq i_2P_T^T$) if and only if the bitstring $i_2 \oplus e$ is correctly decoded into $i_1$ (given the input syndrome $\xi_1 \triangleq i_1P_T^T$).

\textbf{Proof.} As explained in Subsubsection 2.3.1 and in the above definition of the nearest-word-in-code-coset decoder, for the original word $i \in F_2^n$ and the input syndrome $\xi = iP_T^T$, the bitstring $i \oplus e$ is decoded to the nearest word in the code coset $C_\xi \triangleq \{ z \in F_2^n \mid zP_T^T = \xi \}$ (where also $i \in C_\xi$, by definition). Let us denote by $x \in C$ a codeword (in $C$) nearest to $e$, and denote by $y' \in C_\xi$ a word in the code coset that is nearest to $i \oplus e$. We can easily denote $y' = i \oplus y$ (by denoting $y \triangleq y' \oplus i$; let us remember that $y', i \in C_\xi$, so $y \in C$). Then, because $x \in C$ is a codeword nearest to $e$ and $i \oplus y \in C_\xi$ is a word in the code coset nearest to $i \oplus e$, we get:

\begin{align}
    d_H(x, e) = d_H(i \oplus x, i \oplus e) \geq d_H(i \oplus y, i \oplus e) = d_H(y, e) \geq d_H(x, e).
\end{align}
Because the left-hand-side and right-hand-side are equal, it follows that all inequalities must be equalities, so $d_H(x, e) = d_H(y, e)$ and $d_H(i \oplus x, i \oplus e) = d_H(i \oplus y, i \oplus e)$. Therefore, both $x$ and $y$ are codewords nearest to $e$, and both $i \oplus x$ and $i \oplus y$ are words in the code coset $C_x$ that are nearest to $i \oplus e$. This particularly means that the codeword nearest to $e$ is uniquely $x$ if and only if the word in $C_x$ that is nearest to $i \oplus e$ is uniquely $i \oplus x$, in which case $x = y$.

Therefore, the nearest-word-in-code-coset decoder correctly decodes $i \oplus e$ into $i$ if and only if the codeword nearest to $e$ in $C_x$ is uniquely $i \oplus e$, which happens if and only if the codeword nearest to $e$ is uniquely $0$; this proves item 1.

For deducing item 2, we apply item 1 twice, proving that for any two original words $i_1, i_2 \in \mathbb{F}_2^n$, the bitstring $i_1 \oplus e$ is correctly decoded into $i_1$ if and only if the codeword nearest to $e$ is uniquely $0$, which happens if and only if the bitstring $i_2 \oplus e$ is correctly decoded into $i_2$. Therefore, the decoding’s correctness is independent of the codeword ($i_1$ or $i_2$) or the respective input syndromes $\xi_1, \xi_2$, and depends only on the error word $e$. \hfill \Box

Lemmas 1 and 2 imply that for any code $C$, the nearest-word-in-code-coset decoder of QKD gives identical results as the standard nearest-codeword decoder on any error string $e \in \mathbb{F}_2^n$: namely, if any of the two decoders operates on the error string $e$, it will be successful in decoding if and only if the codeword nearest to $e$ is uniquely $0$. Therefore, both decoders work correctly on all error words $e \in \mathbb{F}_2^n$ whose nearest codeword is uniquely $0 \in C$, and both fail on all other error words. (The probability that this decoding fails for a random code is bounded by Corollary 4.)

Second, given a specific word $\ell \in \mathbb{F}_2^n$, we bound the probability that for a randomly chosen code $C$ there is another low-weight word $z$ such that the difference $\ell \oplus z$ is inside $C$:

**Proposition 3.** Let $n,k,t \geq 1$ be integers such that $0 \leq \frac{t}{n} \leq \frac{1}{2}$. For any word $\ell \in \mathbb{F}_2^n$, the probability that a randomly chosen $[n,k]$ binary linear error-correcting code $C$ has another word $z \in \ell + C$ (namely, a word $z \in \mathbb{F}_2^n$ such that $\ell \oplus z \in C$) that satisfies $z \neq \ell$ and $|z| \leq t$ is

$$\Pr_C \left[ \exists z \in \ell + C : z \neq \ell, |z| \leq t \mid \ell \right] \leq 2^{n[H_2(t/n) - r/n]},$$

where $r \triangleq n - k$ and $H_2(x) \triangleq -x \log_2x - (1 - x) \log_2(1 - x)$.

**Proof.** This proof is based on the proof of [18, Lemma 4.15 in page 115]. Let $\ell \in \mathbb{F}_2^n$ be a specific word. If we choose a random $[n,k]$ binary linear code $C$ by choosing a uniformly random $k \times n$ generator matrix $G_C$ over $\mathbb{F}_2$ \(^3\), then for any fixed non-zero vector $u \in \mathbb{F}_2^n \setminus \{0\}$, the resulting product $uG_C$ is uniformly distributed over $\mathbb{F}_2^n \setminus \{0\}$, and so the sum $\ell \oplus uG_C$ is uniformly distributed over $\mathbb{F}_2^n \setminus \{\ell\}$. Therefore:

$$\Pr_C \left[ |\ell \oplus uG_C| \leq t \mid \ell \right] = \sum_{z \in \mathbb{F}_2^n : |z| \leq t} \Pr_C \left[ \ell \oplus uG_C = z \mid \ell \right]$$

$$= \sum_{z \in \mathbb{F}_2^n \setminus \{\ell\} : |z| \leq t} \frac{1}{2^{n - 1}} \leq \frac{1}{2^{n - 1}} \cdot V_2(n, t),$$

\(^2\)If the word in $C_x$ that is nearest to $i \oplus e$ is not unique, then according to our above assumption, the nearest-word-in-code-coset decoder fails.

\(^3\)In fact, it is possible that the randomly chosen generator matrix $G_C$ satisfies $\text{rank}(G_C) < k$, which gives a “bad” code (not a valid $[n,k]$ error-correcting code). In this case, we can assume the process is run again until we get a valid code—namely, until we get a matrix $G_C$ satisfying $\text{rank}(G_C) = k$. 

8
where $V_2(n,t)$ is the number of words $z \in F_2^n$ that satisfy $|z| \leq t$ (see definition in [18, page 95]). In [18, Lemma 4.7 in page 105] it was proved that $V_2(n,t) \leq 2^{nH_2(t/n)}$ (assuming $0 \leq \frac{t}{n} \leq \frac{1}{2}$), so:

$$\Pr_C [\| \ell \oplus uG_C \| \leq t \mid \ell] \leq \frac{1}{2^n-1} \cdot V_2(n,t) \leq \frac{1}{2^n-1} \cdot 2^{nH_2(t/n)}.$$  \hspace{1cm} (10)

We can therefore compute:

$$\Pr_C [\exists z \in \ell + C : z \neq \ell, |z| \leq t \mid \ell] \leq \Pr_C \left[ \exists u \in F_2^k \setminus \{0\} : \| \ell \oplus uG_C \| \leq t \mid \ell \right] \leq \sum_{u \in F_2^k \setminus \{0\}} \Pr_C [\| \ell \oplus uG_C \| \leq t \mid \ell] \leq \sum_{u \in F_2^k \setminus \{0\}} \frac{1}{2^n-1} \cdot 2^{nH_2(t/n)} = 2^{k-1} \cdot \frac{2^n-1}{2^n} \cdot 2^{nH_2(t/n)} \leq 2^k \cdot 2^{nH_2(t/n)} = 2^{nH_2(t/n)-(n-k)} = 2^{n[H_2(t/n)-r/n]}.$$  \hspace{1cm} (11)

Finally, we combine all results to study the failure probability of correcting a specific error $e$ in a randomly chosen code $C$:

**Corollary 4.** Let $n,k,t \geq 1$ be integers such that $0 \leq \frac{t}{n} \leq \frac{1}{2}$. For any error string $e \in F_2^n$ satisfying $|e| \leq t$, the probability that a randomly chosen $[n,k]$ binary linear error-correcting code $C$ cannot correct the error $e$ using a nearest-codeword decoder is

$$\Pr_C [C \text{ cannot correct } e \mid e] \leq 2^{n[H_2(t/n)-r/n]},$$  \hspace{1cm} (12)

where $r \triangleq n - k$ and $H_2(x) \triangleq -x \log_2(x) - (1 - x) \log_2(1 - x)$.

**Proof.** According to Lemmas 1 and 2, the error word $e$ is correctly decoded by a nearest-codeword decoder if and only if the codeword nearest to $e$ is uniquely 0. In other words, $e$ is wrongly decoded by $C$ if and only if there exists a codeword $c \in C \setminus \{0\}$ such that $|e \oplus c| \leq |e| < t$, which implies that $z \triangleq e \oplus c$ satisfies $z \in e + C, z \neq e$, and $|z| \leq t$. Therefore, according to Proposition 3:

$$\Pr_C [C \text{ cannot correct } e \mid e] \leq \Pr_C \left[ \exists z \in e + C : z \neq e, |z| \leq t \mid e \right] \leq 2^{n[H_2(t/n)-r/n]}.$$  \hspace{1cm} (13)

\[ \square \]

### 3 Full Definition of the Generalized BB84 Protocols

The protocols for which we prove security in this paper belong to a generalized class of BB84-like protocols. Below we formally define this general class of protocols. Some of the details in this definition are decided by each specific protocol, but most details are shared by all protocols.
1. Before the protocol begins, Alice and Bob choose some shared (and public) parameters: the integer numbers \( N, n, r, m \) (such that \( r + m \leq n < N \)), the sets \( B \) and \( \{S_b\}_{b \in B} \) and probability distributions over them (decided by the specific protocol) that will control the future choice of bitstrings \( b, s \in F_2^N \), and the testing function \( T \) (decided by the specific protocol).

Formally, for choosing the sets \( B \) and \( \{S_b\}_{b \in B} \) and the corresponding probability distributions, Alice and Bob should choose the set \( B \subseteq F_2^N \) of basis strings, the probabilities \( \Pr(b) \) for all \( b \in B \), the sets \( S_b \subseteq F_2^n \) of \( s \) strings for all \( b \in B \), and the probabilities \( \Pr(s | b) \) for all \( b \in B \) and \( s \in S_b \). We require that \( |s| = n \) for all \( s \in S_b \). (Equivalently, Alice and Bob choose a subset \( R \) of the set \( \{(b, s) | b, s \in F_2^N, |s| = n\} \) and a probability distribution \( \Pr(b, s) \).) The testing function \( T : F_2^{N-n} \times F_2^{N-n} \times F_2^N \rightarrow \{0, 1\} \) must get \((i_T \oplus j_T, b_T, s)\) as inputs and give 0 or 1 as an output. In Section 5 we give examples of protocols and their formal definitions using these notations.

2. Alice randomly chooses an \( N \)-bit string \( i \in F_2^N \) (the choice of \( i \) is uniformly random), an \( N \)-bit string \( b \in B \), an \( N \)-bit string \( s \in S_b \) (that must satisfy \( |s| = n \)), an \( r \times n \) parity-check matrix \( P_C \) (corresponding to a linear error-correcting code \( C \)), and an \( m \times n \) privacy amplification matrix \( P_K \) (representing a linear key-generation function). It is required that all \( r + m \) rows of the matrices \( P_C \) and \( P_K \) put together are linearly independent; both matrices are chosen uniformly at random under this condition, and Alice keeps them secret at the current stage.

Then, Alice sends the \( N \) qubit states \( |i_1 \rangle_{b_1}, |i_2 \rangle_{b_2}, \ldots, |i_N \rangle_{b_N} \), one after the other, to Bob using the quantum channel. (The probability distributions \( \Pr(b), \Pr(s | b) \) and the sets \( B, S_b \) were chosen in Step 1 according to the specific protocol.) Bob keeps each received qubit in a quantum memory, not measuring it yet\(^4\).

3. Alice sends to Bob over the classical channel the bitstring \( b = b_1 \ldots b_N \). Bob measures each of the qubits he saved in the correct basis (namely, when measuring the \( i \)-th qubit, he measures it in the \( z \) basis if \( b_i = 0 \), and he measures it in the \( x \) basis if \( b_i = 1 \)).

The bitstring measured by Bob is denoted by \( j \). The XOR of \( i \) and \( j \) is denoted \( c = i \oplus j \). If there is no noise and no eavesdropping, then \( i = j \) (that is, \( c = 0 \)).

4. Alice sends \( s \) to Bob over the classical channel. The INFO bits (that will be used for creating the final key) are the \( n \) bits with \( s_j = 1 \), while the TEST bits (that will be used for testing) are the \( N - n \) bits with \( s_j = 0 \). We denote the substrings of \( i, j, c, b \) that correspond to the INFO bits by \( i_I, j_I, c_I \), and \( b_I \), respectively; and we denote the substrings of \( i, j, c, b \) that correspond to the TEST bits by \( i_T, j_T, c_T \), and \( b_T \), respectively.

5. Alice and Bob both publish the \( N - n \) bit values they have for all TEST bits (\( i_T \) and \( j_T \), respectively), and they compute their XOR \( c_T = i_T \oplus j_T \). They compute \( T(c_T, b_T, s) \): if it is 0, they abort the protocol; if it is 1, they continue the run of the protocol. (The testing function \( T \) was chosen in Step 1 according to the specific protocol.)

\(^4\)Here we assume that Bob has a quantum memory and can delay his measurement. In practical implementations, Bob usually cannot do that, but he is assumed to choose his own random basis string \( b'' \in B \) and measure in the bases it dictates; later, Alice and Bob discard the qubits measured in the wrong basis. In that case, we need to assume that Alice sends more than \( N \) qubits, so that \( N \) qubits are finally detected by Bob and measured in the correct basis. In Appendix A of [1] it is explained why this change of the protocol does not hurt security.
6. The values of the $n$ remaining bits (the INFO bits, with $s_j = 1$) are kept in secret by Alice and Bob. The bitstring of Alice is $i$, the bitstring of Bob is $j$, and their XOR is $c$.

7. Alice sends to Bob over the classical channel her chosen parity-check matrix $P_C$ (that corresponds to the error-correcting code $C$) and the syndrome of $i$ with respect to $C$, which consists of $r$ bits and is defined as $\xi \triangleq iP^{T}_C$. Bob uses $\xi$ to correct the errors in his $j$ string (so that it should be the same as $i$) using the “code coset” decoding method presented in Subsubsection 2.3.1.

8. Alice sends to Bob over the classical channel her chosen privacy amplification matrix $P_K$. The final key $k$ consists of $m$ bits and is defined as $k \triangleq iP^{T}_{K}$; both Alice and Bob compute it.

4 Bound on the Security Definition for Generalized BB84 Protocols

4.1 The Hypothetical “inverted-INFO-basis” Protocol

Our security proof uses an alternative, hypothetical protocol in which Alice sends to Bob the qubits after inverting the bases of the INFO bits (without changing the bases of the TEST bits). We call this protocol “hypothetical” because it is never actually used by Alice and Bob, and we do not perform any reduction to it (or from it), but we compute probabilities of certain events in the hypothetical protocol for use in our security bound. In particular, we use the error rate in the hypothetical protocol for bounding the trace distance in the security definition of the real protocol.

In the hypothetical protocol, Alice, Bob, and Eve do everything exactly as they would do in the real protocol, except that Alice and Bob use (and publish) the basis string $b^0 \triangleq b \oplus s$ instead of $b$: namely, they use the basis string $b^T$ for the TEST bits and the basis string $b^I$ (the bitwise NOT of $b$) for the INFO bits.

Formally, this hypothetical protocol is defined by replacing Steps 2–3 of the original protocol (as described in Section 3) by the following steps:

2. Alice randomly chooses an $N$-bit string $i \in F_2^N$, an $N$-bit string $b \in B$, an $N$-bit string $s \in S_b$ (that must satisfy $|s| = n$), an $r \times n$ parity-check matrix $P_C$, and an $m \times n$ privacy amplification matrix $P_K$.

Then, Alice computes the $N$-bit string $b^0 \triangleq b \oplus s$, and she sends the $N$ qubit states $|i_1\rangle_{\rho^I_1}, |i_2\rangle_{\rho^I_2}, \ldots, |i_N\rangle_{\rho^I_N}$, one after the other, to Bob using the quantum channel. Bob keeps each received qubit in a quantum memory, not measuring it yet.

3. Alice sends to Bob over the classical channel the bitstring $b^0 = b^0_1 \ldots b^0_N$. Bob measures each of the qubits he saved in the correct basis (namely, when measuring the $i$-th qubit, he measures it in the $z$ basis if $b^0_i = 0$, and he measures it in the $x$ basis if $b^0_i = 1$).

We notice that in this protocol, Alice chooses $b$ and $s$ in the same way as she would choose them in the real protocol, but uses (and sends to Bob for his use) $b^0$ and $s$ instead.
In the security proof, we will use the notation of $\text{Pr}_{\text{inverted-INFO-basis}}$ for computing the probability of a certain event assuming that Alice and Bob use the hypothetical protocol. In particular, we note that $\text{Pr}_{\text{inverted-INFO-basis}}(\cdot \mid b, s)$ is a conditional probability on Alice choosing the bitstrings $b, s$ (while she actually uses the basis string $b^0$).

We should note that $\text{Pr}_{\text{inverted-INFO-basis}}(\cdot \mid b, s)$ is the same as $\text{Pr}(\cdot \mid b^0, s)$ if both notations are well-defined: namely, the hypothetical protocol given that Alice chooses $b, s$ (and thus uses $b^0, s$) is the same as the real protocol given that Alice chooses $b^0, s$. However, the second notation is not always well-defined, because it may be the case that $b \in B$ while $b^0 \notin B$, or that $s \in S_b$ while $s \notin S_{b^0}$; therefore, it may be the case that $b^0$ is not an allowed basis string for the real protocol. In the standard BB84 protocol (see Subsection 5.3), such problems are impossible, and this is why [1] used the notation of $\text{Pr}(\cdot \mid b^0, s)$ instead of $\text{Pr}_{\text{inverted-INFO-basis}}(\cdot \mid b, s)$; however, in our paper, we discuss generalized BB84 protocols, so we must use the notation of $\text{Pr}_{\text{inverted-INFO-basis}}(\cdot \mid b, s)$.

### 4.2 The General Joint Attack of Eve

Before the QKD protocol is performed (and, thus, independently of $i, b,$ and $s$), Eve chooses a specific joint attack she wants to perform. In a joint attack, all qubits are attacked using one giant probe (ancillary state) kept by Eve. Eve saves her probe in a quantum memory and can keep it indefinitely, even after the final key is used by Alice and Bob; and she can perform, at any time of her choice, an optimal measurement of her giant probe, chosen based on all the information she has at the time of the measurement (including the classical information sent during the protocol, and including information she acquires when Alice and Bob use the final key).

Given that Alice sends to Bob the state $|i\rangle_b \triangleq \bigotimes_{j=1}^N |i_j\rangle_{b_j}$ (namely, the $N$-bit string is $i$ and the $N$-bit basis string is $b$), Eve attaches a probe state $|0\rangle_E$ and applies a unitary operator $U$ of her choice to the compound system $|0\rangle_E|i\rangle_b$. Then, Eve keeps to herself (in a quantum memory) her probe state, and she sends to Bob the $N$-qubit quantum state sent from Alice to Bob (which may have been modified due to her attack $U$).

The most general joint attack $U$ of Eve is

$$U|0\rangle_E|i\rangle_b = \sum_{j \in F^N_2} |E'_{i,j}\rangle_b|j\rangle_b,$$

where $|E'_{i,j}\rangle_b$ are non-normalized states in Eve’s probe system. We note that

$$\langle E'_{i,j}|E'_{i',j'}\rangle_b = \text{Pr}(j \mid i, b, s) = \text{Pr}(j \mid i, b, s, P_C, P_K).$$

Writing the INFO and TEST bits of Alice and Bob separately ($i_i, i_T$ instead of $i$, and $j_i, j_T$ instead of $j$), we can denote $|E'_{i,j}\rangle_b$ by $|E'_{i,i_T,j,i_T}\rangle_b$.

---

5It is also possible that $\text{Pr}(b, s) \neq \text{Pr}(b^0, s)$, in which case the use of $b^0, s$ in the real protocol does not happen with the same probability as the use of $b^0, s$ in the hypothetical protocol. However, if both probabilities $\text{Pr}(b, s), \text{Pr}(b^0, s)$ are non-zero, then $\text{Pr}_{\text{inverted-INFO-basis}}(\cdot \mid b, s) = \text{Pr}(\cdot \mid b^0, s)$.

6In [1], no probabilities were conditioned on $P_C, P_K$, because these matrices were assumed to be constant and public. In our paper we assume $P_C, P_K$ to be randomly chosen, so it appears all probability distributions based on [1] should have been conditioned on $P_C, P_K$; however, in our paper Alice chooses $P_C, P_K$ uniformly at random, keeps them secret, and does not use them until Eve’s attack is over, so probabilities related only to Alice’s other random choices $(i, b, s)$ and Eve’s attack $(j, c \triangleq i \oplus j)$ are completely independent of $P_C, P_K$. 

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Subsection 3.4 of [1] introduced the notation of $|E_{i,i,j} \rangle_{b,s}$; this notation is useful because it treats $i_T$ and $j_T$ as constants, assuming their values to be known (since they are ultimately published by Alice and Bob, and then they are known to Eve). It is defined as

$$|E_{i,i,j} \rangle_{b,s} \triangleq \frac{1}{\sqrt{\text{Pr}(j_T | i_T, i, b, s)}} |E'_{i,i,j,j_T} \rangle_{b}. \quad (16)$$

We note that $|E_{i,i,j} \rangle_{b,s}$ also depends on $i_T, j_T$ (and not only on $i, j, b, s$). According to Equations (3.22)–(3.23) of [1], assuming that Alice chooses $i, i_T, b, s$ and Bob measures $j_T$, the normalized state of Eve and Bob is

$$|\psi_i \rangle = \sum_{j_T \in F_2^s} |E_{i,i,j} \rangle_{b,s}|j_T \rangle_b, \quad (17)$$

and it also holds that

$$\langle E_{i,i,j} | E_{i,i,j} \rangle_{b,s} = \text{Pr}(j_T | i_T, i, j_T, b, s). \quad (18)$$

Let us define $\left( \rho_{i,i,j}^{b,s} \right)_E$ (which also depends on $i_T, j_T$) to be the normalized state of Eve if Alice chooses $i, i_T, b, s$ and Bob measures $j_T$. In other words, $\left( \rho_{i,i,j}^{b,s} \right)_E$ is the normalized density matrix of both $|E_{i,i,j} \rangle_{b,s}$ and $|E'_{i,i,j} \rangle_{b}$, so

$$\left( \rho_{i,i,j}^{b,s} \right)_E \triangleq \frac{|E_{i,i,j} \rangle_{b,s}\langle E_{i,i,j} |_{b,s}}{\text{Pr}(j_T | i_T, i, j_T, b, s)} = \frac{|E'_{i,i,j} \rangle_{b}\langle E'_{i,i,j} |_{b}}{\text{Pr}(j_T | i_T, i, j_T, b, s)} = \frac{|E'_{i,i,j} \rangle_{b}\langle E'_{i,i,j} |_{b}}{\text{Pr}(j_T | i_T, i, j_T, b, s, P_C, P_K)}. \quad (19)$$

Eve’s state after her attack (tracing out Bob) is

$$\left( \rho_i^h \right)_E \triangleq \text{tr}_{\text{Bob}}(|\psi_i \rangle \langle \psi_i |) = \sum_{j_T \in F_2^s} |E_{i,i,j} \rangle_{b,s}\langle E_{i,i,j} |_{b,s} = \sum_{j_T \in F_2^s} \text{Pr}(j_T | i_T, i, j_T, b, s) \left( \rho_{i,i,j}^{b,s} \right)_E, \quad (20)$$

and according to Equation (4.2) of [1], we define the purification $|\varphi_i \rangle_E$ of $\left( \rho_i^h \right)_E$ (so that $\left( \rho_i^h \right)_E$ is a partial trace of $|\varphi_i \rangle_E$) as

$$|\varphi_i \rangle_E \triangleq \sum_{j_T \in F_2^s} |E_{i,i,j} \rangle_{b,s}|j_i \oplus j_T \rangle. \quad (21)$$

We also define a “Fourier basis” of non-normalized states $\{|\eta_\ell \rangle \rangle_{\ell} \in F_2^s$ (according to Definition 4.2 in [1]) as

$$|\eta_\ell \rangle_E \triangleq \frac{1}{2^n} \sum_{i_T \in F_2^s} (-1)^{i \cdot \ell} |\varphi_i \rangle_E, \quad (22)$$

define the two notations $d_\ell$ and $| \hat{\eta}_\ell \rangle_E$ (according to the same definition in [1]) as

$$d_\ell \triangleq \sqrt{\langle \eta_\ell | \eta_\ell \rangle_E}, \quad (23)$$

$$| \hat{\eta}_\ell \rangle_E \triangleq \frac{|\eta_\ell \rangle_E}{d_\ell}, \quad (24)$$

and deduce (according to Equation (4.4) in [1]) that

$$|\varphi_i \rangle_E = \sum_{\ell \in F_2^s} (-1)^{i \cdot \ell} |\eta_\ell \rangle_E, \quad (25)$$

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4.3 The Symmetrized Attack of Eve

In [1], the most general joint attack was not directly analyzed: for simplicity, it was assumed that Eve applies a process named symmetrization, resulting in a symmetrized attack. The process of symmetrization is always beneficial to Eve (it does not change the error rate, and we prove in Proposition 9 that it does not decrease Eve’s information), so security against all symmetrized attacks implies security against all possible joint attacks.

In Eve’s original attack, she has her own probe subsystem E; in the symmetrization process, Eve adds another probe subsystem M, in the initial state of $|0\rangle_M \triangleq \frac{1}{\sqrt{2N}} \sum_{m \in F_2} |m\rangle_M$. Given the original attack $U$ (applied to Alice’s qubits and the probe E), the symmetrized attack $U^{\text{sym}}$ (applied to Alice’s qubits and both probes E and M) is defined by

$$U^{\text{sym}} \triangleq (I_E \otimes S^\dagger)(U \otimes I_M)(I_E \otimes S),$$

where $S$ is a unitary operation applied to Alice’s qubits and to the probe $M$, and it operates as follows:

$$S|i\rangle_b\langle m|_M = (-1)^{i \oplus b \cdot m} |i \oplus m\rangle_b\langle m|_M. $$

Intuitively, Eve first XORs Alice’s bit values with a random string $m$ (kept by her), then applies her original attack, and finally reverses the XOR with $m$. Full definition and explanations are available in Subsection 3.1 of [1].

In this paper we use several properties of the symmetrized attack. First of all, the “Basic Lemma of Symmetrization” (Lemma 3.1 of [1]) gives the expression for $|E^{\text{sym}}_{i,j}\rangle_b$ (of the symmetrized attack) as a function of $|E'_{i,j}\rangle_b$ (of the original attack):

$$|E^{\text{sym}}_{i,j}\rangle_b = \frac{1}{\sqrt{2N}} \sum_{m \in F_2} (-1)^{i \oplus j \cdot m} |E'_{i \oplus m,j \oplus m}\rangle_b\langle m|_M. $$

The second property we use, proved in Corollary 3.3 of [1], is the fact that the probabilities of error strings $c_I$ and $c_T$ (if not conditioning on $i$) are not affected by the symmetrization. Namely,

$$\Pr^{\text{sym}}(c_I, c_T | b, s) = \Pr(c_I, c_T | b, s). $$

This property is true for all basis strings $b$. In particular, it is true for the basis string $b^0 \triangleq b \oplus s$ used in the hypothetical “inverted-INFO-basis” protocol defined in Subsection 4.1; therefore,

$$\Pr^{\text{sym}}_{\text{inverted-INFO-basis}}(c_I, c_T | b, s) = \Pr_{\text{inverted-INFO-basis}}(c_I, c_T | b, s).$$

The third property we use, proved in Lemma 3.8 of [1], is the fact that the probabilities for errors in the TEST bits are not affected by the bases used for the INFO bits:

$$\Pr^{\text{sym}}(j_T | i_T, b, s) = \Pr^{\text{sym}}(j_T | j_T, b_T, s). $$

In particular, since the only difference between the hypothetical “inverted-INFO-basis” protocol and the real protocol is the basis string used for the INFO bits ($\overline{b}_T$ and $b_T$, respectively), this property means that the probabilities of errors in the TEST bits are identical for both protocols:

$$\Pr^{\text{sym}}_{\text{inverted-INFO-basis}}(j_T | i_T, b, s) = \Pr^{\text{sym}}(j_T | i_T, b, s).$$
The fourth property we use, proved in Corollary 3.6 of [1], is the fact that the probability of any string of INFO bits \( i_1 \) is uniform (that is, \( \frac{1}{2^n} \)) even when conditioning on the four parameters \( i_T, j_T, b, s \) that are ultimately known to Eve (we note that \( j_T \) is affected by Eve’s attack). Namely,

\[
\Pr^{\text{sym}}(i_1 | i_T, j_T, b, s) = \frac{1}{2^n}.
\]

(33)

In particular, the same is true when conditioning on \( P_C, P_K \) (because, as noted in Subsection 4.2, the randomly chosen matrices \( P_C, P_K \) are completely independent of \( i, j, b, s \)). Thus,

\[
\Pr^{\text{sym}}(i_1 | i_T, j_T, b, s, P_C, P_K) = \Pr^{\text{sym}}(i_1 | i_T, j_T, b, s) = \frac{1}{2^n}.
\]

(34)

Given specific \( P_C, P_K \), we can thus compute the probability of each specific syndrome \( \xi \) (namely, the probability that for a specific value of \( \xi \in \mathbb{F}_2^n \), the chosen value of \( i_1 \) satisfies \( i_1 P_C^T = \xi \)):

\[
\Pr^{\text{sym}}(\xi | i_T, j_T, b, s, P_C, P_K) = \sum_{i_1 | i_1 P_C^T = \xi} \Pr^{\text{sym}}(i_1 | i_T, j_T, b, s, P_C, P_K) = \sum_{i_1 | i_1 P_C^T = \xi} \frac{1}{2^n} = 2^{n-r} \cdot \frac{1}{2^n} = \frac{1}{2^r}
\]

(35)

(because the matrix \( P_C \) has rank \( r \), so for each specific syndrome \( \xi \in \mathbb{F}_2^n \) there are exactly \( 2^{n-r} \) values of \( i_1 \) satisfying \( i_1 P_C^T = \xi \)). In particular,

\[
\Pr^{\text{sym}}(\xi | i_T, j_T, b, s) = \sum_{P_C, P_K} \Pr^{\text{sym}}(P_C, P_K | i_T, j_T, b, s) \cdot \Pr^{\text{sym}}(\xi | i_T, j_T, b, s, P_C, P_K) = \sum_{P_C, P_K} \Pr^{\text{sym}}(P_C, P_K | i_T, j_T, b, s) \cdot \frac{1}{2^r} = \frac{1}{2^r},
\]

(36)

so

\[
\Pr^{\text{sym}}(\xi | i_T, j_T, b, s, P_C, P_K) = \frac{1}{2^r} = \Pr^{\text{sym}}(\xi | i_T, j_T, b, s).
\]

(37)

In addition, we can find that for any \( i_1, P_C, P_K \) and \( \xi \triangleq i_1 P_C^T \), according to Equations (34) and (35):

\[
\Pr^{\text{sym}}(i_1 | i_T, j_T, b, s, \xi, P_C, P_K) = \frac{\Pr^{\text{sym}}(i_1 | i_T, j_T, b, s, P_C, P_K)}{\Pr^{\text{sym}}(\xi | i_T, j_T, b, s, P_C, P_K)} = \frac{1/2^n}{1/2^r} = \frac{1}{2^{n-r}}.
\]

(38)

### 4.4 Main Result: Connecting Information to Disturbance

Our security proof of the generalized BB84 protocols is very similar to the security proof of BB84 itself, that was detailed in [1]. Most parts of the proof are not affected at all by the changes made to BB84 to get the generalized BB84 protocols (changes detailed in Section 3 of the current paper), because these parts assume fixed strings \( b, s \) and fixed matrices \( P_C, P_K \); therefore, in some parts of our current proofs, we refer to results from [1].

However, in this Subsection we improve a pivotal result of [1], resulting in a significant improvement of the allowed error rate derived from our security proof.

Given the error-correction parity-check matrix \( P_C \) (which corresponds to the error-correcting code \( C \)) and the privacy amplification matrix \( P_K \), we denote the rows of \( P_C \) as the vectors
\(v_1, \ldots, v_r\) in \(F_2^n\), and the rows of \(P_K\) as the vectors \(v_{r+1}, \ldots, v_{r+m}\); these vectors must all be linearly independent. We choose \(n - r - m\) additional vectors \(\{v_{r+m+1}, \ldots, v_n\}\) that extend \(\{v_1, \ldots, v_{r+m}\}\) to a basis of \(F_2^n\). We thus also denote, for any \(1 \leq r' \leq r + m\),

\[
V_{r'} \triangleq \text{Span}\{v_1, \ldots, v_{r'}\}, \\
V^c_{r'} \triangleq \text{Span}\{v_{r'+1}, \ldots, v_n\}, \\
V^\text{exc}_{r'} \triangleq \text{Span}\{v_1, \ldots, v_{r'-1}, v_{r'+1}, \ldots, v_{r+m}\},
\]

which means that \(V_{r'}\) and \(V^c_{r'}\) are two complementary vector spaces satisfying \(F_2^n = V^c_{r'} \oplus V_{r'}\) (namely, each vector \(\ell\) in \(F_2^n\) has a unique representation \(\ell = x \oplus y\) with \(x \in V^c_{r'}, y \in V_{r'}\)), and \(V^\text{exc}_{r'}\) is the \((r + m - 1)\)-dimensional vector space that spans all error correction and privacy amplification vectors except \(v_{r'}\).

For a 1-bit final key \(k \in \{0, 1\}\) (that is, for \(m = 1\)), and given a symmetrized attack of Eve, we define \(\rho_k^\text{sym}\) to be the state of Eve corresponding to the final key \(k\), given that she knows \(\xi\), averaging over the matrices \(P_C, P_K\) (which are ultimately known to Eve, and are thus included in the state as classical information that will eventually be available to Eve) and the bitstring \(i_1\) (which is not ultimately known to Eve, and is forgotten by Alice and Bob—who will only remember the final key \(k\)—and is thus not included as classical information). Thus,

\[
\rho_k^\text{sym} \triangleq \frac{1}{2r-t} \sum_{P_C, P_K, i, k} \Pr^\text{sym}(P_C, P_K) \cdot \left(\rho^E\right)^\text{sym}_E \otimes |P_C, P_K\rangle_C \langle P_C, P_K|_C,
\]

where \((\rho^E)^\text{sym}_E\), as defined in Equation (20), is Eve’s state after the (symmetrized) attack, given that Alice sent the INFO bitstring \(i_1\) (and given the bitstrings \(i_T, j_T, b, s\), that are ultimately known to Eve).

We now prove the following pivotal result (based on Lemma 4.5 and Proposition 4.6 of [1], which are proved in Subsection 4.4 and Appendix D.2 of [1]), which is the heart of our security proof—linking the information obtained by Eve to the disturbance she would cause in the hypothetical “inverted-INFO-basis” protocol:

**Theorem 5.** In the case of a 1-bit final key (i.e., \(m = 1\)), for any symmetrized attack and any integer \(0 \leq t \leq \frac{g}{2}\),

\[
\frac{1}{2} \text{tr} \left| \rho^\text{sym}_0 - \rho^\text{sym}_1 \right| \leq 2 \sqrt{\Pr^\text{sym}_{\text{inverted-INFO-basis}}[|C_1| \geq t \mid i_T, j_T, b, s] + n[H_2(t/n) - (r - t - 1)/n]},
\]

where \(C_1\) is the random variable whose value equals \(c_1 \triangleq i_1 \oplus j_1\); \(\Pr^\text{sym}_{\text{inverted-INFO-basis}}\) means that the probability is taken over the hypothetical “inverted-INFO-basis” protocol defined in Subsection 4.1 (to which Eve applies the same symmetrized attack that she applies to the real protocol); and \(H_2(x) \triangleq -x \log_2(x) - (1-x) \log_2(1-x)\).

**Proof.** This proof is mainly based on Appendix D.2 of [1], with some important improvements.

We first define the state \(\rho_k^\text{sym}\), by giving Eve the purification \(\varphi^\text{sym}_i_E\) of \((\rho^E)^\text{sym}\) (that was defined in Equation (21)) instead of \((\rho^E)^\text{sym}\) itself:

\[
\rho_k^\text{sym} \triangleq \frac{1}{2n-r-1} \sum_{P_C, P_K, i, k} \Pr^\text{sym}(P_C, P_K) \cdot \varphi^\text{sym}_i_E \otimes |P_C, P_K\rangle_C \langle P_C, P_K|_C.
\]
The state \( \tilde{\rho}_k^{\text{sym}} \) is a lift-up of \( \tilde{\rho}_k^{\text{sym}} \) — namely, \( \tilde{\rho}_k^{\text{sym}} \) is a partial trace of \( \tilde{\rho}_k^{\text{sym}} \). (Note that \( \tilde{\rho}_k^{\text{sym}} \) was defined in Equation (4.10) of [1], but was denoted there as \( \rho_k(v_{r+1}, \xi) \).) According to [16, Theorem 9.2 and page 407], and using the fact that \( \tilde{\rho}_k^{\text{sym}} \) is a partial trace of \( \tilde{\rho}_k^{\text{sym}} \), we find out that

\[
\frac{1}{2} \text{tr} | \tilde{\rho}_0^{\text{sym}} - \tilde{\rho}_1^{\text{sym}} | \leq \frac{1}{2} \text{tr} | \rho_0^{\text{sym}} - \rho_1^{\text{sym}} |.
\]

(45)

Now, for each \( \xi \in F_2^n \), we denote \( i_\xi \in F_2^n \) as an arbitrary, fixed word having the syndrome \( \xi \) (namely, satisfying \( i_\xi P_C^T = \xi \)). For any \( i \in F_2^n \) that has the same syndrome \( \xi \) (namely, that satisfies \( i P_C^T = \xi \)), it must hold that:

\[
(i - i_\xi) P_C^T = \xi - \xi = 0.
\]

(46)

Thus, because \( V_r \) is the row space of \( P_C \), we have \( (i - i_\xi) \cdot y = 0 \) for any \( y \in V_r \). Based on this observation, Equation (25), and the fact that each vector \( \ell \in F_2^n \) has a unique representation \( \ell = x \oplus y \) with \( x \in V_r, y \in V_r \), we find:

\[
| q_{i_\xi}^{\text{sym}} \rangle_E = \sum_{x \in V_r} (-1)^{\langle x \rangle} | \eta_x \rangle_E = \sum_{x \in V_r} \sum_{y \in V_r} (-1)^{\langle (x,y) \rangle} | \eta_{x,y} \rangle_E
\]

\[
= \sum_{x \in V_r} (-1)^{\langle x \rangle} \sum_{y \in V_r} (-1)^{\langle y \rangle} | \eta_{x,y} \rangle_E = \sum_{x \in V_r} (-1)^{\langle x \rangle} \sum_{y \in V_r} (-1)^{\langle y \rangle} (i_\xi \cdot y) | \eta_{x,\xi,y} \rangle_E
\]

\[
= \sum_{x \in V_r} (-1)^{\langle x \rangle} | \eta_x^{\text{sym}} \rangle_E,
\]

(47)

where we define, for any \( x \in V_r^n \):

\[
| \eta_x^{\text{sym}} \rangle_E \triangleq \sum_{y \in V_r} (-1)^{\langle y \rangle} | \eta_{x,\xi,y} \rangle_E,
\]

\[
d_x^{\text{sym}} \triangleq \sqrt{\langle \eta_x^{\text{sym}} | \eta_x^{\text{sym}} \rangle_E},
\]

\[
| \eta_x^{\text{sym}} \rangle_E \triangleq \frac{| \eta_x^{\text{sym}} \rangle_E}{d_x^{\text{sym}}},
\]

(48)

(49)

(50)

(These quantities depend on \( \xi \), but they do not depend on the entire \( i_\xi \).) According to Proposition 4.3 of [1], it holds for symmetrized attacks that \( \langle \eta_{\ell}^{\text{sym}} | \eta_{\ell'}^{\text{sym}} \rangle = 0 \) for any \( \ell \neq \ell' \); therefore,

\[
d_x^{\text{sym}} = \sqrt{\langle \eta_x^{\text{sym}} | \eta_x^{\text{sym}} \rangle_E} = \sqrt{\sum_{y \in V_r} (-1)^{\langle y \rangle} \langle \eta_y^{\text{sym}} | \eta_{x,y}^{\text{sym}} \rangle_E} = \sqrt{\sum_{y \in V_r} (d_y^{\text{sym}})^2},
\]

(51)

where the last equality is according to Equation (23).

We can now use Equation (47) to compute \( \tilde{\rho}_k^{\text{sym}} \): (we notice that the set \( \{ i_\xi | i_\xi P_C^T = \xi \} \) is actually the code coset \( i_\xi + C \triangleq \{ i_\xi \oplus a | a \in C \} \) corresponding to the syndrome \( \xi \)

\[
\tilde{\rho}_k^{\text{sym}} \triangleq \frac{1}{2^{n-r-1}} \sum_{P_C,P_K,i \in F_2^n} \Pr^{\text{sym}}(P_C,P_K) \cdot | q_{i_\xi}^{\text{sym}} \rangle_E \langle \eta_{i_\xi}^{\text{sym}} | E \otimes | P_C, P_K \rangle_C \langle P_C, P_K | C
\]

\[
= \frac{1}{2^{n-r-1}} \sum_{P_C,P_K,a \in C} \Pr^{\text{sym}}(P_C,P_K) \cdot | q_{i_\xi \oplus a}^{\text{sym}} \rangle_E \langle \eta_{i_\xi \oplus a}^{\text{sym}} | E \otimes | P_C, P_K \rangle_C \langle P_C, P_K | C
\]

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\[
\sum_{x,x' \in V_r^c} (-1)^{(i_x \oplus a) \cdot (x \oplus x')} |\eta_x^{sym}\rangle_E \langle \eta_{x'}^{sym}| \otimes |P_C, P_K|_C \langle P_C, P_K|_C.
\]

Therefore, the difference between \(\tilde{\rho}_0^{sym}\) and \(\tilde{\rho}_1^{sym}\) is:

\[
\tilde{\rho}_0^{sym} - \tilde{\rho}_1^{sym} = \sum_{k \in \{0,1\}} (-1)^k \cdot \tilde{\rho}_k^{sym}
\]

\[
= \frac{1}{2^{n-r-1}} \sum_{P_C, P_K, a \in C} \Pr^{sym}(P_C, P_K) \cdot (-1)^{(i_x \oplus a) \cdot v_{r+1}}
\]

\[
\cdot \sum_{x,x' \in V_r^c} (-1)^{(i_x \oplus a) \cdot (x \oplus x')} |\eta_x^{sym}\rangle_E \langle \eta_{x'}^{sym}| \otimes |P_C, P_K|_C \langle P_C, P_K|_C
\]

\[
= \frac{1}{2^{n-r-1}} \sum_{P_C, P_K, x,x' \in V_r^c} \Pr^{sym}(P_C, P_K) \cdot (-1)^{(i_x \oplus a) \cdot (x \oplus x')} \cdot \left( \sum_{a \in C} (-1)^{a \cdot (x \oplus x' \oplus v_{r+1})} \right) |\eta_x^{sym}\rangle_E \langle \eta_{x'}^{sym}| \otimes |P_C, P_K|_C \langle P_C, P_K|_C
\]

\[
= \frac{1}{2^{n-r-1}} \sum_{P_C, P_K, x,x' \in V_r^c} \Pr^{sym}(P_C, P_K) \cdot (-1)^{(i_x \oplus a) \cdot (x \oplus x' \oplus v_{r+1})} \cdot \left( \sum_{a \in C} (-1)^{a \cdot (x \oplus x' \oplus v_{r+1})} \right) \cdot d_x^{sym} d_{x'}^{sym} |\eta_x^{sym}\rangle_E \langle \eta_{x'}^{sym}| \otimes |P_C, P_K|_C \langle P_C, P_K|_C.
\]

According to Lemma D.1 of [1], assuming that \(C\) is a linear code over \(F^m_2\), for any \(\ell_0 \in F^m_2 \setminus C^\perp\) (where \(C^\perp\) is the dual code of \(C\)) it holds that \(\sum_{a \in C} (-1)^a \ell_0 = 0\).

We notice that in our case \(C^\perp = V_r \triangleq \text{Span}\{v_1, \ldots, v_r\}\), because the dual code \(C^\perp\) is the row space of \(P_C\) (see Subsection 2.3.1). The sum \(\sum_{a \in C} (-1)^{a \cdot (x \oplus x' \oplus v_{r+1})}\) above is thus non-zero if and only if \(x \oplus x' \oplus v_{r+1} \in V_r\), and because it always holds that \(x \oplus x' \oplus v_{r+1} \in V_r^c\) (since \(x, x', v_{r+1} \in V_r^c\)), the sum is non-zero if and only if \(x \oplus x' \oplus v_{r+1} = 0\) (because \(F^m_2 = V_r^c \oplus V_r\), so according to the properties of direct sum, \(V_r^c \cap V_r = \{0\}\)). Therefore, the sum is non-zero if and only if \(x' = x \oplus v_{r+1}\) (in which case the sum is \(\sum_{a \in C} (-1)^{a \cdot (x \oplus x' \oplus v_{r+1})} = \sum_{a \in C} (-1)^a 0 = \sum_{a \in C} (-1)^0 = |C| = 2^k = 2^{n-r}\), so:

\[
\tilde{\rho}_0^{sym} - \tilde{\rho}_1^{sym} = 2 \sum_{P_C, P_K, x \in V_r^c} \Pr^{sym}(P_C, P_K) \cdot d_x^{sym} d_{x \oplus v_{r+1}}^{sym} |\eta_x^{sym}\rangle_E \langle \eta_{x \oplus v_{r+1}}^{sym}| \otimes |P_C, P_K|_C \langle P_C, P_K|_C.
\]

Now, according to the definitions of \(V_r^c \triangleq \text{Span}\{v_{r+1}, \ldots, v_n\}\) and \(V_{r+1}^c \triangleq \text{Span}\{v_{r+2}, \ldots, v_n\}\) and the linear independence of \(v_1, \ldots, v_n\), we get the fact

\[
V_r^c = V_{r+1}^c \cup \{x \oplus v_{r+1} \mid x \in V_r^c\}\]
\( (a \text{ disjoint union}), \) from which we obtain the following result:

\[
\tilde{\rho}_0^{sym} - \tilde{\rho}_1^{sym} = 2 \sum_{P_C, P_K, x \in V_{r+1}} \Pr^{sym}(P_C, P_K) \cdot d_x^{sym} d_{x \oplus v_{r+1}}^{sym} \\
\cdot \left[ \tilde{\eta}_x^{sym} |E(\tilde{\eta}_x^{sym} | \hat{\eta}_x^{sym}) |E + |\tilde{\eta}_x^{sym} |E(\hat{\eta}_x^{sym} | \hat{\eta}_x^{sym}) |E \right] \otimes |P_C, P_K|_C \langle P_C, P_K |_C.
\]

\( (56) \)

Using Equation (45), we deduce:

\[
\frac{1}{2} \text{tr} |\tilde{\rho}_0^{sym} - \tilde{\rho}_1^{sym}| \leq \frac{1}{2} \text{tr} |\tilde{\rho}_0^{sym} - \tilde{\rho}_1^{sym}|
\]

\[
= \text{tr} \left[ \sum_{P_C, P_K, x \in V_{r+1}} \Pr^{sym}(P_C, P_K) \cdot d_x^{sym} d_{x \oplus v_{r+1}}^{sym} \\
\cdot \left[ \tilde{\eta}_x^{sym} |E(\tilde{\eta}_x^{sym} | \hat{\eta}_x^{sym}) |E + |\tilde{\eta}_x^{sym} |E(\hat{\eta}_x^{sym} | \hat{\eta}_x^{sym}) |E \right] \otimes |P_C, P_K|_C \langle P_C, P_K |_C \\
\leq \sum_{P_C, P_K, x \in V_{r+1}} \Pr^{sym}(P_C, P_K) \cdot d_x^{sym} d_{x \oplus v_{r+1}}^{sym} \\
\cdot \text{tr} \left[ \tilde{\eta}_x^{sym} |E(\tilde{\eta}_x^{sym} | \hat{\eta}_x^{sym}) |E + |\tilde{\eta}_x^{sym} |E(\hat{\eta}_x^{sym} | \hat{\eta}_x^{sym}) |E \right] \otimes |P_C, P_K|_C \langle P_C, P_K |_C \\
\leq \sum_{P_C, P_K, x \in V_{r+1}} \Pr^{sym}(P_C, P_K) \cdot 2d_x^{sym} d_{x \oplus v_{r+1}}^{sym}.
\]

\( (57) \)

We now use the general inequality \( 2xy \leq \alpha x^2 + \frac{y^2}{\alpha} \) for any \( \alpha > 0 \) and \( x, y \in \mathbb{R} \) (derived in Appendix D.2 of [1]) to get:

\[
\frac{1}{2} \text{tr} |\tilde{\rho}_0^{sym} - \tilde{\rho}_1^{sym}| \leq \sum_{P_C, P_K, x \in V_{r+1}} \Pr^{sym}(P_C, P_K) \cdot 2d_x^{sym} d_{x \oplus v_{r+1}}^{sym}
\]

\[
= \sum_{P_C, P_K, x \in V_{r+1}} \Pr^{sym}(P_C, P_K) \cdot 2d_x^{sym} d_{x \oplus v_{r+1}}^{sym} \\
+ \sum_{P_C, P_K, x \in V_{r+1}} \Pr^{sym}(P_C, P_K) \cdot 2d_x^{sym} d_{x \oplus v_{r+1}}^{sym}
\]

\[
\leq \sum_{P_C, P_K, x \in V_{r+1}} \Pr^{sym}(P_C, P_K) \cdot \left[ \alpha (d_x^{sym})^2 + \frac{(d_{x \oplus v_{r+1}}^{sym})^2}{\alpha} \right]
\]

\[
+ \sum_{P_C, P_K, x \in V_{r+1}} \Pr^{sym}(P_C, P_K) \cdot \left[ \alpha (d_x^{sym})^2 + \frac{(d_{x \oplus v_{r+1}}^{sym})^2}{\alpha} \right]
\]

\[
= \alpha \sum_{P_C, P_K, x \in V_{r+1}} \Pr^{sym}(P_C, P_K) \cdot (d_x^{sym})^2 \\
+ \alpha \sum_{P_C, P_K, x \in V_{r+1}} \Pr^{sym}(P_C, P_K) \cdot (d_x^{sym})^2 \\
+ \frac{1}{\alpha} \sum_{P_C, P_K, x \in V_{r+1}} \Pr^{sym}(P_C, P_K) \cdot (d_x^{sym})^2
\]

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using the disjoint union $V_{r+1}^c = V_r^c \cup \{x \oplus v_{r+1} \mid x \in V_r^c \}$ from Equation (55).

In addition, the last two sums in Equation (58) are bounded as follows:

\[
\begin{align*}
\frac{1}{\alpha} \sum_{P_C, P_K, x \in V_r^c} & \Pr_{x \oplus v_{r+1} \mid d_H(x,v_r)<t} \Pr_{\sym}(P_C, P_K) \cdot (d_{x \oplus v_{r+1}}^\sym)^2 \\
+ \alpha \sum_{P_C, P_K, x \in V_r^c} & \Pr_{x \oplus v_{r+1} \mid d_H(x,v_r)<t} \Pr_{\sym}(P_C, P_K) \cdot (d_{x \oplus v_{r+1}}^\sym)^2 \\
\leq \frac{1}{\alpha} \sum_{P_C, P_K, x \in V_r^c} & \Pr_{x \oplus v_{r+1} \mid d_H(x,v_r)<t} \Pr_{\sym}(P_C, P_K) \cdot (d_{x \oplus v_{r+1}}^\sym)^2 \\
+ \frac{1}{\alpha} \sum_{P_C, P_K, x \in V_r^c} & \Pr_{x \oplus v_{r+1} \mid d_H(x,v_r)<t} \Pr_{\sym}(P_C, P_K) \cdot (d_{x \oplus v_{r+1}}^\sym)^2 \\
+ \frac{1}{\alpha} \sum_{P_C, P_K, x \in V_r^c} & \Pr_{x \oplus v_{r+1} \mid d_H(x,v_r)<t} \Pr_{x \oplus v_{r+1} \mid d_H(x,v_r)<t} \Pr_{\sym}(P_C, P_K) \cdot (d_{x \oplus v_{r+1}}^\sym)^2. 
\end{align*}
\]  

Using, again, the disjoint union $V_r^c = V_{r+1}^c \cup \{x \oplus v_{r+1} \mid x \in V_r^c \}$ from Equation (55) (for the first two sums in Equation (60)), and using the fact that $\{x \oplus v_{r+1} \mid x \in V_r^c \} \subseteq V_r^c$ (for the third sum in Equation (60)), we get:

\[
\begin{align*}
\leq \frac{1}{\alpha} \sum_{P_C, P_K, x \in V_r^c} & \Pr_{x \oplus v_{r+1} \mid d_H(x,v_r)<t} \Pr_{\sym}(P_C, P_K) \cdot (d_{x \oplus v_{r+1}}^\sym)^2 \\
+ \frac{1}{\alpha} \sum_{P_C, P_K, x \in V_r^c} & \Pr_{x \oplus v_{r+1} \mid d_H(x,v_r)<t} \Pr_{\sym}(P_C, P_K) \cdot (d_{x \oplus v_{r+1}}^\sym)^2. 
\end{align*}
\]  

Substituting Equations (59) and (61) into Equation (58), and using the identity $d_x^\sym = \sqrt{\sum_{y \in V_r^c} (d_{x \oplus y}^\sym)^2}$ for any $x \in V_r^c$ from Equation (51), we find:

\[
\frac{1}{2} \tr (\hat{\rho}_0 - \hat{\rho}_1^\sym) \leq \alpha \sum_{P_C, P_K, x \in V_r^c} \Pr_{\sym}(P_C, P_K) \cdot (d_x^\sym)^2
\]
computing the three sums in Equation (62):

We can now use the fact that $F_{\ell\ell\ell}$ until Eve’s attack is over.

For the second sum in Equation (62), we find:

1. For the first sum in Equation (62), we find:

According to Proposition 4.3 of [1], it holds for symmetrized attacks that $|\eta_{\ell}^{\text{sym}}\rangle = 0$ for any $\ell \neq \ell'$; therefore, according to Equations (23) and (25), the normalized state $|\varphi_{i}^{\text{sym}}\rangle_{E}$ satisfies:

$$1 = \langle \varphi_{i}^{\text{sym}} | \varphi_{i}^{\text{sym}} \rangle_{E} = \sum_{\ell,\ell' \in \mathbb{F}_{2}} (-1)^{i}\langle \ell \oplus \ell' | \eta_{\ell}^{\text{sym}} | \eta_{\ell'}^{\text{sym}} \rangle_{E} = \sum_{\ell \in \mathbb{F}_{2}} \langle \eta_{\ell}^{\text{sym}} | \eta_{\ell}^{\text{sym}} \rangle_{E} = \sum_{\ell \in \mathbb{F}_{2}} (d_{\ell}^{\text{sym}})^{2}. \quad (64)$$

Substituting Equation (64) into Equation (63), we find the first sum in Equation (62) to be:

$$\alpha \sum_{P_{c},P_{k},x \in V_{r}} \Pr^{\text{sym}}(P_{c},P_{k}) \cdot (d_{x}^{\text{sym}})^{2} = \alpha \sum_{\ell \in \mathbb{F}_{2}} (d_{\ell}^{\text{sym}})^{2}. \quad (65)$$

2. For the second sum in Equation (62), we find:

This is correct because $d_{\ell}^{\text{sym}}$ depends on $|\eta_{\ell}^{\text{sym}}\rangle_{E}$, on $\{ |\varphi_{k}^{\text{sym}}\rangle_{E} \}_{k \in \mathbb{F}_{2}}$, and on $\{ |\ell_{k,h}^{\text{sym}}\rangle_{k,h \in \mathbb{F}_{2}} \}$, which are all independent of $P_{c}, P_{k}$, because Alice chooses $P_{c}, P_{k}$ uniformly at random, and she does not use them and keeps them secret until Eve’s attack is over.
3. For the third sum in Equation (62), we find: (based on the fact that for any \( x \in V_r, y \in V_r \) and the corresponding \( \ell \triangleq x \oplus y \in F_2^n \), it holds that \( d_H(x, V_r) = d_H(\ell, V_r) \) and \( d_H(x \oplus v_{r+1}, V_r) = d_H(\ell \oplus v_{r+1}, V_r) \))

\[
\frac{1}{\alpha} \sum_{P_{C,P_k}, x \in V_r, y \in V_r \mid (d_H(x \oplus v_{r+1}, V_r) < t) \wedge (d_H(x, V_r) < t)} \Pr_{\text{sym}}(P_{C,P_k}) \cdot (d_{x \oplus y}^\text{sym})^2 \\
= \frac{1}{\alpha} \sum_{P_{C,P_k}, \ell \in F_2^n \mid |\ell| \geq t} \Pr_{\text{sym}}(P_{C,P_k}) \cdot (d_{\ell}^\text{sym})^2 \\
= \frac{1}{\alpha} \sum_{|\ell| \geq t} \sum_{P_{C,P_k}} \Pr_{\text{sym}}(P_{C,P_k}) \cdot (d_{\ell}^\text{sym})^2 \\
= \frac{1}{\alpha} \sum_{|\ell| \geq t} (d_{\ell}^\text{sym})^2 = \frac{1}{\alpha} \sum_{|\ell| \geq t} (d_{\ell}^\text{sym})^2. \quad (66)
\]

where the probability is actually taken over the randomly-chosen matrices \( P_{C} \in F_2^{r \times n}, P_{K} \triangleq (v_{r+1}) \in F_2^{1 \times n} \), which constitute together an \((r + 1) \times n\) generator matrix for an \((r + 1)\)-dimensional random error-correcting code over \( F_2^n \). This random code, in fact, is \( C_{\text{ext}}^\perp \triangleq \text{Span}\{C^\perp, v_{r+1}\} \), where \( C^\perp = V_r \triangleq \text{Span}\{v_1, \ldots, v_r\} \) is the \( r \)-dimensional dual code (see Subsubsection 2.3.1) of the error-correcting code \( C \) belonging to the parity-check matrix \( P_{C} \), and \( v_{r+1} \) is the only row of the matrix \( P_{K} \). Therefore:

\[
C_{\text{ext}}^\perp \triangleq \text{Span}\{C^\perp, v_{r+1}\} = \text{Span}\{V_r, v_{r+1}\} = \text{Span}\{v_1, \ldots, v_r, v_{r+1}\}. \quad (68)
\]

Now, applying Proposition 3 to this \((r + 1)\)-dimensional random code \( C_{\text{ext}}^\perp = V_{r+1} \) (the parameters for the Proposition are our \( n, k = r + 1 \), our \( 0 \leq t \leq \frac{n}{2} \), and our \( \ell \in F_2^n \)), we find that the probability that there exists a word \( z \triangleq \ell \oplus v_{r+1} \oplus y \in \ell + C_{\text{ext}}^\perp \) which satisfies \( z \neq \ell \) (because \( \{V_r, v_{r+1}\} \) are linearly independent, so \( v_{r+1} \in V_r^c \setminus \{0\} \)) and \( |z| \triangleq |\ell \oplus v_{r+1} \oplus y| \leq t \) is bounded by:

\[
\Pr_{\text{sym}}[\exists y \in V_r : |\ell \oplus v_{r+1} \oplus y| < t \mid \ell] \leq \Pr_{\text{sym}}[\exists z \in \ell + C_{\text{ext}}^\perp : z \neq \ell, |z| \leq t \mid \ell]
\]
Substituting Equations (69) and (64) into Equation (67), we find the third sum in Equation (62) to be:

\[
\frac{1}{\alpha} \sum_{\ell \in F_2^n} \Pr_{\rho_C,P_K} \left( |\rho_{\ell}^{\alpha} - \hat{\rho}_{\ell^{\alpha}}| \right) \leq \frac{1}{\alpha} \sum_{|\ell| \geq t} (d_{\ell^{\alpha}}^{\alpha})^2 + \frac{1}{\alpha} \cdot 2^n[H_2(t/n)-(n-r-1)/n].
\]

Choosing \( \alpha = \sqrt{\sum_{|\ell| \geq t}(d_{\ell^{\alpha}}^{\alpha})^2 + 2^n[H_2(t/n)-(n-r-1)/n]} \), we obtain:

\[
\frac{1}{2} \text{tr} \left| \rho_0 - \rho_1 \right| \leq 2 \sqrt{\sum_{|\ell| \geq t}(d_{\ell^{\alpha}}^{\alpha})^2 + 2^n[H_2(t/n)-(n-r-1)/n]}.
\]

According to Lemma 4.4 of [1], for any \( c_i \in F_2^n \) it holds that:

\[
(d_{c_i^{\alpha}}^{\alpha})^2 = \Pr_{\text{inverted-INFO-basis}}^{\text{sym}}[C_i = c_i | i_T,j_T,b,s].
\]

Therefore, we finally get:

\[
\frac{1}{2} \text{tr} \left| \rho_0 - \rho_1 \right| \leq 2 \sqrt{\Pr_{\text{inverted-INFO-basis}}^{\text{sym}}[|C_i| \geq t | i_T,j_T,b,s] + 2^n[H_2(t/n)-(n-r-1)/n]}.
\]

\( \square \)

### 4.5 Bounding the Differences Between Eve’s States

So far, we have discussed a 1-bit key. We will now discuss a general \( m \)-bit key \( k \). We define \( \hat{\rho}_k^{\text{sym}} \) to be the state of Eve corresponding to the final key \( k \), given that she knows \( \xi \):

\[
\hat{\rho}_k^{\text{sym}} \triangleq \frac{1}{2^{n-r-m}} \sum_{P_C,P_K : h^{P_C}_{\xi = \xi} \otimes |P_C,P_K}_{C;C}. \Pr_{P_C,P_K}^{\text{sym}} \left( |\rho_{h^{P_C}_k^{\xi}}^{\alpha} - \hat{\rho}_{h^{P_C}_k^{\xi}}^{\alpha} |^{\alpha} \right).
\]

We note (for use in Subsection 4.6) that if we substitute \( (\rho_{h^{P_C}_k^{\xi}}^{\alpha})^{\text{sym}} \) from Equation (20), we get

\[
\hat{\rho}_k^{\text{sym}} = \frac{1}{2^{n-r-m}} \sum_{P_C,P_K : h^{P_C}_{\xi = \xi} \otimes |P_C,P_K}_{C;C}. \Pr_{P_C,P_K}^{\text{sym}} \left( |\rho_{h^{P_C}_k^{\xi}}^{\alpha} - \hat{\rho}_{h^{P_C}_k^{\xi}}^{\alpha} |^{\alpha} \right) \cdot \left( \rho_{h^{P_C}_k^{\xi}}^{b,s} \right)^{\text{sym}} \otimes |P_C,P_K}_{C;C}.
\]
Proposition 6. For any two keys \( k, k' \) of \( m \) bits, any symmetrized attack, and any integer \( 0 \leq t \leq n \),

\[
\frac{1}{2} \text{tr} |\rho^\text{sym}_k - \rho^\text{sym}_{k'}| \leq 2m \sqrt{\Pr^\text{sym}_\text{inverted-INFO-basis}[||C_1|| \geq t | i_T, j_T, b, s] + 2^n[H_2(t/n) - (n - r - m)/n]},
\]

where \( C_1 \) is the random variable whose value equals \( c_1 \triangleq i_1 \oplus j_1 \), and \( H_2(x) \triangleq -x \log_2(x) - (1 - x) \log_2(1 - x) \).

Proof. We define the key \( k_j \), for \( 0 \leq j \leq m \), to consist of the first \( j \) bits of \( k' \) and the last \( m-j \) bits of \( k \). This means that \( k_0 = k, k_m = k' \), and \( k_{j-1} \) differs from \( k_j \) at most on a single bit (the \( j \)-th bit).

First, we find a bound on \( \frac{1}{2} \text{tr} |\rho^\text{sym}_{k_{j-1}} - \rho^\text{sym}_{k_j}| \): since \( k_{j-1} \) differs from \( k_j \) at most on a single bit (the \( j \)-th bit, given by the formula \( i_1 \cdot v_{r+j} \)), we can use the same proof that gave us Equation (74), attaching the other (identical) \( m-1 \) key bits to \( \xi \) of the original proof (for the same parameter \( t \)); and we find out that

\[
\frac{1}{2} \text{tr} |\rho^\text{sym}_{k_{j-1}} - \rho^\text{sym}_{k_j}| \leq 2 \sqrt{\Pr^\text{sym}_\text{inverted-INFO-basis}[||C_1|| \geq t | i_T, j_T, b, s] + 2^n[H_2(t/n) - (n - r - m)/n]}.
\]

Now we use the triangle inequality for norms to find

\[
\frac{1}{2} \text{tr} |\rho^\text{sym}_k - \rho^\text{sym}_{k'}| = \frac{1}{2} \text{tr} |\rho^\text{sym}_{k_0} - \rho^\text{sym}_{k_m}| \leq \sum_{j=1}^m \frac{1}{2} \text{tr} |\rho^\text{sym}_{k_{j-1}} - \rho^\text{sym}_{k_j}|
\]

\[
\leq 2m \sqrt{\Pr^\text{sym}_\text{inverted-INFO-basis}[||C_1|| \geq t | i_T, j_T, b, s] + 2^n[H_2(t/n) - (n - r - m)/n]}.
\]

(79)

We would now like to bound the expected value (namely, the average value) of the trace distance between two states of Eve corresponding to two final keys. However, we should take into account that if the test fails, no final key is generated, in which case we define the distance to be 0. We thus define the random variable \( \Delta^\text{sym}_{\text{Eve}}(k, k') \) for any two final keys \( k, k' \):

\[
\Delta^\text{sym}_{\text{Eve}}(k, k' | i_T, j_T, b, s, \xi) \triangleq \begin{cases} \frac{1}{2} \text{tr} |\rho^\text{sym}_k - \rho^\text{sym}_{k'}| & \text{if } T(i_T \oplus j_T, b, s) = 1 \\ 0 & \text{otherwise} \end{cases}.
\]

(80)

We need to bound the expected value \( \langle \Delta^\text{sym}_{\text{Eve}}(k, k') \rangle \), that is given by:

\[
\langle \Delta^\text{sym}_{\text{Eve}}(k, k') \rangle = \sum_{i_T, j_T \in \mathbb{F}_2^{n_r}, b \in B, s \in S_b, \xi \in \mathbb{F}_2^{n'}} \Delta^\text{sym}_{\text{Eve}}(k, k' | i_T, j_T, b, s, \xi) \cdot \Pr^\text{sym}(i_T, j_T, b, s, \xi).
\]

(81)

(In Subsection 4.6 we prove that this expected value is indeed the quantity we need to bound for proving fully composable security.)
Theorem 7. For any two final keys $k, k'$ and any integer $0 \leq t \leq \frac{n}{2}$,

$$\langle \Delta_{\text{Eve}}(k, k') \rangle \leq 2m \sqrt{\Pr_{\text{inverted-INFO-basis}} \left[ \left( \left\lceil \frac{|C_1|}{n} \right\rceil \geq \frac{t}{n} \right) \land (T = 1) \right]} + 2^n [H_2(t/n) - (n-r-m)/n],$$

(82)

where $\left\lceil \frac{|C_1|}{n} \right\rceil$ is a random variable whose value is the error rate on the INFO bits; $T$ is a random variable whose value is 1 if the test passes and 0 otherwise; and $H_2(x) \triangleq -x \log_2(x) - (1 - x) \log_2(1 - x)$. We note that the protocol considered for the probability in the right-hand-side is the hypothetical “inverted-INFO-basis” protocol defined in Subsection 4.1, in which Alice and Bob use the basis string $b^0 \triangleq b \oplus s$ instead of $b$; we also note that the probability in the right-hand-side is the probability for the original (non-symmetrized) attack.

Proof. We use convexity of $x^2$—namely, the fact that for all $\{p_i\}_i$ satisfying $p_i \geq 0$ and $\sum_i p_i = 1$, it holds that $(\sum_i p_i x_i)^2 \leq \sum_i p_i x_i^2$. We also use the fact that

$$\Pr^{\text{sym}}_{\text{inverted-INFO-basis}}(i_T, j_T, b, s) = \Pr^{\text{sym}}_{\text{inverted-INFO-basis}}(i_T, b, s) \cdot \Pr^{\text{sym}}_{\text{inverted-INFO-basis}}(j_T | i_T, b, s)$$

$$= \Pr^{\text{sym}}(i_T, j_T, b, s),$$

(83)

which is correct because $i_T, b, s$ are all chosen in the same way in both the hypothetical “inverted-INFO-basis” protocol and the real protocol (even though different basis strings are used in these protocols), and because according to the third property of the symmetrized attack (Equation (32)), $\Pr^{\text{sym}}_{\text{inverted-INFO-basis}}(j_T | i_T, b, s) = \Pr^{\text{sym}}(j_T | i_T, b, s)$.

In addition, we use the result

$$\Pr^{\text{sym}}_{\text{inverted-INFO-basis}}([|C_1| \geq r] \land (T = 1) | b, s) = \Pr^{\text{sym}}_{\text{inverted-INFO-basis}}([|C_1| \geq r] \land (T = 1) | b, s),$$

(84)

which is correct because according to the second property of the symmetrized attack (Equation (30)), $\Pr^{\text{sym}}_{\text{inverted-INFO-basis}}(c_T, c_r | b, s) = \Pr^{\text{sym}}_{\text{inverted-INFO-basis}}(c_T, c_r | b, s)$, and because the random variable $T$ depends only on the random variable $C_T$ and the parameters $b_T, s$.

We also use the result

$$\Pr^{\text{sym}}_{\text{inverted-INFO-basis}}(b, s) = \Pr^{\text{sym}}_{\text{inverted-INFO-basis}}(b, s),$$

(85)

which is correct because Alice’s random choice of $b, s$ is independent of Eve’s attack.

We find out that:

$$\langle \Delta_{\text{Eve}}(k, k') \rangle^2$$

$$= \left[ \sum_{i_T, j_T, b, s, \xi} \Delta^{\text{sym}}_{\text{Eve}}(k, k' | i_T, j_T, b, s, \xi) \cdot \Pr^{\text{sym}}(i_T, j_T, b, s, \xi) \right]^2$$

(by (81))

$$\leq \sum_{i_T, j_T, b, s, \xi} \left[ \Delta^{\text{sym}}_{\text{Eve}}(k, k' | i_T, j_T, b, s, \xi) \right]^2 \cdot \Pr^{\text{sym}}(i_T, j_T, b, s, \xi)$$

(by convexity of $x^2$)

$$= \sum_{i_T, j_T, b, s, \xi \mid T = 1} \left( \frac{1}{2} \cdot \text{tr} \left| \hat{\rho}^{\text{sym}}_k - \hat{\rho}^{\text{sym}}_{k'} \right| \right)^2 \cdot \Pr^{\text{sym}}(i_T, j_T, b, s, \xi)$$

(by (80))
\[ \leq 4m^2 \cdot \sum_{i_T,j_T,b,s,\xi:T=1} \left( \Pr_{\text{sym}}^\text{inverted-INFO-basis} \left[ |C_1| \geq t \mid i_T, j_T, b, s \right] + 2^{n[H_2(t/n)-(n-r-m)/n]} \right) \]

\[ \cdot \Pr_{\text{sym}}(i_T, j_T, b, s, \xi) \]  
\[ = 4m^2 \cdot \sum_{i_T,j_T,b,s:T=1} \left( \Pr_{\text{sym}}^\text{inverted-INFO-basis} \left[ |C_1| \geq t \mid i_T, j_T, b, s \right] + 2^{n[H_2(t/n)-(n-r-m)/n]} \right) \]

\[ \cdot \Pr_{\text{sym}}(i_T, j_T, b, s) \]  
\[ = 4m^2 \cdot \sum_{i_T,j_T,b,s:T=1} \left( \Pr_{\text{sym}}^\text{inverted-INFO-basis} \left[ |C_1| \geq t \mid i_T, j_T, b, s \right] + 2^{n[H_2(t/n)-(n-r-m)/n]} \cdot \Pr_{\text{sym}}(i_T, j_T, b, s) \right) \]

\[ = 4m^2 \cdot \left( \sum_{i_T,j_T,b,s} \Pr_{\text{sym}}^\text{inverted-INFO-basis} \left[ |C_1| \geq t \mid (T = 1) \mid i_T, j_T, b, s \right] \right) \]

\[ \cdot \Pr_{\text{sym}}(i_T, j_T, b, s) + 2^{n[H_2(t/n)-(n-r-m)/n]} \cdot \Pr_{\text{sym}}(T = 1) \]

\[ \leq 4m^2 \cdot \left( \sum_{i_T,j_T,b,s} \Pr_{\text{sym}}^\text{inverted-INFO-basis} \left[ |C_1| \geq t \mid (T = 1) \mid i_T, j_T, b, s \right] \right) \]

\[ \cdot \Pr_{\text{sym}}^\text{inverted-INFO-basis}(i_T, j_T, b, s) + 2^{n[H_2(t/n)-(n-r-m)/n]} \]  
\[ \text{(by (83))} \]

\[ = 4m^2 \cdot \left( \sum_{b,s} \Pr_{\text{sym}}^\text{inverted-INFO-basis} \left[ |C_1| \geq t \mid (T = 1) \mid b, s \right] \right) \]

\[ \cdot \Pr_{\text{sym}}^\text{inverted-INFO-basis}(b, s) + 2^{n[H_2(t/n)-(n-r-m)/n]} \]  
\[ \text{(by (84)–(85))} \]

\[ = 4m^2 \cdot \left( \sum_{b,s} \Pr_{\text{inverted-INFO-basis}} \left[ |C_1| \geq t \mid (T = 1) \mid b, s \right] \right) \]

\[ \cdot \Pr_{\text{inverted-INFO-basis}}(b, s) + 2^{n[H_2(t/n)-(n-r-m)/n]} \]  
\[ \text{(by (86))} \]

4.6 Bound for Fully Composable Security

We now prove a crucial part of the claim that generalized BB84 protocols satisfy the definition of composable security for a QKD protocol: we derive an upper bound for the expression

\[ \frac{1}{2} \text{tr} |\rho_{\text{ABE}} - \rho_U \otimes \rho_E|, \]

where \( \rho_{\text{ABE}} \) is the actual joint state of Alice, Bob, and Eve at the end of the protocol; \( \rho_U \) is an ideal (random, secret, and shared) key distributed to Alice and Bob; and \( \rho_E \) is the partial trace of \( \rho_{\text{ABE}} \) over the system AB. In other words, we upper-bound the trace distance between the system after the real QKD protocol and the system after an ideal key.
distribution protocol (which first performs the real QKD protocol and then magically distributes to Alice and Bob a random, secret, and shared key).

The states $\rho_{\text{ABE}}$ and $\rho_{\text{U}}$ are

$$
\rho_{\text{ABE}} = \sum_{i,j,b,s,i_T,j_T,b_C,s_C} \Pr(i,j,b,s,P_C,P_K) \cdot |k\rangle_A \langle k|_A \otimes |k^B\rangle_B \langle k^B|_B
$$

$$
\otimes \left( \rho_{i,h,j,T}^{b,s} \right)_E \otimes |i_T, j_T, b, s, \xi, P_C, P_K\rangle_C \langle i_T, j_T, b, s, \xi, P_C, P_K|_C,
$$

(87)

$$
\rho_{\text{U}} = \frac{1}{2^m} \sum_k |k\rangle_A \langle k| \otimes |k\rangle_B \langle k|_B,
$$

(88)

where $\left( \rho_{i,h,j,T}^{b,s} \right)_E$ was defined in Equation (19) to be Eve’s normalized quantum state if Alice chooses $i_h,i_T,b,s$ and Bob measures $j_T,j_T$. All the other states in Equations (87)–(88) actually represent classical information: subsystems A and B represent the final keys held by Alice ($k \equiv i_T P_C^T$) and Bob (his key $k^B$ is computed from $j_T$, $\xi \equiv i_T P_C^T$, and $P_C,P_K$), respectively, and subsystem $C$ represents the information published in the unjammable classical channel during the protocol (this information is known to Alice, Bob, and Eve)—namely, $i_T,j_T$ (all the TEST bits), $b$ (the basis string), $s$ (the string representing the partition into INFO and TEST bits), $\xi \equiv i_T P_C^T$ (the syndrome), and $P_C,P_K$ (the error correction and privacy amplification matrices).

We note that in the definition of $\rho_{\text{ABE}}$, we sum only over the events in which the test is passed (namely, in which the protocol is not aborted by Alice and Bob): in such cases, Alice and Bob generate an $m$-bit key. The cases in which the protocol aborts do not exist in the sum—namely, they are represented by the zero operator, which matches standard definitions of composable security (see [11, Subsection 6.1.2]). Thus, $\rho_{\text{ABE}}$ is a non-normalized state, and $\text{tr}(\rho_{\text{ABE}})$ is the probability that the test is passed.

To help us bound the trace distance, we define the following intermediate state:

$$
\sigma_{\text{ABE}} \equiv \sum_{i,j,b,s,i_T,j_T,b_C,s_C} \Pr(i,j,b,s,P_C,P_K) \cdot |k\rangle_A \langle k|_A \otimes |k\rangle_B \langle k|_B
$$

$$
\otimes \left( \rho_{i,h,j,T}^{b,s} \right)_E \otimes |i_T, j_T, b, s, \xi, P_C, P_K\rangle_C \langle i_T, j_T, b, s, \xi, P_C, P_K|_C.
$$

(89)

This state is identical to $\rho_{\text{ABE}}$, except that Bob holds Alice’s final key ($k$) instead of his own calculated final key ($k^B$). In particular, the similarity between $\rho_{\text{ABE}}$ and $\sigma_{\text{ABE}}$ means that, by definition, $\rho_E \equiv \text{tr}_{\text{AB}}(\rho_{\text{ABE}})$ and $\sigma_E \equiv \text{tr}_{\text{AB}}(\sigma_{\text{ABE}})$ are identical—that is, $\rho_E = \sigma_E$.

A similar intermediate state $\sigma_{\text{ABE}}^{\text{sym}}$ is defined in case Eve uses the symmetrized attack:

$$
\sigma_{\text{ABE}}^{\text{sym}} \equiv \sum_{i,j,b,s,i_T,j_T,b_C,s_C} \Pr_{\text{sym}}(i,j,b,s,P_C,P_K) \cdot |k\rangle_A \langle k|_A \otimes |k\rangle_B \langle k|_B
$$

$$
\otimes \left( \rho_{i,h,j,T}^{b,s} \right)_E^{\text{sym}} \otimes |i_T, j_T, b, s, \xi, P_C, P_K\rangle_C \langle i_T, j_T, b, s, \xi, P_C, P_K|_C.
$$

(90)

**Proposition 8.** For any symmetrized attack and any integer $0 \leq t \leq n$, it holds that

$$
\frac{1}{2} \text{tr} \left| \sigma_{\text{ABE}}^{\text{sym}} - \rho_{\text{U}} \otimes \sigma_{\text{E}}^{\text{sym}} \right|

\leq 2m \sqrt{\Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C|}{n} \geq \frac{t}{n} \right) \wedge (T = 1) \right] + 2^n \left[ H_2(t/n) - (n - r - m)/n \right]},
$$

(91)
for \( \sigma_{\text{sym}}^{\text{ABE}} \) and \( \rho_U \) defined in Equations (90) and (88), respectively; the partial trace state \( \sigma_{E}^{\text{sym}} \triangleq \text{tr}_{AB} (\sigma_{\text{sym}}^{\text{ABE}}) \); and \( H_2(x) \triangleq -x \log_2(x) - (1 - x) \log_2(1 - x) \). We note that the probability in the right-hand-side is the probability for the original (non-symmetrized) attack.

**Proof.** We notice that only two terms in \( \sigma_{\text{sym}}^{\text{ABE}} \) depend directly on \( i, j \) (and not only on \( k, \xi \)): the probability \( \Pr(\text{sym}(i, j, b, s, P_C, P_K) \) and Eve’s state \( \left( \rho_{i, j}^{b, s} \right)^{\text{sym}}_E \). For any \( i, j, b, s, P_C, P_K \) and \( \xi \triangleq i \rho_i^{PT} \), the probability can be reformulated as

\[
\Pr(\text{sym}(i, j, b, s, P_C, P_K)) = \Pr(\text{sym}(i_T, j_T, b, s, P_C, P_K)) \cdot \Pr(\text{sym}(\xi | i_T, j_T, b, s, P_C, P_K))
\]

\[
\cdot \Pr(\text{sym}(i | i_T, j_T, b, s)) \cdot \Pr(\text{sym}(j | i_T, j_T, b, s)) \cdot \frac{1}{2^{m-r}} \cdot \Pr(\text{sym}(j | i_T, j_T, b, s)) \cdot \frac{1}{2^{n-r-m}} \cdot \Pr(\text{sym}(j | i_T, j_T, b, s)).
\]

(This is correct according to the conclusions of the fourth property of the symmetrized attack—Equations (37) and (38); and because as noted in Subsection 4.2, the matrices \( P_C, P_K \) are completely independent of \( i, i_T, j, j_T, b, s \), and therefore \( \Pr(\text{sym}(i_T, j_T, b, s, P_C, P_K) = \Pr(\text{sym}(P_C, P_K) \cdot \Pr(\text{sym}(i | i_T, j_T, b, s)) \cdot \Pr(\text{sym}(j | i_T, j_T, b, s)). \)

Therefore, the state \( \sigma_{\text{sym}}^{\text{ABE}} \) from Equation (90) takes the following form:

\[
\sigma_{\text{sym}}^{\text{ABE}} = \frac{1}{2^m} \sum_{k, i, j, b, s, \xi} \Pr(\text{sym}(i_T, j_T, b, s, \xi)) \cdot |k\rangle_A \langle k|_A \otimes |k\rangle_B \langle k|_B
\]

\[
\otimes \left[ \frac{1}{2^{n-r-m}} \sum_{P_C, P_K} \Pr(\text{sym}(P_C, P_K)) \cdot \Pr(\text{sym}(j | i_T, j_T, b, s)) \cdot \left( \rho_{i, j}^{b, s} \right)^{\text{sym}}_E \otimes |P_C, P_K \rangle_C \langle P_C, P_K |_C \right] \cdot \langle i_T, j_T, b, s, \xi |_C \langle i_T, j_T, b, s, \xi |_C
\]

\[
= \frac{1}{2^m} \sum_{k, i, j, b, s, \xi} \Pr(\text{sym}(i_T, j_T, b, s, \xi)) \cdot |k\rangle_A \langle k|_A \otimes |k\rangle_B \langle k|_B
\]

\[
\otimes \hat{\rho}_{k}^{\text{sym}} \otimes |i_T, j_T, b, s, \xi |_C \langle i_T, j_T, b, s, \xi |_C.
\]

(This expression for \( \hat{\rho}_{k}^{\text{sym}} \) was found in Equation (76).)

The partial trace \( \sigma_{E}^{\text{sym}} \triangleq \text{tr}_{AB} (\sigma_{\text{sym}}^{\text{ABE}}) \) is

\[
\sigma_{E}^{\text{sym}} = \frac{1}{2^m} \sum_{k, i, j, b, s, \xi} \Pr(\text{sym}(i_T, j_T, b, s, \xi))
\]

\[
\cdot \hat{\rho}_{k}^{\text{sym}} \otimes |i_T, j_T, b, s, \xi |_C \langle i_T, j_T, b, s, \xi |_C,
\]

and the state \( \rho_U \otimes \sigma_{E}^{\text{sym}} \) is thus (using Equation (88))

\[
\rho_U \otimes \sigma_{E}^{\text{sym}} = \frac{1}{2^{2m}} \sum_{k, k', i, j, b, s, \xi} \Pr(\text{sym}(i_T, j_T, b, s, \xi)) \cdot |k\rangle_A \langle k|_A \otimes |k\rangle_B \langle k|_B
\]
Since $\rho_{U} \otimes \sigma_{E}^{\text{sym}}$ are almost identical (except the difference between Eve’s states: $\rho_{k}^{\text{sym}}$ and $\rho_{k}^{\text{sym}}$), we can use the triangle inequality for norms, the definitions of $\Delta_{\text{Eve}}(k, k')$ (Equation (80)) and $\langle \Delta_{E}^{\text{sym}}(k, k') \rangle$ (Equation (81)), and Theorem 7 to get:

$$
\frac{1}{2} \text{tr} |\sigma_{ABE}^{\text{sym}} - \rho_{U} \otimes \sigma_{E}^{\text{sym}}| \\
\leq \frac{1}{2^{2m}} \sum_{k,k',i_{T},j_{T},b,s,\xi} \text{Pr}^{\text{sym}}(i_{T},j_{T},b,s,\xi) \cdot \frac{1}{2} \text{tr} |\rho_{k}^{\text{sym}} - \rho_{k'}^{\text{sym}}| \\
= \frac{1}{2^{2m}} \sum_{k,k'} \text{Pr}^{\text{sym}}(i_{T},j_{T},b,s,\xi) \cdot \Delta_{\text{Eve}}^{\text{sym}}(k, k') \\
= \frac{1}{2^{2m}} \sum_{k,k'} \langle \Delta_{\text{Eve}}^{\text{sym}}(k, k') \rangle \\
\leq 2m \sqrt{\text{Pr}_{\text{inverted-INFO-basis}} \left( \left( \frac{C_{i} - t}{n} \right) \land (T = 1) \right)} + 2^{n} |H_{2}(t/n - (n - r - m)/n)|. \tag{96}
$$

\[\square\]

**Proposition 9.** For any attack, it holds that

$$
\frac{1}{2} \text{tr} |\sigma_{ABE}^{\text{sym}} - \rho_{U} \otimes \sigma_{E}^{\text{sym}}| \leq \frac{1}{2} \text{tr} |\sigma_{ABE}^{\text{sym}} - \rho_{U} \otimes \sigma_{E}^{\text{sym}}|, \tag{97}
$$

for $\sigma_{ABE}$, $\sigma_{ABE}^{\text{sym}}$, and $\rho_{U}$ defined in Equations (89), (90), and (88), respectively, and the partial trace states $\sigma_{E} \triangleq \text{tr}_{AB} (\sigma_{ABE})$ and $\sigma_{E}^{\text{sym}} \triangleq \text{tr}_{AB} (\sigma_{ABE}^{\text{sym}})$.

**Proof.** First, we have to find an expression for $(\rho_{b,s}^{b,s})^{\text{sym}}_{E}$. According to Equation (19),

$$
(\rho_{b,s}^{b,s})^{\text{sym}}_{E} = \left[ \frac{|E_{i,j}^{\text{sym}}\rangle_{b}\langle E_{i,j}^{\text{sym}}|_{b}}{\text{Pr}_{\text{sym}}^{\text{sym}}(j | i, b, s, P_{C}, P_{K})} \right]_{E}, \tag{98}
$$

and according to the “Basic Lemma of Symmetrization” (see Equation (28)),

$$
|E_{i,j}^{\text{sym}}\rangle_{b} = \frac{1}{\sqrt{2^{N}}} \sum_{m \in F_{2}^{N}} (-1)^{(i \oplus j) \cdot m} |E_{i \oplus m, j \oplus m}'\rangle_{b} |m\rangle_{M}. \tag{99}
$$

Therefore,

$$
(\rho_{b,s}^{b,s})^{\text{sym}}_{E} = \frac{1}{2^{N}} \sum_{m,m' \in F_{2}^{N}} (-1)^{(i \oplus j) \cdot (m \oplus m')} \left[ \frac{|E_{i \oplus m, j \oplus m}'\rangle_{b}\langle E_{i \oplus m, j \oplus m}'|_{b} \otimes |m\rangle_{M} \langle m' |_{M}}{\text{Pr}_{\text{sym}}^{\text{sym}}(j | i, b, s, P_{C}, P_{K})} \right]_{E}. \tag{100}
$$

Substituting this result into Equation (90), we find:

$$
\sigma_{ABE}^{\text{sym}} = \frac{1}{2^{N}} \sum_{i,j,b,s, P_{C}, P_{K}, m,m' | T = 1} \text{Pr}_{\text{sym}}^{\text{sym}}(i,j,b,s, P_{C}, P_{K}) \cdot |k\rangle_{A} \langle k|_{A} \otimes |k\rangle_{B} \langle k|_{B}
$$

29
\begin{align*}
\frac{(-1)^{(i \otimes j)} \langle m \rangle \langle m' \rangle}{\Pr^{\text{sym}}(j \mid i, b, s, P_C, P_K)} \otimes |i_T, j_T, b, s, \xi, P_C, P_K\rangle_C &
\otimes \frac{1}{2^N} \sum_{i, j, b, s, P_C, P_K} \Pr^{\text{sym}}(i, b, s, P_C, P_K) \cdot |k\rangle_A \langle k \otimes k \rangle_B \langle k \rangle_B

\otimes (-1)^{(i \otimes j)} \langle m \rangle \langle m' \rangle \frac{\langle E'_{i \otimes m, j \otimes m} b \langle E'_{i \otimes m, j \otimes m} |_b \otimes |m\rangle \langle m' \rangle |_M}{\Pr^{\text{sym}}(j \mid i, b, s, P_C, P_K)}

\otimes |i_T, j_T, b, s, \xi, P_C, P_K\rangle_C
\end{align*}

(101)

We define a unitary operator $V$: given the states $|m\rangle_M$ (held by Eve) and $|s, P_C, P_K\rangle_C$ (public information sent over the classical channel), the unitary operator $V$ performs a XOR of all states in subsystems $A$, $B$, and $C$ with the relevant parts of $m$. Namely, if we define $m_I$ and $m_T$ as the INFO bits and the TEST bits of $m$, respectively (which also depend on $s$, of course), and we define $k_m \equiv m_I P_K^T$ and $\xi_m \equiv m_T P_K^T$ (which also depend on $P_C, P_K$), then

\begin{align*}
V |k\rangle_A |k\rangle_B &\!=\! |E\rangle |m\rangle_M \langle k \otimes k \rangle_B \langle k \rangle_B
\langle i_T, j_T, b, s, \xi, P_C, P_K\rangle_C
\otimes (i_T, m_T, j_T, m_T, b, s, \xi, \xi_m, P_C, P_K) \langle i_T, m_T, j_T, m_T, b, s, \xi, \xi_m, P_C, P_K \rangle_C.
\end{align*}

Therefore (also using the fact that $\Pr^{\text{sym}}(i, b, s, P_C, P_K) = \Pr(i, b, s, P_C, P_K) = \Pr(i \otimes m, b, s, P_C, P_K)$),

\begin{align*}
V^{\text{sym}}_{ABE} &\!=\! \frac{1}{2^N} \sum_{i, j, b, s, P_C, P_K} \Pr(i \oplus m, b, s, P_C, P_K)
\otimes |k \oplus k_m\rangle_A |k \oplus k_m\rangle_B |k \oplus k_m\rangle_B
\otimes (-1)^{(i \otimes j)} \langle m \rangle \langle m' \rangle \langle E'_{i \otimes m, j \otimes m} b \langle E'_{i \otimes m, j \otimes m} |_b \otimes |m\rangle \langle m' \rangle |_M
\otimes |i_T \oplus m_T, j_T \oplus m_T, b, s, \xi, \xi_m, P_C, P_K\rangle_C
\langle i_T \oplus m_T, j_T \oplus m_T, b, s, \xi, \xi_m, P_C, P_K \rangle_C.
\end{align*}

(102)

Tracing out subsystem $M$ (which is part of Eve’s probe), we get

\begin{align*}
\text{tr}_M \left[ V^{\text{sym}}_{ABE} V^\dagger \right] &\!=\! \frac{1}{2^N} \sum_{i, j, b, s, P_C, P_K} \Pr(i \oplus m, b, s, P_C, P_K)
\otimes |k \oplus k_m\rangle_A |k \oplus k_m\rangle_B |k \oplus k_m\rangle_B
\otimes \langle E'_{i \otimes m, j \otimes m} b \langle E'_{i \otimes m, j \otimes m} |_b \otimes |m\rangle \langle m' \rangle |_M
\otimes |i_T \oplus m_T, j_T \oplus m_T, b, s, \xi, \xi_m, P_C, P_K \rangle_C
\langle i_T \oplus m_T, j_T \oplus m_T, b, s, \xi, \xi_m, P_C, P_K \rangle_C.
\end{align*}

(103)

Now we change the indexes: we denote $i' \equiv i \oplus m$ and $j' \equiv j \oplus m$ (for a fixed $m$), and according to the definitions, we immediately get $i_T' = i_T \oplus m_T$, $j_T' = j_T \oplus m_T$, $k' \equiv k_P^T = (i \oplus m_i) P_K^T = k \oplus k_m$, and similarly $\xi' \equiv i \xi P_K^T = \xi \oplus \xi_m$. We also notice that $T$ gets $(i_T \oplus j_T, b, s)$ as inputs, and that they all stay the same (because $i_T' = (i_T \oplus m_T) \oplus (j_T \oplus m_T) = i_T \oplus j_T$), and therefore the change of indexes does not impact the condition $T = 1$. Therefore,

\begin{align*}
\text{tr}_M \left[ V^{\text{sym}}_{ABE} V^\dagger \right] &\!=\! \frac{1}{2^N} \sum_{i', j', b, s, P_C, P_K} \Pr(i', b, s, P_C, P_K) \cdot |k'\rangle_A \langle k' \otimes k' \rangle_B \langle k' \rangle_B
\end{align*}

(104)
\[ \rho \]

where the final equality is according to the definition (Equation (89)).

From Equation (19), we get

\[ \text{Pr} \left( \rho^b_s \right) = \sum_{i,j,b,s} \text{Pr} \left( i', j', b, s, P_C, P_K \right) \langle k' \rangle_A \langle k' \rangle_B \]

Using the relation \( \left( \rho_{i,j}^{b,s} \right)_E = \left( \left| E_{i,j}^{b,s} \right| \right)_E \) from Equation (19), we get

\[ \text{tr}_M \left[ V \sigma_{\text{ABE}} \sigma_{\text{sym}} V^\dagger \right] = \sum_{i,j,b,s} \text{Pr} \left( i', j', b, s, P_C, P_K \right) \langle k' \rangle_A \langle k' \rangle_B \]

where the final equality is according to the definition (Equation (89)).

To sum up, we get the result \( \sigma_{\text{ABE}} = \text{tr}_M \left[ V \sigma_{\text{ABE}} \sigma_{\text{sym}} V^\dagger \right] \); a very similar proof gives us the result \( \rho_U \otimes \sigma_E = \text{tr}_M \left[ V \left( \rho_U \otimes \sigma_E \right) \sigma_{\text{sym}} V^\dagger \right] \). Since the trace distance is preserved under unitary operators [16, Equation (9.21) in page 404] and does not increase under partial trace [16, Theorem 9.2 and page 407], we get

\[ \frac{1}{2} \text{tr} \left| \sigma_{\text{ABE}} - \rho_U \otimes \sigma_E \right| = \frac{1}{2} \text{tr} \left| \text{tr}_M \left[ V \left( \sigma_{\text{ABE}} - \rho_U \otimes \sigma_E \right) \right] \right| \]

\[ \leq \frac{1}{2} \text{tr} \left| \text{tr}_M \left[ V \left( \sigma_{\text{ABE}} - \rho_U \otimes \sigma_E \right) \sigma_{\text{sym}} V^\dagger \right] \right| \]

\[ = \frac{1}{2} \text{tr} \left| \sigma_{\text{ABE}} - \rho_U \otimes \sigma_{\text{sym}} \right|. \]  \hspace{1cm} (107)

\[ \blacksquare \]

**Proposition 10.** For any attack,

\[ \frac{1}{2} \text{tr} \left| \sigma_{\text{ABE}} - \rho_{\text{ABE}} - \sigma_{\text{ABE}} \right| \leq \text{Pr} \left( \left( k \neq k^B \right) \land \left( T = 1 \right) \right), \hspace{1cm} (108) \]

where \( \rho_{\text{ABE}} \) and \( \sigma_{\text{ABE}} \) were defined in Equations (87) and (89), respectively; \( k \) is the final key computed by Alice; and \( k^B \) is the final key computed by Bob.

**Proof.**

\[ \rho_{\text{ABE}} - \sigma_{\text{ABE}} = \sum_{i,j,b,s,P_C,P_K} \text{Pr} \left( i,j,b,s,P_C,P_K \right) \]

\[ = \sum_{i,j,b,s,P_C,P_K} \text{Pr} \left( i,j,b,s,P_C,P_K \right) \]

\[ \times \left( \left| E_{i,j}^{b,s} \right| \right)_E \]

\[ \leq \frac{1}{2} \text{tr} \left| \sigma_{\text{ABE}} - \rho_{\text{ABE}} - \sigma_{\text{ABE}} \right|. \]
\[ \sum_{i,j,b,s,P_{C},P_{K}} \Pr[i,j,b,s,P_{C},P_{K} | (k \neq k^{B}) \wedge (T = 1)] \]
\[ \cdot |k \rangle_{A} \langle k |_{A} \otimes [ |k^{B} \rangle_{B} \langle k^{B} |_{B} - |k \rangle_{B} \langle k |_{B} ] \otimes (\rho^{b,s}_{i,j})_{E} \]
\[ \times |i_{T},j_{T},b,s,\xi,\xi_{i},\xi_{j},\xi_{b},\xi_{s},P_{C},P_{K} \rangle_{C} \langle i_{T},j_{T},b,s,\xi,\xi_{i},\xi_{j},\xi_{b},\xi_{s},P_{C},P_{K} |_{C}. \] (109)

The trace distance between any two normalized states is bounded by 1. Therefore,
\[ \frac{1}{2} \text{tr} |\rho_{ABE} - \sigma_{ABE}| \leq \Pr[(k \neq k^{B}) \wedge (T = 1)]. \] (110)

**Corollary 11.** For any attack and any integer \( 0 \leq t \leq \frac{n}{2} \),
\[ \frac{1}{2} \text{tr} |\rho_{ABE} - \rho_{U} \otimes \rho_{E}| \]
\[ \leq \Pr[(k \neq k^{B}) \wedge (T = 1)] \]
\[ + 2m \sqrt{\Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C|}{n} \geq \frac{t}{n} \right) \wedge (T = 1) \right]} \]
\[ + 2n[H_{2}(t/n) - (n-r-m)/n], \] (111)

for \( \rho_{ABE} \) and \( \rho_{U} \) defined in Equations (87) and (88), respectively; the partial trace state \( \rho_{E} \triangleq \text{tr}_{AB} (\rho_{ABE}) \); and \( H_{2}(x) \triangleq -x \log_{2}(x) - (1-x) \log_{2}(1-x) \).

**Proof.** Using Propositions 8, 9, and 10 and the fact that \( \rho_{E} = \sigma_{E} \), we get:
\[ \frac{1}{2} \text{tr} |\rho_{ABE} - \rho_{U} \otimes \rho_{E}| \]
\[ \leq \frac{1}{2} \text{tr} |\rho_{ABE} - \sigma_{ABE}| + \frac{1}{2} \text{tr} |\sigma_{ABE} - \rho_{U} \otimes \sigma_{E}| \]
\[ \leq \frac{1}{2} \text{tr} |\rho_{ABE} - \sigma_{ABE}| + \frac{1}{2} \text{tr} |\sigma_{ABE}^{\text{sym}} - \rho_{U} \otimes \sigma_{E}^{\text{sym}}| \]
\[ \leq \Pr[(k \neq k^{B}) \wedge (T = 1)] \]
\[ + 2m \sqrt{\Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C|}{n} \geq \frac{t}{n} \right) \wedge (T = 1) \right]} \]
\[ + 2n[H_{2}(t/n) - (n-r-m)/n]. \] (112)

We have thus found an upper bound for the expression \( \frac{1}{2} \text{tr} |\rho_{ABE} - \rho_{U} \otimes \rho_{E}| \). In Section 5 we prove this upper bound to be exponentially small in \( n \) for specific protocols.

## 5 Full Security Proofs for Specific Protocols

Below we prove full security for specific important examples of generalized BB84 protocols.
5.1 Hoeffding’s Theorem

The rest of our security proof consists mainly of applications of the following Theorem, proven by Hoeffding in [19, Section 6]:

**Theorem 12** (Hoeffding’s Theorem). \(X_1, \ldots, X_n\) be a random sample without replacement taken from a population \(c_1, \ldots, c_N\) such that \(a \leq c_j \leq b\) for all \(1 \leq j \leq N\). (That is, each \(X_i\) gets the value of a random \(c_j\) such that the same \(j\) is never chosen for two different variables \(X_i, X_f\).) If we denote \(\bar{X} \triangleq \frac{X_1 + \ldots + X_n}{n}\) and \(\mu \triangleq E[\bar{X}]\) (namely, \(\mu\) is the expected value of \(\bar{X}\)), then:

1. For any \(\varepsilon > 0\),
   \[
   \Pr[\bar{X} - \mu \geq \varepsilon] \leq e^{-\frac{2\varepsilon^2}{(b-a)^2}}. \tag{113}
   \]

2. \(\mu = \frac{1}{N} \sum_{i=1}^{N} c_i\); namely, the expected value of \(\bar{X}\) is the average value of the population.

The following Corollary of Hoeffding’s theorem is useful for proving security:

**Corollary 13.** Given an \((n+n')\)-bit string \(c = c_1 \ldots c_{n+n'}\), assume we randomly and uniformly choose a partition of \(c\) into two substrings, \(c_A\) of length \(n\) and \(c_B\) of length \(n'\). (Formally, this is a random partition of the index set \([1, \ldots, n+n']\) into two disjoint sets \(A, B\) satisfying \(|A| = n\), \(|B| = n'\), and \(A \cup B = \{1, \ldots, n+n'\}\).) Then, for any \(p > 0\) and \(\varepsilon > 0\),
   \[
   \Pr \left[ \left( \left\lfloor \frac{|C_A|}{n} \right\rfloor \geq p + \varepsilon \right) \cap \left( \left\lfloor \frac{|C_B|}{n'} \right\rfloor \leq p \right) \right] \leq e^{-2 \left( \frac{n'}{n+n'} \right)^2 \varepsilon^2}, \tag{114}
   \]
   where \(C_A, C_B\) are random variables whose values equal to \(c_A\) and \(c_B\), respectively, and \(|C_A|, |C_B|\) are the corresponding numbers of 1-bits in these bitstrings.

**Proof.** The random and uniform partition of \(c\) into two substrings, \(c_A\) of length \(n\) and \(c_B\) of length \(n'\), is actually a sample of size \(n\) without replacement from the population \(c_1, \ldots, c_{n+n'} \in \{0, 1\}\). (The sampled \(n\) bits are the bits of \(c_A\), while the other \(n'\) bits are the bits of \(c_B\).) Therefore, we can apply Hoeffding’s theorem (Theorem 12) to this sampling.

Let \(\bar{X}\) be the average of the sample, and let \(\mu\) be the expected value of \(\bar{X}\) (so, according to Theorem 12, \(\mu\) is the average value of the population), then
\[
\bar{X} = \frac{|C_A|}{n}, \tag{115}
\]
\[
\mu = \frac{|C_A| + |C_B|}{n+n'}. \tag{116}
\]

The condition \(\left\lfloor \frac{|C_B|}{n'} \right\rfloor \leq p\) is thus equivalent to \((n+n')\mu - n\bar{X} \leq n' \cdot p\), which is equivalent to \(n \cdot (\bar{X} - \mu) \geq n' \cdot (\mu - p)\). Therefore, the conditions \(\left\lfloor \frac{|C_A|}{n} \right\rfloor \geq p + \varepsilon\) and \(\left\lfloor \frac{|C_B|}{n'} \right\rfloor \leq p\) rewrite to
\[
(\bar{X} - \mu + \varepsilon + p - \mu) \cap \left( \frac{n}{n'} \cdot (\bar{X} - \mu) \geq \mu - p \right). \tag{117}
\]

This condition implies \((1 + \frac{n}{n'}) (\bar{X} - \mu) \geq \varepsilon\), which is equivalent to \(\bar{X} - \mu \geq \frac{n'}{n+n'} \varepsilon\). Using Hoeffding’s theorem (Theorem 12) with \(a = 0, b = 1\), we get
\[
\Pr \left[ \left( \left\lfloor \frac{|C_A|}{n} \right\rfloor \geq p + \varepsilon \right) \cap \left( \left\lfloor \frac{|C_B|}{n'} \right\rfloor \leq p \right) \right] \leq \Pr \left[ \bar{X} - \mu \geq \frac{n'}{n+n'} \varepsilon \right] \leq e^{-2 \left( \frac{n'}{n+n'} \right)^2 \varepsilon^2}. \tag{118}
\]
\(\square\)
We are allowed to use Corollary 13 for comparing the error rates in different sets of qubits (e.g., INFO and TEST bits), under the condition that the random and uniform sampling occurs only after the qubits are sent by Alice and measured by Bob. In other words, the sampling cannot affect the bases in which the qubits are sent and measured, and it cannot affect Eve’s attack. This means, in particular, that Alice’s choice of $s$ must be independent of $i$ and kept in secret from Bob and Eve until Bob completes his measurement. (Some independence between $s$ and $b$ must also be assumed, but we discuss this condition separately for each protocol.)

Similar uses of Hoeffding’s theorem for proving security of QKD are available in [1, 20]. We also use another Theorem, proven by Hoeffding in [19, Section 2, Theorem 1]:

**Theorem 14.** Let $X_1, \ldots, X_N$ be independent random variables with finite first and second moments, such that $0 \leq X_i \leq 1$ for all $1 \leq i \leq N$. If $\bar{X} \triangleq \frac{X_1 + \cdots + X_N}{N}$ and $\mu \triangleq E[\bar{X}]$ is the expected value of $\bar{X}$, then for any $\epsilon > 0$,

$$\Pr[\bar{X} - \mu \geq \epsilon] \leq e^{-2N\epsilon^2}, \quad (119)$$

and, in a similar way (see [19, Section 1]),

$$\Pr[\mu - \bar{X} \geq \epsilon] \leq e^{-2N\epsilon^2}. \quad (120)$$

We will use the following Corollary of Theorem 14 for proving the security of the “efficient BB84” protocol, described in Subsection 5.4:

**Corollary 15.** Let $0 \leq p \leq 1$ be a parameter, and let $b = b_1 \ldots b_N$ be an $N$-bit string, such that each $b_i$ is chosen probabilistically and independently out of $\{0, 1\}$, with $\Pr(b_i = 0) = p$ and $\Pr(b_i = 1) = 1 - p$. Then:

$$\Pr\left(|b| \leq \frac{(1-p)N}{2}\right) \leq e^{-\frac{1}{2}N(1-p)^2}, \quad (121)$$

$$\Pr\left(|\bar{b}| \leq \frac{pN}{2}\right) \leq e^{-\frac{1}{2}Np^2}. \quad (122)$$

*Proof.* Let us define $X_i = b_i$ for all $1 \leq i \leq N$. Then $X_i$ are independent random variables with finite first and second moments, such that $0 \leq X_i \leq 1$ for all $1 \leq i \leq N$ and $\mu \triangleq E[\bar{X}] = 1 - p$. Therefore, using Theorem 14, we get the two following results:

$$\Pr\left[(1-p) - X \geq \frac{1-p}{2}\right] \leq e^{-\frac{1}{2}N(1-p)^2}, \quad (123)$$

$$\Pr\left[\bar{X} - (1-p) \geq \frac{p}{2}\right] \leq e^{-\frac{1}{2}Np^2}. \quad (124)$$

We notice that $\bar{X} = \frac{|b|}{N} = 1 - \frac{|\bar{b}|}{N}$. Substituting this result, we get

$$\Pr\left[-\frac{|b|}{N} \geq -\frac{1-p}{2}\right] \leq e^{-\frac{1}{2}N(1-p)^2}, \quad (125)$$

$$\Pr\left[1 - \frac{|\bar{b}|}{N} - 1 \geq -\frac{p}{2}\right] \leq e^{-\frac{1}{2}Np^2}, \quad (126)$$

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and, therefore,
\[
\Pr \left[ |b| \leq \frac{(1-p)N}{2} \right] \leq e^{-\frac{1}{2}N(1-p)^2},
\]
(127)
\[
\Pr \left[ |\tilde{b}| \leq \frac{pN}{2} \right] \leq e^{-\frac{1}{2}Np^2}.
\]
(128)

5.2 The BB84-INFO-z Protocol

In the BB84-INFO-z protocol, all INFO bits are sent by Alice in the z basis, while the TEST bits are sent in both the z and the x bases. This means that \( b \) and \( s \) together define a random partition of the set of indexes \( \{1, 2, \ldots, N\} \) into three disjoint sets:

- \( I \) (INFO bits, where \( s_j = 1 \) and \( b_j = 0 \)) of size \( n \);
- \( T_z \) (TEST-Z bits, where \( s_j = 0 \) and \( b_j = 0 \)) of size \( n_z \); and
- \( T_x \) (TEST-X bits, where \( s_j = 0 \) and \( b_j = 1 \)) of size \( n_x \).

Formally, Alice and Bob agree on parameters \( n, n_z, n_x \) (such that \( N = n + n_z + n_x \)), and we choose \( B = \{ b \in \mathbb{F}_2^N \mid |b| = n_x \} \) and \( S_b = \{ s \in \mathbb{F}_2^N \mid (|s| = n) \land (|s \oplus b| = n + n_x) \} \) (namely, \( s \in S_b \) if it consists of \( n \) 1-bits that do not overlap with the \( n_x \) 1-bits of \( b \)) for all \( b \in B \). The probability distributions \( \Pr(b) \) and \( \Pr(s \mid b) \) are all uniform—namely, \( \Pr(b, s) \) is identical for all \( b \in B \) and \( s \in S_b \).

Alice and Bob also agree on error rate thresholds, \( p_{a,z} \) and \( p_{a,x} \) (for the TEST-Z and TEST-X bits, respectively). The testing function \( T \) is defined as follows:

\[
T(i_T \oplus j_T, b_T, s) = 1 \iff (|i_{T_z} \oplus j_{T_z}| \leq n_z \cdot p_{a,z}) \land (|i_{T_x} \oplus j_{T_x}| \leq n_x \cdot p_{a,x}).
\]
(129)

Namely, the test passes if and only if the error rate on the TEST-Z bits is at most \( p_{a,z} \) and the error rate on the TEST-X bits is at most \( p_{a,x} \).

From Corollary 11 we get the following bound for any integer \( 0 \leq t \leq \frac{n}{2} \):

\[
\frac{1}{2} \text{tr} |\rho_{ABE} - \rho_U \otimes \rho_E| \leq \Pr [(k \neq k^B) \land (T = 1)]
\]
\[
+ 2m \sqrt{\Pr_{\text{inverted-INFO-basis}} \left[ \left( \left| \frac{C_i}{n} \right| \geq \frac{t}{n} \right) \land (T = 1) \right] + 2^n[H_2(t/n) -(n-r-m)/n]}.
\]
(130)

Below we prove the two probabilities in the right-hand-side to be exponentially small in \( n \):

**Theorem 16.** For any \( n, n_z, n_x > 0, p_{a,z}, p_{a,x} > 0, \) and \( \varepsilon_{\text{sec}} > 0 \) such that \( t \triangleq n(p_{a,x} + \varepsilon_{\text{sec}}) \) is an integer and \( p_{a,x} + \varepsilon_{\text{sec}} \leq \frac{1}{2} \), it holds for the BB84-INFO-z protocol that

\[
\Pr_{\text{inverted-INFO-basis}} \left[ \left( \left| \frac{C_i}{n} \right| \geq \frac{t}{n} \right) \land (T = 1) \right] \leq e^{-2(n/2p_{a,x})^2 n\varepsilon_{\text{sec}}^2}.
\]
(131)
Proof. Because \( \frac{t}{n} \equiv p_{a,x} + \varepsilon_{sec} \), it holds that

\[
\Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_i|}{n} \geq \frac{t}{n} \right) \land (T = 1) \right] \\
= \Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_i|}{n} \geq p_{a,x} + \varepsilon_{sec} \right) \land \left( \frac{|C_{Tz}|}{n_z} \leq p_{a,z} \right) \land \left( \frac{|C_{Tx}|}{n_x} \leq p_{a,x} \right) \right] \\
\leq \Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_i|}{n} \geq p_{a,x} + \varepsilon_{sec} \right) \land \left( \frac{|C_{Tx}|}{n_x} \leq p_{a,x} \right) \right]. 
\]

(132)

In the hypothetical "inverted-INFO-basis" protocol, the INFO and TEST-X bits are sent and measured in the x basis, while the TEST-Z bits are sent and measured in the z basis. Therefore, the random and uniform sampling of the n INFO bits out of the \( n + n_z \) bits sent in the x basis (assuming that the TEST-Z bits have already been chosen) does not affect the bases in the hypothetical protocol. This means that we can apply Corollary 13 to this sampling, and we get

\[
\Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_i|}{n} \geq p_{a,x} + \varepsilon_{sec} \right) \land \left( \frac{|C_{Tx}|}{n_x} \leq p_{a,x} \right) \right] \leq e^{-2\left( \frac{n_z}{n_x} \right)^2 \varepsilon_{sec}^2}. \]

(133)

\[
\text{Theorem 17. For any } n, n_z, n_x > 0, p_{a,z}, p_{a,x} > 0, \text{ and } \varepsilon_{rel} > 0 \text{ such that } t_{rel} \equiv n(p_{a,z} + \varepsilon_{rel}) \text{ is an integer and } p_{a,z} + \varepsilon_{rel} \leq \frac{1}{2}, \text{ it holds for the BB84-INFO-z protocol that}
\]

\[
\Pr \left[ (k \neq k^B) \land (T = 1) \right] \leq e^{-2\left( \frac{n_z}{n_x} \right)^2 \varepsilon_{rel}^2} + 2^n[H_2(p_{a,z} + \varepsilon_{rel}) - r/n],
\]

(134)

where \( H_2(x) \equiv -x \log_2(x) - (1 - x) \log_2(1 - x) \).

Proof. If \( k \neq k^B \), Alice and Bob have different final keys, and this means that the error correction stage did not succeed. The \( r \times n \) parity-check matrix \( P_C \) is chosen uniformly at random and is completely independent of the error word \( c_i \equiv i_1 \oplus j_1 \) (because Alice sends it to Bob only after Eve’s attack); therefore, applying Corollary 4, we can find that if there are at most \( t_{rel} \equiv n(p_{a,z} + \varepsilon_{rel}) \) errors in the error word \( c_i \), then the probability a random error-correcting code fails to correct the error word is at most \( 2^n[H_2(t_{rel}/n) - r/n] = 2^n[H_2(p_{a,z} + \varepsilon_{rel}) - r/n] \).

Therefore, failure of the error correction stage (\( k \neq k^B \)) must mean that either there are more than \( t_{rel} \) errors in the INFO bits (namely, \( \frac{|C_i|}{n} \geq \frac{t_{rel}}{n} \equiv p_{a,z} + \varepsilon_{rel} \)) or there are at most \( t_{rel} \) errors in the INFO bits but the error-correcting code failed in correcting the error string. Therefore,

\[
\Pr \left[ (k \neq k^B) \land (T = 1) \right] = \Pr \left[ (k \neq k^B) \land \left( \frac{|C_{Tz}|}{n_z} \leq p_{a,z} \right) \land \left( \frac{|C_{Tx}|}{n_x} \leq p_{a,x} \right) \right] \\
\leq \Pr \left[ (k \neq k^B) \land \left( \frac{|C_{Tz}|}{n_z} \leq p_{a,z} \right) \right] \\
\leq \Pr \left[ \left( \frac{|C_i|}{n} \geq p_{a,x} + \varepsilon_{rel} \right) \land \left( \frac{|C_{Tz}|}{n_z} \leq p_{a,z} \right) \right] + 2^n[H_2(p_{a,z} + \varepsilon_{rel}) - r/n].
\]

(135)

In the real protocol, the INFO and TEST-Z bits are sent and measured in the z basis, while the TEST-X bits are sent and measured in the x basis. Therefore, the random and uniform
sampling of the $n$ INFO bits out of the $n + n_z$ bits sent in the $z$ basis (assuming that the TEST-X bits have already been chosen) does not affect the bases in the real protocol. This means that we can apply Corollary 13 to this sampling, and we get

$$\Pr \left( \left( \frac{|C_1|}{n} \geq p_{a,z} + \epsilon_{\text{rel}} \right) \land \left( \frac{|C_{Tz}|}{n_z} \leq p_{a,z} \right) \right) \leq e^{-2 \left( \frac{n_z}{n} \right)^2 n_{\epsilon_{\text{rel}}}^2}.$$

We thus conclude:

$$\Pr \left( (k \neq k^B) \land (T = 1) \right) \leq e^{-2 \left( \frac{n_z}{n} \right)^2 n_{\epsilon_{\text{rel}}}^2 + 2n[H_2(p_{a,z} + \epsilon_{\text{rel}}) - r/n]}.$$

Combining the results of Corollary 11, Theorem 16, and Theorem 17, we get:

**Corollary 18.** For any $n, n_z, n_x > 0$, $p_{a,z}, p_{a,x} > 0$, and $\epsilon_{\text{sec}}, \epsilon_{\text{rel}} > 0$ such that $n(p_{a,x} + \epsilon_{\text{sec}})$ and $n(p_{a,z} + \epsilon_{\text{rel}})$ are both integers, $p_{a,x} + \epsilon_{\text{sec}} \leq \frac{1}{2}$, and $p_{a,z} + \epsilon_{\text{rel}} \leq \frac{1}{2}$, it holds for the BB84-INFO-$z$ protocol that

$$\frac{1}{2} \text{tr} \left| \rho_{\text{ABE}} - \rho_U \otimes \rho_E \right| \leq e^{-2 \left( \frac{n_z}{n} \right)^2 n_{\epsilon_{\text{rel}}}^2 + 2n[H_2(p_{a,z} + \epsilon_{\text{rel}}) - r/n]}
+ 2m \sqrt{e^{-2 \left( \frac{n_z}{n} \right)^2 n_{\epsilon_{\text{sec}}}^2 + 2n[H_2(p_{a,x} + \epsilon_{\text{sec}}) - (n-r-m)/n]},}$$

where $H_2(x) \triangleq -x \log_2(x) - (1 - x) \log_2(1 - x)$.

This bound is exponentially small in $n$ under the two following conditions:

$$H_2(p_{a,z} + \epsilon_{\text{rel}}) < \frac{r}{n}, \quad H_2(p_{a,x} + \epsilon_{\text{sec}}) < \frac{n - r - m}{n},$$

which are equivalent to:

$$H_2(p_{a,z} + \epsilon_{\text{rel}}) < \frac{r}{n}, \quad H_2(p_{a,x} + \epsilon_{\text{sec}}) + H_2(p_{a,z} + \epsilon_{\text{rel}}) < \frac{n - m}{n} = 1 - \frac{m}{n}.$$

We can thus obtain the following upper bound on the bit-rate:

$$R_{\text{secret}} \triangleq \frac{m}{n} < 1 - H_2(p_{a,x} + \epsilon_{\text{sec}}) - H_2(p_{a,z} + \epsilon_{\text{rel}}).$$

To get the asymptotic error rate thresholds, we require $R_{\text{secret}} > 0$, and we get the asymptotic condition

$$H_2(p_{a,x}) + H_2(p_{a,z}) < 1.$$  

The secure asymptotic error rate thresholds zone is shown in Figure 1 (it is below the curve). Note the trade-off between the error rate thresholds $p_{a,z}$ and $p_{a,x}$. Also note that in the case of $p_{a,z} = p_{a,x}$, we get the same threshold as in BB84, which is 11%.
Figure 1: The secure asymptotic error rates zone for BB84-INFO-z (below the curve)
5.3 The Standard BB84 Protocol

In the standard BB84 protocol, the strings \( b \) and \( s \) are chosen randomly (except that we demand \( |s| = n \)) and independently, and \( N = 2n \). In other words, there are \( n \) INFO bits and \( n \) TEST bits (chosen randomly), and for each one of them, the basis (\( z \) or \( x \)) is chosen randomly and independently.

Formally, in BB84, we choose \( N = 2n, B = \mathbb{F}_2^N \), and \( S_b = \{ s \in \mathbb{F}_2^N \mid |s| = n \} \) for all \( b \in B \). The probability distributions \( \text{Pr}(b) \) and \( \text{Pr}(s \mid b) = \text{Pr}(s) \) are all uniform—namely, \( \text{Pr}(b, s) \) is identical for all \( b \in B \) and \( s \in S_b \).

Given the parameter \( p_a \) agreed by Alice and Bob, the testing function \( T \) is

\[
T(i_T \oplus j_T, b_T, s) = 1 \iff |i_T \oplus j_T| \leq n \cdot p_a.
\]

Namely, the test passes if and only if the error rate on the TEST bits is at most \( p_a \).

**Proposition 19.** In the standard BB84 protocol, for any integer \( 0 \leq t \leq \frac{n}{2} \),

\[
\text{Pr}_{\text{inverted-INFO}} \left[ \left( \frac{|C_i|}{n} \geq \frac{t}{n} \right) \land (T = 1) \right] = \text{Pr} \left[ \left( \frac{|C_i|}{n} \geq \frac{t}{n} \right) \land (T = 1) \right].
\]

**Proof.**

\[
\begin{align*}
\text{Pr}_{\text{inverted-INFO}} \left[ \left( \frac{|C_i|}{n} \geq \frac{t}{n} \right) \land (T = 1) \right] &= \sum_{b, s} \text{Pr}_{\text{inverted-INFO}} \left[ \left( \frac{|C_i|}{n} \geq \frac{t}{n} \right) \land (T = 1) \mid b, s \right] \cdot \text{Pr}(b, s) \\
&= \sum_{b, s} \text{Pr} \left[ \left( \frac{|C_i|}{n} \geq \frac{t}{n} \right) \land (T = 1) \mid b^0, s \right] \cdot \text{Pr}(b^0, s) \\
&= \text{Pr} \left[ \left( \frac{|C_i|}{n} \geq \frac{t}{n} \right) \land (T = 1) \right]
\end{align*}
\]

(\textit{where } \( b^0 \triangleq b \oplus s \)).

The security of the standard BB84 protocol is now easily obtained:

**Theorem 20.** For any \( n > 0 \), \( p_a > 0 \), and \( \varepsilon_{\text{sec}}, \varepsilon_{\text{rel}} > 0 \) such that \( n(p_a + \varepsilon_{\text{sec}}) \) and \( n(p_a + \varepsilon_{\text{rel}}) \) are both integers, \( p_a + \varepsilon_{\text{sec}} \leq \frac{1}{2} \), and \( p_a + \varepsilon_{\text{rel}} \leq \frac{1}{2} \), it holds for the standard BB84 protocol that

\[
\frac{1}{2} \text{tr} |\rho_{\text{ABE}} - \rho_U \otimes \rho_E| \leq e^{-\frac{1}{2}n\varepsilon_{\text{sec}}^2} + 2^n[H_2(p_a + \varepsilon_{\text{rel}}) - \frac{r}{n}] + 2m\sqrt{e^{-\frac{1}{2}n\varepsilon_{\text{sec}}^2} + 2^n[H_2(p_a + \varepsilon_{\text{sec}}) - \frac{(n-r-m)}{n}]},
\]

\( \text{where } H_2(x) \triangleq -x \log_2(x) - (1-x) \log_2(1-x). \)

**Proof.** Using Corollary 11 and Proposition 19, we get the following bound for the standard BB84 protocol, choosing the integer \( t \triangleq n(p_a + \varepsilon_{\text{sec}}) \):

\[
\frac{1}{2} \text{tr} |\rho_{\text{ABE}} - \rho_U \otimes \rho_E|\]
\[ \leq \Pr \left( (k \neq k^B) \land (T = 1) \right) \\
+ 2m \sqrt{\Pr \left( \left( \left| C_I \right| \geq \frac{1}{n} \right) \land (T = 1) \right) + 2^n[H_2(t/n) - (n - r - m)/n]} \\
= \Pr \left( (k \neq k^B) \land \left( \frac{|C_T|}{n} \leq p_a \right) \right) \\
+ 2m \sqrt{\Pr \left( \left( \frac{|C_I|}{n} \geq p_a + \varepsilon_{\text{rel}} \right) \land \left( \frac{|C_T|}{n} \leq p_a \right) \right] + 2^n[H_2(p_a + \varepsilon_{\text{rel}}) - (n - r - m)/n]. \quad (149) \]

Because the event \( k \neq k^B \) implies that either the error rate on the INFO bits is higher than \( p_a + \varepsilon_{\text{rel}} \) or the error correction step fails (which, according to Corollary 4, happens with probability at most \( 2^n[H_2(p_a + \varepsilon_{\text{rel}}) - r/n] \); see the similar proof of Theorem 17 for details), we get:

\[ \Pr \left( (k \neq k^B) \land \left( \frac{|C_T|}{n} \leq p_a \right) \right] \leq \Pr \left( \left( \frac{|C_I|}{n} \geq p_a + \varepsilon_{\text{rel}} \right) \land \left( \frac{|C_T|}{n} \leq p_a \right) \right] + 2^n[H_2(p_a + \varepsilon_{\text{rel}}) - r/n]. \quad (150) \]

All bits in the protocol are sent in random and independent bases. Therefore, the random and uniform sampling of the \( n \) INFO bits out of the \( N = 2n \) bits does not affect the bases (in the real protocol). We can thus apply Corollary 13 to this sampling, and we get

\[ \Pr \left( \left( \frac{|C_I|}{n} \geq p_a + \varepsilon_{\text{rel}} \right) \land \left( \frac{|C_T|}{n} \leq p_a \right) \right] \leq e^{-\frac{1}{2}n \varepsilon_{\text{rel}}^2}, \quad (151) \]
\[ \Pr \left( \left( \frac{|C_I|}{n} \geq p_a + \varepsilon_{\text{sec}} \right) \land \left( \frac{|C_T|}{n} \leq p_a \right) \right] \leq e^{-\frac{1}{2}n \varepsilon_{\text{sec}}^2}. \quad (152) \]

Combining Equations (149)–(152), we get

\[ \frac{1}{2} \left| \text{Tr} \rho_{\text{ABE}} - \rho_U \otimes \rho_E \right| \leq e^{-\frac{1}{2}n \varepsilon_{\text{rel}}^2} + 2^n[H_2(p_a + \varepsilon_{\text{rel}}) - r/n] + 2m \sqrt{e^{-\frac{1}{2}n \varepsilon_{\text{sec}}^2} + 2^n[H_2(p_a + \varepsilon_{\text{sec}}) - (n - r - m)/n]. \quad (153) \]

We point out that this is a finite-key bound, matching pretty closely the state-of-the-art for security of BB84 (e.g., \([6, 8]\)), except the superfluous “\( m \)” factor (resulting from Proposition 6) that we intend to remove in a future version of this paper.

For finding the asymptotic error rate, we note that this bound is exponentially small in \( n \) under the two following conditions:

\[ H_2(p_a + \varepsilon_{\text{rel}}) < \frac{r}{n}, \quad (154) \]
\[ H_2(p_a + \varepsilon_{\text{sec}}) < \frac{n - r - m}{n}, \quad (155) \]

which are equivalent to:

\[ H_2(p_a + \varepsilon_{\text{rel}}) < \frac{r}{n}, \quad (156) \]
\[ H_2(p_a + \varepsilon_{\text{sec}}) + H_2(p_a + \varepsilon_{\text{rel}}) < \frac{n-m}{n} = 1 - \frac{m}{n}. \]  

We can thus obtain the following upper bound on the bit-rate:

\[ R_{\text{secret}} \triangleq \frac{m}{n} < 1 - H_2(p_a + \varepsilon_{\text{sec}}) - H_2(p_a + \varepsilon_{\text{rel}}). \]  

To get the asymptotic error rate threshold, we require \( R_{\text{secret}} > 0 \), and we get the asymptotic condition

\[ 2H_2(p_a) < 1. \]  

This condition gives an asymptotic error rate threshold of 11%.

### 5.4 The “Efficient BB84” Protocol

In the “efficient BB84” protocol (suggested by [15]), the bitstring \( \mathbf{b} \) is chosen probabilistically, but *not uniformly*: each qubit is sent in the \( z \) basis with probability \( p \) (and in the \( x \) basis with probability \( 1-p \)), where \( 0 < p \leq \frac{1}{2} \). Then, the bitstring \( \mathbf{s} \) is chosen such that there are \( n_z \) TEST-Z bits and \( n_x \) TEST-X bits. In other words, as in BB84-INFO-Z, the strings \( \mathbf{b} \) and \( \mathbf{s} \) together define a random partition of the set of indexes \( \{1,2,\ldots,N\} \) into three disjoint sets:

- \( \{1\} \) (INFO bits, where \( s_j = 1 \) of size \( n \). However, unlike BB84-INFO-Z, this set consists of both \( z \) qubits and \( x \) qubits; therefore, it can be divided to two disjoint subsets:
  - \( I_Z \) (INFO-Z bits, where \( s_j = 1 \) and \( b_j = 0 \)); and
  - \( I_X \) (INFO-X bits, where \( s_j = 1 \) and \( b_j = 1 \)).
- \( T_Z \) (TEST-Z bits, where \( s_j = 0 \) and \( b_j = 0 \)) of size \( n_z \); and
- \( T_X \) (TEST-X bits, where \( s_j = 0 \) and \( b_j = 1 \)) of size \( n_x \).

Formally, in “efficient BB84”, Alice and Bob agree on parameters \( n, n_z, n_x \) (such that \( N = n + n_z + n_x \)) and on a parameter \( 0 < p \leq \frac{1}{2} \), and we choose \( B = \mathcal{F}_2^N \) and \( S_b = \{ \mathbf{s} \in \mathcal{F}_2^N \mid |\mathbf{s}| = n \} \cap (\mathcal{F}_2^N \setminus \mathbf{b}) \) for all \( \mathbf{b} \in B \) (namely, it is required that there are \( n \) INFO bits, \( n_z \) TEST-Z bits, and \( n_x \) TEST-X bits). This time, the probability distribution \( \text{Pr}(\mathbf{b}) \) is *not* uniform: it holds that \( \text{Pr}(\mathbf{b}) = (1-p)^{|\mathbf{b}|} \cdot p^{N-|\mathbf{b}|} \), because the probability of each bit to be in the \( x \) basis is \( 1-p \). On the other hand, the probability distribution \( \text{Pr}(\mathbf{s} \mid \mathbf{b}) \) is uniform.

**Remark 21.** A subtle point is that for some values \( \mathbf{b} \in \mathcal{F}_2^N \) (for example, for \( \mathbf{b} = 00\ldots0 \)), the set \( S_b \) is empty: no \( \mathbf{s} \) can be agreed by Alice and Bob for such values of \( \mathbf{b} \). In that case, as assumed in [15, Section 4.3], the protocol aborts, and different values of \( \mathbf{b} \) and \( \mathbf{s} \) are randomly chosen; this is equivalent to assuming that Alice is not allowed to choose those values of \( \mathbf{b} \). Therefore, to be more precise, we must re-define

\[ B = \{ \mathbf{b} \in \mathcal{F}_2^N \mid S_b \neq \emptyset \} = \{ \mathbf{b} \in \mathcal{F}_2^N \mid (|\mathbf{b}| \geq n_x) \cap (|\mathbf{b}| \geq n_z) \}, \]

and we must normalize the probabilities by defining \( \text{Pr}_0(\mathbf{b}) \triangleq (1-p)^{|\mathbf{b}|} \cdot p^{N-|\mathbf{b}|} \) (the original probability of each \( \mathbf{b} \)), \( N_p \triangleq \sum_{\mathbf{b} \in B} \text{Pr}_0(\mathbf{b}) \) (the sum of all the original probabilities for all the allowed values of \( \mathbf{b} \in B \)), and then the real probability of each \( \mathbf{b} \in B \) is

\[ \text{Pr}(\mathbf{b}) = \frac{\text{Pr}_0(\mathbf{b})}{N_p} = \frac{(1-p)^{|\mathbf{b}|} \cdot p^{N-|\mathbf{b}|}}{N_p}. \]  

This guarantees that the sum of probabilities of all the allowed values \( \mathbf{b} \in B \) is 1.
Alice and Bob also agree on an error rate threshold, \( p_a \) (applied both to the TEST-Z bits and to the TEST-X bits). The testing function \( T \) is defined as follows:

\[
T(i_T \oplus j_T, b_T, s) = 1 \iff (|i_T \oplus j_T| \leq n_z \cdot p_a) \land (|i_T \oplus j_T| \leq n_x \cdot p_a).
\] (162)

Namely, the test passes if and only if the error rate on the TEST-Z bits is at most \( p_a \) and the error rate on the TEST-X bits is at most \( p_a \).

In this security proof, instead of analyzing all the INFO bits together, we analyze the INFO-Z and the INFO-X bits separately. We define the following random variables:

- \( C_{l_Z} \) and \( C_{l_X} \) are the random variables corresponding to the error strings on the INFO-Z bits and on the INFO-X bits, respectively.
- \( N_{l_Z} \) and \( N_{l_X} \) are random variables equal to the numbers of INFO-Z and INFO-X bits, respectively. (We note that the parameters \( n, n_z, n_x \) are deterministically chosen by Alice and Bob, while \( N_{l_Z} \) and \( N_{l_X} \) are determined by the probabilistic choice of \( b \). We also note that, necessarily, \( n = N_{l_Z} + N_{l_X} \).)

**Proposition 22.** For any \( \varepsilon > 0 \),

\[
\Pr\left( \frac{|C_t|}{n} \geq p_a + \varepsilon \right) \land (T = 1) \leq \Pr\left( \frac{|C_{l_Z}|}{N_{l_Z}} \geq p_a + \varepsilon \right) \land (T = 1) + \Pr\left( \frac{|C_{l_X}|}{N_{l_X}} \geq p_a + \varepsilon \right) \land (T = 1).
\] (163)

Equation (163) similarly applies to the hypothetical “inverted-INFO-basis” protocol, too (namely, it applies even if \( \Pr \) is replaced by \( \Pr_{\text{inverted-INFO-basis}} \)).

**Proof.** We observe that if the error rate on all the INFO bits together is at least \( p_a + \varepsilon \), then at least one of the error rates (on the INFO-Z bits or on the INFO-X bits) must be at least \( p_a + \varepsilon \). (Equivalently, if both error rates on the INFO-Z bits and on the INFO-X bits are lower than \( p_a + \varepsilon \), then the error rate on the INFO bits is necessarily lower than \( p_a + \varepsilon \).) Namely,

\[
\left( \frac{|C_t|}{n} \geq p_a + \varepsilon \right) \implies \left( \frac{|C_{l_Z}|}{N_{l_Z}} \geq p_a + \varepsilon \right) \lor \left( \frac{|C_{l_X}|}{N_{l_X}} \geq p_a + \varepsilon \right).
\] (164)

In particular, the corresponding probabilities satisfy

\[
\Pr\left( \frac{|C_t|}{n} \geq p_a + \varepsilon \right) \land (T = 1) \leq \Pr\left( \frac{|C_{l_Z}|}{N_{l_Z}} \geq p_a + \varepsilon \right) \land (T = 1) + \Pr\left( \frac{|C_{l_X}|}{N_{l_X}} \geq p_a + \varepsilon \right) \land (T = 1).
\] (165)

This result applies both to the real protocol and to the hypothetical “inverted-INFO-basis” protocol.

**Proposition 23.** For any \( \varepsilon > 0 \) and \( \delta > 0 \),

\[
\Pr\left( \frac{|C_{l_Z}|}{N_{l_Z}} \geq p_a + \varepsilon \right) \land (T = 1)
\]

42
and

$$\Pr \left[ \sum_{t_{x}} \sum_{z} \max_{\delta \leq t_{x} \leq n} \Pr \left( \left\{ \frac{\|C_{x}\|}{N_{x}} \geq p_{a} + \varepsilon \right\} \land (T = 1) \mid N_{i_{x}} = t_{x} \right) \leq \delta \right]$$

Equations (166)–(167) similarly apply to the hypothetical “inverted-INFO-basis” protocol, too (namely, they apply even if \( \Pr \) is replaced by \( \Pr_{\text{inverted-INFO-basis}} \)).

Proof. First, we prove Equation (166):

$$\Pr \left[ \left( \frac{\|C_{x}\|}{N_{x}} \geq p_{a} + \varepsilon \right) \land (T = 1) \right] = \sum_{t_{x}} \Pr \left[ \left( \frac{\|C_{x}\|}{t_{x}} \geq p_{a} + \varepsilon \right) \land (T = 1) \mid N_{i_{x}} = t_{x} \right] \cdot \Pr (N_{i_{x}} = t_{x})$$

$$= \sum_{t_{x}} \Pr \left[ \left( \frac{\|C_{x}\|}{t_{x}} \geq p_{a} + \varepsilon \right) \land (T = 1) \mid N_{i_{x}} = t_{x} \right] \cdot \Pr (N_{i_{x}} = t_{x})$$

$$+ \sum_{\delta \leq t_{x} \leq n} \Pr \left[ \left( \frac{\|C_{x}\|}{t_{x}} \geq p_{a} + \varepsilon \right) \land (T = 1) \mid N_{i_{x}} = t_{x} \right] \cdot \Pr (N_{i_{x}} = t_{x})$$

$$\leq \Pr (N_{i_{x}} \leq \delta) + \max_{\delta \leq t_{x} \leq n} \Pr \left[ \left( \frac{\|C_{x}\|}{t_{x}} \geq p_{a} + \varepsilon \right) \land (T = 1) \mid N_{i_{x}} = t_{x} \right].$$

The proof of Equation (167) is similar. Both proofs apply both to the real protocol and to the “inverted-INFO-basis” protocol.

**Theorem 24.** For any \( 0 < p \leq 1/2 \), \( n > 0 \), \( 0 < n_{z} < n^{\frac{N}{2}} \), \( 0 < n_{x} < \frac{(1-p)N}{2} \), \( p_{a} > 0 \), and \( \varepsilon_{\text{sec}}, \varepsilon_{\text{rel}} > 0 \) such that \( t \triangleq n(p_{a} + \varepsilon_{\text{sec}}) \) and \( t_{\text{rel}} \triangleq n(p_{a} + \varepsilon_{\text{rel}}) \) are both integers, \( p_{a} + \varepsilon_{\text{sec}} \leq 1/2 \), and \( p_{a} + \varepsilon_{\text{rel}} \leq 1/2 \), it holds for the “efficient BB84” protocol that

$$\frac{1}{2} \text{tr} |\rho_{\text{ABE}} - \rho_{U} \otimes \rho_{E}| \leq e^{-\frac{1}{2}Np^{2}} + e^{-2 \left( \frac{n_{x}}{n + n_{z}} \right)^{2} (\frac{2N}{n} - n_{z})} \varepsilon_{\text{rel}}^{2}$$

$$+ e^{-\frac{1}{2}N(1-p)^{2}} + e^{-2 \left( \frac{n_{x}}{n + n_{z}} \right)^{2} (\frac{(1-p)N}{n} - n_{x})} \varepsilon_{\text{sec}}^{2} + 2^{n} [H_{2}(p_{a} + \varepsilon_{\text{rel}}) - r/n]$$

$$+ 2m \sqrt{e^{-\frac{1}{2}Np^{2}} + e^{-2 \left( \frac{n_{x}}{n + n_{z}} \right)^{2} (\frac{2N}{n} - n_{z})} \varepsilon_{\text{rel}}^{2} + e^{-\frac{1}{2}N(1-p)^{2}} + e^{-2 \left( \frac{n_{x}}{n + n_{z}} \right)^{2} (\frac{(1-p)N}{n} - n_{x})} \varepsilon_{\text{sec}}^{2}}$$

$$+ \frac{2n[H_{2}(p_{a} + \varepsilon_{\text{sec}}) - (n-r-m)/n]}{2^{n}[H_{2}(p_{a} + \varepsilon_{\text{sec}}) - (n-r-m)/n]}.$$  

(169)

where \( H_{2}(x) \triangleq -x \log_{2}(x) - (1-x) \log_{2}(1-x) \).
Proof. By using Corollary 11 and Proposition 22 (and also Corollary 4, similarly to the proof of Theorem 17), we get the following bound:

\[
\frac{1}{2} \text{tr} |\rho_{\text{ABE}} - \rho_U \otimes \rho_E| \leq \Pr \left[ (k \neq k^B) \land (T = 1) \right]
\]

\[+ 2m \sqrt{\Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_l|}{n} \geq \frac{t}{n} \right) \land (T = 1) \right] + 2^n[H_2(t/n) - (n - r - m)/n]} \]

\[\leq \Pr \left[ \left( \frac{|C_l|}{n} \geq p_a + \epsilon_{\text{rel}} \right) \land (T = 1) \right] + 2^n[H_2(p_a + \epsilon_{\text{rel}}) - r/n] \]

\[+ 2m \sqrt{\Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_x|}{N_{l_x}} \geq p_a + \epsilon_{\text{sec}} \right) \land (T = 1) \right] + 2^n[H_2(p_a + \epsilon_{\text{sec}}) - (n - r - m)/n] \]

\[\Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_l|}{N_{l_x}} \geq p_a + \epsilon_{\text{sec}} \right) \land (T = 1) \right] + \]

Proposition 23 and the definition of T give us the following bounds:

\[\Pr \left[ \left( \frac{|C_l|}{N_{l_x}} \geq p_a + \epsilon_{\text{rel}} \right) \land (T = 1) \right] \leq \Pr \left( N_{l_x} \leq \frac{pN}{2} - n_z \right) \]

\[+ \max_{\frac{1}{2} - n_x \leq t_x \leq n} \Pr \left[ \left( \frac{|C_l|}{t_z} \geq p_a + \epsilon_{\text{rel}} \right) \land \left( \frac{|C_{l_z}}{n_z} \leq p_a \right) \mid N_{l_z} = t_z \right] \]

\[\leq \Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_{l_z}|}{N_{l_z}} \geq p_a + \epsilon_{\text{rel}} \right) \land (T = 1) \right] \]

\[\leq \Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_{l_z}|}{N_{l_z}} \geq p_a + \epsilon_{\text{rel}} \right) \land (T = 1) \right] \]

\[\leq \Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_{l_x}|}{N_{l_x}} \geq p_a + \epsilon_{\text{rel}} \right) \land (T = 1) \right] \]

\[\leq \Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_{l_x}|}{N_{l_x}} \geq p_a + \epsilon_{\text{sec}} \right) \land (T = 1) \right] \]

\[\leq \Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_{l_x}|}{N_{l_x}} \geq p_a + \epsilon_{\text{sec}} \right) \land (T = 1) \right] \]

\[\leq \Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_{l_x}|}{N_{l_x}} \geq p_a + \epsilon_{\text{sec}} \right) \land (T = 1) \right] \]

\[\leq \Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_{l_x}|}{N_{l_x}} \geq p_a + \epsilon_{\text{sec}} \right) \land (T = 1) \right] \]
\[ \Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_{Iz}|}{t_z} \geq p_a + \epsilon_{rel} \right) \land \left( \frac{|C_{Tz}|}{n_z} \leq p_a \right) \mid N_{Ix} = t_x \right] \geq e^{-\frac{1}{2} N (1-p)^2} \text{.} \] (174)

For each one of Equations (171)–(174), we need to upper-bound two probabilities. For bounding the first set of probabilities, we use the results of Corollary 15:

\[ \Pr \left( \frac{|b|}{2} \leq \frac{(1-p)N}{2} \right) \leq e^{-\frac{1}{2} N (1-p)^2} \text{,} \] (175)

\[ \Pr \left( \frac{|b|}{2} \leq \frac{pN}{2} \right) \leq e^{-\frac{1}{2} N p^2} \text{.} \] (176)

We notice that \(|b| = N_{Ix} + n_x\) and \(|b| = N_{Iz} + n_z\); therefore,

\[ \Pr \left( N_{Ix} \leq \frac{(1-p)N}{2} - n_x \right) \leq e^{-\frac{1}{2} N (1-p)^2} \text{,} \] (177)

\[ \Pr \left( N_{Iz} \leq \frac{pN}{2} - n_z \right) \leq e^{-\frac{1}{2} N p^2} \text{,} \] (178)

\[ \Pr_{\text{inverted-INFO-basis}} \left( N_{Ix} \leq \frac{(1-p)N}{2} - n_x \right) \leq e^{-\frac{1}{2} N (1-p)^2} \text{,} \] (179)

\[ \Pr_{\text{inverted-INFO-basis}} \left( N_{Iz} \leq \frac{pN}{2} - n_z \right) \leq e^{-\frac{1}{2} N p^2} \text{.} \] (180)

For bounding the second set of probabilities, given specific values of \(N_{Iz} = t_z\) and \(N_{Ix} = t_x\), we use Corollary 13:

In the real protocol, the INFO-Z and TEST-Z bits are sent and measured in the \(z\) basis, while the INFO-X and TEST-X bits are sent and measured in the \(x\) basis. Therefore, the random and uniform sampling of the \(t_z\) INFO-Z bits out of the \(t_z + n_z\) bits sent in the \(z\) basis (assuming that the INFO-X and TEST-X bits have already been chosen) does not affect the bases in the real protocol; similarly, the random and uniform sampling of the \(t_x\) INFO-X bits out of the \(t_x + n_x\) bits sent in the \(x\) basis (assuming that the INFO-Z and TEST-Z bits have already been chosen) does not affect the bases in the real protocol. We note that these samplings are uniform, because the probability \(\Pr(s \mid b)\) is uniform for all the allowed values of \(b\) and \(s\). This means that we can apply Corollary 13 to both of these samplings, and we get

\[ \Pr \left[ \left( \frac{|C_{Iz}|}{t_z} \geq p_a + \epsilon_{rel} \right) \land \left( \frac{|C_{Tz}|}{n_z} \leq p_a \right) \mid N_{Iz} = t_z \right] \leq e^{-2 \left( \frac{p_a}{(t_z + n_z)} \right)^2 t_z \epsilon_{rel}^2} \text{,} \] (181)

\[ \Pr \left[ \left( \frac{|C_{Ix}|}{t_x} \geq p_a + \epsilon_{rel} \right) \land \left( \frac{|C_{Tx}|}{n_x} \leq p_a \right) \mid N_{Ix} = t_x \right] \leq e^{-2 \left( \frac{p_a}{(t_x + n_x)} \right)^2 t_x \epsilon_{rel}^2} \text{.} \] (182)

Maximizing over \(t_z\) and \(t_x\), we get:

\[
\max_{\frac{n_z}{2} - n_z \leq t_z \leq n} \Pr \left[ \left( \frac{|C_{Iz}|}{t_z} \geq p_a + \epsilon_{rel} \right) \land \left( \frac{|C_{Tz}|}{n_z} \leq p_a \right) \mid N_{Iz} = t_z \right] \leq e^{-2 \left( \frac{p_a}{n_z + n_z} \right)^2 (n_z + n_z) \epsilon_{rel}^2} \text{,} \] (183)

\[
\max_{\frac{n_x}{2} - n_x \leq t_x \leq n} \Pr \left[ \left( \frac{|C_{Ix}|}{t_x} \geq p_a + \epsilon_{rel} \right) \land \left( \frac{|C_{Tx}|}{n_x} \leq p_a \right) \mid N_{Ix} = t_x \right] \]
\[
\leq e^{-2 \left( \frac{n_z}{n_x + n_z} \right)^2 \left( \frac{(1-p)N}{2} - n_x \right) e_{\text{rel}}^2}.
\] (184)

In the hypothetical “inverted-INFO-basis” protocol, the INFO-X and TEST-Z bits are sent and measured in the \( z \) basis, while the INFO-Z and TEST-X bits are sent and measured in the \( x \) basis. Therefore, the random and uniform sampling of the \( t_x \) INFO-X bits out of the \( t_x + n_x \) bits sent in the \( x \) basis (assuming that the INFO-Z and TEST-X bits have already been chosen) does not affect the bases in the hypothetical protocol; similarly, the random and uniform sampling of the \( t_z \) INFO-Z bits out of the \( t_z + n_z \) bits sent in the \( z \) basis (assuming that the INFO-X and TEST-Z bits have already been chosen) does not affect the bases in the hypothetical protocol. We note that these samplings are uniform, because the probability \( \Pr(b) \) depends only on \( |b| \) and is invariant to permutations. This means that we can apply Corollary 13 to both of these samplings, and we get

\[
\Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_{Iz}|}{t_z} \geq p_a + \varepsilon_{\text{sec}} \right) \land \left( \frac{|C_{TX}|}{n_x} \leq p_a \right) | N_{Iz} = t_z \right] \leq e^{-2 \left( \frac{n_z}{n_x + n_z} \right)^2 \left( \frac{(1-p)N}{2} - n_x \right) e_{\text{rel}}^2},
\] (185)

\[
\Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_{Ix}|}{t_x} \geq p_a + \varepsilon_{\text{sec}} \right) \land \left( \frac{|C_{TZ}|}{n_z} \leq p_a \right) | N_{Iz} = t_x \right] \leq e^{-2 \left( \frac{n_z}{n_x + n_z} \right)^2 \left( \frac{(1-p)N}{2} - n_z \right) e_{\text{sec}}^2}.
\] (186)

Maximizing over \( t_z \) and \( t_x \), we get:

\[
\max_{\frac{1}{2}N - n_z \leq t_z \leq n} \Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_{Iz}|}{t_z} \geq p_a + \varepsilon_{\text{sec}} \right) \land \left( \frac{|C_{TX}|}{n_x} \leq p_a \right) | N_{Iz} = t_z \right] \leq e^{-2 \left( \frac{n_z}{n_x + n_z} \right)^2 \left( \frac{(1-p)N}{2} - n_z \right) e_{\text{sec}}^2},
\] (187)

\[
\max_{\frac{1}{2}N - n_x \leq t_x \leq n} \Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_{Ix}|}{t_x} \geq p_a + \varepsilon_{\text{sec}} \right) \land \left( \frac{|C_{TZ}|}{n_z} \leq p_a \right) | N_{Iz} = t_x \right] \leq e^{-2 \left( \frac{n_z}{n_x + n_z} \right)^2 \left( \frac{(1-p)N}{2} - n_z \right) e_{\text{sec}}^2}.
\] (188)

Substituting Equations (177)--(180), (183)--(184), and (187)--(188) into Equations (171)--(174), and substituting the results into Equation (170), we get the desired bound:

\[
\frac{1}{2} \text{tr} |\rho_{\text{ABE}} - \rho_U \otimes \rho_E| \leq e^{-\frac{1}{2} N p^2} + e^{-2 \left( \frac{n_z}{n_x + n_z} \right)^2 \left( \frac{(1-p)N}{2} - n_z \right) e_{\text{sec}}^2} + e^{-\frac{1}{2} N (1-p)^2} + e^{-2 \left( \frac{n_x}{n_x + n_z} \right)^2 \left( \frac{(1-p)N}{2} - n_x \right) e_{\text{rel}}^2} + 2^n[H_2(p_a + \varepsilon_{\text{sec}}) - r/n]
\]

\[
+ 2m \sqrt{e^{-\frac{1}{2} N p^2} + e^{-2 \left( \frac{n_z}{n_x + n_z} \right)^2 \left( \frac{(1-p)N}{2} - n_z \right) e_{\text{sec}}^2} + e^{-\frac{1}{2} N (1-p)^2} + e^{-2 \left( \frac{n_x}{n_x + n_z} \right)^2 \left( \frac{(1-p)N}{2} - n_x \right) e_{\text{rel}}^2} + 2^n[H_2(p_a + \varepsilon_{\text{sec}}) - (n-r-m)/n]}.
\] (189)

Similarly to the standard BB84 protocol (see Subsection 5.3), we can obtain the following upper bound on the bit-rate:

\[
R_{\text{secret}} \triangleq \frac{m}{n} < 1 - H_2(p_a + \varepsilon_{\text{sec}}) - H_2(p_a + \varepsilon_{\text{rel}}).
\] (190)
To get the asymptotic error rate threshold, we require $R_{\text{secret}} > 0$, and we get the asymptotic condition $2H_2(p_a) < 1$. This condition gives an asymptotic error rate threshold of 11%.

5.5 The “Modified Efficient BB84” Protocol

A relatively minor property of the definition of the “efficient BB84” protocol in [15] (and in Subsection 5.4) makes both the security bound and the security proof pretty complicated. In this Subsection, we describe a modified protocol that has an easier security proof. The only modification in this protocol is setting the number of INFO-Z and INFO-X bits to be fixed, rather than letting them vary probabilistically. This change simplifies the description of the protocol, because it is no longer needed to set the probability $p$ and to treat illegal choices of $b, s$ (see Remark 21); and it also simplifies the security proof, because it is no longer needed to probabilistically analyze the numbers of INFO-Z and INFO-X bits (as done in Subsection 5.4).

In the “modified efficient BB84” protocol, the strings $b$ and $s$ together define a random partition of the set of indexes $\{1, 2, \ldots, N\}$ into four disjoint sets:

- $I_Z$ (INFO-Z bits, where $s_j = 1$ and $b_j = 0$) of size $t_z$;
- $I_X$ (INFO-X bits, where $s_j = 1$ and $b_j = 1$) of size $t_x$;
- $T_Z$ (TEST-Z bits, where $s_j = 0$ and $b_j = 0$) of size $n_z$; and
- $T_X$ (TEST-X bits, where $s_j = 0$ and $b_j = 1$) of size $n_x$.

Formally, in “modified efficient BB84”, Alice and Bob agree on parameters $t_z, t_x, n_z, n_x$ (such that $N = n + n_z + n_x$ and $n = t_z + t_x$), and we choose $B = \{b \in F_2^N \mid |b| = t_z + n_x\}$ and $S_b = \{s \in F_2^N \mid (|s| = n) \wedge (s \wedge b = n_s)\}$ for all $b \in B$ (namely, it is required that there are $t_z$ INFO-Z bits, $t_x$ INFO-X bits, $n_z$ TEST-Z bits, and $n_x$ TEST-X bits). The probability distributions $\Pr(b)$ and $\Pr(s \mid b)$ are uniform (because $|b|$, which is the only parameter that affects $\Pr(b)$ in Subsection 5.4, is fixed in the modified protocol).

Alice and Bob also agree on an error rate threshold, $p_a$ (applied both to the TEST-Z bits and to the TEST-X bits). The testing function $T$ is defined as follows:

$$T(i_T \oplus j_T, b_T, s) = 1 \iff (|i_T \oplus j_T| \leq n_z \cdot p_a) \wedge (|i_X \oplus j_X| \leq n_x \cdot p_a). \quad (191)$$

Namely, the test passes if and only if the error rate on the TEST-Z bits is at most $p_a$ and the error rate on the TEST-X bits is at most $p_a$.

**Proposition 25.** For any $\varepsilon > 0$,

$$\Pr \left[ \left( \frac{|C_l|}{n} \geq p_a + \varepsilon \right) \wedge (T = 1) \right] \leq \Pr \left[ \left( \frac{|C_l|}{t_z} \geq p_a + \varepsilon \right) \wedge (T = 1) \right] + \Pr \left[ \left( \frac{|C_l|}{t_x} \geq p_a + \varepsilon \right) \wedge (T = 1) \right]. \quad (192)$$

*Equation (192) similarly applies to the hypothetical “inverted-INFO-basis” protocol, too (namely, it applies even if $\Pr$ is replaced by $\Pr_{\text{inverted-INFO-basis}}$).*

**Proof.** The same proof as Proposition 22. \qed
Theorem 26. For any \( n, t_z, t_x, n_z, n_x > 0, p_a > 0, \) and \( \varepsilon_{sec}, \varepsilon_{rel} > 0 \) such that \( t \triangleq n(p_a + \varepsilon_{sec}) \) and \( t_{rel} \triangleq n(p_a + \varepsilon_{rel}) \) are both integers, \( p_a + \varepsilon_{sec} \leq \frac{1}{2}, p_a + \varepsilon_{rel} \leq \frac{1}{2}, \) and \( n = t_z + t_x, \) it holds for the “modified efficient BB84” protocol that

\[
\frac{1}{2} \mathbb{E} \left[ |\rho_{ABE} - \rho_U \otimes \rho_E| \right] \\
\leq e^{2 \left( \frac{n \varepsilon_{rel}}{n} \right)^2 t_z^2} e^{2 \left( \frac{n \varepsilon_{rel}}{n} \right)^2 t_x^2} + 2^{n \left[ H_2(p_a + \varepsilon_{rel}) - \frac{r-n}{n} \right]} \\
+ 2m \sqrt{e^{2 \left( \frac{n \varepsilon_{sec}}{n} \right)^2 t_z^2} e^{2 \left( \frac{n \varepsilon_{sec}}{n} \right)^2 t_x^2} + 2^{n \left[ H_2(p_a + \varepsilon_{sec}) - \frac{r-n}{n} \right]}},
\]

(193)

where \( H_2(x) \triangleq -x \log_2(x) - (1-x) \log_2(1-x) \).

Proof. By using Corollary 11 and Proposition 25 (and also Corollary 4, similarly to the proof of Theorem 17), we get the following bound:

\[
\frac{1}{2} \mathbb{E} \left[ |\rho_{ABE} - \rho_U \otimes \rho_E| \right] \\
\leq \Pr \left[ (k \neq K^B) \land (T = 1) \right] \\
+ 2m \sqrt{\Pr_{inverted-INFO-basis} \left[ \left( \frac{|C_I|}{n} \geq \frac{t}{n} \right) \land (T = 1) \right] + 2^{n \left[ H_2(t/n) - \frac{n-r-m}{n} \right]}} \\
\leq \Pr \left[ \left( \frac{|C_I|}{n} \geq p_a + \varepsilon_{sec} \right) \land (T = 1) \right] + 2^{n \left[ H_2(p_a + \varepsilon_{sec}) - \frac{n-r-m}{n} \right]} \\
+ 2m \sqrt{\Pr_{inverted-INFO-basis} \left[ \left( \frac{|C_I|}{n} \geq p_a + \varepsilon_{sec} \right) \land (T = 1) \right] + 2^{n \left[ H_2(p_a + \varepsilon_{sec}) - \frac{n-r-m}{n} \right]}},
\]

(194)
For bounding these probabilities, we use Corollary 13:

In the real protocol, the INFO-Z and TEST-Z bits are sent and measured in the $z$ basis, while the INFO-X and TEST-X bits are sent and measured in the $x$ basis. Therefore, the random and uniform sampling of the $t_z$ INFO-Z bits out of the $t_z + n_z$ bits sent in the $z$ basis (assuming that the INFO-X and TEST-X bits have already been chosen) does not affect the bases in the real protocol; similarly, the random and uniform sampling of the $t_x$ INFO-X bits out of the $t_x + n_x$ bits sent in the $x$ basis (assuming that the INFO-Z and TEST-Z bits have already been chosen) does not affect the bases in the real protocol. This means that we can apply Corollary 13 to both of these samplings, and we get

$$
\Pr \left[ \left( \frac{|C_{lz}}{t_z} \geq p_a + \epsilon_{rel} \right) \land \left( \frac{|C_{tz}}{n_z} \leq p_a \right) \right] \leq e^{-2 \left( \frac{n_z}{n_z + t_z} \right)^2 t_z \epsilon_{rel}^2}, \tag{195}
$$

$$
\Pr \left[ \left( \frac{|C_{lx}}{t_x} \geq p_a + \epsilon_{rel} \right) \land \left( \frac{|C_{tx}}{n_x} \leq p_a \right) \right] \leq e^{-2 \left( \frac{n_x}{n_x + t_x} \right)^2 t_x \epsilon_{rel}^2}. \tag{196}
$$

In the hypothetical “inverted-INFO-basis” protocol, the INFO-X and TEST-Z bits are sent and measured in the $z$ basis, while the INFO-Z and TEST-X bits are sent and measured in the $x$ basis. Therefore, the random and uniform sampling of the $t_z$ INFO-Z bits out of the $t_z + n_z$ bits sent in the $z$ basis (assuming that the INFO-Z and TEST-X bits have already been chosen) does not affect the bases in the hypothetical protocol; similarly, the random and uniform sampling of the $t_x$ INFO-Z bits out of the $t_x + n_x$ bits sent in the $x$ basis (assuming that the INFO-X and TEST-Z bits have already been chosen) does not affect the bases in the hypothetical protocol. This means that we can apply Corollary 13 to both of these samplings, and we get

$$
\Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_{lz}}{t_z} \geq p_a + \epsilon_{sec} \right) \land \left( \frac{|C_{tx}}{n_x} \leq p_a \right) \right]
\leq e^{-2 \left( \frac{n_x}{n_x + t_x} \right)^2 t_x \epsilon_{sec}^2}, \tag{197}
$$

$$
\Pr_{\text{inverted-INFO-basis}} \left[ \left( \frac{|C_{lx}}{t_x} \geq p_a + \epsilon_{sec} \right) \land \left( \frac{|C_{tz}}{n_z} \leq p_a \right) \right]
\leq e^{-2 \left( \frac{n_z}{n_z + t_z} \right)^2 t_z \epsilon_{sec}^2}. \tag{198}
$$

Substituting Equations (195)–(198) into Equation (194), we get the following bound:

$$
\frac{1}{2} \text{tr} \left| \rho_{ABE} - \rho_U \otimes \rho_E \right|
\leq e^{-2 \left( \frac{n_x}{n_x + t_x} \right)^2 t_x \epsilon_{rel}^2} + e^{-2 \left( \frac{n_z}{n_z + t_z} \right)^2 t_z \epsilon_{rel}^2} + 2n[H_2(p_a + \epsilon_{rel}) - r/n]
+ 2m \sqrt{e^{-2 \left( \frac{n_x}{n_x + t_x} \right)^2 t_x \epsilon_{sec}^2} + e^{-2 \left( \frac{n_z}{n_z + t_z} \right)^2 t_z \epsilon_{sec}^2} + 2n[H_2(p_a + \epsilon_{sec}) - (n-r-m)/n]}. \tag{199}
$$

Similarly to the standard BB84 and “efficient BB84” protocols (see Subsections 5.3 and 5.4), we can obtain the following upper bound on the bit-rate:

$$
R_{\text{secret}} \equiv \frac{m}{n} < 1 - H_2(p_a + \epsilon_{sec}) - H_2(p_a + \epsilon_{rel}). \tag{200}
$$

To get the asymptotic error rate threshold, we require $R_{\text{secret}} > 0$, and we get the asymptotic condition $2H_2(p_a) < 1$. This condition gives an asymptotic error rate threshold of 11%.
6 Conclusion

To sum up, we have found a new way for proving the composable security of BB84 and of similar QKD protocols. This proof is relatively simple and is mostly self-contained.

Acknowledgments

The authors thank Eli Biham for useful discussions. The work of T.M. and R.L. was partly supported by the Israeli MOD Research and Technology Unit. The work of R.L. was also partly supported by the Canada Research Chair Program, the Technion’s Helen Diller Quantum Center (Haifa, Israel), the Government of Spain (FIS2020–TRANQI and Severo Ochoa CEX2019–000910–S), Fundació Cellex, Fundació Mir–Puig, Generalitat de Catalunya (CERCA program), and the EU NextGen Funds.

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