Proper Actions of Groupoids on $C^*$-Algebras

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Proper Actions on Spaces

- If $G$ is locally compact Hausdorff group acting on locally compact Hausdorff space $X$, define $\Phi : G \times X \to X \times X$ by $(g, x) \mapsto (gx, x)$. The action is called **proper** if $\Phi^{-1}(K)$ is compact for all compact $K \subset X \times X$.
- If $G$ acts properly on $X$ then orbit space $G \backslash X$ is locally compact Hausdorff.

**Theorem (Green 1977)**

*If $G$ acts freely and properly on a space $X$, then*

$$C_0(X) \rtimes_{lt} G \ (= C_0(X) \rtimes_{lt,r} G) \text{ is Morita equivalent to } C_0(G \backslash X).$$

- It is very compelling to try to generalize the notion of proper action to dynamical systems $(A, G, \alpha)$ for which $A$ in not necessarily commutative.
Definition (Rieffel 1990)

Let \( \alpha \) be an action of a locally compact Hausdorff group \( G \) on a \( C^*\)-algebra \( A \). We say the action is \textit{proper} if there exists a dense \( \alpha \)-invariant \(*\)-subalgebra \( A_0 \) of \( A \) such that

1. for any \( a, b \in A_0 \) the functions \( E \langle a, b \rangle (t) = \Delta(t)^{-1/2} a \alpha_t(b^*) \) and \( t \mapsto a \alpha_t(b^*) \) are integrable.

2. \( \forall \ a, b \in A_0 \ \exists! \text{ element} \ \langle a, b \rangle_D \in M(A_0)^\alpha = \{ d \in M(A) \mid dA_0 \subset A_0 \text{ and } \alpha_t(d) = d \} \), such that \( \forall \ c \in A_0 \) we have

\[
\int_G c \alpha_t(a^*b) dt = c \langle a, b \rangle_D.
\]
Morita Equivalence

- This definition sets up a bimodule structure on $A_0$ and gives candidates for inner products.
- (Rieffel 1990) $A_0$ does indeed complete to an imprimitivity bimodule.
- Rieffel defined $A^\alpha := \text{span} \{ \langle a, b \rangle_D | a, b \in A_0 \}$ to be the generalized fixed point algebra for the action.

**Theorem (Rieffel 1990)**

Let $G$ act properly on a $C^*$-algebra $A$ (via $\alpha$) with respect to the dense subalgebra $A_0$. Then the generalized fixed point algebra for this action $A^\alpha$ is Morita equivalent to an ideal, $I$, of $A \rtimes_{\alpha,r} G$.

- Rieffel’s theorem most closely resembles Green’s (and is most useful) when $I = A \rtimes_{\alpha,r} G$. Rieffel calls this situation **saturated**.
A groupoid can be defined as a small category with inverses. Groupoids are like groups except that the multiplication is only partially defined. Every groupoid $G$ has a unit space $G^{(0)}$ whose elements act trivially.

Every groupoid has two surjections $r, s : G \to G^{(0)}$ such that $r(\gamma)\gamma = \gamma$ and $\gamma s(\gamma) = \gamma$.

Groupoids generalize groups, group actions and equivalence relations.
A topological groupoid is a groupoid endowed with a topology making the groupoid operations continuous.

This implies that \( r, s : G \to G^{(0)} \) are continuous.

All groupoids in this talk are assumed to be locally compact Hausdorff.

There is a natural (continuous) action of groupoid \( G \) on its unit space \( G^{(0)} \) given by

\[
\gamma \cdot s(\gamma) = r(\gamma).
\]

I will assume throughout this talk that each groupoid is endowed with a Haar system \( \{ \lambda_u \}_{u \in G^{(0)}} \).

Note: Not all groupoids have Haar systems.
Motivation
Objects
Results

Path Groupoid (Kumjian et. al. 1997)

Suppose that $E = (E^0, E^1, r, s)$ is a directed graph. Let $P(E)$ be the set of infinite paths in $E$.

- For $x, y \in P(E)$, $x \sim_k y$ if $x_i = y_{i+k}$ eventually.
- $G(E) := \{(x, k, y) \in P(E) \times \mathbb{Z} \times P(E) : x \sim_k y\}$.
- $G(E)$ is a groupoid under the operations:
  \[ (x, k, y) \cdot (y, l, z) := (x, k + l, z) \]
  \[ (x, k, y)^{-1} := (y, -k, x) \]
  \[ r_G(x, k, y) = (x, 0, x) \quad s_G(x, k, y) = (y, 0, y). \]

- So the unit space of $G(E)$ can be identified with $P(E)$.
- The sets $Z(\alpha, \beta) := \{(x, |\beta| - |\alpha|, y) : x = \alpha x', \ y = \beta x'\}$ form a basis for a locally compact Hausdorff topology for $G(E)$, such that counting measure defines a Haar system for $G(E)$. 
Upper Semicontinuous $C^*$-Bundles

**Definition (Upper Semicontinuous $C^*$-Bundle)**

An *upper semicontinuous (usc) $C^*$-bundle* $\mathcal{A}$ over $X$ is a continuous surjection $p : \mathcal{A} \to X$ such that $A(x) := p^{-1}(x)$ is a $C^*$-algebra for every $x \in X$ and such that the norms of $A(x)$ vary upper semicontinuously.

**Theorem (Hofmann, Dupré & Gillette)**

There is a one to one correspondence between upper semicontinuous $C^*$-Bundles over $X$ and $C_0(X)$-algebras, given by $A = \Gamma_0(X, \mathcal{A})$, the set of continuous sections of $\mathcal{A}$ vanishing at infinity.
Definition (Groupoid Dynamical System)

Suppose $G$ is a groupoid and $\mathcal{A}$ is an upper semicontinuous $C^*$-bundle over $G^{(0)}$ and $A = \Gamma_0(G^{(0)}, \mathcal{A})$, then an action of $G$ on $A$ is a family of $*$-isomorphisms $\{\alpha_\gamma\}_{\gamma \in G}$ such that

- $\alpha_\gamma : A(s(\gamma)) \to A(r(\gamma))$,
- $\alpha_\gamma \alpha_\eta = \alpha_{\gamma \eta} \ \forall \ \gamma, \eta$ such that $s(\gamma) = r(\eta)$,
- the map $(\gamma, a) \mapsto \alpha_\gamma(a)$ is continuous on $G \ast \mathcal{A}$.
**Definition of Proper Actions**

**Definition (B 2008)**

Let $\alpha$ be an action of a locally compact Hausdorff groupoid $G$ on a upper semicontinuous $C^*$-bundle $\mathcal{A}$. Let $A = \Gamma_0(G^{(0)}, \mathcal{A})$. We say the action is **proper** if $\exists$ a dense $*$-subalgebra $A_0$ of $A$ such that

1. for any $a, b \in A_0$ the function $E \langle a, b \rangle (\gamma) = a(r(\gamma))\alpha_\gamma(b(s(\gamma)))^*$ is integrable.

2. $\forall$ $a, b \in A_0$ $\exists$! element $\langle a, b \rangle_D \in M(A_0)^\alpha$ where

$$M(A_0)^\alpha = \{d \in M(A) | dA_0 \subset A_0 \text{ and } \alpha_\gamma(d(s(\gamma))) = d(r(\gamma))\}$$

such that $\forall$ $c \in A_0$ we have

$$\int_G c(r(\gamma))\alpha_\gamma(a^* b(s(\gamma)))d\lambda^u(\gamma) = c \langle a, b \rangle_D (u).$$
Just as in the group case the trick here is to show that $A_0$ with the actions and inner products defined on the previous slide completes to an imprimitivity bimodule.

Analogously, I call $A^\alpha := \text{span}\{\langle a, b \rangle_D \mid a, b \in A_0\}$ the generalized fixed point algebra.

**Theorem (B 2008)**

Let $G$ be a groupoid acting properly on a $C^*$-algebra $A$ (via $\alpha$) with respect to the subalgebra $A_0$. Then the generalized fixed point algebra $A^\alpha$ for this action is Morita equivalent to a subalgebra, $E$, of the reduced crossed product.

Analogously to Rieffel theorem, I am most interested in the case when $E = A \rtimes_{\alpha,r} G$. Similarly, I call this situation *saturated*. 
(B 2008) If a groupoid $G$ acts freely and properly on $G^{(0)}$ then the action of $G$ on $C_0(G^{(0)})$ is proper and saturated (with respect to the dense subalgebra $C_c(G^{(0)})$).

- In this case $A^\alpha = C_0(G \setminus G^{(0)})$.
- The proof of saturation uses an approximate unit argument, following [Rieffel 1982].
- This implies $C_0(G^{(0)}) \rtimes_{lt,r} G \sim C_0(G \setminus G^{(0)})$.

(B 2008) More generally if $G$ as above acts on $A$ then the action of $G$ on $A$ is proper and saturated (with respect to the dense subalgebra $C_c(G^{(0)}) \cdot A$).

- The proof of saturation uses a partition of unity argument, following [an Huef et al 2005].
- Uses a new averaging argument to overcome the fact that translations of open sets in groupoids are not necessarily open.
Note: $C_0(G^{(0)}) \rtimes_r G \cong C_r^*(G)$.

Thus if $G$ acts freely and properly on $G^{(0)}$ then $C_r^*(G)$ is Morita equivalent to $C_0(G \backslash G^{(0)})$, hence $C_r^*(G)$ has continuous trace with trivial Dixmier-Douady invariant.

This recovers (Muhly Williams 1990) since (A-D R 2000) implies that if $G$ acts freely and properly on $G^{(0)}$ then $G$ is amenable.
Examples of Proper Principle Groupoids

**Proposition**

Let $E$ be a row finite directed graph, then its path groupoid $G(E)$ acts freely and properly on $P(G)$ if $E$ contains no directed cycles and only finitely many undirected cycles.

Thus if $E$ is a row finite directed graph with no directed cycles and only finitely many undirected cycles then $C^*(E) (= C^*(G(E)))$ is Morita equivalent to $C_0(P(E)/ \sim)$.

If $R \subset X \times X$ is a closed equivalence relation then $R$ is a groupoid under the operations:

\[
(x, y) \cdot (y, z) = (x, z) \quad (x, y)^{-1} = (y, x)
\]

\[
r(x, y) = (x, x) \quad s(x, y) = (y, y).
\]

In this case $R$ acts freely and properly on its unit space.
I would like to find a condition characterizing when the path groupoid is principle and proper.

I would like to find more interesting examples of proper groupoid actions.

Continuous Trace

The simple application of my theory to show $C^*(G)$ has continuous trace suggests there may be an application of proper actions to show continuous trace for a more general class of groupoid crossed product $C^*$-algebras.