Drinfeld Modular Polynomials in Higher Rank

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Abstract

We study modular polynomials classifying cyclic isogenies between Drinfeld modules of arbitrary rank over the ring $\mathbb{F}_q[T]$.

1 Introduction and Results

Let $F$ be an algebraically closed field which contains the ring $A := \mathbb{F}_q[T]$. We denote by $\text{End}_{\mathbb{F}_q}(\mathbb{G}_a,F)$ the ring of $\mathbb{F}_q$-linear endomorphisms of the additive group over $F$, it is isomorphic to the non-commutative ring of $\mathbb{F}_q$-linear polynomials in $X$ with coefficients in $F$ and multiplication defined by composition of polynomials. A Drinfeld module $\rho : A \to \text{End}_{\mathbb{F}_q}(\mathbb{G}_a,F)$ of rank $r$ in generic characteristic is uniquely determined by

$$\rho(T)(X) = TX + g_1(\rho)X^q + \cdots + g_{r-1}(\rho)X^{q^{r-1}} + \Delta(\rho)X^{q^r}$$

with coefficients $g_1(\rho), \ldots, g_{r-1}(\rho), \Delta(\rho) \in F$ and $\Delta(\rho) \neq 0$. We refer the reader to [4, chapter 4] for an overview of Drinfeld modules.

The coefficients $g_1(\rho), \ldots, g_{r-1}(\rho), \Delta(\rho)$ describe the isomorphism class of $\rho$ in the following way (see [7]). Let $g_1, \ldots, g_{r-1}, \Delta$ be indeterminants and define

$$j_k = \frac{g_k^{(q^r-1)/(q^{\gcd(k,r)-1})}}{\Delta^{(q^k-1)/(q^{\gcd(k,r)-1})}} \text{ for } k = 1, \ldots, r-1.$$ 

Let in addition $u_1, \ldots, u_{r-1}$ satisfy the Kummer equations

$$u_k^{(q^r-1)/(q^{\gcd(k,r)-1})} = j_k \text{ for } k = 1, \ldots, r-1.$$ 

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Consider the ring $A[u_1, \ldots, u_{r-1}]$. The group $G = \mathbb{F}_{q^r}^*/\mathbb{F}_q^*$ acts on it by

$$u_k^\beta = \beta^{q^k-1} \cdot u_k \text{ for } \beta \in \mathbb{F}_{q^r}^*.$$  

The subring of invariant elements, which is denoted by $A[u_1, \ldots, u_{r-1}]^G$, plays the crucial part in the following isomorphism problem:

Two Drinfeld modules $\rho$ and $\tilde{\rho}$ are isomorphic over $F$ if and only if $I(\rho) = I(\tilde{\rho})$ for each invariant $I \in A[u_1, \ldots, u_{r-1}]^G$. In other words the affine space

$$M^r = \text{Spec}(A[u_1, \ldots, u_{r-1}]^G)$$

is the coarse moduli space for Drinfeld modules of rank $r$ and no level structure.

Let $C_\infty$ denote the completion of an algebraic closure of $\mathbb{F}_q((\frac{1}{T}))$, and denote by

$$\Omega^r = \mathbb{P}^{r-1} \setminus \{\text{Linear subvarieties defined over } \mathbb{F}_q((\frac{1}{T}))\}$$

the Drinfeld upper half-space. Then the moduli space $M^r$ is given analytically by (see [2])

$$M^r(C_\infty) \cong \text{GL}_r(A) \backslash \Omega^r.$$  

Let again $\rho : A \to \text{End}_{\mathbb{F}_q}(\mathbb{G}_{a,r})$ be a Drinfeld module of rank $r$. For $n \in A$ consider all the isogenies $\rho \to \rho_j$, $j \in J(n)$, which are cyclic of degree $n$, i.e. whose kernel is isomorphic to $A/nA$. The invariants $I(\rho_j)$ for $I \in A[u_1, \ldots, u_{r-1}]^G$ are algebraic over $A[u_1, \ldots, u_{r-1}]^G$, and again one gets indeterminants $I_j$ such that $I_j(\rho) = I(\rho_j)$.

Let $I \in A[u_1, \ldots, u_{r-1}]^G$ be an invariant, then the polynomial

$$P_{I,n}(X) := \prod_{j \in J(n)} (X - I_j)$$

is called the modular polynomial of the invariant $I$ and the level $n$.

Let $f \in F[u_1, \ldots, u_{r-1}]$, we denote by $w(f)$ the weighted degree of $f$, where each monomial is assigned the weight

$$w(u_1^{\alpha_1} \cdots u_{r-1}^{\alpha_{r-1}}) := \sum_{k=1}^{r-1} \alpha_k \frac{q^k - 1}{q^r - 1}.$$  

Let $K$ be the quotient field of $A$, i.e. $K = \mathbb{F}_q(T)$. For a polynomial $n \in A$ we denote $|n| = q^{\deg n}$.

The aim of this paper is to prove the following result:
Theorem 1.1 Let $I \in A[u_1, \ldots, u_{r-1}]^G$ be an invariant of weighted degree $w(I)$, and $n \in A$ monic. Then

1. We have $P_{I,n}(X) \in K[u_1, \ldots, u_{r-1}]^G[X]$, which has degree

$$\# J(n) = |n|^{r-1} \prod_{p|n} \frac{|p|^r}{|p|^r - |p|^{r-1}}$$

in $X$, and is irreducible in $\mathbb{C}_\infty[u_1, \ldots, u_{r-1}][X]$.

2. The weighted degree of the coefficient $a_i \in K[u_1, \ldots, u_{r-1}]^G$ of $X^i$ in $P_{I,n}(X)$ is bounded by:

$$w(a_i) \leq \left( |n|^{2(r-1)} \prod_{p|n} \frac{|p|^r}{|p|^r - |p|^{r-1}} - i \right) w(I).$$

Example. When $r = 2$ we need only the usual $j$-invariant,

$$j = u_1^{q+1} = \frac{g_1^{q+1}}{\Delta}.$$  

We have

$$P_{j,n}(X) = \Phi_n(X, j) = \Phi_n(X, u_1^{q+1}),$$

where $\Phi_n(X, Y) \in A[X, Y]$ is the modular polynomial of level $n$ constructed by Bae [1].

Outline of the paper. We define in §2.2 a parameter $q_{\Lambda_{r-1}}(z_r)$, which may be viewed as a local parameter at the cusp of the compactification of $M^r$, although we make no use of this interpretation. Section 2 is devoted to computing the expansions of various lattice invariants in terms of this parameter.

In Section 3 we study sublattices with cyclic quotients, as these correspond to cyclic isogenies. After expanding invariants of sublattices in terms of the parameter, we count the number of such sublattices in §3.2, prove a non-cancelation result for pole orders in §3.3 and finally prove Theorem 1.1 in §3.4.

2 Lattice invariants

In this section we compute various lattice invariants and express them as Laurent series in a parameter “at the cusp”.

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2.1 Eisenstein Series

Let $F = \mathbb{C}_\infty$. There is a one-to-one correspondence between Drinfeld modules $\rho$ of rank $s$ and $A$-lattices $\Lambda \subset \mathbb{C}_\infty$ of rank $s$ given by the following relation (see e.g. [4, Chapter 4]):

Let

$$e_\Lambda(z) = z \prod_{0 \neq \lambda \in \Lambda} \left( 1 - \frac{z}{\lambda} \right) = \sum_{i=0}^{\infty} e_{q^i}(\Lambda) z^{q^i}$$

be the exponential function for $\Lambda$. Then the corresponding Drinfeld module $\rho_\Lambda$ determined by

$$\rho_\Lambda(T)(X) = TX + g_1(\Lambda)X^q + \cdots + g_{s-1}(\Lambda)X^{q^{s-1}} + \Delta(\Lambda)X^{q^s}$$

satisfies the equation

$$e_\Lambda(Tz) = \rho_\Lambda(T)(e_\Lambda(z)).$$

Equation (3) and the definitions (1) and (2) yield for each $k \geq 1$ the equation

$$(T^{q^k} - T)e_{q^k}(\Lambda) = g_k(\Lambda) + \sum_{j=1}^{k-1} g_j(\Lambda)e_{q^{k-j}}(\Lambda)q^j.$$

Hence there is a polynomial $F_k \in A[X_1, \ldots X_k]$, which is independent of the lattice $\Lambda$ and which can be computed recursively, such that

$$g_k(\Lambda) = F_k(e_q(\Lambda), \ldots, e_{q^k}(\Lambda)).$$

Since $e'_\Lambda(z) = 1$, we get for the logarithmic derivative of $e_\Lambda(z)$

$$\frac{1}{e_\Lambda(z)} = \sum_{\lambda \in \Lambda} \frac{1}{z - \lambda} = \frac{1}{z} - \sum_{i=0}^{\infty} \left( \sum_{0 \neq \lambda \in \Lambda} \lambda^{-(i+1)} \right) z^i.$$

Let

$$E_k(\Lambda) = \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^k}$$

be the $k$-th Eisenstein series of $\Lambda$. $E_k(\Lambda) = 0$ if $k$ is not divisible by $q - 1$. Moreover, we have $E_{kq}(\Lambda) = E_k(\Lambda)^q$. Using the expansion (1) of $e_\Lambda(z)$ and equation (6) we get

$$\left( \sum_{i=0}^{\infty} e_{q^i}(\Lambda)z^{q^i-1} \right) \left( \sum_{j=0}^{\infty} E_j(\Lambda)z^j \right) = -1.$$

Here we define $E_0(\Lambda) = -1$. Comparing coefficients yields in particular for each $k \geq 1$

$$e_{q^k}(\Lambda) = E_{q^{k-1}}(\Lambda) + \sum_{i=1}^{k-1} e_{q^i}(\Lambda)E_{q^{k-1-i}}(\Lambda)^{q^i}. $$
Here we see that there is a polynomial $G_k \in \mathbb{F}_q[X_1, \ldots, X_k]$, which is independent of the lattice $\Lambda$ and which can be computed recursively, such that

$$e_{q^k}(\Lambda) = G_k(E_{q^{k-1}}(\Lambda), \ldots, E_{q^k-1}(\Lambda)). \quad (9)$$

Equations (5) and (9) show that for each $k \geq 1$ there is a polynomial $H_k \in A[X_1, \ldots, X_k]$ which is independent of the lattice $\Lambda$ such that

$$g_k(\Lambda) = H_k(E_{q^{k-1}}(\Lambda), \ldots, E_{q^k-1}(\Lambda)). \quad (10)$$

Remark. The expression (10) can also be obtained using recursion formulas involving $g_i$ and $E_{q^i-1}$ directly (see e.g. [2, §II.2]).

2.2 Parameter at the cusp

The following calculations are deeply influenced by Goss [5].

Let $\Lambda_{r-1}$ be an $A$-lattice in $\mathbb{C}_\infty$ of rank $r-1$ with $\Delta(\Lambda_{r-1}) = 1$. In addition let $z_r \in \mathbb{C}_\infty$ such that $\Lambda_r = \Lambda_{r-1} \oplus Az_r$ is a lattice of rank $r$. We consider

$$q_{\Lambda_{r-1}}(z_r) = \frac{1}{e_{\Lambda_{r-1}}(z_r)}$$

as a local parameter “at the cusp”. Our first aim is to expand various lattice functions of $\Lambda_r$ in terms of this parameter.

To begin with the calculations, we take the logarithmic derivative of $e_{\Lambda_{r-1}}(z_r)$ and get

$$q_{\Lambda_{r-1}}(z_r) = \frac{e'_{\Lambda_{r-1}}(z_r)}{e_{\Lambda_{r-1}}(z_r)} = \sum_{\lambda \in \Lambda_{r-1}} \frac{1}{\lambda + z_r}.$$

In this section we want to express for each $i$ the sums $\sum_{\lambda \in \Lambda_{r-1}} (\lambda + z_r)^{-i}$ in terms of $q_{\Lambda_{r-1}}(z_r)$. Let $x$ be arbitrary, we get

$$\frac{1}{e_{\Lambda_{r-1}}(x) - e_{\Lambda_{r-1}}(z_r)} = \frac{1}{e_{\Lambda_{r-1}}(x - z_r)} = \sum_{\lambda \in \Lambda_{r-1}} \frac{1}{x - z_r - \lambda}. \quad (11)$$

Using the definition of the local parameter this yields

$$-q_{\Lambda_{r-1}}(z_r) \frac{1}{1 - e_{\Lambda_{r-1}}(x) q_{\Lambda_{r-1}}(z_r)} = \sum_{\lambda \in \Lambda_{r-1}} \frac{1}{x - (z_r + \lambda)}. \quad (12)$$
The Taylor expansion in terms of $x$ of the right side of (12) equals
\[ \sum_{i=0}^{\infty} \left( - \sum_{\lambda \in \Lambda_{r-1}} \left( \frac{1}{z_r + \lambda} \right)^{i+1} \right) x^i. \]

Let the left side of (12) be given with
\[ \frac{1}{1 - e_{\Lambda_{r-1}}(x) q_{\Lambda_{r-1}}(z_r)} = \sum_{i=0}^{\infty} A_i x^i, \quad (13) \]
then comparing both sides of (12) yields
\[ \sum_{\lambda \in \Lambda_{r-1}} \left( \frac{1}{z_r + \lambda} \right)^{i+1} = q_{\Lambda_{r-1}}(z_r) A_i. \quad (14) \]

Therefore we have to expand $A_i$ in terms of the local parameter. Let as before
\[ e_{\Lambda_{r-1}}(x) = \sum_{j=0}^{\infty} e_{q^j}(\Lambda_{r-1}) x^{q^j}. \]
Then (13) says
\[ \left( \sum_{i=0}^{\infty} A_i x^i \right) \left( 1 - q_{\Lambda_{r-1}}(z_r) \sum_{j=0}^{\infty} e_{q^j}(\Lambda_{r-1}) x^{q^j} \right) = 1. \]
This yields $A_0 = 1$ and the following recurrences for $i \geq 1$
\[ A_i = q_{\Lambda_{r-1}}(z_r) A_{i-1} + q_{\Lambda_{r-1}}(z_r) \sum_{j \geq 1, q^j \leq i} A_{i-q^j} e_{q^j}(\Lambda_{r-1}). \quad (15) \]

We see that $A_i = P_i(q_{\Lambda_{r-1}}(z_r))$ where $P_i$ is a monic polynomial of degree $i$, which is divisible by $q_{\Lambda_{r-1}}(z_r)$ if $i \geq 1$ and whose coefficients are elements of $F_q[e_{q^j}(\Lambda_{r-1}) \mid q^j \leq i]$. Hence (14) can be written as
\[ \sum_{\lambda \in \Lambda_{r-1}} \left( \frac{1}{z_r + \lambda} \right)^{i+1} = q_{\Lambda_{r-1}}(z_r) P_i(q_{\Lambda_{r-1}}(z_r)). \quad (16) \]

### 2.3 $q_{\Lambda_{r-1}}(z_r)$-expansions of Eisenstein Series

In this section we want to expand the Eisenstein series $E_k(\Lambda_r)$ and the Drinfeld coefficients $g_k(\Lambda_r)$ in terms of the local parameter $q_{\Lambda_{r-1}}(z_r)$. We calculate
\[ E_k(\Lambda_r) = E_k(\Lambda_{r-1}) + \sum_{0 \neq a \in A} \sum_{\lambda \in \Lambda_{r-1}} \left( \frac{1}{\lambda + az_r} \right)^k \]
\[ = E_k(\Lambda_{r-1}) + \sum_{0 \neq a \in A} q_{\Lambda_{r-1}}(az_r) P_{k-1}(q_{\Lambda_{r-1}}(az_r)). \quad (17) \]
using (16) with the polynomials $P_k - 1$.

Let $\rho_{\Lambda r - 1}$ be the Drinfeld module corresponding to the lattice $\Lambda r - 1$. Since $\Delta(\Lambda r - 1) = 1$ by assumption, we get for $a \in A$ with leading coefficient $l(a)$

$$\rho_{\Lambda r - 1}(a)(X) = aX + \ldots + l(a)X^{q(r - 1) \deg a},$$

where all the coefficients are elements in $A[g_1(\Lambda r - 1), \ldots, g_{r - 2}(\Lambda r - 1)]$. The fundamental relation

$$q_{\Lambda r - 1}(az_r)^{-1} = e_{\Lambda r - 1}(az_r) = \rho_{\Lambda r - 1}(a)(e_{\Lambda r - 1}(z_r)) = \rho_{\Lambda r - 1}(a)(q_{\Lambda r - 1}(z_r)^{-1})$$

yields a power series expansion

$$q_{\Lambda r - 1}(az_r) = l(a)^{-1} q_{\Lambda r - 1}(z_r)^{q(r - 1) \deg a} + \sum_{i > q(r - 1) \deg a} a_i(\Lambda r - 1) q_{\Lambda r - 1}(z_r)^i$$  (18)

where the coefficients $a_i(\Lambda r - 1)$ are in $A[g_1(\Lambda r - 1), \ldots, g_{r - 2}(\Lambda r - 1)]$. This expansion (18), equation (17) and the properties of the polynomials $P_k$ yield the following expansion of the Eisenstein series

$$E_k(\Lambda r) = E_k(\Lambda r - 1) + \sum_{i=1}^{\infty} b_i^{(k)}(\Lambda r - 1) q_{\Lambda r - 1}(z_r)^i$$  (19)

where the coefficients $b_i^{(k)}(\Lambda r - 1)$ are elements of $A[e_q(\Lambda r - 1), \ldots, e_{q^{r - 2}}(\Lambda r - 1)]$. Here we used in addition the fact, that the $g_i$ are polynomials in $e_{q^i}$ (cf. (13)).

In §2.1 (cf. (10)) we saw that the elements $g_k(\Lambda)$ are polynomials $H_k(E_{q - 1}(\Lambda), \ldots, E_{q^{r - 1}}(\Lambda))$ in the Eisenstein series, where the $H_k$’s are independent of the lattice $\Lambda$. Therefore (19) yields the expansion

$$g_k(\Lambda r) = g_k(\Lambda r - 1) + \sum_{i=1}^{\infty} c_i^{(k)}(\Lambda r - 1) q_{\Lambda r - 1}(z_r)^i$$  (20)

with coefficients $c_i^{(k)}(\Lambda r - 1) \in A[e_q(\Lambda r - 1), \ldots, e_{q^{r - 2}}(\Lambda r - 1)]$.

### 2.4 Expansions of $\Delta(\Lambda r)$

We want to expand the discriminant $\Delta(\Lambda r)$. This can be done using equation (20) and the facts $\Delta(\Lambda r) = g_r(\Lambda r)$ and $g_r(\Lambda r - 1) = 0$. Then one sees immediately that the $q_{\Lambda r - 1}(z_r)$-order of $\Delta(\Lambda r)$ is positive. To get the exact value of this order one has to evaluate the
coefficients \( c_i^{(r)}(\Lambda_{r-1}) \) in detail. We proceed in a different way, using a product expansion of \( \Delta(\Lambda_r) \) due to Gekeler [3] and Hamahata [6].

The key ingredient is the fundamental relation for each lattice \( \Lambda : \)

\[
\rho_\Lambda(T)(X) = \Delta(\Lambda) \cdot \prod_{z \in T^{-1}\Lambda/\Lambda} (X - e_\Lambda(z)). \tag{21}
\]

We get immediately for \( \Lambda = \Lambda_r \)

\[
\Delta(\Lambda_r) = T \cdot \prod_{0 \neq z \in T^{-1}\Lambda_r/\Lambda_r} \frac{1}{e_{\Lambda_r}(z)}. \tag{22}
\]

So it is enough to expand \( \prod_{0 \neq z \in T^{-1}\Lambda_r/\Lambda_r} e_{\Lambda_r}(z) \) in terms of the local parameter. It is not difficult to show (see [6, Lemma 1]) that

\[
e_{\Lambda_r}(z) = e_{\Lambda_{r-1}}(z) \prod_{0 \neq a \in A} \frac{e_{\Lambda_{r-1}}(z + az_r)}{e_{\Lambda_{r-1}}(az_r)}. \tag{23}
\]

We decompose \( z \in T^{-1}\Lambda_r/\Lambda_r \) as \( z = z' + z_r \) with \( z' \in T^{-1}\Lambda_{r-1}/\Lambda_{r-1} \) and \( \varepsilon \in \mathbb{F}_q \). In view of (22) and (23) we have to evaluate

\[
\prod_{0 \neq z \in T^{-1}\Lambda_r/\Lambda_r} e_{\Lambda_{r-1}}(z + y) = \left( \prod_{0 \neq z' \in T^{-1}\Lambda_{r-1}/\Lambda_{r-1}} e_{\Lambda_{r-1}}(z' + y) \right) \cdot \prod_{\varepsilon \neq 0} \left( \prod_{z' \in T^{-1}\Lambda_{r-1}/\Lambda_{r-1}} e_{\Lambda_{r-1}}(z' + \frac{\varepsilon}{T} z_r + y) \right)
\]

for \( y = 0 \) and \( y = az_r \). We use the fundamental relation (21) now for \( \Lambda = \Lambda_{r-1} \), where we always assume \( \Delta(\Lambda_{r-1}) = 1 \), and get

\[
\prod_{0 \neq z' \in T^{-1}\Lambda_{r-1}/\Lambda_{r-1}} e_{\Lambda_{r-1}}(z' + y) = \frac{\rho_{\Lambda_{r-1}}(T)(e_{\Lambda_{r-1}}(y))}{e_{\Lambda_{r-1}}(y)} = \frac{e_{\Lambda_{r-1}}(T y)}{e_{\Lambda_{r-1}}(y)},
\]

this equals \( T \) if \( y = 0 \), and

\[
\prod_{z' \in T^{-1}\Lambda_{r-1}/\Lambda_{r-1}} e_{\Lambda_{r-1}}(z' + \frac{\varepsilon}{T} z_r + y) = \rho_{\Lambda_{r-1}}(T)(e_{\Lambda_{r-1}}(\frac{\varepsilon}{T} z_r + y)) = e_{\Lambda_{r-1}}(\varepsilon z_r + T y).
\]

If we apply these two formulas to (24) we get with (22)

\[
\frac{1}{\Delta(\Lambda_r)} = \left( \prod_{\varepsilon \neq 0} e_{\Lambda_{r-1}}(\varepsilon z_r) \right) \cdot \prod_{0 \neq a \in A} \prod_{\varepsilon} \frac{e_{\Lambda_{r-1}}(aT + \varepsilon) z_r}{e_{\Lambda_{r-1}}(az_r)^{q-1}}
\]

\[
= - \frac{1}{q_{\Lambda_{r-1}}(z_r)^{q-1}} \cdot \prod_{0 \neq a \in A} \prod_{\varepsilon} \frac{q_{\Lambda_{r-1}}(az_r)^{q-1}}{q_{\Lambda_{r-1}}(aT + \varepsilon) z_r}. \tag{25}
\]
We have used the fact that $\prod_{\varepsilon \neq 0} \varepsilon = -1$. In equation (18) we got the expansion of $q_{\Lambda_r-1}(bz_r)$ in terms of the parameter $q_{\Lambda_r-1}(z_r)$ for each $b \in A$. If we use this, we can evaluate

$$\frac{q_{\Lambda_r-1}(a z_r) q^{r-1}}{q_{\Lambda_r-1}((a) + \varepsilon) z_r} = 1 + \sum_{i=1}^{\infty} d_i(\Lambda_{r-1}) q_{\Lambda_r-1}(z_r)^i$$

with $d_i(\Lambda_{r-1}) \in A[e_q(\Lambda_{r-1}), \ldots, e_{q^{r-2}}(\Lambda_{r-1})]$. Now (25) and (26) give the final result

$$\Delta(\Lambda_r) = -q_{\Lambda_r-1}(z_r)^{q-1} + \sum_{i=q}^{\infty} f_i(\Lambda_{r-1}) q_{\Lambda_r-1}(z_r)^i$$

with coefficients $f_i(\Lambda_{r-1}) \in A[e_q(\Lambda_{r-1}), \ldots, e_{q^{r-2}}(\Lambda_{r-1})]$.

### 2.5 Expansions of $u_k(\Lambda_r)$

Now we want to expand the invariants of Drinfeld modules in terms of the local parameter. Let as above

$$u_k^{(q^r-1)/(q^{gcd(k,r)}-1)} = j_k = \frac{g_k^{(q^r-1)/(q^{gcd(k,r)}-1)}}{\Delta^{(q^r-1)/(q^{gcd(k,r)}-1)}}.$$

For a lattice $\Lambda$ of rank $r$ we set

$$u_k(\Lambda) = \frac{g_k(\Lambda)}{\Delta^{(q^r-1)/(q^r-1)}}.$$

This definition is not canonical, it could be changed by an $(q^r-1)/(q^k-1)$-th root of unity. But since we are interested in the invariants rather than the $u_k$’s themselves, our setting is ultimately independent of the various choices.

If we combine (20) and (27) we get for $k = 1, \ldots, r-1$

$$u_k(\Lambda_r) = (-1)^{(q^k-1)/(q^r-1)} g_k(\Lambda_{r-1}) q_{\Lambda_r-1}(z_r)^{-(q-1)(q^{k-1}-1)/(q^r-1)} + \sum_{i>-(q-1)(q^{k-1})/(q^r-1)} h_i^{(k)}(\Lambda_{r-1}) q_{\Lambda_r-1}(z_r)^i$$

with coefficients $h_i^{(k)}(\Lambda_{r-1}) \in A[e_q(\Lambda_{r-1}), \ldots, e_{q^{r-2}}(\Lambda_{r-1})]$.

### 3 Sublattices

In view of the main theorem we want to expand invariants of sublattices of $\Lambda_r$ in terms of the local parameter.
3.1 Invariants of sublattices

Let \( n \in A \) be a monic polynomial and let \( \tilde{\Lambda}_r \subset \Lambda_r \) be a sublattice with cyclic quotient \( \Lambda_r/\tilde{\Lambda}_r \cong A/nA \). Then \( \tilde{\Lambda}_{r-1} := \tilde{\Lambda}_r \cap \Lambda_{r-1} \) is a sublattice of \( \Lambda_{r-1} \) with \( \Lambda_{r-1}/\tilde{\Lambda}_{r-1} \cong A/n_2A \) where \( n = n_1 \cdot n_2 \) is a decomposition into monic factors. In addition we find a basis element \( w_r = n_1 z_r + \lambda \) with \( \lambda \in \Lambda_{r-1} \) such that \( \tilde{\Lambda}_r = \tilde{\Lambda}_{r-1} \oplus A\alpha w_r \).

If we want to expand the invariants of \( \tilde{\Lambda}_r \) as in \([2.5]\) we have to normalize \( \tilde{\Lambda}_{r-1} \) such that its discriminant equals 1. Choose \( \alpha \in \mathbb{C}_\infty \) such that \( \Delta(\alpha \tilde{\Lambda}_{r-1}) = 1 \) and consider the lattice \( \alpha \tilde{\Lambda}_r = \alpha \tilde{\Lambda}_{r-1} \oplus A\alpha w_r \). Then we get with \((28)\) for \( k = 1, \ldots, r - 1 \)

\[
\begin{align*}
  u_k(\tilde{\Lambda}_r) = u_k(\alpha \tilde{\Lambda}_r) &= (-1)^{(q^k-1)/(q^r-1)} g_k(\alpha \tilde{\Lambda}_{r-1}) q_{\alpha \tilde{\Lambda}_{r-1}}(\alpha w_r)^{-(q-1)(q^k-1)/(q^r-1)} \\
  &+ \sum_{i>-(q-1)(q^k-1)/(q^r-1)} h_i^{(k)}(\alpha \tilde{\Lambda}_{r-1}) q_{\alpha \tilde{\Lambda}_{r-1}}(\alpha w_r)^i.
\end{align*}
\]

We want to find \( \alpha \), express the parameter \( q_{\alpha \tilde{\Lambda}_{r-1}}(\alpha w_r) \) in terms of \( q_{\alpha \tilde{\Lambda}_{r-1}}(z_r) \) and compute the coefficients \( g_k(\alpha \tilde{\Lambda}_{r-1}) \), \( h_i^{(k)}(\alpha \tilde{\Lambda}_{r-1}) \) with formulas involving \( \Lambda_{r-1} \).

Since \( \Lambda_{r-1}/\tilde{\Lambda}_{r-1} \cong A/n_2A \), we have \( n_2 \Lambda_{r-1} \subset \tilde{\Lambda}_{r-1} \). We consider the polynomial

\[
P(X) = n_2 X \prod_{0 \neq \lambda \in n_2^{-1} \tilde{\Lambda}_{r-1}/\Lambda_{r-1}} \left( 1 - \frac{X}{e_{\Lambda_{r-1}}(\lambda)} \right).
\]

This is a polynomial of degree \( q^{(r-2) \deg n_2} = |n_2^{r-2}| \), and its coefficients are elements of \( A[q_{\Lambda_{r-1}}(\lambda) \mid \lambda \in n_2^{-1} \Lambda_{r-1}] \). These values \( q_{\Lambda_{r-1}}(\lambda) \) are independent of \( q_{\Lambda_{r-1}}(z_r) \). \( P(X) \) describes the isogeny corresponding to \( n_2 \Lambda_{r-1} \subset \tilde{\Lambda}_{r-1} \) and as usual (cf. \([1, \S 4.7]\) we get

\[
P(e_{\Lambda_{r-1}}(X)) = e_{\tilde{\Lambda}_{r-1}}(n_2 X)
\]

and

\[
P(\rho_{\Lambda_{r-1}}(T)(X)) = \rho_{\tilde{\Lambda}_{r-1}}(T)(P(X)).
\]

Let \( c_p \) be the leading coefficient of \( P(X) \), then comparing leading coefficients in \((31)\) yields

\[
\Delta(\tilde{\Lambda}_{r-1}) = c_p^{-q^{r-1}+1}.
\]

Hence we take \( \alpha = c_p^{-1} \) and calculate

\[
\Delta(\alpha \tilde{\Lambda}_{r-1}) = \alpha^{-q^{r-1}+1} \Delta(\tilde{\Lambda}_{r-1}) = 1.
\]
Since \( q_\Lambda(z) = e_\Lambda(z)^{-1} \), equation (30) can be written as
\[
P \left( \frac{1}{q_{\Lambda r^{-1}}(X)} \right) = \frac{1}{q_{\Lambda r^{-1}}(n_2 X)},
\]
which yields an expansion
\[
q_{\Lambda r^{-1}}(n_2 X) = \alpha q_{\Lambda r^{-1}}(X)^{q(r-2)\deg n_2} + \sum_{i > q(r-2)\deg n_2} k_i q_{\Lambda r^{-1}}(X)^i
\]
where \( k_i \in A[q_{\Lambda r^{-1}}(\alpha w_r)] \). We apply this formula to the parameter \( q_{\alpha \Lambda r^{-1}}(\alpha w_r) \) and get
\[
q_{\alpha \Lambda r^{-1}}(\alpha w_r) = \alpha^{-1} q_{\Lambda r^{-1}}(w_r) = \alpha^{-1} q_{\Lambda r^{-1}}(n_2 \frac{w_r}{n_2}) = q_{\Lambda r^{-1}}(n_2 \frac{w_r}{n_2}) + \sum_{i > q(r-2)\deg n_2} \alpha^{-1} k_i q_{\Lambda r^{-1}}(n_2 \frac{w_r}{n_2})^i.
\]
(32)

Since \( w_r = n_1 z_r + \lambda \) with \( \lambda \in \Lambda_{r-1} \), we get
\[
q_{\Lambda r^{-1}}(n_2 \frac{w_r}{n_2}) = \left( q_{\Lambda r^{-1}}(n_1 \frac{z_r}{n_2})^{-1} + q_{\Lambda r^{-1}}(\frac{\lambda}{n_2})^{-1} \right)^{-1}
\]
\[
= q_{\Lambda r^{-1}}(n_1 \frac{z_r}{n_2}) + \sum_{i > 1} l_i q_{\Lambda r^{-1}}(n_1 \frac{z_r}{n_2})^i.
\]
(33)

On the other hand we calculate with formula (18)
\[
q_{\Lambda r^{-1}}(\frac{z_r}{n})^{q(r-1)\deg n_2^2} + \sum_{i > q(r-1)\deg n_2^2} a_i(\Lambda_{r-1}) q_{\Lambda r^{-1}}(\frac{z_r}{n})^i.
\]
(34)

Now applying (33) and (34) to (32) we get
\[
q_{\alpha \Lambda r^{-1}}(\alpha w_r) = q_{\Lambda r^{-1}}(\frac{z_r}{n})^{n_1^{2r-2}n_2^{r-2}} + \sum_{i > |n_1^{2r-2}n_2^{r-2}|} m_i q_{\Lambda r^{-1}}(\frac{z_r}{n})^i
\]
(35)

with coefficients \( m_i \in A[q_{\Lambda r^{-1}}(\lambda) \mid \lambda \in n_2^{-1} \Lambda_{r-1}] \).

The coefficients \( g_k(\alpha \Lambda r^{-1}) \) and \( h_i^{(k)}(\alpha \Lambda r^{-1}) \) in (29) are elements of the ring \( A[e_q(\alpha \Lambda r^{-1}), \ldots, e_{q^{r-2}}(\alpha \Lambda r^{-1})] \). Equation (30) shows that this is a subring of \( A[e_q(\Lambda r^{-1}), \ldots, e_{q^{r-2}}(\Lambda r^{-1})][q_{\Lambda r^{-1}}(\lambda) \mid \lambda \in n^{-1} \Lambda_{r-1}] \).
Proposition 3.1 We have

\[ u_k(\tilde{\Lambda}_r) = (-1)^{(q^k-1)/(q'-1)} g_k(\alpha \tilde{\Lambda}_r-1) q_{\Lambda_r-1}(\tilde{z}_r) - n_{1}^{2^{r-2} n_{2}^{r-2}} (q-1)(q^k-1)/(q'-1) \]

\[
+ \sum_{i>-|n_{1}^{2^{r-2} n_{2}^{r-2}} (q-1)(q^k-1)/(q'-1)}} r_i^{(k)}(\alpha \tilde{\Lambda}_r-1) q_{\Lambda_r-1}(\tilde{z}_r)^i \tag{36}
\]

This remark and equations (29) and (35) show that \( u_k(\tilde{\Lambda}_r) \) can be expanded in a series

\[ u_k(\tilde{\Lambda}_r) = (-1)^{(q^k-1)/(q'-1)} g_k(\alpha \tilde{\Lambda}_r-1) q_{\Lambda_r-1}(\tilde{z}_r) - n_{1}^{2^{r-2} n_{2}^{r-2}} (q-1)(q^k-1)/(q'-1) \]

\[
+ \sum_{i>-|n_{1}^{2^{r-2} n_{2}^{r-2}} (q-1)(q^k-1)/(q'-1)}} r_i^{(k)}(\alpha \tilde{\Lambda}_r-1) q_{\Lambda_r-1}(\tilde{z}_r)^i \tag{36}
\]

with coefficients \( g_k(\alpha \tilde{\Lambda}_r-1) \) and \( r_i^{(k)}(\alpha \tilde{\Lambda}_r-1) \) in the ring

\[ A[e_q(\Lambda_r-1), \ldots, e_{q^r-2}(\Lambda_r-1)] [q_{\Lambda_r-1}(\lambda) | \lambda \in n^{-1} \Lambda_{r-1}] \]

3.2 Counting sublattices

Let \( \Lambda_r \) be an \( A \)-lattice of rank \( r \). For \( n \in A \) we denote by \( f(n, r) \) the number of sublattices \( \tilde{\Lambda}_r \subset \Lambda_r \) with \( \Lambda_r/\tilde{\Lambda}_r \simeq A/nA \). Then we get

**Proposition 3.1** We have

\[ f(n, r) = |n|^{r-1} \prod_{p|n} \frac{|p|^r - 1}{|p|^r - |p|^{r-1}} \]

where the product is taken over all monic irreducible divisors \( p \) of \( n \), and where \( |m| = q^{\deg m} \) for each \( m \in A \).

**Proof.** It is obvious that \( f(n, r) \) is multiplicative, i.e. \( f(n, r) \cdot f(m, r) = f(n \cdot m, r) \) if \( \gcd(n, m) = 1 \). We use induction to prove the formula for \( n = p^s \). For \( s = 1 \) we have to count the number of upper triangular matrices \( M \) with coefficients in \( A \) and determinant \( p \) which are in reduced form (this means that the only non-zero entries \( x \) above the diagonal lie above the \( p \) and satisfy \( |x| < |p| \)). We get

\[ f(p, r) = \sum_{i=1}^{r} |p|^{i-1} = |p|^{r-1} \frac{|p|^r - 1}{|p|^r - |p|^{r-1}}. \]

Let \( s \geq 2 \) and let \( \tilde{\Lambda}_r \subset \Lambda_r \) with \( \Lambda_r/\tilde{\Lambda}_r \simeq A/p^sA \). Then there is a unique lattice \( \Lambda'_r \) with \( \tilde{\Lambda}_r \subset \Lambda'_r \subset \Lambda_r \) and \( \Lambda_r/\Lambda'_r \simeq A/p^{s-1}A \). On the other hand if \( \Lambda'_r \) is any sublattice of \( \Lambda_r \) with \( \Lambda_r/\Lambda'_r \simeq A/p^{s-1}A \), we can choose bases of \( \Lambda_r \) and \( \Lambda'_r \) such that their connecting matrix is diagonal of the form \( \text{diag}(1, \ldots, 1, p^{s-1}) \). Any upper triangular matrix \( M_p \) of determinant \( p \) in reduced form gives a sublattice \( \tilde{\Lambda}_r \) of \( \Lambda'_r \) of index \( p \) with connecting matrix \( M_p \cdot \text{diag}(1, \ldots, 1, p^{s-1}) \) to the chosen basis of \( \Lambda_r \). One sees immediately that
\[ \Lambda_r / \tilde{\Lambda}_r \cong A/p^s A \text{ if and only if the } (r, r)\text{-th coefficient of } M_p \text{ is equal to } p. \] Hence we have \(|p|^{r-1}\) choices for \(M_p\). Thus we get by induction
\[
f(p^s, r) = |p|^{r-1} f(p^{s-1}, r) = |p|^{r-1} \frac{|p|^{r-1} - 1}{|p|^r - |p|^{r-1}}
\]
which proves the formula for \(n = p^s\).

\[\square\]

### 3.3 Non-cancelation

The coefficients of the modular polynomial \(P_{l,n}(X)\) are polynomials in the basic invariants. We want to study the weighted degree of these polynomials by comparing it with its order as a series expansion in the local parameter. Therefore we need to show that the leading terms in these series do not cancel. For this we will need the following technical lemma.

We consider the homothety classes of lattices \(\Lambda_s\) of rank \(s\) as points in the moduli space \(M^s(\mathbb{C}_\infty)\) equipped with the analytic topology, i.e. where closed sets are the zero-loci of sets of analytic functions.

For \(k = 1, \ldots, r-1\) let \(v_k\) be functions on \(M^r(\mathbb{C}_\infty)\); suppose that their \(q_{\Lambda_{r-1}}(\frac{z}{n})\)-expansions are of the form
\[
v_k(\Lambda_{r-1} + Az_r) = a_k(\Lambda_{r-1}) q_{\Lambda_{r-1}}(\frac{z}{n})^{c(q^k-1)/(q^r-1)} + \text{higher terms},
\]
where the \(a_k\)'s are algebraically independent analytic functions on \(M^{r-1}(\mathbb{C}_\infty)\) and where \(c\) does not depend on \(\Lambda_{r-1}\) or on \(k\).

Let \(f \in \mathbb{C}_\infty[X_1, \ldots, X_{r-1}]\) be a polynomial of weighted degree \(w(f)\), where the weighted degree of a monomial is given by \(w(X_1^{\alpha_1} \cdots X_{r-1}^{\alpha_{r-1}}) = \sum_{k=1}^{r-1} \alpha_k \frac{q^k-1}{q^r-1}\).

**Lemma 3.2** There exists a non-empty open subset \(S \subset M^{r-1}(\mathbb{C}_\infty)\) such that for any \(\Lambda_{r-1} \in S\) the weighted degree and the order of the \(q_{\Lambda_{r-1}}(\frac{z}{n})\)-expansion satisfy
\[
\text{ord}_{q_{\Lambda_{r-1}}(\frac{z}{n})} \left( f(v_1(\Lambda_{r-1} + Az_r), \ldots, v_{r-1}(\Lambda_{r-1} + Az_r)) \right) = -c \ w(f).
\]

**Proof.** Let
\[
f(X_1, \ldots, X_{r-1}) = \sum_{(\alpha_1, \ldots, \alpha_{r-1})} b_{(\alpha_1, \ldots, \alpha_{r-1})} X_1^{\alpha_1} \cdots X_{r-1}^{\alpha_{r-1}}.
\]
Then the leading term of the \( q_{\Lambda_{r-1}}(\frac{z_r}{n}) \)-expansion of \( f(v_1(\Lambda_{r-1} + Az_r), \ldots, v_{r-1}(\Lambda_{r-1} + Az_r)) \) is given by

\[
\left( \sum_{(\alpha_1, \ldots, \alpha_{r-1})} b(\alpha_1, \ldots, \alpha_{r-1}) a_1(\Lambda_{r-1})^{\alpha_1} \cdots a_{r-1}(\Lambda_{r-1})^{\alpha_{r-1}} \right) q_{\Lambda_{r-1}}(\frac{z_r}{n})^{-cw(f)},
\]

where the sum is taken over all indices \((\alpha_1, \ldots, \alpha_{r-1})\) satisfying

\[
\sum_{k=1}^{r-1} \alpha_k \frac{q^k - 1}{q^r - 1} = w(f).
\]

The coefficient is zero only if \(a_1(\Lambda_{r-1}), \ldots, a_{r-1}(\Lambda_{r-1})\) satisfy a polynomial relation. The locus of \(\Lambda_{r-1} \in M^{r-1}(\mathbb{C}_\infty)\) for which this relation holds is a proper closed set, since the \(a_k\)'s are algebraically independent. The result follows. \(\Box\)

**Remark.** We can apply Lemma 3.2 to \(v_k = u_k\) resp. \(v_k = u_k(\tilde{\Lambda}_{r-1})\) in view of (28) resp. (36) and due to the fact that \(g_1, \ldots, g_{r-1}\) are algebraically independent on \(M^{r-1}(\mathbb{C}_\infty)\).

### 3.4 Proof of the Main Result

**Proof of Theorem 1.1.** (1) The value for the degree follows from Proposition 3.1. Let \(\rho\) be a Drinfeld module with invariant \(I\). The \(I_j\)'s correspond to cyclic submodules of \(\rho[n] \cong (A/nA)^r\) of order \(n\), which are permuted transitively by \(\text{GL}_r(A)\). It follows that the coefficients of \(P_{I,n}(X)\) are functions on \(\text{GL}_r(A)\backslash \Omega^r \cong M^r(\mathbb{C}_\infty)\), and so \(P_{I,n}(X)\) is an irreducible polynomial with coefficients in \(\mathbb{C}_\infty[u_1, \ldots, u_{r-1}]^G\).

We next show how to replace \(\mathbb{C}_\infty\) by \(K\). Let \(u'_1, \ldots, u'_{r-1} \in K\) be arbitrary. These values correspond to a Drinfeld module \(\rho\) defined over \(k\) with \(u_k = u'_k\) and invariant \(I \in K\). The absolute Galois group \(\text{Gal}(K^{\text{sep}}/K)\) permutes the set of cyclic submodules of \(\rho[n]\) of order \(n\), hence permutes the \(I_j\)'s. Thus the coefficients of \(P_{I,n}(X)\), when specialized to \((u'_1, \ldots, u'_{r-1}) \in K^{r-1}\), lie in \(K\). Since \((u'_1, \ldots, u'_{r-1}) \in K^{r-1}\) is arbitrary, it follows that the coefficients of \(P_{I,n}(X)\) lie in \(K[u_1, \ldots, u_{r-1}]^G\).

(2) Let \(\tilde{\Lambda}_r \subset \Lambda_r\) be the sublattice corresponding to \(j \in J(n)\), then \(I_j = I(\tilde{\Lambda}_r)\). Equation (36) and Lemma 3.2 show, for suitably chosen \(\Lambda_{r-1}\), that

\[
\text{ord}_{q_{\Lambda_{r-1}}(\frac{z_r}{n})}(u_k(\tilde{\Lambda}_r)) = -|n_1^{2r-2}n_2^{r-2}|(q - 1)\frac{q^k - 1}{q^r - 1}
\]

(for \(k = 1, \ldots, r - 1\)) implies

\[
\text{ord}_{q_{n_{r-1}}(\frac{z_r}{n})}(I_j) = -(q - 1)|n_1^{2r-2}n_2^{r-2}|w(I) \geq -(q - 1)|n|^{2(r-1)}w(I).
\]
Now let $a_i \in A[u_1, \ldots, u_{r-1}]^G$ be the coefficient of $X^i$ in $P_{I,n}(X)$. Then
\[ a_i = (-1)^d \sum_{(j_1, \ldots, j_d)} I_{j_1} \cdots I_{j_d}, \quad \text{where } d = \#J(n) - i. \]

Hence we get
\[ \text{ord}_{q^{\Lambda_{r-1}(z_n)}}(a_i) \geq -(q-1)(\#J(n) - i)|n|^{2(r-1)}w(I). \]  \hspace{1cm} (37)

On the other hand, since $a_i \in A[u_1, \ldots, u_{r-1}]^G$, using Lemma 3.2, (18) and (28), we get
\[ \text{ord}_{q^{\Lambda_{r-1}(z_n)}}(a_i) = |n|^{r-1}\text{ord}_{q^{\Lambda_{r-1}(z_n)}}(a_i) = -(q-1)|n|^{r-1}w(a_i). \]  \hspace{1cm} (38)

Now (37), (38) and Proposition 3.1 yield
\[ w(a_i) \leq |n|^{2(r-1)} \left( \prod_{p \mid n} \frac{|p|^r}{|p|^r - |p|^{r-1} - i} \right) w(I). \]

\[ \square \]

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