Bad (w) IS HYPERPLANE ABSOLUTE WINNING

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ABSTRACT. In 1998 Kleinbock conjectured that any set of weighted badly approximable $d \times n$ real matrices is a winning subset in the sense of Schmidt’s game. In this paper we prove this conjecture in full for vectors in $\mathbb{R}^d$ in arbitrary dimensions by showing that the corresponding set of weighted badly approximable vectors is hyperplane absolute winning. The proof uses the Cantor potential game played on the support of Ahlfors regular absolutely decaying measures and the quantitative non-divergence estimate for a class of fractal measures due to Kleinbock, Lindenstrauss and Weiss.

Dedicated to Anna Nesharim

1. Introduction

As is well known, the rational points are dense in the real space $\mathbb{R}^d$, meaning that $\mathbb{R}^d$ can be covered by cubes in $\mathbb{R}^d$ of an arbitrarily small fixed sidelength $\varepsilon > 0$ centred at rational points. Various quantitative aspects of this basic property are studied within the theory of Diophantine approximation. For instance, by Dirichlet’s theorem, $\mathbb{R}^d$ can be covered by cubes in $\mathbb{R}^d$ of sidelength $2q^{-(d+1)/d}$ centred at rational points (not necessarily written in the lowest terms) with arbitrarily large denominators $q \in \mathbb{N}$. One of the fundamental concepts studied in Diophantine approximation is that of badly approximable points. These are precisely the points in $\mathbb{R}^d$ that cannot be covered by the cubes arising from Dirichlet’s theorem when 2 is replaced by any positive constant. In the more general case one considers coverings by parallelepipeds with different sidelengths controlled by $d$ real parameters referred to as weights. This more general setup gives rise to the notion of weighted badly approximable points that will be the main object of study in this paper.

In what follows $d \in \mathbb{N}$ and $W_d$ denotes the collection of all $d$-dimensional weights:

$$W_d = \{ w = (w_1, \ldots, w_d) \in \mathbb{R}^d : w_1, \ldots, w_d \geq 0, w_1 + \ldots + w_d = 1 \}.$$ 

For $w \in W_d$, a vector $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ is called badly approximable with respect to $w$ if there exists $c > 0$ such that for every $q \in \mathbb{N}$ and $p = (p_1, \ldots, p_d) \in \mathbb{Z}^d$ there exists $1 \leq i \leq d$ satisfying

$$|x_i - \frac{p_i}{q}| \geq \frac{c}{q^{1+w_i}}.$$ 

Let $\text{Bad}(w)$ be the set of badly approximable vectors in $\mathbb{R}^d$ with respect to $w$.

One of the motivations for studying the set of weighted badly approximable vectors comes from its connection to a conjecture of Littlewood – a famous open problem from the 1930s. Let us briefly recall this connection.

Conjecture 1 (Littlewood’s conjecture, 1930s). Every $x = (x_1, x_2) \in \mathbb{R}^2$ satisfies

$$\inf_{q \in \mathbb{N}, p \in \mathbb{Z}^2} q |qx_1 - p_1| |qx_2 - p_2| = 0.$$ 

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It was noted by Schmidt [Sch83] that if $x \notin \text{Bad (w)}$ for some $w \in W_2$ then $x$ satisfies (1). In particular, if the intersection of the sets $\text{Bad (w)}$ over all $w \in W_2$ was the empty set, then Littlewood’s conjecture would follow. However, Schmidt doubted that using only two weights would be sufficient, if his observation can be used to verify (1) at all. Specifically, Schmidt formulated the following problem that has inspired many researchers in Diophantine approximation and Homogeneous dynamics.

**Conjecture 2** (Schmidt’s conjecture, 1982). For every $w_1, w_2 \in W_2$ we have that

\[ \text{Bad}(w_1) \cap \text{Bad}(w_2) \neq \emptyset. \]  

Almost three decades later Schmidt’s conjecture was verified by Badziahin, Pollington, and Velani in the tour de force [BPV11], which opened the way to many exciting new developments.

The more general version of Schmidt’s conjecture deals with arbitrary finite and, furthermore, countable intersections of $\text{Bad (w)}$. Already in [BPV11] arbitrary finite intersections were considered. In fact, the main result of [BPV11] implies that

\[ \bigcap_{n=1}^{\infty} \text{Bad}(w_n) \neq \emptyset \]

if the countably many weights $w_1, w_2, \ldots \in W_2$ satisfy the condition that

\[ \liminf_{n \to \infty} \min w_n > 0. \]

Using different techniques condition (4) was independently removed by An [An13] and the second named author [Nes13], who both established (5) for arbitrary countable intersections. Indeed, An [An13] showed a stronger dimension statement.

Schmidt’s conjecture can also be considered in higher dimensions. In this generality it was verified by the first named author [Ber15]. Similarly to the two dimensional result of [BPV11], (3) was established in [Ber15] for any sequence of weights $w_1, w_2, \ldots \in W_d$ satisfying (4). Condition (4) was finally removed by the third named author in [Yan19]. In should be noted that all the aforementioned papers go the extra mile to give a full dimension statement for the intersection appearing in (3) and enable to restrict the left hand side of (3) to non-degenerate curves and manifolds.

Two natural frameworks for proving the countable intersection property of the sets $\text{Bad (w)}$ are offered by topology and measure theory. Indeed, if $X$ is a complete metric space or a measure space and $S_1, S_2, \ldots \subseteq X$ are $G_\delta$ dense, or, respectively, full measure sets, then $\bigcap_{n=1}^{\infty} S_n$ is $G_\delta$ dense, respectively, a set of full measure, and in particular, nonempty. However, the set $\text{Bad (w)}$ is neither comeagre nor conull. In fact, $\text{Bad (w)}$ is a countable union of closed sets whose Lebesgue measure is zero, hence it is both meagre and null.

An alternative framework to establish the countable intersection property is offered by game theory. This was first articulated by Schmidt [Sch66] who introduced a variant of the Banach-Mazur game, now called Schmidt’s game, and its corresponding winning sets. Ever since other variants of Schmidt’s game have been proposed by many authors for various purposes. We refer the reader to Section 2 for the definitions of winning, absolute winning, hyperplane absolute winning (abbr. HAW) and Cantor winning, which will be mentioned throughout this introduction.
The study of winning properties of $\text{Bad}(w)$ has a long history. Schmidt proved in $[Sch66]$ that $\text{Bad}(1)$ is winning, where it was also mentioned that the analogous theorem holds for $w = w_d := \left(\frac{1}{d}, \ldots, \frac{1}{d}\right)$ for every $d$. Indeed, the full proof of this can be found in Schmidt’s monograph $[Sch80]$. McMullen $[McM10]$ proved that $\text{Bad}(1)$ is absolute winning. Later Broderick, Fishman, Kleinbock, Reich and Weiss $[BFK+12]$ proved that $\text{Bad}(w_d)$ is HAW for any $d \geq 1$.

However, the study of weighted badly approximable points turned out to be much harder. Indeed, the following natural problem that was raised by Kleinbock $[Kle98$, Section 8] over two decades ago remains open with the exception of one special case that will shortly be mentioned.

**Problem** (Kleinbock, 1998). Is it true that $\text{Bad}(w)$ is winning for every weight $w$?

The first breakthrough came about with the paper of An $[An16]$ who settled it for $d = 2$. Based on $[An16]$, Simmons and the second named author $[NS14]$ proved that $\text{Bad}(w)$ is HAW for any $w \in \mathcal{W}_2$. In higher dimensions, the only known result towards Kleinbock’s problem is due to Guan and Yu $[GY19]$ who proved that for weights $w \in \mathcal{W}_d$ satisfying the condition

$$w_1 = \cdots = w_{d-1} \geq w_d,$$

$\text{Bad}(w)$ is HAW. The goal of this paper is to resolve Kleinbock’s problem in full. Our main result read as follows.

**Theorem 3.** For any $w \in \mathcal{W}_d$ the set $\text{Bad}(w)$ is HAW. In particular, it is winning.

The HAW property implies more than just the countable intersection property. For example, we have the following corollary, which follows from Theorem 3 on applying properties of HAW sets established in $[BFK+12]$ (see Section 2 for the definition of Ahlfors regular and absolutely decaying measures).

**Corollary 4.** For any sequence of weights $w_1, w_2, \ldots \in \mathcal{W}_d$ and any sequence $f_1, f_2, \ldots$ of $C^1$ diffeomorphisms of $\mathbb{R}^d$, the set

$$\bigcap_{n=1}^{\infty} f_n(\text{Bad}(w))$$

is HAW. In particular, for every Ahlfors regular absolutely decaying measure $\mu$ on $\mathbb{R}^d$ we have that

$$\dim \left(\bigcap_{n=1}^{\infty} f_n(\text{Bad}(w)) \cap \text{supp} \mu\right) = \dim(\text{supp} \mu),$$

where $\dim$ stands for Hausdorff dimension.

Theorem 3 is proved by passing to the following equivalent formulation.

**Theorem 5.** For any $w \in \mathcal{W}_d$ and any compactly supported Ahlfors regular absolutely decaying measure $\mu$ on $\mathbb{R}^d$ we have that

$$\text{Bad}(w) \cap \text{supp} \mu \neq \emptyset.$$
Over the last two decades Schmidt’s conjecture motivated significant amount of research concerning badly approximable points in fractals, starting with Pollington and Velani [PV02] and Kleinbock and Weiss [KW05]. Initial progress towards Theorem 5 was made in [KW05] for \( w = w_d \) and in [KTV06], where (5) was proved for product measures \( \mu = \mu_1 \times \cdots \times \mu_d \) with each \( \mu_i \) being Ahlfors regular. Other notable developments include those by Fishman [Fis09] and Kleinbock and Weiss [KW10].

The tools used in the proof of Theorems 3 and 5 are the Cantor potential game which was introduced by Badziahin, Harrap, Simmons and the second named author [BHNS18], and the quantitative nondivergence estimate for “friendly” measures due to Kleinbock, Lindstrauuss and Weiss [KLW04], albeit, within this paper, the latter is only applied in the context of Ahlfors regular absolutely decaying measures.

In order to shed some light on the new ideas involved in the proof of Theorem 3 it is useful to compare the results in this paper to those of [BNY20] and several preceding publications, which deal with badly approximable points on nondegenerate curves in \( \mathbb{R}^d \). For simplicity we restrict our discussion to analytic nondegenerate curves. Let \( f : I_0 \to \mathbb{R}^d \) be an analytic nondegenerate map defined on an interval \( I_0 \subset \mathbb{R} \). By definition, this means that the coordinate functions \( f_1, \ldots, f_d \) are analytic and together with the constant function 1 are linearly independent over \( \mathbb{R} \). The map \( f \) should be understood as the parameterisation of a curve \( C \subset \mathbb{R}^d \), namely \( C = f(I_0) \).

In this case, the set \( f^{-1}(\text{Bad}(w)) \) precisely consists of the parameters \( x \in I_0 \) for which the corresponding point \( f(x) \) on the curve \( C \) is badly approximable with respect to the weight \( w \). For \( d = 2 \) Badziahin and Velani [BV14] proved that \( f^{-1}(\text{Bad}(w)) \) is Cantor winning for every \( w \in \mathcal{W}_2 \). This property was then improved to ‘winning’ by An, Velani and the first named author [ABV18]. In fact, the ‘winning’ property can be strengthened to ‘absolute winning’ on applying [Nes13, Appendix B], see also [ABV18, Remark 7]. For higher dimensions, the first named author [Ber15] proved that for every \( w \in \mathcal{W}_d \) the set \( f^{-1}(\text{Bad}(w)) \) is Cantor winning (see also [BH17, Theorem B]). This result was then improved by the third named author [Yan19] in the following manner. By Definition 18 a Cantor winning set in \( \mathbb{R}^d \) is \( \alpha \)-Cantor winning for some \( 0 \leq \alpha < d \). In [Ber15] the parameter \( \alpha \) depends on \( w \), while in [Yan19] it was shown that \( f^{-1}(\text{Bad}(w)) \) is \( \alpha \)-Cantor winning for some \( 0 \leq \alpha < d \) that depends only on \( d \). Eventually, the argument of [BNY20] strengthened the conclusions of [Yan19] to completely remove the dependence of \( \alpha \) on \( d \). While it does not do it explicitly, it does so essentially by allowing the Cantor potential game to be played on the support of any Ahlfors regular measure on \( I_0 \). By [BHNS18, Theorem 1.5] this implies that \( f^{-1}(\text{Bad}(w)) \) is absolute winning.

Organisation of this paper. In Section 2 we recall the relevant variants of Schmidt’s game, definitions in fractal measure theory and establish the equivalence between Theorem 3 and Theorem 5. In Section 3 we recall the Dani correspondence and the Kleinbock, Lindenstrauss, Weiss quantitative nondivergence. The proof of Theorem 5 is finally given in Section 4.

Notation. Throughout this paper, we will use the following notation. Given a metric space \((X, d)\), any \( S \subseteq X \) and \( r > 0 \), we denote the closed \( r \) neighborhood of \( S \) by

\[
B(S, r) := \{ x \in X : d(x, S) \leq r \}.
\]

For any \( x \in \mathbb{R}^d \), \( r > 0 \) and \( c > 0 \), we let

\[
cB(x, r) := B(x, cr).
\]
2. Schmidt games and intersections with fractals

Schmidt’s game is a quantitative version of the Banach-Mazur game played on a complete metric space. Its corresponding winning sets are dense and often have large Hausdorff dimension. Moreover, by definition, the collection of all $\alpha$-winning sets is stable under taking countable intersections, where $\alpha \in (0, 1)$ is a certain parameter of Schmidt’s games. Schmidt’s winning sets are also stable under affine transformations, although the parameter $\alpha$ may change. Schmidt’s game was innovated in [Sch66] and used to strengthen and simplify earlier results in Diophantine approximation. There are several modifications of Schmidt’s game resulting in alternative notions of winning sets. These include the notions of absolute winning sets [McM10], HAW sets [BFK+12], and Cantor winning sets [BH17], which are described below. For a detailed survey of the various winning sets, their properties and the connections between them, see [BHNS18]. To begin with, we describe the hyperplane absolute game and its winning sets.

**Definition 6** (See [BFK+12]). The hyperplane absolute game on $\mathbb{R}^d$ is played by two players, say Alice and Bob, who take turns making their moves. Bob starts by choosing a parameter $0 < \beta < 1$, which is fixed throughout the game, and a ball $B_0 \subseteq \mathbb{R}^d$ of radius $r_0 > 0$. Subsequently for $n = 0, 1, 2, \ldots$, first, Alice chooses a neighborhood $A_{n+1}$ of some hyperplane in $\mathbb{R}^d$ of radius $\beta^{n+1}r_0$; and second, Bob chooses a ball $B_{n+1}$ of radius $\beta^{n+1}r_0$ which is contained in $B_n \setminus A_{n+1}$. If there is no such ball the game stops and Alice wins by default. Otherwise, the outcome of the game is the unique point in $\bigcap_{n \geq 0} B_n$.

A set $S \subseteq \mathbb{R}^d$ is called hyperplane absolute winning (abbr. HAW) if Alice has a strategy which ensures that she either wins by default or the outcome lies in $S$.

In [BFK+12], it is proved that HAW sets are winning. In order to reduce Theorem 3 to Theorem 5, let us recall the definitions of Ahlfors regular measures and absolutely decaying measures, which can be found, for instance, in [BFK+12].

**Definition 7.** Let $X$ be a metric space. Given $\alpha > 0$, a Borel measure $\mu$ on $X$ is $\alpha$-Ahlfors regular if there exist $A, \rho_0 > 0$ such that for every $x \in \text{supp} \mu$

\begin{equation}
A^{-1}r^\alpha < \mu(B(x, r)) < Ar^\alpha \quad \text{for all } 0 < r \leq \rho_0.
\end{equation}

We say that $\mu$ is Ahlfors regular if it is $\alpha$-Ahlfors regular for some $\alpha > 0$.

**Definition 8.** A Borel measure $\mu$ on $\mathbb{R}^d$ is called absolutely decaying if there exist $D, \delta > 0$ and $\rho_0 > 0$ such that for every $x \in \text{supp} \mu$, $0 < r \leq \rho_0$, every hyperplane $H \subseteq \mathbb{R}^d$ and $r' > 0$ we have that

\begin{equation}
\mu \left( B(H, r') \cap B(x, r) \right) < D \left( \frac{r'}{r} \right)^\delta \mu(B(x, r)).
\end{equation}

The following proposition allows us to reduce Theorem 3 to Theorem 5. This proposition is already hinted in [BHNS18] remark 4.5.

**Proposition 9.** If $S \subseteq \mathbb{R}^d$ is HAW then $S \cap \text{supp} \mu \neq \emptyset$ for any Ahlfors regular absolutely decaying measure $\mu$ on $\mathbb{R}^d$. Conversely, if $S$ is Borel and $S \cap \text{supp} \mu \neq \emptyset$ for any compactly supported Ahlfors regular absolutely decaying measure $\mu$ on $\mathbb{R}^d$, then $S \subseteq \mathbb{R}^d$ is HAW.

Proposition 9 has the following equivalent formulation, stated as Proposition 11, which does not use measures and is slightly easier to prove. First, recall the following definition.
Definition 10. A nonempty closed subset $K \subseteq \mathbb{R}^d$ is called hyperplane diffuse if there exists $\beta > 0$ and $r_0 > 0$ such that for every $x \in K$, $0 < r \leq r_0$ and every hyperplane $H \subseteq \mathbb{R}^d$ we have that

$$K \cap (B(x, r) \setminus B(H, \beta r)) \neq \emptyset.$$  

(8)

Proposition 11. If $S \subseteq \mathbb{R}^d$ is HAW then $S \cap K \neq \emptyset$ for any hyperplane diffuse set $K \subseteq \mathbb{R}^d$. Moreover, if $S$ is Borel then the converse also holds.

The equivalence between Propositions 9 and 11 follows from the fact that if $\mu$ is absolutely decaying then $\text{supp} \mu$ is hyperplane diffuse [BFK+12, Proposition 5.1]. On the other hand, if $K$ is hyperplane diffuse then there exists an Ahlfors regular absolutely decaying measure $\mu$ for which $\text{supp} \mu \subseteq K$. The latter can be shown on modifying the proof of [BFK+12, Proposition 5.5], where an absolutely decaying measure $\mu$ is constructed. Formally, we have the following statement.

Proposition 12. Let $K \subseteq \mathbb{R}^d$ be hyperplane diffuse. Then there exists a compactly supported absolutely decaying Ahlfors regular measure $\mu$ on $\mathbb{R}^d$ such that $\text{supp} \mu \subseteq K$.

To summarise above discussion, in order to fully justify our claim that Theorem 3 follows from Theorem 5, it remains to give formal proofs to Propositions 11 and 12. To begin with, we deal with the former, and start by stating two auxiliary statements that will be used in the proof of Proposition 11.

Lemma 13. For any $\beta > 0$ there exists $0 < \beta' < \beta$ and $N$ such that, for every ball $B = B(x, \rho) \subseteq \mathbb{R}^d$ there is a collection of at most $N$ hyperplanes $H_B$ such that for any hyperplane $H'$ there exists $H \in H_B$ for which

$$B(x, \rho) \cap B(H', \beta' \rho) \subseteq B(H, \beta \rho).$$

Proof. The statement of this lemma is a specific case of Assumption C.6 in [FSU18], where $\beta' = \frac{2}{3}$. In the case of hyperplanes (Lemma 13) it is verified as part (2) of Observation C.7 in [FSU18].

The following is a slightly simplified version of Lemma 4.3 in [BFK+12].

Lemma 14. Let $K \subseteq \mathbb{R}^d$ be hyperplane diffuse. Then there exist $0 < \beta_0 < \frac{1}{3}$ and $r_0 > 0$ such that for any $0 < r \leq r_0$, any $x \in K$ and any hyperplane $H$ there exists $x' \in K$ such that

$$B(x', \beta_0 r) \subseteq B(x, r) \setminus B(H, \beta_0 r).$$

Proof of Proposition 11. Assume that $S$ is hyperplane absolute winning and $K$ is hyperplane diffuse. Let $(B_n)_{n\geq 0}$ be the sequence of balls arising from the absolute game, see Definition 6. Since Alice has a winning strategy, Alice can force the unique point in $\bigcap_{n \geq 0} B_n$ to lie in $S$, no matter how Bob plays. In turn, since $K$ is hyperplane diffuse, by Lemma 14 Bob can force the centres of the balls $B_n$ to lie in $K$, no matter how Alice plays. Indeed, for the latter Bob has to choose $\beta$ and the radius of $B_0$ sufficiently small so that the requirements of Lemma 14 were satisfied. Clearly, this is possible since Bob makes the first move in the game and both $\beta$ and $B_0$ are at his disposal. Consequently, since the centres of $B_n$ are all in $K$, the unique point in $\bigcap_{n \geq 0} B_n$ lies in $K$, and as we have argued above it also lies in $S$. This ensures that $S \cap K \neq \emptyset$ and completes the proof of the first part of Proposition 11.

For converse, assume that $S$ is a Borel set which is not HAW. Then, by Borel determinacy theorem for the absolute game appearing in [FLS14, Theorem 1.6], Bob
has a winning strategy, which will be fixed for the rest of the proof. Let $\beta$ and $B_0$ be chosen on the first move of Bob according to his winning strategy. Define $B_0 = \{B_0\}$ and continue by induction to construct collections of closed balls $B_n$ as follows. Given $B_n$, for every $B \in B_n$ let $H_B$ be the collection of hyperplanes arising from Lemma 13. Define $B_{n+1}(B)$ to be the collection of all of Bob’s responses according to the winning strategy while considering the hyperplanes in $H_B$ as possible moves of Alice. Note that $B_{n+1}(B)$ is always non-empty. Define
\[
(10) \quad K = \bigcap_{n \geq 0} \bigcup_{B \in B_n} B.
\]
By Lemma 13, every $x \in K$ is an outcome of the hyperplane absolute game played according to Bob’s winning strategy. Therefore, $x \not\in S$. Since $x$ is an arbitrary point of $K$, we have that $K \cap S = \emptyset$.

It is left to verify that $K$ is hyperplane diffuse. Indeed, we will show that it is $\frac{\beta\beta}{2}$ hyperplane diffuse for $\beta'$ as in Lemma 13. Assume $x \in K$, $0 < r \leq r_0$ and $H' \subseteq \mathbb{R}^d$ is a hyperplane, where $r_0$ is the radius of $B_0$. Let $n$ be the unique positive integer such that
\[
(11) \quad 2\beta^n r_0 \leq r < 2\beta^{n-1} r_0,
\]
which clearly exists since $0 < \beta < 1$. Since $x \in K$, by (10), there exists a ball $B = B(x_0, \beta^n r_0) \in B_n$ such that $x \in B$. The left hand side of (11) implies that $B \subseteq B(x, r)$. By Lemma 13 applied with $\rho = \beta^n r_0$, there exists $H \in H_B$ such that
\[
B \cap B(H', \beta\beta^n r_0) \subseteq B(H, \beta^{n+1} r_0).
\]
The right hand side of (11) implies that $\frac{\beta\beta}{2} r < \beta\beta^n r_0$ and hence
\[
B \cap B(H', \frac{\beta\beta}{2} r) \subseteq B(H, \beta^{n+1} r_0).
\]
By the definition of $B_{n+1}(B)$, there exists a ball $B' \in B_{n+1}(B)$ such that $B' \cap B(H, \beta^{n+1} r_0) = \emptyset$. Since the collections $B_{n+1}(B)$ are always non-empty, by (10), we have that $K \cap B' \neq \emptyset$. Since $\emptyset \neq K \cap B' \subseteq K \cap B \subseteq K \cap B(x_0, r)$, we have $K \cap B(x_0, r) \not\subseteq B(H', \frac{\beta\beta}{2} r)$. Hence, $K \cap \left(B(x_0, r) \setminus B(H', \frac{\beta\beta}{2} r)\right) \neq \emptyset$. This verifies Definition 10 for the set $K$ and thus completes the proof.$\square$

2.1. Proof of Proposition 12. The proof of Proposition 12 relies on a standard construction of Ahlfors regular measures in $\mathbb{R}^d$ via decreasing collections of disjoint balls. For this construction we follow [KW05, Section 7.2]. Assume that $0 < \beta < 1$, $r_0 > 0$, and that $N > 1$ is some fixed integer. Assume that $B_0$ is a closed ball and that $(B_n)_{n \geq 0}$ is a sequence of collections of closed balls such that $B_0 = \{B_0\}$, any $B \in B_n$ is a ball of radius $\beta^n r_0$, and for any $B \in B_n$ the collection
\[
B_{n+1}(B) = \{B' \in B_{n+1} : B' \subseteq B\}
\]
has exactly $N$ disjoint balls for any $n \geq 0$. Define
\[
(12) \quad K = \bigcap_{n \geq 0} \bigcup_{B \in B_n} B.
\]
Define a sequence of measures
\[
\mu_n = \frac{1}{\#B_n} \sum_{B \in B_n} \frac{\lambda|_B}{\lambda(B)}
\]
for every $n \geq 0$, where $\lambda$ is the Lebesgue measure on $\mathbb{R}^d$ and $\lambda|_B$ is the restriction of $\lambda$ to $B$ which is defined by the formula $\lambda|_B(A) = \lambda(A \cap B)$ for any Lebesgue measurable set $A$. Let $\mu$ be the weak limit of $\mu_n$ and set
\begin{equation}
\alpha = \frac{\log N}{\log \beta}.
\end{equation}

**Proposition 15.** Let $K \subseteq \mathbb{R}^d$, $\mu$ and $\alpha$ be as in the above discussion. Then $\text{supp } \mu = K$ and $\mu$ is $\alpha$-Ahlfors regular.

Proposition 15 is proved in [KW05, Proposition 7.1] for $\mathbb{R}^d$ with the supremum norm. For completeness we repeat their proof with the Euclidean norm.

**Proof.** Assume $x \in K$ and $0 < r \leq 2r_0$. Let $n$ be the unique integer for which
\begin{equation}
2\beta^{n+1}r_0 < r \leq 2\beta^n r_0. \tag{13}
\end{equation}
Since $x \in K$ there exists a unique ball $B \in \mathcal{B}_{n+1}$ such that $x \in B$. The left hand side of (13) implies that $B \subseteq B(x, r)$. So, by (12) and the right hand side of (13) this implies that
\[
\mu(B(x, r)) \geq \mu(B) = \frac{1}{N^{n+1}} = \beta^{\alpha(n+1)} \geq \left(\frac{\beta}{2r_0}\right)^\alpha r^\alpha.
\]
On the other hand, by the right hand side of (13), there exists a constant $C \geq 1$ depending only on $\beta$ and $d$ such that
\[
\# \{B \in \mathcal{B}_n : B \cap B(x, r) \neq \emptyset\} < C.
\]
Therefore, by (12) and the left hand side of (13) this implies that
\[
\mu(B(x, r)) \leq \frac{C}{N^n} = C\beta^{\alpha n} < C \left(\frac{1}{2r_0}\right)^\alpha r^\alpha.
\]
So (6) is verified with $A = \max \left\{ \left(\frac{2r_0}{\beta}\right)^\alpha, C \left(\frac{1}{2r_0}\beta\right)^\alpha \right\}$.

The proof of Proposition 12 is based on the construction described above, for a particular choice of balls which stay far from appropriate neighborhoods of hyperplanes in each level. The argument used for the proof of [BFK+12, Proposition 5.5] provides such a choice. It is based on the following lemma.

**Definition 16.** Say that $d$ points in $\mathbb{R}^d$ are in **general position** if they lie on a unique hyperplane. If $x_1, \ldots, x_d \in \mathbb{R}^d$ are in general position denote this hyperplane by $H(x_1, \ldots, x_d)$.

**Lemma 17.** [BFK+12, Lemma 5.6] Given $\beta_0 > 0$ there exists a positive $\beta' \leq \beta_0$ such that, for every $x \in \mathbb{R}^d$, $\rho > 0$, and $x_1, \ldots, x_d \in B(x, \rho)$ in general position such that the balls $B(x_i, \beta_0 \rho)$ are contained in $B(x, \rho)$ for every $1 \leq i \leq d$ and are pairwise disjoint, if a hyperplane $H$ intersects $B(x_i, \beta' \rho)$ for every $1 \leq i \leq d$ then
\[
B(x, \rho) \cap B(H, \beta' \rho) \subseteq B(H(x_1, \ldots, x_d), \beta_0 \rho).
\]

Lemma 17 is stated in [BFK+12] with the general position assumption implicit. We repeat the proof that appears in [BFK+12] for completeness.
Proof. Without loss of generality assume that $x = 0$ and $ρ = 1$. By contradiction, assume that for every integer $k \geq 1$ there are $x_{1,k}, \ldots, x_{d,k}$ in general position and a hyperplane $H_k$ that intersects $B\left(x_{i,k}, \frac{1}{k}\right)$ for each $1 \leq i \leq d$ but

$$B(0,1) \cap B\left(H_k, \frac{1}{k}\right) \not\subseteq B\left(H(x_{1,k}, \ldots, x_{d,k}), β_0\right).$$

By compactness of $B(0,1)$ there are subsequences $(x_{1,k_j}, \ldots, x_{d,k_j})$ and $H_{k_j}$ that converge, say to $(x_1, \ldots, x_d)$ and $H$ respectively. Then necessarily $x_1, \ldots, x_d \in H$ and, therefore, any $j$ large enough satisfies

$$B(0,1) \cap B\left(H_{k_j}, \frac{β_0}{3}\right) \subseteq B(0,1) \cap B\left(H(x_{1,k_j}, \ldots, x_{d,k_j}), β_0\right)
\subseteq B(0,1) \cap B\left(H(x_{1,k_j}, \ldots, x_{d,k_j}), β_0\right).$$

Choosing $j$ large enough so that it also satisfies $\frac{1}{k_j} \leq \frac{β_0}{3}$ gives a contradiction to (13). \qed

Proof of Proposition 12. We follow the proof of [BFK+12 Proposition 5.5]. Assume $K$ is hyperplane absolutely decaying. The goal is to construct an Ahlfors regular absolutely decaying measure supported on a subset of $K$. Let $β_0$ and $r_0$ be as in Lemma 13. Let $β'$ be as in Lemma 17 and set

$$β = \frac{β'}{2}.$$

Let $x_0 \in K$ be any point, and set $B_0 = B(x_0, r_0)$ and $B_0 = \{B_0\}$. Recursively construct the collections $B_{n+1}(B)$ for every integer $n \geq 0$ and every $B \in B_n$ as follows: Construct by recursion a collection of $d + 1$ points in $K \cap B$. Assume $x_1, \ldots, x_i \in K \cap B$ are already defined for some $0 \leq i \leq d$, and let $H$ be any hyperplane that passes through $x_1, \ldots, x_i$. By (9) there exists a point $x_{i+1}$ such that

$$B\left(x_{i+1}, β_0 β^n r_0\right) \subseteq B \setminus B\left(H, β_0 β^n r_0\right).$$

Define $B_{n+1}(B) = \{B\left(x_1, β^{n+1} r_0\right), \ldots, B\left(x_{d+1}, β^{n+1} r_0\right)\}$. Since $β < β_0$ this is a collection of $d + 1$ disjoint balls contained in $B$. Let $μ$ be as defined in the beginning of this section. Then $supp μ ⊆ K$ since for every $n \geq 0$ every $B \in B_n$ is a ball centered in $K$. Proposition 15 guarantees that $μ$ is Ahlfors regular. It is left to verify that $μ$ is absolutely decaying.

Assume $r \leq r_0$, $x \in supp μ$ and $r' > 0$, and let $H$ be any hyperplane. Let $n \geq 0$ be the unique integer satisfying

$$2β^{n+1} r_0 \leq r < 2β^n r_0$$

Since $x \in supp μ$ there are balls $B \subseteq B'$ with $B \in B_{n+1}$ and $B' \in B_n$ such that $x \in B$. The left hand side of (17) implies $B \subseteq B(x, r)$. On the other hand, equation (16) implies that

$$d\left(B', B''\right) \geq 2(β_0 - β) β^{n-1} r_0,$$

for any $B' \neq B'' \in B_n$, therefore, since $β = \frac{β'}{2} \leq \frac{β_0}{3}$, the right hand side of (17) implies that $B(x, r) \cap B'' = \emptyset$ for any $B' \neq B'' \in B_n$. So, $B(x, r) \cap supp μ \subseteq B'$.
It is enough to verify (17) for every $r'$ small enough. Assume that $r' < \frac{1}{2} \beta r$ and let $m \geq 1$ be the unique integer satisfying
\begin{equation}
\frac{1}{2} \beta^{m+1} r \leq r' < \frac{1}{2} \beta^m r.
\end{equation}
The right hand side of both (17) and (18) imply that $r' < \beta^{m+n} r_0$. Therefore, for every $1 \leq k \leq m$ and every $B'' \in B_{n+k-1}$, the hyperplane neighborhood $B(H, r')$ intersects at most $d$ balls from $B_{n+k}(B'')$. Indeed, recall that
\[ B_{n+k}(B'') = \left\{ B\left(x_1, \beta^{n+k-1} r_0\right), \ldots, B\left(x_d+1, \beta^{n+k-1} r_0\right) \right\}. \]
If $B(H, r') \cap B\left(x_i, \beta^{n+k-1} r_0\right) \neq \emptyset$ for every $1 \leq i \leq d+1$ then (17) implies that $H \cap B\left(x_i, \beta^{n+k-1} r_0\right) \neq \emptyset$ for every $1 \leq i \leq d+1$. By construction, the points $x_1, \ldots, x_d$ are in general position, so Lemma 17 gives
\[ B'' \cap B\left(H, \beta^k \beta^{n+k-1} r_0\right) \subseteq B\left(H(x_1, \ldots, x_d), \beta_0^k \beta^{n+k-1} r_0\right). \]
In particular, $x_{d+1} \in B\left(H(x_1, \ldots, x_d), \beta_0^k \beta^{n+k-1} r_0\right)$, which contradicts (16). The upshot is that $B(H, r')$ intersects at most $d^n m$ balls in $B_{n+m}$, therefore,
\[
\mu(B(x, r) \cap B(H, r')) \leq \left(\frac{d}{d+1}\right)^m \mu(B') = \left(\frac{d}{d+1}\right)^m (d+1) \mu(B) \leq \left(\frac{d}{d+1}\right)^m (d+1) \mu(B(x, r)).
\]
By the left hand side of (18), this verifies (7) with $\delta = \frac{\log d}{\log \beta}$ and $D = \left(\frac{2}{\beta}\right)^\delta (d+1)$.
\[\square\]

2.2. Cantor potential game. In order to prove Theorem 5 we will use the Cantor potential game introduced in [BHNS18]. The game and its corresponding winning sets are defined as follows.

**Definition 18.** Let $X$ be a complete metric space and fix $\alpha \geq 0$. The $\alpha$-Cantor potential game is played by two players, say Alice and Bob, who take turns making their moves. Bob starts by choosing a parameter $0 < \beta < 1$, which is fixed throughout the game, and a ball $B_0 \subseteq \mathbb{R}^d$ of radius $r_0 > 0$. Subsequently for $n = 0, 1, 2, \ldots$, first, Alice chooses collections $\mathcal{A}_{n+1,i}$ of at most $\beta^{-\alpha(i+1)}$ balls in $B_n$ of radius $\beta^{n+1} r_0$ for every $i \geq 0$. Then, Bob chooses a ball $B_{n+1} \subseteq B_{n+1}$ of radius $\beta^{n+1} r_0$ which is contained in $B_n$ and disjoint from $\bigcup_{0 \leq \ell \leq n} \bigcup_{A \in \mathcal{A}_{n+1-i, \ell}} A$. If there is no such ball the game stops and Alice wins by default. Otherwise, the outcome of the game is the unique point in $\bigcap_{n \geq 0} B_n$.

A set $S \subseteq X$ is called $\alpha$-Cantor winning if Alice has a strategy which ensures that she either wins by default or the outcome lies in $S$. If $X$ is the support of an $\alpha$-Ahlfors regular measure then $S \subseteq X$ is called Cantor winning if it is $\alpha'$-Cantor winning for some $0 \leq \alpha' < \alpha$.

It is proved in [BHNS18] that this definition of $\alpha$-Cantor winning sets agrees with the original definition given in [BH17]. Here the convention regarding $\alpha$ is opposite to the the one used in [BH17]. For example, in our convention $0$-Cantor winning sets are absolute winning. This convention allows the definition of the $\alpha$-Cantor
potential game to be independent of the space $X$. This comes at the price that some properties of Cantor winning subsets do depend on $X$. We will use the following fact about Cantor winning sets.

**Theorem 19** (See [BHNS18, Theorems 3.4, 4.1]). Let $X$ be the support of an $\alpha$-Ahlfors regular measure and let $S \subseteq X$ be Cantor winning. Then $S \neq \emptyset$.

We finish this section by stating an auxiliary lemma about efficient covers for Ahlfors regular measures, which will be used in Section 4.

**Lemma 20.** Let $\mu$ be an Ahlfors regular measure on $X$, let $A, \alpha, r_0$ be as in Definition 4 and let $S \subseteq X$ be any measurable set. If $\mu(B(S, r)) < \infty$ then for every $0 < r \leq r_0$ there exists a cover of $S \cap \text{supp} \mu$ with balls of radius $3r$ of cardinality at most

$$\frac{A \mu(B(S, r))}{r^\alpha}.$$  

**(Proof.)** Assume $S \subseteq X$ is measurable and let $r > 0$. Choose a finite collection of points $U \subseteq \text{supp} \mu$ such that $\{B(x, r) : x \in U\}$ are pairwise disjoint balls and

$$S \cap \text{supp} \mu \subseteq \bigcup_{x \in U} B(x, 3r).$$

Indeed, such a cover can be constructed recursively: Given $x_1, \ldots, x_n \in S \cap \text{supp} \mu$ let $x_{n+1}$ be any point in $S \cap \text{supp} \mu$ such that $B(x_{n+1}, r) \cap B(x_i, r) = \emptyset$ for every $1 \leq i \leq n$, and set $U = \{x_1, \ldots, x_n\}$ if there is no such point. By the left hand side of (19), since $B(x, r) \subseteq B(S, r)$ for every $x \in U$ and by assumption $\mu(B(S, r)) < \infty$, the construction of $U$ must end after finitely many steps. To show that $\bigcup_{x \in U} B(x, 3r)$ covers $S \cap \text{supp} \mu$, suppose that $x \in S \cap \text{supp} \mu$ satisfies $x \notin B(x', 3r)$ for every $x' \in U$. Then $B(x, r) \cap B(x', r) = \emptyset$ for every $x' \in U$, contradicting the construction of $U$.

Now, pairwise disjointness of the balls $B(x, r)$ for $x \in U$ implies that

$$\#U \times \frac{r^\alpha}{A} \leq \sum_{x \in U} \mu(B(x, r)) \leq \mu\left(\bigcup_{x \in U} B(x, r)\right) \leq \mu(B(S, r)),$$

which gives the upper bound (19) on the number of elements in $U$. \qed

**3. Homogeneous dynamics and quantitative nondivergence**

The connection between Diophantine approximation and homogeneous dynamics is well known as the Dani correspondence. In this context there is a beautiful relation between bounded orbits and badly approximable vectors. Throughout, $\text{diag}(b_1, \ldots, b_d)$ denotes the $d \times d$ diagonal matrix with diagonal entries $b_1, \ldots, b_d$.

Let $G := \text{SL}_{d+1}(\mathbb{R})$ and $\Gamma := \text{SL}_{d+1}(\mathbb{Z})$. The homogeneous space $X_{d+1} := G/\Gamma$ can be identified with the moduli space of unimodular lattices in $\mathbb{R}^{d+1}$ via the following identification:

$$g\Gamma \in X_{d+1} \mapsto g\mathbb{Z}^{d+1}.$$  

Given $w = (w_1, \ldots, w_d) \in W_d$ and $b > 1$, for any $n \in \mathbb{Z}$ we let

$$a_n := \begin{pmatrix} b^n & b^{-w_1 n} & \cdots & b^{-w_d n} \\ & & & \\ & & & \\ & & & b^{-w_d n} \end{pmatrix} \in G.$$  

(20)
Further, for any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ let

$$u_x := \begin{pmatrix} 1 & x_1 & \cdots & x_d \\ 1 & & \ddots & \\ & & & 1 \end{pmatrix} \in G.$$  \hfill (21)

For $\varepsilon > 0$ define the set

$$K_\varepsilon := \{ \Lambda \in X_{d+1} : \|v\| \geq \varepsilon \text{ for any } v \in \Lambda \setminus \{0\} \},$$

where $\|v\|$ is the Euclidean norm of $v$. Then, as is well known for any $x \in \mathbb{R}^d$ we have that

$$x \in \text{Bad} (w) \iff \exists \varepsilon > 0 \text{ such that } a_n u_x \mathbb{Z}^{d+1} \in K_\varepsilon \text{ for every } n \in \mathbb{N}. \tag{22}$$

See [BPV11, Appendix] and [Ber15, Appendix A] for detailed explanation of this equivalence.

Recall that by Mahler’s criterion, the sets $K_\varepsilon$ give a basis for the topology at $\infty$ in $X_{d+1}$, so (22) may be rephrased as $x \in \text{Bad} (w)$ if and only if $\{a_n u_x \mathbb{Z}^{d+1} : n \in \mathbb{N}\}$ is bounded in $X_{d+1}$.

It is straightforward to verify that for every $x' \in \mathbb{R}^d$ we have that

$$a_n u_{x'} a_n^{-1} = u_{\text{diag}(b^{(1+w_1)n}, \ldots, b^{(1+w_d)n})} x'. \tag{23}$$

Note that if $x = x_0 + x'$ then $u_x = u_{x'} u_{x_0}$ and therefore

$$a_n u_x \mathbb{Z}^{d+1} = a_n u_{x'} a_n^{-1} a_n u_{x_0} \mathbb{Z}^{d+1} \overset{(23)}{=} u_{\text{diag}(b^{(1+w_1)n}, \ldots, b^{(1+w_d)n})} x' a_n u_{x_0} \mathbb{Z}^{d+1}. \tag{23}$$

Thus, on letting $\Lambda = a_n u_{x_0} \mathbb{Z}^{d+1}$ and $y = \text{diag} (b^{(1+w_1)n}, \ldots, b^{(1+w_d)n}) x'$ we see that the set of parameters $y \in \mathbb{R}^d$ for which $u_y \Lambda \in K_\varepsilon$ plays a role in the study of bounded orbits of $u_x \mathbb{Z}^{d+1}$ under the actions by $a_n$. The Dani-Kleinbock-Margulis quantitative nondivergence estimate (see [Dan86, KM98]) gives a sharp and uniform upper bound on the Lebesgue measure of the set of $y$ for which $u_y \Lambda \notin K_\varepsilon$ under some conditions on the lattice $\Lambda$. Later this was generalised to “friendly” measures by Kleinbock, Lindenstrauss and Weiss [KLM04]. Within this paper we will use the following direct consequence of Theorem 5.11 in [BNY20], which in turn is a consequence of the results of [KLM04].

**Theorem 21.** Assume $\mu$ is an Ahlfors regular absolutely decaying measure on $\mathbb{R}^d$.

Then for any $z \in \text{supp} \mu$ there exists an open ball $B(z)$ centred at $z$ and constants $C, \gamma > 0$ such that for any ball $B \subset B(z)$ centred in $\text{supp} \mu$, any diagonal matrix $g \in \text{SL}_{n+1} (\mathbb{R})$ and any $0 < \rho \leq 1$ at least one of the following two conclusions holds:

(i) for all $\varepsilon > 0$

$$\mu \left( \left\{ x \in B : gu_x \mathbb{Z}^{d+1} \notin K_\varepsilon \right\} \right) \leq C \left( \frac{\varepsilon}{\rho} \right)^{\gamma} \mu (B); \tag{24}$$

(ii) there exists $v = v_1 \wedge \cdots \wedge v_j \in \wedge^j (\mathbb{Z}^{d+1})$ with $1 \leq j \leq d$ such that

$$\sup_{x \in B} \|gu_x v\| < \rho.$$

In order to use Theorem 21 some notation related to the action of $\text{SL}_{d+1} (\mathbb{R})$ on the exterior algebra of $\mathbb{R}^{d+1}$ is set up in the rest of this section.
Let $e_+ := (1, 0, \ldots, 0)$ and $e_i := (0, 1, \ldots, 0)$ where for every $1 \leq i \leq d$ the $i$ + 1st coordinate is one and the rest are zero be the standard basis of $\mathbb{R}^{d+1}$. For any $I \subseteq \{+, 1, \ldots, d\}$ let $e_I = \bigwedge_{i \in I} e_i$ be the wedge product of basis elements with indices in $I$. For any $1 \leq j \leq d$ the collection $\{e_I : \# I = j\}$ is a basis of $\bigwedge^j (\mathbb{R}^{d+1})$.

Define an inner product on $\bigwedge^j (\mathbb{R}^{d+1})$ by setting $\langle e_I, e_J \rangle = \delta_{I,J}$ (where $\delta_{I,J} := 1$ if $I = J$ and $\delta_{I,J} := 0$ otherwise) and extending linearly. Let $\| \cdot \|$ be the euclidean norm which is derived from this inner product. Note that this notation is consistent with the notation of Theorem 21.

For every $1 \leq j \leq d$ define the subspaces $V_+ := \text{span}_\mathbb{R}\{e_I : + \in I\}$ and $V_- := \text{span}_\mathbb{R}\{e_I : I \subseteq \{1, \ldots, d\}\}$. Each vector $v \in \bigwedge^j (\mathbb{R}^{d+1})$ decomposes uniquely into $v = v_+ + v_-$ with $v_+ \in V_+$ and $v_- \in V_-$. Let $\text{SL}_{d+1}(\mathbb{R})$ act on $\bigwedge^j (\mathbb{R}^{d+1})$ by linear transformations defined on wedge products as follows: for any $g \in \text{SL}_{d+1}(\mathbb{R})$ and $v = v_1 \wedge \ldots \wedge v_j \in \bigwedge^j (\mathbb{R}^{d+1})$ we define

$$g v = g v_1 \wedge \ldots \wedge g v_j.$$  

**Proposition 22.** Assume $x \in \mathbb{R}^d$, $h \in \mathbb{Z}$, $h \geq 0$ and $v = v_1 \wedge \ldots \wedge v_j \in \bigwedge^j (\mathbb{R}^{d+1})$, $1 \leq j \leq d + 1$. Then

$$u_x v = v + e_+ \wedge \left( \sum_{i=1}^j (-1)^{i+1} \langle v_i, x \rangle \wedge v_{i,-} \right) ;$$  

$$\|a_{-h} v_+\| \leq b^{-w_d h} \|v_+\| \quad \text{and} \quad \|a_{-h} v_-\| \leq b^h \|v_-\|,$$

where $w_d$ is assumed to be the smallest weight and $\langle \cdot, \cdot \rangle$ denotes the standard inner product and the norm is Euclidean.

**Proof.** Both (26) and (27) are elementary to prove. Indeed, (26) is an immediate consequence of definition (25) and the easily verified equation $u_x v_i = v_i + \langle v_i, x \rangle$ together with the alternating property of the wedge product and, in particular, the fact that $e_+ \wedge e_+ = 0$. In turn, since the standard basis $e_I$ of $\bigwedge^j (\mathbb{R}^{d+1})$, where $I \subseteq \{+, 1, \ldots, d\}$ and $\# I = j$, is orthonormal and each of $e_I$ is an eigenvector of $a_{-h}$, it suffices to verify (27) for the basis vectors $e_I$. The latter is a trivial job done by inspecting (27). We leave further computational details, which are straightforward, to the reader. \hfill $\square$

When applying Theorem 21 in section 4 we will use the following simple bound.

**Lemma 23.** For every ball $B \subseteq \mathbb{R}^d$, every diagonal matrix $g = \text{diag}(b_+, b_1, \ldots, b_d)$ such that $b_+ \geq 1$ and $0 < b_1, \ldots, b_d \leq 1$ such that $b_1 b_2 \cdots b_d = 1$, and every $v = v_1 \wedge \ldots \wedge v_j \in \bigwedge^j (\mathbb{Z}^{d+1})$ with $1 \leq j \leq d + 1$ such that $v \neq 0$ we have that

$$\sup_{x \in B} \|g u_x v\| \geq \min\{1, r_B\},$$

where $r_B$ is the Euclidean radius of $B$.

**Proof.** The case of $j = d + 1$ is trivial since in this case we have that $\|g u_x v\| = 1$ for all $x$. Let $1 \leq j \leq d$, $v = v_1 \wedge \ldots \wedge v_j \in \bigwedge^j (\mathbb{Z}^{d+1})$ and $v \neq 0$. Let $\tilde{x} = (1, x_1, \ldots, x_d)$
and write each $v_i = (v_{i,+}, v_{i,1}, \ldots, v_{i,d})$. Then

$$g u_x v = \bigwedge_{i=1}^{j} \left( \begin{array}{c} b_i \langle \tilde{x}, v_i \rangle \\ b_1 v_{i,1} \\ \vdots \\ b_d v_{i,d} \end{array} \right).$$

(29)

Since $v \neq 0$, there exists a collection $\{\ell_2, \ldots, \ell_d\} \subseteq \{1, \ldots, d\}$ such that the rows $(b_{\ell_k} v_{1,\ell_k}, \ldots, b_{\ell_k} v_{j,\ell_k})$ $(2 \leq k \leq d)$ are linearly independent. It follows that the determinant

$$\det \left( \begin{array}{ccc} b_1 \langle \tilde{x}, v_1 \rangle & \ldots & b_1 \langle \tilde{x}, v_j \rangle \\ b_{\ell_2} v_{1,\ell_2} & \ldots & b_{\ell_2} v_{j,\ell_2} \\ \vdots & \ddots & \vdots \\ b_{\ell_j} v_{1,\ell_j} & \ldots & b_{\ell_j} v_{j,\ell_j} \end{array} \right) = \prod_{k=1}^{j} b_{\ell_k} \times \det \left( \begin{array}{ccc} \langle \tilde{x}, v_1 \rangle & \ldots & \langle \tilde{x}, v_j \rangle \\ v_{1,\ell_2} & \ldots & v_{j,\ell_2} \\ \vdots & \ddots & \vdots \\ v_{1,\ell_j} & \ldots & v_{j,\ell_j} \end{array} \right),$$

where $\ell_1 = +$, is not identically zero. Here we used the obvious fact that the functions $\langle \tilde{x}, v_1 \rangle, \ldots, \langle \tilde{x}, v_j \rangle$ are linearly independent over $\mathbb{R}$, which follows from the linear independence of $v_1, \ldots, v_j$. Observe that the above determinant is one of the coordinates of $g u_x v$. Furthermore, since all the vectors $v_i$ are integer, it is of the form $\prod_{k=1}^{j} b_{\ell_k} f(x)$, where $f(x) = a_0 + a_1 x_1 + \cdots + a_d x_d$ for some integer coefficients $a_0, \ldots, a_d$, not all zeros. Since the norm of $g u_x v$ is at least the absolute value of any of its coordinates, using the assumptions that $b_+ \geq 1$ and $b_+ b_1 \cdots b_d = 1$ gives

$$\|g u_x v\| \geq \prod_{k=1}^{j} b_{\ell_k} f(x) \geq |f(x)|.$$

If $a_1 = \cdots = a_d = 0$, then the r.h.s. of (30) is a non-zero integer and therefore is at least 1. Otherwise, $a_k \neq 0$ for some $1 \leq k \leq d$. Then, take the points $x_{\pm 1} = x_0 \pm r_B e_k$, where $x_0$ is the centre of $B$. Then, $|f(x_{\pm 1}) - f(x_{-1})| = |2a_k r_B| \geq 2 r_B$. Consequently, using the triangle inequality, we get that $\sup_{x \in B} |g u_x v| \geq \max\{|f(x_{+1})|, |f(x_{-1})|\} \geq r_B$. The proof is complete.

4. PROOF OF THEOREM 5

Let $w$ be any weight, $\mu$ be a compactly supported Ahlfors regular absolutely decaying measure on $\mathbb{R}^d$ and let $A, \alpha$ and $\rho_0$ be as in (6). For every $z \in \text{supp} \mu$ let $B(z)$ be the ball arising from Theorem 21. Clearly, $\left\{ \frac{1}{2} B(z) : z \in \text{supp} \mu \right\}$ is an open cover of $\text{supp} \mu$. Since $\text{supp} \mu$ is compact, there is a finite subcover $\frac{1}{2} B(z_u)$, $1 \leq u \leq U$, of $\text{supp} \mu$. Thus,

$$\text{supp} \mu \subset \bigcup_{u=1}^{U} \frac{1}{2} B(z_u).$$

(31)

Let $C$ and $\gamma$ satisfy (24) for every ball $B$ centred in $\text{supp} \mu$ that is contained in one of the balls $B(z_u)$, $1 \leq u \leq U$. Clearly, $C$ and $\gamma$ exist since we have a finite collection of balls $B(z_u)$. We need to show that $\text{Bad}(w) \cap \text{supp} \mu \neq \emptyset$. By Theorem 19 with $X = \text{supp} \mu$ and $S = \text{Bad} (w) \cap \text{supp} \mu$ (since $\mu$ is Borel, its support is closed in $\mathbb{R}^d$, and thus complete), it suffices to show that $\text{Bad} (w) \cap \text{supp} \mu$ is $\alpha'$-Cantor winning for some $0 \leq \alpha' < \alpha$.

Without loss of generality assume that $1 \leq t \leq d$ is such that

$$w_1 = \ldots = w_t > w_{t+1} \geq \ldots \geq w_d > 0,$$
and that \( \beta > 0 \) is small enough (to be determined according to (50) and (71)). Let \( B_0 \) be Bob’s first move. Recall that \( B_0 \) is a closed ball in \( X := \text{supp} \mu \) defined by its centre \( x_0 \) and radius \( r_0 \). Without loss of generality we can assume that \( r_0 \) is smaller than the radius of every ball \( \frac{1}{4}B(z_u) \), \( 1 \leq u \leq U \). This can be done as a result of Alice playing arbitrarily for several moves until the condition is met. Let \( u_0 \) be such that \( x_0 \in \frac{1}{4}B(z_{u_0}) \). Then using the triangle inequality and the above condition on \( r_0 \) we conclude that

\[
2B(x_0, r_0) \subseteq B(z), \quad \text{where } z = z_{u_0}.
\]

Here \( B(x_0, r_0) \) is the ball in \( \mathbb{R}^d \) of radius \( r_0 \) centred at \( x_0 \). On re-scaling the measure \( \mu \) if necessary we can assume without loss of generality that

\[
\mu(B_0) = 1.
\]

Also without loss of generality we can assume that Bob’s first move \( B_0 \) is a ball of radius 1 as otherwise we can re-scale the metric on \( X \) appropriately.

We will describe a winning strategy for Alice for the \( \alpha' \) Cantor potential game. Let \( b > 1 \) be such that \( \beta = b^{-(1 + w_1)} \). For any \( n \geq 0 \) denote Bob’s \( n \)th move by \( B_n = B(x_n, \beta^n) \). For any integer \( \ell \in \mathbb{Z} \) define

\[
d_{\ell} := \begin{pmatrix}
\beta^{\ell t} \\
\beta^{-\frac{(1+\ell)\epsilon}{2}} \\
\vdots \\
\beta^{-\frac{(1+\ell)\epsilon}{2}} \\
\beta^{\ell t}
\end{pmatrix}.
\]

The following lemma is the key for Alice’s winning strategy:

**Lemma 24.** For any quintuple of nonnegative integers \( (h, k, \ell, m, n) \), if

\[
d_{\ell+m}a_ku_{x_n}\mathbb{Z}^{d+1} \subseteq K \sqrt{2\pi} \beta^{\frac{m}{d+1}}
\]

then for any \( 0 < \varepsilon < 1 \) the set \( A_{\varepsilon} = \{ x \in B_n : d_{\ell+m}a_ku_x \notin K_{\varepsilon} \} \) satisfies

\[
\mu(A_{\varepsilon}) < C\varepsilon^3 \mu(B_n).
\]

If \( n = 0 \) and \( k \geq \frac{1+w_1}{w_1} \ell \) then (36) holds without assuming (34) and (35). Moreover, if \( r > 0 \) satisfies the inequality

\[
\rho = \sqrt{2d} \left( 1 + \max \left\{ \beta^{\ell-k}, b^{(1+\ell+1)k} \right\} \right) \varepsilon < 1
\]

then

\[
\mu(B(A_{\varepsilon}, r)) < C\rho^3 \mu(B_n).
\]

**Proof.** Write \( x = x_n + x' \) with \( \| x' \| < \beta^n \). Conjugating \( u_x \) by \( d_{\ell+m}a_k \) gives

\[
d_{\ell+m}a_ku_{x_n} = u(\beta^{-k}x'_1, \ldots, \beta^{-k}x'_{\ell}, b^{(1+w_1)k}x'_{\ell+1}, \ldots, b^{(1+w_1)k}x'_{d}) d_{\ell+m}a_ku_{x_n}.
\]
In order to apply Theorem 21 let $1 \leq j \leq d$ and $0 = \textbf{v} = \textbf{v}_1 \wedge \cdots \wedge \textbf{v}_j \in \bigwedge^j (\mathbb{Z}^{d+1})$. Denote $\textbf{v}' = d_{\ell}a_k u_{x_n} \textbf{v}$ and $\textbf{v}'_i = d_{\ell}a_k u_{x_n} \textbf{v}_i$ for every $1 \leq i \leq j$. Assume towards contradiction that

\begin{equation}
\max_{\|\mathbf{x}'\| < \beta^n} \left\| u\left(\beta^{\ell-k}x_1', \ldots, \beta^{\ell-k}x_t', b^{(1+w_{t+1})k}x_{t+1}', \ldots, b^{(1+w_d)k}x_d'\right) \textbf{v}' \right\| < 1.
\end{equation}

Since (40) applied at $\mathbf{x}' = 0$ implies that $\|\textbf{v}'\| < 1$, by Minkowski’s convex body theorem assume without loss of generality that

\begin{equation}
\|\textbf{v}_1\| < 1.
\end{equation}

If $|v_{i,i}'| < \beta^m$ for all $1 \leq i \leq t$ then $\|d_{\ell}m \textbf{v}_1\| < \sqrt{d+1}\beta^{\ell-m}$, so $d_{\ell+m}a_k u_{x_n} \mathbb{Z}^{d+1} \notin K_{\sqrt{d+1}\beta^{\ell-m}}$ — a contradiction to (33).

Otherwise, there exists $1 \leq i_0 \leq t$ for which

\begin{equation}
|v_{i_0,i_0}'| \geq \beta^m.
\end{equation}

It is enough to use (40) for $\mathbf{x}'$ of the form $\mathbf{x}' = (0, \ldots, x_{i_0}', \ldots, 0)$ where the only nonzero entry is in the $i_0$th coordinate. In this case, denote

$$
\tilde{\textbf{v}} = \sum_{i=1}^{j} (-1)^{i+1} v_{i,i_0}' \wedge \textbf{v}_{h,-}'.
$$

Then

\begin{equation}
\tilde{u}\left(\beta^{\ell-k}x_1', \ldots, \beta^{\ell-k}x_t', b^{(1+w_{t+1})k}x_{t+1}', \ldots, b^{(1+w_d)k}x_d'\right) \textbf{v}' = \textbf{v}' + \beta^{\ell-k}x_{i_0}' \mathbf{e}_+ \wedge \tilde{\textbf{v}}.
\end{equation}

Equations (40) and (43) applied for all $\mathbf{x}' = (0, \ldots, x_{i_0}', \ldots, 0)$ with $|x_{i_0}'| < \beta^n$ give

\begin{equation}
\|\tilde{\textbf{v}}\| < \beta^{k-\ell-n}.
\end{equation}

On the other hand, taking wedge product with $\textbf{v}_1'$ gives

\begin{equation}
\textbf{v}_1' \wedge \tilde{\textbf{v}} = (-1)^{i_0+1} v_{1,i_0}' \textbf{v}_{-}'.
\end{equation}

so equations (41), (42), (44) and (45) yield

\begin{equation}
\|\textbf{v}'\| \leq |v_{1,i_0}'|^{-1} \|\textbf{v}_1\| \|\tilde{\textbf{v}}\| < \beta^{k-\ell-m-n}.
\end{equation}

Applying (27) gives

\begin{equation}
\|a-h\textbf{v}'_+\| < \beta^{k-\ell-m-n-\frac{h}{1+w_1}}.
\end{equation}

and, since $\|\textbf{v}'_+\| < 1$, 

\begin{equation}
\|a-h\textbf{v}'_-\| < b^{-hw_d}.
\end{equation}

The upshot is that

\begin{equation}
\|d_{\ell}a_{k-h} u_{x_n} \textbf{v}\| = \|a-h\textbf{v}'\| < \sqrt{2} \max \left\{ \beta^{k-\ell-m-n-\frac{h}{1+w_1}}, b^{-hw_d} \right\}.
\end{equation}

Since $d_{\ell}a_{k-h} u_{x_n}$ has determinant one (40) is a contradiction to (33). Therefore, Theorem 21 implies (33).

If $n = 0$ and $k \geq \frac{1+w_1}{w_1} \ell$ then $d_{\ell}a_k = \text{diag}(b)$ with $b_+ \geq 1$ and $b_1 \leq 1$ for every $1 \leq i \leq d$, so Theorem 21 together with Lemma 23 immediately implies (30).

To see (38), assume $r$ satisfies (37). If $\textbf{x} \in A_c$ then there exists $\textbf{v} \in \mathbb{Z}^{d+1}$ such that $\|d_{\ell}a_k u_{x} \textbf{v}\| < \varepsilon$. Assume that $\|y-x\| < r$ and denote $\textbf{x}' = y - \textbf{x}$ Using conjugation of $u_{\textbf{x}'}$ by $d_{\ell}a_k$ as in (39) gives that

$$
\|d_{\ell}a_k u_{\textbf{x}'} \textbf{v}\| < \sqrt{2d} \left(1 + \max \left\{ \beta^{\ell-k}, b^{(1+w_{t+1})k} \right\} \right) r < \rho.
$$
Therefore, $B(A_\varepsilon, r) \subseteq A_\rho$ and an application of (36) with $\varepsilon$ replaced with $\rho$ gives \textup{(38)}. \qed

To complete the proof, denote
\begin{align}
  s &:= \max \left\{ \left[ \frac{1 + w_1}{w_1 - w_{t+1}} \right], \left[ \frac{2(1 + w_1) + 1}{w_1} \right] \right\}, \\
  \eta &:= \min \left\{ \frac{1}{4(d+1)}, \frac{w_d s}{w_{1}s - 2(1 + w_1)}, \frac{3\alpha}{\gamma} \right\}, \\
  \alpha' &:= \alpha - \frac{\gamma \eta}{3},
\end{align}
where it is agreed, if needed, that $w_{d+1} = 0$. Note that (47) and (48) imply that $\eta > 0$ and $0 \leq \alpha' < \alpha$. Assume that $\beta$ is small enough that it satisfies
\begin{align}
  \beta^{-\frac{w_1 s - 2(1 + w_1)}{2d(1 + w_1)}} &\geq 2^{2d}, \quad \beta^{-\frac{w_d s}{2d(1 + w_1)}} \geq 2^{2d} \quad \text{and} \quad \beta^{-\frac{3\alpha}{\gamma}} \geq \sqrt{d + 1}, \\
  \beta^{-\frac{\gamma \eta}{3}} &\geq \max \left\{ 2, A3^n C \left( 2\sqrt{2d} \right)^\gamma \beta^{-\frac{\gamma \eta}{3}} \right\}.
\end{align}

Let us give the winning strategy for Alice. We will keep notation used in Definition 18.

Define $A_{n+1,i} \subseteq B_n$ as follows: For every $i \geq 0$,\textup{(52)}
$$A_{1,i} := \bigcup_{n \geq 0, \ell \geq \frac{n+1}{s}, n+(s-1)\ell = i} \left\{ x \in B_0 : d_{\ell}a_{n+1+s\ell}u_x \mathbb{Z}^{d+1} \notin K_{\beta^{n\ell}} \right\}.$$For every $n \geq 1$ and $0 < \ell < \frac{n+1}{s}$,\textup{(53)}
$$A_{n+1,(s-1)\ell} := \left\{ x \in B_n : d_{\ell}a_{n+1+s\ell}u_x \mathbb{Z}^{d+1} \notin K_{\beta^{n\ell}} \right\},$$and $A_{n+1,i} := \emptyset$ for every $0 \leq i \notin \left\{ (s-1)\ell : 0 < \ell < \frac{n+1}{s} \right\}$.

For every $n, i \geq 0$ let $A_{n+1,i}$ be an efficient cover (as described in Lemma 20) of $A_{n+1,i}$ with balls of radius $\beta^{n+1+i}$. It follows from (52) and (53) that if $x$ is an outcome of the game then
$$d_1a_{n+1+s\ell}u_x \mathbb{Z}^{d+1} \in K_{\beta^{n\ell}}$$for every $n \geq 0$, hence,
$$a_{n+1+s\ell}u_x \mathbb{Z}^{d+1} \in K_{\beta^{n\ell + \frac{w_1 s}{2d(1 + w_1)}}}.$$By Dani correspondence \textup{(22)} this means that $x \in \text{Bad}(\mathbf{w})$. In order to complete the proof that $\text{Bad}(\mathbf{w})$ is Cantor winning in supp $\mu$ it is left to show that Alice’s strategy is legal, i.e., that every $n, i \geq 0$ satisfy
\begin{align}
  \#A_{n+1,i} &< \beta^{-\alpha'(i+1)}.
\end{align}

The plan is to use Lemma 24 in order to get a measure estimate for small neighborhoods of the sets $A_{n+1,i}$ and then to apply Lemma 20.

First deal with $A_{1,i}$. For any $i \geq 0$, for any $n \geq 0$ and $\ell \geq \frac{n+1}{s}$ such that $i = n + (s-1)\ell$, apply Lemma 24 with $(0, n + 1 + s\ell, \ell, 0, n)$, $\varepsilon = \beta^{n\ell}$, $r = \beta^{i+1}$. In this case using (37) and (47) gives
$$r = \sqrt{2d} \left( 1 + \max \left\{ \beta^{-(n+1+s\ell)}, b^{(1+w_1)(n+1+s\ell)} \right\} \beta^{n+1+(s-1)\ell} \right) \beta^{n\ell} = 2\sqrt{2d} \beta^{n\ell}.$$
Therefore,
\begin{equation}
\mu(B(A_{1,i}, \beta^{i+1})) < \sum_{n \geq 0, \ell \geq 1} C \left( 2\sqrt{2d} \right)^{\gamma} \beta^{\gamma n}\beta^{\eta \ell} \\
\leq C \left( 2\sqrt{2d} \right)^{\gamma} \beta^{-\gamma n} (i+1) \left( \beta^{\gamma n} \right)^{i+1}.
\end{equation}

For $A_{n+1,i}$ with $n \geq 1$, let us assume $i = (s-1)\ell$ where $1 \leq \ell < \frac{n+1}{s}$ (otherwise $A_{n+1,i} = \emptyset$ and there’s nothing to prove). Apply Lemma 24 with $(s\ell, n+1+s\ell, \ell, s, n)$, $\varepsilon = \beta^n((s-1)\ell+1)$ and $r = \beta^{n+1+(s-1)\ell}$. Using (37) and (17) as above gives
\begin{equation}
\rho = 2\sqrt{2d} \left( 1 + \max \left\{ \beta^{-n+1+(s-1)\ell}, b(1+w_{t+1})(n+1+s\ell) \right\} \beta^{n+1+(s-1)\ell} \right) \beta^{n((s-1)\ell+1)} = 2\sqrt{2d} \beta^n((s-1)\ell+1),
\end{equation}

Since $B_n$ is a legal move, we have
\begin{align*}
B_n \cap (A_{1,n+(s-1)\ell} \cup A_{1,n-\ell}) &= \emptyset & \text{if} \quad \frac{n+1}{2s} &\leq \ell, \\
B_n \cap (A_{1,n+(s-2)\ell} \cup A_{n+1-s\ell,(s-1)\ell}) &= \emptyset & \text{if} \quad \frac{n+1}{3s} &\leq \ell < \frac{n+1}{2s}, \\
B_n \cap (A_{n+1-s\ell,2(s-1)\ell} \cup A_{n+1-s\ell,(s-1)\ell}) &= \emptyset & \text{if} \quad \ell &< \frac{n+1}{3s}.
\end{align*}
Hence, in any case above, we have,
\begin{align*}
d_{2\ell \alpha_{n+1+s\ell} u_{n} Z^{d+1}} & \in K_{\beta^{2\eta \ell}} \subseteq K_{\sqrt{d+1} \beta^{\eta \ell}} , \\
d_{\ell \alpha_{n+1} u_{n} Z^{d+1}} & \in K_{\beta^{\eta \ell}} \subseteq K_{2^{\frac{1}{2} \beta^{\eta \ell}} \max \left\{ 1+(s-2)\ell, (n+1+s\ell) \right\}},
\end{align*}
where the containments on the right hand side follow from (48) and (50). Thus, conditions (34) and (35) are satisfied. By (33) and (38) we have that
\begin{equation}
\mu(B(A_{n+1,(s-1)\ell}, \beta^{n+1+(s-1)\ell})) < C \left( 2\sqrt{2d} \right)^{\gamma} \beta^{\gamma n((s-1)\ell+1)}.
\end{equation}
Combining (55) and (56) together with Lemma 20 applied with $r = \frac{\beta^{n+1}}{3}$ gives that for every $n, i \geq 0$
\begin{align*}
\#A_{n+1,i} &< A^{3\alpha} \mu(B(A_{n+1,i}, \beta^{n+i+1})) \\
&< A^{3\alpha} \left( 2\sqrt{2d} \right)^{\gamma} \beta^{-\gamma n} (i+1) \beta^{-(\alpha-\gamma n)(i+1)} \\
&< \beta^{-\alpha i(i+1)},
\end{align*}
where the last inequality follows from (49) and (51). This shows that the collection \{$A_{n+1,i} : n, i \in \mathbb{N}$\} is a legal move for Alice. By the argument in the beginning of this section, this completes the proof.

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