Lecture Note on LCSSX’s Lower Bounds for Non-Adaptive Distribution-free Property Testing

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Abstract
In this lecture note we give Liu-Chen-Servedio-Sheng-Xie’s (LCSSX) lower bound for property testing in the non-adaptive distribution-free model [2].

1 Introduction
Here we give the following LCSSX’s lower bound (Zhengyang Liu, Xi Chen, Rocco A. Servedio, Ying Sheng, and Jinyu Xie. Distribution-free junta testing.)

Theorem 1. [2] Let $k \geq 10$. Let $C$ be a class of boolean functions $f : \{0,1\}^n \to \{0,1\}$ that contains all the $k$-junta functions where $n \geq 15 + 2 \log \log |C|$. Any non-adaptive algorithm that distribution-free $(1/3)$-tests $C$ must have query complexity at least

$$q = \frac{1}{8(1 + 2\lambda)^{k/2}} \cdot 2^{k/2},$$

where

$$\lambda = \sqrt{\frac{5 + \ln \ln |C| + k/2}{n}}.$$ 

In particular, when $\log \log |C| = o(n)$ then

$$q = (2 - o_n(1))^{k/2}.$$ 

The proof in this note is the same as of LCSSX [2].

For the definition of the model and other definitions, read from [2] Subsection “Distribution-free property testing” in Section 1 and Section 2 and Subsection “Junta and literals” in Section 2. For other results when $C$ is the set of $k$-juntas read the introduction in [1].

Because $C$ contains all the $k$-junta functions, if $\log \log |C| = o(n)$ then $k = o(n)$
2 Notations

We follow the same notations as in [2]. Denote \([n] = \{1, 2, \ldots, n\}\). For \(X \subseteq [n]\) we denote by \(\{0, 1\}^X\) the set of all binary strings of length \(|X|\) with coordinates indexed by \(i \in X\). For \(x \in \{0, 1\}^n\) and \(X \subseteq [n]\) we write \(x_X \in \{0, 1\}^X\) to denote the projection of \(x\) over coordinates in \(X\).

Given a sequence \(Y = (y^{(i)} : i \in [q])\) of \(q\) strings in \(\{0, 1\}^n\) and a Boolean function \(\phi : \{0, 1\}^n \to \{0, 1\}\), we write \(\phi(Y)\) to denote the \(q\)-bit string \(\alpha\) with \(\alpha_i = \phi(y^{(i)})\) for \(i \in [q]\). For a distribution \(D\), we write \(y \leftarrow D\) to denote that \(y\) is a draw from the distribution \(D\) and \(Y = (y^{(i)} : i \in [q]) \leftarrow D^q\) to denote a sequence of \(q\) independent draws from the same probability distribution \(D\).

For convenience, we refer to an algorithm as a \(q\)-query algorithm if it makes \(q\) sample queries and \(q\) black-box queries each. Such algorithms are clearly at least as powerful as those that make \(q\) queries in total.

3 Preliminary Results

In this section we give some preliminary results.

3.1 Chernoff Bound

We will use the following version of Chernoff Bound

**Lemma 2. Chernoff’s Bound.** Let \(X_1, \ldots, X_m\) be independent random variables taking values in \(\{0, 1\}\). Let \(X = \sum_{i=1}^{m} X_i\) denotes their sum and let \(\mu = \mathbb{E}[X]\) denotes the sum’s expected value. Then

\[
\Pr[X > (1 + \eta)\mu] \leq \begin{cases} 
    e^{-\frac{\eta^2\mu}{2}} & \text{if } 0 < \eta \leq 1 \\
    e^{-\frac{\eta\mu}{2}} & \text{if } \eta > 1
\end{cases}.
\]

(1)

For \(0 \leq \eta \leq 1\) we have

\[
\Pr[X < (1 - \eta)\mu] \leq e^{-\frac{\eta^2\mu}{2}}.
\]

(2)

3.2 Some Results in Probability

Let \(D\) be a probability distribution over a finite set \(\Omega\). We will use the following (In the following two lemmas \(\Pr = \Pr_D\))

**Lemma 3.** Let \(A, B \subseteq \Omega\) where \(B \neq \emptyset\). Then

\[
\Pr[A|B] - \Pr[B] \leq \Pr[A] \leq \Pr[A|B] + \Pr[B].
\]

Proof. We have

\[
\Pr[A] = \Pr[A|B]\Pr[B] + \Pr[A|B^c]\Pr[B^c] \\
\leq \Pr[A|B] + \Pr[B]
\]

and

\[
\Pr[A] = 1 - \Pr[\overline{A}] \\
\geq 1 - \Pr[\overline{A}|B] - \Pr[B] = \Pr[\overline{A}|B] - \Pr[B].
\]

\(\square\)
Lemma 4. Let $A, B, W \subseteq \Omega$ where $W \neq \emptyset$. If $\Pr[A|W] \leq \Pr[B|W]$ then $\Pr[A] \leq \Pr[B] + \Pr[W]$.

Proof. We have
\[
\Pr[A] = \Pr[A|W]\Pr[W] + \Pr[A\mid W]\Pr[W] \\
\leq \Pr[B|W]\Pr[W] + \Pr[W] \leq \Pr[B] + \Pr[W].
\]

Lemma 5. Birthday Paradox: Let $X$ be a finite set and let $Y$ be a set obtained by making $r$ draws from $X$ uniformly at random with replacement. Then
\[
\Pr[|Y| \neq r] \leq \frac{r^2}{2|X|}.
\]

Proof. Since for $x_1, \ldots, x_j \in [0, 1], (1 - x_1) \cdots (1 - x_j) \geq 1 - (x_1 + \cdots + x_j)$, we have
\[
\Pr[|Y| \neq r] = 1 - \prod_{i=1}^{r-1} \left(1 - \frac{i}{|X|}\right) \leq \frac{r^2}{2|X|}.
\]

3.3 Total Variation Distance

Let $D_1$ and $D_2$ be two probability distributions over a finite set $\Omega$. The total variation distance between $D_1$ and $D_2$ (also called statistical distance) is
\[
\|D_1 - D_2\|_{tv} := \frac{1}{2} \sum_{\omega \in \Omega} |\Pr_{D_1}[\omega] - \Pr_{D_2}[\omega]|.
\]

The following lemmas are well known and easy to prove

Lemma 6. The total variation distance between $D_1$ and $D_2$ is
\[
\|D_1 - D_2\|_{tv} = \max_{E \subseteq \Omega} |\Pr_{D_1}[E] - \Pr_{D_2}[E]|.
\]

Lemma 7. Let $X : \Omega \to [0, 1]$ be a random variable. Then
\[
\left|\mathbb{E}_{D_1}[X] - \mathbb{E}_{D_2}[X]\right| \leq \|D_1 - D_2\|_{tv}.
\]

Lemma 8. Let $W$ be an event such that $\Pr_{D_1}[\omega] = \Pr_{D_2}[\omega|W]$ for all $\omega \in \Omega$. Then
\[
\|D_1 - D_2\|_{tv} = \Pr_{D_2}[W].
\]

Proof. First, we have $\Pr_{D_1}[W] = \Pr_{D_2}[W|W] = 1$. Now
\[
\|D_1 - D_2\|_{tv} = \max_{E \subseteq \Omega} |\Pr_{D_1}[E] - \Pr_{D_2}[E]| \\ \geq \Pr_{D_2}[W]
\]
and by Lemma 3, for any $E$,
\[
|\Pr_{D_1}[E] - \Pr_{D_2}[E]| = |\Pr_{D_2}[E|W] - \Pr_{D_2}[E]| \leq \Pr_{D_2}[W].
\]

□

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Lemma 9. Let $W$ be an event such that $\Pr_{D_1}[\omega|W] = \Pr_{D_2}[\omega|W]$ for all $\omega \in \Omega$. Then

$$\|D_1 - D_2\|_{tv} \leq \Pr_{D_1}[W] + \Pr_{D_2}[\overline{W}].$$

Proof. Let $D_3$ be the conditional distribution of $D_1$ given $W$. Then $\Pr_{D_2}[\omega|W] = \Pr_{D_1}[\omega|W] = \Pr_{D_3}[\omega]$. By Lemma 8 $\|D_1 - D_3\|_{tv} = \Pr_{D_1}[W]$ and $\|D_2 - D_3\|_{tv} = \Pr_{D_2}[\overline{W}]$ and therefore

$$\|D_1 - D_2\|_{tv} \leq \|D_1 - D_3\|_{tv} + \|D_2 - D_3\|_{tv} = \Pr_{D_1}[W] + \Pr_{D_2}[\overline{W}].$$

Lemma 10. Let $D_1$ and $D_2$ be two probability distributions over $\Omega_1 \times \Omega_2$. If for every $\omega \in \Omega_1$, $\Pr_{(\omega_1, \omega_2) \sim D_1}[\omega_1 = \omega] = \Pr_{(\omega_1, \omega_2) \sim D_2}[\omega_1 = \omega]$ then the total variation distance between the distributions $D_1$ and $D_2$ is less than or equal to the maximum over $\omega_1 \in \Omega_1$ of the total variation distance between the distributions of $\omega_2$ conditioning on $\omega_1 = \omega_1$ in $D_1$ and $D_2$.

3.4 Lower Bound Technique

Our goal is to show that there exists no $q$-query non-adaptive (randomized) algorithm that distribution-free $(1/3)$-tests $C$.

We can think of a distribution-free $(1/3)$-tester for $C$ as a randomized algorithm $T$ that receives as an input a pair $(\phi, D)$ where $\phi : \{0,1\}^n \to \{0,1\}$ and $D$ is a probability distribution over $\{0,1\}^n$. If $\phi \in C$ then $T$ accepts with probability at least $2/3$ and if $f$ is $(1/3)$-far from every function in $C$ with respect to $D$ then it rejects with probability at least $2/3$.

The (folklore) technique introduced here shows that it is enough to focus on $q$-query non-adaptive deterministic algorithms. Such an algorithm $A$ consists of two deterministic maps $A_1$ and $A_2$ works as follows. Upon an input pair $(\phi, D)$, where $\phi : \{0,1\}^n \to \{0,1\}$ and $D$ is a probability distribution over $\{0,1\}^n$, the algorithm receives in the first phase a sequence $Y = (y(i) : i \in [q])$ of $q$ strings (which should be thought of as samples from $D$) and a binary string $\alpha = \phi(Y)$ of length $q$. In the second phase, the algorithm $A$ uses the first map $A_1$ to obtain a sequence of $q$ strings $Z = (z(i) : i \in [q]) = A_1(Y, \alpha)$ and feeds them to the black-box oracle. Once the query results $\beta = \phi(Z)$ are back, $A_2(Y, \alpha, \beta)$ returns either 0 or 1 in which cases the algorithm $A$ either rejects or accepts, respectively. Notice that we do not need to include $Z$ as an input of $A_2$, since it is determined by $Y$ and $\alpha$. A randomized algorithm $T$ works similarly and consists of two similar maps $T_1$ and $T_2$ but both are randomized. The following are the two algorithms $A$ and $T$. The (infinite length) strings $s_1$ and $s_2$ are two random seeds.
| Deterministic Algorithm $A$ | Randomized Algorithm $T$ |
|----------------------------|---------------------------|
| 1. Input $(\phi, D)$       | 1. Input $(\phi, D)$      |
| 2. Get $Y = (y(i) : i \in [q])$ | 2. Get $Y = (y(i) : i \in [q])$ |
| 3. $\alpha = \phi(Y)$      | 3. $\alpha = \phi(Y)$     |
| 4. $Z = (z(i) : i \in [q]) = A_1(Y, \alpha)$ | 4. $Z = (z(i) : i \in [q]) = T_1(Y, \alpha, s_1)$ |
| 5. $\beta = \phi(Z)$       | 5. $\beta = \phi(Z)$      |
| 6. Output $A_2(Y, \alpha, \beta)$ | 6. Output $T_2(Y, \alpha, \beta, s_2)$ |

Given the above deterministic algorithm, unlike typical deterministic algorithms, whether $A$ accepts or not depends on not only $(\phi, D)$ but also the sample strings $Y \leftarrow D^q$ it draws. Formally, we have

$$\Pr[A \text{ accepts } (\phi, D)] = \Pr_{Y \leftarrow D^q}[A \text{ accepts } (\phi, D)]$$

For the randomized algorithm $T$ we have

$$\Pr[T \text{ accepts } (\phi, D)] = \Pr_{s_1, s_2, Y \leftarrow D^q}[T \text{ accepts } (\phi, D)]$$

We now prove

**Lemma 11.** [2] Let $\mathcal{YES}$ and $\mathcal{NO}$ be probability distributions over pairs $(\phi, D)$, where $\phi : \{0, 1\}^n \rightarrow \{0, 1\}$ is a Boolean function over $n$ variables and $D$ is a distribution over $\{0, 1\}^n$. For clarity, we use $(f, D)$ to denote pairs in the support of $\mathcal{YES}$ and $(g, D)$ to denote pairs in the support of $\mathcal{NO}$. Suppose $\mathcal{YES}$ and $\mathcal{NO}$ satisfy

- **C1:** Every $(f, D)$ in the support of $\mathcal{YES}$ satisfies that $f$ is in $C$.
- **C2:** With probability at least $12/13$, $(g, D) \leftarrow \mathcal{NO}$ satisfies that $g$ is $(1/3)$-far from every function in $C$ with respect to $D$.
- **C3:** Any $q$-query non-adaptive deterministic algorithm must behave similarly when it is run on $(f, D) \leftarrow \mathcal{YES}$ versus $(g, D) \leftarrow \mathcal{NO}$: That is, any $q$-query deterministic algorithm $A$ satisfies

$$\left| \mathbb{E}_{(f, D) \leftarrow \mathcal{YES}}[\Pr[A \text{ accepts } (f, D)]] - \mathbb{E}_{(g, D) \leftarrow \mathcal{NO}}[\Pr[A \text{ accepts } (g, D)]] \right| \leq \frac{1}{4}.$$
Proof. Assume for a contradiction that there exists a \( q \)-query non-adaptive randomized algorithm \( T_{s_1, s_2} \) that distribution-free \( (1/3) \)-tests \( C \) where \( s_1 \) and \( s_2 \) are the random seeds of the algorithm. Then, by \( C_1 \), for every \((f, \mathcal{D})\) in the support of \( \mathcal{Y\mathcal{E}\mathcal{S}} \) we have \( \Pr[T_{s_1, s_2} \text{ accepts } (f, \mathcal{D})] \geq 2/3 \). Therefore,
\[
\mathbb{E}_{(f, \mathcal{D}) \leftarrow \mathcal{Y\mathcal{E}\mathcal{S}}} \left[ \Pr[T_{s_1, s_2} \text{ accepts } (f, \mathcal{D})] \right] \geq \frac{2}{3},
\]
(3)

Define \( U := [g \text{ is } (1/3)\text{-far from every function in } C \text{ with respect to } \mathcal{D}] \) and \( W := \Pr[T_{s_1, s_2} \text{ accepts } (g, \mathcal{D})] \). Then, by \( C_2 \),
\[
\mathbb{E}_{(g, \mathcal{D}) \leftarrow \mathcal{NO}} [W] = \mathbb{E}_{(g, \mathcal{D}) \leftarrow \mathcal{NO}} [W|U] \Pr[(g, \mathcal{D}) \leftarrow \mathcal{NO}|U] + \mathbb{E}_{(g, \mathcal{D}) \leftarrow \mathcal{NO}} [W|\overline{U}] \Pr[(g, \mathcal{D}) \leftarrow \mathcal{NO}|\overline{U}]
\leq \mathbb{E}_{(g, \mathcal{D}) \leftarrow \mathcal{NO}} [W|U] + \Pr[(g, \mathcal{D}) \leftarrow \mathcal{NO}|\overline{U}]
\leq \frac{1}{3} + \frac{1}{13} < \frac{5}{12}.
\]
(4)

By (3) and (4) we have that
\[
\mathbb{E}_{(f, \mathcal{D}) \leftarrow \mathcal{Y\mathcal{E}\mathcal{S}}} \left[ \Pr[T_{s_1, s_2} \text{ accepts } (f, \mathcal{D})] \right] - \mathbb{E}_{(g, \mathcal{D}) \leftarrow \mathcal{NO}} \left[ \Pr[T_{s_1, s_2} \text{ accepts } (g, \mathcal{D})] \right] > \frac{2}{3} - \frac{5}{12} = \frac{1}{4}.
\]

Since,
\[
\Pr[T_{s_1, s_2} \text{ accepts } (\phi, \mathcal{D})] = \Pr_{s_1, s_2, \mathcal{Y} \leftarrow \mathcal{D}_q} [T_{s_1, s_2} \text{ accepts } (\phi, \mathcal{D})]
\leq \mathbb{E}_{s_1, s_2} \left[ \Pr_{\mathcal{Y} \leftarrow \mathcal{D}_q} [T_{s_1, s_2} \text{ accepts } (\phi, \mathcal{D})] \right]
\leq \frac{1}{4},
\]
we have
\[
\mathbb{E}_{s_1, s_2} \left[ \mathbb{E}_{(f, \mathcal{D}) \leftarrow \mathcal{Y\mathcal{E}\mathcal{S}}} \left[ \Pr[T_{s_1, s_2} \text{ accepts } (f, \mathcal{D})] \right] - \mathbb{E}_{(g, \mathcal{D}) \leftarrow \mathcal{NO}} \left[ \Pr[T_{s_1, s_2} \text{ accepts } (g, \mathcal{D})] \right] \right] > \frac{1}{4}.
\]

Thus, there exist \( s_1' \) and \( s_2' \), and therefore a \( q \)-query nonadaptive deterministic algorithm \( A = T_{s_1', s_2'} \), that satisfies
\[
\mathbb{E}_{(f, \mathcal{D}) \leftarrow \mathcal{Y\mathcal{E}\mathcal{S}}} \left[ \Pr[A \text{ accepts } (f, \mathcal{D})] \right] - \mathbb{E}_{(g, \mathcal{D}) \leftarrow \mathcal{NO}} \left[ \Pr[A \text{ accepts } (g, \mathcal{D})] \right] > \frac{1}{4}.
\]
A contradiction to \( C_3 \). \qed

4 The \( \mathcal{Y\mathcal{E}\mathcal{S}} \) and \( \mathcal{NO} \) Distributions

Given \( J \subseteq [n] \), we partition \( \{0, 1\}^n \) into sections (with respect to \( J \)) where the \( z \)-section, \( z \in \{0, 1\}^J \), consists of those \( x \in \{0, 1\}^n \) that have \( x_J = z \). We write \( \mathcal{JUNTA}_J \) to denote the uniform distribution over all juntas over \( J \). More precisely, a Boolean function \( h : \{0, 1\}^n \to \{0, 1\} \) drawn from \( \mathcal{JUNTA}_J \) is generated as follows: For each \( z \in \{0, 1\}^J \), a bit \( b(z) \) is chosen independently and uniformly at random, and for each \( x \in \{0, 1\}^n \) the value of \( h(x) \) is set to \( b(x_J) \). That is, if \( x \) is in the \( z \)-section then \( f(x) = b(z) \).
We now define two probability distributions: Let
\[ m = 18 \ln |C|. \]

### The probability distribution \( \mathcal{YES} \)
A pair \((f, D)\) drawn from \( \mathcal{YES} \) is generated as follows:
1. Draw a subset \( J \) of \([n]\) of size \( k \) uniformly at random.
2. Draw a subset \( S \) of \( \{0, 1\}^n \) of size \( m \) uniformly at random.
3. Draw \( f \leftarrow \text{JUNTA}_J \).
4. Set \( D \) to be the uniform distribution over \( S \).

### The probability distribution \( \mathcal{NO} \)
A pair \((g, D)\) drawn from \( \mathcal{NO} \) is generated as follows:
1. Draw a subset \( J \) of \([n]\) of size \( k \) uniformly at random.
2. Draw a subset \( S \) of \( \{0, 1\}^n \) of size \( m \) uniformly at random.
3. Draw \( h \leftarrow \text{JUNTA}_J \). We usually refer to \( h \) as the background junta.
4. Draw a map \( \gamma : S \rightarrow \{0, 1\} \) uniformly at random by choosing a bit independently and uniformly at random for each string in \( S \).
5. The distribution \( D \) is set to be the uniform distribution over \( S \), which is the same as \( \mathcal{YES} \).
6. The function \( g : \{0, 1\}^n \rightarrow \{0, 1\} \) is defined using \( h, S \) and \( \gamma \) as follows:

\[
g(x) = \begin{cases} 
    \gamma(x) & x \in S \\
    h(x) & x \notin S, (\forall y \in S) x_J \neq y_J \text{ or } d(x, y) > (0.5 - \lambda)n \\
    \gamma(y) & x \notin S, (\exists y \in S) x_J = y_J \text{ and } d(x, y) \leq (0.5 - \lambda)n 
\end{cases}
\]

\[ (*) \] The choice of the tie-breaking rule here is not important; we can, for example, order the elements of \( S \) in a lexicographic order \((s^{(i)} : i \in [m])\) and define \( g(x) = \gamma(s^{(i)}) \) for the smallest \( i \) that satisfies \( x_J = s_J^{(i)} \) and \( d(x, s^{(i)}) \leq (0.5 - \lambda)n \). This makes \( g \) well defined.

For technical reasons that will become clear in the sequel we use \( \mathcal{YES}^* \) to denote the probability distribution supported over triples \((f, D, J)\), with \((f, D, J) \leftarrow \mathcal{YES}^* \) being generated by the same steps above. So, the only difference is that we include \( J \) in elements of \( \mathcal{YES}^* \). Similarly, we let \( \mathcal{NO}^* \) denote the distribution supported on triples \((g, D, J)\) as generated above.

To understand the intuition behind the above definitions, read subsubsection The lower bound in subsection 1.2 and the last paragraph in page 1:17 in [2] (when \( C \) is the class of all \( k \)-juntas).

## 5 The Proofs of C1 and C2

In this section we prove...
C1: Every \((f, D)\) in the support of \(\mathcal{YES}\) satisfies that \(f\) is in \(C\).

C2: With probability at least 12/13, \((g, D) \leftarrow \mathcal{NO}\) satisfies that \(g\) is \((1/3)\)-far from every function in \(C\) with respect to \(D\).

**Proof of C1:** By the definition of \(\mathcal{YES}\) we have that \(f\) is \(k\)-junta. Since \(C\) contains all the \(k\)-juntas we have that \(f\) is in \(C\). \(\square\)

**Proof of C2:** Let \(\beta \in C\). Since \(D\) is the uniform distribution over \(S\), we have that \(\text{dist}_D(g, \beta)\) is equal to the fraction of strings \(z \in S\) such that \(\gamma(z) \neq \beta(z)\). By the union bound, we have

\[
\Pr_{(g, D) \leftarrow \mathcal{NO}}[\text{dist}(g, C) < 1/3] = \Pr_{(g, D) \leftarrow \mathcal{NO}}[(\exists \beta \in C) \text{ dist}(g, \beta) < 1/3] \leq |C| \cdot \max_{\beta \in C} \Pr_{(g, D) \leftarrow \mathcal{NO}}[\Pr_{z \leftarrow D}[g(z) \neq \beta(z)] < 1/3]. \tag{5}
\]

Now let \(1_{g \neq \beta}(z)\) be the indicator random variable of \(g(z) \neq \beta(z)\), i.e., \(1_{g \neq \beta}(z) = 1\) if \(g(z) \neq \beta(z)\) and zero otherwise. Since each bit \(\gamma(z)\), \(z \in S\), is drawn independently and uniformly at random, we have that, for every \(z \in S\),

\[
\mathbb{E}_{(g, D) \leftarrow \mathcal{NO}}[1_{\gamma \neq \beta}(z)] = \frac{1}{2}.
\]

Then, by Chernoff bound in Lemma 2 \((m = 18 \ln |C|, k \geq 10, C\) contains all \(k\)-Junta functions and therefore \(|C| \geq 2^k > 13)\),

\[
\Pr_{(g, D) \leftarrow \mathcal{NO}} \left[ \Pr_{z \leftarrow D}[g(z) \neq \beta(z)] < \frac{1}{3} \right] = \Pr_{(g, D) \leftarrow \mathcal{NO}} \left[ \sum_{z \in S} 1_{g \neq \beta}(z) < \frac{1}{3}m \right] \leq e^{-m/9} \leq \frac{1}{13|C|}.
\]

Therefore

\[
|C| \cdot \max_{\beta \in C} \Pr_{(g, D) \leftarrow \mathcal{NO}}[\Pr_{z \leftarrow D}[g(z) \neq \beta(z)] < \frac{1}{3}] \leq \frac{1}{13}. \tag{6}
\]

By (5) and (6) we get

\[
\Pr_{(g, D) \leftarrow \mathcal{NO}}[\text{dist}(g, C) < 1/3] \leq \frac{1}{13}. \quad \square
\]

6 The Proof of C3

In this section we prove

**C3:** Any \(q\)-query non-adaptive deterministic algorithm must behave similarly when it is run on \((f, D) \leftarrow \mathcal{YES}\) versus \((g, D) \leftarrow \mathcal{NO}\): That is, any \(q\)-query deterministic algorithm \(A\) satisfies

\[
\left| \mathbb{E}_{(f, D) \leftarrow \mathcal{YES}}[\Pr[A \text{ accepts } (f, D)]] - \mathbb{E}_{(g, D) \leftarrow \mathcal{NO}}[\Pr[A \text{ accepts } (g, D)]] \right| \leq \frac{1}{4}.
\]
Let $A$ be a $q$-query non-adaptive deterministic algorithm where

$$q = \frac{1}{8(1 + 2\lambda)k/2} \cdot 2^{k/2},$$

and

$$\lambda = \sqrt{\frac{5 + \ln \ln |C| + k/2}{n}}.$$

We will use the following definition. Let $Y = (y_i : i \in [q])$ be a sequence of $q$ strings in $\{0,1\}^n$, $\alpha$ be a $q$-bit string, and $J \subset [n]$ be a set of size $k$. We say that $(Y, \alpha, J)$ is consistent if $\alpha_i = \alpha_j$ for all $i, j \in [q]$ with $y_i(J) = y_j(J)$.

Given a consistent triple $(Y, \alpha, J)$, we write $\text{JUNTA}_{Y, \alpha, J}$ to denote the uniform distribution over all juntas $h$ over $J$ that are consistent with $(Y, \alpha)$. More precisely, a draw of $h \leftarrow \text{JUNTA}_{Y, \alpha, J}$ is generated as follows: For each $z \in \{0,1\}^{\lvert J \rvert}$, if there exists a $y_i$ such that $y_i(J) = z$, then $h(x)$ is set to $\alpha_i$ for all $x \in \{0,1\}^n$ with $x_J = z$; if no such $y_i$ exists, then a uniform random bit $b(z)$ is chosen independently and $h(x)$ is set to $b(z)$ for all $x$ with $x_J = z$.

To prove C3, we first derive from $A$ the following randomized algorithm $A'$ that works on triples $(\phi, D, J)$ from the support of either $\text{YES}^*$ or $\text{NO}^*$. Again for clarity we use $\phi$ to denote a function from the support of $\text{YES}/\text{YES}^*$ or $\text{NO}/\text{NO}^*$, $f$ to denote a function from $\text{YES}/\text{YES}^*$ and $g$ to denote a function from $\text{NO}/\text{NO}^*$.

| Deterministic Algorithm $A$ | Randomized Algorithm $A'$ |
|-----------------------------|-----------------------------|
| 1. Input $(\phi, D)$       | 1. Input $(\phi, D, J)$     |
| 2. $Y \leftarrow D^q$      | 2. $Y \leftarrow D^q$;     |
| 3. $\alpha = \phi(Y)$      | 3. $\alpha = \phi(Y)$      |
|                           | If $(Y, \alpha, J)$ is not consistent reject |
| 4. $Z = A_1(Y, \alpha)$    | 4. $Z = A_1(Y, \alpha)$    |
| 5. $\beta = \phi(Z)$       | 5. Draw $h' \leftarrow \text{JUNTA}_{Y, \alpha, J}$; $\beta = h'(Z)$ |
| 6. Output $A_2(Y, \alpha, \beta)$ | 6. Output $A_2(Y, \alpha, \beta)$ |

From the description of $A'$ above, we have

$$\Pr[A' \text{ accepts } (\phi, D, J)] = \Pr_{Y, h'}[(Y, \alpha, J) \text{ is consistent and } A_2(Y, \alpha, h'(Z)) = 1].$$

To prove C3 we will prove the following

**C3.1** $A'$ behaves similarly on $\text{YES}^*$ and $\text{NO}^*$, i.e.,

$$\left| \mathbb{E}_{(f, D, J) \leftarrow \text{YES}^*} [\Pr[A' \text{ accepts } (f, D, J)]] - \mathbb{E}_{(g, D, J) \leftarrow \text{NO}^*} [\Pr[A' \text{ accepts } (g, D, J)]] \right| \leq \frac{1}{8}.$$
C3.2 $A$ and $A'$ behave identically on $\mathcal{ES}$ and $\mathcal{ES}^*$, respectively. i.e,

$$\mathbb{E}_{(f, D, J) \leftarrow \mathcal{ES}^*}[\Pr[A' \text{ accepts } (f, D, J)]] = \mathbb{E}_{(f, D, J) \leftarrow \mathcal{ES}^*}[\Pr[A \text{ accepts } (f, D)]]$$

C3.3 $A'$ and $A$ behave similarly on $\mathcal{NO}$ and $\mathcal{NO}^*$, respectively. i.e,

$$\left| \mathbb{E}_{(g, D, J) \leftarrow \mathcal{NO}^*}[\Pr[A' \text{ accepts } (g, D, J)]] - \mathbb{E}_{(g, D, J) \leftarrow \mathcal{NO}^*}[\Pr[A \text{ accepts } (g, D)]] \right| \leq \frac{1}{8}.$$

Obviously, C3.1-C3.3 imply C3.

### 6.1 Proof of C3.1

In this subsection we prove $A'$ behaves similarly on $\mathcal{ES}$ and $\mathcal{NO}$, i.e,

$$\left| \mathbb{E}_{(f, D, J) \leftarrow \mathcal{ES}^*}[\Pr[A' \text{ accepts } (f, D, J)]] - \mathbb{E}_{(g, D, J) \leftarrow \mathcal{NO}^*}[\Pr[A' \text{ accepts } (g, D, J)]] \right| \leq \frac{1}{8}.$$

We say $Y$ is scattered by $J$ if there is no $i \neq j$ such that $y_i(J) = y_j(J)$. The following claim shows that $Y$ is scattered by $J$ with high probability.

**Claim 1.** We have that $Y$ is scattered by $J$ with probability at least $15/16$
Proof. We fix $J$ and show that $Y$ is scattered by $J$ with probability at least $15/16$. We now define the following distributions $D_1$ and $D_2$ for $Y$.

1. $D_1$: Draw a subset $S$ of $\{0,1\}^n$ of size $m$ uniformly at random. Then choose $q$ strings $Y = (y(i) : i \in [q])$ independently and uniformly at random from $S$ with replacement.

2. $D_2$: Choose $q$ strings $Y = (y(i) : i \in [q])$ independently and uniformly at random from $\{0,1\}^n$ with replacement.

Let $F$ be the event: $Y$ is not scattered by $J$. We need to show that

$$\Pr_{Y \leftarrow D_1}[F] \leq \frac{1}{16}.$$ 

Let $U$ be the event that the strings in $Y$ are distinct. It is clear that for any event $E$ we have that $\Pr_{Y \leftarrow D_1}[E|U] = \Pr_{Y \leftarrow D_2}[E|U]$. By Lemma 9 and Lemma 5, the total variation distance between $D_1$ and $D_2$ is $(q \leq 2^{k/2-3}$ and $m = 18 \ln |C| \geq 2^k$)

$$\|D_1 - D_2\|_{tv} \leq \Pr_{Y \leftarrow D_1[\overline{U}]} + \Pr_{Y \leftarrow D_2[\overline{U}]} \leq \frac{q^2}{2m} + \frac{q^2}{2^{n+1}} \leq \frac{q^2}{m} \leq \frac{1}{32}.$$ 

Since, by Lemma 7 $\Pr_{Y \leftarrow D_1}[F] \leq \Pr_{Y \leftarrow D_2}[F] + 1/32$, it remains to show that $\Pr_{Y \leftarrow D_2}[F] \leq 1/32$. Since $(y(i) : i \in [q])$ are chosen independently and uniformly at random from $\{0,1\}^n$ with replacement, we have that $(y_j(i) : i \in [q])$ are chosen independently and uniformly at random from $\{0,1\}^k$ with replacement. Thus, by Lemma 5 $(q \leq 2^{k/2-3})$,

$$\Pr_{Y \leftarrow D_2}[F] \leq \frac{q^2}{2^{k+1}} \leq \frac{1}{32}$$ 

and the result follows.

Since $A'$ runs on $(Y, \alpha, J)$, by Lemma 7 it suffices to show that the distributions of $(Y, \alpha, J)$ induced from $\mathcal{Y}\mathcal{E}\mathcal{S}^*$ and $\mathcal{N}\mathcal{O}^*$ have total variation distance less than or equal to $1/8$. For this purpose, we first note that the distributions of $(Y, J)$ induced from $\mathcal{Y}\mathcal{E}\mathcal{S}^*$ and $\mathcal{N}\mathcal{O}^*$ are identical: In both cases, $Y$ and $J$ are independent; $J$ is a random subset of $[n]$ of size $k$; $Y$ is obtained by first sampling a subset $S$ of $\{0,1\}^n$ of size $m$ and then drawing a sequence of $q$ strings from $S$ with replacement.

Fix any $(Y, J)$ in the support of $(Y, J)$. By Lemma 10 it is enough to show that the total variation of the distributions of $\alpha$ conditioning on $(Y, J) = (Y, J)$ in the $\mathcal{Y}\mathcal{E}\mathcal{S}^*$ case and the $\mathcal{N}\mathcal{O}^*$ case is less than $1/8$.

Fix any $(Y, J)$ in the support of $(Y, J)$ such that $Y$ is scattered by $J$. By Claim 1 and Lemma 2 it is enough to show that the distributions of $\alpha$ conditioning on $(Y, J) = (Y, J)$ in the $\mathcal{Y}\mathcal{E}\mathcal{S}^*$ case and the $\mathcal{N}\mathcal{O}^*$ case are identical.

For $Y = (y(i) : i \in [q])$ the string $\alpha = (\alpha_i : i \in [q])$ is uniform over strings of length $q$ in both cases. This is trivial for $\mathcal{N}\mathcal{O}^*$. For $\mathcal{Y}\mathcal{E}\mathcal{S}^*$ note that $\alpha$ is determined by the random $k$-junta $f \leftarrow \mathcal{J}\mathcal{U}\mathcal{N}\mathcal{T}\mathcal{A}_J$; the claim follows from the assumption that $Y$ is scattered by $J$. 

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6.2 Proof of C3.2

In this subsection we prove

C3.2 \(A\) and \(A'\) behave identically on \(\mathcal{YES}\) and \(\mathcal{YES}^*\), respectively. i.e,

\[
\mathbb{E}_{(f, D, J) \leftarrow \mathcal{YES}^*} [\Pr[A' \text{ accepts } (f, D, J)]] = \mathbb{E}_{(f, D, J) \leftarrow \mathcal{YES}^*} [\Pr[A \text{ accepts } (f, D)]].
\]

\begin{tabular}{|ll|}
\hline
Algorithm \(A'\) & Algorithm \(A\) \\
\hline
1. \((f, D, J) \leftarrow \mathcal{YES}^*\) & 1. \((f, D, J) \leftarrow \mathcal{YES}^*\) \\
2. \(Y \leftarrow D^q\) & 2. \(Y \leftarrow D^q\) \\
3. \(\alpha = f(Y)\) & 3. \(\alpha = f(Y)\) \\
\quad If \((Y, \alpha, J)\) is not consistent reject & \\
4. \(Z = A_1(Y, \alpha)\) & 4. \(Z = A_1(Y, \alpha)\) \\
5. Draw \(h' \leftarrow \text{JUNTA}_{Y, \alpha, J};\) \\
\quad \(\beta = h'(Z)\) & 5. Let \(\beta = f(Z)\) \\
6. Output \(A_2(Y, \alpha, \beta)\) & 6. Output \(A_2(Y, \alpha, \beta)\) \\
\hline
\end{tabular}

For the first expectation in C3.2, since the triple \((Y, \alpha, J)\) on which we run \(A'\) is always consistent, we can rewrite it as the probability that

\[
A_2(Y, \alpha, h'(A_1(Y, \alpha))) = 1,
\]

where \((f, D, J) \leftarrow \mathcal{YES}^*\), \(Y \leftarrow D^q\), \(\alpha = f(Y)\) and \(h' \leftarrow \text{JUNTA}_{Y, \alpha, J}\).

The second expectation is equal to the probability that

\[
A_2(Y, \alpha, f(A_1(Y, \alpha))) = 1
\]

where \((f, D, J) \leftarrow \mathcal{YES}^*\), \(Y \leftarrow D^q\) and \(\alpha = f(Y)\).

To show that these two probabilities are equal, we first note that the distributions of \((Y, \alpha, J)\) are identical. Fixing any triple \((Y, \alpha, J)\) in the support of \((Y, \alpha, J)\), which must be consistent, we claim that the distribution of \(f\) conditioning on \((Y, \alpha, J) = (Y, \alpha, J)\) is exactly \(\text{JUNTA}_{Y, \alpha, J}\). This is because, for each \(z \in \{0,1\}^J\), if \(y^{(i)}_j = z\) for some \(y^{(i)}_j\) in \(Y\), then we have \(f(x) = \alpha_i\) for all strings \(x\) with \(x_J = z\); otherwise, we have \(f(x) = b(z)\) for all \(x\) with \(x_J = z\), where \(b(z)\) is an independent and uniform bit. This is the same as how \(h' \leftarrow \text{JUNTA}_{Y, \alpha, J}\) is generated. It follows directly from this claim that the two probabilities are the same. This finishes the proof of C3.2.

6.3 Proof of C3.3

In this subsection we prove

C3.3 \(A'\) and \(A\) behave similarly on \(\mathcal{NO}\) and \(\mathcal{NO}^*\), respectively. i.e,

\[
\left| \mathbb{E}_{(g, D, J) \leftarrow \mathcal{NO}^*} [\Pr[A' \text{ accepts } (g, D, J)]] - \mathbb{E}_{(g, D, J) \leftarrow \mathcal{NO}^*} [\Pr[A \text{ accepts } (g, D)]] \right| \leq \frac{1}{8}.
\]
We remind the reader that $(Y, \alpha, J)$ is good if it satisfies the following three conditions: Here $Z = A_1(Y, \alpha)$ and $S$ is the support of $D$

$E_0 : Y$ is scattered by $J$.

$E_1 :$ Every $z$ in $Z$ and every $x \in S \setminus \{y^{(i)} : i \in [q]\}$ have $d(x, z) > (0.5 - \lambda)n$.

$E_2 :$ If a string $z$ in $Z$ satisfies $z_J = y_J$ for some $y$ in $Y$, then we have $d(y, z) \leq (0.5 - \lambda)n$.

We delay the proof of the following claim to the end.

**Claim 2.** We have that $(Y, \alpha, J, D)$ is good with probability at least $7/8$.

Fix any good $(Y, \alpha, J, D)$ in the support and let $Z = A_1(Y, \alpha)$. We first show that since $Y$ is scattered by $J$ we have that $(Y, \alpha, J)$ is consistent. Let $i, j \in [q]$ with $y_J^{(i)} = y_J^{(j)}$. Since $Y$ is scattered by $J$ we have $i = j$ and therefore $\alpha_i = g(y^{(i)}) = g(y^{(j)}) = \alpha_j$. Therefore $(Y, \alpha, J)$ is consistent.

We finish the proof by showing that the distribution of $g(Z)$, a binary string of length $q$, conditioning on $(Y, \alpha, J, D) = (Y, \alpha, J, D)$ is the same as that of $h'(Z)$ with $h' \leftarrow \text{JUNTA}_Y, \alpha, J$.

| Algorithm $A'$ | Algorithm $A$ |
|---------------|---------------|
| 1. $(g, D, J) \leftarrow \mathcal{N}O^*$ | 1. $(g, D, J) \leftarrow \mathcal{N}O^*$ |
| 2. $Y \leftarrow D^q$ | 2. $Y \leftarrow D^q$ |
| 3. $\alpha = g(Y)$ | 3. $\alpha = g(Y)$ |
| If $(Y, \alpha, J)$ is not consistent reject | |
| 4. $Z = A_1(Y, \alpha)$ | 4. $Z = A_1(Y, \alpha)$ |
| 5. Draw $h' \leftarrow \text{JUNTA}_Y, \alpha, J$; $\beta = h'(Z)$ | 5. Let $\beta = g(Z)$ |
| 6. Output $A_2(Y, \alpha, \beta)$ | 6. Output $A_2(Y, \alpha, \beta)$ |

The first expectation in C3.3 is equal to the probability of

$$(Y, \alpha, J) \text{ is consistent } \Rightarrow A_2(Y, \alpha, h'(A_1(Y, \alpha))) = 1,$$

where $(g, D, J) \leftarrow \mathcal{N}O^*$, $Y \leftarrow D^q$, $\alpha = g(Y)$, and $h' \leftarrow \text{JUNTA}_Y, \alpha, J$.

The second expectation is the probability of

$$A_2(Y, \alpha, g(A_1(Y, \alpha))) = 1,$$

where $(g, D, J) \leftarrow \mathcal{N}O^*$ and $\alpha = g(Y)$.

The distributions of $(Y, \alpha, J, D)$ in the two cases are identical.
This combined with Lemmas 8, 10 and Claim 2 implies that the difference of the two probabilities has absolute value at most 1/8. To see this is the case, we partition strings of $Z$ into $Z_w$, where each $Z_w$ is a nonempty set that contains all $z$ in $Z$ with $z_j = w \in \{0,1\}^J$. For each $Z_w$, we consider the following two cases:

**Case I.** There exists $y^{(i)}$ in $Y$ with $y^{(i)}_j = w$. By $E_0$, $y^{(i)}$ is the only string $y$ in $Y$ that satisfies $y_J = w$. By $E_2$, every $z \in Z_w$ satisfies $d(z, y^{(i)}) \leq (0.5 - \lambda)n$. By $E_1$, every $z \in Z_w$ and every $y \in S \setminus \{y^{(i)} : i \in [q]\}$ we have $d(x, z) > (0.5 - \lambda)n$. Therefore, the only $y$ in $S$ that satisfies $y_J = w$ and $d(y, z) \leq (0.5 - \lambda)n$ is $y^{(i)}$. Therefore, for every $z \in Z_w$ we have $g(z) = \gamma(y^{(i)}) = \alpha_i$. On the other hand, for every $z \in Z_w$ and $h' \leftarrow JUNTA_{Y,\alpha,J}$ we have $h'(z) = \alpha_i$.

**Case II.** There exists no $y$ in $Y$ with $y_J = w$. By $E_1$ for every $z \in Z_w$ and every $x \in S \setminus \{y^{(i)} : i \in [q]\}$ we have that $d(x, z) \geq (0.5 - \lambda)n$. Therefore for every $z \in Z_w$ and every $x \in S$ we have that $x_J \neq z_J$ or $d(x, z) \geq (0.5 - \lambda)n$. Thus, for every $z \in Z_w$ we have that $g(z) = h(z) = b(w)$ for some uniform bit $b(w)$. The same is true for $h' \leftarrow JUNTA_{Y,\alpha,J}$.

So the conditional distribution of $g(Z)$ is identical to that of $h'(Z)$ with $h' \leftarrow JUNTA_{Y,\alpha,J}$. This finishes the proof of C3.

Now to prove Claim 2 we show that $\Pr[E_0] \leq 1/16$ and $\Pr[E_1], \Pr[E_2] \leq 1/32$. By the union bound we get

$$\Pr[E_1 \text{ and } E_2 \text{ and } E_3] \geq 1 - \Pr[E_0] - \Pr[E_1] - \Pr[E_2] \geq \frac{7}{8}.$$

### 6.4 The Proof for $E_0$ and $E_1$

From Claim 1 we have

$$\Pr[E_0] \leq \frac{1}{16}.$$

We now prove that with probability at most 1/32, $E_1$: There exists $z$ in $Z$ and $x \in S \setminus \{y^{(i)} : i \in [q]\}$ such that $d(x, z) \leq (0.5 - \lambda)n$.

To prove that $\Pr[E_1] \leq 1/32$, we fix a pair $(Y, \alpha)$ in the support and let $\ell \leq q$ be the number of distinct strings in $Y$ and $Z = A_1(Y, \alpha)$. Conditioning on $Y = Y, S \setminus Y$ is a uniformly random subset of $\{0,1\}^n \setminus Y$ of size $m - \ell$. Instead of working with $S \setminus Y$, we let $T$ denote a set obtained by making $m - \ell$ draws from $\{0,1\}^n$ uniformly at random (with replacements). On the one hand, by Lemma 8 the total variation distance between $S \setminus Y$ and $T$ is exactly the probability that either (1) $T \setminus Y$ is nonempty or (2) $|T| > m - \ell$. By two union bounds, (1) happens with probability $1 - (1 - \ell/2^m)^{m - \ell} \leq (m - \ell) \cdot (\ell/2^m) \leq mq/2^n$ and, by Lemma 3 (2) happens with probability at most $m^2/2^n$. As a result, the total variation distance is at most $(mq + m^2)/2^n$. On the other hand, by Chernoff bound 2 in Lemma 2 the probability that one of the strings of $T$ has distance at most $(0.5 - \lambda)n$ with one of the strings of $Z$ is at most $mq \exp(-\lambda^2 n)$. Thus, by union bound ($n \geq 15 + 2 \log \log |C|$, $q \leq 2^{k/2-3}$ and $m = 18 \log |C|$)

$$\Pr[E_2] \leq \frac{mq + m^2}{2^n} + mq \cdot e^{-\lambda^2 n} \leq \frac{m^2}{2^n - 1} + mq \cdot e^{-\lambda^2 n} \leq \frac{1}{64} + \frac{1}{64} \leq \frac{1}{32}.$$
6.5 The Proof for $E_2$

We now prove that with probability at most $1/32$,

$E_2$: There exists two strings $z$ in $Z$ and $y$ in $Y$ that satisfies $z_J = y_J$ and $d(y, z) > (0.5 - \lambda)n$.

Fix a pair $(Y, \alpha)$ in the support and let $Z = A_2(Y, \alpha)$. Because $J$ is independent from $(Y, \alpha)$, it remains a subset of $[n]$ of size $k$ drawn uniformly at random. For each pair $(y, z)$ with $y$ from $Y$ and $z$ from $Z$ that satisfy $d(y, z) > (0.5 - \lambda)n$, the probability of $y_J = z_J$ is at most

$$\binom{(0.5 + \lambda)n}{k} \leq (0.5 + \lambda)^k.$$

Then

$$\Pr[E_2] \leq q^2 \cdot \binom{(0.5 + \lambda)n}{k} \leq q^2(0.5 + \lambda)^k \leq \frac{1}{32}.$$

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