On a class of anharmonic oscillators II. General case

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\textbf{A B S T R A C T}

In this work we study a class of anharmonic oscillators on $\mathbb{R}^n$ corresponding to Hamiltonians of the form $A(D) + V(x)$, where $A(\xi)$ and $V(x)$ are $C^\infty$ functions enjoying some regularity conditions. Our class includes fractional relativistic Schrödinger operators and anharmonic oscillators with fractional potentials. By associating a Hörmander metric we obtain spectral properties in terms of Schatten-von Neumann classes for their negative powers and derive from them estimates on the rate of growth for the eigenvalues of the operators $A(D) + V(x)$. This extends the analysis in the first part [1], where the case of polynomial $A$ and $V$ has been analysed.

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\textbf{R É S U M É}

Dans cet travail nous étudions une classe de Hamiltoniens sur $\mathbb{R}^n$ de la forme $A(D) + V(x)$, où $A(\xi)$ et $V(x)$ sont des fonctions $C^\infty$ satisfaissant quelques conditions de régularité. Cette classe contient des opérateurs fractionnaires relativistes de Schrödinger et des oscillateurs anharmoniques avec de potentiel fractionnaire. En associant une métrique de Hörmander nous obtenons de propriétés spectrales en termes des classes

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de Schatten-von Neumann pour leur puissances negatives et
derivons le taux de croissance des valeurs proches de l’opérateur
$A(D) + V(x)$. Ceci étend l’analyse dans la première partie [1], où le cas polynomial pour $A$ et $V$ a été considéré.
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1. Introduction

In this manuscript we obtain spectral properties and in particular estimates for the
rate of growth of eigenvalues for a class of operators on $\mathbb{R}^n$ of the form $A(D) + V(x)$,
where $A(\xi)$ and $V(x)$ are appropriate smooth functions. Some important examples in
this class include fractional relativistic Schrödinger operators, the special cases of relativistic
Schrödinger operators $\sqrt{I - \Delta} + V(x)$, and anharmonic oscillators with fractional
power potentials $(-\Delta)^{\ell} + |x|^{2\kappa}$, where $\ell$ is a positive integer and $\kappa > 0$. Our class also
allows lower order terms with respect to each symbol $A(\xi)$ and $V(x)$; notably real-valued
bounded from below potentials. Hamiltonians in relativistic quantum mechanics, and
quantum field theory are not, in general, partial differential operators (as in nonrelativis-
tic quantum mechanics), but pseudo-differential operators. Herein, we frame our class
within the setting of the Weyl-Hörmander calculus by introducing a suitable Hörmander
metric and obtaining a substantial extension of the class considered in [1].

The mother case of the Hamiltonian $-\Delta + V(x)$ ($\ell = 1$), corresponding to the
Schrödinger equation is one of the milestone objects of study in mathematical physics,
specially since the study of energy levels for the Schrödinger equation is reduced to
the eigenvalue problem associated the Hamiltonian. One of the most basic examples of
anharmonic oscillators of polynomial form $-\Delta + |x|^{2k}$, is the one dimensional quartic
oscillator ($n = 1$, $k = 2$) which is an important model in quantum physics and has been
intensively studied in the last 50 years. However, the exact solution for the eigenvalue
problem of this model is unknown (cf. [2], [3]). It is clear that the situation is even more
subtle for non-polynomial Hamiltonians. This kind of fact is a further motivation for
the investigation of different approximative and qualitative properties for the operators
$A(D) + V(x)$.  

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The case of polynomial Hamiltonians has been widely studied. A class of anharmonic oscillators arises in the form

$$\frac{d^{2\ell}}{dx^{2\ell}} + x^{2k} + p(x),$$

where $p(x)$ is a polynomial of order $2k - 1$ on $\mathbb{R}$ and with $k, \ell$ integers $\geq 1$. The spectral asymptotics of such operators have been analysed by B. Helffer and D. Robert [4–6]. The authors have recently studied anharmonic oscillators on $\mathbb{R}^n$ in [1], where a prototype operator is of the form

$$(-\Delta)^\ell + |x|^{2k}, \quad (1.1)$$

where $k, \ell$ are integers $\geq 1$. By considering $A(\xi) = |\xi|^{2\ell}$ and $V(x) = |x|^{2k}$, these symbols will be absorbed by our new class considered in this paper. Spectral properties for the type (1.1) in the case $k = \ell$ can be found in [7] and [8]. The case of a fractional Laplacian and a quartic potential has been considered by S. Durugo and J. Lörinczi in [9]. Bochner–Riesz means and spectral analysis for the one-dimensional anharmonic oscillator $-\frac{d^{2\ell}}{dx^{2\ell}} + |x|$ has been recently studied in [10]. Physical models related to the operator $\sqrt{-\Delta + m^2}$ have been intensely studied in the last 30 years and there exists a huge literature on the spectral properties of relativistic Hamiltonians, most of it strongly influenced by the works of Lieb on the stability of relativistic matter (cf. [11], [12], [13]). On the other hand, the study of general fractional relativistic Schrödinger operators $(I - \Delta)^\gamma + V(x)$, $\gamma > 0$, has also attracted the interest in the last decades. The operators $(I - \Delta)^\gamma$ are also related to the so-called relativistic $\gamma$-stable process. See for instance [14], [15] and the references therein for recent investigation on the fractional relativistic Schrödinger operators.

Herein we study the operator $A(D) + V(x)$ within the setting of Hörmander’s $S(m, g)$ classes by introducing a Hörmander metric $g^{(A, V)}$ in Section 3. In Section 4 we give an intrinsic formulation of such classes:

For $m \in \mathbb{R}$, appropriate functions $A(\xi), V(x)$ on $\mathbb{R}^n$ and suitable values of $\gamma, \kappa$ and $q > 0$, the class $\Sigma^{(m, g)}_{A, V}$ consists of all $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ functions enjoying the property that for all multi-indices $\alpha, \beta$ there exists a constant $C_{\alpha\beta}$ such that

$$|\partial_\xi^\beta \partial_\xi^\alpha a(x, \xi)| \leq C_{\alpha\beta}(q + V(x) + A(\xi))^{m - \frac{\lvert \alpha \rvert}{2\kappa} - \frac{\lvert \beta \rvert}{4\kappa}}, \quad (1.2)$$

holds all $x, \xi \in \mathbb{R}^n$.

By looking at the negative powers of our operators $A(D) + V(x)$ and studying the corresponding Schatten-von Neumann properties within the setting of Hörmander $S(m, g)$ classes, we will deduce the rate of decay of the eigenvalues for those negative powers. Thus, from the inverses of these negative powers we obtain estimates for the rate of growth of eigenvalues of the operators $A(D) + V(x)$. The investigation of such properties
within these classes started with Hörmander [16]. Other works on Schatten-von Neumann classes within the Weyl-Hörmander calculus can be found in [17], [18]. See also [19], [20], [21], [22] for several symbolic and kernel criteria on different types of domains. For the spectral theory of non-commutative versions of the harmonic oscillator we refer the reader to the works of Parmeggiani et al. [23], [24], [25], [26], [27] and [28].

The main results of this work give the order of the corresponding Schatten-von Neumann class for the negative powers of the operators $A(D) + V(x)$. The special case of the trace class is also distinguished. That is the contents of Theorem 5.5 and Corollary 5.7. From those we derive estimates for the rate of growth of eigenvalues of our operators $A(D) + V(x)$ in (5.8). In Theorem 5.10 we give a sharp condition to guarantee the membership of the negative powers of our main examples to the Schatten-von Neumann classes. At the end of the section we also provide examples for the special case of the fractional relativistic Schrödinger operators with fractional potentials

2. Weyl-Hörmander calculus and Schatten-von Neumann classes

In this section we briefly review some basic elements of the Weyl-Hörmander calculus and the Schatten-von Neumann classes. For a comprehensive study on the Weyl-Hörmander calculus, we refer the interested reader to [29], [30], [31].

Let us briefly recall the main concepts and ideas that will be involved in this work. Before doing so, let us note that, in the sequel, we shall use capital latin letters, e.g. $X,Y,T$ to denote elements in the phase space. These do, in turn, correspond to pairs of the form $(x,\xi)$, $(y,\eta)$, $(t,\tau)$, where Latin and Greek letters have been used for elements in the configuration space $\mathbb{R}^n$, and its dual $(\mathbb{R}^n)^*$, respectively.

For $a = a(x,\xi) \in S'(\mathbb{R}^n \times \mathbb{R}^n)$ $(x,\xi \in \mathbb{R}^n)$ and $t \in \mathbb{R}$, we define the $t$-quantization of the symbol $a$ as being the operator $a_t(x,D): S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ given by the formula

$$a_t(x,D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\xi} a(tx + (1-t)y,\xi)u(y)d\xi dy.$$ 

The operator $a_t(x,D)$, or for simplicity $a(x,D)$, is known as the Kohn-Nirenberg quantization, while the operator $a_{1/2}(x,D)$, denoted also by $a^w(x,D)$, is called the Weyl quantization. In the particular cases of operators that we consider in this work, the corresponding symbols that appear stay invariant (up to lower under terms) under different quantizations.

To define Weyl-Hörmander classes of symbols the following notions shall be recalled: if $g_X(\cdot)$ is a positive definite quadratic form on the phase space, then $g(\cdot)$ is a Hörmander’s metric if the following three conditions are satisfied:
1. **Continuity or slowness**- There exists a constant $C > 0$ such that

\[ g_X(X - Y) \leq C^{-1} \Rightarrow \left( \frac{g_X(T)}{g_Y(T)} \right)^{\pm 1} \leq 1, \quad \forall T. \]

2. **Uncertainty principle**- We say that $g$ satisfies the *uncertainty principle* if

\[ \lambda_g(X) = \inf_{T \neq 0} \left( \frac{g_X(T)}{g_Y(T)} \right)^{1/2} \geq 1, \quad \forall X, T, \]

where $g_X(T) := \sup_{Y \neq 0} \{ \sigma(Y, T)^2 / g_X(Y) \}$, and $\sigma(Y, T) := t \cdot \eta - y \cdot \tau$. We put $h_g(X) = (\lambda_g(X))^{-1}$. 

3. **Temperateness**- We say that $g$ is temperate if there exist $\bar{C} > 0$ and $J \in \mathbb{N}$ such that

\[ \left( \frac{g_X(T)}{g_Y(T)} \right)^{\pm 1} \leq \bar{C}(1 + g_X^\sigma(X - Y))^J, \quad \forall X, Y, T. \quad (2.1) \]

For $g$ being a fixed Hörmander metric, and $M$ being a positive function on the phase space we say that $M$ is a *$g$-weight*, if the following are satisfied:

1. **$g$-continuous**, if there exists $\tilde{C} > 0$ such that

\[ g_X(X - Y) \leq \frac{1}{\tilde{C}} \Rightarrow \left( \frac{M(X)}{M(Y)} \right)^{\pm 1} \leq \tilde{C}, \quad \forall X, Y. \]

2. **$g$-temperate**, if there exist $\tilde{C} > 0$ and $N \in \mathbb{N}$ such that

\[ \left( \frac{M(X)}{M(Y)} \right)^{\pm 1} \leq \tilde{C}(1 + g_Y^\sigma(X - Y))^N, \quad \forall X, Y. \]

For $M, g$ being as above, we define the *set of symbols* $S(M, g)$ as the set of smooth functions $a$ on the phase space such that for any integer $k$ there exists $C_k > 0$, such that for all $X, T_1, ..., T_k \in \mathbb{R}^n \times \mathbb{R}^n$ we have

\[ |a^{(k)}(X; T_1, ..., T_k)| \leq C_k M(X) \prod_{i=1}^k g_X^{1/2}(T_i). \quad (2.2) \]

The notation $a^{(k)}$ stands for the $k^{th}$ derivative of $a$ and $a^{(k)}(X; T_1, ..., T_k)$ denotes the $k^{th}$ derivative of $a$ at $X$ in the directions $T_1, ..., T_k$. For $a \in S(M, g)$ we define

\[ \|a\|_{k, S(M, g)} := \inf \{ C_k : C_k \text{ satisfies } (2.2) \}. \]

The family of seminorms $\| \cdot \|_{k, S(M, g)}$ endows $S(M, g)$ with the topology of a Fréchet space.
Finally let us discuss some basic definitions of the Schatten-von Neumann classes of operators; for a more detailed exposition of the theory we refer to [32], [33], [34], [35].

Let $H$ be a separable Hilbert space over $\mathbb{C}$ endowed with the inner product $(\cdot, \cdot)$, and let $T$ be some compact (linear) operator from $H$ to itself. We denote by $|T| := (T^*T)^{1/2}$ the absolute value of $T$, and by $s_n(T)$ (of course, $n \in \mathbb{N}$) the singular values of $T$; that is the eigenvalues of $|T|$, that correspond to the eigenfunctions of the latter given by the spectral theorem. The operator $T$ belongs to the Schatten-von Neumann class of operators $S_p(H)$, where $1 \leq p < \infty$, if

$$
\|T\|_{S_p} := \left( \sum_{k=1}^{\infty} (s_k(T))^p \right)^{1/p} < \infty.
$$

The space $S_p$ is a Banach space if endowed with the natural norm $\|\cdot\|_{S_p}$. In particular, the Banach space $S_1(H)$ is the space of trace-class operators, while for $T \in S_1$ the quantity

$$
\text{Tr}(T) := \sum_{n=1}^{\infty} (T\phi_n, \phi_n),
$$

where $(\phi_n)$ is an orthonormal basis in $H$, is well-defined and shall be called the trace $\text{Tr}(T)$ of $T$. Moreover, the space $S_2(H)$ is identified with the space of Hilbert-Schmidt operators on $H$.

### 3. The class of operators and the metric

In this section we start with the study of the specific class of operators that we consider here that are regarded to be of the form $A(D) + V(x)$. We first introduce a suitable class of functions for the terms $A(\xi), V(x)$ of their symbols.

**Definition 3.1.** Let $\Gamma : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. We say that $\Gamma$ is a $\tau$-function, for some $\tau > 0$, if the following conditions are satisfied:

(a) there exists $q > 0$ such that $\Gamma(x) + \frac{q}{4} > 0$ for all $x \in \mathbb{R}^n$;

(b) there exist $\tau > 0$ such that

$$
C_1 \left| \frac{q}{2} + \Gamma(x) \right|^\frac{1}{\tau} - \left( \frac{q}{2} + \Gamma(y) \right)^{\frac{1}{\tau}} \right| \leq |x - y|, \text{ for all } |x|, |y| \geq R,
$$

for some $C_1 > 0, R > 0$.

**Example 1.** As some examples of functions satisfying Definition 3.1 we can consider potentials of the form:

(i) $V(x) = |x|^{2k} +$ lower order, with $\kappa > 0$ and choosing $\tau = \kappa$;
(ii) \( V(x) = |x|^{2k} + \) lower order, where \( k \) is an integer \( \geq 1 \), and choosing \( \tau = k \).

Similarly for the symbol \( A(\xi) \) of the operator \( A(D) \) we can consider:

(i') \( A(\xi) = (\xi)^{2\gamma} + \) lower order, with \( \tau = \gamma > 0 \);
(ii') \( A(\xi) = |\xi|^{2\ell} + \) lower order, where \( \ell \) is an integer \( \geq 1 \) and \( \tau = \ell \).

More generally we define the following class of operators we are going to consider.

**Definition 3.2.** The \((\gamma, \kappa)\)-class Let \( \gamma, \kappa > 0 \). We say that an operator of the form \( A(D) + V(x) \) belongs to the \((\gamma, \kappa)\)-class, if \( A(\xi) \) is a \( \gamma \)-function, and \( V(x) \) is a \( \kappa \)-function.

**Example 2.** Special examples of operators in the \((\gamma, \kappa)\)-class.

(i) **Anharmonic oscillators** of the form

\[
(-\Delta)^\ell + |x|^{2k},
\]

where \( k, \ell \) integers \( \geq 1 \), belong to the \((\ell, \kappa)\)-class.

Spectral properties for this class of operators have been recently analysed by the authors in [1] within the setting of Weyl-Hörmander calculus. Another example of operators in the \((\ell, \kappa)\)-class are those expressed by the formula \(-\frac{d^2t}{dx^2} + x^{2k} + p_1(x)\), where \( p_1 \) is a suitable polynomial of order \( 2k-1 \). Those have been studied by Helffer and Robert (cf. [6], [5]).

(ii) **Relativistic Schrödinger operators** of the form

\[
\sqrt{I - \Delta} + V(x),
\]

where \( V(x) \) satisfies the Definition 3.1 for a suitable \( \kappa > 0 \), belong the \((1, \kappa)\)-class.

(iii) **Anharmonic oscillators with fractional potential**

\[
(-\Delta)^\ell + (x)^{2\kappa},
\]

where \( \ell \) is an integer \( \geq 1 \) and \( \kappa > 0 \), are in the \((\ell, \kappa)\)-class.

We associate to an operator \( A(D) + V(x) \) in the \((\gamma, \kappa)\)-class, the following metric:

\[
g_{(A,V)}^{(x,\xi)} = \frac{dx^2}{(q + V(x) + A(\xi))^{\frac{1}{2}}} + \frac{d\xi^2}{(q + V(x) + A(\xi))^{\frac{1}{2}}} \tag{3.1}
\]

We choose a constant \( q > 0 \) in (3.1) to be such that \( q + V(x) + A(\xi) \geq 1 \), for all \( x, \xi \in \mathbb{R}^n \).

Our first aim is to prove that the metric introduced in (3.1) is a Hörmander metric. To this end, let us first recall the auxiliary lemma that will be used later in the proof.
of the continuity property. For the proof of it, the interested reader, can consult [29, Theorem 18.4.2].

**Lemma 3.3.** Let $g$ be a Riemannian metric on the phase space. The following statements are equivalent:

(i) $g$ is continuous.

(ii) There exists a constant $C \geq 1$ such that

$$g_X(X - Y) \leq C^{-1} \implies g_Y \leq Cg_X.$$ 

(iii) there exists a constant $C \geq 1$ such that

$$g_X(Y) \leq C^{-1} \implies g_{X+Y} \leq Cg_X.$$

We now consider the metric $g^{(A,V)}$ in detail.

**Theorem 3.4.** The metric $g = g^{(A,V)}$ defined by (3.1) is a Hörmander metric.

**Proof.** Regarding the uncertainty parameter we have

$$\lambda_g(X) = (q + V(x) + A(\xi))^\frac{n}{2n + 2},$$

and it is clear that, due the choice of $q$ in (3.1), we have $\lambda_g(X) \geq 1$ for all $X$.

For the proof of the continuity of the metric $g^{(A,V)}$, we will use the characterisation (ii) in Lemma 3.3; i.e., we aim to prove that

$$\frac{|x - y|^2}{(q + V(x) + A(\xi))^\frac{2}{2n + 2}} + \frac{|\xi - \eta|^2}{(q + V(x) + A(\xi))^\frac{2}{2n + 2}} \leq C^{-1} \implies \frac{|t|^2}{(q + V(y) + A(\eta))^\frac{2}{2n + 2}} + \frac{|\tau|^2}{(q + V(x) + A(\xi))^\frac{2}{2n + 2}} \leq C,$$

for all $X = (x, \xi), Y = (y, \eta), T = (t, \tau) \in \mathbb{R}^n \times \mathbb{R}^n$.

The proof can be reduced to proving that

$$\frac{|x - y|^2}{(q + V(x) + A(\xi))^\frac{2}{2n + 2}}, \quad \frac{|\xi - \eta|^2}{(q + V(x) + A(\xi))^\frac{2}{2n + 2}} \leq C^{-1} \implies \frac{(q + V(x) + A(\xi))^\frac{1}{2}}{(q + V(y) + A(\eta))^\frac{1}{2}} \leq C.$$
\[
\frac{|x - y|}{(q + V(x) + A(\xi))^{\frac{1}{2}}} \cdot \frac{|\xi - \eta|^2}{(q + V(x) + A(\xi))^{\frac{1}{2}}} \leq C^{-1} \implies \frac{q + V(x) + A(\xi)}{q + V(y) + A(\eta)} \leq C. \tag{3.2}
\]

We will assume that the LHS of (3.2) holds for a constant \( C > 0 \) to be chosen later on. Since
\[
C_1 \left( \frac{q}{2} + V(x) \right)^{\frac{1}{2}} - \left( \frac{q}{2} + V(y) \right)^{\frac{1}{2}} \leq |x - y|, \quad \text{for all } |x|, |y| \geq R;
\]
and analogously for \( A(\xi) \), we have
\[
C_1 \left( \frac{q}{2} + V(x) \right)^{\frac{1}{2}} \leq C^{-1} \left( q + V(x) + A(\xi) \right)^{\frac{1}{2}} + C_1 \left( \frac{q}{2} + V(y) \right)^{\frac{1}{2}}, \tag{3.3}
\]
for all \( |x|, |y| \geq R, \) and
\[
C_1 \left( \frac{q}{2} + A(\xi) \right)^{\frac{1}{2}} \leq C^{-1} \left( q + V(x) + A(\xi) \right)^{\frac{1}{2}} + C_1 (q + A(\eta))^{\frac{1}{2}}, \tag{3.4}
\]
for all \( |\xi|, |\eta| \geq R \). We note that due to the continuity of \( A \) and \( V \) we can also assume that the above inequalities hold for \( |x|, |y| \leq 2R \) and \( |\xi|, |\eta| \leq 2R \).

By taking powers \( 2\kappa \) and \( 2\gamma \) of (3.3) and (3.4) respectively, there exists a constant \( C_2 > 1 \) only dependent on \( \kappa \) and \( \gamma \) such that
\[
V(x) \leq C^{-2\kappa} C_2 (q + V(x) + A(\xi)) + C_2 V(y), \tag{3.5}
\]
\[
A(\xi) \leq C^{-2\gamma} C_2 (q + V(x) + A(\xi)) + C_2 A(\eta). \tag{3.6}
\]
By adding (3.5) and (3.6), taking \( \kappa_0 = \min\{\kappa, \gamma\} \) and \( C \) large enough, we obtain
\[
q + V(x) + A(\xi) \leq \tilde{C} (q + V(y) + A(\eta)),
\]
for some \( \tilde{C} > 0 \). Hence, we obtain
\[
\frac{q + V(x) + A(\xi)}{q + V(y) + A(\eta)} \leq \tilde{C},
\]
and the last inequality shows the continuity of the metric \( g^{(A,V)} \).

We now prove the temperateness. According to (2.1), we need to prove that there exist \( C > 0 \) and \( N \in \mathbb{N} \) such that
\[
\frac{|t|^2}{(q + V(x) + A(\xi))^{\frac{1}{2}}} + \frac{|\tau|^2}{(q + V(x) + A(\xi))^{\frac{1}{2}}} \leq C \left( \frac{|t|^2}{(q + V(y) + A(\eta))^{\frac{1}{2}}} + \frac{|\tau|^2}{(q + V(y) + A(\eta))^{\frac{1}{2}}} \right) \times \left( 1 + (q + V(x) + A(\xi))^{\frac{1}{2}} |x - y|^2 + (q + V(x) + A(\xi))^{\frac{1}{2}} |\xi - \eta|^2 \right)^N,
\]
and

$$\frac{|t|^2}{(q + V(y) + A(\eta))^{\frac{2}{n}}} + \frac{|\tau|^2}{(q + V(y) + A(\eta))^{\frac{2}{n}}} \leq C \left( \frac{|t|^2}{(q + V(x) + A(\xi))^{\frac{2}{n}}} + \frac{|\tau|^2}{(2 + V(x) + A(\xi))^{\frac{2}{n}}} \right) \times$$

$$\times \left( 1 + (q + V(x) + A(\xi))^{\frac{1}{2}}|x - y|^2 + (q + V(x) + A(\xi))^{\frac{1}{2}}|\xi - \eta|^2 \right)^N,$$

for all $t, \tau \in \mathbb{R}^n$.

We will only prove the first inequality since the second one can be proven in a similar way.

We now observe that we can reduce the proof of the first inequality to the proof of the following two inequalities:

$$\frac{(q + V(y) + A(\eta))^{\frac{1}{2}}}{(q + V(x) + A(\xi))^{\frac{1}{2}}} \leq \frac{(q + V(y) + A(\xi))^{\frac{1}{2}}}{(q + V(x) + A(\xi))^{\frac{1}{2}}} \leq C \left( 1 + (q + V(x) + A(\xi))^{\frac{1}{2}}|x - y|^2 + (q + V(x) + A(\xi))^{\frac{1}{2}}|\xi - \eta|^2 \right)^N,$$

and

$$\frac{(q + V(y) + A(\eta))^{\frac{3}{2}}}{(q + V(x) + A(\xi))^{\frac{3}{2}}} \leq \frac{(q + V(y) + A(\xi))^{\frac{3}{2}}}{(q + V(x) + A(\xi))^{\frac{3}{2}}} \leq C \left( 1 + (q + V(x) + A(\xi))^{\frac{1}{2}}|x - y|^2 + (q + V(x) + A(\xi))^{\frac{1}{2}}|\xi - \eta|^2 \right)^N.$$

We note that it is enough to prove the first inequality, which, in turn, can be reduced to the following one

$$\frac{(q + V(y) + A(\eta))^{\frac{1}{n}}}{(q + V(x) + A(\xi))^{\frac{1}{n}}} \leq \frac{(q + V(y) + A(\xi))^{\frac{1}{n}}}{(q + V(x) + A(\xi))^{\frac{1}{n}}} \leq C \left( 2 + (q + V(x) + A(\xi))^{\frac{1}{2}}|x - y| + (q + V(x) + A(\xi))^{\frac{1}{2}}|\xi - \eta| \right)^N. \quad (3.7)$$

Now (3.7) can be obtained from the inequalities below

$$\frac{\left( \frac{q}{2} + V(y) \right)^{\frac{1}{n}}}{(q + V(x) + A(\xi))^{\frac{1}{n}}} \leq \frac{\left( \frac{q}{2} + V(y) \right)^{\frac{1}{n}}}{(q + V(x) + A(\xi))^{\frac{1}{n}}} \leq C \left( 2 + (q + V(x))^{\frac{1}{2}}|x - y| \right)^N, \quad (3.8)$$

$$\frac{\left( \frac{q}{2} + A(\eta) \right)^{\frac{1}{n}}}{(q + V(x) + A(\xi))^{\frac{1}{n}}} \leq \frac{\left( \frac{q}{2} + A(\eta) \right)^{\frac{1}{n}}}{(q + V(x) + A(\xi))^{\frac{1}{n}}} \leq C \left( 2 + (q + V(x))^{\frac{1}{2}}|\xi - \eta| \right)^N. \quad (3.9)$$
In order to verify (3.8) we observe that, by choosing an integer $N$ such that \( \frac{N}{2\gamma} \geq 1 \), and applying the assumption (b) of Definition 3.1 we obtain for a suitable $C > 0$:

\[
\left( \frac{q}{2} + V(y) \right)^{\frac{1}{\gamma}} \leq C \left( |x - y| + \left( \frac{q}{2} + V(x) \right)^{\frac{1}{\gamma}} \right) \leq C \left( 1 + \left( \frac{q}{2} + V(x) \right)^{\frac{1}{\gamma}} |x - y| \right)^N \left( 1 + V(x) \right)^{\frac{1}{\gamma}}
\]

The proof of inequality (3.9) follows similarly, and the proof of Theorem 3.4 is complete. \( \square \)

We have shown that $g = g^{(A,V)}$ is a Hörmander metric, thus the $S(M, g)$ classes are defined for any $g$-weight $M$ and a Weyl-Hörmander calculus is available to our disposal.

We now make some essential observations regarding our symbol $V(x) + A(\xi)$.

**Theorem 3.5.** Let $g = g^{(A,V)}$ be the metric defined by (3.1). Then

(i) $q + V(x) + A(\xi)$ is a $g$-weight.

(ii) There exists a regular weight $m$ equivalent to $M(X) = q + V(x) + A(\xi)$, i.e., we have

\[
\left( \frac{m(Y)}{M(Y)} \right)^{\pm 1} \leq C, \text{ for all } Y, \text{ and } m \in S(m, g). \text{ Consequently, } S(m, g) = S(M, g).
\]

**Proof.** (i) A look at the proof of the continuity and temperateness of the metric $g$ shows that $q + V(x) + A(\xi)$ is a $g$-weight. We also give a different and more general argument showing this fact.

For the metric $g = g^{(A,V)}$ the uncertainty parameter is given by

\[
\lambda_g(X) = (q + V(x) + A(\xi))^{\frac{2\kappa + \gamma}{2\kappa + \gamma}}.
\]

In general the uncertainty parameter of a Hörmander metric $G$ is a $G$-weight (cf. [29]). So, by taking the power $\frac{2\kappa + \gamma}{2\kappa + \gamma}$ of $\lambda_g$, we obtain that $q + V(x) + A(\xi)$ is a $g$-weight. (ii) For any weight $M$ one can construct an equivalent regular weight $\tilde{M}$ as a consequence of the existence of suitable $g$-partitions of unity (cf. [30], [31], [36]). Since $q + V(x) + A(\xi)$ is a $g$-weight, in particular, there exists a regular weight $m$ equivalent to $M = q + V(x) + A(\xi)$. It is clear that $S(m, g) = S(M, g)$. \( \square \)

For some purposes it is enough to work with the equivalent weight $\tilde{M}$. In our case with the perspective of establishing Schatten-von Neumann properties we will assume some further regularity conditions that will be introduced in Section 5. In particular, it will be assumed that $A(\xi) + V(x)$ is a symbol in the class $S(q + V(x) + A(\xi), g^{(A,V)})$, a fact that holds for the main examples we are going to present.

**Example 3.** As an example we observe the relativistic Schrödinger operator $\sqrt{T - \Delta} + V(x)$ with $V$ satisfying the Definition 3.1 for some $\kappa > 0$ and $V \in S(q + V(x) + A(\xi), g^{(A,V)})$ with $A(D) = \sqrt{T - \Delta}$. In this case we have $\gamma = \frac{1}{2}$ and
\[ (\xi) + V(x) \in S(\lambda_g^{\frac{\kappa}{\kappa+\frac{1}{2}}}, g^{(A,V)}), \]

i.e. the relativistic Schrödinger operator is of order \( \frac{\kappa}{\kappa+\frac{1}{2}} \) with respect to \( g^{(A,V)} \).

### 4. An intrinsic definition of the classes \( S(\lambda_g^m, g) \)

Let \( g = g^{(A,V)} \) be the Hörmander metric defined by (3.1). Since we have already a guarantee of the existence of a corresponding pseudo-differential calculus, we can now define the class \( S(\lambda_g^m, g) \), for \( m \in \mathbb{R} \), in an equivalent and intrinsic way without referring explicitly to the metric \( g \).

**Definition 4.1.** Let \( m \in \mathbb{R} \), and let \( A(\xi), V(x) \) be a \( \gamma \)-function and a \( \kappa \)-function, respectively. For \( a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \), we will say that \( a \in \Sigma^m_{A,V} \), if for some \( q > 0 \) large enough, and for all multi-indices \( \alpha, \beta \) there exists a constant \( C_{\alpha,\beta} \) such that

\[
|\partial^\beta_x \partial^\alpha_\xi a(x,\xi)| \leq C_{\alpha,\beta}(q + V(x) + A(\xi))^{m - \frac{\|\beta\|}{\kappa} - \frac{\|\alpha\|}{\kappa}}, \tag{4.1}
\]

for all \( x, \xi \in \mathbb{R}^n \).

We point out that the definition above corresponds indeed to the one for the metric \( g^{(A,V)} \).

**Proposition 4.2.** Let \( m \in \mathbb{R} \) and let \( g = g^{(A,V)} \) be the metric defined by (3.1). Then

\[ \Sigma^m_{A,V} = S(\lambda_g^{m(\frac{2\gamma}{\kappa+\gamma})}, g). \]

**Proof.** Observe that for \( g = g^{(A,V)} \) we have

\[ q + V(x) + A(\xi) \sim \lambda_g^{\frac{2\gamma}{\kappa+\gamma}}. \]

Therefore, by (4.1), we have

\[ S(\lambda_g^{m(\frac{2\gamma}{\kappa+\gamma})}, g) \supset \Sigma^m_{A,V}. \]

On the other hand, if \( a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \in S(\lambda_g^{m(\frac{2\gamma}{\kappa+\gamma})}, g) \), then by Definition 2.2 and by taking canonical directional derivatives for every pair of multi-indices \( \alpha, \beta \) there exists a constant \( C_{\alpha,\beta} > 0 \) such that

\[
|\partial^\beta_x \partial^\alpha_\xi a(x,\xi)| \leq C_{\alpha,\beta}(q + V(x) + A(\xi))^{m - \frac{\|\beta\|}{\kappa} - \frac{\|\alpha\|}{\kappa}},
\]

for all \( x, \xi \in \mathbb{R}^n \).

Therefore \( S(\lambda_g^{m(\frac{2\gamma}{\kappa+\gamma})}, g) \supset \Sigma^m_{A,V} \), and this completes the proof. \( \Box \)
We now consider the composition formula in the setting of the classes \( \Sigma_{A,V}^m \) as a consequence of the corresponding one in the \( S(M,g) \) calculus.

**Theorem 4.3.** Let \( m_1, m_2 \in \mathbb{R} \). If \( a \in \Sigma_{A,V}^{m_1}, b \in \Sigma_{A,V}^{m_2} \). There exists \( c \in \Sigma_{A,V}^{m_1+m_2} \) such that \( a(x,D) \circ b(x,D) = c(x,D) \) and
\[
c(x, \xi) \sim \sum_{\alpha} (2\pi i)^{-|\alpha|} \partial_\xi^\alpha a(x, \xi) \partial_x^\alpha b(x, \xi),
\]
i.e., for all \( N \in \mathbb{N} \)
\[
c(x, \xi) - \sum_{|\alpha| < N} (2\pi i)^{-|\alpha|} \partial_\xi^\alpha a(x, \xi) \partial_x^\alpha b(x, \xi) \in \Sigma_{k,\ell}^{m_1+m_2-N\left(\frac{n+k}{2\pi}\right)}.
\]

The \( L^2 \) boundedness of operators in the class \( \Sigma_{A,V}^0 \) is clearly obtained from the corresponding one for the \( S(1,g) \) class (cf. [29], [30]).

**Theorem 4.4.** Let \( a \in \Sigma_{A,V}^0 \). Then, \( a(x,D) \) extends to a bounded operator \( a(x,D) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \). Moreover, the operator norm \( ||a(x,D)||_{L(L^2)} \) is a continuous seminorm in \( \Sigma_{A,V}^0 \).

5. Schatten-von Neumann classes and spectral properties

As an application of the construction of the above classes, we can now deduce some spectral properties for negative powers of operators of the form \( q + A(D) + V(x) \) in relation with their behaviour in the Schatten-von Neumann classes of operators. The main aim of this section is to provide a simple proof for the main term of the spectral asymptotics of these operators, to give the calculations for the important case of the fractional relativistic Schrödinger operators \( (I - \Delta)^\gamma + \langle x \rangle^{2\kappa} \), and in particular deduce an estimate for the rate of growth of eigenvalues of this type of operators, as explained by Examples 4 and 5.

In the context of the Weyl-Hörmander calculus it is useful to recall the following result by Toft [17]. See also [37] for further developments.

**Theorem 5.1.** Let \( g \) be a split Hörmander metric, and let \( M \) be a g-weight. For \( 1 \leq r < \infty \), if \( h_g^\frac{n}{2} M \in L^r(\mathbb{R}^n \times \mathbb{R}^n) \) for some \( k \geq 0 \), then, for any \( a \in S(M,g) \), we have
\[
a_t(x,D) \in S_r(L^2(\mathbb{R}^n)), \text{ for all } t \in \mathbb{R}.
\]

In order to obtain the desired spectral properties for the negative powers of our operator \( q + A(D) + V(x) \), we are going to assume two additional conditions, that hold for relevant cases, and in particular for our main examples. Precisely, the following definition introduces these extra assumptions:
**Definition 5.2.** We define the following conditions on the pair of functions \((A(x), V(\xi))\):

(C1) \(A, V\) are \(C^\infty\) functions on \(\mathbb{R}^n\) and \(A(\xi) + V(x) \in \Sigma_{A,V}^1\).

(C2) There exist \(\delta > 0\) and \(C > 0\) such that \(q + V(x) + A(\xi) \geq C(1 + |x| + |\xi|)^\delta\).

(C3) There exists \(\mu > 0\) such that
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (q + V(x) + A(\xi))^{-\mu} < \infty. \tag{5.1}
\]

**Remark 5.3.** Let us highlight, that condition (C2) implies condition (C3); indeed, one can choose \(\mu > \frac{2n}{\delta}\) in (C3), where \(\delta\) is the one appearing in (C2). In particular, given condition (C1), condition (C2) implies, one the one hand, that the symbol \(q + V(x) + A(\xi)\) is \(g\)-elliptic with respect to the metric \(g = g^{(A,V)}\) in the sense of [38], and, on the other hand, the integrability condition (5.1). However, condition (5.1) can also be achieved for \(\mu \leq \frac{2n}{\delta}\). This is why we chose to refer to condition (C3) explicitly, and not to include it in condition (C2). In particular, as we will see for the fractional relativistic Schrödinger operators \((I - \Delta)\gamma + \langle x \rangle^{2\kappa}\), one can choose in condition (C3) some \(\mu > \mu_0\), where \(\mu_0 = \frac{n(\kappa + \gamma)}{2 \kappa \gamma}\). For this type of operators the natural choice for \(\delta\) is \(\delta = \min\{2\kappa, 2\gamma\}\) in order to guarantee (C2). Now, comparing the lower bounds, \(\frac{2n}{\delta}\) and \(\frac{n(\kappa + \gamma)}{2 \kappa \gamma}\), for \(\mu\), we see that, for \(\kappa \neq \gamma\), we have that
\[
\frac{2n}{\delta} = \frac{2n}{\min\{2\kappa, 2\gamma\}} > \frac{n(\kappa + \gamma)}{2 \kappa \gamma}.
\]

In the case where \(\kappa = \gamma\), the two lower bounds are identical.

Therefore the exponent \(\mu_0 = \frac{n(\kappa + \gamma)}{2 \kappa \gamma}\), gives a better condition for \(\mu\) in (C3) and moreover one can show that it is sharp; see Theorem 5.10 and Remark 5.11.

Regarding the condition (C1), we give a mild sufficient condition guaranteeing such membership.

**Lemma 5.4.** Let \(m \in \mathbb{R}\), and let \(A, V\) be a \(\gamma\)-function and a \(\kappa\)-function, respectively. Let \(B = B(x), W = W(\xi)\) be \(C^\infty\) complex-valued functions defined on \(\mathbb{R}^n\) such that:

(i) for all multi-indices \(\beta\) there exists a constant \(C_\beta\) such that
\[
|\partial^\beta_x B(x)| \leq C_\beta(q + V(x) + A(\xi))^{m - \frac{|\beta|}{2n}}, \text{ for all } x \in \mathbb{R}^n; \tag{5.2}
\]

(ii) for all multi-indices \(\alpha\) there exists a constant \(C_\alpha\) such that
\[
|\partial^\alpha_\xi W(\xi)| \leq C_\alpha(q + V(x) + A(\xi))^{m - \frac{|\alpha|}{2n}}, \text{ for all } \xi \in \mathbb{R}^n. \tag{5.3}
\]

Then, \(W(\xi) + B(x) \in \Sigma_{A,V}^m\).
Proof. The assumptions (i) and (ii) ensure that $B$ and $W$ belong to $\Sigma_{A,V}^m$ according to Definition 4.1. Hence, by the general theory, we also have $W(\xi) + B(x) \in \Sigma_{A,V}^m$. $\square$

Example 4. We note that the pairs $(A(x), V(\xi))$ listed in Example 1, also satisfy the conditions given by Definition 5.2. Indeed, the condition (C1) follows from Lemma 5.4, while the condition (C2) holds by choosing $\delta = \min\{2\kappa, 2\gamma\}$. Condition (C3) then follows from condition (C2) as explained above.

For instance, let us consider the fractional relativistic Schrödinger operator $(I - \Delta)^\gamma + \langle x \rangle^{2\kappa}$, where $\gamma > 0$. We take $A(\xi) = \langle \xi \rangle^{2\gamma}, V(x) = \langle x \rangle^{2\kappa}$ and observe that $V(x), A(\xi)$ satisfy (i) and (ii), respectively, in Lemma 5.4 with $m = 1$. Therefore, $\langle \xi \rangle^{2\gamma} + \langle x \rangle^{2\kappa} \in \Sigma_{(\xi)^{2\gamma},(x)^{2\kappa}}^1$.

We can now formulate some consequences for Schatten-von Neumann classes:

Theorem 5.5. Let $g = g^{(A,V)}$ be the metric defined in (3.1), $1 \leq r < \infty$, and assume that (C3) holds for some $\mu > 0$ as in (5.1). Then, for $a \in S((q + V(x) + A(\xi))^{-\frac{\mu}{2}}, g)$, we have

$$a_t(x, D) \in S_r(L^2(\mathbb{R}^n)), \quad \text{for all } t \in \mathbb{R}.$$  

Proof. We observe that condition (5.1) on $\mu$ implies that

$$(q + V(x) + A(\xi))^{-\frac{\mu}{2}} \in L^r(\mathbb{R}^n \times \mathbb{R}^n).$$

Hence, an application of Theorem 5.1 with $M = (q + V(x) + A(\xi))^{-\frac{\mu}{2}}$, and for $k = 0$, yields that for $a \in S((q + V(x) + A(\xi))^{-\frac{\mu}{2}}, g)$, we have

$$a_t(x, D) \in S_r(L^2(\mathbb{R}^n)), \quad \text{for all } t \in \mathbb{R},$$

proving the theorem. $\square$

We now recall some basic properties regarding the Hilbert-Schmidt and trace class operators, see e.g. [39].

Remark 5.6. If $a^w$ extends to a Hilbert-Schmidt operator on $L^2(\mathbb{R}^n)$, then one has

$$\|a^w\|_{S_2} = (2\pi)^{-\frac{n}{2}}\|a\|_{L^2(\mathbb{R}^{2n})},$$

while if $a^w$ extends to a trace-class operator on $L^2(\mathbb{R}^n)$, then one has

$$\text{Tr}(a^w) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} a(x, \xi) dx \, d\xi.$$
Corollary 5.7. Let \( g = g^{(A,V)} \) be the metric defined by (3.1).

(a) Let \( a \in S((q + V(x) + A(\xi))^{-\nu}, g) \). Then, the following hold true:

(i) For all \( \nu \geq \frac{\mu}{2} \), where \( \mu \) is the one appearing in (5.1), and for all \( t \in \mathbb{R} \), the operator \( a_t(x,D) \) extends to a Hilbert-Schmidt operator on \( L^2(\mathbb{R}^n) \), and in particular we have

\[
\|a_t(x,D)\|_{S_2} = (2\pi)^{-\frac{n}{2}} \|a\|_{L^2(\mathbb{R}^{2n})}.
\]

(ii) For all \( \nu \geq \mu \), where \( \mu \) is the one appearing in (5.1), and for all \( t \in \mathbb{R} \), the operator \( a_t(x,D) \), extends to a trace-class operator on \( L^2(\mathbb{R}^n) \), and in particular we have

\[
\text{Tr}(a_t(x,D)) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x,\xi) \, dx \, d\xi.
\]

(b) Let us additionally assume that the pair \((A(x),V(\xi))\) satisfies the conditions (C1), (C2), and (C3) for some \( \mu > 0 \). Then, for \( 1 \leq r < \infty \) and \( \nu \geq \frac{\mu}{r} \), we have

\[
T^{-\nu} \equiv (q + A(D) + V(x))^{-\nu} \in S_r(L^2(\mathbb{R}^n)).
\]

(c) We keep the pair \((A(x),V(\xi))\) satisfying the conditions (C1), (C2) and (C3) for some \( \mu > 0 \). Then, the following hold true:

(i) The operator \( T^{-\frac{\mu}{2}} \) extends to a Hilbert-Schmidt operator on \( L^2(\mathbb{R}^n) \), and there exists a constant \( C > 0 \) such that

\[
\|T^{-\frac{\mu}{2}}\|_{S_2} \leq C \|(q + V(x) + A(\xi))^{-\frac{\mu}{2}}\|_{L^2(\mathbb{R}^{2n})}.
\]

(ii) The operator \( T^{-\mu} \) extends to a trace-class operator on \( L^2(\mathbb{R}^n) \), and there exists a constant \( C > 0 \) such that

\[
|\text{Tr}(T^{-\mu})| \leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (q + V(x) + A(\xi))^{-\mu} \, dx \, d\xi.
\]

Proof of Corollary 5.7. The proof of part (a) follows by Theorem 5.5 for \( r = 1,2 \) and by Remark 5.6, since, as mentioned above, condition (C2) implies condition (C3). We now prove part (b). We note that, on the one hand, we have \( A(D) = A^w(x,D) \), while on the other hand, the multiplication operator \( V(x) \) has the potential \( V(x) \) itself as is Kohn-Nirenberg symbol, with its Weyl symbol being equal to \( V(x) + \text{lower order} \). The last is due to the asymptotic formula for comparing the corresponding symbols with respect to different \( t \)-quantizations (cf. formula (2.3.29) of Theorem 2.3.18 in [30]).
Moreover, the $g$-ellipticity of the symbol $q + V(x) + A(\xi)$ ensures that, the negative powers $(q + A(D) + V(x))^{-\nu}$ for all $\nu > 0$, are well defined, and their symbols belong to the class $S((q + V(x) + A(\xi))^{-\nu}, g)$. Now an application of Theorem 5.5 to the symbol of $T^{-\nu}$ concludes the proof of part (b). The proof of part (c) follows from part (b), and Remark 5.6, since, arguing as in the proof of part (b), we see that the symbol $\sigma_1$ (resp. $\sigma_2$) of $T^{-\frac{\mu}{2}}$ (resp. $T^{-\mu}$) is in the class $(q + V(x) + A(\xi))^{-\frac{\mu}{2}}, g)$ (resp. $S((q + V(x) + A(\xi))^{-\mu}, g)$). The latter means that there exists some constant $C$ such that

$$|\sigma_1(x, \xi)| \leq C(q + V(x) + A(\xi))^{-\mu}, \quad \text{resp.} \quad \sigma_2 \leq C(q + V(x) + A(\xi))^{-\frac{\mu}{2}}.$$ 

The proof is now completed. \qed

We now derive some consequences for the singular values of the operators we considered. In the sequel we denote by $\lambda_j(K)$ the eigenvalues, in decreasing order, of the compact operator $K$ on the space $H$ as above, and by $s_j(K)$ the singular values of it.

**Corollary 5.8.** Let $g = g^{(A,V)}$ be the metric defined by (3.1) and let $1 \leq r < \infty$.

(i) If $a \in S((q + V(x) + A(\xi))^{-\nu}, g)$, assume (C3) holds for some $\mu > 0$ as in (5.1) and let $\nu \geq \frac{\mu}{r}$. Then

$$s_j(a_t(x,D)) = o(j^{-\frac{1}{2}}), \quad \text{as} \quad j \to \infty,$$

for every $t \in \mathbb{R}$. Consequently, also

$$\lambda_j(a_t(x,D)) = o(j^{-\frac{1}{2}}), \quad \text{as} \quad j \to \infty,$$

for every $t \in \mathbb{R}$.

(ii) If we assume that the pair $(A, V)$ satisfies conditions (C1) and (C2), then as a particular case of (i), we get that for $T^{-\nu} = (q + A(D) + V(x))^{-\nu}$, we have that

$$\lambda_j(T^{-\nu}) = o(j^{-\frac{1}{2}}), \quad \text{as} \quad j \to \infty,$$

for all $\nu \geq \frac{\mu}{r}$, where $1 \leq r < \infty$.

**Proof.** As Corollary 5.7 implies we have $a_t(x,D) \in S_r(\mathbb{R}^n)$ for $\nu$ in the range as in the statement. Hence, as it is well known, one can get for the membership of the above class that

$$s_j(a_t(x,D)) = o(j^{-\frac{1}{2}}), \quad \text{as} \quad j \to \infty.$$

Moreover from the Weyl inequality one has

$$\lambda_j(a_t(x,D)) = o(j^{-\frac{1}{2}}), \quad \text{as} \quad j \to \infty.$$
This proves (i), while similar arguments can be used to prove (ii) having into account that $T^{-\nu}$ is positive definite. □

We now derive an immediate consequence on the rate of growth of the eigenvalues. First we note that by Corollary 5.8 (ii), for $\nu = 1$ and $r \geq \mu$ we get

$$\lambda_j((q + A(D) + V(x))^{-1}) = o(j^{-\frac{1}{r}}), \quad \text{as } j \to \infty. \quad (5.5)$$

From this we obtain the following estimate for the rate of growth of the eigenvalues of $A(D) + V(x)$:

For every $L \in \mathbb{N}$ there exists $L_0 \in \mathbb{N}$ such that

$$L j^{\frac{1}{r}} \leq \lambda_j(A(D) + V(x)), \quad \text{for } j \geq L_0. \quad (5.6)$$

Summarising we have obtained the following corollary.

**Corollary 5.9.** Let $r \geq \mu$ with $\mu > 0$ as in (5.1). Then, for every $L \in \mathbb{N}$ there exists $L_0 \in \mathbb{N}$ such that

$$L j^{\frac{1}{r}} \leq \lambda_j(A(D) + V(x)), \quad \text{for } j \geq L_0. \quad (5.7)$$

Thus, the eigenvalues $\lambda_j(A(D) + V(x))$ have a growth of order at least

$$j^{\frac{1}{r}}, \quad \text{as } j \to \infty. \quad (5.8)$$

We now consider a special class of pairs $(A, V)$ satisfying the following condition:

Let $A, V$ satisfy Definition 3.1 respectively for $\gamma, \kappa$.

(C*) There exist constants $C_1, C_2 > 0$ such that:

$$A(\xi) \geq C_1|\xi|^{2\gamma} \quad \text{for } |\xi| \geq C_2; \quad V(x) \geq C_1|x|^{2\kappa} \quad \text{for } |x| \geq C_2.$$

We observe that condition (C*) implies (C2). Moreover, will see that for this type of pairs we can obtain a sharp $\mu$ satisfying (C3). We also note that all pairs $(A, V)$ taken from the list we provided in the Example 1 satisfy (C*).

**Theorem 5.10.** Let $(A, V)$ be a pair satisfying condition (C*). Then we have

$$I_\mu := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (q + A(\xi) + V(x))^{-\mu} dx d\xi < \infty, \quad (5.9)$$

provided $\mu > \frac{n(\kappa + \gamma)}{2n\kappa \gamma}$.

Consequently, if additionally (C1) holds, and $T^{-\nu} = (q + A(D) + V(x))^{-\nu}$, we have

$$T^{-\nu} \in S_r(L^2(\mathbb{R}^n)), \quad (5.10)$$
provided $\nu > \frac{n(\kappa + \gamma)}{2r\kappa\gamma}$ for $1 \leq r < \infty$.

**Proof.** Since the functions $A, V$ are continuous, they are measurable, and by (C*) there exists a constant $C > 0$ such that:

$$I_\mu = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (q + A(\xi) + V(x))^{-\mu} dx d\xi \geq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1 + |x|^{2\kappa} + |\xi|^{2\gamma})^{-\mu} dx d\xi \quad (5.10)$$

Now the integral

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1 + |x|^{2\kappa} + |\xi|^{2\gamma})^{-\mu} dx d\xi$$

can be estimated on the subset $B = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n : \xi_i > 0, i = 1, \ldots, n \}$. Without loss of generality, we can assume that $\kappa \geq \gamma$. The change of variable in $B$ given by $(x, \xi) \to (x_1, \ldots, x_n, \xi_1^{\frac{\kappa}{\gamma}}, \ldots, \xi_n^{\frac{\kappa}{\gamma}})$ lead us to

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (1 + |x|^{2\kappa} + |\xi|^{2\gamma})^{-\mu} dx d\xi = C \int_{B} \int (1 + |x|^{2\kappa} + |\xi|^{2\gamma})^{-\mu} \xi_1^{\frac{\kappa}{\gamma} - 1} \cdots \xi_n^{\frac{\kappa}{\gamma} - 1} dX$$

$$\leq C \int_{B} \int (1 + |x|^{2} + |\xi|^{2})^{-\mu\kappa} |\xi|^{(\frac{\kappa}{\gamma} - 1)n} dX$$

$$\leq C \int_{B} \int (1 + |x|^{2} + |\xi|^{2})^{-\mu\kappa} |(x, \xi)|^{(\frac{\kappa}{\gamma} - 1)n} dX$$

$$\leq C \int_{B} \int |(x, \xi)|^{(\frac{\kappa}{\gamma} - 1)n - 2\mu\kappa} dX$$

$$< \infty,$$

provided $(\frac{\kappa}{\gamma} - 1)n - 2\mu\kappa < -2n$. Now, the condition $(\frac{\kappa}{\gamma} - 1)n - 2\mu\kappa < -2n$ for the convergence of the integral is reduced to $\mu > \frac{n(\kappa + \gamma)}{2r\kappa\gamma}$.

For the second part, since (C*) implies (C2), and (C3) holds with $\mu > \frac{n(\kappa + \gamma)}{2r\kappa\gamma}$ from the above estimate, an application of Corollary 5.7 (b) concludes the proof of the theorem, since $\nu > \frac{n(\kappa + \gamma)}{2r\kappa\gamma}$ implies $r\nu > \frac{n(\kappa + \gamma)}{2r\kappa\gamma}$. \(\square\)

**Remark 5.11.** It is clear that the condition $\mu > \frac{n(\kappa + \gamma)}{2r\kappa\gamma}$ is sharp, as it can be tested on the case of the symbol of the harmonic oscillator and by using the classical integrability criteria for negative powers of $\langle (x, \xi) \rangle$ of dimension $2n$.

**Example 5.** Theorem 5.10 applied to the particular case of the fractional relativistic Schrödinger operator $(I - \Delta)^\gamma + (x)^{2\kappa}$, where $\gamma > 0$, yields $((I - \Delta)^\gamma + (x)^{2\kappa})^{-\nu} \in S_r(L^2(\mathbb{R}^n))$, for $\nu > \frac{n(\kappa + \gamma)}{2r\kappa\gamma}$. Moreover, by taking $\nu = 1$, if
we can deduce from (5.7) that the growth of the eigenvalues $\lambda_j$ of $(I - \Delta)^\gamma + \langle x \rangle^{2\kappa}$ is at least of order

$$j^{\frac{1}{r}}, \text{ as } j \to \infty.$$ 

**Remark 5.12.** Still regarding the example of the fractional relativistic Schrödinger operator as above, we note that the condition in Example 5, can also be obtained if one uses a different line of arguments: if $N(\lambda)$ is the eigenvalue counting function of $(I - \Delta)^\gamma + \langle x \rangle^{2\kappa}$, then, by Theorem 3.2 of Chapter 2 in [40], we have $N(\lambda) \lesssim \int \int \langle \xi \rangle^{2\gamma + \langle x \rangle^{2\kappa} < \lambda} dx d\xi$ for large values of $\lambda$. Indeed, by the change of variable $\xi = \lambda^{\frac{1}{2\gamma}} \xi'$ and $x = \lambda^{\frac{1}{2\kappa}} x'$ we can estimate

$$N(\lambda) \lesssim \int \int \int \langle \xi' \rangle^{2\gamma + |x'|^{2\kappa} < 1} dx' d\xi' \lesssim \lambda^n \left( \frac{1}{2\gamma} + \frac{1}{2\kappa} \right), \lambda \to \infty.$$ 

From the last estimate one can deduce that $((I - \Delta)^\gamma + \langle x \rangle^{2\kappa})^{-\nu} \in S_r(L^2(\mathbb{R}^n))$ for $\nu > \frac{n(\kappa + \gamma)}{2\kappa \gamma r}$, and our claim follows.

**Declaration of competing interest**

The authors have no conflict of interest to declare.

**Data availability**

No data was used for the research described in the article.

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References

[1] M. Chatzakou, J. Delgado, M. Ruzhansky, On a class of anharmonic oscillators, J. Math. Pures Appl. 153 (9) (2021) 1–29.
[2] V.I. Osherov, V. Ushakov, The Stokes multipliers and quantization of the quartic oscillator, J. Phys. A 44 (36) (2011) 365202–365214.
[3] E. Bender, S. Boettcher, Quasi-exactly solvable quartic potential, J. Phys. A 31 (14) (1998) L263–L277.
[4] B. Helffer, D. Robert, Comportement asymptotique précis du spectre d’opérateurs globalement elliptiques dans $\mathbb{R}^n$, Goulaouic-Meyer-Schwartz Seminar, 1980–1981, École Polytech., Palaiseau 23 (2) (1981) 1–22.
[5] B. Helffer, D. Robert, Asymptotique des niveaux d’énergie pour des hamiltoniens a un degré de liberté, Duke Math. J. 49 (4) (1982) 853–868.
[6] B. Helffer, D. Robert, Propriétés asymptotiques du spectre d’opérateurs pseudodifferentiels sur $\mathbb{R}^n$, Commun. Partial Differ. Equ. 7 (1982) 795–882.
[7] B. Helffer, Théorie Spectrale Pour des Opérateurs Globalement Elliptiques. (Spectral Theory for Globally Elliptic Operators), With an English summary, Astérisque, vol. 112, Société Mathématique de France, 1984.
[8] D. Robert, Autour de l’approximation semi-classique (French) (On semiclassical approximation) Progress in Mathematics, vol. 68, Birkhäuser, Boston, MA, 1987.
[9] S.O. Durugo, J. Larinzi, Spectral properties of the massless relativistic quartic oscillator, J. Differ. Equ. 264 (5) (2018) 3775–3809.
[10] P. Cheng, W. Hebisch, A. Sikora, Bochner–Riesz profile of anharmonic oscillator $L = -\frac{d^2}{dx^2} + |x|$, J. Funct. Anal. 271 (2016) 3186–3241.
[11] E.H. Lieb, H.-T. Yau, The stability and instability of relativistic matter, Commun. Math. Phys. 118 (2) (1987) 177–213.
[12] E.H. Lieb, H.-T. Yau, The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics, Commun. Math. Phys. 112 (1) (1987) 147–174.
[13] E.H. Lieb, M. Loss, Analysis, Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 1997.
[14] V. Ambrosio, The nonlinear fractional relativistic Schrödinger equation: existence, multiplicity, decay and concentration results, Discrete Contin. Dyn. Syst. 41 (12) (2021) 5659–5705.
[15] M.M. Fall, V. Felli, Unique continuation properties for relativistic Schrödinger operators with a singular potential, Discrete Contin. Dyn. Syst. 35 (12) (2015) 5827–5867.
[16] L. Hörmander, On the asymptotic distribution of the eigenvalues of pseudodifferential operators in $\mathbb{R}^n$, Ark. Mat. 17 (2) (1979) 297–313.
[17] J. Toft, Schatten-von Neumann properties in the Weyl calculus, and calculus of metrics on symplectic vector spaces, Ann. Glob. Anal. Geom. 30 (2) (2006) 169–209, https://doi.org/10.1007/s10455-006-9027-7.
[18] J. Toft, Schatten properties for pseudo-differential operators on modulation spaces, in: Pseudo-Differential Operators, in: Lecture Notes in Math., vol. 1949, Springer, Berlin, 2008, pp. 175–202, https://doi.org/10.1007/978-3-540-68268-4_5.
[19] J. Delgado, M. Ruzhansky, Schatten classes on compact manifolds: Kernel conditions, J. Funct. Anal. 267 (2014) 772–798.
[20] J. Delgado, M. Ruzhansky, Schatten–von Neumann classes of integral operators, J. Math. Pures Appl. 154 (9) (2021) 1–29.
[21] J. Delgado, M. Ruzhansky, Schatten classes and traces on compact Lie groups, Math. Res. Lett. 24 (2017) 979–1003, arXiv:1303.3914, http://arxiv.org/abs/1303.3914.
[22] J. Delgado, M. Ruzhansky, $L^p$-nuclearity, traces, and Grothendieck-Lidskii formula on compact Lie groups, J. Math. Pures Appl. 102 (2014) 153–172, arXiv:1303.4792.
[23] A. Parmeggiani, Spectral Theory of Non-Commutative Harmonic Oscillators: An Introduction, Lecture Notes in Mathematics, Springer, 2010.
[24] A. Parmeggiani, Non-commutative harmonic oscillators and related problems, Milan J. Math. 82 (2) (2014) 343–387.
[25] A. Parmeggiani, On the spectrum of certain non-commutative harmonic oscillators and semiclassical analysis, Commun. Math. Phys. 279 (2) (2008) 285–308.
[26] A. Parmeggiani, On the spectrum of certain noncommutative harmonic oscillators, Ann. Univ. Ferrara, Sez. 7: Sci. Mat. 52 (2) (2006) 431–456.
[27] A. Parmeggiani, M. Wakayama, Non-commutative harmonic oscillators I, Forum Math. 14 (4) (2002) 539–604.
[28] A. Parmeggiani, M. Wakayama, Oscillator representations and systems of ordinary differential equations, Proc. Natl. Acad. Sci. USA 98 (1) (2001) 26–30.
[29] L. Hörmander, The Analysis of Linear Partial Differential Operators, vol. III, Springer-Verlag, 1985.
[30] N. Lerner, Metrics on the Phase Space and Non-Selfadjoint Pseudo-Differential Operators, Pseudo-Differential Operators, Birkhäuser, Basel, 2010.
[31] J.-M. Bony, B. Lerner, Quantification asymptotique et microlocalisation d’ordre supérieur I, Ann. Sci. Éc. Norm. Supér. 22 (1989) 377–433.
[32] I.C. Gohberg, M.G. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, Translated from the Russian by A. Feinstein, Translations of Mathematical Monographs, vol. 18, American Mathematical Society, Providence, R.I., 1969.
[33] M. Reed, B. Simon, Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.
[34] B. Simon, Trace Ideals and Their Applications, London Mathematical Society Lecture Note Series, vol. 35, Cambridge University Press, Cambridge, 1979.
[35] R. Schatten, Norm Ideals of Completely Continuous Operators, Second Printing, Ergebnisse der Mathematik und Ihrer Grenzgebiete, vol. 27, Springer-Verlag, Berlin, 1970.
[36] J.M. Bony, J.Y. Chemin, Espaces fonctionnels associés au calcul de Weyl-Hörmander, Bull. Soc. Math. Fr. 122 (1994) 77–118.
[37] E. Buzano, J. Toft, Schatten-von Neumann properties in the Weyl calculus, J. Funct. Anal. 259 (12) (2010) 3080–3114, https://doi.org/10.1016/j.jfa.2010.08.021.
[38] E. Buzano, F. Nicola, Complex powers of hypoelliptic pseudodifferential operators, J. Funct. Anal. 245 (2007) 353–378.
[39] F. Nicola, L. Rodino, Global-Pseudo-Differential Calculus on Euclidean Spaces. Vol. 4 of Pseudo-Differential Operators. Theory and Applications, Lecture Notes in Mathematics, vol. 1862, Birkhäuser, Basel, 2010.
[40] P. Boggiatto, E. Buzano, L. Rodino, Global Hypoellipticity and Spectral Theory, Mathematical Research, vol. 92, Akademie Verlag, Berlin, 1996.