Quantum communication using a quantum switch of quantum switches

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Abstract

The quantum switch describes a quantum operation in which the order of application of two or more quantum channels is controlled by the state of a quantum system. The state of the control system can be suitably chosen to create a quantum superposition of the causal orders of the quantum channels, which can now perform communication tasks that are impossible within the framework of the standard quantum Shannon theory. In this paper, we consider the scenario of one-shot heralded qubit communication and ask whether there exist protocols using a given quantum switch or switches that nonetheless could outperform the given ones. We answer this question in the affirmative. Specifically, we define a higher-order quantum switch composed of two quantum switches, where the order of the quantum switches is controlled by another quantum system. We then show that the quantum switches placed in a quantum superposition of their alternative causal orders can transmit a qubit, without any error, with a probability strictly higher than that achievable with the individual switches. We discuss three examples demonstrating this communication advantage over both kinds of quantum switches: those that are already useful as a resource and those that are useless. However, we also show that there are situations where no communication advantage can be had over the individual switches.

1 Introduction

Quantum theory allows for a novel causal structure in which the causal ordering between the events is controlled by a quantum system. This, in turn, leads to the quantum superposition of causal orders. The quantum switch [1], in particular, combines two or more quantum channels where the order in which they are applied on a quantum system is determined by the state of an order quantum system. For example, if the quantum switch is composed of two quantum channels $\mathcal{E}$ and $\mathcal{F}$, then the order in which they are applied is controlled by the state of an order qubit. If the state of the order qubit is $|0\rangle$, then $\mathcal{F}$ is applied before $\mathcal{E}$, but if the order qubit is $|1\rangle$, then $\mathcal{E}$ is applied before $\mathcal{F}$. However, if the order qubit is in a

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superposition state of $|0\rangle$ and $|1\rangle$, then the quantum switch creates a superposition of the two alternative orders $EF$ and $FE$. In this case, one can no longer determine the order of application of the channels. This is why the quantum switch is said to exhibit indefinite causal order, also known as causal nonseparability \cite{2, 3, 4}. It is important to note here that the quantum switch can neither be realized by a circuit with a definite causal order between the quantum channels nor by a statistical mixture of circuits using the quantum channels with definite causal orders \cite{5}. Recently, experimental realizations of the quantum switch have also been reported in various photonic set-ups \cite{6, 7, 8, 9, 10, 11, 12}.

Recent results have also established that indefinite causal order is a bonafide resource for quantum information processing tasks, such as winning nonlocal games \cite{2}, testing properties of quantum channels \cite{5, 13}, reducing communication complexity \cite{14}, and quantum communication \cite{14, 15, 16, 17, 18, 19, 20, 25}. The examples demonstrating the communication advantage, however, are perhaps the most counterintuitive: The quantum switch from two completely depolarizing channels can transmit classical information \cite{15}, even though the completely depolarizing channel has zero capacity, and the quantum switch from two completely dephasing channels can transfer quantum information with nonzero probability although, a completely dephasing channel cannot \cite{16}. Note that, in each case, the input qubit travels through the channels placed in a superposition of their alternative orders.

The present paper aims towards understanding the strengths and limitations of the quantum switch. Specifically, we consider the task of one-shot qubit communication and investigate which other ways the quantum switch could be employed to gain an advantage. Let us begin by considering the standard protocol using a quantum switch. The given quantum switch might have been composed of different quantum channels or identical copies of the same quantum channel. We further assume that the quantum channels are noisy, so the perfect transfer of qubit is not possible with any nonzero probability. In the standard protocol, a qubit simply traverses the quantum channels in a superposition of their orders; in some cases, this leads to the perfect transfer with a nonzero probability, but in some others, it does not. For example, the quantum switch from two completely depolarizing channels is useless for the reliable transmission of a qubit. So, in general, a quantum switch may or may not provide the advantage we seek, and it is not immediate a priori in which cases it would and in which it would not.

One might be tempted to think the standard protocol is perhaps the only way to obtain a communication advantage, if any, using a given quantum switch. And any other protocol using the same quantum switch can only be as good as the standard protocol but not better. However, we will show that this is not always the case. There, in fact, exists a simple protocol using a given switch or switches that could outperform the standard protocol.

Specifically, we define a higher-order quantum switch that is both simple and intuitively satisfying. The definition follows from two observations: The underlying principle of the quantum switch is the coherent control of the order of quantum channels, and the quantum switch itself is a higher-order quantum channel, a bilinear supermap \cite{16, 21, 22}. So, one could, in principle, replace the quantum channels with quantum switches. This is what we do in this paper. The higher-order quantum switch takes two quantum
switches as inputs and creates a new quantum channel, where the order of the switches is controlled by an order qubit. Then, by choosing the state of the order qubit appropriately, one can place the quantum switches in a quantum superposition of their alternative causal orders. The protocols that we discuss exploit this indefinite causal order arising from the quantum superposition of quantum switches.

We present three examples in which the higher-order quantum switch outperforms the constituent quantum switches. The first two demonstrate the communication advantage over useful quantum switches. Here we show that the probability of perfect transfer of an input qubit is strictly more than that achievable with the constituent switches. The third example shows the advantage when both the quantum switches are useless. We also briefly discuss a possible way to experimentally realize the higher-order quantum switch using a photonic set-up.

We wish to point out here that it is not a priori clear that a higher-order quantum switch, as described above, could indeed provide an advantage over its constituent quantum switches. The definition seemed natural, and that communication advantage could be had seemed plausible, but there was no guarantee. Indeed, we will show that there are instances in which no communication advantage could be had.

One could also consider even higher-order quantum switches. We specifically discuss the complexity associated with the quantum switches of even higher-order and the difficulties one would face if one wishes to employ them for any particular task, not necessarily the one considered here.

2 Quantum switch

Quantum channels describe the evolution of quantum systems. Mathematically, a quantum channel is a completely positive, trace-preserving linear map that transforms quantum states into quantum states. The action of the channels $\mathcal{E}$ and $\mathcal{F}$ on a quantum state $\rho$ can be expressed as:

$$\mathcal{E} (\rho) = \sum_i E_i \rho E_i^\dagger,$$

$$\mathcal{F} (\rho) = \sum_j F_j \rho F_j^\dagger,$$

where $\{E_i\}$ and $\{F_j\}$ are the Krauss operators satisfying $\sum_i E_i^\dagger E_i = \sum_j F_j^\dagger F_j = I$, $I$ being the identity operator. Suppose the channels are now applied sequentially. This gives rise to two possible orders:

$$\mathcal{F} \mathcal{E} (\rho) = \sum_{j,i} F_j E_i \rho E_i^\dagger F_j^\dagger,$$

$$\mathcal{E} \mathcal{F} (\rho) = \sum_{i,j} E_i F_j \rho F_j^\dagger E_i^\dagger.$$

Note that, in each of the above scenarios the order in which the channels are applied to the target state $\rho$ remains fixed. That is, in (3), $\rho$ is first subjected to $\mathcal{E}$ followed by $\mathcal{F}$, whereas in (4), it is just the opposite.
The quantum switch is a higher-order quantum channel constructed from $\mathcal{E}$, $\mathcal{F}$, and an ancilla $\alpha$—the order qubit, which is accessible only to the receiver. Note that the order qubit is assumed to be part of the communication channel. It is defined as \cite{16,21,22}:

$$S(\mathcal{E}, \mathcal{F}, \alpha)(\rho) = \sum_{i,j} K_{ij}(\rho \otimes \omega) K^\dagger_{ij},$$

(5)

where $\omega = |\omega\rangle\langle\omega|$ is the state of the order qubit and $\{K_{ij}\}$ are the Krauss operators

$$K_{ij} = E_i F_j \otimes |0\rangle\langle0| + F_j E_i \otimes |1\rangle\langle1|,$$

(6)

where $\{|0\rangle, |1\rangle\}$ is an orthonormal basis of $\mathbb{C}^2$. Note that (5) is independent of the Krauss operators $\{E_i\}$ and $\{F_j\}$ \cite{1}.

It is now evident from (6) that the order in which the channels $\mathcal{E}$ and $\mathcal{F}$ act is determined by the state of the order qubit. In particular, if $|\omega\rangle = |0\rangle$, first $\mathcal{F}$ and then $\mathcal{E}$ is applied to $\rho$, whereas if $|\omega\rangle = |1\rangle$ they are applied in the reverse order. However, if the order qubit is initially in a superposition state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, the quantum switch creates a superposition of the two orders $\mathcal{E}\mathcal{F}$ and $\mathcal{F}\mathcal{E}$. This way a quantum switch exhibits indefinite causal order. We now take the idea of the quantum switch forward and define a higher-order quantum switch constructed from two quantum switches.

## 3 Quantum switch of quantum switches

### 3.1 Definition

Consider the quantum channels $\mathcal{E}^{(x)}$ with $\{E_i^{(x)}\}$ being the corresponding set of Krauss operators for $x = 1, 2, 3, 4$. Let $S(\mathcal{E}^{(1)}, \mathcal{E}^{(2)}, \alpha) \equiv S_1$ and $S(\mathcal{E}^{(3)}, \mathcal{E}^{(4)}, \alpha) \equiv S_2$ denote the quantum switches from the pairs $(\mathcal{E}^{(1)}, \mathcal{E}^{(2)})$ and $(\mathcal{E}^{(3)}, \mathcal{E}^{(4)})$ respectively. Following (5) they are defined as

$$S_1(\rho) = \sum_{i,j} K^{(1)}_{ij}(\rho \otimes \omega) K^{(1)\dagger}_{ij},$$

(7)

$$S_2(\rho) = \sum_{i,j} K^{(2)}_{ij}(\rho \otimes \omega) K^{(2)\dagger}_{ij},$$

(8)

where

$$K^{(1)}_{ij} = E_i^{(1)} E_j^{(2)} \otimes |0\rangle\langle0| + E_j^{(2)} E_i^{(1)} \otimes |1\rangle\langle1|,$$

(9)

$$K^{(2)}_{ij} = E_i^{(3)} E_j^{(4)} \otimes |0\rangle\langle0| + E_j^{(4)} E_i^{(3)} \otimes |1\rangle\langle1|.$$
Switch from Switch 1 and Switch 2

Figure 1: The mechanism of the higher-order quantum switch constructed from two quantum switches. Observe that the order-ancilla consists of two qubits \( a' \) and \( a \). The first qubit controls the order of the switches whereas the second one controls the order of the channels making up the individual switches.

The quantum switch constructed from the quantum switches \( S_1 \) and \( S_2 \) is now defined as:

\[
S \left( S_1, S_2, a' \right) (\rho) = \sum_{i,j,k,l} K_{ijkl} (\rho \otimes \Omega) K_{ijkl}^\dagger,
\]

where \( a' \) is the order qubit controlling the order of the switches, \( \Omega = |\Omega\rangle \langle \Omega| \) is the joint state of \( aa' \), \( a \) being the order qubit associated with both \( S_1 \) and \( S_2 \), and \( \{K_{ijkl}\} \) are the Krauss operators that are defined as

\[
K_{ijkl} = K_{ij}^{(1)} K_{kl}^{(2)} \otimes |0\rangle \langle 0| + K_{kl}^{(2)} K_{ij}^{(1)} \otimes |1\rangle \langle 1|.
\]

Observe that the order-ancilla \( aa' \) is now a two-qubit system. These two qubits are accessible only to the receiver.

**Remark.** Since our configuration employs two order qubits, one has the freedom to choose any initial two-qubit state, which can even be entangled. Our examples, however, use specific product states, and we did not find any particular advantage using maximally entangled states. But we suspect this may not be the case always. So in situations (like ours or similar) where the control system is composite, one may try to see if an entangled control has an advantage over a product one.

One can also construct a quantum switch from the four channels \( E^{(x)}, x = 1, 2, 3, 4 \) (see, e.g., [19, 23]). In this case, if we allow all possible relative orders, then the dimension of the order-system will be equal to 4!. Our definition (11) of the higher-order quantum switch \( S \), however, can be viewed as a natural
extension of the quantum switch, where the quantum channels are now replaced with quantum switches (see Figure 1).

4 Quantum communication using a higher-order quantum switch

In this section, first we will discuss three examples demonstrating the relative outperformance of the higher-order quantum switch over the constituent quantum switches. Then we will discuss an example in which no communication advantage can be had using the higher-order quantum switch.

4.1 Examples demonstrating communication advantage

Example 1. Consider the Pauli channel $\mathcal{P}$ with the following Krauss operators:

$$
P_0 = \sqrt{p_0}I, \quad P_1 = \sqrt{p_1}\sigma_y, \quad P_2 = \sqrt{p_2}\sigma_z,
$$

where $(p_0, p_1, p_2)$ is a probability vector with $0 < p_0, p_1, p_2 < 1$ and $\sum_{i=0}^2 p_i = 1$, and $\sigma_y$ and $\sigma_z$ are the Pauli $y$ and $z$ matrices respectively. The action of the Pauli channel on a qubit in state $\rho$ is given by

$$
\mathcal{P}(\rho) = p_0\rho + p_1\sigma_y\rho\sigma_y + p_2\sigma_z\rho\sigma_z.
$$

From (14), it is clear that if the channel $\mathcal{P}$ is used only once, error-free qubit communication is not possible.

Following (5), the quantum switch from two Pauli channels $\mathcal{P}$ is defined as

$$
\mathbb{S}(\mathcal{P}, \mathcal{P}, a) (\rho) = \sum_{i,j=0}^2 M_{ij} (\rho \otimes \omega) M_{ij}^\dagger,
$$

where the Krauss operators $\{M_{ij}\}$ are given by

$$
M_{ij} = P_i P_j \otimes \ket{0}\bra{0} + P_j P_i \otimes \ket{1}\bra{1},
$$

and $\omega$ is the state of the order qubit $a$. After simplification, one finds that

$$
\mathbb{S}(\mathcal{P}, \mathcal{P}, a) (\rho) = q_1 \rho_1 \otimes \omega + q_2 \rho_2 \otimes \sigma_z\omega\sigma_z,
$$

where

$$
q_1 = 1 - 2p_1p_2 \quad \rho_1 = \frac{1}{q_1} \left[ \left( p_0^2 + p_1^2 + p_2^2 \right) \rho + 2p_0p_1\sigma_y\rho\sigma_y + 2p_0p_2\sigma_z\rho\sigma_z \right];
$$

$$
q_2 = 2p_1p_2 \quad \rho_2 = \sigma_x\rho\sigma_x.
$$
where $\sigma_x$ is the Pauli $x$ matrix. Note that, (17) holds for any $\omega$, so we have the freedom to choose $\omega$ appropriately.

Suppose the order qubit is initially in the state $\omega = |+\rangle \langle +|$, then $\sigma_z \omega \sigma_z = |-\rangle \langle -|$, where $|\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$. From the output expression (17) it is now easy to see that measuring the order qubit in the $\{|\pm\rangle\}$ basis, the outcome $+ will herald the presence of $\rho_1$, whereas the outcome $- will herald the presence of $\rho_2$. The first outcome, which occurs with probability $q_1$, is not of any particular interest. But in the case of the second outcome, which occurs with probability $q_2 = 2p_1p_2$, an application of $\sigma_x$ on the target qubit leads to an intact transmission of the input state. So the quantum switch defined by (15) offers a clear advantage over the Pauli channel. Note that (17) is valid for any $\omega$. Hence, the probability $q_2$ of successful transmission cannot be exceeded for any input state.

We now describe the quantum switch composed of two identical Pauli switches $S(\mathcal{P}, \mathcal{P}, a)$. It is defined as [ following (11)]:

$$S[S(\mathcal{P}, \mathcal{P}, a), S(\mathcal{P}, \mathcal{P}, a), a'] (\rho) = \sum_{i,j,k,l=0}^{2} K_{ijkl} (\rho \otimes \Omega) K_{ijkl}^\dagger, \quad (18)$$

where $\Omega = |\Omega\rangle \langle \Omega|$ is the joint state of the order qubits $a$ and $a'$, and $\{K_{ijkl}\}$ are the Krauss operators given by

$$K_{ijkl} = M_{ij}M_{kl} \otimes |0\rangle \langle 0| + M_{kl}M_{ij} \otimes |1\rangle \langle 1|. \quad (19)$$

Let us choose $|\Omega\rangle = |++\rangle$. One now obtains that

$$S[S(\mathcal{P}, \mathcal{P}, a), S(\mathcal{P}, \mathcal{P}, a), a'] (\rho) = \sum_{i=1}^{4} q_i \rho_i \otimes \Omega_i, \quad (20)$$

where

|   | $q_i$           | $\rho_i$          | $\Omega_i$        |
|---|----------------|-------------------|-------------------|
| 1 | $1 - \sum_{i=2}^{4} q_i$ | $(1 - u_1 - u_2) \rho + u_1 \sigma_y \rho \sigma_y + u_2 \sigma_z \rho \sigma_z$ | $\Omega_1 = \Omega$ |
| 2 | $8p_0^2p_1p_2$ | $\sigma_x \rho \sigma_x$ | $\Omega_2 = (I \otimes \sigma_z) \Omega (I \otimes \sigma_z)$ |
| 3 | $4p_1p_2 (p_0^2 + p_1^2 + p_2^2)$ | $\sigma_x \rho \sigma_x$ | $\Omega_3 = (\sigma_z \otimes I) \Omega (\sigma_z \otimes I)$ |
| 4 | $8p_0p_1p_2 (p_1 + p_2)$ | $\sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z$ | $\Omega_4 = (\sigma_z \otimes \sigma_z) \Omega (\sigma_z \otimes \sigma_z)$ |
\[ u_1 = \frac{4p_0p_1 \left( p_0^2 + p_1^2 + p_2^2 \right)}{(p_0 + p_1)^4 + 4p_0p_2 \left( p_0^2 + p_1^2 + p_2^2 \right) + 2p_2^2 \left( 3p_0^2 + 2p_0p_1 + 3p_1^2 \right) + p_2^4}, \quad (21) \]
\[ u_2 = \frac{4p_0p_2 \left( p_0^2 + p_1^2 + p_2^2 \right)}{(p_0 + p_1)^4 + 4p_0p_2 \left( p_0^2 + p_1^2 + p_2^2 \right) + 2p_2^2 \left( 3p_0^2 + 2p_0p_1 + 3p_1^2 \right) + p_2^4}, \quad (22) \]

Now observe that: \( \Omega_2 = |+\rangle \langle +|, \Omega_3 = |-+\rangle \langle -+|, \) and \( \Omega_4 = |--\rangle \langle --| \). Therefore, measuring each order qubit in the \( \{ |\pm\rangle \} \) basis, either of the outcomes \(+−\) and \(−+\) will herald the presence of \( \sigma_x \rho \sigma_x \). So the receiver will be able to recover the input state by applying \( \sigma_x \). This happens with probability
\[ q_{23} = q_2 + q_3 = 4p_1p_2 \left( 3p_0^2 + p_1^2 + p_2^2 \right). \quad (23) \]

Now recall that, using the Pauli-switch only once one could achieve noiseless transmission of the input qubit with probability \( q_2 = 2p_1p_2 \). So the higher-order quantum switch obtained from two Pauli switches will do better than that if there exist nonzero \( p_0, p_1, p_2 \) satisfying
\[ q_{23} = 4p_1p_2 \left( 3p_0^2 + p_1^2 + p_2^2 \right) > q_2 = 2p_1p_2, \quad (24) \]
where \( \sum_{i=0}^2 p_i = 1 \). The above condition is equivalent to
\[ 3p_0^2 + p_1^2 + p_2^2 > \frac{1}{2}, \quad (25) \]
where \( \sum_{i=0}^2 p_i = 1 \). The readers can easily convince themselves that solutions do exist. For example, suppose that \( 3p_0^2 = \frac{1}{2} \). Then we need to satisfy
\[ p_1^2 + p_2^2 > 0, \quad (26) \]
\[ \text{such that } p_1 + p_2 = 1 - \frac{1}{\sqrt{6}}. \quad (27) \]

Therefore, all pairs of \( p_1, p_2 > 0 \) satisfying \( (27) \) are legitimate solutions.

To find the complete set of solutions one may proceed as follows. Define
\[ \Delta = q_{23} - 2p_1p_2 \]
\[ = 2p_1p_2 \left[ 2 \left( 3p_0^2 + p_1^2 + p_2^2 \right) - 1 \right] \]
\[ = 2p_1p_2 \left[ 2 \left\{ 3 (1 - p_1 - p_2)^2 + p_1^2 + p_2^2 \right\} - 1 \right], \quad (28) \]
\[ \text{where to arrive at the last line we have used } p_0 = 1 - p_1 - p_2. \text{ Therefore, all possible pairs } (p_1, p_2), \text{ where } 0 < p_1, p_2 < 1 \text{ and } p_1 + p_2 < 1, \text{ satisfying } \Delta > 0 \text{ are admissible solutions (see, Figure 2).} \]
Figure 2: The shaded region depicts the possible values of $p_1$ and $p_2$ for which a single use of the quantum switch constructed from two Pauli switches enables the noiseless transfer of an arbitrary qubit with probability higher than the Pauli switch.

**Example 2.** Consider the bit flip channel $B$ with the Krauss operators

\[
B_0 = \sqrt{1-r}I, \quad B_1 = \sqrt{r}\sigma_x,
\]

where $0 < r < 1$, and the phase flip channel $G$ with the Krauss operators

\[
G_0 = \sqrt{1-s}I, \quad G_1 = \sqrt{s}\sigma_z,
\]

where $0 < s < 1$. The actions of the channels on a qubit state $\rho$ can be written as

\[
B(\rho) = (1-r)\rho + r\sigma_x \rho \sigma_x,
\]

\[
G(\rho) = (1-s)\rho + s\sigma_z \rho \sigma_z.
\]

It is clear from (31) and (32) that for single use of the above channels error-free transfer of a qubit state is not possible.

Consider now the quantum switch constructed from $B$ and $G$, which is defined as

\[
S(B, G, a)(\rho) = \sum_{i,j=0}^1 T_{ij} (\rho \otimes \omega) T_{ij}^\dagger,
\]
where

$$T_{ij} = B_i G_j \otimes |0\rangle \langle 0| + G_j B_i \otimes |1\rangle \langle 1|$$  \hspace{1cm} (34)

are the Krauss operators.

Simplifying (33) one obtains

$$S (B, G, a) (\rho) = [(1 - r) (1 - s) \rho + r (1 - s) \sigma_x \rho \sigma_x + s (1 - r) \sigma_z \rho \sigma_z] \otimes \omega$$

$$+ rs (\sigma_y \rho \sigma_y) \otimes (\sigma_z \omega \sigma_z)$$  \hspace{1cm} (35)

If we choose $\omega = |+\rangle \langle +|$, then we have $\sigma_z \omega \sigma_z = |\rangle \langle -|$. From (35) it then follows that measuring the order qubit in the $\{|\pm\rangle\}$ basis, the outcome $-$ is obtained with probability $rs$. In this case, the target qubit ends up in the state $\sigma_y \rho \sigma_y$ and can be subsequently recovered by applying $\sigma_y$. Thus perfect transfer of an arbitrary qubit is possible with probability $rs$ using the quantum switch $S (B, G, a)$ only once. Note that (35) holds for any initial state $\omega$ of the order qubit. Hence, the probability with which one achieves this perfect transfer cannot be exceeded for any input qubit.

Let us now consider the quantum switch of two identical quantum switches $S (B, G, a)$. Following (11), it is defined as:

$$S [S (B, G, a), S (B, G, a), a'] (\rho) = \sum_{i,j,k,l=0}^1 K_{ijkl} (\rho \otimes \Omega) K_{ijkl}^\dagger$$  \hspace{1cm} (36)

where $\Omega = |\rangle \langle \Omega|$ is the joint state of the order qubits $a$ and $a'$, and $\{K_{ijkl}\}$ are the Krauss operators given by

$$K_{ijkl} = T_{ij} T_{kl} \otimes |0\rangle \langle 0| + T_{kl} T_{ij} \otimes |1\rangle \langle 1|.$$  \hspace{1cm} (37)

Let us now choose $|\Omega) = |++\rangle$. Then the output is given by

$$S [S (B, G, a), S (B, G, a), a'] (\rho) = \sum_{i=1}^4 q_i' g_i' \otimes \Omega_i'$$  \hspace{1cm} (38)

where
Figure 3: The shaded region depicts the possible values of $r$ and $s$ for which a single use of the quantum switch from two quantum switches $S(B, G, a)$ enables the noiseless transfer of an arbitrary qubit with probability higher than $S(B, G, a)$.

| $q'_i$ | $q'_i$ | $\Omega'_i$ |
|--------|--------|-------------|
| $q'_1 = 1 - \sum_{i=2}^{4} q'_i$ | $q'_1 = (1 - v_1 - v_2) \rho + v_1 \sigma_x \rho \sigma_x + v_2 \sigma_z \rho \sigma_z$ | $\Omega'_1 = \Omega$ |
| $q'_2 = 2rs (1 - r) (1 - s)$ | $q'_2 = \sigma_y \rho \sigma_y$ | $\Omega'_2 = (I \otimes \sigma_z) \Omega (I \otimes \sigma_z)$ |
| $q'_3 = q'_2$ | $q'_3 = \sigma_y \rho \sigma_y$ | $\Omega'_3 = (\sigma_z \otimes I) \Omega (\sigma_z \otimes I)$ |
| $q'_4 = 2rs (r + s - 2rs)$ | $q'_4 = \frac{1}{(r+s-2rs)} [s (1 - r) \sigma_x \rho \sigma_x + r (1 - s) \sigma_z \rho \sigma_z]$ | $\Omega'_4 = (\sigma_z \otimes \sigma_z) \Omega (\sigma_z \otimes \sigma_z)$ |

with

\[
v_1 = \frac{2r (1 - r) (1 - s)^2}{1 - 2rs (2 - r - s)}, \quad (39)
\]

\[
v_2 = \frac{2s (1 - s) (1 - r)^2}{1 - 2rs (2 - r - s)}, \quad (40)
\]

Now: $\Omega'_2 = |++ \rangle \langle ++|$, $\Omega'_3 = |++ \rangle \langle --|$, and $\Omega'_4 = |-- \rangle \langle --|$. Therefore, measuring each order qubit in the $\{|\pm\}\}$ basis will herald the presence of $\sigma_y \rho \sigma_y$ for each of the outcomes $++$ and $--$ and the receiver can now recover the input state by applying $\sigma_y$. Hence, with probability

\[
q'_{23} = q'_2 + q'_3 = 4rs (1 - r) (1 - s) \quad (41)
\]

the higher-order quantum switch achieves error-free transfer of the input state.

Once again, this whole exercise will yield something meaningful provided we could find $0 < r, s < 1$.
such that \( q'_{23} > rs \), that is,

\[
4rs (1 - r) (1 - s) > rs
\]  

(42)

or, equivalently,

\[
(1 - r) (1 - s) > \frac{1}{4}.
\]  

(43)

Whenever the above inequality is satisfied the higher-order quantum switch given by (36) will transfer an arbitrary qubit without any error with a probability higher than that achievable using the quantum switch (33). One can now easily see that there exist \( 0 < r, s < 1 \) so that the above inequality is indeed satisfied. For example, for all \( 0 < r, s < \frac{1}{2} \), (43) will be satisfied. This is, of course, a subset of all possible solutions, the complete set of admissible solutions is depicted in Figure 3.

**Example 3.** Consider now two quantum switches, the first one constructed from two identical bit flip channels \( B \) defined in (29) and (31), and the second one from two identical phase flip channels \( G \) defined in (30) and (32):

\[
S (B, B, a) (\rho) = \sum_{i,j=0}^1 V_{ij} (\rho \otimes \omega) V_{ij}^\dagger,
\]  

(44)

\[
S (G, G, a) (\rho) = \sum_{i,j=0}^1 W_{ij} (\rho \otimes \omega) W_{ij}^\dagger,
\]  

(45)

where

\[
V_{ij} = B_i B_j \otimes |0\rangle \langle 0| + B_j B_i \otimes |1\rangle \langle 1|
\]  

(46)

\[
W_{ij} = G_i G_j \otimes |0\rangle \langle 0| + G_j G_i \otimes |1\rangle \langle 1|
\]  

(47)

are the corresponding Krauss operators.

Simplifying (46) and (47) one obtains

\[
S (B, B, a) (\rho) = [(1 - b) \rho + b\sigma_x \rho \sigma_x] \otimes \omega,
\]  

(48)

\[
S (G, G, a) (\rho) = [(1 - g) \rho + g\sigma_z \rho \sigma_z] \otimes \omega,
\]  

(49)

where \( b = 2r (1 - r) \) and \( g = 2s (1 - s) \). Since (48) and (49) hold for an arbitrary \( \omega \), we conclude that neither switch when used only once can transfer a qubit without error.
Let us now consider the quantum switch of $S(B, B, a)$ and $S(G, G, a)$. Following (11) it is defined as:

$$S[S(B, B, a), S(G, G, a), a'] (\rho) = \sum_{i,j,k,l=0} K_{ijkl} (\rho \otimes \Omega) K_{ijkl}^\dagger,$$

(50)

where $\Omega = |\Omega\rangle \langle \Omega|$ is the joint state of the order qubits $a$ and $a'$, and $\{K_{ijkl}\}$ are the Krauss operators given by

$$K_{ijkl} = V_{ij} W_{kl} \otimes |0\rangle \langle 0| + W_{kl} V_{ij} \otimes |1\rangle \langle 1|.$$

(51)

Let us now choose $|\Omega\rangle = |++\rangle$. Then, we get

$$S[S(B, B, a), S(G, G, a), a'] (\rho) = \sum_{i=1}^2 q''_i \rho''_i \otimes \Omega''_i,$$

(52)

where

| $q''_i$ | $\rho''_i$ | $\Omega''_i$ |
|--------|--------|--------|
| $q''_1 = 1 - q''_2$ | $\rho''_1 = (1 - w_1 - w_2) \rho + w_1 \sigma_x \rho \sigma_x + w_2 \sigma_x \rho \sigma_z$ | $\Omega''_1 = \Omega$ |
| $q''_2 = 4rs (1 - r)(1 - s)$ | $\rho''_2 = \sigma_y \rho \sigma_y$ | $\Omega''_2 = (I \otimes \sigma_z) \Omega (I \otimes \sigma_z)$ |

with

$$w_1 = \frac{2[2 (1 - s) s - 1] (1 - r) r}{4rs (1 - r)(1 - s) - 1},$$

(53)

$$w_2 = \frac{2[2 (1 - r) r - 1] (1 - s) s}{4rs (1 - r)(1 - s) - 1}.$$  

(54)

Since $\Omega''_2 = |+\rangle \langle +|$ measuring each of the two order qubits in the $\{|\pm\rangle\}$ basis will herald the presence of $\sigma_y \rho \sigma_y$ whenever the outcome is $+$. This will happen with probability $q''_2 = 4rs (1 - r)(1 - s)$ and when it does, the input state can be recovered completely by applying $\sigma_y$. So with probability $q''_2$ one achieves error-free transfer of a qubit for single use of the higher-order switch.

Now, recall that the two switches $S(B, B, a)$ and $S(G, G, a)$ are completely useless for they cannot perfectly transfer a qubit with a nonzero probability. But, as we have just shown, the higher-order switch can perform this task with a nonzero probability for all $0 < r, s < 1$. So a higher-order quantum switch, even if composed of useless quantum switches, can function as a resource.
4.2 No communication advantage using a higher-order switch

The higher-order quantum switch, however, does not always provide an advantage over its constituent quantum switches. We now present an example in which no advantage could be gained over the constituent switches. In particular, the higher order switch does no better (nor worse) than the constituent switches.

Consider the bit flip channel \( B^{1/2} \) and the phase flip channel \( G^{1/2} \) defined by (29) and (30) for \( r = 1/2 \) and \( s = 1/2 \), respectively. While the channels are useless for noiseless transfer of a qubit, the quantum switch \( S \left( B^{1/2}, G^{1/2}, a \right) \) constructed from \( B^{1/2} \) and \( G^{1/2} \) is useful as a resource. In particular, it allows for noiseless transfer of a qubit in the single-shot case with probability \( 1/4 \) [16].

Consider the quantum switch of two identical quantum switches \( S \left( B^{1/2}, G^{1/2}, a \right) \). Following (36), it is defined as:

\[
S \left[ S \left( B^{1/2}, G^{1/2}, a \right), S \left( B^{1/2}, G^{1/2}, a \right), a' \right] (\rho) = \sum_{i,j,k,l=0}^{1} K_{ijkl}^{1/2} (\rho \otimes \Omega) K_{ijkl}^{1/2\dagger},
\]

where \( \Omega = |\Omega\rangle \langle \Omega| \) is the joint state of the order qubits \( a \) and \( a' \), and \( \{ K_{ijkl}^{1/2} \} \) are the Krauss operators. It can be shown that the probability for noiseless transfer of an input qubit does not exceed \( 1/4 \) for any choice of the joint state \( |\Omega\rangle \) of the order qubits. So, in this situation, the higher-order quantum switch fails to outperform the constituent quantum switches.

5 Discussions

In this section, we briefly discuss two things. First, the complexity associated with the quantum switches of even higher-orders and next, a possible way to experimentally implement the higher-order quantum switch discussed in this paper.

5.1 Quantum switches of even higher-orders

The definition of the higher-order quantum switch presented here can easily be generalized to construct even higher-order quantum switches. Clearly, the dimension of the order quantum system will grow, and so will the number of Krauss operators. So, there is a hierarchy, at least, theoretically, but, as explained below, the complexity will grow pretty quickly as we move up, making the general analysis extremely difficult if not intractable.

To see this, let us begin from the beginning. Consider two quantum channels, each with \( m \) Krauss operators, where \( m \geq 2 \). The quantum switch composed of these quantum channels will therefore be described in terms of \( m^2 \) Krauss operators. Let us denote this switch by \( S \). The first higher-order quantum switch, the one discussed in this paper, is composed of two quantum switches of the kind \( S \). This higher-
order switch will therefore be described by $m^4$ Krauss operators. One could now go up this hierarchy, and it is easy to see that the $n^{th}$ higher-order quantum switch will be described by $m^{4n}$ Krauss operators. So the number of the Krauss operators required for a higher-order quantum switch grows exponentially. For example, if one simply moves one step up from the situation we have considered, the resulting second higher-order quantum switch will require 256 Krauss operators.

Besides, the dimension of the ancilla system will grow as well. And that leads to a problem of a different kind. Recall that our definition involves two order qubits. One of them is required to control the order of the quantum switches, and the other does the same for the channels of the constituent switches. Assuming this could be extended similarly, the $n^{th}$ higher-order quantum switch will require an ancilla system comprising $(n + 1)$ qubits. That means, to analyze such a situation with full generality, one must consider all possible $(n + 1)$ qubit initial states.

For example, even if we start with quantum channels with two Krauss operators, the number of Krauss operators will grow as $2^{4n}$, where $n$ is the order of the quantum switch, and such a higher-order quantum switch will also require an ancilla system of dimension $2^{n+1}$. While the higher-order quantum switches could indeed provide the communication advantage over the lower-order ones, it is not clear whether it is worth considering them, for the communication advantage, if any, could well be incremental or even capped.

5.2 Experimental realization of the higher-order quantum switch

The results presented here can be implemented by suitably modifying and generalizing the techniques adopted in the recent experiments [6, 7, 8, 9, 10, 11, 12]. For example, following [12], the input qubit can be realized through the internal polarization degree of freedom associated with a photon. The effects of the different types of channels considered in this paper can then be implemented using liquid-crystal wave plates that can rapidly implement different polarization rotations [12]. Each quantum switch can be created through a Mach-Zehnder interferometer that creates a quantum superposition of the path degrees of freedom using a 50 : 50 beam splitter. Here the order-qubit is realized via the path degrees of freedom. In particular, the states $|0\rangle$ and $|1\rangle$ of the order qubit correspond to a photon travelling the first arm and the second arm of the interferometer, respectively. Then to realize a quantum switch the interferometer needs to be folded into two loops, so the photon can now travel through the two channels in the two alternative orders in each arm of the interferometer (see, [12] for details). Therefore, in order to create quantum switches $S_1$ and $S_2$, two folded Mach-Zehnder interferometers $I_1$ and $I_2$ are necessary. The quantum switch of two quantum switches can now be implemented using a third folded Mach-Zehnder interferometer $I_3$. Here, the two quantum switches (the Mach-Zehnder interferometers $I_1$ and $I_2$) need to appear in alternating order in each of the arms of the interferometer $I_3$. To summarize, the input qubit can be encoded in the polarization degrees of freedom, and the order-ancilla can be realized via the path degrees of freedom. Finally, each of the order qubits can be measured in the Hadamard basis by suitably setting relative phases between the different arms before recombining them at the end of the
interferometers.

6 Conclusions

The quantum switch leads to a novel causal structure where two quantum channels act in a quantum superposition of their possible causal orders [1]. It has also been shown that the indefinite causal order manifested in a quantum switch is a resource for quantum communication [15, 16, 17, 18, 19]. In this paper, we discussed a higher-order quantum switch. Here, a quantum state could pass through two quantum switches in a superposition of different causal orders, where the order of the quantum switches is controlled by an order qubit. We showed that in one-shot heralded quantum communication this higher-order quantum switch can perform better than the individual quantum switches. In particular, two quantum switches placed in a quantum superposition of their alternative causal orders can transmit a qubit without any error with a probability higher than that of the individual quantum switches. We discussed three examples in detail. The first two showed this outperformance over useful quantum switches whereas, the last one showed that a higher-order quantum switch becomes useful even when constructed from two useless quantum switches. However, this outperformance is not something that is given. We have discussed a specific example where the higher-order quantum switch does not outperform the constituent quantum switches. We also discussed a way to realize the higher-order quantum switch in an experiment and the complexity of even higher-order quantum switches.

There, however, are situations where the communication advantage using a quantum switch can also be obtained using coherently controlled quantum channels without requiring indefinite causal order [24, 25], although this is not always the case [18]. So it would be interesting to find out whether our results or similar ones can also be reproduced in a set-up of coherently controlled quantum channels without involving indefinite causal order. We do not know the answer either way and leave it for future considerations.

There is another problem one might consider. It has been shown there exist noisy quantum channels that can act as a perfect quantum communication channel when used to form a quantum switch [17]. However, these examples are unique up to unitary freedom. In a practical situation, it could be the case that particular quantum channels are not available. This stipulates the question: How to improve the efficacy of a quantum switch from two arbitrary noisy quantum channels? We answered this question partially by showing that the performances of certain quantum switches can be improved by a higher-order quantum switch. But there could be other ways to achieve the same, so other ideas also need to be explored. So, it would be interesting to know whether a higher-order quantum switch from two “noisy” quantum switches could behave as a perfect quantum communication channel.

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