On Generalizations of Theorems of MacMahon and Subbarao

Darlison Nyirenda and Beaullah Mugwangwavari

Abstract. In this paper, we consider various theorems of P.A. MacMahon and M.V. Subbarao. For a non-negative integer \( n \), MacMahon proved that the number of partitions of \( n \) wherein parts have multiplicity greater than 1 is equal to the number of partitions of \( n \) in which odd parts are congruent to 3 modulo 6. We give a new bijective proof for this theorem and its generalization, which consequently provides a new proof of Andrews’ extension of the theorem. Considering Subbarao’s finitization of Andrews’ extension, we generalize this result of Subbarao. Our generalization is based on Glaisher’s extension of Euler’s mapping for odd-distinct partitions and as a result, a bijection given by Sellers and Fu is also extended. Unlike in the case of Sellers and Fu where two residue classes are fixed, ours takes into consideration all possible residues. Furthermore, some arithmetic properties of related partition functions are derived.

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1. Introduction

For a non-negative integer \( n \), a partition of \( n \) is a representation \((\lambda_1, \lambda_2, \ldots, \lambda_\ell)\) where \( \lambda_i \in \mathbb{Z}_{\geq 1} \), called parts, satisfy the conditions

\[ \sum_{j=1}^{\ell} \lambda_i = n \quad \text{and} \quad \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_\ell. \]

The size of a partition \( \lambda \) is denoted by \( |\lambda| \) is actually the sum of parts of \( \lambda \). In the above case, the size is \( n \). Another fruitful notation of a partition is the multiplicity notation, where parts are written with their multiplicities, e.g., \((5^3, 4^2, 3, 1)\). In this partition, 5 appears 3 times, 4 appears twice, etc. Given two partitions \( \lambda \) and \( \mu \), the union \( \lambda \cup \mu \) is essentially the multiset union of \( \lambda \) and \( \mu \) (where partitions are viewed as multisets).
L. Euler was able to show that partitions of $n$ into distinct parts are equinumerous with partitions of $n$ into odd parts. This simple but powerful theorem of Euler has had deep implications in the theory of integer partitions. It is quite clear that one side of the theorem statement places emphasis on multiplicity of parts and the other side gives specification of the residue class to which parts belong. Of similar construction are some theorems of P. A. MacMahon and M. V. Subbarao which are the focus of this study. More explicitly, P.A MacMahon found the following theorem:

**Theorem 1.1.** (MacMahon, [7]). The number of partitions of $n$ in which odd multiplicities are greater than 1 is equal to the number of partitions of $n$ in which odd parts are congruent to 3 (mod 6).

This theorem was proved via generating functions. In 2007, Andrews, Eriksson, Petrov and Romik gave a simple explicit bijection for the theorem (see [3]). The theorem was also generalised by Andrews as follows:

**Theorem 1.2.** (Andrews, [1]). The number of partitions of $n$ in which odd multiplicities are greater than or equal to $2r + 1$ is equal to the number of partitions of $n$ in which odd parts are congruent to $2r + 1$ (mod 4$r + 2$).

Andrews’ generalisation above attracted a lot of interest and M. V. Subbarao gave a finitization of Theorem 1.2 as follows:

**Theorem 1.3.** (Subbarao’s finitization, [9]). Let $m > 1$, $r \geq 0$ be integers, and let $C_{m,r}(n)$ denote the number of partitions of $n$ such that all even multiplicities of the parts are less than $2m$, and all odd multiplicities are at least $2r + 1$ and at most $2(m + r) - 1$. Let $D_{m,r}(n)$ be the number of partitions of $n$ in which parts are either odd and congruent to $2r + 1$ (mod $4r + 2$) or even and not congruent to 0 (mod $2m$). Then $C_{m,r}(n) = D_{m,r}(n)$.

Bijective proofs have been given for Theorems 1.2 and 1.3 (see [4–6]).

Our goal in this paper is threefold: to provide a new explicit bijection for Theorem 1.2, to generalize Theorem 1.3 and consequently extend the bijective maps of Sellers and Fu [4], Rajesh, Kanna, and Dharmendra [6] and to derive some congruence and recurrence relations for related partition functions.

### 2. A New Bijection for Theorems 1.1 and 1.2

Let $A(n, 1)$ denote the set of partitions of $n$ in which odd multiplicities are greater than 1. Denote by $C(n, 1)$ the set of partitions of $n$ in which odd parts are congruent to 3 (mod 6). Define the map $\beta_1 : A(n, 1) \rightarrow C(n, 1)$ as follows. Let $(\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots) \in A(n, 1)$. If $\lambda_i \equiv 0$ (mod 2), then
If \( \lambda \equiv 1 \pmod{2} \), then
\[
\lambda_i^{m_i} \mapsto \begin{cases} 
\lambda_i^{m_i - (2\lambda_i)^{-v+2}}, (2\lambda_i)^{v+v+1}, & \text{if } m_i \equiv 2v + 1 \pmod{2r+1}, \\
\lambda_i^{m_i - 2v}, (2\lambda_i)^v, & \text{if } m_i \equiv 2v \pmod{2r+1},
\end{cases}
\]

The map \( \beta_1 \) is a bijection for Theorem 1.1. This bijection is new and has not appeared in the literature. As one would have it, \( \beta_1 \) can be generalised to a map that establishes Theorem 1.2. To do that, let \( A(n,r) \) denote the set of partitions of \( n \) in which odd multiplicities are greater than or equal to \( 2r + 1 \). Further, denote by \( C(n,r) \) the set of partitions of \( n \) in which odd parts are congruent to \( 2r + 1 \pmod{2r+2} \).

For \( r \geq 1 \), define the map \( \beta_r : A(n,r) \to C(n,r) \) as follows. Let \( (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots) \in A(n,r) \).

If \( \lambda_i \equiv 0 \pmod{2} \), then
\[
\lambda_i^{m_i} \mapsto \begin{cases} 
\lambda_i^{m_i - (2\lambda_i)^{-v+2}}, (2\lambda_i)^{v+v+1}, & \text{if } m_i \equiv 2v + 1 \pmod{2r+1}, \\
\lambda_i^{m_i - 2v}, (2\lambda_i)^v, & \text{if } m_i \equiv 2v \pmod{2r+1},
\end{cases}
\]
Table 1. The map $A(n, r) \rightarrow C(n, r)$ for $r = 3, n = 17$

| $A(n, r)$          | $\rightarrow$ | $C(n, r)$   |
|-------------------|---------------|-------------|
| $(5^2, 1^7)$      | $\mapsto$     | (10, 7)    |
| $(4^2, 1^9)$      | $\mapsto$     | (8, 7, 2)  |
| $(3^2, 2^2, 1^7)$ | $\mapsto$     | (7, 6, 4)  |
| $(3^2, 1^{11})$   | $\mapsto$     | (7, 6, 2^2) |
| $(2^4, 1^9)$      | $\mapsto$     | (7, 4^2, 2) |
| $(2^2, 1^{13})$   | $\mapsto$     | (7, 4^2, 3) |
| $(1^{17})$        | $\mapsto$     | (7, 2^5)   |

If $\lambda_i \equiv 1 \pmod{2}$, then

$\lambda_i^{m_i} \mapsto \begin{cases}
((2r+1)\lambda_i)^{m_i-(2r+2v+2)\frac{m_i}{2r+1}}, (2\lambda_i)^{r+v+1}, & \text{if } m_i \equiv 2v + 1 \pmod{2r+1}, \ 0 \leq v \leq r-1 \\
((2r+1)\lambda_i)^{m_i-2v\frac{m_i}{2r+1}}, (2\lambda_i)^v, & \text{if } m_i \equiv 2v \pmod{2r+1}, \ 0 \leq v \leq r.
\end{cases}$

The image of $(\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots)$ under the map $\beta_r$ is then given by

$\bigcup_{i \geq 1} \beta_r(\lambda_i^{m_i})$.

It is not difficult to see that $\beta_r$ defines a bijection for Theorem 1.2 and that setting $r = 1$ in $\beta_r$ gives rise to the mapping $\beta_1$.

Table 1 shows an example for $r = 3$ and $n = 17$.

The inverse of $\beta_r$ is described as follows:

Let $\mu = (\mu_1^{m_1}, \mu_2^{m_2}, \ldots) \in C(n, r)$. Then

$\mu_i^{m_i} \mapsto \begin{cases}
\left(\frac{\mu_i}{2r+1}\right)^{(2r+1)m_i}, & \mu_i \equiv 2r + 1 \pmod{4r+2} \\
\mu_i^{(2r+1)\frac{m_i}{2r+1}}, (\mu_i)^{2(m_i-(2r+1)\frac{m_i}{2r+1})}, & \mu_i \equiv 0 \pmod{2}.
\end{cases}$

Then map $\mu$ to $\bigcup_{i \geq 1} \beta_r^{-1}(\mu_i^{m_i})$.

3. Generalization of Theorem 1.3

Unless otherwise specified, we assume that $a$ and $p$ are positive integers such that $\gcd(a, p) = 1$.

For integers $m, r \geq 1$, let $B_{p,r,a,m}(n)$ denote the set of partitions of $n$ in which multiplicities which are congruent to $ja \pmod{p}$ are greater than or equal to $j(pr + a)$ and less than or equal to $j(pr + a) + p(m - 1)$ where $j = 0, 1, 2, \ldots, p-1$.

Furthermore, let $E_{p,r,a,m}(n)$ denote the set of partitions of $n$ wherein parts divisible by $p$ are not divisible by $pm$ and those not divisible by $p$ are congruent
to $-s(pr + a) \pmod{p^2r + pa}$, where $s = 1, 2, \ldots, p - 1$. Then we have the following theorem.

**Theorem 3.1.** For all integers $n \geq 0$,

$$|B_{p,r,a,m}(n)| = |E_{p,r,a,m}(n)|.$$

**Proof.** Observe that, if a part of a partition in $E_{p,r,a,m}(n)$ is divisible by $p$, then it is not divisible by $pm$. To account for such, we have the generating function:

$$\prod_{j \geq 1, j \equiv 0 \pmod{p}, j \not\equiv 0 \pmod{pm}} \frac{1}{1 - q^j}.$$

To account for other parts which are not divisible by $p$, we use

$$\prod_{s=1}^{p-1} \prod_{n=1}^{\infty} \frac{1}{1 - q^{(p^2r+ap)n - s(pr+a)}}.$$

Hence, we have

$$\sum_{n=0}^{\infty} |E_{p,r,a,m}(n)|q^n = \prod_{s=1}^{p-1} \prod_{n=1}^{\infty} \frac{1}{1 - q^{(p^2r+ap)n - s(pr+a)}} \prod_{j \geq 1, j \equiv 0 \pmod{p}, j \not\equiv 0 \pmod{pm}} \frac{1}{1 - q^j}.$$

The multiplicity of a part of a partition in $B_{p,r,a,m}(n)$ is of the form

$$\alpha(pr + a) + \beta p$$

where $\alpha = 0, 1, 2, \ldots, p - 1$ and $\beta = 0, 1, 2, \ldots, m - 1$.

So we have the generating function:

$$\sum_{n=0}^{\infty} |B_{p,r,a,m}(n)|q^n$$

$$= \prod_{n=1}^{\infty} \left(1 + q^{pn} + q^{2pn} + q^{3pn} + \cdots + q^{(m-1)pn} + q^{(pr+a)n} + q^{(pr+a+p)n} + q^{(pr+a+2p)n} + q^{(pr+a+3p)n} + \cdots + q^{(pr+a+(m-1)p)n} + q^{2(pr+a)n} + q^{(2(pr+a)+p)n} + q^{(2(pr+a)+2p)n} + q^{(2(pr+a)+3p)n} + \cdots + q^{(2(pr+a)+(m-1)p)n} + q^{3(pr+a)n} + q^{(3(pr+a)+p)n} + q^{(3(pr+a)+2p)n} + q^{(3(pr+a)+3p)n} + \cdots + q^{(3(pr+a)+(m-1)p)n} + \cdots + q^{(p-1)(pr+a)n} + q^{((p-1)(pr+a)+p)n} + q^{((p-1)(pr+a)+2p)n} + \cdots \right).$$
\[
+ q \left( (p-1)(pr+a)+(m-1)p \right) +
\prod_{n=1}^{\infty} \left( \sum_{j=0}^{m-1} q^{jpn} + q^{(pr+a)n} \sum_{j=0}^{m-1} q^{jpn} + \ldots + q^{(p-1)(pr+a)n} \sum_{j=0}^{m-1} q^{jpn} \right)
\]
\[
= \prod_{n=1}^{\infty} \left( \sum_{j=0}^{m-1} q^{jpn} + q^{(pr+a)n} \sum_{j=0}^{m-1} q^{jpn} \right)
\]
\[
= \prod_{n=1}^{\infty} \sum_{j=0}^{m-1} q^{j(pr+a)n} \sum_{i=0}^{m-1} q^{ipn}
\]
\[
= \prod_{n=1}^{\infty} \frac{1 - q^{p(pr+a)n}}{1 - q^{p(pr+a)n}} \prod_{j \geq 1, j \equiv 0 \pmod{pm}} \frac{1}{1 - q^j}
\]
\[
= \prod_{s=1}^{p-1} \prod_{n=1}^{\infty} \frac{1 - q^{(p^2r+ap)n-s(pr+a)}}{1 - q^{(p^2r+ap)n-s(pr+a)}} \prod_{j \geq 1, j \equiv 0 \pmod{pm}} \frac{1}{1 - q^j}
\]
\[
= \sum_{n=0}^{\infty} |E_{p,r,a,m}(n)|q^n.
\]

\[\square\]

**Remark 3.1.** Note that with \(p = 2, a = 1\), Theorem 3.1 reduces to Subbarao’s finitization, Theorem 1.3.

Introducing the following notation for a positive integer \(v\),

\[\text{ord}_v(j) := \max\{i \in \mathbb{Z}_{\geq 0} : v^i | j\},\]

we proceed to describe a bijection for Theorem 3.1.

Let \(\mu = (\mu_1^\omega, \mu_2^\omega, \ldots, \mu_t^\omega) \in E_{p,r,a,m}(n)\). Define a map \(\gamma : E_{p,r,a,m}(n) \rightarrow B_{p,r,a,m}(n)\) as follows.

**Case 1:** \(\mu_i \not\equiv 0 \pmod{pr+a}\).

We write \(\omega_i\) as

\[\omega_i = m^{\rho_1} + m^{\rho_2} + m^{\rho_3} + \cdots + m^{\rho_t} + \eta,\]  

where \(\rho_1 \geq \rho_2 \geq \cdots \geq \rho_t > 0\) and \(0 \leq \eta < m\). The representation of \(\omega_i\) in (3.1) is unique and arises from the \(m\)-ary expansion of \(\omega_i\). For instance, if \(\omega_i = 1 + 2 \cdot 4 + 2 \cdot 4^2 + 3 \cdot 4^4\), we rewrite \(\omega_i\) as \(1 + (4 + 4) + (4^2 + 4^2) + (4^4 + 4^4 + 4^4)\) so that \(\eta = 1, \rho_1 = \rho_2 = \rho_3 = 4, \rho_4 = \rho_5 = 2, \rho_6 = \rho_7 = 1\).

Then construct a partition

\[x_i = \left( m^{\rho_1} \times \frac{\mu_i}{p} \right)^p \cup \left( m^{\rho_2} \times \frac{\mu_i}{p} \right)^p \cup \cdots \cup \left( m^{\rho_t} \times \frac{\mu_i}{p} \right)^p.\]
Thus,
\[ \mu_i^\omega \mapsto \begin{cases} 
  x_i, & \text{if } \eta = 0; \\
  x_i \cup \left( \frac{\mu_i}{p} \right)^{p\eta}, & \text{if } 0 < \eta < m.
\end{cases} \]

Case 2: \( \mu_i \equiv 0 \pmod{pr + a} \). We write \( \omega_i \) as
\[ \omega_i = p^{\rho_1} + p^{\rho_2} + p^{\rho_3} + \cdots + p^{\rho_l} + \zeta, \tag{3.2} \]
where \( \rho_1 \geq \rho_2 \geq \cdots \geq \rho_l > 0 \) and \( 0 \leq \zeta < p \). Construct a partition
\[ y_i = \left( p^{\rho_1} \times \frac{\mu_i}{pr + a} \right)^{pr+a} \cup \left( p^{\rho_2} \times \frac{\mu_i}{pr + a} \right)^{pr+a} \cup \cdots \cup \left( p^{\rho_l} \times \frac{\mu_i}{pr + a} \right)^{pr+a}. \]

Thus,
\[ \mu_i^\omega \mapsto \begin{cases} 
  y_i, & \text{if } \zeta = 0; \\
  y_i \cup \left( \frac{\mu_i}{pr + a} \right)^{(pr+a)\zeta}, & \text{if } 0 < \zeta < p.
\end{cases} \]

The image is then defined as
\[ \gamma(\mu) = \bigcup_{i \geq 1} \gamma(\mu_i^\omega). \]

To prove that \( \gamma(\mu) \in B_{p, r, a, m}(n) \), we consider a couple of things. First, note that \( \mu \) and \( \gamma(\mu) \) have the same size. To see why this is the case, assume that
\[ \mu_i \not\equiv 0 \pmod{pr + a} \] for \( i = 1, 2, 3, \ldots, h \)
and
\[ \mu_i \equiv 0 \pmod{pr + a} \] for \( i = h + 1, h + 2, \ldots, t \).

Then
\[ |\gamma(\mu)| = \sum_{i=1}^{t} |\gamma(\mu_i^\omega)| = \sum_{i=1}^{h} |\gamma(\mu_i^\omega)| + \sum_{i=h+1}^{t} |\gamma(\mu_i^\omega)|. \]

Considering the two sums on the preceding right-hand side separately, we have
\[ \sum_{i=1}^{h} |\gamma(\mu_i^\omega)| = \sum_{i=1}^{h} \left| x_i \cup \left( \frac{\mu_i}{p} \right)^{p\eta} \right| = \sum_{i=1}^{h} |x_i| + p\eta \frac{\mu_i}{p} = \sum_{i=1}^{h} |x_i| + \eta \mu_i. \]
\[
\begin{align*}
&= \sum_{i=1}^{h} \left[ p \left( \sum_{j=1}^{\ell} \frac{\mu_i}{p} m^{p_j} \right) + \eta \mu_i \right] \\
&= \sum_{i=1}^{h} \mu_i \left( \sum_{j=1}^{\ell} m^{p_j} + \eta \right) \\
&= \sum_{i=1}^{h} \mu_i \omega_i \quad \text{(by (3.1)).}
\end{align*}
\]
and
\[
\begin{align*}
\sum_{i=h+1}^{t} |\gamma(\mu_i^{\omega_i})| &= \sum_{i=h}^{t} \left| y_i \cup \left( \frac{\mu_i}{pr + a} \right)^{pr+a} \right| \\
&= \sum_{i=h}^{t} |y_i| + (pr + a)\zeta \frac{\mu_i}{pr + a} \\
&= \sum_{i=h}^{t} |y_i| + \zeta \mu_i \\
&= \sum_{i=h}^{t} \left( (pr + a) \left( \sum_{j=1}^{\ell} \frac{\mu_i}{pr + a} p^{p_j} \right) + \zeta \mu_i \right) \\
&= \sum_{i=h}^{t} \mu_i \left( \sum_{j=1}^{\ell} p^{p_j} + \zeta \right) \\
&= \sum_{i=h}^{t} \mu_i \omega_i \quad \text{(by (3.2)).}
\end{align*}
\]
Thus,
\[
|\gamma(\mu)| = \sum_{i=1}^{h} \mu_i \omega_i + \sum_{i=h}^{t} \mu_i \omega_i \\
&= \sum_{i=1}^{t} \mu_i \omega_i \\
&= |\mu|.
\]

Second, we need to show that the multiplicities of parts in \(\gamma(\mu)\) indeed satisfy the description of multiplicities for partitions in \(B_{p,r,a,m}(n)\). Using (3.1) and (3.2), it is not difficult to see that the multiplicity of \(m^{p_j} \times \frac{\mu_i}{p}\) is \(cp\) where \(c\) is the coefficient of \(m^{p_j}\) in the base \(m\) expansion of \(\omega_i\). Thus \(c = 0, 1, 2, \ldots, m-1\). Similarly, the multiplicity of \(p^{p_j} \times \frac{\mu_i}{pr + a}\) is \(d(pr + a)\) where \(d\) is the coefficient of \(p^{p_j}\) in the base \(p\) expansion of \(\omega_i\). Hence, \(d = 0, 1, 2, \ldots, p - 1\). Clearly, the multiplicity of parts are thus \(\equiv ja \mod p\), at least \(j(pr + a)\) and at most \(j(pr + a) + p(m - 1)\) for some \(0 \leq j \leq p - 1\).
Finally, the uniqueness of the representation of $\omega_i$ in (3.1) and (3.2) implies that $\gamma$ is injective. In fact $\gamma$ is surjective and we construct its inverse in the following section.

3.1. The Inverse of $\gamma$

We now give the inverse of $\gamma$, i.e. $\gamma^{-1}$. Let $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_1}, \ldots) \in B_{p,r,a,m}(n)$. Define a map $\gamma^{-1} : B_{p,r,a,m}(n) \to E_{p,r,a,m}(n)$ as follows.

Case I: $m_i \equiv 0 \pmod{p}$

$\lambda_i^{m_i} \mapsto \left( \left( \frac{p\lambda_i}{m^r} \right)^{m_i} \right)^r$, where $r = \text{ord}_m(\lambda_i)$.

Case II: $m_i \not\equiv 0 \pmod{p}$

Thus $m_i \equiv ja \pmod{p}$ for some $j \in \{1, 2, \ldots, p-1\}$. We have

$$\lambda_i^{m_i} \mapsto \begin{cases} \left( \left( \frac{(pr + a)\lambda_i}{p^j} \right)^j \lambda_i^{m_i - j(pr + a)} \right), & \text{if } \lambda_i \not\equiv 0 \pmod{p} \\ \left( \left( \frac{(pr + a)\lambda_i}{p^j} \right)^j \lambda_i^{m_i - j(pr + a)} \right), & \text{if } \lambda_i \equiv 0 \pmod{p} \end{cases}$$

where $t = \text{ord}_p(\lambda_i)$.

In Case II, if $m_i - j(pr + a) > 0$, you apply Case I to the subpartition $\lambda_i^{m_i - j(pr + a)}$.

The image is then defined as

$$\gamma^{-1}(\lambda) = \bigcup_{i \geq 1} \gamma^{-1}(\lambda_i^{m_i}).$$

Note that $\gamma^{-1}(\lambda) \in E_{p,r,a,m}(n)$ because of the following:

In Case I, since $\frac{\lambda_i}{m^r}$ is not divisible by $m$, it follows that the image parts $p\lambda_i^{m_i}$ are divisible by $p$, but not divisible by $pm$. Such parts define subpartitions in $E_{p,r,a,m}(n)$.

In Case II, if $\lambda_i \not\equiv 0 \pmod{p}$, then $(pr + a)\lambda_i = (pr + a)(pq - s)$ for some $s = 1, 2, \ldots, p - 1$ and $q \geq 1$. Thus,

$$(pr + a)\lambda_i = (pr + a)pq - s(pr + a) = (p^2r + ap)q - s(pr + a) \equiv -s(pr + a) \pmod{p^2r + ap}.$$ 

On the other hand, if $\lambda_i \equiv 0 \pmod{p}$, then $\frac{\lambda_i}{p^j} \not\equiv 0 \pmod{p}$, and by the previous arguments, we must have $(pr + a)\frac{\lambda_i}{p^j} \equiv -s(pr + a) \pmod{p^2r + ap}$.

Clearly, in any case the image parts are subpartitions in $E_{p,r,a,m}(n)$. This in turn means $\gamma^{-1}(\lambda) \in E_{p,r,a,m}(n)$.

Remark 3.2. Theorem 3.1 generalises Theorem 4.1 in [8] and the map $\gamma$ is an extension of $\tau$ in [8].

Henceforth, we shall denote the cardinality $|B_{p,r,a,m}(n)|$ by $b_{p,r,a,m}(n)$ and $|E_{p,r,a,m}(n)|$ by $e_{p,r,a,m}(n)$. Furthermore, let

$$b_{p,r,a,\infty}(n) = \lim_{m \to \infty} b_{p,r,a,m}(n),$$
\[ e_{p,r,a,\infty}(n) = \lim_{m \to \infty} e_{p,r,a,m}(n). \]

Observe that:

\[ b_{p,r,a,\infty}(n) \] is the number of partitions of \( n \) in which multiplicities that are congruent to \( ja \mod p \) are actually greater than or equal to \( j(pr + a) \) where \( j = 0, 1, 2, \ldots, p - 1 \). Let the set of such partitions be denoted by \( B_{p,r,a,\infty}(n) \).

Also, \( e_{p,r,a,\infty}(n) \) is the number of partitions of \( n \) wherein parts not divisible by \( p \) are congruent to \( -s(pr + a) \mod (p^2r + pa) \) where \( s = 1, 2, \ldots, p - 1 \). Let the set of such partitions be denoted by \( E_{p,r,a,\infty}(n) \). Consequently, we have the following result.

**Corollary 3.1.** For all \( n \geq 0 \), we have

\[ b_{p,r,a,\infty}(n) = e_{p,r,a,\infty}(n). \]

It is immediately noticeable that this corollary is a new extension of Andrews’ theorem, Theorem 1.2 (set \( p = 2, a = 1 \)). We give a bijective proof which extends Sellers’ bijection in [4].

**Proof.** Let \( \lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots) \in B_{p,r,a,\infty}(n) \). The bijection is given as follows.

- If \( m_i \) is congruent to 0 (mod \( p \)), then
  \[ \lambda_i^{m_i} \mapsto (p\lambda_i)^{\frac{m_i}{p}}. \]
  Each of these new parts is congruent to 0 (mod \( p \)).

- If \( m_i \) is congruent to \( ja \) (mod \( p \)) for some \( 1 \leq j \leq p - 1 \), then do the following:
  We know that \( m_i \geq j(pr + a) \). Thus, we split off \( j(pr + a) \) copies of the part \( \lambda_i \) and combine any of the remaining as was done in the previous step of the algorithm. This now leaves us with \( j(pr + a) \) copies of each of the parts \( \lambda_i \) which had multiplicity \( ja \) (mod \( p \)) in the original partition. We now take \( j \) copies of each such part and realize that these define a subpartition wherein parts appear at most \( p - 1 \) times. We now apply Glaisher’s map to obtain a subpartition wherein parts are not divisible by \( p \). Finally, to get back the size of \( n \), we multiply each of the parts in this subpartition wherein parts are not divisible by \( p \) by \( pr + a \).

To reverse the transformation, let \( \mu = (\mu_1^{\omega_1}, \mu_2^{\omega_2}, \ldots, \mu_t^{\omega_t}) \in E_{p,r,a,\infty}(n) \). Then

- If \( \mu_i \) is congruent to 0 (mod \( p \)), then
  \[ \mu_i^{\omega_i} \mapsto \left( \frac{\mu_i}{p} \right)^{\omega_i}. \]
  Each new part will have multiplicity congruent to 0 (mod \( p \)).

- If \( \mu_i \) congruent to \( -s(pr + a) \mod (p^2r + pa) \), then we divide each part by \( pr + a \) and realize that these define a subpartition wherein parts are not divisible by \( p \). We now apply Glaisher’s map to obtain a subpartition wherein parts appear at most \( p - 1 \) times. Finally, to get back the size of \( n \), we repeat each part \( pr + a \) times in the subpartition wherein parts appear at most \( p - 1 \) times.

\( \square \)
In the next section, we consider a more general set $B_{v,p,r,a,m}(n)$, where $m, v, p, a, r \geq 1$ are integers, $\gcd(a, p) = 1$ and $v \leq p$. We define $B_{v,p,r,a,m}(n)$ to be the set of partitions of $n$ in which multiplicities which are congruent to $ja \mod p$ are greater than or equal to $j(pr + a)$ and less than or equal to $j(pr + a) + p(m - 1)$, where $j = 0, 1, 2, \ldots, v - 1$.

We also let $b_{v,p,r,a,m}(n) = |B_{v,p,r,a,m}(n)|$. Note that $B_{p,r,a,m}(n)$ in Theorem 3.1 is actually $B_{p,p,r,a,m}(n)$.

4. Arithmetic Properties

We recall the following identities in which $|q| < 1$:

$$\sum_{j=0}^{\infty} p(j)q^j = \prod_{j=1}^{\infty} \frac{1}{1 - q^j},$$  \hspace{1cm} (4.1)

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j+1)/2} = \prod_{j=1}^{\infty} (1 - q^j)$$  \hspace{1cm} (4.2)

and

$$\sum_{j=0}^{\infty} (-1)^j(2j + 1)q^{j(j+1)/2} = \prod_{j=1}^{\infty} (1 - q^j)^3.$$  \hspace{1cm} (4.3)

**Theorem 4.1.** For an integer $n$, If $\gcd(v, p) \nmid n$, then

$$b_{v,p,r,a,m}(n) = \sum_{j=1}^{n} (-1)^{j+1} b_{v,p,r,a,m}(n - w(j)),$$

where $w(j) = \frac{(pr + a)j(3j+1)}{2}$ for $1 \leq j \leq n$ and $b_{v,p,r,a,m}(0) := 1$ and $b_{v,p,r,a,m}(n) = 0$ for all $n < 0$.

**Proof.** By a similar manipulation as in the proof of Theorem 3.1, one can show that

$$\sum_{n=0}^{\infty} b_{v,p,r,a,m}(n)q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{v(pr+a)n})(1 - q^{pmn})}{(1 - q^{pr+a)n})(1 - q^{pmn})},$$

so that

$$\prod_{n=1}^{\infty} (1 - q^{(pr+a)n}) \sum_{n=0}^{\infty} b_{v,p,r,a,m}(n)q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{v(pr+a)n})(1 - q^{pmn})}{1 - q^{pmn}}.$$

By invoking (4.2), we have

$$\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{(pr+a)n(3n+1)/2}}{2}\right) \sum_{n=0}^{\infty} b_{v,p,r,a,m}(n)q^n$$

$$= \prod_{n=1}^{\infty} \frac{(1 - q^{v(pr+a)n})(1 - q^{pmn})}{1 - q^{pmn}}.$$  \hspace{1cm} (4.4)
The exponents of $q$ in the power series representation of the right-hand side of (4.4) are all divisible by $\gcd(v,p)$. Using the notation $[q^n] f(q)$ for the coefficient of $q^n$ in the power series representation of $f(q)$, observe that

$$[q^n] \left(1 + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(pr+a)(3n+1)}{2}} \sum_{n=0}^{\infty} b_{v,p,r,a,m}(n) q^n \right) = 0$$

for all $n$ not divisible by $\gcd(v,p)$. □

**Theorem 4.2.** Let $p > 3$ be prime. Then

$$b_{2,p,r,a,m}(pn + t) \equiv 0 \pmod{2}, \hspace{1em} n \geq 0,$$

where $24ta^{-1} + 1$ is a quadratic nonresidue modulo $p$. Here, $a^{-1}$ is the inverse of a modulo $p$.

**Proof.** From the proof of Theorem 4.1, the generating function of the sequence $b_{2,p,r,a,m}(0), b_{2,p,r,a,m}(1), \ldots$ is

$$\sum_{n=0}^{\infty} b_{2,p,r,a,m}(n) q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{(pr+a)n})}{(1 - q^{pn})} = \prod_{n=1}^{\infty} \frac{(1 + q^{(pr+a)n})(1 - q^{pmn})}{1 - q^{pn}} \equiv \prod_{n=1}^{\infty} \frac{(1 - q^{(pr+a)n})(1 - q^{pmn})}{1 - q^{pn}} \pmod{2}$$

$$= \sum_{s=-\infty}^{\infty} (-1)^s q^{\frac{(pr+a)(3s+1)}{2}} \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{jpm(3j+1)}{2}} \sum_{k=0}^{\infty} p(k) q^{pk} \quad \text{(by (4.1) and (4.2))}$$

$$= \sum_{s=-\infty}^{\infty} q^{\frac{(pr+a)(3s+1)}{2}} \sum_{j=-\infty}^{\infty} q^{\frac{jpm(3j+1)}{2}} \sum_{k=0}^{\infty} p(k) q^{pk} \pmod{2}. \quad (4.5)$$

Comparing the exponents, we have

$$pn + t = \frac{s(pr+a)(3s+1)}{2} + pmj(3j+1) + pk,$$

where $s, j \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$. Reducing the equation modulo $p$, we get

$$t \equiv \frac{sa(3s+1)}{2} \pmod{p}.$$

We have

$$\frac{2t}{3a} \equiv s^2 + \frac{s}{3} \equiv \left(s + \frac{1}{6}\right)^2 - \frac{1}{36} \pmod{p},$$

i.e.

$$\frac{24t}{a} \equiv (6s+1)^2 - 1 \pmod{p},$$

i.e.

$$24ta^{-1} + 1 \equiv (6s+1)^2 \pmod{p}.$$
So if $24ta^{-1} + 1$ is a quadratic nonresidue modulo $p$, then the coefficient of $q^n$ in the right-hand side of (4.5) must be 0. This completes the proof. □

**Theorem 4.3.** Let $p \geq 5$ be prime. Then

$$b_{4,p,r,a,m}(pn + t) \equiv 0 \pmod{2}, \ n \geq 0,$$

where $8ta^{-1} + 1$ is a quadratic nonresidue modulo $p$.

**Proof.** Observe that

$$\sum_{n=0}^{\infty} b_{4,p,r,a,m}(n)q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{4(pr+a)n})(1 - q^{pnm})}{(1 - q^{(pr+a)n})(1 - q^{pn})},$$

$$\equiv \prod_{n=1}^{\infty} \frac{(1 - q^{(pr+a)n})^4(1 - q^{pnm})}{(1 - q^{(pr+a)n})(1 - q^{pn})} \pmod{2},$$

$$= \prod_{n=1}^{\infty} \frac{(1 - q^{(pr+a)n})^3(1 - q^{pnm})}{1 - q^{pn}}$$

$$= \sum_{n=0}^{\infty} (-1)^n(2n + 1)q^{(pr+a)n(n+1)/2} \sum_{j=-\infty}^{\infty} (-1)^j q^{jpm(3j+1)/2} \sum_{k=0}^{\infty} p(k)q^{pk},$$

(by (4.1) (4.2) and (4.3))

from which the result follows by a similar reasoning as in the proof of Theorem 4.2. □

**Data availability** The authors can confirm that this manuscript has no associated data.

**Declarations**

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Darlison Nyirenda and Beaullah Mugwangwavari
School of Mathematics
University of the Witwatersrand
P. O. Wits 2050
Johannesburg
South Africa
e-mail: darlison.nyirenda@wits.ac.za

Beaullah Mugwangwavari
e-mail: 712040@students.wits.ac.za

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