Entanglement dynamics of three-qubit states in noisy channels

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Abstract

We study entanglement dynamics of the three-qubit system which is initially prepared in pure Greenberger-Horne-Zeilinger (GHZ) or W state and transmitted through one of the Pauli channels \( \sigma_x, \sigma_z, \sigma_y \) or the depolarizing channel. With the help of the lower bound for three-qubit concurrence we show that the W state preserves more entanglement than the GHZ state in transmission through the Pauli channel \( \sigma_z \). For the Pauli channels \( \sigma_x, \sigma_y \) and the depolarizing channel, however, the entanglement of the GHZ state is more resistant against decoherence than the W-type entanglement. We also briefly discuss how the accuracy of the lower bound approximation depends on the rank of the density matrix under consideration.

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I. INTRODUCTION

Entanglement is the fundamental property of quantum systems which has been found as a resource for many potential applications, such as quantum teleportation, cryptography and superdense coding [1], to name just a few. Besides the other manifestations of entanglement, quantum teleportation still holds central place in communications protocols for quantum information theory, due to a few remarkable recent experimental achievements [2–4]. A protocol for quantum teleportation was originally suggested in the seminal works of Ekert [5] and of Bennett et al. [6] to facilitate communication between two partners. The protocol enables one to transmit an unknown quantum state to the remote recipient by means of an initially shared two-qubit entangled (Bell’s) state. Soon thereafter, communication protocols for quantum teleportation between several partners, that are based on many-qubit entangled states, were suggested [7, 8]. In particular, Karlsson and Bourennane [7] suggest protocols for quantum teleportation between two and three partners with three-qubit GHZ state

\[
|GHZ\rangle = \frac{1}{\sqrt{2}} \left( |000\rangle + |111\rangle \right).
\]

Later Agrawal and Pati suggested using three-qubit W state

\[
|W\rangle = \frac{1}{2} \left( \sqrt{2} |001\rangle + |010\rangle + |100\rangle \right),
\]

in protocols for quantum teleportation between two partners and superdense coding [8]. Nowadays, protocols for quantum teleportation, that are based on many-qubit entangled states, are widely used in order to construct quantum networks [9, 10].

While protocols for quantum teleportation require pure entangled states, in practice one has to deal with entangled states that are mixed due to imperfect control during their transmission through communication channels. It is known that a pure entangled state can be extracted from several copies of mixed entangled states, if a so-called protocol for entanglement purification is applied to the mixed states [11, 12]. To provide the protocol on received mixed states it is necessary to know how much entanglement is preserved in a single mixed state after transmission through a (noisy) communication channel.

The entanglement dynamics of many-qubit entangled states under influence of different types of environment has been intensively discussed during last decade [15–18]. For instance, entanglement dynamics of three-qubit GHZ and W states for special types of environment coupling, such as a thermal bath at zero and infinite temperatures as well as the dephasing (i.e. the Pauli \( \sigma_z \)) channel, was discussed in Ref. [15]. In particular, with the help of the lower bound for three-qubit concurrence it was shown that GHZ-type entanglement is more fragile under these types of environment coupling than the entanglement of the W state.

In this contribution we also study the entanglement dynamics of three-qubit GHZ [1] and W [2] states under the influence of its environment. We compliment results which were obtained in Ref. [15] by considering entanglement evolution of the states [11, 2] in transmission through various noisy channels, namely, the Pauli channels \( \sigma_x, \sigma_z \) and \( \sigma_y \) as well as the depolarizing channel [1]. Although the entanglement dynamics of the three-qubit states in the Pauli channel \( \sigma_z \) was considered earlier in Ref. [15] we include this channel in our discussion in order to complete the investigation of the Pauli channels.

The time evolution of three-qubit (open) quantum sys-
tems which are initially prepared in the pure states \(1\)-\(2\) and transmitted through one of the Pauli channels or the depolarizing channel was recently analyzed by Jung et al. \[19\]. For each of these channels, the time evolution of the GHZ and W states is given by mixed state density matrices which were obtained by analytical solving of corresponding master equations in Lindblad form. With the help of the analytic expressions for the mixed state density matrices we describe the entanglement evolution of the mixed GHZ and W states. To quantify the entanglement of the mixed states we use the lower bound for the three-qubit concurrence which was recently suggested by Li et al. \[20\]. We show that the W state preserves more entanglement than the GHZ state in transmission through the Pauli channel \(\sigma_z\). Surprisingly, for the Pauli channels \(\sigma_x\) and \(\sigma_y\) as well as the depolarizing channel GHZ-type entanglement is more resistant against decoherence than W-type entanglement.

In addition, we briefly discuss how accuracy of the lower bound approximation for three-qubit concurrence depends on the rank of the density matrix. In general, to quantify entanglement of a three-qubit mixed state, the convex roof extension for three-qubit concurrence is already known \[21\]. Unfortunately the calculation of the convex roof implies an optimization procedure which still has no analytic solution. Of course, the optimization can be done numerically. However, the calculation of the convex roof for a density matrix with rank \(r\) allows us to optimize over \(r^3\) free parameters \[16\] which is indeed quite a formidable task. In practice moreover, one is mostly interested in the minimal (nontrivial) amount of entanglement which is preserved in a mixed state. With regard to this practical requirement several analytical lower bound approximations for the concurrence were recently suggested \[16, 20, 22\] and were justified to be proper entanglement measures. It is important to know how accurate the approximation is and from which parameters of the density matrix this accuracy depends on? As we have already mentioned in this work we use the lower bound approximation for the concurrence from Ref. \[20\] in order to describe entanglement evolution of the mixed states \(1\)-\(2\). We found that for the density matrices with rank \(r\leq 4\) the lower bound coincides with the convex roof for the concurrence. For the density matrices with higher rank the application of the lower bound becomes restricted: the lower bound can not describe long-time evolution of the mixed state.

The paper is organized as follows. In the next section we present the necessary theoretical background to describe the time evolution of the states \(1\)-\(2\) in the Pauli channels \(\sigma_z, \sigma_x, \sigma_y\) as well as the depolarizing channel. In Subsection II B we discuss the convex roof extension for many-qubit concurrence and introduce the lower bound to be used. With the help of presented technique we show the entanglement dynamics of the GHZ and the W states in transmission through the Pauli and the depolarizing channels in Sections III and IV respectively. In Section V we summarize the results which were obtained in the previous two sections. We also briefly discuss the relation between the accuracy of the lower bound approximation and the rank of the density matrix.

II. THEORY

A. Time evolution of states transmitted through noisy channels

Decoherence is a well-known quantum phenomenon that occurs for all open systems, i.e. if they are coupled to their environment. In order to classify and better understand this (often undesired) interaction of a system with its surroundings, a large number of noise models has been investigated during the last decades \[23, 24\], including various thermal bathes at either zero or infinite temperature, the phase and the amplitude damping, or just the Pauli channels \(\sigma_x, \sigma_y, \sigma_z\), etc. Following similar lines, Jung et al. \[19\] have recently analyzed the time evolution of the three-qubit GHZ \(1\) and W \(2\) states, if they are transmitted through one of the Pauli or the depolarizing channel, in order to explore the efficiency of the ‘two-sided’ teleportation protocols that are based on these entangled states. In this work \[19\], in more detail, an initially pure entangled state \(\rho(0)\) was supposed to be transmitted through (one of) these channels for the time \(t\), and its time evolution \(\rho(t)\) obtained as solution of a (Lindblad-type) master equation

\[
\frac{d\rho}{dt} = -i[H_S, \rho] + \sum_{i,\alpha} \left( L_{i,\alpha} \rho L_{i,\alpha}^\dagger - \frac{1}{2} [L_{i,\alpha}^\dagger L_{i,\alpha}, \rho] \right) \tag{3}
\]

In this master equation, the (Lindblad) operators \(L_{i,\alpha}\) were assumed to act independently upon the \(i\)-th qubit; for example, the operator \(L_{1,1} \equiv \sqrt{k} \sigma_z \otimes 1 \otimes 1\) describes the decoherence of the first qubit under a phase-flip \(\sigma_z\), and where the coupling constant \(k\) is approximately inverse to the decoherence time with regard to such a phase-flip. Later we shall also refer the Pauli channels \(\sigma_z, \sigma_x, \sigma_y\) to bit-flip and bit-phase-flip coupling of the three-qubit system to the environment \(1\). For any given Pauli channel \(\sigma_{\alpha}\), therefore, the master equation \[3\] only includes three Lindblad operators, \(L_{1,\alpha}, L_{2,\alpha}\) and \(L_{3,\alpha}\), while nine of these operators are needed for the depolarizing channel, \(L_{i,\alpha} (i = 1, 2, 3, \alpha = x, y, z)\). In the latter case, each of the qubits can be affected with equal coupling strength by all three Pauli channels simultaneously \[1\].

B. Concurrence for three-qubit mixed states

Knowing the time evolution of the three-qubit GHZ \(1\) and W \(2\) states in transmission throw the Pauli and the depolarizing channels, we still need to quantify the remaining entanglement of the mixed states after they have been passed through some of these noisy channels discussed above. For mixed states, in fact, any exact
quantification of their entanglement has been found difficult, and no general solution is known until now \cite{21} apart from two-qubit systems. In the latter case, Wooters concurrence \cite{22} provides a very powerful measure but, despite of its rather simple form for just two qubits, there is not unique generalization of this measure available even for mixed bipartite states, if the dimensions of the associated Hilbert (sub-)spaces are larger than two \cite{22, 20, 27}. The generalization of the concurrence for mixed many-partite states still remains an open problem. However, several useful lower bounds for the concurrence have been proposed in the literature \cite{13, 16, 20, 26} in order to characterize the entanglement of many-partite states. Here, we shall not discuss these suggested ‘measures’ in further detail but make use of a recent work by Li et al. \cite{20} who suggested the lower bound for the concurrence for three-qubit states which is based on the positive partial transpose and realignment separability criteria.

Before we shall further discuss the entanglement of mixed states, let us first consider the concurrence of some pure state $|\psi\rangle$. For any (pure) three-qubit state, for example, the concurrence can be expressed as \cite{20}

$$C_3(|\psi\rangle) = \sqrt{\frac{1}{3} \left( 3 - \text{Tr} \rho_1^2 - \text{Tr} \rho_2^2 - \text{Tr} \rho_3^2 \right)}, \quad (4)$$

and where the reduced density matrices $\rho_i = \text{Tr}_{jk} |\psi\rangle \langle \psi|$ with $i \neq j \neq k$ are obtained by tracing out the remaining two qubits. Using the definition (4), we can easily calculate the concurrence $C_3(|\text{GHZ}\rangle) = 1/\sqrt{2}$ and $C_3(|\text{W}\rangle) = \sqrt{3}/8$ for pure GHZ \cite{11} and pure W \cite{2} states which are considered in the present work. Although both states are known to be fully entangled \cite{21}, the definition (4) results in two different values; therefore, we shall re-normalize the expression (4) for each state in such a way, that we have $C_3(|\text{GHZ}\rangle) = C_3(|\text{W}\rangle) = 1$. This re-normalization is justified from experimental viewpoint, since a three-qubit maximally entangled state can be viewed as a single unit of a quantum teleportation protocol.

Of course, any mixed state can be expressed also as a convex sum of pure states $\{ |\psi_i\rangle \}$: $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$. Using this representation, definition (4) for the entanglement of pure states can be generalized for mixed states and gives rise to the so-called convex roof (extension) \cite{21}

$$C_3(\rho) = \min \sum_i p_i C_3(|\psi_i\rangle), \quad (5)$$

where the minimum has to be found with regard to all possible decompositions of $\rho$ into pure states $|\psi_i\rangle$ (and with positive coefficients $p_i$). The maximal number $i_{\text{max}}$ of pure states in the ensemble is called the cardinality of the decomposition. This cardinality is not fixed by the rank $r$ of the density matrix, though it is usually assumed to be $i_{\text{max}} \leq r^2$ \cite{16, 28}. In practice, therefore, one would need to search for all decompositions of $\rho$ into pure states in order to just evaluate the convex roof for a three-qubit density matrix, implying an optimization procedure with $r \times r^2 \sim r^3$ free parameters \cite{10}. This is indeed quite a formidable task already for three qubits, and the situation becomes worse since no numerical algorithm could guarantee to find the global minimum for the expression on the rhs of Eq. (4). An analytical solution is known for this optimization task but refer to a very special case \cite{29}.

Instead of finding the exact minimum for the three-qubit concurrence (5), the rather simple lower bound to this measure was suggested by Li et al. \cite{20}

$$\tau_3(\rho) \equiv \frac{1}{3} \sum_{i=1}^{6} (C_i^{123})^2 + (C_i^{132})^2 + (C_i^{231})^2, \quad (6)$$

which is given in terms of three bipartite concurrences that correspond to possible bipartite cuts of the three-qubit system. The bipartite concurrence was originally introduced in Ref. \cite{30} and was proved to be an entanglement measure. The bipartite concurrence $C_{123}^{i}$ for qubits 12 and 3 is given by sum of six terms $C_i^{123}$ in Eqn. (6): each term is expressed as

$$C_i^{123} = \max\{0, \lambda_i^1 - \lambda_i^2 - \lambda_i^3 - \lambda_i^4\}, \quad (7)$$

where the $\lambda_i^k$, $k = 1, 4$ (for given $i$) are the square roots of the four nonvanishing eigenvalues in decreasing order of the matrix $\rho \rho_i^{123}$. These matrices are not hermitian and are formed by the density matrix $\rho$ and its complex conjugate $\rho^*$, and which is further transformed by the operators $\{ S_i^{123} = L_i^{12} \otimes L_0^3, \quad i = 1...6 \}$ as: $\rho_i^{123} = S_i^{123} \rho^* S_i^{123}$. Details about the construction of these operators and the mathematics behind can be found in Ref. \cite{20}. In this notation, moreover, $L_0^3$ is the single generator of the group SO(2), while the $L_i^{12}$ are the six generators of SO(4). When the generator $L_0^3$ simply equal to the second Pauli matrix $\sigma_y$, the generators $L_i^{12}$ can be expressed by means of the totally antisymmetric Levi-Cevita symbol in four dimensions, i.e. $(L_k)_{mn} = -i \epsilon_{klmn}$; $k, l, m, n = 1, 4$ \cite{31, 32}. The bipartite concurrences $C_{12}^{i}$ and $C_{23}^{i}$ are defined in the same way as above.

With these remarks about the lower bound for the concurrence (5), we now have two possibilities to describe the time evolution of the entanglement $C_3(\rho(t))$ if an initial state $\rho(0)$ is passed through a noisy channel. To determine the convex roof (5) as function of time $t$, we would first need to split time $t$ into a set of steps, $0 < t_1 < ... < t_N$, and optimize at each time step $C_3(\rho(t_i))$ with regard to $\sim r^3$ parameters, where $r$ is the rank of the density matrix. Finally, an interpolation for all times ‘in between’ these steps need to be performed.

If, in contrast, we make use of the lower bound (6), we can evaluate this bound for the density matrix $\rho(t)$ analytically for all times $t$. In the next two sections, we shall therefore apply the lower bound (6) to the concurrence.
\[ \tau_3(\rho(t)) \] to analyze the decay of the entanglement for an initially pure GHZ (1) and pure W state (2) separately. However, since the lower bound (6) is only an approximation for the convex roof (5), we shall explore the validity of this approach if the density matrix \( \rho(t) \) departs more and more from a pure state due to its interaction with the noisy channels. In rather simple case when the density matrix has rank \( r \leq 4 \) we shall compare the lower bound for the concurrence with actual value of the convex roof obtained numerically. In Section V we discuss how the accuracy of this lower-bound approximation is related to the rank of the density matrix which differs in dependence of the initial state and considered channel.

III. ENTANGLEMENT EVOLUTION OF AN INITIAL GHZ STATE UNDER NOISE

A. Pauli channel \( \sigma_x \)

If an initially pure GHZ state (1) is transmitted through the Pauli channel \( \sigma_x \), its time evolution is obtained as solution of the master equation (3) with Lindblad operators \( \{L_{1,z}, L_{2,z}, L_{3,z}\} \) and can be expressed in terms of the rank-2 density matrix (9)

\[
\rho(t) = \frac{1}{2} \left( |00\rangle \langle 00| + |111\rangle \langle 111| \right) + \frac{1}{2} e^{-6kt} \left( |00\rangle \langle 00| + |111\rangle \langle 111| + |000\rangle \langle 000| \right) .
\]  

(8)

For this mixed state, the lower bound (6) to the three-qubit concurrence is a monotonous function of time,

\[ \tau_3(\rho(t)) = e^{-6kt} . \]

(9)

Since the rank of the density matrix (8) is two, the convex roof extension (5) for this density matrix can be evaluated analytically (15). In this case, the convex roof is shown to follow the behavior of the nondiagonal elements (up to the normalization factor). In fact, the convex roof for the density matrix (8) coincides with the lower bound (9).

B. Pauli channel \( \sigma_y \)

If the GHZ state (1) is instead transmitted through the Pauli channel \( \sigma_y \), its time evolution is given by the rank-4 density matrix (10)

\[
\rho(t) = \frac{1}{8} \left( \begin{array}{cccccccc}
\alpha_+ & 0 & 0 & 0 & 0 & 0 & 0 & \beta_1 \\
0 & \alpha_- & 0 & 0 & 0 & 0 & -\beta_2 & 0 \\
0 & 0 & \alpha_- & 0 & 0 & -\beta_2 & 0 & 0 \\
0 & 0 & 0 & \alpha_- & -\beta_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_- & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_- & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \alpha_- & \beta_1 \\
\alpha_+ & 0 & 0 & 0 & 0 & 0 & \alpha_+ & 0 \\
\end{array} \right),
\]  

(10)

with

\[ \alpha_+ = 1 + 3e^{-4kt} \quad \text{and} \quad \alpha_- = 1 - e^{-4kt} . \]

In this case, the lower bound (6) to the three-qubit concurrence becomes

\[ \tau_3(\rho(t)) = e^{-4kt} , \]  

(11)

i.e. the entanglement of the initial state decays less quickly for a bit-flip coupling of the three qubits to the environment than for a phase-flip.

For the rank-4 density matrix (10) we also calculated numerically the convex roof (5). We found that the lower bound (11) coincides with the numerical values of the convex roof.

C. Pauli channel \( \sigma_y \)

For a transmission of the GHZ (1) state through the Pauli channel \( \sigma_y \), the density matrix (10) has full rank (i.e. rank 8), with the two functions

\[ \beta_1 = 3e^{-2kt} + e^{-6kt} \quad \text{and} \quad \beta_2 = e^{-2kt} - e^{-6kt} , \]

respectively. For this matrix, the lower bound (6) to the concurrence gives rise to

\[ \tau_3(\rho(t)) = \max\{0, \frac{1}{4} \left( 3e^{-2kt} + e^{-4kt} + e^{-6kt} - 1 \right) \} , \]

(13)

or, in other words, this lower bound vanishes already after some finite time. Using the positive partial transpose separability criteria (21), we verified however that the state (12) becomes separable only asymptotically for \( t \to \infty \), which implies that the lower bound (13) does not describe the long-term behavior of the entanglement of an initial GHZ state if its is affected by bit-phase-flip noise.

For the rank-8 density matrix (12) the numerical calculation of the convex roof (5) requires optimization over \( 8^3 = 512 \) free parameters. The numerical value of the convex roof (5) for the rank-8 density matrix (12) as well as for other rank-8 density matrices discussed below has not been obtained by us yet.

D. Depolarizing channel

If the state (1) is transmitted through the depolarizing channel, its density matrix has also rank-8 and takes the
The lower bound (6) for the three-qubit concurrence $\tau_3$ as function of time $t$ for an initial GHZ state (1), if transmitted through various noisy channels: Pauli channels $\sigma_\alpha$ (solid red), $\sigma_x$ (dashed green), $\sigma_y$ (dotted blue) and the depolarizing channel (solid black).

FIG. 1: (Color online) The lower bound (6) for the three-qubit concurrence $\tau_3$ as function of time $t$ for an initial GHZ state (1), if transmitted through various noisy channels: Pauli channels $\sigma_\alpha$ (solid red), $\sigma_x$ (dashed green), $\sigma_y$ (dotted blue) and the depolarizing channel (solid black).

The depolarizing coupling of the three-qubit system to the channel is the most destructive for the entanglement. It is also remarkable that for density matrices with rank-2 and rank-4, the lower bound coincides with the convex roof and describes the entanglement evolution for all times, while this bound is not applicable for the long-time description of density matrices with rank-8 (the Pauli $\sigma_y$ and the depolarizing channels) for which it vanishes at a finite time.

IV. ENTANGLEMENT EVOLUTION OF AN INITIAL W STATE UNDER NOISE

A. Pauli channel $\sigma_\alpha$

A similar analysis as in Section III can be made if the system is initially prepared in a W state. If the state (2) is transmitted through the channel $\sigma_\alpha$, its time evolution is described by the rank-three density matrix [19]

$$\rho(t) = \frac{1}{8} \begin{pmatrix}
\alpha_+ & 0 & 0 & 0 & 0 & 0 & 0 & \gamma \\
0 & \alpha_- & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_- & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_- & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_- & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_- & 0 & 0 \\
\gamma & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_+
\end{pmatrix}, \quad (14)$$

with

$$\alpha_+ = 1 + 3e^{-8kt}, \quad \alpha_- = 1 - e^{-8kt} \quad \text{and} \quad \gamma = 4e^{-12kt}.$$  

Here, again, the lower bound (6) to the entanglement vanishes already after some finite time due to the condition

$$\tau_3(\rho(t)) = \max\{0, \frac{1}{4}(4e^{-12kt} + e^{-8kt} - 1)\}. \quad (15)$$

Fig. 1 displays the time-dependent lower bound (6) for initial GHZ state (1) if it’s transmitted through the different channels. In all cases, this lower bound decays exponentially due to the noise of the channel; in transmission throw the Pauli channels $\sigma_x$ and $\sigma_y$ the entanglement of the GHZ state decreases slowly comparing to the Pauli channels $\sigma_\alpha$. The depolarizing coupling of the three-qubit system to the channel is the most destructive for the entanglement. It is also remarkable that for density matrices with rank-2 and rank-4, the lower bound coincides with the convex roof and describes the entanglement evolution for all times, while this bound is not applicable for the long-time description of density matrices with rank-8 (the Pauli $\sigma_y$ and the depolarizing channels) for which it vanishes at a finite time.

B. Pauli channels $\sigma_x$ and $\sigma_y$

If the (initially prepared) W state is transmitted through the Pauli channels $\sigma_x$ or $\sigma_y$, a full rank-8 density matrix is obtained for its time evolution [19]

$$\rho(t) = \frac{1}{16} \begin{pmatrix}
2\alpha_2 & 0 & 0 & \pm\sqrt{2}\alpha_2 & 0 & \pm\sqrt{2}\alpha_2 & \pm\alpha_2 & 0 \\
0 & 2\alpha_1 & \sqrt{2}\alpha_1 & 0 & \sqrt{2}\alpha_1 & 0 & 0 & \pm\alpha_3 \\
0 & \sqrt{2}\alpha_1 & 2\beta_- & 0 & \alpha_1 & 0 & 0 & \pm\sqrt{2}\alpha_3 \\
\pm\sqrt{2}\alpha_2 & 0 & 0 & 2\beta_- & 0 & \alpha_4 & \sqrt{2}\alpha_4 & 0 \\
0 & \sqrt{2}\alpha_1 & \alpha_1 & 0 & 2\beta_+ & 0 & 0 & \pm\sqrt{2}\alpha_3 \\
\pm\sqrt{2}\alpha_2 & 0 & 0 & \alpha_4 & 0 & 2\beta_- & \sqrt{2}\alpha_4 & 0 \\
\pm\alpha_2 & 0 & 0 & \sqrt{2}\alpha_4 & 0 & \sqrt{2}\alpha_4 & 2\alpha_4 & 0 \\
0 & \pm\alpha_3 & \pm\sqrt{2}\alpha_3 & 0 & \pm\sqrt{2}\alpha_3 & 0 & 0 & 2\alpha_3
\end{pmatrix}, \quad (18)$$

and this gives rise to the lower bound

$$\tau_3(\rho(t)) = e^{-4kt} \quad (17)$$

for the evolution of the entanglement, which moreover coincides with the convex roof (6) as we verified numerically.
and where the + sign refers to the $\sigma_x$ and − to the $\sigma_y$ channel, respectively. The time-dependent parameters in expression (18) are given by

\[
\begin{align*}
\alpha_1 &= 1 + e^{-2kt} + e^{-4kt} + e^{-6kt} \\
\alpha_2 &= 1 + e^{-2kt} - e^{-4kt} - e^{-6kt} \\
\alpha_3 &= 1 - e^{-2kt} - e^{-4kt} + e^{-6kt} \\
\alpha_4 &= 1 - e^{-2kt} + e^{-4kt} - e^{-6kt} \quad \text{and } \beta_{\pm} = 1 \pm e^{-6kt}.
\end{align*}
\]

Since two density matrices $\rho(t)_{\pm}$ have the same structure of matrix elements, the lower bounds for these density matrices coincide. Unfortunately, achieved analytic expression for the lower bound for the density matrix (18) has no compact form and, thus, we do not show it here explicitly. At Fig. 2 the lower bound is shown with blue dashed line. As for all rank-8 density matrices above the lower bound for the density matrix (18) vanishes after finite time.

C. Depolarizing channel

Finally, if the W state (2) is transmitted through the depolarizing channel, the density matrix $\rho(t)$ has also rank-8 and is given by (19)

\[
\frac{1}{8} \left( \begin{array}{cccccccc}
\tilde{\alpha}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \tilde{\alpha}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{2}\tilde{\gamma}_+ & 0 & \sqrt{2}\tilde{\gamma}_+ & 0 & 0 & 0 \\
0 & 0 & \beta_+ & 0 & \tilde{\gamma}_+ & 0 & 0 & 0 \\
0 & 0 & \tilde{\gamma}_- & 0 & \beta_- & 0 & 0 & 0 \\
0 & 0 & \sqrt{2}\tilde{\gamma}_- & 0 & \sqrt{2}\tilde{\gamma}_- & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{2}\tilde{\gamma}_- & 0 & \sqrt{2}\tilde{\gamma}_- & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \tilde{\alpha}_3 & 0
\end{array} \right)
\]

where

\[
\begin{align*}
\tilde{\alpha}_1 &= 1 + e^{-4kt} + e^{-8kt} + e^{-12kt}, \\
\tilde{\alpha}_2 &= 1 + e^{-4kt} - e^{-8kt} - e^{-12kt}, \\
\tilde{\alpha}_3 &= 1 - e^{-4kt} - e^{-8kt} + e^{-12kt}, \\
\tilde{\alpha}_4 &= 1 - e^{-4kt} + e^{-8kt} - e^{-12kt}, \\
\tilde{\beta}_{\pm} &= 1 \pm e^{-12kt} \quad \text{and } \tilde{\gamma}_{\pm} = e^{-8kt} \pm e^{-12kt}.
\end{align*}
\]

The time-dependent lower bound (6) for initial W state (2) transmitted through the different channels is shown at Fig. 2. As in the case of the GHZ state the lower bounds for the W state decay exponentially due to the noise of the channels. In contrast to the GHZ state, the entanglement of the W state decreases slowly in transmission throw the Pauli channel $\sigma_x$ comparing to the Pauli channels $\sigma_z$ and $\sigma_y$. However, the depolarizing coupling of the three-qubit system to the channel is again the most destructive for the entanglement. For the rank-3 density matrix, moreover, the lower bound coincides with the convex roof and describes the time evolution of the entanglement for all times, while this bound is not suitable for the long-time description of density matrices with rank eight (the Pauli $\sigma_x$ and $\sigma_y$ as well as depolarizing channels).

V. RESULTS AND DISCUSSION

In previous two sections we showed explicitly the entanglement evolution of the three-qubit system which is prepared in pure GHZ (1) or W (2) state and transmitted for the time $t$ through one of the Pauli channels $\sigma_z$, $\sigma_x$, $\sigma_y$ or the depolarizing channel. Having these results we can investigate entanglement of which state between the GHZ and the W is more resistant against decoherence in transmission through the noisy channels. For the Pauli channel $\sigma_z$ the lower bounds for the GHZ and the W states are given by Eqs. (9) and (17) respectively and are shown with red solid lines at Figs. 1, 2. In fact, the W state preserves more entanglement than the GHZ state in transmission through the Pauli channel $\sigma_z$ for all times $t$. This result was obtained earlier in Ref. [15]. For the Pauli channels $\sigma_x$, $\sigma_y$ and the depolarizing channel, in contrast, the lower bounds for the GHZ state, that are shown at Fig. 1, are always higher than corresponding lower bounds for the W state at Fig. 2. The entanglement of the GHZ state is thus more resistant against decoherence than W-type entanglement in transmission through the Pauli channels $\sigma_x$, $\sigma_y$ and the depolarizing channel.

Our result extend the investigation of the entanglement dynamics of the three-qubit GHZ and W states in transmission through noisy channels, which was started in Ref. [15]. As it has been already known, the W state preserves more entanglement than the GHZ state if it is coupled to the thermal bath at zero or infinite temperature, or the Pauli channel $\sigma_z$ [13]. We showed that the W state preserves less entanglement than the GHZ state if it is coupled to the Pauli channels $\sigma_x$ or $\sigma_y$, or the
depolarizing channel.

To describe entanglement evolution of the mixed GHZ and W states we used the lower bound \( \text{[6]} \) which is an approximation for the convex roof for three-qubit concurrence \( \text{[5]} \). We showed that the lower bound coincides with the convex roof for the density matrices with rank \( r \leq 4 \). This statement, however, is true only for the density matrices which were considered in Subsections \( \text{III A} \) and \( \text{IV A} \). Whether the lower bound coincides with the convex roof for an arbitrary density matrix with rank \( r \leq 4 \) introduces an open question of great importance. We like to investigate this question in the nearest future.

For the rank-8 density matrices discussed in this work the lower bound for the concurrence vanishes after the finite time. Vanishing of the lower bound, however, does not mean vanishing of the entanglement of the mixed states. We verified with the positive partial transpose separability criteria \( \text{[21]} \) that the mixed states become separable only asymptotically for the limit \( t \to \infty \). For the rank-8 density matrices the lower bound for the concurrence can describe entanglement evolution of the mixed state for a restricted time \( t < t_0 \) only, where \( t_0 \) denotes the time when the lower bound vanishes. The comparison of the lower bound with the convex roof for the rank-8 density matrices has not been yet performed, since the calculation of the convex roof introduce quite difficult problem which requires the numerical optimization over \( 8^3 = 512 \) free parameters. This comparison is left for future research.

The lower bound approximation for the concurrence introduces a powerful simple tool to describe the entanglement of an arbitrary N-partite mixed state. Systematic investigation of the accuracy of the approximation has not already been done. The fact that the accuracy of the lower bound depends on the rank of the density matrix is intuitively understandable, since the complexity of the calculation of the convex roof also depends on the rank. In this work we started investigation of the accuracy of the lower bound approximation with regard to the rank of the density matrix. On particular examples we showed that the lower bound \( \text{[6]} \): i. coincides with the convex roof \( \text{[5]} \) for the density matrices with rank \( r \leq 4 \); and ii. vanishes after the finite time for the rank-8 density matrices. In future we like to investigate in more details how the accuracy of the lower bound approximation for three-qubit concurrence depends on the rank of the density matrix.

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