REAL FUNDAMENTAL CHEVALLEY INVOLUTIONS AND CONJUGACY CLASSES

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Abstract. Let $G$ be a connected reductive linear algebraic group defined over $\mathbb{R}$, and let $C : G \to G$ be a fundamental Chevalley involution. We show that for every $g \in G(\mathbb{R})$, $C(g)$ is conjugate to $g^{-1}$ in the group $G(\mathbb{R})$. Similar result on the Lie algebras is also obtained.

1. Introduction

Let $F$ be a field of characteristic zero, and let $G$ be a connected reductive linear algebraic group defined over $F$. When $F$ is algebraically closed, or when $G$ is a general linear group, a unitary group, an orthogonal group, or a symplectic group, it is respectively shown in [Lu, Proposition 2.6] and [MVW, Proposition 4.I.2] that there exists an involutive algebraic automorphism $\varphi : G \to G$ defined over $F$ with the following remarkable property: for every $g \in G(F)$,

\begin{equation}
\varphi(g) \text{ and } g^{-1} \text{ are conjugate to each other in } G(F).
\end{equation}

However, when $F$ is a $p$-adic field, no such automorphism exists for some $G$. For example, it was pointed out by D. Prasad ([Pr], see also [Ad, Example 2]) that no such automorphism exists when $G$ is a split exceptional group of type $G_2$, $F_4$ or $E_8$, and it was shown in [LST, Proposition 1.3] that no such automorphism exists when $G$ is a quaternionic classical group of rank $\geq 5$. In this note, we will show that such an automorphism exists when $F = \mathbb{R}$.

From now on we assume that $F = \mathbb{R}$. Recall that a Cartan subgroup of $G$ is said to be fundamental if its split rank is minimal among all Cartan subgroups of $G$. Up to conjugation by $G(\mathbb{R})$, there exists a unique fundamental Cartan subgroup of $G$ (see [Kn, Proposition 6.61] for example).

Definition 1.1. An involutive algebraic automorphism $C : G \to G$ defined over $\mathbb{R}$ is called a fundamental Chevalley involution of $G$ if there exists a fundamental Cartan subgroup $H$ of $G$ such that $C(h) = h^{-1}$ for all $h \in H(\mathbb{R})$.

Let $C : G \to G$ be a fundamental Chevalley involution. The following result, which asserts the existence and uniqueness of fundamental Chevalley involutions, is proved by Jeffrey Adams in [Ad, Theorem 1.2].
Theorem 1.2. There exists a fundamental Chevalley involution \( C \) of \( G \). If \( C' \) is another fundamental Chevalley involution of \( G \), then

\[
C' = \text{Ad}_g \circ C \circ \text{Ad}_{g^{-1}}
\]

for some \( g \in G(\mathbb{R}) \), where \( \text{Ad}_g : G \to G \) denotes the conjugation by \( g \).

Let \( C : G \to G \) be a fundamental Chevalley involution. In [Ad, Theorem 1.2], Adams also proves that for every semisimple element \( g \in G(\mathbb{R}) \), \( C(g) \) and \( g^{-1} \) are conjugate to each other in \( G(\mathbb{R}) \). Following Adams ([Ad]), we will prove the following generalization of this result.

Theorem 1.3. For every element \( g \in G(\mathbb{R}) \), \( C(g) \) and \( g^{-1} \) are conjugate to each other in \( G(\mathbb{R}) \).

Let \( g \) denote the Lie algebra of \( G \). By differentiation, the fundamental Chevalley involution \( C : G \to G \) induces an involutive automorphism \( C : g \to g \). We will also prove the following Lie algebra analogue of Theorem 1.3.

Theorem 1.4. For every element \( X \in g \), \( C(X) \) and \( -X \) are conjugate to each other by \( G(\mathbb{R}) \).

2. Semisimple elements

We continue with the notation of the Introduction.

Recall that an element \( g \in G(\mathbb{R}) \) is said to be semisimple (or unipotent) if for every algebraic representation of \( G \) on every finite dimensional real vector space \( V \), \( g \) acts on \( V \) by a semisimple (or unipotent) linear operator. An element \( X \in g \) is said to be semisimple (or nilpotent) if for every aforementioned representation \( V \), \( X \) acts on \( V \) by a semisimple (or nilpotent) operator.

As we already mentioned, the following lemma is proved in [Ad, Theorem 1.2].

Lemma 2.1. Theorem 1.3 holds for all semisimple elements \( g \in G(\mathbb{R}) \).

The following lemma is well-known.

Lemma 2.2. Let \( T \) be an algebraic torus defined over \( \mathbb{R} \). Then there is some \( t \in T(\mathbb{R}) \) such that the cyclic group \( \langle t \rangle \) generated by \( t \) is Zariski-dense in \( T(\mathbb{C}) \).

Proof. We sketch a proof for the convenience of the reader. Note that there are only countably many proper algebraic subgroups of \( T \) that are defined over \( \mathbb{R} \). Thus the set

\[
\bigcup \mathcal{S}(\mathbb{R})
\]

is a proper subset of \( T(\mathbb{R}) \). Pick an element \( t \) in the complementary set, and the lemma follows.

Lemma 2.3. Theorem 1.4 holds for all semisimple elements \( X \in g \).
Proof. Suppose that \( X \in \mathfrak{g} \) is semisimple. Then there is a Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) that contains \( X \). Write \( H \) for the centralizer of \( \mathfrak{h} \), which is a Cartan subgroup of \( G \) defined over \( \mathbb{R} \). By Lemma 2.2, there is an element \( t \in H(\mathbb{R}) \) such that the cyclic group \( \langle t \rangle \) is Zariski dense in \( H(\mathbb{C}) \). By Lemma 2.1, there is an element \( g_0 \in G(\mathbb{R}) \) such that \( C(t) = \text{Ad}_{g_0}(t^{-1}) \).

The Zariski dense property implies that \( C(h) = \text{Ad}_{g_0}(h^{-1}) \) for all \( h \in H(\mathbb{C}) \).

By taking differential, we know that \( C(Y) = \text{Ad}_{g_0}(-Y) \) for all \( Y \in \mathfrak{h} \).

This proves the lemma. \( \square \)

3. Unipotent elements

Write \( C(\mathfrak{g}) \) for the set of \( G(\mathbb{R}) \)-orbits in \( \mathfrak{g} \) under the adjoint action. Write \( C_{ss}(\mathfrak{g}) \) and \( C_{nil}(\mathfrak{g}) \) for the subsets of \( C(\mathfrak{g}) \) consisting of the semisimple orbits and the nilpotent orbits, respectively. Define a map

\[ \iota : C_{nil}(\mathfrak{g}) \rightarrow C_{ss}(\mathfrak{g}), \]

the class of \( E \in \mathfrak{g} \) \( \mapsto \) the class of \( E - F \in \mathfrak{g} \),

where \( E \) is a nilpotent element in \( \mathfrak{g} \) such that \( (E, [E, F], F) \) forms an \( \mathfrak{sl}_2 \)-triple. This map is well-defined and injective ([CM, Theorem 9.2.3, Theorem 9.4.6]).

The involutive automorphism \( C : \mathfrak{g} \rightarrow \mathfrak{g} \) descends to a map

\[ C : C(\mathfrak{g}) \rightarrow C(\mathfrak{g}). \]

By restriction one has the maps \( C : C_{ss}(\mathfrak{g}) \rightarrow C_{ss}(\mathfrak{g}) \) and \( C : C_{nil}(\mathfrak{g}) \rightarrow C_{nil}(\mathfrak{g}) \).

The involutive map \( J : \mathfrak{g} \rightarrow \mathfrak{g}, \ X \mapsto -X \) also descends to a map

\[ J : C(\mathfrak{g}) \rightarrow C(\mathfrak{g}). \]

By restriction one also has the maps \( J : C_{ss}(\mathfrak{g}) \rightarrow C_{ss}(\mathfrak{g}) \) and \( J : C_{nil}(\mathfrak{g}) \rightarrow C_{nil}(\mathfrak{g}) \).

Lemma 3.1. Theorem 1.4 holds for all nilpotent elements \( X \in \mathfrak{g} \).

Proof. It is directly verified that \( C \) and \( J \) both commutes with \( \iota : C_{nil}(\mathfrak{g}) \rightarrow C_{ss}(\mathfrak{g}) \).

As \( \iota \) is injective and \( C = J \) on \( C_{ss}(\mathfrak{g}) \) by Lemma 2.3 one has \( C = J \) on \( C_{nil}(\mathfrak{g}) \) and the lemma follows. \( \square \)

Lemma 3.2. Theorem 1.3 holds for all unipotent elements \( g \in G(\mathbb{R}) \).

Proof. Suppose that \( g \in G(\mathbb{R}) \) is a unipotent element. Then there is a unique nilpotent element \( X \in \mathfrak{g} \) such that \( \exp(X) = g \). By Lemma 3.1

\[ C(X) = \text{Ad}_k(-X) \]

for some \( k \in G(\mathbb{R}) \).
Thus
\[ C(g) = C(\exp(X)) = \exp(C(X)) = \exp(\text{Ad}_k(-X)) = \text{Ad}_k(\exp(-X)) = \text{Ad}_k(g^{-1}). \]
This proves the lemma.

\[ \square \]

4. General elements

Let \( s \in G(\mathbb{R}) \) be a semisimple element. Write \( Z \) for the centralizer of \( s \) in \( G \), which is a reductive linear algebraic group defined over \( \mathbb{R} \). Denote by \( Z^\circ \) the identity connected component of \( Z \), which is a connected reductive linear algebraic group defined over \( \mathbb{R} \). Let \( T \) be a fundamental Cartan subgroup of \( Z^\circ \).

Lemma 4.1. The element \( s \) belongs to \( T(\mathbb{R}) \).

Proof. Note that \( s \in Z(\mathbb{C}) \). As \( s \) is semisimple, it lies in some Cartan subgroup of \( G(\mathbb{C}) \) (see [Sp, Theorem 6.4.5]). Thus \( s \in Z^\circ(\mathbb{C}) \), and hence \( s \in Z^\circ(\mathbb{R}) \). Moreover, \( s \) is in the center of \( Z^\circ \), and hence \( s \in T(\mathbb{R}) \). \( \square \)

Lemma 4.2. There exists an element \( g \in G(\mathbb{R}) \) such that
\[ C(x) = \text{Ad}_g(x^{-1}) \quad \text{for all } x \in T(\mathbb{R}). \]

Proof. Let \( t \in T(\mathbb{R}) \) be as in Lemma 2.2. By Lemma 2.1 there exists an element \( g \in G(\mathbb{R}) \) such that
\[ C(t) = \text{Ad}_g(t^{-1}). \]
This clearly implies the lemma. \( \square \)

Let \( g \in G(\mathbb{R}) \) be as in Lemma 4.2.

Lemma 4.3. The automorphism
\[ \text{Ad}_{g^{-1}} \circ C : G \to G \]
stabilizes the algebraic subgroup \( Z^\circ \).

Proof. Lemma 4.1 implies that
\[ (\text{Ad}_{g^{-1}} \circ C)'(s) = s^{-1}. \]
This clearly implies the lemma. \( \square \)

Let \( C' : Z^\circ \to Z^\circ \) be a fundamental Chevalley involution.

Lemma 4.4. There is an element \( k \in Z^\circ(\mathbb{R}) \) such that
\[ (\text{Ad}_{g^{-1}} \circ C)|_{Z^\circ} = \text{Ad}_{k^{-1}} \circ C' : Z^\circ \to Z^\circ. \]
Proof. By the uniqueness of the fundamental Cartan subgroups, there exists an element $k_0 \in Z^o(\mathbb{R})$ such that
\[ C'(\text{Ad}_{k_0}(x)) = \text{Ad}_{k_0}(x^{-1}) \quad \text{for all } x \in T(\mathbb{R}). \]
Then the automorphism
\[ (\text{Ad}_{g^{-1}} \circ C) \circ (\text{Ad}_{k_0^{-1}} \circ C' \circ \text{Ad}_{k_0}) : Z^o \to Z^o \]
fixes all elements of $T(\mathbb{R})$. By [Ad] Lemma 3.4, this automorphism equals $\text{Ad}_t : Z^o \to Z^o$ for some $t \in T(\mathbb{R})$. Hence
\[
(\text{Ad}_{g^{-1}} \circ C)|_{Z^o} = \text{Ad}_t \circ (\text{Ad}_{k_0^{-1}} \circ C' \circ \text{Ad}_{k_0}) = \text{Ad}_t \circ (\text{Ad}_{k_0^{-1}} \circ \text{Ad}_{C'(k_0)} \circ C') = \text{Ad}_{t^{-1}k_0^{-1}C'(k_0)} \circ C'.
\]
This proves the lemma. □

**Lemma 4.5.** For every unipotent element $u \in G(\mathbb{R})$ that commutes with $s$, there exists an element $g_0 \in G(\mathbb{R})$ such that
\[ C(su) = \text{Ad}_{g_0}(s^{-1}u^{-1}). \]

Proof. Note that $u \in Z^o(\mathbb{R})$. By Lemma [3.2] there is an element $k' \in Z^o(\mathbb{R})$ such that
\[ C'(u) = \text{Ad}_{k'}(u^{-1}). \]
Let $k \in Z^o(\mathbb{R})$ be as in Lemma [4.4]. Then we have that
\[
(\text{Ad}_{g^{-1}} \circ C)(su) = s^{-1} \cdot (\text{Ad}_{k_0^{-1}} \circ C')(u) \quad \text{(by [4.2] and Lemma [4.4])}
\]
\[ = s^{-1} \cdot \text{Ad}_{k_{-1}k'}(u^{-1})
\]
\[ = \text{Ad}_{k_{-1}k'}(s^{-1}u^{-1})
\]
This proves the lemma. □

By using the Jordan decomposition (see [W] Theorem 9.2), Lemma [4.5] implies Theorem [1.3]. The same method also proves Theorem [1.4].

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