Spin-density-wave order in cuprates

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Abstract

We study the nature of the two-dimensional quantum critical point separating two phases with and without long-range spin-density-wave order, which has been recently observed in cuprate superconductors. We consider the Landau-Ginzburg-Wilson Hamiltonian associated with the spin-density critical modes, perform a mean-field analysis of the phase diagram, and study the corresponding renormalization-group flow in two different perturbative schemes at five and six loops, respectively. The analysis supports the existence of a stable fixed point in the full theory whose basin of attraction includes systems with collinear spin-density-wave order, as observed in experiments. The stable fixed point is characterized by an enlarged $O(4) \otimes O(3)$ symmetry. The continuous transition observed in experiments is expected to belong to this universality class. The corresponding critical exponents are $\nu = 0.9(2)$ and $\eta = 0.15(10)$.

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I. INTRODUCTION

In the last few decades several aspects of cuprate superconductors (SCs) have been studied and many efforts have been spent to understand the unique and complex phase diagram exhibited by this class of materials; see, e.g., Ref. 1. Superconductivity in cuprates appears to be due to a mechanism analogous to the BCS one in ordinary superconductors. However, superconductivity is only one of the characteristic features of these materials. There are many other new properties that require more complex mechanisms and can be understood only if the interplay between BCS and additional order parameters is considered. For instance, at $T \approx 0$, La$_{2-x}$Sr$_x$CuO$_4$ at very low doping $\delta$ is an insulator with long-range magnetic order. Increasing $\delta$, at $\delta \approx 0.055$ an insulator-superconductor first-order transition takes place, giving rise to a superconducting state in which spins are still magnetically ordered. At $\delta \approx 0.14$ another phase transition occurs, and, for $\delta \gtrsim 0.14$, the material shows no magnetic order—it is paramagnetic—but is still superconducting. Neutron-scattering experiments suggested that this transition is continuous.

Moreover, in the ordered phase $\delta \lesssim 0.14$, they revealed the presence of collinearly polarized spin-density waves (SDWs) with wavevectors

$$ K_1 = \frac{2\pi}{a} \left( \frac{1}{2} - \theta, \frac{1}{2} \right), \quad K_2 = \frac{2\pi}{a} \left( \frac{1}{2}, \frac{1}{2} - \theta \right), \quad (1.1) $$

where $\theta$ is a function of the doping concentration and $a$ is the lattice spacing. The wave vectors $K_i$ are two-dimensional since cuprates are supposed to be made of weakly interacting planes and thus behave approximately as two-dimensional systems. Following Ref. 4, we assume that superconductivity is not relevant at the transition which is instead driven by the interaction among the SDW degrees of freedom. Since $T \approx 0$ one should take into account the quantum nature of the system. Quantum phase transitions can be studied by introducing a supplementary dimension parametrized by an imaginary time variable $\tau$. The relevant order parameter is the spin field which is parametrized as

$$ S_i(r, \tau) = \text{Re}[e^{iK_1 \cdot r} \Phi_{1i}(r, \tau) + e^{iK_2 \cdot r} \Phi_{2i}(r, \tau)], \quad (1.2) $$

where $\Phi_{ai}$ are complex amplitudes. There are two interesting limiting cases. The first one is when the order parameter can be written as $\Phi_a(r, \tau) = e^{i\alpha_a} n_a$, which corresponds to collinearly polarized SDWs. The second one is when $\Phi_a(r, \tau) = n_{a,1} + i n_{a,2}$, with $n_{a,1} \cdot n_{a,2} = 0$ and $|n_{a,1}| = |n_{a,2}|$, which corresponds to circularly polarized SDWs. In cuprates experiments indicate that the ground state shows a collinear behavior.

The standard strategy for writing down an effective Hamiltonian for a given physical system consists in considering all polynomials of the order parameter of order less than or equal to four that are compatible with the expected symmetries. In the SDW-SC–to–SC phase transition the order parameter is the complex field $\Phi_{ai}(r, \tau)$, with $a = 1, 2$ and $i = 1, 2, 3$. The corresponding symmetries are the following: (i) SO(3) spin rotations: $\Phi_{ai} \rightarrow O_{ij} \Phi_{aj}$; (ii) Translational symmetry of the spin waves: $\Phi_{ai} \rightarrow e^{i\alpha_a} \Phi_{ai}$; (iii) Spatial inversion: $\Phi_{ai} \rightarrow \Phi^{*}_{ai}$; (iv) Interchange of the $\hat{1}$ and $\hat{2}$ axes: $\Phi_{1i} \leftrightarrow \Phi_{2i}$ and $x \leftrightarrow y$. The most general Hamiltonian with these symmetries is

$$ H = \sum_{a, i} \left[ \frac{1}{2} M_a \left( \partial^2 \Phi_{ai} \partial \tau^2 \right) + \text{Tr} \left( \Phi^{*}_{ai} \right) \right], $$

where $M_a$ is a constant.
\[ \mathcal{H} = \int d^2 r \, dr \left\{ |\partial_\tau \Phi_1|^2 + v_1^2 |\partial_x \Phi_1|^2 + v_2^2 |\partial_y \Phi_1|^2 + |\partial_\tau \Phi_2|^2 + \\
+ v_2^2 |\partial_\tau \Phi_2|^2 + v_1^2 |\partial_y \Phi_2|^2 + r(|\Phi_1|^2 + |\Phi_2|^2) + \\
+ \frac{u_{1,0}}{2} (|\Phi_1|^4 + |\Phi_2|^4) + \frac{u_{2,0}}{2} (|\Phi_1|^2 + |\Phi_2|^2)^2 + \\
+ w_{1,0} |\Phi_1|^2 |\Phi_2|^2 + w_{2,0} |\Phi_1| |\Phi_2|^2 + w_{3,0} |\Phi_1^*| |\Phi_2|^2 \right\}, \tag{1.3} \]

where \( v_1 \) and \( v_2 \) are parameters called SDW velocities. Terms such as \( \Phi_a^* \cdot \partial_x \Phi_a \) are forbidden by spatial inversion symmetry and terms like \( i \Phi_a^* \cdot \partial_x \Phi_a \), even if permitted by all symmetries, can be eliminated by redefining the fields as \( \Phi_a \rightarrow e^{i q x} \Phi_a \). Hamiltonian (1.3) admits several different ground states depending on the values of the parameters. They are classified in App. A. In particular, there is the possibility that both fields correspond to collinearly polarized SDWs as observed in experiments: \( \Phi_1 = e^{i \alpha^1} \mathbf{n}_1 \) and \( \Phi_2 = e^{i \alpha^2} \mathbf{n}_2 \), where the vectors \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \) satisfy either \( \mathbf{n}_1 = \mathbf{n}_2 \) or \( \mathbf{n}_1 \cdot \mathbf{n}_2 = 0 \).

In this paper we investigate the nature of the fixed points (FPs) of the renormalization-group (RG) flow of the effective Hamiltonian (1.3). If a stable FP exists and its attraction domain includes systems with collinearly polarized SDWs, then the SDW-SC-to-SC transition may be continuous. Otherwise, it must be of first order. In our study, we consider only the case \( v_1 = v_2 \) that simplifies the analysis and allows us to perform a high-order perturbative analysis. Therefore, we consider the theory

\[ \mathcal{H} = \int d^d x \left\{ \sum_{\mu} \left( |\partial_\mu \Phi_1|^2 + |\partial_\mu \Phi_2|^2 \right) + r(|\Phi_1|^2 + |\Phi_2|^2) + \frac{u_{1,0}}{2} (|\Phi_1|^4 + |\Phi_2|^4) + \\
+ \frac{u_{2,0}}{2} (|\Phi_1|^2 + |\Phi_2|^2)^2 + w_{1,0} |\Phi_1| |\Phi_2|^2 + w_{2,0} |\Phi_1| |\Phi_2|^2 + w_{3,0} |\Phi_1^*| |\Phi_2|^2 \right\}, \tag{1.4} \]

where the field \( \Phi_{ai} \) is a complex \( 2 \times N \) matrix, \( a = 1, 2, i = 1, \ldots, N \). The physically relevant case is \( N = 3 \).

We first perform a standard analysis close to four dimensions,\(^6\) computing the RG functions in powers of \( \epsilon \equiv 4 - d \). A one-loop analysis indicates that a stable FP exists only for \( N \gtrsim 42.8 \). Apparently, this result casts doubts on the existence of a stable FP in three dimensions. However, in three dimensions there may exist FPs that are absent for \( \epsilon \ll 1 \). This is indeed what happens in the Ginzburg-Landau model of superconductors, in which a complex scalar field couples to a gauge field\(^7\) and in \( O(2) \otimes O(n) \) symmetric models.\(^8,9\)

Thus, a more careful investigation of the RG flow in three dimensions calls for strictly three-dimensional perturbative schemes. For this purpose we consider two field-theoretical perturbative approaches: the minimal-subtraction scheme without \( \epsilon \) expansion\(^10\) (in the following we will indicate it as 3d-\text{MS} scheme) and the massive zero-momentum (MZM) renormalization scheme.\(^11\) The use of two different schemes is crucial, since the comparison of the corresponding results provides a nontrivial check on the reliability of our conclusions. In the 3d-\text{MS} scheme one considers the massless (critical) theory in dimensional regularization,\(^12\) determines the RG functions from the divergences appearing in the perturbative expansion of the correlation functions, and finally sets \( \epsilon \equiv 4 - d = 1 \) without expanding in powers of \( \epsilon \) (this scheme therefore differs from the standard \( \epsilon \) expansion\(^6\)). In the MZM scheme one...
considers instead the three-dimensional massive theory in the disordered (high-temperature) phase. We compute the \( \beta \) functions to five loops in the 3d-MS scheme and to six loops in the MZM scheme. We use a symbolic manipulation program that generates the diagrams (approximately one thousand at six loops) and computes their symmetry and group factors, and the compilation of Feynman integrals of Refs. 13, 14. The series are available on request. The perturbative expansions are then resummed using the known large-order behavior.

The perturbative analysis of the RG flow in the full theory is not sufficiently stable to provide reliable results. Therefore, we have focused on the stability of the FPs that occur in specific submodels of Hamiltonian (1.4). The analysis of the perturbative series indicates the stability of the \( \text{O}(4) \otimes \text{O}(3) \) collinear FP that occurs in the model with \( w_{1,0} = u_{1,0} - u_{2,0} \) and \( w_{2,0} = w_{3,0} = u_{2,0} < 0 \). Moreover, its basin of attraction includes systems with collinear SDWs. Therefore, we expect the continuous transition observed experimentally in cuprates to belong to this universality class. This implies an effective enlargement of the symmetry at the transition point. The corresponding critical exponents would be

\[
\nu = 0.9(2), \quad \eta = 0.15(10). \tag{1.5}
\]

The paper is organized as follows. In Sec. II we discuss the possible ordered phases that occur in model (1.3) in the mean-field approximation. Details are given in App. A. In Sec. III we discuss the FP structure close to four dimensions in the standard \( \epsilon \) expansion. Sec. IV contains the main results of this work. We consider three different submodels (Sec. IV A) and then investigate the stability properties of the FPs occurring in each of them (Sections IV B, IV C, and IV D). Conclusions are presented in Sec. V. In App. B and C we give some technical details.

II. MEAN-FIELD ANALYSIS

The phase diagram of Hamiltonian (1.3) can be studied in the mean-field approximation. Due to the large number of couplings the analysis is quite complex. We have limited our considerations to the case \( N \leq 3 \). We summarize here the results that are derived in App. A. For \( r > 0 \) the system is disordered and \( \Phi_1 = \Phi_2 = 0 \). For \( r = 0 \) a continuous phase transition occurs followed by a magnetized phase with \( r < 0 \). The nature of the ordered phase depends on the values of the quartic parameters. The analysis reported in App. A shows that there are seven possibilities:

1. \( \Phi_1 \) is a collinear SDW (\( \Phi_1 = e^{i \alpha_1 \mathbf{n}} \), \( \mathbf{n} \) real) while \( \Phi_2 = 0 \).

2. \( \Phi_1 \) is a circularly polarized SDW (\( \Phi_1 = e^{i \alpha_1 (\mathbf{n}_1 + i \mathbf{n}_2)} \), \( \mathbf{n}_1 \) and \( \mathbf{n}_2 \) real, \( |\mathbf{n}_1| = |\mathbf{n}_2| \), \( \mathbf{n}_1 \cdot \mathbf{n}_2 = 0 \)) while \( \Phi_2 = 0 \).

3. \( \Phi_1 \) and \( \Phi_2 \) correspond to collinear SDWs with the same axis and amplitude: \( \Phi_1 = e^{i \alpha_1 \mathbf{n}} \), \( \Phi_2 = e^{i \alpha_2 \mathbf{n}} \), \( \mathbf{n} \) real.

4. \( \Phi_1 \) and \( \Phi_2 \) correspond to collinear SDWs with orthogonal axes and same amplitude: \( \Phi_1 = e^{i \alpha_1 \mathbf{n}_1} \), \( \Phi_2 = e^{i \alpha_2 \mathbf{n}_2} \), \( \mathbf{n}_1, \mathbf{n}_2 \) real, \( \mathbf{n}_1 \cdot \mathbf{n}_2 = 0 \), \( |\mathbf{n}_1| = |\mathbf{n}_2| \).
(5) \( \Phi_1 \) and \( \Phi_2 \) are circularly polarized SDWs with the same rotation plane and amplitude: \( \Phi_1 = e^{i\alpha_1}(n_1 + in_2), \Phi_2 = e^{i\alpha_2}(n_1 + in_2), \) \( n_1 \) and \( n_2 \) real, \( |n_1| = |n_2|, n_1 \cdot n_2 = 0 \).

(6) \( \Phi_1 \) is a collinear SDW and \( \Phi_2 \) is a circularly polarized SDW. The rotation plane of \( \Phi_2 \) is orthogonal to the axis of \( \Phi_1 \). Explicitly: \( \Phi_1 = e^{i\alpha_1}n_1, \Phi_2 = e^{i\alpha_2}(n_2 + in_3), \) \( n_i \) real, \( |n_2| = |n_3|, n_i \cdot n_j = 0 \).

(7) \( \Phi_1 \) and \( \Phi_2 \) are elliptically polarized SDWs with different rotation planes but with the same amplitude, \( |\Phi_1| = |\Phi_2| \).

For cuprates the relevant solutions are (3) and (4). Necessary conditions to obtain (3) are \( w_{2,0} + w_{3,0} < 0 \) and

\[
\begin{align*}
    w_{1,0} + w_{2,0} + w_{3,0} - u_{1,0} < u_{2,0} < \min\{u_{1,0}-w_{1,0}-w_{2,0}-w_{3,0}, -w_{2,0}, -w_{3,0}\},
\end{align*}
\]  

(2.1) while (4) requires \( w_{2,0} + w_{3,0} > 0 \) and

\[
\begin{align*}
    w_{1,0} - u_{1,0} < u_{2,0} < \min\{u_{1,0} - w_{1,0}, w_{2,0}, w_{3,0}\}.
\end{align*}
\]  

(2.2)

These conditions are not sufficient, since for some values of the parameters satisfying Eqs. (2.1) or (2.2) the ordered phase is given by solutions (6) or (7). Note that the sign of \( u_{2,0} \) is not the relevant parameter that selects the collinear SDWs among all possible solutions.

It is interesting to note that the mean-field solution predicts either \( \Phi_1 \parallel \Phi_2 \) or \( \Phi_1 \perp \Phi_2 \) in the case of collinear SDWs. This result is easy to understand. If both fields correspond to collinear SDWs, then one can take \( \Phi_1 \) and \( \Phi_2 \) real. In this case the only term of the Hamiltonian that contains a scalar product of the two fields is \( (w_{2,0} + w_{3,0})(\Phi_1 \cdot \Phi_2)^2 \) that forces the two fields to be either parallel or orthogonal, depending on the sign of \( w_{2,0} + w_{3,0} \). Note that this also holds if we add additional higher-order terms to the Hamiltonian, as long as the transition is continuous. Indeed, for a continuous transition \( \Phi_s \to 0 \) at the transition (\( \Phi_s = 0 \) in the disordered phase) and thus higher-order terms do not play any role. On the other hand, this relation may not be valid if the transition is of first order. Also the coupling to the charge-density waves (CDWs) that are present in cuprates\(^{15,4} \) does not change this conclusion, since they couple to the scalars \( \Phi_2^2, |\Phi_2|^2 \).

Solutions (3) and (4) also satisfy \( |\Phi_1| = |\Phi_2| \). This property does not necessarily hold if we take into account the CDWs (see Refs. 15, 4 for an extensive discussion). Indeed, let \( \phi_1 \) and \( \phi_2 \) be the complex amplitudes of the CDWs coupled respectively to \( \Phi_1^2 \) and \( \Phi_2^2 \). In the absence of the CDW-SDW coupling, for some values of the CDW Hamiltonian parameters, the ordered solution corresponds to \( |\phi_1| \neq 0, \phi_2 = 0 \). If now the CDW-SDW coupling is included, one may obtain a ground state with \( |\phi_1| \neq |\phi_2| \neq 0 \) and \( |\Phi_1| \neq |\Phi_2| \neq 0 \).

**III. RG FLOW CLOSE TO FOUR DIMENSIONS**

The RG flow close to four dimensions can be investigated perturbatively in \( \epsilon \equiv 4 - d \). In the minimal-subtraction (\( \overline{\text{MS}} \)) the one-loop \( \beta \) functions are:
\[
\begin{align*}
\beta_{u_1} &= -\epsilon u_1 + (N + 4)u_1^2 + 4u_1u_2 + 4u_2^2 + Nw_1^2 + w_1^2 + w_2^2 + 2w_1w_2 + 2w_1w_3, \\
\beta_{u_2} &= -\epsilon u_2 + 6u_1u_2 + Nu_2^2 + 2w_2w_3, \\
\beta_{w_1} &= -\epsilon w_1 + 2w_1^2 + w_2^2 + w_3^2 + 2(N + 1)u_1w_1 + 4u_2w_1 + 2u_1w_2 + 2u_1w_3, \\
\beta_{w_2} &= -\epsilon w_2 + Nu_2^2 + 2u_1w_2 + 4u_2w_3 + 4w_1w_2 + 2w_2w_3, \\
\beta_{w_3} &= -\epsilon w_3 + Nu_3^2 + 2u_1w_3 + 4u_2w_3 + 4w_1w_3 + 2w_2w_3,
\end{align*}
\]

where \(u_i, w_i\) are the renormalized quartic couplings corresponding to the quartic Hamiltonian parameters \(u_{i,0}, w_{i,0}\). They are normalized so that, at tree level, \(g = g_0\mu^{-\epsilon}/A_d\), where \(g\) and \(g_0\) label the renormalized and Hamiltonian parameters respectively and \(A_d \equiv 2^{d-1}\pi^{d/2}\Gamma(d/2)\). The FPs of the RG flow are the common zeroes of the \(\beta\) functions. For \(N = 3\) there are 4 FPs while for \(N = 2\) there are 7 FPs: they are all unstable. Only for \(N \gtrsim 42.8\) does a stable FP exist. It has \(u_2 = w_2 = w_3\) (for \(N \to \infty\) we obtain \(u_1 = u_2 = w_2 = w_3 = \epsilon/N, w_1 = 0\)), so that at the FP the symmetry becomes \(O(4) \otimes O(N)\). This FP is the chiral FP that occurs in \(O(M) \otimes O(N)\) in the large-\(N\) limit.\(^{16}\)

In order to determine the behavior in three dimensions, one should extend the computation to higher order in \(\epsilon\) and determine the function \(N_c(\epsilon) = 42.8 + O(\epsilon)\) such that the chiral FP point identified above exists for \(N > N_c(\epsilon)\) and is no longer present for smaller values of \(N\). We have not pursued this approach for several reasons. First, the analogous five-loop computation that was performed in the \(O(N) \otimes O(2)\) model\(^{17,16,18}\) was not able to explain the correct physics of these models for \(N = 2,3\) (see Sec. II.D in Ref. 8). Moreover, this calculation is only concerned with the stable FP that is present for \(\epsilon = 0\) (in the present case the chiral \(O(4) \otimes O(N)\) FP), while in \(d = 3\) the stable FP may be different, an unstable or even a new FP. The analysis that will be presented in the next Section favors this last possibility.

IV. SUBMODELS AND THEIR STABILITY

The three-dimensional properties of the RG flow are determined by its FPs. Some of them can be identified by considering particular cases in which some of the quartic parameters vanish. The corresponding FPs are also FPs of the general theory. In this section, we identify some of them, and then determine their stability with respect to the complete theory.

A. Some particular cases

For particular values of the couplings Hamiltonian (1.4) reduces to that of simpler models. Three cases have already been extensively studied in the literature:\(^{19}\)

(1) For \(w_{1,0} = w_{2,0} = w_{3,0} = 0\) there is no interaction between the two SDWs and Hamiltonian (1.4) reduces to that of two identical decoupled \(O(2) \otimes O(N)\)-symmetric models. The general \(O(m) \otimes O(n)\)-symmetric model is defined by the Hamiltonian density\(^{17,19}\)

\[
\frac{1}{2} \sum_{ai} \left[ \sum_{\mu} (\partial_{\mu} \phi_{ai})^2 + r \phi_{ai}^2 \right] + \frac{g_{1,0}}{4!} \left( \sum_{ai} \phi_{ai}^2 \right)^2
\]
\[ + \frac{g_{2,0}}{4!} \left[ \sum_{i,j} \left( \sum_{a} \phi_{ai} \phi_{aj} \right)^2 - \left( \sum_{ai} \phi_{ai}^2 \right)^2 \right], \tag{4.1} \]

where \( \phi_{ai} \) is a real \( n \times m \) matrix field \( (a = 1, \ldots, n \text{ and } i = 1, \ldots, m) \). Hamiltonian (4.1) is obtained from Eq. (1.4) by setting \( \Phi_{ai} = \phi_{ai}^{(a)} + i\phi_{2i}^{(a)} \) and

\[ u_{1,0} = g_{1,0}/3 - g_{2,0}/6, \quad u_{2,0} = g_{2,0}/6, \quad w_{1,0} = w_{2,0} = w_{3,0} = 0. \tag{4.2} \]

The properties of \( O(2) \otimes O(N) \) models are reviewed in Refs. 8, 17, 19, 20. In three dimensions perturbative calculations within the MZM scheme\(^{21,22} \) and within the 3d-\( \overline{\text{MS}} \) scheme\(^8 \) indicate the presence of a stable chiral FP with attraction domain in the region \( g_{2,0} > 0 \) for all values of \( N \) (only for \( N = 6 \) the evidence is less clear, since the MZM analysis does not apparently support it). For \( N = 2 \), these conclusions have been recently confirmed by a Monte Carlo simulation.\(^8 \) A stable collinear FP for \( g_2 < 0 \) exists for \( N \leq 4 \).\(^{9,23} \) Apart from the collinear FP for \( N = 2 \), these FPs do not exist close to four dimensions. For \( N = 2 \) the collinear FP is equivalent to an XY FP and corresponds to \( g_1^* = g_{XY}^*, \quad g_2^* = -g_{XY}^* \), where \( g_{XY}^* \) is the FP value of the renormalized coupling in the \( O(2) \phi^4 \) model.

(2) For \( w_{1,0} = u_{1,0} - u_{2,0} \) and \( w_{2,0} = w_{3,0} = u_{2,0} \), Hamiltonian (1.4) reduces to (4.1) with \( m = 4 \) and \( n = N \). The correspondence is given by

\[ \Phi_{1i} = \frac{\phi_{1i} + i\phi_{2i}}{\sqrt{2}}, \quad \Phi_{2i} = \frac{\phi_{3i} + i\phi_{4i}}{\sqrt{2}}, \tag{4.3} \]

where \( \phi_{ei} \) is a \( 4 \times N \) matrix, and

\[ g_{1,0} = 3(u_{1,0} + u_{2,0}), \quad g_{2,0} = 6u_{2,0}. \tag{4.4} \]

We have already discussed the FPs of the \( O(4) \otimes O(2) \) theory. The \( O(4) \otimes O(3) \) theory does not present stable FPs for \( g_2 > 0 \).\(^{24} \) Analyses of the available six-loop series in the MZM scheme and five-loop series in the 3-d \( \overline{\text{MS}} \) scheme indicate the presence of a stable collinear FP for \( g_2 < 0 \).\(^{25} \) This FP does not exist close to four dimensions.

(3) For \( u_{2,0} = w_{2,0} = w_{3,0} = 0 \) we obtain the \( mn \) model with \( n = 2 \) and \( m = 2N \). The so-called \( mn \) model is defined by the Hamiltonian density\(^{26,19} \)

\[ \frac{1}{2} \sum_{ai} \left[ \sum_{\mu} (\partial_{\mu} \phi_{ai})^2 + r \phi_{ai}^2 \right] + \frac{g_{1,0}}{4!} \left( \sum_{ai} \phi_{ai}^2 \right)^2 + \frac{g_{2,0}}{4!} \sum_{aij} \phi_{ai}^2 \phi_{aj}^2, \tag{4.5} \]

where \( \phi_{ai} \) is a real \( n \times m \) matrix, i.e., \( a = 1, \ldots, n \) and \( i = 1, \ldots, m \). The correspondence is obtained by setting

\[ \Phi_{ai} = (\phi_{ai} + i\phi_{a,i+N})/\sqrt{2}, \quad g_{1,0} = 3w_{1,0}, \quad g_{2,0} = 3(u_{1,0} - w_{1,0}). \tag{4.6} \]

A stable FP is the \( O(m) \) FP with \( g_1 = 0 \) and \( g_2 = g_{O(m)}^* \), where \( g_{O(m)}^* \) is the FP value of the renormalized coupling in the \( O(m) \)-symmetric vector model. In three dimensions the analysis of five- and six-loop series\(^{27} \) indicates the presence of a second stable FP with \( g_2 < 0 \) for \( n = 2 \) and \( m = 2, 3, \) and 4.
Beside these three models, there are two other submodels for which no results are available:

(a) For \( w_{2,0} = w_{3,0} = 0 \) we obtain two chiral models coupled by an energy-energy term. Note that in this model the RG flow does not cross the planes \( w_2 = 0 \) and \( w_1 = 0 \).

(b) For \( w_{2,0} = w_{3,0} = w_0 \) we obtain a model with an additional \( U(1) \) symmetry: \( \Phi_1 \rightarrow \Phi_1^* \), \( \Phi_2 \rightarrow \Phi_2^* \). In this model the RG flow does not cross the plane \( w = 0 \).

Finally, note an additional symmetry of Hamiltonian (1.4). It is invariant under \( \Phi_1 \rightarrow \Phi_1^* \), \( \Phi_2 \rightarrow \Phi_2^* \), \( w_{2,0} \rightarrow w_{3,0} \) and \( w_{3,0} \rightarrow w_{2,0} \), while the other couplings are unchanged. This implies that the RG flow in the space of renormalized couplings does not cross the plane \( w_2 = w_3 \) and that, for any FP with \( w_2 > w_3 \) there is an equivalent one with \( w_2 < w_3 \). In particular, we can limit our considerations to \( w_2 \geq w_3 \).

In order to study the RG flow of the theory one can start by discussing the stability in the full theory of the FPs of the models (1), (2), and (3) discussed above.

For \( N = 2 \) and \( N = 3 \), the only cases we consider, model (1) has two FPs:

(1a) the chiral FP, in which \( g_1 = g_{1,\text{ch}}^* \) and \( g_2 = g_{2,\text{ch}}^* \); correspondingly \( u_1^* = g_{1,\text{ch}}^*/3 - g_{2,\text{ch}}^*/6 \), \( u_2^* = g_{2,\text{ch}}^*/6 > 0 \), \( w_1^* = w_2^* = w_3^* = 0 \);

(1b) the collinear FP, in which \( g_1 = g_{1,\text{cl}}^* \) and \( g_2 = g_{2,\text{cl}}^* \); correspondingly \( u_1^* = g_{1,\text{cl}}^*/3 - g_{2,\text{cl}}^*/6 \), \( u_2^* = g_{2,\text{cl}}^*/6 < 0 \), \( w_1^* = w_2^* = w_3^* = 0 \).

Here \( g_{i,\text{ch}} \) and \( g_{i,\text{cl}} \) are the chiral and collinear FPs of the \( O(2) \otimes O(N) \) theory. The analogous FPs are present in model (2):

(2a) the chiral FP, in which \( g_1 = g_{1,\text{ch}}^* \) and \( g_2 = g_{2,\text{ch}}^* \); correspondingly \( u_1^* = g_{1,\text{ch}}^*/3 - g_{2,\text{ch}}^*/6 \), \( u_2^* = g_{2,\text{ch}}^*/6 < 0 \), \( w_1^* = u_1^* - u_2^* \), \( w_2^* = w_3^* = u_2^* \); It does not exist for \( N = 3 \). This is the FP that is relevant for \( N > N_c(\epsilon) \approx 42.8 + O(\epsilon) \) close to four dimensions;

(2b) the collinear FP, in which \( g_1 = g_{1,\text{cl}}^* \) and \( g_2 = g_{2,\text{cl}}^* \); correspondingly \( u_1^* = g_{1,\text{cl}}^*/3 - g_{2,\text{cl}}^*/6 \), \( u_2^* = g_{2,\text{cl}}^*/6 < 0 \), \( w_1^* = u_1^* - u_2^* \), \( w_2^* = w_3^* = u_2^* \). It exists for both \( N = 2 \) and \( N = 3 \).

Here \( g_{i,\text{ch}} \) and \( g_{i,\text{cl}} \) are the chiral and collinear FPs of the \( O(4) \otimes O(N) \) theory. Finally, the \( mn \) theory gives two FPs:

(3a) the \( O(2N) \) FP. This is unstable in the full theory, being already unstable in model (1);

(3b) the \( mn \) FP \( g_1 = g_{1,mn} \), \( g_2 = g_{2,mn} \); correspondingly \( u_1^* = g_{1,mn}^*/3 + g_{2,mn}^*/3 \), \( w_1^* = g_{1,mn}^*/3 \), \( u_2^* = w_2^* = w_3^* = 0 \). It exists only for \( N = 2 \).

In the following we study the stability of these FPs in the complete theory (1.4). For this purpose, using the \( \beta \) functions of the general theory we have computed the stability matrices of the FPs at six and five loops respectively in the MZM and \( 3d-\overline{\text{MS}} \) schemes. The perturbative series have been resummed by using the conformal-mapping method described, e.g., in Ref. 28. For a FP belonging to a submodel, the large-order behavior needed for the conformal-mapping summation is the same as that characterizing all series of that submodel. For all submodels we consider, the large-order behavior is already known.\(^{21,24,29}\)
B. Stability of the decoupled $O(2) \otimes O(N)$ fixed points

We want to establish the stability properties of the decoupled $O(2) \otimes O(N)$ FPs (1a) and (1b) in the complete theory (1.4). For this purpose we need the RG dimensions of the operators present in Hamiltonian (1.4) that break the symmetry of model (1), i.e., of the operators associated with the quartic couplings $w_i$. It is useful to rewrite them as

$$w_{1,0}|\Phi_1|^2|\Phi_2|^2 + w_{2,0}|\Phi_1 \cdot \Phi_2|^2 + w_{3,0}|\Phi_1^* \cdot \Phi_2|^2 = W_{00}P_{00} + W_{11}P_{11} + W_{02}P_{02},$$

where

$$W_{00} = w_{1,0} + \frac{1}{N}(w_{2,0} + w_{3,0}), \quad W_{11} = -w_{2,0} + w_{3,0}, \quad W_{02} = w_{2,0} + w_{3,0},$$

$$P_{00} \equiv O_{00}^{(1)}O_{00}^{(2)}, \quad P_{11} \equiv O_{11,ij}^{(1)}O_{11,ij}^{(2)}, \quad P_{02} \equiv \sum_{ij} O_{02,ij}^{(1)}O_{02,ij}^{(2)},$$

and, using the correspondence (4.2),

$${O^{(a)}_{00} = \sum_{ei} \phi_e^{(a)} \phi_e^{(a)}, \quad O^{(a)}_{11,ij} = \phi_{1i}^{(a)} \phi_{2j}^{(a)} - \phi_{i1}^{(a)} \phi_{2j}^{(a)}, \quad O^{(a)}_{02,ij} = \sum_{e} \phi_e^{(a)} \phi_e^{(a)} - \frac{1}{N} \sum_{ek} \delta_{ij} \sum_{ek} \phi_{ek}^{(a)} \phi_{ek}^{(a)}.}$$

The quadratic operator $O_{ml}^{(a)}$ transform as a spin-$m$ and a spin-$l$ representation with respect to the $O(2)$ and $O(N)$ groups, respectively. Since $P_{00}$, $P_{11}$, and $P_{02}$ belong to different irreducible representations, they do not mix under RG transformations at the decoupled $O(2) \otimes O(N)$ FPs. Their RG dimensions $Y_{ml}$ can be derived from the RG dimensions $y_{ml}$ of the quadratic operators $O_{ml}^{(a)}$ at the $O(2) \otimes O(N)$ FP, using the relation

$$Y_{ml} = 2y_{ml} - 3.$$  \hspace{1cm} (4.11)

The quadratic term $O_{00}^{(a)}$ corresponds to the energy operator and thus $y_{00} = 1/\nu$ and $Y_{00} = \alpha/\nu$, where $\alpha$ and $\nu$ are the specific-heat and correlation-length critical exponents of the given $O(2) \otimes O(N)$ FP. The RG dimensions $y_{11}$ and $y_{02}$ were computed in Ref. 23 (there, they are named $y_1$ and $y_3$ respectively).

At the chiral FP (1a) we obtain: \(30\)

$$Y_{00} = 0.3(3) \quad Y_{11} = 1.6(3) \quad Y_{02} = 0.04(8) \quad \text{for } N = 3;$$

$$Y_{00} = 0.2(3) \quad Y_{11} = 1.9(4) \quad Y_{02} = -0.4(2) \quad \text{for } N = 2.$$  

At the collinear FP (1b) we obtain: \(30\)

$$Y_{00} = 0.3(2) \quad Y_{11} = -0.6(2) \quad Y_{02} = 1.0(3) \quad \text{for } N = 3;$$

$$Y_{00} = -0.2182(8) \quad Y_{11} = -0.022(8) \quad Y_{02} = 0.92(11) \quad \text{for } N = 2.$$  

These results show that the decoupled $O(2) \otimes O(N)$ FPs are unstable in the complete theory (1.4) for both $N = 3, 2$.

It is also interesting to discuss submodels (a) and (b) mentioned in Sec. IV A. In model (a) one should only consider $P_{00}$. The numerical results apparently indicate that the FPs are always unstable (but, with the present errors, we cannot really exclude the opposite possibility), except in one case. For $N = 2$, the collinear FP is stable. In model (b) one should consider $P_{00}$ and $P_{02}$. For $N = 2, 3$, all FPs are unstable.
FIG. 1. Distribution of the results for $Y_1$ (left) and $Y_2$ (right) obtained by varying the resummation parameters $\alpha$ and $b$ as a function of the number of loops in the MZM and 3d-MS schemes. Here $N = 3$.

C. Stability of the $O(4) \otimes O(N)$ FPs

Here we investigate the stability of FPs (2a) (it does not exist for $N = 3$) and (2b). For this purpose we must compute the RG dimensions of the perturbations of the $O(4) \otimes O(N)$ model appearing in the complete theory. This is done in App. B. There are two relevant operators with RG dimensions $Y_1$ and $Y_2$. The corresponding perturbative series are reported in App. B. They are analyzed using the conformal mapping method.\textsuperscript{28,31} The errors we will report takes into account the variation of the estimates when changing the resummation parameters $b, \alpha$ defined in Ref. 31—we use $b = 3, \ldots, 18$ and $\alpha = 0, \ldots, 4$—and the uncertainty of the FP coordinates.

The analysis of the six-loop series in the MZM scheme and of the five-loop 3d-\overline{MS} series gives the following results at the collinear FP (2b):\textsuperscript{25}
\[ Y_1 = -0.4(4) \quad Y_2 = -0.95(7) \quad \text{for } N = 2 \ (\text{MZM}), \]
\[ Y_1 = -0.6(9) \quad Y_2 = -1.2(1.0) \quad \text{for } N = 2 \ (3d-\text{MS}), \]
\[ Y_1 = -1.5(1.2) \quad Y_2 = -0.42(10) \quad \text{for } N = 3 \ (\text{MZM}), \]
\[ Y_1 = -0.8(1.5) \quad Y_2 = -0.1(2) \quad \text{for } N = 3 \ (3d-\text{MS}). \]  

(4.12)

For \( N = 3 \) the MZM and 3d-\( \overline{\text{MS}} \) results are consistent and apparently indicate that \( Y_1 \) and \( Y_2 \) are negative, though with somewhat large errors. A better understanding of the relevance of the two operators can be obtained from Fig. 1, where we give the distributions of the estimates of \( Y_1 \) and \( Y_2 \) obtained by varying the parameters \( \alpha \) and \( b \). For \( Y_1 \) low-order calculations predict \( Y_1 > 0 \). However, as the number of loops increases, \( Y_1 \) decreases. The six-loop MZM results indicate that \( Y_1 < 0 \), a result that is also supported by the trend observed in the 3d-\( \overline{\text{MS}} \) results. Though they give \( Y_2 < 0 \), there is a trend towards larger values of \( Y_2 \). Overall, these results support the stability of the collinear FP (2b) in the complete theory for \( N = 3 \).

Similar conclusions hold for \( N = 2 \). For completeness, we report here the corresponding critical exponents:  
\[ \nu = 0.71(7) \quad \eta = 0.12(1) \quad \text{for } N = 2 \ (\text{MZM}); \]
\[ \nu = 0.76(10) \quad \eta = 0.11(6) \quad \text{for } N = 2 \ (3d-\overline{\text{MS}}); \]
\[ \nu = 0.89(16) \quad \eta = 0.18(3) \quad \text{for } N = 3 \ (\text{MZM}); \]
\[ \nu = 0.88(22) \quad \eta = 0.10(10) \quad \text{for } N = 3 \ (3d-\overline{\text{MS}}). \]  

(4.13)

For \( N = 2 \) we also study the stability of the chiral FP (2a). Using the results of Refs. 8, 22 for the FP we have:  
\[ Y_1 = -0.03(7) \quad Y_2 = 0.9(2) \quad \text{(MZM)}; \]
\[ Y_1 = -0.2(3) \quad Y_2 = 0.73(15) \quad \text{(3d-MS)}. \]  

(4.14)

The chiral FP is clearly unstable.

Finally, note that the same discussion also applies to submodel (b), since the stability of the FP depends on the same operators with RG dimensions \( Y_1 \) and \( Y_2 \). For submodel (a) one should only consider \( Y_1 \). In this case also the chiral FP (2b) might be stable.

**D. Stability of the \( mn \) FP for \( N = 2 \)**

Here we investigate the stability of FP (3b) for \( N = 2 \) (it does not exist for \( N = 3 \)). For this purpose we must compute the RG dimensions of the perturbations of the \( mn \) FP appearing in the complete theory. This is done in App. C. There are two relevant operators with RG dimensions \( Y_1 \) and \( Y_2 \). The analysis of the perturbative series in the MZM scheme gives

\[ Y_1 = -4.0(2.6) \quad Y_2 = -0.6(2). \]  

(4.15)

The results in the 3d-\( \overline{\text{MS}} \) scheme are very imprecise, although negative values for \( Y_1 \) and \( Y_2 \) seem to be favored. These results indicate, although with limited confidence, that the \( mn \) FP present for \( N = 2 \) may be stable in the complete theory.
V. CONCLUSIONS

In this paper we have studied the quantum phase transition that occurs in two-dimensional systems that exhibit an ordered phase with SDW order. The effective Hamiltonian of the relevant critical modes $\Phi_{\alpha i}$ is given in Eq. (1.3). A detailed mean-field analysis shows that in some parameter region Hamiltonian (1.3) has a continuous transition separating a spin disordered phase from an ordered phase characterized by two collinearly polarized SDWs. There are two different possibilities for the axes of these SDWs: either $\Phi_1 \parallel \Phi_2$ or $\Phi_1 \perp \Phi_2$. We have then investigated the role of fluctuations in a simplified model in which the two SDWs have the same velocity. For this purpose we have generated six-loop perturbative series in the MZM scheme and five-loop series in dimensional regularization with minimal subtraction. Close to four dimensions, an analytic $\epsilon$-expansion calculation shows no presence of stable FPs. However, past experience indicates that a FP may exist in three dimensions and be absent for $\epsilon \ll 1$. Therefore, we have considered two strictly three-dimensional schemes. We have analyzed the stability of some FPs that belong to known submodels. The analysis of the perturbative series supports the stability of the $O(4) \otimes O(3)$ collinear FP. The analyses of the MZM and $3d$-MS expansions do not provide sufficiently stable results for the RG flow in the full theory, i.e. in the general space of its five quartic couplings. In particular, they do not allow us to draw any definite conclusion on the existence of other stable FPs.

In any case, even without the analysis of the full flow, simple considerations (reported in App. A) show that systems with collinear SDWs with the same axis (the mean-field solution (3) reported in Sec. II) are in the attraction domain of the $O(4) \otimes O(3)$ collinear FP.

It should be remarked that our RG analysis is only valid for $v_1 = v_2$. In order to extend the results to the generic case $v_1 \neq v_2$ one should also consider the operator

$$O_v = |\partial_x \Phi_1|^2 - |\partial_y \Phi_1|^2 - |\partial_x \Phi_2|^2 + |\partial_y \Phi_2|^2$$

and determine its RG dimension $y_v$ at the $O(4) \otimes O(3)$ collinear FP. If $y_v < 0$ the previous conclusions are unchanged. On the other hand, if $y_v > 0$ the $O(4) \otimes O(3)$ collinear FP is unstable with respect to the perturbation $O_v$. In this case the transition may be of first order or continuous depending on the existence and attraction domain of a stable FP with $v_1 \neq v_2$. Note that, from a practical point of view, our results are of interest even if $y_v > 0$.

Indeed, one expects the SDW velocities $v_1$ and $v_2$ to be close in magnitude, of the order of the spin-wave velocity of the Néel state of the undoped insulator. Therefore, the RG flow always starts very close to the stable FP of the theory with $v_1 = v_2$, and thus the critical behavior is controlled by this FP except in a narrow interval around the critical doping.

Experiments indicate that the SDW-SC–to–SC transition is continuous and is associated with collinear SDWs. It is thus natural to conjecture that its critical behavior is controlled by the $O(4) \otimes O(3)$ collinear FP, since this FP is stable in model (1.4) and its basin of attraction includes systems with collinear SDWs. The corresponding critical exponents are then predicted to be $\nu = 0.9(2)$, $\eta = 0.15(10)$. 

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APPENDIX A: GROUND-STATE CONFIGURATIONS

In this Appendix we compute the possible ground-state configurations of Hamiltonian (1.4), that allow us to identify the possible symmetry-breaking patterns. We consider translation-invariant configurations and the space-independent Hamiltonian density

\[ H(\Phi_1, \Phi_2) = r(|\Phi_1|^2 + |\Phi_2|^2) + H_4(\Phi_1, \Phi_2), \]  

(A1)

where \( H_4 \) is the part of the Hamiltonian that is quartic in the fields. Since \( H_4 \geq 0 \) for stability, for \( r > 0 \) the ground state always corresponds to \( \Phi_1 = \Phi_2 = 0 \). For \( r < 0 \), \( \Phi_1 = \Phi_2 = 0 \) is a local maximum of \( H \) and thus the ground state is nontrivial. The value \( r = 0 \) corresponds to a second-order transition point in the mean-field approximation. In order to determine the ground states for \( r < 0 \), we will first determine all stationary points of \( H \); the ground state is the one with the lowest energy. Note that, if \( \Phi_1, \Phi_2 \) is a stationary point, then

\[ H(\Phi_1, \Phi_2) = \frac{r}{2}(|\Phi_1|^2 + |\Phi_2|^2) = -H_4(\Phi_1, \Phi_2). \]  

(A2)

This relation is quite general. Indeed, assume \( H \) to be of the form

\[ H = \sum_{ij} r_{ij} \Phi_i \Phi_j + \sum_{ijkl} g_{ijkl} \Phi_i \Phi_j \Phi_k \Phi_l. \]  

(A3)

Then

\[ H = \frac{1}{4} \sum_i \Phi_i \frac{\partial H}{\partial \Phi_i} + \frac{1}{2} \sum_{ij} r_{ij} \Phi_i \Phi_j = \frac{1}{2} \sum_i \Phi_i \frac{\partial H}{\partial \Phi_i} - \sum_{ijkl} g_{ijkl} \Phi_i \Phi_j \Phi_k \Phi_l. \]  

(A4)

On a stationary solution, the derivative vanishes, proving Eq. (A2).

The calculation of the ground states also allows us to determine the stability domain of the Hamiltonian. Indeed, a point in the coupling space does not belong to the stability domain if there is a field such that \( H_4 < 0 \). Being \( H_4 \) homogeneous, it is not restrictive to consider only fields such that \( |\Phi_1|^2 + |\Phi_2|^2 = 1 \). Thus, the determination of the minima of \( H_4 \) is equivalent to the determination of the minima of \( (H - r) \) where \( r \) is now interpreted as a Lagrange multiplier. Eq. (A2) shows that \( H_4 \) can be negative only for \( r > 0 \). Thus, the stability domain of \( H_4 \) is obtained by determining the stationary points of \( H \) for \( r \) positive.

In order to determine the minima, we can use the symmetry of the Hamiltonian. Using the \( O(N) \) symmetry we can always write

\[ \text{Re} \Phi_1 = (a, 0, \ldots), \quad \text{Im} \Phi_1 = (b, c, 0, \ldots). \]  

(A5)
Then, by using the U(1) symmetry we can also fix $b = 0$. Indeed, we first perform an O(2) rotation on the first two components:

$$
\psi_1' = \psi_1 \cos \theta + \psi_2 \sin \theta \quad \psi_2' = -\psi_1 \sin \theta + \psi_2 \cos \theta,
$$

(A6)

where $\psi$ is either Re $\Phi_1$ or Im $\Phi_1$. Then, we apply a U(1) rotation, $\Phi'' = e^{ia}\Phi'$, if we choose

$$
\tan 2\theta = \frac{2bc}{a^2 + b^2 - c^2} \quad \tan \alpha = \frac{a \sin \theta}{b \sin \theta - c \cos \theta},
$$

(A7)

the transformed field has the form (A5) with $b = 0$. Once $\Phi_1$ has been fixed we can use O($N - 2$) and U(1) rotations to write

$$
\Phi_2 = (d + ie, g + if, l + ih, im, 0, \ldots).
$$

(A8)

If $N \leq 3$, one can use U(1) rotations to set $l = 0$.

The analysis of the minima is nontrivial due to the complexity of the stationarity equations. We have only consider the case $N \leq 3$ that is relevant experimentally. Other ground states are present for $N \geq 4$. We found seven relevant minima (only five of them occur for $N = 2$):

1) $a^2 = -r/u_{12}, \quad H = -r^2/(2u_{12})$.

2) $a^2 = c^2 = -r/u_1,\quad H = -r^2/(2u_1)$.

3) $a^2 = d^2 = -r/(u_{12} + w_1 + w_+), \quad H = -r^2/(u_{12} + w_1 + w_+)$.

4) $a^2 = f^2 = -r/(u_{12} + w_1), \quad H = -r^2/(u_{12} + w_1)$.

5a,b) $a^2 = c^2 = d^2 = f^2 = -r/(2u_1 + 2w_1 + w_+ \pm w_-), \quad H = -2r^2/(2u_1 + 2w_1 + w_+ \pm w_-)$.

6) $[N \geq 3]$ $c^2 = -r(u_1 - w_1)/\Delta_6, \quad d^2 = h^2 = -r(u_{12} - w_1)/(2\Delta_6), \quad H = -r^2(u_1 + u_{12} - 2w_1)/(2\Delta_6), \quad \Delta_6 = u_{11}u_{12} - w_1^2$.

7) $[N \geq 3, w_+ \neq 0]$ $a^2 = d^2 \neq 0, \quad c, f, h \neq 0, \quad ad/(cf) = w_+/w_-, \quad a^2 + c^2 = d^2 + f^2 + h^2$, with energy $H = -r^2(u_2+w_2) + w_2w_3)/\Delta_7$, with $\Delta_7 = (u_{12} + w_1)w_2w_3 + w_+u_2(u_1 + w_1)$. Alternatively, if we define the four vectors $t_1 = \text{Re } \Phi_1, \quad t_2 = \text{Im } \Phi_1, \quad t_3 = \text{Re } \Phi_2, \quad t_4 = \text{Im } \Phi_2$, and $t_{ij} = t_i \cdot t_j$, the solution can be characterized more geometrically as follows: $t_{11} = t_{22} = t_{33} = t_{44} = H/(2r), \quad t_{12} = t_{34}, \quad t_{13} = t_{24}, \quad t_{14} = t_{23}$, with

$$
t_{12}^2 = r^2u_2^2w_3^2/\Delta_7^2 \quad t_{13}^2 = r^2u_2^2w_2^2/\Delta_7^2 \quad t_{14}^2 = r^2u_2^2w_3^2/\Delta_7^2.
$$

(A9)

Whenever a component is not explicitly written, it vanishes. Moreover, we defined $u_{12} \equiv u_1 + u_2, \quad w_+ \equiv w_2 + w_3, \quad w_- \equiv w_2 - w_3$ and we simplified the notation writing $u_1$ instead of $u_{11,0}$, etc. Beside the seven solutions reported above, for $w_- \neq 0$ we also found stationary points with $e = g = h = 0$ and

$$
\frac{df}{dc} = -4u_2^2w_2^2 + w_+^2 \pm \sqrt{(4u_2^2 + w_2^2 - w_+^2)^2 - 16u_2^2w_-^2}/4u_2w_-.
$$

(A10)
A numerical analysis indicates that they are never absolute minima of the Hamiltonian and thus they are never relevant for the ground-state calculation. For this reason, these solutions have not been included above. The computation of all stationary points is quite straightforward, except for solution 7. We shall now briefly sketch how it is derived. Assume that \( e = g = 0 \) and \( a, c, d, f, h \neq 0 \) and define \( E_a = (1/a)\partial H/\partial a \), etc. Then,

\[
\frac{f}{c}(E_f - E_h) = -aw_- + cfw_+ = 0. \tag{A11}
\]

Using this relation, we can rewrite \( E_a, E_c, E_d, \) and \( E_f \) as linear equations in \( a^2, c^2, d^2, f^2, \) and \( h^2 \). Considering also \( a^2d^2w_-^2 = c^2f^2w_+^2 \) that follows from Eq. (A11), we obtain a system of equations that allows us to determine all components.

Given the list of solutions, we can determine the stability domain of the Hamiltonian. Using solutions 1-5, we obtain the necessary conditions

\[
\begin{align*}
    u_1 &> 0, & u_{12} &> 0, & u_{12} + w_1 &> 0, \\
    u_{12} + w_1 + w_+ &> 0, & u_1 + w_1 + \frac{1}{2}(w_+ \pm w_-) &> 0. \tag{A12}
\end{align*}
\]

These conditions are sufficient for \( N = 2 \). For \( N \geq 3 \) we must also consider solutions 6 and 7. Solution 6 gives the necessary condition

\[
w_1 > -\sqrt{u_1 u_{12}}. \tag{A13}
\]

Numerically, we find that solution 7 is also relevant for stability, although we have not been able to write down an easy condition.

For cuprates the relevant solutions are 3 and 4. In view of the possibility that the \( \text{O}(4) \otimes \text{O}(3) \) FP is stable it is important to understand to which ground state of the \( \text{O}(4) \otimes \text{O}(3) \) Hamiltonian (4.1) they correspond. For generic \( n \) and \( m \), \( m \geq n \), model (4.1) is stable for \( g_{1,0} \neq 0 \) and \( ng_{1,0} - (n-1)g_{2,0} \neq 0 \) and has two ground states depending on the sign of \( g_{2,0} \): for \( g_{2,0} > 0 \) the ground state is chiral, while for \( g_{2,0} < 0 \) the ground state is collinear.\(^{17} \) The corresponding energies are:

\[
H = -\frac{3nr^2}{2[ng_{1,0} - (n-1)g_{2,0}]} \quad (\text{chiral}); \tag{A14}
\]

\[
H = -\frac{3r^2}{2g_{1,0}} \quad (\text{collinear}).
\]

Using \( u_{12} = g_{1,0}/3, u_2 = g_{2,0}/6, w_1 = (g_{1,0} - g_{2,0})/3, w_+ = g_{2,0}/3, w_- = 0, \) we immediately see that for \( m = 4 \) and \( n = 2 \) and \( n = 3 \) solutions 1 and 3 correspond to the collinear case. For \( n = 3 \) solutions 6 and 7 correspond to the chiral case, while solutions 2, 4, and 5 correspond to a stationary state that is never a ground state in the chiral theory. For \( n = 2 \) instead, solutions 2, 4, and 5 are those corresponding to the chiral case. This result is relevant to identify the attraction domain of the \( \text{O}(4) \otimes \text{O}(3) \) collinear FP in the full theory. Indeed, it shows that the attraction domain of this FP includes systems whose ground state is given by solutions 1 and 3 (and therefore two collinearly polarized SDWs). Nothing can be said on the other solutions: in this case an analysis of the RG flow of the full theory is needed.
APPENDIX B: RENORMALIZATION-GROUP DIMENSIONS OF THE PERTURBATIONS AT THE $O(4) \otimes O(N)$ FIXED POINTS

We need to classify the operators that break

$$O(4) \otimes O(N) \to (U(1) \oplus U(1)) \otimes O(N) \cong (SO(2) \oplus SO(2)) \otimes O(N).$$

This is essentially discussed in Ref. 23. There are, however, two differences: first, we have only SO(2) symmetry, instead of O(2) symmetry; second, there is an additional exchange symmetry that forbids the appearance of spin-2 operators. In the notations of Ref. 23 ($M$ and $N$ of Ref. 23 correspond to $N$ and 4 respectively) we define

$$P_1 \equiv O^{(4,4)}_{1133} + O^{(4,4)}_{1144} + O^{(4,4)}_{2233} + O^{(4,4)}_{2244},$$

$$P_2 \equiv O^{(4,r)}_{1313} + O^{(4,r)}_{1414} + O^{(4,r)}_{2323} + O^{(4,r)}_{2424},$$

$$P_3 \equiv O^{(4,r)}_{1234},$$

where $\phi_{ai}$ is the real field defined in Eq. (4.3). Note that $P_3$ would be forbidden if we had $O(2)$ invariance instead of SO(2) invariance. Moreover, $P_2$ and $P_3$ correspond to different components of the same operator, so that they have the same RG dimension.

Hamiltonian (1.4) can then be written as

$$\mathcal{H} = \int d^d x \sum_a \frac{1}{2} \left[ (\nabla \phi_a)^2 + \phi_a^2 \right] + t_1 (\sum_a \phi_a^2)^2 + t_2 \sum_{a,b} \left[ (\phi_a \cdot \phi_b)^2 - \phi_a^2 \phi_b^2 \right] + t_3 P_1 + t_4 P_2 + t_5 P_3,$$

where

$$t_1 = \frac{1}{24} (2u_1 + 2u_2 + w_1 + w_2 + w_3),$$
$$t_2 = \frac{1}{36} (u_1 + 4u_2 - w_1 + 2w_2 + 2w_3),$$
$$t_3 = \frac{1}{12} (-u_1 - u_2 + w_1 + w_2 + w_3),$$
$$t_4 = \frac{1}{12} (-2u_1 + 4u_2 + 2w_1 - w_2 - w_3),$$
$$t_5 = \frac{1}{2} (w_3 - w_2).$$

Since all operators are irreducible with respect to $O(4) \otimes O(N)$ transformations, if the couplings belong to the $O(4) \otimes O(N)$ theory, the stability matrix defined with respect to the couplings $t_i$ has the form

$$\Omega = \begin{pmatrix}
\Omega_{11} & \Omega_{21} & 0 & 0 & 0 \\
\Omega_{21} & \Omega_{22} & 0 & 0 & 0 \\
0 & 0 & \Omega_1 & 0 & 0 \\
0 & 0 & 0 & \Omega_2 & 0 \\
0 & 0 & 0 & 0 & \Omega_2
\end{pmatrix}.$$
Here $Y_1 = -\Omega_1$ is the RG dimension of $\mathcal{O}^{(4,4)}$ and $Y_2 = -\Omega_2$ is the RG dimension of $\mathcal{O}^{(4,r)}$.

In the MZM scheme for $N = 2$ we find:

\[
\Omega_1 = 1 - (0.238732 \, u_1 + 0.31831 \, u_2) +
+ (0.0324838 \, u_1^2 + 0.0614494 \, u_1 \, u_2 + 0.0342428 \, u_2^2) +
- (0.00570145 \, u_1^3 + 0.016556 \, u_1^2 \, u_2 + 0.0218393 \, u_1 \, u_2^2 + 0.00451785 \, u_2^3) +
+ (0.0156398 \, u_1^4 + 0.0052904 \, u_1^3 \, u_2 + 0.0097641 \, u_1^2 \, u_2^2 + 0.00663506 \, u_1 \, u_2^3 +
\quad + 0.000911563 \, u_2^4) - (0.0045218 \, u_1^5 + 0.00188296 \, u_1^4 \, u_2 + 0.00448727 \, u_1^3 \, u_2^2 +
\quad + 0.00487778 \, u_1^2 \, u_2^3 + 0.00209129 \, u_1 \, u_2^4 + 0.000250855 \, u_2^5) + (0.000158666 \, u_1^6 +
\quad + 0.000747711 \, u_1^5 \, u_2 + 0.00212573 \, u_1^4 \, u_2^2 + 0.00311938 \, u_1^3 \, u_2^3 + 0.00230887 \, u_1^2 \, u_2^4 +
\quad + 0.000759722 \, u_1 \, u_2^5 + 0.0000778366 \, u_2^6),
\]

(B6)

\[
\Omega_2 = 1 - (0.238732 \, u_1 - 0.159155 \, u_2) +
+ (0.0324838 \, u_1^2 - 0.0145415 \, u_1 \, u_2 - 0.0164178 \, u_2^2) +
- (0.00570145 \, u_1^3 - 0.00426208 \, u_1^2 \, u_2 - 0.00228256 \, u_1 \, u_2^2 + 0.00294388 \, u_2^3) +
+ (0.0156398 \, u_1^4 - 0.000683968 \, u_1^3 \, u_2 + 9.94721 \cdot 10^{-6} \, u_1^2 \, u_2^2 + 0.000950042 \, u_1 \, u_2^3 +
\quad - 0.000152361 \, u_2^4) - (0.0045218 \, u_1^5 - 0.000214037 \, u_1^4 \, u_2 + 0.00011481 \, u_1^3 \, u_2^2 +
\quad + 0.000428198 \, u_1^2 \, u_2^3 - 0.0000869867 \, u_1 \, u_2^4 - 0.000326009 \, u_2^5) + (0.000158666 \, u_1^6 +
\quad - 0.000038529 \, u_1^5 \, u_2 + 0.000128979 \, u_1^4 \, u_2^2 + 0.000233286 \, u_1^3 \, u_2^3 - 0.0000225715 \, u_1^2 \, u_2^4 +
\quad - 0.0000164528 \, u_1 \, u_2^5 + 8.9658 \cdot 10^{-6} \, u_2^6),
\]

(B7)

For $N = 3$ we obtain:

\[
\Omega_1 = 1 - (0.238732 \, u_1 + 0.397887 \, u_2) +
+ (0.0378782 \, u_1^2 + 0.0687202 \, u_1 \, u_2 + 0.0413963 \, u_2^2) +
- (0.00623158 \, u_1^3 + 0.0193945 \, u_1^2 \, u_2 + 0.0290399 \, u_1 \, u_2^2 + 0.00434652 \, u_2^3) +
+ (0.0020362 \, u_1^4 + 0.00594275 \, u_1^3 \, u_2 + 0.0126485 \, u_1^2 \, u_2^2 + 0.00940724 \, u_1 \, u_2^3 +
\quad + 0.000356974 \, u_2^4) - (0.000572885 \, u_1^5 + 0.00220348 \, u_1^4 \, u_2 + 0.00583754 \, u_1^3 \, u_2^2 +
\quad + 0.0069323 \, u_1^2 \, u_2^3 + 0.00263869 \, u_1 \, u_2^4 + 0.000106569 \, u_2^5) + (0.000227195 \, u_1^6 +
\quad + 0.000881969 \, u_1^5 \, u_2 + 0.00277192 \, u_1^4 \, u_2^2 + 0.00439833 \, u_1^3 \, u_2^3 + 0.00335526 \, u_1^2 \, u_2^4 +
\quad + 0.000841582 \, u_1 \, u_2^5 + 0.0000352387 \, u_2^6),
\]

(B8)

\[
\Omega_2 = 1 - (0.238732 \, u_1 - 0.0795775 \, u_2) +
+ (0.0378782 \, u_1^2 - 0.00727073 \, u_1 \, u_2 - 0.0314283 \, u_2^2) +
- (0.00623158 \, u_1^3 - 0.00255553 \, u_1^2 \, u_2 - 0.00424767 \, u_1 \, u_2^2 + 0.00445854 \, u_2^3) +
- (0.0020362 \, u_1^4 - 0.000329962 \, u_1^3 \, u_2 - 0.000353201 \, u_1^2 \, u_2^2 + 0.00115746 \, u_1 \, u_2^3 +
\quad - 0.000132244 \, u_2^4) - (0.000572885 \, u_1^5 - 0.000151507 \, u_1^4 \, u_2 + 0.000153088 \, u_1^3 \, u_2^2 +
\quad + 0.000587068 \, u_1^2 \, u_2^3 - 0.0000150878 \, u_1 \, u_2^4 - 0.0000901012 \, u_2^5) + (0.000227195 \, u_1^6 +
\quad - 0.0000200211 \, u_1^5 \, u_2 + 0.000205186 \, u_1^4 \, u_2^2 + 0.000321211 \, u_1^3 \, u_2^3 + 0.000031431 \, u_1^2 \, u_2^4 +
\quad - 0.0000663864 \, u_1 \, u_2^5 + 0.000193321 \, u_2^6).
\]

(B9)
In the 3d-MS we find for $N = 2$:

$$\Omega_1 = 1 - (6 u_1 + 8 u_2) +$$
$$+(30.5 u_1^2 + 58 u_1 u_2 + 32 u_2^2) +$$
$$-(327.297 u_1^3 + 928.74 u_1^2 u_2 + 1185.79 u_1 u_2^2 + 258.397 u_2^3) +$$
$$+(5835.31 u_1^4 + 20132.4 u_1^3 u_2 + 35648.3 u_1^2 u_2^2 + 23296.2 u_1 u_2^3 +$$
$$+4377.86 u_2^4) - (123668 u_1^5 + 506531 u_1^4 u_2 + 1.1389 \cdot 10^6 u_1^3 u_2^2 +$$
$$+1.1852 \cdot 10^6 u_1^2 u_2^3 + 552355 u_1 u_2^4 + 85949 u_2^5), \quad (B10)$$

$$\Omega_2 = 1 - (6 u_1 - 4 u_2) +$$
$$+(30.5 u_1^2 - 14 u_1 u_2 - 16 u_2^2) +$$
$$-(327.297 u_1^3 - 219.798 u_1^2 u_2 - 163.596 u_1 u_2^2 + 103.301 u_2^3) +$$
$$+(5835.31 u_1^4 - 3001.7 u_1^3 u_2 - 2298.63 u_1^2 u_2^2 + 1415.85 u_1 u_2^3 +$$
$$+198.484 u_2^4) - (123668 u_1^5 - 48887.2 u_1^4 u_2 - 24466.2 u_1^3 u_2^2 +$$
$$+34690.3 u_1^2 u_2^3 - 12396.4 u_1 u_2^4 - 4256.31 u_2^5). \quad (B11)$$

and finally for $N = 3$:

$$\Omega_1 = 1 - (6 u_1 + 10 u_2) +$$
$$+(35.5 u_1^2 + 65 u_1 u_2 + 38.5 u_2^2) +$$
$$-(369.646 u_1^3 + 1082.64 u_1^2 u_2 + 1548.49 u_1 u_2^2 + 251.598 u_2^3) +$$
$$+(7381.82 u_1^4 + 23673. u_1^3 u_2 + 45677.7 u_1^2 u_2^2 + 31394.4 u_1 u_2^3 +$$
$$+3819.3 u_2^4) - (169602 u_1^5 + 615152 u_1^4 u_2 + 1.48093 \cdot 10^6 u_1^3 u_2^2 +$$
$$+1.61998 \cdot 10^6 u_1^2 u_2^3 + 719990 u_1 u_2^4 + 76742 u_2^5), \quad (B12)$$

$$\Omega_2 = 1 - (6 u_1 - 2 u_2) +$$
$$+(35.5 u_1^2 - 7 u_1 u_2 - 30.5 u_2^2) +$$
$$-(369.646 u_1^3 - 130.399 u_1^2 u_2 - 306.317 u_1 u_2^2 + 164.3 u_2^3) +$$
$$+(7381.82 u_1^4 - 1861.44 u_1^3 u_2 - 4739.46 u_1^2 u_2^2 + 2138.26 u_1 u_2^3 +$$
$$+581.765 u_2^4) - (169602 u_1^5 - 33233.6 u_1^4 u_2 - 55928.8 u_1^3 u_2^2 +$$
$$+53320.3 u_1^2 u_2^3 - 13607.5 u_1 u_2^4 - 6933.24 u_2^5). \quad (B13)$$

**APPENDIX C: RENORMALIZATION-GROUP DIMENSIONS OF THE PERTURBATIONS AT THE MN FIXED POINTS**

The analysis of the perturbations at the $mn$ FP is quite simple. In our case $m = 2N$, $n = 2$ and the relevant symmetry group is O(2N), which is broken by the terms proportional to $u_2$, $u_2$, and $u_3$. If $\phi_\alpha$ is the field defined in Eq. (4.6), $a = 1, 2$, $i = 1, \ldots, 2N$, we define the following spin-2 and spin-4 operators that transform irreducibly under O(2N):
\[ V^{(2)}_{a,i,j} \equiv \phi_a \phi_{aj} - \frac{1}{2N} \delta_{ij} \phi_a^2 \]  
(C1)

\[ V^{(4)}_{a,i,j,k,l} \equiv \phi_a \phi_{aj} \phi_{ak} \phi_{al} - \frac{1}{2(N+2)} \phi_a^2 (\delta_{ij} \phi_{ak} \phi_{al} + 5 \text{ perm.}) \]
\[ + \frac{1}{4(N+1)(N+2)} \phi_a^2 (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \]  
(C2)

where \( \phi_a^2 \equiv \sum_i \phi_{ai}^2 \). Then, the relevant operators are:

\[ P_1 \equiv \sum_{a=1}^2 \sum_{\alpha=0}^1 \sum_{ij=1}^N V^{(4)}_{a,i+\alpha N,i+\beta N,j+\alpha N,j+\beta N}, \]  
(C3)

\[ P_2 \equiv \sum_{\alpha=0}^1 \sum_{ij=1}^N V^{(2)}_{1,i+\alpha N,j+\alpha N} V^{(2)}_{2,i+\beta N,j+\beta N}, \]  
(C4)

\[ P_3 \equiv \sum_{\alpha=0}^1 \sum_{ij=1}^N \epsilon_{\alpha \beta \gamma \delta} V^{(2)}_{1,i+\alpha N,j+\beta N} V^{(2)}_{2,i+\gamma N,j+\delta N}, \]  
(C5)

where \( \epsilon_{01} = -\epsilon_{10} = 1 \) and \( \epsilon_{00} = \epsilon_{11} = 0 \). These operators give rise to different breakings of \( O(2N) \):

\[ O(2N) \xrightarrow{P_1} [O(N) \otimes O(2)] \oplus [O(N) \otimes O(2)] \xrightarrow{P_2} O(N) \otimes [O(2) \oplus O(2)] \]
\[ \xrightarrow{P_3} O(N) \otimes S[O(2) \oplus O(2)]. \]  
(C6)

In terms of \( P_1 \), \( P_2 \), and \( P_3 \) Hamiltonian (1.4) can then be written as

\[ \mathcal{H} = \int d^d x \sum_a \frac{1}{2} [\nabla \phi_a]^2 + \phi_a^2 + t_1 \sum_a (\phi_a^2)^2 + t_2 \phi_1^2 \phi_2^2 \]
\[ + t_3 P_1 + t_4 P_2 + t_5 P_3, \]  
(C7)

where

\[ t_1 = \frac{u_1}{2} + \frac{u_2}{N+1}, \quad t_2 = w_1 + \frac{1}{N} (w_2 + w_3), \quad t_3 = u_2, \quad t_4 = w_2 + w_3, \quad t_5 = w_3 - w_2. \]  
(C8)

Since all operators are irreducible with respect to \( O(2N) \) transformations, if the couplings belong to the \( mn \) theory, the stability matrix defined with respect to the couplings \( t_i \) has the form

\[ \Omega = \begin{pmatrix} \Omega_{11} & \Omega_{21} & 0 & 0 & 0 \\ \Omega_{21} & \Omega_{22} & 0 & 0 & 0 \\ 0 & 0 & \Omega_1 & 0 & 0 \\ 0 & 0 & 0 & \Omega_2 & 0 \\ 0 & 0 & 0 & 0 & \Omega_2 \end{pmatrix} \]  
(C9)

Note that two eigenvalues are degenerate, since \( P_2 \) and \( P_3 \) are different components of the same irreducible operator \( V^{(2)}_{1,i,j} V^{(2)}_{2,k,l} \). The corresponding RG dimensions are \( Y_1 = -\Omega_1 \), \( Y_2 = -\Omega_2 \).
REFERENCES

1. S. Sachdev, Rev. Mod. Phys. 75, 913 (2003).
2. S. Wakimoto, G. Shirane, Y. Endoh, K. Hirot, S. Ueki, K. Yamada, R.J. Birgeneau, M.A. Kastner, Y.S. Lee, P.M. Gehring, and S.H. Lee, Phys. Rev. B 60, R769 (1999); S. Wakimoto, R.J. Birgeneau, Y.S. Lee, and G. Shirane, Phys. Rev. B 63, 172501 (2001).
3. G. Aeppli, T.E. Mason, S.M. Hayden, H.A. Mook, and J. Kulda, Science 278, 1432 (1997).
4. Y. Zhang, E. Demler, and S. Sachdev, Phys. Rev. B 66, 094501 (2002).
5. Y.S. Lee, R.J. Birgeneau, M.A. Kastner, Y. Endoh, S. Wakimoto, K. Yamada, R.W. Erwin, S.-H. Lee, and G. Shirane, Phys. Rev. B 60, 3643 (1999).
6. K.G. Wilson and M.E. Fisher, Phys. Rev. Lett. 28, 240 (1972).
7. See, for example, S. Mo, J. Hove, and A. Sudbø, Phys. Rev. B 65, 104501 (2002), and references therein.
8. P. Calabrese, P. Parruccini, A. Pelissetto, and E. Vicari, Phys. Rev. B 70, 174439 (2004).
9. M. De Prato, A. Pelissetto, and E. Vicari, Nucl. Phys. B 607, 605 (2001).
10. A. Pelissetto and E. Vicari, Cond. Matt. Phys. (Ukraine) 8, 87 (2005) [hep-th/0409214].
11. J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, fourth edition (Clarendon Press, Oxford, 2001).
12. H. Kleinert and V. Schulte-Frohlinde, Critical Properties of $\phi^4$-Theories (World Scientific, Singapore, 2001).
13. A. Aharony, In Phase Transitions and Critical Phenomena, Vol. 6, edited by C. Domb and M.S. Green (New York, Academic, 1976).
14. A. Pelissetto and E. Vicari, Phys. Rev. B 62, 6393 (2000).
15. At the chiral FP we use (Refs. 8, 23): $\nu = 0.60(8), y_{11} = 2.3(2), y_{02} = 1.52(6)$ ($N = 3$); $\nu = 0.63(9), y_{11} = 2.45(25), y_{02} = 1.30(15)$ ($N = 2$). At the collinear FP, for $N = 3$ we use...
(Refs. 9, 23): \( \nu = 0.60(5), y_{11} = 1.20(15), y_{02} = 2.0(2) \). For \( N = 2 \), we use the mapping with the XY model (Refs. 17, 23) and the results of M. Campostrini, M. Hasenbusch, A. Pelissetto, P. Rossi, and E. Vicari, Phys. Rev. B 63, 214503 (2001), \( \nu = 0.67155(27), y_{11} = 1.489(6), y_{02} = 1.9620(8) \).

31 J. M. Carmona, A. Pelissetto, and E. Vicari, Phys. Rev. B 61, 15136 (2000).
32 We use \( u_1 = -1.1(5) \) and \( u_2 = 6.1(4) \) (MZM), \( u_1 = 0.03(2) \) and \( u_2 = 0.215(13) \) (3d-\( \overline{\text{MS}} \)).
33 It is interesting to note that these exponents are also relevant for quantum chromodynamics (the theory of strong interactions) with two quarks at finite temperature if the anomaly contribution at the critical point is small [F. Basile, A. Pelissetto, and E. Vicari, Proceedings of the Symposium on Lattice Field Theory 2005, PoS (LAT2005) 199, hep-lat/0509018, and J. High Energy Phys. 02, 044 (2005); A. Butti, A. Pelissetto, and E. Vicari, J. High Energy Phys. 08, 029 (2003)].