The mathematical modeling of the incomplete algebraic eigenvector and eigenvalue problem for obtaining the reduced frequency equation and its solution

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Abstract. This Paper studies the application of the Sequential Frequency–Dynamic Condensation Method developed by the authors for reducing the frequency equations obtained on the basis of the Finite Element Method (FEM) in the form of a Classical Mixed Method.

1. Introduction
It is typical for FEM–based frequency equation to have a very high order for complex structures. Due to this, solving the equation as a complete algebraic eigenvalues (EVA) and eigenvectors (EVE) problem is hardly possible for the reason of mathematical complexity and excessive use of computer time.

On the other hand, it suffices to know a limited number of the entire EVA and EVE spectrum; usually, knowing the smallest EVA and EVE is enough. This is why developing methods to solve the incomplete EVA and EVE problem has been a popular research topic.

At the same time, for most practical problems it is sufficient to know some portion of the EVA/EVE spectrum; as a rule, the least EVA and their corresponding EVE will suffice. This is why methods for solving the incomplete EVA/EVE problem have been extensively developed.

Note that due to the high matrix order, the conventional iterative methods [1–7] are inefficient for solving the EVA problem, which is why large problems are solved by methods that take into account matrix sparseness and symmetry. It means that all numerical operations are made with non–zero matrix elements that are stored in computer memory in a compact form. This is why these methods can be used to solve high–order equation systems. Even the methods that are developed specifically to solve high–order EVA problems are vulnerable to poorly determined problems. This results in a considerable deterioration of their convergence. Many authors have attempted to develop various modifications to improve the convergence rate and the efficiency of these methods in this or that way.

The monograph [8] presents a classification and a detailed review of most existing reduction methods.

On the other hand, methods for obtaining reduced partial equation and its solution can be divided into two main groups. The first one consists of methods based on the initial frequency equation written as a complete algebraic EVA and EVE problem and a purely mathematical (algebraic) transformation thereof into a reduced form by means of transition matrices [9], static condensation, [10–12], or dynamic condensation, [9, 13, 14].
The second one comprises methods based on using physical reduction models: the superelement (substructure) method [15–18], interpolation methods (sparse–grid methods, spline methods) [19, 20, 21], subcomponent shape synthesis method (modal synthesis method) [22–27], Ritz methods [28] and their mathematical modeling on a sequence of condensing finite–element grids or condensation nodes.

The most effective among them is the Sequential Frequency–Dynamic Condensation Method set out in [20, 29, etc.], developed and applied in a number of publications. Frequency–dynamic condensation (FDC) is used in all these papers to reduce the FEM–based frequency equations in the form of a displacement method and method of forces.

We consider the application of the Sequential FDC method developed by the authors for reducing the frequency equations obtained on the basis of the FEM in the form of a classical mixed method in this paper.

2. The calculation algorithm by the Sequential FDC method with preliminary static condensation

The calculation algorithm by the Sequential FDC method with preliminary static condensation consists of the following operations:

1. The first stage of static condensation is performed.

Similar to FEM displacement calculation, we divide all the degrees of the current system freedom into the primary (at the first stage – kinematic) – \( k \) and secondary (at the first stage – force) – \( f \) following the key static condensation method idea. The frequency equation will be as follows:

\[
\begin{bmatrix}
    r_{kk} & r_{kf} \\
    \delta_{rk} & \delta_{rf}
\end{bmatrix}
\begin{bmatrix}
    q_k \\
    \varphi_f
\end{bmatrix} - \lambda
\begin{bmatrix}
    m_{kk} & m_{kf} \\
    m_{fk} & m_{ff}
\end{bmatrix}
\begin{bmatrix}
    q_k \\
    \varphi_f
\end{bmatrix} = 0,
\]

where:

\[
\begin{bmatrix}
    r_{kk} & r_{kf} \\
    \delta_{rk} & \delta_{rf}
\end{bmatrix}
= D_{cm}
\text{ is the static matrix of system (design) responses;}
\]

\[
\begin{bmatrix}
    m_{kk} & m_{kf} \\
    m_{fk} & m_{ff}
\end{bmatrix}
= \left[ m \right]
\text{ is the consistent mass matrix;}
\]

\[
\begin{bmatrix}
    q_k \\
    \varphi_f
\end{bmatrix}
\text{ is the main unknown vector (kinematic } q_k \text{ and force } \varphi_f \text{);}\]

\[
\begin{bmatrix}
    r_{kk} \\
    \delta_{rk}
\end{bmatrix}
\text{ is the rotation angle vector at the finite element grid's units.}
\]

Let's write the equation (1) in expanded form:

\[
\begin{align}
1) \quad r_{kk}q_k + r_{kf}\varphi_f - \lambda \left( m_{kk}q_k + m_{kf}\varphi_f \right) &= 0, \\
2) \quad \delta_{rk}q_k + \delta_{rf}\varphi_f - \lambda \left( m_{rk}q_k + m_{rf}\varphi_f \right) &= 0.
\end{align}
\]

Introducing the assumption that the mass inertia forces \( m_{rk} \) and, \( m_{rf} \) as referring to secondary (force) degrees of freedom s are small to negligible in comparison with the mass inertia forces \( m_{kk} \) and \( m_{ff} \), referring to the primary (kinematic) degrees of freedom \( k \), we obtain from the second equation in (2) the static equation of deformation compatibility

\[
\begin{bmatrix}
    \delta_{rk} \\
    \delta_{rf}
\end{bmatrix}q_k + \delta_{rf}\varphi_f = 0,
\]

where the connection between the primary and secondary unknowns follows:
\[
\tilde{q}_j = -\delta_{\beta\beta}^{-1} \tilde{\delta}_\beta q_k.
\]  

(4)

As a general principle, the relationship between units’ linear displacements and rotation angles of nodal sections is established by the differentiation operator, i.e. \( \varphi_j = L(\tilde{q}_j) \).

With that knowledge and the dependence (4) in mind, the first equation in (2) is reduced to the form

\[
(D_{cm}' = r_k - r_{ij} \delta_{\beta\beta}^{-1} \tilde{\delta}_\beta)
\]

is the statically condensed response matrix to the main (kinematic) degrees of freedom,

\[
\mathbf{m}^* = \mathbf{m}_{ik} - \mathbf{m}_{ij} L[\delta_{\beta\beta}^{-1} \tilde{\delta}_\beta]
\]

is the condensed mass matrix to the main degrees of freedom.

2. The condensation units in the main system are assigned with kinematic degrees of freedom remaining after the first stage of static condensation.

As well as in the first stage, all system's degrees of freedom represented by the finite elements assembly are divided into primary (associated with the condensation units) and secondary ones.

Let us denote \( N \) — the total number of degrees of freedom, \( n \) — the number of freedom basic degrees \( \bar{n} = N - n \).

The frequency equation can be written as:

\[
\begin{bmatrix}
D_{cm,nn} & D_{cm,nn} & \ldots \\
D_{cm,nn} & D_{cm,nn} & \ldots \\
\vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
q_n \\
q_n \\
\vdots
\end{bmatrix}
- \lambda
\begin{bmatrix}
\mathbf{m}_{nn}^* \\
\mathbf{m}_{nn}^* \\
\vdots
\end{bmatrix}
\begin{bmatrix}
q_n \\
q_n \\
\vdots
\end{bmatrix}
= 0.
\]

(5)

3) The static condensation of the frequency equation (5) to the primary degrees of freedom is performed and the condensed response and mass matrices are found. The frequency equation of the second level of static condensation has the form:

\[
\begin{bmatrix}
\bar{D}_{cm} \\
\bar{D}_{cm}
\end{bmatrix}
\begin{bmatrix}
q_n \\
q_n
\end{bmatrix}
- \lambda \begin{bmatrix}
\bar{m} \\
\bar{m}
\end{bmatrix}
\begin{bmatrix}
q_n \\
q_n
\end{bmatrix}
= 0,
\]

(6)

where:

\[
\begin{bmatrix}
\bar{D}_{cm} \\
\bar{D}_{cm}
\end{bmatrix}
= D_{cm,nn} - D_{cm,nn} D_{cm,nn}^{-1} D_{cm,nn}.
\]

In order to avoid high–order matrix \( D_{cm,nn}^{-1} \) inversion therein, matrix construction \( D_{cm,nn}^{-1} \) is performed by addressing of separate problems for unit values of the conditional load vector component for the matrix \( D_{cm,nn}^{-1} \).

4) The EVA and the EVE are for the static condensed equation (6).

5) All secondary degrees of freedom \( \bar{n} \) are divided into \( T \) groups: \( \sum_{t=1}^{T} \bar{n}_t = \bar{n} \), using a layer–by–layer principle. The degrees of freedom number should not exceed the primary degrees of freedom number \( n \) in each of the groups.

6) We will compose the frequency equation for a partial system \( t \) having a degrees of freedom number \( N_t = n + \bar{n} \) (the partial system is obtained by adding to the masses \( n \) in the calculated \( \bar{n} \) mass units from the number of secondary, surrounding masses in the calculated units by selecting a \( t \) group from these \( T \) groups.

7) For each of the partial systems, the complete EVA and EVE problem is solved.

For example, for the first partial system consisting of \( n \) condensing units (settlement units) with masses and secondary units \( \bar{n}_1 \) of the first group with the corresponding masses, an extended frequency equation with an upper index \( I \) is composed:

\[
\begin{bmatrix}
(D_{cm,ss} - D_{cm,ss} D_{cm,ss}^{-1} D_{cm,ss}) & \ldots \\
\vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_1 \\
\vdots
\end{bmatrix}
- \lambda \begin{bmatrix}
\mathbf{m}_s \\
\mathbf{m}_s \\
\vdots
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_1 \\
\vdots
\end{bmatrix}
= 0,
\]

(7)

\( s = 1, 2, \ldots, n, n+1, \ldots, \bar{n}_1; \tilde{s} = \bar{n}_1 + 1, \bar{n}_1 + 2, \ldots, N \).
We find all \((n + \tilde{n})\) EVA for this partial system and EVE after solving this equation.

8) We find an adjusted mass matrix \(\tilde{m} = \bar{m} + \Delta m_i\) which all \(n\) the eigenvalues and eigenvectors coincide with \(n\) for, the lowest eigen frequencies and the corresponding eigenvibration vector components found for the extended frequency equation (7) by returning to the frequency equation (6) with determined condensed response and masses matrices.

9) We obtain the following equation by substituting into the static condensed frequency equation (6) the matrix \(\bar{m}\) instead of the matrix \(m\) and the first eigenvalues \(\lambda_{i}^{1}, \lambda_{2}^{1}, \ldots, \lambda_{n}^{1}\) from the spectrum lower part with the corresponding eigenvectors components \(v_{k,i}^{1}\) (where \(k = 1, 2, \ldots, n; \ i – \) the units numbers) found for (7):

\[
\left[ \tilde{D}_{cm} \right] - \left\{ \lambda_i^{1} \right\} \left[ \tilde{m} \right] \left[ v_{k,i}^{1} \right] = 0. \tag{8}
\]

We obtain the following by solving equation (8):

\[
\left[ \bar{m} \right] = \left[ \tilde{D}_{cm} \right]^{-1} \left\{ \lambda_i^{1} \right\} \left[ v_{k,i}^{1} \right]^{-1}. \tag{9}
\]

The condensation additives matrix from the first group of secondary masses

\[
\Delta m_i^{1} = \bar{m} - \tilde{m}. \tag{10}
\]

10) We obtain the reduced mass matrix of the considered construction, condensed to the selected \(n\) units by summing the additive for all partial systems:

\[
\bar{m}_{\text{ed}} = \bar{m} + \sum_{i=1}^{r} \Delta m_i^{1}. \tag{11}
\]

11) We solve the obtained reduced frequency equation by substituting \(\bar{m}\) the mass matrix \(\bar{m}_{\text{ed}}\) in (6)

\[
\left[ \tilde{D}_{cm} \right] \{ q_s \} - \lambda \left[ \bar{m}_{\text{ed}} \right] \{ q_s \} = 0. \tag{12}
\]

As a result, we find the spectrum of eigenvalues and eigenvectors reduced to the selected condensation units. We will obtain a sequence of solutions on hold by increasing the number of condensation units. As it is shown in [1, 2], preliminary static condensation substantially reduces the error of the obtained solution without using it.

3. Conclusions
The results analysis obtained by this algorithm revealed the following:

1. The consistent mass matrix has no advantages over the inconsistent mass matrix with respect to the calculation results.

Therefore, we can specify

\[
\left[ m \right] = \begin{bmatrix} m_i & 0 \\ 0 & m_i \end{bmatrix}
\]

2. The influence of the bulk masses obtained by reducing the finite elements distributed over the masses area is insignificant and can be neglected.

3. Compared to the method of static condensation, the results close to the exact ones are obtained for 50–60% of the eigenfrequencies from the total number in the reduced spectrum, while static condensation provides an acceptable result only for the minimum eigenfrequencies.

4. The unit group arrangement of freedom secondary degrees with respect to the condensation units does not significantly affect the precision of the results.
5. The study made it possible to identify the optimal limits for obtaining a reduced frequency spectrum with a desired degree of precision for different ratios of basic and secondary degrees of freedom number, the freedom number degrees of the partial system.

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