Twin Quantum Cheshire Cats
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In an experiment with both pre- and post-selection one can find a photon (the cat) in one place and its polarization (the smile) in another. In [1] Aharonov et al. asked whether more than two degrees of freedom could be separated in the same way. We show that this is possible and that the separation of properties from objects that carry them is in some situations even stronger.

I. INTRODUCTION

Properties are related in common understanding to objects that carry them, for example slowness belongs to the turtle. Yet in his novel Alice’s Adventures in Wonderland C. S. Lewis mischievously mentioned a smile with no cat. Aharonov et al. showed that quantum mechanics indeed allows such situations to happen in experiments with pre- and post-selection on both path and polarization degrees of freedom of a photon [1]. We generalize this result to four degrees of freedom by completely dissociating the position and the orthogonal polarizations of an entangled pair of photons. First, we retrieve the ‘one smile no cat’ result. Second, we show that the disembodiment of physical properties can be even stronger, since in some situations smiles cannot be traced back to any definite cat.

II. ONE SMILE WITH NO CAT

A pair of entangled photons after BS11 and BS21 is in the following state (Fig. 1):

\[ |\psi\rangle = \frac{1}{2}(|13\rangle + |14\rangle + |23\rangle + |24\rangle) \otimes |A\rangle, \]

where

\[ |A\rangle = \frac{|H,V\rangle + |V,H\rangle}{\sqrt{2}}, \]

and \(|H\rangle\) and \(|V\rangle\) denote horizontally and vertically polarized light respectively.

The Mach-Zender interferometers are so tuned that both detectors in the pair \(D11,D22\) or in the pair \(D12,D21\) always click, whereas D13 and D23 never click. Using post-selection [2], we first forbid the entangled photons to travel through arms (1,4) and (2,3). This can be done by introducing a half-wave plate (HWP) between M2 and BS11, and another HWP between M2 and BS21. The HWPs transform polarization \(|V\rangle\) to \(-|V\rangle\) and post-select the state:

\[ |\phi_1\rangle = \frac{1}{2} [(|13\rangle + |24\rangle) \otimes |A\rangle + (|14\rangle - |23\rangle) \otimes |B\rangle], \]

where

\[ |B\rangle = \frac{|H,H\rangle - |V,H\rangle}{\sqrt{2}}. \]

Post-selection (3) indeed constrains the photons to go through arms (1,3) or (2,4), for if we suppose that the photons went through different sides, then \(|\psi\rangle\) would project to \(\frac{|14\rangle + |23\rangle \otimes |A\rangle}{\sqrt{2}}\) and \(|\phi_1\rangle\) to \(\frac{|14\rangle - |23\rangle \otimes |B\rangle}{\sqrt{2}}\). Since \(\langle A|B\rangle = 0\), these two states are orthogonal and post-selection would thus lead to a contradiction.

We now define position measurement operators:

\[ \Pi_{ij} = |ijHV\rangle\langle VHIj| + |ijVH\rangle\langle HVji|, \]

\[ \Pi_i = \Pi_{1i} + \Pi_{2i}, \]

\[ \Pi_j = \Pi_{1j} + \Pi_{2j}, \]
where \( i \) (resp. \( j \)) denotes the arm of the left (right) interferometer on which measurement is performed. The dot \( \cdot \) represents a measurement carried on both arms of the corresponding interferometer, which is equivalent to tracing out one of the entangled particles.

Consider the \( \{|+\rangle, |-\rangle\} \) basis where \( |+\rangle = (|H\rangle + |V\rangle)/\sqrt{2} \) and \( |-\rangle = (|H\rangle - |V\rangle)/\sqrt{2} \) denote left- and right-circularly polarized light respectively. Define in this basis polarization measurement operators:

\[
\sigma_{zz} = \sigma_i \otimes \sigma_z
\]

\[
= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

\[
= |+\rangle\langle+| - |-\rangle\langle-|
\]

\[
- |+\rangle\langle-| + |-\rangle\langle+|.
\]

A measurement of polarization in arms \( i \) and \( j \) \((i = 1, 2; j = 3, 4)\) corresponds to the operator:

\[
\sigma_{zz}^{ij} = \Pi_{ij}\sigma_{zz}.
\]

Trace out one of the particles:

\[
\sigma_{zz}^i = \Pi_i \sigma_{zz},
\]

\[
\sigma_{zz}^j = \Pi_j \sigma_{zz}.
\]

Define the weak value of an operator \( O \) with pre- and post-selected states \( |\psi\rangle \) and \( |\phi\rangle \) as \[2 \, 3\]:

\[
\langle O \rangle_w = \frac{\langle \phi | O | \psi \rangle}{\langle \phi | \psi \rangle}.
\]

After a calculation (see Appendix) we get the following results for the weak values of the position measurement operators:

\[
\langle \Pi_{13} \rangle_w = \langle \Pi_{24} \rangle_w = 1/2,
\]

\[
\langle \Pi_{14} \rangle_w = \langle \Pi_{23} \rangle_w = 0,
\]

\[
\langle \Pi_1 \rangle_w = \langle \Pi_2 \rangle_w = \langle \Pi_3 \rangle_w = \langle \Pi_4 \rangle_w = 1/2.
\]

These equations correspond to one photon pair going through the arms (1,3) or (2,4):

\[
\langle \Pi_{13} + \Pi_{24} \rangle_w = \langle \Pi_{13} \rangle_w + \langle \Pi_{24} \rangle_w = 1,
\]

and one photon travelling through each interferometer:

\[
\langle \Pi_1 + \Pi_2 \rangle_w = \langle \Pi_3 \rangle_w + \langle \Pi_4 \rangle_w = 1.
\]

Similarly we compute weak values of the polarization measurement operators (see Appendix):

\[
\langle \sigma_{zz}^{13} \rangle_w = \langle \sigma_{zz}^{24} \rangle_w = 1/2,
\]

\[
\langle \sigma_{zz}^1 \rangle_w = \langle \sigma_{zz}^2 \rangle_w = \langle \sigma_{zz}^3 \rangle_w = \langle \sigma_{zz}^4 \rangle_w = 1/2.
\]

We can make simultaneous weak measurements of \( \sigma_{zz}^{13} \) and \( \sigma_{zz}^{24} \), hence getting the total weak value:

\[
\langle \sigma_{zz}^{13} + \sigma_{zz}^{24} \rangle_w = \langle \sigma_{zz}^{13} \rangle_w + \langle \sigma_{zz}^{24} \rangle_w = 1,
\]

which together with equation (19) is consistent with the post-selection condition. We also have:

\[
\langle \sigma_{zz}^1 + \sigma_{zz}^2 \rangle_w = \langle \sigma_{zz}^3 \rangle_w + \langle \sigma_{zz}^4 \rangle_w = 1,
\]

which is consistent with the fact that there is one photon in each interferometer.

An interesting effect in considering a pair of entangled photons appears when weak polarization measurements for the photons are carried along orthogonal directions. Consider the following polarization measurement operator:

\[
\sigma_{zx} = \sigma_z \otimes \sigma_x
\]

\[
= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}
\]

\[
= |+\rangle\langle+| + |-\rangle\langle-|
\]

\[
- |+\rangle\langle-| - |-\rangle\langle+|.
\]

Weak values results for the different \( \sigma_{zx}^{ij} \) operators are:

\[
\langle \sigma_{zx}^{13} \rangle_w = \langle \sigma_{zx}^{24} \rangle_w = 0,
\]

\[
\langle \sigma_{zx}^{14} \rangle_w = \langle \sigma_{zx}^{23} \rangle_w = 1/2,
\]

\[
\langle \sigma_{zx}^1 \rangle_w = \langle \sigma_{zx}^2 \rangle_w = \langle \sigma_{zx}^3 \rangle_w = \langle \sigma_{zx}^4 \rangle_w = 1/2.
\]

We conclude that though the photons of the pair go through arms (1,3) or (2,4):

\[
\langle \Pi_{13} \rangle_w = \langle \Pi_{24} \rangle_w = 0,
\]

polarizations don’t:

\[
\langle \sigma_{zx}^{13} \rangle_w = \langle \sigma_{zx}^{24} \rangle_w = 0.
\]

This is an example of quantum Cheshire cat \[1\].

### III. TWO SMILES WITH NO CATS

In the previous section we saw a polarization travelling along a different arm than the photon with which it was associated. Now we shall separate all polarizations from all photons.
Consider a post-selected state:

$$|\phi_2\rangle = \frac{1}{2} \left[ |13\rangle \otimes |A\rangle + (|14\rangle - |23\rangle - |24\rangle) \otimes |B\rangle \right]. \quad (29)$$

This post-selection can be done by inserting a suitable unitary operator along the arms of the Mach-Zender interferometers. It constrains the photons of the pair to travel through arms 1 and 3, one photon travelling through each interferometer.

We compute the same quantities as before:

$$\langle \Pi_{13} \rangle_w = 1, \quad (30)$$

$$\langle \Pi_{24} \rangle_w = \langle \Pi_{14} \rangle_w = \langle \Pi_{23} \rangle_w = 0, \quad (31)$$

$$\langle \Pi_1 \rangle_w = \langle \Pi_3 \rangle_w = 1, \quad (32)$$

$$\langle \Pi_2 \rangle_w = \langle \Pi_4 \rangle_w = 0. \quad (33)$$

This confirms that the photons of the pair went through arms 1 and 3.

$$\langle \sigma_{zz}^{13} \rangle_w = 1, \quad (34)$$

$$\langle \sigma_{zz}^{24} \rangle_w = \langle \sigma_{zz}^{14} \rangle_w = \langle \sigma_{zz}^{23} \rangle_w = 0, \quad (35)$$

$$\langle \sigma_{zz}^1 \rangle_w = \langle \sigma_{zz}^3 \rangle_w = 1, \quad (36)$$

$$\langle \sigma_{zz}^2 \rangle_w = \langle \sigma_{zz}^4 \rangle_w = 0. \quad (37)$$

Measuring polarizations along the same direction for both photons yields no interesting results. On the contrary, if measurement is carried out along orthogonal directions, then:

$$\langle \sigma_{zz}^{13} \rangle_w = 0, \quad (38)$$

$$\langle \sigma_{zz}^{24} \rangle_w = 1, \quad (39)$$

$$\langle \sigma_{zz}^{14} \rangle_w = -\langle \sigma_{zz}^{23} \rangle_w = -1, \quad (40)$$

$$\langle \sigma_{zz}^1 \rangle_w = -\langle \sigma_{zz}^3 \rangle_w = -1, \quad (41)$$

$$\langle \sigma_{zz}^2 \rangle_w = 2, \quad (42)$$

$$\langle \sigma_{zz}^4 \rangle_w = 0. \quad (43)$$

Hence the following conclusion: though the photons of the pair went with certainty through arms 1 and 3 of the interferometers, as confirmed by (30), no polarization pair traveled through these arms, as shown in (38). The polarization pair traveled along arms (2,4) and (2,3) with certainty, as shown in (39) and (40), and quantum mechanics compensates for the polarization pairs excess by letting -1 pair travel through arms (1,4)! The other quantities are consistent with the fact that there is one photon in each interferometer:

$$\langle \sigma_{zz}^1 \rangle_w + \langle \sigma_{zz}^2 \rangle_w + \langle \sigma_{zz}^3 \rangle_w + \langle \sigma_{zz}^4 \rangle_w = 1, \quad (44)$$

so one polarization travelled through each interferometer. It is worth noting that $$\langle \sigma_{zz}^2 \rangle_w = 2$$, which means that any system weakly interacting with arm 2 would feel the presence of two polarizations whereas only one photon entered the interferometer! Still, this does not contradict the statement that only one polarization travelled through the considered interferometer since a measurement on arm 1 will find -1 polarization, bringing the total polarization in the interferometer to 1. The question “which photon polarization?” makes no sense at all, unlike the Cheshire cat experiment in section 1.

**IV. CONCLUSIONS**

We have shown that the disembodiment of physical properties from objects to which they supposedly belong in pre- and post-selected experiments can be generalized to more than two degrees of freedom. It appears even stronger since the correspondence between objects and properties is completely lost in some situations. The results, though concerning polarisation of entangled photons, can be extended to more general situations like those cited in [1]: for example, a separation of the spin from the charge of an electron, or the mass of an atom from the atom itself. An interesting task would be to retrieve these results in the two-state vector formalism [4] and to understand how the behaviour of the polarization of each state vector can give rise to the paradoxical effects due to post-selection.

[1] Y.Aharonov, S.Popescu, and P.Skrzypczyk [arxiv:1202.0631v1]
[2] Y.Aharonov, S.Popescu, D.Rohrlich, and L.Vaidman, Phys. Rev. A 48, 4084 (1993)
[3] Y.Aharonov, A.Botero, S.Popescu, B.Reznik, and J.Tollaksen, Phys. Lett. A 301, 130 (2002)
[4] Y.Aharonov and L.Vaidman [arXiv:quant-ph/0105101v2]
We compute the weak value of $\Pi_{ij}$:

\[
\Pi_{ij} |\psi\rangle = (|ijHV\rangle \langle VHji| + |ijVH\rangle \langle HVji|) \frac{[13] + [14] + [23] + [24]}{2} \otimes |A\rangle
\]

\[
= \frac{[ij]}{2}(|HV\rangle \langle VH|A\rangle + |VH\rangle \langle HV|A\rangle)
\]

\[
= \frac{[ij]}{2}|A\rangle.
\]

We have $\langle \phi_1 | \psi \rangle = \frac{1}{2}$ and $\langle \phi_2 | \psi \rangle = \frac{1}{2}$, and a projection of $\Pi_{ij} |\psi\rangle$ on $\langle \phi_1 \rangle$ or $\langle \phi_2 \rangle$ will depend on $i$ and $j$, leading to equations (13) to (15) and (30) to (33).

We now compute the weak value of $\sigma_{zz}^{ij}$:

\[
\sigma_{zz}^{ij} = \Pi_{ij} \sigma_{zz} = (|ijHV\rangle \langle VHji| + |ijVH\rangle \langle HVji|)
\]

\[
(|++\rangle \langle ++| + |+-\rangle \langle + -| + |-+\rangle \langle -+| + |--\rangle \langle - -|)
\]

\[
= |ij\rangle \langle ij| \langle HV| \langle VH| |++\rangle \langle ++| + |+-\rangle \langle + -| + |-+\rangle \langle -+| + |--\rangle \langle - -|.
\]

Compute the scalar products:

\[
\langle HV|--\rangle = \langle HV|--\rangle = \langle VH|+-\rangle = \langle VH|--\rangle = -\frac{1}{2},
\]

the other scalar products being equal to $\frac{1}{2}$.

Inserting these values we get:

\[
\sigma_{zz}^{ij} = \frac{[ij]}{2}(\langle HV| \langle + + | + |VH| \langle + + | + |HV| \langle - + | - |VH| \langle - - |\rangle.
\]

Projecting on $|\psi\rangle$ we get $\sigma_{zz}^{ij} |\psi\rangle = \frac{[ij]}{2} |A\rangle$, and a further projection on $\langle \phi_1 \rangle$ or $\langle \phi_2 \rangle$ will depend on $i$ and $j$. Hence the results of equations (18) to (20) and (34) to (37).

Similarly, we calculate the weak value of $\sigma_{zz}^{ij}$:

\[
\sigma_{zz}^{ij} = \Pi_{ij} \sigma_{zz} = (|ijHV\rangle \langle VHji| + |ijVH\rangle \langle HVji|)
\]

\[
(|++\rangle \langle ++| + |+-\rangle \langle + -| + |-+\rangle \langle -+| + |--\rangle \langle - -|)
\]

\[
= |ij\rangle \langle ij| \langle HV| \langle VH| |++\rangle \langle ++| + |+-\rangle \langle + -| + |-+\rangle \langle -+| + |--\rangle \langle - -|.
\]

We insert the calculated values for the scalar products:

\[
\sigma_{zz}^{ij} = \frac{[ij]}{2}(\langle HV| \langle + + | + |VH| \langle + + | + |HV| \langle - + | - |VH| \langle - - |\rangle.
\]

Projecting on $|\psi\rangle$ we get $\sigma_{zz}^{ij} |\psi\rangle = -\frac{[ij]}{2} |B\rangle$, and a further projection on $\langle \phi_1 \rangle$ or $\langle \phi_2 \rangle$ will depend on $i$ and $j$. Hence the results of equations (24) to (26) and (38) to (43).