Grassmannians of lines defined in the geometry of a pseudo-polarity

K. Prażmowski and M. Żynel

May 5, 2014

Abstract

The regular point-line geometry with respect to a pseudo-polarity is introduced. It is weaker than the underlying metric-projective geometry. The automorphism group of this geometry is determined. This geometry can be also expressed as the geometry of regular lines and planes.

Mathematics Subject Classification (2010): 51A50, 51F20, 51A45.
Keywords: pseudo-polarity, regular subspaces, Grassmannian.

Introduction

Projective geometry and affine geometry over fields with even characteristic, though have some “strange” properties do not differ so strongly from geometries over arbitrary field. In particular, most of standard methods used to characterize geometry of the Grassmann spaces associated with them can be applied here.

The situation changes when we pass to the orthogonal geometry. Let $\mathcal{P}$ be a projective space coordinatized by a field $\mathcal{F}$ and let $\perp$ be a projective conjugacy (projective polarity) defined on $\mathcal{P}$; then $\mathcal{P}$ equipped with $\perp$ is referred to as a metric-projective space.

Some standard derivatives are associated with the space $(\mathcal{P}, \perp)$. The first is a polar space, which consists of selfconjugate (absolute) points and isotropic (singular or absolute) lines (cf. [11]). One of fundamental results which states that the underlying metric-projective space can be reconstructed in terms of the associated polar space, is valid in all cases except exactly one: when $\text{char}(\mathcal{F}) = 2$ and $\perp$ is a pseudo-polarity. In [13, Subsection 2.4.18] a polarity $\perp$ is said to be a pseudo-polarity if the set $\mathcal{H}$ of selfconjugate points with respect to $\perp$ is a proper (possibly empty) subspace of $\mathcal{F}$. In this paper we additionally assume that $\mathcal{H}$ is a hyperplane (this means that $\mathcal{F}$ is perfect, see also [5, Subsection 2.1.5]). Pseudo-polarities are related to pseudo-quadrics and pseudo-quadratic forms (cf. [2, Section 10.2], [12, Chapter 6]). So, in the aforementioned point the geometry of a pseudo-polarity is exceptional, though the geometry of its polar space is, generally, known.

In a metric projective geometry an important role is played by the family of regular subspaces i.e. subspaces with the trivial radical.

If the conjugacy $\perp$ is not symplectic then the structure of regular points and regular lines is equivalent to the underlying metric geometry (cf. [10]). If $\perp$ is
symplectic then one cannot have a space with regular points and lines. If $\perp$ is a pseudo-polarity then, admittedly char($\mathfrak{F}$) = 2, but $(\mathfrak{P}, \perp)$ does contain regular points and lines. They yield a new geometry a subject of our paper. This geometry is stronger than the affine geometry $\mathfrak{A}$ obtained by deleting nonregular points i.e. a projective hyperplane $H$ from $\mathfrak{P}$, but weaker than the underlying metric-projective geometry in that one can reconstruct the space $\mathfrak{P}$ in terms of regular points and lines, (cf. 3.13), but it is impossible to reconstruct the polarity $\perp$ in these terms (cf. 3.15). In this point the geometry of a pseudo-polarity is essentially exceptional.

In our paper we also discuss more properties of the regular geometry of a pseudo-polarity, in particular, we determine the automorphism group of this geometry (cf. 3.22, 3.24).

The regular geometry of a pseudo-polarity, considered on the varieties of the regular subspaces, presents some more oddities: the incidence system of the regular subspaces contains isolated objects; there are also flags of regular objects whose end-object may be contained in (or contains) no regular successor (or predecessor respectively). In geometry of incidence structures isolated objects generally are trash: nothing can be defined in terms of them and they are undefinable in general, as each automorphism of the structure can arbitrarily permute them. So, it is a good reason to remove them from considerations. Since there are many combinations of isolated objects we get various structures of Grassmannians over regular subspaces. Without entering into details of the general theory of $k$-Grassmannians we show how our apparatus works in case of structures over regular lines. It turns out (cf. 3.20, 4.1, 5.14) that all they are definitionally equivalent to the regular point-line geometry. This, in particular, let us determine the automorphisms. We are convinced that analogous results can be achieved for arbitrary $k$.

1 Basic notions

Let $\perp$ be a projective polarity in a finite dimensional metric projective space $(\mathfrak{P}, \perp)$ coordinatized by a vector space $V$ over a perfect field $\mathfrak{F}$ with even characteristic. In a suitable coordinate system the form $\xi$ which determines $\perp$ has its matrix of one of the following forms (cf. [5, Subsection 2.1.5]). Let us set

$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \nabla = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad \nabla' = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

It is seen that $\det(\nabla), \det(\nabla') \neq 0$. Then the matrix of $\xi$ is one of the following

$$
\begin{bmatrix}
1 & 0 & \cdots & \cdots & 0 \\
0 & \nabla & O & \cdots & O \\
\vdots & O & \nabla' & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & O \\
0 & O & \cdots & O & \nabla \\
\end{bmatrix} \quad \text{Type (1)} \quad \begin{bmatrix}
\nabla' & O & \cdots & O \\
O & \nabla & \ddots & \vdots \\
\vdots & \ddots & \ddots & O \\
O & \cdots & O & \nabla \\
\end{bmatrix} \quad \text{Type (2)} \quad \begin{bmatrix}
\nabla & O & \cdots & O \\
O & \nabla & \ddots & \vdots \\
\vdots & \ddots & \ddots & O \\
O & \cdots & O & \nabla \\
\end{bmatrix} \quad \text{Type (3)}
$$
For a subspace $U$ we write $U^\perp = \{p: p \perp U\}$, $\text{Rad}(U) = U \cap U^\perp$, and $\text{rdim}(U) = \dim(\text{Rad}(U))$. A subspace $U$ is called regular when $\text{Rad}(U) = \emptyset$. The dimension $\dim(U)$ of a subspace $U$ coincides with the dimension of the vector subspace $W$ of $V$ such that $U = \{\langle u \rangle: u \in W, u \neq \theta\}$. In particular, a point has dimension 1 and a line has dimension 2. Frequently, a point $p$ will be identified with the one-element subspace $\{p\}$ which consists of $p$; consequently, we frequently write $U \cap U' = \{p\}$ instead of $U \cap U' = \{\langle p \rangle\}$. We write $R_k$ for the class of regular $k$-dimensional subspaces of $(\mathcal{P}, \perp)$. In the sequel we are interested in the incidence structure (cf. [3])

$$\mathcal{R} := (R_k: 0 \leq k \leq \dim(\mathcal{P}) + 1),$$

which is more closely investigated for indices 1, 2 and 2, 3. Various geometries arise when we delete isolated objects in such structures.

### 1.1 Grassmannians

Grassmann spaces frequently appear in the literature, just to mention [3], [8]. The most general definition could be probably as follows. Let $X$ be a nonempty set and let $\mathcal{P}$ be a family of subsets of $X$. Assume that there is a dimension function $\dim: \mathcal{P} \to \{0, \ldots, n\}$ such that $I = \langle \mathcal{P}, \subset, \dim \rangle$ is an incidence geometry. Write $\mathcal{P}_k$ for the set of all $U \in \mathcal{P}$ with $\dim(U) = k$. For $H \in \mathcal{P}_{k-1}$ and $B \in \mathcal{P}_{k+1}$ with $H \subset B$ the set

$$\mathcal{p}(H, B) := \{U \in \mathcal{P}_k: H \subset U \subset B\}$$

is called a pencil; $\mathcal{G}_k$ stands for the family of all such pencils. If $0 < k < n$, then a $k$-th Grassmann space over $I$ is a point-line geometry

$$\mathcal{P}_k(I) := \langle \mathcal{P}_k, \mathcal{G}_k \rangle.$$

This is the most common understanding of a Grassmann space. We used to call it a space of pencils for its specific lineset and to distinguish it from a closely related point-line geometry consisting of $\mathcal{P}_k$ as points and $\mathcal{P}_{k+1}$ as lines, namely

$$\mathcal{G}_k(I) := \langle \mathcal{P}_k, \mathcal{P}_{k+1}, \subset \rangle$$

which we call a $k$-th Grassmannian over $I$ (cf. [7, Section 1.1.2]).

In our settings, we write $\mathcal{H}_k$ for the set of $k$-dimensional subspaces of $\mathcal{P}$. Then a projective pencil is a set (4) with $\mathcal{P} = \mathcal{H}$ and a regular pencil is a set (4) with $\mathcal{P} = \mathcal{R}$.

### 1.2 Methodological issues

Most of the results of our paper consists in various definabilities/undefinabilities of particular structures, derived from the incidence geometry $\mathcal{R}$, in terms of other structures of the same type. Clearly, in a structure like this various notions can be introduced and some notions can originate in the underlying projective and metric projective geometry. But proving our results we must be strict: in each particular case we must be sure that the respective definition can be expressed entirely in terms of the language with names only for primitive notions of the structure in which the second one is defined. The safest way to ensure that is to write down respective definitions as “formal formulas” in a formal language, and we do follow this convention.
Our structures are, primarily, point-line geometries and the only primitive notion used to characterize their geometry is the incidence relation. We use the symbol $\in$ to name the incidence relation between points and lines. Note that $l = \in$ in $P_k(I)$ while $l = \subset$ in $G_k(I)$.

By a triangle in a point-line geometry we mean three points, called vertices, and three lines, called sides, where every vertex is incident to exactly two sides (or dually, every side is incident to exactly two vertices).

1.3 $\perp$ is symplectic

The form is of type (3). Consequently, $\dim(V) = 2m$ for some integer $m$. In this case each point is selfconjugate; the point set of the respective polar space is the point set of $P$.

1.4 $\perp$ is a pseudo-polarity

The form is of type (1) or (2). Then the set of selfconjugate points under $\perp$ is a subspace $H$ of $\mathcal{P}$. We assume that $H$ is a hyperplane; then the pole $b$ of $H$ is a point. The set $R_1$ of regular points is the hyperplane complement: the complement of $H$. One can imagine the geometry of regular points and regular lines as a fragment of an affine geometry.

The restriction $\perp_H$ of the conjugacy $\perp$ to $H$ determines on $H$ a (possibly degenerate) symplectic polarity. If $\operatorname{Rad}(H) = \emptyset$, this polarity is nondegenerate. In general, $\dim(\operatorname{Rad}(H)) \leq 1$.

Again, two types of geometry may occur.

1.4.1 $b$ lies on $H$

In this case $\operatorname{Rad}(H) = b$, so the polarity $\perp$ restricted to $H$ is a degenerate symplectic polarity. Then $n = \dim(V) = 2k$ for some integer $k$ and the form $\xi$ has form (2); it may be written as

$$\xi([x_0, x_1, \ldots, x_{2k-1}], [y_0, y_1, \ldots, y_{2k-1}]) = x_0 y_0 + \sum_{i=0}^{k-1} (x_{2i} y_{2i+1} + x_{2i+1} y_{2i})$$

(cf. [5]). The hyperplane $H$ and its pole $b$ are characterized by the equations

$$H: x_0 = 0, \quad b = [0, 1, 0, \ldots, 0].$$

1.4.2 $b$ does not lie on $H$

Then $n = \dim(V) = 2k + 1$ and the form $\xi$ has form (1); it may be written as

$$\xi([x_0, x_1, \ldots, x_{2k}], [y_0, y_1, \ldots, y_{2k}]) = x_0 y_0 + \sum_{i=1}^{k} (x_{2i-1} y_{2i} + x_{2i} y_{2i-1})$$

(cf. [5]). The hyperplane $H$ and its pole $b$ are characterized by the equations

$$H: x_0 = 0, \quad b = [1, 0, \ldots, 0].$$

Note that for $x \in H$ we have $x^\perp = b + (x^\perp \cap H)$. Therefore, to determine $x^\perp$ it suffices to know the restriction $\perp_H$ of the conjugacy $\perp$ to $H$. On the other hand, this restriction determines on $H$ a symplectic polarity and this polarity is nondegenerate as $\operatorname{Rad}(H) = \emptyset$. 


2 Results, general

Let $\perp$, $\mathbf{H}$, and $\mathbf{b}$ be like in Section 1.4. Two families of specific regular subspaces arise:

$$A_k = \{ A \in \mathcal{R}_k : A \not\subset \mathbf{H} \}, \quad A_k^\circ = \{ A \in \mathcal{R}_k : \mathbf{b} \not\subset A \}.$$

If $A$ is a subspace of $\mathcal{P}$ we write $A^\infty := A \cap \mathbf{H}$ for the set of selfconjugate points of $A$. Note that $\text{Rad}(A)$ is always a (possibly empty) set of selfconjugate points, i.e. $\text{Rad}(A) \subset A^\infty \subset \mathbf{H}$.

We begin with a simple but very significant fact.

**Fact 2.1** ([9, Cor. 1.3]). If a subspace $A$ contains a regular hyperplane or $A$ is a hyperplane of a regular subspace, then $\text{rdim}(A) \leq 1$.

**Fact 2.2.** If $A$ is a subspace not contained in $\mathbf{H}$, then $\text{dim}(A \cap (A^\infty)^\perp) \geq 1$.

**Proof.** Set $k := \text{dim}(A)$, $n := \text{dim}(A \cap (A^\infty)^\perp)$. Then $\text{dim}(A^\infty) = k - 1$ and $n \geq \text{dim}(A + (A^\infty)^\perp) = k + (n - (k - 1)) - m = n + 1 - m$, which yields our claim. \(\square\)

The subspace $A \cap (A^\infty)^\perp$ will be denoted by $\text{Hrd}(A)$ and will play an essential role (slightly similar to the role of the radical $\text{Rad}(A)$ of $A$). Fact 2.2 says that $\text{Hrd}(A)$ is at least a point for every subspace $A$ not contained in $\mathbf{H}$.

**Proposition 2.3.** Let $A$ be a subspace of $\mathcal{P}$ not contained in $\mathbf{H}$. The following conditions are equivalent:

(i) the subspace $A$ is regular;

(ii) the subspace $\text{Hrd}(A)$ is a point (cf. 2.2);

(iii) $\text{rdim}(A^\infty) \leq 1$ and

(a) either the subspace $A^\infty$ is regular in the (possibly degenerated) symplectic projective geometry induced on $\mathbf{H}$;

(b) or $\text{Rad}(A^\infty)$ is a point $p$ and $A \not\subset p^\perp$.

**Proof.** From 2.2, $\text{Hrd}(A)$ is at least a point.

(i) $\implies$ (iii): Since $A^\infty$ is a hyperplane in $A$, by 2.1, $\text{rdim}(A^\infty) \leq 1$. If $\text{rdim}(A^\infty) = 0$, then $A^\infty$ is regular. If $\text{rdim}(A^\infty) = 1$, then $\text{Rad}(A^\infty)$ is a point, say $p$. The point $p$ could be the only point in $\text{Rad}(A)$. As $A$ is regular, we have $p \not\in \text{Rad}(A)$ i.e. $p \not\in A$ or, in other words, $A \not\subset p^\perp$.

(iii) $\implies$ (ii): Let $A^\infty$ be regular. Suppose that $\text{Hrd}(A)$ contains a line $K$. Then $K \subset A$. As $A^\infty$ is a hyperplane in $A$ so, $K$ and $A^\infty$ share a point $q$. Note that $q \in A^\infty \cap (A^\infty)^\perp = \text{Rad}(A^\infty)$ which is impossible. Consequently, $\text{Hrd}(A)$ is a point.

Now, let $\text{Rad}(A^\infty)$ be a point $p$ and $A \not\subset p^\perp$. Then $p \in \text{Hrd}(A)$. Suppose that there is $q \in \text{Hrd}(A)$, $q \neq p$. If $q \in A^\infty$, then $q \in \text{Rad}(A^\infty)$, which is impossible. So $q \in A \setminus A^\infty$. On the other hand $p \in A^\infty \cap (A^\infty)^\perp$ and $q \in (A^\infty)^\perp$, hence $p \perp A^\infty + q = A$ which contradicts our assumption that $A \not\subset p^\perp$. So, $\text{Hrd}(A)$ is a point.

(ii) $\implies$ (i): Let $\text{Hrd}(A)$ be a point $p$ and suppose that $q \in \text{Rad}(A)$. Then $q \in A$ and $q \in A^\perp \subset (A^\infty)^\perp$, so $q = p$. Hence $p \perp A$ and $p \in A^\infty$, which gives $p \perp A + (A^\infty)^\perp$. Note that $\text{dim}(A + (A^\infty)^\perp) = n$, so $p \perp V$ a contradiction. \(\square\)
**Note.** If $p = \text{Rad}(A^\infty)$, then $p = \text{Hrd}(A)$.

**Note.** Let $A$ be a regular subspace not contained in $H$ and $k = \dim(A)$. Then the restriction $\perp_A$ of $\perp$ to $A$ is a pseudo-polarity and $\text{Hrd}(A)$ is the pole of $A^\infty$ within $A$. Thus

if $2 | k$, then $\text{Hrd}(A) \in A^\infty$, and if $2 \nmid k$, then $\text{Hrd}(A) \notin A^\infty$.

Recall that the induced geometry on $H$ is symplectic. Therefore, $A^\infty$ may be regular iff $2 | \dim(A^\infty)$ i.e. iff $2 \nmid \dim(A)$. This gives

**Corollary 2.4.** Let $A$ be a subspace not contained in $H$ with $k = \dim(A)$. If $2 | k$ then $A$ is regular iff $\text{Rad}(A^\infty)$ is a point $p$ and $A \setminus H$ misses $p^\perp$. If $2 \nmid k$ then $A$ is regular iff $A^\infty$ is regular.

As a particular instance of 2.3 we obtain a series of criterions of regularity.

**Lemma 2.5.** A line $L$ not contained in $H$ is regular iff it is not contained in the hyperplane $(L^\infty)^\perp$.

**Corollary 2.6.** Let $p$ be a regular point. A line $L$ through $p$ is regular iff it misses $p^\perp \cap H$.

**Proof.** Let $q := L \cap H$. By 2.5 $L$ is regular iff $p \notin q^\perp$ i.e. iff $q \notin p^\perp$, as required. \(\square\)

Note that every nonregular line on $H$ is totally isotropic.

**Lemma 2.7.** If $b \in H$, then each affine line with direction $b$ is regular and no line on $H$ through $b$ is regular. If $b \notin H$, then no affine line through $b$ is regular.

**Proof.** Let $b \in H$. In view of 2.5 it suffices to note that $b^\perp = H$ and affine line are those not contained in $H$.

Let $b \notin H$. Then for each $p \in H$ we have $p \in \text{Rad}(\overline{p, b})$ so, the line $\overline{p, b}$ is not regular. \(\square\)

A direct consequence of 2.3 and 2.4 is

**Lemma 2.8.** Let $A$ be a plane not contained in $H$. Clearly, $A^\infty$ is a line. The following conditions are equivalent:

(i) the plane $A$ is regular;

(ii) the subspace $\text{Hrd}(A)$ is a point (cf. 2.2);

(iii) the line $A^\infty$ is regular in the (possibly degenerated) symplectic projective geometry induced on $H$.

**Lemma 2.9.** Let $A$ be a plane contained in $H$. Then either

(i) $\text{Rad}(A)$ is a point and all non-regular lines on $A$ form a pencil through that point, or

(ii) $\text{Rad}(A)$ is a line and no line on $A$ is regular.

(iii) $\text{Rad}(A)$ is the entire plane $A$, i.e. $A$ is totally isotropic.
Proof. It is clear that \( 1 \leq \text{rdim}(A) \) as \( \text{dim}(A) = 3 \) and \( A \subseteq H \).

In case \( p := \text{Rad}(A) \) is a point then \( A \subseteq p^\perp \), so \( p \) is the radical of every line on \( A \) through \( p \). Note that if \( q \) is the radical of some line on \( A \) not through \( p \), then \( A \subseteq q^\perp \) and we would have \( p, q = \text{Rad}(A) \) which is impossible.

Now, if \( L := \text{Rad}(A) \) is a line, then every line of \( A \) crosses \( L \) and thus is non-regular.

\[ \square \]

Remark. Let \( A \) be a plane not contained in \( H \). If \( \text{Hrd}(A) \) is a point not on \( H \) then \( A \) is regular, but no line through \( p \) contained in \( A \) is regular.

Further we assume that:

\[ \text{lines of } \mathfrak{P} \text{ are of size at least } 6, \]

which means that the ground field of \( \mathfrak{P} \) is not GF(2) and not GF(4). Most of our reasonings remain true for GF(4) and those few which fail will be indicated.

3 Grassmannians of regular points and lines

3.1 Regular point-line geometry

In what follows we shall pay attention to the Grassmannian of regular points and lines of the pseudo-polarity \( \perp \), namely

\[ \mathfrak{G}_1 := \mathfrak{G}_1(R) = \langle R_1, R_2, \subset \rangle. \]

Observe, first, that the set \( R_1 \) is simply the point-complement of the hyperplane \( H \). Let \( \mathfrak{A} = \langle R_1, L \rangle \) be the affine space obtained from \( \mathfrak{P} \) by deleting the hyperplane \( H \); then \( A_2 \subseteq L \).

\[ \text{Fact 3.1. If } b \in R_1, \text{ then } b \text{ is an isolated point in } \mathfrak{G}_1. \text{ If } L \in R_2 \text{ and } L \subseteq H, \text{ then } L \text{ is an isolated line in } \mathfrak{G}_1. \]

Proof. The first statement restates 2.7, while the other is trivial. \( \square \)

Let \( \mathfrak{B}_1 \) be the structure obtained from \( \mathfrak{G}_1 \) by deleting its isolated points and lines. Then

\[ \mathfrak{B}_1 = \langle A_1^\circ, A_2, \in \rangle \]

Note that \( \mathfrak{B}_1 \) is a substructure of \( \mathfrak{A} \). This structure, primarily, and related structures will be investigated in this section.

A plane \( A \) of \( \mathfrak{P} \) not contained in \( H \) is a plane of \( \mathfrak{A} \) and will be referred to as an affine plane; \( A^\infty \) is the set of its improper points. Similarly, a line \( L \) of \( \mathfrak{P} \) not contained in \( H \) is a line of \( \mathfrak{A} \) and will be referred to as an affine line; \( L^\infty \) is its improper point.

So, let \( A \) be an affine plane and set \( L := A^\infty \).

\[ \text{Fact 3.2. Through each } q \in L \text{ there passes a nonregular affine line } M \text{ contained in } A \text{ (in every direction in } A \text{ there is a nonregular line contained in } A). \]

Proof. It suffices to consider \( A \cap q^\perp \), which is at least a line. \( \square \)

\[ \text{Fact 3.3. If } A \text{ contains two parallel nonregular lines then } A \text{ is nonregular as well.} \]
Proof. Let \( M_1 \parallel M_2 \) be nonregular, \( M_1, M_2 \subset A \). Take \( q = M_1^\infty \). Then \( q \perp M_1, M_2 \), so \( q \perp M_1 + M_2 = A \) i.e. \( q \in \text{Rad}(A) \). □

**Fact 3.4.** If \( A \) contains a triangle with all its sides nonregular then \( \text{Rad}(A) = A^\infty \) and no affine line contained in \( A \) is regular.

Proof. Let \( a_1, a_2, a_3 \) be the vertices of a required triangle and \( q_i = a_j, a_k^\infty \) for \( \{i, j, k\} = \{1, 2, 3\} \). Then \( a_i \perp q_j, q_k \), so \( a_i \perp A^\infty \) for each \( i \). Thus \( A \perp A^\infty \). □

**Fact 3.5.** The following possibilities may occur.

\( L \) is nonregular: Then \( L \) is isotropic i.e. \( L \perp L, A \) is nonregular, and we have two cases.

1. \( \text{Rad}(A) \) is the line \( L \): Then \( L \perp A \) and \( A \) contains no regular line. In this case \( \text{Hrd}(A) = A \).
2. \( \text{Rad}(A) \) is a point \( q \): Then \( q \in L \). An affine line \( K \) on \( A \) is nonregular iff \( K^\infty = q \). In this case \( \text{Hrd}(A) = L \).

\( L \) is regular: Then \( A \) is regular and \( \text{Hrd}(A) \) is an affine point \( p \) on \( A \). An affine line \( K \) on \( A \) is nonregular iff \( p \in K \).

Proof. The claim is nearly evident. It only remains to prove the above characterization of nonregular lines on \( A \). Let \( K \) be an affine line on \( A \) and \( x = K^\infty \).

Let \( A \perp L \); then \( L \ni x \perp A \supset K \) and thus \( K \) is not regular.

Let \( \text{Rad}(A) = q \) and \( L \perp L \); clearly, \( q \in L \). Let \( q \in K \); as above, \( q \perp A \supset K \) and thus \( q \perp K \). Let \( q \notin K \) and suppose that \( K \) is not regular. Then \( x \neq q \) and \( x \perp L + K = A \), which gives \( A \perp q + x = L \). The obtained contradiction yields that \( K \) is regular.

Let \( A \) be regular. From definition, \( p \perp L \) and thus \( p \perp x \); with \( x \perp x \) from \( p \in K \) we obtain \( x \perp K \), so \( K \) is not regular. Assume that \( K \) misses \( p \) and suppose that \( K \) is not regular. Then \( K \) and \( \overline{x, p} \) are two parallel nonregular lines on \( A \) and thus \( A \) is not regular. This contradiction yields that \( K \) must be regular. □

As a consequence we get

**Fact 3.6.** Let \( A \) be an affine plane with \( r\text{dim}(A) \leq 1 \). In view of 3.5 the nonregular lines on \( A \) form a pencil. Its vertex will be denoted by \( q(A) \). This pencil is

proper and \( q(A) \) is an affine point when \( r\text{dim}(A) = 0 \), or

parallel and \( q(A) \) is a point on \( A^\infty \) when \( r\text{dim}(A) = 1 \).

**Lemma 3.7.** Let \( M_1, M_2 \) be two parallel regular lines in \( L \) and let \( A \) be the affine plane that contains them. Then either \( A \) is regular or \( \text{Rad}(A) \) is a point. In both cases \( A \) contains a pair \( K_1, K_2 \) of regular lines which intersect in an affine point such that \( K_i \) crosses \( M_j \) in an affine point for all \( i, j \).

Proof. Let \( a_1 \) be any point on \( M_1 \) and \( K_0 \) be the unique nonregular line through \( a_1 \) (in above notation, either \( K_0 = \overline{a_1, q} \) or \( K_0 = \overline{a_1, p} \), resp.). Let \( y = M_2 \cap K_0 \) and \( a_2 \) be an affine point on \( M_2 \) distinct from \( y \); take \( K_1 = \overline{a_1, a_2} \). Let \( b \in K_1 \) be an affine point distinct from \( a_1, a_2 \) and \( K_2 = \overline{b, y} \). □
Corollary 3.8. The formula
\[
M_1 \parallel M_2 \iff (\exists K_1, K_2)(\exists p, a_1, a_2, b_1, b_2)[K_1 \neq K_2 \land p \parallel K_1, K_2 \\
a_1 \parallel K_1, M_1 \land a_2 \parallel K_1, M_2 \land b_1 \parallel K_2, M_1 \land b_2 \parallel K_2, M_2 \land p \parallel M_1, M_2] \\
\land \neg \exists a[a \parallel M_1, M_2]
\] (5)
defines the parallelism of regular lines in terms of the geometry of \(\mathcal{B}_1\).

Lemma 3.9. Let \(A\) be an affine plane with \(\text{rdim}(A) \leq 1\). Then \(A\) contains a triangle \(\Delta\) with regular sides and, moreover, through each affine point on \(A\) distinct from \(\mathfrak{q}(A)\) there passes a regular line which crosses the sides of \(\Delta\) in at least two affine points.

Proof. In view of 3.6 the nonregular lines on \(A\) form a pencil with the vertex \(\mathfrak{q}(A)\): a proper one if \(\mathfrak{q}(A)\) is an affine point \(p\) or a parallel one when \(\mathfrak{q}(A)\) is a point \(q\) on \(A^\infty\). Thus the existence of a required triangle \(\Delta\) is evident. Let \(a_1, a_2, a_3\) be the vertices of \(\Delta\) and \(x\) be an arbitrary affine point on \(A\). The only lines through \(x\) that may not cross appropriately the sides of \(\Delta\) are the following \(x, a_1 \parallel a_2, a_3, x, a_2 \parallel a_1, a_3, x, a_3 \parallel a_1, a_2\), and \(x, \mathfrak{q}(A)\). (Note that from the Fano axiom valid in \(\mathfrak{A}\), the lines through \(a_i\) parallel to \(a_j, a_l, \{i, j, l\} = \{1, 2, 3\}\) have indeed a common point \(x\).) From assumptions, there are at least 5 lines through \(x\) contained in \(A\) and thus there exists a line required as well. \(\square\)

In \(\mathcal{B}_1\) for a triangle \(\Delta\) with the sides \(L_1, L_2, L_3\) we define
\[
\pi(\Delta) := \{x: (\exists K)(\exists a, b)[x, a, b \parallel K \land a \neq b \land \\
((a \parallel L_1 \land b \parallel L_2) \lor (a \parallel L_2 \land b \parallel L_3) \lor (a \parallel L_1 \land b \parallel L_3))\}\}. \quad (6)
\]

Let \(\mathcal{P}\) be the set of planes of \(\mathfrak{A}\), and let \(\mathcal{P}_i\) be the set of planes in \(\mathcal{P}\) with \(\text{rdim} = i\). Then \(\mathcal{P}_0 := \mathcal{P}_0 \cup \mathcal{P}_1\) is the set of planes with \(\text{rdim} \leq 1\). For \(A \in \mathcal{P}_0\) write \([A] := A \setminus \{\mathfrak{q}(A)\}\).

Corollary 3.10. If \(A \in \mathcal{P}_1\), then \([A] = A\). If \(A \in \mathcal{P}_0\), then \([A]\) is an affine plane with one point deleted. Moreover, we have
\[
\{[A]: A \in \mathcal{P}_0\} = \{\pi(\Delta): \Delta \text{ is a triangle in } \mathcal{B}_1\}. \quad (7)
\]

Lemma 3.11. Let \(L\) be a nonregular affine line through an affine point \(p\) in \(\mathfrak{A}\). Then \(q := L^\infty \neq b\). Let \(A \in \mathcal{P}_0\) contain \(L\) and \(M := A^\infty\), so \(A = L + M\) with \(q \in M \subseteq \mathcal{H}\). Then one of the following holds.

(i) \(M \cap q^\perp = q (M \not\subset q^\perp)\). In this case \(M\) is regular, so \(A\) is regular as well. Clearly, \(\mathfrak{q}(A) \in L\), so \(M \subset q(A)^\perp\). To have \(p \neq \mathfrak{q}(A)\) we need \(M \not\subset p^\perp\).

(ii) \(M \subset q^\perp\). In this case either \(M \not\perp L\) or \(b\) is an affine point on \(L\).

Proof. Case (i) is evident. If \(q \in M \subset q^\perp\), then \(M \perp M\). To have \(M \neq \text{Rad}(A)\) we must have \(x \not\perp M\) for each \(x \in A \setminus M\); this is equivalent to \(L \not\perp M\). Assume that \(L \perp M\). Then \(M = \text{Rad}(A)\) and \(L \perp q^\perp \cap \mathcal{H}\). Comparing dimensions we get \(L^\perp = q^\perp \cap \mathcal{H}\) which gives \(L = q + b\). \(\square\)
**Lemma 3.12.** Let \( L \) be a nonregular line and \( a_1, a_2, a_3 \) be affine points on \( L \) in \( \mathfrak{A} \). If \( b \neq a_1, a_2, a_3 \), then there are distinct \( A_1, A_2 \in P_{01} \) with \( a_1, a_2, a_3 \in [A_1] \cap [A_2] \).

**Proof.** In view of 3.11 it suffices to find two lines \( M_1, M_2 \subset H \) through \( q = L^\perp \) such that \( M_1, M_2 \not\subset q^\perp, a_1^\perp, a_2^\perp, a_3^\perp \). The required lines exist as we have at least 6 lines in a projective pencil. \( \square \)

**Theorem 3.13.** The affine space \( \mathfrak{A} \) is definable in terms of \( \mathfrak{B}_1 \) and, consequently, \( \mathfrak{P} \) is definable in terms of \( \mathfrak{B}_1 \) as well.

**Proof.** Let \( L_0 \) be the ternary collinearity relation on the set of affine points distinct from \( b \) on nonregular lines. By 3.12 this relation is definable in terms of the geometry of \( \mathfrak{B}_1 \). On the other hand \( L_0 \) is a ternary equivalence relation in the sense of [1, §4.5]. So, let \( L_0 \) be the family of equivalence classes of \( L_0 \). If \( b \) is an improper point we are through, as \( L_0 \) consists of the affine nonregular lines. So, assume that \( b \) is an affine point. Then \( L_0 \) consists of the nonregular lines not through \( b \) and the sets \( L \setminus \{b\} \) where \( L \) is nonregular through \( b \). For every triangle \( \Delta \) of \( \mathfrak{B}_1 \) we set \( \pi(\Delta) = \bigcup \{L \in A_2 \cup L_0: |\pi(\Delta) \cap L| \geq 2\} \).

Let \( L \in L_0 \) and \( L' \) be the nonregular line of \( \mathfrak{A} \) that contains \( L \). Either \( b \in L' \) and then every plane containing \( L \) also contains \( b \), or \( b \not\in L' \) and then there is exactly one plane \( L + b \) containing \( L \) and \( b \). Therefore, if no plane through a line \( L \) is an affine plane, then \( L \in L_0 \) and we set \( L' := L \cup \{b\} \); otherwise we set \( L' := L \). After that \( L_0' := \{L': L \in L_0\} \) is the set of all nonregular lines of \( \mathfrak{A} \) and \( L = A_2 \cup L_0' \). \( \square \)

For a set \( X \) of affine points we write \( \overline{X} \) for the least subspace of \( \mathfrak{P} \) which contains \( X \).

**Lemma 3.14.** Let \( q \in H \). Then the set

\[
[q] := \{a: a \text{ is a point of } \mathfrak{A}, \overline{a}, q \notin A_2\}
\]

is the set of affine points on \( q^\perp \), and thus \( \overline{q} = q^\perp \).

Similarly, if \( a \) is a point of \( \mathfrak{A} \) then the set \( [a] = \{K^\perp: a \triangleright K, K \in L \setminus A_2\} \) coincides with the set \( a^\perp \cap H \); but not with the set \( a^\perp \), unhappily.

**Theorem 3.15.** The metric projective space \( (\mathfrak{P}, \perp) \) is not definable in terms of the geometry of \( \mathfrak{B}_1 \).

**Proof.** Let \( W \) be the subspace of \( V \) with \( H = \{\langle h \rangle: h \in W, h \neq \theta\} \), and \( \xi \) be the form defined on \( V \) which determines the conjugacy \( \perp \). Write \( b = \langle e_0 \rangle \) for a vector \( e_0 \). There are two cases to consider.

- **\( b \not\in H \):** Let \( \xi_0 \) be the restriction of \( \xi \) to \( W \). Then \( \xi_0 \) is a nondegenerate symplectic form. Write \( \varepsilon = \xi(e_0, e_0) \). Let \( e_1, \ldots, e_n \) be a basis of \( H \); then the family \( \mathcal{E} = \{e_i: i = 0, \ldots, n\} \) is a basis of \( V \). We have \( e_0 \perp h \) for every \( h \in H \) and thus the formula defining the form \( \xi \) is the following

\[
\xi(h_1 + \alpha_1 e_0, h_2 + \alpha_2 e_0) = \xi_0(h_1, h_2) + \alpha_1 \alpha_2 \varepsilon,
\]

where \( h_1, h_2 \in W \) and \( \alpha_1, \alpha_2 \) are scalars of the coordinate field. Note that, conversely, for every nondegenerate symplectic form \( \xi_0 \) defined on \( W \) and every scalar
\[ \varepsilon \neq 0 \] the formula (9) defines a nondegenerate bilinear form \( \xi = \xi_\varepsilon \). Indeed, if \( M \) is the matrix of \( \xi_0 \) in the given basis then
\[
M_\varepsilon = \begin{bmatrix}
\varepsilon & 0 & \cdots & 0 \\
0 & & & \\
\vdots & & M & \\
0 & & & 
\end{bmatrix}
\]
is the matrix of \( \xi_\varepsilon \) and \( \det(M_\varepsilon) \neq 0 \). In particular, for \( h, h_1 \in W \) and a scalar \( \alpha \) we have
\[
\xi_\varepsilon(h_1, h + \alpha e_0) = \xi_0(h_1, h) \quad \text{and} \quad \xi_\varepsilon(h + \alpha e, h + \alpha e_0) = \alpha^2 \varepsilon.
\]
Let us write \( \perp_\varepsilon \) for the conjugacy determined by \( \xi_\varepsilon \) and \( \mathfrak{B}_\varepsilon \) for the induced structure of regular points and lines wrt. the conjugacy \( \perp_\varepsilon \). Let \( \varepsilon_1, \varepsilon_2 \) be any two nonzero scalars. Then the following holds
\[
q' \perp_{\varepsilon_1} q'' \iff q' \perp_{\varepsilon_2} q'', \quad (10)
\]
and
\[
q' \perp_{\varepsilon_1} a \iff q' \perp_{\varepsilon_2} a, \quad (11)
\]
for all \( q', q'' \in H, a \notin H \). From (9), (10) and (11) we derive that the set of points selfconjugate under \( \xi_\varepsilon \) is \( H \), and a line of \( \mathfrak{A} \) is regular under \( \perp_{\varepsilon_1} \) iff it is regular under \( \perp_{\varepsilon_2} \). This yields \( \mathfrak{B}_{\varepsilon_1} \subseteq \mathfrak{B}_{\varepsilon_2} \).

Let us take any \( h_1, h_2 \in W \) with \( \xi_0(h_1, h_2) \neq 0 \). Set \( \varepsilon_1 = \xi_0(h_1, h_2), \; a_i = \langle h_i + e_0 \rangle \) for \( i = 1, 2 \), and let \( \varepsilon_2 \) be a nonzero scalar \( \neq \varepsilon_1 \). From (9) we directly compute that \( a_1 \perp_{\varepsilon_1} a_2 \) and \( a_1 \not\perp_{\varepsilon_2} a_2 \). This yields that \( \perp_{\varepsilon_i} \) cannot be defined in terms of \( \mathfrak{B}_{\varepsilon_i} \).

**b ∈ H:** Let \( \omega \notin W \). The form \( \xi | W \) is a degenerate symplectic form. Set \( Y = \omega^\perp \cap W \), then \( e_0, \omega \notin Y \). Let \( e_1, \ldots, e_{n-1} \) be a basis of \( Y \); then \( (e_0, \ldots, e_{n-1}) \) is a basis of \( W \) and \( (\omega, e_0, \ldots, e_{n-1}) \) is a basis of \( \mathbb{V} \). Let \( y \in Y \), then
\[
\xi(y, \omega) = 0, \quad \xi(y, e_0) = 0, \quad \text{and} \quad \xi(e_0, e_0) = 0.
\]
Take any two vectors \( y_i + \alpha_i e_0 + \beta_i \omega, \; (y_i \in Y, \; i = 1, 2) \) of \( \mathbb{V} \). We have
\[
\xi(y_1 + \alpha_1 e_0 + \beta_1 \omega, y_2 + \alpha_2 e_0 + \beta_2 \omega) = \xi_0(y_1, y_2) + (\alpha_1 \beta_2 + \alpha_2 \beta_1) \lambda + \beta_1 \beta_2 \mu, \quad (12)
\]
where \( \lambda = \xi(e_0, \omega) \neq 0, \; \mu = \xi(\omega, \omega) \neq 0 \), and \( \xi_0 \) is the restriction of \( \xi \) to \( Y \).

For any scalars \( \lambda, \mu \neq 0 \) let \( \xi_{\mu, \lambda} \) be a bilinear form defined on \( \mathbb{V} \) by formula (12). Let \( M \) be the matrix of \( \xi_0 \) in the given basis. Note that \( \xi_0 \) is a nondegenerate symplectic form. Then
\[
M_{\mu, \lambda} = \begin{bmatrix}
\mu & \lambda & 0 & \cdots & 0 \\
\lambda & 0 & 0 & \cdots & 0 \\
0 & 0 & & & \\
\vdots & \vdots & M & \\
0 & 0 & & & 
\end{bmatrix}
\]
is the matrix of \( \xi_{\mu, \lambda} \) in our basis. Clearly, \( \det(M_{\mu, \lambda}) \neq 0 \), so \( \xi_{\mu, \lambda} \) is nondegenerate. Let \( \perp_{\mu, \lambda} \) be the conjugacy determined by the form \( \xi_{\mu, \lambda} \) and \( \mathfrak{B}_{\mu, \lambda} \) be the induced structure of regular points and lines.
Since $\xi_{\mu,\lambda}(\omega, \omega) \neq 0$, the form $\xi_{\mu,\lambda}$ is not symplectic. Easy computation gives that $\xi_{\mu,\lambda}(y + \alpha e_0, y + \alpha e_0) = 0$ for each $y \in Y$ and each scalar $\alpha$. Consequently, $H$ is the set of points selfconjugate under $\perp_{\mu,\lambda}$. From (12) we compute
\[
\xi_{\mu,\lambda}(y + \alpha e_0, y + \alpha e_0 + \beta \omega) = \xi_0(y, y) + \lambda \alpha_1 \beta
\]
for all $y, y_1 \in Y$ and scalars $\alpha, \alpha_1, \beta$. This yields that for any nonzero scalars $\lambda, \lambda_1, \mu, \mu_1$ and $q', q'' \in H$, $a \notin H$ we have
\[
q' \perp_{\mu,\lambda} q'' \iff q' \perp_{\mu_1,\lambda_1} q'', \quad (13)
q' \perp_{\mu,\lambda} a \iff q' \perp_{\mu_1,\lambda} a. \quad (14)
\]
Let us take any $y_1, y_2 \in Y$ with $\xi_0(y_1, y_2) \neq 0$ and let $\mu_1 = \xi_0(y_1, y_2)$. Write $a_i = \langle y_i + \omega \rangle$. Finally, let $\mu_2 \neq 0, \mu_1$ be a scalar. Then $B_{\mu_1,\lambda} = B_{\mu_2,\lambda}, a_1 \perp_{\mu_1,\lambda} a_2$, and $a_1 \notin B_{\mu_2,\lambda}$ $a_2$. Consequently, $\perp_{\mu_1,\lambda}$ cannot be defined in $B_{\mu_1,\lambda}$.

Since in any case, the conjugacy $\perp$ cannot be defined in terms of the geometry of $B_1$, our proof is complete.

Gathering together 3.13 and 3.14 we conclude with

**Corollary 3.16.** The structure of the form
\[
\langle \text{points of } A, \text{lines of } A, \text{points of } P, \text{lines of } P, H (= \text{the horizon of } A), \perp_H, \perp \cap (R_1 \times H) \rangle
\]
is definable in terms of the structure $B_1$.

Finally, taking into account 2.4 from 3.16 we obtain

**Proposition 3.17.** For each integer $k$ the family $R_k$ is definable in terms of the structure $B_1$. Consequently, the family
\[
\{ A: A \text{ is a subspace of } A, \overline{A} \in R_k \}
\]
is definable in $B_1$ as well.

Then from 3.15 we conclude with

**Corollary 3.18.** Let $1 \leq k < \dim(V)$ be an integer. The underlying metric projective space $(P, \perp)$ cannot be defined neither in terms of the structure $G_k(R) = (R_k, R_k, \subset)$ nor in terms of the structure $P_k(R) = (R_k, G_k(R))$, where $G_k(R)$ is the family of regular $k$-pencils.

In view of 3.18, the two geometries: of regular subspaces of $(P, \perp)$ and of the metric projective space $(P, \perp)$, are distinct.
3.2 More regular point-line geometry

Now, let us have a look at the incidence structure $\langle \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \rangle$. From 2.7 and 2.8 we see that

**Fact 3.19.** The lines through $b$ are either nonregular, when $b \notin H$, or isolated as there are no regular planes containing such lines when $b \in H$.

This is the reason to investigate a new lineset

$$\mathcal{L}_r := \mathcal{A}_2 \cap \mathcal{A}_2^0$$

of regular affine lines not through $b$. This lineset gives rise to a new geometry

$$\mathfrak{C}_1 := \langle \mathcal{A}_1^0, \mathcal{L}_r, \subset \rangle,$$

which is a substructure of $\mathfrak{B}_1$ (and of $\mathfrak{G}_1$). This slight difference between $\mathfrak{B}_1$ and $\mathfrak{C}_1$ has no impact on the validity of 3.9, 3.10 and 3.13 with $\mathfrak{B}_1$ replaced by $\mathfrak{C}_1$. The respective proofs for $\mathfrak{C}_1$ become a bit more complex but are based on the same ideas. Actually we can state even more:

**Theorem 3.20.** The structures $\mathfrak{B}_1$ and $\mathfrak{C}_1$ are mutually definable. Consequently, the affine space $\mathfrak{A}$, and thus the projective space $\mathfrak{P}$, is definable in terms of $\mathfrak{C}_1$.

3.3 Automorphism group of regular point-line geometry

In view of 3.13, $\text{Aut}(\mathfrak{B}_1)$ is a subgroup of $\text{Aut}(\mathfrak{A})$. Let $f \in \text{Aut}(\mathfrak{A})$ and let $f^\infty$ be its action on the horizon $H$ of $\mathfrak{A}$.

If $f \in \text{Aut}(\mathfrak{B}_1)$ then $f^\infty$ must be an automorphism of the induced metric projective symplectic geometry on $H$.

Moreover, in view of 3.14, $f$ must preserve the family of hyperplanes $\{[q]: q \in H\}$.

The following is simple, though quite useful.

**Lemma 3.21.** Let $f \in \text{Aut}(\mathfrak{A})$. The following conditions are equivalent.

(i) $f \in \text{Aut}(\mathfrak{B}_1)$

(ii) $f^\infty \in \text{Aut}(\langle H, \perp_H \rangle)$ and $(f, f^\infty)$ preserves $\perp \cap (\mathcal{R}_1 \times H)$.

**Proof.** Immediate by 3.16. □

**Proposition 3.22.** Let $W$ be the subspace of $V$ with $H = \{[u]: u \in W, u \neq \theta\}$ and let $\xi_0$ be the restriction of $\xi$ to $W$. Then, clearly, $\xi_0$ determines $\perp_H$. Assume that $b \notin H$. Then

$$\text{Aut}(\mathfrak{B}_1) = \{ \varphi \in \Gamma L(W): \varphi \text{ preserves } \perp_H \}.$$  \hspace{2cm} (15)

**Proof.** Note that $\mathfrak{A}$ can be presented as the affine space $\mathfrak{A}(W)$ over $W$. $b$ is the unique point of $\mathfrak{A}$ such that each line through it is nonregular and thus $b$ remains invariant under automorphisms of $\mathfrak{B}_1$. One can coordinatize $W$ so as $b$ is the origin of the coordinate system and thus each automorphism $\varphi$ of $\mathfrak{B}_1$ is a semilinear bijection of $W$. Since $H$ is the horizon of $\mathfrak{A}$, from 3.14 we get that $\varphi$ preserves $\perp_H$. A direct computation based on (9) justifies that if $\varphi \in \Gamma L(W)$ preserves the conjugacy defined on $H$ by the symplectic form $\xi_0$ then $\varphi$ preserves the class of regular lines. This closes our proof. □
The technique used in the proof of 3.15 enables us to formulate a more elementary definition of the structure \( \mathfrak{B}_1 \).

**Proposition 3.23.** Let \( W \) be the subspace of \( V \) with \( H = \{ (u) : u \in W, u \neq 0 \} \) and let \( \xi_0 \) be the restriction of \( \xi \) to \( W \). Then, clearly, \( \xi_0 \) determines \( \perp_H \). Moreover, one can represent \( \mathfrak{B}_1 \) as a line reduc of the affine space \( \mathfrak{A} = A(W) \) and \( H \) is the horizon of \( \mathfrak{A} \). Assume that \( b \notin H \); consequently, \( b \) is a point of \( \mathfrak{B}_1 \).

(i) Let \( L_* \) be the class of lines in the symplectic polar space determined by the conjugacy \( \perp_H \) on \( H \), let \( a \) be a point of \( \mathfrak{A} \), and \( P^o_a \) be the set of planes of the form \( a + L \) with \( L \in L_* \). Finally, write \( L_*^o = \{ L \in L : L \subset A \text{ for some } A \in P^o_a \} \). Then \( A_2 = L \setminus L_*^o \). Consequently, for an arbitrary affine point \( a \) we have \( \mathfrak{B}_1 \cong \langle W, L \setminus L_*^o \rangle \).

(ii) Let \( u, v \in W \). Then \( u, v \) lie on a nonregular line iff \( \xi_0(u, v) = 0 \).

Now, we pass to the case \( b \in H \). Let us adopt the coordinate system as in subs. 1.4.1 and let \( \xi_0 \) be the restriction of \( \xi \) to \( W \).

**Proposition 3.24.** The following conditions are equivalent.

(i) \( f \in \text{Aut}(\mathfrak{B}_1) \)

(ii) There are \( \varphi \in \Gamma L(W) \) and a vector \( \omega \in W \) such that \( f(x) = \varphi(x) + \omega \) for each \( x \in W \), \( \varphi \) preserves \( \perp_H \), and the following holds

\[
\xi_0(x, y) = \pi_1(y) \implies \xi_0(\varphi(x), \varphi(y)) + \xi_0(\omega, \varphi(y)) = \pi_1(\varphi(y)),
\]

for all \( x, y \in W \), where \( \pi_1 \) is the projection on the 1st coordinate.

**Proof.** It is clear that each automorphism \( f \) of \( \mathfrak{B}_1 \) is a composition of a semilinear map \( \varphi \) and a translation on a vector \( \omega \). In the projective coordinates we can write \( f([1, x]) = [1, \varphi(x) + \omega] \) and \( f([0, y]) = [0, \varphi(y)] \). The map \( f \) of such a form is an automorphism of \( \mathfrak{B}_1 \) iff it preserves \( \perp_H \) and it preserves regular lines. From 3.21, \( f \) preserves regular lines iff it preserves suitable restriction of the polarity. To complete the proof it suffices to note that

\[
\xi([1, x], [0, y]) = \pi_1(y) + \xi_0(x, y)
\]

for all \( x, y \in W \). \( \square \)

Suppose that \( \varphi \in GL(W) \) and \( \xi_0(\varphi(x), \varphi(y)) = c\xi_0(x, y) \) for some \( c \neq 0 \) and all \( x, y \in W \) (then, clearly, \( \varphi \) preserves \( \perp_H \)). The conditions of 3.24 yield \( \xi_0(\omega, \varphi(y)) = \pi_1(cy + \varphi(y)) \) for each \( y \in W \). In particular, if \( \varphi = \text{id} \) we obtain \( \omega \perp W \) and thus \( \omega \parallel b \). If \( \varphi \) is a homothety \( x \mapsto \alpha x \) with \( \alpha \neq 0 \) then \( c = \alpha^2 \) and the condition of 3.24 is read as \( \xi_0(\omega, y) = (\alpha + 1)\pi_1(y) \) for all \( y \in W \); this yields \( \omega \perp \{ y : \pi_1(y) = 0 \} \).

4 Grassmannians of regular secunda and hyperplanes

Note that \( \text{Rad}(U) = \text{Rad}(U^\perp) \). This yields that

**Remark 1.** The mapping \( \perp \) is a correlation in \( \mathfrak{P} \) which maps regular subspaces to regular subspaces.
Now set \( n := \dim(V) = \dim(\mathfrak{P}) + 1 \). In this case \( R_k^\perp = R_{n-k} \) and, clearly, \( A_k^\perp = A_{n-k}^\circ \) for each \( k, 1 \leq k \leq n \). Consider the Grassmannian of regular secunda and hyperplanes

\[
\mathfrak{G}_{n-2} := G_{n-2}(R) = (R_{n-2}, R_{n-1}, \subset).
\]

It can be easily seen that \( \mathfrak{G}_1 \cong \mathfrak{G}^\perp_{n-2} \). So, based on 3.13 we can state that the dual of the affine space \( \mathfrak{A} \) can be defined in terms of \( \mathfrak{G}_{n-2} \). In the dual of \( \mathfrak{A} \) we can define \( \mathfrak{P} \) as well as in \( \mathfrak{A} \). These observation can be summarized in the following

**Theorem 4.1.** The structure \( \mathfrak{B}_1 \) can be reconstructed in terms of \( \mathfrak{G}_{n-2} \).

The case where \( n = 4 \), i.e. \( \dim(\mathfrak{P}) = 3 \), seems quite interesting. Then \( \mathfrak{H} \) is a plane and \( b \in \mathfrak{H} \). In this case the Grassmannian \( \mathfrak{G}_1 \) of regular points and lines is dual isomorphic to the Grassmannian \( \mathfrak{G}_2 \) of regular lines and planes, i.e. \( \mathfrak{G}_1 \cong \mathfrak{G}^\perp_2 \).

Since \( A_2^\perp = A_2^\circ \), we have \( \mathfrak{B}_1^\perp \cong (A_2^\circ, R_3, \subset) \). The latter structure is studied in the next section for arbitrary \( n \).

## 5 Grassmannians of regular lines and planes

It is a quite complex, though more or less routine, job to define \( \mathfrak{B}_1 \) in terms of \( P_k(R) \) or \( G_k(R) \). In this section we shall discuss one particular case of this problem where \( k = 2 \). It seems, however, that the techniques used here can be applied generally.

Consider the Grassmannian of regular lines and planes

\[
\mathfrak{G}_2 := G_2(R) = (R_2, R_3, \subset).
\]

In view of 3.17 the structure \( \mathfrak{G}_2 \) is definable in \( \mathfrak{B}_1 \). Note by 3.19 that there are isolated points in \( \mathfrak{G}_2 \) if \( b \in \mathfrak{H} \). When we get rid of these isolated points we get a new structure

\[
\mathfrak{B}_2 := (A_2^\circ, R_3, \subset)
\]

of the regular lines not through \( b \) and regular planes, a substructure of \( \mathfrak{G}_2 \). Note that \( \mathfrak{B}_2 \) is the dual of \( \mathfrak{B}_1 \) when \( \dim(\mathfrak{P}) = 3 \).

As we already know the incidence structure \( (R_1, R_2, R_3) \) of regular points, lines and planes contains isolated objects: the point \( b \), when \( b \notin \mathfrak{H} \), regular lines on \( \mathfrak{H} \) and regular lines through \( b \) when \( b \in \mathfrak{H} \). So, now we introduce the structure

\[
\mathfrak{C}_2 := (L_r, R_3, \subset)
\]

of the regular affine lines not through \( b \) and regular planes, a substructure of \( \mathfrak{B}_2 \). Note that, when \( \dim(\mathfrak{P}) = 3 \), we have \( L_r^\perp = L_r \) and thus \( \mathfrak{C}_2 \) is the dual of \( \mathfrak{C}_1 \).

**Remark 2.** There is no regular plane on \( \mathfrak{H} \).

This means that the points and lines of \( \mathfrak{C}_2 \) are respectively lines and planes of the affine space \( \mathfrak{A} = (R_1, L) \) obtained from \( \mathfrak{P} \) by deleting the hyperplane \( \mathfrak{H} \).

Recall a known fact

**Fact 5.1.** In the structure \( G_2(\mathfrak{P}) = (\mathfrak{H}_2, \mathfrak{H}_3, \subset) \) consider a triangle with the vertices \( L_0, L_1, L_2 \) and the sides \( A_0, A_1, A_2 \) labelled so as \( L_i \not\subset A_i \) for \( i = 0, 1, 2 \). Then the lines \( L_i \) have a common point \( p \). Moreover

\[
\{ L \in \mathfrak{H}_2 : L \subset A_0 \land \exists A \in \mathfrak{H}_3 [L, L_0 \subset A] \} = \{ L \in \mathfrak{H}_2 : p \in L \subset A_0 \} = p(A_0) \in \mathfrak{H}_2(\mathfrak{P}), \quad (16)
\]
where $S_2(\mathcal{P})$ is the set of projective planar pencils of lines of $\mathcal{P}$. Consequently, the relation

$$L_{\mathcal{P}}(L_1, L_2, L_3) \iff (\exists L_0 \in \mathcal{H}(2))(\exists A, A_1, A_2, A_3 \in \mathcal{H}(3)[L_0 \not\in A \land \bigwedge_{i=1}^{3}(L_i \subset A \land L_i \not\subset L_0 \cap A_i)]$$

(17)

is defined for arbitrary lines $L_1, L_2, L_3$ coincides with the collinearity relation in the space $P_2(\mathcal{P})$ of pencils of lines.

The following is evident by 3.6 and inspection of possible cases.

**Lemma 5.2.** Let $A \in \mathcal{R}_3$ and $p$ be a point on $A$. Then $q(A)$ is an affine point and

$$p(p, A) \cap \mathcal{A}_2 = \begin{cases} \emptyset, & \text{when } p = q(A), \\ p(p, A) \setminus \{p, q(A)\}, & \text{when } p \neq q(A). \end{cases}$$

$$p(p, A) \cap \mathcal{L}_i = \begin{cases} \emptyset, & \text{when } p = q(A), \\ p(p, A) \setminus \{p, q(A)\}, & \text{when } p \neq q(A) \text{ and } p \not\in A^\infty, \\ p(p, A) \setminus \{p, q(A), A^\infty\}, & \text{when } p \in A^\infty, \end{cases}$$

Consequently, if $p \not\in A^\infty$ (or equivalently $p \not\in H$), then $p(p, A) \cap \mathcal{A}_2 = p(p, A) \cap \mathcal{R}_2$. Moreover, $p(p, A) \cap \mathcal{R}_2 = p(p, A) \cap \mathcal{A}_2$ and $p(p, A) \cap \mathcal{A}_2 = p(p, A) \cap \mathcal{L}_i$ as $b \not\in A^\infty$.

**Lemma 5.3.** Let the relation $L_{\mathcal{E}_2}$ be defined by the formula (17) with $\mathcal{H}_2$ replaced by $\mathcal{L}_r, \mathcal{H}_3$ replaced by $\mathcal{R}_3$, and let $L_1, L_2, L_3 \in \mathcal{L}_r$. Then the relation $L_{\mathcal{E}_2}(L_1, L_2, L_3)$ holds iff $L_1, L_2, L_3 \in p(p, A)$ for some point $p$ and some $A \in \mathcal{R}_3$.

**Proof.** $\Rightarrow$: Straightforward by 5.1.

$\Leftarrow$: Let $A \in \mathcal{R}_3$, $p$ be a point on $A$, and $L_1, L_2, L_3 \in \mathcal{L}_r$ be three lines on $A$ through $p$. Set $M := A^\infty$; then $M \in \mathcal{R}_2$. To close the proof we need to find a line $L_0 \in \mathcal{L}_r$ through $p$ but not on $A$ such that the planes $A_1 = L_0 + L_1, A_2 = L_0 + L_2$ and $A_3 = L_0 + L_3$ are all regular. There are two cases to consider.

(i) $p$ is an affine point i.e. $p \not\in M$.

Note that $M \cap M^\perp = \emptyset$. Write $q_i := M \cap L_i$ for $i = 1, 2, 3$. Since the $L_i$ are regular, $q_i \not\in p^\perp$ and thus $p \not\perp M$. Therefore, the intersection $M \cap p^\perp$ is a single point $q_0 \neq q_1, q_2, q_3$.

Take $x \in M$ with $x \not\in q_i$ for $i = 0, \ldots, 3$ and $y \in M^\perp \cap H + x$ with $y \neq x$ and $y \not\in M^\perp$. Then $y \not\in b$ and $y \in x + m = x, m$ for some (uniquely determined by $y$) point $m \in M^\perp$. We have $y \not\in M$, since otherwise $M = x, y \ni m$, so $m \in \text{Rad}(M)$.

Suppose that $y \in q_i^\perp$ for some $i = 1, 2, 3$. Evidently, $q_i \perp M^\perp$ and $q_i \not\perp q_j$; comparing dimensions we get $q_i^\perp = q_i + M^\perp$. From $y \in q_i^\perp$ we get that $y \in q_i + m_i$ for some $m_i \in M^\perp$. Since $M$ and $M^\perp$ are skew, we get $x = q_i$, which is impossible.

Suppose that $y \in p^\perp$ and write $K = y + q_0$; then $K \subset p^\perp$. For each $i = 1, 2, 3$ the line $K$ crosses $q_i + M^\perp$ in a point $y_i$ and $y_i \neq y$. Note: $y_i \perp p$. There are $m_i \in M^\perp$ such that $y_i \in q_i + m_i$; it is seen that $m_i = m$, as otherwise the lines $M$ and $m, m_i \subset M^\perp$ are contained in the plane $M + K$ and thus they have a common
point. Consider another point \( y' \) on \( m + x \). If there were \( y' \in p^\perp \) we would obtain another point \( y'_1 \) on \( m + q_1 \) with \( p \perp y'_1 \). This leads, contradictory, to \( q_i \perp p \).
Without loss of generality we can assume that \( y \notin p^\perp \).

Take \( L_0 := \overline{p, y} \); then \( L_0 \in \mathcal{L} \) as \( p \) is an affine point. Take \( A_i := L_0 + L_i \) for \( i = 1, 2, 3 \). Then \( A_i^\infty = \overline{q_i, y} \). Since \( q_i \perp y \), the lines \( A_i^\infty \) are regular and thus \( A_i \in \mathcal{R}_3 \).

(ii) \( p \in M, M \neq L_1, L_2, L_3 \) i.e. all the \( L_i \) are affine lines.

By assumption, \( M \) is regular. Let \( D \) be a plane on \( H \) containing the line \( M \). By 2.9 the radical of \( D \) is a point, say \( u \). The line \( L := \overline{p, u} \) is nonregular. Now, in the three space \( A + D \) we take a line \( L_0 \) through \( p \) that is not contained in any of the planes \( L_1 + L \) and is not contained in \( p^\perp \). This is doable thanks to assumption that there are at least 5 lines distinct from \( M \) through \( p \) on every plane containing \( M \). Note that \( L_0 \) is regular by 2.5.

Let \( A_i := L_i + L_0 \). By 2.8 each of the planes \( A_i \) is regular as it meets \( D \) in a line distinct from \( L \) thus, in a regular one.

This completes the reasoning. \( \square \)

**Corollary 5.4.** Let the relation \( \mathcal{L}_{\mathcal{R}_2} \) be defined by the formula (17) with \( \mathcal{R}_2, \mathcal{R}_3 \) replaced by \( \mathcal{A}_2, \mathcal{A}_3 \) respectively and let \( L_1, L_2, L_3 \in \mathcal{A}_2^\infty \). The relation \( \mathcal{L}_{\mathcal{R}_2}(L_1, L_2, L_3) \) holds iff \( L_1, L_2, L_3 \in \mathfrak{p}(p, A) \) for some regular point \( p \) and some \( A \in \mathcal{R}_3 \).

**Proof.** \( \Rightarrow \): Clear by 5.1.

\( \Leftarrow \): Let \( A \in \mathcal{R}_3, p \) be a point on \( A \), and \( L_1, L_2, L_3 \in \mathcal{A}_2^\infty \) be three lines on \( A \) through \( p \). Set \( M := A^\infty \); then \( M \in \mathcal{A}_2^\infty \). We have three cases

(i) \( p \) is an affine point i.e. \( p \notin M \),

(ii) \( p \in M, M \neq L_1, L_2, L_3 \) i.e. all the \( L_i \) are affine lines,

(iii) \( p \in M = L_i \) for some \( i \in \{1, 2, 3\} \).

Cases (i) and (ii) follow directly by 5.3 (though in case (ii) there is a simple independent proof here that do not require 6 lines in a projective pencil). In the remaining case (iii) without loss of generality we take \( i = 1 \). Let \( D \) be a plane through \( M \) contained in \( H \). By 2.9, since \( D \) contains a regular line \( M \), \( \text{Rad}(D) \) is a point \( q \). If there was \( p = q \) we would have \( p \perp D \), which gives, contradictory, \( p \in \text{Rad}(L_1) \). The unique nonregular line through \( p \) on \( A \) is \( K_1 := \overline{p, q(\mathfrak{A})} \) and the unique nonregular line through \( p \) on \( D \) is \( K_2 := \overline{p, q} \). Write \( A_0 := K_1 + K_2 \); since \( p \perp K_1, K_2 \) we have \( p \perp A_0 \). Set \( Y := D + A \); then \( Y \in \mathcal{K}_4 \). Note that \( \text{rdim}(Y) \leq 1 \); if \( \text{Rad}(Y) \neq \emptyset \) then \( \text{Rad}(Y) = q \). Since, either \( Y \subset p^\perp \), which gives, contradictory, \( p \perp L_1 \), or \( Y \cap p^\perp \) is a plane, we obtain \( Y \cap p^\perp = A_0 \).

Take \( M_3 \subset D \) with \( p \in M_3 \neq K_2, M \); then \( M_3 \) is regular. For \( A_3 = L_3 + M_3 \) we have \( A_3 \in \mathcal{R}_3 \), because \( A_3^\infty = M_3 \). From \( p \in A_3, A_0 \subset Y \) we get that \( K_0 := A_3 \cap A_0 \) is a line.

Consider the plane \( B = L_2 + K_2 \); then \( B \) and \( A_3 \) have a common line \( K_4 \). Let \( L_0 \) be a line in \( \mathfrak{p}(p, A_3) \) distinct from \( L_3, K_4, K_0, M_3 \). Then \( L_0 \not\subset A_0 \) and thus \( L_0 \) is regular. Set \( A_2 = L_2 + L_0 \); then \( p \in A_2^\infty \) and \( A_2^\infty = A_2 \cap H = A_2 \cap D \neq K_2 \), which gives that \( A_2 \) is regular. Finally, we take \( A_1 = L_0 + L_1 \). Then \( A_1^\infty = M \) and thus \( A_1 \in \mathcal{R}_3 \). \( \square \)
The family of equivalence classes of the relation \( L_{c_{2}} \) is the set
\[
\{ p_{2}^{af}(p, A) : p \in A \in \mathcal{R}_{3} \} \setminus \{ \emptyset \}, \quad \text{where} \quad p_{2}^{af}(p, A) = \{ L \in L_{r} : p \in L \subset A \}.
\]
So, we get the space of affine regular pencils \( A_{2}(\mathcal{R}) \) with regular affine lines not through \( b \), i.e. elements of \( L_{r} \), as points and affine regular pencils \( p_{2}^{af}(p, A) \) as lines. Note that among pencils \( p_{2}^{af}(p, A) \) we have proper pencils, those with \( p \notin H \), and parallel pencils, those with \( p \in H \).

The family of equivalence classes of the relation \( L_{b_{2}} \), is the set
\[
\{ p_{1}(p, A) : p \in A \in \mathcal{R}_{3} \} \setminus \{ \emptyset \}, \quad \text{where} \quad p_{1}(p, A) = \{ L \in A_{2}^{*} : p \in L \subset A \},
\]
and we have the space of regular pencils \( P_{2}(\mathcal{R}) \) with regular lines not through \( b \) as points and regular pencils \( p_{1}(p, A) \) as lines. The space of affine regular pencils \( A_{2}(\mathcal{R}) \) is a substructure of the space of regular pencils \( P_{2}(\mathcal{R}) \) in the sense that points of \( A_{2}(\mathcal{R}) \) are points of \( P_{2}(\mathcal{R}) \) and, in view of 5.2, lines of \( P_{2}(\mathcal{R}) \) are a bit "richer".

Loosely speaking, we have proved that the family of regular pencils is definable in both \( c_{2} \) and in \( b_{2} \) (cf. [9]).

Recall that 2.7 and 2.8 together say the following.

**Fact 5.5.** Let \( b \in H \). If \( A \) is a plane not contained in \( H \) and \( b \in A \), then \( A \) is not regular (comp. 2). Consequently, no regular pencil exists that contains a line through \( b \in H \).

This means that there are no pencils, neither in \( A_{2}(\mathcal{R}) \) nor in \( P_{2}(\mathcal{R}) \), with a vertex \( b \) when \( b \in H \), though there are regular affine lines through \( b \).

For points \( L_{1}, L_{2}, L_{3} \) of a point-line geometry \( X \) we write
\[
\Delta_{X}(L_{1}, L_{2}, L_{3}) \iff L_{1}, L_{2}, L_{3} \text{ are the vertices of a triangle in } X.
\]

From common projective geometry (cf. 5.1) it follows that if \( \Delta_{X}(L_{1}, L_{2}, L_{3}) \) holds and \( X \) is \( c_{2} \), \( b_{2} \) or \( c_{2} \) and \( p \in L_{1}, L_{2} \), then \( p \in L_{3} \) as well.

We are going to identify points of \( B_{1} \) with stars of lines in \( b_{2} \) as well as with stars of lines in \( c_{2} \). The star of regular lines through a point \( p \) is the set
\[
S_{r}(p) = \{ L \in \mathcal{R}_{2} : p \in L \}
\]
and the star of regular affine lines through a point \( p \) is
\[
S_{r}^{af}(p) = \{ L \in A_{2}^{*} : p \in L \}.
\]
Note that
\[
S_{r}(p) = \begin{cases} S_{r}^{af}(p), & p \notin H, \\ S_{r}^{af}(p) \cup \{ L \in \mathcal{R}_{2} : p \in L \subset H \}, & p \in H. \end{cases}
\]
To express the notion of a star of lines purely in terms of the geometry of \( b_{2} \) or \( c_{2} \) for a given pencil \( Q = p_{r}(a, A) \neq \emptyset \) (and respectively for \( Q = p_{r}^{af}(a, A) \neq \emptyset \)) we write
\[
S_{\Delta}(Q) := \{ L \in \mathcal{R}_{2} : (\exists L_{1}, L_{2} \in Q) (L_{1} \neq L_{2}) \},
\]
\[
S_{s}(Q) := \{ L \in \mathcal{R}_{2} : (\exists L_{1}, L_{2} \in S_{\Delta}(Q)) [L_{1} \neq L_{2} \land L(L_{1}, L_{2})] \},
\]
\[
S(Q) := Q \cup S_{\Delta}(Q) \cup S_{s}(Q),
\]
where $\Delta = \Delta_{\mathbb{A}_2}$ and $L = L_{\mathbb{A}_2}$ (or respectively $\Delta = \Delta_{\mathbb{E}_2}$ and $L = L_{\mathbb{E}_2}$). The set $S_{\Delta}(Q)$ contains those lines $L$ with $L^\perp \not\subset A^\perp$ while we need $S_L(Q)$ for the other lines $L$ with $L^\perp \subset A^\perp$. It will become more apparent later in 5.6 and 5.7.

Let $\sim$ be the binary collinearity in $\mathfrak{G}_2$, i.e. for $L_1, L_2 \in R_2$ we write

$$L_1 \sim L_2 \text{ iff there is } A \in R_3 \text{ with } L_1, L_2 \subset A, \text{ and } L_1 \not\sim L_2 \text{ when } L_1 \sim L_2 \text{ does not hold.}$$

Following 2.7 recall that

$$S_t^a(b) = S_t(b) \begin{cases} = \emptyset & \text{if } b \notin H; \text{ there are } L_1, L_2 \in S_t(b) \text{ with } L_1 \sim L_2, L_1 \not\sim L_2, \\ \neq \emptyset & \text{if } b \in H; \text{ } L_1 \not\sim L_2 \text{ for all } L_1, L_2 \in S_t(b). \end{cases}$$

**Lemma 5.6.** Let $Q = p_t(p, A) \neq \emptyset$ with $p \in A \in R_3$ and $p \notin H$, and let $L$ be a regular line not in $Q$. Write $q := L^\perp$ and $M := A^\perp$.

If $p \in L$ and $q \not\subset b$, then one of the following holds:

(i) $q \not\subset M^\perp$. Then there are distinct $L_1, L_2 \in Q$ such that $\Delta_{\mathbb{A}_2}(L_1, L_1, L_2)$.

(ii) $q \subset M^\perp$. Then there are distinct regular lines $L', L''$ through $q$ such that $L_{\mathbb{A}_2}(L, L', L'')$ and $L^{\perp \perp}, L''^{\perp \perp} \not\subset M^\perp$, so for both $L'$ and $L''$ the condition (i) holds.

Conversely, if a line $L$ satisfies (i) or (ii) then $p \in L$.

**Proof.** Note that $M$ is a regular line and $x_1 := p^\perp \cap M$ is a point, since otherwise $p \perp M$ and then no line in $p_t(p, A)$ is regular. The line $L$ is regular so, $p \not\subset q$.

Let $q \not\subset M^\perp$. Then $x_2 := M \cap q^\perp$ is a point, for if not then $q \perp M$, so $q \subset M^\perp$.

Take distinct $y_1, y_2 \in M$ such that $y_1, y_2 \neq x_1, x_2$ and set $L_i = p, q_i$, for $i = 1, 2$.

From construction, $y_1, y_2 \not\subset p, q$. Then $L_1, L_2 \in p_t(a, A)$ by 2.5. In view of 2.9 applied for the plane $M + q$ the lines $M_i = q, y_i$, $i = 1, 2$ are regular. So, the plane $p + M_i$ is regular for $i = 1, 2$, which gives $\Delta_{\mathbb{A}_2}(L, L_1, L_2)$. This completes the proof of (i).

Let $q \subset M^\perp$. The nonregular lines contained in $H$ through $q$ are all contained in the hyperplane $q^\perp \cap H$ of $H$ (the assumption $q \not\subset b$ turns out to be essential here). It is impossible to decompose $H$ into the union of three proper subspaces $q^\perp \cap H, p^\perp \cap H, M^\perp \cap H$, so there is a point $q' \in H$ with $q' \not\subset q^\perp, M^\perp, p^\perp$.

Then the line $K := q, q' \subset H$ is regular and $p \not\subset K$. Let $z := K \cap p^\perp$ and $q'' \in M, q'' \neq q'$.

Then $q'' \not\subset p$ and thus the lines $L' := p, q'$ and $L'' := p, q''$ are regular.

Write $B = K + p$; then $B \in R_3$ and, evidently, $L', L'' \in p_t(p, B)$. Moreover, $q'' \not\subset M^\perp$, as $M$ is regular by 2.8. Since $L^{\perp \perp} = q'$ and $L''^{\perp \perp} = q''$, the proof in case (ii) is complete.

Now, let $L$ be an arbitrary regular line. From 5.1 we get that (i) implies $p \in L$

In case when (ii) holds we get $p \in L', L''$, which directly gives $p \in L$.

Since $p \not\subset H$ in 5.6, then in view of 5.2 we have

**Corollary 5.7.** Let $Q = p_t^a(p, A) \neq \emptyset$ with $p \in A \in R_3$ and $p \not\subset H$, and let $L$ be a regular affine line not in $Q$. Write $q := L^{\perp \perp}$ and $M := A^{\perp \perp}$.

If $p \in L$ and $q \not\subset b$, then one of the following holds:

(i) $q \subset M^\perp$. Then there are distinct $L_1, L_2 \in Q$ such that $\Delta_{\mathbb{E}_2}(L, L_1, L_2)$. 

\[ q \not\subset M. \]
(ii) \( q \in M^1 \). Then there are distinct regular affine lines \( L', L'' \) through \( p \) such that \( \Delta_{\mathcal{E}_2}(L, L', L'') \) and \( L'^\infty, L''^\infty \notin M^1 \), so for both \( L' \) and \( L'' \) the condition (i) holds.

Conversely, if a line \( L \) satisfies (i) or (ii) then \( p \in L \).

**Lemma 5.8.** Let \( Q = p^\partial_t(p, A) \neq \emptyset \) with \( p \in A \in \mathcal{R}_3, p \in \mathcal{H} \) and let \( L \notin Q \) be a regular affine line. Then \( p \in L \) iff there are distinct \( L_1, L_2 \in Q \) such that \( \Delta_{\mathcal{E}_2}(L, L_1, L_2) \) holds.

**Proof.** By 2.8 and our assumptions the line \( M := A^\infty \) is regular. Hence \( p \notin b \). By 5.2 the projective pencil \( p(p, A) \) contains exactly two lines that are not in \( p^\partial_t(p, A) \), namely \( M \) and \( K_0 := p, q(A) \).

\[ \implies \text{ We have } L \notin H \text{ as } L \text{ is affine. Since } A + L \text{ is a projective 3-space, } D := (A + L) \cap H \text{ is a plane. Note that } M \subset D \text{ so, by 2.9 } q := \text{Rad}(D) \text{ is a point such that } q \notin M. \]

Set \( K_1 = p, q \). Consider the line \( K_2 := (K_0 + L) \cap D \). Take two lines \( M_1, M_2 \) on \( D \) through \( p \) with \( M_1, M_2 \neq M, K_1, K_2 \). So, \( M_i \) is regular, and by 2.8 the plane \( L + M_i \) is regular for \( i = 1, 2 \). Now take \( L_i := (L + M_i) \cap A \), \( i = 1, 2 \).

Observe that \( L_1, L_2 \notin K_0 \). Suppose that \( L_i \) is nonregular. Then by 2.5 we have \( p \notin L_i \) which is impossible as the plane \( A = L_1 + L_2 \) is regular. Clearly, \( L_1, L_2 \in Q \) and \( \Delta_{\mathcal{E}_2}(L_1, L_2) \).

\[ \iff \text{ A direct consequence of 5.1.} \]

**Corollary 5.9.** Let \( Q = p^\partial_t(p, A) \neq \emptyset \) with \( p \in A \in \mathcal{R}_3, p \in \mathcal{H} \), and let \( L \notin Q \) be a regular line. Then \( p \in L \) iff there are distinct \( L_1, L_2 \in Q \) such that \( \Delta_{\mathcal{B}_2}(L, L_1, L_2) \) holds.

**Proof.** \( \implies \text{ There are two cases to consider here: (i) } L \subset H \text{ and (ii) } L \notin H. \)

In case (i) it suffices to take any two affine lines \( L_1, L_2 \in Q \) so, we have two new regular planes \( L + L_1, L + L_2 \) by 2.8 and \( A = L_1 + L_2 \) which means that \( \Delta_{\mathcal{B}_2}(L, L_1, L_2) \) is valid.

The case (ii) follows from 5.8.

\[ \iff \text{ A direct consequence of 5.1.} \]

Note that when \( p \in H \), then in \( S(Q) \) there is no line \( L \) with \( L^\infty \perp A^\infty \) as such a line is nonregular. Thus \( S_L(Q) = \emptyset \) and consequently \( S(Q) = Q \cup S_{\Delta}(Q) \). Taking this into account and summing up 5.6 together with 5.9 and 5.7 together with 5.8 we get

**Proposition 5.10.** Let \( p \in A \in \mathcal{R}_3 \). If \( p \notin H \) and \( b \notin H \), or \( p \in H \) and \( p \neq b \), then

\[ S(p_t(p, A)) = S_t(p) \quad \text{and} \quad S(p^\partial_t(p, A)) = S^\partial_t(p). \]

If \( p \notin H \) and \( b \in H \), then

\[ S(p_t(p, A)) = S(p^\partial_t(p, A)) = S_t(p) \setminus \{p, b\} = S^\partial_t(p) \setminus \{p, b\}. \]

Roughly speaking, all the points of \( \mathcal{P} \) except \( b \) can be reconstructed in terms of \( \mathcal{B}_2 \) as well as in terms of \( \mathcal{C}_2 \). Actually, this is expectable since no plane through improper \( b \) is regular (comp. 2 and 5.5).

Now we will try to distinguish regular and nonregular points of \( \mathcal{P} \) in terms of \( \mathcal{B}_2 \) and \( \mathcal{C}_2 \). To do that we need a convenient characterization of binary collinearity.
(adjacency) in these two structures. Basically two regular affine lines are adjacent if they are coplanar and the plane is regular. Note that if a regular line from $\mathbf{H}$ and a regular affine line are coplanar then the plane is always regular, while regular lines on $\mathbf{H}$ are never adjacent by 2. The next lemma sheds yet more light on (non)adjacency of regular lines.

**Lemma 5.11.** Let $L_1, L_2$ be distinct regular lines through a point $p$. Then $L_1 \not\sim L_2$ iff one of the following holds

(i) $p \notin \mathbf{H}$ and $L_1^\infty \perp L_2^\infty$;

(ii) $p \in \mathbf{H}$ and

a) either $L_1, L_2 \subset \mathbf{H}$,

b) or $L_1, L_2 \not\subset \mathbf{H}$ and $p \perp (L_1 + L_2)^\infty$

**Proof.** Clearly, $A := L_1 + L_2$ is a plane. Therefore, $L_1 \not\sim L_2$ iff $A$ is not regular.

In case $p \notin \mathbf{H}$ the plane $A$ is affine and, in view of 2.8, it is nonregular iff $A^\infty = L_1^\infty \cap L_2^\infty$ is nonregular i.e. iff $L_1^\infty \perp L_2^\infty$.

In case $p \in \mathbf{H}$ there are three possibilities. If $L_1, L_2 \subset \mathbf{H}$, then $A \subset \mathbf{H}$, so $A$ is nonregular by 2. If $L_1 \subset \mathbf{H}$ and $L_2 \not\subset \mathbf{H}$, then $A^\infty = L_1$, so $A$ is regular by 2.8. Finally, if $L_1, L_2 \not\subset \mathbf{H}$, then $A$ is nonregular iff $A^\infty \perp A^\infty$, which is equivalent to $p \perp A^\infty$.

**Lemma 5.12.** Assume that dim($\mathcal{V}$) > 4. Let $Q = p_i(p, A)$ or $Q = p_i^A(p, A)$ with $Q \neq \emptyset$ and $A \in \mathcal{R}_3$. The following conditions are equivalent:

(i) $p \in \mathbf{H}$;

(ii) for all $L, L_1, L_2 \in S(Q)$ from $L \not\sim L_1, L_2$ it follows that $L_1 \not\sim L_2$.

**Proof.** Let $L \in S(Q)$. Then $p \in L$ by 5.10. In view of 5.11 it suffices to consider the following cases.

(i) Let $p \notin \mathbf{H}$. Set $q := L^\infty$. Take $q_1 \in q^\perp \cap \mathbf{H}$ with $q_1 \notin p^\perp$ and $q_2 \in q^\perp \cap \mathbf{H}$ with $q_2 \notin p^\perp, q_1^\perp$. The later is doable when dim($\mathcal{V}$) > 4. So, we have $q_i \neq b$, $i = 1, 2$ for if not we would have $q_1 \perp q_2$, a contradiction. Set $L_i := p + q_i$ for $i = 1, 2$. Then $L \not\sim L_1, L_2 \in S(Q)$ and $L_1 \sim L_2$.

(ii) Let $p \in \mathbf{H}$, $L \not\subset \mathbf{H}$, and $L \not\sim L_1, L_2 \in S(Q)$. From 5.11 we get $L_1, L_2 \not\subset \mathbf{H}$. Set $A_i := L + L_i$ and $M_i := A_i^\infty$. From assumption, the $M_i$ are nonregular lines on $\mathbf{H}$ and therefore $p \perp M_i$; this yields $p \perp M_1 + M_2$. If $M_1 = M_2$ then $(L_1 + L_2)^\infty = M_1$, so $L_1 \not\sim L_2$. Assume that $M_1 \neq M_2$. Then there is a line $M_0 := (L_1 + L_2) \cap (M_1 + M_2)$. Clearly, $M_0 = (L_1 + L_2)^\infty$ and $p \in M_0 \perp p$, so $M_0$ is nonregular. Consequently, $L_1 \not\sim L_2$.

(iii) Let $p \in \mathbf{H}$, $L \subset \mathbf{H}$, and $L \not\sim L_1, L_2 \in S(Q)$. From 5.11 we get $L_1, L_2 \subset \mathbf{H}$, and then $L_1 \not\sim L_2$ follows.

This closes our proof.

**Lemma 5.13.** If dim($\mathcal{V}$) = 4, then condition (ii) in 5.12 is always valid.
Proof. Let \( Q \) be as assumed in 5.12 and let \( L, L_1, L_2 \in S(Q) = S_{1}(p) \) with \( L \not\sim L_1, L_2 \). Set \( q := L^\infty, q_i := L_i^\infty \) for \( i = 1, 2 \). By 5.11, \( q \perp q_1, q_2 \) which, in view of 2.8, means that planes \( L + L_1, L + L_2 \) are nonregular.

If \( p = b \), then the plane \( L_1 + L_2 \) is nonregular as no plane through \( b \) is regular, hence \( L_1 \not\sim L_2 \).

Now assume that \( p \not\in H \). According to the definition of \( S(Q) \) there is always a regular plane \( A' \) such that \( L \subset A' \). Hence \( q \not\in b \) by 5.5. In 3-space \( \mathcal{P} \) the subspace \( K := q^\perp \cap H \) is a line on \( H \). Note that \( q_1, q_2 \in K \) and also \( q \in K \) as \( q \in H \) and thus \( q \in q^\perp \). This means that \( K \) is isotropic so, \( q_1 \perp q_2 \). From 5.11 we get \( L_1 \not\sim L_2 \).

Finally, if \( b \not\in p \in H \), then we have three cases:

1. Let \( L_1, L_2 \subset H \). Clearly \( L_1 \not\sim L_2 \).
2. Let \( L_1 \subset H \) and \( L_2 \not\subset H \). If \( L \subset H \), then the plane \( L + L_2 \) is regular by 2.8 which contradicts our assumptions that \( L \not\sim L_2 \). If \( L \not\subset H \), then the plane \( L + L_1 \) is regular, again a contradiction.
3. Let \( L_1, L_2 \not\subset H \). If \( L \subset H \), then both planes \( L + L_1, L + L_2 \) are regular which is impossible by our assumptions that \( L \not\sim L_1, L_2 \). If \( L \not\subset H \), then we have two distinct isotropic lines \( M_i := (L + L_i) \cap H, i = 1, 2 \) on \( H \). Since \( p \in M_1, M_2 \) and \( M_1 \cap M_2 = b \), a contradiction and the proof is complete.

As it has been shown in 5.13 we need to treat the case \( \dim(V) = 4 \) separately. Recall that \( \mathcal{B}_2 \cong \mathcal{B}_1 \) and \( \mathcal{C}_2 \cong \mathcal{C}_1 \). So, we can state that the dual of \( \mathcal{A} \) is definable in \( \mathcal{B}_2 \) and \( \mathcal{C}_2 \) by 3.20 and 3.13. Hence \( \mathcal{P} \) is definable in \( \mathcal{B}_2 \). Moreover, the horizon \( (H, \perp_H) \) is also definable, so is \( \mathcal{B}_1 \).

In 5.10 we have defined points of \( \mathcal{P} \) and thanks to 5.12 we are able to distinguish regular and nonregular points, all strictly in languages of \( \mathcal{B}_2 \) and \( \mathcal{C}_2 \). So, we have \( (H, \perp_H) \) reconstructed. Gathering all together we get

Theorem 5.14. The structure \( \mathcal{B}_1 \) can be defined in terms of both \( \mathcal{B}_2 \) and \( \mathcal{C}_2 \).

References

[1] Bachmann, F., Aufbau der Geometrie aus dem Spiegelungsbegriff, 2. Auflage, Springer-Verlag, Berlin, 1973.
[2] Buekenhout J., Cohen A., Diagram Geometry, (to appear).
[3] Cohen M. A. Point-line spaces related to buildings. In Handbook of incidence geometry, F. Buekenhout, Ed. North-Holland, Amsterdam, 1995, pp. 647–737.
[4] Hall, J. I., Classifying copolar spaces and graphs, Quart. J. Math. Oxford (2), 33 (1982), 421–449.
[5] Hirschfeld J. W. W. P., Projective Geometries over Finite Fields, Clarendon Press, Oxford, 1998.
[6] Lefèvre-Percsy C., Copolar spaces fully embedded in projective spaces, Ann. Discrete Math. 18 (1983), 553–566.
[7] Pankov M., Grassmannians of classical buildings, Algebra and Discrete Mathematics Vol. 2, World Scientific, New Jersey, 2010.
[8] Pankov M., Prażmowski K., Źyźnel M., Geometry of polar Grassmann spaces, Demonstratio Math. 39 (2006), no. 3, 625–637.
[9] Prażmowska M., Prażmowski K., Żyśel M., Grassmann spaces of regular subspaces, J. Geom. 97 (2010), 99-123.
[10] Prażmowski K., Żyśel M., Orthogonality of subspaces in metric-projective geometry, Adv. Geom. 11 (2011), no. 1, 103-116.
[11] Tits, J. Buildings of spherical type and finite BN-pairs, vol. 386 of Lecture Notes in Mathematics. Springer, Berlin, 1974.
[12] Ueberberg J., Foundations of Incidence Geometry: Projective and Polar Spaces, Springer-Verlag, Berlin, 2011.
[13] Van Maldeghem H., Generalized Polygons, Birkhäuser, Basel, 1998.

Authors’ address:
Krzysztof Prażmowski, Mariusz Żyśel
Institute of Mathematics, University of Białystok
ul. Akademicka 2, 15-267 Białystok, Poland
krzypraz@math.uwb.edu.pl, mariusz@math.uwb.edu.pl