Orbit functions of SU(n) and Chebyshev polynomials

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Orbit functions of a simple Lie group/Lie algebra $L$ consist of exponential functions summed up over the Weyl group of $L$. They are labeled by the highest weights of irreducible finite dimensional representations of $L$. They are of three types: $C$-, $S$- and $E$-functions. Orbit functions of the Lie algebras $A_n$, or equivalently, of the Lie group SU($n+1$), are considered. First, orbit functions in two different bases – one orthonormal, the other given by the simple roots of SU($n$) – are written using the isomorphism of the permutation group of $n$ elements and the Weyl group of SU($n$).

Secondly, it is demonstrated that there is a one-to-one correspondence between classical Chebyshev polynomials of the first and second kind, and $C$- and $S$-functions of the simple Lie group SU(2).

It is then shown that the well-known orbit functions of SU($n$) are straightforward generalizations of Chebyshev polynomials to $n - 1$ variables. Properties of the orbit functions provide a wealth of properties of the polynomials.

Finally, multivariate exponential functions are considered, and their connection with orbit functions of SU($n$) is established.

1 Introduction

The history of the Chebyshev polynomials dates back over a century. Their properties and applications have been considered in many papers. We refer to [19, 20] as a basic reference. Studies of polynomials in more than one variable were undertaken by several authors, namely [2–4, 13, 15, 21, 22]. Of these, none follow the path we have laid down here.

In this paper, we demonstrate that the classical Chebyshev polynomials in one variable are naturally associated with the action of the Weyl group of SU(2), or equivalently with the action of the Weyl group $W(A_1)$ of the simple Lie algebra of type $A_1$. The association is so simple that it has been ignored so far. However, by making $W(A_1)$ the cornerstone of our rederivation of Chebyshev polynomials, we have gained insight into the structure of the theory of polynomials. In particular, the generalization of Chebyshev polynomials to any number of variables was a straightforward task. It is based on the Weyl group $W(A_n)$, where $n < \infty$. This only recently became possible, after the orbit functions of simple Lie algebras were introduced as useful special functions [18] and studied in great detail and generality [8,9,11].

We proceed in three steps. In Section 2, we exploit the isomorphism of the group of permutations of $n + 1$ elements $S$ and the Weyl group of SU($n+1$), or equivalently of $A_n$, and define the orbit functions of $A_n$. This opens the possibility to write the orbit functions in two rather different bases, the orthonormal basis, and the basis determined by the simple roots of $A_n$, which considerably alters the appearance of the orbit functions. In the paper, we use the non-orthogonal basis because of its direct generalization to simple Lie algebras of other types than $A_n$.

In Section 3 we consider classical Chebyshev polynomials of the first and second kind, and compare them with the $C$- and $S$-orbit functions of $A_1$. We show that polynomials of the
first kind are in one-to-one correspondence with \(C\)-functions. Polynomials of the second kind coincide with the appropriate \(S\)-function divided by the unique lowest non-trivial \(S\)-function.

We point out that polynomials of the second kind can be identified as irreducible characters of finite dimensional representations of \(SU(2)\). Useful properties of Chebyshev polynomials can undoubtedly be traced to that identification, because the fundamental object of representation theory of semisimple Lie groups/algebras is character. In principle, all one needs to know about an irreducible finite dimensional representation can be deduced from its character. An important aspect of this conclusion is that characters are known and uniformly described for all simple Lie groups/algebras.

In Section 4 we provide details of the recursive procedure from which the analog of the trigonometric form of Chebyshev polynomials in \(n\) variables can be found. Thus there are \(n\) generic recursion relations for \(A_n\), having at least \(n + 2\) terms, and at most \(\left(\frac{n+1}{[(n+1)/2]}\right) + 1\) terms. Irreducible polynomials are divided into \(n + 1\) exclusive classes with the property that monomials within one irreducible polynomial belong to the same class. This follows directly from the recognition of the presence and properties of the underlying Lie algebra.

In subsection 4.2, the simple substitution \(z = e^{2\pi i x}, x \in \mathbb{R}^n\), is used in orbit functions to form analogs of Chebyshev polynomials in \(n\) variables in their non-trigonometric form. It is shown that, in the case of 2 variables, our polynomials coincide with those of Koorwider [13](III), although the approach and terminology could not be more different, ours being purely algebraic, having originated in Lie theory.

In Section 5, we present the orbit functions of \(A_n\) disguised as polynomials built from multivariate orbit functions of the symmetric group. In Section 2, such a possibility is described in terms of related bases, one orthonormal (symmetric group), the other non-orthogonal (simple roots of \(A_n\) and their dual \(\omega\)-basis). Both forms of the same polynomials appear rather different but may prove useful in different situations.

The last section contains a few comments and some questions related to the subject of this paper that we find intriguing.

## 2 Preliminaries

This section is intended to fix notation and terminology. We also briefly recall some facts about \(S_{n+1}\) and \(A_n\), dwelling particularly on various bases in \(\mathbb{R}^{n+1}\) and \(\mathbb{R}^n\). In Section 2.3, we identify elementary reflections that generate the \(A_n\) Weyl group \(W\), with the permutation of two adjacent objects in an ordered set of \(n + 1\) objects. And, finally, we present some standard definitions and properties of orbit functions.

### 2.1 Permutation group \(S_{n+1}\)

The group \(S_{n+1}\) of order \((n + 1)!\) transforms the ordered number set \([l_1, l_2, \ldots, l_n, l_{n+1}]\) by permuting the numbers.

We introduce an orthonormal basis in the real Euclidean space \(\mathbb{R}^{n+1}\),

\[
e_i \in \mathbb{R}^{n+1}, \quad \langle e_i, e_j \rangle = \delta_{ij}, \quad 1 \leq i, j \leq n + 1,
\]

and use the \(l_k\)’s as the coordinates of a point \(\mu\) in the \(e\)-basis:

\[
\mu = \sum_{k=1}^{n+1} l_k e_k, \quad l_k \in \mathbb{R}.
\]

The group \(S_{n+1}\) permutes the coordinates \(l_k\) of \(\mu\), thus generating other points from it. The set of all distinct points, obtained by application of \(S_{n+1}\) to \(\mu\), is called the orbit of \(S_{n+1}\). We
denote an orbit by $W_\lambda$, where $\lambda$ is a unique point of the orbit, such that

$$l_1 \geq l_2 \geq \cdots \geq l_n \geq l_{n+1}.$$ 

If there is no pair of equal $l_k$'s in $\lambda$, the orbit $W_\lambda$ consists of $(n+1)!$ points.

Further on, we will only consider points $\mu$ from the $n$-dimensional subspace $\mathcal{H} \subset \mathbb{R}^{n+1}$ defined by the equation

$$\sum_{k=1}^{n+1} l_k = 0. \quad (2)$$

### 2.2 Lie algebra $A_n$

Let us recall basic properties of the simple Lie algebra $A_n$ of the compact Lie group $SU(n+1)$. Consider the general value $(1 \leq n < \infty)$ of the rank. The Coxeter-Dynkin diagram, Cartan matrix $\mathcal{C}$, and inverse Cartan matrix $\mathcal{C}^{-1}$ of $A_n$ are as follows:

$$\mathcal{C} = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 2
\end{pmatrix},$$

$$\mathcal{C}^{-1} = \frac{1}{n+1} \begin{pmatrix}
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\n\end{pmatrix}.$$

The simple roots $\alpha_i$, $1 \leq i \leq n$ of $A_n$ form a basis ($\alpha$-basis) of a real Euclidean space $\mathbb{R}^n$. We choose them in $\mathcal{H}$:

$$\alpha_i = e_i - e_{i+1}, \quad i = 1, \ldots, n.$$ 

This choice fixes the lengths and relative angles of the simple roots. Their length is equal to $\sqrt{2}$ with relative angles between $\alpha_k$ and $\alpha_{k+1}$ ($1 \leq k \leq n-1$) equal to $\frac{2\pi}{n}$, and $\frac{\pi}{2}$ for any other pair.

In addition to $e$- and $\alpha$-bases, we introduce the $\omega$-basis as the $\mathbb{Z}$-dual basis to the simple roots $\alpha_i$:

$$\langle \alpha_i, \omega_j \rangle = \delta_{ij}, \quad 1 \leq i, j \leq n.$$ 

It is also a basis in the subspace $\mathcal{H} \subset \mathbb{R}^{n+1}$ (see (2)). The bases $\alpha$ and $\omega$ are related by the Cartan matrix:

$$\alpha = \mathcal{C}\omega, \quad \omega = \mathcal{C}^{-1}\alpha.$$ 

Throughout the paper, we use $\lambda \in \mathcal{H}$. Here, we fix the notation for its coordinates relative to the $e$- and $\omega$-bases:

$$\lambda = \sum_{i=1}^{n+1} l_i \omega_i =: (l_1, \ldots, l_{n+1})_e = \sum_{i=1}^{n} \lambda_i \omega_i =: (\lambda_1, \ldots, \lambda_n)_\omega, \quad \sum_{i=1}^{n+1} l_i = 0.$$
Consider a point \( \lambda \in \mathcal{H} \) with coordinates \( l_j \) and \( \lambda_i \) in the \( e \)- and \( \omega \)-bases, respectively. Using \( \alpha = e^i \omega_i \), i.e. \( \omega_i = \sum_{k=1}^n (e^{-1})_{ik} \alpha_k \), we obtain the relations between \( \lambda_i \) and \( l_j \):

\[
\begin{align*}
  l_1 &= \sum_{k=1}^n \lambda_k e^{-1}_{k1}, \quad l_{n+1} = -\sum_{k=1}^n \lambda_k e^{-1}_{kn}, \\
  l_j &= \lambda_1 (e^{-1}_{1j} - e^{-1}_{1,j-1}) + \lambda_2 (e^{-1}_{2j} - e^{-1}_{2,j-1}) + \cdots + \lambda_n (e^{-1}_{nj} - e^{-1}_{n,j-1}), \quad j = 2, \ldots, n.
\end{align*}
\]

or explicitly,

\[
\lambda_i = l_i - l_{i+1}, \quad i = 1, 2, \ldots, n.
\]

The inverse formulas are much more complicated

\[
l = A \lambda,
\]

where \( l = (l_1, \ldots, l_{n+1}) \), \( \lambda = (\lambda_1, \ldots, \lambda_n) \), and \( A \) is the \((n+1) \times n\) matrix:

\[
A = \frac{1}{n+1} \begin{pmatrix}
  n & n-1 & n-2 & \cdots & 2 & 1 \\
  -1 & n-1 & n-2 & \cdots & 2 & 1 \\
  -1 & -2 & n-2 & \cdots & 2 & 1 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  -1 & -2 & -3 & \cdots & -(n-1) & 1 \\
  -1 & -2 & -3 & \cdots & -(n-1) & -n
\end{pmatrix}.
\]

### 2.3 The Weyl group of \( A_n \)

The Weyl group \( W(A_n) \) of order \((n + 1)!\) acts in \( \mathcal{H} \) by permuting coordinates in the \( e \)-basis, i.e. as the group \( S_{n+1} \). Indeed, let \( r_i \), \( 1 \leq i \leq n \) be the generating elements of \( W(A_n) \), i.e. reflections with respect to the hyperplanes perpendicular to \( \alpha_i \) and passing through the origin. Let \( x = \sum_{k=1}^{n+1} x_k e_k = (x_1, x_2, \ldots, x_{n+1})_e \) and \( \langle \cdot, \cdot \rangle \) denote the inner product. We then have the reflection by \( r_i \):

\[
\begin{align*}
r_i x &= x - \frac{2}{\langle \alpha_i, \alpha_i \rangle} (x, \alpha_i) \alpha_i = (x_1, x_2, \ldots, x_{n+1})_e - (x_i - x_{i+1})(e_i - e_{i+1}) \\
&= (x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_{n+1})_e.
\end{align*}
\]

Such transpositions generate the full permutation group \( S_{n+1} \). Thus, \( W(A_n) \) is isomorphic to \( S_{n+1} \), and the points of the orbit \( W_\lambda(S_{n+1}) \) and \( W_\lambda(A_n) \) coincide.

### 2.4 Definitions of orbit functions

The notion of an orbit function in \( n \) variables depends essentially on the underlying semisimple Lie group \( G \) of rank \( n \). In our case, \( G = SU(n+1) \) (equivalently, Lie algebra \( A_n \)). Let the basis of the simple roots be denoted by \( \alpha \), and the basis of fundamental weights by \( \omega \).

The weight lattice \( P \) is formed by all integer linear combinations of the \( \omega \)-basis,

\[
P = \mathbb{Z} \omega_1 + \mathbb{Z} \omega_2 + \cdots + \mathbb{Z} \omega_n.
\]

In the weight lattice \( P \), we define the cone of dominant weights \( P^+ \) and its subset of strictly dominant weights \( P^{++} \)

\[
P \supset P^+ = \mathbb{Z}^\geq \omega_1 + \cdots + \mathbb{Z}^\geq \omega_n \supset P^{++} = \mathbb{Z}^> \omega_1 + \cdots + \mathbb{Z}^> \omega_n.
\]

Hereafter, \( W^e \subset W \) denotes the even subgroup of the Weyl group formed by an even number of reflections that generate \( W \). \( W_\lambda \) and \( W^e_\lambda \) are the corresponding group orbits of a point \( \lambda \in \mathbb{R}^n \).

We also introduce the notion of fundamental region \( F(G) \subset \mathbb{R}^n \). For \( A_n \) the fundamental region \( F \) is the convex hull of the vertices \( \{0, \omega_1, \omega_2, \ldots, \omega_n\} \).
**Definition 1.** The \( C \) orbit function \( C_\lambda(x), \lambda \in P^+ \) is defined as

\[
C_\lambda(x) := \sum_{\mu \in W_\lambda(G)} e^{2\pi i \langle \mu, x \rangle}, \quad x \in \mathbb{R}^n.
\]

**Definition 2.** The \( S \) orbit function \( S_\lambda(x), \lambda \in P^{++} \) is defined as

\[
S_\lambda(x) := \sum_{\mu \in W_\lambda(G)} (-1)^{p(\mu)} e^{2\pi i \langle \mu, x \rangle}, \quad x \in \mathbb{R}^n,
\]

where \( p(\mu) \) is the number of reflections necessary to obtain \( \mu \) from \( \lambda \). Of course the same \( \mu \) can be obtained by different successions of reflections, but all routes from \( \lambda \) to \( \mu \) will have a length of the same parity, and thus the salient detail given by \( p(\mu) \), in the context of an \( S \)-function, is meaningful and unchanging.

**Definition 3.** We define \( E \) orbit function \( E_\lambda(x), \lambda \in P^e \) as

\[
E_\lambda(x) := \sum_{\mu \in W_\lambda(G)} e^{2\pi i \langle \mu, x \rangle}, \quad x \in \mathbb{R}^n,
\]

where \( P^e := P^+ \cup r_i P^+ \) and \( r_i \) is a reflection from \( W \).

If we always suppose that \( \lambda, \mu \in P \) are given in the \( \omega \)-basis, and \( x \in \mathbb{R}^n \) is given in the \( \alpha \) basis, namely \( \lambda = \sum_{j=1}^n \lambda_j \omega_j, \mu = \sum_{j=1}^n \mu_j \omega_j, \lambda_j, \mu_j \in \mathbb{Z} \) and \( x = \sum_{j=1}^n x_j \alpha_j, x_j \in \mathbb{R} \), then the orbit functions of \( A_n \) have the following forms

\[
C_\lambda(x) = \sum_{\mu \in W_\lambda} e^{\sum_{j=1}^n 2\pi i \mu_j x_j} = \sum_{\mu \in W_\lambda} \prod_{j=1}^n e^{2\pi i \mu_j x_j}, 
\]

\[
S_\lambda(x) = \sum_{\mu \in W_\lambda} (-1)^{p(\mu)} e^{\sum_{j=1}^n 2\pi i \mu_j x_j} = \sum_{\mu \in W_\lambda} (-1)^{p(\mu)} \prod_{j=1}^n e^{2\pi i \mu_j x_j},
\]

\[
E_\lambda(x) = \sum_{\mu \in W_\lambda^e} e^{\sum_{j=1}^n 2\pi i \mu_j x_j} = \sum_{\mu \in W_\lambda^e} \prod_{j=1}^n e^{2\pi i \mu_j x_j}.
\]

### 2.5 Some properties of orbit functions

For \( S \) functions, the number of summands is always equal to the size of the Weyl group. Note that in the 1-dimensional case, \( C \)-, \( S \)- and \( E \)-functions are respectively a cosine, a sine and an exponential functions up to the constant.

All three families of orbit functions are based on semisimple Lie algebras. The number of variables coincides with the rank of the Lie algebra. In general, \( C \)-, \( S \)- and \( E \)-functions are finite sums of exponential functions. Therefore they are continuous and have continuous derivatives of all orders in \( \mathbb{R}^n \).

The \( S \)-functions are antisymmetric with respect to the \((n-1)\)-dimensional boundary of \( F \). Hence they are zero on the boundary of \( F \). The \( C \)-functions are symmetric with respect to the \((n-1)\)-dimensional boundary of \( F \). Their normal derivative at the boundary is equal to zero (because the normal derivative of a \( C \)-function is an \( S \)-function).

For simple Lie algebras of any type, the functions \( C_\lambda(x), E_\lambda(x) \) and \( S_\lambda(x) \) are eigenfunctions of the appropriate Laplace operator. The Laplace operator has the same eigenvalues on every exponential function summand of an orbit function with eigenvalue \(-4\pi \langle \lambda, \lambda \rangle\).
2.5.1 Orthogonality

For any two complex squared integrable functions $\phi(x)$ and $\psi(x)$ defined on the fundamental region $F$, we define a continuous scalar product

$$\langle \phi(x), \psi(x) \rangle := \int_F \overline{\phi(x)} \psi(x) dx. \quad (12)$$

Here, integration is carried out with respect to the Euclidean measure, the bar means complex conjugation and $x \in F$, where $F$ is the fundamental region of either $W$ or $W^e$ (note that the fundamental region of $W^e$ is $F^e = F \cup r_i F$, where $r_i \in W$).

Any pair of orbit functions from the same family is orthogonal on the corresponding fundamental region with respect to the scalar product (12), namely

$$\langle C_\lambda(x), C_{\lambda'}(x) \rangle = |W_\lambda| \cdot |F| \cdot \delta_{\lambda\lambda'}, \quad (13)$$
$$\langle S_\lambda(x), S_{\lambda'}(x) \rangle = |W| \cdot |F| \cdot \delta_{\lambda\lambda'}, \quad (14)$$
$$\langle E_\lambda(x), E_{\lambda'}(x) \rangle = |W^e_\lambda| \cdot |F^e| \cdot \delta_{\lambda\lambda'}, \quad (15)$$

where $\delta_{\lambda\lambda'}$ is the Kronecker delta, $|W|$ is the order of the Weyl group, $|W_\lambda|$ and $|W^e_\lambda|$ are the sizes of the Weyl group orbits (the number of distinct points in the orbit), and $|F|$ and $|F^e|$ are volumes of fundamental regions. The volume $|F|$ was calculated in [6].

Proof. Proof of the relations (13,14,15) follows from the orthogonality of the usual exponential functions and from the fact that a given weight $\mu \in P$ belongs to precisely one orbit function. □

The families of $C$-, $S$- and $E$-functions are complete on the fundamental domain. The completeness of these systems follows from the completeness of the system of exponential functions; i.e., there does not exist a function $\phi(x)$, such that $\langle \phi(x), \phi(x) \rangle > 0$, and at the same time $\langle \phi(x), \psi(x) \rangle = 0$ for all functions $\psi(x)$ from the same system.

2.5.2 Orbit functions of $A_n$ acting in $\mathbb{R}^{n+1}$

Relations (4) allow us to rewrite variables $\lambda$ and $x$ in an orbit function in the $e$-basis. Therefore we can obtain the $C$-, $S$- and $E$- functions acting in $\mathbb{R}^{n+1}$

$$C_\lambda(x) = \sum_{s \in S_{n+1}} e^{2\pi i (s(\lambda), x)}, \quad (16)$$
$$C_\lambda(x) = \sum_{s \in S_{n+1}} (\text{sgn } s) e^{2\pi i (s(\lambda), x)}, \quad (17)$$
$$E_\lambda(x) = \sum_{s \in \text{Alt}_{n+1}} e^{2\pi i (s(\lambda), x)}, \quad (18)$$

where $\langle \cdot, \cdot \rangle$ is a scalar product in $\mathbb{R}^{n+1}$, $\text{sgn } s$ is the permutation sign, and $\text{Alt}_{n+1}$ is the alternating group acting on an $(n + 1)$-tuple of numbers. Note that variables $x$ and $\lambda$ are in the hyperplane $H$.

Using the identity $\langle \lambda, r_i x \rangle = \langle r_i \lambda, x \rangle$ for the reflection $r_i$, $i = 1, \ldots, n$, it can be verified that

$$C_\lambda(r_i x) = C_{r_i \lambda}(x) = C_\lambda(x), \quad \text{and} \quad S_{r_i \lambda}(x) = S_\lambda(r_i x) = -S_\lambda(x). \quad (19)$$

Note that it is easy to see for generic points that $E_\lambda(x) = \frac{1}{2} (C_\lambda(x) + S_\lambda(x))$, and from the relations (19), we obtain

$$E_{r_i \lambda}(x) = E_\lambda(r_i x) = \frac{1}{2} (C_\lambda(x) - S_\lambda(x)) = E_\lambda(x). \quad (20)$$

A number of other properties of orbit functions are presented in [8,9,11].
3 Orbit functions and Chebyshev polynomials

We recall known properties of Chebyshev polynomials [19] in order to be subsequently able to make an unambiguous comparison between them and the appropriate orbit functions.

3.1 Classical Chebyshev polynomials

Chebyshev polynomials are orthogonal polynomials which are usually defined recursively. One distinguishes between Chebyshev polynomials of the first kind \( T_n \):

\[
T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n - T_{n-1}, \quad (21)
\]

hence \( T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \ldots \) (22)

and Chebyshev polynomials of the second kind \( U_n \):

\[
U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) = 2xU_n - U_{n-1}, \quad (23)
\]

in particular \( U_2(x) = 4x^2 - 1, \quad U_3(x) = 8x^3 - 4x, \) etc. (24)

The polynomials \( T_n \) and \( U_n \) are of degree \( n \) in the variable \( x \). All terms in a polynomial have the parity of \( n \). The coefficient of the leading term of \( T_n \) is \( 2^n - 1 \) and \( 2^n \) for \( U_n, n = 1, 2, 3, \ldots \).

The roots of the Chebyshev polynomials of the first kind are widely used as nodes for polynomial interpolation in approximation theory. The Chebyshev polynomials are a special case of Jacobi polynomials. They are orthogonal with the following weight functions:

\[
\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n(x)T_m(x)dx = \begin{cases} 
0, & n \neq m, \\
\pi, & n = m = 0, \\
\frac{\pi}{2}, & n = m \neq 0, 
\end{cases} \quad (25)
\]

\[
\int_{-1}^{1} \sqrt{1-x^2} U_n(x)U_m(x)dx = \begin{cases} 
0, & n \neq m, \\
\frac{\pi}{2}, & n = m. 
\end{cases} \quad (26)
\]

There are other useful relations between Chebyshev polynomials of the first and second kind.

\[
\frac{d}{dx} T_n(x) = nU_{n-1}(x), \quad n = 1, 2, 3, \ldots \quad (27)
\]

\[
T_n(x) = \frac{1}{2}(U_n(x) - U_{n-2}(x)), \quad n = 2, 3, \ldots \quad (28)
\]

\[
T_{n+1}(x) = xT_n(x) - (1 - x^2)U_{n-1}, \quad n = 1, 2, 3, \ldots \quad (29)
\]

\[
T_n(x) = U_n(x) - xU_{n-1}, \quad n = 1, 2, 3, \ldots \quad (30)
\]

3.1.1 Trigonometric form of Chebyshev polynomials

Using trigonometric variable \( x = \cos y \), polynomials of the first kind become

\[
T_n(x) = T_n(\cos y) = \cos(ny), \quad n = 0, 1, 2, \ldots \quad (31)
\]

and polynomials of the second kind are written as

\[
U_n(x) = U_n(\cos y) = \frac{\sin((n+1)y)}{\sin y}, \quad n = 0, 1, 2, \ldots \quad (32)
\]

For example, the first few lowest polynomials are

\[
T_0(x) = T_0(\cos y) = \cos(0y) = 1, \quad T_1(x) = T_1(\cos y) = \cos(y) = x,
\]

\[
T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \ldots
\]
From (33) and (31) it directly follows that polynomials generated from doubled Chebyshev polynomials $T_2(x) = T_2(\cos y) = \cos(2y) = \cos^2 y - \sin^2 y = 2\cos^2 y - 1 = 2x^2 - 1$;

$U_0(x) = U_0(\cos y) = \frac{\sin y}{\sin y} = 1$, \quad $U_1(x) = U_1(\cos y) = \frac{\sin(2y)}{\sin y} = 2\cos y = 2x$,

$U_2(x) = U_2(\cos y) = \frac{\sin(3y)}{\sin y} = \frac{\sin(2y)\cos y + \sin y\cos(2y)}{\sin y} = 4\cos^2 y - 1 = 4x^2 - 1$.

### 3.2 Orbit functions of $A_1$ and Chebyshev polynomials

Let us consider the orbit functions of one variable. There is only one simple Lie algebra of rank 1, namely $A_1$. Our aim is to build the recursion relations in a way that generalizes to higher rank groups, unlike the standard relations of the classical theory presented above.

#### 3.2.1 Orbit functions of $A_1$ and trigonometric form of $T_n$ and $U_n$

The orbit of $\lambda = m\omega_1$ has two points for $m \neq 0$, namely $W_\lambda = \{(m), (-m)\}$. The orbit of $\lambda = 0$ has just one point, $W_0 = \{0\}$.

One-dimensional orbit functions have the form (see (9), (10), (11))

$$C_\lambda(x) = e^{2\pi imx} + e^{-2\pi imx} = 2\cos(2\pi mx) = 2\cos(my), \quad \text{where} \quad y = 2\pi x, \ m \in \mathbb{Z}^>0;$$

$$S_\lambda(x) = e^{2\pi imx} - e^{-2\pi imx} = 2i\sin(2\pi mx) = 2i\sin(my), \quad \text{for} \quad m \in \mathbb{Z}^>0;$$

$$E_\lambda(x) = e^{2\pi imx} = y^m, \quad \text{where} \quad y = e^{2\pi ix}, \ m \in \mathbb{Z}.$$  

From (33) and (31) it directly follows that polynomials generated from $C_m$ functions of $A_1$ are doubled Chebyshev polynomials $T_m$ of the first kind for $m = 0, 1, 2, \ldots .$

Analogously, from (34) and (32), it follows that polynomials $\frac{S_m}{S_1}$ are Chebyshev polynomials $U_m$ of the second kind for $m = 0, 1, 2, \ldots .$

The polynomials generated from $E_m$ functions of $A_1$, form a standard monomial sequence $y^m, \ m = 0, 1, 2 \ldots , \text{which is the basis for the vector space of polynomials.}$

$C$- and $S$-orbit functions are orthogonal on the interval $F = [0, 1]$ (see (13) and (14)) what implies the orthogonality of the corresponding polynomials.

Comparing the properties of one-dimensional orbit functions with properties of Chebyshev polynomials, we conclude that there is a one-to-one correspondence between the Chebyshev polynomials and the orbit functions.

#### 3.2.2 Orbit functions of $A_1$ and their polynomial form

In this subsection, we start a derivation of the $A_1$ polynomials in a way which emphasizes the role of the Lie algebra and, more importantly, in a way that directly generalizes to simple Lie algebras of any rank $n$ and any type, resulting in polynomials of $n$ variables and of a new type for each algebra. In the present case of $A_1$, this leads us to a different normalization of the polynomials and their trigonometric variables than is common for classical Chebyshev polynomials. No new polynomials emerge than those equivalent to Chebyshev polynomials of the first and second kind. Insight is nevertheless gained into the structure of the problem, which, to us, turned out to be of considerable importance. We are inclined to consider the Chebyshev polynomials, in the form derived here, as the canonical polynomials.

The underlying Lie algebra $A_1$ is often denoted $sl(2, \mathbb{C})$ or $su(2)$. In fact, this case is so simple that the presence of the Lie algebras has never been acknowledged.

The orbit functions of $A_1$ are of two types (33) and (34); in particular, $C_0(x) = 2$, and $S_0(x) = 0$ for all $x$. 

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The simplest substitution of variables to transform the orbit functions into polynomials is $y = e^{2\pi ix}$, monomials in such a polynomial are $y^m$ and $y^{-m}$. Instead, we introduce new (‘trigonometric’) variables $X$ and $Y$ as follows:

\begin{align*}
X &:= C_1(x) = e^{2\pi ix} + e^{-2\pi ix} = 2\cos(2\pi x), \\
Y &:= S_1(x) = e^{2\pi ix} - e^{-2\pi ix} = 2i\sin(2\pi x).
\end{align*}

(36) (37)

We can now start to construct polynomials recursively in the degrees of $X$ and $Y$, by calculating the products of the appropriate orbit functions. Omitting the dependence on $x$ from the symbols, we have

\begin{align*}
X^2 &= C_2 + 2, & C_2 &= X^2 - 2, \\
XC_2 &= C_3 + X, & C_3 &= X^3 - 3X, \\
XC_m &= C_{m+1} + C_{m-1}, & C_{m+1} &= XC_m - C_{m-1}, & m \geq 3.
\end{align*}

(38)

Therefore, we obtain the following recursive polynomial form of the $C$-functions

$$C_0 = 2, \quad C_1 = X, \quad C_2 = X^2 - 2, \quad C_3 = X^3 - 3X, \quad C_4 = X^4 - 4X^2 + 2, \ldots.$$  

(39)

After the substitution $z = \frac{1}{2}X$ we have

$$C_0 = 2, \quad C_1 = 2z, \quad C_2 = 2(2z^2 - 1), \quad C_3 = 2(4z^3 - 3z), \quad C_4 = 2(8z^4 - 8z^2 + 1), \ldots.$$  

Hence we conclude that $C_m = 2T_m$, for $m = 0, 1, \ldots$.

**Remark 1.**

In our opinion, the normalization of orbit functions is also more ‘natural’ for the Chebyshev polynomials. For example, the equality $C_2^2 = C_4 + 2$ does not hold for $T_2$ and $T_4$.

**Remark 2.**

Each $C_m$ also can be written as a polynomial of degree $m$ in $X, Y$ and $S_{m-1}$. It suffices to consider the products $YS_m$, e.g., $C_2 = Y^2 + 2$, $C_3 = YS_2 + X$, etc. Equating the polynomials obtained in such a way with the corresponding polynomials from (38), we obtain a trigonometric identity for each $m$. For example, we find two ways to write $C_2$, one from the product $X^2$ and one from $Y^2$. Equating the two, we get

$$X^2 - Y^2 = 4 \iff \sin^2(2\pi x) + \cos^2(2\pi x) = 1$$

because $Y$ is defined in (37) to be purely imaginary.

Just as the polynomials representing $C_m$ were obtained above, it is possible to to find polynomial expressions for $S_m$ for all $m$.

Fundamental relations between the $S$- and $C$-orbit functions follow from the properties of the character $\chi_m(x)$ of the irreducible representation of $A_1$ of dimension $m + 1$.

The character can be written in two ways: as in the Weyl character formula and also as the sum of appropriate $C$-functions. Explicitly, we have the $A_1$ character:

$$\chi_m(x) = \frac{S_{m+1}(x)}{S_1(x)} = C_m(x) + C_{m-2}(x) + \cdots + \begin{cases} C_2(x) + 1 \quad &\text{for } m \text{ even,} \\
C_3(x) + C_1(x) \quad &\text{for } m \text{ odd.}
\end{cases}$$

Let us write down a few characters

$$\chi_0 = \frac{S_1(x)}{S_1(x)} = 1, \quad \chi_1 = \frac{S_2(x)}{S_1(x)} = C_1 = X, \quad \chi_2 = \frac{S_3(x)}{S_2(x)} = C_2 = C_0 = X^2 - 1,$$

$$\chi_3 = \frac{S_4(x)}{S_3(x)} = C_3 + C_1 = X^3 - 2X, \quad \chi_4 = \frac{S_5(x)}{S_4(x)} = C_4 + C_2 + C_0 = X^4 - 3X^2 + 1, \ldots$$

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Again, the substitution $z = \frac{1}{2}X$ transforms these polynomials into the Chebyshev polynomials of the second kind $\frac{S_{m+1}}{S_1} = U_m$, $m = 0, 1, \ldots$, indeed
\[
\frac{S_1(x)}{S_1(x)} = 1, \quad \frac{S_2(x)}{S_1(x)} = 2z, \quad \frac{S_3(x)}{S_1(x)} = 4z^2 - 1, \quad \frac{S_4(x)}{S_1(x)} = 8z^3 - 4z, \quad \frac{S_5(x)}{S_1(x)} = 16z^4 - 12z^2 + 1, \ldots
\]

Remark 3.
Note that in the character formula we used $C_0 = 1$, while above (see (11) and (39)) we used $C_0 = 2$. It is just a question of normalization of orbit functions. For some applications/calculations it is convenient to scale orbit functions of non-generic points on the factor equal to the order of the stabilizer of that point in the Weyl group $W(A_1)$.

4 Orbit functions of $A_n$ and their polynomials

This section proposes two approaches to constructing orthogonal polynomials of $n$ variables based on orbit functions. The first comes from the decomposition of Weyl orbit products into sums of orbits. Its result is the analog of the trigonometric form of the Chebyshev polynomials. The second approach is the exponential substitution in [8].

4.1 Recursive construction

Since the $C$- and $S$- functions are defined for $A_n$ of any rank $n = 1, 2, 3, \ldots$, it is natural to take $C$-functions and the ratio of $S$-functions as multidimensional generalizations of Chebyshev polynomials of the first and second kinds respectively
\[
T_\lambda(x) := C_\lambda(x), \quad x \in \mathbb{R}^n,
\]
\[
U_\lambda(x) := \frac{S_{\lambda+\rho}(x)}{S_\rho(x)}, \quad \rho = \omega_1 + \omega_2 + \ldots + \omega_n = (1, 1, \ldots, 1)_\omega, \quad x \in \mathbb{R}^n,
\]

where $\lambda$ is one of the dominant weights of $A_n$.

The functions $T_\lambda$ and $U_\lambda$ can be constructed as polynomials using the recursive scheme proposed in Section 3.2.2. In the $n$-dimensional case of orbit functions of $A_n$, we start from the $n$ orbit functions labeled by the fundamental weights,
\[
X_1 := C_{\omega_1}(x), \quad X_2 := C_{\omega_2}(x), \quad \ldots, \quad X_n := C_{\omega_n}(x), \quad x \in \mathbb{R}^n.
\]

By multiplying them and decomposing the products into the sum of orbit functions, we build the polynomials for any $C$- and $S$-function.

The generic recursion relations are found as the decomposition of the products $X_{\omega_j}C_{(a_1, a_2, \ldots, a_n)}$ with ‘sufficiently large’ $a_1, a_2, \ldots, a_n$. Such a recursion relation has $(\frac{n+1}{j}) + 1$ terms, where $(\frac{n+1}{j})$ is the size of the orbit of $\omega_j$.

An efficient way to find the decompositions is to work with products of Weyl group orbits, rather than with orbit functions. Their decomposition has been studied, and many examples have been described in [5]. It is useful to be aware of the congruence class of each product, because all of the orbits in its decomposition necessarily belong to that class. The congruence number # of an orbit $\lambda$ of $A_n$, which is also the congruence number of the orbit functions $C_\lambda$ and $S_\lambda$, specifies the class. It is calculated as follows,
\[
\#(C_{(a_1, a_2, \ldots, a_n)}(x)) = \#(S_{(a_1, a_2, \ldots, a_n)}(x)) = \sum_{k=1}^{n} ka_k \mod (n + 1). \tag{40}
\]

In particular, each $X_j$, where $j = 1, 2, \ldots, n$, is in its own congruence class. During the multiplication, congruence numbers add up $\mod n + 1$.

Polynomials in two and three variables originating from orbit functions of the simple Lie algebras $A_2$, $C_2$, $G_2$, $A_3$, $B_3$, and $C_3$ are obtained in the forthcoming paper [17].
4.2 Exponential substitution

There is another approach to multivariate orthogonal polynomials, which is also based on orbit functions. Such polynomials can be constructed by the continuous and invertible change of variables

$$y_j = e^{2\pi i x_j}, \quad x_j \in \mathbb{R}, \quad j = 1, 2, \ldots, n. \quad (41)$$

Consider an $A_n$ orbit function $C_\lambda(x)$, $S_\lambda(x)$ or $E_\lambda(x)$, when $\lambda$ is given in the $\omega$-basis and $x$ is given in the $\alpha$-basis. Each of these functions consists of summands $\prod_{j=1}^n e^{2\pi i \mu_j x_j}$, where $\mu_j \in \mathbb{Z}$ are coordinates of an orbit point $\mu$. Then the summand is transformed by (41) into a monomial of the form $\prod_{j=1}^n y_j^{\mu_j}$. It is convenient to label these polynomials by non-negative integer coordinates $(m_1, m_2, \ldots, m_n)$ of the point $\lambda = m_1 \omega_1 + m_2 \omega_2 + \ldots + m_n \omega_n$ and to denote the polynomial obtained from the orbit function $C_\lambda$ as $P^C_{(m_1, \ldots, m_n)}$ (analogously for $S$ and $E$ functions). Polynomials of two variables obtained from the orbit functions by the substitution (41) are already described in the literature [13], where they are derived from very different considerations. The detailed comparison is made in the following example.

**Example 1.** Consider the $A_2$ Weyl orbits of the lower weights $(0, m)_\omega$, $(m, 0)_\omega$ and the orbit of the generic point $(m_1, m_2)_\omega$, $m, m_1, m_2 \in \mathbb{Z}^{>0}$

$$W_{(0,m)}(A_2) = \{(0, m), (-m, 0), (m, -m)\}, \quad W_{(m,0)}(A_2) = \{(m, 0), (-m, m), (0, -m)\},$$

$$W_{(m_1,m_2)}(A_2) = \{(m_1, m_2)^+, (-m_1, m_1 + m_2)^-, (m_1 + m_2, -m_2)^-,$$

$$(-m_2, -m_1)^-, (-m_1 - m_2, m_1)^+, (m_2, -m_1 - m_2)^+\}.$$

Suppose $x = (x_1, x_2)$ is given in the $\alpha$-basis, then the orbit functions assume the form

$$C_{(0,0)}(x) = 1, \quad C_{(0,m)}(x) = \overline{C_{(m,0)}(x)} = e^{-2\pi i m x_1} + e^{2\pi i m x_1} e^{-2\pi i m x_2} + e^{2\pi i m x_2},$$

$$C_{(m_1,m_2)}(x) = e^{2\pi i m_2 x_1} e^{2\pi i m_2 x_2} + e^{-2\pi i m_1 x_1} e^{2\pi i (m_1 + m_2) x_2} + e^{2\pi i (m_1 + m_2) x_1} e^{-2\pi i m_2 x_2} +$$

$$- e^{-2\pi i m_1 x_1} e^{-2\pi i (m_1 + m_2) x_2} + e^{2\pi i (m_1 + m_2) x_1} e^{2\pi i m_2 x_2} e^{-2\pi i (m_1 + m_2) x_2}, \quad (42)$$

$$S_{(m_1,m_2)}(x) = e^{2\pi i m_1 x_1} e^{2\pi i m_2 x_2} - e^{-2\pi i m_1 x_1} e^{2\pi i (m_1 + m_2) x_2} - e^{2\pi i (m_1 + m_2) x_1} e^{-2\pi i m_2 x_2} -$$

$$e^{-2\pi i m_2 x_1} e^{-2\pi i (m_1 + m_2) x_2} + e^{2\pi i (m_1 + m_2) x_1} e^{2\pi i m_2 x_2} e^{-2\pi i (m_1 + m_2) x_2}.$$

Using (41) we have the following corresponding polynomials

$$P^C_{(0,0)} = 1, \quad P^C_{0,m} = \overline{P^C_{0,m}} = y_1^{-m} + y_1^{m} y_2^{-m} + y_2^{-m},$$

$$P^C_{(m_1,m_2)} = y_1^{m_1} y_2^{m_2} + y_1^{-m_1} y_2^{(m_1 + m_2)} + y_1^{(m_1 + m_2)} y_2^{-m_2} +$$

$$y_1^{-m_1} y_2^{-m_2} + y_1^{(m_1 + m_2)} y_2^{m_1} + y_1^{m_1} y_2^{(m_1 + m_2)},$$

$$P^S_{(m_1,m_2)} = y_1^{m_1} y_2^{m_2} - y_1^{-m_1} y_2^{(m_1 + m_2)} - y_1^{(m_1 + m_2)} y_2^{-m_2} -$$

$$y_1^{-m_1} y_2^{-m_2} + y_1^{(m_1 + m_2)} y_2^{m_1} + y_1^{m_1} y_2^{(m_1 + m_2)}.$$

The polynomials $e^+$ and $e^-$ given in (2.6) of [13](III) coincide with those in (42) whenever the correspondence $\sigma = 2\pi x_1$, $\tau = 2\pi x_2$ is set up. So, both the orbit functions polynomials of $A_2$ and $e^\pm$ are orthogonal on the interior of Steiner’s hypocycloid.

It is noteworthy that the regular tessellation of the plane by equilateral triangles considered in [13] is the standard tiling of the weight lattice of $A_2$. The fundamental region $R$ of [13]
the elementary symmetric polynomials, see [8, 9].

Furthermore, continuing the comparison with the paper [13], we want to point out that orbit functions are eigenfunctions not only of the Laplace operator written in the appropriate basis, e.g. in $\omega$-basis, the corresponding eigenvalues bring $-4\pi^2\langle\lambda, \lambda\rangle$, where $\lambda$ is the representative from the dominant Weyl chamber, which labels the orbit function. This property holds not only for the Lie algebra $A_n$ and its Laplace operator, but also for the differential operators built from the elementary symmetric polynomials, see [8, 9].

An independent approach to the polynomials in two variables is proposed in [22], and the generalization of classical Chebyshev polynomials to the case of several variables is also presented in [4]. A detailed comparison would be a major task because the results are not explicit and contain no examples of polynomials.

5 Multivariate exponential functions

In this section, we consider one more class of special functions, which, as it will be shown, are closely related to orbit functions of $A_n$. Such a relation allows us to view orbit functions in the orthonormal basis, and to represent them in the form of determinants and permanents. At the same time, we obtain the straightforward procedure for constructing polynomials from multivariate exponential functions.

**Definition 4.** [12] For a fixed point $\lambda = (l_1, l_2, \ldots, l_{n+1})_\epsilon$, such that $l_1 \geq l_2 \geq \cdots \geq l_{n+1}$, $\sum_{k=1}^{n+1} l_k = 0$, the symmetric multivariate exponential function $D^+_\lambda$ of $x = (x_1, x_2, \ldots, x_{n+1})_\epsilon$ is defined as follows

$$D^+_\lambda(x) := \det^+ \begin{pmatrix} e^{2\pi i l_1 x_1} & e^{2\pi i l_2 x_2} & \cdots & e^{2\pi i l_{n+1} x_{n+1}} \\ e^{2\pi i l_2 x_1} & e^{2\pi i l_1 x_2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ e^{2\pi i l_{n+1} x_1} & e^{2\pi i l_{n+1} x_2} & \cdots & e^{2\pi i l_{n+1} x_{n+1}} \end{pmatrix}.$$ (43)

Here, $\det^+$ is calculated as a conventional determinant, except that all of its monomial terms are taken with positive sign. It is also called permanent [14] or antideterminant.

It was shown in [12] that it suffices to consider $D^+_\lambda(x)$ on the hyperplane $x \in H$ (see (2)). Furthermore, due to the following property of the permanent

$$\det^+(a_{ij})_{i,j=1}^m = \sum_{s \in S_m} a_{1,s(1)}a_{2,s(2)}\cdots a_{m,s(m)} = \sum_{s \in S_m} a_{s(1),1}a_{s(2),2}\cdots a_{s(m),m}$$

we have

$$D^+_\lambda(x) = \sum_{s \in S_{n+1}} e^{2\pi i l_1 x_{s(1)}} \cdots e^{2\pi i l_m x_{s(n+1)}} = \sum_{s \in S_{n+1}} e^{2\pi i \langle\lambda, s(x)\rangle} = \sum_{s \in S_{n+1}} e^{2\pi i (s(\lambda), x)}.$$

**Proposition 1.** For all $\lambda, x \in H \subset \mathbb{R}^{n+1}$, we have the following connection between the symmetric multivariate exponential functions in $n + 1$ variables, and $C$ orbit functions of $A_n$, $D^+_\lambda(x) = kC_\lambda(x)$, where $k = \frac{|W_{\lambda}|}{|W|}$, $|W|$ and $|W_{\lambda}|$ are sizes of the Weyl group and Weyl orbit respectively. In particular, for generic points, $k = 1$.

**Proof.** Proof follows from the definitions of the functions $C$ and $D^+$ (definitions 1 and 4 respectively) and properties of orbit functions formulated in Section 2.5.2. □
**Definition 5.** [12] For a fixed point \( \lambda = (l_1, l_2, \ldots, l_{n+1})_e \), such that \( l_1 \geq l_2 \geq \cdots \geq l_{n+1} \), \( \sum_{k=1}^{n+1} l_k = 0 \), the antisymmetric multivariate exponential function \( D^-_\lambda \) of \( x = (x_1, x_2, \ldots, x_{n+1})_e \in \mathcal{H} \) is defined as follows

\[
D^-_\lambda (x) := \det \begin{pmatrix}
e^{2\pi i l_1 x_1} & e^{2\pi i l_1 x_2} & \cdots & e^{2\pi i l_1 x_{n+1}} \\
e^{2\pi i l_2 x_1} & e^{2\pi i l_2 x_2} & \cdots & e^{2\pi i l_2 x_{n+1}} \\
\vdots & \vdots & \ddots & \vdots \\
e^{2\pi i l_{n+1} x_1} & e^{2\pi i l_{n+1} x_2} & \cdots & e^{2\pi i l_{n+1} x_{n+1}} 
\end{pmatrix} = \sum_{s \in S_{n+1}} (\text{sgn } s)e^{2\pi i (s(\lambda), x)},
\]

(44)

where \text{sgn} is the permutation sign.

**Proposition 2.** For all generic points \( \lambda \in \mathcal{H} \subset \mathbb{R}^{n+1} \), we have the following connection \( D^-_\lambda (x) = S_\lambda(x) \).

The antisymmetric multivariate exponential functions \( D^- \), and \( S \) orbit functions, equal zero for non-generic points.

**Proof.** Proof directly follows from the definitions of functions \( S \) and \( D^- \) (definitions 2 and 5 respectively), and properties of \( S \) functions formulated in Section 2.5.2.

**Definition 6.** [7] The alternating multivariate exponential function \( D^{\text{Alt}}_\lambda(x) \), for \( x = (x_1, \ldots, x_{n+1})_e \), \( \lambda = (l_1, \ldots, l_{n+1})_e \), is defined as the function

\[
D^{\text{Alt}}_\lambda (x) := \text{sdet} \begin{pmatrix}
e^{2\pi i l_1 x_1} & e^{2\pi i l_1 x_2} & \cdots & e^{2\pi i l_1 x_{n+1}} \\
e^{2\pi i l_2 x_1} & e^{2\pi i l_2 x_2} & \cdots & e^{2\pi i l_2 x_{n+1}} \\
\vdots & \vdots & \ddots & \vdots \\
e^{2\pi i l_{n+1} x_1} & e^{2\pi i l_{n+1} x_2} & \cdots & e^{2\pi i l_{n+1} x_{n+1}} 
\end{pmatrix},
\]

(45)

where \( \text{Alt}_{n+1} \) is the alternating group (even subgroup of \( S_{n+1} \)) and

\[
\text{sdet} \left( e^{2\pi i l_k x_k} \right)_{j,k=1}^{n+1} := \sum_{w \in \text{Alt}_{n+1}} e^{2\pi i l_1 x_{w(1)} + 2\pi i l_2 x_{w(2)} + \cdots + 2\pi i l_{n+1} x_{w(n+1)}} = \sum_{w \in \text{Alt}_{n+1}} e^{2\pi i (\lambda, w(x))}.
\]

Here, \((\lambda, x)\) denotes the scalar product in the \((n + 1)\)-dimensional Euclidean space.

Note that \( \text{Alt}_m \) consists of even substitutions of \( S_m \), and is usually denoted as \( A_m \); here we change the notation in order to avoid confusion with simple Lie algebra \( A_n \) notations.

It was shown in [7] that it is sufficient to consider the function \( D^{\text{Alt}}_\lambda(x) \) on the hyperplane \( \mathcal{H} : x_1 + x_2 + \cdots + x_{n+1} = 0 \) for \( \lambda \), such that \( l_1 \geq l_2 \geq l_3 \geq \cdots \geq l_{n+1} \).

Alternating multivariate exponential functions are obviously connected with symmetric and antisymmetric multivariate exponential functions. This connection is the same as that of the cosine and sine, with the exponential function of one variable \( D^{\text{Alt}}_\lambda(x) = \frac{1}{2}(D^+_\lambda(x) + D^-_\lambda(x)) \).

**Proposition 3.** For all generic points \( \lambda \in \mathcal{H} \subset \mathbb{R}^{n+1} \), the following relation between the alternative multivariate exponential functions \( D^{\text{Alt}}_\lambda \) and \( E_\lambda \)-orbit functions of \( A_n \) holds true:

\[
D^{\text{Alt}}_\lambda(x) = E_\lambda(x).
\]

For non-generic points \( \lambda \), we have \( E_\lambda(x) = C_\lambda(x) \) and, therefore, \( E_\lambda(x) = kD^+_\lambda(x) \), where

\[
k = \frac{|W_\lambda|}{|W_\lambda|}.
\]

**Proof.** Proof directly follows from definitions 3 and 6, from the relation \( E = \frac{1}{2}(C + S) \), and from the properties of orbit functions formulated in Section 2.5.2. \(\square\)
6 Concluding remarks

1. Consequences of the identification of W-invariant orbit functions of compact simple Lie groups and multivariable Chebyshev polynomials merit further exploitation. It is conceivable that Lie theory may become a backbone of a segment of the theory of orthogonal polynomials of many variables.

Some of the properties of orbit functions translate readily into properties of Chebyshev polynomials of many variables. However there are other properties whose discovery from the theory of polynomials is difficult to imagine. As an example, consider the decomposition of the Chebyshev polynomial of the second kind into the sum of Chebyshev polynomials of the first kind. In one variable, it is a familiar problem that can be solved by elementary means. For two and more variables, the problem turns out to be equivalent to a more general question about representations of simple Lie groups. In general the coefficients of that sum are the dominant weight multiplicities. Again, simple specific cases can be worked out, but a sophisticated algorithm is required to deal with it in general [16].

In order to provide a solution for such a problem, extensive tables have been prepared [1] (see also references therein).

2. Our approach to the derivation of multidimensional orthogonal polynomials hinges on the knowledge of appropriate recursion relations. The basic mathematical property underlying the existence of the recursion relation is the complete decomposability of products of the orbit functions. Numerous examples of the decompositions of products of orbit functions, involving also other Lie groups than SU(n), were shown elsewhere [8, 9]. An equivalent problem is the decomposition of products of Weyl group orbits [5].

3. Possibility to discretize the polynomials is a consequence of the known discretization of orbit functions. For orbit functions it is a simpler problem, in that it is carried out in the real Euclidean space \( \mathbb{R}^n \). In principle, it carries over to the polynomials. But variables of the polynomials happen to be on the maximal torus of the underlying Lie group. Only in the case of \( A_1 \), the variables are real (the imaginary unit multiplying the \( S \)-functions can be normalized away). For \( A_n \) with \( n > 1 \) the functions are complex valued. Practical aspects of discretization deserve to be thoroughly investigated.

4. For simplicity of formulation, we insisted throughout this paper that the underlying Lie group be simple. The extension to compact semisimple Lie group and their Lie algebras is straightforward. Thus, orbit functions are products of orbit functions of simple constituents, and different types of orbit functions can be mixed.

5. Polynomials formed from \( E \)-functions by the same substitution of variables should be equally interesting once \( n > 1 \). We know of no analogs of such polynomials in the standard theory of polynomials with more than one variable. Intuitively, they would be formed as ‘halves’ of Chebyshev polynomials although their domain of orthogonality is twice as large as that of Chebyshev polynomials [11].

6. Orbit functions have many other properties [8,9,11] that can now be rewritten as properties of Chebyshev polynomials. Let us point out just that they are eigenfunctions of appropriate Laplace operators with known eigenvalues.

7. Notions of multivariate trigonometric functions [10] lead us to the idea of new, yet to be defined classes of \( W \)-orbit functions based on trigonometric sine and cosine functions, hence also to new types of polynomials.
8. Analogs of orbit functions of Weyl groups can be introduced also for the finite Coxeter groups that are not Weyl groups of a simple Lie algebra. Many of the properties of orbit functions extend to these cases. Only their orthogonality, continuous or discrete, has not been shown so far.

9. Our choice of the \( n \)-dimensional subspace \( \mathcal{H} \) in \( \mathbb{R}^{n+1} \) by requirement (2), is not the only possibility. A reasonable alternative appears to be setting \( l_{n+1} = 0 \) (orthogonal projection on \( \mathbb{R}^n \)).

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