BEHAVIOR OF SOLUTION OF STOCHASTIC DIFFERENCE EQUATION WITH CONTINUOUS TIME UNDER ADDITIVE FADING NOISE

LEONID SHAIKHET
Department of Mathematics, Ariel University
Ariel 40700, Israel
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ABSTRACT. Effect of additive fading noise on a behavior of the solution of a stochastic difference equation with continuous time is investigated. It is shown that if the zero solution of the initial stochastic difference equation is asymptotically mean square quasistable and the level of additive stochastic perturbations is given by square summable sequence, then the solution of a perturbed difference equation remains to be an asymptotically mean square quasitrivial. The obtained results are formulated in terms of Lyapunov functionals and linear matrix inequalities (LMIs). It is noted that the study of the situation, when an additive stochastic noise fades on the infinity not so quickly, remains an open problem.

1. Introduction Stochastic difference equations with continuous time are very popular in research (see [1-18]). But a problem that is considered here has not yet been encountered in the literature for this type of equations. Effect of additive noise, fading on the infinity, on a behavior of a solution of stochastic difference equation with continuous time is investigated. It is clear that by the presence of an additive noise the zero solution of the initial system ceases to be a solution of the perturbed equation. The question is discussed how quickly this noise must decay in order that the solution of the perturbed equation remains asymptotically mean square quasitrivial. Similar problem for differential and difference equations with multiplicative fading stochastic perturbations was considered by the author in [19-22].

Following [17] consider the necessary below notations, definitions and auxiliary statements.

Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ be a complete probability space, $\{\mathcal{F}_t\}_{t \geq 0}$ be a nondecreasing family of sub-$\sigma$-algebras of $\mathcal{F}$, i.e., $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$, $\mathbb{E}$ be the mathematical expectation with respect to the measure $\mathbb{P}$, $\mathbb{E}_t = \mathbb{E}\{./\mathcal{F}_t\}$ be the conditional mathematical expectation.

Consider the stochastic difference equation with continuous time

$$x(t + \tau) = a_1(t, x(t), x(t - h_1), x(t - h_2), ...) + a_2(t, x(t), x(t - h_1), x(t - h_2), ...)\xi(t + \tau), \quad t > t_0 - \tau,$$

(1.1)

and the initial condition

$$x(\theta) = \phi(\theta), \quad \theta \in \Theta = [t_0 - h, t_0], \quad h = \tau + \sup_{j \geq 1} h_j.$$

(1.2)

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Here \( x(t) \in \mathbb{R}^n, \tau, h_1, h_2, \ldots \) are positive constants, the functionals \( a_1 \in \mathbb{R}^n \) and \( a_2 \in \mathbb{R}^{n \times k} \) satisfy the conditions

\[
|a_1(t, x_0, x_1, x_2, \ldots)|^2 \leq \sum_{j=0}^{\infty} a_{1j}|x_j|^2, \quad \|a_2(t, x_0, x_1, x_2, \ldots)\|^2 \leq \sum_{j=0}^{\infty} a_{2j}|x_j|^2, \\
A = \sum_{l=1}^{2} \sum_{j=0}^{\infty} a_{lj} < \infty,
\]

(1.3)

\(|| \cdot ||_0\) are the Euclidean norm in \( \mathbb{R}^n \) and a matrix norm respectively, \( \phi(\theta), \theta \in \Theta, \) is a \( \mathcal{F}_t \)-measurable function, the perturbation \( \xi(t) \in \mathbb{R}^k \) is a \( \mathcal{F}_t \)-measurable stationary stochastic process such that \( \xi(t) \) is independent on \( \mathcal{F}_s \) for \( s \leq t - \tau, \)

\[
E_t \xi(t + \tau) = 0, \quad E_t \xi(t + \tau)\xi'(t + \tau) = I_k, \quad t > t_0 - \tau,
\]

(1.4)

where \( I_k \in \mathbb{R}^{k \times k} \) is the identical matrix.

The solution of the equation (1.1), (1.2) is a \( \mathcal{F}_t \)-measurable process \( x(t) = x(t; t_0, \phi), \) that equals to the initial function \( \phi(t) \) from (1.2) for \( t \leq t_0 \) and with probability 1 is defined by the equation (1.1) for \( t > t_0. \)

**Definition 1.1.** The zero solution of the equation (1.1), (1.2) is called mean square stable if for any \( \varepsilon > 0 \) and \( t_0 \) there exists a \( \delta = \delta(\varepsilon, t_0) > 0 \) such that

\[
E[x(t; t_0, \phi)]^2 < \varepsilon, \quad \text{for all} \quad t \geq t_0 \text{ if } \|\phi\|^2 = \sup_{\theta \in \Theta} E|\phi(\theta)|^2 < \delta.
\]

**Definition 1.2.** The solution of the equation (1.1), (1.2) is called asymptotically mean square trivial if \( \lim_{t \to \infty} E|x(t; t_0, \phi)|^2 = 0. \)

**Definition 1.3.** The solution of the equation (1.1), (1.2) is called asymptotically mean square quasitrivial if \( \lim_{t \to \infty} E|x(t + j \tau; t_0, \phi)|^2 = 0 \) for each \( t \in [t_0, t_0 + \tau). \)

**Definition 1.4.** The zero solution of the equation (1.1) is called asymptotically mean square stable if it is mean square stable and for each initial function \( \phi \) the solution of the equation (1.1) is asymptotically mean square trivial.

**Definition 1.5.** The zero solution of the equation (1.1) is called asymptotically mean square quasitrivial if it is mean square stable and for each initial function \( \phi \) the solution of the equation (1.1) is asymptotically mean square quasitrivial.

**Remark 1.1.** If the solution of the equation (1.1), (1.2) is asymptotically mean square trivial then it is also asymptotically mean square quasitrivial but the inverse statement is not true (see [17], p.229).

Below it is supposed that the functional \( V(t) = V(t, x(t), x(t - h_1), x(t - h_2), \ldots) \) equals zero if and only if \( x(t) = x(t - h_1) = x(t - h_2) = \ldots = 0, \)

\[
\Delta V(t) = V(t + \tau) - V(t),
\]

the inequality \( P > 0 (P < 0) \) means that \( P \) is a positive (negative) definite matrix.

**Theorem 1.1** [17] Let there exists a nonnegative functional

\[
V(t) = V(t, x(t), x(t - h_1), x(t - h_2), \ldots)
\]

and positive numbers \( c_1, c_2, \) such that

\[
EV(s) \leq c_1 \sup_{\theta \in \Theta} E|\phi(\theta)|^2, \quad s \in (t_0 - \tau, t_0],
\]

\[
E \Delta V(t) \leq -c_2 E|x(t)|^2, \quad t \geq t_0.
\]
Then the zero solution of the equation (1.1), (1.2) is asymptotically mean square quasistable.

2. Additive fading noise. Let us suppose that the equation (1.1) is exposed by additive stochastic perturbations, i.e.,

\[ x(t + \tau) = a_1(t, x(t), x(t - h_1), x(t - h_2), ...) + a_2(t, x(t), x(t - h_1), x(t - h_2), ...)\xi(t + \tau) + b(t)\eta(t + \tau), \]  

(2.1)

where \( b(t) \in \mathbb{R}^{n \times m} \), the perturbation \( \eta(t) \in \mathbb{R}^m \) is a \( \mathcal{F}_t \)-measurable stationary stochastic process, which is independent on \( \mathcal{F}_s \) for \( s \leq t - \tau \) and satisfies the conditions of the type (1.4), the processes \( \eta(t) \) and \( \xi(t) \) are mutually independent.

Note that in difference from the equation (1.1) the equation (2.1) does not have the zero solution.

**Theorem 2.1.** Let there exists a nonnegative functional \( V(t) = V(t, x(t), x(t - h_1), x(t - h_2), ...) \) and positive numbers \( c_1, c_2 \), such that

\[ E\|V(t)\| \leq c_1 \sup_{t \leq s} E|\phi(\theta)|^2, \quad s \in (t_0 - \tau, t_0], \]  

(2.2)

\[ E\Delta V(t) \leq -c_2 E|x(t)|^2 + \gamma(t), \quad t \geq t_0, \]  

(2.3)

where \( \gamma(t) \) is summable for each \( t \in [t_0, t_0 + \tau) \), i.e.,

\[ \sum_{j=0}^{\infty} \gamma(t + j\tau) < \infty. \]  

(2.4)

Then the solution of the equation (2.1), (1.2) is asymptotically mean square quasitrivial.

**Proof.** Rewrite the condition (2.3) in the form

\[ E\Delta V(t + j\tau) \leq -c_2 E|x(t + j\tau)|^2 + \gamma(t + j\tau), \quad t \in [t_0, t_0 + \tau), \quad j = 0, 1, \ldots. \]

Summing this inequality from \( j = 0 \) to \( j = i \), we obtain

\[ EV(t + (i + 1)\tau) - EV(t) \leq -c_2 \sum_{j=0}^{i} E|x(t + j\tau)|^2 + \sum_{j=0}^{i} \gamma(t + j\tau). \]

From this and (2.4) it follows that for each \( t \in [t_0, t_0 + \tau) \)

\[ c_2 \sum_{j=0}^{\infty} E|x(t + j\tau)|^2 \leq EV(t) + \sum_{j=0}^{\infty} \gamma(t + j\tau) < \infty, \]

i.e., the solution of the equation (2.1), (1.2) is mean square summable for each \( t \in [t_0, t_0 + \tau) \) and therefore asymptotically mean square quasitrivial. The proof is completed.

**Remark 2.1** Note that by the condition \( \gamma(t) \equiv 0 \) (which holds if the condition \( b(t) \equiv 0 \) holds) Theorem 2.1 coincides with Theorem 1.1. It means that if the zero solution of the equation (1.1) is asymptotically mean square quasistable then after
appearing of quickly fading additive noise the solution of the equation (2.1) remains asymptotically mean square quasitrivial.

3. Linear system. Consider the linear stochastic difference equation

\[ x(t + \tau) = \sum_{i=0}^{k} A_i x(t - i\tau) + \sum_{i=0}^{k} B_i x(t - i\tau)\xi(t + \tau) + C(t)\eta(t + \tau), \quad t > -\tau, \quad \tau > 0, \]

where \( x(t) \in \mathbb{R}^n \), \( A_i, B_i \in \mathbb{R}^{n \times n} \), \( C(t) \in \mathbb{R}^{n \times m}, \xi(t) \in \mathbb{R} \) and \( \eta(t) \in \mathbb{R}^m \) are mutually independent \( \mathcal{F}_t \)-measurable stationary stochastic processes satisfying conditions of the type (1.4).

Consider the matrix \( \Psi \in \mathbb{R}^{n(k+1) \times n(k+1)} \) such that

\[
\Psi = \begin{bmatrix}
A_0'PA_0 + B_0'PB_0 \quad P + \sum_{i=1}^{k} R_i & A_0'P\Phi(A) + B_0'P\Phi(B) \\
\Phi'(A)P\Phi(A) + \Phi'(B)P\Phi(B) - R
\end{bmatrix},
\]

(3.2)

where \( P, R_1, \ldots, R_k \in \mathbb{R}^{n \times n} \), the sign "*" means a symmetric element of a matrix, "\( \tau \)" is the sign of transpose and

\[
\Phi(A) = [A_1 \ldots A_k] \in \mathbb{R}^{n \times nk}, \quad \Phi(B) = [B_1 \ldots B_k] \in \mathbb{R}^{n \times nk},
\]

(3.3)

Theorem 3.1. Let there exist positive definite matrices \( P, R_1, \ldots, R_k \in \mathbb{R}^{n \times n} \) and the function \( \gamma(t) \), satisfying the condition (2.4), such that for the matrix \( \Psi \), defined in (3.2), (3.3), the LMI \( \Psi < 0 \) holds and \( Tr[C'(t)PC(t)] \leq \gamma(t) \). Then the solution of the equation (3.1) is asymptotically mean square quasitrivial.

Proof. Following the general method of Lyapunov functionals construction [17] we will construct the Lyapunov functional for the equation (3.1) in the form \( V_1 + V_2 \), where \( V_1 = x'(t)Px(t) \), \( P > 0 \), and the additional functional \( V_2 \) will be chosen below. Using the conditions (1.4) for the processes \( \xi(t) \) and \( \eta(t) \) for the functional \( V_1 \) and the equation (3.1) we have

\[
\mathbb{E} \Delta V_1(t) = \mathbb{E}[x'(t + \tau)Px(t + \tau) - x'(t)Px(t)]
\]

\[
= \mathbb{E} \left[ \sum_{i=0}^{k} A_i x(t - i\tau) + \sum_{i=0}^{k} B_i x(t - i\tau)\xi(t + \tau) + C(t)\eta(t + \tau) \right]'
\]

\[
\times P \left[ \sum_{j=0}^{k} A_j x(t - j\tau) + \sum_{j=0}^{k} B_j x(t - j\tau)\xi(t + \tau) + C(t)\eta(t + \tau) \right]
\]

\[
- \mathbb{E} x'(t)Px(t) \]

(3.4)

\[
= \mathbb{E} \left[ \sum_{i=0}^{k} x'(t - i\tau)A_i'P \sum_{j=0}^{k} A_j x(t - j\tau) + \sum_{i=0}^{k} x'(t - i\tau)B_i' \right]
\]

\[
\times P \sum_{j=0}^{k} B_j x(t - j\tau) + Tr[C'(t)PC(t)] - \mathbb{E} x'(t)Px(t).
\]
Putting
\[ y_i(t) = x(t - i\tau), \quad i = 1, \ldots, k, \quad y(t) = \text{col}(y_1(t), \ldots, y_k(t)) \in \mathbb{R}^{uk}, \tag{3.5} \]
we obtain
\[
\sum_{i=0}^{k} x'(t - i\tau)A_i'P \sum_{j=0}^{k} A_jx(t - j\tau) = \left( x'(t)A_0' + \sum_{i=1}^{k} x'(t - i\tau)A_i' \right) P \left( A_0x(t) + \sum_{j=1}^{k} A_jx(t - j\tau) \right) \tag{3.6}
\]
and similarly
\[
\sum_{i=0}^{k} x'(t - i\tau)B_i'P \sum_{j=0}^{k} B_jx(t - j\tau) = x'(t)B_0'PB_0x(t) + 2x'(t)B_0'P\Phi(\Phi(A)y(t)) + y'(t)\Phi'(A)P\Phi(B)y(t) \tag{3.7}
\]
From (3.4), (3.6), (3.7) it follows that
\[
E\Delta V_1(t) = x'(t)(A_0'PA_0 + B_0'PB_0 - P)x(t) + 2x'(t)(A_0'P\Phi(A) + B_0'P\Phi(B))y(t) \tag{3.8}
\]
\[
+ y'(t)(\Phi'(A)P\Phi(\Phi(A) + \Phi'(B)P\Phi(B))y(t) + Tr[C'(t)PC(t)]
\]
For the additional functional \( V_2 = \sum_{l=1}^{k} \sum_{i=1}^{l} x'(t - i\tau)R_ix(t - i\tau), \) \( R_i > 0, \) we have
\[
\Delta V_2(t) = \sum_{l=1}^{k} \left( \sum_{i=1}^{l} x'(t + \tau - i\tau)R_ix(t + \tau - i\tau) \right)
\]
\[
- \sum_{i=1}^{l} x'(t - i\tau)R_ix(t - i\tau)
\]
\[
= \sum_{l=1}^{k} \left( \sum_{i=0}^{l-1} x'(t - i\tau)R_ix(t - i\tau) - \sum_{i=1}^{l} x'(t - i\tau)R_ix(t - i\tau) \right) \tag{3.9}
\]
\[
= \sum_{l=1}^{k} (x'(t)R_ix(t) - x'(t - l\tau)R_lx(t - l\tau)) \]
\[
= x'(t) \sum_{l=1}^{k} R_lx(t) - y'(t)R_lg(t),
\]
where \( R \) and \( y(t) \) are defined respectively in (3.3) and (3.5). As a result from (3.8), (3.9) for the functional \( V = V_1 + V_2 \) we obtain
\[
E\Delta V(t) = [x'(t) y'(t)]\Psi[x'(t) y'(t)]' + Tr[C'(t)PC(t)] \leq [x'(t) y'(t)]\Psi[x'(t) y'(t)]' + \gamma(t).
\]
From this and the LMI $\Psi < 0$ it follows that the constructed above Lyapunov functional $V$ satisfies the conditions (2.2), (2.3). From Theorem 2.1 it follows that the solution of the equation (3.1) is asymptotically mean square quasitrivial. The proof is completed.

4. **Scalar case.** For the scalar linear difference equation

$$
x(t + \tau) = \sum_{l=0}^{k} a_l x(t - l\tau) + \sum_{l=0}^{k} b_l x(t - l\tau) \xi(t + \tau) + \sigma(t) \eta(t + \tau),
$$

where \(t > -\tau, \tau > 0\) \hspace{1cm} (4.1)

the matrix (3.2) with \(P = 1\) is $\Psi_0 \in \mathbb{R}^{(k+1)\times(k+1)}$, where

$$
\Psi_0 = \begin{bmatrix}
a_0^2 + b_0^2 - 1 + \sum_{l=1}^{k} r_l & a_0 a_1 + b_0 b_1 & \ldots & a_0 a_k + b_0 b_k \\
* & a_1^2 + b_1^2 - r_1 & \ldots & a_1 a_k + b_1 b_k \\
* & * & \ldots & * \\
* & * & \ldots & a_k^2 + b_k^2 - r_k
\end{bmatrix}.
$$

So, from Theorem 3.1 we obtain the following statement.

**Corollary 4.1.** Let there exist positive numbers \(r_1, \ldots, r_k\) such that for the matrix $\Psi_0$, defined in (4.2), the LMI $\Psi_0 < 0$ holds and $\sum_{j=0}^{\infty} \sigma^2(t + j\tau) < \infty$ for each $t \geq t_0$.

Then the solution of the equation (4.1) is asymptotically mean square quasitrivial.

To get more simple condition for asymptotic mean square quasitriviality of the solution of the equation (4.1) note that the condition (3.8) with $P = 1$ takes the form

$$
E\Delta V_1(t) = E \left( \sum_{l=0}^{k} a_l x(t - l\tau) \right)^2 + E \left( \sum_{l=0}^{k} b_l x(t - l\tau) \right)^2 + \sigma^2(t) - Ex^2(t).
$$

From this via simple inequalities

$$
\left( \sum_{l=0}^{k} a_l x(t - l\tau) \right)^2 \leq a \sum_{l=0}^{k} |a_l| x^2(t - l\tau),
$$

$$
\left( \sum_{l=0}^{k} b_l x(t - l\tau) \right)^2 \leq b \sum_{l=0}^{k} |b_l| x^2(t - l\tau),
$$

where

$$
a = \sum_{l=0}^{k} |a_l|, \quad b = \sum_{l=0}^{k} |b_l|,
$$

it follows that

$$
E\Delta V_1(t) \leq (a|a_0| + b|b_0| - 1)Ex^2(t) + \sum_{l=1}^{k} (a|a_l| + b|b_l|)Ex^2(t - l\tau) + \sigma^2(t).
$$
Using now the condition (3.9) with \( r_1 = a|a| + b|b| \), for the functional \( V = V_1 + V_2 \) we obtain
\[
E \Delta V(t) \leq -(1 - a^2 - b^2)E x^2(t) + \sigma^2(t).
\]
From this via Theorem 2.1 we obtain the following statement.

**Corollary 4.2.** Let be \( \sum_{j=0}^{\infty} \sigma^2(t + j\tau) < \infty \) for each \( t \geq t_0 \) and \( a, b \) are defined in (4.3). If
\[
a^2 + b^2 < 1
\]
then the solution of the equation (4.1) is asymptotically mean square quasitrivial.

**Remark 4.1.** Note that the condition (4.4) is only simpler than the LMI \( \Psi_0 < 0 \), but it is more conservative. For simplicity let us show this for the case \( k = 1 \). Really, from (4.2) for \( k = 1 \) it follows that the matrix \( \Psi_0 \) is negative definite if and only if
\[
\text{Tr}(\Psi_0) = a_0^2 + a_1^2 + b_0^2 + b_1^2 - 1 < 0,
\]
\[
\text{det}(\Psi_0) = (a_0^2 + b_0^2 - 1 + r)(a_1^2 + b_1^2 - r) - (a_0a_1 + b_0b_1)^2 > 0.
\]
Maximising the \( \text{det}(\Psi_0) \) via \( r = \frac{1}{2} [1 - (a_0^2 + b_0^2) + (a_1^2 + b_1^2)] \), from (4.5) we obtain the inequality
\[
1 > a_0^2 + a_1^2 + b_0^2 + b_1^2 + 2|a_0a_1 + b_0b_1|,
\]
which is equivalent to the condition
\[
1 + 2(|a_0a_1| + |b_0b_1| - |a_0a_1 + b_0b_1|) > a^2 + b^2,
\]
and therefore it is less conservative than (4.4).

**Example 4.1.** Let in the equation (4.1) be \( k = 1 \), \( a_0 = a_1 = b_0 = 0.45, b_1 = -0.45 \). Then via (4.3) \( a = b = 0.9 \) and the condition (4.4) does not hold: \( a^2 + b^2 = 1.62 > 1 \) but the condition (4.6) holds: \( 1 > 4 \times 0.45^2 + 0 = 0.81 \). In Fig. 1 50 trajectories of the solution of the equation (4.1) are shown for the given values of the parameters, \( \tau = 1 \), \( \sigma(t) = m \) for \( t \leq m \) and \( \sigma(t) = m \) for \( t > m, m = 15 \).

All trajectories converge to zero that corresponds to the statement that the solution of the equation (4.1) is asymptotically mean square quasitrivial. Putting \( b_1 = 0.45 \), we obtain that the condition (4.6) coincides with the condition (4.4) and does not hold. Trajectories in this case show absolutely another picture (see Fig. 2). Comparison with Fig.1 allows to say that in this case the trajectories do not converge to zero, i.e., that the solution of (4.1) is not asymptotically mean square quasitrivial.

An interesting question remains about a possibility to weaken the condition of quadratic summability of the function \( \sigma(t) \) that is the rate of fading of an additive stochastic noise. The situation with not square summable \( \sigma(t) \) is shown in Fig. 3. One can see that trajectories do not converge to zero though the observation time has increased essentially. It means that the solution is not asymptotically mean square quasitrivial. So, the question about a possibility to weaken the condition of quadratic summability of the function \( \sigma(t) \) remains to be open.
For Figs 1-3

Fig. 1. 50 trajectories (green) of the solution of the equation (1.1) by \( k = 1, \tau = 1, a_0 = a_1 = b_0 = 0.45, b_1 = -0.45 \) and \( \sigma(t) \) (red)

Fig. 2. The same as Fig. 1 besides of \( b_1 = 0.45 \)

Fig. 3. The same as Fig. 1 besides of \( \sigma(t) = \frac{m}{\sqrt{t + \tau - m}}, t > m \)

**Conclusions.** Stochastic difference equation with continuous time is considered. It is supposed that the zero solution of the equation under consideration is asymptotically mean square quasistable and the equation is influenced by additive fading stochastic noise. It is shown that if the level of this noise fades on the infinity
quickly enough, in particular, if it is given by square summable function, then the solution of the perturbed equation remains asymptotically mean square qua-trivial.

The open problem remains for future investigation: what will be the behavior of a solution of the perturbed equation if an additive noise will fade on the infinity not so quickly. For instance, if the level of an additive noise converges to zero but is not square summable. Another problem for future research: an extension of the results obtained for the system (3.1) on a system with non commensurated delays.

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*E-mail address:* leonid.shaikhet@usa.net