The $\eta$–meson light nucleus resonances and quasi–bound states

S. A. Rakityansky$^{1,2}$, S. A. Sofianos$^1$, M. Braun$^1$, V. B. Belyaev$^2$, and W. Sandhas$^3$

$^1$ Physics Department, University of South Africa, P.O.Box 392, Pretoria 0001, South Africa
$^2$ Joint Institute for Nuclear Research, Dubna, 141980, Russia
$^3$ Physikalisches Institut, Universität Bonn, D-53115 Bonn, Germany

(October 23, 2018)

Abstract

The position and movement of poles of the amplitude for elastic $\eta$–meson scattering off the light nuclei $^2$H, $^3$H, $^3$He, and $^4$He are studied. It is found that, within the existing uncertainties for the elementary $\eta N$ interaction, all these nuclei can support a quasi–bound state. The values of the $\eta$–nucleus scattering lengths corresponding to the critical $\eta N$–interaction which produces a quasi–bound state are given.

PACS numbers: 25.80.-e, 21.45.+v, 25.10.+s

Since meson factories cannot produce $\eta$–meson beams, these particles are available for experimental investigations only as products of certain nuclear reactions where they appear as final–state particles. Therefore, final–state interaction effects are the only source of information about the $\eta$–meson interaction with nucleons. In this connection, $\eta$–nucleus systems can play an important role in investigating the $\eta N$–dynamics, especially if they can form quasi–bound states. In this case, the final–state $\eta$–mesons can be trapped for a relatively long time, and thus the properties of the $\eta N$–interaction can be studied.

Estimations, obtained in the framework of the optical model approach [1,2], put a lower bound on the atomic number $A$ for which an $\eta$–nucleus bound state could exist, namely $A \geq 12$. In Ref. [3], the formation of $\eta$–nucleus states has been investigated, using the standard Green’s function method of many–body problems. There it was found that an $\eta^{16}O$ bound state should be possible. Experimentally the cross sections of pion-collisions with lithium, carbon, oxygen, and aluminum, however, gave no evidence for the existence of $\eta$ bound states with these nuclei [4].

A new theoretical analysis of the problem [3] predicted a binding of the $\eta$–meson to $^{12}C$ and heavier nuclei, however, with rather large widths. The formation of an $\eta^4He$ bound state was studied in a more recent work by Wycech et al. [4], using a modified multiple scattering theory. These authors obtained a comparatively large negative value for the real part of the $\eta$–nucleus scattering length, which was interpreted as an indication that an $\eta$–nucleus bound state could exist. We note that previous results of ours, concerning the $\eta$ scattering lengths with light nuclei [4,5], showed that the $\eta – ^4He$ scattering length can have an even
larger (negative) real part than that of Ref. [6].

In Ref. [10], a preliminary investigation on the possibility of η–meson binding in the $d$, $t$, $^3He$, and $^4He$ systems was made within the framework of the Finite-Rank Approximation (FRA) of the nuclear Hamiltonian [11,12]. The FRA approach treats the motion of the projectile (η–meson) and of the nucleons inside the nucleus separately. As a result the internal dynamics of the nucleus enters the theory only via the nuclear wave function. In [10], these wave functions were approximated by simple Gaussian forms, which reproduce the nuclear sizes only. In the present work, we perform calculations with more realistic nuclear wave functions, obtained via the so–called Integro–Differential Equation Approach (IDEA) [13–17]. We study, in particular, the position and movement of poles of the elastic amplitude of η–meson scattering off the light nuclei $^2H$, $^3H$, $^3He$, and $^4He$.

The approximate few–body equations in the FRA approach enable us to calculate the η–nucleus T–matrix

$$T(\vec{k}', \vec{k}; z) = <\vec{k}', \psi_0|T(z)|\vec{k}, \psi_0>, \quad (1)$$

at any complex energy. That is, we can locate the poles of the T–matrix in the complex momentum plane $p = \sqrt{2\mu z}$. Here, $\vec{k}$ is the η–nucleus momentum, $z$ the total energy of the system, $\mu$ the η–nucleus reduced mass, and $\psi_0$ the nuclear ground-state wave function.

For the low energies and the light nuclei with only one bound state, being considered, it appears justified to approximate the target Hamiltonian $H_A$ by its discrete spectrum

$$H_A \approx \mathcal{E}_0|\psi_0><\psi_0|. \quad (2)$$

Here $|\psi_0>$ stands for the $^2H$, $^3H$, $^3He$, $^4He$ bound states, respectively, and $\mathcal{E}_0$ for the corresponding binding energies.

As a result, we obtain [8] for the T–matrix the following equation

$$T(z) = \sum_{i=1}^A T^0_i(z) + \sum_{i=1}^A T^0_i(z)|\psi_0> \frac{\mathcal{E}_0}{(z - H_0)(z - H_0 - \mathcal{E}_0)} <\psi_0|T(z), \quad (3)$$

where $H_0$ is the η–nucleus kinetic energy operator, $A$ the number of nucleons. The $T^0_i(z)$ are Faddeev–type components of an auxiliary T-operator, which obey the system of coupled equations

$$T^0_i(z) = t_i(z) + t_i(z)\frac{1}{(z - H_0)}\sum_{j \neq i} T^0_j(z) . \quad (4)$$

Here $t_i$ describes the scattering of the η–meson off a nucleon at point $\vec{r}_i$, where $\vec{r}_i$ is the vector from the nuclear center of mass, which can be expressed in terms of the relative Jacobi vectors $\{\vec{r}\}$ of the nucleons. In mixed representation, the operator $t_i$ is given by

$$t_i(\vec{k}', \vec{k}; \vec{r}; z) = t_{\eta N}(\vec{k}', \vec{k}; z) \exp \left[i(\vec{k} - \vec{k}') \cdot \vec{r}_i\right], \quad (5)$$

where $t_{\eta N}$ is the η–nucleon overlap integral.
with \( t_{\eta N}(\vec{k}', \vec{k}; z) \) being the off-shell \( \eta N \) amplitude.

Thus, to calculate the T–matrix (1) for any fixed value of the complex parameter \( z = p^2/2\mu \), we have to proceed in three steps. First, the coupled integral equations

\[
T^0_i(\vec{k}', \vec{k}; z) = t_i(\vec{k}', \vec{k}; z) + \int \frac{d^3k''}{(2\pi)^3} \, t_i(\vec{k}', \vec{k}''; \vec{r}; z) \sum_{j \neq i} T^0_j(\vec{k}'', \vec{k}; z) \tag{5}
\]

are to be solved for a number of points \( \vec{r} \) in configuration space, sufficient to perform in a second step the integration

\[
<\vec{k}', \psi_0| \sum_{i=1}^A T^0_i(z)|\vec{k}, \psi_0> = \int d^3r |\psi_0(r)|^2 \sum_{i=1}^A T^0_i(\vec{k}', \vec{k}; z). \tag{6}
\]

Having determined these matrix elements, it remains, as a final step, to solve the integral equation

\[
T(\vec{k}', \vec{k}; z) = <\vec{k}', \psi_0| \sum_{i=1}^A T^0_i(z)|\vec{k}, \psi_0> = \mathcal{E}_0 \int \frac{d^3k''}{(2\pi)^3} \frac{<\vec{k}', \psi_0| \sum_{i=1}^A T^0_i(z)|\vec{k}'', \psi_0>}{(z - \frac{k''^2}{2\mu})(z - \mathcal{E}_0 - \frac{k''^2}{2\mu})} T(\vec{k}'', \vec{k}; z). \tag{7}
\]

Note that after partial–wave decomposition both equations (5) and (7) become one–dimensional. As an input information, we need the ground-state wave functions \( \psi_0 \) of the nuclei involved and the two-body T–matrix \( t_{\eta N} \).

The \( \psi_0(\vec{r}) \) for \( A=3 \) and \( A = 4 \) were obtained by means of the IDEA [13]. In this method the \( A \)–body bound state wave function is expanded in Faddeev-type components,

\[
\Psi(\vec{r}) = \sum_{i<j \leq A} \psi_{ij}(\vec{r}), \tag{8}
\]

given as solutions of

\[
(T - E)\psi_{ij}(\vec{r}) = -V(r_{ij}) \sum_{k<l \leq A} \psi_{kl}(\vec{r}), \tag{9}
\]

where \( \vec{r}_{ij} = \vec{r}_i - \vec{r}_j \). The IDEA is then introduced using the ansatz

\[
\psi_{ij}(\vec{r}) = H_{[L_m]}(\vec{r})P(\zeta_{ij}, \rho)/\rho^{(D-1)/2}, \tag{10}
\]

with \( \rho = \left[ 2/A \sum_{ij} r_{ij}^2 \right]^{1/2} \) being the hyperradius, \( D = 3(A-1) \), and \( H_{[L_m]}(\vec{r}) \) the harmonic polynomial of minimal degree \( [L_m] \) [18]. For \([L_m] = 0 \) the IDEA reads

\[
\left[ T + \frac{A(A-1)}{2} V_0(\rho) - E \right] P(\zeta_{ij}, \rho)/\rho^{(D-1)/2} = -[V(r_{ij}) - V_0(\rho)] \sum_{k<l \leq A} P(\zeta_{kl}, \rho)/\rho^{(D-1)/2}, \tag{11}
\]
where $V_0(\rho)$ is the so-called hypercentral potential [18]. Projecting Eq. (11) onto the $r_{ij}$-space provides us, for spin dependent nucleon-nucleon potentials, with two coupled integrodifferential equations for the symmetric $S$ and mixed symmetric $S'$ components of the function $P^n(\zeta_{ij}, \rho)$, $n = S, S'$. More details and explicit equations are found in Refs. [13,15].

For the nuclear ground states, we use the fully symmetric S-wave components obtained with the semi–realistic Malfliet–Tjon I–III (MT I–III) nucleon-nucleon potential [19]. The corresponding two-, three-, and four-body binding energies are 2.272 MeV, 8.936 MeV, and 30.947 MeV, while the root mean square (r.m.s.) radii are 1.976 fm, 1.685 fm and 1.431 fm, respectively. The omission of Coulomb effects and of the mixed symmetry components makes $^3H$ and $^3He$ indistinguishable. In order to compensate partly for this omission, we use in Eq. (7) the experimental values for masses and binding energies of the nuclei [20].

At low energies the $\eta N$ interaction is dominated by the $N^*(1535)\ S_{11}$ - resonance. For the $\eta N$–amplitude we, therefore, choose the separable form

$$t_{\eta N}(k', k; z) = \frac{\lambda}{(k'^2 + \alpha^2)(z - E_0 + i\Gamma/2)(k^2 + \alpha^2)}$$

(12)

with $E_0 = 1535$ MeV $- (m_N + m_\eta)$ and $\Gamma = 150$ MeV [21]. To find the range parameter $\alpha$, we use the results of Refs. [22,23]. There the same $\eta N \rightarrow N^*\$ vertex function $(k^2 + \alpha^2)^{-1}$ was employed, and $\alpha$ was determined via a two-channel fit to the $\pi N \rightarrow \pi N$ and $\pi N \rightarrow \eta N$ experimental data.

Three different values for the range parameter $\alpha$ are available in the literature, namely, $\alpha = 2.357$ fm$^{-1}$ [22], $\alpha = 3.316$ fm$^{-1}$ [23], and $\alpha = 7.617$ fm$^{-1}$ [22]. Since there is no criterion for singling out one of them, we use all three in our calculations. The remaining parameter $\lambda$ is chosen to provide the correct zero-energy on-shell limit, i.e., to reproduce the $\eta N$ scattering length $a_{\eta N}$,

$$t_{\eta N}(0, 0, 0) = -\frac{2\pi}{\mu_{\eta N}} a_{\eta N}.$$  

(13)

Different analyses provided values for the real part $Re\ a_{\eta N}$ in the range $0.27 \div 0.98$ fm and for the imaginary part $Im\ a_{\eta N}$ in the range $0.19 \div 0.37$ fm [24]. To examine at which value of $a_{\eta N}$ within the above ranges an $\eta$–nucleus bound state exists, we parametrize the scattering length as follows

$$a_{\eta N} = (g 0.55 + ig' 0.30) \text{ fm},$$

(14)

where $g$ and $g'$ are adjustable parameters.

Since $a_{\eta N}$ is complex, the $\eta$–nucleus Hamiltonian is non–Hermitian and its eigenvalues are generally complex. In this case, eigenvalues attributed to resonances and quasi–bound states are located in the second–quadrant of the complex $p$–plane [20]. The energy $E_0 = p_0^2/2\mu$ corresponding to a pole at $p = p_0$,

$$E_0 = \frac{1}{2\mu} \left[(Re\ p_0)^2 - (Im\ p_0)^2 + 2i(Re\ p_0)(Im\ p_0)\right],$$

(15)
has a negative real part, \( \text{Re} E_0 < 0 \), only if \( p_0 \) is above the diagonal of this quadrant. Such a pole is related to a quasi-bound state. For \( p_0 \) below the diagonal we have \( \text{Re} E_0 > 0 \) and the pole is then attributed to a resonance. Therefore this diagonal is critical: when crossing it from below a resonance pole becomes a quasi-bound pole.

Fixing \( g \) and \( g' \) of Eq. (14) to \( g = g' = 1 \) and varying the complex momentum \( p = \sqrt{2\mu z} \), we located the poles close to the origin \( p = 0 \). The results obtained are given in Table 1. For one choice of the range parameter, namely \( \alpha = 2.357 \text{ fm}^{-1} \), the positions of the poles found are shown by the open circles in Fig. 1. It is seen that for the \( \eta d \), \( \eta t \), and \( \eta^3\text{He} \) systems, these poles lie in the resonance region, while for the \( \eta^4\text{He} \) system the pole is in the quasi-bound region.

Increasing \( g \) while keeping \( g' = 1 \), the resonance poles are moving towards, and finally cross, the diagonal. In the deuteron case, the corresponding trajectory is depicted in Fig. 1 by the solid curve which crosses the diagonal when \( g = 1.6536 \).

To find the relationship of poles above the diagonal to physical bound states, we gradually removed the imaginary part of \( a_{\eta N} \) by fixing \( g \) and decreasing \( g' \) in Eq. (14) to zero. The imaginary part of the Breit-Wigner factor in Eq. (12) was also decreased, using the same parameter \( g' \), so that it goes over into \( (z - E_0 + ig'\Gamma/2)^{-1} \). For \( g' = 0 \) the Hamiltonian becomes Hermitian and, hence, the bound state poles in this case must be on the positive imaginary axis. The dashed curve in Fig. 1 is the trajectory of the \( \eta d \) bound state pole (with \( g = 2 \)) when \( g' \) decreases from 1 to 0. It is seen that the final position of the pole lies on the positive imaginary axis. This supports our interpretation of poles above the diagonal as quasi-bound states.

By varying the enhancing factor \( g \) for each of the \( \eta \)--nucleus systems under consideration, we found the values which generate quasi-bound states on the diagonal. They are given in the Table 2. These values correspond to an \( \eta N \) attraction, which generates an \( \eta \)--nucleus binding with \( \text{Re} E_0 = 0 \). Further increase of \( g \) moves the poles up and to the right, enhancing the binding and reducing the widths of the states. The value of \( \text{Re} a_{\eta N} \) which provides the critical binding lies within the range \([0.27, 0.98] \text{ fm}\) used in the literature. Therefore, an \( \eta \)--nucleus quasi-bound states may exist for \( A \geq 2 \). If this is not the case, then at least near-threshold resonances (poles just below the diagonal) could exist. However, as can be seen in Table 1 and 2, the widths of such quasi-bound and resonance states could be small only for the \( \eta^4\text{He} \) system.

In conclusion, we have shown that the uncertainties in the \( \eta N \) scattering length allow
for choices of parameters in the $\eta N$-amplitude that may generate poles in the $\eta$-nucleus amplitudes considered, which can be attributed to quasi-bound states.

ACKNOWLEDGMENTS

Financial support from the University of South Africa and the Joint Institute for Nuclear Research, Dubna, is appreciated.
REFERENCES

[1] Q. Haider and L. C. Liu, Phys. Lett., 172 B, 257 (1986).
[2] L. C. Liu and Q. Haider, Phys. Rev., C 34, 1845 (1986).
[3] G. L. Li, W. K. Cheng, and T. T. S. Kuo, Phys. Lett., 195 B, 515 (1987).
[4] R. E. Chrien et al., Phys. Rev. Lett., 60, 2595 (1988).
[5] H. C. Chiang, E. Oset, and L. C. Liu, Phys. Rev. C 44, 738 (1991).
[6] S. Wycech, A. M. Green, and J. A. Niskanen, preprint of University of Helsinki: HU–TFT–95–8 (1995).
[7] S. A. Rakityansky, S. A. Sofianos, and V. B. Belyaev, in Symposium on Effective Interactions in Quantum Systems, Ed. S. A. Sofianos, December 1994, UNISA, Pretoria.
[8] S. A. Rakityansky, S. A. Sofianos, W. Sandhas, and V. B. Belyaev, Physics Letter B 359, 33 (1995).
[9] V. B. Belyaev, S. A. Rakityansky, S. A. Sofianos, W. Sandhas, and M. Braun, Few–Body Systems Suppl. 8, 312 (1995).
[10] S. A. Rakityansky, S. A. Sofianos, V. B. Belyaev, and W. Sandhas, in: The International Conference Mesons and Light Nuclei (MESONS–95), Prague, 3–7 July 1995; In press in Few–Body Systems Suppl.
[11] V. B. Belyaev and J. Wrzecionko, Sov. Journal of Nucl. Phys. 28, 78 (1978).
[12] V. B. Belyaev, in: Lectures on the theory of few-body systems, Springer-Verlag, Heidelberg, 1990.
[13] M. Fabre de la Ripelle, H. Fiedeldey, and S. A. Sofianos, Phys. Rev. C 38, 449 (1988).
[14] M. Fabre de la Ripelle, H. Fiedeldey, and S. A. Sofianos, Few-Body Systems 6, 157 (1989).
[15] W. Oehm, S. A. Sofianos, H. Fiedeldey, and M. Fabre de la Ripelle, Phys. Rev. C 44, 81 (1991).
[16] W. Oehm, S. A. Sofianos, H. Fiedeldey, and M. Fabre de la Ripelle, Phys. Rev. C 43, 25 (1991).
[17] W. Oehm, S. A. Sofianos, H. Fiedeldey, and M. Fabre de la Ripelle, Phys. Rev. C 42, 2322 (1990).
[18] M. Fabre de la Ripelle, in Models and Methods in Few-Body Physics, Vol. 273 of Lecture Notes in Physics, edited by L.S. Fereira, A. C. Fonseca, and L. Streit (Springer-Verlag, New York, 1987), pp 283-323.
[19] R. A. Malfliet and J.A. Tjon, Nucl. Phys. A 127, 161 (1969); Ann. Phys. (N.Y.) 61, 425 (1970).
[20] A. H. Wapstra and G. Audi, Nucl. Phys. 432, 1 (1985).
[21] Particle Data Group, Phys. Rev. D 50(3), 1319 (1994).
[22] R. S. Bhalerao and L. C. Liu, Phys. Rev. Lett., 54, 865 (1985).
[23] C. Bennhold and H. Tanabe, Nucl. Phys., A 530, 625 (1991).
[24] M. Batinic, A. Svarc, Los-Alamos e-print archive: nucl-th/9503020, (1995),
[25] C. Wilkin, Phys. Rev., C 47, R938 (1993).
[26] W. Cassing, M. Stingl, and A. Weiguny, Phys. Rev., C 26, 22 (1982).
Table 1. Positions of poles $p_0 = \sqrt{2\mu E_0}$ of the $\eta$–nucleus amplitudes with $g = g' = 1$ for the three values of the range parameter $\alpha$.

|   | $p_0$ (fm$^{-1}$) | $E_0$ (MeV) | $\alpha$ (fm$^{-1}$) |
|---|-----------------|-------------|---------------------|
| $\eta d$ | $-0.90259 + i0.35870$ | $31.456 - i29.691$ | 2.357 |
|     | $-0.84594 + i0.32195$ | $28.061 - i24.976$ | 3.316 |
|     | $-0.82460 + i0.30423$ | $26.935 - i23.006$ | 7.617 |
| $\eta t$ | $-0.56045 + i0.23859$ | $10.906 - i11.341$ | 2.357 |
|     | $-0.55511 + i0.26826$ | $10.015 - i12.630$ | 3.316 |
|     | $-0.51725 + i0.27896$ | $8.0456 - i12.238$ | 7.617 |
| $\eta^3He$ | $-0.54692 + i0.24478$ | $10.143 - i11.354$ | 2.357 |
|     | $-0.50815 + i0.30402$ | $7.0305 - i13.102$ | 3.316 |
|     | $-0.48310 + i0.33948$ | $5.0099 - i13.909$ | 7.617 |
| $\eta^4He$ | $-0.16504 + i0.27876$ | $-2.0540 - i3.7447$ | 2.357 |
|     | $-0.20215 + i0.38726$ | $-4.4403 - i6.3718$ | 3.316 |
|     | $-0.25931 + i0.45846$ | $-5.8175 - i9.6766$ | 7.617 |

Table 2. The parameter $g$ generating the $\eta$–nucleus amplitude poles $p_0 = \sqrt{2\mu E_0}$ on the diagonal for the three values of the range parameter $\alpha$ and $g' = 1$.

|   | $g$ | $p_0$ (fm$^{-1}$) | $E_0$ (MeV) | $\alpha$ (fm$^{-1}$) |
|---|-----|-----------------|-------------|---------------------|
| $\eta d$ | 1.6536 | $-0.32527 + i0.32527$ | $-i9.7026$ | 2.357 |
|     | 1.5605 | $-0.33541 + i0.33541$ | $-i10.317$ | 3.316 |
|     | 1.5260 | $-0.33670 + i0.33670$ | $-i10.397$ | 7.617 |
| $\eta t$ | 1.3624 | $-0.33515 + i0.33515$ | $-i9.5266$ | 2.357 |
|     | 1.3055 | $-0.35190 + i0.35190$ | $-i10.503$ | 3.316 |
|     | 1.2436 | $-0.35186 + i0.35186$ | $-i10.500$ | 7.617 |
| $\eta^3He$ | 1.3306 | $-0.34034 + i0.34034$ | $-i9.8239$ | 2.357 |
|     | 1.2171 | $-0.36267 + i0.36267$ | $-i11.155$ | 3.316 |
|     | 1.1421 | $-0.37631 + i0.37631$ | $-i12.010$ | 7.617 |
| $\eta^4He$ | 0.86222 | $-0.20641 + i0.20641$ | $-i3.4679$ | 2.357 |
|     | 0.80813 | $-0.26522 + i0.26522$ | $-i5.7255$ | 3.316 |
|     | 0.79578 | $-0.35215 + i0.35215$ | $-i10.094$ | 7.617 |

Table 3. The $\eta$–nucleus scattering lengths for the parameter $g$ of Table 2, which generate the condition for binding ($\Re E = 0$).

|   | $\alpha = 2.357$ (fm$^{-1}$) | $\alpha = 3.316$ (fm$^{-1}$) | $\alpha = 7.617$ (fm$^{-1}$) |
|---|-----------------|-------------|---------------------|
| $\eta d$ | $0.171 + i5.99$ | $-0.198 + i4.57$ | $-0.318 + i3.52$ |
| $\eta t$ | $-3.65 + i3.49$ | $-2.91 + i3.02$ | $-2.19 + 2.70$ |
| $\eta^3He$ | $-3.49 + i3.67$ | $-2.66 + i3.31$ | $-1.96 + i2.86$ |
| $\eta^4He$ | $-3.43 + i2.60$ | $-2.81 + i2.14$ | $-2.30 + i1.72$ |
FIG. 1. The $\eta$–nucleus elastic scattering amplitude pole positions in the complex $p$–plane. The open circles correspond to $g=1$. The solid curve is the $\eta d$–amplitude pole trajectory when $g$ increases from $g=1$ to $g=2$. The dashed curve shows the trajectory of the $\eta d$ pole with $g=2$ and with $g'$ varied until the $\eta N$ interaction becomes real.