Rearrangements with supporting Trees, Isomorphisms and Combinatorics of coloured dyadic Intervals *

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Abstract

We determine a class of rearrangement operators acting on dyadic intervals that admit a supporting tree. This condition implies that the associated rearrangement operator has a bounded vector valued extension to $L^p_E$, where $E$ is a UMD space.

We prove the existence of a large subspace $X_p \subset L^p$ on which a bounded rearrangement operator acts as an isomorphism.

Moreover, we study winning strategies for a combinatorial two person game played with coloured collections of dyadic intervals.

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Key words: Haar system, Rearrangement Operators, UMD spaces, 2-person games

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1 Introduction

We study the connections between rearrangement operators of the Haar system and martingale transforms. We restrict to rearrangements $\tau$ acting on dyadic intervals such that $|\tau(I)| = |I|$ and operators given by
\[ T(h_I) = h_{\tau(I)}. \]

Our chief interest lies in obtaining workable and directly verifyable descriptions of those rearrangements $\tau$ for which $T \otimes \text{Id}_E$ is bounded on $L^p_E$ for all Banach spaces $E$ in the UMD class.

We isolate first a purely combinatorial condition on $\tau$ that implies the boundedness of $T \otimes \text{Id}_E$ on $L^p_E$ for $1 < p < \infty$, and $E$ in the UMD class. Thereby we convert the analytic question into a combinatorial problem. We let $\mathcal{F}$ be a collection of dyadic intervals and $\{A_I : I \in \mathcal{F}\}$ be a tree of measurable sets such that
\[ A_I \cap A_J \neq \emptyset \implies A_I \subseteq A_J \quad \text{or} \quad A_J \subseteq A_I. \]

We say that $\tau : \mathcal{F} \rightarrow \tau(\mathcal{F})$ admits the supporting tree $\{A_I : I \in \mathcal{F}\}$ if there exists $c > 0, \delta > 0$ so that $|A_I| \leq c |I|$ and
\[ |I \cap A_I| \geq \delta |I| \quad \text{and} \quad |\tau(I) \cap A_I| \geq \delta |I|, \quad I \in \mathcal{F}. \tag{1.1} \]

If $\tau : \mathcal{F} \rightarrow \tau(\mathcal{F})$ admits a supporting tree, then, by an application of Stein's martingale inequality, $T \otimes \text{Id}_E$ is bounded on
\[ \text{span}_{L^p_E} \{h_I x_I : x_I \in E, I \in \mathcal{F}\}, \quad 1 < p < \infty. \]

Hence if we ask for the $L^p_E$ boundedness of $T \otimes \text{Id}_E$ the ensuing combinatorial problem consists in decomposing the collection of all dyadic intervals into $\mathcal{F}_1, \ldots, \mathcal{F}_N$ so that the restrictions $\tau : \mathcal{F}_i \rightarrow \tau(\mathcal{F}_i)$ admit a supporting tree. For such a decomposition to exist it is necessary that $T$ (scalar valued) is an isomorphism on some $L^p$ with $1 < p \neq 2 < \infty$. The results of this paper are all related to the open problem whether this is also a sufficient condition. That is, we are concerned with the following extension problem for rearrangement operators:

**Conjecture 1.1** Assume that the scalar valued rearrangement operator $T$ is an isomorphism on $L^p$, $p \neq 2$. Is it true that $T \otimes \text{Id}_E$ is an isomorphism on $L^p_E$, for $1 < p < \infty$ and any $E$ in the UMD class.

The vector valued extension problem for rearrangement operators as formulated above is meaningful only within the class of isomorphisms on $L^p$. Indeed [9] and [20] contain examples of bounded rearrangement operators on $L^p$, $1 < p < 2$ and associated examples of UMD spaces $E$ so that the vector valued extension $T \otimes \text{Id}_E$ is not a bounded operator on $L^p_E$.

Semenov’s theorem [22] provides an intrinsic criterion for $\tau$ so that $T$ is an $L^p$ isomorphism. Thus by considering $T$ together with its vector valued extension $T \otimes \text{Id}_E$ we are led to the following combinatorial problem concerning tree structures and rearrangements.

**Conjecture 1.2** Does the hypothesis
\[ \bigcup_{J \subseteq I} \tau(J) \leq C |I| \quad \text{and} \quad \bigcup_{J \subseteq I} \tau^{-1}(J) \leq C |I|, \]

for any dyadic $I$ imply, that the entire collection of dyadic intervals can be decomposed into $\mathcal{F}_1, \ldots, \mathcal{F}_N$, with $N = N(C)$, so that the restrictions

$$\tau : \mathcal{F}_i \rightarrow \tau(\mathcal{F}_i)$$

admit a supporting tree?

In this paper we give partial solutions to the conjectures 1.1 and 1.2. Our results are connected to a wider set of problems as follows:

1. Seeking understanding of scalar valued operators by studying simultaneously their vector valued extensions is a central line of investigation in Banach space theory. Classical and authoritative accounts thereof are [5] by T. Figiel and [25] by A. Pełczyński. In the context of rearrangement operators, the search for supporting trees is just motivated by our attempts to prove the boundedness of vector valued rearrangements.

2. Section 4.7. in [26] by G. Pisier contains the question of describing the class of $L^p$ bounded operators $R$ so that $R \otimes \text{Id}_E$ is bounded on $L^p_E$, for any UMD space $E$. Calderon-Zygmund integrals and martingale transforms share this property. The theorems in the present paper aim at a description of the rearrangement operators in this class.

3. The rearrangement problems arising with the unconditionality of the Franklin system and its generalisations [6], [14] are special cases respectively model cases for the problems treated in Section 4.

4. The recent extrapolation theorems [9], [20], for bounded and invertible vector valued rearrangement operators on the $L^p_E$ $(1 < p < \infty)$ scale provide the structural support for the conjectures 1.1 and 1.2. Indeed, if the latter were true the former extrapolation results would simply follow from the well known scalar case.

5. The examples in [9] and [20] of $L^p$ bounded rearrangement operators $T$ for which

$$\|T \otimes \text{Id}_E\|_{L^p_E} = \infty \quad \text{and} \quad \|T^{-1} \otimes \text{Id}_E\|_{L^p_E} = \infty,$$

provide the motivation for proving the subspace theorem in Section 3. Depending on $\tau$ we determine a large subspace $X_p \subseteq L^p$ on which nevertheless the restricted operator $T|_{X_p} \otimes \text{Id}_E$ acts as an isomorphism so that,

$$\left\|T|_{X_p} \otimes \text{Id}_E\right\|_{L^p_E} \cdot \left\|T^{-1}|_{\tau(X_p)} \otimes \text{Id}_E\right\|_{L^p_E} < \infty,$$

for $1 < p < \infty$ and any UMD space $E$.

The organisation of the paper is as follows. In Section 2 we review the concepts and theorems used in the paper. In Section 3 we prove a subspace theorem for rearrangement operators. For an $L^p$ bounded rearrangement operator $T$ we determine a block basis $\{\tilde{h}_I\}$, equivalent to the entire Haar system, such that $T$ acts as an isomorphism on the subspaces spanned by $\{\tilde{h}_I\}$. This done by constructing a tree that simultaneously supports $\{\tilde{h}_I\}$ and its image $\{T(\tilde{h}_I)\}$, Section 4 treats special shift operators $S$. We show that $S \otimes \text{Id}_E$ is $L^p_E$ bounded for $E$ in the UMD class, provided that the associated shift parameters form a decomposable sequence in the sense of Definition 4.5. In Section 5 we study thoroughly the problem of finding $(d, \eta)$ homogeneous decompositions of a collection $\mathcal{C}$ of dyadic intervals, that preserves a pre-existing $(d, \eta)$ homogenous decomposition of a fixed subcollection $\mathcal{C}'$ in $\mathcal{C}$. We obtain conditions for its existence and determine examples for which this problem is without solution. While -we think- this is a combinatorial problem interesting in itself we present it here since it should support the construction of trees for rearrangements.

3
2 Preliminaries

Dyadic intervals and the Haar system. See e.g. [19]. We let \( \mathcal{D} \) denote the collection of (half-open) dyadic intervals contained in \([0,1]\). Thus

\[
\mathcal{D} = \left\{ ([k-1]2^{-n}, k2^{-n}], \quad 1 \leq k \leq 2^n, \quad n \in \mathbb{N} \right\}.
\]

For \( n \in \mathbb{N} \) write \( \mathcal{D}_n = \{ I \in \mathcal{D} : |I| = 2^{-n} \} \). For a collection of dyadic intervals \( \mathcal{E} \) we use the \( * \) notation to denote the pointset covered by \( \mathcal{E} \) thus

\[
\mathcal{E}^* = \bigcup_{I \in \mathcal{E}} I.
\]

For \( I \in \mathcal{D} \) denote by \( Q(I) \) the collection of all dyadic intervals that are contained in \( I \).

Denote by \( \{ h_I : I \in \mathcal{D}\} \) the \( L^\infty \)–normalized Haar system, where \( h_I \) is supported on \( I \) and

\[
h_I = \begin{cases} 
    1 & \text{on the left half of } I; \\
    -1 & \text{on the right half of } I. 
\end{cases}
\]

The Haar system is an unconditional basis in \( L^p \), \( 1 < p < \infty \). For \( f \in L^p \) we define its dyadic square function as

\[
S(f) = \left( \sum_{I \in \mathcal{D}} \langle f, h_I \rangle^2 1_I \right)^{1/2}.
\]

The Marcinkiewicz [15] interpretation of R.E.A.C. Paley’s theorem [23] asserts that \( c_p \| f \|_{L^p} \leq \| S(f) \|_{L^p} \leq C_p \| f \|_{L^p} \), where \( C_p \sim p^2/(p-1) \) and \( c_p = C_p^{-1} \). Recall also the definition of dyadic \( H^1 \), defined by the relation \( f \in H^1 \) if \( S(f) \in L^1 \) and

\[
\| f \|_{H^1} = \| S(f) \|_{L^1}.
\]

The Haar system in Bochner-Lebesgue spaces. Let \( 1 < p < \infty \). For a Banach space \( E \) we denote by \( L^p_E \), the Bochner Lebesgue space of \( E \) valued and \( p \)–integrable functions on the unit interval. We say that a Banach space \( E \) satisfies the UMD property if there exists \( C_p > 0 \) so that for any finite collection \( x_I \in E \)

\[
\left\| \sum \pm x_I h_I \right\|_{L^p_E} \leq C_p \left\| \sum x_I h_I \right\|_{L^p_E}.
\]

Kahane’s principle of contraction and Kahane’s inequality. Let \( \{ r_n \} \) denote the sequence of independent \( \{ +1, -1 \} \) valued Rademacher functions. Let \( x_n \in E \) be a sequence in a Banach space \( E \) and let \( a_n \in \mathbb{R} \) so that \( |a_n| \leq 1 \). Then, 

\[
\int_0^1 \left\| \sum_{n=1}^N r_n(t) a_n x_n \right\|_{E} dt \leq \int_0^1 \left\| \sum_{n=1}^N r_n(t) x_n \right\|_{E} dt.
\]

where \( C > 0 \) is independent of \( N \). We apply the above principle of contraction in combination with the Kahane’s inequality asserting that

\[
\left( \int_0^1 \left\| \sum_{n=1}^N r_n(t) x_n \right\|_{E}^p dt \right)^{1/p} \leq C \int_0^1 \left\| \sum_{n=1}^N r_n(t) x_n \right\|_{E} dt, \quad 1 < p < \infty.
\]
A Martingale Inequality. [19] [8]. The following vector valued martingale inequality is due to Bourgain and known as Bourgain’s version of E.M. Stein’s martingale inequality. It asserts the following. Let $E$ be a Banach space with the UMD property. For any increasing sequence of $\sigma$—algebras $\mathcal{F}_n$ and any sequence $f_n \in L^p_E$ with $1 < p < \infty$,

$$
\int_0^1 \left\| \sum_{n=1}^N r_n(t) \mathbb{E}(f_n | \mathcal{F}_n) \right\|_{L^p_E} dt \leq C_p \int_0^1 \left\| \sum_{n=1}^N r_n(t)f_n \right\|_{L^p_E} dt.
$$

where $\mathbb{E}(f|\mathcal{F})$ denotes conditional expectation with respect to $\mathcal{F}$ and where as above $\{r_n\}$ denotes the sequence of independent Rademacher functions.

The Theorem of Mazur. [28]. Let $E$ be Banach space and $(x_n)$ be a sequence in $E$ with weak limit $x \in E$. Then there exist a sequence $a_{n,j} \in [0,1]$ $N_j \in \mathbb{N}$ so that

$$
\sum_{n=1}^{N_j} a_{n,j} = 1 \quad \text{and} \quad y_j = \sum_{n=1}^{N_j} a_{n,j}x_n, \quad j \in \mathbb{N},
$$

converges to $x$ in norm of $E$, that is, $\lim_{j \to \infty} \|y_j - x\|_E = 0$.

Semenov’s Theorem. [22]. Let $\tau : \mathcal{D} \to \mathcal{D}$ be bijective satisfying

$$
|\tau(I)| = |I|, \quad I \in \mathcal{D}. \tag{2.1}
$$

The induced rearrangement operator is the linear extension of the map

$$
T : h_I \to h_{\tau(I)}.
$$

Let $1 < p < 2$. Semenov’s theorem asserts that $T$ is bounded on $L^p$ if there exists $C > 0$ so that

$$
|\tau(Q(I))^\ast| \leq C|I|, \quad I \in \mathcal{D}. \tag{2.2}
$$

Moreover, condition (2.2) characterizes the boundedness of $T$ on $H^1$. Specializing further we consider shifts defined by

$$
\tau(I) = I + |I|m(|I|), \quad I \in \mathcal{D}.
$$

For shifts, Semenov’s condition specializes as follows: Shift operators are bounded on $L^p$ if there exists $K$ so that the set $\tau(Q(I))^\ast$ can be covered by $K$ dyadic intervals of the same length as $I$. The best known of these shifts are those used by T. Figiel [6, 7] to study of Calderon-Zygmund operators,

$$
\tau_m(I) = I + m|I|, \quad I \in \mathcal{D}.
$$

Their norm estimates are given by T.Figiel’s theorem [6, 7]. Below we apply it for fixed and small values of $m$.

**Theorem 2.1** The linear extension of $T h_I = h_{\tau_m(I)}$ defines a bounded operator on $L^p_E$ ($1 < p < \infty$) for each Banach space $E$ with UMD property, and

$$
\|T : L^p_E \to L^p_E\| \leq C_p(E)(1 + \log m).
$$
**Dyadic trees.** Let $\mathcal{D}$ be a collection of dyadic intervals. We say that $\{E_I : I \in \mathcal{D}\}$ is a dyadic tree of sets if the following conditions hold:

1. Each of the sets $E_I$ is a finite union of dyadic intervals.
2. There exists $C > 0$ so that
   $$|I|/C \leq |E_I| \leq C|I|. \quad (2.3)$$
3. If $I_1$ is the left half of $I$ and $I_2$ is its right half then
   $$E_{I_1} \cup E_{I_2} \subseteq E_I \quad \text{and} \quad E_{I_1} \cap E_{I_2} = \emptyset. \quad (2.4)$$

Let $\{E_I : I \in \mathcal{D}\}$ be a dyadic tree and $\alpha \in \mathbb{R}$. Then we also use the term dyadic tree for the translates $\{\alpha + E_I : I \in \mathcal{D}\}$. We say that a collection of measurable sets $\{H_I : I \in \mathcal{D}\}$ supports a dyadic tree $\{E_I : I \in \mathcal{D}\}$ if there exists $\delta > 0$ so that for $I \in \mathcal{D}$

$$H_I \subseteq E_I \quad \text{and} \quad |H_I| \geq \delta |E_I|. \quad (2.5)$$

**Trees and nested collections.** Let $\mathcal{F}$ be a subset of all dyadic intervals. We say that $\{A_I : I \in \mathcal{F}\}$ is a tree (or equivalently a nested collection) of measurable sets if for $I, J \in \mathcal{F}$,

$$A_I \cap A_J \neq \emptyset \quad \text{implies that} \quad A_I \subseteq A_J \quad \text{or} \quad A_J \subseteq A_I. \quad (2.6)$$

**Rearrangements with supporting trees.** Recall that the rearrangements we consider satisfy $|\tau(I)| = |I|$. We say that $\tau : \mathcal{F} \to \mathcal{D}$ admits the supporting tree $\{A_I : I \in \mathcal{F}\}$ if there exists $C > 0, \delta > 0$ so that

$$|A_I| \leq C|I|,$$

$$|I \cap A_I| \geq \delta |I| \quad \text{and} \quad |\tau(I) \cap A_I| \geq \delta |I|, \quad I \in \mathcal{F}.$$

The interest in the notion of rearrangements admitting a supporting tree comes from the following observation, obtained by merging [12, Proposition 6.9] with [6, 7].

**Theorem 2.2** Let $\tau : \mathcal{F} \to \mathcal{D}$ be a rearrangement admitting a supporting tree (with constants $C > 0, \delta > 0$.) Let

$$X_p = \text{span}_{L^p(E)}\{x_Ih_I, I \in \mathcal{F}, x_I \in E\} \quad \text{and} \quad Y_p = \text{span}_{L^p(E)}\{x_Ih_{\tau(I)}, I \in \mathcal{F}, x_I \in E\}.$$

Then

$$T_\tau h_I = h_{\tau(I)}$$

extends to an isomorphism

$$T_{|X_p \otimes \text{Id}_E} : X_p \to Y_p$$

so that

$$\|T_{|X_p \otimes \text{Id}_E}\| \cdot \|T_{Y_p \otimes \text{Id}_E}^{-1}\| \leq f(C, \delta, E).$$
Proof. The proof is based on the contraction principle and Stein’s martingale inequality. By hypothesis \( \tau : \mathcal{F} \to \mathcal{D} \) has a supporting tree, say \( \{A_I : I \in \mathcal{F}\} \). Let \( N \in \mathbb{N} \) and define two families of increasing \( \sigma \)-algebras,

\[
A_N = \sigma \{A_I : |I| = 2^{-N}\} \quad \text{and} \quad \mathcal{F}_N = \sigma \{I : |I| = 2^{-N}\}.
\]

We translate the hypothesis into pointwise estimates for conditional expectations. Let \( I \in \mathcal{F} \) with \(|I| = 2^{-N}\), then

\[
1_{\tau(I)} \leq \delta^{-1} \mathbb{E}(1_{A_I \cap \tau(I)}|\mathcal{F}_N) \quad \text{and} \quad 1_{A_I \cap \tau(I)} \leq 1_{A_I}.
\]

Now fix \( x_I \in E \) for \( I \in \mathcal{F} \). For \( I \in \mathcal{D} \setminus \mathcal{F} \) put \( x_I = 0 \). With the UMD property on \( E \), the contraction principle, and Bourgain’s version of Stein’s Martingale inequality we get,

\[
\int_0^1 \left\| \sum_{N \in \mathbb{N}} \sum_{I \in \mathcal{D}_N} r_I(t)x_I 1_{\tau(I)} \right\|_{L^p_E} dt \leq C_p \delta^{-1} \int_0^1 \left\| \sum_{N \in \mathbb{N}} \sum_{I \in \mathcal{D}_N} r_I(t)x_I \mathbb{E}(1_{A_I \cap \tau(I)}|\mathcal{F}_N) \right\|_{L^p_E} dt
\]

\[
\leq C_p \delta^{-1} \int_0^1 \left\| \sum_{N \in \mathbb{N}} \sum_{I \in \mathcal{D}_N} r_I(t)x_I 1_{A_I} \right\|_{L^p_E} dt.
\]

(2.7)

Next exploit the second part of the hypothesis

\[
1_{A_I} \leq C \delta^{-1} \mathbb{E}(1_{A_I \cap I}|\mathcal{A}_N) \quad \text{and} \quad 1_{A_I \cap I} \leq 1_I,
\]

and continue again with the contraction principle and the martingale inequality.

\[
\int_0^1 \left\| \sum_{N \in \mathbb{N}} \sum_{I \in \mathcal{D}_N} r_I(t)x_I 1_{A_I} \right\|_{L^p_E} dt \leq C_p \delta^{-1} \int_0^1 \left\| \sum_{N \in \mathbb{N}} \sum_{I \in \mathcal{D}_N} r_I(t)x_I \mathbb{E}(1_{A_I \cap I}|\mathcal{A}_N) \right\|_{L^p_E} dt
\]

\[
\leq C_p \delta^{-1} \int_0^1 \left\| \sum_{N \in \mathbb{N}} \sum_{I \in \mathcal{D}_N} r_I(t)x_I 1_I \right\|_{L^p_E} dt.
\]

(2.8)

The reverse estimate follows in a similar fashion.

\[\blacksquare\]

3 Subspace Theorems for Rearrangement Operators

Subspace theorems are concerned with the following phenomenon. For a well behaved linear transformation \( T \) on a Banach space \( E \) there exists a –large– subspace \( F \subseteq E \) on which \( T \) is much better behaved. The best known examples of subspace theorems include bounded and non weakly compact operators on the spaces \( C(K), A, L^\infty, H^\infty \), and embedding operators on \( L^p \) spaces.

1. If \( T \) is a bounded operator on \( C(K) \) and not weakly compact, then there exists a subspace \( F \subseteq C(K) \) isomorphic to \( c_0 \) so that \( T|_F \) is an isomorphism. \([3]\) The same holds for the disk algebra \( A \). \([3]\) If \( T \) is a bounded operator on \( L^\infty \) and not weakly compact, then there exists a subspace \( F \subseteq L^\infty \) isomorphic to \( \ell^\infty \) so that \( T|_F \) is an isomorphism. \([27]\) The same assertion holds for operators on the space \( H^\infty \). \([2]\)
2. Another class of subspace theorems concern embedding operators $T : L^p \to L^p$ with $1 \leq p < \infty$. To any such embedding there exist a subspace $F \subseteq L^p$ so that $F$ and $T(F)$ are complemented in $L^p$, and $F$ is isomorphic to the ambient space $L^p$. [8], [19]. Extensions of this theorem hold for rearrangement invariant Banach spaces in which the Haar system is an unconditional basis. [12]

In this section we prove a subspace theorem for a rearrangement operator acting on the Haar system

$$T(h_I) = h_{\tau(I)}$$

where $\tau : D \to D$ is bijective satisfying

$$|\tau(I)| = |I|, \quad I \in D.$$ (3.1)

**Theorem 3.1** Assume that $T$ is bounded on $L^{p_0}$ for some $1 < p_0 \neq 2 < \infty$. Then for any $1 < p < \infty$ (sic!) there exists a closed subspace $X_p \subseteq L^p$ isomorphic to $L^p$, so that

$$T|_{X_p} : (X_p, \| \cdot \|_{L^p}) \to (T(X_p), \| \cdot \|_{L^p})$$

is an isomorphism,

$$\| T|_{X_p} \|_{L^p} : \| T^{-1}|_{T(X_p)} \|_{L^p} < \infty, \quad \text{for} \quad (1 < p < \infty).$$

The subspaces $X_p$ and $T(X_p)$ are complemented in $L^p$.

Theorem 3.1 is a direct consequence of Theorem 3.4 and Theorem 3.5. In the course of its proof we use martingale techniques, most notably the inequalities of Paley respectively Stein (in Burkholder’s respectively Bourgain’s version, [8], [19]). Thus the method yields extensions to vector valued rearrangement operators so that

$$\| T|_{X_p} \otimes \text{Id}_E \|_{L^p_E} : \| T^{-1}|_{T(X_p)} \otimes \text{Id}_E \|_{L^p_E} < \infty, \quad \text{for} \quad (1 < p < \infty),$$

whenever $E$ satisfies the UMD property. The significance of this remark is connected with the examples in [9] and [20] of scalar valued $L^p$ bounded rearrangement operator $T$ satisfying

$$\| T \otimes \text{Id}_E \|_{L^p_E} = \infty \quad \text{and} \quad \| T^{-1} \otimes \text{Id}_E \|_{L^p_E} = \infty.$$ 

**The combinatorial core.** The following result, was developed for the proof that the spaces $L^p, 1 \leq p < \infty$ are primary. We refer to the work of Enflo and Starbird [4], Johnson, Maurey, Schechtman, and Tzafriri [12], and Enflo via Maurey [16]. It is the main combinatorial tool by which we find two tree structures (one in the domain, another in the range of the operator $T$) that are compatible with the action of rearrangement operators.

**Proposition 3.2** Let $\nu$ be a measure on $[0,1]$ taking values in $\{ f \in L^1[0,1] : f \geq 0 \}$. Assume that

$$\int_0^1 \max_{I \in \mathcal{D}_N} \nu(I) dt \geq C^{-1} \quad \text{and} \quad \int_0^1 \nu(I) dt \leq C|I|, \quad I \in \mathcal{D}. \quad (3.2)$$

Then there exist dyadic trees $\{ G_I : I \in \mathcal{D} \}$ and $\{ F_I : I \in \mathcal{D} \}$ so that

$$\int_{G_I} \nu(F_I) dt \geq \delta |I|, \quad I \in \mathcal{D}, \quad (3.3)$$

where $\delta = \delta(C)$. 


Our definition of a dyadic tree (as given in the preliminaries section) includes the requirement that $G_I$ and $F_I$ can be written as finite unions of dyadic intervals. Lemma 9.8 in [12] states just that $F_I$ is a finite union of dyadic intervals. However, the proof of [12, Lemma 9.8] can easily be modified to yield that also $G_I$ is a finite union of dyadic intervals. See [18]. In the applications of Proposition 3.2 the vector valued measure $\nu$ carries well structured information on weak limits of non linear functionals [12], [1], [18]. The non linearities arise by composing the linear operator under investigation with the dyadic square function.

The hypothesis of Proposition 3.2 are easily verified with the following criterion [12, Chapter 9].

**Proposition 3.3** Let $\nu$ be a measure on $[0, 1]$ taking values in $\{f \in L^1([0, 1] : f \geq 0\}$. Assume that

$$0 \leq \nu(I) \leq 1, \quad \nu([0, 1]) = 1_{[0,1]}, \quad (3.4)$$

and

$$\int_0^1 \sqrt{\nu(I)} dt \leq C|I|, \quad I \in D, \quad (3.5)$$

then

$$\int_0^1 \max_{I \in D_N} \nu(I) dt \geq C^2. \quad (3.6)$$

**Proof.** First observe that by (3.4) and the additivity of the vector measure,

$$1_{[0,1]} = \left( \sum_{I \in D_N} \nu(I) \right)^{1/2}. \quad (3.7)$$

Next by arithmetic and the Cauchy-Schwarz inequality,

$$1 = \int_0^1 \left( \sum_{I \in D_N} \nu(I) \right)^{1/2} dt$$

$$\leq \int_0^1 \left( \max_{I \in D_N} \nu(I)^{1/2} \right)^{1/2} \left( \sum_{I \in D_N} \nu(I)^{1/2} \right)^{1/2} dt$$

$$\leq \left( \int_0^1 \max_{I \in D_N} \nu(I)^{1/2} dt \right)^{1/2} \left( \int_0^1 \sum_{I \in D_N} \nu(I)^{1/2} dt \right)^{1/2}. \quad (3.7)$$

By (3.5) we get for the second term in the above expression

$$\int_0^1 \sum_{I \in D_N} \nu(I)^{1/2} dt \leq C \sum_{I \in D_N} |I|$$

$$= C. \quad (3.8)$$

Combining (3.7) with (3.8) and using Hoelder’s inequality gives

$$1 \leq C^{1/2} \left( \int_0^1 \max_{I \in D_N} \nu(I)^{1/2} dt \right)^{1/2}$$

$$\leq C^{1/2} \left( \int_0^1 \max_{I \in D_N} \nu(I) dt \right)^{1/4} \quad (3.9)$$
Theorem 3.4: To each rearrangement $\tau : \mathcal{D} \to \mathcal{D}$ satisfying Semenov’s condition

$$|\tau(I)| = |I| \quad \text{and} \quad |\tau(Q(I))| \leq C|I|, \quad I \in \mathcal{D}. \quad (3.10)$$

there exist pairwise disjoint collections of dyadic intervals $\{\mathcal{H}_I : I \in \mathcal{D}\}$ so that

1. The family $\mathcal{H}_I$ consists of pairwise disjoint dyadic intervals of equal length, that is, if $J_1, J_2 \in \mathcal{H}_I$ then $|J_1| = |J_2|$ and $J_1 \cap J_2 = \emptyset$.

2. The families $\{\tau(\mathcal{H}_I)^* : I \in \mathcal{D}\}$ respectively $\{\mathcal{H}_I^* : I \in \mathcal{D}\}$ support dyadic trees.

Proof. In the course of selecting the families $\{\mathcal{H}_I : I \in \mathcal{D}\}$ we exploit Proposition 3.2 and take advantage of the fact that we are working with rearrangement operators for which $(3.10)$ holds.

Selecting two trees. We review the construction of the non negative $L^2$ valued vector measure describing the limiting behavior of the operator $T$. Here we refer to Chapter 9 of [12].

Fix $I \in \mathcal{D}$, $n \in \mathbb{N}$. Let

$$\mathcal{D}_n(I) = \{J \in \mathcal{D} : J \subseteq I, |J| = 2^{-n}\}, \quad \text{and} \quad d_n(I) = \sum_{J \in \mathcal{D}_n(I)} 1_{\tau(J)}.$$

Since $0 \leq d_n(I) \leq 1$, the sequence $\{d_n(I) : n \in \mathbb{N}\}$ has a weak cluster point in $L^2$. By a diagonal argument there exists a subsequence $(n_k)$ so that for $I \in \mathcal{D}$ the sequence $\{d_{n_k}(I)\}$ converges weakly in $L^2$. For $I \in \mathcal{D}$ define $\nu(I)$ as $L^2$–weak limit,

$$d_{n_k}(I) \overset{\sigma}{\rightharpoonup} \nu(I)$$

Using that $\tau$ is bijective and $|\tau(I)| = |I|$ it is straightforward to observe that

$$\nu([0, 1]) = 1, \quad \nu(I) \leq 1, \quad \text{and} \quad \int_0^1 \nu(I) = |I|. \quad (3.11)$$

Since $\tau$ satisfies Semenov’s condition, the linear extension of $T(h_I) = h_{\tau(I)}$ defines a bounded operator on $H^1$. We use the boundedness of $T$ on $H^1$ to prove that

$$\int_0^1 \sqrt{\nu(I)} \leq \|T\|_{H^1} |I|. \quad (3.12)$$

Mazur’s theorem asserts that there exist $N \in \mathbb{N}$, $a_n \in [0, 1]$ so that $\sum_{n \in A_I} a_n = 1$ and

$$\|\sum_{n=1}^N a_n d_n(I) - \nu(I)\|_{L^2} \leq \epsilon^2 |I|. \quad (3.13)$$

Define next

$$k_I = \sum_{n=1}^N a_n^{1/2} \sum_{J \in \mathcal{D}_n(I)} h_J.$$

Since $d_n(I)^2 = d_n(I)$, we have the identity

$$S(Tk_I) = \left(\sum_{n=1}^N a_n d_n(I)\right)^{1/2}. \quad (3.14)$$
By (3.13) and (3.14) we get immediately
\[ \int_0^1 \sqrt{\nu(I)} \leq (1 + \epsilon) \int_0^1 S(Tk_I). \]
Invoking that \( \|k_I\|_{H^1} = |I| \) we obtain that and \( \|Tk_I\|_{H^1} \leq \|T\|_{H^1} |I| \) hence
\[ \int_0^1 \sqrt{\nu(I)} \leq (1 + \epsilon) \|T\|_{H^1} |I|. \]  \( (3.15) \)
Since \( \epsilon > 0 \) is arbitrary we obtain (3.12) as claimed. Combining (3.11) and (3.12) yields these estimates
\[ \int_0^1 \max_{I \in D} \nu(I) dt \geq \|T\|_{H^1}^{-2} \] and \[ \int_0^1 \nu(I) dt \leq |I|. \]  \( (3.16) \)
Since \( \nu \) is a finitely additive set function satisfying \( \int_0^1 \nu(I) dt \leq |I| \), we may extend the mapping
\[ \nu : D \to L^2([0,1]), \quad I \to \nu(I)\]
to an absolutely continuous vector measure on the \( \sigma \)-algebra generated by \( D \) so that (3.16) holds and
\[ |A_n| \to 0, \quad \text{implies} \quad \int_0^1 \nu(A_n) dt \to 0, \quad A_n \in \sigma(D) \]
By Proposition 3.2 there exist two dyadic trees \( \{G_I : I \in D\} \) and \( \{F_I : I \in D\} \) so that
\[ \int_{G_I} \nu(F_I) dt \geq \|T\|_{H^1}^{-2} |I|, \quad I \in D. \]  \( (3.17) \)
This completes our summary of the selection process in [12].

**Defining \( \mathcal{H}_I \).** We turn to drawing consequences of (3.17). Here we exploit that the operator generating the vector measure \( \{\nu(I) : I \in D\} \) is defined as a rearrangement
\[ T(h_I) = h_{\tau(I)} \quad \text{where} \quad |\tau(I)| = |I|. \]

There exists \( N(I) \in \mathbb{N} \) so that \( G_I \) is a finite union of intervals in \( D_{N(I)} \). We test the weak limit \( \nu(F_I) \) by integrating against the function \( 1_{G_I} \). There exists \( M(I) \geq N(I) \) so that
\[ \int_{G_I} d_M(F_I) dt \geq \frac{1}{2} \int_{G_I} \nu(F_I) dt. \]  \( (3.18) \)
Define now the collection
\[ \mathcal{H}_I = \{J \in D_{M(I)} : J \subseteq F_I, \tau(J) \subseteq G_I\}. \]  \( (3.19) \)
**Supporting trees.** We first verify that \( \{\tau(H_I)^*: I \in D\} \) supports the tree \( \{G_I : I \in D\} \). Recall that by definition of \( H_I \) we have the inclusion
\[
\tau(H_I)^* \subseteq G_I.
\]
It remains for verify the measure estimate: Here we use the identity
\[
\int_{G_I} dM(I)(F_I) = \sum \{|\tau(J)| : J \in H_I\},
\]
together with (3.17) and (3.18). This gives
\[
|\tau(H_I)^*| = \sum \{|\tau(J)| : J \in H_I\}
\]
\[
\geq \frac{1}{2} \|T\|_{H^1}^2 |I|.
\]
Next we verify that \( \{H_I^* : I \in D\} \) supports the tree \( \{F_I : I \in D\} \). Note first the inclusion
\[
H_I^* \subseteq F_I.
\]
Next using the identity \( |\tau(I)| = |I| \) we get a reduction to (3.20) treated above:
\[
|H_I^*| = \sum \{|J| : J \in H_I\}
\]
\[
= \sum \{|\tau(J)| : J \in H_I\}
\]
\[
= |\tau(H_I)^*|
\]
\[
\geq \frac{1}{2} \|T\|_{H^1}^2 |I|.
\]

The collections \( \{H_I : I \in D\} \) are chosen so that Stein’s Martingale Inequality yields the estimates of Theorem 3.5 below. As a result the scalar valued estimates of Theorem 3.5 remain true when the coefficients are taken from a Banach space with the UMD property.

**Theorem 3.5** Let \( \tau : D \to D \) be a rearrangement satisfying \( |\tau(I)| = |I|, \quad I \in D \) and define
\[ T(h_I) = h_{\tau(I)}. \]
Assume that \( \{H_I : I \in D\} \) are pairwise disjoint collections of dyadic intervals so that

1. If \( J_1, J_2 \in H_I \) then \( |J_1| = |J_2| \) and \( J_1 \cap J_2 = \emptyset \).

2. The families \( \{H_I^* : I \in D\} \) and \( \{\tau(H_I)^* : I \in D\} \) support dyadic trees of sets.

Then the blocked system
\[ \tilde{h}_I = \sum_{J \in H_I} h_J, \quad I \in D, \]
satisfies the following estimates: For \( 1 < p < \infty \), and any choice of \( x_I \in \mathbb{R} \),
\[
c_p \left\| \sum x_I \tilde{h}_I \right\|_{L^p} \leq \left\| \sum x_I T(h_I) \right\|_{L^p} \leq C_p \left\| \sum x_I \tilde{h}_I \right\|_{L^p}, \tag{3.23}
\]
\[
c_p \left\| \sum x_I h_I \right\|_{L^p} \leq \left\| \sum x_I \tilde{h}_I \right\|_{L^p} \leq C_p \left\| \sum x_I h_I \right\|_{L^p}, \tag{3.24}
\]

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**Proof.** Let \( \{F_I\} \) be the dyadic tree supported by \( \{\mathcal{H}_I^* : I \in \mathcal{D}\} \) and \( \{G_I\} \) be the dyadic tree supported by \( \{\tau(\mathcal{H}_I)^* : I \in \mathcal{D}\} \). Without loss of generality we may assume that the tree of sets \( \{F_I\} \) is contained in the interval \([0, 1]\) and that \( \{G_I\} \) is contained in \([1, 2]\). Let \( \mathcal{E}_N \) be the \( \sigma \)-algebra generated by the sets
\[
\{F_I \cup G_I, \ I \in \mathcal{D}_N\}.
\]
Since \( \{F_I\} \) and \( \{G_I\} \) are trees and moreover \( G_I \cap F_I = \emptyset, I, J \in \mathcal{D} \) it follows that \( \{\mathcal{E}_N, N \in \mathbb{N}\} \) is an increasing sequence of \( \sigma \)-algebras. Let \( \mathbb{E}_N \) denote the conditional expectation operator induced by \( \mathcal{E}_N \). Note that the following pointwise estimates hold true.
\[
1_{G_I} + |\tilde{h}_I| \leq C\mathbb{E}_N(|\tilde{h}_I|) \quad I \in \mathcal{D}_N. \tag{3.25}
\]

Exploiting the estimate (3.25) together with the unconditionality of the Haar system and Bourgain’s version of E.M. Stein’s martingale inequality we get the following estimates. Let \( x_I \in \mathbb{R} \) with \( I \in \mathcal{D} \) be a sequence with finitely many entries different from zero.
\[
\| \sum_{N \in \mathbb{N}} \sum_{I \in \mathcal{D}_N} x_I T(\tilde{h}_I) \|_{L^p} \leq C_p \int_0^1 \| \sum_{N \in \mathbb{N}} \sum_{I \in \mathcal{D}_N} r_I(t)x_I|T(\tilde{h}_I)| \|_{L^p} dt
\]
\[
\leq C_p \int_0^1 \| \sum_{N \in \mathbb{N}} \sum_{I \in \mathcal{D}_N} r_I(t)x_I1_{G_I} \|_{L^p} dt
\]
\[
\leq C_p \int_0^1 \| \sum_{N \in \mathbb{N}} \sum_{I \in \mathcal{D}_N} r_I(t)x_I (1_{G_I} + |\tilde{h}_I|) \|_{L^p} dt \tag{3.26}
\]
\[
\leq C_p \int_0^1 \| \sum_{N \in \mathbb{N}} \sum_{I \in \mathcal{D}_N} r_I(t)x_I \mathbb{E}_N(|\tilde{h}_I|) \|_{L^p} dt
\]
\[
\leq C_p \| \sum_{N \in \mathbb{N}} \sum_{I \in \mathcal{D}_N} x_I \tilde{h}_I \|_{L^p}.
\]

Observe also the pointwise estimates.
\[
1_{F_I} + |T(\tilde{h}_I)| \leq C\mathbb{E}_N(|T(\tilde{h}_I)|), \quad I \in \mathcal{D}_N. \tag{3.27}
\]
By (3.27) together with the unconditionality of the Haar system and Bourgain’s version of E.M. Stein’s martingale inequality we get similarly,
\[
\| \sum_{N \in \mathbb{N}} \sum_{I \in \mathcal{D}_N} x_I \tilde{h}_I \|_{L^p} \leq C_p \int_0^1 \| \sum_{N \in \mathbb{N}} \sum_{I \in \mathcal{D}_N} r_I(t)x_I1_{F_I} \|_{L^p} dt
\]
\[
\leq C_p \int_0^1 \| \sum_{N \in \mathbb{N}} \sum_{I \in \mathcal{D}_N} r_I(t)x_I (1_{F_I} + |T(\tilde{h}_I)|) \|_{L^p} dt \tag{3.28}
\]
\[
\leq C_p \int_0^1 \| \sum_{N \in \mathbb{N}} \sum_{I \in \mathcal{D}_N} r_I(t)x_I \mathbb{E}_N(|T(\tilde{h}_I)|) \|_{L^p} dt
\]
\[
\leq C_p \| \sum_{N \in \mathbb{N}} \sum_{I \in \mathcal{D}_N} x_I T(\tilde{h}_I) \|_{L^p}.
\]

This proves (3.23).
To prove (3.24), let \( \mathcal{F}_N \) be the \( \sigma \)-algebra generated by the sets

\[ \{ F_I, \ I \in D_N \}. \]

Since \( \{ F_I \} \) is a tree \( \{ \mathcal{F}_N, N \in \mathbb{N} \} \) is an increasing sequence of \( \sigma \)-algebras. Let \( \mathbb{F}_N \) denote the conditional expectation operator induced by \( \mathcal{F}_N \). The collection \( \{ \mathcal{H}_I : I \in D \} \) supports the tree \( \{ \mathcal{F}_I, I \in D \} \). Therefore the following pointwise estimates hold true.

\[
|\tilde{h}_I| \leq 1_{F_I} \leq C\mathbb{F}_N(|\tilde{h}_I|), \quad I \in D_N.
\]  

(3.29)

\[
\| \sum_{N \in \mathbb{N}} \sum_{I \in D_N} x_I \tilde{h}_I \|_{L^p} \leq C_p \int_0^1 \| \sum_{N \in \mathbb{N}} \sum_{I \in D_N} r_I(t) x_I 1_{F_I} \|_{L^p} dt 
\]

\[
\leq C_p \int_0^1 \| \sum_{N \in \mathbb{N}} \sum_{I \in D_N} r_I(t) x_I \mathbb{F}_N(|\tilde{h}_I|) \|_{L^p} dt
\]

(3.30)

\[
\leq C_p \| \sum_{N \in \mathbb{N}} \sum_{I \in D_N} x_I \tilde{h}_I \|_{L^p}
\]

\[ \|
\sum_{N \in \mathbb{N}} \sum_{I \in D_N} r_I(t) x_I 1_{I} \|_{L^p} \leq C_p \int_0^1 \| \sum_{N \in \mathbb{N}} \sum_{I \in D_N} r_I(t) x_I 1_{F_I} \|_{L^p} dt 
\]

(3.31)

\[
\leq C_p \int_0^1 \| \sum_{N \in \mathbb{N}} \sum_{I \in D_N} r_I(t) x_I 1_{I} \|_{L^p} dt.
\]

Finally since both \( \{ I \} \) and \( \{ F_I \} \) are trees of sets we obtain by a measure preserving transformation that

\[
\int_0^1 \| \sum_{N \in \mathbb{N}} \sum_{I \in D_N} r_I(t) x_I 1_{I} \|_{L^p} dt \leq C_p \int_0^1 \| \sum_{N \in \mathbb{N}} \sum_{I \in D_N} r_I(t) x_I 1_{F_I} \|_{L^p} dt 
\]

(3.32)

\[
\leq C_p \int_0^1 \| \sum_{N \in \mathbb{N}} \sum_{I \in D_N} r_I(t) x_I 1_{I} \|_{L^p} dt.
\]

The estimates of Theorem 3.5 assert that in \( L^p, 1 < p < \infty \) the blocked system is equivalent to the Haar system, hence the subspace \( X_p \subseteq L^p \) defined to be the \( L^p \)-closure of span \( \{ \tilde{h}_I : I \in D \} \) is isomorphic to the ambient space \( L^p \). And furthermore on the subspace \( X_p \) the rearrangement operator \( T \) acts as an isomorphism.

**Proof of Theorem 3.1** : Consider first \( 1 < p_0 < 2 \). By Semenov’s theorem \( \| T \|_{p_0} < \infty \) implies that

\[
|\tau(Q(I)^*)| \leq C|I|, \quad I \in D.
\]

Apply Theorem 3.4 to the rearrangement \( \tau : D \rightarrow D \). Let \( \{ \mathcal{H}_I : I \in D \} \) denote the collections of dyadic intervals satisfying the conclusion of Theorem 3.4. Put

\[
X = \text{span} \{ \tilde{h}_I : I \in D \} \quad \text{where} \quad \tilde{h}_I = \sum_{J \in \mathcal{H}_I} h_J.
\]

Next fix \( 1 < p < \infty \). Let \( X_p \) denote the closure of \( X \) in \( L^p \), and similarly let \( (T(X))_p \) be the closure of \( T(X) \) in \( L^p \). Theorem 3.5 asserts that \( X_p \) is isomorphic to \( L^p \), and that the map

\[
T(\tilde{h}_I) = \sum_{J \in \mathcal{H}_I} h_{\tau(J)}
\]
extends uniquely to an isomorphism with domain \( X_p \) and range \((T(X))_p\). Denoting the extension still by \( T \) we have \((T(X))_p = T(X_p)\), and
\[
\left\| T|_{X_p} \right\|_{L_p} \cdot \left\| T^{-1}|_{T(X_p)} \right\|_{L_p} < \infty.
\]
Next we turn to the case \( 2 < p_0 < \infty \). Note that for rearrangement operators the transposed operator coincides with the inverse \( T^{-1} \) defined by \( \tau^{-1} : D \to D \). Thus \( S = T^* = T^{-1} \) is a bounded operator on \( L^{p_0}_\infty \) where \( 1/p_0 + 1/p_0 = 1 \). Now \( p_0^* < 2 \). By the first part of the theorem applied to the rearrangement operator \( S \), for \( 1 < p < \infty \) there exists \( X_p \) isomorphic to \( L^p \), so that
\[
\left\| S|_{X_p} \right\|_{L_p} \cdot \left\| S^{-1}|_{S(X_p)} \right\|_{L_p} < \infty.
\]
Since \( S = T^{-1}, S^{-1} = T \) with \( Y_p = S(X_p) \) we obtain
\[
\left\| T|_{Y_p} \right\|_{L_p} \cdot \left\| T^{-1}|_{T(Y_p)} \right\|_{L_p} < \infty.
\]

\[\boxed{\text{4 Shift operators}}\]

We consider rearrangements of \( D \) of particular type, namely shift operators. Let
\[
I_{j,k} = \left[ \frac{k - 1}{2^j}, \frac{k}{2^j} \right) \quad \text{for} \quad k = 1, \ldots, 2^j,
\]
and let \( M = \{m_j, j \geq 1\} \) be a sequence of integers satisfying \( |m_j| \leq 2^j \). Consider \( \tau_M : D \to D \) given by
\[
\tau_M(I_{j,k}) = I_{j,k+m_j} = I + m_j I \quad \text{for} \quad I \in D_j,
\]
where \( k + m_j \) is understood mod \( 2^j \). This rearrangement is called \textit{shift generated by} \( M \).

In this section, we give a version of Semenov’s theorem suitably adapted to the special nature of shift operators, see Proposition \[\boxed{4.1}\]. We isolate a class of shift operators for which Conjecture \[\boxed{1.1}\] and Conjecture \[\boxed{1.2}\] hold true, Theorems \[\boxed{4.3}\] and \[\boxed{4.6}\]. We prove directly, without using the machinery of Section \[\boxed{3}\] a subspace theorem for shift operators, Theorem \[\boxed{4.4}\].

\[\boxed{4.1 \text{ Semenov condition for shifts.}}\]

We next give a version of Semenov’s criterion that holds specifically for shift operators. Let \( j \in \mathbb{N} \) and put
\[
x_j = \frac{m_j}{2^j},
\]
so that
\[
\tau_M(I) = I + x_j \quad \text{for} \quad |I| = 2^{-j}.
\]
Then define
\[
N_j(M) = |\{k : I_{j,k} \cap \{x_l, l \geq j\} \neq \emptyset\}|.
\]
The next Proposition \[\boxed{4.1}\] relates Semenov’s condition to the boundedness of the sequence \( N_j(M) \).
Proposition 4.1 Let $\tau_M$ be the shift generated by the sequence $M = \{m_j, j \geq 1\}$. Then we have for $I \in \mathcal{D}_j$

$$ \frac{N_j(M)}{2}|I| \leq |\tau_M(Q(I))^*| \leq 2N_j(M)|I|. \quad (4.1) $$

Consequently, $\tau_M$ satisfies Semenov’s condition if and only if there is a constant $K > 0$ such that $N_j(M) \leq K$ for all $j \geq 1$.

**Proof.** Let $N_j = N_j(M)$. For fixed $j$, let

$$ 1 \leq k_1 < k_2 < \ldots < k_{N_j} \leq 2^j $$

be the enumeration of the set of indices $\{k : I_{j,k} \cap \{x_l, l \geq j\} \neq \emptyset\}$. Take $I \in \mathcal{D}_j$ and $l \geq j$. Let $k_i$ be such that $x_l \in I_{j,k_i}$. Then we have

$$ \tau_M(Q(I) \cap \mathcal{D}_l)^* = I + x_l \subset (I + \frac{k_i - 1}{2^j}) \cup (I + \frac{k_i}{2^j}), $$

and consequently

$$ |\tau_M(Q(I))^*| \leq 2N_j|I|. $$

To prove the other estimate, let $J, L \in \{I_{j,k}, 1 \leq i \leq N_j\}$ be such that $\text{dist}(J, L) > 0$. Then for $l_1, l_1$ such that $x_{l_1} \in J$ and $x_{l_2} \in L$ we have

$$ \tau_M(Q(I) \cap \mathcal{D}_{l_1})^* \cap \tau_M(Q(I) \cap \mathcal{D}_{l_2})^* = \emptyset. $$

Clearly, for each $l \geq j$ we have $|\tau(Q(I) \cap \mathcal{D}_l)^*| = |I|$. Combining these observations we get

$$ |\tau_M(Q(I))^*| \geq \frac{N_j}{2}|I|. $$

\[\square\]

4.2 Shifts and nested collections.

Proposition 4.1 implies that if $\tau_M$ satisfies Semenov’s condition, then the collection of accumulation points of sequence $X = \{x_j, j \geq 1\}$ is finite. Therefore, without loss of generality we assume $\lim_{j \to \infty} x_j = 0$.

We turn our attention to sequences $M$ with $\limsup_{j \to \infty} N_j(M) \leq 2$. We are able to describe the structure of such sequences:

Proposition 4.2 Let $M = \{m_j, j \geq 1\}$ and $X = \{x_j = \frac{m_j}{2^j}, j \geq 1\}$ be such that $\lim_{j \to \infty} x_j = 0$ and $\limsup_{j \to \infty} N_j = 2$. Then exist sequences

$$ \{a_k, k \geq 1\} \quad \text{and} \quad \{j_k, k \geq 0\} $$

with the following properties: $\{j_k, k \geq 0\}$ is increasing and if

$$ j_{k-1} \leq j < j_k $$

then

$$ x_j = 0 \quad \text{or} \quad |a_k - x_j| < \frac{1}{2^j}, $$

and if

$$ j \geq j_k $$

then

$$ x_j \leq 2^{-j_k + 1}. $$
Proof. First, note that if \( N_j = 1 \) then \( 0 \leq x_l < \frac{1}{2^j} \) for all \( l \geq j \). Therefore, in case \( \limsup_{j \to \infty} N_j = 1 \) the above condition is clearly satisfied.

Consider the case \( \limsup_{j \to \infty} N_j = 2 \). We construct sequences \( \{a_k, k \geq 1\} \) and \( \{j_k, k \geq 0\} \) inductively. We put an additional requirement that \( N_{j_k} = 2 \).

\[
\begin{align*}
    j_0 &= \min\{j : N_i \leq 2 \text{ for all } i \geq j \text{ and } N_j = 2\}, \\
    n_1 &= \max\{j \geq j_0 : x_j > \frac{1}{2^{j_0}}\}, \\
    j_1 &= \min\{j > n_1 : N_j = 2\}.
\end{align*}
\]

For \( n_1 < j < j_1 \) we have \( N_j = 1 \), and consequently \( \{x_l, l \geq j\} \subset [0, \frac{1}{2^{j_0}}) \). Taking \( j = j_1 - 1 \) we get \( x_l \leq \frac{1}{2^{j_0}} \) for \( l \geq j_1 \).

Then, take \( l \) such that \( j_0 \leq l \leq n_1 \). Since \( N_l = 2 \), there are only two intervals from \( D_l \) which have nonempty intersection with \( \{x_i, i \geq l\} \): one of them is \( I_{l,1} \), and the other contains \( x_{n_1} \). This implies that either \( x_l \in I_{l,1} \) (in such case, \( x_l = 0 \)), or \( x_l \) and \( x_{n_1} \) must be in the same interval from \( D_l \), which implies \( |x_{n_1} - x_l| < \frac{1}{2^{j_0}} \). Therefore, it is enough to put \( a_1 = x_{n_1} \). Note that \( N_{j_1} = 2 \).

Having defined \( j_k \) and \( a_k \), we define \( j_{k+1} \) and \( a_{k+1} \). Since \( N_{j_k} = 2 \), there is \( j \geq j_k \) such that \( x_j > \frac{1}{2^{j_k}} \). We put

\[
\begin{align*}
    n_{k+1} &= \max\{j \geq j_k : x_j > \frac{1}{2^{j_k}}\}, \\
    j_{k+1} &= \min\{j > n_{k+1} : N_j = 2\}, \\
    a_{k+1} &= x_{n_{k+1}}.
\end{align*}
\]

Arguments analogous to the above one show that Proposition 4.2 holds for this choice of \( j_k \), \( j_{k+1} \) and \( a_{k+1} \).

\[\square\]

**Theorem 4.3** Let \( M \) be a sequence such that \( \limsup_{j \to \infty} N_j(M) \leq 2 \). Then the operator \( T_M h_I = h_{T_M(I)} \) extends to an isomorphism of \( L^p_E \), for each Banach space \( E \) with UMD property and \( 1 < p < \infty \).

**Proof.** Without loss of generality, we assume that on each \( D_j \) the shift is nontrivial, so that \( m_j \neq 0 \) and consequently \( x_j \neq 0, j \geq 1 \). By Proposition 4.2 the hypothesis

\[ \limsup_{j \to \infty} N_j(M) \leq 2 \]

implies that there exist

\[ \{a_k, k \geq 1\} \text{ and } \{j_k, k \geq 0\}, \]

where \( j_k, k \geq 0 \) is increasing and satisfies the following properties.

If \( j_{k-1} \leq j < j_k \) then \( |a_k - x_j| < \frac{1}{2^j} \), \hspace{1cm} (4.2)

and

\[ j \geq j_k \text{ implies } x_j \leq 2^{-j_k+1}. \] \hspace{1cm} (4.3)
Applying an additional shift by no more than 1 unit, we can assume that for each $j_{k-1} \leq j < j_k$ and $I \in \mathcal{D}_j$

$$|(x_j + I) \cap (a_k + I)| \geq \frac{|I|}{2},$$

while for $j \geq j_k$ we have $x_j \leq \frac{a}{2^k}$. Recall that by Theorem 2.1, the additional shift applied above induces an isomorphism of $L^p_{\mathcal{E}}$, with uniform bounds.

The idea is to take a suitable splitting of level $\mathcal{D}_{j_{k-1}}$, and for $\mathcal{D}_j$ with $j_{k-1} \leq j < j_k$ take a splitting induced by the splitting of $\mathcal{D}_{j_{k-1}}$; two intervals from $\mathcal{D}_j$ will be put into the same collection in the splitting of $\mathcal{D}_j$ iff their dyadic ancestors in $\mathcal{D}_{j_{k-1}}$ are put into the same collection in the splitting of $\mathcal{D}_{j_{k-1}}$. Then we would like to take $I \cup (a_k + I)$ as our building block for the elements in the nested collection for $I \in \mathcal{D}_j$ with $j_{k-1} \leq j < j_k$. In order to make this work, we need to produce space between the sets $I \cup (a_k + I)$. Therefore we perform now one additional preparatory operation. Define $\phi : \mathcal{D} \to \mathcal{D}$ by the following procedure: given $I \in \mathcal{D}$, let $r(I)$ be the right endpoint of $I$ and $s(I)$ be its midpoint. Then define $\phi(I)$ uniquely by the following relation:

$$|\phi(I)| = \frac{1}{4}|I| \quad \text{and} \quad r(\phi(I)) = s(I).$$

That is, $\phi : \mathcal{D} \to \mathcal{D}$ be the mapping assigning to $I \in \mathcal{D}_j$ an interval $\phi(I) \in \mathcal{D}_{j+2}$ with right endpoint coinciding with the midpoint of $I$. The reason we use the mapping $\phi$ is that the intervals in the range of $\phi$ satisfy the following property:

If $J_1, J_2 \in \phi(\mathcal{D})$ with $|J_1| \leq |J_2|$, then $J_1 \subset J_2$ or $\text{dist}(J_1, J_2) \geq |J_1|$.

(4.4)

Let $\psi : \mathcal{D} \to \mathcal{D}$ be the mapping assigning to $I \in \mathcal{D}_j$ the unique interval $\psi(I) \in \mathcal{D}_{j-2}$ such that $I \subset \psi(I)$. The operations $T_\phi$ and $T_\psi$ are bounded on $L^p_{\mathcal{E}}$. Let

$$j'_k = j_k + 2.$$  

(4.5)

Let $\sigma$ be a shift such that for $I \in \mathcal{D}_j$ with $j_{k-1} \leq j < j'_k$

$$|\sigma(I) \cap (a_k + I)| \geq \frac{|I|}{2}.$$ 

Put $\tau' = \psi \circ \sigma \circ \phi$. Note that $\tau$ can be obtained from $\tau'$ by applying an additional shift by at most 1. Therefore, to estimate the norm of $T_\tau$, it is enough to consider $\sigma$ restricted to $\phi(\mathcal{D})$. In the next paragraph, we will split $\phi(\mathcal{D})$ as

$$\phi(\mathcal{D}) = \bigcup_{r=1}^{6} \bigcup_{l=1}^{512} \mathcal{F}(r, l)$$

so that the restricted rearrangements $\sigma : \mathcal{F}(r, l) \to \mathcal{D}$ are supporting nested collections. Combining this with Theorem 2.2 proves the statement of Theorem 4.3

**Horizontal splitting at stage $k$.** In this paragraph we fix $k$. We will obtain a splitting of the collection

$$\phi \left( \bigcup_{j=j_{k-1}}^{j_{k-1}} \mathcal{D}_j \right).$$
We obtain it by first decomposing the collection of top level intervals \( \phi(\mathcal{D}_{j_{k-1}}) \) and then simply pushing it down to later levels. (Hence the name horizontal splitting.) On \( \mathcal{D}_{j_{k-1}} \), \( \tau \) is a shift by \( m_{j_{k-1}} \). We split \( \mathcal{D}_{j_{k-1}} \) as

\[
\mathcal{D}_{j_{k-1}} = \bigcup_{l=1}^{L} \mathcal{D}_{j_{k-1},l}, \quad L \leq 8^3
\]

so that if

\[
I \in \mathcal{D}_{j_{k-1},l}
\]

then

\[
I \pm |I| \notin \mathcal{D}_{j_{k-1},l}, \quad \tau(I), \tau(I) \pm |I| \notin \mathcal{D}_{j_{k-1},l}, \quad \text{and} \quad \tau(I) \pm 2|I| \notin \mathcal{D}_{j_{k-1},l}, \quad (4.6)
\]

This is done in a straightforward manner by consecutive separation along the orbits of the three shifts \( m_{j_{k-1}} \) and \( m_{j_{k-1}} \pm 2 \). The number of the collections obtained this way admits an universal bound \( L \leq 8^3 \).

We next employ the collections \( \mathcal{D}_{j_{k-1},l} \) to split each of collections \( \mathcal{D}_j \cap \phi(\mathcal{D}) \) with \( j'_{k-1} \leq j < j''_{k-1} \). That is, we define \( \mathcal{B}_{j,l} \) as the collection of intervals in \( \mathcal{D}_j \cap \phi(\mathcal{D}) \) whose dyadic predecessor in \( \mathcal{D}_{j_{k-1}} \) is actually contained in \( \mathcal{D}_{j_{k-1},l} \). Thus

\[
\mathcal{B}_{j,l} = \{ J \in \mathcal{D}_j \cap \phi(\mathcal{D}) : J \subset \mathcal{D}_{j_{k-1},l} \}. \quad (4.7)
\]

Now we fix \( l \leq L \) and \( k \) as above together with \( j'_{k} = j_k + 2 \) and \( j''_{k-1} = j_{k-1} + 2 \), to analyze the joint properties of the intervals in the collection

\[
\mathcal{G} = \bigcup_{j'_{k-1}}^{j''_{k-1}} \mathcal{B}_{j,l}.
\]

If \( I, J \in \mathcal{G} \) with \( |I| \leq |J| \), then by (4.4) we have:

Either \( I \subset J \) or \( \text{dist}(I, J) \geq |I| \).

Clearly by shifting \( I, J \) with \( a_k \) this implies that either \( a_k + I \subset a_k + J \), or \( \text{dist}(a_k + I, a_k + J) \geq |I| \). Next we exploit the condition (4.6). Together with (4.7) condition (4.6) gives

\[
\text{dist}(I, a_k + J) \geq \frac{1}{2^{j_{k-1}}} \quad \text{and} \quad \text{dist}(a_k + I, J) \geq \frac{1}{2^{j_{k-1}}}.
\]

Recall that \( \frac{1}{2^{j_{k-1}}} \) is in fact the length of the largest interval in \( \mathcal{G} \). Therefore, the distance between \( I \) and \( J + a_k \) is always larger than 4 times the length of the largest interval in \( \mathcal{G} \). Hence if \( I \cup (I + a_k) \) is not contained in \( J \cup (J + a_k) \), then \( I \cup (I + a_k) \) and \( J \cup (J + a_k) \) are separated by more than \( \min\{|I|, |J|\} \). Summing up we arrived at the following alternative for \( I, J \in \mathcal{G} \) with \( |I| \leq |J| \) then:

Either \( I \cup (I + a_k) \subset J \cup (J + a_k) \) or \( \text{dist}(I \cup (a_k + I), J \cup (a_k + J)) \geq |I| \) \quad (4.8)

Next we compare the above separation condition with the diameter of the smallest interval containing \( I \) and \( I + a_k \). By (4.2) we have

\[
\text{diam}(I \cup a_k + I) \leq \frac{m_{j_{k-1}} + 1}{2^{j_{k-1}}} = x_{j_{k-1}} + \frac{1}{2^{j_{k-1}}}.
\]

Later in the proof, we will exploit that the separation (4.8) at stage \( k \) is much wider than the diameter in (4.9) at stages \( k + 5 \) and following. It is only this implication which makes
our construction work. It is here where we rely on the strong dichotomy expressed by our hypothesis (4.2)-(4.3). Above \( k \in \mathbb{N} \) and \( l \leq L \) were fixed. We write now

\[
G(k, l) = G.
\]

Thus we obtained the decomposition

\[
\phi \left( \bigcup_{j=j_{h-1}}^{j_{h-1}} D_j \right) = \bigcup_{l=1}^{L} G(k, l), \quad L \leq 8^3. \tag{4.10}
\]

**Consequences of (4.9).** Here we specify the form of the diameter estimates (4.9) at stages later than \( k \). Fix as before \( l \leq L \). Then for \( h > k \) form \( G(h, l) = \bigcup_{j=j_{h-1}}^{j_{h}} B_{j,l} \).

Note that if \( j_{h-1}' \leq j < j_{h}' \) then \( x_j \leq \frac{3}{2^{2h-1}} \).

Therefore by (4.9) if \( I \in G(h, l) \) we get

\[
\text{diam}(I \cup a_h + I) \leq \left[ \frac{4}{2^{h-k-2}} \right] \frac{2}{2^k}.
\]

Next observe that if \( h \) is larger than \( k + 6 \) then the left hand factor on the right hand side of the above estimate is bounded by 1/4. Hence for \( h \geq k + 6 \) and \( I \in G(h, l) \) we get

\[
\text{diam}(I \cup a_h + I) \leq \frac{1}{4} \frac{2}{2^k} \tag{4.11}
\]

**Vertical splitting.** It follows from (4.10) that

\[
\phi(D) = \bigcup_{l=1}^{L} \bigcup_{k=0}^{\infty} G(k, l).
\]

Now, we split the sequence \( \{j_k, k \geq 1\} \) into 6 subsequences \( \{j_{r+6s}, s \geq 0\}, r = 1, \ldots, 6 \). We put

\[
\mathcal{F}_s(r, l) = G(r + 6s, l),
\]

thus

\[
\phi(D) = \bigcup_{l=1}^{L} \bigcup_{r=1}^{6} \bigcup_{s=0}^{\infty} \mathcal{F}_s(r, l).
\]

**Construction of a tree.** Fix \( 1 \leq r \leq 6 \) and \( 1 \leq l \leq L \). Let

\[
\mathcal{F}_s = \mathcal{F}_s(r, l) \quad \text{and} \quad k_s = r + 6s.
\]

For \( I \in \mathcal{F}_s \) and \( n \geq s \), we construct two sets \( B_n(I) \) and \( C_n(I) \), with the following properties:

\[
I \subset B_n(I) \quad \text{and} \quad a_{k_s} + I \subset C_n(I), \tag{4.12}
\]
and inversely

\[ B_n(I) \subset \{ t : \text{dist}(t, I) \leq F_{n-s}|I| \} \quad \text{and} \quad C_n(I) \subset \{ t : \text{dist}(t, a_k + I) \leq F_{n-s}|I| \}, \quad (4.13) \]

where \( F_{n-s} = 2 \sum_{i=1}^{n-s} 4^{-i} \). Thus \( B_n(I) \) is contained in an interval with the same midpoint as \( I \) and diameter bounded by \((1 + 2F_{n-s}) \cdot |I|\), and the same for \( C_n(I) \) and \( a_k + I \). This will give us a nested collection defined by

\[ A_n(I) = B_n(I) \cup C_n(I), \quad I \in \mathcal{F}_0 \cup \cdots \cup \mathcal{F}_n \]

so that for \( I, J \in \mathcal{F}_0 \cup \cdots \cup \mathcal{F}_n \) and \(|I| \leq |J|\) we have:

Either \( A_n(I) \subset A_n(J) \) or \( \text{dist}(A_n(I), A_n(J)) \geq \frac{2}{2^{k_n}} \). (4.14)

We recall that the factor \( \frac{2}{2^{k_n}} \) is the length of the shortest intervals in \( \mathcal{F}_n \).

The construction is inductive.

**Step** \( n = 0 \). For \( I \in \mathcal{F}_0 \), put

\[ B_0(I) = I, \quad C_0(I) = a_{k_0} + I, \quad A_0(I) = B_0(I) \cup C_0(I). \]

**Step** \( n + 1 \). In step \( n \), we defined \( B_n(I), C_n(I) \), for \( I \in \bigcup_{s=0}^{n} \mathcal{F}_s \). For \( J \in \mathcal{F}_{n+1} \) we put

\[ B_{n+1}(J) = J, \quad C_{n+1}(J) = a_{k_{n+1}} + J, \quad A_{n+1}(J) = B_{n+1}(J) \cup C_{n+1}(J). \]

Now we define \( B_{n+1}(I) \) and \( C_{n+1}(I) \) for \( I \in \bigcup_{s=0}^{n} \mathcal{F}_s \) by updating \( B_{n}(I), C_{n}(I) \). To this end we define the index sets

\[ K_{n+1}(I) = \{ J \in \mathcal{F}_{n+1} : \text{dist}(B_n(I), A_{n+1}(J)) \leq \frac{2}{2^{k_{n+1}}} \}, \]

\[ L_{n+1}(I) = \{ J \in \mathcal{F}_{n+1} : \text{dist}(C_n(I), A_{n+1}(J)) \leq \frac{2}{2^{k_{n+1}}} \}, \]

and do the updating

\[ B_{n+1}(I) = B_n(I) \cup \bigcup_{J \in K_{n+1}(I)} A_{n+1}(J), \]

\[ C_{n+1}(I) = C_n(I) \cup \bigcup_{J \in L_{n+1}(I)} A_{n+1}(J), \]

We complete the definition of the tree by putting

\[ A(I) = \bigcup_{n=s}^{\infty} A_n(I) \quad \text{for} \quad I \in \mathcal{F}_s. \]

It follows by (4.13) that if \( I, J \in \mathcal{F} \) and \(|I| \leq |J|\), then either \( A(I) \subset A(J) \) or \( A(I) \cap A(J) = \emptyset \). Moreover, as a consequence of (4.12)-(4.13) we have

\[ I \subset A(I), \quad |\sigma(I) \cap A(I)| \geq \frac{1}{2}|I| \quad \text{and} \quad 2|I| \leq |A(I)| \leq \frac{20}{3}|I|. \]
Verification of \((4.12)-(4.14)\). It remains to check \((4.12)-(4.14)\). The proof is inductive.

For \(n = 0\) and \(I,J \in \mathcal{F}_0\), \((4.14)\) follows by \((4.8)\), while \((4.12)-(4.13)\) are immediate consequences of the definition of \(B_0(I), C_0(I)\).

Assume that \((4.12)-(4.14)\) hold at stage \(n\). To verify them at stage \(n + 1\), recall that for \(J \in \mathcal{F}_{n+1}\) we have \(\text{diam} A_{n+1}(J) \leq \frac{1}{4} 2^{k_n}\), cf. \((4.11)\). For \(I \in \mathcal{F}_n\) we have \(|I| \geq \frac{2}{2^{k_n}}\). Since \(k_n = k_s + 6(n-s)\) it follows that \(j_{k_n}^n \geq 6(n-s) + j_{k_s}^s\). Therefore, using the induction hypothesis on \(B_n(I)\) we get for \(t \in A_{n+1}(J)\) with \(J \in K_{n+1}(I)\)

\[
\text{dist}(t, I) \leq \text{diam} A_{n+1}(J) + \frac{2}{2^{j_{k_n}^n}} + 2 \sum_{i=1}^{n-s} 4^{-i}|I| \leq 2 \sum_{i=1}^{n+1-s} 4^{-i}|I|.
\]

Condition (i) for \(C_{n+1}(I)\) is checked in the same way.

To check condition (ii), we consider several cases:

(a) If \(I,J \in \mathcal{F}_{n+1}\) and \(I \neq J\), then \(\text{dist}(A_{n+1}(I), A_{n+1}(J)) \geq \frac{2}{2^{j_{k_n}^n}}\) by \((4.8)\).

(b) If \(I \in \mathcal{F}_s\) with \(s \leq n\) and \(J \in \mathcal{F}_{n+1}\) is such that \(J \in K_{n+1}(I) \cup L_{n+1}(I)\), then \(A_{n+1}(J) \subset A_{n+1}(I)\), by definition of \(A_{n+1}(I)\).

(c) If \(I \in \mathcal{F}_s\) with \(s \leq n\) and \(J \in \mathcal{F}_{n+1}\) is such that \(J \not\in K_{n+1}(I) \cup L_{n+1}(I)\), then \(\text{dist}(A_n(I), A_{n+1}(J)) \geq \frac{2}{2^k_{k_n+1}}\) by the definition of \(K_{n+1}(I)\) and \(L_{n+1}(I)\). If \(J' \in K_{n+1}(I) \cup L_{n+1}(I)\), then \(\text{dist}(A_{n+1}(J), A_{n+1}(J')) \geq \frac{2}{2^k_{k_n+1}}\) by \((4.8)\). Therefore

\[
\text{dist}(A_{n+1}(I), A_{n+1}(J)) \geq \frac{2}{2^k_{k_n+1}}.
\]

(d) If \(I, J \in \bigcup_{s=0}^n \mathcal{F}_s\) are such that \(A_n(I) \subset A_n(J)\) then \(K_{n+1}(I) \cup L_{n+1}(I) \subset K_{n+1}(J) \cup L_{n+1}(J)\), and consequently \(A_{n+1}(I) \subset A_{n+1}(J)\).

(e) Finally, let \(I, J \in \bigcup_{s=0}^n \mathcal{F}_s\) be such that \(\text{dist}(A_n(I), A_n(J)) \geq \frac{2}{2^k_{k_n}}\). Recall that for \(J' \in K_{n+1}(I) \cup L_{n+1}(I)\) or \(J' \in K_{n+1}(J) \cup L_{n+1}(J)\) we have \(\text{diam} A_{n+1}(J') \leq \frac{1}{4} 2^{j_{k_n}^n}\). Since \(k_{n+1} = k_n + 6\) we have \(j_{k_{n+1}}^n \geq j_{k_n}^n + 6\), and we get

\[
\text{dist}(A_{n+1}(I), A_{n+1}(J)) \geq \text{dist}(A_n(I), A_n(J)) - 2(\frac{2}{2^k_{k_n+1}} + \frac{1}{4} \frac{2}{2^k_{k_n}}) \geq \frac{1}{2^k_{k_n}} - \frac{4}{2^k_{k_n+1}} \geq \frac{2}{2^k_{k_n+1}}.
\]

This completes the construction of a tree for \(\sigma\) restricted to \(\mathcal{F}\).

Subspace theorem for shifts. As immediate application of the above theorem we prove now that the subspace theorem holds for shift operators in a very peculiar way. Without assuming that the shift operator itself is bounded on \(L^p\) we are able to find a subspace of \(L^p\) on which the operator acts as an isomorphism. Moreover our argument here does not use any of the construction developed in Section 3.
Theorem 4.4 Let $M = \{m_j, j \geq 1\}$ be a sequence of integers satisfying $|m_j| \leq 2^j$, and let $\tau_M$ be the associated shift on $\mathcal{D}$. Then there is a sequence $\{j_k, k \geq 1\}$ such that $T_M h_I = h_{\tau_M(I)}$ extends to an isomorphism of $\text{span}_{L^p_E} \{x_I h_I, I \in \bigcup_{k=1}^\infty D_{j_k}, X_I \in E\}$ for each Banach space $E$ with UMD property and $1 < p < \infty$.

Proof. Observe that the sequence $x_n = \frac{m_n}{2^n}$ has an accumulation point. Without loss of generality we can assume that it has a subsequence $n_l$ such that $\lim_{l \to \infty} x_{n_l} = 0$ and $x_{n_l} \neq 0$. The sequence $(j_k, k \geq 1)$ is defined inductively: $j_1 = n_1$. When $j_1, \ldots, j_k$ are already defined, we put

$$j_{k+1} = \min\{n_l > j_k : 0 \leq x_{n_l} < \frac{1}{2^j_{k+1}} \text{ for all } s \geq l\}.$$  

Consider sequence $M' = \{m'_j, j \geq 1\}$ defined by $m'_j = m_j$ if $j = j_k$ for some $k$ and $m'_j = 0$ otherwise. By the definition of the sequence $j_k$ we have $N_{j_k}(M') = 2$ and $N_j(M') = 1$ for $j \neq j_k$. Application of Theorem 4.3 completes the proof.  

Decomposable sequences. Let us fix a shift operator given by $M = \{m_j, j \geq 1\}$. By the following condition we attempt to capture the essence of the conclusion in Proposition 4.2 and at the same time we would like to allow for a higher degree of flexibility. The condition is chosen so that the proof given for Theorem 4.3 actually shows that shifts satisfying the condition below induce isomorphisms on $L^p_E$, for each UMD-space $E$ (See Theorem 4.6)

Definition 4.5 Let $M = \{m_j, j \geq 1\}$ and $X = \{x_j = \frac{m_j}{2^j}, j \geq 1\}$ be such that $\lim_{j \to \infty} x_j = 0$. We say that the sequence $M$ is decomposable if there are $w_1, w_2 \geq 1$, a sequence $\{a_k, k \geq 1\}$ and an increasing sequence of natural numbers $\{j_k, k \geq 1\}$ such that

$$x_j \leq \frac{w_1}{2^j} \quad \text{or} \quad |a_k - x_j| < \frac{w_2}{2^j} \quad \text{for} \quad j_{k-1} \leq j < j_k,$$

and

$$x_j \leq \frac{w_1}{2^{j_k}} \quad \text{for} \quad j \geq j_k.$$  

As stated above, the argument given in the course of proving Theorem 4.3 can be adapted in a straightforward way to provide the proof of the following result.

Theorem 4.6 Let $M$ be a decomposable sequence. Then the operator $T_M h_I = h_{\tau_M(I)}$ extends to an isomorphism of $L^p_E$, for each Banach space $E$ with UMD property and $1 < p < \infty$.

Proposition 4.2 states that a sequence $M$ with $\lim \sup_{j \to \infty} N_j(M) \leq 2$ is decomposable in the sense that it satisfies the condition of Definition 4.5. To close this section, we formulate the following conjecture:

Conjecture 4.7 Each shift satisfying the Semenov-type condition

$$\sup N_j(M) < \infty$$

can be written as a finite composition of shifts satisfying the hypothesis of Definition 4.5.

By Theorem 4.6, the positive answer to this conjecture implies the positive answer to Conjecture 1.1 for shift operators.
5 Combinatorics of coloured intervals

The intricacies of the extension problem as formulated in Conjecturies 1.1 and 1.2 give rise to the following two-person game of general combinatorial interest. The game is played by two players with collections of coloured dyadic intervals in

\[ D_j = \{ I \in D : |I| = 2^{-j} \} \text{ for a fixed } j \in \mathbb{N}. \]

It starts by fixing \( \eta > 0 \), \( d \in \mathbb{N} \), and a subcollection 

\[ C(0) \subset D_j \]

with with an \((\eta, d)\)-homogeneous colouring

\[ C_1(0), \ldots, C_d(0). \]

(see Definition 5.1 below). The rules of the game are as follows: Throughout the game, \( j \in \mathbb{N} \) is fixed.

1. In the first stage, Player A chooses a collection \( C(1) \supseteq C(0) \) and \( C(1) \subset D_j \). Player B determines an \((\eta, d)\)-homogeneous colouring of \( C(1) \) that preserves the colours of \( C(0) \).

2. In the second stage, Player A chooses \( C(2) \supseteq C(1) \) and \( C(2) \subset D_j \). Player B determines an \((\eta, d)\)-homogeneous colouring of \( C(2) \) that preserves the colours of \( C(1) \).

3. At stage \( n \), Player A chooses \( C(n) \supseteq C(n-1) \) and \( C(n) \subset D_j \). Player B determines an \((\eta, d)\)-homogeneous colouring of \( C(n) \) preserving the colours of \( C(n-1) \).

4. The game stops at stage \( n \) if \( C(n-1) = D_j \), and then Player B is the winner, or if there does not exist an \((\eta, d)\)-homogeneous colouring of \( C(n) \) that preserves the colours of \( C(n-1) \). In the second case, Player A is the winner.

Defining homogeneous colourings. For a collection \( C \subset D_j \), consider its partition into \( d \) subcollections \( C = C_1 \cup \ldots \cup C_d \). Such partition we call colouring of \( C \). Colour of \( \Gamma \in C \) means the (unique) index \( i \) such that \( \Gamma \in C_i \).

Let \( L \in D \), \( |L| \geq \frac{1}{2^j} \). Denote

\[
\rho(C, L) = |\{ \Gamma \in C : \Gamma \subset L \}|, \quad \rho_i(C, L) = |\{ \Gamma \in C_i : \Gamma \subset L \}|. \tag{5.1}
\]

**Definition 5.1** Let \( C \subset D_j \), and fix \( d \in \mathbb{N}, 0 < \eta \leq \frac{1}{2} \). Let \( C = C_1 \cup \ldots \cup C_d \) be some decomposition of \( C \). This decomposition is called \((\eta, d)\)-homogeneous colouring of \( C \) if for each \( L \in D \), \( |L| \geq \frac{1}{2^j} \) one of the following holds:

- **Either** \( \rho(C, L) \leq d \), and then
  \[ \rho(C_i, L) \leq 1 \quad \text{for each} \quad 1 \leq i \leq d, \tag{5.2} \]

- **Or else** \( \rho(C, L) > d \), and then
  \[ \eta \max_{1 \leq i \leq d} \rho_i(C, L) \leq \min_{1 \leq i \leq d} \rho_i(C, L). \tag{5.3} \]
Remark: We remark that for each (uncoloured) $C \subset D_j$, $d \in \mathbb{N}$ and $\eta = \frac{1}{2}$ there is always a $(\eta, d)$-homogeneous decomposition that can be obtained as follows: Enumerate the intervals in $C$ from left to right, and simply put

$$C_r = \{ \Gamma_l \in C : l = r \mod d \}, \quad 1 \leq r \leq d.$$ 

The problem we treat in this section consists of finding $(\eta, d)$-homogeneous decompositions of $C \cup U$ under the condition that a previously given $(\eta, d)$-homogeneous decomposition of $C$ is preserved. More precisely, given an $(\eta, d)$-homogeneous decompositions of $C$ and given $U$ disjoint from $C$ we seek to determine a splitting of $U$ as

$$U = U_1 \cup \cdots \cup U_d,$$

which induces an $(\eta, d)$-homogeneous decomposition of $U \cup C$ as

$$U \cup C = U_1 \cup C_1 \cup \cdots \cup U_d \cup C_d.$$ 

We refer to this question as the decompositon/allocation problem for $U$. Note when $U$ contains one element only our problem is just concerned with allocation. It should be pointed out that the smaller $U$ is, the harder it is to find a suitable splitting. Clearly the following possibilities may arise.

1. The solution to the decomposition/allocation problem is unique. That is, there exists just one decomposition of $U$ so that

$$U_1 \cup C_1, \ldots, U_d \cup C_d$$

is an $(\eta, d)$-homogeneous decomposition of $U \cup C$.

2. The decompositon/allocation problem for $U$ does not have a solution.

3. There are conditions on $U$ and $C$ implying that the decompositon/allocation problem for $U$ has a solution. Here it is important that any such condition refers just to $C$ and not to the given and fixed decomposition $\{C_i\}$.

In this section we address these three possibilities and examine the transition from one case to the next.

1. We isolate a condition on $U$ and $C$ (previsibility; see Definition 5.2) implying that the decompositon problem for $U$ has a solution. See Theorem 5.3 which gives rise to winning strategies for Player B.

2. We give examples where the decomposition/allocation problem for $U$ has just one solution. Moreover, we give examples (of $C$, its decomposition $\{C_i\}$ and $U$) for which the decomposition problem does not have a solution. See Proposition 5.4. This translates to an initial configuration of the game, where Player A has a winning strategy.
5.1 Winning Strategies for Player B.

In the following definition we isolate a criterion under which Player B can always make his/her move. Recall that for a dyadic interval $L \in \mathcal{D}$, we say that the intervals $L', L'' \in \mathcal{D}$ are its dyadic successors if $L = L' \cup L''$, $|L'| = |L''| = \frac{1}{2}|L|$.

**Definition 5.2** Let $\mathcal{C}, \mathcal{U} \subset \mathcal{D}_j$, $\mathcal{C} \cap \mathcal{U} = \emptyset$. Let $d \in \mathbb{N}$. The collection $\mathcal{U}$ is called $d$-previsible with respect to the collection $\mathcal{C}$ if for every $L \in \mathcal{D}$ with $|L| \geq \frac{1}{2^j}$ and its dyadic successors $L', L''$, the following holds:

$$\rho(\mathcal{U} \cup \mathcal{C}, L') < d \quad \text{and} \quad \rho(\mathcal{U} \cup \mathcal{C}, L'') \geq d \quad \text{implies} \quad \rho(\mathcal{U}, L'') = 0.$$  

Now, we have the following Theorem that imposes restrictions on the game – specifically on the feasible choices of moves for Player A – under which Player B can build a winning strategy.

**Theorem 5.3** Fix $d \in \mathbb{N}$ and $\eta$, $0 < \eta \leq \frac{1}{2}$. Let $\mathcal{C} \subset \mathcal{D}_j$, and let $\{C_i, 1 \leq i \leq d\}$ be a fixed $(\eta, d)$-homogeneous decomposition of $\mathcal{C}$. Let $\mathcal{U} \subset \mathcal{D}_j$ be $d$-previsible with respect to $\mathcal{C}$. Then there is a decomposition $\{U_i, 1 \leq i \leq d\}$ of $\mathcal{U}$ such that $\{C_i \cup U_i, 1 \leq i \leq d\}$ is an $(\eta, d)$-homogeneous decomposition of $\mathcal{C} \cup \mathcal{U}$.

**Proof.** Denote $\mathcal{H} = \mathcal{C} \cup \mathcal{U}$. We are going to define partition of $\mathcal{U}$ by an inductive argument. Let $\alpha$ be such that $2^{\alpha} \leq d < 2^{\alpha+1}$. Let us observe that if $\frac{1}{2^\alpha} \leq |L| \leq \frac{1}{2^{\alpha+1}}$, then $\rho(\mathcal{H}, L) \leq 2^{\alpha} \leq d$. Thus, if the homogeneity conditions $[5.2]$ respectively $[5.3]$ are satisfied for $L \in \mathcal{D}$ with $|L| \geq \frac{1}{2^j}$, then they are satisfied for each $L \in \mathcal{D}$ with $|L| \geq \frac{1}{2^j}$. Therefore, in our procedure of colouring $\mathcal{U}$ we consider only $L \in \mathcal{D}_k$ with $k \leq j - \alpha$.

**I.** $L \in \mathcal{D}_{j-\alpha}$. Then either $\rho(\mathcal{H}, L) < d$ or $\rho(\mathcal{H}, L) = d$.

**I.1.** If $\rho(\mathcal{H}, L) < d$, then intervals from $\mathcal{U}$ included in $L$ are left uncoloured.

**I.2.** If $\rho(\mathcal{H}, L) = d$, then also $\rho(\mathcal{C}, L) \leq d$, which implies that $\rho_i(\mathcal{C}, L) \leq 1$ for each $1 \leq i \leq d$. In such case it is possible to colour intervals from $\mathcal{U}$ included in $L$ so that $\rho_i(\mathcal{H}, L) = 1$ for each $i$, $1 \leq i \leq d$.

**II.** $L \in \mathcal{D}_\nu$ with $\nu < j - \alpha$. Then $L = L' \cup L''$ with $L', L'' \in \mathcal{D}_{\nu+1}$, $\nu + 1 \leq j - \alpha$. Each interval from $\mathcal{C}$ or $\mathcal{U}$ included in $L$ is included in $L'$ or $L''$, so we have

$$\rho(\mathcal{C}, L) = \rho(\mathcal{C}, L') + \rho(\mathcal{C}, L''),$$

$$\rho(\mathcal{U}, L) = \rho(\mathcal{U}, L') + \rho(\mathcal{U}, L''),$$

$$\rho(\mathcal{H}, L) = \rho(\mathcal{H}, L') + \rho(\mathcal{H}, L'').$$

**Induction hypothesis:** Let $K \in \mathcal{D}_{\nu+1}$. If $\rho(\mathcal{H}, K) < d$, then intervals from $\mathcal{U}$ included in $K$ are still uncoloured. If $\rho(\mathcal{H}, K) \geq d$, then all intervals from $\mathcal{U}$ included in $K$ are coloured, hence the counting parameters $\rho_i(\mathcal{H}, K)$ are well defined by $[5.1]$. They satisfy $\rho_i(\mathcal{H}, K) \geq 1$ and

$$\eta \max_{1 \leq i \leq d} \rho_i(\mathcal{H}, K) \leq \min_{1 \leq i \leq d} \rho_i(\mathcal{H}, K). \quad (5.4)$$

Then we have two main cases:

**II.1.** $\rho(\mathcal{H}, L) < d$. Clearly, then also $\rho(\mathcal{H}, L'), \rho(\mathcal{H}, L'') < d$, and intervals from $\mathcal{U}$ included in $L', L''$ are uncoloured.
If $\nu > 0$, then leave intervals from $\mathcal{U}$ included in $\mathcal{L}$ still uncoloured.

If $\nu = 0$, then $\mathcal{L} = [0, 1]$, and the induction ends. This means that $|\mathcal{H}| < d$, and it is enough to assign elements of $\mathcal{U}$ to colours different from colours of elements of $\mathcal{C}$.

II.2. $\rho(\mathcal{H}, L) \geq d$. Then we have next four subcases:

II.2.1. $\rho(\mathcal{H}, L') \geq d$ and $\rho(\mathcal{H}, L'') \geq d$. Then by induction hypothesis all intervals from $\mathcal{U}$ included in $L'$ and $L''$ are already coloured, i.e. all intervals from $\mathcal{U}$ included in $L$ are coloured. Moreover, for each $1 \leq i, k \leq d$

$$\eta \rho_i(\mathcal{H}, L) = \eta \rho_i(\mathcal{H}, L') + \eta \rho_i(\mathcal{H}, L'') \leq \rho_k(\mathcal{H}, L') + \rho_k(\mathcal{H}, L'') = \rho_k(\mathcal{H}, L).$$

Of course, we have also $\rho_i(\mathcal{H}, L) \geq 1$.

II.2.2. $\rho(\mathcal{H}, L') < d$ and $\rho(\mathcal{H}, L'') < d$. Then by induction hypothesis all intervals from $\mathcal{U}$ included in $L'$ and $L''$ are uncoloured, but the intervals from $\mathcal{C}$ carry their colours.

Now, we need to colour all intervals from $\mathcal{U}$ included in $L', L''$. To simplify notation, let $m = \rho(\mathcal{C}, L')$, $n = \rho(\mathcal{C}, L'')$, $x = \rho(\mathcal{U}, L')$, $y = \rho(\mathcal{U}, L'')$. We have $0 \leq m, n \leq d - 1$, $0 \leq m + x, n + y \leq d - 1$ and $d \leq \rho(\mathcal{H}, L) = m + x + n + y \leq 2(d - 1)$.

First consider the case $m + n < d$. Then $0 \leq \rho_i(\mathcal{C}, L) \leq 1$ for each $i$. For simplicity, assume that intervals from $\mathcal{C}$ included in $L'$ have colours $1, \ldots, m$, and intervals from $\mathcal{C}$ included in $L''$ have colours $m + 1, \ldots, m + n$. Now, we colour intervals from $\mathcal{U}$. First, colour intervals from $\mathcal{U}$ included in $L'$ using colours $m + n + 1, \ldots, d$, and then, if necessary (i.e. $x > d - (m + n)$), continuing with $x - (d - (m + n))$ colours from $m + 1, \ldots, m + n$; since $m + x < d$, in this way we assign colours to all intervals from $\mathcal{U}$ included in $L'$. Next, we assign colours to intervals from $\mathcal{U}$ included in $L''$. If $m + n + x < d$, then assign first colours $m + n + x + 1, \ldots, d$, then continue with colours $1, \ldots, m$, and then if necessary with colours $m + n + 1, \ldots, m + n + x$. If $m + n + x \geq d$, then just choose $y$ different colours from $1, \ldots, m$ and $m + n + 1, \ldots, d$.

With such colouring of intervals from $\mathcal{U}$ included in $L'$ and $L''$ we find that both $\rho_i(\mathcal{H}, L') \leq 1$ and $\rho_i(\mathcal{H}, L'') \leq 1$. This implies that for each $K \subset L'$ or $K \subset L''$ we have $\rho(\mathcal{H}, K) < d$ and $\rho_i(\mathcal{H}, K) \leq 1$. Moreover, we get $1 \leq \rho_i(\mathcal{H}, L) \leq 2$, which implies

$$\eta \max_{1 \leq i \leq d} \rho_i(\mathcal{H}, L) \leq \frac{1}{2} \max_{1 \leq i \leq d} \rho_i(\mathcal{H}, L) \leq \min_{1 \leq i \leq d} \rho_i(\mathcal{H}, L).$$

It remains to consider the case $m + n \geq d$. Then the homogeneity assumption on the decomposition of $\mathcal{C}$ in (5.2) for $L', L''$ and (5.3) for $L$ implies $1 \leq \rho_i(\mathcal{C}, L) = \rho_i(\mathcal{C}, L') + \rho_i(\mathcal{C}, L'') \leq 2$. For simplicity, assume that intervals from $\mathcal{C}$ included in $L'$ have colours $1, \ldots, m$ and intervals from $\mathcal{C}$ included in $L''$ have colours $m + 1, \ldots, d$ and $1, \ldots, m + n - d$. To colour intervals from $\mathcal{U}$ included in $L'$ choose $x$ colours from $m + 1, \ldots, d$. To colour intervals from $\mathcal{U}$ included in $L''$ choose $y$ colours from $m + n - d + 1, \ldots, m$. This is possible since $m + x < d$ and $n + y < d$. Observe that in this way we get $0 \leq \rho_i(\mathcal{H}, L'), \rho_i(\mathcal{H}, L'') \leq 1$ and $1 \leq \rho_i(\mathcal{H}, L) \leq 2$. Therefore, for each $K \subset L'$ or $K \subset L''$ we have $\rho_i(\mathcal{H}, K) \leq 1$, while for $L$ we have

$$\eta \max_{1 \leq i \leq d} \rho_i(\mathcal{H}, L) \leq \frac{1}{2} \max_{1 \leq i \leq d} \rho_i(\mathcal{H}, L) \leq \min_{1 \leq i \leq d} \rho_i(\mathcal{H}, L).$$

II.2.3. $\rho(\mathcal{H}, L') < d$ and $\rho(\mathcal{H}, L'') \geq d$. Then by induction hypothesis all intervals from $\mathcal{U}$ included in $L'$ are uncoloured, but the intervals from $\mathcal{C}$ included in $L'$ carry their colours. Since $\mathcal{U}$ is $d$-previsble with respect to $\mathcal{C}$, we have $\rho(\mathcal{U}, L'') = 0$. Therefore, $\rho(\mathcal{H}, L'') = \rho(\mathcal{C}, L'')$,
and by condition (5.3) of the \((\eta, d)\)-homogeneity for \(C\), we get \(\rho_t(H, L'') = \rho_t(C, L'') \geq 1\) and \(\rho_t(H, L'')\) satisfy (5.4).

If \(\rho(U, L') = 0\) as well, then all intervals from \(H\) included in \(L\) come from \(C\), and there is nothing to do.

Let \(\rho(U, L') = x > 0\). We need to colour \(x\) intervals from \(U\) included in \(L'\). To simplify notation, let \(m = \rho(C, L')\). Note that \(1 \leq m + x < d\). Let \(S = \{i : \rho_t(C, L') = 1\}\) and \(T = \{i : \rho_t(C, L') = 0\}\). Let \(t_1, \ldots, t_{d-m}\) be an ordering of \(T\) such that

\[
\rho_{t_1}(C, L'') \leq \ldots \leq \rho_{t_{d-m}}(C, L'').
\]  

(5.5)

Since \(x < d - m\), there are more colours in \(T\) than intervals in \(U\) that are included in \(L'\). Now attach the colours \(t_1, \ldots, t_x\), bijectively, to intervals in \(U\) contained in \(L'\). Then \(\rho_t(H, L') \leq 1\).

It remains to check that \(\rho_t(H, L)\) satisfy (5.4). By assumption on partition of \(C\) we have

\[
\eta \max_{1 \leq i \leq d} \rho_t(C, L'') \leq \min_{1 \leq i \leq d} \rho_t(C, L''),
\]  

(5.6)

\[
\eta \max_{1 \leq i \leq d} \rho_t(C, L) \leq \min_{1 \leq i \leq d} \rho_t(C, L).
\]  

(5.7)

Moreover,

\[
\begin{align*}
\rho_t(H, L) &= \rho_t(C, L) = \rho_t(C, L'') + 1 & \text{for } i \in S, \\
\rho_t(H, L) &= \rho_t(C, L) + 1 = \rho_t(C, L'') + 1 & \text{for } i = t_1, \ldots, t_x, \\
\rho_t(H, L) &= \rho_t(C, L) = \rho_t(C, L'') & \text{for } i = t_{x+1}, \ldots, t_{d-m}.
\end{align*}
\]  

(5.8)

Observe that if \(\max_i \rho_t(H, L) = \rho_t(H, L)\), then \(k \in S\) or \(k \in T\); in case \(k \in T\) we have \(k = t_x\) or \(k = t_{d-m}\), because of ordering (5.5). If \(k \in S\), then (5.4) is satisfied for \(L\) and \(H\) because of (5.7) and the first line of (5.8). If \(k = t_{d-m}\), then (5.4) is satisfied for \(L\) and \(H\) because of (5.6) and the last line of (5.8). If \(k = t_x\) and \(\rho_{t_x}(H, L) > \rho_{t_{d-m}}(H, L)\) then we check inequality

\[
\eta \rho_{t_x}(H, L) \leq \rho_t(H, L) \quad \text{for } 1 \leq i \leq d.
\]  

(5.9)

For \(i \in S\) inequality (5.9) is satisfied because of (5.6) and the first two lines of (5.8). For \(i = t_1, \ldots, t_x\) inequality (5.9) is satisfied because of (5.7) and the second line of (5.8). When \(\rho_{t_x}(H, L) > \rho_{t_{d-m}}(H, L)\), then the two last lines of (5.8) and the ordering (5.5) imply \(\rho_{t_x}(C, L'') = \rho_{t_{x+1}}(C, L'') = \ldots = \rho_{t_{d-m}}(C, L'')\). This implies that inequality (5.9) is satisfied, even with \(\frac{1}{2}\) on the left-hand-side, for \(i = t_{x+1}, \ldots, t_{d-m}\).

II.2.4. \(\rho(H, L') \geq d\) and \(\rho(H, L'') < d\). This case is analogous to II.2.3.

This completes the proof of Theorem 5.3.

5.2 Winning Strategies for Player A.

Here we analyze the role of the previsibility assumption in Theorem 5.3. We do this by defining an initial configuration of the two-person game so that Player A has a strategy to win in exactly \(n\) moves. This corresponds to a sequence of examples for which the decomposition/allocation
We start with \( C(0) \), its initial decomposition and \( U(0) \) in such a way that the decomposition problem for \( U(0) \) has just one solution. This uniquely determined solution defines the decomposition for \( C(1) = U(0) \cup C(0) \). This and the given \( U(1) \) determines a decomposition problem for which we will see that it again has only one solution. This solution in turn determines a splitting of \( C(2) = U(1) \cup C(1) \) which again leads to a decomposition problem with a unique solution. This will go on until we reach \( C(n-1) \) and its decomposition that has been determined uniquely by \( C(0) \) its initial decomposition and by our choice of \( U(0), \ldots, U(n-2) \). Then we change the situation and choose the collection \( U(n-1) \) that forces the decomposition problem in \( U(n-1) \cup C(n-1) \) to be without solution.

Throughout this section we take \( d = 2^a \), \( a \in \mathbb{N} \), and \( \eta = \frac{1}{n} \) with \( n \in \mathbb{N} \) and \( j \geq n + a + 1 \).

**Proposition 5.4** There exist

\[ C(0) \subset D_j \]

with \((\eta,d)\)-homogeneous decomposition

\[ C_1(0), \ldots, C_d(0) \]

and an increasing chain of collections

\[ C(0) \subset C(1) \subset \ldots \subset C(n) \subset D_j \]

so that for

\[ U(k) = C(k+1) \setminus C(k) \quad \text{with} \quad 1 \leq k \leq n - 1, \]

the following conditions hold:

(A) Stage 0. There exists exactly one splitting of \( U(0) \) as \( U_1(0), \ldots, U_d(0) \) so that

\[ U_1(0) \cup C_1(0), \ldots, U_d(0) \cup C_d(0) \]

is an \((\eta,d)\)-homogeneous decomposition of \( U(0) \cup C(0) \), hence of \( C(1) \).

(B) Stage \( k \), \( 1 \leq k \leq n - 2 \), let

\[ C_1(k), \ldots, C_d(k) \]

be the unique \((\eta,d)\)-homogeneous decomposition of \( C(k) \), obtained at stage \( k - 1 \). There exists exactly one splitting of \( U(k) \) as \( U_1(k), \ldots, U_d(k) \) so that

\[ U_1(k) \cup C_1(k), \ldots, U_d(k) \cup C_d(k) \]

is an \((\eta,d)\)-homogeneous decomposition of \( U(k) \cup C(k) \), hence of \( C(k+1) \).

(C) Stage \( n - 1 \). Let

\[ C_1(n-1), \ldots, C_d(n-1) \]

be the unique \((\eta,d)\)-homogeneous decomposition of \( C(n-1) \), obtained at stage \( n - 2 \). There does not exist a splitting of \( U(n-1) \) as \( U_1(n-1), \ldots, U_d(n-1) \) so that

\[ U_1(n-1) \cup C_1(n-1), \ldots, U_d(n-1) \cup C_d(n-1) \]

is an \((\eta,d)\)-homogeneous decomposition of \( C(n) = U(n-1) \cup C(n-1) \).
Proof. Observe that for each $j$, the testing levels for $C \subset D_j$ are $D_j, D_{j-1}, \ldots, D_1, D_0$. Since $j \geq n + a + 1$, there are at least $n + 2$ testing levels. Take a chain of dyadic intervals
\[ L_1 \subset L_2 \subset \ldots \subset L_{n+2}, \quad L_i \in D_{j-a-i+1}. \]
Then $|L_i| = \frac{1}{2}|L_{i+1}|$, and let $P_i$ be the dyadic brother of $L_i$ in $L_{i+1}$, $i = 1, \ldots, n + 1$. Thus $P_i = L_{i+1} \setminus L_i$.

Now, take two sets of intervals from $D_j$:
\[ I_1, \ldots, I_{d-1} \in D_j \text{ such that } I_i \subset L_1 \text{ for each } i = 1, \ldots, d-1, \]
\[ J_1, \ldots, J_{n+1} \in D_j \text{ such that } J_i \subset P_i \text{ for each } i = 1, \ldots, n + 1. \]
Consider the following sequence of collections:
\[ C(0) = \{ I_1, \ldots, I_{d-1} \} \cup \{ J_{n+1} \}, \]
\[ C(k) = \{ I_1, \ldots, I_{d-1} \} \cup \{ J_{n-k+1}, \ldots, J_{n+1} \}. \]
Observe that
\[ U(k) = \{ J_{n-k} \}, \quad k = 0, \ldots, n - 1, \]
and
\[ J_i \in C(n+1-i), \quad (5.10) \]
and since our chain of collections $C(0), \ldots, C(n)$ is increasing, $J_i$ is also contained in $C(k)$ with $k \geq n + 1 - i$.

Initialization — verification of (A). Consider possible colourings of $C(0)$. Take $L_{n+2}$ as a testing interval. Observe that $\rho(C(0)) = \rho(C(0), L_{n+2}) = d$, so if we want to have $(\eta, d)$-homogeneity, we must have (5.2) and therefore $\rho_i(C(0), L_{n+2}) = 1$ for each $i = 1, \ldots, d$. Without loss of generality we can assume that $J_{n+1}$ has colour 1, and each $I_i$ has colour $i + 1$, $i = 1, \ldots, d-1$. Therefore for $C(0)$ and each testing interval $L \subset L_{n+2}$ we have $\rho_i(C(0), L) \leq 1$ for each $1 \leq i \leq d$.

The basic observation. Our example is based on iterating systematically the following basic observation. Let $k \leq n$. Assume that $C(k)$ has an $(\eta, d)$-homogeneous decomposition as
\[ C_1(k), \ldots, C_d(k), \]
so that
\[ C_1(0) \subset C_1(k), \ldots, C_d(0) \subset C_d(k). \]
Then necessarily
\[ J_{n-k+1} \text{ must have colour 1.} \quad (5.11) \]

Verification of (5.11). We know already that $J_{n+1}$ has to have colour 1. To check the claim for $J_{n-k+1}, k = 1, \ldots, n$ we consider the pair of collections $C(0) \subset C(k)$:
\[ C(k) = C(0) \cup \{ J_{n-k+1}, \ldots, J_n \}. \]
and testing interval $L_{n-k+2}$. Elements of $C(0)$ included in $L_{n-k+2}$ are $I_1, \ldots, I_{d-1}$. In addition, $J_{n-k+1} \subset I_{n-k+1} \subset L_{n-k+2}$, while $J_{n-k+2}, \ldots, J_n \not\subset L_{n-k+2}$. Therefore we have
\[ \rho(C(0), L_{n-k+2}) = d - 1, \quad \rho(C(k), L_{n-k+2}) = d, \]
\[ \rho_1(C(0), L_{n-k+2}) = 0 \quad \text{and} \quad \rho_i(C(0), L_{n-k+2}) = 1 \text{ for } i = 2, \ldots, d. \]
Therefore, (5.2) of the $(\eta, d)$-homogeneity condition for $C(k)$ implies that $J_{n-k+1}$ is of colour 1.
Verification of (B). Recall that $0 \leq k \leq n$

$$\mathcal{C}(k) = \{I_1, \ldots, I_{d-1}\} \cup \{J_{n-k+1}, \ldots J_{n+1}\}.$$ 

Moreover, by (5.11), the only possible $(\frac{1}{n}, d)$-homogeneous decomposition of $\mathcal{C}(k)$ is 

$$\mathcal{C}_1(k) = \{J_{n-k+1}, \ldots J_{n+1}\}, \quad \mathcal{C}_i(k) = \{I_{i-1}\} \quad \text{for} \quad 2 \leq i \leq d$$

Let’s check that for $0 \leq k \leq n - 1$, the above decomposition of $\mathcal{C}(k)$ is indeed $(\frac{1}{n}, d)$-homogeneous. We present the detailed proof for $k = n - 1$, since the cases $k \leq n - 1$ are fully analogous.

First, take as a testing interval $L_s$, $s = 3, \ldots n + 2$. Then elements of $\mathcal{C}(n - 1)$ included in $L_s$ are $I_1, \ldots, I_{d-1}$ and $J_2, \ldots, J_{s-1}$. Therefore

$$\rho(\mathcal{C}(n - 1), L_s) = s + d - 3,$$

and

$$\rho_1(\mathcal{C}(n - 1), L_s) = s - 2, \quad \rho_i(\mathcal{C}(n - 1), L_s) = 1 \quad \text{for} \quad i = 2, \ldots d.$$ 

Therefore

$$\frac{1}{n} \max_{1 \leq i \leq d} \rho_i(\mathcal{C}(n - 1), L_s) \leq \min_{1 \leq i \leq d} \rho_i(\mathcal{C}(n - 1), L_s), \quad s = 3, \ldots, n + 2.$$ 

Next take as a testing interval $L_2$. Then elements of $\mathcal{C}(n - 1)$ included in $L_2$ are $I_1, \ldots, I_{d-1}$, so $\rho(\mathcal{C}(n - 1), L_2) = d - 1$,

$$\rho_1(\mathcal{C}(n - 1), L_2) = 0, \quad \rho_i(\mathcal{C}(n - 1), L_2) = 1 \quad \text{for} \quad i = 2, \ldots d.$$ 

Therefore $L_2$ also satisfies (5.2) of the $(\frac{1}{n}, d)$-homogeneity condition for $\mathcal{C}(n - 1)$. Consequently, $L_1, P_1 \subset L_2$ also satisfy these conditions.

Finally, take as a testing interval $P_k$, $k = 2, \ldots, n + 1$. The only element of $\mathcal{C}(n - 1)$ included in $P_k$ is $J_k$, so $\rho(\mathcal{C}(n - 1), P_k) = 1$, and more precisely

$$\rho_1(\mathcal{C}(n - 1), P_k) = 1, \quad \rho_i(\mathcal{C}(n - 1), P_k) = 0 \quad \text{for} \quad i = 2, \ldots d.$$ 

Thus, $P_k$ (and consequently, each testing interval included in $P_k$) satisfies (5.2) of the $(\frac{1}{n}, d)$-homogeneity condition for $\mathcal{C}(n - 1)$.

Verification of (C). Consider $\mathcal{C}(n - 1)$ and $\mathcal{C}(n) = \mathcal{C}(n - 1) \cup \mathcal{U}(n - 1)$. Recall that

$$\mathcal{C}(n) = \{I_1, \ldots, I_{d-1}\} \cup \{J_1, J_2, \ldots J_{n+1}\}.$$ 

Take $L_{n+2}$ as a testing interval. All intervals from $\mathcal{C}(n)$ are included in $L_{n+2}$, and the colouring yields

$$\rho_1(\mathcal{C}(n), L_{n+2}) = n + 1, \quad \rho_i(\mathcal{C}(n), L_{n+2}) = 1 \quad \text{for} \quad i = 2, \ldots d.$$ 

For $\mathcal{C}(n)$ and $L_{n+2}$ we have to consider (5.3) of the $(\frac{1}{n}, d)$-homogeneity condition. But the above formulae mean that for $\mathcal{C}(n)$ and testing interval $L_{n+2}$, the condition (5.3) is satisfied with $\eta' = \frac{1}{n+1}$, but not with $\eta = \frac{1}{n}$.

\[\square\]

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Remark. For $0 \leq k \leq n - 1$, the collection $U(k)$ is not previsible with respect to $C(k)$. Nevertheless, the colouring problem has a solution for $0 \leq k \leq n - 2$.

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