INTERVAL INCIDENCE COLORING OF SUBCUBIC GRAPHS

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Abstract

In this paper we study the problem of interval incidence coloring of subcubic graphs. In [14] the authors proved that the interval incidence 4-coloring problem is polynomially solvable and the interval incidence 5-coloring problem is \(\mathcal{NP}\)-complete, and they asked if \(\chi_{ii}(G) \leq 2\Delta(G)\) holds for an arbitrary graph \(G\). In this paper, we prove that an interval incidence 6-coloring always exists for any subcubic graph \(G\) with \(\Delta(G) = 3\).

Keywords: interval incidence coloring, incidence coloring, subcubic graph.

2010 Mathematics Subject Classification: 05C15, 05C85, 05C69.

1. Introduction

In the paper we consider simple nonempty graphs, and we use the standard notation of graph theory. Let \(G = (V, E)\) be a simple graph, and let \(X \subset V\) be a non-empty set. By \(N_G(X) = \{v \in V : \exists u \in X \{v, u\} \in E\}\) we mean the open

\footnote{This project has been partially supported by Narodowe Centrum Nauki under contract DEC-2011/02/A/ST6/00201.}
Guiduli in [9] who observed that Paley graphs have incidence coloring number at

In what follows we use degree of a vertex $\Delta(G)$ and by

neighborhood of $X$, by $G[X]$ we mean the subgraph of $G$ induced by the set $X$, and by $G \setminus X$ we mean the graph $G[V \setminus X]$. We say that $X$ is a dominating set of $G$ if $V = N_G(X) \cup X$, and we say that $X$ is a total dominating set if $V = N_G(X)$. In what follows we use $N_G(v)$ instead of $N_G(\{v\})$. Let $\deg_G(v) = |N_G(v)|$ be the degree of a vertex $v \in V(G)$. By $n(G)$, $\Delta(G)$ and $\delta(G)$ we denote the number of vertices of $G$, the maximum and the minimum degree of a vertex of $G$, respectively. By a subcubic graph $G$ we mean a graph with $\Delta(G) \leq 3$. By an isolated vertex (in a graph $G$) we mean a vertex $v \in V(G)$ with $\deg_G(v) = 0$, and by an isolated edge (in a graph $G$) we mean an edge $e = \{u, v\}$ such that $\deg_G(u) = \deg_G(v) = 1$. We say that $X \subset V(G)$ is an independent set if each vertex of $G[X]$ is isolated in $G[X]$. By a pendant vertex we mean a vertex of degree 1.

For a given graph $G = (V, E)$, we define an incidence as a pair $(v, e)$, where vertex $v \in V$ is one of the endpoints of edge $e \in E$, i.e., $v \in e$. The set of all incidences of $G$ will be denoted by $I(G)$, thus $I(G) = \{(v, e) : v \in V \wedge e \in E \wedge v \in e\}$. We say that two incidences $(v, e)$ and $(w, f)$ are adjacent if one of the following holds: (1) $v = w$ and $e \neq f$; (2) $e = f$ and $v \neq w$; (3) $e = \{v, w\}$, $f = \{w, u\}$ and $v \neq u$.

By an incidence coloring of $G$ we mean a function $c : I(G) \to \mathbb{N}$ such that $c((v, e)) \neq c((w, f))$ for any two adjacent incidences $(v, e)$ and $(w, f)$. The incidence coloring number of $G$, denoted by $\chi_i(G)$, is the smallest number of colors in an incidence coloring of $G$. In what follows we use the simplified notation $c(v, e)$ instead of $c((v, e))$.

A finite nonempty set $A \subset \mathbb{N}$ is an interval if it contains all integers between $\min A$ and $\max A$. For a given incidence coloring $c$ of graph $G$ and $v \in V(G)$ let $A_c(v) = \{c(v, e) : e \in V \wedge e \in E(G)\}$. By an interval incidence coloring of a graph $G$ we mean an incidence coloring $c$ of $G$ such that for each vertex $v \in V(G)$ the set $A_c(v)$ is an interval. By an interval incidence $k$-coloring we mean an interval incidence coloring using all colors from the set $\{1, \ldots, k\}$. The interval incidence coloring number of $G$, denoted by $\chi_{ii}(G)$, is the smallest number of colors in an interval incidence coloring of $G$.

1.1. Background and previous results

Alon et al. [1] defined the problem of partitioning a graph into the minimal number of star forests. Brualdi and Massey [3] formulated a model of incidence coloring of graphs with references to certain models of coloring of graphs, such as strong edge and vertex coloring of graphs. Guiduli [9] observed that the problem of incidence coloring of graphs is a special case of the problem of partitioning a symmetric digraph into directed star forests.

In [3] the authors conjectured that $\chi_i(G) \leq \Delta(G) + 2$ holds for every graph $G$ (incidence coloring conjecture, shortly ICC). This conjecture was disproved by Guiduli in [9] who observed that Paley graphs have incidence coloring number at
least $\Delta + \Omega(\log \Delta)$. In fact, he used the crucial result from [1]. For many classes of graphs it is shown that the incidence coloring number is at most $\Delta + 2$, e.g., trees and cycles [3], complete graphs [3], complete bipartite graphs [3] (proof corrected in [19]), planar graphs with girth at least 11 or with girth at least 6 and maximum degree at least 5 [5], partial 2-trees (i.e., $K_4$-minor free graphs) [4], hypercubes [18], complete $k$-partite graphs [15].

In [17] the author proved that ICC holds for subcubic graphs. The incidence 4-colorability problem is $\mathcal{NP}$-complete for semicubic graphs (i.e., subcubic graphs with vertex degrees equal to 1 or 3) [16] and for semicubic bipartite graphs [15].

In this paper we consider a restriction of the problem of incidence coloring of graphs in which the colors of incidences at a vertex form an interval. Interval incidence coloring is a new concept arising from a well-studied model of interval edge-coloring (see, e.g., [2, 6, 8]), which can be applied to the open-shop scheduling problem [6, 7]. In [11] the authors introduced the concept of interval incidence coloring that models a message passing flow in networks, and in [12] the authors studied applications in one-multicast transmission in multifiber WDM networks.

In [13] the authors proved that the problem of interval incidence $k$-coloring of bipartite graphs is polynomial for each $k \leq 6$ and $\Delta \leq 3$, polynomial for $k = 5$ and $\Delta = 4$, and $\mathcal{NP}$-complete for $k = 6$ and $\Delta = 4$. In [14] the authors proved certain lower and upper bounds on the interval incidence coloring number, e.g., $\Delta(G) + 1 \leq \chi_{ii}(G) \leq \chi(G) \cdot \Delta(G)$ for an arbitrary graph $G$, and they determined the exact values of $\chi_{ii}$ for some basic classes of graphs (e.g., complete $k$-partite graphs). In [14] the authors also studied the complexity of the interval incidence coloring problem for subcubic graphs for which they showed that the problem of deciding whether $\chi_{ii} \leq 4$ is easy, and $\chi_{ii} \leq 5$ is $\mathcal{NP}$-complete. The problem of interval incidence 6-coloring of subcubic graphs remained unsolved.

1.2. Main results

Our main result in the paper is Theorem 21 which states $\chi_{ii}(G) \leq 6$ for every subcubic graph $G$. To prove it, we state and prove Theorem 8: in any subcubic graph $G$ with $\delta(G) \geq 2$ there is a maximal induced bipartite subgraph of $G$ without isolated vertices, or equivalently, $G$ has a total dominating set $S$ such that $G[S]$ is a bipartite graph.

2. Maximal Induced Bipartite Subgraphs Without Isolated Vertices

In this section we prove (in Theorem 8) that any subcubic graph $G$ with $\delta(G) \geq 2$ contains a maximal induced bipartite subgraph without isolated vertices.
2.1. Introductory properties

By $H \subseteq G$ we mean that $H$ is a subgraph of $G$. By $H \sqsubseteq G$ we mean that $H$ is an induced subgraph of $G$, i.e., $H = G[V(H)]$.

**Observation 1.** If $G_1 \sqsubseteq G_2$ and $G_2 \sqsubseteq G_3$, then $G_1 \sqsubseteq G_3$.

**Observation 2.** Let $G_1 \sqsubseteq G$ and $G_2 \sqsubseteq G$. If $G_1 \subseteq G_2$, then $G_1 \sqsubseteq G_2$.

Let $B(G) = \{H \subseteq G : N_G(V(H)) = V(G) \land H \text{ is bipartite}\}$, i.e., the set of all induced bipartite subgraphs of a given graph $G$ such that $V(H)$ is a total dominating set of $G$. If $H \in B(G)$, then $V(H)$ is a total dominating set of $G$ and, obviously, $H$ has no isolated vertices.

In the following, let $G$ be any graph. Let $\hat{B}(G)$ be the subfamily of $B(G)$ consisting of all the elements (graphs) in $B(G)$ that are maximal with respect to the subgraph relation ($\subseteq$).

**Observation 3.** If $H \in B(G)$, then there is $H' \in \hat{B}(G)$ such that $H \subset H'$.

By Observations 2 and 3 we have

**Observation 4.** Let $H \in B(G)$. Then, $H \in \hat{B}(G)$ if and only if for each $v \in V(G) \setminus V(H)$ the subgraph $G[V(H) \cup \{v\}]$ is not bipartite.

**Observation 5.** If $H \in B(G) \setminus \hat{B}(G)$, then there is a vertex $v \in V(G) \setminus V(H)$ such that $G[V(H) \cup \{v\}] \in B(G)$.

Since any dominating set $S \subseteq V(G)$ is a total dominating set if and only if $G[S]$ has no isolated vertices, we have

**Observation 6.** Let $G$ be an arbitrary graph and let $H \subseteq G$. Then, $H \in \hat{B}(G)$ if and only if $H$ is a maximal induced bipartite subgraph (of $G$) without isolated vertices.

Let $G^2_3$ be the family of subcubic graphs without isolated and pendant vertices, i.e., each vertex in a graph of this family has degree 2 or 3. Let $\mathcal{M}^2_3$ be the subfamily of $G^2_3$ consisting of all the graphs for which there is no maximal induced bipartite subgraph without isolated vertices. Let us denote by $\mathcal{M}$ the set of elements in $\mathcal{M}^2_3$ that are minimal with respect to the subgraph relation ($\subseteq$). By Observation 6 we have

**Observation 7.** Let $G \in G^2_3$. Then, $G \in \mathcal{M}^2_3 \iff B(G) = \emptyset \iff \hat{B}(G) = \emptyset$. 
2.2. Main Theorem

Theorem 8. Let $G$ be a subcubic graph with $\delta(G) \geq 2$. Then, $G$ has a maximal induced bipartite subgraph without isolated vertices.

By Observation 7, Theorem 8 is equivalent to $\mathcal{M} = \emptyset$. First, we prove some structural properties of graphs from $\mathcal{M}$.

Lemma 9. Let $G \in \mathcal{M}$. Then, $G$ is a connected graph and $\Delta(G) = 3$.

Proof. Let $G \in \mathcal{M}$. Let us assume to the contrary that $G = G_1 \cup G_2$, where $G_1$ and $G_2$ are disjoint graphs (without common vertices). Since $G_1 \subseteq G \in \mathcal{M}$ and $G_i \in G_3^2$, we have $G_i \notin M_3^2$, for $i \in \{1, 2\}$. Hence, there exist $H_1 \in \mathcal{B}(G_1)$ and $H_2 \in \mathcal{B}(G_2)$. Thus, $H_1 \cup H_2 \in \mathcal{B}(G)$, a contradiction.

Since every cycle is either a bipartite graph or it becomes a bipartite graph after deleting an arbitrary vertex, $G$ is not a cycle, which implies $\Delta(G) = 3$. ■

Lemma 10. Let $G \in \mathcal{M}$ and let $v$ be a vertex of degree 2 in $G$. Then, every neighbor of $v$ in $G$ has degree 3.

Proof. Let $G \in \mathcal{M}$. Suppose to the contrary that there are two adjacent vertices of degree 2. Since $G$ is not a cycle (by Lemma 9), there is a subgraph $P$ of $G$ with vertex set $\{v_0, \ldots, v_k\}$ and edges $\{v_i, v_{i+1}\}$, for $i \in \{0, \ldots, k\}$, such that $deg_G(v_0) = deg_G(v_{k+1}) = 3$, and $deg_G(v_i) = 2$ for $i \in \{1, \ldots, k\}$, where $k \geq 2$.

Suppose $v_0 \neq v_{k+1}$. Since $G' = G \setminus \{v_1, \ldots, v_k\} \subseteq G \in \mathcal{M}$ and $G' \in G_3^2$, we have $G' \notin M_3^2$. Hence, there exists $H' \in \mathcal{B}(G')$, and $H' \subseteq G$ by Observation 1. If $v_0 \in V(H')$, then let $H = G[V(H') \cup \{v_1, \ldots, v_{k-1}\}]$, otherwise, let $H = G[V(H') \cup \{v_1, \ldots, v_k\}]$. In both cases, $H \subseteq G$, $H$ is a bipartite graph, and $V(H)$ is a total dominating set, i.e., $H \in \mathcal{B}(G)$. By Observation 7 we get a contradiction.

Suppose $v_0 = v_{k+1}$. Since $deg_G(v_0) = 3$, there is $c \in N_G(v_0) \setminus \{v_1, v_k\}$. If $deg_G(c) = 3$, then let $G' = G \setminus \{v_0, \ldots, v_k\}$. If $deg_G(c) = 2$, then let $G' = G \setminus \{v_0, \ldots, v_k, c\}$. In both cases, $G' \subseteq G$ and $G \neq G' \in G_3^2$. Hence, there is $H' \in \mathcal{B}(G')$. Let $H = G[V(H') \cup \{v_0, \ldots, v_{k-1}\}]$. Thus, $H \in \mathcal{B}(G)$, a contradiction. ■

Lemma 11. If $G \in G_3^2$ contains $G_0$ as a subgraph (see Figure 1), where vertices $v_2, v_3 \in V(G_0)$ are of degree 2 in $G$, then $G \notin \mathcal{M}$.

![Figure 1. The subgraph $G_0$ of a graph $G$.](image)
Proof. Suppose to the contrary that \( G \in \mathcal{M} \). Suppose \( G_0 \subset G \). The other possible edges in \( G \) are marked by the dotted lines (in Figure 1).

By \( \deg_G(v_2) = \deg_G(v_3) = 2 \), from Lemma 10 we have \( \deg_G(v_1) = \deg_G(v_4) = 3 \). Since \( G' = G \setminus \{v_3\} \in \mathcal{G}_2^3 \setminus \mathcal{M}_2^3 \), there is \( H' \in \mathcal{B}(G') \). Hence, \( v_1 \in V(H') \) or \( v_4 \in V(H') \). Thus, \( H' \in \mathcal{B}(G) \), a contradiction.

Lemma 12. Let \( G \in \mathcal{M} \) and let \( v \) be a vertex of degree 3 in \( G \). Then, at most one neighbor of \( v \) has degree 2.

Proof. Let \( G \in \mathcal{M} \) and let \( N_G(v) = \{x, y, z\} \). Suppose to the contrary that at least two vertices from \( N_G(v) \) have degree 2. Let \( \deg_G(x) = \deg_G(y) = 2 \). Let \( \{v_x\} = N_G(x) \setminus \{v\} \) and \( \{v_y\} = N_G(y) \setminus \{v\} \). By Lemma 10, \( \deg_G(v_x) = \deg_G(v_y) = 3 \).

Suppose \( \deg_G(z) = 2 \). Let \( \{v_z\} = N_G(z) \setminus \{v\} \). By Lemma 10, \( \deg_G(v_z) = 3 \).

If any two of the vertices \( v_x, v_y, v_z \) are equal, then by Lemma 11 (i.e., because \( G_0 \subset G \)) we get a contradiction. Hence, vertices \( v_x, v_y, v_z \) are different. Since \( G' = G \setminus \{x, y, z, v\} \in \mathcal{G}_2^3 \setminus \mathcal{M}_2^3 \), there is \( H' \in \mathcal{B}(G') \). Thus, \( G[V(H') \cup \{v, x\}] \in \mathcal{B}(G) \), a contradiction.

Suppose \( \deg_G(z) = 3 \). If \( v_x = v_y \), then by Lemma 11 we get a contradiction. Hence, \( v_x \neq v_y \). Suppose \( z = v_x \) (the case \( z = v_y \) can be treated analogously). Since \( G_x = G \setminus \{x\} \in \mathcal{G}_2^3 \setminus \mathcal{M}_2^3 \), there is \( H_x \in \mathcal{B}(G_x) \). Since \( H_x \) is maximal in \( \mathcal{B}(G) \), we have \( v \in V(H_x) \) or \( z \in V(H_x) \). Thus, \( H_x \in \mathcal{B}(G) \), a contradiction. Then, vertices \( v_x, v_y, z \) are different. Since \( G' = G \setminus \{x, y, v\} \in \mathcal{G}_2^3 \setminus \mathcal{M}_2^3 \), there is \( H' \in \mathcal{B}(G') \). If \( z \in V(H') \), then let \( A = V(H') \cup \{v\} \). If \( z \notin V(H') \), then let \( A = V(H') \cup \{v, x\} \). In both cases, \( G[A] \in \mathcal{B}(G) \), a contradiction.

Let \( G \) be any subcubic graph. We say that \( H \subset G \) is a Q-cycle (of \( G \)) if:

(q1) for each \( v \in V(H) \), \( \deg_G(v) = 3 \), and

(q2) \( H \subset G \) and \( H \) is isomorphic to a cycle, i.e., \( H \) is an induced cycle, and

(q3) for each vertex \( v \in V(G) \setminus V(H) \), \( |N_G(v) \cap V(H)| \leq 1 \).

Lemma 13. Let \( G \in \mathcal{M} \). Let \( v \in V(G) \) have all neighbors of degree 3. Then, for each \( x \in N_G(v) \) there is a Q-cycle \( C_x \) such that \( x \in V(C_x) \), \( v \notin V(C_x) \) and \( N_G(v) \cap V(C_x) = \{x\} \).

Proof. Let \( G \in \mathcal{M} \) and let \( v \in V(G) \) be a vertex with all neighbors of degree 3. Since \( G' = G \setminus \{v\} \in \mathcal{G}_2^3 \setminus \mathcal{M}_2^3 \), there is \( H' \in \mathcal{B}(G') \). Hence, \( N_G(v) \cap V(H') = \emptyset \).

Let \( x \in N_G(v) \) and let \( N_G(x) = \{a, b, v\} \). Since \( H' \) is bipartite and maximal in \( \mathcal{B}(G) \), we have that \( a \) and \( b \) belong to the same connected component of \( H' \), and the length of each path in \( H' \) from \( a \) to \( b \) is odd. Let \( P \subset H' \) be a path joining \( x_1 = a \) and \( x_{s-1} = b \) (\( s \) is odd), with vertex set \( \{x_1, \ldots, x_{s-1}\} \) and edges \( \{x_i, x_{i+1}\} \), for \( i \in \{1, \ldots, s-2\} \). Let \( x_0 = x \) and let \( C_x \) be the graph with \( V(C_x) = V(P) \cup \{x_0\} \), and \( E(C_x) = E(P) \cup \{\{x_{s-1}, x_0\}, \{x_0, x_1\}\} \). Since \( P \subset H' \), we have \( N_G(v) \cap V(C_x) = \{x\} \), and \( v \notin V(C_x) \).
Claim 14. For each \( i \in \{1, \ldots, s-1\} \), the following properties are satisfied:

\( p_1 \) \( \deg_G(x_i) = 3 \), 

\( p_2 \) \( N_G(x_i) = \{a_i, x_{(i-1) \mod s}, x_{(i+1) \mod s}\} \), where \( a_i \in V(H') \setminus V(C_x) \), 

\( p_3 \) \( N_G(a_i) \cap V(H') = \{x_i\} \).

Proof. We proceed by induction on \( i \). Suppose \( i = 1 \). Let \( X = V(H') \setminus \{x_1\} \cup \{x, v\} \). Hence, \( G[X] \) is bipartite. If \( \deg_G(x_1) = 2 \) or \( N_G(a_i) \cap V(H') \neq \{x_i\} \), then \( G[X] \in \mathcal{B}(G) \), a contradiction. If \( a_i \notin V(H') \setminus V(C_x) \) or \( a_i \in V(C_x) \). If \( a_i \notin V(H') \), then \( N_G(a_i) \cap V(H') \neq \{x_i\} \) (otherwise \( H' \) is not maximal in \( \mathcal{B}(G) \)), a contradiction. If \( a_i \in V(C_x) \), then \( G[X] \in \mathcal{B}(G) \), a contradiction.

Suppose the properties \( (p_1), (p_2), (p_3) \) hold for \( 1, \ldots, i-1 \) (\( 2 \leq i \leq s-1 \)). Hence, each path joining \( x_1 \) and \( x_{s-1} \) in \( H' \) contains \( x_1, \ldots, x_i \). Let \( X = V(H') \setminus \{x_i\} \cup \{x, v\} \). Hence, \( G[X] \) is bipartite. The rest of the proof of properties \( (p_1), (p_2), (p_3) \) for \( i \) is literally the same as in the case \( i = 1 \).

We show that \( C_x \) is a \( Q_z \)-cycle. Since \( \deg_G(x) = 3 \), by \( (p_1) \) we have \( (q_1) \). Since \( v \notin V(C_x) \) and \( a_i \notin V(C_x) \) (by \( (p_2) \)), for \( i \in \{1, \ldots, s-1\} \), we have that \( C_x \) is an induced cycle of \( G \). Since \( a_i \in V(H') \) (by \( (p_2) \)), we have \( a_i \neq v \). Thus, by \( (p_3) \) we get \( |N_G(a_i) \cap V(C_x)| \leq 1 \), for \( i \in \{1, \ldots, s-1\} \).

We say that \( H \) is a \( Q_z \)-cycle (of \( G \)) if \( H \) is a \( Q_z \)-cycle of \( G \), and it holds \( (q_4) \) for each \( v \in N_G(V(H)) \setminus V(H), \deg_G(v) = 2 \).

Lemma 15. Let \( G \in \mathcal{M} \) and let \( C \) be a \( Q_z \)-cycle of \( G \). Then, \( C \) is a \( Q_z \)-cycle.

Proof. Let \( G \in \mathcal{M} \). Let \( C \) be a \( Q_z \)-cycle of \( G \) with the vertex set \( \{x_0, \ldots, x_{s-1}\} \), and edges \( \{x_0, x_1\}, \ldots, \{x_{s-2}, x_{s-1}\}, \{x_{s-1}, x_0\} \). Let \( S = \{0, \ldots, s-1\} \). Let \( \{a_i\} = N_G(x_i) \setminus V(C) \), for \( i \in S \). If \( \deg_G(a_i) = 2 \), then let \( \{b_i\} = N_G(a_i) \setminus \{x_i\} \).

Hence, \( b_i \notin V(C) \). By Lemma 10 we have \( \deg_G(b_i) = 3 \). Let \( G' = G \setminus (V(C) \cup \{a_i: \deg_G(a_i) = 2 \land i \in S\}) \). Since \( G' \in \mathcal{G}_2^3 \setminus \mathcal{M}_3^2 \), there is \( H' \in \mathcal{B}(G') \).

Suppose to the contrary that \( C \) is not a \( Q_z \)-cycle, i.e., there exists \( r \in S \) such that \( \deg_G(a_r) = 3 \). Let \( f: V(G') \to \{0, 1\} \) be the characteristic function of \( V(H') \), i.e., \( f(u) = 1 \) if and only if \( u \in V(H') \). Let us consider two cases.

(i) For each \( i \in S \): \( \deg_G(a_i) = 2 \Rightarrow f(b_i) = 0 \) and \( \deg_G(a_i) = 3 \Rightarrow f(a_i) = 0 \).

(ii) For some \( t \in S \): \( \deg_G(a_t) = 2 \land f(b_t) = 1 \) or \( \deg_G(a_t) = 3 \land f(a_t) = 1 \).

We construct a function \( \tilde{f}: V(G) \to \{0, 1\} \) such that \( \tilde{f}(u) = f(u) \) for each \( u \in V(G') \). Let \( u \in V(G) \setminus V(G') \). We define \( \tilde{f}(u) \) depending on cases (i), (ii).

(i) Let \( \tilde{f}(x_r) = 0 \) and let \( \tilde{f}(x_j) = 1 \), for each \( j \in S \setminus \{r\} \). For each \( j \in S \), if \( \deg_G(a_j) = 2 \), then \( \tilde{f}(a_j) = 1 \),
(ii) Take any \( t \in S \), if exists, such that \( \deg_G(a_t) = 2 \wedge f(b_t) = 1 \) and let \( \tilde{f}(a_t) = 1 \).

Then, for each \( j \in S, j \neq t \), if \( \deg_G(a_j) = 2 \), then \( \tilde{f}(a_j) = 1 - f(b_j) \). Next, for each \( j \in S \), if \( \deg_G(a_j) = 2 \wedge f(b_j) = 0 \), then \( \tilde{f}(x_j) = 1 \). Finally, for each \( j \in S \), if \( \deg_G(a_j) = 3 \) or \( \deg_G(a_j) = 2 \wedge f(b_j) = 1 \), then \( \tilde{f}(x_j) = 1 - \tilde{f}(a_{(j+1) \mod s}) \).

Let \( H = G[\{ u \in V(G) : \tilde{f}(u) = 1 \}] \). In the case (i), \( x_r \notin V(H) \cap V(C) \). Consequently, there is \( t \in S \) such that \( \deg_G(a_t) = 3 \wedge f(a_t) = 1 \), so finally \( \tilde{f}(a_t) = 1 \) for some \( t \in S \). Hence, there is \( p \in S \) such that \( \tilde{f}(x_p) = 0 \). Thus, \( V(C) \cap V(H) \neq \emptyset \).

Let us remind that for each \( i \in S \setminus \{ t \} \), if \( \deg_G(a_i) = 2 \) and \( \tilde{f}(a_i) = 1 \), then \( f(b_i) = 0 \). Let \( X = \{ i \in S : \deg_G(a_i) = 3 \wedge f(a_i) = 1 \} \cup \{ t \} \). Suppose that for some two \( i, j \in X \), there is a path in \( H \) between \( a_i \) and \( a_j \) with successive vertices \( x_i, x_{(i+1) \mod s}, \ldots, x_j \). Hence, \( \tilde{f}(x_i) = \tilde{f}(x_{(i+1) \mod s}) = \cdots = \tilde{f}(x_j) = 1 \), which implies that \( \tilde{f}(a_{(i+1) \mod s}) = 0, \tilde{f}(a_{(i+2) \mod s}) = 0, \ldots, \tilde{f}(a_j) = 0 \), a contradiction. Thus, \( H \) is a bipartite graph.

For every \( j \in S \) we have \( N_G(a_j) \cap V(H) \neq \emptyset \), and \( \tilde{f}(a_j) = 1 \) or \( \tilde{f}(a_j) = 0 \wedge \tilde{f}(x_{(j-1) \mod s}) = 1 \). Hence, we get \( N_G(x_j) \cap V(H) \neq \emptyset \). Thus, \( V(H) \) is a total dominating set and \( H \in B(G) \), a contradiction.

By Lemmas 10, 12, 13 and Lemma 15, and by the definition of \( Q_2 \)-cycle we have the following corollary.

**Corollary 16.** Let \( G \in \mathcal{M} \) and \( v \in V(G) \). The following properties are satisfied:

(i) \( \deg_G(v) = 2 \) if and only if vertex \( v \) has all neighbors of degree 3,
(ii) \( \deg_G(v) = 3 \) if and only if exactly one neighbor of \( v \) has degree 2,
(iii) if \( \deg_G(v) = 3 \), then there is exactly one \( Q_2 \)-cycle containing \( v \),
(iv) if \( \deg_G(v) = 2 \), then vertex \( v \) has two neighbors from disjoint \( Q_2 \)-cycles.

By Corollary 16 we have the next corollary.

**Corollary 17.** Let \( G \in \mathcal{M} \). The graph \( G \) satisfies the following properties:

(i) there is an integer \( q \geq 1 \) such that \( V(G) = D \cup \bigcup_{i=1}^{q} V(C_i) \), where for each \( i \in \{1, \ldots, q\} \) the graph \( C_i \) is a \( Q_2 \)-cycle and \( D \) is the set of all vertices of degree 2,
(ii) \( E(G) = \{ \{u, v\} : \exists i \in \{1, \ldots, q\} (\{u, v\} \in E(C_i) \lor (u \in V(C_i) \land v \in D)) \} \).

**Proof of Theorem 8.** Suppose to the contrary that \( G \in \mathcal{M} \).
By Corollary 17, there is $q \geq 1$ such that $V(G) = D \cup \bigcup_{i=1}^{q} V(C_i)$, where for each $i \in \{1, \ldots, q\}$ the graph $C_i$ is a $Q_2$-cycle and $D$ is the set of all vertices of degree 2, and

$$E(G) = \{\{u, v\} : \exists i \in \{1, \ldots, q\} \{\{u, v\} \in E(C_i) \lor (u \in V(C_i) \land v \in D)\}\}.$$

Let $Q = (D \cup \bigcup_{i=1}^{q} \{c_i\}, E_Q)$, where for each $i \in \{1, \ldots, q\}$ vertex $c_i$ corresponds to the cycle $C_i$ and

$$E_Q = \{\{v, c_i\} : i \in \{1, \ldots, q\} \land v \in D \land \exists x \in V(C_i)\{v, x\} \in E(G)\}.$$

By Corollary 16 and Corollary 17 we have that $Q$ is a simple bipartite graph with partitions $D$ and $C = \bigcup_{i=1}^{q} \{c_i\}$. Obviously, for all vertices $v \in D$ and $c \in C$ we have that $\deg_Q(v) = 2 < \deg_Q(c)$. Thus, by Hall’s Marriage Theorem [10] there is a matching $S$ in $Q$ covering all vertices from partition $C$.

Let $S' = \{\{v, x\} \in E(G) : v \in D \land \exists i \in \{1, \ldots, q\}\{v, c_i\} \in S \land x \in V(C_i)\}$ and let

$$V' = \left\{x \in \bigcup_{i=1}^{q} V(C_i) : \exists e \in S \; x \in e\right\}.$$

Let $H = G[V(G) \setminus (D \cup V')]$. For each $i \in \{1, \ldots, q\}$ there is $x$ such that $\{x\} = V(C_i) \cap V'$ and $N_G(x) \cap V(H) \neq \emptyset$. If $y \in V(C_i)$ and $x \neq y$, then $N_G(y) \cap V(H) \neq \emptyset$. Hence, $H$ is an induced bipartite graph without isolated vertices. Since for each $v \in D$ at most one neighbor of $v$ belongs to $V'$, we have $N_G(v) \cap V(H) \neq \emptyset$. Thus, $N_G(V(H)) = V(G)$ and $H \in \mathcal{B}(G)$, a contradiction.

3. INTERVAL INCIDENCE 6-COLORING OF SUBCUBIC GRAPHS

In this section we prove our main result, i.e., Theorem 21, which states $\chi_{ii}(G) \leq 2\Delta(G)$ for each subcubic graph $G$. By Theorem 8 we have the following lemma.

**Lemma 18.** Let $G$ be a connected graph and $G \in \mathcal{G}_3^2$. Let $H \in \hat{\mathcal{B}}(G)$ and let $A, B \subset V(H)$ be any partition of $V(H)$, such that $A$ and $B$ are disjoint independent sets and $A \cup B = V(H)$. Then, $A$ and $B$ are disjoint independent dominating sets, and the graph $G[V(G) \setminus V(H)]$ has only isolated vertices and isolated edges.

**Proof.** Let $v \in V(G) \setminus V(H)$. If $N_G(v) \cap V(H) \subset A$ or $N_G(v) \cap V(H) \subset B$, then $G[V(H) \cup \{v\}]$ is a bipartite graph, a contradiction. Thus, $N_G(v) \cap A \neq \emptyset$ and $N_G(v) \cap B \neq \emptyset$. Let $v \in A$ ($v \in B$). Since $H$ is an induced graph without isolated vertices, we have $v \in N_G(B)$ ($v \in N_G(A)$). Hence, $A$ and $B$ are disjoint independent dominating sets.
Since $G$ is subcubic and $|N_G(v) \cap V(H)| \geq 2$ for any $v \in V(G) \setminus V(H)$, graph $G[V(G) \setminus V(H)]$ has only isolated vertices and isolated edges. 

Lemma 19. Let $G$ be a subcubic non-bipartite graph with $\Delta(G) = 3$. Then, there is a vertex coloring $c: V(G) \to \{1, 2, 3, 4\}$ such that for each $v \in V(G)$ the following properties hold:

(i) if $\deg_G(v) = 1$, then $c(v) \in \{1, 4\}$,
(ii) if $\deg_G(v) \geq 2$ and $c(v) \neq p$, then $\deg_G(v) \geq 1$, for $p \in \{1, 4\}$,
(iii) $a_i(v) \leq |c(v) - i|$, for $i \in \{1, 2, 3, 4\}$,

where $a_i(v) = |\{w \in N_G(v) : c(w) = i\}|$, for $i \in \{1, 2, 3, 4\}$.

Proof. If $\delta(G) = 1$, then we successively remove pendant vertices from graph $G$, until there is no pendant vertex. Let us denote the resulting graph by $G'$. Obviously, $\delta(G') \geq 2$. Let us observe that we cut off all trees attached to $G$.

By Theorem 8 we have $\mathcal{B}(G') \neq \emptyset$. Let $H$ be any element of $\mathcal{B}(G')$ with the largest possible number of vertices.

Let $A, B \subset V(H)$ be any two partite sets of $V(H)$, i.e., $A$ and $B$ are disjoint independent sets and $A \cup B = V(H)$. By Lemma 18, $A$ and $B$ are disjoint independent dominating sets of $G'$, and the graph $G[V(G') \setminus V(H)]$ has only isolated vertices and isolated edges. Let $I_i \subset V(G') \setminus V(H)$ be the set of all vertices of degree $i$ in $G'$, for $i \in \{2, 3\}$. Let us define the partition $I_3 = I_3^A \cup I_3^B \cup I_3^C$:

- $I_3^A = \{v \in I_3 : |N_{G'}(v) \cap A| = 2 \land |N_{G'}(v) \cap B| = 1\}$,
- $I_3^B = \{v \in I_3 : |N_{G'}(v) \cap A| = 1 \land |N_{G'}(v) \cap B| = 2\}$,
- $I_3^C = \{v \in I_3 : |N_{G'}(v) \cap A| = 1 \land |N_{G'}(v) \cap B| = 1\}$.

Note that $I_2, I_3^A, I_3^B$ are independent sets in $G'$, each vertex $v \in I_3^C$ belongs to an isolated edge in $G'[I_3^C]$, and each vertex from $I_2$ has neighbors from $A$ and $B$.

Let us define a coloring $c: V(G) \to \{1, 2, 3, 4\}$ in the following steps.

(C1) If $v \in A$, then $c(v) = 1$, and if $v \in B$, then $c(v) = 4$.

(C2) If $v \in I_3^B$, then $c(v) = 2$, and if $v \in I_3^A$, then $c(v) = 3$.

(C3) For each successive $v \in I_2$ we assign a color following the algorithm: if $c(v)$ is not determined, then let $\{u\} = N_{G'}(v) \cap A$. If there is $x \in N_{G'}(u)$ such that $c(x) = 2$, then let $c(v) = 3$. Otherwise, for each vertex $x \in N_{G'}(u)$ either $c(x) \in \{3, 4\}$ or $c(x)$ is not determined, and then let $c(v) = 2$.

(C4) For each successive $\{v, w\} \in E(G'[I_3^C])$ we assign colors to both $v$ and $w$ following the algorithm: if $c(v)$ and $c(w)$ are not determined, then let $\{u\} = N_{G'}(v) \cap A$. If there is $x \in N_{G'}(u)$ such that $c(x) = 2$, then let $c(v) = 3$ and $c(w) = 2$. Otherwise, for each vertex $x \in N_{G'}(u)$ either $c(x) \in \{3, 4\}$ or $c(x)$ is not determined, and then let $c(v) = 2$ and $c(w) = 3$.

(C5) For each $v \in V(G')$ such that $\deg_G(v) \leq \deg_G(v)$, there is a tree $T_v$ such that $V(T_v) \subset V(G) \setminus V(G')$ and let $\{w\} = V(T_v) \cap N_G(v)$. Let $d: V(T_v) \to$
Let $\{a, b\}$ be a 2-coloring of $T_v$ such that $d(w) = a$. Suppose $c(v) \leq 2$. For each $u \in V(T_v)$, if $d(u) = a$, then let $c(u) = 4$, and if $d(u) = b$, then let $c(u) = 1$. Suppose $c(v) \geq 3$. For each $u \in V(T_v)$, if $d(u) = a$, then let $c(u) = 1$, and if $d(u) = b$, then let $c(u) = 4$.

In step $(C_1)$ we colored $V(H) = A \cup B$ with colors 1 and 4, in steps $(C_2)$–$(C_4)$ we colored vertices from $I_2 \cup I_3$ with colors 2 or 3, and in step $(C_5)$ we colored vertices from $V(G) \setminus V(G')$ with colors 1 or 4. Since vertices colored with an arbitrary color form an independent set, $c$ is a vertex 4-coloring of $G$.

Let $v \in V(G)$ and let $\deg_G(v) = 1$. Then, $v \in V(G) \setminus V(G')$ and, by $(C_5)$, $c(v) \in \{1, 4\}$. Thus, we get the property (i). Let $\deg_G(v) \geq 2$. If $v \in V(G) \setminus V(G')$, then, by $(C_5)$, the property (ii) holds. Let $v \in V(G')$. Since $A$ and $B$ are disjoint independent dominating sets of $G'$, the property (ii) holds.

Since $c$ is a proper coloring of $G$, there is $a_{c(v)}(v) = 0$ for each $v \in V(G)$.

Let $v \in V(G) \setminus V(G')$. By step $(C_5)$, $c(v) \in \{1, 4\}$. If $c(v) = 1$, then $a_2(v) = 0$, $a_3(v) \leq 1$ and $a_4(v) \leq 3$. If $c(v) = 4$, then $a_3(v) = 0$, $a_2(v) \leq 1$ and $a_1(v) \leq 3$.

Let $v \in V(G') \setminus V(H)$. If $v \in I_2^3$, then $c(v) = 3$, $a_1(v) = 2$, $a_2(v) = 0$, $a_4(v) = 1$. If $v \in I_1^2$, then $c(v) = 2$, $a_1(v) = 1$, $a_3(v) = 0$, $a_4(v) = 2$. If $v \in I_2$, then $c(v) \in \{2, 3\}$. If $\deg_{G'}(v) = \deg_G(v)$, then $a_1(v) = a_4(v) = 1$, and $a_2(v) = a_3(v) = 0$. If $\deg_{G'}(v) < \deg_G(v)$, then if $c(v) = 2$, then $a_1(v) = 1$, $a_2(v) = a_3(v) = 0$, $a_4(v) = 2$, and if $c(v) = 3$, then $a_1(v) = 2$, $a_2(v) = a_3(v) = 0$, $a_4(v) = 1$. If $v \in I_2^3$, then $c(v) \in \{2, 3\}$. If $c(v) = 2$, then $a_1(v) = a_3(v) = a_4(v) = 1$. If $c(v) = 3$, then $a_1(v) = a_2(v) = a_4(v) = 1$.

Let $v \in A \cup B$. Since $A$ and $B$ are disjoint dominating sets of $G'$ and $H \in B(G')$, it suffices to prove that if $c(v) = 1$, then $a_2(v) \leq 1$, and if $c(v) = 4$, then $a_3(v) \leq 1$.

Suppose to the contrary that $c(v) = 1$ and $a_2(v) = 2$ for some $v \in A$. The case $c(v) = 4$ and $a_3(v) = 2$, for some $v \in B$, is analogous. Let $x, y \in N_{G'}(v)$ such that $c(x) = c(y) = 2$. Since $B$ is a dominating set of $G'$, there is $w \in N_{G'}(v) \cap B$ with $c(w) = 4$. By the definition of coloring $c$, we have $v, x, y, w \in V(G')$ and $v, w \in V(H)$.

Since $c(x) = c(y) = 2$, we have $a_1(x) = a_1(y) = 1$, $a_3(x) \leq 1$, $a_4(y) \leq 1$, $1 \leq a_4(x) \leq 2$ and $1 \leq a_4(y) \leq 2$. Let us consider the following cases:

- $x \notin N_{G'}(w)$ and $y \notin N_{G'}(w)$. If edge $\{v, w\}$ is isolated in $H$, then let $W = V(H) \cup \{x, y\}$. Otherwise, let $W = V(H) \cup \{x, y\} \setminus \{v\}$.
- $x \in N_{G'}(w)$ or $y \in N_{G'}(w)$. Let $W = V(H) \cup \{x, y\} \setminus \{v\}$.

In both cases, the graph $G'[W] \in B(G')$ and $|V(G'[W])| > |V(H)|$, a contradiction. Thus, the coloring $c$ satisfies the property (iii).

\textbf{Proposition 20.} \cite{14} \textit{For any graph $G$, $\Delta(G) + 1 \leq \chi_{ii}(G) \leq \chi(G) \cdot \Delta(G)$.}

We prove that an interval incidence 6-coloring always exists for any subcubic graph $G$ with $\Delta(G) = 3$. 
Theorem 21. Let $G$ be a subcubic graph. Then, $\chi_{ii}(G) \leq 2\Delta(G)$.

Proof. If $G$ is a subcubic bipartite graph, then by Proposition 20 we have $\chi_{ii}(G) \leq 2\Delta(G)$. If $\Delta(G) = 2$, then one can easily construct an interval incidence 4-coloring. Thus, $\chi_{ii}(G) \leq 2\Delta(G)$. Let $G$ be a subcubic non-bipartite graph with $\Delta(G) = 3$. By Lemma 19, there is a vertex coloring $c: V(G) \to \{1, 2, 3, 4\}$ satisfying the properties (i), (ii), (iii) from Lemma 19.

We construct an incidence coloring $f: I(G) \to \{1, 2, 3, 4, 5, 6\}$ in three steps.

In the first step, using the coloring $c$, we define the interval $A_f(v)$ for each vertex $v \in V(G)$, as follows. If $\deg_G(v) = 2$ and $c(v) \in \{2, 3\}$, then let $A_f(v) = \{3, 4\}$. If $c(v) = 4$ and $\deg_G(v) = 1$, then $A_f(v) = \{6\}$. If $c(v) = 4$ and $\deg_G(v) = 2$, then $A_f(v) = \{5, 6\}$. In the other cases, let $A_f(v) = \{c(v), \ldots, c(v) + \deg_G(v) - 1\}$. Thus, by Lemma 19 (i)–(iii) we get

(a1) if $\deg_G(v) = 1$, then $c(v) \in \{1, 4\}$ and $A_f(v) = \{c(v)\}$,

(a2) if $\deg_G(v) = 2$, then if $c(v) \in \{1, 3\}$, then $A_f(v) = \{c(v), c(v) + 1\}$ and if $c(v) \in \{2, 4\}$, then $A_f(v) = \{c(v) + 1, c(v) + 2\}$,

(a3) if $\deg_G(v) = 3$, then $A_f(v) = \{c(v), c(v) + 1, c(v) + 2\}$.

In the second step, for each $v \in V(G)$, we construct a sequence $L_f(v)$ (i.e., a linear ordered set) from elements of $N_G(v)$, as follows (see Figure 2).

(l1) Suppose $\deg_G(v) = 1$. If $N_G(v) = \{x\}$, then let $L_f(v) = (x)$.

(l2) Suppose $\deg_G(v) = 2$. Let $N_G(v) = \{x, y\}$, where $c(x) \leq c(y)$. Then,

- if $c(v) \in \{1, 4\}$, then let $L_f(v) = (x, y)$,

- if $c(v) \in \{2, 3\}$, then let $L_f(v) = (y, x)$.

(l3) Suppose $\deg_G(v) = 3$. Let $N_G(v) = \{x, y, z\}$, where $c(x) \leq c(y) \leq c(z)$. Then,

- if $c(v) \in \{1, 4\}$, then let $L_f(v) = (x, y, z)$,

- if $c(v) = 2$, then let $L_f(v) = (y, z, x)$,

- if $c(v) = 3$, then let $L_f(v) = (z, x, y)$.

By $v_i$ we mean the $i$-th element of the sequence $L_f(v)$, i.e., $L_f(v) = (v_1, \ldots)$.

In the final step, for each vertex $v$, we define the incidence coloring $f$ as follows: $f(v, \{v, v_i\}) = \min A_f(v) + i - 1$, for $i \in \{1, \ldots, \deg_G(v)\}$.

In Figure 2 the white vertex is the vertex $v$, and the list above is $L_f(v)$. By Lemma 19 (i)–(iii), the set of all possible values of $c$ of a vertex is as given in the curly brackets below the vertex. The colors of incidences at the white vertex (i.e., $v$) are given at the edges adjacent to $v$.

Obviously, all the incidences at vertex $v$ are colored with different colors from $A_f(v)$. Observe that the set of colors $A_f(v)$ is an interval of integers.

We prove that the coloring $f$ is an incidence coloring. It is enough to prove that for each vertex $v \in V(G)$ and each vertex $w \in N_G(v)$ we have $f(v, \{v, w\}) \notin A_f(w)$, or, equivalently, $f(v, \{v, w\}) < \min A_f(w)$ or $f(v, \{v, w\}) > \max A_f(w)$. 


Thus, \( f(v) = 1 \). Then, \( A_f(v) \subset \{1, 2, 3\} \) and \( \min A_f(v) = 1 \). By the construction of \( L_f(v) \) we have: if \( \deg_G(v) \geq 1 \), then \( c(v_1) \in \{2, 3, 4\} \), and if \( \deg_G(v) = 2 \), then \( c(v_2) = 4 \), and if \( \deg_G(v) = 3 \), then \( c(v_2) \in \{3, 4\} \) and \( c(v_3) = 4 \) (see Figure 2). Hence, for each \( i \in \{1, \ldots, \deg_G(v)\} \) we have \( f(v, \{v, v_1\}) = \min A_f(v) + i - 1 < i + 1 \leq \min A_f(v_i) \).

Suppose that \( c(v) = 2 \). Then, \( A_f(v) \subset \{2, 3, 4\} \). Let \( \deg_G(v) = 3 \). Hence, \( \min A_f(v_2) = 2 \), and \( c(v_1) \in \{3, 4\} \) and \( c(v_2) = 4 \cap c(v_3) = 1 \). Thus, \( f(v, \{v, v_1\}) = \min A_f(v) + i - 1 = i + 1 < i + 2 \leq \min A_f(v_i) \), for \( i \in \{1, 2\} \), and \( f(v, \{v, v_3\}) = \min A_f(v) + 2 = 4 > 3 \geq \max A_f(v_3) \). Let \( \deg_G(v) = 2 \). Hence, \( \min A_f(v_2) = 3 \), and \( c(v_1) = 4 \) and \( c(v_2) = 1 \). Thus, \( f(v, \{v, v_1\}) = \min A_f(v) = 3 < 4 \leq \min A_f(v_1) \) and \( f(v, \{v, v_2\}) = 4 > 3 \geq \max A_f(v_2) \).

Suppose that \( c(v) = 3 \). Then, \( A_f(v) \subset \{3, 4, 5\} \) and \( \min A_f(v) = 3 \). Let \( \deg_G(v) = 3 \). Hence, \( c(v_1) = 4 \) and \( c(v_2) = 1 \) and \( c(v_3) \in \{1, 2\} \). Thus, \( f(v, \{v, v_1\}) = \min A_f(v) = 3 < 4 \leq \min A_f(v_1) \), and \( f(v, \{v, v_1\}) = \min A_f(v) + i - 1 > i + 1 \geq \max A_f(v_i) \), for \( i \in \{2, 3\} \). Let \( \deg_G(v) = 2 \). Hence, \( c(v_1) = 4 \) and \( c(v_2) = 1 \). Thus, \( f(v, \{v, v_1\}) = 3 < 4 \leq \min A_f(v_1) \) and \( f(v, \{v, v_2\}) = 4 > 3 \geq \max A_f(v_2) \).

Suppose that \( c(v) = 4 \). Then, \( A_f(v) \subset \{4, 5, 6\} \). Let \( \deg_G(v) = 3 \). Hence, \( c(v_1) = 1 \) and \( c(v_2) \in \{1, 2\} \) and \( c(v_3) \in \{1, 2, 3\} \) and \( c(v_2) \leq c(v_3) \). Thus, \( f(v, \{v, v_1\}) = \min A_f(v) + i - 1 < i + 3 < i + 2 \geq \max A_f(v_i) \), for \( i \in \{1, 2, 3\} \). Let \( \deg_G(v) = 2 \). Hence, \( c(v_1) = 1 \) and \( c(v_2) \in \{1, 2, 3\} \), and \( A_f(v) = \{5, 6\} \). Thus, \( f(v, \{v, v_1\}) = 5 > \max A_f(v_1) \) and \( f(v, \{v, v_2\}) = 6 > \max A_f(v_2) \). Let \( \deg_G(v) = 1 \). Hence, \( c(v_1) \in \{1, 2, 3\} \). Thus, \( f(v, \{v, v_1\}) = 6 > 5 \geq A_f(v_1) \).

In all the cases we proved that \( f(v, \{v, v_1\}) \notin A_f(v) \) for each \( v_1 \in N_G(v) \). Thus, \( f \) is an interval incidence 6-coloring of \( G \). ■
A. Małafiejska and M. Małafiejski

4. Summary

In this paper we proved that for any subcubic graph $G$, $\chi_{ii}(G) \leq 2\Delta(G)$. In [14] we proved that the upper bound of $2\Delta(G)$ on $\chi_{ii}(G)$ holds for each complete $k$-partite graph $G$ and this bound is valid for other classes of graphs. Thus, we state the following

Conjecture 22 [Interval Incidence Coloring Conjecture (IICC)]. For any graph $G$, $\chi_{ii}(G) \leq 2\Delta(G)$.

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Received 9 February 2016
Revised 12 January 2017
Accepted 12 January 2017