THE TOTAL ZERO-DIVISOR GRAPH OF COMMUTATIVE RINGS

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ABSTRACT. In this paper we initiate the study of the total zero-divisor graphs over commutative rings with unity. These graphs are constructed by both relations that arise from the zero-divisor graph and from the total graph of a ring. We characterize Artinian rings with the connected total zero-divisor graphs and give their diameters. Moreover, we compute major characteristics of the total zero-divisor graphs of the ring \( \mathbb{Z}_m \) of integers modulo \( m \) and prove that the total zero-divisor graphs of \( \mathbb{Z}_m \) and \( \mathbb{Z}_n \) are isomorphic if and only if \( m = n \).

1. INTRODUCTION

Recently, the interplay between properties of algebraic structures and their relationship graphs has been studied extensively. To further the understanding of the structure of zero-divisors in semigroups, several various graphs were introduced.

In 1988, Beck [5] introduced the coloring properties of a graph, whose vertices were all the elements of the ring and two vertices were adjacent if their product was 0. This definition was simplified in 1999 by Anderson and Livingston [4] to the zero-divisor graph. The vertices of the zero-divisor graph are all nonzero zero-divisors and distinct vertices \( x \) and \( y \) are adjacent if and only if \( xy = 0 \). Compared with the Beck’s graph, they omitted 0 and all vertices which are not zero-divisors in the graph, and thus the properties of the zero-divisors in the ring were more clearly reflected. Zero-divisor graphs were largely studied in the last two decades (see e.g. [3, 6] for survey papers). In 2002, Mulay [10] defined a compressed zero-divisor graph, in which the vertices are the equivalence classes of zero-divisors of the ring. Compressed graphs are smaller hence easier to investigate than the zero-divisor graphs. All those graphs were extensively studied over structures with one or two operations.

In [2] Anderson and Badawi introduced the total graph of a commutative ring \( R \) as the graph with vertices of all elements of \( R \) and with edges \( x \sim y \) for distinct \( x, y \in R \) if \( x + y \) is a zero-divisor in \( R \). The total graph of the ring engages both ring operations instead of studying only multiplication as in...
the zero-divisor graph and therefore reflects more structure of the subset of zero-divisors in a given ring.

In this paper we initiate the study of both relationships to reveal more information about the commutative rings. Denote the set of all zero divisors of $R$ by $Z(R)$. The total zero-divisor graph $ZT(R)$ of a commutative unital ring $R$ is a simple graph whose vertex set is the set of all nonzero zero divisors of $R$, and where distinct vertices $u$ and $v$ are adjacent if and only if

$$uv = 0 \text{ and } u + v \in Z(R).$$

In Section 3 we investigate the connectedness of a total zero-divisor graph of a commutative unital ring. In Theorem 3.1 we characterize Artinian rings with a connected total zero-divisor graph. On the contrary to the zero-divisor graph or to the total graph, this characterization highly depends on the associated prime ideals of the ring. Moreover, we provide the diameter of the graph (see Theorem 3.2), which depends on the maximal ideals of the ring.

In Section 4 we compute the main characteristics of the total zero-divisor graph of the ring $\mathbb{Z}_m$ of integers modulo $m$, such as girth, degrees, chromatic number, domination number and metric dimension. Since $\mathbb{Z}_m$ is a prototype for certain classes of rings, we believe some of the results might be generalized. In Corollary 4.3 we show that the total zero-divisor graphs of $\mathbb{Z}_m$ and $\mathbb{Z}_n$ are isomorphic if and only if $m = n$. Moreover, we characterize all acyclic total zero-divisor graphs in Corollary 4.1.

2. Preliminaries

Throughout the paper, let $R$ be a commutative ring with unity. Let $Z(R)$ denote the set of zero-divisors in $R$ and $\mathcal{U}(R)$ the set of invertible elements in $R$. If $I$ is a nilpotent ideal in $R$, we call the smallest positive integer $\ell$ such that the product of any $\ell$ elements of $I$ is 0, a nilindex of $I$.

For an $R$-module $M$ and a subset $S \subseteq M$ we denote by

$$\text{Ann}_R S = \{ r \in R; rs = 0 \text{ for all } s \in S \}$$

the annihilator of $S$ in $R$. We will omit index assuming that it is $R$ if not stated otherwise. We say that a prime ideal $P$ of $R$ is an associated prime of $M$ if $P = \text{Ann}(m)$ for some $m \in M$. The set of all associated primes of $M$ is denoted by $\text{Ass}(M)$. If $R$ is Noetherian and $M$ a finitely generated $R$-module, then $\text{Ass}(M)$ is finite (see e.g. [9, Proposition (7.G)])

We say that an element $r \in R$ is a zero-divisor for $M$ if $rm = 0$ for some non-zero $m \in M$. Recall that if $R$ is Noetherian ring and $M$ a finitely generated non-zero $R$-module, then the set of zero-divisors for $M$ is the union of all the associated primes of $M$ [9, Proposition (7.B)]. In the special case where we treat $R$ as an $R$-module, we have

$$Z(R) = \bigcup_{P \in \text{Ass}(R)} P.$$

Our notation for the graphs is the following. For a graph $G = (V(G), E(G))$ we denote its order by $|G| = |V(G)|$. The sequence of edges $x_0 \sim x_1 \sim \ldots \sim$
A path \( x_{k-1} \sim x_k \) in a graph is called a \textit{path of length} \( k \). The distance between two vertices is the length of the shortest path between them and the diameter \( \text{diam}(G) \) is the largest distance between any two vertices of the graph.

The \textit{open neighbourhood} of a vertex \( v \) is denoted by \( N_G(v) \) and consists of all vertices at distance 1 from vertex \( v \) and the \textit{closed neighbourhood} of a vertex \( v \) is denoted by \( N_G[v] = N_G(v) \cup \{v\} \) and consists of all vertices at distance at most 1 from vertex \( v \). When stated without any qualification, a neighbourhood is assumed to be open.

A path \( x_0 \sim x_1 \sim \ldots \sim x_k \sim x_0 \) is called a \textit{cycle}. A graph with a cycle will be called a \textit{cyclic} graph and \textit{acyclic} otherwise. A \textit{Hamiltonian cycle} of a graph \( G \) is a cycle that contains every vertex of \( G \). A graph is called \textit{Hamiltonian} if it contains a Hamiltonian cycle. A graph \( G \) is \textit{Eulerian} if it contains a cycle that consists of all the edges of \( G \). A complete graph on \( n \) vertices will be denoted by \( K_n \) and a path with \( n \) vertices will be denoted by \( P_n \).

3. Connectedness of the total zero-divisor graph

The zero-divisor graphs are always connected with the diameter at most three [4]. On the other hand, the total graph of a commutative ring is not connected if the set of zero-divisors forms an ideal [2]. We will see that the connectedness of the total zero-divisor graph of the ring \( R \) depends on the existence of a maximal associated prime \( P \triangleleft R \), such that \( P \cap \text{Ann} P = \{0\} \). Note that for example in the ring \( \mathbb{Z}_6 \) every prime ideal intersects its annihilator trivially, but in \( \mathbb{Z}_{16} \) no prime ideal intersects its annihilator trivially.

First we prove the following lemma, which will give us a necessary condition for the graph \( ZT(R) \) to be connected. We say that \( P \) is a maximal associated prime of \( R \) if it is maximal among all associated primes of \( R \).

\textbf{Lemma 3.1.} \textit{Let} \( R \text{ a Noetherian ring with unity and assume} Z(R) \neq \{0\}. If there exists a maximal associated prime} \( P \text{ of} R \text{ such that} P \cap \text{Ann} P = \{0\} \text{ then the graph} ZT(R) \text{ is not connected.}

\textbf{Proof.} \textit{Let} \( P \text{ be a maximal associated prime of} R \text{ such that} P \cap \text{Ann} P = \{0\}. Since} \( R \text{ is Noetherian, the set} \text{Ass}(R) \text{ is finite. Due to the maximality of} P, \text{ the prime avoidance lemma} \text{ (see e.g. [8] Lemma 3.3) implies that the set} P \setminus (\bigcup_{Q \in \text{Ass}(R)} \{P\} Q) \text{ is nonempty. If} P \setminus (\bigcup_{Q \in \text{Ass}(R)} \{P\} Q) = \{0\}, \text{ then} \text{Ass}(R) = \{P\} \text{ and} P = 0. \text{ By (I) this implies} Z(R) = \{0\}, \text{ a contradiction. Hence} P \setminus (\bigcup_{Q \in \text{Ass}(R)} \{P\} Q) \neq \{0\}. \text{Take any} 0 \neq x \in P \setminus (\bigcup_{Q \in \text{Ass}(R)} \{P\} Q). \text{ By (I),} x \text{ is a nonzero zero-divisor. We will show that} x \text{ is an isolated vertex of} ZT(R). \text{ Let} y \neq x \text{ be a nonzero zero-divisor and suppose that} x \text{ and} y \text{ are adjacent in} ZT(R). \text{ By (I) this implies} x + y \in Q \text{ for some} Q \in \text{Ass}(R). \text{ Multiplying} x + y \text{ by} x \text{ and} y \text{ respectively and taking into account that} xy = 0 \text{ gives us} x, y \in Q, \text{ since} Q \text{ is prime. By definition of} x \text{ we have} Q = P, \text{ so} y \in P. \text{ If we view the ideal} (y) \text{ as an} R\text{-module, then} x \text{ is a zero-divisor for} (y), \text{ so} x \in \bar{P} \text{ for some associated prime} \bar{P} \text{ of} (y). \text{ In particular,} \bar{P} = \text{Ann}(z) \text{ for}
some \( z \in (y) \subseteq P \). But \( \hat{P} \) is also an associated prime of \( R \), hence \( \hat{P} = P \) by definition of \( x \). We conclude that \( z \in P \cap \text{Ann} P = 0 \), which implies \( P = \text{Ann}(z) = R \), a contradiction since \( P \) is prime. This shows that \( x \) is an isolated vertex.

To conclude the proof we need to show that \( \text{ZT}(R) \) contains at least two vertices. Assume on the contrary that \( \text{ZT}(R) \) has a single vertex \( x \). Then \( Z(R) = \{0, x\} \). This implies that \( \{0, x\} \) is the only associated prime of \( R \), hence \( P = \{0, x\} \). By assumption \( P \cap \text{Ann} \ P = \{0\} \) which implies \( x \notin \text{Ann} P \). Consequently \( x^2 \neq 0 \). But \( x^2 \) is a zero-divisor, so \( x^2 = x \) which implies \( x(1-x) = 0 \). Thus \( 1-x \) is a non-zero zero-divisor as well. Hence \( 1-x = x \) which implies \( 2x=1 \), a contradiction since \( x \) is a zero-divisor. \( \square \)

In the next example we show that Lemma 3.1 is not valid for non-Noetherian rings.

**Example 3.1.** Let \( R \) be the ring of all real valued functions on \([0,1]\). To show that \( R \) is not Noetherian, let \( I_S \subseteq R \) denote a subring that contains functions which vanish on a subset \( S \subseteq [0,1] \). For any infinite decreasing chain \( S_0 \supseteq S_1 \supseteq \cdots \) of subsets of \([0,1]\), we get an infinite increasing chain \( I_{S_0} \subsetneq I_{S_1} \subsetneq \cdots \) of ideals of \( R \).

Now, define \( f(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases} \) Let \( P = \text{Ann}(f) = \{g \mid g(0) = 0\} \) and note that \( \text{Ann} P = \{h \mid h(x) = 0 \text{ for all } x \neq 0\} \), hence \( P \cap \text{Ann} P = \{0\} \). Since \( R/P \cong \mathbb{R} \), it follows that \( P \) is maximal associated prime. Since the zero-divisors in \( R \) are the functions which are equal to 0 on a set with a non-zero measure, it is possible to construct a path between any two vertices in \( \text{ZT}(R) \). Therefore, \( \text{ZT}(R) \) is connected.

We will prove that in the case of a Noetherian and Artinian ring \( R \), the graph \( \text{ZT}(R) \) is connected if and only if \( P \cap \text{Ann} P \neq \{0\} \) for all maximal associated prime ideals \( P \triangleleft R \). First we need a few technical lemmas.

**Lemma 3.2.** Let \( R \) be an Artinian ring and choose \( a_1, a_2 \in R \), such that the ideals \( Q_1 = \text{Ann}(a_1) \) and \( Q_2 = \text{Ann}(a_2) \) are distinct prime ideals in \( R \). It follows that \( a_1 a_2 = 0 \).

**Proof.** Since \( R \) is Artinian, we can suppose \( R = R_1 \times R_2 \times \cdots \times R_k \), for some local rings \( R_1, R_2, \ldots, R_k \).

Let us first show that if \( P \) is a prime ideal in \( R \), then \( P = P_1 \times P_2 \times \cdots \times P_r \), where \( P_i \) is a prime ideal in \( R_i \) for exactly one index \( i \) and \( P_j = R_j \) for all \( j \neq i \). Assume the opposite and without loss of generality suppose that \( P_1 \neq R_1 \) and \( P_2 \neq R_2 \). Choose \( x = (p_1, 1, 0, 0, \ldots, 0) \) and \( y = (1, p_2, 0, 0, \ldots, 0) \), where \( p_i \in P_i \). It follows that \( xy \in P \), but neither \( x \) nor \( y \) is an element of the ideal \( P \), a contradiction.

Let \( Q_1 = \text{Ann}(a_1) \) have a proper prime ideal at \( j_1 \)-th position and let \( Q_2 = \text{Ann}(a_2) \) have a proper prime ideal at position \( j_2 \), i.e. for \( i = 1, 2 \) we have \( Q_i = R_1 \times \cdots R_{j_i-1} \times P_i \times R_{j_i+1} \times \cdots \times R_k \), where \( P_i \) is a prime ideal in \( R_{j_i} \). Since \( Q_k = \text{Ann}(a_k) \), \( k = 1, 2 \), it follows that \( a_1 = (0, \ldots, 0, p_1, 0, \ldots, 0) \) and \( a_2 = (0, \ldots, 0, p_2, 0, \ldots, 0) \), where \( P_k = \text{Ann}_{R_{j_k}}(p_k) \subseteq R_{j_k} \). If \( j_1 \neq j_2 \), then clearly \( a_1 a_2 = 0 \). If \( j_1 = j_2 \), assume without loss of generality that \( j_1 = j_2 = 1 \). If neither \( p_1 \) nor \( p_2 \) is a zero-divisor in \( R_1 \), they are both
invertible in $R_1$ and hence $Q_1 = Q_2$, a contradiction. Otherwise, assume that $p_1$ is a zero-divisor. Since $R_1$ is Artinian and local, we have $p_1^n = 0 \in P_2$ for some positive integer $n$. Since $P_2$ is prime, we have $p_1 P_2 = \Ann(p_2)$, and therefore $p_1p_2 = 0$ and $a_1a_2 = 0$. Similarly we argue that $a_1a_2 = 0$ in the case $p_2$ is a zero-divisor. □

**Lemma 3.3.** Let $R$ be an Artinian ring and $I$ ideal generated by

$$S = \bigcup_{P \in \Ass(R)} P \cap \Ann P.$$  

Then the subgraph of $ZT(R)$ induced by vertices in $I$ is a complete graph.

Proof. Let us first show that $S^2 = 0$. Take arbitrary $s_1 \in P_1 \cap \Ann P_1$ and $s_2 \in P_2 \cap \Ann P_2$ for some $P_1, P_2 \in \Ass(R)$. Thus there exist $a_1, a_2 \in R$, such that $P_1 = \Ann(a_i), i = 1, 2$, and so $s_i \in \Ann(a_i) \cap \Ann(\Ann(a_i))$. It follows that $s_i^2 = 0$ and $\Ann(a_i) \subset \Ann(s_i), i = 1, 2$.

If $P_1 = P_2$, then $s_1, s_2 \in \Ann(a) \cap \Ann(\Ann(a))$ and therefore $s_1s_2 = 0$. If $P_1$ and $P_2$ are distinct, it follows by Lemma 3.2 that $a_1a_2 = 0$. This implies that $a_1 \in \Ann(a_2) \subset \Ann(s_2)$, so $a_1s_2 = 0$. Thus, $s_2 \in \Ann(a_1) \subset \Ann(s_1)$ and so $s_1s_2 = 0$.

If $I$ is the ideal, generated by $S$, notice that $S^2 = 0$ gives us $I^2 = 0$. Hence, $xy = 0$ and $(x + y)^2 = 0$ for any $x, y \in I$ and thus the vertices from $I$ form a clique in $ZT(R)$. □

**Lemma 3.4.** Let $R$ be an Artinian ring. If $P \cap \Ann P \neq \{0\}$ for each maximal associated prime ideal $P$ of $R$, then $ZT(R)$ is connected with $\diam(ZT(R)) \leq 3$.

Proof. Take any nonzero $x \in Z(R)$. By 1 there exists $P' \in \Ass(R)$, such that $x \in P'$. Take maximal $P \in \Ass(R)$ that includes $P'$. By assumption $P \cap \Ann P \neq \{0\}$ there exists $z_x \in P \cap \Ann P$, which implies $xz_x = 0$. Since $P$ is an ideal, we have $x + z_x \in P \subset Z(R)$, and thus $x \sim z_x$ is an edge in $ZT(R)$. By Lemma 3.3, vertex $z_x$ is a vertex of a clique in $ZT(R)$. Therefore, for any distinct nonzero $x, y \in Z(R)$, there exists a path $x \sim z_x \sim z_y \sim y$ in $ZT(R)$ and the result follows. □

Recall that a ring $R$ is Artinian if and only if it is Noetherian and all the prime ideals in $R$ are maximal [8 Theorem 2.14]. Hence, as a corollary of Lemma 3.1 and Lemma 3.4 we have the following.

**Theorem 3.1.** For an Artinian ring $R$, graph $ZT(R)$ is connected if and only if $P \cap \Ann P \neq \{0\}$ for all maximal associated prime ideals $P \in \Ass(R)$.

In the case graph $ZT(R)$ is connected, it is possible to classify $R$ with respect to diameter of $ZT(R)$.

**Theorem 3.2.** Let $R$ be an Artinian ring, such that $ZT(R)$ is connected and not empty. Then

1. If $R$ is not local, then $\diam(ZT(R)) = 3$.
2. If $R$ is local and $m \triangleleft R$ is its unique maximal ideal with nilindex $n \geq 3$, then $\diam(ZT(R)) = 2$.
3. If $R$ is local and $m \triangleleft R$ is its unique maximal ideal such that $m^2 = 0$, $m \neq \{0\}$, then $\diam(ZT(R)) = 1$. 
Proof. If $R$ is an Artinian ring that is not local, then $R = R_1 \times R_2 \times \cdots \times R_n$, where $k \geq 2$ and $R_1, R_2, \ldots, R_k$ are local rings. Denote $x = (0, 1, 1, \ldots, 1)$ and $y = (1, 0, 0, \ldots, 0)$. Note that $x$ and $y$ are not adjacent, since $x + y = (1, 1, \ldots, 1) \notin Z(R)$. Furthermore, if $(a_1, a_2, \ldots, a_k) \in N(x) \cap N(y)$, then $a_i = 0$, $i = 1, 2, \ldots, k$, but $(0, 0, \ldots, 0) \notin V(ZT(R))$. Therefore $d(x, y) \geq 3$ and by Theorem 3.4 it follows that diam($ZT(R)$) = 3.

Suppose now $R$ is local Artinian with unique maximal ideal $m$. Then $Z(R) = m$ and so $V(ZT(R)) = m \setminus \{0\}$. In the case $m^2 = \{0\}$, we have $x + y \in Z(R)$ and $xy = 0$ for any $x, y \in m$. Therefore diam($ZT(R)$) = 1 and nonzero elements of $m$ form a clique.

Suppose now, there exists nonzero $u \in m^{n-1}$ and $m^n = 0$ for some $n \geq 3$. Thus, for any nonzero $x \in m \setminus \{u\}$, we have $xu = 0$ and $x + u \in m$, so $x$ is adjacent to $u$ in $ZT(R)$. Hence, diam($ZT(R)$) = 2. We would like to show that $ZT(R)$ is not complete graph. Since $n \geq 3$, there exist $v, w \in m$ such that $vw \neq 0$. If $v \neq w$, then $v \sim w$ and thus diam($ZT(R)$) = 2. Otherwise, if $vw = 0$ for every two distinct $v, w \in m$, there exists $v \in m$ such that $v^2 \neq 0$. This implies $u(u + w) = u^2 \neq 0$ for every $w \in m$. If $u + w$ is nonzero and distinct from $u$, it implies $u \sim u + w$, hence diam($ZT(R)$) = 2. In the case $u + w = 0$ for some $w \in m$, we have $u^2 = 0$, a contradiction. Therefore, for every $w \in m$ we have $u + w = u$, which implies $m = \{0, u\}$ and so $u^2 = u$. This contradicts the fact that $u^n = 0$.

We can easily apply Theorems 3.1 and 3.2 to the case $R = \mathbb{Z}_m$.

Corollary 3.1. The total zero-divisor graph $ZT(\mathbb{Z}_m)$ is connected for $m = p_1^{m_1}p_2^{m_2} \cdots p_n^{m_n}$, where $p_1 < p_2 < \cdots < p_n$ are prime numbers, if and only if $m_i \geq 2$ for $i = 1, 2, \ldots, n$.

Moreover, if $ZT(\mathbb{Z}_m)$ is connected, then

1. diam($ZT(\mathbb{Z}_m)$) = 3 if $n \geq 2$.
2. diam($ZT(\mathbb{Z}_m)$) = 2 if $m = p^k$, $k \geq 3$.
3. diam($ZT(\mathbb{Z}_m)$) = 1 and $ZT(\mathbb{Z}_m) = K_{p-1}$ if $m = p^2$.

4. The total zero-divisor graph of $\mathbb{Z}_m$

If not stated differently, an integer $m$ will be factorized as $m = p_1^{m_1}p_2^{m_2} \cdots p_n^{m_n}$, where $p_1 < p_2 < \cdots < p_n$ are prime numbers.

We will often make use of the following lemma, which follows from Corollary 3.1 in the case $R = \mathbb{Z}_m$.

Lemma 4.1. If the total zero-divisor graph $ZT(\mathbb{Z}_m)$ is connected, the vertices in $ZT(\mathbb{Z}_m)$ of the form

$$r \prod_{i=1}^{n} p_i^{[\frac{m_i}{p}]}$$

where $1 \leq r \leq p_1^{[\frac{m_1}{p}]} \cdots p_n^{[\frac{m_n}{p}]} - 1$, form a clique of size $p_1^{[\frac{m_1}{p}]} \cdots p_n^{[\frac{m_n}{p}]} - 1$.

From this fact we can compute some other parameters of the total zero-divisor graph.
Lemma 4.2. Graph $\text{ZT}(\mathbb{Z}_m)$ is acyclic if $m$ is equal to $2^2, 3^2, 4^2, m = 2^2p$ or $m = pq$, for any distinct prime numbers $p, q$. Otherwise, girth $\text{ZT}(\mathbb{Z}_m) = 3$.

Proof. By Lemma 4.1 we have that girth $\text{ZT}(\mathbb{Z}_m) = 3$ if the graph is connected and $m$ distinct from $2^2, 3^2$ and $4^2$. Note that $\text{ZT}(\mathbb{Z}_{2^2}) = K_1$, $\text{ZT}(\mathbb{Z}_{3^2}) = K_2$, $\text{ZT}(\mathbb{Z}_{4^2}) = P_3$, which are all acyclic. In $\text{ZT}(\mathbb{Z}_{3^2})$ we have $3 \sim 9 \sim 18 \sim 3$ and so girth $\text{ZT}(\mathbb{Z}_{3^2}) = 3$.

If $p$ and $q$ are distinct primes, $m = pq$, then $\text{ZT}(\mathbb{Z}_{pq})$ is an empty graph with $p + q - 2$ vertices. In the case $m = 2^2p$, observe that every edge in the graph is incident with $2p$, hence $\text{ZT}(\mathbb{Z}_{2^2p})$ is acyclic. However, if $m = p^{m_1}q^{m_2}$, where $m_1 \geq 2$, and $m \neq 2^2p$, then $q^{m_1-1} \sim \frac{m}{q} \sim (q-1)\frac{m}{q} \sim q^{m_2-1}$ if $q > 2$ and $2^{m_1-1} \sim \frac{m}{q} \sim \frac{m}{2q} \sim 2^{m_2-1}$ if $q = 2$. In both cases, girth $\text{ZT}(\mathbb{Z}_{p^{m_1}q^{m_2}}) = 3$. If $n \geq 3$, then $\frac{a}{a'} \sim \frac{a}{a'}$ for all $1 \leq i, j \leq n$ and so girth $\text{ZT}(\mathbb{Z}_m) = 3$.

Corollary 4.1. The only acyclic total zero-divisor graphs of rings $\mathbb{Z}_m$ are $P_1$, $P_2$, $P_3$, $K_1$, $K_1 \cup 2K_1$ and $(p + q - 2)K_1$ for any distinct prime numbers $p$ and $q$.

Question 4.1. Theorems 3.1 and 3.2 give us some necessary conditions for a graph $G$ to be the total zero-divisor graph of a Noetherian and Artinian ring $R$. Is it possible to classify all acyclic total zero-divisor graphs of such rings $R$? Or more generally, is it possible to classify all graphs that are the total zero-divisor graph of a Noetherian and Artinian ring $R$?

4.1. Associated elements in $\mathbb{Z}_m$. We say that two elements $a, b \in \mathbb{Z}_m$ are associated if $b = au$ for some invertible element $u \in \mathcal{U}(\mathbb{Z}_m)$. We name the set of all $\{au: u \in \mathcal{U}(\mathbb{Z}_m)\}$ the associatedness class of $a$ in $\mathbb{Z}_m$. Note that for associated zero-divisors $a$ and $b$, $N(a) \setminus \{b\} = N(b) \setminus \{a\}$ holds, hence associated vertices are indistinguishable. While in the zero-divisor graph the only way for vertices to be indistinguishable is to be associated, in the total zero-divisor graph there are vertices that are indistinguishable even though they are not associated. Namely, $p_i^{m_i}$ and $p_i^{m_i-1}$ where $m_i \geq 2$, are such vertices.

We use the notation $w|_m v$ to denote that $w$ is a divisor of $v$ in $\mathbb{Z}_m$, i.e. there exists $x \in \mathbb{Z}_m$ such that $wx \equiv v$ modulo $m$. The symbol "$\equiv$" denotes congruence modulo $m$, i.e. equality in ring $\mathbb{Z}_m$.

Lemma 4.3. Every associatedness class of $\mathbb{Z}_m$ contains exactly one positive divisor of $m$.

Proof. We are going to show that for an arbitrary $a = p_1^{\ell_1}p_2^{\ell_2}\cdots p_n^{\ell_n}s_a \in \mathbb{Z}_m$, $s_a \in \mathcal{U}(\mathbb{Z}_m)$, there is a divisor of $m$ which is associated to $a$. Let us denote $b(a) = \{i: \ell_i > m_i\} \subseteq \{1, 2, \ldots, n\}$. Then, we have

$$a \equiv a - m \prod_{j \notin b(a)} p_j = p_1^{\ell_1}p_2^{\ell_2}\cdots p_n^{\ell_n}s_a - m \prod_{j \notin b(a)} p_j =$$

$$= \prod_{i \notin b(a)} p_i^{m_i} \cdot \prod_{j \notin b(a)} p_j^{\ell_j} \left( \prod_{i \notin b(a)} p_i^{\ell_i - m_i} \cdot s_a - \prod_{j \notin b(a)} p_j^{m_j - \ell_j + 1} \right)$$

$$= v \cdot u,$$
Lemma 4.4. Let \(G\) be connected but not complete graph. Then

\[
\Delta (\text{ZT} (Z_m)) = \deg \left( \frac{m}{p_i} \right) = \frac{m}{p_1} - 2 \quad \text{and} \quad \delta (\text{ZT} (Z_m)) = \deg (p_1) = p_1 - 1.
\]

Proof. Consider the vertex \(q_i = \frac{m}{p_i}\) for any \(i \in \{1,\ldots,n\}\) and its closed neighbourhood \(N[q_i] = \{q_i r; 1 \leq r \leq \frac{m}{p_i} - 1\}\). We have \(\deg (q_i) = \frac{m}{p_i} - 2\). By Corollary 4.2, for an arbitrary vertex \(q\) in \(\text{ZT} (Z_m)\) there exists \(k \in \{1,\ldots,n\}\) such that \(q \in N[q_k]\). Hence, \(N[q] \subseteq N[q_k]\) and so we have \(\deg (q) \leq \deg (q_k) = \frac{m}{p_k} - 2 \leq \frac{m}{p_1} - 2\).

Now, consider the vertex \(p_i\) for any \(i \in \{1,\ldots,n\}\). Its open neighbourhood is equal to \(N(p_i) = \{r \cdot \frac{m}{p_i}; 1 \leq r \leq p_i - 1\}\) and hence \(\deg (p_i) = p_i - 1\). For an arbitrary vertex \(v = p_1^{t_1}p_2^{t_2}\ldots p_n^{t_n}s_v \in V (\text{ZT} (Z_m))\), there is \(l \in \{1,\ldots,n\}\) such that \(v_l \geq 1\). Hence, \(N[p_i] \subseteq N[v]\) and thus \(\deg (v) \geq \deg (p_i) = p_i - 1 \geq p_1 - 1\).

If \(G\) is a connected but not complete graph, isomorphic to \(\text{ZT} (Z_m)\) for some \(m\), then by Lemma 4.4 we have \(m = (\Delta (G) + 2)(\delta (G) + 1)\). Therefore, we have the following property of the total zero-divisor graph.

Corollary 4.3. If \(\text{ZT} (Z_m) \cong \text{ZT} (Z_n)\), then \(m = n\).
Corollary 4.4. A connected graph $ZT(\mathbb{Z}_m)$ is regular if and only if it is complete.

Proof. Assume $ZT(\mathbb{Z}_m)$ be connected but not complete graph. If it is regular, we have $\Delta(ZT(\mathbb{Z}_m)) = \delta(ZT(\mathbb{Z}_m))$. By Lemma 4.4 it follows that $\frac{m}{p_1} - 2 = p_1 - 1 \Rightarrow \frac{m}{p_1} = p_1 + 1$. By Corollary 3.1, connectedness of $ZT(\mathbb{Z}_m)$ implies $m_1 \geq 2$ and thus $p_1 \mid \left(\frac{m}{p_1}\right)$, a contradiction. So, $ZT(\mathbb{Z}_m)$ is not regular graph. Since every complete graph is regular, we have the equivalence. \qed

Corollary 4.5. For a pair of integers $(x, y)$, there is a positive integer $m$ such that

(a) graph $ZT(\mathbb{Z}_m)$ is connected but not complete,
(b) $\Delta(ZT(\mathbb{Z}_m)) = x$ and $\delta(ZT(\mathbb{Z}_m)) = y$;

if and only if the following conditions hold:

(1) $x > y > 0$,
(2) $y + 1$ is the least prime divisor of $x + 2$,
(3) if $q \mid (x + 2)(y + 1)$ then $q^2 \mid (x + 2)(y + 1)$ for all prime $q$.

Proof. Assume that there are positive integers $m, x, y$ such that graph $ZT(\mathbb{Z}_m)$ satisfies conditions (a) and (b). By Lemma 4.3, $m$ is uniquely determined as $m = (x + 2)(y + 1)$. Since $ZT(\mathbb{Z}_m)$ is connected, Corollary 3.1 implies (3) and moreover that $y > 0$. In particular, $p_1 \mid \frac{m}{p_1}$ and $p_1$ is the smallest prime divisor of $\frac{m}{p_1}$. Since $ZT(\mathbb{Z}_m)$ is not complete, Proposition 4.4 implies $x > y$ and thus (1) and (2).

Conversely, assume that $(x, y)$ is a pair of integers such that conditions (1), (2) and (3) hold. Take $m = (x + 2)(y + 1)$. Corollary 3.1 and (3) imply that the graph $ZT(\mathbb{Z}_m)$ is connected. By (2), $y + 1$ is the least prime divisor of $m$ and thus by Proposition 4.4 it follows that $\Delta(ZT(\mathbb{Z}_m)) = \frac{m}{p_1} - 2 = x$ and $\delta(ZT(\mathbb{Z}_m)) = (y + 1) - 1 = y$. Moreover, by (1), $ZT(\mathbb{Z}_m)$ is not regular thus by 4.4 not complete. \qed

4.2. Colorings of $ZT(\mathbb{Z}_m)$. In this subsection we will investigate two different colorings of the total zero-divisor graph.

The smallest number of colors needed to color the vertices of graph $G$, such that any two adjacent vertices have different colors, is called the chromatic number of $G$ and is denoted by $\chi(G)$.

Every nonempty bipartite graph has the chromatic number equal to 2. Thus by Corollary 4.4 we have that the $\chi(ZT(\mathbb{Z}_m)) = 2$ if and only if $m = 2^2, 2^3, 3^2, m = 2^ip$ or $m = pq$, for any distinct prime numbers $p$ and $q$. In the following proposition we constructively find the chromatic number of any total zero-divisor graph $ZT(\mathbb{Z}_m)$.

Proposition 4.1. In the case $ZT(\mathbb{Z}_m)$ is connected, let $o(m)$ denote the number of odd exponents $m_i$. Then the chromatic number of graph $ZT(\mathbb{Z}_m)$ is equal to

$$\chi(ZT(\mathbb{Z}_m)) = p_1^{\left\lfloor \frac{m}{p_1} \right\rfloor} \cdots p_n^{\left\lfloor \frac{m}{p_n} \right\rfloor} + o(m) - 1.$$
If $\text{ZT}(\mathbb{Z}_m)$ is not connected and not empty, then let $I \subseteq \{1, \ldots, n\}$ be the set of all indices $i$ such that $m_i \geq 2$ and let $J = \{1, \ldots, n\} \setminus I$. Then the chromatic number of $\text{ZT}(\mathbb{Z}_m)$ is equal to

$$\chi(\text{ZT}(\mathbb{Z}_m)) = \prod_{i \in I} \left\lceil \frac{m_i}{2} \right\rceil + |J|.$$ 

**Proof.** Suppose first that $\text{ZT}(\mathbb{Z}_m)$ is connected. By Lemma 4.1 we need at least $\prod_{i=1}^{n} \left\lceil \frac{m_i}{2} \right\rceil - 1$ colors to color the vertices of the form $s \prod_{i=1}^{n} \frac{m_i}{p_i}$, $1 \leq r \leq p_1^{m_1} \cdots p_n^{m_n} - 1$, which form a clique $A_0$ in $\text{ZT}(\mathbb{Z}_m)$. We will construct subsets $A_0, A_1, \ldots, A_{o(m)}$ of $V(\text{ZT}(\mathbb{Z}_m))$ in the following way.

Let $k_1 < k_2 < \ldots < k_{o(m)}$ denote indices such that $m_{k_j}$ is odd for $j = 1, 2, \ldots, o(m)$. Now, define the sets $A_j$ to consist of all zero-divisors of the form $s \prod_{i=1}^{n} \frac{m_i}{p_i^{m_i}}$, $1 \leq i \leq \left\lceil \frac{m_{k_j}}{2} \right\rceil$, where $s$ is coprime to $p_{k_j}$ and which are not already in $A_0, A_1, A_2, \ldots, A_{j-1}$. Note that there are no edges within any of the sets $A_j$ and by definition they are disjoint. However, vertices

$$v_j = \frac{1}{p_{k_j}} \prod_{i=1}^{n} \frac{m_i}{p_i^{m_i}} \in A_j,$$

$j = 1, 2, \ldots, o(m)$, form a clique and each of them is adjacent to every vertex in $A_0$. Therefore,

$$\chi(A_0 \cup A_1 \cup \ldots \cup A_{o(m)}) \geq \chi(A_0) + o(m) = \prod_{i=1}^{n} \left\lceil \frac{m_i}{2} \right\rceil + o(m) - 1. \tag{2}$$

Now, let $A_{o(m)+1} = V(\text{ZT}(\mathbb{Z}_m)) \setminus \bigcup_{i=0}^{o(m)} A_i$. Note that if $o(m) = n$, then $A_{o(m)+1} = \emptyset$. Otherwise, for every vertex $w = p_1^{s_1} \cdots p_n^{s_n}$ in $A_{o(m)+1}$, there exists $s_j$ such that $m_j$ is even and $s_j < \frac{m_j}{2}$ and that $s_i \geq \left\lceil \frac{m_i}{2} \right\rceil$ if $m_i$ is odd.

Clearly $w$ is not adjacent to $u = p_j^{m_j} \prod_{i \neq j} p_i^{m_i} \in A_0$ and therefore, $w$ can be colored with the same color as $u$. Notice that $w$ and $w'$ which get the same color in that way, are not adjacent to each other. Therefore, by (2)

$$\chi(\text{ZT}(\mathbb{Z}_m)) = \chi(A_0 \cup A_1 \cup \ldots \cup A_{o(m)}) = \prod_{i=1}^{n} \left\lceil \frac{m_i}{2} \right\rceil + o(m) - 1. \tag{3}$$

If $\text{ZT}(\mathbb{Z}_m)$ is not connected, then $J \neq \emptyset$. Moreover, if $\text{ZT}(\mathbb{Z}_m)$ has an edge, observe that vertices of the form

$$\prod_{i \in I, i \geq 0} \frac{m_i}{p_i^{t_i}} \prod_{j \in J} p_j,$$

together with vertices of the form $\frac{m_j}{p_j}$, where $j \in J$, form a clique $A$ of size $\prod_{i \in I} \left\lceil \frac{m_i}{2} \right\rceil + |J|$. So, at least that many colors is needed. Let us show that coloring the remaining vertices does not require any additional colors.

Any vertex $x = p_1^{r_1} \cdots p_n^{r_n}$ not in $A$ must have at least one of these two properties:

1. there exists $i \in I$ such that $r_i < \left\lceil \frac{m_i}{2} \right\rceil$, or
(2) there exists \( j \in J \) such that \( r_j = 0 \), i.e. \( p_j \) does not divide \( x \).

If \( r_i < \lceil \frac{m_i}{2} \rceil \) for some \( i \in I \), then \( x \) is not adjacent to \( u = m p_i^{-\lceil \frac{m_i}{2} \rceil} \in A \), thus can be colored with the same color as \( u \). In the second case, \( r_j = 0 \) for some \( j \in J \), then \( x \) is not adjacent to \( v = \frac{m}{p_j} \in A \) and can be colored with the same color as \( v \). If \( x = p_1^{s_1} \cdots p_n^{s_n} \) and \( y = p_1^{s_1} \cdots p_n^{s_n} \) are not in \( A \) and would they take the color of the same vertex in \( A \), then either there is \( i \in I \) such that \( r_i < \lceil \frac{m_i}{2} \rceil \) and \( s_i < \lceil \frac{m_i}{2} \rceil \), or there is \( j \in J \) such that \( p_j = r_j = 0 \). In any case \( x \) is not adjacent to \( y \) and thus we found the coloring of \( ZT(Z_m) \) and proved that \( \chi(ZT(Z_m)) = \prod_{i \in I} \lceil \frac{m_i}{2} \rceil + |J| \).

\( \square \)

**Example 4.1.** Let us illustrate the algorithm presented in the proof of Proposition 4.1 in the case \( m = 2^3 \cdot 3^2 \cdot 5 = 360 \).

The set \( A_0 \) contains vertices: \( 2^2 \cdot 3 \cdot 5 \), \( 2^2 \cdot 3 \cdot 5 \cdot 2 \), \( 2^2 \cdot 3 \cdot 5 \cdot 3 \), \( 2^2 \cdot 3 \cdot 5 \cdot 4 \), \( 2^2 \cdot 3 \cdot 5 \cdot 5 \). They form a \( K_5 \). The set \( A_1 \) is the union of the sets \( \{ 2^2 \cdot 3^1 \cdot 5 \cdot 2^1 \cdot 5 \cdot 2^1 \cdot 7 \cdot 2 \} \) and \( \{ 3^5 \cdot 2 \cdot 2 \cdot 9 \cdot 2^1 \cdot 5 \cdot 2 \cdot 9 \cdot 2^1 \cdot 5 \cdot 2 \cdot 9 \cdot 2^1 \cdot 5 \cdot 2 \} \). The vertex \( v_1 = 2^1 \cdot 15 \) is adjacent to every vertex in \( A_0 \) and \( v_1 \sim v_2 \). Moreover, \( A_2 = \{ 5^0 \cdot 4 \cdot 5^0 \cdot 8 \cdot 5^0 \cdot 12, \ldots, 5^0 \cdot 356 \} \) and \( v_2 = 2^2 \cdot 3^1 \cdot 5^0 \in A_2 \) is adjacent to all the vertices in \( A_0 \). Lastly, \( A_3 = \{ 2^2 \cdot 3^0 \cdot 5^1 \cdot 2^3 \cdot 3^0 \cdot 5^1 \cdot 2^3 \cdot 3^0 \cdot 5^1 \cdot 2^3 \cdot 3^0 \cdot 5^1 \cdot 17 \} \). For every vertex in \( A_3 \), there is a vertex in \( A_0 \), which is not adjacent to it \( 2^2 \cdot 3^1 \cdot 5^1 \sim 2^2 \cdot 3^1 \cdot 5^1 \), \( 2^3 \cdot 3^0 \cdot 5^1 \sim 2^3 \cdot 3^0 \cdot 5^1 \). The graph vertices require 5 colors to color \( A_0 \) and all other vertices except \( v_1 \) and \( v_2 \). Since \( v_1 \sim v_2 \) and they are adjacent to all the vertices in \( A_0 \), we have \( \chi(ZT(Z_{360})) = 7 \).

The edge coloring of a graph requires that no two adjacent edges have the same color and the smallest number of colors for the edge coloring is called the chromatic index of \( G \) and is denoted by \( \chi'(G) \).

**Proposition 4.2.** If \( ZT(Z_m) \) is a connected but not complete graph, then its chromatic index is equal to \( \chi'(ZT(Z_m)) = \frac{m}{p_1} - 2 \).

**Proof.** Recall that by Vizing’s Theorem and Lemma 4.4 we have that \( \chi'(Z_m) = \frac{m}{p_1} - 2 \) or \( \Delta(Z_m) + 1 = \frac{m}{p_1} - 1 \). Let us prove that \( \frac{m}{p_1} - 2 \) colors is sufficient to color \( ZT(Z_m) \).

By Lemma 4.4, the vertices with the maximal degree are the ones associated to \( \frac{m}{p_1} \). They are of the form \( s \cdot \frac{m}{p_1} \) for \( s = 1, 2, \ldots, p_1 - 1 \), hence there is \( p_1 - 1 \) of them.

Firstly, we color all the edges incident to \( \frac{m}{p_1} \). Then, for \( s = 2, \ldots, p_1 - 1 \) we color all the uncolored edges incident to \( s \cdot \frac{m}{p_1} \) in a way that for edge \( s \cdot \frac{m}{p_1} \sim v \), we take a color different from the colors of \( i \cdot \frac{m}{p_1} \sim v \) for \( i = 1, 2, \ldots, s - 1 \). For each \( v \) which is adjacent to the associatedness class of \( \frac{m}{p_1} \), edges of the form \( s \cdot \frac{m}{p_1} \sim v \) are colored using at most \( p_1 - 1 \) colors. Inequality \( \frac{m}{p_1} - 2 > p_1 - 1 \) is guaranteed by Corollary 4.4. Coloring the rest of the edges with no additional colors is possible because each vertex not in the associatedness class of \( \frac{m}{p_1} \) has degree less than \( \frac{m}{p_1} - 2 \). \( \square \)

4.3. **Cycles in** \( ZT(Z_m) \). To the best of our knowledge, there is no full description of rings with Hamiltonian zero-divisor graphs. In [1] the authors proved that if the total graph of a finite commutative ring is connected then
it is also a Hamiltonian graph. In the next proposition we characterize Hamiltonian total zero-divisor graphs and prove that the total zero-divisor graph is Hamiltonian if and only if it is complete with at least 4 vertices.

**Proposition 4.3.** The total zero-divisor graph $ZT(\mathbb{Z}_m)$ is Hamiltonian if and only if $m = p^2$, where $p \geq 5$.

*Proof.* Assume first $m = p^2$, where $p \geq 5$. By Corollary 3.13, we have $ZT(\mathbb{Z}_m) \cong K_{p-1}$, which is Hamiltonian.

If $ZT(\mathbb{Z}_m)$ is not connected, it is clearly not Hamiltonian. Moreover, if $m = 2^2$ or $m = 3^2$ then $ZT(\mathbb{Z}_m)$ is $K_1$ or $K_2$ and hence not Hamiltonian. So, suppose that $n \geq 2$ or $m_1 \geq 3$ and note that $\frac{m}{p_1} > p_1$. For

$$S = \left\{ r \cdot \frac{m}{p_1}; 1 \leq r < p_1 \right\} \quad \text{and} \quad T = \left\{ p_1 s; 1 \leq s < \frac{m}{p_1}, \gcd(s, m) = 1 \right\}$$

we have $|S| = p_1 - 1$. Moreover, $r \leq p_1 - 1 \leq \frac{m}{p_1} - 1$ implies $\gcd(r, m) = 1$, so $|S| \leq |T|$. Since $\gcd\left(\frac{m}{p_1} - 1, m\right) = 1$ and $\frac{m}{p_1} - 1 > p_1 - 1$ it follows that $|S| < |T|$.

Notice that vertices in $T$ have no neighbours in $V(ZT(\mathbb{Z}_m)) - S$. Therefore, the number of components of $ZT(\mathbb{Z}_m) - S$ is at least equal to $|T| - |S|$, so $ZT(\mathbb{Z}_m)$ is not Hamiltonian. \qed

As a corollary of Lemma 4.4 we have the following property of the total zero-divisor graph.

**Corollary 4.6.** For any positive integer $m$, the total zero-divisor graph $ZT(\mathbb{Z}_m)$ is not Eulerian.

*Proof.* If $ZT(\mathbb{Z}_m)$ is not connected, obviously it is not Eulerian. If $ZT(\mathbb{Z}_m)$ is connected, recall that it is Eulerian if and only if every vertex has an even degree. In the case $ZT(\mathbb{Z}_m)$ is complete, by Corollary 3.13, $ZT(\mathbb{Z}_m) \cong K_{p_1 - 1}$. Therefore, every vertex has a degree $p_1 - 2$, which is either 0 or odd. Thus, $ZT(\mathbb{Z}_m)$ is not Eulerian. Otherwise, if $ZT(\mathbb{Z}_m)$ is connected but not complete, we make a use of Lemma 4.4. If $p_1 = 2$, then $\delta(ZT(\mathbb{Z}_m)) = 1$, and otherwise if $p_1 > 2$, $m$ is odd and so $\Delta(ZT(\mathbb{Z}_m)) = \frac{m}{p_1} - 2$ is odd as well. In both cases we have that $ZT(\mathbb{Z}_m)$ is not Eulerian. \qed

### 4.4. Domination number

Recall that a dominating set for a graph $G$ is a subset $D \subseteq V(G)$ such that every vertex not in $D$ is adjacent to at least one member of $D$. The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set for $G$.

It was proved in [12] that the domination number $\gamma(R)$ is equal to the number of distinct maximal ideals of a finite commutative ring with identity $R$, if $R \neq \mathbb{Z}_2 \times F$ for any field $F$ and $R$ is not a domain. The following proposition shows the same is true for the total zero-divisor graph of $\mathbb{Z}_m$.

**Proposition 4.4.** If $ZT(\mathbb{Z}_m)$ is a connected graph, then the domination number is equal to

$$\gamma(ZT(\mathbb{Z}_m)) = n.$$
we assume that \( w \) for some \( ZT(\mathbb{Z}_m) \). Therefore, if \( v \) is in a dominating set \( D \) for \( ZT(\mathbb{Z}_m) \), \( D \cup \{ \frac{m}{p_k} \} - \{ v \} \) is dominating set as well. Hence \( \gamma(ZT(\mathbb{Z}_m)) \geq n \).

We claim that \( D = \{ \frac{m}{p_i}, i = 1, 2, \ldots, n \} \) is a dominating set. Namely, for an arbitrary nonzero zero-divisor \( u = p_1^{m_1}p_2^{m_2} \cdots p_n^{m_n}s_u \) there exists \( k \) such that \( u_k \geq 1 \). By assumption that \( ZT(\mathbb{Z}_m) \) is connected, it follows that \( m_k \geq 2 \), thus \( u \cdot \frac{m}{p_k} \equiv 0 \) and \( p_k \mid (u + \frac{m}{p_k}) \). Therefore \( u + \frac{m}{p_k} \in Z(\mathbb{Z}_m) \), hence \( u \) is adjacent to \( \frac{m}{p_k} \in D \). So, \( \gamma(ZT(\mathbb{Z}_m)) = n. \)

\[ \square \]

4.5. Metric dimension. The metric dimension \( \text{dim}_M(G) \) of a graph \( G \) is the minimum cardinality of a set \( S \subseteq V(G) \) such that all other vertices are uniquely determined by their distances to the vertices in \( S \). A set \( S \) is known as a resolving set. Determining the metric dimension of a graph is known to be an NP-complete problem.

In [11] and [7] the authors give certain bounds on the metric dimension of a zero divisor graph and of the total graph in some specific ring. For the total zero-divisor graph of \( \mathbb{Z}_m \) we are able to compute its metric dimension and the proof allows us to apply the result also for the zero-divisor graph of \( \mathbb{Z}_m \).

**Proposition 4.5.** If \( ZT(\mathbb{Z}_m) \) is connected, then

\[ \text{dim}_M(ZT(\mathbb{Z}_m)) = \begin{cases} m - \varphi(m) - \tau(m) + n + 1, & n \geq 2, \\ m - \varphi(m) - \tau(m) + 1, & n = 1. \end{cases} \]

**Proof.** Recall that vertices \( a \) and \( b \) are indistinguishable in a simple graph if \( N(a) \setminus \{b\} = N(b) \setminus \{a\} \). Note that a simple graph \( G \) is invariant to permutation of indistinguishable vertices. So, complement of a resolving set of \( G \) cannot contain a pair of indistinguishable vertices because, otherwise, these would have the same ordered set of distances from the elements of resolving set. Therefore, complement of a resolving set can contain at most one element from each class of indistinguishable vertices.

Assume first \( n \geq 2 \). Clearly, associates are indistinguishable in \( ZT(\mathbb{Z}_m) \). For every \( i = 1, 2, \ldots, n \), vertices \( p_i^{m_i} \) and \( p_i^{m_i-1} \) are also indistinguishable in \( ZT(\mathbb{Z}_m) \); clearly, \( N(p_i^{m_i-1}) \setminus \{p_i^{m_i}\} \subset N(p_i^{m_i}) \setminus \{p_i^{m_i-1}\} \); if \( w \in N(p_i^{m_i}) \setminus \{p_i^{m_i-1}\} \), then \( p_i \mid w \) because of \( p_i^{m_i} + w \in Z(\mathbb{Z}_m) \), and \( \frac{m}{p_i} \) because of \( p_i^{m_i} \cdot w \equiv 0 \), but then \( \text{lcm}(p_i, \frac{m}{p_i}) = \frac{m}{p_i} \) is also a divisor of \( w \) so \( w \in N(p_i^{m_i}) \setminus \{p_i^{m_i-1}\} \). Next we show that those two are the only types of indistinguishability in \( ZT(\mathbb{Z}_m) \).

Let \( u \) and \( v \) be vertices which are neither associates nor \( p_i^{m_i} \) and \( p_i^{m_i-1} \) for some \( i \in \{1, 2, \ldots, n\} \). By Lemma 4.3 there is no loss of generality if we assume that \( u \mid m \) and \( v \mid m \). Take \( u = p_1^{m_1}p_2^{m_2} \cdots p_n^{m_n} \) and \( v = p_1^{n_1}p_2^{n_2} \cdots p_n^{n_n} \).

If \( u \neq v \) then there exists \( k \) such that \( u_k \neq v_k \). Without loss of generality assume \( u_k < v_k \). If \( v_k < m_k \) then \( w = \frac{m}{p_k} \) is adjacent to \( v \) but it is not
adjacent to \( u \). Hence, \( u \) and \( v \) are not indistinguishable. Else if \( v_k = m_k \), then from the assumption above that \( u \) and \( v \) are not \( p_i^{m_i} \) and \( p_i^{m_i-1} \) for some \( i \in \{1, 2, \ldots, n\} \), we get \( u_k < m_k - 1 \). Then \( w = \frac{m}{p_k^{m_k}} \) is adjacent to \( v \) but it is not adjacent to \( u \). Again, \( u \) and \( v \) are not indistinguishable.

By the above arguments, the number of different classes of indistinguishable vertices is equal to the number of associatedness classes minus the number of prime factors of \( m \), which is \( \tau (m) - 2 - n \) by Lemma 4.3.

Since the complement of a resolving set can contain at most one element from each class of indistinguishable vertices, a resolving set must have at least

\[
|V (ZT (\mathbb{Z}_m))| - (\tau (m) - 2 - n) = m - \varphi (m) - \tau (m) + n + 1
\]
elements.

Consider the set

\[
B = \left\{ p_i^{b_1} p_2^{b_2} \cdots p_n^{b_n} \mid \text{\( b_i \neq b_j \) and \( b_i \neq b_1 \) for any \( i \neq j \)} \right\} \cup \{ x_i^{m_i}; \ i = 1, 2, \ldots, n \}
\]
with

\[
|B| = (m - \varphi (m) - 1) - (\tau (m) - 2) + n = m - \varphi (m) - \tau (m) + n + 1
\]
and whose complement contains all the vertices corresponding to the divisors of \( m \) except those in set \( \{ x_i^{m_i}; \ 1 \leq i \leq n \} \). We are going to show that \( B \) is a resolving set. Let \( z = p_1^{x_1} p_2^{x_2} \cdots p_n^{x_n} \) and \( y = p_1^{y_1} p_2^{y_2} \cdots p_n^{y_n} \) be different elements of \( V (ZT (\mathbb{Z}_m)) \) \( B \). Then \( x \) and \( y \) are not indistinguishable by definition of \( B \). So, without loss of generality assume there exists \( z \in N (x) \ \setminus \ N (y) \). If \( z \in B \), then \( d (x, z) = 1 \neq d (y, z) \) so ordered set of distances to elements of \( B \) differ for \( x \) and \( y \). Else, \( z \notin B \) so \( z \mid m \) and \( z \neq p_i^{m_i} \) for any \( 1 \leq i \leq n \). Since \( ZT (\mathbb{Z}_m) \) is connected and \( n \geq 2 \), we have \( m \geq 36 \) and thus there exists \( s \in \mathcal{U} (R) \ \setminus \ \{ 1 \} \). Take \( z' = z \cdot s \in B \) and observe that \( d (x, z') = 1 \neq d (y, z') \). So, \( B \) is a resolving set and the metric dimension of \( ZT (\mathbb{Z}_m) \) is equal to \( m - \varphi (m) - \tau (m) + n + 1 \).

Now, let \( n = 1 \), or equivalently \( m = p_1^{m_1} \). The only indistinguishable elements are the associates since \( p_1^{m_1} = 0 \). It follows that \( \dim_H (ZT (\mathbb{Z}_m)) = m - \varphi (m) - \tau (m) + 1 \).

\[\Box\]

**Remark 4.1.** Note that the same proof would work to show that the metric dimension of a connected zero-divisor graph is equal to \( m - \varphi (m) - \tau (m) + 1 \).

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