Quantum computations starting with computational basis states and involving only Clifford operations, are classically simulable despite the fact that they generate highly entangled states; this is the content of the Gottesman-Knill theorem. Here we isolate the ingredients of the theorem and provide generalisations of some of them with the aim of identifying new classes of simulable quantum computations. In the usual construction, Clifford operations arise as projective normalisers of the first and second tensor powers of the Pauli group. We consider replacing the Pauli group by an arbitrary finite subgroup $G$ of $U(d)$. In particular we seek $G$ such that $G \otimes G$ has an entangling normaliser. Via a generalisation of the Gottesman-Knill theorem the resulting normalisers lead to classes of quantum circuits that can be classically efficiently simulated. For the qubit case $d = 2$ we exhaustively treat all finite irreducible subgroups of $U(2)$ and find that the only ones (up to unitary equivalence and trivial phase extensions) with entangling normalisers are the groups generated by $X$ and the $n^{th}$ root of $Z$ for $n \in \mathbb{N}$.

Keywords:

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1 Introduction

The identification of classes of quantum computations that can be classically efficiently simulated is a basic tool for studying the relationship between classical and quantum computational power. One of the earliest results in this context is the Gottesman-Knill (GK) theorem [1, 2]. It initially arose in the study of the stabiliser formalism for quantum error correcting codes and has a rich variety of mathematical ingredients. In this paper we isolate these ingredients and develop generalisations of some of them with an aim of identifying further new classes of simulable quantum computations. (Other directions of generalisation of the GK theorem were considered in [5]).

To motivate our proposed generalisations we first outline the ingredients of the
GK theorem and the notion of stabiliser state that usually forms the basis of the theorem. Let $X$ and $Z$ denote the standard qubit Pauli operations. The Pauli group is defined by $\mathcal{P} = \langle X, Z, iI \rangle$ (where the pointed brackets denote the group generated by the enclosed elements). The Pauli group on $n$ qubits is defined as the $n$-fold tensor power $\mathcal{P}_n = \mathcal{P}^\otimes n$ which is a finite group of size $|\mathcal{P}_n| = O(4^n)$. If $|\psi\rangle$ is any $n$ qubit state define its stabiliser as

$$\text{Stab}(|\psi\rangle) = \{ g \in \mathcal{P}_n : g |\psi\rangle = |\psi\rangle \}.$$

Clearly $\text{Stab}(|\psi\rangle)$ is always a subgroup of $\mathcal{P}_n$ (albeit the trivial subgroup for many $|\psi\rangle$’s). $|\psi\rangle$ is a stabiliser state if it is uniquely characterised by its stabiliser i.e. it is the only state left invariant by all $g \in \text{Stab}(\psi)$. Next note that for any $U$, $\text{Stab}(U|\psi\rangle) = U\text{Stab}(|\psi\rangle)U^\dagger$. Thus if $|\psi\rangle$ is a stabiliser state and $U\mathcal{P}_n U^\dagger = \mathcal{P}_n$ then $U|\psi\rangle$ will again be a stabiliser state. Correspondingly for each $n$ we introduce the so-called Clifford group $\mathcal{C}_n$ defined by

$$\mathcal{C}_n = \{ C \in U(2^n) : C\mathcal{P}_n C^\dagger = \mathcal{P}_n \}$$

i.e. $\mathcal{C}_n$ is the (group-theoretic) normaliser of the group $\mathcal{P}_n$ (within the unitary group). Let $H$ denote the Hadamard operation, let $P$ denote the $\pi/4$-phase gate

$$P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

and let $CZ$ denote the 2-qubit controlled-Z gate. Then we have the following full explicit characterisation [4, 1] of Clifford operations.

**Lemma 1** $\mathcal{C}_1 = \langle H, P \rangle$ and $\mathcal{C}_2 = \langle \mathcal{C}_1 \otimes \mathcal{C}_1, CZ \rangle$. For $n \geq 3$ an $n$-qubit gate $U$ is in $\mathcal{C}_n$ iff it can be expressed as a circuit of gates from $\mathcal{C}_1$ and $\mathcal{C}_2$.

In terms of these structures we can give a precise statement of the GK theorem:

**Theorem 1** (Gottesman-Knill theorem). Consider any polynomial-time quantum computation of the following sort. The starting state $|\psi_0\rangle$ is a stabiliser state and each computational step is one of the following:

(a) a measurement on a qubit in the Z basis, or

(b) application of a gate from $\mathcal{C}_1$ or $\mathcal{C}_2$ (not depending on measurement outcomes in (a)), or

(c) application of a gate from $\mathcal{C}_1$ or $\mathcal{C}_2$ chosen adaptively depending on a previous measurement outcome from (a).

Finally the output is the result of a measurement of the first qubit in the Z basis.

Then the computation may be classically efficiently simulated.

The standard proof of this result (see e.g. [2] or [5] for a more recent improved algorithm) proceeds by updating the stabiliser description of the state through the course of the computation. The update procedure for (b) (and (c) once the measurement result is given) is via the normalising property of the Clifford group in relation to the Pauli group $\mathcal{P}_n$ containing the generators. This purely group-theoretic relationship in itself, may be entertained for any group $G$ replacing the Pauli group.
the other hand the stabiliser update rules for (a) (as elaborated for example in [2] page 463) depend on further features specific to the Pauli group, such as the fact that in this group every two elements either commute or anti-commute.

The starting point for our generalisations is an alternative simpler proof in the absence of the adaptively chosen steps in (c): instead of forwardly propagating the state description we will backwardly propagate the final measurement allowing us in particular even to free the simulation from requiring stabiliser states. Thus let C now be any circuit of Clifford operations on starting state \( |\psi_0\rangle \) which is now not required to be a stabiliser state. If the final measurement on the first qubit has outputs 0,1 with probabilities \( p_0, p_1 \) then \( p_0 - p_1 \) is given by the expectation value of \( Z_1 = Z \otimes I \otimes \ldots \otimes I \) in the final state \( C |\psi_0\rangle \):

\[
p_0 - p_1 = \langle \psi_0 | C^\dagger Z_1 C |\psi_0\rangle.
\]

(1)

This computation suffices to simulate the output (as we also have \( p_0 + p_1 = 1 \)). Now \( Z_1 \in \mathcal{P}_n \) so \( C^\dagger Z_1 C \) has the product form \( P_{i_1} \otimes \ldots \otimes P_{i_n} \) for Pauli operators \( P_{i_k} \). Hence if \( |\psi_0\rangle \) is any product state \( |\psi_0\rangle = |a_1\rangle \ldots |a_n\rangle \) then we get

\[
p_0 - p_1 = \prod_{k=1}^{n} \langle a_k | P_{i_k} | a_k \rangle
\]

(2)

which can clearly be calculated classically in linear time \( O(n) \). Similarly the commuting of the successive one and two qubit Clifford gates through \( Z_1 \) can also be done in time linear in the size of the circuit giving a linear time classical simulation of the quantum computation’s output.

This approach to the simulation of Clifford circuits may also be extended to allow for measurement steps (of type (a) above) so long as subsequent gates are not chosen adaptively (as they are in (c) above, with stabiliser starting states). To achieve this we replace each measurement step by the following: for each measurement on a qubit \( i \) adjoin an extra initial qubit in state \( |0\rangle \) and replace the measurement step by a (Clifford) CNOT operation with control and target being the \( i \)th and new qubits respectively. The newly introduced qubit is not used in any other way by the computation so its presence serves to decohere the \( i \)th qubit into the post-measurement mixture i.e. each measurement step of the form (a) is replaced by a CNOT step of the form (b) and the final output is unchanged. We may ask if a further such trick could allow efficient simulation of the output of the process with adaptively chosen gates (as in (c)) in addition to just measurements (a) themselves, for the scenario of Clifford circuits on arbitrary product starting states. Such further generalisation is not likely to be possible for the following reason: if we allow arbitrary product state inputs and adaptive Clifford gate choices then we could (as shown in [6]) implement the \( \pi/8 \)-phase gate

\[
S = \begin{pmatrix}
1 & 0 \\
0 & e^{i\pi/4}
\end{pmatrix}.
\]

To see how this is achieved let \( |\alpha\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/4}|1\rangle) \). Then for any qubit \( |\psi\rangle \) apply CNOT to \( |\psi\rangle |\alpha\rangle \) and measure the second qubit. If the outcome is 0 then the
post-measurement state is $S|\psi\rangle|0\rangle$. If the outcome is 1 then the post-measurement state is $S^\dagger|\psi\rangle|1\rangle e^{i\pi/4}$ so applying $P$ to the first qubit gives $S|\psi\rangle|1\rangle$ up to an overall phase. Thus we implement $S$ in either case by responding adaptively to the measurement outcome. A supply of $|\alpha\rangle$ states can be provided as an extension of the input product state. Now it is known that $S$ together with $C_2$ is a universal set of gates for quantum computation so we would get an efficient simulation of all poly-time quantum computation, which is generally believed not to be possible.

Our discussion above generalises the GK theorem by allowing arbitrary product state inputs but on the other hand restricts the original form by not allowing adaptive choices of gates. Its virtue is that it relies only on the group-theoretic normaliser relationship between Clifford and Pauli groups and may thus be immediately generalised to having arbitrary unitary matrix groups $G$ replacing the Pauli group as the starting point. We require no associated subgroup structure to support a stabiliser state formalism nor any consideration of stabiliser states themselves.

Let $G$ be any finite matrix subgroup of $U(d)$ and let $G \otimes G \subset U(d^2)$ denote the subgroup of tensor products $G \otimes G = \{g_1 \otimes g_2 : g_1, g_2 \in G\}$. Introduce the (linear) normalisers of $G$ and $G \otimes G$:

\[ \mathcal{N}(G) = \{U \in U(d) : UGU^\dagger = G\} \]
\[ \mathcal{N}(G \otimes G) = \{U \in U(d^2) : U(G \otimes G)U^\dagger = G \otimes G\}. \]

We will be interested in using normaliser operations as circuit gates and we can therefore allow extra overall phases to be generated in the above relations. Thus we introduce the notion of projective normaliser:

\[ \mathcal{PN}(G) = \{U \in U(d) : \forall g \in G, UgU^\dagger = cg'\text{ for some } g' \in G\text{ and } c \in S_1\} \]
\[ \mathcal{PN}(G \otimes G) = \{U \in U(d^2) : \forall g \in G \otimes G, UgU^\dagger = cg'\text{ for some } g' \in G \otimes G\text{ and } c \in S_1\} . \]

where $S_1 = \{e^{i\theta} : 0 \leq \theta < 2\pi\}$.

Remark. The significance of projective normalisers is illustrated by the following example. If $G$ is the Pauli group $\langle X, Z, iI \rangle$ then $\mathcal{N}(G) = \mathcal{PN}(G)$ and it contains the phase gate $P$. But if $G$ is the group $\langle X, Z \rangle$ comprising matrices with only real entries then $\mathcal{N}(G)$ does not contain the phase gate $P$ but $\mathcal{PN}(G)$ does capture this gate remedying the absence of complex elements in the centre of this smaller real number version of the Pauli group. □

Mimicking our previous discussion we will especially seek examples of groups $G$ such that $\mathcal{PN}(G \otimes G)$ contains an entangling gate (such as CNOT in the case of $G$ being the Pauli group). Otherwise all normaliser circuits will preserve product states and be computationally uninteresting. Furthermore in view of eq. (1) it is desirable that $G$ contains a Hermitian element (such as $Z$) which can be associated with a measurement. Then we will be able to efficiently calculate its expectation value in the final state of any normaliser circuit (with product state input) forming the basis of our classical simulation procedure. Even if $G$ does not contain a Hermitian element we may use the Hermitian matrix $A + A^\dagger$ for any $A \in G$ and similarly apply the arguments following eq. (1) to simulate an associated measurement expectation.
value. In this vein we also note that if $G$ acts irreducibly on $\mathbb{C}^d$ (e.g., as is the case for the usual Pauli qubit group) then any $d \times d$ matrix may be expressed as a linear combination of the matrices of $G$ (c.f. [18] p. 48) so we may efficiently compute the expectation value for any von Neumann measurement on a single qudit or more generally on $O(\log n)$ qudits.

1.1 Teleportation groups

In this paper we will consider only subgroups $G$ of $U(d)$ that act irreducibly on $\mathbb{C}^d$. In addition to facilitating the mathematical analysis at various stages such groups have an extra significance as prospective generalised substitutes for the Pauli group as follows. Recall that another fundamental appearance of the Pauli group is in quantum teleportation, providing the set of Bob’s “correction operators”. In measurement based quantum computation [7], which can be viewed from the perspective of teleportation [8] the associated Clifford operations have a special role of being parallelisable to depth 1 in this formalism (where depth is the minimum number of layers of simultaneous measurements that are needed [7]). Thus we may ask what other sets of operators may appear as Bob’s correction operators in generalised teleportation schemes and then ask for their normalisers. This will again lead to classes of computations that are parallelisable in the corresponding generalised measurement based computational model. In this regard, irreducible subgroups of $U(d)$ play an important role.

Let us define a generalised teleportation scheme as follows. Alice and Bob share the maximally entangled 2-qudit state $|\phi\rangle = \frac{1}{\sqrt{d}} \sum |i\rangle |i\rangle$ and Alice also has a 1-qudit state $|\alpha\rangle$. Let $\mathcal{M} = \{A_1, \ldots, A_r\}$ be any 2-qudit generalised measurement (POVM). Suppose Alice applies $\mathcal{M}$ to the first two qudits of $|\alpha\rangle_1 |\phi\rangle_{23}$ obtaining measurement outcome $i$. Let $\rho_i$ be Bob’s post-measurement state. (We may without loss of generality take the full post-measurement state to be $\sqrt{A_i} |\alpha\rangle |\phi\rangle$ renormalised, and $\rho_i$ is obtained by tracing out the first two qudits). This comprises a generalised teleportation scheme if there exists a family of unitary operators $U_i$ parameterised by the measurement outcomes, such that for all $|\alpha\rangle$ and all $i$, $\rho_i$ is the pure state $U_i |\alpha\rangle$ i.e. $U_i^\dagger$ functions as Bob’s correction operator for measurement outcome $i$. In the case that $\{U_1, \ldots, U_r\}$ also forms a group we have the following.

**Lemma 2** Let $G = \{U_1, \ldots, U_r\}$ be any finite subgroup of $U(d)$ that acts irreducibly on $\mathbb{C}^d$. Then there exists a generalised teleportation scheme with $G$ comprising Bob’s correction operators.

**Proof:** Define $|a_i\rangle = U_i^\dagger \otimes I |\phi\rangle$ for $i = 1, \ldots, r$ and introduce the positive rank 1 operators $A_i = \frac{d^2}{|G|} |a_i\rangle\langle a_i|$. Then using Schur’s lemma (by virtue of the irreducible action of $G$) we can see that $\sum_i A_i = I_{d^2}$ so $\{A_1, \ldots, A_r\}$ is a (rank 1) POVM. Furthermore a straightforward calculation gives

$$\sqrt{A_i} |\alpha\rangle_1 |\phi\rangle_{23} = \frac{1}{\sqrt{|G|}} |a_i\rangle_{12} U_i |\alpha\rangle_3$$
and thus Bob’s post-measurement state is $U_i |\alpha\rangle$ as required. Also each measurement outcome occurs with equal probability $1/|G|$. □

In view of this result we introduce the term *teleportation group* to refer to any finite subgroup of $U(d)$ that acts *irreducibly* on $\mathbb{C}^d$.

### 1.2 Outline of the paper

Returning to our primary motivation of classical simulation we would ideally wish to find all teleportation groups $G$ in $U(d)$, compute the projective normalisers of $G$ and $G \otimes G$ seeking especially the cases of $G$ such that $\mathcal{PN}(G \otimes G)$ contains an entangling gate. We refer to such groups as *entangling teleportation groups*.

In the case of the Pauli group the projective normalisers are known explicitly analytically. However the derivation is lengthy and rests on many properties special to the Pauli operators. We are not able to similarly explicitly analytically characterise projective normalisers for general teleportation groups (even for $d = 2$) and we resort to exhaustive methods using various computer algebra packages. In the qubit case $d = 2$ we will be able to treat exhaustively all possible teleportation groups. In section 2 we will describe our algorithm for computing normalisers and projective normalisers of $G$ and $G \otimes G$ for any given teleportation group. In these methods it will be important to cut down wherever conveniently possible, the range of various cases that needs to be considered to allow the computer algebra to terminate in a reasonable time. In this respect it is important to note that the centre $Z(G)$ of any teleportation group $G$ (which by Schur’s lemma comprises only phase multiples of the identity) plays no role in extending or limiting the existence of projective normalisers. Hence in section 3 we will describe how a search over all teleportation groups in $U(d)$ for entangling ones, can be reduced to the study of normalisers of projectively inequivalent projective representations of the so-called base groups in $U(d)$ which are defined to be the central quotients of teleportation groups.

Next, in section 4 we will apply our methods to identify all possible entangling teleportation groups in the qubit case of subgroups of $U(2)$. We prove the following result.

**Theorem 2** The only finite irreducible subgroups $G$ of $U(2)$ (up to unitary equivalence) such that $G \otimes G$ has an entangling projective normaliser, are $\langle X, Z^{1/n}, cI \rangle$ for $n \in \mathbb{N}$ and $c$ a root of unity (the usual Pauli group corresponding to $n = 1$ and $c = i$).

Finally in section 5 we will make some concluding remarks and identify some avenues for further developments.

### 2 Algorithm for determining normalisers and projective normalisers

In this section we describe a procedure for computing the linear and projective normaliser elements of a teleportation group $G = \{ U_j \} \subset U(d)$.

Let $Gen(G) \subset G$ be a set of generators for $G$. For each $U \in Gen(G)$ let $U' \in G$ denote the image of $U$ under conjugation with some $N \in \mathcal{N}(G)$:

$$NU N^\dagger = U'.$$  \hspace{1cm} (3)
Let us rewrite the normaliser matrix $N$ as a $(d^2 \times 1)$-column vector $\vec{n}$ where

$$\vec{n}_{dj+k} = N_{j,k}.$$  \hfill (4)

Then eq. (3) becomes

$$(I \otimes U^T - U' \otimes I)\vec{n} = 0$$ \hfill (5)

where $I$ is the $(d \times d)$ identity matrix.

By specifying the values of one such pair $U$ and $U'$ and treating the entries in the vector $\vec{n}$ as unknowns we can obtain from eq. (5) $d^2$ simultaneous equations in $d^2$ unknowns. If we assign members of $G$ as the images of all the elements of $\text{Gen}(G)$ then we can solve these equations simultaneously by finding the null space of $(I \otimes U^T - U' \otimes I)$ for each $U \in \text{Gen}(G)$ and its chosen image $U'$. If a non-trivial solution exists for $\vec{n}$ simultaneously for all $U$ then this gives us a solution for $N \in \mathcal{N}(G)$.

This provides us with the basis for an algorithmic approach to computing the elements of the normaliser. We enumerate all the possible choices of images of the generators of $G$ and solve the simultaneous equations, discarding the trivial solutions.

In order to improve the performance of this approach we observe that all the mappings on $G$ induced by conjugation with a normaliser element $N$ are constrained by the fact that each image must have the same order as the generator and any pairwise choice of images must preserve the group relations of the corresponding generators.

Thus we get the following algorithm for computing the normaliser elements of a teleportation group.

**Procedure 1 - Compute $\mathcal{N}(G)$**

1. For each $U$ in $G$ compute $\text{Order}(U)$, the elements of $G$ with the same order as $U$, $\text{Comm}(U)$, the elements of $G$ that commute with $U$ and $\text{NComm}(U)$, the elements of $G$ that do not commute with $U$.
2. Take a minimal set of generators $\text{Gen}(G) = \{U_1, \ldots U_r\}$ of $G$ and find the set of all pairs that commute.
3. Calculate all possible images of $\text{Gen}(G)$ by considering all the choices given in steps 4, 5 and 6.
4. For the images of the first generator, $U_1$, iterate through the set $\text{Order}(U_1)$.
5. The set of possible images of each subsequent generator $U_j$ is formed by starting with the set $\text{Order}(U_j)$ and then repeatedly intersecting with $\text{Comm}(U_k)$ if $U_j$ and $U_k$ commute and with $\text{NComm}(U_k)$ otherwise for each $k < j$.
6. For each choice of possible images $\{U'_1, \ldots U'_r\}$ of $\{U_1, \ldots U_r\}$ compute the combined null space of $(I \otimes U_j^T - U'_j \otimes I)$ for $j \in \{1, \ldots r\}$. Any non trivial solution corresponds to a normaliser element of $G$. 

We can apply the same procedure to compute \( N(G \otimes G) \). In addition we may also test to see if a normaliser gate is entangling using the following result. A 2-qudit unitary operator \( V \in U(d^2) \) is said to be entangling if for all \( A, B \in U(d) \) it is true that \( V \neq A \otimes B \) and \( V \neq \text{SWAP}(A \otimes B) \) (where the SWAP operation is defined by \( \text{SWAP} |i\rangle |j\rangle = |j\rangle |i\rangle \)). Then we have the following characterisation [9]: \( V \) is not entangling if and only if one of the two following conditions holds for every \( i, j, k, l, \bar{i}, \bar{j}, \bar{k}, \bar{l} \in \{0, \ldots, d-1\} \):

1. \( V_{ij,kl}V_{ij,kl} = V_{ij,kl}V_{ij,kl} \)
2. \( V_{ij,kl}V_{ij,kl} = V_{ij,kl}V_{ij,kl} \)

By checking these simple algebraic conditions we can readily identify if a given operation is entangling or not.

### 2.1 Algorithm for projective normaliser elements

To develop an algorithm for determining projective normalisers of a group \( G \) we first show that any such element can be found as a linear normaliser of a group \( G' \) generated by adding suitable additional central elements to \( G \).

Let \( N \) be any (fixed, chosen) projective normaliser element for \( G \). Then for all \( U \in G \)

\[
NUN^\dagger = cV \quad \text{with} \quad c \in S_1 \text{ and } V \in G
\]

(6)

Since any \( U \in G \) has \( U|G| = I \), \( c \) must be a \(|G|\)th root of unity. Thus if \( G' \) is the group obtained by including all such roots into \( G \) we see that any operator is a projective normaliser of \( G \) iff it is a linear normaliser of \( G' \). In practice (especially when treating larger groups such as \( G \otimes G \)) this extension of \( G \) to \( G' \) becomes too large to be manageable for subsequent application of exhaustive enumerations in procedure 1. Thus we develop more refined restrictions on \( c \) to further limit its possible values.

Note first that there is ambiguity in the choice of \( c \) and \( V \) in eq. (6) due to central phases that may already exist in \( G \). This is remedied using the following lemma.

**Lemma 3** If \( G \) is any teleportation group then the centres of \( G \) and \( G \otimes G \) are both cyclic, comprising phase multiples of \( I \).

**Proof.** The claim follows from Schur’s lemma (since \( G \), and hence also \( G \otimes G \), are irreducible). \( \square \)

Now let \( \omega_sI \) with \( \omega_s = e^{2\pi i/s} \) be the minimal phase element of \( Z(G) \). Then in eq. (6) we choose \( c = e^{i\theta} \) such that \( 0 \leq \theta < 2\pi/s \) which fixes \( c \) and \( V \) uniquely. Furthermore with this choice, the unique correspondence between \( U \) and \( V \) means that we can view \( c \) as a function of \( V \) (rather than \( U \)):

\[
NUN^\dagger = f(V)V \quad \text{with} \quad 0 \leq \arg(f(V)) < 2\pi/s.
\]

(7)

(The function \( f \) will also depend on the choice of \( N \) but we omit explicit inclusion of this parameter for notational clarity.) We will also refer to the association of phase
values \( f(V) \) to \( V \in G \) as a *phase function* for \( G \). Introduce

\[
\Gamma = \{ f(V) V : V \in G \} = NGN^\dagger.
\]

Thus \( \Gamma \) is a unitary matrix group isomorphic to \( G \) and \( Z(\Gamma) = Z(G) \).

**Lemma 4** Let \( \{ U_1, \ldots, U_r \} \) be any set of generators for \( G \).

(a) Then \( \{ \omega_s, f(U_1)U_1, \ldots, f(U_r)U_r \} \) generates \( \Gamma \).

(b) If \( U_i \) has order \( n_i \) then \( f(U_i) \) has the form \( \omega_{sn_i}^k \) for some \( 0 \leq k < sn_i \).

**Proof.** (a) For any \( V \in G \) we have \( V = U_1 U_2 \ldots U_r = \prod_k U_{i_k} \). Also from eq. (7) \( f : G \to \mathbb{S}_1 \) has the multiplicative property: \( f(V_1 V_2) = f(V_1) f(V_2) z \) for some \( z \in Z(G) \). Hence \( f(V)^V = z \prod_k f(U_{i_k}) U_{i_k} \) for some \( z \in Z(G) \). Thus \( \omega_s \) together with \( f(U_i) U_i \) for \( i = 1, \ldots, r \) generates \( \Gamma \).

(b) We have \( f(U_i)^{s_i} U_i^{n_i} = f(U_i)^{n_i} I \) which is thus in \( Z(\Gamma) = Z(G) \). Hence \( f(U_i)^{n_i} = \omega_{s_i}^k \) so \( f(U_i) \) is a power of \( \omega_{s_i} \). \( \square \)

For any projective normaliser \( N \) of \( G \), lemma 4 provides restrictions on the values that the phases \( f(V) \) in eq. (7) can possibly take. Define \( \Phi(G) \) to be the set of all choices of \( f(U_1), \ldots, f(U_r) \) satisfying the conditions (a) and (b) of the lemma. Extending \( G \) by new central elements \( \Phi(G) \) will then give a group \( G' \) whose linear normalisers are precisely the projective normalisers of \( G \). In many practical examples \( |Z(G)| = s \) and the generator orders \( n_i \) are small compared to \( |G| \) so the resulting extension to \( G' \) can be far smaller than that obtained by simply adding all \( |G|^\text{th} \) roots of unity to \( G \). Correspondingly we introduce the following computational procedures.

**Procedure 2 - Compute \( \Phi(G) \)**

1. Take a set of generators \( \{ U_1, \ldots, U_r \} \) of \( G \). Let the orders of the generators be \( \{ n_1, \ldots, n_r \} \).

2. Take a generating element \( z \) of the centre of \( G \).

3. For each possible combination of \( j_t \in \{ 0, \ldots, n_t s - 1 \} \) for \( t = 1, \ldots, r \) perform steps 4 and 5.

4. Let \( f(U_i) = \omega_{j_t}^{j_t} \).

5. If \( \langle z, f(U_t) U_t : t \in \{ 1, \ldots, r \} \rangle \cong G \) then add each \( f(U_i) \) to the set \( \Phi(G) \).

6. Output \( \Phi(G) \).

We note that, for small groups, step 5 can be performed relatively quickly using a computational package such as GAP[15].

This provides us with an algorithm that produces a group \( G' \) such that the projective normalisers of \( G \) are the linear normalisers of \( G' \) and hence a means to compute the projective normaliser elements of a teleportation group and its tensor square.

**Procedure 3 - Compute \( PN(G) \)**

1. Compute \( \Phi(G) \) using procedure 2.
2. Compute $G' = \langle \phi I, U : \phi \in \Phi(G), U \in G \rangle$.

3. Compute and output the linear normaliser elements of $G'$ using procedure 1.

Procedure 3 can be used to compute if teleportation groups are projectively entangling in the following manner. We compute the two-qudit matrix group $G \otimes G$ from $G$ and use procedure 3 applied to the group $G \otimes G$ to find $\mathcal{PN}(G \otimes G)$ (recalling that the centre of $G \otimes G$ is cyclic). Each projective normaliser element found can then be tested to see if it is entangling.

3 Base groups and their projective representations

Our principal aim is to apply the procedures of the preceding section to systematically study classes of teleportation groups. In the next section we will exhaustively treat the qubit case of all teleportation subgroups of $U(2)$. There are infinitely many teleportation groups in $U(d)$ but (at least for small $d$) they are known to fall into regular families. Instead of directly enumerating these we will adopt a different approach with a view to reducing the amount of computer algebra required. It is clear from the definition of $\mathcal{PN}(G)$ that the group’s centre $Z(G)$ plays no role in restricting or enabling new projective normalisers.

For any teleportation group $G$ we introduce the central quotient $B = G/Z(G)$. These central quotients are called base groups of $U(d)$. (Our term “base group” is synonymous with “finite collineation group” in [10]). Let $T : B \to G$ be any chosen transversal of $Z(G)$ in $G$ i.e. a choice of element in each coset of $Z(G)$ in $G$. We also require that $I$ is chosen from $Z(G)$ itself. By slight abuse of notation we will also use $T$ to denote the set of matrices $\{T(b) : b \in B\}$. Thus $T$ defines a faithful projective representation of $B$ i.e. the elements of $T$ are projectively inequivalent, and for all $b_1, b_2 \in B$ $T(b_1)T(b_2) = cT(b_1b_2)$ for some $c \in S_1$. Conversely any faithful projective representation $\rho$ of $B$ arises as a transversal of a representation of some $G$ with $B = G/Z(G)$. (Indeed $G$ may be generated by the matrices of $\rho$ together with extra central phases $c$ from $\rho(b_1)\rho(b_2) = c\rho(b_1b_2)$).

Lemma 5 Let $T$ be any faithful projective representation of a base group $B$. Let $G \subset U(d)$ be a group such that $B = G/Z(G)$ and $T$ is a transversal of $Z(G)$ in $G$. Then $N$ is a projective normaliser of $G$ (resp. $G \otimes G$) iff $N$ is a projective normaliser of the set $T$ (resp. $T \otimes T = \{A \otimes B : A, B \in T\}$).

Proof: immediate from the fact that every $U \in G$ has the form $U = cV$ for some $V \in T$ and $c \in S_1$ (and similarly for $G \otimes G$ and $T \otimes T$). □

Let $G_1$ and $G_2$ be teleportation groups with isomorphic central quotients $B$. Choose transversals $T_1$ and $T_2$ giving projective representations of the central quotients. We say that $G_1$ and $G_2$ are projectively equivalent if the projective representations $T_1, T_2$ (for some hence any choice of transversals) are projectively equivalent as projective representations of $B$ i.e. there is $A \in U(d)$ and $c(b) \in S_1$ such that $T_1(b) = c(b)AT_2(b)A^\dagger$ for all $b \in B$.

Recall that $G$ is called entangling if $\mathcal{PN}(G \otimes G)$ contains an entangling operation.
Lemma 6 Let $G_1, G_2$ be projectively equivalent teleportation groups. Then $G_1$ is entangling iff $G_2$ is entangling.

Proof: Let $T_1, T_2, A$ be as above. Suppose $N$ is an entangling projective normaliser for $G_1$. Introduce $M = (A \otimes A)N(A^\dagger \otimes A^\dagger)$. Since $T_i$ is a transversal of the centre of $G_i$ every member $U$ of $G_i$ has the form $cV$ for $V \in T_i$ and $c \in S_1$. Thus a straightforward calculation shows that $M$ is a projective normaliser for $G_2 \otimes G_2$. Also $M$ is locally equivalent to $N$ so it is also entangling. □

In view of the above lemmas, to find all entangling teleportation groups $G$ it suffices to look at a complete set of all projectively inequivalent (faithful) projective representations $T$ of all base groups $B$ and determine if the projective normaliser of $T \otimes T$ contains an entangling operation or not. $G$ up to unitary equivalence is then generated by the matrices of $T$ and further central phases $cI$. Actually even a complete list of projectively inequivalent projective representations may involve redundancies as the normaliser structure we seek is a property of a set of matrices irrespective of how the set represents a group. Then note that it is possible for two (projective) representations of a group $B$ to be inequivalent yet comprise the same overall set of matrices (up to overall phases) which are then associated with the elements of $B$ in different ways (c.f eq. (24) later for a non-projective example).

To access the full list of base groups for $d = 2$ (and also for $d = 3, 4$) we note that the full list of subgroups of $SU(2), SU(3), SU(4)$ are known and provided in [10] (with the latter two cases considered in more detail in [11] and [12] respectively.)

Remark. These lists are complete up to abstract group isomorphism but not complete up to unitary equivalence. In this regard it is important to note that the normaliser group $N(G)$ is not a property of an abstract group but of a given representation i.e. two unitarily inequivalent representations of the same group will generally have different normalisers. □

To pass from base groups of $SU(d)$ to those of $U(d)$ we have the following.

Lemma 7 The sets of central quotients of finite subgroups of $SU(d)$ and $U(d)$ are the same.

Proof If $G \subseteq SU(d)$ then $G \subseteq U(d)$ so its central quotient is in both sets. Conversely if $G \subseteq U(d)$ then the (finite) group $G'$ generated by $det(g)^{-1/d} g$ for all $g \in G$ (and any choice of $d^{th}$ root) has $G' \subseteq SU(d)$ and the same central quotient as $G$ i.e. $G/Z(G)$ also appears in both lists. □

Hence the full list of base groups of $U(2)$ (as abstract groups) is obtained from the central quotients of the lists in [10].

In order to find all the projectively inequivalent irreducible projective representations of a finite group we may use the concept of a covering group [14, 13] and it suffices to calculate the inequivalent irreducible linear representations of this group.

Definition 1 (page 361 [14]) A covering group $G^*G^*$ of a finite group $G$ is a finite

Both ‘covering group’ [14] and ‘representation-group’ [13] are used in the literature as translations of Schur’s ‘Darstellungsgruppe’. We prefer ‘covering group’ so as to avoid the confusion of constructing
group which is an extension of \( G \) with kernel contained in the centre of \( G^* \) such that every projective representation of \( G \) is equivalent to one which can be lifted to a linear representation of \( G^* \).

In this construction the linear representations of \( G^* \) arise from the projective representations of \( G \) by inclusion of further central elements (phase multiples of the identity).

The existence of such groups is then established by the following theorem.

**Theorem 3** Every finite group \( G \) of order \( n \) has at least one covering group of order \( nm \) where \( m \) is the size of the Schur multiplier (2nd cohomology group) of \( G \).

**Proof** Originally due to Schur. See Karpilovksy [13]. \( \square \)

**Corollary 1** In each dimension \( d \geq 2 \) there is a finite number of projectively inequivalent irreducible unitary projective representations of each finite group \( G \). Each of these can be lifted to an irreducible unitary linear representation of a covering group of \( G \).

**Proof** This follows directly from theorem 3 specialised to irreducible unitary representations of a particular dimension. \( \square \)

Now we are in the position, for a given dimension \( d \), to find all the inequivalent teleportation groups using the following procedure.

**Procedure 4 - Find projectively inequivalent teleportation groups in** \( U(d) \)

1. Let \( S \) be a set of finite subgroups of \( SU(d) \) up to isomorphism.
2. Calculate all base groups of \( U(d) \) as the set of central quotients of the elements of \( S \).
3. Let \( BC \) denote a set of covering groups for the base groups.
4. A complete set of projectively inequivalent teleportation groups of \( U(d) \) is generated by the set of inequivalent irreducible unitary linear representations of the elements of \( BC \).

We now provide some additional practical information for the steps in procedure 4.

1. The lists of finite subgroups of \( SU(d) \) for \( d = 2, 3, 4 \) are provided in the literature.
2. In practice central quotients may be calculated using a computational package such as GAP [15]. The equivalence of the base groups of \( SU(d) \) and \( U(d) \) is shown in lemma 7.
3. The existence of the covering groups in step 3 comes from theorem 3 and in practice we use GAP to calculate them.

---

a group representation of the representation-group.
4. The set of inequivalent irreducible unitary representations of the covering groups can be produced in GAP.

5. For each resulting representation $G$ we apply procedures 1, 2, 3 to determine whether $G \otimes G$ has an entangling projective normaliser or not.

### 3.1 Teleportation groups represented in $GL(d, \mathbb{C})$

In applying the above procedures we perform computations using the GAP computational system [15] and in particular the REPSN [16] representation theory package to find the irreducible representations of finite groups. For some groups REPSN does not produce a unitary representation but a general linear representation in $GL(d, \mathbb{C})$.

In this section we relate the previous results concerning the normaliser and projective normaliser of faithful irreducible unitary representations to their general linear counterparts and show that the algorithms we provide may be performed with faithful irreducible representations of a finite group in $GL(d, \mathbb{C})$ and the results applied to the faithful irreducible unitary representations.

Two general linear representation $\rho_1^L, \rho_2^L : G \rightarrow GL(d, \mathbb{C})$ of a finite group $G$ are said to be equivalent if there exists $M \in GL(d, \mathbb{C})$ such that for all $g \in G$

$$\rho_1^L(g) = M \rho_2^L(g) M^{-1}. \quad (8)$$

**Theorem 4** For any finite group $G$ and faithful irreducible representation $\rho_L : G \rightarrow GL(d, \mathbb{C})$ of $G$ there exists a faithful irreducible unitary representation $\rho_U : G \rightarrow U(d)$ and a matrix $E \in GL(d, \mathbb{C})$ such that for all $g \in G$

$$\rho_U(g) = E \rho_L(g) E^{-1}. \quad (9)$$

**Proof** see [17] page 74. □

Furthermore if $\rho_L$ in the theorem ranges over a full list of inequivalent linear representations then $\rho_U$ ranges over a full list of unitarily inequivalent representations. For general linear representations we can define linear and projective normalisers as follows.

$$\mathcal{N}_{GL(d, \mathbb{C})}(\rho_L) = \{ N \in GL(d, \mathbb{C}) : N \rho_L(g) N^{-1} \in \rho_L(G) \text{ for all } g \in G \}, \quad (10)$$

$$\mathcal{PN}_{GL(d, \mathbb{C})}(\rho_L) = \{ N \in GL(d, \mathbb{C}) : \forall g \in G \ \exists g' \in G, c \in \mathbb{C} : N \rho_L(g) N^{-1} = c \rho_L(g') \} \quad (11)$$

Theorem 4 immediately gives:

**Corollary 2** Given the two representations $\rho_L$ and $\rho_U$ of theorem 4 then every normaliser element $N \in \mathcal{N}_{U(d)}(\rho_U)$ defines a normaliser element $M = E^{-1} N E \in \mathcal{N}_{GL(d, \mathbb{C})}(\rho_L)$.

Note that in eq. (11) $c$ must actually be in $\mathbb{S}_1$ since $G$ is finite. Thus our previous procedures 1, 2, 3 can be used unchanged to compute the normalisers and projective normalisers of $\rho_L$ and $\rho_L \otimes \rho_L$. If the latter fails to be entangling we can conclude by the corollary that $\rho_U$ is also not entangling, without having to carry out the translation from $\rho_L$ to $\rho_U$ explicitly.
4 Application to teleportation groups in $U(2)$

We now apply our procedures to consider all teleportation groups in $U(2)$ and show that the entangling ones are exactly those which have a central quotient isomorphic to a dihedral group of order $4m$ for some integer $m$. Alternatively these teleportation groups may be described as unitary equivalents of groups obtained by adjoining additional central elements to the matrix groups $\langle X, Z^{1/n} \rangle$ for $n \in \mathbb{N}$.

We use a labelling system for small finite groups that is used in the GAP computational package. This system assigns two numbers to a group. The first is the order of the group and the second is a unique index for each group of a particular order. As examples we have $[4, 1]$ for the cyclic group of order 4, $[4, 2]$ for the Klein four group, $[12, 3]$ for the alternating group on 4 elements etc.

4.1 Base groups of $U(2)$

We use the list of base groups of $SU(2)$ which are given in Blichfeldt [10] as the finite collineation groups and which are also the base groups of $U(2)$. These consist of two infinite families of groups and three ‘special’ groups. The infinite families are the cyclic groups, which have no irreducible representations in $U(2)$, and the dihedral groups $D_{2n}$ of order $2n$. The three special groups are the tetrahedral group $A_4 \cong [12, 3]$, the octahedral (or cube) group $S_4 \cong [24, 12]$ and the dodecahedral (or icosahedral) group $A_5 \cong [60, 5]$. First we deal with the three special groups and finally the dihedral case.

4.2 The tetrahedral group as base group

A covering group of the tetrahedral group $A_4 \cong [12, 3]$ is $[24, 3]$. We use the computational package GAP to compute three inequivalent representations in $U(2)$ which is the maximum number of inequivalent representations. These are the matrix groups $M_1, M_2$ and $M_3$ where

\[
M_1 = \left\langle \frac{1}{\sqrt{2}} \begin{pmatrix} \omega_8 & \omega_8 \\ \omega_8^3 & \omega_8 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right\rangle
\]

\[
M_2 = \left\langle \frac{1}{\sqrt{2}} \begin{pmatrix} \omega_{11} & \omega_{11} \\ \omega_{24}^1 & \omega_{24} \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right\rangle
\]

\[
M_3 = \left\langle \frac{1}{\sqrt{2}} \begin{pmatrix} \omega_{24} & \omega_{24} \\ \omega_{24} & \omega_{24} \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right\rangle.
\]

Using an implementation of the algorithm to find normalisers given in section 2 it can be seen that $M_1, M_2$ and $M_3$ are not entangling. To see that these three representations are not projectively entangling the ranges of the possible phase functions of the matrix groups $M_1, M_2$ and $M_3$ are calculated using the algorithm described in section 2.1. From this we find that the possible phase functions take values in $\{\omega_j^3 : j = 0, 1, 2\}$. When we add the central elements corresponding to these phases to the matrix groups $M_1, M_2$ and $M_3$ we get matrix groups isomorphic to
This group has only one irreducible representation in $U(2)$ up to equivalence. This can be represented as

$$\left\langle \frac{1}{\sqrt{2}} \begin{pmatrix} \omega_{11}^{11} \\ \omega_{14}^{11} \\ \omega_{24}^{11} \end{pmatrix}, \begin{pmatrix} \omega_{12} \\ 0 \\ \omega_{12} \end{pmatrix} \right\rangle$$

and we compute that it is not entangling. Hence no teleportation group in $U(2)$ with central quotient isomorphic to the tetrahedral group is entangling.

### 4.3 The octahedral group as base group

The octahedral group $S_4 \cong [24, 12]$ has [48, 29] as a covering group. This has one representation in $U(2)$ up to equivalence which is not entangling. The generators we used are

$$\left\langle \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} \omega_8^3 \\ \omega_3^8 \\ \omega_3^8 \end{pmatrix} \right\rangle$$

All possible phase functions take values in $\{1, i\}$ so to test if this group is projectively entangling we must test if any linear representation of $Z_4 \otimes [48, 29] \cong [96, 192]$ is entangling. Up to equivalence there is only one faithful representation in $U(2)$ and that has generators

$$\left\langle \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} \omega_8^4 \\ \omega_8^3 \\ \omega_8^5 \end{pmatrix} \right\rangle$$

We have computed that this matrix group is not entangling. We conclude that no teleportation group with central quotient isomorphic to the octahedral group is entangling.

### 4.4 The dodecahedral group as base group

The dodecahedral group $A_5 \cong [60, 5]$ has unique covering group $[120, 5]$. There are two faithful irreducible representations of this group up to equivalence in $U(2)$. The computational system gap gives us representations in $GL(2, \mathbb{C})$ which by the results of section 3.1 will suffice for our calculations. The first representation has the following two generators

$$\begin{pmatrix} \omega_{15} - \omega_{15}^2 + \omega_{15}^4 - \omega_{15}^8 - \omega_{15}^{11} - \omega_{15}^{14} & -2\omega_{15} - 2\omega_{15}^4 - \omega_{15}^7 - \omega_{15}^{13} + \omega_{15}^{14} \\ -\omega_{15}^4 + \omega_{15}^{11} - \omega_{15}^{14} & -\omega_{15} - \omega_{15}^4 - \omega_{15}^7 + \omega_{15}^{11} + \omega_{15}^{13} + \omega_{15}^{14} \end{pmatrix}$$

$$\begin{pmatrix} -\omega_{15} + \omega_{15}^2 - \omega_{15}^4 + \omega_{15}^8 + \omega_{15}^{11} + \omega_{15}^{14} \\ \omega_{15}^2 + \omega_{15}^8 + \omega_{15}^{11} + \omega_{15}^{14} \end{pmatrix}, \begin{pmatrix} \omega_{15} + \omega_{15}^4 \\ \omega_{15}^1 + \omega_{15}^7 + \omega_{15}^{11} + \omega_{15}^{13} + \omega_{15}^{14} \end{pmatrix}$$

The second representation has the following two generators

$$\frac{1}{2} \begin{pmatrix} -2\omega_{15} - 2\omega_{15}^4 - \omega_{15}^7 - \omega_{15}^{11} - \omega_{15}^{13} + \omega_{15}^{14} & 2\omega_{15}^2 + \omega_{15}^7 + \omega_{15}^8 + \omega_{15}^{11} + \omega_{15}^{13} + \omega_{15}^{14} \\ -\omega_{15}^4 + \omega_{15} + \omega_{15}^7 + \omega_{15}^{11} + \omega_{15}^{13} + \omega_{15}^{14} \end{pmatrix}$$
\[
\frac{1}{2} \left( \omega_{15} + \omega_{15}^{2} + \omega_{15}^{3} + \omega_{15}^{4} + 2\omega_{15}^{5} + \omega_{15}^{6} + 2\omega_{15}^{7} + \omega_{15}^{8} + \omega_{15}^{9} + \omega_{15}^{10} + \omega_{15}^{11} + \omega_{15}^{12} + \omega_{15}^{13} + \omega_{15}^{14} + \omega_{15}^{15} - \omega_{15}^{2} - \omega_{15}^{4} - \omega_{15}^{7} - \omega_{15}^{8} - \omega_{15}^{13} - \omega_{15}^{15} \right). 
\]

(21)

We have computed that these two representations are not entangling. Furthermore, we have computed that all phase functions of both representations are trivial and so all two-qubit projective normaliser elements of are linear normaliser elements. This implies that the two representations of \([120, 5]\) given above are not projectively entangling. Hence no teleportation group in \(U(2)\) which has central quotient isomorphic to the dodecahedral group is entangling.

### 4.5 A dihedral group as base group

Since the family of dihedral groups comprises an infinite list we approach the study of their projectively inequivalent projective representations analytically.

The Schur multiplier of a group [13] is key in calculating covering groups. In particular when the Schur multiplier of a group is the trivial group then the group is its own covering group. The Schur multiplier \(M(D_{2n})\) of the dihedral group \(D_{2n}\) is given in [19] as

\[
M(D_{2n}) = \mathbb{Z}_{\gcd(2, n)}. 
\]

(22)

This splits the analysis of the dihedral group \(D_{2n}\) into the case of odd \(n\) where \(D_{2n}\) covers itself and even \(n\) where we obtain the so called binary dihedral groups as covering groups.

#### 4.5.1 \(D_{2n}\) when \(n\) is odd

We claim that there are no entangling teleportation groups with central quotient isomorphic to \(D_{2n}\) when \(n\) is odd.

As noted above, \(D_{2n}\), for odd \(n\), is its own covering group. The only representations we need to consider in looking for entangling teleportation groups are the the irreducible linear ones of \(D_{2n}\). When \(D_{2n}\) is presented as

\[
D_{2n} = \langle a, b | a^n = 1, b^2 = 1, bab = a^{-1} \rangle 
\]

(23)

we find, from [18], the \(r\)th irreducible representation \(\rho_r\) with \(0 < r < \frac{(n-1)}{2}\) may be taken to be

\[
\rho_r(a) = \left( \begin{array}{cc} \omega_r^n & 0 \\ 0 & \omega_n^{-r} \end{array} \right), \quad \rho_r(b) = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right). 
\]

(24)

In particular we are only interested in the faithful representations when \(\gcd(r, n) = 1\). We prove that the teleportation group \(G\) for \(r = 1\) is not projectively entangling and the argument for general \(r\) follows similarly. Indeed for each \(r\) we get the same set of matrices (but associated to group elements in different ways) and the normalising property is a property of the collection of matrices only as a set.

**Proposition 1** Let \(G = \langle A = \left( \begin{array}{cc} \omega_n^n & 0 \\ 0 & \omega_n^{-n} \end{array} \right), B = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \rangle \cong D_{2n}\) for odd \(n\). Then every phase function of \(G\) takes values in \(\{1, -1\}\).
Proof For any phase function $f$ of $G$ we must have that
\[ f(A) = \omega_n^j \text{ and } f(B) = (-1)^k \]
by the orders of $A$ and $B$ respectively. This defines the value of $f$ on all elements of $G$. It is easily verified that if $j = 0$ and $k = 1$ then $f$ defines a valid phase function of $G$ but that for $j \neq 0$ the group $\langle f(A), f(B) \rangle$ is never isomorphic to $G$. 

Since every phase function takes values in $\{1, -1\}$ we must now show that $G'$ is not entangling where $G'$ is generated by adding the element \((-1 \ 0 \ 0 \ 1)\) to the generators of $G$. So we have $G' = \langle C = \begin{pmatrix} \omega_{2n} & 0 \\ 0 & \omega_{2n}^{-1} \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle \cong D_{4n}$. 

Proposition 2 $G'$ is not entangling for any odd $n \geq 3$.

The proof of the above is split into three lemmas from which the result follows.

Lemma 8 Every two-qubit normaliser gate for $G'$ is a generalised permutation matrix. That is it has exactly one non-zero entry for each row and each column.

Proof Let us write a general two-qubit normaliser gate of $G'$ as $N$ with matrix entries $N_{jk}$. We write $C_1 = C \otimes I$ and $C_2 = I \otimes C$ with $C$ as in eq. (26). For any $j \in \{0, 1, 2, 3\}$ we have
\[(NC_1N^\dagger)_{jj} = \omega_{2n}(|N_{j1}|^2 + |N_{j2}|^2) + \omega_{2n}^{-1}(|N_{j3}|^2 + |N_{j4}|^2).\] 

Since $NC_1N^\dagger \in G'$ each entry $(NC_1N^\dagger)_j$ must be zero or a power of $\omega_{2n}$ and since $n \geq 3$ we must have either
\[|N_{j1}|^2 + |N_{j2}|^2 = 0 \text{ or } |N_{j3}|^2 + |N_{j4}|^2 = 0.\] 

Similarly
\[(NC_2N^\dagger)_{jj} = \omega_{2n}(|N_{j1}|^2 + |N_{j2}|^2) + \omega_{2n}^{-1}(|N_{j3}|^2 + |N_{j4}|^2)\]
from which we conclude that either
\[|N_{j1}|^2 + |N_{j3}|^2 = 0 \text{ or } |N_{j2}|^2 + |N_{j4}|^2 = 0.\] 

From equations 28 and 30 we see that $N$ must have exactly one non-zero entry per row and since $N$ must be unitary we conclude it is a generalised permutation matrix.

Lemma 9 If there exists an entangling two-qubit normaliser gate for $G'$ then there also exists a diagonal entangling two-qubit normaliser gate for $G'$.

Proof Suppose that $N$ is any entangling 2-qubit normaliser for $G'$. From lemma 8 we have $N = DP$ where $D$ is diagonal and $P$ is a permutation. The three 2-qubit normaliser gates $I \otimes B, B \otimes I$ and $B \otimes B$ (with $B$ as in eq. (26)) are permutations that interchange 00 with 01, 10 and 11 respectively. Hence if $R$ is a suitably chosen one of these three, then we get an entangling 2-qubit normaliser $N' = NR = DP'$.
where $P'$ is a permutation that leaves 00 fixed. If $S$ denotes the swap gate (which is a
2-qubit normaliser for any group) and $C_X$ denotes the controlled NOT gate, then the
six possible choices of $P'$ can be written as $I, S, C_X, S C_X, C S X$ and $S C X S$. If $P' = I$
then $N'$ is diagonal. If $P' = S$ then $S N'$ is a diagonal entangling normaliser. For
$P' = C_X$ recall $C_2 = I \otimes C$ (with $C \in G'$ as in eq. (26)). Then a direct calculation
shows
\begin{equation}
N' C_2 N'^\dagger = \text{diag}(\omega_{2n}, \omega_{2n}^{-1}, \omega_{2n}^{-1}, \omega_{2n}) \notin G'.
\end{equation}
Hence we cannot have $P' = C_X$. Similarly if $P'$ were $S C_X, C X S$ or $S C X S$ we could
pre- and/or post-multiply $N'$ by $S$ to obtain a normaliser again of the form $N'' = D'C_X$ with $D'$
diagonal. Hence these three cases of $P'$ are also excluded and in all allowed cases, the existence of $N$
implies the existence of a diagonal 2-qubit entangling normaliser.

**Lemma 10** No diagonal two-qubit normaliser gates for $G'$ are entangling.

**Proof** Let us take an arbitrary diagonal two-qubit normaliser gate of $G'$ which we
may write up to phase as $D = \text{diag}(1, a, b, c)$. We then see that for $DB_1 D^\dagger$ to be in $G'$
there must be $j, k$ such that
\begin{equation}
DB_1 D^\dagger = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 \\
0 & c & 0 & 0
\end{pmatrix} = C^j B \otimes C^k = 
\begin{pmatrix}
0 & 0 & \omega_{2n}^{j+k} & 0 \\
0 & 0 & 0 & \omega_{2n}^{j-k} \\
\omega_{2n}^{-j+k} & 0 & 0 & 0 \\
0 & \omega_{2n}^{-j-k} & 0 & 0
\end{pmatrix}.
\end{equation}
Eliminating $b$ from the above we see that since $n$ is odd we must have $k = 0$ or $k = n$.
Similarly from $DB_2 D^\dagger$ we have $l, m$ such that
\begin{equation}
DB_2 D^\dagger = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 \\
0 & 0 & c & 0
\end{pmatrix} = C^l B \otimes C^m = 
\begin{pmatrix}
0 & \omega_{2n}^{l+m} & 0 & 0 \\
0 & 0 & 0 & \omega_{2n}^{l-m} \\
\omega_{2n}^{l-m} & 0 & 0 & 0 \\
0 & 0 & \omega_{2n}^{-l+m} & 0
\end{pmatrix}
\end{equation}
and eliminating $a$ gives $l = 0$ or $l = n$. By solving for $a, b, c$ in the four cases of
$k, l = 0, n$ we see that in each case $D$ can be written as a tensor product of two gates
and hence is not entangling.

We have now completed the proof of proposition 2. This completes the result
that no dihedral group $D_{2n}$ when $n$ is odd forms the base group of an entangling
teleportation group.

### 4.5.2 $D_{2n}$ when $n$ is even

We claim that every dihedral group $D_{2n}$, where $n$ is even, is isomorphic to the central
quotient of an entangling teleportation group. Indeed introduce $m$ defined by $n = 2m$
and
\begin{equation}
Z^{1/m} = \begin{pmatrix} 1 & 0 \\ 0 & \omega_{2m} \end{pmatrix}, \quad G_m = \langle X, Z^{1/m} \rangle.
\end{equation}
A straightforward calculation shows that \( \rho \) defined by \( \rho(a) = Z^{1/m} \) and \( \rho(b) = X \) provides a projective representation of \( D_{2n} \) as presented in eq. (23) and \( G_m/Z(G_m) \cong D_{2n} \).

For \( m = 1 \) the projective normalisers are clearly just those of the Pauli group given in lemma 1.

For \( m \geq 2 \) we find that \( Z^{1/2m} = \text{diag}(1, \omega_{4m}) \) is a normaliser of \( G_m \). This is the generalisation of the phase gate \( P \) from the Pauli group normaliser (and now \( H \) is no longer a normaliser). Also \( CZ \) is a normaliser of \( G_m \otimes G_m \) so \( G_m \) is entangling for all \( m \in \mathbb{N} \). Our computer algebra procedures for small \( m \) values showed that all normalisers of \( G_m \otimes G_m \) are generated from \( CZ \) with SWAP and \( G_m \otimes G_m \) included.

Finally we show that there are no other teleportation groups \( G \subset U(2) \) with central quotient \( D_{2n} \) \((n\text{ even})\) that are not unitary equivalents of central extensions of \( G_m \) above. Indeed the covering group of \( D_{2n} \) \((n\text{ even})\) is the binary dihedral group \( Q_{4n} \) with presentation[20]

\[
Q_{4n} = \{a, b : a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1}\}
\]

and a complete set of faithful irreducible representations on \( \mathbb{C}^2 \) is given by[20]

\[
\rho_r(a) = \begin{pmatrix} \omega_{4n}^r & 0 \\ 0 & \omega_{2n}^{-r} \end{pmatrix}, \quad \rho_r(b) = \begin{pmatrix} 0 & (-1)^r \\ 1 & 0 \end{pmatrix}
\]

with \( \text{gcd}(r, 2n) = 1 \).

The case \( r = 1 \) reproduces \( G_m \) above and for each \( r \), \( \rho_r \) comprises the same set of matrices (up to phase multiples) so we get no new projective normaliser structures as \( r \) varies. This completes the proof of theorem 2. \( \square \)

5 Conclusions

We have identified all entangling teleportation groups in \( U(2) \) and seen that they comprise only a mild generalisation of the standard qubit Pauli group. Also the associated projective normalisers, apart from extra roots of \( Z \), are already present for the Pauli group case. Thus the qubit case appears to be of limited scope in generating new classes of classically simulatable circuits, but there are yet further cases and generalisations worthy of investigation which we list as open questions.

Firstly we may consider higher values of \( d \). The Pauli groups may be naturally generalised to arbitrary dimension \( d \) and the associated normaliser groups were analytically characterised for all prime \( d \) in [21]. In this case it is found that an analogue of lemma 1 holds (with \( H, P \) and \( CZ \) being replaced by suitable one and two qudit operations as given in [21]). We applied our computational programs to a further few chosen examples of teleportation groups in \( U(3) \) but did not find any further interesting entangling ones. We were unable to exhaustively treat all base groups of \( U(3) \) because of the increased size of the groups involved. Thus it would be advantageous to further develop the study of computational procedures for projective normalisers, inventing algorithms that search over more restricted spaces of values.

We have considered the projective normaliser structure of \( G \) and \( G \otimes G \) only for matrix groups \( G \) that act irreducibly. But more generally if a matrix group acts
reducibly it is not clear how its normaliser structure relates to that of its irreducible parts. This may provide an avenue for constructing further interesting examples of entangling normalisers. For example, for the case \( d = 2 \) that we have considered exhaustively, any reducible group is diagonal. Hence \( G \otimes G \) is diagonal too and thus any diagonal matrix will be a normaliser.

Our exhaustive qubit results indicate that the Pauli matrix group is very special in possessing a suitably rich variety of normalisers. In this vein it would be particularly interesting to identify a mathematical “signature” property of a given matrix group \( G \) whose validity signals the existence of non-trivial projective normalisers.

Finally we point out that lemma 1 asserts a remarkable structural property of normalisers of the Pauli group \( \mathcal{P} \) and its tensor powers \( \mathcal{P}^{\otimes n} \) viz. that for levels \( n \geq 3 \) there are no new normalisers beyond those generated from circuits of \( n = 1 \) and 2 normalisers. (This is also true of the generalised Pauli groups in prime dimension[21]). The proof of this property utilises many extra properties special to the Pauli matrices. Thus we may ask: are there teleportation groups \( G \) (even in dimension \( d = 2 \)) such that \( G^{\otimes n} \) for \( n \geq 3 \) has normalisers that are not expressible as composites of \( n = 1, 2 \) normalisers? In our computational analyses we have considered projective normalisers only up to tensor square \( G \otimes G \) and it remains open whether or not there may exist entangling normalisers for \( n \geq 3 \), even in the case that they are absent for \( n = 2 \).

Any mathematical signature property of the kind mentioned above would be helpful in addressing this fundamental issue.

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References

1. D. Gottesman, Stabilizer Codes and Quantum Error Correction, PhD thesis, California Institute of Technology, Pasadena, CA, 1997.
2. M. A. Nielsen and I. Chuang, Quantum Computation and Information, CUP 2000.
3. R. Jozsa and N. Linden, On the role of entanglement in quantum-computational speedup, Proc. Roy. Soc. Lond. A 459, 2011-2032 (2003). arXiv:quant-ph/0201143.
4. D. Gottesman, Course on quantum error correction, Perimeter Institute, Waterloo http://perimeterinstitute.ca/people/researchers/dgottesman/CO639-2004/index.html
5. S. Aaronson and D. Gottesman, Improved simulation of stabiliser circuits, Phys. Rev. A 70:052328, 2004. quant-ph/0406196.
6. S. Bravyi and A. Kitaev, Universal quantum computation with ideal clifford gates and noisy ancillas, Phys. Rev. A 71, 022316 (2005).
7. R. Raussendorf and H. J. Briegel, A one-way quantum computer. Phys. Rev. Lett., 86, 5188–5191, 2001. arXiv:quant-ph/0010033; R. Raussendorf, D. E. Browne, and H. J. Briegel. Measurement-based quantum computation with cluster states Phys. Rev. A, 68, 022312, 2003. arXiv:quant-ph/0301052.
8. R. Jozsa, An introduction to measurement based quantum computation, Proc. NATO-ASI vol. 199, Quantum Information Processing from theory to experiment, ed. D Angelakis et
9. J. Brylinski and R. Brylinski. Universal quantum gates. *quant-ph/0108062*, August 2001.
10. H. F. Blichfeldt. *Finite Collineation Groups*. University of Chicago press, 1st edition, 1917.
11. W. M. Faibairn, T. Fulton, and W. H. Clink. Finite and disconnected subgroups of $su_3$ and their spectrum application to the elementary-particle spectrum. *Journal of mathematical physics*, 5(8), August 1964.
12. A. Hanany and Y. H. He. A monograph on the classification of the discrete subgroups of $su(4)$. *Journal of High Energy Physics*, 2001(02):027, 2001.
13. G. Karpilovsky. *Group Representations*. North-Holland mathematics studies, 1st edition, 1992.
14. C. W. Curtis and I. Reiner. *Representation theory of finite groups and associative algebras*. Wiley, 1st edition, 1962.
15. The GAP Group. Gap — groups, algorithms, and programming. [http://www.gap-system.org](http://www.gap-system.org), 2005. Version 4.4.6.
16. V. Dabbaghian-Abdoly. An algorithm to construct representations of finite groups. *Ph.D. thesis, Dept. Mathematics, Univ. Carleton*, 2003. Available as part of GAP software.
17. E. P. Wigner. *Group theory and its application to the quantum mechanics of atomic spectra*. Academic Press Inc., expanded and improved edition, 1959.
18. J. P. Serre. *Linear Representations of Finite Groups*. Springer-Verlag, 2nd edition, 1977.
19. B. Feng, A. Hanany, Y. H. He, and N. Prezas. Discrete torsion, non-abelian orbifolds and the schur multiplier. *Journal of High Energy Physics*, (01):033, 2001.
20. W. Malfait and A. Szczepanski. The structure of the (outer) automorphism group of a Bieberbach group. *Composito Mathematica*, 136, p89-101 (2003).
21. S. Clark, Valence bond solid formalism for d-level one way quantum computation, J. Phys. A: Math. Gen. 39, 2701-2721(2006). *quant-ph/0512155*. 