Optimal Allocation of Limited Funds in Quadratic Funding

Ricardo A. Pasquini *

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Abstract

We examine the allocation of a limited pool of matching funds to public good projects using Quadratic Funding. In particular, we consider a variation of the Capital Constrained Quadratic Funding (CQF) mechanism proposed by Buterin, Hitzig and Weyl (2019) where only funds in the matching pool are distributed among projects. We show that this mechanism achieves a socially optimal allocation of limited funds.

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Quadratic Funding (QF) is a target matching funds allocation rule for public goods financing proposed by Buterin, Hitzig, and Weyl (2019) (BHW). Denoting by $c^p_i$ the amount committed by an individual contributor to project $p$, the QF rule proposes funding project $p$ in the amount $F^p$ according to the formula that squares the sum of the square roots of contributions from all contributors (i.e., $F^p = (\sum_{i\in I} \sqrt{c^p_i})^2$). BHW shows that the QF rule has a powerful theoretically property. Under this mechanism individual contributors choose (in a decentralized way) a socially efficient allocation of funds. Indeed, a social planner would balance the marginal cost of an additional dollar invested in a project against the sum of the marginal benefits for all the community members, and this is what QF achieves in its decentralized design. 

*Facultad de Ciencias Empresariales, Universidad Austral, Mariano Acosta 1611, B1630FHB Pilar, Buenos Aires. The views expressed in the paper do not necessarily reflect those of the institutions I am affiliated with. rpasquini@austral.edu.ar

1See Buterin, Hitzig, and Weyl (2019) (BHW), Proposition 3, pp. 5175
In practice, implementing the QF rule fully will probably prove difficult. This is because matching funds requirements scale rapidly (quadratically), and available matching funds (typically provided by external donors) are limited. In Gitcoin Grants (a website that has implemented QF to fund projects), for example, requirements reach the total matching pool within the first days of a round (Pasquini 2020). As a consequence, what is actually implemented in practice is a variant of QF, named by BHW the Capital Constrained QF (CQF). By this rule what a project obtains is not the full amount from the QF rule, but a linear combination between the target QF and actual funds committed by individual contributors.

Once we recognize that matching funds will be limited in general, it is worth consider if CQF is the best rule for the allocation of a limited pool of funds. An optimal allocation of limited funds should follow the condition that next dollar invested by the mechanism should go to the project with the highest marginal social benefit. Therefore, investments should equalize marginal social benefits across projects. It turns out that this is not perfectly achieved by CQF. Differences between marginal social benefits across projects will be larger: i) the lesser funds are available in the matching pool relative to matching requirements, ii) the higher the variability in the supporting preferences across projects (e.g., more equally invested projects imply higher marginal benefits than more concentrated projects), and iii) the higher the number of contributors (Pasquini 2020).

Here we propose examining the allocation of limited funds to public good projects using a slight variation of the CQF rule. In this version:

- Projects do not receive the funds committed directly by contributors. They do receive a percentage of the total funds a project should receive according to the QF rule. In other words, only the funds in the pool of matching funds are distributed.

- Individual contributions are kept by the mechanism, and could be summed to the matching pool of funds in subsequent financing rounds.

We show that, under perfect information, this mechanism tends to a social optimal allocation of limited funds, namely it equalizes marginal social benefits across projects. Individual contributors have incentives to contribute (and change the allocation of funds), if the marginal social benefit from a project of interest is greater than the (public goods)
weighted marginal social benefit they would receive from all (other) projects. This incentive tends to disappear as marginal social benefits tend to equalize across projects.

Notice that because of quadratic funding, individual contributors fully internalize the decisions on the contributions from the rest of the contributors (i.e., the community). This is the main innovation from QF. In this particular case, differently from the BHW’s CQF, by removing the effect of private commitments to projects, marginal social benefits tend to equalize without biases.

With limited funds, we can think of contributors as paying for the right to distribute the matching pool of funds in favor of their project of interest, but without the security of fully delivering their committed funds to their project of interest. With enough matching funds, the QF rule determines an amount to a project that is always greater than individual commitments, but with limited funds, in this design, the mechanism could deliver a share of those.

We will also note that the price contributors need to pay for distributing matching funds towards their projects of interest increases as the financing round progresses. This is because this cost is higher the higher are QF-rule fund requirements relative to actual funds in the matching pool. So any contribution to the mechanism increases this price. As a result, contributors eventually cease to invest in reallocating funds.

**The mechanism**

Assume that \( p \in P \) indexes public good projects competing to receive funding. Also \( i \in I \) indexes individual contributors. An individual \( i \) supports a project \( p \) by committing an amount of money \( c_i^p \). In addition, assume there is a pool of funds provided by donors, that we will denote \( D \), and that will be used to match individual contributions.

The mechanism promises, for each project \( p \), an investment of:

\[
F^p = \begin{cases} 
F_p^{p,QF}, & \text{if } \sum_p F_p^{p,QF} \leq D \\
\frac{F_p^{p,QF}}{\sum_p F_p^{p,QF}} D, & \text{if } \sum_p F_p^{p,QF} > D 
\end{cases}
\]

Where \( F_p^{p,QF} \) denotes the target amount of funds according to the QF Rule (i.e., \( F_p^{p,QF} = (\sum \sqrt{c_i^p})^2 \)).
If contributions to projects are such that \( \sum p F_{p,QF} \leq D \), then the mechanism is essentially the *capital unconstrained* version of QF that, as already discussed, leads to a socially efficient outcome (See BHW).

In case \( \sum p F_{p,QF} \geq D \), the mechanism is restricted to distribute \( D \) across projects, according to the shares \( \frac{F_p}{\sum p F_p} p \in P \). This essentially implies that projects compete for a fixed amount of funding \( D \). By investing in a project \( p \), contributors change the share of funds \( p \) will receive, at the cost of invested funds (n.b. increasing invested funds do not change \( D \)).

Notice that contributions \( c_i^p \) are not distributed by the mechanism. Since in practice QF rounds are regularly repeated, funds raised by the mechanism in a given round could be used as part of a the pool of matching funds in a subsequent round.

**Individual contributor problem**

As in Buterin, Hitzig, and Weyl (2019), let \( V_i^p \) be the currency-equivalent utility a citizen \( i \) receives from a public good \( p \). Utilities from different public goods are assumed to be independent, and it is assumed a setting of complete information.

A contributor \( i \) chooses how much to contribute to each project \( p \in P \). We denote such decisions as \( \{c_i^p\}_{p \in P} \). We will assume that contributors assume there will be limited matching funds available. Contributor’s \( i \) optimization is defined by

\[
\max \left\{ \sum_{p' \in P} V_i^{p'} \left( \frac{D}{\sum_{p' \in P} F_{p',QF}} F_{p',QF} \right) - c_i^{p'} \right\}
\]

Notice that (differently from BHW) here we consider the sum of utilities across all projects. Because there are limited funds, contributing decisions cannot be independently considered.

The F.O.C. for \( c_i^p \) is

\[
V_i^{p'}(c_i^p) \left( \frac{\sqrt{c_i^p}}{\sum_{p' \in P} \sqrt{c_i^{p'}}} \frac{D}{\sum_{p' \in P} F_{p',QF}} \right) - \frac{D}{\left( \sum_{p' \in P} F_{p',QF}\right)^2} \sum_{i} \sqrt{c_i^p} - 1 + \sum_{p' \in P, p' \neq p} V_i^{p'}(c_i^p) \left( -\frac{D}{\left( \sum_{p' \in P} F_{p',QF}\right)^2} \sum_{i} \sqrt{c_i^{p'}} F_{p',QF} \right) = 0
\]

The first term gives the direct effect of contributing in \( p \) plus an indirect effect caused by the fact that increasing \( c_i^p \) also increases the pool of matching requirements \( \sum_{p' \in P} F_{p',QF} \).
The second term gives the indirect effect of increasing the pool of matching requirements in all the remaining projects ($p' \neq p$).

Indirect effects can be grouped

$$V'_{i}^p(.) \left( \frac{\sum_{i} \sqrt{c_i}}{\sqrt{c_i}} \frac{D}{\sum_{p' \in P} F_{p',QF}} \right) + \sum_{p' \in P} V'_{i}^{p'}(.) \left( - \frac{D}{(\sum_{p' \in P} F_{p',QF})^2} \frac{\sum_{i} \sqrt{c_i}}{\sqrt{c_i}} \frac{F_{p',QF}}{D} \right) = 1$$

Then, grouping common factors yield

$$\frac{\sum_{i} \sqrt{c_i}}{\sqrt{c_i}} \left( \frac{D}{\sum_{p' \in P} F_{p',QF}} \right) \left[ V'_{i}^p(.) - \sum_{p' \in P} V'_{i}^{p'}(.) \frac{F_{p',QF}}{(\sum_{p' \in P} F_{p',QF})} \right] = 1$$

(Equation 1)

Equation 1 can be interpreted as follows: Contributing to $p$ increases the marginal benefit from project $p$ (i.e., $V'_{i}^p$), but this is scaled, according to the QF rule, by the relative contributors of all other individuals (i.e., $\frac{\sum_{i} \sqrt{c_i}}{\sqrt{c_i}}$). Because there are limited funds, this benefit is also scaled-down by the ratio of available (to target) funds (i.e., $\frac{D}{\sum_{p' \in P} F_{p',QF}}$).

In addition, contributing to project $p$ now reduces the availability of funds to all other projects, so there is a negative effect that weights the marginal disutility of each project by the relative size of target funding.

Rearranging terms yields

$$V'_{i}^p(.) - \sum_{p' \in P} V'_{i}^{p'}(.) \frac{F_{p',QF}}{(\sum_{p' \in P} F_{p',QF})} = \frac{\sum_{p' \in P} F_{p',QF}}{D} \frac{\sum_{i} \sqrt{c_i}}{\sqrt{c_i}} \left( \frac{\sum_{p' \in P} F_{p',QF}}{D} \right)$$

Summing each side of the equation across individuals yields

$$\sum_{i} V'_{i}^p(.) - \sum_{i} \sum_{p' \in P} V'_{i}^{p'}(.) \frac{F_{p',QF}}{(\sum_{p' \in P} F_{p',QF})} = \frac{\sum_{p' \in P} F_{p',QF}}{D}$$

Or equivalently

$$\sum_{i} V'_{i}^p(.) - \sum_{p' \in P} \frac{F_{p',QF}}{(\sum_{p' \in P} F_{p',QF})} \sum_{i} V'_{i}^{p'}(.) = \frac{\sum_{p' \in P} F_{p',QF}}{D}$$

(Equation 2)

It is useful to get an intuition on Equation 2. Notice that $\sum_{i} V'_{i}^p(.)$ is the sum of the marginal utilities across individuals from public good $p$. The second term, is a (public-good sized) weighted average of the sum of marginal utilities. If the sum of the marginal utilities from $p$ is greater than the weighted average from all projects, individuals will be incentivized to invest in $p$ (i.e., there will be a reallocation of limited funds towards $p$). But investments in $p$ will only take place if such difference is greater than the ratio $\frac{\sum_{p' \in P} F_{p',QF}}{D}$. 


Observation 1. The marginal cost of reallocating funds by contributing to a project is \( \sum_{p' \in P} \frac{F_{p', QF}}{D} \), so this cost increases as more funds are contributed.

Indeed, we can see this latter ratio (the RHS in Equation 2) as the marginal cost of contributing to reallocate limited funds. This design has the property that the cost of reallocating funds increases as more contributions are made.

It is also useful to rewrite Equation 2 as

\[
\sum_i V_i^{p'}(.) = \sum_{p' \in P} \frac{F_{p', QF}}{D} + \sum_{p' \in P} \left(\sum_{p' \in P} \frac{F_{p', QF}}{D}\right) \sum_i V_i^{p'}(.)
\]

(3)

Notice that the LHS of Equation 3 is the sum of the marginal utility across individuals of increasing the size of the public good \( p \). Also notice that the RHS is constant for every project \( p \). Notice that Equation 3 implies

\[
\sum_i V_i^{p}(.) = \sum_i V_i^{p'}(.) \forall p, p'
\]

Which is the condition for the socially optimal allocation for limited funds.

Observation 2. The mechanism tends to equalize the sum of marginal utilities across public goods.

Once the sum of marginal benefits are equalized enough across projects, there are no further incentives to invest (n.b. under limited funds contributors pay to reallocate funds but total funds are not further increased). Indeed, by Equation 2 contributions will only take place as reallocating funds cover the cost \( \sum_{p' \in P} \frac{F_{p', QF}}{D} \).

This result implies another powerful property of QF. In addition to the efficiency result in BHW, where the mechanism requires enough matching funds to efficiently allocate funds for all projects, this shows that the mechanism can also stimulate a social efficient allocation on the margin with available funding.

Concluding notes

One of the most notorious characteristics of QF is that this mechanism is able to solve the planner’s problem in a decentralized way. Even a benevolent, efficient social planner, will
face the problem of knowing the preferences of the community. By allowing the community to signal their preferences, there seems to be an obvious advantage for a decentralized arrangement.

Buterin, Hitzig, and Weyl (2019) (BHW) show that, when there are no limits on matching funds, the decentralized allocation achieves the socially efficient allocation. In a more realistic, limited funds scenario, a question of interest is whether this mechanism achieves an socially optimal allocation of limited funds. In the variation of the mechanism we have proposed here, we have shown that this is the case.

It is worth mentioning that here we have assumed full information, but a decentralized arrangement as the one that we have examined here, might face some coordination problems related to available information. An individual contributor invests on the basis of the contributions that have been made by others, and their expectation on the final distribution of the matching pool. Therefore such a contributor needs to learn about the preferences of the community as they evaluate when and how much to contribute.

With limited matching funds (in a realistic, limited-information case), some contributions could have the unpleasant return of increasing the public good of interest in less than the money invested. This seems to imply that having information on the preferences of the community will be even more valuable in this case. Also, with limited funds, the mechanism has an auction flavor, where the cost of reallocating funds increases as the round progresses.

If contributors learn from the contributions of others as the round unfolds, increasing the amount of funds in the donor pool might provide more time for learning without incurring costs. On the good side, all allocated funds to the mechanism will be part of a new round of financing, as part of the new round matching pool, so inefficient investments could be re-distributed.

References

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