EXISTENCE OF INvariant VOLUMES IN NONHOLONOMIC SYSTEMS

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Abstract. We derive sufficient conditions for a nonholonomic system to preserve a smooth volume form; these conditions become necessary when the density is assumed to only depend on the configuration variables. Moreover, this result can be extended to geodesic flows for arbitrary metric connections and the sufficient condition manifests as integrability of the torsion. As a consequence, volume-preservation of a nonholonomic system is closely related to the torsion of the nonholonomic connection. This result is applied to the Suslov problem for left-invariant systems on Lie groups (where the underlying space is Poisson rather than symplectic).

1. Introduction. This work is motivated by Liouville’s theorem which asserts that all (unconstrained) Hamiltonian systems preserve the symplectic form (and, consequently, the induced volume form). However, nonholonomic systems are not symplectic (which follows from the fact that nonholonomic systems are not variational). As such, the question of volume-preservation becomes nontrivial. A famous example of this is the Chaplygin sleigh; this system, although energy-preserving, experiences “dissipation” (cf. [31] for a general discussion on stability of nonholonomic systems or [25] for an interpretation via impact systems).

The purpose of this work is to construct a systematic way to determine whether or not a nonholonomic system preserves volume. In particular, we present necessary and sufficient conditions on when there exists an invariant volume form with density depending only on the configuration variables, i.e. $f = \pi_Q^*g : T^*Q \to \mathbb{R}$ where $g : Q \to \mathbb{R}$ and $\pi_Q : T^*Q \to Q$ is the standard cotangent projection.

Theorem 1.1 (Main Result). Let $L : TQ \to \mathbb{R}$ be a natural Lagrangian (i.e. the kinetic energy is induced by a Riemannian metric) and $\mathcal{D} \subset TQ$ be a regular distribution (each fiber has constant dimension). Then, there exists an invariant

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volume with density depending only on the configuration variables if and only if there exists \( \rho \in \Gamma(D^0) \) such that \( \vartheta_\varepsilon + \rho \) is exact where

\[
\vartheta_\varepsilon = m_{\alpha\beta} \cdot \mathcal{L}_{W^\alpha} \eta^\beta.
\]

Here, \( D^0 \subset T^*Q \) is the annihilator of \( D \subset TQ \), \( \{ \eta^\alpha \} \) is a frame for \( D^0 \), \( W^\alpha = FL^{-1}(\eta^\alpha) \) are dual vector fields, and \( m^{\alpha\beta} = \eta^\alpha(W^\beta) \).

In particular, suppose that \( \vartheta_\varepsilon + \rho = dg \). Then the following volume form is preserved:

\[
\exp(\pi_Q^*g) \cdot \mu_\varepsilon,
\]

where \( \mu_\varepsilon \) is the nonholonomic volume form (cf. Definition 4.1). Preliminaries on nonholonomic systems are presented in Section 2. Section 3 presents the construction of the “global nonholonomic vector fields” which allow us to work on the whole manifold \( T^*Q \) and restrict to the constraint distribution after the calculations are performed. The divergence calculation for a nonholonomic system is performed in Section 4. The main result, Theorem 1.1, is proved in Section 5 (cf. Theorem 5.3). Section 6 shows that this 1-form, \( \vartheta_\varepsilon \) is intimately connected to the torsion of the nonholonomic connection, which seems to be a new observation. Section 7 shows how this result is applicable to the Suslov problem and to the problem of invariant volumes on Poisson manifolds. This paper concludes with examples in Section 8.8.

This paper is a continuation of the work done in [6] and, as such, many of the results below can be found there. Related results can be found in [8], cf. Theorem 4.2 therein. However, there exists a few key differences. Firstly, [8] constructs an almost-Poisson structure on \( D^* = FL(D) \subset T^*Q \) and studies its modular class. This differs from our treatment as we define a global nonholonomic vector field on the whole of \( T^*Q \) and restrict to \( D^* \) at the end. This has the advantage of avoiding local coordinates and allowing greater freedom in choosing how to express the constraints. Additionally, [8] requires \( Q \) to be orientable while we make no such assumption; cf. §8.8 where we consider the case where \( Q \) is the Möbius strip.

2. Preliminaries.

2.1. Unconstrained Mechanics. We will first briefly cover the case of unconstrained mechanical systems before discussing nonholonomic systems. A smooth (finite-dimensional) manifold \( Q \) is called the configuration space, the tangent bundle \( TQ \) is called the state space, and the cotangent bundle \( T^*Q \) is called the phase space.

2.1.1. Lagrangian Systems. Lagrangian systems take place on the state space and are given by a smooth Lagrangian function, \( L : TQ \to \mathbb{R} \). A Lagrangian function generates dynamics on \( TQ \) via Hamilton’s principle which leads to the Euler-Lagrange equations.

**Proposition 1.** Hamilton’s principle is equivalent to the condition that the curve \( q(t) \) satisfies the Euler-Lagrange equations

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.
\]
2.1.2. Hamiltonian Systems. Contrary to Lagrangian systems, Hamiltonian systems take place on the phase space $T^*Q$. Given a Lagrangian, there is an identification between $TQ$ and $T^*Q$ called the fiber derivative.

$$\mathbb{F}L : TQ \rightarrow T^*Q$$

$$\mathbb{F}L(v)(w) := \left. \frac{d}{dt} \right|_{t=0} L(q, v + tw).$$

**Definition 2.1.** A Lagrangian, $L : TQ \rightarrow \mathbb{R}$, is hyperregular if the fiber derivative is a global diffeomorphism between $TQ$ and $T^*Q$. Furthermore, a Lagrangian is natural if it has the form

$$L(q, v) = \frac{1}{2} g_q(v, v) - V(q)$$

where $g$ is a Riemannian metric on $Q$ and $V : Q \rightarrow \mathbb{R}$ is a smooth function called the potential.

For the most part, we will be dealing with natural Lagrangians (which are hyperregular) and the fiber derivative in this case takes the form

$$\mathbb{F}L(v)(w) = g(v, w).$$

For a given (hyperregular) Lagrangian, we define the Hamiltonian $H : T^*Q \rightarrow \mathbb{R}$ via the Legendre transform:

$$H(q, p) = \langle p, v \rangle - L(q, v), \quad p = \mathbb{F}L(v).$$

While $L$ generates dynamics on $TQ$ variationally, $H$ generates dynamics on $T^*Q$ symplectically.

**Definition 2.2.** A symplectic form is a closed, nondegenerate 2-form. A pair $(M, \omega)$ where $\omega$ is a symplectic form on the manifold $M$ is called a symplectic manifold. A vector field, $X_H$, on $M$ is called Hamiltonian if

$$i_{X_H} \omega = dH,$$

for some energy $H : M \rightarrow \mathbb{R}$. Here, $i_X \omega = \omega(X, \cdot)$ is the contraction.

We can construct Hamiltonian vector fields on $T^*Q$ via the natural symplectic form $\omega = dq^i \wedge dp_i \in \Omega^2(T^*Q)$. With this symplectic form, the Lagrangian and Hamiltonian formulations are equivalent.

**Proposition 2** (cf. Theorem 3.6.2 in [1]). Let $L$ be a natural Lagrangian on $Q$ and $H$ its Legendre transform. Then the integral curves of (1) are mapped to the integral curves of (2) under $\mathbb{F}L$. Furthermore, both systems have the same base integral curves.

Given a symplectic form $\omega$, the $n$-fold wedge product $\omega^n$ is a volume form. An important feature of Hamiltonian systems is that they are always volume-preserving.

**Theorem 2.3** (Liouville’s theorem). Hamiltonian dynamics preserve the symplectic form and, additionally, preserve the volume form $\omega^n$.

**Proof.** This follows immediately from Cartan’s magic formula:

$$\mathcal{L}_{X_H} \omega = di_{X_H} \omega + i_{X_H} d\omega.$$

The first term vanishes as $di_{X_H} \omega = ddH$ and the second vanishes as $\omega$ is closed. □

The main goal of this work is to extend Liouville’s theorem to nonholonomic systems.
2.2. Constraint Distributions. Suppose that a Lagrangian system \( L : TQ \to \mathbb{R} \) is subject to certain constraints, i.e. a figure skater who cannot slide perpendicular to the direction of her skate. Constraints involving the velocities of the system are known as nonholonomic constraints (holonomic constraints involve only the positions; this distinction will be made precise below). Everything below that holds true for nonholonomic constraints also works for holonomic constraints, so we will treat everything as nonholonomic and not worry about the distinction. For the most part, we will assume that the constraints are linear in the velocities.

Nonholonomic constraints are normally described as specifying a submanifold \( D \subset TQ \) that describes the restricted motion. When the constraints are linear in the velocities, the submanifold \( D \) is a distribution.

**Definition 2.4.** A smooth distribution on a manifold \( Q \) is the assignment to each \( x \in Q \) of a subspace \( D_x \subset T_x Q \), i.e. \( D \subset TQ \) is a vector sub-bundle. A distribution \( D \) is involutive if for any two vector fields \( X, Y \) on \( M \) with values in \( D \), \([X,Y]\) also has values in \( D \). A distribution \( D \) is regular if \( \dim(D_x) \) is the same for every \( x \in M \).

**Theorem 2.5** (Frobenius’ Theorem). \( D \) is involutive if and only if there is a foliation on \( Q \) whose tangent bundle equals \( D \).

If \( D \) is involutive, it is said to be integrable and the constraints are called holonomic. When \( D \) is not involutive, it is nonintegrable and the constraints are nonholonomic.

Constraint distributions are usually described by a family of 1-forms \( \eta^\alpha \).

\[
D = \bigcap_{\alpha = 1}^m \ker \eta^\alpha, \quad \eta^\alpha \in \Omega^1(Q).
\]

In this situation, the distribution is integrable if the 1-forms \( \eta^\alpha \) can be chosen such that they are all closed: \( d\eta^\alpha = 0 \).

2.3. Hamiltonian Nonholonomic Systems. It is important to note that nonholonomic systems are not described by variational principles (on the Lagrangian side) nor are they symplectic (on the Hamiltonian side). Rather than obeying Hamilton’s principle, nonholonomic systems follow the Lagrange-d’Alembert principle. In the Hamiltonian setting, this manifests as (see [16, 22] and §5.8 in [3]):

\[
i_{X_H^\omega} \omega = dH + \lambda_\alpha \pi_Q^* \eta^\alpha,
\]

where \( \pi_Q : T^*Q \to Q \) is the cotangent projection and the \( \lambda_\alpha \) are multipliers to enforce the constraints.

Let \( g \) be the Riemannian metric underlying the natural Hamiltonian, \( H \) (a Hamiltonian is natural if it comes from a natural Lagrangian). For each constraining 1-form \( \eta^\alpha \), let \( W^\alpha \in \mathfrak{X}(Q) \) be the vector field such that \( g(W^\alpha, \cdot) = \eta^\alpha \) (equivalently, \( W^\alpha = FL^{-1} \eta^\alpha \)). The constraint distribution \( D \subset TQ \) on the cotangent side becomes

\[
D^\ast = FL(D) = \{(x,p) \in T^*Q : P(W^\alpha)(x,p) = 0\}.
\]

The function \( P(W) : T^*Q \to \mathbb{R} \) is the momentum of the vector field \( W \) given by

\[
P(W)(q,p) = \langle p, W(q) \rangle.
\]

The multipliers \( \lambda_\alpha \) in (3) are chosen such that \( X_H^\omega \) is tangent to \( D^\ast \subset T^*Q \).
3. Global Nonholonomic Vector Fields. Given a constraint distribution, $D^* \subset T^*Q$, we can determine the nonholonomic vector field $X^D_H \in \mathfrak{X}(D^*)$ via (3). Commonly local, noncanonical, coordinates are chosen for $D^*$ (cf. §5.8 in [3] and [26]). However, we will instead work with the entire manifold $T^*Q$ and define a global vector field $X^D_{H}^{glb} \in \mathfrak{X}(T^*Q)$ such that $X^D_{H}^{glb}|_{D^*} = X^D_H$. This section outlines an intrinsic (albeit non-unique) way to determine such a vector field.

**Definition 3.1.** For a given constraint submanifold $D^* \subset T^*Q$ ($D^*$ need not be a distribution), a realization of $D^*$ is an ordered collection of functions $\mathcal{C} := \{g_i : T^*Q \to \mathbb{R}\}$ such that zero is a regular value of $G = g_1 \times \ldots \times g_m$ and

$$D^* = \bigcap_i g_i^{-1}(0).$$

If the functions $g_i$ are given by momenta, i.e. $g_i = P(X^i)$, then the realization is called natural.

**Remark 1.** Under the case where the Lagrangian is natural (which provides a Riemannian metric on $Q$) and the constraint submanifold is a distribution, we can choose the realization to be natural:

$$\mathcal{C} = \{P(W^1), \ldots, P(W^m)\},$$

where $W^i = P^{-1}(\eta^i) = (\eta^i)^{\xi}.$

By replacing $D^*$ with a realization $\mathcal{C}$, we can extend the nonholonomic vector field to a vector field on $T^*Q$ that preserves the constraining functions $g_i$. Recall that the form of the nonholonomic vector field is $i_{X^D_H} \omega = dH + \lambda_\alpha \pi^\alpha_Q \eta^\alpha$. We construct the global nonholonomic vector field, $\Xi^\mathcal{C}_H$, by requiring that:

\begin{align*}
& (\text{NH.1}) \quad i_{\Xi^\mathcal{C}_H} \omega = dH + \lambda_\alpha \pi^\alpha_Q \eta^\alpha \
& (\text{NH.2}) \quad \mathcal{L}_{\Xi^\mathcal{C}_H} g_i = 0 \quad \text{for all } g_i \in \mathcal{C}.
\end{align*}

Under reasonable compatibility assumptions on $\mathcal{C}$ (cf. §3.4.1 in [23]), such a vector field exists and is unique. However, given two different realizations, $\mathcal{C}$ and $\mathcal{C}'$, of the same constraint distribution $D^*$, it is not generally true that $\Xi^\mathcal{C}_H = \Xi^{\mathcal{C}'}_H$, however $\Xi^\mathcal{C}_H|_{D^*} = \Xi^{\mathcal{C}'}_H|_{D^*}$. When both the Hamiltonian and realization are natural, the global field can be explicitly computed via the constraint mass matrix defined below.

**Remark 2.** The constraint manifold is given by the joint zero level-sets of the $g_i$ while the realization provides additional irrelevant information off of the constraint manifold. This is why $\Xi^\mathcal{C}_H \neq \Xi^{\mathcal{C}'}_H$ but they agree once restricted.

**Definition 3.2.** For a natural realization $\mathcal{C} = \{P(W^1), \ldots, P(W^m)\}$ and natural Hamiltonian (so $(Q,g)$ is Riemannian), the constraint mass matrix, $(m^{\alpha\beta})$, is given by orthogonally pairing the constraints, i.e.

$$m^{\alpha\beta} = g(W^\alpha, W^\beta) = \eta^\alpha(W^\beta).$$

Additionally, its inverse will be denoted by $(m^{\alpha\beta})^{-1} = (m^{\alpha\beta})^{-1}$.

**Lemma 3.3.** The constraint mass matrix is symmetric and positive-definite so long as all the constraints are linearly independent.

**Proof.** This follows from the fact that $(m^{\alpha\beta})$ is a Gram matrix for a nondegenerate inner product. \qed
We can now write down a formula for $\Xi^\mathcal{C}_H$. Using (NH.1) and (NH.2), we get that (where $\{\cdot,\cdot\}$ is the standard Poisson bracket)

$$\mathcal{L}_{\Xi^\mathcal{C}_H} P(W^\beta) = i_{X_{P(W^\beta)}} \omega(\Xi^\mathcal{C}_H) = -i_{\Xi^\mathcal{C}_H} \omega(X_{P(W^\beta)}) = -dH(X_{P(W^\beta)}) - \lambda_\alpha \pi^*_Q \eta^\alpha(X_{P(W^\beta)}) = \{P(W^\beta), H\} - \lambda_\alpha \eta^\alpha(W^\beta) = 0 \implies \{P(W^\beta), H\} = m^{\alpha\beta} \lambda_\alpha.$$

Due to the constraint mass matrix being nondegenerate, the multipliers have a unique solution and the global nonholonomic vector field is given by

$$i_{\Xi^\mathcal{C}_H} \omega = dH - m^{\alpha\beta} \{H, P(W^\alpha)\} \pi^*_Q \eta^\beta \quad (4)$$

**Remark 3.** The global nonholonomic vector field given by (4) can be extended to the case of nonlinear constraints via Chetaev’s rule (which is not necessarily the correct procedure, cf. [19] for a discussion), which will give equivalent results to those in [17] where the “almost-tangent” structure of the tangent bundle is utilized. For Lagrangian systems, Chetaev’s rule states that if we have a nonlinear constraint $f(q, \dot{q}) = 0$, then the constraint force takes the following form:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \lambda \cdot \frac{\partial f}{\partial \dot{q}} = \lambda \cdot S^* df,$$

and $S : T(TQ) \to T(TQ)$ is its almost-tangent structure. However as we are instead on the cotangent bundle, the object we will use will be the related to the almost-tangent structure through the fiber derivative,

$$C : T(T^*Q) \to T(T^*Q)$$

$$C^* (\alpha dx^i + \beta^i dp_j) = g_{ij} \beta^j dx^i.$$

For a constraint realization $\mathcal{C} = \{g^1, \ldots, g^m\}$, the nonholonomic vector field is given by

$$i_{\Xi^\mathcal{C}_H} \omega = dH - m^{\alpha\beta} \{H, g^\alpha\} C^* dg^\beta,$$

where the multipliers are given by $m^{\alpha\beta} = C^* dg^\beta (X_{g^\alpha})$.

**Definition 3.4.** The 1-form given by

$$\nu^\mathcal{C}_H := dH - m^{\alpha\beta} \{H, P(W^\alpha)\} \pi^*_Q \eta^\beta,$$

is called the nonholonomic 1-form with respect to the realization $\mathcal{C}$.

**Proposition 3.** Given two different natural realizations, $\mathcal{C}$ and $\mathcal{C}'$, the global nonholonomic vector fields given by (4) agree on $D^*$.

**Proof.** Suppose that there is only a single constraint and $\mathcal{C} = \{P(W)\}$ and $\mathcal{C}' = \{fP(W)\}$ for some smooth $f$. By Leibniz’s rule,

$$i_{\Xi^\mathcal{C}'_H} \omega = dH + \frac{1}{f^2 g(W, W)} \{fP(W), H\} f \pi^*_Q \eta = dH + \frac{1}{f^2 g(W, W)} \left[ f \{P(W), H\} + P(W) \{f, H\} \right] f \pi^*_Q \eta = dH + \frac{1}{g(W, W)} \{P(W), H\} \pi^*_Q \eta + \frac{P(W)}{fg(W, W)} \{f, H\} \pi^*_Q \eta.$$
Therefore, we have
\[ i_{\Xi^C} \omega - i_{\Xi^C} \omega = \frac{P(W)}{f g(W, W)} \{ f, H \} \pi^*_Q \eta, \]
which vanishes on $\mathcal{D}^*$. A similar argument works for multiple constraints.

Throughout the rest of this work, we will assume that $\mathcal{C}$ is a natural realization. This, in turn, requires that the constraints are linear in the velocities / momenta.

4. Nonholonomic Volume. An invariant measure is a powerful tool for understanding the asymptotic nature of a dynamical system. In the case of nonholonomic systems, a smooth invariant measure offers two key insights. The first is the usual case in dynamical systems where an invariant measure allows for the use of the Birkhoff Ergodic Theorem (cf. e.g. 4.1.2 in [13]) as well as for recurrence. The other is unique to nonholonomic systems; even though nonholonomic systems are not Hamiltonian, “nonholonomic systems which do preserve volume are in a quantifiable sense closer to Hamiltonian systems than their volume changing counterparts,” [9] (see also [2] and [5]). Therefore, being able to find an invariant measure for a nonholonomic system allows for ergodic-like understanding of its asymptotic behavior as well as provide a way to “Hamiltonize” a nonholonomic system.

There has already been work done in finding invariant measures in systems where symmetries are present: Chaplygin systems are studied in, e.g. [11, 14, 23, 24], Euler-Poincaré-Suslov systems are studied in, e.g. [3, 12], systems with internal degrees of freedom are studied in, e.g. [3, 4, 30], and [8] studies the case of symmetric kinetic systems where the dimension assumption does not hold. Related work on asymptotic dynamics may be found in [29]. This work, rather, uses an altogether different approach where no symmetries will be used. Additionally, in §5.2, we provide necessary and sufficient conditions for when an invariant measure exists whose density depends only on the base variables, i.e. $f = \pi^*_Q g$ for some $g \in C^\infty(Q)$.

4.1. Nonholonomic Volume form. The symplectic manifold $T^*Q$ has a canonical volume form $\omega^n$. However, the nonholonomic flow takes place on a submanifold $\mathcal{D}^* \subset T^*Q$ which is $2n - m$ dimensional. Therefore, $\omega^n$ is not a volume form on $\mathcal{D}^*$. Here, we construct a volume form on $\mathcal{D}^*$ which is unique up to the choice of realization. The derivation of this will be similar to the construction of the volume form on an energy surface in §3.4 of [1]. For the realization $\mathcal{C} = \{ P(W^1), \ldots, P(W^m) \}$, define the $m$-form
\[ \sigma^\mathcal{C} := dP(W^1) \wedge \ldots \wedge dP(W^m). \]

Definition 4.1. If we denote the inclusion map by $\iota : \mathcal{D}^* \hookrightarrow T^*Q$, then a nonholonomic volume, $\mu^\mathcal{C}$, is given by
\[ \mu^\mathcal{C} = \iota^* \varepsilon, \quad \sigma^\mathcal{C} \wedge \varepsilon = \omega^n. \]

Proposition 4. Given an ordered collection of constraints, $\mathcal{C}$, the induced volume form $\mu^\mathcal{C}$ is unique.

Proof. Suppose that $\varepsilon$ and $\varepsilon'$ are two forms satisfying $\sigma^\mathcal{C} \wedge \varepsilon = \omega^n$. Then
\[ \varepsilon - \varepsilon' = \alpha, \quad \sigma^\mathcal{C} \wedge \alpha = 0. \]
Now let $\iota : \mathcal{D}^* \hookrightarrow T^*Q$ be the inclusion. Then from the above, we see that
\[ \iota^* \varepsilon = \iota^* \varepsilon' + \iota^* \alpha. \]
The result will follow so long as $\iota^*\alpha = 0$. Suppose that $\iota^*\alpha \neq 0$ and choose vectors $v^1, \ldots, v^{2n-m} \in T_xD^* \subset T_xT^*Q$ such that $\alpha(v^1, \ldots, v^{2n-m}) \neq 0$. Complete this collection of vectors to a basis of $T_xT^*Q$: $v^1, \ldots, v^{2n-m}, v^{2n-m+1}, \ldots, v^{2n}$ such that $\sigma_{\iota}(v^{2n-m+1}, \ldots, v^{2n}) \neq 0$. Then we have

$$\sigma_{\iota} \wedge \alpha(v^1, \ldots, v^{2n}) = (-1)^{(2n-m)m} \alpha(v^1, \ldots, v^{2n-m}) \cdot \sigma_{\iota}(v^{2n-m+1}, \ldots, v^{2n}) \neq 0,$$

which is a contradiction.

Remark 4. Notice that for an ordered collection of constraints the volume form is unique. However, changing the order of the constraints changes the sign of the induced volume form and rescaling constraints rescales the volume form. In this sense, $\mathcal{C}$ uniquely determines $\mu_{\iota C}$, but $D^*$ only determines $\mu_{\iota C}$ up to a multiple.

While examining the failure of Liouville’s theorem (Theorem 2.3) for nonholonomic systems, we will see when $\mu_{\iota C}$ is preserved under the flow of $X_{DH}$. More generally, we will consider the existence of a smooth density $f \in C^\infty(D^*)$ when $f\mu_{\iota C}$ is preserved.

4.2. Divergence. Let $\omega = dq^i \wedge dp_i$ be the standard symplectic form on $T^*Q$. This in turn induces a volume form $\omega^n$. It is a known result that Hamiltonian flows preserve this measure, however, nonholonomic flows generally do not. A measure of how much a flow fails to preserve a volume form is described by its divergence. Below, we first discuss some basics of the divergence before applying it to nonholonomic systems.

4.2.1. Divergence Preliminaries. To understand volume preservation, we will use the notion of the divergence of a vector field (cf. §2.5 of [1] or §5.1 in [13]).

Definition 4.2. Let $M$ be an orientable manifold with volume form $\Omega$ and $X$ a vector field on $M$. Then the unique function $\text{div}_{\Omega}(X) \in C^\infty(M)$ such that $\mathcal{L}_X\Omega = \text{div}_{\Omega}(X)\Omega$ is called the divergence of $X$. The vector field $X$ is called incompressible iff $\text{div}_{\Omega}(X) = 0$.

This definition of divergence generalizes the familiar one from multivariate calculus in which $M = \mathbb{R}^n$ and $\Omega = dx^1 \wedge \ldots \wedge dx^n$. Indeed,

$$\mathcal{L}_X\Omega = d_i X^i \Omega = d \left[ \sum_{i=1}^n (-1)^{i-1} X^i \cdot dx^1 \wedge \ldots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \ldots \wedge dx^n \right]$$

$$= \left[ \sum_{i=1}^n \frac{\partial X^i}{\partial x^i} \right] \cdot \Omega.$$

Studying the divergence is a useful test to check volume preservation via the following proposition.

Proposition 5 (2.5.25 in [1]). Let $M$ be a manifold with volume $\Omega$ and vector field $X$. Then $X$ is incompressible iff every flow box of $X$ is volume preserving.

Liouville’s theorem in this language states that for an unconstrained Hamiltonian system, $\text{div}_{\omega^n}(X_H) = 0$. That is, Hamiltonian systems preserve the volume induced by the symplectic form. This is, in general, not the case for nonholonomic systems.
4.2.2. **Divergence of a nonholonomic system.** We now proceed with computing the divergence of a nonholonomic vector field, \( \text{div}_{\mu}(X_H^P) \). When this is nonzero, we will be interested in finding a density, \( f \), such that \( \text{div}_{f \mu}(X_H^P) = 0 \). This problem will be addressed in §5.

Before we begin with the divergence calculation, we first present a helpful lemma which allows us to relate the divergence of the global nonholonomic vector field with the corresponding restricted vector field.

**Lemma 4.3.** If \( \mathcal{D} \) is a (natural) realization of a constraint \( \mathcal{D}^* \subset T^*Q \), then
\[
\text{div}_\omega(\mathcal{E}H)\big|_{\mathcal{D}^*} = \text{div}_{\mu}(X_H^P).
\]

**Proof.** Leibniz’s rule for the Lie derivative provides
\[
\mathcal{L}_{\mathcal{E}H}(\omega^n) = \left(\mathcal{L}_{\mathcal{E}H}(\sigma_\xi \wedge \varepsilon)\right) \wedge \varepsilon + \sigma_\xi \wedge \mathcal{L}_{\mathcal{E}H}(\varepsilon).
\]
However, \( \mathcal{L}_{\mathcal{E}H}(\sigma_\xi) = 0 \) because the constraints are preserved under the flow. Applying this, we see that
\[
\mathcal{L}_{\mathcal{E}H}(\omega^n) = \sigma_\xi \wedge \left(\mathcal{L}_{\mathcal{E}H}(\varepsilon)\right),
\]
which gives
\[
(\text{div}_\omega(\mathcal{E}H)) \sigma_\xi \wedge \varepsilon = \sigma_\xi \wedge \left(\mathcal{L}_{\mathcal{E}H}(\varepsilon)\right).
\]
Due to the fact that the Lie derivative commutes with restriction, the result follows.

This lemma allows for us to calculate the divergence of the global nonholonomic vector field and to restrict to the constraint distribution afterwards.

Before we compute the divergence of arbitrary nonholonomic systems, we first consider the simplified case where there is only a single constraint present, i.e. \( \mathcal{D} = \{\mathcal{C}(W)\} \). Here, we make the normalization \( \eta(W) = 1 \) to simplify equation (4).

The divergence of \( X_H^P \) is given by
\[
\mathcal{L}_{X_H^P}\mu = \text{div}_{\mu}(X_H^P)\mu_\xi.
\]
In order to compute this, we will invoke Cartan’s magic formula as well as Lemma 4.3 (restricting to \( \mathcal{D}^* \) will occur at the end):
\[
\mathcal{L}_{\mathcal{E}H}(\omega^n) = i_{\mathcal{E}H}d\omega^n + d(i_{\mathcal{E}H}\omega^n)
\]
\[
= n \cdot d \left(i_{\mathcal{E}H}\omega \wedge \omega^{n-1}\right)
\]
\[
= n \cdot d \left(i_{\mathcal{E}H}\omega\right) \wedge \omega^{n-1} - n \cdot \left(i_{\mathcal{E}H}\omega\right) \wedge d\omega^{n-1}
\]
\[
= n \cdot \left(d\omega^{n-1}\right) \wedge \omega^{n-1}.
\]
The problem of computing the divergence collapses to calculating \( d(i_{\mathcal{E}H}\omega) \) (which captures how “non-symplectic” the flow is). Let \( N \) be difference between the nonholonomic and Hamiltonian vector fields:
\[
N = \{H, P(W)\} \eta_k \frac{\partial}{\partial p_k}.
\]
Then, from Hamilton’s equations, we obtain
\[
i_{\mathcal{E}H}\omega = dH + i_N\omega.
\]
Returning to the divergence calculation, \( d_i \Xi C H \omega = d_i N \omega \) where \( i_N \omega = -\eta_i \{ H, P(W) \} dq^i \).

Applying the exterior derivative yields:

\[
d_i N \omega = \left[ -\frac{\partial \eta_i}{\partial q^k} \{ H, P(W) \} - \eta_i \frac{\partial}{\partial q^k} \{ H, P(W) \} \right] dq^k \wedge dq^i
- \eta_i \frac{\partial}{\partial p^\ell} \{ H, P(W) \} dp^\ell \wedge dq^i
\]

Notice that when we wedge \( d_i N \omega \) with \( \omega^{n-1} \), the entire first line vanishes and only the diagonal on the second survives. Combining everything, we see that

\[
\text{div}_{\mu^C} (X_D H) = n \cdot \eta_i \frac{\partial}{\partial p^i} \{ H, P(W) \}.
\]  
(5)

The exact same procedure can be carried out when there are an arbitrary number of constraints. The divergence is then simply

\[
\text{div}_{\mu^C} (X_D H) = n \cdot m_{\alpha \beta} \cdot \eta^\alpha_i \frac{\partial}{\partial p^k} \{ H, P(W^\beta) \}.
\]  
(6)

4.2.3. An intrinsic form of the divergence. This section concludes with an intrinsic way to interpret (5) and (6). Recall the cotangent projection \( \pi_Q : T^* Q \to Q \) and the fact that

\[
d\pi_Q \cdot X_f = \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i}.
\]

The divergence formula (5) becomes:

\[
\text{div}_{\mu^C} (X_H^P) = n \cdot \pi_Q^* \eta \left( X_{\{ H, P(W) \}} \right)
= -n \cdot \pi_Q^* \eta \left( [X_H, X_P(W^\beta)] \right).
\]  
(7)

This can also be carried over to the multiple constraint case.

\[
\text{div}_{\mu^C} (X_H^P) = -n \cdot m_{\alpha \beta} \cdot \pi_Q^* \eta^\alpha \left( [X_H, X_P(W^\beta)] \right).
\]  
(8)

The formulas (7) and (8) have a structure similar to the curvature of an Ehresmann connection. This is because these formulas have the structure of a projection composed with a vector field bracket. The main difference is that while the curvature of an Ehresmann connection is vertical-valued, these formulas are real-valued. It turns out that the divergence is closely related to the torsion of the nonholonomic connection as will be discussed in §6.

Remark 5. In the same way that the nonholonomic 1-form can be extended to the case of nonlinear constraints via Chetaev’s rule, see Remark 3, the divergence described above by (8) can also be extended to the case of nonlinear constraints. The divergence is given by

\[
\text{div}_{\mu^C} (X_H^P) = -n \cdot m_{\alpha \beta} \cdot C^* dg^\alpha \left( [X_H, X_P(W^\beta)] \right).
\]  

5. Invariant Volumes and the Cohomology Equation. In general, the divergence of a nonholonomic system does not vanish as (8) shows. When does there exist a different volume form on \( D^* \) that is invariant under the flow? i.e. does there exist a density \( f > 0 \) such that \( \text{div}_{f \mu^C} (X_H^P) = 0 \)? Finding such an \( f \) requires solving a certain type of partial differential equation which is known as the smooth dynamical cohomology equation. Solving this PDE is generally quite difficult, but if we add the assumption that \( f = \pi_Q^* g \) for some \( g : Q \to \mathbb{R} \), then the problem...
becomes much more tractable and reduces to studying a 1-form called the density form.

5.1. The Cohomology Equation. What conditions need to be met for $f$ such that $f_{\mu e}$ is an invariant volume form? Using the formula for the divergence as well as the fact that the Lie derivative is a derivation yields:

$$\text{div}_{f_{\mu e}} (X^D_H) = \text{div}_{\mu e} (X^D_H) + \frac{1}{f} \mathcal{L}_{X^D_H} (f).$$

Therefore the density, $f$, yields an invariant measure if and only if

$$\frac{1}{f} \mathcal{L}_{X^D_H} (f) = - \text{div}_{\mu e} (X^D_H).$$

Notice that the left hand side of (9) can be integrated to

$$\frac{1}{f} \mathcal{L}_{X^D_H} (f) = d (\ln f) (X^D_H).$$

Calling $g = \ln f$, we have the following proposition.

**Proposition 6.** For a nonholonomic vector field, $X^D_H$, there exists a smooth invariant volume, $f_{\mu e}$, if there exists an exact 1-form $\alpha = dg$ such that

$$\alpha (X^D_H) = - \text{div}_{\mu e} (X^D_H).$$

Then the density is (up to a multiplicative constant) $f = e^g$.

Therefore the existence of invariant volumes boils down to finding global solutions to the PDE (10). The remainder of this section deals with uniqueness of solutions and a necessary condition for solutions to exist.

**Remark 6.** PDEs of the form $dg(X) = f$ for a given smooth function $f$ and vector field $X$ are called cohomology equations [10, 18]. Thus the equation (10) is a cohomology equation.

5.1.1. Uniqueness. The problem of existence is quite difficult in general and we postpone that discussion until the next subsection where we assume that the solution has the form $f = \pi^*_Q g$. In the meantime, assuming that there exists a function $g \in C^\infty (D^*)$ that solves (10), do there exist other solutions? Suppose that $g_1$ and $g_2$ both solve (10). Then their difference must be a first integral of the system:

$$\mathcal{L}_{X^D_H} (g_1 - g_2) = 0.$$ Solutions of (10) are then unique up to constants of motion. i.e. if $g$ solves (10), then every invariant density has the form (again, up to a multiplicative constant)

$$f = \exp (g + \text{constant of motion}).$$

Therefore invariant measures can be thought of as an affine space with dimension being equal to the number of first integrals of the nonholonomic system.

5.2. Special Case: Densities depending only on configuration. In general, solving the cohomology equation (10) is quite difficult. It turns out, however, that it is relatively easy to determine necessary and sufficient conditions on the solvability when the density is assumed to depend only on the configuration variables.

**Definition 5.1.** A density $f : T^*Q \to \mathbb{R}$ is said to depend only on configuration if $f = \pi^*_Q g$ for some $g : Q \to \mathbb{R}$. 
Under this assumption, (8) can be presented in a surprisingly nice way. In this case, the divergence can be described by an equivalence class of 1-forms. The density form, defined below, is a representative element from this class.

**Definition 5.2.** Let \( \mathcal{C} \) be a (natural) realization of \( D^* \subset T^*Q \). Then, define the **density form** to be the following 1-form

\[
\vartheta_{\mathcal{C}} = m_{\alpha \beta} \cdot \mathcal{L}_{W^*} \eta^\alpha.
\]

Studying the 1-form, \( \vartheta_{\mathcal{C}} \), provides necessary and sufficient conditions for the existence of densities depending only on configuration. Recall that \( D^0 = \text{Ann}(D) \subset T^*Q \) is the annihilator of \( D \subset TQ \) and \( \Gamma(D^0) \) are its sections.

**Theorem 5.3.** There exists an invariant density depending on configuration if and only if there exists \( \rho \in \Gamma(D^0) \) such that \( \vartheta_{\mathcal{C}} + \rho \) is exact.

**Proof.** To show this, we will prove that \(-n \cdot \pi_Q^* \vartheta_{\mathcal{C}}(X^P_H) = \text{div}_{\mu_{\mathcal{C}}}(X^P_H)\). Recall that the differential of a 1-form is given by \( d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]) \) and that \( \pi_Q^* \eta^\beta(X_H)|_{D^*} = 0 \). Returning to (8), we have

\[
\text{div}_{\mu_{\mathcal{C}}}(X^P_H) = -n \cdot \pi_Q^* \eta^\alpha \left( [X_H, X_{P(W^\beta)}] \right) \\
= -n \cdot m_{\alpha \beta} \cdot \left( X_H m^{\alpha \beta} - \pi_Q^* d\eta^\alpha(X_H, X_{P(W^\beta)}) \right) \\
= -n \cdot m_{\alpha \beta} \cdot (dm^{\alpha \beta}(\dot{q}) - d\eta^\alpha(\dot{q}, W^\beta)) \\
= -n \cdot m_{\alpha \beta} \cdot (di_{W^\beta} \eta^\alpha + i_{W^\beta} d\eta^\alpha) (\dot{q}) \\
= -n \cdot m_{\alpha \beta} \cdot \mathcal{L}_{W^\beta} \eta^\alpha(\dot{q}).
\]

This computation shows that \( \text{div}_{\mu_{\mathcal{C}}}(X^P_H) = -n \cdot \vartheta_{\mathcal{C}}(\dot{q}) \), but \( \dot{q} \) cannot be arbitrary as it must lie within \( D \). Therefore, we can add on an element of \( D^0 \) to \( \vartheta_{\mathcal{C}} \) without changing its value on \( \dot{q} \): \( \text{div}_{\mu_{\mathcal{C}}}(X^P_H) = -n \cdot (\vartheta_{\mathcal{C}} + \rho)(\dot{q}) \) for any \( \rho \in D^0 \). Hence, a solution exists depending only on configuration if \( \vartheta_{\mathcal{C}} + \rho \) can be integrated, i.e. it is exact.

This theorem allows for a straightforward algorithm to find invariant volumes in nonholonomic systems; one only needs to compute the 1-form \( \vartheta_{\mathcal{C}} \) and determine whether or not it can be made exact by appending constraints to it. This procedure will be carried out on multiple examples in §8.8.

**Remark 7.** In the pure kinetic energy case discussed in [8], it is proved that if the system admits an (arbitrary) invariant volume, then one can always find another invariant volume form whose density function depends only on the (reduced) configuration variables.

The above shows that exactness of \( \vartheta_{\mathcal{C}} \) determines the existence of a density depending on configuration. How does this depend on the choice of \( \mathcal{C} \) to realize the constraints? It turns out the answer is independent of the choice of realization.

**Theorem 5.4.** Let \( \mathcal{C} \) and \( \mathcal{C}' \) both be natural realizations of the constraint \( D^* \). If \( \vartheta_{\mathcal{C}} + \rho \) is exact, then there exists \( \rho' \) such that \( \vartheta_{\mathcal{C}'} + \rho' \) is too. Moreover, if \( \vartheta_{\mathcal{C}} + \rho = df \) and \( \vartheta_{\mathcal{C}'} + \rho' = df' \), then \( e^f \cdot \mu_{\mathcal{C}} = e^{f'} \cdot \mu_{\mathcal{C}'} \), modulo a constant of motion.
Proof. Suppose, as was in the proof of Proposition 3, that there is a single constraint such that \( \mathcal{C} = \{ P(W) \} \) and \( \mathcal{C}' = \{ h \cdot P(W) \} \). Computing \( \vartheta_{\mathcal{C}'} \) gives:

\[
\vartheta_{\mathcal{C}'} = \frac{1}{h^2 m} \mathcal{L}_{\mathcal{H}W} (h\eta) = \frac{1}{h^2 m} (h^2 \cdot \mathcal{L}W \eta + hm \cdot dh + h \cdot dh(W) \eta) = \frac{1}{m} \mathcal{L}W \eta + \frac{1}{h} dh(W) \eta = \vartheta_{\mathcal{C}} + d[\ln h] + \alpha \cdot \eta.
\]

This shows that \( \vartheta_{\mathcal{C}'} \) and \( \vartheta_{\mathcal{C}} \) differ by something exact and something living in \( \mathcal{D}^0 \). The component in \( \mathcal{D}^0 \) can be disregarded as it is absorbed into \( \rho' \). Integrating gives \( f' = f + \ln h \) and it remains to prove that \( h\mu_{\mathcal{C}'} \equiv \mu_{\mathcal{C}} \). Recalling Definition 4.1, we have \( \sigma_{\mathcal{C}} = dP(W) \) and \( \sigma_{\mathcal{C}'} = P(W)dh + hdP(W) \), so

\[
dP(W) \wedge \varepsilon = (P(W)dh + hdP(W)) \wedge \varepsilon' = \omega^n, \quad \mu_{\mathcal{C}} = \iota^* \varepsilon, \quad \mu_{\mathcal{C}'} = \iota^* \varepsilon'.
\]

Using the fact that \( P(W) = 0 \) under the pullback of \( \iota \), this component can be ignored and we have \( \mu_{\mathcal{C}} = h\mu_{\mathcal{C}'} \).

\[ \square \]

**Remark 8.** It is only possible for \( \varepsilon^{f} \cdot \mu_{\mathcal{C}} \) and \( \varepsilon^{f'} \cdot \mu_{\mathcal{C}'} \) to be off by a constant of motion if there exists an exact form in \( \Gamma(\mathcal{D}^0) \). This only happens if the constraints are not completely nonintegrable.

A reason why studying \( \vartheta_{\mathcal{C}} \) is insightful is that it immediately demonstrates why holonomic systems are measure-preserving. This can be shown with the help of a useful lemma.

**Lemma 5.5.** \( m_{\alpha \beta} \cdot dmn^{\alpha \beta} = d[\ln \det (m^{\alpha \beta})] \).

**Proof.** It suffices to check along a curve in the manifold. Let \( \gamma : I \to Q \) be a curve and let \( A(t) = (m^{\alpha \beta}) \circ \gamma(t) \) be the mass matrix along the curve. Note that \( A(t) \) is positive-definite and changes smoothly with \( t \). We have

\[
\frac{d}{dt} \ln \det A(t) = \frac{A^{\dagger} \det A(t)}{det A(t)} = \sum_{i=1}^{m} \frac{A_i(t)}{det A(t)}
\]

where \( A_i(t) \) is obtained from \( A(t) \) by differentiating the \( i \)-th row and leaving all other rows intact, i.e.

\[
A_i(t) = \begin{pmatrix}
a_{11}(t) & \cdots & a_{1m}(t) \\
\vdots & \ddots & \vdots \\
a_{(i-1)1}(t) & \cdots & a_{(i-1)m}(t) \\
a'_{11}(t) & \cdots & a'_{1m}(t) \\
a_{(i+1)1}(t) & \cdots & a_{(i+1)m}(t) \\
\vdots & \ddots & \vdots \\
a_{m1}(t) & \cdots & a_{mm}(t)
\end{pmatrix}.
\]

Expanding \( \det A_i(t) \) along the \( i \)-th row:

\[
\det A_i(t) = \sum_{j=1}^{m} (-1)^{i+j-1} a'_{ij}(t) \det A_{ij}(t),
\]

\[ \footnote{We thank Dr. Alexander Barvinok for help with this proof.} \]
where $A_{ij}(t)$ is the $(m-1) \times (m-1)$ matrix obtained from $A_i(t)$ and hence from $A(t)$ by crossing out the $i$-th row and $j$-th column.

Next, observe that $(-1)^{i+j-1} \det A_{ij}/ \det A(t)$ is the $(j,i)$-th entry of the inverse matrix $A^{-1}(t) = (b_{ij})(t)$, and since $A(t)$ is symmetric, is also the $(i,j)$-th entry of $(b_{ij})(t)$. Summarizing,

$$
d dt \ln \det A(t) = \sum_{i,j=1}^m a'_{ij}(t)b_{ij}(t).$$

\[\square\]

**Proposition 7.** If the constraints are holonomic, then there exists a $\rho \in \Gamma(D^0)$ such that $\vartheta_C + \rho$ is exact. In particular, if $C$ is chosen such that all $\eta^\alpha$ are closed, $\vartheta_C$ is exact.

**Proof.** When the constraints are holonomic, the 1-forms $\eta^\alpha$ can be chosen such that they are closed. Then the density form is

$$\vartheta_C = m_{\alpha\beta} \left( dW_{\beta} \eta^\alpha + i_{W_{\beta}} d\theta^\alpha \right)$$

which is exact by Lemma 5.5. \[\square\]

5.3. **Example: The Chaplygin Sleigh.** As an example of Theorem 5.3, we will prove that no invariant volumes exist for the Chaplygin sleigh (where the density depends only on the configuration variables).

The Chaplygin sleigh is a nonholonomic on the configuration $Q = SE_2$, the special Euclidean group, and has the following Lagrangian

$$L = \frac{1}{2} \left( m\dot{x}^2 + m\dot{y}^2 + (I + ma^2) \dot{\theta}^2 - 2ma\dot{x}\sin \theta + 2ma\dot{y}\cos \theta \right),$$

where $(x, y) \in \mathbb{R}^2$ is the coordinate of the contact point, $\theta \in SO_2$ is its orientation, $m$ is the sleigh’s mass, $I$ is the moment of inertia about the center of mass, and $a$ is the distance from the center of mass to the contact point (cf. §1.7 in [3]).

The nonholonomic constraint is that the sleigh can only slide in the direction it is pointing and is given by

$$\dot{y}\cos \theta - \dot{x}\sin \theta = 0,$$

which corresponds to the 1-form $\eta = (\cos \theta) dy - (\sin \theta) dx$.

We wish to compute $\vartheta_C$ for the Chaplygin sleigh and show that no volumes depending on configuration exist. For this example,

$$W = \frac{ma^2 + I}{I m} \left[ \cos \theta \frac{\partial}{\partial y} - \sin \theta \frac{\partial}{\partial x} \right] - a \frac{\partial}{I \partial \theta}, \quad \eta = (\cos \theta) dy - (\sin \theta) dx.$$

This gives us

$$\vartheta_C = \frac{1}{\eta(W)} \mathcal{L}_W \eta = \frac{ma}{ma^2 + I} \left[ (\sin \theta) dy + (\cos \theta) dx \right].$$

As a consequence of this, the divergence of the Chaplygin sleigh is given by

$$\text{div}_{\mu_C}(X^\eta_H) = -\frac{3ma}{I + ma^2}, \quad v = \dot{x}\cos \theta + \dot{y}\sin \theta. \quad (11)$$
We want to show that for any $\tilde{\eta} \in \Gamma(D^0)$, $\vartheta_\varphi + \tilde{\eta}$ is not exact. Because there is only one constraint, it suffices to show that there does not exist a smooth $k$ such that $\vartheta_\varphi + k \cdot \eta$ is exact, i.e. it requires the following to be zero:

\[
d(\vartheta_\varphi + k \cdot \eta) = \frac{ma}{ma^2 + I} \left[ (\cos \theta) d\theta \wedge dy - (\sin \theta) d\theta \wedge dx \right] + \left( \frac{\partial k}{\partial x} \cos \theta + \frac{\partial k}{\partial y} \sin \theta \right) dx \wedge dy + \left( \frac{\partial k}{\partial \theta} \cos \theta - k \sin \theta \right) d\theta \wedge dx - \left( \frac{\partial k}{\partial \theta} \sin \theta + k \cos \theta \right) d\theta \wedge dx.
\]

Separating the above, we need the following three to vanish:

\[
\begin{align*}
0 &= \frac{\partial k}{\partial x} \cos \theta + \frac{\partial k}{\partial y} \sin \theta, \\
0 &= \frac{\partial k}{\partial \theta} \cos \theta - k \sin \theta + \frac{ma}{ma^2 + I} \cos \theta, \\
0 &= \frac{\partial k}{\partial \theta} \sin \theta + k \cos \theta + \frac{ma}{ma^2 + I} \sin \theta.
\end{align*}
\]

The second two lines of (12) are overdetermined for $k$ in the $\theta$-direction and are inconsistent (unless $a = 0$ and we obtain the trivial solution $k \equiv 0$). Therefore, there does not exist a smooth $k$ such that $\vartheta_\varphi + k \cdot \eta$ is closed. We note that this is compatible with the known result that when $a = 0$, no asymptotically stable dynamics occur.

6. Connections with the Nonholonomic Connection. It turns out that the divergence of a nonholonomic system, in particular the density form, is encoded in the nonholonomic connection. This interpretation seems to be new.

Let $(L, Q, \varphi)$ be a natural nonholonomic Lagrangian. The nonholonomic connection for this system is given by (cf. §5.3 in [3] and [27]):

\[
\nabla^\varphi_X Y = \nabla_X Y + W^i \cdot m_{ij} \left[ X \eta^j(Y) - \eta^j(\nabla_X Y) \right].
\]

The equations of motion can then be described via

\[
\nabla^\varphi_{\dot{q}} \dot{q} = F,
\]

where $F$ contains the forces (including the potential forces).

6.1. Torsion. The nonintegrability of the constraints appears in the torsion of the connection. Computing this, we see

\[
T^\varphi(X, Y) = \nabla^\varphi_X Y - \nabla^\varphi_Y X - [X, Y]
= W^i \cdot m_{ij} \left[ X(\eta^j(Y)) - Y(\eta^j(X)) - \eta^j(\nabla_X Y - \nabla_Y X) \right]
= W^i \cdot m_{ij} \left[ X(\eta^j(Y)) - Y(\eta^j(X)) - \eta^j([X, Y]) \right]
= W^i \cdot m_{ij} \cdot d\eta^j(X, Y).
\]

The torsion can be written as

\[
T^\varphi = m_{\alpha\beta} \cdot W^\alpha \otimes d\eta^\beta.
\]
Indeed, if the constraining 1-forms $\eta^i$ are all closed (so holonomic) then the torsion vanishes. It is worth pointing out that the torsion is vertical-valued; if $X, Y \in D$, then $T^e(X, Y) \in D^\perp$.

Due to the fact that the torsion is a $(1,2)$-tensor, its trace will be a $(0,1)$-tensor. Therefore, the trace of the nonholonomic torsion will be a 1-form:

$$\text{tr} T^e = m_{\alpha\beta} \cdot i_{W^\alpha} d\eta^\beta.$$ 

Returning to the density form, we see that

$$\text{tr} T^e + d \ln \det (m^{\alpha\beta}) = \vartheta^e,$$

i.e. the trace of the torsion differs from the density form by something exact. This leads to the following theorem.

**Theorem 6.1.** A natural nonholonomic system $(\mathcal{Q}, L, \mathcal{C})$ has an invariant volume of the form $(\pi^*_Q f) \cdot \mu^e$ if and only if there exists a $\rho \in \Gamma(D^0)$ such that

$$\text{tr} T^e + \rho$$

is exact.

**Remark 9.** The vanishing of the torsion shows that the constraints are integrable while the integrability of the (trace of the) torsion shows that a volume is preserved.

In the case of nonholonomic systems, the nonholonomic connection is compatible with the metric but has nonzero torsion. This idea extends to arbitrary, metric-compatible connections as the following theorem states.

**Theorem 6.2.** Let $\bar{\nabla}$ be an affine connection compatible with the metric with torsion $\bar{T}$. There exists an invariant volume with density of the form $\pi^*_Q f$ if and only if $\text{tr} \bar{T}$ is exact.

**Proof.** Consider the volume form on $TQ$ given by

$$\Omega = \det g \cdot dx^1 \wedge \ldots \wedge dx^n \wedge dv^1 \wedge \ldots \wedge dv^n.$$

We want to compute $\mathcal{L}_X \Omega$ where $X$ is the geodesic spray given by

$$X = v^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk} v^j v^k \frac{\partial}{\partial v^i}.$$

The Lie derivative is then

$$\mathcal{L}_X \Omega = d_i X \Omega = \frac{1}{\det g} \Omega,$$

and therefore the divergence is given by

$$\text{div}_\Omega(X) = d [\ln \det g](v) - \left( \Gamma^i_{jk} + \Gamma^i_{kj} \right) v^k. \quad (13)$$

We will now use the fact that the connection is compatible with the metric:

$$\frac{\partial g_{jk}}{\partial x^l} = \Gamma_{lk}^j + g_{jl} \Gamma_{ik}^l.$$

This implies that

$$g_{jk} \frac{\partial g_{ij}}{\partial x^l} = g_{ik} \Gamma_{lj}^j + g_{jl} \Gamma_{ik}^l = \delta^l_i \Gamma_{lj}^j + \delta^l_k \Gamma_{ij}^j = 2 \Gamma_{ik}^l.$$
Integrating the left-hand side above gives
\[ d \ln \det g_i^j(v) = 2\Gamma^i_{kj} v^k. \quad (14) \]
Substituting (14) into (13), we get
\[ \text{div}_\Omega(X) = (\Gamma^i_{kj} - \Gamma^j_{ik}) v^k. \]
It remains to show that this is the trace of the torsion. Indeed,
\[ \bar{T} = (\Gamma^k_{ij} - \Gamma^k_{ji}) \frac{\partial}{\partial x^k} \otimes dx^i \otimes dx^j \]
\[ \implies \text{tr} \bar{T} = (\Gamma^i_{ki} - \Gamma^i_{ik}) dx^k. \]
We conclude that
\[ \text{div}_\Omega(X) = \text{tr} \bar{T}(v). \]

This shows that a way to interpret the torsion of a connection is by measuring
how much the geodesic spray fails to preserve volume.

7. Application to the Suslov Problem. The last section demonstrated how the
density form can be interpreted arising from the torsion of a connection. This section
examines how the density form applies to the Suslov problem of a nonholonomic
system evolving on a Lie group.

7.1. Invariant Volumes in Poisson Manifolds. Our starting point for examin-
ing invariant volumes for nonholonomic systems was Liouville’s theorem (Theorem
2.3). However, when dealing with a general Poisson manifold, this theorem no
longer applies. This stems from the fact that symplectic manifolds have a distin-
guished volume form, \( \omega_n \), while Poisson manifolds do not. We recall the definition
of a Poisson manifold. These results are similar to previous results on mechanical
systems on Lie algebroids, cf. [20].

Definition 7.1. Let \( P \) be a manifold and consider a bracket operation denoted by
\[ \{ \cdot, \cdot \} : C^\infty(P) \times C^\infty(P) \to C^\infty(P). \]
The pair \( (P, \{ \cdot, \cdot \}) \) is called a Poisson manifold if the bracket is \( \mathbb{R} \)-bilinear, anti-
commutative, satisfies Jacobi’s identity, and satisfies Leibniz’ rule.

Notice that, unlike symplectic manifolds, the bracket is allowed to be degenerate.
The degeneracy of the bracket is what inhibits the creation of a distinguished volume
form.

In order to study volume preservation in Poisson manifolds, we look at the “mod-
ular vector field” [28].

Definition 7.2. Let \( \mu \in \Omega^{\dim P}(P) \) be a volume form. Let us define the derivation
\( \mathcal{M}_\mu : C^\infty(P) \to C^\infty(P) \) via
\[
\begin{array}{ccc}
C^\infty(P) & \xrightarrow{\cdot} & \mathfrak{X}(P) \xrightarrow{\cdot, f} C^\infty(P) \\
f & \mapsto & \{\cdot, f\} \\
X & \mapsto & \text{div}_\mu(X).
\end{array}
\]
As \( \mathcal{M}_\mu \) is a derivation, it is a vector field called the modular field. If the modular
field is Hamiltonian, the Poisson manifold is said to be unimodular.
Proposition 8. Hamiltonian systems are volume preserving if and only if the Poisson manifold is unimodular.

Proof. Suppose that there exists a density $f \in C^\infty(P)$ such that all Hamiltonian flows preserve $f\mu$. Therefore,

$$0 = \text{div}_{f\mu}(X_h) = \text{div}_{\mu}(X_h) + \frac{1}{f} \mathcal{L}_X_h f$$

This tells us that

$$\text{div}_{\mu}(X_h) = -d(\ln f)(X_h) = \{h, \ln f\},$$

and therefore the modular field is Hamiltonian. The opposite direction follows. \hfill \square

7.2. Lie-Poisson Equations. A primary class of examples of Poisson manifolds which are not symplectic are the dual spaces to Lie algebras.

Proposition 9. The dual space $\mathfrak{g}^*$ is a Poisson manifold with the following bracket

$$\{f, k\}_\lambda(p) = -\langle p, [df, dk]\rangle = -p_k \cdot c_{ij}^k \cdot \frac{\partial f}{\partial p_i} \cdot \frac{\partial k}{\partial p_j},$$

where $\mathfrak{g}$ is identified with $\mathfrak{g}^{**}$.

For a Hamiltonian function, $h : \mathfrak{g}^* \to \mathbb{R}$, the equations of motion (the Lie-Poisson equations) are given by

$$\frac{dp}{dt} = \text{ad}_{\mu}^* dh.$$

(15)

It turns out that volume-preservation of the Lie-Poisson equations can be understood by the algebraic properties of $\mathfrak{g}$.

Definition 7.3. A Lie algebra, $\mathfrak{g}$, is called unimodular if $\text{tr ad} : \mathfrak{g} \to \mathbb{R}$ vanishes.

Unimodular Lie algebras and Poisson manifolds are closely related as shown in the following well-known result.

Theorem 7.4. $\mathfrak{g}^*$ is unimodular as a Poisson manifold if and only if $\mathfrak{g}$ is unimodular as a Lie algebra.

Proof. We wish to show, first, that the modular vector field is given by the constant vector field $\text{tr ad} \in \mathfrak{g}^*$. Choose a basis $\{e^k\} \in \mathfrak{g}^*$ where $p = p_k e^k$ and let $\mu = dp_1 \wedge \ldots \wedge dp_n$ be our volume form. For a function $f : \mathfrak{g}^* \to \mathbb{R}$, its Hamiltonian vector field is given by

$$X_f = p_k \cdot c_{ij}^k \cdot \frac{\partial f}{\partial p_i} \cdot \frac{\partial}{\partial p_j}.$$

The divergence of $X_f$ is given by

$$\text{div}_{\mu}(X_f) = \frac{\partial}{\partial p_j} \left(p_k \cdot c_{ij}^k \cdot \frac{\partial f}{\partial p_i}\right) = c_{ij}^k \cdot \frac{\partial f}{\partial p_i} + p_k \cdot c_{ij}^k \cdot \frac{\partial^2 f}{\partial p_i \partial p_j}.$$

However, the second term vanishes as mixed partials are equal and $c_{ij}^k = -c_{ji}^k$. We have

$$\text{div}_{\mu}(X_f) = \text{tr ad}(df).$$

It remains to show that if $\text{tr ad}(df) = \{h, f\}_\lambda$, then $\text{tr ad} \equiv 0$. Assume that $\text{tr ad} = \nu \in \mathfrak{g}^*$, then for every $f$ we have

$$\langle p, [dh, df]\rangle = \langle \nu, df \rangle.$$
Choose \( f \) such that \( \langle \nu, df \rangle(0) \neq 0 \). Then we have
\[
0 = -\langle 0, [dh, df] \rangle = \langle \nu, df \rangle \neq 0,
\]
which is a contradiction. \( \square \)

7.3. **Euler-Poincaré-Suslov Equations.** The Lie-Poisson equations (15) describe unconstrained left-invariant systems on a Lie group. Introducing left-invariant constraints yield the Euler-Poincaré-Suslov equations [7]. We will make the assumption that \( h \) is natural, i.e.
\[
h = \frac{1}{2} I_{ij} p_i p_j,
\]
where \( (I_{ij}) \) is assumed to be positive-definite and symmetric. Suppose we have a constraint distribution (a subspace) of the form
\[
D = \bigcap \ker \eta^\alpha \subset g, \quad \eta^\alpha \in g^*.
\]
The equations of motion have the form
\[
\frac{dp}{dt} = \text{ad}^* dh p + \lambda_\alpha \eta^\alpha.
\]
The multipliers can be explicitly solved (reminiscent of (4)) to obtain the following
\[
\frac{dp}{dt} = \text{ad}^* dh p + m_{\alpha\beta} \{ h, P(W^\alpha) \}_\lambda \eta^\beta
\]
where \( W^\alpha \in g \) are related to \( \eta^\alpha \in g^* \) by the fiber derivative (as \( h \) is assumed to be natural) and \( m_{\alpha\beta} = \eta^\alpha (W^\beta) \).

For the divergence calculation, we already know the component corresponding to \( \text{ad}^*_h p \), so we will focus on the reaction forces. In coordinates,
\[
f_\ell = -m_{\alpha\beta} \cdot p_k \cdot c^k_{ij} \cdot \frac{\partial h}{\partial p_i} \cdot W^{\alpha\cdot j} \cdot \eta^\beta.
\]
\[
\sum_\ell \frac{\partial f_\ell}{\partial p_\ell} = -m_{\alpha\beta} \cdot c^k_{ij} \cdot \frac{\partial h}{\partial p_i} \cdot W^{\alpha\cdot j} \cdot \eta^\beta - m_{\alpha\beta} \cdot p_k \cdot c^k_{ij} \cdot \frac{\partial^2 h}{\partial p_i \partial p_\ell} \cdot W^{\alpha\cdot j} \cdot \eta^\beta
\]
\[
= -m_{\alpha\beta} \cdot \eta^\beta ([dh, W^\alpha]) - m_{\alpha\beta} \cdot p_k \cdot c^k_{ij} \cdot [I^{\ell \cdot} \cdot W^{\alpha\cdot j} \cdot \eta^\beta
\]
\[
= -m_{\alpha\beta} \cdot \eta^\beta ([dh, W^\alpha]) - m_{\alpha\beta} \cdot p ( [W^\beta, W^\alpha] )
\]
Calling \( Y^{\alpha\beta} := [W^\alpha, W^\beta] \), and noting that \( I: g \to g^* \), we have the following result. Notice that \( m_{\alpha\beta} Y^{\alpha\beta} = 0 \) as \( Y^{\alpha\beta} \) is skew.

**Theorem 7.5.** The divergence of (16) is given by
\[
\text{div}_\mu (X_h) = \text{tr ad}(v) + m_{\alpha\beta} \cdot \text{ad}^*_{W^\alpha} \eta^\beta(v)
\]
Moreover, volume is preserved if and only if
\[
\vartheta := \text{tr ad} + m_{\alpha\beta} \cdot \text{ad}^*_{W^\alpha} \eta^\alpha \in D^0.
\]
A corollary of this is the well-known result of Kozlov [15] (see also [12] and [30]).

**Corollary 1** ([15]). Let \( G \) be compact, \( \kappa: g \times g \to \mathbb{R} \) be the Killing form, and \( \kappa^\beta: g^* \to g \) be the induced isomorphism. If there is a single constraint \( \eta \in g^* \), then there exists and invariant volume if and only if
\[
[I^{-1} \eta, \kappa^\beta \eta] = a \cdot \kappa^\beta \eta,
\]
i.e. \( \kappa^\beta \eta \) is an eigenvector of \( \text{ad}_{I^{-1}}. \eta \).
Proof. As $G$ is compact, $\mathfrak{g}$ is unimodular. Additionally, as there is only a single constraint, the divergence (17) has the form

$$\text{div}_\mu(X_h) = \frac{1}{\eta(\mathcal{I}^{-1}\eta)} \cdot \text{ad}^*_{\mathcal{I}^{-1}\eta}\eta(v).$$

Therefore, there exists an invariant volume if and only if

$$\text{ad}^*_{\mathcal{I}^{-1}\eta}\eta \in D^0 = \mathbb{R} \cdot \eta.$$

Using the fact that the Killing form is associative, we have

$$\text{ad}^*_{\mathcal{I}^{-1}\eta}\eta(v) = \eta \left( [\mathcal{I}^{-1}\eta, v] \right) = \kappa \left( [\kappa^2\eta, \mathcal{I}^{-1}\eta, v] \right) = \kappa \left( [\kappa^2\eta,\mathcal{I}^{-1}\eta], v \right).$$

It follows that $\text{ad}^*_{\mathcal{I}^{-1}\eta}\eta = a \cdot \eta$ if and only if $[\mathcal{I}^{-1}\eta,\kappa^\#\eta] = a \cdot \kappa^\#\eta$. \hfill $\square$

**Corollary 2.** Consider the case in Corollary 1 above except there are multiple constraints $\{\eta^\alpha\} \in \mathfrak{g}^*$. Then there exists an invariant volume if and only if

$$m_{\alpha\beta} \left[ \mathcal{I}^{-1}\eta^\alpha, \kappa^\#\eta^\beta \right] \in \text{span} \left\{ \kappa^\#\eta^\alpha \right\}.$$

In the theory of spinning tops, a totally symmetric top corresponds to a bi-invariant metric on $\mathfrak{g}$. When this happens, the Euler-Poincaré-Suslov equations are volume preserving independent of the choice of $\mathcal{D}$.

**Corollary 3.** Suppose that $\mathcal{I}$ is bi-invariant. Then for any subspace, $\mathcal{D} \subset \mathfrak{g}$, there exists an invariant volume.

Proof. A theorem of Milnor (cf. lemma 7.5 in [21]) states that Lie algebras admitting a bi-invariant metric are unimodular. Additionally, as the metric is bi-invariant, it is associative. The divergence becomes

$$\text{div}_\mu(X_h) = m_{\alpha\beta} \cdot \text{ad}^*_{\mathcal{I}^{-1}\eta^\alpha,\mathcal{I}^{-1}\eta^\beta}(v)$$

$$= m_{\alpha\beta} \cdot \eta^\beta \left( [\mathcal{I}^{-1}\eta^\alpha, v] \right)$$

$$= m_{\alpha\beta} \cdot \mathcal{I}(W^\beta)([\mathcal{I}^{-1}\eta^\alpha, v])$$

$$= m_{\alpha\beta} \cdot \mathcal{I}(W^\beta,W^\alpha)(v).$$

This vanishes because $m_{\alpha\beta} \cdot [W^\beta,W^\alpha] = 0$. \hfill $\square$

7.4. **Example: The Chaplygin Sleigh.** Let us revisit the Chaplygin sleigh as a Euler-Poincaré-Suslov system rather than a general nonholonomic system as earlier to compare results. Recall that the configuration space for this system is $G = \text{SE}(2)$.

The constraint is left-invariant because

$$g = \begin{bmatrix} \cos \theta & -\sin \theta & x \\ \sin \theta & \cos \theta & y \\ 0 & 0 & 1 \end{bmatrix}, \quad g^{-1}\dot{g} = \begin{bmatrix} 0 & -\omega & \dot{x} \cos \theta + \dot{y} \sin \theta \\ \omega & 0 & \dot{y} \cos \theta - \dot{x} \sin \theta \\ 0 & 0 & 0 \end{bmatrix},$$

and the constraint is $\dot{y} \cos \theta - \dot{x} \sin \theta = 0$, i.e. $\mathcal{D} = \ker e_3^\#. \text{ Notice that we cannot utilize Corollary 1 as SE}(2)$ is not compact (but is unimodular). To compute the equations of motion, we translate the Lagrangian to $\mathfrak{se}(2)$ and we obtain

$$\ell = \frac{1}{2} \left( \dot{u}^2 + \dot{v}^2 + (I + ma^2)\omega^2 \right) - ma\omega v,$$
which gives the moment of inertia tensor to be
\[ I = \begin{bmatrix} I + ma^2 & 0 & -ma \\ 0 & m & 0 \\ -ma & 0 & m \end{bmatrix}. \]

The constraint (and dual vector) are given by
\[ \eta = e_3^*, \quad W = \frac{a}{I} e_1 + \frac{I + ma^2}{Im} e_3. \]

Computing the divergence gives
\[ \vartheta = \frac{1}{\eta(W)} \text{ad}^*_W \eta = -\frac{ma}{I + ma^2} e_2^* \notin D^0. \]

Therefore, no invariant volumes exist (which is compatible with the observation in §5.3). Also note that the divergence here agrees with the divergence in (11),
\[ \vartheta(\dot{q}) = -\frac{mav}{I + ma^2}. \]

8. Examples. We end this work with applying Theorem 1.1 to various nonholonomic systems. The idea is to compute \( \vartheta_\varepsilon \) for the examples below to determine whether or not an invariant volume exists. Each example will come equipped with a Lagrangian, \( L \), and a collection of constraining 1-forms, \( \eta^\alpha \). Recall that \( \vartheta_\varepsilon \) need not be exact to guarantee the existence of an invariant volume, merely that there exists a collection of smooth \( f_\alpha \) such that
\[ \vartheta_\varepsilon + f_\alpha \cdot \eta^\alpha, \]
is exact. These examples are taken from [3] and [23].

8.1. The Vertical Rolling Disk. The first example we will consider is that of the vertical rolling disk. The Lagrangian is given by
\[ L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} J \dot{\phi}^2 \]
with constraints
\[ \eta^1 = dx - R (\cos \varphi) d\theta, \]
\[ \eta^2 = dy - R (\sin \varphi) d\theta. \]
Here, \( m \) is the mass of the disk, \( I \) is the moment of inertia of the disk about the axis perpendicular to the plane of the disk, \( J \) is the moment of inertia about an axis in the plane of the disk, and \( R \) is the radius of the disk. The Lagrangian for this system is only the kinetic energy. The corresponding vector fields are given by

\[
W^1 = \frac{1}{m} \frac{\partial}{\partial x} - \frac{1}{I} R \cos \varphi \frac{\partial}{\partial \theta},
\]

\[
W^2 = \frac{1}{m} \frac{\partial}{\partial y} - \frac{1}{I} R \sin \varphi \frac{\partial}{\partial \theta}.
\]

The constraint mass matrix is given by

\[
(m^\alpha_{\beta}) = \begin{bmatrix}
\frac{1}{m} + \frac{1}{I} R^2 \cos^2 \varphi & \frac{1}{I} R^2 \cos \varphi \sin \varphi \\
\frac{1}{I} R^2 \cos \varphi \sin \varphi & \frac{1}{m} + \frac{1}{I} R^2 \sin^2 \varphi
\end{bmatrix}.
\]

The four Lie derivatives are

\[
\mathcal{L} W^1 \eta_1 = -\frac{R^2}{I} (\cos \varphi \sin \varphi) \, d\varphi,
\]

\[
\mathcal{L} W^2 \eta_1 = \frac{R^2}{I} (\cos^2 \varphi) \, d\varphi,
\]

\[
\mathcal{L} W^1 \eta_2 = -\frac{R^2}{I} (\sin^2 \varphi) \, d\varphi,
\]

\[
\mathcal{L} W^2 \eta_2 = \frac{R^2}{I} (\cos \varphi \sin \varphi) \, d\varphi.
\]

This leads to \( \vartheta = 0 \) and therefore volume is preserved for the vertical rolling disk.

8.2. The Falling Rolling Disk. The next example is a physical extension of the previous example where the disk is now allowed to tilt. The Lagrangian is

\[
L = \frac{m}{2} \left[ \left( \xi - R \left( \dot{\varphi} \sin \theta + \dot{\psi} \right) \right)^2 + \eta^2 \sin^2 \theta + \left( \eta \cos \theta + R \dot{\theta} \right)^2 \right] + \frac{1}{2} \left[ J \left( \ddot{\varphi} + \dot{\varphi}^2 \cos^2 \theta \right) + I \left( \dot{\varphi} \sin \theta + \dot{\psi} \right)^2 \right] - mgR \cos \theta,
\]
where
\[ \xi = \dot{x} \cos \varphi + \dot{y} \sin \varphi + R \dot{\psi}, \quad \eta = -\dot{x} \sin \varphi + \dot{y} \cos \varphi. \]

The constraints are
\[
\eta^1 = \cos \varphi \cdot dx + \sin \varphi \cdot dy + R \cdot d\psi, \\
\eta^2 = -\sin \varphi \cdot dx + \cos \varphi \cdot dy.
\]

Here \( m, R, I, \) and \( J \) are all the same as in the vertical rolling disk. As the disk is now allowed to fall, a potential energy term is added where \( g \) is the acceleration due to gravity. The corresponding vector fields are
\[
W^1 = \frac{1}{m} \cos \varphi \frac{\partial}{\partial x} + \frac{1}{m} \sin \varphi \frac{\partial}{\partial y} + \frac{R}{I} \frac{\partial}{\partial \psi}, \\
W^2 = \frac{J + mR^2}{Jm + m^2 R^2 \sin^2 \theta} \left[ -\sin \varphi \frac{\partial}{\partial x} + \cos \varphi \frac{\partial}{\partial y} \right] - \frac{R \cos \theta}{Jm + m^2 R^2 \sin^2 \theta} \frac{\partial}{\partial \theta}. 
\]

The constraint mass matrix for these constraints is diagonal,
\[
(m^{\alpha \beta}) = \begin{bmatrix}
\frac{1}{m} + \frac{R^2}{I} & 0 \\
0 & \frac{J + mR^2}{Jm + m^2 R^2 \sin^2 \theta}
\end{bmatrix}.
\]

Due to the fact that the constraint mass matrix is diagonal, we only care about two Lie derivatives:
\[
\mathcal{L}_{W^1} \eta^1 = 0, \\
\mathcal{L}_{W^2} \eta^2 = -\frac{R^2(J + mR^2) \sin(2\theta)}{(J + mR^2 \sin^2 \theta)^2}.
\]

The corresponding density form is
\[
\vartheta_C = -\frac{2mR^2 \sin(2\theta)}{2J + mR^2 - mR^2 \cos(2\theta)} d\theta,
\]
which, although nonzero, is exact. Therefore \textbf{volume is preserved for the falling disk with density}
\[
\rho = \frac{K}{J + mR^2 \sin^2 \theta},
\]
where \( K \) is some constant.

8.3. \textbf{The Rolling Ball}. The next example is that of the (homogeneous) rolling ball. The Lagrangian is the kinetic energy and is given by
\[
L = \frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 + k^2 \left( \dot{\theta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2 \dot{\varphi} \dot{\psi} \cos \theta \right) \right),
\]
with constraints
\[
\eta^1 = dx - r \sin \psi \cdot d\theta + r \sin \theta \cos \psi \cdot d\varphi, \\
\eta^2 = dy + r \cos \psi \cdot d\theta + r \sin \theta \sin \psi \cdot d\varphi,
\]
The roller racer, cf. §1.11 in [3].

where $r$ is the radius of the ball and $k$ contains the inertial terms (cf. §3.2.2 in [23]). The corresponding vector fields are

$$W^1 = \frac{\partial}{\partial x} - \frac{r}{k^2} \sin \psi \frac{\partial}{\partial \theta} + \frac{r \cos \psi}{k^2 \sin \theta} \frac{\partial}{\partial \phi} - \frac{r \cos \psi \cos \theta}{k^2 \sin \theta} \frac{\partial}{\partial \psi},$$

$$W^2 = \frac{\partial}{\partial y} + \frac{r}{k^2} \cos \psi \frac{\partial}{\partial \theta} + \frac{r \sin \psi}{k^2 \sin \theta} \frac{\partial}{\partial \phi} - \frac{r \sin \psi \cos \theta}{k^2 \sin \theta} \frac{\partial}{\partial \psi}.$$

The constraint mass matrix for this system is

$$\left( m^{\alpha \beta} \right) = \begin{bmatrix} 1 + \frac{r^2}{k^2} & 0 \\ 0 & 1 + \frac{r^2}{k^2} \end{bmatrix},$$

i.e. the constraints are “orthogonal.” In this case, we only need to calculate two Lie derivatives. These are given by

$$\mathcal{L}_{W^1} \eta^1 = \mathcal{L}_{W^2} \eta^2 = 0.$$

Therefore, $\vartheta = 0$ and volume is preserved for the rolling ball.

8.4. The Heisenberg System. The next example is the Heisenberg system which is associated with the homonymous Lie algebra. The Lagrangian is the standard kinetic energy on $\mathbb{R}^3$,

$$L = \frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right),$$

with the constraint

$$\eta = ydx - xdy - dz.$$

The corresponding vector field is

$$W = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} - \frac{\partial}{\partial z}.$$

The density form is given by

$$\vartheta = \frac{1}{1 + x^2 + y^2} \mathcal{L}_W \eta = 0.$$

Therefore, volume is preserved for the Heisenberg system.
8.5. The Roller Racer. This example is the roller racer, a tricycle-like mechanical system. The Lagrangian is the kinetic energy of the system,

\[ L = \frac{m}{2} (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_2 \left( \dot{\theta} + \dot{\varphi} \right)^2, \]

with the constraints

\[ \eta^1 = dx - \cos \theta \left( \frac{d_1 \cos \varphi + d_2}{\sin \varphi} \cdot d\theta + \frac{d_2}{\sin \varphi} \cdot d\varphi \right), \]

\[ \eta^2 = dy - \sin \theta \left( \frac{d_1 \cos \varphi + d_2}{\sin \varphi} \cdot d\theta + \frac{d_2}{\sin \varphi} \cdot d\varphi \right). \]

Here, \( m \) is the mass of the first body, and the moments of inertia of the two bodies are given by \( I_i \). The corresponding vector fields are

\[ W^1 = \frac{1}{m} \frac{\partial}{\partial x} - \frac{d_1 \cos \varphi \cos \theta}{I_1 \sin \varphi} \frac{\partial}{\partial \theta} - \cos \theta \cdot \frac{I_1 d_2 - I_2 d_1 \cos \varphi}{I_1 I_2 \sin \varphi} \frac{\partial}{\partial \varphi}, \]

\[ W^2 = \frac{1}{m} \frac{\partial}{\partial y} - \frac{d_1 \cos \varphi \sin \theta}{I_1 \sin \varphi} \frac{\partial}{\partial \theta} - \sin \theta \cdot \frac{I_1 d_2 - I_2 d_1 \cos \varphi}{I_1 I_2 \sin \varphi} \frac{\partial}{\partial \varphi}. \]

The density form is calculated to be

\[ \vartheta_\varphi = \frac{m}{\lambda} (I_1 d_2 - I_2 d_1 \cos \varphi)(1 + d_2 \cos \varphi)d\theta \]

\[ - \frac{m}{\lambda} \cos \varphi \left( I_2 d_1^2 - I_2 d_1 d_2 \cos \varphi + 2I_1 d_2^2 \right) d\varphi, \]

where

\[ \lambda = \sin \varphi \left( I_2 m d_1^2 \cos^2 \varphi + I_1 m d_2^2 - I_1 I_2 \cos^2 \varphi + I_1 I_2 \right). \]

The density form is clearly not exact (nor is it closed). This is consistent with the asymptotic nature of the roller racer, cf. §8.4.5 in [3]. However, this does not necessarily rule out the existence of an invariant volume. Indeed, we need to look at all 1-forms \( \vartheta_\varphi + \rho \) where \( \rho \in \Gamma(D^0) \). Disappointingly, this is quite difficult as we need to solve a coupled transport equation. We will avoid this computation in this work.

8.6. The Chaplygin Sphere. This example is the Chaplygin sphere, a non-homogeneous rolling ball. The Lagrangian is the kinetic energy,

\[ L = \frac{1}{2} I_1 \left( \dot{\theta} \cos \psi + \dot{\varphi} \sin \psi \sin \theta \right)^2 + \frac{1}{2} I_2 \left( -\dot{\theta} \sin \psi + \dot{\varphi} \cos \psi \sin \theta \right)^2 \]

\[ + \frac{1}{2} I_3 \left( \dot{\psi} + \dot{\varphi} \cos \theta \right)^2 + \frac{1}{2} M (\dot{x}^2 + \dot{y}^2), \]

with constraints

\[ \eta^1 = dx - \sin \varphi \cdot d\theta + \cos \varphi \sin \theta \cdot d\psi, \]

\[ \eta^2 = dy + \cos \varphi \cdot d\theta + \sin \varphi \sin \theta \cdot d\psi, \]

where it is assumed that \( r = 1 \). It is interesting that in this case \( \vartheta_\varphi \) is closed. We have \( \vartheta_\varphi = A \cdot d\theta + B \cdot d\psi \) where

\[ A = \frac{M \sin(2\theta) [J_1 + J_2 \sin^2 \psi]}{2 (J_3 + J_4 \sin^2 \theta + J_5 \sin^2 \theta \sin^2 \psi)} \]

\[ B = \frac{M J_3 \sin(2\psi) \sin^2 \theta}{2 (J_3 + J_4 \sin^2 \theta + J_5 \sin^2 \theta \sin^2 \psi)}. \]
The form $\vartheta_\psi$ is closed because
\[
\frac{\partial A}{\partial \psi} = \frac{\partial B}{\partial \theta}.
\]
Integrating yields (done with Matlab symbolic):
\[
\int A \, d\theta = \frac{M(J_4 + J_2 \sin^2 \psi)}{J_4 + J_5 \sin^2 \psi} \cdot \text{arctanh} \left( \frac{\sin^2 \theta (J_4 + J_5 \sin^2 \psi)}{2J_3 + J_4 \sin^2 \theta + J_5 \sin^2 \theta \sin^2 \psi} \right) + K(\psi)
\]
\[
\int B \, d\psi = \frac{J_2 M}{J_5} \cdot \text{arctanh} \left( \frac{J_5 \sin^2 \psi \sin^2 \theta}{2J_3 + 2J_4 \sin^2 \theta + J_5 \sin^2 \theta \sin^2 \psi} \right) + L(\theta)
\]
Comparing the “constants” yields to a solution $\vartheta_\psi = df$ where
\[
f = \frac{J_2 M}{2J_5} \ln (J_3 + (J_3 + J_4) \tan^2 \theta) - \frac{1}{2} \ln (\sec^2 \theta)
\] + \[
\frac{J_2 M}{J_5} \cdot \text{arctanh} \left( \frac{J_5 \sin^2 \psi \sin^2 \theta}{2J_3 + 2J_4 \sin^2 \theta + J_5 \sin^2 \theta \sin^2 \psi} \right).
\]
The invariant density is given by $\rho = \exp(f)$ and is therefore
\[
\rho = \left( \frac{1 + \beta}{1 - \beta} \right)^{\frac{1}{2}} \cdot \cos \theta \cdot (J_3 + (J_3 + J_4) \tan^2 \theta)^{\frac{1}{2}},
\]
where
\[
\beta = \frac{J_5 \sin^2 \theta \sin^2 \psi}{2J_3 + 2J_4 \sin^2 \theta + J_5 \sin^2 \theta \sin^2 \psi}.
\]
Therefore, volume is preserved for the Chaplygin Sphere.

8.7. Chaplygin Sleigh with an Oscillator. The Lagrangian is given by
\[
L = \frac{m}{2} \left( \dot{x}^2 + \dot{y}^2 + \dot{r}^2 + r^2 \dot{\theta}^2 + 2\dot{r} (\dot{x} \cos \theta + \dot{y} \sin \theta) + 2r \dot{\theta} (\dot{y} \cos \theta - \dot{x} \sin \theta) \right)
\] + \[
\frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} M (\dot{x}^2 + \dot{y}^2) - U(r)
\]
with constraint
\[
\eta = \cos \theta \cdot dy - \sin \theta \cdot dx
\]
Remark 10. Notice that potential terms to not enter into the computation at all; the only thing we need are the constraints and the metric.

The dual vector field is
\[
W = \frac{I + mr^2}{Mmr^2 + I(m + M)} \left( - \sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y} \right) - \frac{mr}{Mmr^2 + I(m + M)} \frac{\partial}{\partial \theta},
\]
and the normalizing term is
\[
\eta(W) = \frac{I + mr^2}{I(m + M) + Mmr^2}
\]
Notice that this is not a constant and we therefore have

\[ \vartheta_C = \frac{1}{\eta(W)} \cdot (i_W d\eta + d_W \eta) \]

\[ di_W \eta = \frac{2Im^2r}{(I(m + M) + Mm^2)^2} dr \]

\[ i_W d\eta = \frac{mr \cos \theta}{Mmr^2 + I(m + M)} dx + \frac{mr \sin \theta}{Mmr^2 + I(m + M)} dy \]

\[ \vartheta_C = \frac{mr}{I + mr^2} (\cos \theta dx + \sin \theta dy) + \frac{2Im^2r}{(I + mr^2)(I(m + M) + Mm^2)} dr \]

We can integrate the shape component to obtain

\[ \delta(r) = 2 \arctan \left( \frac{m^2r^2}{2I(m + M) + (2Mm + m^2)r^2} \right). \]

However, form the same reasoning from the Chaplygin sleigh above, we cannot make \( \vartheta_C \) exact. Therefore, there does not exist an invariant volume with density depending only on configuration variables for the Chaplygin sleigh with an oscillator.

8.8. A non-orientable example: Möbius strip. Consider the Möbius strip paramaterized in \( \mathbb{R}^3 \) by

\[ x = \left( 1 + v \cdot \cos \left( \frac{u}{2} \right) \right) \cdot \cos(u), \]

\[ y = \left( 1 + v \cdot \cos \left( \frac{u}{2} \right) \right) \cdot \sin(u), \]

\[ z = v \cdot \sin \left( \frac{u}{2} \right), \]

for \( 0 \leq u < 2\pi \) and \( -1/2 < v < 1/2 \). The Euclidean metric on the Möbius strip is

\[ g = \left( 4v \cos \frac{u}{2} + 2v^2 \cos \frac{u}{2} + \frac{v^2}{2} + 2 \right) du^2 + 2dv^2. \]

When \( \dim Q = 2 \) any single constraint is automatically holonomic. As a consequence, volume will always be preserved. To make this problem more interesting, let us “thicken” the strip by \( w \) so the metric becomes

\[ g_{\text{thick}} = \left( 4v \cos \frac{u}{2} + 2v^2 \cos \frac{u}{2} + \frac{v^2}{2} + 2 \right) du^2 + 2dv^2 + dw^2. \]

For the sake of this example, let us impose the nonholonomic constraint

\[ \eta = dv + \sin(u) \cdot dw, \quad W = \frac{1}{2} \frac{\partial}{\partial v} + \frac{1}{2} \sin(u) \frac{\partial}{\partial w}. \]

The density form is then

\[ \vartheta_C = \frac{\sin(u) \cos(u)}{1 + \sin^2(u)} du, \]

which is exact. Therefore, the exponential of its integral is an invariant density and is given by

\[ \rho = \sqrt{1 + \sin^2(u)}. \]

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