Damage spreading and dynamic stability of kinetic Ising models

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We investigate how the time evolution of different kinetic Ising models depends on the initial conditions of the dynamics. To this end we consider the simultaneous evolution of two identical systems subjected to the same thermal noise. We derive a master equation for the time evolution of a joint probability distribution of the two systems. This equation is then solved within an effective-field approach. By analyzing the fixed points of the master equation and their stability we identify regular and chaotic phases.

The question to what extent the time evolution of a physical system depends on its initial conditions is one of the central questions in nonlinear dynamics that have lead to the discovery of chaotic behavior [1]. In more recent years analogous concepts have been applied to the stochastic time evolution of interacting systems with a macroscopic number of degrees of freedom. Among the simplest of such many-body systems are kinetic Ising models where the above question has been investigated by means of so-called "damage spreading" simulations [2,3]. In these Monte-Carlo simulations two identical systems with different initial conditions are subjected to the same thermal noise, i.e. the same random numbers are used in the Monte-Carlo procedure. The differences in the microscopic configurations of the two systems are then used to characterize the dynamic stability.

Later the name "damage spreading" has also been applied to a different though related type of investigations in which the two systems are not identical but differ in that one or several spins in one of the copies are permanently fixed in one direction. Therefore the equilibrium properties of the two systems are different and the microscopic differences between the two copies can be related to certain thermodynamic quantities [2,3]. Note that in this type of simulations the use of identical noise (i.e. random numbers) for the two systems is not essential but only a convenient method to reduce the statistical error. Whereas this second type of damage spreading is well understood and established as a method to numerically calculate equilibrium correlation functions, much less is known about the original problem of dynamic stability. In particular, the transition between regular and chaotic behavior (called the "spreading transition") is not understood and it is unknown under which conditions it coincides with the equilibrium phase transition of the Ising model. (In numerical simulations it seems to depend on the dynamical algorithm whether the two transitions coincide or not. Glauber dynamics usually gives a spreading temperature which is slightly lower than the equilibrium critical temperature whereas the two are identical for heat-bath dynamics [3,4].)

In this Letter we therefore concentrate on the original question of the stability of the stochastic dynamics in kinetic Ising models. To this end we investigate the time evolution of two identical systems with different initial conditions which are subjected to the same thermal noise. We derive a master equation for the joint probability distribution of the two systems and solve it within an effective-field approach. By analyzing the fixed points of this equation we identify regular and chaotic phases. We find that the location of these phases in the phase diagram is sensitive to the choice of the dynamic algorithm. In particular, Glauber dynamics and heat-bath dynamics give very different dynamical phase diagrams.

We consider two identical kinetic Ising models with $N$ sites, described by the Hamiltonian

$$H = -\frac{1}{2} \sum_{ij} J_{ij} S_i S_j - h \sum_i S_i$$

(1)

where $S_i$ is an Ising variable with the values $\pm 1$. The dynamics is given by one of the two following stochastic maps which describe Glauber dynamics

$$S_i(t+1) = \text{sign}\left\{v[h_i(t)] - \frac{1}{2} + S_i(t)\left[\xi_i(t) - \frac{1}{2}\right]\right\}$$

(2a)

and heat-bath dynamics

$$S_i(t+1) = \text{sign}\{v[h_i(t)] - \xi_i(t)\}$$

(2b)

with

$$v(h) = e^{h/T} / (e^{h/T} + e^{-h/T}).$$

(3)

Here $h_i(t) = \sum_j J_{ij} S_j(t) + h$ is the local magnetic field at site $i$ and (discretized) time $t$, $\xi_i(t) \in [0,1]$ is a random number which is identical for both systems, and $T$ denotes the temperature. Note that Glauber- and heat-bath algorithm differ only in the way how the random numbers are used to update the configuration. The transition probabilities $v$ are identical for both algorithms.

In order to describe the simultaneous time evolution of two systems $H^{(1)}$ and $H^{(2)}$ with Ising spins $S_i^{(1)}$ and...
Using this, the master equation (5) reduces to an equation of motion for the single-site distribution $P_\nu$,

$$\frac{d}{dt} P_\nu = \sum_{\mu \neq \nu} [ - P_\nu W(\nu \to \mu) + P_\mu W(\mu \to \nu) ], \quad (7)$$

where

$$W(\mu \to \nu) = \langle w(\mu \to \nu) \rangle_p$$

is the transition probability averaged over the states $\nu_i$ of all sites according to the distribution $P_\nu$. Note that the average magnetizations $m^{(1)}$, $m^{(2)}$ of the two systems and the Hamming distance (also called the damage)

$$D = \frac{1}{2N} \sum_{i=1}^{N} |S^{(1)}_i - S^{(2)}_i|$$

which measures the distance between the two systems in phase space can be easily expressed in terms of $P_\nu$,

$$m^{(1)} = P_{++} + P_{+-} - P_{-+} - P_{--}, \quad (10a)$$

$$m^{(2)} = P_{++} - P_{+-} + P_{-+} - P_{--}, \quad (10b)$$

$$D = P_{++} + P_{--}. \quad (10c)$$

So far the considerations have been rather general, to be specific we will now concentrate on a two-dimensional system on a hexagonal lattice with a nearest neighbor interaction of strength $J$. The external magnetic field $h$ is set to zero. To solve the master equation (7) for the single-site distribution $P$ we first calculate the transition probabilities $w(\mu \to \nu)$ between the states of a spin pair from one of the stochastic maps (2a) or (2b) and then average these probabilities over the states of the three neighboring sites of a certain reference site with respect to the yet unknown distribution $P$. This yields the transfer rates $W(\mu \to \nu)$ which enter (7). The calculations involved are quite tedious but straightforward, they will be presented in some detail elsewhere.

The resulting system of non-linear equations for the variables $P_{++}$, $P_{+-}$, $P_{-+}$, and $P_{--}$ can first be used to calculate the thermodynamics. As expected, Glauber and heat-bath dynamics give the same results. In particular, there is a ferromagnetic phase transition at a temperature $T_c$ determined by

$$\tanh \frac{3J}{T_c} + \tanh \frac{J}{T_c} = \frac{4}{3}, \quad (11)$$

which gives $T_c/J \approx 2.11$. In the ferromagnetic phase the magnetization is given by

$$m^2 = \frac{3}{4}(\tanh 3J/T + \tanh J/T) - 1 - \frac{3}{4}\tanh 3J/T. \quad (12)$$

We now discuss the time evolution of the Hamming distance $D$ between the two systems which characterizes the stability of the dynamics. In contrast to the thermodynamics Glauber and heat-bath algorithms give very different results for the Hamming distance. We first consider the Glauber case.
The equation of motion of the Hamming distance can easily be derived from (7) and (10c). In the paramagnetic phase we obtain after some algebra

\[ \frac{d}{dt} D = \frac{1}{2} (D - 3D^2 + 2D^3) \tanh \frac{3J}{T}. \tag{13} \]

This equation has three stationary solutions, i.e. fixed points, \( D^s \), viz. \( D_1^s = 0 \) which corresponds to the two systems being identical, \( D_2^s = 1 \) where \( S^{(1)} = -S^{(2)} \) for all sites, and \( D_3^s = 1/2 \) which corresponds to completely uncorrelated configurations. To investigate the stability of these fixed points we linearize (13) in \( d = D - D^s \). The linearized equation has a solution \( d \propto e^{-\lambda t} \) with \( \lambda_1 = \lambda_2 = \frac{1}{2} \tanh 3J/T \) and \( \lambda_3 = -\frac{1}{4} \tanh 3J/T \). Consequently, the only stable fixed point is \( D_3^s = 1/2 \). Thus, in the paramagnetic phase the Glauber dynamics is chaotic, since two systems, starting close together in phase space \( (D \) small initially) will become separated exponentially fast with a Lyapunov exponent \( \lambda_1 \), eventually reaching a stationary state with an asymptotic Hamming distance being zero for temperatures smaller than the spreading temperature \( T \), viz. \( \lambda \frac{T}{J} \) as functions of temperature for the Glauber Ising model. Below \( T_s \) the curve for \( D \) has two branches corresponding to the two systems being in the same or in different free energy valleys.

We now turn to the ferromagnetic phase. In order to find the fixed points of the master equation (7) we can set the magnetizations of both systems to their equilibrium values (12) from the outset. In doing so we exclude, however, all phenomena connected with the behavior after a quench from high temperatures to temperatures below \( T_c \). These phenomena require an investigation of the early time behavior and will be analyzed elsewhere. For \( m^{(1)} = m^{(2)} = m \) the equation of motion for the Hamming distance reads

\[ \frac{d}{dt} D = \frac{1}{2} (D - 3D^2 + 2D^3) \tanh \frac{3J}{T} \]

\[ -\frac{3}{4} m^2 \left( 2D \tanh \frac{J}{T} - D^2 \tanh \frac{J}{T} + D^2 \tanh \frac{3J}{T} \right). \tag{14} \]

This equation has two fixed points \( D^s \) in the interval \( [0,1] \). The first, \( D_1^s = 0 \) exists for all temperatures. The second fixed point \( D_3^s \) with \( 0 < D_3^s < \frac{1}{2} \) exists only for \( T > T_s \) where the spreading temperature \( T_s \) is determined by

\[ 3m^2 \tanh \frac{J}{T_s} = \tanh \frac{3J}{T_s}. \tag{15} \]

This gives \( T_s \approx 1.74 \approx 0.82T_c \). The stability analysis shows that \( D_1^s = 0 \) is stable for \( T < T_s \) and unstable for \( T > T_s \) with a Lyapunov exponent \( \lambda_1 = \frac{1}{2} \tanh 3J/T - \frac{1}{4} m^2 \tanh J/T. \) The fixed point \( D_3^s \) which exists only for \( T > T_s \) is always stable. Consequently, we find that the Glauber dynamics is regular with the asymptotic Hamming distance being zero for temperatures smaller than the spreading temperature \( T_s \) but chaotic for \( T > T_s \). Close to the spreading temperature the asymptotic Hamming distance increases linearly with \( T - T_s \) which corresponds to the spreading transition being of 2nd order. In contrast to the paramagnetic phase, where the two systems become eventually completely uncorrelated, for \( T_s < T < T_c \) the asymptotic Hamming distance \( D \) is always smaller than 1/2 so that the two systems remain partially correlated. Directly at the spreading point the term linear in \( D \) in (14) vanishes. For small Hamming distances the equation of motion now reads \( dD/dt \propto -D^2 \) which gives a power-law behavior \( D(t) \propto t^{-\gamma} \).

Analogously, for \( m^{(1)} = -m^{(2)} = m \) we find two fixed points, \( D_2^s = 1 \) which exists for all temperatures and \( D_3^s \) with \( \frac{1}{2} < D_3^s < 1 \) which exists for \( T > T_s \) only. \( D_2^s \) is stable for temperatures \( T < T_s \) and unstable for \( T > T_s \) whereas \( D_3^s \) is always stable if it exists. The results for damage spreading in the Glauber Ising model within our effective-field approximation are summarized in Fig. 1.

We now investigate the time evolution of the damage for the Ising model with heat-bath dynamics (2b). After calculating the averaged transition rates \( W(\mu \to \nu) \) and inserting them into (7), we obtain the equation of motion for the Hamming distance \( D \). In the paramagnetic phase it reads

\[ \frac{d}{dt} D = \frac{3D^2}{4} \left[ \tanh \frac{3J}{T} + \tanh \frac{J}{T} - 4/3 \right] \]

\[ -\frac{3}{4} m^2 \left[ \tanh \frac{J}{T} + \tanh \frac{J}{T} \right] + \frac{D^3}{4} \left[ \tanh \frac{3J}{T} + 3 \tanh \frac{J}{T} \right]. \tag{16} \]

This equation has only a single fixed point in the physical interval \([0,1]\), viz. \( D_1^s = 0 \). It is stable everywhere in the paramagnetic phase. Consequently, the asymptotic
The equation of motion is given by
\[ \frac{dD}{dt} = \frac{3}{4} \left[ \left( 1 + m^2 \right) \tanh \frac{3J}{T} + \left( 1 - 3m^2 \right) \tanh \frac{J}{T} - 4/3 \right] \]
\[ - \frac{3D^2}{4} \left[ \tanh \frac{3J}{T} + \tanh \frac{J}{T} \right] + \frac{D^3}{4} \left[ \tanh \frac{3J}{T} + 3 \tanh \frac{J}{T} \right]. \]

Here we also obtain only one fixed point \( D_1^* = 0 \) which is stable for all temperatures. The Lyapunov exponent is given by \( \lambda_1 = \frac{3}{4} \left( 1 + m^2 \right) \tanh \frac{3J}{T} + \frac{3}{4} \left( 1 - 3m^2 \right) \tanh \frac{J}{T} - 1 < 0 \). Thus, the Lyapunov exponent is zero for all temperatures smaller than a spreading temperature \( T_c \). The Lyapunov exponent is given by
\[ \lambda = \frac{1}{2} \left( 1 + m^2 \right) \tanh \frac{3J}{T} + \frac{1}{2} \left( 1 - 3m^2 \right) \tanh \frac{J}{T} - 1 < 0. \]

As with any mean-field theory we have, of course, to discuss in which parameter region it correctly describes the physics of our system. Since we treated the fluctuations in a very simplistic way, viz. treating fluctuations at different sites as independent, our effective-field theory will be reliable if the fluctuations are small, i.e. away from the critical point. Therefore our theory correctly describes the high- and low-temperature behavior whereas it might misrepresent the details close to the critical point. In particular, the questions under which conditions the spreading temperature coincides with the equilibrium critical temperature and whether the spreading transition is of 1st or 2nd order cannot be considered solved. Further open questions are connected with the influence of external magnetic fields, long-range interactions and disorder. Some investigations along these lines are in progress.

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