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An index theorem for manifolds with boundary

by Paulo Carrillo Rouse and Bertrand Monthubert

Abstract

In [2] II.5, Connes gives a proof of the Atiyah-Singer index theorem for closed manifolds by using deformation groupoids and appropriate actions of these on $\mathbb{R}^N$. Following these ideas, we prove an index theorem for manifolds with boundary.

Résumé

Dans [2] II.5, Connes donne une preuve du théorème d’Atiyah-Singer pour des variétés fermées en utilisant des groupoïdes de déformation et des actions appropriées de ceux-ci dans $\mathbb{R}^N$. Nous suivons ces idées pour montrer un théorème d’indice pour des variétés à bord.

Version française abrégée

Dans [2], II.5, Alain Connes donna une preuve du théorème d’Atiyah-Singer pour une variété entièrement fondée sur l’utilisation de groupoïdes, grâce à une action du groupoïde tangent de la variété sur $\mathbb{R}^N$. L’idée centrale est de remplacer des groupoïdes qui ne sont pas (Morita) équivalents à des espaces, par des groupoïdes obtenus par produit croisé et qui possèdent cette propriété, ce qui permet ensuite de donner une formule.

Si $X$ est une variété à bord $\partial X$, nous construisons le groupoïde $T_{b}X := \left< ad_{G_{\partial X}} \times \mathbb{R} \right> \cup \partial T_{X}$ en recollant $ad_{G_{\partial X}} \times \mathbb{R}$ avec $T_{X}$ le long de leur bord commun $T_{\partial X} \times \mathbb{R} \times (0, 1)$ est le groupoïde adiabatique). Nous le recollons alors avec le groupoïde tangent de l’intérieur de $X$, $T_{\circ}X = T X \cup T X \times \{0\}$ : $T_{\circ}X := T_{\circ}X \cup \{0\} \times \mathbb{R}$.

Il existe un homomorphisme $T_{\circ}G_{X} \xrightarrow{h} \mathbb{R}^{\infty}$ induit par un plongement de $X$ dans $\mathbb{R}^{\infty}$, tel que $\partial X$ se plonge dans $\mathbb{R}^{\infty} \times \{0\}$ et $X$ se plonge dans $\mathbb{R}^{\infty} \times \mathbb{R}^{*}$. Le produit croisé de $T_{\circ}G_{X}$ par $h$ (noté $T_{\circ}(G_{X})h$) est un groupoïde propre dont les groupes d’isotropie sont triviaux, il est donc Morita-équivalent à son espace d’orbites.

Soit $V(\hat{\circ}X)$ le fibré normal de $\hat{\circ}X$ dans $\mathbb{R}^{N}$, et $V(\partial X)$ le fibré normal de $\partial X$ dans $\mathbb{R}^{N-1}$ : soit enfin $V(X) = V(\hat{\circ}X) \cup V(\partial X)$. En notant $\mathcal{D}_0 = V(\partial X) \times \{0\} \cup \mathbb{R}^{N-1} \times (0, 1)$ et $\mathcal{D}_0 = V(\hat{\circ}X) \times \{0\} \cup \mathbb{R}^{N} \times (0, 1)$ les déformations au cône normal, on construit les espaces $\mathcal{D}_0 := V(X) \cup \mathcal{D}_0$ et $\mathcal{D}_0 := \mathcal{D}_0 \cup \mathcal{D}_0$.

**Proposition 0.1.** Le groupoïde $(T_{\circ}G_{X})h$ est Morita équivalent à l’espace $\mathcal{D}$.

Soit

$$ind_{\mathcal{D}} := (e_1)_* \circ (e_0)^{-1} : K^0(\mathcal{D}) \longrightarrow K^0(\hat{\circ}X \times X) \cong \mathbb{Z}.$$

**Définition 0.1** (Indice topologique pour une variété à bord). Soit $X$ une variété à bord. L’indice topologique de $X$ est le morphisme

$$ind_{\mathcal{D}} : K^0(\mathcal{D}) \longrightarrow \mathbb{Z}$$

défini comme la composition des trois morphismes suivants.
(1) L’isomorphisme de Connes-Thom $CT_0$ suivi de l’équivalence de Morita $\mathcal{M}_0$ :

$$K^0(T_bX) \xrightarrow{CT_0} K^0((T_bX)_{h_0}) \xrightarrow{\partial_h} K^0(\mathcal{B}_0),$$

où $(T_bX)_{h_0}$ est le produit croisé de $T_bX$ par $h_0$ (l’homomorphisme $h$ en $t = 0$).

(2) Le morphisme indice de l’espace de déformation $\mathcal{B} : K^0(\mathcal{B}_0) \xrightarrow{(e_0)_*} K^0(\mathcal{B}) \xrightarrow{(e_1)_*} K^0(\mathbb{R}^N)$

(3) Le morphisme de périodicité de Bott : $K^0(\mathbb{R}^N) \xrightarrow{Bott} \mathbb{Z}$.

Theorem 0.2. Pour toute variété à bord, on a l’égalité

$$\text{ind}_{\mathbb{R}^N} = \text{ind}_{\mathbb{R}}.$$

1. Actions of $\mathbb{R}^N$

   All the groupoids considered here will be continuous family groupoids $\mathcal{M}$. Hence we can consider their convolution and $C^*$-algebras without any problem. If $G$ is such a groupoid, we will denote by $K^0(G)$ the $K$-theory group of its $C^*$-algebra (unless explictely written otherwise). This is consistent with the usual notion when $G$ is a space (a groupoid made only of units). In the sequel, given a smooth manifold $N$, we will denote by $\alpha_k G_N : T_N \times \{0\} \coprod N \times \mathbb{R} \xrightarrow{\approx} N \times \mathbb{R}$, the deformation to normal cone of $N$ in $N \times \mathbb{R}$ (for complete details about this deformation functor see $\mathcal{M}$). At each time, we will need to restrict it to some interval, e.g. $[0,1]$ gives the tangent groupoid, and $[0,1]$ gives the adiabatic groupoid.

   Let $G \xrightarrow{h} M$ be a groupoid and $h : G \rightarrow \mathbb{R}^N$ a (smooth or continuous) homomorphism of groupoids, $(\mathbb{R}^N)$ as an additive group). Connes defined the semi-direct product groupoid $G_h = G \times \mathbb{R}^N \xrightarrow{\approx} M \times \mathbb{R}^N$ (II.5) with structure maps $t(\gamma, X) = (t(\gamma), X)$, $s(\gamma, X) = (s(\gamma), X + h(\gamma))$ and product $(\gamma, X) \circ (\eta, Y + h(\gamma)) = (\gamma \circ \eta, X)$ for composable arrows.

   At the level of $C^*$-algebras, $C^*(G_h)$ can be seen as the crossed product algebra $C^*(G) \rtimes \mathbb{R}^N$ where $\mathbb{R}^N$ acts on $C^*(G)$ by automorphisms by the formula: $\alpha(\gamma) = e^{t(X, h(\gamma))}$, $\forall f \in C_c(G)$, (see II.5.7 for details). In particular, in the case $N$ is even, we have a Connes-Thom isomorphism in $K$-theory $K^0(G) \xrightarrow{\approx} K^0(G_h)$ (II.2).

   Using this groupoid, Connes gives a conceptual, simple proof of the Atiyah-Singer Index theorem for closed smooth manifolds. Let $M$ be a smooth manifold, $G_M = M \times M$ its groupoid, and consider the tangent groupoid $T_G_M$. It is well known that the index morphism provided by this deformation groupoid is precisely the analytic index of Atiyah-Singer, $\mathcal{M}$. In other words, the analytic index of $M$ is the morphism

$$K^0(TM) \xrightarrow{(e_0)_{-1}} K^0(TG_M) \xrightarrow{(e_1)_*} K^0(M \times M) = K^0(\mathcal{X}(L^2(M))) \approx \mathbb{Z},$$

where $e_t$ are the obvious evaluation algebra morphisms at $t$. As discussed by Connes, if the groupoids appearing in this interpretation of the index were equivalent to spaces then we would immediately have a geometric interpretation of the index. Now, $M \times M$ is equivalent to a point (hence to a space), but the other fundamental groupoid playing a role is not, that is, $TM$ is a groupoid whose fibers
are the groups $T\Sigma M$, which are not equivalent (as groupoids) to a space. The idea of Connes is to use an appropriate action of the tangent groupoid in some $\mathbb{R}^N$ in order to translate the index (via a Thom isomorphism) in an index associated to a deformation groupoid which will be equivalent to some space.

2. Groupoids and Manifolds with Boundary

Let $X$ be a manifold with boundary $\partial X$. We denote, as usual, $\overset{\circ}{X}$ the interior which is a smooth manifold. Let $X_\partial$ be the smooth manifold obtained by glueing $\overset{\circ}{X}$ with $\partial X \times \{0, 1\}$ along their common boundary, $\partial X \sim \partial X \times \{0\}$. Set $TX := TX_\partial|_X$, and consider the smooth manifold $T_bX := (\text{adj}G_{\partial X} \times \mathbb{R}) \cup_b TX$ obtained by glueing $\text{adj}G_{\partial X} \times \mathbb{R}$ and $TX$ along their common boundary $\partial TX \times \mathbb{R}$ ($\text{adj}G_{\partial X} = T\partial X \cup \partial X \times \partial X \times (0, 1)$ is the adiabatic groupoid). Now, we have a continuous family groupoid over $X_\partial$: $T_bX \rightrightarrows X_\partial$. As a groupoid it is the union of the groupoids $T\partial X \rightrightarrows \partial X \times \{0, 1\}$ and $TX \rightrightarrows X$. For the topology, it is very easy to see that all the groupoid structures are compatible with the gluings we considered.

We are going to consider a deformation groupoid $\overset{\circ}{T}G_X$ (§). This will be a natural generalisation of the Connes tangent groupoid of a smooth manifold, to the case with boundary. The space of arrows $\overset{\circ}{T}G_X := T_bX \cup_b \overset{\circ}{T}G_X\overset{\circ}{\rightrightarrows}$ is obtained by glueing at $TX$ ($TX \times \{0\} \subset \overset{\circ}{T}G_X\overset{\circ}{\rightrightarrows}$ is closed). The space of units $X_{\partial 0} := X_\partial \cup_\partial \overset{\circ}{X} \times [0, 1]$ is obtained by glueing $\overset{\circ}{X} \sim \overset{\circ}{X} \times \{0\} (\overset{\circ}{X} \times \{0\} \subset \overset{\circ}{X} \times [0, 1]$ is closed). Using the groupoid structures of $T_bX \rightrightarrows X_\partial$ and of $\overset{\circ}{T}G_x \rightrightarrows \overset{\circ}{X} \times [0, 1]$, we have a continuous family groupoid $\overset{\circ}{T}G_X \rightrightarrows X_{\partial 0}$. Again, all the groupoid structures are compatible with the considered gluings.

To define a homomorphism $\overset{\circ}{T}G_X \xrightarrow{h} \mathbb{R}^N$ we will need as in the nonboundary case an appropriate embedding. It is possible to find an embedding $i : X \hookrightarrow \mathbb{R}^{N-1} \times \mathbb{R}_+$ such that its restrictions to the interior and to the boundary are (smooth embeddings) of the following form $i_\partial : \overset{\circ}{X} \hookrightarrow \mathbb{R}^{N-1} \times \mathbb{R}_+$ and $i_\partial : \partial X \hookrightarrow \mathbb{R}^{N-1} \times \{0\}$. We define the homomorphism $h : \overset{\circ}{T}G_X \rightarrow \mathbb{R}^N$ as follows.

(2) \[
\begin{aligned}
h(x, X, 0) &= d_xi_\partial(X) \text{ and } h(x, y, \epsilon) = \frac{d_xi_\partial(x) - i_\partial(y)}{\epsilon} \text{ on } \overset{\circ}{T}G_X \\
h(x, \xi, 0, \lambda) &= (d_xi_\partial(\xi), \lambda) \text{ and } h(x, y, \epsilon, \lambda) = \frac{d_xi_\partial(x) - i_\partial(y) - \lambda}{\epsilon} \text{ on } \text{adj}G_{\partial X} \times \mathbb{R} \\
h(x, X) &= d_xi_\partial(X) \text{ on } TX
\end{aligned}
\]

Since all these morphisms are compatible with the gluings, one has:

Proposition 2.1. With the formulas defined above, $h : \overset{\circ}{T}G_X \rightarrow \mathbb{R}^N$ defines a homomorphism of continuous family groupoids.

The action of $\overset{\circ}{T}G_X$ on $\mathbb{R}^N$ defined by $h$ is free because $i$ is an immersion. It is not necessarily proper (in the case of Connes [2] II.5 it is since $M$ was supposed closed), however we can prove the following:

Proposition 2.2. The groupoid $(\overset{\circ}{T}G_X)_h$ is a proper groupoid with trivial isotropy groups.

Notice that the groupoid $G_h$ is not the action groupoid (if not, the properness of the action would be equivalent to the properness of the groupoid). It is very important that the units of the groupoid $G_h$ be the units of $G$ times $\mathbb{R}^N$.
As an immediate consequence of the propositions above, the groupoid \( (T^*G_X)_h \) is Morita equivalent to its space of orbits. Let us specify this space.

Let \( V(\tilde{X}) \) be the total space of the normal bundle of \( \tilde{X} \) in \( \mathbb{R}^N \). Similarly, let \( V(\partial X) \) be the total space of the normal bundle of \( \partial X \) in \( \mathbb{R}^{N-1} \). Observe that they have the same fiber vector dimension. In fact, their union \( V(X) = V(\tilde{X}) \cup V(\partial X) \) is a vector bundle over \( X \), the normal bundle of \( X \) in \( \mathbb{R}^N \).

Take \( \mathcal{B}_\partial = V(\partial X) \times \{0\} \bigcup \mathbb{R}^{N-1} \times (0,1) \) the deformation to the normal cone associated to the embedding \( \partial X \hookrightarrow \mathbb{R}^{N-1} \). We consider the space \( \mathcal{B}_\partial := V(X) \bigcup \mathcal{B}_\partial \). Take over their common boundary \( V(\partial X) \sim V(\partial X) \times \{0\} \). On the other hand, take \( \mathcal{B}_\partial = V(\tilde{X}) \times \{0\} \bigcup \mathbb{R}^N \times (0,1) \) the deformation to the normal cone associated to the embedding \( \tilde{X} \hookrightarrow \mathbb{R}^N \). We consider the space \( \mathcal{B} := \mathcal{B}_\partial \bigcup \mathcal{B}_0 \) glued over \( V(\tilde{X}) \) by the identity map.

**Proposition 2.3.** The space of orbits of the groupoid \( (T^*G_X)_h \) is \( \mathcal{B} \).

We can give the explicit homeomorphism. The orbit space of \( (T^*G_X)_h \) is a quotient of \( X_{g_0} \times \mathbb{R}^N \). To define a map \( \Psi : X_{g_0} \times \mathbb{R}^N \rightarrow \mathcal{B} \) it is enough to define it for each component of \( X_{g_0} \). Let

\[
\Psi : \begin{cases}
\partial X \times (0,1) \times \mathbb{R}^{N-1} \times \mathbb{R} \rightarrow \mathbb{R}^{N-1} \times (0,1) \\
\Psi(a, t, \xi, \lambda) := \left( \frac{t \alpha(a)}{\xi}, \xi, t \right)
\end{cases}
\]

\[
\begin{cases}
\tilde{X} \times (0,1) \times \mathbb{R} \rightarrow \mathbb{R}^{N} \times (0,1) \\
\Psi(x, t, X) := \left( \frac{t \alpha(x)}{X + t}, X, t \right)
\end{cases}
\]

\[
\begin{cases}
\tilde{X} \times (0,1) \times \mathbb{R} \rightarrow \mathbb{R}^{N} \times (0,1) \\
\Psi(x, t, X) := \left( \frac{t \alpha(x)}{X + t}, X, t \right)
\end{cases}
\]

where \( \xi \) denotes the class in \( V_\alpha(\partial X) := \mathbb{R}^{N-1}/T_{\alpha(a)} \partial X \) (resp. \( \tilde{X} \) denotes the class in \( V_\alpha(\tilde{X}) := \mathbb{R}^{N}/T_{\alpha(x)} \tilde{X} \)). This gives a continuous map \( \Psi : X_{g_0} \times \mathbb{R}^N \rightarrow \mathcal{B} \) that passes to the quotient into a homeomorphism \( \overline{\Psi} : (X_{g_0} \times \mathbb{R}^N)/\sim \rightarrow \mathcal{B} \), where \( (X_{g_0} \times \mathbb{R}^N)/\sim \) is the orbit space of the groupoid \( (T^*G_X)_h \).

### 3. The Index Theorem for Manifolds with Boundary

Deformation groupoids induce index morphisms. The groupoid \( T^*G_X \) is naturally parametrized by the closed interval \([0,1]\). Its algebra comes equipped with evaluations to the algebra of \( T_0 M \) (at \( t=0 \)) and to the algebra of \( X \times \tilde{X} \) (for \( t \neq 0 \)). We have a short exact sequence of \( C^* \)-algebras

\[
\begin{array}{c}
0 \rightarrow C^*(\tilde{X} \times \tilde{X} \times (0,1]) \rightarrow C^*(T^*G_X) \rightarrow C^*(T_0 M) \rightarrow 0
\end{array}
\]

where the algebra \( C^*(\tilde{X} \times \tilde{X} \times (0,1]) \) is contractible. Hence applying the \( K \)-theory functor to this sequence we obtain an index morphism

\[
ind_{\tilde{X}} = (e_1)_{*} \circ (e_0)^{-1} : K^0(T_0 M) \rightarrow K^0(\tilde{X} \times \tilde{X}) \approx \mathbb{Z}.
\]

The morphism \( h : T^*G_X \rightarrow \mathbb{R}^N \) is by definition also parametrized by \([0,1]\), i.e., we have morphisms \( h_0 : T_0 M \rightarrow \mathbb{R}^N \) and \( h_t : \tilde{X} \times \tilde{X} \rightarrow \mathbb{R}^N \), for \( t \neq 0 \). We can consider the associated groupoids, which satisfy the same properties as in proposition [2].
(in fact, for proving such proposition it is better to do it for each \( t \), and to check all the compatibilities).

**Définition 3.1.** [Topological index morphism for a manifold with boundary] Let \( X \) be a manifold with boundary. The topological index morphism of \( X \) is the morphism

\[
\text{ind}_t^X : K^0(T_t X) \longrightarrow \mathbb{Z}
\]

defined (using an embedding as above) as the composition of the following three morphisms

1. The Connes-Thom isomorphism \( CT \) followed by the Morita equivalence \( M \):

\[
K^0(T_t X) \xrightarrow{CT} K^0((T_t X)_{h_0}) \xrightarrow{M} K^0(B_\partial)
\]

2. The index morphism of the deformation space \( B \):

\[
K^0(B_\partial) \xrightarrow{(e_0)_*} K^0(B) \xrightarrow{(e_1)_*} K^0(\mathbb{R}^N)
\]

3. The usual Bott periodicity morphism: \( K^0(\mathbb{R}^N) \xrightarrow{\text{Bott}} \mathbb{Z} \).

**Remark 1.** The topological index defined above is a natural generalisation of the topological index theorem defined by Atiyah-Singer. Indeed, in the boundaryless case, they coincide. The index of the deformation space \( B \) is quite easy to understand because we are dealing now with spaces (as groupoids the product is trivial), then the group \( K^0(B) \) is the K-theory of the algebra of continuous functions vanishing at infinity \( C_0(B) \) and the evaluation maps are completely explicit. In particular, if we identify \( B_\partial \) with an open subset of \( \mathbb{R}^N \) (in the natural way), then the morphism (ii) above correspond to the canonical extension of functions of \( C_0(B_\partial) \) to \( C_0(\mathbb{R}^N) \).

The following diagram, in which the morphisms \( CT \) and \( M \) are the Connes-Thom and Morita isomorphisms respectively, is trivially commutative:

\[
\begin{array}{cccc}
K^0(T_t X) & \xrightarrow{e_0} & K^0(G_X) & \xrightarrow{e_1} & K^0(\tilde{X} \times \tilde{X}) \\
\xrightarrow{CT} & = & \xrightarrow{CT} & = & \xrightarrow{CT} \\
K^0((T_t X)_{h_0}) & \xrightarrow{e_0} & K^0(G_X_{h_0}) & \xrightarrow{e_1} & K^0(\tilde{X} \times \tilde{X}_{h_1}) \\
\xrightarrow{\#} & = & \xrightarrow{\#} & = & \xrightarrow{\#} \\
K^0(B_{\partial}) & \xrightarrow{e_0} & K^0(B) & \xrightarrow{e_1} & K^0(\mathbb{R}^N).
\end{array}
\]

The left vertical line gives the first part of the topological index map. The bottom line is the morphism induced by the deformation space \( B \). And the right vertical line is precisely the inverse of the Bott isomorphism \( \mathbb{Z} = K^0(\{pt\}) \approx K^0(\tilde{X} \times \tilde{X}) \rightarrow K^0(\mathbb{R}^N) \). Since the top line gives \( \text{ind}_f^X \), we obtain the following result:

**Theorem 3.1.** For any manifold with boundary \( X \), we have the equality of morphisms

\[
\text{ind}_f^X = \text{ind}_t^X.
\]
4. Perspectives

As discussed in [3, 4, 5], the index map \( \text{ind}_I^X \) computes the Fredholm index of a fully elliptic operator in the \( b \)-calculus of Melrose. We shall use the result proven here to give a formula in relation to that of Atiyah-Patodi-Singer (\[6\]).

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