Stochastic dynamics of a point charge under electromagnetic squeezed vacuum fluctuations

Tai-Hung Wu, Jen-Tsung Hsiang and Da-Shin Lee

Department of Physics, National Dong Hwa University, Hualien, Taiwan, R.O.C.

Abstract

The effect of the electromagnetic squeezed vacuum on the dynamics of a nonrelativistic charge is studied. We derive the Langevin equation under the classical approximation by taking the variation of the coarse-grained stochastic effective action that is obtained within the context of the closed-time-path formalism. It is found that the nonstationary quantum noise, arising from the squeezed vacuum fluctuations of the electromagnetic fields, may give rise to reduction in charge’s velocity dispersion when it barely moves. We show that the velocity dispersion can be even below the value solely given by the normal vacuum state of the electromagnetic fields. Then, the full-time evolution of a charged oscillator, influenced by the electromagnetic squeezed vacuum fluctuations, is also investigated within the dipole approximation. In contrast, it shows that reduction in charge’s velocity dispersion can not always be achieved by manipulating the nonstationary quantum noise via the choice of the squeeze parameters. The implications are discussed.

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I. INTRODUCTION

The phenomena of particle creation from the vacuum state under a time-dependent background have been an active research topic. Particle creation has been shown to arise from the time-dependent mass term of the quantum field as a consequence of its coupling to the dynamical background [1]. Starting from an initial vacuum state with respect to which the creation and annihilation operators are defined, this vacuum state may finally evolve into a squeezed vacuum state where the expectation value of the particle number operator, defined in the initial state, becomes nonzero. This can be interpreted as particle creation. The best known example is inflationary amplification of vacuum fluctuations through the processes of particle creation in the early universe in either the strong gravitational field, or the interacting inflaton field background. This will lead to entropy production characterized by the squeeze parameter of the end-of-inflation state [2, 3, 4, 5, 6].

In quantum optics, the squeezed state of light can be produced by a variety of nonlinear processes, such as degenerate parametric amplification [7]. Within the context of improving the sensitivity of the laser-interferometric gravitational-wave detector, the technique of squeezed light has been applied to reduce the quantum noise on the test mass [8]. Thus, the full-time evolution of a classical object, which couples to squeezed quantum fluctuations, to some extent, may become of great importance in this regard. Very little has been studied based upon our knowledge. Nevertheless, the squeezed vacuum state is of interest in its own right because the corresponding renormalized energy density can be negative in some spacetime region. The existence of the negative energy density may raise exotic phenomena such as the traversable wormholes [9]. Recently, it has also been proposed that the application of the electromagnetic squeezed vacuum fluctuations to a stationary two-level-atom system can possibly enhance laser cooling of the neutral atom [10, 11].

Thus, in this article we wish to explore the nature of quantum fluctuations of the electromagnetic squeezed vacuum by its influence on motion of a charged particle. Its velocity dispersion will be studied in details. The effects of dissipative backreaction and fluctuations of the quantized electromagnetic fields on the dynamics of the charge will be incorporated in a consistent manner. To do so, we treat the charged particle as the system of interest, and the degrees of freedom of the fields as the environment. The influence of the fields on the charge can be investigated by integrating out field variables within the context of the closed-
time-path formalism \cite{12, 13, 14, 15, 16, 17, 18, 19}. Then, the Langevin equation of the charged particle can be derived under the classical approximation. In previous works \cite{19, 20}, the effects of the electromagnetic vacuum fluctuations on motion of a nonrelativistic charge in the presence of the conducting boundary has been studied. They show that the modified vacuum fluctuations always enhance charge’s velocity dispersion irrespective of the orientation of the boundary as well as the state of charge’s motion. Thus, reduction in velocity fluctuations can not be achieved by placing the conducting plate in the neighborhood of the charge. Here a simple generalization of previous work is offered. We consider that a charged particle couples to the electromagnetic squeezed vacuum, and study under what circumstances its velocity dispersion can be possibly reduced to a level below the result due to the normal vacuum of the electromagnetic fields.

Our presentation is organized as follows. In Sec. \textbf{II} we briefly introduce the closed-time-path formalism to describe the evolution of the density matrix of a nonrelativistic point charge coupled to the quantized electromagnetic fields in the squeezed vacuum state. We trace out field variables to obtain the coarse-grained stochastic effective action from which the Langevin equation is derived in the classical approximation. The method of solving the Langevin equation in the dipole approximation has been introduced in Ref. \cite{19}. The velocity dispersion of the charge is introduced in Sec. \textbf{III}. In addition, the full-time evolution of the velocity dispersion is studied both numerically and analytically. The results are summarized and the extension of our present study is discussed in Sec. \textbf{IV}.

The Lorentz-Heaviside units and the convention $\hbar = c = 1$ will be used unless otherwise mentioned. The signature of the metric is $\text{diag}(+1, -1, -1, -1)$.

\section*{II. INFLUENCE FUNCTIONAL AND LANGEVIN EQUATION}

The dynamics of a nonrelativistic charged particle interacting with the quantized electromagnetic fields can be described by the Lagrangian

$$ L[q, A_T] = \frac{1}{2} m \dot{q}^2 - V(q) - \frac{1}{2} \int d^3x \, d^3y \, \varrho(x; q) G(x, y) \varrho(y; q) + \int d^3x \left[ \frac{1}{2} (\partial_\mu A_T)^2 + j \cdot A_T \right], \quad (1) $$

in terms of the transverse components of the gauge potential $A_T$, and the position $q$ of the charge in the Coulomb gauge, $\nabla \cdot A = 0$. The instantaneous Coulomb Green’s function
\( G(x, y) \) satisfies the Gauss law. The charge and current densities take the form, respectively,

\[
\begin{align*}
\varrho(x; q(t)) &= e \delta^{(3)}(x - q(t)), \\
\mathbf{j}(x; q(t)) &= e \dot{q}(t) \delta^{(3)}(x - q(t)).
\end{align*}
\]

Let \( \hat{\rho}(t) \) be the density matrix of the charge-field system, and then it evolves unitarily according to the functional Liouville equation. The effects of the field on the dynamics of the charged particle can be realized by the diagonal elements of the reduced density matrix \( \rho_r \), which is obtained by tracing out field variables in the density matrix \( \hat{\rho}(t) \). If we assume that at a given initial time \( t = t_i \) the system and environmental fields are uncorrelated,

\[
\hat{\rho}(t_i) = \hat{\rho}_e(t_i) \otimes \hat{\rho}_{A_T}(t_i),
\]

then the reduced density matrix at time \( t_f \) is given by

\[
\rho_r(q_f, \tilde{q}_f; t_f) = \int dA_T \langle q_f, A_T | \rho(t_f) | q_f, A_T \rangle
\]

\[
= \int d^3q_1 d^3q_2 J(q_f, \tilde{q}_f; q_1, q_2, t_i) \rho_e(q_1, q_2, t_i),
\]

where the propagating function \( J(q_f, \tilde{q}_f; q_1, q_2, t_i) \) is defined as

\[
J(q_f, \tilde{q}_f; q_1, q_2, t_i) = \int_{q_1}^{q_f} Dq^+ \int_{q_2}^{\tilde{q}_f} Dq^- \exp \left[ i \int_{t_i}^{t_f} dt \left( L_e[q^+] - L_e[q^-] \right) \right] F[j^+_T, j^-_T],
\]

and charge’s Lagrangian \( L_e[q] \) takes the form

\[
L_e[q] = \frac{1}{2} m \dot{q}^2 - V(q) - \frac{1}{2} \int d^3x d^3y \ \varrho(x; q) G(x, y) \varrho(y; q).
\]

Here we introduce the influence functional \( F[j^+_T, j^-_T] \),

\[
F[j^+_T, j^-_T] = \text{Tr}_{A_T} \{ U(t_f, t_i; j^+_T) \hat{\rho}_{A_T}(t_i) U^{-1}(t_f, t_i; j^-_T) \},
\]

and \( U(t_f, t_i; j_T) \) is an evolution operator of the free \( A_T \) field driven by a classical current density \( j_T \). Initially the quantized electromagnetic fields are assumed to be in the squeezed vacuum states,

\[
\hat{\rho}_{A_T}(t_i) = \prod_k S(\zeta_k) \langle 0 | S(\zeta_k),
\]

where the squeeze operator \( S(\zeta_k) \) is defined by

\[
S(\zeta_k) = \exp \left[ \frac{\zeta_k^*}{2} a^2_{\lambda_k} - \frac{\zeta_k}{2} a^*_{\lambda_k} \right], \quad \text{and} \quad \zeta_k = r_k e^{i\theta_k}, \quad (2)
\]
and the wave vector $k$ labels the modes of the squeezed vacuum states. The squeeze parameter $\zeta_k = r_k e^{i\theta_k}$ is an arbitrary complex number with $r_k \geq 0$ and $\theta_k \in \mathbb{R}$. The creation and the annihilation operators satisfy the canonical commutation relations

$$\left[ a_{\lambda k}, a_{\lambda' k'}^\dagger \right] = \delta_{\lambda \lambda'} \delta^{(3)}(k - k') ,$$

and the plane-wave expansion of the vector potential is of the form

$$A_T(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega}} \sum_{\lambda=1,2} \hat{\epsilon}_{\lambda k} a_{\lambda k} e^{ik\cdot x - i\omega t} + \text{h.c.} , \quad (3)$$

with $\omega = |k|$. The polarization unit vectors $\hat{\epsilon}_{\lambda k}$ obey the transversality condition,

$$\sum_{\lambda=1,2} \hat{\epsilon}^i_{\lambda k} \hat{\epsilon}^j_{\lambda k} = \delta^{ij} - \frac{k^i k^j}{|k|^2} .$$

With the help of the Baker-Campbell-Hausdorff formula, we readily find that the unitary transformations of the creation and the annihilation operators due to the squeeze operator $S(\zeta_k)$ are,

$$S^\dagger(\zeta_k) a_{\lambda k} S(\zeta_k) = \mu_k a_{\lambda k} - \nu_k a_{\lambda k}^\dagger , \quad \text{and} \quad S^\dagger(\zeta_k) a_{\lambda k}^\dagger S(\zeta_k) = \mu_k a_{\lambda k}^\dagger - \nu_k^* a_{\lambda k} ,$$

with $\mu_k = \cosh r_k$, $\nu_k = \sinh r_k e^{i\theta_k}$, $\eta_k = |\nu_k|$, and $\mu_k^2 - |\nu_k|^2 = \mu_k^2 - \eta_k^2 = 1$. The expectation values of the creation and the annihilation operators in the squeezed vacuum state are given by

$$\langle 0 | S^\dagger(\zeta_k) a_{\lambda k} S(\zeta_k) |0 \rangle = 0 , \quad \langle 0 | S^\dagger(\zeta_k) a_{\lambda k}^\dagger S(\zeta_k) |0 \rangle = 0 .$$

Moreover, we have

$$\langle 0 | S^\dagger(\zeta_k) a_{\lambda k}^2 S(\zeta_k) |0 \rangle = -\mu_k \nu_k , \quad \langle 0 | S^\dagger(\zeta_k) a_{\lambda k}^{\dagger 2} S(\zeta_k) |0 \rangle = -\mu_k \nu_k^* ,$$

$$\langle 0 | S^\dagger(\zeta_k) a_{\lambda k} a_{\lambda k} S(\zeta_k) |0 \rangle = \eta_k^2 . \quad (4)$$

From Eq. (4), it is seen that the squeezed vacuum state is not a normal vacuum state and it contains $\eta_k^2$ photons on average for each mode $k$.

Since the charge interacts with the fields via a linear coupling, the field variables can be integrated out exactly. The laborious derivations of the influence functional can be found in Ref. [5]. The expression of $F[j^+, j^-]$ is then written in terms of real-time Green’s functions of the vector potentials,

$$F[j^+, j^-] = \exp \left\{ -\frac{1}{2\hbar^2} \int d^4x \int d^4x' \left[ j^+_i(x; q^+(t)) \langle A^+_T(x) A^+_T(x') \rangle j^+_i(x'; q^+(t')) \right] \right\}.$$
we obtain the Langevin equation of the charged particle (see Ref. [19] for more detailed derivation)

\[
\begin{aligned}
-j_i^+(x; q^+(t)) \langle A_T^{+i}(x) A_T^{-j}(x') \rangle j_j^-(x'; q^-(t')) \\
-j_i^-(x; q^-(t)) \langle A_T^{-i}(x) A_T^{+j}(x') \rangle j_j^+(x'; q^+(t')) \\
+ j_j^-(x; q^-(t)) \langle A_T^{-i}(x) A_T^{-j}(x') \rangle j_j^+(x'; q^+(t'))
\end{aligned}
\]

and it contains full information about the influence of the environmental fields. Here the explicit \(\hbar\) dependence is restored. The Green’s functions are respectively defined by

\[
\begin{aligned}
\langle A_T^{+i}(x) A_T^{+j}(x') \rangle &= \langle A_T^{+i}(x) A_T^{+j}(x') \rangle \theta(t-t') + \langle A_T^{+i}(x') A_T^{+j}(x) \rangle \theta(t'-t)
\end{aligned}
\]

\[
\begin{aligned}
\langle A_T^{-i}(x) A_T^{-j}(x') \rangle &= \langle A_T^{-i}(x') A_T^{-j}(x) \rangle \theta(t-t') + \langle A_T^{+i}(x) A_T^{-j}(x') \rangle \theta(t'-t)
\end{aligned}
\]

\[
\begin{aligned}
\langle A_T^{+i}(x) A_T^{-j}(x') \rangle &= \langle A_T^{+i}(x') A_T^{-j}(x) \rangle \equiv \text{Tr} \{ \rho_A T_i A_T^{+j}(x') \}
\end{aligned}
\]

\[
\begin{aligned}
\langle A_T^{-i}(x) A_T^{+j}(x') \rangle &= \langle A_T^{+i}(x) A_T^{-j}(x') \rangle \equiv \text{Tr} \{ \rho_A T_i A_T^{+j}(x) \}
\end{aligned}
\]

The classical approximation on the dynamics of the charge can be employed if we have ignored charge’s intrinsic quantum fluctuations, and that holds as long as the resolution of the measurement on length scales is greater than charge’s position uncertainty. Thus, we obtain the Langevin equation of the charged particle (see Ref. [19] for more detailed derivation),

\[
\begin{aligned}
\frac{m}{\hbar} \ddot{q}^i + \nabla^i V(q(t)) + e^2 \nabla^i G[q(t), q(t)] + e^2 \left( \delta^{ii} \frac{d}{dt} - \dot{q}^i(t) \nabla_i \right) \\
\times \int_{-\infty}^{\infty} dt' \ G_{R}^{ij}(q(t), q(t'); t-t') \dot{q}^j(t') = -\hbar e \left( \delta^{ii} \frac{d}{dt} - \dot{q}^i(t) \nabla_i \right) \xi^i(t)
\end{aligned}
\]

(5)

with the stochastic noise \(\xi^i(t)\) having the correlation,

\[
\langle \xi^i(t) \rangle = 0, \quad \langle \xi^i(t) \xi^j(t') \rangle = \frac{1}{\hbar} G_{R}^{ij}(q(t), q(t'); t, t').
\]

(6)

This Langevin equation encompasses the effects of fluctuations and dissipative backreaction on charge’s motion from the quantized electromagnetic fields via the kernels \(G_{H}^{ij}\) and \(G_{R}^{ij}\), which are respectively defined by

\[
\hbar G_{H}^{ij}(x, x') = \frac{1}{2} \langle \{ A_T^{+i}(x), A_T^{-j}(x') \} \rangle = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \left( \delta^{ij} - \frac{k^i k^j}{\omega^2} \right) \left\{ (2\eta_k^2 + 1) e^{-i\omega(t-t')} e^{i\mathbf{k} \cdot [q(t) - q(t')]} \\
- 2\eta_k \mu_k e^{i\mathbf{k} \cdot [q(t) - t']} e^{i\mathbf{k} \cdot [q(t) + q(t')]} \right\} + \text{c.c.}
\]

(7)

\[
= \hbar G_{H, st}(x, x') + \hbar G_{H, ns}(x, x'),
\]
\[ \hbar G^{ij}_{R}(x, x') = i \theta(t - t') \langle [A^i_T(x), A^j_T(x')] \rangle \]
\[ = i \theta(t - t') \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \left( \delta^{ij} - \frac{k^i k^j}{\omega^2} \right) \left\{ e^{-i\omega(t-t')} e^{i\mathbf{k} \cdot \mathbf{q}(t)-\mathbf{q}(t')} \right\} + \text{c.c.} \] (8)

Apparently, there are two distinct contributions to the Hadamard function \( G^{ij}_{R}(x, x') \) in the squeezed vacuum state. The first term in the curly brackets comes from the stationary component of the noise. However, there exists a nonstationary component due to the fact that this quantum state is presumably generated by nonlinear processes of particle creation in which the time-translational invariance is broken. In contrast, the retarded Green’s function, accounting for the dissipative backreaction, is independent of the state of environmental fields, and will lead to the known self-force, which is proportional to a third-order time derivative of charge’s position.

It is worth mentioning that the nonstationary feature of the squeezed vacuum state plays a crucial role in having the negative energy density. Let us consider the renormalized energy density of a single-mode squeezed vacuum state with the wavenumber \( \bar{k} \) and the frequency \( \bar{\omega} = |\bar{k}| \). The energy density operator \( \varrho(x) \) can be expressed as the sum of the square of the electric field \( \mathbf{E} \) and of the magnetic field \( \mathbf{B} \)
\[ \varrho(x) = \frac{1}{2} \left[ \mathbf{E}^2(x) + \mathbf{B}^2(x) \right] , \] (9)
and in the Coulomb gauge the fields \( \mathbf{E} \) and \( \mathbf{B} \) are given by the transverse part of the vector potential, that is, \( \mathbf{E} = -\dot{\mathbf{A}}_T \) and \( \mathbf{B} = \nabla \times \mathbf{A}_T \). Then the renormalized energy density can be obtained by subtracting the vacuum contributions of all modes, and we arrive at
\[ \varrho(x) = \frac{\bar{\omega}}{\sqrt{\eta_k}} \left[ \eta_k + \mu_k \cos(2\bar{\omega}t - \mathbf{k} \cdot \mathbf{x} - \theta_k) \right] . \] (10)

In the vicinity of the coordinate origin, where \( \mathbf{k} \cdot \mathbf{x} \simeq 0 \), Eq. (10) reduces to an expression proportional to \( \eta_k [\eta_k + \mu_k \cos(2\bar{\omega}t - \theta_k)] \), which is always greater than \( \eta_k^2 - \mu_k \eta_k > -1/2 \). Thus the renormalized energy density may become negative for some moments. However it can not be arbitrarily negative, and is found bounded from below. The corresponding vacuum energy density can be obtained by the replacements \( \eta_k^2 \rightarrow 1/2 \) and \( \mu_k \rightarrow 0 \). After adding this part back to the renormalized energy density, the total energy density still remains positive.

This observation brings us the hope that the presence of the nonstationary noise due to the squeezed vacuum, may reduce the velocity dispersion of the charge below the value
exclusively from the vacuum state. Later, it will be shown that this reduction is only possible when the charge barely moves. The nonstationary noise just leads to transient effects on the dynamics of the charge over the time scales determined by the frequency bandwidth of the squeezed modes. At later times, the contributions of the nonstationary noise to the velocity dispersion asymptotically vanish. On the other hand, since the dissipative backreaction is significant as the motion of the charge evolves into the relaxation regime, the stationary part of the Hadamard function becomes crucial to dynamically stabilize this nonequilibrium system, and thus we may derive the fluctuation-dissipation relation that links the retarded Green's function to the stationary component of the Hadamard function,

$$G_{ij,sl}[q(t), q(t');\omega] = (2\eta_k^2 + 1) [\theta(\omega) - \theta(-\omega)] \text{Im} G_{ij,R}[q(t), q(t');\omega].$$  

(11)

### III. VELOCITY DISPERSION OF THE CHARGED OSCILLATOR

When a charged particle interacts with the quantized electromagnetic fields, the quantum fluctuations of the fields are expected to drive the particle into irregular movement. This behavior may be perceivable by observing its velocity dispersion. Consider a charge undergoing a simple harmonic motion along the direction $i$ with angular frequency $\omega_0$, dictated by some external potential. The velocity dispersion of the charge due to the electromagnetic quantum fluctuations is given by (see Ref. [19] for more detailed derivation)

$$\langle \Delta v_i^2(t) \rangle = \frac{e^2}{m^2} \int_0^t du \int_0^t du' \hat{K}(t-u)\hat{K}(t-u')G_{ii,H}[q(u), q(u'); u, u'],$$

(12)

where $K(\tau)$ is the kernel function of charge's equation of motion. The time derivative of the kernel function $\dot{K}$ is

$$\dot{K}(\tau) = Z e^{-\Gamma \tau} \cos(\Omega \tau + \alpha),$$

(13)

with the resonance frequency $\Omega$ and the decay constant $\Gamma$ given by

$$\Omega \sim \omega_0 + \frac{\text{Re} \Sigma(\omega_0)}{2\omega_0}, \quad Z \sim \left[1 - \frac{\partial \text{Re} \Sigma(\Omega)}{\partial \Omega^2}\right]^{-1},$$

$$\Gamma \sim Z \frac{\text{Im} \Sigma(\Omega)}{2\Omega}, \quad \alpha \sim Z \frac{\partial \text{Im} \Sigma(\Omega)}{\partial \Omega^2}.$$  

(14)

The kernels $\text{Re} \Sigma(\omega)$ and $\text{Im} \Sigma(\omega)$ are defined by

$$\tilde{\Sigma}(s = i \omega + 0^\pm) = \text{Re} \Sigma(\omega) \pm i \text{Im} \Sigma(\omega),$$
\[ \Sigma(s) = \frac{e^2}{m} s^2 \left\{ \widetilde{G}^{ii}_{R}[q = 0; s] - \frac{\Lambda}{3\pi^2} \right\}. \]

The function \( \widetilde{G}^{ii}_{R}[q = 0; s] \) is the Laplace transformation of the retarded Green’s function in the dipole approximation, \( e^{ik\cdot q} \approx 1 \). Its ultraviolet divergence is absorbed by appropriate mass renormalization in which \( \Lambda \) is an ultraviolet frequency cutoff. Thus, up to order \( e^2 \), we obtain

\[ \text{Re} \Sigma(\omega) = 0, \quad \text{Im} \Sigma(\omega) = \frac{e^2}{4\pi m} \text{sgn}(\omega) \frac{2\omega^3}{3}. \tag{15} \]

The weak coupling leads to \( \Gamma \ll \Omega \). We assume that only the modes within a certain band are excited to the squeezed vacuum state, and the rest of them remain in the vacuum state. Let the mean frequency of the band be \( \Xi \) and the bandwidth \( \Delta \), and the excited modes be distributed over an narrow solid angle \( d\Omega_s \) about a certain direction \( k^i \) in the momentum space. Moreover, the squeeze parameters are assumed to be mode-independent within the band; thus their subscript \( k \) can be dropped hereafter. Then, Eq. (12) can be further simplified,

\[ \delta \langle \Delta v_i^2(t) \rangle_{st} = \frac{e^2}{m^2} A(d\Omega_s) \eta^2 \int_{\Xi - \Delta/2}^{\Xi + \Delta/2} d\omega \frac{\omega^3}{2} \int_0^t \int_0^t d\tau d\tau' \dot{K}(\tau) \dot{K}(\tau') e^{-i\omega(\tau - \tau')} + c.c., \tag{16} \]

\[ \delta \langle \Delta v_i^2(t) \rangle_{ns} = \frac{e^2}{m^2} A(d\Omega_s) \mu \eta \int_{\Xi - \Delta/2}^{\Xi + \Delta/2} d\omega \frac{\omega^3}{2} \int_0^t \int_0^t d\tau d\tau' \dot{K}(\tau) \dot{K}(\tau') e^{i\theta - i\omega(2t - \tau - \tau')} + c.c., \tag{17} \]

where,

\[ A(d\Omega_s) \equiv \frac{d\Omega_s}{(2\pi)^3} \left( \delta^{ii} - \frac{k^{i2}}{\omega^2} \right) \]

denotes the angular contribution over the solid angle \( d\Omega_s \). In Eqs. (16) and (17), the contributions from the vacuum fluctuations of the electromagnetic fields has been subtracted out. Thus, they respectively represent the difference in the stationary and the nonstationary component of the velocity fluctuations between the squeezed vacuum and the normal vacuum of the electromagnetic fields.

After integrating over \( \tau \) and \( \tau' \) variables, we arrive at the expressions for the stationary component of the velocity fluctuations

\[ \delta \langle \Delta v_i^2(t) \rangle_{st} = Z^2 \frac{e^2}{m^2} A(d\Omega_s) \eta^2 \int_{\Xi - \Delta/2}^{\Xi + \Delta/2} d\omega \frac{\omega^3}{4} \left\{ L_1(\omega) + L_2(\omega) + L_3(\omega) + L_4(\omega) \right\}, \tag{18} \]
where

\[
L_1(\omega) = \frac{1}{2\Gamma(\Omega + i\Gamma)(\omega - \Omega - i\Gamma)} \times \left[ -i\left(\Omega + i\Gamma\right) \left(1 + e^{-2\Gamma t} - e^{-\Gamma t + i\omega t + i\Omega t} - e^{-\Gamma t - i\omega t + i\Omega t}\right) + e^{2i\alpha} \Gamma \left(1 - e^{-\Gamma t - i\omega t + i\Omega t} - e^{\Gamma t + i\omega t + i\Omega t} + e^{-2\Gamma t + i2\Omega t}\right) \right],
\]

\[
L_2(\omega) = \frac{1}{2\Gamma(\Omega - i\Gamma)(\omega - \Omega + i\Gamma)} \times \left[ +i\left(\Omega - i\Gamma\right) \left(1 + e^{-2\Gamma t} - e^{-\Gamma t + i\omega t - i\Omega t} - e^{-\Gamma t - i\omega t + i\Omega t}\right) + e^{-2i\alpha} \Gamma \left(1 - e^{-\Gamma t - i\omega t - i\Omega t} - e^{\Gamma t + i\omega t + i\Omega t} + e^{-2\Gamma t + i2\Omega t}\right) \right],
\]

\[
L_3(\omega) = \frac{1}{2\Gamma(\Omega + i\Gamma)(\omega + \Omega + i\Gamma)} \times \left[ -i\left(\Omega + i\Gamma\right) \left(1 + e^{-2\Gamma t} - e^{-\Gamma t - i\omega t - i\Omega t} - e^{-\Gamma t + i\omega t + i\Omega t}\right) - e^{2i\alpha} \Gamma \left(1 - e^{-\Gamma t + i\omega t + i\Omega t} - e^{\Gamma t - i\omega t + i\Omega t} + e^{-2\Gamma t - i2\Omega t}\right) \right],
\]

\[
L_4(\omega) = \frac{1}{2\Gamma(\Omega - i\Gamma)(\omega + \Omega - i\Gamma)} \times \left[ +i\left(\Omega - i\Gamma\right) \left(1 + e^{-2\Gamma t} - e^{-\Gamma t - i\omega t + i\Omega t} - e^{-\Gamma t + i\omega t + i\Omega t}\right) - e^{-2i\alpha} \Gamma \left(1 - e^{-\Gamma t + i\omega t - i\Omega t} - e^{\Gamma t - i\omega t - i\Omega t} + e^{-2\Gamma t - i2\Omega t}\right) \right],
\]

and for the nonstationary component

\[
\delta\langle A(t)^2 \rangle_{ns} = Z^2 \frac{e^2}{m^2} A(d\Omega_s) \mu \eta \int_{\Xi-\Delta/2}^{\Xi+\Delta/2} d\omega \frac{\omega^3}{4} \left\{ J_1(\omega) + J_2(\omega) + J_3(\omega) \right\},
\]

where

\[
J_1(\omega) = e^{-2i\alpha} \left[ -\frac{e^{-2i\omega t + i\theta}}{2(\omega + i\Gamma - \Omega)^2} + \frac{e^{-\Gamma t - i\omega t + i\Omega t + i\theta}}{(\omega + i\Gamma - \Omega)^2} - \frac{e^{-2\Gamma t - 2i\Omega t + i\theta}}{2(\omega + i\Gamma - \Omega)^2} \right] + e^{2i\alpha} \left[ -\frac{e^{2i\omega t - i\theta}}{2(\omega - i\Gamma - \Omega)^2} + \frac{e^{\Gamma t - i\omega t + i\Omega t - i\theta}}{(\omega - i\Gamma - \Omega)^2} - \frac{e^{-2\Gamma t + 2i\Omega t - i\theta}}{2(\omega - i\Gamma - \Omega)^2} \right],
\]

\[
J_2(\omega) = \left[ -\frac{e^{-2\Gamma t - i\theta}}{(\omega - i\Gamma)^2 - \Omega^2} - \frac{e^{2i\omega t - i\theta}}{2(\omega - i\Gamma)^2 - \Omega^2} + \frac{e^{-\Gamma t + i\omega t + i\Omega t + i\theta}}{(\omega - i\Gamma)^2 - \Omega^2} + \frac{e^{-\Gamma t + i\omega t - i\Omega t + i\theta}}{(\omega - i\Gamma)^2 - \Omega^2} \right] \left[ -\frac{e^{-2\Gamma t + i\theta}}{(\omega + i\Gamma)^2 - \Omega^2} - \frac{e^{2i\omega t + i\theta}}{2(\omega + i\Gamma)^2 - \Omega^2} + \frac{e^{-\Gamma t - i\omega t + i\Omega t - i\theta}}{(\omega + i\Gamma)^2 - \Omega^2} + \frac{e^{-\Gamma t - i\omega t - i\Omega t - i\theta}}{(\omega + i\Gamma)^2 - \Omega^2} \right],
\]

\[
J_3(\omega) = e^{-2i\alpha} \left[ -\frac{e^{2i\omega t - i\theta}}{2(\omega - i\Gamma + \Omega)^2} + \frac{e^{-\Gamma t + i\omega t - i\Omega t - i\theta}}{(\omega - i\Gamma + \Omega)^2} - \frac{e^{-2\Gamma t - 2i\Omega t - i\theta}}{2(\omega - i\Gamma + \Omega)^2} \right].
\]
respectively. The nonstationary component reveals a much faster oscillatory behavior in frequency $\omega$ as compared with the stationary component. This can be seen by examining the nonstationary component (17), say, at early time interval $\Omega^{-1} \ll t \ll \Gamma^{-1}$,

$$
\int_0^t d\tau \int_0^t d\tau' K(\tau) K(\tau') \left[ e^{i\theta - i\omega(2t-\tau-\tau')} + e^{-i\theta + i\omega(2t-\tau-\tau')} \right]
$$

$$
= \frac{4}{\omega^2 - \Omega^2} \sin \left[ \frac{\omega - \Omega}{2} t \right] \sin \left[ \frac{\omega + \Omega}{2} t \right] \cos(\omega t - \theta) + \frac{2}{(\omega - \Omega)^2} \sin^2 \left[ \frac{\omega - \Omega}{2} t \right] \cos[(\omega + \Omega)t - \theta + 2\alpha] + \frac{2}{(\omega + \Omega)^2} \sin^2 \left[ \frac{\omega + \Omega}{2} t \right] \cos[(\omega - \Omega)t - \theta - 2\alpha],
$$

among which the second term has the most dominant contributions as a consequence of resonance. If we compare this expression with its counterpart in the stationary component,

$$
\frac{2}{(\omega - \Omega)^2} \sin^2 \left[ \frac{\omega - \Omega}{2} t \right],
$$

we see that the nonstationary component has an additional factor $\cos[(\omega + \Omega)t - \theta + 2\alpha]$, which is rapidly oscillating in both frequency $\omega$ and time $t$. This in turn may cause heavy cancellation as we perform an integration over the finite bandwidth of $\omega$ at later times. The time scale after which the nonstationary component starts to die out is determined by $t \simeq \Delta^{-1}$ when the modes inside the band evolve out of coherence. Besides, the nonstationary component may still undergo fast oscillation with time for $t \gg \Omega^{-1}$ such that the measurement of it over a finite period of time tends to yield smaller values. We may expect that in comparison with $\delta\langle \Delta v_i^2(t) \rangle_{st}$, the values of $\delta\langle \Delta v_i^2(t) \rangle_{ns}$ can be significant only in the much earlier stage of motion where the evolution time is much shorter than the oscillation period $t \ll \Omega^{-1}$.

**A. earlier times $t \ll \Omega^{-1}$**

Here we consider the velocity dispersion of a charged oscillator at extremely early time, $t \ll \Omega^{-1}$, when the charge barely moves. Thus, by letting $\Omega t \to 0$, $\Gamma t \to 0$, $\alpha \simeq 0$, and $Z \simeq 1$ in Eqs. (18) and (19), we have

$$
\delta\langle \Delta v_i^2(t) \rangle = \frac{e^2}{m^2 A(d\Omega_s)} \int_{\Xi-\Delta/2}^{\Xi+\Delta/2} d\omega 4\omega \left[ \eta^2 + \mu \eta \cos(\omega t - \theta) \right] \sin^2 \frac{\omega t}{2}.
$$
Carrying out the \( \omega \)-integral gives

\[
\int_{\Xi - \Delta/2}^{\Xi + \Delta/2} d\omega \omega \eta \left[ \eta + \mu \cos(\omega t - \theta) \right] \sin^2 \frac{\omega t}{2} = \frac{\Delta \Xi}{4} \eta \left[ 2\eta - \mu \cos \theta \right] + \frac{\eta}{8t^2} \sin \frac{\Delta}{2} t \left[ 8\eta \sin \Xi t - 8\mu \sin(\Xi t - \theta) + 2\mu \cos \frac{\Delta}{2} t \sin(2\Xi t - \theta) \right] - \frac{\eta}{8t} \left\{ \left[ \Xi + \frac{\Delta}{2} \right] \mu \sin(2\Xi t + \Delta t - \theta) - 4\mu \sin(\Xi t + \frac{\Delta}{2} t - \theta) + 4\eta \sin(\Xi t + \frac{\Delta}{2} t) \right\}.
\]

Except for the expression in the first pair of square brackets, the rest reveal a power-law decay in time. It shows that the relevant frequencies which will give rise to any significant effect on charge’s dynamics should be such that \( \Xi \gg \Delta \gg \Omega \). Hence, we find that when \( \Xi^{-1} \ll \Delta^{-1} \ll t \ll \Omega^{-1} \), the change of the velocity dispersion is dominated by the constant

\[
\delta \langle \Delta v_i^2(t) \rangle = 2A(d\Omega_s) \left( \frac{\Xi}{m} \right) \left( \frac{e^2 \Xi}{m} \right) \left( \frac{\Delta}{\Xi} \right) \left[ \eta^2 - \frac{1}{2} \mu \eta \cos \theta \right].
\]  

(20)

The nonstationary component of the velocity fluctuations, derived from the corresponding component of the noise-noise correlation function, exhibits an interesting feature that it may change the sign with an appropriate choice of squeeze parameters \( r \) and \( \theta \). The expression in the squared brackets of Eq. (20) is always greater than \( \eta^2 - \mu \eta/2 \), which is bounded below,

\[
\eta^2 - \frac{1}{2} \mu \eta \geq -\frac{2 - \sqrt{3}}{4} > -\frac{1}{2}.
\]  

(21)

Thus, it implies that the change of the velocity fluctuations can be negative, leading to reduction in the velocity dispersion. In particular, the velocity dispersion can be reduced to a level lower than what it would otherwise be due to the vacuum state of the electromagnetic fields with the same frequency band. The corresponding velocity dispersion due to the vacuum fluctuations is given by replacing \( \eta^2 \) with 1/2 and \( \mu \) with 0 respectively in Eq. (20),

\[
\langle \Delta v_i^2(t) \rangle_0 = 2A(d\Omega_s) \left( \frac{\Xi}{m} \right) \left( \frac{e^2 \Xi}{m} \right) \left( \frac{\Delta}{\Xi} \right) \times \frac{1}{2}.
\]  

(22)

Therefore, the ratio of \( \delta \langle \Delta v_i^2(t) \rangle \) to \( \langle \Delta v_i^2(t) \rangle_0 \) defines a function \( R(r, \theta) \),

\[
R(r, \theta) = 2\eta^2 - \mu \eta \cos \theta,
\]  

(23)

whose dependence on the squeeze parameters is shown in Fig. 1. The function \( R(r, \theta) \) is negative in the region shaded in darker gray and encircled by the curve \( R(r, \theta) = 0 \). Three
contours of $R(r, \theta)$ with different fixed values of $r$ are highlighted by thick solid curves, among which the foremost one passes through the minimum of $R(r, \theta)$. It is clearly seen that with larger values of $r$, the interval of $\theta$ in which $R(r, \theta)$ can be negative becomes increasingly narrower.

Since the vacuum fluctuations of the environmental fields are omnipresent, it would be interesting if we could somehow remove the very last bit of this background noise as much as possible. From Eq. (21), we see that the minimum value of $\eta^2 - \mu \eta / 2$ is still greater than $-1/2$. It indicates that although the noise of the background can be reduced, it can not be completely removed. Hence, charge’s velocity dispersion can be maximally reduced by a factor $\sqrt{3}/2 \sim 0.866$ from what solely results from the vacuum fluctuations. The corresponding photon number is $\bar{n} = \eta^2 = (2\sqrt{3} - 3)/6 \ll 1$, namely, there is less than one photon in each mode $k$. Besides, from Eq. (21), the amount of reduction is bounded from below to ensures that the total velocity dispersion, the sum of Eqs. (20) and (22), is always positive definite, as is required by definition.

On the other hand, in the even earlier stage of the evolution, that is, $\Xi^{-1} \ll t \ll \Delta^{-1} \ll \Omega^{-1}$, the change of the velocity dispersion is

$$
\delta\langle \Delta v_i^2(t) \rangle = 4A(d\Omega_n) \left( \frac{\Xi m}{\Xi} \right) \left( \frac{\Xi}{m} \right) \left( \frac{\Delta}{\Xi} \right) \left[ \eta^2 + \mu \eta \cos(\Xi t - \theta) \right] \sin^2 \frac{\Xi t}{2}.
$$

(24)
It is straightforward to show that the squared brackets in Eq. (24) has the minimum

\[ \eta^2 + \mu \eta \cos(\Xi t - \theta) \geq \eta^2 - \mu \eta = -\frac{1}{2} + e^{-2r} > -\frac{1}{2}. \]

At first sight, we might conclude that the squeezed vacuum state in this time regime is more effective to eliminate the background noise as long as the appropriate squeeze parameters are chosen at specific moments. However, since maximum reduction is to occur at some specific moments, the time uncertainty associated with any measurement has to be greatly reduced for the required precision. This action in turns gives rise to tremendously unwanted quantum uncertainty in momentum or energy of the particle, which may increase the velocity dispersion unexpectedly. Besides, any finite time uncertainty in the measurement tends to average out the nonstationary component. Hence, reduction of the velocity dispersion is far from being optimal.

In the following, we will study the full-time evolution of the charged oscillator influenced by the electromagnetic squeezed vacuum fluctuations.

**B. later times** \( t \gg \Omega^{-1} \)

When the charge sets into oscillatory motion, we notice that the integrands of the \( \omega \)-integral in Eqs. (18) and (19) have a Breit-Wigner feature of the narrow resonance due to weak coupling. The resonance peaks at frequency \( \Omega \) and has width \( \max\{2\pi/t, \Gamma\} \), which indicates the resonance width decreases with time as \( t^{-1} \) and then approaches to the value \( \Gamma \) as the motion evolves into the relaxation regime. Thus, the dissipative backreaction from the electromagnetic fields renders the charged oscillator into under-damped motion. In addition, the presence of the resonance implies that the effects from the squeezed vacuum state on the charge should depend on whether the band of excited modes overlaps with the resonance peak. Typically, the velocity dispersion will be expected to be much greater in the on-resonance case where the resonance peak lies within the band of the squeezed modes, than in the off-resonance case. In the following sections the on-resonance and off-resonance cases will be discussed separately.

Here we will consider the narrow band situation, \( \Delta \ll \Xi \). The extension of our study to the broad band case is straightforward if more details about the mode dependence of the squeeze parameters are known. Since each mode influences charge’s evolution independently,
we may split the broad band into many narrow bands within which the squeeze parameters are frequency-independent but may take different values. Thus the result of the broadband case with frequency-dependent squeeze parameters amounts to adding up all effects from the narrow bands. So, the narrow band example is simple enough to have analytical expressions, but sufficiently sophisticated to capture all essences about how the dynamics of the charged oscillator is affected by the electromagnetic squeezed vacuum state.

1. on-resonance

We first consider the on-resonance case that the resonance frequency of the charged oscillator lies within the band of the squeezed vacuum modes. For simplicity, let us assume that the mean frequency of the band \( \Xi \) coincides with the resonance frequency \( \Omega \), namely \( \Xi = \Omega \), and consider the situation that the bandwidth \( \Delta \) is much smaller than the mean frequency \( \Xi \) in such a way that \( \Gamma \ll \Delta \ll \Xi = \Omega \) holds.

At early time \( \Omega^{-1} \ll t \ll \Delta^{-1} \), the resonance width, which is of the order \( t^{-1} \) is greater than the bandwidth of the squeezed vacuum modes \( \Delta \), that is, \( \Delta t \ll 1 \). So the integrands in Eqs. (18) and (19) slowly varies inside the bandwidth, and can be pull out of the \( \omega \)-integral by substituting \( \omega \) with \( \Xi \). As such, this result at the early time regime bears a strong similarity to that of the single-mode case. For narrow resonance under consideration, the most dominant contributions of the stationary component come from \( L_1(\Omega) \) and \( L_2(\Omega) \) because the sum of them contains terms of the form \( \left[ (\omega - \Omega)^2 + \Gamma^2 \right]^{-1} \) around the resonance frequency. Similarly, the nonstationary component is dominated by the term \( J_1(\Omega) \) since it varies like \( (\omega - \Omega)^{-2} \) in the neighborhood of the resonance peak. By combining the stationary and nonstationary components of \( \delta \langle \Delta v_i^2(t) \rangle \), we find that, at early time \( \Omega^{-1} \ll t \ll \Delta^{-1} \ll \Gamma^{-1} \), the change of the velocity dispersion is given by

\[
\delta \langle \Delta v_i^2(t) \rangle \simeq A(d\Omega_s) \left( \frac{\Omega}{m} \right) \left( \frac{e^2\Omega}{m} \right) \left( \frac{1}{4} \Delta \Omega \right) \left[ \eta^2 + \mu \eta \cos(2\Omega t - \theta + 2\alpha) \right] t^2,
\]

which grows quadratically in time, and depends on the bandwidth of the squeezed vacuum modes \( \Delta \).

As the time progresses to the regime \( \Delta^{-1} \ll t \ll \Gamma^{-1} \), the resonance width gradually decreases to a value about the same order of magnitude as \( \Gamma \), which is smaller than the bandwidth \( \Delta \). Thus, it implies that there may exist a transition or crossover of the time
dependence of the velocity dispersion at about $t \simeq \Delta^{-1}$. In addition, when $t \gg \Delta^{-1}$, the squeezed modes may start to evolve out of phase with each other. It may lead to cancellation between the contributions to the nonstationary component from different modes, and slow down the growth of the velocity dispersion. Therefore, analytically we find that the stationary component of $\delta \langle \Delta v_i^2(t) \rangle$ in this time regime, $\Delta^{-1} \ll t \ll \Gamma^{-1}$, increases linearly in time,

$$
\delta \langle \Delta v_i^2(t) \rangle_{st} \simeq \eta^2 A(d\Omega_s) \left( \frac{\Omega}{m} \right) \left( \frac{\epsilon^2 \Omega}{m} \right) \frac{\pi}{2} \Omega t .
$$

(26)

Heuristically this evolution behavior can also be obtained from the stationary component in Eq. (25) by replacing the bandwidth $\Delta$ with the resonance width, which is of order $t^{-1}$.

This can be understood by the fact that when the bandwidth is wider than the resonance width, the result of the velocity dispersion should not explicitly depend on bandwidth [19]. In contrary, the nonstationary component flattens out, and behaves like

$$
\delta \langle \Delta v_i^2(t) \rangle_{ns} \simeq \mu \eta A(d\Omega_s) \left( \frac{\epsilon^2 \Omega}{m} \right) \left( \frac{\Omega}{\Delta} \right) \cos(2\Omega t - \theta + 2\alpha) .
$$

(27)

It is seen that the growth rate of the envelope of the nonstationary component starts falling behind that of the stationary component since coherence between the squeezed vacuum modes is gradually lost. This cancellation effect will be more significant as the evolution moves to the relaxation regime.

Finally, at much later times, $t \gg \Gamma^{-1}$, the nonstationary component falls off with time as $t^{-1}$, and vanishes eventually

$$
\delta \langle \Delta v_i^2(t) \rangle_{ns} \simeq -\mu \eta A(d\Omega_s) \left( \frac{\epsilon^2 \Omega}{m} \right) \left( \frac{\Omega}{\Delta^2 t} \right) \sin(\Delta t) \cos(2\Omega t - \theta + 2\alpha) ,
$$

(28)

but the stationary component of $\delta \langle \Delta v_i^2(t) \rangle$ saturates to a time-independent constant as a result of the fluctuation-dissipation relation. The saturated value is obtained as

$$
\delta \langle \Delta v_i^2(t) \rangle_{st} \simeq \eta^2 \frac{\pi}{4} A(d\Omega_s) \left( \frac{\epsilon^2 \Omega}{m} \right) \left( \frac{\Omega}{\Gamma} \right) .
$$

(29)

Since the nonstationary component vanishes at asymptotical times, the velocity dispersion can not be reduced by manipulating the nonstationary component via squeeze parameters in the same ways as is done in the extremely early time regime, and its values is solely determined by the stationary component. The corresponding modification in the effective temperature is given by

$$
\delta T_{eff} \sim \eta^2 \frac{3\pi^2}{2} A(d\Omega_s) \frac{\hbar \Omega}{k_B} \sim \bar{n} A(d\Omega_s) \left( \frac{\Omega}{10^9 \text{s}^{-1}} \right) K ,
$$

(30)
FIG. 2: The full-time evolution of the change of the velocity dispersion of a charged oscillator for the case $\Gamma \ll \Delta \ll \Omega = \Xi$ is with $\Gamma/\Omega = 0.004$ and $\Delta/\Omega = 0.015$ is drawn. The nonstationary component of $\delta\langle \Delta v^2_i(t) \rangle$ oscillates rapidly at early moments and then vanishes eventually. The stationary component grows at early time and reaches saturation at asymptotical times.

where $\hbar$ has been put back and $k_B$ is the Boltzmann constant. To arrive at Eq. (30), we have made an substitution of $\Gamma = (e^2/12\pi m) \Omega^2$ in Eq. (29). Compared with the result in Ref. [19] where the vacuum state of electromagnetic fields is considered, the effective temperature we obtain here is found to depend not only on the oscillation frequency of the charged oscillator, but also on the number of photons $\bar{n} = \eta^2$ in each squeezed vacuum mode.

The full-time evolution of the velocity dispersion of a charged oscillator in the on-resonance case is shown in Fig. 2. The evolution of $\delta\langle \Delta v^2_i(t) \rangle$ in time regime $t \ll \Gamma^{-1}$ has been zoomed up. The vertical axis is normalized by the late-time values of the change of the velocity fluctuations $\delta\langle \Delta v^2_i(\infty) \rangle$.

2. off-resonance

When the band of the squeezed vacuum modes does not overlap with the resonance peak, we consider the situation $\Xi - \Delta/2 \gg \Omega + \Gamma/2$, which may have greater effects on charge’s dynamics than the other situation, that is, the mean frequency of the band much smaller
than the resonance frequency. In the off-resonance cases, the stochastic force will not drive
the motion of the charged oscillator into resonance. We may expect that at later times the
velocity dispersion should be much smaller than in the on-resonance case. Since each of the
$L_i$ terms for the stationary component has a comparable contribution, we have to take all
$L_i$ into considerations. The same is true for the nonstationary component.

At early time $\Omega^{-1} \ll t \ll \Delta^{-1} \ll \Gamma^{-1}$, the most dominant contributions of the stationary
component for the change of the velocity fluctuations is

$$
\delta \langle \Delta v_i^2(t) \rangle_{st} = \eta^2 A(d\Omega_s) \left( \frac{\Xi}{m} \right) \left( \frac{e^2 \Xi}{m} \right) \left( \frac{\Delta}{\Xi} \right) \left[ \sin^2 \left( \frac{-\Xi - \Omega}{2} t \right) + \sin^2 \left( \frac{-\Xi + \Omega}{2} t \right) 
- \left( \cos \Xi t - \cos \Omega t \right) \cos(\Omega t + 2\alpha) \right],
$$

and that of the nonstationary component goes as follows

$$
\delta \langle \Delta v_i^2(t) \rangle_{ns} = \mu \eta A(d\Omega_s) \left( \frac{\Xi}{m} \right) \left( \frac{e^2 \Xi}{m} \right) \left( \frac{\Delta}{\Xi} \right) \left[ - \left( \cos \Xi t - \cos \Omega t \right) \cos(\Omega t - \theta) 
+ \sin^2 \left( \frac{-\Xi - \Omega}{2} t \right) \cos(\Xi t + \Omega t - \theta + 2\alpha) + \sin^2 \left( \frac{-\Xi + \Omega}{2} t \right) \cos(\Xi t - \Omega t - \theta - 2\alpha) \right].
$$

The velocity dispersion remains oscillatory in this regime, instead of growing with time as in
the on-resonance case. Then, at later moments $\Delta^{-1} \ll \Gamma^{-1} \ll t$, the stationary component
of $\delta \langle \Delta v_i^2(t) \rangle$ reaches a constant, given by

$$
\delta \langle \Delta v_i^2(t) \rangle_{st} \simeq \eta^2 A(d\Omega_s) \left( \frac{\Xi}{m} \right) \left( \frac{e^2 \Xi}{m} \right) \left( \frac{\Delta}{\Xi} \right),
$$

where $Z \simeq 1$ and $\cos \alpha \simeq 1$ have been applied. The nonstationary component vanishes in
the end as

$$
\delta \langle \Delta v_i^2(t) \rangle_{ns} \simeq -\mu \eta A(d\Omega_s) \left( \frac{\Xi}{m} \right) \left( \frac{e^2 \Xi}{m} \right) \left( \frac{\sin(\Delta t)}{\Xi t} \right) \cos(2\Xi t - \theta).
$$

In comparison with the results of the on-resonance case [29], the magnitude of the velocity
dispersion in Eq. (31) is in general much smaller due to the presence of the large factor
$\Omega/\Gamma$ in Eq. (29). This can be understood by the fact that in the case of off-resonance, the
motion of the charged particle does not evolve in synchronization with the driving force so
that the energy flows in and out of the charge constantly, and its velocity dispersion does
not accumulate steadily as in the resonance situation.

However, in principle, this off-resonance result might be comparable to that in the on-
resonance case, that is,

$$
\left( \frac{\Xi}{m} \right) \left( \frac{e^2 \Xi}{m} \right) \left( \frac{\Delta}{\Xi} \right) \simeq \left( \frac{\Omega}{m} \right) \left( \frac{e^2 \Omega}{m} \right) \left( \frac{\Omega}{\Gamma} \right),
$$
if the mean frequency of the band is sufficiently high and satisfies the condition

\[
\frac{\Xi}{\Omega} \gtrsim \frac{\Omega^2}{\Gamma \Delta} \sim \mathcal{O}\left(\frac{c}{r_c \Delta}\right),
\]

(33)

where \(r_c\) is the classical radius of the charge. If we consider the charge as an electron, then \(r_c \sim 2.82 \times 10^{-15}\) m, and \(\Gamma/\Omega \approx 10^{-23} \times \Omega\), which is an extremely small value for the typical choice of \(\Omega\) in the unit of \(s^{-1}\). Hence, for the electron, \(\Gamma \ll \Delta\) is a plausible assumption. Then we find that the mean frequency of the band \(\Xi\) must be about the order \(10^{23} \times (\Omega/\Delta)\) s\(^{-1}\) to satisfy the condition (33). Since the ratio \(\Delta/\Omega\) is assumed to be small, the mean frequency of the excited squeezed vacuum states must be much greater than \(10^{23}\) s\(^{-1}\), which is very unlikely if the squeezed light is produced by some nonlinear process in a medium because this frequency value has been much higher than the plasma frequency of typical materials. Therefore under normal circumstances, the velocity dispersion at late time should be much smaller in the off-resonance case than in the on-resonance case.

IV. SUMMARY AND PROSPECTS

Here the nature of the quantum fluctuations of the electromagnetic squeezed vacuum state is studied. We derive the Langevin equation of the charge moving in such an environment. This stochastic approach consistently incorporates both fluctuations and dissipation backreaction of the quantized electromagnetic field on the charge. The squeezed vacuum state brings in two distinct contributions to the quantum noise. Apart from the stationary noise, there exists a new feature of the nonstationary noise due to the fact that this quantum state is presumably generated from nonlinear processes of particle creation where the time translational invariance is not respected.

We then study the full-time evolution of the velocity dispersion of the charge when it undergoes harmonic motion and couples to the squeezed vacuum modes of the electromagnetic fields. We find that the presence of the nonstationary noise can reduce the velocity dispersion of the charge from the value solely given by the vacuum state of the electromagnetic fields for an appropriate choice of the squeeze parameters. This reduction is especially significant when the charge is just about to move. Nonetheless, the dissipative backreaction effect, characterized by the decay constant \(\Gamma\), is increasingly important when charge’s motion becomes oscillatory. We first consider the on-resonance case, where the frequency
of the charged oscillator lies within the band of the squeezed vacuum modes. We assume that the mean frequency of the band $\Xi$ is equal to the resonance frequency $\Omega$, and that the bandwidth $\Delta$ is much smaller than the mean frequency $\Xi$ such that $\Gamma \ll \Delta \ll \Xi = \Omega$ is satisfied. The change of the velocity fluctuations in general grows with time at early moments $\Omega^{-1} \ll t \ll \Gamma^{-1}$. In this time regime, although the stationary component and the envelope of the nonstationary component start growing quadratically with time in the regime $\Omega^{-1} \ll t \ll \Delta^{-1}$, their evolution behaviors go through a transition at $t \simeq \Delta^{-1}$, where the excited squeezed vacuum modes gradually go out of phase with each other. After the transition time, in the regime $\Delta^{-1} \ll t \ll \Gamma^{-1}$, the increase of the stationary component slows down, and becomes linearly in time, while the nonstationary component falls off much more quickly so its envelope flattens out. Thus the nonstationary component becomes less significant than the stationary component. At later times $t \gg \Gamma^{-1}$, the nonstationary component vanishes like $t^{-1}$ but the stationary component saturates. Therefore, charge’s velocity fluctuations will reach a time-independent constant, solely determined by its stationary component as a result of the fluctuation-dissipation relation. It also indicates that the squeezed vacuum fluctuations are not always effective in reducing velocity dispersion of the charged particle. The effectiveness thus depends on the state of its motion. Moreover, the velocity dispersion of the charged oscillator, driven by the squeezed vacuum fluctuations, has been shown to be greater in the on-resonance situation than in the off-resonance case.

It is quite straightforward to extend the current approach to the case of neutral atoms, which couple to the electromagnetic squeezed vacuum state by the atomic dipole moment. The intrinsic quantum coherence of the squeezed vacuum may introduce the colored noise, which in turns affects the evolution of the atom. Therefore it would be of interest to take the effects of colored noise into account during the processes of the laser cooling of neutral atoms. In addition, the atom in such a fluctuating environment, driven by any external force, will inevitably move in such a way that the extension beyond the static atom assumption is needed in order to assess the full evolution of its velocity dispersion owing to squeezed vacuum fluctuations [2, 4]. This will be tackled in the near future.
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