ON THE SEMITOPOLOGICAL EXTENDED BICYCLIC SEMIGROUP  
WITH ADJOINED ZERO

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It is shown that every Hausdorff locally compact semigroup topology on the extended bicyclic semigroup with adjoined zero $\mathbb{C}_Z^0$ is discrete. At the same time, on $\mathbb{C}_Z^0$, there exist different Hausdorff locally compact shift-continuous topologies. In addition, on $\mathbb{C}_Z^0$, we construct a unique minimal shift-continuous topology and a unique minimal inverse semigroup topology.

Keywords: extended bicyclic semigroup, locally compact semitopological semigroup, topological semigroup, minimal topological semigroup, discrete.

Introduction and Preliminaries

We follow the terminology of [13, 14, 17, 31]. In the present paper, we assume that all spaces are Hausdorff. By $\mathbb{Z}$, $\mathbb{N}_0$, and $\mathbb{N}$ we denote the sets of all integers, nonnegative integers, and positive integers, respectively.

A semigroup is a nonempty set with binary associative operation. A semigroup $S$ is called inverse if every $a \in S$ possesses a unique inverse in $S$, i.e. if there exists a unique element $a^{-1} \in S$ such that

\[ a \cdot a^{-1} \cdot a = a \quad \text{and} \quad a^{-1} \cdot a \cdot a^{-1} = a^{-1}. \]

A map that associates any element of an inverse semigroup with its inverse is called the inversion.

For a semigroup $S$, by $E(S)$ we denote the subset of all idempotents in $S$. If $E(S)$ is closed under multiplication, then we refer to $E(S)$ as the band of $S$. The semigroup operation on $S$ determines the following partial order $\preceq$ on $E(S)$: $e \preceq f$ if and only if $ef = fe = e$. This order is called the natural partial order on $E(S)$. A semilattice is a commutative semigroup of idempotents. A semilattice $E$ is called linearly ordered or a chain if its natural partial order is a linear order. The maximal chain of a semilattice $E$ is defined as a chain, which is not properly contained in any other chain of $E$.

The axiom of choice implies the existence of maximal chains in every partially ordered set. According to [30, Definition II.5.12], a chain $L$ is called an $\omega$-chain if $L$ is order-isomorphic to $\{0, -1, -2, -3, \ldots\}$ with the ordinary order $\leq$ or, equivalently, if $L$ is isomorphic to $(\mathbb{N}_0, \max)$.

A bicyclic semigroup (or bicyclic monoid) $\mathcal{C}(p,q)$ is a semigroup with identity 1 generated by two elements $p$ and $q$ subjected only to the condition $pq = 1$. The bicyclic monoid $\mathcal{C}(p,q)$ is a combinatorial bi-simple $F$-inverse semigroup (see [28]). It plays an important role in the algebraic theory of semigroups and in

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the theory of topological semigroups. Thus, the well-known Andersen’s result [2] states that a (0-)simple semigroup is completely (0-)simple if and only if it does not contain the bicyclic semigroup. The bicyclic semigroup cannot be embedded into stable semigroups [27].

A (semi)topological semigroup is a topological space with (separately) continuous semigroup operation. An inverse topological semigroup with continuous inversion is called a topological inverse semigroup. A topology \( \tau \) on a semigroup \( S \) is called:

- shift-continuous if \( (S, \tau) \) is a semitopological semigroup;
- semigroup if \( (S, \tau) \) is a topological semigroup;
- inverse semigroup if \( (S, \tau) \) is a topological inverse semigroup.

The bicyclic semigroup admits solely the discrete semigroup topology and if a topological semigroup \( S \) contains it as a dense subsemigroup, then \( C(p,q) \) is an open subset of \( S \) [16]. Bertman and West [12] extended this result to the case of Hausdorff semitopological semigroups. Stable and \( \Gamma \)-compact topological semigroups do not contain the bicyclic semigroup [3, 25]. The problem of embedding of the bicyclic monoid into compact-like topological semigroups was studied in [4, 5, 10, 24]. Moreover, in [18] it was proved that the discrete topology is the unique topology on the extended bicyclic semigroup \( C_\mathbb{Z} \) such that the semigroup operation on \( C_\mathbb{Z} \) is separately continuous. An unexpected dichotomy for the bicyclic monoid with adjoined zero \( C^0 = C(p,q) \cup \{0\} \) was found in [20]: every Hausdorff locally compact semitopological bicyclic semigroup with adjoined zero \( C^0 \) is either compact or discrete.

The indicated dichotomy was extended by Bardyla [7] to locally compact \( \lambda \)-polycyclic semitopological monoids. In [8], this dichotomy was extended to locally compact semitopological graph inverse semigroups. Further, in [21], it was extended by the authors to locally compact semitopological interassociates of the bicyclic monoid with adjoined zero. Moreover, in [19], it was extended to locally compact semitopological 0-bisimple inverse \( \omega \)-semigroups with compact maximal subgroups. The lattice of all weak shift-continuous topologies on \( C^0 \) was described in [9].

On the Cartesian product \( C_\mathbb{Z} = \mathbb{Z} \times \mathbb{Z} \) we define the semigroup operation as follows:

\[
(a,b)(c,d) = \begin{cases} 
(a-b+c,d), & b < c, \\
(a,d), & b = c, \\
(a,d+b-c), & b > c, 
\end{cases}
\]  

for \( a,b,c,d \in \mathbb{Z} \). The set \( C_\mathbb{Z} \) with the operation defined above is called the extended bicyclic semigroup [33].

The algebraic properties of \( C_\mathbb{Z} \) were described in [18]. It was proved that every nontrivial congruence \( \mathcal{C} \) on the semigroup \( C_\mathbb{Z} \) is a group congruence and, moreover, the quotient semigroup \( C_\mathbb{Z}/\mathcal{C} \) is isomorphic to a cyclic group. It was shown that the semigroup \( C_\mathbb{Z} \), as a Hausdorff semitopological semigroup, admits only the discrete topology. Furthermore, the closure \( \text{cl}_T(C_\mathbb{Z}) \) of the semigroup \( C_\mathbb{Z} \) in a topological semigroup \( T \) was also studied in the same work.

In [22], we proved that the group \( \text{Aut}(C_\mathbb{Z}) \) of automorphisms of the extended bicyclic semigroup \( C_\mathbb{Z} \) is isomorphic to the additive group of integers.

By \( C^0_\mathbb{Z} \) we denote the extended bicyclic semigroup \( C_\mathbb{Z} \) with adjoined zero 0.
In the present paper, we show that every Hausdorff locally compact semigroup topology on the semigroup $\mathbb{C}_Z^0$ is discrete. At the same time, on $\mathbb{C}_Z^0$ there exist $c$ different Hausdorff locally compact shift-continuous topologies. Moreover, on $\mathbb{C}_Z^0$, we construct the unique minimal shift-continuous topology and the unique minimal inverse semigroup topology.

1. Locally Compact Shift-Continuous Topologies on the Extended Bicyclic Semigroup

We need the following simple statement:

**Proposition 1** [18, Proposition 2.1(viii)]. For every integer $n$ the set

$$\mathbb{C}_Z[n] = \{(a,b): a \geq n \wedge b \geq n\}$$

is an inverse subsemigroup of $\mathbb{C}_Z$ isomorphic to the bicyclic semigroup $\mathbb{C}(p,q)$ by the map

$$h: \mathbb{C}_Z[n] \rightarrow \mathbb{C}(p,q), \quad (a,b) \mapsto q^{a-n}p^{b-n}.$$

Proposition 1 implies the following corollary:

**Corollary 1.** For every integer $n$ the set $\mathbb{C}_Z^0[n] = \mathbb{C}_Z[n] \cup \{0\}$ is an inverse subsemigroup of $\mathbb{C}_Z^0$ isomorphic to the bicyclic monoid $\mathbb{C}_0$ with adjoined zero by the map $h: \mathbb{C}_Z^0[n] \rightarrow \mathbb{C}_0$, $(a,b) \mapsto q^{a-n}p^{b-n}$, and $0 \mapsto 0$.

**Lemma 1.** Let $\tau$ be a nondiscrete Hausdorff shift-continuous topology on $\mathbb{C}_Z^0$. Then $\mathbb{C}_Z^0[n]$ is a nondiscrete subsemigroup of $(\mathbb{C}_Z^0,\tau)$ for any integer $n$.

**Proof.** First we observe that, by Theorem 1 from [18], all nonzero elements of the semigroup $\mathbb{C}_Z^0$ are isolated points in $(\mathbb{C}_Z^0,\tau)$.

On the contrary, suppose that there exist a nondiscrete Hausdorff shift-continuous topology $\tau$ on $\mathbb{C}_Z^0$ and an integer $n$ such that $\mathbb{C}_Z^0[n]$ is a discrete subsemigroup of $(\mathbb{C}_Z^0,\tau)$. We fix an arbitrary open neighborhood $U(0)$ of zero 0 in $(\mathbb{C}_Z^0,\tau)$ such that $U(0) \cap \mathbb{C}_Z^0[n] = \{0\}$. Then the separate continuity of the semigroup operation in $(\mathbb{C}_Z^0,\tau)$ implies that there exists an open neighborhood $V(0) \subseteq U(0)$ of zero 0 in $(\mathbb{C}_Z^0,\tau)$ such that $(n,n) \cdot V(0) \cdot (n,n) \subseteq U(0)$. Our assumption implies that every open neighborhood $W(0) \subseteq U(0)$ of zero 0 in $(\mathbb{C}_Z^0,\tau)$ contains infinitely many points $(x,y)$ such that $x \leq n$ or $y \leq n$. Then, for any nonzero $(x,y) \in V(0)$, by virtue of relation (1), we have

$$(n,n) \cdot (x,y) \cdot (n,n) = (n,n-x+y) \cdot (n,n) = \begin{cases} (n+x-y,n), & y \leq x, \\ (n,n-x+y), & y \geq x. \end{cases}$$
and, hence, \( (n,n) \cdot V(0) \cdot (n,n) \cap C^0_Z[n] \neq \emptyset \) which contradicts the assumption \( U(0) \cap C^0_Z[n] = \{0\} \). The obtained contradiction implies the statement of the lemma.

For any nonzero element \((a,b) \in C^0_Z\), we denote

\[
\uparrow_{\preceq} (a,b) = \{(x,y) \in C^0_Z : (a,b) \preceq (x,y)\},
\]

where \( \preceq \) is the natural partial order on \( C^0_Z \). It is obvious that

\[
\uparrow_{\preceq} (a,b) = \{(x,y) \in C^0_Z : a-b = x-y, x \leq a \text{ in } (\mathbb{Z}, \preceq)\}.
\]

**Lemma 2.** Let \((a,b),(c,d),(e,f) \in C^0_Z\) be such that \((a,b) \cdot (c,d) = (e,f)\). Then the following statements hold:

(i) if \( b \leq c \) then \((x,y) \cdot (c,d) = (e,f)\) for any \((x,y) \in \uparrow_{\preceq} (a,b)\) and, moreover, there exists a minimal element \((\hat{a},\hat{b}) \preceq (a,b)\) in \( C^0_Z \) such that \((\hat{a},\hat{b}) \cdot (c,d) = (e,f)\); furthermore, there exist no other elements \((x,y) \in C^0_Z\) with the property \((x,y) \cdot (c,d) = (e,f)\);

(ii) if \( b \geq c \) then \((a,b) \cdot (x,y) = (e,f)\) for any \((x,y) \in \uparrow_{\preceq} (c,d)\) and, moreover, there exists a minimal element \((\hat{c},\hat{d}) \preceq (c,d)\) in \( C^0_Z \) such that \((a,b) \cdot (\hat{c},\hat{d}) = (e,f)\); in addition, there are no other elements \((x,y) \in C^0_Z\) with the property \((a,b) \cdot (x,y) = (e,f)\).

**Proof.** (i). Since \( b \leq c \), the semigroup operation of \( C^0_Z \) implies that \((b,b) \cdot (c,d) = (c,d)\). Further, if \((a,b) \preceq (x,y)\), then it follows from Lemma 1.4.6(5) in [28] that

\[
(x,y) \cdot (b,b) = (x,y) \cdot (a,b)^{-1} \cdot (a,b) = (a,b),
\]

and, therefore, we conclude that

\[
(x,y) \cdot (c,d) = (x,y) \cdot ((b,b) \cdot (c,d)) = ((x,y) \cdot (b,b)) \cdot (c,d) = (a,b) \cdot (c,d) = (e,f).
\]

We set \((\hat{a},\hat{b}) = (a-b+c,c)\). Thus, \((\hat{a},\hat{b}) \preceq (a,b)\) and relation (1) implies that the element \((\hat{a},\hat{b})\) is required.

The last statement follows from Proposition 2.1 in [18] and relation (1).

The proof of statement (ii) is similar.

**Lemma 3.** Let \( \tau \) be a nondiscrete Hausdorff shift-continuous topology on \( C^0_Z \). Then the natural partial order \( \preceq \) is closed on \((C^0_Z, \tau)\) and \( \uparrow_{\preceq} (a,b) \) is an open-and-closed subset of \((C^0_Z, \tau)\) for any nonzero element \((a,b) \) of \( C^0_Z \).
Proof. By Theorem 1 in [18] all nonzero elements of the semigroup $C^0_Z$ are isolated points in $(C^0_Z, \tau)$. Since $0 \preceq (a,b)$ for any $(a,b) \in C^0_Z$, this yields the first statement of the lemma.

The definition of the natural partial order $\preceq$ on $C^0_Z$ and the separate continuity of the semigroup operation on $(C^0_Z, \tau)$ imply the second statement because

$$\uparrow_{\preceq} (a,b) = \{(x,y) \in C^0_Z : (a,a) \cdot (x,y) = (a,b)\}.$$

Proposition 2. Assume that the semigroup $C^0_Z$ admits a nondiscrete Hausdorff locally compact shift-continuous topology $\tau$. Then the following statements hold:

(i) for any open neighborhood $U(0)$ of zero, there exists a compact-and-open neighborhood $V(0) \subseteq U(0)$ of 0 in $(C^0_Z, \tau)$;

(ii) the set $\uparrow_{\preceq} (a,b) \cap U(0)$ is finite for any compact-and-open neighborhood $V(0) \subseteq U(0)$ of the zero 0 in $(C^0_Z, \tau)$ and any nonzero element $(a,b)$ of $C^0_Z$;

(iii) for any open neighborhood $U(0)$ of zero in $(C^0_Z, \tau)$ and any integer $n$, the set $U(0) \setminus C^0_Z[n]$ is finite.

Proof. Statement (i) follows from Theorem 1 of [18] and the local compactness of the space $(C^0_Z, \tau)$.
Statement (ii) follows from Lemma 3 and Theorem 1 in [18].

(iii). It is obvious that $C^0_Z[n] = (n,n) \cdot C^0_Z \cdot (n,n)$ for any integer $n$. This implies that $C^0_Z[n]$ is a closed subset of $(C^0_Z, \tau)$ because $C^0_Z[n]$ is a retract of the space $(C^0_Z, \tau)$ and, hence, by Corollary 3.3.10 from [17], it is locally compact. Since the topology $\tau$ is nondiscrete, Lemma 1 and Theorem 1 in [20] imply that $C^0_Z[n]$ is a compact subspace of $(C^0_Z, \tau)$. Finally, we apply Theorem 1 from [18].

Further, we construct an example of nondiscrete Hausdorff locally compact shift-continuous topology on the semigroup $C^0_Z$, which is neither compact nor discrete.

Example 1. Let $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ be two increasing sequences of positive integers with the following properties: $x_1, y_1 > 1$ and

$$x_n + 1 < x_{n+1}, \quad 2 < y_n + 1 < y_{n+1}$$

for any $n \in \mathbb{N}$.

We denote

$$A_0 = \uparrow_{\preceq} (0,0) \cup \bigcup_{i=1}^{x_1-1} \uparrow_{\preceq} (0,-i) \cup \bigcup_{i=1}^{y_1-1} \uparrow_{\preceq} (-j,0).$$
and
\[
A_n^d = \bigcup_{i=x_n}^{x_{n+1}-1} (-x_n, -i), \quad A_n^\ell = \bigcup_{j=y_n}^{y_{n+1}-1} (-j, -y_n)
\]
for any positive integer \( n \).

Further, we set
\[
D = A_0 \cup \bigcup_{i \in \mathbb{N}} (A_i^d \cup A_i^\ell).
\]

For finitely many \((a_1, b_1), \ldots, (a_k, b_k) \in \mathbb{C}_Z\), we denote
\[
U_{(a_1, b_1), \ldots, (a_k, b_k)} = \mathbb{C}_Z^0 \setminus (D \cup \bigcup_{i=1}^{k} (a_1, b_1) \cup \cdots \cup (a_k, b_k)).
\]

We define a topology \( \tau_{\{x_n\}} \) on the semigroup \( \mathbb{C}_Z^0 \) in the following way:

(i) all nonzero elements of \( \mathbb{C}_Z^0 \) are isolated points;

(ii) the family
\[
B_{\tau_{\{x_n\}}}^0 = \{U_{(a_1, b_1), \ldots, (a_k, b_k)} : (a_1, b_1), \ldots, (a_k, b_k) \in \mathbb{C}_Z, k \in \mathbb{N}\}
\]

is the base of the topology \( \tau_{\{x_n\}} \) at zero 0.

**Proposition 3.**

(i) The set \( \bigcup_{i \in \mathbb{N}} (a,b) \setminus D \) is finite for any \((a,b) \in \mathbb{C}_Z\);

(ii) \( D \) is a compact subset of the space \((\mathbb{C}_Z^0, \tau_{\{x_n\}})\);

(iii) the space \((\mathbb{C}_Z^0, \tau_{\{x_n\}})\) is locally compact and Hausdorff.

**Proof.** (i) The statement is trivial for \((a,b) \in D\). We assume that \((a,b) \notin D\) and consider the following cases:

(a) If \( a = b \), then \( \bigcup_{i \in \mathbb{N}} (a,b) \setminus D = \{(1,1), \ldots, (a,a)\} \).

(b) Suppose that \( a < b \). Then either there exists a positive integer \( i \geq 1 \) such that either \( y_i \leq b - a < y_{i+1} \) or \( b - a < y_1 \). In the first case we have
\[ \uparrow \leq (a,b) \setminus D = \left\{ (-i+1-b+a,-i+1), \ldots, (a,b) \right\} \]
\[ = \bigcup \{(k-b+a,k) : k = -i+1, \ldots \}. \]

In the second case we conclude that \( b > 0 \) and, hence,
\[ \uparrow \leq (a,b) \setminus D = \{(1-b+a,1), \ldots, (a,b)\} = \bigcup \{(k-b+a,k) : k = 1, \ldots \}. \]

(c) Suppose that \( a > b \). Then either there exists a positive integer \( j \geq 1 \) such that \( x_j \leq a - b < x_{j+1} \) or \( a - b < x_1 \). In the first case, we get
\[ \uparrow \leq (a,b) \setminus D = \left\{ (-j+1,-j+1-a+b), \ldots, (a,b) \right\} \]
\[ = \bigcup \{(k-a+b,k) : k = -j+1, \ldots, a \}. \]

In the second case we have \( a > 0 \) and, therefore,
\[ \uparrow \leq (a,b) \setminus D = \{(1,1-a+b), \ldots, (a,b)\} = \bigcup \{(k,k-a+b) : k = 1, \ldots, a \}. \]

Statement (i) is proved. Statement (ii) now follows from (i).

Since all nonzero elements of \( C_Z^0 \) are isolated points in \( (C_Z^0, \tau_{\{y_n\}}^{\{x_n\}}) \), statement (iii) follows from (ii).

For any nonzero element \( (a,b) \) of \( C_Z^0 \) we denote
\[ S^{b \uparrow} = \{(x,y) \in C_Z : y \geq b\} \cup \{0\}, \]
\[ S^{\rightarrow a} = \{(x,y) \in C_Z : x \geq a\} \cup \{0\}. \]

It is obvious that \( (a,b)C_Z^0 = S^{\rightarrow a} \) and \( C_Z^0 (a,b) = S^{b \uparrow} \) for any nonzero \( (a,b) \in C_Z^0 \).

**Theorem 1.** \( (C_Z^0, \tau_{\{y_n\}}^{\{x_n\}}) \) is a semitopological semigroup.

**Proof.** By the definition of the topology \( \tau_{\{y_n\}}^{\{x_n\}} \), it is sufficient to prove that the left and right shifts of \( C_Z^0 \) are continuous at zero 0.

We fix an arbitrary nonzero element \( (a,b) \in C_Z^0 \) and an arbitrary basic open neighborhood \( U(a_1, b_1), \ldots, (a_k, b_k) \) of zero 0 in \( (C_Z^0, \tau_{\{y_n\}}^{\{x_n\}}) \).

The definition of the topology \( \tau_{\{y_n\}}^{\{x_n\}} \) implies that there exist finitely many nonzero elements \( (e_1, f_1), \ldots, \),
\((e_m, f_m)\) of the semigroup \(C_\mathbb{Z}^0\) with \(e_1, \ldots, e_m \geq a\) such that

\[
U(a_1, b_1) \cap \hat{S}^{a} = \hat{S}^{a} \setminus (\uparrow \preceq (e_1, f_1) \cup \ldots \cup \uparrow \preceq (e_m, f_m)).
\]

Since \((a, b)C_\mathbb{Z}^0 = \hat{S}^{a}\), by virtue of Lemma 2(ii), there exist minimal elements \((\hat{c}_1, \hat{d}_1), \ldots, (\hat{c}_m, \hat{d}_m)\) in \(C_\mathbb{Z}\) such that

\[
(a, b) \cdot (\hat{c}_1, \hat{d}_1) = (e_1, f_1), \ldots, (a, b) \cdot (\hat{c}_m, \hat{d}_m) = (e_m, f_m).
\]

Thus, the last equalities imply that

\[
(a, b) \cdot U(\hat{c}_1, \hat{d}_1), \ldots, (\hat{c}_m, \hat{d}_m) \subseteq U(a_1, b_1), \ldots, (a_k, b_k).
\]

Similarly, there exist finitely many nonzero elements \((e_1, f_1), \ldots, (e_p, f_p)\) of the semigroup \(C_\mathbb{Z}^0\) with \(f_1, \ldots, f_p \geq b\) such that

\[
U(a_1, b_1) \cap S^{b^\uparrow} = S^{b^\uparrow} \setminus (\uparrow \preceq (e_1, f_1) \cup \ldots \cup \uparrow \preceq (e_p, f_p)).
\]

Since \(C_\mathbb{Z}(a, b) = S^{b^\uparrow}\), by Lemma 2(i) there exist minimal elements \((\hat{c}_1, \hat{d}_1), \ldots, (\hat{c}_p, \hat{d}_p)\) in \(C_\mathbb{Z}\) such that

\[
(\hat{c}_1, \hat{d}_1) \cdot (a, b) = (e_1, f_1), \ldots, (\hat{c}_p, \hat{d}_p) \cdot (a, b) = (e_p, f_p).
\]

Then the last equalities imply that

\[
U((\hat{c}_1, \hat{d}_1), \ldots, (\hat{c}_p, \hat{d}_p)) \cdot (a, b) \subseteq U(a_1, b_1), \ldots, (a_k, b_k),
\]

which completes the proof of separate continuity of the semigroup operation in \((C_\mathbb{Z}^0, \tau_{\{x_n\}})\).

If, in Example 1, we set \(x_i = y_i\) for any \(i \in \mathbb{N}\) and denote \(\tau_{\{x_n\}} = \tau_{\{y_n\}}\), then we get

\[
(U(a_1, b_1), \ldots, (a_k, b_k))^{-1} = U(b_1, a_1), \ldots, (b_k, a_k)
\]

for any \(a_1, b_1, \ldots, a_k, b_k \in \mathbb{Z}\). This and Theorem 1 yield the following corollary:

**Corollary 2.** \((C_\mathbb{Z}^0, \tau_{\{x_n\}})\) is a Hausdorff locally compact semitopological semigroup with continuous inversion.
Theorem 1 implies that, on the semigroup \( C^0_Z \), there exist \( \epsilon \) Hausdorff locally compact shift-continuous topologies. However, Lemma 1 implies the following counterpart of Corollary 1 from [20]:

**Corollary 3.** Every Hausdorff locally compact semigroup topology on the semigroup \( C^0_Z \) is discrete.

2. Minimal Shift-Continuous and Inverse Semigroup Topologies on \( C^0_Z \)

The concept of minimal topological group was independently introduced in the early 1970s by Doïtchinov [15] and Stephenson [32]. Both authors were motivated by the theory of minimal topological spaces, which was well understood at that time (cf. [11]). More than 20 years earlier, Nachbin [29] had studied minimality in the context of division rings. Moreover, Banaschewski [6] investigated minimality in a more general setting of topological algebras. The concept of minimal topological semigroup was introduced in [23].

**Definition 1** [23]. A Hausdorff semitopological (respectively, topological or topological inverse) semigroup \((S, \tau)\) is called minimal if no Hausdorff shift-continuous (respectively, semigroup or semigroup inverse) topology on \( S \) is strictly contained in \( \tau \). If \((S, \tau)\) is minimal semitopological (respectively, topological or topological inverse) semigroup, then \( \tau \) is called the minimal shift-continuous (respectively, semigroup or semigroup inverse) topology.

It is obvious that every Hausdorff compact shift-continuous (respectively, semigroup or semigroup inverse) topology on a semigroup \( S \) is a minimal shift-continuous (respectively, semigroup or semigroup inverse) topology on \( S \). However, an infinite semigroup of matrix units admits a unique compact shift-continuous topology, as well as noncompact minimal semigroup and inverse semigroup topologies [23]. Similar results were obtained in [9] for the bicyclic monoid with adjoined zero \( C^0 \).

**Example 2.** For finitely many \( (a_1,b_1), \ldots, (a_k,b_k) \in C^0_Z \), we denote

\[
U^\uparrow_{(a_1,b_1), \ldots, (a_k,b_k)} = C^0_Z \setminus \bigcup_{=\pm} (a_1,b_1) \cup \ldots \cup \bigcup_{=\pm} (a_k,b_k).
\]

We define a topology \( \tau_{\text{min}}^{\text{sh}} \) on the semigroup \( C^0_Z \) in the following way:

1. all nonzero elements of \( C^0_Z \) are isolated points;
2. the family

\[
B^0_{\tau_{\text{min}}^{\text{sh}}} = \{ U^\uparrow_{(a_1,b_1), \ldots, (a_k,b_k)} : (a_1,b_1), \ldots, (a_k,b_k) \in C^0_Z, k \in \mathbb{N} \}
\]

is the base of the topology \( \tau_{\text{min}}^{\text{sh}} \) at zero 0.

Note that, by Lemma 3, the space \((C^0_Z, \tau_{\text{min}}^{\text{sh}})\) is Hausdorff, 0-dimensional and scattered. Hence, it is regul-
lar. Since the base $B^{0}_{\tau_{\text{min}}}^\text{sh}$ is countable, by the Urysohn metrization theorem (see [26, p. 123, Theorem 16]), the space $(C_{Z}, \tau_{\text{min}}^\text{sh})$ is metrizable and, therefore, by Corollary 4.1.13 from [17], it is perfectly normal.

**Proposition 4.** $(C_{Z}^{0}, \tau_{\text{min}}^\text{sh})$ is a minimal semitopological semigroup with continuous inversion.

**Proof.** The definition of the topology $\tau_{\text{min}}^\text{sh}$ implies that it is sufficient to prove that the left and right shifts of $C_{Z}^{0}$ are continuous at zero 0.

We fix any nonzero element $(a, b) \in C_{Z}^{0}$ and any basic open neighborhood $U_{(a_{1}, b_{1}), \ldots, (a_{k}, b_{k})}^{\uparrow}$ of zero 0 in $(C_{Z}^{0}, \tau_{\text{min}}^\text{sh})$.

The definition of the topology $\tau_{\text{min}}^\text{sh}$ implies that there exist finitely many nonzero elements $(e_{1}, f_{1}), \ldots, (e_{m}, f_{m})$ of the semigroup $C_{Z}^{0}$ with $e_{1}, \ldots, e_{m} \geq a$ such that

$$U_{(a_{1}, b_{1}), \ldots, (a_{k}, b_{k})}^{\uparrow} \cap S^{\rightarrow} = S^{\rightarrow} \setminus (\uparrow \leq (e_{1}, f_{1}) \cup \ldots \cup \uparrow \leq (e_{m}, f_{m})).$$

Since $(a, b)C_{Z}^{0} = S^{\rightarrow}$, by Lemma 2(ii) there exist minimal elements $(\hat{c}_{1}, \hat{d}_{1}), \ldots, (\hat{c}_{m}, \hat{d}_{m})$ in $C_{Z}$ such that

$$(a, b) \cdot (\hat{c}_{1}, \hat{d}_{1}) = (e_{1}, f_{1}), \ldots, (a, b) \cdot (\hat{c}_{m}, \hat{d}_{m}) = (e_{m}, f_{m}).$$

Then the last equalities imply that

$$(a, b) \cdot U_{(\hat{c}_{1}, \hat{d}_{1}), \ldots, (\hat{c}_{m}, \hat{d}_{m})}^{\uparrow} \subseteq U_{(a_{1}, b_{1}), \ldots, (a_{k}, b_{k})}^{\uparrow}.$$

Further, in a similar way, we conclude that there exists finitely many nonzero elements $(e_{1}, f_{1}), \ldots, (e_{p}, f_{p})$ of the semigroup $C_{Z}^{0}$ with $f_{1}, \ldots, f_{p} \geq b$ such that

$$U_{(a_{1}, b_{1}), \ldots, (a_{k}, b_{k})}^{\uparrow} \cap S^{\leftarrow} = S^{\leftarrow} \setminus (\uparrow \leq (e_{1}, f_{1}) \cup \ldots \cup \uparrow \leq (e_{p}, f_{p})).$$

Since

$$C_{Z}^{0}(a, b) = S^{\leftarrow},$$

Lemma 2(i) implies that there exist minimal elements $(\hat{c}_{1}, \hat{d}_{1}), \ldots, (\hat{c}_{p}, \hat{d}_{p})$ in $C_{Z}$ such that

$$(\hat{c}_{1}, \hat{d}_{1}) \cdot (a, b) = (e_{1}, f_{1}), \ldots, (\hat{c}_{p}, \hat{d}_{p}) \cdot (a, b) = (e_{p}, f_{p}).$$

Then the last equalities imply that
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\[ U^+_{(c_1,d_1),\ldots,(c_p,d_p)} \cdot (a,b) \subseteq U_{(a_1,b_1),\ldots,(a_k,b_k)}, \]

which completes the proof of separate continuity of the semigroup operation in \((C^0_Z, \tau^{sh}_{\min})\).

In addition, since

\[ \left( U^+_{(a_1,b_1),\ldots,(a_k,b_k)} \right)^{-1} = U^+_{(b_1,a_1),\ldots,(b_k,a_k)} \]

for any \((a_1,b_1),\ldots,(a_k,b_k) \in C_Z\), the inversion is also continuous in \((C^0_Z, \tau^{sh}_{\min})\).

Lemma 3 implies that \(\tau^{sh}_{\min}\) is the coarsest Hausdorff shift-continuous topology on \(C^0_Z\) and, therefore, \((C^0_Z, \tau^{sh}_{\min})\) is the minimal semitopological semigroup.

**Example 3.** We define a topology \(\tau^{i}_{\min}\) on the semigroup \(C^0_Z\) in the following way:

1. All nonzero elements of \(C^0_Z\) are isolated points in the topological space \((C^0_Z, \tau^{i}_{\min});\)

2. The family

\[ B^0_{\tau^{i}_{\min}} = \{ S_{\alpha}^\rightarrow \cap S_{\beta}^\leftarrow : a, b \in Z \} \]

is the base of the topology \(\tau^{i}_{\min}\) at zero \(0\).

It is obvious that the space \((C^0_Z, \tau^{sh}_{\min})\) is Hausdorff, 0-dimensional, and scattered. Hence, it is regular. Since the base \(B^0_{\tau^{i}_{\min}}\) is countable, by analogy with Example 2, we conclude that the space \((C^0_Z, \tau^{i}_{\min})\) is metrizable.

**Proposition 5.** \((C^0_Z, \tau^{i}_{\min})\) is a minimal topological inverse semigroup.

**Proof.** We conclude that, for any \(a, b \in Z\) and any nonzero element \((x, y) \in C^0_Z\), there exists an integer \(n\) such that

\[ (x, y) \in C^0_Z[n] \quad \text{and} \quad S_{\alpha}^\rightarrow \cap S_{\beta}^\leftarrow \subseteq C^0_Z[n]. \]

By Corollary 1, the semigroup \(C^0_Z[n]\) is isomorphic to the bicyclic monoid with adjoined zero \(C^0\). Moreover, it is obvious that the topology \(\tau^{sh}_{\min}\) induces the topology \(\tau\) on \(C^0_Z[n]\) such that \(\tau\) generates, by the map \(h: C^0_Z[n] \to C^0\), \((a,b) \to q^{a-n} p^{b-n}\), and \(0 \to 0\), the topology \(\tau_{\min}\) on \(C^0\) [9]. Then the proof of Lemma 2 in [1] implies that \((C^0, \tau_{\min})\) is a Hausdorff topological semigroup. This and the arguments presented above imply that \((C^0_Z, \tau^{i}_{\min})\) is a topological inverse semigroup. The minimality of \((C^0_Z, \tau^{i}_{\min})\) as a topological inverse...
semigroup follows from Lemma 3 because

\[
C^0_Z \setminus (S^a \cap S^b) = \{(x,y) : (x,y) \cdot (x,y)^{-1} \in \uparrow_{\leq} (a-1,a-1) \}
\cup \{(x,y) : (x,y)^{-1} \cdot (x,y) \in \uparrow_{\leq} (b-1,b-1) \}.
\]

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REFERENCES

1. O. V. Gutik, “Any topological semigroup is topologically isomorphically embedded into a simple path-connected topological semigroup”, in: Algebra and Topology” [in Ukrainian], Lviv Univ. Press, Lviv (1996), pp. 65–73.
2. O. Andersen, Ein Bericht Über die Struktur Abstrakter Halbgruppen, PhD Thesis, Hamburg (1952).
3. L. W. Anderson, R. P. Hunter, and R. J. Koch, “Some results on stability in semigroups,” Trans. Amer. Math. Soc., 117, 521–529 (1965); https://doi.org/10.1090/S0002-9947-1965-0171869-7.
4. T. Banakh, S. Dimitrova, and O. Gutik, “Embedding the bicyclic semigroup into countably compact topological semigroups,” Topology Appl., 157, No. 18, 2803–2814 (2010); https://doi.org/10.1016/j.topol.2010.08.020.
5. T. Banakh, S. Dimitrova, and O. Gutik, “The Rees–Suschkiewitsch theorem for simple topological semigroups,” Mat. Studii, 31, No. 2, 211–218 (2009).
6. B. Banaschewski, “Minimal topological algebras,” Math. Ann., 211, No. 2, 107–114 (1974); https://doi.org/10.1007/BF01344165.
7. S. Bardyla, “Classifying locally compact semitopological polycyclic monoids,” Mat. Visn. NTSh, 49, No. 1, 19–28 (2018); https://doi.org/10.15330/ms.49.1.19-28.
8. S. Bardyla and O. Gutik, “On the lattice of weak topologies on the bicyclic monoid with adjoined zero”, Algebra Discr. Math., 30, No. 1, 26–43 (2020); https://doi.org/10.12958/adm1459.
9. S. Bardyla and A. Ravsky, “Closed subsets of compact-like topological spaces”, Appl. General Topol., 21, No. 2, 201-214, (2020); https://doi.org/10.4995/agt.2020.12258.
10. M. P. Berri, J. R. Porter, and Jr. R. M. Stephenson, “A survey of minimal topological spaces,” in: S. P. Franklin, Z. Frolik, V. Kouulnik (editors), General Topology and Its Relations to Modern Analysis and Algebra: Proc. of the Kanpur Topological Conference (1968), Academia Publishing House, Czechoslovak Academy of Sciences, Prague (1971), pp. 93–114.
11. M. O. Bertman and T. T. West, “Conditionally compact bicyclic semitopological semigroups,” Proc. Roy. Irish Acad., A76, No. 21-23, 219–226 (1976).
12. J. H. Carruth, J. A. Hildebrant, and R. J. Koch, The Theory of Topological Semigroups, Marcell Dekker, New York etc.; Vol. 1 (1983), Vol. 2 (1986).
13. A. H. Clifford and G. B. Preston, The Algebraic Theory of Semigroups, Vol. 1, Amer. Math. Soc., Providence, RI (1961), Vol. 2 (1972).
14. D. Dolitchinov, “Produits de groupes topologiques minimaux,” Bull. Sci. Math. Sér. 2, 97, 59–64 (1972).
15. C. Eberhart and J. Selden, “On the closure of the bicyclic semigroup,” Trans. Amer. Math. Soc., 144, 115–126 (1969); https://doi.org/10.1090/S0002-9947-1969-0252547-6.
16. R. Engelking, General Topology, Heldermann, Berlin (1989).
17. I. R. Fihel and O. V. Gutik, “On the closure of the extended bicyclic semigroup,” Karp. Mat. Publ., 3, No. 2, 131–157 (2011).
18. O. Gutik, “On locally compact semitopological 0-bisimple inverse to-semigroups,” Topol. Algebra Appl., 6, 77–101 (2018); https://doi.org/10.1515/taa-2018-0008.
19. O. Gutik, “On the dichotomy of a locally compact semitopological bicyclic monoid with adjoined zero,” Visn. Lviv. Univ. Ser. Mekh.-Mat., Issue 80 (2015), pp. 33–41.
20. O. Gutik and K. Maksymyk, “On semitopological interassociates of the bicyclic monoid,” Visn. Lviv. Univ. Ser. Mekh.-Mat., Issue 82 (2016), pp. 98–108.
21. O. Gutik and K. Maksymyk, “On variants of the bicyclic extended semigroup,” Visn. Lviv. Univ. Ser. Mekh.-Mat., Issue 84 (2017), pp. 22–37.
22. O. Gutik and K. Pavlyk, “On topological semigroups of matrix units,” Semigroup Forum, 71, No. 3, 389–400 (2005); https://doi.org/10.1007/s00233-005-0530-0.
24. O. Gutik and D. Repovš, “On countably compact 0-simple topological inverse semigroups,” Semigroup Forum, 75, No. 2, 464–469 (2007); https://doi.org/10.1007/s00233-007-0706-x.
25. J. A. Hildebrant and R. J. Koch, “Swelling actions of Γ-compact semigroups,” Semigroup Forum, 33, 65–85 (1986); https://doi.org/10.1007/BF02573183.
26. J. L. Kelley, General Topology, Springer-Verlag, New York (1975).
27. R. J. Koch and A. D. Wallace, “Stability in semigroups,” Duke Math. J., 24, No. 2, 193–195 (1957); https://doi.org/10.1215/S0012-7094-57-02425-0.
28. M. V. Lawson, Inverse Semigroups. The Theory of Partial Symmetries, World Scientific, Singapore (1998).
29. L. Nachbin, “On strictly minimal topological division rings,” Bull. Amer. Math. Soc., 55, No. 12, 1128–1136 (1949); https://doi.org/10.1090/S0002-9904-1949-09339-4.
30. M. Petrich, Inverse Semigroups, John Wiley & Sons, New York (1984).
31. W. Ruppert, Compact Semitopological Semigroups: An Intrinsic Theory, Lect. Notes Math., 1079, Springer, Berlin (1984).
32. Jr. R. M. Stephenson, “Minimal topological groups,” Math. Ann., 192, No. 3, 193–195 (1971); https://doi.org/10.1007/BF02052870.
33. R. J. Warne, “I-bisimple semigroups,” Trans. Amer. Math. Soc., 130, No. 3, 367–386 (1968); https://doi.org/10.1090/S0002-9947-1968-0223476-8.