Dominated and Bounded Convergence Results of Sequential Henstock Stieltjes Integral in Real Valued Space

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Abstract: In this paper, we prove the dominated and bounded convergence results for real-valued Sequential Henstock Stieltjes integral.

Keywords: Sequential Henstock integrable, Increasing functions, Guages, Dominated and bounded convergence, Uniform-integrability

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1 Introduction and Preliminaries

In 1955 and 1957 respectively, R. Henstock and J. Kursweil independently gave a Riemann-type integral called the Henstock integral. It is a kind of non-absolute integral which includes the Riemann, Improper Riemann, Newton and Lebesgue integral. Many authors have studied Henstock integral which is now known as the Kursweil-Henstock integral, since Kursweil defined the same integral though they went different ways in developing and applying the theory. For simplicity, we shall refer the Kursweil-Henstock integral and its general form as the Henstock integral in this paper, see [1]-[6]. It is well known that the Henstock integral is equivalent to the Denjoy integral, Perron integral and Denjoy-Perron integrals. The equivalence of the Henstock integral and Sequential Henstock integral has been discussed in [5]. In this paper, we prove the dominated and bounded convergence theorems for the Sequential Henstock Stieltjes integral.

The symbols used in this paper are as follows: \( \mathbb{R} \) and \( \mathbb{N} \) for a set of real and natural numbers respectively, \( \{\delta_n(x)\}_{n=1}^{\infty} \) as set of gauge functions of \( x \in [a,b] \), and \( P_n \) as set of partitions of subintervals of a compact interval \( [a,b] \) for \( n = 1,2,3,\cdots \).

The following useful definitions of Sequential Henstock integral are needed.

Definition 1.1. [5](Sequential Henstock Integral). A function \( f : [a,b] \to \mathbb{R} \) is Sequential Henstock integrable on \( [a,b] \) if there exists a number \( \alpha \in \mathbb{R} \) and a sequence of gauge functions \( \delta_n(x) \in \{\delta_n(x)\}_{n=1}^{\infty} \) such that for each \( \delta_n(x) \)-fine tagged partitions \( P_n = \{(u_{(i-1)}n, u_in), t_in\} \) we have

\[
U(f, P_n) = \sum_{i=1}^{n} f(t_in) (u_in - u_{(i-1)}n) \to \alpha, n \to \infty,
\]

i.e. \( \alpha = \int_{[a,b]} f \).

Sequel to Definition 1.1, we give a new definition as follows for Sequential Henstock Stieltjes integral.

Definition 1.2. (Sequential Henstock Stieltjes Integral). A function \( f : [a,b] \to \mathbb{R} \) is Sequential Henstock Stieltjes integrable on \( [a,b] \) with respect to an increasing function \( g : [a,b] \to \mathbb{R} \) if there exists a number \( \alpha \in \mathbb{R} \) such that for \( \varepsilon > 0 \) there exists a sequence of gauge functions \( \delta_{\mu}(x) \in \{\delta_n(x)\}_{n=1}^{\infty} \) such that \( \mu \leq n \) and for every \( \delta_n(x) \)-fine tagged partitions \( P_n = \{(u_{(i-1)}n, u_in), t_in\} \) and \( u_{(i-1)}n \leq t_in \leq u_in \) we have

\[
U(f, g, P_n) = \sum_{i=1}^{n} f(t_in) [g(u_in) - g(u_{(i-1)}n)] \to \alpha, \text{ as } n \to \infty.
\]
Dominated and Bounded Convergence Results of Sequential Henstock Stieltjes Integral in Real Valued Space

We say that $\alpha \in \mathbb{R}$ is the Sequential Henstock Stieltjes integral of $f$ with respect to $g$ on $[a,b]$ with $\alpha = \int_a^b f \, dg$

**Definition 1.3.** [5] (Uniform Integrability). Let $f_k : [a, b] \to \mathbb{R}$ be a sequence of functions for $k \in \mathbb{N}$ and a function $g : [a, b] \to \mathbb{R}$. Then $f_k$ is uniformly Sequential Henstock Stieltjes integrable with respect to $g$ on $[a, b]$ if

i. the integral $\int_a^b f_k \, dg$ exists for each $k \in \mathbb{N}$,

ii. for $\varepsilon > 0$ there exists a sequence of gauges $\delta_n(x) = \sup_{\mu \in \mathbb{N}} \{\delta_n(x)\}^{\infty}$ and $n \geq \mathbb{N}$ on $[a, b]$ such that the inequality

$$\left| \int_a^b f_k \, dg - U(f_k, dg, P_n) \right| < \varepsilon,$$

(1.1)

holds for each $\delta_n(x)$- fine partition $P_n$ of $[a, b]$ for $n = 1, 2, 3, \ldots$.

Now, we state the following lemma which was proved in [4], and useful in the proof of our main theorems.

**Lemma 1.1.** [4] Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of functions and $g : [a, b] \to \mathbb{R}$ be an increasing function satisfying the following conditions:

i. The integral $\int_a^b f_n \, dg$ exists for each $k \in \mathbb{N}$.

ii. $\lim_{k \to \infty} f_k(x) = f(x)$ for all $x \in [a, b]$.

iii. There exist $\beta, \gamma \in \mathbb{R}$ such that the inequalities

$$\beta \leq \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} f_s \, dg \leq \gamma,$$

holds for all partitions $P_n$ of $[a, b]$ and all $s_1, s_2, \ldots, s_n \in \mathbb{N}$. Then, $f_k$ is uniformly Sequential Henstock Stieltjes integrable with respect to $g$. Then, the integral $\int_a^b f \, dg$ exists and

$$\lim_{k \to \infty} \int_a^b f_k \, dg = \int_a^b f \, dg.$$

Moreover, we have

$$\lim_{k \to \infty} \left( \sup_{t \in [a, b]} \left| \int_a^t f_k \, dg - \int_a^t f \, dg \right| \right) = 0.$$

2 Main Results

We state and give the proof of theorems in our main results.

**Theorem 2.1.** (Dominated Convergence). Let $f_k : [a, b] \to \mathbb{R}$ be a sequence of functions which is Sequential Henstock Stieltjes integrable with respect to an increasing function $g : [a, b] \to \mathbb{R}$ and is satisfying the following conditions:

i. The integral $\int_a^b f_k \, dg$ exists for each $k \in \mathbb{N}$,

ii. $\lim_{k \to \infty} f_k(x) = f(x)$ for all $x \in [a, b]$.

iii. There exist Sequential Henstock Stieltjes integrable functions $h_1, h_2 : [a, b] \to \mathbb{R}$ such that $\int_a^b h_1 \, dg$ and $\int_a^b h_2 \, dg$ exist, where $h_1 \leq f_k \leq h_2$ on $[a, b]$ for each $k \in \mathbb{N}$.
exists and
Then, \( f_k \) is uniformly Sequential Henstock Stieltjes integrable with respect to \( g \). Then, the integral \( \int_a^b f dg \) exists and
\[
\lim_{k \to \infty} \int_a^b f_k dg = \int_a^b f dg.
\]
Moreover, we have
\[
\lim_{k \to \infty} \left( \sup_{t \in [a,b]} \left| \int_a^t f_k dg - \int_a^t f dg \right| \right) = 0.
\]

**Proof.** From Lemma 1.1, following from condition (iii). Let \( \beta = U(h_1, dg, P_n) \) and \( \gamma = U(h_2, dg, P_n) \).
If \( P_n \) is a sequence of divisions on \([a, b]\) and \( s_1, s_2, \cdots, s_n \in \mathbb{N} \), i.e. \( P_n = (u_{(i-1)}n, u_{in}) \in [a, b] \) for \( n = 1, 2, 3, \cdots \), then
\[
\beta = U(h_1, dg, P_n) \leq \sum_{i=1}^n \int_{u_{(i-1)}n}^{u_{in}} f_s dg \leq U(h_2, dg, P_n) = \gamma.
\]
This shows that the assumption of Lemma 1.1 is satisfied and the proof is complete.

**Theorem 2.2.** (Bounded Convergence). Let \( f_n : [a, b] \to \mathbb{R} \) be a sequence of function and \( g : [a, b] \to \mathbb{R} \) be an increasing function satisfying the following conditions:

i. The integral \( \int_a^b f_k dg \) exists for each \( k \in \mathbb{N} \)

ii. \( \lim_{k \to \infty} f_k(x) = f(x) \) for all \( x \in [a, b] \),

iii. There exist a constant \( M \geq 0 \) such that \( |f_k(x)| \leq M \) for all \( k \in \mathbb{N} \) and \( x \in [a, b] \). Then, \( f_k \) is uniformly Sequential Henstock Stieltjes integrable with respect to \( g \), the integral \( \int_a^b f dg \) exists and
\[
\lim_{k \to \infty} \int_a^b f_k dg = \int_a^b f dg.
\]
Moreover, we have
\[
\lim_{k \to \infty} \sup_{t \in [a,b]} \left( \left| \int_a^t f_n dg - \int_a^t f dg \right| \right) = 0.
\]

**Proof.** If \( P_n \) is a sequence of partitions on \([a, b]\) and \( s_1, s_2, \cdots, s_n \in \mathbb{N} \) where \( P_n = (u_{(i-1)}n, u_{in}) \in [a, b] \) for \( n = 1, 2, 3, \cdots \), then
\[
\left| \sum_{i=1}^n \int_{u_{(i-1)}n}^{u_{in}} f_s dg \right| \leq \sum_{i=1}^n \int_{u_{(i-1)}n}^{u_{in}} f_n dg \leq \sum_{i=1}^n M \text{var}_{u_{(i-1)}n}u_{in} g = M \text{var}_a^b g.
\]
by the assumptions of Lemma 1.1, which is also satisfied with
\[
-M \text{var}_a^b g \leq \left| \sum_{i=1}^n \int_{u_{(i-1)}n}^{u_{in}} f_k dg \right| \leq M \text{var}_a^b g.
\]
This completes the proof.

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