Truncated convolution of the Möbius function and multiplicative energy of an integer $n$

by

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1. Introduction. Let $\mu(\cdot)$ be the Möbius function and consider

$$M(n, z) := \sum_{d|n} \mu(d).$$

The function $M(n, z)$ has been studied by various authors (see [4], [2], [6], [7], [5], [9], [1] for example). In [4] it is established that

$$\left| \sum_{d|n} \mu(d) \right| \leq \left( \omega(n) \right)_{\lfloor \omega(n)/2 \rfloor} (1 \leq a \leq b \leq n)$$

where $\omega(n)$ is the number of distinct prime divisors of $n$. A very interesting tool, known as symmetrical chains, is used to establish a generalization of this property in [2].

In the present paper, we are interested in the average size of $M(n, z)$ over $1 \leq z \leq n$. More precisely, we consider the quantity

$$L_t(n) := \int_1^n M(n, z)^t \, dz$$

for integer values of $t \geq 1$. Let us remark that $L_t(n) = L_t(\gamma(n))$, where $\gamma(n) := \prod_{p|n} p$. From what we know, only the value of $L_1(n)$, which is $-\prod_{p|n} (1-p)$ for $n \geq 2$, is easy to evaluate. Write $\log_+ x := \log \max(x, 2)$ and $\delta_{i,j}$ for the Kronecker delta.

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**Theorem 1.1.** Let $t \geq 2$ be an integer and $n \geq 1$ be a squarefree integer. Then
\[
|L_t(n)| \leq (1 + \delta_{2,t}) n \exp\left(C t^{1-1/t} \frac{\omega(n)^{1-1/t}}{(1 - 1/t) \log^{1/t} \omega(n)}\right),
\]
where $C = 1.07073472\ldots$ is defined in the statement of Lemma 2.4.

**Theorem 1.2.** Let $t \geq 2$ be an integer and $n \geq 1$ be a squarefree integer. Then
\[
|L_t(n)| \leq \begin{cases} 
2n \exp(t\omega(n)^{1-1/t}) & \text{if } t = 2 \text{ and } \omega(n) \leq 55, \\
\exp(t\omega(n)^{1-1/t}) & \text{otherwise}.
\end{cases}
\]

We will use Theorem 1.2 to get some control over the quantity
\[
H_\theta(n) := |\{j \in [1, n] \cap \mathbb{N} : |M(n, j)| \geq 2^{\theta \omega(n)}\}| \quad (\theta \in (0, 1]).
\]
To express our result, we need to define a function $W : [e, \infty) \to [1, \infty)$ implicitly by
\[
\frac{\exp(W(x))}{W(x)} = x.
\]
This function is linked to the Lambert $W$-function by the relation $W(x) = -W(-1/x)$ in which we take the solution larger than 1.

**Corollary 1.3.** Fix $\theta \in (0, 1]$ and write
\[
\alpha := W\left(\frac{e}{\theta \log 2}\right).
\]
Let also $n \geq 2$ be a fixed squarefree integer. Then, assuming that $\alpha - 1 < \log \omega(n)$ and $\omega(n) \geq 56$, we have
\[
H_\theta(n) \leq n \exp\left(-\frac{\theta \log 2}{\alpha} \omega(n) \log \omega(n) + \left(1 - \frac{\alpha - 1}{\log \omega(n)}\right)^3 \frac{\omega(n)}{\exp\left(-\frac{\alpha - 1}{\log \omega(n)}\right) \log \omega(n)}\right).
\]

We record some approximate values of $\alpha = \alpha(\theta)$ in Table 1.

| $\theta$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 |
|---------|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|
| $\alpha(\theta)$ | 5.34 | 4.47 | 3.94 | 3.54 | 3.23 | 2.96 | 2.72 | 2.50 | 2.30 | 2.11 |

In [11], it has been shown that the number of divisors function, denoted $\tau(\cdot)$, satisfies the inequality
\[
\tau(n) \leq \left(\frac{\log(n^\gamma(n))}{\omega(n)}\right)^{\omega(n)} \beta(n) \quad (n \geq 2) \quad \text{where} \quad \beta(n) := \prod_{p \mid n} \frac{1}{\log p}.
\]
This inequality has been extensively worked out in the author’s Ph.D. thesis [3]. It is worth mentioning that the function $\beta(n)$ is intimately linked to the value of $\tau(n)$ in more than one way. In particular, it follows from [13, Theorem 5.3, p. 491] that

$$\frac{(\log z)^{\omega(n)}}{\omega(n)!} \beta(n) \leq \left|\{1 \leq j \leq z : \gamma(j) \mid \gamma(n)\}\right| \leq \frac{(\log(z\gamma(n)))^{\omega(n)}}{\omega(n)!} \beta(n)$$

so that

$$\tau(n, z) := \sum_{d \mid n \atop d \leq z} 1 \leq \frac{(\log(z\gamma(n)))^{\omega(n)}}{\omega(n)!} \beta(n).$$

In the special case where $n$ is squarefree, one prefers the estimate

$$\tau(n, z) \leq \sum_{0 \leq j \leq D(n, z)} \binom{\omega(n)}{j} \quad \text{where} \quad D(n, z) := \max_{d \mid n \atop d \leq z} \omega(d).$$

For comparison, the argument in [4] allows one to establish that

$$-\max_{0 \leq j \leq D(n, z)} \binom{\omega(n) - 1}{j} \leq M(n, z) \leq \max_{2 \mid j \atop 0 \leq j \leq D(n, z)} \binom{\omega(n) - 1}{j}$$

for every integer $n \geq 2$.

Let $s \geq 1$ be a fixed integer. For any integer $n \geq 1$ we write $d(n) := \{d : d \mid n\}$. We define the $s$th multiplicative energy of $n$ to be

$$E_s(n) := |\{(d_1, \ldots, d_s, d_{s+1}, \ldots, d_{2s}) \in d(n)^{2s} : d_1 \cdots d_s = d_{s+1} \cdots d_{2s}\}|.$$

In particular, we trivially have $E_s(n) \leq (\tau(n))^{2s-1}$. In what follows, $A(i, j)$ are the Eulerian numbers of the first kind, which can be computed by using the formula

$$A(i, j) = \sum_{v=0}^{j} \binom{i+1}{v} (-1)^v (j+1-v)^i \quad (0 \leq j \leq i-1, i, j \in \mathbb{Z}).$$

**Theorem 1.4.** Let $s, n \geq 2$ be positive integers. Then

$$\tau(n)^{2s-1} \left(\frac{A(2s-1, s-1)}{(2s-1)!}\right)^{\omega(n)} < E_s(n) \leq \tau(n)^{2s-1} \left(\frac{1}{2^{2s-1}}\binom{2s}{s}\right)^{\omega(n)}.$$

**Remark 1.5.** It is possible to establish that

$$\frac{A(2s-1, s-1)}{(2s-1)!} \sim \sqrt{\frac{3}{\pi s}} \quad \text{and} \quad \frac{1}{2^{2s-1}}\binom{2s}{s} \sim \sqrt{\frac{4}{\pi s}} \quad (s \to \infty).$$

The first relation is deduced from the identity

$$\int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x}\right)^{2s} \, dx = \pi \frac{A(2s-1, s-1)}{(2s-1)!} \quad (s \in \mathbb{N}).$$
The upper bound in (1.5) is in fact an equality in the case where \( n \) is squarefree. In this direction, we will see that the proof gives a much more general result.

Throughout the paper, we denote the \( k \)th prime number by \( p_k \). Also, for each \( k \geq 0 \), we denote by \( n_k \) the number \( \prod_{j=1}^{k} p_j \) (and set \( n_0 = 1 \)).

2. Preliminary lemmas

**Lemma 2.1.** Let \( i \geq 0 \) and \( j \geq 1 \) be integers. Denote by \( S_{i,j} \) the number of surjections from a set of \( i \) elements to a set of \( j \) elements. Then

\[
S_{i,j} = \sum_{v=0}^{j} (-1)^{j-v} \binom{j}{v} v^i.
\]

*Proof.* This is a well known result. We remark that it implies that

\[
\sum_{v=0}^{j} (-1)^{j-v} \binom{j}{v} v^i = \begin{cases} 0 & \text{for } i = 0, \ldots, j - 1, \\ i! & \text{for } i = j. \end{cases}
\]

**Lemma 2.2.** Let \( 0 \leq u_1 < \cdots < u_\ell \) and \( 0 < x_1 < \cdots < x_\ell \) be sequences of real numbers. Then the generalized Vandermonde determinant satisfies

\[
\begin{vmatrix}
x_1^{u_1} & x_1^{u_2} & \cdots & x_1^{u_\ell} \\
x_2^{u_1} & x_2^{u_2} & \cdots & x_2^{u_\ell} \\
\vdots & \vdots & \ddots & \vdots \\
x_\ell^{u_1} & x_\ell^{u_2} & \cdots & x_\ell^{u_\ell}
\end{vmatrix} > 0.
\]

*Proof.* This is a known result of Mitchell [10]. A modern proof uses [8, Lemma A2].

For \( x \in \mathbb{R} \), we define the sign function by

\[
\text{sgn}(x) := \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{otherwise}. \end{cases}
\]

**Lemma 2.3.** Let \( \lambda \geq 1 \) and \( m \geq 0 \) be integers. Let \( F_{\lambda,m}(x) \) be the polynomial of minimal degree that satisfies

\[
F_{\lambda,m}(j) := \text{sgn}(j) j^m \quad (j \in \{-\lambda, \ldots, \lambda\}).
\]

Assume that \( 0 \leq m \leq 2\lambda - 1 \).

- If \( m \) is even then \( F_{\lambda,m}(x) \) is an odd function of degree \( 2\lambda - 1 \) with leading term of sign \((-1)^{(2\lambda-m-2)/2}\).
- If \( m \) is odd then \( F_{\lambda,m}(x) \) is an even function of degree \( 2\lambda \) with leading term of sign \((-1)^{(2\lambda-m-1)/2}\).
Proof. We first assume that \( m \geq 1 \) is odd. From Lagrange interpolation with \( 2\lambda + 1 \) points, we have \( \deg F_{\lambda,m}(x) \leq 2\lambda \). Now, the polynomial
\[
G_{\lambda,m}(x) := F_{\lambda,m}(x) - F_{\lambda,m}(-x)
\]
has at least \( 2\lambda + 1 \) roots, so that \( G_{\lambda,m}(x) \) is identically 0 and we deduce that \( F_{\lambda,m}(x) \) is an even function. Therefore, we search for a polynomial of the type
\[
F_{\lambda,m}(x) := \sum_{j=1}^{\lambda} a_{2j}x^{2j}.
\]
We get the linear system
\[
\begin{pmatrix}
1^2 & 1^4 & \cdots & 1^{2\lambda} \\
2^2 & 2^4 & \cdots & 2^{2\lambda} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda^2 & \lambda^4 & \cdots & \lambda^{2\lambda}
\end{pmatrix}
\begin{pmatrix}
a_2 \\
a_4 \\
\vdots \\
a_{2\lambda}
\end{pmatrix}
= \begin{pmatrix}
1^m \\
2^m \\
\vdots \\
\lambda^m
\end{pmatrix}.
\]
By Cramer’s rule,
\[
\left|\begin{array}{cccc}
1^2 & 1^4 & \cdots & 1^{2\lambda-2} & 1^m \\
2^2 & 2^4 & \cdots & 2^{2\lambda-2} & 2^m \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda^2 & \lambda^4 & \cdots & \lambda^{2\lambda-2} & \lambda^m
\end{array}\right| = a_{2\lambda}
\]
so that we deduce from Lemma 2.2 that \( \text{sgn}(a_{2\lambda}) = (-1)^{(2\lambda-m-1)/2} \). The result follows from the fact that there is a unique such interpolating polynomial of degree at most \( 2\lambda \).

When \( m \geq 0 \) is even, simply observe that \( F_{\lambda,m}(x) := F_{\lambda,m+1}(x)/x \).

Let us define
\[
\eta(n,t) := \prod_{p\mid n} \left(1 + \frac{1}{p^{1/t}}\right).
\]

Lemma 2.4. Let \( t \geq 2 \) and \( k \geq 1 \) be positive integers. Then
\[
\eta(n_k,t) \begin{cases} 
\exp(k^{1-1/t}) & \text{if } t = 2 \text{ and } k \leq 55, \\
\exp(k^{1-1/t} - \frac{\log t}{t}) & \text{otherwise}.
\end{cases}
\]
Also,
\[
\eta(n_k,t) \leq \exp\left(C \frac{k^{1-1/t}}{(1 - 1/t) \log_{1/t} k} - (1 - \delta_{2,t}) \frac{\log t}{t}\right)
\]
with \( C = 1.07073472 \ldots \). The constant \( C \) is best possible and is attained only at \( t = 2 \) and \( k = 2149 \).
Proof. We prove (2.1) by induction on $k$ for each value of $t \geq 2$. For $t = 2, \ldots, 99$ we verify the result with a computer for $k = 1, \ldots, 56$. For $t \geq 100$ and those $k$ there is no need to verify since

\begin{equation}
\frac{\log t}{t} + \sum_{j=1}^{k} \log \left(1 + \frac{1}{p_j^{1/t}} \right) \leq \frac{\log 100}{100} + k \log 2 < 0.74k,
\end{equation}

while $k^{1-1/t} \geq 0.74k$ for $k = 1, \ldots, 56$.

Now let $k \geq 57$, consider $t \geq 2$ as fixed, and assume that the result holds for $k - 1$. We will establish that

\begin{equation}
(k - 1)^{1-1/t} + \log \left(1 + \frac{1}{k^{1/t}} \right) < k^{1-1/t},
\end{equation}

which is clearly enough for the induction step with this value of $t$. We see that (2.4) holds if

\[ \frac{1}{p_k^{1/t}} < k^{1-1/t} - (k - 1)^{1-1/t} \]

\[ \uparrow \]

\begin{equation}
\frac{1}{p_k^{1/t}} < \frac{1 - 1/t}{k^{1/t}}
\end{equation}

from the mean value theorem. Now, it is known that $p_k > k \log k$ for each $k \geq 1$ (see [12]). Using this inequality, we deduce that (2.5) holds if

\[ \frac{1}{(k \log k)^{1/t}} < \frac{1 - 1/t}{k^{1/t}} \iff \frac{1}{\log k} < \left(1 - \frac{1}{t}\right)^t \]

\[ \iff \frac{1}{\log k} < \frac{1}{4} \iff k \geq 57. \]

We have used the fact that the function $(1 - 1/x)^x$ is strictly increasing for $x > 1$. This concludes the proof of (2.4) and thus the induction step of (2.1).

We now turn to the proof of (2.2). The argument is very similar, that is, we proceed by induction for each value of $t \geq 2$. For $t = 2, \ldots, 99$ we verify the result with a computer for all $k$ from 1 to the value given in Table 2.

\begin{table}[h]
\centering
\begin{tabular}{lcccccccc}
\hline
$t$ & 2 & 3 & 4 & 5 & 6 & 7 & 8 to 99 \\
$k$ & 3750230 & 1936 & 155 & 44 & 20 & 12 & 8 \\
\hline
\end{tabular}
\end{table}

For each $t \geq 100$, by using (2.3), it is enough to have

\[ 0.74k < \frac{C}{1 - 1/t} \frac{k^{1-1/t}}{\log^{1/t} k} \iff (k \log_+ k)^{1/t} < \frac{C}{(1 - 1/t) \cdot 0.74}, \]

which is easily seen to hold for $k = 1, \ldots, 8$. 
Consider \( t \geq 2 \) as fixed and assume that
\[
\sum_{j=1}^{J} \log \left( 1 + \frac{1}{p_{j}^{1/t}} \right) < C \frac{J^{1-1/t}}{(1-1/t) \log^{1/t} J} - (1 - \delta_{2,t}) \frac{\log t}{t}
\]
holds at \( J = k - 1 \); we want to show that it holds with \( J = k \) (\( \geq 9 \)). It is enough to show that
\[
\frac{1}{p_{k}^{1/t}} < C \frac{k^{1-1/t}}{(1-1/t) \log^{1/t} k} - C \frac{(k-1)^{1-1/t}}{(1-1/t) \log^{1/t}(k-1)}
\]
\[
\uparrow
\]
\[
\frac{1}{p_{k}^{1/t}} < C \frac{1}{1-t} \left( \frac{1-1/t}{(k \log k)^{1/t}} - \frac{1/t}{k^{1/t} \log^{1+1/t} k} \right)
\]
from the mean value theorem and the fact that \( f_{t}(x) := x^{1-1/t}/\log^{1/t} x \) satisfies \( f_{t}''(x) < 0 \) if \( x \geq 6 > \exp \left( \frac{2/(t-1) + 5/4}{\sqrt{5-4/t}} \right) \) for each \( t \geq 2 \). Again, by using \( p_{k} > k \log k \) for each \( k \geq 1 \), we deduce that (2.7) holds if
\[
\frac{1}{(k \log k)^{1/t}} < C \frac{1}{1-t} \left( \frac{1-1/t}{(k \log k)^{1/t}} - \frac{1/t}{k^{1/t} \log^{1+1/t} k} \right)
\]
\[
\updownarrow
\]
\[
1 < C \left( 1 - \frac{1}{(t-1) \log k} \right)
\]
\[
\updownarrow
\]
\[
\log k > \frac{C}{(t-1)(C-1)}
\]
which holds for \( k \) greater than the corresponding value in Table 2 if \( t = 2, \ldots, 99 \), and for \( k \geq 9 \) if \( t \geq 100 \). This completes the inductive step for the fixed value of \( t \geq 2 \).

3. Proof of Theorems 1.1 and 1.2. We write
\[
L_{t}(n) = \int_{1}^{n} \sum_{d_{1},\ldots,d_{t}|n, d_{1},\ldots,d_{t} \leq z} \mu(d_{1}) \cdots \mu(d_{t}) \ dz
\]
\[
= \sum_{d_{1},\ldots,d_{t}|n} \mu(d_{1}) \cdots \mu(d_{t}) \int_{1}^{n} \chi(d_{1}, z) \cdots \chi(d_{t}, z) \ dz
\]
\[
= \sum_{d_{1},\ldots,d_{t}|n} \mu(d_{1}) \cdots \mu(d_{t})(n - \max(d_{1}, \ldots, d_{t}))
\]
where

\[ \chi(d, z) := \begin{cases} 0 & \text{if } d < z \\ 1 & \text{otherwise} \end{cases} \quad (d \in \mathbb{N}, z \in \mathbb{R}). \]

Now, for \( n \geq 2 \), we rearrange the terms according to the number \( j = 1, \ldots, t \) of \( d_i \) that realize the maximum and we use the fact that \( M(n, n) = 0 \) to get

\[
L_t(n) = - \sum_{d_1, \ldots, d_t | n} \mu(d_1) \cdots \mu(d_t) \max(d_1, \ldots, d_t)
\]

\[
= - \sum_{j=1}^{t} \binom{t}{j} \sum_{d | n} \mu(d)^j d \sum_{d_1, \ldots, d_{t-j} | n, d_1, \ldots, d_{t-j} < d} \mu(d_1) \cdots \mu(d_{t-j})
\]

\[
= - \sum_{j=1}^{t} \binom{t}{j} \sum_{d | n} \mu(d)^j d \left( \sum_{e | n, e < d} \mu(e) \right)^{t-j}
\]

\[
= - \sum_{d | n} d \left( \left( \sum_{e | n, e \leq d} \mu(e) \right)^t - \left( \sum_{e | n, e < d} \mu(e) \right)^t \right)
\]

\[
= (-1)^t n \sum_{d | n} \frac{1}{d} \left( \left( \sum_{e | n, e \geq n/d} \mu(e) \right)^t - \left( \sum_{e | n, e > n/d} \mu(e) \right)^t \right).
\]

Thus, let \( 1 = d_1 < d_2 < \cdots < \omega(n) = n \) be the sequence of divisors of \( n \). We write

\[ J_{\rho}(n) := \sum_{i=1}^{2^\omega(n)} \frac{i^\rho}{d_i} \quad (\rho \in \mathbb{R}_{\geq 0}). \]

Now, we deduce from (3.1) that

\[ |L_t(n)| \leq n \sum_{i=1}^{2^\omega(n)} \frac{i^t - (i-1)^t}{d_i} \leq tn^t J_{t-1}(n). \]

Also, for any integer value of \( \rho \geq 0 \) and \( \sigma \in \mathbb{R}_{\geq 0} \), we can write

\[ J_{\rho}(n) = \sum_{d | n} \frac{\tau_{\rho}(n, d)}{d} \leq \sum_{d | n} \frac{1}{d} \left( \sum_{e | n} \left( \frac{d}{e} \right)^\sigma \right)^\rho = \sum_{d | n} d^{\rho \sigma - 1} \left( \sum_{e | n} \frac{1}{e^\sigma} \right)^\rho \]

\[ = \prod_{p | n} (1 + p^{\rho \sigma - 1}) \left( 1 + \frac{1}{p^\sigma} \right)^\rho = \prod_{p | n} \left( 1 + \frac{1}{p^{\rho + 1}} \right)^{\rho + 1} \]

\[ = \eta^{\rho + 1}(n, \rho + 1) \]
Truncated convolution of the Möbius function

where we have used $\sigma = \frac{1}{\rho + 1}$. We thus get

$$|L_t(n)| \leq t\eta^t(n, t) \leq t\eta^t(n_{\omega(n)}, t).$$

The results then follow from Lemma 2.4.

**Remark 3.1.** The function $J_\rho(\cdot)$ satisfies

$$J_\rho(n) \leq J_\rho(n_{\omega(n)}) \quad (n \in \mathbb{N}).$$

Indeed, let $n = q_1 \cdots q_{\omega(n)}$ with $q_1 < \cdots < q_{\omega(n)}$ be the factorization of $n$. Since $p_{r_1} \cdots p_{r_l} \leq q_{r_1} \cdots q_{r_l}$, it follows that the $i$th term in the ordered sequence of divisors of $n_{\omega(n)}$ is at most equal to the $i$th term in the corresponding sequence for $n$.

4. **Proof of Corollary 1.3.** Since $M(n, z)$ is constant for $z \in [j, j + 1)$ ($j \in \mathbb{Z}_{\geq 0}$) and $M(n, n) = 0$ for $n \geq 2$, we deduce that

$$H_{\theta}(n) \leq \frac{L_t(n)}{2\theta \omega(n)} \quad \text{ (if } 2 \mid t).$$

From Theorem 1.2 and the hypothesis $\omega(n) \geq 56$, we have

$$H_{\theta}(n) \leq n \exp \left( t\omega(n) \left( \frac{1}{\omega(n)^{1/t}} - \theta \log 2 \right) \right).$$

Now, the idea is simply to optimize this inequality over the even integers $t \geq 2$. Our strategy is to find the exact value $t_0 \in (1, \infty)$ and to estimate the variation caused by $t = t_0 + \xi$ with $|\xi| \leq 1$. We write

$$f(x) := x \left( \theta \log 2 - \frac{1}{\omega(n)^{1/x}} \right),$$

so that

$$f'(x) = \theta \log 2 - \frac{1}{\omega(n)^{1/x}} - \frac{\log \omega(n)}{x\omega(n)^{1/x}} \quad \text{ and } \quad f''(x) = -\frac{\log^2 \omega(n)}{x^3 \omega(n)^{1/x}}.$$

Let $t_0 = c \log \omega(n)$. Then $f'(t_0) = 0$ if and only if

$$\omega(n)^{1/t_0} \theta \log 2 = 1 + \frac{\log \omega(n)}{t_0} \iff \exp(1/c)\theta \log 2 = 1 + \frac{1}{c}$$

$$\iff \frac{\exp(1 + 1/c)}{1 + 1/c} = \frac{e}{\theta \log 2}$$

$$\iff 1 + \frac{1}{c} = \alpha = W \left( \frac{e}{\theta \log 2} \right)$$
so that $t_0 = \frac{\log \omega(n)}{\alpha - 1}$, which is strictly larger than 1 by hypothesis. We can verify that $f(t_0) = \frac{\theta \log 2}{\alpha} \log \omega(n)$. Now,

$$|f(t) - f(t_0)| \leq \sup_{z \in (t_0-1, t_0+1)} |f'(z)| = \sup_{z \in (t_0-1, t_0+1)} |f'(z) - f'(t_0)| \leq \sup_{\zeta \in (t_0-1, t_0+1)} |f''(\zeta)|$$

from the mean value theorem applied twice. The result follows from the estimate

$$\sup_{\zeta \in (t_0-1, t_0+1)} |f''(\zeta)| < \frac{\log^2 \omega(n)}{(t_0 - 1)^3 \omega(n)^{1/(t_0+1)}},$$

which holds since $t_0 > 1$. ■

5. **Proof of Theorem 1.4.** We assume throughout the proof that $s \geq 2$ is a fixed integer. The function $E_s(n)$ is multiplicative, so it will be enough to show that

$$\tau(p^\alpha)^2 s - 1 \left( \frac{A(2s - 1, s - 1)}{(2s - 1)!} \right) < E_s(p^\alpha) \leq \tau(p^\alpha)^2 s - 1 \left( \frac{1}{2^{2s-1}} \binom{2s}{s} \right)$$

for any prime $p$.

For a fixed $p$, the function $E_s(p^\alpha)$ counts the number of solutions to the system

$$R_s(\alpha) := \{(\alpha_1, \ldots, \alpha_{2s}) \in \{0, \ldots, \alpha\}^{2s} : \alpha_1 + \cdots + \alpha_s = \alpha_{s+1} + \cdots + \alpha_{2s}\}. $$

We clearly have $R_s(0) = 1$ and also

$$R_s(1) := \sum_{j=0}^{s} \binom{s}{j}^2 = \binom{2s}{s},$$

an identity that follows from $(x + 1)^{2s} = (x + 1)^s (x + 1)^s$. In general, $R_s(\alpha)$ is the coefficient of $x^0$ in the expansion of

$$N_{s, \alpha}(x) := \left(1 + x + \cdots + x^\alpha\right) \left(1 + \frac{1}{x} + \cdots + \frac{1}{x^\alpha}\right)^s = \left(\frac{x^{\alpha+2} + x^{-\alpha} - 2x}{(1 - x)^2}\right)^s = (x^{\alpha+2} + x^{-\alpha} - 2x)^s \sum_{j \geq 0} \binom{2s - 1 + j}{2s - 1} x^j,$$

from which we deduce that

$$R_s(\alpha) = \sum_{a+b+c=s \atop (a+2)a - ab + c \leq 0} \binom{s}{a, b, c} (-2)^c \binom{2s - 1 - (\alpha + 2)a + \alpha b - c}{2s - 1}$$

$$= \sum_{i=0}^{s} \sum_{0 \leq j \leq s - i \atop (\alpha + 1)(i - j) + s \leq 0} \binom{s}{i, j, s - i - j} (-2)^{s - i - j} \binom{(\alpha + 1)(j - i) + s - 1}{2s - 1}.$$
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\[ \begin{aligned}
&= \sum_{i=0}^{s} \sum_{-i \leq v \leq s} \left( \binom{s}{i, v + i, s - v - 2i} (-2)^{s-v-2i} \frac{((\alpha + 1)v + s - 1)}{2s - 1} \right)
&= \sum_{v=1}^{s} (-1)^{s-v} \binom{(\alpha + 1)v + s - 1}{2s - 1} \sum_{i=0}^{s} \binom{s}{i, v + i, s - v - 2i} 2^{s-v-2i}
&= \sum_{v=1}^{s} (-1)^{s-v} \binom{2s}{s-v} \binom{(\alpha + 1)v + s - 1}{2s - 1} =: P_s(\alpha + 1).
\end{aligned} \]

The last expression follows from
\[ \begin{aligned}
&= \sum_{i=0}^{s} \left( \binom{s}{i, v + i, s - v - 2i} 2^{s-v-2i} = \binom{2s}{s-v}, \right)
\end{aligned} \]

which can be shown by using the identity \((x + 1)^{2s} = (x^2 + 2x + 1)^s\).

Now, the idea of the proof is to show that \(P_s(x)\) is an odd function with strictly positive coefficients (of \(x^j\) with \(j\) odd) so that it is clear that the function \(P_s(x)/x^{2s-1}\) has a strictly negative derivative. With this in mind, we write
\[ \begin{aligned}
P_s(x) &= \sum_{v=1}^{s} (-1)^{s-v} \binom{2s}{s-v} \binom{vx + s - 1}{2s - 1} \\
&= \frac{x}{(2s - 1)!} \sum_{v=1}^{s} (-1)^{s-v} \binom{2s}{s-v} v \prod_{j=1}^{s-1} (v^2x^2 - j^2)
\end{aligned} \]

so that
\[ \begin{aligned}
\frac{(2s - 1)!P_s(ix)}{ix} &= \sum_{v=1}^{s} (-1)^{v-1} \binom{2s}{s-v} v \prod_{j=1}^{s-1} (v^2x^2 + j^2) \\
&= \sum_{v=1}^{s} (-1)^{v-1} \binom{2s}{s-v} v \sum_{r=0}^{s} b_{2r} v^{2r} x^{2r} \\
&= \sum_{r=0}^{s-1} b_{2r} x^{2r} \sum_{v=1}^{s} (-1)^{v-1} \binom{2s}{s-v} v^{2r+1} 
\end{aligned} \]

where each \(b_{2r}\) with \(r = 0, \ldots, s - 1\) is strictly positive. By writing
\[ c_r := \sum_{v=1}^{s} (-1)^{v-1} \binom{2s}{s-v} v^{2r+1}, \]
we deduce that it is enough to show that \(\text{sgn}(c_r) = (-1)^r\). We write
\[ c_r = -\sum_{v=0}^{s} (-1)^{s-v} \binom{2s}{v} (s-v)^{2r+1} = (-1)^{s+1} \sum_{v=0}^{2s} (-1)^{2s-v} \binom{2s}{v} Q_{s, 2r+1}(v) \]
where $Q_{s,2r+1}(x)$ is the Lagrange polynomial of degree at most $2s$ for which

$$Q_{s,2r+1}(v) = \begin{cases} (s-v)^{2r+1}, & v = 0, \ldots, s, \\ 0, & v = s+1, \ldots, 2s. \end{cases}$$

From Lemma 2.1 and the remark in its proof, we deduce that $\text{sgn}(c_r) = (-1)^{s+1} \text{sgn}(e_{2s})$ where $e_{2s}$ is the leading term of $Q_{s,2r+1}(x)$. Now,

$$Q_{s,2r+1}(x+s) + Q_{s,2r+1}(-x+s) = F_{s,2r+1}(x),$$

the function in Lemma 2.3. We deduce that $\text{sgn}(2)e_{2s} = \text{sgn}(e_{2s}) = (-1)^{s-r-1}$ so that $\text{sgn}(c_r) = (-1)^r$ as desired.

6. Concluding remark. Let us consider the quantity

$$T_s(\alpha) := \{|(\alpha_1, \ldots, \alpha_s) \in \{-\alpha, \ldots, \alpha\}^s : \alpha_1 + \cdots + \alpha_s = 0\}|.$$  

The methods used in the proof of Theorem 1.4 also apply to $T_s(\alpha)$. That is, the function $T_s(\alpha)/(2\alpha + 1)^{s-1}$ is strictly decreasing for integer values of $\alpha \geq 0$ when $s \geq 3$, and it is constant for $s = 1$ or 2.

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