In this note we prove that for every integer $d \geq 1$, there exists an explicit constant $B_d$ such that the following holds. For all primes $p > B_d$ and all primes $q > \max\{d-1, 5\}$, if $E$ is an elliptic curve defined over $K$ with $[K : \mathbb{Q}] = d$ such that $E$ has potentially multiplicative reduction at all primes above $q$, then $E$ has an irreducible mod $p$ Galois representation. This result has Diophantine applications within the “modular method”. We present one such application in the form of an Asymptotic version of Fermat’s Last Theorem that has not been covered in the existing literature.

1. Introduction

Throughout this article $K$ will denote a number field, $G_K = \text{Gal}(\overline{K}/K)$ its absolute Galois group and $E$ an elliptic curve defined over $K$. For a rational prime $p$, we are going to write $\overline{\rho}_{E,p}$ for the representation $\overline{\rho}_{E,p} : G_K \rightarrow \text{Aut}(E[p]) \cong \text{GL}_2(\mathbb{F}_p)$ arising from the action of $G_K$ on the $p$-torsion points in $E(\overline{K})$. In the language of Galois representations, $E$ having a $p$-isogeny defined over $K$ is equivalent to $\overline{\rho}_{E,p}$ being reducible.

When $K = \mathbb{Q}$, it follows from Mazur’s Theorem [13, Theorem 1] that for $p > 163$ the representation $\overline{\rho}_{E,p}$ is irreducible for all elliptic curves $E$. For a general number field $K$, the question of whether there is a constant $B_K$ such that $\overline{\rho}_{E,p}$ is irreducible for all primes $p > B_K$ and all elliptic curves $E$ without complex multiplication is an active topic of research. The existence of such a constant $B_K$ has not been proved for any number field other than $\mathbb{Q}$.

The “modular approach” is a powerful method for showing that certain Diophantine equations do not have solutions using Galois representations of elliptic curves [17]. Absolute irreducibility of the mod $p$ Galois representations associated to Frey elliptic curves is a necessary hypothesis for successful applications of this method. With these type of Diophantine applications in mind, Freitas and Siksek [6] proved the following theorem.

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Theorem 1.1 ([6, Theorem 2]). Let $K$ be a totally real Galois field. There is an explicit effective constant $B_K$, depending only on $K$ such that for any rational prime $p > B_K$ and elliptic curve $E$ over $K$ semistable at all $p | p$, the representation $\overline{\rho}_{E,p}$ is irreducible.

Remark. When $K$ is totally real and $p$ is odd, the existence of a complex conjugation in $G_K$ whose image under $\overline{\rho}_{E,p}$ is similar to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} implies that if $\overline{\rho}_{E,p}$ is irreducible, then it is absolutely irreducible. This is not true if $E$ is defined over a general number field $K$.

Recent progress in (potential) modularity over number fields that are not totally real sparked an interest into attacking Diophantine equations over number fields via the modular method. In [4, 16] the authors discuss the Fermat equation with prime exponent over general number fields $K$

$$a^p + b^p + c^p = 0,$$

where $a, b, c \in K$ and $p$ is a rational prime, assuming two conjectures from the Langlands programme. The results in the aforementioned papers are in the direction of the “Asymptotic Fermat’s Last Theorem”. To be precise, under various hypotheses on $K$, the authors of these papers prove that for $p$ larger than a constant $C_K$, the equation (1) does not have non-trivial solutions. Under the same two conjectures, Kara and Özman [9] proved asymptotic versions of the so called "Generalized Fermat equation" over number fields.

In [18] and [19], the second author considered the Fermat equation over quadratic imaginary fields of class number one, assuming [19, Conjecture 2.2]. The latter is commonly called "Serre’s modularity conjecture" and is one of the two conjectures assumed in [4, 16].

The hypotheses required for applying [19, Conjecture 2.2] include the absolute irreducibility of the mod $p$ Galois representations $\overline{\rho}_{E,p}$. Proving results such as Theorem 1.1 for general number fields $K$ is a hopeless task. This is due to the possible presence of elliptic curves with complex multiplication that are defined over $K$. The representations $\overline{\rho}_{E,p}$ of such curves are reducible for infinitely many (half of the) values of $p$ and irreducible but absolutely reducible for almost all the remaining ones. Our main results in this paper are as follows.

Theorem 1.2. Suppose an elliptic curve $E$ over a quadratic field $K$ has multiplicative reduction at all primes $q$ above $q > 5$. Then $E$ has an irreducible mod $p$ representation for all $p > 71$.

For number fields of degree which is larger than two, we obtained the subsequent theorem.

Theorem 1.3. Suppose an elliptic curve $E$ over a number field $K$ of degree $d$ has multiplicative reduction at all primes $q$ above some rational prime $q$ with $q > d - 1$. Then $E$ has an irreducible mod $p$ representation for all $p > 65(2d)^6$.

Recall that $E/K$ has potentially multiplicative reduction at a prime $q$ in $K$ if and only if $v_q(j(E)) < 0$. The results above imply that if $\overline{\rho}_{E,p}$ is reducible for $p$ larger than a constant $C_d$, depending only on the degree $d = [K : \mathbb{Q}]$, then $j(E)$ is integral at certain infinite set of primes such as all the inert primes of $K$.

We now explain that, when carrying out the modular approach to Diophantine equations, results such as the ones above are very valuable. A Frey elliptic curve $E$ is
semistable outside a fixed finite set of primes. That is, $E$ has good or multiplicative reduction at prime ideals outside this set. For $p > C_K$ and $\rho_{E,p}$ (absolutely) irreducible one can proceed with the modular method as usual. In the case that $\rho_{E,p}$ is reducible, together with some additional hypothesis, the theorems above imply that the primes of multiplicative reduction for $E$ belong to a finite fixed set. The $j$-invariant of $E$ is hence integral outside this set. Moreover, there are explicit formulas for $j(E)$ depending on the solution to the Diophantine equation in question. Knowing that the denominators of $j(E)$ are supported on a restricted set of primes can therefore lead to a complete resolution of the equation.

To emphasize this phenomenon, recall that if $a, b, c \in \mathcal{O}_K$ are coprime and satisfy the Fermat equation (1) one can construct the Frey elliptic curve

$$E := E_{a,b,c,p} : Y^2 = X(X - a^p)(X + b^p).$$

The $j$-invariant of this elliptic curve has the formula

$$j(E) = \frac{2^4(b^{2p} - a^p c^p)}{(abc)^{2p}}.$$

Knowledge that $j(E)$ is integral outside a finite set of primes implies that $abc$ is actually supported only on that set.

In [19], the second author proved the following result.

**Theorem 1.4** ([19, Theorem 1.3]). Let $K$ be a quadratic imaginary number field of class number one. Assume [19, Conjecture 2.2] holds over $K$. Then, for any prime $p \geq 19$, the equation $a^p + b^p + c^p = 0$ does not have solutions in coprime $a, b, c \in \mathcal{O}_K \setminus \{0\}$ such that $2 \mid abc$.

**Remark.** The only reason for which the restriction $2 \mid abc$ appears in the statement of the theorem is as follows. Let $E$ be the Frey curve associated to the putative solution to (1) as in (2). The author of [19] could only prove that $\rho_{E,p}$ is absolutely irreducible for $p \geq 19$ such that $2 \mid abc$. The absolute irreducibility is required in the hypothesis of Serre’s modularity conjecture (see [19, Conjecture 2.2]) and, if one shows it, the rest of the proof of the theorem above goes through.

**Remark.** Previous results on the Asymptotic Fermat’s Last Theorem (see [16, Theorems 1.1 and 1.2] or the theorems in [4]) are all stated for number fields $K$ which contain a prime $q$ with residue field $\mathbb{F}_2$ above 2. One of the reasons for this restriction is that, over general $K$, the authors of the previously mentioned works had to assume that $E$ has a fixed prime of potentially multiplicative reduction $q$ in order to prove that $\rho_{E,p}$ is irreducible for $p$ larger than a constant $C_{K,q}$. If $q$ has residue field $\mathbb{F}_2$, considering $a^p + b^p + c^p = 0 \pmod{q}$, one sees that $q \mid abc$. Using the formula (3) it can be easily deduced that $v_q(j(E)) < 0$ for $p$ large enough and $q$ is the desired prime of multiplicative reduction for $E$.

As an application to Theorem 1.2, we prove the following version of the Asymptotic Fermat’s Last Theorem for at least half of the prime exponents $p$.

**Theorem 1.5.** Let $K$ be a quadratic imaginary field of class number one and suppose that Conjecture 2.2 in [19] holds for $K$. Fix $S$, a finite set of rational primes containing 2, 3 and 5. There exists a constant $C_{K,S}$, depending only on $K$ and $S$, such that for all $p > C_{K,S}$, if the following hold:

1. $p \equiv 1 \pmod{3}$ or $p$ is a prime that splits in $K$ and such that $p \equiv 3 \pmod{4}$;
(2) \(a, b, c \in \mathcal{O}_K \setminus \{0\}\) are coprime such that \(a^p + b^p + c^p = 0\);

(3) If \(l \notin S\) is a rational prime that divides \(\text{Norm}(abc)\), then all the primes \(l\) above \(l\) divide \(abc\) in \(\mathcal{O}_K\);

then \(K = \mathbb{Q}(\sqrt{-3})\) and the triple \((a, b, c)\) is, up to reordering, \((1, \epsilon, \epsilon^2)\), where \(\epsilon\) is a non-trivial third root of unity.

We note that the larger \(S\) is taken, the less restrictive hypothesis (3) becomes. However, this comes at the cost of (probably) increasing the constant \(C_{K,S}\).

This result is aligned with a more general Fermat Conjecture. We refer the interested reader to the discussion on page 2 of [4], where the authors point out that the Fermat Conjecture is a consequence of the abc-conjecture for number fields.

2. Formal immersions and the proof of Theorems 1.2 and 1.3

In this section we prove Theorems 1.2 and 1.3. In particular, we prove that if an elliptic curve has multiplicative reduction at all primes above some rational prime \(q\), then it cannot have an isogeny of large prime degree \(p\).

The idea is essentially to use the method going back to Mazur and Kamienny (see for example [8]), as modified by Merel [14] (see also [15]). All the aforementioned papers use the fact that if the elliptic curve \(E\) has a point of large order \(n\) then it forces the curve to have a prime \(q\) such that the curve has multiplicative reduction over it and all of its Galois conjugates. Then it follows that the reduction mod \(q\) a putative non-cuspidal point on \(X_0(p)\) corresponding to this curve will be the same as the reduction mod \(q\) of a cusp (the same is true for all Galois conjugates of \(q\)). This is then shown to be impossible by proving that a certain map from a symmetric power into a quotient (the Eisenstein quotient in Mazur’s and Kamienny’s papers and the winding quotient in Merel’s) of the Jacobian \(J_0(n)\) of \(X_0(n)\), which is of rank 0 over \(\mathbb{Q}\), is a formal immersion at the aforementioned cusp modulo \(q\).

This method cannot be applied to elliptic curves with isogenies (without assuming anything else about the curve), as the existence of an isogeny of arbitrarily large degree does not force multiplicative reduction at any prime over number fields, as is easily seen on elliptic curves with complex multiplication. However, if one supposes both multiplicative reduction at all primes above a rational prime that is not very small (as compared to the degree) and an isogeny of large prime degree \(p\) simultaneously, then one can arrive at a contradiction using basically the same argument as before.

Let \(X_0(p)\) be the classical modular curve of level \(p\), whose non-cuspidal \(K\)-rational points represent isomorphism classes of pairs \((E, C)\) of elliptic curves \(E\) together with a \(K\)-rational subgroup \(C\) of order \(p\). The curve \(X_0(p)\) has two cusps, 0 and \(\infty\). We follow the convention (as in [2]) that \(\infty\) is the cusp which is unramified under the \(j\)-map \(X_0(p) \to X_0(1)\). This map is ramified of degree \(p\) at the cusp 0.

Fix the following notation for this section: let \(E/K\) have multiplicative reduction at all primes \(q\) lying above the rational prime \(q\), and a \(K\)-rational subgroup \(C\) of order \(p\). Let \(\sigma_1, \ldots, \sigma_d\) be the embeddings of \(K\) into \(\bar{K}\). Let \(x \in X_0(p)(K)\) be the point corresponding to \((E, C)\), and let \(y = (x^{\sigma_1}, \ldots, x^{\sigma_d}) \in X_0^{(d)}(p)(\mathbb{Q})\) be the point on the symmetric \(d\)-th power of \(X_0^d\).

**Proposition 2.1.** The point \(y \in X_0^{(d)}(p)(\mathbb{Q})\) reduces to \((\infty, \ldots, \infty)_{\bar{\mathbb{Q}}_q}\) after applying an appropriate Atkin-Lehner involution.
Proof. The proof proceeds exactly as in [8, Lemma 3.2], after in loc. cit. one proves that the elliptic curve has multiplicative reduction modulo \( q \) and all the Galois conjugates of \( q \).

The fact that one should apply an Atkin-Lehner involution to map the reduction into \( (\infty, \ldots, \infty) \) modulo \( q \) is explained in the remark after [3, Proposition A.4]. □

Let \( J_0^S(p) \) be the winding quotient of \( J_0(p) \), see [14] for the definition.

**Proposition 2.2** ([14, Proposition 1]). The rank of the winding quotient \( J_0^S(p)(\mathbb{Q}) \) is 0.

Define \( f_d : X_0^d(p) \to J_0^S(p) \) to be the composition of the natural map

\[
X_0^d(p) \to J_0(p)
\]

\[
(\alpha_1, \ldots, \alpha_d) \mapsto \left[ \sum_{i=1}^{d} \alpha_i - d\infty \right]
\]

and the quotient map \( J_0(p) \to J_0^S(p) \).

**Proposition 2.3.** Suppose \( d > 2, q > d - 1 \) and \( p > 65(2d)^6 \). Then the map \( f_d : X_0^d(p) \to J_0^S(p) \) is a formal immersion at \((\infty, \ldots, \infty)_{\mathbb{F}_q}\).

Proof. This follows from [15, Theorem 4.18 and Section 5] (see also [3, Appendix A]). □

For \( d = 2 \) we can use the Mazur’s Eisenstein quotient and Kamienny’s results instead, as the results are more explicitly stated using the Eisenstein quotient in this case, and the results we get, as we will see, are best possible.

Let \( J_p \) be the Eisenstein quotient of \( J_0(p) \) (see [12] for the definition).

**Proposition 2.4.** ([11, Theorem 4]) The rank of \( J_p(\mathbb{Q}) \) is zero.

Define now \( f_d \) to be as defined before, with the difference that the quotient map maps to \( J_p \) instead of \( J_0^S(p) \).

**Proposition 2.5** ([7, Proposition 3.2]). Let \( d = 2 \) and \( q > 5, p > 71 \). Then \( f_d \) is a formal immersion at \((\infty, \infty)_{\mathbb{F}_q}\).

We now have all the ingredients needed to prove the theorems.

**Proof of Theorems 1.2 and 1.3**. To prove our claims, we use the following standard argument. Suppose that an elliptic curve \( E \) with a \( p \)-isogeny (where \( p \) satisfies the assumptions of the theorems) over a number field \( K \) has bad reduction at all primes \( q \) of \( K \) above \( q \). It corresponds to a non-cuspidal point \( x \in X_0^d(p)(\mathbb{Q}) \). Let \( y = (x^{a_1}, \ldots, x^{a_d}) \in X_0^d(p)(\mathbb{Q}) \).

The map \( f_d \) is a formal immersion at \((\infty, \ldots, \infty)_{\mathbb{F}_q}\) by Propositions 2.5 and 2.3. Then it follows that \( f_d^{-1}(\infty, \ldots, \infty)_{\mathbb{F}_q} \) has one element (see for example [3, Lemma 3.1]). Since we know that \( f_d((\infty, \ldots, \infty)) = (\infty, \ldots, \infty)_{\mathbb{F}_q} \), and by Proposition 2.1 we have that \( f_d(y) = (\infty, \ldots, \infty)_{\mathbb{F}_q} \), which is now a contradiction. □

**Remark 2.6.** We claim that in Theorem 1.2 the bound of 71 is best possible. To see this, note that \( X_0(71) \) is a hyperelliptic curve with hyperelliptic involution \( w_{71} \). Hence we obtain the quotient map \( X_0^+(71) \to X_0^+(71) \) with \( X_0^+(71) \) being of genus 0. Note that in [1] it is proved that all elliptic curves with 71-isogenies
over quadratic fields are obtained in this way. Hence we have that the $j$-function 
$j_{71} \in \mathbb{Q}(X_0(71))$ is quadratic (over $\mathbb{Q}(X_0^+(71)) \simeq \mathbb{Q}(t)$). So there exists a quadratic polynomial $f \in \mathbb{Q}(t)[y]$ such that $f(j_{71}(t)) = 0$. All the $j$-invariants $j_E$ of elliptic curves with $71$-isogenies over quadratic fields are obtained as roots of this polynomial (when specialized in $t$). Finally, one can make the denominator of $j_E(t)$ divisible by an arbitrarily large inert prime by taking an appropriate $t$.

3. Mod $p$ Galois representations and the proof of Theorem 1.5

Before giving the proof for our Diophantine result, let us bring to the reader’s attention the following theorem of Larson and Vaintrob.

**Theorem 3.1** ([10] Theorem 1). Let $K$ be a number field. There exists a finite set of primes $M_K$, depending only on $K$, such that for any prime $p \notin M_K$ and any elliptic curve $E/K$ for which $\mathcal{P}_{E,p} \otimes \mathbb{F}_p \cong \left( \begin{array}{cc} \lambda & * \\ 0 & \lambda' \end{array} \right)$ where $\lambda, \lambda' : G_K \to \mathbb{F}_p^\times$ are characters, one of the following happens.

1. There exists a CM elliptic curve $E'/K$, whose CM field is contained in $K$, 
   with $\mathcal{P}_{E,p} \otimes \mathbb{F}_p \cong \left( \begin{array}{cc} \theta & 0 \\ 0 & \theta' \end{array} \right)$ and such that $\lambda^{12} = \theta^{12}$.
2. The Generalized Riemann Hypothesis fails for $K(\sqrt{-p})$, and $\phi^{12} = \chi_p^6$. 
   Moreover, in this case $\mathcal{P}_{E,p}$ is already reducible over $\mathbb{F}_p$ and $p \equiv 3 \pmod{4}$.

Quadratic imaginary fields of class number one are $K = \mathbb{Q}(\sqrt{-d})$, where $d$ is one of $1, 2, 3, 7, 11, 19, 43, 67$ or $163$. The cases $d \in \{1, 2, 7\}$ follow by the more general results in [15]. From now on we will assume that $K$ is one of the six remaining fields in which the prime $2$ is inert.

Suppose that a triple $a, b, c \in \mathcal{O}_K \setminus \{0\}$ which satisfies the hypothesis (2) and (3) of Theorem 1.5 exists and assume that $p \geq 19$. From Theorem 1.5 we know the following:

- The prime ideal $q = 2\mathcal{O}_K$ does not divide $abc$.
- If $E : Y^2 = X(X - a^2)(X + b^2)$ is the Frey elliptic curve attached to this solution, then $E$ has additive potentially good reduction at $q := 2\mathcal{O}_K$ and $E$ is semi-stable at every prime ideal $q' \neq q$.
- From the proof of the aforementioned theorem, it follows that the mod $p$ Galois representation $\mathcal{P}_{E,p}$ is absolutely reducible and unramified outside the primes above $2p$.

To avoid discussing any of the exceptional subgroups of $GL_2(\mathbb{F}_p)$, let us assume that $C_{K,S} \geq 53$. This implies that if $\mathcal{P}_{E,p} \otimes_{\mathbb{F}_p} \mathbb{F}_p$ is diagonalisable, then the image $\mathcal{P}_{E,p}(G_K)$ is contained in a Cartan subgroup of $GL_2(\mathbb{F}_p)$. We distinguish two cases.

Suppose first that $\mathcal{P}_{E,p}$ is irreducible, but absolutely reducible. In this situation, it follows that the image $\mathcal{P}_{E,p}(G_K)$ is contained in a Cartan non-split subgroup. Up to conjugation, we have that $\mathcal{P}_{E,p} \otimes_{\mathbb{F}_p} \mathbb{F}_p \cong \left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda^p \end{array} \right)$, where $\lambda : G_K \rightarrow \mathbb{F}_p^\times$ is a character. The latter is not $\mathbb{F}_p$-valued and $\lambda^{p+1} = \chi_p$, where $\chi_p : G_K \rightarrow \mathbb{F}_p^\times$ is the mod $p$ cyclotomic character.

From [5] Lemma 3.7 we know that $3$ divides the order of $\mathcal{P}_{E,p}(I_q)$. There exists an element $g \in I_q$ such that $\mathcal{P}_{E,p}(g)$ has order exactly $3$ by Cauchy’s theorem. This implies that $\lambda(g)$ has order $3$ in $\mathbb{F}_p^\times$. On the other hand, it is known that $\chi_p$ is
unramified outside the places above \( p \) and since \( g \in I_q \), we have \( \lambda^{p+1}(g) = 1 \). The latter implies that \( 3 \mid p + 1 \) and this, combined with the hypothesis of our theorem, gives that \( p \equiv 3 \pmod 4 \) and that it must split in \( K \).

Let us now apply Theorem 3.1 to our Frey curve \( E \). To make sure the hypothesis of the aforementioned theorem is satisfied, we will assume that \( C_{K,S} \) and hence \( p \) is larger than any of the primes in \( M_K \). As we assume that \( \rho_{E,p} \) is not reducible but only absolutely reducible, the first case of Theorem 3.1 applies.

There is an elliptic curve \( E'/K \) with CM by an order in \( K \) such that

\[
\rho_{E',p} \otimes \mathbb F_p \simeq \begin{pmatrix} \theta & 0 \\ 0 & \theta' \end{pmatrix} \quad \text{and} \quad \lambda^{12} = \theta^{12}.
\]

We recall that as \( p \) is supposed to be split in \( K \) from the theory of elliptic curves with CM, we know that the image \( \rho_{E',p}(G_K) \) is contained inside a split Cartan subgroup. The character \( \theta : G_K \to \mathbb F_p^* \) is in fact \( \mathbb F_p \)-valued and has order dividing \( p - 1 \). It follows easily that the order of \( \theta^{12} \) must divide \( (p - 1)/2 \).

If needed, we now increase \( C_{K,S} \) such that \( C_{K,S} > 163 \) to ensure that \( p \) is unramified in \( K \). Let \( \mathfrak p \) be any of the two primes lying above \( p \). Since the order of \( \chi_p | G_K \) is \( p - 1 \), it is not hard to deduce that \( \text{ord} \left( \lambda | G_K \right) = 2k(p - 1) \), where \( 2k \) is a divisor of \( p + 1 \). We get that

\[
\text{ord} \left( \lambda^{12} | G_K \right) = \frac{2k(p - 1)}{\gcd(12, 2k(p - 1))} = \frac{k}{\gcd(3, k)} \cdot \frac{p - 1}{2}
\]

and this quantity is a divisor of \( \text{ord}(\theta^{12}) \). The latter implies that \( k \in \{1, 3\} \), hence the character \( \lambda^6 | G_K \) has order \( p - 1 \) and is \( \mathbb F_p \)-valued.

Let us choose \( \sigma \in G_K \) such that \( \chi_p(\sigma) \) is a generator of \( \mathbb F_p^* \). Observe that \( \chi_p(\sigma) = (\lambda^6(\sigma))^\frac{p+1}{6} \). This is a contradiction since on one hand \( \frac{p+1}{6} \) is even and on the other \( \chi_p(\sigma) \) is not a square in \( \mathbb F_p^* \) due to the choice of \( \sigma \).

We just showed that if \( \rho_{E,p} \) is not absolutely irreducible, then it is reducible. Recall that \( S \) is a finite set of rational primes containing \( 2, 3 \) and \( 5 \). The hypothesis (3) of our theorem implies that if \( l \notin S \) then \( E \) has the same type of reduction, good or multiplicative, at all prime ideals \( l \) above \( l \). From Theorem 1.2 it follows that the reduction is good at such primes \( l \). This means that \( j(E) \) is integral outside of \( S \), in particular that \( abc \) is supported only on primes lying above the ones in \( S \).

The Fermat equation can be written as \((-a/c)^p + (-b/c)^p = 1\). Observe that \((-a/c)^p \) and \((-b/c)^p \) are solutions to the \( S \)-unit equation

\[
(4) \quad x + y = 1, \quad \text{where} \quad x, y \in \mathcal O_{K,S}^*.
\]

Due to the famous Siegel’s theorem, we know that (4) has finitely many solutions. Suppose that one of \( a, b \) or \( c \) is not a unit. Without losing generality, we can assume that \( a \) is divisible by some prime ideal \( \mathfrak l \) of \( K \). As (4) has finitely many solutions, the possible valuations of \( v_\mathfrak l(x) \) belong to a finite set. A contradiction can be reached by increasing the exponent \( p \) and this shows that there is a bound \( B_{K,S} \) such that if \( p > B_{K,S} \) then \( a, b, c \) are units in \( \mathcal O_K \).

Let us now assume that \( C_{K,S} > B_{K,S} \). For each one of the number fields \( K \), there are finitely many units in \( \mathcal O_K \). By checking all the possibilities, we find that if \( p \) and \( a, b, c \) are as in the hypothesis of our theorem, the only possible solutions arise when \( K = \mathbb Q(\sqrt{-3}) \) and these are \((a, b, c) = (1, \epsilon, \epsilon^2)\), up to reordering.
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