A Norm-Bounded based MPC strategy for uncertain systems under partial state availability

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Abstract
A robust model predictive control scheme for a class of constrained norm-bounded uncertain discrete-time linear systems is developed under the hypothesis that only partial state measurements are available for feedback. Off-line calculations are devoted to determining an admissible, though not optimal, linear memoryless controller capable to formally address the input rate constraint; then, during the on-line operations, predictive capabilities complement the off-line controller by means of $N$ steps free control actions in a receding horizon fashion. These additive control actions are obtained by solving semi-definite programming problems subject to linear matrix inequalities constraints.

1 Introduction
Model Predictive Control (MPC) is an optimization-based control technique, also popular because of its capability to handle constraints in an efficient manner [1–7]. Despite the usual unavailability of the full state measurement in real world applications, most of the MPC literature deals with the state feedback scenario. In practice, the state-feedback MPC controller is then implemented by replacing measured states with their estimation (see the recent survey [8]), with the unavoidable requirement that the controller must be robust against estimation errors and dynamics. For this reason, contributions on the output-feedback MPC share the common feature of guaranteeing the stability of the closed-loop system including the observer dynamics. In [9] a moving horizon observer was developed for the model uncertainty free case, whereas in [10] an output feedback MPC scheme based on a dual mode approach has been proposed for linear discrete-time plants subject to input constraints, bounded disturbances and measurement noise. A high gain observer is designed in [11] and the control law is obtained by minimizing, over admissible control sequences, a cost index subject to a terminal constraint in the absence of uncertainties. In [12], the overall controller consists of two components, a stable, possibly time-varying, state estimator and a tube based robustly stabilizing model predictive controller. In [13], a robust output-feedback MPC strategy for a class of open-loop stable systems, having non-vanishing output disturbances, hard constraints and linear-time invariant model uncertainties subject to state and input constraints, has been developed. The design is based on the interesting idea of
incorporating a novel closed-loop stability test and minimizes a quadratic upper bound on the MPC cost function at each time step. Along similar lines are the contributions proposed in [14] and [15], where the Unscented Kalman filter is used. In [16] a constrained output-feedback predictive controller, having the same small-signal properties as a preexisting linear time invariant output-feedback controller, is presented. Specifically, the method provides a systematic way to select the most adequate (non-unique) state observer realization instrumental to recast an offset-free reference tracking controller into the combination of an observer, a reference pre-filter, a steady-state target calculator and a predictive controller. Moreover in [17] the proposed MPC output-feedback approach recast the state estimation and control law design into a unique min-max optimization problem. Other relevant solutions follow a similar philosophy; e.g. a dual mode approach for linear discrete-time plants, subject to input constraints and bounded disturbance/measurement noises is adopted in [10]; deterministic state estimation, min-max optimization and ellipsoidal set techniques in [18]; dilated LMIs (Linear Matrix Inequalities) and off-line computations in [19]. Finally it is worth to mention that in [20], [21], [22], the output feedback MPC methodology has been successfully applied in different contexts: quasi-Linear Parameter Varying (LPV) systems with bounded disturbances, plant descriptions subject to stochastic disturbances and constrained LPV systems, respectively.

Starting from the literature background illustrated so far, in this paper we develop a discrete-time output-feedback MPC strategy for constrained uncertain systems subject to structured norm-bounded uncertainties. The following improvements and additional developments are provided with respect to preliminary results presented in [23, 24]: an off-line two-step procedure to properly address the presence of limited input rates; a formal derivation of the LMI conditions characterizing an upper-bound on the cost function, the prescribed state/input constraints and the convex outer approximation of the set of states where the non measurable state components lie; detailed proofs of the theoretical results.

The proposed solution takes its cue from the ideas reported in [25] and [26]: the first contribution develops robust state-feedback MPC with respect to norm-bounded uncertainties, whereas the second one deals with the output-feedback scenario by designing an off-line solution in terms of a controller/observer structure. The main drawback of such an approach lies on the fact that a non-convex Bilinear Matrix Inequalities (BMI) optimization problem results: as it is well-known local optimization algorithms can be applied with the consequence that solutions highly depends on the starting point.

Notice that, when the full state is not available, there is a technical complication to formally take care of non-measurable state variables and input rate constraints during the off-line phase which is one of the key points for ensuring recursive feasibility of an underlying MPC scheme. We pursue a guaranteed approach by embedding, via prescribed state constraints, the region where state trajectories must lie. This avoids to design further units besides the MPC controller, so reducing the computational burden in view of real-time applications. As input rate constraints are considered, a two-step procedure has been defined to build up a stabilizing controller and the associated robust positively invariant region. The capability of combining partial state availability with input rate constraints into a unique off-line framework represents a novelty traced along the arguments of [27]. In particular, non measurable state components are
1 Introduction

considered as additional sources of uncertainty. To this end, first an outer convex approximation of the state space region to which the non measurable states belong is derived; then, a two step procedure to off-line achieve an admissible static output feedback controller and the related robust positively invariant set is conceived with the main aim to satisfy hard constraints, i.e. input saturations, state restrictions and input rate requirements. This idea straightforwardly translates into a computable algorithm by using the S-Procedure machinery \[28\]. The on-line control action is then obtained as the solution of a sequence of convex problems in terms of LMI feasibility conditions solvable via standard SemiDefinite Programming (SDP) algorithms.

Notation and preliminary results

Given a symmetric matrix \(H \in \mathbb{R}^{n \times n}\), \(\sigma(H)\) denotes the maximum singular value of \(H\), whereas \(\bar{\lambda}(H)\) the largest eigenvalue.

\(I_n \in \mathbb{R}^{n \times n}\) and \(0_{n \times m}\) denote the identity matrix and the zero entries matrix, respectively.

Given a symmetric matrix \(P = P^T \in \mathbb{R}^{n \times n}\), the inequality \(P > 0 (P \geq 0)\) denotes matrix positive definiteness (semi-definiteness). Given two symmetric matrices \(P, Q\), the inequality \(P > Q (P \geq Q)\) indicates that \(P - Q > 0, (P - Q \geq 0)\).

Given a vector \(x \in \mathbb{R}^n\), the standard 2-norm is denoted by \(|x|_2 = x^T x\) whereas \(|x|_p^p \triangleq x^T P x\) denotes the vector \(P\)-weighted 2-norm.

The notation \(\hat{v}_k(t) \triangleq v(t + k|t), \, k \geq 0\) will be used to denote the \(k\)-steps ahead prediction of a generic system variable \(v\) from \(t\) onward under specified initial state and input scenario.

\textbf{S-procedure} \[29\]: Let the quadratic forms \(F_0, \ldots F_p\) be defined by

\[
F_i(x) \triangleq x^T P_i x + 2u_i^T x + v_i, \quad i = 0, \ldots, p
\]

where \(x \in \mathbb{R}^n\) and \(P_i = P_i^T\). The condition

\(F_0(x) \geq 0\) for all \(x\) such that \(F_i(x) \geq 0, \quad i = 1, \ldots, p\)

is satisfied if there exist reals \(\tau_i \geq 0, i = 1, \ldots, p\) such that

\[
\begin{bmatrix}
P_0 & u_0 \\
u_0^T & v_0
\end{bmatrix}
- \sum_{i=1}^{p} \tau_i
\begin{bmatrix}
P_i & u_i \\
u_i^T & v_i
\end{bmatrix} \geq 0
\tag{1}
\]

Note that (1) is also necessary if \(p = 1\) and \(F_1(x) > 0\) for at least one vector \(x\) (see \[29\]).

\textbf{Schur complements}: The following pairs of matrix inequalities

\[
X - YZ^{-1}Y^T > 0, \quad Z > 0 \\
Z - Y^TX^{-1}Y > 0, \quad X > 0
\]

are equivalent to

\[
\begin{bmatrix}
X & Y \\
Y^T & Z
\end{bmatrix} > 0
\]
Consider the following discrete-time linear system with uncertainties (or perturbations) appearing in a feedback loop

\[
\begin{align*}
\dot{x}(t+1) &= \Phi x(t) + G u(t) + B_p p(t) \\
y(t) &= C x(t) \\
q(t) &= C_q x(t) + D_q u(t) \\
p(t) &= \Delta(t) q(t)
\end{align*}
\]

(2)

with \( x \in \mathbb{R}^{n_x} \) denoting the state, \( u \in \mathbb{R}^{n_u} \) the control input, \( y \in \mathbb{R}^{n_y} \) the output, \( p, q \in \mathbb{R}^{n_p} \) additional variables accounting for the uncertainty. Operator \( \Delta \) may represent either a memoryless, possibly time-varying, matrix with \( \|\Delta(t)\|_2 = \bar{\sigma}(\Delta(t)) \leq 1 \) \( \forall t \geq 0 \), or a convolution operator with norm, induced by the truncated \( \ell_2 \)-norm, less than 1 viz.

\[
\sum_{j=0}^{t} p(j)^T p(j) \leq \sum_{j=0}^{t} q(j)^T q(j), \forall t \geq 0
\]

For a more extensive discussion about this type of uncertainty see [29]. It is further assumed that the plant is subject to the following ellipsoidal constraints

\[
u(t) \in \Omega_u, \quad \Omega_u \triangleq \{ u \in \mathbb{R}^{n_u} : u^T u \leq \bar{u}_{max}^2 \}
\]

(3)

\[
x(t) \in \Omega_x, \quad \Omega_x \triangleq \{ x \in \mathbb{R}^{n_x} : (C x)^T (C x) \leq \bar{x}_{max}^2 \}
\]

(4)

with \( \bar{u}_{max} > 0 \) and \( \bar{x}_{max} > 0 \).

In the sequel, relevant technical results concerning the state-feedback regulation problem for the constrained uncertain system (2)-(4) are provided. The family of systems (2) is said to be robustly quadratically stabilizable if there exists a constant state-feedback control law \( u = F x \) such that all closed-loop trajectories asymptotically converge to zero for all admissible realizations of \( \Delta(t) \). It is well known, see e.g. [29], that a linear state-feedback control law can quadratically stabilize the uncertain linear system (2) and provides an upper-bound to the following quadratic performance index

\[
J(x(0), u(\cdot)) \triangleq \max_{p(t) \in \mathcal{S}(t)} \sum_{t=0}^{\infty} \left\{ \| x(t) \|_{R_x}^2 + \| u(t) \|_{R_u}^2 \right\}
\]

(5)

with \( R_x \geq 0, R_u > 0 \) given symmetric matrices, if there exist a matrix \( \Pi = \Pi^T > 0 \), and a scalar \( \lambda > 0 \) such that the following LMI is satisfied

\[
\begin{bmatrix}
\Phi_F^T \Pi F - \Pi + F^T R_u F + R_x + \lambda C_F^T C_F & \Phi_F^T \Pi B_p \\
B_p^T \Pi \Phi_F & B_p^T \Pi B_p - \lambda I_{n_x}
\end{bmatrix} \leq 0
\]

where

\[
\Phi_F \triangleq \Phi + G F, \ C_F \triangleq C_q + D_q F, \ \lambda > 0
\]

Accordingly, the sets

\[
\mathcal{S}(t) \triangleq \{ p \mid \| p \|_2^2 \leq \| C_F x(t) \|_2^2 \}
\]

(6)
represent plant uncertainty domains at each time instant \( t \). Then, a bound on (5) is as follows

\[
J(x(0), u(\cdot)) \leq x(0)^T \Pi x(0)
\]

while the ellipsoidal set

\[
C(\Pi, \xi) \doteq \{ x \in \mathbb{R}^n \mid x^T \Pi x \leq \xi \}
\]

is a robust positively invariant region for the state evolutions of the closed-loop system, viz. \( x(0) \in C(\Pi, \xi) \) implies that \( \Phi^T_{t} F x(0) + B_{p} p(t) \in C(\Pi, \xi), \forall p(t) \in S(t) \) and \( \forall t \). In presence of input and state constraints \( u(t) \in \Omega_u \) and \( x(t) \in \Omega_x \), all the above setup holds true, provided that the pair \( (\Pi, \xi) \) and \( F \) are chosen so that \( x(0) \in C(\Pi, \xi) \) with \( FC(\Pi, \xi) \subset \Omega_u \) and \( C(\Pi, \xi) \subset \Omega_x \).

# 2 Problem Formulation

Consider the class of constrained uncertain systems (2) and the assumption that the state is partially available (measured), i.e.

\[
x(t) := [x_a^T(t) \ x_{na}^T(t)]^T, \ y(t) := Cx(t) = x_a(t)
\]  

(7)

where \( x_{na}(t) \) accounts for non measurable state components and, without loss of generality, \( C := [I_{n_y} \ 0_{n_y \times (n_x - n_y)}] \).

The following constraint on the input rate is also prescribed:

\[
\|u(t + 1) - u(t)\|_2^2 \leq \bar{\delta} u_{max}^2 \quad \text{with} \quad u(t) \in \Omega_u \quad \forall t \geq 0.
\]  

(8)

Then, we state the following problem:

**Constrained Output Feedback Stabilization (COFS)**

Given the plant model (2), find an output memoryless feedback control strategy

\[
u(t) = g(y(t))
\]  

(9)

such that the prescribed constraints (3), (4) and (8) are always fulfilled and the closed-loop system is asymptotically stable.

COFS problem will be addressed by resorting to the dual-mode Receding Horizon Control (RHC) approach proposed in [25] for the full-state feedback case. There, the key idea was the following: in the off-line phase, a stabilizing RHC law compatible with (3), (4) and (8) is computed; then, during the on-line operations, an MPC control law with a control horizon \( N \) is designed in order to improve the overall control performance.

By following the same modus operandi, a customization to the proposed output-feedback framework prescribes that two critical problems are formally addressed within the controller design phase: to take care of the unmeasured state components and to deal with input rate constraints (8).

**Remark 1** - Notice that the prescribed constraints (3), (4) and (8) could be directly considered in a component-wise fashion as follows:
• input:

\[ u(t) \in \Omega_u, \Omega_u \triangleq \{ u \in \mathbb{R}^{n_u} : |u_i(t + k)| \leq \bar{u}_{i,\text{max}}, \forall k \geq 0, \bar{u}_{i,\text{max}} \in \mathbb{R}^+, i = 1, \ldots, n_u \} \]  \hspace{1cm} (10)

• state:

\[ x(t) \in \Omega_x, \Omega_x \triangleq \{ x \in \mathbb{R}^{n_x} : |x_j(t + k)| \leq \bar{x}_{j,\text{max}}, \forall k \geq 0, \bar{x}_{j,\text{max}} \in \mathbb{R}^+, j = 1, \ldots, n_x \} \]  \hspace{1cm} (11)

• input rate:

\[ u(t) \in \Omega_{\delta u} \subseteq \Omega_u, \Omega_{\delta u} \triangleq \{ u \in \Omega_u : |u_i(t + k + 1) - u_i(t + k)| \leq \delta u_{i,\text{max}}, \forall k \geq 0, \delta u_{i,\text{max}} \in \mathbb{R}^+, i = 1, \ldots, n_u \} \]  \hspace{1cm} (12)

Although, for practical applications, formulation (10)-(12) may appear more natural w.r.t. (3), (4) and (8), it is worth to underline that such a description leads to a significant increase of the number of constraints to be accounted for.

3 Off-line robust MPC design

In what follows, the RHC scheme developed in [27] is adapted to the framework outlined in the problem statement section and the following argument are used to take care of unmeasured state components. Let \( S \in \mathbb{R}^{n_x-n_y \times n_x-n_y} \) be a symmetric matrix such that region

\[ D(S) \triangleq \{ x \in \mathbb{R}^{n_x} \mid x^TH^TSHx \leq 1, H = [0_{(n_x-n_y)} \times n_y \ I_{(n_x-n_y)}], S \geq 0 \} \]  \hspace{1cm} (13)

describes a convex outer approximation of the set of states where \( x_{na}(t) \) lies \( \forall t \geq 0 \). First a pair (\( \bar{P}, \bar{K} \)), \( \bar{K} \) being a linear output feedback matrix gain complying with input and state constraints (3)-(4) is computed by minimizing the cost (5) under the condition that \( x(t) \in D(S) \); then, the input rate constraints (8) are taken into account by fixing the output feedback gain \( K \) and deriving an admissible subset \( \chi \) of the RPI region

\[ \bar{\chi} \triangleq \{ x \in \mathbb{R}^{n_x} \mid x^T \bar{Q}^{-1}x \leq 1 \} = \{ x \in \mathbb{R}^{n_x} \mid x^T \bar{P}x \leq \bar{\rho}, \bar{P} = \bar{\rho} \bar{Q}^{-1} \} \]

In order to find an admissible, though not optimal, solution to the COFS problem, we first determine a pair (\( \bar{P}, \bar{K} \)) compatible with input and state constraints by overlooking requirement (8). Let us consider the following cost index:

\[ J(x(0), u(\cdot)) \triangleq \max_{p(t) \in \mathcal{P}(t), x(t) \in D(S)} \sum_{t=0}^{\infty} \{ ||x(t)||^2_{R_x} + ||u(t)||^2_{R_u} \} \]  \hspace{1cm} (14)

where

\[ \mathcal{P}(t) \triangleq \{ p : ||p||_2^2 \leq \|(C_y + D_y KC)x(t)||^2_2 \} \]  \hspace{1cm} (15)

is the set accounting for model uncertainty and an upper bound to (14) is given by

\[ J(x(0), u(\cdot)) \triangleq \max_{x(0) \in D(S)} x(0)^T \mathcal{P} x(0) \]  \hspace{1cm} (16)
Lemma 1: Let \( x(t) = [x_u(t)^T \  x_{na}(t)^T]^T \) be the current state of the uncertain system (2) subject to (3) and (4) at each sampling time \( t \), with \( x_{na}(t) \) the unmeasurable part of the state (7) such that \( x(t) \in D(S) \). Then, the gain matrix \( K \) of the constant output feedback control law

\[
    u_k(t) = Ky_k(t), \ k \geq 0
\]

minimizing the upper bound (16), over the prediction time \( k \), with an initial state \( x(t) \) is obtained as the solution of the following SDP problem:

\[
    \min \ \bar{Q}_1, \bar{Q}_2, Y_1, \bar{\rho}, \bar{\tau}, \lambda, \tilde{\rho} \quad \text{(18)}
\]

subject to

\[
    \begin{bmatrix}
    \bar{Q} & Y^T R_a^{1/2} & \bar{Q} \bar{R}_e^{1/2} & \bar{Q} C_q^T + Y^T D_q^T & \bar{Q} \Phi^T + Y^T G^T \\
    * & \tilde{\rho} I_n & 0_{n_u \times n_x} & 0_{n_u \times n_p} & 0_{n_u \times n_x} \\
    * & * & \tilde{\rho} I_n & 0_{n_p \times n_p} & 0_{n_p \times n_x} \\
    * & * & * & \lambda I_n & 0_{n_p \times n_x} \\
    * & * & * & * & \bar{Q} - \lambda B_p B_p^T
    \end{bmatrix} \geq 0 \quad \text{(19)}
\]

where \( \bar{\lambda} > 0 \)

\[
    \begin{bmatrix}
    1 - \bar{\tau} & x_u^T(t) \\
    * & \tilde{Q}_1
    \end{bmatrix} \geq 0 \quad \text{(20)}
\]

\[
    \begin{bmatrix}
    \bar{\tau} S & I_{n_x - n_y} \\
    * & \tilde{Q}_2
    \end{bmatrix} \geq 0 \quad \text{(21)}
\]

\[
    \begin{bmatrix}
    \bar{u}_{\max}^2 I_{n_u} & Y_1 \\
    * & \tilde{Q}_1
    \end{bmatrix} \geq 0 \quad \text{(22)}
\]

\[
    \tilde{Q}_1 \leq \bar{u}_{\max}^2 I_{n_y} \quad \text{(23)}
\]

where

\[
    \tilde{Q} = \begin{bmatrix}
    \bar{Q}_1 & 0_{n_y \times (n_x - n_y)} \\
    0_{(n_x - n_y) \times n_y} & \bar{Q}_2
    \end{bmatrix} > 0, \quad Y = \begin{bmatrix}
    Y_1 & 0_{n_u \times (n_x - n_y)}
    \end{bmatrix}
\]

with \( \bar{Q}_1 \in \mathbb{R}^{n_y \times n_y} \) and \( \bar{Q}_2 \in \mathbb{R}^{(n_x - n_y) \times (n_x - n_y)} \), \( Y_1 \in \mathbb{R}^{n_u \times n_y} \), \( \lambda > 0, \bar{\rho} > 0, \bar{\tau} > 0 \) and \( K = Y_1 \tilde{Q}_1^{-1} \).

Proof - LMIs (19), (22) and (23) can be obtained by exploiting the same arguments as in [27] under control strategy (17).

Requirements (20) and (21) come out because of the unknown state components \( x_{na}(t) \) such that \( x(t) \in D(S) \), \( \forall t \geq 0 \), i.e.

\[
    \begin{bmatrix}
    x_{na}(t)^T & 1
    \end{bmatrix}^T \begin{bmatrix}
    -S & 0 \\
    0 & 1
    \end{bmatrix} \begin{bmatrix}
    x_{na}(t) \\
    1
    \end{bmatrix} \geq 0 \quad \text{(24)}
\]

This ensures that \( x(t) \in \tilde{\zeta}, \forall t \geq 0 \), if the following statement holds true:

\[
    \begin{bmatrix}
    x_{na}(t)^T & 1
    \end{bmatrix}^T \begin{bmatrix}
    -\tilde{Q}_2^{-1} & 0 \\
    0 & 1 - x_u(t)^T \tilde{Q}_1^{-1} x_u(t)
    \end{bmatrix} \begin{bmatrix}
    x_{na}(t) \\
    1
    \end{bmatrix} \geq 0 \quad \text{(25)}
\]
Following \textit{S-procedure} arguments, implications (24) and (25) are fulfilled if and only if there exists a positive scalar $\bar{\tau}$ such that the following inequality is satisfied

\[
\begin{bmatrix}
-Q_2^{-1} + \bar{\tau}S & 0 \\
0 & 1 - \bar{\tau} - x_a(t)^T \bar{Q}_1^{-1} x_a(t)
\end{bmatrix} \geq 0
\] (26)

that, by using \textit{Schur complements}, straightforwardly gives rise to (20) and (21).

The next result is aimed at achieving an RPI region capable to take care of the input rate constraints (8).

\textbf{Lemma 2:} Let the output feedback gain $K$ and the RPI region $\bar{\zeta}$ be given, then the ellipsoidal set

\[
\zeta := \{ x \in \mathbb{R}^{nx} \mid x^T Q^{-1} x \leq 1 \} = \{ x \in \mathbb{R}^{nx} \mid x^T P x \leq \rho, P = \rho Q^{-1} \}
\] (27)

compatible with (8) and such that

\[
\zeta \subseteq \bar{\zeta}
\] (28)

is an admissible RPI region for (2) subject to (3), (4) and (8) if the following optimization problem has a solution:

\[
\min_{\rho, \bar{\rho}, \tau, \lambda} \rho
\] (29)

subject to

\[
\begin{bmatrix}
\Phi_K^T P \Phi_K - P + K^T R_u K + R_x + \lambda C_K^T C_K & \Phi_K^T P B_p \\
B_p^T P \Phi_K & B_p^T P B_p - \lambda I_{nx}
\end{bmatrix} \leq 0
\] (30)

where $\lambda > 0$

\[
\frac{\rho}{\bar{\rho}} \bar{P} \leq P
\] (31)

\[
\rho - \tau - x_a^T P_1 x_a \geq 0
\] (32)

\[
-P_2 + \tau S \geq 0
\] (33)

\[
P \geq \rho T^{-1}
\] (34)

where $\tau > 0$,

\[
P = \begin{bmatrix}
P_1 & 0_{nx \times (nx - ny)} \\
0_{nx - ny \times ny} & P_2
\end{bmatrix} > 0
\]

\[
\Phi_K = \Phi + GKC, \quad C_K = C_q + D_qKC,
\]

\[
\bar{C} = K \bar{C}, \quad \bar{A} = C(\Phi_K - I_{nx}), \quad \bar{B} = C \bar{B}_p,
\]

\[
T = \delta u_{\max}^{-2} (\bar{A}^T \bar{A} + \delta C_K^T C_K + \bar{A}^T \bar{B}(-\bar{B}^T \bar{B} + \delta I_{ny})^{-1} \bar{B}^T \bar{A})
\]
and

\[ \dot{\sigma} \triangleq \arg \min_{\sigma \geq 0} \lambda \left( \delta u_{\text{max}}^2 (\bar{A}^T \bar{A} + \sigma C_K^T C_K + \bar{A}^T \bar{B} (\bar{B}^T + \sigma I_{n_p})^{-1} \bar{B}^T \bar{A}) \right) \]

subject to

\[ -\bar{B}^T \bar{B} + \sigma I_{n_p} > 0 \]

**Proof** - First, the set inclusion (28) can be straightforwardly recast into the matrix inequality (31), while (30) and (32)-(33) account for quadratic stabilizability and positive invariance requirements, respectively (see proof of Lemma 1).

Then, by considering the generic control action \( \delta u_k(t) = u_k(t+1) - u_k(t) \) the satisfaction of (8) translates into the following statement:

\[
\begin{bmatrix}
    p_k(t)^T & 1 \\
    -\bar{B}^T \bar{B} & -\bar{B}^T \bar{A} x_k(t) \\
    * & -\bar{B} x_k(t)^T \bar{A}^T x_k(t) \\
    0_{n_p} & 1
\end{bmatrix} \begin{bmatrix}
    p_k(t) \\
    1
\end{bmatrix} \geq 0
\]

holds true for all vectors \( p_k(t) \) such that

\[
\begin{bmatrix}
p_k(t)^T & 1 \\
-I_{n_p} & x_k(t)^T C_K^T C_K x_k(t)
\end{bmatrix} \begin{bmatrix}
p_k(t) \\
1
\end{bmatrix} \geq 0
\]

Note that, via *S-procedure arguments*, this implication is valid if and only if there exists a positive scalar \( \hat{\sigma} \) such that the following LMI condition is satisfied

\[
\begin{bmatrix}
    -\bar{B}^T \bar{B} + \hat{\sigma} I_{n_p} & -\bar{B}^T \bar{A} x_k(t) \\
    * & -\bar{B} x_k(t)^T (\bar{A}^T \bar{A} + \hat{\sigma} C_K^T C_K) x_k(t)
\end{bmatrix} \geq 0
\]

Hence by means of Schur complements, it results that if

\[-\bar{B}^T \bar{B} + \hat{\sigma} I_{n_p} > 0 \text{ and } T \geq 0\]

one obtains

\[ x_k(t)^T T x_k(t) \leq 1 \]

Therefore, the set inclusion (28) is valid if matrix inequality (34) holds. Finally, (35) follows *mutatis mutandis* the same lines exploited in [25].

As a conclusion, the above results allow to achieve a feasible, though not optimal, solution to the **COFS** problem providing:

- a stabilizing output feedback gain \( K \) computed by solving the SDP (18)-(23);
- an admissible RPI region \( \zeta \) obtained via the solution of the SDP (29)-(34).

**Remark 1** - Notice that the *a-priori* knowledge of the initial state component \( x_a(0) \) has been assumed only for the sake of clarity. When such an assumption does not hold true and one only knows e.g. that \( x_a(0) \in X_a \), with \( X_a \) a given polytopic or ellipsoidal compact set, the off-line phase can straightforwardly be generalized, see [30] for technical details.
4 LMI based Output MPC control strategy

In this section, a set of LMI conditions that allows to improve the control performance pertaining to the controller (17) are derived. To this end, we consider the following family of virtual commands

$$ u(\cdot|t) := \begin{cases} K\hat{y}_k(t) + c_k(t), & k = 0, \ldots, N - 1 \\ K\hat{y}_k(t), & k \geq N \end{cases} $$

(36)

where the vectors $c_k(t)$ provide $N$ free perturbations to the action of the output feedback law $K\hat{y}_k(t)$, $\hat{y}_k = C\hat{x}_k$, with

$$ \hat{x}_k(t) := \Phi_K^k x(t) + \sum_{i=0}^{k-1} \Phi_K^{k-1-i}(Gc_i(t) + Bp_i(t)) $$

(37)

the convex set-valued state predictions such that $p_i(t) \in \mathcal{P}_i(t), i = 0, \ldots, k - 1$

$$ \mathcal{P}_i(t) \triangleq \left\{ p_i : \|p_i\|_2^2 \leq \max_{\hat{x}_i(t)} \|C_K \hat{x}_i(t) + Dqc_i(t)\|_2^2 \right\} $$

(38)

Since, by virtue of (37), it follows that

$$ \begin{cases} \hat{x}_{k+1}(t) = \Phi_K \hat{x}_k(t) + Gc_k(t) + Bp_k(t), & \forall p_k(t) \in \mathcal{P}_k(t), \\ \hat{y}_k(t) = C\hat{x}_k(t) \end{cases} $$

(39)

an upper bound to the cost (14) is given by the following quadratic index $V := V(x(t), P, c_k(t)) :$

$$ V := \max_{x(t) \in D(S)} \|x(t)\|_{R_x}^2 + \|c_0(t)\|_{R_z}^2 + \sum_{k=1}^{N-1} \left( \max_{\hat{x}_k(t) \in D(S)} \|\hat{x}_k(t)\|_{R_x}^2 + \|c_k(t)\|_{R_z}^2 \right) + \max_{\hat{x}_N(t) \in D(S)} \|\hat{x}_N(t)\|_{R_z}^2 $$

(40)

Then, at each time instant $t$, the sequence of $N$ perturbations $c_k(t), k = 0, \ldots, N - 1$, complying with COFS requirements is obtained by solving the following optimization problem

$$ \{c_k^*(t)\}_{k=0}^{N-1} \triangleq \arg \min_{c_k(t)} V $$

(41)

subject to

$$ K\hat{y}_k(t) + c_k(t) \in \Omega_u, \ k = 0, \ldots, N - 1, $$

(42)

$$ \|K\hat{y}_0(t) + c_0(t) - u(t-1)\|_2^2 \leq \bar{\delta}u_{max}^2 $$

(43)

$$ \|K\hat{y}_k(t) + c_k(t) - K\hat{y}_{k-1}(t) - c_{k-1}(t)\|_2^2 \leq \bar{\delta}u_{max}^2, \ k = 1, \ldots, N - 1, $$

(44)
\[ \dot{x}_k(t) \in \Omega_x, \quad k = 1, \ldots, N, \]  

\[ \dot{x}_k(t) \in D(S), \quad k = 1, \ldots, N, \]  

\[ \dot{x}_N(t) \subset \zeta, \]  

where \( \zeta \) is the RPI set under \( K \triangleq Y_1 \bar{Q}^{-1}_1 \) with \((P, Q, \rho)\) solutions of the SDPs (18)-(23) and (29)-(34).

LMI feasibility conditions characterizing a suitable upper-bound to the quadratic cost (40) and complying with the prescribed constraints on inputs, input rates, outputs (42)-(47) are now derived to develop a computable MPC algorithm. For the sake of simplicity, we omit the dependency for \( c_k, p_k, \hat{x}_k, x, y, \hat{P}_k \). Moreover we will denote \( I(\cdot) = I \) and \( 0(\cdot) = 0 \).

### 4.1 Cost upper bound

Let \( J_0, \ldots, J_{N-1} \) be a sequence of non-negative scalars such that, for arbitrary \( P, K \) and \( c_k, k = 0, \ldots, N - 1 \), the following inequalities hold true

\[
\max_{p_0 \in P_0} \max_{x \in D(S)} \dot{x}_1^T R_x \dot{x}_1 + c_0^T R_u c_0 \leq J_0
\]

\[
\max_{p_i \in P_i} \max_{x \in D(S)} \dot{x}_{k+1}^T R_x \dot{x}_{k+1} + c_k^T R_u c_k \leq J_k,
\]

\[
\max_{i=0, \ldots, N-1} \max_{x \in D(S)} \dot{x}_N^T P \dot{x}_N + c_{N-1}^T R_u c_{N-1} \leq J_{N-1},
\]

then it results that

\[
V \leq x^T R_x x + J_0 + J_1 + \ldots + J_{N-1}.
\]

Inequalities (48)-(50) can be recast into LMIs. Let us start with (48) for a generic triplet \((x, c_0, J_0)\) that is satisfied if

\[
(\Phi_K x + Gc_0 + B_p p_0)^T R_x (\Phi_K x + Gc_0 + B_p p_0) + c_0^T R_u c_0 \leq J_0
\]

holds true for all \( p_0 \) such that

\[
p_0^T p_0 \leq (C_K x + D_q c_0)^T (C_K x + D_q c_0)
\]

and for all \( x \in D(S) \) such that

\[
x^T H^T S H x \leq 1
\]
According to (7), inequalities (52)-(54) can be rearranged as

\[
\begin{bmatrix}
  x_{na}^T & P_0^T & 1
\end{bmatrix}
\begin{bmatrix}
  -F_0 & -D_0 & y \\
  * & J_0 - [y^T(t) c_0^T] E_0 & y(t)
\end{bmatrix}
\begin{bmatrix}
  x_{na} \\
  p_0 \\
  1
\end{bmatrix} \geq 0
\quad (55)
\]

\[
\begin{bmatrix}
  x_{na}^T & P_0^T & 1
\end{bmatrix}
\begin{bmatrix}
  Z_0 & M_0 & y \\
  * & N_0 & y(t)
\end{bmatrix}
\begin{bmatrix}
  x_{na} \\
  p_0 \\
  1
\end{bmatrix} \geq 0
\quad (56)
\]

\[
\begin{bmatrix}
  x_{na}^T & P_0^T & 1
\end{bmatrix}
\begin{bmatrix}
  \bar{S}_0 & 0 & 0 \\
  * & 1 & \bar{S}_0
\end{bmatrix}
\begin{bmatrix}
  x_{na} \\
  p_0 \\
  1
\end{bmatrix} \geq 0
\quad (57)
\]

where

\[
D_0 \triangleq \begin{bmatrix}
  \Phi_{K,na}^T R_x \Phi_{K,a} & \Phi_{K,na}^T R_x G \\
  B_p^T R_x \Phi_{K,a} & B_p^T R_x G
\end{bmatrix},
\]

\[
E_0 \triangleq \begin{bmatrix}
  \Phi_{K,a}^T R_x \Phi_{K,a} & \Phi_{K,a}^T R_x G \\
  * & G^T R_x G + R_u
\end{bmatrix},
\]

\[
F_0 \triangleq \begin{bmatrix}
  \Phi_{K,na}^T R_x \Phi_{K,na} & \Phi_{K,na}^T R_x B_p \\
  * & B_p^T R_x B_p
\end{bmatrix},
\]

\[
M_0 \triangleq \begin{bmatrix}
  C_{K,na}^T C_{K,a} & C_{K,na}^T D_q \\
  0 & 0
\end{bmatrix},
\]

\[
N_0 \triangleq \begin{bmatrix}
  C_{K,a}^T C_{K,na} & C_{K,na}^T D_q & D_q^T D_q \\
  * & * & -I
\end{bmatrix},
\]

\[
Z_0 \triangleq \begin{bmatrix}
  C_{K,na}^T C_{K,na} & 0 \\
  * & -I
\end{bmatrix},
\]

\[
\bar{S}_0 \triangleq \begin{bmatrix}
  -S & 0 \\
  * & 0
\end{bmatrix}
\]

with

\[
\Phi_K = \begin{bmatrix}
  \Phi_{K,a} & \Phi_{K,na} \\
  C_{K,a} & C_{K,na}
\end{bmatrix}, \Phi_{K,a} \in \mathbb{R}^{n_x \times n_y}, \Phi_{K,na} \in \mathbb{R}^{n_x \times (n_x - n_y)}
\]

\[
C_K = \begin{bmatrix}
  C_{K,a} & C_{K,na}
\end{bmatrix}, C_{K,a} \in \mathbb{R}^{n_x \times n_y}, C_{K,na} \in \mathbb{R}^{n_x \times (n_x - n_y)}
\]

Then the implication (52) holds true for all \( p_0 \) satisfying (53) and for all \( x \) satisfying (54) this implication can be shown to be true, via the S-procedure, if there exist two scalars \( \tau_0^0 \geq 0 \) and \( \tau_1^0 \geq 0 \) such that the inequality

\[
\begin{bmatrix}
  -F_0 - \tau_0^0 Z_0 - \tau_1^0 \bar{S}_0 \\
  * & J_0 - \tau_1^0 - [y^T c_0^T] (E_0 + \tau_0^0 N_0) [y] \\
\end{bmatrix}
\begin{bmatrix}
  y \\
  c_0
\end{bmatrix} \geq 0
\quad (59)
\]
holds true for \((y, c_0, J_0)\). By Schur complements, positive semidefiniteness of (59) is equivalent to the satisfaction of the following conditions

\[
J_0 - \tau_1^0 - 
\begin{bmatrix}
  y \\
  c_0 
\end{bmatrix}^T (E_0 + \tau_0^0 N_0 + A_0^T (-F_0 - \tau_0^0 Z_0 - \tau_1^0 S_0)^{-1} A_0 \begin{bmatrix}
  y \\
  c_0 
\end{bmatrix} \geq 0 
\]

(60)

\[
-F_0 - \tau_0^0 Z_0 - \tau_1^0 S_0 > 0 
\]

(61)

being \(A_0 = (D_0 + \tau_0^0 M_0)\). Notice that (61) can be satisfied regardless of the specific triplet \((y, c_0, J_0)\) by selecting a sufficiently large \(\tau_1^0\). Then, under (61), (60) characterizes a suitable class of triplets \((y, c_0, J_0)\) which makes (59) positive semidefinite. In order to enlarge this class, a convenient choice is

\[
[\tau_0^0, \tau_1^0] = \arg \min_{\tau_0^0 \geq 0, \tau_1^0 \geq 0} \overline{\lambda}(E_0 + \tau_0^0 N_0 + A_0^T (-F_0 - \tau_0^0 Z_0 - \tau_1^0 S_0)^{-1} A_0) 
\]

subject to

\[
-F_0 - \tau_0^0 Z_0 - \tau_1^0 S_0 > 0 
\]

Finally, by performing the following Cholesky factorization:

\[
\hat{L}_0^T \hat{L}_0 = E_0 + \tau_0^0 N_0 + (D_0 + \tau_0^0 M_0)^T (-F_0 - \tau_0^0 Z_0 - \tau_1^0 S_0)^{-1} (D_0 + \tau_0^0 M_0) 
\]

(63)

(60) can be rearranged into the following LMI condition

\[
\Sigma_0 \triangleq \begin{bmatrix}
  J_0 - \tau_1^0 - [y^T c_0^T \hat{L}_0^T] \\
  I 
\end{bmatrix} \geq 0 
\]

(64)

which is linear in terms of \(y, c_0\) and \(J_0\).

The same reasoning can be applied for (49) and (50). Specifically, consider the inequality (49) for the generic \(k = 1, \ldots, N - 2\). By defining vectors

\[
L_k \triangleq [c_0^T c_1^T \cdots c_k^T]^T \in \mathbb{R}^{(k+1)n_x}, \quad P_k \triangleq [p_0^T p_1^T \cdots p_k^T]^T \in \mathbb{R}^{(k+1)n_p}
\]

and matrices

\[
\Phi_k \triangleq \Phi_k^k \in \mathbb{R}^{n_x \times n_x}, \quad G_k \triangleq \Phi_k G \Phi_k^{-1} G \cdots \Phi_k G G \in \mathbb{R}^{n_x \times (k+1)n_x}, \\
\bar{B}_k \triangleq [\Phi_k B_p \Phi_k^{-1} B_p \cdots \Phi_k B_p B_p] \in \mathbb{R}^{n_x \times (k+1)n_x}
\]

one obtains

\[
\Sigma_k \triangleq \begin{bmatrix}
  J_k - \hat{\tau}_{k+1}^k - [y^T c_k^T \hat{L}_k^T] \\
  I 
\end{bmatrix} \geq 0 
\]

(65)

with \(\hat{L}_k\) the Cholesky factor of

\[
\hat{L}_k^T \hat{L}_k = E_k + \sum_{i=0}^k \hat{\tau}_i^k N_i^k 
\]

(66)
Finally, the following LMI condition:

\[ \Sigma_{N-1} = \begin{bmatrix} J_{N-1} - \hat{r}_N^{N-1} & -[y^T \ell_{N-1}^T] \hat{L}_{N-1}^T \\ \ast & 0 \end{bmatrix} \geq 0 \]  

provides a sufficient condition for (50) to hold true, with \( \hat{L}_{N-1}^T \) the Cholesky factor of

\[ \hat{L}_{N-1}^T \hat{L}_{N-1} = E_{N-1} + \sum_{i=0}^{N-1} \hat{z}_i^{N-1} N_i^{N-1} + \]

\[ + \mathcal{L}^T (-F_{N-1} - \sum_{i=0}^{N-1} \hat{r}_i^{N-1} Z_i^{N-1} - \hat{r}_N^{N-1} \bar{S}_{N-1})^{-1} \mathcal{L} \]  

being \( \mathcal{L} = (D_{N-1} + \sum_{i=0}^{N-1} \hat{z}_i^{N-1} M_i^{N-1}) \)

\[ [\hat{r}_0^{N-1}, \hat{r}_1^{N-1}, \ldots, \hat{r}_N^{N-1}] = \arg \min_{\hat{r}_i^{N-1} \geq 0, i = 0, \ldots, N} \lambda (\hat{L}_{N-1}^T \hat{L}_{N-1}) \]  

subject to

\[ -F_{N-1} - \sum_{i=0}^{N-1} \hat{r}_i^{N-1} Z_i^{N-1} - \hat{r}_N^{N-1} \bar{S}_{N-1} > 0 \]

where \( D_{N-1}, E_{N-1}, F_{N-1} \) are reported in the Appendix, while \( M_i^{N-1}, N_i^{N-1}, Z_i^{N-1} \) and \( \bar{S}_{N-1} \) are achieved by simply considering \( k = N - 1 \).

The above developments are straightforwardly collected in the following result.

Lemma 3: Let the initial measurement \( y_i \), the stabilizing output control law \( K \) and the input increments \( e_k, k = 0, \ldots, N - 1 \), be given. Then, the set of all non-negative variables \( J_0, \ldots, J_{N-1} \) satisfying LMI conditions (64), (65) and (68)

\[ \Sigma_k \geq 0, k = 0, \ldots, N - 1 \]

provide an upper-bound to the cost \( V \).

The same technicalities will be exploited for dealing with the constraints (42)-(47) and, therefore, the associated matrix manipulations and derivations will be ruled out, see [25] for details.
4.2 Input constraints

The following LMI conditions allow to enforce the quadratic input constraint (42):

\[
\Gamma_0 \triangleq \begin{bmatrix}
\bar{u}_{\text{max}}^2 & -(Ky + c_0)^T
\end{bmatrix} \geq 0
\]

\[
\Gamma_k \triangleq \begin{bmatrix}
\bar{u}_{\text{max}}^2 & -\hat{\alpha}_k^T
\end{bmatrix} \geq 0
\]

where \( \hat{\alpha}_k \) is the Cholesky factor of

\[
\hat{\alpha}_k = N_k^T + \sum_{i=0}^{k-1} \hat{\alpha}_i^k \hat{M}_i^T - \hat{F}_k - \sum_{i=0}^{k-1} \hat{\alpha}_i^k \hat{Z}_i^k - \hat{\alpha}_k^k \hat{S}_{k-1}^{-1} ( \hat{D}_k + \sum_{i=0}^{k-1} \hat{\alpha}_i^k \hat{M}_i^T )
\]

and

\[
[\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_k] = \arg \min_{\alpha_j \geq 0, j=0, \ldots, k} \bar{\lambda}(U_k^T U_k)
\]

subject to

\[-\hat{F}_k - \sum_{i=0}^{k-1} \hat{\alpha}_i^k \hat{Z}_i^k - \hat{\alpha}_k^k \hat{S}_{k-1} > 0 \]

\[\bar{u}_{\text{max}}^2 - \hat{\alpha}_k > 0\]

where \( N_k, \hat{D}_k, \hat{F}_k, \hat{Z}_i^k, \hat{M}_i^k \) are reported in the Appendix.

The following lemma summarizes these developments.

**Lemma 4:** Let the initial measurement \( y \) and the stabilizing output control law \( K \) be given. Then, all vectors \( c_k \) which, along with \( J_k, k = 0, \ldots, N - 1, \) satisfy LMI conditions (71) and (72)

\[\Gamma_k \geq 0, \ k = 0, \ldots, N - 1\]

fulfill input constraints (42) along the state predictions.

4.3 Input rate constraints

The input rate constraints (43) can be straightforwardly recast in the following LMI conditions

\[
\Psi_0 \triangleq \begin{bmatrix}
\bar{u}_{\text{max}}^2 & -(Ky + c_0 - u(t - 1))^T
\end{bmatrix} \geq 0
\]

\[
\Psi_k \triangleq \begin{bmatrix}
\bar{u}_{\text{max}}^2 & -\hat{\beta}_k^T
\end{bmatrix} \geq 0, \ k = 1, \ldots, N - 1
\]
Lemma 5: The above analysis gives rise to the following result.

Let the initial measurement $y$, the control input $u(t - 1)$ and the stabilizing output control law $K$ be given. Then, all vectors $c_k$ which, along with $J_k$, $k = 0, \ldots, N - 1$, satisfy the LMI conditions (75)-(76)

$$\Psi_k \geq 0, \quad k = 0, \ldots, N - 1$$

fulfill the input rate constraints (43)-(44) along the state predictions.

### 4.4 Output constraints

The following LMIs allow to enforce the quadratic output constraints (45):

$$\chi_k \triangleq \begin{bmatrix} \bar{x}_{\text{max}}^2 - \hat{\theta}_k^k & -[y^T \bar{c}^T_{k-1}]^T \hat{T}_k^T \\ 0 & 0 \end{bmatrix} \geq 0$$  \hspace{1cm} (79)

with $\hat{T}_k^T$ the Cholesky factor of

$$\hat{T}_k^T \hat{T}_k = \bar{E}_k + \sum_{i=0}^{k-1} \hat{\theta}_i^k \bar{N}_i^{k-1}$$

$$+(\hat{D}_k + \sum_{i=0}^{k-1} \hat{\theta}_i^k \hat{M}_i^{k-1})^T \bar{F}_k + \sum_{i=0}^{k-1} \hat{\theta}_i^k \sum_{i=0}^{k-1} \hat{\theta}_i^k \bar{S}_{k-1} \bar{F}_k + \sum_{i=0}^{k-1} \hat{\theta}_i^k \hat{M}_i^{k-1})$$

and

$$[\hat{\theta}_0^k, \hat{\theta}_1^k, \ldots, \hat{\theta}_k^k] = \min_{\hat{\theta}_j^k \geq 0, j=0 \ldots k} \bar{\lambda} (T_k^T T_k)$$  \hspace{1cm} (81)

subject to

$$-\bar{F}_k - \sum_{i=0}^{k-1} \theta_i^k \bar{Z}_i^{k-1} - \theta_k^k \bar{S}_{k-1} > 0, \bar{x}_{\text{max}}^2 - \theta_k^k > 0$$

where $\hat{D}_k, \hat{E}_k, \hat{F}_k$ are reported in the Appendix.

The above analysis gives rise to the following result.

The following result finally holds.
Lemma 6: Let the initial measurement $y$ and the stabilizing output control law $K$ be given. Then, all vectors $c_k$ which, along with $J_k$, $k = 0, \ldots, N - 1$, satisfy the LMI conditions (79)

$$
\chi_k \geq 0, \quad k = 1, \ldots, N
$$

fulfill the input constraints (45) along the state predictions.

4.5 Non measurable state constraints

Constraints (46) can be recast as follow

$$
\Xi_k \triangleq \begin{bmatrix}
1 - \hat{\eta}_k^T & -[y^T T_{k-1} \hat{W}_k^T]
\end{bmatrix} \geq 0
$$

(82)

with $\hat{W}_k^T$ the Cholesky factor of

$$
\hat{W}_k^T W_k = \hat{E}_k + \sum_{i=0}^{k-1} \hat{e}_i^T N_i^{k-1}
$$

$$
+ (\hat{D}_k + \sum_{i=0}^{k-1} \hat{e}_i^T M_i^{k-1})^T (\hat{F}_k - \sum_{i=0}^{k-1} \hat{e}_i^T Z_i^{k-1} - \hat{e}_k^T S_{k-1})^{-1} (\hat{D}_k + \sum_{i=0}^{k-1} \hat{e}_i^T M_i^{k-1})
$$

and

$$
[\hat{\eta}_0^k, \hat{\eta}_1^k, ..., \hat{\eta}_k^k] = \arg \min_{\eta_j^i \geq 0, j=0, \ldots, k} \bar{\lambda} (W_k^T W_k)
$$

(83)

subject to

$$
-\hat{F}_k - \sum_{i=0}^{k-1} \hat{e}_i^T Z_i^{k-1} - \hat{e}_k^T S_{k-1} > 0, \quad 1 - \hat{\eta}_k^k > 0
$$

where $\hat{D}_k$, $\hat{E}_k$, $\hat{F}_k$ are reported in the Appendix.

The following lemma summarizes the above developments.

Lemma 7: Let the initial measurement $y$ and the stabilizing output control law $K$ be given. Then, all vectors $c_k$ which, along with $J_k$, $k = 0, \ldots, N - 1$, satisfy the LMI conditions (82)

$$
\Xi_k \geq 0, \quad k = 1, \ldots, N
$$

fulfill the non measurable state constraints (46) along the state predictions.

4.6 Terminal constraint

Finally, the terminal constraint (47) can be satisfied by means of the following LMI:

$$
\Sigma_N \triangleq \begin{bmatrix}
\rho - \hat{\tau}_N^T & -[y^T T_{N-1} \hat{E}_N^T]
\end{bmatrix} \geq 0
$$

(85)
with \( \hat{L}_N \) the Cholesky factor of

\[
\hat{L}_N^T \hat{L}_N = E_N + \sum_{i=0}^{N-1} \hat{\tau}_i^N N_i^{N-1} + C^T (-F_N - \sum_{i=0}^{N-1} \hat{\tau}_i^N Z_i^{N-1} - \hat{\tau}_N S_{N-1}^{-1}) C
\]

being \( C = (D_N + \sum_{i=0}^{N-1} \hat{\tau}_i^N M_i^{N-1}) \) and

\[
[\hat{\tau}_0^N, \hat{\tau}_1^N, \ldots, \hat{\tau}_N^N] = \arg \min_{\tau_j^N \geq 0, j = 0, \ldots, N} \lambda (L_N^T L_N)
\]

subject to

\[
-F_{N-1} - \sum_{i=0}^{N-1} \tau_i^N Z_i^{N-1} - \tau_N^N S_{N-1}^{-1} > 0, \rho - \tau_N^N > 0
\]

where \( D_N, E_N, F_N \) are reported in the Appendix.

### 4.7 Output MPC Algorithm

Hereafter, the following constraints on the scalars resulting from the solution of the GEVPs \((62), (67), (70), (74), (78), (81), (84)\) and \((87)\) are taken into account:

\[
\begin{align*}
\hat{\tau}_{h-1}^{k-1} &\leq \hat{\tau}_h^k & k = 1, \ldots, N - 1 & h = 1, \ldots, k + 1 \\
\hat{\alpha}_{h-1}^{k-1} &\leq \hat{\alpha}_h^k & k = 1, \ldots, N - 1 & h = 1, \ldots, k \\
\hat{\beta}_{h-1}^{k-1} &\leq \hat{\beta}_h^k & k = 1, \ldots, N - 1 & h = 1, \ldots, k \\
\hat{\theta}_{h-1}^{k-1} &\leq \hat{\theta}_h^k & k = 2, \ldots, N & h = 1, \ldots, k \\
\hat{\eta}_{h-1}^{k-1} &\leq \hat{\eta}_h^k & k = 2, \ldots, N & h = 1, \ldots, k
\end{align*}
\]

Such extra requirements are mandatory in order to ensure the recursive feasibility of the underlying MPC strategy, i.e. the existence of a solution at time \( \bar{t} \) implies the existence of solutions for all future time instants \( t \geq \bar{t} \). The relevance of \((88)\) will be soon clarified in the proof of the next Theorem 1.

All above developments allows one to write down the following computable scheme:

**Output Model Predictive Control Algorithm (OUT-MPC)**

**OFF-LINE PHASE:**

A1: **Given** the current partial state measurement \( y = x_a(t) \);

A2: **Compute** the stabilizing output feedback gain \( K \) by solving \((18)-(23)\);

A3: **Compute** the RPI set \( \zeta \) by solving \((29)-(34)\);
A4: Compute $\hat{\tau}^k_h$, $h = 0, \ldots, k + 1, k = 0, \ldots, N - 1$, by solving the GEVPs (62), (67), (70);

A5: Compute $\hat{\tau}^N_k$, $k = 0, \ldots, N$, by solving (87) subject to (88);

A6: Compute $\hat{\alpha}^k_h$, $h = 0, \ldots, k, k = 0, \ldots, N - 1$, by solving (74) subject to (88);

A7: Compute $\hat{\beta}^k_h$, $h = 0, \ldots, k, k = 0, \ldots, N - 1$, by solving (78) subject to (88);

A8: Compute $\hat{\theta}^k_h$, $h = 0, \ldots, k, k = 1, \ldots, N$, by solving (81) subject to (88);

A9: Compute $\hat{\eta}^k_h$, $h = 0, \ldots, k, k = 1, \ldots, N$, by solving (84) subject to (88);

A10: Store scalars $\{\hat{\tau}^k_h\}_{k=0}^{N-1}$, $\{\hat{\alpha}^k_h\}_{k=0}^{N-1}$, $\{\hat{\beta}^k_h\}_{k=0}^{N-1}$, $\{\hat{\theta}^k_h\}_{k=0}^{N-1}$, $\{\hat{\eta}^k_h\}_{k=0}^{N-1}$.

**ON-LINE PHASE:**

B1: Given $y(t)$ at each time instant, Solve

$$[J^*_k(t), c^*_k(t)] = \arg \min_{J_k, c_k} \sum_{k=0}^{N-1} J_k$$

subject to

- $\Sigma_k \geq 0$, $k = 0, \ldots, N - 1$
- $\Gamma_k \geq 0$, $k = 0, \ldots, N - 1$
- $\Psi_k \geq 0$, $k = 0, \ldots, N - 1$
- $\chi_k \geq 0$, $k = 1, \ldots, N$
- $\Xi_k \geq 0$, $k = 1, \ldots, N$
- $\Sigma_N \geq 0$

B2: Feed the plant with $u(t) = Ky(t) + c^*_0(t)$;

B3: $t \leftarrow t + 1$ and goto Step B1.

Feasibility and closed-loop stability properties of the OUT-MPC scheme are proved below.

**Theorem 1:** Let the OUT-MPC-Algorithm have solution at time $t$, then it has solution at time $t + 1$, satisfies prescribed constraints (3), (4) and (8) and yields a quadratically stable closed-loop system.
Proof - Let \((J_k^x(t), c_k^x(t))\), \(k = 0, \ldots, N - 1\), be the optimal solution from the computation Step B1 at time \(t\). We will prove recursive feasibility by showing that the following sequence

\[
(J_1^x(t), c_1^x(t)), (J_2^x(t), c_2^x(t)), \ldots, (J_N^x(t), c_{N-1}^x(t)), (J_N^x(t), 0_{n_u})
\]  

(90)
is an admissible, though possibly non optimal, solution for Step B1 at time \(t + 1\). First, we prove that \(\Sigma_k(t + 1) \geq 0, \ k = 0, \ldots, N - 1\) if, at the optimum, \(\Sigma_k(t) \geq 0\).

In fact this inequality can be equivalently rewritten as

\[
J_k^x(t) - \max_{\tilde{x}_{k+1}(t)} \{\tilde{x}_{k+1}^T(t) R_x \tilde{x}_{k+1}(t)\} - c_k^x(t)^T R_u c_k^x(t) - \\
\tilde{x}_{k+1}^T \max(1 - x(t)^T H_x^T S H_x(t) - \\
- \sum_{i=0}^{k} \tilde{x}_{k+1}^T \max_{\tilde{x}_i(t)} \{C_K \tilde{x}_i(t) + D_q c_i^x(t)\}^T (C_K \tilde{x}_i(t) + D_q c_i^x(t)) - p_i^T(t)(p_i(t)) \geq 0
\]

(91)

which holds true \(\forall x(t) \in D(S)\) and \(\forall p_i(t) \in \mathbb{R}^{n_r}\), as guaranteed by the S-procedure. Since (91) is feasible for all \(x(t) \in \mathbb{R}^{n_x}\) and \(\forall p_i(t) \in \mathbb{R}^{n_r}\) if it is feasible for \(x(t) = 0_{n_x}\) and \(p_i(t) = 0_{n_r}\), we can limit the analysis to these two values of the state and the uncertain parameter. Then, at the next time instant \(t + 1\), condition \(\Sigma_{k-1}(t + 1) \geq 0\), for a generic pair \((J_{k-1}(t + 1), c_{k-1}(t + 1))\), is equivalent to

\[
J_{k-1}(t + 1) - \max_{\tilde{x}_{k+1}(t+1)} \{\tilde{x}_{k+1}^T(t+1) R_x \tilde{x}_{k+1}(t+1)\} - \\
c_{k-1}(t+1)^T R_u c_{k-1}(t+1) - \tilde{x}_{k+1}^{k-1} - \\
- \sum_{i=1}^{k} \tilde{x}_{k+1}^{i-1} \max_{\tilde{x}_i(t+1)} \{M_i^T M_i\} \geq 0
\]

(92)

being \(M = (C_K \tilde{x}_{i-1}(t+1) + D_q c_{i-1}(t+1))\). We want to show that (92) is fulfilled under the following substitutions

\[
J_{k-1}(t + 1) \leftarrow J_k^x(t), \ c_{i-1}(t+1) \leftarrow c_i^x(t), \ i = 1, \ldots, N - 1, \\
J_{N-1}(t + 1) \leftarrow J_{N-1}^x(t), \ c_{N-1}(t+1) \leftarrow 0_{n_u}
\]

(93)

Observe that the following inclusions

\[
x(t + 1) \in \hat{x}_1(t), \ \hat{x}_1(t + 1) \subset \hat{x}_2(t), \ldots, \hat{x}_N(t + 1) \subset C(P, \rho)
\]

are satisfied along the state predictions under (90) and ensure that each term (viz. the one multiplying \(\tilde{x}_{k+1}^T\) and the others multiplying \(\tilde{x}_i^T\)) in the summation of (91) is greater than or equal to the corresponding term (viz. the one multiplying \(\tilde{x}_{k+1}^{k-1}\) and \(\tilde{x}_{i-1}^{k-1}\)) in (92). Then feasibility holds true because \(\tilde{x}_{k+1}^{k-1} \leq \tilde{x}_{k+1}^{k-1}\) and \(\tilde{x}_{i-1}^{k-1} \leq \tilde{x}_{i-1}^{k-1}\).

These reasoning lines, and the same arguments exploited in [25] apply mutatis mutandis to show feasibility of \(\Sigma_{N-1}(t + 1) \geq 0\), where it is further used the fact that \(\hat{x}_N(t + 1) \subset \Phi_K \hat{x}_N(t), \ \Sigma_{N}(t + 1) \geq 0 \ \Gamma_k \geq 0, \ k = 0, \ldots, N, \ \Psi_k, \ k = 0, \ldots, N - 1, \ \chi_k, \ k = 1, \ldots, N - 1, \ \text{and} \ \Xi_k, \ k = 1, \ldots, N - 1\).
As the closed-loop stability issue is concerned, we shall consider as a candidate Lyapunov function the cost (40) with \( V(t) \) the numerical value at the time instant \( t \) corresponding to the optimal solution \( c_k^*(t), k = 0, \ldots, N - 1 \). Denoting with \( \bar{V}(t + 1) \) the value of the cost at \( t+1 \) under the feasible solution \( \{c_1^*(t), c_2^*(t), \ldots, c_{N-1}^*(t), 0_{n_u}\} \), by a direct substitution, and exploiting the fact that \( \|x(t + 1)\|_{R_x}^2 - J_0^*(t) \leq -\|c_0^*(t)\|_{R_u}^2 \), one finds that the following inequalities hold true

\[
V(t + 1) \leq \bar{V}(t + 1) \leq V(t) - \|x(t)\|_{R_x}^2 - \|c_0^*(t)\|_{R_u}^2
\]

Therefore, one derives that \( \lim_{t \to \infty} V(t) = V(\infty) < \infty \) and

\[
\sum_{t=0}^{\infty} \|x(t)\|_{R_x}^2 + \|c_0^*(t)\|_{R_u}^2 \leq V(0) - V(\infty) < \infty.
\]

As a consequence, \( \lim_{t \to \infty} x(t) = 0_{n_x} \) and \( \lim_{t \to \infty} c_0^*(t) = 0_{n_u} \), thanks to \( R_x > 0 \) and \( R_u > 0 \). \( \square \)
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**Appendix**

**Cost upper bound**

\[ D_k := \begin{bmatrix} \Phi^T_{k,na} R_x \Phi_{k,a} & \Phi^T_{k,na} R_x \tilde{G}_k \\ \Phi^T_{k,a} R_x \Phi_{k,a} & \Phi^T_{k,a} R_x \tilde{G}_k \end{bmatrix} \]

\[ E_k := \begin{bmatrix} \Phi^T_{k,a} R_x \Phi_{k,a} & \Phi^T_{k,a} R_x \tilde{G}_k \\ \Phi^T_{k,a} R_x \Phi_{k,a} & \Phi^T_{k,a} R_x \tilde{G}_k \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & R_u \end{bmatrix} \]

\[ F_k := \begin{bmatrix} \Phi^T_{k,na} R_x \Phi_{k,na} & \Phi^T_{k,na} R_x \tilde{B}_k \\ \Phi^T_{k,na} R_x \Phi_{k,na} & \Phi^T_{k,na} R_x \tilde{B}_k \end{bmatrix} \]

\[ \Phi_k := [\Phi_{k,a} \Phi_{k,na}], \Phi_{k,a} \in \mathbb{R}^{n_x \times n_y}, \Phi_{k,na} \in \mathbb{R}^{n_x \times n_x - n_y} \]

\[ M_i^k := \begin{bmatrix} C^a_{k,i} & C^a_{k,i} & C^a_{k,i} \\ C^a_{k,i} & B^T_{k,i} G_{k,i} \end{bmatrix}, \quad N_i^k := \begin{bmatrix} C^a_{k,i} & C^a_{k,i} & C^a_{k,i} \\ C^a_{k,i} & B^T_{k,i} G_{k,i} \end{bmatrix} \]

\[ Z_i^k = \begin{bmatrix} C^a_{k,i} & C^a_{k,i} & C^a_{k,i} \\ C^a_{k,i} & B^T_{k,i} G_{k,i} \end{bmatrix}, \quad C^a_{k,i} := \begin{bmatrix} C_{K} \Phi_{i-1,a}, i > 0 \\ C_{K,a}, i = 0 \end{bmatrix} \]

\[ C^a_{k,i} := \begin{bmatrix} C_{K} \Phi_{i-1,a}, i > 0 \\ C_{K,na}, i = 0 \end{bmatrix} \]

\[ B_{k,i} := \begin{bmatrix} C_{K} \Phi^{-1}_{i-1} B_p & \Phi^{-2}_{i-1} B_p & \cdots & \Phi^{-i}_{i-1} B_p & B_p & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \quad i > 0 \]

\[ G_{k,i} := \begin{bmatrix} C_{K} \Phi^{-1}_{i} G & \Phi^{-2}_{i} G & \cdots & \Phi_{i} G & G & 0 & 0 & D_q \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \quad i > 0 \]

\[ H_i^k := \begin{bmatrix} 0 & 0 & 0 \\ * & I & 0 \\ * & * & 0 \end{bmatrix}, \quad S_k := \begin{bmatrix} -S & 0 \\ * & 0 \end{bmatrix}, \]
4 LMI based Output MPC control strategy

\[
D_{N-1} := \begin{bmatrix} \Phi_{N-1,na}^T P \Phi_{N-1,a} & \Phi_{N-1,na}^T P \bar{G}_{N-1} \\ * & B_{N-1}^T P \bar{G}_{N-1} \end{bmatrix}
\]

\[
E_{N-1} := \begin{bmatrix} \Phi_{N-1,na}^T P \Phi_{N-1,a} & \Phi_{N-1,na}^T P \bar{G}_{N-1} \\ * & \bar{G}_{N-1}^T P \bar{G}_{N-1} + \begin{bmatrix} 0 & 0 \\ 0 & R_a \end{bmatrix} \end{bmatrix}
\]

\[
F_{N-1} := \begin{bmatrix} \Phi_{N-1,na}^T P \Phi_{N-1,na} & \Phi_{N-1,na}^T P \bar{B}_{N-1} \\ * & B_{N-1}^T P \bar{B}_{N-1} \end{bmatrix}
\]

**Input constraints**

\[
\hat{D}_k := \begin{bmatrix} \Phi_{k-1,na}^T C^T K^T K C \Phi_{k-1,a} & \Phi_{k-1,na}^T C^T K^T \begin{bmatrix} KC & G_{k-1} & I \end{bmatrix} \\ * & B_{k-1}^T C^T K^T \begin{bmatrix} KC & G_{k-1} & I \end{bmatrix} \end{bmatrix}
\]

\[
\hat{E}_k := \begin{bmatrix} \Phi_{k-1,a}^T C^T K^T K C \Phi_{k-1,a} & \Phi_{k-1,a}^T C^T K^T \begin{bmatrix} KC & G_{k-1} & I \end{bmatrix} \\ * & G_{k-1}^T C^T K^T K C G_{k-1} \end{bmatrix}
\]

\[
\hat{F}_k := \begin{bmatrix} \Phi_{k-1,na}^T C^T K^T K C \Phi_{k-1,na} & \Phi_{k-1,na}^T C^T K^T K C \bar{B}_{k-1} \\ * & B_{k-1}^T C^T K^T K C \bar{B}_{k-1} \end{bmatrix}
\]

\[
\hat{Z}_i := \begin{bmatrix} C_{k,i}^n \Phi_{k-1,a}^T C_{k,i} & C_{k,i}^n \Phi_{k-1,a}^T \bar{G}_{k-1} \\ * & B_{k-1}^T \Phi_{k-1,a}^T \bar{B}_{k-1,i} - H_{k-1,i} \end{bmatrix}
\]

\[
\hat{M}_i := \begin{bmatrix} C_{k,i}^n \Phi_{k-1,a}^T C_{k,i} & C_{k,i}^n \Phi_{k-1,a}^T \bar{G}_{k-1} \\ * & B_{k-1,i}^T \bar{G}_{k-1,i} \end{bmatrix}
\]

**Input rate constraints**

\[
\hat{D}_k := \begin{bmatrix} \Phi_{k-1,na}^T \Phi_{k-1,a} & \Phi_{k-1,na}^T \bar{G}_{k-1} \\ * & B_{k-1}^T \bar{G}_{k-1} \end{bmatrix}
\]

\[
\hat{E}_k := \begin{bmatrix} \Phi_{k-1,a}^T \Phi_{k-1,a} & \Phi_{k-1,a}^T \bar{G}_{k-1} \\ * & G_{k-1}^T \bar{G}_{k-1} \end{bmatrix}
\]

\[
\hat{F}_k := \begin{bmatrix} \Phi_{k-1,na}^T \Phi_{k-1,na} & \Phi_{k-1,na}^T \bar{B}_{k-1} \\ * & B_{k-1}^T \bar{B}_{k-1} \end{bmatrix}
\]

**Output constraints**

\[
\hat{D}_k := \begin{bmatrix} \Phi_{k-1,na}^T C^T C \Phi_{k-1,a} & \Phi_{k-1,na}^T C^T C \bar{G}_{k-1} \\ * & B_{k-1}^T C^T C \bar{G}_{k-1} \end{bmatrix}
\]

\[
\hat{E}_k := \begin{bmatrix} \Phi_{k-1,a}^T C^T C \Phi_{k-1,a} & \Phi_{k-1,a}^T C^T C \bar{G}_{k-1} \\ * & G_{k-1}^T C^T C \bar{G}_{k-1} \end{bmatrix}
\]

\[
\hat{F}_k := \begin{bmatrix} \Phi_{k-1,na}^T C^T C \Phi_{k-1,na} & \Phi_{k-1,na}^T C^T C \bar{B}_{k-1} \\ * & B_{k-1}^T C^T C \bar{B}_{k-1} \end{bmatrix}
\]
Non measurable state constraints

\[ \hat{D}_k := \begin{bmatrix} \hat{\Phi}_T^T k-1,na H^T S H \hat{\Phi}_k-1,na & \hat{\Phi}_T^T k-1,na H^T S H \hat{G}_k-1 \\ * & B^T_{k-1} H^T S H \hat{G}_k-1 \end{bmatrix} \]

\[ \hat{E}_k := \begin{bmatrix} \hat{\Phi}_T^T k-1,a H^T S H \hat{\Phi}_k-1,a & \hat{\Phi}_T^T k-1,a H^T S H \hat{G}_k-1 \\ * & G^T_{k-1} H^T S H \hat{G}_k-1 \end{bmatrix} \]

\[ \hat{F}_k := \begin{bmatrix} \hat{\Phi}_T^T k-1,na H^T S H \hat{\Phi}_k-1,na & \hat{\Phi}_T^T k-1,na H^T S H \hat{B}_k-1 \\ * & B^T_{k-1} H^T S H \hat{B}_k-1 \end{bmatrix} \]

Terminal Constraint

\[ D_N := \begin{bmatrix} \hat{\Phi}_T^T N-1,na P \hat{\Phi}_N-1,na & \hat{\Phi}_T^T N-1,na P \hat{G}_N-1 \\ * & \hat{B}_N^{-1} P \hat{G}_N-1 \end{bmatrix} \]

\[ E_N := \begin{bmatrix} \hat{\Phi}_T^T N-1,a P \hat{\Phi}_N-1,a & \hat{\Phi}_T^T N-1,a P \hat{G}_N-1 \\ * & \hat{G}_N^{-1} P \hat{G}_N-1 \end{bmatrix} \]

\[ F_N := \begin{bmatrix} \hat{\Phi}_T^T N-1,na P \hat{\Phi}_N-1,na & \hat{\Phi}_T^T N-1,na P \hat{B}_N-1 \\ * & \hat{B}_N^{-1} P \hat{B}_N-1 \end{bmatrix} \]