Invariance of the restricted $p$-power map on integrable derivations under stable equivalences

Leonard Rubio y Degrassi

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Abstract

We show that the $p$-power maps in the first Hochschild cohomology space of finite-dimensional selfinjective algebras over a field of prime characteristic $p$ commute with stable equivalences of Morita type on the subgroup of classes represented by integrable derivations. We show, by giving an example, that the $p$-power maps do not necessarily commute with arbitrary transfer maps in the Hochschild cohomology of symmetric algebras.

1 Introduction

Let $k$ be a field of prime characteristic $p$. For symmetric $k$-algebras, it is shown in [5] that the Gerstenhaber bracket in Hochschild cohomology commutes with the transfer maps introduced in [7]. Zimmermann proved in [10] that the $p$-power map on (the positive part of) Hochschild cohomology commutes with derived equivalences. We show in this paper that the $p$-power map, restricted to the classes of integrable derivations, commutes with stable equivalences of Morita type between finite-dimensional selfinjective algebras. We also show, by giving an example, that $p$-power maps need not commute with arbitrary transfer maps in the Hochschild cohomology of symmetric algebras.

Let $A$ be a finite-dimensional selfinjective $k$-algebra. For $r$ a positive integer, we denote by $\text{Aut}_r(A[[t]])$ the subgroup of $k[[t]]$-algebra automorphism of $A[[t]]$ which induce the identity on $A[[t]]/t^rA[[t]]$. If $\alpha \in \text{Aut}_r(A[[t]])$, then there is a unique $k[[t]]$-linear map $\bar{\mu}$ on $A[[t]]$ such that $\alpha(a) = a + t^r \bar{\mu}(a)$ for all $a \in A[[t]]$. An easy verification (see Proposition 3.5) shows that the map $\bar{\mu}$ induced by $\mu$ on the quotient $A[[t]]/tA[[t]] \cong A$ is a derivation; any such derivation is called $r$-integrable. We denote by $\text{HH}_1^r(A)$ the image in $\text{HH}_1(A)$ of all $r$-integrable derivations. Let $A$, $B$ be finite-dimensional selfinjective $k$-algebras, $M$ an $A$-$B$-bimodule and $N$ a $B$-$A$-bimodule. Following Broué [2], we say that $M$ and $N$ induce a stable equivalence of Morita type between $A$ and $B$ if $M$, $N$ are finitely generated projective as left and right modules with the property that $M \otimes_B N \cong A \oplus X$ for some projective $A$-$A$-bimodule $X$ and $N \otimes_A M \cong B \oplus Y$ for some projective $B$-$B$-bimodule $Y$. If $A$, $B$ are symmetric then $N$ can by replaced by $M^\vee$.

**Theorem 1.1.** Let $A$, $B$ be finite-dimensional selfinjective $k$-algebras with separable semisimple quotients, and let $M$, $N$ be an $A$-$B$-bimodule, $B$-$A$-bimodule,
respectively, inducing a stable equivalence of Morita type between \( A \) and \( B \). For any positive integer \( r \), the \( p \)-power map sends \( \text{HH}^i(A) \) to \( \text{HH}^i_{rp}(A) \), and we have a commutative diagram of maps

\[
\begin{array}{ccc}
\text{HH}^i(A) & \xrightarrow{\sim} & \text{HH}^i(B) \\
[\mu] & & [\nu] \\
\text{HH}^i_{rp}(A) & \xrightarrow{\sim} & \text{HH}^i_{rp}(B)
\end{array}
\]

where the horizontal isomorphisms are induced by the functor \( N\otimes_A \otimes_A M \), and where the vertical maps are the \( p \)-power maps.

In Section 2, we recall some basic results. In Section 3 we prove the main results concerning \( r \)-integrable derivation that allow us to prove in Section 4 the Theorem \[\text{[L3]}\]. In the last section we provide an example of when the \( p \)-power map does not commute with a transfer map between the Hochschild cohomology of two symmetric algebras.

2 Background

Let \( A \) be a finite-dimensional algebra over \( k \). For any integer \( n \geq 0 \) and any \( A\otimes_k A^{op} \)-module \( M \) the Hochschild cohomology of degree \( n \) of \( A \) with coefficients in \( M \) is denoted by \( \text{HH}^n(A; M) \). In particular \( \text{HH}^n(A) = \text{HH}^n(A; A) \). It is well known that \( \text{HH}^0(A) = Z(A) \) and \( \text{HH}^1(A) \) is the space of derivations modulo inner derivations. The direct sum \( \bigoplus_{n \geq 0} \text{HH}^n(A) \) is a Gerstenhaber algebra, in particular \( \text{HH}^1(A) \) is a Lie algebra. In addition, if the characteristic of \( k \) is positive, there is a map \( [p] : \text{HH}^1(A) \to \text{HH}^1(A) \), called \( p \)-power map. This map is induced by the map sending a derivation \( f \) to \( f^p \) that is \( f \) composed \( p \)-times with itself. Then \( \text{HH}^1(A) \) endowed with the \( p \)-power becomes a restricted Lie algebra. Let \( A[[t]] \) be the formal power series with coefficients in \( A \). By \[\text{[L2.1]}\] the canonical map \( A[[t]] \to A[[t]]/[t^r]A[[t]] \) induces an isomorphism

\[
\text{HH}^n(A[[t]]; A[[t]]/[t^r]A[[t]]) \cong \text{HH}^n(A[[t]]/[t^r]A[[t]]).
\]

for all \( n \geq 0 \) and \( r > 0 \). The following is well known:

**Lemma 2.1.** Let \( A \) be a finite-dimensional algebra over \( k \) and let \( A[[t]] \) be the formal power series with coefficients in \( A \). Then the multiplication in \( A[[t]] \) induce a \( k[[t]] \)-algebra isomorphism \( k[[t]] \otimes_k A \cong A[[t]] \).

**Proof.** The isomorphism sends \( \sum_{i \geq 0} \lambda_i t^i \otimes a \) to \( \sum_{i \geq 0} \lambda_i a t^i \) where \( \lambda_i \in k \) and \( a \in A \). In order to show that this is an isomorphism, we construct its inverse as follows: let \( \sum_{i \geq 0} a_i t^i \in A[[t]] \) and let \( \{ e_j \}_{1 \leq j \leq n} \) be a \( k \)-basis of \( A \). Write \( a_i = \sum_{j=1}^n \mu_{ij} e_j \) for every non-negative integer \( i \) where \( \mu_{ij} \in k \). The inverse map sends \( \sum_{i \geq 0} a_i t^i \) to \( \sum_{j=1}^n \left( \sum_{i \geq 0} \mu_{ij} t^i \otimes e_j \right) \). \( \square \)

**Corollary 2.2.** Let \( A \) be a finite-dimensional algebra over \( k \) and let \( r \) be a positive integer. Then the canonical map \( Z(A[[t]]) \to Z(A[[t]]/[t^r]A[[t]]) \) is surjective.
Let $n$ be an integer. We recall that if

$$0 \rightarrow X \xrightarrow{\tau} Y \xrightarrow{\sigma} Z \rightarrow 0$$

is a short exact sequence of cochain complexes with differentials $\delta, \epsilon, \zeta$ respectively, then this induces a long exact sequence

$$\cdots \rightarrow H^n(X) \xrightarrow{H^n(\tau)} H^n(Y) \xrightarrow{H^n(\sigma)} H^n(Z) \xrightarrow{d^n} H^{n+1}(X) \rightarrow \cdots$$

depending functorially on the short exact sequence, where $d^n$ is called the connecting homomorphism which is obtained in the following way: let $\bar{z} = z + \text{Im}(\epsilon^{n-1}) \in H^n(Z)$ for some $z \in \text{Ker}(\epsilon^n) \subseteq Z^n$. Since $\epsilon$ is surjective in each degree there is $y \in Y^n$ such that $\epsilon^n(y) = z$. Then $\epsilon(y) \in Y^{n+1}$ satisfies

$$\sigma^{n+1}(\epsilon^n(y)) = \zeta^n(\epsilon^n(y)) = \zeta^n(z) = 0 \quad (2)$$

Hence $\epsilon(y) \in \text{Ker}(\sigma^{n+1}) = \text{Im}(\epsilon^{n+1})$. Thus the is an $x \in X^{n+1}$ such that $\tau^{n+1}(x) = \epsilon^n(y)$. It is easy to check that $x \in \text{Ker}(\delta^{n+1})$ and the class $\bar{x} = x + \text{Im}(\delta^n) \in H^{n+1}(X)$ depends only in the class $\bar{z}$ of $z$ in $H^n(Z)$. The connecting homomorphism sends $\bar{z}$ to $\bar{x}$.

For the next two sections all the tensor products are over $k$ unless otherwise specified.

### 3 Integrable derivations of degree $r$

**Definition 3.1.** (cf. [S 1.1]) Let $A$ be a finite-dimensional $k$-algebra. A higher derivation $D$ of $A$ is a sequence $D = (D_i)_{i \geq 0}$ of $k$-linear endomorphisms $D_i : A \rightarrow A$ such that $D_0 = \text{Id}$ and $D_n(ab) = \sum_{i+j=n}D_i(a)D_j(b)$ for all $n \geq 1$ and all $a, b \in A$.

For a fixed positive integer $r$ we denote by $\text{Aut}_r(A[t][t])$ the group of all $k[t]$-algebra automorphisms of $A[[t]]$ which induce the identity on $A[[t]]/t^rA[[t]]$. Clearly we have an inclusion $\text{Aut}_r(A[[t]]) \subseteq \text{Out}_1(A[[t]])$ for every $r \geq 1$.

Following [S] any higher derivation $D = (D_i)_{i \geq 0}$ of $A$ determines a unique automorphism $\alpha \in \text{Aut}_r(A[[t]])$ satisfying $\alpha(a) = \sum_{i \geq 0}D_i(a)t^i$ for all $a \in A$ and vice versa. Note that any $k[[t]]$-ring endomorphism of $A[[t]]$ is determined by its restriction to $A$. We denote by $\text{Out}_r(A[[t]])$ the image of the canonical map $\varphi : \text{Aut}_r(A[[t]]) \rightarrow \text{Out}(A[[t]])$ and by $\text{Der}(A)$ the set of derivations over $A$.

**Lemma 3.2.** Let $A$ be a finite-dimensional $k$-algebra. Let $r$ be a positive integer. Then $\text{Out}_r(A[[t]])$ is the kernel of the canonical group homomorphism

$$\psi : \text{Out}(A[[t]]) \rightarrow \text{Out}(A[[t]]/(t^rA[[t]])). \quad (3)$$

**Proof.** Clearly $\text{Out}_r(A[[t]]) \subseteq \text{Ker}(\psi)$. Let $\alpha$ be a representative of an element in the kernel of $\psi$. Then $\psi(\alpha)$ is given by conjugation with an invertible element $u = u + t^rA[[t]]$ in $A[[t]]/t^rA[[t]]$ where $u \in A[[t]]$. If we denote by $A[[t]] = A[[t]]/t^rA[[t]]$, we have $A[[t]] = A[[t]]u$. Then we can lift it to $A[[t]] = A[[t]]u + t^rA[[t]]$. By Nakayama’s Lemma we have $A[[t]] =
For the second part we let $\bar{z}$ be an element $\bar{z}$ such that $\bar{t} \cdot \bar{z} = 1 + t \cdot a$ and we divide by $t$. The class of $\bar{z}$ is an inner automorphism induced by conjugation by an element $\bar{d}$ for some $d \in A$; that is $\bar{d}$ is a inner derivation.

Proposition 3.4. Let $A$ be a finite-dimensional $k$-algebra. Let $r$ be a positive integer, let $\alpha \in \text{Aut}(A[[t]])$ and let $\mu : A[[t]] \to A[[t]]$ be the unique $k[[t]]$-linear map such that $\alpha(a) = a + t^r \mu(a)$ for all $a \in A[[t]]$. Then the following hold:

(a) The map $\bar{\mu} : A \cong A[[t]]/tA[[t]] \to A \cong A[[t]]/tA[[t]]$ induced by $\mu$ is a derivation.

(b) The class of $\bar{\mu} \in \text{HH}^1(A)$ depends only on the class of $\alpha \in \text{Out}(A[[t]])$.

Proof. Let $a, b \in A[[t]]$, since $\alpha$ is an automorphism we have $\alpha(ab) = ab + t^r \mu(ab)$ is equal to $\alpha(a)\alpha(b) = ab + t^r \mu(a) + t^r \mu(b) + t^{2r} \mu(a)\mu(b)$ hence we obtain $\mu(ab) = a\mu(b) + \mu(a)b + t^r \mu(a)\mu(b)$. Reducing modulo $t^r$ we have $\mu(ab) = a\mu(b) + \mu(a)b$.

Consequently $\bar{\mu}$ is a derivation on $A$.

Now suppose that $\alpha$ is an inner automorphism induced by conjugation by an element $c \in A[[t]]$ such that $\alpha(a) = cac^{-1}$. Since $\alpha$ induces the identity on $A[[t]]/t^r A[[t]]$ then taking the projection of $\alpha$ in $A[[t]]/t^r A[[t]]$ we have $\bar{c}a\bar{c} = \bar{a}$, that is $\bar{c}a = a\bar{c}$ hence $\bar{c} \in Z(A[[t]]/t^r A[[t]])$. Since the map $Z(A[[t]]) \to Z(A[[t]]/t^r A[[t]])$ is surjective then there is an element $z \in Z(A)$ such that $\bar{z} = \bar{c}$ hence such that $cz^{-1} \in 1 + t^r A[[t]]$. So if we replace $c$ by $cz^{-1}$ we have $c = 1 + t^r d$ for some $d \in A[[t]]$. If we take an $a \in A[[t]]$ we have $cac^{-1} = a\alpha(a) = a + t^r \mu(a)$ hence $ca = ac + t^r \mu(a)c$, that is $[c, a] = t^r \mu(a)c$. Now if we replace $c$ by $1 + t^r d$ and we divide by $t^r$ we obtain $[d, a] = \mu(a) + t^r \mu(a)d$. Hence $\mu$ is a derivation on $A$.

Consequently $[\bar{d}, \bar{a}] = \bar{\mu}(\bar{a})$ hence the result.

For the second part we let $\alpha_1, \alpha_2$ be two representatives in $\text{Out}(A[[t]])$ with...
induced derivations $\mu_1, \mu_2$. Since $\alpha_1 \circ \alpha_2^{-1} \in \text{Inn}(A[[t]])$ then using Proposition 3.8 and first part of the Proposition we have that $\mu_1 - \mu_2 \in \text{Inn}(A)$. Hence the result.

An equivalent definition of $r$-integrable can be deduced from the following: let $\alpha \in \text{Aut}_r(A[[t]])$ and let $a = \sum_{i=0}^{\infty} a_i t^i$. Then $\alpha(a) = \sum_{i,n \geq 0} D_n(a_i) t^{i+n} = a + t^r \sum_{k \in \mathbb{N}} \sum_{n,i \geq 1,t+n-k} D_n(a_j) t^{i-n}$ since $D_i = 0$ for $1 \leq i \leq r - 1$. Hence we can write $\alpha$ as $\alpha(a) = a + t^r \mu(a)$ where $\mu$ is an linear endomorphism of $A[[t]]$. From Proposition 3.5 the map $\bar{\mu} : A \to A$ induced by $\mu$ is a derivation over $A$, in fact, $\bar{\mu}$ is exactly $D_r$. Hence a derivation $D$ on $A$ is $r$-integrable if there is an algebra automorphism of $A[[t]]$, say $\alpha$, and a $k[[t]]$-linear endomorphism $\mu$ of $A[[t]]$ such that $\alpha(a) = a + t^r \mu(a)$ for all $a \in A[[t]]$ and such that $D$ is equal to the map $\bar{\mu}$ induced by $\mu$ on $A \cong A[[t]]/tA[[t]]$.

**Proposition 3.6.** Let $A$ be a finite-dimensional $k$-algebra and let $\alpha \in \text{Aut}_1(A[[t]])$. Let $(D_i)_{i \geq 0}$ be a higher derivation satisfying $\alpha(a) = \sum_{i \geq 0} D_i(a) t^i$ for $a \in A$. The map that sends $\alpha$ to $\sum_{i \geq 0} D_i t^i$ induces a group homomorphism $\phi : \text{Aut}_1(A[[t]]) \to (\text{End}_k(A[[t]]))^\times$.

**Proof.** Let $\beta \in \text{Aut}_1(A[[t]])$. For $l \geq 0$ let $E_l \in \text{End}_k(A)$ such that $\beta(a) = \sum_{i \geq 0} E_i(a) t^i$. For all $a \in A$ let $\{e_j\}_{j \leq n}$ be a $k$-basis of $A$. For every $i \geq 0$ define $\mu_{ij} : A \to k$ such that $D_i(a) = \sum_{j=1}^n \mu_{ij}(a) e_j$ where $a \in A$. On one side we have:

$$
(\beta \circ \alpha)(a) = \beta\left( \sum_{i \geq 0} D_i(a) t^i \right) = \sum_{j=1}^n \beta\left( \sum_{i \geq 0} \mu_{ij}(a) t^i e_j \right)
$$

$$
= \sum_{j=1}^n \sum_{i \geq 0} \mu_{ij}(a) t^i \beta(e_j) = \sum_{l \geq 0} \sum_{j=1}^n \mu_{ij}(a) E_l(e_j) t^{i+l}
$$

where the third equation holds since $\beta$ is an automorphism over $k[[t]]$. If we fix a degree $m \in \mathbb{N}$ we have

$$
\sum_{l,t} \sum_{i+l=m}^n \mu_{ij}(a) E_l(e_j) t^{i+l} = \sum_{l,t} E_l(\sum_{j=1}^n \mu_{ij}(a) e_j) t^m
$$

$$
= \sum_{l,t} E_l(D_i(a)) t^m
$$

Hence $\phi(\beta \circ \alpha)$ in degree $m$ is equal to $\sum_{l,t \geq 0} E_l \circ D_i t^m$. This is clearly equal to the coefficient at $t^m$ of $\phi(\beta \circ \alpha)$.

**Definition 3.7.** Let $A$ be a finite-dimensional $k$-algebra. Let $r$ be a positive integer then by $\text{HH}^1_r(A)$ we denote the quotient $\text{Der}_r(A)/\text{Inn}_r(A)$.

Clearly $\text{HH}^1_r(A)$ can be identified with a subgroup of $\text{HH}^1(A)$.

**Proposition 3.8.** Let $A$ be a finite-dimensional algebra over $k$. Let $r$ be a positive integer and let $\alpha \in \text{Aut}_r(A[[t]])$. Let $\mu$ the unique $k[[t]]$-linear map on $A[[t]]$ such that $\alpha(a) = a + t^r \mu(a)$ for all $a \in A[[t]]$. We denote by $\bar{\mu}$ the derivation induced on $A$ by $\mu$. 

5
(a) The derivation $\bar{\mu}$ is inner if and only if $\alpha$ induces an inner automorphism in $A[[t]]/t^{r+1}A[[t]]$.

(b) We have the following short exact sequence of groups:

$$1 \longrightarrow \text{Out}_{r+1}(A[[t]]) \longrightarrow \text{Out}_r(A[[t]]) \longrightarrow \text{HH}_1^r(A) \longrightarrow 1$$

Proof. Let assume that $\bar{\mu}$ is inner derivation so $\bar{\mu} = [\bar{d},-]$ for some $d \in A[[t]]$. We can take $c = 1 + t'$ as in the proof of Proposition 3.5. Then from Equation 6 we can choose $\tau(a) = -\mu(a)d$ so we have $[d,a] = \mu(a) - t'\tau(a)$ and since $c = 1 + t'd$ then

$$[c,a] = [1 + t'd,a] = t'\cdots[a]$$

So $[c,a] = t'[a] = t'\mu(a) - t^{2r}\tau(a)$. Hence $t'\mu(a) = [c,a] + t^{2r}\tau(a)$. Consequently $cac^{-1} = a + t'\mu(a)c^{-1} - t^{2r}\tau(a)c^{-1}$. Using the fact that $\alpha(a) = a + t'\mu(a)$ it follows that $\alpha(a) - cac^{-1} = \mu(a)(1 - c^{-1}) + t^{2r}\tau(a)c^{-1}$. Since $c$ belongs to $1 + t'\A[[t]]$, we have $c^{-1} \in 1 + t'\A[[t]]$ hence $1 - c^{-1} \in t'\A[[t]]$. This shows that $\alpha(a) - cac^{-1} \in t^{2r}\A[[t]] \subset t^{r+1}\A[[t]]$. Consequently $\alpha$ induces an inner automorphism on $A[[t]]/t^{r+1}A[[t]]$.

Conversely, suppose that $\alpha$ acts as an inner automorphism on $A[[t]]/t^{r+1}A[[t]]$. Using the same argument as in Lemma 3.2 we may assume that $\alpha$ acts as identity on $A[[t]]/t^{r+1}A[[t]]$ hence it induces an inner derivation on $A[[t]]/t^{r+1}A[[t]]$. Hence we can assume $\alpha$ such that $\alpha \in \text{Aut}_{r+1}(A[[t]])$. Hence $\alpha(a) = a + t^{r+1}\mu'(a)$ for some $\mu'(a) \in A[[t]]$, which gives the equality $\mu(a) = t\mu'(a)$. Consequently we have that $\mu$ induces the zero map on $\A$. For the second part let $\beta \in \text{Aut}_r(A)$ such that $\beta(a) = a + t^r\nu(a)$ for all $a \in A[[t]]$ and for some linear morphism $\nu$ on $A[[t]]$. From Proposition 3.3 and Proposition 3.5 we have that the class determined by $\beta \circ \alpha$ in $\text{HH}_1^r(A)$ is the class determined by $\bar{\mu} + \bar{\nu}$. 

A way to understand the action of the $p$-power map on the integrable derivations is by studying it on $\text{Aut}_1(A[[t]])$ and then using the homomorphism $\phi : \text{Aut}_1(A[[t]]) \to (\text{End}_k(A[[t]])^\times$. 

Proposition 3.9. Let $D$ be a higher derivation and let $l, n$ be positive integers. The term at $t^l$ in $\left(\sum_{i \geq 0} D_it^i\right)^n$ is equal to

$$\sum_{i=1}^l \binom{n}{c} \prod_{i_1, \ldots, i_c \geq 1} D_{i_j}$$

(10)

Proof. The term at $t^l$ in $\left(\sum_{i \geq 0} D_it^i\right)^n$ is given by

$$\sum_{i_1, \ldots, i_n \geq 0} \prod_{j=1}^n D_{i_j}$$

(11)

Let $c$ be a positive integer. Then for each $c$-tuple $(i_1', i_2', \ldots, i_c')$ which has non-zero components and such that $\sum_{j=1}^c i_j' = l$, there are $\binom{n}{c}$ different $n$-tuples $(i_1, i_2, \ldots, i_n)$ which have the $c$ non-zero components of the $c$-tuple $(i_1', i_2', \ldots, i_c')$.
and rest equal to zero. Since \( D_0 = \text{Id} \) then \( \prod_{j=1}^c D_{i_j} = \prod_{j=1}^c D_{j_j} \). For a fixed \( c \) the Equation (11) is given by \( \sum_{i_1,\ldots,i_c \geq 1}^{(n)} \prod_{j=1}^c D_{i_j} = \prod_{j=1}^c \sum_{i_1,\ldots,i_c \geq 1}^{(n)} D_{i_j} \). If we sum over all \( c \) we have the result.

**Corollary 3.10.** Let \( A \) be a finite-dimensional \( k \)-algebra and let \( \alpha \in \text{Aut}_r(A[[t]]) \) for some positive integer \( r \). Then \( \alpha^p \in \text{Aut}_{rp}(A[[t]]) \). The \( p \)-power map sends \( \text{HH}^1_r(A) \) to \( \text{HH}^1_{rp}(A) \), and \( \text{Out}_r(A[[t]]) \) to \( \text{Out}_{rp}(A[[t]]) \) and we have a commutative diagram

\[
\begin{array}{ccc}
\text{Out}_r(A[[t]]) & \xrightarrow{(\cdot)^p} & \text{Out}_{rp}(A[[t]]) \\
\downarrow & & \downarrow \\
\text{HH}^1_r(A) & \xrightarrow{[p]} & \text{HH}^1_{rp}(A)
\end{array}
\]

where the vertical maps are from Proposition 3.8 (b), \( (\cdot)^p \) is the \( p \)-fold composition and \( [p] \) is the \( p \)-power map.

**Proof.** Let \( \alpha \in \text{Aut}_r(A[[t]]) \) and let \( D_r \) the derivation in \( \text{Der}_r(A) \). Let \( D'_r \) be the higher derivation associated to \( \alpha^p \). Using Proposition 3.9 in degree \( l \leq p - 1 \) we have:

\[
\sum_{c=1}^{l} \binom{p}{c} \sum_{i_1,\ldots,i_c \geq 1}^{(n)} \prod_{j=1}^c D_{i_j} t^l = 0
\]

since the binomial coefficient give us multiples of \( p \). For \( l \geq p \)

\[
\sum_{c=1}^{l} \binom{p}{c} \sum_{i_1,\ldots,i_c \geq 1}^{(n)} \prod_{j=1}^c D_{i_j} t^l = \sum_{i_1,\ldots,i_p \geq 1}^{(n)} \prod_{j=1}^p D_{i_j} t^l
\]

Now we know that each \( D_{i_j} \) is zero for \( i = 1,\ldots,r - 1 \) so in order to have an element different from zero we should impose that each \( i_j \) be at least \( r \). Therefore the sum \( i_1 + \cdots + i_p = rp \) that is \( l = rp \) hence the first non-zero coefficient is \( D^p \). Consequently the diagram commutes.

### 4 A cohomological interpretation of \( r \)-integrable derivations

Integrable derivation can also being interpreted using a cohomological point of view. Starting from the short exact sequence of \( A[[t]] \)-\( A[[t]] \)-bimodules:

\[
0 \longrightarrow A[[t]] \xrightarrow{t^r} A[[t]] \longrightarrow A[[t]]/t^rA[[t]] \longrightarrow 0
\]

after dividing by \( tA[[t]] \) and twisting on the right by the automorphism \( \alpha \in \text{Aut}_r(A[[t]]) \) we obtain the short exact sequence:
0 \longrightarrow A[[t]]/t A[[t]] \xrightarrow{t^r} (A[[t]]/t^{r+1} A[[t]]), \alpha \longrightarrow A[[t]]/t^r A[[t]] \longrightarrow 0

since \( \alpha \) induces the identity on \( A[[t]]/t^r A[[t]] \) hence also on \( A[[t]]/t A[[t]] \).

The following proposition is an adaptation of \[4.1\] to the situation under consideration.

**Proposition 4.1.** Let \( A \) be a finite-dimensional algebra over \( k \). Set \( \hat{A} = A[[t]] \)
and set \( \hat{A}^r = \hat{A} \otimes_{k[[t]]} \hat{A}^{\text{op}} \). Let \( \alpha \in \text{Aut}_r(\hat{A}) \). Let \( r \) be a positive integer and let \( \mu : A \to A \) be the unique linear map satisfying \( \alpha(a) = a + t^r \mu(a) \). Let \( P \) be a projective resolution of \( \hat{A} \) as \( \hat{A}^r \)-module. Applying the functor \( \text{Hom}_{\hat{A}}(P, -) \) to the exact sequence of \( \hat{A}^r \)-modules

\[
0 \longrightarrow \hat{A}/t \hat{A} \xrightarrow{t^r} (\hat{A}/t^{r+1} \hat{A}), \alpha \longrightarrow \hat{A}/t^r \hat{A} \longrightarrow 0
\]

yields a short exact sequence of cochain complexes

\[
0 \longrightarrow \text{Hom}_{\hat{A}}(P, A) \xrightarrow{t^r} \text{Hom}_{\hat{A}}(P, (\hat{A}/t^{r+1} \hat{A}), \alpha) \longrightarrow \text{Hom}_{\hat{A}}(P, \hat{A}/t^r \hat{A}) \longrightarrow 0
\]

The first non-trivial connecting homomorphism can be identified with a map

\[
\text{End}_{\hat{A}}(\hat{A}/t^r \hat{A}) \to \text{HH}^1(A)
\]

and this map sends \( \text{Id}_{\hat{A}/t^r \hat{A}} \) to the class of the derivation induced by \( \mu \) on \( A \).

**Proof.** We take as a projective resolution the bar resolution \( P \) of \( \hat{A} \) where the tensor products are over \( k[[t]] \):

\[
\cdots \longrightarrow \hat{A} \otimes_{k[[t]]} \hat{A} \otimes_{k[[t]]} \cdots \otimes_{k[[t]]} \hat{A} \longrightarrow \hat{A} \longrightarrow \cdots
\]

which is given by \( \delta_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^n (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \).

The last non-zero differential is the map \( \delta_1 : \hat{A} \otimes A \to \hat{A} \otimes A \) which sends \( a \otimes b \otimes c \) to \( ab \otimes c - a \otimes bc \) for \( a, b, c \in \hat{A} \). We have the following identifications:

\[
\begin{align*}
\text{H}^0(\text{Hom}_{\hat{A}}(P, \hat{A}/t^r \hat{A})) & = \text{HH}^0(\hat{A}, \hat{A}/t^r \hat{A}) \\
& \cong \text{HH}^0(\hat{A}/t^r \hat{A}) = \text{End}_{\hat{A}}(\hat{A}/t^r \hat{A})
\end{align*}
\]

The identity map in \( \text{End}_{\hat{A}}(\hat{A}/t^r \hat{A}) \) corresponds to the homomorphism

\[
\zeta : \hat{A} \otimes_{k[[t]]} \hat{A} \to \hat{A}/t^r \hat{A}
\]

\[
a \otimes b \mapsto \zeta(a \otimes b) = ab + t^r \hat{A}
\]

for all \( a, b \in A[[t]] \). This lifts to an \( \hat{A}^r \)-homomorphism

\[
\bar{\zeta} : \hat{A} \otimes_{k[[t]]} \hat{A} \to (\hat{A}/t^{r+1} \hat{A}), \alpha
\]

\[
a \otimes b \mapsto \bar{\zeta}(a \otimes b) = a\alpha(b) + t^{r+1} \hat{A}
\]
for $a, b \in \hat{A}$ since $\alpha$ induces the identity on $\hat{A}/t' \hat{A}$.

Since $\zeta \in \text{Hom}_{\hat{A}}(\hat{A} \otimes \hat{A}, (\hat{A}/t^{r+1} \hat{A})_{\alpha})$ we need to apply the first non-zero differential

$$
\epsilon : \text{Hom}_{\hat{A}}(\hat{A}^{\otimes 2}, (\hat{A}/t^{r+1} \hat{A})_{\alpha}) \to \text{Hom}_{A}(\hat{A}^{\otimes 3}, (\hat{A}/t^{r+1} \hat{A})_{\alpha})
$$

which is given by composing with $-\delta_1$. Hence in $\hat{A}/t^{r+1} \hat{A}$ we have:

$$
(-\bar{\zeta} \circ \delta_1)(a \otimes b \otimes c) = -\bar{\zeta}(ab \otimes c + a \otimes bc) = -abc(a) + ac(b) = a(ab(b) - a) \alpha(c) = t' \alpha(b) \alpha(c).
$$

for all $a, b, c \in \hat{A}$. We observe that $t' \alpha(b) \alpha(c) + t^{r+1} \hat{A} \in \hat{A}/t^{r+1} \hat{A}$ is the image, under $t' : A/tA \to (A/t^{r+1} \hat{A})_{\alpha}$, of the map $\psi : \hat{A}^{\otimes 3} \to \hat{A}/tA$, that is we have the following commutative diagram:

\[
\begin{array}{ccc}
\hat{A}^{\otimes 3} & \xrightarrow{\psi} & \hat{A}/tA \\
\downarrow{\bar{\zeta} \circ \delta} & & \downarrow{t'} \\
(A/t^{r+1} \hat{A})_{\alpha} & & \end{array}
\]

where $\psi$ sends $a \otimes b \otimes c$ to $a \alpha(b) \alpha(c) + t^{r+1} \hat{A}$ which is equal to $a \alpha(b) \alpha(c) + t^{r+1} \hat{A}$ since $a \alpha(b) \alpha(c) - c \in t' \hat{A} \subseteq \hat{A}$. Consequently $\psi$ induces a map $\hat{A}^{\otimes 3} \to A$ which sends $a \otimes b \otimes c$ to $a \alpha(b) \alpha(c)$ that can be restricted to the map $\psi : A \to \hat{A}$ that sends $b$ to $\bar{\zeta} \circ \delta_1$. Using (1) the result follows.

\[\square\]

5 Proof of Theorem 1.1

The proof of Theorem 1.1 requires the following result, which is a variation of [3, 5.1]:

**Theorem 5.1.** Let $A, B$ be finite-dimensional selfinjective $k$-algebras with separable semisimple quotients. Let $r$ be a positive integer and let $M, N$ be an $A$-$B$-bimodule, $B$-$A$ bimodule, respectively, inducing a stable equivalence of Morita type between $A$ and $B$. Then for any $\alpha \in \text{Aut}_r(A[[t]])$ there is $\beta \in \text{Aut}_r(B[[t]])$ such that $\alpha_{[\alpha]} M[[t]] \cong M[[t]]_{[\beta]}$ as $A[[t]]$-$B[[t]]$-bimodules. This correspondence induce a group isomorphism $\text{Out}_r(A[[t]]) \cong \text{Out}_r(B[[t]])$ making the following diagram commutative:

\[
\begin{array}{ccc}
\text{Out}_r(A[[t]]) & \cong & \text{Out}_r(B[[t]]) \\
\downarrow{\cong} & & \downarrow{\cong} \\
HH^1_r(A) & \cong & HH^1_r(B)
\end{array}
\]

where the vertical maps are from Proposition 3.8 and the lower horizontal isomorphism is induced by the functor $N \otimes_A - \otimes_A M$.
Proof. By the Lemma \[6, 4.2\] we have that the upper horizontal map is a group isomorphism. Let \( \alpha \in \text{Aut}_A(A[[t]]) \), \( \beta \in \text{Aut}_B(B[[t]]) \) such that \( \alpha^{-1} M[[t]] \cong M[[t]] \beta \) as \( A[[t]]-B[[t]] \)-bimodules. We also have that \( \alpha \) is such that \( \alpha(a) = a + t' \mu(a) \) for all \( a \in A[[t]] \) and \( \beta \) such that \( \beta(b) = b + t' \nu(b) \) for all \( b \in B[[t]] \) for some \( k[[t]] \)-linear endomorphisms \( \mu, \nu \). We denote by \( \bar{\mu} \) and \( \bar{\nu} \) the classes in \( \text{HH}_1(A) \) and \( \text{HH}_1(B) \) respectively determined by the canonical group homomorphism \( \text{Out}_A(A[[t]]) \to \text{HH}_1(A) \) and \( \text{Out}_B(B[[t]]) \to \text{HH}_1(B) \). Set \( \bar{M} = M[[t]] \). By the assumptions, tensoring by \( M \) yields a stable equivalence of Morita type between \( A \) and \( B \). In particular we have:

\[
\text{HH}_1(A) \cong \text{Ext}^1_{A \otimes_k B^{op}}(M, M) \cong \text{HH}_1(B)
\]

induced by the functors \( - \otimes_A M \) and \( M \otimes_B - \). In addition since \( B[[t]] \) is isomorphic to \( \bar{N} \otimes_{A[[t]]} \bar{M} \) in the relatively \( k[[t]] \)-stable category of \( B[[t]] \otimes_k [[t]] \)-modules, it follows that the isomorphism

\[
\text{HH}_1(A) \cong \text{HH}_1(B)
\]

given by the composition of the two previous isomorphisms is induced by the functor \( N \otimes_A - \otimes_A M \). The functors \( M \otimes_B - \), \( - \otimes_A M \) also induce algebra homomorphisms

\[
\text{End}_{A^e}(A) \to \text{End}_{A \otimes B^{op}}(M) \leftarrow \text{End}_{B^e}(B)
\]

where \( A^e = A \otimes_k A^{op} \) and similarly for \( B^e \). Tensoring the following two exact sequence

\[
0 \to A \to (A[[t]]/t^{r+1}A[[t]])_\alpha \to A[[t]]/t^r A[[t]] \to 0
\]

and

\[
0 \to B \to (B[[t]]/t^{r+1}B[[t]])_\alpha \to B[[t]]/t^r B[[t]] \to 0
\]

by \( - \otimes_{A[[t]]} \bar{M} \) and \( \bar{M} \otimes_{B[[t]]} - \) yields short exact sequences of the form

\[
0 \to M \to \alpha^{-1}(M[[t]]/t^{r+1}M[[t]]) \to M \to 0
\]

\[
0 \to M \to (M[[t]]/t^{r+1}M[[t]])_\beta \to M \to 0
\]

By the naturality properties of the connecting homomorphism and from the description of \( \bar{\mu}, \bar{\nu} \) in Proposition \[4.4\] the image of \( \bar{\mu} \otimes \text{Id}_M \) and \( \text{Id}_M \otimes \bar{\nu} \) in \( \text{Ext}^1_{\bar{A} \otimes_k \bar{B}^{op}}(M, M) \) are equal to the images of \( \text{Id}_{\bar{M}} \) under the two connecting homomorphisms

\[
\text{End}_{\bar{A} \otimes_k \bar{B}^{op}}(\bar{M}) \to \text{Ext}^1_{\bar{A} \otimes_k \bar{B}^{op}}(M, M)
\]

obtained after applying the functor \( \text{Hom}_{\bar{A}[[t]] \otimes \bar{B}[[t]]^{op}}(\bar{M}, -) \) to the short exact sequences using the same identification used in Proposition \[4.4\]. By the Lemma \[6, 4.3\] the two exact sequences are equivalent, consequently the connecting homomorphism are equal. Hence the two images of \( \text{Id}_M \) coincide. This shows that the group isomorphism \( \text{HH}_1(B) \cong \text{HH}_1(A) \) induced by \( \text{Out}_B(B[[t]]) \cong \text{Out}_B(A[[t]]) \) is equal to the one determined by the functor \( N \otimes_A - \otimes_A M \). Hence the result.
Proof of Theorem 1.1 We show first that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Out}_r(A[[t]]) & \cong & \text{Out}_r(B[[t]]) \\
\downarrow & & \downarrow \\
\text{Out}_{rp}(A[[t]]) & \cong & \text{Out}_{rp}(B[[t]])
\end{array}
\]

where the horizontal maps are from Theorem 5.1 and the vertical maps are p-fold compositions. Let \(\alpha \in \text{Aut}_r(A[[t]])\) and \(\beta \in \text{Aut}_r(B[[t]])\) such that \(\alpha^{-1} M[[t]] \cong M[[t]] \beta\). Let \(\mu, \nu\) be the unique linear maps on \(A[[t]]\) such that \(\alpha(a) = a + t \mu(a)\) and \(\beta(b) = b + t \nu(b)\) respectively. By Corollary 5.10 we have \(\alpha^p \in \text{Aut}_{rp}(A[[t]])\), \(\beta^p \in \text{Aut}_{rp}(A[[t]])\) and also that the maps \(\mu, \nu\) induced by \(\alpha, \beta\) on \(A\), is sent under the p-power map to \(\hat{\mu}^p\) and \(\hat{\nu}^p\) respectively. Hence we have the commutativity of the diagram above since \(\alpha^{-p} M[[t]] \cong M[[t]] \beta^p\). Using the commutative diagram above and Theorem 5.1 we have that the class of \(\hat{\nu}^p\) is sent though the isomorphism defined in Theorem 1.1 to the class of \(\hat{\mu}^p\). Hence we have the commutativity of the diagram of the Theorem 1.1.

\[\square\]

6 Example

The purpose of the following example is to show that p-power maps do not commute in general with transfer maps in the Hochschild cohomology of symmetric algebras.

Let \(H = \{1, (123), (132)\} \cong C_3 \leq S_3\) and \(M = kS_3\) considered as a \(kS_3 \otimes kC_3\) bimodule. By \(\langle \cdot , \cdot \rangle\) we mean the standard bilinear form for the group algebra \(kH\). We choose \(\{1, t = (12)\}\) as set of representatives of \(S_3 / H\). We note that \(M\) is finitely generated and projective as a right \(kC_3\)-module, since \([G : H] = 2\), so there exist \(x_i \in \text{Hom}_{kC_3}(kS_3, kC_3)\) with \(1 \leq i \leq 2\) such that for any \(x \in M\), \(x = \sum_i x_i \varphi_i(x)\). Explicitly:

\[
\begin{align*}
\varphi_1(1) &= 1, \varphi_1((123)) = (132), \\
\varphi_2((132)) &= (132), \varphi_2(g) = 0
\end{align*}
\]

for every other \(g \in G\). Similarly we define:

\[
\begin{align*}
\varphi_1(12) &= 1, \varphi_1((13)) = (132), \varphi_1((23)) = (123), \varphi_1(g) = 0
\end{align*}
\]

for every other \(g \in G\). Since \(C_3\) is commutative then \(HH^1(kC_3) = \text{Der}_k(kC_3)\) which is generated by \(\{f_0, f_1, f_2\}\) such that \(f_0((123)) = 1, f_1((123)) = (123)\) and \(f_2((123)) = (132)\). In this case the explicit formula of the transfer map by [5.2.5] is given by:

\[
\begin{align*}
\text{tr}^M(f) &= \sum_{h \in H} \langle h^{-1}, f(\varphi_1(a)) \rangle h + \langle h^{-1}t, f(\varphi_1(a)) \rangle th + \\
&\quad \langle h^{-1}, f(\varphi_1(at)) \rangle ht + \langle h^{-1}, f(\varphi_1(at)) \rangle th
\end{align*}
\]
where \( f \in \text{Der}_k(kC_3) \). In particular for \( a = (123) \) we have:

\[
\text{tr}^M(f_0)((123)) = \sum_{h \in H} \left( \left\langle h^{-1}, f_0((123)) \right\rangle + \left\langle h, f_0((132)) \right\rangle \right) = 1 - (123)
\]

(similarly we have:

\[
\text{tr}^M(f_0)(132) = 1 - (123).
\]

We can note now that \( \text{tr}^M(f_0^{[3]}) = 0 \) since \( f_0^{[3]} = 0 \), so \( \text{tr}^M(f_0^{[3]})(132) = 0 \). On the other hand \( \text{tr}^M(f_0^{[3]}((132)) = \text{tr}^M(f_0)\circ\text{tr}^M(f_0)(1 - (123)) = \text{tr}^M(f_0)(-1 + (132)) = 1 - (123) \). Since the transfer maps send elements on \( \text{HH}^1(B) \) to elements \( \text{HH}^1(A) \) it should exists a inner derivation in \( S_3 \) which sends \( (132) \) to \( 1 - (123) \) if we require the commutativity of the diagram. But there is no element in \( a \in kS_3 \) such that \( [a, (132)] = 1 \). Hence in this case the \( p \)-power map does not commute with the transfer map.

**Remark 6.1.** This shows that the \( p \)-power map cannot be expressed in terms of the \( BV \)-operator, as this is invariant under transfer maps, by [5 10.7].

**References**

[1] D. J. Benson, *Representations and Cohomology: Volume 1, Basic Representation Theory of Finite Groups and Associative Algebras*. Cambridge Studies Adv. Math. 30 (1991), Cambridge University Press.

[2] M. Broué, *Equivalences of blocks of group algebras*. In *Finite dimensional algebras and related topics*, Kluwer (1994), 1–26.

[3] M. Gerstenhaber, *The cohomology structure of an associative ring*. Ann. of Math. (2) 78 (1963), 267–288.

[4] N. Jacobson, *Restricted Lie algebras of characteristic \( p \)*, Trans. Amer. Math. Soc. 50 (1941), no.1, 15–25.

[5] S. Koenig, Y. Liu, and G. Zhou, *Transfer maps in Hochschild (co)homology and applications to stable and derived invariants and to the Auslander-Reiten Conjecture*. Trans. Amer. Math. Soc. 364 (2012), no. 1, 195–232.

[6] M. Linckelmann, *Integrable derivations and stable equivalences of Morita type*, Preprint (2015), Arxiv.org/abs/1506.04676.

[7] M. Linckelmann, *Transfer in Hochschild Cohomology of Blocks of Finite Groups*, Algebras and Representation Theory 2 (1999), 107–135.

[8] H. Matsumura, *Integrable derivations*, Math. J. 87 (1982), 227–245.

[9] C. A. Weibel, *An introduction to homological algebra*. Cambridge Studies Adv. Math. 38 (1994), Cambridge University Press.

[10] A. Zimmermann *Fine Hochschild invariants of derived categories for symmetric algebras*, Journal of Algebra 308 (2007), 350–367.