1. Introduction

A recurring theme in number theory is that multiplicative and additive properties of integers are more or less independent of each other, the classical result in this vein being Dirichlet’s theorem on primes in arithmetic progressions. Since the set of primitive roots to a given modulus is a union of arithmetic progressions, it is natural to study the distribution of prime primitive roots. Results concerning upper bounds for the least prime primitive root to a given modulus $q$, which we denote by $g^*(q)$, have hitherto been of three types. There are conditional bounds: assuming the Generalized Riemann Hypothesis, Shoup [11] has shown that

$$g^*(q) \ll \left( \omega(\phi(q)) \log 2 \omega(\phi(q)) \right)^4 \left( \log q \right)^2,$$

where $\omega(n)$ denotes the number of distinct prime factors of $n$. There are also upper bounds that hold for almost all moduli $q$. For instance, one can show [9] that for all but $O(Y^\varepsilon)$ primes up to $Y$, we have

$$g^*(p) \ll (\log p)^C(\varepsilon)$$

for some positive constant $C(\varepsilon)$. Finally, one can unimaginatively apply a uniform upper bound for the least prime in a single arithmetic progression. The best uniform result of this type, due to Heath-Brown [7], implies that $g^*(q) \ll q^{5.5}$. However, there is not at present any stronger unconditional upper bound for $g^*(q)$ that holds uniformly for all moduli $q$. The purpose of this paper is to provide such an upper bound, at least for primitive roots that are “almost prime”.

The methods herein will actually apply for any modulus $q$, not just those $q$ whose group $\mathbb{Z}_q^\times$ of reduced residue classes is cyclic (which occurs exactly when $q = 2$, 4, an odd prime power, or twice an odd prime power). We say that an integer $n$, coprime to $q$, is a $\lambda$-root (mod $q$) if it has maximal order in $\mathbb{Z}_q^\times$. We see that $\lambda$-roots are generalizations of primitive roots, and we extend the notation $g^*(q)$ to represent the least prime $\lambda$-root (mod $q$) for any integer $q \geq 2$.

We also recall that a $P_k$ integer is one that has at most $k$ prime factors, counted with multiplicity. For any integer $k \geq 1$, we let $g^*_k(q)$ denote the least $P_k$ $\lambda$-root (mod $q$) (so that $g^*_1(q) = g^*(q)$). We may now state our main theorem.

**Theorem 1.** For all integers $q$, $r \geq 2$ and all $\varepsilon > 0$, we have

$$g^*_r(q) \ll \varepsilon q^{1/r \log q + 1/(4(r-1-\delta_r)) + \varepsilon},$$

1991 Mathematics Subject Classification. 11N69.
where $q_c$ is the largest odd cubefree divisor of $q$, and one can take

$$\delta_2 = 0.0044560, \quad \delta_3 = 0.074267, \quad \delta_4 = 0.103974,$$

and $\delta_r = 0.1249$ for any $r \geq 5$.

In the context of primitive roots, Theorem 1 can be improved to the extent of replacing the largest cubefree divisor of the modulus with the largest squarefree divisor. The result is as follows:

**Theorem 2.** Let $p$ be an odd prime, and let $q$ be a power of $p$ or twice a power of $p$. Then

$$g_2^*(q) \ll p^{1/2+1/873},$$

$$g_3^*(q) \ll p^{3/8+1/207},$$

$$g_4^*(q) \ll p^{1/3+1/334},$$

$$g_r^*(q) \ll_r p^{1/4+O(1/r)}.$$

(The exponents here are simply approximations to the corresponding exponents in Theorem 1.) By comparison, from the work of Mikawa on small $P_2$s in almost all arithmetic progressions [10], one can easily derive that

$$g_2^*(q) \ll q (\log q)^5 \frac{\phi(q)}{\phi(\phi(q))},$$

which is majorized by the above theorems. We remark that the $\lambda$-roots we find to establish Theorems 1 and 2 are squarefree and have no small prime factors (“small” here meaning up to some fixed power of $q_c$).

The proof of Theorem 1 uses the weighted linear sieve, specifically results due to Greaves [2], [3] (see equation (18) below). We note that conjecturally, there is some choice of weight function $W$ in the weighted linear sieve which would allow us to take $\delta_r$ arbitrarily small in Theorem 1; this would also allow us to replace the first three exponents in Theorem 2 by $1/2 + \varepsilon, 3/8 + \varepsilon$, and $1/3 + \varepsilon$ respectively. We also note that if the generalized Lindelöf hypothesis for the $L$-functions corresponding to certain characters (the ones in the subgroup $G$ defined in Lemma 4 below) were true, we could employ much stronger character sum estimates than Lemma 7 below, allowing us to improve Theorem 1 to $g_2^*(q) \ll q^{1/2+\varepsilon} q_c^\varepsilon$.

We would of course like to be able to show the existence of small prime primitive roots rather than $P_2$ primitive roots. In his work on the analogous problem of finding primes in arithmetic progressions, Heath-Brown [6] first treats the case where the $L$-function corresponding to a real Dirichlet character has a real zero very close to $s = 1$. Although it is certainly believed that these “Siegel zeros” do not exist, disposing of this case allowed Heath-Brown in [7] to work with a better zero-free region for Dirichlet $L$-functions than is known unconditionally.

Similarly, if we assume the existence of a sufficiently extreme Siegel zero, we can show the existence of small prime primitive roots, as the following theorem asserts.

**Theorem 3.** Let $\varepsilon > 0$, let $q$ be an odd prime power or twice an odd prime power, and let $\chi_1$ denote the nonprincipal quadratic Dirichlet character (mod $q$). Suppose that $L(s, \chi_1)$ has
a real zero \( \beta \) satisfying
\[
\beta > 1 - \frac{1}{A(\varepsilon) \log q},
\]
where \( A(\varepsilon) \) is some sufficiently large real number depending on \( \varepsilon \) (but not on \( q \)). Then
\[
g^*(q) \ll_{\varepsilon} p^{3/4+\varepsilon}.
\]

It is plausible that in the weighted linear sieve, one can derive a corresponding upper bound of the same order of magnitude as the lower bound. In this case, the exponent \( 3/4 + \varepsilon \) in Theorem \( 3 \) could be replaced by \( 1/2 + 1/873 \), the exponent associated with \( g^*_2(q) \) in Theorem \( 2 \).

It is a pleasure to thank Trevor Wooley and Hugh Montgomery for their many helpful suggestions on improving this paper, and for their guidance in general. The author would also like to thank Andrew Granville, George Greaves, Ram Murty, and Amora Nongkynrih for valuable comments regarding recent progress in areas relevant to this paper. This material is based upon work supported under a National Science Foundation Graduate Research Fellowship.

For the remainder of this paper, the constants implicit in the \( \ll \) and \( O \)-notations may depend on \( \varepsilon, \eta, \) and \( r \) where appropriate. We denote the cardinality of the set \( S \) by \( |S| \). As before, \( \omega(n) \) is the number of distinct prime factors of \( n \), and \( \Omega(n) \) is the number of prime factors of \( n \) counted with multiplicity, so that an integer \( n \) is a \( P_r \) precisely when \( \Omega(n) \leq r \).

2. The characteristic function of \( \lambda \)-roots

Let \( \gamma \) be the characteristic function of those integers that are \( \lambda \)-roots \((\mod q)\). Since \( \gamma \) is periodic with period \( q \) and is supported on reduced residue classes, \( \gamma \) can be written as a linear combination of Dirichlet characters. The following lemma exhibits this linear combination explicitly. Let \( E(q) \) denote the exponent of the group \( \mathbb{Z}_q^* \), so that an integer \( n \), coprime to \( q \), is a \( \lambda \)-root \((\mod q)\) precisely when the multiplicative order of \( n \) is \( E(q) \). Let \( S(q) \) denote the largest squarefree divisor of \( E(q) \). We notice that \( \phi(q) \), \( E(q) \), and \( S(q) \) all have exactly the same prime divisors.

**Lemma 4.** Let \( G \) be the subgroup of characters \((\mod q)\) given by
\[
G = \{\chi^{E(q)/S(q)} : \chi \pmod{q}\}.
\]
For every prime \( p \) dividing \( \phi(q) \), let \( m(p) \) denote the number of independent characters of order \( p \) in \( G \). For every character \( \chi \pmod{q} \), let \( \sigma(\chi) \) denote the order of \( \chi \), and define
\[
c_\chi = \begin{cases} 
\prod_{p|\sigma(\chi)} \left( \frac{-1}{p^{m(p)}} \right) \prod_{p|\phi(q)} \left( 1 - \frac{1}{p^{m(p)}} \right) & \text{if } \chi \in G, \\
0 & \text{otherwise}.
\end{cases}
\]
Then, for any integer \( n \),
\[
\gamma(n) = \sum_{\chi \pmod{q}} c_\chi \chi(n).
\]
Remarks: For simplicity we write $c_0$ for $c_{\chi_0}$. We note that the exponent of $G$ is $S(q)$, the number of characters in $G$ whose order divides $p$ is $p^m(p)$, and the number of characters in $G$ whose order equals $d$ is
\[ \prod_{p|d} \left( p^{m(p)} - 1 \right). \] (3)

We note that $c_0$ is the probability that a randomly chosen element of $\mathbb{Z}_q^\times$ has order $E(q)$, i.e., the number of $\lambda$-roots (mod $q$) less than $q$ is $c_0\phi(q)$. We also note that $\mathbb{Z}_q^\times$ is cyclic if and only if $E(q) = \phi(q)$. When $\mathbb{Z}_q^\times$ is cyclic, the definition (1) of $c_\chi$ reduces to
\[ c_\chi = \frac{\phi(\phi(q)) \mu(\sigma(\chi))}{\phi(\sigma(\chi))}. \]

In particular, $c_0 = \phi(\phi(q))/\phi(q)$ in this case.

Proof: The lemma clearly holds for $(n, q) > 1$, since both sides of equation (2) are zero, and thus for the remainder of the proof, we assume that $(n, q) = 1$. From the standard properties of characters, for every prime $p$ dividing $\phi(q)$ we have
\[ \sum_{\chi \in G} \chi(n) = \begin{cases} p^{m(p)} & \text{if } n^{E(q)/p} \equiv 1 \pmod{q}, \\ 0 & \text{otherwise}, \end{cases} \]
since the number of characters being summed over is $p^{m(p)}$, as noted above. We rewrite this as
\[ \left(1 - \frac{1}{p^{m(p)}}\right) \chi_0(n) - \frac{1}{p^{m(p)}} \sum_{\chi \in G} \chi(n) = \begin{cases} 1 & \text{if } n^{E(q)/p} \not\equiv 1 \pmod{q}, \\ 0 & \text{otherwise}. \end{cases} \] (4)

Now $n$ is a $\lambda$-root if and only if for every prime $p$ dividing $\phi(q)$, we have $n^{E(q)/p} \not\equiv 1 \pmod{q}$. Therefore, by using the relation (3) for all primes dividing $\phi(q)$, we see that
\[ \prod_{p|\phi(q)} \left( 1 - \frac{1}{p^{m(p)}} \right) \chi_0(n) - \frac{1}{p^{m(p)}} \sum_{\chi \in G} \chi(n) = \gamma(n). \]

When we expand this product, the only characters that appear are those in $G$. For such a character $\chi$, the coefficient is
\[ \prod_{p|\sigma(\chi)} \left( -\frac{1}{p^{m(p)}} \right) \prod_{p|\phi(q)} \left( 1 - \frac{1}{p^{m(p)}} \right), \]
which is the same as the definition (1) of $c_\chi$. This establishes the lemma. \qed

Lemma 5. Let $c_\chi$ be defined as in (1). Then
\[ \sum_{\chi \pmod{q}} |c_\chi| = 2^{\omega(\phi(q))}c_0. \] (5)
In particular, for any \( \varepsilon > 0 \),

\[
\sum_{\chi \pmod{q}} |c_\chi| \ll (S(q))^\varepsilon.
\] (6)

**Proof:** From the definition (4) of \( c_\chi \), we have

\[
\sum_{\chi \pmod{q}} |c_\chi| = \sum_{d \mid S(q)} \prod_{p \mid d} \left( \frac{-1}{p^{m(p)}} \right) \prod_{p \mid \phi(q)} \left( 1 - \frac{1}{p^{m(p)}} \right) \prod_{p \mid d} \left( p^{m(p)} - 1 \right) \sum_{\chi \in \mathcal{G} \atop \sigma(\chi) = d} 1.
\]

Using the remark (3) to evaluate the inner sum, this becomes

\[
\sum_{\chi \pmod{q}} |c_\chi| = \sum_{d \mid S(q)} \prod_{p \mid d} \left( 1 - \frac{1}{p^{m(p)}} \right) \prod_{p \mid \phi(q)} \left( 1 - \frac{1}{p^{m(p)}} \right) \prod_{p \mid d} \left( p^{m(p)} - 1 \right) \sum_{\chi \in \mathcal{G} \atop \sigma(\chi) = d} 1
\]

\[
= c_0 \cdot 2^{\omega(S(q))}.
\]

This establishes the first assertion (3) of the lemma, since \( \omega(S(q)) = \omega(\phi(q)) \), and also the second assertion (3), since \( c_0 \leq 1 \) and \( 2^{\omega(n)} \ll n^\varepsilon \) for all \( n \).

Let \( q_\alpha \) denote the largest odd cubefree divisor of \( q \). For any prime \( p \) and any nonzero integer \( n \), let \( \text{ord}_p(n) \) denote the largest integer \( r \) such that \( p^r \) divides \( n \). Let \( \alpha = \max \{ 3, \text{ord}_2(q) \} \), and let \( \tilde{q}_\alpha = 2^\alpha q_\alpha \). Thus \( \tilde{q}_\alpha \) is almost the largest cubefree divisor of \( q \), except that we allow \( 8 \) to divide \( \tilde{q}_\alpha \) if it divides \( q \).

**Lemma 6.** Every \( \chi \in G \) is induced by a character \( \pmod{\tilde{q}_\alpha} \).

**Proof:** Since every \( \chi \in G \) is the \( (E(q)/S(q)) \)-th power of a character \( \pmod{q} \), it suffices to show the following: for every character \( \chi \pmod{q} \), \( \chi^{E(q)/S(q)} \) is periodic with period dividing \( \tilde{q}_\alpha \). For the remainder of the proof, let \( \chi \) denote any character \( \pmod{q} \).

Since \( \mathbf{Z}_{\tilde{q}_\alpha}^{\times \text{ord}_p(q)} \) is a subgroup of \( \mathbf{Z}_q^{\times} \), we certainly have \( \text{ord}_p(E(q)) \geq \text{ord}_p(E(p^{\text{ord}_p(q)})) = \text{ord}_p(q) - 1 \) for all odd primes \( p \) dividing \( q \), and also \( \text{ord}_2(E(q)) \geq \text{ord}_2(E(2^{\text{ord}_2(q)})) \geq \text{ord}_2(q) - 2 \). This implies that \( \text{ord}_p(E(q)/S(q)) \geq \max \{ 0, \text{ord}_p(q) - 2 \} \) for odd primes \( p \) dividing \( q \), and \( \text{ord}_2(E(q)/S(q)) \geq \max \{ 0, \text{ord}_2(q) - 3 \} \). In particular,

\[
\text{ord}_p(\tilde{q}_\alpha \cdot E(q)/S(q)) \geq \text{ord}_p(q) \quad \text{for all } p \mid q.
\] (7)

If \( p \) is a prime and \( r \) a positive integer, we note that whenever \( m \equiv n \pmod{p^r} \), we also have \( m^p \equiv n^p \pmod{p^{r+1}} \); one can see this by using the binomial theorem to expand \( \{ m + (n - m) \}^p - m^p \). Using this fact \( \text{ord}_p(E(q)/S(q)) \) times for each prime \( p \) dividing \( q \), and using the inequality (4), we can conclude that

\[
m \equiv n \pmod{\tilde{q}_\alpha} \Rightarrow m^{E(q)/S(q)} \equiv n^{E(q)/S(q)} \pmod{q}.
\]

This tells us that for every \( m \equiv n \pmod{\tilde{q}_\alpha} \), we have

\[
\chi^{E(q)/S(q)}(m) = \chi(m^{E(q)/S(q)}) = \chi(n^{E(q)/S(q)}) = \chi^{E(q)/S(q)}(n),
\]

which establishes the lemma. \( \Box \)
Lemma 7. For every $\chi \in G$, $N \geq 1$, and $0 < \eta < 1$, we have
\[
\sum_{n \leq N} \chi(n) \ll N \left( N^{-1} q_c^{1/4 + \eta} \right)^\eta.
\]

Proof: From Lemma 6, $\chi$ is induced by a character (mod $\tilde{q}_c$). However, $\tilde{q}_c$ is divisible by all of the primes which divide $q$, and so $\chi$ is exactly equal to a character (mod $\tilde{q}_c$). Write $\chi = \chi_1 \chi_2$, where $\chi_1$ is a character (mod $q_c$) and $\chi_2$ is a character (mod 8). Then
\[
\sum_{n \leq N} \chi(n) = \sum_{n \leq N} \chi_1(n) \chi_2(n) = \sum_{i=1,3,5,7} \chi_2(i) \sum_{n \leq N, n \equiv i \pmod{8}} \chi_1(n) \ll \max_{i=1,3,5,7} \left| \sum_{m \leq N/8} \chi_1(8m + i) \right|.
\]
We choose $k$ so that $8k \equiv 1 \pmod{q_c}$, and multiply the inner sum through by $\chi_1(k)$; this doesn’t change the size of the expression, since $|\chi_1(k)| = 1$. Therefore
\[
\sum_{n \leq N} \chi(n) \ll \max_{j=k,3k,5k,7k} \left| \sum_{j < n \leq j + N/8} \chi_1(n) \right|.
\]
Now $\chi_1$ is a character (mod $q_c$), and $q_c$ is cube-free. By Burgess’ character sum estimates [1, Theorem 2] for cube-free moduli, for any integers $r \geq 2$ and $J$ and any $\varepsilon > 0$, we have
\[
\sum_{J < n \leq J + N/8} \chi_1(n) \ll (N/8)^{1-1/r} q_c^{(r+1)/(4r^2)+\varepsilon} \ll N \left( N^{-1} q_c^{1/4+1/(4r)+\varepsilon} \right)^{1/r}.
\]
If we stipulate that $r > \eta^{-1}$ and set $\varepsilon = 3\eta/(4r)$, the bounds (8) and (9) establish the lemma. \(\square\)

3. Sieve results

In this section we cite the sieve results needed in the later arguments. Most of the notation is standard for the theory of sieves: let $A$ be a set of positive integers, $X > 1$ a real number, and $\rho(d)$ a multiplicative function, and define the “remainder terms” $R_d$ (which are intended to be small, at least on average over $d$) by
\[
R_d = \sum_{\substack{a \in A \atop d \mid a}} 1 - \frac{\rho(d)}{d} X.
\]
Let $L$ and $y$ be positive numbers, and suppose that the following conditions hold:
\[
1 \ll 1 - \frac{\rho(p)}{p} \leq 1 \quad \text{for all primes $p$};
\]
\[
-L < \sum_{w \leq p < z} \frac{\rho(p) \log p}{p} - \log \frac{z}{w} < O(1) \quad \text{for all $2 \leq w < z$};
\]
\[
\sum_{d \leq y} \mu^2(d)3^{\omega(d)} |R_d| \ll X/(\log X)^2.
\] (12)

Define \(P(z) = \prod_{p < z} p\). Then we have
\[
\sum_{a \in A \atop (a, P(\sqrt{y}))=1} 1 \ll X \prod_{p < \sqrt{y}} \left(1 - \frac{\rho(p)}{p}\right); \tag{13}
\]
also, for all \(\varepsilon > 0\) and all \(z \geq 2\) such that \(z^{2+\varepsilon} \ll y\), we have
\[
\sum_{a \in A \atop (a, P(z))=1} 1 \geq X \prod_{p < z} \left(1 - \frac{\rho(p)}{p}\right) \left(\Delta(\varepsilon) + O \left(\frac{L}{(\log y)^{1/5}}\right)\right), \tag{14}
\]
This follows, for example, from Theorems 4.1 and 8.4 of Halberstam-Richert \[4\], where in Theorem 4.1, we take \(y = z^2\) in their notation, and in Theorem 8.4, we take \(y = X^\alpha\) in their notation and subsume the quantity \(f(\cdot)\) into the constant \(\Delta(\varepsilon)\).

We now describe the results of Greaves on the weighted linear sieve that we will employ. Let \(A, X, \rho, R_d, L,\) and \(y\) be as before (so that conditions (10) through (12) are satisfied), and let \(g\) be a positive number such that \(1 \leq a \leq y^g\) for all \(a \in A\).

Let \(r \geq 2\) be an integer and \(U, V\) be real numbers satisfying
\[
V < 1/4, \quad 1/2 < U < 1, \quad V + rU \geq g, \tag{16}
\]
and define \(m = \max\{V, (1 - U)/2\}\). Let \(W\) be a nondecreasing “weight function”, defined on the positive real numbers, satisfying
\[
0 \leq W(t) \leq \begin{cases} 
U - V & \text{if } U < t, \\
V - t & \text{if } 1/3 \leq t \leq U, \\
V - m & \text{if } m < t \leq 1/3
\end{cases} \tag{17}
\]
and also
\[
W(t) \leq 9(U - 1/3)t^2 \quad \text{for all } t > 0.
\]
Define \(\omega(z, n)\) to be the number of prime factors of \(n\), where a multiple prime factor \(p\) is counted multiply only if \(p \geq z\), so that \(\omega(\infty, n) = \omega(n)\) and \(\omega(2, n) = \Omega(n)\). Also define \(p(n)\) to be the least prime factor of \(n\). Then
\[
\sum_{a \in A \atop \omega(y^{U/a}, a) \leq r} W \left(\frac{\log p(a)}{\log y}\right) \geq 2 \gamma X \prod_{p < y} \left(1 - \frac{\rho(p)}{p}\right) \left(M(W) + O \left(\left(\frac{L}{\log y}\right)^{1/5}\right)\right), \tag{18}
\]
where \(\gamma\) is Euler’s constant and \(M(W)\) is a constant depending on \(W\). This is essentially Theorem 1 of Greaves \[3\], although we have already dealt with the remainder term through the condition (12).

Furthermore, there exist \(U, V\) satisfying condition (16) and a weight function \(W\) satisfying conditions (17) and (18) such that \(M(W)\) is positive, as long as \(g \leq r - \delta_r\), where the values in Theorem 1 are permissible values for \(\delta_r\). (The permissible value when \(r \geq 5\) comes from Greaves’ earlier work \[2\].)
4. Proof of Theorem 1

The following lemma provides an upper bound for a remainder term sum we will encounter in our applications of the sieve results in Section 3. We recall that \( c_0 \) is a shorthand for \( c_{\chi_0} \).

**Lemma 8.** Given a real number \( x > 1 \) and coprime integers \( q \geq 2 \) and \( d \) satisfying \( 1 \leq d < x \), let the quantity \( R_d \) be defined by

\[
R_d = \sum_{n < x \atop d \mid n} \gamma(n) - \frac{c_0 \phi(q)}{q} x \frac{1}{d},
\]

(19)

Then for any \( \varepsilon, \eta > 0 \), we have

\[
R_d \ll \frac{c_0 \phi(q)}{q} x \frac{2^\varepsilon}{d \epsilon_c} \left( \frac{d \epsilon_c^{1/4+\eta}}{x} \right)^\eta.
\]

**Proof:** Using Lemma 4, we can write

\[
R_d = \sum_{n < x \atop d \mid n} \sum_{\chi \in G} c_\chi \chi(n) - \frac{c_0 \phi(q)}{q} x \frac{1}{d},
\]

(20)

For the term corresponding to the principal character, we note that for any \( T > 1 \), we have

\[
\sum_{n < T} \chi_0(n) = \sum_{n < T \atop (n, q) = 1} 1 = \sum_{n < T} \sum_{\mu(f) = 1} \sum_{f \mid q} \sum_{m < T / f} 1
\]

\[
= T \sum_{f \mid q} \frac{\mu(f)}{f} + O \left( \sum_{f \mid q} |\mu(f)| \right)
\]

\[
= T \frac{\phi(q)}{q} + O \left( 2^\omega(q) \right).
\]

Thus

\[
c_0 \chi_0(d) \sum_{m < x / d} \chi(m) = \frac{c_0 \phi(q)}{q} x \frac{1}{d} + O \left( c_0 \cdot 2^\omega(q) \right),
\]

(21)

the first term of which will cancel the last term of equation (20). For the other terms in the sum over \( \chi \) in (21), we apply Lemma 4 to the inner sums to see that

\[
\sum_{m < x / d} \chi(m) \ll \frac{x}{d} \left( \frac{d \epsilon_c^{1/4+\eta}}{x} \right)^\eta
\]

(22)

for any \( \varepsilon, \eta > 0 \). We use equations (21) and (22) in equation (20) to get

\[
R_d \ll 2^\omega(q) c_0 + \sum_{\chi \in G \atop \chi \neq \chi_0} |c_\chi| \frac{x}{d} \left( \frac{d \epsilon_c^{1/4+\eta}}{x} \right)^\eta
\]

\[
\ll \frac{c_0 \phi(q)}{q} x \frac{2^\varepsilon}{d \epsilon_c} \left( \frac{d \epsilon_c^{1/4+\eta}}{x} \right)^\eta,
\]
since $\sum_{\chi \in G} |c_\chi| = 2^{\omega(\phi(q))} c_0$ by Lemma 8, and both $q/\phi(q)$ and $2^{\omega(\phi(q))}$ are $\ll q^\varepsilon$. This establishes the lemma.

Lemma 9. Let $q$, $x$, and $\eta$ be as in Lemma 8, and define

$$y = x^{1-\eta}/q_c^{1/4+3\eta}.$$  

(23)

Then

$$\sum_{d \leq y} \mu^2(d) 3^\omega(d) R_d \ll \frac{c_0 \phi(q)}{q} x^{1-\eta^2/2}.$$  

In particular, condition (14) is satisfied.

Proof: Let $\varepsilon = \eta^2$. For any $n < x$, we have $3^\omega(n) \ll x^{\varepsilon/2}$, and so

$$\sum_{d \leq y} \mu^2(d) 3^\omega(d) R_d \ll x^{\varepsilon/2} \sum_{d \leq y} |R_d|.$$  

(24)

When $(d, q) > 1$, we have $R_d = 0$; and so we may use equation (24), Lemma 8, and the definition (23) of $y$ to see that

$$\sum_{d \leq y} \mu^2(d) 3^\omega(d) R_d \ll x^{\varepsilon/2} \sum_{d \leq y} \frac{c_0 \phi(q)}{q} x^{2\varepsilon} \left( \frac{d}{x^q^{1/4+\eta}} \right)^\eta$$

$$\ll x^{\varepsilon/2} \frac{c_0 \phi(q)}{q} x^{2\varepsilon} (y q^{1/4+\eta})^\eta$$

$$= x^{\varepsilon/2} \frac{c_0 \phi(q)}{q} x^{2\varepsilon} (x^{-\eta} q_c^{-2\eta})^\eta = \frac{c_0 \phi(q)}{q} x^{1-\eta^2/2},$$

which establishes the lemma.  

We now use the weighted linear sieve to deduce the following quantitative version of Theorem 1.

Theorem 10. Let $q$, $r \geq 2$ be integers, $0 < \eta < 1/12$ a real number, and $x$ a real number satisfying

$$x \geq q_c^{1/4+1/(4(r-1-\delta_r))}+15\eta,$$  

(25)

where the $\delta_r$ are given in Theorem 7. Then for some positive constant $C = C(r, \eta)$, we have

$$\sum_{\substack{n < x \\ n \equiv \lambda \pmod{q}}} \gamma(n) \geq \frac{c_0 \phi(q)}{q} \frac{x}{\log x} \left( C + O \left( \frac{\log \log 3 x}{\log x} \right)^{1/5} \right).$$

Proof: We apply the lower bound (18), with $A$ being the set of all $\lambda$-roots (mod $q$) less than $x$, and with the various parameters taken as follows:

$$X = \frac{c_0 \phi(q)}{q} x; \quad g = r - \delta_r; \quad \rho(p) = \begin{cases} 1 & \text{if } p \nmid q, \\
0 & \text{if } p \mid q; \end{cases}$$

$$y = x^{1-\eta}/q_c^{1/4+3\eta}; \quad L \ll \log \log 3 q_c.$$

(26)

We note that for $d$ coprime to $q$, the remainder term $R_d$ takes the form given by equation (19), while for $d$ not coprime to $q$, we have $R_d = 0$. It is straightforward to verify the
conditions (10), (11), and (15) for this choice of parameters, the requirement (25) being crucial to the validity of (15). Moreover, we see from Lemma 9 that condition (12) holds as well. Therefore, equation (18) gives us

$$\sum_{n \leq x} \gamma(n) W \left( \frac{\log p(n)}{\log y} \right) \geq \frac{c_0 \phi(q)}{q} \frac{x}{\log x} \left( \mathcal{M}(W) + O \left( \left( \frac{\log \log 3x}{\log x} \right)^{1/5} \right) \right). \tag{27}$$

where we have used Mertens’ formula for the product over primes less than $y$ and the fact that, from the definition (26) of $y$ and the restriction (25) on $x$, we have

$$\log x \geq \log y \gg \log x \text{ and } \log \log q \ll \log \log x.$$  

The terms counted by the sum in (27) are not necessarily $P_s$; we now assure that the contribution of those integers that are not $P_s$ is negligible. First we estimate the order of magnitude of the factor $c_0 \phi(q)/q$. By the definition of $c_0$, we see that $c_0 \geq \phi(\phi(q))/\phi(q)$. Since $n/\phi(n) \ll \log \log n$ for any positive integer $n$, the restriction (25) implies that

$$q/(c_0 \phi(q)) \ll (\log \log 3\phi(q))(\log \log 3q) \ll (\log \log 3x)^2.$$  

Notice that, by the condition (17) on $W$ and the fact that $W$ is nondecreasing, any integer counted with a positive weight by the sum in (27) has no prime factors less than $y^m$. Since $W(t) \leq W(1)$ for all $t > 0$, we may define $C = \mathcal{M}(W)/W(1)$ and divide both sides of equation (27) by $W(1)$ to see that

$$\sum_{n \leq x} \gamma(n) \geq \frac{c_0 \phi(q)}{q} \frac{x}{\log x} \left( C + O \left( \left( \frac{\log \log 3x}{\log x} \right)^{1/5} \right) \right).$$

Now the number of squarefree integers less than $x$ whose smallest prime factor exceeds $y^m$ is at most

$$\sum_{p>y^m} \frac{x}{p^2} \ll \frac{x}{y^m}.$$  

Thus

$$\sum_{n \leq x} \mu^2(n) \gamma(n) \geq \frac{c_0 \phi(q)}{q} \frac{x}{\log x} \left( C + O \left( \left( \frac{\log \log 3x}{\log x} \right)^{1/5} + \frac{(\log x)(\log \log 3x)^2}{y^m} \right) \right), \tag{28}$$

and again the second error term is negligible, since the definition (26) of $y$ and the restriction (25) on $x$ insure (for $\eta < 1/3$, for instance) that $y \geq x^\eta$. But if $n$ is squarefree, then $\omega(z, n) = \Omega(n)$ for any real $z$. Thus the only integers counted by the sum in (28) are $P_s$, which establishes the theorem.

5. Proof of Theorem 2

Theorem 2 is almost a corollary of Theorem 10, except that we must argue that some of the small almost-prime primitive roots (mod $p$) are also primitive roots (mod $p^2$). The following lemma was established by Kruswijk [8]; we provide a proof for the sake of completeness.
Lemma 11. Let $p$ be a prime, and for every real $x > 1$, define the set
\[ B(x) = \{b : b \leq x : b \text{ is a } p^{th} \text{ power } (\text{mod } p^2)\} \]
Let $B(x) = |B(x)|$. Then, uniformly for all positive integers $m$, we have
\[ B(p^{1/m}) \leq p^{1/(2m)} \exp\left( O\left( \frac{\log p}{\log \log p} \right) \right) . \]

Proof: Fix an integer $m \geq 1$ and consider the set
\[ \mathcal{C} = \{b_1 \cdots b_{2m} : b_i \in B(p^{1/m}), 1 \leq i \leq 2m\} . \]

On one hand, every element of $\mathcal{C}$ is a $p^{th}$ power (mod $p^2$) and is at most $p^2$ in size; thus $|\mathcal{C}| \leq p$; the total number of $p^{th}$ powers (mod $p^2$) up to $p^2$. On the other hand, the total number of products of $2m$ elements of $B(x)$ is of course $B(x)^{2m}$. Moreover, the number of ways to write an integer $n$ as a product of $2m$ elements of $B(x)$ is bounded by the number of ways to decompose $n$ generally as a product of $2m$ factors; and this in turn is at most
\[ (2m)^\Omega(n) , \]
as we can assign each of the $\Omega(n)$ prime divisors of $n$ to any of the $2m$ factors in the decomposition, and this covers all cases. When we take $x = p^{1/m}$, the fact that $n \leq p^2$ for all $n \in \mathcal{C}$ together with a standard upper bound for $\Omega(n)$ gives us
\[ \Omega(n) \ll \frac{\log p^2}{\log \log p^2} . \]
Thus we can say that
\[ p \geq |\mathcal{C}| \geq \frac{B(p^{1/m})^{2m}}{(2m)^{O(\log p/\log \log p)}} , \]
or
\[ B(p^{1/m}) \leq p^{1/(2m)} \exp\left( O\left( \frac{\log m \log p}{m \log \log p} \right) \right) . \]
Since $\log m/m$ is uniformly bounded, this establishes the lemma.

We can now establish Theorem 8. First, when $q = p$ is an odd prime, Theorem 8 is an immediate corollary of Theorem 14 using the values of $\delta_r$ given in Theorem 11. Now the only way a primitive root (mod $p$) can fail to be a primitive root (mod $p^2$) is if it is a $p^{th}$ power (mod $p^2$). Applying Lemma 11 with various values of $m$, we see that
\[ B(p^{1/2+1/873}) \leq B(p) \ll p^{1/2+\varepsilon} ; \]
\[ B(p^{1/3+1/334}) \leq B(p^{3/8+1/207}) \leq B(p^{1/2}) \ll p^{1/4+\varepsilon} ; \]
\[ B(p^{1/4+O(1/r)}) \leq B(p^{1/3}) \ll p^{1/6+\varepsilon} . \]
In all cases, the number of $p^{th}$ powers (mod $p^2$) is of a lower order of magnitude than the number of primitive roots (mod $p$). This establishes Theorem 8 for $q = p^2$. Finally, it is well-known that any primitive root (mod $p^2$) is also a primitive root (mod $p^r$) for every $r \geq 3$, and that any odd primitive root (mod $p^2$) is also a primitive root (mod $2p^r$) for every $r \geq 1$. Since the primitive roots counted by Theorem 10 are odd, this establishes Theorem 8 in its entirety.
6. Siegel zeros and prime primitive roots: Proof of Theorem

In this section we only consider moduli $q$ that admit primitive roots, i.e., $q$ is an odd prime power or twice an odd prime power (we ignore $q = 2$ and $q = 4$). Every primitive root (mod $q$) is certainly a quadratic nonresidue. Thus those primitive roots with an even number of prime factors (counted with multiplicity) must have a prime factor that is a quadratic residue.

On the other hand, suppose that for such a modulus $q$, the $L$-function of the unique non-principal quadratic character $\chi_1$ has a Siegel zero. We know, by the prime number theorem for arithmetic progressions, that this makes small primes that are quadratic residues very rare, and so small primitive roots with an even number of prime factors are correspondingly rare. Thus if we can show the existence of $P_2$ primitive roots of a certain type, then assuming the existence of a Siegel zero, we might expect to be able to argue that most of them must in fact be primes.

To do so, we use the lower bound linear sieve to produce $P_2$ primitive roots, and then a simple upper bound sieve to show that the contribution from primitive roots divisible by a quadratic residue is small. First we need to consider the remainder sum we will encounter while employing the upper bound sieve.

Lemma 12. Let $q$ be an odd prime and $x > 1$ a real number, and let $\chi_1$ be the nonprincipal quadratic character (mod $q$). Define the quantity

$$H = \sum_{x^{1/3} < p < x^{2/3}} \frac{1}{p},$$

Given an integer $d$, coprime to $q$ and satisfying $1 \leq d < x^{1/3}$, let the quantity $R_d$ be defined by

$$R_d = \sum_{x^{1/3} < p < x^{2/3}} \sum_{n < x/p \atop d | n} \gamma(pn) - \frac{c_0 \phi(q) x}{q} d H,$$  \hspace{1cm} (29)

Then for any $\varepsilon, \eta > 0$, we have

$$R_d \ll \frac{\phi(q-1) x}{q} d H q^{2\varepsilon} \left( \frac{d}{x^{1/3} q^{1/4 + \eta}} \right)^\eta.$$

Proof: Since the argument parallels the proof of Lemma 8, we provide only an outline of the proof. We have

$$R_d = \sum_{\chi \in G} c_\chi \chi(d) \sum_{x^{1/3} < p < x^{2/3}} \chi(p) \sum_{m < x/pd} \chi(m) - \frac{\phi(q-1) x}{q} d H.$$

For the principal character we have

$$c_0 \chi_0(d) \sum_{x^{1/3} < p < x^{2/3}} \chi_0(p) \sum_{m < x/pd} \chi_0(m) - \frac{\phi(q-1) x}{q} d H \ll \frac{\phi(q-1)}{q} - 1 \sum_{x^{1/3} < p < x^{2/3}} \sum_{\chi_1(p) = 1} 2^{\omega(q-1)}$$

$$\ll \frac{\phi(q-1)}{q} q^{2/3} H,$$
while for \( \chi \neq \chi_0 \) we have
\[
c_{\chi} \chi(d) \sum_{x^{1/3} < p < x^{2/3}} \chi(p) \sum_{m < x/pd} \chi(m) \ll |c_{\chi}| \sum_{x^{1/3} < p < x^{2/3}} \frac{x}{dp} \left( \frac{dp}{x} q^{1/4+\eta} \right)^{\eta}
\]
\[
\ll |c_{\chi}| \frac{x}{d} H \left( \frac{d}{x^{1/3} q^{1/4+\eta}} \right)^{\eta}.
\]
Thus
\[
R_d \ll \frac{\phi(q-1)}{q} \frac{x}{d} H q^{2\varepsilon} \left( \frac{d}{x^{1/3} q^{1/4+\eta}} \right)^{\eta},
\]
which establishes the lemma.

\[\square\]

**Lemma 13.** Let \( q, x, \) and \( \eta \) be as in Lemma [12], and define
\[
y = x^{1/3-\eta}/q^{1/4+3\eta}.
\]
Then
\[
\sum_{d \leq y} \mu^2(d) 3^{\omega(d)} R_d \ll \frac{\phi(q-1)}{q} x^{1-\eta^2/2} H.
\]
In particular, condition (12) is satisfied.

**Proof:** Let \( \varepsilon = \eta^2 \). For any \( n < x \), we have \( 3^{\omega(n)} \ll x^{\varepsilon/2} \), and so
\[
\sum_{d \leq y} \mu^2(d) 3^{\omega(d)} R_d \ll x^{\varepsilon/2} \sum_{d \leq y} |R_d|.
\]
We may now apply Lemma [12] to see that
\[
\sum_{d \leq y} \mu^2(d) 3^{\omega(d)} |R_d| \ll x^{\varepsilon/2} \sum_{d \leq y} \phi(q-1) \frac{x}{d} H q^{2\varepsilon} \left( \frac{d}{x^{1/3} q^{1/4+\eta}} \right)^{\eta}
\]
\[
\ll x^{\varepsilon/2} \phi(q-1) x H q^{2\varepsilon} \left( \frac{y}{x^{1/3} q^{1/4+\eta}} \right)^{\eta}
\]
\[
\ll x^{\varepsilon/2} \phi(q-1) x H q^{2\varepsilon} (x^{-\eta} q^{-2\eta}) \eta = \frac{\phi(q-1)}{q} x^{1-\eta^2/2} H,
\]
which establishes the lemma. \[\square\]

We may now establish the following quantitative version of Theorem [3].

**Theorem 14.** Let \( q \) be an odd prime power or twice an odd prime power, and let \( \chi_1 \) be the nonprincipal quadratic character \( \mod q \). Suppose that \( L(s, \chi_1) \) has a real zero \( \beta \) of the form
\[
\beta = 1 - \frac{1}{\alpha \log q}
\]
with \( \alpha \geq 3 \). Then for any real number \( 0 < \eta < 1/52 \), and any real number \( x \) satisfying \( q^{3/4+13\eta} \leq x \leq q^{500} \), we have
\[
\sum_{p < x} \gamma(p) \geq \frac{\phi(\phi(q))}{q} \frac{x}{\log x} \left( \Delta(\eta) + O \left( (\log x)^{-1/14} + (\log \alpha)^{-1/2} \right) \right),
\]
where $\Delta(\eta)$ is a positive constant depending on $\eta$.

**Proof:** First we assume that $q$ is an odd prime. We let $\mathcal{A}$ be the set of all primitive roots (mod $q$) less than $x$, and we choose

$$X = \frac{\phi(q-1)}{q} x; \quad \rho(d) = 1 \text{ for all } d < q; \quad y = x^{1-\eta}/q^{1/4+3\eta}; \quad z = x^{1/3}; \quad L \ll 1. \quad (30)$$

Again, conditions (10) and (11) are easy to verify, and the special case of Lemma 8 where $q$ is an odd prime establishes that condition (12) holds. In addition, the definition (30) of $y$, together with the restrictions $x \geq q^{3/4+13\eta}$ and $\eta < \frac{1}{52}$, insure that $z^{2+\eta} \leq y$. Therefore we may apply the linear sieve (14) to obtain

$$\sum_{a \in \mathcal{A}} \left( \frac{\Phi(\frac{x}{3})}{\Phi(q)} \right) \prod_{p \nmid a, p \nmid x} \left( 1 - \frac{1}{p} \right) \geq \frac{\phi(q-1)}{q} \frac{x}{2e^\gamma \log x^{1/3}} \left( \Delta(\eta) + O \left( (\log x)^{-1/14} \right) \right), \quad (31)$$

where we have again used Mertens’ formula for the product over primes. We notice that all the integers counted by this sum are $P_2$s, and that the lower bound is all the more true if we replace the denominator $2e^\gamma \log x^{1/3}$ by $\log x$.

We now show that the contribution to the sum (31) from products of two primes is negligible. We write this contribution as

$$T = \sum_{p_1 p_2 < x} \gamma(p_1 p_2).$$

When $p_1 p_2$ is a primitive root, which must be the case for $p_1 p_2$ to contribute to this sum, exactly one of $p_1$ or $p_2$ is a quadratic residue (mod $q$). Thus this sum becomes

$$T = \sum_{x^{1/3} < p_1 < x^{2/3}} \sum_{x^{1/3} < p_2 < x/p_1} \gamma(p_1 p_2).$$

Relaxing the restrictions on the integer $p_2$ will allow us to estimate this sum more easily. Let $z$ be a parameter satisfying $2 \leq z \leq x^{1/3}$. Instead of summing over primes in the range $x^{1/3} < p < x/p_1$ which are quadratic nonresidues, we will simply sum over all integers $n < x/p_1$ whose prime divisors all exceed $z$. Thus we have

$$T \leq T(z) = \sum_{x^{1/3} < p < x^{2/3}} \sum_{n < x/p} \gamma(pn). \quad (32)$$

We can now apply the upper bound sieve (13) with $\mathcal{A}$ being the set of all integers $n < x^{2/3}$, counted with multiplicity equal to the number of primes $p$, meeting the criteria of the outer sum in (32), such that $pn$ is a primitive root (mod $q$). We take the various sieve parameters as follows:

$$X = \frac{c_0 \phi(q)}{q} x H; \quad \rho(d) = 1 \text{ for all } d < q; \quad y = x^{1/3-\eta}/q^{1/4+3\eta}; \quad L \ll 1.$$
Now the remainder term $R_d$ takes the form given by equation (29), and it is again straightforward to verify the conditions (10) and (11) and to note that Lemma 13 establishes condition (12) as well. Therefore, equation (13) gives us

$$T \leq T(\sqrt{y}) \ll \frac{\phi(q - 1)}{q} \frac{x}{\log x} H,$$

which, together with the lower bound (31), yields

$$\sum_{p \in A \atop p > x^{1/3}} 1 \geq \frac{\phi(q - 1)}{q} \frac{x}{\log x} \left( \Delta(\eta) + O \left( (\log x)^{-1/14} + H \right) \right), \quad (33)$$

Heath-Brown has shown [5, Lemma 3] that for $x \leq q^{500}$, we have

$$\sum_{p < x \atop \chi_1(p) = 1} \frac{\log p}{p} \ll (\log q)(\log \alpha)^{-1/2},$$

from which it follows that $H \ll (\log \alpha)^{-1/2}$, since $\log x \gg \log q$. Using this fact in the lower bound (33) establishes the theorem when $q$ is an odd prime. The arguments used in Section 4 to justify Theorem 2 for composite moduli apply here as well to establish the theorem in its full generality. 

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