Robin conditions on the Euclidean ball

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Abstract
Techniques are presented for calculating directly the scalar functional determinant on the Euclidean \(d\)-ball. General formulae are given for Dirichlet and Robin boundary conditions. The method involves a large mass asymptotic limit which is carried out in detail for \(d = 2\) and \(d = 4\) incidentally producing some specific summations and identities. Extensive use is made of the Watson-Kober summation formula.

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1. Introduction

For specially symmetric situations it is often possible to relate the eigenvalue problem on a bounded region, \( \mathcal{M} \), to one on a covering space, the method of images being an example. As Pockels [1] has pointed out, in the case of the boundary condition, \( (\mathbf{n}.\nabla \phi - h\phi) \big|_{\partial \mathcal{M}} = 0 \), this technique is no longer applicable and one has to proceed directly. In classical physics such conditions arise in heat conduction [2].

In the present work we wish to analyse some cases of these Robin boundary conditions, evaluating the coefficients in the asymptotic expansion of the heat-kernel and also the corresponding functional determinant. The reason is that it is often useful to have explicit results for special cases in order to put restrictions on the general situation. In addition, Robin conditions enter into the mixed boundary conditions required for higher dimensional vector bundles.

The domain of concern here is the Euclidean \( d \)-ball. The determinants on such regions have already been found, in the massless case, by a conformal transformation method but this technique is limited, at the moment, to \( d \leq 4 \) by the availability of the necessary transformation law. A more direct method is therefore needed.

The recent determination by Branson, Gilkey and Vassilevich [3] of the \( C_{5/2} \) coefficient is not sufficient to allow the extension to \( d = 5 \) as it is restricted to totally geodesic boundaries.

Other calculations involving relevant material that should be mentioned are those by D’Eath and Esposito [4,5], Schleich [6] and Louko [7]. A parallel calculation by Bordag et al has recently appeared, [8].

2. The interval

We work on \( I = [0, 1] \) and consider the simplest equation

\[
-\frac{d^2 \phi}{dx^2} = \lambda \phi
\]

with

\[
\frac{d\phi}{dx} \bigg|_1 = -h_1 \phi_1, \quad \frac{d\phi}{dx} \bigg|_0 = h_0 \phi_0.
\]

This eigenvalue problem is discussed in Carslaw and Jaeger. We write the eigenfunctions

\[
\phi(x) = N \sin(\alpha x + \delta)
\]
where $\lambda = \alpha^2$ and $\tan \delta = \alpha/h_0$. The eigenvalue equation is derived to be
\[
\frac{\tan \alpha}{\alpha} = \frac{h_1 + h_0}{\alpha^2 - h_1 h_0}.
\] (1)

The normalisation, $N$ is easily worked out.

If $h_0 = h_1 = h$, (1) factorises into
\[
\left(2h \cos(\alpha/2) + \alpha \sin(\alpha/2)\right)\left(h \frac{\sin(\alpha/2)}{\alpha} - 2 \cos(\alpha/2)\right) = 0.
\] (2)

An eigenvalue equation similar to (1), but with the right-hand side reversed in sign, occurs in the theory of the three-ball with Neumann conditions at the surface (Rayleigh [9] I.p265, eqn.(2)).

The eigenvalues are non-degenerate and, although easily computed, are not needed here since the Mittag-Leffler theorem can be applied
\[
(h_0 h_1 - z^2)^{\sin z} + (h_1 + h_0) \cos z = (h_0 h_1 + h_0 + h_1) \prod_{\alpha} \left(1 - \frac{z^2}{\alpha^2}\right)
\] (3)

where the product is over the positive roots.

Incidentally, logarithmic differentiation, expansion in positive powers of $z$ and the equating of coefficients allows one to evaluate the sums of inverse powers of the roots. (This technique goes back to Euler, cf Rayleigh [9] II p.279. Rayleigh does not differentiate but just expands the logarithms. See also Watson [10] 15.5, 15.51.)

It is found that, for example,
\[
\sum \frac{1}{\alpha^2} = \frac{h_0 h_1 + 3(h_0 + h_1) + 6}{6(h_0 h_1 + h_0 + h_1)},
\]
\[
\sum \frac{1}{\alpha^4} = \frac{h_0^2 h_1^2 + 6(h_0^2 h_1 + h_0 h_1^2) + 15(h_0^2 + h_1^2) + 30h_0 h_1 + 60(h_0 + h_1) + 90}{90(h_0 h_1 + h_0 + h_1)^2}
\]

which can be confirmed numerically. Defining the $\zeta$-function by
\[
\zeta(s) = \sum \frac{1}{\lambda^s}
\]
we see that the values $\zeta(n)$, $n \in \mathbb{Z}^+$, can be determined.

One of our aims is to find the coefficients in the heat-kernel expansion. These are related to the residues at the poles of $\zeta(s)$ and to the values $\zeta(-n)$, $n \in \mathbb{Z}^+$. The
method we use here involves the time-honoured one of the asymptotic behaviour of the resolvent

\[ R(m^2) = \sum_{\lambda} \frac{1}{\lambda + m^2} \]  

(4)
as \( m^2 \to \infty \). (cf Dikii [11], and [12,13]).

More generally we can define

\[ \zeta(s, m^2) = \sum_{\lambda} \frac{1}{(\lambda + m^2)^s}, \]  

(5)

and it is straightforward to derive the large, positive \( m \) asymptotic expansions (cf Dowker and Critchley [14], Voros [12], Elizalde [15]),

\[ \zeta(s, m^2) \sim \frac{1}{(4\pi)^{d/2}} \sum_{n=0,1/2,1,...} (m^2)^{d/2-n-s} \frac{\Gamma(s-d/2+n)}{\Gamma(s)} C_n. \]  

(6)

and

\[ \zeta'(0, m^2) \sim \frac{1}{(4\pi)^{1/2}} \sum_{n=0,1,3/2,...} \left( C_n \Gamma(n-1/2)m^{-2n+1} - C_{1/2} \ln m^2 \right). \]  

(7)

If \( d = 1 \) it is sufficient to set \( s = 1 \) in (6) to get

\[ R(m^2) \sim \frac{1}{(4\pi)^{1/2}} \sum_{n=0,1/2,1,...} \frac{\Gamma(1/2+n)}{(m^2)^{1/2+n}} C_n. \]  

(8)

Comparison with (3) shows that \( z = im \) and that

\[ R(m^2) = -\frac{d}{dm^2} \ln \left( \frac{(h_0 h_1 + m^2)^{\frac{\sinh m}{m}} + (h_1 + h_0) \cosh m}{m} \right). \]  

(9)

The large positive \( m \) limit is easy to find,

\[ R(m^2) \sim \frac{1}{2m} \left( 1 + \frac{1}{m+h_1} + \frac{1}{m+h_0} - \frac{1}{m} \right) \]

\[ = \frac{1}{2m} \left( 1 + \frac{1}{m} + \sum_{k=1}^{\infty} (-1)^k \frac{h_1^k + h_0^k}{m^{k+1}} \right) \]

and the coefficients \( C_n \) can be read off as

\[ C_0 = 1, \]

\[ C_{1/2} = \sqrt{\pi}, \]

\[ C_n = \frac{(-1)^{2n} \sqrt{\pi}}{\Gamma(n + 1/2)} (h_1^{2n-1} + h_0^{2n-1}); \quad n \geq 1. \]  

(10)
These check with known forms and we are now in possession of the values \( \zeta(-n) \).

Our next objective is the functional determinant,

\[
D(-m^2) = \exp \left( -\zeta'(0, m^2) \right)
\]  

(11)
in particular \( D(0) \).

From (3)

\[
\zeta'(0, 0) - \zeta'(0, m^2) = \ln F(m^2) - \ln \left( h_0 h_1 + h_0 + h_1 \right),
\]  

(12)

where \( F \) is the factor in (9). We can thus write

\[
\zeta'(0, 0) = \lim_{m \to \infty} \left( \zeta'(0, m^2) + \ln F(m^2) - \ln \left( h_0 h_1 + h_0 + h_1 \right) \right).
\]

From (9)

\[
\ln F(m^2) \to m + \ln m - \ln 2 + \ln \left( \left( 1 + \frac{h_0}{m} \right) \left( 1 + \frac{h_1}{m} \right) \right),
\]  

(13)

and using the asymptotic form (7) one finds that

\[
\zeta'(0, 0) = -\ln 2 - \ln \left( h_0 h_1 + h_1 + h_0 \right)
\]  

(14)

and so \( D(0) = 2(h_0 h_1 + h_1 + h_0) \). The same result emerges from the method of Barvinsky et al. [16].

As a check, expanding the logarithm in (13) gives the previous expressions, (10), for the coefficients.

The Neumann (and Dirichlet) value for \( \zeta'(0, 0) \) is \( -\ln 2 \). But the Neumann case, \( h_1 = h_0 = 0 \), has to be treated afresh since there are zero modes. The vanishing of the determinant, \( i.e. \ h_0 h_1 + h_1 + h_0 = 0 \), is related to the appearance of a zero eigenvalue but the corresponding mode does not exist.

Turning equation (12) around, one gets the massive functional determinant,

\[
D(-m^2) = 2 \left( (h_0 h_1 + m^2) \frac{\sinh m}{m} + (h_1 + h_0) \cosh m \right).
\]  

(15)

Other, miscellaneous boundary conditions, [2], can be treated in the same way with similar results.
3. Heat kernel coefficients on Euclidean balls.

The higher dimensional interval, *i.e.* the ball, is treated by the same approach. The operator is the Laplacian and the standard mode expressions, which will not be repeated here, are outlined by Moss [13]. (See also [17,18].) The eigenvalues \( \lambda = \alpha^2 \) in (5) are given by the roots of

\[
J_p(\alpha) = 0, \quad \text{Dirichlet}
\]

\[
\beta J_p(\alpha) + \alpha J'_p(\alpha) = 0, \quad \text{Robin},
\] (16)

where \( \beta = h + 1 - d/2 \). The degeneracy for a given Bessel order is, for even \( d \)

\[
N_p^{(d)} = \frac{2}{(d-2)!} p^2 (p^2 - 1) \cdots (p^2 - (d/2 - 2)^2),
\] (17)

while for odd \( d \),

\[
N_p^{(d)} = \frac{2}{(d-2)!} p(p^2 - 1/4) \cdots (p^2 - (d/2 - 2)^2).
\] (18)

The Mittag-Leffler theorem can again be employed. Since there are no normalisable zero modes, the zero roots of (16) must be removed. We therefore write

\[
z^{-p} F_p(z) = \gamma \prod_\alpha \left( 1 - \frac{z^2}{\alpha^2} \right)
\] (19)

where

\[
\gamma = \frac{1}{2p!}, \quad \text{Dirichlet}
\]

\[
= \frac{(\beta + p)}{2^{p+1} p!}, \quad \text{Robin} \quad (\beta, p) \neq (0, 0).
\]

If \( \beta = 0 \), then \( p = 0 \) is a special case, \( F_0 = zJ'_0 = -zJ_1 \), and

\[
z^{-2} F_0(z) = -\frac{1}{2} \prod_\alpha \left( 1 - \frac{z^2}{\alpha^2} \right).
\] (20)

Moss [13] was concerned to evaluate the heat-kernel coefficients, and we present here an equivalent treatment. The coefficients can be read off from the asymptotic expansion (6) with \( s = (d + 2)/2 \) if \( d \) is even and \( s = (d + 1)/2 \) if \( d \) is odd. The requisite expression for the left-hand side can be obtained by repeated differentiation of the resolvent (4) or, equivalently, of the product (19), which is rewritten with \( z = im \) as

\[
m^{-p} F_p(m) = \gamma \prod_\alpha \left( 1 + \frac{m^2}{\alpha^2} \right)
\] (21)
where $F_p$ is now given by

$$\begin{align*}
F_p(m) &= I_p(m), \quad \text{Dirichlet} \\
&= \beta I_p(m) + mI'_p(m), \quad \text{Robin},
\end{align*}
$$

in terms of the modified Bessel function. (See also D’Eath and Esposito [4,5].)

The required quantity is then (cf [13])

$$
\zeta(q, m^2) = -\frac{1}{(q-1)!} \left( -\frac{\partial}{\partial m^2} \right)^q \sum_p N_p \ln \left( m^{-p} F_p(m) \right).
$$

(23)

The development of the logarithm follows from Olver’s asymptotic expansion of the Bessel functions. In particular

$$
\ln \left( m^{-p} I_p(m) \right) \sim -\frac{1}{2} \ln(2\pi) + \epsilon - p \ln(p + \epsilon) - \frac{1}{2} \ln \epsilon + \ln \left( 1 + \sum_{n=1}^{\infty} \frac{U_n(t)}{p^n} \right)
$$

(24)

and

$$
\ln \left( m^{-p+1} I_p'(m) \right) \sim -\frac{1}{2} \ln(2\pi) + \epsilon - p \ln(p + \epsilon) + \frac{1}{2} \ln \epsilon + \ln \left( 1 + \sum_{n=1}^{\infty} \frac{V_n(t)}{p^n} \right)
$$

(25)

where $\epsilon = \sqrt{p^2 + m^2}$ and $t = p/\epsilon$. It is sometimes more convenient to use Olver’s variable, $z = m/\epsilon$, and we will use the same kernel symbols for the polynomials whether they are written as functions of $t$ or of $z$.

The polynomials $U_n(t)$ satisfy the recursion relation [19]

$$
U_{n+1} = \frac{1}{2} t^2 (1 - t^2) \frac{dU_n}{dt} + \frac{1}{8} \int_0^t (1 - 5t^2) U_n dt,
$$

(26)

and the $V_n$ are related to the $U_n$ by

$$
V_n = U_n - \frac{1}{2} t(1 - t^2)(U_{n-1} + 2t \frac{dU_{n-1}}{dt}).
$$

Equation (24) is relevant for the Dirichlet case. The Robin expansion is

$$
\ln \left( m^{-p} F_p \right) \sim \epsilon - p \ln(p + \epsilon) - \frac{1}{2} \ln(2\pi) + \frac{1}{2} \ln \epsilon + \ln \left( 1 + \sum_{n=1}^{\infty} \frac{W_n}{p^n} \right),
$$

(27)

where $W_n = V_n + \beta t U_{n-1}$ with $U_0 = 1$. 

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From the recursion (26) it is clear that $U_n(t)$, $V_n(t)$ and $W_n(t)$ each have a factor of $t^n$. Extracting this factor, we write e.g.

\[
\frac{U_n(t)}{p^n} = \frac{U_n(t)}{e^n},
\]

so that \( W_n = V_n + \beta U_{n-1} \).

The systematic treatment of the summations in (24) and (27) involves the cumulant, or cluster, expansion, written generally as

\[
\ln \left( 1 + \sum_{n=1}^{\infty} \frac{Q_n(t)}{e^n} \right) = \sum_{n=1}^{\infty} \frac{S_n(t)}{e^n}
\]

with

\[
S_n = \sum_{\{l_j\}} (-1)^{\delta-1} (\delta - 1)! \prod_{j=1}^{n} \frac{Q_{l_j}}{l_j!},
\]

(28)

where \( \delta = \sum_{j=1}^{n} l_j \) and the summation is over sets of positive integers \( l_j \) including zero such that \( \sum_{j=1}^{n} j l_j = n \).

For example,

- \( S_1 = Q_1 \)
- \( S_2 = Q_2 - \frac{1}{2} Q_1^2 \)
- \( S_3 = Q_3 - Q_2 Q_1 + \frac{1}{3} Q_1^3 \)
- \( S_4 = Q_4 - Q_3 Q_1 - \frac{1}{2} Q_2^2 + Q_2 Q_1^2 - \frac{1}{4} Q_1^4 \)
- \( S_5 = Q_5 - Q_4 Q_1 - Q_3 Q_2 + Q_3 Q_1^2 - Q_2 Q_1^3 + Q_2 Q_2^2 Q_1 + \frac{1}{5} Q_1^5 \)

the number of terms increasing rapidly with the order \( n \).

Specific evaluation shows agreement with ref. [13], e.g. in the Dirichlet case, (24)

\[
T_1(t) = -\frac{1}{12} + \frac{5}{24} z^2,
\]

\[
T_2(t) = -\frac{1}{4} z^2 + \frac{5}{16} z^4
\]

\[
T_3(t) = \frac{1}{360} + \frac{61}{240} z^2 - \frac{221}{192} z^4 + \frac{1105}{1152} z^6
\]

\[
T_4(t) = -\frac{1}{4} z^2 + 3 z^4 - \frac{113}{16} z^6 + \frac{565}{128} z^8
\]

\[
T_5(t) = -\frac{1}{1260} + \frac{125}{504} z^2 - \frac{493}{72} z^4 + \frac{1621}{48} z^6 - \frac{82825}{1536} z^8 + \frac{82825}{3072} z^{10}
\]
while, for Robin conditions, the corresponding polynomials are

\[ R_1(\beta, t) = \beta - \frac{1}{12} - \frac{7}{24} z^2, \]
\[ R_2(\beta, t) = -\frac{1}{2} \beta^2 + \frac{1}{4} (2\beta + 1) z^2 - \frac{7}{16} z^4 \]
\[ R_3(\beta, t) = \frac{1}{360} + \frac{1}{3} \beta^3 - \frac{59}{240} z^2 - \frac{1}{2} \beta^2 z^2 + \frac{259}{192} z^4 + \frac{7}{8} \beta z^4 - \frac{1463}{1152} z^6. \] (30)

Related expansions for the spin-half case were computed by D’Eath and Esposito [4,5].

The next step is the differentiation in (23) followed by the sum over \( p \), which is the hardest section of the analysis.

To check the method, and for illustrative purposes, we first look at the Dirichlet 2-disc, \( q = 2 \) and \( N_p(2) = 2 - \delta_{p0} \) in (23). Algebra produces

\[ \left( \frac{d}{dm^2} \right)^2 \ln (m^{-p} I_p) \sim \frac{1}{2m^2} \left( \frac{p - \epsilon}{m^2} + \frac{1}{2\epsilon} \right) + \frac{1}{4\epsilon^4} + \left( \frac{d}{dm^2} \right)^2 \sum_{n=1}^{\infty} \frac{T_n(p/\epsilon)}{\epsilon^n} \] (31)

for \( p > 0 \).

Since the bracketed term in (31) converges overall as \( p \to \infty \) it is legitimate to regularise it. Moss writes

\[ \left( \frac{p - \epsilon}{m^2} + \frac{1}{2\epsilon} \right) = \lim_{s \to 0} \left( \frac{p - \epsilon}{\epsilon^{2s} m^2} + \frac{1}{2\epsilon^{2s+1}} \right), \]

which introduces the series

\[ \zeta_2(s, m^2) = \sum_{p=0}^{\infty} \frac{p}{(p^2 + m^2)^s} \] (32)

and

\[ \zeta_1(s, m^2) = \sum_{p=0}^{\infty} \frac{1}{(p^2 + m^2)^s} \] (33)

so that

\[ \sum_{p=0}^{\infty} \frac{N_p(2)}{(p^2 + m^2)^s} = 2\zeta_1(s, m^2) - \frac{1}{m^{2s}} \equiv 2\zeta(s, m^2). \] (34)

Then

\[ \sum_{p=0}^{\infty} N_p(2) \left( \frac{p - \epsilon}{m^2} + \frac{1}{2\epsilon} \right) = \frac{2}{m^2} \lim_{s \to 0} \left( \zeta_2(s, m^2) - \zeta_1(s - \frac{1}{2}, m^2) + \frac{m^2}{2} \zeta_1(s + \frac{1}{2}, m^2) \right). \] (35)
Although (32) and (33) are very standard we proceed from first principles and write

\[ \zeta_2(s, m^2) = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \sum_p pe^{-(p^2+m^2)t} \]

which is related to the \( \zeta \)-function on a two-sphere. The asymptotic expansion of the heat-kernel, related to Mulholland’s result, is known to be

\[ \sum_p pe^{-p^2t} \sim \frac{1}{4\pi t} \sum_{n=0,1...} C_n t^n \]

(36)

where

\[ C_0 = 2\pi, \quad C_n = 2\pi \frac{(-1)^n}{n!} B_{2n}, \quad n > 0 \]

in terms of Bernoulli numbers. Of course the coefficients in (36) are best deduced by applying a summation formula to the \( \zeta \)-function, but it is convenient here to consider them as given (e.g. [20]).

The asymptotic expansion of \( \zeta_2(s, m^2) \) is then given by (6),

\[
\zeta_2(s, m^2) \sim \frac{m^{2-2s}}{2(s-1)} + \sum_{n=1} \frac{(m^2)^{1-n-s} \Gamma(s-1+n)}{2\Gamma(s)n!} (-1)^n B_{2n}
\]

and,

\[
= \frac{m^{2-2s}}{2(s-1)} - \frac{m^{-2s}}{12} + \ldots
\]

and, similarly,

\[ \zeta_1(s, m^2) \sim \frac{\sqrt{\pi}}{2} \frac{\Gamma(s-1/2)}{\Gamma(s)} m^{1-2s}. \]

(37)

In (35) the first term in brackets is immediately finite,

\[ \zeta_2(0, m^2) = -\frac{1}{12} - \frac{m^2}{2}, \]

and the other two can be combined, asymptotically, to a finite expression using (37),

\[
\zeta_1(s+1/2, m^2) - \frac{2}{m^2} \zeta_1(s-1/2, m^2)
\]

\[ \sim \frac{\sqrt{\pi}}{2} m^{-2s} \left( \frac{\Gamma(s)}{2\Gamma(s+1/2)} - \frac{\Gamma(s-1)}{\Gamma(s-1/2)} \right) \]

\[ = \frac{\sqrt{\pi}}{2} m^{-2s} \frac{s\Gamma(s)}{(1-s)\Gamma(s+1/2)}, \]

which equals 1/2 at \( s = 0 \).
We therefore find,
\[
\left( \frac{d}{dm^2} \right)^2 \sum_{p=0}^{N_p^2} (\epsilon - p \log(p + \epsilon)) \sim -\frac{1}{12m^4} - \frac{1}{4m^2},
\]
and, further,
\[
\left( \frac{d}{dm^2} \right)^2 \sum_{p=0}^{N_p^2} (-\frac{1}{2} \log(\epsilon)) \sim \frac{\pi}{8m^3}.
\]

Consider now the polynomial part of (31). The \(n = 1, 2\) and 3 contributions are
\[
\left( \frac{d}{dm^2} \right)^2 T_1 = \frac{25m^2}{32\epsilon^7} + \frac{11}{16\epsilon^5},
\]
\[
\left( \frac{d}{dm^2} \right)^2 T_2 = \frac{15m^4}{4\epsilon^{10}} - \frac{21m^2}{4\epsilon^8} + \frac{13}{8\epsilon^6},
\]
\[
\left( \frac{d}{dm^2} \right)^2 T_3 = \frac{12155m^6}{512\epsilon^{13}} - \frac{11271m^4}{256\epsilon^{11}} + \frac{771m^2}{32\epsilon^9} - \frac{57}{16\epsilon^7}.
\]

After multiplying by \(N_p^2\), the summations can be effected using (34) and (37).

Going back to (6), and writing out the first few terms,
\[
\zeta(2, m^2) \sim \frac{1}{4\pi} \left( \frac{C_0}{m^2} + \frac{\sqrt{\pi} C_{1/2}}{2m^3} + \frac{C_1}{m^4} + \frac{3\sqrt{\pi} C_{3/2}}{4m^5} + \ldots \right),
\]

elementary arithmetic confirms the heat-kernel expansion derived by Stewartson and Waechter [21], and extended recently by Berry and Howls [22] to many more terms. We note that Stewartson and Waechter’s method also uses knowledge of just the asymptotic behaviour of the eigenfunctions, but the information is organised differently.

The Robin case is similar to the Dirichlet one. The polynomials are,
\[
\left( \frac{d}{dm^2} \right)^2 R_1 = 13 + 12\beta - \frac{35m^2}{16\epsilon^5} - \frac{32\beta \epsilon^7}{2}
\]
\[
\left( \frac{d}{dm^2} \right)^2 R_2 = -\frac{8\beta^2 + 16\beta + 15}{8\epsilon^6} + \frac{3m^2 4\beta + 9}{\epsilon^8} - \frac{21m^4}{4\epsilon^{10}}
\]
\[
\left( \frac{d}{dm^2} \right)^2 R_3 = \frac{20\beta^3 + 40\beta^2 + 68\beta + 63}{16\epsilon^7} - m^2 \frac{140\beta^2 + 532\beta + 917}{32\epsilon^9} + m^4 \frac{3528\beta + 14217}{256\epsilon^{11}} - m^6 \frac{16093}{512\epsilon^{13}}.
\]

In two dimensions \(\beta = h\).
The Robin coefficients, derived from a comparison with (40), agree with those found by Moss [13] and so will not be given here. There is no difficulty in carrying out the evaluation to any desired order.

Waechter used the method of [21] to discuss the case \( d = 3 \), \( i.e. \) the three-ball. His result was corrected by Kennedy [23] who extended the calculation up to \( d = 6 \).

The present method also extends to higher dimensions without the need for any special summation techniques. Since the complications are simply arithmetic ones, symbolic manipulation can be used to advantage.

A method has also recently been developed by Bordag, Elizalde and Kirsten [24] that allows any number of coefficients to be calculated in higher dimensions. Their method is similar to that of Barvinsky et al [16,25].

4. The functional determinant.

In the general case, Voros [12] develops the connection between the two quantities \( D(z^2) \) and \( \Delta(z^2) \), where \( D \) is given in (11) and the Weierstrassian product is

\[
\Delta(z^2) = \prod_{\lambda} (1 - \frac{z^2}{\lambda}) \exp \sum_{k=1}^{[\mu]} \frac{1}{k} \left( \frac{z^2}{\lambda} \right)^k,
\]

which amounts to subtracting the first \( [\mu] \) terms of the Taylor expansion of \( \ln(1 - z^2/\lambda) \). Formally

\[
\ln \Delta(z^2) = \frac{1}{[\mu]!} \int_0^1 d\tau (1 - \tau)^{[\mu]} \left( \frac{d}{d\tau} \right)^{[\mu]+1} \sum_{\lambda} \ln \left( 1 - \frac{z^2\tau}{\lambda} \right).
\]

(See also Quine, Heydari and Song [26].)

Voros’ relation is

\[
\zeta'(0,0) = \ln \left( \frac{\Delta(z^2)}{D(z^2)} \right) - \sum_{k=1}^{[\mu]} FP\zeta(k,0) \frac{z^{2k}}{k} - \sum_{k=2}^{[\mu]} C_{d/2-k} \left( 1 + \ldots + \frac{1}{k+1} \right) \frac{z^{2k}}{k!} \quad (43)
\]

where \( \mu = d/2 \) is the order of the sequence of eigenvalues and \( FP \) indicates use of the finite part prescription. When \( \mu = 1/2 \), (43) is equivalent to (12).

From this equation it is clear that to find \( \zeta'(0,0) \) it is only necessary to pick out the \( m \)-independent part of the right-hand side which can be done in the infinite \( m \) limit. We note from (6) that \( \ln D(-m^2) = -\zeta'(0,m^2) \) contains no constant part in this limit and that the two summations in (43) are also not needed. Hence

\[
\zeta'(0,0) = \lim_{m \to \infty} \ln \Delta(-m^2), \quad (44)
\]
and this is our starting formula. Once \( \zeta'(0,0) \) has been found, \( \zeta'(0,m^2) \) can be computed numerically from (43). A power series expansion in \( m^2 \) is easily derived.

5. Determinant on the disc.

As an example of the use of this technique consider Dirichlet conditions on the disc, for which \( \mu = 1 \). In this case (41) reads (the degeneracy is implied),

\[
\Delta(-m^2) = \prod_{p,\alpha_p} (1 + \frac{m^2}{\alpha_p^2}) \exp\left(-\frac{m^2}{\alpha_p^2}\right) = \prod_{p} (p! 2^p m^{-p} I_p) \exp\left(-\frac{m^2}{4(1 + p)}\right), \tag{45}
\]

where we have used Rayleigh’s explicit results on the sums of inverse powers of the zeros of Bessel functions rather than the formal subtraction (42).

From (45) and (24), asymptotically,

\[
\ln \Delta(-m^2) \sim \sum_{p=0}^{\infty} N_p^{(2)} \left( p \ln 2 + \ln p! - \ln \sqrt{2\pi} + \epsilon - p \ln(p + \epsilon) \right) - \ln \sqrt{\epsilon} + \sum_{n} \frac{T_n(t)}{\epsilon^n} - \frac{m^2}{4(1 + p)} \tag{46}
\]

We would like to do the sum over \( p \) and take the large \( m^2 \) limit. The integral representation of \( \log p! \) is employed to give

\[
\ln \Delta(-m^2) \sim -\ln \sqrt{2\pi} + m - \frac{1}{2} \ln m + \sum_{p=0, n=2} N_p^{(2)} \frac{T_n(t)}{\epsilon^n}
- \frac{1}{12} \sum_{p=0}^\infty N_p^{(2)} \left( \frac{1}{\epsilon} - \frac{5m^2}{2\epsilon^3} \right) + 2 \sum_{p=1}^\infty \left[ p \ln \frac{2p}{p + \epsilon} + \epsilon - p \right] \tag{47}
- \frac{m^2}{4p} - \frac{1}{2} \ln \frac{\epsilon}{p} + \int_0^\infty \left( \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-tp}}{t} dt
\]

where some initial terms have been extracted for convenience of doing the summation and taking the asymptotic limit. It is readily confirmed by expanding in \( m/p \) that the sum over \( p \) converges.

Consider firstly,

\[
\lim_{m \to \infty} \frac{5m^2}{24} \sum_{p=0}^\infty \frac{N_p^{(2)}}{\epsilon^3}
\]

which can be evaluated using (37) to give 5/12.
Next look at

\[
\sum_{p=1}^{\infty} \left( \int_0^{\infty} \left( \frac{1}{2} - \frac{1}{t} + \frac{1}{t} \frac{1}{e^t - 1} \right) \frac{e^{-tp}}{t} dt - \frac{1}{12\epsilon} \right).
\]

The sum over the integral diverges because of the behaviour at \( t = 0 \). Adding and subtracting the offending term, and performing the sum, yields

\[
\int_0^{\infty} \left( \frac{1}{2} - \frac{1}{t} - \frac{t}{12} + \frac{1}{e^t - 1} \right) \frac{1}{t(e^t - 1)} dt + \frac{1}{12} \sum_{p=1}^{\infty} \left( \frac{1}{p} - \frac{1}{\epsilon} \right).
\]

Dealing with the final summation first, we make use of the standard formula [27]

\[
\sum_{p=1}^{\infty} \left( \frac{1}{p} - \frac{1}{\epsilon} \right) = \log m + \frac{1}{2m} - 2 \sum_{p=1}^{\infty} K_0(2\pi mp) + \gamma - \log 2
\]

the last two terms of which are the required constant contribution. The integral term can be regularised as

\[
\lim_{s \to 0} \int_0^{\infty} \left( \frac{1}{2} - \frac{1}{t} - \frac{t}{12} + \frac{1}{e^t - 1} \right) \frac{t^{s-1}}{e^t - 1} dt = 2\zeta'_{\mathcal{R}}(-1) + \frac{1}{2} \ln \sqrt{2\pi} - \frac{1 + \gamma}{12}.
\]  

(49)

The remaining terms in the summation in (47) are now evaluated in piecemeal fashion.

The \( \ln(\epsilon/p) \) term in (47) can be found by differentiating the particular Watson-Kober formula [28,27]

\[
\zeta_R(2s) = \sum_{p=1}^{\infty} \left( \frac{1}{p^{2s}} - \frac{1}{e^{2s}} \right) + \frac{2\pi^{s}m^{(1-2s)/2}}{\Gamma(s)} \sum_{n=1}^{\infty} n^{(2s-1)/2} K_{(2s-1)/2}(2\pi nm) - \frac{m^{-2s}}{2} + \frac{m^{-2s}}{2} \Gamma(s-1/2) \sqrt{\pi} \]

valid for \( \text{Re} \ s > -1/2 \), in particular near \( s = 0 \). We find

\[
\sum_{p=1}^{\infty} \left( \ln \epsilon - \ln p \right) = \frac{\pi m}{2} - \ln \sqrt{2\pi} - \frac{1}{2} \ln m + \frac{1}{2} \ln \left(1 - e^{-2\pi m}\right). \quad (51)
\]

Alternatively, the asymptotic part only of this follows by differentiating

\[
\sum_{p=1}^{\infty} \left( \frac{1}{p^{2s}} - \frac{1}{e^{2s}} \right) = \zeta_R(2s) - \zeta_1(s) + \frac{1}{2m^{2s}}
\]
and using (37). The same goes for (48).

To treat the \((\epsilon - p)\) term, twice the following term, i.e. \(m^2/2p^2\), could be replaced by \(m^2/2\epsilon^2\), plus a known correction, using (48), so allowing the earlier result (35) to be applied. However it is preferred here to use directly Watson’s general formula,

\[
\sum_{p=1}^{\infty} \left( \frac{1}{(p^2 + m^2)^s} - \sum_{n=0}^{M-1} \left( \frac{-s}{n} \right) \frac{m^{2n}}{p^{2s+2n}} \right) = \frac{\sqrt{\pi} \Gamma(s-1/2) m^{1-2s}}{2 \Gamma(s)} - \frac{1}{2} m^{-2s} \\
+ \frac{2\pi^s}{\Gamma(s)} m^{(1-2s)/2} \sum_{p=1}^{\infty} p^{(2s-1)/2} K_{(2s-1)/2}(2\pi mp) (52)
\]

which is valid for \(\text{Re } s > 1/2 - M\) and plays an important role in the sequel. If \(M\) is set equal to 1, one gets (50) and the \(s \to 1/2\) limit then yields (48). We now need to take \(M = 2\). This gives

\[
\sum_{p=1}^{\infty} \left( \frac{1}{\epsilon^{2s}} - \frac{1}{p^{2s}} + s \frac{m^2}{p^{2s+2}} \right) = \frac{\sqrt{\pi} \Gamma(s-1/2) m^{1-2s}}{2 \Gamma(s)} - \frac{1}{2} m^{-2s} \\
+ \frac{2\pi^s}{\Gamma(s)} m^{(1-2s)/2} \sum_{p=1}^{\infty} p^{(2s-1)/2} K_{(2s-1)/2}(2\pi mp) (53)
\]

valid for \(\text{Re } s > -3/2\). In particular at \(s = -1/2\)

\[
\sum_{p=1}^{\infty} (\epsilon - p - \frac{m^2}{2p}) = \frac{m^2}{4} (2 \ln 2 + 1) - \frac{1}{2} \gamma m^2 - \frac{1}{2} m - \frac{1}{4} m^2 \ln m^2 - \frac{m}{\pi} \sum_{p=1}^{\infty} \frac{K_1(2\pi mp)}{p} + \frac{1}{12} (54)
\]

which, in conjunction with (48), provides an alternative derivation of (35) and (38).

Incidently, the integral representation

\[
\Gamma(1/2 - \nu)K_\nu(z) = \sqrt{\pi}(z/2)^{-\nu} \int_1^{\infty} e^{-zt} (t^2 - 1)^{-\nu-1/2} dt (55)
\]

allows the summation to be performed,

\[
-\frac{m}{\pi} \sum_{p=1}^{\infty} \frac{K_1(2\pi mp)}{p} = -2m^2 \int_1^{\infty} \frac{1}{e^{2\pi mt} - 1} (t^2 - 1)^{1/2} dt \\
= \frac{m}{\pi} \int_1^{\infty} \ln \left(1 - e^{-2\pi mt}\right) \frac{t}{(t^2 - 1)^{1/2}} dt. (56)
\]
Equation (54) then agrees with a formula derived by Elizalde, Leseduarte and Zerbini [29] in a different connection by a longer method.

Harking back to (47), all terms have now been dealt with except for

$$\sum_{p=1}^{\infty} \left( p \ln \frac{2p}{p+\epsilon} + \frac{m^2}{4p} \right).$$

If this expression is differentiated, (48) and (54) employed and the result integrated we find,

$$\sum_{p=1}^{\infty} \left( p \ln \frac{2p}{p+\epsilon} + \frac{m^2}{4p} \right) = \frac{1}{8} (m^2 + \frac{1}{3}) \ln m^2 + \frac{1}{4} m^2 (\gamma - \ln 2)$$

$$+ \frac{m}{\pi} \sum_{p} K_1(2\pi mp) + \frac{1}{2\pi^2} \sum_{p} K_0(2\pi mp) \frac{1}{p^2} - \frac{1}{12} \ln 2 - \zeta_R'(1).$$

(57)

where the constant of integration has been found by setting $m$ to zero. This is allowed because (57) is exact. From (54)

$$\lim_{m \to 0} m \frac{1}{\pi} \sum_{p=1}^{\infty} \frac{K_1(2\pi mp)}{p} = \frac{1}{12} \zeta_R'(1)$$

(58)

which can be derived from (56) and also by replacing $K_1(z)$ by its leading term, $\sim 1/z$, as $z \to 0$.

Replacing $K_0(z)$ by its leading term as $z \to 0$, \textit{i.e.} $-\ln(z/2) - \gamma$, leads to

$$\lim_{m \to 0} \frac{1}{2\pi^2} \sum_{p=1}^{\infty} \frac{K_0(2\pi mp)}{p^2} = -\frac{1}{12} (\ln(\pi m) + \gamma) + \frac{1}{2\pi^2} \zeta_R'(1)$$

(59)

$$= \zeta_R'(-1) + \frac{1}{12} (\ln 2 - 1 - \ln m)$$

where we have used

$$\frac{1}{2\pi^2} \zeta_R'(1) = \zeta_R'(-1) + \frac{1}{12} (\ln 2 + \gamma - 1).$$

These results enable equation (57) to be completed.

As an aside, a useful expression arises from the Fourier transform representation

$$K_\nu(az) = \pi^{-1/2}(2z/a)^\nu \Gamma(\nu + 1/2) \int_0^{\infty} \frac{\cos at}{(t^2 + z^2)^{\nu+1/2}} dt.$$ 

(60)
Thus
\[ \sum_p \frac{K_0(2\pi mp)}{p^2} = \int_0^\infty \sum_1^\infty \frac{\cos pt}{p^2} \frac{dt}{(t^2 + 4\pi^2m^2)^{1/2}}. \]

The sum over \( p \) gives a Bernoulli polynomial in the range \( 0 \leq t \leq 2\pi \) and splitting up the integral into \( 2\pi \) lengths we find, after rescaling,
\[ \frac{1}{\pi^2} \sum_{p=1}^\infty \frac{K_0(2\pi mp)}{p^2} = \int_{-1}^1 B_2(t) \sum_{n=1,3,}^\infty \frac{dt}{(t+n)^2 + 4m^2)^{1/2}} \]
where
\[ B_2(t) = \frac{3t^2 - 1}{12}. \]

The integrals can be performed, leading to the summation
\[ \frac{1}{\pi^2} \sum_{p=1}^\infty \frac{K_0(2\pi mp)}{p^2} = \frac{1}{24} \sum_{n=1,3,}^\infty \left[ 2(3n^2 - 1) \ln \left( \frac{(n+1)^2 + 4m^2)^{1/2} + n - 1}{(n-1)^2 + 4m^2)^{1/2} + n - 1} \right] \\
- 3((3n - 1)\sqrt((n+1)^2 + 4m^2) - (3n + 1)\sqrt((n-1)^2 + 4m^2)) \right], \]
which is convenient numerically as a means of checking the foregoing expansions.

These results can be generalised to sums of the form \( \sum_p K_0(zp)/p^{2n} \). We give only the formula
\[ \lim_{m \to 0} \sum_{p=1}^\infty \frac{K_0(2\pi mp)}{p^{2n}} = \frac{(-1)^{n+1}2^{2n-1}\pi^{2n}}{(2n)!} \left( B_{2n} \left( \ln 2 - \ln m + \gamma - \psi(2n) \right) + 2n\zeta'_R(1 - 2n) \right) \]
and also note that
\[ \lim_{m \to 0} \frac{m^l}{\pi^n} \sum_{p=1}^\infty \frac{K_l(2\pi mp)}{p^n} = 2^{n+l-2} \frac{(l-1)!}{(n+l)!} B_{n+l} \]
when \( l + n \) is even.

Incidentally, taking the \( m \to 0 \) limit of (62) and comparing with (59) gives an expression for \( \zeta'_R(-1) \)
\[ \zeta'_R(-1) = \frac{1}{24} \sum_{n=3,5,}^\infty \left( (3n^2 - 1) \ln \left( \frac{n+1}{n-1} \right) - 6n \right) - \frac{1}{6} \]
\[ = \frac{1}{6} \sum_{k=3,5,}^\infty \frac{k - 1}{k(k + 2)} \left( (1 - 2^{-k})\zeta_R(k) - 1 \right) - \frac{1}{6}. \]
Numerically, from (65) we find \( \zeta'_R(-1) \approx -0.165421143700045092139 \). A summation for \( \zeta'_R(-1) \) is given by Elizalde and Romeo [30]. Similar summations can be found for \( \zeta'_R(1 - 2n) \).

All parts of (47) have now been evaluated. Collecting the constant terms we arrive at

\[
\zeta'_{2\text{ball}}(0, 0) = 2\zeta'_R(-1) + \frac{1}{6} \ln 2 + \frac{5}{12} - \frac{1}{2} \ln \pi \approx 0.773714,
\]

in agreement with Weisberger [31].

As a tactical point we note that the equations we have developed contain far more information than needed for the specific purpose of picking out the \( m \)-independent part and so, from this point of view, the method is not very economical. A more streamlined version is presented in section 8 where only the essentials are retained.

6. The Robin case.

In this case the appropriate expansion is (27) and we firstly need \( \sum 1/\alpha^2 \). It is straightforward to derive,

\[
\sum_{\alpha} \frac{1}{\alpha^2} = \frac{1}{4(p+1)} \left( 1 + \frac{2}{\beta + p} \right)
\]

\[
= \frac{1}{8}, \quad \beta = 0, \quad p = 0.
\]

We remark that Lamb’s formula, [32]

\[
\sum_{\alpha} \frac{1}{\alpha^2 + \beta^2 - p^2} = \frac{1}{2(\beta + p)}, \quad \beta > 0,
\]

is not enough to prove results like (67) but does provide a test when \( \beta = p \). A rough, numerical check of (67), and similar formulae, can be obtained in the \( p = 0 \) case by using the roots given in Table III of Carslaw and Jaeger [2].

Instead of (45) we have

\[
\Delta(-m^2) = \prod_{p=0}^{\infty} \left( \frac{p! 2^p m^{-p}}{\beta + p} F_p \right) \exp \left( -\frac{m^2}{4(1 + p)} \left( 1 + \frac{2}{\beta + p} \right) \right)
\]
and so

\[ \ln \Delta(-m^2) \sim \sum_{p=0}^{\infty} \left( p \ln 2 + \ln p! - \ln \sqrt{2\pi} - \ln(\beta + p) + \epsilon - \ln(p + \epsilon) \right. \\
\left. - \ln \sqrt{\epsilon} + \sum_n \frac{S_n}{p^n} - \frac{m^2}{4(1 + p)}(1 + \frac{2}{\beta + p}) \right). \]

Corresponding to (47) we find (\(\beta \neq 0\))

\[ \ln \Delta(-m^2) \sim -\ln \sqrt{(2\pi)} - \ln \beta + m - \frac{1}{2} \ln m + \sum_{p=0, n=2} N_p^{(2)} \frac{R_n(t)}{e^n} \\
+ \frac{1}{12} \sum_{p=0} \left( \frac{12 \beta - 1}{e} - \frac{7m^2}{2e^3} \right) + 2 \sum_{p=1} \left[ p \ln \frac{2p}{p + \epsilon} + \epsilon - p \right. \\
\left. - \frac{m^2}{4p} \left( 1 + \frac{2}{\beta + p - 1} \right) + \frac{1}{2} \ln \frac{\epsilon}{p} - \ln(1 + \beta/p) \right. \\
\left. + \int_0^\infty \left( \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-tp}}{t} \, dt \right] \]  

which can be reorganised in the same way that (47) was.

The only sum not covered by previously given expressions is

\[ \sum_{p=1}^{\infty} \left( \frac{\beta}{\epsilon} - \ln \left( 1 + \frac{\beta}{p} \right) \right) \]

which, using (48), can be converted to

\[ \sum_{p=1}^{\infty} \left( \frac{\beta}{p} - \ln \left( 1 + \frac{\beta}{p} \right) \right) = \ln \Gamma(1 + \beta) + \gamma \beta. \]

Putting all the pieces together and selecting the \(m\)-independent part yields, if \(\beta \neq 0\),

\[ \zeta_R^\prime(0, 0) = 2\zeta_R^\prime(-1) - \frac{7}{12} - \frac{5}{6} \ln 2 - \frac{1}{2} \ln \pi + 2 \ln \Gamma(1 + \beta) + 2\beta \ln 2 - \ln \beta. \]

The case when \(\beta\) is zero must be treated separately because of the appearance of extra zero modes. The \(p = 0\) term contributes an additional \(\ln 2\), which is clear from the fact that \(J_0(z) = -zJ_1(z) \sim z^2/2\) and the result is the Neumann determinant,

\[ \zeta_N^\prime(0, 0) = 2\zeta_R^\prime(-1) - \frac{7}{12} + \frac{1}{6} \ln 2 - \frac{1}{2} \ln \pi. \]
7. The four-ball.

We will pursue the calculation of the functional determinant in detail only for $B^4$. In this case the order $\mu$ is two and equation (41) reads

$$
\Delta(-m^2) = \prod_{p,\alpha_p} \left( 1 + \frac{m^2}{\alpha_p^2} \right) \exp \left( -\frac{m^2}{\alpha_p^2} + \frac{m^4}{2\alpha_p^4} \right) 
= \prod_p \left( p!2^p m^{-p} I_p \right) \exp \left( -\frac{m^2}{4(1+p)} + \frac{m^4}{32(1+p)^2(2+p)} \right)
$$

and (46) is

$$
\ln \Delta(-m^2) \sim \sum_{p=1}^{\infty} p^2 \left( p \ln 2 + \ln p! - \ln \sqrt{2\pi} + \epsilon - p \ln(p+\epsilon) \right)
\left( -\frac{m^2}{4(1+p)} + \frac{m^4}{32(1+p)^2(2+p)} \right).
$$

The situation now is that more terms in the summation over the polynomials $T_n$ become relevant.

(47) is

$$
\ln \Delta(-m^2) \sim \sum_{p=1, n=4}^{\infty} p^2 \frac{T_n(t)}{e^n} + \sum_{p=1}^{\infty} \left( \frac{5m^2}{24\epsilon} - \frac{5m^4}{24\epsilon^3} + \frac{9m^4}{16\epsilon^4} - \frac{5m^6}{16\epsilon^6} - \frac{m^2}{4\epsilon^2} \right)
+ \frac{1}{360} \left( \frac{1}{\epsilon} - \frac{1}{p} \right) + \frac{181m^2}{720\epsilon^3} - \frac{1349m^4}{960\epsilon^5} + \frac{2431m^6}{1152\epsilon^7} - \frac{1105m^8}{1152\epsilon^9} 
+ \sum_{p=1}^{\infty} p^2 \left[ p \ln \frac{2p}{p+\epsilon} + \epsilon - p \frac{m^2}{4(1+p)} + \frac{m^4}{32(1+p)^2(2+p)} \right] 
- \frac{1}{2} \ln \frac{\epsilon}{p} + \frac{1}{12} \left( \frac{1}{p} - \frac{1}{\epsilon} \right) 
+ \int_0^{\infty} \left( \frac{1}{2} - \frac{1}{t} \right) \frac{t^3}{120} + \frac{t}{e^t-1} \frac{e^{-tp}}{t} dt.
$$

Rearranging the summations of the Rayleigh terms for future use,

$$
\sum_{p=1}^{\infty} p^2 \left( -\frac{m^2}{4(1+p)} + \frac{m^4}{32(1+p)^2(2+p)} \right)
= \frac{1}{4} \sum_{p=1}^{\infty} \left[ m^2 \left( 1 - p - \frac{1}{p(1+p)} \right) + \frac{1}{8} m^4 \left( \frac{1}{p} - \frac{3}{p(1+p)} + \frac{1}{p^2(1+p)} \right) \right] 
= \frac{1}{4} \sum_{p=1}^{\infty} \left[ m^2 \left( 1 - p - \frac{1}{p} \right) + \frac{1}{8p} m^4 \right] + \frac{m^2}{4} + m^4 \left( \frac{\pi^2}{6} - 4 \right)
$$
The convergent summations would be needed for a finite mass calculation but are irrelevant here and (75) can thus be replaced by

\[-\frac{1}{4} \sum_{p=1}^{\infty} \left[ m^2 \left( p - 1 + \frac{1}{p} \right) - \frac{1}{8} m^4 \right] + O(m^2).\]

The formulae needed to analyse (74) are extensions of those used earlier. We set $M$ equal to 3 in (52), to get

\[
\sum_{p=1}^{\infty} \left( \frac{1}{\epsilon^{2s}} - \frac{1}{p^{2s}} + s \frac{m^2}{p^{2s+2}} - \frac{s(s+1)}{2} \frac{m^4}{p^{2s+4}} \right) = \frac{\sqrt{\pi} \Gamma(s - 1/2) m^{1-2s}}{2 \Gamma(s)} - \frac{1}{2} m^{-2s} \\
+ \frac{2 \pi^s}{\Gamma(s)} m^{(1-2s)/2} \sum_{p=1}^{\infty} p^{(2s-1)/2} K^{(2s-1)/2}(2\pi mp) \\
- \zeta_R(2s) + sm^2 \zeta_R(2s + 2) - \frac{s(s+1)}{2} m^4 \zeta_R(2s + 4)
\]

and then let $s$ tend to $-3/2$, yielding

\[
\sum_{p=1}^{\infty} \left( \epsilon^3 - p^3 + \frac{3m^2}{2} p - \frac{3m^4}{8} \right) = -\zeta(-3) + \frac{1}{8} m^2 - \frac{1}{2} m^3 - \frac{3}{16} m^4 \ln m^2 - \frac{3}{8} m^4 \gamma \\
+ \frac{3}{32} m^4 (4 \ln 2 + 3) + \sum_{p=1}^{\infty} \frac{K_2(2\pi mp)}{p^2}.
\]

Another necessary equation follows upon differentiating (53) with respect to $s$ and then setting $s = -1$,

\[
\sum_{p} \left( 2 \epsilon^2 (\ln p - \ln \epsilon) + m^2 \right) = m^2 \ln m^2 - \frac{1}{2} m^2 - 2m^2 \ln \sqrt{2\pi} - \frac{2\pi}{3} m^3 \\
- \frac{m}{\pi} \sum_{p} \frac{e^{-2\pi mp}}{p^2} - \frac{1}{2\pi^2} \sum_{p} \frac{e^{-2\pi mp}}{p^3} - 2 \zeta_R'(-2).
\]

As a small check, we note that $\zeta_R'(-2) = -\zeta_R(3)/4\pi^2$ and the right hand side correctly vanishes when $m$ is zero.
The final equation we need is
\[
\sum_{p=1}^{\infty} \left( p^3 \ln \left( \frac{2p}{p+\epsilon} \right) + \frac{m^2 p}{4} - \frac{3m^4}{32p} \right) = \frac{m^4}{32} (3\gamma - 3 \ln 2 + 1) + \frac{1}{12} m^3 + \frac{7}{48} m^2 \\
- \frac{1}{2} \zeta_R(-3) \ln m^2 - \frac{11}{64} m^4 \ln m^2 - \frac{m^3}{\pi} \sum_{p=1}^{\infty} \frac{K_3(2m\pi p)}{p} \\
- \frac{3m}{4\pi^3} \sum_{p=1}^{\infty} \frac{K_1(2m\pi p)}{p^3} - \frac{3}{4\pi^4} \sum_{p=1}^{\infty} \frac{K_0(2m\pi p)}{p^4} + \frac{1}{120} \ln 2 - \zeta'_R(-3),
\]
which is obtained in a similar way to (57) and involves using (63) and (64).

The integral is again regularised
\[
\lim_{s \to 0} \int_0^{\infty} \left( \frac{1}{2} - \frac{1}{t} - \frac{t}{12} + \frac{t^3}{720} + \frac{1}{e^t - 1} \right) t^{s-1} \frac{d^2}{dt^2} \frac{1}{e^t - 1} dt.
\]
(80)
The power terms are most easily dealt with by partial integration which gives
\[
\int_0^{\infty} t^\mu t^{s-1} \frac{d^j}{dt^j} \frac{1}{e^t - 1} dt = (-1)^j \Gamma(\mu + s) \zeta_R(\mu - j + s).
\]
(81)
For the specific case here, \( j = 2 \) and the finite part of the power terms is
\[
\zeta'_R(-3) + \frac{1}{2} \zeta'_R(-2) - \frac{1}{120} \gamma + \frac{7}{360}.
\]
(82)
The remaining piece of (80) is
\[
\int_0^{\infty} \frac{1}{e^t - 1} t^{s-1} \frac{d^2}{dt^2} \frac{1}{e^t - 1} dt = \int_0^{\infty} t^{s-1} \left( \frac{2}{(e^t - 1)^4} + \frac{3}{(e^t - 1)^3} + \frac{1}{(e^t - 1)^2} \right) dt.
\]
(83)
These integrals are examples of a particular Barnes zeta function,
\[
\zeta_r(s, a) = \frac{i \Gamma(1-s)}{2\pi} \int_L \frac{e^{z(r-a)}(-z)^{s-1}}{(e^z - 1)^r} dz = \sum_{n=0}^{\infty} \left( \frac{n+r-1}{r-1} \right) \frac{1}{(a+n)^s}, \quad \text{Re } s > r.
\]
(84)
In the present case \( a = r \) and the lower limit can be adjusted so that
\[
\zeta_r(s, r) = \sum_{n=1}^{\infty} \left( \frac{n-1}{r-1} \right) \frac{1}{n^s}.
\]
Expanding the binomial coefficient as a polynomial in \( n \) using Stirling numbers

\[
\binom{n-1}{r-1} = \frac{1}{(r-1)!} \sum_{k=1}^{r} S_r^{(k)} n^{k-1}
\]
gives

\[
\zeta_r(s, r) = \frac{1}{(r-1)!} \sum_{k=1}^{r} S_r^{(k)} \zeta_R(s - k + 1)
\]
in terms of the Riemann \( \zeta \)–function.

Explicitly for \( d = 4 \) we need

\[
\begin{align*}
\zeta_2(s, 2) &= \zeta_R(s - 1) - \zeta_R(s) \\
\zeta_3(s, 3) &= \frac{1}{2} \zeta_R(s - 2) - \frac{3}{2} \zeta_R(s - 1) + \zeta_R(s) \\
\zeta_4(s, 4) &= \frac{1}{6} \zeta_R(s - 3) - \frac{2}{3} \zeta_R(s - 2) + \frac{11}{6} \zeta_R(s - 1) - \zeta_R(s).
\end{align*}
\]

Of course, one could legitimately leave the answer in terms of the Barnes \( \zeta \)–function and the multiple \( \Gamma \)–function.

The finite part of (83) is evaluated at \( s = 0 \) to give

\[
2\zeta_4'(0, 4) + 3\zeta_3'(0, 3) + \zeta_2'(0, 2) - \gamma(2\zeta_4(0, 4) + 3\zeta_3(0, 3) + \zeta_2(0, 2)) = \\
\frac{1}{3} \zeta_R'(-3) - \frac{1}{2} \zeta_R'(-2) + \frac{1}{6} \zeta_R'(-1) + \frac{1}{90} \gamma
\]
and this has to be added to (82) to get the total contribution from the integral term.

With these and the earlier equations it is possible to evaluate the summations over \( p \) in (74) and to extract the \( m \)–independent part. A point to note is that the convergent terms coming from the \( T_3 \) polynomial all contribute to this constant term using (37).

Regarding this, a minor calculational point should be remarked. Because of the \( p^2 \) factor, the sum over \( p \) can be extended to \( p = 0 \) when convenient. This means that a factor of \( N_p^{(2)} \) can be inserted without penalty so allowing the simple asymptotic form (37) to be used. A more refined approach is given in section 8.

Combining all \( m \)-independent contributions by hand we obtain

\[
\zeta_{4\text{ball}}'(0) = \frac{1}{3} \zeta_R'(-3) - \frac{1}{2} \zeta_R'(-2) + \frac{1}{6} \zeta_R'(-1) + \frac{1}{90} \ln 2 + \frac{173}{30240} \approx 0.002869,
\]
agreeing with a previous result [33].

We note that there is cancellation of the \( \zeta_R' \) terms in (82) with those from the summations, leaving just those in (86). The \( \gamma \) terms always cancel. These facts are generic as will now be shown.
8. The d-Ball.

With the experience gained in the previous sections, the calculation can be readily extended to arbitrary dimensions.

The degeneracy is expanded as usual, (for \( d > 2 \))

\[
N_p^{(d)} = \sum_{j=1}^{d-2} H_j^{(d)} p^j,
\]

which is an odd polynomial for \( d \) odd and even for \( d \) even. The coefficients are Stirling numbers.

For even balls, it suffices to discuss

\[
\ln \Delta(-m^2) \sim \sum_{p=1}^{\infty} p^{2\nu} \left( p \ln 2 + \ln p! - p \ln(\pi) \right) - \ln \sqrt{\epsilon} + \sum_{n=1}^{\infty} \frac{T_n(t)}{\epsilon^n} - \text{Ray}(m, p) \bigg). \]

\[
= \sum_{p=1}^{\infty} p^{2\nu} \left[ p \ln \frac{2p}{p+\epsilon} + \epsilon - p - \frac{1}{2} \ln \frac{\epsilon}{p} + \sum_{n=1}^{\infty} \frac{T_n(t)}{\epsilon^n} - \text{Ray}(m, p) \right. + \left. \int_0^\infty \left( \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) e^{-tp} \frac{dt}{t} \right],
\]

where Ray\((m, p)\) are the ‘Rayleigh’ terms.

It is enough to decide what terms are irrelevant for the \( m \)-independent part of the large \( m \) limit and we first note that the Rayleigh terms are so irrelevant. Thus it is not necessary to find the sums of inverse powers of the eigenvalues, interesting though these may be.

Again we go through the various contributions. The analysis is somewhat involved and fragmented, but not without interest. Much of the difficulty is bookkeeping.

First consider

\[
p^{2\nu}(\epsilon^\alpha - p^\alpha) = (\epsilon^2 - m^2)^\nu(\epsilon^\alpha - p^\alpha)
\]

\[
= \sum_{q=0}^{\nu} \left( \begin{array}{c} \nu \\ q \end{array} \right) (-1)^{\nu-q} m^{2\nu-2q}(\epsilon^{2q+\alpha} - p^{2q+\alpha}) + \sum_{q=0}^{\nu-1} \left( \begin{array}{c} \nu \\ q \end{array} \right) (-1)^{\nu-q} m^{2\nu-2q} p^{2q+\alpha}
\]

for \( \alpha = 1 \).
From (52) the relevant part of $\sum p^2 (\epsilon^{2q+1} - p^{2q+1})$ is just $-\zeta_R(-2q-1)$ and so that of $\sum p^{2\nu} (\epsilon - p)$ equals $-\zeta_R(-2\nu-1)$ since terms involving $m$ can be ignored and the second sum in (89) is ultimately irrelevant.

The next contribution is

$$-\frac{1}{2} \sum_p p^{2\nu} \ln \frac{\epsilon}{p}$$

whose relevant part, $-\zeta_R'(2\nu)/2$, is obtained by differentiating (52) and setting $s = -\nu$, after using $\epsilon^2 = p^2 + m^2$.

Now consider

$$\sum p^{2\nu+1} \ln \frac{2p}{p+\epsilon}$$

which is the hardest sum to evaluate. Our method involves differentiation with respect to $m^2$, use of the Watson-Kober formula (52) and then a back integration, the relevant term emerging as a constant of integration.

It is clear from the examples in previous sections that only the Bessel functions in (52) ultimately contribute to the relevant term. However we write out (52) in general form, setting $M = q + 2$ and $s = -(2q+1)/2$ $(q \in \mathbb{Z})$ to give

\[
\sum_{p=1}^{\infty} \left[ (\epsilon^{2q+1} - p^{2q+1}) - \sum_{l=1}^{q+1} \binom{q+1/2}{l} m^{2l} p^{2l+1+2q} \right] = -\zeta_R(-2q-1) \\
- \frac{1}{2} m^{2q+1} - \frac{\Gamma(q+1/2)}{2\sqrt{\pi}} \frac{(1+2q)}{(q+1)!} m^{2q+2} \left( \gamma - \ln 2 + \frac{1}{2} \sum_{k=1}^{1+q} \left( \frac{1}{2k+1} - \frac{1}{k} \right) \right) \\
- \sum_{l=1}^{q} \binom{q+1/2}{l} m^{2l} \zeta_R(2l-2q-1) + \frac{2\pi^{q-1/2} m^{1+q}}{\Gamma(-q-1/2)} \sum_{p=1}^{\infty} \frac{K_{1+q}(2\pi mp)}{p^{1+q}},
\]

only the last term of which is actually needed.

Returning to the required contribution, (90), sufficient members ($\nu$ in number) of the Taylor expansion are subtracted to render the summation finite. We denote this regularisation by $\sum^*$. Then we find, using (89),

$$\frac{d}{dm^2} \sum^* p^{2\nu+1} \ln \frac{2p}{p+\epsilon} = \frac{1}{2} \sum_{q=0}^{\nu-1} (-1)^{\nu-q} \binom{\nu+1}{q+1} m^{2\nu-2q-2} \sum_p (\epsilon^{2q+1} - p^{2q+1}) + \frac{1}{2m^2} \sum_p (\epsilon^{2\nu+1} - p^{2\nu+1}) - (-1)^\nu \frac{m^{2\nu}}{2} \sum_p \frac{1}{(\epsilon - \frac{1}{p})},$$

where $\sum^* (\epsilon^{2q+1} - p^{2q+1})$ is just the left-hand side of equation (91).
Integrating (92) we see that all terms in (91), apart possibly from the Bessel function ones, will produce irrelevant contributions and so the problem is reduced to evaluating the massless limit of the summations

$$\sum_{p=1}^{\infty} \int m^{2\nu-q} \frac{K_{1+q}(2\pi mp)}{p^{1+q}} \, dm$$

(93)

where $q$ ranges from $-1$ to $\nu$.

The integrations can be performed using a recursion given by Watson [10] which reduces the $2\nu$ exponent to zero in steps of 2 finally allowing use of the integral

$$\int x^{-q} K_{1+q}(x) \, dx = -x^{-q} K_q(x).$$

The result of the recursion is given in terms of terminating hyperbolic Lommel functions [18], 7.14.1 (7),

$$\int x^{2\nu-q} K_{1+q}(x) \, dx = -2\nu x S_{2\nu-q-1,q}(x) K_{1+q}(x) - x S_{2\nu-q,q+1}(x) K_q(x).$$

(94)

Written out, the Lommel functions are

$$S_{2\nu-q-1,q}(x) = x^{2\nu-q-2} \left( 1 + \frac{4(\nu-1)(\nu-q-1)}{x^2} \right. + \frac{4^2(\nu-1)(\nu-2)(\nu-q-1)(\nu-q-2)}{x^4} + \ldots \left. \right)$$

and

$$S_{2\nu-q,q+1}(x) = x^{2\nu-q-1} \left( 1 + \frac{4\nu(\nu-q-1)}{x^2} \right. + \frac{4^2\nu(\nu-1)(\nu-q-1)(\nu-q-2)}{x^4} + \ldots \left. \right).$$

Since the massless limit has to be taken, some simplification can be made at this point. The leading term in $K_q(x)$, $q > 0$, as $x \to 0$ is $2^{q-1}(q-1)!/x^q$, while $K_0(x) \sim -(\ln(x/2) + \gamma)$. For $q = -1$ and $q = 0$ it is easily checked that the $K_0$ terms in (94) go out as $x \to 0$ and that, for $0 \leq q \leq (\nu-1)$, so does the $K_q$ term. The explicit form of the Lommel function then yields

$$\lim_{x \to 0} \int x^{2\nu-q} K_{1+q}(x) = -2^{q-1}(\nu-q-1)! \nu! \, , \quad -1 \leq q \leq \nu - 1.$$  

(95)
When \( q = \nu \), the inverse power terms in the series expansion of the Bessel functions cancel on the right-hand side of (94) leaving just the logarithm terms and positive powers which yield the limit

\[
\lim_{x \to 0} \int x^{\nu} K_{1+\nu}(x) \sim -2^{\nu}\nu! \left( \ln(x/2) + \gamma - \frac{1}{2} \sum_{k=1}^{\nu} \frac{1}{k} \right), \quad \nu > 0.
\] (96)

To regain (93) we set \( x = 2\pi mp \). As \( m \to 0 \), (96) diverges as \( \ln m \) but this will be cancelled by other, irrelevant terms that we do not need to calculate (except as a check).

Up to this log term, we find that the massless limit of the Bessel terms obtained by substituting (91) into (92) and integrating with respect to \( m^2 \) is

\[
\frac{\nu!}{2\pi^{2\nu+2}} \zeta_R(2\nu + 2) \sum_{q=-1}^{\nu-1} \frac{\pi^{1/2}}{\Gamma(-q - 1/2)} \frac{(-1)^{\nu-q}}{(q+1)! (q-\nu)}
\]

\[
+ \frac{\nu! \pi^{1/2}}{\pi^{2\nu+2} \Gamma(-\nu - 1/2)} \left( (\ln \pi + \gamma - \frac{1}{2} \sum_{k=1}^{\nu} \frac{1}{k}) \zeta_R(2\nu + 2) - \zeta'_R(2\nu + 2) \right)
\]

which, in view of the identities,

\[
(-1)^\nu \frac{\nu! 2^{2\nu+1} \sqrt{\pi}}{(2\nu + 1)! \Gamma(-\nu - 1/2)} = -1.
\]

and

\[
\frac{2^{2\nu+1} (\nu!)^2}{(2\nu + 1)!} + \sum_{q=0}^{\nu-1} \frac{2^{2\nu-2q}}{\nu - q} \binom{2q + 1}{q} \binom{2\nu + 1}{\nu} = 2 \sum_{k=1}^{2\nu+1} \frac{1}{k} - \sum_{k=1}^{\nu} \frac{1}{k},
\]

simplifies considerably to

\[
\zeta'_R(-2\nu - 1) + \frac{(-1)^\nu}{2(\nu + 1)} B_{2\nu+2} \ln 2
\]

(98)

with no fractional part. The ultimate simplicity of this result suggests the existence of an easier route.

When integrating (92), the constant of integration necessary to give a zero value for the regulated sum, \( \sum_p p^{2\nu+1} \ln (2p/(p+\epsilon)) \), in the massless limit is the negative of (98) and, finally, this is the relevant term coming from (90).

The remaining contribution in (88) is

\[
\sum_{p=1}^{\infty} p^{2\nu} \left( \sum_{n=1}^{\infty} \frac{T_n(t)}{e^n} + \int_0^\infty \left( \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-tp}}{t} dt \right). \] (99)
The first thing to note is that for \( n > 2\nu + 1 \), the sum over \( p \) of \( p^{2\nu}T_n/e^n \) converges and the result is irrelevant. This can be shown by expanding the binomial \( p^{2\nu} = (\epsilon^2 - m^2)^{2\nu} \) and noting that, according to (37), only terms like \( m^r/\epsilon^{r+1} \) sum to a relevant part in the asymptotic limit. Alternatively, the limit, [13],

\[
\sum_{p=0}^{\infty} \frac{t^{2r}}{e^n} \sim \frac{\Gamma(r + 1/2)\Gamma((n - 1)/2)}{2\Gamma(r + n/2)} m^{1-n}, \quad r > 0, \ n > 1
\]  

(100)
can be used.

If the bracket in the integrand in (99) is completely expanded and the integration performed, one gets the standard asymptotic formula,

\[
\int_0^\infty \left( \frac{1}{2} - \frac{1}{t + 1} \right) e^{-tp} dt \sim \sum_{k=1}^{\infty} (-1)^{k+1} \frac{B_{2k}}{2k(2k-1)} \frac{1}{p^{2k-1}}
\]

where the last equality comes from Olver’s relation between the \( U_n(1) \) and the coefficients in the Stirling approximation, [19] (2.15), (2.16). This shows that \( T_n(1) \) vanishes when \( n \) is even and that \( T_{2k-1}(1) = (-1)^kB_{2k}/2k(2k - 1) \).

If the \( n \) summation is restricted to the possibly relevant \( n \)-values and appropriate terms added to and subtracted from the integral, one finds for the possibly relevant part of (99)

\[
\sum_{p=1}^{\infty} p^{2\nu} \sum_{n=1}^{2\nu+1} \left( \frac{T_n(t)}{e^n} - \frac{T_n(1)}{p^n} \right) + \int_0^\infty \left( \frac{1}{2} - \frac{1}{t + 1} + \sum_{k=1}^{\nu+1} (-1)^k B_{2k} \frac{t^{2k-1}}{(2k)!} + \frac{1}{e^t - 1} \right) \frac{e^{-tp}}{t} dt.
\]

(101)
The two components of this expression are looked at in turn. Consider first

\[
\sum_{p=1}^{\infty} p^{2\nu} \sum_{n=1}^{2\nu+1} \left( \frac{T_n(t)}{e^n} - \frac{T_n(1)}{p^n} \right).
\]

(102)
The \( T_n(t) \) are polynomials in \( t \). Set

\[
T_n(t) = T_n(1) + T_n'(t).
\]

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Hence one has for (102)

$$\sum_{p=1}^{\infty} \sum_{n=1,3,} 2^{\nu+1} T_n(1)(\varepsilon^2 - m^2)^\nu \left(\frac{1}{\varepsilon} \frac{1}{p^n} \right) + \sum_{p=1}^{\infty} \sum_{n=1}^{2^{\nu+1}} \frac{1}{\varepsilon^n} (\varepsilon^2 - m^2)^\nu T_n'(t).$$

Most of the terms in the second summation can be dismissed as irrelevant – if they are divergent because of the $m$-dependence in $T'$ and if convergent because of (37). The only relevant contribution is when $n = 2\nu + 1$,

$$\sum_{p=1}^{\infty} \frac{t^{2\nu} T_{2\nu+1}(t)}{\varepsilon}.$$  

A convenient computational way of finding the asymptotic limit of this sum is to extract a factor of $(1 - t^2) = m^2/\varepsilon^2$ from $T'_n$, expand the resulting polynomial,

$$T'_n(t) = \frac{m^2}{\varepsilon^2} T''_n(t) = \frac{m^2}{\varepsilon^2} \sum_{i=0}^{\infty} a_i^{(n)} t^{2i},$$

and then use formula (100) to get

$$\sum_{p=1}^{\infty} \frac{t^{2\nu} T_{2\nu+1}(t)}{\varepsilon} = \sum_{i=0}^{\infty} a_i^{(2\nu+1)} m^2 \sum_{p=1}^{\infty} \frac{t^{2i+2\nu}}{\varepsilon^3} \sim \sum_{i=0}^{\infty} a_i^{(2\nu+1)} \frac{2i}{2i + 2\nu + 1}.$$  

In the first summation in (103) we can write, using (89),

$$\sum_{p=1}^{\infty} (\varepsilon^2 - m^2)^\nu \left(\frac{1}{\varepsilon} \frac{1}{p^n} \right) \sim \sum_{q=0}^{\nu} \left(\frac{\nu}{q}\right) (-1)^{\nu-q} m^{2\nu-2q} \sum_{p} (\varepsilon^{2q-n} - p^{2q-n}).$$

For nominally divergent terms to be relevant, there must be no factors of $m$. (Refer to (91).) This means that $\nu = q$ and then all $n$ values are divergent. These yield

$$\sum_{n=1,3,}^{2\nu+1} T_n(1) \sum_p^* (\varepsilon^{2\nu-n} - p^{2\nu-n}) \sim$$

$$T_{2\nu+1}(1) (\ln 2 - \gamma) - \sum_{k=1}^{\nu} T_{2k-1}(1) \zeta_R(2k - 2\nu - 1),$$

(107)
since the relevant part of $\sum_p^q (e^{2\nu-n} - p^{2\nu-n})$ is $-\zeta_R(n - 2\nu)$, for $n < 2\nu + 1$, according to (91), and $\ln 2 - \gamma$ if $n = 2\nu + 1$ according to (48).

For $n = 2\nu + 1$ there are also convergent terms in (106) which can be evaluated using (37) to give a relevant contribution,

$$T_{2\nu+1}(1) \sum_{q=1}^{\nu} \left( \frac{\nu}{q} \right) (-1)^q \sum_p m^{2q} \frac{e^{2q+1}}{e^{2q+1}} \sim T_{2\nu+1}(1) \sum_{q=1}^{\nu} \left( \frac{\nu}{q} \right) (-1)^q \sqrt{\pi} \Gamma(q) \frac{2\Gamma(q + 1/2)}{2\Gamma(2\nu + 1)}. \quad (108)$$

The combination of (107) and (108) is

$$- \sum_{k=1}^{\nu} \frac{(-1)^k B_{2k}}{2k(2k - 1)} \zeta_R(2k - 2\nu - 1) + T_{2\nu+1}(1) \left( \ln 2 - \gamma + \frac{1}{2} \sum_{q=1}^{\nu} \frac{(-1)^q \sqrt{\pi} \nu!}{q(\nu - q)! \Gamma(q + 1/2)} \right). \quad (109)$$

The integral in (101) remains to be evaluated. Performing the sum over $p$ and introducing a regularisation, it becomes

$$\lim_{s \to 0} \int_0^\infty \left( \frac{1}{2} - \frac{1}{t} + \sum_{k=1}^{\nu+1} (-1)^k B_{2k} \frac{t^{2k-1}}{2k(2k - 1)} + \frac{1}{e^t - 1} \right) t^{s-1} d^{2\nu} \frac{1}{dt^{2\nu}} e^t - 1 \, dt. \quad (110)$$

The power terms have been dealt with earlier in (81). The remaining part is

$$\lim_{s \to 0} \int_0^\infty \frac{1}{e^t - 1} t^{s-1} \frac{d^{2\nu}}{dt^{2\nu}} \frac{1}{e^t - 1} \, dt. \quad (111)$$

Writing

$$(-1)^j \frac{d^j}{dt^j} \frac{1}{e^t - 1} = \sum_{l=1}^{j+1} \frac{A_t^{(j)}}{(e^t - 1)^l}$$

with

$$A_t^{(j)} = lA_t^{(j-1)} + (l - 1)A_t^{(j-1)}, \quad 2 \leq l \leq j$$

$$A_t^{(j)} = j!$$

$$A_t^{(1)} = 1$$

(which is easily iterated by machine) we find for (111)

$$\lim_{s \to 0} \sum_{l=1}^{2\nu+1} A_t^{(2\nu)} \Gamma(s) \zeta_{l+1}(s, l + 1) \quad (113)$$
the finite part of which is
\[
\sum_l A_l^{(2\nu)} (\zeta'_{l+1}(0,l+1) - \gamma \zeta_{l+1}(0,l+1)).
\] (114)

Equation (85) allows the Barnes \(\zeta\)-functions to be rewritten in terms of the Riemann \(\zeta\)-function. Combining the two parts of the integral term gives
\[
\lim_{s \to 0} \left( \frac{1}{2} \Gamma(s) \zeta_R(s - 2\nu) - \Gamma(s - 1) \zeta_R(s - 2\nu - 1) - \sum_{k=1}^{\nu+1} (-1)^{k+1} \frac{B_{2k}}{(2k)!} \Gamma(s + 2k - 1) \zeta_R(s + 2k - 2\nu - 1) + \Gamma(s) \sum_{l=1}^{2\nu+1} \sum_{k=1}^{l+1} A_l^{(2\nu)} \frac{S_{l+1}^{(k)}}{l!} \zeta_R(s - k + 1) \right).
\] (115)

The cancellation of the individual divergences is a check of the analysis and implies the identity
\[
\zeta_R(-2\nu - 1) - \frac{(-1)^{\nu} B_{2\nu+2}}{(2\nu + 2)!} \Gamma(2\nu + 1) + \sum_{l=1}^{2\nu+1} \sum_{k=1}^{l+1} A_l^{(2\nu)} \frac{S_{l+1}^{(k)}}{l!} \zeta_R(-k + 1) = 0,
\] (116)

where \(k\) will actually be even.

The finite part is
\[
\frac{1}{2} \zeta_R'(-2\nu) + \zeta_R'(-2\nu - 1) - (\gamma - 1) \zeta_R(-2\nu - 1) - \gamma \frac{(-1)^{\nu} B_{2\nu+2}}{(2\nu + 2)!} \Gamma(2\nu + 1) - \sum_{k=1}^{\nu} (-1)^{k+1} \frac{B_{2k}}{(2k)!} \Gamma(2k - 1) \zeta_R(2k - 2\nu - 1) + T_{2\nu+1}(1) \psi(2\nu + 1)
\]
\[
+ \sum_{l=1}^{2\nu+1} \sum_{k=1}^{l+1} A_l^{(2\nu)} \frac{S_{l+1}^{(k)}}{l!} (\zeta_R'(-k + 1) - \gamma \zeta_R(-k + 1))
\]
which can be reduced using (116) leaving,
\[
\frac{1}{2} \zeta_R'(-2\nu) + \zeta_R'(-2\nu - 1) + \zeta_R(-2\nu - 1) + T_{2\nu+1}(1) (\psi(2\nu + 1) + 2\gamma)
\]
\[
- \sum_{k=1}^{\nu} (-1)^{k+1} \frac{B_{2k}}{(2k)!} \Gamma(2k - 1) \zeta_R(2k - 2\nu - 1) + \sum_{l=1}^{2\nu+1} \sum_{k=1}^{l+1} A_l^{(2\nu)} \frac{S_{l+1}^{(k)}}{l!} \zeta_R'(-k + 1).
\] (117)

A certain amount of simplification occurs if (117) is combined with (109). The first summation in (117) goes out and a \(\gamma\) term cancels so that the \(\psi\) can combine
with the last γ term to remove any remaining γ-dependence. Including also the relevant parts of \( \sum_p p^{2\nu} (\epsilon - p) \) and \(-\frac{1}{2}\sum_p p^{2\nu} \ln(\epsilon/p) \) obtained earlier, we find

\[
\zeta'_R(-2\nu - 1) + \sum_{k=1}^{2\nu+1} \sum_{l=1}^{l+1} A_l^{(2\nu)} \frac{S_l^{(k)}}{l!} \zeta'_R(-k + 1) \\
+ T_{2\nu+1}(1) \left( \ln 2 + \sum_{k=1}^{2\nu} \frac{1}{k} + \sum_{q=1}^{\nu} \frac{(-1)^q \sqrt{\pi \nu}}{2q(\nu - q)!\Gamma(q + 1/2)} \right). \tag{118}
\]

Added to (104) and (98), this would be the final answer for (88), but first we manipulate (118) a little. Specifically, the order of summation is changed,

\[
\sum_{l=1}^{2\nu+1} \sum_{k=1}^{l+1} A_l^{(2\nu)} \frac{S_l^{(k)}}{l!} \zeta'_R(-k + 1) \\
= \sum_{k=1}^{2\nu} M_k^{(2\nu)} \zeta'_R(-k) + \frac{1}{2\nu + 1} \zeta'_R(-2\nu - 1) = \sum_{k=1}^{2\nu+1} M_k^{(2\nu)} \zeta'_R(-k)
\]

where

\[
M_k^{(2\nu)} \equiv \sum_{l=k}^{2\nu+1} A_l^{(2\nu)} \frac{S_l^{(k+1)}}{l!}.
\]

We have used the Stirling number value \( S_r^{(r)} = 1 \). We have also used the values \( S_l^{(1)} = (-1)^l l! \), and the sum rule,

\[
\sum_{l=1}^{2\nu+1} (-1)^l A_l^{(2\nu)} = 0,
\]

(which is easily proved from the recursion relation (112)) to alter a lower summation limit from \( k = 0 \) to \( k = 1 \) in the intermediate algebra.

(118) now becomes

\[
\zeta'_R(-2\nu - 1) + \sum_{k=1}^{2\nu+1} M_k^{(2\nu)} \zeta'_R(-k) \\
+ T_{2\nu+1}(1) \left( \ln 2 + \sum_{k=1}^{2\nu} \frac{1}{k} + \sum_{q=1}^{\nu} \frac{(-1)^q \sqrt{\pi \nu}}{2q(\nu - q)!\Gamma(q + 1/2)} \right). \tag{119}
\]

The final contribution to (88) is the asymptotic form (105). Adding this to (119) and subtracting the Bessel function contribution (98), we obtain the complete
Dirichlet expression,
\[
\sum_{k=1}^{2\nu+1} M_k^{(2\nu)} \zeta_R'(-k) - \frac{(-1)^\nu B_{2\nu+2}}{2\nu+1} \ln 2 + \int_0^1 t^{2\nu} T_{2\nu+1}''(t) \, dt \\
+ T_{2\nu+1}(1) \left( \sum_{k=1}^{2\nu} \frac{1}{k} + \sum_{q=1}^\nu \frac{(-1)^q \sqrt{\pi \nu}}{2q(\nu-q)! \Gamma(q+1/2)} \right).
\]

(120)

The particular values
\[
A_j^{(j)} = \frac{1}{2} (j + 1)! , \quad S_r^{(r-1)} = -\frac{1}{2} r(r-1)
\]
are readily obtained and give
\[
M_{2\nu}^{(2\nu)} = -\frac{1}{2}.
\]
Also \(M_{2\nu+1}^{(2\nu)} = 1/(2\nu + 1)\).

The final step to finding \(\zeta'_R(0)\) on an even ball is to compound (120) with the expansion of the degeneracy (17),
\[
N_p^{(d)} = \frac{2}{(d-2)!} \sum_{\nu=1}^r S_r^{(\nu)} p^{2\nu} , \quad r = 1 + (d/2 - 2)^2.
\]

(121)

The explicit evaluation for any specific dimension \(d\) is straightforward, all but the rational part being quickly done. The integral term in (120) is the hardest to find since it involves the cumulant expansion, (28).

The examples that have been calculated agree with those exhibited by Bordag \textit{et al} [8] and in [33] for \(d = 4\). The \(d = 6\) expression is repeated here for completeness,
\[
\zeta'_R^{\text{ball}}(0) = -\frac{4027}{6486480} - \frac{1}{756} \ln 2 + \frac{1}{60} \zeta'_R(-5) \\
- \frac{1}{24} \zeta'_R(-4) + \frac{1}{24} \zeta'_R(-2) - \frac{1}{60} \zeta'_R(-1) \\
\approx -0.000392.
\]

We also give the coefficients of the \(\zeta'_R(-k)\) terms for \(d = 8\) in the form of a vector labelled by \(k\),
\[
\begin{bmatrix}
- \frac{1}{135}, - \frac{7}{144}, \frac{79}{2160}, \frac{5}{72}, - \frac{31}{648}, - \frac{1}{30}, \frac{19}{990}
\end{bmatrix}.
\]
9. Robin conditions.

The Robin case proceeds in very similar fashion. The starting point is the expression

\[
\ln \Delta (-m^2) \sim \sum_{p=1}^{\infty} p^{2\nu} \left[ p \ln \frac{2p}{p + \epsilon} + \epsilon - p + \frac{1}{2} \ln \frac{\epsilon}{p} - \ln(1 + \beta/p) + \sum_{n=1}^{\infty} \frac{R_n(\beta,t)}{\epsilon^n} \right]
\]

\[
- \text{Ray}(\beta, m, p) + \int_0^\infty \left( \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) e^{-tp} dt
\]

(122)

Again, so far as relevant terms go, the Rayleigh contribution can be ignored, as can the \( n > 2\nu + 1 \) terms in the summation.

From the formulae in section 3, it is not difficult to derive the relation

\[
\ln \left( \sum_{n=0}^{\infty} \frac{W_n(1)}{p^n} \right) = \ln (1 + \beta/p) + \ln \left( \sum_{n=0}^{\infty} \frac{T_n(1)}{p^n} \right).
\]

which implies (cf [8])

\[
R_n(\beta, 1) = T_n(1) + \frac{(-1)^{n+1}}{n} \beta^n.
\]

This allows one to add and subtract terms to rewrite the effective part of (122) as

\[
\ln \Delta (-m^2) \sim \sum_{p=1}^{\infty} p^{2\nu} \left( p \ln \frac{2p}{p + \epsilon} + \epsilon - p + \frac{1}{2} \ln \frac{\epsilon}{p} - \ln(1 + \beta/p) \right)
\]

\[
+ \sum_{p=1}^{\infty} p^{2\nu} \left[ \sum_{n=1}^{2\nu+1} \frac{R_n(\beta,t) - R_n(\beta,1)}{\epsilon^n} + \sum_{n=1}^{2\nu+1} R_n(\beta,1) \left( \frac{1}{\epsilon^n} - \frac{1}{p^n} \right) \right]
\]

\[
+ \int_0^\infty \left( \frac{1}{2} - \frac{1}{t} - \sum_{k=1}^{\nu+1} (-1)^{k+1} B_{2k} \frac{t^{2k-1}}{(2k)!} \right) + \frac{1}{e^t - 1} e^{-tp} dt
\]

(123)

The only really new term is \( \sum_{p=1}^{\infty} p^{2\nu} \ln(1 + \beta/p) \) which is easily reduced to

\[
\sum_{p=1}^{\infty} p^{2\nu} \ln(1 + \beta/p) = \gamma \beta^{2\nu + 1} - \int_0^\beta \beta^{2\nu} \psi(1 + \beta) d\beta.
\]

(124)

Most of the other terms are identical to the Dirichlet case but the cancellations
are not as complete and we find for the total expression corresponding to (120),

\[
\zeta_R'(-2\nu) + \sum_{k=1}^{2\nu+1} M_k^{(2\nu)} \zeta_R'(-k) + \frac{1}{2\nu + 1} (\beta^{2\nu+1} - (-1)^\nu B_{2\nu+1}) \ln 2
\]

\[+ \int_0^\beta \beta^{2\nu} \psi(1 + \beta) d\beta + \int_0^1 t^{2\nu} R''_{2\nu+1}(\beta, t) dt + T_{2\nu+1}(1) \sum_{k=1}^{2\nu} \frac{1}{k} - \frac{1}{4\nu} \beta^{2\nu}
\]

\[+ \sum_{k=1}^{\nu} \frac{\beta^{2k-1}}{2k-1} \zeta_R(2k - 2\nu - 1) + (T_{2\nu+1}(1) + \frac{\beta^{2\nu+1}}{2\nu + 1}) \sum_{q=1}^{\nu} \frac{(-1)^q \nu!}{2q(q - q)!\Gamma(q + 1/2)}.
\]

\[(125)\]

Again this has to be compounded with the expansion of the degeneracy.

For \(d = 6\) we find

\[
\zeta_6'(0) = -\frac{9479}{32432400} - \frac{11}{315} \beta + \frac{6517}{15120} \beta^3 - \frac{41}{1512} \beta^3 - \frac{3}{160} \beta^4 - \frac{1}{45} \beta^5
\]

\[+ \left[ -\frac{1}{756} + \frac{1}{180} \beta^3 (5 - 3\beta^2) \right] \ln 2 + \frac{1}{12} \int_0^1 \beta^2 (\beta^2 - 1) \psi(1 + \beta) d\beta
\]

\[+ \frac{1}{60} \zeta_R'(-5) + \frac{1}{24} \zeta_R'(-4) - \frac{1}{24} \zeta_R'(-2) - \frac{1}{60} \zeta_R'(-1)
\]

again agreeing with [8] after some rearrangement.

The coefficients of the \(\zeta_R'(-k)\) for \(d = 8\) are

\[
\left[ -\frac{1}{135}, \frac{7}{144}, \frac{79}{2160}, -\frac{5}{72}, -\frac{31}{648}, \frac{1}{30}, \frac{19}{990} \right].
\]

10. The massive case.

Although our main interest is with massless fields, we indicate in this section a means of computing the finite mass determinant, \(\ln D(-m^2) = -\zeta'(0, m^2)\). This will follow from the basic formula (43) which is repeated here

\[
\zeta'(0, m^2) = \zeta'(0, 0) - \ln \Delta(-m^2)
\]

\[+ FP \zeta(k, 0) \frac{z^{2k}}{k} - \sum_{k=2}^{[\nu]} \frac{C_{d/2-k}}{k} (1 + \ldots + \frac{1}{k+1}) \frac{z^{2k}}{k!}.
\]

(126)

\(\ln \Delta(-m^2)\) can be found from the properties of the Bessel functions but the finite part

\[
FP \zeta(k, 0) \equiv \zeta(k, 0) \quad (k \text{ not a pole})
\]

\[
\equiv \lim_{\epsilon \to 0} \left( \zeta(k + \epsilon, 0) - \frac{R}{\epsilon} \right) \quad (k \text{ is a pole}),
\]

(127)
is a transcendental quantity and its direct computation would involve an analytic continuation of the $\zeta$-function. However, since we know, in principle, the precise large-$m$ dependence of the other quantities in (126), $FP\zeta(k, 0)$ can simply be read off. This procedure is now illustrated for the disc, $[\mu] = 1$.

From the known asymptotic behaviour of $\zeta'(0, m^2)$ in terms of the heat-kernel coefficients, and from the $m^2$ dependence of the quantities in (47) obtained in section 5, one can confirm that the only term in $\ln (\Delta(-m^2)/D(-m^2))$ that does not cancel is proportional to $m^2$. Therefore, since $\zeta(s, 0)$ has a pole at $s = 1$ of residue $1/4$,

$$\lim_{s \to 1} \left( \zeta(s, 0) - \frac{1}{4(s - 1)} \right) = \frac{1}{2} (\gamma - 1 - \ln 2).$$  \hspace{1cm} (128)$$

On the disc, inserting the power series for $I_p$ into (45),

$$\ln \Delta(-m^2) = \sum_{p=0}^{\infty} \sum_{l=2}^{\infty} \sum_{k=1}^{\infty} \frac{N_p^{(2)} p!}{k!(p+k)!} (\frac{m}{k})^{2k} - \frac{m^2}{4(1+p)};$$  \hspace{1cm} (129)$$

with the extension to higher dimensions being obvious.

The Rayleigh terms can be found from the Taylor series (cumulant) expansion of the logarithm and, making this expansion, one obtains a power series in $m^2$ that begins with the first term not subtracted. Then, in terms of the well-known Rayleigh functions defined by

$$\sigma^{(l)}(p) = \sum_{\alpha_p} \frac{1}{\alpha_p^{2l}};$$  \hspace{1cm} (129)$$

(129) becomes

$$\ln \Delta(-m^2) = \sum_{l=2}^{\infty} \sum_{p=0}^{\infty} \sum_{k=1}^{\infty} N_p^{(2)} \sigma^{(l)}(p) m^{2l} = \sum_{l=2}^{\infty} \zeta(l, 0) m^{2l}. $$

In $d$-dimensions, the $l$-sum starts at $[d/2] + 1$.

The expressions for the Rayleigh functions allow the sum over $p$ to be performed. The first few terms are

$$\zeta'(0, m^2) = \zeta'(0, 0) - \frac{1}{2} (\gamma - 1 - \ln 2)m^2 - \frac{1}{192} (2\pi^2 - 15)m^4$$

$$- \frac{1}{3072} (35 - 6\pi^2 + 24\zeta_R(3)) m^6 + \ldots \hspace{1cm} (130)$$

$$\approx \zeta'(0, 0) + 0.557966 m^2 - 0.0246834 m^4 + 0.0020103 m^6 + \ldots$$
Of course, this series follows more directly by expanding the definition (5) but it would then be necessary to find the remainder (128) by an independent calculation.

For comparison, the large positive \( m \) expansion is

\[
\zeta'(0, m^2) \sim -\frac{1}{4} m^2 (\ln m^2 - 1) + \frac{\pi}{2} m - \frac{1}{6} \ln m^2 + \frac{\pi}{128m} + \frac{2}{315m^2} + \frac{37\pi}{214m^3} + \ldots
\]  

(131)

and a graphical analysis shows very good agreement between this and (130) for \( m^2 \) in the range 0.6 to 0.8. We should remark that Elizalde [15] has performed numerical calculations on similar expansions.

11. Comments

In this paper, which is a mainly technical one, we have presented methods for the evaluation of the functional determinants on the \( d \)-ball for Dirichlet and Robin boundary conditions applied to a massless scalar field.

The information required is the asymptotic behaviour of the eigenfunctions, the eigenvalue properties being introduced via the Mittag-Leffler theorem. The method involves extracting the mass-independent terms from a large-mass asymptotic limit.

For the disc and 4-ball, the calculation is done in more detail than required. In these cases, complete expansions are obtained which allow the necessary cancellations of the mass-dependent terms to be confirmed. The particular summations encountered in these explicit evaluations may have independent interest.

For arbitrary dimension (actually only even here) a systematic procedure has been presented and an explicit formula given for the functional determinant in terms of coefficients evaluated by recursion.

The method of [24] has been also used by Bordag, Geyer, Kirsten and Elizalde [8] recently to evaluate the functional determinants on balls directly from properties of Bessel functions in an effective and systematic fashion. There are many similarities in the intermediate expressions to our work but the details are different. They give precise results as far as \( d = 6 \) and indicate a general method.

The spin-1/2 values are available for \( d = 2, 3 \) and 4 (J.S.Apps) by conformal methods and one anticipates only technical differences in extending the present approach to this case.

The corresponding calculation on spherical caps should also be possible. (See Moss [13] and Barvinsky et al [16].)
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