A Quadruple Definite Integral Expressed in Terms of the Lerch Function

Robert Reynolds * and Allan Stauffer

Department of Mathematics and Statistics, York University, 4700 Keele Street, Toronto, ON M3J 1P3, Canada; stauffer@yorku.ca
* Correspondence: milver@my.yorku.ca

Abstract: A quadruple integral involving the logarithmic, exponential and polynomial functions is derived in terms of the Lerch function. Special cases of this integral are evaluated in terms of special functions and fundamental constants. Almost all Lerch functions have an asymmetrical zero-distribution. The majority of the results in this work are new.

Keywords: Lerch function; quadruple integral; contour integral; logarithmic function

1. Significance Statement

Quadruple definite integrals are widely used in a vast number of areas spanning mathematics and physics, from integrating over a four-dimensional volume, integrating over a Lagrangian density in field theory and four-dimensional Fourier transforms of a function of spacetime \((x, y, z, t)\).

Some interesting areas where these integrals are used are in asymptotic expansion [1], calculating the mean distance between two independent points within a circle [2], providing a classical derivation of the Compton effect [3], the radiation impedance computations of a square piston in a rigid infinite baffle [4], the acoustic radiation impedance of a rectangular panel [5], the statistical basis for the theory of stellar scintillation [6], modelling in three dimensions of a guiding center plasma within the purview of gyroelastic magneto-hydrodynamics [7], and the formulation of an axisymmetric potential problem for a plane circular electrode [8].

After perusing the current literature, the authors found many applications of quadruple integrals. In some cases these integrals were separable and in some cases asymptotic expansions were used to attain a solution. To the best of our knowledge the authors were unable to find quadruple definite integrals involving the logarithmic, exponential and polynomial functions derived in terms of a closed form solution.

In this present work we provide a formal derivation for a quadruple integral not present in the current literature. This integral features a kernel with the product of the logarithmic, exponential and polynomial functions. The log term mixes the variables so that the integral is not separable except for special values of \(k\).

In this work our goal is to expand upon the current literature of definite quadruple integrals by providing a formal derivation in terms of the Lerch function.

2. Introduction

In this paper we derive the quadruple definite integral given by

\[
\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty (t+z)^{-m}(x+y)^{m-1}e^{-p(x+z)-q(t+y)} \log k \left( \frac{a(x+y)}{t+z} \right) dxdydzdt
\]

where the parameters \(k, a, p, q\) and \(m\) are general complex numbers. This definite integral will be used to derive special cases in terms of special functions and fundamental constants.
The derivations follow the method used by us in [9]. This method involves using a form of the generalized Cauchy’s integral formula given by

\[
\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_C \frac{e^{yw}}{w^{k+1}} \, dw.
\]  

(2)

where C is in general an open contour in the complex plane where the bilinear concomitant has the same value at the end points of the contour. We then multiply both sides by a function of \(x, y, z\) and \(t\), then take a definite quadruple integral of both sides. This yields a definite integral in terms of a contour integral. Then we multiply both sides of Equation (2) by another function of \(x, y, z\) and \(t\) and take the infinite sums of both sides such that the contour integral of both equations are the same.

3. Definite Integral of the Contour Integral

We use the method in [9]. The variable of integration in the contour integral is \(a = w + m\). The cut and contour are in the second quadrant of the complex \(a\)-plane. The cut approaches the origin from the interior of the second quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using a generalization of Cauchy’s integral formula we form the quadruple integral by replacing \(y\) by \(\log \left( \frac{a(\pi)}{t+z} \right)\) and multiplying by \((t+z)^{m}(x+y)^{m-1}e^{-p(x+z)-q(t+y)}\) then taking the definite integral with respect to \(x \in [0, \infty), y \in [0, \infty), z \in [0, \infty)\) and \(t \in [0, \infty)\) to obtain

\[
\frac{1}{\Gamma(k+1)} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty (t+z)^{-m}(x+y)^{m-1}e^{-p(x+z)-q(t+y)} \log^k \left( \frac{a(\pi)}{t+z} \right) \, dx \, dy \, dz \, dt = \frac{1}{2\pi i} \int_C \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty (t+z)^{-m}(x+y)^{m-1}e^{-p(x+z)-q(t+y)} \log^k \left( \frac{a(\pi)}{t+z} \right) \, dx \, dy \, dz \, dt \, dw
\]  

from Equation (3.1.3.7) in [10] where \(0 < \text{Re}(w + m)\) and using the reflection Formula (8.334.3) in [11] for the Gamma function. We are able to switch the order of integration over \(a, x, y, z\) and \(t\) using Fubini’s theorem since the integrand is of bounded measure over the space \(\mathbb{C} \times [0, \infty) \times [0, \infty) \times [0, \infty)\).

4. The Lerch Function and Infinite Sum of the Contour Integral

In this section we use Equation (2) to derive the contour integral representations for the Lerch function.

4.1. The Lerch Function

The Lerch function has a series representation given by

\[
\Phi(z, s, v) = \sum_{n=0}^\infty (v + n)^{-s}z^n
\]  

(4)

where \(|z| < 1, v \neq 0, -1, \ldots\) and is continued analytically by its integral representation given by

\[
\Phi(z, s, v) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-vt}}{1 - ze^{-t}} \, dt = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t - z} \, dt
\]  

(5)

where \(\text{Re}(v) > 0\), and either \(|z| \leq 1, z \neq 1\), \(\text{Re}(s) > 0\), or \(z = 1, \text{Re}(s) > 1\).

4.2. Derivation of the First Contour Integral

In this section we will derive the contour integral given by

\[
\frac{1}{2\pi i} \int_C \frac{\pi a^m e^{w-k-1} \csc(\pi(m + w))}{p(p - q)^2} \, dw
\]  

(6)
Using Equation (2) and replacing \( y \) by \( \log(a) + i\pi(2y + 1) \) then multiplying both sides by \(-\frac{2i\pi e^{\pi i(m+2y+1)}}{p(p-q)^2}\) taking the infinite sum over \( y \in [0, \infty) \) and simplifying in terms of the Lerch function we obtain

\[
- \frac{(2i\pi)^{k+1}e^{\pi i}\Phi\left(2im\pi - k, \frac{\pi - i\log(a)}{\pi}\right)}{\eta(k+1)(p-q)^2} \\
- \frac{1}{2\pi i} \sum_{y=0}^{\infty} \int_{C} \frac{2i\pi e^{\pi i(m+2y+1)(m+w)}}{p(p-q)^2} dw \\
= - \frac{1}{2\pi i} \int_{C} \sum_{y=0}^{\infty} 2i\pi w^{k-1} e^{\pi i(2y+1)(m+w)} dw \\
= 2\pi i \int_{C} \pi w^{k-1} \text{csc}(\pi(m+w)) dw
\]

from Equation (1.232.3) in [11] and \( Im(w + m) > 0 \) in order for the sum to converge.

4.3. Derivation of the Second Contour Integral

In this section we will derive the contour integral given by

\[
\frac{1}{2\pi i} \int_{C} \pi a^{m} w^{k-1} \text{csc}(\pi(m+w)) dw
\]

Using Equation (2) and replacing \( y \) with \( \log(a) + i\pi(2y + 1) \) then multiplying both sides by \(-\frac{2i\pi e^{\pi i(m+2y+1)}}{q(p-q)^2}\) taking the infinite sum over \( y \in [0, \infty) \) and simplifying in terms of the Lerch function we obtain

\[
- \frac{(2i\pi)^{k+1}e^{\pi i}\Phi\left(2im\pi - k, \frac{\pi - i\log(a)}{\pi}\right)}{\eta(k+1)(p-q)^2} \\
- \frac{1}{2\pi i} \sum_{y=0}^{\infty} \int_{C} \frac{2i\pi e^{\pi i(m+2y+1)(m+w)}}{q(p-q)^2} dw \\
= - \frac{1}{2\pi i} \int_{C} \sum_{y=0}^{\infty} 2i\pi w^{k-1} e^{\pi i(2y+1)(m+w)} dw \\
= \frac{1}{2\pi i} \int_{C} \pi w^{k-1} \text{csc}(\pi(m+w)) dw
\]

from Equation (1.232.3) in [11] and \( Im(w + m) > 0 \) in order for the sum to converge.

4.4. Derivation of the Third Contour Integral

In this section we will derive the contour integral given by

\[
- \frac{1}{2\pi i} \int_{C} \pi a^{m} w^{k-1} p^{m+w-1} q^{-m-w} \text{csc}(\pi(m+w)) dw
\]

Using Equation (2) and replacing \( y \) with \( \log(a) + \log(p) - \log(q) + i\pi(2y + 1) \) then multiplying both sides by \(\frac{2i\pi p^{m-1} q^{-m} e^{\pi i(m+2y+1)}}{(p-q)^2}\) taking the infinite sum over \( y \in [0, \infty) \) and simplifying in terms of the Lerch function we obtain

\[
- \frac{(2i\pi)^{k+1}e^{\pi i}p^{m-1} q^{-m} \Phi\left(2im\pi - k, \frac{\pi - i\log(a)}{\pi}\right)}{\eta(k+1)(p-q)^2} \\
= \frac{1}{2\pi i} \sum_{y=0}^{\infty} \int_{C} \frac{2i\pi e^{\pi i(m+2y+1)(m+w)}}{(p-q)^2} dw \\
= \frac{1}{2\pi i} \int_{C} \sum_{y=0}^{\infty} 2i\pi w^{k-1} e^{\pi i(2y+1)(m+w)} dw \\
= - \frac{1}{2\pi i} \int_{C} \pi w^{k-1} p^{m+w-1} q^{-m-w} \text{csc}(\pi(m+w)) dw
\]

from Equation (1.232.3) in [11] and \( Im(w + m) > 0 \) in order for the sum to converge.
4.5. Derivation of the Fourth Contour Integral

In this section we will derive the contour integral given by

\[-}\frac{1}{2\pi i} \int_C \pi q w^{-k-1} p - m - w q^m + w^{-1} \csc(\pi(m + w)) \, dw \tag{12}\]

Using Equation (2) and replacing \(y\) by \(\log(a) - \log(p) + \log(q) + i\pi(2y + 1)\) then multiplying both sides by \(\frac{2\pi i}{(p - q)^2}\) taking the infinite sum over \(y \in [0, \infty)\) and simplifying in terms of the Lerch function we obtain

\[
\frac{(2\pi)^{k+1} e^{s^2\pi^2 p - m - q^m - 1}}{(p - q)^2} \cdot \Phi \left( \frac{2i\sin(-k, -i \log(a) + i \log(p) - i \log(q) + \pi)}{\pi \pi} \right)
\]

\[
= \frac{1}{2\pi i} \sum_{y=0}^{\infty} \int_C \frac{2i\pi^{-k-1} p - m - w q^m - 1 \exp(iw(a) - \log(p) + \log(q)) + i\pi(2y + 1)(m + w)) \, dw}{(p - q)^2}
\]

\[
= -\frac{1}{2\pi i} \int_C \frac{\pi q w^{-k-1} p - m - w q^m + w^{-1} \csc(\pi(m + w)) \, dw}{(p - q)^2}
\]

from Equation (1.232.3) in [11] and \(Im(w + m) > 0\) in order for the sum to converge.

5. Definite Integral in Terms of the Lerch Function

**Theorem 1.** For all \(k, a, p, q \in \mathbb{C}, -1 < Re(m) < 1\),

\[
\int_0^\infty \int_0^\infty \int_0^\infty (t + z)^{-m} (x + y)^{m-1} e^{-p(x+y) - q(t+z)} \log^k \left( \frac{a(x+y)}{y} \right) \, dxdydt \tag{14}
\]

\[
= \frac{1}{(p - q)^2} \left( \frac{2i\pi)^{k+1} e^{s^2\pi^2 p - m - q^m - 1} \exp(iw(a) - \log(p) + \log(q)) + i\pi(2y + 1)(m + w)) \, dw\right)
\]

\[
\frac{1}{2\pi i} \sum_{y=0}^{\infty} \int_C \frac{2i\pi^{-k-1} p - m - w q^m - 1 \exp(iw(a) - \log(p) + \log(q)) + i\pi(2y + 1)(m + w)) \, dw}{(p - q)^2}
\]

\[
= -\frac{1}{2\pi i} \int_C \frac{\pi q w^{-k-1} p - m - w q^m + w^{-1} \csc(\pi(m + w)) \, dw}{(p - q)^2}
\]

**Proof.** Since the right-hand side of Equation (3) is equal to the addition of the right-hand side of Equations (7), (9), (11) and (13) we can equate the left-hand sides and simplify the gamma function to obtain the stated result. \(\square\)

**Corollary 1.**

\[
\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-p(x+y) - q(t+z)} \log^k \left( \frac{a(x+y)}{y} \right) \, dxdydt \tag{15}
\]

\[
= \frac{1}{2\pi i} \sum_{y=0}^{\infty} \int_C \frac{2i\pi^{-k-1} p - m - w q^m - 1 \exp(iw(a) - \log(p) + \log(q)) + i\pi(2y + 1)(m + w)) \, dw}{(p - q)^2}
\]

\[
= \frac{1}{2\pi i} \sum_{y=0}^{\infty} \int_C \frac{2i\pi^{-k-1} p - m - w q^m - 1 \exp(iw(a) - \log(p) + \log(q)) + i\pi(2y + 1)(m + w)) \, dw}{(p - q)^2}
\]

**Proof.** Use Equation (14) and set \(m = 1/2\) and simplify in terms of the Hurwitz zeta function \(\zeta(s, v)\) using entry (4) below the Table in [12]. \(\square\)

**Corollary 2.**

\[
\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty (t + z)^{-m} (x + y)^{m-1} e^{-p(x+y) - q(t+z)} \, dxdydt \tag{16}
\]

\[
= -\frac{1}{2\pi i} \sum_{y=0}^{\infty} \int_C \frac{2i\pi^{-k-1} p - m - w q^m - 1 \exp(iw(a) - \log(p) + \log(q)) + i\pi(2y + 1)(m + w)) \, dw}{(p - q)^2}
\]

**Proof.** Use Equation (14) and set \(k = 0\) and simplify using entry (2) below Table in [12]. \(\square\)
Corollary 3.

\[ \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty}(t + z)^{-m}(x + y)^{m-1} \log\left(\frac{x+y}{t+z}\right)e^{-p(x+z)-q(t+y)} \, dx \, dy \, dz \, dt = \frac{1}{(1+2\pi i \nu)(p-q)^2} 4\pi e^{2\pi i \nu} p^{-m-1} e^{-\pi \cos(\pi m)} (p^m - q^m) (pq)^m \]  

(17)

Proof. Use Equation (14) and set \( k = 1 \) and simplify using entry (3) in the Table below (64:12.7) [12].

Corollary 4.

\[ \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} (1-x-y+z)^{-m}(x+y)^{m-1} \log\left(\frac{x+y}{t+z}\right)e^{-p(x+z)-q(t+y)} \, dx \, dy \, dz \, dt = \frac{1}{p^{m^{2}}q^{m^{2}}(p-q)^{2}} \pi^{k+1} (p+q) (-2\sqrt{p}\sqrt{q}\zeta(-k, \frac{1}{4}) + 2\sqrt{p}\sqrt{q}\zeta(-k, \frac{3}{4}) + p\zeta(-k, \frac{1}{4}) + q\zeta(-k, \frac{3}{4})) \]  

(18)

Proof. Use Equation (18) and form a second equation by replacing \( m \rightarrow n \) and take their difference. Using the resulting equation set \( m = 1/2, n = -1/2, a = 1 \) and simplify in terms of the Hurwitz zeta function \( \zeta(s, v) \) using entry (4) in the Table below (64:12.7) in [12].

Lemma 1.

\[ \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} e^{-\frac{1}{2}x+y-z(t-x-y+z)} \log\left(\frac{x+y}{t+z}\right) \, dx \, dy \, dz \, dt = 3i \left( 4\pi + \sqrt{2} \left( -H - \frac{i}{2} \log(2) + H - \frac{i}{2} \log(2) + 2H - \frac{i}{2} \log(2) - 2H \frac{i}{2} \log(2) \right) \right) \]  

(19)

Proof. Use Equation (18) apply l’Hôpital’s rule to the right-hand side as \( k \to -1 \) and set \( p = 1, q = 1/2 \) and simplify in terms of the Harmonic number function \( H_n \) using Equations (44:1.1) and (64:4.1) in [12].

Lemma 2.

\[ \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} e^{-2t-x-y-z} \log\left(\frac{x+y}{t+z}\right) \, dx \, dy \, dz \, dt = \pi \left( \frac{47}{16} - \frac{\sqrt{2}}{\log^2(2)} \right) \]  

(20)

and

Lemma 3.

\[ \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} e^{-2t-x-y-z} \log\left(\frac{x+y}{t+z}\right) \, dx \, dy \, dz \, dt = 0 \]  

(21)

Proof. Use Equation (15) and set \( k = -2, a = -1, p = 1, q = 2 \) and simplify by rationalizing the denominator and comparing real and imaginary parts and using Equation (9521.1) in [11]. Note the integrand in Equation (21) is highly oscillatory.

Corollary 5.

\[ \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} e^{-p(x+z)-q(t+y)} \log\left(\frac{x+y}{t+z}\right) \, dx \, dy \, dz = \frac{\pi \left( \frac{p(p^2+2pq+2q^2)}{(p-q)^2} \right)}{24pq(p-q)^2} \]  

(22)
Proof. Use Equation (15) and set $k = -2, a = -1$ and simplify using Equation (9.521.1) in [11]. □

Corollary 6.

\[
\int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-t-2x-y-2z}(t-x-y+z)}{\sqrt{x+z+y}} \, dx \, dy \, dt = \frac{1}{8} \left( \frac{3}{4} \pi + \sqrt{2} \left( - H - \frac{1}{4} \log(2) + \frac{1}{4} \log(4) + \frac{1}{4} \log(2) - 2 H - \frac{1}{4} \log(2) \right) \right)
\]

Proof. Use Equation (14) and form a second equation by replacing $m \rightarrow n$ and take their difference. Next, using the resulting equation set $k = -1, a = n = 1/2, n = -1/2, p = 2, q = 1$ and simplify using Equations (44:1:1) and (64:4:1) in [12]. □

Corollary 7.

\[
\int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-2(i+y-x-z)} \log\left(\frac{-i+y}{-i+x-z}\right)}{\sqrt{1+z+y}} \, dx \, dy \, dt = \frac{i}{2} \left( \log(8) + \sqrt{2} \left( - \log\Gamma\left(-\frac{i}{2}\right) - \Gamma\left(-\frac{i}{2}\right) \right) - \log\Gamma\left(-\frac{i}{2}\right) - \log\Gamma\left(-\frac{i}{2}\right) - 2 \log(\log(2)) \right)
\]

Proof. Use Equation (14) and set $m = 1/2$ and simplify in terms of the Hurwitz zeta function $\zeta(s, v)$ using entry (4) in the Table below (64:12:7) in [12]. Next set $a = -1, p = 1, q = 2$ and simplify. □

Example 1.

\[
\int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-t-2x-y-z}}{\sqrt{1+z+y}} \log\left(\frac{-i+y}{-i+x-z}\right) \, dx \, dy \, dt
\]

Proof. Use Equation (24) take the first partial derivative with respect to $k$ and set $k = 0$ and simplify using Equation (64:10:2) in [12]. Note the integrand is highly oscillatory. □

Lemma 4.

\[
\int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-t-2x-y-z}}{\sqrt{1+z+y}} \log\left(\frac{x+y}{x+y}\right) \, dx \, dy \, dt = \log(64) + \sqrt{2} \left( - H - \frac{1}{4} \log(2) + H + \frac{1}{4} \log(2) + \frac{1}{4} \log(2) \right)
\]

and

Lemma 5.

\[
\int_0^\infty \int_0^\infty \int_0^\infty \frac{e^{-t-2x-y-z}}{\sqrt{1+z+y}} \log^2\left(\frac{x+y}{x+y}\right) \, dx \, dy \, dt = 0
\]

Proof. Use Equation (15) apply l’Hopital’s rule as $k \rightarrow -1$ and set $a = -1, p = 1, q = 2$ and simplify by rationalizing the denominator and comparing real and imaginary parts and using Equation (9.521.1) in [11]. □
Lemma 6.

\[
\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \left( \log \left( \frac{x+y+z}{x+y} \right) - \pi^2 \right) e^{-p(x+z)-q(t+y)} \sqrt{t+z} \sqrt{x+y} (\log \left( \frac{x+y}{x+y} \right) + \pi^2)^2 \ dx \ dy \ dz \ dt = \pi \left( \frac{715}{6} - \frac{8\sqrt{6}}{\log^2 \left( \frac{1}{2} \right)} \right)
\]

(28)

and

Lemma 7.

\[
\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{\pi \log \left( \frac{x+y}{x+y} \right) e^{-p(x+z)-q(t+y)} \sqrt{t+z} \sqrt{x+y} (\log \left( \frac{x+y}{x+y} \right) + \pi^2)^2}{24(p-q)^2} \ dx \ dy \ dz \ dt = 0
\]

(29)

Proof. Use Equation (15) set \( k = -2, a = -1 \) and simplify using Equation (64:3:5) in [12]. □

Lemma 8.

\[
\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-2(t+y)-3(x+z)} \left( \log \left( \frac{x+y}{x+y} \right) - \pi^2 \right) \sqrt{t+z} \sqrt{x+y} (\log \left( \frac{x+y}{x+y} \right) + \pi^2)^2 \ dx \ dy \ dz \ dt = \frac{1}{24} \pi \left( \frac{715}{6} - \frac{8\sqrt{6}}{\log^2 \left( \frac{1}{2} \right)} \right)
\]

(30)

Proof. Use Equation (28) set \( p = 3, q = 2 \) and simplify. Note the integrand is highly oscillatory. □

Lemma 9.

\[
\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-x-y-z} \left( \log \left( \frac{x+y}{x+y} \right) - \pi^2 \right)^2 \sqrt{t+z} \sqrt{x+y} (\log \left( \frac{x+y}{x+y} \right) + \pi^2)^2 \ dx \ dy \ dz \ dt = \pi \left( \frac{47}{2} - \frac{8\sqrt{2}}{\log^2 \left( \frac{1}{2} \right)} \right)
\]

(31)

Proof. Use Equation (28) set \( p = 1, q = 1/2 \) and simplify. Note the integrand is highly oscillatory. □

6. Discussion

In the current work, the authors use their contour integration method to derive a quadruple integral based on the Lerch function that does not exist in the current literature. The formulae derived in this work use our method [9], which can be used to derive other quadruple integrals. The authors will use their method in future work to generate more multiple definite integrals. Wolfram Mathematica was used to verify numerical values of the parameters in the integral formulae.

Author Contributions: Conceptualization, R.R.; methodology, R.R.; draft preparation, R.R.; funding acquisition, A.S.; supervision, A.S. Both authors have read and agreed to the published version of the manuscript.

Funding: This research is supported by NSERC Canada under Grant 504070.

Data Availability Statement: Not applicable.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.
References
1. McClure, J.P.; Wong, R. Asymptotic expansion of a quadruple integral involving a Bessel function. *J. Comput. Appl. Math.* 1990, 33, 199–215. [CrossRef]
2. Watson, G.N. A Quadruple Integral. *Math. Gaz.* 1959, 43, 280–283. [CrossRef]
3. Raman, C.V. A classical derivation of the Compton effect. *Indian J. Phys.* 1928, 3, 357–369.
4. Lee, J.; Seo, I. Radiation impedance computations of a square piston in a rigid infinite baffle. *J. Sound Vib.* 1996, 198, 299–312. [CrossRef]
5. Davy, J.L.; Larner, D.J.; Wareing, R.R.; Pearse, J.R. The acoustic radiation impedance of a rectangular panel. *Build. Environ.* 2015, 92, 743–755. [CrossRef]
6. Chandrasekhar, S. A Statistical Basis for the Theory of Stellar Scintillation. *Mon. Not. R. Astron. Soc.* 1952, 112, 475–483. [CrossRef]
7. Kerbel, G.D. *Gyroelastic Fluids*; Lawrence Livermore Lab.: Livermore, CA, USA, 1981. [CrossRef]
8. Temme, N.M; de Bruin, R. *Quadruple Integral Equations for the Charged Disc and Coplanar Annulus*; Toegepaste Wiskunde; Stichting Mathematisch Centrum: Amsterdam, The Netherlands, 1981.
9. Reynolds, R.; Stauffer, A. A Method for Evaluating Definite Integrals in Terms of Special Functions with Examples. *Int. Math. Forum* 2020, 15, 235–244. [CrossRef]
10. Prudnikov, A.P.; Brychkov, Y.A.; Marichev, O.I. *Integrals and Series, More Special Functions*; USSR Academy of Sciences: Moscow, Russia, 1990; Volume 1.
11. Gradshteyn, I.S.; Ryzhik, I.M. *Tables of Integrals, Series and Products*, 6th ed.; Academic Press: Cambridge, MA, USA, 2000.
12. Oldham, K.B.; Myland, J.C.; Spanier, J. *An Atlas of Functions: With Equator, the Atlas Function Calculator*, 2nd ed.; Springer: New York, NY, USA, 2009.