Cascade of 3D canard doublets

E Shchepakina¹ and V Sobolev¹

¹Samara National Research University, Moskovskoe shosse 34, Samara, Russia, 443086

e-mail: shchepakina@ssau.ru

Abstract. This paper deals with a special kind of trajectories of a singularly perturbed system which includes the so-called canard doublets. We call these trajectories the cascades of canard doublets. By use of a biological model a new approach to the canard constructing in 3D is described.

1. Introduction

In this paper the trajectories of the singularly perturbed systems that contain a stable/unstable path of the slow motion are considered. An example of such objects is a canard. The term “canard” had been originally given by French mathematicians to the intermediate periodic trajectories of the van der Pol equation between the small and the large orbits due to their special shapes [1, 2]. Later the canards were investigated for other types of the singularly perturbed systems including the ones of higher dimensions, i.e. when the phase variables are vectors (see, for example, [3]–[14] and the references therein).

From the viewpoint of the geometrical theory of singular perturbations a canard may be considered as a result of gluing stable and unstable slow invariant manifolds at one point of the breakdown surface. This is possible due to the availability of an additional scalar parameter in the differential system. This approach was first proposed in [15, 16] and was then applied in [6], [17]–[21].

Other examples of the stable/unstable trajectories may be a canard cascade [22, 23] and a canard doublet [24]. Both of these objects are the results of gluing stable (attractive) and unstable (repulsive) slow invariant manifolds at several points of the breakdown surface. To make this gluing possible, several additional parameters are required.

In this paper we consider a new type of the stable/unstable trajectories that contain canard doublets separated from each other by the paths of the fast movement. We call these trajectories the cascades of canard doublets. An approach to the construction of the cascade of canard doublets are discussed via a competition model in population biology. This approach can be extended to the canards construction in 3D.

2. Canards, canard doublets, canard cascades

Consider an autonomous singularly perturbed system

\[ \frac{dx}{dt} = f(x, y, \alpha, \varepsilon), \]

(1)
\[ \frac{dy}{dt} = g(x, y, \alpha, \varepsilon), \]  

(2)

where \( x \) and \( y \) are vectors in Euclidean spaces \( \mathbb{R}^n \) and \( \mathbb{R}^m \), respectively, \( \varepsilon \) is a small positive parameter, vector-functions \( f \) and \( g \) are sufficiently smooth and their values are comparable to unity. The slow and fast subsystems are represented by (1) and (2), respectively. If we put \( \varepsilon = 0 \) into the fast subsystem we get the degenerate equation

\[ g(x, y, \alpha, 0) = 0, \]  

(3)

which describes the slow surface (or slow curve) \( S \) of the system (1), (2). The slow surface is a zero-order approximation (\( \varepsilon = 0 \)) of the slow invariant manifold \( S_\varepsilon \) of the system (1), (2). The slow invariant manifold is defined as an invariant surface of slow motions and its dimension is equal to \( n \) \([10, 23, 25]\).

Let a function \( y = \varphi(x, \alpha) \) be an isolated root of the equation (3). The subset \( S^s \) of \( S \) is stable (or attractive) if the spectrum of the Jacobian matrix

\[ J = \frac{\partial g}{\partial y}(x, \varphi(x, \alpha), \alpha, 0) \]  

(4)

is located in the left open complex half-plane. If there is at least one eigenvalue of the Jacobian matrix (4) with a positive real part then the subset of the slow surface is unstable (or repulsive). We will consider the case when only one eigenvalue of the Jacobian matrix can change its sign.

The stable and unstable parts of the slow surface are separated by a breakdown surface (curve or points) at which \( \det J = 0 \).

As noted above the slow surface can be considered as a zero-order approximation of the slow invariant manifold, hence, in an \( \varepsilon \)–neighborhood of a stable (unstable) subset of the slow surface there exists a stable (unstable) slow invariant manifold.

In the case of the scalar variables \( x \) and \( y \), the stable \( S^s \) and unstable \( S^u \) subsets of the slow curve \( S \) for which \( g_y < 0 \) and \( g_y > 0 \), respectively, are separated by the breakdown point(s) at which \( g_y = 0 \).

The presence of the additional scalar parameter \( \alpha \) provides the possibility of gluing the stable and unstable invariant manifolds at the breakdown point to form a single trajectory, a canard. Recall, that canards are trajectories of (1), (2) which at first move along the stable slow invariant manifold and then continue for a while along the unstable slow invariant manifold \([10, 23, 25]\). Trajectories which at first move along the unstable slow invariant manifold and then continue for a while along the stable slow invariant manifold are called false canards.

From a mathematical point of view, this gluing procedure means the following. The canard and the corresponding parameter value \( \alpha = \alpha^\ast(\varepsilon) \) allow for asymptotic expansions in powers of the small parameter \( \varepsilon \):

\[ y = h(x, \alpha^\ast(\varepsilon), \varepsilon) = h_0(x, \alpha_0) + \varepsilon h_1(x, \alpha_0, \alpha_1) + \varepsilon^2 h_2(x, \alpha_0, \alpha_1, \alpha_2) + O(\varepsilon^3), \]  

(5)

\[ \alpha^\ast(\varepsilon) = \alpha_0 + \varepsilon \alpha_1 + \varepsilon^2 \alpha_2 + O(\varepsilon^3). \]  

(6)

We can calculate the functions \( h_0, h_1, h_2, \) etc. from the invariance equation:

\[ \varepsilon \frac{\partial h}{\partial x} f(x, h(x, \alpha^\ast(\varepsilon), \varepsilon), \alpha^\ast(\varepsilon), \varepsilon) = g(x, h(x, \alpha^\ast(\varepsilon), \varepsilon), \alpha^\ast(\varepsilon), \varepsilon), \]

which follows from the system (1), (2) and the asymptotic expansions (5) and (6). However, all functions in (5) have a discontinuity at the breakdown point. A proper choice of \( \alpha_0, \alpha_1, \alpha_2, \) etc. provides to avoid this discontinuity. This choice of the coefficients in (6) means that we glue the
stable and unstable slow invariant manifolds at the breakdown point step by step in their zero-order approximation, first-order approximation, etc. The outcome of this procedure is a canard. The reader can find the algorithm and justifications of this procedure in [15, 16, 17, 25]. In the case of the scalar variables \(x\) and \(y\), the canards are exponentially close to each other near the slow curve and have the same asymptotic expansion (5) in powers of \(\varepsilon\). An analogous assertion is true for corresponding parameter values (6). Namely, any two values of the parameter \(\alpha\) for which canards exist have the same asymptotic expansions, and the difference between them is given by \(\exp\left(-\frac{1}{c\varepsilon}\right)\), where \(c\) is some positive number.

If it is necessary to glue stable and unstable slow invariant manifolds at several breakdown points, we need several additional parameters and as a result we obtain a one-dimensional slow invariant manifold with multiple change of stability. A canard cascade [22] and a canard doublet [23, 24, 25] are the examples of such slow invariant manifold.

A canard cascade is the continuous one-dimensional slow invariant manifold of (1), (2) which contains at least two canards or false canards. Trajectories which at first pass along an attractive part of a slow curve, then continue for a while along a repulsive part of the slow curve and after that jump in the direction of another attractive part of the slow curve, pass along this attractive part of the slow curve, then continue for a while along another repulsive part of the slow curve are called canard doublets.

The gluing procedure described above can be extended to the 3D case [6, 10] and the vector case, i.e., when \(\dim x = n\), \(\dim y = m\), see, for example, [19, 26, 27].

In the next section, a new approach to the canard chase in 3D will be proposed via a predator–prey model.

3. Two predator – one prey competition model
Consider the competition problem of two predators for a single prey in a constant and uniform environment which is described by a system of ordinary differential equations of the form [28]

\[
\dot{x}_1 = \frac{m_1 x_1 S}{a_1 + S} - d_1 x_1, \tag{7}
\]

\[
\dot{x}_2 = \frac{m_2 x_2 S}{a_2 + S} - d_2 x_2, \tag{8}
\]

\[
\dot{S} = \gamma S \left(1 - \frac{S}{K}\right) - \frac{m_1 x_1 S}{y_1 a_1 + S} - \frac{m_2 x_2 S}{y_2 a_2 + S}. \tag{9}
\]

Here \(x_i\) is the population density of the \(i\)th predator \((i = 1, 2)\); \(S\) is the population density of the prey; \(m_i > 0\) is the maximal growth or birth rate of the \(i\)th predator; \(d_i > 0\) is the death rate of the \(i\)th predator; \(y_i\) is the yield factor for the \(i\)th predator feeding on the prey; \(a_i\) is the half-saturation constant for the \(i\)th predator, i.e., the prey density at which the functional response of the predator is half maximal; \(\gamma > 0\) and \(K > 0\) are the intrinsic rate of growth of the prey and the carrying capacity of the prey, respectively. The term \((m_i/y_i)S/(a_i + S)\) is the functional response of the per capita rate at which the predator \(x_i\) captures prey \(S\) \((i = 1, 2)\).

Following [29, 30], we assume that the prey population has a fast dynamics while the predators dynamics are slow. Let us introduce new variables and parameters:

\[
\varepsilon = \frac{1}{\gamma}, \quad \beta_1 = \frac{a_1}{K}, \quad \beta_2 = \frac{a_2}{K}, \quad x = \frac{x_1}{\gamma y_1 K}, \quad y = \frac{x_2}{\gamma y_2 K}, \quad z = \frac{S}{K},
\]

where \(\varepsilon\) is the small positive parameter. The model (7)–(9) transforms to its singular perturbed form

\[
\dot{x} = x \left(\frac{m_1 z}{\beta_1 + z} - d_1\right), \tag{10}
\]
Figure 1. The slow curve of the system (13), (14). The parts $S_1^s$ and $S_2^s$ ($S_1^u$ and $S_2^u$) are stable (unstable), $A_1$ and $A_2$ are the breakdown points.

\[
\dot{y} = y \left( \frac{m_2 z}{\beta_2 + z} - d_2 \right),
\]

\[
\varepsilon \dot{z} = z \left( 1 - z - \frac{m_1 x}{\beta_1 + z} - \frac{m_2 y}{\beta_2 + z} \right).
\]

3.1. 2D canard doublets

Consider the case of the absence of one of the predators, i.e., when, for example, $y \equiv 0$. In this case the system (10)–(12) takes the form

\[
\dot{x} = x \left( \frac{m_1 z}{\beta_1 + z} - d_1 \right),
\]

\[
\varepsilon \dot{z} = z \left( 1 - z - \frac{m_1 x}{\beta_1 + z} \right).
\]

The slow curve $S$ of (13), (14) is described by the expression

\[
z \left( 1 - z - \frac{m_1 x}{\beta_1 + z} \right) = 0,
\]

and consists of the straight line $z = 0$ and the parabola. Two breakdown points,

\[
A_1 \left( x = \beta_1 \frac{1}{m_1}, z = 0 \right), \quad A_2 \left( x = \frac{(1 + \beta_1)^2}{4m_1}, z = \frac{1 - \beta_1}{2} \right),
\]

are divided $S$ into the stable subsets ($S_1^s$ and $S_2^s$) and unstable subsets ($S_1^u$ and $S_2^u$), see Figure 1. It should be noted that the straight line $z = 0$ is the exact solution of (13), (14), hence, it is the exact canard of the system.

In an $\varepsilon$–neighborhood of the stable (unstable) subset $S_2^s$ ($S_2^u$) of the slow curve there exists the stable (unstable) slow invariant manifold $S_2^{s,x}$ ($S_2^{u,x}$). We can glue together the stable and unstable slow invariant manifolds at the point $A_2$, using the standard procedure discussed above, to get a canard doublet. For this goal we consider $d_1$ as a gluing parameter and for the canard point $d_1 = d_1^c$, where

\[
d_1^c = \frac{m_1 (1 - \beta_1)}{1 + \beta_1} - \varepsilon \frac{\beta_1^2 (1 + \beta_1)}{2(1 - \beta_1)^2} + O(\varepsilon^2),
\]

at which the system (13), (14) has the canard doublet, see Figure 2.
Figure 2. The slow curve (red) and the canard doublet (blue) of the system (13), (14); $\beta_1 = 0.1$, $m_1 = 0.5$, $d_1 = 0.408498356365369948123245707727$, $\epsilon = 0.1$, $x(0) = 0.75$, $z(0) = 0.9647815074935001578968473210$.

Figure 3. The slow curve (red) and the canard doublet (blue) of the system (15), (16); $\beta_2 = 0.13$, $m_2 = 0.4$, $d_2 = 0.307288368583724272284580038651$, $\epsilon = 0.1$, $y(0) = 0.17$, $z_0 = 0.93622350340310351326816248256$.

It should be noted that we do not need to glue together the manifolds at the point $A_1$. In this special case, when the system has the exact canard, the trajectories of the system starting in the basin of attraction of the stable part of the straight line $x = 0$ will continue their movement for a while along its unstable part, see, for instance, [24].

Note that the system (10)–(12) has the competitive symmetry between the two predators, therefore the results of this section can be extended to the case $x \equiv 0$, i.e., for the system

$$\dot{y} = y \left( \frac{m_2 z}{\beta_2 + z} - d_2 \right),$$

$$\epsilon \dot{z} = z \left( 1 - z - \frac{m_2 y}{\beta_2 + z} \right).$$

Similar reasoning gives the value of the canard point $d_2^c$ of the parameter $d_2$, where

$$d_2^c = \frac{m_2(1 - \beta_2)}{1 + \beta_2} - \epsilon \frac{\beta_2^2 (1 + \beta_2)}{2(1 - \beta_2)^2} + O(\epsilon^2),$$

at which the system (15), (16) has the canard doublet, see Figure 3.

3.2. 3D canard doublets

We now return to the system (7)–(9). Substituting the canard values of the parameters $d_1$ and $d_2$ into the system (7)–(9), we get a cascade of canard doublets in 3D, which are shown in Figure 4.

We can slightly modify this cascade of canard doublets. We can transform the canard doublet to a shape of a canard without head on one plane ($xOz$ or $yOz$) keeping the canard doublet on the other plane to obtain the trajectories in 3D shown in Figures 5 and 6. If we destroy the gluing of the stable and unstable slow invariant manifolds at the point $A_2$ on one plane only,
Figure 4. The cascade of 3D canard doublets of the system (7)–(9); $\beta_1 = 0.1$, $\beta_2 = 0.13$, $m_1 = 0.5$, $m_2 = 0.4$, $d_1 = 0.408498356365369948123245707727$, $d_2 = 0.307288368583724272284580038651$, $\epsilon = 0.1$.

Figure 5. The cascade of 3D canard doublets of the system (7)–(9); $\beta_1 = 0.1$, $\beta_2 = 0.13$, $m_1 = 0.5$, $m_2 = 0.4$, $d_1 = 0.408515869461855289714590007273$, $d_2 = 0.307288368583724272284580038651$, $\epsilon = 0.1$. 
Figure 6. The cascade of 3D canard doublets of the system (7)–(9): \( \beta_1 = 0.1, \beta_2 = 0.13, m_1 = 0.5, m_2 = 0.4, d_1 = 0.408498356365369948123245707727, d_2 = 0.307310783464160722412785784682, \epsilon = 0.1 \).

Figure 7. The cascade of 3D canard doublets of the system (7)–(9): \( \beta_1 = 0.1, \beta_2 = 0.13, m_1 = 0.5, m_2 = 0.4, d_1 = 0.402439024390243902439024390244, d_2 = 0.307310783464160722412785784682, \epsilon = 0.1 \).
Figure 8. The cascade of 3D canard doublets of the system (7)–(9); $\beta_1 = 0.1, \beta_2 = 0.13, m_1 = 0.5, m_2 = 0.4, d_1 = 0.40849837869880315350610765232, d_2 = 0.305131128848346636259977194983, \epsilon = 0.1$.

keeping the canard doublet on the other plane, we obtain the trajectories in 3D shown in Figures 7 and 8.

We can destroy the gluing of the stable and unstable slow invariant manifolds at the point $A_2$ on both planes $xOz$ and $yOz$ to obtain the 3D canard shown in Figure 9. Keeping the gluing of the stable and unstable slow invariant manifolds at the point $A_2$ on both plane for initial points outside of the basin of attraction of the stable invariant manifolds $S_{1,x}$ we obtain the 3D canards similar to that shown in Figures 10.

All these trajectories describe various oscillations in the populations of predators and prey and are the object of interest for population dynamics.

It should be noted that the considered situation when the system has the exact slow invariant manifold with various stability is usual for many 3D biological models. Thus, this approach for canard chase in 3D can by applied to their investigation.

4. Conclusions
In this paper we suggested a new approach to the canard constructing in 3D. The essence of the approach is based on the canards constructing of 2D–projections of the differential system. The input of the obtained canards values for the corresponding parameters in the complete differential system gives a 3D canard. This approach was applied to the two predator – one prey competition model. The presence of the exact trivial slow invariant manifold with a change of stability made it possible to obtain a new kind of trajectories with a multiple change of stability.

5. References
[1] Diener M 1979 Nessie et Les Canards (Strasbourg: Publication IRMA)
[2] Benoit E, Callot J L, Diener F and Diener M 1981-1982 Chasse au canard Collect. Math. 31-32 37-119
[3] Benoit E and and Lobry C 1982 Les canards de R3 C.R. Acad. Sc. Paris 294 483-488
**Figure 9.** The 3D canard of the system (7)–(9); $\beta_1 = 0.1$, $\beta_2 = 0.13$, $m_1 = 0.5$, $m_2 = 0.4$, $d_1 = 0.402439024390243902439024390244$, $d_2 = 0.305131128848346636259977194983$, $\epsilon = 0.1$.

**Figure 10.** The 3D canard of the system (7)–(9); $\beta_1 = 0.1$, $\beta_2 = 0.13$, $m_1 = 0.5$, $m_2 = 0.4$, $d_1 = 0.408510257704258279454891174419$, $d_2 = 0.307301700325550614555441030367$, $\epsilon = 0.1$. 
(4) Benoit E 1983 Systèmes lents-rapides dans $R^3$ et leurs canards Société Mathématique de France, Astérisque 109-110 159-191

[5] Mishchenko E F, Kolesov Yu S, Kolesov A Yu and Rozov N Kh 1995 Asymptotic Methods in Singularly Perturbed Systems (New York: Plenum Press)

[6] Sobolev V A and Shchepakina E A 1996 Duck trajectories in a problem of combustion theory Differential Equations 32 1177-1186

[7] Szmolyan P and Wechselberger M 2001 Canards in R3 J. Diff. Eq. 177 419-453

[8] Wechselberger M 2005 Existence and bifurcation of canards in R3 in the case of a folded node SIAM J. Appl. Dyn. Syst 4(1) 101-139

[9] Xie F, Han M and Zhang W 2005 Canard Phenomena in Oscillations of a Surface Oxidation Reaction J. Nonlinear Sci. 15 363-386

[10] Shchepakina E and Sobolev V 2005 Black Swans and Canards in Laser and Combustion Models (Singular Perturbation and Hysteresis) (Philadelphia: SIAM) 207-255

[11] Xie F, Han M and Zhang W 2006 The persistence of canards in 3-D singularly perturbed systems with two fast variables Asympt. Anal. 47(1) 95-106

[12] Marino F, Marin F, Balle S and Piro O 2007 Chaotically spiking canards in an excitable system with 2D inertial fast manifolds Phys. Rev. Lett. 98 074104

[13] Desroches M, Krauskopf B and Osinga H M 2010 Numerical continuation of canard orbits in slow-fast dynamical systems Nonlinearity 23 739-765

[14] Tchizawa K 2014 On the two methods for finding 4-dimensional duck solution Applied Mathematics 5 16-24

[15] Gorelov G N and Sobolev V A 1991 Mathematical modelling of critical phenomena in thermal explosion theory Combust. Flame 87 203-210

[16] Gorelov G N and Sobolev V A 1992 Duck–trajectories in a thermal explosion problem Appl. Math. Lett. 5 3-6

[17] Sobolev V A and Shchepakina E A 1993 Self–ignition of laden medium J. Combustion, Explosion and Shock Waves 29 378-381

[18] Gol’dshhtein V, Zinoviev A, Sobolev V and Shchepakina E 1996 Criterion for thermal explosion with reactant consumption in a dusty gas Proc. London Roy. Soc. Ser. A. 452 2103-2119

[19] Shchepakina E 2003 Black swans and canards in self–ignition problem Nonlinear Analysis: Real World Applications 4 45-50

[20] Schneider K, Shchepakina E and Sobolev V 2003 A new type of travelling wave Mathematical Methods in the Applied Sciences 26 1349-1361

[21] Gorelov G N, Sobolev V A and Shchepakina E A 2006 Canards and critical behaviour in autocatalytic combustion models Journal of Engineering Mathematics 56 143-160

[22] Sobolev V 2013 Canard Cascades Discr. and Cont. Dynam. Syst. B 18 513-521

[23] Shchepakina E, Sobolev V and Mortell M P 2014 Singular Perturbations. Introduction to system order reduction methods with applications Lect. Notes in Math. 2114

[24] Pokrovskii A, Shchepakina E and Sobolev V 2008 Canard doublet in a Lotka-Volterra type model Journal of Physics: Conference Series 138 012019

[25] Shchepakina E and Sobolev V 2016 Invariant surfaces of variable stability Journal of Physics: Conference Series 727 012016

[26] Shchepakina E and Sobolev V 2001 Integral manifolds, canards and black swans Nonlinear Analysis A 44 897-908
Acknowledgment
This work was funded by RFBR and Samara Region according to the research project 16-41-630529 and the Ministry of Education and Science of the Russian Federation under the Competitiveness Enhancement Program of Samara University (2013-2020).

[27] Shchepakina E (2002) Slow integral manifolds with stability change in the case of a fast vector variable Differential Equations 38 1146-1452
[28] Hsu S-B, Hubbell S P and Waltman P 1978 Competing predators SIAM J Appl Math 35 617-625
[29] Muratori S and Rinaldi S 1989 Remarks on competitive coexistence SIAM J. Appl. Math. 49 1462-1472
[30] Liu W, Xiao D and Yi Y 2003 Relaxation oscillations in a class of predatorprey systems J. Differential Equations 188 306-331