Bose–Hubbard Models Coupled to Cavity Light Fields

A. O. Silver, M. Hohenadler, M. J. Bhaseen, and B. D. Simons

1University of Cambridge, Cavendish Laboratory, Cambridge, CB3 0HE, UK.
2OSRAM Opto Semiconductors GmbH, 93055 Regensburg, GER.
(Dated: March 5, 2010)

Recent experiments on strongly coupled cavity quantum electrodynamics present new directions in “matter–light” systems. Following on from our previous work [Phys. Rev. Lett. 102, 135301 (2009)] we investigate Bose–Hubbard models coupled to a cavity light field. We discuss the emergence of photo-exitations or “polaritons” within the Mott phase, and obtain the complete variational phase diagram. Exploiting connections to the superradiance transition in the Dicke model we discuss the nature of polariton condensation within this novel state. Incorporating the effects of carrier superfluidity, we identify a first order transition between the superradiant Mott phase and the single component atomic superfluid. The overall predictions of mean field theory are in excellent agreement with exact diagonalization and we provide details of superfluid fractions, density fluctuations, and finite size effects. We highlight connections to recent work on coupled cavity arrays.

PACS numbers: 03.75.Mn, 03.75.Hh, 67.85.-d, 05.30.Jp

I. INTRODUCTION

Over the last few years there has been tremendous activity in studying the coherent interaction of matter and radiation in a rich variety of “matter–light” systems. Recent experiments have combined cavity quantum electrodynamics (cavity QED) with cold atomic gases, and allow access to the strongly coupled regime. This has led to pioneering work on Bose–Einstein condensates (BECs) in ultra high finesse optical cavities [1], and with optical fibers on atom chips [2]. It has also stimulated advances in cavity opto-mechanics and condensate dynamics [3–4]. More recently, strong matter–light coupling has been achieved for ion crystals [5, 6], with potential applications in quantum information processing. These developments offer a wealth of possibilities, at the interface between quantum optics, cold atoms and condensed matter physics. The light field serves not only as a probe of the many-body system [7], but may also support interesting cavity mediated phenomena and phases. It may further provide routes to simulate strongly correlated quantum systems, with proposals based on coupled cavity arrays [8–22] and nonlinear optical fibers [24].

Allied advances in solid state devices include cavity QED experiments with superconducting qubits in microwave resonators [25]. This has provided clean realizations of the paradigmatic Jaynes–Cummings [26] and Dicke models [27], describing two-level systems coupled to radiation. It has also led to remarkable observations of the Lamb shift [28]. This is complemented by the quest for polariton condensates in semiconductor microcavities [29–31], where the hybridization of an exciton and a photon yields low effective mass polaritons. This offers the prospect of higher transition temperatures than for exciton BEC, and gives access to coherence properties via the cavity light field.

Motivated by this broad spectrum of activity, we examine the impact of cavity radiation on bosonic Hubbard models [32]. Following on from our previous work [33], we focus on a two-band model in which photons induce transitions between two internal states or Bloch bands. This is a natural generalization of the much studied two-level systems coupled to radiation, and may serve as a useful paradigm in other contexts. In Ref. [34] we discussed the interplay of Mott physics, photo-excitation, and Bose condensation promoted by carrier itinerancy. In particular, we provided evidence for a novel Mott phase with photo-exitations analogous to polaritons. In this work we study this problem in more detail, with emphasis on the nature of the polariton condensate. We also highlight connections to coupled cavity arrays described by the Jaynes–Cummings–Hubbard model and its variants [35–38, 22]. Additional directions in cold atoms include recent work on excitons [39], generalized Dicke models [38], and light propagation in atomic Mott insulators [33–40].

The outline of this paper is as follows. We begin in section II with an introduction to the two-band Bose–Hubbard model coupled to quantum light [36]. In section III we discuss the zero hopping limit of this model and anchor the phase diagram. We study the evolution with the strength of the matter–light coupling and highlight the connections to the Dicke model and the superradiance transition [41–43]. In section IV we use a variational approach to obtain the overall phase diagram in the presence of hopping. We corroborate our findings in section V using numerical simulations. We discuss the relation to the bosonic BEC–BCS crossover and to other problems of current interest in section VI. We conclude in section VII and provide technical appendices.

II. MODEL

Let us consider a two-band Bose–Hubbard model coupled to the quantum light field of an optical cavity in the
rotating wave approximation [36]:
\[
H_0 = \sum_{\alpha} \epsilon_{\alpha} n_{i\alpha} + \omega \psi^\dagger \psi + \sum_{\alpha \alpha'} U_{\alpha \alpha'} : n_{i\alpha} n_{i\alpha'} : \\
- \sum_{\langle ij \rangle} J_\alpha (\alpha_i^\dagger \psi + \text{h.c.}) + g \sum_i (b_i^\dagger a_i \psi + \text{h.c.}),
\]
where \( i \) labels lattice sites, and \( \alpha = a, b \) are bosons obeying canonical commutation relations \([\alpha_i, \alpha_j^\dagger] = \delta_{ij}\). These might be states of different orbital or spin angular momentum, and we assume the cavity radiation field, \( \psi \), may induce photo-excitations between these states; see Fig. 1. Here, \( \epsilon_{\alpha} \) effects the band splitting \( \omega_0 \equiv \epsilon_b - \epsilon_a \), and \( U_{\alpha \alpha'} \) are interactions, where :: indicates normal ordering. This yields: \( n_{i\alpha} : n_{i\alpha} \equiv n_{i\alpha} (n_{i\alpha} - 1) \) for like species and : \( n_{i\alpha} n_{i\alpha'} \equiv n_{i\alpha} n_{i\alpha'} \) for distinct species. \( J_\alpha \) are nearest neighbor hopping parameters, and \( \omega \) is the frequency of the cavity mode. For simplicity we consider just a single mode which couples uniformly to the bands. The coupling \( g \) is the strength of the matter–light interaction. In view of the box normalization of the photon field, the dipole coupling strength is proportional to \( 1/\sqrt{V} \), where \( V \) is the volume of the cavity. With a fixed density of lattice sites, \( \rho = N/V \), it is convenient to denote \( g = g/\sqrt{N} \), where \( N \) is the total number of lattice sites. We work in units where the half-band splitting \( \omega_0/2 = (\epsilon_b - \epsilon_a)/2 = 1 \).

An important feature of the Hamiltonian (1) is that the individual atom and photon numbers are not conserved due to the matter–light interaction. However, the total number of atomic carriers
\[
N_1 \equiv \sum_i (n_i^a + n_i^b),
\]
and the total number of photo-excitations
\[
N_2 \equiv \psi^\dagger \psi + \frac{1}{2} \sum_i (n_i^b - n_i^a + 1),
\]
are conserved and commute with \( H_0 \). The latter counts the total number of photons plus particle–hole pairs, and we refer to these composite excitations as “polaritons” — see Fig. 1. These conservation laws reflect the global \( U(1) \times U(1) \) symmetry of \( H_0 \) such that
\[
a \rightarrow e^{i\vartheta} a, \quad b \rightarrow e^{i\varphi} b, \quad \psi \rightarrow e^{-i(\vartheta - \varphi)} \psi,
\]
where \( \vartheta, \varphi \in \mathbb{R} \). In general this symmetry involves mixing between the matter–light sectors. Moreover, it also allows for the simultaneous coexistence of Mott behavior and condensation, corresponding to an unbroken \( U(1) \) and broken \( U(1) \) symmetry respectively. This symmetry will have a direct manifestation in the phase diagram, and will suggest implications for other multicomponent problems. We will work in the grand canonical ensemble:
\[
H = H_0 - \mu_1 N_1 - \mu_2 N_2.
\]

III. ZERO HOPPING LIMIT

Before embarking on a detailed examination of the model (1), we investigate the zero hopping limit. As in the single-band Bose–Hubbard model [43, 44] this anchors the topology of the general phase diagram. In the present case this is particularly informative since the zero hopping phase diagram evolves with the matter–light coupling, \( g \). In addition, the global photon mode couples all the sites, even in this zero hopping limit. To gain a handle on this reduced Hamiltonian, we find it convenient to develop two complementary approaches. These help illuminate different aspects of the more general itinerant problem, and provide a platform for extensions. In section IIIA we begin with a variational approach using a coherent state ansatz for the photons. This will enable us to derive an effective single site Jaynes–Cummings model [17] for the \( a, b \) bosons, and proceed with the minimum of technical input. In section IIIB we instead map the \( a, b \) bosons on to effective spins. This yields the paradigmatic Dicke model [41], describing many spins coupled to radiation. The Jaynes–Cummings and Dicke models are familiar in atomic physics and quantum optics and we provide a brief overview of these closely related Hamiltonians in Appendix A. Both approaches yield equivalent results and indicate a novel quantum phase transition occurring within the lowest Mott lobe [36]. This takes place from a conventional Mott state with no photons, to a state with a non-vanishing population of photo-excitations, as the strength of the matter–light coupling is increased. This quantum phase transition coincides with the well known superradiance transition in the Dicke model [41, 44], and we interpret the results in this framework. The combination of perspectives will be useful in the generalization to the itinerant problem discussed in section IV. For other proposals of Dicke models and superradiance transitions in cold atomic gases see Refs. [45–50].
A. Variational Approach

In the zero hopping and $U_{aa} \to \infty$ limit, the Hamiltonian (3) may be written in the form

$$H = \varepsilon_a \sum_i n_i^a + \varepsilon_b \sum_i n_i^b + \omega \psi^\dagger \psi + g \sum_i \left( \hat{b}^\dagger a_i \psi + \psi a_i^\dagger \hat{b} \right),$$

(6)

where the site occupancy of $a$-atoms is limited to $0, 1$. We absorb the chemical potentials in to the definitions

$$\epsilon_a \equiv \epsilon_a - \mu_1 + \mu_2/2, \quad \epsilon_b \equiv \epsilon_b - \mu_1 - \mu_2/2,$$

(7)

$$\bar{\omega} \equiv \omega - \mu_2.$$

This bosonic Hamiltonian is analogous to the fermionic problem of localized excitons in a microcavity [51, 52]. An instructive way to analyze the problem is to consider a coherent state for the cavity light field:

$$|\gamma\rangle \equiv e^{-\bar{\omega}^2/2 + \gamma \psi^\dagger} |0\rangle,$$

(8)

where $\gamma$ is a variational parameter to be determined. As usual, this is an eigenstate of the annihilation operator, $\psi$, with eigenvalue $\gamma$. The coherent state assumption is expected to become exact in the thermodynamic limit, and this is borne out by the complementary approach in section III B. The expectation value

$$\langle \gamma | H | \gamma \rangle \equiv \sum_i H_{\text{eff}},$$

(9)

yields an effective single site problem for the $a$ and $b$ atoms which is readily diagonalized:

$$H_{\text{eff}} = \epsilon_a n^a + \epsilon_b n^b + \bar{\omega} \gamma^2 + g \gamma (b^\dagger a + a^\dagger b),$$

(10)

where $\bar{\omega} \equiv \bar{\omega}/N$. This is a significant merit of the current approach, since it is directly tractable without technical input. In view of the hardcore constraint on the $a$ atoms, this describes a single two-level system coupled to an effective “radiation field” of $b$ atoms. This paradigm is described by the much studied Jaynes–Cummings model [17, 18], discussed in Appendix A. The eigenstates of (10) are superpositions in the upper and lower bands. Focusing on Mott states with total occupation $n_a + n_b = n$, we consider admixtures of $|0, n\rangle$ and $|1, n-1\rangle$ in the $|n_a, n_b\rangle$ basis. The lowest eigenstate has energy

$$E_n = \bar{\omega} \gamma^2 + n \epsilon_b - \bar{\omega}_{0} - g^2 \bar{\omega}^2/4 + 4 g^2 \gamma^2 n,$$

(11)

where $\bar{\omega}_{0} \equiv \bar{\omega} - \epsilon_a$ is the effective band splitting. The nonlinear dependence on $\sqrt{n}$ is a notable feature of the Jaynes–Cummings eigenstates and has recently been seen in circuit QED experiments [19,20]. (Analogous dependence is also seen in BECs [1, 2] as a function of the number of atoms in the cavity.) Minimizing with respect to $\gamma$, one obtains the variational self-consistency condition

$$\frac{\partial E_n}{\partial \gamma} = 2 \gamma \left( \bar{\omega} - \frac{g^2 n}{\sqrt{\bar{\omega}^2 + 4 g^2 \gamma^2 n}} \right) = 0,$$

(12)

Depending on the parameters, one may therefore obtain either the trivial solution $\gamma = 0$, corresponding to zero photon occupancy, or the non-trivial solution

$$\gamma_{\text{var}}^2 = \frac{1}{4} \left( \frac{g^2 n}{\bar{\omega}^2} - \frac{\bar{\omega}_{0}^2}{g^2 n} \right) = \langle \psi^\dagger \psi \rangle,$$

(13)

corresponding to a finite photon occupancy. This latter solution is supported in the region where $\gamma_{\text{var}}^2 > 0$, or when the matter–light coupling exceeds the critical value

$$\bar{g}_c \equiv \bar{g} \sqrt{n} = \sqrt{\bar{\omega}_{0} \bar{\omega}},$$

(14)

where for simplicity we consider $\bar{\omega}_{0} > 0$. As we shall discuss in section III B, this onset of the photon field corresponds to the well known superradiance transition in the Dicke model [31–34]. Indeed, it is readily seen from equation (13) that for $\bar{g} \geq \bar{g}_c$, the photon field has the characteristic variation

$$\frac{\langle \psi^\dagger \psi \rangle}{N} = \frac{1}{4 \bar{\omega}^2} \left( \frac{\bar{g}_1^4 - \bar{g}_4^4}{\bar{g}_2^4} \right),$$

(15)

where $\bar{g} \equiv \bar{g} \sqrt{n}$. See for example Table 1 of Ref. [54]. We shall give an alternative derivation of these results in section III B. This complementary approach becomes asymptotically exact in the thermodynamic limit and helps justify the coherent state ansatz [3].

Knowledge of the eigenvalues (11), together with the variational consistency condition (12), enables one to construct the zero hopping phase diagram depicted in Fig. 2. The lower boundary consists of two segments. The first corresponds to the transition from the vacuum to the Mott state with no photons, and is given by the condition $E_1 (\gamma = 0) \leq 0$ or $\mu_1 \geq \epsilon_a + \mu_2/2$. The second segment corresponds to the transition to the photon rich Mott state and occurs when $E_1 (\gamma_{\text{var}}) \leq 0$ where

$$E_n (\gamma_{\text{var}}) = n \epsilon_b - \bar{\omega}_{0} - \frac{n}{4 \omega} \left( \frac{\bar{g}_1^4 + \bar{g}_4^4}{\bar{g}_2^4} \right).$$

(16)

Explicitly this locus is given by

$$\mu_1 \geq \epsilon_a + \epsilon_b - \frac{1}{4 \omega} \left( \frac{\bar{g}_1^4 + \bar{\omega}^2 (\omega - \mu_2)^2}{\bar{g}_2^4} \right).$$

(17)

Noting that $\bar{g}_c$ depends on $n$ according to (13) we see that $E_n (\gamma_{\text{var}}) \to n (\epsilon_b - \bar{g}_0^2/4 \omega)$ as $n \to \infty$. For $\epsilon_b - \bar{g}_0^2/4 \omega \leq 0$, or chemical potentials satisfying

$$\mu_1 \geq \epsilon_b - \mu_2/2 - \frac{\bar{g}_0^2}{4 (\omega - \mu_2)},$$

(18)

it is energetically favorable to macroscopically populate the $b$ levels. This corresponds to the upper boundary in...
FIG. 2: (Color online). Zero hopping phase diagram in the $U_{ab} \to \infty$ limit with $U_{bb} = U_{ab} = 0$. We set $\epsilon_b = -\epsilon_a = \bar{g} = 1$ and $\omega = 1$ corresponding to $\omega < \omega_0$, where $\omega_0 \equiv \epsilon_b - \epsilon_a$. The vertical line, $\bar{g} = \bar{g}_c$, is the superradiance transition in the Dicke model, and separates a Mott insulator (MI) with $n_a + n_b = 1$ and $\langle \psi^\dagger \psi \rangle = 0$, from a superradiant Mott insulator (SRMI) with $\langle \psi^\dagger \psi \rangle \neq 0$. Outside of these regions are the vacuum state, and the unstable region corresponding to macroscopic population of the $b$ states. The total density is fixed within both Mott phases, $n_a + n_b = 1$, the individual $a$ and $b$ populations vary in the superradiant phase as shown in Fig. 3. Inset: $\epsilon_b = -\epsilon_a = \bar{g} = 1$ and $\omega = 3$. For $\omega > \omega_0$ the upper and lower boundaries may cross and terminate the lobe, since for $\mu_2 > \omega_0$ the lowest stable state is the vacuum; see Appendix B for a derivation of this.

Fig. 2. Indeed, in the absence of the matter–light coupling equation (15) yields $\bar{g}_b \leq 0$ and is naturally associated with population of the $b$ states; we provide the corresponding $g = 0$ phase diagram for comparison in Appendix B. In contrast to the classical light case (where $\psi$ is treated as a fixed c-number) coupling to the fluctuating quantum field, $\psi$, eliminates the higher Mott lobes corresponding to integer increases in the $b$-populations; the higher lobes corresponding to the $a$-particles have been explicitly eliminated by the hardcore constraint. We confirm this reduction analytically in Appendix C.

Accompanying the superradiance transition is a change in the relative atomic populations or “magnetization”

$$\mathcal{M} = \frac{1}{2} \left( \langle n_b \rangle - \langle n_a \rangle \right),$$

(19)

corresponding to photo-excitation into the upper band. Indeed, the possibility of such magnetic phase transitions is a strong motivation for studying multicomponent problems, even without matter–light coupling. The magnetization (19) is readily computed from the Jaynes–Cummings eigenstates (A0) with $n = 1$. It also serves as an order parameter for this continuous transition:

$$\mathcal{M} = \begin{cases} -\frac{1}{2}; \quad \bar{g} \leq \bar{g}_c, \\ -\frac{1}{2} \left( \frac{\bar{g}}{\bar{g}_c} \right)^2; \quad \bar{g} \geq \bar{g}_c. \end{cases}$$

(20)

This corresponds to $\langle n_a \rangle = 1$ and $\langle n_b \rangle = 0$, for $\bar{g} \leq \bar{g}_c$, and a non-trivial imbalance, $\langle n_b \rangle - \langle n_a \rangle$, for $\bar{g} > \bar{g}_c$.

B. Dicke Model

As in the fermionic cases considered elsewhere, an alternative way to view the zero hopping Hamiltonian (6) is as an effective spin-boson model. Within the subspace of fixed density, $n_a + n_b = n$, we introduce effective spins for \textit{a priori} possible Mott lobes

$$| \psi \rangle = |1, n-1\rangle, \quad | \phi \rangle = |0, n\rangle,$$

(21)

where we denote bosonic states as $|n_a, n_b\rangle$. The operators

$$S^+ = b^\dagger \frac{a}{\sqrt{n}}, \quad S^- = a^\dagger \frac{b}{\sqrt{n}}, \quad S^z = \frac{1}{2} - n_a,$$

(22)

form a representation of $su(2)$ on this restricted Hilbert space. In the lowest Mott lobe with $n = 1$ this reduces to the usual Schwinger boson construction. In the representation (22) the Hamiltonian (6) becomes

$$H = \tilde{\omega}_0 \sum_{i=1}^N S_i^z + \tilde{\omega} \psi^\dagger \psi + \bar{G} \sum_{i=1}^N (S_i^+ \psi + h.c.) + c_n,$$

(23)

where $\bar{G} \equiv g\sqrt{n}$ and $c_n = N(n\bar{e}_b - \tilde{\omega}_0)/2$. This is the much studied spin-1/2 Dicke model, describing $N$ two-level systems coupled to radiation; see Appendix A for a brief review. This model is integrable via the so-called superradiant phase when

$$\bar{G}_c \equiv \sqrt{\bar{G}_c} \sqrt{N} = \sqrt{\bar{G}_c},$$

(24)

and in the thermodynamic limit it has a quantum phase transition to a so-called superradiant phase when
in agreement with our previous result \cite{14}. In this context, the term superradiance indicates the onset of many-body or cooperative effects involving the photon coupled to many atoms. This mapping not only helps justify our variational approach, but will also provide insights into polariton condensation and matter–light coherence at the superradiance transition.

The thermodynamic limit of the Dicke model \cite{23} may be analyzed using collective spin operators \cite{52-53}:

$$J = \sum_i N S_i, \quad (25)$$

where we exploit site independence of the global photon field, and $N \equiv 2S$ plays the role of a large spin. This motivates an asymptotically exact semiclassical treatment based on the Holstein–Primakoff transformation \cite{54-56}:

$$J^+ = c^\dagger (2S - c^\dagger c)^{1/2},$$
$$J^- = (2S - c^c c)^{1/2} c,$$
$$J^z = c^c c - S,$$

where $c$ is a canonical boson. The superradiance transition is associated with condensation of this auxiliary boson. Most crucially, this is related to condensation of polaritons, and not the $a, b$ bosons themselves. The Dicke model \cite{23} becomes

$$H = \tilde{\omega}_0 \left( c^\dagger c - S \right) + \tilde{\omega} \psi^\dagger \psi + \frac{\tilde{G}}{\sqrt{N}} \left( \psi^\dagger (2S - c^\dagger c)^{1/2} c + c^\dagger (2S - c^c c)^{1/2} \psi \right), \quad (27)$$

where we drop the ‘constant’ $c_n$. Within a semiclassical $1/S$ expansion around the thermodynamic limit \cite{59} we may proceed by introducing coherent states for the photon and auxiliary boson

$$| \gamma \rangle \equiv e^{-\tilde{\omega}^2/2 + \gamma \psi^\dagger} | 0 \rangle, \quad | \zeta \rangle \equiv e^{-\tilde{\omega}^2/2 + \zeta_1^c c^c} | 0 \rangle, \quad (28)$$

where $\gamma$ and $\zeta$ are classical $c$-numbers. The energy density reads

$$\mathcal{E} \equiv (H)/N = \tilde{\omega}_0 (\tilde{\zeta}^2 - 1/2) + \tilde{\omega}^2 \gamma^2 + 2 \tilde{G} \tilde{\gamma} \zeta (1 - \tilde{\zeta}^2)^{1/2}, \quad (29)$$

where $\tilde{\gamma} \equiv \gamma/\sqrt{N}, \quad \tilde{\zeta} \equiv \zeta/\sqrt{N}$. Minimizing over $\tilde{\gamma}$ gives

$$\tilde{\gamma} = -\frac{\tilde{G} \tilde{\zeta}(1 - \tilde{\zeta}^2)^{1/2}}{\tilde{\omega}}. \quad (30)$$

Substituting back in \cite{29} yields

$$\mathcal{E} = \frac{\tilde{G}^2}{\tilde{\omega}} \tilde{\zeta}^4 + \left( \tilde{\omega}_0 - \frac{\tilde{G}^2}{\tilde{\omega}} \right) \tilde{\zeta}^2 - \frac{\tilde{\omega}_0}{2}, \quad (31)$$

and we acquire an expectation value $\langle c \rangle \neq 0$ when the quadratic term becomes negative. This corresponds to the superradiance quantum phase transition at $\tilde{G}_c = \sqrt{\tilde{\omega}_0}$ in agreement with the previous results. Minimizing \cite{31} over $\tilde{\zeta}^2$ and substituting into \cite{30} one obtains

$$\tilde{\zeta}^2 = \frac{\tilde{G}^2 - \tilde{G}_c^2}{2\tilde{G}^2}, \quad \tilde{\zeta}^2 = \frac{\tilde{G}^4 - \tilde{G}_c^4}{4\tilde{G}^2 \tilde{G}^2}; \quad \tilde{G} \geq \tilde{G}_c. \quad (32)$$

The onset of the photon field agrees with equation \cite{15}. In addition, the expectation value $\langle J^+ \rangle = \zeta (2S - \tilde{\zeta}^2)^{1/2}$ tracks the condensation of the Holstein–Primakoff boson, $c$, via its coherent state parameter $\zeta$:

$$\frac{\langle J^+ \rangle}{N} = \zeta (1 - \tilde{\zeta}^2)^{1/2} = -\frac{\tilde{\omega} \gamma}{\tilde{G}}. \quad (33)$$

These results show that the superradiance transition is accompanied by condensation of the Holstein–Primakoff boson, $\langle c \rangle \neq 0$. As we will discuss in section IV,B in the two-band Bose–Hubbard model coupled to light, this corresponds to condensation of the polaritonic bilinear $(b^\dagger a)$ above the Mott background, and not the $a, b$ bosons themselves. In addition to the onset of photons described by equations \cite{15} and \cite{32}, the magnetization $\mathcal{M} \equiv \langle J^+ \rangle/N$ reproduces the previous results.

The correspondence between the variational approach outlined in section III,A and the Dicke model analysis is clearly encouraging. In section IV we shall extend the variational approach to include the important effects of itinerancy and carrier superfluidity.

IV. VARIATIONAL APPROACH FOR HARDCORE ATOMS

A. Phase Diagram

Having confirmed a zero hopping Mott phase, with $n_a + n_b = 1$, we consider itinerancy and carrier superfluidity. Within this lowest lobe we may take hardcore $a$ and $b$ bosons. This will also be convenient for the numerical simulations in section V. Whilst this does not affect physics within the lobe, the zero hopping upper boundary is modified by the restriction on the $b$-atom population. In this case we only need retain the states $|0, 0\rangle, |1, 1\rangle$ and the admixtures of $|0, 1\rangle$ and $|1, 0\rangle$. The zero hopping diagram shown in Fig. 2 is replaced by Fig. 4. The lower boundary remains unchanged because the same eigenstates are involved, and the upper boundary becomes a mirror reflection of the lower one. This is quite natural since the $a$ and $b$ operators now appear on an equal footing, modulo the effects of the band splitting. More formally, this may be traced to the invariance of the hardcore Hamiltonian \cite{49} (up to a constant term) under the particle–hole transformation $a \rightarrow a^\dagger, b \rightarrow b^\dagger$, together with $\mu_1 \rightarrow \epsilon_a + \epsilon_b + U_{ab} - \mu_1$, and the interchange of the $a$ and $b$ operators and the hopping parameters $J_a$ and $J_b$: since hardcore bosons obey onsite anticommutation relations, $n \rightarrow 1 - n$, under particle–hole transformation. For $\epsilon_a = -1, \epsilon_b = 1$ and $U_{ab} = 0$ this involves $\mu_1 \rightarrow -\mu_1$. The particle–hole transformation also accounts for the change in the vacuum, $|0, 0\rangle \rightarrow |1, 1\rangle$. 
To incorporate itinerancy and superfluidity we augment the variational analysis for two component bosons in an optical lattice \(^{56}\) with a coherent state for light:

\[
|\gamma\rangle \otimes \prod_i \left[ \cos \theta_i (\cos \chi_i a_i^\dagger + \sin \chi_i b_i^\dagger) + \sin \theta_i (\cos \eta_i + \sin \eta_i b_i a_i^\dagger) \right]|0\rangle,
\]

where \(|\gamma\rangle\) is the coherent state introduced previously, and \(\theta, \chi, \eta, \gamma\) are to be determined. The corresponding order parameters are given by \(\langle a \rangle = \frac{1}{2} \sin 2\theta \cos (\chi - \eta)\), \(\langle b \rangle = \frac{1}{2} \sin 2\theta \sin (\chi + \eta)\) and \(\langle \psi \rangle = \gamma\). The first term in brackets in equation \((34)\) describes the Mott state, and the second superfluidity. For \(\theta = 0\) this coincides with the variational approach for localized excitons coupled to light \(^{33}\), and as we will discuss in section \(\text{IVB}^{13}\) reproduces the previous results for \(J_a = 0\). More generally, \(\text{III}^{33}\) takes real hopping into account, involving site vacancies and interspecies double occupation. It provides a useful starting point to identify the boundaries between the Mott and superfluid regions. We consider spatially uniform phases with energy density \(\mathcal{E} \equiv \langle \mathcal{V} | H | \mathcal{V} \rangle / N\):

\[
\mathcal{E} = (\tilde{\epsilon}_+ - \tilde{\epsilon}_- \cos 2\chi) \cos^2 \theta + (2\tilde{\epsilon}_+ + U_{ab}) \sin^2 \eta \sin^2 \theta - \frac{z}{4} \left[ J_a \cos^2 (\chi - \eta) + J_b \sin^2 (\chi + \eta) \right] \sin^2 2\theta + \tilde{g} \gamma^2 + \tilde{g} \gamma \cos^2 \theta \sin 2\chi, \tag{35}
\]

where \(z\) is the coordination and \(\tilde{\epsilon}_\pm \equiv (\tilde{\epsilon}_b \pm \tilde{\epsilon}_a)/2\). Minimizing on \(\gamma\) gives \(\gamma = \langle \psi \rangle / \sqrt{N} = -\tilde{g} \cos \theta \sin 2\chi / 2\tilde{\omega}\). This may be eliminated from \(\text{III}^{33}\) to yield

\[
\mathcal{E} = (\tilde{\epsilon}_+ - \tilde{\epsilon}_- \cos 2\chi) \cos^2 \theta + (2\tilde{\epsilon}_+ + U_{ab}) \sin^2 \eta \sin^2 \theta - \frac{z}{4} \left[ J_a \cos^2 (\chi - \eta) + J_b \sin^2 (\chi + \eta) \right] \sin^2 2\theta - \frac{\tilde{g}^2 \cos^4 \theta \sin^2 2\chi}{4\tilde{\omega}}, \tag{36}
\]

The symmetries of \(\text{III}^{33}\) determine the domain of minimization. It is invariant under \(\theta \rightarrow \theta + \pi\) and \(\theta \rightarrow -\theta\), and may be minimized over \(\theta \in [0, \pi/2]\). Likewise, it is invariant under \(\chi \rightarrow \chi + \pi\) and \(\eta \rightarrow \eta + \pi\). Since the hopping contribution favors \(\chi\) and \(\eta\) taking the same sign, we may also minimize \(\chi, \eta \in [0, \pi/2]\). To begin with we set \(J_a = J_b = J\) and the expression may be further reduced using \(J_a \cos^2 (\chi - \eta) + J_b \sin^2 (\chi + \eta) = J (1 + \sin 2\chi \sin 2\eta)\). Minimizing over the restricted domain yields the phase diagram in Fig. 5. For the chosen parameters, we have up to four distinct phases in the interval \(\mu_1 < 0\): (i) a Mott state with \(\langle a \rangle = \langle b \rangle = \langle \psi^\dagger \psi \rangle = 0\), (ii) a superradiant Mott state with \(\langle a \rangle = \langle b \rangle = 0\) and \(\langle \psi^\dagger \psi \rangle \neq 0\), (iii) a single component superfluid with \(\langle a \rangle \neq 0\) and \(\langle b \rangle = \langle \psi^\dagger \psi \rangle = 0\), and (iv) a superradiant superfluid \(\langle a \rangle \neq 0\), \(\langle b \rangle \neq 0\), \(\langle \psi^\dagger \psi \rangle \neq 0\). Indeed, the Hamiltonian displays a U(1) \(\times\) U(1) symmetry and these may be broken independently. The phase diagram reflects this pattern of symmetry breaking. In particular, the superradiant Mott state corresponds to an unbroken U(1) in the matter sector (corresponding to a pinned density and phase fluctuations) but a broken U(1) (or phase coherent condensate) for photo-excitations. As we shall discuss below, the expectation value of the bilinear, \(\langle b^\dagger a \rangle \neq 0\), corresponds to

Fig. 4: (Color online). Variational zero hopping diagram for hardcore \(a\) and \(b\) atoms. The symmetry about \(\mu_1 = 0\) reflects the combined particle–hole and species interchange symmetry of the Hamiltonian \(\text{III}^{33}\) in the hardcore limit; see text.

Fig. 5: (Color online). Variational phase diagram with \(J_a = J_b = J\) and \(\epsilon_a = -1, \epsilon_b = 1, \omega = \tilde{g} = 1, U_{ab} = 0\). The phases are (i) a Mott insulator (MI, dark blue), (ii) a superradiant Mott state supporting a condensate of photo-excitations (SR-MI, light blue), (iii) a superradiant superfluid (SR-SF, light red), and (iv) an a-type superfluid (a-SF, dark red). As shown by the solid line, the tetracritical line ultimately bifurcates into two bicritical points connected by a first order transition. For \(\mu_1 > 0\), the a-type superfluid is replaced by a b-type owing to the particle–hole and species interchange symmetry of the hardcore Hamiltonian \(\text{III}^{33}\) when \(J_a = J_b\); see text.
the onset of coherence in the Dicke model. This novel phase may thus be regarded as a form of supersolid in which photo-excitation condense on the background of a Mott insulator. Although the lattice precludes spontaneous translational symmetry breaking (at least with this periodicity) the excitations may be thought of as mobile defects in an otherwise ordered background.

As in the zero hopping case, we may extend Fig.[5] into the region $\mu_1 > 0$ by exploiting symmetries of [6]. This is reflected in the variational energy by using $\sin^2 \eta = (1 - \cos 2\eta)/2$ to combine the $\tilde{\epsilon}_a$ contributions:

$$
E = \frac{1}{2}(\epsilon_a + \epsilon_b) - \mu_1 - \frac{1}{2}(\epsilon_b - \epsilon_a - \mu_2) \cos 2\chi \cos^2 \theta
+ \frac{1}{2}(U_{ab} - (\epsilon_a + \epsilon_b + U_{ab} - 2\mu_1) \cos 2\eta) \sin^2 \theta
- \frac{zJ}{4} (1 + \sin 2\chi \sin 2\eta) \sin^2 2\theta
- \frac{\tilde{g}^2 \cos^4 \theta \sin^2 2\chi}{4(\omega - \mu_2)}. \tag{37}
$$

Since $J_a = J_b = J$ this is invariant (up to a constant) under $\mu_1 \rightarrow \epsilon_a + \epsilon_b + U_{ab} - \mu_1$ and $\eta \rightarrow \pi/2 - \eta$, which interchanges the superfluid order parameters (a) and (b). For the parameters used in Fig.[5], the a-type superfluid observed for $\mu_1 < 0$ is replaced with a $b$-type for $\mu_1 > 0$.

As may be seen from Fig.[5], the locus of the tetracritical line may be determined from the intersection of the superradiance transition with the Mott insulator to a-type superfluid phase boundary. This yields $\tilde{\epsilon}_a + zJ = 0$ as discussed in Appendix[6]. This may also be seen from a Landau type expansion of $E$. In view of the non-linear relationship, it is convenient to expand in the angles $\theta, \chi, \eta$ as opposed to the order parameters $\langle a \rangle, \langle b \rangle, \langle \psi \rangle$:

$$
E = \tilde{\epsilon}_a - (\tilde{\epsilon}_a + zJ)\theta^2 - \left(\frac{\tilde{g}^2 - \tilde{\omega}_0^2}{\tilde{\omega}}\right)\chi^2 + \ldots \tag{38}
$$

The quadratic “mass” terms vanish when $\tilde{\epsilon}_a + zJ = 0$ and the superradiance condition $\tilde{g} = \sqrt{\tilde{\omega} \tilde{\omega}_0}$ is met. In some simple cases $E$ may be expanded directly in the order parameters. For example, throughout the entire Mott lobe where $\langle a \rangle = \langle b \rangle = 0$ we may set $\theta = 0$. Returning to equation (38) and minimizing over $\chi$ yields $2\tilde{\chi} = -2\tilde{g}\tilde{\gamma}/\tilde{\omega}_0$. This gives $E = \tilde{\epsilon}_a + \tilde{\omega}\tilde{\gamma}^2 - \sqrt{\tilde{\omega}_0^2 / 4 + \tilde{g}^2 \tilde{\gamma}^2}$, where $\tilde{\gamma} = \langle \psi \rangle/\sqrt{N}$. This agrees with our variational zero hopping result (14) in the lowest lobe with $n = 1$.

In Fig.6 we present a cross section of the phase diagram for $\mu_1 = -0.6$. We indicate continuous transitions by single lines and first order transitions by dotted lines. In general, the transition from the superradiant Mott state to the non-superradiant a-type superfluid involves a discontinuous jump in the photon density. This results in a first order transition as indicated by the discontinuity in the first derivative of the energy and the order parameters as shown in Fig.[5]. This first order line connects two bicritical points as indicated in Fig.[6]. The length of this first order segment changes with $\mu_1$, and emanates from the tetracritical line as shown in Fig.[5]. These overall features are also exhibited for more general hopping parameters as shown in Fig.[8] where a-type and $b$-type superfluids emerge for large hopping asymmetry.

### B. Coherence in Superradiant Mott Phase

An interesting aspect of the superradiant Mott phase is that polariton condensation coexists with the Mott character. Within both Mott phases $\theta = 0$ and our variational wavefunction (36) becomes

$$
|\psi\rangle = |\gamma\rangle \otimes \prod_i \cos \chi_i a_i^+ + \sin \chi_i b_i^+ |0\rangle. \tag{39}
$$

To describe photo-excitations above the filled Mott state, we introduce a change of vacuum $|\Omega\rangle \equiv \prod_i a_i^+ |0\rangle$ so that

$$
|\psi\rangle = |\gamma\rangle \otimes \prod_i \cos \chi_i \prod_i \left(1 + \tan \chi_i b_i^+ a_i^+\right) |\Omega\rangle. \tag{40}
$$

Since we are dealing with hardcore bosons this may be exponentiated, and for homogenous parameters

$$
|\psi\rangle = |\gamma\rangle \otimes e^{N \ln \cos \chi e^{\tan \chi \sum_i b_i^+ a_i^+}} |\Omega\rangle. \tag{41}
$$

This is already reminiscent of a coherent state for the bilinears, although one needs to be careful since we are dealing with hardcore bosons. Instead, we may examine condensation properties directly by computing expectation values using (39). Using the Schwinger boson repre-
behavior and polariton condensation may coexist.

Since matter–light coherence in the Dicke model translates into polariton and not carrier condensation, Mott condensation of the individual bosonic carriers $\langle a_\alpha^n \rangle$ accompanied by $\langle a_\alpha^\dagger a_\alpha \rangle$ in the dual formulation as evidenced by equation (33). Crucially, condensation of the Holstein–Primakoff boson $\langle a_\alpha^\dagger a_\alpha \rangle$ has a discontinuity above the Mott background with $\langle a_\alpha a_\alpha \rangle \neq 0$.

In (I) First order transition from the superradiant Mott state to the $a$-type superfluid, (II) continuous transition from the $a$-type superfluid to the superradiant superfluid. Inset: Evolution of the variational energy (relative to the zero hopping Mott phase) across the transitions in Fig. 6 for fixed values of $\mu_2$. The solid line at $\mu_2 = 0.45$ has a discontinuity in the first derivative and indicates a first order transition in passing from the superradiant Mott insulator to the $a$-type superfluid. The remaining transitions are continuous.

V. NUMERICAL SIMULATIONS

We now analyze the Hamiltonian $H$ by exact diagonalization, imposing a maximum number of photons, $\langle \psi^\dagger \psi \rangle \leq M_\psi$, in addition to the hardcore $a,b$ constraints. We consider a one-dimensional system with $N$ lattice sites, with the basis of tensor product states $|\phi_\alpha^{(\nu_{a,b})} \otimes |\phi_\nu^{(\nu_{o})} \rangle \otimes |\phi_\nu^{(\nu_{b})} \rangle$, where $\nu_{a,b} = 1, \ldots, 2^N$ and $\nu_\psi = 0, 1, \ldots, M_\psi$. Here $|\phi_\alpha^{(\nu_{a})} \rangle = |n_{\alpha,1}^{(\nu_{a})}, \ldots, n_{\alpha,N}^{(\nu_{a})} \rangle$ where $n_{\alpha,i}^{(\nu_{a})} \in \{0, 1\}$, and the photon states are given by $|\phi_\nu^{(\nu_{\psi})} \rangle = |n_{\nu}^{(\nu_{\psi})} \rangle$ where $n_{\nu}^{(\nu_{\psi})} \in \{0, 1, \ldots, M_{\psi}\}$. The total Hilbert space has dimension $D = 2^N 2^N (M_\psi + 1)$. In the largest case considered, with $N = 8$ sites and $M_\psi = 64$ photons, this corresponds to a matrix dimension $D \approx 4E6$. Increasing $M_\psi$ further has only minor influence.

Applying periodic boundary conditions in real space, we diagonalize the sparse matrix representation of $H$ to obtain the groundstate $|\Phi_0 \rangle$ and its energy $E_0$. We compute the atom and photon densities, and the density fluctuations $\sigma_\alpha = \sqrt{\langle n_{\alpha}^2 \rangle - \langle n_{\alpha} \rangle^2}$. To obtain the superfluid fraction, $f_{s}^\alpha$, of the $\alpha$ atoms, we impose a phase twist $\Theta = \pi$ by means of a Peierls factor $a_\alpha^\dagger a_\alpha \rightarrow a_\alpha^\dagger a_\alpha e^{-i\Theta/N}$, and calculate the change in the ground state energy [62]:

$$f_{s}^\alpha = \frac{N}{J_{\alpha} \langle n_{\alpha} \rangle} \frac{E_0(\Theta) - E_0}{\Theta^2}.$$  \hspace{1cm} (45)

For the single-component Bose–Hubbard model, this...
quantity is zero deep in the Mott insulator, and approaches unity far in the superfluid. Note that in our case, the hardcore constraint always provides an effective interaction even for large hopping, so that $f_a < 1$. A total superfluid density can be obtained by imposing the phases on both species $a, b$ and calculating

$$f_a = \frac{N}{\sum_{\alpha} J_{\alpha}^{(n_{\alpha})}} \frac{E_{\alpha}^{(n_{\alpha})} - E_0}{\Theta^2}.$$  

We supplement these superfluid diagnostics with the zero momentum occupations $n_{\alpha}(k = 0) = N^{-1} \sum_{\rho \sigma} \langle \alpha_p \alpha_q \rangle$.

A. Small Hopping and Dicke Superradiance

To verify the existence of a superradiance transition within the Mott phase we begin our numerical treatment in the limit of small non-zero hopping. Figure 9 shows the Mott lobe with density $n = 1$ at $zJ = 0.1$, for two different values of $M_0$. For sufficiently large $M_0$ the results are in good agreement with the variational zero-hopping phase diagram shown in Fig. 4 and the overall features depend only weakly on the number of sites. In order to examine the onset of superradiance, we track the evolution of the polariton and photon densities in Fig. 10. At zero hopping, equation (24) yields the critical coupling $\kappa_c = \omega_0 = (\omega - \mu_2)(\epsilon_b - \epsilon_a - \mu_2)$. For the chosen parameters at fixed $\bar{g} = 1$, the transition occurs at a critical chemical potential, $\mu_2^* = (3 - \sqrt{5})/2 \approx 0.382$. This onset is well reproduced at small finite hopping, as shown in Fig. 10. For $N = 1$ (where the Dicke model reduces to the Jaynes–Cummings model) we see quantized steps with fixed integer polariton density, $N_2/N$. These form the basis of the Mott lobes observed in Jaynes–Cummings–Hubbard models [9, 23]. Even in this extreme limit, the densities closely track the thermodynamic Dicke model results. As $N$ increases, the step sizes are reduced by a factor of $1/N$ and we approach the variational thermodynamic results. This behavior in our itinerant boson model closely mirrors direct finite $N$ simulations of the Dicke model; see Fig. 10. A notable difference between the Dicke and Jaynes–Cummings–Hubbard models [9, 23] (both with $N$ sites) is reflected in their zero hopping eigenstates. In the latter, eigenstates are tensor products of superpositions of two states which differ in photon number by one, whereas in the Dicke model the global photon mode leads to a coherent photon state for large $N$. In the Dicke model the higher Mott lobes are eliminated in favor of a continuous photon onset, whereas in the Jaynes–Cummings–Hubbard models [9, 23] they remain in tact.

B. General Phase Diagram

Turning to the construction of the overall phase diagram, we first consider a fixed value $\mu_1 = 0$ and present

![FIG. 9: Exact diagonalization results for the phase diagram at small hopping, $zJ = 0.1$, for hardcore $a, b$ atoms and $\epsilon_a = -1$, $\epsilon_b = 1$, $\omega = \bar{g} = 1$, $U_{ab} = 0$. The dashed (solid) lines correspond to $N = 8$ and $M_0 = 16$ ($M_0 = 64$) photons, and have been obtained from spline fits to the contours at $n = 0.99$ and $n = 1.01$. The vertical line indicates the location of the superradiance transition as determined from $\langle \psi^\dagger \psi \rangle/N \geq 0.01$. The results are in good qualitative agreement with the variational zero hopping results in Fig. 4 showing the continuity of this behavior into the Mott phase.](image)

![FIG. 10: Exact diagonalization results with $N = 1, 8$ sites, $J_a = J_b = J$ and $zJ = 0.1$. We set $\epsilon_a = -1$, $\epsilon_b = 1$, $\omega = \bar{g} = 1$, $\mu_1 = U_{ab} = 0$. The profiles show the onset of superradiance within the Mott phase, and evolve with increasing system size towards those for the thermodynamic limit of the Dicke model. Evolution of (a) the photon density $\langle \psi^\dagger \psi \rangle/N$, (b) the polariton density, $N_2/N$. For $N = 1$ the Dicke model reduces to the Jaynes–Cummings model and exhibits quantized integer steps in the polariton density [9, 23]. More generally these are quantized in units of $1/N$. The value of $M_0$ is of minor importance in the range of $\mu_2$ considered as it sets the upper limit on the number of excitations available.](image)
results in the \((zJ, \mu_2)\) plane; see Figures 11 and 12. So far we have set \(U_{ab} = 0\) in order to expose the essential details. Following our original work \[30\], we now take \(U_{ab} = 1\) to help illustrate the generality of the overall results. The interspecies interaction also helps stabilize the Mott region up to larger values of the hopping.

In Fig. 11 we show density plots of atom and photon densities, and density fluctuations. In Fig. 12 we plot superfluid fractions. From these data, we can identify the four different phases present in Fig. 5. The Mott–superfluid transition is visible from the deviation of the total atom density from unity in Fig. 11(a). The superradiance transition corresponds to the onset of photon density in Fig. 11(b). Panels (c) and (d) demonstrate the existence of the single-component superfluid in the region where \(n \neq 1\) and \(n_a \neq 0\), while \(n_b = 0\). This is also visible in the individual superfluid fractions shown in Fig. 12. We also see the onset of a non-trivial population imbalance, \(n_b - n_a\), or existence of \(b\)-bosons, accompanying the superradiance transition. Specifically, \(n_a = 1, n_b = 0\) in the left of the white region of panel (b), but \(n_a < 1, n_b > 0\) above; see panels (e), (d) and (g). The nature of the two Mott phases (normal and superradiant) is illustrated by panels (e)–(f). We observe an onset of local fluctuations \(\sigma_{\alpha}\) with increasing hopping. Since the Mott state is characterized by \(n = n_a + n_b = 1\), the individual atom density fluctuations \(\sigma_{\alpha}\) in the superradiant Mott phase are much larger than \(\sigma\). In the normal Mott phase, \(n_b = 0\) so that the Mott insulator consists purely of \(a\)-atoms and hence \(\sigma_{\alpha} = \sigma_{\beta} = 0\). Finally, the Mott–superfluid transition is also apparent in our results for the superfluid fractions as shown in Fig. 12.

Exact diagonalization of small clusters yields an approximation to the critical value \(J_c\) for the Mott–superfluid transition by monitoring the onset of \(f_s\), as in Fig. 12(a), or the local fluctuations \(\sigma\) of the total atom density, as in Fig. 11(e). Both \(\sigma\) and \(f_s\) are also nonzero below \(J_c\) for finite \(N\) \[33\]. While \(f_s\) is expected to scale to zero for \(J < J_c\), as \(N \to \infty\), the local density fluctuations remain nonzero in the Mott phase due to virtual hopping processes. This is illustrated in Fig. 13. The system size \(N\) has a noticeable impact on the onset of correlations related to superfluidity. Since a finite size scaling analysis requires much larger system sizes, we provide approximate phase boundaries obtained from the contour lines \(\sigma(zJ, \mu_2) = 0.3\) (alternatively \(f_s(zJ, \mu_2) = \text{const.}\)) to indicate the onset of superfluidity, and \(\langle \psi^\dagger \psi \rangle / N = 0.01\) to indicate superradiance. The numerical constants for the contours have been chosen in order to obtain phase boundaries that match those suggested by the data in Figs. 11 and 12. See Fig. 13. Figure 13(c) shows the zero-momentum occupation \(n(k = 0) = n_a(k = 0) + n_b(k = 0)\) for the atoms. The latter is expected to diverge as a function of system size in the SF phase, and such behavior can indeed be seen above \(zJ \approx 1.1\).

Repeating the above procedure for different \(\mu_1\) values we build up a picture of the overall phase diagram as shown in Fig. 14. This may be compared to the varia-

FIG. 11: Exact diagonalization results for \(N = 8\) sites and \(M_b = 16\) photons. We set \(\mu_1 = 0, J_a = J_b = J\) and \(\epsilon_a = -1, \epsilon_b = 1, \omega = g = U_{ab} = 1\). The panels show density plots of (a) total atom density, (b) photon density, (c) \(a\)-atom density, (d) \(b\)-atom density, (e) fluctuations of the total atom density, (f) fluctuations of the \(a\)-atom density, (g) fluctuations of the \(b\)-atom density. In particular (a) reveals the Mott–superfluid transition and (b) the superradiance transition.
where sense of excitations. In the spatially uniform case of (a) total atomic superfluid fraction, (b) a-atom superfluid fraction, and (c) b-atom superfluid fraction. The Mott–superfluid transition is evident from the onset in panel (a), and the single component a-type superfluid in panels (b) and (c).

The agreement between the numerical simulations and mean field theory is remarkable given the enhanced role of fluctuations in low dimensions. This mirrors the success of mean field theory in other bosonic systems and may be assisted by the long range cavity photons.

VI. RELATION TO OTHER PROBLEMS

A feature not addressed by the present mean field theory, but captured in Fig. 14 is the dispersion of the superradiance transition with $J$; in the Mott phase, $\theta = 0$, and $J$ drops out of the variational energy \( \mathcal{E} \). One way to understand this is to recast the matter contribution as

\[
|\psi_M \rangle = \prod_i (\cos \chi_i + \sin \chi_i b_i^\dagger a_i) |\Omega \rangle,
\]

where \( |\Omega \rangle \equiv \prod_i a_i^\dagger |0 \rangle \) is the filled Mott state in the absence of excitations. In the spatially uniform case

\[
|\psi_M \rangle = \mathcal{N} e^{\lambda \sum_i a_i^\dagger a_i} |\Omega \rangle,
\]

where $\lambda \equiv \tan \chi$ and $\mathcal{N} \equiv (1 + \lambda^2)^{-N/2}$. This only accommodates local particle-hole pairs above the filled Mott background, and may thus be regarded as the BEC limit of the BEC–BCS crossover problem. By analogy with the fermionic BCS approach to exciton insulators \cite{64, 65}, and the crossover problem in \cite{40, 66}, one expects that excitons may lower their energy by spreading out in real space and pairing in momentum space. In this regard, the “BCS” pairing phenomenon in Bose gases has a long history, with the Valatin–Butler wavefunction \cite{67} playing the role of the BCS state. This has been developed in a series of early works motivated by liquid $^4$He \cite{68, 72} and biexciton formation \cite{74, 76}. This later emerged in studies of High-$T_c$ superconductivity \cite{47} and two-component Bose gases \cite{78, 79}. The interesting possibility of bound states of three or four particles has also been explored \cite{80, 51}. Central to these studies is the requirement to stabilize attractive Bose gases against collapse, with the aid of short-range repulsion or internal structure. When this condition is met, and at sufficiently low densities where exciton overlap is negligible, it has been argued...
that such pairing states may exist \cite{73}. In the present context, stabilization of the paired state and the absence of carrier condensation is brought about by the interplay of Mott physics and photoexcitation. It would be interesting to explore this problem in more detail, and we leave these refinements to future studies.

As noted in our previous work, the connection to the matter-light coupling. As such we incorporate $U_{ab}$ as in Ref. 54. Within this variational approach, this interaction has no impact on the Mott states with $\theta = 0$, as evident from the energy density \cite{55}. Nonetheless, the presence of the matter-light coupling stabilizes the non-trivial phases in Fig. 5 and provides good agreement with the numerical simulations. However, as noted by Söyler et al. \cite{54} and previous works \cite{78–81}, analogous pairing phases may be supported in the two-component Bose–Hubbard model, without matter-light coupling, through a more sophisticated treatment of $U_{ab}$ itself. Indeed, onsite repulsive interactions, $U_{ab}n_an_b$, favor a particle of one species and a hole of the other on the same site. Treating this pairing in a BCS approach, one may replace $n_i^a n_i^b$ by $|\Delta_i|^2 + (\Delta_i b_i^a a_i + \text{h.c.})$, where $\Delta_i \equiv \langle a_i^b b_i^a \rangle$, is to be determined self-consistently. This field acts as a local “photon”, and a similar mean field phenomenology may ensue. Such pairing also occurs in fermionic models \cite{87}. Although our discussion has focused on a single local photon, the symmetry analysis is more general. This is supported by studies of the two-band Bose–Hubbard model for equal fillings and commensurate densities \cite{91}.

In closing we note that the classical light limit of equation \ref{eq:1} (where $\psi$ is replaced by a c-number) may be simulated in optical superlattices \cite{92} where the coupling $g$ represents tunnelling between different wells, $a$ and $b$.

values of $\mu_1$ owing to the particle–hole and species interchange symmetry involving $\mu_1 \rightarrow \epsilon_a + \epsilon_b + U_{ab} - \mu_1$. In contrast to the single species mean field theory \cite{82–84}, this two species case supports an atomic superfluid, since condensation of one carrier no longer induces an effective field. Moreover, condensation may leave a $U(1)$ symmetry intact, which allows the coexistence of Mott and phase coherent behavior.

In deriving \eqref{eq:35} and the phase diagram, we are primarily concerned with the matter-light coupling. As such we incorporate $U_{ab}$ as in Ref. 54. Within this variational approach, this interaction has no impact on the Mott states with $\theta = 0$, as evident from the energy density \cite{55}. Nonetheless, the presence of the matter-light coupling stabilizes the non-trivial phases in Fig. 5 and provides good agreement with the numerical simulations. However, as noted by Söyler et al. \cite{54} and previous works \cite{78–81}, analogous pairing phases may be supported in the two-component Bose–Hubbard model, without matter-light coupling, through a more sophisticated treatment of $U_{ab}$ itself. Indeed, onsite repulsive interactions, $U_{ab}n_an_b$, favor a particle of one species and a hole of the other on the same site. Treating this pairing in a BCS approach, one may replace $n_i^a n_i^b$ by $|\Delta_i|^2 + (\Delta_i b_i^a a_i + \text{h.c.})$, where $\Delta_i \equiv \langle a_i^b b_i^a \rangle$, is to be determined self-consistently. This field acts as a local “photon”, and a similar mean field phenomenology may ensue. Such pairing also occurs in fermionic models \cite{87}. Although our discussion has focused on a single local photon, the symmetry analysis is more general. This is supported by studies of the two-band Bose–Hubbard model for equal fillings and commensurate densities \cite{91}.

In closing we note that the classical light limit of equation \ref{eq:1} (where $\psi$ is replaced by a c-number) may be simulated in optical superlattices \cite{92} where the coupling $g$ represents tunnelling between different wells; see Fig. 14. In the case of hard core $a$-atoms and softcore $b$-atoms this provides an analogue of the Jaynes–Cummings–Hubbard models considered in Refs. \cite{8, 23}. This geometry may also be useful in realizing other “matter–bath” systems.
VII. CONCLUSIONS

We have investigated the two-band Bose–Hubbard model coupled to a single mode of a cavity light field. We combine analytical and numerical techniques and find good agreement between the approaches. The model displays a novel phase in which “polaritons” condense on the platform of a bosonic Mott insulator. We extend our previous work in several directions including an investigation of the overall phase diagram, and the nature of polariton condensation. In particular, we use the framework of the Dicke model to discuss how polariton condensation emerges in the absence of carrier condensation. In terms of numerical results, we have presented superfluid fractions, atom density fluctuations and zero-momentum occupation, and analyzed the phase diagram under variation of $\mu_1$. In addition, we have addressed the effects of finite size and the photon cutoff. This helps illustrate connections to work on Jaynes–Cummings–Hubbard models and coupled cavity arrays. This topic has broad connections to other problems of current interest including atom–molecule mixtures and the BEC-BCS crossover in bosonic systems.

There are many avenues for further research including non-equilibrium aspects and collective excitations. It would be interesting to develop numerical techniques to explore finite temperature polariton condensation and the onset of phase coherence in the Mott phase. It would also be worthwhile to examine the phase diagram with softcore bosons with finite $U_{aa}$ and $U_{bb}$.

Acknowledgments

We are grateful to G. Conduit, N. Cooper, J. Keeling, and M. Köhl for helpful discussions. MJB, AOS, and BDS acknowledge EPSRC grant no. EP/E018130/1.

Appendix A: Dicke Models

Throughout this manuscript we make use of the Dicke model and its reductions. In the literature this appears in various guises and with different names so we present a brief guide. The Dicke model describes $N$ two-level systems or “spins” coupled to radiation

$$H = \omega_0 \sum_i S_i^z + \omega \psi^\dagger \psi + \frac{\bar{g}}{\sqrt{N}} \sum_i (\psi^\dagger S_i^- + S_i^+ \psi), \quad (A1)$$

where $S_i$ is a spin-1/2 operator and $\psi$ is a canonical boson. Here $\omega_0$ is the level-splitting between the two-level systems, $\omega$ is the frequency of the cavity mode, and $\bar{g}$ is the strength of the matter–light coupling. In quantum optics the Dicke model is also known as the Tavis–Cummings model. In the special case of a single spin, $N = 1$, the Dicke model is often referred to as the Jaynes–Cummings model. The Hamiltonian (A1) is written in the so-called rotating wave approximation in which terms of the form $S_{i}^{\pm} \psi^{\pm}$ are excluded. In the absence of these terms, the model is integrable and exhibits a superradiance quantum phase transition when $\bar{g} = \sqrt{\omega \omega_0}$ [41, 44]; for a discussion of the model including counter-rotating terms see Ref. [53]. Depending on the application these models are often recast in different representations and we gather a few results below.

1. Jaynes–Cummings

The simplest case is the Dicke model with $N = 1$, or the Jaynes–Cummings model, describing a single two-level system coupled to radiation. It plays a central role in the variational analysis presented in section III A, and recent problems in matter–light systems, e.g. [42, 23]. A convenient representation is the single site Hamiltonian

$$H = \epsilon_a n_a + \epsilon_b n_b + g (b^\dagger a + a^\dagger b), \quad (A2)$$

where $a$ is a hardcore boson (restricted to occupancy zero or one) and $b$ may take arbitrary occupancy. We may construct exact eigenstates of total integer occupation, $n = n_a + n_b \geq 1$, as superpositions

$$|n\rangle = \alpha |1, n - 1\rangle + \beta |0, n\rangle, \quad (A3)$$

where the first and second entries of the states are the occupations of the $a$ and $b$ particles respectively. In matrix form this yields the eigenvalue problem

$$\begin{pmatrix} \epsilon_a + (n - 1)\epsilon_b & g \sqrt{n} \\ g \sqrt{n} & n \epsilon_b \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = E_n \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad (A4)$$

where we recall that $b|n\rangle = \sqrt{n} |n - 1\rangle$ and $b^\dagger |n\rangle = \sqrt{n + 1} |n + 1\rangle$ for bosons. The eigenvalues are given by

$$E_n^\pm = n \epsilon_b - \omega_0/2 \pm \sqrt{\omega_0^2/4 + g^2 n}, \quad (A5)$$

where $\omega_0 \equiv \epsilon_b - \epsilon_a$. This agrees with the spectrum of the Jaynes–Cummings model given by Greentree et al [9], where we identify $\epsilon_a$, $\epsilon_b$, and $g$ with their parameters $\epsilon$, $\omega$, and $\beta$. The coefficients in the corresponding normalized eigenvectors $|n\rangle_\pm$ may be written

$$\alpha_\pm = \frac{-\omega_0/2 \pm \chi(n)}{\sqrt{2\chi^2(n) + \omega_0\chi(n)}}, \quad \beta_\pm = \frac{g \sqrt{n}}{\sqrt{2\chi^2(n) + \omega_0\chi(n)}}, \quad (A6)$$

where $\chi(n) \equiv \sqrt{\omega_0^2/4 + g^2 n}$. We note that there is a minor typing error in the eigenstates given in the methods section of Ref. [3], where the basis states $|1, n - 1\rangle$ and $|0, n\rangle$ are erroneously reversed. Equivalently, one may reverse the sign of $\Delta \equiv \omega_0$ in their expression.
The generic Dicke model (A1) is integrable for arbitrary $N$ \[42, 44, 58\], but is most conveniently analyzed in the thermodynamic limit, $N \to \infty$ \[50\]. However, in numerical simulations we must deal with the effects of finite $N$ and truncations of the photon Hilbert space. In Appendix B: Zero Hopping Phase Diagram with $g = 0$ we approach the variational results. Inset: Evolution of the polariton density, $\frac{\langle \psi^\dagger \psi \rangle}{N}$, where $\mu_2 \to \infty$, in the absence of matter–light interaction, with $\epsilon_a = -1$, $\epsilon_b = 1$, $\omega = 1$, $g = 0$. The loci $\tilde{\epsilon}_a = 0$, $\tilde{\epsilon}_b = 0$, $\tilde{\omega} = 0$, indicated by the lower, upper and vertical lines respectively, delineate the regions of population onset for $a, b, \psi$ in the grand canonical ensemble. When $g$ is switched on the horizontal boundaries bend downwards and evolve into those shown in Fig. 2.

**Appendix C: Variational Phase Boundaries**

The variational energy for the finite hopping problem allows some analytic progress with the phase boundaries, and highlights connections to the Dicke model. In the absence of competition from other phases, the transition between the non-superradiant insulator ($\theta = \chi = \tilde{\gamma} = 0$) and the $a$-type superfluid ($\theta \neq 0$, $\chi = \eta = \tilde{\gamma} = 0$) for example occurs when $\tilde{\epsilon}_a + z J = 0$. This may be seen by explicit computation of the energies of each phase. The energy density in the generic Mott phase is

$$E_{\theta=0} = (\tilde{\epsilon}_+ - \tilde{\epsilon}_- \cos 2\chi) - \frac{\tilde{g}^2 \sin^2 2\chi}{4\tilde{\omega}}. \quad (C1)$$

Minimizing on $\chi$ yields either $\chi = 0$, corresponding to an ordinary Mott insulator, with $E_{\text{MI}} = \tilde{\epsilon}_+ - \tilde{\epsilon}_-$, or

$$\cos 2\chi = \frac{2\tilde{\omega} \tilde{\epsilon}_-}{\tilde{g}^2} \equiv \frac{\tilde{\omega} \tilde{\omega}_0}{\tilde{g}^2}, \quad (C2)$$

corresponding to the superradiant Mott insulator. At this non-trivial stationary point

$$\frac{\partial^2 E}{\partial \chi^2} = \frac{2(\tilde{g}^4 - \tilde{\omega}^2 \tilde{\omega}_0^2)}{\tilde{\omega} \tilde{g}^2}. \quad (C3)$$

This corresponds to a minimum (with $\tilde{\omega} > 0$) provided $\tilde{g} > \sqrt{\tilde{\omega} \tilde{\omega}_0}$. This coincides with the superradiance transition in the Dicke model \[41, 44\]. The energy density in the superradiant Mott phase is

$$E_{\text{SRMI}} = E_{\text{MI}} - \frac{(\tilde{g}^2 - \tilde{\omega} \tilde{\omega}_0)^2}{4\tilde{g}^2 \tilde{\omega}}. \quad (C4)$$

Similarly, in the $a$-type superfluid

$$E_{a\text{SF}} = (\tilde{\epsilon}_+ - \tilde{\epsilon}_- \cos^2 \theta - \frac{z J}{4} \sin^2 2\theta). \quad (C5)$$

**Appendix B: Zero Hopping Phase Diagram with $g = 0$**

An instructive way to think about the topology of the zero hopping phase diagram in Fig. 2 is in the absence of the matter–light coupling. In Fig. 17 we plot the loci $\tilde{\epsilon}_a = 0$, $\tilde{\epsilon}_b = 0$, $\tilde{\omega} = 0$, corresponding to population transitions in the Hamiltonian \[9\] for $g = 0$. When $g$ is switched on, the horizontal boundaries bend downwards and continuously evolve into those shown in Fig. 2.

**Fig. 16:** Evolution of the polariton density, $N_2/N$, in the Dicke model for $N = 1, 2$ sites and comparison to the $N \to \infty$ thermodynamic limit. For $N = 1$ the Dicke model reduces to the Jaynes–Cummings model and the quantized polariton steps correspond to the Mott lobes discussed in Ref. \[3\]. In the Dicke model these steps are quantized in units of $1/N$ and as $N \to \infty$ we approach the variational results. Inset: Evolution of the photon density. The magnetization (or population imbalance) may be obtained by subtraction. All figures show the onset of superradiance at $\mu_2 = (3 - \sqrt{5})/2 \approx 0.382$, for $\omega \to \omega = 1 - \mu_2$, $\omega_0 \to \tilde{\omega}_0 = 2 - \mu_2$, and $\tilde{g} = 1$.

**Fig. 17:** Zero hopping phase diagram for the Hamiltonian \[9\] in the absence of matter–light interaction, with $\epsilon_a = -1$, $\epsilon_b = 1$, $\omega = 1$, $g = 0$. The loci $\tilde{\epsilon}_a = 0$, $\tilde{\epsilon}_b = 0$, $\tilde{\omega} = 0$, indicated by the lower, upper and vertical lines respectively, delineate the regions of population onset for $a, b, \psi$ in the grand canonical ensemble. When $g$ is switched on the horizontal boundaries bend downwards and evolve into those shown in Fig. 2.
Minimizing on $\theta$ yields either $\theta = 0$ or $\cos 2\theta = -\frac{(\varepsilon - \varepsilon_\text{SF})}{2}$. The latter yields
\[ \mathcal{E}_{\text{SF}} = -\frac{(\varepsilon_+ - \varepsilon_- - zJ)^2}{4zJ}. \tag{C6} \]

The condition $\mathcal{E}_{\text{MI}} = \mathcal{E}_{\text{SF}}$ yields the MI–aSF boundary $\varepsilon_+ + \varepsilon_- + zJ = 0$ or equivalently $\varepsilon_+ + zJ = 0$. This meets the superradiance onset $\bar{g}^2 = \bar{\omega}\bar{\omega}_0$ at $\bar{g}^2 = \bar{\omega}(\bar{\varepsilon}_b + zJ)$ or
\[ \mu_1 = \bar{g}^2 - \frac{\mu_2}{2} = \frac{\bar{g}^2}{\omega} + zJ. \tag{C7} \]

In a similar fashion, the transition between the superradiant Mott state and the a-type superfluid occurs when $\mathcal{E}_{\text{SR}} = \mathcal{E}_{\text{SF}}$ or
\[ zJ(\bar{g}^2 - \bar{\omega}_0)^2 = \bar{g}^2\bar{\omega}(\bar{\varepsilon}_a + zJ)^2. \tag{C8} \]

Appendix D: Absence of Higher Lobes in Coupling to Quantum Light

In Fig. 2, we see that only the Mott lobe with atomic density, $n = 1$, is supported due to the coupling to the quantum light field. To prove that only this lobe survives we compare the energies of the higher states to those depicted in Fig. 2. Without loss of generality we may focus on the stable region defined by $\bar{\varepsilon}_b - \bar{g}^2/4\bar{\omega} > 0$. Within this domain $E^-_{\text{SF}}(\gamma_{\text{var}})$ and $E^{-a}_{\text{SF}}(\gamma = 0)$ are both smoothly increasing functions of $n$. Furthermore, whilst the superradiance condition [14] holds, $E^{-a}_{\text{SF}}(n) \geq E^{-a}_{\text{SF}}(\gamma_{\text{var}})$. We immediately conclude that whenever the $n = 1$ superradiance condition is met ($\bar{g}^2 > \bar{\omega}_0^2/2\bar{\varepsilon}_b$), the ground state energy is either $E^{-a}_{\text{SF}}(\gamma_{\text{var}})$ or 0, the vacuum energy. However, outside of this region we must examine whether the lowest energy superradiant state is lower in energy than either $E^{-a}_{\text{SF}}(0)$ or the vacuum.

To address this, let us consider superradiant states with $n > 1$. These are candidates for the ground state within the region, $\bar{g}^4 < \bar{\omega}_0^2 < n^2\bar{\omega}^2$, where the upper bound follows from the superradiance condition and the lower bound precludes the region in which $E^{-a}_{\text{SF}}(\gamma_{\text{var}})$ is supported. If $\bar{\omega}_0 > 0$, taking roots of the inequality $\bar{\omega}_0 > \bar{\omega}_0^2/n\bar{g}^2$ and $\bar{\omega}_0 > \bar{g}^2/\bar{\omega}$. In conjunction with the condition for stability, we see that $E^{-a}_{\text{SF}}(\gamma_{\text{var}}) > \bar{\varepsilon}_a = E^{-a}_{\text{SF}}(0)$. That is to say, the ground state never has $n > 1$.

If $\bar{\omega}_0 \leq 0$ a stronger statement can be made. Invoking the superradiance condition, $\bar{\omega}_0 = -\bar{\omega}_0^2/n\bar{g}^2$, it follows that $E^{-a}_{\text{SF}}(\gamma_{\text{var}}) > 0$ for all $n$. Hence the vacuum must be the ground state when $\bar{\omega}_0 \leq 0$. (The latter implies $E^{-a}_{\text{SF}}(0) = \bar{\varepsilon}_a \geq \bar{\varepsilon}_b > 0$, where the last inequality has followed from the stability requirement). We see that a sufficient condition for termination of the Mott lobe is $\mu_2 > \bar{\omega}_0$. However, since $\mu_2 < \omega$ in general, such regimes are only stable for $\omega_0 < \omega$; see Fig. 2.

[1] F. Brennecke, T. Donner, S. Ritter, T. Bourdel, M. Köhl, and T. Esslinger, Nature 450, 268 (2007).
[2] Y. Colombe, T. Steinmetz, G. Dubois, F. Linke, D. Hunger, and J. Reichel, Nature 450, 272 (2007).
[3] F. Brennecke, S. Ritter, T. Donner, and T. Esslinger, Science 322, 235 (2008).
[4] S. Ritter, F. Brennecke, K. Baumann, T. Donner, C. Guerlin, and T. Esslinger, Appl. Phys. B 95, 213 (2009).
[5] P. F. Herskind, A. Dantan, J. P. Marler, M. Albert, and M. Drewsen, Nat. Phys. 5, 494 (2009).
[6] W. Lange, Nat. Phys. 5, 455 (2009).
[7] I. B. Mekhov, C. Mascher, and H. Ritsch, Nat. Phys. 3, 319 (2007).
[8] M. J. Hartmann, F. G. S. L. Brandão, and M. B. Plenio, Nat. Phys. 2, 849 (2006).
[9] A. D. Greentree, C. Tahan, J. H. Cole, and L. C. L. Hollenberg, Nat. Phys. 2, 856 (2006).
[10] D. G. Angelakis, M. F. Santos, and S. Bose, Phys. Rev. A 76, 031805 (2007).
[11] Y. C. N. Na, S. Utsunomiya, L. Tian, and Y. Yamamoto, Phys. Rev. A 77, 031803(R) (2008).
[12] M. J. Hartmann, F. G. S. L. Brandão, and M. B. Plenio, Laser and Photon. Rev. 2, 527 (2008).
[13] J. Cho, D. G. Angelakis, and S. Bose, Phys. Rev. Lett. 101, 246808 (2008).
[14] F. Illuminati, Nat. Phys. 2, 803 (2006).
[15] D. Rossini and R. Fazio, Phys. Rev. Lett. 99, 186401 (2007).
[16] S.-C. Lei and R.-K. Lee, Phys. Rev. A 77, 033827 (2008).
[17] M. I. Makin, J. H. Cole, C. Tahan, L. C. L. Hollenberg, and A. D. Greentree, Phys. Rev. A 77, 053819 (2008).
[18] M. Aichhorn, M. Hohenadler, C. Tahan, and P. B. Littlewood, Phys. Rev. Lett. 100, 216401 (2008).
[19] P. Pippan, H. G. Evertz, and M. Hohenadler, Phys. Rev. A 80, 033612 (2009).
[20] M. Grochol, Phys. Rev. B 79, 205306 (2009).
[21] S. Schmidt and G. Blatter, [arXiv:0905.3344].
[22] A. Mering, P. Ivanov, K. Singer, and M. Fleischhauer, [arXiv:0907.1876].
[23] J. Koch and K. L. Hur, Phys. Rev. A 80, 023811 (2009).
[24] D. E. Chang, V. Gritsev, G. Morici, V. Vuletic, M. D. Lukin, and E. A. Demler, Nat. Phys. 4, 884 (2008).
[25] A. Wallraff, D. I. Schuster, A. Blais, L. Frunzio, R.-S. Huang, J. Majer, S. Kumar, S. M. Girvin, and R. J. Schoelkopf, Nature 431, 162 (2004).
[26] J. M. Fink, M. Göppl, M. Baur, R. Bianchetti, P. J. Leek, A. Blais, and A. Wallraff, Nature 454, 315 (2008).
[27] J. M. Fink, R. Bianchetti, M. Baur, M. Göppl, L. Steffen, S. Filipp, P. J. Leek, A. Blais, and A. Wallraff, Phys. Rev. Lett. 103, 083601 (2009).
[28] A. Fragner, M. Göppl, J. M. Fink, R. Bianchetti, P. J. Leek, A. Blais, and A. Wallraff, Science 322, 1357 (2008).
[29] J. Kasprzak, M. Richard, S. Kundermann, A. Baas, P. Jeambrun, J. M. J. Keeling, F. M. Marchetti, M. H. Szymańska, R. André, J. L. Staehli, et al., Nature 443, 409 (2006).
[30] H. Deng, G. Weihs, C. Santori, J. Bloch, and Y. Ya-
