**BETTI NUMBERS OF SKELETONS**

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**ABSTRACT.** We demonstrate that the Betti numbers associated to an \(\mathbb{N}_0\)-graded minimal free resolution of the Stanley-Reisner ring \(S/I_{\Delta^{(d-1)}}\) of the \((d-1)\)-skeleton of a simplicial complex \(\Delta\) of dimension \(d\) can be expressed as a \(\mathbb{Z}\)-linear combination of the corresponding Betti numbers of \(\Delta\). An immediate implication of our main result is that the projective dimension of \(S/I_{\Delta^{(d-1)}}\) is at most one greater than the projective dimension of \(S/I_{\Delta}\), and it thus provides a new and direct proof of this. Our result extends immediately to matroids and their truncations. A similar result for matroid elongations can not be hoped for, but we do obtain a weaker result for these. The result does not apply to generalized skeleton ideals.

1. **INTRODUCTION**

In this paper we investigate certain aspects of the relationship between an \(\mathbb{N}_0\)-graded minimal free resolution of the Stanley-Reisner ring of a simplicial complex and those associated to its skeletons. Our main result is Theorem 3.1 which says that each of the Betti numbers associated to an \(\mathbb{N}_0\)-graded minimal free resolution of \(S/I_{\Delta^{(d-1)}}\), where \(I_{\Delta^{(d-1)}}\) is the ideal generated by monomials corresponding to nonfaces of the \((d-1)\)-skeleton of a finite simplicial complex \(\Delta\), can be expressed as a \(\mathbb{Z}\)-linear sum of the Betti numbers associated to \(S/I_{\Delta}\).

Previous results on the Stanley-Reisner rings of skeletons include the classic [10, Corollary 2.6] which states that

\[
\text{depth } S/I_{\Delta} = \max \{ j : \Delta^{(j-1)} \text{ is Cohen-Macaulay} \}.
\]

This result was later generalized to arbitrary monomial ideals in [7, Corollary 2.5] (we shall return to this in our final section, where we give a counterexample showing that our main result can not be generalized in the same way). By the Auslander-Buchsbaum identity, it follows from (1) that

\[
p. d. \ I_{\Delta} \leq p. d. \ S/I_{\Delta^{(d-1)}} \leq 1 + p. d. \ S/I_{\Delta}.
\]

From the latter of these inequalities it is easily demonstrated, again by using the Auslander-Buchsbaum identity, that every skeleton of a Cohen-Macaulay simplicial complex is Cohen-Macaulay - a fact which was proved in [10, Corollary 2.5] as well.
That p. d. $S/I_{\Delta(d-1)} \leq 1 + \text{p. d. } S/I_{\Delta}$ can also be seen as an immediate consequence of our main result, and Theorem 3.2 thus provides a new and direct proof of this and therefore also of the fact that the Cohen-Macaulay property is inherited by skeletons.

The projective dimension of Stanley-Reisner rings has seen recent research interest. Most notably, it was demonstrated in [16, Corollary 3.33] that

$$\text{p. d. } S/I_{\Delta} \geq \max\{|C| : C \text{ is a circuit of the Alexander dual } \Delta^* \text{ of } \Delta\},$$

with equality if $S/I_{\Delta}$ is sequentially Cohen-Macaulay.

Our main result extends immediately to a matroid $M$ and its truncations. Such matroid truncations have themselves seen recent research interest. Examples of this are [13], which contains the strengthening of a result by Brylawski [5, Proposition 7.4.10] concerning the representability of truncations, and [4, Proposition 15], where it is demonstrated that the Tutte polynomial of $M$ determines that of its truncation $M^{(1)}$.

Corresponding to our main result applied to matroid truncations, we give a considerably weaker result concerning matroid elongations. It says that the Betti table associated to the elongation of $M$ to rank $r(M) + 1$ is equal to the Betti table obtained by removing the second column from the Betti table of $S/I_M$ - but only in terms of zeros and nonzeros.

1.1. Structure of this paper.
- In Section 2, we provide definitions and results used later on.
- In Section 3, we demonstrate that the Betti numbers associated to a $\mathbb{N}_0$-graded minimal free resolution of the Stanley-Reisner ring of a skeleton can be expressed as a $\mathbb{Z}$-linear combination of the corresponding Betti numbers of the original complex. This leads immediately to a new and direct proof that the property of being Cohen-Macaulay is inherited from the original complex.
- In Section 4, we see how our main result applies to truncations of matroids. We also explore whether a similar result can be obtained for matroid elongations.
- In Section 5, we give a counterexample demonstrating that our main result does not hold for the generalized skeleton ideals constructed in [8] and [7].

2. Preliminaries

2.1. Simplicial complexes.

Definition 2.1. A simplicial complex $\Delta$ on $E = \{1, \ldots, n\}$ is a collection of subsets of $E$ that is closed under inclusion.
We refer to the elements of $\Delta$ as the faces of $\Delta$. A facet of $\Delta$ is a face that is not properly contained in another face, while a nonface is a subset of $E$ that is not a face.

**Definition 2.2.** If $X \subseteq E$, then $\Delta|_X = \{ \sigma \subseteq X : \sigma \in \Delta \}$ is itself a simplicial complex. We refer to $\Delta|_X$ as the restriction of $\Delta$ to $X$.

**Definition 2.3.** Let $m$ be the cardinality of the largest face contained in $X \subseteq E$. The dimension of $X$ is $\dim(X) = m - 1$.

In particular, the dimension of a face $\sigma$ is equal to $|\sigma| - 1$. We define $\dim(\Delta) = \dim(E)$, and refer to this as the dimension of $\Delta$.

**Definition 2.4 (The $i$-skeleton of $\Delta$).** For $0 \leq i \leq \dim(\Delta)$, let the $i$-skeleton $\Delta^{(i)}$ be the simplicial complex

$$\Delta^{(i)} = \{ \sigma \in \Delta : \dim(\sigma) \leq i \}.$$

In particular, we have $\Delta^{(d)} = \Delta$. The 1-skeleton $\Delta^{(1)}$ is often referred to as the underlying graph of $\Delta$.

**Remark.** Whenever $\sigma \in \mathbb{N}_0^n$ the expression $|\sigma|$ shall signify the sum of the coordinates of $\sigma$. When, on the other hand, $\sigma \subseteq \{1, \ldots, n\}$, the expression $|\sigma|$ denotes the cardinality of $\sigma$.

### 2.2. Matroids

There are numerous equivalent ways of defining a matroid. It is most convenient here to give the definition in terms of independent sets. For an introduction to matroid theory in general, we recommend e.g. [17].

**Definition 2.5.** A matroid $M$ consists of a finite set $E$ and a non-empty set $I(M)$ of subsets of $E$ such that:

- $I(M)$ is a simplicial complex.
- If $I_1, I_2 \in I(M)$ and $|I_1| > |I_2|$, then there is an $x \in I_1 \setminus I_2$ such that $I_2 \cup x \in I(M)$.

The elements of $I(M)$ are referred to as the independent sets (of $M$). The bases of $M$ are the independent sets that are not contained in any other independent set: in other words, the facets of $I(M)$. Conversely, given the bases of a matroid, we find the independent sets to be those sets that are contained in a basis. We denote the bases of $M$ by $B(M)$. It is a fundamental result that all bases of a matroid have the same cardinality, which implies that $I(M)$ is a pure simplicial complex.

The dual matroid $\overline{M}$ is the matroid on $E$ whose bases are the complements of the bases of $M$. Thus

$$B(\overline{M}) = \{ E \setminus B : B \in B(M) \}.$$
Definition 2.6. For $X \subseteq E$, the rank function $r_M$ of $M$ is defined by

$$r_M(X) = \max\{|I| : I \in I(M), I \subseteq X\}.$$ 

Whenever the matroid $M$ is clear from the context, we omit the subscript and write simply $r(X)$. The rank $r(M)$ of $M$ itself is defined as $r(M) = r_M(E)$. Whenever $I(M)$ is considered as a simplicial complex we thus have $r(X) = \dim(X) + 1$ for all $X \subseteq E$, and $r(M) = \dim(I(M)) + 1$.

Definition 2.7. If $X \subseteq E$, then $\{I \subseteq X : I \in I(M)\}$ form the set of independent sets of a matroid $M_{|X}$ on $X$. We refer to $M_{|X}$ as the restriction of $M$ to $X$.

In [14] the $i$th generalized Hamming weight of a linear code is generalized to matroids as follows.

Definition 2.8. For $1 \leq i \leq n - r(M)$, the $i$th higher weight of $M$ is

$$d_i(M) = \min\{|X| : X \subseteq E \text{ and } |X| - r(X) = i\}.$$ 

We refer to $\{d_i(M)\}$ as the higher weights of $M$.

Definition 2.9 (Truncation). The $i$th truncation $M^{(i)}$ of $M$ is the matroid on $E$ whose independent sets consist of the independent sets of $M$ that have rank less than or equal to $r(M) - i$. In other words

$$I(M^{(i)}) = \{X \subseteq E : r(X) = |X|, r(X) \leq r(M) - i\}.$$ 

Observe that $M^{(i)} = I(M)^{(r(M) - i - 1)}$, whenever $I(M)$ is considered as a simplicial complex. That is, the $i$th truncation corresponds to the $(d-i)$-skeleton.

Definition 2.10 (Elongation). For $0 \leq i \leq n - r(M)$, let $M_{(i)}$ be the matroid whose independent sets are $I(M_{(i)}) = \{\sigma \in E : n(\sigma) \leq i\}$.

Since $r(M_{(i)}) = r(M) + i$, the matroid $M_{(i)}$ is commonly referred to as the elongation of $M$ to rank $r(M) + i$. It is straightforward to verify that for $i \in [0, \ldots, n - r(M)]$ we have $M_{(i)} = \overline{M^{(i)}}$.

2.3. The Stanley-Reisner ideal, Betti numbers, and the reduced chain complex. Let $\Delta$ be an abstract simplicial complex on $E = \{1, \ldots, n\}$. Let $\mathbb{k}$ be a field, and let $S = \mathbb{k}[x_1, \ldots, x_n]$. By employing the standard abbreviated notation

$$x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} = x^a$$

for monomials, we establish a $1-1$ connection between monomials of $S$ and vectors in $\mathbb{N}_0^n$. Furthermore, identifying a subset of $E$ with its indicator vector in $\mathbb{N}_0^n$ (as is done in Definition 2.11 below) thus provides a $1-1$ connection between squarefree monomials of $S$ and subsets of $E$. 


**Definition 2.11.** Let \( I_\Delta \) be the ideal in \( S \) generated by monomials corresponding to nonfaces of \( \Delta \). That is, let

\[
I_\Delta = \langle x^\sigma : \sigma \notin \Delta \rangle.
\]

We refer to \( I_\Delta \) and \( S/I_\Delta \), respectively, as the **Stanley-Reisner ideal** and **Stanley-Reisner ring** of \( \Delta \).

Being a (squarefree) monomial ideal, the Stanley-Reisner ideal, and thus also the Stanley-Reisner ring, permits both the standard \( \mathbb{N}_0 \)-grading and the standard \( \mathbb{N}_0^n \)-grading. For \( \mathbf{b} \in \mathbb{N}_0^n \) let \( S_\mathbf{b} \) be the 1-dimensional \( \mathbb{k} \)-vector space generated by \( \mathbf{x}^\mathbf{b} \), and let \( S(\mathbf{a}) \), \( S \) shifted by \( \mathbf{a} \), be defined by \( S(\mathbf{a})_\mathbf{b} = S_{\mathbf{a} + \mathbf{b}} \). Analogously, for \( j \in \mathbb{N}_0 \) let \( S_j \) be the \( \mathbb{k} \)-vector space generated by monomials of degree \( i \), and let \( S(j) \) be defined by \( S(j)_i = S_{i+j} \). For the remainder of this section let \( N \) be an \( \mathbb{N}_0^n \)-graded \( S \)-module.

**Definition 2.12.** An \((\mathbb{N}_0^n \text{ or } \mathbb{N}_0)-\text{graded minimal free resolution} \) of \( N \) is a left complex

\[
0 \leftarrow F_0 \leftarrow \phi_1 F_1 \leftarrow \phi_2 F_2 \leftarrow \cdots \leftarrow \phi_i F_i \leftarrow 0
\]

with the following properties:

- \( F_i = \left\{ \bigoplus_{\mathbf{a} \in \mathbb{N}_0^n} S(-\mathbf{a}) \beta_{i,a}, \mathbb{N}_0^n \text{-graded resolution} \right\}
  \cup \left\{ \bigoplus_{j \in \mathbb{N}_0} S(-j) \beta_{i,j}, \mathbb{N}_0 \text{-graded resolution} \right\}

- \( \text{im} \phi_i = \ker \phi_{i-1} \) for all \( i \geq 2 \), and \( F_0 / \text{im} \phi_1 \cong N \) (Exact)

- \( \text{im} \phi_i \subseteq \mathfrak{m} F_{i-1} \) (Minimal)

\[
\phi_i ((F_i)_a) \subseteq (F_{i-1})_a \quad \text{(Degree preserving, } \mathbb{N}_0^n \text{-graded case)}
\]

\[
\phi_i ((F_i)_j) \subseteq (F_{i-1})_j \quad \text{(Degree preserving, } \mathbb{N}_0 \text{-graded case)}.
\]

It follow from [9, Theorem A.2.2] that the Betti numbers associated to a \((\mathbb{N}_0 \text{ or } \mathbb{N}_0^n)-\text{graded minimal free resolution} \) are unique, in that any other minimal free resolution must have the same Betti numbers. We may therefore without ambiguity refer to \( \{ \beta_{i,a}(N; \mathbb{k}) \} \) and \( \{ \beta_{i,j}(N; \mathbb{k}) \} \), respectively, as the \( \mathbb{N}_0^n \)-graded and \( \mathbb{N}_0 \)-graded Betti numbers of \( N \) (over \( \mathbb{k} \)). Observe that

\[
\beta_{i,j}(N; \mathbb{k}) = \sum_{|a|=j} \beta_{i,a}(N; \mathbb{k})
\]

where \( |a| = a(1) + a(2) + \cdots + a(n) \) (see Remark 2.1 above). Note also that for an \( \mathbb{N}_0^n \)-graded (that is, monomial) ideal \( I \subseteq S \), we have \( \beta_{i,\sigma}(S/I; \mathbb{k}) = \beta_{i-1,\sigma}(I; \mathbb{k}) \) for all \( i \geq 1 \), and \( \beta_{0,\sigma}(S/I; \mathbb{k}) = \begin{cases} 1, \sigma = \emptyset \\ 0, \sigma \neq \emptyset \end{cases} \).
The $\mathbb{N}_0$-graded Betti numbers of $N$ may be compactly presented in a so-called Betti table:

\[
\begin{array}{cccc}
\beta [N; \mathbb{K}] & 0 & 1 & \cdots & l \\
      j & \beta_{0,j}(N; \mathbb{K}) & \beta_{1,j+1}(N; \mathbb{K}) & \cdots & \beta_{1,j+l}(N; \mathbb{K}) \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
      k & \beta_{0,k}(N; \mathbb{K}) & \beta_{1,k+1}(N; \mathbb{K}) & \cdots & \beta_{1,k+l}(N; \mathbb{K})
\end{array}
\]

By the (graded) Hilbert Syzygy Theorem we have $F_i = 0$ for all $i \geq n$. If $F_i \neq 0$ but $F_i = 0$ for all $i > l$, we refer to $l$ as the length of the minimal free resolution. It can be seen from e.g. [6, Corollary 1.8] that the length of a minimal free resolution of $N$ equals its projective dimension (p. d. $N$).

A sequence $f_1, \ldots, f_r \in \langle x_1, x_2, \ldots, x_n \rangle$ is said to be a regular $N$-sequence if $f_{i+1}$ is not a zero-divisor on $N/(f_1N + \cdots + f_iN)$.

**Definition 2.13.** The depth of $N$ is the common length of a longest regular $N$-sequence. Whenever $N$ is $\mathbb{N}_0$-graded the polynomials may be assumed to be homogeneous.

In general we have depth $N \leq \dim N$, where $\dim N$ denotes the Krull dimension of $N$. The following is a particular case of the famous Auslander-Buchsbaum Theorem.

**Theorem 2.1** (Auslander-Buchsbaum).

\[\text{p. d. } N + \text{depth } N = n.\]

**Proof.** See e.g. [9, Corollary A.4.3].

Note that the Krull dimension $\dim S/I_\Delta$ of $S/I_\Delta$ is one more than the dimension of $\Delta$ (see [9, Corollary 6.2.2]). The simplicial complex $\Delta$ is said to be Cohen-Macaulay if depth $S/I_\Delta = \dim S/I_\Delta$. That is, if $S/I_\Delta$ is Cohen-Macaulay as an $S$-module.

**Definition 2.14.** Let $\mathcal{F}_i(\Delta)$ denote the set of $i$-dimensional faces of $\Delta$. That is, $\mathcal{F}_i(\Delta) = \{ \sigma \in \Delta : |\sigma| = i + 1 \}$. Let $\mathbb{K}\mathcal{F}_i(\Delta)$ be the free $\mathbb{K}$-vector space on $\mathcal{F}_i(\Delta)$. The (reduced) chain complex of $M$ over $\mathbb{K}$ is the complex

\[0 \leftarrow \mathbb{K}\mathcal{F}_{i-1}(\Delta) \xleftarrow{\delta_i} \ldots \xleftarrow{\delta_i} \mathbb{K}\mathcal{F}_i(\Delta) \xleftarrow{\delta_i} \mathbb{K}\mathcal{F}_{i+1}(\Delta) \xleftarrow{\delta_i} \cdots \xleftarrow{\delta_{\dim(\Delta)}} \mathbb{K}\mathcal{F}_{\dim(\Delta)}(\Delta) \xleftarrow{\delta_i} 0,'\]

where the boundary maps $\delta_i$ are defined as follows: With the natural ordering on $E$, set $\text{sign}(j, \sigma) = (-1)^{r-1}$ if $j$ is the $r$th element of $\sigma \subseteq E$, and let

\[\delta_i(\sigma) = \sum_{j \in \sigma} \text{sign}(j, \sigma) \sigma \setminus j.\]
Extending $\delta_i \otimes \mathbb{k}$-linearly, we obtain a $\mathbb{k}$-linear map from $\mathbb{k}^F_i(\Delta)$ to $\mathbb{k}^F_{i-1}(\Delta)$.

**Definition 2.15.** The $i$th reduced homology of $\Delta$ over $\mathbb{k}$ is the vector space $\tilde{H}_i(\Delta; \mathbb{k}) = \ker(\delta_i) / \text{im}(\delta_{i+1})$.

The following is one of the most celebrated results in the intersection between algebra and combinatorics.

**Theorem 2.2 (Hochster’s formula).**

$$\beta_{i,j}(S/I_\Delta; \mathbb{k}) = \beta_{i-1,j}(I_\Delta; \mathbb{k}) = \dim \mathbb{k}^H_i(\Delta_{|\sigma}; \mathbb{k}).$$

**Proof.** See [15, Corollary 5.12] and [9, p. 81].

### 3. Betti numbers of $i$-skeletons

Let $\Delta$ be a $d$-dimensional simplicial complex on $\{1, \ldots, n\}$, and let $\mathbb{k}$ be a field. In this section we shall demonstrate how each of the Betti numbers of $S/I_{\Delta(d-1)}$ can be expressed as a $\mathbb{Z}$-linear combination of the Betti numbers of $S/I_\Delta$.

#### 3.1. The first rows of the Betti table.

**Lemma 3.1.**

$$\tilde{H}_i(\Delta_{|\sigma}; \mathbb{k}) = \tilde{H}_i(\Delta^{(d-1)}_{|\sigma}; \mathbb{k})$$

for all $0 \leq i \leq d - 2$.

**Proof.** By the definition of a skeleton we have $\mathcal{F}_i(\Delta_{|\sigma}) = \mathcal{F}_i(\Delta^{(d-1)}_{|\sigma})$ and thus also $\mathbb{k}^F_i(\Delta_{|\sigma}) = \mathbb{k}^F_i(\Delta^{(d-1)}_{|\sigma})$, for all $-1 \leq i \leq d - 1$. In other words, the reduced chain complexes of $\Delta_{|\sigma}$ and $\Delta^{(d-1)}_{|\sigma}$ are identical except for in homological degree $d$. The result follows.

**Proposition 3.1.** For all $i$ and $j \leq d + i - 1$ we have

$$\beta_{i,j}(S/I_\Delta; \mathbb{k}) = \beta_{i,j}(S/I_{\Delta^{(d-1)}_d}; \mathbb{k}).$$

**Proof.** If $j \leq d + i - 1$ then $j - i - 1 \leq d - 2$. By Theorem [2.2] and Lemma [3.1] then, we have

$$\beta_{i,j}(S/I_\Delta; \mathbb{k}) = \sum_{|\sigma|=j} \beta_{i,j}(S/I_\Delta; \mathbb{k})$$

$$= \sum_{|\sigma|=j} \dim \mathbb{k}^H_i(\Delta_{|\sigma}; \mathbb{k})$$

$$= \sum_{|\sigma|=j} \dim \mathbb{k}^H_i(\Delta^{(d-1)}_{|\sigma}; \mathbb{k})$$

$$= \sum_{|\sigma|=j} \beta_{i,j}(S/I_{\Delta^{(d-1)}_d}; \mathbb{k})$$

$$= \beta_{i,j}(S/I_{\Delta^{(d-1)}_d}; \mathbb{k}).$$
3.2. The final row of the Betti table. The Hilbert series of \( S/I_\Delta \) over \( \mathbb{k} \) is \( H(S/I_\Delta) = \sum_{i \in \mathbb{Z}} \dim_{\mathbb{k}}(S/I_\Delta)_i t^i \). Let \( f_i(\Delta) = |\mathcal{F}_i(\Delta)| \). By [9] Section 6.1.3, Equation (6.3) we have

\[
H(S/I_\Delta) = \frac{\sum_{i=0}^{n} (-1)^i \sum_j \beta_{i,j}(S/I_\Delta; \mathbb{k})}{(1-t)^n}.
\]

On the other hand, we see from [9, Proposition 6.2.1] that

\[
H(S/I_\Delta) = \frac{\sum_{i=0}^{d+1} f_{i-1}(\Delta) t^i (1-t)^{d+1-i}}{(1-t)^{d+1}}.
\]

Combined, these two equations imply

\[
\sum_{i=0}^{d+1} f_{i-1}(\Delta) t^i (1-t)^{n-i} = \sum_{i=0}^{n} (-1)^i \sum_j \beta_{i,j}(S/I_\Delta; \mathbb{k}) t^i,
\]

and

\[
\sum_{i=0}^{d} f_{i-1}(\Delta(d-1)) t^i (1-t)^{n-i} = \sum_{i=0}^{n} (-1)^i \sum_j \beta_{i,j}(S/I_{\Delta(d-1)}; \mathbb{k}) t^i.
\]

Remark. From here on we shall employ the convention that \( i! = 0 \) for \( i < 0 \), and that \( \binom{j}{l} = 0 \) if one or both of \( j \) and \( k \) is negative.

Differentiating both sides of equation (2) \( n-d-1 \) times, we get

\[
\sum_{i=0}^{d+1} f_{i-1}(\Delta) \sum_{l=0}^{n-d-1} (-1)^l \binom{n-d-1}{l} \frac{i!(n-i)!}{(i-n+d+1+l)! (n-i-l)!} t^{i-n+d+1+l} (1-t)^{n-i-l}
\]

\[
= \sum_{i=0}^{n} (-1)^i \sum_j \beta_{i,j}(S/I_\Delta; \mathbb{k}) \frac{j!}{(j-(n-d-1))!} t^{j-n+d+1}.
\]

When evaluated at \( t = 1 \), the left side of the above equation is 0 except when \( i = d+1 \) and \( l = n-d-1 \). Thus, we have

\[
(-1)^{n-d-1} (n-d-1)! f_d(\Delta) = \sum_{i=0}^{n} (-1)^i \sum_{j \geq n-d-1} \beta_{i,j}(S/I_\Delta; \mathbb{k}) \frac{j!}{(j-(n-d-1))!},
\]

and

\[
f_d(\Delta) = \sum_{i=0}^{n} (-1)^{n+d+i+1} \sum_{j \geq n-d-1} \binom{j}{n-d-1} \beta_{i,j}(S/I_\Delta; \mathbb{k}).
\]

Lemma 3.2. For all \( i \) and \( j \geq d+i+2 \) we have

\[
\beta_{i,j}(S/I_\Delta; \mathbb{k}) = 0.
\]
Proposition 3.2. Let \( |\sigma| \geq d + i + 2 \), then \( |\sigma| - i - 1 \geq \dim(\Delta) + 1 \), which implies

\[
\dim_k \tilde{H}_{|\sigma|-i-1}(\Delta; \mathbb{k}) = 0.
\]

So by Hochster’s formula we have that if \( j \geq d + i + 2 \) then

\[
\beta_{i,j}(S/I_\Delta; \mathbb{k}) = \sum_{|\sigma|=j} \beta_{i,\sigma}(S/I_\Delta; \mathbb{k}) = \sum_{|\sigma|=j} \dim_k \tilde{H}_{|\sigma|-i-1}(\Delta; \mathbb{k}) = 0.
\]

\( \square \)

According to Proposition 3.1 and Lemma 3.2 and because \( f_i(\Delta) = f_i(\Delta^{(d-1)}) \) for all \( i \neq d \), subtracting equation (3) from equation (2) yields

\[
f_d(\Delta) t^{d+1} (1-t)^{n-d-1} = \sum_{i=0}^{n} (-1)^i (\beta_{i,d+i}(S/I_\Delta; \mathbb{k}) - \beta_{i,d+i}(S/I_{\Delta^{(d-1)}}; \mathbb{k})) t^{d+i}
\]

\[
+ \sum_{i=0}^{n} (-1)^i \beta_{i,d+i+1}(S/I_\Delta; \mathbb{k}) t^{d+i+1}.
\]

Let \( 1 \leq u \leq n \). Differentiating both sides of the above equation \( d + u \) times yields

\[
f_d(\Delta) \sum_{l=0}^{d+u} (-1)^l \binom{d+u}{l} \frac{(d+1)! (n-d-1)!}{(l-u+1)! (n-d-1-l)!} t^{l-u+1} (1-t)^{n-d-1-l}
\]

\[
= \sum_{i=u}^{n} (-1)^i \left( \beta_{i,d+i}(S/I_\Delta; \mathbb{k}) - \beta_{i,d+i+1}(S/I_{\Delta^{(d-1)}}; \mathbb{k}) \right) \frac{(d+i)!}{(i-u)!} t^{i-u}
\]

\[
+ \sum_{i=u-1}^{n-1} (-1)^i \beta_{i,d+i+1}(S/I_\Delta; \mathbb{k}) \frac{(d+i+1)!}{(i-u+1)!} t^{i-u+1}.
\]

Evaluating at \( t = 0 \), we get

\[
\delta' = \left\{
\begin{array}{ll}
1, & 1 \leq u \leq n - d \\
0, & u > n - d
\end{array}
\right.
\]

where

\[
\delta' = \left\{
\begin{array}{ll}
1, & 1 \leq u \leq n - d \\
0, & u > n - d
\end{array}
\right.
\]

Summarizing the above:

**Proposition 3.2.** For \( 1 \leq u \leq n \), we have

\[
\beta_{u,d+u}(S/I_{\Delta^{(d-1)}}; \mathbb{k}) = \beta_{u,d+u}(S/I_\Delta; \mathbb{k}) - \beta_{u-1,d+u}(S/I_\Delta; \mathbb{k}) + \binom{n-d-1}{u-1} \delta,
\]

\[
\beta_{u,d+u}(S/I_{\Delta^{(d-1)}}; \mathbb{k}) = \beta_{u,d+u}(S/I_\Delta; \mathbb{k}) - \beta_{u-1,d+u}(S/I_\Delta; \mathbb{k}) + \binom{n-d-1}{u-1} \delta,
\]

\[
\beta_{u,d+u}(S/I_{\Delta^{(d-1)}}; \mathbb{k}) = \beta_{u,d+u}(S/I_\Delta; \mathbb{k}) - \beta_{u-1,d+u}(S/I_\Delta; \mathbb{k}) + \binom{n-d-1}{u-1} \delta,
\]
where
\[
\delta = \begin{cases} 
  f_d(\Delta) = \sum_{i=0}^{n} (-1)^{n+d+i+1} \sum_{j\geq n-d-1} \binom{n-j}{n-d-1} \beta_{i,j}(S/I_\Delta; \mathbb{K}), & 1 \leq u \leq n-d \\
  0, & u > n-d.
\end{cases}
\]

Bringing together Propositions 3.1 and 3.2, we get

**Theorem 3.1.** For all \( i \geq 1 \), we have

\[
\beta_{i,j}(S/I_{\Delta^{(d-1)}}; \mathbb{K}) = \begin{cases} 
  \beta_{i,j}(S/I_\Delta; \mathbb{K}), & j \leq d + i - 1 \\
  \beta_{i,d+j}(S/I_\Delta; \mathbb{K}) - \beta_{i-1,d+j}(S/I_\Delta; \mathbb{K}) + (n-d-1) \delta, & j = d + i, \\
  0, & j \geq d + i - 1
\end{cases}
\]

where
\[
\delta = \begin{cases} 
  f_d(\Delta) = \sum_{k=0}^{n} (-1)^{n+d+k+1} \sum_{l\geq n-d-1} \binom{n-l}{n-d-1} \beta_{k,l}(S/I_\Delta; \mathbb{K}), & 1 \leq i \leq n-d \\
  0, & i > n-d
\end{cases}
\]

**Example 3.1.** Let \( T \) be one of the two irreducible triangulations of the real projective plane (see [1]) – namely the one corresponding to an embedding of the complete graph on 6 vertices. Clearly then, we have \( n = 6 \) and \( d = 2 \). The Betti table of \( S/I_T \) over \( \mathbb{F}_3 \) is

\[
\beta[S/I_T; \mathbb{F}_3] = \begin{bmatrix}
0 & 1 & 2 & 3 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
3 & 0 & 10 & 15 & 6
\end{bmatrix}.
\]

In this case
\[
f_d(\Delta) = \binom{4}{3} \beta_{1,4}(S/I_T; \mathbb{F}_3) - \binom{5}{3} \beta_{2,5}(S/I_T; \mathbb{F}_3) + \binom{6}{3} \beta_{3,6}(S/I_T; \mathbb{F}_3) = 10.
\]

By Theorem 3.1 the Betti numbers of \( S/I_{T(1)} \) are

\[
\beta_{1,4}(S/I_{T(1)}; \mathbb{F}_3) = \beta_{1,4}(S/I_T; \mathbb{F}_3) + \binom{3}{0} \delta = 10 + 10.
\]

\[
\beta_{2,5}(S/I_{T(1)}; \mathbb{F}_3) = \beta_{2,5}(S/I_T; \mathbb{F}_3) - \beta_{1,5}(S/I_T; \mathbb{F}_3) + \binom{3}{1} \delta = 15 + 30.
\]

\[
\beta_{3,6}(S/I_{T(1)}; \mathbb{F}_3) = \beta_{3,6}(S/I_T; \mathbb{F}_3) - \beta_{2,6}(S/I_T; \mathbb{F}_3) + \binom{3}{2} \delta = 6 - 0 + 30.
\]

\[
\beta_{4,7}(S/I_{T(1)}; \mathbb{F}_3) = \beta_{4,7}(S/I_T; \mathbb{F}_3) - \beta_{3,7}(S/I_T; \mathbb{F}_3) + \binom{3}{3} \delta = 0 - 0 + 10.
\]
Betti Numbers of Skeletons

\[
\beta[S/I_T; \mathbb{F}_3] =
\begin{array}{c|cccc}
   & 0 & 1 & 2 & 3 \\
\hline
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
3 & 0 & 20 & 45 & 36 \\
\end{array}
\]

Remark. Observe that as

\[
\beta[S/I_T; \mathbb{F}_2] =
\begin{array}{c|cccc}
   & 0 & 1 & 2 & 3 \\
\hline
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
3 & 0 & 10 & 15 & 6 \\
4 & 0 & 0 & 1 & 0 \\
\end{array}
\]

the simplicial complex \(T\) of Example 3.1 is an example of a pure simplicial complex whose Betti numbers depend upon the field \(\mathbb{F}_2\) – as opposed to what is the case for matroids.

3.3. The projective dimension of skeletons. Let \(p. d. S/I_\Delta\) denote the projective dimension of \(S/I_\Delta\). By Auslander-Buchsbaum Theorem we have

\[
p. d. S/I_\Delta = n - \text{depth } S/I_\Delta \\
\geq n - \text{dim } S/I_\Delta \\
= n - (d + 1),
\]

so \(n - d - 1 \leq p. d. S/I_\Delta \leq n\).

As for the skeletons, we have

**Corollary 3.1.**

\[
p. d. S/I_{\Delta^{(d-1)}} \leq 1 + p. d. S/I_\Delta.
\]

**Proof.** Let \(p = p. d. S/I_\Delta\). By Proposition 3.1 it suffices to show that

\[
\beta_{p+2, d+p+2}(S/I_{\Delta^{(d-1)}}; \mathbb{F}_2) = 0.
\]

But by Theorem 3.2, we have

\[
\beta_{p+2, d+p+2}(S/I_{\Delta^{(d-1)}}; \mathbb{F}_2) = \beta_{p+2, d+p+2}(S/I_\Delta; \mathbb{F}_2) - \beta_{p+1, d+p+2}(S/I_\Delta; \mathbb{F}_2) + \delta \\
= 0 - 0 - \delta = 0,
\]

where the last equality is due to \(p + 2 > n - d\).

**Corollary 3.2.** If \(\Delta\) is Cohen-Macaulay, then so is \(\Delta^{(d-1)}\).

**Proof.** Let \(\Delta\) be a simplicial complex with \(\text{dim}(\Delta) = d\) and \(\text{depth } S/I_\Delta = \text{dim } S/I_\Delta\). As \(\text{dim } S/I_{\Delta^{(d-1)}} = d\), we only need to prove that \(\text{depth } S/I_{\Delta^{(d-1)}} = d\) as well.
Since depth \( S/I_{\Delta(d-1)} \leq \dim S/I_{\Delta(d-1)} = d \), we have by the Auslander-Buchsbaum Theorem that p.d. \( S/I_{\Delta(d-1)} \geq n - d \). On the other hand, since
\[
p.d. S/I_{\Delta} = n - \text{depth } S/I_{\Delta} = n - \dim S/I_{\Delta} = n - (d + 1),
\]
we see from Corollary [3.1] that p.d. \( S/I_{\Delta(d-1)} \leq n - d \). We conclude that
\[
p.d. S/I_{\Delta(d-1)} = n - d
\]
and, by Auslander-Buchsbaum again, that depth \( S/I_{\Delta(d-1)} = d \). 

4. BETTI NUMBERS OF TRUNCATIONS AND ELONGATIONS OF MATROIDS

Let \( M \) be a matroid on \( \{1, \ldots, n\} \), with \( r(M) = k \). As was established in [3], the dimension of \( \tilde{H}_i(M; \mathbb{k}) \) is in fact independent of the field \( \mathbb{k} \). Thus for matroids, the \((\mathbb{N}_0\text{-} or \mathbb{N}_0^n\text{-}graded)\) Betti numbers are not only unique, but independent of the choice of field. We shall therefore omit referring to or specifying a particular field \( \mathbb{k} \) throughout this section. By a slight abuse of notation we shall denote the Stanley-Reisner ideal associated to the set of independent sets \( I(M) \) of \( M \) simply by \( I_M \).

4.1. Truncations. Note that the \( i \)th truncation of \( M \) corresponds to the \((k - i - 1)\)-skeleton of \( I(M) \), a fact which enables us to invoke Theorem [3.1]. In addition, it follows from [12, Corollary 3(b)] that the minimal free resolutions of \( S/I_M \) have length \( n - k \). We thus have

**Proposition 4.1.** For all \( i \), we have
\[
\beta_{i,j}(S/I_{M^{(1)}}) = \begin{cases} 
\beta_{i,j}(S/I_M), & j \leq k + i - 2. \\
\beta_{i,k+i-1}(S/I_M) - \beta_{i-1,k+i-1}(S/I_M) + \binom{n-k}{i-1} \left( \sum_{u=0}^{n-k} (-1)^{n+k+u} \sum_{v \geq n-k} \binom{n-k}{v} \beta_{u,v}(S/I_M) \right), & j = k + i - 1. \\
0, & j \geq k + i.
\end{cases}
\]

**Corollary 4.1.** For all \( 1 \leq i \leq n - k + 1 \), we have
\[
d_i(M^{(1)}) = \min \{d_i(M), k + i - 1\}.
\]

**Proof.** By [12, Theorem 4] we have \( d_i(M^{(1)}) = \min \{j : \beta_{i,j}(S/I_{M^{(1)}}) \neq 0\} \). The result now follows immediately from Proposition 4.1. \( \square \)
4.2. **Elongations.** When it comes to elongations, the Betti numbers of $M$ provide far less information about the Betti numbers of $M_{(1)}$ than what was the case with truncations. We do however have the following.

**Proposition 4.2.** For $i \geq 1$,

$$\beta_{i,j}(I_{M_{(l)}}) \neq 0 \iff \beta_{i-1,j}(I_{M_{(l+1)}}) \neq 0.$$ 

**Proof.** According to [12, Theorem 1], we have that

$$\beta_{i,\sigma}(I_{M}) \neq 0 \iff \sigma \text{ is minimal with the property that } n_{M}(\sigma) = i + 1.$$ 

Since $\beta_{i,j} = \sum_{|\sigma| = j} \beta_{i,\sigma}$, we see that

$$\beta_{i,j}(I_{M_{(l)}}) \neq 0 \iff \exists \sigma \text{ such that } |\sigma| = j \text{ and } \sigma \text{ is minimal with the property that } n_{M_{(l)}}(\sigma) = i + 1$$

$$\iff \exists \sigma \text{ such that } |\sigma| = j \text{ and } \sigma \text{ is minimal with the property that } n_{M_{(l+1)}}(\sigma) = i$$

$$\iff \beta_{i-1,j}(I_{M_{(l+1)}}) \neq 0.$$ 

$\square$

In terms of Betti tables, this implies that when it comes to zeros and nonzeros the Betti table of $I_{M_{(l+1)}}$ is equal to the table you get by deleting the first column from the table of $I_{M}$. As the following counterexample (computed using MAGMA [2]) demonstrates, there can be no result for elongations analogous to Theorem [3,1].

Let $M$ and $N$ be the matroids on $\{1, \ldots, 8\}$ with bases

$$B(M) = \{\{1,3,4,6,7\}, \{1,2,3,6,8\}, \{1,2,3,4,8\}, \{1,2,3,5,8\}, \{1,2,5,6,8\},$$

$$\{1,2,3,4,7\}, \{1,2,3,5,7\}, \{1,2,5,6,7\}, \{1,3,4,5,7\}, \{1,3,4,6,8\},$$

$$\{1,2,4,6,8\}, \{1,2,4,6,7\}, \{1,3,4,5,8\}, \{1,2,4,5,7\}, \{1,4,5,6,7\},$$

$$\{1,2,3,6,7\}, \{1,3,5,6,7\}, \{1,4,5,6,8\}, \{1,3,5,6,8\}, \{1,2,4,5,8\}\}$$

and

$$B(N) = \{\{1,3,4,6,7\}, \{1,2,3,4,8\}, \{1,2,3,5,8\}, \{1,2,5,6,8\}, \{1,2,3,4,7\},$$

$$\{1,2,3,5,7\}, \{1,2,5,6,7\}, \{1,3,4,5,7\}, \{1,3,4,6,8\}, \{1,2,4,6,8\},$$

$$\{1,2,4,6,7\}, \{1,3,4,5,8\}, \{1,2,4,5,7\}, \{1,3,4,5,6\}, \{1,2,4,5,6\},$$

$$\{1,3,5,6,7\}, \{1,2,3,5,6\}, \{1,2,3,4,6\}, \{1,3,5,6,8\}, \{1,2,4,5,8\}\}.$$
Both $I_M$ and $I_N$ have Betti table

|   | 0 | 1 | 2 |
|---|---|---|---|
| 2 | 1 | 0 | 0 |
| 3 | 0 | 0 | 0 |
| 4 | 1 | 4 | 0 |
| 5 | 0 | 5 | 4 |

but while $I_{M(1)}$ has Betti table

|   | 1 | 2 |
|---|---|---|
| 5 | 1 | 0 |
| 6 | 5 | 5 |

the ideal $I_{N(1)}$ has Betti table

|   | 1 | 2 |
|---|---|---|
| 5 | 2 | 0 |
| 6 | 3 | 4 |

This shows that the Betti numbers associated to a matroid do not determine those associated to its elongation.

5. THE $i$th SKELETON IDEAL

As mentioned in the introduction, the Stanley-Reisner ideal of a skeleton is generalized in [8] and [7] to arbitrary monomial ideals. We shall briefly describe the construction as it is found in the above papers, and present a counterexample showing that our main result does not extend to these ideals.

For $a, b \in \mathbb{N}_0^n$ we say that $a \leq b$ if $a(i) \leq b(i)$ for $1 \leq i \leq n$. Clearly, this constitutes a partial order on $\mathbb{N}_0^n$. Let $I, J \subseteq S$ be monomial ideals with (unique) minimal generating sets $\{x^{a_1}, \ldots, x^{a_r}\}$ and $\{x^{b_1}, \ldots, x^{b_s}\}$, respectively, and let $g \in \mathbb{N}_0^n$ be such that $a_i \leq g$ and $b_j \leq g$ for all $1 \leq i \leq r$, $1 \leq j \leq s$. Define the characteristic poset $P^g_{J/I}$ of $J/I$ with respect to $g$ to be

$$P^g_{J/I} = \{ b \in \mathbb{N}_0^n : b \leq g, b \geq b_j \text{ for some } j, b \nless a_i \text{ for all } i \}.$$

For $b \in \mathbb{N}_0^n$, let $\rho(b) = |i : b(i) = g(i)|$. It is demonstrated in [8, Corollary 2.6] that $\dim J/I = \max \{ \rho(b) : b \in P^g_{J/I} \}$.

The $j$th generalized skeleton ideal $I_j$ is the ideal generated by $\{x^{a_1}, \ldots, x^{a_r}\} \cup \{x^b : b \in \mathbb{N}_0^n, \rho(b) > j\}$. By [7, Corollary 2.5] these ideals form a chain $I = I_d \subseteq I_{d-1} \subseteq \cdots \subseteq I_0 \subseteq S$ with the property that $S/I_j$ is Cohen-Macauley for all $j \leq \text{depth } S/I$, and depth $S/I = \max \{ j : S/I_j \text{ is Cohen-Macauley} \}$. In other words, these ideals successfully generalize (1) from the introduction.
Furthermore, in the special case $J = S$, $I = I_\Delta$, and $g = (1, 1, \ldots, 1)$, we have $I_j = I_{\Delta(j)}$.

Now, let $M, N$ be the matroids on $\{1, \ldots, 6\}$ with

$$B(M) = \{\{1, 3, 6\}, \{1, 3, 5\}, \{4, 5, 6\}, \{1, 3, 4\}, \{2, 3, 6\}, \{1, 2, 5\},$$
$$\quad \{2, 4, 6\}, \{1, 4, 6\}, \{3, 5, 6\}, \{2, 3, 4\}, \{1, 2, 3\}, \{1, 5, 6\},$$
$$\quad \{3, 4, 5\}, \{1, 4, 5\}, \{1, 2, 4\}, \{2, 5, 6\}, \{2, 3, 5\}, \{3, 4, 6\}\}$$

and

$$B(N) = \{\{1, 3, 6\}, \{1, 3, 5\}, \{4, 5, 6\}, \{1, 3, 4\}, \{1, 2, 6\}, \{2, 3, 6\},$$
$$\quad \{1, 2, 5\}, \{2, 4, 6\}, \{3, 5, 6\}, \{2, 3, 4\}, \{1, 2, 3\}, \{1, 5, 6\},$$
$$\quad \{3, 4, 5\}, \{1, 4, 5\}, \{1, 2, 4\}, \{2, 5, 6\}, \{2, 4, 5\}, \{3, 4, 6\}\}.$$

Then

$$\beta[S/I_M] = \begin{array}{cccc}
0 & 1 & 2 & 3 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
2 & 0 & 2 & 0 & 0 \\
3 & 0 & 9 & 18 & 8
\end{array} = \beta[S/I_N],$$

for all base fields $\mathbb{K}$. However, if we take $g = (1, 2, 1, 1, 1, 1)$ and $J = S = \mathbb{Q}[x_1, \ldots, x_n]$ in the above construction, we get

$$\beta[S/(I_M)_{(1)}; \mathbb{Q}] = \begin{array}{cccc}
0 & 1 & 2 & 3 & 4 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 12 & 30 & 12 & 2 \\
3 & 0 & 17 & 24 & 24 & 8
\end{array},$$

while

$$\beta[S/(I_N)_{(1)}; \mathbb{Q}] = \begin{array}{cccc}
0 & 1 & 2 & 3 & 4 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 11 & 27 & 9 & 1 \\
3 & 0 & 18 & 27 & 27 & 9
\end{array}.$$ We conclude that the statement of Theorem 3.1 does not necessarily hold if one replaces the Stanley-Reisner ideals of skeletons with generalized skeleton ideals.

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