Pricing FX Options under Intermediate Currency

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Abstract

We introduce a new pricing mechanism for FX options, which is based on the idea of an intermediate pseudo-currency market. This approach allows us to price options on all FX markets simultaneously under the same risk-neutral measure which ensures consistency of FX option prices across all markets. In particular, it is sufficient to calibrate a model to the volatility smile on the domestic market as, due to the consistency of pricing formulas, the model automatically reproduces the correct smile for the inverse pair (the foreign market). We first consider the case of two currencies and then we extend the pricing mechanism to the multi-currency setting. We illustrate the new pricing mechanism by applying it to the Heston and SABR stochastic volatility models, to the model in which exchange rates are described by an extended skewed normal distribution, and also to the model-free approach of option pricing.

Keywords: foreign exchange market, FX option pricing, foreign-domestic symmetry, multi-currency options, skewed normal distribution.

1 Introduction

In the commonly used approach to risk-neutral pricing of foreign exchange (FX) options, an arbitrage price of an FX option depends on whether the option is priced with respect to the domestic or foreign numeraire of the currency pair, because there is no measure which is risk-neutral simultaneously for both markets (see e.g. [4, 26] and also Section 2 here). For stochastic volatility models, this dependence on numeraire results in violation of the foreign-domestic symmetry, which is a fundamental relationship between prices of put and call FX options on the two markets as dictated by the no-arbitrage assumption [21]. This can be seen as an asymmetry between the different market views due to the different choice of numeraires. To address this asymmetry of FX option pricing, currency pair conventions are usually used in practice in order to standardize option price quotations for each specific currency pair [21 26]. The question addressed in our paper is whether it is possible to have a pricing mechanism which works for pricing FX options on both markets simultaneously in a consistent way.

Further, it is of practical importance to be able to price options within a multi–currency setting of the global FX market in a consistent fashion. With a large number \( N \) of currencies, the existence of a consistent FX model is not trivial as a suitable model must preserve relationships between all \( N \) currencies and consistency of volatility smiles between all \( N(N - 1)/2 \) cross pairs. Moreover, a model has to be also consistent so that all exchange rates are described by a stochastic process of the same type. To address these problems of consistent FX modelling, in [10] (see also [12 11 8]) the concept of intrinsic currency [10 11] or artificial currency [8] was introduced. The approach of [10] is based on the idea that each currency has an ‘intrinsic value’, which is a description of the value of a currency in relation to other currencies. In the intrinsic

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currency-valuation framework of [10, 11] one models the $N$ intrinsic values of $N$ currencies rather than modelling the $N - 1$ exchange rates. In [11] Doust extends his original idea of the intrinsic currency-valuation framework to a SABR-type model, which allows to capture the observed volatility smile on the FX market in a multi-currency setting. On a FX market with $N$ currencies, he describes the market with $N$ intrinsic currency values and chooses one (without loss of generality) as the valuation currency and its associated risk–neutral measure, which produces the usual risk–neutral processes for all exchange rates. For option pricing, this approach results in a closed form solution similar to the original SABR model by Hagan et al. [17] adapted to the intrinsic currency–valuation framework, which allows to price FX vanilla options on one currency pair considering the correlation effects of all $N$ currencies. In [8] $N$ exchange rates between an artificial currency and $N$ real currencies are modelled under a risk-neutral measure associated with the artificial currency so that all relationships (in particular, the inversion property that the exchange rate for a pair of real currencies and for their inverse satisfy SDEs of a similar form) between $N$ currencies are satisfied.

Our FX option pricing approach is based on the concept of an intermediate pseudo-currency. The main difference with [10, 12, 11, 8] is that the pseudo-currency is explicitly defined via exchange rates of real currencies, while in [10, 12, 11, 8] exchange rates of real currencies are described via an artificial currency. Consequently, we naturally model $N - 1$ exchange rates, not $N$ as in [10, 12, 11, 8]. Further, we can use three modelling approaches. The first one is the traditional modelling way in Financial Mathematics, where we start from a stochastic model for $N - 1$ exchange rates under a ‘market’ measure and then we introduce a pseudo-currency market which, as we show, has a risk-neutral measure. Under this risk-neutral measure (the intermediate pseudo-currency is used as the numeraire) we can price FX products on all currency markets simultaneously which guarantees consistency of volatility smiles and other natural relationships between currencies (e.g., the foreign-domestic symmetry). This approach allows us to start with popular stochastic volatility models (e.g., Heston or SABR) written under a ‘market’ measure and derive the corresponding consistent models on the pseudo-currency market. Alternatively, in the second approach, from the start we model exchange rates under a risk-neutral measure or under a forward measure associated with the pseudo-currency market. The third approach is model-free (see [1, 13, 2, 9] and references therein), where we reconstruct a risk-neutral measure or a forward measure from volatility smiles.

The rest of the paper is organized as follows. In Section 2 we recall that there is no measure which is simultaneously risk-neutral for both domestic and foreign FX markets and also recall the foreign-domestic symmetry. The intermediate pseudo-currency and new pricing mechanism are introduced in Section 3, which is done for clarity of the exposition in the case of a single currency pair. We extend this new concept to the multi-currency setting in Section 4. In Section 5 we illustrate the new pricing mechanism by first applying it to the Heston model [18] and SABR [17]. Then, for further illustration, we model the spot exchange rate using an extended skewed normal distribution. This exchange rate model is an illustration of how one can describe the observed fat-tailed distribution of the log exchange rate (compared to the assumption of log normal). The considered extended skewed normal distribution is constructed by combining one normal and two shifted half-normal distributed random variables and it allows a flexible control of the tails of the spot exchange rate distribution. We note that the use of the extended skewed normal distribution in pricing FX options is somewhat new. Further, we illustrate our FX option pricing mechanism on the model-free approach. We provide some calibration examples in Section 6.

2 Preliminaries

In this section we recall (see e.g. [4, 26]) that there is no measure which is simultaneously risk-neutral for both the domestic and the foreign market. We also state the foreign-domestic
symmetry. For definiteness, in this section and in Section 3 we use the EUR-USD and USD-EUR pairs.

Let us recall that the EUR-USD spot exchange rate at time $t$

$$f(t) := S_{\text{e}/\text{s}}(t)$$

is quoted as

units of USD

one EUR,

and

$$S_{\text{s}/\text{e}}(t) = \frac{1}{S_{\text{e}/\text{s}}(t)} = \frac{1}{f(t)}.$$  

In currency pairs (e.g. EUR-USD), the first mentioned currency is known as the foreign (or base) currency, while the second is known as the domestic currency (or numeraire) [5, 26].

Within the standard option pricing setting, we assume that the currency market under a ‘market’ measure is described by the system:

$$dB_{\text{s}} = r_{\text{s}}(t)B_{\text{s}}dt,$$

$$dB_{\text{e}} = r_{\text{e}}(t)B_{\text{e}}dt,$$

$$df = \mu(t)fdt + \sigma(t)fW(t),$$

where $B_{\text{s}}(t), B_{\text{e}}(t)$ and $r_{\text{s}}(t), r_{\text{e}}(t)$ are USD and EUR bank accounts with their short interest rates, respectively; $\sigma(t) > 0$ is a volatility, $\mu(t)$ is a drift; and $W(t)$ is a standard Wiener process. It is assumed that the coefficients $r_{\text{s}}(t), r_{\text{e}}(t), \sigma(t),$ and $\mu(t)$ are stochastic processes adapted to a filtration $\mathcal{F}_t$ to which $W(t)$ is also adapted (typically, in stochastic volatility models $\mathcal{F}_t$ is larger than the natural filtration of $W(t)$), and they have bounded second moments. We also require that $\sigma(t)$ satisfies Novikov’s condition.

On the USD market, the foreign currency EUR is paid for by USD (the domestic currency) and the risky asset is

$$Y_{\text{e}/\text{s}}(t) = S_{\text{e}/\text{s}}(t)B_{\text{e}}(t),$$

while on the EUR market the risky asset is

$$Y_{\text{s}/\text{e}}(t) = S_{\text{s}/\text{e}}(t)B_{\text{s}}(t).$$

Following the classical theory of pricing, we have to find equivalent (local) martingale measures (EMMs) $Q^\$ and $Q^\e$ under which the corresponding discounted risky assets are (local) martingales. By standard arguments we arrive at the SDEs for $f(t)$ and $g(t) := 1/f(t)$ written under the corresponding EMMs:

$$df = (r_{\text{s}}(t) - r_{\text{e}}(t))fdt + \sigma(t)fW^{Q^\$}(t),$$

$$dg = (r_{\text{e}}(t) - r_{\text{s}}(t))gdt - \sigma(t)gW^{Q^\e}(t),$$

where $W^{Q^\$}(t)$ is a standard Wiener process under $Q^\$ and $W^{Q^\e}(t)$ is a standard Wiener process under $Q^\e$. We can see (cf. (2.1) and (2.2)-(2.3)) that the market prices of risk on the two markets differ:

$$\gamma_{\text{e}}(t) = \frac{\mu(t) + r_{\text{e}}(t) - r_{\text{s}}(t)}{\sigma(t)} \neq \frac{\sigma^2(t) - \mu(t) + r_{\text{s}}(t) - r_{\text{e}}(t)}{-\sigma(t)} = \gamma_{\text{s}}(t)$$

(recall that $\sigma(t) > 0$). Thus,

$$Q^\$ \neq Q^\e,$$

i.e., there is no measure which is simultaneously risk-neutral for the EUR domestic market and for the USD domestic market in this rather general setting.
Note that the SDE for $f$ under the measure $Q^e$ takes the form

$$df = (r_S(t) - r_e(t) + \sigma^2(t))fdt + \sigma(t)f dW^Q_e.$$  

(2.5)

Intuitively, one could think that the drift for the exchange rate $g(t) = 1/f(t)$ in (2.3) should be the negative of the drift of $f(t)$ under the same measure, i.e. $-(r_e(t) - r_S(t)) = r_S(t) - r_e(t)$. However, as we can see in (2.5), this is not the case. This is related to the phenomenon known as Siegel's paradox [24], which is due to the convexity of the function $1/f$.

Let us also recall [21, 26] that under the no-arbitrage assumption (and other standard conditions like no transaction costs, etc.), there is the so-called foreign-domestic symmetry for FX options which we formulate in the following theorem. This symmetry is the key requirement for a model to be consistent for a currency pair and its inverse pair (see e.g. [10][11][8][15] and references therein).

**Theorem 2.1.** Under the no-arbitrage assumption, there is the following relationship (called **Foreign-Domestic Symmetry**) for FX options

$$C_{e/S}(0, T, K) = S_{e/S}(0) K P_{S/e}(0, T, 1/K),$$  

(2.6)

where $C_{e/S}(0, T, K)$ is the call option price (in $S$) at time 0 to buy one EUR for $S K$ at time $T$; $P_{S/e}(0, T, 1/K)$ is the put option price (in $e$) at time 0 to sell one USD for $1/e K$ at time $T$.

Let us emphasise that the proof of this theorem is solely based on the no-arbitrage argument, and hence it states a fundamental property of the FX market. However, since the risk neutral measures are different on the two markets (2.4), stochastic volatility models (including popular models such as the Heston and SABR) are not compatible with this property (2.6) (see e.g. [10][11]): the risk-neutral pricing leads to different calibrated parameters depending on the choice of a pair. We note that for the SABR model there is a one-to-one mapping between the parameters obtained for USD-EUR and the parameters of the inverted world (i.e., EUR-USD), still the parameters are different for the direct and inverted worlds.

In the next section we propose a pricing mechanism based on an intermediate pseudo-currency, which is consistent with the foreign-domestic symmetry, in particular, within this new approach, calibration of a stochastic volatility model using FX data from one of the domestic markets guarantees replication of volatility smiles by the model on both domestic markets. This is achieved within our new pricing mechanism because on the introduced intermediate market we can price FX products from both EUR-USD and USD-EUR domestic markets under the same EMM in contrast to the traditional approach discussed earlier in this section, where FX products are priced using two different EMMs depending on which market products are traded.

### 3 FX option pricing via intermediate pseudo-currency

As we discussed in the previous section, the commonly used approach to pricing of FX options lacks consistency. To address this problem, we introduce an intermediate pseudo-currency market in this section. We note that the intermediate currency market is virtual and is only used as a proxy to derive the new pricing formula while calibration of the intermediate currency market is done using the usual FX data. We start with the definition of the pseudo-currency, then (Section 3.1) we consider pricing under an EMM $Q^X$ on the pseudo-market and (Section 3.2) under the T-forward measure $Q^X_T$ equivalent to $Q^X$.

**Definition 3.1.** Let $S_{e/S}(t) = f(t)$ be the EUR-USD exchange rate at time $t$. An **intermediate pseudo-currency** $X$ is a currency with exchange rate EUR-$X$, $S_{e/X}(t) = \sqrt{f(t)}$, and the exchange rate USD-$X$, $S_{S/X}(t) = \frac{1}{\sqrt{f(t)}}$.
We observe the natural relationship for the intermediate currency

\[ S_{e/X}(t) \cdot \frac{1}{S_{s/X}(t)} = f(t). \]  

(3.1)

We note the following symmetry:

\[ S_{e/X}(t) = \sqrt{f(t)} = \frac{1}{S_{s/X}(t)} = S_{X/s}(t) \]

and

\[ S_{s/X}(t) = \frac{1}{\sqrt{f(t)}} = \frac{1}{S_{e/X}(t)} = S_{X/e}(t). \]

We also introduce the money market account \( B_X \) for the intermediate currency \( X \) with its respective interest rate \( r_X(t) \):

\[ dB_X = r_X(t)B_X dt. \]  

(3.2)

In the next section we first establish that for a sufficiently broad class of models for \( f(t) \) there is an EMM \( Q^X \) on the pseudo-market and then, assuming existence of an EMM \( Q^X \), we derive a pricing formula.

### 3.1 An EMM for the intermediate market

Consider the virtual market which domestic currency is \( X \). On this market we have two risky assets: USD paid by \( X \) and EUR paid by \( X \):

\[ Y_{e/X}(t) = S_{e/X}(t)B_e(t), \quad Y_{s/X}(t) = S_{s/X}(t)B_s(t). \]  

(3.3)

Assume that EUR-USD exchange rate \( f(t) \) satisfies the model \( \text{(2.1)} \). Based on \( \text{(2.1)} \), we can write the SDEs under market measure for \( Y_{e/X}(t) \) and \( Y_{s/X}(t) \):

\[ dY_{e/X} = \frac{1}{2} \left( \mu(t) + 2r_e(t) - \frac{\sigma^2(t)}{4} \right) Y_{e/X} dt + \frac{\sigma(t)}{2} Y_{e/X} dW(t), \]

\[ dY_{s/X} = \frac{1}{2} \left( -\mu(t) + 2r_s(t) + \frac{3\sigma^2(t)}{4} \right) Y_{s/X} dt - \frac{\sigma(t)}{2} Y_{s/X} dW(t). \]

If we choose the intermediate currency interest rate \( r_X(t) \) equal to

\[ r_X(t) = \frac{r_s(t) + r_e(t)}{2} + \frac{\sigma^2(t)}{8}, \]  

(3.4)

then there is an EMM \( Q^X \) for the pseudo-currency market with the following market price of risk \( \gamma(t) \):

\[ \gamma(t) = \frac{\mu(t) - \frac{\sigma^2(t)}{2} + r_e(t) - r_s(t)}{\sigma(t)}, \]  

(3.5)

i.e.

\[ dY_{e/X} = r_X(t)Y_{e/X} dt - \frac{\sigma(t)}{2} Y_{e/X} dW^{Q^X}, \]

\[ dY_{s/X} = r_X(t)Y_{s/X} dt + \frac{\sigma(t)}{2} Y_{s/X} dW^{Q^X}, \]

where \( W^{Q^X} \) is the standard Wiener process under \( Q^X \). Thus, we have shown that the intermediate pseudo-currency market can be arbitrage free within a sufficiently broad setting. We summarise this result in the following statement.
Theorem 3.2. Assume that the EUR-USD currency market under a ‘market’ measure is described by the model \([2,7]\). Then there is the unique intermediate currency interest rate \(r_X(t)\) defined in \([3,4]\) and an EMM \(Q^X\) for the intermediate pseudo-currency market with the market price of risk \(\gamma(t)\) from \([3,7]\), i.e., under \([3,4]\) the market is arbitrage-free.

We see from \([3,4]\) that even if the short rates \(r_s(t)\) and \(r_e(t)\) are assumed to be constant, the intermediate currency interest rate \(r_X(t)\) is non-constant if the volatility \(\sigma(t)\) is time-dependent. In particular, if \(\sigma(t)\) is a stochastic process, then so is the short rate \(r_X(t)\).

Example 3.1 (An analogue of the Garman-Kohlhagen formula). Assume that the exchange rate between EUR and USD \(f(t) = S_{E/S}(t)\) satisfies the model \([2,1]\) with constant coefficients: \(\sigma(t) = \sigma, r_e(t) = r_e\) and \(r_s(t) = r_s\). Note that in this simplified case (the geometric Brownian motion case) the intermediate currency interest rate \(r_X(t)\) is constant. Analogously, to the standard derivation of the Garman-Kohlhagen formula, we can find option prices for a pseudo-currency market investor. For a European call option (priced in \(X\)) to buy 1 EUR for \(\sqrt{f(t)/K}\), \(X\) we have:

\[
C_{e/X}(0, T, f(0), K, r_s, r_e) = e^{-r_X T} E_{Q^X} \left[ \left( \sqrt{f(T)} - \frac{K}{\sqrt{f(T)}} \right)_+ \right]
\]

\[= \sqrt{f(0)} e^{-r_e T} N \left( \frac{\log \frac{f(0)}{K} + (r_s - r_e + \frac{\sigma^2}{2}) T}{\sigma \sqrt{T}} \right) - \frac{K}{\sqrt{f(0)}} e^{-r_s T} N \left( \frac{\log \frac{f(0)}{K} + (r_s - r_e - \frac{\sigma^2}{2}) T}{\sigma \sqrt{T}} \right).\]

And, similarly for a European put option (priced in \(X\)) to sell 1 USD for \(\sqrt{f(t)/K}\), \(X\) we have:

\[
P_{s/X}(0, T, \frac{1}{f(0)}, \frac{1}{K}, r_e, r_s) = e^{-r_X T} E_{Q^X} \left[ \left( \frac{1}{\sqrt{f(t)}} - \frac{1}{\sqrt{f(t)}} \right)_+ \right]
\]

\[= \frac{\sqrt{f(0)}}{K} e^{-r_s T} N \left( \frac{\log \frac{f(0)}{K} + (r_s - r_e + \frac{\sigma^2}{2}) T}{\sigma \sqrt{T}} \right) - \frac{1}{\sqrt{f(0)}} e^{-r_e T} N \left( \frac{\log \frac{f(0)}{K} + (r_s - r_e - \frac{\sigma^2}{2}) T}{\sigma \sqrt{T}} \right),\]

From \([3,6]\) and \([3,7]\), we can deduce prices for the call \(C_{e/s}\) and put \(P_{s/e}\). To this end, we first observe that the in-the-money payoff of the call \(C_{e/X}\) (priced in \(X\)) is equivalent to buying €1 for $K. Indeed, this call’s payoff is equal to the amount of \(X\)

\[
\left( \sqrt{f(T)} - \frac{K}{\sqrt{f(T)}} \right)_+
\]

which is equivalent to the amount of USD

\[
\sqrt{f(T)} \left( \sqrt{f(T)} - \frac{K}{\sqrt{f(T)}} \right)_+ = (f(T) - K)_+
\]

as we can exchange \(X\) for USD at the rate \(\sqrt{f(T)}\). Analogously, the in-the-money payoff payoff of \(P_{s/X}\) is equivalent to selling €1/K.

Further, by multiplying the price of \(C_{e/X}\) priced in \(X\) by \(\sqrt{f(0)}\), we covert its option price in \(X\) to the price in USD, and by multiplying the price of \(P_{s/X}\) priced in \(X\) by \(1/\sqrt{f(0)}\), we convert its price to EUR. Hence

\[
C_{e/s}(0, T, f(0), K, r_s, r_e) = \sqrt{f(0)} C_{e/X}(0, T, f(0), K, r_s, r_e), \quad (3.8)
\]
for the USD and EUR money markets,
particular model for
Introduce the discounting factor \( D \) constant. We assume that that there is a broad class of models for which \( Q \) intermediates currency \( X \) is domestic (note that at the start of this subsection we demonstrated that there is a broad class of models for which \( Q^X \) exists). Assume that the distribution of \( f(t) \) is such that \( f(t) \) and \( 1/f(t) \) have second moments. We remark that we do not assume a particular model for \( f(t) \) in the pricing part of this section. For simplicity, let the interest rates for the USD and EUR money markets, \( r_S \) and \( r_E \), be constant. As we mentioned earlier, the intermediate currency interest rate \( r_X(t) \) is, in general, not constant even when \( r_S \) and \( r_E \) are constant. We assume that \( r_X(t) \) is adapted to the same filtration \( \mathcal{F}_t \) and

\[
B_X(t) = \exp \left( \int_0^t r_X(s) ds \right). \tag{3.9}
\]

Introduce the discounting factor \( D_X(t, T) \) related to the intermediate currency interest rate and the intermediate currency non-defaultable zero-coupon bond price \( P_X(t, T) \):

\[
D_X(t, T) = \exp \left( - \int_t^T r_X(s) ds \right) \tag{3.10}
\]

and

\[
P_X(t, T) = \mathbb{E}_{Q^X} \left[ D_X(t, T) | \mathcal{F}_t \right], \tag{3.11}
\]

where we assumed that \( D_X(t, T) \) has finite moments. Since \( Q^X \) is EMM, the discounted \( Y_\epsilon/X(t) \) and \( Y_S/X(t) \),

\[
D_X(0, t) Y_\epsilon/X(t) = D_X(0, t) S_\epsilon/X(t) B_\epsilon(t) = D_X(0, t) \sqrt{f(t)} B_\epsilon(t) \]

and

\[
D_X(0, t) Y_S/X(t) = D_X(0, t) S_S/X(t) B_s(t) = D_X(0, t) \frac{1}{\sqrt{f(t)}} B_s(t),
\]

are \( Q^X \)-martingales. Hence we obtain for any \( t \geq 0 \)

\[
\sqrt{f(0)} = e^{r_E t} \mathbb{E}_{Q^X} \left[ D_X(0, t) \sqrt{f(t)} \right],
\]

\[
\frac{1}{\sqrt{f(0)}} = e^{r_S t} \mathbb{E}_{Q^X} \left[ \frac{D_X(0, t)}{\sqrt{f(t)}} \right].
\]

Therefore, to obey the no-arbitrage condition, the distribution of \( f(t) \), \( t \geq 0 \), under \( Q^X \) should be so that

\[
\mathbb{E}_{Q^X} \left[ D_X(0, t) \sqrt{f(t)} \right] = e^{(r_S - r_E) t} f(0). \tag{3.12}
\]

In option pricing we will consider the following natural class of scalable payoff functions \( g(x; K) \), where \( x > 0 \) denotes a price of the underlier and \( K \geq 0 \) has the meaning of a strike.
**Definition 3.3.** Scalable payoff functions satisfy the property so that for any $a > 0$:

$$a \cdot g(x; K) = g(ax; aK). \quad (3.13)$$

It is clear that e.g. plain vanilla puts and calls satisfy (3.13). For definiteness, assume that $g(x; K)$ is a payoff of an option written on one EUR, where $x$ has the meaning of EUR-USD exchange rate, and $K$ and $g$ are denominated in USD. As in the case of a call (see Example 3.1), the amount of USD $g(x; K)$ is equivalent to the amount $G(x; K)$ in $X$:

$$G(x; K) := \frac{1}{\sqrt{x}} g(x; K) = g \left( \frac{K}{\sqrt{x}} \right),$$

where $1/\sqrt{x}$ has the meaning of the exchange rate USD-X (cf. Definition 3.1) and $G(x; K)$ and $K/\sqrt{x}$ are denominated in $X$. According to the risk-neutral pricing theory, we can write the value of the European option $V_{e/X}(t)$ with payoff $g(\sqrt{f(T)}; K/\sqrt{f(T)})$ and maturity $T$ at time $t \leq T$ as

$$V_{e/X}(t) = \mathbb{E}_{Q^X} \left[ D_X(t, T) g \left( \sqrt{f(T)}; \frac{K}{\sqrt{f(T)}} \right) \bigg| \mathcal{F}_t \right].$$

Note that this is an option on EUR priced in $X$. The price in dollars for this option is

$$V_{e/\$/}(t) = \sqrt{f(t)} \mathbb{E}_{Q^X} \left[ D_X(t, T) g \left( \frac{1}{\sqrt{f(T)}}, \frac{K}{\sqrt{f(T)}} \right) \bigg| \mathcal{F}_t \right]. \quad (3.14)$$

Analogously, we can derive a formula for an option on USD priced in EUR:

$$V_{S/e}(t) = \frac{1}{\sqrt{f(t)}} \mathbb{E}_{Q^X} \left[ D_X(t, T) g \left( \frac{1}{\sqrt{f(T)}}, \sqrt{f(T)} K \right) \bigg| \mathcal{F}_t \right], \quad (3.15)$$

where $g(y; K)$ is a payoff of an option written on one USD, $y$ has the meaning of USD-EUR exchange rate, and $K$ and $g$ are denominated in EUR. We summarise this result in the following theorem.

**Theorem 3.4.** Assume that the EUR-USD exchange rate $f(t)$ satisfies a model for which the no-arbitrage condition (3.12) holds. Then the arbitrage price of a European option on EUR with a scalable payoff $g(x; K)$ and maturity time $T$ is given by (3.14) and the arbitrage price of an option on USD is given by (3.15).

It is not difficult to show that the foreign-domestic symmetry (2.6) holds when we use the pricing formulas (3.14) and (3.15) based on the intermediate currency.

### 3.2 T-forward measure for the intermediate market

Introduce the T-forward measure $Q^X_T$ equivalent to $Q^X$ on $\mathcal{F}_T$ with the Radon-Nikodym derivative

$$\frac{Q^X_T}{Q^X} = \frac{1}{P_X(0, T) B_X(T)} \quad (3.16)$$

and for $t > 0$

$$E_Q \left[ \frac{Q^X_T}{Q^X} \bigg| \mathcal{F}_t \right] = \frac{P_X(t, T)}{P_X(0, T) B_X(t)}. \quad (3.17)$$

Under this forward measure, we get [19] [13] (see also [4]):

$$\sqrt{f(0)} = e^{\epsilon T} \mathbb{E}_{Q^X} \left[ D_X(0, T) \sqrt{f(T)} \right] = e^{\epsilon T} P_X(0, T) \mathbb{E}_{Q^X_T} \left[ \sqrt{f(T)} \right], \quad (3.18)$$
Then the no-arbitrage condition (3.12) becomes
\[
\frac{1}{\sqrt{f(0)}} = e^{rsT \mathbb{E}_{Q^X}} \left[ \frac{D_X(0,T)}{\sqrt{f(T)}} \right] = e^{rsT} P_X(0,T) \mathbb{E}_{Q^X} \left[ \frac{1}{\sqrt{f(T)}} \right].
\]

Then the no-arbitrage condition (3.12) becomes
\[
\frac{1}{\sqrt{f(0)}} = e^{r_S T \mathbb{E}_{Q^X}} \left[ \frac{D_X(0,T)}{\sqrt{f(T)}} \right] = e^{(r_S - r_E) T} f(0).
\]

Further, (3.18) implies that the bond price \( P_X(0,T) \) should satisfy
\[
P_X(0,T) = e^{-r_E T} \frac{\sqrt{f(0)}}{\mathbb{E}_{Q^X} \sqrt{f(T)}} = e^{-r_S T} \frac{1}{\mathbb{E}_{Q^X} \left[ \frac{1}{\sqrt{f(T)}} \right]}.
\]

Note that \( f(0) \) is the current EUR-USD exchange rate and hence it is observable as well as \( r_S \) and \( r_E \). The current forward EUR-USD exchange rate
\[
F_{e/S}(0,T) = e^{(r_S - r_E) T} f(0)
\]
is also observable on the USD market.

We remark that the forward EUR-X and USD-X exchange rates,
\[
F_{e/X}(t,T) = e^{-r_E (T-t)} \frac{\sqrt{f(t)}}{P_X(t,T)} \quad \text{and} \quad F_{S/X}(t,T) = e^{-r_S (T-t)} \frac{1}{P_X(t,T) \sqrt{f(t)}},
\]
are both \( Q^X \) martingales. For convenience, we recall that if \( r_X(t) \) is deterministic then the two measures \( Q^X \) and \( Q^X_T \) coincide.

It is also not difficult to show that
\[
\sqrt{f(t)} = e^{r_S (T-t) \mathbb{E}_{Q^X}} \left[ D_X(t,T) \sqrt{f(T)} \mid \mathcal{F}_t \right] = e^{r_S (T-t)} P_X(t,T) \mathbb{E}_{Q^X_T} \left[ \sqrt{f(T)} \mid \mathcal{F}_t \right],
\]
\[
\frac{1}{\sqrt{f(t)}} = e^{r_S (T-t) \mathbb{E}_{Q^X}} \left[ \frac{D_X(t,T)}{\sqrt{f(T)}} \mid \mathcal{F}_t \right] = e^{r_S (T-t)} P_X(t,T) \mathbb{E}_{Q^X_T} \left[ \frac{1}{\sqrt{f(T)}} \mid \mathcal{F}_t \right].
\]

Then
\[
P_X(t,T) = e^{-r_E (T-t)} \frac{\sqrt{f(t)}}{\mathbb{E}_{Q^X_T} \left[ \frac{1}{\sqrt{f(T)}} \right]} = e^{-r_S (T-t)} \frac{1}{\mathbb{E}_{Q^X_T} \left[ \frac{1}{\sqrt{f(T)}} \right]}.
\]

The pricing formula (3.14) under the T-forward measure \( Q^X_T \) becomes
\[
V_{e/S}(t) = \sqrt{f(t)} \mathbb{E}_{Q^X} \left[ D_X(t,T) g \left( \sqrt{f(T)}; \frac{K}{\sqrt{f(T)}} \right) \mid \mathcal{F}_t \right] = \sqrt{f(t)} P_X(t,T) \mathbb{E}_{Q^X_T} \left[ g \left( \sqrt{f(T)}; \frac{K}{\sqrt{f(T)}} \right) \mid \mathcal{F}_t \right]
\]
\[
= \frac{1}{\mathbb{E}_{Q^X_T} \left[ \frac{1}{\sqrt{f(T)}} \right]} \mathbb{E}_{Q^X_T} \left[ g \left( \frac{1}{\sqrt{f(T)}}; \frac{K}{\sqrt{f(T)}} \right) \mid \mathcal{F}_t \right],
\]
where in the last line we used (3.23). Analogously we have (see (3.15));
\[
V_{S/e}(t) = \frac{e^{-r_E (T-t)}}{\mathbb{E}_{Q^X_T} \left[ \frac{1}{\sqrt{f(T)}} \mid \mathcal{F}_t \right]} \mathbb{E}_{Q^X_T} \left[ g \left( \frac{1}{\sqrt{f(T)}}; \sqrt{f(T)} K \right) \mid \mathcal{F}_t \right].
\]

We summarize this result in the next theorem.
Theorem 3.5. Assume that the EUR-USD exchange rate \( f(t) \) satisfies a model for which the no-arbitrage condition (3.12) or (3.19) holds. Then the arbitrage price of an option on EUR with a scalable payoff \( g(x; K) \) and maturity time \( T \) is given by (3.24) and the arbitrage price of an option on USD is given by (3.25).

The benefit of (3.24) and (3.25) vs (3.14) and (3.15) is that in (3.24) and (3.25) we do not need to compute the intermediate currency interest rate \( r_X(t) \).

Example 3.2. The prices of the call for buying \( €1 \) for \( $K \) and of the put for selling \( $1 \) for \( €1/K \) are equal to

\[
C_{€/8}(0, T, K) = \frac{1}{\mathbb{E}_{Q_X} \left[ \frac{1}{\sqrt{f(T)}} \right]} \mathbb{E}_{Q_X} \left[ \left( \sqrt{f(T)} - \frac{K}{\sqrt{f(T)}} \right)_+ \right], \quad (3.26)
\]

\[
P_{$/€}(0, T, 1/K) = \frac{1}{\mathbb{E}_{Q_X} \sqrt{f(T)}} \mathbb{E}_{Q_X} \left[ \left( \frac{\sqrt{f(T)}}{K} - \frac{1}{\sqrt{f(T)}} \right)_+ \right].
\]

We see that these pricing formulas satisfy the foreign-domestic symmetry (2.6):

\[
C_{€/8}(0, T, K) = f(0) \cdot K \cdot P_{$/€} \left( 0, T, \frac{1}{K} \right).
\]

To conclude, we derived the consistent pricing formulas for FX options. Although the new pricing formulas are derived using the virtual X market, their evaluation depends on parameters of the USD and EUR markets only. When we are interested in option prices at the current time \( t = 0 \), they are valid for any distribution (i.e., we do not need to explicitly define the process \( f(t) \) of the exchange rate \( f(T) \) which satisfies (3.19)). We will demonstrate this observation in illustrations of the new pricing formulas in Section 5.

4 Extension to the multi–currencies case

In this section we extend the approach of pricing FX options developed in the previous section to the multi-currencies case. Let us assume we have \( N \) currencies \( c_i \), where \( i = 1, \ldots, N \). Fixing one currency, for definiteness \( i = N \), we can introduce the \( N - 1 \) exchange rates

\[
f_j = S_{c_i/c_N} > 0, \quad j = 1, \ldots, N - 1,
\]

which denote the exchange rates between the currency \( c_N \) to all other currencies \( c_i, \ i = 1, \ldots, N - 1 \).

Now we introduce the intermediate currency \( X \) by defining the \( N \) exchange rates \( S_{c_i/X} \) as follows

\[
S_{c_i/X} = f_1^{b_{i1}} \times f_2^{b_{i2}} \times \cdots \times f_N^{b_{iN-1}}, \quad i = 1, \ldots, N,
\]

where \( b_{ij} \in \mathbb{R} \) are so that

\[
b_{ii} = 1 - \alpha_i, \quad i = 1, \ldots, N,
\]

\[
b_{ij} = -\alpha_j, \quad i \neq j, \quad i, j = 1, \ldots, N.
\]

By symmetry arguments (see also Remark 4.2 below), we choose

\[
\alpha_i = \frac{1}{N}, \quad i = 1, \ldots, N - 1.
\]

Note that \( S_{c_i/X} \) is the exchange rate between the observable currency \( c_i \) and the introduced intermediate currency \( X \) and hence it is the worth of 1 unit of currency \( c_i \) in the intermediate currency \( X \).
We assume that the currency market under a ‘market’ measure \(P\) is described by the system:

\[
\begin{align*}
    df_j &= \mu_j(t)f_j(t)dt + \sigma_j(t)f_j(t)d\overline{W}_j, \quad j = 1, \ldots, N - 1, \\
    d\overline{W}_j d\overline{W}_k &= d\overline{W}_k d\overline{W}_l = \rho_{ik}(t)dt, \quad l, k = 1, \ldots, N - 1, \\
\end{align*}
\]

(4.4)

and

\[
    dB_i = r_i(t)B_i dt, \quad i = 1, \ldots, N,
\]

(4.5)

where \(B_i(t)\) describes the bank account of currency \(c_i\) with its short rate \(r_i(t)\); \(\sigma_j(t) > 0\) is the volatility of the exchange rate \(f_j(t)\), \(\mu_j(t)\) is its drift; and \(W(t) = (\overline{W}_1(t), \ldots, \overline{W}_{N-1}(t))^T\) is an \(N - 1\)-dimensional correlated Wiener process with the correlation matrix \(R(t) \in \mathbb{R}^{N - 1 \times N - 1}\) which components we denote by \(\rho_{ij}(t)\) (obviously \(\rho_{ii} = 1\)). It is assumed that \(r_i(t), \sigma_j(t), \mu_j(t)\) are stochastic processes adapted to a filtration \(\mathcal{F}_t\) to which \(W(t)\) is also adapted, and they have bounded second moments and \(\sigma_j(t)\) satisfy Novikov’s condition. Furthermore, let us assume that the matrix \(R\) is symmetric strictly positive definite. Then using the Cholesky decomposition, we can represent \(R = LL^T\), where \(L \in \mathbb{R}^{N - 1 \times N - 1}\) is a lower triangular matrix with entries \(L_{ij}\). Using this decomposition, we can rewrite the SDEs (4.4) as

\[
    df_j = \mu_j(t)f_j(t)dt + \sigma_j(t)f_j(t) \sum_{k=1}^{j} L_{jk}(t)dW_k, \quad j = 1, \ldots, N - 1,
\]

(4.6)

where

\[
    L_{ii}(t) = \sqrt{1 - \sum_{k=1}^{i-1} L^2_{ik}(t)}, \quad L_{ij}(t) = \frac{\rho_{ij} - \sum_{k=1}^{i-1} L_{jk}(t)L_{ik}(t)}{L_{ii}(t)}, \quad \text{for } i < j,
\]

and \(W(t) = (\overline{W}_1(t), \ldots, \overline{W}_{N-1}(t))^T\) is an \(N - 1\)-dimensional standard Wiener process. We first show that the intermediate currency introduced in (4.2) permits an arbitrage-free market involving all \(N\) currencies.

**Theorem 4.1.** Assume that \(N - 1\) exchange rates \(f_j\) between the currency \(c_N\) to all other currencies \(c_i, i = 1, \ldots, N - 1,\) under a ‘market’ measure are described by the model (4.6) together with (4.5). Consider the intermediate currency \(X\) introduced in (4.2). There is the unique intermediate currency interest rate \(r_X(t)\) defined by

\[
    r_X(t) = \frac{1}{N} \sum_{i=1}^{N} r_i(t) + \frac{1}{2N} \left(1 - \frac{1}{N}\right) \sum_{i=1}^{N-1} \sigma_i^2(t) - \frac{1}{N^2} \sum_{k=1}^{N-1} \sum_{j=1}^{N-1-k} \sigma_j(t)\sigma_k(t)\rho_{jk}(t)
\]

(4.7)

and there is an EMM \(Q_X\) for the intermediate pseudo-currency market, i.e., under (4.7) this market is arbitrage-free.

**Proof.** Applying the Ito formula to (4.2), we obtain the SDEs for the exchange rates \(S_{c_i/X}\):

\[
    \frac{dS_{c_i/X}}{S_{c_i/X}} = \left[ \frac{1}{N} \sum_{j=1}^{N-1} \left( \frac{1}{2} \left( \frac{1}{N} + 1 \right) \sigma_j^2 - \sigma_j \mathbb{1}_{i \neq j} \sigma_j \rho_{ij} + \frac{1}{N} \sum_{k=1}^{j-1} \sigma_k \rho_{kj} - \mu_j \right) + \mu_i \mathbb{1}_{i \neq N} \right] dt
\]

\[
    - \frac{1}{N} \sum_{j=1}^{N-1} \sum_{k=1}^{j} \sigma_j L_{jk} dW_k + \sigma_i \mathbb{1}_{i \neq N} \sum_{k=1}^{i} L_{ij} dW_k, \quad i = 1, \ldots, N.
\]

On the considered market the risky assets have the prices \(Y_{c_i/X} = S_{c_i/X}B_i, i = 1, \ldots, N\). Introduce the discounted risky assets’ prices in the usual way:

\[
    \hat{Y}_{c_i/X}(t) = \frac{S_{c_i/X}(t)B_i(t)}{B_X(t)}, \quad i = 1, \ldots, N.
\]

(4.8)
Recall that we chose to use Remark 4.2. Theorem 4.1 is proved. The found \( \gamma_i \) (4.10) over equations in 'preference' to a particular currency(ies).

\[
\frac{d\tilde{Y}_{c_i/X}}{Y_{c_i/X}} = [r_i - r_X] \, dt \\
\quad + \left[ \frac{1}{N} \sum_{j=1}^{N-1} \left( \frac{1}{2} \left( \frac{1}{N} + 1 \right) \sigma_j^2 - \sigma_i \mathbf{1}_{i \neq N} \sigma_j \rho_{ij} + \frac{1}{N} \sigma_j \sum_{k=1}^{j-1} \sigma_k \rho_{kj} - \mu_j \right) + \mu_i \mathbf{1}_{i \neq N} \right] \, dt \\
- \frac{1}{N} \sum_{j=1}^{N-1} \sum_{k=1}^{j} \sigma_j L_{jk} \gamma_k + \sigma_i \mathbf{1}_{i \neq N} \sum_{k=1}^{i} L_{ik} \gamma_k, \quad i = 1, \ldots, N.
\]

The no-arbitrage condition requires existence of an EMM \( Q^X \) under which all \( \tilde{Y}_{c_i/X} \) are martingales. This implies that for \( Q^X \) to exist the following system of \( N \) simultaneous linear algebraic equations in \( N \) unknown variables (which are the market prices of risk \( \gamma_k, k = 1, \ldots, N - 1 \), and \( r_X \)) should have a solution:

\[
r_i - r_X + \frac{1}{N} \sum_{j=1}^{N-1} \left( \frac{1}{2} \left( \frac{1}{N} + 1 \right) \sigma_j^2 - \sigma_i \mathbf{1}_{i \neq N} \sigma_j \rho_{ij} + \frac{1}{N} \sigma_j \sum_{k=1}^{j-1} \sigma_k \rho_{kj} - \mu_j \right) + \mu_i \mathbf{1}_{i \neq N}
\]

\[= - \frac{1}{N} \sum_{j=1}^{N-1} \sum_{k=1}^{j} \sigma_j L_{jk} \gamma_k + \sigma_i \mathbf{1}_{i \neq N} \sum_{k=1}^{i} L_{ik} \gamma_k, \quad i = 1, \ldots, N.\]

Subtracting the equation (4.9) with \( i = N \) from the equations (4.9) for \( i \neq N \), we obtain

\[
r_i - r_N + \mu_i - \frac{1}{N} \sigma_i \sum_{k=1}^{N-1} \sigma_k \rho_{ik} = \sigma_i \sum_{k=1}^{i} L_{ik} \gamma_k, \quad i = 1, \ldots, N - 1. \tag{4.10}
\]

Using (4.10), we recurrently find the market prices of risk:

\[
\gamma_i = \frac{r_i - r_N + \mu_i - \frac{1}{N} \sigma_i^2 - \frac{1}{N} \sigma_i \sum_{k=1}^{N-1} \sigma_k \rho_{ik} - \sigma_i \sum_{k=1}^{i} L_{ik} \gamma_k}{\sigma_i L_{i,i}}, \quad i = 1, \ldots, N - 1, \tag{4.11}
\]

which are well defined because due to our assumptions \( \sigma_i > 0 \) and \( L_{i,i} > 0 \). Further, sum up (4.10) over \( i \) from \( i = 1 \) to \( N - 1 \) and substitute the result in (4.9) with \( i = N \) to confirm (4.7).

The found \( \gamma_i, i = 1, \ldots, N - 1 \), from (4.11) and \( r_X \) from (4.7) together with Girsanov’s theorem ensure that there is an EMM \( Q^X \) under which all \( \tilde{Y}_{c_i/X} \) are martingales. Hence, the considered market is arbitrage free. Theorem 4.1 is proved.

**Remark 4.2.** Recall that we chose to use \( \alpha_1 = \cdots = \alpha_{N-1} = \frac{1}{N} \) in (4.2). If we repeat the proof of Theorem 4.1 for arbitrary \( 0 < \alpha_j < 1 \) then we arrive at the following intermediate currency interest rate \( r_X \):

\[
r_X = \left( 1 - \sum_{j=1}^{N-1} \alpha_j \right) r_N + \sum_{j=1}^{N-1} \alpha_j r_j + \sum_{j=1}^{N-1} \alpha_j (1 - \alpha_j) \sigma_j^2 - \sum_{j=1}^{N-1} \sum_{k=1}^{j-1} \sigma_j \alpha_j \alpha_k \sigma_k \rho_{jk}. \tag{4.12}
\]

ensuring that there is an EMM in this market. We see that the choice \( \alpha_j = \frac{1}{N} \) results in the symmetry so that each \( r_j \) enters (4.12) with the same weight. Other choices of \( \alpha_j \) give a ‘preference’ to a particular currency(ies).

Let us extend Definition 3.3 to be suitable for the multi-currencies case.

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Definition 4.3. Scalable payoff functions satisfy the property so that for any $a > 0$:

$$a \cdot g(x_1, \ldots, x_{N-1}; K) = g(ax_1, \ldots, ax_{N-1}; aK).$$  \hfill (4.13)

Most of multi-currency options (e.g. basket options [6]) have payoffs belonging to this class. Consider a European-type option with maturity time $T$ and payoff in the currency $c_N$:

$$g(T) := g(f_1(T), \ldots, f_{N-1}(T); K).$$

Its equivalent value in the intermediate currency $X$ is equal to (see (4.2)):

$$G(T) := S_{c_N/X}(T) \cdot g(T) = g \left( f_1(T)S_{c_N/X}(T), \ldots, f_{N-1}(T)S_{c_N/X}(T); K \cdot S_{c_N/X}(T) \right)$$

$$= g \left( S_{c_1/X}(T), \ldots, S_{c_{N-1}/X}(T); K \cdot S_{c_N/X}(T) \right) = g \left( f_1(T), \ldots, f_{N-1}(T); K' \right),$$

where $K' = K \cdot S_{c_N/X}(t)$ is the equivalent strike in $X$. It is not difficult to see that at the maturity time $T$ the option holder is indifferent between receiving $g(T)$ in currency $c_N$ or $G(T)$ in currency $X$ as he can obtain the same amount by exchanging $G(T)$ to $c_N$:

$$\frac{G(T)}{S_{c_N/X}(T)} = \frac{1}{S_{c_N/X}(T)} g \left( S_{c_1/X}(T), \ldots, S_{c_{N-1}/X}(T); K \cdot S_{c_N/X}(T) \right)$$

$$= g \left( f_1(T), \ldots, f_{N-1}(T); K \right).$$

Example 4.1 (Basket option). Consider a basket option on the $c_N$ market written on all $N-1$ exchange rates $f_i(t), i = 1, \ldots, N-1$, which pay-off function is of the form [6]:

$$g(x_1, \ldots, x_{N-1}; K) = \left( \sum_{i=1}^{N-1} \omega_i x_i - K \right)_+,\,$$

where $x_i, i = 1, \ldots, N - 1$, and $K$ are denominated in the currency $c_N$ and $\omega_i \geq 0, i = 1, \ldots, N - 1$, are some weights. The equivalent pay-off on the $X$ currency market at the maturity $T$ is equal to

$$G(T) = S_{c_N/X}(T) \cdot g(f_1(T), \ldots, f_{N-1}(T); K)$$

$$= S_{c_N/X}(T) \left( \sum_{i=1}^{N-1} \omega_i f_i(T) - K \right)_+ = \left( \sum_{i=1}^{N-1} \omega_i S_{c_i/X}(t) - K \cdot S_{c_N/X}(t) \right)_+$$

$$= \left( \sum_{i=1}^{N-1} \omega_i S_{c_i/X}(t) - K' \right)_+,\,$$

where $S_{c_i/X}(t)$ and $K'$ are denominated in the intermediate currency $X$.

As in the case of a single FX pair (see Theorem 3.2), we have demonstrated by Theorem 4.1 that there is a sufficiently broad class of models for which there is an EMM $Q^X$ with an appropriate choice of the intermediate currency interest rate $r_X(t)$. We now generalize the pricing formulas of Theorems 3.4 and 3.5 from a single FX pair to the multi-currency case.

Let the exchange rates $f_i(t)$ between the currency $c_N$ to all other currencies $c_i, i = 1, \ldots, N - 1$, be defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q^X)$, where $Q^X$ is an EMM corresponding to the virtual market for which the intermediate currency $X$ is domestic. Assume that $f_i(t), i = 1, \ldots, N - 1$, and the exchange rates $S_{c_i/X}$ between the pseudo-currency $X$ to all the currencies $c_i, i = 1, \ldots, N$, defined in (4.2), (4.3) have second moments. Also, assume that $r_X(t)$ is adapted to the same filtration $\mathcal{F}_t$ and recall the expressions and assumptions for the money
market account $B_X(t)$ (see (3.9)), the discounting factor $D_X(t,T)$ related to the intermediate currency interest rate (see (3.10)) and the intermediate currency zero-coupon bond price $P_X(t,T)$ (see (3.11)).

Since $Q^X$ is an EMM, the discounted $Y_{c_i/X}(t)$ for all $i = 1, \ldots, N$,

$$
\hat{Y}_{c_i/X} = D_X(0,t)Y_{c_i/X}(t) = D_X(0,t)c_i(t), \quad i = 1, \ldots, N,
$$

are $Q^X$-martingales. Hence we obtain

$$
S_{c_i/X}(0) = e^{r_i t}E_{Q^X} \left[ D_X(0,t)c_i(X(t)) \right], \quad i = 1, \ldots, N.
$$

Therefore, for all $i = 1, \ldots, N - 1$ and $t > 0$, we have

$$
\frac{E_{Q^X} \left[ D_X(0,t)c_i(X(t)) \right]}{E_{Q^X} \left[ D_X(0,t)c_{cN/X}(X(t)) \right]} = e^{(r_i - r_N)t} \frac{S_{c_i/X}(0)}{S_{c_{cN/X}}(0)} = e^{(r_N - r_i)t} f_i(0). \quad (4.16)
$$

Thus, to obey the no-arbitrage condition, the distributions of $S_{c_i/X}(t), t > 0$, under $Q^X$ should be so that (4.16) holds.

Consider a European option with maturity $T$ and pay-off function $G(T)$ on the intermediate currency market. Its price in $X$ is equal to

$$
V_X(t) = E_{Q^X} \left[ D_X(0,T)G(T) | \mathcal{F}_t \right]. \quad (4.17)
$$

Using (4.14), we obtain the price for this option in the currency $c_i$:

$$
V_{c_i}(t) = \frac{1}{S_{c_{cN/X}}(t)} E_{Q^X} \left[ D_X(0,T) \cdot g(S_{c_i/X}(T), \ldots, S_{c_{cN-1}/X}(T); KS_{c_{cN/X}}(T)) | \mathcal{F}_t \right]. \quad (4.18)
$$

Then the analog of Theorem 3.4 is as follows.

**Theorem 4.4.** Assume that the exchange rates $f_i(t), i = 1, \ldots, N - 1$, (or $S_{c_i/X}(t), i = 1, \ldots, N$) satisfy a model for which the no-arbitrage condition (4.16) holds. Then on the $c_N$ market the arbitrage price $V_{c_N}(t)$ of a European option on $c_1, \ldots, c_{N-1}$ currencies with a scalable payoff $g(x_1, \ldots, x_{N-1}; K)$ and maturity time $T$ is given by (4.18).

Introduce the T-forward measure $Q^X_T$ equivalent to $Q^X$ on $\mathcal{F}_T$ with the Radon-Nikodym derivative as in (3.16) (see also (3.17)). Under this forward measure, we get

$$
S_{c_i/X}(0) = e^{r_i T}E_{Q^X_T} \left[ D_X(0,T)c_i(X(T)) \right] = e^{r_i T}P_X(0,T)E_{Q^X_T} \left[ S_{c_i/X}(T) \right], \quad i = 1, \ldots, N. \quad (4.19)
$$

Then the no-arbitrage conditions (4.16) become

$$
\frac{E_{Q^X_T} \left[ S_{c_i/X}(T) \right]}{E_{Q^X_T} \left[ S_{c_{cN/X}}(T) \right]} = e^{(r_N - r_i)T} f_i(0), \quad i = 1, \ldots, N - 1. \quad (4.20)
$$

Here $F_{c_i/c_N}(0) = e^{(r_N - r_i)T} f_i(0)$ is the current forward $c_i$-$c_N$ exchange rate. It follows from (4.20) that for any $j = 1, \ldots, N$

$$
\frac{E_{Q^X_T} \left[ S_{c_i/X}(T) \right]}{E_{Q^X_T} \left[ S_{c_j/X}(T) \right]} = e^{(r_N - r_i)T} f_i(0), \quad i = 1, \ldots, N, i \neq j. \quad (4.21)
$$

We remark that the no-arbitrage condition does not depend on the choice of $c_N$ used in (4.1).

Further, (4.19) implies that the bond price $P_X(0,T)$ should satisfy

$$
P_X(0,T) = e^{-r_i T} \frac{S_{c_i/X}(0)}{E_{Q^X_T} \left[ S_{c_i/X}(T) \right]}, \quad i = 1, \ldots, N. \quad (4.22)
$$
We observe that the relationships (4.20) ensure that (4.22) holds for all $i = 1, \ldots, N$. Note that $f_i(0)$ are the current $c_i/c_N$ exchange rates and hence $S_{c_i/X}(0)$ (see (4.2)) are observable as well as all $r_i$. Similarly to (3.23), we also have

$$P_X(t, T) = e^{-r_i(T-t)} \frac{S_{c_i/X}(t)}{\mathbb{E}_{Q^X_T}[S_{c_i/X}(T)|\mathcal{F}_t]}, \quad i = 1, \ldots, N.$$ 

Analogously to (3.25), the pricing formula (4.18) under the T-forward measure $Q^X_T$ becomes

$$V_{c_N}(t) = \frac{e^{-r_N(T-t)}}{\mathbb{E}_{Q^X_T}[S_{c_N/X}(T)|\mathcal{F}_t]} \mathbb{E}_{Q^X_T}[g(S_{c_1/X}(T), \ldots, S_{c_{N-1}/X}(T); K_{S_{c_N/X}(T)})|\mathcal{F}_t].$$

Then the analog of Theorem 3.5 is as follows.

**Theorem 4.5.** Assume that the exchange rates $f_i(t), i = 1, \ldots, N - 1$, (or $S_{c_i/X}(t), i = 1, \ldots, N$) satisfy a model for which the no-arbitrage condition (4.20) (or (4.16)) holds. Then on the $c_N$ market the arbitrage price $V_{c_N}(t)$ of a European option on $c_1, \ldots, c_{N-1}$ currencies with a scalable payoff $g(x_1, \ldots, x_{N-1}; K)$ and maturity time $T$ is given by (4.23).

It is clear that the pricing formula (4.23) remains true if we replace the currency $c_N$ with any other $c_j$ and the scalable payoff $g(x_1, \ldots, x_{j-1}, x_j, \ldots, x_N; K)$ is denominated in $c_j$. We now return to Example 4.1.

**Example 4.2** (Basket option pricing). Let us make the same assumptions as in Example 4.1 and find the arbitrage price of a European option with pay-off $G(T)$ at maturity $T$ on the $X$ currency market given by

$$G(t) = \left( \sum_{i=1}^{N-1} \omega_i S_{c_i/X}(t) - K \right)_+,$$

where $S_{c_i/X}(t)$ and $K$ are denominated in $X$. Following Theorem 4.5, the price of this basket option at time 0, denominated in currency $c_N$, is equal to

$$BasketOption_{c_N}(0) = \frac{e^{-r_N T}}{\mathbb{E}_{Q^X_T}[S_{c_N/X}(T)]} \mathbb{E}_{Q^X_T} \left[ \left( \sum_{i=1}^{N-1} \omega_i S_{c_i/X}(T) - K \right)_+ \right].$$

To conclude, we derived consistent pricing formulas (4.18) and (4.23) for FX options in the multi-currency case. As it was in Section 3 for a single FX pair, here in the multi-currency case, although the pricing formulas (4.18) and (4.23) are derived using the virtual $X$ market, their evaluation depends on parameters of the real $c_i, i = 1, \ldots, N$, markets only. The distinguishing feature of our approach in comparison with the others is that we can price all FX options regardless from their domestic market using the same measure which in turn guarantees that all natural relationships between exchange rates and FX options are automatically fulfilled.

5 Illustrations

For illustrative purposes, we consider four examples in this section. The first example (Section 5.1) illustrates the use of FX pricing from Section 3 in the case when the EUR-USD exchange rate $f(t)$ is described by the Heston model [18] while the second example (Section 5.2) deals with the SABR model [17]. In these two examples we follow the traditional route: we start with models written under a ‘market’ measure, then find an EMM $Q^X$ on the intermediate currency market and use Theorem 3.4 for pricing FX options. The third example presented in Section 5.3 follows a different route: we suggest a distribution for an exchange rate at maturity
time $T$, e.g. for EUR-USD, under a forward measure $Q_X$ on the intermediate currency market so that the no-arbitrage condition (3.19) is satisfied. Then we use Theorem 3.5 or Theorem 4.5 for pricing FX options. To this end, in Section 5.3 we assume that the EUR-USD exchange rate $f(T)$ has a skew normal distribution. We remark that the use of the considered extended skew normal model for FX pricing is novel. In Section 5.4 we illustrate the results of Sections 3 and 4 in the case of the model-free approach.

5.1 Heston model

For simplicity, let the interest rates for the USD and EUR money markets, $r_\$\,$ and $r_€$, be constant. Consider the Heston stochastic volatility model for the EUR-USD exchange rate $S_{€/\$}(t) = f(t)$ written under a ‘market’ measure [18]:

$$df = \mu f dt + \sqrt{v} f \left( \sqrt{1 - \rho^2} dW_1(t) + \rho dW_2(t) \right), \quad f(0) = f_0,$$

$$dv = \kappa (\theta - v) dt + \sqrt{v} dW_2(t), \quad v(0) = v_0,$$

where $W_1(t)$ and $W_2(t)$ are independent standard Wiener processes; $\sigma(t) = \sqrt{v(t)}$ is a (stochastic) volatility; $\theta, \kappa, \delta, f_0$ and $v_0$ are positive constants, satisfying

$$2\kappa \theta \geq \delta^2; \quad (5.2)$$

and the correlation coefficient $\rho \in (-1, 1)$. Recall that the condition (5.2) guarantees that zero is unattainable by $v(t)$ in finite time.

Following Section 3.1, to re-write (5.1) under $Q_X$, we need to find the market prices of risk, $\gamma_1(t)$ and $\gamma_2(t)$, so that (cf. (3.5)):

$$\sqrt{1 - \rho^2} \gamma_1(t) + \rho \gamma_2(t) = \frac{\mu - v(t)/2 + r_€ - r_\$}{\sqrt{v(t)}}.$$  \quad (5.3)

As it is standard for the Heston model [18], to deal with incompleteness of the market, we choose

$$\gamma_2(t) = \lambda \sqrt{v(t)},$$  \quad (5.4)

where $\lambda$ is a constant. Therefore, we have

$$d\sqrt{f} = (r_X(t) - r_€) \sqrt{f} dt + \frac{\sqrt{v}}{2} \sqrt{f} \left( \sqrt{1 - \rho^2} dW_1^Q(t) + \rho dW_2^Q(t) \right),$$

$$d\frac{1}{\sqrt{f}} = (r_X(t) - r_\$) \frac{1}{\sqrt{f}} dt - \frac{\sqrt{v}}{2} \frac{1}{\sqrt{f}} \left( \sqrt{1 - \rho^2} dW_1^Q(t) + \rho dW_2^Q(t) \right),$$

$$dv = \kappa (\theta - v) dt + \delta \sqrt{v} dW_2^Q, \quad v(0) = v_0,$$

where, as before (see (3.4)),

$$r_X(t) = \frac{r_\$ + r_€}{2} + \frac{v(t)}{8}$$

and, without changing the notation, the new $\kappa$ and $\theta$ in (5.5) are equal to $\kappa + \lambda\delta$ and $\kappa\theta/(\kappa + \lambda\delta)$, respectively, in terms of the old $\kappa$ and $\theta$ from (5.1). Then (see Theorem 3.4, e.g. the price of the call (in USD) for buying $€1$ for $\$K$ is equal to

$$C_{€/\$}(0, T, K) = \sqrt{f(0)} E_{Q_X} \left[ D_X(0, T) \left( \sqrt{f(T)} - \frac{K}{\sqrt{f(T)}} \right)_+ \right],$$

where $\sqrt{f(T)}$ and $1/\sqrt{f(T)}$ are from (5.5).
Now we will rewrite (5.5) under the \( T \)-forward measure \( Q_T \) using the results of Section 3.2. By (5.6), we have

\[
P_X(t, T) = \mathbb{E}_{Q_X} \left[ D_X(t, T) | \mathcal{F}_t \right] = \mathbb{E}_{Q_X} \left[ \exp \left( - \int_t^T r_X(s) ds \right) | \mathcal{F}_t \right] = \exp \left( - \frac{r_s + r_e}{2} (T - t) \right) \mathbb{E}_{Q_X} \left[ \exp \left( - \int_t^T \frac{v(s)}{8} ds \right) | v(t) \right].
\]

The stochastic \( X \) short rate \( r_X(t) \) defined by (5.6) with \( v(t) \) from (5.5) possesses an affine term structure (see e.g. [4]):

\[
P_X(t, T) = \exp \left( - r_s + r_e (T - t) + A(T - t) - C(T - t) v(t) \right), \tag{5.8}
\]

where

\[
A(t) = \frac{2 \kappa \theta}{\delta^2} \ln \left( \frac{2 \beta e^{(\beta + \kappa) t/2}}{\beta + \kappa} \right),
\]

\[
C(t) = \frac{e^{\beta t} - 1}{4 (\beta + \kappa) (e^{\beta t} - 1) + 2 \beta},
\]

with

\[
\beta = \sqrt{\kappa^2 + \delta^2/4}.
\]

We note that

\[
dP_X = r_X(t) P_X dt - \delta C(T - t) \sqrt{P_X} dW^{Q_X}_2(t).
\]

Then we obtain

\[
\mathbb{E}_{Q_X} \left[ \frac{dQ_X}{dQ_X} | \mathcal{F}_t \right] = \frac{P_X(t, T)}{P_X(0, T) B_X(t)} = \exp \left( - \frac{1}{2} \int_0^t C^2(T - s) \delta^2 v(s) ds - \int_0^t C(T - s) \delta \sqrt{v(s)} dW^{Q_X}_2(s) \right).
\]

Hence

\[
dW^{Q_X}_2 = dW^{Q_X}_2 + C(T - t) \delta \sqrt{v(t)} dt.
\]

To complete the change of measure, we need to look at \( W^{Q_X}_1(t) \). To this end, we recall that both forward EUR-X and USD-X exchange rates,

\[
F_{e/X}(t, T) = e^{-r_e(T-t)} \frac{\sqrt{f(t)}}{P_X(t, T)}
\]

and

\[
F_{s/X}(t, T) = e^{-r_s(T-t)} \frac{1}{P_X(t, T) \sqrt{f(t)}}
\]

should be \( Q_X \)-martingales. It is not difficult to check that to achieve the above no-arbitrage requirement, we need

\[
dW^{Q_X}_1(t) = dW^{Q_X}_1(t),
\]

which is natural since the change of measure (5.9) does not depend on \( W^{Q_X}_1(t) \).
Thus, applying Theorem 3.5 to the Heston model setting, we can price, e.g. the call option as (see (3.26)):

\[
C_{\mathcal{E}/\mathcal{S}}(0, T, K) = \frac{e^{-r_s T}}{\mathbb{E}_{Q_T^X} \left[ \frac{1}{\sqrt{f(T)}} \right]} \mathbb{E}_{Q_T^X} \left[ \left( \sqrt{f(T)} - \frac{K}{\sqrt{f(T)}} \right)_+ \right],
\]

and (see (3.4))

\[
C_{\mathcal{S}/\mathcal{E}}(0, T, K) = \frac{e^{-r_e T}}{\mathbb{E}_{Q_T^X} \left[ F_{\mathcal{E}/\mathcal{X}}(T, T) \right]} \mathbb{E}_{Q_T^X} \left[ (F_{\mathcal{S}/\mathcal{X}}(T, T) - K F_{\mathcal{E}/\mathcal{X}}(T, T))_+ \right],
\]

and

\[
C_{\mathcal{S}/\mathcal{E}}(0, T, K) = \frac{e^{-r_e T}}{\mathbb{E}_{Q_T^X} \left[ F_{\mathcal{E}/\mathcal{X}}(T, T) \right]} \mathbb{E}_{Q_T^X} \left[ (F_{\mathcal{S}/\mathcal{X}}(T, T) - K F_{\mathcal{E}/\mathcal{X}}(T, T))_+ \right].
\]

where

\[
dF_{\mathcal{E}/\mathcal{X}}(t, T) = \frac{\sqrt{\nu}}{2} F_{\mathcal{E}/\mathcal{X}}(t, T) \left( \sqrt{1 - \rho^2} dW_1^{Q_X}(t) + \rho dW_2^{Q_X}(t) \right) + \delta C(T - t) \sqrt{\nu} F_{\mathcal{E}/\mathcal{X}}(t, T) dW_2^{Q_X},
\]

\[
dF_{\mathcal{S}/\mathcal{X}}(t, T) = -\frac{\sqrt{\nu}}{2} F_{\mathcal{S}/\mathcal{X}}(t, T) \left( \sqrt{1 - \rho^2} dW_1^{Q_X}(t) + \rho dW_2^{Q_X}(t) \right) + \delta C(T - t) \sqrt{\nu} F_{\mathcal{S}/\mathcal{X}}(t, T) dW_2^{Q_X},
\]

\[
dv = (\kappa - C(t - t) \delta^2) \left( \frac{\kappa \theta}{\kappa - C(t - t) \delta^2} - v \right) dt + \delta \sqrt{\nu} dW_2^{Q_X}, \quad v(0) = v_0,
\]

and we require that for \(0 \leq t \leq T\)

\[
\kappa/\delta^2 > C(t).
\]

The prices (5.10) and (5.11) satisfy the foreign-domestic symmetry (see Theorem 2.1).

We note that in comparison with the classical Heston model (5.1), the model (5.12) has time dependence in the coefficients. For other time-dependent Heston models, see e.g. [3, 16] and references therein.

### 5.2 SABR model

For simplicity again, let the interest rates for the USD and EUR money markets, \(r_s\) and \(r_e\), be constant. Following Section 3.1, we can re-write the classical SABR model [17] for EUR-USD exchange rate \(f(t)\) that under the measure \(Q_X\), and the corresponding SDEs for \(S_{\mathcal{E}/\mathcal{X}} = \sqrt{f}\) and \(S_{\mathcal{S}/\mathcal{X}} = 1/\sqrt{f}\) take the form

\[
d\sqrt{f} = (r_X(t) - r_e) \sqrt{f} dt + \frac{\sigma(t)}{\sqrt{f}}\sqrt{1 - \rho^2} dW_1^{Q_X}(t) + \rho dW_2^{Q_X}(t),
\]

\[
d\frac{1}{\sqrt{f}} = (r_X(t) - r_s) \frac{1}{\sqrt{f}} dt + \frac{\sigma(t)}{\sqrt{f}}\sqrt{1 - \rho^2} dW_1^{Q_X}(t) + \rho dW_2^{Q_X}(t),
\]

\[
d\sigma = \nu \sigma dW_2^{Q_X}(t), \quad \sigma(0) = \alpha,
\]

where \(W_1^{Q_X}(t)\) and \(W_2^{Q_X}(t)\) are independent standard Wiener processes under \(Q_X\), \(\rho \in (-1, 0)\) is the correlation coefficient, \(\nu > 0\) is the volatility of the volatility \(\sigma(t)\), \(\alpha\) is a positive constant, and (see (3.4))

\[
r_X(t) = \frac{r_s + r_e}{2} + \frac{\sigma^2(t)}{8}.
\]
Note that the parameter known as $\beta$ in the classical SABR model is taken to be equal to 1 here, which is the typical requirement for FX modelling as it ensures that the SDE for the exchange rate for the inverse pair $1/f$ has the same form as for $f$.

By Theorem 3.4 e.g. the price of the call (in USD) for buying €1 for $K$ is equal to

$$C_{\mathcal{E}/S}(0, T, K) = \sqrt{f(0)\mathbb{E}_Q} \left[ D_X(0, T) \left( \frac{\sqrt{F(T)} - \frac{K}{\sqrt{f(T)}}}{\sqrt{f(T)}} \right) \right], \quad (5.16)$$

where $\sqrt{f(T)}$ and $1/\sqrt{f(T)}$ satisfy (5.14), (5.15).

### 5.3 Extended skew normal model

In this section we consider another illustration of Theorem 3.5. Here we start not with a model under a ‘market’ measure but with a direct assumption on the distribution of the exchange rate under a forward measure $Q^X_T$ on the intermediate market.

We assume that under a $T$-forward measure $Q_T^X$ the EUR-USD exchange rate $f(T)$ can be written as

$$f(T) = \tilde{F} e^Z,$$  \hspace{1cm} (5.17)

where $\tilde{F} > 0$ is a constant and $Z$ is a random variable such that $\mathbb{E}[e^Z]$ exists and the no-arbitrage condition (3.19) is satisfied by $f(T)$. Here the no-arbitrage condition (3.19) implies that

$$\tilde{F} = F \frac{\mathbb{E}[e^{Z/2}]}{\mathbb{E}[e^{Z/2}]}, \quad (5.18)$$

where we neglect the full notation $\mathbb{E}_{Q^X_T}[]$ and write $\mathbb{E}[]$ instead as in this section we work with the measure $Q_T^X$ only and we also write here $F$ instead of $F_{\mathcal{E}/S}(0, T)$ for the current forward EUR-USD exchange rate (see (3.21)). We use this simplified notation throughout this section and in the Appendix A which should not cause any confusion. The interest rates for the USD and EUR money markets, $r_S$ and $r_E$, are assumed to be constant.

Further, (3.26) (i.e. Theorem 3.5, (5.17) and (5.18) imply that the price (in USD) of the European call to buy €1 for $K$ at the maturity $T$ is

$$C_{\mathcal{E}/S}(0, T, K) = e^{-r_S T} \mathbb{E} \left[ \frac{1}{\sqrt{f(T)}} \left( \frac{\sqrt{F} - \frac{K}{\sqrt{f(T)}}}{\sqrt{f(T)}} \right) \right] \mathbb{1}_{F e^Z > K}, \quad (5.19)$$

where $z_0 = \log(K/\tilde{F})$ and

$$M(t) = \mathbb{E}[e^{tZ}] \text{ and } M(t, z_0) = \mathbb{E} \left[ e^{tZ} \mathbb{1}_{Z > z_0} \right], \quad (5.20)$$

and write

$$C_{\mathcal{E}/S}(0, T, K) = e^{-r_S T} \mathbb{E} \left[ \frac{1}{\sqrt{f(T)}} \left( \frac{\sqrt{F} - \frac{K}{\sqrt{f(T)}}}{\sqrt{f(T)}} \right) \mathbb{1}_{F e^Z > K} \right].$$
which are the moment generating function (MGF) and the restricted MGF for \( Z \), respectively. Analogous to (5.19), we can derive the pricing formulas for the put and also for the call and put for the inverse pair:

\[
P_{\epsilon/S}(0, T, K) = \frac{e^{-r_s T}}{E[\frac{1}{\sqrt{f(t)}}]} E \left[ \left( \frac{K}{\sqrt{f(t)}} - \sqrt{f(T)} \right)^+ \right] = e^{-r_s T} \left( K \frac{M^*(\alpha, z_0) - F \sqrt{f(T)}}{M(\alpha, z_0)} \right),
\]

\[
C_{\epsilon/S}(0, T, \frac{1}{K}) = \frac{e^{-r_s T}}{E[\frac{1}{\sqrt{f(t)}}]} E \left[ \left( \frac{1}{\sqrt{f(t)}} - \sqrt{f(T)} \right)^+ \right] = e^{-r_s T} \left( \frac{1}{F} \frac{M^*(\alpha, z_0) - 1}{M(\alpha, z_0)} \right),
\]

\[
P_{\epsilon/S}(0, T, \frac{1}{K}) = \frac{e^{-r_s T}}{E[\frac{1}{\sqrt{f(t)}}]} E \left[ \left( \frac{\sqrt{f(T)}}{K} - \frac{1}{\sqrt{f(T)}} \right)^+ \right] = e^{-r_s T} \left( \frac{1}{K} \frac{M(\alpha, z_0)}{M(\alpha, z_0)} - \frac{1}{F} \frac{M(-\alpha, z_0)}{M(-\alpha, z_0)} \right),
\]

where

\[M^*(t, z_0) = E \left[ e^{Z \mathbb{1}_{Z<z_0}} \right].\]

It is not difficult to verify that these pricing formulas satisfy the foreign-domestic symmetry (2.6) as expected.

Now we will propose a skew normal model for the random variable \( Z \). To this end, we start by introducing a new random variable \( V \), which is a combination of one normal and two shifted half-normal distributed random variables:

\[V := X + \alpha_1 \max(\beta_1 - Y, 0) + \alpha_2 \max(Y - \beta_2, 0),\] (5.21)

where \( X \) and \( Y \) are independent random variables with the standard normal distribution and \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R} \) are parameters. The parameters \( \beta_1 \) and \( \beta_2 \) describe the support domains of the two half-normal distributions which, from the modeling perspective, should not overlap. Consequently, we are only interested in the case:

\[0 < \beta_1 \leq \beta_2.\]

Since we use the random variable \( Z \) in (5.17), similarly to as a Gaussian random variable is used in the geometric Brownian motion model for \( f(T) \), we then define it as follows

\[Z = a V,\] (5.22)

where \( a = \sigma \sqrt{T} \) with \( \sigma \) having the meaning of volatility and \( T \) of the maturity time. The benefit of using \( Z \) instead of a Gaussian random variable is that \( Z \) can have heavier tails and can be successfully used for describing the volatility smile effect. At the same time, \( Z \) still has a very simple distribution which makes the model (5.17), (5.21), (5.22) very practical as it allows fast calibration. Indeed, the MGFs (5.20), which we need for pricing calls and puts (see (5.19)), can be found analytically for this \( Z \). The corresponding expressions are given in the next proposition, which proof can be found in the Appendix A.

**Proposition 5.1.** For \( 0 < \beta_1 \leq \beta_2 \), the MGF \( M(t) \) and the restricted MGF \( M(t, z_0) \) from (5.20) are equal to

\[M(t) = e^{\frac{(at)^2}{2}} (N(\beta_2) - N(\beta_1)) + e^{\frac{t(a^2 + \alpha_2^2 - 2a\alpha_2\beta_2)}{2}} N(ta\alpha_2 - \beta_2)\] (5.23)
of call options and make a conclusion about how far volatility is from a constant.

where \(N(\cdot)\) is the cdf of the standard normal distribution and \(N_2(\cdot, \cdots; \rho)\) is the cdf of the bivariate normal distribution with zero mean, unit variance, and correlation \(\rho\).

Using the parameters \(\alpha_1, \alpha_2, \beta_1, \) and \(\beta_2\), we can manipulate with the distribution of \(Z\) defined in (5.21), (5.22) and, in particular, its skew and kurtosis, which are equal to

\[
\text{skew}_Z = \frac{M_V^{(3)}(0) - 3M_V^{(1)}(0)M_V^{(2)}(0) + 2[M_V^{(1)}(0)]^3}{\left( M_V^{(2)}(0) - [M_V^{(1)}(0)]^2 \right)^{3/2}},
\]

\[
\text{kurtosis}_Z = \frac{M_V^{(4)}(0) - 4M_V^{(1)}(0)M_V^{(3)}(0) + 6[M_V^{(1)}(0)]^2 M_V^{(2)}(0) - 3[M_V^{(1)}(0)]^4}{\left( M_V^{(2)}(0) - [M_V^{(1)}(0)]^2 \right)^2},
\]

where \(M_V^{(i)}(0)\) are \(i\)-th derivatives of the MGF for the random variable \(V\), which are given in Appendix B. By putting \(\alpha_1 = \alpha_2 = 0\) in (5.21), the random variable \(Z\) becomes normal with zero mean and variance \(\sigma^2\), and the considered model (5.17), (5.21), (5.22) is reduced to the geometric Brownian motion whose one of the critical deficiencies is a flat (constant) volatility. In this case \(Z\) has \(\text{skew} = 0\) and \(\text{kurtosis} = 3\). In Figure 5.1 one can see the difference of distribution of \(Z\) (blue area) compared to a standardized normal distribution (red line). It can be seen that a parameter set with \(\alpha_1 < 0\) and \(\alpha_2 = 0\) results in a bigger left tail in distribution and a skew in the resulting volatility smile (\(\text{skew} \approx -1.6\)). Similarly, in Figure 5.2 it can be observed that using \(\alpha_1 < 0\) to adjust the left tail and \(\alpha_2 > 0\) to adjust the right tail of the smile, we can get an asymmetric distribution and an asymmetric smile. A smaller \(\alpha_2\) results in a smaller right tail and hence in a flatter smile. As seen in these figures, by adjusting the parameters \(\alpha_1, \alpha_2, \beta_1, \beta_2\), the shape of the distribution and of the smile can be changed in various ways and it can be associated with the resulting skew and kurtosis of the log exchange rate. Hence, after calibrating the parameters of \(Z\) to FX market data, we can compare \(\text{skew}_Z\) with zero skew and \(\text{kurtosis}_Z\) with the kurtosis of 3 in the geometric Brownian motion case and make a conclusion about how far volatility is from a constant.

### 5.4 Model-free approach

The model-free approach to pricing derivatives has become popular in recent years [1, 13, 2, 9] (see also references therein). The main idea of the approach is to construct a density or distribution function of risky assets under a risk-neutral measure using observed prices of plain-vanilla options. For clarity of the exposition how this approach works within our intermediate currency framework, we start with the case of two currencies. Then we will extend the consideration to the three-currencies case where we will exploit ideas from [1] (see also [2]).

In this section we will work under a T–forward measure \(Q_T\). Assume that we know prices of call options \(C_{E/S}(0; K)\) for all strikes \(K > 0\) and let \(\rho(x; T)\) be the density of the EUR-USD
Figure 5.1: Effect of the parameters on the distribution of $Z$ and the corresponding smile: the case of $\alpha_1 < 0$ and $\alpha_2 = 0$.

Figure 5.2: Effect of the parameters on the distribution of $Z$ and the corresponding smile: the case of $\alpha_1 < 0$ and $\alpha_2 > 0$. 
Using the pricing formula (4.23), we get

\[ V_{\varepsilon/\$}(0) = \frac{e^{-r_\varepsilon T}}{E_{Q^X_T} \left[ \sqrt{f(T)} \right]} \mathbb{E}_{Q^X_T} \left[ g \left( \sqrt{f(T)}; \frac{K}{\sqrt{f(T)}} \right) \right] \]

\[ = \frac{e^{-r_\varepsilon T}}{E_{Q^X_T} \left[ \sqrt{f(T)} \right]} \int_{0}^{\infty} \frac{1}{\sqrt{x}} g(x; K) \rho(x; T) dx \]

\[ = \int_{0}^{\infty} g(x; K) \frac{\partial^2}{\partial x^2} C_{\varepsilon/\$}(0; x) dx, \]

as simple calculations give

\[ \frac{e^{-r_\varepsilon T}}{\sqrt{f(T)}} \rho(K; T) = \frac{\partial^2}{\partial K^2} C_{\varepsilon/\$}(0; K). \]

(5.26)

Typically, observed data are expressed via volatility smile data \( \sigma(K) \) and from (3.8) we have

\[ C_{\varepsilon/\$}(0; K) = F_{\varepsilon/\$} e^{-r_\varepsilon T} N \left( \log \frac{F_{\varepsilon/\$}}{K} + \frac{\sigma^2(K)T/2}{\sigma(K)\sqrt{T}} \right) - K e^{-r_\varepsilon T} N \left( \log \frac{F_{\varepsilon/\$}}{K} - \frac{\sigma^2(K)T/2}{\sigma(K)\sqrt{T}} \right). \]

(5.27)

Combining (5.25) with (5.27) and given \( \sigma(K) \), we can price any FX derivative \( V_{\varepsilon/\$}(0) \) and analogously any derivative \( V_{\$/\varepsilon}(0) \) based on a smile from one of the markets. Note that the smile data computed from \( C_{\varepsilon/\$}(0; K) \) coincide with smile data computed from \( C_{\$/\varepsilon}(0; K) \) and that the prices \( V_{\varepsilon/\$}(0) \) and \( V_{\$/\varepsilon}(0) \) are consistent with each other thanks to using the intermediate currency framework.

Now we progress to the three-currencies case. Let us assume that we are interested in the GBP-USD-EUR currency triangle, where we denote GBP as currency 1, USD as currency 2, and EUR as currency 3. As before, the interest rates for the GBP, USD and EUR money markets, \( r_\varepsilon, r_\$ \) and \( r_\$, are assumed to be constant.

Consider a best-of option on the EUR market which payoff is equal to

\[ b(T) = \max \left\{ \frac{(S_{L/\varepsilon}(T) - K_1)}{K_1}, \frac{(S_{\$/\varepsilon}(T) - K_2)}{K_2} \right\}. \]

(5.28)

As it is known \( \Pi \), the value of a best-of option is arbitrary close to values of plain-vanilla calls on \( S_{L/\varepsilon}(T) \) or \( S_{\$/\varepsilon}(T) \) or to a vanilla option on the cross \( S_{L/\$}(T) \). Hence, a model used for FX pricing should price a best-of option and plain-vanilla options in a consistent manner.

By (4.23) we have

\[ S_{L/X} = S_{L/\varepsilon}^{2/3}(T)S_{\$/\varepsilon}^{-1/3}(T), \quad S_{\$/X} = S_{L/\varepsilon}^{-1/3}(T)S_{\$/\varepsilon}^{2/3}(T), \quad S_{\varepsilon/X} = S_{L/\varepsilon}^{-1/3}(T)S_{\$/\varepsilon}^{-1/3}(T). \]

(5.29)

Using the pricing formula (4.23), we get

\[ V_{\varepsilon}(0) = \frac{e^{-r_\varepsilon T}}{E_{Q^X_T} \left[ S_{\varepsilon/X}(T) \right]} \mathbb{E}_{Q^X_T} \left[ S_{\varepsilon/X}(T) g(T) \right] \]

\[ = \frac{e^{-r_\varepsilon T}}{E_{Q^X_T} \left[ S_{\varepsilon/X}(T) \right]} \mathbb{E}_{Q^X_T} \left[ \left( S_{L/\varepsilon}(T) - K_1 \right)_{+}, \left( S_{\$/\varepsilon}(T) - K_2 \right)_{+} \right] \]

where \( g(T) \) is an arbitrary payoff on the EUR market. Hence, for the best-of option we have

\[ v_{\varepsilon}(0) = \frac{e^{-r_\varepsilon T}}{E_{Q^X_T} \left[ S_{\varepsilon/X}(T) \right]} \mathbb{E}_{Q^X_T} \left[ \left( S_{L/\varepsilon}(T) - K_1 \right)_{+}, \left( S_{\$/\varepsilon}(T) - K_2 \right)_{+} \right] \]

23
GBP-USD exchange rate is equal to EUR exchange rate, respectively, and where motions under a T-forward measure we need to express the current price below for completeness of the exposition should be used. To complete, the model-free pricing, in the context of our intermediate currency approach, the Garman-Kohlhagen formulas given where

\[ \sigma \]

and from (4.23) we do not assume in (5.33) that \( g \) consistency of FX option pricing across different markets. Note that in comparison with (4.23) also into account (5.29), if we can evaluate (5.31) from market data, then we can price any where \( 1 \) \( ( \) \( K \)

In accordance with the no-arbitrage condition (5.32), we set

\[ V_{cn}(0) = \frac{e^{-rNT}}{\mathbb{E}_{Q^X} \left[ S_{c_n/X}(T) \right]} \mathbb{E}_{Q^X} \left[ S_{c_n/X}(T)g(T) \right], \]

where \( N \) can be any of the three currencies with \( g(T) \) being in the currency \( N \). Therefore (taking also into account (5.29)), if we can evaluate (5.31) from market data, then we can price any FX derivatives on any of the three markets using the same \( \rho(K_1, K_2; T) \) and, thus, ensuring consistency of FX option pricing across different markets. Note that in comparison with (4.23) we do not assume in (5.33) that \( g(T) \) is scalable.

Market data in the case of three currencies are typically presented via three volatility smiles: \( \sigma_1(K) \) and \( \sigma_2(K) \) from vanilla options on GBP-EUR and USD-EUR, respectively, and \( \sigma_3(K) \) from the cross, GBP-USD. To compute values on the smile curves from observed option prices in the context of our intermediate currency approach, the Garman-Kohlhagen formulas given below for completeness of the exposition should be used. To complete, the model-free pricing, we need to express the current price \( v(0) \) of the best-of option via the three volatility smiles. To this end, we need to find \( v(0) \) assuming that the exchange rates follow geometrical Brownian motions under a T-forward measure \( Q^X \), which coincides with the EMM \( Q^X \).

In accordance with the no-arbitrage condition (5.32), we set

\[ S_{L/e}(T) = F_{L/e} \exp \left( -\frac{T}{6} \left[ \sigma^2 - 2\sigma_1\sigma_2\rho_{12} \right] + \sigma_1 \sqrt{T} X_1 \right), \]

\[ S_{\$/e}(T) = F_{\$/e} \exp \left( -\frac{T}{6} \left[ \sigma^2 - 2\sigma_1\sigma_2\rho_{12} \right] + \sigma_2 \sqrt{T} X_2 \right), \]

where \( F_{L/e} = F_{L/e}(0, T) \) and \( F_{\$/e} = F_{\$/e}(0, T) \) are the current forward GBR-EUR and USD-EUR exchange rate, respectively, and \( X_i \sim N(0, 1) \) with correlation coefficient \( \rho_{12} \). Then the GBP-USD exchange rate is equal to

\[ S_{L/\$/e}(T) = \frac{S_{L/e}(T)}{S_{\$/e}(T)} = F_{L/\$/e} \exp \left( -\frac{T}{6} \left[ \sigma_3^2 - 2\sigma_2\sigma_3\rho_{23} \right] + \sigma_3 \sqrt{T} X_3 \right) \]

(5.35)
where
\[
F_{L/\epsilon} = \frac{F_{L/\epsilon}}{F_{S/\epsilon}},
\]
\[
\sigma_3^2 = \sigma_1^2 - 2\sigma_1\sigma_2\rho_{13} + \sigma_2^2.
\]
and \(X_3 \sim N(0, 1)\) with the correlation coefficients
\[
\rho_{13} = \frac{\sigma_1^2 + \sigma_3^2 - \sigma_2^2}{2\sigma_1\sigma_3}, \quad \rho_{23} = \frac{\sigma_2^2 + \sigma_3^2 - \sigma_1^2}{2\sigma_2\sigma_3}
\]
with \(X_1\) and \(X_2\), respectively.

We have
\[
\frac{S_{\epsilon/X}(T)}{\mathbb{E}_{Q_T^X}[S_{\epsilon/X}(T)]} = \frac{\exp \left( -\frac{1}{3}\sigma_1\sqrt{T}X_1 - \frac{1}{3}\sigma_2\sqrt{T}X_2 \right)}{\mathbb{E}_{Q_T^X}[\exp \left( -\frac{1}{3}\sigma_1\sqrt{T}X_1 - \frac{1}{3}\sigma_2\sqrt{T}X_2 \right)]} = \exp \left( -\frac{T}{18} [\sigma_1^2 2\sigma_1\sigma_2\rho + \sigma_2^2] - \frac{1}{3}\sigma_1\sqrt{T}X_1 - \frac{1}{3}\sigma_2\sqrt{T}X_2 \right),
\]
and it is not difficult to show that the above expression is the Radon-Nikodym derivative \(\frac{dQ_T^\epsilon}{dQ_T^X}\) of the T-forward measure \(Q_T^\epsilon\) on the EUR market with respect to \(Q_T^X\). Then
\[
V_\epsilon(0) = \frac{e^{-r_T}}{\mathbb{E}_{Q_T^X}[S_{\epsilon/X}(T)]} \mathbb{E}_{Q_T^X}[S_{\epsilon/X}(T)g(T)] = e^{-r_T} \mathbb{E}_{Q_T^\epsilon}[g(T)].
\]

Hence, the corresponding Garman-Kohlhagen formulas for calls are given by (see, e.g. [4]):
\[
C_{L/\epsilon}(0; K) = F_{L/\epsilon}e^{-r_T}N \left( \frac{\ln(F_{L/\epsilon}/K) + \sigma_1^2 T/2}{\sigma_1\sqrt{T}} \right) - K e^{-r_T}N \left( \frac{\ln(F_{L/\epsilon}/K) - \sigma_1^2 T/2}{\sigma_1\sqrt{T}} \right),
\]
\[
C_{S/\epsilon}(0; K) = F_{S/\epsilon}e^{-r_T}N \left( \frac{\ln(F_{S/\epsilon}/K) + \sigma_2^2 T/2}{\sigma_2\sqrt{T}} \right) - K e^{-r_T}N \left( \frac{\ln(F_{S/\epsilon}/K) - \sigma_2^2 T/2}{\sigma_2\sqrt{T}} \right).
\]

Analogously
\[
C_{L/S}(0; K) = \frac{e^{-r_T}}{\mathbb{E}_{Q_T^X}[S_{L/X}(T)]} \mathbb{E}_{Q_T^X}[S_{L/X}(T)(S_{S/L}(T) - K) +]
\]
\[
= e^{-r_T} \mathbb{E}_{Q_T^\epsilon}[(S_{S/L}(T) - K) +]
\]
\[
= F_{L/S}e^{-r_T}N \left( \frac{\ln(F_{L/S}/K) + \sigma_3^2 T/2}{\sigma_3\sqrt{T}} \right) - K e^{-r_T}N \left( \frac{\ln(F_{L/S}/K) - \sigma_3^2 T/2}{\sigma_3\sqrt{T}} \right).
\]

We also have (see [22, 23, 1]):
\[
v_\epsilon(0) = \frac{e^{-r_T}}{\mathbb{E}_{Q_T^X}[S_{\epsilon/X}(T)]} \times \mathbb{E}_{Q_T^\epsilon} \left[ S_{\epsilon/X}(T) \max \left\{ \frac{(S_{L/\epsilon}(T) - K_1)_+}{K_1}, \frac{(S_{S/\epsilon}(T) - K_2)_+}{K_2} \right\} \right]
\]
\[
= e^{-r_T} \mathbb{E}_{Q_T^\epsilon} \left[ \max \left\{ \frac{(S_{L/\epsilon}(T) - K_1)_+}{K_1}, \frac{(S_{S/\epsilon}(T) - K_2)_+}{K_2} \right\} \right].
\]

(5.36)
\[ e^{-rT} \left[ \frac{F_L/\varepsilon}{K_1} N(d_1^+, d_3^+; \rho_{13}) + \frac{F_S/\varepsilon}{K_2} N(d_2^+, d_3^+; \rho_{23}) + N(-d_1^-, -d_2^-; \rho_{12}) - 1 \right], \]

where

\[ d_i^\pm = \frac{\ln(F_i/K_i) \pm \sigma_i^2 T/2}{\sigma_i \sqrt{T}} \]

and \( F_1 = F_L/\varepsilon, F_2 = F_S/\varepsilon, \) and \( F_3 = F_L/S. \)

Now we put implied volatility smiles \( \sigma_i(K_i), i = 1, 2, 3, \) with \( K_3 = K_1/K_2 \) into (5.36) and evaluate the left-hand side of (5.31). As a result, we obtain for \( \psi_{\varepsilon}(0) = \psi_{\varepsilon}(0; K_1, K_2, \sigma_1(K_1), \sigma_2(K_2), \sigma_3(K_1/K_2)) \) from (5.36) (see [2, Ch. 11]):

\[ U(K_1, K_2) = \left[ 1 + K_1 \frac{\partial}{\partial K_1} + K_2 \frac{\partial}{\partial K_2} \right] \psi_{\varepsilon} \]

\[ = e^{-rT} \left[ N(-d_1^-, -d_2^-; \rho_{12}) + \left[ K_1 \sigma'_1(K_1) \frac{\partial}{\partial \sigma_1} + K_2 \sigma'_2(K_2) \frac{\partial}{\partial \sigma_2} \right] \psi_{\varepsilon} \right] \]

\[ = e^{-rT} \left[ N(-d_1^-, -d_2^-; \rho_{12}) + K_1 \sqrt{T} \sigma'_1(K_1) N'(d_1^-) N \left( \frac{d_2^- \rho_{12} - d_1^-}{\sqrt{1 - \rho_{12}^2}} \right) \right. \]

\[ + K_2 \sqrt{T} \sigma'_2(K_2) N'(d_2^-) N \left( \frac{d_2^- \rho_{12} - d_1^-}{\sqrt{1 - \rho_{12}^2}} \right) - 1 \right]. \]

Let us recall how the model-free approach is used in practice: (i) for observed plain-vanilla prices, compute values of the implied volatilities \( \sigma_i(K_i) \) by inverting the Garman-Kohlhagen formulas; (ii) smoothly interpolate the implied values to obtain three smiles \( \sigma_i(K_i) \); (iii) plug-in the smiles in (5.37); (iv) use \( U(K_1, K_2) \) (cf. (5.31) and (5.37)) together with (5.32) to price options on all the three markets by the pricing formula (5.33). The step (iv) can be either realized via integration by parts (see Example 5.1 below) or by further differentiation to get

\[ \frac{e^{-rT}}{E_{\bar{Q}^{\varepsilon}}[S_{\varepsilon/X}(T)]} K_1^{-1/3} K_2^{-1/3} \rho(K_1, K_2; T) = \frac{\partial^2}{\partial K_1 \partial K_2} U(K_1, K_2). \]

We emphasise that thanks to the intermediate currency approach we can consistently price products for all the six pairs based on a single calibration.

We remark that the no arbitrage condition imposes the following asymptotic requirements on smiles [20, 1, 2]:

\[ \sigma_i^2(K) = o(|\ln K|) \text{ as } K \to 0, \infty. \]  

(5.38)

Also, to ensure that \(-1 < \rho_{ij}(K, K_1) < 1\) the smiles should satisfy [1, 2]:

\[ \sigma_1(K_1) + \sigma_2(K_2) > \sigma_3(K_1/K_2), \]

\[ \sigma_2(K_2) + \sigma_3(K_1/K_2) > \sigma_1(K_1), \]

\[ \sigma_1(K_1) + \sigma_3(K_1/K_2) > \sigma_2(K_2). \]

(5.39)

**Example 5.1.** Consider basket pricing as in Example 4.2. Doing integration by parts twice, we get [2, Ch. 11]:

\[ BasketOption_{\varepsilon}(0) = \frac{e^{-rT}}{E_{\bar{Q}^{\varepsilon}}[S_{\varepsilon/X}(T)]} \times E_{\bar{Q}^{\varepsilon}} \left[ S_{\varepsilon}^{-1/3}(T) S_{S/\varepsilon}^{-1/3}(T) (K - \omega_1 S_{L/\varepsilon}(T) - \omega_2 S_{S/\varepsilon}(T))_+ \right] \]

\[ = \int_{0}^{\infty} \int_{0}^{\infty} (K - \omega_1 x - \omega_2 y)_+ \frac{\partial^2}{\partial x \partial y} U(x, y) dx dy \]
\[ \int_0^K U \left( \frac{z}{\omega_1}, \frac{K - z}{\omega_2} \right) \, dz. \]

We can also obtain

\[
\text{BasketOption}_E(0) = e^{-rT} \mathbb{E}_{Q^E} \left[ S_{E/\bar{X}}(T) \left( K - \omega_1 S_{E/\bar{X}}(T) - \omega_2 S_{\bar{E}/E}(T) \right) \right] (5.41)
\]

\[
= S_{E/\bar{X}}(0) e^{-rT} \mathbb{E}_{Q^E} \left[ S_{2/3}^{2/3}(T) S_{1/3}^{1/3}(T) \left( K - \frac{\omega_1}{S_{E/\bar{X}}(T)} - \frac{\omega_2}{S_{\bar{E}/E}(T)} \right) \right] + \frac{\partial^2}{\partial x \partial y} U(x, y) \, dx \, dy
\]

\[
= S_{E/\bar{X}}(0) \int_0^\infty \int_0^\infty U \left( \infty, \frac{z}{\omega_2} \right) - U \left( \frac{z \omega_2 + \omega_1}{K}, \frac{z}{\omega_2} \right) \, dz,
\]

where

\[
U(\infty, K_2) = e^{-rT} \left[ N(-d_2^*) + K_2 \sqrt{T} \sigma_2(K_2) N'(d_2) - 1 \right].
\]

We note that if we set one of \( \omega_i \) to zero in (5.40), then the formula gives the EUR price of a put on GBP or USD. Substituting \( U \) from (5.37) in (5.40) with one of \( \omega_i \) being zero, we can recover the Black-Scholes price of the corresponding put which means that the pricing formula (5.40) (or what is the same, (5.33)) exactly reproduces the plain vanilla data to which the calibration is made. See a calibration illustration in the next section.

6 Examples of calibration

In this section we present calibration examples for the models from Section 5.3 and we illustrate the model-free approach of Section 5.4. We also confirm via the calibration examples that using the proposed FX option pricing formulas allows us to retain the foreign-domestic symmetry.

We recall \[6, 23\] that the FX market is different to other financial markets in terms of volatility smile construction and quoting mechanisms used. FX options are quoted in implied volatility \( \sigma \), delta \( \Delta \) instead of strike \( K \), and maturity \( T \). The market convention is to quote three currency pair-specific most commonly traded options. Their choice depends on a delta hedging and ATM convention \[23, 7\] and typically 25\( \Delta \) options are among the considered options. Occasionally, one also uses 10\( \Delta \) put/call options, as they are widely available but not as liquid as 25\( \Delta \) options \[6\]. The option prices are inverted to calculate the corresponding volatility values, which are used for constructing the volatility smile. The data we use in this section for calibration are given in Table 6.1.

|                  | GBP-EUR | USD-EUR | GBP-USD |
|------------------|---------|---------|---------|
| \( \sigma_{25\text{Put}}^{\Delta} \) | 12.435% | 9.005%  | 11.000% |
| \( \sigma_{\text{ATM}} \)        | 10.945% | 9.250%  | 13.072% |
| \( \sigma_{25\text{Call}}^{\Delta} \)| 10.345% | 10.265% | 9.972%  |

6.1 Calibration: extended skew normal model

In this subsection we calibrate the model (5.17), (5.21), (5.22) from Section 5.3 to market data for two currency pairs. The use of just three options in calibration of volatility smiles leads to another typical (and which is in contrast to other markets) feature of the FX market that the volatility smile should interpolate the given three data points. Hence FX calibration is usually
done via a root-finding numerical algorithm, while on other markets, where a large number of option prices are available for constructing volatility curves, one normally uses least-square type algorithms for this purpose.

Figure 6.1: Calibration results for the GBP-EUR currency pair (left) and the inverse pair EUR-GBP (right) with $T = 1$, $r_L = 0.0025$, $r_E = 0.00$, $S_{L/E}(0) = 1.2935$.

Table 6.2: The results of calibration for GBP-EUR and EUR-GGBP.

| parameter    | GBP-EUR/EUR-GGBP | GBP-EUR       | EUR-GGBP       |
|--------------|------------------|---------------|---------------|
| $a$          | 0.06297173       | -0.87012308   | 0.87012308    |
| $\alpha_1$  | -3.18990817      |               |               |
| $\alpha_2$  | 1.57557895       |               |               |
| $\beta_1$   | -0.5             | 4.94244079    | 4.94244079    |
| $\beta_2$   | 0.5              |               |               |

The calibration was done in MatLab R2016a, where we use the MatLab function \texttt{fsolve} (which by default uses the built-in trust-region-dogleg algorithm) to match the option price data (three points per currency pair). We fixed the (free) parameters $\beta_1 = -0.5$ and $\beta_2 = 0.5$. For the calibration of the GBP-EUR pair, we use $a = \sigma_{ATM}$, $\alpha_1 = -3.0$ and $\alpha_2 = 1.0$ as initial values, as the negative skew of the volatility smile suggests a larger left tail (of the the distribution of $Z$). The calibration on a standard Desktop computer (Windows 7, 64-bit, Intel(R) Core(TM) i5-6500 CPU@3.20GHz, 16GB RAM) takes 0.11 seconds.

The calibration results for the GBP-EUR pair are given in Figure 6.1 and Table 6.2. One can see that the proposed pricing mechanism (see Theorem 3.5 and also (5.19)) together with the exchange rate model (5.17), (5.21), (5.22) preserves the volatility smile symmetry as skew, kurtosis (neglecting natural sign changes) and the model parameters stay the same. We also confirm that it is sufficient to calibrate the model using the GBP-EUR data and that the model reproduces both GBP-EUR and EUR-GGBP smiles with the same parameters $a$, $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$. Moreover, it can been seen that the resulting skew of 0.547 and kurtosis of 4.527 indicate the difference of the resulting distribution $Z$ to a normal distribution ($skew = 0$, $kurtosis = 3.0$).

The calibration results for the USD-EUR pair are given in Figure 6.2 and Table 6.3. The same observations as above for the GBP-EUR pair can be made here as well.

6.2 Illustration of the model-free approach

In this subsection, we demonstrate how we can approximate the scaled density function in the model-free approach of Section 5.4 from market data for three currencies. We recall that thanks
Figure 6.2: Calibration for the USD-EUR currency pair (left) and the inverse pair EUR-USD (right) with $T = 1.0$, $r_S = 0.0025$, $r_E = 0.00$, $S_{S/E}(0) = 0.8968$.

Table 6.3: The results of calibration for USD-EUR and EUR-USD.

| parameter | USD-EUR/EUR-USD |
|-----------|-----------------|
| $a$       | 0.05259980      |
| $\alpha_1$ | -1.94011846    |
| $\alpha_2$ | 2.90433341     |
| $\beta_1$ | -0.5            |
| $\beta_2$ | 0.5             |

| parameter | USD-EUR | EUR-USD |
|-----------|---------|---------|
| skew      | 0.53740761 | -0.53740761 |
| kurtosis  | 4.52666183 | 4.52666183 |

to the intermediate currency approach we can use the same density function to price options on all three markets. We retrieve the scaled density by differentiating $U(K_1, K_2)$ twice:

$$
\frac{\partial^2}{\partial K_1 \partial K_2} U(K_1, K_2) = \frac{e^{-r E T}}{E^{Q_X} S_{E/X}(T)} K_1^{-1/3} K_2^{-1/3} \rho(K_1, K_2; T).
$$

We use the same market data as before, for the three currency pairs GBP-EUR, USD-EUR and GBP-USD, which can be found in Table 6.1. We can find the corresponding strikes by inverting the Garman-Kohlhagen formula for all three pairs. As we need the volatility smiles to satisfy the growth condition (5.38), we fit a 2nd order polynomial with the three parameters $p_j^{(i)} \in \mathbb{R}$, $j = 1, 2, 3$, to the implied volatility data transformed by $\exp\left[\sigma_i^2(K)\right]$. Then we obtain the interpolated implied volatilities as

$$
\tilde{\sigma}_i(K) = \sqrt{\log \left[ p_i^{(1)} K^2 + p_i^{(2)} K + p_i^{(3)} \right]}.
$$

The results of the interpolation for the implied volatility smiles can be seen in Figure 6.3. The partial derivative with respect to $K_1$ and $K_2$ of $U(K_1, K_2)$ can be found by numerically differentiating (5.37) on a fine grid of $K_1$ and $K_2$. We use the MATLAB function `diff` to compute the point-wise $\frac{\partial^2}{\partial K_1 \partial K_2} U(K_1, K_2)$ surface for a range of strikes $K_1$ and $K_2$. Note that $K_3 = \frac{K_1}{K_2}$. The resulting surface and contour plots are given in Figure 6.4. We remark that $\frac{\partial^2}{\partial K_1 \partial K_2} U(K_1, K_2)$ is positive for the whole range of strikes considered as required.

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Figure 6.3: Implied volatility interpolation for GBP-EUR, USD-EUR and GBP-USD pairs with $T = 1.0$, $r_S = 0.0025$, $r_E = 0.0025$, $r_L = 0.0025$, $r_E = 0.0025$, $S_{S/E}(0) = 0.8968$, $S_{L/E}(0) = 1.2935$, $S_{L/S} = 1.4423$.

Figure 6.4: Implied scaled density surface and contour plot for the three currency pairs for a range of strikes $K_1$ and $K_2$. 
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\[ \beta \]

\[ \text{For which can be viewed as a complement to } M \]

Consider the following restricted MGF for \( Z \):

\[ M_{\alpha \max(\beta_1 - Y, 0) + \alpha_2 \max(Y - \beta_2, 0)}(t) = e^{\frac{\beta}{2}(\alpha^2 + 2\alpha_1 \beta_1)} N(t \alpha_1 + \beta_1) + N(\beta_2) - N(\beta_1) \quad (A.1) \]

\[ + e^{\frac{\beta}{2}(\alpha_2^2 - 2\alpha_2 \beta_2)} N(t \alpha_2 - \beta_2). \]

Using \((A.1)\), the fact that \( V \) is a combination of two independent random variables, \( X \) and \( \alpha \max(\beta_1 - Y, 0) + \alpha_2 \max(Y - \beta_2, 0) \), and the convolution theorem, we obtain the MGF for \( V \):

\[ M_V(t) = M_X(t) \times M_{\alpha \max(\beta_1 - Y, 0) + \alpha_2 \max(Y - \beta_2, 0)}(t) \]

\[ = e^{\frac{\alpha^2}{2}} (N(\beta_2) - N(\beta_1)) + e^{\frac{\beta}{2}(1 + \alpha^2 - 2\alpha_2 \beta_2)} N(t \alpha_2 - \beta_2) \]

\[ + e^{\frac{\beta}{2}(1 + \alpha_1^2 + 2\alpha_2 \beta_1)} N(t \alpha_1 + \beta_1). \]

Making use of basic properties of MGFs leads to the resulting formula \((5.23)\):

\[ M(t) = M_Z(t) = M_{aV}(t) = M_V(at) \]

\[ = e^{\frac{\alpha^2}{2}} (N(\beta_2) - N(\beta_1)) + e^{\frac{\beta}{2}(1 + \alpha^2 - 2\alpha_2 \beta_2)} N(t \alpha_2 - \beta_2) \]

\[ + e^{\frac{\beta}{2}(1 + \alpha_1^2 + 2\alpha_2 \beta_1)} N(t \alpha_1 + \beta_1). \]

**Appendix A  Proof of Proposition 5.1**

**Derivation of the MGF \( M(t) \).** Consider the MGF \( M_{\alpha \max(\beta_1 - Y, 0) + \alpha_2 \max(Y - \beta_2, 0)}(t) \) for the random variable \( \alpha \max(\beta_1 - Y, 0) + \alpha_2 \max(Y - \beta_2, 0) \). We have for \( \beta_1 \leq \beta_2 \):

\[ M_{\alpha \max(\beta_1 - Y, 0) + \alpha_2 \max(Y - \beta_2, 0)}(t) = e^{\frac{\beta}{2}(\alpha^2 + 2\alpha_1 \beta_1)} N(t \alpha_1 + \beta_1) + N(\beta_2) - N(\beta_1) \]

\[ + e^{\frac{\beta}{2}(\alpha_2^2 - 2\alpha_2 \beta_2)} N(t \alpha_2 - \beta_2). \]

Using \((A.1)\), the fact that \( V \) is a combination of two independent random variables, \( X \) and \( \alpha \max(\beta_1 - Y, 0) + \alpha_2 \max(Y - \beta_2, 0) \), and the convolution theorem, we obtain the MGF for \( V \):

\[ M_V(t) = M_X(t) \times M_{\alpha \max(\beta_1 - Y, 0) + \alpha_2 \max(Y - \beta_2, 0)}(t) \]

\[ = e^{\frac{\alpha^2}{2}} (N(\beta_2) - N(\beta_1)) + e^{\frac{\beta}{2}(1 + \alpha^2 - 2\alpha_2 \beta_2)} N(t \alpha_2 - \beta_2) \]

\[ + e^{\frac{\beta}{2}(1 + \alpha_1^2 + 2\alpha_2 \beta_1)} N(t \alpha_1 + \beta_1). \]

**Derivation of the restricted MGF \( M(t, z_0) \).** To obtain the formula \((5.24)\) for \( M(t, z_0) \), we consider the following restricted MGF for \( Z \):

\[ M_Z^*(t, z_0) := \mathbb{E}[e^{tZ} \mathbf{1}_{Z < z_0}], \]

which can be viewed as a complement to \( M(t, z_0) \) as \( M(t, z_0) = M(t) - M_Z^*(t, z_0) \) (note that \( M_Z^*(t, z_0) \) is naturally used for pricing puts). We start with deriving the restricted MGF for \( V \):

\[ M_V(t, v_0) := \mathbb{E}[e^{tV} \mathbf{1}_{V < v_0}]. \]

By splitting up the integration domain into three regions and calculating each integral separately, we obtain for \( \beta_1 \leq \beta_2 \):

\[ M_V(t, v_0) = \mathbb{E}[e^{tV} \mathbf{1}_{V < v_0}] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t(x + \alpha_1 \max(\beta_1 - Y, 0) + \alpha_2 \max(Y - \beta_2, 0))} \mathbf{1}_{\{x + \alpha_1 \max(\beta_1 - Y, 0) + \alpha_2 \max(Y - \beta_2, 0) < v_0\}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \]

\[ \times \mathbf{1}_{\{x + \alpha_1 \max(\beta_1 - Y, 0) + \alpha_2 \max(Y - \beta_2, 0) < v_0\}} dy. \]
moments of The derivatives of the MGF $M$

Using basic properties of MGFs, we get

$$
M_e^2(t_0, z_0) = M_e^2(t, z_0) = \mathbb{E} \left[ e^{\alpha Z_2} Z < z_0 \right] = \mathbb{E} \left[ e^{atV} V < z_0 \right] = M_V \left( at, \frac{z_0}{\alpha} \right)
$$

$$
e^{-\frac{\alpha^2}{2}} N \left( \frac{z_0}{\alpha} - at \right) (N(\beta_2) - N(\beta_1))
$$

Using basic properties of MGFs, we get

$$
M(t, z_0) = M_Z^*(t, z_0) = \mathbb{E} \left[ e^{\alpha Z} Z < z_0 \right] = \mathbb{E} \left[ e^{atV} V < z_0 \right] = M_V \left( at, \frac{z_0}{\alpha} \right)
$$

$$
= e^{-\frac{\alpha t^2}{2}} N \left( \frac{z_0}{\alpha} - at \right) (N(\beta_2) - N(\beta_1))
$$

We can simplify the following expression

$$
M(t, z_0) = M(t) - M^*(t, z_0)
$$

$$
= e^{-\frac{\alpha t^2}{2}} N \left( \frac{z_0}{\alpha} - at \right) (N(\beta_2) - N(\beta_1))
$$

$$
+ e^{-\frac{\alpha t^2}{2}} N \left( \frac{z_0}{\alpha} - at \right) N_2 \left( t a_1 + \beta_1, \frac{z_0}{\alpha} - at - \alpha_1 (\beta_1 + \alpha_1 t) \right)
$$

$$
+ e^{-\frac{\alpha t^2}{2}} N \left( \frac{z_0}{\alpha} - at \right) N_2 \left( t a_2 - \beta_2, \frac{z_0}{\alpha} - t + \alpha_2 (\beta_2 - \alpha_2 t) \right)
$$

which gives [5.24].

**Appendix B  Moments of the random variable V**

The derivatives of the MGF $M_V^{(i)}(0)$ the random variable $V$ from [5.21] (i.e., the first four moments of $V$) are equal to

$$
M_V^{(1)}(0) = \alpha_1 \beta_1 N(\beta_1) - \alpha_2 \beta_2 N(-\beta_2) + \frac{\alpha_1}{\sqrt{2\pi}} e^{-\frac{\beta^2}{2}} + \frac{\alpha_2}{\sqrt{2\pi}} e^{-\frac{\beta^2}{2}},
$$

$$
M_V^{(2)}(0) = N(\beta_1) \left[ (\alpha_1 \beta_1)^2 + \alpha_1^2 \right] + N(\beta_2) + N(\beta_2) \left[ (\beta_2 - \alpha_2)^2 + 1 + \alpha_2^2 \right]
$$

$$
+ \alpha_1 \beta_1 \frac{\alpha_1}{\sqrt{2\pi}} e^{-\frac{\beta^2}{2}} - \alpha_2 \beta_2 \frac{\alpha_2}{\sqrt{2\pi}} e^{-\frac{\beta^2}{2}}.
$$
\[
M_V^{(3)}(0) = N(\beta_1) \left[ 3\alpha_1 \beta_1 (1 + \alpha_1^2) + (\alpha_1 \beta_1)^3 \right] + N(-\beta_2) \left[ -3\alpha_2 \beta_2 (1 + \alpha_2^2) - (\alpha_2 \beta_2)^3 \right] \\
+ \frac{\alpha_1}{\sqrt{2\pi}} e^{-\frac{\beta_1^2}{2}} \left[ (\alpha_1 \beta_1)^2 + 3 + 2\alpha_1^2 \right] + \frac{\alpha_2}{\sqrt{2\pi}} e^{-\frac{\beta_2^2}{2}} \left[ (\alpha_2 \beta_2)^2 + 3 + 2\alpha_2^2 \right],
\]

\[
M_V^{(4)}(0) = 3N(\beta_2) - 3N(\beta_1) + N(-\beta_2) \left[ 3(1 + \alpha_2^2)(2(\alpha_2 \beta_2)^2 + 1 + \alpha_2^2) + (\alpha_2 \beta_2)^4 \right] \\
+ N(\beta_1) \left[ 3(1 + \alpha_1^2)(2(\alpha_1 \beta_1)^2 + 1 + \alpha_1^2) + (\alpha_1 \beta_1)^4 \right] \\
+ \frac{\alpha_1}{\sqrt{2\pi}} e^{-\frac{\beta_1^2}{2}} \left[ \alpha_1 \beta_1 (6 + 5\alpha_1^2 + (\alpha_1 \beta_1)^2) \right] + \frac{\alpha_2}{\sqrt{2\pi}} e^{-\frac{\beta_2^2}{2}} \left[ -\alpha_2 \beta_2 (6 + 5\alpha_2^2 + (\alpha_2 \beta_2)^2) \right].
\]