A new explicit expression for the Korteweg-De Vries hierarchy

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October 14, 1997

We derive an improved fully explicit expression for the right-hand sides of the matrix KdV hierarchy using the relation to the heat kernel of the one-dimensional Schrödinger operator. Our method of "matrix elements" produces, moreover, an explicit expression for the powers of a Schrödinger-like differential operator of any order.

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1 Introduction

The Korteweg-de Vries (abbreviated KdV henceforth) equation

\[ \frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u \frac{\partial u}{\partial x} \]  

for a function \( u = u(x,t) \) of position \( x \) and time \( t \) naturally extends to an infinite sequence of partial differential equations of solitonic character \[\text{[1]}\]

\[ \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} G_k[u] \quad (k = 1, 2, \ldots), \]  

where each \( G_k = G_k[u] \) is some differential polynomial in \( u \) with respect to \( x \), i.e. a polynomial in \( u \) and its derivatives

\[ u_n := \frac{\partial^n u}{\partial x^n}, \quad (n = 1, 2, \ldots). \]  

The sequence begins with

\[ \begin{align*}
G_1 &= u, \\
G_2 &= u_2 + 3u^2, \\
G_3 &= u_4 + 10uu_2 + 5u^2_1 + 10u^3, \quad \ldots
\end{align*} \]  

While usually the KdV hierarchy is defined through a recursive rule for the \( G_k \), we derived in our papers \[\text{[2, 3]}\] an explicit expression for the latter. The main aim of the present paper is to improve this expression by decomposing it in homogeneous parts. Thus, the rest of implicitness in \[\text{[2, 3]}\] is overcome. We derive the fully explicit expression by means of a "method of matrix elements", developed in \[\text{[4, 5]}\], which will be described shortly in section 2. The same method produces an expression for the powers \( L^k \) of the one-dimensional Schrödinger operator

\[ L = \frac{\partial^2}{\partial x^2} + u \]  

and gives some knowledge on the differential operators \( A_k \) in the Lax representation

\[ \frac{\partial L}{\partial t} = [A_k, L], \quad (k = 1, 2, \ldots) \]  

of the KdV hierarchy \[\text{[1]}\]. More generally, the method gives, as a side-result, an expression for the powers \( L^k \) of an \( r \)-th order Schrödinger-like operator

\[ L_r = \frac{\partial^r}{\partial x^r} + u. \]  

A particularly convenient formula, in terms of the so called Bell differential polynomials, is valid for \( r = 1 \).
While the paper [2] treated a scalar potential \( u = u(x,t) \) only, the derivation in [3] and in the present paper holds for the matrix case too, where \( u = u(x,t) \) is a \( N \times N \) matrix valued function. Notice that the matrix KdV hierarchy (55) has solitonic character if \( u \) is a Hermitian (in particular, a real symmetric) matrix: the papers \([6, 7, 8, 9, 10]\) showed that the inverse scattering method for solving Cauchy’s problem works well then.

It is well known (cf., e.g., \([4, 3, 11, 12]\)) that the heat kernel \( K = K(s;x,x') \), i.e. the fundamental solution of the heat equation

\[
\left( \frac{\partial}{\partial s} - L \right) K = 0,
\]

to the one dimensional Schrödinger operator (3) has an asymptotic expansion as \( t \to 0 \) of the form:

\[
K = K(s;x,x') \sim (4\pi s)^{-1/2} \exp \left( -\frac{(x-x')^2}{4s} \right) \sum_{k=0}^{\infty} \frac{1}{k!} a_k(x,x')s^k.
\]

The coefficients \( a_k = a_k(x,x') \) satisfy the recursive differential equations

\[
\left( 1 + \frac{1}{k} D \right) a_k = L a_{k-1}, \quad (k = 1, 2, \ldots), \quad a_0 := 1,
\]

where we abbreviate

\[
D := (x-x') \frac{\partial}{\partial x}.
\]

It is further known \([13, 14, 3]\) that the right-hand sides of the KdV hierarchy are determined by the diagonal values of the heat kernel coefficients:

\[
G_k[u](x) = \frac{1}{2} \binom{2k}{k} a_k(x,x).
\]

We will evaluate the recursion system (10) in order to calculate the diagonal values \( a_k(x,x) \). In fact, we do more and calculate all Taylor coefficients

\[
<n|a_k> := \frac{\partial^n}{\partial x^n} a_k(x,x') \big|_{x=x'}.
\]

## 2 The method of matrix elements

Let us briefly describe a ”method of matrix elements” as developed in \([3, 4, 12]\). Let us fix a point \( x' \) on the real line \( \mathbb{R} \) and consider a sufficiently small interval around \( x' \), say \( I = [x'-R, x'+R] \), where \( R \) is a sufficiently small positive constant. We consider the space \( C^\omega(I) \) of analytic functions \( f = f(x,x') \) of the real variable \( x \in I \) and their Taylor series

\[
f(x,x') = \sum_{n=0}^{\infty} \frac{1}{n!} (x-x')^n \frac{\partial^n}{\partial x^n} f(x,x') \big|_{x=x'}.
\]
We rewrite the Taylor formula (14) in the form
\[ f \equiv |f> = \sum_{n=0}^{\infty} |n><n|f>, \tag{15} \]
where
\[ |n> := \frac{1}{n!}(x - x')^n, \quad (n = 0, 1, 2, \ldots), \tag{16} \]
\[ <n|f> := \left. \frac{\partial^n f(x, x')}{\partial x^n} \right|_{x=x'}. \tag{17} \]

In a sense, the functions \(|n>\) form a basis of the space \(C^\omega(I)\) and the \(<n|\) are the corresponding dual linear functionals \(<n| : C^\omega(I) \to \mathbb{R}\). Taylor’s theorem can formally be expressed as a completeness relation
\[ \sum_{n=0}^{\infty} |n><n| = 1 \tag{18} \]
where \(1\) denotes the identical operator. The duality of the systems \(<m|\) and \(|n>\) is expressed by
\[ <m|n> = \delta_{mn}, \tag{19} \]
where \(\delta_{mn}\) is the Kronecker symbol.

Let us consider a linear differential operator of order \(r\) with respect to \(x \in I\)
\[ L = \sum_{j=1}^{r} u_j(x, x') \frac{\partial^j}{\partial x^j} \]
with analytic coefficients \(u_j = u_j(x, x')\). Such an operator can be characterized by its matrix elements
\[ <m|L|n> :<m|(L|n>) = \sum_{j=1}^{\min(r, n)} \binom{m}{n-j} <m-n+j|u_j> \tag{20} \]
Namely, there holds
\[ L = \sum_{n,m \geq 0} |m><m|L|n><n|, \tag{21} \]
which is just an abbreviation of
\[ L|f> = \sum_{n,m \geq 0} |m><m|L|n><n|f>. \tag{22} \]
Note that for differential operators of order \(r\) the matrix elements \(<m|L|n>\) vanish for \(n \geq m + r + 1:\)
\[ <m|L|n> = 0 \quad \text{for } n \geq m + r + 1. \tag{23} \]
Therefore, the sum over \( n \) in (21) is always finite! The infinite sum over \( m \) converges because of the analyticity of the coefficients \( u_j \) of the operator \( L \). This is a special property of differential operators and it is not necessarily valid for pseudodifferential operators.

Formally, (21) is obtained by application of the identity operator (18) on both sides. By inserting the completeness relation (18) one can show that the matrix elements of the product of two operators \( L_1 L_2 \) result from the matrix multiplication of the matrix elements of \( L_1 \) and \( L_2 \):

\[
<m|L_2 L_1|n> = \sum_{l=0}^{\infty} <m|L_2|l><l|L_1|n>. \tag{24}
\]

Again, from the property (23), we see that this sum is finite and there is no problem of convergence. The same argument applies to the product \( L_k \cdots L_2 L_1 \) of operators \( L_1, L_2, \cdots, L_k \):

\[
<m|L_k \cdots L_2 L_1|n> = \sum_{n_1, n_2, \ldots, n_k \geq 0} <m|L_k|n_{k-1}><n_{k-1}|L_{k-1}|n_{k-2}> \cdots \times \cdots <n_2|L_2|n_1><n_1|L_1|n>. \tag{25}
\]

Let us note that the method of matrix elements applies also to the more general case of smooth functions, i.e. to the space \( C^\infty(I) \). In this case the Taylor series does not, in general, converge but is an asymptotic expansion. Thus, all the equations of this section hold also for operators with smooth, not necessarily analytic, coefficients, if one replaces the sign = by the sign for asymptotic equality \( \sim \) in the Taylor expansion. The equations between Taylor coefficients remain exactly valid even in this case.

Let us consider now some examples of linear operators.

1. We already know that the identical operator \( I \) has the matrix elements

\[
<m|I|n> = \langle m|n> = \delta_{mn}. \tag{26}
\]

2. The special first-order operator

\[
D := (x - x') \frac{\partial}{\partial x} \tag{27}
\]

has the Taylor basis (13) as a complete system of eigenfunctions:

\[
D|n> = n|n> \quad \text{for } n = 0, 1, 2, \ldots \tag{28}
\]

As a conclusion, the matrix elements of \( D \) read

\[
<m|D|n> = n\delta_{mn}; \quad \tag{29}
\]

hence,

\[
D = \sum_{n=0}^{\infty} n|n><n|. \tag{30}
\]
More generally, for an arbitrary polynomial $P$ we have

$$P(D) = \sum_{n=0}^{\infty} P(n) |n > < n|,$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} (31)

This operator can be inverted provided the polynomial does not have roots at the integer positive points, i.e. if $P(n) \neq 0$ for $n = 0, 1, 2, \ldots$ then

$$P(D)^{-1} = \sum_{n=0}^{\infty} \frac{1}{P(n)} |n > < n|$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} (32)

The polynomial values $P(n)$ in the denominator even improve the convergence if $P(D)^{-1}$ is applied to an analytic function.

3. The one-dimensional Schrödinger operator

$$L := \frac{d^2}{dx^2} + u, \quad u = u(x),$$ \hspace{1cm} \hspace{1cm} \hspace{1cm} (33)

has the matrix elements

$$< m | L | n > = \binom{m}{n} u_{m-n} \quad \text{for} \quad n \leq m,$$ \hspace{1cm} \hspace{1cm} \hspace{1cm} (34)

$$< m | L | m + 1 > = 0,$$ \hspace{1cm} \hspace{1cm} \hspace{1cm} (35)

$$< m | L | m + 2 > = 1,$$ \hspace{1cm} \hspace{1cm} \hspace{1cm} (36)

$$< m | L | n > = 0 \quad \text{for} \quad n \geq m + 3,$$ \hspace{1cm} \hspace{1cm} \hspace{1cm} (37)

where

$$u_n := \frac{d^n u}{dx^n}.$$ \hspace{1cm} \hspace{1cm} \hspace{1cm} (38)

The result can also be compactly expressed by:

$$< m | L | n > = \binom{m}{n} u_{m-n} + < m + 2 | n >,$$ \hspace{1cm} \hspace{1cm} \hspace{1cm} (39)

if we set

$$\binom{m}{n} := 0, \quad u_n := 0 \quad \text{for} \quad n < 0.$$ \hspace{1cm} \hspace{1cm} \hspace{1cm} (40)
It is also helpful to arrange the matrix elements to an infinite matrix

\[ \mathbf{L} \equiv \langle m | L | n \rangle = \begin{pmatrix}
    u_0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
    u_1 & u_0 & 0 & 1 & 0 & 0 & \cdots \\
    u_2 & \left( \frac{3}{1} \right) u_1 & u_0 & 0 & 1 & 0 & \cdots \\
    u_3 & \left( \frac{3}{1} \right) u_2 & \left( \frac{3}{2} \right) u_1 & u_0 & 0 & 1 & \cdots \\
    u_4 & \left( \frac{5}{1} \right) u_3 & \left( \frac{4}{2} \right) u_2 & \left( \frac{4}{3} \right) u_1 & u_0 & 0 & \cdots \\
    u_5 & \left( \frac{5}{1} \right) u_4 & \left( \frac{5}{2} \right) u_3 & \left( \frac{5}{3} \right) u_2 & \left( \frac{5}{4} \right) u_1 & u_0 & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix} \tag{41} \]

This matrix can be expanded as an infinite sum of constant matrices with shifted diagonals multiplied by derivatives of \( u \):

\[ \mathbf{L} = I_{-2} + u_0 I_0 + u_1 I_1 + \cdots = \sum_{j \geq 0} u_{j-2} I_{j-2} \tag{42} \]

where we formally set

\[ u_{-2} := 1, \quad u_{-1} := 0, \quad u_{j-2} := 0, \quad j \geq 1 \tag{43} \]

and the proper meaning of each matrix \( I_j \) is just defined by (41).

4. More generally, let us consider a linear differential operator of the form

\[ L_r = \frac{\partial^r}{\partial x^r} + u, \quad u = u(x) \tag{44} \]

where \( r \) is a positive integer. We will call it a higher-order Schrödinger-like operator. The one-dimensional Schrödinger operator is the special case \( r = 2 \). The calculation of the matrix elements in the preceding item can be extended to general \( r \). We obtain

\[ \langle m | L_r | n \rangle = \binom{m}{n} u_{m-n} + \langle m + r | n \rangle, \tag{45} \]

if we adopt the convention (41). This defines the matrix representation of \( L_r \)

\[ L_r = \langle m | L_r | n \rangle = \sum_{j \geq 0} u_{j-r} I_{j-r} \tag{46} \]

where now we alternatively set

\[ u_{-r} = 1, \quad u_{-r-1} = \cdots = u_{-1} = 0, \tag{47} \]

and the matrix \( I_{-r} \) is defined by

\[ \langle m | I_{-r} | n \rangle = \langle m + r | n \rangle \tag{48} \]
5. Finally, let us consider the powers $L_r^k$ of the Schrödinger-like operator (44). Clearly, the matrix elements $\langle m|L_r^k|n \rangle$ of the powers of the operator $L_r$ are given just by the $k$-th power of the infinite matrix $L_r$ (46). Thus
\[
L_r^k = \sum_{m,n \geq 0} |m> < m|L_r^k|n > < n|,
\]
where
\[
\langle m|L_r^k|n \rangle = \sum_{j_1,\ldots,j_k \geq 0} h(j)u_{j_1-r} \cdots u_{j_k-r}.
\]
Here $j = (j_1,\ldots,j_k)$ is a multi-index formed from non-negative integers $j_1, j_2, \ldots, j_k$ and the constants $h(j)$ are determined by the products of the matrices $I_j$:
\[
h(j) = \langle m|I_{j_1-r} \cdots I_{j_k-r}|n \rangle
\]

3 Derivation of the KdV polynomials

Let us apply a special multi-index formalism in order to present our result: we set
\[
m := (m_1,m_2,\ldots,m_d),
\]
\[
|m|_p := m_1 + m_2 + \cdots + m_p, \quad \text{for } p = 1,2,\ldots,d,
\]
\[
|m| := |m|_d = m_1 + m_2 + \cdots + m_d,
\]

**Theorem 1** The Taylor coefficients of the heat kernel coefficients $a_k = a_k(x,x')$ to the one-dimensional Schrödinger operator (3) read
\[
\langle n|a_k \rangle = \sum_{d=1}^k \sum_{|m|=n+2k-2d} c(m)u_{m_d} \cdots u_{m_2} u_{m_1}
\]
where the inner sum runs over integers $m_1 \geq 0, m_2 \geq 0, \ldots, m_d \geq 0$ such that
\[
|m| \equiv m_1 + m_2 + \cdots + m_d = n + 2k - 2d
\]
and where the numerical coefficients read
\[
c(m) \equiv c(m_1,m_2,\ldots,m_d)
\]
\[
= \sum_{i_1,\ldots,i_{d-1}} \prod_{p=1}^d \left( i_p \right) \left( |m|_p - 2i_{p-1} + 2p \right) \left( |m|_p - i_{p-1} + 2p + 1 \right)^{-1}.
\]
The sum in (57) runs over integers $i_1, i_2, \ldots, i_{d-1}$ such that
\[
0 \equiv i_0 < i_1 < i_2 < \cdots < i_{d-1} < i_d \equiv k,
\]
\[
2i_p \leq |m|_{p+1} + 2p.
\]
Proof. The recursive differential equations (10) have the symbolic solution

\[
a_k = \left(1 + \frac{1}{k} D\right)^{-1} L \cdots \left(1 + \frac{1}{2} D\right)^{-1} L \left(1 + \frac{1}{1} D\right)^{-1} L \cdot 1
\]  

(60)

This can be translated, by use of the eq. (32), into a formula in terms of matrix elements:

\[
< n | a_k > = \sum_{n_1, \ldots, n_{k-1}} \left(1 + \frac{n_k}{k}\right)^{-1} \cdots \left(1 + \frac{n_2}{2}\right)^{-1} \left(1 + \frac{n_1}{1}\right)^{-1} 
\times < n_k | L | n_{k-1} > \cdots < n_2 | L | n_1 > < n_1 | L | 0 > ,
\]  

(61)

where \( n_k \equiv n \) and the summation indices \( n_1, n_2, \ldots, n_{k-1} \) run through 0, 1, 2, \ldots.

Recall the values (34)-(37) of the matrix elements of \( L \). Although the sum in (61) is formally an infinite one, it is effectively finite. Let us call a matrix element \( < m | L | n > \) essential if \( n \leq m \) and non-essential if \( n \geq m + 1 \). The number of essential elements in a term of (61) equals the degree \( d \) of the differential monomial in \( u \). Let us analyze a typical term of degree \( d = N + 1 \) in (61). Clearly, \( < n_1 | L | 0 > \) is essential since \( n_1 \geq 0 \). Let

\[
< n_{i_N+1} | L | n_{i_N} >, \ldots, < n_{i_1+1} | L | n_1 >, < n_{i_1} | L | 0 > .
\]  

(62)

denote all the essential elements, numerated from the right to the left. The space between two essential elements is filled by a chain of unessential matrix elements, which can be written as Kronecker symbols. The unessential chain between \( < n_{i_1+1} | L | n_{i_1} > \) and \( < n_1 | L | 0 > \) reads

\[
< n_{i_1} + 2 | n_{i_1} > \cdots < n_3 + 2 | n_2 > < n_2 + 2 | n_1 > .
\]  

(63)

This product of Kronecker symbols cancels the summations over \( n_2, n_3, \ldots, n_{i_1} \) by fixing these running indices:

\[
n_2 = n_1 - 2 , \quad n_3 = n_2 - 2 = n_1 - 4 , \quad \ldots \quad n_{i_1} = n_{i_1 - 2} = \ldots = n_1 - 2(i_1 - 1) .
\]  

(64)

The chain under consideration produces the numerical coefficient

\[
c_0 := \left(1 + \frac{n_{i_1}}{i_1}\right)^{-1} \cdots \left(1 + \frac{n_3}{3}\right)^{-1} \left(1 + \frac{n_2}{2}\right)^{-1} \left(1 + \frac{n_1}{1}\right)^{-1}
\equiv \frac{i_1}{n_{i_1} + i_1} \cdot \frac{3}{n_3 + 3} \cdot \frac{2}{n_2 + 2} \cdot \frac{1}{n_1 + 1} .
\]  

(65)

By virtue of (64), it becomes

\[
c_0 = \left(\frac{n_1 + 1}{i_1}\right)^{-1} .
\]  

(66)
The next unessential chain between $< n_{i+1} | L | n_i >$ and $< n_{i+1} | L | n_i >$ reads

$$< n_2 + 2n_1 > \cdots < n_{i+1} + 2n_{i+2} > < n_{i+2} + 2n_{i+1} >$$

(67)

It cancels the summation over $n_{i+2}, n_{i+3}, \ldots, n_{i+2}$ by fixing

$$n_{i+2} = n_{i+1} - 2,$$
$$n_{i+3} = n_{i+2} - 2 = n_{i+1} - 4,$$
$$\ldots$$
$$n_{i+2} = n_{i+1} - 2 = \ldots = n_{i+1} - 2(i_2 - i_1 - 1)$$

(68)

and produces the numerical coefficient

$$c_1 := \left( 1 + \frac{n_2}{i_2} \right)^{-1} \cdots \left( 1 + \frac{n_{i+3}}{i_1 + 3} \right)^{-1} \left( 1 + \frac{n_{i+2}}{i_1 + 2} \right)^{-1} \left( 1 + \frac{n_{i+1}}{i_1 + 1} \right)^{-1}$$

$$= \frac{i_2}{n_2 + i_2} \cdots \frac{3}{n_{i+3} + i_1 + 3} \cdot \frac{i_1 + 2}{n_{i+2} + i_1 + 2} \cdot \frac{i_1 + 1}{n_{i+1} + i_1 + 1}$$

(69)

By virtue of (68), the latter becomes

$$c_1 = \left( \frac{i_2}{i_1} \right) \left( \frac{n_{i+1} + i_1 + 1}{i_2 - i_1} \right)^{-1}.$$  

(70)

The procedure carries on this way; the last unessential chain fixes

$$n_{i+2} = n_{i+1} - 2,$$
$$n_{i+3} = n_{i+2} - 2 = n_{i+1} - 4,$$
$$\ldots$$
$$n_{k-1} = n_{k-2} - 2 = \ldots = n_{i_N+1} - 2(k - i_N - 2)$$
$$n_k \equiv n = \ldots = n_{i_N+1} - 2(k - i_N - 1)$$

(71)

and produces the numerical coefficient

$$c_N = \left( \frac{k}{i_N} \right) \left( \frac{n_{i_N+1} + i_N + 1}{k - i_N} \right)^{-1}.$$ 

(72)

After the reduction procedure the typical term in (61) reads

$$c_N \cdots c_1 c_0 < n_{i_N+1} | L | n_i > \cdots < n_{i+1} | L | n_1 > < n_1 | L | 0 >$$

(73)

and the summation indices are now $N, i_1, i_2, \ldots, i_N, n_1, n_{i_1+1}, n_{i_2+1}, \ldots, n_{i_N+1}$. Notice, further, that $n_k = n$ is fixed.
In order to get a better expression for the essential matrix elements it is convenient to pass to new summation indices

\[ m_1 := n_1, \]
\[ m_2 := n_{i+1} - n_i = n_{i+1} - n_1 + 2(i_1 - 1), \]
\[ m_3 := n_{i+2} - n_i = n_{i+2} - n_{i+1} + 2(i_2 - i_1 - 1), \]
\[ \ldots \]
\[ m_{N+1} := n_{i+N} - n_i = n_{i+N} - n_{i+N-1} + 2(i_N - i_{N-1} - 1). \] (74)

The old summation indices are expressed in terms of the new ones by

\[ n_1 = m_1, \]
\[ n_{i+1} = m_1 + m_2 - 2i_1 + 2, \]
\[ n_{i+2} = m_1 + m_2 + m_3 - 2i_2 + 4, \]
\[ \ldots \]
\[ n_{i+N} = m_1 + m_2 + \cdots + m_{N+1} - 2i_N + 2N. \] (75)

Hence

\[ c_0 = \binom{m_1 + 1}{i_1}^{-1}, \]
\[ c_1 = \binom{i_2}{i_1} \binom{m_1 + m_2 - i_3 + 3}{i_2 - i_1}^{-1}, \]
\[ \ldots \]
\[ c_N = \binom{k}{i_N} \binom{m_1 + \cdots + m_{N+1} - i_N + 2N + 1}{k - i_N}^{-1}. \] (76)

Some restriction for \( m_1, m_2, \ldots, m_{N+1} \) is to be considered:

\[ m_1 + m_2 + \cdots + m_{N+1} = n + 2(k - N - 1); \] (77)

it follows from (74). Besides, all \( m_i \) are non-negative and we have to require

\[ 0 \leq 2i_p \leq m_1 + m_2 + \cdots + m_{p+1} + 2p. \] (78)

Finally, we express the essential matrix elements in terms of \( m_1, m_2, \ldots, m_{N+1} \):

\[ < n_1 | L | 0 > = \binom{m_1}{0} u_{m_1}, \]
\[ < n_{i+1} | L | n_i > = \binom{m_1 + m_2 - 2i_1 + 2}{m_2} u_{m_2}, \]
\[ < n_{i+2} | L | n_i > = \binom{m_1 + m_2 + m_3 - 2i_2 + 4}{m_3} u_{m_3}, \]
\[ \ldots \]
\[ < n_{i+N} | L | n_i > = \binom{m_1 + \cdots + m_{N+1} - 2i_N + 2N}{m_{N+1}} u_{m_{N+1}}. \] (79)
We complete the proof by setting $d := N + 1, i_0 := 0, i_d := k$ and summation over $d = 1, 2, \ldots, k$. The asserted formula (52)–(57) follows.

The outer summation in (53) is a decomposition into homogeneous parts. Note, further, that the differential polynomial $< n|a_k >$ in $u$ is isobaric, that means every term has the same weight

$$|m| \equiv m_1 + m_2 + \cdots + m_d = n + 2(k - d).$$  \hspace{1cm} (81)

Theorem 1 is a very special one-dimensional case of the general formula for the Taylor coefficients of the heat kernel coefficients to an arbitrary second-order partial differential operator of Laplace type obtained by one of the authors (I.G. A.) in [5, 4]. We present here for the first time the complete proof; in [5, 4] the combinatorial details were omitted.

Specializing to

$$< 0|a_k > = a_k(x, x),$$ \hspace{1cm} (82)

and considering (12) we obtain

**Theorem 2** The right-hand sides of the KdV hierarchy (3) are given by

$$G_k[u] = \frac{1}{2} \binom{2k}{k} \sum_{d=1}^{k} \sum_{|m| = 2k - 2d} c(m) u_{m_d} \cdots u_{m_2} u_{m_1}$$ \hspace{1cm} (83)

where the multiindex $m$ and the numerical coefficients $c(m)$ have the same meaning as in Theorem 1.

Notice that (83) is valid for the matrix KdV hierarchy, since our proof did not make use of a commutative law. The conventional scalar KdV hierarchy, where commutativity is assumed, emerges as a special case.

## 4 On the structure of the KdV polynomials

The explicit expression (83) is well suited to be evaluated by means of computer algebra. It is not practical to calculate higher terms by hand. Also, (83) conceals that some terms in $G_k[u]$ have a relatively simple structure, as we are going to discuss now. The asymptotic expansion of the heat kernel diagonal

$$K(s; x, x) \sim (4\pi s)^{-1/2} \sum_{k \geq 0} \frac{1}{k!} a_k(x, x) s^k = (4\pi s)^{-1/2} \sum_{k \geq 0} \frac{k!}{(2k)!} G_k[u] s^k$$ \hspace{1cm} (84)

is a generating function for the sequence of the KdV polynomials $G_k$. It is well known, that the diagonal value $K(s; x, x)$ of the of the heat kernel obeys

$$K(s; x, x) = e^{s \overline{K}}(s; x, x),$$ \hspace{1cm} (85)
where \( \tilde{K} \) is the heat kernel diagonal calculated for \( u_o = 0 \). As a consequence, there holds

\[
a_k = \sum_{i=0}^{k} \binom{k}{i} u_0^i \tilde{a}_{k-i},
\]

and

\[
G_k = \sum_{i=0}^{k} \binom{k - \frac{1}{2}}{i} (4u_0)^i \tilde{G}_{k-i},
\]

where

\[
\tilde{a}_k = a_k \big|_{u_o=0}, \quad \tilde{G}_k = G_k \big|_{u_o=0}.
\]

Notice that

\[
\binom{k - \frac{1}{2}}{i} = \frac{\Gamma \left( k + \frac{1}{2} \right)}{i! \Gamma \left( k - i + \frac{1}{2} \right)} = \frac{1}{2^{2i}} \binom{2k}{2i} \binom{2i}{i} \binom{k}{i}^{-1}.
\]

Herefrom we find the terms of highest degree and lowest order in \( G_k \) as far as we want:

\[
60 \binom{2k}{k}^{-1} G_k = 30u_0^k + 10 \binom{k}{2} u_0^{k-2} u_2 + 3 \binom{k}{3} u_0^{k-3} (u_4 + 5u_1^3) + \cdots
\]

The terms of lowest degree and highest order are found by means of some other general relation. Namely, we have shown in [3] that for the matrix KdV hierarchy the recursion

\[
G_k = 2 \sum_{i=1}^{k} Z_{2i-1} G_{k-i}, \quad (k = 1, 2, \ldots),
\]

holds, where the differential polynomials \( Z_k = Z_k[u] \) on their side are recursively defined by

\[
Z_{k+1} := \frac{d}{dx} Z_k + \sum_{i=1}^{k-1} Z_i Z_{k-i},
\]

\[
Z_1 := u_0,
\]

(The proof is far from being trivial in the non-commutative case.) This enabled us in [3] to find

\[
G_k = u_{2k-2} + \sum_{i=0}^{2k-4} \left\{ \binom{2k-2}{i+1} + (-1)^i \right\} u_i u_{2k-4-i} + \cdots,
\]

where the points indicate terms of higher degree.

Let us add here another result on the structure of the KdV hierarchy: we find, in the commutative case, all monomials in \( G_k \) which are built solely from \( u_0, u_1 \) and \( u_2 \).

The proof uses again diagonal values of the heat kernel as generating function and the fact that \( u_0, u_1 u_2 \) commute with each other. Besides, we will use the Bernoulli numbers \( B_n, \ (n = 0, 1, 2, \ldots) \), (cf., e.g., [13]) defined recursively by

\[
\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0, \quad B_0 := 1.
\]

The first Bernoulli numbers read: \( B_1 = -\frac{1}{2}, \ B_2 = -\frac{1}{6}, \ B_3 = 0, \ B_4 = -\frac{1}{30} \). Generally, \( B_{2k+1} = 0, \ B_{2k} = (-1)^{k+1} |B_{2k}| \neq 0 \) for \( k \geq 1 \).
Theorem 3  For the scalar KdV hierarchy there holds

\[ G_k[u] \equiv \frac{(2k)!}{2k!k!} a_k(x, x) = \sum_{N=1}^{k} \frac{(2k)!}{2k!N!} \sum_{|m|=k} \frac{b_{m_1}}{m_1!} \cdots \frac{b_{m_N}}{m_N!} + \ldots \]  \hspace{1cm} (96)

where the inner sum runs over integer \( m_1 \geq 1, \ldots, m_N \geq 1 \) such that \( |m| \equiv m_1 + \cdots + m_N = k \), the monomials \( b_k \) are defined by

\[ b_1 = u_0, \]  \hspace{1cm} (97)

\[ b_{2n} = \frac{1}{n} 2^{3n-2} |B_{2n}| u_2^n, \]  \hspace{1cm} (98)

\[ b_{2n-1} = \frac{1}{n} 2^{n-1} (2^{2n} - 1) |B_{2n}| u_2^{n-2} u_1^2, \quad (n \geq 2), \]  \hspace{1cm} (99)

and the dots indicate terms which effectively contain higher derivatives \( u_3, u_4 \) etc. of \( u \).

Proof.  Actually, we have to evaluate the asymptotic expansion of the heat kernel for a quadratic potential (harmonic oscillator)

\[ u(x) = u_0(x') + u_1(x')(x - x') + \frac{1}{2} u_2(x')(x - x')^2. \]  \hspace{1cm} (100)

The heat kernel diagonal in this particular case reads (up to exponentially small terms that do not contribute to the generating function)

\[ K(s; x, x) \sim (4\pi s)^{-1/2} \exp \left\{ su_0 + \Phi(s) + \frac{1}{4} s^3 u_1^2 \Psi(s) \right\} \]  \hspace{1cm} (101)

where

\[ \Phi(s) = -\frac{1}{2} \log \left( \frac{\sinh (2s\omega)}{2s\omega} \right), \]  \hspace{1cm} (102)

\[ \Psi(s) = \frac{s\omega - \tanh (s\omega)}{(s\omega)^3} \]  \hspace{1cm} (103)

and

\[ \omega \equiv \sqrt{-u_2/2} \]  \hspace{1cm} (104)

Actually, this is the well-known heat kernel diagonal of a the one-dimensional harmonic oscillator, which can be read from the literature (cf., e.g. [11, 12]). Expanding the heat kernel diagonal one finds all the coefficients \( a_k(x, x) \) in the considered approximation. The Taylor expansions of the functions \( \Phi(t) \) and \( \Psi(t) \) read

\[ \Phi(s) = \sum_{n \geq 1} \frac{2^{3n-2} |B_{2n}|}{n(2n)!} u_2^n s^{2n}, \]  \hspace{1cm} (105)
\[ \Psi(s) = \sum_{n \geq 2} \frac{2^{n+2} (2^{2n} - 1) |B_{2n}| u_2^{n-2} s^{2n-4}}{(2n)!}, \]  
\[ (106) \]

where \( B_n \) are the Bernoulli numbers. Obviously, these series have only positive coefficients. We obtain the heat kernel diagonal in the form

\[ K(s;x,x) \sim (4\pi s)^{-1/2} \exp \left( \sum_{n \geq 1} \frac{s^n}{n!} b_n \right), \]
\[ (107) \]

where the coefficients \( b_n \) are defined by (97)-(99). Note that all \( b_n \) are positive. The heat kernel coefficients \( a_k(x,x) \) and, therefore, the differential polynomials \( G_k \) (84) are determined by the expansion of the exponent, which completes the proof.

Let us note that the heat kernel coefficients \( a_k(x,x) \), in the above approximation, are given by the so-called Bell polynomials \( B_k \). Namely, the Bell polynomials are defined by their generating function

\[ \sum_{k=0}^{\infty} \frac{s^k}{k!} B_k = \exp \left( \sum_{n=1}^{\infty} \frac{s^n}{n!} y_n \right). \]
\[ (108) \]

Therefore, the formula (90) can be also rewritten in the form

\[ G_k[u] = \frac{1}{2} \binom{2k}{k} B_k (b_1, \ldots, b_k). \]
\[ (109) \]

Several explicit expressions for the Bell polynomials are known, in particular,

\[ B_n(y_1, \ldots, y_d) = \sum_{d=1}^{n} \sum_{|m| = n} \prod_{p=1}^{d} \left( \binom{|m|_p}{|m|_{p-1}} - 1 \right) y_{m_p} \]
\[ (110) \]

for \( n \geq 1 \) [17]. Here, as usual, \( m = (m_1, \ldots, m_d) \) with non-negative integers \( m_1, \ldots, m_d \) and

\[ |m|_p := m_1 + \cdots + m_p, \quad \text{for } p = 1, \ldots, d \]
\[ (111) \]

\[ |m|_0 := 0, \quad |m| := |m|_d \]
\[ (112) \]

The expression (110) is valid for generally non-commuting variables \( y_1, y_2, \ldots \). Another expression, which holds only in the commutative case, is due to Faa di Bruno (cf. e.g. [17]):

\[ B_n(y_1, \ldots, y_n) = n! \sum_{|m| = n} \prod_{j=1}^{n} \frac{1}{m_j} \left( \frac{y_j}{j!} \right)^{m_j} \]
\[ (113) \]

where the sum runs over all non-negative integers \( m_1, \ldots, m_n \) such that

\[ |m| := m_1 + 2m_2 + \cdots + nm_n = n \]
\[ (114) \]

One should point out here that the theorem 3. is a particular case of a general result [16] of one of the authors (I.G.A.). In that paper the heat kernel diagonal to a Laplace
type operator $L = -g^{\mu\nu}\nabla_\mu \nabla_\nu + Q$ acting on sections of a vector bundle $V$ over a flat $n$-dimensional manifold with a metric $g$ and the bundle connection $\nabla^V$ was studied and the part of the heat kernel diagonal that depends solely on the curvature $R_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$ of the bundle connection, the potential $Q$ and its first $\nabla_\mu Q$ and second $\nabla_\mu \nabla_\nu Q$ derivatives is calculated in a closed form. Note that there is a misprint in the eq. (3.31) of the paper [16].

5 Powers of Schrödinger-like operators

The matrix elements of powers of a Schrödinger-like operator are already given in compact form by (50). Let us present here a more explicit expression for the powers of the proper Schrödinger operator (44).

**Theorem 4** The matrix elements of the $k$-th power of the $r$-th order Schrödinger-like operator (44) read

$$< m | L^k_r | n > = < m + rk | n > + \sum_{d=1}^{k} \sum_{|\tilde{m}| = m - n + r(k - d)} h_{r,k}(\tilde{m}) u_{m_d} \cdots u_{m_1}$$

where the inner sum here runs over non-negative integers $m_1 \geq 0, \ldots, m_d \geq 0$, such that

$$|\tilde{m}| \equiv m_1 + \cdots + m_d = m - n + r(k - d)$$

and the numerical coefficients read

$$h_{r,k}(m) \equiv h_{r,k}(m_1, m_2, \ldots, m_d) = \sum_{i_1, \ldots, i_d} \prod_{p=1}^{d} \left( |m|_p + n - r(i_p - p + 1) \right).$$

The sum in (117) runs over non-negative integers $i_1, \ldots, i_d$ such that

$$0 \leq i_1 < i_2 < \cdots < i_{d-1} < i_d < k,$$

$$|m|_p + n - r(i_p - p + 1) \geq 0.$$  

The proof is performed by the method of matrix elements, analogous to the proof of the Theorem 1 and, hence, is omitted.

Let us translate the result into the conventional notation.

**Corollary 1** The $k$-th power of the $r$-th order Schrödinger-like operator (44) applied to a function $f$ gives

$$L^k_r f = f_{rk} + \sum_{d=1}^{k} \sum_{|\tilde{m}| = r(k - d)} h_{r,k}(\tilde{m}) u_{m_d} \cdots u_{m_1} f_{m_0}$$
where \( f_m := \frac{d^m}{dx^m} f \), \( \bar{m} \equiv (m_0, m) \equiv (m_0, m_1, \ldots, m_d) \), the inner sum here runs over non-negative integers \( m_0 \geq 0, m_1 \geq 0, \ldots, m_d \geq 0 \), such that
\[
|\bar{m}| \equiv m_0 + m_1 + \cdots + m_d = r(k - d) \quad (121)
\]
and the numerical coefficients read
\[
h_{r,k}(\bar{m}) \equiv h_{r,k}(m_0, m_1, m_2, \ldots, m_d) = \sum_{i_1, \ldots, i_d} \prod_{p=1}^{d} \left( \frac{|\bar{m}|_p - r(i_p - p + 1)}{m_p} \right). \quad (122)
\]
The sum in (122) is the same as in the theorem 4.

**Proof.** There holds, with a mixture of matrix element notation and conventional notation,
\[
L_r^k f = \sum_{0 \leq n \leq rk} <0|L_r^n|n>n f_n. \quad (123)
\]
Here we insert \(<0|L_r^n|n>n\) from the theorem 4, i.e. we set \( m = 0 \) there, rename \( m_0 := n \) and define a new multiindex \( \bar{m} = (m_0, m) \).

The numerical coefficients in (120) for \( d = 1 \) and for \( d = k \) exhibit a simple structure, namely,
\[
h_{r,k}(m_0, m_1) = \sum_{i=0}^{k-1} \binom{r i}{m_1}, \quad (124)
\]
\[
h_{r,k}(m_0, m_1, \ldots, m_k) = h_{r,k}(0, \ldots, 0) = 1. \quad (125)
\]
Thus
\[
L_r^k f = f_{rk} + \sum_{m_0 + m_1 = r(k-1)} \sum_{i=0}^{k-1} \binom{r i}{m_1} u_{m_0} f_{m_0} + \cdots + u^k f \quad (126)
\]
where the dots indicate the terms of degree in \( u \) greater than 1 and less than \( k \).

More explicit formulas for \( L_r^k f \) are available for \( r = 1 \). The powers of the first order operator
\[
L_1 = \frac{d}{dx} + u \quad (127)
\]
are given by [17]
\[
L_r^k f = \sum_{l=0}^{k} \binom{k}{l} B_l(u_0, \ldots, u_{l-1}) f_{k-l}, \quad (128)
\]
where \( B_l = B_l(y_1, \ldots, y_l) \) \((l = 0, 1, 2, \ldots)\) denotes the Bell polynomials, which we already discussed above.

**Acknowledgements**

This work was supported by the Deutsche Forschungsgemeinschaft.
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