1 Adjoint orbits of the generalized real orthogonal group

In this section we define the generalized real orthogonal group and introduce some concepts needed to classify its adjoint orbits on its Lie algebra.

First we give a description of the generalized real orthogonal group in terms of matrices. Let $\tilde{G}$ be the Gram matrix of a nondegenerate inner product $\tilde{\gamma}$ on $\mathbb{R}^n$. The real orthogonal group $O(\mathbb{R}^n, \tilde{G})$ on $(\mathbb{R}^n, \tilde{G})$ is the set of $n \times n$ real matrices $\tilde{A}$ such that $\tilde{A}^T \tilde{G} \tilde{A} = \tilde{G}$. It is a Lie group whose Lie algebra $o(\mathbb{R}^n, \tilde{G})$ is $\{ \tilde{\xi} \in \mathfrak{gl}(n, \mathbb{R}) \mid \tilde{\xi}^T \tilde{G} + \tilde{G} \tilde{\xi} = 0 \}$. Let $\{ e_0, e_1, \ldots, e_n \}$ be the standard basis for $\mathbb{R}^{n+1}$. Suppose that $\gamma$ is symmetric bilinear form on $\mathbb{R}^{n+1}$ whose Gram matrix is $G = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{G} \end{pmatrix}$. The bilinear form $\gamma$ is degenerate since $\ker G = \text{span}\{ e_0 \}$. The generalized real orthogonal group $O(\mathbb{R}^{n+1}, G)_{e_0}$ on $(\mathbb{R}^{n+1}, G)$ is $\{ A \in O(\mathbb{R}^{n+1}, G) \mid Ae_0 = e_0 \}$, that is,

\[
\left\{ \begin{pmatrix} 1 & b^T \\ 0 & A \end{pmatrix} \in \text{Gl}(n+1, \mathbb{R}) \mid b \in \mathbb{R}^n, \ A \in O(\mathbb{R}^n, \tilde{G}) \right\}.
\]

$O(\mathbb{R}^{n+1}, G)_{e_0}$ is a Lie group with Lie algebra $o(\mathbb{R}^{n+1}, G)_{e_0}$ given by

\[
\left\{ \begin{pmatrix} 0 & b^T \\ 0 & \tilde{\xi} \end{pmatrix} \in \mathfrak{gl}(n+1, \mathbb{R}) \mid b \in \mathbb{R}^n, \ \tilde{\xi} \in o(\mathbb{R}^n, \tilde{G}) \right\}.
\]

We now give a more abstract description of the generalized real orthogonal group. Let $\tilde{V}$ be a finite dimensional real vector space with a symmetric bilinear form $\tilde{\gamma}$, which is nondegenerate. Let $O(\tilde{V}, \tilde{\gamma})$ be the set of real linear mappings $\tilde{A}$ of $\tilde{V}$ into itself such that $\tilde{\gamma}(\tilde{A}\tilde{u}, \tilde{A}\tilde{v}) = \tilde{\gamma}(\tilde{u}, \tilde{v})$ for every $\tilde{u}, \tilde{v} \in \tilde{V}$. $O(\tilde{V}, \tilde{\gamma})$ is called the real orthogonal group. It is a Lie group whose Lie algebra $o(\tilde{V}, \tilde{\gamma})$ is the set of real linear maps $\tilde{\xi} : \tilde{V} \to \tilde{V}$ such that $\tilde{\gamma}(\tilde{\xi}\tilde{u}, \tilde{v}) + \tilde{\gamma}(\tilde{u}, \tilde{\xi}\tilde{v}) = 0$ for every $\tilde{u}, \tilde{v} \in \tilde{V}$. Suppose that $\gamma$ is a symmetric
bilinear form on a real finite dimensional vector space $V$, whose kernel is the span of a nonzero vector $v^0$. Let $O(V, \gamma)_{v^0}$ be the set of bijective real linear mappings $A : V \rightarrow V$ such that $Av^0 = v^0$ and $\gamma(Au, Av) = \gamma(u, v)$ for every $u, v \in V$. $O(V, \gamma)_{v^0}$ is called the generalized real orthogonal group. It is a Lie group whose Lie algebra $o(V, \gamma)_{v^0}$ consists of real linear maps $\xi : V \rightarrow V$ such that $\xi v^0 = 0$ and $\gamma(\xi u, v) + \gamma(u, \xi v) = 0$ for every $u, v \in V$.

2 Basic concepts

The goal of this paper is to find a unique representative (= normal form) for each orbit of the adjoint action of $O(V, \gamma)_{v^0}$ on $o(V, \gamma)_{v^0}$.

We begin by defining some basic concepts, which follow [1] and [2]. A pair $(\tilde{\xi}|\tilde{W}, \tilde{W}; \tilde{\gamma}|\tilde{W})$ is a proper $\tilde{\xi}$-invariant subspace $\tilde{W}$ of $V$ such that $\tilde{\gamma}$ is nondegenerate on $\tilde{W}$ and $\tilde{\xi} \in o(\tilde{W}, \tilde{\gamma}|\tilde{W})$. We say that two pairs $(\xi|\tilde{W}, \tilde{W}; \tilde{\gamma}|\tilde{W})$ and $(\tilde{\xi}^\prime|\tilde{W}^\prime, \tilde{W}^\prime; \tilde{\gamma}^\prime|\tilde{W}^\prime)$ are equivalent if there is a bijective real linear mapping $P : \tilde{W} \rightarrow \tilde{W}^\prime$ such that $\tilde{\gamma}^\prime(\tilde{P}u, \tilde{P}v) = \tilde{\gamma}(u, v)$ and $(\tilde{P} \circ \tilde{\xi})\tilde{w} = (\tilde{\xi}^\prime P)\tilde{w}$ for every $u, v, \tilde{w} \in \tilde{W}$. Being equivalent is an equivalence relation on the set of pairs. An equivalence class $\Delta$ of pairs is called a type. If $\tilde{W}_i$, $i = 1, 2$, are proper $\tilde{\xi}$-invariant, and $\tilde{\gamma}$-orthogonal subspaces of $\tilde{W}$, whose direct sum is $\tilde{W}$ and on which $\tilde{\gamma}|\tilde{W}_i$ is nondegenerate, then the type $\Delta$, represented by $(\xi|\tilde{W}_1, \tilde{W}_1; \tilde{\gamma}|\tilde{W}_1)$ and $\Delta_2$ represented by $(\xi|\tilde{W}_2, \tilde{W}_2; \tilde{\gamma}|\tilde{W}_2)$, respectively. We write $\Delta = \Delta_1 + \Delta_2$. A type, which cannot be written as the sum of two types, is indecomposable. Indecomposable types have been classified in [1].

A triple $(W, \xi|W, v^0; \gamma|W)$ is a proper, $\xi$-invariant subspace $W$ of $V$ containing $v^0$ such that $\ker \gamma|W = \text{span}\{v^0\}$ and $\xi|W \in o(W, \gamma|W)_{v^0}$. Two triples $(W, \xi|W, v^0; \gamma|W)$ and $(W^\prime, \xi^\prime|W^\prime, (v^\prime)^0; \gamma^\prime|W^\prime)$ are equivalent if there is a bijective real linear mapping $P : W \rightarrow W^\prime$ such that $Pv^0 = (v^\prime)^0$, $\gamma^\prime(Pu, Pv) = \gamma(u, v)$ and $(\xi^\prime P)w = (P \circ \xi)w$ for every $u, v, w \in W$. Being equivalent is an equivalence relation on the set of triples. An equivalence class $\Delta$ of triples is called a special type. If $(W, \xi|W, v^0; \gamma|W)$ is a triple, representing $\Delta$, with $\xi$ nilpotent, then $\Delta$ is a nilpotent special type. If $W_i$, $i = 1, 2$, are proper $\xi$-invariant, $\gamma$-orthogonal subspaces of $W$ with $v^0 \in W_i$, $W = W_1 \oplus W_2$ and $\ker \gamma|W_1 = \text{span}\{v^0\}$, then the special type $\Delta_i$, represented by $(W, \xi|W, v^0; \gamma|W)$, is the sum of the special type $\Delta_i^\prime$, represented by $(W_1, \xi|W_1, v^0; \gamma|W_1)$, and the type $\Delta$ represented by $(\xi|W_2, W_2; \gamma|W_2)$. We write $\Delta = \Delta_i^\prime + \Delta$. A special type is indecomposable if it cannot be written as the sum of a special type and a type.
For $\xi \in o(V, \gamma)$, let $V_0$ be the direct sum of generalized eigenspaces of $\xi$ each corresponding to an eigenvalue 0 and let $U$ be the direct sum of generalized eigenspaces of $\xi$ each corresponding to a nonzero eigenvalue of $\xi$. Then $V_0$ and $U$ are $\xi$-invariant, $\gamma$-orthogonal subspaces of $V$ such that $V = V_0 \oplus U$. Here $v^0 \in V_0$, since $\xi v^0 = 0$, and $\ker \gamma|V_0 = \text{span}\{v^0\}$. Thus $\gamma|U$ is nondegenerate. Therefore the special type $\underline{\Delta}$ represented by $(V, \xi, v^0; \gamma)$, is the sum of the special type $\underline{\Delta}_0$, represented by $(V_0, \xi|V_0, v^0; \gamma|V_0)$, and the type $\Delta$, represented by the pair $(\xi|U; \gamma|U)$, that is, $\underline{\Delta} = \underline{\Delta}_0 + \Delta$. Since $\xi|V_0$ is nilpotent, $\underline{\Delta}_0$ is a nilpotent special type.

So we have proved

**Proposition 1.** Every special type $\underline{\Delta}$ may be written uniquely as the sum of a nilpotent special type $\underline{\Delta}_0$ and a type $\Delta$.

Let $(V, \xi, v^0; \gamma)$ be a triple representing the special type $\underline{\Delta}$. Since $v^0 \in \ker \xi$, the vector $v^0$ lies at the end of a Jordan chain in $V$. Let $h$ be the largest nonnegative integer such that there is a vector $v \in V$ with $\xi^h v = v^0$. Then a longest Jordan chain in $V$ ending at $v^0$ has length $h + 1$, since $\xi^{h+1}v = \xi v^0 = 0$. We call $h$ the *special height* of $\underline{\Delta}$ and denote it by $sht \underline{\Delta}$.

The special height of $\underline{\Delta}$ does not depend on the choice of triple representing the special type $\underline{\Delta}$.

**Lemma 2.** Let $\underline{\Delta}$ be a special type with special height $h$. Suppose that

$$\underline{\Delta} = \underline{\Delta}' + \Delta,$$

where $\underline{\Delta}'$ is a special nilpotent type and $\Delta$ is a type. Then $sht \underline{\Delta}' = sht \underline{\Delta}$ and $\text{ht} \Delta < sht \underline{\Delta}'$. Here $\text{ht} \Delta$ is the height of the type $\Delta$, see [1].

**Proof.** Suppose that the triple $(V, \xi, v^0; \gamma)$ represents the special type $\underline{\Delta}$. Since $sht \underline{\Delta} = h$, there is a vector $v \in V$ such that $\xi^h v = v^0$. Let $(V_1, \xi|V_1, v^0; \gamma|V_1)$ be a triple representing the special type $\underline{\Delta}'$ and $(\xi|V_2, V_2; \gamma|V_2)$ be a pair representing the type $\Delta$. Then $V = V_1 \oplus V_2$, $V_i$ are $\xi$-invariant, and $v^0 \in V_1$. Write $v = v_1 + v_2 \in V_1 \oplus V_2$. Then $v^0 = \xi^h v = \xi^h v_1 + \xi^h v_2 \in V_1 \oplus V_2$. But $v^0 \in V_1$. So $\xi^h v_2 = 0$, which implies that the height $\text{ht} \Delta$ of $\Delta$ is less than $h$. Clearly $sht \underline{\Delta}' \leq sht \underline{\Delta} = h$. But $v^0 = \xi^h v_1$. This implies $sht \underline{\Delta}' \geq h$.

Using lemma 2 repeatedly we may assume that $\underline{\Delta}'$ in (1) is an *indecomposable* nilpotent special type.

Suppose that $\underline{\Delta}$ is a special type, represented by $(V, \xi, v^0; \gamma)$, which is *not* equal to the nilpotent special type $\underline{\Delta}_0$ represented by $(\text{span}\{v^0\}, 0, v^0; 0)$. Let
\[ V = V / \text{span}\{v^0\} \] with projection map \( \pi : V \to V \) \( v \mapsto v \). Since \( \Delta \neq \tau \), it follows that \( V \neq \{0\} \). Let \( \gamma \) be the bilinear form on \( V \) defined by \( \gamma(v, w) = \gamma(v, w) \), where \( v \in V \) and \( w \in \overline{V} \). The form \( \gamma \) is well defined and is symmetric. In fact it is nondegenerate. For if \( \gamma(v, w) = 0 \) for every \( w \in V \), then \( 0 = \gamma(v, w) \) for every \( w \in V \). Therefore, \( v \in \ker \gamma = \text{span}\{v^0\} \), that is, \( v = \mu v^0 \) for some \( \mu \in \mathbb{R} \). So \( \pi = \overline{0} \). Since \( \xi v^0 = 0 \), the map \( \xi \in o(V, \gamma) \) induces a linear map \( \xi : \overline{V} \to \overline{V} : v \mapsto \xi v \). Because
\[ \gamma(\xi v, w) = \gamma(\xi v, w) = -\gamma(v, \xi w) = -\gamma(\overline{v}, \overline{\xi w}), \]
for every \( v, w \in V \), it follows that \( \xi \in o(\overline{V}, \gamma) \). Therefore \( (\xi, \overline{V}; \gamma) \) represents a type \( \overline{\Delta} \), called the type induced by the special type \( \Delta \).

**Lemma 3.** Suppose that \( \Delta \) is a special type, represented by \( (V, \xi, v^0; \gamma) \), with special height \( h + 1 \). Then the induced type \( \overline{\Delta} \) has height \( h \).

**Proof.** Because \( \text{sht} \Delta \) is \( h + 1 \), there is a vector \( v \in V \) such that \( \xi^{h+1} v = v^0 \). Since \( \xi^h v \notin \text{span}\{v^0\} \), it follows that \( \overline{\xi^h v} \neq \overline{0} \). But \( \overline{\xi^{h+1} v} = \overline{\xi^h v} = \overline{v^0} = \overline{0} \). Thus the induced type \( \overline{\Delta} \), represented by \( (\xi, \overline{V}; \gamma) \), has height \( h \). \( \square \)

## 3 Classification of indecomposable nilpotent special types

In this section we classify indecomposable nilpotent special types. Our argument shows that an indecomposable nilpotent special type is uniquely determined by its induced type.

### 3.1 A rough classification

In this subsection we give a rough classification of indecomposable nilpotent special types.

1. We begin by treating the case when the indecomposable nilpotent special type \( \Delta \), represented by \( (V, N, v^0; \gamma) \), has special height 0. By results of [4], the underlying gl-type \( \Delta \), represented by the pair \( (V, N) \), is the unique sum of indecomposable gl-types \( \Delta_0 + \Delta_1 + \cdots + \Delta_r \) with heights \( h_0 \leq \cdots \leq h_r \), respectively. For \( 0 \leq i \leq r \) let \( (V_i, N|V_i) \) be a pair representing the gl-type \( \Delta_i \). Since the special height of \( \Delta \) is 0, the longest Jordan chain in \( V \) which ends at \( v^0 \) is \( \{v^0\} \). Clearly the pair \( (V_0 = \text{span}\{v^0\}, 0) \) represents an indecomposable gl-type of height 0. Therefore \( h_0 = 0 \) and \( (\text{span}\{v^0\}, 0) \in \Delta_0 \). Write \( V = V_0 \oplus U \), where \( U = V_1 \oplus \cdots \oplus V_r \). Then \( V_0 \) and \( U \) are
\(N\)-invariant with \(v^0 \in V_0\). Since \(\ker \gamma = \text{span}\{v^0\}\), it follows that \(V_0\) and \(U\) are \(\gamma\)-orthogonal and \(\gamma|U\) is nondegenerate. Therefore \(\Delta\) is the sum of the nilpotent special type \(\tau\), represented by \((\text{span}\{v^0\}, 0, v^0; 0)\), and a type represented by \((N|U, U; \gamma|U)\). But by hypothesis \(\Delta\) is indecomposable. Therefore \(\Delta = \tau\). This completes alternative 1.

2. Suppose that \((V, N, v^0; \gamma)\) represents an indecomposable special nilpotent type \(\Delta\) of special height \(h + 1\). Let \(\{w, Nw, \ldots, N^{h+1}w = v^0\}\) be a longest Jordan chain ending at \(v^0\) which lies in \(V\). There are two possibilities.

a. \(\gamma(w, N^{h}w) \neq 0\). This hypothesis implies that \(h\) is even. If not, then \(h\) is odd and

\[
\gamma(w, N^{h}w) = (-1)^{h} \gamma(N^{h}w, w) = -\gamma(w, N^{h}w).
\]

Therefore \(\gamma(w, N^{h}w) = 0\), which contradicts our hypothesis. Since \(N^{h}w \notin \text{span}\{v^0\}\), it follows that \(\overline{N^{h}}w \neq 0\). Also \(\overline{N^{h+1}}w = 0\), because \(N^{h+1}w = v^0\). So \(\{\overline{w}, \overline{Nw}, \ldots, \overline{N^{h}w}\}\) is a Jordan chain in \(\overline{V}\) of length \(h + 1\). Let \(\overline{U}\) be the space spanned by the elements of this chain. Let \(\overline{G}\) be the Gram matrix of \(\gamma\) on \(\overline{U}\). If \(i + j \geq h + 1\), then

\[
\overline{\gamma}(\overline{N^{i}w}, \overline{N^{j}w}) = (-1)^{i} \overline{\gamma}(\overline{w}, \overline{N^{i+j}w}) = 0,
\]

since \(\overline{N^{h+1}}w = 0\). Therefore all the entries of \(\overline{G}\) below the antidiagonal are equal to 0. All the entries of \(\overline{G}\) on the antidiagonal are nonzero, because

\[
\overline{\gamma}(\overline{N^{i}w}, \overline{N^{h-i}w}) = (-1)^{i} \overline{\gamma}(\overline{w}, \overline{N^{h}w}) \neq 0, \quad \text{for } 0 \leq i \leq h,
\]

since \(\overline{\gamma}(\overline{w}, \overline{N^{h}w}) = \gamma(w, N^{h}w) \neq 0\) by hypothesis. Therefore \(\det \overline{G} \neq 0\), that is, \(\overline{\gamma}\) on \(\overline{U}\) is nondegenerate. Since \(\overline{U}\) is spanned by a Jordan chain of length \(h + 1\), it follows that \(\overline{\Delta}\) is a uniform type of height \(h\), that is, \(\ker \overline{N^{h}} = \overline{N^{h}}\overline{U} = \overline{NU}\). Therefore \(\overline{\gamma}\) induces a nondegenerate bilinear form \(\widehat{\gamma}\) on \(\widehat{U} = \overline{U}/\overline{NU}\) defined by \(\widehat{\gamma}(\widehat{w}, \widehat{\gamma}) = \overline{\gamma}(\overline{w}, \overline{N^{h}w})\), where \(\widehat{w} \in \overline{\gamma}, \overline{\gamma} \in \overline{w}\). The bilinear form \(\widehat{\gamma}\) is symmetric, since \(h\) is even. Because \(\dim \widehat{U} = 1\), we can choose a vector \(\widehat{w} \in \widehat{U}\) so that \(\widehat{\gamma}(\widehat{w}, \widehat{w}) = \varepsilon\) with \(\varepsilon^{2} = 1\). In other words, with respect to the basis \(\{\widehat{w}\}\) of \(\widehat{U}\) the Gram matrix of \(\widehat{\gamma}\) is \((\varepsilon)\). From the classification of indecomposable types in \(\Pi\) we see that the induced type \(\overline{\Delta}\), represented by \((\overline{N}[\overline{U}, \overline{U}; \overline{\gamma}]\overline{U})\), is \(\Delta_{\varepsilon}^{h}(0)\). This completes case \(a\).

b. \(\gamma(w, N^{h}w) = 0\). Since \(N^{h}w \notin \text{span}\{v^0\}\), it follows that \(\overline{N^{h}}w \neq \overline{0}\). Moreover, \(\overline{N^{h+1}}w = \overline{N^{h+1}w} = v^0 = \overline{0}\). Therefore \(\{\overline{w}, \overline{Nw}, \ldots, \overline{N^{h}w}\}\) is a Jordan chain in \(\overline{V}\). Because \(\overline{\gamma}\) is nondegenerate on \(\overline{V}\), there is a
nonzero vector $z \in V$ such that $\gamma(z, N^h w) \neq 0$. Therefore $N^h z \neq 0$; for otherwise $0 = \gamma(N^h z, w) = \gamma(z, N^h w)$, which is a contradiction. Let $U = \text{span}\{z, \ldots, N^h z, w, \ldots, N^h w\} \subseteq V$. Since $\text{ht} \Delta = h$, it follows that $N^{h+1} z = 0$. Therefore $U$ is $N$-invariant. Consider the basis $\{z, w, Nz, Nw, \ldots, N^h z, N^h w\}$ of $U$. Since $\gamma(N^i w, N^j w) = 0$, if $i + j \geq h + 1$, $\gamma(N^i z, N^j z) = 0$, if $i + j \geq h + 1$, $\gamma(N^i w, N^j z) \neq 0$, if $i + j = h$, the Gram matrix $G$ of $\gamma|U$ has zero entries below the antidiagonal and nonzero entries on the antidiagonal. Therefore $\gamma$ is nondegenerate on $U$. Consequently the pair $(N|U, U; \gamma|U)$ represents a type $\Delta$, which is nilpotent of height $h$. Since $U$ is spanned by two Jordan chains of length $h + 1$, it follows that $\Delta$ is a uniform type of height $h$, that is, $\ker N^h = N U = U$. Therefore $\gamma$ induces a nondegenerate bilinear form $\hat{\gamma}$ on $\hat{U} = U/NU$ defined by $\hat{\gamma}(\hat{z}, \hat{w}) = \gamma(z, N^h w)$, where $\hat{z} \in \hat{V} = \overline{V}$, $\hat{w} \in \overline{V}$. The bilinear form $\hat{\gamma}$ is symmetric if $h$ is even and skew symmetric if $h$ is odd. Since $\dim \hat{U} = 2$, we can choose vectors $\hat{z}, \hat{w} \in \hat{U}$ so that the Gram matrix of $\hat{\gamma}$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ if $h$ is even and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ if $h$ is odd. Thus $\hat{\gamma}(\hat{z}, \hat{w}) = 1$ in both cases. From the classification of indecomposable types in $\prod \Delta$ is $\Delta^+_h(0) + \Delta^-_h(0)$ if $h$ is even and $\Delta_h(0,0)$ if $h$ is odd. This completes case b.

The rough classification of indecomposable nilpotent special types is now established. □

### 3.2 Fine classification

In this subsection we give a fine classification of indecomposable nilpotent special types.

**Theorem 5.** An indecomposable nilpotent special type $\Delta$, represented by the triple $(V, N, v_0; \gamma)$, is exactly one of the following.

1. $\tau$. Here $\text{sh} \tau = 0$. Then $V = \text{span}\{v^0\}$. With respect to this basis the matrix of $\gamma$ is 0 and the matrix of $N$ is zero.

2. $\Delta_{h+1}^\varepsilon(0), \alpha > 0$. Here $h$ is even, $\varepsilon = \pm$, and $\text{sh} \Delta_{h+1}^\varepsilon(0) = h + 1$. We have $v^0 = \alpha N^{h+1} w$ with $\alpha > 0$. We call $\alpha$ a *modulus*. If $h = 0$, then
\( \mathfrak{f} = \{ \alpha Nw; w \} \) is a basis of \( V \). With respect to \( \mathfrak{f} \) the Gram matrix of \( \gamma \) is \( \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon \end{pmatrix} \) and the matrix of \( N \) is \( \begin{pmatrix} 0 & \alpha^{-1} \\ 0 & 0 \end{pmatrix} \). When \( h \) is nonzero

\[
\mathfrak{f} = \{ \alpha N^{h+1}w; N^{h/2-1}w, N^{h/2-2}w, \ldots, w; N^{h/2}w; \delta N^{h/2+1}w, -\delta N^{h/2+2}w, \ldots; \delta N^h w \}
\]

is a basis of \( V \) with \( \delta = \varepsilon (-1)^{h/2} \), where \( \gamma(w, N^h w) = \varepsilon = \pm 1 \). With respect to the basis \( \mathfrak{f} \) the Gram matrix of \( \gamma \) is

\[
\begin{pmatrix} 0 \\ J_{h, \delta} \end{pmatrix}, \quad \text{where} \quad J_{h, \delta} = \begin{pmatrix} 0 & 0 & I_{h/2} \\ 0 & 0 & 0 \end{pmatrix}
\]

(2)

The index of \( \gamma \) is \( h/2 \) if \( \delta = 1 \) and \( h/2 + 1 \) if \( \delta = -1 \). The matrix of \( N \) with respect to the basis \( \mathfrak{f} \) is

\[
\begin{pmatrix} 0 \\ J_{h/2} \\ e_1^T \\ -\delta e_1 \\ -J_{h/2}^T \end{pmatrix}
\]

(3)

Here \( J_{h/2} \) is the \( h/2 \times h/2 \) upper Jordan block.

3. \( \Delta_{h+1}(0,0) \) if \( h \) is odd, or \( \Delta_{h+1}^+(0) + \Delta_{h+1}^-(0) \) if \( h \) is even. Here \( \text{sht} \Delta_{h+1}(0,0) = \text{sht}(\Delta_{h+1}^+(0) + \Delta_{h+1}^-(0)) = h + 1 \). We have \( v^0 = N^{h+1}v \).

\[ \mathfrak{f} = \{ N^{h+1}v; v, Nv, \ldots N^{h}v; N^{h}w, -N^{h-1}w, \ldots; (-1)^h w \}, \]

is a basis of \( V \). The Gram matrix of \( \gamma \) with respect to \( \mathfrak{f} \) is

\[
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I_{h+1} \\ 0 & I_{h+1} & 0 \end{pmatrix}
\]

(4)

and the matrix of \( N \) is

\[
\begin{pmatrix} 0 & e_{h+1}^T \\ J_{h+1}^T \\ -J_{h+1} \end{pmatrix}
\]

(5)

where \( J_{h+1} \) is an \((h+1) \times (h+1)\) upper Jordan block.
Proof. Suppose that \((V, N, v^0; \gamma)\) represents an indecomposable nilpotent special type \(\Delta\), whose induced type \(\overline{\Delta}\) is given.

1. The case when \(\Delta = \tau\) has been dealt with in the first case of the rough classification in §3.1. This establishes alternative 1.

2. By case a of the rough classification, the induced type \(\Delta\) of height \(h\), where \(h\) is even, and is represented by the pair \((N, \overline{V}; \overline{\gamma})\). We can choose \(w \in \overline{V}\) so that \(\overline{V}\) has a basis
\[
\{\overline{N}^{h/2-1}w, \overline{N}^{h/2-2}w, \ldots, w, \overline{N}^{h/2}w, -\delta \overline{N}^{h/2+1}w, \delta \overline{N}^{h/2+2}w, \ldots, \delta \overline{N}^{h}w\}. \tag{6}
\]
Here \(\overline{\gamma}(\overline{N}^iw, \overline{N}^{h-i}w) = \varepsilon(-1)^i\) for \(0 \leq i \leq h\) and the value of \(\overline{\gamma}\) is 0 on any other pair of vectors in the basis (3). Also \(\delta = \varepsilon(-1)^{h/2}\).

We now reconstruct the special type \(\Delta\) from its induced type \(\overline{\Delta}\). Since \(\text{sht } \overline{\Delta} = h + 1\), the vector space \(V\) has a basis \(\{w, Nw, \ldots, N^{h}w, N^{h+1}w\}\), where \(w \in \overline{w} \in \overline{V}\). Therefore \(V\) is spanned by a Jordan chain of length \(h + 2\). Because \(\overline{N}^{h+1}w = 0\), there is a real number \(\mu\) such that \(N^{h+1}w = \mu v^0\). Moreover, we can choose \(w \in \overline{w}\) so that \(\gamma(N^iw, N^{h-i}w) = (-1)^i\varepsilon\) for \(0 \leq i \leq h\) and the value of \(\gamma\) is 0 on any other pair of vectors in the basis \(\mathfrak{f}\). If \(\mu = 0\), then the triple \((V, N, v^0; \gamma)\) does not represent a nilpotent special type of special height \(h + 1\) because there is no Jordan chain in \(V\) of length \(h + 2\). Therefore \(\mu \neq 0\). Let \(\alpha = \mu^{-1}\). With respect to the basis
\[
\{\alpha N^{h+1}w; N^{(h-1)/2}w, N^{(h-2)/2}w, \ldots, w; N^{h/2}w; \delta N^{h/2+1}w, -\delta N^{h/2+2}w, \ldots, \delta N^{h}w\}, \tag{7}
\]
the Gram matrix of \(\gamma\) is (2). Changing the sign of the vector \(w\) does not change the Gram matrix of \(\gamma\). Therefore we may select the modulus \(\alpha\) so that it is positive. It is straightforward to see that the matrix of \(N\) with respect to the basis (7) is (3).

3. By case b of the rough classification, the induced type \(\overline{\Delta}\) has height \(h\) and is represented by the pair \((\overline{N}, \overline{V}; \overline{\gamma})\). We now reconstruct the special type \(\Delta\) from its induced type \(\overline{\Delta}\). From case 3 of the rough classification in §3.1 we can choose \(\overline{z} \in \overline{z} \in \overline{V}\) and \(\overline{w} \in \overline{w} \in \overline{V}\) so that
\[
\{\overline{z}, \overline{N}^i\overline{z}, \ldots, \overline{N}^{h}\overline{z}, \overline{w}, \overline{N}^i\overline{w}, \ldots, \overline{N}^{h}\overline{w}\} \tag{8}
\]
is a basis of \(\overline{V}\) such that \(\overline{\gamma}(\overline{N}^i\overline{z}, \overline{N}^{h-i}\overline{w}) = (-1)^i\) for \(0 \leq i \leq h\) and the value of \(\overline{\gamma}\) on any other pair of vectors in the basis (8) is zero.
Let \( z \in \mathbb{V} \) and \( w \in \mathbb{V} \). Since \( N^{h+1}z = N^{h+1}w = 0 \), there are real numbers \( \lambda \) and \( \mu \) such that \( N^{h+1}z = \lambda v^0 \) and \( N^{h+1}w = \mu v^0 \). The vectors
\[
\{z, Nz, \ldots, N^{h+1}z = \lambda v^0; w, Nw, \ldots, N^{h+1}w = \mu v^0\}
\] span \( V \). Suppose that \( \lambda = \mu = 0 \) in (9). Then there is no Jordan chain in \( V \) of length \( h + 2 \). This contradicts the fact that the special height of \( \Delta \) is \( h + 1 \). Therefore not both \( \lambda \) and \( \mu \) are zero. Interchanging \( z \) and \( w \) in (9), if necessary, we may suppose that \( \mu \neq 0 \). Let \( \eta = \mu z - \lambda w \) and \( \zeta = \mu^{-1}w \). Then \( N^{h+1}\eta = 0 \) and \( N^{h+1}\zeta = v^0 \). Note that
\[
\{\eta, N\eta, \ldots, N^{h} \eta; \zeta, N\zeta, \ldots, N^{h+1} \zeta = v^0\}
\] is a basis of \( V \). Therefore \( V \) is spanned by two Jordan chains one of length \( h + 2 \), which ends at \( v^0 \), and the other of length \( h + 1 \).

We now calculate the Gram matrix of \( \gamma \) with respect to the basis (10). When \( i + j = h \) we have
\[
\gamma(N^i\eta, N^j\zeta) = \gamma(\mu N^i z - \lambda N^i w, \mu^{-1} N^j w) = (-1)^i \gamma(z, N^h w) = (-1)^i;
\]
while when \( i + j \neq h \) we obtain
\[
\begin{align*}
\gamma(N^i\eta, N^j\zeta) &= (-1)^i \gamma(z, N^{i+j} w) = 0; \\
\gamma(N^i\eta, N^j\eta) &= (-1)^i \gamma(\eta, N^{i+j} \eta) \\
&= (-1)^i \left[ \mu^2 \gamma(z, N^{i+j} z) - 2\lambda \mu \gamma(z, N^{i+j} w) + \lambda^2 \gamma(w, N^{i+j} w) \right] \\
&= 0; \\
\gamma(N^i\zeta, N^j\zeta) &= (-1)^i \gamma(w, N^{i+j} w) = 0.
\end{align*}
\]
Thus the Gram matrix of \( \gamma \) with respect to the basis
\[
\{N^{h+1} \eta = v^0; \eta, N\eta, \ldots, N^h \eta; N^h \zeta, -N^{h-1} \zeta, \ldots, (-1)^h \zeta\}
\] is given by (10). The matrix of \( N \) with respect to the above basis in given by (11).

Suppose that the nilpotent special type \( \Delta \) is decomposable, that is \( \Delta = \Delta' + \Delta \), where \( \Delta' \) is a nilpotent special type and \( \Delta \) is a type. Using lemma 2 we may assume that \( \Delta' \) is indecomposable and has the same special height \( h + 1 \) as \( \Delta \). The type \( \Delta' \) induced by the special type \( \Delta \) is in case a equal to \( \Delta_{h+1}(0) \) and in case b is either \( \Delta_{h+1}(0, 0) \) if \( h \) is odd or \( \Delta_{h+1}^+(0) + \Delta_{h+1}^-(0) \) if \( h \) is even.
by construction. Because the special height of the special types $\Delta' \alpha$ and $\Delta$ are equal to $h + 1$, we can repeat the construction given in case $a$ or $b$ on the special type $\Delta'$ induced from the special type $\Delta$. Hence $\Delta'$ in case $a$ is $\Delta_{h+1}(0)$ and in case $b$ is $\Delta_{h+1}(0,0)$ if $h$ is even or $\Delta_{h+1}(0) + \Delta_{h+1}(0)$ if $h$ is odd. Since the nilpotent special types $\Delta$ and $\Delta'$ are each uniquely determined by their respective induced types, which we have shown are equal, it follows that $\Delta = \Delta'$. Hence the nilpotent special type $\Delta$ is indecomposable.

This completes the proof of the second and third items of theorem 5 and thereby the classification of indecomposable nilpotent special types. □

**Proposition 6.** Let $\Delta$ be a special type of special height $h + 1$. Suppose that

$$\Delta = \Delta' + \Delta, \quad (11)$$

where $\Delta'$ is an indecomposable nilpotent special type of special height $h + 1$ and $\Delta$ is a type of height less than or equal to $h$. Then the decomposition (11) is unique up to reordering of the summands in the type $\Delta$.

**Proof.** The decomposition (11) yields the decomposition $\Delta = \Delta' + \Delta$ for the induced types. By results of [1] the induced type $\Delta$ is the unique sum of indecomposable types $\Delta_i$ for $1 \leq i \leq s$ of height $h_i$ where $h_1 \leq h_2 \leq \cdots \leq h_s$. Therefore

$$\Delta = \Delta_1 + \cdots + \Delta_s + \Delta'. \quad \text{(Note that $\Delta = \Delta_1$. This follows because any type has a representative pair $(N| \tilde{V}, \tilde{V})$, where $v^0 \not\in \tilde{V}$. Therefore the projection map $\pi$ in the definition of induced type, when restricted to $\tilde{V}$, is a bijective linear map, which defines an equivalence of pairs. So $\Delta = \Delta_1 + \cdots + \Delta_s$ is the unique decomposition of $\Delta$ into indecomposable types. Using the classification of indecomposable nilpotent special types, we see the indecomposable nilpotent special type $\Delta'$ in the decomposition (11) is uniquely determined by its induced type $\Delta'$. Thus the decomposition (11) is unique up to reordering of the summands in the type $\Delta$. □)

Combining the results of lemma 2 and 6 we obtain

**Theorem 7.** Every special type may be written as a sum of an indecomposable nilpotent special type and a sum of indecomposable types. This decomposition is unique up to reordering of the indecomposable type summands.

Recall that indecomposable types have been classified in [1].
Theorem 7 solves the problem of finding a unique representative for each conjugacy class of elements of $o(V, \gamma)_{v,0}$ under the adjoint action of $O(V, \gamma)_{v,0}$.

References

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