String-like Lagrangians from a generalized geometry

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Abstract

This note will use Hitchin’s generalized geometry and a model of axionic gravity developed by Warren Siegel in the mid-nineties to show that the construction of Lagrangians based on the inner product arising from the pairing of a vector and its dual can lead naturally to the low-energy Lagrangian of the bosonic string.

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1. Introduction

String theory is one of the most popular theories of quantum gravity. It is widely believed to predict gravity, while remaining quantum mechanically finite. Nevertheless, unlike relativity, or gauge theory, it cannot be derived from a simple postulate. This note, inspired by Hitchin’s generalized geometry and a model of axionic gravity developed by Warren Siegel in the mid-nineties, will argue that the low-energy Lagrangian of the bosonic string follows naturally from the use of an inner product based on the pairing between a vector and its dual, rather than the, arguably less fundamental, standard Riemannian metric.

In the generalized geometries developed by Nigel Hitchin and his students [1–3] geometric objects that are normally defined solely on the tangent bundle $T$, or cotangent bundle $T^*$, are redefined on the vector bundle $T \oplus T^*$. These geometries are endowed with a natural metric arising from the inner product between elements of a vector space and its dual. This inner product has, on a $d$-dimensional manifold, an $O(d, d)$ symmetry similar to that of the $T$-dualities found in string theory. The choice of subspaces of $T \oplus T^*$ that are positive, or negative, definite with respect to the natural metric breaks the symmetry to $O(d) \times O(d)$ and in turn leads to the generation of a positive definite metric, corresponding to the standard metric on a Riemannian manifold, and a $b$-field [2, 3].

A model of axionic gravity having similar properties was developed by Warren Siegel in the mid-nineties [4]. It was formulated by adding a second vielbein to an Einstein–Cartan theory of gravity, and has the Lagrangian of the closed, oriented bosonic string at low energies.
The extra vielbein combines with the standard one to form an object transforming as a vector of $O(d, d)$. This note will reappraise Siegel’s model in the light of the recent development of generalized geometries. It will argue that the vielbeins can be understood as a set of $d$ sections of $T \oplus T^*$, coupling in the same way and having the same transformation properties and relationship to the metric and $b$-field.

In addition to an $O(d, d)$ duality symmetry, the Lagrangian describing Siegel’s model has a $GL(d)$ gauge symmetry that leaves the physical metric and $b$-field unchanged. The gauge potential for this symmetry is a combination of the vielbeins rather than an independent field. The Lagrangian also contains a scalar field, corresponding to the dilaton, required to make the measure duality invariant, and terms constructed from the vielbeins corresponding to the Ricci scalar and $H^2$ terms from the low-energy action for the bosonic string (the $H^2$ term accounts for the behaviour of the $b$-field via the relation $H = db$). The Ricci scalar can be regarded as a curvature term for the $GL(d)$ gauge symmetry and the $H^2$ term as the analogue of the $F_{\mu \nu} F^{\mu \nu}$ term arising in conventional theories of vector fields, but with the ‘vectors’ living in $T \oplus T^*$ rather than $T$ alone. These are the terms one might expect, given the model’s field content, implying that the low-energy Lagrangian of the bosonic string follows naturally from the use of an inner product based on the pairing of elements of the tangent and cotangent spaces in place of an inner product based on a Riemannian metric. This new inner product is arguably more fundamental than the original one; it exists on any differential manifold and does not require the existence of a Riemannian metric.

2. Axionic gravity and generalized geometry

In 1993 Warren Siegel proposed a model of axionic gravity inspired by string theory and based, for a $d$-dimensional manifold, on objects transforming as vectors of $O(d, d)$ [4]. He began with the observation that the action for bosonic string theory can be written as

$$ S = \int \partial_+ X^m \partial_- X^a e_{mn}, $$

(2.1)

where $\partial_\pm$ are lightlike derivatives on the string’s worldsheet, and

$$ e_{mn} = g_{mn} + b_{mn}, $$

(2.2)

g and $b$ being the usual metric and 2-form respectively. He goes on to note that

$$ e_{mn} = e_{ma} e^a_m, $$

(2.3)

where $e^a_m$ and $e_{ma}$ can be thought of as right- and left-handed vielbeins. This expression is invariant under $GL(d)$ transformations of the form [4]

$$ e_{ma} \rightarrow \Lambda_b^a e_{mb}, \quad e^a_m \rightarrow (\Lambda^{-1})^a_b e^b_m. $$

(2.4)

When constructing his theory, Siegel found it more useful to use $e_{am}$ and the inverse of $e^a_m$, $e^m_a$ as fundamental fields. They can be combined to form an object

$$ E_a = \left( \begin{array} {c} e^a_m \\ e^m_a \end{array} \right) $$

(2.5)

that transforms as a vector of $O(d, d)$ (on the $m$ index) and the $GL(d)$ transformations given above (on the $a$ index).

2 The form of $E_a$ given here differs slightly from that given in [4]; the order of the two vielbeins has been reversed to make comparisons with generalized geometry easier.
It is also useful to define an indefinite metric that reflects the $O(d,d)$ structure:

$$ L = \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix}. $$

(2.6)

A similar metric is found in the generalized geometries recently developed by Nigel Hitchin and his students (a good introduction to generalized geometry, which I have followed here, can be found in [2]). Consider a vector space $V$ of dimension $d$ and its dual $V^*$. It is possible to define a new vector space $V \oplus V^*$, which is naturally endowed with an inner product

$$ \langle X + \xi, Y + \eta \rangle = \frac{1}{2} \left( \xi(Y) + \eta(X) \right), $$

(2.7)

where $X, Y \in V$ and $\xi, \eta \in V^*$. It defines a metric $L$ on $V \oplus V^*$ and we can write, in a basis where indices ranging from 1 to $d$ lie in $V$ and those ranging from $d+1$ to $2d$ lie in $V^*$,

$$ L = \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix}. $$

(2.8)

A generalized geometry takes $V$ to be the tangent space $T$ of a manifold, and $V^*$ its dual, the cotangent space $T^*$. If the identifications

$$ X^m_a = e^m_a, \quad \xi_{ma} = e_{ma} $$

(2.9)

are made, with $X_a$ being a set of $d$ vectors and $\xi$ being a set of $d$ 1-forms, the metric $L$ is the same as that in equation (2.6), the first hint that Siegal’s model can be understood in terms of a theory based on sections of $T \oplus T^*$.

It is also possible to define a positive definite metric

$$ G = LM, $$

(2.10)

where $g^{ab}$ being the inverse of

$$ g_{ab} = \frac{1}{2} E_a^T L E_b. $$

(2.11)

This equation, which differs from that in [4] because I have used a different definition of $E_a$, follows [4] from the fact that $M$ can be written in terms of vielbeins $V$ obeying

$$ M = V V^T, \quad L = V L V^T. $$

(2.12)

The eigenvalues of $I_d + L$ are 2 and 0, both of which have $d$ corresponding eigenvectors. Then

$$ M_{mn} + L_{mn} = 2\epsilon_{mn}^a \epsilon^T_{an}, $$

(2.13)

where $\epsilon_{ma}$ is some $2d \times d$ matrix obeying $G \epsilon L = L \epsilon$. Equation (2.10) can then be derived using the $GL(d)$ symmetry (2.4), the equation above corresponding to the case $g_{ab} = \frac{1}{2} \delta_{ab}$.

Like the indefinite metric $L$, $M$ can be found in generalized geometry. If $C_+$ is a positive definite (with respect to $L$) subspace of $T \oplus T^*$, and $C_-$ its negative definite orthogonal compliment, the positive definite metric [2, 3] is

$$ M = \langle \cdot , \cdot \rangle_{C_+} - \langle \cdot , \cdot \rangle_{C_-}. $$

(2.14)

Using the indefinite metric $L$ to identify $T \oplus T^*$ and its dual, it is possible to show that $G = LM$ can be regarded as a map from $T \oplus T^*$ to itself. The $+1$ ($-1$) eigenspace of this
map is $C_+ (C_-)$, which enables an expression for $G$ in terms of symmetric and antisymmetric tensors to be found, the origin of the metric and $b$-field in generalized geometry. It is [2]
\[
G = \begin{pmatrix}
g^{-1} b & g^{-1} \\
g^{-1} b & b g^{-1} \end{pmatrix}.
\] (2.15)
It is also possible [2] to show that within $C_\pm$, $\xi = (b \pm g) X$. (2.16)

This is consistent with the identifications made in (2.9), the eigenvectors of $\Pi_{2d} + L$ with eigenvalue 2 corresponding to the $+1$ eigenvectors of $G$. Equation (2.16) is then the same as (2.3).

In Siegel’s model, the derivative $\partial_a$ is coupled to $E$ by defining
\[
\partial_M = \begin{pmatrix} 0 \\ \partial_m \end{pmatrix}, \quad e_a = E_a^M \partial_M = e^m_a \partial_a.
\] (2.17)

Derivatives transform as 1-forms, so this is the coupling that would be expected were the relationship between Siegel’s model and generalized geometry outlined above to hold.

It is also useful to define
\[
f_{abc} = \frac{1}{2} E^T_c L e_a E_b, \quad f_{abcd} = \frac{1}{2} (e_a E^T_b) L (e_c E_d).
\] (2.18)

$f_{ab}$ has the same transformation properties as a $GL(d)$ connection, so it is possible [4] to associate the $GL(d)$ symmetry found in Siegel’s model with a covariant derivative
\[
\nabla_a = e_a - f_{ab}^b.
\] (2.19)

The minus sign ensures that the metric is covariantly constant. Note that a theory with a $GL(d)$ gauge symmetry can be constructed from $E_a$’s alone; there is no need to introduce a separate $GL(d)$ gauge potential. To do so, one would need to replace $e_a$ with the covariant derivative (2.19) in (2.18), leading to [4]
\[
F_{abc} = \frac{1}{2} E^T_c L \nabla_a E_b = 0
\] (2.20)
\[
F_{abcd} = \frac{1}{2} (\nabla_a E^T_b) L (\nabla_c E_d) = f_{abcd} - f_{ab}^e f_{ede}.
\] (2.21)

Siegel relates his theory to a version of Cartan’s theory of gravity itself based upon a $GL(d)$ gauge theory. In that theory the fundamental gravitational fields are a vielbein $e$ and an independent tangent space metric, which is required to be covariantly constant. It can be identified with the theory of axionic gravity given above by taking $e^m_a$ to be the vielbein while regarding $e_a$ as a kind of matter field. The tangent space metric $g_{ab}$ can then be related to the standard one $g_{mn}$ via [4]
\[
g_{ab} = e^m_a e^n_b g_{mn}.
\] (2.22)

The torsion and curvature tensors, $T_{abc}$ and $R_{abcd}$, can be found using
\[
[\nabla_a, \nabla_b] = T_{ab}^c \nabla_c + R_{abc}^d G_d^c,
\] (2.23)

which, using the covariant derivative given in (2.19), implies [4]
\[
R_{abcd} = -F_{(a|d|b|c)}, \quad T_{ab}^c = c_{ab}^c - f_{(ab)}^c,
\] (2.24)

where $c_{ab}^c$ is defined via the relationship $[e_a, e_b] = c_{ab}^c e_c$.

$R$ can be thought of as a $GL(d)$ field strength, something that can be seen by defining
\[
\omega_{mab} = -\frac{1}{2} E^T_b L \partial_M E_a,
\] (2.25)
so that \( \omega_{abc} = -f_{abc} = e^m a \omega_{mbc} \), and regarding \( \omega_{mab} \) as a 1-form valued in the Lie algebra of \( GL(d) \), a standard \( GL(d) \) gauge field. Then

\[
R_{abc}^d = e^m a e^n b R_{mnc}^d,
\]

(2.26)

where

\[
R_{mnc}^d = \delta \omega + \omega \wedge \omega
\]

(2.27)

is the expected gauge field strength. It can also be shown [4] that

\[
H_{abc} = \frac{1}{2} e^m a e^n b \partial_m b_{np} = \frac{1}{2} \epsilon_{[abc]} - f_{[abc]}.
\]

(2.28)

The low-energy field-theory Lagrangian of the closed, oriented bosonic string is [5–7]

\[
L = \phi^2 \left( \bar{R} - \frac{1}{12} H^2 \right) + 4 g^{mn} \partial_m \phi \partial_n \phi,
\]

(2.29)

where \( \bar{R} \) is the torsion free version of the curvature scalar and \( \phi \) is the dilaton. It is possible to show, after a series of algebraic manipulations detailed in [4], that the corresponding action can be written in terms of the fields given earlier:

\[
S = \int d^d x \sqrt{g} L = \int d^d x 4 \left\{ \left[ \nabla \phi + \frac{1}{2} (1 \cdot \nabla) \phi \right]^2 + \phi^2 \left( F_{[ab]}^b + F_{[ab]}^{ab} \right) \right\}.
\]

(2.30)

The field \( \Phi = g^{1/4} \phi \) absorbs the measure and renders it duality invariant. \( F_{[ab]}^{cd} \) is an analogue of the \( F_{\mu \nu} F^{\mu \nu} \) term for the vector field \( E_a \). \( F_{[a] [b]} \) can similarly be regarded as a \( GL(d) \) curvature term. A \( GL(d) \) analogue of \( F_{\mu \nu} F^{\mu \nu} \) would be fourth order in derivatives of \( E_a \), but this problem can be avoided because the fact that the metric \( g_{ab} \) transforms under \( GL(d) \) means that a \( GL(d) \) invariant scalar can be constructed from a single copy of the curvature.

3. Conclusions

So, within the context of a generalized geometry in which objects that would have been defined on a manifold’s tangent, or cotangent bundles, \( T \) or \( T^* \), are instead defined on their direct sum \( T \oplus T^* \), it is possible to construct a simple model, involving only scalar and vector fields, whose Lagrangian is the field theory Lagrangian of the closed, oriented bosonic string at low energies. This model is based on one constructed by Warren Siegel in the mid-nineties. In Siegel’s model, the metric and \( b \)-field found in string theory are constructed from two independent vielbeins which combine to form objects transforming as vectors of \( O(d, d) \). I have argued that these vielbeins can be thought of as a set of sections of \( T \oplus T^* \), which also transforms naturally as a vector under \( O(d, d) \) and has the same relation to the metric and \( b \)-field. The model has a separate \( GL(d) \) symmetry, which can be realized locally without having to introduce a separate \( GL(d) \) gauge field because there exists a combination of the vielbeins that transforms in the same way. The model also contains a scalar field, a kind of dilaton, which is introduced to compensate for the duality transformations of the measure. The Lagrangian contains a curvature term for \( GL(d) \), which corresponds to the Ricci scalar in other gravitational theories, and a term analogous to both the \( F_{\mu \nu} F^{\mu \nu} \) term found in standard theories of vector fields, but applied to sections of \( T \oplus T^* \) instead of \( T \) alone, and the \( H^2 \) term of the Lagrangian for the bosonic string. These terms are not overly complex, on the contrary, they are the ones that would be expected given the symmetries of the model. The coupling between vectors and 1-forms around which this geometry is formulated is arguably more fundamental than the standard one between vectors alone involving a metric, and so necessitating the introduction of a new field, even though it may be less familiar.
Acknowledgments

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