Abstract. Co-t-structures were introduced about ten years ago as a type of mirror image of t-structures. Like t-structures, they permit to divide an object in a triangulated category \( \mathcal{T} \) into a “left part” and a “right part”, but there are crucial differences. For instance, a bounded t-structure gives rise to an abelian subcategory of \( \mathcal{T} \), while a bounded co-t-structure gives rise to a so-called silting subcategory.

This brief survey will emphasise three philosophical points. First, bounded t-structures are akin to the canonical example of “soft” truncation of complexes in the derived category. Secondly, bounded co-t-structures are akin to the canonical example of “hard” truncation of complexes in the homotopy category.

Thirdly, a triangulated category \( \mathcal{T} \) may be skewed towards t-structures or co-t-structures, in the sense that one type of structure is more useful than the other for studying \( \mathcal{T} \). In particular, we think of derived categories as skewed towards t-structures, and of homotopy categories as skewed towards co-t-structures.

0. Introduction

The notion of co-t-structure in a triangulated category \( \mathcal{T} \) was introduced independently by Bondarko and Pauksztello, see Definition 2.1. It is a mirror image of the classic notion of t-structure due to Beilinson, Bernstein, and Deligne, see Definition 1.1.

Given an object \( t \in \mathcal{T} \), both types of structure give a way to divide \( t \) into a “left part” and a “right part”. This is exemplified by dividing a complex of modules into a left and a right part by “soft” or “hard” truncation, see Figures 1 and 2.

Each case gives a triangle \( u \to t \to v \). Crucially, for a t-structure, \( u \) is the left part of \( t \) and \( v \) the right part; for a co-t-structure, vice versa. This reversal leads to a number of differences, and the theories of t-structures and co-t-structures are far from being simple mirrors of each other. For instance, while a bounded t-structure induces an abelian subcategory of \( \mathcal{T} \), a bounded co-t-structure induces a so-called silting subcategory; see Definition 2.3.

In the decade since their inception, the theory of co-t-structures has grown considerably. This brief survey is far from encyclopedic, but has the important goal of communicating three philosophical points:

(i) Bounded t-structures are akin to soft truncation in the bounded derived category of an abelian category.

(ii) Bounded co-t-structures are akin to hard truncation in the bounded homotopy category of an additive category.

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(iii) A triangulated category $\mathcal{T}$ may be skewed in the direction of t-structures or co-t-structures, in the sense that one type of structure is more useful than the other for studying $\mathcal{T}$.

We explain these points in the next three subsections. Throughout, $\mathcal{T}$ is a triangulated category with Hom-spaces $\mathcal{T}(\cdot, \cdot)$ and suspension functor $\Sigma$. If $\mathcal{M}$ is an abelian category then $\mathcal{D}^b(\mathcal{M})$ is the derived category of bounded complexes over $\mathcal{M}$, and if $\mathcal{P}$ is an additive category then $\mathcal{K}^b(\mathcal{P})$ is the homotopy category of bounded complexes over $\mathcal{P}$.

(i) **Bounded t-structures are akin to soft truncation in the bounded derived category.** Let $R$ be a ring, $\mathcal{M} = \text{Mod} R$ the category of left modules over $R$. Each complex $t \in \mathcal{D}^b(\mathcal{M})$ has soft truncations $a$ and $b$ as shown in Figure 1 and there is a triangle $a \to t \to b$ in $\mathcal{D}^b(\mathcal{M})$. The full subcategories $\mathcal{A}$ and $\mathcal{B}$ consisting of complexes isomorphic to such truncations satisfy Definition 1.1 and hence form a bounded t-structure, sometimes known as the standard t-structure, see Example 1.2.

Up to isomorphism, the heart $\mathcal{H} = \mathcal{A} \cap \Sigma \mathcal{B}$ consists of complexes concentrated in degree 0. The heart is an abelian subcategory of $\mathcal{D}^b(\mathcal{M})$ which is equivalent to $\mathcal{M}$. Each $t \in \mathcal{D}^b(\mathcal{M})$ permits a “tower” as shown in Proposition 3.1 due to Beilinson, Bernstein, and Deligne. This expresses how to build $t$ from objects of the form $\Sigma^i h$ with $h \in \mathcal{H}$. The objects can be taken to be $\Sigma^i H^{-i}(t)$ and then the tower shows how $t$ is built from its cohomology modules, see Example 3.3.

In general, if $(\mathcal{A}, \mathcal{B})$ is a bounded t-structure in a triangulated category $\mathcal{T}$, then each $t \in \mathcal{T}$ still permits a triangle $a \to t \to b$ with $a \in \mathcal{A}$, $b \in \mathcal{B}$. The heart $\mathcal{H}$ is still an abelian subcategory of $\mathcal{T}$ by Theorem 1.3 due to Beilinson, Bernstein, and Deligne, and each $t \in \mathcal{T}$ still has the tower in Proposition 3.1.

Working in such a setup is akin to working with soft truncation in $\mathcal{D}^b(\mathcal{M})$.

(ii) **Bounded co-t-structures are akin to hard truncation in the bounded homotopy category.** Let $R$ be a ring, $\mathcal{P} = \text{Prj} R$ the category of projective left modules over $R$. Each complex $t \in \mathcal{K}^b(\mathcal{P})$ has hard truncations $x$ and $y$ as shown in Figure 2 and there is a triangle $x \to t \to y$ in $\mathcal{K}^b(\mathcal{P})$. The full subcategories $\mathcal{X}$ and $\mathcal{Y}$ consisting of
complexes isomorphic to such truncations satisfy Definition 2.1 and hence form a bounded co-t-structure, sometimes known as the standard co-t-structure, see Example 2.2.

Up to isomorphism, the co-heart \( \mathcal{C} = \mathcal{X} \cap \Sigma^{-1}\mathcal{Y} \) consists of complexes concentrated in degree 0. The co-heart is an additive subcategory of \( \mathcal{K}^{b}(\mathcal{P}) \) which is equivalent to \( \mathcal{P} \).

The co-heart is not in general abelian, but it does have the strong property of being a so-called silting subcategory of \( \mathcal{K}^{b}(\mathcal{P}) \), see Definition 2.3 and Theorem 2.4 due to Mendoza Hernández et al. Such subcategories are structurally important.

Each \( t = \cdots \rightarrow P^{-3} \rightarrow P^{-2} \rightarrow P^{-1} \rightarrow P^{0} \rightarrow P^{1} \rightarrow P^{2} \rightarrow \cdots \) in \( \mathcal{K}^{b}(\mathcal{P}) \) permits a “tower” as shown in Proposition 3.2 due to Bondarko. This expresses how to build \( t \) from objects of the form \( \Sigma^{i}c \) with \( c \in \mathcal{C} \). The objects can be taken to be \( \Sigma^{i}P^{-i} \) and then the tower shows how \( t \) is built from its constituent modules, see Example 3.4.

In general, if \( (\mathcal{X}, \mathcal{Y}) \) is a bounded co-t-structure in a triangulated category \( \mathcal{I} \), then each \( t \in \mathcal{I} \) still permits a triangle \( x \rightarrow t \rightarrow y \) with \( x \in \mathcal{X}, y \in \mathcal{Y} \). The co-heart \( \mathcal{C} \) is still a silting subcategory of \( \mathcal{I} \) by Theorem 2.4 and each \( t \in \mathcal{I} \) still has the tower in Proposition 3.2.

Working in such a setup is akin to working with hard truncation in \( \mathcal{K}^{b}(\mathcal{P}) \).

(iii) Triangulated categories skewed in the direction of t-structures or co-t-structures. When studying a given triangulated category, bounded co-t-structures may be more useful than bounded t-structures, simply because there are none of the latter. See Section 4 for a toy example.

Section 5 shows a subtler skewing phenomenon. Let \( \Lambda \) be a finite dimensional \( \mathbb{C} \)-algebra, \( \mathcal{M} = \text{mod } \Lambda \) the category of finite dimensional left modules over \( \Lambda \), and \( \mathcal{P} = \text{prj } \Lambda \) the category of finite dimensional projective left modules over \( \Lambda \). Theorem 5.2 due to König and Yang, shows a bijection between all bounded co-t-structures in \( \mathcal{K}^{b}(\mathcal{P}) \) and the bounded t-structures in \( \mathcal{D}^{b}(\mathcal{M}) \) whose hearts are length categories (these are, in a sense, the “algebraic” t-structures; for instance, they permit a nice mutation theory).

In itself, this does not imply that either category has more t-structures than co-t-structures or vice versa, but we think of it as indicating that when going from derived to homotopy categories, the role played by t-structures is taken over by co-t-structures.

Summing up, Sections 1 and 2 show the definitions of t-structures and co-t-structures, emphasising the similarities with soft and hard truncation of complexes of modules. Section
Section 6 shows the “towers” whereby general objects can be built from objects in the (co-)heart. Sections 4 and 5 illustrate how a triangulated category may be skewed towards t-structures or co-t-structures.

Section 6 shows the silting mutation of Aihara and Iyama which mutates between bounded co-t-structures. It permits the definition of the so-called silting quiver which is a combinatorial picture of how the bounded co-t-structures fit inside a fixed triangulated category.

1. t-structures

The following definition was made in [4, def. 1.3.1].

**Definition 1.1 (Beilinson, Bernstein, and Deligne).** A t-structure in the triangulated category \( \mathcal{T} \) is a pair \((\mathcal{A}, \mathcal{B})\) of full subcategories, closed under isomorphisms, direct sums, and direct summands, which satisfy the following conditions.

(i) \( \Sigma \mathcal{A} \subseteq \mathcal{A} \) and \( \Sigma^{-1} \mathcal{B} \subseteq \mathcal{B} \).

(ii) \( \mathcal{T}(\mathcal{A}, \mathcal{B}) = 0 \).

(iii) For each object \( t \in \mathcal{T} \) there is a triangle \( a \to t \to b \) with \( a \in \mathcal{A}, b \in \mathcal{B} \).

The heart is \( \mathcal{H} = \mathcal{A} \cap \Sigma \mathcal{B} \).

The t-structure is called bounded if

\[
\bigcup_{i \in \mathbb{Z}} \Sigma^i \mathcal{A} = \bigcup_{i \in \mathbb{Z}} \Sigma^i \mathcal{B} = \mathcal{T}.
\]

The objects \( a \) and \( b \) in Definition 1.1(iii) depend functorially on \( t \). The resulting functor \( t \mapsto a \) is a right-adjoint to the inclusion \( \mathcal{A} \hookrightarrow \mathcal{T} \). Similarly, \( t \mapsto b \) is a left adjoint to the inclusion \( \mathcal{B} \hookrightarrow \mathcal{T} \). See [4, prop. 1.3.3].

The following is the canonical example of a t-structure.

**Example 1.2 (The standard t-structure).** Let \( R \) be a ring, \( \mathcal{M} = \text{Mod } R \) the category of left modules over \( R \). Let \( \mathcal{A} \) and \( \mathcal{B} \) be the isomorphism closures in the bounded derived category \( D^b(\mathcal{M}) \) of the subsets

\[
\{ \cdots \to M^{-2} \to M^{-1} \to M^0 \to 0 \to 0 \to 0 \to \cdots | M^i \in \mathcal{M}, M^i = 0 \text{ for } i \ll 0 \},
\]

\[
\{ \cdots \to 0 \to 0 \to 0 \to M^1 \to M^2 \to M^3 \to \cdots | M^i \in \mathcal{M}, M^i = 0 \text{ for } i \gg 0 \}.
\]

We will show that if \( \mathcal{A} \) and \( \mathcal{B} \) are viewed as full subcategories, then \((\mathcal{A}, \mathcal{B})\) is a bounded t-structure with heart \( \mathcal{H} \) equivalent to \( \mathcal{M} \).

It is easy to show

\[
\mathcal{A} = \{ M \in D^b(\mathcal{M}) | H^i(M) = 0 \text{ for } i \geq 1 \},
\]

\[
\mathcal{B} = \{ M \in D^b(\mathcal{M}) | H^i(M) = 0 \text{ for } i \leq 0 \}.
\]

(1.1)

This description clearly implies that \( \mathcal{A} \) and \( \mathcal{B} \) are closed under isomorphisms, direct sums, and direct summands.
We next check the conditions in Definition 1.1. Condition (i) is immediate. Condition (ii) requires that if objects
\[ a = \cdots \rightarrow M^{-2} \rightarrow M^{-1} \rightarrow M^0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots, \]
\[ b = \cdots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow M^1 \rightarrow M^2 \rightarrow M^3 \rightarrow \cdots \]
in \( \mathcal{A} \) and \( \mathcal{B} \) are given, then \( \text{Hom}_{\mathcal{D}}(b)(a) = 0 \). This can be shown by noting that \( a \) has a projective resolution
\[ p = \cdots \rightarrow P^{-2} \rightarrow P^{-1} \rightarrow P^0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots, \]
and that \( \text{Hom}_{\mathcal{D}}(b)(a) = \text{Hom}_{K}(p, b) = 0 \). Here \( K(M) \) is the homotopy category of complexes over \( M \), and the second \( = \) holds because in each degree, either the complex \( p \) or the complex \( b \) is zero. The triangle in Definition 1.1(iii) can be obtained by soft truncation of the object \( t = \cdots \rightarrow M^{-1} \rightarrow M^0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots \) in \( \mathcal{D} \), see Figure 1 in the introduction.

It is immediate that the t-structure \( (\mathcal{A}, \mathcal{B}) \) is bounded.

Finally, it follows from Equation (1.1) that the heart
\[ \mathcal{H} = \mathcal{A} \cap \Sigma \mathcal{B} \]
is an abelian subcategory of \( \mathcal{T} \). The following pivotal result is one of the motivations for the definition of t-structures. It was proved in [4, thm. 1.3.6].

**Theorem 1.3** (Beilinson, Bernstein, and Deligne). Let \( (\mathcal{A}, \mathcal{B}) \) be a t-structure in \( \mathcal{T} \). Then the heart \( \mathcal{H} = \mathcal{A} \cap \Sigma \mathcal{B} \) is an abelian subcategory of \( \mathcal{T} \).

**Example 1.4.** Let \( \Lambda = \mathbb{C}A_2 \) be the path algebra of the quiver
\[ A_2 = 1 \rightarrow 2 \]
and let \( \mathcal{M} = \text{mod} \Lambda \) be the category of finite dimensional left modules over \( \Lambda \).

There is a bounded t-structure \( (\mathcal{A}, \mathcal{B}) \) in \( \mathcal{D}(\mathcal{M}) \) defined by Equation (1.1); this is shown by the same method as in Example 1.2.

There are three isomorphism classes of indecomposable objects in \( \mathcal{M} \) given by the following representations of the quiver \( A_2 \).
\[ 0 \rightarrow \mathbb{C}, \quad \mathbb{C} \rightarrow \mathbb{C}, \quad \mathbb{C} \rightarrow 0 \]
They induce isomorphism classes \( x_0, x_1, x_2 \) of indecomposable objects in \( \mathcal{D}(\mathcal{M}) \), and we define further isomorphism classes recursively by \( \Sigma x_i = x_{i+3} \) for \( i \in \mathbb{Z} \).

The Auslander–Reiten quiver of \( \mathcal{D}(\mathcal{M}) \) looks as follows, where red and green vertices show \( \mathcal{A} \) and \( \mathcal{B} \).

\[
\begin{array}{cccccccccccc}
\cdots & x_{-3} & \rightarrow & x_{-2} & \rightarrow & x_{-1} & \rightarrow & x_0 & \rightarrow & x_1 & \rightarrow & x_2 & \rightarrow & x_3 & \rightarrow & x_4 & \rightarrow & x_5 & \rightarrow & \cdots \\
\end{array}
\]

Note that if \( t \in \mathcal{D}(\mathcal{M}) \) is indecomposable, then \( t \) is in \( \mathcal{A} \) or in \( \mathcal{B} \), so the triangle in Definition 1.1(iii) is trivial in the sense that it reads \( t \rightarrow t \rightarrow 0 \) or \( 0 \rightarrow t \rightarrow t \).
The heart $\mathcal{H} = \mathcal{A} \cap \Sigma \mathcal{B}$ is determined by
\[ \mathcal{H} = \text{add}(x_0 \oplus x_1 \oplus x_2). \]

\[ \square \]

2. Co-t-structures

The following definition was made in [5, def. 1.1.1] and [13, def. 2.4].

**Definition 2.1** (Bondarko and Pauksztello). A co-t-structure in $\mathcal{T}$ is a pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories, closed under isomorphisms, direct sums, and direct summands, which satisfy the following conditions.

(i) $\Sigma^{-1} \mathcal{X} \subseteq \mathcal{X}$ and $\Sigma \mathcal{Y} \subseteq \mathcal{Y}$.

(ii) $\mathcal{T}(\mathcal{X}, \mathcal{Y}) = 0$.

(iii) For each object $t \in \mathcal{T}$ there is a triangle $x \to t \to y$ with $x \in \mathcal{X}$, $y \in \mathcal{Y}$.

The co-heart is $\mathcal{C} = \mathcal{X} \cap \Sigma^{-1} \mathcal{Y}$.

The co-t-structure is called bounded if
\[ \bigcup_{i \in \mathbb{Z}} \Sigma^i \mathcal{X} = \bigcup_{i \in \mathbb{Z}} \Sigma^i \mathcal{Y} = \mathcal{T}. \] \[ \square \]

In contrast to t-structures, the objects $x$ and $y$ in Definition 2.1(iii) do not in general depend functorially on $t$, see [5, rmk. 1.2.2].

The following is the canonical example of a co-t-structure.

**Example 2.2** (The standard co-t-structure). Let $R$ be a ring, $\mathcal{P} = \text{Prj} R$ the category of projective left-modules over $R$. Let $\mathcal{X}$ and $\mathcal{Y}$ be the isomorphism closures in the bounded homotopy category $\mathcal{K}^b(\mathcal{P})$ of the subsets
\[ \{ \cdots \to 0 \to 0 \to P^0 \to P^1 \to P^2 \to \cdots \mid P^i \in \mathcal{P}, P^i = 0 \text{ for } i \gg 0 \}, \]
\[ \{ \cdots \to P^{-3} \to P^{-2} \to P^{-1} \to 0 \to 0 \to 0 \to \cdots \mid P^i \in \mathcal{P}, P^i = 0 \text{ for } i \ll 0 \}. \]

We will show that if $\mathcal{X}$ and $\mathcal{Y}$ are viewed as full subcategories, then $(\mathcal{X}, \mathcal{Y})$ is a bounded co-t-structure with co-heart $\mathcal{C}$ equivalent to $\mathcal{P}$.

Recall that $\mathcal{X}$ and $\mathcal{Y}$ are required to be closed under isomorphisms, direct sums, and direct summands. The two former properties are immediate, and the latter follows from the results in [14, secs. 3 and 4].

We next check the conditions in Definition 2.1. Conditions (i) and (ii) are clear. The triangle in Definition 2.1(iii) can be obtained by hard truncation of the object $t = \cdots \to P^{-2} \to P^{-1} \to P^0 \to P^1 \to \cdots$ in $\mathcal{K}^b(\mathcal{P})$, see Figure 2 in the introduction.

It is immediate that the co-t-structure $(\mathcal{X}, \mathcal{Y})$ is bounded.

Finally, it follows from [14, cor. 4.11] that the coheart $\mathcal{C} = \mathcal{X} \cap \Sigma^{-1} \mathcal{Y}$ is equivalent to $\mathcal{P}$. \[ \square \]

The term silting set was coined in [9]. The following definition was made in [11, def. 2.1].

**Definition 2.3.** A silting subcategory $\mathcal{C}$ of $\mathcal{T}$ is a full subcategory, closed under isomorphisms, direct sums, and direct summands, which satisfies
(i) $\mathcal{T}(\mathcal{C}, \Sigma^{>0}\mathcal{C}) = 0$.

(ii) Each object in $\mathcal{T}$ can be obtained from $\mathcal{C}$ by taking finitely many (de)suspensions, triangles, and direct summands.

A silting object $s$ of $\mathcal{T}$ is an object such that $\text{add}(s)$ is a silting subcategory. \hfill $\square$

The following was proved in [12, cor. 5.9].

**Theorem 2.4** (Mendoza Hernández et.al.). The map

$$(\mathcal{X}, \mathcal{Y}) \mapsto \mathcal{C} = \mathcal{X} \cap \Sigma^{-1}\mathcal{Y}$$

is a bijection between bounded co-t-structures and silting subcategories of $\mathcal{T}$.

**Remark 2.5.** The inverse map sends a silting subcategory $\mathcal{C}$ to a pair $(\mathcal{X}, \mathcal{Y})$ where $\mathcal{X}$ is the smallest full subcategory, closed under isomorphisms, direct sums, and direct summands, which is closed under $\Sigma^{-1}$ and contains $\mathcal{C}$. Similarly, $\mathcal{Y}$ is the smallest full subcategory, closed under isomorphisms, direct sums, and direct summands, which is closed under $\Sigma$ and contains $\Sigma\mathcal{C}$. \hfill $\square$

**Example 2.6.** We continue Example 1.4 so $\Lambda = \mathbb{C}A_2$ is the path algebra of the quiver $A_2$ from Equation (1.2) and $\mathcal{P} = \text{prj}\, \Lambda$ is the category of finite dimensional projective left modules over $\Lambda$.

There is a bounded co-t-structure $(\mathcal{X}, \mathcal{Y})$ in $\mathbb{K}^b(\mathcal{P})$ where $\mathcal{X}$ and $\mathcal{Y}$ are the isomorphism closures in $\mathbb{K}^b(\mathcal{P})$ of the subsets in Equation (2.1); this is shown by the same method as in Example 2.2.

Recall that $\Lambda$ also has a bounded derived category $\mathbb{D}^b(\mathcal{M})$, see Example 1.4. Since $\Lambda$ has global dimension 1, the triangulated categories $\mathbb{K}^b(\mathcal{P})$ and $\mathbb{D}^b(\mathcal{M})$ are equivalent, so $\mathbb{K}^b(\mathcal{P})$ has the Auslander–Reiten quiver shown in Equation (1.3). We redraw the quiver, this time with red and green vertices showing $\mathcal{X}$ and $\mathcal{Y}$.

Note that $x_2$ is neither in $\mathcal{X}$ nor $\mathcal{Y}$. Indeed, if we abuse notation to confuse isomorphism classes with individual objects, then we can set $t = x_2$ and the triangle in Definition 2.1(iii) becomes $x_1 \rightarrow x_2 \rightarrow x_3$.

The coheart $\mathcal{C} = \mathcal{X} \cap \Sigma^{-1}\mathcal{Y}$ is determined by

$$\mathcal{C} = \text{add}(x_0 \oplus x_1).$$

Theorem 2.4 implies that $\mathcal{C}$ is a silting subcategory of $\mathbb{K}^b(\mathcal{P})$. The corresponding isomorphism class of silting objects is $x_0 \oplus x_1$. \hfill $\square$

3. **Towers which build an arbitrary object from objects of the (co-)heart**

The following two results were proved in [3, p. 34] and [5, prop. 1.5.6]. A wavy arrow $s \rightsquigarrow t$ denotes a morphism $s \to \Sigma t$. 

Proposition 3.1 (Beilinson, Bernstein, and Deligne). Let \((\mathcal{A}, \mathcal{B})\) be a bounded t-structure in \(\mathcal{T}\) with heart \(\mathcal{H}\). For each object \(t \in \mathcal{T}\), there is an integer \(n \geq 1\) and a diagram consisting of triangles,

\[
0 = t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow \cdots \rightarrow t_{n-1} \rightarrow t_n = t,
\]

with \(h_m \in \mathcal{H}\) for each \(m\) and \(i_1 > i_2 > \cdots > i_n\).

Proposition 3.2 (Bondarko). Let \((\mathcal{X}, \mathcal{Y})\) be a bounded co-t-structure in \(\mathcal{T}\) with co-heart \(\mathcal{C}\). For each object \(t \in \mathcal{T}\), there is an integer \(n \geq 1\) and a diagram consisting of triangles,

\[
0 = t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow \cdots \rightarrow t_{n-1} \rightarrow t_n = t,
\]

with \(c_m \in \mathcal{C}\) for each \(m\) and \(i_1 < i_2 < \cdots < i_n\).

Example 3.3. Consider the t-structure in Example 1.2. If \(t \in \mathcal{D}^b(\mathcal{M})\) is given, then there is a diagram as in Proposition 3.1 where the objects \(\Sigma^m h_m\) are of the form \(\Sigma^i H^{-i}(t)\). The diagram expresses that \(t\) can be built from its cohomology modules \(H^{-i}(t)\).

Example 3.4. Consider the co-t-structure in Example 2.2. If \(t = \cdots \rightarrow P^{-2} \rightarrow P^{-1} \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots\) in \(\mathcal{K}^b(\mathcal{P})\) is given, then there is a diagram as in Proposition 3.2 where the objects \(\Sigma^m c_m\) are of the form \(\Sigma^i P^{-i}\). The diagram expresses that \(t\) can be built from its constituent modules \(P^{-i}\).

4. Categories skewed towards t- or co-t-structures

In this section, \(d\) is a fixed integer.

Definition 4.1. If \(\mathcal{T}\) is \(\mathbb{C}\)-linear, then an object \(s \in \mathcal{T}\) is called \(d\)-spherical if there is an isomorphism

\[
\mathcal{T}(s, \Sigma^* s) \cong \mathbb{C}[X]/(X^2)
\]

of graded algebras where \(\deg X = d\).

Remark 4.2. If \(d \neq 0\), then \(s\) is \(d\)-spherical if and only if there are isomorphisms of \(\mathbb{C}\)-vector spaces

\[
\mathcal{T}(s, \Sigma^i s) \cong \begin{cases} 
\mathbb{C} & \text{for } i \in \{0, d\}, \\
0 & \text{otherwise}.
\end{cases}
\]

The following was proved in [10, thm. 2.1].

Theorem 4.3 (Keller, Yang, and Zhou). There is a triangulated category \(\mathcal{I}_d\) which is algebraic, \(\mathbb{C}\)-linear with finite dimensional Hom-spaces, and contains a \(d\)-spherical object \(s\) such that each object in \(\mathcal{I}_d\) can be obtained from \(s\) by taking finitely many (de)suspensions, triangles, and direct summands.

Up to triangulated equivalence, \(\mathcal{I}_d\) is unique.

The following was proved in [7, thm. A]. It clearly implies that if \(d \leq 0\), then bounded co-t-structures are more useful than bounded t-structures for the study of \(\mathcal{I}_d\).
Theorem 4.4 (Holm, J, and Yang). If \( d \leq 0 \) then \( S_d \) has no bounded t-structures. It has one family of bounded co-t-structures, all of which are (de)suspensions of a canonical one.

If \( d \geq 1 \) then \( S_d \) has no bounded co-t-structures. It has one family of bounded t-structures, all of which are (de)suspensions of a canonical one.

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Definition 5.1. A simple minded collection in \( \mathcal{T} \) is a set \( \{t_1, \ldots, t_n\} \) of objects of \( \mathcal{T} \) with the following properties.

(i) \( \mathcal{T}(t_i, \Sigma \leq 0 t_j) = 0 \) for all \( i \) and \( j \).

(ii) \( \mathcal{T}(t_i, t_i) \) is a division ring for each \( i \) and \( \mathcal{T}(t_i, t_j) = 0 \) when \( i \neq j \).

(iii) Each object in \( \mathcal{T} \) can be obtained from \( t_1, \ldots, t_n \) by taking finitely many (de)suspensions, triangles, and direct summands.

\[ \] Let \( \Lambda \) be a finite dimensional \( \mathbb{C} \)-algebra, \( \mathcal{M} = \operatorname{mod} \Lambda \) the category of finite dimensional left modules over \( \Lambda \), and \( \mathcal{P} = \operatorname{prj} \Lambda \) the category of finite dimensional projective left modules over \( \Lambda \). The following was proved in [11, thm. (6.1)]. We interpret it as indicating that the role of t-structures in derived categories is taken over by co-t-structures in homotopy categories.

Theorem 5.2. There are bijections between the following sets.

(i) Bounded t-structures in \( \mathcal{D}^b(\mathcal{M}) \) whose hearts are length categories ("length category" means that each object has finite length).

(ii) Bounded co-t-structures in \( \mathcal{K}^b(\mathcal{P}) \).

(iii) Isomorphism classes of simple minded collections in \( \mathcal{D}^b(\mathcal{M}) \).

(iv) Isomorphism classes of basic silting objects in \( \mathcal{K}^b(\mathcal{P}) \) (basic means no repeated indecomposable summands).

Remark 5.3. There is an extensive body of work on the bijections of Theorem 5.2 which predates [11], see [1], [2], [3], [5], [6], [8], [9], and [12]. The contributions of these papers to Theorem 5.2 are explained in the introduction to [11].

Remark 5.4. The proof of Theorem 5.2 occupies a significant part of [11], and we only show how some of the bijections are defined.

(i) to (iii): Let \( (\mathcal{A}, \mathcal{B}) \) be a bounded t-structure in \( \mathcal{D}^b(\mathcal{M}) \) whose heart \( \mathcal{H} = \mathcal{A} \cap \Sigma \mathcal{B} \) is a length category. Take a simple object from each isomorphism class of simple objects in \( \mathcal{H} \). This gives a simple minded system in \( \mathcal{D}^b(\mathcal{M}) \), see [11, sec. 5.3].

(ii) to (iv): Let \( (\mathcal{X}, \mathcal{Y}) \) be a bounded co-t-structure in \( \mathcal{K}^b(\mathcal{P}) \). The co-heart \( \mathcal{C} = \mathcal{X} \cap \Sigma^{-1} \mathcal{Y} \) is a silting subcategory by Theorem 2.4 and there is a silting object \( s \) such that \( \mathcal{C} = \operatorname{add}(s) \), see [11, sec. 5.2].

(i) to (ii): Let \( (\mathcal{X}, \mathcal{Y}) \) be a bounded co-t-structure in \( \mathcal{K}^b(\mathcal{P}) \). Set

\[ \mathcal{A} = \{a \in \mathcal{D}^b(\mathcal{M}) \mid \operatorname{Hom}_{\mathcal{D}^b(\mathcal{M})}(x, a) = 0 \text{ for each } x \in \mathcal{X}\}, \]

\[ \mathcal{B} = \{b \in \mathcal{D}^b(\mathcal{M}) \mid \operatorname{Hom}_{\mathcal{D}^b(\mathcal{M})}(y, b) = 0 \text{ for each } y \in \mathcal{Y}\} \]
and view these two sets as full subcategories of $\mathcal{D}^b(\mathcal{M})$. Then $(\mathcal{A}, \mathcal{B})$ is a bounded t-structure in $\mathcal{D}^b(\mathcal{M})$ with length heart, see [11, sec. 5.7].

6. The silting mutation of Aihara and Iyama

Silting mutation is an operation which changes one silting subcategory into another. By virtue of Theorem 2.4 it can be viewed as changing one bounded co-t-structure into another. This leads to the definition of the so-called silting quiver of the triangulated category $\mathcal{T}$ which shows how silting subcategories, and hence co-t-structures, fit together inside $\mathcal{T}$.

In this section, $\mathcal{T}$ is $\mathbb{C}$-linear with finite dimensional Hom-spaces and split idempotents, and $m = m_0 \oplus m_1$ is a basic silting object of $\mathcal{T}$ with $m_0$ indecomposable. The following definition and theorem are special cases of [1, sec. 2.4].

**Definition 6.1** (Aihara and Iyama). Let $r \xrightarrow{\rho} m_0 \xrightarrow{\lambda} \ell$ be a minimal right $\text{add}(m_1)$-approximation and a minimal left $\text{add}(m_1)$-approximation of $m_0$. Complete these morphisms to triangles

$$m_0 \xrightarrow{\sim} r \xrightarrow{\rho} m_0 \quad m_0 \xrightarrow{\lambda} \ell \xrightarrow{\mu} m_0^\dagger$$

in $\mathcal{T}$ and set

$$\mu^-(m, m_1) = m_0^\sim \oplus m_1 \quad \mu^+(m, m_1) = m_0^\dagger \oplus m_1.$$

These are called right and left silting mutations of $m$. □

**Theorem 6.2** (Aihara and Iyama).

(i) The silting mutations $\mu^-(m, m_1)$ and $\mu^+(m, m_1)$ are basic silting objects of $\mathcal{T}$.

(ii) Right and left silting mutations are inverse in the sense that

$$\mu^-(\mu^+(m, m_1), m_1) \cong m \quad \mu^+(\mu^-(m, m_1), m_1) \cong m.$$

The following is a special case of [1, def. 2.41].

**Definition 6.3** (Aihara and Iyama). The silting quiver of $\mathcal{T}$ has a vertex for each isomorphism class of basic silting objects of $\mathcal{T}$, and an arrow $[m] \to [m^*]$ if $m^*$ is a left silting mutation of $m$, where square brackets denote isomorphism class. □

**Remark 6.4.** The silting quiver gives a picture of how silting mutation moves from one silting object to another, hence from one silting subcategory to another. By virtue of Theorem 2.4 it gives a picture of how silting mutation moves from one bounded co-t-structure to another. □

**Example 6.5.** We continue Example 2.6, so $\Lambda = \mathbb{C}A_2$ is the path algebra of the quiver $A_2$ from Equation (1.2). The bounded homotopy category $\mathcal{K}^b(\mathcal{P})$ has the Auslander–Reiten quiver shown in Equation (2.2).

Recall from Example 2.6 that $\mathcal{K}^b(\mathcal{P})$ has the isomorphism class of silting objects $x_0 \oplus x_1$. Indeed, $x_0 \oplus x_1$ is a vertex in the silting quiver of $\mathcal{K}^b(\mathcal{P})$. The full quiver was determined in [1, exa. 2.45], see Figure 3. As the quiver shows, there is a left silting mutation of $x_0 \oplus x_1$ which gives $x_0 \oplus x$. The isomorphism classes of silting objects $x_0 \oplus x_1$ and $x_0 \oplus x_4$ give rise to silting subcategories $\text{add}(x_0 \oplus x_1)$ and $\text{add}(x_0 \oplus x_4)$ which, under the bijection of Theorem 2.4, correspond to two bounded co-t-structures. The first of these is shown on the AR quiver.
Figure 3. The silting quiver of the bounded homotopy category $\mathcal{K}_b(\mathcal{P})$ where $\mathcal{P}$ is the category of finite dimensional projective modules over $\mathbb{C}A_2$.

of $\mathcal{K}_b(\mathcal{P})$ in Equation (2.2). The second co-t-structure $(\mathcal{X}', \mathcal{Y}')$ can be shown as follows, where the red and green vertices show $\mathcal{X}'$ and $\mathcal{Y}'$.

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