Serre’s “formule de masse” in prime degree

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Abstract. For a local field $F$ with finite residue field of characteristic $p$, we describe completely the structure of the filtered $\mathbb{F}_p[G]$-module $K^\times/K^{\times p}$ in characteristic 0 and $K^\times/\mathcal{O}(K^\times)$ in characteristic $p$, where $K = F^{(\sqrt[p]{F^\times})}$ and $G = \text{Gal}(K|F)$. As an application, we give an elementary proof of Serre’s mass formula in degree $p$. We also determine the compositum $C$ of all degree-$p$ separable extensions with solvable galoisian closure over an arbitrary base field, and show that $C$ is $K^{(\sqrt[p]{K^\times})}$ or $K^{(\mathcal{O}^{-1}(K))}$ respectively, in the case of the local field $F$. Our method allows us to compute the contribution of each character $G \to \mathbb{F}_p^\times$ to the degree-$p$ mass formula, and, for any given group $\Gamma$, the contribution of those degree-$p$ separable extensions of $F$ whose galoisian closure has group $\Gamma$.

1. Introduction. — Let $p$ be a prime number and let $F$ be a local field with finite residue field $k$ of characteristic $p$ and cardinality $q = p^f$, so that the characteristic $\text{car}(F)$ of $F$ either $p$ or 0 and indeed $F = k((\pi))$ in the characteristic-$p$ case and $F$ is a finite extension of $\mathbb{Q}_p$ of residual degree $f$ and ramification index $e = [F : \mathbb{Q}_p]/f$ in the characteristic-0 case; put $e = +\infty$ in the former case.

For a totally ramified separable extension $E$ of $F$ of degree $n$ and discriminant $\delta_{E|F}$ of valuation $v(\delta_{E|F})$, put

$$c(E) = v(\delta_{E|F}) - (n - 1),$$

so that $c(E) = 0$ if and only if $E|F$ is tamely ramified or equivalently $n$ is prime to $p$. Serre’s mass formula says that when $E$ runs through such extensions (contained in a fixed separable algebraic closure of $F$), then

$$\sum_E q^{-c(E)} = n.$$

Keywords: Formule de masse de Serre, Serre’schen Maßformel, Serre’s mass formula.
This beautiful formula is easy to prove in the tame case when \( n \) is prime to \( p \) but lies much deeper when \( p | n \). The main purpose of this Note is to give an elementary proof of this formula in the case \( n = p \) and to develop the algebraic and arithmetic ingredients on which this proof relies.

(It turns out that the method adopted here leads to several refinements of the mass formula. For instance, one can compute the contribution of those \( E \) which are cyclic over \( F \), or those \( E \) whose galoisian closure \( \overline{E} \) over \( F \) is of the form \( EF' \) for some unramified extension \( F'|F \) (depending on \( E \)), or those \( E \) for which the group \( \text{Gal}(\overline{E}|F) \) is isomorphic to a given group \( \Gamma \). See \( \S 9 \) for details.)

There are two sources of inspiration for our method. The first one is our recent reworking \([7]\) of a standard technique used in proofs of the local \([3, \text{p. 155}]\) or the global \([13, \text{p. 110}]\) Kronecker-Weber theorem. It allowed us to compute the contribution of degree-\( p \) cyclic extensions to the mass formula \([7], \S 6\).

The idea was that in characteristic \( p \), the set of such extensions is in bijection with the set of \( \mathbb{F}_p \)-lines in \( F/\varphi(F) \), where \( \varphi : x \mapsto x^p - x \), and that in characteristic 0 it is in bijection with the set of \( \mathbb{F}_p \)-lines in the \( \omega \)-eigenspace for the action of \( \Delta \) on \( F(\zeta)^\times/F(\zeta)^{\times p} \), where \( \zeta \) is a primitive \( p \)-th root of 1, \( \Delta = \text{Gal}(F(\zeta)|F) \), and \( \omega : \Delta \to \mathbb{F}_p^\times \) is the cyclotomic character giving the action of \( \Delta \) on the \( p \)-th roots of 1. Roughly speaking, what we did for \( \omega \) there, we need to do here for all characters \( \chi : G \to \mathbb{F}_p^\times \), where \( G = \text{Gal}(K|F) \) and \( K = F(\sqrt[p]{K^\times}) \).

The results and methods of Del Corso and Dvornicich \([8]\) provided the second source of inspiration and ideas. They study the action of \( G \) on \( K^\times/K^{\times p} \) in characteristic 0 and prove that the compositum of all degree-\( p \) extensions of \( F \) is \( K(\sqrt[p]{K^\times}) \). The characteristic-\( p \) analogue of their main theorem, stating that the compositum of all degree-\( p \) separable extensions of \( F \) is \( K(\varphi^{-1}(K)) \), where \( K \) is still \( F(\sqrt[p]{K^\times}) \), can be found below (prop. 36). We also study (\( \S 8 \)) the compositum of all degree-\( p' \) extensions of \( F \) for a prime \( p' \neq p \), which turns out to be \( K'(\sqrt[p']{K'^\times}) \), where \( K' = F(\zeta') \) and \( \zeta' \) is a primitive \( p' \)-th root of 1.

Our first task is thus to extend the methods and the main result of \([8]\) to characteristic-\( p \) local fields. They turn out to behave exactly as their characteristic-0 counterparts would have if \( e = +\infty \) and \( \omega = 1 \). We give a unified presentation of the two cases which is perhaps more intrinsic and conceptual than the treatment of the characteristic-0 case in \([8]\), to which our debt should however be clear to the reader.

Our main arithmetic contribution is thus an explicit description of the structure of the filtered \( G \)-module \( K^\times/K^{\times p} \) (resp. \( K^+/\varphi(K^+) \)). It is \( \mathbb{F}_p[G]-\)
isomorphic to

\[ \mathbb{F}_p\{\omega\} \oplus k[G]e \oplus \mathbb{F}_p \quad (\text{resp. } \mathbb{F}_p \oplus k[G] \oplus k[G] \oplus \cdots) \]

(where \(\mathbb{F}_p\{\omega\}\) denotes an \(\mathbb{F}_p\)-line on which \(G\) acts via \(\omega\)) with a specific filtration to be described in detail in §6. For degree-\(p'\) extensions, the analogue is the filtered \(\mathbb{F}_p'[G']\)-module \(\mathbb{F}_p'\{\omega'\} \oplus \mathbb{F}_p'\), where \(\omega'\) is the mod-\(p'\) cyclotomic character of \(G' = \text{Gal}(K'/F)\): it is as if \(e = 0\). We then use this description to give an elementary proof of Serre’s mass formula in prime degree; it is uniformly applicable to all three cases: \(0 < e < +\infty\), \(e = +\infty\), and \(e = 0\).

Let us give a sketch of the proof in the degree-\(p\) case. Every separable degree-\(p\) extension \(E\) of \(F\) becomes cyclic when translated to \(K\). It corresponds therefore to an \(\mathbb{F}_p\)-line \(D\) in \(K^\times/K^\times p\) (resp. \(K^+/\varphi(K^+))\). Such lines are stable under the action of \(G\), and every \(G\)-stable line arises from some \(E\). Two such extensions \(E, E'\) give rise to the same \(D\) if and only if they are conjugate over \(F\). If \(E|F\) is not cyclic, then it has \(p\) conjugates.

The invariant \(c(E)\) of \(E\) can be recovered from the “level” of \(D\) in the filtration on \(K^\times/K^\times p\) (resp. \(K^+/\varphi(K^+)\)) — the integer \(m\) such that \(D \subset \bar{U}_m\) but \(D \not\subset \bar{U}_{m+1}\) (resp. \(D \subset \mathfrak{p}^m\) but \(D \not\subset \mathfrak{p}^{m+1}\)), where \((\bar{U}_n)_{n \in \mathbb{N}}\) (resp. \((\mathfrak{p}^n)_{n \in \mathbb{Z}}\)) is the induced filtration — by the Schachtelungssatz. The number \(r\) of \(E\) giving rise to \(D\) can be read off from the character \(\chi : G \to \mathbb{F}_p^\times\) through which \(G\) acts on \(D\); we have \(r = 1\) if \(\chi = \omega\) and \(r = p\) if \(\chi \neq \omega\), where \(\omega\) is the mod-\(p\) cyclotomic character in characteristic 0 and \(\omega = 1\) by convention in characteristic \(p\). So the sum over all ramified separable degree-\(p\) extensions \(E\) of \(F\) gets replaced by a sum over all \(G\)-stable lines \(D\) in \(K^\times/K^\times p\) or \(K^+/\varphi(K^+)\) other than the line \(\bar{U}_pe\) in characteristic 0 and other than the line \(\mathfrak{o}\) in characteristic \(p\). As we have already determined (§6) the structure of the filtered \(\mathbb{F}_p[G]\)-module \(K^\times/K^\times p\) (resp. \(K^+/\varphi(K^+)\)), the contribution from these \(D\) can be computed level by level, leading to the result. See §7 for the complete proof and §9 for an extended summary along with some refinements.

This proof may be thought of a generalisation from the case \(p = 2\) ([4, prop. 67], [7, prop. 14]) where it amounted to a trivial identity because every separable quadratic extension is kummerian — of the form \(F(x)\) with \(x^2 \in F^\times\) (resp. \(x^2 - x \in F\)). The reader may also wish to contrast this proof with the much easier case of degree-\(p'\) extensions for a prime \(p' \neq p\) outlined in §8.

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While this application to the degree-\(p\) case of Serre’s mass formula (§7) for local fields of residual characteristic \(p\) was our main motivation,
the contribution of this Note is not confined to this proof. We have first developed (§4) the purely algebraic theory of solvable extensions of prime degree (def. 11). Observations such as Remark 13 and lemmas 17–20 are of independent interest. The arithmetic theory of filtered galoisian modules (§§6,8) has been developed to the fullest (in the case at hand) in an intrinsic manner without distinction between the mixed-characteristic, the equicharacteristic, and the $p' \neq p$ cases.

The general algebraic theory (§4) and the local arithmetic theory (§§6,8) have other applications. For instance, the degree-$p$ mass formula can be thought of as the sum of the contributions of various characters $\xi : G \to \mathbf{F}_p^\times$. Our method lets us compute the contribution of each $\xi$, which allows us to compute the contribution of various natural classes of degree-$p$ separable extensions of $F$ (see §9).

2. Contents. — §3 recalls some basic facts about groups and their representations which are applied in §4 to study degree-$p$ solvable extensions (def. 11) of an arbitrary base field, culminating in the determination of the compositum of all such extensions (lemma 20). What is needed for the subsequent §§ is summarised at the end of §4.

We return to our local field $F$ in §5 and work out the consequences of the algebraic theory in this particular case. We undertake in §6 a detailed study of the filtered $G$-module $K^\times/K^{\times p}$ (resp. $K^+/\wp(K^+)$) of $\mathbf{F}_p$-dimension $2 + (p - 1)^2ef$ (resp. $+\infty$), where $G = \text{Gal}(K/F)$ and $K = F((p-1)^{1/2})$, leading to the determination of the compositum of all separable extensions of $F$ of degree $p$ (prop. 36), and to the proof of Serre’s degree-$p$ mass formula (th. 39) in §7. The method is supple enough to compute the contributions of individual characters $\chi \in F^\times/F^{\times p-1}$, as explained in §9. Finally, we indicate (§8) how simple the whole theory becomes when dealing with degree-$p'$ extensions for some prime $p' \neq p$ ($K$ gets replaced by $K' = F(\sqrt[p]{T})$ and the $\mathbf{F}_{p'}$-space $K'^{\times}/K'^{\times p'}$ is 2-dimensional).

3. Groups and their representations. — All we need are some fairly standard results but we include some proofs.

First, let $P \subset \mathfrak{S}_p$ be a subgroup of order $p$ and $N$ its normaliser; identify $\text{Aut} P$ with $\mathbf{F}_p^\times$. For every $\eta \in N$, conjugation by $\eta$ induces an automorphism $\text{Int}(\eta) \in \mathbf{F}_p^\times$ of $P$ which depends only on the class $\bar{\eta} \in N/P$, leading to a map $N/P \to \mathbf{F}_p^\times$.

Lemma 1. — The map $N/P \to \mathbf{F}_p^\times$ is an isomorphism : we have a split short exact sequence $1 \to P \to N \to \mathbf{F}_p^\times \to 1$. 
In other words, \( N = P \times \iota \mathbf{F}_p^\times \), where \( \iota : \sigma \mapsto (\sigma \mapsto \sigma^a) \) is the canonical isomorphism \( \mathbf{F}_p^\times \to \text{Aut} P \). Consequently, a subgroup \( \Gamma \subset \mathfrak{S}_p \) containing \( P \) and contained in \( N \) is transitive and solvable, and indeed \( \Gamma \) is the canonical split extension of a subgroup of \( \mathbf{F}_p^\times \) by \( P \). There is a converse.

**Lemma 2 (Galois).** — A transitive subgroup \( \Gamma \subset \mathfrak{S}_p \) is solvable if and only if it contains a unique Sylow \( p \)-subgroup \( P \) (of order \( p \)).

This was guessed at from the reference to [1, ch. 3, th. 7] in [8]; Robin Chapman and Jack Schmidt provided short arguments on MathOverflow, reproduced below, and Matthew Emerton indicated the provenance.

The order of a transitive subgroup \( \Gamma \subset \mathfrak{S}_p \) is divisible by \( p \) (because the orbit has \( p \) elements) but not by \( p^2 \) (because the order \( p! \) of \( \mathfrak{S}_p \) is not). Therefore \( \Gamma \) has a Sylow \( p \)-subgroup \( P \) of order \( p \). If it is unique, then \( P \subset \Gamma \) is normal, and hence \( \Gamma \) is contained in the normaliser \( N \) of \( P \) in \( \mathfrak{S}_p \). By lemma 1, \( \Gamma \) is solvable and the result follows in this case. We show next that this is the only case : \( P \) is unique if \( \Gamma \) is solvable.

Suppose that \( \Gamma \) has Sylow \( p \)-subgroups other than \( P \); they have to be conjugates of \( P \) in \( \Gamma \) (Sylow). If \( H \subset \Gamma \) is a normal subgroup of order \( > 1 \), then \( H \) must be transitive since otherwise the orbits of \( H \) would form a partition of \( \{1, \cdots, p\} \) invariant under the action of \( \Gamma \), which is impossible because \( p \) is prime. So \( H \) contains some (and hence every) conjugate of \( P \). It follows that \( \Gamma \) is not solvable, as there are no normal subgroups containing only one conjugate of \( P \). A standard reference is [11, p. 163].

Next, let \( 1 \to C \to \Gamma \to G \to 1 \) be any short exact sequence of finite groups in which \( C \) is commutative.

**Lemma 3 (Schur-Zassenhaus).** — If the orders of \( C \) and \( G \) are mutually prime, then sections \( G \to \Gamma \) do exist, and any two sections \( G \to \Gamma \) differ by an inner automorphism \( \text{Int}(\gamma) : \Gamma \to \Gamma \) for some \( \gamma \in C \).

Associating to the image \( \bar{\gamma} \in G \) of \( \gamma \in \Gamma \) the automorphism \( \sigma \mapsto \gamma \sigma \gamma^{-1} \) of \( C \) (which depends only on \( \bar{\gamma} \) because \( C \) is commutative), we get an action \( \theta : G \to \text{Aut} C \) of \( G \) on \( C \). The group \( H^2(G, C)_\theta \) (the action \( \theta \) is often omitted from the notation) classifies extensions of \( G \) by \( C \) in which \( G \) acts on \( C \) via \( \theta \); the twisted product \( C \times_\theta G \) is the neutral element of \( H^2(G, C)_\theta \). For every subgroup \( G' \subset G \), we have the natural maps

\[
\text{Res} : H^2(G, C)_\theta \to H^2(G', C)_{\theta'}, \quad \text{Cor} : H^2(G', C)_{\theta'} \to H^2(G, C)_\theta,
\]

of restriction and corestriction whose composite \( \text{Cor} \circ \text{Res} \) is multiplication by the index \( (G : G') \). Upon taking \( G' = \{1\} \), we see that \( H^2(G, C)_\theta \) is
killed by the order of $G$. But it is also killed by the order of $C$. As these orders are mutually prime by hypothesis, we have $H^2(G, C)_\theta = 0$, and hence $\Gamma = C \times_\theta G$ is a twisted product, which admits sections.

Now, $C$-conjugacy classes of sections $G \to C \times \theta G$ are classified by the group $H^1(G, C)_\theta$, which can be shown to vanish exactly as above. I thank Joseph Oesterlé for supplying this argument at short notice. See [2] for details and [14, IV, 2, cor. 4] for a direct proof in the case needed below, in which $C$ is a $p$-group and $G$ is commutative of exponent dividing $p - 1$.

**Remark 4.** When moreover $G$ is commutative and the action $\theta$ is trivial, then there is a unique section $G \to C \times G$, namely the inclusion.

**Remark 5.** A solvable transitive subgroup $\Gamma \subset \mathfrak{S}_p$ has exactly $p$ subgroups of index $p$, unless $\Gamma$ is commutative (in which case it is in fact cyclic of order $p$). Indeed, $P$ being the unique Sylow $p$-subgroup of $\Gamma$ (lemma 2), $\Gamma/P$ is a subgroup of $F_p^\times$ (lemma 1), hence there are $p$ sections $\Gamma/P \to \Gamma$ unless $\Gamma = P$ (lemma 3).

**Remark 6.** Let $\bar{\tau} \in F_p^\times$ be a generator of $\Gamma/P$ and $g$ its order (so that $g \mid p - 1$). The groups $P$ and $\Gamma$ admit the presentations

$$P = \langle \sigma \mid \sigma^p = 1 \rangle, \quad \Gamma = \langle \sigma, \tau \mid \sigma^p = 1, \tau^g = 1, \tau \sigma \tau^{-1} = \sigma \bar{\tau} \rangle,$$

and we have $\Gamma = P \iff \bar{\tau} = 1 \iff g = 1$. If $\Gamma \neq P$, then the $p$ index-$p$ subgroups $G_i \subset \Gamma$ are generated respectively by $\sigma^i \tau \sigma^{-i}$ as $i$ runs through $F_p$. For $i \neq j$, we have $G_i \cap G_j = \{1\}$ and $G_i \cup G_j$ generates $\Gamma$.

**Remark 7.** A version of lemma 3 remains valid for profinite groups. We omit the details (*), for $\Gamma$ is finite in our main application to local fields. The profinite version would be convenient for the proof of lemma 16 where $1 \to C \to \Gamma \to G \to 1$ comes from a galoisian tower $L|K|F$ in which $L|K$ is cyclic of degree $p$ and $K|F$ is abelian of exponent dividing $p - 1$ (but of possibly infinite degree). Writing $L = K(x)$, taking $K'$ to be a finite galoisian extension of $F$ in $K$ containing the coefficients of the minimal polynomial $f \in K[T]$ of $x$ and such that $K'(x)$ is cyclic (of degree $p$) over $K'$ and galoisian over $F$, lemma 3 as stated above is applicable to the short exact sequence associated to the galoisian tower $K'(x)|K'|F$; this will suffice.

Finally, we need a few facts about representations of a commutative group $G$ of exponent $d$ over a field $k$ of characteristic prime to the order of $G$ and containing all $d$-th roots of 1. So let $V$ be a $k$-space on which $G$

(*) (2011/03/15) See for example Iwasawa (Transactions AMS, 80 (1955), 448–469, lemma 5)
acts by \(k\)-automorphisms; in other words, \(V\) is a \(k[G]\)-module. For every character \(\chi : G \to k^\times\), we denote by \(V(\chi)\) the \(\chi\)-eigenspace for the action of \(G\) on \(V\); it consists of all \(x \in V\) such that \(\sigma.x = \chi(\sigma)x\) for every \(\sigma \in G\).

**Lemma 8.** — The \(k\)-space \(V\) is the internal direct sum of its subspaces \(V(\chi)\), indexed by all characters \(\chi : G \to k^\times\). Every \(G\)-stable subspace \(W \subset V\) has a \(G\)-stable supplement \(W'\), and the canonical map \(W' \to V/W\) is an isomorphism of \(k[G]\)-modules.

Indeed, \(k[G]\) is a semisimple algebra and the only simple \(k[G]\)-modules are \(k\)-lines on which \(G\) acts through some character \(G \to k^\times\).

A particularly interesting example occurs when \(G = \text{Gal}(l/k)\) for \(l/k\) a finite abelian extension of exponent \(d\) prime to the characteristic of \(k\) and such that \(k\) contains a primitive \(d\)-th root of \(1\), and \(V = l\). The normal basis theorem — the \(k[G]\)-module \(l\) is free of rank \(1\) — implies

**Lemma 9.** — For every character \(\chi : G \to k^\times\) of \(G\), the \(k\)-space \(l(\chi)\) is of dimension \(1\).

A direct proof in the special case we need can be found in prop. 30.

**Remark 10.** — For a version of the normal basis theorem applicable to \(l/k\) of possibly infinite degree, see [12]. The ring \(k[G]\) is replaced by \(k[[G]]\) — the inverse limit of \(k[G/H]\) as \(H\) runs through open normal subgroups of \(G\), and \(l\) is replaced by the inverse limit of \(l^H\) under the trace maps; as a \(k[[G]]\)-module, the latter limit is free of rank \(1\).

**4. Fields and their extensions.** — We work over an arbitrary (commutative) field \(F\) whose characteristic \(\text{car}(F)\) may or may not be equal to our fixed prime \(p\). If \(\text{car}(F) \neq p\), denote by \(\omega\) the mod-\(p\) cyclotomic character (giving the action on the \(p\)-th roots of \(1\)) and take \(\omega = 1\) to be the trivial character if \(\text{car}(F) = p\), so that \(\omega\) has values in \(F^\times_p\).

**Definition 11.** — A degree-\(p\) extension \(E\) of \(F\) is called solvable if it is separable and if the group \(\text{Gal}(\bar{E}/F)\) of \(F\)-automorphisms of its galoisian closure \(\bar{E}\) over \(F\) is solvable.

The terminology is not standard (unless \(\bar{E} = E\)) but unlikely to confuse.

**Lemma 12.** — For every degree-\(p\) solvable extension \(E|F\), there exists a cyclic extension \(F'|F\) of degree dividing \(p - 1\) such that \(EF'\) is cyclic (of degree \(p\)) over \(F'\) and galoisian over \(F\). If \(E|F\) is not cyclic, then it has exactly \(p\) conjugates over \(F\).
Let \( \tilde{E} \) be the galoisian closure of \( E \) and \( \Gamma = \text{Gal}(\tilde{E}|F) \); the group \( \Gamma \) is solvable (def. 11). Also, \( \Gamma \) operates transitively on the set of \( F \)-embeddings of \( E \) (in any fixed but arbitrary separable algebraic closure of \( F \)), so \( \Gamma \subset S_p \).

Lemma 2 then furnishes a (unique) order-\( p \) subgroup \( P \subset \Gamma \). It is then clear that we may take \( F' = \tilde{E}^P \). Indeed, \( F'|F \) is cyclic of group \( \Gamma/P \) of order dividing \( p-1 \) (lemma 1), and \( EF' = \tilde{E} \) is cyclic over \( F' \) and galoisian over \( F \). Finally, Remark 5 says that if the group \( \Gamma \) is not cyclic, then it has \( p \) index-\( p \) subgroups, so the extension \( E \) has precisely \( p \) conjugates over \( F \) if it is not cyclic — something which is also otherwise clear.

Remark 13. — If \( E', E'' \) are two distinct conjugates of \( E \), then \( E'E'' = \tilde{E} \) (Remark 6). Consequently, if a solvable separable irreducible degree-\( p \) polynomial over \( F \) has two roots in an extension \( R \) of \( F \), then all its roots are in \( R \). The case \( R = R[1, p] \) is sometimes attributed to Kronecker: (Wenn eine irreductible Gleichung mit ganzzahligen Coefficients auflösbar und der Grad derselben eine ungrade (sic) Primzahl ist, so sind entweder alle ihre Wurzeln oder nur eine reell (Werke, IV, p. 25), but, as Kronecker himself points out, Galois had the general result: Pour qu’une équation de degré premier soit résoluble par radicaux, il faut et il suffit que deux quelconques de ces racines étant connues, les autres s’en déduisent rationnellement (Bulletin de M. Férussac, XIII (avril 1830), p. 271). Note that Kronecker’s observation applies to every \( R \) because Galois’s observation applies to every solvable transitive group of degree \( p \).

Lemma 12 admits a converse:

**Lemma 14.** — Let \( F'|F \) be cyclic of degree dividing \( p-1 \), and \( L|F' \) cyclic of degree \( p \). If \( L|F \) is galoisian, then there exists a (solvable) degree-\( p \) extension \( E|F \) such that \( L = EF' \), any two such extensions are conjugate over \( F \), and every conjugate of \( E \) is contained in \( L \).

Suppose that \( L|F \) is galoisian of group \( \Gamma = \text{Gal}(L|F) \). We then have an exact sequence \( 1 \to P \to \Gamma \to G \to 1 \), with \( P = \text{Gal}(L|F') \) of order \( p \) and \( G = \text{Gal}(F'|F) \) (cyclic) of order dividing \( p-1 \). The Schur-Zassenhaus theorem (lemma 3) then implies that \( \Gamma \) has a subgroup \( G' \) of index \( p \), and that any two such subgroups are conjugate in \( \Gamma \). It follows that \( E = L^{G'} \) is a solvable (def. 11) degree-\( p \) extension of \( F \) in \( L \) such that \( L = EF' \), and that any two such extensions are conjugate over \( F \). Finally, every \( F \)-conjugate of \( E \) is contained in \( L \) because \( L|F \) is galoisian and contains \( E \).

Let \( K \) be the compositum of all cyclic extensions of \( F \) of degree dividing \( p-1 \), so that \( K \) is the maximal abelian extension of \( F \) of exponent dividing \( p-1 \).
Lemma 15. — For every degree-$p$ solvable extension $E$ of $F$, the compositum $EK$ is cyclic over $K$ and galoisian over $F$.

This follows from lemma 12, the definition of $K$, and def. 11.

Which cyclic extensions $L|K$ arise as $L = EK$ for some (degree-$p$, solvable) extension $E|F$? If $L$ arises from $E$, then $L$ would be galoisian over $F$ (lemma 15). Conversely,

Lemma 16. — If a degree-$p$ cyclic extension $L$ of $K$ is galoisian over $F$, then there is a degree-$p$ solvable extension $E|F$ such that $L = EK$; two such extensions $E, E'$ give rise to the same $L$ if and only if they are conjugate over $F$, and every conjugate of $E$ is contained in $L$.

This follows from the Schur-Zassenhaus theorem (lemma 3) exactly in the same way as lemma 14 does (cf. Remark 7).

But the great thing about degree-$p$ cyclic extensions $L|K$ is that they correspond bijectively to lines $D$ in the $F_p$-space $K^\times/K^{\times p}$ in case the characteristic of $F$ is $\neq p$ because $K^\times$ contains a primitive $p$-th root of 1, and in the space $K^+/\varphi(K^+)$ in the characteristic-$p$ case. When is the (degree-$p$, cyclic) extension of $K$ corresponding to $D$ galoisian over $F$? Precisely when $D$ is stable under the $G$-action on these spaces, where $G = \text{Gal}(K|F)$.

Lemma 17. — Let $D$ be a line in $K^\times/K^{\times p}$ (resp. $K^+/\varphi(K^+)$) and let $L = K(\sqrt[p]{D})$ (resp. $L = K(\varphi^{-1}(D))$) be the corresponding cyclic extension of degree $p$. Then $L|F$ is galoisian if and only if $D$ is $G$-stable. If $G$ acts on $D$ via the character $\chi : G \to F_p^\times$, then it acts on $P = \text{Gal}(L|K)$ via the character $\omega \chi^{-1}$, where $\omega : G \to F_p^\times$ is the cyclotomic character if $\text{car}(F) \neq p$ and $\omega = 1$ if $\text{car}(F) = p$.

Note first of all that when $L|F$ is galoisian of group $\Gamma = \text{Gal}(L|F)$, we have a short exact sequence $1 \to P \to \Gamma \to G \to 1$ which provides an action of $G$ on $P$ by conjugation and hence a character $\xi : G \to F_p^\times$. It is being asserted that $\xi = \omega \chi^{-1}$. Note also the corollary that $L|F$ is abelian if and only if $G$ acts on $D$ via $\omega$.

We have $D = \ker(\iota : K^\times/K^{\times p} \to L^\times/L^{\times p})$ in characteristic $\neq p$ and $D = \ker(\iota : K^+/\varphi(K^+) \to L^+/\varphi(L^+))$ in characteristic $p$, where $\iota$ is induced by the inclusion of $K$ in $L$.

If $L|F$ is galoisian of group $\Gamma$, then $\gamma \circ \iota = \iota \circ \gamma$ for every $\gamma \in \Gamma$, which shows that $D = \gamma(D)$ and hence $D$ is $G$-stable. Conversely, if $D$ is $G$-stable, then $g(a) \in D$ for every $a \in D$ and every $g \in G$. Therefore $L$ contains a
Suppose now that \( L/F \) is galoisian and that \( G \) acts on \( D \) via \( \chi \). Let \( 1 \to P \to \Gamma \to G \to 1 \) be the associated short exact sequence of groups. Let us first show the final assertion in characteristic \( \neq p \). Write \( L = K(x) \) for some \( x \in L^\times \) such that \( a = x^p \) is in \( K^\times \) and \( \bar{a} \in D \) is a generator. Also choose a generator \( \sigma \in P \) and denote by \( \zeta \in \mu_p \) the \( p \)-th root of 1 such that \( \sigma(x) = \zeta x \); we have to show that \( \gamma \sigma \gamma^{-1} = \sigma^{\chi^{-1}(\bar{\gamma})} \) for every \( \gamma \in \Gamma \) of image \( \bar{\gamma} \in G \), for which it is sufficient to show that \( \gamma \sigma \gamma^{-1}(x) = \zeta^{\omega \chi^{-1}(\bar{\gamma})} x \).

All that is given to us is that \( \bar{\gamma}(\bar{a}) = \bar{a}^{\chi(\bar{\gamma})} \) for every \( \gamma \in \Gamma \). As \( \gamma(x)^p = \gamma(x^p) = \gamma(a) \), we must have \( \gamma(x) = b_\gamma x^{\chi(\bar{\gamma})} \) for some \( b_\gamma \in K^\times \) with \( \gamma^{-1}(b_\gamma) b_\gamma^{\chi(\bar{\gamma})} = 1 \) (modulo \( K^{\times p} \)), to ensure that \( \gamma^{-1} \gamma(x) = x \). Now,

\[
\gamma \sigma \gamma^{-1}(x) = \gamma \sigma(b_{\gamma^{-1}} x^{\chi^{-1}(\bar{\gamma})}) \\
= \gamma(b_{\gamma^{-1}} \zeta^{\chi^{-1}(\bar{\gamma})} x^{\chi^{-1}(\bar{\gamma})}) \\
= \gamma(b_{\gamma^{-1}}) \zeta^{\omega \chi^{-1}(\bar{\gamma})} b_{\gamma^{-1}}^{\chi^{-1}(\bar{\gamma})} x \\
= \zeta^{\omega \chi^{-1}(\bar{\gamma})} x,
\]

at least modulo \( K^{\times p} \), proving the result in this case. A similar argument works in characteristic \( p \) upon replacing the multiplicative notation with the additive notation.

Let’s do the transcription. Write \( L = K(x) \) for some \( x \in L \) such that \( a = x^p - x \) is in \( K \) and \( \bar{a} \in D \) is a generator. Let \( \sigma \in P \) be the generator such that \( \sigma(x) = x + 1 \); we have to show that \( \gamma \sigma \gamma^{-1} = \sigma^{\chi^{-1}(\bar{\gamma})} \) for every \( \gamma \in \Gamma \), for which it is sufficient to show that \( \gamma \sigma \gamma^{-1}(x) = x + \chi^{-1}(\bar{\gamma}) \).

We are given that \( \bar{\gamma}(\bar{a}) = \chi(\bar{\gamma}) \bar{a} \). As \( \varphi(\gamma(x)) = \gamma(x)^p - \gamma(x) = \gamma(a) \), we must have \( \gamma(x) = \chi(\bar{\gamma}) x + b_\gamma \) for some \( b_\gamma \in K \), with \( \gamma^{-1}(b_\gamma) + \chi(\bar{\gamma}) b_{\gamma^{-1}} = 0 \) (modulo \( \varphi(K) \)) in order to ensure that \( \gamma^{-1} \gamma(x) = x \). Now, at least modulo \( \varphi(K) \), we have

\[
\gamma \sigma \gamma^{-1}(x) = \gamma(\chi(\bar{\gamma}) x + b_{\gamma^{-1}}) \\
= \gamma(\chi^{-1}(\bar{\gamma}) x + \chi(\bar{\gamma}) + b_{\gamma^{-1}}) \\
= x + \chi^{-1}(\bar{\gamma}) b_\gamma + \chi^{-1}(\bar{\gamma}) + \gamma(b_{\gamma^{-1}}) \\
= x + \chi^{-1}(\bar{\gamma})
\]

We have seen that solvable degree-\( p \) extensions \( E \) of \( F \) give rise to degree-\( p \) cyclic extensions \( L \) of \( K \) which are galoisian over \( F \) (lemma 15) or equivalently to a \( G \)-stable line \( D \) in \( K^\times/K^{\times p} \) or \( K^+/\varphi(K^+) \) (lemma 17), making it galoisian over \( F \).
but \( E \) is not determined by \( D \) or \( L \) unless \( E|F \) is cyclic (lemma 12); it is determined only up to \( F \)-conjugacy.

But it should be possible to determine from \( L \) or \( D \) invariants of \( E \) which depend only on the \( F \)-conjugacy class of \( E \). We shall see some examples in the next §; here we shall see how to recover the galoisian closure \( \tilde{E} \) of \( E \).

Recall (lemma 12) that \( \tilde{E} \) is a degree-\( p \) cyclic extension of some cyclic extension \( F' \) of \( F \) of degree dividing \( p-1 \). Which degree-\( p \) cyclic extensions of \( K \) come from some degree-\( p \) cyclic extension of a given \( F' \)?

**Lemma 18.** — *Let \( F' \) be an extension of \( F \) in \( K \) and \( G' = \text{Gal}(K|F') \). A degree-\( p \) cyclic extension \( L \) of \( K \) of group \( P = \text{Gal}(L|K) \) comes from a degree-\( p \) cyclic extension \( E' \) of \( F' \) if and only if \( L \) is galoisian over \( F' \) and the resulting action of \( G' \) on \( P \) by conjugation is trivial.*

In terms of the line \( D \) corresponding to \( L \), the condition is that \( D \) be \( G' \)-stable and that \( G' \) should act on \( D \) via the cyclotomic character \( \omega \) (lemma 17). The proof is similar to that of lemma 16 (cf. Remark 4 and [7], §2).

Now let \( D \) be in the \( \chi \)-eigenspace for some character \( \chi \) of \( G \). For \( L = K(\sqrt[p]{D}) \) (in characteristic \( \neq p \)) or \( L = K(\varphi^{-1}(D)) \) (in characteristic \( p \)) to come from a degree-\( p \) cyclic extension of \( F' \), a necessary condition is that \( \chi|_{G'} = \omega \) (lemmas 17–18), so the smallest \( F' \) which would do is \( F_\chi = K^{G_\chi} \), where \( G_\chi = \text{Ker}(\omega \chi^{-1}) \).

Define the short exact sequence \( 1 \to P \to \Gamma_\chi \to G_\chi \to 1 \) by restricting \( 1 \to P \to \Gamma \to G \to 1 \) along the inclusion \( G_\chi \to G \). As the action of \( G_\chi \) on \( P \) is trivial (and their orders mutually prime), there is a canonical section \( G_\chi \to \Gamma_\chi \) (Remark 4) using which we identify \( G_\chi \) with a subgroup of \( \Gamma \). Here is the diagram

\[
\begin{array}{cccccc}
1 & \to & P & \longrightarrow & \Gamma & \longrightarrow & G & \to & 0 \\
& = & \uparrow & \uparrow & \uparrow & \subset & & & \\
1 & \to & P & \longrightarrow & \Gamma_\chi & \longrightarrow & G_\chi & \to & 0.
\end{array}
\]

With the identification \( G_\chi \subset \Gamma \) we have \( EF_\chi = L^{G_\chi} \), so we have proved :

**Lemma 19.** — *Let \( E \) be a degree-\( p \) solvable extension of \( F \) and \( \chi \) the character through which \( G \) acts on the corresponding line \( D \). Then \( L^{G_\chi} \) is the galoisian closure of \( E \), and we have \( L^{G_\chi} = EF_\chi \).*
In summary, we have the following commutative diagram of fields:

\[
\begin{array}{cccccc}
E & \overset{I_\chi}{\longrightarrow} & EF & \overset{G_\chi}{\longrightarrow} & L = EK & G_\chi = \text{Ker}(\omega \chi^{-1}) \\
\uparrow & & \uparrow P & & \uparrow P & \quad P = \text{Gal}(L|K) \\
F & \overset{I_\chi}{\longrightarrow} & F_\chi & \overset{G_\chi}{\longrightarrow} & K & I_\chi = \text{Im}(\omega \chi^{-1});
\end{array}
\]

in which the names above the arrows are not maps but the relative automorphism groups (in addition to \(\Gamma = \text{Gal}(L|F), \Gamma_\chi = \text{Gal}(L|F_\chi)\) and \(G = \text{Gal}(K|F)\)); the extension \(E|F\) is not galoisian unless \(I_\chi = \{1\}\), which happens precisely when \(\chi = \omega\).

**Lemma 20.** — The compositum \(C\) of all degree-\(p\) solvable extensions of \(F\) is the maximal abelian extension \(M'\) of exponent dividing \(p\) of \(K'\), where \(K'\) is the compositum of all \(F_\chi\) such that the \(\chi\)-eigenspace \(K^*/K^{\times p}(\chi)\) or \(K^+/\varphi(K^+)(\chi)\) (respectively) is \(\neq \{1\}\) or \(\neq \{0\}\), where \(\chi : G \to \mathbf{F}_p^\times\) runs through characters of \(G = \text{Gal}(K|F)\).

It is clear that \(F_\chi \subset C\) for every \(\chi : G \to \mathbf{F}_p^\times\) such that the \(\chi\)-eigenspace is \(\neq \{1\}\) or \(\neq \{0\}\) (lemma 19). By definition, \(K'\) is the compositum of all such \(F_\chi\), so \(K' \subset C\). It is also clear that that \(C \subset M'\) (cf. lemma 15). We may thus ask for the subspace \(G_{M'|C} = \text{Gal}(M'|C)\) of the \(F_p\)-space \(G_{M'|K'} = \text{Gal}(M'|K')\). Here is the picture of the various fields and their relative automorphism groups:

\[
\begin{array}{cccccc}
F & \overset{G'}{\longrightarrow} & F_\chi & \overset{G_{M'|C}}{\longrightarrow} & K' & \overset{G_{M'|K'}}{\longrightarrow} & M', & \overset{G'}{\longrightarrow} & K.
\end{array}
\]

Notice that, as each \(F_\chi|F\) is galoisian, so is \(K'|F\). By maximality, so is \(M'|F\). Now, \(C\) is the compositum of all degree-\(p\) cyclic extensions \(L\) of \(K'\) (in \(M'\)) which are galoisian over \(F\); the set of such \(L\) is in natural bijection \(L = M'^H\) with the set of \(G'\)-stable hyperplanes \(H \subset G_{M'|K'}\), where the action of \(G'\) on \(G_{M'|K'}\) comes from the short exact sequence

\[1 \to G_{M'|K'} \to \text{Gal}(M'|F) \to G' \to 1,\]

which itself comes from the galoisian tower \(M'|K'|F\). So the subspace in question \(G_{M'|C} \subset G_{M'|K'}\) is the intersection of all \(G'\)-stable hyperplanes in \(G_{M'|K'}\). But this intersection is trivial (lemma 8), so \(G_{M'|C} = \{1\}\) and hence \(C = M'\).
Examples 21. — When the field $F$ is finite, we have $K' = F$. Indeed, the only character $\chi$ of $G$ for which $K^\times/K^{\times p}(\chi)$ (resp. $K^+/(\varphi(K^+))(\chi)$) is $\neq \{1\}$ (resp. $\neq \{0\}$) is $\chi = \omega$ in characteristic $\neq p$ (resp. the trivial character $\chi = 1$ in characteristic $p$, because the trace gives an isomorphism $K^+/(\varphi(K^+)) = \mathbb{F}_p$), and we have $F_\chi = F$ in both cases. When $F$ is a local field of residual characteristic $p$, we have $K' = K = F(\sqrt[p]{1})$ and $C = K(\sqrt[p]{K^\times})$ if $\text{car}(F) = 0$ and $C = K(\sqrt[p]{K})$ if $\text{car}(F) = p$ (prop. 36). When $F$ is a local field of residual characteristic $\neq p$, we have $K' = F(\sqrt[p]{1})$ and $C = K'(\sqrt[p]{1})$ ($\S 8$).

Summary of the algebraic ingredients. — We extract from the foregoing what is relevant for the following $\S\S$. We have a prime $p$, a field $F$, and the maximal abelian extension $K$ of $F$ of exponent dividing $p - 1$; if the characteristic of $F$ is $\neq p$, then $K$ contains a primitive $p$-th root of 1. Write $G = \text{Gal}(K/F)$.

Solvable extensions $E$ of $F$ of degree $p$ give rise to $G$-stable lines $D$ in $K^\times/K^{\times p}$ or $K^+/(\varphi(K^+))$, and every $G$-stable line $D$ arises from some $E$. Two such extensions $E$, $E'$ give rise to the same $D$ if and only if they are $F$-conjugate; each $E$ has exactly $p$ conjugates, unless $E|F$ is cyclic. Invariants of $E$ which depend only on its $F$-conjugacy class (such as the galoisian closure $\tilde{E}$) can be recovered from the $G$-module $D$. For example, $\tilde{E} = L^G_{\chi}$, where $\chi : G \to \mathbb{F}_p^\times$ is the character through which $G$ acts on $D$, $L = K(\sqrt[p]{D})$ or $L = K(\varphi^{-1}(D))$, and $G_\chi = \text{Ker}(\omega \chi^{-1})$ has been identified with a subgroup of $\Gamma = \text{Gal}(L|F)$. Also, $E|F$ is cyclic if and only if $\chi = \omega$, and the number of $F$-conjugates of $E$ is $p$ if $\chi \neq \omega$.

5. The case of local fields. — We now return definitively to our local field $F$ with finite residue field $k$ of characteristic $p$ and cardinality $q = p^f$, and consider degree-$p$ separable extensions of $F$. (In $\S 8$, we discuss extensions of $F$ of prime degree $p' \neq p$.) Let us record the special features of this case in a series of remarks, of which 23–25 summarise the essential content of [9]. We denote the normalised valuation of $F$ by $v : F^\times \to \mathbb{Z}$.

Remark 22. — Every degree-$p$ separable extension $E$ of $F$ is solvable. Indeed, the group $\Gamma = \Gamma_{-1}$ of the galoisian closure $\tilde{E}|F$ comes equipped with the ramification filtration $(\Gamma_i)_{i \in \mathbb{N}}$ (in the lower numbering) the successive quotients of which are commutative.

Remark 23. — It is simpler to show that $\tilde{E}$ is a degree-$p$ cyclic extension of a cyclic extension $F'$ of $F$ of degree dividing $p - 1$; we don’t need to invoke lemma 2, as lemma 1 suffices. Indeed, we may assume that $E|F$ is ramified; the ramification subgroup $\Gamma_1$ then has order $> 1$, for otherwise $\tilde{E}|F$ would be tamely ramified whereas the ramification index of $E|F$ is $p$. As the order
of $\mathfrak{S}_p$ is not divisible by $p^2$, $\Gamma_1$ must have order $p$. As the subgroup $\Gamma_1 \subset \Gamma$ is normal, the quotient $\Gamma/\Gamma_1$ is a subgroup of $F_p^\times$ (lemma 1).

**Remark 24.** — When $E|F$ is ramified, the unique ramification break $b$ of $\Gamma_1 = \text{Gal}(\tilde{E}|F')$ (the integer $b$ such that $\Gamma_b = \Gamma_1$, $\Gamma_{b+1} = \{1\}$), and the order $pt$ of the inertia group $\Gamma_0$ determine the valuation $v(\delta_{E|F})$ of the discriminant of $E|F$ by computing $v(\delta_{E|F})$ in two different ways, using the Schachtelungssatz along the two towers $\tilde{E}|F'|F$, $\tilde{E}|E|F$. The situation is summarised in the following diagram in which the ramification indices (resp. residual degrees) are indicated outside (resp. inside) the square

$$
\begin{array}{c c c}
E & \xrightarrow{t} & \tilde{E} \\
p & \uparrow 1 & 1 \uparrow p \\
F & \xrightarrow{r} & F'.
\end{array}
$$

Indeed, $v_{F'}(\delta_{\tilde{E}|F'}) = (p-1)(1+b)$. The extension $F'|F$ is tamely ramified of group $\Gamma/\Gamma_1$ of some order $tr$ and inertia subgroup $\Gamma_0/\Gamma_1$ of order $t$, so the ramification index is $t$ and the residual degree is $r$, whence $v(\delta_{F'|F}) = (t-1)r$. The extension $\tilde{E}|E$ has the same ramification index and residual degree as $F'|F$, so $v_E(\delta_{\tilde{E}|E}) = (t-1)r$. This leads to the equality

$$(p-1)(1+b)r + (t-1)r.p = (t-1)r + v(\delta_{E|F}).tr,$$

which leads to $v(\delta_{E|F}) = (p-1)(b+t)/t$. A similar double application of the Schachtelungssatz is needed later (prop. 37).

**Remark 25.** — We are in the presence of two natural embeddings $\iota, \theta_0 : \Gamma_0/\Gamma_1 \to F_p^\times$. The first one comes from the conjugation action $\sigma \tau^{-1} = \sigma^{\iota(\tau)}$ of $\Gamma_0$ on $\Gamma_1$ (and the identification $F_p^\times = \text{Aut} \Gamma_1$). The second one comes from the galoisian action $\tau(\pi) = \theta_0(\tau)\pi$ of $\Gamma_0/\Gamma_1$ on the $t$-th roots $\pi$ of a uniformiser $\varpi$ of the maximal unramified extension $F'_0$ of $F$ in $F'$ such that $\varpi \in F'^\times t$ (and the identification $F_p^\times \subset F'_0^\times$ with the group of $(p-1)$-th roots of $1$): $\theta_0$ is independent of the choice of $\varpi$. We also have an embedding $\theta_b : \Gamma_b/\Gamma_{b+1} \to U_b/U_{b+1}$ (where $U_i (i > 0)$ is the group of principal units of $F'$ of level at least $i$). But $\Gamma_b = \Gamma_1$ and $\Gamma_{b+1} = \{1\}$, so $\theta_b$ is an embedding of $\Gamma_1$. Now, the compatibility relation $\theta_b(\tau\sigma\tau^{-1}) = \theta_b(\sigma)^{\theta_0(\tau)b}$ for every $\sigma \in \Gamma_1$ and every $\tau \in \Gamma_0$ [14, IV, prop. 9] implies that $\iota(\tau) = \theta_0(\tau)b$ for every $\tau \in \Gamma_0/\Gamma_1$. In other words, the two embeddings $\iota, \theta_0$ differ by the automorphism $( )^b$ of $\Gamma_0/\Gamma_1$. In particular, $\gcd(b,t) = 1$ (where $t$ is the order of $\Gamma_0/\Gamma_1$).
Remark 26. — In principle, it should now be possible to prove Serre’s degree-$p$ mass formula by computing the contribution of each such $F'$; when $F' = F$, then $E|F$ is cyclic, and the contribution of these has been computed in [7, prop. 14–16]. The number of $F'$ can be deduced from [10, Kap. 16] or [5, Lecture 18], and equals the number of cyclic subgroups of $F^\times/F^{\times p−1}$.

Remark 27. — The maximal abelian extension $K$ of $F$ of exponent dividing $p − 1$ equals $F(√−p/F^\times)$. Indeed, $F$ contains a primitive $(p − 1)$-th root of 1. The group $G = Gal(K|F)$ is dual to $F^\times/F^{\times p−1}$ under the pairing

$G \times (F^\times/F^{\times p−1}) \to F_p^\times \quad (\sigma, \bar{x}) \mapsto \frac{\sigma(y)}{y} \quad (y^{p−1} = x)$

in which $F_p^\times \subset F^\times$ has been identified with the group of $(p − 1)$-th roots of 1.

In the characteristic-0 case, denote by $\omega : G \to F_p^\times$ the cyclotomic character giving the action of $G$ on the $p$-th roots of 1; in characteristic $p$, let $\omega = 1$ be the trivial character. Note that $K$ is a finite extension of $F$ of ramification index and residual degree $p − 1$. Denote its ring of integers by $\mathfrak{o}$, the unique maximal ideal of $\mathfrak{o}$ by $\mathfrak{p}$, and the residue field by $l = \mathfrak{o}/\mathfrak{p}$. Finally, let $U_n = 1 + p^n$ for $n > 0$.

Remark 28. — In characteristic 0, the character $\omega$ corresponds to the class $-\bar{p} \in F^\times/F^{\times p−1}$ under biduality. In other words, we have to show that $\sigma(y)/y \equiv \omega(\sigma) \pmod{p}$ for every $\sigma \in G$ and every $(p − 1)$-th root $y \in K^\times$ of $−p$. Let $\zeta \in K^\times$ be a primitive $p$-th root of 1, so that $\sigma(\zeta) = \zeta^{\omega(\sigma)}$ for every $\sigma \in G$. We may take $y = \eta\pi$, where $\pi = 1 − \zeta$ and $\eta \in U_1$; we have $\sigma(\pi)/\pi \equiv \omega(\sigma) \pmod{p}$, from which the claim follows. Cf. [4, prop. 25].

Remark finally that the space $K^\times/K^{\times p}$ or $K^+/\wp(K^+)$ carries a natural filtration, and that the discriminant of a degree-$p$ separable extension $E$ of $F$ can be computed from the “level” of the corresponding line $D \subset K^\times/K^{\times p}$ (resp. $D \subset K^+/\wp(K^+)$) in this filtration. See the next § for a description of the filtration and the definition of the “level” of a line, and prop. 37 for the computation.

6. Filtered galoisian modules. — We keep the notation $F, K, G$ from §5. We have seen (§4) that the $G$-modules $K^\times/K^{\times p}$ or $K^+/\wp(K^+)$ (respectively) play an important role in the study of degree-$p$ separable extensions of $F$. These modules come with a natural filtration, which we discuss next.

Denote by $(\bar{U}_n)_{n>0}$ the filtration on $\bar{U}_0 = K^\times/K^{\times p}$ by units of various levels. Similarly, in the characteristic-$p$ case, let $\overline{p^n}$ be the image of $p^n$ in
the $\mathbb{F}_p$-space $K^+ / \varphi(K^+)$, where $\mathfrak{p}$ is the unique maximal ideal of the ring of integers $\mathfrak{o}$ of $K$. For some background on these $\mathbb{F}_p$-space (without the $\mathbb{G}$-action), see [4] in characteristic $0$ and [6] in characteristic $p$.

Examples of $\mathbb{G}$-stable lines are provided, in the characteristic-$0$ case, by $\bar{U}_{pe}$ and $\bar{\mu}$ (the image of the torsion subgroup $\mu \subset \mathfrak{o}^\times$), on both of which $\mathbb{G}$ acts via the cyclotomic character $\omega$. In the characteristic-$p$ case, the line $\bar{\mathfrak{o}} = \mathbb{F}_p$ is $\mathbb{G}$-stable and the action of $\mathbb{G}$ is in fact trivial.

In general, the subspaces $\bar{U}_i$ for $i \in [0, pe]$ in characteristic $0$ (resp. $\bar{p}^i$ for $i \in -N$ in characteristic $p$) are $\mathbb{G}$-stable, essentially because there is a unique extension of the valuation from $F$ to $K$. We have seen that $\bar{U}_{pi+1} = \bar{U}_{pi}$ except for $i = 0, e$ ([4, prop. 42]) and that $\bar{p}^{pi+1} = \bar{p}^{pi}$ except for $i = 0$ ([6, prop. 11]), respectively.

The codimension is $1$ in the three exceptional cases, namely $\bar{U}_1 \subset \bar{U}_0$ and $\bar{U}_{pe+1} \subset \bar{U}_{pe}$ in characteristic $0$, and $\bar{p} \subset \bar{\mathfrak{o}}$ in characteristic $p$. In characteristic $0$, we have $\bar{U}_{pe+1} = \{1\}$ and $\bar{U}_{pe}$ is a stable line on which $\mathbb{G}$ acts via $\omega$, and the valuation $v_K$ provides an isomorphism $\bar{U}_0 / \bar{U}_1 \rightarrow \mathbb{Z} / p\mathbb{Z}$. In characteristic $p$, we have $\bar{p} = 0$ and the trace $S_l|_{\mathbb{F}_p}$ induces an isomorphism $\bar{\mathfrak{o}} \rightarrow \mathbb{F}_p$, where $l$ is the residue field of $K$.

For $i \in [1, pe]$ prime to $p$ (in characteristic $0$) or $i < 0$ prime to $p$ (in characteristic $p$), the codimension equals the absolute degree $[l : \mathbb{F}_p]$, and indeed the quotients are canonically isomorphic to $U_i / U_{i+1}$ (resp. $p^i / p^{i+1}$) ([4, prop. 42], [6, prop. 11]). Thus they are not merely $\mathbb{F}_p$-spaces but $k$-spaces (of $k$-dimension $p - 1$). The pictures in [7], §5 summarise some of these facts.

**Proposition 29.** — For $i \in [1, pe]$ prime to $p$ (in characteristic $0$) or for $i < 0$ prime to $p$ (in characteristic $p$), the natural maps

$$\bar{U}_i / \bar{U}_{i+1} \rightarrow U_i / U_{i+1} \rightarrow p^i / p^{i+1}, \quad (\text{resp. } \bar{p}^i / p^{i+1} \rightarrow p^i / p^{i+1})$$

of $k$-spaces are $\mathbb{G}$-equivariant $k$-isomorphisms.

*Beweis:* Klar.

So we need to study the $\mathbb{G}$-modules $p^i / p^{i+1}$, for which a preliminary study of the $\mathbb{G}$-module $l$ is useful.

**Proposition 30.** — Let $k$ be a finite field, $q = \text{Card} k$, and $l|k$ any extension of degree dividing $q - 1$. For every character $\chi : \text{Gal}(l|k) \rightarrow k^\times$, the $\chi$-eigenspace $l(\chi)$ is a $k$-line in $l$.

Although this is a special case of lemma 9, we give a short direct proof. Let $\varphi$ (Frobenius) be the canonical generator $x \mapsto x^q$ of $\text{Gal}(l|k)$, and put
a = \chi(\varphi); let m = [l : k] be the order of \varphi, so that the order of a \in k^\times divides m. The \chi-eigenspace consists of all x \in l such that \varphi(x) = ax; such x are roots of the binomial Tq - aT = T(Tq-1 - a), which has at most q roots. As the map x \mapsto x^q - ax defined by this binomial is a linear endomorphism of the k-space l, it is sufficient to prove that a has a (q-1)-th root in l.

We have said that the order of a in k^\times/k^\times q-1 = k^\times divides m. Therefore the degree of the extension k(q^{-1}\sqrt[n]{a})|k divides m, and hence k(q^{-1}\sqrt[n]{a}) \subset l. This shows that the k-endomorphism x \mapsto x^q - ax of l is not injective, and hence its kernel l(\chi) is a k-line in l. Incidentally, if \chi is trivial, then a = 1 and l(\chi) = k.

Momentarily let K be any galoisian extension of F of group G, and suppose that there is a uniformiser \pi of K such that \varpi = \pi^s is in F for some s > 0.

**Proposition 31.** — For every integer i \in \mathbb{Z}, “multiplication by \varpi” gives an isomorphism \mathfrak{p}^i/\mathfrak{p}^{i+1} \rightarrow \mathfrak{p}^{i+s}/\mathfrak{p}^{i+s+1} of k[G]-modules.

More precisely, the reduction modulo \mathfrak{p} of the \sigma-linear isomorphism x \mapsto \varpi x : \mathfrak{p}^i \rightarrow \mathfrak{p}^{i+s} is G-equivariant. But this is clearly the case : \sigma(\varpi x) = \varpi \sigma(x) for every \sigma \in G, because \varpi, F, and \sigma are F-linear.

Let us now return to our K = F(p^{-\sqrt[p]{\mathbb{F}}}p) and G = Hom(F^x/F^x p^{-1}, F^x p), so that the group of characters of G is Hom(G, F^x p) = F^x/F^x p^{-1}. The cyclotomic character \omega corresponds to \mathfrak{p} in characteristic 0, and \omega = 1 by convention in characteristic p). Each character \chi therefore has a “valuation” \bar{\chi} = \mathbb{Z}/(p - 1)\mathbb{Z}, coming from the valuation v : F^x \rightarrow \mathbb{Z}.

Unramified characters — those which are trivial on the inertia subgroup G_0 — are the same as characters of the quotient G/G_0 = Gal(l/k). They get identified with the kernel \sigma_F^\times/\sigma_F^x p^{-1} = k^\times/k^\times p^{-1} of \bar{v}. Indeed, the short exact sequence 1 \rightarrow G_0 \rightarrow G \rightarrow G/G_0 \rightarrow 1 gives rise, upon taking duals ( )^\vee = Hom( , F^x p), to a short exact sequence which fits into the commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & (G/G_0)^\vee & \rightarrow & G^\vee & \rightarrow & G_0^\vee & \rightarrow & 0 \\
\sim & \downarrow & \sim & \downarrow & \sim & \\
1 & \rightarrow & k^\times/k^\times p^{-1} & \rightarrow & F^\times/F^x p^{-1} & \stackrel{\bar{v}}{\rightarrow} & \mathbb{Z}/(p - 1)\mathbb{Z} & \rightarrow & 0.
\end{array}
$$

Notice that the map \bar{x} \mapsto x^{(a-1)/(p-1)} (x \in k^\times) identifies k^\times/k^\times p^{-1} with
In characteristic 0, the cyclotomic character \( \omega \) is unramified if and only if \( p - 1 \mid e \), for \( \omega \) corresponds to \(-p\) and \( \bar{\omega}(-p) \equiv e \pmod{p - 1} \).

Let us decompose the \( k[G] \)-modules \( p^i/p^{i+1} \) as an internal direct sum of \( \chi \)-eigenspaces for various \( \chi : G \to \mathbf{F}_p^\times \). For \( i = 0 \), the \( k[G] \)-module \( p^i/p^{i+1} \) is in fact the \( k[G/G_0] \)-module \( l \). We have seen that for every unramified character \( \chi \) of \( G \), the \( \chi \)-eigenspace \( l(\chi) \) is a \( k \)-line (lemma 9 or prop. 30). It follows that for every ramified character \( \chi \), we have \( l(\chi) = 0 \); there is room only for so many, and unramified characters have used it all up.

Let us provide the details of the notion of twisting a \( k[G] \)-module \( m \) by a character \( \xi : G \to \mathbf{F}_p^\times \). Denote by \( m\{\xi\} \) the \( k[G] \)-module whose underlying \( k \)-space is \( m \), but the new action \( \star_\xi \) is defined by \( \sigma \star_\xi x = \xi(\sigma)\sigma(x) \) for every \( \sigma \in G \) and every \( x \in m \), so that if \( \xi = 1 \) is the trivial character, then \( \sigma \star_1 x = \sigma(x) \) and \( m\{1\} \) is the \( k[G] \)-module \( m \). It is clear that \( m\{\xi_1\xi_2\} = m\{\xi_1\}\{\xi_2\} \) for any two characters \( \xi_1, \xi_2 \) of \( G \), and that \( (m_1 + m_2)\{\xi\} = m_1\{\xi\} + m_2\{\xi\} \) for any two \( k[G] \)-modules \( m_1, m_2 \). In this process, the \( \chi \)-eigenspace of \( m \) gets converted into the \( \xi\chi \)-eigenspace of \( m\{\xi\} \), for every character \( \chi \) of \( G \). The same discussion applies to \( \mathbf{F}_p[G] \)-modules.

It is easy to see that \( l\{\xi\} \) is \( k[G] \)-isomorphic to \( l \) for every unramified character \( \xi \) of \( G \). Indeed, by prop. 30, \( l \) is the direct sum of the \( k \)-lines \( l(\chi) \) indexed by the unramified characters \( \chi \in \text{Hom}(G/G_0, \mathbf{F}_p^\times) \) of \( G \), so \( l\{\xi\} \) is the direct sum of the \( k \)-lines \( l(\xi\chi) \). But as \( \chi \) runs through \( \text{Hom}(G/G_0, \mathbf{F}_p^\times) \), so does \( \xi\chi \), and the two direct sums are \( k[G] \)-isomorphic.

It follows that \( l\{\xi\} \) depends only on \( \bar{\xi}(\xi) \in \mathbf{Z}/(p - 1)\mathbf{Z} \) (up to \( k[G] \)-isomorphism). We denote by \( l[i] \) (for \( i \in \mathbf{Z} \)) the \( k[G] \)-modules \( l\{\xi\} \) for any \( \xi \) such that \( \bar{\xi}(\xi) \equiv i \pmod{p - 1} \), so that \( l[0] = l \).

**Proposition 32.** For every \( i \in \mathbf{Z} \) and every character \( \chi : G \to \mathbf{F}_p^\times \), the \( \chi \)-eigenspace in the \( k[G] \)-module \( l[i] \) is a \( k \)-line if \( \bar{\xi}(\chi) \equiv i \pmod{p - 1} \); it is reduced to 0 otherwise.

This is just prop. 30 in the case \( i = 0 \), and the general case follows from this by our discussion of twisting. An immediate consequence is the following result.

**Proposition 33.** For \( i \in \mathbf{Z} \), the \( k[G] \)-module \( p^i/p^{i+1} \) is isomorphic to \( l[i] \).

Choose any uniformiser \( \pi \) of \( K \) such that \( \pi^{p-1} \) is (a uniformiser) in \( F \); this is possible. It is easy to see that, by taking \( \pi^i \) as an \( \sigma \)-basis of \( p^i \), the resulting \( k \)-linear map \( p^i/p^{i+1} \to l\{\xi^i\} \) is an isomorphism of \( k[G] \)-modules, where \( \xi \) is the character such that \( \sigma(\pi) = \xi(\sigma)\pi \) for every \( \sigma \in G \). As we
have $\bar{v}(\xi) \equiv 1$, this shows that $\mathfrak{p}^i/\mathfrak{p}^{i+1}$ is $k[G]$-isomorphic to $l[i]$.

**Corollary 34.** — Put $W_1 = \mathfrak{p}^{-1}/\mathfrak{p}^0 \oplus \mathfrak{p}^{-2}/\mathfrak{p}^{-1} \oplus \cdots \oplus \mathfrak{p}^{-(p-1)}/\mathfrak{p}^{-(p-1)+1}$. For every $\chi : G \to \mathbf{F}_p^\times$, the $\chi$-eigenspace in the $k[G]$-module $W_1$ is a $k$-line.

In other words, and thanks to prop. 33, $W_1 = l[-1] \oplus \cdots \oplus l[-(p-1)]$. The point is that we have endowed $W_1$ with the filtration for which the successive quotients are, for example when $p = 5$,

$$l[-1], l[-2], l[-3], l[-4],$$

in that specific order, rather than in some other order. If we twist it by the cyclotomic character (which we will soon need to) to get $W_1\{\omega\}$, and if $\omega$ has “valuation” $\bar{v}(\omega) \equiv 1 \pmod{4}$ (for example when $F|\mathbb{Q}_5$ is unramified), then the successive quotients of the filtered $k[G]$-module $W_1\{\omega\}$ are

$$l[-4], l[-1], l[-2], l[-3],$$

which are circularly shifted one step to the right (or three steps to the left, which comes to the same because $1 \equiv -3 \pmod{4}$). We need to keep track of both the filtration and the $G$-action. There is no difference when $\omega$ is unramified, for then the shift is by 0 steps.

In a similar vein, define $W_2 = \mathfrak{p}^{-(p+1)}/\mathfrak{p}^p \oplus \cdots \oplus \mathfrak{p}^{-(p-1)}/\mathfrak{p}^{-(p-1)+1}$, and think of this $k[G]$-module as being endowed with the filtration for which the successive quotients are $l[-(p+1)], \ldots, l[-(2p-1)]$ (cf. prop. 33), in that specific order, so that when $p = 5$, they are

$$l[-2], l[-3], l[-4], l[-1].$$

As in prop. 34, we see that the $\chi$-eigenspace in the $k[G]$-module $W_2$ is a $k$-line for every $\chi : G \to \mathbf{F}_p^\times$.

For every $i \in \mathbb{N}$, put $W_{i+1} = l[-(ip + 1)] \oplus \cdots \oplus l[-(ip + p - 1)]$. These $k[G]$-modules are all isomorphic to each other (and free of rank 1, to boot) because the $\chi$-eigenspace in each $W_{i+1}$ is a $k$-line for every character $\chi : G \to \mathbf{F}_p^\times$, but they carry different filtrations. The filtered $k[G]$-modules $W_m, W_n$ are isomorphic if and only if $m \equiv n \pmod{p-1}$.

Get back to the $k[G]$-modules $\bar{U}_i/\bar{U}_{i+1}$ (resp. $\bar{p}^i/\bar{p}^{i+1}$) for appropriate $i$.

**Proposition 35.** — The $k[G]$-module $\bar{U}_i/\bar{U}_{i+1}$ for $0 < i < p$ prime to $p$ in the characteristic-0 case (resp. $\bar{p}^i/\bar{p}^{i+1}$ for $i < 0$ prime to $p$ in the characteristic-$p$ case) is isomorphic to $l[i]$. 

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Indeed, $\bar{U}_i/\bar{U}_{i+1}$ (resp. $p^i/p^{i+1}$) is $k[G]$-isomorphic to $p^i/p^{i+1}$ (prop. 29), which is $k[G]$-isomorphic to $l[i]$ (prop. 33).

Let us record all this information in pictures, for I have still not got over the fact that the quotients are $k[G]$-modules for appropriate $i$, instead of merely being $F_p[G]$-modules. In characteristic 0, the first picture of the filtered $F_p[G]$-module $K^\times/K^{\times p} = \bar{U}_0$ is

$$
\{1\} \subset \bigoplus_{l|pe} \bar{U}_{pe} \subset \bar{U}_{pe-1} \ldots
\bigoplus_{l|pj} \bar{U}_{pj+1} = \bigoplus_{l|pj-1} \bar{U}_{pj} \subset \bar{U}_{pj-1} \ldots
\bigoplus_{l|1} \bar{U}_1 \subset \bar{U}_0,
$$

with successive quotients indicated below the inclusion signs and $j \in [1, e]$, whereas in characteristic $p$ the picture of the $F_p[G]$-module $K^\times/K^{\times p}$ goes on for ever ($j < 0$):

$$
\{0\} \subset \bigoplus_{l|-1} p^{-1} \ldots \subset \bigoplus_{l|pj+1} p^{pj+1} = \bigoplus_{l|pj-1} p^{pj} \subset \bigoplus_{l|1} p^{pj-1} \ldots \subset K^\times/K^{\times p} = K^\times/\varphi(K^+).
$$

The beauty of this can reduce even the most hardened criminal to tears.

The analogy can be further improved. First, declare $\omega$ to be the trivial character in the characteristic-$p$ case. Secondly, the two pictures will look even more similar if the first one is shifted to the right by $pe$ steps. The problem is that the $k[G]$-modules $l[pe-1]$ and $l[-1]$ are not isomorphic, unless $\omega$ is unramified, which is equivalent to $e \equiv 0 \pmod{p-1}$ More precisely, $l[pe-1]$ is the sum of $G$-stable $k$-lines indexed by the characters of “valuation” $\bar{v}(\omega) - 1$, whereas $l[-1]$ is the sum of $G$-stable $k$-lines indexed by the characters of “valuation” $-1$. But this can be easily remedied if we twist the latter by $\omega$.

So $l[-1]\{\omega\}$ is the same $k[G]$-module as $l[pe-1]$. Suppressing the terms indexed by multiples of $p$ other than 0 (and $pe$ in characteristic 0), and exploiting the fact that any $G$-stable subspace has a $G$-stable supplement (lemma 8), the filtered $F_p[G]$-module $K^\times/K^{\times p}$ (resp. $K^\times/\varphi(K^+)$) is

$$
F_p\{\omega\} \oplus l[-1]\{\omega\} \oplus \cdots \oplus l[-b^{(n)}]\{\omega\} \oplus \cdots \oplus \bigoplus_{F_p}
$$

where the middle terms are indexed by the sequence $b^{(n)}$ of prime-to-$p$ integers, for every $n > 0$ in characteristic $p$ but only for $n \in [1, (p-1)e]$ in...
characteristic 0, which is also when the last parenthetical term appears. This picture keeps track of both the filtration and the G-action.

Recall that $b^{(n)} = n + \lfloor (n - 1)/(p - 1) \rfloor$, and that if $n = (p - 1)i + j$ with $i \in \mathbb{N}$ and $j \in [1, p - 1]$ (sic), then $b^{(n)} = pi + j$. Clearly, $b^{(\cdot)} : \mathbb{N}^* \rightarrow \mathbb{N}^*$ is the unique strictly increasing function whose image is the set of integers $> 0$ prime to $p$; we put $b^{(0)} = 0$.

Define $V_i$ ($i > 0$) to be the filtered $k[G]$-module $W_i \{\omega\}$, so that for example when $p = 5$ and $e = 1$, the successive quotients of $V_2$ are

$l[-1], l[-2], l[-3], l[-4]$.

Grouping together $p - 1$ middle terms at a time, and recalling the definitions of $W_i$ and $V_i = W_i \{\omega\}$, we see from the above description that the filtered $\mathbb{F}_p[G]$-module $K^x/K^{xp}$ is $\mathbb{F}_p[G]$-isomorphic to $(*)$

$$\mathbb{F}_p \{\omega\} \oplus (V_1 \oplus 0) \oplus \cdots \oplus (V_{e-1} \oplus 0) \oplus V_e \oplus \mathbb{F}_p$$

if $e < +\infty$ (resp. $K^+/\varphi(K^+)$ is isomorphic to $\mathbb{F}_p \oplus (V_1 \oplus 0) \oplus (V_2 \oplus 0) \cdots$ if $e = +\infty$), where we have inserted some 0s to indicate the cases of equality. From now on, such subtleties will be ignored.

Henceforth we use the opposite filtration on these $\mathbb{F}_p[G]$-modules $(*)$, so that it becomes an increasing filtration indexed by $[0, pe]$ in characteristic 0 and by $\mathbb{N}$ in characteristic $p$. Concretely, the filtration

$$\mathbb{F}_p \{\omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{pe-1} \subset \mathcal{F}_{pe} = K^x/K^{xp}$$

in characteristic 0 is just $\mathcal{F}_i = \bar{U}_{pe-i}$, which justifies the convention $\bar{U}_0 = K^x/K^{xp}$. In characteristic $p$, the filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$ on $K^+/\varphi(K^+)$ is just $\mathcal{F}_i = \overline{p^{-i}}$. In both cases, $\mathcal{F}_{pt} = \mathcal{F}_{pt-1}$ for every $t \in [1, e]$ and indeed for every $t > 0$ in characteristic $p$. Also, $\mathcal{F}_{pt} = \mathbb{F}_p \{\omega\} \oplus V_1 \oplus \cdots \oplus V_t$ for every $t \in [1, e]$ (resp. $t > 0$).

Such isomorphisms hold because for example $\bar{U}_{pe-1}/\bar{U}_{pe}$ is isomorphic to $l[pe - 1]$ (prop. 29 and 33), which is isomorphic to $l[e - 1] = l[-1] \{\omega\}$

(*) (2011/07/14) In the earlier versions, it was asserted that the spaces in question are isomorphic to $\mathbb{F}_p \{\omega\} \oplus V_1 \oplus \mathbb{F}_p$ (resp. $\mathbb{F}_p \oplus V_1 \oplus V_2 \oplus V_1 \oplus \cdots$) as filtered $\mathbb{F}_p[G]$-modules. This was wrong but did not affect the proof in §7, for the lapse merely had the effect of permuting the terms in the sum to be computed.

(*) (2011/07/04) This convention was not adopted in the version of this paper published in the Monatshefte, hence the minus sign in prop. 35 there and the absence of this sign in prop. 37 below.
(recalling that $\bar{v}(\omega) \equiv e \pmod{p-1}$), so $\mathcal{F}_p/\mathcal{F}_0 = W_1\{\omega\} = V_1$ as filtered $k[G]$-modules. Also, when $e > 1$,

$$\mathcal{F}_{p+1}/\mathcal{F}_p = \begin{cases} \frac{p^{se} - (p+1)/p^{se} - p}{p - (p+1)/p - p} & \text{if } \text{car}(F) = 0, \\ l[e-2] & \text{if } \text{car}(F) = p, \end{cases}$$

so $\mathcal{F}_{2p} = \mathcal{F}_p\{\omega\} \oplus W_1\{\omega\} \oplus W_2\{\omega\} = \mathcal{F}_p\{\omega\} \oplus V_1 \oplus V_2$, and so on. Maybe we could have done with fewer letters, but there we are.

Define the level of a line $D \subset K^\times/K^\times p$ or $D \subset K^+/\varphi(K^+)$ to be the integer $i$ such that $D \subset \mathcal{F}_i$ but $D \not\subset \mathcal{F}_{i-1}$. The possible levels in characteristic 0 are $b^{(n)}$ for every $n \in [0, (p-1)e]$, and $pe$. In characteristic $p$, they are $b^{(n)}$ for every $n \in \mathbb{N}$.

**Proposition 36.** — The compositum $C$ of all degree-$p$ separable extensions of $F$ is the maximal abelian extension $M$ of exponent $p$ of $K = F(\sqrt[p-1]{F^\times})$, namely $M = K(\sqrt[p-1]{K^\times})$ or $M = K(\varphi^{-1}(K))$ respectively.

This is the main result of [8] in the characteristic-0 case, and their proof can now be carried over to characteristic $p$.

In view of lemma 20, all we need to show is that $K$ is the compositum of the $F_\chi$ such that the $\chi$-eigenspace $K^\times/K^\times p(\chi)$ or $K^+/\varphi(K^+)(\chi)$ (respectively) is $\neq \{1\}$ or $\neq \{0\}$. The above description shows that such is the case for every $\chi : G \to F_p^\times$, and it is clear that $K$ is the compositum of all cyclic extensions of $F$ of degree dividing $p - 1$.

Interesting as it is, this result will not be needed in the next §; only the algebraic ingredients summarised at the end of §4 and the structural analysis of this § will be needed. (*

7. Serre’s mass formula in prime degree. — It is time to do the counting. The reader will need to refer back frequently to the pictures in §6 in what follows.

Let $E|F$ be a ramified degree-$p$ separable extension, $L = EK$, and let

$$D \subset \begin{cases} \mathcal{F}_p\{\omega\} \oplus V_1 \oplus \cdots \oplus V_e \oplus \mathcal{F}_p & \text{if } \text{car}(F) = 0, \\ \mathcal{F}_p \oplus V_1 \oplus V_2 \oplus V_3 \oplus \cdots & \text{if } \text{car}(F) = p, \end{cases}$$

be the $G$-stable line corresponding to $L|K$, so that $D$ is the line we associate to $E$. Denote the “level” of $D$ by $d(D)$, so that $d(D) = b^{(n)}$

(*

(2011/07/03) Notice that in characteristic 0, the degree $[M : F]$ is finite and hence $F$ has only finitely many extensions of degree $p$, which implies that it has only finitely many extensions of any given degree.
for some \( n \in [1, (p-1)e] \) (resp. \( n > 0 \)) in the \textit{peu ramifié} case and \( d(D) = pe \) in the \textit{très ramifié} case (which occurs only in characteristic 0).

Denote by \( b(L|K) \) the unique ramification break of \( L|K \), and recall that \( c(E) = v(\delta_{E|F}) - (p-1) \) measures the wild ramification of \( E|F \).

**Proposition 37.** — We have \( b(L|K) = c(E) = d(D) \).

We have already seen that \( b(L|K) = d(D) \) in [4, prop. 60] in the characteristic-0 case (note that the numbering convention here is shifted by \( pe \) steps and involves a sign change) and in [6, prop. 14] in the characteristic-\( p \) case. Using the \textit{Schachtelungssatz} twice in the diagram

\[
\begin{array}{c@{\quad}c@{\quad}c}
E & p-1 & L \\
p & p-1 & 1 \\
F & p-1 & K \\
\end{array}
\]

in which the ramification indices (resp. residual degrees) are indicated outside (resp. inside) the square, we get \( c(E) = b(L|K) \). Cf. Remark 24.

**Corollary 38.** — The invariant \( c(E) \) is prime to \( p \) except for \( c(E) = pe \) in characteristic 0.

Indeed, \( d(D) \) is prime to \( p \) for every line \( D \subset K^+/\wp(K^+) \) other than \( \bar{o} \) [6, prop. 11], and also for every \( D \subset K^x/K^{x_p} \) other than \( \bar{U}_{pe} \), unless \( D \not\subset \bar{U}_1 \), in which case \( d(D) = pe \) [4, prop. 42]. It is also clear that every \( m > 0 \) (resp. \( 0 < m < pe \)) prime to \( p \) occurs as \( c(E) \) for some (ramified) \( E \).

**Proof of the mass formula in degree \( p \).** — The number of \( E \) which give rise to the same \( D \) is 1 if the character \( \chi \) through which \( G \) acts on \( D \) is \( \omega \), and \( p \) if \( \chi \neq \omega \) (§4). So the contribution of such \( E \) to Serre’s mass formula is (prop. 37)

\[
\sum_{E \rightarrow D} q^{-c(E)} = \begin{cases} 
q^{-d(D)} & \text{if } \chi = \omega, \\
pe^{-d(D)} & \text{if } \chi \neq \omega.
\end{cases}
\]

Thus the sum over all ramified separable degree-\( p \) extensions \( E \) of \( F \) gets replaced by a sum over all \( G \)-stable lines \( D \) in \( K^x/K^{x_p} \) or \( K^x/\wp(K^+) \) other than the level-0 line \( \bar{U}_{pe} = \mathbf{F}_p\{\omega\} \) (resp. \( \bar{o} = \mathbf{F}_p \)) in characteristic 0 (resp. \( p \)).
For every character \( \chi : G \to \mathbb{F}_p^\times \), the dimension of the \( \chi \)-eigenspace \((\mathbb{F}_p\{\omega}\oplus V_1 \oplus \cdots \oplus V_i)(\chi)\) (for all \( i \in \mathbb{N} \) in characteristic \( p \), for \( i \in [0,e] \) in characteristic 0) is (§6)

\[
\dim_{\mathbb{F}_p}(\mathbb{F}_p\{\omega}\oplus V_1 \oplus \cdots \oplus V_i)(\chi) = \begin{cases} 1 + if & \text{if } \chi = \omega, \\ if & \text{if } \chi \neq \omega, \end{cases}
\]

so the number of points in this space is \( pq^i \) if \( \chi = \omega \) and \( q^i \) if \( \chi \neq \omega \). Therefore the number of lines in \((\mathbb{F}_p\{\omega}\oplus V_1 \oplus \cdots \oplus V_{i+1})(\chi)\) which are not in \((\mathbb{F}_p\{\omega}\oplus V_1 \oplus \cdots \oplus V_i)(\chi)\)

\[
\frac{pq^{i+1} - 1}{p - 1} - \frac{pq^i - 1}{p - 1} \quad \left( \text{resp. } \frac{q^{i+1} - 1}{p - 1} - \frac{q^i - 1}{p - 1} \right)
\]

according as \( \chi = \omega \) or \( \chi \neq \omega \). The “level” of such a line \( D \) is \( d(D) = pi + j \), where \( j \in [1,p[ \) is determined by

\[
\bar{v}(\chi) \equiv \bar{v}(\omega) - (pi + j) \pmod{p - 1},
\]

and therefore depends only on \( \bar{v}(\chi) \) and \( i \pmod{p - 1} \) (*). This follows from the fact that the \( k[G]\)-module \( l[n] \) is the direct sum of \( k\)-lines \( k\{\xi\} \) for those characters \( \xi : G \to \mathbb{F}_p^\times \) for which \( \bar{v}(\xi) \equiv n \pmod{p - 1} \) (prop. 32).

Hence the contribution of all \( G \)-stable lines for given \( i, \chi, \) and \( j \) (with \( \bar{v}(\chi) \equiv \bar{v}(\omega) - (i + j) \)) is

\[
p \left( \frac{q^{i+1} - q^i}{p - 1} \right) q^{-(pi + j)}
\]

irrespective of whether \( \chi = \omega \) or \( \chi \neq \omega \), as we saw above. As there are \( p - 1 \) characters having a given “valuation”, the contribution of all \( G \)-stable lines for a given \( i \) is

\[
p(q^{i+1} - q^i)q^{-pi}Q, \quad Q = \sum_{j=1}^{p-1} q^{-j}.
\]

Now all that remains to be done in the characteristic-\( p \) case is to sum over all \( i \in \mathbb{N} \) and use the fact that

\[
\sum_i (q^{i+1} - q^i)q^{-pi} = (q - 1) \sum_i q^{i-pi} = (q - 1) \frac{q^{p-1}}{q^{p-1} - 1} = \frac{1}{Q},
\]

(* \( 2011/07/09 \) After having been so scrupulous about the twist \( \{\omega\} \) by \( \omega \) hitherto, it was inadvertently omitted at this point in the earlier versions. \( \) This omission merely had the effect of permuting the terms.\)
proving the formula in this case. In the characteristic-0 case, the sum extends only over \( i \in [0,e] \) to give the contribution of \( \text{peu ramifiées} \) extensions

\[
pQ \sum_{i=0}^{e-1} (q^{i+1} - q^i)q^{-pi} = p(1 - q^{(1-p)e}).
\]

But to this must be added the contribution of the \( \text{très ramifiées} \) extensions, which correspond to lines in the 1-eigenspace \( F_{pe}(1) = (K^\times/K^\times p)(1) \) which are not in the hyperplane \( F_{pe-1}(1) = \bar{U}_1(1) \). The dimension of the ambient space is

\[
\dim_{F_p}(F_p(\omega) \oplus V_1 \oplus \cdots \oplus V_e \oplus F_p)(1) = \begin{cases} 
  ef + 1 & \text{if } 1 \neq \omega, \\
  ef + 2 & \text{if } 1 = \omega,
\end{cases}
\]

giving the following contributions in the two cases \( 1 \neq \omega \), resp. \( 1 = \omega \):

\[
p \left( \frac{pq^e - 1}{p - 1} - \frac{q^e - 1}{p - 1} \right) q^{-pe}, \quad \left( \frac{p^2 q^e - 1}{p - 1} - \frac{pq^e - 1}{p - 1} \right) q^{-pe}.
\]

But these two are the same and equal \( pq^{(1-p)e} \). The total comes to \( p \) as expected, proving the degree-\( p \) mass formula in characteristic 0 as well.

(Note that our method computes the contributions \( p(1 - q^{(1-p)e}) \) and \( p.q^{(1-p)e} \) of \( \text{peu ramifiées} \) and \( \text{très ramifiées} \) extensions separately, not just their sum \( p \).)

This proof is reminiscent of the \textit{Führerdiskriminantenproduktformel}, as applied before lemma 6 in [6]. We thus obtain the following prime-degree case of Serre’s mass formula:

**Theorem 39** (Serre, 1978). — Let \( F \) be a local field with finite residue field of characteristic \( p \) and cardinality \( q \). When \( E \) runs through ramified separable degree-\( p \) extensions of \( F \) (contained in a fixed separable algebraic closure \( \bar{F} \) of \( F \)), then

\[
\sum_E q^{-c(E)} = p,
\]

where \( c(E) = v(\delta_{E|F}) - (p - 1) \), and \( v(\delta_{E|F}) > (p - 1) \) is the valuation of the discriminant \( \delta_{E|F} \) of \( E|F \) (**).

The above proof is summarised in §9. Some refinements of the degree-\( p \) mass formula are also presented there.

(**) (2011/03/15) Clearly, our method can be used to count the number of degree-\( p \) separable extensions \( E \) of \( F \) (contained in \( \bar{F} \)) with a given \( c(E) \), and also the number of \( F \)-conjugacy classes of such \( E \). These numbers were determined by Krasner in all degrees; using them, he gave a different proof of Serre’s mass formula (Remarques au sujet d’une note de J.-P. Serre :
8. Tame extensions of prime degree. — Let us end, for the sake of completeness and contrast, with a word about the compositum $C'$ of all degree-$p'$ extensions $E'$ of $F$, where $p'$ is a prime $\neq p$. The reader who has made it so far should have no difficulty in supplying proofs modelled on §4. In any case, the assertions here are nothing but translations of standard facts about tamely ramified extensions of $F$ into the language of §6; they serve to further illustrate the general theory of §4.

Let $K' = F(\zeta')$, where $\zeta'$ is a primitive $p'$-th root of $1$, $G' = \text{Gal}(K'|F)$ and let $\omega' : G' \to F_{p'}^\times$ be the cyclotomic character, so that $\sigma'(\zeta') = \zeta'^{\omega'\sigma}$ for every $\sigma' \in G'$. The extension $K'|F$ is unramified of degree equal to the order of $q$ in $F_{p'}^\times$, so $K' = F \iff G' = \{1\} \iff \omega' = 1 \iff p' | (q - 1)$.

The $F_{p'}$-space $\overline{K'}^\times = K'^\times/K'^\times_{p'}$ is of dimension $2$ and contains the line $\overline{\omega'}^\times$ on which $G'$ acts via $\omega'$, where $\omega'$ is the ring of integers of $K'$; this line is canonically $G'$-isomorphic to $k'^\times/k'^\times_{p'}$, where $k'$ is the residue field of $K'$ and $G'$ has been identified with $\text{Gal}(k'|k)$. The valuation provides a $G'$-isomorphism $\overline{K'^\times}/\overline{\omega'}^\times = F_{p'}$, so that $\overline{K'^\times}$ is isomorphic to $\overline{\omega'}^\times \oplus F_{p'} = F_{p'}\{\omega'\} \oplus F_{p'}$ as a filtered $F_{p'}[G']$-module. Consequently, the only characters $\chi' : G' \to F_{p'}^\times$ for which $\overline{K'^\times}(\chi') \neq \{1\}$ are $\chi' = \omega'$ and $\chi' = 1$.

Every ramified $E'$ is of the form $F(\sqrt[p']{\pi})$ for some uniformiser $\pi$ of $F$, and hence $E'K'$ is cyclic (of degree $p'$) over $K'$ (and galoisian over $F$); it thereby gives rise to a $G'$-stable line $D' \subset \overline{K'^\times}$. Conversely, every $G'$-stable line $D' \subset \overline{K'^\times}$ comes from some $E'$; $\overline{\omega'}^\times$ comes from the unramified $E'$. If $\chi' : G' \to F_{p'}^\times$ is the character through which $G'$ acts on $D'$, then $D'$ comes from one (resp. $p'$) $E'$ if $\chi' = \omega'$ (resp. $\chi' \neq \omega'$). More precisely, we have established a bijection between the set of $G'$-stable lines in $\overline{K'^\times}$ and the set of $F$-conjugacy classes of degree-$p'$ extensions of $F$.

Let $M' = K'((\sqrt[p']{K'^\times}))$ be the maximal abelian extension of $K'$ of exponent $p'$. It is easily seen that $C' = M'$. Indeed, we have just seen that $C' \subset M'$. Next, $K' \subset C'$ because $K'$ is contained in the galoisian closure of any ramified $E'$. As $C'|K'$ is abelian of exponent $p'$, there is a subspace $T' \subset \overline{K'^\times}$ such that $C' = K'((\sqrt[p']{T}))$. As $T'$ contains every $G'$-stable line, we must have $T' = \overline{K'^\times}$ and $C' = M'$.

Life would be dry if everything had been so tame.

"Une 'formule de masse' pour les extensions totalement ramifiées de degré donné d'un corps local" : une démonstration de la formule de M. Serre à partir de mon théorème sur le nombre des extensions séparables d'un corps valué localement compact, qui sont d'un degré et d'une différente donnés, Comptes Rendus 288 (1979) 18, pp. A863–A865.)
Remark 40. — The foregoing can be used to prove Serre’s mass formula in degree \( p' \), just as §6 was used in §7. Indeed, the only character \( \chi' : G' \to \mathbf{F}_p^\times \) for which there is a line \( D' \neq \mathfrak{o}^{\times} \) in \( \mathbf{K}_{\mathfrak{K}}^{\times}(\chi') \) is \( \chi' = 1 \). There are \( p' \) (resp. 1) such \( D' \) if \( 1 = \omega' \) (resp. \( 1 \neq \omega' \)) and each \( D' \) comes from 1 (resp. \( p' \)) ramified degree-\( p' \) extension \( E' \) of \( F \) if \( 1 = \omega' \) (resp. \( 1 \neq \omega' \)). This balancing, similar to what we saw in §7 for \( \text{très ramifiées} \) extensions of degree \( p \), leads to the result.

9. Some refined mass formulae. — Let us summarise the proof of the degree-\( p \) mass formula and show how the same strategy leads to certain refinements. These consist mainly in computing the contribution of those degree-\( p \) separable extensions \( E \) of \( F \) for which the group \( G = \text{Gal}(K|F) \) acts on the order-\( p \) group \( \text{Gal}(E_K|K) \) via a given character \( G \to \mathbf{F}_p^\times \).

Summary of the proof (2011/07/03). — There is a map \( E \mapsto E_K \) (with \( K = F(\sqrt[p]{\mathfrak{o}}) \)) from the set \( S_p(F) \) of degree-\( p \) separable extensions of \( F \) (in some fixed separable algebraic closure of \( F \)) into the set \( C_p(K) \) of degree-\( p \) cyclic extensions of \( K \); its image is the set \( C_p(K,F) \) of those \( L \in C_p(K) \) which are galoisian over \( F \), and if we go modulo the relation \( \sim \) of \( F \)-conjugacy in \( S_p(F) \), we get a bijection \( S_p(F)/\sim \to C_p(K,F) \).

There is a natural bijection of \( C_p(K) \) with the set \( \mathbf{P}(K^\times/K^{\times p}) \) or \( \mathbf{P}(K^+/\varphi(K^+)) \) of lines in the \( \mathbf{F}_p \)-space \( K^\times/K^{\times p} \) or \( K^+/\varphi(K^+) \) respectively. The subset \( C_p(K,F) \) corresponds to the set \( \mathbf{P}(K^\times/K^{\times p})^G \) or \( \mathbf{P}(K^+/\varphi(K^+))^G \) of \( G \)-stable lines for \( G = \text{Gal}(K|F) \). Under the composite map

\[
\Phi : S_p(F) \to \mathbf{P}(\ )^G, \quad \Phi(E) = D, \quad (E_K = K(\sqrt[p]{\mathfrak{d}}) \text{ or } E_K = K(\varphi^{-1}(D))),
\]

the set \( C_p(F) \) of degree-\( p \) cyclic extensions of \( F \) is in bijection with the set \( \mathbf{P}((\ )|\omega) \) of lines in the \( \omega \)-eigenspace for the action of \( G \), where \( \omega : G \to \mathbf{F}_p^\times \) is the mod-\( p \) cyclotomic character (resp. \( \omega = 1 \) is the trivial character, so that \( K^+/(\varphi)(K^+)(\omega) = F/\varphi(F) \)). On the complement of \( \mathbf{P}((\ )|\omega) \), the fibres of the map \( \Phi \) have \( p \) elements each, mutually conjugate as extensions of \( F \). Note finally that \( \Phi \) sends the unramified degree-\( p \) extension \( F_p \) of \( F \) to the line \( \mathbf{U}_{pe} \) in characteristic 0 (where \( e \) is the absolute ramification index of \( F \)) or \( \mathbf{U} \) in characteristic \( p \). Here, \( \mathbf{U} \) is the image of \( \mathfrak{o} \) in \( K^+/\varphi(K^+) \), where \( \mathfrak{o} \) is the ring of integers of \( K \), and \( \mathbf{U}_{pe} \) is the image of \( U_{pe} = 1 + p^{pe} \mathfrak{p} \) in \( K^\times/K^{\times p} \), where \( \mathfrak{p} \) is the unique maximal ideal of \( \mathfrak{o} \).

Some of these facts are summarised in the following commutative
in which \(( )\) is to be replaced by \(K^{\times}/K^{\times p}\) or \(K^{+}/\wp(K^{+})\) respectively, and "\(p : 1\)" means that the map is \(p\)-to-1 except on the subset \(C_{p}(F)\), on which it is bijective. In other words, every \(D \in \mathbf{P}(\ )^{G}\) has \(p\) preimages in \(S_{p}(F)\) unless the character through which \(G\) acts on \(D\) is \(\omega\) or \(\bar{\wp}\) respectively, where \(\bar{\wp}\) is isomorphic to \(\mathbb{F}_{p}\) or \(\mathbb{F}_{q}(K^{+})\) respectively.

The measure \(c(E) = v(\delta_{E[F]}) - (p - 1)\) of wild ramification of a ramified \(E \in S_{p}(F)\) equals \(d(D)\), the "level" of the line \(D\) corresponding to \(E\) — the first step in the filtration on \(K^{\times}/K^{\times p}\) or \(K^{+}/\wp(K^{+})\) to which \(D\) belongs, when the "level" of \(\bar{\wp}\) is declared to be 0 and the increasing filtration is indexed by \([0, p]\) (resp. \(\mathbb{N}\)). In other words, the stratification on \(S_{p}(F)\) given by \(c\) corresponds to the stratification on \(\mathbf{P}(\ )^{G}\) coming from the filtration on \(K^{\times}/K^{\times p}\) or \(K^{+}/\wp(K^{+})\) respectively.

The filtered \(F_{p}[G]\)-module \(K^{\times}/K^{\times p}\) or \(K^{+}/\wp(K^{+})\) was shown in §6 to be isomorphic to

\[
F_{p}(\omega) \oplus l[e - 1] \oplus \cdots \oplus l[e - b^{(p-1)e}] \oplus F_{p}
\]

or to \(F_{p} \oplus l[-1] \oplus \cdots \oplus l[-b^{(n)}] \oplus \cdots\) respectively, where \(l = \wp/p\) is the residue field of \(K\), \(l[i]\) denotes the \(k[G]\)-module \(p^{i}/p^{i+1}\) (where \(k\) is the residue field of \(F\)) and \(\{\omega\}\) denotes twist by the character \(\omega\). Also, \(b^{(n)}\) is the \(n\)-th prime-to-\(p\) integer (so that \(b^{(p-1)e} = pe - 1\)); here, \(n\) runs over all integers \(> 0\) in characteristic \(p\) but only from 1 to \((p - 1)e\) in characteristic 0, which is also when the last term \(F_{p}\) appears, accounting for \(très\ ramifiées\) extensions.

Put \(V_{i+1} = l[-(ip + 1)]\{\omega\} \oplus \cdots \oplus l[-(ip + p - 1)]\{\omega\}\). The filtered \(F_{p}[G]\)-module \(K^{\times}/K^{\times p}\) is isomorphic to \(F_{p}\{\omega\} \oplus V_{1} \oplus \cdots \oplus V_{e} \oplus F_{p}\) in characteristic 0 (resp. \(K^{+}/\wp(K^{+})\) is isomorphic to \(F_{p} \oplus V_{1} \oplus V_{2} \oplus \cdots\) in characteristic \(p\)).

Everything is now in place for proving the degree-\(p\) mass formula by rewriting it as an appropriate sum over \(D \neq \bar{\wp}\) (resp. \(D \neq \bar{\wp}\)) in \(\mathbf{P}(K^{\times}/K^{\times p})^{G}\) (resp. \(\mathbf{P}(K^{+}/\wp(K^{+}))^{G}\) as in §7. This is done by successively
computing the contribution of G-stable lines in \( F_p \{ \omega \} \oplus V_1 \oplus \cdots \oplus V_{i+1} \) which are not in \( F_p \{ \omega \} \oplus V_1 \oplus \cdots \oplus V_i \), and adding them all up for \( i \in [0, e[ \) in characteristic 0 (resp. \( i \in \mathbb{N} \) in characteristic \( p \)). Finally, in characteristic 0, one adds the contribution of G-stable lines in \( K^x/K^{x,p} \) which are not in \( \bar{U} \).

**Remark 41** (2011/07/14). — As the “level” of the line \( \bar{U}_{pe} \) or \( \bar{o} \) is 0, logic dictates that we pose \( c(F_p) = 0 \) for the unramified degree-\( p \) extension \( F_p \) of \( F \). The degree-\( p \) mass formula then becomes \( \sum_{E \in S_p(F)} q^{-c(E)} = 1 + p \).

More interestingly, there is a map \( S_p(F) \to F^x/F^{x,p} \) which sends every \( E \) to the character in \( \text{Hom}(G, F^x) = F^x/F^{x,p} \) through which \( G \) acts on the G-stable \( F_p \)-line \( D \in \mathbf{P}( )^G \) corresponding to \( E \). Our analysis makes it possible to compute \( \sum_{E \to \chi} q^{-c(E)} \) for any given \( \chi \in F^x/F^{x,p} \).

Fix a character \( \chi : G \to F^x_p \) and an index \( i \in [0, e[ \) (resp. \( i \in \mathbb{N} \)), and define \( j_{\chi,i} \in [1, p[ \) by the requirement \( \bar{v}(\chi) \equiv \bar{v}(\omega) - (i + j_{\chi,i}) \pmod{p-1} \).

We have seen that every \( F_p \)-line
\[
D \subset (F_p \{ \omega \} \oplus V_1 \oplus \cdots \oplus V_{i+1})(\chi), \quad D \not\subset (F_p \{ \omega \} \oplus V_1 \oplus \cdots \oplus V_i)(\chi),
\]
has “level” \( pi + j_{\chi,i} \) (cf. prop. 32) and their contribution to the mass formula is
\[
p\left( \frac{q^{i+1} - q^i}{p-1} \right) q^{-(pi + j_{\chi,i})}.
\]
So the contribution of \( \chi \) — it turns out to depend only on \( \bar{v}(\chi) \) — is
\[
\frac{p(q - 1)}{p - 1} \sum_i q^{i - (pi + j_{\chi,i})} \begin{cases} i \in [0, e[ & \text{if } \text{car}(F) = 0, \\ i \in \mathbb{N} & \text{if } \text{car}(F) = p, \end{cases}
\]
except when \( \chi = 1 \) is the trivial character and \( \text{car}(F) = 0 \), in which case très ramifiées extensions contribute a further \( p/q^{(p-1)e} \). This sum can be evaluated upon remarking that \( j_{\chi,i} = j_{\chi,i'} \) if \( i \equiv i' \pmod{p-1} \).

**Remark 42** (2011/07/14). — Reconciling these results with the formulae (1)–(3) in [7, prop. 14–16] for \( \chi = \omega \) amounts to checking the identity
\[
p \bar{v} + j_{\omega,i} = (p - 1)b^{(i+1)}
\]
for all \( i \in \mathbb{N} \). Upon writing \( i = (p - 1)n_i + r_i \) (with \( r_i \in [0, p - 1[, n_i \in \mathbb{N} \)), we have \( j_{\omega,i} = p - 1 - r_i \) and \( b^{(i+1)} = i + 1 + n_i \), and hence the identity.

**Example 43** (2011/10/06). — Take \( F = F_3(\pi) \) so that \( p = q = 3 \), \( e = +\infty \), and \( F^x/F^{x,2} = \{ 1, -1, \pi, -\pi \} \). For \( \chi \in F^x/F^{x,2} \) and \( i \in \mathbb{N} \),
\[
\bar{v}(\chi) = 0 \quad \Rightarrow \quad j_{\chi,i} = \begin{cases} 2 & \text{if } i \equiv 0 \pmod{2} \\ 1 & \text{if } i \equiv 1 \pmod{2} \end{cases}
\]
29
so the contribution of each of the unramified characters (namely $\bar{1}$ and $\bar{-1}$) is $9/20$. Similarly,

$$\bar{v}(\chi) = \bar{1} \implies j_{\chi,i} = \begin{cases} 1 & \text{if } i \equiv 0 \pmod{2} \\ 2 & \text{if } i \equiv 1 \pmod{2}, \end{cases}$$

so the contribution of each of the ramified characters (namely $\bar{\pi}$ and $\bar{-\pi}$) is $21/20$.

Example 44 (2011/10/07) Take $p = 5$ and $F = k((\pi))$. For any given $w \in \mathbb{Z}/4\mathbb{Z}$, a character of “valuation” $w$ contributes $5(q - 1)CA_w/4$, where $C = \sum_{a \in \mathbb{N}} q^{-16a} = q^{16}/(q^{16} - 1)$ and

$$A_{\bar{0}} = q^{-4} + q^{-7} + q^{-10} + q^{-13},$$
$$A_{\bar{3}} = q^{-1} + q^{-8} + q^{-11} + q^{-14},$$
$$A_{\bar{2}} = q^{-2} + q^{-5} + q^{-12} + q^{-15},$$
$$A_{\bar{1}} = q^{-3} + q^{-6} + q^{-09} + q^{-16}.$$ (2011/10/08) Using Remark 41, it is easy to determine the contribution of cyclic extensions to the degree-$p$ mass formula for odd $p$ (the contribution is 2 for $p = 2$). Let’s do it more generally for any any character of “valuation” $0$ first in the easier case $F = k((\pi))$.

Proposition 45. — Suppose that $p \neq 2$ and that $F = k((\pi))$, and let $\chi \in F^\times/F^{\times p-1}$ be a character of “valuation” $\bar{v}(\chi) \equiv 0 \pmod{p - 1}$. The contribution of $\chi$ to the degree-$p$ mass formula is

$$\sum_{E \mapsto \chi} q^{-c(E)} = \frac{pq^{p-2}(q - 1)(q^{p-2}(p-1) - 1)}{(p - 1)(q^{p-2} - 1)(q^{(p-1)^2} - 1)}$$ (sauf erreur). In particular, for $\chi = 1$, this is the contribution of cyclic extensions.

It is a matter of evaluating the sum $\sum_{i \in \mathbb{N}} q^{i-(pi+j_{\chi,i})}$ (cf. Remark 41), where $j_{\chi,i} \in [1, p - 1]$ is subject to $i + j_{\chi,i} \equiv 0 \pmod{p - 1}$. Write $i = (p - 1)n + r$ ($r \in [0, p - 2]$, $n \in \mathbb{N}$), and notice that $j_{\chi,i} = p - 1 - r$. 30
so that
\[
\sum_{i \in \mathbb{N}} q^{i-(p^i+j_{\chi,i})} = \sum_{r=0}^{p-2} \left( \sum_{n \in \mathbb{N}} q^{-(p-1)^2 n-(p-2)r-(p-1)} \right)
\]
\[
= \frac{q^{-(p-1)}}{q^{(p-1)^2} - 1} \sum_{r=0}^{p-2} q^{-(p-2)r} \sum_{n \in \mathbb{N}} q^{-(p-1)^2 n}
\]
\[
= q^{-(p-1)} \frac{q^{(p-1)^2}}{q^{(p-1)^2} - 1} \sum_{r=0}^{p-2} q^{-(p-2)r}
\]
\[
= \frac{q^{p-2} (q^{(p-2)} - 1)}{(q^{p-2} - 1)(q^{(p-1)^2} - 1)}.
\]

Plugging this value into the formula in Remark 41 gives the result.

**Corollary 46.** — For \( p \) odd and \( F = k((\pi)) \), the contribution of all ramified \( E \in S_p(F) \) for which there is an unramified extension \( F' \mid F \) (depending on \( E \)) such that \( EF' \mid F \) be galoisian is

\[
\frac{pq^{p-2}(q-1)(q^{(p-2)}-1)}{(q^{p-2} - 1)(q^{(p-1)^2} - 1)}.
\]

Let \( \chi : G \to F_p^\times \) be the character through which \( G \) acts on the line \( D \in \mathbf{P}(K^+ / \varphi(K^+))^G \) corresponding to \( E \). It suffices to show that \( EF' \) is galoisian over \( F \) for some unramified extension \( F' \) of \( F \) if and only if \( \bar{v}(\chi) \equiv 0 \) (mod. \( p-1 \)).

Now, in the notation of lemma 19, the galoisian closure of \( E \) over \( F \) is \( EF_\chi \) (where \( F_\chi = K^{\ker(\chi^{-1})} \)). But \( F_\chi \mid F \) is unramified if and only if \( \chi \) is unramified \( \iff \chi(G_0) = \{1\} \iff \bar{v}(\chi) \equiv 0 \), for \( \text{Gal}(F_\chi \mid F) = \text{Im}(\chi^{-1}) \), so \( \text{Gal}(F_\chi \mid F)_0 = \chi^{-1}(G_0) \). As there are \( p-1 \) unramified characters, the corollary follows from prop. 44.

(2011/10/11) More generally, let \( \chi \in F_p^\times / F_p^{\times p-1} \) be any character and let \( a \in [0, p-2] \) be such that \( \bar{v}(\chi) \equiv -a \) (mod. \( p-1 \)). Notice that if \( i-a = (p-1)n + r \) for some \( r \in [0, p-2] \) and some \( n \in \mathbb{N} \), then \( j_{\chi,i} = p-1-r \), as follows from the defining requirements \( j_{\chi,i} \in [1, p-1] \) and \( \bar{v}(\chi) \equiv -(i+j_{\chi,i}) \) (mod. \( p-1 \)). This simple device helps in evaluating the sum \( \sum_{i \in \mathbb{N}} \) by rewriting it as \( \sum_{i=0}^{a-1} + \sum_{r=0}^{p-2} \sum_{n \in \mathbb{N}} \), where the first sum is empty if \( a = 0 \) as in prop. 45. The case of ramified characters \((a > 0)\) is treated in the following proposition.
Proposition 47. — Suppose that \( p \neq 2 \) and that \( F = k((\pi)) \). For a character \( \chi \in \mathbb{F}_p^\times / \mathbb{F}_p^{\times p-1} \) of \( G = \text{Gal}(K|F) \), define \( a \in [0, p - 2] \) by \( \bar{v}(\chi) \equiv -a \pmod{p - 1} \). Then the contribution \( \sum_{E \rightarrow \chi} q^{-c(E)} \) of \( \chi \) to the degree-p mass formula is \( p(q-1)/(p-1) \times \frac{q^{p-2}(q^{(p-2)a} - 1)}{q^{(p-1)a}(q^{p-2} - 1)} + \frac{q^{p-2}(q^{(p-2)(p-1)} - 1)}{q^{(p-1)a}(q^{p-2} - 1)(q^{(p-1)^2} - 1)} \).

[Notice that for \( a = 0 \) the first term is 0 and we retrieve the expression in prop. 45.] We have to compute \( \sum_{i \in \mathbb{N}} q_{i-(pi+j_{\chi,i})} \). The first term in the above expression corresponds to \( i \in [0, a] \), interval in which \( j_{\chi,i} = a - i \). The second term corresponds to \( i \in [a, +\infty[ ; \) to evaluated it, write \( i-a = (p-1)n+r \) (with \( r \in [0, p-2] \), \( n \in \mathbb{N} \)), and note that \( j_{\chi,i} = p-1-r \), as explained above. So

\[
\sum_{i \in [a, +\infty[} q_{i-(pi+j_{\chi,i})} = \sum_{r=0}^{p-2} \left( \sum_{n \in \mathbb{N}} q^{-(p-1)^2n-(p-2)r-(p-1)(a+1)} \right) = q^{-(p-1)(a+1)} \sum_{r=0}^{p-2} \left( q^{-(p-2)r} \sum_{n \in \mathbb{N}} q^{-(p-1)^2n} \right) = q^{-(p-1)(a+1)} \frac{q^{(p-1)^2} - 1}{q^{(p-1)^2} - 1} \sum_{r=0}^{p-2} q^{-(p-2)r} = \frac{q^{p-2}(q^{(p-2)(p-1)} - 1)}{q^{(p-1)a}(q^{p-2} - 1)(q^{(p-1)^2} - 1)}.
\]

Remark 48 (2011/10/14). — Evaluating the sum

\[
\sum_{a=0}^{p-2} \frac{(q^{(p-2)a} - 1)(q^{(p-1)^2} - 1) + (q^{(p-2)(p-1)} - 1)}{q^{(p-1)a}}
\]

gives \( (q^{p-2} - 1)(q^{(p-1)^2} - 1)/q^{p-2}(q - 1) \), which serves as a check on the computations by reproving the degree-p mass formula.

Example 49 (2011/07/14). — Take \( F = \mathbb{Q}_3 \), so that \( p = q = 3 \), \( e = 1 \), and the respective contributions of the four characters in \( \mathbb{F}_p^\times / \mathbb{F}_p^{\times 2} = \{1, -1, 3, -3\} \) are \( 1 + \frac{1}{3}, 1, \frac{1}{3}, \frac{1}{3} \).

Example 50 (2011/07/14). — Take \( p = 5 \), \( e = 5 \), and \( \chi \in \mathbb{F}_p^\times / \mathbb{F}_p^{\times 4} \) with \( \chi \neq 1 \) but \( \bar{v}(\chi) = 0 \). Then \( j_{\chi,i} = 1, 4, 3, 2, 1 \) respectively for \( i \in [0, 5] \), and the contribution of \( \chi \) is \( 5(q-1)(q^{-1} + q^{-8} + q^{-11} + q^{-14} + q^{-17})/4 \). Also, the contribution of \( \omega \) is \( 5(q-1)(q^{-4} + q^{-7} + q^{-10} + q^{-13} + q^{-20})/4 \).
Let us now treat the characteristic-0 case, so that $F$ is a finite extension of $Q_p$ ($p \neq 2$). This case is different from the characteristic-$p$ case in three ways: $e$ is finite, $\bar{\nu}(\omega) \equiv e$ may very well be $\neq 0$ (mod. $p - 1$), and finally, there are très ramifiées extensions. Nevertheless, as we have seen in §6, there is enough structural similarity between the $F_p[G]$-module $K^+$/\(\varphi(K^+)\) in characteristic $p$ and the $F_p[G]$-module $K^x/K^{xP}$ in characteristic 0 for us to carry out the same analysis. The only essential difference is that $\sum_{i \in \mathbb{N}}$ gets replaced by $\sum_{i \in [1, e[}$. 

As an illustration, we will compute the contribution of cyclic extensions to the degree-$p$ mass formula. More generally, we have

**Proposition 51.** — *Suppose that $F|Q_p$ is a finite extension ($p \neq 2$). For every $\chi \in F^x/F^{xP-1}$ of “valuation” $\bar{\nu}(\chi) \equiv \bar{\nu}(\omega)$, the contribution $\sum_{E \to \chi} q^{-c(E)}$ of $\chi$ to the degree-$p$ mass formula is*

$$\frac{p(q - 1)}{p - 1} (I + J)$$

*(where $I$ and $J$ are described below) except for $\chi = 1$, when très ramifiées extensions contribute a further $p/q(p-1)e$. In particular, these are the contributions of cyclic extensions in the cases $\omega \neq 1$, $\omega = 1$ respectively.*

We have to compute $\sum_{i \in [0, e]} q^{-i(pi + jx, i)}$. Write $e - 1 = (p - 1)N + R$ (with $R \in [1, p - 2]$ and $N \in \mathbb{N}$), and split the sum over $i \in [0, e]$ into $i \in [0, (p - 1)N]$ and $i \in [(p - 1)N, e]$. Notice that if $i = (p - 1)n + r$ (with $r \in [0, p - 2]$ and $n \in \mathbb{N}$), then $jx, i = p - 1 - r$. We have

$$I = \sum_{i=0}^{(p-1)N-1} q^{-i(pi + jx, i)} = \sum_{r=0}^{p-2} \left( \sum_{n=0}^{N-1} q^{-(p-1)^2n - (p-2)r - (p-1)} \right)$$

$$= q^{-(p-1)} \sum_{r=0}^{p-2} \left( q^{-(p-2)r} \sum_{n=0}^{N-1} q^{-(p-1)^2n} \right)$$

$$= q^{-(p-1)} q^{(p-1)^2} q^{(p-1)^2N} - 1 \sum_{r=0}^{p-2} q^{-(p-2)r}. $$

Notice that $I = 0$ if $N = 0$. The second term is

$$J = \sum_{i=(p-1)N}^{e-1} q^{-i(pi + jx, i)} = \sum_{r=0}^{R} q^{-(p-1)^2N - (p-2)r - (p-1)}$$

$$= q^{-(p-1)^2N - (p-1)} \sum_{r=0}^{R} q^{-(p-2)r}. $$

It is left to the reader to do a similar analysis for other $\chi \in F^x/F^{xP-1}$. 

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Corollary 52. — With the same notation, the contribution of those ramified \( E \in \mathcal{S}_p(F) \) for which the maximal tamely ramified extension \( \hat{E}_1 \) of \( F \) in the galoisian closure \( \hat{E} \) of \( E|F \) is unramified over \( F \) is \( p(q-1)(1+J) \).

We know that the maximal tamely ramified extension of \( F \) in \( \hat{E} \) is \( F_\chi \) (cf. lemma 19), where \( F_\chi \) is the fixed field of \( \text{Ker}(\omega \chi^{-1}) \) and \( \chi : G \to F^\times \) is the character through which \( G \) acts on the line \( D \in \mathbf{P}(K^\times/K^{x^p})^G \) corresponding to \( E \). For \( F_\chi \) to be unramified over \( F \), it is necessary and sufficient that \( \omega \chi^{-1} \) be unramified, which is equivalent to \( \bar{v}(\chi) = \bar{v}(\omega) \). As there are \( p-1 \) such \( \chi \), the result follows from prop. 51.

Remark 53. — Let us return to the general case of a finite extension \( F \) of \( \mathbb{Q}_p \) or of \( F_p((\pi)) \), where the prime \( p \) is odd, and fix an extension \( F' \) of \( F \) in \( K \). One can similarly compute the contribution of all \( E \in \mathcal{S}_p(F) \) such that \( E|F' \) is galoisian over \( F \); the case of cyclic extensions corresponds to the choice \( F' = F \) and the degree-\( p \) mass formula is the case \( F' = K \). We could also require that \( F' \) be the maximal tamely ramified extension \( \hat{E}_1 \) of \( F \) in the galoisian closure \( \hat{E} \) of \( E \) over \( F \). The contributions in question are the sum of the contributions of all \( \chi \in F^\times/F^{x^p-1} \) such that \( F_\chi \subset F' \) (resp. \( F_\chi = F' \)), in the notation of lemma 19. The basic case occurs when \( F'|F \) is cyclic of degree dividing \( p-1 \) (cf. lemma 12 or Remark 23); the number of such \( F' \) can be easily computed [10, Kap. 16] or [5, Lecture 18].

Remark 54 (2011/10/18). — Similarly, given a group \( \Gamma \), extension of a subgroup of \( F^\times_p \) of order \( n \) by a group of order \( p \) (cf. Remark 23), one can compute the contribution of all \( E \in \mathcal{S}_p(F) \) such that the group \( \text{Gal}(\hat{E}|F) \) is isomorphic to \( \Gamma \). As \( F^\times_p \) has \( \varphi(p-1) \) subgroups, there are \( \varphi(p-1) \) possibilities for \( \Gamma \).

Example 55 (2011/10/20). — For example, when \( n = 2 \), so that \( \Gamma \) is the dihedral group of order \( 2p \), and \( F = k((\pi)) \), there are three characters in \( \chi \in F^\times/F^{x^p-1} \) for which \( \text{Im}(\chi^{-1}) \) has order 2, namely \( \chi = \bar{g}^m, \bar{\pi}^m, (\bar{g}\bar{\pi})^m \), where \( g \) is a generator of \( k^\times \) and \( m = (p-1)/2 \). The contribution of those \( E \in \mathcal{S}_p(F) \) whose galoisian closure \( \hat{E} \) over \( F \) has group \( \text{Gal}(\hat{E}|F) \) isomorphic to \( \Gamma \) is the sum of the contributions of these three \( \chi \), the first of which has “valuation” \( \equiv 0 \) and the other two \( \equiv m \) (mod. \( p-1 \)).

Remark 56 (2011/07/14). — Notice finally that when \( e < +\infty \), we can similarly compute the number of \( E \in \mathcal{S}_p(F) \) mapping to any given \( \chi \in F^\times/F^{x^p-1} \) under the composite

\[
\gamma : \mathcal{S}_p(F) \to \mathbf{P}(K^\times/K^{x^p})^G \to \text{Hom}(G, F^\times_p) \to F^\times/F^{x^p-1}
\]

where the second map sends a \( G \)-stable line \( D \) to the character through which \( G \) acts on \( D \); the cases \( \chi = 1, \omega \) would need a separate treatment.
The isomorphism of $K^\times/K^{\times p}$ with $F_p\{\omega\} \oplus V_1 \oplus \cdots \oplus V_e \oplus F_p$ as a filtered $F_p[G]$-module is all that is needed. The same method works for $e = +\infty$ if we restrict to $E$ with bounded $c(E)$; here the case $\chi = 1$ would be special.

Remark 57 (2011/10/18). — For a given cyclic extension $F'$ of $F$ in $K$ of degree dividing $p-1$, one can similarly compute the number of $E \in S_p(F)$ whose galoisian closure $\tilde{E}$ is $EF'$, or for which the group $\text{Gal}(\tilde{E}|F)$ is isomorphic to a given group $\Gamma$ as in Remark 54.

Remark 58 (2011/10/19). — There are two naturally defined functions on the set $S_p(F)$ of degree-$p$ separable extensions of $F$. The first one $c : S_p(F) \to \mathbb{N}$ is a measure of wild ramification and the second one $\gamma : S_p(F) \to F^\times/F^{\times p-1}$ doesn’t seem to have a name in the literature. I believe that these are the only natural functions on this set, and that subsets defined in terms of $c$ and $\gamma$ are the only natural subsets of $S_p(F)$ (*). Be that as it may, the methods of this paper allow us to compute the mass or the cardinal of any such subset.

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(*) This is not literally true. Some quadratic extensions of $\mathbb{Q}_2$ can be embedded into quartic cyclic extensions whereas others cannot; this behaviour cannot be detected by $c$. More generally, if $F^\times$ has an element of order $p$ (in which case $\omega = 1$) but no element of order $p^2$, then some degree-$p$ cyclic extensions of $F$ (so $\gamma = \omega$) can be embedded in a degree-$p^2$ cyclic extensions and some cannot; I don’t think this can be detected by $c$ alone.
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