Action of $w_0$ on $V^L$ for orthogonal and exceptional groups

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January 25, 2022

In this note, we present some results that partially answer the following question. Let $G$ be a simple real Lie group; what is the set of representations $V$ of $G$ in which the longest element $w_0$ of the restricted Weyl group $W$ acts nontrivially on the subspace $V^L$ of $V$ formed by vectors that are invariant by $L$, the centralizer of a maximal split torus of $G$? We give a conjectural answer to that question, as well as the experimental results that back this conjecture, when $G$ is either an orthogonal group (real form of $SO_n(\mathbb{C})$ for some $n$) or an exceptional group.

1 Introduction and motivation

1.1 Basic notations and statement of problem

Let $G$ be a semisimple real Lie group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{g}^\mathbb{C}$ the complexification of $\mathfrak{g}$. We start by establishing the notations for some well-known objects related to $\mathfrak{g}$.

- We choose in $\mathfrak{g}$ a Cartan subspace $\mathfrak{a}$ (an abelian subalgebra of $\mathfrak{g}$ whose elements are diagonalizable over $\mathbb{R}$ and which is maximal for these properties).
- We choose in $\mathfrak{g}^\mathbb{C}$ a Cartan subalgebra $\mathfrak{h}^\mathbb{C}$ (an abelian subalgebra of $\mathfrak{g}^\mathbb{C}$ whose elements are diagonalizable and which is maximal for these properties) that contains $\mathfrak{a}$.
- We denote $L := Z_G(\mathfrak{a})$ the centralizer of $\mathfrak{a}$ in $G$, $\mathfrak{l}$ its Lie algebra.
- Let $\Delta$ be the set of roots of $\mathfrak{g}^\mathbb{C}$ in $(\mathfrak{h}^\mathbb{C})^\ast$. We shall identify $(\mathfrak{h}^\mathbb{C})^\ast$ with $\mathfrak{h}^\mathbb{C}$ via the Killing form. We call $\mathfrak{h}(\mathbb{R})$ the $\mathbb{R}$-linear span of $\Delta$; it is given by the formula $\mathfrak{h}(\mathbb{R}) = \mathfrak{a} \oplus i\mathfrak{a}^\perp$.
- We choose on $\mathfrak{h}(\mathbb{R})$ a lexicographical ordering that “puts $\mathfrak{a}$ first”, i.e. such that every vector whose orthogonal projection onto $\mathfrak{a}$ is positive is itself positive. We call $\Delta^+$ the set of roots in $\Delta$ that are positive with respect to this ordering, and we let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be the set of simple roots in $\Delta^+$ (where $r$ is the rank of $\mathfrak{g}^\mathbb{C}$). Let $\varpi_1, \ldots, \varpi_r$ be the corresponding fundamental weights.
• We introduce the **dominant Weyl chamber**

\[ \mathfrak{h}^+ := \{ X \in \mathfrak{h} \mid \forall i = 1, \ldots, r, \quad \alpha_i(X) \geq 0 \}, \]

and the **dominant restricted Weyl chamber**

\[ a^+ := \mathfrak{h}^+ \cap a. \]

• We introduce the **restricted Weyl group** \( W := N_G(a)/Z_G(a) \) of \( G \). Then \( a^+ \) is a fundamental domain for the action of \( W \) on \( a \). We define the **longest element** of the restricted Weyl group as the unique element \( w_0 \in W \) such that \( w_0(a^+) = -a^+ \).

• For each **dominant integral weight** \( \lambda \) of \( \mathfrak{g}^\mathbb{C} \) (i.e. linear combination of the fundamental weights \( \varpi_i \) with nonnegative integer coefficients), we denote by \( V_\lambda \) the irreducible representation of \( \mathfrak{g} \) with highest weight \( \lambda \).

Our goal is to study the action of \( W \), and more specifically of \( w_0 \), on various representations \( V \) of \( G \). Note however that this action is ill-defined: indeed if we want to see the abstract element \( w_0 \in W = N_G(a)/Z_G(a) \) as the projection of some concrete map \( \tilde{w}_0 \in N_G(a) \in G \), then \( \tilde{w}_0 \) is defined only up to multiplication by an element of \( Z_G(a) = L \), whose action on \( V \) can of course be nontrivial.

This naturally suggests the idea of restricting to \( L \)-invariant vectors. Given a representation \( V \) of \( \mathfrak{g} \), we denote

\[ V^L := \{ v \in V \mid \forall l \in L, \quad l \cdot v = v \} \]

the \( L \)-invariant subspace of \( V \): then \( W \), and in particular \( w_0 \), has a well-defined action on \( V^L \).

Our goal is to characterize, for a given semisimple real Lie group \( G \), the representations \( V \) of \( G \) for which the action of \( w_0 \) on \( V^L \) is nontrivial. This problem naturally splits into two subproblems (see [Smi20a] for a more extended discussion):

**Problem 1.1.** Given a semisimple Lie algebra \( \mathfrak{g} \) and a dominant integral weight \( \lambda \), give a simple necessary and sufficient condition for having \( V_\lambda^L \neq 0 \).

**Problem 1.2.** Given a simple Lie algebra \( \mathfrak{g} \) and a dominant integral weight \( \lambda \), assuming that \( V_\lambda^L \neq 0 \), give:

(i) a simple necessary and sufficient condition for having \( w_0|_{V_\lambda^L} = \pm \text{Id} \);

(ii) a criterion to determine the actual sign.

In [Smi22b], we have already completely solved problem \[1.1\]. In [LFS18], we have solved problem \[1.2\] in the case where \( \mathfrak{g} \) is split. In this note, we shall present our recent work on this latter problem.
1.2 Background and motivation

These two problems arose from the author's work in geometry. The interest of this particular algebraic property is that it furnishes a sufficient, and presumably necessary, condition for another, geometric property of $V$. Namely, the author obtained the following result:

**Theorem 1.3.** [Smir] Let $G$ be a semisimple real Lie group, $V$ a representation of $G$. Suppose that the action of $w_0$ on $V^L$ is nontrivial. Then there exists, in the affine group $G \ltimes V$, a subgroup $\Gamma$ whose linear part is Zariski-dense in $G$, which is free of rank at least 2, and acts properly discontinuously on the affine space corresponding to $V$.

He, and other people, also proved the converse statement in some special cases:

**Theorem 1.4.** The converse holds, for irreducible $V$:

- [Smir] if $G$ is split, but not of type $A_n$ ($n \geq 2$), $D_{2n+1}$ or $E_6$;
- [Smir] if $G$ is split, has one of these types, and $V$ satisfies a very restrictive additional assumption (see [Smir] for the precise statement);
- [AMS11] if $G = \text{SO}(p,q)$ for arbitrary $p$ and $q$, and $V = \mathbb{R}^{p+q}$ is the standard representation.

Moreover, it seems plausible that, by combining the approaches of [Smir] and [AMS11], we might prove the converse in all generality. This geometric property is related to the so-called Auslander conjecture [Aus64], which is an important conjecture that has stood for more than fifty years and generated an enormous amount of work: see e.g. [Mil77, Mar83, FG83, AMS02, DGK16] and many many others. For the statement of the conjecture as well as a more comprehensive survey of past work on it, we refer to [DDGS22].

1.3 Statement of main results

We have run some numerical experiments that allow us to conjecture the answer to Problem 1.2(ii) in the case where $g^C$ is of type $B_r$, $D_r$ or exceptional. Indeed, our numerical experiments prove this conjecture in some particular cases (see theorem 1.6 below).

In the case where $g^C$ is of type $A_r$ or $C_r$, we have also made some computations in low rank. Unfortunately, the data we have is not sufficient to be able to predict the general pattern (and we do not have enough computational power to generate more); see also the final remark in [Smir] for more details.

**Conjecture 1.5.** Assume that $g^C$ is of type $B_r$ (for some $r \geq 1$), $D_r$ (for some $r \geq 3$), $E_6$, $E_7$, $E_8$, $F_4$ or $G_2$. Let $\lambda$ be a dominant integral weight of $g^C$, $V_\lambda$ the irreducible (complex) representation of $g$ with highest weight $\lambda$. Then:

(i) If $\lambda$ is one of the weights listed in Table 1, then $w_0|_{V_\lambda} = \pm \text{Id}$. 
(ii) If \( V^L_\lambda \neq 0 \) (this can be looked up in \[\text{Smi22b, Table 1}\]) but \( \lambda \) does not occur in Table \[\text{A}\] then \( w_0|_{V^L_\lambda} \neq \pm \text{Id} \).

Here are the cases in which we have checked this conjecture. (In fact, in a few special cases where we judged it to be useful and found it feasible, we have actually gone a bit farther than the cutoff figures listed below; but these details would be too tedious to list.)

**Proposition 1.6.**

- Conjecture 1.5.(ii) holds for all real forms of \( B_r \) with \( r \leq 7 \), of \( D_r \) with \( r \leq 9 \), and of all exceptional algebras.

- Conjecture 1.5.(i) holds for all the algebras in the same list, for weights \( \lambda = \sum_{i=1}^{r} c_i \varpi_i \) satisfying \( c_i \leq 3p_i \) for all coefficients \( c_i \), where \( p_i \) is the least positive integer such that \( V^L_{p_i \varpi_i} \neq 0 \). (In the case of \( i \in \{2p, 2p + 1\} \) for \( g = \mathfrak{so}(p,q) \) with \( p = \frac{r+q-2}{4} \), this definition technically makes \( p_i \) infinite, but we take the convention \( p_i = 1 \) instead.)

The proof of this proposition relies on the additivity property \[\text{Smi20a, Proposition 1.(iii)}\], which reduces it to a finite number of computations; and an algorithm to compute the restriction of \( w_0 \) to \( V^L \) that the author has recently developed and implemented in the LiE software \[\text{vLCL00}\]. The details of that algorithm will be published in a subsequent paper.

### 1.4 Acknowledgements

The author is supported by the European Research Council (ERC) under the European Union Horizon 2020 research and innovation programme (ERC starting grant DiGGeS, grant agreement No. 715982).
Table 1: Values of $\lambda$ for which $w_0|_{V_L} = \pm \text{Id}$, for various algebras $g$. The fundamental weights $\varpi_i$ are numbered using the Bourbaki ordering [Bou68]. The coefficients $k$, $l$ and $m$ range in the nonnegative integers. Note that the lists may contain duplicates.

Table 1a – Real forms of $B_r = \mathfrak{so}_{2r+1}(\mathbb{C})$ ($r \geq 1$). In $\mathfrak{so}(p, q)$, we always assume $p \leq q$.

| The algebra $g$ | Weights $\lambda$ | Conditions on indices | Conditions on coefficients |
|-----------------|-------------------|-----------------------|---------------------------|
| $\mathfrak{so}(p, q)$ | $\lambda = k\varpi_i + l\varpi_{2p}$ | $i = 1$ or $2p - 1$ | any $k$, any $l$ |
| $p \leq \frac{p+q}{4}$ | $p + q$ odd | $2 = i = 2p - 2$ | any $k$, any $l$ |
| | | $i = 2$ or $2p - 2$ | $k \leq 2$, any $l$ |
| | | $2 < i < 2p - 2 \land 2|i$ | $k \leq 1$, any $l$ |
| $\mathfrak{so}(p, q)$ | $\lambda = k\varpi_i + l\varpi_{q-p}$ | $i = 1$ | any $k; l \leq 2$ |
| $p = \frac{p+q+1}{4}$ | | $2 = i = q - p - 1$ | any $k; l \leq 2$ |
| | | $2 = i < q - p - 1$ | $k \leq 2$, $l \leq 2$ |
| | | $2 < i < q - p - 1 \land 2|i$ | $k \leq 1$, $l \leq 2$ |
| | | $i = q - p$ | any $k$, any $l$ |
| | | $2 < i = q - p - 1$ | $k \leq 2$, $l \leq 2$ |
| $\mathfrak{so}(p, q)$ | $\lambda = k\varpi_i + l\varpi_{q-p}$ | $i = 1$ | any $k; l \leq 1$ |
| $p > \frac{p+q+1}{4}$ | | $2 = i < q - p$ | $k \leq 2$, $l \leq 1$ |
| | | $2 < i < q - p \land 2|i$ | $k \leq 1$, $l \leq 1$ |
| $p + q$ odd | | | |
| $\mathfrak{so}(p, q)$ | $\lambda = k\varpi_i$ | $2 = i = \frac{p+q-1}{2}$ | any $k$ |
| $p > \frac{p+q+1}{4}$ | | $q - p < i = 2$ | $k \leq 2$ |
| | | $q - p < i$ | $k \leq 1$ |
| | | $q - p + 1 = i = \frac{p+q-1}{2}$ | $k \leq 4$ |
| | | $q - p < i = \frac{p+q-1}{2}$ | $k \leq 2$ |
Table 1b – Real forms of $D_r = \mathfrak{so}_{2r}(\mathbb{C})$ ($r \geq 3$). In $\mathfrak{so}(p,q)$, we always assume $p \leq q$; and we denote by $r := \frac{p+q}{4}$ the (complex) rank.

| The algebra $\mathfrak{g}$ | Weights $\lambda$ | Conditions on indices | Conditions on coefficients |
|---------------------------|-------------------|-----------------------|---------------------------|
| $\mathfrak{so}(p,q)$     |                   |                       |                           |
| $p \leq \frac{p+q}{4} - 1$ |                   | same as for $p \leq \frac{p+q}{4}$ in the $B_r$ case |                           |
| $p + q$ even              |                   |                       |                           |
| $\mathfrak{so}(p,q)$     |                   | same as for $p \leq \frac{p+q}{4}$ in the $B_r$ case, with $\varpi_{2p}$ replaced by $(\varpi_{2p} + \varpi_{2p+1})$ |                           |
| $p = \frac{p+q}{4}$      |                   | same as for $p \leq \frac{p+q}{4}$ in the $B_r$ case, with $\varpi_{2p}$ replaced by $\varpi_{2p-1}$ |                           |
| $p > \frac{p+q}{4}$      |                   |                       |                           |
| $p + q \equiv 0 \pmod{4}$ |                   |                       |                           |
| $\lambda = k\varpi_i$    | $i = 1$          | $2 < i < r - 1 \land 2|i$ | $k \leq 2$               |
|                           | $i = 2$          | $i \in \{r-1,r\} \land r = 4$ | $k \leq 1$               |
|                           |                  | $i \in \{r-1,r\} \land p = \frac{p+q}{4} + 1$ | $k \leq 4$               |
|                           |                  | $i \in \{r-1,r\}$ | $k \leq 2$               |
| $\mathfrak{so}^*(6)$     | $\lambda = k\varpi_i + l\varpi_i$ | $i \in \{2,3\}$ | any $k$, any $l$          |
| $\mathfrak{so}^*(8)$     |                   |                       |                           |
| $\mathfrak{so}^*(10)$    | $\lambda = k\varpi_i$ | $i \in \{1,4,5\}$ | any $k$                  |
| $\mathfrak{so}^*(12)$    | $\lambda = k\varpi_i$ | $i \in \{1,2,6\}$ | any $k$                  |
|                           |                  | $i = 4$               | $k \leq 1$               |
|                           |                  | $i = 5$               | $k \leq 2$               |
| $\mathfrak{so}^*(2r)$    | $\lambda = k\varpi_i$ | $i = 1$               | any $k$                  |
| $r > 5$, $r$ odd         |                   |                       |                           |
|                           | $i = 1$          | $2 < i < r - 1 \land 2|i$ | $k \leq 2$               |
|                           | $i \in \{r-1,r\}$ | $k \leq 1$               |
| $\mathfrak{so}^*(2r)$    | $\lambda = k\varpi_i$ | $i = 1$               | any $k$                  |
| $r > 6$, $r$ even        |                   |                       |                           |
|                           | $i = 2$          | $2 < i < r - 1 \land 2|i$ | $k \leq 2$               |
|                           | $i \in \{r-1,r\}$ | $k \leq 1$               |
### Table 1c – Real forms of exceptional algebras.

| The algebra $\mathfrak{g}$ | Weights $\lambda$ | Conditions on indices | Conditions on coefficients |
|--------------------------|------------------|----------------------|---------------------------|
| E I, E II                | $\lambda = 0$    |                      |                           |
| E III                   | $\lambda = k\omega_i$ | $i \in \{1, 6\}$ | any $k$                  |
| E IV                    | $\lambda = k\omega_2 + l\omega_i$ | $i \in \{1, 3, 5, 6\}$ | any $k$, any $l$          |
| E V, E VI               | $\lambda = k\omega_i$ | $i = 1$             | $k \leq 2$              |
|                         | $\lambda = k\omega_i$ | $i \in \{6, 7\}$  | $k \leq 1$              |
| E VII                   | $\lambda = k\omega_i$ | $i = 1$             | any $k$                  |
|                         | $\lambda = k\omega_i$ | $i \in \{6, 7\}$  | $k \leq 1$              |
| E VIII, E IX            | $\lambda = k\omega_i$ | $i = 1$             | $k \leq 1$              |
|                         | $\lambda = k\omega_i$ | $i = 8$             | $k \leq 2$              |
| F I                     | $\lambda = k\omega_i$ | $i \in \{1, 4\}$  | $k \leq 2$              |
| F II                    | $\lambda = k\omega_1 + l\omega_2 + m\omega_i$ | $i \in \{3, 4\}$ | any $k$, any $l$, any $m$ |
| G                       | $\lambda = k\omega_i$ | $i \in \{1, 2\}$  | $k \leq 2$              |

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