Quartetting in Nuclear Matter

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A general theory for the condensation of strongly bound quartets in infinite nuclear matter is presented. Critical temperatures for symmetric and asymmetric nuclear matter are evaluated. A fully nonlinear theory for the quartet order parameter, based on an analogy of the Gorkov approach to pairing, is presented and solved. The strong qualitative difference with pairing is pointed out.

\section{Introduction}

Since about one decade, there has been a renewed and strong interest in gas like $\alpha$ particle states in nuclei, as, e.g. the famous Hoyle state in $^{12}$C. This stems essentially from the fact that in 2001 the prediction was made that besides the Hoyle state with three $\alpha$’s, other similar states with more $\alpha$’s can exist and that those states can be considered as being of $\alpha$ particle condensate type.\textsuperscript{1} Of course, in finite systems, like nuclei, neither pairing nor quartetting can show phase locking as in macroscopic systems. They can be considered as precursors to condensation phenomena in macroscopic systems, like neutron or nuclear matter. The proposed THSR wave function\textsuperscript{1} for the description of the $\alpha$ gas states is very successful for nuclei. It is a number conserving ansatz, in spirit similar to a number projected BCS wave function, which cannot be extended to the infinite matter case. We, therefore, will discuss in this article how $\alpha$ particle condensation can be described for the case of infinite nuclear matter. This may be relevant for compact stellar objects. However, the general theory may also serve for other systems where quartet correlations are important as, e.g., in a gas of excitons where bi-excitons may be formed. Also in the physics of cold atoms the possibility of trapping four different fermions and, thus, of quartet condensation, may exist in the future. The paper is organised as follows. In §2 we will describe how to determine the critical temperature in symmetric and asymmetric matter. In §3 a general formula for the quartet order parameter will be presented and applied to symmetric nuclear matter using a Gorkov type of formalism. Finally in §4 we give some perspectives and conclude.

\section{Critical temperature of quartetting and $\alpha$ particle condensation}

In this section, we will discuss the critical temperature of quartet condensation in symmetric and asymmetric nuclear matter. A comparison with the pairing case will
also be given. For this, we start out to establish an in medium four body equation.

In medium four particle correlation for example appears if one adds an $\alpha$-particle on top of a double magic nucleus such as $^{208}\text{Pb}$, or in semiconductors where in the gas of excitons, i.e. $p-h$ bound states, may appear bi-excitons, i.e. bound state of two $p-h$ pairs. The effective wave equation contains, in mean field approximation, the Hartree-Fock self-energy shift of the single-particle energies as well as the Pauli blocking of the interaction. We consider for the derivation the Equation of Motion Method (EMM), well known from the Random Phase Approximation (RPA)\textsuperscript{2}

\[
\langle [\delta Q, [H, \delta Q^\dagger_\alpha]] \rangle = E_\rho \langle [\delta Q, Q^\dagger_\alpha] \rangle, \tag{2.1}
\]

where

\[
Q^\dagger_\alpha = \sqrt{\frac{1}{4!}} \sum_{1234} \psi_{4,\alpha}(1234) a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger, \tag{2.2}
\]

is the quartet creator. Of course, we also could consider analogously a quartet destructor. Considering a fermion Hamiltonian with two body interaction, we evaluate the double commutator in (2.1) and perform the expectation value with the Hartree-Fock ground state, which is, as we know, the standard procedure to obtain RPA equations in linearised form. The result is\textsuperscript{3,4}

\[
(E_\rho - \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4) \psi_{1,\rho}(1234) = [1 - f_1 - f_2] V_{121'}2' \psi_{1,\rho}(1'2'34) + \text{permutations}, \tag{2.3}
\]

where $\epsilon_i$ is the single particle energy, i.e. the kinetic energy plus HF shift. A similar equation can be obtained for three particles, i.e. $A = 3$. This equation has exactly the same structure as a free four body Schrödinger equation with the only difference that the two body matrix elements $V_{ijij'}$ are premultiplied with the phase space factor $f_i f_j - f_i f_j = 1 - f_i - f_j$ where the $f_i$ are the Fermi-Dirac functions, and $f_i = 1 - f_i$. The Fermi function is equal to the step function at $T = 0$, i.e. $f_i = \Theta(\mu - \epsilon_i)$, but immediately generalisable to finite temperature with $f_i = [1 + e^{(\epsilon_i - \mu)/T}]^{-1}$, a result which could have been derived employing Matsubara Green functions. In (2.3) the $\epsilon_i$ are as usual the HF single particle energies and $V_{ijij'} = \langle ij | V | ij' \rangle - \langle jj | V | jj' \rangle$ is the antisymmetrized matrix element of the two body interaction. We realise that the phase space factor in (2.3) is exactly the same as in the two particle RPA\textsuperscript{2}. It turns out to be a general rule in deriving higher $pp$ or $ph$ RPA’s that the $pp$ and $ph$ matrix elements are always premultiplied with the same phase space factors as in the corresponding standard $pp$ or $ph$ RPA’s.

We are interested in an example of nuclear physics where the $\alpha$-particle constitutes a particularly strongly bound cluster of four nucleons. One can ask the question how, for a fixed temperature, the binding energy of the $\alpha$-particle varies with increasing temperature.

The effective wave equation has been solved using separable potentials for $A = 2$ by integration. For $A = 3, 4$ we can use a Faddeev approach.\textsuperscript{5} The shifts of binding energy can also be calculated approximately via perturbation theory. In Fig. 1 we show the shift of the binding energy of the light clusters (deuteron ($d$), triton
(t), helion (h) and α) in symmetric nuclear matter as a function of density for temperature $T = 10$ MeV.

It is found that the cluster binding energy decreases with increasing density. Finally, at the Mott density $\rho_{A,n}^{\text{Mott}}(T)$ the bound state is dissolved. The clusters are not present at higher densities, merging into the nucleonic medium. It is found that the α particle already dissolves at a density $\rho_{\alpha}^{\text{Mott}} \approx \rho_0/10$, see Fig. 1. For a given cluster type characterized by $A, n$, we can also introduce the Mott momentum $P_{A,n}^{\text{Mott}}(\rho, T)$ in terms of the ambient temperature $T$ and nucleon density $\rho$, such that the bound states exist only for $P \geq P_{A,n}^{\text{Mott}}(\rho, T)$. We do not present an example here, but it is intuitively clear that a cluster with high c.o.m. momentum with respect to the medium is less affected by the Pauli principle than a cluster at rest.

Since Bose condensation only is of relevance for deuterons (d) and α’s, and the fraction of $d$, tritons (t) and helions (h) becomes low compared with that of α’s with increasing density, we can neglect the contribution of the latter to the equation of state. Consequently, if we further neglect the contribution of the four-particle scattering phase shifts in the different channels, we can now construct an equation of state $\rho(T, \mu) = \rho_{\text{free}}(T, \mu) + \rho_{d}^{\text{bound}}(T, \mu) + \rho_{\alpha}^{\text{bound}}(T, \mu)$ such that α-particles determine the behavior of symmetric nuclear matter at densities below $\rho_{\alpha}^{\text{Mott}}$ and temperatures below the binding energy per nucleon of the α-particle. The formation of deuteron clusters alone gives an incorrect description because the deuteron binding energy is small, and, thus, the abundance of $d$-clusters is small compared with that of α-clusters. In the low density region of the phase diagram, α-matter emerges as an adequate model for describing the nuclear-matter equation of state.

With increasing density, the medium modifications – especially Pauli blocking – will lead to a deviation of the critical temperature $T_c(\rho)$ from that of an ideal Bose gas of α-particles (the analogous situation holds for deuteron clusters, i.e., in the isospin-singlet channel).

Symmetric nuclear matter is characterized by the equality of the proton and neutron chemical potentials, i.e., $\mu_p = \mu_n = \mu$. Then an extended Thouless condition based on the relation for the four-body $T$-matrix (in principle equivalent to (2.3) at eigenvalue $4\mu$)

$$T_4(1234, 1''2''3''4'') = \sum_{1'2'3'4'} \left[ \frac{V_{12'1'2'}[1 - f_1 - f_2]}{4\mu - \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4} \delta(3, 3')\delta(4, 4') + \text{cycl.} \right]$$

Fig. 1. Shift of binding energy of the light clusters (d - dash dotted, t/h - dotted, and α - dashed: perturbation theory, full line: non-perturbative Faddeev-Yakubovski equation) in symmetric nuclear matter as a function of density for given temperature $T = 10$ MeV.
serves to determine the onset of Bose condensation of $\alpha$-like clusters, noting that the existence of a solution of this relation signals a divergence of the four-particle correlation function. An approximate solution has been obtained by a variational approach, in which the wave function is taken as Gaussian incorporating the correct solution for the two-particle problem\(^3\).

On the other hand, Eq. (2.4), respectively (2.3) at eigenvalue $4\mu$, has also been solved numerically exactly by the Faddeev-Yakubovsky method employing the Malfliet-Tjon force\(^5\). The results for the critical temperature of $\alpha$-condensation is presented in Fig. 2 as a function of the chemical potential $\mu$. The exact solution could only be obtained for negative $\mu$, i.e. when there exists a bound cluster. It is, therefore, important to try yet another approximate solution of the in-medium four-body equation. Since the $\alpha$-particle is strongly bound, we make a momentum projected mean field ansatz for the quartet wave function\(^2\),\(^7\)

$$\Psi_{1234} = (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \times \prod_{i=1}^{4} \varphi(\mathbf{k}_i) \chi^{ST}, \quad (2.5)$$

where $\chi^{ST}$ is the spin-isospin function which we suppose to be the one of a scalar ($S = T = 0$). We will not further mention it from now on. We work in momentum space and $\varphi(\mathbf{k})$ is the as-yet unknown single particle $0S$ wave function. In position space, this leads to the usual formula\(^2\) $\Psi_{1234} \rightarrow \int d^3R \prod_{i=1}^{4} \tilde{\varphi}(\mathbf{r}_i - \mathbf{R})$ where $\tilde{\varphi}(\mathbf{r}_i)$ is the Fourier transform of $\varphi(\mathbf{k}_i)$. If we take for $\varphi(\mathbf{k}_i)$ a Gaussian shape, this gives: $\Psi_{1234} \rightarrow \exp[-c \sum_{1 \leq i < k \leq 4} (\mathbf{r}_i - \mathbf{r}_k)^2]$ which is the translationally invariant ansatz often used to describe $\alpha$-clusters in nuclei. For instance, it is also employed in the $\alpha$-particle condensate wave function of Tohsaki, Horiuchi, Schuck, Röpke (THSR) in Ref. 1).

Inserting the ansatz (2.5) into (2.3) and integrating over superfluous variables, or minimizing the energy, we arrive at a Hartree-Fock type of equation for the single particle $0S$ wave function $\varphi(k) = \varphi(|\mathbf{k}|)$ which can be solved. However, for a general two body force $V_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}'_1,\mathbf{k}'_2}$, the equation to be solved is still rather complicated. We, therefore, proceed to the last simplification and replace the two body force by a

![Fig. 2. Critical temperature of alpha and deuteron condensations as functions of (a) chemical potential and (b) density of free nucleon. Crosses (×) correspond to calculation of Eq. (2.3) with the Malfliet-Tjon interaction (MT I-III) using the Faddeev-Yakubovsky method.](image-url)
unique separable one, that is
\[ V_{k_1k_2,k'_1k'_2} = \lambda e^{-k^2/k_0^2} e^{-k'^2/k_0^2} (2\pi)^3 \delta^{(3)}(K - K'), \] (2.6)
where \( k = (k_1 - k_2)/2, \ k' = (k'_1 - k'_2)/2, \ K = k_1 + k_2, \) and \( K' = k'_1 + k'_2. \) This means that we take a spin-isospin averaged two body interaction and disregard that in principle the force may be somewhat different in the \( S,T = 0,1 \) or \( 1,0 \) channels. It is important to remark that for a mean field solution the interaction only can be an effective one, very different from a bare nucleon-nucleon force. This is contrary to the usual gap equation for pairs, to be considered below, where, at least in the nuclear context, a bare force can be used as a reasonable first approximation.

We are now ready to study the solution of (2.3) and (2.4) for the critical temperature \( T_{c\alpha} \), defined by the point where the eigenvalue equals 4\( \mu \). For later comparison, the deuteron (pair) wave function at the critical temperature is also deduced from the Thouless criterion for the onset of pairing with (2.6) to be
\[ \phi(k) = \frac{1 - 2f(\varepsilon(k))}{k^2/m - 2\mu} \lambda e^{-k^2/k_0^2} \int \frac{d^3k'}{(2\pi)^3} e^{-k'^2/k_0^2} \phi(k'), \] (2.7)
where \( \phi(k) \) is the relative wave function of two particles given by \( \Psi_{12} \rightarrow \phi(\sqrt{\frac{1 - k_1k_2}{2}}) \delta^{(3)}(k_1 + k_2), \) and \( \varepsilon(k) = k^2/(2m) - \mu. \) We also neglected the momentum dependence of the Hartree-Fock mean field shift in Eq. (2.7). It, therefore, can be incorporated into the chemical potential \( \mu. \) With Eq. (2.7), the critical temperature of pair condensation is obtained from the following equation:
\[ 1 = -\lambda \int \frac{d^3k}{(2\pi)^3} \frac{1 - 2f(\varepsilon(k))}{k^2/m - 2\mu} e^{-k^2/k_0^2}. \] (2.8)

In order to determine the critical temperature for \( \alpha \)-particle condensation, we have to adjust the temperature so that the eigenvalue of (2.3) and (2.4) equals 4\( \mu. \) The result is shown in Fig. 2(a). In order to get an idea how this converts into a density dependence, we use for the moment the free gas relation between the density \( n(0) \) of uncorrelated nucleons and the chemical potential
\[ n(0) = 4 \int \frac{d^3k}{(2\pi)^3} f(\varepsilon). \] (2.9)
We are well aware of the fact that this is a relatively gross simplification, for instance at the lowest densities, and we intend to generalize our theory in the future so that correlations are included into the density. This may be done along the work of Nozières and Schmitt-Rink.\(^8\) The two open constants \( \lambda \) and \( k_0 \) in Eq. (2.6) are determined so that binding energy (\(-28.3 \text{ MeV}\)) and radius (1.71 fm) of the free \((f_i = 0) \alpha\)-particle come out right. The adjusted parameter values are: \( \lambda = -992 \) MeV fm\(^3\), and \( k_0 = 1.43 \text{ fm}^{-1}. \) The results of the calculation are shown in Fig. 2.

In Fig. 2 the maximum of critical temperature \( T_{c\alpha}^{\text{max}} \) is at \( \mu = 5.5 \text{ MeV}, \) and the \( \alpha \)-condensation can exist up to \( \mu_{\text{max}} = 11 \text{ MeV}. \) It is very remarkable that the results obtained with (2.5) for \( T_{c\alpha} \) very well agree with the exact solution of (2.3) and
using the Malfliet-Tjon interaction (MT I-III) with the Faddeev-Yakubovski method also shown by crosses in Fig. 2 (the numerical solution only could be obtained for negative values of \( \mu \)). This indicates that \( T^c_{\alpha} \) is essentially determined by the Pauli blocking factors.

In Fig. 2 we also show the critical temperature for deuteron condensation derived from Eq. (2.8). In this case, the bare force is adjusted with \( \lambda = -1305 \text{ MeV fm}^3 \) and \( k_0 = 1.46 \text{ fm}^{-1} \) to get experimental energy \((-2.2 \text{ MeV})\) and radius \((1.95 \text{ fm})\) of the deuteron. It is seen that at higher densities deuteron condensation wins over the one of \( \alpha \)-particles. The latter breaks down rather abruptly at a critical positive value of the chemical potential. Roughly speaking, this corresponds to the point where the \( \alpha \)-particles start to overlap substantially. This behavior stems from the fact that Fermi-Dirac distributions in the four body case, see (2.4), can never become step-like, as in the two body case, even not at zero temperature, since the pairs in an \( \alpha \)-particle are always in motion. Therefore, no threshold effect occurs as with pairing for Cooper pairs at rest. As a consequence, \( \alpha \)-condensation generally only exists as a BEC phase and the weak coupling regime is absent.

An important consequence of this study is that at the lowest temperatures, Bose-Einstein condensation occurs for \( \alpha \) particles rather than for deuterons. As the density increases within the low-temperature regime, the chemical potential \( \mu \) first reaches \(-7 \text{ MeV} \), where the \( \alpha \)'s Bose-condense. By contrast, Bose condensation of deuterons would not occur until \( \mu \) rises to \(-1.1 \text{ MeV} \).

The “quartetting” transition temperature sharply drops as the rising density approaches the critical Mott value at which the four-body bound states disappear. At that point, pair formation in the isospin-singlet deuteron-like channel comes into play, and a deuteron condensate will exist below the critical temperature for BCS pairing up to densities above the nuclear-matter saturation density \( \rho_0 \), as described in the previous Section. Of course, also the well known and studied isovector \( n-n \) and \( p-p \) pairing develops. Usually, \( p-n \) pairing in the isoscalar channel is stronger. However, the competition between the different pairing channels is not well known. The critical density at which the \( \alpha \) condensate disappears is estimated to be \( \rho_0/3 \). Therefore, \( \alpha \)-particle condensation primarily only exists in the Bose-Einstein-Condensed (BEC) phase and there does not seem to exist a phase where the quartets acquire a large extension as Cooper pairs do in the weak coupling regime. However, the variational approaches of Ref. [3] and of Eq. (2.5) on which this conclusion is based represent only first attempts at the description of the transition from quartetting to pairing. The detailed nature of this fascinating transition remains to be clarified. Many different questions arise in relation to the possible physical occurrence and experimental manifestations of quartetting: Can we observe the hypothetical “\( \alpha \)-condensate” in nature? What about thermodynamic stability? What happens with quartetting in asymmetric nuclear matter? Are more complex quantum condensates possible? What is their relevance for finite nuclei? As discussed, the special type of microscopic quantum correlations associated with quartetting may be important in nuclei, its role in these finite inhomogeneous systems being similar to that of pairing[1].

On the other hand, if at all, \( \alpha \)-condensation in compact star occurs at strongly
asymmetric matter. It is, therefore, important to generalize the above study for symmetric nuclear matter to the asymmetric case. This can be done straightforwardly, again using our momentum projected mean field ansatz (2.5) generalized to the asymmetric case. This implies to introduce two chemical potentials, one for neutrons and for protons. We also have to distinguish two single particle wave functions in our product ansatz which now read:

$$\psi_{1234} \rightarrow \varphi_p(k_1)\varphi_p(k_2)\varphi_n(k_3)\varphi_n(k_4)\chi_0(2\pi)^3\delta(k_1 + k_2 + k_3 + k_4) \quad (2.10)$$

where $\varphi_\tau(k_i) = \varphi_\tau(|k_i|)$ is the s-wave single particle wave functions for protons ($\tau = p$) and neutrons ($\tau = n$), respectively. $\chi_0$ is the spin-isospin singlet wave function. This now leads to two coupled equations of the Hartree-Fock type for $\varphi_n$ and $\varphi_p$. For the force we use the same as in the symmetric case.

Fig. 3(a) shows the critical temperature of $\alpha$ condensation as a function of the total chemical potential $\mu_{\text{total}} = \mu_p + \mu_n$. We see that $T_c$ decreases as the asymmetry, given by the parameter

$$\delta = \frac{n_n - n_p}{n_n + n_p} \quad (2.11)$$

increases. This is in analogy with the deuteron case (also shown) which already had been treated in Refs. 10,11. On the other hand, in Fig. 3(b), it is also interesting to show $T_c$ as a function of the free density which is

$$n_{\text{total}}^{(0)} = n_p^{(0)} + n_n^{(0)} \quad (2.12)$$

$$n_{p,n}^{(0)} = 2 \int \frac{d^3k}{(2\pi)^3} f_{p,n}(k) \quad (2.13)$$

where the factor two in front of the integral comes from the spin degeneracy, and $f_{p,n}(k) = [1 + \exp(k^2/2m - \mu_{p,n})]^{-1}$. It should be emphasized, however, that in the above relation between density and chemical potential, the free gas relation is used and correlations in the density have been neglected. In this sense the dependence of $T_c$ on density only is indicative, more valid at the higher density side. The very low density part where the correlations play a more important role, will be treated in the future. For instance, it will be important to recover the critical temperature corresponding to an ideal Bose gas in that limit. Techniques similar to
the one of Nozières and Schmitt-Rink may be employed. It should, however, be
stressed that the dependence of $T_c$ on the chemical potential as in Fig. 3(a), stays
unaltered. It is only the relation between the chemical potential and the (correlated)
density which changes.

The fact that for more asymmetric matter the transition temperature decreases,
is natural, since as the Fermi levels become more and more unequal, the proton-
neutron correlations will be suppressed. For small $\delta$’s, i.e., close to the symmetric
case, $\alpha$ condensation (quartetting) breaks down at smaller density (smaller chemical
potential) than deuteron condensation (pairing). This effect has already been dis-
cussed above for symmetric nuclear matter. For $\delta$’s close to one, i.e. strong asym-
metries, the behavior is opposite, i.e., deuteron condensation breaks down at smaller
densities than $\alpha$ condensation, because the small binding energy of the deuteron can
not compensate the difference of the chemical potentials.

More precisely, for small $\delta$’s, the deuteron with zero center of mass momentum
is only weakly influenced by the density or the total chemical potential as can be
seen in Fig. 3. However, as $\delta$ increases, the different chemical potentials for protons
and neutrons very much hinders the formation of proton-neutron Cooper pairs in the
isoscalar channel for rather obvious reasons. The point to make here is that because
of the much stronger binding per particle of the $\alpha$-particle, the latter is much less
influenced by the increasing difference of the chemical potentials. For the strong
asymmetry $\delta = 0.9$ in Fig. 3 then finally $\alpha$-particle condensation can exist up to
$n_{\text{total}} = 0.02$ fm$^{-3}$ ($\mu_{\text{total}} = 9.3$ MeV), while the deuteron condensation exists only
up to $n_{\text{total}} = 0.005$ fm$^{-3}$ ($\mu_{\text{total}} = 6.0$ MeV).

Overall, the behavior of $T_c$ is more or less as expected. We should, however,
remark that the critical temperature for $\alpha$-particle condensation stays quite high,
even for the strongest asymmetry considered here, namely $\delta = 0.9$. This may be
of importance for the possibility of $\alpha$-particle condensation in neutron stars and
supernovae explosions.

In conclusion the $\alpha$-particle (quartet) condensation was investigated in homo-
geeous symmetric nuclear matter as well as in asymmetric nuclear matter. We found
that the critical density at which the $\alpha$-particle condensate appears is estimated to
be around $\rho_0/3$ in the symmetric nuclear matter, and the $\alpha$-particle condensation
can occur only at low density. This result is consistent with the fact that the Hoyle
state ($0^+_2$) of $^{12}$C also has a very low density $\rho \sim \rho_0/3$. On the other hand, in
asymmetric nuclear matter, the critical temperature $T_c$ for the $\alpha$-particle condensa-
tion was found to decrease with increasing asymmetry. However, $T_c$ stays relatively
high for very strong asymmetries, a fact of importance in the astrophysical context.
The asymmetry affects deuteron pairing more strongly than $\alpha$-particle condensation.
Therefore, at high asymmetries, if at all, $\alpha$-particle condensate seems to dominate
over pairing at all possible densities.

§3. ‘Gap’ equation for the quartet order parameter

For macroscopic $\alpha$ condensation it is, of course, not conceivable to work with a
number projected $\alpha$ particle condensate wave function as we did when in finite nuclei
only a couple of $\alpha$ particles were present. We rather have to develop an analogous procedure to BCS theory but generalized for quartets. In principle a number non-conserving wave function of the type $|\alpha\rangle = \exp[\sum_{1234} z_{1234} c_{1}^+ c_{2}^+ c_{3} c_{4}^+] |\text{vac}\rangle$ would be the ideal generalization of the BCS wave function for the case of quartets. However, unfortunately, it is unknown so far (see, however, Ref. 12)) how to treat such a complicated many body wave function mathematically in a reasonable way. So, we rather attack the problem from the other end, that is with a Gorkov type of approach, well known from pairing but here extended to the quartet case. Since, naturally, the formalism is complicated, we only will outline the main ideas and refer for details to the literature.

Actually one part of the problem is written down easily. Let us guide from a particular form of the gap equation in the case of pairing. We have at zero temperature

$$\left(\epsilon_1 + \epsilon_2\right)\kappa_{12} + \left(1 - \rho_1 - \rho_2\right)\frac{1}{2} \sum_{1'2'} \tilde{V}_{121'2'} \kappa_{1'2'} = 2\mu\kappa_{12}, \quad (3.1)$$

where $\kappa_{12} = \langle c_1 c_2 \rangle$ is the pairing tensor, $\rho_i = \langle c_i^+ c_i \rangle = \frac{1}{2} \left[1 - \frac{\epsilon_i - \mu}{E_i}\right]$ with $E_i = \sqrt{\left(\epsilon_i - \mu\right)^2 + \Delta_i^2}$ the quasi-particle energy are the BCS occupation numbers, and $\tilde{V}_{121'2'}$ denotes the antisymmetrized matrix element of the two-body interaction. The $\epsilon_i$ are the usual mean field energies. Equation (3.1) is equivalent to the usual gap equation in the case of zero total momentum and opposite spin, i.e. in short hand: $2 = \bar{I}$ where the bar stands for ‘time reversed conjugate’. The extension of (3.1) to the quartet case is formally written down without problem and can be derived with EMM, slightly extending (2.3) to correlated occupation numbers at $T=0$.

$$\left(\epsilon_{1234} - 4\mu\right)\kappa_{1234} = \left(1 - \rho_1 - \rho_2\right)\frac{1}{2} \sum_{1'2'} \tilde{V}_{121'2'} \kappa_{1'2'34}$$

$$+ \left(1 - \rho_1 - \rho_3\right)\frac{1}{2} \sum_{1'3'} \tilde{V}_{131'3'} \kappa_{1'23'4} + \text{all permutations}. \quad (3.2)$$

with $\kappa_{1234} = \langle c_1 c_2 c_3 c_4 \rangle$ the quartet order parameter. This is formally the same equation as in Eq. (2.3) with, however, the Fermi-Dirac occupation numbers replaced by the zero temperature quartet correlated single particle occupation numbers, similar to the BCS case. For the quartet case, the crux lies in the determination of those occupation numbers. Let us again be guided by BCS theory or rather by the equivalent Gorkov approach. In the latter, there are two coupled equations, one for the normal single particle Green’s function (GF) and the other for the anomalous GF. Eliminating the one for the anomalous GF in inserting it into the first equation leads to a Dyson equation with a single particle mass operator:

$$M_{1;1'}^{\text{BCS}}(\omega) = \sum_{2} \frac{\Delta_{12} \Delta_{1'2'}}{\omega + \epsilon_2} \quad (3.3)$$

Fig. 4. Graphic representation of the BCS mass operator in Eq. (3.3)
with
\[ \Delta_{12} = -\frac{1}{2} \sum_{34} \bar{V}_{12,34} \kappa_{43} \quad (3.4) \]

This can be graphically represented as in Fig. 4 where \( \langle cc \rangle \) stands for the order parameter \( \kappa_{12} \) and the dot for the two body interaction.

The generalization to the quartet case is considerably more complicated but schematically the corresponding mass operator in the single particle Dyson equation can be represented graphically as in Fig. 5 with the quartet order parameter \( \langle cccc \rangle \).

In Fig. 5 we show the level density at zero temperature \( f(\omega) = \theta(-\omega) \), where it is calculated with the proton mass \( m = 938.27 \text{ MeV} \). Two cases have to be considered, chemical potential \( \mu \) positive or negative. In the latter case we have binding of the quartet. Let us first discuss the case \( \mu > 0 \). We remark that in this case, the 3h level density goes through zero at \( \omega = 0 \), i.e., since we are measuring energies with respect to the chemical potential \( \mu \), just in the region where the quartet
correlations should appear. This is at strong variance with the pairing case where the $1h$ level density, $g_{1h}(\omega) = \int \frac{d^3k}{(2\pi)^3} |\tilde{f}_k + f_k\rangle \delta(\omega + \varepsilon(k))$, does not feel any influence from the medium and, therefore, the corresponding level density varies (neglecting the mean field for the sake of the argument) like in free space with the square root of energy. In particular, this means that the level density is finite at the Fermi level. This is a dramatic difference with the quartet case and explains why Cooper pairs can strongly overlap whereas for quartets this is impossible as we will see below. We also would like to point out that the $3h$ level density is just the mirror to the $3p$ level density which has been discussed in Ref. [14].

For the case where $\mu < 0$ where anyway the $f_i$’s are zero at $T=0$, there is nothing very special, besides the fact that the three hole level density only is non-vanishing for negative values of $\omega$ and that the upper boundary is given by $\omega = 3\mu$. Therefore, the level density of Eq. (3.5) is zero for $\omega > 3\mu$. Therefore, in the BEC regime ($\mu < 0$), there is no marked difference between the pairing and quartetting cases.

With these preliminary but crucial considerations we now pass to the evaluation of the s.p. mass operator with quartet condensation. Its expression corresponding to Fig. 6 can be shown to be of the following form

$$M_{1;1}^{\text{quartet}}(\omega) = \sum_{234} \tilde{\Delta}_{1234} (f_2 \tilde{f}_3 f_4 + f_2 f_3 \tilde{f}_4) \tilde{\Delta}^*_{1234}$$

$$\omega + \varepsilon_{234}$$

$$\Delta_{1234} = \sum_{1'2'3'4'} \frac{1}{2} V_{12,1'2'} \delta_{33'} \delta_{44'} \kappa_{1'2'3'4'}$$

Again, comparing the quartet s.p. mass operator (3.6) with the pairing one (3.3), we notice the presence of the phase space factors in the former case while in Eq. (3.3) they are absent. As already indicated above, this fact implies in the
quartet case that only the Bose-Einstein condensation phase is born out whereas a 'BCS phase' (long coherence length) is absent, since the three hole level density is zero at $3\mu$ due to the non vanishing phase space factors.

The complexity of the calculation in Eq. (3.6) is much reduced using for the order parameter $\langle cccc \rangle$ our mean field ansatz projected on zero total momentum, as it was already very successfully employed with Eq. (2.5),

$$\langle c_1 c_2 c_3 c_4 \rangle \rightarrow \phi_{k_1 k_2 k_3 k_4} \chi_0,$$

$$\phi_{k_1 k_2 k_3 k_4} = \varphi(|k_1|)\varphi(|k_2|)\varphi(|k_3|)\varphi(|k_4|)(2\pi)^3 \delta(k_1 + k_2 + k_3 + k_4), \quad (3.8)$$

where $\chi_0$ is the spin-isospin singlet wave function. It should be pointed out that this product ansatz with four identical $0S$ single particle wave functions is typical for a ground state configuration of the $\alpha$ particle. Excited configurations with wave functions of higher nodal structures may eventually be envisaged for other physical situations. We also would like to mention that the momentum conserving $\delta$ function induces strong correlations among the four particles and (3.8) is, therefore, a rather non trivial variational wave function.

For the two-body interaction of $V_{k_1 k_2 k_3 k_4}$ in Eq. (3.7), we employ the same separable form (2.6) as done already for the quartet critical temperature.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig7}
\caption{\textit{Im$M^{\text{quartet}}(k, \omega + i\eta)$} in Eq. (3.6) as a function of $\omega$ for $\mu = -4.9\text{MeV}$ (left) and for $\mu = 0.55\text{MeV}$ (right) at zero temperature.}
\end{figure}

At first let us mention that in this pilot application of our selfconsistent quartet theory, we only will consider the zero temperature case. As a definite physical example, we will treat the case of nuclear physics with the particularly strongly bound quartet, the $\alpha$ particle. It should be pointed out, however, that if scaled appropriately all energies and lengths can be transformed to other physical systems. For the nuclear case it is convenient to measure energies in Fermi energies $\varepsilon_F = 35 \text{ MeV}$ and lengths in inverse Fermi momentum $k_F^{-1} = 1.35^{-1} \text{ fm}$.

We are now in a position to solve, as in the BCS case, the two coupled equations (3.1) for the quartet order parameter and the single particle occupation numbers from the single particle Dyson equation with single particle self energy (3.6) selfconsistently. The single particle wave functions and occupation numbers
obtained from the above cycle are shown in Fig. 8. We also insert the Gaussian wave function with same r.m.s. momentum as the single particle wave function in the left figures in Fig. 8. As shown in Fig. 8 the single particle wave function is sharper than a Gaussian.

We could not obtain a convergent solution for $\mu > 0.55$ MeV. This difficulty has precisely it’s origin in the fact that the three hole level density goes through zero at $3\mu > 0$, just where the four body correlations should build up, as this was discussed above. In the r.h.s. panels of Fig. 8 we also show the corresponding occupation numbers. We see that they are very small. However, they increase for increasing values of the chemical potential. For $\mu = 0.55$ MeV the maximum of the occupation still only attains 0.35 what is far away from the saturation value of one. What really happens for larger values of the chemical potential, remains unclear.

The situation in the quartet case is also in so far much different, as the $3h$
Green’s function produces a considerable imaginary part of the mass operator.

Figure 7 shows the imaginary part of the approximate quartet mass operator of Eq. (3.6) for \( \mu < 0 \) and \( \mu > 0 \). These large values of the damping rate imply a strong violation of the quasiparticle picture. In Fig. 9 we show the spectral function of the single particle GF. Contrary to the pairing case with its sharp quasiparticle pole, we here only find a very broad distribution, implying that the quasiparticle picture is completely destroyed. How to formulate a theory which goes continuously from the quartet case into the pairing case is, as mentioned, an open question. One solution could be to start right from the beginning with an in medium four body equation which contains a superfluid phase. When the quartet phase disappears, the superfluid phase may remain. Such investigations shall be done in the future.

§4. Discussion, perspective, and conclusion

The considerations in §3 are adequate for a situation where the uncorrelated Fermi gas directly goes over into the \( \alpha \)-particle condensed phase. However, in reality, in the low density phase of nuclear matter, the process may go via bound states of tritons or helions. One may imagine a mixture of deuterons, tritons, helions, and nucleons. The capture of a further nucleon of the triples can lead to \( \alpha \) condensation. The mass operator of such a process is depicted in Fig. 10(a). Since the triples now have a definite c.o.m. momentum, stable quasi particles can form again. One has to describe such a process as with usual pairing but generalised to the case of two fermions with unequal masses. Another possible channel for \( \alpha \) condensation is its formation out of two deuterons, see Fig. 10(b). This is similar to pairing of two bosons. Which of the various processes will in the end win has to be evaluated in the future. Another open question is how the quartet order parameter behaves as a function of asymmetry and temperature. The dependence on asymmetry can partially be deduced from our study of the critical temperature in §3. However, even the latter is incomplete, since only valid at the upper end of the density. In the limit as the density goes to zero we have to generalise the study of the critical temperature in such a way that the one of the ideal Bose gas is obtained. In the pairing case this
Quartetting in Nuclear Matter

Fig. 10. (a) Schematic representation of single particle mass operator with $\alpha$ particle condensate pairing up a nucleon (n,p) with a trion ($^{3}\text{He}, t$); (b) mass operator for deuteron propagation pairing up with another deuteron of opposite momentum.

has been achieved by Nozières and Schmitt-Rink. It is certainly a challenge to incorporate such an approach to quartet condensation. Coulomb repulsion between the $\alpha$-particles may eventually favor an $\alpha$ crystal structure. Competition between Bose condensed phase and crystalization may be investigated in future. Quartetting is not confined to nuclear physics. As mentioned several times, there exists the possibility of the formation of a gas of bi-excitons in semiconductors. Also, one may speculate about the possibility that with cold atoms four different species of fermions will be trapped in the future. If they attract one another as the four nucleons in nuclear physics, certainly the quartet condensation can also be studied in such systems. First theoretical investigations have already appeared. The theory of quartetting in infinite systems is certainly not complete. More theoretical investigations are needed to get a full picture of the process.

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