Multidimensional cosmological solutions of Friedmann type in dilaton gravity

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Abstract

In $D$-dimensional dilaton gravitational model with the central charge deficit $\Lambda$ the generalized Friedmann-type cosmological solutions (spatially homogeneous and isotropic) are obtained and classified.

Introduction

One of the consequences of the string theory in its low-energy limit is the scalar dilaton field $\phi(x)$ in the space-time with the (critical) dimension $D$. The dilaton was involved in various gravitational models. In particular, it makes non-trivial the $1 + 1$-dimensional (2D) gravity that has fruitfully progressed during the last ten years [3, 4, 5].

On the other hand the dilaton field and other lowest-order corrections to Einstein gravity from the string theory were intensively used in cosmology [6] – [23]. Some of the mentioned authors [13] – [23] searched 3 + 1-dimensional solutions with the Friedmann-Robertson-Walker metric keeping in mind some form of compactification of other $D - 4$ dimensions. The dilaton field was recognized as a mechanism for the inflation [24] that was required for solving some cosmological problems. In Refs. [13, 16, 19, 21] the dilaton was used in attempts to avoid the Big Bang singularity.

A. Tseytlin [11] obtained a set of cosmological solutions for the maximally symmetric space of an arbitrary dimension $D$ with the dilaton, the central charge and the covariantly constant antisymmetric tensor field (for $D = 3$ or 4).

The latter approach is used in the present paper. We consider and classify cosmological solutions in $D$-dimensional dilaton gravity with the central charge deficit $\Lambda$ for the case of an arbitrary $D$. We use the $D$-dimensional generalization of FRW metric corresponding to homogeneous and isotropic spatial part with the constant curvature ($k = 0, \pm 1$). All spatial dimensions are equal as it may be before the compactification. The purpose of our studying is to order the known 2$D$ and higher $D$ dilaton cosmological solutions (with adding new ones) and to classify them exhaustively.

In this paper we don’t consider the terms in the action with the antisymmetric tensor fields so the condition of the maximal spatial symmetry restricts the possible dimensions within $D = 3$ and $D = 4$. In these cases the problem was studied by many authors [3, 11, 17, 18, 20, 22].

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1. Model and equations

The dilaton gravitational model with the following action \[ S = \int \sqrt{|g|} e^{-2\phi(x)} \left[R + 4(\nabla \phi)^2 + \Lambda\right] d^{n+1}x. \] (1)

Here \( \phi(x) \) is the scalar dilaton field in \( n+1 \)-dimensional \( (D = n+1) \) pseudoriemann manifold \( M_{1,n} \) with coordinates \( x^\mu \) and the metric tensor \( g_{\mu\nu} \), \( g = \text{det} |g_{\mu\nu}| \), \( R \) is the scalar curvature, \( (\nabla \phi)^2 = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \).

We use here and below the string frame for the action (1) keeping in mind that it may be conformally transformed \( g_{\mu\nu} \rightarrow e^{-\nu \phi} g_{\mu\nu} \), \( r = 4/(D - 2) \) to the Einstein frame [1, 11, 12].

Equations of evolution for this model

\[ R_{\mu\nu} - \frac{1}{2}(R + \Lambda) g_{\mu\nu} + 2\phi_{,\mu\nu} = g_{\mu\nu} \left[2\phi^{;\lambda}_{,\lambda} - 2(\nabla \phi)^2\right], \]

\[ \frac{1}{4}(R + \Lambda) + \phi^{;\lambda}_{,\lambda} - (\nabla \phi)^2 = 0 \]

are obtained from action (1) by the variation of the gravitational field \( g_{\mu\nu} \) and the dilaton field \( \phi \) accordingly. Here \( R_{\mu\nu} \) is the Ricci tensor, \( \phi_{,\mu\nu} \) is the covariant derivative.

Transform the latter equations to the equivalent system

\[ R_{\mu\nu} + 2\phi_{,\mu\nu} = 0, \]
\[ R + 4(\nabla \phi)^2 = \Lambda. \] \hspace{1cm} (2)

We search cosmological Friedmann’s type solutions of Eqs. (2). It means that \( M_{1,n} = R \times S^n_k \) where \( S^n_k \) is the homogeneous and isotropic manifold described by the scale factor \( a(t) \). The manifold \( S^n_k \) has one of the following forms: \( n \)-dimensional sphere \( S^n_1 \), pseudosphere \( S^n_{-1} \), flat space \( S^n_0 \equiv R^n \). In the spherical \((k = 1)\) case \( a \) is the radius of \( S^n_1 \). The metric in \( M_{1,n} = R \times S^n_k \) may be taken in the form [25]

\[ ds^2 = dt^2 - a^2(t)[(dx^1)^2 + c_k(x^1)(dx^2)^2 + \ldots + c_k(x^n) \cdot c_k(x^2) \cdot \ldots \cdot c_k(x^{n-1})(dx^n)^2], \] \hspace{1cm} (3)

where \( c_k(x) = \begin{cases} \cos x, & k = 1, \\ 1, & k = 0, \\ \cosh x, & k = -1. \end{cases} \)

We substitute the metric (3) and \( \phi = \phi(t) \) into Eqs. (2) taking into account the following expressions (for all \( k \)):

\[ R_0^0 = -naa/\dot{a}^2, \quad R_1^1 = R_2^2 = \ldots = R_n^n = -[\ddot{a}a + (n - 1)(\dot{a}^2 + k)]/a^2, \]
\[ R = -n[(2a\ddot{a} + (n - 1)(\dot{a}^2 + k)]/a^2, \quad \dot{a} = da/dt. \]

It reduces system (2) to three nontrivial equations

\[ -n\ddot{a}/a + 2\ddot{\phi} = 0, \] \hspace{1cm} (4)
\[ -\ddot{a} - (n - 1)(\dot{a}^2 + k)/a + 2a\ddot{\phi} = 0, \] \hspace{1cm} (5)
\[ n(n - 1)(\dot{a}^2 + k)/a^2 + 4\dot{\phi}^2 - 4n\dot{\phi}\dot{a}/a = \Lambda. \] \hspace{1cm} (6)

These three equations are not independent. Eq. (3) is the “zero energy” constraint that is conserved as the consequence of Eqs. (4) and (5) and hence gives only a restriction on the initial values of \( a, \dot{a}, \phi, \dot{\phi} \). [10]
One may express $\dot{\phi}$ from Eq. (3)

$$
\dot{\phi} = \frac{1}{2} \left[ \frac{\ddot{a}}{a} + \frac{n-1}{a \ddot{a}} (\dot{a}^2 + k) \right]
$$

and substitute it into Eq. (3). The resulting second-order equation (it is quadratic one with respect to $\ddot{a}$) may be transformed through the substitution

$$
\dot{a} = q(a)
$$

to the following form:

$$
\frac{dq}{da} = \frac{q}{a} - k \frac{n-1}{aq} \pm \sqrt{\frac{nq^2}{a^2} - kn \frac{n-1}{a^2} + \Lambda}.
$$

To classify all solutions of the latter equation and system (3)–(6) we consider some cases separately.

2. Flat universe: $k = 0$

In this case Eq. (3) is homogeneous one. It may be integrated for all values of $\Lambda$:

$$
q = a \frac{C^2 a^2 + 2\sqrt{n}}{2} - \frac{\Lambda}{n}.
$$

It results in the following evolution of the scale factor:

$$
a(t) = a_0 \cdot \left\{ \begin{array}{l}
\left[ \tanh\left( \frac{1}{2} \sqrt{\Lambda} \tau \right) \right]^{\pm 1/\sqrt{n}}, \quad \Lambda > 0, \\
\tau^{\pm 1/\sqrt{n}}, \quad \Lambda = 0, \\
\left[ \tan\left( \frac{1}{2} \sqrt{-\Lambda} \tau \right) \right]^{\pm 1/\sqrt{n}}, \quad \Lambda < 0.
\end{array} \right.
$$

Here $C$ or its equivalent $a_0 (a_0^{\mp 2\sqrt{n}} = nC^2/|\Lambda|$ for $\Lambda \neq 0$) is the constant of integration in (3). We use in Eq. (10) and below the time parameter

$$
\tau = \pm (t - t_0) = |t - t_0|,
$$

because all solutions of system (3)–(6) are invariant with respect to time translation ($t_0$ is the constant of integration) and time reflection. The signs $\pm$ in Eqs. (11) and (10) (the latter is connected with the sign in Eq. (3)) are independent.

The expression for $\phi$ corresponding to (10) results from Eq. (7)

$$
e^{2\phi - 2\phi_0} = \left\{ \begin{array}{l}
\left[ \sinh\left( \frac{1}{2} \sqrt{\Lambda} \tau \right) \right]^{-1 \pm \sqrt{n}} \left[ \cosh\left( \frac{1}{2} \sqrt{\Lambda} \tau \right) \right]^{-1 \pm \sqrt{n}}, \quad \Lambda > 0, \\
\tau^{1 \pm \sqrt{n}}, \quad \Lambda = 0, \\
\left[ \sin\left( \frac{1}{2} \sqrt{-\Lambda} \tau \right) \right]^{-1 \pm \sqrt{n}} \left[ \cos\left( \frac{1}{2} \sqrt{-\Lambda} \tau \right) \right]^{-1 \pm \sqrt{n}}, \quad \Lambda < 0.
\end{array} \right.
$$

The functions $a(t)$ (10) and $\phi(t)$ (12) are exact solutions not only of Eqs. (3) and (6) but also of Eq. (4). This situation is more general. It is the non-trivial fact connected with the constraint character of Eq. (3) that all solutions of Eqs. (3) for all $k$, $\Lambda$ and $n$ satisfy Eq. (4).
It is necessary to supplement the family (10), (12) by the linear dilaton solution [7, 21]

\[ a = \text{const}, \quad \phi = \phi_0 \pm \frac{1}{2} \sqrt{\Lambda} t, \quad \Lambda \geq 0, \quad k = 0, \]

(13)

that was lost because of dividing on \( \dot{a} \) in Eq. (7).

The solutions (10), (12) are well known in the 4D or \( n = 3 \) case [16, 20, 21] and were considered for \( \Lambda > 0 \) and an arbitrary \( D \) in Ref. [11]. They coincide with the solutions of Veneziano [9] for the flat manifold with different scale factors generalize Eqs. (10), (12).

One can see that qualitative picture of solutions (10), (12) and the structure of branches are similar for all \( n > 1 \). In the cases \( \Lambda \geq 0 \) for all \( n \) we have two different branches of solutions (or 4 branches if we take two symmetric with respect to the change “+” to “−” in (11) solutions as different ones [16]). And there are only one branch for \( \Lambda < 0 \) due to the symmetry properties of (11).

The upper sign in Eqs. (10) and (12) corresponds to an expansion (with increasing \( \tau \)) of the flat dilatonic universe from the singular point \( \tau = 0 \) with \( \phi \) growing up from \( -\infty \), the opposite sign describes an evolution without the singularity \( a = 0 \) but with \( a \to \infty, \phi \to +\infty \) at \( \tau \to 0 \). The value \( \Lambda = 0 \) always corresponds to the power-law expressions for \( a \) and \( \phi \) generalizing ones [16] for the case \( n = 3 \). The presence of \( \Lambda > 0 \) results in slowing down the above mentioned expansion and stabilization of the scale factor: \( a \to a_0 = \text{const}, \phi \to -\infty \) at \( \tau \to \infty \). In the case \( \Lambda < 0 \) we, v.v. have acceleration of evolution that is finished during finite time.

Thus the described above role of \( \Lambda \)-term in the dilaton cosmology is in some sense opposite [11] to that in usual Friedmann type \( n + 1 \)-dimensional cosmology. In the latter case the evolution of the flat universe (\( k = 0 \)) with dust matter and with Einstein’s gravitational Lagrangian \( \sqrt{|g|} (R + 2\Lambda) \) has the form [25]

\[
\begin{cases}
    \text{[sinh (} \sqrt{\frac{n\Lambda}{2(n-1)}} \tau \text{)} \text{]}^{2/n}, & \Lambda > 0, \\
    \text{[sin (} \sqrt{\frac{n\Lambda}{2(n-1)}} \tau \text{)} \text{]}^{2/n}, & \Lambda < 0,
\end{cases}
\]

and \( \Lambda > 0 \) accelerates the expansion.

Note that for the 2D dilaton model (\( n = 1 \)) all “cosmological” solutions are exhausted by Eqs. (10) – (13). It is naturally so the 1-dimensional space has no an intrinsic curvature, hence \( k \) vanishes in Eqs. (5), (6) if \( n = 1 \). In this case the \( n = 1 \) solution (10), (12) coincides with the \( D = 2 \) string “black hole” solution [4, 12, 26] (taking the Euclidean form after the analytic continuation \( \tau \to ir, \Lambda \to -\Lambda \) [10]).

3. Curved space \( k = \pm 1 \) and \( \Lambda = 0 \)

For any value of \( k \) all solutions of system (4) – (6) may be obtained by integrating Eq. (9). It was mentioned above that Eq. (4) in this case is satisfied too. But the most convenient

1Below for counting a number of the branches we don’t differ two solutions connected only by the time reversal (11). Similarly, describing an expansion we always keep in mind that the corresponding solution with the contraction exists. If one differ these solutions he should multiply any mentioned number of branches by two.

2In action (1) all matter is in “exotic” (dilaton and \( \Lambda \)-term) form.
way to classify various types of the solutions for $\Lambda = 0$ is to exclude the scale factor $a$ from
Eq. (4) – (6). For this purpose we take the linear combination of these equations with the
coefficients $-1, n/a$ and 1. The resulting expression

$$\frac{\dot{a}}{a} = \frac{1}{n} \left( 2\dot{\phi} - \frac{\ddot{\phi}}{\phi} \right)$$

(with $\Lambda = 0$) is integrable:

$$a^n = \tilde{C} e^{2\phi/\dot{\phi}}.$$  

Here $\tilde{C}$ is an arbitrary constant.

Before using the expressions (14) or (15) we are to consider all solutions of system (4) –
(6) with $\dot{\phi} = 0$. For all $k, \Lambda, n > 1$ we have only two solutions of this type: the trivial case
of (13) $a = a_0, \phi = \phi_0, k = 0, \Lambda = 0$ and the solution with linear grows of $a$

$$a = a_0 \pm t, \quad \phi = \phi_0, \quad k = -1, \quad \Lambda = 0.$$  

Substituting Eq. (14) or (15) into Eq. (4) we obtain the differential equation

$$n \frac{d}{dt} \ddot{\phi} = \left( 2\dot{\phi} - \frac{\ddot{\phi}}{\phi} \right)^2.$$  

Introducing the notation

$$p = \frac{\ddot{\phi}}{\dot{\phi}}^2$$

we integrate Eq. (17) in the following form:

$$\frac{d\phi}{dt} = \frac{1}{Q} \left| p + \frac{2}{\sqrt{n-1}} \right|^{\frac{-\sqrt{n}}{\pi(\sqrt{n-1})}} \left| p - \frac{2}{\sqrt{n+1}} \right|^{\frac{-\sqrt{n}}{\pi(\sqrt{n+1})}}.$$  

Here $Q$ is the constant of integration that can take various values in three intervals resulting
from the division of the axis $-\infty < p < \infty$ by the singular points

$$p_1 = -\frac{2}{\sqrt{n-1}}, \quad p_2 = \frac{2}{\sqrt{n+1}}.$$  

Taking into account that $p dt = -d (\dot{\phi})^{-1}$ we obtain the relation

$$dt = -\frac{Qn}{n-1} \left| p - p_1 \right|^{\frac{2}{\pi(\sqrt{n-1})}} \left| p - p_2 \right|^{\frac{2}{\pi(\sqrt{n+1})}} \text{sign} (p - p_1) \text{sign} (p - p_2) dp.$$  

It expresses $t$ through the parameter $p$ in quadratures in each of the intervals $(-\infty, p_1),
(p_1, p_2), (p_2, +\infty)$.

Eqs. (13), (18) and (21) let us express $\phi$ and $a$ through $p$:

$$a(p) = C_a \left| p - p_1 \right|^{\frac{1}{\pi(\sqrt{n+1})}} \left| p - p_2 \right|^{\frac{1}{\pi(\sqrt{n+1})}},$$

$$\phi(p) = \phi_0 + \frac{\sqrt{n}}{4} \ln \left| \frac{p - p_1}{p - p_2} \right|.$$  

Here the constants $C_a, \phi_0$ and $\tilde{C}$ in Eq. (13) are connected: $C_a = |\tilde{C} Q e^{2\phi_0}|^{1/n}$.
Expressions \((20) - (22)\) satisfy Eqs. \((1) - (3)\) only under the following condition:

\[
(nQ/C_a)^2 = -k \text{ sign } (p - p_1) \cdot \text{ sign } (p - p_2). \tag{23}
\]

It means that the solution \((20) - (22)\) for the spherical case of \(k = 1\) exist only if \(p \in (p_1, p_2)\) and for the case \(k = -1\) it takes place if \(p \in (-\infty, p_1) \cup (p_2, +\infty)\).

So for \(k = 1\), \(\Lambda = 0\) there is only one branch of solutions (we count the branches in accordance with the footnote in Sect. 2) represented in Fig. 1 as the evolution of the scale factor \(a\) in time \((11)\) for various \(n = D - 1\) (solid lines). The graphs for \(\phi = \phi(\tau)\) with \(\phi_0 = 3\) are drawn by the dashed lines for the cases \(n = 2\) and \(n = 9\). For any \(n\) these solutions describe the expanding universe \((a\) grows up from the singular value 0 to \(+\infty\) and \(\phi\) rolls up from \(-\infty\) to \(+\infty\) when \(p\) changes between the points \((11)\) \(p_1\) and \(p_2\)) during the finite time

\[
T = \int_{p_1}^{p_2} \frac{dt}{dp} = Qn \left( \frac{4\sqrt{n}}{n - 1} \right)^{-1} \frac{1}{\Gamma \left( \frac{\sqrt{n}}{2(n - 1)} \right) \Gamma \left( \frac{\sqrt{n}}{2(n + 1)} \right) / \Gamma \left( \frac{1}{n - 1} \right). \tag{24}
\]

Here \(\Gamma(x)\) is the Euler gamma function. In Fig. 1 the graphs \(a = a(\tau)\) and \(\phi = \phi(\tau)\) with the equal values \(T = 1\) for different \(n\) are drawn. Note that the solutions \((20) - (21)\) for any fixed \(n > 1\) and \(k = \pm 1\) constitute the two-parameter family: the first parameter \(t_0\) in \((11)\) is trivial \((a\) translation in \(t)\), the second is \(C_a\), or \(Q\), or \(T\) (they are connected by Eqs. \((23)\) and \((24)\)). The third parameter \(\phi_0\) contains Eq. \((22)\).

In the pseudospherical case \(k = -1\) the solutions \((20) - (22)\) are divided into two branches which are shown in Fig. 2. The first one (Fig. 2a) describes the expansion of \(a\) and growing \(\phi: 0 < a < +\infty, -\infty < \phi < \phi_0, p < p_1\). The second branch with \(p > p_2\) (Fig. 2b) corresponds to evolution of \(a\) without the singularity \(a = 0\): the value \(a\) decreases from \(+\infty\) to the minimum \(a_{\text{min}} = C_a[\sqrt{n}p_1(\sqrt{n}p_2)^{1/2}]^{-1/4}\) and then it grows up to \(+\infty\) when \(\phi\) monotonically decreases from \(+\infty\) to \(\phi_0\). In these both cases the asymptotic \(\tau \to \infty\) behavior is \(\phi \to \phi_0 = \text{const}, a \to \text{const} + \tau, \text{i.e. in this limit all solutions } (20) - (21) \text{ for } k = 1 \text{ tend to the linear evolution } (19)\).

In the opposite (singular) limit \(p \to p_1\) or \(p \to p_2\) for \(k = \pm 1\) and for all branches we have an asymptotic dependence of the following two types:

\[
a \simeq \text{const} \cdot \tau^{1/\sqrt{n}}, \quad e^{2\phi} \simeq \text{const} \cdot \tau^{-1+\sqrt{n}}, \quad \tau \to +0, \tag{25}
\]

\[
a \simeq \text{const} \cdot \tau^{-1/\sqrt{n}}, \quad e^{2\phi} \simeq \text{const} \cdot \tau^{-1-\sqrt{n}}, \quad \tau \to +0. \tag{26}
\]

Eq. \((25)\) corresponds to the case \(p \to p_1\) and \(a \to 0\) that takes place for the branches represented in Figs. 1 \((k = 1)\) and 2a \((k = -1)\). The case \((26)\) is realized for the solutions in Fig. 2b and in the limit \(t \to T\) in Fig. 1.

We see that the asymptotic behavior of the solutions \((20) - (22)\) \((k = \pm 1, \Lambda = 0)\) and \((11) - (12)\) \((k = 0)\) is identical. So one can easily set the correspondence between the equivalent branches with the same limits \((24)\) or \((26)\).

There is only one\(^3\) dimension \(n = 4\) (or \(D = 5\)) where an explicit solution \((20) - (22)\) exists:

\[
a(t) = \left( \frac{\tau^3 - T^3}{\tau} \right)^{1/2}, \quad \phi(t) = C_\phi + \frac{1}{2} \ln \left| \frac{\tau^3 - T^3}{\tau^3} \right|. \tag{27}
\]

\(^3\)For the critical dimensions \(D = 10\) or \(D = 26\) the exponents in \((20)\) are rational but this expression isn’t integrable through any Euler’s substitution.
Here $T = 8Q/\sqrt{3}$ in accordance with (24), $\tau = \pm(t - t_0)$ may be negative (unlike (11)).

Expression (27) is rather simple and may illustrate the described above solution (20) - (22). The constant $t_0$ in Eq. (27) is chosen so that the point $\tau = 0$ is the image of $p = p_2 = 2/3$ and $\tau = T$ is the image of $p = p_1 = -2$ for the continuous mapping the $p$-axis onto $\tau$-axis. So the points $\tau = 0$ and $\tau = T$ divide the $\tau$-axis into 3 parts, and the interval $0 < \tau < T$ corresponds to the case $k = 1$.

4. Curved space $k = \pm 1$ and $\Lambda \neq 0$

In these cases we study various types of evolution $a(t), \phi(t)$ by solving the system (8), (9) that may be rewritten as

$$\dot{a} = q, \quad \dot{q} = a^{-1}\left[q^2 - k(n-1) \pm q\sqrt{n[q^2 - k(n-1)] + \Lambda a^2}\right].$$

(28)

To consider solutions of this system we use in this Section, in particular, numerical analysis so we have explicit solutions only in special cases.

For example, for the dimensionality $n = 4$ or $D = 5$ (specified in the previous Section) and $\Lambda > 0$ we obtained the exact particular solutions with the power-law evolution of the scale factor

$$a = a_0\sqrt{\tau}, \quad e^{2\phi - 2\phi_0} = \tau \cdot e^{k\sqrt{\Lambda}\tau}, \quad a_0 = \sqrt{36/\Lambda}, \quad n = 4, \quad k = \pm 1, \quad \Lambda > 0.$$  

(29)

For other values $n$ the analogs of the solutions (29) also exist but the expressions

$$a \simeq a_0\sqrt{\tau}, \quad e^{2\phi - 2\phi_0} \simeq \tau^{(n-2)/2} \cdot e^{k\sqrt{\Lambda}\tau}, \quad a_0^2 = 2(n-1)/\sqrt{\Lambda}, \quad k = \pm 1, \quad \Lambda > 0.$$  

(30)

are only asymptotic at $\tau \to \infty$ [11]. In the opposite limit $\tau \to 0$ these solution have the asymptotic form (23) for $k = \pm 1$ or (26) for $k = -1$.

We’ll see below that the behavior of the mentioned solutions (29) and (30) in the $t \to \infty$ limit radically differs from that of all other solutions in the positive curvature case $k = 1, \Lambda > 0$. Otherwise, in the case $k = -1, \Lambda > 0$ all solutions have the asymptotic form (30).

Analyzing the solutions of the system (28) we are to note that there is the domain

$$nq^2 + \Lambda a^2 - kn(n-1) < 0,$$

(31)

in the phase $aq$-plane where the radicand in (28) or (3) is negative. Hence, the graphs of the solutions in phase plane can not pass through the domain (31) and cross its border — the 2-nd order curve.

For $k = 1$ and an arbitrary fixed value $\Lambda$ the graph of an every solution of Eqs. (28) in the phase $aq$-plane contacts with the mentioned border of the domain (31), that is this border (the ellipse if $\Lambda > 0$ or the hyperbola if $\Lambda < 0$) is the envelope of the graphs family. At the contact point (we denote its coordinates $a_0, q_0$) for each solution the sign before the radical in Eq. (28) changes from “−” to “+”. So any graph of solution in the case $k = 1$ consists of two parts corresponding to these signs and smoothly connected at the point $(a_0, q_0)$.

In Fig. 3 the graphs of the considered solutions with $k = 1$ in the $ta$-plane are represented for the case $n = 3$ and various $\Lambda$ (Fig. 3a), and for the case $n = 4$ with fixed value $\Lambda = 1$ (Fig. 3b). The parameter $t_0$ in Eq. (11) is chosen so the value $a = 0$ corresponds to $t = 0$. Under these conditions the curves in Fig. 3b are the one parameter family, and one may choose the value $a_0$ (if $q_0 \geq 0$) as the parameter numerating the curves in the family. The
values $a_0$ are shown near corresponding lines in Fig. 3b. The curves marked by symbols “4” and “5” in Fig. 3b correspond to the values $q_0 < 0$ at the point of sign reversal in Eq. (28); these numbers are maximal values of the scale factor $a$ for these solutions.

In Fig. 3a for the case $n = 3$ the solutions $a = a(t)$ with the same value $a_0 = 1$ for various values $\Lambda$ (marked near the graphs) are shown. For the curve with $\Lambda = 10$ the value $a = 1$ is the maximum of $a$. We see that presence of $\Lambda < 0$ accelerates the expansion in comparison with the case $\Lambda = 0$ (Fig. 1) and $\Lambda > 0$ — decelerates. But with increasing $\Lambda$ (without exceeding the certain limit) the expansion remains irreversible — the values $a$ and $\phi$ grows up to $+\infty$ during the finite time $T(a_0, \Lambda)$. The asymptotic behavior $a(t)$ at $t \to T$ takes the form (28) or

$$a \simeq \text{const} \cdot (T - t)^{-1/\sqrt{n}}, \quad e^{2\phi} \simeq \text{const} \cdot (T - t)^{-1-\sqrt{n}}, \quad t \to T.$$  

If $\Lambda$ reaches some critical value (depending on $a_0$) the expansion becomes unlimited in time with the power-law behavior (30) at $t \to \infty$. In Fig. 3a ($n = 3$, $a_0 = 1$) this critical value slightly exceeds $\Lambda = 2.732$. For larger $\Lambda$ the type of evolution changes — the expansion in some time interval is replaced by compression that is finished in the finite time $T = T(a_0, \Lambda)$ with vanishing $a$. The value $\phi(t)$ during the time $T$ grows from $-\infty$ up to some maximum and then decreases to $-\infty$. In the limit $t \to T$ the asymptotic relation (23) takes place.

Fig. 3b illustrates that the critical power-law solution (it takes the form (29) for $n = 4$) may be obtained for fixed value $\Lambda > 0$ by choosing the value $a_0$. Any small deviation from the critical value results in an evolution within the framework of any of the two mentioned types with finite lifetime $T$.

For the case of negative spatial curvature $k = -1$ and $\Lambda < 0$ each graph of solution in the phase $aq$-plane contacts with the border of the forbidden domain (31) with the sign reversal in Eq. (28) (similarly to the case $k = 1$). The family of these graphs $a = a(t)$ for fixed $n = 4$, $\Lambda = -1$ and various values $a_0$ is shown in Fig. 4a. The coordinates of the contact point $a_0$ numerate the curves as in Fig. 3b.

All the solutions with $\Lambda < 0$ (Fig. 4a) have finite lifetimes $T(a_0, \Lambda)$ and the behavior (32). But their behavior at $t \to 0$ depends on $a_0$ and $\Lambda$. If $a_0$ does not exceed some critical value $a_0 = a_{cr}(\Lambda)$ then $a, \phi \to +\infty$ at $t \to 0$ in accordance with (28) (similarly to the case $\Lambda = 0$ in Fig. 2b). In the opposite case $a_0 > a_{cr}$ this limit is $a, e^{2\phi} \to 0$ (27) (compare with Fig. 2a). These two cases are divided by the solution with the critical value $a_0 = a_{cr}$ (Fig. 4a) that is close to the linear expansion (11) $a \simeq t, \phi \simeq \phi_0$ for $t \ll T$.

In the case $k = -1$, $\Lambda > 0$ there is no real forbidden domain (31) but the solutions with different signs in Eq. (28) are connected by the time reversal $t \to -t$, $q \to -q$. So we have one family of solutions with the infinite lifetime and $a \sim \sqrt{T}$ (28) asymptotic behavior at $t \to \infty$ (11), that is shown in Fig. 4b for $\Lambda = 1$ and $n = 0$. The values $q = \dot{a}$ at the point $a = a_0 = 1$ numerate the curves. For values $q < q_{cr}(a_0, \Lambda)$ (here $q_{cr} \simeq 1$) the solutions have the $t \to 0$ limit (23) and for $q > q_{cr}$ — the limit (25). The critical solution with $q = q_{cr}$ (Fig. 4b) has $a \simeq t, \phi \simeq \phi_0$ dependence for small $t$.

**Conclusion**

The Friedmann type cosmological solutions in $D$-dimensional dilaton gravity with the action (1) are investigated in this paper. Different values of spatial curvature $k = 0, \pm 1$ and various values $\Lambda$ are considered. For all cases the solutions of the system (1) — (3) (cosmological solutions) are described and classified. They include as the expressions (10) — (12), (13), (16) (particular cases or generalizations of them were obtained in Refs. 3 — 11), as new analytic solutions (20) — (22), (27), (23).
To classify various types of solutions we represent them in the following table:

| $\Lambda$ | $\Lambda = 0$ | $\Lambda > 0$ |
|-----------|---------------|---------------|
| $k = 1$   | $0 \to \infty$ | $0 \to a \approx a_0 \tau^{1/\sqrt{n}}$ |
| $k = 0$   | $0 \to \infty$ | $\infty \to a \approx a_0 \tau^{1/\sqrt{n}}$ |
| $k = -1$  | $\infty \to \infty$ | $\infty \to a \approx a_0 \sqrt{\tau}$ |

Here the symbol "0" denotes the beginning or the end of an evolution with the asymptotic behavior $(a, e^{2\phi} \to 0)$. The symbol "\infty" corresponds to the asymptotics $a, \phi \to +\infty$ of the type $(26)$ or $(32)$. Note that for $\Lambda < 0$ a lifetime of the dilatonic universe is always finite — the beginning and the end of the evolution are marked in Table 1 by the mentioned symbols (for $k = -1$ we have a critical solution with close to linear $a \approx \tau, \phi \approx \phi_0$ beginning of the expansion and the finite lifetime).

Other symbols describing the end of an evolution in Table 1 correspond to an infinite lifetime and power-law evolution within one of the following types: $(30)$ $(a \sim \sqrt{\tau})$, $(13)$, $(10) - (12)$ $(k = 0, \Lambda \geq 0)$, $(16)$, $(20) - (22)$ $(k = -1, a \sim \tau)$.

For $k = -1, \Lambda > 0$ all solutions have the asymptotics $a_0 \sqrt{\tau}$ $(30)$, but for $k = 1, fixed \Lambda > 0$ and $t_0$ only one critical solution (dividing different types of solutions) has such a behavior.

Note that the considered cosmological solutions have identical qualitative features for various dimensionalities $n \geq 2 (n = 1$ or $D = 2$ is reduced to the case $k = 0$). In this sense the classification in Table 1 is universal for all $D$.

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