POLYNOMIAL VECTOR FIELDS
WITH ALGEBRAIC TRAJECTORIES

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Abstract. It is known after Jouanolou that a general holomorphic foliation of
degree \( \geq 2 \) in projective space has no algebraic leaf. We give formulas for the
degrees of the subvarieties of the parameter space of one-dimensional foliations
that correspond to foliations endowed with some invariant subvariety of degree
1 or 2 and dimension \( \geq 1 \).

Introduction

Holomorphic foliations are an offspring of the geometric theory of polynomial
differential equations. Following the trend of many branches in Mathematics, in-
terest has migrated to global aspects. Instead of focusing on just one curve, or
surface, or metric, or differential equation, try and study their family in a suitable
parameter space. The geometry within the parameter space of the family acquires
relevance. For instance, the family of hypersurfaces of a given degree correspond
to points in a suitable projective space; geometric conditions imposed on hyper-
surfaces, e.g., to be singular, usually correspond to interesting subvarieties in the
parameter space, e.g., the discriminant. Hilbert schemes have their counterpart in
the theory of polynomial differential equations, to wit, the spaces of foliations.

Let us recall that, while a general surface of degree \( d \geq 4 \) in \( \mathbb{P}^3 \) contains no line
–in fact, only complete intersection curves are allowed, those that do contain some line
–correspond to a subvariety of codimension \( d - 3 \) and degree \( (d+1)(3d^3 + 6d^3 + 17d^2 + 22d + 24)/4! \) in a suitable \( \mathbb{P}^N \).

Similarly, while a general holomorphic foliation, say in \( \mathbb{P}^2 \), of degree \( d \geq 2 \) has no
algebraic leaf, those that do have, say an invariant line, correspond to a subvariety of codimension \( d - 1 \) and degree \( 3(d+3)/4 \) in a suitable \( \mathbb{P}^N \).

Our goal is to give similar closed formulas for the degrees of the subvarieties of
the parameter space of one-dimensional foliations of degree \( d \) on \( \mathbb{P}^n \) that correspond
to foliations endowed with some invariant subvariety of degree 1 or 2 and dimension
\( \geq 1 \). Imposing a linear invariant subvariety is easy, essentially due to the absence
of degenerations. The classical spaces of complete quadrics help us handle the
quadratic case. For higher degree, we don’t know the answer.

1. The space of foliations

The main reference for this matterial is Jouanolou [7]. We call a vector field of
degree \( d \) in \( \mathbb{P}^n \) a global section of \( T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(d - 1) \), for some \( d \geq 0 \).

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Denote by $S_d$ the space $\text{Sym}_d((\mathbb{C}^{n+1})^\vee)$ of homogeneous polynomials of degree $d$ in the variables $z_0, \ldots, z_n$. We write $\partial_i = \partial/\partial z_i$, thought of as a vector field basis for $\mathbb{C}^{n+1} = S_1$. Recalling Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(d-1) \to \mathcal{O}_{\mathbb{P}^n}(d) \times \mathbb{C}^{n+1} \to \mathbb{T}_{\mathbb{P}^n}(d-1) \to 0$$

and taking global sections we get the exact sequence

$$0 \to S_{d-1} \to S_d \to V_d := H^0(\mathbb{P}^n, \mathbb{T}_{\mathbb{P}^n}(d-1)) \to 0.$$  

Here $\iota(H) = HR$, with $R = \sum z_i \partial_i$ the radial vector field. Any degree $d$ vector field $\mathcal{X} \in V_d$ can be written in homogeneous coordinates as

$$(2) \quad \mathcal{X} = F_0 \partial_0 + \cdots + F_n \partial_n,$$

where the $F_i$’s denote homogeneous polynomials of degree $d$, modulo multiples of the radial vector field. A vector field $\mathcal{X}$ induces a distribution of directions in $\mathbb{T}_{\mathbb{P}^n}$. Any nonzero multiple of $\mathcal{X}$ yields the same distribution.

The space of foliations of degree $d$ in $\mathbb{P}^n$ is the projective space

$$(3) \quad \mathbb{P}^N = \mathbb{P}(V_d)$$

dimension

$$N = (n+1)\left(\frac{d+n}{n}\right) - \left(\frac{d+n-1}{n}\right) - 1.$$  

1.1. Invariant subvarieties. Let $\mathcal{X}$ be a foliation in $\mathbb{P}^n$. An irreducible subvariety $Z \subset \mathbb{P}^n$ is said to be invariant by $\mathcal{X}$ if

$$\mathcal{X}(p) \in T_pZ$$

for all $p \in Z \setminus (\text{Sing}(Z) \cup \text{Sing}(\mathcal{X}))$. If $Z$ is reducible, it is invariant by $\mathcal{X}$ if and only if each irreducible component of $Z$ is invariant by $\mathcal{X}$. If $Z$ is defined by a saturated ideal $I_Z := \langle G_1, \ldots, G_r \rangle$, invariance means

$$dG_i(\mathcal{X}) = \mathcal{X}(G_i) \in I_Z$$

for all $i = 1, \ldots, r$. The hypothesis of saturation is necessary, see [3, p.5]. It can be easily checked that the condition above does not depend on the representative of $\mathcal{X}$ in $S_d \otimes S_1$.

2. Foliations with an invariant k-plane

We show that the locus in $\mathbb{P}^N$ corresponding to foliations with an invariant k-plane is the birational image of a natural projective bundle over the grassmannian of k-planes in $\mathbb{P}^n$.

2.1. Consider the tautological exact sequence of vector bundles over the grassmannian $\mathbb{G} := \mathbb{G}(k,n)$ of projective k-planes in $\mathbb{P}^n$,

$$(4) \quad 0 \to \mathcal{T} \to \mathbb{G} \times \mathbb{C}^{n+1} \to \mathcal{Q} \to 0$$

where $\mathcal{T}$ is of rank $k+1$. The projectivization

$$\mathbb{P}(\mathcal{T}) = \{(W, p) \in \mathbb{G} \times \mathbb{P}^n \mid p \in W\}$$
is the universal k-plane. Write the projection maps
\[ \mathbb{P}(T) \subset G \times \mathbb{P}^n \]
\[ G \xrightarrow{p_1} \mathbb{P}(T) \xrightarrow{p_2} \mathbb{P}^n. \]
We denote by \( N \) the normal bundle to \( \mathbb{P}(T) \) in \( G \times \mathbb{P}^n \). We have the exact sequence over \( \mathbb{P}(T) \),
\[ 0 \to T_{\mathbb{P}(T)/G} \to p_2^*T\mathbb{P}^n \to N \to 0. \]
It is easy to see that \( N = p_1^*Q \otimes O_T(1) \).

2.2. Proposition. Notation as in (1) and (4), there exists a vector subbundle
\[ \mathcal{E} \subset G \times V_d^n \]
such that
(i) \( \mathbb{P}(\mathcal{E}) = \{(W, \mathcal{X}) \in G \times \mathbb{P}^N \mid W \text{ is invariant by } \mathcal{X}\} \).

Set \( \mathcal{Y} := q_2(\mathbb{P}(\mathcal{E})) \), where \( q_2 : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^N \) is the projection. Then
(ii) the codimension of \( \mathcal{Y} \) in \( \mathbb{P}^N \) is \((n - k)((k + d) - (k + 1))\) and
(iii) the degree of \( \mathcal{Y} \) is given by the top-dimensional Chern class,
\[ c_g(Q \otimes \text{Sym}_d(T^\vee)) \]
where \( g := \dim G \).

Proof. Consider the following diagram of maps of vector bundles over \( \mathbb{P}(T) \),
\[ T_{\mathbb{P}(T)/G}(d - 1) \]
\[ \xrightarrow{ev} p_2^*T\mathbb{P}^n(d - 1) \]
\[ \xrightarrow{\varphi} p_1^*Q \otimes O_T(d). \]
The map of evaluation yields a surjective map of vector bundles\[ \varphi : V_d^n \twoheadrightarrow p_1^*Q \otimes O_T(d). \]
The kernel of \( \varphi \) is the vector bundle
\[ \ker \varphi = \{(W, p, \mathcal{X}) \mid \mathcal{X}(p) \in T_pW \}. \]
Observe that a k-plane \( W \) is invariant by \( \mathcal{X} \) if and only if \( (W, p, \mathcal{X}) \in \ker \varphi \) for all \( p \in W = p_1^{-1}(W) \). Put in other words, we ask \( \varphi \) to vanish along the fibers of \( p_1 : \mathbb{P}(T) \to G \). It follows from [1, p.16] that there exists a map of vector bundles over \( G \),
\[ \varphi^\flat : V_d^n \xrightarrow{\varphi^\flat} p_1^*(p_1^*Q \otimes O_T(d)) = Q \otimes \text{Sym}_d(T^\vee) \]
such that
\[ \mathcal{E} := \ker(\varphi^\flat) = \{(W, \mathcal{X}) \mid \mathcal{X}(p) \in T_pW \forall p \in W \}. \]
The projectivization $\mathbb{P}(E) \subset \mathbb{G} \times \mathbb{P}^N$ is clearly as stated in (i). Let $q_1: \mathbb{P}(E) \to \mathbb{G}$ and $q_2: \mathbb{P}(E) \to \mathbb{P}^N$ be the maps induced by projection. It can be shown that $q_2: \mathbb{P}(E) \to \mathbb{P}^N$ is generically injective: the general vector field of degree $d \geq 2$ with an invariant $k$-plane has exactly one invariant $k$-plane. Write $H$ for the hyperplane class of $\mathbb{P}^N$. Set $u = \dim Y = \dim \mathbb{P}(E)$. We have $q_2^*(H) = c_1(\mathcal{O}_E(1)) =: h$. We may compute
\[
\deg Y = \int_{\mathbb{P}^N} H^u \cap Y = \int_{\mathbb{P}(E)} h^u = \int_{\mathbb{G}} q_1^*(h^u) = \int_{\mathbb{G}} s_\mathcal{E}.
\]
The assertions (ii) and (iii) follow noticing that the Segre class satisfies
\[
s_\mathcal{E}(E) = c_\mathcal{E}(Q \otimes \text{Sym}_d(T^\vee))
\]
in view of the exact sequence arising from \((\bigtriangledown)\).
\[
\mathcal{E} \rightarrow V_d^u \rightarrow Q \otimes \text{Sym}_d(T^\vee).
\]

In the case of invariant hyperplane (i.e., $k = n - 1$) we have explicitly

2.3. Theorem. The degree of the subvariety $Y$ of the space of foliations in $\mathbb{P}^n$ that admit an invariant hyperplane is given by
\[
\deg Y = \binom{d+n}{n}.
\]

Proof. We have the following exact sequence over $\mathbb{G}(n-1,n) = \mathbb{P}^n$,
\[
0 \rightarrow S_{d-1} \otimes Q^\vee \rightarrow S_d = \text{Sym}_d(\mathbb{C}^{n+1})^\vee \rightarrow \text{Sym}_d(T^\vee) \rightarrow 0.
\]
Twisting by $Q = \mathcal{O}_{\mathbb{P}^n}(1)$ we obtain:
\[
0 \rightarrow S_{d-1} \rightarrow S_d \otimes Q \rightarrow \text{Sym}_d(T^\vee) \otimes Q \rightarrow 0.
\]
From this we can compute $c_n(\text{Sym}_d(T^\vee) \otimes Q) = c_n(S_d \otimes Q)$. Setting $H = c_1(Q)$, the hyperplane class in $\mathbb{P}^n$, the sought for degree is just the coefficient of $H^n$ in $(1 + H)^{d+n}$. \hfill \Box

To compute $c_g(Q \otimes \text{Sym}_d(T^\vee))$ for any fixed $k,n$, see the script in §5.

3. Foliations with an invariant conic in $\mathbb{P}^2$

We construct a compactification of the space of foliations that leave invariant a smooth conic. This compactification is obtained as the birational image of a projective bundle over the variety of complete conics.

3.1. The incidence variety. The parameter space for the family of conics is $\mathbb{P}^5 = \mathbb{P}(S_2)$. We have the natural trilinear map
\[
S_2 \otimes S_d \otimes S^\vee \rightarrow S_{d+1}
\]
\[
G \otimes F \otimes \partial_i \mapsto (\partial_i G) F.
\]
It induces the map of vector bundles over $\mathbb{P}^5$,
\[
\varphi: \mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_d \otimes S^\vee \rightarrow S_{d+1}
\]
given by $\varphi(G, \mathcal{X}) := \mathcal{X}(G) = \sum F_i \partial_i G$, where $G$ is the equation of the conic and $\mathcal{X} = \sum F_i \partial_i$, $F_i \in S_d$.

Notice that $\varphi(G, HR) = 2GH$, for all $H \in S_{d-1}$. Recalling (1),
\[
V_d^u = S_d \otimes S^\vee / S_{d-1} R
\]
we see that \( \varphi \) induces a map

\[
\psi : \mathcal{O}_{\mathbb{P}^5}(-1) \otimes V^n_d \longrightarrow \frac{S_{d+1}}{\mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_{d-1}}.
\]

This maps fits into the commutative diagram,

\[
\begin{array}{cccc}
\mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_{d-1} & \longrightarrow & \mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_d \otimes S_1^\vee & \longrightarrow & \mathcal{O}_{\mathbb{P}^5}(-1) \otimes V^n_d \\
\cong & \downarrow & \varphi & \downarrow & \psi \\
\mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_{d-1} & \longrightarrow & S_{d+1} & \longrightarrow & S_{d+1} \\
\end{array}
\]

Twisting by \( \mathcal{O}_{\mathbb{P}^5}(1) \) we obtain

\[
\begin{array}{cccc}
\mathcal{M} & \cong & \tilde{\mathcal{M}} & \cong & \mathcal{M} \\
\mathcal{O}_{\mathbb{P}^5}(1) \otimes S_{d+1} & \longrightarrow & \mathcal{O}_{\mathbb{P}^5}(1) \otimes S_{d+1} & \longrightarrow & \mathcal{O}_{\mathbb{P}^5}(1) \otimes S_{d+1} \\
\cong & \downarrow & \theta & \downarrow & \tilde{\theta} \\
S_{d-1} & \longrightarrow & \mathcal{O}_{\mathbb{P}^5(1) \otimes S_{d+1}} & \longrightarrow & \mathcal{O}_{\mathbb{P}^5(1) \otimes S_{d+1}} \\
\cong & \downarrow & \cong & \downarrow & \cong \\
S_{d-1} & \longrightarrow & \mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_{d-1} & \longrightarrow & \mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_{d-1} \\
\end{array}
\]

where

\[
\mathcal{M} := \ker \theta \cong \tilde{\mathcal{M}} := \ker \tilde{\theta}.
\]

The map \( \theta : S_d \otimes S_1^\vee \to \mathcal{O}_{\mathbb{P}^5}(1) \otimes S_{d+1} \) appearing in (7) is surjective only over the open subset consisting of smooth conics. In fact, for \( G \in \mathbb{P}^5 \) the rank of the image of \( \theta_G \) depends on the singularities of the conic:

\[
\text{rank } \theta_G = \begin{cases} 
\text{total, } \binom{d+3}{2}, & \text{if } G \text{ is smooth;} \\
\binom{d+3}{2} - 1, & \text{if } G \text{ is a line pair;} \\
\text{rank } \text{Sym}_d = \binom{d+2}{2}, & \text{if } G \text{ is a double line.}
\end{cases}
\]

The minimal rank, \( r = \binom{d+2}{2} \), is achieved along

\[
\mathcal{V} := \nu_2(\mathbb{P}(S_1)) \subset \mathbb{P}(S_2),
\]

the Veronese variety of double lines. It turns out that over the open subset

\[
U \subset \mathbb{P}^5 \setminus \mathcal{V}
\]

of smooth conics, the restriction \( \tilde{\mathcal{M}}_U \) is a vector subbundle of the trivial bundle \( V^n_d \). The projectivization \( \mathbb{P}(\tilde{\mathcal{M}}_U) \) is the incidence variety

\[
\{(C, \mathcal{X}) \mid C \text{ is invariant by } \mathcal{X} \} \subset U \times \mathbb{P}^N.
\]

Let us denote by \( \overline{\mathcal{V}} \) the closure of its image in \( \mathbb{P}^N \). We see that \( \overline{\mathcal{V}} \) consists of (limits of) foliations that admit an invariant smooth conic. Our strategy to find its degree is summarized in the following.
3.2. Theorem. Let \( \pi : \mathcal{B} \to \mathbb{P}^5 \) be the blowup of \( \mathbb{P}^5 \) along the veronese \( V \). There exists a subbundle

\[ \mathcal{E} \subset \mathcal{B} \times V^d \]

such that the restriction \( \mathcal{E}_{\pi^{-1}(U)} \) coincides with \( \pi^* \tilde{\mathcal{M}}_U \) (cf. 7). In particular, the image of \( \mathbb{P}(\mathcal{E}) \) in \( \mathbb{P}^N = \mathbb{P}(V^d) \) is equal to \( \mathcal{Y} \).

Proof. Consider the pullback by \( \pi \) of the maps \( \varphi \) and \( \psi \) (cf. 6)

\[ \varphi_B : \pi^*(\mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_d \otimes S_1^\vee) \to \pi^* S_{d+1}, \]
\[ \psi_B : \pi^*(\mathcal{O}_{\mathbb{P}^5}(-1) \otimes V^d) \to \pi^* \left( \frac{S_{d+1}}{S_{d-1} \mathcal{O}_{\mathbb{P}^5}(-1)} \right). \]

By Lemma 3.3 below it’s enough to prove that the \( k \times k \) minors of \( \varphi_B \) are locally principal for all \( k \geq 1 \). Indeed, in this case \( \mathcal{J} := \text{Im} \varphi_B \) is locally free. Therefore we obtain a factorization of \( \varphi_B \),

\[ \pi^*(\mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_d \otimes S_1^\vee) \xrightarrow{\varphi} \pi^* S_{d+1} \]
\[ \xrightarrow{\varphi_B} \mathcal{J}, \]

and \( \pi^*(\mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_{d-1}) = \varphi_B(\pi^*(\mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_{d-1} \cdot R)) \subset \mathcal{J} \). This factorization induces a factorization of \( \psi_B \)

\[ \pi^*(\mathcal{O}_{\mathbb{P}^5}(-1) \otimes V^d) \xrightarrow{\psi} \pi^* \left( \frac{S_{d+1}}{\mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_{d-1}} \right) \]
\[ \xrightarrow{\psi_B} \tilde{\mathcal{J}}, \]

where

\[ \tilde{\mathcal{J}} := \frac{\mathcal{J}}{\mathcal{O}_{\mathbb{P}^5}(-1) \otimes \pi^*(S_{d-1})} \]

is a vector bundle. Define

\[ \mathcal{E} := \pi^* \mathcal{O}_{\mathbb{P}^5}(1) \otimes \text{ker} \tilde{\psi}. \]

(8)

It follows that \( \mathcal{E} \) is a subbundle of \( \mathcal{B} \times V^d \) that coincides with \( \pi^* \tilde{\mathcal{M}} \) over the open set \( \pi^{-1}(U) \), where \( U \subset \mathbb{P}^5 \) is the open set of smooth conics. Indeed, over \( \pi^{-1}(U) = U \) the map

\[ \iota : \tilde{\mathcal{J}} \to \pi^* \left( \frac{S_{d+1}}{\mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_{d-1}} \right) \]

is an isomorphism. Therefore \( \text{ker} \tilde{\psi} = \text{ker} \psi_B \) over \( U \).

To prove that the \( k \times k \) minors of a local matrix representation of \( \varphi_B \) are principal for all \( k \geq 1 \), consider

\[ \varphi_0 : \mathcal{O}_{\mathbb{P}^5}(-1) \otimes S_1^\vee \to S_1, \]

the universal symmetric map that gives the matrix of the conic. We are blowing-up the Veronese, which is the scheme of zeros of the ideal of \( 2 \times 2 \)-minors of \( \varphi_0 \). Therefore, up on \( \mathcal{B} \), we have that the \( 2 \times 2 \) minors of \( \varphi_{0B} \) are locally principal, say generated by \( t \). Thus we can write the matrix of \( \varphi_{0B} \) locally in the form \( A = \begin{pmatrix} 1 & t_0 & 0 \\ 0 & 0 & t_2 \end{pmatrix} \) (cf. [10]). Let \( C = Z(z_0^2 + tz_1^2 + tsz_2^2) \) be the associated conic.
Choosing an appropriate ordering of the basis of $S_d \otimes S_1^\vee$ and of $S_{d+1}$ the matrix of $\varphi_C$ can be put in the following form:

$$A_d = \begin{pmatrix}
2I_{n(d)} & 0 & 0 & B_1 & B_3 \\
0 & 2tI_{d+1} & 0 & 0 & B_4 \\
0 & 0 & 2ts & 0 & 0
\end{pmatrix},$$

where all the entries of $B_1$ are multiples of $t$, and the entries of $B_3, B_4$ are multiples of $ts$. Here $I_n$ stands for the identity matrix of size $m$, and $n(d) = \binom{d+2}{2}$. It follows that the ideals $J_i$ of $i \times i$-minors of $A_d$ are:

$$\begin{cases}
J_i = \langle 1 \rangle \text{ for } i = 1, \ldots, n(d); \\
J_{n(d)+j} = \langle t^j \rangle \text{ for } j = 1, \ldots, d+1; \\
J_{n(d+1)} = \langle t^{d+2}s \rangle.
\end{cases}$$

In particular these minors are principal as claimed. 

### 3.3. Lemma

Let $R$ be a local Noetherian domain, and $\varphi : R^n \to R^m$ a homomorphism of free, finitely generated $R$-modules. Suppose that the ideals $\langle k \times k \text{ minors of } \varphi \rangle$ are principal for all $k$. Then $J := \text{Im} \varphi$ is free.

**Proof.** Let

$$A = \begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots \\
a_{m1} & \cdots & a_{mn}
\end{pmatrix}$$

be the $m \times n$ matrix associated to $\varphi$, i.e. the columns of $A$ generate $J$. By hypotheses for $k = 1$, the ideal of coefficients of $A$ is principal:

$$\langle a_{11}, \ldots, a_{ij}, \ldots, a_{mn} \rangle = \langle b \rangle$$

We may assume $b \neq 0$. Let $b_{ij} := \frac{a_{ij}}{b}$. We may suppose $a_{11} = b$. Let $J'$ be the module generated by the columns of

$$B = \begin{pmatrix}
1 & \cdots & b_{1n} \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots \\
b_{m1} & \cdots & b_{mn}
\end{pmatrix}.$$ 

Equivalently (by elementary operations) $J'$ is generated by the columns of

$$\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & B'
\end{pmatrix},$$

where

$$B' = \begin{pmatrix}
b_{22} & \cdots & b_{2m} \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots \\
b_{mn2} & \cdots & b_{mn}
\end{pmatrix}.$$ 

Applying the inductive hypotheses, we have that $\text{Im} B'$ is free. Thus $J'$ is free. Since $R$ is a domain we have $J = bJ' \simeq J'$ and we conclude that $J$ is free. 

### 3.4. Remark

The variety $\mathcal{B}$ is the well known variety of complete conics (see [9], [11]). It is constructed to solve the indeterminacies of the map

$$e_1 : \mathbb{P}^5 \dashrightarrow \mathbb{P}^5$$

sending a conic to the envelope of its tangent lines. Equivalently $e_1(A) = \wedge A$, where $A$ is the symmetric matrix of the conic. We have that $\mathcal{B}$ is equal to the closure of the graph of $e_1$ in $\mathbb{P}^5 \times \mathbb{P}^5$.

### 3.5. A parameter space for foliations with invariant smooth conic

Consider the projective bundle associated to $\mathcal{E}$ cf. (3.2, p. 6), and let $q_2 : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^N$ denote the projection. Then

$$\mathbb{Y} = q_2(\mathbb{P}(\mathcal{E})) \subset \mathbb{P}^N$$

is a compactification for the parameter space of foliations with an invariant smooth conic. It’s not difficult to show that $q_2$ is generically injective, so the degree of $\mathbb{Y}$
is given by the top dimensional Segre class \(s_5(\mathcal{E})\). It will be computed using Bott’s formula.

3.6. Theorem. Notation as above, let \(Y\) be the compactification for the parameter space of 1-dimensional foliations of degree \(d\) on \(\mathbb{P}^2\) with an invariant smooth conic. Then the degree of \(Y\) is given by

\[
\frac{1}{2^55!}(d-1)d(d+1)(d^2 + 25d^6 + 231d^5 + 795d^4 + 1856d^3 + 2468d^2 + 2256d + 768)
\]

and its codimension is equal to \(2(d-1)\).

Proof. From the definition of \(\mathcal{E}\) (cf. 3.2) we see that \(\text{rank}(\mathcal{E}) = d(d+2)\). As \(q_2\) is generically injective it is easy to see that \(\text{cod} Y = 2(d-1)\).

To compute the degree we use Bott’s formula,

\[
\int s_5(\mathcal{E}) \cap [B] = \sum_{p \in BT} \frac{s_T^T(\mathcal{E}_p) \cap [p]}{c^T_5(T_pB)}
\]

(cf. [2]) where the torus \(T := \mathbb{C}^*\) acts on \(B\) with isolated fixed points. We recall that each \(T\)-equivariant Chern class \(c^T_1\) appearing in the left hand side is simply the \(i\)-th symmetric function on the weights of the fiber of the vector bundle at the fixed point. The action of \(T\) on \(B\) is induced by the action of \(T\) on \(S_1\), given by

\[
t \cdot z_i = t^{w_i} z_i
\]

for a suitable choice of weights \(w_0, w_1, w_2 \in \mathbb{Z}\). This action induces actions on \(\mathbb{P}(S_1), \mathbb{P}(S_2) = \mathbb{P}^5\) and on \(\mathbb{P}(\text{Sym}_2 S_1) = \mathbb{P}^5\) in such a way that the map \(c_1 : \mathbb{P}^5 \to \mathbb{P}^5\) (see 3.4) is \(T\)-equivariant. Thus we obtain an action of \(T\) on \(B = \text{closure of Graph } c_1 \subset \mathbb{P}^5 \times \mathbb{P}^5\).

It is easy to see that if we choose the weights in such a way that all sums \(w_i + w_j\) with \(0 \leq i \leq j \leq 2\) are pairwise distinct, we obtain the following six isolated fixed points in \(\mathbb{P}^5\):

\[
z_0^2, z_0 z_1, z_0 z_2, z_1^2, z_1 z_2, z_2^2.
\]

It remains to find the fixed point on the fiber of \(\pi : B \to \mathbb{P}^5\) over each fixed point in \(\mathbb{P}^5\). If \(A \in \{z_0 z_1, z_0 z_2, z_1 z_2\}\), then \(\pi^{-1}(A)\) has just one point. For example, \(\pi^{-1}(z_0 z_1) = (z_0, z_1, z_2^2)\). Here we put \(z_i\) for the dual basis.

Take \(A \in \{z_0^2, z_1^2, z_2^2\}\), say \(A = z_0^2\). Recall that the exceptional divisor of the blowup is \(E := \mathbb{P}(\mathcal{N})\) where \(\mathcal{N} = N_Y \mathbb{P}^5\) stands for the normal bundle. We have an explicit description for \(\mathcal{N}\) see [11, Proposition 4.4.]. Notation as in (4, p. 2) with \(k = 1, n = 2\), we have

\[
N_Y \mathbb{P}^5 = O_{\mathbb{P}^2}(2) \otimes \text{Sym}_2 (T^\vee).
\]

The fiber of \(O_{\mathbb{P}^2}(2)\) (resp. \(\text{Sym}_2 (T^\vee)\)) over \(z_0\) is the dual space \((z_0^2)\) (resp. \((z_1^2, z_1 z_2, z_2^2)\)). Thus we get

\[
\pi^{-1}(z_0^2) = E_{z_0} = \mathbb{P}((z_0^2) \otimes (z_1^2, z_1 z_2, z_2^2)).
\]

We see there are three fixed points in the fiber of each fixed point \(A \in V\). For \(A = z_0^2\) these points are

\[
z_0^2 \otimes z_1^2, z_0^2 \otimes z_1 z_2, z_0^2 \otimes z_2^2.
\]

Summarizing, we have twelve fixed points in \(B\):

1. \((z_iz_j, z_k^2)\) with \(0 \leq i < j \leq 2\); \(k \neq i, j\); these three lie off \(E\);
2. \((z_i^2, z_j^2 \otimes z_k)\) with \(j, k \neq i; j < k\);
(3) \((z_i^2, z_j^2 \otimes z_k^2)\) with \(i \neq j\).

Next we compute the fibers of \(E\) over each type of fixed point. Suppose that \(B \in \mathbb{B}\) is a fixed point. The strategy is to take a curve \(B(t) \in \mathbb{B}\) such that
\[
\lim_{t \to 0} B(t) = B
\]
and such that \(A(t) := \pi(B(t)) \in \mathbb{P}^5\) is a curve of smooth conics for \(t \neq 0\). Therefore \(E_B\) will be obtained as the limit of \(E_{B(t)} = \widetilde{M}_{A(t)}\),
\[
\lim_{t \to 0} \widetilde{M}_{A(t)} = E_B.
\]

This enables us to use the well known space of vector fields of degree \(d\) that leave invariant a smooth conic \(C = Z(G)\) (see [3]). This space is
\[
\{F_{ij}(\frac{\partial G}{\partial z_i} - \frac{\partial G}{\partial z_j}) \mid F_{ij} \in S_{d-1}\},
\]
modulo multiples of the radial vector field.

We will adopt the following notation: for each subset \(J := \{v_0, \ldots, v_k\} \subset \{z_0, z_1, z_2\}\) we set
\[
M_m(J) = \{v_0^m \cdot v_0^{m-1} v_1 \ldots v_k^m\},
\]
the canonical monomial basis of \(\text{Sym}_m(J)\). We write simply \(M_m\) for \(M_m(\{z_0, z_1, z_2\})\).

Set \(X_{ij} := z_i \partial_j - z_j \partial_i\). Notice this is a vector of weight 0, since \(t \cdot z_i = t^{w_i} z_i\) whereas \(t \cdot \partial_i = t^{-w_i} \partial_i\).

We now describe suitable 1-parameter families of smooth conics abutting each type of fixed point.

(1) \(B_1 = z_0 z_1\). We take \(A(t) = z_0 z_1 + t z_2^2 \in \mathbb{P}^5\). Using the characterization (♠) we obtain that the space \(M_{A(t)}\) of vector fields leaving \(A(t)\) invariant is given by
\[
\{F_{10}(z_1 \partial_1 - z_0 \partial_0), F_{20}(z_1 \partial_2 - 2 t z_2 \partial_0), F_{21}(z_0 \partial_2 - 2 t z_2 \partial_1) \mid F_{ij} \in S_{d-1}\}.
\]

Taking limit as \(t \to 0\), we find a basis for \(E_{B_1}\):
\[
\{F_1 X_{0, 1}, F_2 \partial_2 \mid F_1 \in M_{d-1}, F_2 \in M_d \setminus \{z_2^1\}\},
\]
Clearly this basis consists of \(T\)-eigenvectors.

(2) \(B_2 = (z_0^2, z_0^2 \otimes z_1 z_2)\). In this case, we take \(A(t) = z_0^2 + t z_1 z_2\). With the same procedure as above, we obtain the following basis (of eigenvectors) for \(E_{B_2}\):
\[
\{F_1 z_0 \partial_1, F_2 z_0 \partial_2, F_3 X_{1, 2} \mid F_1, F_2 \in M_{d-1}, F_3 \in M_{d-1}(\{z_1, z_2\})\}.
\]

(3) \(B_3 = (z_0^2, z_0^2 \otimes z_1^2)\). In this case a curve of smooth conics that approximates \(B_3\) is \(A(t) = z_0^2 + t z_1^2 + t^2 z_2^2\). As before, we obtain the following basis of eigenvectors for \(E_{B_3}\):
\[
\{F_1 z_0 \partial_1, F_2 \partial_2 \mid F_1 \in M_{d-1}, F_2 \in M_d \setminus \{z_2^d\}\}.
\]

In order to handle the denominator in Bott’s formula (9) we obtain, for each fixed point \(B\), a base consisting of eigenvectors of \(T_B \mathbb{B}\).

If \(B \notin E\) then \(T_B \mathbb{B} \simeq T_{\pi(B)} \mathbb{P}(S_2)\). For example, for \(B_1 = z_0 z_1\) we have
\[
T_{B_1} \mathbb{B} \simeq \langle z_0 z_1 \rangle^\vee \otimes \langle z_0^2, z_0 z_2, z_1 z_2, z_2^2 \rangle.
\]
If \(B \in E\), then \(B = (A, [v])\) with \(A \in \mathbb{V}\) and \(v \in \mathcal{N}_A, v \neq 0\). Now
\[
T_B \mathbb{B} = T_A \mathbb{V} \oplus \text{Hom}(\langle v \rangle, \mathcal{N}_A) \oplus \langle v \rangle.
\]
as \( C^* \)-modules. For \( B_2 = (z_0^2, z_0^2 \otimes z_1 z_2) \) we have:

\[
T_{B_2} \mathbb{B} = T_{z_0^2} \mathbb{V} \oplus (z_0^2 \otimes z_1 z_2) \otimes (z_0^2 \otimes z_1 z_2) \oplus (z_0^2 \otimes z_1 z_2)
\]

where \( T_{z_0^2} \mathbb{V} = (z_0^2 \otimes z_0 z_1 z_2) \). Similarly, for \( B_3 = (z_0^2, z_0^2 \otimes z_1^2) \) we have

\[
T_{B_3} \mathbb{B} = T_{z_0^2} \mathbb{V} \oplus (z_0^2 \otimes z_1^2) \otimes (z_0^2 \otimes z_0 z_1 z_2, z_0^2 \otimes z_1^2) \oplus (z_0^2 \otimes z_1^2).
\]

The explicit calculation in Bott’s formula is better left for a script (see § 5). Finally we use Lemma 3.7 below which enables us to restrict the computation just for the first sixteen values of \( d = 2, \ldots, 17 \) and then interpolate the answers obtained.

3.7. Lemma. The sum in the right hand side of Bott’s formula

\[
\int s_5(\mathcal{E}(d)) \cap [\mathbb{B}] = \sum_{B \in \mathfrak{B}^T} \frac{s_5^B(\mathcal{E}(d)_{B}) \cap [B]}{c_5^B(T_{B} \mathbb{B})},
\]

is a combination of \( w_i \)'s with polynomial coefficients in \( d \) of degree \( \leq 15 \).

Proof. For each fixed point \( B \) let \( \{\xi_1(d), \ldots, \xi_m(d)\} \) denote the set of weights of \( \mathcal{E}(d)_{B} \). Since \( s_5^B(\mathcal{E}(d)_{B}) \) is a polynomial in \( \{c_k^B(\mathcal{E}(d)_{B}) \mid k = 1, \ldots, 5\} \) it’s enough to prove that each

\[
c_k^B(\mathcal{E}(d)_{B}) = \sigma_k(\xi_1(d), \ldots, \xi_m(d)(d))
\]

is a combination of \( w_i \)'s with polynomial coefficients in \( d \) of degree \( \leq 3k \).

Recalling Newton’s identities

\[
k\sigma_k = \sum_{i=1}^k (-1)^{i+1} \sigma_{k-i} p_i
\]

where

\[
p_k(\xi_1(d), \ldots, \xi_m(d)) := \sum_{i=1}^m \xi_i(d)^k
\]

we see that it suffices to prove that \( p_k(\xi_1(d), \ldots, \xi_m(d)(d)) \) is a combination of \( w_i \)'s with polynomial coefficients in \( d \) of degree \( \leq k + 2 \).

On the other hand, a careful analysis of the weights appearing in the basis of \( \mathcal{E}(d)_{B} \) at each fixed point shows that these weights can be separated into sets of the form

\[
\{ \text{weights of } M_e(J) \} \text{ or } \{ \text{weights of } M_e(J) \} + w
\]

where \( w \) is a (fixed) combination of \( w_i \)'s; \( e = d, d - 1 \) and \( J = (z_0, z_1, z_2), J = (z_i, z_j), i \neq j \). From this the reader may convince her(him)self that it’s enough to prove the following

Claim: Let \( m = m(d, n) := \binom{d+n}{n} \) and \( \{\xi_{n,1}(d), \ldots, \xi_{n,m}(d)\} \) be the weights associated to the basis \( M_d(\{z_0, \ldots, z_n\}) \) of \( \text{Sym}_d(\{z_0, \ldots, z_n\}) \). Then

\[
p_k^m(\xi)(d) := \sum_{i=1}^m \xi_{n,i}(d)^k
\]

is a combination of \( w_i \)'s with polynomial coefficients in \( d \) of degree \( \leq k + n \).

To prove the claim we proceed by induction on \( n \geq 1 \) and on \( k \geq 0 \).

For \( n = 1 \),

\[
M_d(\{z_0, z_1\}) = \{z_0^d, z_0^{d-1} z_1, \ldots, z_0 z_1^{d-1}, z_1^d\}
\]
By induction we conclude that

\[ p_k^1(d) = \sum_{i=1}^{d+1} \xi_{i,i}(d)k = \sum_{i=0}^{d} (i w_0 + (d - i) w_1)k \]

\[ = \sum_{i=0}^{d} (i (w_0 - w_1) + dw_1)k = \sum_{i=0}^{d} \sum_{j=1}^{k} \binom{k}{j} (i (w_0 - w_1))^j (dw_1)^{k-j} \]

\[ = \sum_{j=1}^{k} \binom{k}{j} (dw_1)^{k-j} (w_0 - w_1)^j \sum_{i=0}^{d} i^j. \]

The sum \( \sum_{i=0}^{d} i^j \) is polynomial in \( d \) of degree \( j + 1 \), therefore \( p_k^1(d) \) is a polynomial in \( d \) of degree \( \leq k + 1 \).

For \( k = 0 \), we have \( p_0^0(d) = m(d, n) \), a polynomial in \( d \) of degree \( n \).

For the general case, decompose the basis \( M_d(\{z_0, \ldots, z_n\}) \) as the union

\[ z_0 M_{d-1}(\{z_0, \ldots, z_n\}) \cup z_1 M_{d-1}(\{z_1, \ldots, z_n\}) \cup z_2 M_{d-1}(\{z_2, \ldots, z_n\}) \cup \cdots \cup \{z_n\}. \]

Then the weights are:

\[ w_0 + \{\xi_{n,i}(d-1)\} \cup w_1 + \{\xi_{n-1,i}(d-1)\} \cup w_2 + \{\xi_{n-2,i}(d-1)\} \cup \cdots \cup \{dw_n\}. \]

Hence we can write

\[ p_k^n(d) = \sum_{i=1}^{m(d,n)} (\xi_{n,i}(d))^k = \sum_{i=1}^{m(d-1,n)} (w_0 + \xi_{n,i}(d-1))^k + \]

\[ \sum_{i=1}^{m(d-1,n-1)} (w_1 + \xi_{n-1,i}(d-1))^k + \sum_{i=1}^{m(d-1,n-2)} (w_2 + \xi_{n-2,i}(d-1))^k + \cdots + (dw_n)^k \]

\[ = \sum_{j=0}^{k} \binom{k}{j} w_0^j p_k^{n-j}(d-1) + \sum_{j=0}^{k} \binom{k}{j} w_1^j p_{k-j}^{n-1}(d-1) + \]

\[ \sum_{j=0}^{k} \binom{k}{j} w_2^j p_{k-j}^{n-2}(d-1) + \cdots + (dw_n)^k. \]

By induction we conclude that \( p_k^n(d) - p_0^n(d-1) \) is a polynomial in \( d \) of degree \( \leq k + n - 1 \), and this implies that \( p_k^n(d) \) is a polynomial in \( d \) of degree \( \leq k + n \).

4. Comments

Using the varieties of complete quadrics of any dimension in \( \mathbb{P}^n \) (see [11]) it is possible to find a compactification of the space of dimension one foliations in \( \mathbb{P}^n \) that leave invariant a smooth quadric.

For example, in the case of conics in \( \mathbb{P}^3 \) we obtain the following.

4.1. Theorem. Let \( \mathcal{Y} \) denote the closure in \( \mathbb{P}^N \) of the variety of dimension one foliations in \( \mathbb{P}^3 \) that have an invariant smooth conic. The degree of \( \mathcal{Y} \) is given by

\[ \frac{4}{852}(d - 1)d [207d^{14} + 2763d^{13} + 15447d^{12} + 54395d^{11} + 114847d^{10} + 207891d^9 + 256737d^8 + 225801d^7 + 164937d^6 + 182101d^5 + 38993d^4 + 316221d^3 + 248856d^2 - 118908d - 332640] \]
and its codimension is equal to \(4(d - 1)\).

For invariant quadric surfaces in \(\mathbb{P}^3\) we find:

4.2. Theorem. Let \(\mathcal{Y}\) denote the closure in \(\mathbb{P}^N\) of the variety of dimension one foliations in \(\mathbb{P}^3\) that have an invariant quadric. The degree of \(\mathcal{Y}\) is given by

\[
\deg(\mathcal{Y}) = \frac{1}{9!}(3!)^9(d - 1)d(d + 1) \left( d^2 - 287493287072d^9 - 388532146832d^8 - 1115680433472d^7 - 4477695012864d^6 + 826426536528d^5 + 8139069775872d + 4334215495680 \right)
\]

and its codimension is equal to \((d - 1)(d + 5)\).

5. Scripts

The script below uses SCHUBERT [8] to compute the formula for the degree of the variety of 1-dimensional foliations of degree \(d\) in \(\mathbb{P}^n\) with some invariant \(k\)-plane.

```plaintext
with(schubert);

deg:=proc(k,n)local Ec, SrE, F;
grass(n-k, n+1, c);
Ec:=dual((n+1)-Qc);
SrE:=Symm(d, Ec);
F:=SrE*Qc;
integral(chern(DIM,F));end;

def factor(deg(1,2)); #1/8*d*(d+3)*(d+2)*(d+1)
```

5.1. Singular script. The calculations for Bott’s formula uses SINGULAR [6].

```plaintext
//download from www.mat.ufmg.br/~israel/Projetos/myprocs.ses
//save and load
<"myprocs.ses"; //more procedures; save and load

//Sum fractions
proc sumfrac(list ab, list cd){
def l= (ideal(ab[1]*cd[2]+ab[2]*cd[1],
ab[2]*cd[2]));def p=gcd(l[1],l[2]);
return(list(l[1]/p,l[2]/p));}

//Subtract fractions
proc subfrac(list ab, list cd){
def l= (ideal(ab[1]*cd[2]-ab[2]*cd[1],
ab[2]*cd[2]));def p=gcd(l[1],l[2]);
return(list(l[1]/p,l[2]/p));}

//Multiply fractions
proc mulfrac(list ab, list cd){
def l= ideal(ab[1]*cd[1],
ab[2]*cd[2]);
def p=gcd(l[1],l[2]);
return(list(l[1]/p,l[2]/p));}

//Inverse of a fraction
proc invfrac(list ab){
return(list(ab[2],ab[1]));}

//Divide a fraction
proc divfrac(list ab, list cd){
return(mulfrac(ab, invfrac(cd)));}
```
// If vs is a poly then return its monomials,
// else return the degree d monomials
proc mon(vs,d)
{if(typeof(vs)=="poly"){
def vvs=pol2id(vs);} else {def vvs=vs;}
return(vvs^d);}

// Compute f^k
proc pot(f,k){def p=f;
for(int i=1;i<=k-1;i++){p=reduce(p*f,mt);} return(p);}

// Substitute the weights by given values P[i]
proc pesos(H){ //H= poly or list
list res;
def mm=matrix(vars(1..(nvars(basering))));
def ty@=typeof(H); if(ty@=="poly"){
  for(int i=1;i<size(H);i++){
def i_=H[i]; def iv=leadexp(i_);
res[i]=(mm*iv)[1,1];} return(mymapn(ideal(x(0..n)),ideal(P[1..n+1]),res));}
else{if(ty@=="list" and size(H)==2) {
def num,den=H[1..2];
res=pesos(H[1]);
def denn=pesos(den);
for(int i=1;i<size(res);i++) {res[i]=res[i]-denn[1];}
return(res);}
else{if(ty@=="ideal"{res=pesos(H[1]);
for(int i=2;i<=size(matrix(H));i++) {res+=pesos(H[i]);}
return(res);}}}

// Symmetric functions in the weights of H
proc cherns(H){
poly sigma= 1;
for(int j=1;j<=min(rk,size(H));j++){
sigma=reduce((1+H[j]*t)*sigma,std(t^(d+1)));}
kill j;def j=coeffs(sigma,t);
while(size(j)<d+1){j=transpose(concat(transpose(j),[0]));}
return(ideal(j));}

/Interpolation
proc interpola(lista){
//list(list(a,b),list(c,d),..)
def s=size(lista);
if (s>1) {
  for (int j= 2;j<= s;j++){
poly f=1;
  for (int i = 1;i<= j;i++){
f=f*(t-lista[i][1]);}
  if (gcd(f,diff(f,t))==1) {
poly g=0;for(i= 1;i<=j;i++){def p=f/(t-lista[i][1]);
g +=lista[i][2] / subst(p,t,lista[i][1])*p;}
  int ii;
  for(i=1;i<s;i++){
if(subst(g,t,lista[i][1])<>lista[i][2]) {ii++;}
  see(j,ii,s,deg(g));
if (ii==0) { break;}}
else {ERROR("WRONG DATA1");}
  if (deg(g)==s-1){print(g);return("work harder");}
else {
print(myfact(g));
return(g);} 
}

else {ERROR("WRONG DATA2");}

// end of procedures

// load dimension of P^n

int n=2;
// dimension of conics in P^2
int d=(n+3)*n/2;kill r;
ring r=(0),(x(0..n),t,c(1..d)),dp;
def mt=std(t^(d+1));
//d-Chern class
def S(d)=summ(seq("c(i)*t^i",1,d));
//d-Segre class
poly p; for(int i=1;i<=d;i++){
p=p+(-1)^i*pot(S(d),i); } S(d)=p;def p=coeff(S(d),t); def S(d)=row(p,2);kill p; S(d)=S(d)[d..1];
// give values to the weights
intvec P=0,1,3;
//forr(3,n+1,"P[i]=random(2,15)");
def xx=ideal(x(0..n));
poly xxs=summ(seq("x(i)"),0,n));
list sfs;
sfs[2]=summ(mon(xxs,2));
// Define lists TP5, TB=tangent of \B, TV, NV=Normal bundle to the Veronese
// in each fixed point, and find c_{d}(T\B)= product of the weights.
list TP5,TV,NV,TB;
forr(1,size(sfs[2]),"
    def a=sfs[2][i];
    TP5[i]=list(sfs[2]-a,a);
    // test if a is of the form Z_{0}^2 or Z_{0}Z_{1}
    def b=radical(a);
    // if a is of the form Z_{0}^2
    if(b[1]<>a)
    {def p=list(xxs-b[1],b[1]);
     TV[size(TV)+1]=list(p);
     def q=subfrac(TP5[i],p);
     NV[size(NV)+1]=q;
     // For each vector of NV
     for(int j=1;j<=size(q[1]);j++){
     // Do the sum TV+O_{N}(-1)
     def jj=list(q[1][j],q[2]);
     def TT=sumfrac(p,jj);
     // Obtain TB
     TB[size(TB)+1]=list(jj,prod(pesos(TT)));
    }
    }
    // if a is Z_{0}Z_{1} then TB=TP5
    else {TB[size(TB)+1]=list(a,prod(pesos(TP5[i])));}
    }"
}
list respostas;
// compute the weights of \widetilde{N}
// in the fixed points for d=2..17
// to do interpolation.
// dd is the degree of the foliation
for(int dd=2;dd<=17;dd++){
    int rk=int((n+1)*binomial(dd+n,n))-
     binomial(dd+n-1,n));
sfs[dd]=summ(mon(xxs,dd));
sfs[dd-1]=summ(mon(xxs,dd-1));
    poly NN;
for(int ii=1;ii<=size(TB);ii++) {
    def p=TB[ii][1];
    // the fixed point, can be list (in E)
    // or poly (not in E) if the point is not in E
    if(typeof(p)=="poly") {
        // convert product in summ, in order to
        // obtain the factors
        p=dotprod(leadexp(p),xx);
        // obtain the factors
        def a=p[1]; def al=p-a;
        def z=xxs-a-al;
        // Using the basis found in the text
        // compute the weights of
        // the fiber of \( \widetilde{\mathcal{N}} \)
        def w=pesos(list(sfs[dd]-z^dd,z))+pesos(sfs[dd-1]);
        // symmetric functions in the weights,
        // with given values!
        // i.e. \( c(i)^T(\widetilde{\mathcal{N}}) \)
        W=cherns(W);
        // substitute, in \( s_5 \), \( c(i) \) by the
        // values \( c(i)^T(\widetilde{\mathcal{N}}) \)
        def NUM=subst(S(d)[d],c(1),W[1][2]);
        for(int i=2;i<d;i++) {NUM=subst(NUM,c(i),W[1][i+1]);}
        // Do the sum of the Bott's formula
        NN=NN+NUM/TB[ii][2];}
    // If the point is in E, p is a list
    else {def p=TB[ii]; def a=p[1][1]; def z=p[1][2][1];
        // convert prod in summ
        z=dotprod(leadexp(z),xx);
        // test if a is of the form \( q_2 \) or \( q_3 \)
        if(size(z)==1) {
            z=leadmonom(z); def w=xxs-a-z;
            // Using the basis found in the text compute the weights of
            // the fiber of \( \widetilde{\mathcal{N}} \)
            def w=pesos(list(sfs[dd]-w^dd,w))+pesos(list(a*sfs[dd-1],z));
            // symmetric functions in the weights, with given values!
            W=cherns(W);
            // substitute \( c(i) \) in \( s_5 \) by the values \( c(i)^T(\widetilde{\mathcal{N}}) \)
            def NUM=subst(S(d)[d],c(1),W[1][2]);
            for(int i=2;i<d;i++) {NUM=subst(NUM,c(i),W[1][i+1]);}
            // Do the sum in Bott's formula:
            NN=NN+NUM/TB[ii][2];} // matches \( z=leadmon(z) \)
        // If a is of the type \( q_2 \):
        else {def z1=z[1]; z=z[2];
            // Using the basis found in the text compute
            // the weights of the fiber of \( \widetilde{\mathcal{N}} \)
            def w=pesos(list(a*sfs[dd-1],z))+pesos(list(a*sfs[dd-1],z1))+pesos(mon(xxs-a,dd-1));
            W=cherns(W);
            def NUM=subst(S(d)[d],c(1),W[1][2]);
            for(int i=2;i<d;i++) {NUM=subst(NUM,c(i),W[1][i+1]);}
            NN=NN+NUM/TB[ii][2];}}}
    // Save in respostas the values obtained for each degree
    respostas[size(respostas)+1]=list(dd,NN);
    // Interpolate the list respostas
    interpola(respostas);
References

[1] A. Altman, S. Kleiman. Foundations of the Theory of Fano Schemes. Compositio Math. 34, 3–47, 1977.
[2] M. Brion. Equivariant cohomology and equivariant intersection theory. arxiv 9802063.
[3] E. Esteves. The Castelnuovo-Mumford regularity of an integral variety of a vector field on projective space. Math. Res. Lett. 9, no. 1, 1-15, 2002.
[4] V. Ferrer, I. Vainsencher. Polynomial vector fields with algebraic trajectories. arxiv, www.mat.ufmg.br/~israel/, 2010.
[5] W. Fulton. Intersection Theory. Springer-Verlag. New York. 1985.
[6] G.-M. Greuel, G. Pfister, H. Schönemann. SINGULAR 3-1-1 – A Computer Algebra System for Polynomial Computations. http://www.singular.uni-kl.de, 2010.
[7] J.P. Jouanolou. Equations de Pfaff algébriques. Lecture Notes in Math., 708. Springer-Verlag, 1979.
[8] S. Katz, S. A. Strømme. Schubert, A Maple package for Intersection Theory. http://linus.mi.uib.no, 2001.
[9] S. Kleiman, A. Thorup. Complete bilinear forms. [in Algebraic geometry (Sundance, UT, 1986)], 253–320, Lecture Notes in Math., 1311, Springer, Berlin, 1988.
[10] D. Laksov The geometry of complete linear maps. Ark. Mat. 26, no. 2, 231–263, 1988.
[11] I. Vainsencher. Schubert calculus for complete quadrics. [in Enumerative geometry and classical algebraic geometry (Nice, 1981)], pp. 199–235, Progr. Math., 24, Birkhäuser, Boston, Mass. 1982.

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