On the Intuitionistic Background of 
Gentzen’s 1935 and 1936 Consistency Proofs and 
Their Philosophical Aspects

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Abstract

Gentzen’s three consistency proofs for elementary number theory have a com- 
mon aim that originates from Hilbert’s Program, namely, the aim to justify the 
application of classical reasoning to quantified propositions in elementary number 
theory. In addition to this common aim, Gentzen gave a “finitist” interpretation 
to every number-theoretic proposition with his 1935 and 1936 consistency proofs. 
In the present paper, we investigate the relationship of this interpretation with 
intuitionism in terms of the debate between the Hilbert School and the Brouwer 
School over the significance of consistency proofs. First, we argue that the in-
terpretation had the role of responding to a Brouwer-style objection against the 
significance of consistency proofs. Second, we propose a way of understanding 
Gentzen’s response to this objection from an intuitionist perspective.

Key words: proof theory, intuitionism, foundations of mathematics, philosophy of 
logic, history of logic

1. Introduction

Gerhard Gentzen gave three proofs for the consistency of elementary number 
theory. The first proof was included in a paper submitted in 1935 and posthumously

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1 Note that Gentzen intended his consistency proofs to comprehend not only first-order 
Peano arithmetic but also a system that may have as an axiom any finitistically valid 
(quantifier-free) formula. See [Gentzen 1936, §6.2] and [Gentzen 1938b, §1.4].
published in 1974 ([Gentzen 1974]). The second and third proofs were published in 1936 ([Gentzen 1936]) and 1938 ([Gentzen 1938b]), respectively. The third proof is usually called “Gentzen’s consistency proof for elementary number theory.”

Gentzen’s three consistency proofs have a common aim that originates from Hilbert’s Program. In this program, Hilbert aimed to justify both the introduction of ideal elements into mathematics and the application of classical reasoning to these elements by proving the following from his finitary standpoint: No contradiction can be derived in formal systems codifying ideal parts of mathematics. Gentzen also had these aims. At the outset of his attempt to achieve them, Gentzen aimed to justify with his three consistency proofs the application of classical reasoning to quantified proposition of elementary number theory.

In addition to his three consistency proofs’ common aim, Gentzen gave a “finitist” interpretation to every number-theoretic proposition by means of his first two consistency proofs, i.e., the 1935 and 1936 proofs. Even though Hilbert began to investigate interpretations of mathematical propositions for foundational purposes

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2 An English translation of this paper had been published in 1969 as an appendix of [Gentzen 1969, ch.4]. As to the historical exposition for the submission of the 1935 proof, see [Menzler-Trotz 2007, pp.57–62].

3 In fact, the 1935 and 1936 proofs have been studied over the last few decades, as has the 1938 proof. The first reconstruction of Gentzen’s 1935 proof was made by Bernays 1970 and was later refined by Negri 1980. Kreisel 1971 claimed that the bar rule is used in the 1935 proof. Sundholm 1983 explained the relationship between devices of the 1935 proof and infinitary derivations. Coquand 1995, Tait 2005, and von Plato 2009b independently revealed the game theoretic aspect of the 1935 proof. Tait 2005 also pointed out the relationship between the 1935 proof and the no-counterexample interpretation. Akiyoshi 2010 reconstructed the 1935 proof by using Mints-Buchholz’s method of finite notations for infinitary derivations. Sieg 2012 investigated the historical relationship between Hilbert’s proof theory and the 1935 proof by scrutinizing Gentzen’s unpublished manuscripts. Recently, Detlefsen 2015 explained the difference between Gentzen’s formalism and Hilbert’s formalism. Tait 2015 argued that Gentzen did not employ the bar theorem in the 1935 proof. von Plato 2015 investigated the historical background of the 1935 proof by examining Gentzen’s manuscripts. In Appendix of von Plato 2015, Siders and von Plato gave a formulation of the 1935 proof by means of the bar theorem.

As to Gentzen’s 1936 proof, Yasugi 1980 presented a reconstruction and some applications of it. [SP 1995] pointed out that Gödel 1938 sketched the transformation of the 1936 consistency proof into the no-counterexample interpretation. Buchholz 2015 reconstructed the 1936 proof by using Mints-Buchholz’s method of finite notations for infinitary derivations.

4 Hilbert 1926, pp.178–179, [Hilbert 1928, p.74].

5 This reading of the 1935 and 1936 proofs has been adopted by Kreisel 1971, Negri 1980, Coquand 1995, Tait 2005, Tait 2015, Sieg 2012 and AT 2013. Especially, in Tait 2015, p.215], Tait extracted from the 1935 and 1936 proofs the same interpretation as the interpretation (GI) that we state below.
after Gödel’s incompleteness theorems, Hilbert’s 1920s proof theory included no such interpretation. In this respect, Gentzen’s proof theory transcended Hilbert’s proof theory of the 1920s.

Several studies have provided some clues to the investigation into the relationship of Gentzen’s interpretation of number-theoretic propositions with intuitionism. Bernays remarked in [Bernays 1970] that reduction procedures (Reduziervorschrift), a key concept for that interpretation, “involves universal quantification over free choice sequences.” In [Kreisel 1971], Kreisel claimed, “In fact, not the fan theorem, but rather the bar theorem is involved in Gentzen’s [first] proof; or, to be precise, […] Gentzen uses the corresponding rule.”

Here, we investigate the relationship of Gentzen’s interpretation of number-theoretic propositions with intuitionism in terms of the debate between the Hilbert School and the Brouwer School over consistency proofs’ significance. First, we argue that this interpretation functioned as a response to a Brouwer-style objection against consistency proofs’ significance. Brouwer had raised such an objection on the basis of his claim about the interpretation of mathematical propositions; his claim was very close to Hilbert’s claim about the finitist interpretation of existential propositions. Gentzen had accepted Hilbert’s claim; hence, Gentzen took this objection seriously and responded to it. Second, we propose a way to understand the 1935 proof’s response to the Brouwer-style objection from an intuitionist perspective. We formulate Gentzen’s interpretation of number-theoretic propositions by means of spreads, which are infinite trees in intuitionistic mathematics, and then prove the key lemma to his response by monotone bar induction. This means that Gentzen’s interpretation of number-theoretic propositions is admissible from an intuitionist perspective. Note that we do not claim Gentzen himself used monotone bar induction to prove the key lemma.

This paper is structured as follows. In Section 2, we explain both Hilbert’s claim and the Brouwer-style objection mentioned above. In Section 3, we argue that Gentzen’s interpretation of number-theoretic propositions served as a response to the Brouwer-style objection. In Section 4, we provide an intuitionist formulation of reduction procedures–key to this interpretation–by using the notion of spreads. Section 5 gives a proof for the key lemma of Gentzen’s response, and Section 6 concludes the paper.

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6 Cf. [Sieg 2012, §5.5].
7 [Bernays 1970, p.417].
8 [Kreisel 1971, p.262].
9 As to monotone bar induction, see [TD 1988, ch.4, §8]. In [Tait 2015, p.223], Tait gave a proof of this key lemma by means of the principle that corresponds to decidable bar induction.
2. Hilbert and Brouwer on Mathematical Propositions

In this section, we first explain Hilbert’s claim that set the scene for Gentzen’s interpretation of number-theoretic propositions. Next, we argue that Brouwer raised an objection to consistency proofs’ significance and that one of its premises is very close to Hilbert’s claim.

In papers published during the 1920s, Hilbert established the finitary standpoint to provide a firm foundation for the introduction of ideal elements into mathematics, such as irrational numbers and complex numbers. The finitary standpoint admits only facts and concepts about the definite manipulation of concrete symbols. Hilbert aimed to justify the introduction of ideal elements and the application of classical reasoning to these elements by proving the following from the finitary standpoint: No contradiction can be derived in formal systems codifying ideal parts of mathematics. This is the main aim of Hilbert’s Program.

In developing this program, Hilbert made several claims about the finitist interpretation of quantified propositions. Most important here is the finitist interpretation of existential propositions. Hilbert wrote,

In general, from the finitist point of view an existential proposition of the form “There exists a number having this or that property” has a sense (Sinn) only as a partial proposition (Partialaussage), that is, as part of a proposition that is more precisely determined but whose exact content is unessential for many applications. ([Hilbert 1926, p.173])

Here, Hilbert claimed that from the finitary standpoint an existential proposition has a sense only as a “partial proposition.” He provided an example of partial propositions in the following passage:

This proposition [“Between \( p + 1 \) and \( p! + 1 \) there certainly exists a new prime number”] itself, moreover, is completely in conformity with our finitist attitude. For “there exists” here serves merely to abbreviate the proposition:

Certainly \( p + 1 \) or \( p + 2 \) or \( p + 3 \) or . . . or \( p! + 1 \) is a prime number.

But let us go on. Obviously, to say

There exists a prime number that (1) is \( > p \) and (2) is at the same time \( \leq p! + 1 \) \((E)\)

would amount to the same thing, and this leads us to formulate a proposition that expresses only a part of Euclid’s assertion, namely: \((P)\) there exists a prime

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10 [Hilbert 1926, pp.170–171].
11 [Hilbert 1926, pp.178–179], [Hilbert 1928, p.74].
12 The next two translations of [Hilbert 1926] are taken from [van Heijenoort 1967, pp.377–378].
number that is $> p$. ([Hilbert 1926, p.172], italics added)

In this passage, the example of partial propositions is the proposition (P). This proposition is part of the proposition (E), whose content is more exactly determined.

On the basis of the last quotation, we propose the following characterization of partial propositions:

A partial proposition is an existential proposition $\exists x A(x)$ such that it alludes to some effective way of yielding its witness in some definite and finite totality.

Note that the word “definite” means “not hazy.” Moreover, we should mention that $\exists x A(x)$ may contain free variables, so its witness may be a function of those parameters. If we adopt the characterization above, we can ascribe the following claim to Hilbert: From the finitary standpoint, an existential proposition $\exists x A(x)$ means that one possesses some effective way of yielding a witness for $\exists x A(x)$ in some definite and finite totality.

This is the content of Hilbert’s claim that an existential proposition has a sense only as a partial proposition. Of course, one can admit that an existential proposition $\exists x A(x)$ has an ordinary sense, according to which one does not need to possess a way of yielding an object satisfying $A(x)$. Let us call this interpretation of existential propositions the classical interpretation. The above quotations’ point is that the classical interpretation of existential propositions is not admissible from the finitary standpoint: From this standpoint, an existential proposition must be treated as a partial proposition.

According to the reading of Hilbert as an instrumentalist, Hilbert considered that existential propositions without witnesses have no sense from any standpoint and that they are merely useful instruments to prove propositions that have senses.13

In this paper, we are not committed to the question of whether Hilbert really held with instrumentalism or not. As stated in the last paragraph, what we want to emphasize is the following: Hilbert claimed that the classical interpretation of existential propositions is not admissible from the finitary standpoint.

About the interpretation of existential propositions, Brouwer took a position very close to Hilbert’s: The classical interpretation of existential propositions is not admissible in Brouwer’s intuitionism. We ascribe this position to Brouwer on the basis of the following two passages from his writings. First, in his 1933 paper “Willen, Weten, Spreken,” Brouwer wrote,

[...] [S]uppose that human beings with unlimited power of memory recorded their constructions in shortened form in a suitable language, surveyed the strings

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13 This reading is, for example, presented in [Detlefsen 1990, pp.346–347]. As to the reading of Hilbert as a non-instrumentalist, see [Hallett 1990] and [AR 2001, §4].
of their affirmations in this language and then would be able to see the occurrence of the linguistic figures of logical principles in all their mathematical modifications. Careful rational reflection would then show that as far as the principles of identity, contradiction and syllogism are concerned such an occurrence could be expected; as to the linguistic figure of the principle of the excluded middle, this would only occur if one restricted oneself to affirmations concerning parts of a definite, once and for all given, finite mathematical system. Wider applications of the latter principle would never occur since such applications to pure-mathematical affirmations usually lead to verbal complexes devoid of any mathematical sense and therefore of any sense. ([van Stigt 1990, pp.427–428], italics added)

For simplicity’s sake, consider an instance $\exists x A(x) \lor \neg \exists x A(x)$ of the Principle of the Excluded Middle (PEM), where $A(x)$ is a unary predicate. According to Brouwer, if the variable $x$ runs over a finite collection and the predicate $A(x)$ is decidable, the use of this instance leads to a correct conclusion, namely, a conclusion accompanied by some mental constructions. However, if it runs over an infinite collection, its use produces a proposition not accompanied by any construction. For it is not always the case that one possesses a way of yielding either a witness of $\exists x A(x)$ or an operation $f$ such that $f(a)$ is a witness of $\neg A(a)$ for every $a$ in the range of $x$. That is to say, one does not always have some mental constructions witnessing either of the disjuncts.\textsuperscript{14}

In his 1922 paper “Intuitionistische Mengenlehre” ([Brouwer 1922]), Brouwer made the same remark as in the last quotation:

The Principle of the Excluded Middle has only scholastic and heuristic value, so that theorems that in their proof cannot avoid the use of this principle lack all mathematical content (Inhalt). ([Brouwer 1922, pp.949–950])\textsuperscript{15}

If one admits the classical interpretation of existential propositions, it is inevitable to admit the validity of PEM of the form $\exists x A(x) \lor \neg \exists x A(x)$ such that the predicate $A(x)$ is not decidable. However, as we have seen, Brouwer did not admit the validity of such instances, so the classical interpretation of existential propositions is not admissible by Brouwer’s intuitionism.

In our reading, Brouwer’s position above—that the classical interpretation of existential propositions is not admissible—is a premise of his objection to the consistency proofs, for which the Hilbert School asked. Note that the Hilbert School aimed to show the consistency of classical mathematics, in which the use of PEM or a similar classical principle is not restricted. Brouwer’s objection can be summarized as

\textsuperscript{14} As to Brouwer’s conception of the relationship between the content of a mathematical proposition and mental constructions, see [Kaneko 2006].

\textsuperscript{15} This English translation is from [Mancosu 1998, p.23].
follows.

1. The classical interpretation of existential propositions is not admissible.
2. In the intuitionist interpretation, the alternative to the classical interpretation, the existential propositions of classical mathematics proved with substantial use of PEM are incorrect or at least not proved yet.
3. Thus, classical mathematics is incorrect.
4. Moreover, classical mathematics remains incorrect even if its consistency is proved using the Hilbert School’s methods, since such a proof does not make those existential propositions correct in the sense of the intuitionist interpretation.
5. Accordingly, the consistency proofs that the Hilbert School requested are of no significance.

Let us explain our reading above in detail. First of all, we want to call attention to the following passage in Brouwer’s 1923 paper “Über die Bedeutung des Satzes vom ausgeschlossenen Dritten in der Mathematik, insbesondere in der Funktionentheorie.”

The contradictions that, as a result, one repeatedly encountered gave rise to the formalistic critique, a critique which in essence comes to this: the language accompanying the mathematical mental activity is subjected to a mathematical examination. To such an examination the laws of theoretical logic present themselves as operators acting on primitive formulas or axioms, and one sets himself the goal of transforming these axioms in such a way that the linguistic effect of the operators mentioned (which are themselves retained unchanged) can no longer be disturbed by the appearance of the linguistic figure of a contradiction. We need by no means despair of reaching this goal, but nothing of mathematical value will thus be gained: an incorrect theory, even if it cannot be inhibited by any contradiction that would refute it, is none the less incorrect, just as a criminal policy is none the less criminal even if it cannot be inhibited by any court that would curb it. ([Brouwer 1923, pp.2–3])

Here, Brouwer argued that consistency proofs for “an incorrect theory” are of no significance because such theories are incorrect regardless of their consistency. The following passage, which occurs just after the previous quotation, indicates that Brouwer considered classical mathematics an example of such theories:

The following two fundamental properties, which follow from the principle of excluded middle, have been of basic significance for this incorrect “logical” mathematics of infinity (“logical” because it makes use of the principle of excluded

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16 This and the next translations of [Brouwer 1923] are from [van Heijenoort 1967, pp.336–337].
middle), especially for the *theory of real functions* (developed mainly by the Paris school):

1. *The points of the continuum form an ordered point species;*
2. *Every mathematical species is either finite or infinite.*

([Brouwer 1923, p.3])

Details of the “two fundamental properties” are not relevant to our purpose. Rather, what is important is that Brouwer obviously referred to classical mathematics by “this incorrect ‘logical’ mathematics of infinity.” In sum, Brouwer argued in these quotations that consistency proofs for classical mathematics are of no significance because classical mathematics is *incorrect* regardless of its consistency.

Moreover, we can understand Brouwer’s word “incorrect” as follows. For Brouwer, classical mathematics is incorrect because its existential propositions proved with substantial use of PEM either are incorrect or are at least not proved yet. Then, we can read Brouwer’s objection to consistency proofs, which we formulated above, from the passages we quoted in this section.

In the next section, we will see Gentzen’s response to this Brouwer-style objection. In the rest of this section, let us briefly see a response from Hilbert’s 1920s proof theory to this objection.

For Hilbert, the consistency of classical mathematics *suffices* to show its correctness. He treated existential propositions without witnesses as ideal elements of formal systems codifying classical mathematics and claimed that the introduction of these ideal elements are justified by consistency proofs for classical mathematics.\(^\text{17}\)

This means the following: Hilbert, at least in the 1920s, did not consider the finitist interpretation of such ideal elements necessary. He wrote,

> To make it a universal requirement that each individual formula then be interpretable by itself is by no means reasonable; on the contrary, a theory by its very nature is such that *we do not need to fall back upon intuition or meaning in the midst of some argument.* ([Hilbert 1928, p.79], italics added)\(^\text{18}\)

According to this quotation, it is not reasonable to require that each formula in a formal system of classical mathematics has a meaning: It is a crucial feature of a formal system that we do not need to fall back upon meanings of formulas in constructing a derivation in this system. As we will see in the next section, Hilbert’s this remark sharply distinguishes Hilbert’s response to the Brouwer-style objection from Gentzen’s.

\(^{17}\) [Hilbert 1926, pp.178–179], [Hilbert 1928, pp.73–74].

\(^{18}\) This English translation is from [van Heijenoort 1967, p.475].
3. Gentzen’s Response to the Brouwer-style Objection

In this section, we argue that Gentzen’s interpretation of number-theoretic propositions served as a response to the Brouwer-style objection. First of all, we can say that Gentzen clearly recognized the objection and he took it seriously. In the following passage, which appeared in both [Gentzen 1974] and [Gentzen 1936] including the 1935 and 1936 proofs, respectively, Gentzen mentioned the objection and then responded to it:

On the part of the intuitionists, the following objection is raised against the significance of consistency proofs*: even if it had been demonstrated that the disputable forms of inference cannot lead to mutually contradictory results, these results would nevertheless be propositions without sense (sinnlos) and their investigation therefore an idle pastime; real knowledge (wirkliche Erkenntnisse) could be gained only by means of indisputable intuitionist (or finitist, as the case may be) forms of inference.

Let us, for example, consider the existential proposition cited at 10.6, for which the statement of a number whose existence is asserted is not possible. According to the intuitionist view, this proposition is therefore without sense (sinnlos); an existential proposition can after all be significantly asserted only if a numerical example is available.

What can we say to this?

[...] The major part of my consistency proof, however, consists precisely in ascribing (beilegen) a finitist sense to actualist propositions (an-sich-Aussagen), viz.: for every arbitrary proposition, as long as it is provable, a reduction procedure according to 13.6 can be stated, and this fact represents the finitist sense of the proposition concerned and this sense is gained precisely through the consistency proof. ([Footnote]*: For example, cf.: L. E. J. Brouwer, Intuitionistische Betrachtungen über den Formalismus, Sitzungsber. d. Preuß. Akad. d. Wiss., phys.-math. Kl. (1928), S.48–52.) ([Gentzen 1974, pp.117–118], [Gentzen 1936, pp.563–564], [Gentzen 1969, pp.200–201], italics original) ¹⁹

In this quotation’s latter part, Gentzen claimed that the major part of his consistency proof...
proof consists in ascribing a “finitist” interpretation to each theorem of first-order classical number theory. This claim was Gentzen’s response to the Brouwer-style objection. Moreover, Gentzen said that he responded to the objection with “the major part” of his consistency proofs. This indicates that Gentzen took the objection seriously.

Indeed, Gentzen took the Brouwer-style objection seriously because he had followed Hilbert’s claim about the finitist interpretation of existential propositions. Gentzen wrote,

What sense should we concede to a proposition of the form $\exists \mathfrak{F}(x)$? The actualist interpretation that somewhere in the infinite number sequence there exists a number with the property $\mathfrak{F}$ is for us without sense. If, on the other hand, the proposition $\mathfrak{F}(n)$ has been recognized as significant and valid for a definite number $n$, we wish to be able to conclude ($\exists$-introduction): $\exists \mathfrak{F}(y)$. There are no objections to this; the proposition $\exists \mathfrak{F}(x)$ now constitutes only a weakening of the proposition $\mathfrak{F}(n)$ (‘Partialaussage’ for Hilbert, ‘Urteilsabstrakt’ for Weyl) in that it now attests merely that we have found a number $n$ with property $\mathfrak{F}$, although this number itself is no longer mentioned. Thus, $\exists \mathfrak{F}(x)$ acquires in this way a finitary sense. ([Gentzen 1936, p.527], [Gentzen 1969, pp.164–165], italics original)

Gentzen followed Hilbert’s claim that an existential proposition has a sense only as a partial proposition. Thus, he aimed at responding to the Brouwer-style objection, since a very close claim to Hilbert’s is used as a premise in the objection. Gentzen was concerned that the theorems proved with PEM or a similar principle also have no sense from his standpoint for consistency proofs. The 1935 and 1936 proofs have dealt with not only the problem of the consistency of first-order classical number theory, but also this concern.

In the rest of this section, we explain how Gentzen responded to the Brouwer-style objection. Our key claim is as follows: Gentzen aimed to ascribe a finitist meaning to each theorem of first-order classical number theory by giving his own interpretation, according to which all theorems of first-order classical number theory are correct.

In the papers for the 1935 and 1936 proofs, Gentzen manifestly pointed out the possibility of ascribing a finitist interpretation to every number-theoretic proposition, after having rejected the standard interpretation (“the actualist interpretation,” in his term) because of its inadmissibility from his standpoint. He wrote,

Having rejected the actualist interpretation of transfinite propositions, we are still left with the possibility of ascribing a ‘finitist’ sense (ein ‘finiter’ Sinn) to such
propositions, i.e., of interpreting (zu deuten) them in each case as expressions for definite finitely (endlich) characterizable states of affairs. ([Gentzen 1936, p.525], [Gentzen 1969, pp.162–163], italics original)

In the following passage, Gentzen suggested how to give such a finitist interpretation, although he did not present it explicitly.

The concept of the ‘statability of a reduction procedure’ (die Angebbarkeit einer Reduziervorschrift) for a sequent, to be defined below, will serve as the formal replacement (formaler Ersatz) of the contentual concept of correctness (der inhaltliche Richtigkeitsbegriff); it provides us with a special finitist interpretation (finite Deutung) of propositions and takes the place of their actualist interpretation [...] ([Gentzen 1974, p.100], [Gentzen 1936, p.536], [Gentzen 1969, p.173], italics original)\(^{21}\)

According to Gentzen, his interpretation of a number-theoretic proposition is given by the stabability of a reduction procedure for it. The stabability of a reduction procedure serves as an alternative concept of correctness and gives this interpretation by explaining the correctness of number-theoretic propositions.

We extract from the last quotation the following interpretation. Let \(\Gamma \rightarrow A\) be a sequent of first-order arithmetic, where \(\Gamma\) is a finite set of formulas.\(^{22}\) Then,

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\text{(GI) } \Gamma \rightarrow A \text{ is correct if and only if a reduction procedure is statable for } \Gamma \rightarrow A.
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Definition 6 in Section 4 provides an (equivalent) version of Gentzen’s notion of a reduction procedure. Moreover, in Section 5, we explain when a reduction procedure is statable. Here, note that the stabability of a reduction procedure requires the availability of not only this procedure but also a proof for its termination, as Tait pointed out in [Tait 2015, Footnote 4].\(^{23}\) If we stipulate that a reduction procedure

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\(^{21}\) Following [Sieg 2012], we translate “der inhaltliche Richtigkeitsbegriff” as “the contentual concept of correctness.” In [Gentzen 1969] Szabo translates it as “the informal concept of truth.”

\(^{22}\) Note that the formal system used in the 1935 and 1936 proofs is a natural deduction system in sequent-style.

\(^{23}\) For Gentzen, the stabability of a reduction procedure for \(\Gamma \rightarrow A\) gives a sense to \(\Gamma \rightarrow A\) from his finitist standpoint. It is not easy to estimate the exact strength of Gentzen’s finitist standpoint. As far as we know, his standpoint should be constructive in the following sense. First, all infinite totalities must be generated by some finitary rules ([Gentzen 1936, pp.524–525], [Gentzen 1969, p.162]). For example, the totality of all natural numbers is generated from 0 by the successor rule. Second, one must avoid the use of PEM for non-decidable predicates ([Gentzen 1936, pp.527–528], [Gentzen 1969, pp.164–165]).
for a formula $A$ means one for $\rightarrow A$, we obtain the following from (GI):

$A$ is correct
if and only if
a reduction procedure is statable for $A$.

As stated above, Gentzen’s response to the Brouwer-style objection was to define the sense of a number-theoretic proposition, or more generally of a sequent to be that a reduction procedure is statable for it and to show that classically derivable sequents are all correct in that sense. The main lemma of the 1935 proof, which was crucial not only to the proof of the consistency of first-order classical number theory $Z$ but also to Gentzen’s response, is as follows:\textsuperscript{24}

**Main Lemma.** For every sequent $\Gamma \rightarrow A$ of $Z$, if $\Gamma \rightarrow A$ is derivable in $Z$, then there is a reduction procedure for $\Gamma \rightarrow A$.

In Section 5, we see the following: This lemma shows the correctness of each $Z$-derivable sequent $\Gamma \rightarrow A$ in the sense of (GI) because the proof of the lemma gives not only a reduction procedure for $\Gamma \rightarrow A$, but also a proof for its termination. Therefore, all classical theorems of $Z$ are also correct in the sense of (GI).

Let us make the following conjecture about Gentzen’s response, for which we have only incomplete arguments here: Gentzen, by responding to the Brouwer-style objection in the way above, attempted to show that first-order classical number theory is an idealization of first-order intuitionist number theory. The former theory is an idealization of the latter in the following sense. On the one hand, an existential proposition $\exists x A(x)$ in the latter theory means that one possesses some effective way of yielding a witness for $\exists x A(x)$. This often makes a proof of an existential proposition complicated. On the other hand, an existential proposition in first-order classical number theory has the classical interpretation. This makes a proof of an existential proposition simple, since classical reasoning, such as PEM, can be applied to existential propositions in the proof. In sum, first-order classical number theory has an idealized concept of existence, which is simpler than the one in intuitionist number theory.

Indeed, Gentzen wrote as follows in the lecture notes “Die gegenwärtige Lage in der mathematischen Grundlagenforschung” published in 1938.

[... ] Actualist mathematics idealizes, for example, the notion of ‘existence’ by saying: A number exists if its existence can be proved by means of a proof in which the logical deductions are applied to completed infinite totalities in the same form in which they are valid for finite totalities; entirely as if these infinite totalities were actually present quantities. In this way the concept of existence

\textsuperscript{24} [Gentzen 1974, p.103], [Gentzen 1936, p.539], [Gentzen 1969, p.177].
therefore inherits the advantages and the disadvantages of an ideal element: The *advantage* is, above all, that a considerable simplification and elegance of the theory is achieved – since intuitionist existence proofs are, as mentioned, mostly very complicated and plagued by unpleasant exceptions –, the *disadvantage*, however, is that this ideal concept of existence is no longer applicable to the same degree to physical reality as, for example, the constructive concept of existence. ([Gentzen 1938a, p.17], [Gentzen 1969, p.248], italics original)

The last sentence of this quotation indicates that Gentzen, in 1938, considered the concept of existence in first-order classical number theory an idealization of the one in intuitionist number theory. It is not implausible to conjecture that Gentzen already held this claim in 1935, since a similar claim can be found in the paper including the 1935 proof.

The proof [the 1935 proof] certainly reveals that it is possible to reason consistently ‘as if’ everything in the infinite domain of objects were as actualistically determined as in finite domains (cf. §9). ([Gentzen 1974, p.118], [Gentzen 1969, p.201])

Then, it is natural to hold that Gentzen’s interpretation (GI) of number-theoretic propositions served as a *justification* of the above conception of first-order classical number theory: (GI) justified the conception of first-order classical number theory as an idealization of first-order intuitionist number theory. In order to verify that the former is an idealization of the latter, the following must be shown: Classical reasoning can be used in intuitionist number theory via its *reification* that cancels the idealization.\(^{25}\) As said above, the main lemma of the 1935 proof gave such a reification of classical reasoning. This lemma showed that classical reasoning is still sound with respect to (GI) and Gentzen suggested that this interpretation is admissible also for intuitionists.

Let us summarize our arguments in Section 2 and 3. First, we have seen the following claim made by Hilbert: An existential proposition without witnesses has no sense from the finitary standpoint. Next, we have argued that Brouwer made a very close claim to Hilbert’s and raised an objection to the significance of consistency proofs by using his own claim as a premise of the objection. Finally, we have maintained that Gentzen, who followed Hilbert’s claim, responded to the Brouwer-style objection in the following way: He formulated the interpretation (GI) of number-theoretic propositions and ascribed a sense to every theorem of first-order classical number theory by means of it.\(^{26}\) The interpretation (GI), according to which all the-
orems of first-order classical number theory are correct, had the role of responding to the Brouwer-style objection.

4. Formulation of Reduction Procedures with Spreads

In this section, we give our definition of reduction procedures in terms of Brouwer’s notion of a spread, which is an infinite tree in intuitionistic mathematics.

First, we introduce the proof system \( \mathcal{Z} \) of first-order classical arithmetic. The system \( \mathcal{Z} \) is the same as the proof system of the 1935 proof except for some minor points, which we explain later. We assume a language \( \mathcal{L} \) of first-order arithmetic with the following vocabulary: the constant 0 (zero), the unary function symbol \( S \) (successor), some predicate symbols for primitive recursive relations and the logical connectives \( \wedge, \neg, \forall \). Terms and formulas are defined in a usual way. Atomic formulas are formulas of the form \( p(t_1, \ldots, t_n) \), where \( p \) is an \( n \)-ary predicate symbol and \( t_1, \ldots, t_n \) are terms. We use the following syntactic variables possibly with suffixes: \( k, m, n \) for numerals, \( A, B, C, D \) for formulas and \( \Gamma, \Delta \) for finite sets of formulas. We denote the union \( \Gamma \cup \Delta \) of \( \Gamma \) and \( \Delta \) by \( \Gamma, \Delta \) and the union \( \{ A \} \cup \Gamma \) by \( A, \Gamma \) or \( \Gamma, A \).

Sequents are expressions of the form \( \Gamma \rightarrow A \) and we call a formula in \( \Gamma \) an antecedent formula and \( A \) the succedent formula. Sequents are denoted by \( \Sigma \) possibly with suffixes. If \( \theta \) is a term, a formula, a finite set of formulas or a sequent, then \( \text{FV}(\theta) \) denotes the set of all free variables in \( \theta \) and we say \( \theta \) is closed whenever \( \text{FV}(\theta) = \emptyset \).

The system \( \mathcal{Z} \) differs from the proof system of the 1935 proof in the following two respects. First, the connectives \( \exists, \lor \) and \( \supset \) are excluded from the language of \( \mathcal{Z} \). Second, a sequent of \( \mathcal{Z} \) includes not finite sequences of formulas but finite sets of them, so structural rules are omitted.

For readers’ convenience, we define some notions and notations that are not found in Gentzen’s original presentation. The set \( \text{TRUE} \) (resp. \( \text{FALSE} \)) consists of all

---

27 Gentzen did not exclude \( \exists, \lor \) and \( \supset \) from the language of the proof system of the 1935 proof, but he translated \( \exists x A(x) \), \( A \lor B \) and \( A \supset B \) as \( \neg \forall x \neg A(x) \), \( \neg (A \land \neg B) \) and \( \neg (A \land \neg B) \), respectively. See [Gentzen 1974, §12].
closed atomic formulas that are true (resp. false) in a usual sense. We use \( \bot \) as a variable for formulas in \( \mathcal{FACSE} \). When \( \text{FV}(S) \subseteq \{x_0, \ldots, x_k\} \), \( S[x_0 := n_0, \ldots, x_k := n_k] \) denotes the sequent obtained by substituting \( n_i \) for \( x_i \) in \( S \) for \( i = 0, \ldots, k \) and we often abbreviate \( S[x_0 := n_0, \ldots, x_k := n_k] \) as \( S[x := \vec{n}] \). The set \( \mathcal{CSQE} \) is the set of all closed sequents.

Axioms and inference rules of \( Z \) are as follows:

**Logical axioms:**

\[
\Gamma, A \rightarrow A.
\]

**Non-Logical axioms:** Let us assume a primitive recursive set \( \mathcal{AX} \) of some arithmetical axioms. Here, we do not need to specify this set. We require that \( \mathcal{AX} \) includes the defining axioms for each predicate symbol \( p \) and that every sequent in \( \mathcal{AX} \) may have an arbitrary set of formulas as auxiliary antecedent formulas. For example, the following sequents may be included in \( \mathcal{AX} \):

\[
\Gamma \rightarrow t = t \text{ and } \Gamma, S(s) = S(t) \rightarrow s = t.
\]

**Logical rules:**

\[
\begin{align*}
\Gamma \rightarrow A_0 & \quad \Gamma \rightarrow A_1 \quad (\land I) \\
\Gamma & \rightarrow A_0 \land A_1 \\
\Gamma \rightarrow A_i & \quad (\land E) \text{ with } i \in \{0, 1\} \\
\Gamma \rightarrow A(y) & \quad (\forall I) \text{ with } y \notin \text{FV}(\Gamma) \\
\Gamma & \rightarrow \forall x.A(x) \\
\Gamma \rightarrow \neg\neg A & \quad (\text{DNE}) \\
\Gamma, A \rightarrow B & \quad A, \Gamma \rightarrow \neg B \quad (\text{RED})
\end{align*}
\]

**Mathematical Induction:**

\[
\begin{align*}
\Gamma \rightarrow A(0) & \quad \Gamma, A(y) \rightarrow A(S(y)) \quad (\text{IND}) \text{ with } y \notin \text{FV}(\Gamma)
\end{align*}
\]

We can show in a standard way that the left weakening rule is admissible in \( Z \).

Next, we define *reduction steps* (Reduktionsschritte) of the 1935 proof in our manner. Our definition is an application of the method of representing infinitary derivations as functions, which is found in [Mints 1978] and [Buchholz 1991]. We use the notation of [Buchholz 1991] with some modifications. This method enables us

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28 Gentzen also did not specify which arithmetical axioms are included in the proof system of the 1935 proof. See [Gentzen 1936, §6.2].

29 For Gentzen’s definition of reduction steps, see [Gentzen 1974, pp.100–102], [Gentzen 1936, pp.536–537], [Gentzen 1969, pp.173–175].

30 [Sundholm 1983] also gave a reconstruction of reduction procedures in the 1935 proof by using the method of [Mints 1978].
to define reduction procedures by means of spreads. Preliminary to our definition of reduction steps, the set $\mathcal{STEP}$ of the symbols for reduction steps is defined as follows:

$$\mathcal{STEP} := \{Ax\} \cup \{(\forall, A) | A \text{ is of the form } \forall x B(x)\} \cup \{(\forall, k, A) | A \text{ is of the form } \forall x B(x), k \in \mathbb{N}\} \cup \{(\land, A) | A \text{ is of the form } B_0 \land B_1\} \cup \{(\land, i, A) | A \text{ is of the form } B_0 \land B_1, i \in \{0, 1\}\} \cup \{(\neg, i, A) | A \text{ is of the form } \neg B, i \in \{r, l\}\}.$$  

We use $R$ as a variable for elements of $\mathcal{STEP}$.

**Definition 1 (Reduction Steps).** For every $\langle R, \Gamma \rightarrow A \rangle \in \mathcal{STEP} \times \mathcal{CSEQ}$ and every infinite sequence $\langle \Delta_n \rightarrow B_n \rangle_{n \in \mathbb{N}}$ of closed sequents, $\text{RED}(\langle R, \Gamma \rightarrow A \rangle, \langle \Delta_n \rightarrow B_n \rangle_{n \in \mathbb{N}})$ holds if and only if all of the following statements hold:

(i) If $R = Ax$, then either $A \in \mathcal{TRUE}$ holds or both $A \in \mathcal{FALSE}$ and $\Gamma \cap \mathcal{FALSE} \neq \emptyset$ hold,

(ii) if $R = (\forall, \forall x C(x))$, then $A = \forall x C(x)$ holds and for every $n \in \mathbb{N}$, $\Delta_n = \Gamma$ and $B_n = C(n)$ hold,

(iii) if $R = (\forall, k, \forall x C(x))$, then $\forall x C(x) \in \Gamma$, $\Delta_0 = \Gamma \cup \{C(k)\}$ and $B_0 = A \in \mathcal{FALSE}$ hold,

(iv) if $R = (\land, C_0 \land C_1)$, then $A = C_0 \land C_1$ holds and for every $i \in \{0, 1\}$, $\Delta_i = \Gamma$ and $B_i = C_i$ hold,

(v) if $R = (\land, i, C_0 \land C_1)$, then $C_0 \land C_1 \in \Gamma$, $\Delta_0 = \Gamma \cup \{C_i\}$ and $B_0 = A \in \mathcal{FALSE}$ hold,

(vi) if $R = (\neg, r, \neg C)$, then $A = \neg C$, $\Delta_0 = \Gamma \cup \{C\}$ and $B_0 \in \mathcal{FALSE}$ hold,

(vii) if $R = (\neg, l, \neg C)$, then $\neg C \in \Gamma$, $A \in \mathcal{FALSE}$, $\Delta_0 = \Gamma$ and $B_0 = C$ hold.

**Example 2.** It holds that $\text{RED}(\langle \forall, \forall x A(x) \rangle, \Gamma \rightarrow \forall x A(x)), \langle \Gamma \rightarrow A(n) \rangle_{n \in \mathbb{N}})$. This corresponds to the following reduction step:\footnote{Cf. [Gentzen 1974, §13.21].}

$$\Gamma \rightarrow \forall x A(x) \triangleright \Gamma \rightarrow A(n)$$  

for an arbitrarily chosen numeral $n$.

Let us turn to the definition of spreads. The set $\mathbb{N}^\omega$ is the primitive recursive set of (the codes of) all finite sequences of natural numbers. The elements of $\mathbb{N}^\omega$
are denoted by $\vec{u}$, $\vec{v}$ and $\vec{w}$. We also denote the code of the empty sequence by $\langle \rangle$, infinite sequences of natural number by $\alpha$ and the concatenation function for $\vec{u}$ and $\vec{v}$ by $\vec{u} \ast \vec{v}$. The ordering $\leq$ on $\mathbb{N}^{<\omega}$ is defined by

$$\vec{u} \leq \vec{v} \text{ if and only if } \vec{u} \ast \vec{w} = \vec{v} \text{ holds for some } \vec{w},$$

$$\vec{u} < \vec{v} \text{ if and only if } \vec{u} \leq \vec{v} \text{ and } \vec{u} \neq \vec{v} \text{ hold.}$$

For an arbitrary $\alpha$, the initial segment $\bar{\alpha}(n)$ of length $n$ is defined by

$$\bar{\alpha}(n) := \langle \alpha(0), \ldots, \alpha(n-1) \rangle.$$
(iii) \( \varphi \) is well-founded if and only if for every \( \alpha \) there exists \( n \) such that \( \varphi^0(\bar{\alpha}(n)) = \Lambda x \) holds.

(iv) \( \varphi \) is locally correct if and only if \( \varphi \) is monotone and for every \( \bar{u} \),

\[ R\mathcal{E}D(\varphi(\bar{u}), \langle \varphi^1(\bar{u} \cdot \langle n \rangle) \rangle_{n \in \mathbb{N}}) \] holds.

**Example 5.** There is a function \( \varphi \) such that \( \varphi \) is locally correct but not well-founded. A typical example of such a function \( \varphi \) can be represented as the following tree, where \( A(x) \) is of the form \( x = x \).

\[
\begin{align*}
\langle \forall, 3, \forall x A(x) \rangle, & \ A(0), \ A(1), \ A(2), \forall x A(x) \to 0 = 1 \\
\langle \forall, 2, \forall x A(x) \rangle, & \ A(0), \ A(1), \forall x A(x) \to 0 = 1 \\
\langle \forall, 1, \forall x A(x) \rangle, & \ A(0), \forall x A(x) \to 0 = 1 \\
\langle \forall, 0, \forall x A(x) \rangle, & \forall x A(x) \to 0 = 1
\end{align*}
\]

Now, we define reduction procedures as a kind of dressed spreads.

**Definition 6** (Reduction Procedures). For every \( \varphi \) and every \( \Gamma \to A \in \text{CSEQ} \), \( \langle s^*, \varphi \rangle \) is a reduction procedure for \( \Gamma \to A \) if and only if \( \text{End}(\varphi) = \Gamma \to A \), \( \varphi \) is well-founded and locally correct.

When a sequent \( S \) is not closed and \( \text{FV}(S) \subseteq \{x_1, \ldots, x_k\} \), we say there is a reduction procedure for \( S \) if and only if for every \( n_1, \ldots, n_k \), there is a reduction procedure for \( S[x_1 := n_1, \ldots, x_k := n_k] \).

5. **Proof of Main Lemma with Monotone Bar Induction**

In this section, we prove the main lemma of the 1935 proof. This lemma is a key for Gentzen’s response to the Brouwer-style objection, as seen in Section 3. The proof below is given from an intuitionist perspective: We use monotone bar induction, which is a induction principle on a well-founded tree in intuitionistic mathematics.\(^{32}\)

Then, we propose a way to understand Gentzen’s response to the Brouwer-style objection from an intuitionist perspective.

The **rank** \( rk(A) \) of a formula \( A \) is defined by

\[
\begin{align*}
\text{rk}(A) & := 0, \text{ if } A \text{ is an atomic formula,} \\
\text{rk}(A \land B) & := \max(\text{rk}(A), \text{rk}(B)) + 1, \\
\text{rk}(\neg A) & := \text{rk}(A) + 1, \\
\text{rk}(\forall x A(x)) & := \text{rk}(A(x)) + 1.
\end{align*}
\]

\(^{32}\) As we have said in the introduction to this paper, we do not claim that Gentzen himself used monotone bar induction.
Hereafter, we abbreviate “there is a reduction procedure for \( \Gamma \rightarrow A \)” as “\( \Gamma \rightarrow A \) is reducible.” In addition, for simplicity’s sake, we consider closed sequents only, unless indicated otherwise.

**Lemma 7.** The following statements hold:

1. \( \Gamma \rightarrow A(n) \) is reducible for every \( n \in \mathbb{N} \) if and only if \( \Gamma \rightarrow \forall x A(x) \) is reducible,
2. \( \Gamma \rightarrow A_0 \) and \( \Gamma \rightarrow A_1 \) are reducible if and only if \( \Gamma \rightarrow A_0 \land A_1 \) is reducible,
3. If \( \neg A, \Gamma \rightarrow \bot \) is reducible, then \( \neg A, \Gamma \rightarrow \bot \) is reducible,
4. If \( A, \Gamma \rightarrow \bot \) is reducible, then \( \Gamma \rightarrow \neg A \) is reducible,
5. If \( A(n), \forall x A(x), \Gamma \rightarrow B \) is reducible, then \( \forall x A(x), \Gamma \rightarrow B \) is reducible,
6. If \( A_i, A_0 \land A_1, \Gamma \rightarrow B \) is reducible \( (i \in \{0, 1\}) \), then \( A_0 \land A_1, \Gamma \rightarrow B \) is reducible.

**Proof.** It is obvious by the definition of reduction procedures that the statements 1, 2, 3 and 4 hold. The statements 5 and 6 are proved by induction on \( \text{rk}(B) \).

**Lemma 8.** \( \Gamma, A \rightarrow A \) is reducible for every formula \( A \).

**Proof.** By induction on \( \text{rk}(A) \). Apply Lemma 7 in the case of the induction steps.

**Lemma 9.** The following statements hold:

1. If \( A, \Gamma \rightarrow B \) is reducible, then \( \neg \neg A, \Gamma \rightarrow B \) is reducible,
2. \( \Gamma, \neg \neg A \rightarrow A \) is reducible.

**Proof.** (1) By induction on \( \text{rk}(B) \).

(2) The sequent \( A \rightarrow A \) is reducible by Lemma 8, then apply (1).

**Lemma 10** (The Soundness of the Weakening Rule). If \( \Gamma \rightarrow A \) is reducible, then \( \Delta, \Gamma \rightarrow A \) is reducible.

**Lemma 11** (Main Lemma of the 1935 Proof). If \( \Gamma \rightarrow A \) is derivable in \( Z \), then \( \Gamma \rightarrow A \) is reducible.

**Proof.** By induction on the length of the \( Z \)-derivation \( d \) of \( \Gamma \rightarrow A \). For simplicity’s sake, we focus on the interesting cases.

(1) If \( d \) is a logical axiom \( \Gamma, B \rightarrow B \), apply Lemma 8.

(2) Assume that the last rule of \( d \) is

\[
\Gamma \rightarrow A(y) \\
\Gamma \rightarrow \forall x A(x) \quad (\forall I).
\]

Apply Lemma 7.(1).
(3) Assume that the last rule of \( d \) is
\[
\begin{align*}
\Gamma \rightarrow \neg \neg A & \\
\Gamma \rightarrow A & \quad \text{(DNE)}.
\end{align*}
\]
By IH, the sequent \( \Gamma \rightarrow \neg \neg A \) is reducible. On the other hand, by Lemma 9.(2), the sequent \( \neg \neg A \rightarrow A \) is reducible. We assume that the following lemma is proved:

(*) For every \( \Gamma, A \) and \( B \),
if \( \Gamma \rightarrow A \) and \( A, \Gamma \rightarrow B \) are reducible, then \( \Gamma \rightarrow B \) is reducible.

Therefore, \( \Gamma \rightarrow A \) is reducible.

(4) Assume that the last rule of \( d \) is
\[
\begin{align*}
\Gamma \rightarrow A(0) & \\
\Gamma, A(y) \rightarrow A(S(y)) & \quad \text{(IND)}.
\end{align*}
\]
By IH, \( \Gamma \rightarrow A(0) \) and \( \Gamma, A(y) \rightarrow A(S(y)) \) are reducible. Then, \( \Gamma, A(m) \rightarrow A(S(m)) \) is reducible for every \( m \) such that \( m < n \). By (*), \( \Gamma \rightarrow A(n) \) is reducible.

To finish the proof of Lemma 11, it suffices to show that (*) holds. Preliminary to a proof of (*), we formulate monotone bar induction, following [TD 1988, ch.4, §8].

**Monotone Bar Induction.** Let \( P \) and \( Q \) be predicates on \( \mathbb{N}^{<\omega} \). If the following four conditions hold, then \( Q(\langle \rangle) \) holds.

1. for every infinite sequence \( \alpha \) of natural numbers, there is a natural number \( n \) such that \( P(\bar{\alpha}(n)) \) holds,
2. for every \( \vec{u} \) and \( \vec{v} \), if \( P(\vec{u}) \) holds then \( P(\vec{u} \ast \vec{v}) \) holds,
3. for every \( \vec{u} \), if \( P(\vec{u}) \) holds then \( Q(\vec{u}) \) holds,
4. for every \( \vec{u} \), if \( Q(\vec{u} \ast \langle n \rangle) \) holds for all \( n \) then \( Q(\vec{u}) \) holds.

**Lemma 12** (The Soundness of the Cut Rule). If \( \Gamma \rightarrow A \) and \( A, \Gamma \rightarrow B \) are reducible, then \( \Gamma \rightarrow B \) is reducible.

**Proof.** Assume that \( \Gamma \rightarrow A \) and \( A, \Gamma \rightarrow B \) are reducible. First, we use the induction principle on \( rk(A) \). If \( A \in \Gamma \), then \( A, \Gamma \rightarrow B = \Gamma \rightarrow B \), so the assertion holds. In what follows, we consider the case that \( A \notin \Gamma \). For simplicity’s sake, we focus on the case that \( A = \forall x C(x) \). Let \( \psi \) be a given reduction procedure for \( A, \Gamma \rightarrow B \).

We apply monotone bar induction (MBI), setting

\[
\begin{align*}
P(\vec{u}) \text{ if and only if } & \psi^0(\vec{u}) = Ax, \\
Q(\vec{u}) \text{ if and only if } & \text{for every closed } \Delta_0 \text{ and } B_0, \\
& \text{if } \psi^1(\vec{u}) = \forall x C(x), \Delta_0 \rightarrow B_0 \text{ holds, then } \Gamma, \Delta_0 \rightarrow B_0 \text{ is reducible.}
\end{align*}
\]
If we show that all premises of MBI hold, then we can conclude that \( Q(\langle \rangle) \) holds.
Since $\psi^1(\langle \rangle) = \text{End}(\psi) = \forall x C(x), \Gamma \rightarrow B$ holds, it follows that $\Gamma \rightarrow B$ is reducible.

By the well-foundedness and local correctness of $\psi$, it is obvious that Premises 1 and 2 of MBI hold.

First, we show that Premise 3 of MBI holds. Assume that $\psi^0(\vec{u}) = \text{Ax}$ and $\psi^1(\vec{u}) = \forall x C(x), \Delta_0 \rightarrow B_0$ hold. Then, $\psi(\vec{u}) = (\text{Ax}, \forall x C(x), \Delta_0 \rightarrow B_0)$ holds. By the local correctness of $\psi$, $B_0 \in \text{TRUE}$ or $\Delta_0 \cap \text{FALSE} \neq \emptyset$ holds. Define $\varphi$ as

$$\varphi(\vec{u}) := (\text{Ax}, \Delta_0 \rightarrow B_0)$$

for every $\vec{u}$, then it is obvious that $(s^*, \varphi)$ is a reduction procedure for $\Gamma, \Delta_0 \rightarrow B_0$, so $\Gamma, \Delta_0 \rightarrow B_0$ is reducible.

Next, we show that Premise 4 of MBI holds. Assume that for every $n$ and every closed $\Delta_0, B_0$,

$$\text{if } \psi^1(\vec{u} \ast \langle n \rangle) = \forall x C(x), \Delta_0 \rightarrow B_0 \text{ holds, then } \Gamma, \Delta_0 \rightarrow B_0 \text{ is reducible (IH of MBI).}$$

To show that $Q(\vec{u})$ holds, assume that $\psi^1(\vec{u}) = \forall x C(x), \Delta_1 \rightarrow B_1$ holds for arbitrary closed $\Delta_1, B_1$. We have to distinguish the cases according to the value of $\psi^0(\vec{u})$ and consider the most crucial case only.

Suppose that $\psi^0(\vec{u}) = (\forall, k, \forall x C(x))$ and $C(k) \in \Delta_1$. By the local correctness of $\psi$, $\psi^1(\vec{u} \ast \langle 0 \rangle) = \forall x C(x), \Delta_1 \rightarrow B_1$ holds. Hence, by IH of MBI, the sequent $\Gamma, \Delta_1 \rightarrow B_1$ is reducible. Next, consider the case that $C(k) \notin \Delta_1$ holds. By the local correctness of $\psi$ and IH of MBI, the sequent $C(k), \Gamma, \Delta_1 \rightarrow B_1$ is reducible. From the assumption that $\Gamma \rightarrow \forall x C(x)$ is reducible, it follows by Lemma 7.(1) that $\Gamma \rightarrow C(k)$ is reducible, so $\Gamma, \Delta_1 \rightarrow C(k)$ is reducible by Lemma 10. Therefore, $\Gamma, \Delta_1 \rightarrow B_1$ is reducible by IH of the induction on $rk(A)$.

As said in Section 3, the statability of a reduction procedure requires the availability of both a reduction procedure $(s^*, \varphi)$ and a proof for the well-foundedness of $\varphi$: For every reduction procedure $(s^*, \varphi)$ for a closed sequent $S$, $(s^*, \varphi)$ is statable for $S$ if and only if both $(s^*, \varphi)$ and a proof for the well-foundedness of $\varphi$ are obtained. For every sequent $S$ such that $S$ is not closed and $FV(S) \subseteq \{x_1, \ldots, x_k\}$, a reduction procedure is statable for $S$ if and only if for every $n_1, \ldots, n_k$, a reduction procedure is statable for $S[x_1 := n_1, \ldots, x_k := n_k]$.

Then, through the above proof of Lemma 11, we can show that every $\mathcal{Z}$-derivable sequent $S$ is correct in the sense of (GI). Consider a substitution instance $S[x_1 := n_1, \ldots, x_k := n_k]$ of $S$. The above proof of Lemma 11 gave a reduction procedure $(s^*, \varphi)$ for $S[\vec{x} := \vec{n}]$ with a proof for the well-foundedness of $\varphi$. Therefore, a reduction procedure is statable for $S$ and $S$ is correct in the sense of (GI).

This means that the main lemma of the 1935 proof provides each theorem of first-order classical number theory with a sense being admissible to intuitionists. Re-
member that reduction procedures were defined as dressed spreads, which are infinite trees in intuitionistic mathematics. Moreover, in the proof of the main lemma, we avoided PEM for non-decidable predicates and used monotone bar induction as an induction principle that was needed to complete the proof. We showed that every theorem of first-order classical number theory is correct, in a manner being admissible to intuitionists.

6. Concluding Remarks

The use of spreads and monotone bar induction in consistency proofs, as in Sections 4 and 5, results in the use of intuitionistic mathematics in consistency proofs. Brouwer had suggested such use of intuitionistic mathematics. In the paper “Intuitionistische Betrachtungen über den Formalismus” ([Brouwer 1928]), to which Gentzen referred in [Gentzen 1974, p.117] and [Gentzen 1936, p.563], Brouwer discussed the relationship between formalism, namely the Hilbert School, and intuitionism as follows:

**First Insight.** The distinction in the Formalist practice between the construction of “a stock of mathematical formulae” (the Formalist description of mathematics) and the intuitive (contentual) theory of laws of this construction, as well as the recognition that for the latter theory the intuitionist mathematics of the set of natural numbers is indispensable. ([Brouwer 1928, p.375], italics added)\(^3\)

Here, Brouwer claimed that “the intuitionist mathematics of the set (Menge) of natural numbers,” in the original German text “die intuitionistische Mathematik der Menge der natürlichen Zahlen,” is indispensable for Hilbert’s finitary standpoint. Note that Brouwer called spreads *Mengen* in his German writings.\(^4\) Moreover, monotone bar induction is an induction principle on a spread being well-founded, so “die intuitionistische Mathematik der Menge der natürlichen Zahlen” includes this induction principle. Thus, Brouwer actually stated that the branch of intuitionistic mathematics, in which spreads and bar induction are used, is indispensable for Hilbert’s finitary standpoint.

In the present paper, we have argued first that Gentzen’s interpretation (GI) of arithmetical formulas took the role of responding to the Brouwer-style objection, which opposes the significance of consistency proofs. Specifically, to respond to the objection, Gentzen ascribed a sense to each theorem of first-order classical number theory from his finitist standpoint: Gentzen proved that such a theorem is still correct in the sense of (GI). Second, we have proposed a way to understand Gentzen’s

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\(^3\) This English translation is from [Mancosu 1998, p.41].

\(^4\) For example, see [Brouwer 1925, pp.244–245].
response from an intuitionist perspective. On the basis of our definition for reduction procedures by means of spreads, we have proved the key lemma to Gentzen’s response by using monotone bar induction. The present paper’s entire argument showed the following: The role of responding to the Brouwer-style objection was given to Gentzen’s interpretation of arithmetical formulas by the dialogue between Gentzen and intuitionists, and the response is admissible from an intuitionist perspective.

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(Received 2015.12.17; Revised 2017.8.9; Accepted 2017.9.13)