Possible discovery of nonlinear tail and quasinormal modes in black hole ringdown

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Abstract. We discuss the nonlinear evolution of black hole ringdown in the framework of higher-order metric perturbation theory. By solving the initial-value problem of a simplified nonlinear field model analytically as well as numerically, we find that (i) second-order quasinormal modes (QNMs) are indeed excited at frequencies different from those of first-order QNMs, as predicted recently. We also find serendipitously that (ii) late-time evolution is dominated by a new type of power-law tail at late times. This “second-order power-law tail” decays more slowly than any late-time tails known in the first-order (i.e., linear) perturbation theory, and is generated at the wavefront of the first-order perturbation by an essentially nonlinear mechanism. These nonlinear components should be particularly significant for binary black hole coalescences, and could open a new precision science in gravitational wave studies.

1. Introduction

Direct detection of gravitational waves is one of the most exciting challenges in astrophysics today. It will not only enable to verify general relativity, but will also provide a new observational window for studying the universe. Future gravitational wave detectors such as LISA should make it a reality in the near future.

The “ringdown” of black holes is an important target of gravitational-wave observations. It is known theoretically that black holes have characteristic damping oscillation modes, called quasinormal modes (QNMs) \cite{1, 2}. The remarkable property of black hole QNMs is that their frequencies as well as damping rates are uniquely determined by the black hole parameters. This means that the parameters of black holes can be directly measured by observing their ringdown through the gravitational waves.

Usually, black hole ringdown is understood in the framework of linear perturbation theory. Although the linear theory is clearly useful, the results of recent numerical simulations imply that the nonlinearity of the ringdown may be also important. It has been revealed that quasinormal ringing radiated from merged binary black holes carry away a large fraction (~1\%) of the initial rest mass energy of the system (see, e.g., \cite{3}). This suggests that nonlinear components of the radiation will have amplitude as large as ~10\% of the first-order amplitude. This strongly suggests that the second- and higher-order perturbations should contribute the ringdown waveform to that extent. These components will provide us with some additional information on the merger events and the nonlinearity of general relativity.
Despite its potential importance, the nonlinearity of black hole ringdown has been largely ignored. Only recently, one of the authors (K.I.) and his coworker have for the first time applied second-order perturbation theory \[4, 5\] for the ringdown of Schwarzschild black holes \[6, 7\]. Because of the nonlinearity of the Einstein equation, two normal modes in first-order perturbations, say \(\cos(\Omega_1 t)\) and \(\cos(\Omega_2 t)\), couple each other, yielding their “sum tone” \(\cos[(\Omega_1 + \Omega_2) t]\) and “difference tone” \(\cos[(\Omega_1 - \Omega_2) t]\) in second-order ones. The authors found that the sum and difference tones of the first-order QNMs also satisfy the boundary conditions of QNMs, meaning that they are the QNMs in the second-order perturbation theory. They concluded that if these “second-order QNMs” are excited in a binary black hole merger, they will have enough energy to be detected by future gravitational-wave detectors. Detection of the second-order QNMs has many possible applications, such as measuring the distance from the source system, testing the nonlinearity of general relativity, and rejecting fake events in ringdown searches.

Questions have remained unanswered. Are the second-order QNMs really excited in initial value problems? Is it only the second-order QNMs that arise from the second-order perturbations? In this study, we give answers to these questions by solving an initial-value problem for nonlinear black hole ringdown. Our analytic calculation is based on a time-domain analysis developed for linear black hole perturbations \[8, 9\]. As far as we know, this analysis has been never applied to nonlinear black hole perturbations. Also, we employ a simple nonlinear field model having key properties in common with black hole perturbations. This nonlinear model is appealing, since the original equations for black hole perturbations are too complicated to be treated analytically, and since current numerical relativity would not be accurate enough to resolve the nonlinear evolution of black hole ringdown \[6\]. Our model does not only enable to solve the problem analytically, but also allow us to confirm the results with fully nonlinear simulations in a good accuracy.

Using this simple model, we have found that the second-order QNMs are not only actually excited, but are also accompanied by a power-law tail which is essentially different from those known in the linear perturbation theory \[10\]. This tail, to which we shall refer as “second-order power-law tail,” decays more slowly than any of the first-order tails, and even dominates the fully nonlinear evolution of the model field at late times. This means a surprising fact that the first-order theory fails to predict the late-time behavior of the ringdown.

2. Nonlinear scalar field model

Now we introduce our nonlinear field model. To avoid oversimplification, we require the model to satisfy that (i) its first- and second-order perturbations obey \((1+1)\)-dimensional wave equations with a potential barrier; and that (ii) the effective source term for the second-order perturbation is quadratic in the first-order one. These are the essential properties of black hole perturbations \[5\]. As one of the simplest models, we adopt a nonlinear, \((1+1)\)-dimensional scalar field obeying

\[
\left[-\partial_t^2 + \partial_x^2 - V(x)\right] \Phi(t, x) = F(x)\Phi(t, x)^2,
\]

where \(V(x)\) is a potential term and \(F(x)\) is a function that characterizes the nonlinearity of the system. We assume that \(V(x)\) has a peak at \(x = 0\) and vanishes at \(|x| \gg 1\). Since \(x\) corresponds to the tortoise coordinate \(r_*\) for black holes, we refer to the limits \(x \to -\infty\) and \(x \to +\infty\) as “horizon” and “infinity,” respectively. We also assume an observer’s position near infinity, \(x \gg 1\).

It is easily checked that this model meets the above requirements. We expand \(\Phi(t, x)\) in terms of an expansion parameter \(\epsilon\) to write

\[
\Phi(t, x) = \epsilon \phi^{(1)}(t, x) + \epsilon^2 \phi^{(2)}(t, x) + \cdots.
\]
Substituting this into Eq. (1), we obtain an infinite set of perturbation equations, the first two of which read
\[
\begin{align*}
\left[-\partial_t^2 + \partial_x^2 - V(x)\right] \phi^{(1)} &= 0, \\
\left[-\partial_t^2 + \partial_x^2 - V(x)\right] \phi^{(2)} &= S^{(2)}(t, x),
\end{align*}
\]
where \(S^{(2)}(t, x) = F(x)(\phi^{(1)})^2\). It is evident that equations (3) and (4) satisfy the above requirements (i) and (ii).

We choose \(F(x)\) so that the source term \(S^{(2)}\) satisfies the same asymptotic behavior at \(x \to \infty\) as that of second-order black hole perturbations at \(r \to \infty\). As one of the simplest forms, we adopt \(F(x) = (|x| + 1)^{-2}\).

In the following, we consider an initial-value problem of Eq. (1) with the following initial conditions:
\[
\begin{align*}
\Phi(0, x) &= f(x), \\
\partial_t \Phi(0, x) &= g(x),
\end{align*}
\]
where \(f(x)\) and \(g(x)\) are arbitrary functions vanishing far away from the potential peak.

2.1. Analytic calculation
Since all the perturbations \(\phi^{(j)} (j = 1, 2, \ldots)\) obey linear equations with the same operator \(-\partial_t^2 + \partial_x^2 - V(x)\), their evolution is determined by a single Green’s function. The retarded Green’s function \(G(\tau; x, x’)\) for \(\phi^{(j)}\) is the solution to
\[
\left[-\partial_t^2 + \partial_x^2 - V(x)\right] G(t-t'; x, x') = \delta(t-t')\delta(x-x'),
\]
under the retarded boundary condition \(G(\tau; x, x') = 0, \tau < 0\). Using this Green’s function, the solutions to Eqs. (3) and (4) with the initial conditions (5) are formally written as
\[
\begin{align*}
\phi^{(1)}(t, x) &= \int_{-\infty}^{\infty} dt' dx' G(t-t'; x, x') I(t', x'), \\
\phi^{(2)}(t, x) &= \int_{-\infty}^{\infty} dt' dx' G(t-t'; x, x') S^{(2)}(t', x'),
\end{align*}
\]
where \(I(t, x) = -f(x)\delta'(t) - g(x)\delta(t)\).

Unfortunately, the exact expression of the Green’s function \(G(\tau; x, x')\) is unknown so far. We have derived an approximate form of \(G(\tau; x, x')\) with the “asymptotic approximation” developed by Andersson [9]. The detailed calculation of Eqs. (7) and (8) using the approximate Green’s function is shown in reference [11]. Here we show the final results of the calculation. The first-order perturbation \(\phi^{(1)}(t, x)\) near infinity \((x \gg 1)\) is expressed as a superposition of the first-order QNMs truncated at the light cone \(t = x\),
\[
\phi^{(1)}(t, x) \approx \theta(t-x) \sum_{n=0}^{\infty} \left[C_n e^{s_n(t-x)} + C_n^* e^{s_n(t-x)}\right]
\]
where \(\theta\) is the step function, and \(s_n\) and \(C_n\) are the eigenvalues and excitation coefficient of the QNMs, respectively. \(s_n\) is related to a more familiar “quasinormal frequency” \(\omega_n\) by \(\omega_n = is_n\).

The information of initial data \(f(x)\) and \(g(x)\) is implicitly contained in \(C_n\). We have found that the asymptotic solution of the second-order perturbation \(\phi^{(2)}(t, x)\) is of the form
\[
\phi^{(2)}(t, x) \approx \theta(t-x) \left[\phi^{(2)}_{1Q}(t, x) + \phi^{(2)}_{2Q}(t, x) + \phi^{(2)}_{1T}(t, x)\right]
\]
(10)
where \( \phi_{1Q}^{(2)} \) is again a superposition of first-order QNMs, and

\[
\phi_{2Q}^{(2)}(t, x) \approx \sum_{n,n'} \left[ C_n C_{n'} A_{nn'} e^{(s_n+s_{n'})/(t-x)} + C_n C_{n'}^* B_{nn'} e^{(s_n+s_{n'}^*)/(t-x)} + \text{c.c.} \right],
\]

\[
\phi_T^{(2)}(t, x) \approx \sum_{n,n'} \left( \frac{C_n C_{n'}}{s_n + s_{n'}} + \frac{C_n C_{n'}^*}{s_n + s_{n'}^*} + \text{c.c.} \right) \left( \frac{1}{t-x+2} - \frac{1}{t+x+2} \right),
\]

are the contributions from the second-order QNMs and power-law tail, respectively. The excitation coefficients of the second-order QNMs, \( A_{nn'} \) and \( B_{nn'} \), can be expressed as a function of \( s_n \) and \( s_{n'} \) [11]. Equation (12) implies that this tail behaves as \( t^{-1} \) for \( x < t \lesssim 2x \), and as \( t^{-2} \) for \( t \gtrsim 2x \) (note that we have assumed that \( x \gtrsim 1 \), but not assumed that \( t \gtrsim x \)).

We emphasize that the second-order power-law tail \( \phi_T^{(2)} \) is essentially different from the tails known in first-order perturbation theory [10]. First, although the first-order tails are known to decay more quickly than \( t^{-2} \), our tail decays more slowly than \( t^{-2} \). Second, while the behavior of the first-order tails depends on the form of \( V(x) \), that of our tail is independent of it. In fact, it can be shown that the behavior of \( \phi_T^{(2)} \) is completely determined by the long-range behavior of the source term \( S^{(2)}(x) \), or \( F(x) \). This fact manifests that our tail is intrinsically nonlinear. Remember that we have carefully chosen the functional form of \( F(x) \) so that the asymptotic behavior of \( S^{(2)}(x) \) in our model corresponds to that in original black hole perturbations. Therefore, it is concluded that black hole ringdown should possess the same type of second-order tail.

2.2. Numerical calculation

To check the validity of the above analysis, we have performed two types of numerical calculations. In the first type, we integrated the first- and second-order perturbation equations (3) and (4). In the second type, we integrate the full-order equation (1). In this paper, we only show the result of the second type of simulation (for the full results, see reference [11]).

As an initial condition, we chose a momentarily stationary Gaussian wave packet, \( f(x) = \exp[-(2.0x)^2] \), \( g(x) = 0 \). As the potential term \( V(x) \), we adopted the Pöschl-Teller potential \( V_{PT}(x) = V_0 / \cosh^2(Kx) \), where \( V_0 \) and \( K^{-1} \) are the height and the width of the potential barrier, respectively. We set \( V_0 = 5.0 \) and \( K = 1.0 \) in this study. The eigenvalues \( s_n \) of first-order QNMs for \( V_{PT} \) are known analytically [12], and the quasinormal frequency \( \omega_0 = i\delta_0 \) of the least-damped mode is calculated to be \( \omega_0 = 2.18 - 0.50i \) for the above set of parameters. It follows that the frequencies of the least-damped second-order QNMs are \( 4.36 - 1.0i \) ("sum tone" mode) and \( -1.0i \) ("difference tone" mode).

Figure 1 shows the numerical waveform of the full nonlinear field \( \Phi \) observed at \( x = 5.0 \equiv x_{\text{obs}} \). It is observed that the waveform for early times \( (x_{\text{obs}} \lesssim t \lesssim 12) \) is well fitted by the least-damped first-order QNM \( (\omega_0 = 2.18 - 0.50i) \). At late times \( (t \gtrsim 12) \), however, the numerical waveform does not agree with the first-order QNM. Surprisingly, it is the second-order power-law tail [Eq. (12)] that agrees with the waveform (although difficult to see in the figure, the second-order QNMs are also found to be contained in the waveform). This means that the late-time behavior of the nonlinear field is dominated by the second-order perturbation, not by the first-order one. We have also confirmed that the second-order tail agrees with the full-order waveform even at \( t \approx 50 \), which strongly suggests that third and higher-order perturbations are negligible for all times.

3. Conclusion and Discussion

In this study, we have investigated the nonlinear evolution of black hole ringdown using second- and higher-order perturbation theory and a simplified nonlinear field model for black hole metric
perturbations. We have proven that second-order QNMs, whose existence has been predicted by recent works [6, 7], do appear in the evolved second-order perturbation. As a bonus, we have discovered a new type of power-law tail appearing with the second-order QNMs. This power-law tail, to which we have referred as the second-order power-law tail, decays more slowly than any of the tails in the first-order theory, and even dominates the fully nonlinear evolution at late times. In other words, the first-order perturbation theory fails to predict the late-time evolution of the ringdown, and higher-order corrections must be taken into account. Also, we have found that the behavior of the second-order tail is determined by the asymptotic form of an effective source term, i.e., a term expressing the nonlinearity of the system. Since the asymptotic form of the source term in our model is the same as that in black hole perturbations, we conclude that the second-order power-law tail as well as the second-order QNMs will certainly appear in real black hole ringdown. These nonlinear components could open a new precision science in gravitational-wave studies.

Is it possible to detect the second-order power-law tail with LISA? To address this issue, we need to estimate the energy carried out by the second-order tail assuming astrophysically realistic events (e.g., binary black hole mergers), as already done for second-order QNMs [6, 7]. Also, we will need a special data analysis to extract the second-order tail from detector outputs. While the matched filtering technique has been developed for extracting QNMs [13], there seems to be only a few techniques for power-law tails [14]. If the second-order tail is detectable, it could be used to distinguish true signals and spurious ones in black hole ringdown search, where fake reduction and event identification are crucial [7, 13].

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