BOUNDARY RIGIDITY FOR SOME CLASSES OF MEROMORPHIC FUNCTIONS

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Abstract. Sufficient boundary asymptotic conditions are established for a generalized Schur function \( f \) to be identically equal to a given rational function \( g \) unimodular on the unit circle. Similar rigidity statements are presented for generalized Carathéodory and generalized Nevanlinna functions.

1. Introduction

In what follows, we use the following notation:

(1) \( \mathbb{C}, \mathbb{D}, \mathbb{T} \) denote the complex plane, the open unit disk and the unit circle, respectively.
(2) \( \mathcal{S} \) – the Schur class (the closed unit ball of \( H^\infty \)).
(3) \( \mathcal{B}_\kappa \) – the set of all Blaschke products of degree \( \kappa \).
(4) \( \mathcal{B}_p/\mathcal{B}_q \) – the set of all coprime quotients \( g = b/\theta \) with \( b \in \mathcal{B}_p \) and \( \theta \in \mathcal{B}_q \), i.e., the set of all rational functions \( g \) unimodular on \( \mathbb{T} \) and with \( p \) zeros and \( q \) poles in \( \mathbb{D} \) (counted with multiplicities).
(5) \( \mathcal{S}_\kappa \) – the generalized Schur class (introduced in [11]) consisting of all coprime quotients of the form \( f = s/b \) where \( s \in \mathcal{S} \) and \( b \in \mathcal{B}_\kappa \).
(6) \( \mathcal{S}_{\leq \kappa} := \bigcup_{q \leq \kappa} \mathcal{S}_q \) – the set of quotients as in (3), but not necessarily coprime.
(7) \( \mathcal{Z}(f) \) – the zero set of a function \( f \).

It is clear from definitions (4) and (7) that \( \mathcal{B}_p/\mathcal{B}_q \subset \mathcal{S}_{\leq \kappa} \) whenever \( q \leq \kappa \). The following rigidity result was presented in [7] as an intermediate step to obtain a similar statement in the multivariable setting.

Theorem 1.1. Let \( f \in \mathcal{S} \) and let \( f(z) = z + O((z-1)^4) \) as \( z \to 1 \). Then \( f(z) \equiv z \).

Generalizations and further developments can be found e.g., in [1, 3, 4, 8, 9, 10, 12, 13]. Here we recall one from [4].

Theorem 1.2. Let \( f \in \mathcal{S} \), \( g \in \mathcal{B}_d \), let \( t_1, \ldots, t_n \) be \( n \) distinct points on \( \mathbb{T} \) and let

\[
    f(z) = g(z) + o((z-t_i)^{m_i}) \quad \text{for} \quad i = 1, \ldots, n
\]

as \( z \) tends to \( t_i \) nontangentially, where \( m_1, \ldots, m_n \) are nonnegative integers. If

\[
    \left\lfloor \frac{m_1+1}{2} \right\rfloor + \ldots + \left\lfloor \frac{m_n+1}{2} \right\rfloor > d = \text{deg } g,
\]

then \( f \equiv g \). Otherwise, the uniqueness fails.

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In (1.2), \([x]\) denotes the largest integer that does not exceed a real number \(x\). The last statement in Theorem 1.2 means: if condition (1.2) fails for a finite Blaschke product \(g\) and nonnegative integers \(m_1, \ldots, m_n\), then for every choice of \(n\) points \(t_1, \ldots, t_n \in \mathbb{T}\), there are infinitely many functions \(f \in \mathcal{S}\) subject to (1.1). Thus, conditions (1.1) are minimal. Observe that Theorem 1.1 follows from Theorem 1.2 by letting \(n = 1, m_1 = 3, g = id\) and \(t_1 = 1\) in the latter.

Since conditions (1.1) are of interpolation nature, an analog of Theorem 1.2 for generalized Schur functions must exist. It does indeed, as Theorem 2.1 below shows.

### 2. Rigidity for generalized Schur functions

We start with the main result which turns out to a straightforward consequence of Theorem 1.2.

**Theorem 2.1.** Let \(\kappa, r, \ell\) be nonnegative integers and let \(t_1, \ldots, t_n\) be \(n\) distinct points on \(\mathbb{T}\), let \(g \in B_\ell / B_r\) and let us assume that a function \(f \in \mathcal{S}_{\leq \kappa}\) satisfies conditions

\[

f(z) = g(z) + o((z - t_i)^{m_i}) \quad \text{for} \quad i = 1, \ldots, n \tag{2.1}
\]

as \(z\) tends to \(t_i\) nontangentially, for some nonnegative integers \(m_1, \ldots, m_n\). If

\[

\left[ \frac{m_1 + 1}{2} \right] + \ldots + \left[ \frac{m_n + 1}{2} \right] > \kappa + \ell, \tag{2.2}
\]

then \(f \equiv g\).

**Proof:** Substituting coprime quotient representations for \(f\) and \(g\)

\[

f(z) = \frac{s_f(z)}{b_f(z)} \quad (s_f \in \mathcal{S}, \ b_f \in B_\kappa) \quad \text{and} \quad g(z) = \frac{b(z)}{\theta(z)} \quad (b \in B_\ell, \ \theta \in B_r) \tag{2.3}
\]

into (2.1) and then multiplying both sides in (2.1) by \(b_f \cdot \theta \in B_{\kappa + r}\), we get

\[

s_f(z)\theta(z) = b(z)b_f(z) + o((z - t_i)^{m_i}) \quad \text{for} \quad i = 1, \ldots, n. \tag{2.4}
\]

Since \(s_f \cdot \theta \in \mathcal{S}, b \cdot b_f \in B_{\kappa + \ell}\) and since by (2.2),

\[

\sum_{i=1}^{n} \left[ \frac{m_i + 1}{2} \right] > \kappa + \ell = \deg (b \cdot b_f),
\]

we conclude from (2.4) by Theorem 1.2 that \(s_f \cdot \theta \equiv b \cdot b_f\) which is equivalent, by (2.3), to \(f \equiv g\). \(\square\)

**Remark 2.2.** Observe that the membership \(f \in \mathcal{S}_\kappa\) means that total pole multiplicity of \(f\) does not exceed \(\kappa\). Although we allow \(f\) and \(g\) to have different pole multiplicities, this possibility cannot be realized under conditions (2.2).

Being specialized to the case \(n = 1\), Theorem 2.1 gives the following.

**Corollary 2.3.** Let \(\kappa, r, \ell\) be nonnegative integers, let \(g \in B_\ell / B_r\) and let \(f \in \mathcal{S}_\kappa\) be such that

\[

f(z) = g(z) + o((z - t_0)^{2\kappa + 2\ell + 1}) \tag{2.5}
\]

as \(z\) tends to \(t_0 \in \mathbb{T}\) nontangentially. Then \(f \equiv g\).
For the proof, it is enough to notice that the least integer $m$ satisfying inequality $\left\lfloor \frac{m+1}{2} \right\rfloor > \kappa + \ell$ is $m = 2\kappa + 2\ell + 1$.

We now recall a recent result from \[1\] where rigidity for functions in $\mathcal{S}_\kappa$ was established under a slightly stronger condition than (2.5).

**Theorem 2.4.** Let $t_0$ be a point on $\mathbb{T}$ and let us assume that the numbers $\tau_0, \tau_{k-1}, \ldots, \tau_{2k-1} \in \mathbb{C}$ are such that the matrix $P = \tau_0TB$ is Hermitian, where $T$ is the lower triangular Toeplitz matrix with the bottom row equal $[\tau_{2k-1} \tau_{2k-2} \ldots \tau_{k+1}, \tau_k]$ and $B = [b_{ij}]_{i,j=1}^k$ is the $k \times k$ right lower triangular matrix with the entries

$$b_{ij} = \begin{cases} 0, & \text{if } 2 \leq i + j \leq k, \\ (1)^{j-1}(j + 1)^{i-1}t_0^{j+k-1}, & \text{if } k + 1 \leq i + j \leq 2k. \end{cases}$$

Let $g(z)$ be the function defined by

$$g(z) = \frac{a(z)x + b(z)}{c(z)x + d(z)} \quad (2.6)$$

where $x \in \mathbb{T} \setminus \{\tau_0\}$, $\begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix} = I_2 - \left(\frac{1-z\tau_0}{1-\tau_0}\right)p(z) \left[ \begin{array}{cc} 1 & -\tau_0 \\ \tau_0 & -1 \end{array} \right]$ where $z_0 \neq t_0$ is an arbitrary point on $\mathbb{T}$ and $p(z)$ is the polynomial (note that the matrix $P$ is invertible by construction) given by $p(z) = (1 - z\tau_0)^kR(z)P^{-1}R(z_0)^*$ where

$$R(z) = \left[ \frac{1}{1-z\tau_0} \frac{z}{(1-z\tau_0)^2} \ldots \frac{z^{k-1}}{(1-z\tau_0)^k} \right].$$

Then

1. The function $g$ is the quotient of two finite Blaschke products with $r$ poles in $\mathbb{D}$ (where $r$ is the number of negative eigenvalues of the matrix $P$) and with the following Taylor expansion at $t_0$:

$$g(z) = \tau_0 + \sum_{i=k}^{2k-1} \tau_i(z - t_0)^i + O((z - t_0)^{2k}). \quad (2.7)$$

2. If $f \in \mathcal{S}_r$ satisfies that $f(z) = g(z) + O((z - t_0)^{2k+2})$, then $f \equiv g$. \quad (2.8)

To embed Theorem 2.4 into our framework we first recall that for every quotient of two finite Blaschke products with the Taylor expansion (2.7), the matrix $P$ constructed in the theorem is necessarily Hermitian and $\tau_0 = g(t_0)$ is unimodular (see \[6\] Section 2). On the other hand, it follows from general results from \[2\] Section 21] that formula (2.6) parametrizes all unimodular functions $g \in \mathcal{B}_{k-r}/\mathcal{B}_r$. Therefore, the rigidity part in Theorem 2.4 can be reformulated equivalently in the following more compact form.

**Theorem 2.5.** Let $g \in \mathcal{B}_{k-r}/\mathcal{B}_r$ admit the Taylor expansion (2.7) at $t_0 \in \mathbb{T}$. If $f \in \mathcal{S}_r$ satisfies the nontangential asymptotic condition (2.8), then $f \equiv g$.

The main limitation in Theorem 2.5 is that $g$ has quite special Taylor coefficients at $t_0$ ($\tau_1 = \tau_2 = \ldots = \tau_{k-1} = 0$) (observe that the original Burns-Krantz theorem is of a different type, since there we have $\tau_1 = 1$ and $\tau_2 = \tau_3 = 0$; however it was shown in \[1\] Section 4) that Theorem 1.1 can be deduced from Theorem 2.4). Corollary 2.3 shows that rigidity holds for any quotient of finite Blaschke products. Besides, Corollary 2.3 shows that the term $O((z - t_0)^{2k+2})$ in (2.8) can
be relaxed to \( o((z - t_0)^{2k+1}) \), that the order of approximation can be of any parity (not necessarily even) and that rigidity may hold also in case where only a bound for the pole multiplicity of \( f \) is known.

3. Rigidity for generalized Carathéodory and generalized Nevanlinna functions

The generalized Schur class \( S_\kappa \) can be alternatively characterized as the class of all functions \( f \) meromorphic on \( \mathbb{D} \) and such that the kernel \( S_f(z, \zeta) = \frac{1 - f(z)f(\zeta)}{1 - \overline{f(\zeta)}f(z)} \) has \( \kappa \) negative squares on \( \mathbb{D} \cap \text{Dom}(f) \). A related to \( S_\kappa \) is the class \( C_\kappa \) of generalized Carathéodory functions \( h \) which by definition, are meromorphic on \( \mathbb{D} \) and such that the associated kernel \( C_h(z, \zeta) = \frac{h(z) + \overline{h(\zeta)}}{1 - \overline{h(\zeta)}h(z)} \) has \( \kappa \) negative squares on \( \mathbb{D} \cap \text{Dom}(h) \).

It is convenient to include the function \( h \equiv \infty \) into \( C_0 \). Then the Cayley transform

\[
    f \mapsto h = \frac{1 + f}{1 - f} \tag{3.1}
\]

establishes a one-to-one correspondence between \( S_\kappa \) and \( C_\kappa \) and therefore, between \( S_\leq \kappa \) and \( C_\leq \kappa := \bigcup_{r \leq \kappa} C_r \). The representation \( f = s/b \) for an \( f \in S_\kappa \) combined with (3.1) implies that \( h \) belongs to \( C_\leq \kappa \) if and only if it is of the form

\[
    h = \frac{b + s}{b - s} \quad \text{where} \quad b \in B_\kappa, \ s \in S \quad \text{and} \quad Z(s) \cap Z(b) = \emptyset \tag{3.2}
\]

Theorem [2.1] in the present setting looks as follows.

**Theorem 3.1.** Let \( \kappa, r, \ell \) be nonnegative integers and let \( g \) be of the form

\[
    g = \frac{b_2 + b_1}{b_2 - b_1} \quad \text{where} \quad b_1 \in B_{\ell}, \ b_2 \in B_r \quad \text{and} \quad Z(b_1) \cap Z(b_2) = \emptyset. \tag{3.3}
\]

Let us assume that a function \( h \in C_\leq \kappa \) satisfies asymptotic equations

\[
    h(z) = g(z) + o((z - t_i)^{m_i}) \quad \text{for} \quad i = 1, \ldots, n \tag{3.4}
\]

at some points \( t_1, \ldots, t_n \in \mathbb{T} \) and some nonnegative integers \( m_1, \ldots, m_n \) which in turn, are subject to (2.4). Then \( h \equiv g \).

**Proof:** Substituting (1.2) and (1.3) into (3.4) and then multiplying both sides in (3.4) by \((b_2 - b_1)(b - s)\) we eventually get

\[
    s(z)b_2(z) = b(z)b_1(z) + o((z - t_i)^{m_i}) \quad \text{for} \quad i = 1, \ldots, n. \tag{3.5}
\]

Since \( s \cdot b_2 \in S \) and \( b \cdot b_1 \in B_{\kappa + \ell} \), we invoke Theorem [1.2] (as in the proof of Theorem [2.1]) to conclude from (3.5) that \( s \cdot b_2 \equiv b \cdot b_1 \) which implies that \( h \equiv g \), thanks to (1.2) and (3.3). \( \square \)

Another class related to \( S_\kappa \) is the class \( N_\kappa \) of generalized Nevanlinna functions, that is, the functions \( h \) meromorphic on the open upper half-plane \( \mathbb{C}^+ \) and such that the associated kernel \( N_h(z, \zeta) = \frac{h(z) - \overline{h(\zeta)}}{z - \zeta} \) has \( \kappa \) negative squares on \( \mathbb{C}^+ \cap \text{Dom}(h) \).

The function \( h \equiv \infty \) is assumed to be in \( N_0 \). The classes \( N_\kappa \) and \( S_\kappa \) are related by

\[
    h(\zeta) = i \cdot \frac{1 + f(\gamma(\zeta))}{1 - f(\gamma(\zeta))}, \quad \gamma(\zeta) = \frac{\zeta - i}{\zeta + i} \tag{3.6}
\]
which allows us to characterize $N^\kappa$-functions by the fractional representation

$$h = i \cdot \frac{b + s}{b - s}$$

(3.7)

where $s$ (analytic and bounded by one in modulus in $\mathbb{C}^+$) and $b \in B_\kappa$ do not have common zeroes. For the rest of the paper we denote by $B_k(\mathbb{C}^+)$ the set of finite Blaschke products of the form

$$b(\zeta) = \prod_{i=1}^k \frac{\zeta - a_i}{\zeta - \bar{a}_i} \quad (\zeta, a_i \in \mathbb{C}^+).$$

Here is Theorem 3.2 for generalized Nevanlinna functions.

**Theorem 3.2.** Let $\kappa, r, \ell$ be two nonnegative integers, let $g$ be of the form

$$g = i \cdot \frac{b_2 + b_1}{b_2 - b_1} \quad \text{where} \quad b_1 \in B_\ell(\mathbb{C}^+), \ b_2 \in B_r(\mathbb{C}^+), \ Z(b_1) \cap Z(b_2) = \emptyset.$$  

(3.8)

Let $\lambda_1, \ldots, \lambda_n$ be real points, let $m_1, \ldots, m_n$ be nonnegative integers and let us assume that a function $h \in N_{\leq \kappa}$ satisfies the asymptotic equations

$$h(\zeta) = g(\zeta) + o((\zeta - \lambda_j)^{m_j}) \quad \text{for} \quad i = 2, \ldots, n$$

(3.9)

as $\zeta \in \mathbb{C}^+$ tends to $\lambda_i$ nontangentially and the asymptotic equation

$$h(\zeta) = g(\zeta) + o(|\zeta|^{-m_i})$$

(3.10)

as $z$ tend to infinity staying inside the angle $\{z : \epsilon < \arg z < \pi - \epsilon\}$. If the numbers $m_1, \ldots, m_n$ are subject to (2.2), then $h \equiv g$.

**Proof:** Let $z := \gamma(\zeta)$ where $\gamma$ is given in (3.6). Then $t_1 := \gamma(\infty) = 1 \in \mathbb{T}$ and since $\lambda_i \in \mathbb{T}$, we have $t_i := \gamma(\lambda_i) \in \mathbb{T}$ for $i = 2, \ldots, n$. Observe that

$$|z - t_j| = |\gamma(\zeta) - \gamma(\lambda_j)| = \frac{2|\zeta - \lambda_j|}{|\zeta + i(\lambda_j + i)|} = O(|\zeta - \lambda_j|)$$

for $j = 2, \ldots, n$ and $|z - t_1| = |z - 1| = |\gamma(\zeta) - 1| = \frac{2}{|\zeta + i|} = O(|\zeta|^{-1})$. Therefore, and since $\gamma$ maps $\mathbb{C}^+$ onto $\mathbb{D}$ conformally, we can write (3.9) and (3.10) as

$$h(\gamma^{-1}(z)) = g(\gamma^{-1}(z)) + o((z - t_i)^{m_i}) \quad \text{for} \quad i = 1, \ldots, n.$$  

(3.11)

It remains to note that the functions $-ih \circ \gamma^{-1}$ and $-ih \circ \gamma^{-1}$ are generalized Carathéodory functions satisfying the assumptions of Theorem 3.1. Therefore, they are equal identically and thus, $h \equiv g$. \qed

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