Linear solutions for cryptographic nonlinear sequence generators

A. Fúster-Sabater\(^{(1)}\) and P. Caballero-Gil\(^{(2)}\)

(1) Instituto de Física Aplicada, C.S.I.C., Serrano 144, 28006 Madrid, Spain
amparo@iec.csic.es

(2) DEIOC, University of La Laguna, 38271 La Laguna, Tenerife, Spain
pcaballe@ull.es

Abstract

This letter shows that linear Cellular Automata based on rules 90/150 generate all the solutions of linear difference equations with binary constant coefficients. Some of these solutions are pseudo-random noise sequences with application in cryptography: the sequences generated by the class of shrinking generators. Consequently, this contribution shows that shrinking generators do not provide enough guarantees to be used for encryption purposes. Furthermore, the linearization is achieved through a simple algorithm about which a full description is provided.

Keywords: Nonlinear Science, Cellular Automata, Predictability, Cryptanalysis.

"Linearity is the curse of the cryptographer" (J. L. Massey, Crypto’89)

1 Introduction

Secret-key cryptography is commonly divided into block and stream ciphers. As opposed to block ciphers, stream ciphers encrypt each data symbol (as small as a bit) into a ciphertext symbol under a nonlinear dynamical transformation. Stream ciphers are the fastest among the encryption procedures so they are implemented in many practical applications e.g. the algorithms A5 in GSM communications [1], the generator RC4 in Wi-Fi security protocol [2] or the encryption system E0 in Bluetooth specifications [3].

A stream cipher procedure is based on the generation of a long keyed Pseudo-random Noise (PN) sequence and its addition to the original message. In particular, for encryption the sender realizes the bit-wise XOR operation among

\[^{0}\text{Final version published in Physics Letters A Vol. 369, Is. 5-6, 1 Oct. 2007, pp. 432-437 DOI:10.1016/j.physleta.2007.04.103}\]
the bits of the original message or plaintext and the pseudo-random noise sequence, giving rise to the ciphertext. For decryption, the receiver generates the same pseudo-random noise sequence, realizes the same bit-wise XOR operation between the received ciphertext and the pseudo-random noise sequence and recuperates the original message.

Most pseudo-random noise sequence generators are based on either chaotic encryption procedures (see for instance [4], [5] and [6]), or Linear Feedback Shift Registers (LFSRs) [7]. The output sequences of LFSRs have application in multiple areas such as spread spectrum communication, digital ranging, tracking systems, simulation of random processes, computer sequencing and timing schemes. For their use in cryptography, such sequences are combined by means of nonlinear functions. That is the case of combinational generators, nonlinear filters, clock-controlled generators and irregularly decimated generators. All of them produce pseudo-random noise sequences with high linear complexity, long period and good statistical properties (see [8] and [9]).

Cellular Automata (CA) are discrete structures with dynamical behaviour extensively studied and applied in modelling systems in physics, chemistry, biology, computer science and other disciplines. It has been proved [10] that one-dimensional linear CA generate exactly the same pseudo-random noise sequences as those of LFSRs. Regarding more complex generators, this work shows that certain CA generate exactly the same pseudo-random noise sequences as those of nonlinear generators based on LFSRs. Linearity in cipher’s behavior, say the cryptanalysts, is the end of a cipher. It essentially means that information is leaked from the plaintext to the ciphertext. In particular, this letter proves how a well known class of LFSR-based nonlinear generators, the shrinking generators, can be modelled in terms of linear CA. According to the cryptanalytic statement, this class of cryptographic generators has been broken.

This contribution proposes the use of difference equations and cellular automata to predict the dynamic behavior of certain nonlinear noise sequences. The predictability of such sequences is carried out through the linearization of their generator. Furthermore, that linearization process seems to be applicable for more general noise sequence generators such as those based on quantum physics and chaotic processes.

2 The class of shrinking generators

A shrinking generator is a nonlinear binary sequence generator composed by two LFSRs (see [11]): a control register notated $R_1$ that decimates the sequence produced by the other register notated $R_2$. Let $L_j \in N$; ($j = 1, 2$) be their corresponding lengths with $(L_1, L_2) = 1$ and let $P_j(x) \in GF(2)[x] (j = 1, 2)$ be their corresponding characteristic polynomials of degree $L_j$. In practical applications, such polynomials are primitive in order to generate PN-sequences of maximum length. Henceforth, $\{a_i\}$ and $\{b_i\}$ ($i \geq 0$) $a_i, b_i \in GF(2)$ denote the
binary sequences generated by $R_1$ and $R_2$, respectively. The output sequence of the generator (the shrunken sequence) is denoted by \{c_j\} ($j \geq 0$) with $c_j \in GF(2)$. The sequence produced by $R_1$ determines what elements of the sequence produced by $R_2$ are included in the shrunken sequence. The decimation rule is:

1. If $a_i = 1 \implies c_j = b_i$
2. If $a_i = 0 \implies b_i$ is discarded.

A simple example illustrates the behavior of this structure.

**Example 1:** Let us consider the following LFSRs:

1. $R_1$ of length $L_1 = 3$, characteristic polynomial $P_1(x) = 1 + x^2 + x^3$ and initial state $IS_1 = (1, 0, 0)$. The PN-sequence generated by $R_1$ is \{1, 0, 0, 1, 1, 1, 0, 1\} with period $T_1 = 2^{L_1} - 1 = 7$.

2. $R_2$ of length $L_2 = 4$, characteristic polynomial $P_2(x) = 1 + x + x^4$ and initial state $IS_2 = (1, 0, 0, 0)$. The PN-sequence generated by $R_2$ is \{1, 0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1\} with period $T_2 = 2^{L_2} - 1 = 15$.

The output sequence \{c_j\} is given by:

- $\{a_i\} \rightarrow \underline{1} 0 0 1 \underline{1} 1 0 1 0 0 1 1 1 0 1 0 0 1 1 1 0 1$ ....
- $\{b_i\} \rightarrow \underline{1} 0 0 0 \underline{1} 0 0 1 1 0 1 0 1 1 1 0 0 1 0 0 0 1 0 0$ ....
- $\{c_j\} \rightarrow 1 0 1 0 1 1 0 1 1 0 0 1 0$ ....

According to the decimation rule, the underlined bits 0 or 1 in $\{b_i\}$ are discarded. Thus, the sequence produced by the shrinking generator is a decimation of $\{b_i\}$ governed by the bits of $\{a_i\}$. According to [11], the period of the shrunken sequence is $T = (2^{L_2} - 1)2^{(L_1 - 1)}$ and its linear complexity, notated $LC$, satisfies the following inequality

$$L_22^{(L_1 - 2)} < LC \leq L_22^{(L_1 - 1)}. \quad (1)$$

In addition, the shrunken sequence is balanced and has good distributional statistics. Therefore, this scheme is suitable for practical implementation of stream cipher cryptosystems and pattern generators.

### 3 Linear multiplicative polynomial CA

CA are particular forms of finite state machines defined as uniform arrays of identical cells in an $n$-dimensional space (see [12]). The cells change their states (contents) synchronously at discrete time instants. The next state of each cell depends on the current states of the neighbor cells according to its transition
rule. If the transition rules are all linear, so will be the automaton under consideration. In this letter, we will deal with a particular kind of binary CA, the so-called linear multiplicative polynomial cellular automata. They are discrete dynamical systems characterized by:

1. Their underlying topology is one-dimensional, that is they can be represented by a succession of \( L \) cells where \( L \) is an integer that denotes the length of the automaton. The state of the \( i \)-th cell at instant \( n \), notated \( x_i^n \), takes values in a finite field \( x_i^n \in GF(2) \).

2. They are linear cellular automata as the transition rule for each cell is a linear mapping \( \Phi_i : GF(2)^k \to GF(2) \) where

\[
x_i^{n+1} = \Phi_i(x_{i-q}^n, \ldots, x_i^n, \ldots, x_{i+q}^n) \quad (i = 1, \ldots, L)
\]

\( k = 2q + 1 \) being the size of the neighborhood.

3. Each one of these cellular automata is uniquely represented by an \( L \times L \) transition matrix \( M \) over \( GF(2) \). The characteristic polynomial of such matrices is of the form

\[
P_M(x) = (P(x))^p
\]

where \( P(x) = x^r + \sum_{j=1}^{r} c_j x^{r-j} \) denotes a irreducible (primitive) polynomial of degree \( r \) over \( GF(2) \) and \( p \) an integer such that \( L = p \cdot r \).

This letter is concentrated on one-dimensional binary linear CA with neighborhood size \( k = 3 \) and particular transition rules defined as follows:

| Rule 90 | Rule 150 |
|---------|----------|
| \( x_i^{n+1} = x_{i-1}^n \oplus x_{i+1}^n \) | \( x_i^{n+1} = x_{i-1}^n \oplus x_i^n \oplus x_{i+1}^n \) |

where the symbol \( \oplus \) represents the XOR logic operation. Remark that they are linear and very easy transition rules involving just the addition of either two bits (rule 90) or three bits (rule 150).

For a cellular automaton of length \( L = 10 \) cells, configuration rules (90, 150, 150, 150, 90, 90, 150, 150, 150, 90) and initial state (0, 0, 0, 1, 1, 1, 0, 1, 1, 0), Table 1 illustrates the behavior of this structure: the formation of its output sequences (binary sequences read vertically) and the succession of states (binary configurations of 10 bits read horizontally). In addition, cells with permanent null contents are supposed to be adjacent to the array extreme cells.

The characteristic polynomial \( P(x) \) of an arbitrary binary sequence \( \{a_n\} \) specifies its linear recurrence relationship. This means that the \( n \)-th element \( a_n \) can be written as a linear combination of the previous elements:

\[
a_n \oplus \sum_{i=1}^{r} c_i a_{n-i} = 0, \quad n \geq r. \tag{4}
\]
The linear recursion is expressed as a linear difference equation:

\[(E^{r} \oplus \sum_{i=1}^{r} c_i E^{r-i}) a_n = 0, \quad n \geq 0\]  

(5)

where \(E\) is the shifting operator that operates on \(a_n\), i.e. \(Ea_n = a_{n+1}\). If the characteristic polynomial \(P(x)\) is primitive and \(\alpha\) one of its roots, then

\[\alpha, \alpha^2, \alpha^2, \ldots, \alpha^{2^{(r-1)}}\]  

are the \(r\) different roots of such a polynomial as well as primitive elements in \(GF(2^r)\) (see [13]).

Now, if the characteristic polynomial of an arbitrary binary sequence \(\{a_n\}\) is of the form \(P_M(x) = (P(x))^p\) as defined in (3), then its roots will be the same as those of \(P(x)\) but with multiplicity \(p\). The corresponding difference equation will be:

\[(E^{r} \oplus \sum_{i=1}^{r} c_i E^{r-i})^p a_n = 0, \quad n \geq 0\]  

(7)

and its solutions are of the form \(a_n = \sum_{j=0}^{p-1} \sum_{m=0}^{r-1} \binom{n}{m} A_m^j \alpha^{2^j n}\), where \(A_m\) is an arbitrary element in \(GF(2^L)\). Different choices of \(A_m\) will give rise to different sequences \(\{a_n\}\). Consequently, all the binary sequences \(\{a_n\}\) of characteristic polynomial \(P_M(x) = (P(x))^p\) can be generated by linear multiplicative polynomial CA as well as all of them are solutions of the linear difference equation described in (7). Our analysis focuses on all the possible solutions of this equation.

4 Realization of linear multiplicative polynomial CA

In the previous section, algebraic properties of the sequences obtained from multiplicative polynomial CA have been considered. Now the particular form of these automata is analyzed.

A natural way of representation for this type of 90/150 linear CA is a binary \(L\)-tuple \(\Delta = (d_1, d_2, \ldots, d_L)\) where \(d_i = 0\) if the \(i\)-th cell verifies rule 90 while \(d_i = 1\) if the \(i\)-th cell verifies rule 150. The Cattell and Muzio synthesis algorithm [10] presents a method of computing two 90/150 CA corresponding to a given polynomial. Such an algorithm takes as input an irreducible polynomial \(Q(x)\) and computes two reversal \(L\)-tuples corresponding to two different linear CA whose output sequences have \(Q(x)\) as characteristic polynomial. The total number of operations required for this algorithm is linear in the degree of the polynomial and is listed in [10](Table II, page 334). The method is efficient for
all practical applications (e.g. in 1996 finding a pair of length 300 CA took 16 CPU seconds on a SPARC 10 workstation). For cryptographic applications, the degree of the primitive polynomial $P(x)$ is $L_2 \approx 64$, so that the consuming time is negligible. Finally, a list of one-dimensional linear CA of degree through 500 can be found in [14].

Since the characteristic polynomials we are dealing with are of the form $P_M(x) = (P(x))^p$, it seems quite natural to construct a multiplicative polynomial cellular automaton by concatenating $p$ times the automaton whose characteristic polynomial is $P(x)$. The procedure of concatenation is based on the following result.

**Lemma 1.** Let $\Delta = (d_1, d_2, \ldots, d_L)$ be the representation of an one-dimensional binary linear cellular automaton with $L$ cells and characteristic polynomial $P_L(x) = (x + d_1)(x + d_2)\ldots(x + d_L)$. The cellular automaton whose characteristic polynomial is $P_{2L}(x) = (P_L(x))^2$ is represented by:

$$\Delta = (d_1, d_2, \ldots, \overline{d_L}, \overline{d_L}, \ldots, d_2, d_1)$$

where the overline symbol represents bit complementation.

*Proof.* The result follows from the fact that:

$$P_{\overline{T}}(x) = P_L(x) + P_{L-1}(x)$$

where $P_{\overline{T}}(x)$ is the polynomial corresponding to $\Delta = (d_1, d_2, \ldots, \overline{d_L})$. In the same way

$$P_{L+1}(x) = (x + d_L)P_{\overline{T}}(x) + P_L(x)$$

$$P_{L+2}(x) = (x + d_{L-1})P_L(x) + P_{\overline{T}}(x)$$

$$\vdots$$

$$P_{2L}(x) = (x + d_1)P_{2L-1}(x) + P_{2L-2}(x).$$

Thus, by successive substitutions of the previous polynomial into the next one we get:

$$P_{2L}(x) = (x + d_1)P_{2L-1}(x) + P_{2L-2}(x) = (P_L(x))^2.$$  \hspace{1cm} (9)

The result can be iterated for successive exponents. In this way, the concatenation of an automaton and its mirror image allows us to realize linear multiplicative polynomial CA. The complementation is due to the fact that rule 90 (150) at the end of the array is equivalent to two consecutive rules 150 (90) with identical sequences.
5 Shrunken sequences as solutions of linear equations: a simple linearization procedure

Now the result that relates the shrunken sequences from shrinking generators with the sequences obtained from linear multiplicative polynomial cellular automata is introduced.

**Theorem 1.** The characteristic polynomial of the output sequence of a shrinking generator with parameters \(L_j \in \mathbb{N}\) and \(P_j(x) \in GF(2)[x]\) \((j = 1, 2)\) defined as in section (2) is of the form \(P_M(x) = (P(x))^p\), where \(P(x) \in GF(2)[x]\) is a \(L_2\)-degree polynomial and \(p\) is an integer satisfying the inequality \(2^{(L_1-2)} < p \leq 2^{(L_1-1)}\).

Proof. The shrunken sequence can be written as a sequence made out of an unique \(PN\)-sequence starting at different points and repeated \(2^{(L_1-1)}\) times. Such a sequence is obtained from \(\{b_i\}\) taking elements separated a distance \(2^{L_2-1}\), that is the period of the sequence \(\{a_i\}\). As \((2^{L_2-1}, 2^{L_1-1}) = 1\) due to the primality of \(L_2\) and \(L_1\), the result of the decimation of \(\{b_i\}\) is a \(PN\)-sequence whose characteristic polynomial \(P(x)\) of degree \(L_2\) is the characteristic polynomial of the cyclotomic coset \(2^{L_2-1}\), that is \(P(x) = (x + \alpha^N)(x + \alpha^{2^2N})\ldots(x + \alpha^{2^{L_2-1}N})\) being \(N\) an integer given by \(N = 2^0 + 2^1 + \ldots + 2^{L_1-1}\). Moreover, the number of times that this \(PN\)-sequence is repeated coincides with the number of 1’s in \(\{a_i\}\) since each 1 of \(\{a_i\}\) provides the shrunken sequence with \(2^{L_2-1}\) elements of \(\{b_i\}\). Consequently, the characteristic polynomial of the shrunken sequence will be \(P(x)^p\) with \(p \leq 2^{(L_1-1)}\). The lower limit follows immediately from equation (3) and the definition of linear complexity of a sequence as the shortest linear recurrence relationship.

According to its characteristic polynomial, the output sequence of a shrinking generator is a particular solution of a linear difference equation as well as it can be generated by linear multiplicative polynomial CA. Now, the construction of such linear models from the shrinking generator parameters is carried out by the following algorithm:

**Linearization algorithm**

**Input:** A shrinking generator characterized by two LFSRs, \(R_1\) and \(R_2\), with their corresponding lengths, \(L_1\) and \(L_2\), and the characteristic polynomial \(P_2(x)\) of the register \(R_2\).

**Step 1** From \(L_1\) and \(P_2(x)\), compute the polynomial \(P(x)\) as

\[
P(x) = (x + \alpha^N)(x + \alpha^{2^N})\ldots(x + \alpha^{2^{L_2-1}N})
\]

with \(N = 2^0 + 2^1 + \ldots + 2^{L_1-1}\).

**Step 2** From \(P(x)\), apply the Cattell and Muzio synthesis algorithm to determine two linear 90/150 CA, notated \(s_i\), whose characteristic polynomial is \(P(x)\).
Step 3 For each $s_i$ separately, proceed:

3.1 Complement its least significant bit. The resulting binary string is notated $S_i$.

3.2 Compute the mirror image of $S_i$, notated $S_i^*$, and concatenate both strings

$$S'_i = S_i \ast S_i^*.$$  

3.3 Apply steps 3.1 and 3.2 to each $S'_i$ recursively $L_1 - 1$ times.

Output: Two binary strings of length $L = L_2 \cdot 2^{L_1 - 1}$ codifying two CA corresponding to the given shrinking generator.

Remark 1. In this algorithm the characteristic polynomial of the register $R_1$ is not needed. Thus, all the shrinking generators with the same $R_2$ but different registers $R_1$ (all of them with the same length $L_1$) can be modelled by the same pair of one-dimensional linear CA.

Remark 2. It can be noticed that the computation of both CA is proportional to $L_1$ concatenations. Consequently, the algorithm can be applied to shrinking generators in a range of practical application.

Remark 3. In contrast to the nonlinearity of the shrinking generator, the CA-based models that generate the shrunk sequence are linear.

In order to illustrate the previous steps a numerical example is presented.

Example 2:

Input: A shrinking generator characterized by two LFSRs: $R_1$ of length $L_1 = 3$, $R_2$ of length $L_2 = 5$ and characteristic polynomial $P_2(x) = 1 + x + x^2 + x^4 + x^5$.

Step 1 $P(x)$ is the characteristic polynomial of the cyclotomic coset $N = 7$. Thus,

$$P(x) = 1 + x^2 + x^5.$$  

Step 2 From $P(x)$ and applying the Cattell and Muzio synthesis algorithm, two reversal linear CA whose characteristic polynomial is $P(x)$ can be determined. Such CA are written in binary format as:

$$0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1 \ 0$$

Step 3: Computation of the required pair of CA by successive concatenations.

For the first automaton:

$$0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1$$

$(final \ automaton)$
For the second automaton:

\[
\begin{array}{ccccccccccc}
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

*(final automaton)*

For each automaton, the procedure of concatenation has been carried out \(L_1 - 1\) times.

**Output:** Two binary strings of length \(L = L_2 \cdot 2^{(L_1 - 1)} = 20\) codifying the required pair of CA.

In this way, we have obtained a pair of linear CA:

\[
(90,150,150,150,150,90,150,150,150,150,150,150,90,150,150,150,150,150,150,150,150,90) \\
(150,150,150,150,150,150,150,150,150,90,90,150,150,150,150,150,150,150,150,150,150,150)
\]

both of them able to generate the shrunk sequence corresponding to the given shrinking generator. Consequently, the shrinking generator can be expressed in terms of a lineal model based on CA.

### 6 Conclusions

The pseudo-random noise sequence produced by a shrinking generator is a particular solution of a linear difference equation and can be generated by linear multiplicative polynomial cellular automata. In this way, cryptographic generators conceived and designed as nonlinear generators can be linearized in terms of cellular automata, which implies that such cryptographic generators have been broken. The used linearization algorithm is simple and might be applied to more general sequence generators such as those based on quantum physics and chaotic processes.

**Acknowledgements**

This work has been supported by Ministerio de Educación y Ciencia (Spain), Projects SEG2004-02418 and SEG2004-04352-C04-03.

**References**

[1] GSM, *Global Systems for Mobile Communications*, available at http://cryptome.org/gsm-a512.htm

[2] Wi-Fi Alliance, *RC4 Encryption Algorithm*, available at http://www.wifialliance.com

[3] Bluetooth, *Specifications of the Bluetooth system*, available at http://www.bluetooth.com/
Table 1: An one-dimensional linear cellular automaton of 10 cells with rules 90/150 starting at a given initial state. The period of these sequences is $T = 62$

|   | 90 | 150 | 150 | 150 | 90 | 90 | 150 | 150 | 150 | 90 |
|---|----|-----|-----|-----|----|----|-----|-----|-----|----|
|   | 0  | 0   | 0   | 1   | 1  | 1  | 0   | 1   | 1   | 0  |
|   | 0  | 0   | 1   | 0   | 0   | 1  | 0   | 0   | 0   | 1  |
|   | 0  | 1   | 1   | 1   | 1   | 0  | 1   | 0   | 1   | 0  |
|   | 1  | 0   | 1   | 1   | 1   | 0  | 1   | 0   | 1   | 1  |
|   | 0  | 0   | 0   | 1   | 1   | 1   | 0  | 1   | 0   | 0   | 1  |
|   | 0  | 0   | 1   | 0   | 1   | 0   | 1   | 1   | 1   | 0  |
|   | .  | .   | .   | .   | .   | .   | .   | .   | .   | .   |