ON THE THEORY OF A NON-LINEAR NEUTRAL SCALAR FIELD WITH SPONTANEOUSLY BROKEN SYMMETRY *

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On the example of a real scalar field, an approach to quantization of non-linear fields and construction of the perturbation theory with account of spontaneous symmetry breaking is proposed. The method is based on using as the main approximation of the relativistic self-consistent field model, in which the influence of vacuum field fluctuations is taken into account when constructing the one-particle states. The solutions of the self-consistent equations determine possible states, which also include the states with broken symmetries. Different states of the field are matched to particles, whose masses are determined by both parameters of the Lagrangian and vacuum fluctuations. The density of the vacuum energy in these states is calculated. It is shown that the concept of Bogolubov’s quasi-averages can naturally be applied for definition of exact Green functions in the states with broken symmetries. Equations for exact one- and two-point Green functions are obtained.

Key words: scalar field, broken symmetry, self-consistent field, perturbation theory, quasi-average

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INTRODUCTION

The study of states with broken symmetries in quantum field theory was initiated under influence of the works on microscopic theory of superfluidity [1] and superconductivity [2,3], in the works of Nambu and Jona-Lasinio [4], Wax and Larkin [5], Goldstone [6] et al. The effect of spontaneous symmetry breaking is an important element of contemporary theory of elementary particles [7–9]. Most often it is performed by way of including of massive scalar fields with an “incorrect” sign of the square of mass into the Lagrange function (Goldstone mechanism [6]). This mechanism was used in construction of a unified theory of weak and electromagnetic interactions [7]. Another way to describe spontaneous symmetry breaking was suggested by Nambu and Jona-Lasinio [4], Wax and Larkin [5] on the example of chiral symmetry breaking in hadron physics. The method used in the works [4,5] is based upon the analogy with the superconductivity theory, and it got the name of dynamical mechanism of symmetry breaking [10]. Both the Goldstone mechanism and the dynamical one are the ways of description of the same phenomenon.

The aim of the present work is to construct on the example of a real scalar field a common approach for description of the states of non-linear relativistic quantized fields with arbitrary spontaneously broken symmetries. The proposed approach to the description of spontaneous symmetry breaking in the quantum field theory generalizes to the relativistic case the quantum-field approach for microscopic description of non-relativistic many-particle Fermi- and Bose-systems with broken symmetries, that is developed in the works [11–15], and is, in its essence, closer to the dynamical mechanism.

In the developed approach the Lagrangian of a non-linear field is split, as usually, into two parts, one of which describes free particles and the other one their interaction. The peculiarity of the proposed method consists in the way of construction of the Lagrangian of non-interacting particles. This peculiarity consists in choice of Lagrangian that contains the terms not higher than quadratic in field operators, including those terms whose symmetry is lower than the symmetry of the initial Lagrangian. The parameters that enter into the free Lagrangian are selected from the requirement of its maximal proximity to the full Lagrange function. The system of non-linear equations derived based upon this condition, which determines the parameters of the free Lagrangian, in particular the particle masses, can have different solutions that describe the states with different symmetries. Such an approach lets, to a certain extent, account for interaction already in the stage of construction of one-particle states. In particular, the value of the interaction constant determines the particle masses that turn out to be different in different states. The perturbation theory constructed on the basis of the mentioned choice of the main approximation can be also applicable in the case when the interaction constant is not small. It is notable that, when using
the postulated method of construction of one-particle states, the Hamiltonian of interaction automatically takes the form of normally ordered product of field operators. Owing to this, many diagrams are excluded from the perturbation theory expansion, namely all diagrams containing loops, i.e. the lines both ends of which are connected to the same vertex and, therefore, the structure of the perturbation theory is significantly simplified. In the present work the equations for one- and two-point Green functions are also obtained. It is shown that, when determining exact Green functions in the states with spontaneously broken symmetries, it is natural to use the concept of quasi-averages introduced by Bogoljubov [16] in statistical mechanics. Although in this work we used as an example the real scalar field with broken discrete symmetry, the proposed method of description of the states with broken symmetries is applicable also for other, including non-linear fermionic, fields [23].

RELATIVISTIC MODEL OF SELF-CONSISTENT FIELD

Let us consider a Bose system described by the secondary-quantized Hermitian field operator \( \varphi \equiv \varphi (x) \) that depends on space coordinates \( x \) and time \( t \), so that 
\[ x \equiv x_\mu = (x, it), \mu = 1, 2, 3, 4. \]
The system is characterized by the density of Lagrange function
\[ L = -\frac{1}{2} \left[ (\partial_\mu \varphi)^2 + \kappa^2 \varphi^2 \right] - \frac{g}{4!} \varphi^4, \]
(1)
where \( \partial_\mu = \partial / \partial x_\mu \) and the metrics used is \( ab = ab + a_4 b_4 = ab - a_0 b_0 \). It is supposed that the sign of the real parameter \( \kappa^2 \) (that we will call the mass parameter) is arbitrary, and the interaction constant \( g \) is positive. The system of units used in this paper is \( \hbar = c = 1 \). Let us note that, when constructing the quantum field theory, usually in the initial step one considers classical fields, and in the following step one realizes the “quantization” of the classical fields by way of transition to the secondary-quantized operators that obey certain commutation relations. Such an approach, no doubt, is justified in both historical and methodological respects. However, in virtue of the fact that the quantum description is a more profound level of description of the reality, we will start immediately from it. Classical fields in this case must be a limiting case of quantum fields.

For field operators at the same time arguments the standard commutation relations are postulated:
\[
\begin{align*}
[\pi (x, t), \varphi (x', t)] &= -i \delta (x - x'), \\
[\varphi (x, t), \varphi (x', t)] &= [\pi (x, t), \pi (x', t)] = 0,
\end{align*}
\]
(2)
where \( \pi (x, t) = \partial L / \partial \dot{\varphi} (x, t) \) is the canonically conjugated momentum. The field obeys the non-linear operator equation
\[ \partial_\mu \partial_\mu \varphi - \kappa^2 \varphi - \frac{g}{3!} \varphi^3 = 0. \]
(3)
The field operators in (1) are taken in the Heisenberg picture:

$$\varphi (x, t) = e^{iHt} \varphi (x, 0) e^{-iHt}, \quad \pi (x, t) = e^{iHt} \pi (x, 0) e^{-iHt},$$

where $\varphi (x, 0), \pi (x, 0)$ are operators in the Schrödinger picture and the Hamilton operator has the form

$$H = \int dx \left\{ \frac{1}{2} \left[ \pi^2 + (\nabla \varphi)^2 + \kappa^2 \varphi^2 \right] + \frac{g}{4!} \varphi^4 \right\}.$$  \hspace{1cm} (5)

The transition from the field operators to the particle operators for free fields is performed in a well-known way. For this purpose, the field operator is split into the positive-frequency and negative-frequency parts and transition to the Fourier representation is carried out. The coefficients of such expansion have the sense of creation and annihilation operators of non-interacting particles. In the case of non-linear theory the described prescription of introducing the one-particle states cannot be used. In this case the coefficients of Fourier transform do not have the sense of operators of creation and annihilation of states with a correct relativistic law of dispersion. In the initial Lagrangian one could select the part quadratic in the field operators and consider it, as is usually done, as the free particles Lagrangian; and consider the rest part of the Lagrangian as a perturbation. It is worth noting that this “obvious” way of constructing the one-particle states is, in its essence, a tacit agreement. However, such a partition is not uniquely possible. Moreover, it is not effective in the case of states with spontaneously broken symmetries, for whose description the account of the effects caused by nonlinearity of the system is essential. Particularly, such consideration does not have sense when the mass parameter $\kappa^2$ is negative, because in this case the free Lagrangian describes the objects with “incorrect” (tachyonic) law of dispersion. The possibility of introducing of tachyons in physics and of giving them a real status was widely discussed earlier (see the book [17]). A need of introducing of such objects as tachyons is under question, since, if a field is described by non-linear equations, the presence of an “incorrect” sign of the square of mass in the quadratic part of Lagrangian does not at all indicate the existence of particles with exotic properties, although, as was shown by Goldstone [6], this case describes quite a real physical situation. When constructing a non-linear field theory it is worth remembering, that the initial physical sense is only inherent in the full Lagrangian of the system. The splitting of the full Lagrangian into the sum of the Lagrangian of “non-interacting” particles and the interaction Lagrangian is, obviously, ambiguous. Indeed, besides some splitting $L = L_1 + L_2$, one can proceed from another splitting $L = L'_1 + L'_2$, where $L'_1 = L_1 + \Delta L$, $L'_2 = L_2 - \Delta L$ and $\Delta L$ is some operator addition. By fixing the way, in which we construct the part of Lagrangian describing the one-particle states, we in fact give a definition for the
notion of “non-interacting particle” in the framework of the non-linear theory. Now let
us present the full Lagrangian (11) as the sum

\[ L = L_0 + L_C, \]  

(6)

where \( L_0 \) is the Lagrangian that contains the terms not higher than quadratic in the
field operators, including those which violate the symmetry of the initial Lagrangian \( L \).
Practically the only requirement imposed onto the form of the Lagrangian \( L_0 \), besides
its quadraticness, is its Lorentz-invariance. This Lagrangian, by definition, will be
considered as the Lagrangian of “non-interacting” particles. The second term, \( L_C \), is
the Lagrangian that contains the terms not present in \( L_0 \). When constructing the per-
turbation theory we consider the first term in (6) as a non-perturbed Lagrangian, and
the second one – as a perturbation describing the interaction of particles determined
by the first term. In the considered case of a real scalar field

\[ L_0 = -\frac{1}{2} (\partial_\mu \phi)^2 + D\phi^2 + p\phi - \Omega, \]  

(7)

where \( D, p, \Omega \) are real parameters. The interaction Lagrangian, obviously, has the form

\[ L_C = L - L_0 = -\frac{g}{4!} \phi^4 - \left( \frac{\kappa^2}{2} + D \right) \phi^2 - p\phi + \Omega. \]  

(8)

In the main Lagrangian we have added and subtracted the operator term \( \Delta L = D\phi^2 + p\phi - \Omega \) without changing \( L \). The parameters \( D, p, \Omega \) can be to a large degree arbitrary,
and one should choose them using some extra considerations. Let us note that \( L_0 \)
contains a term linear in \( \phi \), that breaks the symmetry relative to the transformation
\( \phi \to -\phi \) inherent in the Lagrangian \( L \). An analogical statement is also true for
\( L_C \). The selected split into the main part and the interaction takes place also for the
Hamiltonian \( H = H_0 + H_C \):

\[ H_0 = \int dx \left[ \frac{\pi^2}{2} + \frac{1}{2} (\nabla \phi)^2 - D\phi^2 - p\phi + \Omega \right], \]  

(9)

\[ H_C = \int dx \left[ \frac{g}{4!} \phi^4 + \left( D + \frac{\kappa^2}{2} \right) \phi^2 + p\phi - \Omega \right]. \]  

(10)

As far as the full Hamiltonian is expressed in the same way both in terms of Heisenberg
and of Schrödinger operators, then \( H_0 \) and \( H_C \) in (9) and (10) can also be expressed
in terms of the field operators in these pictures. Let us consider more in detail the
main approximation given by the Hamiltonian (9), expressed in terms of Schrödinger
operators. This approximation describes the system of “non-interacting”, in our un-
derstanding, particles, in the framework of the self-consistent field model. Let us define
the operators in the interaction picture:

$$\hat\phi (x, t) = e^{iH_0 t} \hat\phi (x, 0) e^{-iH_0 t}, \quad \hat\pi (x, t) = e^{iH_0 t} \hat\pi (x, 0) e^{-iH_0 t}.$$ \(\text{(11)}\)

Simultaneous commutation relations for operators \(\text{(11)}\) have the same form \(\text{(2)}\), as for Heisenberg operators. Obviously, \(H_0\) is expressed in terms of the operators \(\text{(11)}\) in the same way as in terms of Schrödinger operators:

$$H_0 = \int dx \left[ \frac{\hat\pi^2}{2} + \frac{1}{2} (\nabla \hat\phi)^2 - D \hat\phi^2 - p \hat\phi + \Omega \right].$$ \(\text{(12)}\)

Before reducing the Hamiltonian \(\text{(12)}\) to the diagonal form it is necessary to eliminate the linear term from it by carrying out the substitution

$$\hat\phi (x, t) = \hat\psi (x, t) + \chi,$$ \(\text{(13)}\)

where \(\chi\) is a real parameter, \(\hat\psi (x, t)\) an operator obeying the commutation rules analogous to \(\text{(2)}\). The need for a “shift” of the field operator by the value \(\chi\) is, apparently, conditioned by breaking of the symmetry relative to the operation \(\varphi \to -\varphi\). So, following the terminology used in the theory of phase transitions we will call this quantity the order parameter, and the state in which \(\chi \neq 0\) – the ordered state. Following the analogy with the non-relativistic theory of Bose-systems \([13]\), the parameter \(\chi\) can also be called the condensate wave function, and the operator \(\hat\psi (x, t)\) – the over-condensate field operator. The fields \(\hat\phi (x, t)\) and \(\hat\psi (x, t)\) are connected by the unitary transformation \(\hat\phi (x, t) = U_\chi \hat\psi (x, t) U_\chi\), where \(U_\chi = \exp \left\{ -i \chi \int dx' \hat\pi (x') \right\}\) is a unitary operator.

Let us substitute \(\text{(13)}\) into \(\text{(12)}\) and require that the terms linear in the operator \(\hat\psi (x, t)\) drop from the obtained expression. This leads to the condition

$$p + 2\chi D = 0.$$ \(\text{(14)}\)

As a result the Hamiltonian \(\text{(12)}\) takes the form

$$H_0 = \int dx \left[ \frac{\hat\pi^2}{2} + \frac{1}{2} (\nabla \hat\psi)^2 - D \hat\psi^2 + D\chi^2 + \Omega \right].$$ \(\text{(15)}\)

Let \(|0\rangle\) be a vacuum vector of the system with the Hamiltonian \(\text{(15)}\). As far as \(H_0\) does not contain terms linear in \(\hat\psi (x)\) and the Fock space is used, then obviously \(\langle 0 | \hat\psi | 0 \rangle = 0\). Therefore, by virtue of \(\text{(13)}\):

$$\chi = \langle 0 | \hat\phi | 0 \rangle.$$ \(\text{(16)}\)

Thus, the state with broken symmetry is characterized by a nonzero vacuum average of the field operator.
Until now the parameters $D, p, \Omega$ were not fixed in no wise. One should give a way to determine these parameters. By indicating it, we will in fact determine the mode of construction of the one-particle states in the framework of the developed approach. We postulate that the parameters $D, p, \Omega$ are found from the requirement that the approximating Hamiltonian $H_0$ be in some respect maximally close to the exact Hamiltonian (4). In order to formulate this requirement quantitatively, let us define the functional

$$ J \equiv \langle 0 | H - H_0 | 0 \rangle^2 = \langle 0 | H_C | 0 \rangle^2 \quad (17) $$

that characterizes the difference between the exact and approximating Hamiltonians. The Hamiltonian $H_C$ in (17) is taken in the interaction picture. The average over the vacuum state of $H_C$ has the form

$$ \langle 0 | H_C | 0 \rangle = \int dx \left[ \frac{3g}{4!} \left( \langle 0 | \hat{\phi}^2 | 0 \rangle \right)^2 - \frac{2g}{4!} \chi^4 + \left( D + \frac{\kappa^2}{2} \right) \langle 0 | \hat{\phi}^2 | 0 \rangle + p\chi - \Omega \right] . \quad (18) $$

Let us require that the following conditions hold

$$ \frac{\partial J}{\partial \langle 0 | \hat{\phi}^2 | 0 \rangle} = 0, \quad \frac{\partial J}{\partial \chi} = 0, \quad \frac{\partial J}{\partial \Omega} = 0. \quad (19) $$

Here one should consider the parameters $D, p, \Omega$ being independent and, for the present, not take into account the relation (14). From the conditions (19) we obtain

$$ 6g'\langle 0 | \hat{\phi}^2 | 0 \rangle + D + \frac{\kappa^2}{2} = 0, \quad (20) $$

$$ p = 8g'\chi^3, \quad (21) $$

$$ \Omega = g'\langle 0 | \hat{\phi}^4 | 0 \rangle + \left( D + \frac{\kappa^2}{2} \right) \langle 0 | \hat{\phi}^2 | 0 \rangle + p\langle 0 | \hat{\phi} | 0 \rangle, \quad (22) $$

where, for the brevity, we set $g' \equiv g/4!$. Taking into account the relation (16), we find

$$ \langle 0 | \hat{\phi}^2 | 0 \rangle = \langle 0 | \hat{\psi}^2 | 0 \rangle + \chi^2, \quad \langle 0 | \hat{\phi}^4 | 0 \rangle = 3\langle 0 | \hat{\psi}^2 | 0 \rangle^2 + 6\chi^2 \langle 0 | \hat{\psi}^2 | 0 \rangle + \chi^4. $$

Let us introduce the designation for the vacuum average of the square of the over-condensate field operator:

$$ \rho \equiv \langle 0 | \hat{\psi}^2 | 0 \rangle. \quad (23) $$

As is known, vacuum fluctuations of electromagnetic field lead to observable physical effects, such, for example, as Lamb shift and Casimir effect, and therefore must be taken into account in a consistent quantum theory of any field. It is all the more important in the case of states with broken symmetry, which can be characterized by a nonzero vacuum average of the field operator (16). When taking account of the vacuum
average of the field operator, there is no reason to neglect the effects conditioned by
the vacuum average of the square of the field operator. Let us note that taking account
of the averages of the field operator in powers higher than two, over the vacuum state
of the quadratic Hamiltonian \((15)\), does not have sense because such averages are
expressed in terms of the vacuum averages \((16), (23)\). It is also worth noting that the
non-operator term \(\Omega\) in the Hamiltonian \((15)\) and in the Hamiltonian of interaction
\((10)\) is quite substantial. The last of the conditions \((19)\), that leads to the formula
\((22)\), ensures fulfillment of a natural condition
\[
\langle 0|H_C|0 \rangle = 0. \tag{24}
\]
Since the Hamiltonian \(H_C\) describes the interaction of particles, then, naturally, in
the vacuum state the energy of this interaction must be equal zero. Just taking into
account of the non-operator term \(\Omega\) ensures fulfillment of this condition. Moreover, as
will be seen, the energy of the vacuum state is expressed in terms of the quantity \(\Omega\).
From \((21) - (23)\) it follows that
\[
\Omega = -3g' \left( \rho^2 + 2\rho\chi^2 - \chi^4 \right). \tag{25}
\]
With the help of \((14)\) and \((20), (21)\) we obtain the equation that determines the order
parameter \(\chi\):
\[
\chi \left( \kappa^2 + 12g'\rho + 4g'\chi^2 \right) = 0. \tag{26}
\]
This equation has two solutions. One of them, \(\chi = 0\), corresponds to the phase with
unbroken symmetry. The second one, with \(\chi \neq 0\), is found from the requirement of
equality to zero of the expression in parentheses in \((26)\). We will assign the index \(s\) to
the quantities relating to the symmetrical state with \(\chi = 0\), and the index \(b\) – to those
relating to the state with broken symmetry, \(\chi \neq 0\).

**PARTICLE MASSES. PHASE DIAGRAM**

The proposed approach lets us calculate the masses of particles, i.e. express them
in terms of the parameters of the initial Lagrangian \((1)\). From the Hamiltonian \((15)\) it
follows that the parameter \(D\) is connected with the square of the particle mass:
\[
D = -\frac{m^2}{2} = -\frac{\kappa^2}{2} - 6g' \left( \rho + \chi^2 \right). \tag{27}
\]
As we see, particle masses in this approach are determined not only by the mass
parameter \(\kappa^2\) that enters into initial the Lagrangian \((1)\) but also by the effects caused by
quantum fluctuations; moreover, they depend on the value of the interaction constant.
In symmetrical case the mass, according to (27), is defined by the relation

\[ m_s^2 = \kappa^2 + 12g' \rho_s, \]  

(28)

and the non-operator term (25) has the form

\[ \Omega_s = -3g' \rho_s^2. \]  

(29)

When the symmetry is broken, we have the following equation for the order parameter:

\[ \kappa^2 + 12g' \rho_b + 4g' \chi^2 = 0. \]  

(30)

In this case the mass is determined by the value of the order parameter

\[ m_b^2 = 8g' \chi^2 \]  

(31)

and \( \Omega_b \) by the formula (25). Let us note that, if we do not take into account the vacuum average of the square of the field operator, by formally supposing \( \rho = 0 \), then from the formula (30) follows the relation \( \chi^2 = -6\kappa^2/g \) obtained by Goldstone [6].

In the theory of non-relativistic many-particle systems with finite density of number of particles, one understands by a self-consistent field such mean field acting on a given particle, which is created by all other particles of a considered system. In relativistic theory, where the mean density of the number of particles equals zero, the self-consistent field, as is seen from the relations (23), (27), is formed by the vacuum field fluctuations. By using the self-consistent approach for determining the one-particle states, we actually account for the influence of the vacuum fluctuations on the dynamical properties of a particle.

The equation of motion for the over-condensate field operator \( \hat{\psi} \) in the interaction picture has the form of the Klein-Gordon equation

\[ \Delta \hat{\psi}(x) - \frac{\partial^2 \hat{\psi}(x)}{\partial t^2} - m^2 \hat{\psi}(x) = 0. \]  

(32)

Depending on whether in symmetrical or asymmetrical state a system stays, one should understand by \( m^2 \) in (32) either \( m_s^2 \) or \( m_b^2 \). So, unlike Heisenberg field operator that obeys the non-linear operator equation (3), the field operator in the interaction picture obeys the linear Klein-Gordon equation with a positive, as will be shown below, value of the mass square. The over-condensate field operator and also the conjugate momentum
can be presented in the form of expansion into Fourier integral:
\[ 
\hat{\psi}(x) = (2\pi)^{-3/2} \int dq \, (2q_0)^{-1/2} \left[ a(q) e^{iqx} + a^+(q) e^{-iqx} \right], 
\]
\[ 
\hat{\pi}(x) = -i (2\pi)^{-3/2} \int dq \sqrt{q_0/2} \left[ a(q) e^{iqx} - a^+(q) e^{-iqx} \right], 
\] (33)

where \( q = (q, iq_0) \), \( q_0 = \sqrt{m^2 + q^2} \), \( qx = q_\mu \vec{x}_\mu = \vec{q} \cdot \vec{x} - q_0 t \). Commutation relations of operators at coinciding times have the usual for Bose-particles form
\[ 
[a(q), a(q')] = [a^+(q), a^+(q')] = 0, \quad [a(q), a^+(q')] = \delta(q - q'). 
\]

As far as the equation (32) is linear, we can interpret operators \( a^+(q) \) and \( a(q) \) as the operators of creation and annihilation of particles, and the state vector \( a^+(q) |0\rangle \) as one that describes the one-particle state with momentum \( q \) and energy \( q_0 = \sqrt{m^2 + q^2} \).

Using the representation (33), let us calculate the vacuum average (23) that can be presented in the form
\[ 
\rho = \left( \frac{1}{2} \right) (2\pi)^{3/2} \int \frac{dq}{\sqrt{q_0^2 + m^2}} = -\lim_{\epsilon \to +0} \frac{i}{(2\pi)^3} \int \frac{dq}{q^2 + m^2 - i\epsilon}. 
\] (34)

Here \( dq = dq_0 dq \), \( q^2 = q_0^2 - q^2 \). As is known, the integral in (34) diverges at large momenta. It is accepted [18, §73] that this divergence must not cause any trouble, because the square of amplitude in any point is not a measurable quantity. However, as we see, in this approach the vacuum average of the square of the field operator is directly related to an observed characteristic – the particle mass. It seems to be physically grounded, because spreading of a particle in the vacuum occurs against a background of its fluctuations and, consequently, the particle mass must substantially depend on the intensity of such fluctuations. In order to provide the finiteness of particle masses, there appears a need to introduce the cutoff parameter \( \Lambda \) at large momenta. Such parameter was introduced also in earlier works [4, 5], in which the methods of superconductivity were firstly used for construction of models of elementary particles. Let us note that in the self-consistent equation of the superconductivity theory we also have to cut off the integral at large momenta. Here, however, natural characteristic scales are present, such as the average interparticle distance and the effective radius of the interparticle potential. As is known, a certain arbitrariness in the mode of regularization of divergent integrals is present. Let us introduce the cutoff parameter in a relativistically invariant way. When calculating the integral in (34) we perform the Wick rotation of the integration contour and use the substitution \( q_0 = iq_1 \) [9]. As a result we obtain
\[ 
\rho = \Lambda^2 f\left(\bar{m}^2\right) / 2\pi^2, \quad f\left(\bar{m}^2\right) = 1 - \bar{m}^2 \ln \left(1 + \bar{m}^{-2}\right), 
\] (35)
Figure 1: Graphical solution of the equations (36).

I: \[ y = \tilde{m}^2 - \tilde{\kappa}^2 \], II: \[ y = -\tilde{m}^2 / 2 - \tilde{\kappa}^2 \], III: \[ y = g'' \left[ 1 - \tilde{m}^2 \ln \left( 1 + \tilde{m}^2 \right) \right] \].

where \( \tilde{m}^2 = m^2 / \Lambda^2 \). The function \( f (\tilde{m}^2) \) is equal to unity at \( \tilde{m}^2 = 0 \), and monotonically decreases tending to zero at \( \tilde{m}^2 \to \infty \). By substituting the expression for the vacuum average (35) into (28) and (31), we find the self-consistent equations that define the square of mass in symmetrical (a) and asymmetrical (b) phases:

\[
\begin{align*}
\tilde{m}_s^2 - \tilde{\kappa}^2 &= g'' \left[ 1 - \tilde{m}_s^2 \ln \left( 1 + \tilde{m}_s^2 \right) \right], \\
-\frac{\tilde{m}_b^2}{2} - \tilde{\kappa}^2 &= g'' \left[ 1 - \tilde{m}_b^2 \ln \left( 1 + \tilde{m}_b^2 \right) \right],
\end{align*}
\] (36a, 36b)

where \( \tilde{m}_i^2 = m_i^2 / \Lambda^2 \) (\( i = s, b \)), \( \tilde{\kappa}^2 = \kappa^2 / \Lambda^2 \), \( g'' = 6g'/\pi = g/4\pi \). The equations (36) define the square of mass related to the square of the cutoff parameter as a function of the dimensionless mass parameter \( \tilde{\kappa}^2 \) and the interaction constant \( g'' \): \( \tilde{m}_i^2 = \tilde{m}_i^2 (\tilde{\kappa}^2, g'') \). It is noteworthy that, while the particle mass substantially depends on the cutoff parameter, the equations (36) for the dimensionless masses \( \tilde{m}_i^2 \) do not already contain it explicitly.

It is convenient to analyze these equations graphically (see Figure 1). The right parts of the equations (36) are identical, being a function that decreases monotonically from the value \( g'' \) at \( \tilde{m}^2 = 0 \) to zero at \( \tilde{m}^2 \to \infty \) (curve III). Their left parts are functions linear in the variable \( \tilde{m}^2 \), which branch off from the point \( -\tilde{\kappa}^2 \). The point of intersection of the straight line \( y = \tilde{m}^2 - \tilde{\kappa}^2 \) (curve I) with the curve III gives the value of the particle mass in symmetrical phase (s-phase) \( \tilde{m}_s^2 \), and the intersection of the line \( y = -\tilde{m}^2 / 2 - \tilde{\kappa}^2 \) (curve II) with the curve III gives the value of the particle mass in asymmetrical phase (b-phase) \( \tilde{m}_b^2 \). It is obvious from Figure 1 that, depending on the value of the parameter \( -\tilde{\kappa}^2 \), for the s-phase there exists either one solution or none.
For the b-phase the solutions are either absent or there exist one or two solutions. Here we can mark out two particular cases. The first one takes place when the condition $\tilde{\kappa}^2 = -g''$ holds. Then there exist one solution for s-phase with zero mass, and two solutions for b-phase. One of them also corresponds to zero mass, and the other one to a finite mass

$$\tilde{m}_{b^*}^2 = \left(e^{1/2g''} - 1\right)^{-1}. \tag{37}$$

The second particular solution corresponds to the case when the straight line $y = -\tilde{m}^2/2 - \tilde{\kappa}^2$ is a tangent to the curve $y = g''f(\tilde{m}^2)$. The solution of the equation (36b) in the tangency point is determined by the equation

$$(2g'')^{-1} = \ln \left(1 + \tilde{m}_{b^*}^{-2}\right) - \left(1 + \tilde{m}_{b^*}^2\right)^{-1}, \tag{38}$$

and corresponding value of the mass parameter is given by the formula

$$\tilde{\kappa}_{*}^2 = -g''/ \left(1 + \tilde{m}_{b^*}^2\right). \tag{39}$$

The particle mass in the symmetrical phase $\tilde{m}_{s*}^2$ at the value of the mass parameter (39) is defined by the equation (36a), where it is necessary to assume $\tilde{\kappa}^2 = \tilde{\kappa}_{*}^2$. So, the following solutions of the equations (36) are possible:

a) $\tilde{\kappa}_s^2 < \tilde{\kappa}^2 < \infty$: there exists a solution for s-phase only ($\tilde{m}_{s*}^2 > 0$); the solutions for b-phase are absent;

b) $\tilde{\kappa}_s^2 = \tilde{\kappa}^2$: there exists a solution for s-phase ($\tilde{m}_{s*}^2 > 0$) and appears one solution for b-phase ($\tilde{m}_{b*}^2 > \tilde{m}_{s*}^2$);

c) $-g'' < \tilde{\kappa}^2 < \tilde{\kappa}_s^2$: there exist a solution for s-phase ($\tilde{m}_{s*}^2 > 0$) and two solutions for b-phase ($\tilde{m}_{b+}^2 > \tilde{m}_{b-}^2 > \tilde{m}_{s*}^2$), let us call the one with $\tilde{m}_{b+}^2$ a b_+ -phase, and the one with $\tilde{m}_{b-}^2$ - a b_- -phase;

d) $\tilde{\kappa}_s^2 = -g''$: there exist the solution $\tilde{m}_{s}^2 = 0$ for s-phase and two solutions for b-phase, the first one $\tilde{m}_{b}^2 = 0$, and the second one $\tilde{m}_{b*}^2 > 0$ (37);

e) $-\infty < \tilde{\kappa}^2 < -g''$: the solution for s-phase is absent and there exists the only solution $\tilde{m}_{b}^2 > 0$ for b-phase.

It is convenient to present the ranges of parameters, where different solutions exist, on the phase diagram (see Figure 2). The ordinates axis is the value of the interaction constant $g''$ and the absciss axis is the value of the mass parameter $\tilde{\kappa}^2$. In the area S lying rightwards from the curve OD, which is given by the equations (38), (39), there exists the symmetrical phase only. In the area B lying leftwards from the curve OA ($g'' = -\kappa^2$), there exists the asymmetrical phase only. In the area between the curves OA and OD, the existence of both s-phase and two b-phases with different values of the particle masses is possible. In order to determine which of three possible phases is stable in the indicated area, it is necessary to compare the energy densities of their
Figure 2: Areas of existence of phases of a real scalar field.

vacuums. The Hamiltonian (15), with account of the expansions (33), has the form

$$H_0 = \int dq q_0 a^+(q) a(q) + C + V (-4g' \chi^4 + \Omega),$$

(40)

where the constant

$$C = \frac{1}{2} \int dq dq' q_0 \delta (q - q') \delta (q - q')$$

appears when passing to a normal order of operators in the Hamiltonian $H_0$. So, from (40) it follows that the ground state energy of a scalar field (the vacuum) is determined by the formula $E_V = E_0 + C$, where

$$E_0 = V (-4g' \chi^4 + \Omega).$$

(42)

The energy $E_0$ is different in s- and b-phases. With regard to the formulas (25), (29) we find for s-phase

$$\frac{E_{0s}}{V} = -\frac{1}{g''} \varepsilon_0 \left( \tilde{m}_s^2 - \tilde{\kappa}^2 \right)^2,$$

(43)

and for b-phase

$$\frac{E_{0b}}{V} = -\frac{1}{g''} \varepsilon_0 \left[ \left( \tilde{m}_b^2 - \tilde{\kappa}^2 \right)^2 - 3 \tilde{m}_b^4 / 2 \right],$$

(44)

where $\varepsilon_0 = \Lambda^4 / 8\pi^2$. The constant $C$ is infinite, an with regard to the fact that $\delta (0) = V/(2\pi)^3$, it can be presented in the form

$$C = \frac{V}{2 (2\pi)^3} \int dq \sqrt{q^2 + m^2} = \frac{V J}{2 (2\pi)^3},$$

(45)

where $J \equiv \int dq \sqrt{q^2 + m^2}$. Since above there was introduced a cutoff at large momenta, the integral $J$ can be calculated if its regularization is carried out by the same way as in calculating the integral of the vacuum average in (34). By differentiating $J$ with
respect to \( m^2 \), we come to the obtained earlier integral (35), so that

\[
\frac{dJ}{dm^2} = \frac{1}{2} \int dq (q^2 + m^2)^{-1/2} = 4\pi \Lambda^2 \left[ 1 - \bar{m}^2 \ln \left( 1 + \bar{m}^{-2} \right) \right].
\]  \( (46) \)

By integrating the last relation we obtain

\[
C = V\varepsilon_0 \left[ -\bar{m}^4 \ln \left( 1 + \bar{m}^{-2} \right) + \ln \left( 1 + \bar{m}^2 \right) + \bar{m}^2 + c' \right],
\]  \( (47) \)

where \( c' \) is an integration constant, which is not dependent on the system parameters, and because of this fact can be put equal to zero. In this case the parameter \( C \) is always positive and raises monotonically with the increase of \( \bar{m}^2 \). So, the vacuum energy is determined by two contributions. The contribution defined by the parameter \( C \) \( (47) \) is conditioned by the necessity of normal ordering of operators in the free Hamiltonian \( H_0 \). This contribution into the vacuum energy is also present in the model of free fields and is always positive for Bose fields. The contribution of the term \( E_0 \) \( (42) \) is essentially determined by taking account of nonlinearity at the step of construction of the main approximation and, as one can see, gives a negative contribution into the vacuum energy. With regard to the equations (36) one can present the full density of the vacuum energy in the form

\[
\frac{E_V}{V} = \varepsilon_0 \left[ \frac{\kappa^2}{g''} \left( \bar{m}^2 - \bar{\kappa}^2 \right) + \ln \left( 1 + \bar{m}^2 \right) \right].
\]  \( (48) \)

The formula (48) relates both to symmetrical phase and to asymmetrical phases. Given that the particle mass is determined by the parameters of Lagrangian, formula (48) together with equations (36) determines the vacuum energy as a function of Lagrangian parameters \( \kappa^2 \) and \( g \). Also, the vacuum energy substantially depends on the cutoff parameter \( \Lambda \). In Figure 3 the dependence of the vacuum energy on the mass parameter is presented. The vacuum energy is positive at large positive values of the mass parameter and decreases with decrease of \( \bar{\kappa}^2 \). Under some negative value of \( \bar{\kappa}_0^2 \) the vacuum energy vanishes, and becomes negative at \( \bar{\kappa}^2 < \bar{\kappa}_0^2 \). In the range \(-g'' < \bar{\kappa}^2 < \bar{\kappa}_0^2 \) the existence of three phases is possible. They are symmetrical s-phase and two phases with broken symmetry \( b_+ \) and \( b_- \) that correspond to higher and lower masses. That one among them will be stable, which corresponds to the minimal vacuum energy. In the range \( \bar{\kappa}_f^2 < \bar{\kappa}^2 < \bar{\kappa}_*^2 \) the minimal is the energy of s-phase. In the point \( \bar{\kappa}^2 = \bar{\kappa}_f^2 \) (curve OF in Figure 2) a continuous phase transition occurs without a jump of energy, and under \( \bar{\kappa}^2 < \bar{\kappa}_f^2 \) stable is the phase with broken symmetry with a larger value of mass (\( b_+ \)-phase).
Figure 3: Dependence of the vacuum energy density of phases of a real scalar field on the square of the mass parameter.

PERTURBATION THEORY

While constructing the perturbation theory, one frequently postulates that the interaction Hamiltonian must be written in the form of a normal product [19]. In this way zero energy is at once excluded from the theory. However, as was noted above, the effects caused by zero oscillations of field can lead to observable effects and substantially influence the particle dynamics. Notable is the fact that, under the self-consistent mode of introduction of the one-particle states that is described in the present work, the interaction Hamiltonian [10], without supplementary suppositions about normal form of the initial Hamiltonian, takes the form of a normal product of the field operators:

\[ \hat{H}_C(x) = \frac{g}{4!} N(\hat{\psi}^4(x)) + \frac{g}{3!} \chi N(\hat{\psi}^3(x)), \] (49)

where normal products of the field operators taken in one point can be presented in the form

\[ N(\hat{\psi}^4(x)) = \hat{\psi}^4(x) - 6 \hat{\psi}^2(x)\rho + 3\rho^2, \]
\[ N(\hat{\psi}^3(x)) = \hat{\psi}^3(x) - 3 \hat{\psi}(x)\rho. \]

The averages over the vacuum state of the normal products introduced here are equal to zero:

\[ \langle 0|N(\hat{\psi}^4(x))|0 \rangle = \langle 0|N(\hat{\psi}^3(x))|0 \rangle = 0, \]
so that the condition \( \langle 0|\hat{H}_C(x)|0 \rangle = 0 \) is, obviously, satisfied. Thus, in this approach the normal form of the interaction Hamiltonian is not postulated \textit{ab initio} but arises as a consequence of the choice of the self-consistent field model as the main approximation, where the effects of zero fluctuations are already taken into account. As far as the interaction Hamiltonian (49) is a sum of two Hamiltonians, then the contribution of the \( n \)th order into \( S \)-matrix

\[ S = \sum_{n=0}^{\infty} S^{(n)} \] can be written in the form
\[ S^{(n)} = \frac{(-i)^n}{n!} \int dx_1 \ldots dx_n T \left[ \hat{H}_C^{(4)}(x_1) \ldots \hat{H}_C^{(4)}(x_n) + \hat{H}_C^{(3)}(x_1) \ldots \hat{H}_C^{(3)}(x_n) + \sum_{m=1}^{n-1} C_n^m \hat{H}_C^{(4)}(x_1) \ldots \hat{H}_C^{(4)}(x_m) \hat{H}_C^{(3)}(x_{m+1}) \ldots \hat{H}_C^{(3)}(x_n) \right], \]

where \( \hat{H}_C^{(4)}(x) = \frac{g}{4!} N(\hat{\psi}^4(x)) \), \( H_C^{(3)}(x) = \frac{g}{3!} \chi N(\hat{\psi}^3(x)) \), \( C_n^m \) – binomial coefficients, \( T \) – chronological operator. Perturbation theory is constructed in the standard way. Feinman diagrams in this case contain the elements presented in Figure 4. The solid line outgoing from the point \( x \), which describes the creation of a particle with 4-momentum \( q \), corresponds to the expression \( f_q(x) = e^{-iqx}/(2\pi)^{3/2} \sqrt{2q_0} \) (Figure 4a), and the solid line incoming into the point \( x \), which describes the annihilation of a particle with 4-momentum \( q \), corresponds to the expression \( f^*_q(x) = e^{iqx}/(2\pi)^{3/2} \sqrt{2q_0} \) (Figure 4b). The solid line connecting the points \( x_1 \) and \( x_2 \) (Figure 4c) corresponds to the expression \(-G^{(0)}(x_1 - x_2)\), where Green function has the form

\[ G^{(0)}(x) = \lim_{\varepsilon \to 0^+} \frac{i}{(2\pi)^4} \int \frac{dq e^{iqx}}{q^2 + m^2 - i\varepsilon}. \]

Let us note that, according to (51), \( G^{(0)}(0) = -\rho \). The condensedate function \( i\chi \) is put in correspondence to a wavy line with a cross (Figure 4d), and the vertex corresponds to a factor \(-ig\) (Figure 4e). There are two types of elementary vertices. Either four solid lines or three solid and one wavy line can converge in a vertex. If a permutation of \( n \) inner lines does not change the form of the diagram, then when writing a matrix element one should use the factor \( 1/n! \). As far as the interaction Hamiltonian in the proposed variant of perturbation theory has the normal form (49), then the diagrams that contain “loops”, i.e. solid lines whose ends converge in the same vertex, are absent (Figure 4f). Since the factor \( \rho \) is matched to a loop, then it is obvious that the diagrams with loops were in fact taken into account when constructing the main approximation and calculating the particle mass. The circumstance that the interaction \( g \) is already to a certain extent accounted for when constructing the main approximation and the one-particle states, leads to that the perturbation theory formulated in this way remains correct also in the case when the interaction constant is not small. Thus, the proposed method of construction of the perturbation theory can prove to be effective for description of strongly interacting particles. The fact that the area of applicability of the perturbation theory constructed on the basis of the presented approach is much larger than the area of applicability of the standard theory is shown on the example of anharmonic oscillator [20].

It is of interest to compare the proposed variant of the perturbation theory with
Figure 4: Elements of Feynman diagrams of a real scalar field with broken symmetry.

Figure 4: Elements of Feynman diagrams of a real scalar field with broken symmetry.

The perturbation theory constructed on the basis of the Goldstone approach \[6\]. The free particle Lagrangian in the Goldstone model is obtained by carrying out a shift of the field by the value which is found from the condition of minimum of the potential energy of a classical scalar field \[6, 8\]. The Goldstone result can also be obtained by another way, directly from considering a quantized field. For this, in the Lagrangian \[11\] we will go over to a new field by performing the substitution \( \hat{\phi}(x) = \hat{\psi}(x) + \chi \) and require that the coefficient at appeared linear in the field \( \hat{\psi}(x) \) term turns into zero. This leads to the equation that determines the parameter \( \chi \):

\[
\chi \left( \kappa^2 + \frac{g}{6} \chi^2 \right) = 0.
\]

The equation \(52\) differs from \(27\) by the circumstance that the expression in parentheses does not contain a term with the vacuum average of the square of the over-condensate field operator \( \rho \), i.e. the effects caused by vacuum fluctuations are not taken into account. At \( \kappa^2 > 0 \) (remember that \( g > 0 \)) the equation \(52\) has the only solution \( \chi = 0 \). At \( \kappa^2 < 0 \) the solution \( \chi = 0 \) does not describe real particles, since the square of their mass is negative. The particles with a correct sign of the square of mass are described by the second solution of \(52\), \( \chi^2 = -6\kappa^2/g \), that coincides with the result obtained in another approach by Goldstone \[6\], and the square of mass of such particles is \( m^2 = -2\kappa^2 \). Let us remark that here the masses of free particles are determined by only the mass parameter and do not depend on both vacuum fluctuations and the interaction constant. The interaction Hamiltonian in this case coincides with the expression \(49\), if the normal product sign is dropped there, and consequently we must take into account the diagrams containing loops in all orders of the perturbation
theory. Thus, the construction of the perturbation theory on the basis of the choice of the self-consistent field model as the main approximation lets us significantly simplify the perturbation theory, because a very large number of diagrams with loops are already taken into account in the zero approximation.

QUASI-AVERAGES. EXACT GREEN FUNCTIONS

Since the zero approximation Hamiltonian (9) contains the terms linear in the field operators $\hat{\varphi}(x)$, so that its symmetry relative to the operation $\varphi \rightarrow -\varphi$ is broken, there follows the distinction from zero of the vacuum average of the field operator, $\langle 0 | \hat{\varphi} | 0 \rangle = \chi \neq 0$. The interaction Hamiltonian (10) is not invariant under the operation $\varphi \rightarrow -\varphi$ too; however, the full Hamiltonian $H = H_0 + H_C$ remains invariant to this operation. It is easy to show that the average over the vacuum state of a system with the invariant Hamiltonian $H$ equals zero. Indeed, let $|0\rangle$ be the vacuum vector of the system $H$, so that $H |0\rangle = E_0 |0\rangle$, where $E_0$ is the vacuum energy. Also, let Hamiltonian be invariant under some unitary transformation $U H U^+ = H$. Then, obviously, $U |0\rangle$ is also the state vector corresponding to the energy $E_0$. We consider the vacuum state with the energy $E_0$ as unique, i.e. $U |0\rangle$ coincides with $|0\rangle$: $U |0\rangle = e^{i\alpha} |0\rangle$ except for a phase factor. In the considered case of a scalar field with broken symmetry the unitary transformation is such that $U \varphi U^+ = -\varphi$. In consequence of invariance of the Hamiltonian and uniqueness of the vacuum we have $(0 | \varphi | 0) = -(0 | U^+ \varphi U | 0) = -(0 | \varphi | 0) = 0$. So, in order to obtain a state with non-zero average, $(0 | \varphi | 0) \neq 0$, one should break the symmetry of the initial Hamiltonian. Such an approach to the description of states with broken symmetry in statistical mechanics in the framework of the concept of quasi-averages was developed by Bogoljubov [16]. An idea to use a model of self-consistent field for determining quasi-averages was proposed in [21]. Let us agree to use for the description of systems with broken symmetries a Hamiltonian of a more common than $H = H_0 + H_C$ form, namely $H_\nu = H_0 + \nu H_C$, where $\nu$ is a real positive parameter. If $\nu = 0$, then $H_\nu$ coincides with the self-consistent field Hamiltonian, but if $\nu = 1$, then $H_\nu$ coincides with the full Hamiltonian $H$ [5]. By changing the parameter $\nu$ from zero to unity, we “switch on” the interaction between the particles defined in the self-consistent field model. If the parameter $\nu$ is not equal to unity, but is close to it as is wished, then the Hamiltonian $H_\nu$ is as wished close to the Hamiltonian $H$. However, there is a principal difference between these Hamiltonians in the situation when the symmetry is violated i.e. the symmetry of $H_0$ is lower than the symmetry of the initial Hamiltonian $H$. In this case, at $\nu \neq 1$ the Hamiltonian $H_\nu$ contains an additional term that lowers its symmetry to the symmetry of the Hamiltonian $H_0$, that allows us to describe the states with the symmetry lower than that of $H$. The ground state vector, determined by the Schrödinger equation $H_\nu |0_\nu\rangle = E_{0_\nu} |0_\nu\rangle$, as well as the vacuum energy and also the field operators in Heisenberg picture depend on the
parameter $\nu$. We will define the quasi-average over the vacuum state of the operator $A$ by the expression

$$
\langle A \rangle_0 \equiv \lim_{\nu \to 1} \langle 0_\nu | A | 0_\nu \rangle .
$$

(53)

The passage to the limit $\nu \to 1$ must be performed after the passage $V \to \infty$ is done. The limit in the right side of the definition (53), generally speaking, may not coincide with the average $\langle 0 | A | 0 \rangle$ calculated over the system vacuum state with the Hamiltonian $H_\nu$ at $\nu = 1$. So, the quasi-average (53) can be distinct from zero, even if, by virtue of the symmetry properties of the Hamiltonian $H$, the average $\langle 0 | A | 0 \rangle$ is equal to zero.

Now let us give the equations for exact Green functions of a real scalar field. The exact $n$-point over-condensate Green function is determined by the relation

$$
G^{(n)}(x_1 \ldots x_n) = i^n \langle T[\psi(x_1) \ldots \psi(x_n)]\rangle_0 ,
$$

(54)

where we should understand averaging in the sense of quasi-average (53). Let us note that for the states with broken symmetries the Green functions with odd $n$ can be distinct from zero, too. We write the full Hamiltonian in the form of a sum of self-consistent and correlation Hamiltonians in terms of operators in Heisenberg picture. By using the equations of motion for the field operator and the conjugated momentum, and also the commutation relations, we obtain the following equations for the one-point and two-point Green functions:

$$
\left( \frac{\partial^2}{\partial x^2} - m^2 + \frac{g}{2} \rho \right) G^{(1)}(x) + \frac{g}{2} i \chi (G^{(2)}(x) + \rho) + \frac{g}{3!} G^{(3)}(x) = 0 ,
$$

(55)

$$
\left( \frac{\partial^2}{\partial x_1^2} - m^2 + \frac{g}{2} \rho \right) G^{(2)}(x_1,x_2) - i \frac{g}{2} \chi (G^{(3)}(x_1,x_2) + \rho G^{(1)}(x_2)) + \frac{g}{3!} G^{(4)}(x_1,x_1,x_2) = -i \delta(x_1 - x_2) ,
$$

(56)

where Green functions at coinciding arguments are determined by the relation $G^{(n)}(x) \equiv \langle \psi^n(x) \rangle_0$. By introducing the vertex functions $\Gamma^{(3)}(x_1 x_2 x_3), \Gamma^{(4)}(x_1 x_2 x_3 x_4)$, we can present the three- and four-point Green functions in another form:

$$
G^{(3)}(x_1 x_2 x_3) = G^{(2)}(x_1 x_2) G^{(1)}(x_3) + G^{(2)}(x_1 x_3) G^{(1)}(x_2) + G^{(2)}(x_2 x_3) G^{(1)}(x_1) + \int dx'_1 dx'_2 dx'_3 \Gamma^{(3)}(x'_1 x'_2 x'_3) G^{(2)}(x_1 x'_1) G^{(2)}(x_2 x'_2) G^{(2)}(x_3 x'_3) ,
$$

(57)

$$
G^{(4)}(x_1 x_2 x_3 x_4) =
\begin{align*}
&= G^{(2)}(x_1 x_2) G^{(2)}(x_3 x_4) + G^{(2)}(x_1 x_3) G^{(2)}(x_2 x_4) + G^{(2)}(x_1 x_4) G^{(2)}(x_2 x_3) + \\
&+ \int dx'_1 dx'_2 dx'_3 dx'_4 \Gamma^{(4)}(x'_1 x'_2 x'_3 x'_4) G^{(2)}(x_1 x'_1) G^{(2)}(x_2 x'_2) G^{(2)}(x_3 x'_3) G^{(2)}(x_4 x'_4) .
\end{align*}
$$

(58)
With account of the last relations, (55) and (56) take the following form:

\[
\frac{\partial^2}{\partial x^2} - m^2 + \frac{g}{2} \left( G^{(2)}(x) + \rho \right) G^{(1)}(x) + i \frac{g}{2} \chi \left( G^{(2)}(x) + \rho \right) + \\
+ \frac{g}{3!} \int dx_1^* dx_2^* dx_3^* \Gamma^{(3)} \left( x_1^* x_2^* x_3^* \right) G^{(2)}(x x_1^*) G^{(2)}(x x_2^*) G^{(2)}(x x_3^*) = 0,
\]

(59)

\[
\frac{\partial^2}{\partial x_1^2} - m^2 + \frac{g}{2} \left( G^{(2)}(x_1) + \rho \right) G^{(1)}(x_2) - \\
- i \frac{g}{2} \chi \left[ G^{(2)}(x_1) + \rho \right] G^{(1)}(x_2) - i g \chi G^{(2)}(x_1 x_2) G^{(1)}(x_1) - \\
- i \frac{g \chi}{2} \int dx_1^* dx_2^* dx_3^* \Gamma^{(3)} \left( x_1^* x_2^* x_3^* \right) G^{(2)}(x_1 x_1^*) G^{(2)}(x_1 x_2^*) G^{(2)}(x_2 x_3^*) + \\
+ \frac{g}{3!} \int dx_1^* dx_2^* dx_3^* dx_4^* \Gamma^{(4)} \left( x_1^* x_2^* x_3^* x_4^* \right) G^{(2)}(x_1 x_1^*) G^{(2)}(x_1 x_2^*) G^{(2)}(x_1 x_3^*) G^{(2)}(x_2 x_4^*) = \\
= - i \delta(x_1 - x_2).
\]

(60)

The equations for Bose fields under broken symmetry have a more complex structure than those for Fermi fields, owing to the presence of Green functions with odd number of field operators.

**CONCLUSION**

On the example of a real non-linear scalar field with the interaction $\varphi^4$, a method of systematic description of states of quantized fields with spontaneously broken symmetry is proposed. The approach is based upon the use of the self-consistent field model for the description of free particles. This model accounts for the influence on the dynamics of the particles of the average field formed by vacuum fluctuations. As distinct from the Goldstone approach [6] to whose results we come in the present paper when neglecting the contribution of the vacuum average of the square of field operator, the mass of free particles is determined not only by the mass parameter entering into the initial Lagrangian, but also by the interaction constant and the vacuum average from the square of the field operators. As far as the integral which appears in calculating the vacuum average diverges at large momenta, there arises a need of introducing into the theory of a cutoff parameter $\Lambda$. However, the non-linear equations for dimensionless masses $m^2/\Lambda^2$ derived in the work already do not contain explicitly this cutoff parameter, and the dimensionless particle masses are determined in a unique way. There may exist not one, but several solutions of the equations for masses at the same values of parameters of the initial Lagrangian. These solutions correspond to different states (phases) of the considered system. Among the phases, there may exist both symmetrical phases and those with broken symmetries. The most stable among them is the phase with the lowest density of the vacuum energy. The density of the vacuum energy of a scalar field is determined by both the positive contribution...
of zero fluctuations and the negative contribution conditioned by non-linear effects, so that the full density of the vacuum energy of the Bose field can be either positive or negative.

An important advantage of the present approach is the fact that the choice of the self-consistent field model for description of free particles automatically brings us to the normal form of the interaction Lagrangian. Owing to this fact, in all orders of the perturbation theory all diagrams containing loops are excluded from the consideration, and the structure of the perturbation theory is substantially simplified. Also, the difficulties, conditioned by the diagrams which contain the loops connected with one external line, do not appear [22]. The absence of loop diagrams in higher orders of the perturbation theory means that their contribution is already taken into account when constructing the zero approximation Lagrangian that describes independent particles in the self-consistent field model. As is known [19], the theories of scalar field with interactions $\varphi^3$ and $\varphi^4$ are renormalizable. The question of elimination of divergences in higher orders of the perturbation theory, that is constructed on the basis of the presented approach, is not discussed: this should be a subject of a separate study. It should be noted, however, that as far as the cutoff parameter is a substantial parameter of the theory that eliminates the ultraviolet divergences, then, apparently, the proposed approach can be also applied to non-renormalizable theories.

In the present work it is shown that for calculation of averages over the vacuum state of the exact Hamiltonian it is natural to use the concept of quasi-averages that was earlier developed in the framework of statistical mechanics [16, 21]. The equations for exact one- and two-point Green functions are obtained.

The proposed method of description of the states with broken symmetries is quite universal, because it lets consider on the basis of the unified approach both Bose and Fermi fields, and also describe the states with breaking of different symmetries, including several symmetries at once [23].

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