A "METRIC" COMPLEXITY FOR WEAKLY CHAOTIC SYSTEMS

STEFANO GALATOLO

Abstract. We consider the number of Bowen sets which are necessary to cover a large measure subset of the phase space. This introduces some complexity indicator characterizing different kind of (weakly) chaotic dynamics. Since in many systems its value is given by a sort of local entropy, this indicator is quite simple to be calculated. We give some example of calculation in nontrivial systems (interval exchanges, piecewise isometries e.g.) and a formula similar to the Ruelle-Pesin one, relating the complexity indicator to some initial condition sensitivity indicators playing the role of positive Lyapunov exponents.

1. Introduction

Many techniques and results have been developed for the study of smooth hyperbolic systems (systems where the dynamics is given by smooth functions and distances between typical nearby initial conditions expand or contract exponentially fast). In recent times, systems whose dynamics is not regular (sometimes discontinuous) and/or not hyperbolic (no exponential contraction/expansion) are more and more important in various kind of applications (interval exchanges, piecewise isometries: \cite{23}, \cite{1}, Hamiltonian systems with stable islands: \cite{3}, \cite{2}, symbolic systems and automata such as substitutions and similar).

Such systems often have zero entropy, but their dynamics is far to be simple and predictable because there is still a "weak" initial condition sensitivity "slowly" separating nearby starting orbits.

The need to provide complexity indicators which can describe and quantify this "weakly" chaotic behavior lead in the mathematical literature to many definitions and different notions of complexity (or generalized entropies).

The first natural attempt is to repeat the same construction leading to the K-S entropy (considering first a partition of the space, considering the induced symbolic system, and so on...) replacing the usual formula for the Shannon entropy of a symbolic system (\(\sum p_i \log p_i\)) with a different one (here the physical literature is huge, but there are few rigorous results, see e.g. \cite{10}, \cite{32}, \cite{33}). This kind of construction often has the problem that the resulting indicator is not continuous with respect to change of partitions (see, \cite{10}, \cite{32}) thus its physical meaning is compromised and the calculation of the suprema over all partition is difficult.

To overcome this difficulty a more refined construction can be performed (\cite{13}, \cite{21}). This lead to a more stable definition and to an invariant which can be calculated and has nontrivial values on interesting examples. This indicator works in...
the measure-theoretic, ergodic framework and is invariant under measure preserving transformations.

Another, topological approach considers the number of essentially different orbits (orbits whose distance at a certain time is greater than a given resolution \( \epsilon \)) which appear in the system (\([9, 26]\), see also \([3, 2]\) for many variants on this theme) and consider how this number increases with time. This lead to topological complexities which generalizes the topological entropy. The disadvantage of a purely topological approach can be understood comparing topological entropy with Kolmogorov-Sinai entropy. The presence of a physical invariant measure in system gives more weight to the most frequent (and physically relevant) configurations, neglecting the least relevant ones, which on the other hand are not neglected in the topological approach.

Another approach to define complexity is to consider the complexity of single orbits of the system (see e.g. \([12, 26]\)), this complexity indicator is then local, the global behavior can be given by the complexity of a typical orbit, or the average with respect to some invariant measure. The orbit complexity is given by the amount of information (algorithmic information) which is necessary to describe the orbit up to some give accuracy. If the accuracy is given by some partition or by an open cover the notion is more measure theoretic or topology oriented. In this approach the complexity indicators can be easily calculated in many interesting examples, and there are connections with many other features of chaotic dynamics, such as dimension of attractors and so on (see, for example \([35, 24, 6, 27]\)).

In this paper we follow an approach which defines a global indicator of complexity and which is not only topological or measure theoretic. We will define some (more rigid) indicators which are invariant under morphisms which are both continuous and measure preserving.

Many interesting physical coordinate change are continuous and they preserve some physical measure (for example if we are observing and reconstructing a system through some continuous observable, as in the nonlinear time series framework, see e.g. \([20, 29]\)).

We will construct a complexity indicator which is invariant for this kind of morphisms and it is easy to be calculated. Moreover it has connections with the other features of chaos.

Roughly speaking we will consider the number of "important", essentially different orbits which appear in the system. The importance will be given by the measure \( \mu \). More precisely, we will consider the number of Bowen sets which are necessary to cover a large part of \( \mu \) and we will consider how this number increases with time. We will see that under mild assumptions, this indicator is equivalent to the rate of decreasing of the measure of a typical Bowen set (a sort of extension of the Brin-Katok theorem \([11]\)). This will allow an easy calculation of the complexity indicator in nontrivial cases, as interval exchanges, piecewise isometries, the logistic map, and some more examples, which are listed in section 3. In section 4 we will consider a set of numbers describing the geometrical features of the Bowen set, these numbers plays the role of the Lyapunov exponents, describing initial condition sensitivities at different directions and allowing a result similar to the Ruelle-Pesin formula.
2. A "METRIC" COMPLEXITY

We consider a system \((X, T, \mu)\) of the following type: \(X\) is a metric space equipped with a distance \(d\). The dynamics is given by a Borel map \(T : X \to X\) and \(\mu\) is invariant for \(T\).

Let us consider the Bowen set

\[
B(n, x, \epsilon) = \{y \in X : d(T^i(y), T^i(x)) \leq \epsilon \ \forall i \ \text{s.t.} \ 0 \leq i \leq n\}.
\]

\(B(n, x, \epsilon)\) is the set of points “following” the orbit of \(x\) for \(n\) steps at a distance less than \(\epsilon\). As the nearby starting orbits of \((X, T)\) diverges the set \(B(n, x, \epsilon)\) will be smaller and smaller as \(n\) increases. If two points are in the same set we can think that their orbits are similar (up to a resolution given by \(\epsilon\), for \(n\) steps) if two points are in different sets, their orbits are essentially different\(^1\).

We want to consider the number of Bowen sets which is necessary to cover a large (according to the measure \(\mu\)) part of \(X\). This counts how many different "important" orbits appears in the system. Here the notion of importance if provided by the measure \(\mu\), which will give different weight to different parts of \(X\). This complexity depends both on the metric, and ergodic features of the system and the notion is physically relevant when we consider a physical invariant measure. Hence this notion is related to the metric of the system (which induces the Lesbegue measure, which induces the physical measure, see e.g. [34]) for this reason we call it "metric complexity".

Let us hence consider the following

\[
N(n, \epsilon, \epsilon') = \min\{k \in \mathbb{N} | \exists x_1, ..., x_k, \mu(\cup_{0 \leq i \leq k} B(n, x_i, \epsilon)) \geq 1 - \epsilon'\}
\]

that is the number of Bowen sets that is necessary to cover a subset of \(X\) whose measure is bigger than \(1 - \epsilon'\). We want to consider the asymptotic growing rate of \(N(n, \epsilon, \epsilon')\) as \(n\) increases, when \(\epsilon\) and \(\epsilon'\) are small.

To formalize this, for each monotonic function \(f(n)\) with \(\lim_{n \to \infty} f(n) = \infty\) we define an indicator by comparing the asymptotic behavior of \(\log(N(n, \epsilon, \epsilon'))\) with \(f^2\). Hence let us consider

\[
h_{\epsilon, \epsilon'}^f(X, T, \mu) = \limsup_{n \to \infty} \frac{\log(N(n, \epsilon, \epsilon'))}{f(n)}
\]

this quantity is monotonic in \(\epsilon\) and \(\epsilon'\) and hence we can consider the limits

\[
h^f(X, T, \mu) = \lim_{\epsilon' \to 0} \lim_{\epsilon \to 0} h_{\epsilon, \epsilon'}^f(X, T, \mu).
\]

We will see (see proposition \[2\]) that when \(f\) is the identity (\(f(n) = n\)), the quantity \(h^{id}(X, T, \mu)\) equals the Kolmogorov-Sinai entropy for a large family of systems.

Let us now consider the invariance properties of \(h^f\) under isomorphisms of systems.

\(^1\)Counting the number of essentially different orbits needed to cover the whole space \(X\) leads to the notion of topological entropy and to its generalizations which can be called topological complexity of a system.

\(^2\)From now on, in the definition of indicators \(f\) is always assumed to be monotonic and tends to infinity.
Theorem 1. If \((X, T, \mu), (Y, T', \mu')\) are dynamical systems over compact metric spaces \((X, d), (Y, d')\). Let \(\phi\) be a measure preserving homeomorphism such that the following diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & Y \\
\downarrow T & & \downarrow T' \\
X & \xrightarrow{\phi} & Y
\end{array}
\]

commutes, then \(h^f(X, T, \mu) = h^f(Y, T', \mu')\).

Proof. Let us call \(N_1(n, \epsilon, \epsilon')\) the number of Bowen sets that is necessary to cover a large subset of \(X\) as above, and \(N_2(n, \epsilon, \epsilon')\) be the number of Bowen sets that is necessary to cover a large subset of \(Y\). Since the spaces are compact and \(\phi\) is continuous then it is uniformly continuous. Let \(g : \mathbb{R} \to \mathbb{R}\) such that \(d(x_1, x_2) \leq g(\epsilon)\) (with \(x_i \in X\)) implies \(d'(\phi(x_1), \phi(x_2)) \leq \epsilon\). For each \(n\) it holds \(f(B(x, n, g(\epsilon))) \subset B(f(x), n, \epsilon)\) then let us suppose that \(\{B(n, x_1, g(\epsilon)), ..., B(n, x_k, g(\epsilon))\}\) is a minimal cover of a large set \(A \subset U_{0<\epsilon \leq k}B(n, x_i, g(\epsilon))\) with measure \(\mu(A) = 1 - \epsilon'\), this implies that \(f(A) \subset U_{0<\epsilon \leq k}B(n, f(x_i), \epsilon)\). We recall that \(\mu(A) = \mu'(f(A))\). Hence \(N_2(n, \epsilon, \epsilon') \leq N_1(n, g(\epsilon), \epsilon')\). This implies that \(h^{f, \epsilon}_g(X, T, \mu) \geq h^{f, \epsilon'}_g(Y, T', \mu')\) and \(h^f(X, T, \mu) \geq h^f(Y, T', \mu')\). Similarly we can prove the reverse inequality. \(\Box\)

It is useful to consider a version of the Brin-Katok local entropy \([11]\): let us define

\[
\overline{BK}^f(x, \epsilon) = \limsup_{n \to \infty} \frac{-\log(\mu(B(n, x, \epsilon)))}{f(n)},
\]

\[
\underline{BK}^f(x, \epsilon) = \liminf_{n \to \infty} \frac{-\log(\mu(B(n, x, \epsilon)))}{f(n)}.
\]

When \(f(n) = n\) is the identity then \(BK^f\) is the Brin-Katok local entropy. In \([11]\) it is proved that if the system is ergodic \(BK^d(x) = BK^{\text{K-S}}(x) = h_\mu(T)\) (the K-S entropy) for almost each \(x \in X\). Hence \(BK^d(x)\) and \(BK^{\text{K-S}}(x)\) are almost everywhere equal and they are invariant under \(T\).

In the general case however the invariance under \(T\) holds under some mild conditions

Proposition 1. If \(T\) is such that

- i) Almost each point \(x\) has a small neighborhood \(U\) such that \(T|_U : U \to T(U)\) is an homeomorphism
- ii) For each measurable \(A\) it holds \(\mu(T(A)) \leq K\mu(A)\) for some fixed constant \(K\)

then \(BK^f(x) = BK^f(T(x))\) and \(BK^f(x) = BK^f(T(x))\) for \(\mu\) almost each \(x\).

Proof. First let us notice that

\[
B(n, x, \epsilon) = B(x, \epsilon) \cap T^{-1}(B(n-1, T(x), \epsilon))
\]

then it is clear that \(T\) preserves \(\mu\) \(\mu(B(n, x, \epsilon)) \leq \mu(B(n-1, T(x), \epsilon))\) and then \(BK^f(x) \geq BK^f(T(x))\) and \(BK^f(x) \geq BK^f(T(x))\).

For the other inequality, we have that \(T\) is a.e. a local homeomorphism, let \(x\) be a typical point and \(\epsilon' < \epsilon\) such that \(B(T(x), \epsilon') \subset T(B(x, \epsilon))\). Obviously \(B(n-1, T(x), \epsilon') \subset B(T(x), \epsilon')\). Now \(B(n-1, T(x), \epsilon') \subset T(B(n, x, \epsilon))\), this
is true because if $y \in B(n - 1, T(x), \epsilon')$ then there is some $z \in B(x, \epsilon)$ with $T(z) = y$. Now, if $z$ is such that $d(x, z) < \epsilon$, $T(z) \in B(n - 1, T(x), \epsilon')$ with $\epsilon' < \epsilon$ then $d(T^i(z), T^i(x)) < \epsilon$ for each $0 \leq i \leq n$ and then $z \in B(x, \epsilon)$. By ii) $\frac{1}{n} \mu(B(n, x, \epsilon)) \geq \mu(T(B(n, x, \epsilon))) \geq \mu(B(n - 1, T(x), \epsilon'))$, and then $BK(x) \leq BK(T(x))$. \Box

The relation between $BK^f$ and $h^f$ in general is quite natural

**Proposition 2.** If $\overline{BK}^f(x) = BK^f(x) = BK^f$ almost everywhere then $BK^f = h^f(X, T, \mu)$.

**Proof.** Since $\overline{BK}(x) = BK(x)$ almost everywhere, for each $\varepsilon > 0$ there is an $\overline{\pi}$ and a set $A_{\varepsilon, \overline{\pi}}$ such that for each $n \geq \overline{\pi}$ and $x \in A_{\varepsilon, \overline{\pi}}$

$$\mu(B(n, x, \epsilon)) \leq 2^{-(BK(x)g(\epsilon) - \epsilon)f(n)}$$

for some $g$, such that $g(\epsilon) \rightarrow 1$ as $\epsilon \rightarrow 0$. Moreover the sequence $A_{\varepsilon, \overline{\pi}}$ is increasing as $\overline{\pi}$ increases and $\mu(A_{\varepsilon, \overline{\pi}}) \rightarrow 1$ as $\overline{\pi} \rightarrow \infty$. Let us fix an arbitrary small $\varepsilon$ and $\overline{\pi}$ such that

$$\mu(A_{\varepsilon, \overline{\pi}}) \geq \frac{3}{4}.$$ 

Now, let us consider $n \geq \overline{\pi}$ and a set $\{B(n, x_1, \frac{\varepsilon}{2}), \ldots, B(n, x_k, \frac{\varepsilon}{2})\}$ covering a big subset of $X$ as in the definition of $h^f(X, T, \mu)$. More precisely, we can suppose that

$$\mu(\cup_{0 \leq i \leq k} B(n, x_i, \frac{\varepsilon}{2})) \geq \frac{\overline{\pi}}{2}$$

and hence $\mu(\cup_{0 \leq i \leq k} B(n, x_i, \frac{\varepsilon}{2})) = A_{\varepsilon, \overline{\pi}} \geq \frac{3}{4}$.

Now we remark that if $B(n, x_1, \frac{\varepsilon}{2}) \cap A_{\varepsilon, \overline{\pi}} \neq \emptyset$ then there is $x \in A_{\varepsilon, \overline{\pi}}$ such that $B(n, x_1, \frac{\varepsilon}{2}) \subset B(n, x, \epsilon)$, hence $\mu(B(n, x_1, \frac{\varepsilon}{2})) \leq \mu(B(n, x, \epsilon)) \leq 2^{-1}B(K(x)g(\epsilon) - \epsilon)f(n)$.

Since each of these sets $B(x, n, \frac{\varepsilon}{2})$ has small measure and their union has measure greater than $\frac{1}{2}$ then its number must be greater than $2^{(BK(x)g(\epsilon) - \epsilon)f(n)}$ giving that $h^f_{\frac{\varepsilon}{2}, x}(X, T, \mu) \geq BK^f(x, \epsilon)$ almost everywhere, hence $h^f(X, T, \mu) \geq BK^f(x, \epsilon)$ a.e.

For the other inequality, similar as before for each $\varepsilon$ there is an $\overline{\pi}$ and a set $B_{\varepsilon, \overline{\pi}}$ such that for each $n \geq \overline{\pi}$ and $x \in B_{\varepsilon, \overline{\pi}}$ it holds $\mu(B(n, x, \epsilon)) \geq 2^{-(BK(x)g(\epsilon) + \epsilon)f(n)}$ for some $g$, such that $g(\epsilon) \rightarrow 1$ as $\epsilon \rightarrow 0$ and $\mu(B_{\varepsilon, \overline{\pi}}) \rightarrow 1$ as $\overline{\pi} \rightarrow \infty$. Let us consider $C = \{B(n, x_1, \epsilon), \ldots, B(n, x_k, \epsilon)\}$ such that $C$ is made of disjoint Bowen sets, each $x_i$ is contained in $B_{\varepsilon, \overline{\pi}}$ and $C$ is maximal, in the sense that $\forall x \in B_{\varepsilon, \overline{\pi}}$ then $B(n, x, \epsilon) \cap B(n, x_i, \epsilon) \neq \emptyset$ for some $B(n, x_i, \epsilon) \in C$. The set $C$ is finite because by definition of $B_{\varepsilon, \overline{\pi}}$ each $B(n, x_i, \epsilon) \in C$ has a measure greater than $2^{-(BK(x)g(\epsilon) - \epsilon)f(n)}$ and their total measure must be less than 1. Thus the number of such set is less or equal than $2^{(BK(x)g(\epsilon) + \epsilon)f(n)}$. Now we remark that if $C$ is as before, then $C_2 = \{B(n, x_1, 2\epsilon), \ldots, B(n, x_k, 2\epsilon)\}$ is a cover of $B_{\varepsilon, \overline{\pi}}$. Then we proved that there is a cover of some big as wanted subset (with measure let us say, greater than $1 - \epsilon'$) of $X$ made with no more than $2^{(BK(x)g(\epsilon) + \epsilon)f(n)}$ Bowen sets and this proves $h^f_{\epsilon', 2\epsilon}(X, T, \mu) \leq BK^f(x, \epsilon)$ and $h^f(X, T, \mu) \leq BK^f(x) a.e.$ \Box

**Remark 1.** If $\overline{BK}^f \geq BK^f(x) \geq BK^f(x) \geq BK^f$ almost everywhere, the above proof gives that $BK^f \leq h^f(X, T, \mu) \leq \overline{BK}^f$.

---

3This happen for example if $(X, T, \mu)$ is ergodic and it satisfies the assumptions of proposition
For a natural example where $BK^f(x) > BK^f(x)$ a.e. see section 5. Proposition 2 allows to easily calculate $h^f(X,T,\mu)$. If the assumptions of the proposition are verified, instead to construct a global cover of the system by Bowen sets we only need to look the behavior of the measure of a typical Bowen set. To give an example of nontrivial calculation, in next section we calculate the complexity of typical Interval Exchange Transformations, the Logistic map the Feigenbaum point, the Casati-Prose map.

3. Some example

As said before, since the assumptions of Proposition 2 are mild and easy to be verified we can apply it in many cases and estimate $h^f(X,T,\mu)$ by $BK^f(x,\epsilon)$, which is an estimation of initial condition sensitivity at typical points. We give some example of this application on some non trivial examples.

3.1. General piecewise isometries. Let consider a nontrivial family of systems for which we can have an upper estimation for the complexity. Piecewise Isometries (PI) are simple families of dynamical systems that show dynamical complexity while not being hyperbolic in any senses; classical examples in one dimension are, interval exchange transformations (IETs, see also below). PIs have also been found to arise in several applications such as in digital filter models and billiard systems (see [5], [28]).

It is conjectured that the symbolic dynamics of a PI has polynomial complexity (in the sense that the number of different names of subcylinders appearing in the dynamics grow polynomially with the length, for some works on this direction see e.g. [1], [13], [19]). We give an upper bound of our definition of complexity. This correspond to a polynomial bound on the growth of Bowen sets necessary to cover the invariant measure (instead of cylinders).

Let us recall briefly the class of systems we are considering. Let $X = \mathbb{R}^n$, Let us suppose that $P_1, ..., P_m$ is a measurable partition of $X$. A piecewise isometry $T : X \to X$ is a map defined in the following way: let $A_1, ..., A_m : X \to X$ be a set of isometries, then $T(x) = A_i(x) \iff x \in P_i$. The sets $P_i$ are called atoms and most of the literature consider piecewise linear atoms. We will consider a more general situation.

In our piecewise isometries, the only source of initial condition sensitivity is the presence of discontinuities at the boundary of atoms. Let $Y = \bigcup_{i \leq m} \partial P_i$. If for each $i \leq n$ it holds $d(T^i(x),Y) \geq r$ then we know that the Bowen set satisfies $B(x,\epsilon) \supseteq B_r(x)$ for each $\epsilon > r$. Hence the initial condition sensitivity depends on the speed a typical orbit approaches the discontinuity set $Y$.

To estimate this we will use the following simple result ([17] Lemma 2): given $Y \subset X$, let us define the $r$ neighborhood of $Y$ by

$$B_r(Y) = \{ x \in X, d(x,Y) < r \}$$

and consider $d_\mu(Y) = \liminf_{\epsilon \to 0} \frac{\log(\mu(B_\epsilon(Y)))}{\log(\epsilon)}$. We remark that if $Y = x$ is a point this gives the definition of lower local dimension of $\mu$ at $x$. We recall that if $d_\mu(x) = \limsup_{\epsilon \to 0} \frac{\log(\mu(B_\epsilon(x)))}{\log(\epsilon)} = d_\mu(x)$ this is called the local dimension of $\mu$ at $x$. 

Lemma 1. Let \((X,T,\mu)\) be a measure preserving transformation, \(Y \subset X\). If \(\alpha > \frac{1}{d_\mu(Y)}\) then for almost each \(x \in X:\)

\[
\liminf_{n \to \infty} n^\alpha d(T^n(x), Y) = \infty.
\]

Hence we obtain the following

Proposition 3. If \(T\) is an ergodic piecewise isometry as defined above, \(Y = \bigcup_{i \leq m} \partial P_i\), and \(d_\mu(Y) \neq 0\), moreover if the local dimension \(d_\mu(x)\) is well defined and \(a.e.\) constant on \(X\), then

\[
h_\mu^{\log}(T) \leq \frac{d_\mu(x)}{d_\mu(Y)}.
\]

Proof. First we remark that since \(d = d_\mu(Y) \neq 0\) then \(\mu(Y) = 0\). This, together with the other properties of piecewise isometries implies that \(T\) satisfies the assumptions of proposition [11] hence by remark [11] it is sufficient to estimate the behavior of \(\mu(B(x,n,\epsilon))\). First we remark that we can suppose \(\liminf_{n \to \infty} d(T^n(x), Y) = 0\), otherwise the statement is trivial (because the typical orbit never approaches to the discontinuity). In this case, as remarked above, by Lemma [11] we have that for almost each \(x \in X\), small \(\epsilon > 0\) it holds \(B(x,n,\epsilon) \supset B_{\frac{n \epsilon}{n+1}}(x)\) eventually with respect to \(n\). Then if \(n\) is big enough \(\mu(B(x,n,\epsilon)) \geq \mu(B_{\frac{n \epsilon}{n+1}}(x))\). By the assumptions on the local dimension of the system then we have that again, if \(n\) is big enough, if \(\epsilon, \epsilon'\) are small \(\mu(B(x,n,\epsilon)) \geq n^{(d_\mu(x)-\epsilon')/(\frac{n \epsilon}{n+1})}\). Which gives the statement. \(\square\)

3.2. Interval Exchanges. Interval Exchanges are close relatives of surface flows, these maps are particular bijective piecewise isometries of the unit interval, whose atoms are intervals and which preserve the Lesbegue measure. In this section we apply a result of Boshernitzan about a full measure class of uniquely ergodic interval exchanges to estimate their metric complexity. We refer to [8] for generalities on this important class of maps.

Let \(T\) be some interval exchange. Let \(\delta(n)\) be the minimum distance between the discontinuity points of \(T^n\). We say that \(T\) has the property \(\tilde{P}\) if there is a constant \(C\) and a sequence \(n_k\) such that \(\delta(n_k) \geq C/n_k\).

Lemma 2. (by [8]) The set of interval exchanges having the property \(\tilde{P}\) has full measure in the space of interval exchange maps.

From Lemma [11] it easily follows that

Corollary 1. For each interval exchange \(T\) and each \(\epsilon > 0\), for almost each \(x\) the distance from the orbit of \(x\) to the discontinuity set of \(T\) is estimated as follows. If \(y_1, \ldots, y_k\) are the discontinuity points of \(T\) then eventually with respect to \(n\)

\[
\min_{1 \leq i,j \leq k} d(T^i(x), y_j) > n^{-1-\epsilon}.
\]

Since the initial condition sensitivity of interval exchanges is determined by the speed of approaching of starting points to the discontinuities, these results will allow to estimate \(h_\mu^{\log}(T)\). Indeed by the above corollary [11] we know that if \(T\) is ergodic, for almost each \(x\), for each \(\epsilon > 0\) eventually \(\mu(B(n,x,\epsilon)) \geq n^{-1-\epsilon}\). Since an interval exchange satisfies the assumptions of remark [11] then this implies that \(h_\mu^{\log}(T) \leq 1\).
On the other hand the converse estimation follows from the remark that if \( x_0 \) is a discontinuity point, and
\[
\min_{i \leq n, T^i(x) \leq x_o} d(T^i(x), x_0) = l_1(n) \quad \text{and} \quad \min_{i \leq n, T^i(x) \geq x_o} d(T^i(x), x_0) = l_2(n)
\]
(the minimum distance after \( n \) steps of the orbit on the left and on the right side of the discontinuity \( x_0 \)) then for small \( \epsilon \), \( B(n, x, \epsilon) \subseteq (x - l_1(n), x + l_2(n)) \). Now we have to estimate from above the speed of approaching to the discontinuity on both sides. Using property \( \hat{P} \), like in [24] we can obtain the following

**Proposition 4.** Let \( T \) be an IET with property \( \hat{P} \) as before, then \( h^{\log}_P(T) \geq 1 \).

**Proof.** If \( T \) has \( m \) discontinuity points, \( T^n \) has \( nm \) discontinuity points and they will divide the unit segment into \( nm + 1 \) small segments. The total length is 1, then among these small segments there are at least \( \frac{mC}{2} \) ones with length less or equal than \( \frac{2}{nm+1} \). Let us denote by \( J_n \), the union of these segments. By property \( \hat{P} \) there is a sequence \( n_k \) such that the segments in \( J_{n_k} \) are longer than \( \frac{C}{n_k} \) by this \( \mu(J_{n_k}) \geq \frac{mC}{2} \). Hence there is a set \( B \) with positive measure, \( \mu(B) \geq \frac{mC}{2} \) such that if \( x \in B \) then \( x \) is contained in infinitely many \( J_{n_k} \). Let us notice at this point that if \( x \in J_{n_k} \), then the discontinuities of \( T^{n_k} \) near \( x \) are the ends of the small interval \( (y, y_i) \subset J_{n_k} \) containing \( x \), hence for small \( \epsilon \) the Bowen set around \( x \) satisfies \( B(n_k + 1, x, \epsilon) \subseteq (y, y_i) \). Recalling that \( \mu(J_{n_k}) \geq \frac{mC}{2} \) now, we estimate (see eq. 2.4) \( N(n_k + 1, \epsilon, 1 - \frac{mC}{2}) \). To cover a set with measure greater than \( 1 - \frac{mC}{4} \) we need to cover at least half of \( J_{n_k} \), but his intervals (and respective Bowen sets) have measure less or equal than \( \frac{2}{nm_2+1} \), hence we need at least \( \frac{2mC}{4} \) sets, which gives the statement. 

Collecting the above results we have the following estimation of the complexity for typical interval exchanges.

**Proposition 5.** If \( T \) is an IET with property \( \hat{P} \) then \( h^{\log}_P(T) = 1 \).

The situation for nontypical IET in general is much more complicated. We expect arithmetical phenomena like in section 3 to happen.

### 3.3. Casati Prosen map

In this subsection we will consider the Casati Prosen map, the map acts on the unit square, is weakly chaotic and it is not a piecewise isometry. This kind of map was introduced by Casati and Prosen [13] in connection with the mixing properties of flows in certain triangular billiards [14]. We will give an upper estimation of its complexity.

Let us define the map: let \( \theta(q) \) be the discontinuous function over the circle given by \( \theta(q) = -1 \) if \( 0 \leq q \leq 1/2 \) and \( \theta(q) = 1 \) otherwise.

For any \( \alpha, \beta \in [0, 1] \), we define the map \( T_{\alpha, \beta} \) as
\[
T_{\alpha, \beta}(q, p) = (q + p + \beta, p + \alpha \theta(q)) \mod 1.
\]

We remark that \( T_{\alpha, \beta} \) can be written as the composition of three elementary maps,
\[
T_{\alpha, \beta} = B \circ R \circ G_{\alpha},
\]
where \( B \) is represented by the matrix \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) (a skew translation), \( R(q, p) = (q + \beta, p) \) is a translation in the \( q \) direction and \( G_{\alpha} \) is the discontinuous part of the dynamics \( G_{\alpha}(q, p) = (q, p + \alpha \theta(q)) \) this discontinuous map cuts the square
Proof. Let us consider the projections $\pi_q((q,p)) = q, \pi_p((q,p)) = p$. Let us consider an $y$ such that $\forall i \leq n$

$$d(\pi_q(T^n(x)), \pi_q(T^n(y))) \leq \frac{C_i}{4} i^{-\alpha}, \quad d(\pi_p(T^n(x)), \pi_p(T^n(y))) \leq \frac{C_i}{4} i^{-\alpha}. \quad (3.1)$$

Then the orbits of $x$ and $y$ are not separated by the discontinuity at the $n+1$ step. This is true because the orbit of $x$ will stay far away (more than $\frac{C_i}{4} i^{-\alpha}$) enough from $Y$ and after the skew translation $d(\pi_q(B(T^n(x)), \pi_q(B(T^n(y)))) \leq \frac{C_i}{4} i^{-\alpha}$ hence when $G_\alpha$ is applied the two points are near enough to avoid to be separated by the discontinuity.

Now let us estimate the set of points which are near enough to $x$ to satisfy equation (3.1) after $m$ steps. If

$$d(\pi_p(x), \pi_p(y)) = d_p, \quad d(\pi_q(x), \pi_q(y)) = d_q$$

and after $m$ steps, if the orbit of $x$ and $y$ are separated only by the effect of the skew translation we have that $d_q(T^n(x)), \pi_q(T^n(y))) \leq m d_p + d_q$ hence if $m d_p + d_q \leq \frac{C_i}{4} m^{-\alpha}$ the two points are not separated by the discontinuity at next step. Let us suppose $d_p \leq \frac{C_i}{4} m^{-\alpha}$, this gives $d_p \leq \frac{C_i}{4} m^{-\alpha-1}$. By this we obtain that when $m$ is big enough with respect to $\epsilon$

$$B(x, m, \epsilon) \supset R = \{y : d(\pi_q(x), \pi_q(y)) \leq \frac{C_i}{8} m^{-\alpha}, \quad d(\pi_p(x), \pi_p(y)) \leq \frac{C_i}{8} m^{-\alpha-1}\}$$

the measure of the rectangle $R$ on the right side is $\mu(R) = \frac{C_i}{8} m^{-\alpha} \frac{C_i}{8} m^{-\alpha-1} = \frac{C_i^2}{64} m^{-2\alpha-1}$ and $\alpha$ is near to 1 as wanted. This gives the statement.

Proposition 6. If $(X, T_{\alpha, \beta, \lambda})$ is the Casati Prosen map then $h^{\text{top}}(T_{\alpha, \beta}) \leq 3$.

Proof. Let us consider $Y = \rho \cup \rho'$ (the discontinuity set) since we consider the Lesbegue measure we have $d_\lambda(Y) = 1$, hence by lemma $\text{[1]}$ we obtain for each $\alpha > 1$ and almost each $x$ it holds $\lim \inf_{n \to \infty} n^\alpha d(T^n(x), Y) = \infty$. Let also suppose that the orbit of $x$ never meet $Y$. There is a $c$ such that for all $n$ it holds $n^\alpha d(T^n(x), Y) \geq c > 0$.

Let us consider the projections $\pi_q((q,p)) = q, \pi_p((q,p)) = p$. Let us consider an $y$ such that $\forall i \leq n$

$$d(\pi_q(T^n(x)), \pi_q(T^n(y))) \leq \frac{C_i}{4} i^{-\alpha}, \quad d(\pi_p(T^n(x)), \pi_p(T^n(y))) \leq \frac{C_i}{4} i^{-\alpha}. \quad (3.1)$$

Then the orbits of $x$ and $y$ are not separated by the discontinuity at the $n+1$ step. This is true because the orbit of $x$ will stay far away (more than $\frac{C_i}{4} i^{-\alpha}$) enough from $Y$ and after the skew translation $d(\pi_q(B(T^n(x)), \pi_q(B(T^n(y)))) \leq \frac{C_i}{4} i^{-\alpha}$ hence when $G_\alpha$ is applied the two points are near enough to avoid to be separated by the discontinuity.

Now let us estimate the set of points which are near enough to $x$ to satisfy equation (3.1) after $m$ steps. If

$$d(\pi_p(x), \pi_p(y)) = d_p, \quad d(\pi_q(x), \pi_q(y)) = d_q$$

and after $m$ steps, if the orbit of $x$ and $y$ are separated only by the effect of the skew translation we have that $d_q(T^n(x)), \pi_q(T^n(y))) \leq m d_p + d_q$ hence if $m d_p + d_q \leq \frac{C_i}{4} m^{-\alpha}$ the two points are not separated by the discontinuity at next step. Let us suppose $d_p \leq \frac{C_i}{4} m^{-\alpha}$, this gives $d_p \leq \frac{C_i}{4} m^{-\alpha-1}$. By this we obtain that when $m$ is big enough with respect to $\epsilon$

$$B(x, m, \epsilon) \supset R = \{y : d(\pi_q(x), \pi_q(y)) \leq \frac{C_i}{8} m^{-\alpha}, \quad d(\pi_p(x), \pi_p(y)) \leq \frac{C_i}{8} m^{-\alpha-1}\}$$

the measure of the rectangle $R$ on the right side is $\mu(R) = \frac{C_i}{8} m^{-\alpha} \frac{C_i}{8} m^{-\alpha-1} = \frac{C_i^2}{64} m^{-2\alpha-1}$ and $\alpha$ is near to 1 as wanted. This gives the statement.

3.3.1. Logistic map at chaos threshold. Now we calculate the metric complexity of the orbits of this well known dynamical system. First let us recall that the Logistic map at the chaos threshold is a map with zero topological entropy. Nevertheless the topological complexity of the map $T_{\lambda, \alpha}$ is not trivial (see $\text{[20]}$, theorem 22) this means that the total number of essentially different orbits is not bounded as time increases. On the contrary as we will see below, the metric complexity is trivial.

To understand the dynamics of the Logistic map at the chaos threshold let us use a result of $\text{[16]}$ (Theorem III.3.5.)
Lemma 3. The logistic map $T_{\lambda_\infty}$ at the chaos threshold has an invariant Cantor set $\Omega$ with the following properties.

(1) There is a decreasing chain of closed subsets

$$J^0 \supset J^1 \supset J^2 \supset \ldots,$$

each of which contains $1/2$, and each of which is mapped onto itself by $T_{\lambda_\infty}$.

(2) Each $J^i$ is a disjoint union of $2^i$ closed intervals. $J^{i+1}$ is constructed by deleting an open subinterval from the middle of each of the intervals making up $J^i$.

(3) $T_{\lambda_\infty}$ maps each of the intervals making up $J^i$ onto another one; the induced action on the set of intervals is a cyclic permutation of order $2^i$.

(4) $\Omega = \cap_i J^i$. $T_{\lambda_\infty}$ maps $\Omega$ onto itself in a one-to-one fashion. Every orbit in $\Omega$ is dense in $\Omega$.

(5) For each $k \in \mathbb{N}$, $T_{\lambda_\infty}$ has exactly one periodic orbit of period $2^k$. This periodic orbit is repelling and does not belong to $J^{k+1}$. Moreover this periodic orbit belongs to $J^k \setminus J^{k+1}$, and each point of the orbit belongs to one of the intervals of $J^k$.

(6) Every orbit of $T_{\lambda_\infty}$ either lands after a finite number of steps exactly on one of the periodic orbits enumerated in 5, or converges to the Cantor set $\Omega$ in the sense that, for each $k$, it is eventually contained in $J^k$. There are only countably many orbits of the first type.

By this it follows that the metric complexity of this map is trivial, in the following sense:

Theorem 2. In the dynamical system $([0, 1], T_{\lambda_\infty}, \mu)$ if $\mu$ is some invariant measure supported on the attractor $\Omega$, for each $f : h_\mu^f(x) = 0$.

Proof. By point 2 of the above lemma $\exists J^i = \bigcup_{k \leq 2^i} J^i_k$ is the union of $2^i$ intervals, let $\epsilon_i = \max_{k \leq 2^i} (\text{diam}(J^i_k))$. By lemma 3 point 3, if $x, y \in J^i$ then $\sup_{n \geq 0} d(T^n(x), T^n(y)) \leq \epsilon_i$. By this we know that for each $\epsilon \geq \epsilon_i$ and each $x \in J^i_n$ the set $B(x, n, \epsilon)$ contains $J^i_n$ for each $n$. Hence $2^i$ Bowen sets are sufficient to cover $J^i$ for any $n$. Since the support of the measure is contained in each $J^i$ we have the statement. \qed

4. Characteristic exponents

The set $B(t, x, \epsilon)$ and its way of shrinking as $t$ increases describes the initial condition sensitivity of the system around the point $x$.

The set will shrink with different speeds at different directions. For example, the presence of a stable manifold at $x$ will imply that $B(t, x, \epsilon)$ contains for each $n$ a piece of the manifold and does not shrink in the directions parallel to the manifold. We introduce a set of numbers $l_i$ which describes the shrinking rate at the different directions. These numbers are in some sense versions of the positive Lyapunov exponents. In the cases when the geometry of $B(t, x, \epsilon)$ in nice the numbers $l_i$ are related to the metric complexity, by a result which plays the role of the Ruelle-Pesin formula.

For simplicity we suppose that $X$ is an open subset of $\mathbb{R}^n$, the case where $X$ is a manifold is similar. Let us consider the set $S$ of isometries of $\mathbb{R}^n$. Let $P_{\ell_1 \ldots \ell_n} = [-\frac{\ell_1}{2}, \frac{\ell_1}{2}] \times \ldots \times [-\frac{\ell_n}{2}, \frac{\ell_n}{2}]$ be the rectangular parallelepiped with sides $\ell_1 \ldots \ell_n$. Let

$$l_1(B(t, x, \epsilon)) = \inf \{ \ell_1 : \exists \text{ an isometry } A \text{ s.t. } B(t, x, \epsilon) \subset A(P_{\ell_1 \ldots \ell_n}) \}$$
Remark 2. \( l_1(B(t, x, \epsilon)) \) is a minimum.

Proof. This follows by compactness, indeed the space \( S \) and the space of all possible parallelepipeds are locally compact. Moreover, a sequence \( A_i(P_{t_1}...t_n) \) realizing the infimum of \( l_1 \) can be chosen to be a bounded one, hence, by compactness it has a subsequence having limit.

Since each parallelepiped is compact then this limit parallelepiped will contain \( B(t, x, \epsilon) \), conversely a whole neighborhood of the limit parallelepiped should not contain \( B(t, x, \epsilon) \).

By this, let us also define

\[
 l_2(B(t, x, \epsilon)) = \inf \{ l_2 : \exists \text{ an isometry } A \text{ s.t. } B(t, x, \epsilon) \subset A(P_{t_1}t_2...\epsilon_n) \}.
\]

By remark 2 \( l_2 \) is well defined, and then more generally we define \( l_1, ..., l_n \) as

\[
 l_i(B(t, x, \epsilon)) = \inf \{ l_i : \exists \text{ an isometry } A \text{ s.t. } B(t, x, \epsilon) \subset A(P_{t_1}...t_i...\epsilon_n) \}.
\]

Starting from the above defined \( l_1, ..., l_n \) we can define some indicator, characterizing the initial condition sensitivity at different directions.

\[
 l_i'(x, \epsilon) = \limsup_{t \to \infty} \frac{-\log(l_i(B(t, x, \epsilon)))}{f(t)}, \quad l_i''(x, \epsilon) = \liminf_{t \to \infty} \frac{-\log(l_i(B(t, x, \epsilon)))}{f(t)}
\]

\[
 l_i'(x) = \lim_{\epsilon \to 0} l_i'(x, \epsilon), l_i''(x) = \lim_{\epsilon \to 0} l_i''(x, \epsilon).
\]

The numbers \( l_i'(x) \) are in some sense lower estimations of the way of shrinking of \( B(t, x, \epsilon) \) into different directions. We can also consider the upper estimations given by

\[
 L_i(B(t, x, \epsilon)) = \sup \{ l_i : \exists \text{ an isometry } A \text{ s.t. } B(t, x, \epsilon) \supset (A(P_{t_1}...\epsilon_n))' \},
\]

\[
 L_i(B(t, x, \epsilon)) = \sup \{ l_i : \exists \text{ an isometry } A \text{ s.t. } B(t, x, \epsilon) \supset (A(P_{t_1}...t_i...\epsilon_n))' \},
\]

\[
 l_i'(x, \epsilon) = \limsup_{t \to \infty} \frac{-\log(L_i(B(t, x, \epsilon)))}{f(t)}, \quad l_i''(x, \epsilon) = \liminf_{t \to \infty} \frac{-\log(L_i(B(t, x, \epsilon)))}{f(t)}
\]

\[
 l_i'(x) = \lim_{\epsilon \to 0} l_i'(x, \epsilon), l_i''(x) = \lim_{\epsilon \to 0} l_i''(x, \epsilon).
\]

Similar to the traditional Lyapunov exponents the indicators \( l_i' \) and \( L_i \) allows to prove the following inequalities.

Theorem 3. If the system is ergodic, it satisfies the assumptions of proposition 2 and the measure \( \mu \) is invariant and absolutely continuous with bounded density then almost everywhere it holds

\[
 \sum_{i \leq n} l_i'(x) \geq h_{\mu}(X, T) \geq \sum_{i \leq n} l_i''(x).
\]

Proof. As before, by proposition 2 we have to estimate \( \mu(B(t, x, \epsilon)) \) for a typical \( x \). We remark that from remark 2 it follows that there is an isometry \( A \) such that \( B(t, x, \epsilon) \subset A(P_{t_1}...\epsilon_n) \) then \( \mu(B(t, x, \epsilon)) \leq \mu(A(P_{t_1}...\epsilon_n)) \). Since \( \mu \) has bounded density then \( \mu(A(P_{t_1}...\epsilon_n)) \leq \text{Const} \cdot l_1(B(t, x, \epsilon))l_2(B(t, x, \epsilon))...l_n(B(t, x, \epsilon)) \), hence

\[
 \log(\mu(A(P_{t_1}...\epsilon_n))) \leq \text{Const}l_1 + \log(l_1(B(t, x, \epsilon))) + \log(l_2(B(t, x, \epsilon))) + \ldots + \log(l_n(B(t, x, \epsilon))),
\]

from which, dividing by \( f(t) \) and taking the appropriated limits we obtain

\[
 BK_l'(x) \geq \sum_{i \leq n} l_i'(x). \quad \text{The other inequality is similar.} \]

\[
^4\text{By } B^0 \text{ we denote the internal part of } B.
\]
5. Appendix: an example where $\overline{BK}^f(x, \epsilon) \neq \overline{BK}^{f'}(x, \epsilon)$

We will give an example where $\overline{BK}^f(x) \neq \overline{BK}^{f'}(x)$ almost everywhere. For $f(n) = \log(n)$. We remark that by the Brin-katok theorem such an example is not possible when $f(n) = n$.

Let us consider the two dimensional torus $X = [0, 1 \text{ (mod 1)}] \times [0, 1 \text{ (mod 1)}]$.

For simplicity, let us equip it with the sup distance. If $d'$ is the distance on the circle $[0, 1 \text{ (mod 1)}]$ then $d((x_1, y_1), (x_2, y_2)) = \max(d'(x_1, x_2), d'(y_1, y_2))$. Let us also define $d_x((x_1, y_1), (x_2, y_2)) = d'(x_1, x_2)$, $d_y((x_1, y_1), (x_2, y_2)) = d'(y_1, y_2)$.

Let us consider $\alpha = 0.0505000000000005... = \sum_{n=0}^{\infty} \frac{1}{2^{2^n}}$ we have that $\alpha$ is obviously irrational. We define $T : X \to X$ as

$$T = T_1 \circ T_2$$

where

$$T_1(x, y) = (x + \alpha \mod 1, y)$$

$$T_2(x, y) = (x, y + \theta(x) \mod 1)$$

where $\theta(q)$ is the discontinuous function over the unit circle defined in the following way: let us consider the points $\frac{1}{2} \binom{0}{1}$ and $\frac{1}{2} - \alpha$. Such points divide the unit circle into two intervals $I_1, I_2$. $\theta(q) = -\frac{1}{2}$ if $q \in I_1$ and $\theta(q) = \frac{1}{2}$ if $q \in I_2$. $T$ at each step rotates on the $x$ direction and then cuts the torus along the circles $x = \frac{1}{2}$ $x = \frac{1}{2} - \alpha$, rotating the torus in opposite directions along the discontinuity circles.

In this system the Lesbegue measure is invariant, hence let us consider as $(X, T, \mu)$ the system described above with the Lesbegue measure.

Let us consider the first entrance time of the orbit of $x$ in the ball $B(y, r)$ with center $y$ and radius $r$

$$\tau_r(x, y) = \min\{n \in \mathbb{N}, n > 0, T^n(x) \in B(y, r)\}.$$ 

An irrational $\gamma$ is said to be of type $\nu_\gamma$ if

$$\nu_\gamma = \sup \{\beta| \liminf_{n \to \infty} j^\beta (\min_{n \in \mathbb{N}} |j\gamma - n| = 0)\}.$$ 

Lesbegue almost each irrational is of type 1, but there are irrationals with type $> 1$. For example the $\alpha$ defined above has type $\infty$. From the main result of [22] it can be deduced that an irrational rotation with angle $\gamma$ of type $\nu_\gamma > 1$ satisfies

$$\limsup_{r \to 0} \frac{\log \tau_r(x, y)}{-\log r} = \nu_\gamma$$

for almost each $x$, while

$$\liminf_{r \to 0} \frac{\log \tau_r(x, y)}{-\log r} \leq 1 \text{ a.e.}$$

In other words this implies that for almost each $x$ there are real sequences $r_n$ and $r'_n$ such that $\lim_{n \to \infty} \frac{\log \tau_r(x, \frac{1}{2})}{-\log r_n} = \nu_\gamma$ and $\lim_{n \to \infty} \frac{\log \tau_r(x, \frac{1}{2})}{-\log r'_n} = 1$. Since the values of $\tau_r$ selects times $i$ where the distance $d(T^i(x), \frac{1}{2})$ is minimal $(d(T^i(x), \frac{1}{2}) = \min_{i \leq \tau_r} (T^i(x), \frac{1}{2})$). This means that there is a sequence $n_k$ such that $d(T^{nk}(x), \frac{1}{2}) = \ldots$
\[
\min d(T^i(x), \frac{1}{2}) \sim n_k^{-\frac{1}{\alpha}}. \]
Now, coming back to our system we have that \( \nu_\alpha = \infty \), moreover, let us remark that \( d_x(T^i(x), \frac{1}{2} - \alpha) = d_x(T^{i+1}(x), \frac{1}{2}) \), hence \( \min_{i \leq n_k-1} d_x(T^i(x), \frac{1}{2} - \alpha) \geq \min_{i \leq n_k} d_x(T^i(x), \frac{1}{2}) \). This means that if the orbit is far from the discontinuity \( x = \frac{1}{2} \), then it is also far from the other discontinuity. By this let us choose \( \epsilon < \frac{1}{2} \) and estimate \( \mu(B(n, (x_0, y_0), \epsilon)) \) where \( (x_0, y_0) \) is a typical initial condition satisfying the above equation 5.1 with \( y = \frac{1}{2} \). The only source of initial condition sensitivity is the action of the discontinuities, let us consider the discontinuity set \( Y = \{ \begin{pmatrix} x \\ y \end{pmatrix} \in X : x = \frac{1}{2} \text{ or } x = \frac{1}{2} - \alpha \} \) by equation 5.1 for each \( \delta > 0 \) there is a sequence \( n_k \) such that eventually \( n_k^{-\delta} = o(\min_{i \leq n_k} d(T^i(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}), Y)) \) then for each \( \delta > 0 \) it holds \( B(n_k, (x_0, y_0), \epsilon) \supset [x_1 - n_k^{-\delta}, x_1 + n_k^{-\delta}] \times [y_1 - \epsilon, y_1 + \epsilon] \) and
\[
\liminf_{n \to \infty} \frac{-\log(\mu(B(n, (x_0, y_0), \epsilon)))}{\log(n)} = 0 \quad \text{which gives } B^{K^{\log}}(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}) = 0.
\]

For the estimation of \( B^{K^{\log}}(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}) \), let us consider the way the projection on the \( x \) circle of the orbit of the initial point \( \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \) divides the circle. Let us hence consider the sequence \( x_0, x_1 = x_0 + \alpha, x_2 = x_0 + 2\alpha \ldots \) Let us also suppose that the discontinuity points are not included in the sequence \( x_i \) (this is obviously true for a full measure set of initial conditions).

At each time of the form \( n_k = 2^{2^k} \) the unit circle is divided by the sequence \( x_i \) into small intervals with length less or equal than \( \frac{2}{2^{2^k}} \). This is true because \( 2^{2^k} \) is the minimal period of the rotation by the angle \( \alpha_k = \sum_{n=0}^{k} \frac{1}{2^n} \) and this divides the circle into equal pieces of length \( \frac{1}{2^{2^k}} \). Now
\[
2^{2^k} \sum_{n=k+1}^{\infty} \frac{1}{2^{2^n}} < \frac{1}{2^{2^k}}\]
and then the distance of the first \( 2^{2^k} \) steps of the two rotations (with angles \( \alpha \) and \( \alpha_k \)) is smaller than the length of one small interval, giving the required result.

The size of \( B(n_k, (x_0, y_0), \epsilon) \) is then estimated by the length of these small intervals. Indeed we have that, the point \( x = \frac{1}{2} \) is contained in some interval \([x_1, x_2]\). This means that \( T^i(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}) \) is on the left of the discontinuity set at a distance less or equal than \( \frac{1}{2^{2^k}} \), while \( T^j(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}) \) is on the right of the discontinuity line at a distance less or equal than \( \frac{1}{2^{2^k}} \). This means that
\[
B(n_k, (x_0, y_0), \epsilon) \subset [x_1 - n_k^{-1}, x_1 + n_k^{-1}] \times [y_1 - \epsilon, y_1 + \epsilon],
\]
thus \( \limsup \frac{-\log(\mu(B(n, x_0, y_0), \epsilon))}{\log(n)} \geq 1 \) which gives \( BK^{\log}(x, y) \geq 1 \).

Since the initial point can be chosen in a full measure set we have

**Proposition 7.** In the above system, for almost each \( x \) \( BK^{\log}(x) \geq 1 \) while \( BK^{\log}(x) = 0 \).

**References**

[1] Adler R.L., Kitchens B. and Tresser C., *Dynamics of non-ergodic piecewise affine maps of the torus*. Ergod. Th. & Dynam. Sys. 21 (2001) 959-999.

[2] Afraimovich, V., Zaslavsky, G. M.; *Working with complexity functions*. Chaotic dynamics and transport in classical and quantum systems, 73–85, NATO Sci. Ser. II Math. Phys. Chem., 182, Kluwer Acad. Publ., Dordrecht, 2005.

[3] Afraimovich, V; Glebsky, L *Complexity, fractal dimensions and topological entropy in dynamical systems*. Chaotic dynamics and transport in classical and quantum systems, 35–72, NATO Sci. Ser. II Math. Phys. Chem., 182, Kluwer Acad. Publ., Dordrecht, 2005.

[4] Ashwin P., Goetz A. *Polygonal invariant curves for planar piecewise isometry*. Transactions AMS 358 No 1 373-390 (2005)

[5] Ashwin, P. *Non–smooth invariant circles in digital overflow oscillations*. Proceedings of the 4th Int. Workshop on Nonlinear Dynamics of Electronic Systems, Sevilla (1996) 417-422.

[6] Bonanno C; Galatolo, S *Algorithmic information for interval maps with an indifferent fixed point and infinite invariant measure*. Chaos 14 (2004), no. 3, 756–762.

[7] Bonanno C, Isola S, Galatolo S *Recurrence and algorithmic information*, Nonlinearity, num. 3, vol. 17, pp. 1057-1074, 2003

[8] Boshernitzan M D, *A condition for minimal interval exchange maps to be uniquely ergodic*. Duke Math. J. 52 (1985), no. 3, 723–752

[9] Blanchard, F.; Host, B.; Maass, A. *Topological complexity*. Ergodic Theory Dynam. Systems 20 (2000), no. 3, 641–662.

[10] Blume, Frank. *Possible rates of entropy convergence*. Ergodic Theory Dynam. Systems 17 (1997), no. 1, 45–70.

[11] Brin, M.; Katok, A. *On local entropy*. Geometric dynamics (Rio de Janeiro, 1981), 30–38, Lecture Notes in Math., 1007, Springer, Berlin, 1983.

[12] Brudno A.A. *Entropy and the complexity of the trajectories of a dynamical system*. Trans. Moscow Math. Soc. 2 127-151 (1983)

[13] Buzzi J., *Piecewise isometries have zero topological entropy*. Ergod. Th. Dyn. Sys. 21 (2001) 1371-1377

[14] Casati G. and Prosen T., *Mixing property of triangular billiards*, Physical Review Letters, 83, n.23 (1999), 4729-4732.

[15] Casati G. and Prosen T., *The triangle map: a model of quantum chaos*, Physical Review Letters, 85, (2000), 4261.

[16] Collet, Pierre; Eckmann, Jean-Pierre. *Iterated maps on the interval a s dynamical systems*. Progress in Physics, 1. Birkhauser, Boston, Mass., 1980. viii+248 pp.

[17] Esposti, M D; Galatolo, S *Recurrence near given sets and the complexity of the Casati-Prosen map*. Chaos Solitons Fractals 23 (2005), no. 4, 1275–1284.

[18] Ferenczi, S. *Measure-theoretic complexity of ergodic systems*. Israel J. Math. 100 (1997), 189–207.

[19] Kahng B., *Dynamics of symplectic piecewise affine elliptic rotation maps on tori*. Ergod. Th.& Dynam. Sys. 2 (2002) 483-505.

[20] Kantz, H; Schreiber, T *Nonlinear time series analysis*. Second edition. Cambridge University Press, Cambridge, 2004. xvi+369 pp. ISBN: 0-521-82150-9; 0-521-52902-6 62-02

[21] Katok, A ; Thouvenot, J-P *Slow entropy type invariants and smooth realization of commuting measure-preserving transformations*. Ann. Inst. H. Poincaré Probab. Statist. 33 (1997), no. 3, 323–338.

[22] Kim D H , Seo B K *The waiting time for irrational rotations*, Nonlinearity V. 16, N. 5, Sept. 2003
[23] Kocarev L., Wu C.W. and Chua L.O. Complex behaviour in Digital filters with overflow nonlinearity: analytical results. IEEE Trans CAS-II 43 (1996) 234-246.

[24] Galatolo, S Hitting time and dimension in axiom A systems, generic interval exchanges and an application to Birkhoff sums. J. Stat. Phys. 123 (2006), no. 1, 111–124.

[25] Galatolo, S Complexity, initial condition sensitivity, dimension and weak chaos in dynamical systems. Nonlinearity 16 (2003)

[26] Galatolo, S Global and local complexity in weakly chaotic dynamical systems. Discrete Contin. Dyn. Syst. 9 (2003), no. 6, 1607–1624.

[27] Gaspard P., Wang X.-J., Sporadicity: between periodic and chaotic dynamical behaviors, Proc. Natl. Acad. Sci. USA 85, 4591-4595 (1988).

[28] Goetz, A Piecewise isometries—an emerging area of dynamical systems. Fractals in Graz 2001, 135–144, Trends Math., Birkhauser, Basel, 2003.

[29] Ott, W; Yorke, J. Learning about reality from observation. SIAM J. Appl. Dyn. Syst. 2 (2003), no. 3, 297–322 (electronic).

[30] Pesin Y Dimension theory in dynamical systems Chicago lectures in Mathematics (1997).

[31] Tabachnikov S., On the dual billiard problem. Adv. Math. 115 (1995) 221-249.

[32] Takens, Floris; Verbitski, Evgeny. Generalized entropies: Renyi and correlation integral approach. Nonlinearity 11 (1998), no. 4, 771–782.

[33] Tsallis, C.; Plastino, A. R.; Zheng, W.-M. Power-law sensitivity to initial conditions—new entropic representation. Chaos Solitons Fractals 8 (1997), no. 6, 885–891.

[34] Young L.S. What are SRB measures, and which dynamical systems have them? Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays. J. Statist. Phys. 108 (2002), no. 5-6, 733–754.

[35] Zweimüller, Roland Asymptotic orbit complexity of infinite measure preserving transformations. Discrete Contin. Dyn. Syst. 15 (2006), no. 1, 353–366.

Dipartimento di Matematica Applicata, Università di Pisa, via Buonarroti 1 Pisa
E-mail address: galatolo@dm.unipi.it
URL: http://www2.ing.unipi.it/~d80288