Vortex motion of a continuous medium depending on the pressure change

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In this paper, we study the vortex motion of a continuous medium, which is described by forces obtained from the principle of least action. It is shown that in a continuous medium the vortex force components are proportional to the velocity and pressure gradient components. This article gives a description of the 2D vortex motion of air in zones of high and low pressure. If the pressure decreases, the angular velocity of rotation of the continuous medium increases, whereas if the pressure increases, the angular velocity fades. The lifting force is obtained due to the vortex movement of air in the form of a funnel. It is shown that the vortex force contains a vortex term of the Euler hydrodynamic equations with a relative factor equal to the velocity of the continuous medium squared divided by the sound velocity squared. To describe the motion of a continuous medium correctly it is necessary to replace the forces obtained by Euler with the forces obtained from the minimum of action in the equations of motion. It is concluded that vortex motions and turbulence are described by the obtained equations of motion, and not by the Navier–Stokes equations. Most likely, this is related to the Problem of the Millennium description of turbulence announced at the International Congress of Mathematics in 2000.

Subject Index J18

1. Introduction

It is known [1] that the motion of a continuous medium is described by the equations of Euler hydrodynamics, which were obtained by differentiating the velocity field by parts in the equation

\[ \rho \frac{dv_j}{dt} = -\frac{\partial p}{\partial x_j}, \tag{1} \]

determining and proposing that \( dx_i/dt = v_i \):

\[ \rho \frac{dv_j}{dt} + \rho v_i \frac{\partial v_j}{\partial x_i} = -\frac{\partial p}{\partial x_j}, \tag{2} \]

where \( \rho \) is density, \( p \) is pressure, and \( v_j \) is the velocity of the continuous medium.

The vortex term of the Euler equations has the form

\[ f_j = \rho \varepsilon_{ijk} v_i \varepsilon_{kmn} \frac{\partial v_n}{\partial x_m}. \tag{3} \]

Taking into account the convolution of Levi–Civita tensors, it is easy to find that \( \varepsilon_{ijk} \varepsilon_{kmn} = \delta_{jm} \delta_{in} - \delta_{jn} \delta_{im} \), which results in \( \rho \varepsilon_{ijk} v_i \varepsilon_{kmn} \partial v_n/\partial x_m = \rho v_j \partial v_j/\partial x_j - \rho v_i \partial v_j/\partial x_i \). Then the Euler equations have the form

\[ \rho \frac{dv_j}{dt} = -\rho v_i \frac{\partial v_j}{\partial x_i} - \frac{\partial p}{\partial x_j} + \rho \varepsilon_{ijk} v_i \varepsilon_{kmn} \frac{\partial v_n}{\partial x_m}. \tag{4} \]
Applying the curl operator to Eq. (4) and considering that the gradient of a curl is zero, we obtain the known equations of vortex motion [1]:

\[
\rho \frac{\partial}{\partial t} \left( e_{pqj} \frac{\partial v_j}{\partial x_q} \right) = \rho e_{pqj} \frac{\partial}{\partial x_q} \left( e_{ijk} v_i e_{kmn} \frac{\partial v_m}{\partial x_n} \right). \tag{5}
\]

It follows from Eq. (5) that the vortex motions of a continuous medium are described by equations that do not depend on pressure. However, it is known that the vortex motion of air atmosphere is associated with changes in atmospheric pressure. Through numerous observations, meteorologists have learned to associate areas of low pressure with cyclones and areas of high pressure with anticyclones and thus predict the weather.

The associated problem is how to explain the vortex motion in the atmosphere associated with changes in pressure. After all, the equations of vortex motion of a continuous medium (5) do not depend on pressure and, therefore, clearly contradict meteorological observations, which in this case are, in fact, a physical experiment.

Interest in vortex motions in the atmosphere is related to the vortex force obtained in Ref. [2], which in a continuous medium has the form

\[
f^V_j = e_{ijk} v_i e_{kmn} \frac{\partial p}{c^2} \frac{\partial}{\partial x_m}, \tag{6}
\]

where \(c\) is the speed of sound. The vortex force (6), in contrast to the Euler vortex force (3), contains an obvious dependence on the pressure gradient.

In addition, in Ref. [2] the centrally symmetric force acting in a continuous medium was obtained from the gradient symmetry and the principle of least action:

\[
f^C_j = -\rho v_i \frac{\partial v_j}{\partial x_j} - \frac{v_j}{c^2} \frac{\partial p}{\partial t}, \tag{7}
\]

The first term in Eq. (7) corresponds to the potential term of the Euler hydrodynamics equations. The second term in Eq. (7) is irrespective of spatial derivatives. In Ref. [2] it was shown that this force describes homogeneous motions in a continuous medium, \(v_j = v_{j0} \exp(\beta p_0 - \beta p)\), as solutions of the homogeneous equation of motion, \(\rho \partial v_j / \partial t = -c^{-2} v_j \partial p / \partial t\), where \(\beta = c^{-2} \rho^{-1}\) is compressibility. Note that the temporal derivative of the pressure in Eq. (7) is absent in the equations of hydrodynamics [1].

The purpose of this article is to describe the vortex motions in the atmosphere due to the pressure gradient (6) and the temporal derivative of the pressure (7).

Section 2 describes a flat vortex motion of a continuous medium due to high or low pressure. In Sect. 3 the lifting force generated by the vortex motion in the funnel is calculated from Eq. (6). In Sect. 4 it is proved that the vortex force of Euler equations (3) is the vortex force obtained from the principle of least action. Section 5 presents a comparative analysis of the equations obtained from the least action and the Euler, Navier–Stokes, and Friedman equations and a discussion of the Millennium Problem of the impossibility of describing turbulence by the Navier–Stokes equations.
Due to the fundamental difference between the vortex force (6) and the force (3) and the absence of the second component of force (7) in hydrodynamics [1], we recall how the forces (6) and (7) were obtained in Ref. [2].

In 1983, Kadić and Edelen formally introduced two forces similar to those of Coulomb and Lorentz [3,4]:

\[
f_j = p_i \left( -\frac{\partial \nu_i}{\partial x_j} + \frac{\partial A_{ij}}{\partial t} \right), \\
\quad \text{(8)} \\
f_j = -\epsilon_{jqp} \nu_i \epsilon_{eqmn} \frac{\partial A_{mn}}{\partial x_m}. \\
\quad \text{(9)}
\]

Here \( A_{in} \) is the distortion tensor [5], \( \nu_i \) is the velocity field [3,4], and \( \nu_i \) is the velocity of particle flow with momentum \( p_i \). As is known, the distortion tensor \( A_{in} \) appeared in the elasticity theory [5] to determine the dislocation density: \( \rho_{iq} = -\epsilon_{qmn} \partial A_{in}/\partial x_m \). The distortion tensor \( A_{ij} \) was introduced as a generalization, \( \partial u_i/\partial x_n \), where \( u_i \) is the displacement vector. Substituting the expression \( \rho_{iq} = -\epsilon_{qmn} \partial A_{in}/\partial x_m \) in Eq. (9), we obtain a vortex force, which is similar in form to the Lorentz force: \( f_j = \epsilon_{jqp} \nu_i \epsilon_{eqmn} \frac{\partial A_{mn}}{\partial x_m} \). Note that in a solid state it is known as the Peach–Koehler force \( f_j = \epsilon_{jqp} \rho_{iq} \sigma_{ip} \) [3–6]. Also, the velocity field \( \nu_i \) in Refs. [3,4] is a generalization of the derivative \( \partial u_i/\partial t \).

It is easy to verify that the forces (8) and (9) are invariant under gradient transformation: \( A_{ij} \rightarrow A_{ij} + \partial u_i/\partial x_j, \nu_i \rightarrow \nu_i + \partial u_i/\partial t \). This is a property of the intensities of the compensating fields of interaction: \( \nu_i, A_{ij} \). Intensities are the forces that act on a unit charge. For fields \( \nu_i, A_{ij} \), the invariant intensities are the fields \( \epsilon_{ij} = -\partial \nu_i/\partial x_j + \partial A_{ij}/\partial T, \rho_{iq} = -\epsilon_{qmn} \partial A_{in}/\partial x_m \) (8), (9). It is easy to see that \( \epsilon_{ij} \) and \( \rho_{iq} \) are analogues of electric intensity and magnetic induction.

According to the Kadić–Edelen theory [3,4], the momentum \( p_i \) is similar to the electric charge (8), (9) and the compensating fields \( \nu_i, A_{ij} \) are similar to the electromagnetic potentials \( \varphi, A_j \). In the gauge theory of dislocations [3,4] the fields \( \nu_i, A_{ij} \) are linked by a pseudo-Lorentz gauge condition:

\[
\frac{\partial A_{ij}}{\partial x_j} = e^{-2} \frac{\partial \nu_i}{\partial t}. \\
\quad \text{(10)}
\]

The gauge condition (10) has a simple physical meaning. It describes mechanical waves in a continuous medium. Since the gradient transformation is \( A_{ij} \rightarrow A_{ij} + \partial u_i/\partial x_j, \nu_i \rightarrow \nu_i + \partial u_i/\partial t \), from Eq. (10) it follows that mechanical waves exist, and they propagate with the speed of sound:

\[
\frac{\partial^2 u_i}{\partial x_j^2} = e^{-2} \frac{\partial^2 u_i}{\partial t^2}.
\]

In Ref. [7], it was shown that the distortion tensor \( A_{ij} \) is a compensating field of minimal interaction induced by a subgroup of translations. In Ref. [2] the force that acts on the momentum of a particle \( p_i \) in the interaction field \( \nu_i, A_{ij} \) was calculated. As a result of calculations of the principle of least action for the action \( S_{\varphi} = -p_i \int_{a}^{b} (A_{ij} dx_j + \nu_i dt) \) in Ref. [2] the force was obtained, which is exactly equal to the sum of the two forces (8) and (9). This means that the analogy with electrodynamics traced by Kadić–Edelen is true, and the compensating interaction fields \( \nu_i, A_{ij} \) are independent fields.

This means that they exist wherever there is momentum, not just in a solid state. Simply put, in a solid state, the tensor interaction \( A_{ij} \) coincides with the distortion tensor \( A_{ij} \), and its anti-symmetric derivative coincides with the dislocation density \( \rho_{iq} = -\epsilon_{qmn} \partial A_{in}/\partial x_m \) [5]. At the same time, the interaction fields \( \nu_i, A_{ij} \) act on the momentum \( p_i \) (8), (9), and the electromagnetic field acts on the electric charge.
It is known that in a continuous medium the momentum is \( p_i = \rho \nu_i \). Then, from the law of conservation of momentum,

\[
\frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial p_i}{\partial t},
\]  

(11)

and from condition (10) it follows that \( \sigma_{ij} = \rho c^2 A_{ij} \) in a continuous medium.

Indeed, from Eqs. (10) and (11), as from the continuity equations, it follows that the fields \( A_{ij}, \sigma_{ij} \) are flows: \( A_{ij} = -c^{-2} \nu_i \nu_j, \sigma_{ij} = -p_i \nu_j \). In a continuous medium \( p_i = \rho \nu_i \) and \( \sigma_{ij} = -\rho \nu_i \nu_j \), which suggests that \( \sigma_{ij} = \rho c^2 A_{ij} \).

The relation \( \sigma_{ij} = \rho c^2 A_{ij} \) is, in fact, a condition of elasticity of a continuous medium or a generalization of Hooke’s law, since the distortion tensor \( A_{ij} \) is a generalization of the deformation tensor [5]. This, therefore, means that in this model of physical fields \( \nu_i, A_{ij} \) the state of the continuous medium, which is given by the ratio \( p_i = \rho \nu_i \), is elastic: \( \sigma_{ij} = \rho c^2 A_{ij} \).

In a gas, \( \sigma_{ij} = -\delta_{ij} p \). Hence, in a gas, \( A_{ij} = -\rho^{-1} c^{-2} \delta_{ij} p \). Substituting both \( p_i = \rho \nu_i \) and \( A_{ij} = -\rho^{-1} c^{-2} \delta_{ij} p \) in Eq. (9), and considering that in the continuous medium the velocity field \( \nu_i \) coincides with the flow velocity \( \nu_i \), we obtain the vortex force: \( f_j^V = e_{ijk} \nu_l e_{kmn} \frac{\partial \nu_m}{\partial x_l} \) (6).

Substituting \( p_i = \rho \nu_i \) and \( A_{ij} = -\rho^{-1} c^{-2} \delta_{ij} p \) in Eq. (8), we obtain a centrally symmetric force (7) acting in a continuous medium [2]. The first term in Eq. (7), \( -\rho \nu_i \partial \nu_i / \partial x_j \), is a potential term of the Euler equation (4), which is responsible for the change in the kinetic energy of a continuous medium. The second term in Eq. (7), \( f_j^T = -c^{-2} \nu_j \partial p / \partial t \), is the force, which depends on the temporal derivative of the pressure and is responsible for the homogeneous motion of the continuous medium [2].

In this theory, pressure \( p \) acts as an external source for the distortion tensor \( A_{ij} \) and the velocity field \( \nu_i \). Here pressure \( p \) is an external thermodynamic parameter, as well as density \( \rho \).

Note that in the transition from Eqs. (8) and (9) to Eqs. (6) and (7) it is assumed that the density \( \rho \) is the equilibrium constant density of the continuous medium. In this case, the deformation of the continuous medium is given by the distortion tensor \( A_{ij} = -\rho^{-1} c^{-2} \delta_{ij} p \), which has the form \( A_{ij} = -\delta_{ij} A \) [2]. Denoting the change in the density of the continuous medium, \( \rho' = \rho A \), we obtain a known relationship between the change in density and pressure \( \rho' c^2 = \rho' \) during mechanical vibrations [1].

Thus, the expression for forces (6), (7) is a consequence of the basic principles of theoretical physics: symmetry, resulting in minimal interaction with the compensating field \( A_{ij} \) [7], and the principle of least action [2]. Taking into account the condition for the existence of Eqs. (8) and (9) in a continuous medium with density \( \rho, p_i = \rho \nu_i \), forces \( f_j^V, f_j^T \) (6), (7) were obtained, which have not previously been studied in a continuous medium [1].

Note that in this paper we study the vortex motions of a continuous medium that are not significantly affected by the action of the viscosity force (p. 4) and the Coriolis force since they do not depend on pressure. A more detailed study of the effect of these forces on the continuous medium is beyond the scope of this paper.

2. Flat vortex motion of a continuous medium depending on high and low pressure

We use the convolution of the Levi–Civita tensor, \( e_{jik} e_{kmn} = \delta_{im} \delta_{ln} - \delta_{in} \delta_{lm} \), in the expression for the vortex force (6), and obtain an equation for the vortex force (9) in a continuous medium in the
form
\[ f_j^V = \frac{v_i v_j}{c^2} \frac{\partial p}{\partial x_j} - \frac{v_j v_i}{c^2} \frac{\partial p}{\partial x_i}. \] (12)

Let us examine this expression. In the 1D case, when the velocity \( v_j \) and force \( f_j^V \) lie on the same line, expression (12) is equal to zero.

Consider the 2D case of a plane vortex (circular) motion. Then the force (12) will be equal to
\[ f_j = \frac{v_j^2}{c^2} \frac{\partial p}{\partial x_j}; \] (13)
since the vortex motion velocity is perpendicular to the force \( v_i \partial p / \partial x_i = 0 \), the second term in expression (12) is zero for circular motion.

The vortex force (13) is proportional to velocity squared and the pressure gradient. That is, it is zero without vortex motion. In fact, the force (13) contains a multiplier \( v_j^2 / c^2 \) and is directed towards increasing pressure, in contrast to the usual force acting in a continuous medium at a pressure drop: \( f_j = -\partial p / \partial x_j \). At speeds one order of magnitude lower than the speed of sound, up to 30 m/s, the force (13) is two orders of magnitude smaller than the force \( f_j = -\partial p / \partial x_j \) acting in a continuous medium.

Thus, the force (13) makes the vortex movements stable, as it is directed towards increasing pressure. The force (13) in cyclones is directed away from the center, and in anticyclones it is directed towards the center. When a vortex motion appears, this force essentially “pushes” the pressure. The force (13), unlike conventional forces, \( -\partial p / \partial x_j \), which even out the pressure change, creates the well-known picture of circular isobars given on meteorological weather maps.

Consider a plane vortex motion with a pressure distribution in the form of circular isobars. In this case, the force in a continuous medium will be
\[ f_j = \frac{v_j^2}{c^2} \frac{\partial p}{\partial x_j} - \frac{\partial p}{\partial x_j} - \rho \frac{\partial v_j}{\partial x_j} - \frac{v_j \partial p}{c^2} \frac{\partial}{\partial t}. \] (14)

The first term in Eq. (14) is a consequence of the vortex force (12). The second term is a pressure gradient with a minus sign. The third and fourth terms in Eq. (14) are the force (7) in a continuous medium. The potential term \( -\rho v_i \partial v_i / \partial x_j \) (7), (14) is also contained in the Euler equation (4). The fourth term \(-c^{-2} v_j \partial p / \partial t \) of Eq. (7) is a consequence of the gradient invariance (8). This force has not been previously taken into account in the equations of hydrodynamics [1].

Similar to the description of vortex movements in hydrodynamics [1], let us act by the curl operator on the expression (14), taking into account that \( f_j = \rho \partial v_j / \partial t \):
\[ \rho \frac{\partial}{\partial t} \left( e_{pqj} \frac{\partial v_j}{\partial x_q} \right) + e_{pqj} \frac{\partial}{\partial x_q} \left( \frac{v_j \partial p}{c^2} \right) = e_{pqj} \frac{\partial}{\partial x_q} \left( \frac{v_i v_j}{c^2} \frac{\partial p}{\partial x_i} \right). \] (15)

It is immediately evident that this expression contains a dependence on pressure, in contrast to expression (5) for vortex motion obtained from the Euler equations.

Since for the horizontal plane the Levi–Civita symbols in Eq. (15) are equal to \( e_{3qj} \), Eq. (15) has the form
\[ \rho \frac{\partial}{\partial t} \left( \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) + \frac{1}{c^2} \left( \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) \frac{\partial p}{\partial t} + \frac{v_2}{c^2} \frac{\partial p}{\partial t} \frac{\partial}{\partial x_1} - \frac{v_1}{c^2} \frac{\partial p}{\partial t} \frac{\partial}{\partial x_2} \]
The solution of Eq. (17) is the exponent distribution of isobars, when increasing the pressure, the speed of the vortex \( \partial \) the radius. However, initially, in this problem we considered the case of radial pressure distribution: then, according to Bernoulli’s law, the pressure must also change in the direction perpendicular to the pressure in the atmosphere can be measured and is quite stable. Therefore, the question is what continuous medium is distributed depending on the pressure in the general case. It is known that (8) in a continuous elastic medium.

This equation describes the vortex motion of air in the atmosphere in the planar case. We will look for solutions of this equation for the pressure whose isobars in the horizontal plane have the form of concentric circles. Then the pressure will depend only on the radius. In this case, it is convenient to move from the coordinates \( x_1, x_2 \) to spherical coordinates \( r, \alpha \).

We introduce the notation \( v_r = \partial r/\partial t, v_\alpha = \partial \alpha/\partial t \). Then \( v_1 = v_r \cos \alpha - v_\alpha r \sin \alpha, v_2 = v_r \sin \alpha + v_\alpha r \cos \alpha \). Since initially, when writing the force (13), it was assumed that the velocity is directed perpendicular to the radius, let us assume that \( v_r = 0 \), then \( v_1 = -v_\alpha r \sin \alpha, v_2 = v_\alpha r \cos \alpha \). The components of the pressure gradient in spherical coordinates have the form \( \partial p/\partial x_1 = \cos \alpha \partial p/\partial r, \partial p/\partial x_2 = \sin \alpha \partial p/\partial r \), since only the radial pressure distribution is considered here and it is assumed that \( \partial p/\partial \alpha = 0 \).

In this case, the right-hand part of expression (16) will become \(-2r v_\alpha \partial v_\alpha/\partial \alpha \partial p/\partial r\). Changing the angular velocity \( \partial v_\alpha/\partial \alpha \neq 0 \) means that there is a velocity gradient, \( rv_\alpha \), perpendicular to the radius, but then, according to Bernoulli’s law, the pressure must also change in the direction perpendicular to the radius. However, initially, in this problem we considered the case of radial pressure distribution: \( \partial p/\partial \alpha = 0 \). Therefore: \( \partial v_\alpha/\partial \alpha = 0 \).

Thus, in the case of \( \partial p/\partial \alpha = 0 \) and \( v_r = 0 \), the right-hand part of expression (16) is zero. The left-hand part of expression (16) becomes

\[
2p \frac{\partial v_\alpha}{\partial t} + \rho r \frac{\partial}{\partial r} \frac{\partial v_\alpha}{\partial t} + \frac{r}{c^2} \frac{\partial}{\partial r} \frac{\partial p}{\partial t} + 2 \frac{\partial p}{c^2} \frac{\partial}{\partial t} + \frac{\alpha r}{c^2} \frac{\partial}{\partial t} = 0,
\]

or

\[
2 \left( \rho \frac{\partial v_\alpha}{\partial t} + \frac{\alpha}{c^2} \frac{\partial p}{\partial t} \right) + r \frac{\partial}{\partial t} \left( \frac{\partial v_\alpha}{\partial t} + \frac{\alpha}{c^2} \frac{\partial p}{\partial t} \right) = 0,
\]

whence it follows that

\[
\frac{1}{v_\alpha} \frac{\partial v_\alpha}{\partial t} = -\frac{1}{\rho c^2} \frac{\partial p}{\partial t}.
\]

The solution of Eq. (17) is the exponent

\[
v_\alpha = v_{\alpha 0} \exp(\beta p_0 - \beta p).
\]

Solution (18) has a simple physical meaning. For a flat vortex of air movement with a concentric distribution of isobars, when increasing the pressure, the speed of the vortex \( v_\alpha \) decreases and when the pressure drops the speed of the vortex \( v_\alpha \) increases. Indeed, in cyclones, the pressure decreases in the center, so in the center of the cyclone there is a more intense vortex movement of air. In anticyclones, the pressure decreases at the periphery, so there is a vortex motion of air masses. Note that expression (18) is universal and is executed for any time and coordinate.

This result was obtained taking into account the force \( f_j^T = -c^{-2}u_j \partial p/\partial t \) (7) in the equation of motion (14). Note that the temporal derivative of the pressure is not included in the Euler equations, so this result cannot be obtained from the Euler or Navier–Stokes equations. Expression (7) is a consequence of the gradient invariance of the interaction fields \( u_i, A_{ij} \) and is derived from the force (8) in a continuous elastic medium.

As is known, meteorologists are interested in solving the problem of how the velocity of the continuous medium is distributed depending on the pressure in the general case. It is known that the pressure in the atmosphere can be measured and is quite stable. Therefore, the question is what
equations to use to calculate the motion of air masses: Navier–Stokes equations [1], which are based on the Euler equation (4), or equations of motion, given by forces (6) and (7) in a continuous medium:

\[ \rho \frac{\partial \upsilon_j}{\partial t} = -\frac{\partial p}{\partial x_j} - \rho \upsilon_i \frac{\partial \upsilon_j}{\partial x_i} \frac{\partial p}{\partial t} - \frac{1}{c^2} \frac{\partial p}{\partial x_j} - \frac{1}{c^2} \frac{\partial p}{\partial x_i}. \] (19)

The main aim of this paper is to prove that the motion in a continuous medium is given by expression (19) and not by the Euler equation (4). A comparative analysis of Eqs. (4) and (19) is made in Sect. 5.

Note that solution (18) can be obtained directly from the solutions of Eq. (19) for the plane case without the action of the curl operator on the equations of motion (19).

Indeed, we will look for solution (19) in the form \( \upsilon_r = 0, \frac{\partial p}{\partial \alpha} = 0 \) in the planar case. Then we get two equations, the first of which has the form

\[ r \left( \rho \frac{\partial \upsilon_\alpha}{\partial t} + \upsilon_\alpha \frac{\partial p}{c^2} \right) \frac{\partial p}{\partial r} = \left( 1 - \frac{\upsilon_\alpha^2 r^2}{c^2} \right) \frac{\partial p}{\partial r} + \rho \upsilon_\alpha r^2 \frac{\partial \upsilon_\alpha}{\partial r}. \] (20)

The second equation can be obtained from Eq. (20) by replacing \( \frac{\partial p}{\partial r} \) by \( -ctg \alpha \).

Expression (20) is for any angle \( \alpha \) from which follows Eq. (17) and its solution (18). In addition, it follows from Eq. (20) that if a planar vortex motion is independent of time then the ratio

\[ \left( 1 - \frac{\upsilon_\alpha^2 r^2}{c^2} \right) \frac{\partial p}{\partial r} + \rho \upsilon_\alpha r^2 \frac{\partial \upsilon_\alpha}{\partial r} = 0. \] (21)

It follows from expression (21) that, at the velocities of the continuous medium equal to the velocity of sound, when \( \upsilon^2 / c^2 = 1 \) there can be no vortex movements, since there will be no force giving a centripetal acceleration to the continuous medium (14). Equality \( \frac{\partial \upsilon_\alpha}{\partial r} = 0 \) under the condition \( \upsilon^2 / c^2 = 1 \) (21) means that in this case the continuous medium must move as a solid, with the same angular velocity. Indeed, in nature there are no tornadoses at a speed greater than the speed of sound.

### 3. Calculation of the lifting force in the vortex motion of a continuous medium

Let us consider an example of using the vortex force (12) for the 3D case. The force (6), (12) appears in the vortex motion in the pressure gradient field. In stationary continuous motion of a continuous medium, the pressure gradient \( -\partial p/\partial x_j \) is compensated by the force, \( -\rho \upsilon_i \partial \upsilon_j / \partial x_j \), since the air pressure is the energy density of the air, and it decreases with increasing kinetic energy of air flows. This is described by Bernoulli’s law. Note that this follows from both the Euler equation (4) and the force (19) obtained from Eqs. (8) and (9) in a continuous medium. In this case, the projection of the vortex force (6) on the direction of motion of the continuous medium is zero, as well as the projection of the force (3).

Let us consider a tornado as an example of a 3D case of air movement. It is known that a tornado is a vortex motion of air in a horizontal plane in a column of air, which has the form of a funnel, with an extension upwards. We calculate the vertical component of the vortex force (12). According to Eq. (12), component \( f_3 \) has the form

\[ f_3 = \frac{\upsilon_\perp}{c^2} \frac{\partial p}{\partial x_3}, \] (22)

where \( \upsilon_\perp^2 = \upsilon_1^2 + \upsilon_2^2 \), since in the horizontal plane \( \upsilon_1 \partial p / \partial x_1 + \upsilon_2 \partial p / \partial x_2 = 0 \).

Expression (22) represents the lifting force acting in a tornado as the gradient \( \partial p / \partial x_3 \) is directed upwards, due to the opening angle of the tornado funnel with the expansion upwards. From Eq. (22) it follows that the greater the horizontal speed of rotation in a tornado, the greater the lifting force.
It is known that the pressure in a tornado falls to half the atmospheric pressure. A simple calculation shows that the force (22) can be greater than gravity; e.g., at a tornado height of 500 m and a horizontal speed of 100 m/s,

\[ f_3 = \rho \rho_{atm}^{-1} u_3^2 \frac{\partial p}{\partial x_3} \approx \rho \cdot 10^{-5} \cdot 10^4 \cdot 0.5 \cdot 10^5 \cdot 2 \cdot 10^{-3} = \rho \cdot 10^2 M/c^2. \]

Here it was assumed that the vertical pressure changes uniformly, and the atmospheric pressure is equal: \( p_{atm} = \rho c^2 \). These parameters are conditional; e.g., the speed may be less, and the pressure gradient is greater. It is important that the free-fall acceleration can really be achieved for the lift (22) in a tornado.

If gravity and the vertical force (22) are not taken into account, the movement of air along the funnel is possible in either direction, both up and down (the change in atmospheric pressure due to the height of 500 m was not taken into account in these calculations). This movement is similar to the movement of water in a riverbed when narrowing or expanding the channel. When stationary 1D motion forces \(-\partial p/\partial x_3\) and \(-\rho u_3 \partial u_3/\partial x_3\) balance each other, they are equal and opposite-directed.

If we take into account the vortex motion and gravity, then, with an increase in the speed of rotation, the force (22) can compensate for the gravity: \(-\rho g\). As a result of the vortex motion power \(-\rho u_\perp \partial u_\perp/\partial x_3\) in the funnel appears. This force is also directed upwards, as the vortex velocity is greater at the base of the tornado. It is balanced by a pressure gradient, \(-\partial p/\partial x_3\), according to the Bernoulli equation, which leads to an increase in the pressure gradient and a greater lifting force (22).

This schematic description is given here to show the possibilities of using the forces (8) and (9) obtained from the principle of least action for the continuous medium (6) and (7). As shown above, it is necessary to set boundary conditions and to solve Eq. (19).

For liquids, expression (22) explains why the whirlpools are sucked into the funnel. For liquids, though the funnel and the extension are up, the pressure gradient is directed downwards—in the direction of increasing pressure. In this case, it is necessary to take into account the pressure that is created by the liquid column, since the liquid is three orders of magnitude heavier than air.

The lifting vortex force (22) obtained from Eq. (12) has never been taken into account in fluid dynamics before, since the vortex force of Euler (3) does not depend on pressure. The question arises, what is the relationship between vortex force (6), (12) and the vortex force of Euler (3)?

### 4. The vortex force of Euler contained in the vortex force obtained from the principle of least action

The vortex force of Euler hydrodynamics has the form (3)

\[ f_j = \rho \epsilon_{jmn} \epsilon_{npq} \partial u_q / \partial x_p. \]

Let us use the convolution of Levi–Civita \( \epsilon_{jmn} \epsilon_{npq} = \delta_{jp} \delta_{mq} - \delta_{jq} \delta_{mp} \) tensors and write it in the known form [1]:

\[ f_j = \rho \nu_m \frac{\partial \nu_m}{\partial x_j} - \rho \nu_m \frac{\partial \nu_j}{\partial x_m}. \]

(23)

To connect the vortex force (9) with the vortex force of Euler (23), we use the condition (10), which has the form of a continuity equation. From Eq. (10) it follows that the tensor field \( A_{ij} \) can be written as a flow velocity field: \( A_{ij} = -c^{-2} u_i u_j \). In a continuous elastic medium, the flow rate coincides with the velocity field; therefore, the tensor field \( A_{ij} \) has the form \( A_{ij} = -c^{-2} u_i u_j \). Substituting \( p_i = \rho u_i \), \( A_{ij} = -c^{-2} u_i u_j \) in Eq. (9) and taking into account the convolution of Levi–Civita tensors, we obtain
a vortex force in the form of \( f_j = \rho \frac{\nu \nu_m}{c^2} \left( \frac{\partial (\nu_j \nu_m)}{\partial x_j} - \frac{\partial (\nu_i \nu_j)}{\partial x_m} \right) \) or, after differentiation in parts,

\[
f_j = \rho \nu_m \frac{\nu^2}{c^2} \left( \frac{\partial \nu_m}{\partial x_j} - \frac{\partial \nu_j}{\partial x_m} \right) + \rho \frac{\nu_i \nu_m}{c^2} \left( \nu_m \frac{\partial \nu_i}{\partial x_j} - \nu_j \frac{\partial \nu_i}{\partial x_m} \right). \tag{24}\]

The first component of the vortex force in Eq. (24) corresponds to the vortex term of Euler hydrodynamics (23) multiplied by a dimensionless factor \( \nu^2/c^2 \). The second term is not taken into account in the vortex force of Euler.

In a continuous medium, \( p_i = \rho \nu_i, \sigma_{ij} = \rho c^2 A_{ij} \). Hence, in gas, \( \sigma_{ij} = -\delta_{ij} p \) and \( A_{ij} = -\rho^{-1} c^{-2} \delta_{ij} p \). Substituting these expressions in Eq. (9) we obtain (12): \( f_j^V = \frac{\nu_j \nu_p}{c^2} \frac{\partial p}{\partial x_j} - \frac{\nu_i \nu_p}{c^2} \frac{\partial p}{\partial x_i} \).

In Eq. (24) the continuous medium was also taken into account, \( p_i = \rho \nu_i \) and \( A_{ij} = -c^{-2} \nu_i \nu_j \) (10), but the convolution of the stress tensor in the continuous medium \( \sigma_{ij} = -\delta_{ij} p \) was not. Therefore, Eq. (12) depends on the components of the pressure gradient and not on the components of the velocity gradient of the continuous medium.

On the one hand, the vortex force (9) written as Eq. (24) in a continuous medium takes into account the vortex term of the Euler equations, although it does not fully describe Eq. (24). This indicates the continuity of the description of the vortex motion of the continuous medium.

On the other hand, in Eq. (24) a relative multiplier appeared before the Euler vortex term, \( \nu^2/c^2 \), which indicates that the Euler vortex term can give the correct result in only one case, when the velocity of the continuous medium is equal to the velocity of sound. At the same time, in Sect. 2 it was shown by expression (21) that in a continuous medium there can be no vortex movements at a speed equal to the speed of sound.

So, at small velocities, of the order of 1 m/s, the Euler equations (3), (4), and (23) give a vortex force five orders of magnitude greater than the vortex force (6), (12), and (24). In this case, at speeds above 1 km/c, the Euler equation (23) gives a vortex force of an order of magnitude less than the vortex force (24).

Since the Euler equations at low velocities predicted vortex motions of a continuous medium, which are not really present, it was proposed to take into account the viscosity to explain the laminar nature of the motion of a continuous medium at low velocities. Thus, the Navier–Stokes equation appeared:

\[
\frac{\partial \nu_i}{\partial t} = -\rho \frac{\partial \nu_i}{\partial x_j} - \frac{\partial p}{\partial x_j} + \rho \nu_j \nu_k \nu_m \frac{\partial \nu_m}{\partial x_j} + \eta \frac{\partial^2 \nu_j}{\partial x_j \partial x_i}, \tag{25}\]

where \( \eta \) is the viscosity coefficient.

Expression (25) corresponds to the Navier–Stokes equation for an incompressible liquid or gas. The Navier–Stokes equation (25) is written in this form in order to simplify the writing, since the compressibility of the continuous medium does not affect the problems that are considered here.

The Navier–Stokes equations gave an answer to the question of why laminar motion is observed at low velocities in a liquid [1]. However, in gas they do not explain the absence of turbulence at speeds three orders of magnitude lower than the speed of sound.

Indeed, the viscosity coefficient \( \eta \) for air is of the order of \( 10^{-5} \) Pa s. This means that there is no turbulence if the speed of objects in a continuous medium, with a characteristic size \( l \) of about 1 m, is less than a millimeter per hour, since the Reynolds number \( \eta^{-1} \rho v l \) should be an order of magnitude less than unity.

The denominator of the factor \( \nu^2/c^2 \) in Eqs. (12) and (24) gives the same order of magnitude as the viscosity coefficient: \( \eta \approx 10^{-5} \) Pa s. This solves the problem of explaining the vortex-free movement.
of air at low speeds. Without the coefficient $\nu^2/c^2$ in Eq. (24) it is impossible to explain laminar motion in the air at speeds of about 0.1 m/c. It is known that the smoke from a cigarette at such speeds has a laminar motion.

Thus, the vortex force of Euler (3), (23) is part of the vortex force obtained from the principle of least action, which in the continuous medium has the form (24), but with a relative coefficient $\nu^2/c^2$, which leads to fundamentally different results. The coefficient $\nu^2/c^2$ explains the absence of turbulence at low velocities of the continuous medium, along with the viscosity coefficient, and explains the presence of large turbulence at velocities above the speed of sound. In addition, the vortex force (12) contains a clear dependence of the vortex force on pressure, which explains the vortex motion in the atmosphere (13), (22).

The Euler equation (4) does not take into account the speed of sound, $c$. This follows from their derivation (1)–(4). Then the question arises of how the forces (6), (7), (12), (13), (15), (19), and (22) take into account the speed of sound: where did it come from?

The appearance of the speed of sound in the equations of motion of a continuous medium is associated with a minimum interaction [2,7] and a gauge condition (10), which implies that the mechanical waves of the continuous medium extend at the speed of sound. Minimal interaction with compensating fields $\nu_i, A_{ij}$ was constructed as well as the electromagnetic interaction, which is known to be relativistic and contains the speed of light [8].

In connection with the above, there is a question: what equations describe the dynamics of a continuous medium? After all, even if we agree that the Euler equations describe vortex movements in a continuous medium (5) incorrectly, since they do not take into account the relative coefficient $\nu^2/c^2$ and the pressure change in the vortex motion (15), then the question of how to deal with the Euler equations themselves remains. Different equations of motion (4) and (19) cannot describe the same dynamics of a continuous medium.

5. Analysis of the equations of state of the continuous medium obtained from the principle of least action

To compare the derivation of forces (4) in the Euler equations, we present a sequence of steps that resulted in the forces acting in a continuous medium (6), (7) in Eq. (19).

(1) The minimal interaction with the compensating interaction field $\nu_i, A_{ij}$ was constructed from translational spatial symmetry in Landau theory [2,7].

(2) For the compensating fields of interaction $\nu_i, A_{ij}$, gradient-invariant intensities $\epsilon_{iq}, \rho_{iq}$ were constructed, which represent the forces acting on the unit charge of interaction.

(3) Forces (8) and (9), which act on a particle with a momentum $p_j$ in the field of compensating interaction fields, were obtained from the principle of least action, $\nu_i, A_{ij}$ [2].

(4) A special case of a continuous elastic medium is considered, $p_j = \rho \nu_j, \sigma_{ij} = \rho c^2 A_{ij}$, in which the forces (8), (9) have the form (6), (7) and then the equations of motion (19) were written.

Note that the gradient symmetry for the fields $\nu_i, A_{ij}$ is broken in the continuous medium. Such states in the physics of phase transitions are called low-symmetric phases. Indeed, since in a continuous medium the fields $\nu_i, A_{ij}$ are proportional to the conjugate observable fields, $p_j = \rho \nu_j, \sigma_{ij} = \rho c^2 A_{ij}$, the gradient symmetry is broken and the fields $\nu_i, A_{ij}$ become observable. This means that the continuous medium is a low-symmetric state, in which the equations of state for the independent compensating interaction fields $\nu_i, A_{ij}$ are degenerate and
become inseparably related to the conjugate observable fields $p_j, \sigma_{ij}$. Therefore, in expression (19), all forces depend on pressure $p$ and velocity $v_i$, which is inseparably linked to momentum $p_j = \rho v_j$.

Compensating fields and minimal interaction appeared in field theory only in the 20th century [9,10]; they were unknown to either Euler or Navier and Stokes.

As in the Navier–Stokes equation (25) obtained from the Euler equations [1], the summands responsible for the viscosity of the continuous medium can be added to Eq. (19). The viscosity ratios answer the question why there is laminar flow at low speeds (p. 4). However, the question of describing turbulent movements at high speeds is more important and interesting, and there is no obvious answer about the existence of laminar flows in a continuous medium.

As is known, today the motion of air masses in the atmosphere is determined from the Friedman equations [11], which are a generalization of Eq. (5) and the Navier–Stokes equation for a compressible continuous medium:

$$\frac{\partial}{\partial t} \left( e_{pqj} \frac{\partial v_j}{\partial x_q} \right) - e_{pqj} \frac{\partial}{\partial x_q} \left( e_{ijk} v_i e_{kmn} \frac{\partial v_n}{\partial x_m} \right) = \frac{1}{\rho^2} \frac{\partial p}{\partial x_i} \frac{\partial \rho'}{\partial x_j} + e_{pqj} \frac{\partial}{\partial x_q} \left( f_{ij}^M \right),$$

where $f_{ij}^M$ is the strength density of the molecular viscosity.

It follows from Eq. (26) that the pressure in the expression for the vortex motion of air masses is included in the equation as a vector product of the pressure gradient and the density gradient.

It is known that the vortex motion in the form of cyclones and anticyclones is a plane motion parallel to the Earth’s surface, associated with a flat pressure distribution in the form of isobars in the atmosphere. The same plane distribution applies to the distribution of air density affecting cyclones and anticyclones. That is, the density gradient that affects cyclones and anticyclones also lies in the horizontal plane. Therefore, the projection of the vector product of the pressure gradient and density gradient onto the horizontal plane is zero in Eq. (26).

Thus, the Friedman equations (26) do not describe plane vortex motions in the atmosphere, since in the plane case they do not depend on pressure. As is known, cyclones and anticyclones are formed in cases of zones of low and high pressure, respectively. As was shown in Sect. 2, cyclones and anticyclones are described by vortex equations (14)–(19) and forces (6), (7) obtained in Ref. [2]. In Ref. [12] it is shown that, in the stationary case for a compressible continuous medium, cyclones and anticyclones are described by solutions obtained from Eq. (21).

In 2000, the Clay Mathematics Institute called the problem of description of turbulence one of the seven Mathematical Problems of the Millennium. The description of vortex motion in the atmosphere by Eq. (19) (p. 2, p. 3) and the lack of factor $v^2/c^2$ in the vortex term of the Euler equation (p. 4) clearly show that the problem of description of turbulence by the Navier–Stokes equations declared by the Clay Mathematics Institute cannot be solved in principle.

Firstly, the vortex terms of the Euler and Navier–Stokes equations (3), (25) do not contain pressure dependence, which does not allow the description of even the obvious vortex motions in the atmosphere in the form of cyclones and anticyclones using these equations (p. 2). Secondly, the absence of the factor $1/c^2$ in the Euler and Navier–Stokes equations does not allow laminar flows in the atmosphere to be explained. The absence of a relative dependence $v^2/c^2$ for the Euler vortex forces leads to errors in the calculations by six orders of magnitude, in the transition from the velocity of the order of 1 m/s to the velocities of the order of 1 km/s.

This is why the Millennium challenge has arisen. Then, however, the question arises, what forces operate in a continuous medium and how can they be calculated?
We use the definition of the stress tensor in the continuous medium: \( \frac{\partial \sigma_{ij}}{\partial x_i} = f_j \) \[5\]. Then, in gas or liquid, where \( \sigma_{ij} = -\delta_{ij}p \) \[1\], the force has the form \( f_j = -\partial p/\partial x_j \). From Newton’s second law \( dp_j/dt = f_j \) the equation of motion \( dp_j/dt = -\partial p/\partial x_j \) follows, which is a differential form of the law of conservation of momentum or continuity equation \(11\).

It is known that in addition to the force \( f_j = -\partial p/\partial x_j \) in a continuous medium, physical fields that transmit interactions can act on a single volume of liquid or gas. Since liquid and gas have a density, they are affected by the gravitational force: \( f_3 = -\rho g \). If the gas is electrically charged, then it will be affected by electromagnetic force. According to the principle of superposition, the equations of motion are the sum of all forces acting on the unit volume of the continuous medium.

In Ref. \[2\] the force equal to the sum of two forces \(8\), \(9\) was obtained from the principle of least action for a particle with a momentum in the compensating interaction field: \( \nu_i, A_{ij} \). Forces \(8\) and \(9\) are the same fundamental forces as electromagnetic forces, since they were obtained from the minimal interaction due to translational symmetry.

In a continuous medium, \( p_j = \rho \nu_j \), and there is elasticity: \( \sigma_{ij} = \rho c^2 A_{ij} \) \[10\], \[11\]. Substituting these expressions in Eqs. \(8\) and \(9\), taking into account \( \sigma_{ij} = -\delta_{ij}p \), we obtain two forces acting in a continuous medium \(6\), \(7\). Thus, these forces are included in the equations of motion of the continuous medium \(19\). Comparing Eqs. \(4\) and \(19\) we see that the potential force component coincides, the vortex force partly coincides \(p.4\), and the temporal derivative of the pressure is absent in Eq. \(4\).

Indeed, the first term of the force \(7\) corresponds to Bernoulli’s calculations, which were based on the law of conservation of energy of a continuous medium. This force was also written by Euler in Eq. \(4\).

The second term in Eq. \(7\), \( f_j^T = -c^{-2} \nu_j \partial p/\partial t \), is the force that contains the temporal derivative of the pressure. Force \( f_j^T \) was not taken into account earlier in hydrodynamics \[1\]. It is related to the gradient symmetry \( \varepsilon_{ij} \) of the tensor interaction field \(8\). In a continuous medium, this symmetry disappears, so in a continuous medium there is no reason to write the force \(7\). Apparently, the second term of forces \(7\), \( f_j^T = -c^{-2} \nu_j \partial p/\partial t \), cannot be written in continuum mechanics \[1\]. As shown above, this force is responsible for homogeneous motions in the atmosphere \[2\] and for the solution \(18\) describing vortex motions in the atmosphere for radial pressure distribution.

The vortex force \(6\), \(12\) was also not taken into account in the continuous medium in full, since there was no reason to write the vortex intensity \( \rho_{ij} \) \(9\) in the air \(6\), \(12\). Of course, \( \rho_{ij} \) is not the dislocation density, but the anti-symmetric derivative of the tensor interaction field \( A_{ij} \), which is everywhere, not only in solids \[2\]. Based on the calculations made in paragraph 4, we can conclude that part of the vortex force was recorded by Euler in Eq. \(4\), but without the factor \( \nu^2/c^2 \).

Incomplete writing of the vortex force \(24\) by Euler \(3\), \(23\) and the absence of the factor \( \nu^2 c^2 \) in Eq. \(23\) led to a difference between the values of the vortex force obtained by calculations from Eq. \(23\) and real measurements. This difference is measured in orders of magnitude. When the speed of the continuous medium is less than the speed of sound, the difference is less. When the speed of the continuous medium is above the speed of sound, the real force is higher than the vortex force of Euler. As a result, there was a Millennium Problem associated with the problem of describing the turbulence using the Navier–Stokes equations.
Thus, for the correct description of the continuum mechanics it is necessary to replace the Euler forces obtained in Eq. (4) by the forces (6) and (7), obtained from the minimum of the action [2]. In other words, replace Eq. (4) with Eq. (19).

It is clear how the forces (8) and (9) in Ref. [2] and the forces (6) and (7) acting in a continuous medium were obtained from a mathematical point of view. Therefore, there is another question: what interaction describes the compensating fields of interaction $\nu_i, A_{ij}$ [2] and their forces (8) and (9)?

In Ref. [2] it was proved that the charge for the fields $\nu_i, A_{ij}$ is a quantum momentum $p_i = \hbar k_i$. If the charge is known, then the interaction is known. At small distances, the wave vector $k_i$ is very large. Therefore, the smaller the distance, the stronger the charge and the stronger the interaction. The hypothesis is that the fields $\nu_i, A_{ij}$ and the charge $p_i = \hbar k_i$ describe a strong fundamental interaction. Most likely, the strong interaction is described by a three-parameter Abelian group, which is given by the vector $k_i$ and tensor fields of the interaction $\nu_i, A_{ij}$ rather than the non-Abelian group SU(2) and the three-vector compensating Yang–Mills fields.

In this case, the strong interaction will manifest itself not only at distances commensurate with the nucleus of the atom, but wherever there is a quantum momentum. The phenomenon of black holes and high-temperature plasma described in Ref. [13], as well as the correct description of vortex motions in the atmosphere described here, confirms this hypothesis.

The fact that the quantum momentum $p_i = \hbar k_i$ is a charge is proved by the pairing of electrons with equal and opposite-directed quantum momentum in the superconducting state [14]. The proof of this fact is beyond the scope of this article, but the proof scheme is based on the fact that in the 1D case, when quantum momentums lie on the same line, they can be considered as plus and minus charges depending on the direction, and as is known, opposite charges are attracted.

Why, with the minimal interaction, is the momentum quantum; why, in a continuous medium, is the momentum equal to $p_j = \rho \nu_j$; and why is there elasticity $\sigma_{ij} = \rho c^2 A_{ij}$? The answer to this question will be given in a following article, which describes the phase transition of the violation of the gradient symmetry of the field: $\nu_i, A_{ij}$. This phase transition is similar to the phase transition of the violation of the gradient symmetry of the electromagnetic field $\varphi, A_j$ in the superconducting state, when the current density is proportional to the electromagnetic potential [14]: $j_j = -\delta^{-2} A_i$ (here $\delta$ is the London depth of penetration of the magnetic field).

### 6. Conclusion

From the principles of symmetry and of least action, the vortex force $f^V_j = \frac{\nu_j \partial p}{c^2} \frac{\partial \nu_j}{\partial x_j} - \frac{\nu_i \partial p}{c^2} \frac{\partial \nu_i}{\partial x_i}$ (6), (12) acting in a continuous medium is obtained. For a plane circular motion, the vortex force has the form $f^V_j = \frac{\nu_j \partial p}{c^2} \frac{\partial \nu_j}{\partial x_j}$ (13). This force is directed towards increasing pressure. Therefore, when a vortex motion occurs, the force (13) leads to the appearance of a picture of circular isobars. In cyclones, the force (13) is directed from the center of the cyclone, and in anticyclones the force (13) is directed to the center of the anticyclone.

When there is radial distribution of pressure, the plane vortex motion is given by the ratio (18) between the angular velocity and the pressure, $\nu_\alpha = \nu_\alpha^0 \exp(\beta p_0 - \beta p)$, where $\beta = \rho^{-1} c^{-2}$. Solution (18) explains the relationship between vortex motion and the occurrence of a low-pressure region. For cyclones, the area of low pressure is in the center; for anticyclones it is on the periphery. Therefore, there is an intense vortex motion there. When the pressure rises, the vortex motion fades (18).
The solution (18) cannot be obtained from the Euler or Navier–Stokes equations (4), (25), since they do not contain a force \( f^T_j = -c^{-2} \nu_j \partial \rho / \partial t \) (7) dependent on the temporal derivative of the pressure (5), (15), which was also obtained from the principle of least action (7), (8).

Thus, having the dependence of the velocity of the continuous medium on the pressure, it is possible to calculate both the vortex motion of the continuous medium (15) and to study the total motion of the continuous medium due to pressure (19). The expression (19) contains the set of all forces (6), (7) obtained from the principle of least action (8), (9) for the case of a continuous medium depending on pressure and velocity. The expression (19) contains both known potential forces, \(-\partial \rho / \partial x_j - \rho \nu_j \partial \nu_i / \partial x_j\), in the form of gradients that lead to the Bernoulli equation, forces that depend on the temporal derivative of the pressure \( f^T_j = -c^{-2} \nu_j \partial \rho / \partial t \) (7), and the vortex force
\[
 f^V_j = \nu_i \nu_i c^{-2} \partial \rho / \partial x_j - \nu_j \nu_i c^{-2} \partial \rho / \partial x_i \quad (6), (12).
\]

In connection with the obtained results, an analysis of forces that are taken into account in the equations of hydrodynamics of Euler (4) is made. In Sect. 4 it is shown that the vortex force of Euler \( \rho \epsilon_{ijk} \nu_i \epsilon_{kmm} \partial \nu_n / \partial x_m \) (3.23) is contained in the vortex force (24) obtained from the principle of least action (9), in the form of the term \( \nu^2 c^{-2} \rho \epsilon_{ijk} \nu_i \epsilon_{kmm} \partial \nu_n / \partial x_m \), with the coefficient \( \nu^2 / c^2 \). This indicates a fundamental difference in the description of the vortex motion by the forces obtained from the principle of least action (19) and the Euler forces (4).

The relative factor \( \nu^2 / c^2 \) means that the vortex force of Euler (23) coincides with the component of the vortex force obtained from the principle of least action (24) only in the case when the velocity of the continuous medium is equal to the velocity of sound. The relative coefficient \( \nu^2 / c^2 \) is present in all expressions for the vortex force (6), (12), (13), (22). This explains why at low speeds there is no turbulence or vortex motion, and at speeds approaching the speed of sound and above, turbulence is significant.

The problem of describing vortex and turbulent motion by the Euler equations, or Navier–Stokes equations, is called a Millennium Problem by the Clay Mathematics Institute. This is due to the fact that the Euler and Navier–Stokes equations cannot correctly describe even the obvious motion of cyclones and anticyclones in the atmosphere (p. 2). Namely, to link the plane vortex motion with the pressure change (18), not to mention the description of turbulence at high speeds.

The article shows that the problem posed by the Clay Mathematics Institute—the description of the turbulent motion of the Navier–Stokes equations—cannot be solved. Euler equations and Navier–Stokes equations cannot describe the vortex motion and turbulence as they do not take into account the dependence of vortex motion on pressure (5), (15), and do not take into account the ratio of the speed of vortex motion to the speed of sound in the continuous medium: \( \nu^2 / c^2 \) (p. 4). Therefore, nothing can be calculated by these equations as they give deviations of the measured values by orders of magnitude and in different directions at speeds above and below the speed of sound.

Thus, the vortex motion of a continuous medium and turbulence are described by mechanical forces (6), (7), which were obtained from the principle of least action (8), (9) [2], and not by the Navier–Stokes equations [1]. To describe the vortex motion in the atmosphere correctly it is necessary to replace the Euler equation (4) by the equation obtained from the minimum of action (19).

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