Sub-Rate Linear Network Coding
B. Grinboim, I. Shrem and O. Amrani

Abstract—
Increasing network utilization is often considered as the holy grail of communications. In this article, the concept of subrate coding and decoding in the framework of linear network coding (LNC) is discussed for single-source multiple-sinks finite acyclic networks. Sub-rate coding offers an add-on to existing LNC. It allows sinks whose max-flow is smaller than the source message-rate, termed sub-rate sinks, to decode a portion of the transmitted message without degrading the maximum achievable rate of LNC sinks whose max-flow is equal (or greater) than the rate of the source node. The article studies theoretical aspects of sub-rate coding by formulating the conditions a node (and indeed the network) must fulfill so as to qualify as a legitimate sub-rate sink.

I. INTRODUCTION

Network coding is a research field targeting the improvement of communications within a network [3]. The fundamental idea, discussed in [1], is that intermediate nodes in the network serve as computational units rather than merely relays (i.e., assuming computational tasks may be implemented in them), thus supporting increased overall network throughput. Network coding has been shown to potentially increase overall network throughput in single-source multiple-sink communications scenarios (i.e., broadcast of massages), or multi-source networks[6]. One of the fundamental results in this field is that in the case of a single-source, acyclic, and finite network, given that all computations are done over a field large enough, LNC can achieve the maximum capacity to every eligible sink [6], [5]. By eligible sink, we refer to a sink whose max-flow (achievable throughput) is greater or equal to the source transmission-rate.

A simple example demonstrating the strength of a network coding is the butterfly network, a network that transmits two signals a and b to two sinks, 6 and 7 as demonstrated in Figures I.1 and I.2. In Figure I.1 the network does not employ network coding (i.e., intermediate nodes do not execute any coding-related computations) - hence only one of the sinks, 6 or 7, can decode both messages in each network use. In Figure I.2 the network employs network coding. To this end, node 4 can XOR (exclusive-or) both its inputs, which in turn enables both sinks to decode the transmitted messages.

In previous works pertaining to network coding, sinks with smaller max-flow than the transmission rate, (herein termed sub-rate sinks) had been considered ineligible, and so communication to them was avoided. This approach is derived from the classical definition of a multicast: a sink that can not receive all the information (a.k.a. network rate) is considered “useless”. While this holds true for many cases, there are other cases where retrieving some of the transmitted data is valuable; for example when a sub-rate sink can benefit from reduced data “resolution”. Alternatively, one can look at the other way around - the ability to communicate with some network sinks in the multicast using higher rate than other sinks, may enable the higher-rate sinks to retrieve the transmitted information faster than the others (given the communication is not continuous), or use the “spare” capacity for conveying data that is not necessarily required by all the sinks, e.g. monitoring information.

The question therefore: is it possible to configure a network code such that some of the ineligible sinks can still decode a portion of the input data? Apparently, in some cases - it is certainly possible. An example is given in Figures I.3 and I.4. In this example, another sink, 8, is added to the butterfly network, a node whose max-flow (from the source) is only 1. Clearly, it is assumed to be beneficial (hence required) to communicate with node 8 at rate of 1 symbol per network use. In the original configuration of the butterfly network code - node 8 receives a + b - and hence cannot decode any symbol, as shown in Figure I.3. By properly manipulating the transmitted symbols at the source node though, as depicted in Figure I.4 it is possible to configure the network so that node 8 operates at a communication rate of 1 symbol per network use, without affecting the ability of eligible sinks, 6 and 7, to operate at their original rate of 2 symbols per network use.

This work studies the ability to communicate with some of the network nodes as sub-rate max-flow sinks (without affecting the eligible sinks) using tailored linear network codes.

The rest of the paper is organized as follows. Section 2 provides the notations that will be used throughout the paper. Section 3 presents the basic intuition of sub-rate coding with LNC, which motivates the following definitions and theoretical contributions. The methodology of ful sub-rate coding is given in Section 4 along with sufficient conditions for its existence in a linear multicast with a set of sub-rate designated sinks. An algorithm for realizing such a linear multicast is also provided.
is often referred to as scalar network codes, which are elements of a base field $F$. $h_i$ is the positive integer representing the number of symbols created by the source every network use. In this paper, assume that $r$ is constant.

For every node $x \in V \setminus \{s\}$, denote by $In(x)$ and $Out(x)$ the sets of input and output edges to the node $x$, respectively. For convenience, it is customary to add another node $s'$, referred to as the imaginary source, that has $r$ outgoing edges to the original source $s$—referred to as the $r$ imaginary links (namely, $In(s)$ is the set of $r$ imaginary links). Assume that $G'$ includes the imaginary source and edges.

The basic concept of network coding is that all the nodes in the network are able to perform calculations over the field $F$.

The following is a description of a linear network code (LNC) for finite acyclic network that is derived from [8]:

**Definition 1.** An $r$-dimensional $F$-valued linear network code operating in an acyclic communication network is characterized by:

1) **Local Encoding Kernel (LEK)** - $\{k_{d,e}\}$ - A set of scalars $k_{d,e} \in F$ for every adjacent pair of edges $(d, e)$ in the network;

2) **Global Encoding Kernel (GEK)** - $\{f_e\}_{e \in E}$ - A set of $r$-dimensional column vectors $f_e$ for every edge $e$ in the network such that:

   a) For every non-source node $x$, and every $e \in Out(x)$, $f_e = \sum_{d \in In(x)} k_{d,e} f_d$;

   b) For the imaginary links $e \in In(s)$, the set of vectors $\{f_e\}_{e \in In(s)}$ are the $r$ linearly independent vectors that constitute the natural basis of the vector space $F^r$.

The local encoding kernel (LEK) associated with node $x$ refers to a $|In(x)| \times |Out(x)|$ matrix. The vector $f_e$ is called the global encoding kernel (GEK) for edge $e$.

Note that given the local encoding kernels at all the nodes in an acyclic network, the global encoding kernels can be calculated recursively in an upstream-to-downstream order according to the given definition.

**Definition 2.** Let $\{f_e\}_{e \in E}$ denote the global encoding kernels in an $r$-dimensional $F$-valued linear network code on a single-source finite acyclic network. Let $V_x = \text{span}\{f_d|d \in In(x)\}$. Then, the linear network code is a linear multicast if $\text{dim}(V_x) = r$ for every non-source node $x$ with $\text{max} - \text{flow}(x) \geq r$.

In [8], [9], an algorithm for the construction of a linear multicast is provided, requiring the field size $|F|$ to be greater than the size of the set of sinks (whose $\text{max} - \text{flow}(\{T\}) \geq r$) i.e., $|F| > |\{T : \text{max} - \text{flow}(T) \geq r\}|$. This algorithm provides a LNC that allows all target sinks to receive decodable information simultaneously in every network use.

Note that this algorithm does not require that the set of target sinks include all the eligible sinks (satisfying $h_i \geq r$), i.e., it is also possible to use the algorithm to create a generalized linear multicast, which is defined as follows:

**Definition 3.** Let $\{f_e\}_{e \in E}$ denote the global encoding kernels in an $r$-dimensional $F$-valued linear network code on a single-source finite acyclic network. Let $V_x = \text{span}\{f_d|d \in In(x)\}$. Define a set of non-source nodes $T \subseteq V$ as sinks. Then, the linear network code is a generalized linear multicast if $\text{dim}(V_t) = r$ for any sink $t \in T$ with $\text{max} - \text{flow}(t) \geq r$.

For convenience, in this paper any reference to linear multicast refers to the generalized definition.

Note that not all the links in $G'$ need necessarily be employed by the network code in the case of a generalized linear multicast. Hereinafter, we shall only be referring to the network $G$, which is defined as the sub-network of $G'$ consisting only of the links from $E$ that are employed by the network code along with their adjacent nodes. A network node in $G$ that is neither source, nor sink, shall be called an intermediate node. A path is a set of links that provides a connection between two nodes in the network.

As defined in [3], a linear multicast assumes that every network use the source is generating and transmitting $r$ symbols,
and every eligible sink $t_i$ can extract the transmitted symbols from the $h_i$, different symbols it receives. Nevertheless, in this paper, a node $t_i$ could be defined as a sink of the multicast even if $h_i < r$. In this case, the algorithm described in [6], [8] for constructing a linear multicast yields $h_i$, independent paths from $s$ to $t_i$, resulting in $h_i$, linearly independent symbols received by $t_i$ every network use.

Two different approaches to LNC had been proposed: modeling the data units and the coding operations over the finite field $GF(q^k)$ (scalar network coding), or modeling the data units and vector operations over $GF(q)^L$ (vector network coding) [2], [3]. For simplicity, in this manuscript we are focused with scalar network coding, but all the results may be equivalently applicable to vector network coding.

B. Global Encoding Kernel and Matrix

For the standard scenario whereby a network operates at a constant rate $r$, a Global Encoding Kernel (GEK) - defined as a per-node function that determines the information conveyed by each network link as a function of the network input can be derived from the LEK (and the other way around). Consequently, a linear network code is equivalently defined by a Global Encoding Matrix (GEM) $B$, a $r \times m$ matrix over the field $F$, where $m = |E|$. Every column, $b_i$, of $B$ is associated with one network link, such that multiplying an input vector $v \in F^r$ by this column results in the symbol from $F$ that the network code conveys on that link per network input vector $v$. Observing that the $h_i$ links impinging on the sink $t_i$ are employed for decoding the information reaching this sink, one can denote by $B_t$ the $r \times h_t$ matrix whose columns are given by $b_i$ ($i$ is entering $t$ & used for decoding in $t$). $B_t$ will be called the global encoding matrix - GEM - of the sink $t$. Again, from the linear multicast definition, it is guaranteed that the columns of $B_t$ are linearly independent. Particularly, if a sink $t$ satisfies $h_t \geq r$, $B_t$ is an invertible matrix (the sink only uses $\min(h_t, r)$ links for decoding). For convenience, the vector space spanned by the columns of $B_t$ will be denoted by $B_t^r$.

C. Decoding

Each sink $t$ holds a $h_t \times h_t$ decoding matrix, $D_t$, whose entries are defined over the field $F$. The decoding matrix is used for extracting the source information $v$ from the sink input, $vB_t$. When $h_t \geq r$, $B_t$ is invertible, and hence, the natural choice of $D_t$ is given by $D_t = B_t^{-1}$; thus clearly $vB_tD_t = v$. The case $h_t < r$ is generally not decodable, and was not considered before (to the best of our knowledge). This provides the motivation for the following.

D. Precoding

Next, we wish to introduce precoding into the framework of network coding. Precoding is to be executed at the source node, prior to transmission. To this end, the source information $v$ shall be multiplied by an $r \times r$ matrix $P$ over the field $F$, hence termed precoding, such that the input to the network shall be $vP$ rather than $v$. For the data to be decodable by the sinks, $P$ must be invertible (i.e., the $r$ outputs from the source must be linearly independent). Consequently, for each sink $h_t$ with $h_t \geq r$, the choice of decoding matrix $D_t$ will change to $D_t = B_t^{-1}P^{-1}$, thus providing $vPB_tD_t = v$. Given that, the network code and the precoding scheme are independent with respect to one another. Meaning that the construction of the network code is carried out irrespective of the precoding, the precoding process is reversible, so its effect on the information can clearly be mitigated by those sinks satisfying $h_t \geq r$.

E. Sub-Rate Block LNC

Finally, $l$ consecutive network uses may be referred to as a block LNC of length $l \cdot r$. We relate to this approach as the $l$-block linear multicast of $B$. In this paper, the network code and the transmission rate are considered constant within a block. Therefore:

1) A row vector $\hat{v}$ of length $l \cdot r$ is the, so-called, LNC message block at the network input;
2) The precoding matrix $\hat{P}$ is an invertible $l \cdot r \times l \cdot r$ matrix;
3) The global encoding kernel matrix is a $l \cdot r \times l \cdot |E|$ block matrix $\hat{B}$, with $l$ identical blocks $B$ on the diagonal. Accordingly, each sink $t$ is reached via a global encoding matrix $\hat{B}_t$ which is a $l \cdot r \times l \cdot h_t$ block matrix;
4) The decoding matrix of the sink $t$, $\hat{D}_t$, is a $l \cdot h_t \times l \cdot h_t$ matrix. The decoding rate of a sink is defined as $r_t = \frac{|\{\text{decodable symbols}\}|}{l}$. It is fairly easy to see that for a sink $t$ with $h_t \geq r$ (i.e., $B_t$ is invertible), $\hat{B}_t$ is also invertible. Therefore, defining $\hat{D}_t = \hat{B}_t^{-1}\hat{P}^{-1}$ allows $t$ to decode all the transmitted data, thus supporting a decoding rate of $\frac{r_t}{l} = r$, just as in the single-network-use case.

III. Motivation and Basic Approach

Sub-rate coding and decoding is introduced for improving the overall system throughput achievable via linear multicast. Such improvement can be realized if proper coding and decoding approach is found so as to allow those nodes whose max-flow, $h_t$, is smaller than the source rate $r$, to extract at least a portion of $r$ data units in each network use.

In the sequel, a sink $t$ with $h_t < r$, will be referred to as sub-rate sink. According to information theory, a sink cannot operate at rates higher than its max-flow $h_t$ (the max-flow between the source and a sink is essentially the capacity of the corresponding set of links constituting a communication path). It is, therefore, the aim of the proposed approach to provide communication rate as close as possible to the max-flow of a sub-rate sink, while allowing no interference, or rate loss, to be experienced by any eligible sink. Note that the max-flow to a sub-rate sink $t$ is sink dependent, and different sub-rate sinks may have different max-flows.

In this section, a few basic scenarios of sub-rate coding shall be explicitly demonstrated in order to shed light on the proposed coding methodology. All the examples concern a specific case of a network operating at symbol generation rate of $r = 3$. 
A. A Single Sub-Rate Sink

To begin with, the case of a single sub-rate sink $t$ (with $h_t < r$) shall be considered. In this case, the GEM of $t$, $B_t$, is a $3 \times r$ full (column) rank matrix.

**Lemma 4.** Let sink $t$ be a single sub-rate sink, with $h_t = 2$ in a linear multicast with $r = 3$. There exists a linear multicast of rate 3 allowing $t$ to work at decoding rate of 2.

Before formally proving the lemma, the following definition is inserted:

**Definition 5.** A BR-factorization of an arbitrary $m \times n$ matrix $A$, ($m \geq n$), with linearly independent columns, is defined as a multiplication of two matrices $A = BR$, where $B$ is an invertible $m \times m$ matrix, and $R$ is an $m \times n$ matrix whose columns are $n$ out of $m$ (distinct) columns taken from the identity $m \times m$ matrix $I_m$.

It is easy to see that any full rank matrix has a BR-factorization: consider e.g. $B = (A \mid A^c)$; $A^c$ being an $m \times (m - n)$ complimentary matrix such that $B$ spans the $m$-dimensional space over the field $F$, and $R = \text{(The first n columns of I_m)}$.

Note that BR-factorization of a matrix $A$ is not unique.

**Proof:** Using a BR-factorization of $B_t$, we can define $B_t = BR$. By definition, $B$ is invertible. Taking $P = B^{-1}$ provides:

1) Any sink $t'$ with $h_{t'} \geq 3$, can fully decode the transmitted data, with the choice of decoding matrix $D_{t'} = B_{t'}^{-1}P_{t'}^{-1}$, since $P_{B_{t'}B_{t'}} = I_3$.

2) The sub-rate sink $t$ can operate with decoding rate of $h_t = 2$, with the choice of $D_t = I_2$, $P_{B_{t}D_{t}} = B^{-1}BRI_2 = R$. Recall that $R$ is a $3 \times 2$ matrix with columns from $I_3$ – and so $t$ can decode two of the three transmitted symbols.

The case of $h_t = 1$ is simple, and can be treated in exactly the same way. Obviously, the decoding rate of $t$ can only be 1 in this case.

B. Two Sub-Rate Sinks

**Lemma 6.** In the case of two sub-rate sinks $t_1$ and $t_2$, there exists a linear multicast of rate 3 allowing $t_1$ and $t_2$ to work at decoding rates of $h_{t_1}$ and $h_{t_2}$ respectively.

**Proof:** The proof will be given by examining all the possible cases:

1) $h_{t_1} = h_{t_2} = 1$. It is shown that there exists BR-factorizations $B_{t_i} = BR_i$, $i = 1, 2$, that satisfy $B_1 = B_2 = B$.

   a) If the vectors $B_{t_1}, B_{t_2}$ are linearly dependent, $B_{t_1} = a \cdot B_{t_2}$, $B = (B_{t_1}|\text{two CLIV})$, $P = B^{-1}$, with $D_{t_1} = 1$ and $D_{t_2} = a$, provides $P_{B_{t_1}D_{t_1}} = R_{t_1}$ (with CLIV = complimentary linearly independent vectors). In this case, both sub-rate sinks can decode the same symbol.

   b) If the vectors $B_{t_1}, B_{t_2}$ are linearly independent, the choice $B = (B_{t_1}|B_{t_2}|\text{CLIV})$, $D_{t_1} = D_{t_2} = 1$ provides $P_{B_{t_1}D_{t_1}} = R_{t_1}$. In this case, each sub-rate sink decodes a different symbol.

2) $h_{t_1} = 1, h_{t_2} = 2$ (w.l.o.g. $i = 1, j = 2$):

   a) If the vector $B_{t_1}$ is linearly independent of (i.e. not on the “plane” spanned by) the vectors of $B_{t_2}$, then choosing $B = (B_{t_1}|\text{two vectors spanning } B_{t_2})$, $P = B^{-1}$, $D_{t_1} = 1$, $D_{t_2} = I_2$ shall provide $P_{B_{t_1}D_{t_1}} = R_{t_1}$.

   b) If the vector $B_{t_1}$ is on the plane spanned by the vectors of $B_{t_2}$, a slightly different approach is used: essentially, decoding matrix $D_{t_2}$ is required to manipulate the columns of $B_{t_2}$ so that one of them becomes $B_{t_1}$, but the span of the obtained matrix does not change. Stated mathematically, $\text{span}\{B_{t_1}, D_{t_2}\} = \text{span}\{B_{t_2}\}$, while $B_{t_1}$ is one of the columns of $B_{t_2}, D_{t_2}$. Such $D_{t_2}$ necessarily exists since $B_{t_1}$ is in $\text{span}\{B_{t_2}\}$, $D_{t_2}$. Then, the choice of $B = (B_{t_1}|D_{t_2}|\text{CLIV})$, $P = B^{-1}$, $D_{t_1} = 1$ provide $P_{B_{t_1}D_{t_1}} = R_{t_1}$. Notably, $B_{t_2} = BR_{t_1}D_{t_2}$.

3) $h_{t_1} = h_{t_2} = 2$:

   a) In case the vectors in $B_{t_1}$ and $B_{t_2}$ span the same subspace, one can pick any two linearly independent vectors $v_1$ and $v_2$ in that subspace, then choose $B = (v_1|v_2|\text{CLIV})$. Clearly, matrices $D_{t_1}$ and $D_{t_2}$ exist such that for $i = 1, 2$, the columns of $B_{t_i}D_{t_i}$ will be $\langle v_1, v_2 \rangle$. As in the previous case, $B_{t_1} = BR_{t_1}D_{t_1}^{-1}$. Therefore $P = B^{-1}$ alongside the corresponding $D_{t_1}$ provide $P_{B_{t_1}D_{t_1}} = R_{t_1}$.

   b) In case the vectors in $B_{t_1}$ and $B_{t_2}$ span different subspaces, these subspaces define two different planes. Therefore, their intersection forms a line. Let $v$ be a vector on that line. Let $v_1$ and $v_2$ be two vectors in $B_{t_1}$ and $B_{t_2}$ respectively, such that $v_i \in \text{span}\{B_{t_i}\}$ for $i = 1, 2$, and $\langle v, v_1, v_2 \rangle$ is a linearly independent set. Choosing $B = \langle v_1|v_2 \rangle$, decoding matrices $D_{t_1}$ and $D_{t_2}$ exist, such that for $i = 1, 2$, the columns of $B_{t_i}D_{t_i}$ will be $\langle v_i \rangle$. Therefore $P = B^{-1}$ alongside these $D_{t_i}$ provide $P_{B_{t_i}D_{t_i}} = R_{t_i}$.

Based on the understanding of the methodology gained in the aforementioned scenarios, the following lemma is direct.

**Lemma 7.** Let $B \in M(F)_{r \times m}$ be a global encoding matrix for a linear multicast on the graph $G = \{V, E\}$ with a source $s \in V$ and set of sinks $T \subseteq V$, where $m = |E|$. For every sink $t \in T$, denote by $B_t$ the GEM of $t$, by $h_t$ its max-flow from the source set $s$, and by $D_t$ its decoding matrix. Let $T' \subset T$ be a set of sub-rate sinks, \{t_1, . . . , t_d\}. If there exists a matrix $B$, and matrices $D_1$ and $R_i$ for each sub-rate sink $t$, such that $BR_t$ is a BR-factorization of $B_tD_t$, then each sub-rate sink $t \in T'$ can work at a decoding rate $h_t$.

**Proof:** The matrix $B$ is invertible by the definition of the BR-factorization. Hence, the preceding matrix $P$ can be defined as $P = B^{-1}$. The input to a sink $t$ will then be $PB_t$; multiplying this by the decoding matrix of $t$, $D_t$, the output of
t is: \( PB_tD_t = B^{-1}BR_t = R_t \). Since the columns of \( R_t \) are, by the definition of the BR-factorization, different columns of \( I_r \), then given a transmitted data vector \( v \in F^r \), \( vPB_tD_t = vR_t \) - and \( t \) is capable of working at a decoding rate \( h_t \). ■

**Definition 8.** A set of vectors \( V = \{v_1, \ldots, v_d\} \) will be called an exact spanner with respect to the set of matrices \( \mathcal{B} = \{B_1, \ldots, B_k\} \), if for each and every \( r \times h_t \) matrix \( B_t \in \mathcal{B} \), there exists a subset of \( V \) with exactly \( h_t \) vectors \( \{v_1, \ldots, v_{h_t}\} \subset V \), such that \( \text{span}\{v_1, \ldots, v_{h_t}\} = \text{span}\{B_t\} \). Note that an exact spanner \( V \) may contain linearly dependent vectors.

**Corollary 9.** Let \( B \in M(F)_{r \times m} \) be a global encoding matrix in a linear multicast on the graph \( G = (\mathcal{V}, \mathcal{E}) \) with a source \( s \in \mathcal{V} \) and set of sinks \( T \subseteq \mathcal{V} \). For every sink \( t \in T \), denote by \( B_t \) the GEM of \( t \), by \( h_t \) its max-flow from the source \( s \), and by \( D_t \) its decoding matrix. Let \( T' \subset T \) be a set of sub-rate sinks. \( \{t_1, \ldots, t_d\} \), with \( B = \{B_{t_1}, \ldots, B_{t_d}\} \). If there exists a set of \( r \) vectors \( U = \{v_1, \ldots, v_r\} \) such that:

1. \( \{v_1, \ldots, v_r\} \) are linearly independent;
2. \( U \) is an exact spanner with respect to \( B \), then each \( t_i \in T' \) can work at a decoding rate \( h_{t_i} \).

**Proof:** let \( t \in T' \) be a sub-rate sink. The GEM \( B_t \) is a full-rank \( r \times h_t \) matrix, \( r > h_t \). Therefore, for any set of \( h_t \) vectors spanning the same subspace as do the columns of \( B_t \), there exists a \( h_t \times h_t \) matrix \( D \) such that the columns of \( B_tD \) are these vectors. Formally, if \( \text{span}\{v_1, \ldots, v_{h_t}\} = \text{span}\{B_t\} \), then an \( h_t \times h_t \) matrix \( D \) necessarily exists such that \( B_tD = (v_1v_2 \ldots v_{h_t}) \). Since \( U \) is an exact spanner with respect to \( B \), there exists a subset \( U_t \subseteq U \) of size \( h_t \) spanning the same subspace as the columns of \( B_t \). The choice \( D_t = D \), and the matrix created by concatenating the columns of \( U \) as \( B \), we get that for every \( t \in T' \), all the columns of \( B_tD_t \) are also columns in \( B \). The choice of \( R_t \) as the columns from \( I_r \) pointing at the locations of the columns of \( B_tD_t \) in \( B \), we get that \( B_tD_t = B_tR_t \), namely - \( B_tD_t \) is a BR-factorization of \( B_tD_t \) for every \( t \in T' \). Thus, according to Lemma 7 any sub-rate sink \( t \in T' \) can work at a decoding rate \( h_t \). ■

This corollary implies that when given a set of sub-rate sinks, in order to enable all of them to work in their maximum possible decoding rates (while not interfering with the full-rate, a.k.a. eligible, sinks to operate at rate \( r \)), it suffices to find a set \( U \) of \( r \) vectors that: (a) are linearly independent; and (b) \( U \) is an exact spanner with respect to the GEMs of all the sub-rate sinks. Note that these conditions do not imply that the cardinality of \( U \) has to be \( r \). If indeed \( |U| < r \), then \( r - |U| \) linearly independent vectors may be added to \( U \) in order to create a linearly independent exact spanner with cardinality \( r \), as required for the preceding scheme described in Corollary 9.

**C. Three sub-rate sinks**

With the above understanding, the following lemma is fairly easy to prove:

**Lemma 10.** Given three sub-rate sinks \( t_1, t_2, t_3 \), with \( h_{t_i} = 2 \) for \( 1 \leq i \leq 3 \), in a linear multicast with rate \( r = 3 \), if \( \dim(B_{t_1}' \cap B_{t_2}' \cap B_{t_3}') = 0 \), then there exists a linear multicast with rate \( r = 3 \) allowing all three sub-rate sinks to work at decoding rate \( 2 \).

This lemma is best understood by using geometric arguments. Thus, given Corollary 9 for proving this lemma it suffices to find a set \( U \), of 3 linearly independent vectors, such that every “plane” defined by \( B_t \), i.e. \( B_t' \), is spanned by a combination of 2 of the vectors of \( U \). Recalling that every two planes intersect in a line - and given that all three of planes intersect in a dot - we get that there are 3 different lines of intersection - and choosing to be these 3 lines would satisfy the condition of Corollary 9.

**Proof:** First, note that \( B_t', 1 \leq i \leq 3 \), represent 3 different subspaces (otherwise, \( \dim(B_{t_1}' \cap B_{t_2}' \cap B_{t_3}') \geq 1 \), since it is actually an intersection of only two planes). This implies that for every \( 1 \leq \{i,j\} \leq 3 \), \( i \neq j \), one necessarily has \( \dim(B_{t_i}' \cap B_{t_j}') = 1 \); denote by \( v_{ij} \) a vector in \( B_{t_i}' \cap B_{t_j}' \). Since \( \dim(B_{t_1}' \cap B_{t_2}' \cap B_{t_3}') = 0 \), the set \( U = \{v_{12}, v_{13}, v_{23}\} \) is a set of linearly independent vectors, and since every vector is in the intersection of two subspaces, \( U \) is an exact spanner with respect to \( B_{t_i}, 1 \leq i \leq 3 \). Therefore, according to Corollary 9 all 3 sub-rate sinks can work at a decoding rate of their max-flow - 2.

Other cases pertaining to three sub-rate sinks have to be treated carefully, as a set \( U \) with the required characteristics does not necessarily exist. The following example is aimed at demonstrating that.

**Example 11.** (non-existence) Let \( B \) denote a linear multicast of rate \( r = 3 \), and let \( T' = \{t_1, t_2, t_3\} \) be a set of sub-rate sinks with \( h_{t_i} = 2 \) for any \( t \in T' \), with \( B_{t_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B_{t_2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B_{t_3} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). In this case, \( B_{t_1}' \cap B_{t_2}' \cap B_{t_3}' = \text{span}\{0, 0\} \), namely \( \dim(B_{t_i}' \cap B_{t_j}' \cap B_{t_k}') = 1 \). Unfortunately, there is no possible choice of 3 vectors satisfying the conditions for \( U \) in Corollary 9. In fact, to have the subspace \( B_{t_i}' \) spanned by exactly two vectors of a set \( U \) for every \( t \in T' \), the set \( U \) has to consist of at least 4 vectors, making them necessarily linearly dependent.

In order to fully understand the difference between the cases in Lemma 10 and Example 11 examine the cases presented in figures III.1 and III.2. A case similar to the one of Example 11 is presented in figure III.1. In this case, since the 3 planes intersect on a line, there is no possible choice of 3 vectors which is an exact spanner. An exact spanner for this example could be of the vectors \( \{v, u_1, u_2, u_3\} \) marked in the figure. The case shown in figure III.2 on the contrary, in which the three planes intersect in a dot (namely, \( B_{t_1}' \cap B_{t_2}' \cap B_{t_3}' \) - is a case in which a set of 3 linearly independent vectors which is an exact spanner exists, and it is the set of three vectors that are the intersection of each 2 planes. In this case, the vectors marked as \( \{u_1, u_2, u_3\} \) could be a legitimate choice of such an exact spanner.

While the case of example 11 will be dealt with in more details in section V where partial sub-rate decoding is dis-
Definition 13. The commonality degree of a vector \( v \) with respect to the set of matrices \( \tilde{B} = \{ B_1, \ldots, B_k \} \), denoted by \( \text{comd}_B(v) \), is the number of matrices \( B_i \) in \( \tilde{B} \) with \( v \in B_i \), the subspace spanned by \( B_i \).

Definition 12. The commonality set size of a vector \( v \) with respect to the set of matrices \( \tilde{B} = \{ B_1, \ldots, B_k \} \), denoted by \( \text{comss}(\tilde{B}) \), is the number of matrices \( B_i \) in \( \tilde{B} \) with \( v \in B_i \), the subspace spanned by \( B_i \).

Definition 14. The commonality set size of a set of matrices \( \tilde{B} = \{ B_1, \ldots, B_k \} \), denoted by \( \text{comss}(\tilde{B}) \), is the minimum cardinality of a set of vectors \( V \), where \( V \) is an exact spanner of \( \tilde{B} \). Stated formally: \( \text{comss}(\tilde{B}) = \min\{|V| \mid V \text{ is an exact spanner of } \tilde{B}\} \).

Lemma 15. Given a set of \( k \) sub-rate sinks \( T' \) in a linear multicast \( \tilde{B} \), consisting of a set of global encoding matrices \( \tilde{B} = \{ B_1, \ldots, B_k \} \), if \( \text{comss}(\tilde{B}) = \dim(\sum_{i=1}^{k} B_i') \), then all the sub-rate sinks in \( T' \) are fully sub-rate decodable.

Proof: By the definition of the commonality set size of \( \tilde{B} \), there exists an exact spanner \( V \) with cardinality \( |V| = \text{comss}(\tilde{B}) \). Showing that the vectors of \( V \) are linearly independent will have \( V \) fulfill both conditions of Corollary 9 and hence conclude the proof. Since \( V \) is an exact spanner of \( \tilde{B} \), it is clearly also a spanner of \( \sum_{i=1}^{k} B_i' \), meaning that \( \dim(\sum_{i=1}^{k} B_i') = |V| \), where the equality follows from the condition of the lemma. Yet taking into account that any set of vectors satisfies \( |V| = \dim(\sum_{i=1}^{k} B_i') \), necessarily results with \( \dim(\sum_{i=1}^{k} B_i') = |V| \), meaning that all the vectors of \( V \) are linearly independent, and by corollary 9 all the sub-rate sinks in \( T' \) are fully sub-rate decodable.

Notably, finding the value of \( \text{comss}(\tilde{B}) \) when given an arbitrary set of sub-rate sinks \( T' \) - in order to determine whether \( P,\{D_t\}_{t \in T'} \) exist such that all the sinks in \( T' \) are fully sub-rate decodable - may not be a trivial task.

Referring to 15 and taking in account that it is clear that \( \text{comss}(\tilde{B}) \geq \dim(\sum_{i=1}^{k} B_i') \) (see e.g. Example 11), we shall provide sufficient conditions under which a set of matrices \( \tilde{B} \) satisfies \( \text{comss}(\tilde{B}) \leq \dim(\sum_{i=1}^{k} B_i') \). Thus, in the following, we offer a (constructive) method for upper bounding \( \text{comss}(\tilde{B}) \).

Definition 16. The c-commonality set size of a set of matrices \( \tilde{B} = \{ B_1, \ldots, B_k \} \), denoted by \( \text{comss}_B(c) \) is defined as follows:

1) for \( c = k \), \( \text{comss}_B(k) = \dim(\sum_{i=1}^{k} B_i') \)
2) for \( c = k - 1 \),
\[ \text{comss}_B(k - 1) = \sum_{j=1}^{k} \left[ \dim(\sum_{i=1}^{k} B_i') - \dim(\sum_{i,j}^{k} B_i') \right] \]
3) for \( c < k \), denote \( d = (k - c) \), so:
Lemma 19. Given a set of matrices $\tilde{B} = \{B_1, ..., B_k\}$, denoted $\text{compol}_\tilde{B} : \mathbb{N}^k \to \mathbb{N}$, is defined as $\text{compol}_\tilde{B}(i_1, ..., i_k) = \sum_{c=1}^{k} c \cdot i_c$, where $i_c = |\{v \in V \mid \text{com}_\tilde{B}(v) = c\}|$, and $V$ is an arbitrary set of vectors. $\text{compol}_\tilde{B}$ can be thought of as receiving an arbitrary set of vectors $V$, and for every $1 \leq c \leq k$, calculating the number of vectors of commonality degree $c$ in the set, substituting the result as $i_c$.

Defined as such, the outcome of $\text{compol}_\tilde{B}(i_1, ..., i_k)$ would be the commonality degree of $V$ with respect to $\tilde{B}$ - namely $\text{compol}_\tilde{B}(i_1, ..., i_k) = \text{com}_\tilde{B}(V)$.

Note that in order for a set of vectors to be an exact spanner, all entries of $\text{compol}_\tilde{B}$, $i_c$, - must be no greater than the $c$-commonality set size of $\tilde{B}$; hence, when given a set of vectors $V$, one is sure to have $\text{compol}_\tilde{B} : \{0, ..., \text{comss}_\tilde{B}(1)\} \times ... \times \{0, ..., \text{comss}_\tilde{B}(k)\} \to \mathbb{N}$ as long as $V$ is an exact spanner, as shown in the following lemma.

Lemma 19. Given a set of matrices $\tilde{B} = \{B_1, ..., B_k\}$, any vector $\tilde{i} = (i_1, ..., i_k)$, $\tilde{i} \in \{0, ..., \text{comss}_\tilde{B}(1)\} \times ... \times \{0, ..., \text{comss}_\tilde{B}(k)\}$, that satisfies $\text{compol}_\tilde{B}(\tilde{i}) \geq \sum_{c=1}^{k} \text{dim}(B'_c)$, means that there exists a set of vectors $V$ that constitute an exact spanner of $\tilde{B}$, with $|\{v \in V \mid \text{com}_\tilde{B}(v) = c\}| = i_c$ for every $1 \leq c \leq k$. This means that $|V| = \sum_{c=1}^{k} i_c$.

Proof: For simplicity, a specific special case of the lemma will be proven, of when $\tilde{i} = (\text{comss}_\tilde{B}(1), ..., \text{comss}_\tilde{B}(k))$. This case is not the most general case, but its proof is simpler and it is still covering many of the important cases.

First, notice that any subspace $B'_i$ can be presented as a sum of subspaces, each created by the sum of intersections of $B'_i$ with a different number of other subspaces from $\tilde{B}$.

Let $B''_i$ be the subspace that is a sum of all possible $c$-set intersections that include $B'_i$. Clearly, $B''_i = B'_1 \supseteq B''_2 \supseteq ... \supseteq B''_k$.

Denote by $\text{condim}_\tilde{B}(B''_i, c)$ the $c$-commonality dimension of $B''_i$ with respect to $\tilde{B}$, defined as $\text{condim}_\tilde{B}(B''_i, c) = \text{dim}(B''_i) - \text{dim}(B''_{i+1})$. $\text{condim}_\tilde{B}(B''_i, k)$ represents the contribution of the $c$-set intersections (that include $B'_i$) to the dimension of $B'_i$ that had not been added with $(c + 1)$-set intersections. By definition, it is clear that $\sum_{i=1}^{k} \text{condim}_\tilde{B}(B''_i, c) = \text{dim}(B''_i)$.

Note that every $c$-set intersection (with $B'_i$ included) is contributing its “unique” dimension to both $\text{condim}_\tilde{B}(B''_i, c)$, with $\text{comss}_\tilde{B}(c)$, possibly consisting additional $c$-set intersections (those that do not include $B'_i$), leading to the fact that $\text{comss}_\tilde{B}(c) \geq \text{condim}_\tilde{B}(B''_i, c)$.

Since any $c$-set intersection repeats in $c$ sets, we get the relation $c \cdot \text{comss}_\tilde{B}(c) \geq \sum_{i=1}^{k} \text{condim}_\tilde{B}(B''_i, c)$, which in turn results with $\text{compol}_\tilde{B}(\tilde{i}) = \sum_{c=1}^{k} c \cdot \text{comss}_\tilde{B}(c) \geq \sum_{i=1}^{k} \text{condim}_\tilde{B}(B''_i, c) = \sum_{i=1}^{k} \text{dim}(B''_i)$. This means that $\tilde{i} = (\text{comss}_\tilde{B}(1), ..., \text{comss}_\tilde{B}(k))$ is indeed an adequate choice for this lemma.

Consequently, the construction of $V$ can be carried out according to the guidelines below.

Begin with $V = \{\}$. The construction will be realized by running through the commonality degrees $c$, beginning with $c = k$. For $c = k$, choose $\text{comss}_\tilde{B}(k) = \dim(\bigcap_{i=1}^{k} B'_i)$ linearly independent vectors from $\bigcap_{i=1}^{k} B'_i$, and add them to $V$.

For each $k - 1 \geq c \geq 1$, choose $\text{comss}_\tilde{B}(c)$ vectors by creating $c$-set intersections and choosing the number of vectors required for spanning them according to the corresponding element of the sum $\text{comss}_\tilde{B}(c)$ (in Definition 16); this process guarantees that vectors with higher commonality degree required for spanning the intersection are already contained in $V$. Note that by construction, there is necessarily a set of $\text{condim}_\tilde{B}(B''_i, c)$ vectors spanning $B''_i$ for every $c$ (practically, any spanning subset of $\text{condim}_\tilde{B}(B''_i, c)$ vectors from those chosen by at least $c$-intersections including $B'_i$ will be adequate; all of them are in $B'_i$ and there must be more
than \( \text{comdim}_{\hat{B}}(B'_i, c) \) of them). Overall, a total of \( \text{comss}_{\hat{B}}(c) \) vectors of commonality degree of \( c \) are systematically chosen in this step for each value of \( c \). Add them to \( V \).

In conclusion, the process described above ensures that

\[
|\{v \in V \mid \text{cond}_{B}(v) = c\}| = \text{comss}_{\hat{B}}(c) = i_c \quad \text{for every} \quad 1 \leq c \leq k.
\]

Since for every \( c \), one obtains a subset of \( \sum_{\gamma=c}^{k} \text{cond}_{B}(B'_i, \gamma) \) vectors spanning \( B'_i \), one eventually has a set of size \( \dim(B'_i) \) spanning \( B'_i \), meaning that \( V \) is an exact spanner - as required.

**Theorem 20.** The full sub-Rate Decodability Theorem: Given a set of sub-rate sinks \( T' \) in a linear multicast \( B \), consisting of a set of global encoding matrices \( \hat{B} = \{B_{i_1}, ..., B_{i_k}\} \) (every pair of matrices spanning two different subspaces), if there exists a vector \( i = (i_1, ..., i_k) \), \( i \in \{(0, ..., \text{comss}_{\hat{B}}(1)) \times ... \times \{0, ..., \text{comss}_{\hat{B}}(k)\}\} \), so that \( \text{compol}_{\hat{B}}(i) \geq \sum_{c=1}^{k} \dim(B'_c) \)

\[ \quad \text{and} \quad \sum_{c=1}^{k} i_c \leq \dim(\cap_{i=1}^{k} B'_i) \] - then all the sub-rate sinks in \( T' \) are fully sub-rate decodable.

**Proof:** According to Lemma 19, since \( \text{compol}_{\hat{B}}(i) \geq \sum_{c=1}^{k} \dim(B'_c) \), there exists a set of vectors \( V \) that is an exact spanner of \( \hat{B} \), with \( |V| = \sum_{c=1}^{k} i_c \). Since \( V \) is an exact spanner of \( \hat{B} \), \( \text{comss}(\hat{B}) \leq |V| = \sum_{c=1}^{k} i_c \leq \dim(\cap_{i=1}^{k} B'_i) \). Also recall that \( \text{comss}(\hat{B}) \geq \dim(\cap_{i=1}^{k} B'_i) \), and hence \( \text{comss}(\hat{B}) = \dim(\cap_{i=1}^{k} B'_i) \), which according to Lemma 15 means that all the sub-rate sinks in \( T' \) are fully sub-rate decodable.

To sum up this section, in a linear multicast network with rate \( r \), in order to determine whether all the sub-rate sinks in a set \( T' \) with \( \hat{B} = \{B_{i_1}, ..., B_{i_k}\} \) are fully sub-rate decodable, and find the proper linear multicast configuration fulfilling it, we propose the following algorithm:

1) Find the proper domain for the polynomial \( \text{compol}_{\hat{B}} \):
   For every \( 1 \leq c \leq k \), compute \( \text{comss}_{\hat{B}}(c) \) according to the definition. It is a closed form derived by all the possible intersections of the subspaces \( \{B'_i\} \).
2) Find a vector \( i \in \{(0, ..., \text{comss}_{\hat{B}}(1)) \times ... \times \{0, ..., \text{comss}_{\hat{B}}(k)\}\} \) satisfying both conditions:
   \[ \sum_{c=1}^{k} i_c \leq \dim(\cap_{i=1}^{k} B'_i) \quad \text{and} \quad \text{compol}_{\hat{B}}(i) \geq \sum_{c=1}^{k} \dim(B'_c). \]

3) Define an empty set \( V = \{\} \).
4) For \( c = k : -1 : 1 \), proceed as follows:
   a) Find \( i_c \): linearly independent vectors with commonality degree of \( \text{comss}_{\hat{B}}(c) \), denoted by \( V_c \), such that \( V_c \cup V \) is a linearly independent set.
   b) Substitute \( V \leftarrow V \cup V_c \).
5) Since \( |V| = \sum_{c=1}^{k} i_c \leq \dim(\cap_{i=1}^{k} B_i) \), all the vectors in \( V \) are linearly independent. If the number of vectors in \( V \) is smaller than \( r \), find a set of \( r - |V| \) vectors, denoted \( U \), such that \( U + V \) span the \( r \)-dimensional vector space. Accordingly, substitute \( V \leftarrow U + V \).
6) Define \( \hat{B} = \text{mat}(V) \), a matrix whose columns are the vectors of \( V \). \( V \) is an exact spanner of \( \hat{B} \), allowing to find for each \( t \in T' \) an invertible \( h_t \times h_t \) matrix \( D_t \) satisfying \( B_t D_t = \hat{B} R_t \), with \( R_t \) being the columns from \( \hat{B} \) pointing at the locations of the columns of \( B_t D_t \) in \( \hat{B} \) (see Definition 3). Since \( \hat{B} \) is invertible, it is possible to define the precoding matrix \( P = \hat{B}^{-1} \).
7) The result, a linear multicast \( PB \), will allow any of the sub-rate sinks \( t \in T' \) to work at a decoding rate of its max-flow \( h_t \) using the corresponding decoding matrix \( D_t \).

**B. Examples and Corollaries**

In order to provide the reader with some intuition regarding the use of the the full sub-Rate Decodability theorem (Theorem 20) and the algorithm that followed, this sub-section includes an example for the usage of the algorithm, as well as two more general cases that are immediate corollaries from the theorem. Appendix A includes examples for more cases in which the FSRD theorem can be easily applied in order to determine whether a set of sinks are fully sub-rate decodable.

**Example 21.** Let \( T' = \{t_1, t_2, t_3\} \) be three sub-rate sinks with \( h_{t_1} = 2 \), \( 1 \leq i \leq 3 \) in a linear multicast \( B \) with rate \( r = 3 \) over \( GF(3) \). Let \( B = \begin{bmatrix} B_{t_1} & B_{t_2} & B_{t_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \) be the corresponding GEMs of these sub-rate sinks, then all the sub-rate sinks in \( T' \) are fully sub-rate decodable.

**Proof:** Since it is clear that \( \dim(B'_{t_1} \cap B'_{t_2} \cap B'_{t_3}) = 0 \), the example can be proven using Lemma 10. In this section, the algorithm described at the end of the previous section will be used for finding the modifications required in order to find the proper linear multicast enabling full sub-rate decodability.

1) First, the domain for \( \text{compol}_{\hat{B}} \) is computed to be \( i \in \{(0) \times \{0, ..., 3\} \times \{0\}\} \), as follows:
   a) First, for \( c = 3 \), \( \text{comss}_{\hat{B}}(3) = \dim(\bigcap_{i=1}^{3} B'_i) \).
   It can be easily shown that \( \dim(\bigcap_{i=1}^{3} B'_i) = 0 \), and so \( \text{comss}_{\hat{B}}(3) = 0 \). This means that the third entry of \( i \) must be 0.
   b) for \( c = 2 \),
      \[ \text{compol}_{\hat{B}}(2) = \sum_{j=1}^{3} \left[ \dim(\bigcap_{i=1 \neq j}^{3} B'_i) - \dim(\bigcap_{i=1}^{3} B'_i) \right] \]
      \[ = \sum_{j=1}^{3} \dim(\bigcap_{i=1 \neq j}^{3} B'_i) + \dim(B'_{t_1} \cap B'_{t_2}) + \dim(B'_{t_1} \cap B'_{t_3}) + \dim(B'_{t_2} \cap B'_{t_3}) \]
      Since \( B'_{t_2} \cap B'_{t_3} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \) and \( B'_{t_1} \cap B'_{t_3} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \),
      \[ B'_{t_1} \cap B'_{t_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \], \( \text{comss}_{\hat{B}}(2) = 3 \), and so, the
second entry of \( \bar{i}, i_2 \) has to satisfy \( 0 \leq i_2 \leq 3 \).

c) for \( c = 1 \),
\[
\text{comss}_B(1) = \sum_{i_1, i_2=1}^{3} \left[ \dim(B'_{i_1}) - \dim(B'_{i_2} \cap B'_{i_3}) - \dim(B'_{i_1} \cap B'_{i_3}) + \dim(\bigcap_{i=1}^{3} B'_i) \right]
\]
\[
= \sum_{i=1}^{3} \left[ \dim(B'_{i_1}) - \sum_{j=1}^{3} \dim(B'_{i_2} \cap B'_{i_3}) \right].
\]

Since for every \( 1 \leq i \leq 3 \), \( \dim(B'_{i_1}) = 2 \), and for every \( 1 \leq j \leq 3 \), \( j \neq i \), \( \dim(B'_{i_2} \cap B'_{i_3}) = 1 \), we get that \( \text{comss}_B(1) = 0 \). This means the the first entry of \( i \) must also be 0.

2) The vector \( \bar{i} = (0, 3, 0) \) is the only vector in the domain that satisfies both conditions of the FSRD theorem:

a) \( \sum_{c=1}^{k} B'_{c} \) is the 3-dimensional vector space, hence \( \sum_{c=1}^{k} i_c = 3 \leq \dim(\sum_{c=1}^{k} B'_{c}) \).

b) \( \text{compol}_B(\bar{i}) = 2 \cdot 3 = 6 > 2 + 2 = k \).

3) Define \( V = \{ \} \).

4) Since \( i_3, i_1 = 0 \), this stage only requires finding 3 linearly independent vectors with \( \text{comss}_B(2) \). Such vectors can be easily found as the intersection vectors of each pair of sub-spaces. This gives
\[
V_2 = \left\{ \left( \begin{array}{c} 2 \\ 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) \right\},
\]
and hence the output from this stage is
\[
V = \left\{ \left( \begin{array}{c} 2 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{array} \right) \right\}.
\]

5) Since \( |V| = 3 = r \), no vectors need to be added to \( V \) in this stage.

6) Define \( \bar{B} = \text{mat}(V) = \left( \begin{array}{cccc} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{array} \right) \).

a) For each \( 1 \leq i \leq 3 \), we find matrices \( R_{t_1}, D_{t_1} \), so that \( B_{t_1}D_{t_1} = B\bar{R}_{t_1} \):

i) For \( i = 1 \), \( B_{t_1} = \left( \begin{array}{cc} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{array} \right) \), so \( R_{t_1} = \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) \), giving \( B\bar{R}_{t_1} = \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{array} \right) \). The choice of \( D_{t_1} = \left( \begin{array}{c} 1 \\ 1 \\ 0 \\ 1 \end{array} \right) \), will provide
\[
B_{t_1}D_{t_1} = \left( \begin{array}{cccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{array} \right).
\]

ii) Similarly, for \( i = 2 \), \( B_{t_2} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right) \), and so \( R_{t_2} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{array} \right) \) will provide \( B_{t_1}D_{t_2} = B\bar{R}_{t_2} = \left( \begin{array}{c} 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \right) \).

iii) Finally, for \( i = 3 \), \( B_{t_3} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right) \), and so \( R_{t_3} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{array} \right) \) will provide \( B_{t_3}D_{t_3} = B\bar{R}_{t_3} = \left( \begin{array}{c} 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{array} \right) \).

b) The precoding matrix is
\[
P = B^{-1} = \left( \begin{array}{cccc} 1 & 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{array} \right) = \left( \begin{array}{cccc} 1 & 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{array} \right).
\]

7) The result, \( PB \), is the desired linear multicast.

\[ \text{Corollary 22.} \] Let \( B \) denote a linear multicast of rate \( r \). A single sub-rate sink \( T' = \{ t \} \), with \( h_1 < r \) and \( B = \{ B_i \} \), is always fully sub-rate decodable.

\[ \text{Proof:} \] While this case is not difficult to prove directly, the proof we provide herein employs the FSRD theorem, Theorem \[ \text{[20]} \] Since there is only one sub-rate sink \( t \), we have \( \text{comss}_B(1) = h_1 \). The choice of \( \bar{i} \) is trivial - \( \bar{i} = (h_1) \).

Then - \( \text{compol}_B(\bar{i}) = \sum_{j \in 1}^{l} j \cdot \bar{i}(l) = 1 \cdot h_1 = \dim(B_i) \), and
\[ \sum_{j=1}^{l} \bar{i}(j) = h_1 = \dim(B_i) \]. Therefore - according to the FSRD theorem- \( T' \) is fully sub-rate decodable.

\[ \text{Corollary 23.} \] Let \( B \) denote a linear multicast of rate \( r \), and let \( T' \) be a set of \( 2 \leq l \leq r \) sub-rate sinks with max-flow
are not satisfied, since the conditions for the full sub-rate decodability theorem are not met. By \( B' \) following definition precedes the main lemma of this subsection.

\[ B' = \{ B_1, ..., B_l \} \]

The following definition precedes the main lemma of this subsection.

\[ \text{Definition 24.} \] Define a \( k \)-degree BR-factorization of a \( m \times n \) matrix \( A \) (\( m \geq n \geq k \)) as a multiplication of two matrices, \( A = BR \), such that \( B \) is a \( m \times m \) full rank matrix and \( R \) is an \( n \times n \) matrix with \( k \) (out of \( n \)) columns taken from \( I_m \).

\[ \text{Lemma 25.} \] Let \( B \) denote a linear multicast of rate \( r = 3 \), and let \( T' = \{ t_1, t_2, t_3 \} \) be a set of sub-rate sinks with \( h_{t_i} = 2 \) for every \( t_i \in T' \). Denote the corresponding GEM matrices by \( B = \{ B_1, B_2, B_3 \} \). If \( \dim(B'_1 \cap B'_2 \cap B'_3) = 1 \), there exists a linear multicast of rate \( r = 3 \) allowing every \( t_i \in T' \) to work at a decoding rate of \( r_{t_i} = 5/3 \), hence \( t_i \in T' \) are partially sub-rate decodable.

**Proof:** The idea behind this approach is to use a block of 3 network uses (instead of a single use), which enables to find 5-degree BR factorizations with a single full rank matrix for all 3 of the GEM matrices of the sinks in \( T' \). With this factorization given, the idea implemented by Lemma 7 is used in order to partially-decode 5 out of the 6 symbols each sink receives during the 3 network uses - enabling them to work at decoding rate of 5/3. The following paragraph shows that at mathematically.

Let \( v_4 \in B'_1 \cap B'_2 \cap B'_3 \), and for every \( 1 \leq i \leq 3 \), \( v_i \) is a vector so that \( B'_{t_i} = \text{span}\{v_i, v_4\} \). Note that \( V = \{ v_1, ..., v_4 \} \) is an exact spanner of \( B \). Let \( B \) denote the 3-block linear multicast of \( B \) (see Subsection II-E). Denote by \( v_i^j \) the vector obtained by substituting \( v_i \) as the \( j \)-th block, with all the other entries being 0. For example, for \( v_4 = (v_3,1, v_3,2, v_3,3)^T \), we have \( v_4^2 = (0, 0, 0, v_3,1, v_3,2, v_3,3, 0, 0, 0)^T \). Note that the set \( \bar{V} = \{ v_1^1, v_2^1, v_2^2, v_2^3, v_2^4, v_2^5, v_3^1, v_3^2, v_3^3, v_3^4 \} \) is a linearly independent set of vectors, each of length \( l \cdot r = 3 \cdot 3 = 9 \).

Defining \( B \) as the matrix whose columns are the 9 vectors from \( V \), clearly \( B \) is invertible. Employing \( P = B^{-1} \) as a precoding matrix will have each sub-rate sink \( t_i \) receiving \( vPB_{t_i} \) at its input where \( \hat{v} \) denotes an input data vector \( \hat{v} \in F_{q^r} \). Showing that \( \hat{B}R_{t_i} = \hat{B}B_{t_i} \), for some matrices \( R_{t_i} = \bar{B}_{t_i} \cdot \bar{D}_{t_i} \cdot \bar{A}_{t_i} = 9 \times 6 \) matrix with 5 columns taken from \( I_5 \), and \( D_{t_i} \cdot \bar{A}_{t_i} = 6 \times 6 \) matrix will provide the appropriate 5-degree BR factorizations.

Since for every \( 1 \leq i \leq 3 \), \( B'_i = \text{span}\{v_i, v_4\} \), the block form of \( B'_{t_i} \) satisfies \( B'_{t_i} = \text{span}\{v_i^1, v_i^4, v_i^2, v_i^3, v_i^4 \} \). For every \( i \), let \( \bar{D}_{t_i} \) be a basis transformation matrix such that \( \bar{B}_{t_i}D_{t_i} = (v_1^1, v_2^1, v_2^2, v_2^3, v_2^4)^T \) hence, there are exactly 5 columns of \( \bar{B} \) in \( B_{t_i}D_{t_i} \). For every \( i \), let \( \bar{D}_{t_i} \) be a basis transformation matrix such that \( \bar{B}_{t_i}D_{t_i} \) has these 5 columns from \( \bar{B} \), with the sixth column being the zero vector. Then constructing \( \bar{D}_{t_i} \) so that it chooses these 5 columns from \( \bar{B} \) and leaving the sixth column as zero, provides the desired factorization \( \bar{B}R_{t_i} = \bar{B}B_{t_i} \).

With that understanding, by employing \( \bar{D}_{t_i} \) as the decoding matrix for \( t_i \) for every \( i = \hat{v}PB_{t_i}D_{t_i} = \hat{v}PB_{t_i}R_{t_i} = \hat{v}R_{t_i} \). Namely, each sub-rate sink \( t_i \in T' \) can decode 5 symbols every 3 network uses, meaning that it is sub-rate decodable with \( r_{t_i} = 5/3 \).

**Corollary 26.** A linear multicast of rate \( r = 3 \), any set of up to 3 sub-rate sinks with max-flow of \( h_{t_i} = 2 \) can work at decoding rates of at least 5/3.

2) **Partial sub-rate decoding in \( r = 3 \) for 4 sub-rate sinks:** In view of the scenario discussed in the previous subsection one may wonder whether it is also possible to use a similar approach for a larger set \( T' \) of sub-rate sinks. Herein, we answer this in the affirmative by providing an example of achieving partial sub-rate decoding in block linear multicausing for 4 sub-rate sinks.

\[ \text{\it \begin{eqnarray*} \end{eqnarray*}} \]
Lemma 27. Let $B$ denote a linear multicast of rate $r = 3$, and let $T' = \{t_1, t_2, t_3, t_4\}$ be a set of sub-rate sinks with $h_{t_i} = 2$ for any $t_i \in T'$. Denote the corresponding GEM matrices by $\hat{B} = \{B_1, B_2, B_3, B_4\}$. Assuming that $\dim(\bigcap_{i=1}^{3} B_{t_i}) = 0$ for every $\{t_1, t_2, t_3\}$ there exists a linear multicast of rate $r = 3$ allowing every $t_i \in T'$ to work at a decoding rate of $r_{t_i} = 1.5$.

Proof: The proof is quite similar to that of Lemma 25. Since the intersection of any set of 3 subspaces is of dimension 0, it is not difficult to see that an exact spanning set is of minimum size 4. Denote such a spanning set by $\hat{v}_{i1}, \hat{v}_{i2}, \hat{v}_{i3}, \hat{v}_{i4}$, where $\hat{v}_{ij}$ is a vector on the intersection line of the spaces $B_{t_i}$ and $B_{t_j}$. Choosing a 4-block linear multicast $\hat{B}$ with $\hat{V} = \{\hat{v}_{11}, \hat{v}_{12}, \hat{v}_{13}, \hat{v}_{14}, \hat{v}_{21}, \hat{v}_{22}, \hat{v}_{23}, \hat{v}_{24}, \hat{v}_{31}, \hat{v}_{32}, \hat{v}_{33}, \hat{v}_{34}\}$ would enable every sink to decode 6 symbols (the rest of this proof follows the same line of arguments of Lemma 25).

With 6 symbols decoded in 4 network uses, each sub-rate sink $t_i \in T'$ decoding rate is hence $r_{t_i} = 6/4 = 1.5$. $\blacksquare$

B. Existence of partial sub-rate decoding in the general case

From the previous lemmas, it seems that for any set of sub-rate sinks $T'$, there is a linear multicast allowing them to work at some decoding rate $r_{t_i}' \leq h_{t_i}'$. Next, this notion is formally substantiated.

Definition 28. Let $\tilde{B} = \{B_1, ..., B_k\}$ denote a set of $r$-row matrices, each $B_i$ comprises of $h_{t_i}$ linearly independent columns. A set of vectors $\tilde{V} = \{v_{11}, ..., v_{r}\}$, where $v_{ij}$ is a vector on the intersection line of the spaces $B_{t_i}$ and $B_{t_j}$, choosing a 4-block linear multicast $\tilde{B}$ with $\tilde{V} = \{v_{11}, v_{12}, v_{13}, v_{14}, v_{21}, v_{22}, v_{23}, v_{24}, v_{31}, v_{32}, v_{33}, v_{34}\}$ would enable every sink to decode 6 symbols (the rest of this proof follows the same line of arguments of Lemma 25).

With 6 symbols decoded in 4 network uses, each sub-rate sink $t_i \in T'$ decoding rate is hence $r_{t_i} = 6/4 = 1.5$. $\blacksquare$

Lemma 29. Let $B \in M(F)_{r \times m}$ denote a global encoding kernel for a linear multicast of rate $r$ on the graph $G = (\mathcal{V}, \mathcal{E})$ with a source $s \in \mathcal{V}$ and set of sinks $T \subseteq \mathcal{V}$. For every sink $t \in T$, denote by $B_t$ the GEM of $t$, by $h_t$ its max-flow from the source $s$, and by $D_t$ its decoding matrix. Let $T' \subseteq T$ be a set of sub-rate sinks, $\{t_1, ..., t_d\}$. If there exists a set of $r$ linearly independent vectors $U = \{v_{1}, ..., v_{r}\}$ that constitutes a partial exact spanner with respect to $B$ with $\{h_{t_1}', ..., h_{t_d}'\}$ spanning vectors (for each of the $d$ sinks), when for each $t \in T$, $1 \leq h_{t_i}' \leq h_{t_i}$ - then each sink $t \in T'$ can work at decoding rate of $r_{t_i}'$.

Proof: Let $t \in T'$ be a sub-rate sink. Its GEM $B_t$ is a full-rank $r \times h_t$ matrix, $r > h_t$. Therefore, for a set $U_t$ of $h_{t_i}'$ vectors that span a subspace of span$\{\text{columns of } B_{t_i}\}$, there exist a $h_t \times h_{t_i}'$ selection matrix $C_t$ (that selects the $h_{t_i}'$ out of $h_t$ decodable input coordinates) and $h_{t_i}' \times h_{t_i}'$ decoding matrix $D_t$, so that the columns of $B_tC_tD_t$ are the vectors of $U_t$. Since for every $t$, there is a $U_t \subseteq U$ that is such a set of $h_{t_i}'$ vectors, there also exist such matrices $D_t$ and $C_t$. By the choice of $D_{t_i} = C_tC_tD_t$, and the matrix created by concatenating the columns of $U$ as $\hat{B}$, we get that for every $t \in T'$, all the columns of $B_tD_t$ are columns of $\hat{B}$. The choice of $R_t$ as the columns from $I_t$ pointing at the locations of the columns of $B_tD_t$ in $\hat{B}$, yields $B_tD_t = BR_t$, namely $BR_t$ is a $h_{t_i}$-degree BR-factorization of $B_tD_t$ for every $t \in T'$. Therefore, each sub-rate sink $t \in T'$ can work at decoding rate of $r_{t_i}'$. $\blacksquare$

Theorem 30. Let $B \in M(F)_{r \times m}$ be a global encoding kernel for a linear multicast on the graph $G = (\mathcal{V}, \mathcal{E})$ with the source $s \in \mathcal{V}$ and set of sinks $T \subseteq \mathcal{V}$. For every sink $t \in T$, denote by $B_t$ the GEM of $t$, by $h_t$ its max-flow from the source $s$, and by $D_t$ its decoding matrix. Let $T' \subseteq T$ be a set of sub-rate sinks, $\{t_1, ..., t_d\}$. Each sub-rate sink can work at some positive decoding rate.

Proof: Denote by $\hat{B} = \{B_1, ..., B_k\}$ a set of matrices derived from the GEM of a corresponding set of sub-rate sinks; these matrices span different subspaces. Let $B$ be an exact spanner of $\hat{B}$ (choosing the exact spanner with minimal cardinality will provide better results - in the sense of a smaller $l$, i.e. a shorter block - but it is not mandatory). Let $\dim(\text{span}\{V\}) = d(V)$, and $l$ (the block size) be the number of distinct subsets of $V$ that consist $d(V)$ linearly independent vectors. Denote these subsets by $V_i \in \mathcal{V}$ for $1 \leq i \leq l$. For each $V_i$, let $V_i$ be a set of $r$ linearly independent vectors (spanning the $r$-dimensional vector space), with $V_i \in \mathcal{V}$. Let $\hat{B}$ denote the $l$-block linear multicast associated with $B$, and let $\tilde{B}$ be the set of block matrices induced by $B$. Denote by $\tilde{V}_i$ the set of $l \times r$ column vectors created by substituting every $v_{ij} \in V_i$ as the $i$-th block (1 $\leq i \leq l$), with all the other entries being 0. Note that all the vectors in $\{\tilde{V}_1, ..., \tilde{V}_l\}$ are linearly independent. Denote $\tilde{V} = \bigcup_{i=1}^{l} \tilde{V}_i$. For every $t \in T'$, denote by $h_t'$ the number of columns in $\tilde{V}_i$ that are in the subspace spanned by $B_T$ (Obviously, $h_t \geq h_t'$ $\geq 1$, since there have been least $h_t$ vectors spanning $B_t$ in $V$ - and the choice of all the distinct subsets of $V$ that are linearly independent assures that they are included at least once). Therefore, $\tilde{V}$ is a partial exact spanner with respect to $\hat{B}$, with $\{h_t', ..., h_t'\}$. Finally, employing Lemma 29 concludes the proof. $\blacksquare$

1) Topics for Future Work: The above theorem is arguing for the existence of a partial sub-rate decoding methodology, but is leaving two important topics in that respect for discussion:

1) What is the minimum possible block length as a function of the intersections between the sets in $\hat{B}$? Reducing the block size $l$ is of interest, since it shortens the decoding delay, requires smaller amount of memory at the end units and enables all operations to be executed with smaller-size matrices. Note that one may derive the minimum size of a “valid” exact spanner as the minimum value of the sum $\sum_{i=1}^{k} i_c$ when $\vec{i} = (i_1, ..., i_k)$, $\vec{i} \in \{0, ..., \text{comms}_B(1)\} \times ... \times \{0, ..., \text{comms}_B(k)\}$, that is satisfying $\text{comps} \vec{i} \geq \sum_{i=1}^{k} \text{dim}(B'_c)$ (see Lemma 19). Albeit, it can in fact be observed from the lemmas at the beginning of this section that the size of the exact spanner is simply not sufficiently revealing, even when taking into account the dimension of the sum space $\sum_{B_i \in \hat{B}}$.
block length; two different sets of matrices, both with exact spanner of minimum size 4 and dimension sum of 3 - had required two different block lengths - 3 and 4. Intuitively, the block length should indeed be a function of the values of the \(C\)-commonality set sizes of \(B\) for all \(1 \leq c \leq |B|\), yet an exact closed-form solution is not presented in this paper.

2) **What are the maximum possible effective decoding rates in which all the sub-rate sinks can operate simultaneously?** With \(V\) being the minimal exact spanner, the intuition leads to the size \(h_i' \sim \frac{h_i \cdot r}{|V|}\), since in every network use the “effective” transmission rate to the sub-rate sinks is \(\frac{r}{|V|}\), and the max-flow to \(t\) is \(h_t\). The reality is more complicated, as seen from the previous lemmas [25] and [27] for which the values of \(r, h_t, |V|\) are equal in both scenarios, yet the decoding rates are different (5/3 and 1.5, respectively). The effective rate also depends on the values of the \(C\)-commonality set sizes of \(B\) for all \(1 \leq c \leq |B|\); finding the exact dependency is left for future work.

VI. Further Results and Relations to Other Methods

All the results in this paper can be applied to vector network coding, which means operation with blocks of size \(m\) over a binary field instead of the field of size \(2^m\), referred to as scalar network coding.

In this section, more involved combinations of sub-rate coding with other multicast characteristics and methods are discussed.

A. Sub-Rate Coding for Static Linear Network Codes

Often, network configuration may change due to various reasons such as node failures and communication-channel variations to name but a few. A very important method in network coding relates to static linear multicast, aimed at facilitating the communication with all the designated sinks under a few, different, network configurations without having to change the network code when the network configuration changes. This capability comes at the cost of increasing the field size, hence reducing the communication rate, with the number of (predetermined) configurations as a factor of the former increase.

A schematic algorithm for constructing static linear multicast was presented in [5], [7]. First, for every sink \(t \in T\) of the network and every \(\epsilon \in \hat{\epsilon}\) configuration, an virtual sink \(t^\epsilon\) is created, with \(\max - \text{flow}_{\epsilon}(t)\) paths connecting it to the original source, \((\max - \text{flow}_{\epsilon}(t)\) denoting the max-flow from the source to the real sink \(t\) under the network configuration \(\epsilon\)). Then, the linear multicast is designed in the usual manner, with the set of virtual sinks acting as the designated set of sinks, while making sure that all the paths to each virtual sink \(t^\epsilon\) only employ links that are present in network configuration \(\epsilon\). That approach yields \(|\hat{\epsilon}|\hat{\epsilon}\)-GEM matrices, \(\{B_{t,\epsilon}\}_{\epsilon \in \hat{\epsilon}}\) - each representing the linear multicast under one of the link-failure configurations.

There are two different approaches for combining sub-rate coding with a static network code so as to provide static linear multicast:

1) When the different configurations are not a result of spontaneous link failures, but rather expected changes in the network’s topology, it is reasonable to assume that the network source can keep track of the instantaneous configuration in each network use. In this case, for each configuration \(\epsilon\), a set of sub-rate sinks \(T_{\epsilon}'\), each set accompanied by a precoding matrix \(P_{\epsilon}\), can be defined with respect to the \(\epsilon\)-GEM \(B_{t,\epsilon}\), according to the methods presented in earlier sections of this work.

2) When the current network configuration is not monitored by the source, the choice of the set of sub-rate sinks and precoding matrix cannot change per configuration. In that case, a set of designated sub-rate sinks \(T'\) has to be fixed. For each sub-rate sink \(t \in T'\), a specific configuration \(\epsilon_t \in \hat{\epsilon}\) has to be chosen, and the sink’s \(\epsilon\)-GEMs, \(B_{t,\epsilon_t}\), is the matrix considered for all the computations presented in the previous chapters. In this case, the precoding matrix \(P\) is fixed, and is computed as an invertible matrix using the set of matrices \(\{B_{t,\epsilon}\}_{\epsilon \in \hat{\epsilon}}\). In general - each designated sub-rate sink \(t \in T'\) will only be able to apply the sub-rate decoding scheme in the configuration \(\epsilon_t\) (or in other configurations \(\epsilon \in \hat{\epsilon}\) with \(\text{span}\{B_{t,\epsilon}\} = \text{span}\{B_{t,\epsilon_t}\}\)). Note that - as in the non-static case described below - for every designated real sink \(t \in T\), for any configuration \(\epsilon\) where \(\max - \text{flow}_{\epsilon}(t) \geq r\) (\(r\) being the network’s operation rate), \(B_{t,\epsilon}\) is an invertible matrix, ensuring that the precoding does not interfere with its decoding ability.

B. Sub-Rate Coding for Variable-Rate Linear Network Codes

An interesting method to combine with sub-rate coding is the so-called variable-rate network coding [4], [7]. In essence, this method enables to work with the same linear multicast under different rates of symbol-generation from the source (limited by the number of outgoing links from the source). For example, if the source has 5 outgoing links, then linear multicast using variable-rate network coding will allow it to operate at any rate \(r \leq 5\), making the transmission decodable by all sinks whose max-flow is at least \(r\). In contrary to the previous methods, this method does not necessitate a larger field size, hence it entails no reduction in rate.

The construction method of variable rate linear multicast is carried out much the same as linear multicast, only using a slightly different input graph, leading to a change in both the path-finding stage and the values calculation stage of the linear multicast. The algorithm results with a Local Encoding Kernel, but note that the traditional GEM \(B\) can not be well defined, because the number of its rows equals the symbols generation rate \(r\), which is not constant. Alternatively, for every network operation rate \(1 \leq q \leq r\), \(B_q\), a \(q\)-GEM is defined, as a \(q \times |E|\) matrix that represents the network code operation under the rate \(q\). In order to combine sub-rate coding with variable-rate coding, it is necessary to refer to each rate
separately. Namely, for every operation rate \( q \) a different set of sub-rate sinks has to be defined with respect to \( B_q \), and a corresponding \( q \times q \) precoding matrix \( P_q \) has to be derived. Since the network operation rate is determined by the network source, it is only natural that it can also adjust the precoding scheme accordingly.

Note that since it is necessary to adjust the sub-rate coding scheme for each rate, one may choose different sinks for each rate. This makes perfect sense, as for each operation rate, different sinks have max-flow from the source that is smaller than the network operation rate; i.e., the set of potential sub-rate sinks changes with the network operation rate.

C. Sub-Rate Coding as Alternative to Sinks Addition

Finally, we tackle a more intricate issue - which, in a way, generalizes the above two cases. As more nodes in a network get designated as sinks, the larger is the field-size over which calculations need to be performed, hence - in many cases - resulting with reduction in communication rate to all the sinks in the network. An interesting approach to alleviate this problem, while still allowing communication with many (designated) nodes without having to increase the field-size, is to avoid regarding these designated nodes as standard sinks, but rather as consequential sub-rate sinks. To this end, we propose to construct the linear multicast while considering some designated sinks (not as standard sinks but rather) as “intermediate” nodes. Then, using the effective rate allowed by the network code, each such intermediate node will be approached as sub-rate sink and operate at its consequential max-flow, which is basically the number of linearly independent inputs to this node as entailed by the linear multicast.

In certain cases this approach may actually improve the communication rate with the designated nodes compared to the rate achievable by regarding them as (standard) sinks - especially when working with a small set of sinks (hence a small field). Finally, since increased field size always entails reduced communication rate to all the sinks, the proposed approach may prove a viable modification to any design of a network code.

The following claim offers a simple sanity-check for identifying whether a single node should be regarded as a standard sink, or as a consequential sub-rate sink. Just recall that, as explained above, this is not the only consideration in favor of employing the sub-rate coding methodology.

**Claim 31**. Let \( B \in M(F)_{r \times m} \) be a global encoding kernel for a linear multicast on the graph \( G = (\mathcal{V}, E) \) with the source \( s \in \mathcal{V} \) and set of sinks \( T \subseteq \mathcal{V} \). For every node \( t \in \mathcal{V} \), denote by \( h_t \) its max-flow from the source \( s \), and by \( B_t \) the GEM of \( t \), which is the matrix whose columns, taken from \( B \), represent the links entering \( t \), just as the GEM is defined for a “standard” sink. For a node \( t \notin T \) let \( r_t \) denote the consequential max-flow, namely the number of linearly independent columns in \( B_t \) (for standard sinks, \( r_t = h_t \), but in general \( r_t \leq h_t \)). Denote by \( F' \) the field required for considering the set of sinks \( T' = T \cup \{ t \} \). If \( r_t \cdot \frac{1}{\log_2 |F'|} \geq h_t \cdot \frac{1}{\log_2 |F'|} \), then considering \( t \) as a sink would reduce its communication rate compared to considering it as a consequential sub-rate sink.

**Proof**: Consider two different cases:

1) Referring to \( t \) as a consequential sub-rate sink, would allow it to decode \( r_t \) symbols with each network use. Since the network messages under the network code represented by \( B \) are elements of the field \( F \), each symbol is represented by \( \lfloor \log_2 |F'| \rfloor \) bits - meaning that the communication rate to \( t \) as a consequential sub-rate sink is \( r_t \cdot \frac{1}{\log_2 |F'|} \) bits per network use.

2) Considering \( t \) as a sink (i.e. - calculating a new linear multicast, with \( T' \) being the set of sinks) will allow \( t \) to decode at most \( h_t \) symbols per network use. Notably, that the network's max-flow in this case is \( \min\{\max - \text{flow between } s \text{ and } \tau \} \), and may very well be smaller than \( h_t \).

Thus, if \( r_t \cdot \frac{1}{\log_2 |F'|} \geq h_t \cdot \frac{1}{\log_2 |F'|} \), then the number of decoded bits per network use is higher when referring to \( t \) as a consequential sub-rate sink.

Note that, while using claim 31 is straight-forward for a single consequential sub-rate sink, using it for more than one sink requires that the set of potential consequential sub-rate sinks satisfy the conditions given in Section IV of this paper, namely that this set could be fully sub-rate decodable.

Figure VI.1 shows the function \( \frac{\log_2 |F'|}{|F'|} \), termed the rate ratio bound, as a function of the number of sinks. As the requirement for the existence of a linear multicast is that \( |F'| > |T| \), for every \( T \), the cardinality of the field to work with is the smallest prime number greater than \( T \). If the ratio \( \frac{\log_2 |F'|}{|F'|} \) is above the curve in the figure, then considering \( t \) as a sink will actually reduce its communication rate as opposed to considering it as a sub-rate sink.

As seen in the figure, some values of the rate ratio require \( \frac{\log_2 |F'|}{|F'|} \) to be quite higher than others in order for sub-rate sink to be superior to standard sink designation.

VII. Conclusions

Sub-rate coding is introduced and thoroughly studied. It amounts to network (pre)-coding at the source side and de-
coding of partial information by a set of sinks whose max-flow is smaller than the network operation rate. This work is concerned with a single-source, finite acyclic scalar network, although it is straight-forward to apply the results to vector networks.

Sub-rate coding methodology does not impact the transmission rate towards sinks whose max-flow permits full-rate decoding. This makes sub-rate coding a viable add-on to any network. Applications of sub-rate coding include any scenario that employs multicasting to communicate with different users experiencing different channel qualities, or otherwise when they require different volumes of the data. It is especially valuable when a small number of sinks have a wide dynamic range of max-flows.

For achieving full sub-rate decodability, a general and sufficient condition allowing each sub-rate sink to operate at its optimal rate (its max-flow from the source) is introduced. Furthermore, an algorithm is proposed for determining if full sub-rate decodability is realizable for a given set of sub-rate sinks; the necessary modifications to the existing linear network code are detailed, bearing no impact on the original network sinks.

With respect to partial sub-rate decodability, it was proven that for any set of sub-rate sinks there exists an effective sub-rate coding scheme that is based on the transmission of a sequence of messages thus resulting with a linear sub-rate network block-code. A few examples are provided.

Finally, the combination of sub-rate coding with other important linear network codes is briefly discussed. Proper ways to combine sub-rate coding with static linear network codes and with variable-rate linear network codes are presented. It is also shown that in some cases, sub-rate coding can be a preferable solution when additional sinks are to added to an existing network.

APPENDIX A

FSRD Theorem Examples

To demonstrate the usage of the FSRD theorem (Theorem 20 in some more detail, and provide better intuition as to the actual meaning of each of the measures, many examples for possible types of GEMs (given a set of sinks) are summarized in Table I. For each type of GEM set \( \tilde{B} = \{B_{t_1}, ..., B_{t_k}\} \), the values \( \text{comss}(\tilde{B}, c) \) for \( 1 \leq c \leq k \), \( \sum_{c=1}^{k} \text{dim}(B'_c) \) and \( \text{dim}(\sum_{c=1}^{k} B'_c) \) are calculated, and then a vector \( \bar{i} \) that satisfies the FSRD theorem conditions (namely, \( \text{compol}(\tilde{B}, \bar{i}) \geq \sum_{c=1}^{k} \text{dim}(B'_c) \) and \( \sum_{c=1}^{k} i_c \leq \text{dim}(\sum_{c=1}^{k} B'_c) \)) is suggested, if such a vector exists.

REFERENCES

[1] R Alshwede, N Cai, S.-Y Li, and R Yeung. Network information flow: Single source. IEEE Trans. Inform. Theory, 2000.
[2] J Ebrahimi and C Fragouli. Vector network coding algorithms. In 2010 IEEE International Symposium on Information Theory, pages 2408–2412. IEEE, 2010. June.
[3] Muhammad Farooqi and null Zubair. A survey on network coding: From traditional wireless networks to emerging cognitive radio networks. Journal of Network and Computer Applications, 46:166–181, 2014.
[4] S Fong and R Yeung. Variable-rate linear network coding. pages 409–412, 2006. Oct. 2006.
[5] R Koetter and M Medard. An algebraic approach to network coding. IEEE/ACM Trans. Netw, Oct. 2003.
[6] S Li, R Yeung, and N Cai. Linear network coding. IEEE Trans. Inform. Theory, 49:371, Feb. 2003.
[7] QIFU TYLER SUN. On construction of variable-rate and static linear network codes. IEEE Access, 2018.
[8] R Yeung, S.-Y. Li, and N Cai. Network Coding Theory (Foundations and Trends in Communications and Information Theory). Now Publishers, Hanover, MA, 2006.
| GEMs | GEMs intersections | $\text{comss}_B(c)$ for $1 \leq c \leq k$ | $\sum_{c=1}^{k} \dim(B'_c)$ | $\dim(\sum_{c=1}^{k} B'_c)$ | Possible FSRD $i$ |
|------|--------------------|----------------------------------|------------------|-----------------|------------------|
| A vector $B_1 = v_1$ with $r \geq 2$ | $\dim(B'_1) = 1$ | $\text{comss}_B(1) = 1$ | $\dim(B'_1) = 1$ | $\dim(B'_1) = 1$ | $i = (1)$ |
| Two vectors $B_1 = v_1$, $B_2 = v_2$ with $r \geq 2$ | $\dim(B'_1 \cap B'_2) = 0$ | $\text{comss}_B(2) = 0$, $\text{comss}_B(1) = 2$ | $\dim(B'_1) + \dim(B'_2)$ | $\dim(B'_1 + B'_2)$ | $i = (2, 0)$ |
| Three vectors $B_1 = v_1$, $B_2 = v_2$, $B_3 = v_3$ with $r \geq 2$ | $\dim(B'_1 \cap B'_2) = 0$ | $\text{comss}_B(3) = 0$, $\text{comss}_B(2) = 0$, $\text{comss}_B(1) = 3$ | $3$ | $2$ | Impossible - the only eligible $i$ is $i = (3, 0, 0)$, for which $\sum_{c=1}^{3} i_c > 2$ |
| Two 2-column matrices $B_1, B_2$ with $r \geq 3$ | $\dim(B'_1 \cap B'_2) = 1$ | $\text{comss}_B(2) = 1$, $\text{comss}_B(1) = 2$ | $4$ | $3$ | $i = (2, 1)$ |
| Two 2-column matrices $B_1, B_2$ and a vector $B_3 = v_1$ with $r \geq 4$ | $\dim(B'_1 \cap B'_2) = 1$, $\dim(B'_3 \cap B'_1) = 0$ | $\text{comss}_B(3) = 0$, $\text{comss}_B(2) = 1$, $\text{comss}_B(1) = 3$ | $5$ | $4$ (Note that for $r = 3$, $\dim(\sum_{c=1}^{3} B'_c) = 3$, resulting in no FSRD possibility) | $i = (3, 1, 0)$ |
| Four 2-column matrices $B_1, ..., B_4$, with $r = 3$ | For each $i \neq j$, $\dim(B'_i \cap B'_j) = 1$, and for each $i \neq j \neq k$, $\dim(B'_i \cap B'_j \cap B'_k) = 0$ | $\text{comss}_B(4) = 0$, $\text{comss}_B(3) = 0$, $\text{comss}_B(2) = 6$, $\text{comss}_B(1) = 0$ | $8$ | $3$ | Impossible - in order to have $\text{compol}_B(i) \geq 8$, necessarily $i_2 \geq 4$, giving $\sum_{c=1}^{3} i_c > 3$ |
| Three 3-column matrices $B_1, B_2, B_3$ with $r \geq 4$ | $\dim(\bigcap_{k=1}^{3} B'_k) = 1$, and for each $i \neq j$, $\dim(B_i \cap B_j) = 2$ | $\text{comss}_B(3) = 1$, $\text{comss}_B(2) = 3$, $\text{comss}_B(1) = 0$ | $9$ | $4$ | $\tilde{i} = (0, 3, 1)$ |
| Three 3-column matrices $B_1, B_2, B_3$ with $r \geq 4$ | $\dim(\bigcap_{k=1}^{3} B'_k) = 2$, and for each $i \neq j$, $\dim(B_i \cap B_j) = 2$ | $\text{comss}_B(3) = 2$, $\text{comss}_B(2) = 0$, $\text{comss}_B(1) = 3$ | $9$ | $4$ (For $r > 4$, may also be 5) | Only possible if $\dim(\sum_{c=1}^{3} B'_c) = 5$, with $i = (3, 0, 2)$ |
| Four 3-column matrices $B_1, ..., B_4$, with $r \geq 4$ | $\dim(\bigcap_{k=1}^{4} B'_k) = 0$, for each $i$ \[ \dim(\bigcap_{k=1 \neq i}^{4} B'_k) = 1, \text{ and for every} \] $i \neq j$ \[ \dim(B_i \cap B_j) = 2 \] | $\text{comss}_B(4) = 0$, $\text{comss}_B(3) = 4$, $\text{comss}_B(2) = 0$, $\text{comss}_B(1) = 0$ | $12$ | $4$ | $\tilde{i} = (0, 0, 4, 0)$ |
| Four 3-column matrices $B_1, ..., B_4$, with $r \geq 4$ | $\dim(\bigcap_{k=1}^{4} B'_k) = 1$, for each $i$, \[ \dim(\bigcap_{k=1 \neq i}^{4} B'_k) = 1, \text{ and for every} \] $i \neq j$ \[ \dim(B_i \cap B_j) = 2 \] | $\text{comss}_B(4) = 1$, $\text{comss}_B(3) = 0$, $\text{comss}_B(2) = 4$, $\text{comss}_B(1) = 0$ | $12$ | $4$ | Impossible - the only eligible $i$ is $i = (0, 0, 2, 1)$, for which $\sum_{c=1}^{4} i_c > 4$ |
| $n$ matrices with $n - 1$ columns, $B_1, ..., B_n$, with $r = n$ | $\dim(\bigcap_{k=1}^{n} B'_k) = 0$ | $\text{comss}_B(n) = 0$, $\text{comss}_B(n - 1) = n$, $\text{comss}_B(n - 2) = ... = \text{comss}_B(1) = 0$ | $n \cdot (n - 1)$ | $n$ | $\tilde{i} = (0, ..., 0, n, 0)$ |