IDENTIFYING THE OCCURRENCE OR NON OCCURRENCE OF COGNITIVE BIAS IN SITUATIONS RESEMBLING THE MONTY HALL PROBLEM

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Abstract. People reason heuristically in situations resembling inferential puzzles such as Bertrand’s box paradox and the Monty Hall problem. The practical significance of that fact for economic decision making is uncertain because a departure from sound reasoning may, but does not necessarily, result in a “cognitively biased” outcome different from what sound reasoning would have produced. Criteria are derived here, applicable to both experimental and non-experimental situations, for heuristic reasoning in an inferential-puzzle situations to result, or not to result, in cognitively bias. In some situations, neither of these criteria is satisfied, and whether or not agents’ posterior probability assessments or choices are cognitively biased cannot be determined.

1. Introduction

People use heuristic reasoning in decision situations, and thus potentially make “cognitively biased” decisions that deviate from what they would have done if they had reasoned soundly. This article concerns conditions under which a particular type of heuristic Bayesian inference will, or will not, deviate from sound inference in a situation, and provides examples of plans (that is, patterns of evidence-based choices) that result from sound inference in some situations, and from heuristic inference in others, while being demonstrably inconsistent with the other sort of reasoning. This explicit concern with demonstrability (with identifiability, in statistical or econometric parlance), rather than with the simple occurrence or non occurrence of cognitive bias, may distinguish the present research with respect to behavioral-economics research on the whole.

The paradigmatic situation to be studied is the “box paradox” formulated by Bertrand (1889, p. 2). Gardner (1959) and others have subsequently formulated isomorphic problems. Bar-Hillel and Falk (1982) recognized and elucidated the significance of those problems for cognitive psychology. Shimojo and Ichikawa (1989) conducted a pioneering cognitive-psychology experiment to understand better the logic of the heuristic reasoning by which people analyze such situations. Selvin

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(1975), inspired by an eponymous television producer’s adaptation of such a situation for entertainment, formulated the “Monty Hall problem”. The distinctive feature of this problem is that a person is required to make a utility-maximizing choice among a set of alternate gambles, rather than to express a numerical probability judgment. That is, it is a “behavioralistic” (in the sense of Savage (1972)) version of the box paradox. Granberg and Brown (1995), followed by Friedman (1998) and others, have used that problem as the basis for an experimental protocol.

The experiments that have just been mentioned, are designed to show a detectable, observable outcome from which an unobservable cause can be inferred. The outcome is an incorrect probability assessment or a biased decision, and the cause is the subject’s use of heuristic reasoning rather than of sound reasoning. The import of the experiments is that, even though the outcomes of heuristic reasoning are typically not detectable by casual observation of non-experimental situations (but, rather, require an insightfully designed experimental protocol to become apparent), heuristic reasoning is presumably endemic in everyday situations.

Heuristic reasoning is called ‘heuristic’ for a reason: that in some, albeit not all, cases that it is employed, it providentially leads to correct or approximately correct conclusions. Thus, that people endemically employ heuristic inductive logic does not necessarily imply that faulty posterior-probability assessments or misguided choices are endemic. The program of this article is to examine, in the specific context of situations resembling the box paradox and the Monty Hall problem, what are the characteristics of situations in which outcomes (posterior-probability assessments or choices) are, or are not, informative about whether cognitive bias has occurred. That is, the goal is to distinguish among three types of situation.

Type 1 In some situations, including experiments, some outcomes may be observed that demonstrably reflect heuristic reasoning and are inconsistent with sound reasoning. That is, persons (or agents) making those choices exhibit cognitive bias.

Type 2 There may also be situations in which heuristic reasoning will lead demonstrably to the same outcome as sound reasoning would have produced, given identical probability beliefs regarding potentially observable events. That is, even if the agent is reasoning heuristically, no cognitive bias will result from it.

Type 3 Finally, there may be situations in which some observable outcome can be imputed to heuristic reasoning by making one set of assumptions about the agent’s beliefs (and about utilities of available alternatives, if the outcome is a choice or decision), but different assumptions about the agent lead to the conclusion that the same outcome has resulted from sound reasoning. That is, given that outcome, although there is cognitive bias if the first set of assumptions is correct, the bias is not demonstrable because the alternate set of assumptions cannot be ruled out.

1Mark Feldman has mentioned to the authors that the Monty Hall problem is isomorphic to the situation of “restricted choice” in the game of bridge.

2Contemplating heuristic reasoning broadly, some researchers (such as Simon (1955)) have been inclined to believe that such cognitive bias is endemic in fact, and that experimental situations are exceptional only in point of the bias being demonstrable. Others (such as Friedman (1953)) have leaned toward the view that providential outcomes are normal, and that experimental situations are exceptional because cognitive bias occurs at all.
Sections 2 and 3 are devoted to formalizing a broad class of situations resembling the box paradox and Monty Hall problem, and to articulating what it means for heuristic reasoning to be justified by sound reasoning in such a situation. Proposition 1 (in section 4) provides a criterion for posterior beliefs reached by heuristic reasoning to be justifiable by sound reasoning, if specific prior beliefs are imputed to the agent. But the criterion does not rule out the possibility that those beliefs are cognitively biased outcomes of different prior beliefs. That is, a situation that meets the criterion might be of either type 2 or type 3. The trichotomy of situations is studied further in sections 5 and 6. Proposition 2 (in section 6) provides conditions that are sufficient (and, under an auxiliary assumption, necessary) for a situation to be of one or another of the three types. Section 7 concerns the “behavioralistic” framework, in which outcomes of reasoning are taken to be choices rather than posterior-probability assessments. This framework invokes more parsimonious assumptions about what is observable to a researcher than the “verbalistic” framework of sections 3–6 makes. Not surprisingly, it becomes more difficult to infer from outcomes how an agent has reasoned. Nonetheless, example 4 exhibits a pattern of choices that can only arise as an outcome of sound reasoning, while example 6 exhibits a pattern that can only result from heuristic reasoning (and thus is cognitively biased).

2. An example of heuristic inference

Here we formulate, and analyze in ad hoc terms, an example of the sort of heuristic inference that is the subject of this article. It illustrates the type of bias that is routinely observed in the performance of subjects in cognitive-science experiments.

2.1. The broken-fuel-gauge (BFG) problem. Your car has a broken fuel gauge. It always shows either ‘Full’ or ‘Empty’. When the tank is more than 70% full, the gauge always shows ‘Full’. When the tank is less than 30% full, the gauge always shows ‘Empty’. In between, the gauge might be in either state.

You have been on vacation—away from your car—for a month. You no longer recall how far you drove after last having filled the tank. Before reading the gauge, your beliefs about the amount of fuel in the tank correspond to a uniform distribution.

When you look, the gauge shows ‘Empty’. What is now your degree of belief that the tank is at most 30% full? In the notation of probability theory, what is $P[\text{Tank is } \leq 30\%\text{ full } | \text{‘Empty’}]$?

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3The example is formulated to avoid some features of the Monty Hall problem that Granberg and Brown (1995) and Friedman (1998) have identified as being related to other biases—involving revisions of choices and breaking of indifference among alternatives—to which some subjects’ performance might be imputed. The relationship between the example and the Monty Hall problem will be examined in section 7.3.

4Early studies, such as those of Granberg and Brown (1995) and Friedman (1998) established that about 90% of subjects initially give biased responses that would be derived from the heuristic analysis to be specified below, and that roughly 50% of subjects persist in giving those responses after many repetitions of the problem. Subsequent studies, such as the one by Kluger and Friedman (2010) and those that they cite, establish that some experimental treatments can reduce the incidence of cognitively biased responses, but not dramatically so.
2.2. **Heuristic analysis.** Let \( F_x \) denote the event that the tank is at most \( x\% \) full. Then \( P(F_x) = x/100. \)

The *heuristic analysis* of the BFG problem is based on the assumption that the gauge showing ‘Empty’ corresponds to \( F_{70} \). Of course, there are other assumptions that an agent might conceivably substitute for the more complex and subtle assumption that sound reasoning would require, but this particular assumption is one that succeeds in accounting for way that experimental subjects tend to respond to such situations.

To an agent who reasons heuristically, then, a particular configuration of the gauge denotes the set of states of nature in which the gauge can possibly be in that configuration. If you reason heuristically, then, when asked what is \( P(F_{30} | \text{’Empty’}) \), you interpret that conditional probability as being \( P(F_{30} | F_{70}) \). By Bayes’s rule,

\[
(1) \quad P(F_{30} | F_{70}) = \frac{P(F_{30} \cap F_{70})}{P(F_{70})} = \frac{P(F_{30})}{P(F_{70})} = \frac{3}{7}
\]

2.3. **Sound analysis.** A conceptually correct, or *sound*, analysis of the BFG problem proceeds according to the logic articulated by Harsanyi (1967). This analysis emphasizes that your having observed the tank to show ‘Empty’ is a fact about you, rather than being per se a fact about the tank or its contents. Whether the gauge shows ‘Empty’ or ‘Full’ determines your type.

Your type is random, from an *ex ante* point of view. This randomness is modeled as a type-valued function \( \tau : \Phi \rightarrow \{\text{’Empty’}, \text{’Full’}\} \), where \( \Phi \) is the set of states of the world. The gauge showing ‘Empty’ (and you observing that fact) corresponds to the event that the state of the world is in \( \tau^{-1}(\text{’Empty’}) \). Thus, in contrast to the heuristic analysis, \( P[F_{30}|\text{’Empty’}] \) means \( P[F_{30}|\tau^{-1}(\text{’Empty’})] \).

\[
(2) \quad P[F_{30}|\tau^{-1}(\text{’Empty’})] = \frac{P(F_{30} \cap \tau^{-1}(\text{’Empty’}))}{P(\tau^{-1}(\text{’Empty’}))} = \frac{P(F_{30})}{P(\tau^{-1}(\text{’Empty’}))}
\]

Note that \( \tau^{-1}(\text{’Empty’}) = F_{30} \cup (\tau^{-1}(\text{’Empty’}) \cap (F_{70} \setminus F_{30})) \). Implicit in the specification that “The gauge might show either ‘Empty’ or ‘Full’ in event \( F_{70}, \)” is the idea that \( P(\tau^{-1}(\text{’Empty’}) \cap (F_{70} \setminus F_{30})) < P(F_{70} \setminus F_{30}) \). Therefore,

\[
(3) \quad P[F_{30}|\tau^{-1}(\text{’Empty’})] > \frac{P(F_{30})}{P(F_{70})} = \frac{3}{7}
\]

In conclusion, comparing (1) and (3) shows that the sound analysis yields a higher answer than the heuristic analysis does to the question about your posterior belief that the tank is truly near empty after having observed ‘Empty’.

3. **Models of evidence and of beliefs**

Two types of structures will be defined in this section, and how they apply to the BFG problem will be explained. A *model of evidence* formalizes heuristic Bayesian inference. A *model of beliefs* formalizes sound Bayesian inference. According to a model of beliefs, the agent reasons introspectively about the grounds for his/her

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5When this example is formalized below, a third type—corresponding to you not having yet observed the gauge (and thus holding your prior beliefs)—will be added. That change will not affect the present calculation.

6This idea is formalized for heuristic reasoning in the third clause of condition (7) below. By condition (20), the idea extends to sound reasoning also.
beliefs. That is, the agent asks, what determines the relationship of its own cognitive state to the objective facts about the world? An agent whose reasoning is represented by a model of evidence, is not introspecting. Rather, the agent is thinking solely in terms of objective events. Within each framework, the agent revises beliefs (that is, subjective probabilities) according to Bayes’s rule. The question to be addressed is under what conditions a model of evidence reflects—that is, is justified by—a model of beliefs.

3.1. Model of evidence. A model of evidence is a structure, \((\Omega, \mathcal{O}, P, \mathcal{E})\), where

1. \(\Omega\) comprises the states of nature
2. \(\mathcal{O}\) is a \(\sigma\)-field of objective events on \(\Omega\)
3. \(P : \mathcal{O} \to [0, 1]\) is a countably additive probability measure
4. \(\mathcal{E} \subseteq \mathcal{O}\) comprises the evidential events.
   - \(\mathcal{E}\) is countable
   - \(B \in \mathcal{E} \implies P(B) > 0\)
   - \([B \in \mathcal{E} \text{ and } C \in \mathcal{E} \text{ and } C \subsetneq B] \implies P(C \setminus B) > 0\)
   - \(\Omega \in \mathcal{E}\)
   - \(\bigcup(\mathcal{E} \setminus \{\Omega\}) = \mathcal{E}\)

Clearly an agent requires no evidence to be certain that \(\omega \in \Omega\). It will be convenient to have a notation for the non-trivial evidential events, that is, for those in \(\mathcal{E} \setminus \{\Omega\}\). Define

\[
\mathcal{E}' = \mathcal{E} \setminus \{\Omega\}
\]

The assumptions made in condition 7 reflect the focus of this article. Notably the assumptions that \(P(B) > 0\) and that \(P(C \setminus B) > 0\) simplify the analysis of the specific cognitive bias studied here, to which subtle questions that arise concerning conditioning on events of prior probability zero have no apparent relevance. That is, although questions regarding how to extend conditional probability to conditioning events of prior probability zero are crucial for some issues in game theory, they are arcane in the context of belief revision and choice by a single agent.

It is assumed that \(\mathcal{E}\) is countable because, otherwise, that \(P(B) > 0\) for every \(B \in \mathcal{E}\) would be impossible.\(^7\)

The assumption that \(\Omega \in \mathcal{E}\) is a convention that will play a role in defining what it means for a model of beliefs to justify a model of evidence. \(\mathcal{E}'\) is actually the set of entities that formalizes the intuitive notion of a non-trivial evidential event. In principle, there might be some state of the world for which no corroborating evidence could possibly be found. That is, conceivably \(\bigcup \mathcal{E}' \neq \Omega\). A condition, balancedness, will be defined in section 4.1 that will fail if \(P(\bigcup \mathcal{E}') < 1\). Proposition 1 will assert that balancedness of a model of evidence is a necessary and sufficient condition for there to be some model of beliefs that justifies it. Thus it is known that if, the definition of a model of evidence were relaxed to permit that \(P(\bigcup \mathcal{E}') < 1\), then such model would represent a situation of type 1 (in the taxonomy of the introduction). But, rather than complicate the exposition of proposition 1 and other results by explicit consideration of that possibility of un-corrorborable states

\(^7\)It would be possible to define a more general structure that would not require \(\mathcal{E}\) to be countable, analogously to the way that full-support probability distributions on continuously distributed random variables are defined in probability theory, but to do so would greatly complicate the mathematical arguments to be made here without making a corresponding conceptual gain.
of the world (and to avoid arcane complications of dealing with probability-zero events), it has been stipulated that \( \cup E' = \Omega \).

As usual in Bayesian decision theory, the probability space, \((\Omega, \mathcal{O}, P)\) models an agent’s prior beliefs. The events in \(E'\) model observations that the agent might make, on the basis of which evidence the agent would form posterior beliefs. Those beliefs are formed by conditionalization, where conditional probability,

\[
P[A|B] = \frac{P(A \cap B)}{P(B)}
\]

(9)

Let’s see how the heuristic analysis of the BFG problem is formalized as a model of evidence. The description of the problem in section 2.2 is made more simple here, by assuming that there are just three states of nature, \(\Omega = \{e, h, f\}\). State \(e\) represents the situation that the fuel tank is nearly empty \((0 \leq x < 30)\); and \(f\) represents the situation that the fuel tank is nearly full \((70 \leq x \leq 100)\).

**Example 1.** Define \(\Omega, \mathcal{O}, P, \mathcal{E}\) as follows. \(\Omega = \{e, h, f\}\) and \(\mathcal{O} = 2^\Omega\). By additivity, \(P\) is defined by the probabilities of singleton events in \(\mathcal{O}\). Specify that \(P(\{e\}) = P(\{f\}) = 0.3\) and that \(P(\{h\}) = 0.4\). There are two non-trivial evidential events: \(E = \{e, h\}\) and \(F = \{h, f\}\), \(\mathcal{E} = \{\Omega, E, F\}\).

Note that, corresponding to (11) in section 2.2, \(P(\{e\} | E) = 3/7\).

### 3.2. Model of beliefs

Harsanyi (1967) introduced a structure that he called a beliefs space, which consists of a probability space, \((\Phi, \mathcal{B}, Q)\) and a type function, \(\tau: \Phi \rightarrow T\), where \(T\) is an abstract set. The elements of \(\Phi\) are states of the world and the elements of \(T\) are types of the agent. The types in Harsanyi’s structure, per se, are nothing but arbitrary labels. What is meaningful are the inverse images, \(\tau^{-1}(t)\), of the types. These information sets partition the states of the world. In the event that an agent is of type \(t\), then the agent’s posterior belief regarding an event, \(C\), is \(Q[C|T^{-1}(t)]\), where \(Q: \mathcal{B} \times \tau^{-1}(T) \rightarrow [0, 1]\) is defined analogously to (10).

A model of beliefs, \((\Phi, \mathcal{B}, Q, \Omega, \mathcal{E}, \tau)\), is a slight variant of a beliefs space.

(10) \(\Omega\) comprises the states of nature, \(\mathcal{E} \subset 2^\Omega \setminus \{\emptyset\}\) is countable. \(\Omega = \bigcup E' \in \mathcal{E}\).

(11) \(\Phi\) comprises the states of the world. \(\mathcal{B}\) is a \(\sigma\)-field on \(\Phi\) and \(Q: \mathcal{B} \rightarrow [0, 1]\) is a countably additive probability measure.

(12) The type function, \(\tau: \Phi \rightarrow \mathcal{E}'\), maps \(\Phi\) onto \(\mathcal{E}'\). \(\Omega\) is called the prior-beliefs type, and elements of \(\mathcal{E}'\) are called posterior-beliefs types.

(13) \(\tau^{-1}(B) \in \mathcal{B}\) and \(0 < Q(\tau^{-1}(B)) < 1\)

It will be convenient to extend \(\tau^{-1}\), the inverse correspondence of \(\tau\), to a correspondence, \(\beta: \mathcal{E} \rightarrow \mathcal{B}\).

(14) \(\phi \in \beta(B) \iff [B = \Omega \text{ or } \tau(\phi) = B]\)

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8 Throughout this article, \(A\) should be interpreted to range over all of \(\mathcal{O}\), and \(B\) to range only over \(\mathcal{E}\), absent a statement to the contrary.

9 This nomenclature corresponds to standard terminology in decision theory regarding a single agent. In game theory, probability assessments conditioned on a player’s type are generally called interim beliefs.
It is a trivial formal change of Harsanyi’s framework to specify that the agent’s types are sets of states of nature rather than being arbitrary labels, and to introduce a new type that is not realized in any state of the world. Nonetheless, this modification enables a model of evidence and a beliefs space to be compared explicitly as representations of Bayesian inference. Before framing a systematic comparison in the next section, let’s see how a model of belief contrasts with a model of evidence as a representation of inference in the BFG problem. The intuition behind the formalization of the BFG problem as a model of beliefs is as follows. $\Omega$, $P$, and $E$ are as in example 1. Type $\Omega$ represents the agent’s prior beliefs, while types $E$ and $F$ represent posterior beliefs after having observed the gauge to show ‘Empty’ and ‘Full’ respectively. Conditionally on the state of nature being $e$ or $f$, if the agent’s type is not $\Omega$, then it must be $E$ or $F$ respectively. However, if the state of nature is $h$ and the agent’s type is not $\Omega$, then the type may be either $E$ or $F$. Assume that, conditionally on the state of nature being $h$ and the agent’s type not being $\Omega$, the other two types are equally probable.

**Example 2.** Let $(\Omega, B, P, E)$ be as in example 1. $\Phi = \Omega \times E' = \{e, h, f\} \times \{E, F\}$. $B = 2^\Phi$. $Q$ is specified according to the following table. Each cell of the table is a probability. The column labeled ‘$\Omega$’ shows marginal probabilities of $Q$ on $\Omega$. The cell in the row labeled ‘$\omega$’ and the column labeled ‘$B$’, for $B \in E'$, shows the probability of the corresponding state of the world, $(\omega, B)$.

| $Q(\omega, B)$ | $\Omega$ | $E$ | $F$ |
|-----------------|--------|---|---|
| $e$             | .3    | .3| 0 |
| $h$             | .4    | .2| .2|
| $f$             | .3    | 0 | .3|

(15)

The type function is defined by $\tau(\omega, B) = B$ for all $\omega \in \Omega$ and $B \in E'$. Note that $Q(A \times \Omega) = P(A)$ for all $A \in O$, where $P$ is the probability measure constructed in example 1.

If $A$ is a set of states of nature, then the event that the state of nature is in $A$ is $A \times E' \in B$. Thus, as explained above, the posterior probability held by an agent of type $B \in E'$ that the state of nature is in $A$ is $Q[A \times E'| \beta(B)]$. In particular, taking $A = \{e\}$ and $B = E$, the agent’s posterior probability that the state of nature is $e$ is $3/5$. Since it has been shown that $P[\{e\} | E] = 3/7$ in example 1 the formal representations of the BFG problem via a model of evidence and a model of belief reproduce the overall conclusion, inequality (3), of section 2.2.

3.3. Reflection/justification. In the context of comparing examples 1 and 2, it was just suggested that $A \times E'$ is the event in $B$ that is associated with a set, $A \in O$, of states of nature. In fact, this association defines an isomorphism, $A \mapsto A \times E'$, of $O$ with a sub $\sigma$-field of $B$ in the example. Capitalizing on this idea, the relationship between a model of evidence and a model of beliefs that was implicitly defined in that discussion can be stated in an explicit and general way.

In order to make this generalization, an isomorphism of probability spaces must be defined in a slightly more permissive way than the obvious one. In the preceding paragraph, ‘isomorphism’ was used in the obvious way, to denote a mapping that preserves Boolean relationships among sets. However, in general—when the situation does not have the convenient feature that the domain of the isomorphism is a Cartesian factor of its range—such an exact relationship may not hold. Rather, if
(Ξ, C, R) and (Ψ, D, S) are probability spaces, then define \( \alpha : C \to D \) to be a measure isomorphism from (Ξ, C, R) to (Ψ, D, S) iff the following conditions hold:\(^{14}\)

For every \( C \in C \), \( S(\alpha(C)) = R(C) \)

(16) For every countable \( F \subseteq C \), \( S \left( \alpha \left( \bigcup F \right) \triangle \bigcup \{ \alpha(B) \mid B \in F \} \right) = 0 \)

For every \( D \in D \), there exists \( C \in C \) such that \( Q(D \triangle \alpha(C)) = 0 \)

Throughout the remainder of this article, isomorphism refers to a measure isomorphism. When the full specifications of (Ξ, C, R) and (Ψ, D, S) are clear from context, \( \alpha \) will be called an isomorphism from \( C \) to \( D \).

Let \( \Omega \in \mathcal{E} \subseteq 2^E \setminus \{ \emptyset \} \). A model of beliefs, (Φ, B, Q, Ω, E, τ), conforms to a model of evidence, (Ω, O, P, E), via \( \alpha : O \to B \) iff

(17) For some sub \( \sigma \)-field, \( C \), of \( B \), \( \alpha : O \to B \) is a measure isomorphism from (Ω, O, P) to (Φ, C, Q \mid O), and

(18) For all \( B \in E \), \( \beta(B) \subseteq \alpha(B) \)

A model of evidence, (Ω, O, P, E), reflects a model of beliefs, (Φ, B, Q, Ω, E, τ), (equivalently, the model of beliefs justifies the model of evidence) via \( \alpha : O \to B \) under the following conditions:\(^1\)

(19) (Φ, B, Q, Ω, E, τ), conforms to (Ω, O, P, E), via \( \alpha \), and

(20) For all \( A \in O \) and \( B \in E \), \( Q(\alpha(A)|\beta(B)) = P[A|B] \)

Note that, when this relationship holds, there is a clear reason to view the agent’s types as being evidential events rather than mere abstract labels. The agent’s type is the most specific objective event (not only the most specific evidential event) that the agent believes to obtain almost surely. That is, for any \( A \in O \) and \( B \in E \), \( Q(\beta(B) \setminus \alpha(B)) = 0 \) and, if \( P(A) < P(B) \), then \( Q(\beta(B) \setminus \alpha(A)) > 0 \).

Condition (20) is central to this article, because it formalizes what it means for cognitive bias not to occur. An agent is envisioned to have authentic probability beliefs, either fully articulated or inchoate, that are represented by a model of beliefs, (Φ, B, Q, Ω, E, τ). With respect to the events in some sub \( \sigma \)-field of \( B \), at least, the agent’s beliefs are envisioned to be fully articulated. Specifically it is envisioned that there is a model of evidence, (Ω, O, P, E), such that (Φ, B, Q, Ω, E, τ) conforms to (Ω, O, P, E) via some measure isomorphism, \( \alpha \), and that the agent’s authentic probability beliefs about events in the image of \( O \) under \( \alpha \) are fully articulated. That is, condition (19) is satisfied. What determines whether or not (Ω, O, P, E) reflects (Φ, B, Q, Ω, E, τ), is condition (20). If (20) is not satisfied, then (Ω, O, P, E) does not reflect (Φ, B, Q, Ω, E, τ). Intuitively that is the case in which, if the agent reasons heuristically according to the model of evidence, (Ω, O, P, E), then the agent exhibits cognitive bias relative to sound inference from authentic beliefs (that is, relative to inference based soundly on the model of beliefs, (Φ, B, Q, Ω, E, τ)).

**Claim 1.** The model of evidence in example 4 does not reflect any model of beliefs. The model of beliefs in example 2 does not justify any model of evidence.

**Proof.** Suppose that \( \alpha : O \to B \) satisfies conditions (17) and (19), and that either (Ω, O, P, E) is the model of evidence in example 1 or else (Φ, B, Q, Ω, E, τ)
A fact about conformity, to be used later in the proof of proposition 2 is stated and proved now.

**Lemma 1.** For every model of evidence, there is a conforming model of beliefs.

*Proof.* Consider a model of evidence, \( \Omega, \mathcal{O}, P, \mathcal{E} \). Let \( \{B_s\}_{s \in S} \) enumerate \( \mathcal{E}' \). \( S = \{1, \ldots, n\} \) if \( \mathcal{E}' \) has \( n \) elements, and \( S = \{1, 2, 3, \ldots\} \) if \( \mathcal{E}' \) is infinite.) Let \( t = \sum_{s \in S} 2^{-s} \). Define \( \Phi = \Omega \times \mathcal{E}' \) and \( B = \Sigma(\Omega \times 2^{\mathcal{E}'}) \).\(^{12}\) Begin to define \( Q \) by \( Q(A \times B_s) = 2^{-s} P(A)/t \). This definition extends by countable additivity to \( \Sigma(\Omega \times 2^{\mathcal{E}'}) \).\(^{13}\) Define \( \mu: \Omega \to S \) by \( \mu(\omega) = \min \{ s \mid \omega \in B(s) \} \), and define \( \tau: \Phi \to \mathcal{E}' \) by

\[
\tau(\omega, B) = \begin{cases} 
B & \text{if } \omega \in B \\
B_{\mu(\omega)} & \text{if } \omega \notin B
\end{cases}
\]

Define \( \alpha: \mathcal{O} \to \mathcal{B} \) by \( \alpha(A) = A \times \mathcal{E}' \). It is routinely verified that \( (\Phi, B, Q, \Omega, \mathcal{E}, \tau) \) is a model of beliefs that conforms to \( (\Omega, \mathcal{O}, P, \mathcal{E}) \) via \( \alpha \). \( \square \)

4. When does some model of beliefs justify a model of evidence?

In this section, a necessary and sufficient condition will be derived for a model of evidence to reflect some model of beliefs. To set the stage, let us point out a feature of the BFG problem that seems to be conducive for cognitive bias to occur. Consider the model of beliefs presented in example 2. There are only 2 posterior-beliefs types: \( E = \{e, h\} \) and \( F = \{h, f\} \). The image of \( \{h\} \) under \( \alpha \) is split between \( \beta(E) \) and \( \beta(F) \). Its probability mass is correspondingly split in the model of beliefs.

In contrast, \( \alpha(\{e\}) \subseteq \beta(E) \) and \( \alpha(\{f\}) \subseteq \beta(F) \). The probability mass of these states of nature therefore is not split.

Reflection is impossible because probability mass of some states of nature, but not others, must be split. This situation results from a particular state of nature being in both \( E \) and \( F \), while others are only in one of them. A state of nature that belongs to more evidential events than others do, is under weighed relative to those others by \( Q \) within the image under \( \beta \) of each of the evidential events to which it belongs.

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\(^{12}\) If \( \mathcal{C} \subseteq 2^\mathcal{E} \), then \( \Sigma(\mathcal{C}) \) is the smallest \( \sigma \)-field containing \( \mathcal{C} \).

\(^{13}\) Specifically, this extension is a measure by Caratheodory’s theorem. (Cf. Aliprantis and Border (2006, theorem 10.29)).
Clearly this problem of under weighting cannot occur if $E'$ is a partition of $\Omega$. The following example shows that, to avoid under weighting, it is not necessary for $E'$ to be a partition, or even for there to exist a partition of $\Omega$ by elements of $E'$. A condition that the example does satisfy, and that will be generalized below, is that each state of nature belongs to the same number (3, in the example) of evidential events.

**Example 3.** Define $(\Omega, B, P, E)$ by setting $\Omega = \{0, 1, 2\}$, $B = 2^\Omega$, and $P(\{\omega\}) = 1/3$, and by defining $E'$ to be the set of two-element subsets of $2^\Omega$. For each $\omega \in \Omega$, define $\omega' \equiv \omega + 1 \pmod{3}$ and $\omega'' \equiv \omega - 1 \pmod{3}$, and define $E_\omega = \{\omega, \omega'\}$. Define $E' = \{\omega', \omega''\}$.

Define $\Psi = \{0, \ldots, 5\}$, $C = 2^\Psi$, and define $R(\{\psi\}) = 1/6$ for each $\psi$. Let $(\Phi, B, Q)$ be the product of $(\Omega, O, P)$ and $(\Psi, C, R)$. Using the unique representation of $\psi = 2j + k$ (with $0 \leq j \leq 2$, $0 \leq k \leq 1$), define $\tau$ by

$$\tau(\omega, \psi) = E_i, \text{ where } i = \begin{cases} \omega, \text{ if } k = 0 \\ \omega'', \text{ if } k = 1 \end{cases}$$

There is no partition of $\Omega$ by elements of $E'$. Nonetheless, defining $\alpha(A) = A \times \Psi$, $(\Phi, B, Q, \Omega, E, \tau)$ justifies $(\Omega, O, P, E)$.

### 4.1. Balancedness defined.

Define $(\Omega, O, P, E)$ to be balanced iff, for some $\theta$,

$$\theta: E' \rightarrow (0, 1] \quad \text{and} \quad P\left(\left\{\omega \mid \sum_{B \in E'} \theta(B) = 1\right\}\right) = 1$$

Call $\theta$ a balancing function (for $P$ and $E$).

Note that, with $\chi_B: \Omega \rightarrow \{0, 1\}$ being the indicator function of $B$, (24) is equivalent to

$$\theta: E' \rightarrow (0, 1] \quad \text{and} \quad P\left(\left\{\omega \mid \sum_{B \in E'} \theta(B) \chi_B(\omega) = 1\right\}\right) = 1$$

By setting $\theta(B_0) = \theta(B_1) = \theta(B_2) = 1/2$, it is seen that the model of evidence in example 3 is balanced. In contrast, for the model of evidence in example 1 if $\theta: E \rightarrow (0, 1)$, then $\sum_{h \in B} \theta(B) - \sum_{e \in B} \theta(B) = \theta(F) > 0$, so either $\sum_{h \in B} \theta(B) > 1$ or $\sum_{e \in B} \theta(B) < 1$ and therefore (24) cannot hold. In each of these two examples, then, the model of evidence being balanced is equivalent to it reflecting some model of beliefs.

### 4.2. Balancedness and the justifiability of a model of evidence.

The following proposition follows immediately from the two lemmas that are subsequently proved.

**Proposition 1.** A model of evidence is balanced if, and only if, it reflects some model of beliefs.

**Lemma 2.** A model of evidence is balanced, if some model of belief justifies it. 

---

14 The states of nature and the evidential events in example 1 can be embedded in this structure by assigning $e \mapsto 0$, $h \mapsto 1$, $f \mapsto 2$, $E \mapsto B_0$, and $F \mapsto B_1$. 
Then, from Fubini’s theorem and (30) and (32), it follows that, for all
\[Q(\alpha(A) \cap \beta(B)) = P(A|B)Q(\beta(B)).\]

Define \[\theta: E' \to (0, 1)\] by
\[
\theta(B) = \frac{Q(\beta(B))}{P(B)}
\]
Then
\[P(A) = Q(\alpha(A)) = \sum_{B \in E'} Q(\alpha(A) \cap \beta(B)) = \sum_{B \in E'} P(A|B)Q(\beta(B))
\]
\[
= \sum_{B \in E'} \int_A \frac{\chi_B}{P(B)}Q(\beta(B)) \, dP = \int_A \sum_{B \in E'} \theta(B)\chi_B \, dP
\]
Given that (27) holds for all \(A \in \mathcal{O}\), condition (25) is satisfied, so equation (26) defines a balancing function. □

Lemma 3. If a model of evidence is balanced, then some model of beliefs justifies it.

Proof. Let \(\theta\) be a balancing function for a model of evidence, \((\Omega, \mathcal{O}, P, E)\). A model of beliefs, \((\Phi, B, Q, \Omega, E, \tau)\), that justifies \((\Omega, \mathcal{O}, P, E)\) is now constructed.

Let \(\Psi = [0, 1)\) and let \(\mathcal{C} \) and \(R\) be the \(\sigma\)-field of Borel sets on \(\Psi\) and the Lebesgue measure. Specify that \(\Phi = \Omega \times \Psi\), \(\mathcal{B} = \Sigma(\Omega \times \mathcal{C})\), and \(Q = P \times R\). Define an isomorphism, \(\alpha: \mathcal{O} \to B\), by
\[
\alpha(A) = A \times \Psi
\]
Let \(\langle B_s \rangle_{s \in S}\) enumerate \(E'\). For \(n \in \{0\} \cup S\), define
\[
g_n(\omega) = 0 \text{ and } g_{n+1}(\omega) = g_n(\omega) + \theta(B_{n+1})\chi_{B_{n+1}}(\omega)
\]
If \(\langle x_s \rangle_{s \in S}\) is a sequence of numbers, then define \(\lim_{s \to \max S} x_s = x_{\max S}\) if \(S\) is finite and \(\lim_{s \to \max S} x_s = \lim_{s \to \infty} x_s\) if \(S\) is infinite. Define \(N = \{\omega \mid \lim_{s \to \max S} g_s(\omega) \neq 1\}\). Since \(\theta\) is a balancing function,
\[
P(N) = 0
\]
Define \(\tau: \Phi \to E'\) by
\[
\tau(\omega, \psi) = B_s \iff \begin{cases} g_{s-1}(\omega) \leq \psi < g_s(\omega) & \text{if } s \notin N \\ s = 1 & \text{if } s \in N \end{cases}
\]
From this definition, it follows that
\[
\{ (\omega, \psi) \mid \omega \in B_s \text{ and } g_{s-1}(\omega) \leq \psi < g_s(\omega) \} \subseteq \beta(B_s)
\]
\[
\subseteq \{ (\omega, \psi) \mid \omega \in B_s \text{ and } g_{s-1}(\omega) \leq \psi < g_s(\omega) \} \cup N
\]
Then, from Fubini’s theorem and (30) and (32), it follows that, for all \(A \in \mathcal{O}\),
\[
Q(\alpha(A) \cap \beta(B_s)) = \int_{A \cap B_s} \int_{g_{s-1}(\omega)}^{g_s(\omega)} 1 \, dR \, dP
\]
\[
= \int_{A \cap B_s} g_s(\omega) - g_{s-1}(\omega) \, dP = \theta(B_s)P(A \cap B_s)
\]
Conditions (30) and (32) and (33) imply that
\[
Q(\beta(B_s)) = Q(\alpha(B_s) \cap \beta(B_s)) = \theta(B_s)P(B_s)
\]
so, for $A \in \mathcal{O}$ and $B \in \mathcal{E}$,

$$Q[\alpha(A) | \beta(B)] = \frac{Q(\alpha(A) \cap \beta(B))}{Q(B)} = \frac{\theta(B) P(A \cap B)}{\theta(B) P(B)} = P[A|B]$$

That is, $(\Phi, B, Q, \Omega, \mathcal{E}, \tau)$ justifies $(\Omega, \mathcal{O}, P, \mathcal{E})$. $\square$

5. Situations

An undefined term, situation, played an important role in the introductory discussion of alternate views about the import of psychological experiments for economics. In this section, building on the framework introduced in the preceding section, a situation will be formally defined.

In the introduction, it was mentioned that Shimojo and Ichikawa (1989) used a situation isomorphic to Bertrand’s box paradox as the basis for an experiment to exhibit subjects’ cognitive bias. In that situation, there are three states of nature. Shimojo and Ichikawa stipulated that, given the way that the situation was described to subjects in the experimental protocol, their prior beliefs would be that each state of nature has probability $1/3$. (That is, those would be the subjects’ probability assessments after having received the description of the situation, but before having observed evidence that would be presented in the course of the experiment.) However, those researchers did not make any assumption regarding a subject’s beliefs about the correlation between the state of nature and the evidence that would be observed. They did not need to make any such assumption, because most subjects reported posterior probability assessments that were inconsistent with any model of beliefs corresponding to the stipulated prior probability beliefs about states of nature. That is, the outcomes of the experiment were generated by a situation of type 1 according to the trichotomy presented in the introduction.

The idea of a “model of beliefs corresponding to the stipulated prior probability beliefs about states of nature” is formalized by the definition of conformity. Shimojo and Ichikawa’s assumptions about subjects’ beliefs regarding the states of nature can be represented as a model of evidence. As has been discussed following the definition of reflection in the previous section, their discussion of their experiment presupposed that each subject had authentic probability beliefs about the state of the world that could be represented as some model of beliefs or other, but they did not pretend to know anything about that model beyond the fact that it conformed to the stipulated model of evidence. The general form of Shimojo and Ichikawa’s way of thinking about their experiment is captured by the following definition. That is, a situation is a structure that formally describes a researcher’s assumptions regarding both the observable and unobservable aspects of a what that researcher assumes to be a subject’s authentic prior-probability beliefs. Specifically, the research assumes everything that is common to all of the models of belief in the situation.

A situation is a structure, $((\Omega, \mathcal{O}, P, \mathcal{E}), S)$, comprising a model of evidence and a non empty set, $S$, of ordered pairs. Each element of $S$ is of the form $((\Phi, B, Q, \Omega, \mathcal{E}, \tau), \alpha)$, where $(\Phi, B, Q, \Omega, \mathcal{E}, \tau)$ is a model of beliefs that conforms to $(\Omega, \mathcal{O}, P, \mathcal{E})$ via $\alpha : \mathcal{O} \rightarrow B$. Where no confusion will result from abuse of notation, ‘$S$’ will be used to name the situation. Also, a statement such as “Some model of beliefs in $S$ justifies $(\Omega, \mathcal{O}, P, \mathcal{E})$” should be understood as “For some $((\Phi, B, Q, \Omega, \mathcal{E}, \tau), \alpha) \in S$, $(\Phi, B, Q, \Omega, \mathcal{E}, \tau)$ justifies $(\Omega, \mathcal{O}, P, \mathcal{E})$ via $\alpha$. ”
As Shimojo and Ichikawa have done, a researcher may assume nothing at all about a subject’s beliefs, except that those beliefs conform to the model of evidence that is communicated in the experimental protocol. That situation is represented by \(((\Omega, \mathcal{O}, P, \mathcal{E}), \mathcal{S})\), where \((\Omega, \mathcal{O}, P, \mathcal{E})\), is communicated in the protocol and \(\mathcal{S}\) comprises all of the pairs, \((\Phi, \mathcal{B}, Q, \Omega, \mathcal{E}, \tau, \alpha)\), such that \((\Phi, \mathcal{B}, Q, \Omega, \mathcal{E}, \tau)\) conforms to \((\Omega, \mathcal{O}, P, \mathcal{E})\) via \(\alpha\). Such a situation will be called full.

6. Characterizing the types of situation

In terms of the definition of a situation just given, the trichotomy discussed in the introduction is formalized by

\[
(\Omega, \mathcal{O}, P, \mathcal{E}), \mathcal{S} \text{ is of type } \begin{cases} 
1 & \text{if no model of beliefs in } \mathcal{S} \text{ justifies } (\Omega, \mathcal{O}, P, \mathcal{E}) \\
2 & \text{if every model of beliefs in } \mathcal{S} \text{ justifies } (\Omega, \mathcal{O}, P, \mathcal{E}) \\
3 & \text{otherwise}
\end{cases}
\]

If \(((\Omega, \mathcal{O}, P, \mathcal{E}), \mathcal{S})\) is a model of evidence, then define \(\mathcal{E}'\) to be an almost sure partition iff, for every pair of distinct elements, \(C\) and \(D\), of \(\mathcal{E}'\), \(P(C \cap D) = 0\).

**Proposition 2.** Let \(((\Omega, \mathcal{O}, P, \mathcal{E}), \mathcal{S})\) be a situation. Then

\[
\begin{align*}
(37) & \text{ If situation } \mathcal{S} \text{ is not balanced, then it is of type } 1. \\
(38) & \text{ If situation } \mathcal{S} \text{ is full and of type } 1, \text{ then it is not balanced.} \\
(39) & \text{ If } \mathcal{E}' \text{ is an almost sure partition, then situation } \mathcal{S} \text{ is of type } 2. \\
(40) & \text{ If situation } \mathcal{S} \text{ is full and of type } 2, \text{ then } \mathcal{E}' \text{ is an almost sure partition.} \\
(41) & \text{ There exist situations of each of the three types.}
\end{align*}
\]

**Proof.** Consider each of the claims.

37 This follows from proposition 1.
38 Equivalently, if situation \(\mathcal{S}\) is full and balanced, then it is not of type 1. Assume the antecedent. Because the situation is balanced, it reflects some model of beliefs. Because the situation is full, that model of beliefs is in \(\mathcal{S}\). That is, the situation is not of type 1.
39 Suppose that \(\mathcal{E}'\) is an almost sure partition and that \(((\Phi, \mathcal{B}, Q, \Omega, \mathcal{E}, \tau), \alpha) \in \mathcal{S}\). Since \((\Phi, \mathcal{B}, Q, \Omega, \mathcal{E}, \tau)\) conforms to \((\Omega, \mathcal{O}, P, \mathcal{E})\), condition 18 holds. Together with the fact that \(\mathcal{E}'\) is an almost sure partition, 18 implies that, for every \(B \in \mathcal{E}'\), \(Q(\alpha(B) \triangle \beta(B)) = 0\). It follows that \((\Omega, \mathcal{O}, P, \mathcal{E})\) reflects \((\Phi, \mathcal{B}, Q, \Omega, \mathcal{E}, \tau)\). Therefore the situation is of type 2.
40 Equivalently, if the situation is full and \(\mathcal{E}'\) is not an almost sure partition, then the situation is not of type 2. That is, in that case, there is some model of beliefs, \((\Phi, \mathcal{B}, Q, \Omega, \mathcal{E}, \tau)\), in \(\mathcal{S}\) that does not reflect \((\Omega, \mathcal{O}, P, \mathcal{E})\). Since \(\mathcal{E}'\) is not an almost sure partition, there are two distinct elements of \(\mathcal{E}'\), \(C\) and \(D\), such that \(P(C \cap D) > 0\).

If the situation is not balanced, then it is of type 1 by 37, so assume that it is balanced. Let \(\theta\) be a balancing function. A model of evidence that conforms to \((\Omega, \mathcal{O}, P, \mathcal{E})\), but that does not justify \((\Omega, \mathcal{O}, P, \mathcal{E})\), will be constructed. To do so, let \(\Phi, \mathcal{B}, Q, \tau\), and the enumeration of \(\mathcal{E}'\), \((B_s)_{s \in \mathcal{S}}\), be as in the proof of lemma 1.
(Again, suppose that \(S = \{1, \ldots, n\}\) if \(\mathcal{E}'\) has \(n\) elements and that \(S = \{1, 2, 3, \ldots\}\)
if $\mathcal{E}'$ is infinite.) Without loss of generality, assume that $P(B_1 \cap B_2) > 0$ and that $B_2 \not\subseteq B_1$. By (7), $P(B_2 \setminus B_1) > 0$. Since the situation is full, the situation to be constructed is in $\mathcal{S}$, so the situation cannot be of type 2.

To construct the model of beliefs, a type function, $\sigma: \Phi \to \mathcal{E}'$, will be constructed. Begin by noting that $Q(\Omega \times [0, 1/2]) = 1/2$. On $\Omega \times [0, 1/2)$, $\sigma$ will satisfy $\sigma(\omega, \psi) = \tau(\omega, 2\psi)$. This specification ensures that, for each $B \in \mathcal{E}'$, $0 < Q(\sigma^{-1}(B)) < 1$ as required by condition (13) of the definition of a model of beliefs. On $\Omega \times [1/2, 1)$, define $\sigma$ to make $\beta(B)$ as large as condition (13) will permit.

Specifically, define $\mu(\omega) = \min \{s \mid \omega \in B_s\}$, and define $\sigma$ by

$$
\sigma(\omega, \psi) = \begin{cases}
\tau(\omega, 2\psi) & \text{if } \psi < 1/2 \\
\mu(\omega) & \text{otherwise}
\end{cases}
$$

With $\beta(B) = \tau^{-1}(B)$ and $\gamma(B) = \sigma^{-1}(B)$, note that

$$
Q(\gamma(B_2)) = Q((\gamma(B_2) \cap (\Omega \times [0, 1/2))) \cup ((\gamma(B_2) \cap (\Omega \times [1/2, 1))))
$$

$$
= Q((\gamma(B_2) \cap (\Omega \times [0, 1/2]))) + Q((\gamma(B_2) \cap (\Omega \times [1/2, 1])))
$$

$$
> Q((\gamma(B_2) \cap (\Omega \times [0, 1/2])))
$$

and

$$
\alpha(B_1 \cap B_2) \cap \gamma(B_2) = \alpha(B_1 \cap B_2) \cap (\gamma(B_2) \cap (\Omega \times [0, 1/2]))
$$

Also note that, for all $A \in \mathcal{O}$ and $B \in \mathcal{E}'$,

$$
Q(\alpha(A) \cap (\gamma(B) \cap (\Omega \times [0, 1/2]))) = \frac{Q(\alpha(A) \cap \beta(B))}{2}
$$

(and specifically, taking $A = \Omega$, $Q(\gamma(B) \cap (\Omega \times [0, 1/2))) = Q(\beta(B))/2$). Then

$$
Q[\alpha(B_1 \cap B_2) \cap \gamma(B_2)] = \frac{Q(\alpha(B_1 \cap B_2) \cap \gamma(B_2))}{Q(\gamma(B_2))}
$$

$$
< \frac{Q(\alpha(B_1 \cap B_2) \cap \gamma(B_2) \cap (\Omega \times [0, 1/2)))}{Q(\gamma(B_2) \cap (\Omega \times [0, 1/2)))}
$$

$$
= \frac{Q(\alpha(B_1 \cap B_2) \cap \beta(B_2))}{Q(\beta(B_2))}
$$

$$
= Q[\alpha(B_1 \cap B_2) \cap \beta(B_2)]
$$

By the construction of $(\Phi, \mathcal{B}, Q, \Omega, \mathcal{E}, \tau)$ in lemma 3, $Q[\alpha(B_1 \cap B_2) \cap \beta(B_2)] = P[B_1 \cap B_2|B_2]$. Therefore, by (10), $Q[\alpha(B_1 \cap B_2) \cap \gamma(B_2)] < P[B_1 \cap B_2|B_2]$. That is, $(\Phi, \mathcal{B}, Q, \Omega, \mathcal{E}, \sigma)$ conforms to $(\Omega, \mathcal{O}, P, \mathcal{E})$ but it does not justify $(\Omega, \mathcal{O}, P, \mathcal{E})$. Since situation $\mathcal{S}$ is full, $(\Phi, \mathcal{B}, Q, \Omega, \mathcal{E}, \sigma) \in \mathcal{S}$. Therefore, situation $\mathcal{S}$ is not of type 2.

By lemma 4 there is a situation, and hence a full situation, corresponding to every model of evidence. There are models of evidence that are not balanced, so, by (37), there are full situations of type 1. There are models of evidence such that $\mathcal{E}'$ is an almost sure partition, so, by (39), there are situations of type 2. Example 3 is a balanced model of evidence, for which $\mathcal{E}'$ is not an almost sure partition, so, by (38) and (40), the full situation corresponding to that model of evidence is of type 3.
Shimojo and Ichikawa (1989) elicited subjects’ reports of their posterior beliefs. They analyzed that data under the assumptions that (a) their experimental protocol induced specific prior beliefs that the researchers intended subjects to hold, and (b) subjects were capable of reporting precise numerical subjective probabilities of events and were willing to report those probabilities truthfully.

When those assumptions do not hold, another approach must be taken. One such approach, inspired by the characterization of subjective utility provided by Ramsey (1931) and Savage (1972), is to infer subjects’ prior and posterior probability measures from data regarding their choices among alternatives offered in the experiment.

Experimenters studying the Monty Hall problem, such as those of Granberg and Brown (1995) and Friedman (1998), have adopted a hybrid approach. They have assumed subjects to hold particular prior probabilities, but have inferred posterior probabilities from observed choices. In principle, though, some of the reasons to prefer choice-based imputation of posterior probabilities to subjects should apply to prior probabilities also. Subjects could be given opportunities to make choices both before and after having received evidence, with the former choices revealing information about subjects’ prior beliefs and the latter ones revealing information about posterior beliefs.

Such a thoroughly behavioralistic protocol will be considered now. The notion of a plan, to be defined momentarily, will play a cognate role to that of a situation in preceding sections. Essentially, subjects’ choices will be treated as statistics of prior and posterior beliefs. To observe a statistic of a probability distribution is less informative than to observe the distribution directly. Correspondingly, the precise characterization of the various types of situation in proposition 2 will not have a counterpart here. Nonetheless, example 4 will exhibit a plan that can only be chosen by a subject who reasons according to a model of beliefs, while example 6 will exhibit a plan that can only be chosen by a subject who reasons according to a model of evidence.

7.1. Plans and conditional expected utility. Consider an agent who may choose from a set, $A$, of alternatives. Suppose that the set of feasible alternatives does not depend on the state of nature. Let $E$ be a set of evidential events. (As specified in section 3.1, $\Omega \in E \subseteq O \setminus \{\emptyset\} \subseteq 2^\Omega$.) Intuitively, a plan is a correspondence that assigns a non-empty set of alternatives to each evidential event.

A question that it would be typical to pose in decision theory is: what are the conditions under which a plan, $\zeta : E \rightarrow A$, may represent an agent’s choices according to maximization of conditional expected utility? That is, when can $E$ be associated with a probability space, and can state-contingent utilities be imputed to the various alternatives, such that for each $B$, $\zeta(B)$ is the set of alternatives that maximize expected utility conditional on $B$?

However, regarding the decisions of experimental subjects and of other agents, there are really two questions. The probability space with respect to which conditional probabilities are formed might be taken to be either a model of evidence or else a model of beliefs. If a model of evidence, then the agent conditions on the event itself. If a model of beliefs, then the agent conditions on the distinct event, $\beta(B)$, that the evidential event, $B$, is the agent’s type.
The two questions are formulated explicitly as follows.

(47) Under what conditions on \((\Omega, \mathcal{C}, E)\) and \(\zeta\) do there exist a probability space, \((\Psi, \mathcal{C}, P)\), an isomorphism \(\alpha: \mathcal{O} \rightarrow \mathcal{C}\), and a set of (bounded, \(\mathcal{O}\)-measurable, state-contingent) utility functions, \(\{u_a: \Psi \rightarrow \mathbb{R}\}_{a \in A}\), such that, for all \(a \in A\) and \(B \in E\),

\[
\int_{\alpha(B)} u_a \, dP = \max_{b \in A} \int_{\alpha(B)} u_b \, dP \iff a \in \zeta(B)
\]

That is, under what conditions do there exist a model of evidence, \((\Psi, \mathcal{C}, P, \alpha(\mathcal{E}))\), and a set of utility functions that rationalize \(\zeta\)?

(48) Under what conditions on \(E\) and \(\zeta\) do there exist a model of beliefs, \((\Phi, B, Q, \Omega, \mathcal{E}, \tau)\) and a set of bounded utility functions, \(\{v_a: \Phi \rightarrow \mathbb{R}\}_{a \in A}\), such that, for all \(a \in A\) and \(B \in \mathcal{E}\),

\[
\int_{\beta(B)} v_a \, dQ = \max_{b \in A} \int_{\beta(B)} v_b \, dQ \iff a \in \zeta(B)
\]

A model of evidence that satisfies the condition stated in (47) rationalizes \(\zeta\) by evidence. A model of beliefs that satisfies the condition stated in (48) rationalizes \(\zeta\) by beliefs. A plan that is rationalized by some model is called rational with respect to that type of model.

The definition of rationalization by a model of beliefs shows why the prior-beliefs type, \(\Omega\), is needed although it is not in the range of the type function. If the definition of \(\mathcal{E}\) were amended so that \(\Omega \notin \mathcal{E}\), then any plan could be rationalized by beliefs. The reason is that, since \(\{\beta(B) \mid B \in \mathcal{E}'\}\) is a partition of \(\Phi\), functions \(v_a\) can be defined by

\[
v_a(\phi) = \begin{cases} 
1 & \text{if } a \in \zeta(\tau(\phi)) \\
0 & \text{otherwise} 
\end{cases}
\]

However, because the definition of rationality with respect to beliefs requires that (45) must be satisfied also by \(B = \Omega\), (49) does not automatically define utility functions that rationalize \(\zeta\). This observation reflects the basic principle that the force of Bayesian decision theory comes from the relationship between choices based on prior versus posterior beliefs, not solely on relationships among choices conditioned on alternate posterior beliefs.

Section 7.2 concerns an example of a plan that is rational with respect to beliefs, but not with respect to evidence. Section 7.3 concerns an example of a plan that is rational with respect to evidence, but not with respect to beliefs. In fact, this example formalizes the Monty Hall problem that has been studied in various experiments cited earlier.

7.2. Rationality with respect to beliefs does not imply rationality with respect to evidence. In this section, first a model of beliefs will be constructed and will be shown to rationalize a plan. Then, it will be shown that no model of evidence can rationalize that plan.

Example 4. The model of beliefs closely resembles example 2. It is based on a model of evidence that differs from example 1 by the addition of a new evidential event, \(H\). Thus, let \(\Omega = \{e, h, f\}\) and \(\mathcal{E} = \{\Omega, E, H, F\}\), where \(E = \{e, h\}\), \(F = \{h, f\}\), and \(H = \{h\}\). Besides the addition of \(H\) to \(\mathcal{E}\), the other change of the
current example from example[2] is to put greater weight on \( h \) than is specified in that earlier example. The set of states of the world, \( \Phi \), of the model of beliefs will be \( \Omega \times \mathcal{E}' \), and prior beliefs will be specified so that \( Q(\{h\} \times \mathcal{E}') = \frac{3}{5} \) and \( Q(\{f\} \times \mathcal{E}') = Q(\{e\} \times \mathcal{E}') = \frac{1}{5} \).

The specification of the model of beliefs is completed by taking \( B = 2^\Phi \), and \( \tau(\omega, B) = B \), and by fully specifying \( Q \) according to the following table. The cells are interpreted as in table (15).

\[
\begin{array}{cccc}
Q(\omega, B) & \Omega & E & F \\
\hline
e & .2 & 0 & 0 \\
h & .6 & .1 & .4 \\
f & .2 & .2 & 0 \\
\end{array}
\]

Since \( \tau(\omega, B) = B \), \( \beta(B) = B \times \mathcal{E}' \). Specify \( A \) by \( A = \{w, d\} \). Specify that \( \zeta(\Omega) = \zeta(H) = \{w\} \) and \( \zeta(E) = \zeta(F) = \{d\} \).

**Claim 2.** The plan specified in example 4 is rational with respect to beliefs, but not with respect to evidence.

**Proof.** It will be shown that there are utility functions that, together with the model of beliefs constructed in proposition 1, rationalize \( \zeta \) in example 4. However, \( \zeta \) is not rational with respect to evidence.

Intuitively, \( w \) is supposed to be wagering that the state of nature is \( h \) and \( d \) is supposed to be declining to wager. Formally suppose that, for all \( B \in \mathcal{E}' \), \( v_w(h, B) = 10 \) and \( v_w(e, B) = v_w(f, B) = -10 \) and that, for all \( \omega \in \Omega \), \( v_d(\omega) = 0 \).

Then

\[
\int_{\beta(\Omega)} v_w - v_d \, dQ = 2 \quad \text{and} \quad \int_{\beta(H)} v_w - v_d \, dQ = 4 \quad \text{and} \quad \int_{\beta(E)} v_d - v_w \, dQ = 1
\]

so \( \zeta \) is rational with respect to beliefs.

A contradiction will be obtained from supposing that some isomorphism, \( \alpha: \mathcal{O} \to \mathcal{C} \), model of evidence, \( (\Psi, C, R, \alpha(\mathcal{E})) \), and set of utility functions, \( \langle u_a : \Psi \to \mathbb{R} \rangle_{a \in A} \), rationalize \( \zeta \)

\[
\zeta(E) = \{d\}, \quad \text{so} \quad \int_{\alpha(E)} u_d - u_w \, dR > 0 \\
\zeta(H) = \{w\}, \quad \text{so} \quad \int_{\alpha(H)} u_d - u_w \, dR < 0
\]

\[
\text{thus} \quad \int_{\alpha(H) \setminus \alpha(E)} u_d - u_w \, dR > 0
\]

\[
\zeta(F) = \{w\}, \quad \text{so} \quad \int_{\alpha(F)} u_d - u_w \, dR > 0
\]

\[
\int_{\alpha(\Omega)} u_d - u_w \, dR = \int_{\alpha(F)} u_d - u_w \, dR + \int_{\alpha(E) \setminus \alpha(H)} u_d - u_w \, dR > 0
\]

But, that \( \int_{\alpha(\Omega)} u_d - u_w \, dR > 0 \) and \( \zeta(\Omega) = \{w\} \) contradicts condition (48.1) for \( \alpha \), \( (\Psi, C, R, \alpha(\mathcal{E})) \), and \( \langle u_a \rangle_{a \in A} \) to rationalize \( \zeta \). \( \square \)
7.3. Rationality with respect to evidence does not imply rationality with respect to beliefs. The BFG problem is in what Savage (1972) called the “verbalistic” tradition, while the Monty Hall (MH) problem is in the “behavioralistic” tradition. That is, the BFG problem is formulated in terms of eliciting first-person reports of an agent’s probability assessments, while the MH problem is formulated in terms of acquiring evidence about the pattern of the agent’s practically significant decisions. In a situation where it is possible to observe an agent’s choices but not to query the agent about probability assessments, or where it is thought that an agent’s responses to such queries will either over- or under-state the agent’s capacity to act in conformity to expected-utility maximization, the MH problem could be the more advantageous one to consider.

Of course, the BFG problem can be reformulated in a behavioralistic framework. This will be done now. It will be shown that the plan that corresponds naturally to heuristic reasoning is rational with respect to beliefs, as well as with respect to evidence. Thus, in the situations just envisioned, the BFG problem cannot be used to design an experiment, the outcome of which could rule out the possibility that an agent reasons soundly according to a model of beliefs. The plan that corresponds naturally to heuristic reasoning in the MH problem is defined from the same evidential events and alternatives as is the previous plan. That plan is rationalized by the model of evidence presented in example 1, amended so that the prior probabilities are modified so that each is $1/3$, along with the same utility functions by which the heuristic plan for the BFG problem is rationalized. However, it will be shown that the heuristic plan for the MH problem is not rational with respect to evidence.

Consider the behavioralistic formulation of the BFG problem.

**Example 5.** Specify $\Omega$, $O$, and $E$ as in example 1 and let $A = \{e, h, f\}$. Specify that $\zeta(\Omega) = \zeta(E) = \zeta(F) = \{h\}$.

Define $a: \Omega \to A$ by $a(e) = e$, $a(h) = h$, and $a(f) = f$. For $\omega \in \Omega$ and $b \in A$, specify that

$$u_b(\omega) = \begin{cases} 1 & \text{if } b = a(\omega) \\ 0 & \text{if } b \neq a(\omega) \end{cases}$$

In example 1, $P(\{h\}) = 2/5$ and $P(\{e\}) = P(\{f\}) = 3/10$. Consequently $P(\{h\} | E) = P(\{h\} | F) = 4/7$. Thus the model of evidence in example 1 together with the utility functions defined in (53), rationalize $\zeta$ with respect to evidence. That is, on the intuitive understanding of the alternatives that was suggested above, $\zeta$ is the plan that corresponds naturally to heuristic reasoning in the BFG problem.

Plan $\zeta$ is also rational with respect to beliefs. One way of showing that is to appeal to the model of beliefs constructed in example 2 and to specify that, for all $\phi \in \Phi$, $v_e(\phi) = v_f(\phi) = 0$ and $v_h(\phi) = 1$. Rationality with respect to beliefs can also be shown by defining $v_b(\omega, B) = u_b(\omega)$ and by modifying $Q$ from (15) to the following specification, which assigns very high probability to the event that $h$ is the state of nature.

| $Q(\omega, B)$ | $\Omega$ | $E$ | $F$ |
|----------------|---------|-----|-----|
| $e$            | .1      | .1  | 0   |
| $h$            | .8      | .4  | .4  |
| $f$            | .1      | 0   | .1  |
This second way has the feature that the alternatives continue to be given their intuitive interpretations according to the assignment of utilities, rather than being treated as a dominant alternative.

In contrast, the way that subjects are understood to reason in Monte Hall experiments suggests a plan that is rational with respect to evidence, but is not rational with respect to beliefs. In its original form, the MH problem involves an agent making a provisional choice, and subsequently having an opportunity to revise that choice. The specification of of $E'$ in example 1, which is incorporated in following example, corresponds to the revised-choice stage of the MH problem that would follow the agent having made $h$ as the provisional choice.

**Example 6.** The example is identical to example 5 except that

$\zeta(\Omega) = \{A, h\}$  $\zeta(E) = \{e, h\}$  $\zeta(F) = \{h, f\}$

**Claim 3.** The plan specified by (55) in example 6 is rational with respect to evidence, but not with respect to beliefs.

**Proof.** Specify a model of evidence according to example 6 along with the specification that $P(e) = P(h) = P(f) = 1/3$. This model and the utility functions defined by (53) rationalize $\zeta$.

Now it will be shown by contradiction that there do not exist a model of beliefs, $(\Phi, B, Q, \Omega, E, \tau)$ and utility functions $(v_b)_{b \in A}$ that rationalize $\zeta$. By (48.1), since $h \in \zeta(\Omega) \cap \zeta(E)$ and $f \in \zeta(\Omega) \setminus \zeta(E)$,

\[
\int_{\beta(\Omega)} v_h \, dQ = \int_{\beta(\Omega)} v_f \, dQ
\]

and

\[
\int_{\beta(E)} v_h \, dQ > \int_{\beta(E)} v_f \, dQ
\]

Since $\beta(\Omega) = \Omega$, (56) implies that

\[
\int_{\Omega} v_h \, dQ = \int_{\Omega} v_f \, dQ
\]

Because $E' = \{E, F\}$, $\{\beta(E), \beta(F)\}$ is a partition of $\Omega$. Therefore, (57) and (58) imply that

\[
\int_{\beta(F)} v_f \, dQ > \int_{\beta(F)} v_h \, dQ
\]

But, given condition (48.1), inequality (59) contradicts a clause of assumption (55), that $h \in \zeta(F)$. □

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11Granberg and Brown (1995, pp. 711, 712) hypothesize that their subjects reason heuristically as in example 4 in a setting tantamount to that example except that the prior probability of each state of nature is $1/3$. This prior makes the two alternatives consistent with the subject’s type to be equal to one another in expected utility. Subjects cannot express indifference, given the forced-choice protocol of the experiment. Granberg and Brown suggest that “inertia” or some other tie-breaking consideration accounts for subjects’ expressed choices, implying that those choices represent a single-valued selection from an underlying plan that is a multi-valued correspondence. They hypothesize that some subjects may, in fact, be randomizing between the two alternatives.
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