Harmonic analysis of weighted $L^p$-algebras

Yu. N. Kuznetsova       C. Molitor-Braun *

Abstract

Let $G$ be a locally compact, compactly generated group of polynomial growth and let $\omega$ be a weight on $G$. Under proper assumptions on the weight $\omega$, the Banach space $L^p(G, \omega)$ is a Banach $\ast$-algebra. In this paper we give examples of such weighted $L^p$-algebras and we study some of their harmonic analysis properties, such as symmetry, existence of functional calculus, regularity, weak Wiener property, Wiener property, existence of minimal ideals of a given hull.

1 Introduction

Weights and weighted function spaces play an important role in mathematics. In essence, a weight makes it possible to study the behaviour of functions around a certain point, ignoring their oscillations at infinity, or on the contrary, to amplify the asymptotic behaviour of a function. More precisely, introducing a weight means modelling in a quantitative manner the decay of the functions to be studied. This has numerous applications in numerical mathematics and is quite often used for concrete applications (signal theory, Gabor analysis, sampling theory,...), see for instance ([Gr¨ o-Lei], [Da-Fo-Gr¨ o], [Gr¨ o-Lei1], [Fe-Gr¨ o-Lei]).

On the other hand, weights appear naturally in analysis: in inequalities relating the norm of a function to the norm of its derivatives, in extension theorems, etc.; see, e.g., a survey of L. D. Kudryavtsev and S. M. Nikol’sky [Kud-S.Ni]. One of the areas where weighed spaces are applied most intensively is the theory of boundary value problems for partial differential equations (see the surveys [Kud-S.Ni], [Glu-Sav]). By the way, using the Laplace transform also means working in a weighted function space.

In representation theory, which interests us most, weights occur for instance in the following way. If $G$ denotes a locally compact group and $(T,V)$ is a continuous representation of $G$, then the maps $\omega : x \mapsto \|T(x)\|_{op}$ and $\omega : x \mapsto \max(\|T(x)\|_{op}, \|T(x^{-1})\|_{op})$ are weights, the last one being symmetric ($\omega(x^{-1}) = \omega(x), \forall x$). For any one of these weights, the map

$$f \mapsto T(f) := \int_G f(x)T(x)dx$$

is a representation of the weighted function algebra

$$L^1(G, \omega) := \{ f : G \to \mathbb{C} \mid f \text{ measurable and } \int_G |f(x)|\omega(x)dx < +\infty \}.$$  

Subexponential weights like the ones introduced in [1,2,3] below, appear in the context of nilpotent Lie groups. In fact, let $G$ be a connected, nilpotent Lie group. Let $G_1$ be the derived group of $G$, i.e. the closed subgroup generated by the elements of the form $[x,y] = x^{-1}y^{-1}xy$, $x,y \in G$. Let $U$ be a generating neighbourhood of the identity $e$ in $G$ and $V = U \cap G_1$ the corresponding neighbourhood of $e$ in $G_1$. Let

$$|x|_U := \inf\{ n \in \mathbb{N} \mid x \in U^n \} \text{ for } x \neq e, \ |e|_U = 1,$$

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similarly for $|x|_V$. Then it is shown in [Al], that for any weight $\omega$ on $G$ which is submultiplicative, i.e. such that $\omega(xy) \leq \omega(x)\omega(y)$ for all $x, y$,

$$\omega|_{G_1}(x) \leq e^{C|x|_V^\frac{1}{p}}, \forall x \in G_1,$$

for some constant $C$. By the way, on any compactly generated locally compact group, with generating neighbourhood $U$, every submultiplicative weight $\omega$ is exponentially bounded, i.e. satisfies a relation of the form $\omega(x) \leq e^{K|x|_U}$ for $K = \ln \sup_{x \in U} \omega(x)$.

If the weight $\omega$ is submultiplicative, then the weighted function space $L^1(G, \omega)$ is a Banach algebra for convolution, and even a Banach $*$-algebra if the weight is symmetric. The advantage of Banach $*$-algebras over just Banach spaces is clear. They have a much richer structure which may be studied via representation theory and harmonic analysis techniques. In this way, interesting problems arise. Let us just mention the question of their ideal theory, problems of generalized spectral synthesis, of symmetry of the algebra, of invertibility, factorization problems. All these questions make sense for the weighted algebra $L^1(G, \omega)$. Moreover, it is the harmonic analysis properties for algebras, that make the weighted algebras $L^1(G, \omega)$ interesting for some of the concrete applications mentioned in the beginning (see for instance [Gö-Lei1]).

On the other hand, the importance of $L^p$-spaces of the form $L^p(G)$ or $L^p(G, \omega)$, $1 < p < +\infty$, is well known in functional analysis. It would be attractive to extend the theory of convolution algebras to the $L^p$-case, because $L^p$ spaces are reflexive — not a common property among Banach algebras. Unfortunately, if $G$ is not compact and if $p \neq 1$, $L^p(G)$ is not an algebra for convolution. Nevertheless, for appropriate groups $G$ and weights $\omega$, the weighted $L^p$-spaces

$$L^p(G, \omega) := \{ f : G \to \mathbb{C} \mid f \text{ measurable and } \|f\|_{p, \omega} := \left( \int_G |f(x)|^p \omega(x) dx \right)^{\frac{1}{p}} < +\infty \}$$

may be algebras. A sufficient condition on the group $G$ for the existence of weighted $L^p$-algebras, is that the group is $\sigma$-compact. In that case, there are even a lot of such weighted $L^p$-algebras. In [1.2.2] we show that any positive symmetric submultiplicative function multiplied by any $L^p$-algebra weight produces again an $L^p$-algebra weight. This makes it possible to construct $L^p$-algebra weights with all kinds of different growth behaviors.

In the context of weighted $L^p$-algebras, let us mention the works of Wermer [We], Nikol’ski [N.Ni], Feichtinger [Fell], and recently [Kn1]–[Kn4]. Most of these papers concentrate mainly on the question whether the corresponding $L^p$-spaces are algebras. The only well-studied case is $L^p(\mathbb{Z}, \omega)$, see, e.g., a long paper of El-Fallah, Nikol’ski and Zarrabi [Fa-N.Ni-Za]. This is mainly for the reason that in the problems of weighted approximation by polynomials, as initiated by S. N. Bernstein [Be], $L^p(\mathbb{Z}, \omega)$ algebras play a distinguished role [N.Ni]. But this is not the only possible application of weighted $L^p$-algebras. Similarly as for $L^1$-algebras, the weights can be used in numerical mathematics to model the decay of the functions to be used and allow numerical computations. On the other hand, weighted $L^p$-algebras may turn out to be important examples for people working in Banach algebra theory or operator theory. Such algebras have already been used successfully in the interpolation theory and in questions of factorization [Bl-Ka-Ra].

As for possible harmonic analysis properties of an arbitrary Banach $*$-algebra $\mathcal{A}$, several questions have particularly caught the interest of mathematicians:

Is the algebra symmetric, i.e. does every self-adjoint element have a real spectrum?

Is the algebra $\mathcal{A}$ regular, i.e. do the elements of the algebra separate points from closed sets in $\text{Prim}_{\ast}\mathcal{A}$, the space of kernels of topologically irreducible unitary representations?

Does the algebra have the weak Wiener property, i.e. is every proper, closed, two-sided ideal annihilated by an algebraically irreducible representation?

Does the algebra have the Wiener property, i.e. does the previous property hold for topologically irreducible unitary representations?

Do there exist minimal ideals of a given hull?

The list of authors who studied group algebras $L^1(G)$ and their properties is long and is outside the scope of this paper. For weighted group algebras $L^1(G, \omega)$, let us mention among others the
following: In the abelian case, the systematic study of such properties for weighted group algebras $L^1(G, \omega)$ goes back to Beurling ([Beu1], [Beu2]), Domar [Do], and Vretblad [Vr] among others. In the non-abelian case, one may refer to Hulanicki ([Hu1], [Hu2]), Pytlik ([Py1], [Py2]), as well as to more recent studies ([Dz-Lu-Mo], [Fe-Gro-Lei-Lu-Mo] and [Fe-Gro-Lei]). In [Fe-Gro-Lei] for instance, the question of the symmetry for weighted group algebras $L^1(G, \omega)$ is completely solved for compactly generated groups with polynomial growth. Let us mention in this respect, that these abstract problems may be quite important for concrete applications. Hence Gröchenig and Leinert [Grö-Lei] point out that the theory of symmetric group algebras is an important tool to solve problems about Gabor frames, motivated by signal theory.

A systematic study of harmonic analysis properties of weighted $L^p$-algebras $L^p(G, \omega)$ should also be of importance, as well for applications and for more abstract mathematical problems. In [Kn3] some harmonic analysis properties like the regularity are studied in the case of abelian groups. But not much seems to have been done up to now in the non-abelian case. Hence the main purpose of this paper will be to study harmonic analysis properties in the context of non-abelian weighted $L^p$-algebras.

In the present paper, we work on general compactly generated groups with polynomial growth. The weight $\omega$ is supposed to satisfy $\omega^{-q} * \omega^{-q} \leq C \omega^{-q}$ for some constant $C > 0$, where $\frac{1}{q} + \frac{1}{q} = 1$. This ensures $L^p(G, \omega)$ to be an algebra ([We], [Ku1]). We also assume the weight to be submultiplicative. We start by giving examples of $L^p(G, \omega)$ *-algebras, as well for polynomially growing weights as for sub-exponentially growing weights. We then address the questions raised previously: We prove the symmetry of the $L^p$-algebra $L^p(G, \omega)$, if either $G$ is abelian and $\omega$ satisfies the same condition as for the case of $L^1(G, \omega)$ (condition (S), [Fe-Gro-Lei-Lu-Mo], [Fe-Gro-Lei]) or if $\omega$ is polynomial in the sense of Pytlik [Py2] (and $G$ not necessarily abelian). The same hypothesis as in [Dz-Lu-Mo], i. e. the non-abelian Beurling-Domar condition (BDna) allows us to construct a functional calculus on a total subset of the algebra $L^p(G, \omega)$ and to show regularity, as well as the weak Wiener property. If the $L^p$-algebra is moreover symmetric, we also get the Wiener property and the existence of minimal ideals of a given hull. Let us recall that the (BDna) condition is defined as follows in [Dz-Lu-Mo]: Let $G = \bigcup_n U^n$, where $U$ is a relatively compact, generating neighbourhood of $e$ in $G$. We define $s(n) := \sup_{x \in U^n} \omega(x)$. Then the weight $\omega$ satisfies (BDna) if and only if

$$\sum_{n \in \mathbb{N}, n \geq e^n} \frac{(\ln(n))^2 \ln(s(n))}{1 + n^2} < +\infty.$$  

This condition is independent of the choice of the generating neighbourhood $U$ and is only slightly stronger than the conditions used by Domar and Beurling in the abelian case (see [Dz-Lu-Mo] for additional comments). These results relay on the corresponding results for $L^1$-algebras $L^1(G, \omega)$. The question of whether, more generally, condition (S) or the GRS-condition as defined in [Fe-Gro-Lei-Lu-Mo] and [Fe-Gro-Lei] imply the symmetry of the algebra $L^p(G, \omega)$, as they do for $L^1(G, \omega)$, is still an open problem. Finally, let us point out that although these algebras $L^p(G, \omega)$ have certain nice harmonic analysis properties, they are not amenable if $p > 1$ and $G$ non-discrete, as they don’t have bounded approximate identities [Ku2].

Let us also mention, that we assume our weights to be submultiplicative, in order for $L^1(G, \omega)$ to be a *-algebra. Therefore we can rely on the known $L^1(G, \omega)$-results. But there are examples of weights which are not submultiplicative and which produce nevertheless Banach *-algebras $L^p(G, \omega)$ (see for instance [Ku2]). Studying these weights and the properties of the corresponding $L^p$-algebras is still a challenge.

1.1 Assumptions on groups and weights

We suppose in this paper that $G$ is a compactly generated locally compact group. This group $G$ is a group of polynomial growth if there is a relatively compact generating neighbourhood of the identity $U$ such that $|U^n| \leq Cn^Q$ for some constants $C, Q$. It is known that $Q$ does not depend on the choice of $U$. The class of such groups will be denoted by [PG], and for the rest of the paper we assume that $G$ is [PG].
Moreover, it can be shown that the weight $p > 1$ for all $\omega \in G$. By the reasoning of section 1.2.2,

1.2.3 Non-polynomial weight

On every group of polynomial growth, the weight $\omega$ is a relatively compact generating neighbourhood of the identity, we define

$$|x|_U := \inf\{n \mid x \in U^n\}.$$ 

When the choice of $U$ is not important, we write simply $|x|$. In the case $G = \mathbb{R}$, $|x|$ may also denote the absolute value of $x$, and the results of this paper remain correct.

Let

$$\omega : G \rightarrow [1, +\infty]$$

be a measurable function (weight) such that

$$\omega(xy) \leq \omega(x)\omega(y), \quad \forall x, y \in G$$

$$\omega(x) = \omega(x^{-1}), \quad \forall x \in G$$

$$w^{-q} \ast w^{-q} \leq w^{-q},$$

where $\frac{1}{p} + \frac{1}{q} = 1, p > 1$. These conditions are sufficient for $L^p(G, \omega)$ to be a $\ast$-Banach algebra ([We], [Ku]). We will say that $(G, \omega)$ satisfies (LPAlg) ($L^p$-algebra) if these conditions are satisfied. It is often easier to check that $\omega$ satisfies conditions (LPAlg) with some constants $C_1, C_2$: $\omega(xy) \leq C_1\omega(x)\omega(y)$ and $w^{-q} \ast w^{-q} \leq C_2\omega^{-q}$. But a renormalizing $\omega_1 = C\omega$ with $C = \max(C_1, C_2^{1/2})$ gives an equivalent weight satisfying (LPAlg). Since every group in [PG] is amenable, it follows from (LPAlg), by [Ku] Theorem 3.2, that $\omega^{-q} \in L^1(G)$. We may assume, without loss of generality, that the weight $\omega$ is continuous ([Fei]). This will be assumed for the rest of the paper, except for some examples which depend on the discontinuous function $|x| = |x|_U$.

1.2 Examples of weights

1.2.1 Polynomial weights

On every group of polynomial growth, the weight $\omega(x) = (1 + |x|)^D$ satisfies (LPAlg) for $D$ sufficiently large ([Fei]), so $L^p(G, \omega)$ is an algebra.

1.2.2 Products of weights

Let $u$ be a positive submultiplicative function on $G$, $u(x) = u(x^{-1})$, and let $w_1$ be a weight satisfying (LPAlg). Then $w(x) = u(x)w_1(x)$ also satisfies (LPAlg). In particular, any such submultiplicative function $u$, multiplied by $(1 + |x|)^D$ for $D$ sufficiently large, is a (LPAlg)-weight. To prove this, we need to check only the last condition:

$$(w^{-q} \ast w^{-q})(x) = \int_G w^{-q}(y)w^{-q}(y^{-1}x)dy = \int_G u^{-q}(y)w^{-q}(y^{-1}x)w^{-q}_1(y)w^{-q}_1(y^{-1}x)dy.$$ 

From $u(x) \leq u(y)w(y^{-1}x)$ we have $u(x)^{-q} \geq u(y)^{-q}u(y^{-1}x)^{-q}$, so the integral above is bounded by

$$(w^{-q} \ast w^{-q})(x) \leq u^{-q}(x) \int_G w^{-q}_1(y)w^{-q}_1(y^{-1}x)dy \leq u^{-q}(x)w^{-q}_1(x) = w^{-q}(x).$$

1.2.3 Non-polynomial weight

By the reasoning of section 1.2.2, $\omega(x) = e^{|x|^\gamma}(1 + |x|)^D$ with $0 < \gamma \leq 1$ is an $L^p$-algebra weight on any group in [PG] for all $D$ sufficiently large. Moreover, it can be shown that the weight $\omega(x) = e^{|x|^\gamma}$ itself with $0 < \gamma < 1$ satisfies (LPAlg) for all $p > 1$. For $G = \mathbb{R}$ this example is contained in the very first paper of Wermer [We] on $L^p$-algebras.
Let $G = \bigcup U^n$, where $U = U^{-1}$ and $|U^n| \leq Cn^Q$. Denote $U_n = U^n \setminus U^{n-1}$ and assume that $G$ is non-compact. We define
\[ \omega_n = \omega|_{U_n} = e^{n\gamma}, \]
$0 < \gamma < 1$, and show that $L^p(G, \omega)$ is an algebra for every $p > 1$ (the case $p = 1$ is known). For this, we check the sufficient condition $\omega^{-q} * \omega^{-q} \leq C' \omega^{-q}$.

Denote $u = \omega^{-q}$, then
\[ u_n \equiv u|_{U_n} = e^{-qn\gamma}. \]

Take $x \in U_m$ and estimate $\frac{u * u}{u}(x)$.
\[ \frac{u * u}{u}(x) = \frac{1}{u_m} \int_G u(y)u(y^{-1}x)dy = \sum_n \frac{u_n}{u_m} \int_{U_n} u(y^{-1}x)dy. \]

If $y \in U_n = U_n^{-1}$, and $y^{-1}x \in U_k$, then $\max(n-m, m-n) \leq k \leq n+m$. Denote $U_{nk} = U_k \cap (U_n x)$. Then $U_n x = \cup_k U_{nk}$, $U_k = \cup_n U_{nk}$. In particular, $|U_{nk}^x| \leq \min(|U_n|, |U_k|)$. We can rewrite the sum as
\[ \frac{u * u}{u}(x) = \sum_{n,k} \frac{u_n}{u_m} u_k |U_{nk}^x|. \]

Note that $u_n$ decreases and $C_0 = \int_G u < \infty$. Split now the sum into four parts:

1) $n \geq m$. Then $u_n \leq u_m$, and
\[ (\text{sum1}) \leq \sum_{n,k} \frac{u_m}{u_m} u_k |U_{nk}^x| \leq \sum_k u_k |U_k| = \int_G u = C_0. \]

2) Similarly, if $n < m$ but $k \geq m$ then $u_k \leq u_m$ and
\[ (\text{sum2}) \leq \sum_{n,k} \frac{u_m}{u_m} u_n |U_{nk}^x| \leq \sum_n u_n |U_n| = \int_G u = C_0. \]

3) $m/2 < n < m$, $m/2 < k < m$. Then $\max(u_n, u_k) \leq u_{[m/2]+1} \leq \exp(-qm^\gamma/2^\gamma)$;
\[ (\text{sum3}) \leq \sum_{n=[m/2]+1}^m \sum_{k=[m/2]+1}^m e^{q(m^\gamma-2m^\gamma/2^\gamma)}|U_{nk}^x| \]
\[ \leq \sum_{n=1}^m e^{qm^\gamma(1-2^{1-\gamma})}|U_n| \leq e^{qm^\gamma(1-2^{1-\gamma})}Cm^Q \cdot m. \]

Since $2^{1-\gamma} > 1$, the coefficient in the exponent is negative, so this expression tends to zero as $m \to \infty$. Thus, this is bounded by a constant $C_1$.

4) The only complicated case is $n \leq m/2$, $m-n \leq k < m$, and the symmetric case $k \leq m/2$, $m-k \leq n < m$ which reduces to the first one by exchanging $k$ and $n$. Here $u_k \leq u_{m-n}$, so
\[ (\text{sum4}) \leq \sum_{n=0}^{[m/2]} \frac{u_n u_{m-n}}{u_m} \sum_{k=m-n}^m |U_{nk}^x| \leq \sum_{n=0}^{[m/2]} \frac{u_n u_{m-n}}{u_m} |U_n| \]
\[ \leq C \sum_{n=0}^{[m/2]} e^{q(m^\gamma-n^\gamma-(m-n)^\gamma)}n^Q. \]

Denote
\[ f(x) = \int_0^{x/2} t^Q e^{q(x^\gamma-t^\gamma-(x-t)^\gamma)}dt; \]

clearly our sum is bounded for all $m$ if and only if $f$ is bounded on $\mathbb{R}_+$. 

5
By changing the variable to $s = t/x$ we have:

$$f(x) = \int_0^{1/2} x^q s^\gamma (1-s^\gamma -(1-s)^\gamma ) ds = x^{Q+1} \int_0^{1/2} t^q e^{\gamma (1-t^\gamma -(1-t)^\gamma )} dt.$$ 

There is a classical theorem [Erd, §2.4] for integrals of the type

$$F(x) = \int_\alpha^\beta g(t) e^{xh(t)} dt.$$ 

Suppose that:

- $h$ is real-valued and continuous at $t = \alpha$;
- $h'$ exists and is continuous for $\alpha < t \leq \beta$;
- $h' < 0$ for $\alpha < t < \alpha + \eta$ with some $\eta > 0$;
- $h(t) \leq h(\alpha) - \varepsilon$ for some $\varepsilon > 0$ and all $t \in [\alpha + \eta, \beta]$
- $h'(t) \sim -a(t-\alpha)^{\nu-1}$ as $t \to \alpha$, where $\nu > 0$;
- $g(t) \sim b(t-\alpha)^{\lambda-1}$ as $t \to \alpha$, where $\lambda > 0$.

Then

$$f(x) \sim \frac{b}{\nu} \left( \frac{\lambda}{\nu} \right) \left( \frac{\nu}{\alpha x} \right)^{\lambda/\nu} e^{xh(\alpha)}$$

as $x \to \infty$.

We can apply this theorem to

$$F(x) = \int_0^{1/2} t^q e^{\gamma (1-t^\gamma -(1-t)^\gamma )} dt.$$ 

In this case $g(t) = t^Q$, $h(t) = 1 - t^\gamma -(1-t)^\gamma$, $\alpha = 0$, $\beta = 1/2$. The derivative $h'(t) = -\gamma (t^{\gamma-1} -(1-t)^{\gamma-1})$ is negative on $(0, 1/2)$ since $\gamma - 1 < 0$ (so that $t^{\gamma-1}$ decreases) and $t < 1-\nu$.

It follows that $h$ decreases, so all the conditions hold. We have $b = 1$, $\lambda = Q+1$, $a = \nu = \gamma$. Thus,

$$F(x) \sim \frac{1}{\gamma} \Gamma\left( \frac{Q+1}{\gamma} \right) \left( \frac{\gamma}{\gamma x} \right)^{(Q+1)/\gamma} e^{x^\gamma} \equiv C_2 x^{-(Q+1)/\gamma}.$$ 

If we return to $f$, then we get

$$f(x) = x^{Q+1} F(qx^\gamma) \sim C_2 x^{Q+1} \left( qx^{\gamma^\gamma} \right)^{-(Q+1)/\gamma} = C_2 q^{-(Q+1)/\gamma} \equiv C_3.$$ 

It follows that there is a constant $C_4$ such that $f(x) \leq C_4$ for all $x > 0$.

Now, collecting all together, we have

$$\frac{u * u}{u}(x) \leq 2C_0 + C_1 + C_4 \equiv C',$$

which completes the proof.

### 1.2.4 Fast-growing weights

If the weight is submultiplicative, as we always assume, then it can grow at most exponentially. But $L^p(G, \omega)$ is in general not an algebra with the weight $\omega(x) = e^{\nu|x|}$. We will show this for $G = \mathbb{R}$. Take nonnegative $f, g \in L^p(\mathbb{R}, e^{\nu|x|})$, then $F = e^{\nu|x|} f, G = e^{\nu|x|} g$ are in $L^p(\mathbb{R})$:

$$(f * g)(s) = \int_{-\infty}^{\infty} f(t)g(s-t)dt \geq \int_0^{\infty} F(t)e^{\nu|s-t|} e^{-\nu+s} dt = e^{-s} \int_0^{\infty} F(t)G(s-t)dt.$$
Let \( F_+ = F \cdot I_{[0, +\infty)} \), \( G_+ = G \cdot I_{[0, +\infty)} \), where \( I_{[0, +\infty)} \) is the characteristic function of the interval \([0, +\infty)\). If we assume that \( f * g \in L^p(\mathbb{R}, \omega) \), then from the formula above \( F_+ * G_+ \in L^p(\mathbb{R}) \). Since for every \( F_+, G_+ \in L^p([0, +\infty)) \) we have \( |e^{-t}F_+|, |e^{-t}G_+| \in L^p(\mathbb{R}, \omega) \), it follows that \( L^p([0, +\infty)) \) is a convolution algebra if \( L^p(\mathbb{R}, \omega) \) is so. But it is well-known that this is not true, if \( p \neq 1 \).

Nevertheless, by 1.2.1 and 1.2.2, \( L^p(\mathbb{R}, \omega) \) is an algebra for \( \omega(x) = (1 + |x|)^p e^{2|x|} \). There are, moreover, super-exponential weights with which \( L^p(G, \omega) \) is an algebra. One example is \( \omega(x) = e^{x^2} \) on the real line, found by El Kinani [K]. But this weight is not submultiplicative.

## 2 First properties of weighted algebras

### 2.1 Known inequalities

The conditions \( \text{(LPAlg)} \) guarantee that \( L^p(G, \omega) \) is an algebra, that \( L^p(G, \omega) \) is translation-invariant and \( \|f\|_{p, \omega} \leq \omega(x)\|f\|_{p, \omega}, \) where \( x \in \mathbb{R} \), as

\[
\|f\|_{p, \omega}^p = \int |f(x)|^p \omega(x)^p dx = \int |f(y)|^p \omega(xy)^p dy \leq \omega(x)^p \|f\|_{p, \omega}^p.
\]

Also under \( \text{(LPAlg)} \) the following is known:

- \( L^1(G, \omega) \subset L^1(G) \), and \( \|f\|_1 \leq \|f\|_{1, \omega} \) for all \( f \in L^1(G, \omega) \)
- \( L^p(G, \omega) \subset L^1(G) \), and \( \|f\|_1 \leq C\|f\|_{p, \omega} \) for all \( f \in L^p(G, \omega) \), with \( C = \left( \int \omega^{-q}(x) dx \right)^{1/q} \)
- \( L^1(G, \omega) * L^p(G, \omega) \subset L^p(G, \omega) \), and

\[
\|f * g\|_{p, \omega} \leq \|f\|_{1, \omega}\|g\|_{p, \omega}
\]

for all \( f \in L^1(G, \omega), \ g \in L^p(G, \omega) \) \hspace{1cm} (2.1)

If \( \omega(x) = \omega(x^{-1}) \), the usual involution \( f^*(x) = f(x^{-1}) \) is an isometry on \( L^p(G, \omega) \): \( \|f^*\|_{p, \omega} = \|f\|_{p, \omega} \) (recall that every group of polynomial growth is unimodular).

### 2.2 Approximate units

Under the assumption that \( \omega \) is continuous, the proof of \( (\text{Ku2}) \) of the fact that the measurable, bounded functions of compact support are dense in \( L^p(G, \omega) \), shows that the same is also true for the set \( C_c(G) \) of continuous functions with compact support. By \( (\text{Ku2}) \), there exists a net \((f_s)_s\) of measurable, bounded functions with compact support which form a bounded approximate identity in \( L^1(G, \omega) \) and an (unbounded) approximate identity in \( L^p(G, \omega) \). In fact, in the same way it can be proved that if \( V_s \) runs through a basis of compact, symmetric neighbourhoods of the identity \( e \) in \( G \), then every family \( f_s \) such that \( 0 \leq f_s \leq 1, \|f_s\|_1 = 1, \text{supp} f_s \subset V_s \), is an approximate identity with properties as above. It is easy to see that these functions \( f_s \) may be chosen to be continuous and self-adjoint, \( f_s = f^*_s \). Moreover, it will be convenient to have \( V_s \subset K \), where \( K \) is a fixed compact set.

### 2.3 Representations of the weighted algebras

For every non-degenerate \( * \)-representation \( T \) of \( L^p(G, \omega) \) in a Hilbert space \( \mathcal{H} \), there is a unitary continuous representation \( V \) of \( G \) such that

\[
T(f) = \int_G V(x)f(x)dx \hspace{1cm} (2.2)
\]

for all \( f \). This is proved exactly like in [H-R Theorem 22.7], though the assumption (ii) of this theorem does not hold. Let us make the following remarks on the proof.

(1) If \( T \) is cyclic with the cyclic vector \( \xi \), one defines \( V \) as follows: on the dense subspace of vectors of the type \( T(f)\xi, f \in L^p(G, \omega) \), put \( V(x)(T(f)\xi) = T(f)\xi \); it may be easily shown that this is
an isometry, so $V(x)$ extends to a unitary operator on $\mathcal{H}$. It is also straightforward that $V$ is a representation.

(2) To get the equality (2.2), we need to prove that every coefficient $x \mapsto \langle V(x)T(f)\xi, \eta \rangle$, where $f \in L^p(G, \omega)$, and $\xi, \eta \in \mathcal{H}$, is measurable (this is [H-R 22.3]). But we have even more: coefficients of $V$ are continuous since by [Ku2] the mapping $x \mapsto ^*f$ from $G$ to $L^p(G, \omega)$ is continuous for every $f$. From this, by [H-R 22.3] we get some representation $\tilde{T}$ of $L^p(G, \omega)$ defined by

$$\tilde{T}(f) = \int_G V(x)f(x)dx$$

(2.3)

(3) To prove that $\tilde{T} = T$, take a vector $\eta \in \mathcal{H}$ and the cyclic vector $\xi \in \mathcal{H}$. For the linear functional $H(f) = \langle T(f)\xi, \eta \rangle$ on $L^p(G, \omega)$ there is a function $h \in L^q(G, \omega)$ such that $H(f) = \int f(x)h(x)dx$. Then for any $f, g \in L^p(G, \omega)$ we have, with all integrals absolutely converging,

$$\langle \tilde{T}(f)T(g)\xi, \eta \rangle = \int_G \langle V(x)f(x)T(g)\xi, \eta \rangle dx = \int_G \langle V(x)T(g)\xi, \eta \rangle f(x)dx$$

$$= \int_G \langle T^*(g)\xi, \eta \rangle f(x)dx = \int_G \left( \int_G g(y)h(y)dy \right) f(x)dx$$

$$= \int_G \left( \int_G g(x^{-1}y)h(y)dy \right) f(x)dx = \int_G h(y)(f * g)(y)dy$$

$$= \langle T(f * g)\xi, \eta \rangle = \langle T(f)T(g)\xi, \eta \rangle.$$

It follows that $\tilde{T} = T$ on the dense subspace of vectors of the type $T(f)\xi$, $f \in L^p(G, \omega)$, and as a consequence on the whole $\mathcal{H}$.

(4) In general, $T$ can be expanded into a direct sum of cyclic representations $T_\alpha$ [H-R, 21.13]. Every $T_\alpha$ is given by the formula (2.3) with some $V_\alpha$; then $T$ is equal to the same integral (2.3) with $V = \oplus V_\alpha$.

Further, by [H-R 22.6], $T$ and $V$ are irreducible or not simultaneously.

In particular, this gives us the identification $L^p(G, \omega) = \hat{G}$.

3 Symmetry

3.1

The notion of symmetry plays an important role in the theory of Banach $*$-algebras. It may be defined as follows:

Let $\mathcal{A}$ be a Banach $*$-algebra and let $a \in \mathcal{A}$. We will denote the spectrum of $a$ in $\mathcal{A}$ by $\sigma_{\mathcal{A}}(a)$ and the spectral radius of $a$ in $\mathcal{A}$ by $r_{\mathcal{A}}(a)$. Then the algebra $\mathcal{A}$ is said to be symmetric if $\sigma_{\mathcal{A}}(a^*a) \subset [0, +\infty)$ for all $a \in \mathcal{A}$, or, equivalently, if $\sigma_{\mathcal{A}}(a) \subset \mathbb{R}$ for all $a = a^* \in \mathcal{A}$.

For abelian Banach $*$-algebras the symmetry is equivalent to the fact that all the characters of the algebra are unitary.

Let $(G, \omega)$ satisfy (LPAlg). Let us recall the following definitions for the weight $\omega$ (see for instance [Fe-Grö-Lei-Lu-Mo] and [Fe-Grö-Leo]).

**Definition 3.1.** a) The weight $\omega$ on $G$ is said to satisfy the GRS-condition (or GNR-condition), if

$$\lim_{n \to +\infty} \omega(x^n)^{1/n} = 1, \quad \forall x \in G.$$  

GRS

b) The weight $\omega$ is said to satisfy condition (S), if, for every generating, relatively compact neighbourhood $U$ of $G$,

$$\lim_{n \to +\infty} \sup_{x \in U^n} \omega(x)^{1/n} = 1.$$  

S
These conditions are linked to the symmetry of weighted group algebras. Among others, the following results are known:

For $G = \mathbb{Z}$, $l^1(\mathbb{Z}, \omega)$ is symmetric if and only if $\lim_{n\to+\infty} \omega(n)^{\frac{1}{n}} = 1$ ([Na]).

If $G \in [PG]$, then $L^1(G)$ is symmetric ([Lo]).

If $G \in [PG]$ and $\omega$ satisfies condition (S), then $L^1(G, \omega)$ is symmetric ([Fe-Grö-Lei-Lu-Mo]).

The final version of results of this type is due to Fendler, Gröchenig and Leinert ([Fe-Grö-Lei]). They prove:

**Theorem 3.2.** Let $G \in [PG]$ and let $\omega$ be a weight on $G$. Then the following are equivalent:

(i) $\omega$ satisfies the GRS-condition.

(ii) $\omega$ satisfies condition (S).

(iii) $L^1(G, \omega)$ is symmetric.

(iv) $\sigma_{L^1(G, \omega)}(f) = \sigma_{L^1(G)}(f)$, $\forall f \in L^1(G, \omega)$.

The last three results are based on a method developped by Ludwig ([Lu]). Previously, using a result of Hulanicki ([Hu1]), Pytlik ([Py2]) had already proved the following:

**Theorem 3.3.** If the weight $\omega$ satisfies

$$\omega(xy) \leq C(\omega(x) + \omega(y)), \forall x, y \in G$$

for some positive constant $C$ and if $\omega^{-1} \in L^p(G)$ for some $0 < p < +\infty$, then $L^1(G, \omega)$ is symmetric.

Pytlik calls a weight satisfying (3.1) a polynomial weight. In particular, weights of the form

$$\omega(x) = (1 + |x|)^D,$$

for some positive $D$, where

$$|x| := \inf\{n \mid x \in U^n\},$$

satisfy $\omega(xy) \leq C(\omega(x) + \omega(y))$ for all $x, y \in G$, and hence give symmetric weighted group algebras $L^1(G, \omega)$, as Pytlik already noticed in ([Py1]).

By ([Py2]) every weight satisfying $\omega(xy) \leq C(\omega(x) + \omega(y))$ is dominated by a weight of the form $K(1 + |x|)^D$, $K \geq 1, D > 0$. It is then easy to check that all these weights satisfy condition (S).

So the result of Pytlik is a particular case of the result of Fendler, Gröchenig and Leinert.

**3.2**

Our aim is to study symmetry for weighted $L^p$-algebras. For the rest of this section we hence assume that $(G, \omega)$ satisfies (LPAlg), in order to be sure that $L^p(G, \omega)$ is an algebra. The question is whether condition (S) will also imply the symmetry of $L^p(G, \omega)$. We need some preliminary result.

**Lemma 3.4.** The weight $\omega$ satisfies condition (S) if and only if $\omega(x) = O(e^{\varepsilon |x|})$ for all $\varepsilon > 0$, where $|x| = \inf\{n \mid x \in U^n\}$.

**Proof.** Let us assume that $\omega(x) \leq C(\varepsilon)e^{\varepsilon |x|}$ for some constant $C(\varepsilon)$, $\varepsilon > 0$. Then

$$x \in U^k \Rightarrow |x| \leq k \Rightarrow \omega(x) \leq C(\varepsilon)e^{\varepsilon k}$$

and

$$\sup_{x \in U^k} \omega(x)^{\frac{1}{k}} \leq C(\varepsilon)^{\frac{1}{k}}e^{\varepsilon}.$$
Thus \( \omega \) does not satisfy condition (S).

We may now prove the symmetry result for abelian groups:

**Theorem 3.5.** Let \( G \) be an abelian group such that \( (G, \omega) \) satisfies (LPAlg). Let us assume that \( \omega \) satisfies condition (S). Then \( L^p(G, \omega) \) is a symmetric Banach \(*\)-algebra.

**Proof.** According to ([Kn1]), every character \( \chi \) of \( L^p(G, \omega) \) is of the form

\[
\chi(f) = \int_G f(x)\sigma(x)dx = \int_G \left( f(x)\omega(x) \right) \left( \sigma(x)\omega^{-1}(x) \right) dx, \quad \forall f \in L^p(G, \omega),
\]

where \( \sigma \) is a (possibly unbounded) character of the group \( G \). As \( f\omega \in L^p(G) \) is arbitrary, \(|\sigma(x)|\omega(x)^{-1} \in L^q(G)\), with \( \frac{1}{p} + \frac{1}{q} = 1 \), and, as \( \omega(x) \leq C(\varepsilon)e^{\varepsilon|\sigma|} \) for all \( \varepsilon > 0 \),

\[
|\sigma(x)|e^{-\varepsilon|\sigma|} \leq |\sigma(x)|C(\varepsilon)\omega(x)^{-1} \in L^q(G).
\]

Let us assume that \( \sigma \) is not unitary. Then there exists \( x_0 \in G \) and \( \delta > 0 \) such that \(|\sigma(x_0)| > 1 + \delta \).

By continuity, there is a non-empty open subset \( V \) of \( G \) such that \(|\sigma(x)| > 1 + \frac{\delta}{2} \), for all \( x \in V \). Hence, for \( x \in V^n, x = x_1 \cdot x_2 \cdots x_n \) with \( x_j \in V \) for all \( j \), and \(|\sigma(x)| = \prod_{j=1}^n |\sigma(x_j)| \geq (1 + \frac{\delta}{2})^n \).

On the other hand, if \( V \subset U^k \) where \( U \) is a (relatively compact) generating neighbourhood of the identity then \( V^n \subset U^{kn} \), and so \(|\sigma(x)| \leq kn \) for \( x \in V^n \). This implies that, for all \( n \) (and for all \( \varepsilon > 0 \)),

\[
+\infty > \int_G \left( |\sigma(x)|e^{-\varepsilon|\sigma|} \right)^q dx \quad (3.2)
\]

\[
\geq \int_{V^n} |\sigma(x)|^q e^{-\varepsilon q|\sigma|} dx \quad (3.3)
\]

\[
\geq (1 + \frac{\delta}{2})^q \int_{V^n} e^{-\varepsilon qkn} dx \quad (3.4)
\]

\[
\geq \left( (1 + \frac{\delta}{2}) \cdot e^{-\varepsilon k} \right)^q |V| \quad (3.5)
\]

But we can choose \( \varepsilon \) so that \((1 + \frac{\delta}{2}) \cdot e^{-\varepsilon k} > 1\), then \((3.5)\) tends to \(+\infty\) with \( n \). This is a contradiction which shows that \( \sigma \), and hence \( \chi \) are unitary. So \( L^p(G, \omega) \) is symmetric.

**Example:** \( L^2(\mathbb{R}, e^{\sqrt{|\cdot|}}) \) is a symmetric Banach \(*\)-algebra (section [2.4.3]).
3.3

Before studying the non-abelian case, let us first recall the generalized Minkowski relation: Let\(X,Y\) be measure spaces and let\(F\) be a measurable function on\(X \times Y\). Then, for all\(p \geq 1\),
\[
\left(\int_X \left(\int_Y |F(x,y)|^p dy \right)^{\frac{1}{p}} dx\right) \leq \int_Y \left(\int_X |F(x,y)|^p dx \right)^{\frac{1}{p}} dy.
\]

We need the following relation:

**Lemma 3.6.** Let us assume that the weight \(\omega\) satisfies (LPAlg) and is polynomial in the sense of Pytlik, i.e. that it satisfies
\[
\omega(xy) \leq C(\omega(x) + \omega(y)), \quad \forall x, y \in G,
\]
for some constant \(C > 0\). Then
\[
\|f * g\|_{p,\omega} \leq C\left(\|f\|_{p,\omega} \|g\|_1 + \|g\|_{p,\omega} \|f\|_1\right), \quad \forall f, g \in L^p(G,\omega) \subset L^1(G).
\]

**Proof.**
\[
\|f * g\|_{p,\omega} = \left(\int_G \left(\int_G f(y)g(y^{-1}x)dy|\omega(x)|^pdx\right)^{\frac{1}{p}} dy \right)^{\frac{1}{p}}
\]
\[
\leq C\left(\int_G \left(\int_G f(y)g(y^{-1}x)(\omega(y) + \omega(y^{-1}x))dy|\omega(x)|^pdx\right)^{\frac{1}{p}}
\]
\[
\leq C\left(\int_G \int_G f(y)g(y^{-1}x)\omega(y)dy|\omega(x)|^pdx\right)^{\frac{1}{p}} + C\left(\int_G \int_G f(y)g(y^{-1}x)\omega(y^{-1}x)dy|\omega(x)|^pdx\right)^{\frac{1}{p}}
\]
\[
= I + II
\]
by the triangle inequality for \(\|\cdot\|_1\). Then the generalized Minkowski inequality implies that
\[
I \leq C\left(\int_G \left(\int_G f(y)g(y^{-1}x)\omega(y)dy|\omega(x)|^pdx\right)^{\frac{1}{p}}
\]
\[
\leq C\left(\int_G \left(\int_G f(xu^{-1})g(u)|\omega(xu^{-1})|^pdu\right)^{\frac{1}{p}} dy \equiv (y = xu^{-1})
\]
\[
\leq C\int_G \left(\int_G |f(xu^{-1})|^p|g(u)|^p|\omega(xu^{-1})|^pdu\right)^{\frac{1}{p}} du
\]
\[
= C\|f\|_{p,\omega} \|g\|_1
\]
and
\[
II \leq C\left(\int_G \left(\int_G f(y)g(y^{-1}x)\omega(y^{-1}x)dy|\omega(x^{-1})|^pdx\right)^{\frac{1}{p}}
\]
\[
\leq C\int_G \left(\int_G |f(y)|^p|g(y^{-1}x)|^p\omega(y^{-1}x)|^pdx\right)^{\frac{1}{p}} dy
\]
\[
= C\|g\|_{p,\omega} \|f\|_1.
\]

We may now use the methods of Pytlik ([Py1], [Py2]) to show the symmetry of the algebra \(L^p(G,\omega)\) for polynomial weights. This is done via the following lemmas.

**Lemma 3.7.** Let \((G,\omega)\) satisfy (LPAlg). Then, for any \(f \in L^p(G,\omega) \subset L^1(G)\), \(r_1(f) \leq r_{p,\omega}(f)\), where \(r_1(f)\) denotes the spectral radius of \(f\) in \(L^1(G)\) and \(r_{p,\omega}(f)\) denotes the spectral radius of \(f\) in \(L^p(G,\omega)\).
Proof. From
\[ \| f \|_1 = \int_G |f(x)| \omega(x) \, \mathrm{d}x \leq \left( \int_G \omega^{-q}(x) \, \mathrm{d}x \right)^{\frac{1}{q}} \| f \|_{p,\omega} = C \| f \|_{p,\omega} \]
we deduce
\[ \| f^n \|_1 \leq C \| f^n \|_{p,\omega}^\frac{1}{n} , \]
where \( f^n = f \ast f \ast \cdots \ast f \) (\( n \) factors). Hence, for \( n \to +\infty \),
\[ r_1(f) \leq r_{p,\omega}(f). \]

Lemma 3.8. Let \((G,\omega)\) satisfy (LPAlg). Let us assume that \( \omega \) is polynomial in the sense of Pytlik, i.e. that
\[ \omega(xy) \leq C(\omega(x) + \omega(y)) , \quad \forall x, y \in G. \]
Then
\[ r_1(f) = r_{p,\omega}(f), \quad \forall f \in L^p(G,\omega). \]

Proof. By the methods of Pytlik ([Py2], 3.8) gives
\[ \| f \ast f \|_{p,\omega} \leq C \| f \|_{p,\omega} \| f \|_1 \]
and, by induction,
\[ \| f^{2^n} \|_{p,\omega} \leq (2C)^n \| f \|_{p,\omega} \| f \|_{1}^{2^n-1} . \]
So,
\[ r_{p,\omega}(f) = \lim_{n \to +\infty} \| f^{2^n} \|_{p,\omega}^{\frac{1}{2^n}} \leq \lim_{n \to +\infty} (2C)^n \| f \|_{p,\omega} \| f \|_{1}^{\frac{1}{2^n}} = \| f \|_1 . \]
Finally,
\[ r_{p,\omega}(f) = r_{p,\omega}(f^n)^\frac{1}{n} \leq \| f^n \|_{1}^{\frac{1}{n}} , \quad \forall n , \]
and
\[ r_{p,\omega}(f) \leq \lim_{n \to +\infty} \| f^n \|_{1}^{\frac{1}{n}} = r_1(f). \]

We finally get:

Theorem 3.9. Let \((G,\omega)\) satisfy (LPAlg). Let us assume that \( \omega \) is polynomial in the sense of Pytlik, i.e. that
\[ \omega(xy) \leq C(\omega(x) + \omega(y)) , \quad \forall x, y \in G. \]
Then \( L^p(G,\omega) \) is a symmetric Banach \(*\)-algebra.

Proof. This follows from Lemma 3.1 of ([Fe-Grö-Lei]) applied to \( A := L^p(G,\omega) \) and \( B := L^1(G) \), and from the result of Losert ([Lo]) about the symmetry of \( L^1(G) \). As a matter of fact, these results imply that
\[ \sigma_{L^1(G)}(f) = \sigma_{L^p(G,\omega)}(f) , \quad \forall f = f^* \in L^p(G,\omega) , \]
and, as \( L^1(G) \) is symmetric, \( \sigma_{L^1(G)}(f) \subset \mathbb{R} \).

Example: We know that for \( D > 0 \) sufficiently large, \( \omega(x) = (1 + |x|)^D \), where \( |x| = \inf\{n \mid x \in U^n\} \), gives rise to an \( L^p \)-algebra ([1.2.1]). Hence this algebra is symmetric.

One may conjecture that, more generally, \( L^p(G,\omega) \) is a symmetric Banach \(*\)-algebra, if \((G,\omega)\) satisfies (LPAlg) and \( \omega \) satisfies condition (S), or, equivalently, the GRS-condition. But this remains an open question.
4 Functional calculus

4.1

Let \((G, \omega)\) satisfy \((\text{LPAlg})\). The aim of the following section is the construction of functional calculus for all continuous functions \(f\) with compact support such that \(f = f^*,\) where \(f^*(x) = \bar{f}(x^{-1}),\) for all \(x \in G.\) We will follow the method of \((\text{Hi2}, \text{Dz-Lu-Mo})\) and use their results. To use this method, we have to bound

\[
u(nf) := \sum_{k=1}^{\infty} \frac{i^k}{k!} n^k f^{*k}
\]

in \(L^p(G, \omega)\) and show that there are "enough" functions \(\varphi : \mathbb{R} \to \mathbb{R},\) periodic with period \(2\pi,\) with \(\varphi(0) = 0,\) such that

\[
\varphi\{f\} := \sum_{n \in \mathbb{Z}} u(nf)\hat{\varphi}(n)
\]

converges in \(L^p(G, \omega)\). For more details on functional calculus, see among others \((\text{Di}, \text{Dz-Lu-Mo})\).

Let us first recall that, for all continuous functions \(g\) with compact support, \(\|g\|_1 \leq \|g\|_{1, \omega}\) and \(\|g\|_1 \leq C\|g\|_{p, \omega}\) for some positive constant \(C.\) Moreover,

\[
\|u(nf)\| \leq \sum_{k=1}^{+\infty} \frac{1}{k!} n^k \|f\|^k \leq e^n\|f\|
\]

in any convenient norm \(\| \cdot \|\). So for any continuous function \(f\) with compact support such that \(f = f^*,\) the series defining \(u(nf)\) converges in \(L^1(G), L^1(G, \omega)\) and \(L^p(G, \omega)\) to the same element, i. e. the notation \(u(nf)\) represents a function belonging to \(L^1(G, \omega) \cap L^p(G, \omega) \subset L^1(G).\) We will now deduce a bound for \(\|u(nf)\|_{p, \omega}\) from the bound for \(\|u(nf)\|_{1, \omega}\) which was established in \((\text{Dz-Lu-Mo})\). Let us recall the following notations and facts from \((\text{Dz-Lu-Mo})\): There exists a constant \(C > 1\) such that

\[
\omega(x) \leq e^{C|x|}, \quad \forall x \in G,
\]

where \(|x| = \inf\{n \mid x \in U^n\}\) for an arbitrary (relatively compact) generating neighbourhood \(U.\) We denote

\[
s(n) := \sup_{x \in U^n} \omega(x), \quad \forall n \in \mathbb{N}^*
\]

\[
s(0) := 1
\]

\[
\omega_1(x) := s(|x|)
\]

\[
\omega_2(x) := e^{C|x|}
\]

\[
s_2(n) := \sup_{x \in U^n} \omega_2(x) = e^{Cn}.
\]

We also consider an arbitrary increasing function \(r : \mathbb{N} \to \mathbb{N},\) which will be specified later. It is shown in \((\text{Dz-Lu-Mo})\) that there exist positive constants \(C_1, C_2\) such that

\[
\|u(nf)\|_{1, \omega} \leq C_1 (1 + |n|)(1 + |n| r(|n|)) Q s(|n| r(|n|)) e^{C_2 \left(\frac{|n|}{n^{1/2}}\right)}, \quad \forall n \in \mathbb{Z}, \quad (4.1)
\]

where \(Q\) denotes the power appearing in the polynomial growth condition of the group \(G,\) i. e. \(|U^n| \leq Kn^Q,\) for all \(n \in \mathbb{N}^*,\) for some positive constant \(K.\)

From the formal representation \(u(f) = e^{if} - 1\) we get the following identity valid also in the non-unital case:

\[
u(nf) = e^{in} - 1 = e^{(n-1)f}e^{if} - 1 = (u((n-1)f) + 1) \cdot (u(f) + 1) - 1 = \]

\[
= u((n-1)f)*u(f) + u((n-1)f) + u(f).
\]

By induction it follows that
\[ u(nf) = nu(f) + \sum_{k=1}^{n-1} u(kf) * u(f). \]

From (2.11), we get an estimate
\[ \|u(nf)\|_{p,\omega} \leq n\|u(f)\|_{p,\omega} + \sum_{k=1}^{n-1} \|u(kf)\|_{1,\omega}\|u(f)\|_{p,\omega}. \]

Using the bound for \( \|u(kf)\|_{1,\omega} \) obtained in (Dz-Lu-Mo) and recalled in (4.1), we get
\[ \|u(nf)\|_{p,\omega} \leq Kn + K_1 \sum_{k=1}^{n-1} (1 + k)(1 + kr(k)) \frac{k}{\omega s(kr(k)) e^{C_2(\frac{k}{\omega s(kr(k))})}} \]
\[ \leq Kn + K_1 (n + 1)(1 + nr(n)) \frac{n}{\omega s(nr(n))} \sum_{k=1}^{n-1} e^{C_2(\frac{k}{\omega s(nr(n))})}, \]
for some constants \( K, K_1 \), as the functions \( s \) and \( r \) are increasing. (Here \( 1 + k \) could be bounded by \( n \) as well, but we choose \( n + 1 \) to comply with assumptions of (Dz-Lu-Mo). As in (Dz-Lu-Mo), we put \( r(n) := \ln(n) + 1 \), for \( n \geq e^e \). Hence, for \( k \geq \max(e^e, e^C) \),
\[ s_2(r(k)) = e^{C_2(r(k))} \geq e^{C_2 \ln(n)} = (\ln n)^C \]
and
\[ e^{C_2(\frac{k}{\omega s(nr(n))})} \leq e^{C_2(\frac{k}{\ln(n)})} \leq \frac{n}{\omega s(nr(n))}, \]
as the function \( f(x) = \frac{x}{\ln(x)} \) is increasing for \( x \geq e^C \). This allows to estimate the sum over \( k \) by \( ne^{C_2(\ln n)} \). Moreover, as in (Dz-Lu-Mo), for \( n \geq e^e \),
\[ \ln(n) \leq r(n) \leq 2\ln(n) \leq 2n \]
\[ s(nr(n)) \leq s(n)^{r(n)} \leq s(n)^{2\ln(n)}, \]
Finally, noticing that \( nf = (-n)(-f) \), we may compute \( \|u(nf)\|_{p,\omega} \) even for negative \( n \) (by replacing the constants depending on \( f \) by the sup of the corresponding constants for \( f \) and \( -f \)).

So, there exist positive constants \( A_1, A_2 \) (depending on \( f \) and \( \omega \)) such that
\[ \|u(nf)\|_{p,\omega} \leq A_1 (1 + |n|)^2 (1 + n^2) \frac{s(|n|) 2\ln(|n|)}{1 + n^2} e^{A_2(\frac{|n|}{\ln(|n|)^C})} \]
for all \( |n| \geq \max(e^e, e^C) \). We thus obtain a similar bound as for \( \|u(nf)\|_{1,\omega} \), except that the factor \( (1 + |n|) \) has been replaced by \( (1 + |n|)^2 \). Of course the constants are slightly different too. They depend on \( f \) and \( \omega \). We may conclude exactly as in (Dz-Lu-Mo).

4.2

Let us recall the non-abelian Beurling-Domar condition (BDna) given by
\[ \sum_{n \in \mathbb{N}, n \geq e^e} \frac{\ln(n) \ln(s(n))}{1 + n^2} < +\infty. \]

It is independent of the choice of the generating neighbourhood \( U \) used to compute \( s(n) \). See (Dz-Lu-Mo) for more details on that condition. We then have the following result:
Theorem 4.1. Let \((G,\omega)\) satisfy (LPAlg). Let us assume that moreover the weight \(\omega\) satisfies the (BDna) condition. Let \(f = f^*\) be a continuous function with compact support. Then, given \(a, b, \epsilon\) such that \(0 < a < a + \epsilon < b - \epsilon < b < 2\pi\), there exists a function \(\psi : \mathbb{R} \to \mathbb{R}\), continuous, periodic of period \(2\pi\) such that \(\text{supp } \psi \cap [0, 2\pi] \subset [a, b]\), \(\psi \equiv 1\) on \([a + \epsilon, b - \epsilon]\) and
\[
\sum_{n \in \mathbb{Z}} \|u(nf)\|_{p,\omega} |\hat{\psi}(n)| < +\infty.
\]
Hence this defines a function
\[
\psi\{f\} := \sum_{n \in \mathbb{Z}} \hat{\psi}(n)u(nf) \in L^p(G, \omega) \cap L^1(G, \omega)
\]
and the properties of functional calculus are satisfied, i.e.
\[
\chi(\psi\{f\}) = \psi(\chi(f))
\]
for every character \(\chi\) of the abelian Banach \(*\)-subalgebra of \(L^p(G, \omega)\) generated by \(f\),
\[
\pi(\psi\{f\}) = \psi(\pi(f)), \quad \forall \pi \in \hat{L}^p(G, \omega) \equiv \hat{G},
\]
\[
(\varphi \psi)\{f\} = \varphi\{f\} * \psi\{f\},
\]
if the functions \(\varphi\) and \(\varphi \psi\) still have the correct properties to allow functional calculus.

Proof. See ([Dz-Lu-Mo]), pages 337 to 345. Here we use again the argument that if a series converges in \(L^1(G, \omega)\) and \(L^p(G, \omega)\), then it also converges in \(L^1(G)\) and the limit is the same in the three spaces. \qed

4.3

Examples: a) If \(G \in [PG]\) and \(\omega(x) = K(1 + |x|)^D\) for \(K \geq 1\) and \(D > 0\) large enough, then \((G, \omega)\) satisfies (LPAlg) (1.2.1). It is easy to check, that \(\omega\) also verifies (BDna) and so functional calculus exists.
b) Let \((G, \omega)\) satify (LPAlg) and let us assume that \(\omega(xy) \leq C(\omega(x) + \omega(y))\) for all \(x, y \in G\). By ([Py2]), such a weight is bounded by a weight of the form \(K(1 + |x|)^D\) and hence (BDna) is verified. Functional calculus exists.
c) If \(G \in [PG]\), then
\[
\omega(x) := e^{C|x|^\gamma}, \quad 0 < \gamma < 1,
\]
is such that \((G, \omega)\) satisfies (LPAlg) (1.2.3) and the weight \(\omega\) verifies (BDna) ([Dz-Lu-Mo]). Functional calculus exists.

Remarks: a) Condition (BDna) is independent of the choice of the generating neighbourhood \(U\).
b) Condition (BDna) is only slightly more restrictive than the well known Beurling-Domar condition in the abelian case.
c) If \(\omega\) satisfies (BDna), it also verifies condition (S).

See ([Dz-Lu-Mo]) for more details.

Functional calculus is a very useful tool to prove different harmonic analysis properties, as will be shown in the rest of this paper.

5 Regularity

5.1

For abelian Banach algebras, regularity is defined as follows by Šilov [S]. Let \(\mathcal{A}\) be an abelian Banach algebra and let \(\Delta(\mathcal{A})\) denote the space of characters of \(\mathcal{A}\). Then \(\mathcal{A}\) is said to be regular
if, given any \( \varphi \in \Delta(A) \) and any closed set \( F \subset \Delta(A) \) not containing \( \varphi \), there exists \( x \in A \) such that \( \hat{x}(\varphi) = \varphi(x) = 1 \) and \( \hat{x}|_F \equiv 0 \), where \( \hat{x} \) denotes the Gelfand transform of \( x \).

In the non-abelian case \( \Delta(A) \) should be replaced by the space \( \text{Prim}_A \) defined as the set of all kernels of topologically irreducible \( \ast \)-representations of \( A \). The set \( \text{Prim}_A \) is equipped with the hull-kernel topology.

5.2

As previously, we assume that \( (G, \omega) \) satisfies (LPAlg). Let \( C_c(G) \) denote the set of continuous functions with compact support on \( G \). It is obvious that \( C_c(G) \subset L^p(G, \omega) \subset L^1(G) \), that \( C_c(G) \) is dense in \( L^p(G, \omega) \) and in \( L^1(G) \), that for all \( \pi \in \hat{G} \equiv \hat{L}^p(G, \omega) \),

\[
\| \pi(f) \|_{op} \leq \| f \|_1 \leq C \| f \|_{p, \omega}, \quad \forall f \in L^p(G, \omega),
\]

resp. \( \| \pi(f) \|_{op} \leq \| f \|_1 \) for all \( f \in L^1(G) \). Moreover, we have seen in the previous section that functional calculus is possible on the self-adjoint elements of \( C_c(G) \), provided the weight \( \omega \) satisfies (BDna). This implies that the arguments of ([Dz-Lu-Mo], pages 350 and 351) remain valid.

In particular, we have the following results:

**Theorem 5.1.** Let \( (G, \omega) \) satisfy (LPAlg). Let us assume that the weight \( \omega \) verifies (BDna). We then have:

(i) The map

\[
\Psi : \text{Prim}_A \ L^1(G) \rightarrow \text{Prim}_A \ L^p(G, \omega)
\]

\[
\ker \pi \mapsto \ker \pi \cap L^p(G, \omega)
\]

is a homeomorphism.

(ii) In particular, \( \text{Prim}_A \ L^p(G, \omega) \), \( \text{Prim}_A \ L^1(G, \omega) \) and \( \text{Prim}_A \ L^1(G) \) are homeomorphic.

(iii) Given any \( \rho \in \hat{G} \) and any open neighbourhood \( N_1 \) of \( \ker \pi \cap L^p(G, \omega) \) in \( \text{Prim}_A \ L^p(G, \omega) \), there exists \( f \in L^p(G, \omega) \) such that \( \rho(f) \neq 0 \) and \( \pi(f) = 0 \) for all \( \pi \in \hat{G} \setminus N_1 \), resp. for all \( \pi \) such that \( \ker \pi \cap L^p(G, \omega) \in \text{Prim}_A \ L^p(G, \omega) \setminus N_1 \).

**Proof.** See ([Dz-Lu-Mo]). Part of the argument relies heavily on functional calculus and on the \( \ast \)-regularity of groups with polynomial growth. \( \square \)

Point (iii) of the previous theorem, which is often called Domar’s property, corresponds to the regularity of abelian Banach algebras.

6 Weak Wiener property

Let us recall the following definitions:

**Definition 6.1.** Let \( A \) be a Banach algebra.

(i) A representation \( (T, V) \) of \( A \) on a vector space \( V \) is said to be algebraically irreducible, if there are no non-trivial \( T \)-invariant subspaces in \( V \).

(ii) The algebra \( A \) is said to have the weak Wiener property, if every proper closed two-sided ideal of \( A \) is contained in the kernel of an algebraically irreducible representation.

Let \( (G, \omega) \) satisfy (LPAlg) and let \( (f_s)_s \) be an approximate unit of \( L^p(G, \omega) \) with the properties discussed in 5.2.

In ([Dz-Lu-Mo]) it is shown that, provided \( \omega \) satisfies (BDna), there exists a periodic function \( \varphi \) of period \( 2\pi \) with \( \varphi(1) = 1, \varphi \equiv 0 \) in a neighbourhood of \( 0 \), such that \( \varphi\{f_s\} \) is defined in \( L^1(G, \omega) \). By our section on functional calculus in \( L^p(G, \omega) \), the same \( \varphi\{f_s\} \) also converges in \( L^p(G, \omega) \), i.e. \( \varphi\{f_s\} \in L^1(G, \omega) \cap L^p(G, \omega) \) for all \( s \). Moreover, in ([Dz-Lu-Mo]) it is shown that

\[
\| \varphi\{f_s\} \ast f - f \|_{1, \omega} \to 0
\]
for all continuous functions \( f \) with compact support in \( G \). Hence, for any \( f, g \in C_c(G) \), \( f * g \in C_c(G) \subset L^1(G, \omega) \cap L^p(G, \omega) \) and
\[
\| \varphi(f_s) * f * g - f * g \|_{p, \omega} \leq \| \varphi(f_s) * f - f \|_{1, \omega} \| g \|_{p, \omega}.
\]
So
\[
\| \varphi(f_s) * f * g - f * g \|_{p, \omega} \to 0.
\]
This gives the following result:

**Lemma 6.2.** Under the assumptions above, let \( I \) be a proper closed two-sided ideal of \( L^p(G, \omega) \). Then there exists \( s \) such that \( \varphi(f_s) \notin I \).

**Proof.** Let us assume that \( \varphi(f_s) \in I \), for all \( s \). Then \( \varphi(f_s) * f * g \in I \) for all \( s \) and all \( f, g \in C_c(G) \). As \( I \) is closed, the relation (6.1) shows that \( f * g \in I \) for all \( f, g \in C_c(G) \). But this implies that \( I = L^p(G, \omega) \), by density, which is a contradiction. \( \square \)

We are now able to prove the weak Wiener property:

**Theorem 6.3.** Let \((G, \omega)\) be \((L^p)\). Let us also assume that the weight \( \omega \) satisfies \((BDna)\). Then the algebra \( L^p(G, \omega) \) has the weak Wiener property.

**Proof.** The proof is standard, but we repeat it for the sake of completeness. Let \( I \) be a proper, closed, two-sided ideal of \( L^p(G, \omega) \). Let \( s \) and \( \varphi \) be such that \( \varphi(f_s) \notin I \). Let \( \psi \) be another function such that functional calculus \( \psi(f_s) \) is possible and such that \( \psi \equiv 1 \) on the support of \( \varphi \). Such a \( \psi \) exists by theorem 6.1. Then
\[
\psi(f_s) * \varphi(f_s) = (\psi \varphi)(f_s) = \varphi(f_s).
\]
Let us consider the algebra \( \mathcal{A} := L^p(G, \omega)/I \). We have
\[
0 \neq \varphi(f_s) \in \mathcal{A} \quad (\psi(f_s) - 1) * \varphi(f_s) = 0 \quad \text{in } \mathcal{A} \oplus \mathbb{C},
\]
where the dot denotes the equivalence class in the quotient space \( L^p(G, \omega)/I \). Hence, by (Bo-Du) \( \psi(f_s) - 1 \) is not invertible in \( \mathcal{A} \oplus \mathbb{C} \), i.e. \( 1 \in \sigma_{\mathcal{A}}(\psi(f_s)) \). So \( \psi(f_s) \notin \text{rad}(\mathcal{A}) \), where \( \text{rad}(\mathcal{A}) \) denotes the radical of \( \mathcal{A} \). This implies that there exists an algebraically irreducible representation \( (\hat{T}, V) \) of \( \mathcal{A} \) such that \( \hat{T}(\psi(f_s)) \neq 0 \). We then define the non-trivial algebraically irreducible representation \( (T, V) \) of \( L^p(G, \omega) \) by \( T(f) := \hat{T}(f) \). By construction, \( I \subset \ker T \). Hence \( L^p(G, \omega) \) is weakly Wiener. \( \square \)

7 Wiener property

We start with the following definition:

**Definition 7.1.** Let \( \mathcal{A} \) be a Banach \(*\)-algebra.

(i) A representation \((T, V)\) of \( \mathcal{A} \) on a Banach space \( V \) is said to be topologically irreducible, if there are no non-trivial closed \( T \)-invariant subspaces in \( V \). In particular, this definition is applied to unitary representations \((T, \mathcal{H})\) on Hilbert spaces \( \mathcal{H} \).

(ii) The algebra \( \mathcal{A} \) is said to have the Wiener property, if every proper closed two-sided ideal in \( \mathcal{A} \) is contained in the kernel of a topologically irreducible unitary representation of \( \mathcal{A} \).

It is well known that every symmetric Banach \(*\)-algebra which has the weak Wiener property, also has the Wiener property (see [Le1] and [Le2]). This leads us to the following result:
Theorem 7.2. Let \((G, \omega)\) satisfy \((LPAlg)\). Then the algebra \(L^p(G, \omega)\) has the Wiener property in the following cases:

a) \(G\) is abelian and \(\omega\) satisfies \((BDna)\).

b) \(G\) is non abelian and \(\omega(xy) \leq C(\omega(x) + \omega(y))\), for all \(x, y \in G\), for some constant \(C > 0\).

c) \(G\) is non abelian and \(\omega(x) = K(1 + |x|)^p\) for \(D\) large enough, with \(|x| = \inf\{n \mid x \in U^n\}\), where \(U\) is an arbitrary, relatively compact neighbourhood of the identity.

If one could prove that property \((S)\) implies the symmetry of the algebra \(L^p(G, \omega)\) (conjecture), then the property \((BDna)\) would imply the Wiener property.

8 Minimal ideals of a given hull

As always, we assume that \((G, \omega)\) satisfies \((LPAlg)\). We also suppose that \(\omega\) satisfies \((BDna)\). For details of the following we refer to \((Dz-Lu-Mo)\). We will use the following notations:

Let \(\Phi\) the set of functions \(\varphi\) from \(\mathbb{R}\) to \(\mathbb{R}\), periodic of period \(2\pi\), with \(\varphi(0) = 0\), with \(\text{supp} \varphi \cap [0, 2\pi]\) compact contained in \(]0, 2\pi[\), which operate on the set of continuous, self-adjoint functions with compact support in the algebras \(L^1(G, \omega)\) and \(L^p(G, \omega)\). The construction of such functions is described in more details in \((Dz-Lu-Mo)\). See also theorem 4.1

For any closed ideal \(I\) of \(L^p(G, \omega)\), we define the hull of \(I\) by

\[
h(I) := \{\ker \pi \in \text{Prim}_* L^p(G, \omega) \mid I \subset \ker \pi\}.
\]

For any compact subset \(C\) of \(\text{Prim}_* L^p(G, \omega)\) (endowed with the hull-kernel topology), let us define

\[
\hat{C} := \{\pi \in \hat{G} \mid \ker \pi \subset C\}
\]

and \(\|f\|_C := \sup_{\pi \in \hat{C}} \|\pi(f)\|\) for all \(f \in C_c(G)\), \(\|f\|_1 \leq 1\), \(\varphi \in \Phi\), \(\varphi \equiv 0\) on a neighbourhood of \([-\|f\|_C, \|f\|_C]\)

Let \(j(C)\) be the closed two-sided ideal of \(L^p(G, \omega)\) generated by \(m(C)\). As in \((Dz-Lu-Mo)\), one may prove:

Lemma 8.1. The hull of \(j(C)\) is \(C\).

Proof. See \((Dz-Lu-Mo)\). \(\square\)

Theorem 8.2. Let \((G, \omega)\) satisfy \((LPAlg)\). Let the weight \(\omega\) satisfy \((BDna)\). We also assume that the algebra \(L^p(G, \omega)\) is symmetric. Let \(C\) be a closed subset of \(\text{Prim}_* L^p(G, \omega)\) (\(\equiv \text{Prim}_* L^1(G, \omega) \equiv \text{Prim}_* L^1(G)\)). There exists a closed two-sided ideal \(j(C)\) of \(L^p(G, \omega)\) with \(h(j(C)) = C\), which is contained in every two-sided closed ideal \(I\) with \(h(I) = C\).

This is in particular the case if \(G\) is either abelian and \(\omega\) satisfies \((BDna)\) or if \(G\) is non-abelian and \(\omega\) satisfies the relationship \(\omega(xy) \leq C(\omega(x) + \omega(y))\) for all \(x, y \in G\), for some positive constant \(C\).

Proof. See \((Dz-Lu-Mo)\). \(\square\)

9 A symmetric algebra having infinite-dimensional irreducible representations

Using C. Read’s example of a quasi-nilpotent operator on \(\ell_1\) with no nontrivial invariant subspaces, one can construct an example of a weighted algebra \(L^p(\mathbb{R}, \omega)\) for any \(p > 1\) which is symmetric and has infinite-dimensional topologically irreducible representations. Up to our knowledge, the first application of this type is due A. Atzmon \((Atz)\).

Let \(T : \ell_1 \to \ell_1\) be the operator constructed in \((Re)\). It is quasi-nilpotent, so that

\[
\|T\| \leq 1, \quad \|T^n\|^{1/n} \to 0, \quad n \to \infty.
\]
9.1 Definition of the weights and symmetry

We will introduce a family of weights on \( \mathbb{R} \) which will depend on \( p \geq 1 \). For \( p = 1 \), we take as a weight

\[
\omega(x) = \max\{\|e^{xT}\|, \|e^{-xT}\|\}. \tag{9.2}
\]

Obviously, \( \omega \) is submultiplicative. Thus, \( L^1(\mathbb{R}, \omega) \) is an algebra. For \( p > 1 \), we put \( \omega_1(x) = \omega(x)(1+|x|)^2 \); by sections 1.2.1 and 1.2.2, this is an \( L^p \)-algebra weight (possibly after multiplication by a constant).

Next we prove the following estimate:

**Lemma 9.1.** Let \( \omega \) be defined by (9.2). Then \( \omega(x) = \mathcal{O}(\exp(\varepsilon|x|)), \ x \to \infty, \) for any \( \varepsilon > 0 \).

**Proof.** Let \( \varepsilon > 0 \) be given. It is enough to estimate \( \omega(x) \) for \( x > 0 \). By (9.1), there is \( N(\varepsilon) \) such that \( \|T^n\| < \varepsilon^n \) for all \( n \geq N(\varepsilon) \). Separate the series into two parts: \( \omega(x) = \Omega_1(x) + \Omega_2(x) \), where

\[
\Omega_1(x) = \sum_{n < N(\varepsilon)} \frac{\|T^n\|\varepsilon^n}{n!},
\]

\[
\Omega_2(x) = \sum_{n \geq N(\varepsilon)} \frac{\|T^n\|\varepsilon^n}{n!}.
\]

For \( \Omega_2 \), we have:

\[
\Omega_2(x) \leq \sum_{n \geq N(\varepsilon)} \frac{\varepsilon^n x^n}{n!} \leq \exp(\varepsilon x).
\]

For \( \Omega_1 \):

\[
\Omega_1(x) \leq \sum_{n < N(\varepsilon)} \frac{x^n \varepsilon^n}{n!} \leq \varepsilon^{-N(\varepsilon)} \sum_{n < N(\varepsilon)} \frac{x^n \varepsilon^n}{n!} \leq \varepsilon^{-N(\varepsilon)} \exp(\varepsilon x).
\]

Now,

\[
\omega(x) \leq \exp(\varepsilon x) \left(1 + \varepsilon^{-N(\varepsilon)}\right) = C(\varepsilon) \exp(\varepsilon x),
\]

what proves the lemma. \( \square \)

**Corollary 9.2.** The algebra \( L^1(\mathbb{R}, \omega) \) and every algebra \( L^p(\mathbb{R}, \omega_1) \) with \( p > 1 \) are symmetric.

**Proof.** It is easy to see that \( \omega_1(x) = \mathcal{O}(\exp(\varepsilon x)) \) for any \( \varepsilon > 0 \) as well. By lemma 3.4, \( \omega \) and \( \omega_1 \) satisfy condition (S), so by theorems 3.2 and 3.5, the algebras \( L^1(\mathbb{R}, \omega) \) and \( L^p(\mathbb{R}, \omega_1) \) are symmetric. \( \square \)

9.2 An infinite-dimensional irreducible representation

Let \( \mathcal{A} \) stand for \( L^1(\mathbb{R}, \omega) \) if \( p = 1 \) and for \( L^p(\mathbb{R}, \omega_1) \) if \( p > 1 \). Now we can put

\[
U(f) = \int_{\mathbb{R}} \exp(xT)f(x)dx
\]

for any \( f \in \mathcal{A} \). First of all, we will show that this integral converges absolutely. We can estimate

\[
\|U(f)\| \leq \int_{\mathbb{R}} \|\exp(Tx)\| |f(x)|dx \leq \int_{\mathbb{R}} \omega(x)|f(x)|dx.
\]

If \( p = 1 \), this equals \( \|f\|_{1,\omega} \). If \( p > 1 \),

\[
\int_{\mathbb{R}} \omega(x)|f(x)|dx = \int_{\mathbb{R}} \omega_1(x)|f(x)|(1+|x|)^{-2}dx \leq \|\omega_1f\|_p(1+|x|)^{-2}\|q = C_q\|f\|_{p,\omega_1}.
\]

In both cases, we see that \( \|U(f)\| \leq C\|f\|_{\mathcal{A}} \), so \( U \) is continuous. Clearly \( U \) is a homomorphism.
It remains now to show that $U$ is topologically irreducible. This will follow from the fact that closed invariant subspaces of $U$ are invariant under $T$. Suppose that $Z \subset \ell_1$ is a nonzero invariant subspace of $U$. Take $z \in Z$, $z \neq 0$. Let $I_\varepsilon$ be the indicator function of $[0, \varepsilon]$ and let $\xi_\varepsilon = \varepsilon^{-1} I_\varepsilon$. Then

$$U(\xi_\varepsilon) - I = \frac{1}{\varepsilon} \int_0^\varepsilon e^{xT} dx - \frac{1}{\varepsilon} \int_0^\varepsilon \text{id}dx = \frac{1}{\varepsilon} \int_0^\varepsilon (\sum_{n=0}^{\infty} \frac{(xT)^n}{n!} - 1)dx$$

$$= \frac{1}{\varepsilon} \int_0^\varepsilon \sum_{n=1}^{\infty} \frac{(xT)^n}{n!} dx = \frac{1}{\varepsilon} \int_0^\varepsilon (xT + \sum_{n=1}^{\infty} \frac{(xT)^n}{n!})dx.$$

Now, assuming that $0 < \varepsilon < 1$, we have

$$\|U(\xi_\varepsilon) - I - \frac{\varepsilon}{2} T\| = \|\frac{1}{\varepsilon} \int_0^\varepsilon x dx T - \frac{\varepsilon}{2} T + \frac{1}{\varepsilon} \int_0^\varepsilon \sum_{n=2}^{\infty} \frac{(xT)^n}{n!} dx\| \leq \frac{1}{\varepsilon} \int_0^\varepsilon \sum_{n=2}^{\infty} \frac{\|xT\|^n}{n!} dx$$

$$\leq \frac{1}{\varepsilon} \int_0^\varepsilon x^2 \sum_{n=0}^{\infty} \frac{x^n}{(n+2)!} dx \leq \frac{1}{\varepsilon} \int_0^\varepsilon x^2 e^x dx \leq \frac{1}{\varepsilon} \int_0^\varepsilon e x^2 dx = e \frac{e^3}{3} < \varepsilon^2.$$

Thus, $\varepsilon^{-1}(U(\xi_\varepsilon) - I) - \frac{T}{2} \to 0$, as $\varepsilon \to 0$, or $T = \lim_{\varepsilon \to 0} 2\varepsilon^{-1}(U(\xi_\varepsilon) - I)$. If now $Z$ is an invariant subspace for $U$, then its closure $\overline{Z}$ is invariant for $T$, so $\overline{Z}$ is trivial, and we are done.

Remarks. It is clear that this example can be extended to $\mathbb{R}^n$: replace $e^{xT}$ by $e^{\eta(x)T}$, where $\eta$ is a nonzero linear form on $\mathbb{R}^n$. One can show also that $\omega(x) > C \exp(x/\ln x)$, and so the algebras which we construct are not regular.

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Julia Kuznetsova, *Unité de Recherche en Mathématiques, Université du Luxembourg*, 6, rue Coudenhove-Kalergi, L-1359 Luxembourg, Luxembourg, julia.kuznetsova@uni.lu

Carine Molitor-Braun, *Unité de Recherche en Mathématiques, Université du Luxembourg*, 6, rue Coudenhove-Kalergi, L-1359 Luxembourg, Luxembourg, carine.molitor@uni.lu