APPLICATION OF A BERNSTEIN TYPE INEQUALITY TO RATIONAL INTERPOLATION IN THE DIRICHLET SPACE

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Abstract. We prove a Bernstein-type inequality involving the Bergman and the Hardy norms, for rational functions in the unit disc $D$ having at most $n$ poles all outside of $\frac{1}{r}D$, $0 < r < 1$. The asymptotic sharpness of this inequality is shown as $n \to \infty$ and $r \to 1^-$. We apply our Bernstein-type inequality to an effective Nevanlinna-Pick interpolation problem in the standard Dirichlet space, constrained by the $H^2$-norm.

INTRODUCTION

a. Statement of the problems.

Let $D = \{ z \in \mathbb{C} : |z| < 1 \}$ be the unit disc of the complex plane and let $\text{Hol}(D)$ be the space of holomorphic functions on $D$. Let also $X$ and $Y$ be two Banach spaces of holomorphic functions on the unit disc $D$, $X, Y \subset \text{Hol}(D)$. Here and later on, $H^\infty$ stands for the space (algebra) of bounded holomorphic functions in the unit disc $D$ endowed with the norm $\|f\|_\infty = \sup_{z \in D} |f(z)|$.

We suppose that $n \geq 1$ is an integer, $r \in [0, 1)$ and we consider the two following problems.

Problem 1. Let $\mathcal{P}_n$ be the complex space of analytic polynomials of degree less or equal than $n$, and
$$\mathcal{R}_{n, r} = \left\{ \frac{p}{q} : q \in \mathcal{P}_n, \ d^q p < d^p q, \ q(\zeta) = 0 \implies |\zeta| \geq \frac{1}{r} \right\},$$
(where $d^p \rho$ means the degree of any $p \in \mathcal{P}_n$) be the set of all rational functions in $D$ of degree less or equal than $n \geq 1$, having at most $n$ poles all outside of $\frac{1}{r}D$. Notice that for $r = 0$, we get $\mathcal{R}_{n, 0} = \mathcal{P}_{n-1}$. Our first problem is to search for the “best possible” constant $C_{n, r}(X, Y)$ such that
$$\|f'\|_X \leq C_{n, r}(X, Y) \|f\|_Y$$
for all $f \in \mathcal{R}_{n, r}$.

Problem 2. Let $\sigma = \{\lambda_1, ..., \lambda_n\}$ be a finite subset of $D$. What is the best possible interpolation by functions of the space $Y$ for the traces $f|_\sigma$ of functions of the space $X$, in the worst case? The case $X \subset Y$ is of no interest, and so one can suppose that either $Y \subset X$ or $X$ and $Y$ are incomparable. More precisely, our second problem is to compute or estimate the following interpolation constant
$$I(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \inf_{g : g|_\sigma = f|_\sigma} \|g\|_Y.$$
We also define

\[ I_{n,r}(X, Y) = \sup \{ I(\sigma, X, Y) : \text{card } \sigma \leq n, |\lambda| \leq r, \forall \lambda \in \sigma \}. \]

b. Motivations.

**Problem 1.** Bernstein-type inequalities for rational functions are applied

1.1. in matrix analysis and in operator theory (see “Kreiss Matrix Theorem” [LeTr, Sp] or [Z1, Z4] for resolvent estimates of power bounded matrices),

1.2. to “inverse theorems of rational approximation” using the classical Bernstein decomposition (see [Da, Pel, Pek]),

1.3. to effective \( H^\infty \) interpolation problems (see [Z3] and our Theorem B below in Subsection d), and more generally to our Problem 1.

**Problem 2.** We can give three main motivations for Problem 2.

2.1. It is explained in [Z3] (the case \( Y = H^\infty \)) why the classical interpolation problems, those of Nevanlinna-Pick (1908) and Carathéodory-Schur (1916) (see [N2] p.231 for these two problems), on the one hand and Carlesson’s free interpolation problem (1958) (see [N1] p.158) on the other hand, are of the nature of our interpolation problem.

2.2. It is also explained in [Z3] why this constrained interpolation is motivated by some applications in matrix analysis and in operator theory.

2.3. It has already been proved in [Z3] that for \( X = H^2 \) (see Subsection c. for the definition of \( H^2 \)) and \( Y = H^\infty \),

\[
\frac{1}{4\sqrt{2}} \frac{\sqrt{n}}{\sqrt{1-r}} \leq I_{n,r}(H^2, H^\infty) \leq \sqrt{2} \frac{\sqrt{n}}{\sqrt{1-r}}.
\]

The above estimate (1) answers a question of L. Baratchart (private communication), which is part of a more complicated question arising in an applied situation in [BL1] and [BL2]: given a set \( \sigma \subset \mathbb{D} \), how to estimate \( I(\sigma, H^2, H^\infty) \) in terms of \( n = \text{card}(\sigma) \) and \( \max_{\lambda \in \sigma} |\lambda| = r \) only?

c. The spaces \( X \) and \( Y \) considered here.

Now let us define some Banach spaces \( X \) and \( Y \) of holomorphic functions in \( \mathbb{D} \) which we will consider throughout this paper. From now on, if \( f \in \text{Hol}(\mathbb{D}) \) and \( k \in \mathbb{N} \), \( \hat{f}(k) \) stands for the \( k^{th} \) Taylor coefficient of \( f \).

1. The standard Hardy space \( H^2 = H^2(\mathbb{D}) \),

\[
H^2 = \left\{ f \in \text{Hol}(\mathbb{D}) : \|f\|^2_{H^2} = \sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(rz)|^2 \, dm(z) < \infty \right\},
\]
where $m$ stands for the normalized Lebesgue measure on $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$. An equivalent description of the space $H^2$ is

$$H^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \| f \|_{H^2} = \left( \sum_{k \geq 0} |\hat{f}(k)|^2 \right)^{\frac{1}{2}} < \infty \right\}.$$ 

2. The standard Bergman space $L^2_a = L^2_a(\mathbb{D})$,

$$L^2_a = \left\{ f \in \text{Hol}(\mathbb{D}) : \| f \|_{L^2_a}^2 = \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty \right\},$$

where $A$ is the standard area measure, also defined by

$$L^2_a = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \| f \|_{L^2_a} = \left( \sum_{k \geq 0} |\hat{f}(k)|^2 \frac{1}{k+1} \right)^{\frac{1}{2}} < \infty \right\}.$$ 

3. The analytic Besov space $B^\frac{1}{2}_{2, 2}$ (also known as the standard Dirichlet space) defined by

$$B^\frac{1}{2}_{2, 2} = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \| f \|_{B^\frac{1}{2}_{2, 2}} = \left( \sum_{k \geq 0} (k+1) |\hat{f}(k)|^2 \right)^{\frac{1}{2}} < \infty \right\}.$$ 

Then if $f \in B^\frac{1}{2}_{2, 2}$, we have the following equality

$$\| f \|_{B^\frac{1}{2}_{2, 2}}^2 = \| f' \|_{L^2_a}^2 + \| f \|_{H^2}^2,$$

which establishes a link between the spaces $B^\frac{1}{2}_{2, 2}$ and $L^2_a$.

d. The results. Here and later on, the letter $c$ denotes a positive constant that may change from one step to the next. For two positive functions $a$ and $b$, we say that $a$ is dominated by $b$, denoted by $a = O(b)$, if there is a constant $c > 0$ such that $a \leq cb$; and we say that $a$ and $b$ are comparable, denoted by $a \asymp b$, if both $a = O(b)$ and $b = O(a)$ hold.

**Problem 1.** Our first result (Theorem A, below) is a partial case ($p = q = 2$, $s = \frac{1}{2}$) of the following K. Dyakonov’s result [Dy]: if $p \in [1, \infty)$, $s \in (0, +\infty)$, $q \in [1, +\infty]$, then there exists a constant $c_{p, s} > 0$ such that

$$C_{n, r} \left( B^{s-1}_{p, p}, H^q \right) \leq c_{p, s} \sup \| B' \|_{H^q},$$

where $\gamma$ is such that $\frac{s}{\gamma} + \frac{1}{q} = \frac{1}{p}$, and the supremum is taken over all finite Blaschke products $B$ of order $n$ with $n$ zeros outside of $\frac{1}{r} \mathbb{D}$. Here $B^s_{p, p}$ stands for the Hardy-Besov space which consists of
analytic functions \( f \) on \( \mathbb{D} \) satisfying
\[
\|f\|_{B^p_p} = \sum_{k=0}^{n-1} |f^{(k)}(0)| + \int_{\mathbb{D}} (1 - |w|)^{(n-s)p-1} |f^{(n)}(w)|^p \, dA(w) < \infty.
\]

For the (tiny) partial case considered here, our proof is different and the constant \( c_2, \frac{1}{2} \) is asymptotically sharp as \( r \) tends to \( 1^- \) and \( n \) tends to \(+\infty\).

**Theorem A.** Let \( n \geq 1 \) and \( r \in [0, 1) \). We have

(i)

\[
\tilde{a}(n, r) \sqrt{\frac{n}{1 - r}} \leq C_{n, r} \left( L^2_a, H^2 \right) \leq \tilde{A}(n, r) \sqrt{\frac{n}{1 - r}}.
\]

where

\[
\tilde{a}(n, r) \geq \left( 1 - \frac{1 - r}{n} \right)^{\frac{1}{2}} \text{ and } \tilde{A}(n, r) \leq \left( 1 + r + \frac{1}{\sqrt{n}} \right)^{\frac{1}{2}}.
\]

(ii) Moreover, the sequence

\[
\left( \frac{C_{n, r} \left( L^2_a, H^2 \right)}{\sqrt{n}} \right)_{n \geq 1}
\]

is convergent and there exists a limit

\[
\lim_{n \to \infty} \frac{C_{n, r} \left( L^2_a, H^2 \right)}{\sqrt{n}} = \sqrt{\frac{1 + r}{1 - r}}.
\]

for all \( r \in [0, 1) \).

Notice that it has already been proved in [Z2] that there exists a limit

\[
\lim_{n \to \infty} \frac{C_{n, r} \left( H^2, H^2 \right)}{n} = \frac{1 + r}{1 - r},
\]

for every \( r, 0 \leq r < 1 \).

**Problem 2.** Looking at motivation 2.3, we replace the algebra \( H^\infty \) by the Dirichlet space \( B^\frac{1}{2}_{2,2} \). We show that the “gap” between \( X = H^2 \) and \( Y = H^\infty \) (see (1)) is asymptotically the same as the one which exists between \( X = H^2 \) and \( Y = B^\frac{1}{2}_{2,2} \). In other words,

\[
\mathcal{I}_{n, r} \left( H^2, B^\frac{1}{2}_{2,2} \right) \asymp \mathcal{I}_{n, r} \left( H^2, H^\infty \right) \asymp \sqrt{\frac{n}{1 - r}}.
\]
More precisely, we prove the following Theorem B, in which the right-hand side inequality of (10) is a consequence of the right-hand side inequality of (4) in the above Theorem A.

**Theorem B.** Let \( n \geq 1 \), and \( r \in [0, 1) \). Then,

\[
I_{n, r} \left( H^2, B^\frac{3}{2}, \right) \leq \left( I_{n, r} \left( L^2_a, H^2 \right) \right)^2 + 1 \]

Let \( \lambda \in \mathbb{D} \) and the corresponding one-point interpolation set \( \sigma_{n, \lambda} = \{ \lambda, \lambda, ..., \lambda \} \). We have,

\[
I \left( \sigma_{n, \lambda}, H^2, B^\frac{3}{2}, \right) \geq \sqrt{n} \left[ \frac{(1 + |\lambda|)^2 - 2}{2(1 + |\lambda|)} \right]^{\frac{1}{2}}.
\]

In particular,

\[
\left[ \frac{1 + r}{2} \left( 1 - \frac{1}{n} \right) \right]^{\frac{1}{2}} \leq I_{n, r} \left( H^2, B^\frac{3}{2}, \right) \leq \left( 1 + r + \frac{1}{\sqrt{n}} + \frac{1 - r}{n} \right)^{\frac{1}{2}} \sqrt{n} \leq \sqrt{1 + r}.
\]

and

\[
\sqrt{\frac{1 + r}{2}} \leq \liminf_{n \to \infty} \frac{I_{n, r} \left( H^2, B^\frac{3}{2}, \right)}{\sqrt{n}} \leq \limsup_{n \to \infty} \frac{I_{n, r} \left( H^2, B^\frac{3}{2}, \right)}{\sqrt{n}} \leq \sqrt{1 + r}.
\]

\[
\frac{\sqrt{2}}{2} \leq \liminf_{r \to 1^-} \liminf_{n \to \infty} \sqrt{\frac{1 - r}{n}} I_{n, r} \left( H^2, B^\frac{3}{2}, \right) \leq \limsup_{r \to 1^-} \limsup_{n \to \infty} \sqrt{\frac{1 - r}{n}} I_{n, r} \left( H^2, B^\frac{3}{2}, \right) \leq \sqrt{2}.
\]

In the next Section, we first give some definitions introducing the main tools used in the proofs of Theorem A and Theorem B. After that, we prove these theorems.

**Proofs of Theorems A and B**

From now on, if \( \sigma = \{ \lambda_1, ..., \lambda_n \} \subset \mathbb{D} \) is a finite subset of the unit disc, then

\[
B_{\sigma} = \prod_{j=1}^{n} b_{\lambda_j}
\]
is the corresponding finite Blaschke product where \( b_\lambda = \frac{\lambda}{1-\lambda z} \), \( \lambda \in \mathbb{D} \). In Definitions 1, 2, 3 and in Remark 4 below, \( \sigma = \{\lambda_1, \ldots, \lambda_n\} \) is a sequence in the unit disc \( \mathbb{D} \) and \( B_\sigma \) is the corresponding Blaschke product.

**Definition 1.** *Malmquist family.* For \( k \in [1, n] \), we set \( f_k = \frac{1}{1-\lambda_k z} \), and define the family \((e_k)_{1 \leq k \leq n}\), (which is known as Malmquist basis, see [N1, p.117]), by

\[
e_1 = \frac{f_1}{\|f_1\|_2} \quad \text{and} \quad e_k = \left( \prod_{j=1}^{k-1} b_{\lambda_j} \right) \frac{f_k}{\|f_k\|_2},
\]

for \( k \in [2, n] \); we have \( \|f_k\|_2 = (1 - |\lambda_k|^2)^{-1/2} \).

**Definition 2.** *The model space \( K_{B_\sigma} \).* We define \( K_{B_\sigma} \) to be the \( n \)-dimensional space:

\[
K_{B_\sigma} = \left( B_\sigma H^2 \right)^\perp = H^2 \ominus B_\sigma H^2.
\]

**Definition 3.** *The orthogonal projection \( P_{B_\sigma} \) on \( K_{B_\sigma} \).* We define \( P_{B_\sigma} \) to be the orthogonal projection of \( H^2 \) on its \( n \)-dimensional subspace \( K_{B_\sigma} \).

**Remark 4.** The Malmquist family \((e_k)_{1 \leq k \leq n}\) corresponding to \( \sigma \) is an orthonormal basis of \( K_{B_\sigma} \).

In particular,

\[
P_{B_\sigma} = \sum_{k=1}^{n} (\cdot, e_k)_{H^2} e_k,
\]

where \((\cdot, \cdot)_{H^2}\) means the scalar product on \( H^2 \).

**Proof of Theorem A.**

*Proof of (i).* 1) We first prove the right-hand side inequality of (4). Using both Cauchy-Schwarz inequality and the fact that \( \hat{f}'(k) = (k + 1) \hat{f}(k + 1) \) for all \( k \geq 0 \), we get

\[
\|f'\|_{L^2_2}^2 = \sum_{k \geq 0} \frac{\left| \hat{f}'(k) \right|^2}{k + 1} = \sum_{k \geq 0} \frac{(k + 1)^2 \left| \hat{f}(k + 1) \right|^2}{k + 1} = \sum_{k \geq 1} k \left| \hat{f}(k) \right|^2 \leq \left( \sum_{k \geq 1} k^2 \left| \hat{f}(k) \right|^2 \right)^{1/2} \left( \sum_{k \geq 1} \left| \hat{f}(k) \right|^2 \right)^{1/2}
\]

\[
= \|f'\|_{H^2} \|f\|_{H^2} \leq C_{n, r} (H^2, H^2) \|f\|_{H^2}^2,
\]

and hence,

\[
\|f'\|_{L^2_2} \leq \sqrt{C_{n, r} (H^2, H^2)} \|f\|_{H^2},
\]
which means
\[ C_{n, r} (L^2_a, H^2) \leq \sqrt{C_{n, r} (H^2, H^2)}. \]

Then it remains to use [Z2, p.2]:
\[ C_{n, r} (H^2, H^2) \leq \left( 1 + r + \frac{1}{\sqrt{n}} \right) \frac{n}{1 - r}, \]
for all \( n \geq 1 \) and \( r \in [0, 1) \).

2) The proof of the left-hand side inequality of (4) repeats the one of [Z2, (i)] (for the left-hand side inequality) except that this time, we replace the Hardy norm \( \| \cdot \|_{H^2} \) by the Bergman one \( \| \cdot \|_{L^2_a} \). Indeed, we use the same test function \( e_n = \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b_r^{n-1} \) (the \( n^{th} \) vector of the Malmquist family associated with the one-point set \( \sigma_{n, r} = \{ r, r, ..., r \} \) (see Definition 1) and prove by the same changing of variable \( \circ b_r \) (in the integral on the unit disc \( \mathbb{D} \) which defines the \( L^2_a \)-norm) that
\[ \| e_n' \|_{L^2_a}^2 = \frac{n}{1-r} \left( 1 - \frac{1-r}{n} \right)^{\frac{1}{2}}, \]
which gives
\[ C_{n, r} (L^2_a, H^2) \geq \sqrt{\frac{n}{1-r} \left( 1 - \frac{1-r}{n} \right)^{\frac{1}{2}}}. \]

Here are the details of the proof. We have \( e_n \in K_{b_r} \) and \( \| e_n \|_{H^2} = 1 \), (see [N1], Malmquist-Walsh Lemma, p.116). Moreover,
\[
e_n' = \frac{r (1-r^2)^{\frac{1}{2}}}{(1-rz)^{\frac{1}{2}}} b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b_r' b_r^{n-2} = -\frac{r}{(1-r^2)^{\frac{1}{2}}} b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b_r' b_r^{n-2},
\]
since \( b_r' = \frac{r^2-1}{(1-rz)^2} \). Then,
\[
e_n' = b_r' \left[ -\frac{r}{(1-r^2)^{\frac{1}{2}}} b_r^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rz} b_r^{n-2} \right],
\]
and
\[
\| e_n' \|_{L^2_a}^2 = \frac{1}{2\pi} \int_{\mathbb{D}} |b_r' (w)|^2 \left| -\frac{r}{(1-r^2)^{\frac{1}{2}}} (b_r (w))^{n-1} + (n-1) \frac{(1-r^2)^{\frac{1}{2}}}{1-rw} (b_r (w))^{n-2} \right|^2 dm(w) =
\]
\[ \frac{1}{2\pi} \int_{D} |b'_r(w)|^2 |(b_r(w))^{n-2}|^2 \left| - \frac{r}{(1-r^2)^{\frac{1}{2}}} b_r(w) + (n-1) \frac{1-r^2}{1-rw} \right| \, dm(w), \]

which gives, using the variables \( u = b_r(w) \),

\[ \|e'_n\|_{L^2_a}^2 = \frac{1}{2\pi} \int_{D} |u^{n-2}|^2 \left| - \frac{r}{(1-r^2)^{\frac{1}{2}}} u + (n-1) \frac{1-r^2}{1-rb_r(u)} \right|^2 \, dm(u). \]

But \( 1 - rb_r = \frac{1-rz - r(r-\bar{z})}{1-rz} = \frac{1-rz}{1-rz} \) and \( b'_r \circ b_r = \frac{r^2 - 1}{1-rb_r} = -\frac{(1-rz)^2}{1-r^2} \). This implies

\[ \|e'_n\|_{L^2_a}^2 = \frac{1}{2\pi} \int_{D} |u^{n-2}|^2 \left| - \frac{r}{(1-r^2)^{\frac{1}{2}}} u + (n-1) \frac{1-r^2}{1-r^2}(1-ru) \right|^2 \, dm(u) = \]

\[ = \frac{1}{1-r^2} \frac{1}{2\pi} \int_{D} |u^{n-2}|^2 |(ru + (n-1)(1-ru)|^2 \, dm(u), \]

which gives

\[ \|e'_n\|_{L^2_a} = \frac{1}{1-r^2} \|\varphi_n\|_2, \]

where \( \varphi_n = z^{n-2}(-rz + (n-1)(1-rz)) \). Expanding, we get

\[ \varphi_n = z^{n-2}(-rz + n-1 + rz - nrz) = \]

\[ = z^{n-2}(-nrz + n-1) = (n-1)z^{n-2} - nrz^{n-1}, \]

and

\[ \|e'_n\|_{L^2_a}^2 = \frac{1}{1-r^2} \left( \frac{(n-1)^2}{n-1} + \frac{n^2}{n}r^2 \right) = \frac{1}{1-r^2} (n(1+r) - 1) \]

\[ = \frac{n}{(1-r)(1+r)} \left( (1+r) - \frac{1}{n} \right) = \frac{n}{1-r} \left( 1 - \frac{1-r}{n} \right), \]

which gives

\[ C_{n,r}(L^2_a, H^2) \geq \sqrt{\frac{n}{1-r} \left( 1 - \frac{1-r}{n} \right)^{\frac{1}{2}}}. \]

**Proof of (ii).** This is again the same proof as [Z2, (ii)] (the three steps). More precisely in Step 2, we use the same test function

\[ f = \sum_{k=0}^{s+2} (-1)^k e_{n-k}, \]

(where \( s = (s_n) \) is defined in [Z2, p.8]), and the same changing of variable \( \circ b_r \) in the integral on \( D \). Here are the details of the proof.
Step 1. We first prove the right-hand-side inequality:

\[
\limsup_{n \to \infty} \frac{1}{\sqrt{n}} C_{n,r}(L_a^2, H^2) \leq \sqrt{\frac{1 + r}{1 - r}},
\]

which becomes obvious since

\[
\frac{1}{\sqrt{n}} C_{n,r}(L_a^2, H^2) \leq \frac{1}{\sqrt{n}} \sqrt{C_{n,r}(H^2, H^2)},
\]

and

\[
\frac{1}{\sqrt{n}} \sqrt{C_{n,r}(H^2, H^2)} \to \sqrt{\frac{1 + r}{1 - r}},
\]
as \(n\) tends to infinity, see [Z1] p. 2.

Step 2. We now prove the left-hand-side inequality:

\[
\liminf_{n \to \infty} \frac{1}{\sqrt{n}} C_{n,r}(L_a^2, H^2) \geq \sqrt{\frac{1 + r}{1 - r}}.
\]

More precisely, we show that

\[
\liminf_{n \to \infty} \frac{1}{\sqrt{n}} \|D\|_{(K_{\mathbb{P}}, \|\cdot\|_{L^2}) \to H^2} \geq \sqrt{\frac{1 + r}{1 - r}}.
\]

Let \(f \in K_{\mathbb{P}}\). Then,

\[
f'(e_1)_{H^2} = \frac{r}{(1 - rz)} e_1 + \sum_{k=2}^{n} (k-1) (f, e_k)_{H^2} \frac{b_{r}^k}{b_{r}} e_k + r \sum_{k=2}^{n} (f, e_k)_{H^2} \frac{1}{(1 - rz)} e_k =
\]

\[
= r \sum_{k=1}^{n} (f, e_k)_{H^2} \frac{1}{(1 - rz)} e_k + \frac{1 - r^2}{(1 - rz)(z - r)} \sum_{k=2}^{n} (k-1) (f, e_k)_{H^2} e_k =
\]

\[
= \frac{r (1 - rz)\frac{1}{2}}{(1 - rz)^{\frac{3}{2}}} \sum_{k=1}^{n} (f, e_k)_{H^2} b_{r}^{k-1} + \frac{(1 - rz)^{\frac{1}{2}}}{(1 - rz)^2 (z - r)} \sum_{k=2}^{n} (k-1) (f, e_k)_{H^2} b_{r}^{k-1} =
\]

\[
= -b_{r} \left[ \frac{r}{(1 - rz)^{\frac{1}{2}}} \sum_{k=1}^{n} (f, e_k)_{H^2} b_{r}^{k-1} + \frac{(1 - rz)^{\frac{1}{2}}}{z - r} \sum_{k=2}^{n} (k-1) (f, e_k)_{H^2} b_{r}^{k-1} \right].
\]

Now using the change of variables \(v = b_{r}(u)\), we get

\[
\|f''\|^2_{L_{2}^{\mathbb{D}}} = \int_{\mathbb{D}} |b_{r}'(u)|^2 \left[ \frac{r}{(1 - rz)^{\frac{1}{2}}} \sum_{k=1}^{n} (f, e_k)_{H^2} b_{r}^{k-1} + \frac{(1 - rz)^{\frac{1}{2}}}{u - r} \sum_{k=2}^{n} (k-1) (f, e_k)_{H^2} b_{r}^{k-1} \right]^2 du =
\]

\[
= \int_{\mathbb{D}} \left[ \frac{r}{(1 - rz)^{\frac{1}{2}}} \sum_{k=1}^{n} (f, e_k)_{H^2} v^{k-1} + \frac{(1 - rz)^{\frac{1}{2}}}{b_{r}(v) - r} \sum_{k=2}^{n} (k-1) (f, e_k)_{H^2} v^{k-1} \right]^2 dv.
\]
Now, $b_r - r = \frac{r - z - r(1 - rz)}{1 - rz} = \frac{z(r^2 - 1)}{1 - rz}$, which gives
\[
\|f\|_{H^2}^2 = \int_D \left| \frac{r}{(1 - r^2)^2} \sum_{k=1}^{n} (f, e_k)_{H^2} v^{k-1} + \frac{(1 - r^2)^{\frac{3}{2}}}{v(r^2 - 1)} (1 - rv) \sum_{k=2}^{n} (k - 1) (f, e_k)_{H^2} v^{k-1} \right|^2 \, dv =
\]
\[
= \frac{1}{1 - r^2} \int_D r \sum_{k=1}^{n} (f, e_k)_{H^2} v^{k-1} - (1 - rv) \sum_{k=2}^{n} (k - 1) (f, e_k)_{H^2} v^{k-2} \right|^2 \, dv =
\]
\[
= \frac{1}{1 - r^2} \int_D r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^{k} - (1 - rv) \sum_{k=0}^{n-2} (k + 1) (f, e_{k+2})_{H^2} v^{k} \right|^2 \, dv.
\]

Thus,
\[
(16) \quad \frac{1}{\|f\|_{H^2} \sqrt{n(1+r)}} \left[ \left\| (1 - rv) \sum_{k=0}^{n-1} (k + 1) (f, e_{k+1})_{H^2} v^{k} \right\|_{L^2_a} + \left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^{k} \right\|_{L^2_a} \right] \geq
\]
\[
\geq \sqrt{\frac{1 - r}{n} \|f\|_{H^2}^2} \geq
\]
\[
\geq \frac{1}{\|f\|_{H^2} \sqrt{n(1+r)}} \left[ \left\| (1 - rv) \sum_{k=0}^{n-1} (k + 1) (f, e_{k+1})_{H^2} v^{k} \right\|_{L^2_a} - \left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^{k} \right\|_{L^2_a} \right].
\]

Now,
\[
(1 - rv) \sum_{k=0}^{n-2} (k + 1) (f, e_{k+2})_{H^2} v^{k} =
\]
\[
= \sum_{k=0}^{n-2} (k + 1) (f, e_{k+2})_{H^2} v^{k} - r \sum_{k=0}^{n-2} (k + 1) (f, e_{k+2})_{H^2} v^{k+1} =
\]
\[
= \sum_{k=0}^{n-2} (k + 1) (f, e_{k+2})_{H^2} v^{k} - r \sum_{k=1}^{n-1} k (f, e_{k+1})_{H^2} v^{k} =
\]
\[
= (f, e_2)_{H^2} + 2 (f, e_3)_{H^2} v + \sum_{k=2}^{n-2} [(k + 1) (f, e_{k+2})_{H^2} - r k (f, e_{k+1})_{H^2}] v^{k} +
\]
\[
- r [(f, e_2)_{H^2} v + (n - 1) (f, e_n)_{H^2} v^{n-1}] =
\]
\[
= (f, e_2)_{H^2} + [(f, e_3)_{H^2} - r (f, e_2)_{H^2}] v + \sum_{k=2}^{n-2} [(k + 1) (f, e_{k+2})_{H^2} - r k (f, e_{k+1})_{H^2}] v^{k} +
\]
\[
- r(n - 1) (f, e_n)_{H^2} v^{n-1},
\]
which gives

\[(17) \quad \left\| (1 - rv) \sum_{k=0}^{n-2} (k + 1) (f, e_{k+2})_{H^2} v^k \right\|_{L^2_n}^2 = \]

\[= \| (f, e_2)_{H^2} \|^2 + \frac{1}{2} \| (f, e_3)_{H^2} - r (f, e_2)_{H^2} \|^2 + \]

\[+ \frac{1}{n} r^4 (n - 1)^2 \| (f, e_n)_{H^2} \|^2 + \sum_{k=2}^{n-2} \left( (f, e_{k+2})_{H^2} - \frac{rk}{k+1} (f, e_{k+1})_{H^2} \right)^2.\]

On the other hand,

\[(18) \quad \left\| r \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_{L^2_n} \leq r \left( \sum_{k=0}^{n-1} \frac{1}{k+1} \| (f, e_{k+1})_{H^2} \|^2 \right)^{1/2} \leq r \| f \|_{H^2},\]

Now, let \(s = (s_n)\) be a sequence of even integers such that

\[\lim_{n \to \infty} s_n = \infty \text{ and } s_n = o(n) \text{ as } n \to \infty.\]

Then we consider the following function \(f\) in \(K_{b_n^*}\):

\[f = \sum_{k=0}^{s+2} (-1)^k e_{n-k}.\]

Applying (17) with such an \(f\), we get

\[\left\| (1 - rv) \sum_{k=0}^{n-2} (k + 1) (f, e_{k+2})_{H^2} v^k \right\|_{L^2_n}^2 = \]

\[= r^4 \frac{(n - 1)^2}{n} + \]

\[+ \sum_{l=2}^{n-2} (n - l + 1) \left| (f, e_{n-l+2})_{H^2} - \frac{r(n - l)}{n - l + 1} (f, e_{n-l+1})_{H^2} \right|^2,\]

setting the change of index \(l = n - k\) in the last sum. This finally gives

\[\left\| (1 - rv) \sum_{k=0}^{n-2} (k + 1) (f, e_{k+2})_{H^2} v^k \right\|_{L^2_n}^2 = \]
\[ r^4 \frac{(n-1)^2}{n} + \sum_{l=2}^{s+1} (n-l+1) \left( 1 + \frac{r(n-l)}{n-l+1} \right)^2 = \]

\[ = r^4 \frac{(n-1)^2}{n} + \sum_{l=2}^{s+1} (n-l+1) \left[ 1 + r \left( 1 - \frac{1}{n-l+1} \right) \right]^2, \]

and

\[ \left\| (1 - rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2} \geq \]

\[ \geq r^4 \frac{(n-1)^2}{n} + (s + 1 - 2 + 1)(n-(s+1)+1) \left[ 1 + r \left( 1 - \frac{1}{n-(s+1)+1} \right) \right]^2 = \]

\[ = r^4 \frac{(n-1)^2}{n} + s(n-s) \left[ 1 + r \left( 1 - \frac{1}{n-s} \right) \right]^2. \]

In particular,

\[ \left\| (1 - rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2}^2 \geq s(n-s) \left[ 1 + r \left( 1 - \frac{1}{n-s} \right) \right]^2. \]

Now, since \( \|f\|_{H^2}^2 = s_n + 3 \), we get

\[ \liminf_{n \to \infty} \frac{1}{n} \left\| f \right\|_{H^2}^2 \left\| (1 - rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_2 \geq \]

\[ \geq \liminf_{n \to \infty} \frac{1}{n} \left\| f \right\|_{H^2}^2 \left( n - \|f\|_{H^2}^2 \right) \left[ 1 + r \left( 1 - \frac{1}{n-s} \right) \right]^2 = \]

\[ = \lim_{n \to \infty} \left( 1 - \frac{s_n}{n} \right) \left[ 1 + r \left( 1 - \frac{1}{n-s} \right) \right]^2 = (1 + r)^2. \]

On the other hand, applying (18) with this \( f \), we obtain

\[ \lim_{n \to \infty} \frac{1}{n} \left\| f \right\|_{H^2}^2 \left\| \sum_{k=0}^{n-1} (f, e_{k+1})_{H^2} v^k \right\|_{L_a^2} = 0. \]

Thus, we can conclude passing after to the limit as \( n \) tends to \( +\infty \) in (16), that

\[ \liminf_{n \to \infty} \frac{1 - r}{n} \left\| f' \right\|_{L_a^2} = \frac{1}{\sqrt{1+r}} \liminf_{n \to \infty} \frac{1}{\|f\|_{H^2} \sqrt{n}} \left\| (1 - rv) \sum_{k=0}^{n-2} (k+1) (f, e_{k+2})_{H^2} v^k \right\|_{L_a^2} \geq \]
\[ \frac{1 + r}{\sqrt{1 + r}} = \sqrt{1 + r}, \]

and

\[ \liminf_{n \to \infty} \sqrt{\frac{1 - r}{n}} \|D\|_{K_{\varphi}} \to H^2 \geq \liminf_{n \to \infty} \sqrt{\frac{1 - r}{n}} \|f^\prime\|_{L^2_a} \geq \sqrt{1 + r}. \]

**Step 3. Conclusion.** Using both Step 1 and Step 2, we get

\[ \limsup_{n \to \infty} \sqrt{\frac{1 - r}{n}} C_{n, r} (L^2_a, H^2) = \liminf_{n \to \infty} \sqrt{\frac{1 - r}{n}} C_{n, r} (L^2_a, H^2) = 1 + r, \]

which means that the sequence \( \left( \frac{1}{n} C_{n, r} (L^2_a, H^2) \right)_{n \geq 1} \) is convergent and

\[ \lim_{n \to \infty} \frac{1}{\sqrt{n}} C_{n, r} (L^2_a, H^2) = \frac{\sqrt{1 + r}}{1 - r}. \]

\[ \square \]

**Proof of Theorem B.**

*Proofs of inequality (8) and of the right-hand side inequality of (10).* Let \( \sigma \) be a sequence in \( \mathbb{D} \), and \( B = B_\sigma \) the finite Blaschke product corresponding to \( \sigma \). If \( f \in H^2 \), we use the same function \( g \) as in [Z3] which satisfies \( g_{|\sigma} = f_{|\sigma} \). More precisely, let \( g = P_B f \in K_B \) (see Definitions 2, 3 and Remark 4 above for the definitions of \( K_B \) and \( P_B \)). Then \( g - f \in B H^2 \) and using the definition of \( C_{n, r} (L^2_a, H^2) \),

\[ \|g^\prime\|_{L^2_a} \leq \left( C_{n, r} (L^2_a, H^2) \right)^2 \|g\|_{H^2}. \]

Now applying the identity (2) to \( g \) we get

\[ \|g\|_{B^{\frac{1}{2}}_{2, 2}} \leq \left( \left( C_{n, r} (L^2_a, H^2) \right)^2 + 1 \right) \|g\|_{H^2}. \]

Using the fact that \( \|g\|_{H^2} = \|P_B f\|_{H^2} \leq \|f\|_{H^2} \), we finally get

\[ \|g\|_{B^{\frac{1}{2}}_{2, 2}} \leq \left( \left( C_{n, r} (L^2_a, H^2) \right)^2 + 1 \right)^{\frac{1}{2}} \|f\|_{H^2}, \]

and as a result,

\[ I (\sigma, H^2, B^{\frac{1}{2}}_{2, 2}) \leq \left( \left( C_{n, r} (L^2_a, H^2) \right)^2 + 1 \right)^{\frac{1}{2}}. \]

It remains to apply the right-hand side inequality of (4) in Theorem A to prove the right-hand side one of (10).

*Proof of inequality (9).* 1) We use the same test function

\[ f = \sum_{k=0}^{n-1} (1 - |\lambda|^2) \frac{1}{2} b^k (1 - \lambda z)^{-1}, \]
as the one used in the proof of [Z3, Theorem B] (the lower bound, page 11 of [Z3]). \( f \) being the sum of \( n \) elements of \( H^2 \) which are an orthonormal family known as Malmquist’s basis (associated with \( \sigma_{n, \lambda} = \{\lambda, \lambda, \ldots, \lambda\} \), see Remark 4 above or [N1, p.117] ), we have \( \|f\|_{H^2}^2 = n \).

2) Since the spaces \( H^2 \) and \( B_{\frac{1}{2}, 2}^\frac{1}{2} \) are rotation invariant, we have \( I \left( \sigma_{n, \lambda}, H^2, B_{\frac{1}{2}, 2}^\frac{1}{2} \right) = I \left( \sigma_{n, \mu}, H^2, B_{\frac{1}{2}, 2}^\frac{1}{2} \right) \) for every \( \lambda, \mu \) with \( |\lambda| = |\mu| = r \). Let \( \lambda = -r \). To get a lower estimate for \( \|f\|_{B_{\frac{1}{2}, 2}^\frac{1}{2}/\nu B_{\frac{1}{2}, 2}^\frac{1}{2}} \) consider \( g \) such that \( f - g \in \sigma(\lambda)_{\text{Hol}}(D) \), i.e. such that \( f \circ b_{\lambda} - g \circ b_{\lambda} \in z^n \text{Hol}(D) \).

3) First, we notice that

\[
\|g \circ b_{\lambda}\|_{B_{\frac{1}{2}, 2}^\frac{1}{2}}^2 = \left\| (g \circ b_{\lambda})' \right\|_{L^2}^2 + \|g \circ b_{\lambda}\|_{H^2}^2 = \|b_{\lambda} \cdot (g' \circ b_{\lambda})\|_{L^2}^2 + \|g \circ b_{\lambda}\|_{H^2}^2 =
\]

\[
= \int_D |b_{\lambda}(u)|^2 |g'(b_{\lambda}(u))|^2 du + \|g \circ b_{\lambda}\|_{H^2}^2 = \int_D |g'(w)|^2 dw + \|g \circ b_{\lambda}\|_{H^2}^2,
\]

using the changing of variable \( w = b_{\lambda}(u) \). We get

\[
\|g \circ b_{\lambda}\|_{B_{\frac{1}{2}, 2}^\frac{1}{2}}^2 = \|g'\|_{L^2}^2 + \|g \circ b_{\lambda}\|_{H^2}^2 = \|g\|_{B_{\frac{1}{2}, 2}^\frac{1}{2}}^2 + \|g \circ b_{\lambda}\|_{H^2}^2 - \|g\|_{H^2}^2,
\]

and

\[
\|g\|_{B_{\frac{1}{2}, 2}^\frac{1}{2}}^2 = \|g\|_{H^2}^2 + \|g \circ b_{\lambda}\|_{B_{\frac{1}{2}, 2}^\frac{1}{2}}^2 - \|g \circ b_{\lambda}\|_{H^2}^2 =
\]

\[
\geq \|g \circ b_{\lambda}\|_{B_{\frac{1}{2}, 2}^\frac{1}{2}}^2 - \|g \circ b_{\lambda}\|_{H^2}^2.
\]

Now, we notice that

\[
f \circ b_{\lambda} = \sum_{k=0}^{n-1} z^k \frac{(1 - |\lambda|^2)^{\frac{1}{2}}}{1 - \lambda b_{\lambda}(z)} = (1 - |\lambda|^2)^{-\frac{1}{2}} \left( 1 + (1 - \overline{\lambda}) \sum_{k=1}^{n-1} z^k - \overline{\lambda} z^n \right) =
\]

\[
= (1 - r^2)^{-\frac{1}{2}} \left( 1 + (1 + r) \sum_{k=1}^{n-1} z^k + r z^n \right).
\]

4) Next,

\[
\|g \circ b_{\lambda}\|_{B_{\frac{1}{2}, 2}^\frac{1}{2}}^2 - \|g \circ b_{\lambda}\|_{H^2}^2 = \sum_{k \geq 1} k \left| g \circ b_{\lambda}(k) \right|^2 \\
\geq \sum_{k=1}^{n-1} k \left| g \circ b_{\lambda}(k) \right|^2 = \sum_{k=1}^{n-1} k \left| f \circ b_{\lambda}(k) \right|^2,
\]
since \( g \circ b_\lambda(k) = \hat{f} \circ b_\lambda(k), \ \forall \ k \in [0, n - 1] \). This gives

\[
\|g \circ b_\lambda\|_{B_{2,2}^s}^2 - \|g \circ b_\lambda\|_{H^2}^2 \geq \frac{1}{1 - r^2} \left( \frac{(1 + r)^2}{2} \sum_{k=1}^{n-1} k \right) =
\]

\[
\frac{(1 + r)^2 n(n - 1)}{2} = \frac{1 + r}{1 - r} \frac{n(n - 1)}{2} = \frac{1 + r}{1 - r} \frac{n(n - 1)}{2} \|f\|_{H^2}^2,
\]

for all \( n \geq 2 \) since \( \|f\|_{H^2}^2 = n \). Finally,

\[
\|g\|_{B_{2,2}^s}^2 \geq \frac{n}{1 - r} \frac{1 + r}{2} \left( 1 - \frac{1}{n} \right) \|f\|_{H^2}^2.
\]

In particular,

\[
\mathcal{I}_{n, r} \left( H^2, B_{2,2}^s \right) \geq \sqrt{n} \left( 1 - \frac{1}{n} \right)^{\frac{3}{2}}.
\]

\[
\square
\]

Some comments.

a. Extension of Theorem A to spaces \( B_{2,2}^s, s \geq 0 \). Using the techniques developed in the proof of our Theorem A (combined with complex interpolation (between Banach spaces) and a reasoning by induction), it is possible both to precise the sharp numerical constant \( c_{2,s} \) in K. Dyakonov’s result (3) (mentioned above in paragraph d. of the Introduction) and to prove the asymptotic sharpness (at least for \( s \in \mathbb{N} \cup \frac{1}{2} \mathbb{N} \)) of the right-hand side inequality of (3). In the same spirit, we would obtain that there exists a limit:

\[
\begin{aligned}
\lim_{n \to \infty} \frac{C_{n, r} \left( B_{2,2}^{s-1}, H^2 \right)}{n^s} &= \left( \frac{1 + r}{1 - r} \right)^s.
\end{aligned}
\]

Our Theorem A corresponds to the case \( s = \frac{1}{2} \).

b. Extension of Theorem B to spaces \( B_{2,2}^s, s \geq 0 \). The proof of the upper bound in our Theorem B can be extended so as to give an upper (asymptotic) estimate of the interpolation constant \( \mathcal{I}_{n, r} \left( H^2, B_{2,2}^s \right), s \geq 0 \). More precisely, applying K. Dyakonov’s result (3) (mentioned above in paragraph d. of the Introduction) we get

\[
\mathcal{I}_{n, r} \left( H^2, B_{2,2}^s \right) \leq \tilde{c}_s \left( \frac{n}{1 - r} \right)^s, \text{ with } \tilde{c}_s \asymp c_{2,s},
\]

where \( c_{2,s} \) is defined in (3) and precised in (19). Looking at the above comment 1, \( \tilde{c}_s \asymp (1 + r)^s \) for sufficiently large values of \( n \). Our Theorem B corresponds again to the case \( s = \frac{1}{2} \). In this Theorem B, we prove the sharpness of the right-hand side inequality in (20) for \( s = \frac{1}{2} \). However, for the general case \( s \geq 0 \), the asymptotic sharpness of \( \left( \frac{n}{1 - r} \right)^s \) as \( r \to 1^- \) and \( n \to \infty \) is less obvious. Indeed, the key of the proof (for the sharpness) is based on the property that the Dirichlet norm
(the one of $B_{2,2}^{1/2}$) is “nearly” invariant composing by an elementary Blaschke factor $b_\lambda$, as this is the case for the $H^\infty$ norm. A conjecture given by N. K. Nikolski is the following:

$$\mathcal{I}_{n,r} (H^2, B_{2,2}^s) \asymp \begin{cases} \frac{n^s}{\sqrt{1-r}} & \text{if } s \geq \frac{1}{2} \\ \left(\frac{n}{1-r}\right)^s & \text{if } 0 \leq s \leq \frac{1}{2} \end{cases},$$

and is due to the position of the spaces $B_{2,2}^s, s \geq 0$ with respect to the algebra $H^\infty$.

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