Decomposable Norm Minimization with Proximal-Gradient Homotopy Algorithm

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Abstract In this paper we study the convergence rate of proximal-gradient homotopy algorithm for norm-regularized linear least squares problems. Homotopy algorithm reduces regularization parameter in a series of steps, and uses proximal-gradient algorithm to solve the problem at each step. Proximal-gradient algorithm has a linear rate of convergence given that the objective function is strongly convex and the gradient of the smooth component of the objective function is Lipschitz continuous. In general, the objective function in this type of problems is not strongly convex, especially when the problem is high-dimensional. We will show that if the linear sampling matrix and the regularizing norm satisfy certain assumptions, proximal-gradient homotopy algorithm converges with a linear rate even though the objective function is not strongly convex. Our result generalizes results on the linear convergence of homotopy algorithm for $l_1$-regularized least squares problems.

Keywords Proximal-Gradient · Homotopy · Decomposable norm

1 Introduction

In signal processing and statistical regression, problems arise in which the goal is to recover a structured model from a few, often noisy, linear measurements. Well studied examples include recovery of sparse vectors and low rank matrices. These problems can be formulated as non-convex optimization programs. While under suitable conditions the solution of the aforementioned programs can recover the original model, the algorithms exploited for solving them become computationally intractable. One can relax these non-convex problems by utilizing appropriate convex penalty functions, for example $l_1$, $l_{1,2}$ and nuclear norms in sparse vector, group sparse and low rank matrix recovery problems. These relaxations perform very well in many practical applications. Following [9,6,7], there has been a flurry of publications that formalize the condition for recovery of sparse vectors, e.g., [2,33], low rank matrices, e.g., [28,4,11] from linear measurements by solving the appropriate relaxed convex optimization problems. Alongside results for sparse vector and low rank matrix recovery several authors have proposed more general frameworks for structured model recovery problems with linear measurements [5,8,23]. In many problems of interest, to recover the model from linear noisy measurements, one can formulate the following optimization program:

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minimize $\|x\|$ \hspace{1cm} (1)
s.t. $\|Ax - b\|_2^2 \leq \epsilon^2$,

where $b \in \mathbb{R}^m$ is the measurements vector, $A \in \mathbb{R}^{m \times n}$ is the linear measurement matrix, $\epsilon^2$ is the noise energy and $\|\cdot\|$ is a norm on $\mathbb{R}^n$ that promotes the desired structure in the solution.

There has been extensive work on algorithms for solving problem (1) in special cases of $l_1$ and nuclear norms. First order methods have been the method of choice for large scale problems, since each iteration is computationally cheap. Of particular interest is the proximal-gradient method for minimization of composite functions, which can be utilized for solving the regularized version of problem (1), which has the form:

$$\minimize \frac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|,$$  \hspace{1cm} (2)

where $\lambda > 0$ is the regularization parameter. When the smooth component of the objective function has a Lipschitz continuous gradient, proximal-gradient algorithm has a convergence rate of $O(1/t)$, where $t$ is the iteration number. For the accelerated version of proximal-gradient algorithm the convergence rate improves to $O(1/t^2)$. When the objective function is strongly convex as well, proximal-gradient has linear convergence, i.e. $O(\kappa t)$ with $\kappa \in (0, 1)$. However, in instances of problem (2) that are of interest, the number of samples is less than the dimension of the space, hence the matrix $A$ has a non-zero null space which results in an objective function that is not strongly convex. Several algorithms that combine homotopy continuation over $\lambda$ with proximal-gradient steps have been proposed in the literature for problem (2) in the special cases of $l_1$ and nuclear norms [12,35,34,19,32]. Xiao and Zhang [36] have studied an algorithm with homotopy with respect to $\lambda$ for solving $l_1$ regularized least squares problem. Formulating their algorithm based on Nesterov’s proximal-gradient method, they have demonstrated that this algorithm has an overall linear rate of convergence whenever the linear transformation $A$ satisfies the restricted isometry property (RIP) and the final value of the regularizer parameter $\lambda$ is greater than a problem-dependent lower bound.

1.1 Our result

In this paper, we consider the natural extension of the homotopy algorithm studied in [36] to problem (2). We explore the properties of the norm $\|\cdot\|$ that allow generalization from $l_1$ norm case and show that under RIP assumption on $A$, the homotopy algorithm has a linear rate of convergence for norms that satisfy those properties (in particular, for $l_1$, $l_{1,2}$ and nuclear norms). We provide an upper bound on the number measurements sufficient for $A$ to satisfy the RIP condition with high probability when rows of $A$ are sub-Gaussian random vectors. This bound is orderwise the same as the lower bound on the number of measurements for statistical consistency of estimation of the true solution.

1.2 Algorithms for structured model recovery

There has been extensive work on algorithms for solving problems (1) and (2) in the special cases of $l_1$ and nuclear norms. For a detailed review of first order methods we refer the reader to [25] and reference therein. In [36], authors have reviewed sparse recovery and $l_1$ norm minimization algorithms that are related to the homotopy algorithm for $l_1$ norm. We discuss related algorithms mostly focusing on algorithms for other norms including nuclear norm here.

Proximal-gradient method for $l_1$/nuclear norm minimization has a local linear convergence in a neighborhood of the optimal value [13,57,15]. The proximal operator for nuclear norm is soft-thresholding operator on singular values. Several authors have proposed algorithms for low rank matrix recovery and matrix completion problem based on soft- or hard-thresholding operators; see, e.g., [14,13,20,19]. The singular value projection algorithm proposed by Jain et al. has a linear rate; however, to apply the hard-thresholding operator, one should know the rank of $x_0$. While the authors have introduced a heuristic for estimating the rank when it is not known a priori, their convergence results rely upon a known rank [14].
Continuation over $\lambda$ for solving the regularized problem has been utilized in fixed point continuation algorithm (FPC) proposed by Ma et al. [19] and accelerated proximal-gradient algorithm (APGL) by Toh et al. [32]. FPC and APGL both solve a series of regularized problems where in each outer-iteration $\lambda$ is reduced by a factor less than one, the former uses soft-thresholding and the latter uses accelerated proximal-gradient for solving each regularized problem.

Agarwal et al. [1] have proposed algorithms for solving problems (1) and (2) with an extra constraint in the form of $\|x\| \leq \rho$. They have introduced the assumption of decomposability of the norm and given convergence analysis for norms that satisfy that assumption. They establish linear rate of convergence for their algorithms up to a neighborhood of the optimal solutions. However, their algorithm uses the bound $\rho$ which should be selected based on the norm of the true solution. In many problems this quantity is not known beforehand. Jin et al. [15] have proposed an algorithm for $l_1$ regularized least squares that receives $\rho$ as a parameter and has linear rate of convergence. Their algorithm utilizes proximal gradient method but unlike homotopy algorithm reduces $\lambda$ at each step.

By using SDP formulation of nuclear norm, interior point methods can be utilized to solve problems (1) and (2). Interior point methods do not scale as well as first order methods for large scale problems (For example, for a general SDP solver when the dimension exceeds a few hundreds). However, Specialized SDP solvers for nuclear norm minimization can bring down the computational complexity of each iteration to $O(n^3)$ [16].

2 Preliminaries

Let $A \in \mathbb{R}^{m \times n}$. Fix an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$, which is given by $\langle x, y \rangle = x^T B y$ for some positive definite matrix $B$. We equip $\mathbb{R}^m$ with ordinary dot product $\langle v, u \rangle = v^T u$. We denote the adjoint of $A$ with $A^* = B^{-1} A^T$. Note that for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$

$$\langle A x, u \rangle = \langle x, A^* u \rangle. \quad (3)$$

We use $\|\cdot\|_2$ to denote the norms induced by the inner product in $\mathbb{R}^n$ and $\mathbb{R}^m$, that is:

$$\forall x \in \mathbb{R}^n : \|x\|_2 = \sqrt{x^T B x},$$

$$\forall v \in \mathbb{R}^m : \|v\|_2 = \sqrt{v^T v}.$$

We use $\|\cdot\|$ and $\|\cdot\|^*$ to denote a regularizing norm and its dual on $\mathbb{R}^n$. The latter is defined as:

$$\|y\|^* = \sup \{ \langle y, x \rangle | \|x\| \leq 1 \}.$$

Given a convex function $f : \mathbb{R}^n \mapsto \mathbb{R}$, $\partial f (x)$ denotes the set of subgradients of $f$ at $x$, i.e., the set of all $z \in \mathbb{R}^n$ such that

$$\forall y \in \mathbb{R}^n : f(y) \geq f(x) + \langle z, y - x \rangle.$$

When $f$ is differentiable, $\partial f (x) = \{ \nabla f(x) \}$. Note that $\xi \in \partial \|x\|$ if and only if

$$\langle \xi, x \rangle = \|x\|,$$

$$\|\xi\|^* \leq 1.$$

We say $f$ is strongly convex with strong convexity parameter $\mu_f$ when $f(x) - \frac{\mu_f}{2} \|x\|^2_2$ is convex. For a differentiable function this implies that for all $x, y \in \mathbb{R}^n$:

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu_f}{2} \|x - y\|^2_2. \quad (4)$$

We call the gradient of a differentiable function Lipschitz continuous with Lipschitz constant $L_f$, when for all $x, y \in \mathbb{R}^n$: 

...
\[\|\nabla f(x) - \nabla f(y)\|_2 \leq L_f \|y - x\|_2.\] \hfill (5)

For a convex function \(f\), gradient Lipschitz continuity is equivalent to the following inequality [see [20] Lemma 1.2.3. and Theorem 2.1.5]:

\[f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L_f}{2} \|x - y\|_2^2,\] \hfill (6)

for all \(x, y \in \mathbb{R}^n\).

### 3 Properties of the regularizing norm and \(A\)

In this section we introduce our assumptions on the regularizing norm \(||-||\), and derive the properties of the norm based on these assumptions. We will utilize these properties in the analysis of homotopy algorithm. Before stating our assumptions, we add some more definitions to our tool box. Let \(\norm{\cdot}\) be the set of extreme points of the norm ball \(B_{\|\cdot\|} := \{x : \|x\| \leq 1\}\). We assume that \(\mathcal{G}_{\|\cdot\|} \subset S^{n-1}\). The first condition on the norm is the decomposability condition introduced in [5] which was inspired by the assumption introduced in [23].

**Condition 1 (Decomposability)** For all \(x \in \mathbb{R}^n\) there exists a subspace \(T_x\) and a vector \(e_x \in T_x\) such that:

\[\partial \|x\| = \{e_x + v|v \in T_x^\perp, \|v\|^* \leq 1\}.\] \hfill (7)

Note that \(x \in T_x\) for all \(x \in \mathbb{R}^n\) because if \(x \notin T_x\), then \(x = y + z\) with \(y \in T_x\) and \(z \in T_x^\perp - \{0\}\). Let \(z' = z/\|z\|^*\). Since \(e_x + z' \notin \partial \|x\|, \|x\| = \|e_x + z' + y + z\| = \|x\| + \|z\|/\|z\|^*\), which is a contradiction.

The next theorem captures our main results for decomposable norms.

**Theorem 1 (Orthogonal representation)** Suppose \(\mathcal{G}_{\|\cdot\|} \subset S^{n-1}\), then \(||\cdot||\) is decomposable if and only if for any \(x \in \mathbb{R}^n - \{0\}\) and \(a_1 \in \text{argmax}_{a \in \mathcal{G}_{\|\cdot\|}} \langle a, x \rangle\) there exist \(a_2, \ldots, a_k \in \mathcal{G}_{\|\cdot\|}\) such that \(\{a_1, a_2, \ldots, a_k\}\) is an orthogonal set that satisfies the following conditions:

1. There exists \(\{\gamma_i > 0 | i = 1, \ldots, k\}\) such that:
   \[x = \sum_{i=1}^{k} \gamma_i a_i,\] \hfill (8)
   \[\|x\| = \sum_{i=1}^{k} \gamma_i.\] \hfill (9)

2. For any set \(\{\eta_i | \eta_i \leq 1, i = 1, \ldots, l\}\):
   \[\left\| \sum_{i=1}^{k} \eta_i a_i \right\|^* \leq 1.\] \hfill (10)

The proof of Theorem 1 is presented in Appendix B. A byproduct of the proof of Theorem 1 is that \(\|\sum_{i=1}^{k} a_i\| = k\) and

\[e_x = \sum_{i=1}^{k} a_i.\]

We define the function \(k : \mathbb{R}^n \mapsto \{0, 1, 2, \ldots, n\}\) as:

\[k(x) = \|e_x\|^2_2.\]

Note that for every \(x \in \mathbb{R}^n\):
\[ \|x\|^2 = \langle e_x, x \rangle^2 \leq \|e_x\|^2_2 \|x\|^2_2 = k(x)\|x\|^2_2. \] (11)

In the analysis of homotopy algorithm we utilize (11) alongside the structure of the subgradient given by \[7\].

\( l_1, l_{1,2}, \) and nuclear norms are three important examples that satisfy conditions \[1\]. Here we briefly discuss each one of these norms.

- **Nuclear norm** on \( \mathbb{R}^{d_1 \times d_2} \) is defined as:

\[ \|X\|_* = \min \{d_1, d_2\} \sum_{i=1} \sigma_i(X) \]

where \( \sigma_i(X) \) is the \( i \)th largest singularvalue of \( X \) given by the singularvalue decomposition \( X = \sum_{i=1}^{\min\{d_1, d_2\}} \sigma_i(X) u_i v_i^T \). With the trace inner product \( \langle X, Y \rangle = \text{trace}(X^T Y) \), nuclear norm satisfies condition \[1\]. In this case, \( k(X) = \text{rank}(X), \sigma_i(X) \) and \( a_i = u_i v_i^T \) for \( i \in \{1, 2, \ldots, \text{rank}(X)\} \).

- **Weighted \( l_1 \) norm** on \( \mathbb{R}^n \) is defined as:

\[ \|x\|_1 = \sum_{i=1}^n w_i |x_i| \]

where \( w \) is a vector of positive weights. With \( \langle x, y \rangle = \sum_{i=1}^n w_i^2 x_i y_i \), \( l_1 \) norm satisfies condition \[1\]. For \( l_1 \) norm, \( k(x) = |\{i|x_i| \neq 0\}| \), \( \{\gamma_1, \gamma_2, \ldots, \gamma_k\} = \{w_i |x_i| |x_i| > 0, i = 1, \ldots, n\} \).

- **\( l_{1,2} \) norm** \( \mathbb{R}^{d_1 \times d_2} \): For a given inner product \( \langle \cdot, \cdot \rangle : \mathbb{R}^{d_1} \times \mathbb{R}^{d_1} \rightarrow \mathbb{R} \) and its induced norm \( \|\cdot\|_2 \) on \( \mathbb{R}^{d_1} \), We define:

\[ \|X\|_{1,2} = \sum_{i=1}^{d_2} \|X_i\|_2, \]

where \( X_i \) denotes the \( i \)th column of \( X \). With inner product \( \langle X, Y \rangle = \sum_{i=1}^{d_2} \langle X_i, Y_i \rangle \), \( l_{1,2} \) norm satisfies condition \[1\]. For this norm, \( k(X) = |\{i|X_i| \neq 0\}| \) and \( \{\gamma_1, \gamma_2, \ldots, \gamma_k\} = \{|X_i|_2 \ | |X_i|_2 > 0, i = 1, \ldots, d_2\} \).

We introduce a further condition on the norm which is satisfied for the previous examples:

**Condition 2** For all \( x, y \in \mathbb{R}^n \)

\[ k(x + y) \leq k(x) + k(y). \] (12)

Condition \[2\] for \( l_1, l_{1,2} \) and nuclear norm is equivalent to sublinearity of cardinality of vectors, number of non-zero columns and rank of matrices.

3.1 Properties of \( A \)

Restricted Isometry Property was first discussed in \[6\] for sparse vectors. Generalization of that concept to low rank matrices was introduced in \[28\]. Note that if \( k(x) \leq k \), then \( \|x\| \leq \sqrt{k}\|x\|_2 \). Based on this observation we define restricted isometry constants of \( A \in \mathbb{R}^{m \times n} \) as:

**Definition 1** The upper (lower) restricted isometry constant \( \rho_+ (A, k) (\rho_- (A, k)) \) of a matrix \( A \in \mathbb{R}^{m \times n} \) is the smallest (largest) positive constant that satisfies this inequality:

\[ \rho_- (A, k) \|x\|^2_2 \leq \|Ax\|^2_2 \leq \rho_+ (A, k) \|x\|^2_2, \]

whenever \( \|x\|^2 \leq k\|x\|^2_2 \).
Proposition 1 Let $A \in \mathbb{R}^{m \times n}$ and $f(x) = \frac{1}{2} \|Ax - b\|_2^2$. Suppose that $\rho_+(A,k)$ and $\rho_-(A,k)$ are restricted isometry constants corresponding to $A$, then:

\begin{align*}
    f(y) & \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2} \rho_- (A,k) \|x - y\|_2^2, \\
    f(y) & \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2} \rho_+ (A,k) \|x - y\|_2^2,
\end{align*}

for all $x, y \in \mathbb{R}^n$ such that $\|x - y\|^2 \leq k \|x - y\|_2^2$.

Proposition 1 follows from the definition of restricted isometry constants and the following equality:

$$\frac{1}{2} \|A(x - y)\|_2^2 = f(y) - f(x) - \langle \nabla f(x), y - x \rangle.$$ 

4 Proximal-gradient method and homotopy algorithm

We state the proximal-gradient method and the homotopy algorithm for the following optimization problem:

$$\text{minimize } \phi_\lambda(x) = f(x) + \lambda \|x\|,$$

where $f(x) = \frac{1}{2} \|Ax - b\|_2^2$. While, for simplicity, we analyze the homotopy algorithm for the least squares loss function, the analysis can be extended to every function of form $f(x) = g(Ax)$ when $g$ is a differentiable strongly convex function with Lipschitz continuous gradient. The key element in the proximal-gradient method is the proximal operator which was developed by Moreau [22] and later extended to maximal monotone operators by Rockafellar [29]. Nesterov has proposed several variants of the proximal-gradient methods [23]. In this section, we discuss the gradient method with adaptive line search. For any $x, y \in \mathbb{R}^n$ and positive $L$, we define:

$$m_{\lambda,L}(y, x) = f(y) + \langle \nabla f(y), x - y \rangle + \frac{L}{2} \|x - y\|^2 + \lambda \|x\|,$$

$$\text{Prox}_{\lambda,L}(y) = \arg\min_{x \in \mathbb{R}^n} m_{\lambda,L}(y, x)$$

$$\omega_\lambda(x) = \min_{\xi \in \partial \|x\|} \|\lambda \xi + \nabla f(x)\|.$$ 

Xiao and Zhang [30] have considered the proximal-gradient homotopy algorithm for $l_1$ norm. Here we state it for general norms. Algorithm 1, introduces the homotopy algorithm and contains the proximal-gradient method as a subroutine.

The homotopy algorithm reduces the value of $\lambda$ in a series of steps and in each step applies the proximal-gradient method. At step $t$, $\lambda_t = \lambda_{t-1} \eta^t$ and $\epsilon_t = \delta^t \lambda_t$ with $\eta \in (0, 1)$ and $\delta \in (0, 1)$. In the proximal-gradient method and the backtracking subroutine, the parameters $\gamma_{\text{dec}} \geq 1$ and $\gamma_{\text{inc}} > 1$ should be initialized. Since the function $f$ satisfies the inequality (5), it is clear that $L_{\text{min}}$ should be chosen less than $L_f$.

Theorem 5 in [24] states that the proximal-gradient method has a linear rate of convergence when $f$ satisfies (4) and (6) on a restricted set. The proof of this proposition is given in appendix B.

Proposition 2 Let $x^* \in \text{argmin} \, \phi_\lambda$. If for every $t$:

\begin{align*}
    f(x^{(t)}) & \geq f(x^*) + \langle \nabla f(x^*), x^{(t)} - x^* \rangle + \frac{\mu f}{2} \|x^{(t)} - x^*\|^2, \\
    f(x^{(t+1)}) & \geq f(x^{(t)}) + \langle \nabla f(x^{(t)}), x^{(t+1)} - x^{(t)} \rangle + \frac{\mu f}{2} \|x^{(t)} - x^{(t+1)}\|^2, \\
    f(x^{(t+1)}) & \leq f(x^{(t)}) + \langle \nabla f(x^{(t)}), x^{(t+1)} - x^{(t)} \rangle + \frac{L f}{2} \|x^{(t)} - x^{(t+1)}\|^2,
\end{align*}

then

$$\phi_\lambda(x^{(t)}) - \phi_\lambda(x^*) \leq \left(1 - \frac{\mu f \gamma_{\text{inc}}}{4L_f}\right)^t \left(\phi_\lambda(x^{(0)}) - \phi_\lambda(x^*)\right).$$
Suppose the regularizing norm satisfies conditions 1 and 2 introduced in section 3. Before we state the convergence theorem, we introduce an assumption:

**Assumption 1** \( \lambda_{\text{tgt}} \) is such that \( \| A^* z \|^* \leq \frac{\lambda_{\text{tgt}}}{4} \). Furthermore, there exist constants \( k \) and \( \delta \in [0, \frac{1}{4}] \) such that:

\[
\frac{\tilde{k}}{\rho_+ \left( A, 2\tilde{k} \right)} > 36 c k_0 (1 + \gamma) \gamma_{\text{inc}} \rho_- \left( A, c k_0 (1 + \gamma)^2 \right) \tag{22}
\]

\[
L_{\min} \leq \gamma_{\text{inc}} \rho_+ \left( A, 2\tilde{k} \right), \tag{23}
\]

In addition, if

\[
\left\| \nabla f \left( x^{(t)} \right) - \nabla f \left( x^{(t+1)} \right) \right\|^* \leq L'_f \left\| x^{(t)} - x^{(t+1)} \right\|_2 \tag{19}
\]

and

\[
\left\| x^{(t)} - x^{(t+1)} \right\|^* \leq \theta \left\| x^{(t)} - x^{(t+1)} \right\|_2 \tag{20}
\]

for some constants \( \theta \) and \( L'_f \), then

\[
\omega_{\lambda} \left( x^{(t)} \right) \leq \theta \left( 1 + \frac{L'_f}{\rho_f} \right) \sqrt{2 \gamma_{\text{inc}} L_f \left( \phi_{\lambda} \left( x^{(t)} \right) - \phi_{\lambda} \left( x^* \right) \right)} \tag{21}
\]

5 Convergence result

Suppose \( b = Ax_0 + z \), for some \( x_0 \in \mathbb{R}^n \) and \( z \in \mathbb{R}^m \). Also, we define \( k_0 := k(x_0) \) and the constant \( c \):

\[
c := \max_{x \in T_{x_0 - (0)}} \frac{\|x\|^2}{k_0 \|x\|^2}. \]

Note that \( c = 1 \) for \( l_1 \) and \( l_{1,2} \) norms, and \( c = 2 \) for nuclear norm. Through out this section, we assume the regularizing norm satisfies conditions [1] and [2] introduced in section [3] Before we state the convergence theorem, we introduce an assumption:

**Assumption 1** \( \lambda_{\text{tgt}} \) is such that \( \| A^* z \|^* \leq \frac{\lambda_{\text{tgt}}}{4} \). Furthermore, there exist constants \( k \) and \( \delta \in [0, \frac{1}{4}] \) such that:

\[
\frac{\tilde{k}}{\rho_+ \left( A, 2\tilde{k} \right)} > 36 c k_0 (1 + \gamma) \gamma_{\text{inc}} \rho_- \left( A, c k_0 (1 + \gamma)^2 \right) \tag{22}
\]

\[
L_{\min} \leq \gamma_{\text{inc}} \rho_+ \left( A, 2\tilde{k} \right), \tag{23}
\]
where:

\[
\gamma := \frac{\lambda_{\text{tgt}} (1 + \delta) + \|A^* z\|}{\lambda_{\text{tgt}} (1 - \delta) - \|A^* z\|}.
\]  

(24)

The assumption on \( L_{\text{min}} \) is only for convenience. If \( L_{\text{min}} > \gamma_{\text{inc}} \rho_+ (A, 2 \tilde{k}) \), one can replace \( L_{\text{min}} \) in the analysis. In appendix A, we provide an upper bound on the number of measurement needed for (22) to be satisfied with high probability whenever rows of \( A \) are sub-Gaussian random vectors.

The next theorem establishes the linear convergence of the proximal gradient method when \( \omega_{\lambda} (x^{(0)}) = \min_{\xi \in \partial \|x\|} \|\nabla f (x) + \lambda \xi\| \) is sufficiently small, while Theorem 3 establishes the overall linear rate of convergence of homotopy algorithm.

\textbf{Theorem 2} Let \( x^{(t)} \) denote the \( t \)th iterate of ProxGrad\( \phi_{\lambda} (x^{(0)}, L_0, L_{\text{min}}, \epsilon) \), and let \( x^* \in \arg\min \phi_{\lambda} (x) \). Suppose Assumption \( \mathbf{1} \) holds true for some \( \delta \) and \( \tilde{k} \), and \( \lambda \geq \lambda_{\text{tgt}} \). If \( x^{(0)} \) satisfies:

\[
k (x^{(0)}) \leq \tilde{k}, \quad \omega_{\lambda} (x^{(0)}) \leq \delta \lambda,
\]

then:

\[
k (x^{(t)}) \leq \tilde{k},
\]

(25)

\[
\phi_{\lambda} (x^{(t)}) - \phi_{\lambda} (x^*) \leq \left( 1 - \frac{1}{4 \gamma_{\text{inc}} \kappa} \right)^t \left( \phi_{\lambda} (x^{(0)}) - \phi_{\lambda} (x^*) \right),
\]

(26)

and

\[
\omega (x^{(t)}) \leq \left( 1 + \frac{\sqrt{\rho_+ (A, 1) \rho_+ (A, 2 \tilde{k})}}{\rho_- (A, 2 \tilde{k})} \frac{1}{\sqrt{2 \gamma_{\text{inc}} \rho_+ (A, 2 \tilde{k})}} \right) \left( \phi_{\lambda} (x^{(t)}) - \phi_{\lambda} (x^*) \right),
\]  

(27)

where \( \kappa = \frac{\rho_+ (A, 2 \tilde{k})}{\rho_- (A, 2 \tilde{k})} \).

\textbf{Theorem 3} Let \( y^{(t)} \) denote the \( t \)th iterate of Homotopy algorithm, and let \( y^* \in \arg\min \phi_{\lambda_{\text{tgt}}} (y) \). Suppose Assumption \( \mathbf{1} \) holds true for some \( \delta \) and \( \tilde{k} \). Furthermore, suppose that \( \delta' \) and \( \eta \) in the algorithm satisfy:

\[
\frac{1 + \delta'}{1 + \delta} \leq \eta.
\]  

(28)

If \( y^{(0)} \) satisfies:

\[
k (y^{(0)}) \leq \tilde{k}, \quad \omega_{\lambda_0} \leq \delta \lambda_0,
\]

then when \( t = 1, \ldots, N \), the number of proximal-gradient iterations for computing \( y^{(t)} \) is bounded by

\[
\log \left( \frac{C/\delta^2}{\log \left( 1 - \frac{1}{4 \gamma_{\text{inc}} \kappa} \right)} \right)^{1-t},
\]  

(29)

The number of proximal-gradient iterations for computing \( y \) is bounded by

\[
\log \left( \frac{C \lambda_{\text{tgt}}/\epsilon^2}{\log \left( 1 - \frac{1}{4 \gamma_{\text{inc}} \kappa} \right)} \right)^{1-t},
\]  

(30)
where \( C := 6\gamma_{inc}k\delta c k_0 (1 + \gamma) \left( \sqrt{\rho_-(A, 2\tilde{k})} + \sqrt{\rho_+(A, 1)\kappa} \right)^2 / \rho_-(A, c (1 + \gamma)^2 k_0) \) and \( \kappa = \rho_+(A, 2\tilde{k}) / \rho_-(A, 2\tilde{k}) \).

The objective gap of the output \( y \) is bounded by

\[
\phi_{\lambda_{tgt}}(y) - \phi_{\lambda_{tgt}}(y^*) \leq \frac{9ck_0\lambda_{tgt} (1 + \gamma) \epsilon}{\rho_-(A, c (1 + \gamma)^2 k_0)}.
\]

while the total number of iterations for computing \( y \) is bounded by:

\[
\log \left( \frac{CA_{tgt}/c^2}{\rho_-(A, c (1 + \gamma)^2 k_0)} \right) \leq \frac{\log \left( \frac{\lambda_{inc}}{\delta} \right)}{\log (\eta)} \log \left( \frac{C/\delta^2}{\gamma} \right).
\]

The main part of the proof of Theorems 2 and 3 is establishing the fact that \( k(x^{(t)}) \leq \tilde{k} \). Given that \( k(x^{(t)}) \leq \tilde{k} \) for all \( t \), Proposition 1 ensures that hypothesis of Proposition 2, i.e., strong convexity and gradient Lipschitz continuity over a restricted set, are satisfied. We adapt the same strategy as in [36] and prove that \( k(x^{(t)}) \leq \tilde{k} \) in a series of three lemmas. We have written the statement of the lemmas here, while their proofs are given in Appendix B. Lemma 1 states that if \( \omega(x) \) does not exceed a small fraction of \( \lambda \), then \( x \) is close to \( x_0 \).

**Lemma 1** If \( \omega(x) \leq \delta \lambda \) and \( \rho_-(A, c (1 + \gamma)^2 k_0) > 0 \) then:

\[
\max \left\{ \|x - x_0\|, \frac{1}{\delta \lambda} (\phi_{\lambda}(x) - \phi_{\lambda}(x_0)) \right\} \leq \frac{ck_0 (1 + \gamma) \|A^*z\|}{\rho_-(A, c (1 + \gamma)^2 k_0)}.
\]

Note that if \( \lambda \geq 4\|A^*z\| \) and \( \delta \leq \frac{1}{2} \), we can simplify the conclusion of Lemma 1 as

\[
\max \left\{ \|x - x_0\|, \frac{1}{\delta \lambda} (\phi_{\lambda}(x) - \phi_{\lambda}(x_0)) \right\} \leq \frac{3ck_0\lambda (1 + \gamma)}{2\rho_-(A, c (1 + \gamma)^2 k_0)}
\]

While the hypotheses of this lemma is true in the first step of every outer iteration of homotopy algorithm, \( \omega(x^{(t)}) \) may not be decreasing in proximal-gradient algorithm. However, the objective decreases after every iteration of the proximal-gradient algorithm. Thus to conclude that \( x^{(t)} \) is close to \( x_0 \) in all the inner proximal-gradient steps we can use the following lemma:

**Lemma 2** Suppose Assumption 1 holds true, and \( \lambda \geq \lambda_{tgt} \). If

\[
\phi_{\lambda}(x) - \phi_{\lambda}(x_0) \leq \frac{3ck_0\delta \lambda^2 (1 + \gamma)}{2\rho_-(A, c (1 + \gamma)^2 k_0)},
\]

then

\[
\max \left\{ \frac{1}{2\lambda} \|A(x - x_0)\|^2, \|x - x_0\| \right\} \leq \frac{9ck_0\lambda (1 + \gamma)}{2\rho_-(A, c (1 + \gamma)^2 k_0)}.
\]

The proofs of Lemma 1 and Lemma 2 generalize the proofs of the corresponding lemmas in [36] given for \( l_1 \) norm to norms the satisfy condition 1 using the structure of \( \partial\|x_0\| \) given by 7. The last lemma provides an upper bound on \( k(x^*) \), where \( x^* \) is produced via a proximal-gradient step on \( x \), as long as \( x \) satisfies the conclusion of Lemma 2 and Assumption 1 holds. The proof of Lemma 3 uses a slightly different approach than the one given in [36] resulting in a simpler requirement on \( \tilde{k} \) in Assumption 1.

**Lemma 3** Let \( x^* = \text{Prox}_{\lambda, L}(x) \) and suppose Assumption 1 holds true, and \( \lambda \geq \lambda_{tgt} \). If \( L \leq \gamma_{inc}\rho_+(A, 2\tilde{k}) \) and

\[
\max \left\{ \frac{1}{2\lambda} \|A(x - x_0)\|^2, \|x - x_0\| \right\} \leq \frac{9ck_0\lambda (1 + \gamma)}{2\rho_-(A, c (1 + \gamma)^2 k_0)},
\]

then \( k(x^*) \leq \tilde{k} \).
5.1 Proof of Theorem 2

First we show that \( L_t \leq \gamma_{\text{inc}} \rho_+ \left( A, 2\tilde{k} \right) \) and \( k \left( x^{(t)} \right) \leq \tilde{k} \) for all \( k \geq 0 \). The inequalities hold true for \( t = 0 \) by the hypothesis. Suppose \( L_t \leq \gamma_{\text{inc}} \rho_+ \left( A, \tilde{k} \right) \) and \( k \left( x^{(t)} \right) \leq \tilde{k} \) for some \( t > 0 \). Since \( \phi_{\lambda} \left( x^{(t)} \right) \leq \phi_{\lambda} \left( x^{(0)} \right) \), by Lemma 1 we have:

\[
\max \left\{ \frac{1}{2\rho} \left\| A \left( x^{(t)} - x_0 \right) \right\|^2, \left\| x^{(t)} - x_0 \right\| \right\} \leq \frac{9c_k \lambda (1 + \gamma)}{2 \rho - \left( A, c \left( 1 + \gamma \right)^2 k_0 \right)}.
\]

By Lemma 2, Lemma 3 and Condition 2 for any \( L \leq \gamma_{\text{inc}} \rho_+ \left( A, 2\tilde{k} \right) \)

\[
k \left( \text{Prox}_{\lambda,L} \left( x^{(t)} \right) \right) \leq \tilde{k},
\]

\[
k \left( \text{Prox}_{\lambda,L} \left( x^{(t)} \right) - x^{(t)} \right) \leq 2\tilde{k}.
\]

Now we can use Proposition 1 to conclude that \( M_{t+1} \leq \gamma_{\text{inc}} \rho_+ \left( A, 2\tilde{k} \right) \) hence \( L_{t+1} \leq M_{t+1}/\gamma_{\text{dec}} \leq \gamma_{\text{inc}} \rho_+ \left( A, 2\tilde{k} \right) \). In addition, by Lemma 4 \( k \left( x^{(t+1)} \right) = k \left( \text{Prox}_{\lambda,M_{t+1}} \left( x^{(t)} \right) \right) \leq \tilde{k} \).

Since \( \text{Prox}_{\lambda,L} \left( x^* \right) = x^* \) for any \( L > 0 \), by Lemmas 1, 2 and 3 \( k \left( x^* \right) \leq \tilde{k} \). By Condition 2 we have:

\[
k \left( x^{(t+1)} - x^{(t)} \right) \leq 2\tilde{k}, \ k \left( x^{(t)} - x^* \right) \leq 2\tilde{k},
\]

which yields

\[
\left\| A^* A \left( x^{(t+1)} - x^{(t)} \right) \right\|^* = \max_{a \in \mathcal{A}} \left\langle a, A^* A \left( x^{(t+1)} - x^{(t)} \right) \right\rangle = \max_{a \in \mathcal{A}} \left\langle A a, A \left( x^{(t+1)} - x^{(t)} \right) \right\rangle \leq \sqrt{\rho_+ \left( A, 1 \right) \rho_+ \left( A, 2\tilde{k} \right)} \left\| x^{(t+1)} - x^{(t)} \right\|_2. \tag{32}
\]

Now Proposition 1 and 32 ensure that all the hypotheses of Theorem 2 are satisfied with \( \mu_f = \rho_- \left( A, 2\tilde{k} \right), \ L_f = \rho_+ \left( A, 2\tilde{k} \right), \ L'_f = \sqrt{\rho_+ \left( A, 1 \right) \rho_+ \left( A, 2\tilde{k} \right)} \) and \( \theta = 1 \). Thus the conclusion follows from Theorem 2.

5.2 Proof of Theorem 3

Let \( y^*_t \in \text{argmin} \phi_{\lambda_t} \left( y \right) \). For the ease of notation let \( \lambda_{\text{tgt}} \leftrightarrow \lambda_{\text{tgt}} \). First we show that \( \omega_{\lambda_t} \left( y^{(t)} \right) \leq \delta \lambda_t \) and \( k \left( y^{(t)} \right) \leq \tilde{k} \) for \( t = 0, 1, \ldots, N \). When \( t = 0 \) the claims follow from the hypothesis. Suppose \( \omega_{\lambda_{t-1}} \left( y^{(t-1)} \right) \leq \delta \lambda_{t-1} \) and \( k \left( y^{(t-1)} \right) \leq \tilde{k} \). By Theorem 2 we have:

\[
k \left( y^{(t)} \right) \leq \tilde{k}.
\]

Also, the stopping condition in the proximal gradient algorithm forces \( \omega_{\lambda_{t-1}} \left( y^{(t)} \right) \leq \delta' \lambda_{t-1} \). Therefore, by hypothesis 25 we get:

\[
\omega_{\lambda_t} \left( y^{(t)} \right) \leq \left\| A^* \left( A y^{(t)} - b \right) + \lambda_t \xi \right\|^* 
\leq \left\| A^* \left( A y^{(t)} - b \right) + \lambda_{t-1} \xi \right\|^* + \left\| \left( \lambda_t - \lambda_{t-1} \right) \xi \right\|^* 
\leq \omega_{\lambda_{t-1}} \left( x^t \right) + \left( \lambda_{t-1} - \lambda_t \right) \leq ( -1 + (\delta' + 1)/\eta ) \lambda_t \leq \delta \lambda_t.
\]
By Lemma 1 for all \( t = 0, \ldots, N \), we have

\[
\| y^{(t)} - y^*_t \| \leq \| y^{(t)} - x_0 \| + \| y^*_t - x_0 \|
\leq \frac{c k_0 (1 + \gamma) \left( (2 + \delta) \lambda_t + 2 \| A^* z^* \| \right)}{\rho_-(A, c (1 + \gamma)^2 k_0)}
\leq \frac{3ck_0 (1 + \gamma) \lambda_t}{\rho_-(A, c (1 + \gamma)^2 k_0)}.
\]

Hence

\[
\phi_{\lambda_t} \left( y^{(t)} \right) - \phi_{\lambda_t} \left( y^*_t \right) \leq \omega_{\lambda_t} \left( y^{(t)} \right) \| y^{(t)} - y^*_t \|
\leq \frac{3\delta ck_0 (1 + \gamma) \lambda_t^2}{\rho_-(A, c (1 + \gamma)^2 k_0)}.
\]

Now the upper bounds in (29) and (30) on the number of inner iterations follow from the second conclusion in Theorem 2.

By (54), we have

\[
\| y - y^* \| \leq \| y - y_0 \| + \| y_0 - y^* \| \leq \frac{9ck_0 \lambda_{tgt} (1 + \gamma)}{\rho_-(A, c (1 + \gamma)^2 k_0)}.
\]

By convexity of \( \phi_{\lambda_{tgt}} \), we get:

\[
\phi_{\lambda_{tgt}} (y) - \phi_{\lambda_{tgt}} (y^*) \leq \langle \omega_{\lambda_{tgt}} (y), y - y^* \rangle
\leq \frac{9ck_0 \lambda_{tgt} (1 + \gamma) \epsilon}{\rho_-(A, c (1 + \gamma)^2 k_0)}.
\]

### 6 Numerical Experiments

We consider two problems. The details of each problem are summarized in the following table:

| Problem 1 | Problem 2 |
|-----------|-----------|
| Objective | \( \frac{1}{2} \| A \text{vec}(X) + b \|_2^2 + \lambda \| X \|_* \) | \( \frac{1}{2} \| A \text{vec}(X) + b \|_2^2 + \lambda \| X \|_{1,2} \) |
| Dimension of \( X_0 \) | \( 100 \times 100 \) | \( 50 \times 1000 \) |
| \( k(X_0) \) | \( \text{rank}(X_0) = 10 \) | \( \# \text{non-zero columns } X_0 = 50 \) |
| \# of samples | \( m = 4000 \) | \( m = 15000 \) |
| \( b \) | \( A \text{vec}(X_0) + z \) | \( A \text{vec}(X_0) + z \) |
| \( A_{i,j} \text{ sampled from } N(0, 1/\sqrt{m}) \) | \( \{-1/\sqrt{m}, 1/\sqrt{m}\} \) uniformly at rand. |
| \( z_i \text{ sampled from } U(-0.005, 0.005) \) | \( U(-0.005, 0.005) \) |

In the homotopy algorithms, \( \lambda_0 = 5 \| A^T b \|_* \) and \( \lambda_{tgt} = 4 \| A^T z \|_* \), and in the proximal-gradient algorithm \( \lambda = \lambda_{tgt} \).

**Problem 1.** Figure 1 demonstrates the overall linear rate of convergence of homotopy algorithm applied to this problem and compares it with proximal-gradient algorithm. As rank vs. iteration plot demonstrates, the proximal-gradient algorithm speeds up to a linear rate when the rank drops to a certain level, while the homotopy algorithm keeps the rank at a level that ensures a linear rate of convergence.

We examine the performance of homotopy algorithm with three different values of \( \eta \) and \( \delta' \) in Figure 2. For \( \eta \) to satisfy the condition of Theorem 3 it is necessary that \( \eta > 0.5 \). However, as Figure 2 demonstrates,
one can choose $\eta \leq 0.5$ and still get an overall linear rate of convergence. For example, when $\eta = 0.2$, at the beginning of the last stage where $\lambda = \lambda_{tgt}$, $X^{(k)}$ is not low-rank and the algorithm has a sublinear rate of convergence, but nevertheless the algorithm converges faster with $\eta = 0.2$ than $\eta = 0.7$. Homotopy algorithm appears to be even less sensitive to $\delta'$. As $\delta'$ gets closer to 1, the rank of $X^{(k)}$ jumps higher, which can cause a slowdown in convergence specially at the beginning of each stage.

In Figure 3a we have compared recovery error of the following algorithms: SVP, FPC, Proximal-gradient Homotopy, Proximal-gradient and APGL. In SVP we provide the algorithm with the rank of $X_0$, while in SVP2 we use the same heuristic that is proposed in [14] to estimate the rank. We have implemented the FPC algorithm with the backtracking procedure which improves the performance of the algorithm. Both APGL and APGL2 have been implemented with continuation over $\lambda$ with the latter utilizing an extra truncation heuristic proposed in [22]. The method of continuation for APGL is the same as the one proposed in [22]; we reduce $\lambda$ by a factor of 0.7 after three iterations or whenever the stopping criterion is met whichever comes first. In FPC and APGL similar to the homotopy algorithms, $\lambda_0 = 5\|A^T(b)\|^2$ and $\lambda_{tgt} = 4\|A^T(z)\|^2$. In homotopy algorithm $\eta = 0.6$ and $\delta' = 0.2$. In the rest of the algorithms we have used the default values of the parameters. Note that APGL2 has an extra truncation procedure which improves the recovery error. Finally, Figure 3b shows the objective gap for the algorithms for the which the quantity is meaningful.

**Problem 2.** Figure 4 demonstrate the linear convergence of homotopy algorithm for this problem and compares the performance with that of proximal-gradient algorithm. Similar to problem 1, homotopy algorithm keeps the number of non-zero columns below a certain level. In homotopy algorithm $\delta' = 0.2$ and $\eta = 0.6$.

**Appendix A**

In this section we give a lower bound on the number of measurements $m$ that suffice for the existence of $\hat{k}$ in Assumption 1 with high probability when $A$ is sampled from a certain class of distributions. Given a random variable $Z$ the sub-Gaussian norm of $Z$ is defined as:

$$\|Z\|_{\psi_2} = \inf\{c > 0 | E\psi_2\left(\frac{|Z|}{c}\right) \leq 1\},$$

where $\psi_2(x) = e^{x^2} - 1$. For an $n$ dimensional random vector $W \sim P$ the sub-Gaussian norm is defined as

$$\|W\|_{\psi_2} = \sup_{u \in S^{n-1}} \|\langle W, u \rangle\|_{\psi_2}.$$  

$P$ is called isotropic if $E[(W, u)^2] = 1$ for all $u \in S^{n-1}$. Two important examples of sub-Gaussian random variables are Gaussian and bounded random variables. Suppose $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is given by:

$$(Ax)_i = \frac{1}{\sqrt{m}}(A_i, x) \; \forall i \in \{1, 2, \ldots, m\},$$  

(33)

where $A_i$, $1 \leq i \leq m$ are iid samples from an isotropic sub-Gaussian distribution $P$ on $\mathbb{R}^n$. Two important examples are $A_i = B^\top A_i'$ with $A_i' \sim \mathcal{N}(0, I_n)$ and $A_i, j$ Rademacher for all $j$. We want to bound the following probabilities for $\theta \in (0, 1)$:

$$P(\rho_-(A, k) < 1 - \theta) \quad \text{(34)}$$

$$P(\rho_+(A, k) > 1 + \theta). \quad \text{(35)}$$

When $A_i \sim \mathcal{N}(0, I_n)$ for all $i$, one can use the generalization of Slepian’s lemma by Gordon [10] alongside concentration inequalities for Lipschitz function of Gaussian random variable to derive:

$$P(\sqrt{\rho_-(A, k)} < \sqrt{\frac{m}{m+1} - \theta}) \leq e^{-\frac{m\theta^2}{2}},$$

$$P(\sqrt{\rho_+(A, k)} > 1 + \theta) \leq e^{-\frac{m\theta^2}{8}},$$

whenever,

$$\theta \geq \frac{2 \sqrt{\log \|\frac{m+1}{m} \cap S^{n-1}\}}{\sqrt{m}}.$$
Here, for any \( S \subset \mathbb{R}^n \), \( l_*(S) := E \sup_{u \in S} |\langle u, g \rangle| \), where \( g \sim N(0, I_n) \). For sub-Gaussian case, we use a result by Mendelson et al. [21, Theorem 2.3]. Using Talgand’s generic chaining theorem [31, Theorem 2.1.1], the authors have given a result, which similar to the Gaussian case depends on \( l_*(\sqrt{m}B_\| \cap S^{n-1}) \). Their result in our notation states:

**Proposition 3** Suppose \( A \) is given by (33). If \( P \) is an isotropic distribution and \( \| A_1 \|_{\psi_2} \leq \alpha \), then there exist constants \( c_1 \) and \( c_2 \) such that

\[
\begin{align*}
\rho_-(A/\sqrt{m}, k) &\geq 1 - \theta, \\
\rho_+(A/\sqrt{m}, k) &\leq 1 + \theta,
\end{align*}
\]

with probability exceeding \( 1 - \exp(-(c_2 \theta^2 m/\alpha^4)) \) whenever

\[
\theta \geq \frac{ca^2 l_*(\sqrt{m}B_\| \cap S^{n-1})}{\sqrt{m}}.
\]

Suppose \( \lambda_{\text{tgt}} = 4\|A^*z\|_\ast \), which sets \( \gamma = \frac{1 + \sqrt{5}}{4} \). We can state the following proposition based on Proposition 3:

**Proposition 4** Let \( r > 1, \tilde{k} = 36rk_0(1 + \gamma) \), and \( \hat{k} = c_k(1 + \gamma)^2 \). If \( m \geq \frac{\alpha^4}{(r - 1)^2} \left( l_*(\sqrt{m}B_\| \cap S^{n-1})^2 + r^2 l_*(\sqrt{m}B_\| \cap S^{n-1})^2 \right) \), then \( \hat{k} \) satisfies Assumption 7 with probability exceeding \( 1 - \exp(c_2 \theta^2 m/\alpha^2) \).

The proof is a simple adaptation of proof of theorem 1.4 in [21] which we omit here. To compare this with the number of measurements sufficient for successful recovery within a given accuracy, by combining (53) in the proof Lemma 4 and Proposition 3 we get:
Fig. 2: (a), (b): Performance of homotopy algorithm with $\delta' = 0.2$ and three different values of $\eta$, (c), (d): Performance of homotopy algorithm with $\eta = 0.6$ and three different values of $\delta'$

Fig. 3: Comparison between SVP, FPCA, homotopy, proximal-gradient and APGL
Decomposable Norm Minimization with Proximal-Gradient Homotopy Algorithm

(0x0) (0x0)

\[ \phi_k(X^{(k)}) - \phi^*_X \]

(105x556) 0 50 100 150 200 250 300 350

k

10 100 200 300 400 500 600 700 800 900 1000

k

10 100 200 300 400 500 600 700 800 900 1000

k

\[ \|X - X^0\|_2 \]

\[ \|X^0\|_2 \]

(a) Objective gap vs. iteration

(b) # non-zero columns vs. iteration

(c) Recovery error vs. iteration

Fig. 4: Comparison of homotopy and proximal-gradient algorithms for problem 2

Proposition 5 Let \( r > 1, k = ck_0(1+\gamma)^2 \) and \( x^* \in \arg\min \phi_X(x) \). If \( m \geq \frac{c_2\lambda_0^2}{(r-1)^2}l_s(\sqrt{k}B_{\|\|} \cap S^{n-1})^2 \), then \( \|x^* - x_0\|_2 \leq c_2 r\lambda \sqrt{ck_0} \) with probability exceeding \( 1 - \exp(c_2(r-1)^2 m/r^2 \alpha^2) \).

Note that this bound on \( m \) in case of \( l_1, l_1, 2 \) and nuclear norms orderwise matches the lower bounds given by minimax rates in [27], [17] and [30].

Appendix B

B.1 Proof of Theorem 1

Sufficiency. First consider the case where \( k = 1 \) and \( x = \gamma_1 a_1 \) with \( \gamma_1 > 0 \). Note that \( a_1 \in \partial\|x\| = \partial\|a_1\| \) because \( \|a_1\|^* = 1 \) for all \( a_1 \in G_{\|\|} \) and \( \langle a_1, x \rangle = \gamma_1 = \|x\|^* \). Define:

\[ C = \{ \xi - a_1 | \xi \in \partial\|a_1\| \} \]

\[ T_{a_1} = \text{Aff } C \]

where \( \text{Aff } C \) is the affine hall of \( C \). Note that \( C \) is a convex set that contains the origin; therefore, its affine hall is a subspace of \( \mathbb{R}^n \). Moreover, \( C \) is orthogonal to \( a_1 \). We will prove that \( C \) is symmetric and is contained in the dual norm ball. Let \( v \in C \) and \( \xi = a_1 + v \in \partial\|a_1\| \). Since \( \langle a_1, \xi \rangle = \|\xi\|^* = 1 \), by the hypothesis, \( \xi = a_1 + \sum_{i=1} b_i \) with \( \|v\|^* = \max_i b_i \leq 1 \). Let \( \xi' = a_1 - \sum_{i=1} b_i \). By the hypothesis, \( \|\xi'\|^* = \max\{1, \max_i b_i \} = 1 \). Also, \( \langle \xi', a_1 \rangle = 1 \) hence \( \xi' \in \partial\|a_1\| \) and \( -v \in C \).
Let $v \in T^+_a$ with $\|v\|* \leq 1$. Since $C$ is a symmetric convex set and $T^+_a = \text{Aff } C$, there exists $\lambda \in (0, 1]$ such that $\lambda v \in C$. Define $z = a_1 + \lambda v$ which is in $\partial \|a_1\|$. Since $\langle a_1, z \rangle = \|z\|^1 = 1$, we can write $z$ as

$$z = a_1 + \sum_{i=1}^{k'} \nu_i c_i,$$

where $\{a_i | i = 1, \ldots, k'\} \subset \mathcal{G}^\|\|$ and $\{\nu_i \geq 0 | i = 1, \ldots, k'\}$ satisfy the hypothesis of the theorem. In particular, since $v = 1/\lambda \sum_{i=1}^{k'} \nu_i c_i,$ we have $\max_i \nu_i / \lambda \leq 1$. Hence $\|a_1 + \nu v\| = \max\{1, \nu_1 / \lambda, \ldots, \nu_{k'} / \lambda\} = 1$ and $a_1 + v \in \partial \|a_1\|$. Therefore,

$$\partial \|a_1\| = \{a_1 + v | v \in T^+_a, \|v\|^* \leq 1\}.$$ 

Now suppose that $x = \sum_{i=1}^k \gamma_i a_i$ with $k > 1$. Note that $\sum_{i=1}^{k-1} a_i \in \partial \|x\|$ since $\sum_{i=1}^{k-1} a_i = 1$ and $\langle \sum_{i=1}^{k-1} a_i, x \rangle = \sum_{i=1}^k \gamma_i = \|x\|$. Let $\xi \in \partial \|x\|$ and define $v = \xi - \sum_{i=1}^k a_i$. We can write:

$$\|x\| = \sum_{i=1}^k \gamma_i = \langle \xi, x \rangle = \sum_{i=1}^k \gamma_i \langle \xi, a_i \rangle$$

$$\Rightarrow \forall i \in [k] : \langle \xi, a_i \rangle = 1 \Rightarrow \forall i \in [k] : \xi \in \partial \|a_i\|.$$  

Also, since $\sum_{i=1}^k a_i \in \partial \|a_i\|$, (38) results in:

$$\forall i \in [k] : \langle \xi, a_i \rangle = 1 \Rightarrow v \in T^+_a.$$  

(39)

Since $\xi = \sum_{i=1}^k a_i + v \in \partial \|a_1\|$, we have $\sum_{i=1}^k a_i \xi + v \|1\| = 1$ hence $\sum_{i=1}^k a_i + v \in \partial \|a_2\|$. By induction, we conclude that $a_1 + v \in \partial \|a_2\|$. This implies $\|v\|^* \leq 1$.

Let $v' \in \bigcap_{i \in [k]} T^+_a$ with $\|v'\|^* \leq 1$ and define $\xi' = \sum_{i=1}^k a_i + v'$. We will prove that $\|\xi'\|^* \leq 1$ and hence $\xi' \in \partial \|x\|$. Define $z_i = \sum_{i=k-l+1}^k a_i + v$ for $l \in \{1, 2, \ldots, k\}$. Note that $\|z_i\|^* \leq 1$ since $z_1 = a_k + v \in \partial \|a_k\|$. Suppose $\|z_i\|^* \leq 1$ for some $l \leq k$. We have $z_i = \sum_{i=k-l+1}^k a_i \in T^+_a$ because $z_i = \sum_{i=k-l+1}^k a_i \in \partial \|a_k\|$. Since $v' \in T^+_a$, $z_i \in T^+_a$ hence $\|z_i\|^* \leq 1$. Thus $\|\xi'\|^* \leq 1$. We conclude that:

$$\partial \|x\| = \{ \sum_{i=1}^k a_i + v | v \in \bigcap_{i=1}^k T^+_a, \|v\|^* \leq 1 \}. $$

(40)

Necessity. We first introduce a lemma that will be used in the rest of the proof.

**Lemma 4** Suppose $a \in \mathcal{G}^\|\|$ and $y \in T^+_a - \{0\}$. If $z \in \mathcal{B}^\|\|$ is such that $\|y\|^* = \langle y, z \rangle$, then $z \in T^+_a$.

**Proof** Without loss of generality assume that $\|y\|^* = 1$. It suffices to show that if $b \in \mathcal{G}^\|\|$ and $\langle y, b \rangle = 1$, then $b \in T^+_a$. Consider such $b \in \mathcal{G}^\|\|$. By decomposability assumption $\|a + y\|^* = 1$. That results in:

$$1 \geq \langle a + y, b \rangle = \langle a, b \rangle + 1 \Rightarrow 0 \geq \langle a, b \rangle.$$ 

By considering $-y$ and $-b$ we get that $\langle a, b \rangle = 0$. Since $\langle a + y, b \rangle = \|b\| = 1$, we can conclude that $a + y \in \partial \|b\|$. Since $\langle y, b \rangle = 1$ and $\|y\|^* = 1$, $y \in \partial \|b\|$.

Combining these two conclusions, we get:

$$y \in \partial \|b\|, a + y \in \partial \|b\| \Rightarrow a \in T^+_b \Rightarrow \|a + b\|^* \leq 1 \Rightarrow a + b \in \partial \|a\| \Rightarrow b \in T^+_a$$

□

For any $a \in \mathcal{G}^\|\|$, we have:

$$\langle a, a \rangle = 1,$$

$$\forall \mathcal{G}^\|\| : \langle b, a \rangle \leq \|b\| \|a\| \leq 1.$$ 

That implies $\|a\|^* = 1$ and $a \in \partial \|a\|$. Since $a \in \mathcal{T}_a$, we conclude that:

$$\partial \|a\| = \{ a + v | v \in T^+_a, \|v\|^* \leq 1 \}.$$ 

(41)

Take $\gamma_1 = \langle a_1, x \rangle = \|x\|^*$ and let $\Delta_1 = x - \gamma_1 a_1$. If $\Delta_1 = 0$, then take $k = 1$ and $x = \gamma_1 a_1$. Suppose $\Delta_1 \neq 0$. Since $\|\gamma_1 x\|^* = 1$ and $\langle a_1, \gamma_1 x \rangle = \|a_1\|^* = 1$, we can conclude that $\|a_1\|^* = 1$. Furthermore, we have

$$P_{T^+_a}(x) = x - \gamma_1 P_{T^+_a}(\gamma_1 x) = x - \gamma_1 a_1 = \Delta_1$$

(42)
\[ \Rightarrow \Delta_l \in T_{a_{l+1}}^\perp. \]

Suppose that there exist \( l \in \{1, 2, \ldots, k\} \), an orthogonal set \( \{a_i \in G_{\|\cdot\|_1} \mid i = 1, 2, \ldots, l\} \), and a set of coefficients \( \{\gamma_i \geq 0 \mid i = 1, 2, \ldots, l\} \) such that \( x = \sum_{i=1}^l \gamma_i a_i + \Delta_l, \Delta_l \in \cap_{i=1}^l T_{a_i}^\perp \), and:

\[ \partial \left\| \sum_{i=1}^l a_i \right\| = \{ \sum_{i=1}^l a_i + v \mid v \in \bigcap_{i=1}^l T_{a_i}^\perp, \|v\|_E \leq 1 \}. \]

(43)

By Lemma 2 there exists \( a_{l+1} \in G_{\|\cdot\|_1} \) such that \( a_{l+1} \in \bigcap_{i=1}^l T_{a_i}^\perp \) and \( \langle a_{l+1}, \Delta_l \rangle = \|\Delta_l\|_E^2 \). Take \( \gamma_{l+1} = \langle a_{l+1}, \Delta_l \rangle = \|\Delta_l\|_E^2 \) and let \( \Delta_{l+1} = \Delta_l - \gamma_{l+1} a_{l+1} \). We have \( \Delta_{l+1} \in \bigcap_{i=1}^l T_{a_i}^\perp \) because \( \{\Delta_l, a_{l+1}\} \subset \bigcap_{i=1}^l T_{a_i}^\perp \). Since \( \frac{1}{\gamma_{l+1}} \Delta_{l+1} \bigotimes 1 = 1 \) and \( \langle a_{l+1}, \frac{1}{\gamma_{l+1}} \Delta_{l+1} \rangle = \|a_{l+1}\|_E \leq 1 \), we can conclude that \( \frac{1}{\gamma_{l+1}} \Delta_{l+1} \in \partial\|a_{l+1}\| \). Using the same reasoning as in (42), we have \( \Delta_{l+1} \in T_{a_{l+1}}^\perp \), hence \( \Delta_{l+1} \in \bigcap_{i=l+1}^k T_{a_i}^\perp \).

By decomposability assumption there exists \( e \in \mathbb{R}^n \) and a subspace \( T \) such that:

\[ \partial \left\| \sum_{i=1}^{l+1} a_i \right\| = \{ e + v \mid v \in T^\perp, \|v\|_E \leq 1 \}. \]

(44)

We claim that \( e = \sum_{i=1}^{l+1} a_i \) and \( T^\perp = \bigcap_{i=1}^{l+1} T_{a_i}^\perp \). To prove the first claim, it is enough to show that \( \sum_{i=1}^{l+1} a_i \in \partial \left\| \sum_{i=1}^{l+1} a_i \right\| \).

Note that \( \left\| \sum_{i=1}^{l+1} a_i \right\| \leq 1 \), since \( \sum_{i=1}^{l+1} a_i = \sum_{i=1}^{l+1} a_i + a_{l+1} \in \partial \left\| \sum_{i=1}^{l+1} a_i \right\| \), which is given by (43). Now we can write:

\[ l + 1 = \sum_{i=1}^{l+1} a_i + a_{l+1} \leq \left\| \sum_{i=1}^{l+1} a_i \right\| + l + 1 \sum_{i=1}^{l+1} a_i, \quad \left\| \sum_{i=1}^{l+1} a_i \right\| \leq l + 1 \sum_{i=1}^{l+1} a_i. \]

Therefore, \( \sum_{i=1}^{l+1} a_i \in \partial \left\| \sum_{i=1}^{l+1} a_i \right\| \) and \( \|a_{l+1}\| \leq 1 \), since \( \sum_{i=1}^{l+1} a_i \in T_{\sum_{i=1}^{l+1} a_i} \), we conclude that:

\[ \partial \left\| \sum_{i=1}^{l+1} a_i \right\| = \left\{ \sum_{i=1}^{l+1} a_i + v \mid v \in T^\perp, \|v\|_E \leq 1 \right\}. \]

Let \( \xi = e + v \) with \( v \in \bigcap_{i=1}^{l+1} T_{a_i}^\perp \). Note that \( \|a_{l+1} + v\|^2 \leq 1 \) since \( a_{l+1} + v \in \partial\|a_{l+1}\| \). Furthermore, \( a_{l+1} + v \in \bigcap_{i=1}^{l+1} T_{a_i}^\perp \), which in turn implies \( \sum_{i=1}^{l+1} a_i + v \in \partial \left\| \sum_{i=1}^{l+1} a_i \right\| \) hence \( \left\| \sum_{i=1}^{l+1} a_i + v \right\| \leq 1 \). Additionally, we have:

\[ \sum_{i=1}^{l+1} a_i + v \in \bigcap_{i=1}^{l+1} T_{a_i}^\perp \]

Hence \( \xi \in \partial \left\| \sum_{i=1}^{l+1} a_i \right\| \) and \( v \in T^\perp \). Now, let \( \xi' = \left\{ \sum_{i=1}^{l+1} a_i + v' \right\} \in \sum_{i=1}^{l+1} a_i \right\| \). Note that:

\[ \langle \xi', \sum_{i=1}^{l+1} a_i \rangle = \langle \xi', \sum_{i=1}^{l+1} a_i \rangle + \langle \xi', a_{l+1} \rangle = l + 1 \Rightarrow \langle \xi', \sum_{i=1}^{l+1} a_i \rangle \in \partial \left\| \sum_{i=1}^{l+1} a_i \right\| \]
Now the claim follows from Lemma 4.

Let \( l = |\{ \eta_i | \eta_i \neq 0 \}| \). If \( l = 0 \), the statement is trivially true. Suppose the statement is true when \( l = l' - 1 \) for some \( l' \in \{ 1, \ldots, n \} \) and consider the case where \( l = l' \). Suppose that \( |\eta_j| = \max_{i} |\eta_i| \). By proper normalization we can assume that \( \eta_j = 1 \).

Let \( y = \sum_{i \neq j} \eta_i a_i \). We can deduce the following properties for \( y \):

\[
\forall i \neq j : a_i \in T_{a_j} \implies y \in T_{a_j},
\]

\[
\|y\|^* = \max_{i \neq j} |\eta_i| \leq 1.
\]

By the decomposability assumption \( \sum_{i=1}^{k} \eta_i a_i = a_j + y \in \partial |\eta_j| \) hence \( \|\sum_{i=1}^{k} \eta_i a_i\|^* \leq 1 \). Hence \( \|\sum_{i=1}^{k} \eta_i a_i\|^* = 1 \).

Remark 1 Let \( x = \sum_{i=1}^{k(x)} \gamma_i a_i \). Since \( T_{a_j} = \bigcap_{i=1}^{k(x)} T_{a_i} \), a more general version of Lemma 3 holds:

Lemma 5 Suppose \( x \in \mathbb{R}^n \) and \( y \in T_{a_j}^+ - \{0\} \). If \( z \in B_{\| \cdot \|} \) is such that \( \|y\|^* = \langle y, z \rangle \), then \( z \in T_{a_j}^+ \).

We state and prove a dual version of Lemma 5 which will be used in the proof of Lemma 1 and Lemma 2.

Lemma 6 Let \( x \in \mathbb{R}^n \). If \( y \in T_{a_j} \), then there exists \( z \in T_{a_j}^+ \cap B_{\| \cdot \|} \) such that \( \|y\| = \langle y, z \rangle \).

Proof If \( y = 0 \), then the lemma is trivially true. If \( y \neq 0 \), then:

\[
\frac{y}{\|y\|} \in T_{a_j}^+ \cap \{ x | \|x\| = 1 \} \implies \exists z \in T_{a_j}^+ \text{ such that } \frac{y}{\|y\|} \in \arg\max_{a \in T_{a_j}^+ \cap B_{\| \cdot \|}} \langle a, z \rangle.
\]

Therefore, by Lemma 5 we get

\[
\|z\|^* = \max_{a \in T_{a_j}^+ \cap B_{\| \cdot \|}} \langle a, z \rangle \leq (\frac{y}{\|y\|}, z) \leq \|z\|^* \implies (\frac{y}{\|y\|}, z) = \|z\|^* \implies \langle y, z \rangle = \|y\|.
\]

\( \square \)

B.2 Proof of Proposition 2

In iteration \( t + 1 \) when the backtracking procedure stops, the following inequality holds true:

\[
\phi_\lambda(x^{(t+1)}) \leq m_{L+1}(x^{(t)}, x^{(t+1)}) = \min_x f(x^{(t)}) + \langle \nabla f(x^{(t)}), x - x^{(t)} \rangle + \frac{M_{t+1}}{2} \|x - x^{(t)}\|_2^2 + \lambda \|x\|_2^2 \\
\leq \min_x \phi_\lambda(x) + \frac{M_{t+1}}{2} \|x - x^{(t)}\|_2^2.
\]

(46)

On the other hand, by [17], we have

\[
\phi_\lambda(x^{(t+1)}) \leq m_{L_f}(x^{(t)}, x^{(t+1)}),
\]

which ensures \( M_{t+1} \leq \gamma_{\text{inc}} L_f \) since \( m_{L_f}(x^{(t)}, x^{(t+1)}) \) is non-decreasing in \( L \). By [15], we have:

\[
\phi_\lambda(x^{(t)}) \geq \phi_\lambda(x^*) + \frac{M_f}{2} \|x^{(t)} - x^*\|_2^2.
\]

(47)

If we confine \( x \) to \( \{ \alpha x^* + (1 - \alpha) x^{(t)} | 0 \leq \alpha \leq 1 \} \), inequality [16] combined with [17] results in

\[
\phi_\lambda(x^{(t+1)}) \leq \min_{\alpha \in [0,1]} \{ \phi_\lambda(\alpha x^* + (1 - \alpha) x^{(t)}) + \frac{\alpha^2 M_{t+1}}{2} \|x^{(t)} - x^*\|_2^2 \}
\]

\[
\leq \min_{\alpha \in [0,1]} (\alpha \phi_\lambda(x^*) + (1 - \alpha) \phi_\lambda(x^{(t)}) + \frac{\alpha^2 M_{t+1}}{2} \|x^{(t)} - x^*\|_2^2)
\]

\[
\leq \min_{\alpha \in [0,1]} (\alpha \phi_\lambda(x^*) + (1 - \alpha) \phi_\lambda(x^{(t)}) + \frac{\alpha^2 \gamma_{\text{inc}} L_f}{2} (\phi_\lambda(x^{(t)}) - \phi_\lambda(x^*))).
\]

The R.H.S of the above inequality is minimized for \( \alpha^* = \min \{ 1, \frac{M_f}{2 \gamma_{\text{inc}} L_f} \} \). Therefore, we get

\[
\phi_\lambda(x^{(t+1)}) - \phi_\lambda(x^*) \leq (1 - \alpha^*) + \alpha^2 \gamma_{\text{inc}} L_f \phi_\lambda(x^{(t)}) - \phi_\lambda(x^*)) \leq (1 - \frac{M_f}{4 \gamma_{\text{inc}} L_f}) (\phi_\lambda(x^{(t)}) - \phi_\lambda(x^*)).
To prove (21), we first bound \( \omega(x^{(t+1)}) \) by \( \|x^{(t)} - x^{(t+1)}\|^\ast \). Since \( x^{(t+1)} = \arg\min_{x \in \mathbb{R}^n} m_{M_{t+1}}(x^{(t)}, x) \), there exists \( \xi \in \partial \|x^{(t)}\| \) such that \( \nabla f(x^{(t)}) + \xi + M_{t+1}(x^{(t+1)} - x^{(t)}) = 0 \). Therefore, we get
\[
\omega(x^{(t+1)}) \leq \left\| \xi + \nabla f(x^{(t+1)}) \right\|^\ast \leq \left\| \xi + \nabla f(x^{(t)}) \right\|^\ast + \left\| \nabla f(x^{(t+1)}) - \nabla f(x^{(t)}) \right\|^\ast \\
\leq \theta(M_{t+1} + L_f') \left\| x^{(t+1)} - x^{(t)} \right\|_2.
\]

The backtrack stopping criteria ensures
\[
\phi_{\lambda}(x^{(t+1)}) \leq f(x^{(t)}) + \langle \nabla f(x^{(t)}), x^{(t+1)} - x^{(t)} \rangle + \frac{M_{t+1}}{2} \left\| x^{(t+1)} - x^{(t)} \right\|_2^2 + \lambda \left\| x^{(t+1)} \right\|_2
\]
\[
\leq f(x^{(t)}) - \left( M_{t+1}(x^{(t+1)} - x^{(t)}) + \xi \right) \left\| x^{(t+1)} - x^{(t)} \right\|_2^2 + \frac{M_{t+1}}{2} \left\| x^{(t+1)} - x^{(t)} \right\|_2^2 + \lambda \left\| x^{(t+1)} \right\|_2 \\
\leq \phi_{\lambda}(x^{(t)}) - \frac{M_{t+1}}{2} \left\| x^{(t+1)} - x^{(t)} \right\|_2^2.
\]

Note that hypothesis (13) forces \( M_{t+1} \geq \mu_f \). Combining (13) and (14) and using the lower and the upper bounds on \( M_{t+1} \), we get the desired result
\[
\omega(x^{(t+1)}) \leq \theta(M_{t+1} + L_f') \left\| x^{(t+1)} - x^{(t)} \right\|_2 \\
\leq \theta(1 + \frac{L_f'}{M_{t+1}}) \sqrt{2M_{t+1}(\phi_{\lambda}(x^{(t)}) - \phi_{\lambda}(x^{(t+1)}))} \\
\leq \theta(1 + \frac{L_f'}{\mu_f}) \sqrt{2\gamma u c L_f(\phi_{\lambda}(x^{(t)}) - \phi_{\lambda}(x^*))}.
\]

**B.3 Proof of Lemma 1**

By the hypothesis there exists \( \xi \in \partial \|x\| \) such that \( \|A^\ast(Ax - b) + \lambda \xi\|^\ast \leq \delta \lambda \). Therefore, we can write
\[
\delta \lambda \|x - x_0\| \geq \|x - x_0\| \|A^\ast(Ax - b) + \lambda \xi\|^\ast \geq \langle (x - x_0)\), A^\ast(Ax - b) + \lambda \xi \rangle \\
= \langle (x - x_0), A^\ast(A(x - x_0) - A^\ast z + \lambda \xi) \rangle \\
= \|A(x - x_0)\|_2^2 - \langle x - x_0, A^\ast z \rangle + \lambda \langle x - x_0, \xi \rangle \\
\geq \|A(x - x_0)\|_2^2 - \|x - x_0\| \|A^\ast z\|^\ast + \lambda(\|x\| - \|x_0\|).
\]

Now we lower-bound \( \|x\| \):
\[
\|x\| = \|x - x_0 + x_0\| \geq \left\| P_{T_{x_0}} \perp (x - x_0) + x_0 \right\| - \left\| P_{T_{x_0}} (x - x_0) \right\|.
\]

By Lemma 6 there exists \( s \in T_{x_0} \) such that \( \langle s, P_{T_{x_0}} \perp (x - x_0) \rangle = \|P_{T_{x_0}} \perp (x - x_0)\| \) and \( \|s\|^\ast = 1 \). Note that \( e_{x_0} + s \in \partial \|x_0\| \) hence \( \|e_{x_0} + s\|^\ast \leq 1 \). Therefore, we get:
\[
\left\| P_{T_{x_0}} \perp (x - x_0) + x_0 \right\| \geq \langle e_{x_0} + s, P_{T_{x_0}} \perp (x - x_0) + x_0 \rangle \geq \left\| P_{T_{x_0}} \perp (x - x_0) \right\| + \|x_0\|. \\
\|x\| - \|x_0\| \geq \left\| P_{T_{x_0}} \perp (x - x_0) \right\| - \left\| P_{T_{x_0}} (x - x_0) \right\|.
\]

Combining (61) and (60), we get
\[
\delta \lambda \|x - x_0\| \geq \lambda \left( \|P_{T_{x_0}} \perp (x - x_0)\| - \|P_{T_{x_0}} (x - x_0)\| \right) - \|x - x_0\| \|A^\ast z\|^\ast + \|A(x - x_0)\|_2^2.
\]

By applying triangle inequality to \( \|x - x_0\| \), we obtain
\[
(\lambda(1 + \delta) + \|A^\ast z\|^\ast) \left\| P_{T_{x_0}} (x - x_0) \right\| \geq (\lambda(1 - \delta) - \|A^\ast z\|^\ast) \left\| P_{T_{x_0}} \perp (x - x_0) \right\| + \|A(x - x_0)\|_2^2.
\]
That yields

\[
\frac{\|x - x_0\|}{\|x_0\|} \leq \frac{\left\|P_{\mathbb{T}_0}(x - x_0)\right\| + \left\|P_{\mathbb{T}_0^\perp}(x - x_0)\right\|}{\left\|P_{\mathbb{T}_0}(x - x_0)\right\|_2} \leq (1 + \gamma) \frac{\left\|P_{\mathbb{T}_0}(x - x_0)\right\|}{\left\|P_{\mathbb{T}_0}(x - x_0)\right\|_2} \leq (1 + \gamma) \sqrt{\kappa_0}.
\]

Using the definition of the lower restricted isometry constant, we derive

\[
\rho_-(A, c(1 + \gamma)^2k_0) \|x - x_0\|_2^2 \leq \|A(x - x_0)\|_2^2 \leq \frac{\rho_-(A, c(1 + \gamma)^2k_0)}{(1 + \delta)\lambda + \|A^*z\|^*} \left\|P_{\mathbb{T}_0}(x - x_0)\right\|_2 \leq \sqrt{\kappa_0}((1 + \delta)\lambda + \|A^*z\|^*) \left\|P_{\mathbb{T}_0}(x - x_0)\right\|_2 \leq \sqrt{\kappa_0}((1 + \delta)\lambda + \|A^*z\|^*) \|x - x_0\|_2,
\]

which yields the following bounds

\[
\|x - x_0\|_2 \leq \frac{\sqrt{\kappa_0}(1 + \delta)\lambda + \|A^*z\|^*}{\rho_-(A, c(1 + \gamma)^2k_0)} \tag{53}
\]

\[
\|x - x_0\| \leq \frac{c\kappa_0((1 + \delta)\lambda + \|A^*z\|^*)}{\rho_-(A, c(1 + \gamma)^2k_0)}. \tag{54}
\]

By convexity of \(\phi_\lambda\),

\[
\phi_\lambda(x) - \phi_\lambda(x_0) \leq \langle \lambda \xi + A^*(Ax - b), x - x_0 \rangle \leq \frac{c\kappa_0\delta\lambda(1 + \gamma)((1 + \delta)\lambda + \|A^*z\|^*)}{\rho_-(A, c(1 + \gamma)^2k_0)}.
\]

B.4 Proof of Lemma 2

Let \(\Delta = \frac{3\kappa_0\lambda(1 + \gamma)}{2\rho_-(A, c(1 + \gamma)^2k_0)}\). We can write

\[
\phi_\lambda(x) \leq \phi_\lambda(x_0) + \delta\lambda\Delta
\]

\[
\Rightarrow \quad \frac{1}{2}\|Ax - b\|_2^2 - \frac{1}{2}\|Ax_0 - b\|_2^2 \leq \lambda(\|x_0\| - \|x\|) + \delta\lambda\Delta
\]

\[
\leq \lambda\|x_0 - x\| + \delta\lambda\Delta
\]

If \(\|x - x_0\| \leq \Delta\), half of the conclusion is immediate. To get the second half, we can expand the left hand side of (50) to get:

\[
\frac{1}{2}\|A(x - x_0)\|_2^2 \leq \lambda\|x - x_0\| + \langle x - x_0, A^*z \rangle + \delta\lambda\Delta
\]

\[
\leq (\lambda + \|A^*z\|^*)\|x - x_0\| + \delta\lambda\Delta
\]

\[
\leq (\frac{5}{4} + \delta)\lambda\Delta \leq \lambda \frac{3\Delta}{2}.
\]

Suppose \(\|x - x_0\| > \Delta\), then from (50) we get:

\[
f\lambda(\|x\| - \|x_0\|) \leq \frac{1}{2}\|Ax_0 - b\|_2^2 - \frac{1}{2}\|Ax - b\|_2^2 + \delta\lambda\|x - x_0\|
\]

\[
\leq -\frac{1}{2}\|A(x - x_0)\|_2^2 + \langle x - x_0, A^*z \rangle + \delta\lambda\|x - x_0\|
\]

\[
\leq -\frac{1}{2}\|A(x - x_0)\|_2^2 + \|A^*z\|^*\|x - x_0\| + \delta\lambda\|x - x_0\|.
\]

By using (51) and triangle inequality we get:

\[
\lambda(1 + \delta)\|P_{\mathbb{T}_0}(x - x_0)\| \geq \lambda(1 - b')\|P_{\mathbb{T}_0^\perp}(x - x_0)\| \left\|P_{\mathbb{T}_0}(x - x_0)\right\|_2 + \frac{1}{2}\|A(x - x_0)\|_2^2.
\]

Using the same reasoning as in the proof of Lemma 2, we get the desired results.
B.5 Proof of Lemma 3

By first order optimality condition there exists $\xi \in \partial \|x^+\|$ such that:

$$\lambda \xi = \nabla f(x) - \nabla f(x) = L(x - x^+) - A^*(Ax - b) = L(x - x^+) - A^*(A(x - x_0)) + A^*z$$

Note that $\xi = e_x + v$ for some $v \in T_{x^+}S$. By Lemma 4 there exists $v' \in T_{x^+}S \cap B_{\|\cdot\|^*}$ such that $\langle v', v \rangle = \|v\|$. Since $e_x + v' \in \partial \|x^+\|$, $\|e_x + v'\|^* \leq 1$. Therefore, we can write:

$$\|\xi\| = \|e_x + v\| \geq \|e_x + v'\| + \|v\| = \|e_x^+\| \Rightarrow k(x^+) = \|e_x^+\| \leq \|\xi\|.$$

Let $\xi = \sum_{i=1}^l \gamma_i a_i$, where $a_1, \ldots, a_l$ and $\gamma_1, \ldots, \gamma_l$ are given by the orthogonal representation theorem. Since $\gamma_i \leq 1$ for all $i, 1 \geq \|\xi\|$. If $\|\xi\| > \tilde{k}$, we can define $u = \sum_{i=1}^\tilde{k} a_i$ then

$$\tilde{k} \lambda \leq \langle u, \lambda \xi \rangle = \langle u, L(x^+ - x) \rangle - \langle Au, A(x - x_0) \rangle + \langle u, A^*z \rangle \leq L\|x^+ - x\| + \sqrt{\rho_+ (A, \tilde{k}) \|A(x - x_0)\|_2} + \tilde{k}\|A^*\|^*$$

$$\Rightarrow \frac{3\tilde{k} \lambda}{4} \leq L\|x^+ - x\| + \sqrt{\rho_+ (A, \tilde{k}) \|A(x - x_0)\|_2}. \tag{56}$$

Since $\phi_\lambda(x^+) \leq \phi_\lambda(x)$, by Lemma 2 we have:

$$\|x^+ - x\| \leq \|x^+ - x_0\| + \|x - x_0\| \leq \frac{9\kappa_0 k \lambda (1 + \gamma)}{\rho_- (A, c (1 + \gamma)^2 k_0)}$$

$$\|A(x - x_0)\|^2 \leq \frac{9\kappa_0 \lambda^2 (1 + \gamma)}{\rho_- (A, c (1 + \gamma)^2 k_0)}$$

Define

$$\alpha = \gamma \kappa_0 \rho_+ (A, 2\tilde{k}) \frac{9\kappa_0 (1 + \gamma)}{\rho_- (A, c (1 + \gamma)^2 k_0)},$$

$$\beta^2 = \rho_+ (A, \tilde{k}) \frac{9\kappa_0 (1 + \gamma)}{\rho_- (A, c (1 + \gamma)^2 k_0)}.$$

We can rewrite $\tag{56}$ as:

$$\frac{3\tilde{k}}{4} - \alpha - \beta \sqrt{\tilde{\gamma}} < 0 \Rightarrow \sqrt{\tilde{\gamma}} < \frac{2}{3} (\beta + \sqrt{\beta^2 + 3\alpha}) \leq 2\sqrt{\alpha}.$$

But this contradicts Assumption 1 so $\|\xi\| \leq \tilde{k}$ hence $k(x^+) \leq \tilde{k}$.

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