EXISTENCE OF PERIODIC SOLUTION FOR A CAHN-HILLIARD/ALLEN-CAHN EQUATION IN TWO SPACE DIMENSIONS

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Abstract. In this paper, we discuss the existence of the periodic solutions of a Cahn-Hillard/Allen-Cahn equation which is introduced as a simplification of multiple microscopic mechanisms model in cluster interface evolution. Based on the Schauder type a priori estimates, which here will be obtained by means of a modified Campanato space, we prove the existence of time-periodic solutions in two space dimensions. The uniqueness of solutions is also discussed.

1. Introduction. In this paper, we consider the following problem

$$\frac{\partial u}{\partial t} = -\gamma \Delta [\Delta u - \varphi(u)] + [\Delta u - \varphi(u)] + f(x, t), \quad \gamma > 0, \quad x \in \Omega,$$

with the boundary condition

$$u|_{\partial \Omega} = \Delta u|_{\partial \Omega} = 0, \quad t \in (0, T),$$

and the periodicity condition

$$u(x, 0) = u(x, T), \quad x \in \Omega,$$

where $Q_T = \Omega \times (0, T)$, $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, $\varphi(u) = A(t) u^3 - B(t) u$, $A(t)$, $B(t)$ are Hölder continuous functions defined on $\mathbb{R}$ with period $T$, and $f(x, t)$ belongs to the space $C^{1+\alpha, \frac{\alpha}{2}}(Q_T)$. In addition, we assume that $\underline{M} \leq A(t) \leq \overline{M}$, $0 \leq B(t) \leq \frac{1}{\gamma} = K$, $|A'(t)| \leq N_1$, $|B'(t)| \leq N_2$, where $\underline{M}, \overline{M}, K, N_1$ and $N_2$ are positive constants.

The equation (1.1) is introduced as a simplification of multiple microscopic mechanisms model [4] in cluster interface evolution. Karali and Ricciardi in [3] constructed special sequences of solutions to the equation (1.1) with $f(x, t) = 0$, converging to the second order Allen-Cahn equation. They considered the evolution...
equation without boundary, as well as the stationary case on domains with Dirichlet boundary conditions. Karali and Katsoulakis [4] discussed microscopic models describing pattern formation mechanisms for a prototypical model of surface processes that involve multiple microscopic mechanisms. Karali and Nagase [5] proved that the initial-boundary-value problem for (1.1) with $f(x,t) = 0$ admits a global smooth solution.

The time periodic solutions are important for the higher-order parabolic equation. During the past years, many authors have paid much attention to the time periodic solutions of higher order parabolic equations. Liu and Wang [6] proved the existence of time-periodic solutions for a sixth order nonlinear parabolic equation in two space dimensions. Yin et al. [9] considered the existence of time periodic solutions for the Cahn-Hilliard type equation in one dimension, see also [10]. Using the Galerkin method and the Leray-Schauder fixed point theorem, Wang et.al. [8] proved the existence and uniqueness of time-periodic generalized solutions and time-periodic classical solutions to the generalized Ginzburg-Landau model equation in 1D and 2D cases, see also [1]. However, many physical phenomena, such as the diffusion of oil film over a solid surface, need to be discussed in two dimensions and higher dimensions. Therefore, we should study the multi-dimensional case considered not only from mathematics itself but also from physical background. As far as we know, there are few investigations concerned with the time periodic solutions of such kind of equations.

This paper is a step further in the study of the [9]. The purpose is to prove the existence of time periodic solutions of the problem in two space dimensions. The main difficulties for treating the problem (1.1)–(1.3) are caused by the nonlinearity of both $\Delta \varphi(u)$ and $\varphi(u)$. The main method that we use is based on the Schauder type a priori estimates, which here will be obtained by means of a modified Campanato space. After proving the compactness of the operator and some necessary estimates of the solutions, we obtain a fixed point of the operator in a suitable functional space, which is the desired solution of the problem (1.1)–(1.3). Compared the methods of this paper with [6], the $\|u_\sigma\|_\infty$ can be easily obtained in [6]. But in this paper, firstly, we have to get $\|u_\sigma\|_q \leq C(q < \infty)$ and then use the Gagliardo-Nirenberg inequality. On the other hand, to prove the main Theorem, the key step is to get a priori estimates on the Hölder norm of $u$ in [6]. For this purpose, we need to consider the linear problem in [6]

$$\frac{\partial u}{\partial t} - \gamma \Delta^3 u + \nabla \Delta (a(x,t)\nabla u) = \nabla \Delta \tilde{F} + \Delta E(x,t).$$

In this paper, we want to establish the Hölder norm of the $\nabla u$. For this purpose, we consider the other linear problem

$$\frac{\partial u}{\partial t} + \gamma \Delta^2 u - \nabla (a(x,t)\nabla u) = F(x,t).$$

Our main purpose is to find the relation between the Hölder norm of the solution $\nabla u$ and $a(x,t), F(x,t)$. The method used in [6] is not applicable to the present situation. Due to the equation is fourth order equation, however, we want to establish the higher order Hölder norm of the solution than [6], and there are more difficulties. Based on a suitable integral inequality and Campanato spaces, we obtain the Hölder norm of the $\nabla u$. 
The plan of the paper is as follows. We first present a key step for the a priori estimates on the H"older norm of solutions in Section 2, and then give the proof of our main theorem subsequently in Section 3.

Throughout the paper, the norms of $L^\infty(\Omega)$, $L^2(\Omega)$, $L^r(\Omega)$ are denoted by $\| \cdot \|_\infty$, $\| \cdot \|$ and $\| \cdot \|_s$.

2. H"older estimates.

**Theorem 2.1.** The problem (1.1)-(1.3) admits a time-periodic solution $u$ in the space $C^{4+\alpha,1+\frac{\alpha}{2}}(Q_T)$.

We will only prove the existence of weak solutions in the space $C^{2+\alpha, \frac{\alpha}{2}}(Q_T)$, since the regularity follows a quite standard method [7]. To prove the existence of such solutions we employ the Leray-Schauder fixed point theorem which enables us to study the problem by considering the following equation

$$\frac{\partial u_{\sigma}}{\partial t} + \gamma \Delta^2 u_{\sigma} - \Delta u_{\sigma} = \sigma \gamma \Delta g(x,t) - \sigma g(x,t) + \sigma f(x,t), \quad (x,t) \in Q_T, \quad (2.1)$$

with the conditions (1.2)-(1.3), where $\sigma$ is a parameter taking values in the interval $[0,1]$, and $g(x,t) \in C^{2+\alpha, \frac{\alpha}{2}}(Q_T)$ in time $t$ with period $T$. For any given function $g(x,t) \in C^{2+\alpha, \frac{\alpha}{2}}(Q_T)$, from the classical theory, we see that the problem admits a unique solution $u_{\sigma} \in C^{4+\alpha,1+\frac{\alpha}{2}}(Q_T) \subset C^{2+\alpha, \frac{\alpha}{2}}(Q_T)$. Hence, we can define a map $G$ as follows

$$G : C^{2+\alpha, \frac{\alpha}{2}}(Q_T) \times [0,1] \rightarrow C^{2+\alpha, \frac{\alpha}{2}}(Q_T), \quad (g, \sigma) \mapsto u_{\sigma}.$$ 

Obviously, for any given $g \in C^{2+\alpha, \frac{\alpha}{2}}(Q_T)$, $G(g,0) = 0$. By virtue of the Leray-Schauder fixed point theorem, to prove the existence of solutions of the problem (1.1)-(1.3), we need to show that the map $G$ is compact and prove that if $u_{\sigma} = G(g,\sigma)$ admits a fixed point $u_{\sigma}$ in the space $C^{2+\alpha, \frac{\alpha}{2}}(Q_T)$ for some $\sigma \in [0,1]$, then $\| u_{\sigma} \|_{C^{2+\alpha, \frac{\alpha}{2}}(Q_T)} \leq C$ with $C$ being a constant independent of $u_{\sigma}$ and $\sigma$.

**Lemma 2.2.** The map $G : C^{2+\alpha, \frac{\alpha}{2}}(Q_T) \times [0,1] \rightarrow C^{2+\alpha, \frac{\alpha}{2}}(Q_T)$ is compact.

This result can be directly obtained from the compact embedding $C^{4+\alpha,1+\frac{\alpha}{2}}(Q_T) \subset C^{2+\alpha, \frac{\alpha}{2}}(Q_T)$. We omit the details.

**Lemma 2.3.** Let $u_{\sigma}$ be a time periodic solution of the equation

$$\frac{\partial u_{\sigma}}{\partial t} + \gamma \Delta^2 u_{\sigma} - \Delta u_{\sigma} = \sigma \gamma \Delta \varphi(u_{\sigma}) - \sigma \varphi(u_{\sigma}) + \sigma f(x,t), \quad (2.2)$$

subject to the conditions (1.2)-(1.3), where $\sigma \in [0,1]$. Then

$$\| u_{\sigma} \|_\infty \leq C, \quad (2.3)$$

where $C$ is a constant independent of the solution and $\sigma$.

**Proof.** Multiplying (2.2) by $u_{\sigma}$, integrating the result over $Q_T$ and using the conditions (1.2) and (1.3), we have

$$\gamma \iint_{Q_T} |\Delta u_{\sigma}|^2 dxdt + \iint_{Q_T} |\nabla u_{\sigma}|^2 dxdt$$

$$= - \sigma \iint_{Q_T} 3\gamma A(t) u_{\sigma}^4 |\nabla u_{\sigma}|^2 dxdt + \sigma \iint_{Q_T} \gamma B(t) |\nabla u_{\sigma}|^2 dxdt - \sigma \iint_{Q_T} A(t) u_{\sigma}^4 dxdt$$

$$+ \sigma \iint_{Q_T} B(t) u_{\sigma}^4 dxdt + \sigma \iint_{Q_T} f(x,t) u_{\sigma} dxdt.$$
Therefore, we get
\[
\gamma \iint_{Q_T} |\Delta u_\sigma|^2 dxdt + \int_{Q_T} |\nabla u_\sigma|^2 dxdt
\]
\[
\leq -\sigma \iint_{Q_T} 3\gamma A(t) u_\sigma^2 |\nabla u_\sigma|^2 dxdt + \gamma K \iint_{Q_T} |\nabla u_\sigma|^2 dxdt - \sigma \iint_{Q_T} A(t) u_\sigma^4 dxdt
\]
\[
+ K \iint_{Q_T} u_\sigma^2 dxdt + \frac{1}{2} \iint_{Q_T} f^2(x,t)dxdt + \frac{1}{2} \iint_{Q_T} u_\sigma^2 dxdt.
\]
We notice that
\[
\gamma K \iint_{Q_T} |\nabla u_\sigma|^2 dxdt \leq \frac{\gamma}{2} \iint_{Q_T} |\Delta u_\sigma|^2 dxdt + \frac{\gamma K^2}{2} \iint_{Q_T} u_\sigma^2 dxdt.
\]
From the above inequality and the assumptions on \( A(t) \), \( B(t) \), we obtain
\[
\gamma \iint_{Q_T} |\Delta u_\sigma|^2 dxdt + \int_{Q_T} |\nabla u_\sigma|^2 dxdt
\]
\[
\leq \frac{\gamma}{2} \iint_{Q_T} |\Delta u_\sigma|^2 dxdt + \left( \frac{\gamma K^2}{2} + K + \frac{1}{2}\right) \iint_{Q_T} u_\sigma^2 dxdt + \frac{1}{2} \iint_{Q_T} f^2(x,t)dxdt.
\]
The above inequality implies that
\[
\iint_{Q_T} |\nabla u_\sigma|^2 dxdt \leq \left( \frac{\gamma K^2}{2} + K + \frac{1}{2}\right) \iint_{Q_T} u_\sigma^2 dxdt + \frac{1}{2} \iint_{Q_T} f^2(x,t)dxdt, \quad (2.4)
\]
\[
\iint_{Q_T} |\nabla u_\sigma|^2 dxdt \leq \left( \frac{\gamma K^3}{2} + \gamma K^2 + \frac{\gamma K}{2} + K \right) \iint_{Q_T} u_\sigma^2 dxdt + \left( \frac{\gamma K}{2} + \frac{1}{2}\right) \iint_{Q_T} f^2(x,t)dxdt. \quad (2.5)
\]
On the other hand, by the assumptions \( M \leq A(t) \leq \bar{M} \), \( B(t) \leq K \) and (2.5), we have
\[
\frac{M}{2} \iint_{Q_T} u_\sigma^4 dxdt + 3\gamma M \iint_{Q_T} u_\sigma^2 |\nabla u_\sigma|^2 dxdt
\]
\[
\leq \gamma K \iint_{Q_T} |\nabla u_\sigma|^2 dxdt + K \iint_{Q_T} u_\sigma^2 dxdt + \frac{1}{2} \iint_{Q_T} f^2(x,t)dxdt + \frac{1}{2} \iint_{Q_T} u_\sigma^2 dxdt
\]
\[
\leq \left( \frac{\gamma K^3}{2M} + \gamma K^2 + \frac{\gamma K}{2M} + \frac{K}{M} \right) \iint_{Q_T} u_\sigma^2 dxdt + \left( \frac{\gamma K}{2M} + \frac{1}{2}\right) \iint_{Q_T} f^2(x,t)dxdt
\]
\[
\leq \frac{1}{2} \iint_{Q_T} u_\sigma^2 dxdt + \frac{1}{2} \left( \frac{\gamma K^3}{2M} + \frac{\gamma K^2}{2M} + \frac{\gamma K}{2M} + \frac{K}{M} \right)^2 |\Omega| T
\]
\[
+ \left( \frac{\gamma K}{2M} + \frac{1}{2}\right) \iint_{Q_T} f^2(x,t)dxdt, \quad (2.6)
\]
Therefore, we get
\[
\iint_{Q_T} u_\sigma^4 dxdt
\]
\[
\leq \left( \frac{\gamma K^3}{2M} + \gamma K^2 + \frac{\gamma K}{2M} + \frac{K}{M} \right) \iint_{Q_T} u_\sigma^2 dxdt + \left( \frac{\gamma K}{2M} + \frac{1}{2}\right) \iint_{Q_T} f^2(x,t)dxdt
\]
\[
\leq \frac{1}{2} \iint_{Q_T} u_\sigma^2 dxdt + \frac{1}{2} \left( \frac{\gamma K^3}{2M} + \frac{\gamma K^2}{2M} + \frac{\gamma K}{2M} + \frac{K}{M} \right)^2 |\Omega| T
\]
hence
\[
\iint_{Q_T} u_\sigma^4 dxdt \leq \left( \frac{\gamma K^3}{2M} + \gamma K^2 + \frac{\gamma K}{2M} + \frac{K}{M} \right)^2 |\Omega| T
\]
By the Young inequality, we obtain
\[ \int_{Q_T} u_\sigma^2 \, dx \, dt \leq \int_{Q_T} u_\sigma^4 \, dx \, dt + \frac{1}{4} |\Omega| T = A^* + \frac{1}{4} |\Omega| T. \]  

Combining the above estimate with (2.4) and (2.5), we have
\[ \int_{Q_T} |\nabla u_\sigma|^2 \, dx \, dt \leq C, \quad \int_{Q_T} |\Delta u_\sigma|^2 \, dx \, dt \leq C. \]  

Set \( H(t) = \int_{\Omega} \left( \frac{1}{2} |\nabla u_\sigma|^2 + \sigma (F(u_\sigma) + \lambda) \right) \, dx \), where \( F(u, t) = \int_0^u \Delta u \, ds \). Integrating \( H(t) \) over \((0, T)\), we have
\[ \int_0^T H(t) \, dt \leq \int_{Q_T} \left( \frac{1}{2} |\nabla u_\sigma|^2 + \sigma \left[ \frac{A(t) u_\sigma^4}{4} - \frac{B(t) u_\sigma^2}{2} + \lambda \right] \right) \, dx \, dt \]
\[ \leq \frac{1}{2} \int_{Q_T} |\nabla u_\sigma|^2 \, dx \, dt + \frac{M}{4} \int_{Q_T} u_\sigma^4 \, dx \, dt + \frac{K}{2} \int_{Q_T} u_\sigma^2 \, dx \, dt + \lambda |Q_T| \leq C. \]  

On the other hand, integrating by parts and noticing (2.2) and (1.2), we obtain
\[ \frac{dH}{dt} = \int_{\Omega} \nabla u_\sigma \cdot \frac{\partial \nabla u_\sigma}{\partial t} + \sigma \left[ \varphi(u_\sigma) \frac{\partial u_\sigma}{\partial t} + \frac{A'(t) u_\sigma^4}{4} - \frac{B'(t) u_\sigma^2}{2} \right] \, dx \]
\[ = \int_{\Omega} (-\Delta u_\sigma + \sigma \varphi(u_\sigma)) \frac{\partial u_\sigma}{\partial t} + \sigma \int_{\Omega} \left[ \frac{A'(t) u_\sigma^4}{4} - \frac{B'(t) u_\sigma^2}{2} \right] \, dx \]
\[ = \int_{\Omega} (-\Delta u_\sigma + \sigma \varphi(u_\sigma)) \left[ -\gamma (\Delta u_\sigma - \sigma \Delta \varphi(u_\sigma)) + \Delta u_\sigma - \sigma \varphi(u_\sigma) + \sigma f(x, t) \right] \, dx \]
\[ + \sigma \int_{\Omega} \left[ \frac{A'(t) u_\sigma^4}{4} - \frac{B'(t) u_\sigma^2}{2} \right] \, dx. \]

Integrating the above inequality over \((0, T)\) and noticing the periodicity of \( H \), we have
\[ \gamma \int_{Q_T} |\nabla (-\Delta u_\sigma + \sigma \varphi(u_\sigma))|^2 \, dx \, dt + \int_{Q_T} (\Delta u_\sigma - \sigma \varphi(u_\sigma))^2 \, dx \, dt \]
\[ = \sigma \int_{Q_T} (-\Delta u_\sigma + \sigma \varphi(u_\sigma)) f(x, t) \, dx \, dt + \sigma \int_{Q_T} \left[ \frac{A'(t) u_\sigma^4}{4} - \frac{B'(t) u_\sigma^2}{2} \right] \, dx \, dt. \]
Integrating $\int_0^T \frac{dH}{dt} \, dt$ over $(0, T)$, using (2.7), (2.8) and (2.9), we have
\[
\int_0^T \left| \int_Q \nabla (-\Delta u_\sigma + \sigma \varphi(u_\sigma)) \right|^2 \, dx \, dt + \int_Q (\Delta u_\sigma - \sigma \varphi(u_\sigma))^2 \, dx \, dt \\
+ \sigma \int_Q |(-\Delta u_\sigma + \sigma \varphi(u_\sigma)) f(x, t)| \, dx \, dt + \sigma \int_Q \left\{ \frac{A'(t)}{4} u_\sigma^4 - \frac{B'(t)}{2} u_\sigma^2 \right\} \, dx \, dt \\
\leq 2 \sigma \int_Q |(-\Delta u_\sigma + \sigma \varphi(u_\sigma)) f(x, t)| \, dx \, dt \\
+ 2 \sigma \int_Q \left\{ \frac{A'(t)}{4} u_\sigma^4 - \frac{B'(t)}{2} u_\sigma^2 \right\} \, dx \, dt \\
\leq \int_Q |\Delta u_\sigma|^2 \, dx \, dt + \left( 1 + \frac{K}{2} \right) \int_Q f^2 \, dx \, dt + \left( \frac{M}{2} + \frac{1}{2} N_1 \right) \int_Q u_\sigma^3 \, dx \, dt \\
+ \left( \frac{M}{2} \|f\|_\infty + \frac{K}{2} + N_2 \right) \int_Q u_\sigma^2 \, dx \, dt \leq C. \tag{2.11}
\]

By virtue of (2.10) and (2.11), we know that $\|H\|_{W^{1,1}([0, T])} \leq C$, from which we have $H(t) \leq C$. Noticing the definition of $H(t)$, we get
\[
\int_{\Omega} |\nabla u_\sigma|^2 \, dx \leq C. \tag{2.12}
\]

Combining the above estimate with the boundary value conditions, we obtain that
\[
\|u_\sigma(x, t)\|_q \leq C, \quad 0 < q < \infty, \quad (n = 2). \tag{2.13}
\]

Multiplying (2.2) by $\Delta u_\sigma$, integrating the result over $Q_T$ and using the conditions (1.2) and (1.3), we have
\[
\gamma \int_{Q_T} |\nabla \Delta u_\sigma|^2 \, dx \, dt + \int_{Q_T} |\Delta u_\sigma|^2 \, dx \, dt \\
= - \sigma \gamma \int_{Q_T} |\varphi'(u_\sigma)(\Delta u_\sigma)^2 + \varphi''(u_\sigma)| \nabla u_\sigma |2 \, dx \, dt + \sigma \int_{Q_T} \varphi(u_\sigma) \Delta u_\sigma \, dx \, dt \\
- \sigma \int_{Q_T} f(x, t) \Delta u_\sigma \, dx \, dt.
\]

It follows from $\varphi(u_\sigma) = A(t) u_\sigma^3 - B(t) u_\sigma$ and Hölder’s inequality that
\[
\gamma \int_{Q_T} |\nabla \Delta u_\sigma|^2 \, dx \, dt + \int_{Q_T} |\Delta u_\sigma|^2 \, dx \, dt \\
\leq C \int_{Q_T} |\nabla u_\sigma|^4 \, dx \, dt + C \int_{Q_T} |\Delta u_\sigma|^2 \, dx \, dt \\
+ C \int_{Q_T} |\nabla u_\sigma|^2 \, dx \, dt + C.
\]

By the Gagliardo-Nirenberg inequality (noticing that we consider only the two-dimensional case), we can obtain
\[
\|\nabla u_\sigma\|_4 \leq C \|\Delta u_\sigma\|_2 \|\nabla u_\sigma\|_2^\frac{1}{2}. 
\]
Therefore, by (2.9), we have

\[
\iint_{Q_T} |\nabla \Delta u_\sigma|^2 dx dt \leq C. \tag{2.14}
\]

Similarly, multiplying (2.2) by \(\Delta^2 u_\sigma\), integrating the result over \(Q_T\) and using the conditions (1.2) and (1.3), we have

\[
\begin{align*}
\gamma & \iint_{Q_T} |\Delta^2 u_\sigma|^2 dx dt + \int_{Q_T} |\nabla u_\sigma|^2 dx dt \\
& = \sigma \gamma \iint_{Q_T} \Delta \varphi(u_\sigma) \Delta^2 u_\sigma dx dt - \sigma \iint_{Q_T} \varphi(u_\sigma) \Delta^2 u_\sigma dx dt \\
& \quad + \sigma \int_Q f(x,t) \Delta^2 u_\sigma dx dt.
\end{align*}
\]

By Hölder's inequality, we know

\[
\begin{align*}
\gamma & \iint_{Q_T} |\Delta^2 u_\sigma|^2 dx dt + \int_{Q_T} |\nabla u_\sigma|^2 dx dt \\
& \leq C \left( \iint_{Q_T} |u_\sigma|^4 (\Delta u_\sigma)^2 dx dt \right)^{1/2} \left( \iint_{Q_T} |\Delta^2 u_\sigma|^2 dx dt \right)^{1/2} \\
& \quad + C \left( \iint_{Q_T} u_\sigma^2 |\nabla u_\sigma|^4 dx dt \right)^{1/2} \left( \iint_{Q_T} |\Delta^2 u_\sigma|^2 dx dt \right)^{1/2} \\
& \quad + \frac{\gamma}{2} \iint_{Q_T} |\Delta^2 u_\sigma|^2 dx dt + C.
\end{align*}
\]

By the Gagliardo-Nirenberg inequality, we obtain

\[
\begin{align*}
\|u_\sigma\|_\infty & \leq C \|\Delta^2 u_\sigma\|_2^a \|u_\sigma\|_q^{1-a}, \quad a = (1 - 3q/2)^{-1}, \\
\|\nabla u_\sigma\|_4 & \leq C \|\Delta^2 u_\sigma\|_2 \|\nabla u_\sigma\|_2^{\frac{3}{4}}, \\
\|\Delta u_\sigma\|_2 & \leq C \|\Delta^2 u_\sigma\|_2 \|\nabla u_\sigma\|_2^{\frac{1}{2}}.
\end{align*}
\]

Therefore, we have

\[
\iint_{Q_T} |\Delta^2 u_\sigma|^2 dx dt \leq C. \tag{2.15}
\]

Now we set \(H_1(t) = \int_\Omega (\Delta u_\sigma)^2 dx\). Obviously,

\[
\int_0^T |H_1(t)| dt \leq C. \tag{2.16}
\]

On the other hand, by (2.2), (2.9) and (2.13), we have

\[
\begin{align*}
\int_0^T & \frac{dH_1}{dt}(t) dt \\
= & \int_0^T \int_\Omega \Delta^2 u_\sigma \left[ -\gamma (\Delta^2 u_\sigma - \sigma \Delta \varphi(u_\sigma)) + \Delta u_\sigma - \sigma \varphi(u_\sigma) + \sigma f(x,t) \right] dx dt \\
\leq & \ C \iint_{Q_T} (\Delta^2 u_\sigma)^2 dx dt + C \int_{Q_T} (\Delta \varphi(u_\sigma))^2 dx dt + C \leq C.
\end{align*}
\]
By virtue of (2.15) and (2.16), we have \( F_1(t) \leq C \). Noticing the definition of \( H_1(t) \), we get

\[
\int_\Omega (\Delta u_\sigma)^2 \, dx \leq C.
\]

Hence

\[
\|u_\sigma\|_\infty \leq C. \tag{2.17}
\]

The proof of the Lemma 2.3 is complete.

\[ \square \]

**Lemma 2.4.** Let \( u_\sigma \) be a time periodic solution of the problem (2.2), (1.2)-(1.3). Then there exists a constant \( C \) depending only on the known quantities, such that for any \((x_1, t_1), (x_2, t_2) \in Q_T \) and some \( 0 < \alpha < 1 \),

\[
|u_\sigma(x_1, t_1) - u_\sigma(x_2, t_2)| \leq C(|t_1 - t_2|^{\alpha/4} + |x_1 - x_2|^{\alpha}).
\]

**Proof.** From (2.15), for some \( \alpha \), we have

\[
|u_\sigma(x_1, t) - u_\sigma(x_2, t)| \leq C|x_1 - x_2|^{\alpha}. \tag{2.18}
\]

Integrating the equation (2.2) over \( \Omega_y \times (t_1, t_2) \), where \( 0 < t_1 < t_2 < T, \Delta t = t_2 - t_1 \), \( \Omega_y = (y_1, y_1 + (\Delta t)^{1/8}) \times (y_2, y_2 + (\Delta t)^{1/8}) \), we see that

\[
\int_{\Omega_y} [u_\sigma(z, t_2) - u_\sigma(z, t_1)] \, dz
\]

\[
= \int_{t_1}^{t_2} \int_{y_2}^{y_2+(\Delta t)^{1/8}} [F_1(y_1 + (\Delta t)^{1/8}, y, s) - F_1(y_1, y, s)] \, dy \, ds
\]

\[
+ \int_{t_1}^{t_2} \int_{y_1}^{y_1+(\Delta t)^{1/8}} [F_2(y, y_2 + (\Delta t)^{1/8}, s) - F_2(y, y_2, s)] \, dy \, ds
\]

\[
- \sigma \int_{t_1}^{t_2} \int_{\Omega_y} \varphi(u_\sigma(z, s)) \, dz \, ds + \sigma \int_{t_1}^{t_2} \int_{\Omega_y} f(z, s) \, dz \, ds
\]

\[
= \int_{t_1}^{t_2} (\Delta t)^{1/8} [F_1(y_1 + (\Delta t)^{1/12}, y_2 + \theta_1^*(\Delta t)^{1/8}, s)
\]

\[
- F_1(y_1, y_2 + \theta_1^*(\Delta t)^{1/8}, s) + F_2(y_1 + \theta_2^*(\Delta t)^{1/8}, y_2 + (\Delta t)^{1/8}, s)
\]

\[
- F_2(y_1 + \theta_2^*(\Delta t)^{1/8}, y_2, s)] \, ds + \sigma \int_{t_1}^{t_2} \int_{\Omega_y} \varphi(u_\sigma(z, s)) \, dz \, ds
\]

\[
+ \sigma \int_{t_1}^{t_2} \int_{\Omega_y} f(z, s) \, dz \, ds,
\]

where

\[
(-\gamma \nabla \Delta u_\sigma + \nabla u_\sigma + \sigma \gamma \nabla \varphi(u_\sigma))(x, s) = (F_1, F_2).
\]

Set

\[
N(s, y_1, y_2)
\]

\[
= (\Delta t)^{1/8} [F_1(y_1 + (\Delta t)^{1/8}, y_2 + \theta_1^*(\Delta t)^{1/8}, s) - F_1(y_1, y_2 + \theta_1^*(\Delta t)^{1/8}, s)
\]

\[
+ F_2(y_1 + \theta_2^*(\Delta t)^{1/8}, y_2 + (\Delta t)^{1/8}, s) - F_2(y_1 + \theta_2^*(\Delta t)^{1/8}, y_2, s)].
\]
Then (2.19) is converted into
\[
(\Delta t)^{1/4} \int_{t=(0,1) \times (0,1)} [u_\sigma(y + \theta(\Delta t)^{1/8}, t_2) - u_\sigma(y + \theta(\Delta t)^{1/8}, t_1)]d\theta
\]
\[
= \int_{t_1}^{t_2} N(s, y_1, y_2)ds + \sigma \int_{t_1}^{t_2} \int_{\Omega_y} \varphi(u_\sigma(z, s))dzds + \sigma \int_{t_1}^{t_2} \int_{\Omega_y} f(z, s)dzds.
\]
Integrating the above equality over \(\Omega_x\), we get
\[
(\Delta t)^{1/2}(u_\sigma(x^*, t_2) - u_\sigma(x^*, t_1))
\]
\[
= \int_{t_1}^{t_2} \int_{\Omega_x} N(s, y)dyds + \sigma \int_{t_1}^{t_2} \int_{\Omega_x} \varphi(u_\sigma(z, s))dzdyds + \sigma \int_{t_1}^{t_2} \int_{\Omega_y} f(z, s)dzdyds.
\]
Here, we have used the mean value theorem, where \(x^* = y^* + \theta^*(\Delta t)^{1/8}\). Hence by the Hölder inequality, (2.13), (2.14), (2.15), (2.17) and \(f(x, t) \in C^{1+\alpha, 4}(\bar{Q}_T)\), we get
\[
|u_\sigma(x^*, t_2) - u_\sigma(x^*, t_1)| \leq C(\Delta t)^{\alpha/4}, \quad 0 < \alpha < 1.
\]
Hence
\[
|u_\sigma(x_1, t_1) - u_\sigma(x_2, t_2)| \leq C(|x_1 - x_2|^\alpha + |t_1 - t_2|^\frac{\alpha}{4}). \quad (2.20)
\]

3. Proof of the main result. In this section, we give the proof of Theorem 2.1. To prove the Theorem 2.1, the key estimate is the Hölder estimate for \(\nabla u_\sigma\).

We first consider the following linear problem
\[
\frac{\partial u}{\partial t} + \gamma \Delta^2 u - \nabla(a(x, t)\nabla u) = F(x, t), \quad (3.1)
\]
\[
u_{\partial \Omega} = \Delta u_{\partial \Omega} = 0, \quad (3.2)
\]
\[
u(x, 0) = u(x, T). \quad (3.3)
\]
Without loss of generality, we may assume that \(a(x, t)\) and \(F(x, t)\) are sufficiently smooth, otherwise we replace them by their approximation functions. Our main purpose is to find the relation between the Hölder norm of the solution \(\nabla u\) and \(a(x, t), F(x, t)\).

Let \((x_0, t_0) \in \Omega \times (0, T)\) be fixed and define
\[
\varphi(\rho) = \int_{S_\rho} \left( |\nabla u - (\nabla u)_\rho|^2 + \rho^4 |\nabla \Delta u|^2 \right) dx dt, \quad (\rho > 0),
\]
where
\[
S_\rho = B_\rho(x_0) \times (t_0 - \rho^4, t_0 + \rho^4), \quad (\nabla u)_\rho = \frac{1}{|S_\rho|} \int_{S_\rho} \nabla u dx dt
\]
and \(B_\rho(x_0)\) is the ball centred at \(x_0\) and radius \(\rho\).

Let \(u\) be the solution of the problem (3.1), (3.2), (3.3). We split \(u\) on \(S_R\) into \(u = u_1 + u_2\), where \(u_1\) is the solution of the problem
\[
\frac{\partial u_1}{\partial t} + \gamma \Delta^2 u_1 + a(x_0, t_0)\Delta u_1 = 0, \quad (x, t) \in S_R, \quad (3.4)
\]
\[ u_1 = u, \quad \frac{\partial u_1}{\partial n} = \frac{\partial u}{\partial n}, \quad \Delta u_1 = \Delta u, \quad x \in \partial B_R(x_0), \quad (3.5) \]

\[ u_1|_{t=t_0-R^4} = u|_{t=t_0-R^4}, \quad x \in B_R(x_0), \quad (3.6) \]

and \( u_2 \) solves the problem

\[
\frac{\partial u_2}{\partial t} + \gamma \Delta^2 u_2 + a(x_0, t_0) \Delta u_2 = \nabla \left[ (a(x_0, t_0) - a(x, t)) \nabla u \right] + F(x, t), \quad (x, t) \in S_R, \quad (3.7)
\]

\[ u_2 = 0, \quad \frac{\partial u_2}{\partial n} = 0, \quad \Delta u_2 = 0, \quad (x, t) \in \partial B_R(x_0), \quad (3.8) \]

\[ u_2|_{t=t_0-R^4} = 0, \quad x \in B_R(x_0). \quad (3.9) \]

By classical linear theory, the above decomposition is uniquely determined by \( u \).

We need several lemmas on \( u_1 \) and \( u_2 \).

**Lemma 3.1.** Assume that

\[ |a(x, t) - a(x_0, t_0)| \leq a_\sigma \left( |t - t_0|^{\sigma/4} + |x - x_0|^{\sigma} \right), \quad (x, t) \in B_R(x_0) \times J_R(t_0), \]

where \( J_R(t_0) = (t_0 - R^4, t_0 + R^4) \). Then

\[
\sup_{(t_0 - R^4, t_0 + R^4)} \int_{B_R(x_0)} |\nabla u_2|^2 (x, t) \, dx + \int_{S_R} |\nabla \Delta u_2|^2 \, dx dt \\
\leq C R^{2\sigma} \int_{S_R} |\nabla u|^2 \, dx dt + C \sup_{S_R} |F|^2 R^6. \]

**Proof.** Multiply the equation (3.7) by \( \nabla u_2 \) and integrate the resulting relation over \((t_0 - R^4, t) \times B_R(x_0)\). Integrating by parts, we have

\[
\frac{1}{2} \int_{B_R} |\nabla u_2|^2 \, dx + \int_{t_0 - R^4}^{t} ds \int_{B_R} |\nabla \Delta u_2|^2 \, dx \\
+ a(x_0, t_0) \int_{t_0 - R^4}^{t} ds \int_{B_R} (\Delta u_2)^2 \, dx \\
= \int_{t_0 - R^4}^{t} ds \int_{B_R} [a(x_0, t_0) - a(x, t)] \nabla u \cdot \nabla \Delta u_2 \, dx \\
+ \int_{t_0 - R^4}^{t} ds \int_{B_R} F \Delta u_2 \, dx.
\]

Noticing that

\[
\left| \int_{t_0 - R^4}^{t} ds \int_{B_R} [a(x_0, t_0) - a(x, t)] \nabla u \nabla \Delta u_2 \, dx \right| \\
\leq \varepsilon \int_{S_R} |\nabla \Delta u_2|^2 \, ds dx + C \varepsilon a_\sigma R^{2\sigma} \sup_{S_R} |\nabla u|^2 \, dx ds,
\]

and

\[
\left| \int_{t_0 - R^4}^{t} ds \int_{B_R} F \nabla \Delta u_2 \, dx \right| \leq \varepsilon \int_{S_R} |\nabla \Delta u_2|^2 \, dx ds + C \varepsilon R^6 \sup_{S_R} |F|^2,
\]

we obtain the estimate and the proof is complete. \( \square \)
Lemma 3.2. For any \((x_1, t_1), (x_2, t_2) \in S_\rho\),

\[
\frac{|\nabla u_1(t_1, x_1) - \nabla u_1(t_2, x_2)|^2}{|t_1 - t_2|^{1/4} + |x_1 - x_2|} \leq C \int_{S_\rho} (\Delta^2 u_1)^2 \, dx \, dt
\]

+ \(\sup_{(t_0 - \rho^k, t_0 + \rho^k)} \int_{B_\rho(x_0)} (|\nabla u_1(x, t) - (\nabla u_1)_\rho|^2 + \rho^4 |\nabla \Delta u_1|^2) \, dx\).

Proof. By the Sobolev imbedding theorem, we have for any \((x_1, t), (x_2, t) \in S_\rho\),

\[
\frac{|u_1(x_1, t) - u_1(x_2, t)|^2}{|x_1 - x_2|} \leq C \sup_{(t_0 - \rho^k, t_0 + \rho^k)} \int_{B_\rho(x_0)} (|\nabla u_1(x, t) - (\nabla u_1)_\rho|^2 + \rho^4 |\nabla \Delta u_1|^2) \, dx \quad (3.10)
\]

Differentiating Eq. (3.4) gives

\[
\frac{\partial D_{x_1} u_1}{\partial t} + \gamma \Delta^2 D_{x_1} u_1 + a(x_0, t_0) \Delta D_{x_1} u_1 = 0.
\]

Integrating the equation over \(\Omega_y \times (t_1, t_2)\), where \(0 < t_1 < t_2 < T\), \(\Delta t = t_2 - t_1\), \(\Omega_y = (y_1, y_1 + (\Delta t)^{1/8}) \times (y_2, y_2 + (\Delta t)^{1/8})\), we see that

\[
\int_{\Omega_y} [D_{x_1} u_1(z, t_2) - D_{x_1} u_1(z, t_1)] \, dz
\]

= \(\int_{t_1}^{t_2} \int_{y_2}^{y_2 + (\Delta t)^{1/8}} [G_1(y_1 + (\Delta t)^{1/8}, y, s) - G_1(y_1, y, s)] \, dy \, ds\)

+ \(\int_{t_1}^{t_2} \int_{y_1}^{y_1 + (\Delta t)^{1/8}} [G_2(y, y_2 + (\Delta t)^{1/8}, s) - G_2(y, y_2, s)] \, dy \, ds\)

= \(\int_{t_1}^{t_2} (\Delta t)^{1/8} [G_1(y_1 + (\Delta t)^{1/12}, y_2 + \theta_1^*(\Delta t)^{1/8}, s) - G_1(y_1, y_2 + \theta_1^*(\Delta t)^{1/8}, s)] \, ds\)

- \(G_1(y_1, y_2 + \theta_1^*(\Delta t)^{1/8}, s) + G_2(y_1 + \theta_2^*(\Delta t)^{1/12}, y_2 + (\Delta t)^{1/8}, s) - G_2(y_1 + \theta_2^*(\Delta t)^{1/8}, y_2, s)] \, ds\),

where

\[
\gamma \nabla \Delta D_{x_1} u_1(x, s) - a(x_0, t_0) \nabla D_{x_1} u_1(x, s) = (G_1, G_2).
\]

Similar to the proof of Lemma 2.4, integrating the above equality over \(\Omega_x\), we get

\[
|u_1(x^*, t_2) - u_1(x^*, t_1)|
\]

\[
\leq C|t_1 - t_2|^{1/4} \left[ \int_{S_\rho} (\Delta^2 u_1)^2 \, dx \, dt + \int_{S_\rho} (\Delta u_1)^2 \, dx \, dt \right],
\]

where \(x^* = y^* + \theta^*(\Delta t)^{1/8}\). This and (3.10) yield the desired conclusion and the proof is complete. \(\Box\)
Lemma 3.3. (Caccioppoli type inequality)

\[ \sup_{(t_0-(R/2)^4,t_0+(R/2)^4)} \int_{B_R(x_0)} |\nabla u_1(x,t) - (\nabla u_1)_R|^2 \, dx + \int_{S_R/2} |\nabla \Delta u_1|^2 \, dx \, dt \]
\[ \leq \frac{C}{R^4} \int_{S_R} |\nabla u_1(x,t) - (\nabla u_1)_R|^2 \, dx \, dt \]
\[ + \int_{S_R/2} |\Delta u_1|^2 \, dx \, dt \]
\[ \leq \frac{C}{R^4} \int_{S_R} |\nabla u_1|^2 \, dx \, dt \]
\[ \leq \frac{C}{R^4} \int_{S_R} |\nabla \Delta u_1|^2 \, dx \, dt. \]

Proof. For simplicity, we only prove the first inequality, since the other inequality can be shown similarly. Choose a cut-off function \( \chi(x) \) defined on \( (x_0 - R, x_0 + R) \) such that \( \chi(x) = 1 \) in \( (x_0 - \frac{R}{2}, x_0 + \frac{R}{2}) \) and

\[ |\nabla \chi| \leq \frac{C}{R}, \quad |D^2 \chi| \leq \frac{C}{R^2}, \]
\[ |D^3 \chi| \leq \frac{C}{R^3}, \quad |D^4 \chi| \leq \frac{C}{R^4}. \]

Let \( g(t) \in C_0^\infty(0, +\infty) \) with \( 0 \leq g(t) \leq 1, \ 0 \leq g'(t) \leq \frac{C}{R^4} \) and \( g(t) = 1 \) for \( t \geq t_0 - \frac{R}{2} \). Multiplying the equation (3.4) by \( g(t)\nabla [\chi^4(\nabla u_1(x,t) - (\nabla u_1)_R)] \) and then integrating the resulting relation over \( (t_0 - R^4, t) \times B_R(x_0) \), we have

\[ \int_{t_0 - R^4}^t g(s) \, ds \int_{B_R(x_0)} \frac{\partial u_1}{\partial t} \nabla [\chi^4(\nabla u_1(x,t) - (\nabla u_1)_R)] \, dx + \gamma \int_{t_0 - R^4}^t g(s) \, ds \int_{B_R(x_0)} \Delta^2 u_1 \nabla [\chi^4(\nabla u_1(x,t) - (\nabla u_1)_R)] \, dx \]
\[ + a(x_0, t_0) \int_{t_0 - R^4}^t g(s) \, ds \int_{B_R(x_0)} \Delta u_1 \nabla [\chi^4(\nabla u_1(x,t) - (\nabla u_1)_R)] \, dx = 0. \]

It follows from integrating by parts,

\[ \frac{1}{2} \int_{B_R(x_0)} g(s) \chi^4 |\nabla u_1(x,t) - (\nabla u_1)_R|^2 \, dx + \gamma \int_{t_0 - R^4}^t g(s) \, ds \int_{B_R(x_0)} \Delta u_1 \nabla [\chi^4 \Delta u_1 + 4\chi^3 \nabla \chi (\nabla u_1 - (\nabla u_1)_R)] \, dx \]
\[ - a(x_0, t_0) \int_{t_0 - R^4}^t g(s) \, ds \int_{B_R(x_0)} \Delta u_1 \chi^4 \Delta u_1 + 4\chi^3 \nabla \chi (\nabla u_1 - (\nabla u_1)_R) \, dx \]
\[ = \frac{1}{2} \int_{t_0 - R^4}^t g'(s) \, ds \int_{B_R(x_0)} \chi^4 |\nabla u_1(x,t) - (\nabla u_1)_R|^2 \, dx. \]
Thus
\[
\frac{1}{2} \int_{B_R(x_0)} g(s)\chi^4|\nabla u_1(x,t) - (\nabla u_1)_R|^2 \, dx
+ \gamma \int_{t_0}^t g(s) ds \int_{B_R(x_0)} \chi^4|\nabla \Delta u_1|^2 \, dx
+ a(x_0, t_0) \int_{t_0}^t g(s) ds \int_{B_R(x_0)} \chi^4(\Delta u_1)^2 \, dx
+ \gamma \int_{t_0}^t g(s) ds \int_{B_R(x_0)} [8\chi^3\nabla \chi \Delta u_1 \nabla \Delta u_1
+ (4\chi^3 \Delta \chi + 12\chi^2 |\nabla \chi|^2)(\nabla u_1(x,t) - (\nabla u_1)_R) \nabla \Delta u_1] \, dx
+ a(x_0, t_0) \int_{t_0}^t g(s) ds \int_{B_R(x_0)} 4\chi^3 \nabla \chi(\nabla u_1(x,t) - (\nabla u_1)_R) \Delta u_1 \, dx
= \frac{1}{2} \int_{t_0}^t g'(s) ds \int_{B_R(x_0)} \chi^4|\nabla u_1 - (\nabla u_1)_R|^2 \, dx.
\]
By the Cauchy-Schwarz inequality, we have
\[
\left| \frac{1}{2} \int_{t_0}^t \int_{B_R(x_0)} g(s)\gamma \chi^3 \nabla \chi \Delta u_1 \nabla \Delta u_1 \, dxds \right|
\leq \frac{1}{4} \gamma \int_{t_0}^t \int_{B_R(x_0)} g(s)\chi^4|\nabla \Delta u_1|^2 \, dxds
+ C \int_{t_0}^t \int_{B_R(x_0)} g(s)\chi^2|\nabla \chi|^2(\Delta u_1)^2 \, dxds,
\]
and
\[
\left| \int_{t_0}^t \int_{B_R(x_0)} g(s)\gamma (4\chi^3 \Delta \chi + 12\chi^2 |\nabla \chi|^2)(\nabla u_1(x,t) - (\nabla u_1)_R) \nabla \Delta u_1 \, dxds \right|
\leq \frac{1}{4} \gamma \int_{t_0}^t \int_{B_R(x_0)} g(s)\chi^4|\nabla \Delta u_1|^2 \, dxds
+ \frac{C}{R^2} \int_{t_0}^t \int_{B_R(x_0)} (\nabla u_1(x,t) - (\nabla u_1)_R)^2 \, dxds.
\]
Similarly, we obtain
\[
\left| \int_{t_0}^t \int_{B_R(x_0)} g(s)a(x_0, t_0)4\chi^3 |\nabla \chi|^2(\nabla u_1(x,t) - (\nabla u_1)_R) \Delta u_1 \, dxds \right|
\leq \frac{1}{4} a(x_0, t_0) \int_{t_0}^t \int_{B_R(x_0)} g(s)\chi^4(\Delta u_1)^2 \, dxds
+ \frac{C}{R^2} \int_{t_0}^t \int_{B_R(x_0)} |\nabla u_1(x,t) - (\nabla u_1)_R|^2 \, dxds.
\]
Noticing that
\[
\int_{t_0}^t \int_{B_R(x_0)} g(s)\chi^2 |\nabla \chi|^2(\Delta u_1)^2 \, dxds
\]
Lemma 3.3, we have completed.

We obtain immediately the desired first inequality of the lemma and the proof is completed.

Lemma 3.4. Assume that
\[ |a(x,t) - a(x_0,t_0)| \leq a_\sigma \left( |t - t_0|^{\sigma/4} + |x - x_0|^{\sigma} \right), \]
\[ t \in (t_0 - R^4, t_0 + R^4), \quad x \in B_R(x_0). \]

Then for any \( \rho \in (0, R) \),
\[ \frac{1}{\rho^2} \int_{S_\rho} (|\nabla u_1 - (\nabla u_1)_\rho|^2 + \rho^4 |\nabla \Delta u_1|^2) \, dx \, dt \]
\[ \leq C \frac{R^4}{R^2} \int_{S_R} (|\nabla u_1 - (\nabla u_1)_R|^2 + R^4 |\nabla \Delta u_1|^2) \, dx \, dt. \]

Proof. One only needs to check the inequality for \( \rho \leq \frac{R}{2} \). From Lemma 3.2 and Lemma 3.3, we have
\[ \frac{1}{\rho^2} \int_{S_\rho} |\nabla u_1 - (\nabla u_1)_\rho|^2 \, dx \, dt \]
\[ \leq C \sup_{(t_0 - \frac{R}{2})^4, t_0 + \frac{R}{2})^4} \int_{B_{\frac{R}{2}}(x_0)} \left( |\nabla u_1(x,t) - (\nabla u_1)_{R/2}|^2 \right) \]
\[ + R^4 |\nabla \Delta u|^2 \, dx + C \int_{S_{\frac{R}{2}}} |\nabla \Delta^2 u_1|^2 \, dx \, dt \]
$$\leq \frac{C}{R^7} \int_{S_R} (|\nabla u_1 - (\nabla u_1)_R|^2 + R^4 |\nabla \Delta u_1|^2) \, dx \, dt.$$  

On the other hand,

$$\int_{S_R} \int \rho^4 |\nabla u_1|^2 \, dx \, dt \leq R \sup_{(t_0 - (\frac{4}{7})^4, t_0 + (\frac{4}{7})^4) \setminus B_{\frac{4}{7}}(x_0)} |\nabla u_1|^2 \, dx$$

$$\leq C \left( \frac{\rho}{R} \right)^7 \int_{S_{\frac{4}{7}}} R^4 |\nabla \Delta u_1|^2 \, dx \, dt.$$  

The conclusion of the lemma follows at once. \(\square\)

**Lemma 3.5.** For \(\lambda \in (6, 7)\),

$$\varphi(\rho) \leq C_{\lambda} \left( \varphi(R_0) + \sup_{S_{R_0}} |F|^2 \right) \rho^{\lambda}, \quad \rho \leq R_0 = \min \left( \text{dist}(x_0, \partial \Omega), t_0^{1/4} \right),$$

where \(C_{\lambda}\) depends on \(\lambda, R_0\) and the known quantities.

**Proof.** By Lemma 3.4,

$$\varphi(\rho) = \int_{S_{\rho}} (|\nabla u - (\nabla u)_\rho|^2 + \rho^4 |\nabla \Delta u|^2) \, dx \, dt$$

$$= \int_{S_{\rho}} (|\nabla u_1 - (\nabla u_1)_\rho|^2 + \rho^4 |\nabla \Delta u_1|^2) \, dx \, dt$$

$$+ \int_{S_{\rho}} (|\nabla u_2 - (\nabla u_2)_\rho|^2 + \rho^4 |\nabla \Delta u_2|^2) \, dx \, dt$$

$$\leq C \left( \frac{\rho}{R} \right)^7 \int_{S_R} (|\nabla u - (\nabla u)_R|^2 + R^4 |\nabla \Delta u|^2) \, dx \, dt$$

$$+ C \int_{S_R} (|\nabla u_2|^2 + R^4 |\nabla \Delta u_2|^2) \, dx \, dt$$

$$\leq C \left[ \left( \frac{\rho}{R} \right)^7 + R^{2\sigma} \right] \varphi(R) + C \sup_{S_{R_0}} |F|^2 R^{10}.$$  

The conclusion follows immediately from the Proposition 1.3 in [2]. \(\square\)

Similar to the discussion about the Campanato spaces in [2], we first conclude from Lemma 3.5 that

**Theorem 3.6.** Let \(F\) be appropriately smooth function, \(u\) be the smooth solution of the problem (3.1)-(3.3). Then for any \(\alpha \in (0, \frac{1}{2})\), there exists a coefficient \(K\) depending only on \(\alpha, a, b, \int_{Q_T} u^2 \, dx \, dt\) and \(\int_{Q_T} |\nabla u|^2 \, dx \, dt\), such that

$$|\nabla u(x_1, t_1) - \nabla u(x_2, t_2)| \leq K (1 + \sup |F|)(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}}). \quad (3.11)$$

**Proof of Theorem 2.1.** We fist change Eq. (2.2) into the form

$$\frac{\partial u}{\partial t} + \gamma \Delta^2 u - \nabla (a(x, t) \nabla u) = F(x, t),$$
where \( F = \sigma f(x, t) - \sigma \varphi(u_\sigma), \ a(x, t) = \sigma \gamma \varphi'(u_\sigma) + 1. \) Hence, using (2.12), (2.17) and Theorem 3.6, we conclude that

\[
|\nabla u_\sigma(x_1, t_1) - \nabla u_\sigma(x_2, t_2)| \leq C(|x_1 - x_2|^{a/2} + |t_1 - t_2|^{a/12}). \tag{3.12}
\]

Then, it follows from the results in [7] that \( \|u_\sigma\|_{C^{a+\alpha, \frac{a}{2}}(Q_T)} \leq C, \) since we can transform the equation (2.1) into the form

\[
\frac{\partial u_\sigma}{\partial t} + \gamma \Delta^2 u_\sigma - a(x, t)\Delta u_\sigma + \overrightarrow{B}(x, t)\nabla u_\sigma + b(x, t)u_\sigma = \sigma f(x, t),
\]

where the H"older norms on

\[
a(x, t) = \sigma \gamma \varphi'(u_\sigma) + 1, \quad \overrightarrow{B}(x, t) = \sigma \gamma \varphi''(u_\sigma)\nabla u_\sigma,
\]

\[
b(x, t) = \sigma A(t)u_\sigma^2 - B(t)
\]

have been obtained. Therefore, we have \( \|u_\sigma\|_{C^{2+a, \frac{a}{2}}(Q_T)} \leq C, \) where \( C \) is a constant independent of the parameter \( \sigma. \) From Lemma 2.2 and the Leray-Schauder fixed point theorem, \( G(y(\cdot, t), 1) \) admits a fixed point \( u \) in the space \( \mathcal{C}_T(Q_T), \) which is the desired solution of the problem (1.1)-(1.3). The proof is completed. \( \square \)

Now, we prove the uniqueness of solutions.

**Theorem 3.7.** If \( \gamma > 2C_\ast \) and \( \|f\| \) is sufficiently small, where the \( C_\ast \) is a constant in the Poincaré inequality, the problem (1.1)-(1.3) admits at most one time-periodic solution.

**Proof.** To prove the Theorem, firstly, we give a suitable upper bound of the \( L^\infty \) norm of the time periodic solutions. Assume \( u \) is a periodic solution of the problem (1.1)-(1.3). Multiplying (1.1) by \( u, \) integrating the result over \( Q_T \) and using the conditions (1.2) and (1.3), we have

\[
\gamma \int_{Q_T} |\Delta u|^2 dx dt + \int_{Q_T} |\nabla u|^2 dx dt
\]

\[
= - \int_{Q_T} 3\gamma A(t)u^2 |\nabla u|^2 dx dt - \int_{Q_T} \gamma B(t)|\nabla u|^2 dx dt - \int_{Q_T} A(t)u^4 dx dt
\]

\[
+ \int_{Q_T} B(t)u^2 dx dt + \int_{Q_T} f(x, t)u dx dt.
\]

We notice that \( B(t) \leq K = \frac{1}{\gamma} \) and

\[
\int_{Q_T} |\nabla u|^2 dx dt \leq \frac{\gamma}{2} \int_{Q_T} |\Delta u|^2 dx dt + \frac{1}{2\gamma} \int_{Q_T} u^2 dx dt.
\]

From the above inequality and the assumptions on \( A(t), \ B(t), \) we obtain

\[
\gamma \int_{Q_T} |\Delta u|^2 dx dt + \int_{Q_T} |\nabla u|^2 dx dt + M \int_{Q_T} u^4 dx dt
\]

\[
\leq \frac{\gamma}{2} \int_{Q_T} |\Delta u|^2 dx dt + \left( \frac{1}{2\gamma} + K + \frac{1}{2\gamma} \right) \int_{Q_T} u^2 dx dt + \frac{\gamma}{2} \int_{Q_T} f^2 dx dt.
\]

By the Poincaré inequality, we see that

\[
\int_{\Omega} u^2 dx \leq C_\ast \int_{\Omega} |\nabla u|^2 dx.
\]
Therefore, we have
\[ \frac{\gamma}{2} \int_Q |\Delta u|^2 dx dt + \int_Q |\nabla u|^2 dx dt + M \int_Q u^4 dx dt \]
\[ \leq \left( \frac{1}{2\gamma} + K + \frac{1}{2\gamma} C_* \right) \int_Q |\nabla u|^2 dx dt + \frac{\gamma}{2} \int_Q f^2 dx dt. \]

If \( \gamma > 2C_* \), the above inequality implies that
\[ \int_Q |\Delta u|^2 dx dt \leq \int_Q f^2 dx dt, \quad (3.13) \]
\[ \int_Q |u|^4 dx dt \leq \frac{\gamma}{2M} \int_Q f^2 dx dt. \quad (3.14) \]

By the Hölder inequality, we know that
\[ \int_Q |u|^2 dx dt \leq (|\Omega| T)^{1/2} \left( \int_Q |u|^4 dx dt \right)^{1/2}, \quad (3.15) \]
and
\[ \int_Q |\nabla u|^2 dx dt \leq \frac{\gamma}{2} \int_Q f^2 dx dt + \frac{1}{2\gamma} (|\Omega| T)^{1/2} \left( \frac{\gamma}{2M} \int_Q f^2 dx dt \right)^{1/2}. \quad (3.16) \]

Let \( H_2(t) = \int_\Omega \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) dx \), where \( F(u, t) = \int_0^u f(s, t) ds = \frac{A(t)}{4} u^4 - \frac{B(t)}{2} u^2 \).

Integrating \(|H_2(t)|\) over \((0, T)\), using (3.13)-(3.16), we have
\[ \int_0^T |H_2(t)| dt \]
\[ \leq \int_Q \left( \frac{1}{2} |\nabla u|^2 + \left[ \frac{A(t)}{4} u^4 + \frac{B(t)}{2} u^2 \right] \right) dx dt \]
\[ \leq \frac{1}{2} \int_Q |\nabla u|^2 dx dt + \frac{M}{4} \int_Q u^4 dx dt + \frac{K}{2} \int_Q u^2 dx dt \]
\[ \leq C_1(\gamma, M, M, K) \int_Q f^2 dx dt + C_2(\gamma, M, M, K, T) \left( \int_Q f^2 dx dt \right)^{1/2}. \]

Similar to the proof of Lemma 2.3, we have
\[ \int_0^T \left| \frac{dH_2}{dt} \right| dt \leq C_3 \int_Q f^2 dx dt + C_4 \left( \int_Q f^2 dx dt \right)^{1/2}. \]

Hence, we know that \( ||H_2||_{W^{1,1}(0,T)} \leq C(||f||) \), from which we have \( H_2(t) \leq C(||f||) \). Noticing the definition of \( H_2(t) \) and using the Poincaré inequality, we get
\[ \int_\Omega \left[ \frac{1}{2} |\nabla u|^2 + \frac{A(t)}{4} u^4 \right] dx \leq C(||f||) + \int_\Omega \frac{B(t)}{2} u^2 dx \]
\[ \leq C(\|f\|) + \frac{K}{2} \int_{\Omega} u^2 \, dx \leq C(\|f\|) + \frac{K}{2} C_* \int_{\Omega} |\nabla u|^2 \, dx \]
\[ \leq C(\|f\|) + \frac{1}{4} \int_{\Omega} |\nabla u|^2 \, dx. \]

Therefore, we obtain
\[ \int_{\Omega} |\nabla u|^2 \, dx \leq C_5 \int_{Q_T} f^2 \, dx \, dt + C_6 \left( \int_{Q_T} f^2 \, dx \, dt \right)^{1/2}. \tag{3.17} \]

Similar to the proof of (2.17), we have
\[ \int_{\Omega} |\Delta u|^2 \, dx \leq C_7 \int_{Q_T} f^2 \, dx \, dt + C_8 \left( \int_{Q_T} f^2 \, dx \, dt \right)^{1/2}. \tag{3.18} \]

Hence,
\[ \|u\|_\infty \leq C_9 \int_{Q_T} f^2 \, dx \, dt + C_{10} \left( \int_{Q_T} f^2 \, dx \, dt \right)^{1/2}. \tag{3.19} \]

Suppose that \( u(x, t) \) and \( v(x, t) \) are two solutions of (1.1)-(1.3). Let \( w(x, t) = u(x, t) - v(x, t) \), then \( w(x, t) \) satisfies the following problem
\[ \frac{\partial w}{\partial t} = -\gamma \Delta^2 w + \Delta w + \gamma \Delta [\varphi(u) - \varphi(v)] - [\varphi(u) - \varphi(v)], \quad \gamma > 0, \quad x \in \Omega, \tag{3.20} \]
\[ w|_{\partial \Omega} = \Delta w|_{\partial \Omega} = 0, \tag{3.21} \]
\[ w(x, 0) = w(x, T), \quad x \in \Omega. \tag{3.22} \]

Multiplying (3.14) by \( w \), integrating the result over \( \Omega \) and using (3.16)-(3.19), we obtain
\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} w^2 \, dx + \gamma \int_{\Omega} |\Delta w|^2 \, dx + \int_{\Omega} |\nabla w|^2 \, dx \leq C_{11} \|f\| \|\nabla w\|^2. \]

Taking \( \|f\| \) sufficiently small such that \( 1 - C_{11} \|f\| = L > 0 \), and using the Poincaré inequality, one gets from above inequality that
\[ \frac{d}{dt} \|w\|^2 \leq -L \|w\|^2, \]
where \( L = \frac{L_1}{C_*} > 0 \), which implies that for any \( t > 0 \),
\[ \|w(\cdot, t)\|^2 \leq \|w(\cdot, 0)\|^2 e^{-Lt}. \]

Since \( w(x, t) \) is time periodic, for any \( t \in R \) there exists a natural number \( N_0 \) such that \( t + N_0 T > 0 \) and
\[ \|w(\cdot, t)\|^2 = \|w(\cdot, t + N_0 T)\|^2 \leq \|w(\cdot, 0)\|^2 e^{-LNT}, \quad \forall \, N > N_0, \]
which implies that
\[ \|w(\cdot, t)\|^2 = 0, \quad \forall \, t \in R. \]

The proof is completed. \( \square \)

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