DIFFERENTIAL HARNACK INEQUALITIES FOR A FAMILY OF SUB-ELLIPTIC DIFFUSION EQUATIONS ON SASAKIAN MANIFOLDS

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Abstract. We prove a version of differential Harnack inequality for a family of sub-elliptic diffusions on Sasakian manifolds under certain curvature conditions.

1. Introduction

Harnack inequality is one of the most fundamental results in the theory of elliptic and parabolic equations. For linear parabolic equations in divergence form, this was first done in [14]. Since then, numerous developments around this inequality were found. In [13], the so-called, Li-Yau estimate was proved. This is a sharp gradient estimate for linear parabolic equations on Riemannian manifolds with a lower bound on the Ricci curvature. This estimate is also called a differential Harnack inequality since one can recover the Harnack inequality by integrating this estimate along geodesics.

There are many generalizations of the Li-Yau estimate for geometric evolution equations. This includes the evolution equations for hypersurfaces [11, 7, 1], the Yamabe flow [8], the Ricci flow [10] and its Kähler analogue [5]. For a more detailed account of these generalizations as well as further developments, see [15].

There are also generalizations [2] of the Li-Yau estimate to linear parabolic equations of the form

(1.1) \[ \dot{\rho}_t = L \rho_t \]

under certain conditions called curvature-dimension conditions. Here \( L \) is a second order linear elliptic operator without constant term. The curvature-dimension conditions were recently generalized by [3] to obtain Li-Yau type estimates for equations of the form (1.1), where \( L \) is a linear sub-elliptic operator without constant term. The following is one of the main results in [3] when the underlying manifold is Sasakian

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and the equation is the sub-elliptic heat equation (for the definition of Sasakian manifolds and various related notions appeared in Theorem 1.1, see Section 2).

**Theorem 1.1.** \[3\] Assume that the manifold is Sasakian and satisfies $\mathcal{Rc} \geq 0$. Then any positive solution of the sub-elliptic heat equation

$$\dot{\rho}_t = \Delta_{\text{hor}} \rho_t$$

satisfies

$$\left(1 + \frac{3}{n}\right) f_t + \frac{1}{2} |\nabla_{\text{hor}} f_t|^2 + \frac{tn}{3} |\nabla_{\text{ver}} f_t(x)|^2 \leq \frac{2n \left(1 + \frac{3}{n}\right)^2}{t}$$

for all $t \geq 0$ and all $x$ in $M$, where $f_t = -2 \log \rho_t$.

We remark that a local version of the above estimate, which did not take the curvature into account, appeared in an earlier work \[6\].

On the other hand, the author proved in \[12\] another version of the differential Harnack inequality which is different from that of \[2\].

**Theorem 1.2.** \[12\] Let $\rho_t$ be a positive solution of the equation

$$\dot{\rho}_t = \Delta \rho_t + \langle \nabla \rho_t, \nabla U_1 \rangle + U_2 \rho_t$$

on a compact Riemannian manifold of non-negative Ricci curvature. Assume that

$$\Delta \left(-\Delta U_1 - \frac{1}{2} |\nabla U_1|^2 + 2U_2\right) \geq k_3.$$

Then

$$2\Delta \log \rho_t + \Delta U_1 \geq -na_{\frac{k_3}{n}}(t),$$

where

$$a_K(t) = \begin{cases} \sqrt{K} \cot(\sqrt{K} t) & \text{if } K > 0 \\ \frac{1}{t} & \text{if } K = 0 \\ \sqrt{-K} \coth(\sqrt{-K} t) & \text{if } K < 0. \end{cases}$$

In this paper, we combine the ideas from \[3\] and \[12\] to obtain a differential Harnack estimate for (1.1), where $L$ is a linear sub-elliptic operator (possibly with constant term). In fact, we allow $L$ to have a mild non-linearity (see Theorem 2.1 for the detail). In the case when the manifold is Sasakian and $L$ is linear, we have the following result.

**Theorem 1.3.** Assume that the manifold is a compact Sasakian and it satisfies $\mathcal{Rc} \geq 0$. Let $U_1$ and $U_2$ be two smooth functions on $M$ satisfying

1. $V = \Delta_{\text{hor}} U_1 + \frac{1}{2} |\nabla_{\text{hor}} U_1|^2 - 2U_2$
2. $V \leq k_1,$
(3) \[ \Delta_{\text{hor}} V + \frac{n^2}{3\kappa_2} \left( 1 + \frac{3}{n} \right)^2 |\nabla_{\text{ver}} V|^2 \leq \kappa_2, \]
for some positive constants \( \kappa_1 \) and \( \kappa_2 \). Then any positive solution of the equation
\[ \dot{\rho}_t = \Delta_{\text{hor}} \rho_t + \langle \nabla_{\text{hor}} U_1, \nabla_{\text{hor}} \rho_t \rangle + U_2 \rho_t \]
satisfies
\[ \left( 1 + \frac{3}{n} \right) \dot{f}_t(x) + \frac{1}{2} |\nabla_{\text{hor}} f_t|^2 - V(x) \]
\[ + \left( 1 + \frac{n}{3} \right) \sqrt{\frac{n}{2\kappa_2}} \tanh(c_2 t) |\nabla_{\text{ver}} f_t(x)|^2 \leq \frac{\kappa_2}{c_2} \coth(c_2 t) + \frac{3\kappa_1}{n} \]
for all \( t \geq 0 \) and all \( x \) in \( M \), where \( c_2 = \frac{1}{n+3} \sqrt{\frac{n+3}{2}} \) and \( f_t = -2 \log \rho_t - U_1 \).

As usual, we can integrate the estimate in Theorem 1.3 and obtain a Harnack estimate. Let
\[ W(x) = \left( \frac{n}{n+3} \right)^2 V(x) + \frac{3nK_4}{(n+3)^2}. \]
In this case, we consider the following cost function.
\[ c_{s_0, s_1}(x_0, x_1) = \inf_I \int_{s_0}^{s_1} \frac{1}{2} |\dot{\gamma}(s)|^2 + W(\gamma(s)) \, ds \]
where \( I \) ranges over all smooth curves \( \gamma(\cdot) \) such that \( \dot{\gamma}(s) \in D_{\gamma(s)} \) for all \( t \) in \([s_0, s_1]\).

**Corollary 1.4.** Assume that the manifold is compact Sasakian and it satisfies \( \text{Ric} \geq 0 \). Let \( U_1 \) and \( U_2 \) be two smooth functions on \( M \) satisfying
\( (1) \ V = \Delta_{\text{hor}} U_1 + \frac{1}{2} |\nabla_{\text{hor}} U_1|^2 - 2U_2 \)
\( (2) \ V \leq \kappa_1 \)
\( (3) \Delta_{\text{hor}} V + \frac{n^2}{3\kappa_2} \left( 1 + \frac{3}{n} \right)^2 |\nabla_{\text{ver}} V|^2 \leq \kappa_2, \)
for some positive constants \( \kappa_1 \) and \( \kappa_2 \). Then any positive solution of the equation
\[ \dot{\rho}_t = \Delta_{\text{hor}} \rho_t + \langle \nabla_{\text{hor}} U_1, \nabla_{\text{hor}} \rho_t \rangle + U_2 \rho_t \]
satisfies
\[ \frac{\rho_{s_1}(x_1)}{\rho_{s_0}(x_0)} \geq \left( \frac{\sinh(c_2 s_1)}{\sinh(c_2 s_0)} \right)^{(n+3)} \cdot \exp \left( -\frac{1}{2} \left( U(x_1) - U(x_0) + \left( 1 + \frac{3}{n} \right) c_{s_0, s_1}(x_0, x_1) \right) \right) \].
for all \( s_0, s_1 \geq 0 \) and all \( x_0, x_1 \) in \( M \), where \( c_2 = \frac{1}{n+3} \sqrt{\frac{n+2}{2}} \).

The structure of the paper is as follows. The main results of this paper are stated in Section 2. In [12], a moving frame argument was used for the proof of Theorem 1.2 instead of the Bochner formula. The advantage is that a matrix version of Theorem 1.2 generalizing the matrix Hamilton-Li-Yau estimate for the heat equation [9], can be proved using a very similar argument. Although there is no matrix analogue of Theorem 1.3 in this paper, we show that a version of the moving frame argument is possible in the present setting. This is done in Section 3 and 4. The proofs of the main results are given in Section 5. Section 6 is an appendix devoted to some calculations needed in the proofs.

2. The main results

In this section, we give the statements of the main results. First, let us introduce the setup which is essentially the same as that of [3].

Let \( M \) be a Riemannian manifold and let us fix a distribution \( D \) (a vector bundle of the tangent bundle \( TM \)) of rank \( k \). We assume that the orthogonal complement of \( D \) is spanned by \( n-k \) vector fields denoted by \( w_1, \ldots, w_{n-k} \) which satisfy certain symmetry conditions to be specified. We will also assume that the vector field \( X_t \) is the horizontal gradient \( \nabla_{\text{hor}} f_t \) of a one-parameter family of functions \( f_t \) defined on the manifold \( M \) and specialize Lemma 3.4 to this case.

Let us call vectors or vector fields which are contained in the distribution \( D \) horizontal. Let \( \psi_t \) be the flow of a vector field \( w \). Assume that \( \psi_t \) sends horizontal vector fields to horizontal ones and preserves their lengths. If \( X_1 \) and \( X_2 \) are horizontal vector fields, then we have

\[
\langle (\psi_t)_*(X_1), v \rangle = 0 \quad \text{and} \quad \langle (\psi_t)_*(X_1), (\psi_t)_*(X_2) \rangle = \langle X_1, X_2 \rangle.
\]

for any vector field \( v \) which is in the orthogonal complement of \( D \).

If we differentiate the above equations with respect to \( t \), then we obtain

\[
\langle [w, X_1], \tilde{w} \rangle = 0 \quad \text{and} \quad \langle \nabla_{X_1} w, X_2 \rangle + \langle X_1, \nabla_{X_2} w \rangle = 0.
\]

Therefore, we call a vector field \( w \) which satisfies the following two conditions horizontal isometry:

- \([w, X_1]\) is horizontal,
- \(\langle \nabla_{X_1} w, X_2 \rangle + \langle X_1, \nabla_{X_2} w \rangle = 0\),

for all horizontal vector fields \( X_1 \) and \( X_2 \).

Let \( v \) be a tangent vector of \( M \). Then the projections of \( v \) onto the distribution \( D \) and its orthogonal complement \( D^\perp \) are called the
horizontal part $v_{\text{hor}}$ and the vertical part $v_{\text{ver}}$ of $v$, respectively. Let $f : M \to \mathbb{R}$ be a smooth function. For the notation convenience, we also denote the horizontal part and vertical part of the gradient $\nabla f$ by $\nabla_{\text{hor}} f$ and $\nabla_{\text{ver}} f$, respectively. Let $v_1, \ldots, v_k$ be a frame in $D$ which is orthonormal. Then the sub-Laplacian of $f$ is defined by
\[
\Delta_{\text{hor}} f = \sum_i \langle \nabla_{v_i} \nabla f, v_i \rangle.
\]
Recall that the Ricci curvature $Rc(v, v)$ is defined as the trace of the following operator $w \mapsto \langle Rm(w, v)v, w \rangle$. We define the horizontal Ricci curvature $Rc_{\text{hor}}$ by
\[
Rc_{\text{hor}}(v, v) = \sum_i \langle Rm(v_i, v)v, v_i \rangle
\]
and the vertical Ricci curvature $Rc_{\text{ver}}$ by
\[
Rc_{\text{ver}}(v, v) = \sum_i \langle Rm(v_i, u)v, u_i \rangle.
\]
Let $\rho_t$ be a smooth positive solution of the following equation
\[
\dot{\rho}_t = \Delta_{\text{hor}} \rho_t + \langle \nabla U_1, \nabla_{\text{hor}} \rho_t \rangle + U_2 \rho_t + K \rho_t \log \rho_t.
\]
where $U_1, U_2$ are smooth functions on $M$ and $K$ is a constant.
Let $f_t$ be the one-parameter family of smooth functions defined by
\[
f_t = -2 \log \rho_t - U_1.
\]
A computation shows that $f_t$ satisfies the following equation
\[
(2.1) \quad \dot{f}_t + \frac{1}{2} |\nabla_{\text{hor}} f_t|^2 = \Delta_{\text{hor}} f_t + V + K f_t,
\]
where $V = \Delta_{\text{hor}} U_1 + KU_1 + \frac{1}{2} |\nabla_{\text{hor}} U_1|^2 - 2U_2$.
We call a solution $r$ of the problem
\[
\dot{r}(t) = F(r(t)), \quad r(t) \to \infty \text{ as } t \to 0^+
\]
stable if there is a family of solutions $r_\epsilon$ of the following
\[
\dot{r}(t) = F(r(t)) + \epsilon, \quad r_\epsilon(t) \to \infty \text{ as } t \to 0^+
\]
such that $r_\epsilon$ converges pointwise to $r_\epsilon$.
Finally, recall that a distribution is involutive if the Lie bracket of any two sections in the distribution is again in the distribution. The following is the main result of this paper.

**Theorem 2.1.** Assume that the orthogonal complement $D^\perp$ of the distribution $D$ is involutive and is given by the span of $n - k$ horizontal isometries. Assume also that the following conditions hold:
\[
(1) \quad Rc_{\text{hor}}(v, v) + 3Rc_{\text{ver}}(v_{\text{hor}}, v_{\text{hor}}) \geq K_1 |v_{\text{hor}}|^2 + K_2 |v_{\text{ver}}|^2,
\]
(2) $\mathbf{Rc}_{\text{ver}}(v_{\text{hor}}, v_{\text{hor}}) \leq K_3|v_{\text{hor}}|^2$.

(3) $V \leq K_4$.

(4) $\Delta_{\text{hor}}V + K_5|\mathbf{Rc}_{\text{ver}}V|^2 - 2K_1V \leq K_6$.

(5) $a_3(t) + \frac{4K_5}{a_2(t)} \geq 0$.

(6) $\dot{a}_1(t) + \frac{a_1(t)a_4(t)}{k} + (a_1(t) + 1) \left(2K_1 - a_3(t) - \frac{8K_3}{a_2(t)} \right) = 0$.

(7) $\frac{\dot{a}_2(t)}{2} - K_2 - 4K_3 + a_2(t) \left(\frac{a_3(t)}{4K_5} - \frac{a_4(t)}{2} + \frac{a_4(t)}{k} - \frac{K}{2} + K_1 \right) = 0$.

(8) $\dot{a}_3(t) + \frac{2a_4(t)a_4(t)}{k} + (K - a_3(t)) \left(a_3(t) + \frac{8K_3}{a_2(t)} - 2K_1 \right) = 0$.

for some positive constants $K_1, K_2, K_3, K_4, K_5, K_6$.

Let $r(\cdot)$ be a stable solution of

$$
\dot{r}(t) = \frac{a_4(t)^2}{k} + K_6 + 2 \left(a_3(t) + \frac{4K_5}{a_2(t)} \right) K_4
\vspace{0.5em}
+ \left(a_3(t) - \frac{2a_4(t)}{k} + K + \frac{8K_3}{a_2(t)} - 2K_1 \right) r(t)
$$

with the condition $r(t) \to \infty$ as $t \to 0^+$.

Then

$$
\Delta_{\text{hor}}f_t(x) + a_1(t)\dot{f}_t(x) + \frac{a_2(t)}{2}|\nabla_{\text{ver}}f_t(x)|^2 + a_3(t)f_t(x) \leq r(t)
$$

for all $t \geq 0$ and all $x$ in $M$.

In the case $\dot{\rho}_t = \Delta_{\text{hor}}\rho_t$, the proof of Theorem 2.1 gives the following result which is one of the results in [3]. Note that, unlike [3], the curvature conditions of the following result is written using a completely Riemannian notations.

**Corollary 2.2.** Assume that the orthogonal complement $D^\perp$ of the distribution $D$ is involutive and is given by the span of $n-k$ horizontal isometries. Assume also that the following conditions hold:

(1) $\mathbf{Rc}_{\text{hor}}(v, v) + 3\mathbf{Rc}_{\text{ver}}(v_{\text{hor}}, v_{\text{hor}}) \geq K_2|v_{\text{ver}}|^2$,

(2) $\mathbf{Rc}_{\text{ver}}(v_{\text{hor}}, v_{\text{hor}}) \leq K_3|v_{\text{hor}}|^2$,

for some positive constants $K_2, K_3$. Let $\rho_t$ be a smooth positive solution of the equation

$$
\dot{\rho}_t = \Delta_{\text{hor}}\rho_t
$$

and let $f_t = -2\log \rho_t$. Then

$$
\Delta_{\text{hor}}f_t(x) + c\dot{f}_t(x) + 2t \left(\frac{K_2}{c} - \frac{4K_3}{c} \right) |\nabla_{\text{ver}}f_t(x)|^2
\leq \frac{4(c + 1)^2K_3^2}{(cK_2 - 4K_3)(8K_3 - cK_2)t}
$$
for all $c > \frac{4K_3}{K_2}$, all $t \geq 0$, and all $x$ in $M$.

If we set $c = \frac{6K_3}{K_2}$ in Corollary 2.2 then the result further simplified to the following.

**Corollary 2.3.** Suppose that the assumptions in Corollary 2.2 hold. Then

$$\Delta_{\text{hor}} f_t(x) + \frac{6K_3}{K_2} \dot{f}_t(x) + \frac{2K_2}{3} t |\nabla_{\text{ver}} f_t(x)|^2 \leq \frac{k \left(1 + \frac{6K_3}{K_2}\right)^2}{t}$$

for all $t \geq 0$ and all $x$ in $M$.

In the case $\dot{\rho}_t = \Delta_{\text{hor}} \rho_t + \langle \nabla U_1, \nabla_{\text{hor}} \rho_t \rangle + U_2 \rho_t$, Theorem 2.1 gives

**Corollary 2.4.** Assume that the orthogonal complement $D^\perp$ of the distribution $D$ is involutive and is given by the span of $n - k$ horizontal isometries. Assume also that the following conditions hold:

1. $R_{\text{hor}} (v, v) + 3 R_{\text{ver}} (v_{\text{hor}}, v_{\text{hor}}) \geq K_2 |v_{\text{ver}}|^2$,
2. $R_{\text{ver}} (v_{\text{hor}}, v_{\text{hor}}) \leq K_3 |v_{\text{hor}}|^2$,
3. $V \leq K_4$,
4. $\Delta_{\text{hor}} V + K_5 |\nabla_{\text{ver}} V|^2 \leq K_6$,
5. $K_5 = \frac{2k(c+1)^2K_3^2}{cK_6(2K_3 - cK_2)}$,

for some positive constants $K_2, K_3, K_4, K_6$ and $c > \frac{4K_3}{K_2}$. Let $\rho_t$ be a smooth positive solution of the equation

$$\dot{\rho}_t = \Delta_{\text{hor}} \rho_t + \langle \nabla U_1, \nabla_{\text{hor}} \rho_t \rangle + U_2 \rho_t,$$

$$f_t = -2 \log \rho_t - U_1, c_1 = 2 \sqrt{K_5 \left( K_2 - \frac{4K_3}{c} \right)} \text{, and } c_2 = \sqrt{\frac{cK_3 - 4K_3}{cK_5}}.$$  

Then

$$\Delta_{\text{hor}} f_t(x) + c_1 \dot{f}_t(x) + \frac{c_1 \tanh(c_2 t)}{2} |\nabla_{\text{ver}} f_t(x)|^2 \leq \frac{K_6}{c_2 \coth(c_2 t)} + cK_4$$

for all $t \geq 0$ and all $x$ in $M$.

Next, we recall the definition of Sasakian manifolds and show that Theorem 1.3 is a consequence of Corollary 2.4. For a more detail discussion of Sasakian manifolds, see [4].

Let $M$ be a $2n+1$ dimensional manifold. A 1-form $\alpha$ on $M$ is contact if $d\alpha_x$ is a non-degenerate 2-form on the kernel $D$ of $\alpha$ (i.e.

$$D_x = \{v \in T_x M | \alpha(v) = 0\}$$

for each $x$).

Let $J$ be a $(1,1)$-tensor, $w$ be a vector field, and $\alpha$ be a contact 1-form on $M$. The triple $(J, w, \alpha)$ is an almost contact structure of $M$ if the following conditions hold

$$(2.2) \quad J^2(v) = -v, \quad \alpha(w) = 1, \quad J(w) = 0,$$
where \( v \) is any vector in \( D \).

An almost contact structure \((J, w, \alpha)\) is normal if

\[
[J, J](w_1, w_2) + d\alpha(w_1, w_2)w = 0
\]

for any vector fields \( w_1 \) and \( w_2 \) on \( M \), where \([J, J]\) denotes the Nijenhuis tensor defined by

\[
[J, J](w_1, w_2) = J^2[w_1, w_2] + [Jw_1, Jw_2] - J[Jw_1, w_2] - J[w_1, Jw_2].
\]

An almost contact structure \((J, w, \alpha)\) together with a Riemannian metric \(\langle \cdot, \cdot \rangle\) is called a almost contact metric structure if

\[
\langle Jv_1, Jv_2 \rangle = \langle v_1, v_2 \rangle \quad \text{and} \quad \langle w, w \rangle = 1,
\]

where \( v_1 \) and \( v_2 \) are vectors in \( D \).

An almost contact metric structure is a contact metric structure if

\[
\langle w_1, Jw_2 \rangle = d\alpha(w_1, w_2).
\]

A Sasakian manifold is a manifold \( M \) equipped with a contact metric structure which is normal.

An example of Sasakian manifolds is given by the Heisenberg group. The underlying space \( M \) of the Heisenberg group is the \( 2n + 1 \)-dimensional Euclidean space \( \mathbb{R}^{2n+1} \). In this case, the contact form \( \alpha \) is given by

\[
\alpha = dz - \frac{1}{2} \sum_{i=1}^{n} (y_idx_i - x_idy_i)
\]

where \( \{x_1, ..., x_n, y_1, ..., y_n, z\} \) are coordinates on \( \mathbb{R}^{2n+1} \).

In the Heisenberg group, the vector field \( w \) is given by \( w = \partial_z \). Let \( X_i = \partial_{x_i} + \frac{1}{2}y_i \partial_z \) and \( Y_i = \partial_{y_i} - \frac{1}{2}x_i \partial_z \). The Riemannian metric is defined such that \( \{X_1, ..., X_n, Y_1, ..., Y_n, w\} \) is an orthonormal frame. The tensor \( J \) is defined by

\[
JX_i = -Y_i, \quad JY_i = X_i, \quad \text{and} \quad Jw = 0.
\]

Back to the general case, the Riemann curvature tensor \( \mathbf{Rm} \) of a Sasakian manifold satisfies the following properties.

**Proposition 2.5.** Let \((M, J, w, \alpha, \langle \cdot, \cdot \rangle)\) be a Sasakian manifold. Then the followings hold:

1. \((\nabla_{w_1} J)w_2 = \frac{1}{2} (\langle w_1, w_2 \rangle w - \langle w, w_2 \rangle w_1), \)
2. \(\nabla_{w_1} w = -\frac{1}{2} Jw_1, \)
3. \(w\) is a Killing vector field,
4. \(\mathbf{Rm}(w_1, w_2)w = \frac{1}{4} (\langle w, w_2 \rangle w_1 - \langle w, w_1 \rangle w_2), \)

for all tangent vectors \( w_1 \) and \( w_2 \) on \( M \).
A proof of the above proposition can be found in [4]. Note that the definition of the exterior differential $d\alpha$ of a differential 2-form $\alpha$ used in this paper is

$$d\alpha(w_1, w_2) = w_1(\alpha(w_2)) - w_2(\alpha(w_1)) - \alpha([w_1, w_2]).$$

This is different from the one in [4] and so the above formulas are also different from those in [4] by a multiplicative constant.

Recall that the Tanaka connection $\overline{\nabla}$ of a given almost contact metric manifold is given by

$$\overline{\nabla}_{w_1} w_2 = \nabla_{w_1} w_2 + \frac{1}{2} \langle w, w_1 \rangle Jw_2 + \frac{1}{2} \langle w, w_2 \rangle Jw_1 - \frac{1}{2} \langle Jw_1, w_2 \rangle w.$$ 

The corresponding curvature $\overline{\text{Rm}}$ is given by

$$\overline{\text{Rm}}(w_1, w_2)w_3 = \overline{\nabla}_{w_1} \overline{\nabla}_{w_2} w_3 - \overline{\nabla}_{w_2} \overline{\nabla}_{w_1} w_3 - \overline{\nabla}_{[w_1, w_2]} w_3$$

and we denote by $\overline{\text{Rc}}$ the corresponding Ricci curvature

$$\overline{\text{Rc}}(w_1, w_1) = \text{trace}(w_2 \mapsto \langle \overline{\text{Rm}}(w_2, w_1)w_1, w_2 \rangle).$$

The following proposition shows that Sasakian manifolds provide examples to the main results.

**Proposition 2.6.** The followings hold on a Sasakian manifold:

1. $Rc_{\text{ver}}(v, v) = R\overline{c}_{\text{ver}}(v_{\text{hor}}, v_{\text{hor}}) = \frac{1}{4} |v_{\text{hor}}|^2,$
2. $Rc_{\text{hor}}(v, v) + 3Rc_{\text{ver}}(v_{\text{hor}}, v_{\text{hor}}) = R\overline{c}_{\text{hor}}(v, v) + \frac{3}{4} |v_{\text{hor}}|^2$
   $$= \frac{n}{2} |v_{\text{ver}}|^2 + \overline{\text{Rc}}(v_{\text{hor}}, v_{\text{hor}}).$$

**Proof.** Clearly, we have

$$Rc_{\text{ver}}(v, v) = \langle \text{Rm}(w, v)v, w \rangle$$

$$= \langle \text{Rm}(w, v_{\text{hor}})v_{\text{hor}}, w \rangle = R\overline{c}_{\text{ver}}(v_{\text{hor}}, v_{\text{hor}}).$$

By Proposition 2.5, we also have

$$\langle \text{Rm}(w, v_{\text{hor}})v_{\text{hor}}, w \rangle = \langle \text{Rm}(v_{\text{hor}}, w)v_{\text{hor}}, w \rangle = \frac{1}{4} |v_{\text{hor}}|^2.$$

By Proposition 2.5 again, we have

$$\langle \text{Rm}(w_1, v)v, w_2 \rangle = \langle \text{Rm}(w_1, v_{\text{hor}})v_{\text{hor}}, w_2 \rangle$$

$$+ \langle w, v \rangle \langle \text{Rm}(w_1, v)v, w_2 \rangle + \langle w, v \rangle \langle \text{Rm}(w_1, w)v_{\text{hor}}, w_2 \rangle$$

$$= \langle \text{Rm}(w_1, v_{\text{hor}})v_{\text{hor}}, w_2 \rangle + \frac{1}{4} \langle w, v \rangle^2 \langle w_1, w_2 \rangle$$

$$= \langle \text{Rm}(w_1, v_{\text{hor}})v_{\text{hor}}, w_2 \rangle + \frac{|v_{\text{ver}}|^2}{4} \langle w_1, w_2 \rangle.$$
is horizontal for any vector fields $X$ and $Y$. Therefore, by Proposition 2.5

\begin{align*}
\langle \bar{\nabla}_{v_i} \bar{\nabla}_{v_{i\text{hor}}} v_{j\text{hor}}, v_j \rangle &= \langle \bar{\nabla}_{v_i} \bar{\nabla}_{v_{i\text{hor}}} v_{j\text{hor}}, v_j \rangle \\
&= \langle \nabla_{v_i} (\bar{\nabla}_{v_{i\text{hor}}} v_{j\text{hor}} - \bar{\nabla}_{v_{i\text{hor}}} v_{j\text{hor}}), v_j \rangle \\
&= \langle \nabla_{v_i} (\bar{\nabla}_{v_{i\text{hor}}} v_{j\text{hor}} - \bar{\nabla}_{v_{i\text{hor}}} v_{j\text{hor}}), v_j \rangle \\
&= \langle \nabla_{v_i} \bar{\nabla}_{v_{i\text{hor}}} v_{j\text{hor}}, v_j \rangle.
\end{align*}

(2.6)

Here we extend $v$ to a vector field and still call it $v$.

Similarly, we also have

\begin{align*}
\langle \bar{\nabla}_{v_{i\text{hor}}} \bar{\nabla}_{v_i} v_{j\text{hor}}, v_j \rangle &= \langle \bar{\nabla}_{v_{i\text{hor}}} \bar{\nabla}_{v_i} v_{j\text{hor}}, v_j \rangle \\
&= \langle \bar{\nabla}_{v_{i\text{hor}}} (\bar{\nabla}_{v_i} v_{j\text{hor}} - \bar{\nabla}_{v_{i\text{hor}}} v_{j\text{hor}}), v_j \rangle \\
&= \langle \bar{\nabla}_{v_{i\text{hor}}} \bar{\nabla}_{v_i} v_{j\text{hor}}, v_j \rangle + \frac{1}{2} \langle \bar{\nabla}_{v_i} v_{j\text{hor}}, Jv \rangle \langle Jv, v_j \rangle \\
&= \langle \nabla_{v_{i\text{hor}}} \bar{\nabla}_{v_i} v_{j\text{hor}}, v_j \rangle - \frac{1}{4} \langle Jv, v_i \rangle \langle Jv, v_j \rangle.
\end{align*}

(2.7)

By the definition of Tanaka connection, it also follows that

\begin{align*}
\langle \bar{\nabla}_{[v_i, v_{i\text{hor}}]} v_{j\text{hor}}, v_j \rangle &= \langle \bar{\nabla}_{[v_i, v_{i\text{hor}}]} v_{j\text{hor}}, v_j \rangle \\
&= \langle \bar{\nabla}_{[v_i, v_{i\text{hor}}]} v_{j\text{hor}}, v_j \rangle + \frac{1}{2} \langle v_i, [v_{i\text{hor}}] \rangle \langle Jv, v_j \rangle.
\end{align*}

(2.8)

Therefore, by combining (2.6), (2.7), and (2.8), we obtain

\begin{align*}
\langle R_m (v_i, v_{i\text{hor}}) v_{j\text{hor}}, v_j \rangle &= \langle R_m (v_i, v_{i\text{hor}}) v_{j\text{hor}}, v_j \rangle \\
&+ \frac{1}{4} \langle Jv, v_i \rangle \langle Jv, v_j \rangle - \frac{1}{2} \langle v_i, [v_{i\text{hor}}] \rangle \langle Jv, v_j \rangle \\
&= \langle R_m (v_i, v_{i\text{hor}}) v_{j\text{hor}}, v_j \rangle + \frac{3}{4} \langle Jv, v_i \rangle \langle Jv, v_j \rangle.
\end{align*}

(2.9)

Since $\bar{\nabla} XY$ is horizontal, we also have

\begin{equation}
\langle R_m (w, v_{i\text{hor}}) v_{j\text{hor}}, w \rangle = \langle R_m (w, v_{i\text{hor}}) v_{j\text{hor}}, w \rangle = 0
\end{equation}

(2.10)

for any vector field $w$.

Therefore, by (2.5), (2.9), and (2.10), the second assertion follows. \hfill \Box

Proof of Theorem 1.3 It follows immediately from Corollary 2.3 and Proposition 2.6 with $c = \frac{6K_3}{K_2}$, $K_2 = \frac{n}{2}$, $K_3 = \frac{1}{4}$, and $k = 2n$. \hfill \Box

Proof of Corollary 1.4 Let $\gamma(\cdot)$ be a minimizer of the functional

$$\gamma(\cdot) \mapsto \inf I \int_{s_0}^{s_1} \frac{1}{2} |\dot{\gamma}(s)|^2 + W(\gamma(s)) ds,$$
where $I$ ranges over all smooth curves $\gamma(\cdot)$ such that $\gamma(s) \in D_{\gamma(s)}$ for all $t$ in $[s_0, s_1]$.

By Theorem 1.3,

$$\left(1 + \frac{3}{n}\right) \frac{d}{dt} f_t(\gamma(t)) = \left(1 + \frac{3}{n}\right) \dot{f}_t(\gamma(t)) + \left(1 + \frac{3}{n}\right) \langle \nabla_{\text{hor}} f_t(\gamma(t)), \dot{\gamma}(t) \rangle$$

$$\leq -\frac{1}{2} |\nabla_{\text{hor}} f_t|_x^2 + V(x) + \frac{K_6}{c_2} \coth(c_2 t) + \frac{3K_4}{n} + \left(1 + \frac{3}{n}\right) \langle \nabla_{\text{hor}} f_t(\gamma(t)), \dot{\gamma}(t) \rangle.$$

By Young's inequality, we have

$$\left(1 + \frac{3}{n}\right) \frac{d}{dt} f_t(\gamma(t)) \leq V(\gamma(t)) + \frac{K_6}{c_2} \coth(c_2 t) + \frac{3K_4}{n} + \frac{1}{2} \left(1 + \frac{3}{n}\right)^2 |\dot{\gamma}(t)|^2.$$

By integrating the above inequality, we obtain

$$\left(1 + \frac{3}{n}\right) (f_{s_1}(x_1) - f_{s_0}(x_0)) \leq \frac{K_6}{c_2} \ln \left(\frac{\sinh(c_2 s_1)}{\sinh(c_2 s_0)}\right) + \left(1 + \frac{3}{n}\right)^2 c_{s_0, s_1}(x_0, x_1).$$

This result follows from this. □

3. Parallel adapted frames

In this section, we define convenient adapted frames along a path called parallel adapted frames and use it to see how the linearization of a flow changes.

Let $M$ be a compact Riemannian manifold of dimension $n$ equipped with a distribution $D$ (i.e. a sub-bundle of the tangent bundle) of rank $k$. An orthonormal frame $v_1, \ldots, v_k, u_1, \ldots, u_{n-k}$ in the tangent space $T_x M$ at a point $x$ is an adapted frame if $v_1, \ldots, v_k$ is contained in the space $D_x$.

**Lemma 3.1.** Let $\gamma : [0, T] \to M$ be a smooth path in $M$. Then there exists a 1-parameter family of adapted frames

$$\{v_1(t), \ldots, v_k(t), u_1(t), \ldots, u_{n-k}(t)\}$$

along $\gamma(t)$ such that

1. $\dot{v}_i(t)$ is in the orthogonal complement of $D_{\gamma(t)}$ for $1 \leq i \leq k$,
2. $\dot{u}_j(t)$ is in $D_{\gamma(t)}$ for $1 \leq j \leq n - k$.

Here $\dot{v}$ denotes the covariant derivative of $v(\cdot)$ along $\gamma(\cdot)$.

Moreover, if $\{\bar{u}_1(t), \ldots, \bar{u}_{n-k}(t), \bar{v}_1(t), \ldots, \bar{v}_k(t)\}$ is another such frame, then there are orthogonal matrices $O^{(1)}$ and $O^{(2)}$ of sizes $k \times k$ and
Here, we use $\bar{d}$-tensor $\bar{w}$-tensor. Let $N$ precisely, the $\nabla_\phi \gamma$ which starts from frames to the above lemma, there is a unique 1-parameter family of adapted distribution. we have a smoothly varying flags of subspaces instead of just one distribution. The construction of the above frame can be generalized to a more complicated setting where we have a smoothly varying flags of subspaces instead of just one distribution.

**Remark 3.2.** The notion of parallel adapted frame is a generalization of parallel transported frame in Riemannian geometry to the present setting (see the end of this section for detail). The construction of the above frame can be generalized to a more complicated setting where we have a smoothly varying flags of subspaces instead of just one distribution.

Let $\varphi_t$ be the flow of a vector field $X_t$ defined by $\dot{\varphi}_t = X_t(\varphi_t)$ and $\varphi_0(x) = x$. The linearization $d\varphi_t$ of $\varphi_t$ satisfies

$$\frac{d}{dt}d\varphi_t(w) = \nabla_{d\varphi_t(w)}X_t.$$ 

Here, we use $\frac{d}{dt}$ to denote covariant derivative along $t \mapsto \varphi_t(x)$.

Therefore, the change in $d\varphi_t$ is completely determined by the $(1, 1)$-tensor $w \mapsto \nabla_w X_t$. We will investigate the equation satisfied by this tensor. Let $N(t)$ be the matrix representation of the $(1, 1)$-tensor $w \mapsto \nabla_w X_t$ with respect to the above parallel adapted frame at time $t$. More precisely, the $ij$-th entry $N_{ij}(t)$ of $N(t)$ is defined by

$$N_{ij}(t) = \begin{cases} 
\langle \nabla_{u_i(t)} X_t, u_j(t) \rangle & \text{if } i \leq n-k \text{ and } j \leq n-k \\
\langle \nabla_{u_i(t)} X_t, v_{j-n+k}(t) \rangle & \text{if } i \leq n-k \text{ and } j > n-k \\
\langle \nabla_{v_{i-n+k}(t)} X_t, u_j(t) \rangle & \text{if } i > n-k \text{ and } j \leq n-k \\
\langle \nabla_{v_{i-n+k}(t)} X_t, v_{j-n+k}(t) \rangle & \text{if } i > n-k \text{ and } j > n-k.
\end{cases}$$

Similarly, let $R(t)$ and $M(t)$ be the matrix representations of the bilinear form $w \mapsto \langle R_m(w, X_t) X_t, w \rangle$ and the $(1, 1)$-tensor

$$w \mapsto \nabla_w (\dot{X}_t + \nabla X_t X_t),$$
respectively, with respect to the given parallel adapted frame at time $t$. Finally, let
\[ \mathfrak{W}(t) = \begin{pmatrix} 0_{(n-k)\times(n-k)} & W(t) \\ -W(t)^T & 0_{k\times k} \end{pmatrix}, \]
where $W(t)$ is the $(n-k) \times k$ matrix with $ij$-th entry equal to $W_{ij}(t) = \langle \dot{u}_i(t), v_j(t) \rangle$.

\textbf{Lemma 3.3.} The 1-parameter family of matrices $N(t)$ satisfies the following matrix Riccati equation
\[ \dot{N}(t) = -N(t)^2 - N(t)\mathfrak{W}(t) - \mathfrak{W}(t)^T N(t) - R(t) + M(t). \]

Finally, we split each of $N(t)$, $R(t)$, and $M(t)$ into four pieces
\[ N(t) = \begin{pmatrix} N_{00}(t) & N_{01}(t) \\ N_{10}(t) & N_{11}(t) \end{pmatrix}, \quad R(t) = \begin{pmatrix} R_{00}(t) & R_{01}(t) \\ R_{10}(t) & R_{11}(t) \end{pmatrix}, \quad M(t) = \begin{pmatrix} M_{00}(t) & M_{01}(t) \\ M_{10}(t) & M_{11}(t) \end{pmatrix}, \]
where $N_{00}(t)$, $R_{00}(t)$, and $M_{00}(t)$ are of size $(n-k) \times (n-k)$.

The following, which will be used in the later sections, is an immediate consequence of Lemma 3.3.

\textbf{Lemma 3.4.} The 1-parameter family of matrices $N(t)$ satisfies the following
\[ \dot{N}_{11}(t) = -N_{11}(t)^2 - N_{10}(t)N_{01}(t) - N_{10}(t)W(t) - W(t)^T N_{01}(t) - R_{11}(t) + M_{11}(t). \]

\textbf{Proof of Lemma 3.4.} Let $(\bar{v}_1(t), \ldots, \bar{v}_k(t), \bar{u}_1(t), \ldots, \bar{u}_{n-k}(t))$ be a 1-parameter family of adapted frames along the path $\gamma(t)$. Let
\[ (v_1(t), \ldots, v_k(t), u_1(t), \ldots, u_{n-k}(t)) \]
be any other such family. Then
\[ v_i(t) = \sum_{l=1}^k O_{il}(t)\bar{v}_l(t), \quad 1 \leq i \leq k, \]
where $O(t)$ are orthogonal matrices of size $k \times k$.

Let $A(t)$ be the $k \times k$ matrix with $ij$-th entry equal to $\langle \dot{v}_i(t), \bar{v}_j(t) \rangle$. Then $\langle \dot{v}_i(t), v_j(t) \rangle = 0$ for each $1 \leq i, j \leq k$ if and only if
\[ \dot{O}(t) = -O(t)A(t). \]

Since $A(t)$ is skew symmetric, we have a solution $O(t)$ to the above ODE. Moreover, any solution is determined by its initial condition.
This proves the result for \( v_i(\cdot) \). A similar procedure gives the result for \( u_i(\cdot) \). \(\square\)

**Proof of Lemma 3.3**

Let

\[
V(t) = (u_1(t), \ldots, u_{n-k}(t), v_1(t), \ldots, v_k(t))^T
\]

and

\[
\dot{V}(t) = (\dot{u}_1(t), \ldots, \dot{u}_{n-k}(t), \dot{v}_1(t), \ldots, \dot{v}_k(t))^T,
\]

where \( \{u_1(t), \ldots, u_{n-k}(t), v_1(t), \ldots, v_k(t)\} \) is a parallel adapted frame defined along the curve \( t \mapsto \varphi_t(x) \) as in Lemma 3.1.

By Lemma 3.1, we have

\[
\dot{V}(t) = \left( \begin{array}{cc} 0_{(N-k) \times (N-k)} & W(t) \\ -W(t)^T & 0_{k \times k} \end{array} \right) V(t) = \mathcal{W}(t)V(t).
\]

If we differentiate the above equation once more, then we obtain

\[
\ddot{V}(t) = \mathcal{W}(t)V(t) + \mathcal{W}(t)\dot{V}(t) = (\mathcal{W}(0) + 2\mathcal{W}(t))V(t).
\]

Let \( \Phi_t = (d\varphi_t(v_0(0)), \ldots, d\varphi_t(v_2n(0)))^T \) and let \( A(t) \) be the matrices defined by

\[
\Phi_t = A(t)V(t).
\]

It follows that

\[
\frac{D}{dt}\Phi_t = (\dot{A}(t) + A(t)\mathcal{W}(t))V(t)
\]

and

\[
\frac{D}{dt}\frac{D}{dt}\Phi_t = \left( \dot{A}(t) + 2\dot{A}(t)\mathcal{W}(t) + A(t)\mathcal{W}(t) + A(t)\mathcal{W}(t)^2 \right) V(t).
\]

On the other hand, if we let \( \gamma(s) \) be a path such that \( \gamma'(0) = v_i(0) \), then

\[
\frac{D}{dt}d\varphi_t(v_i(0)) = \frac{D}{ds}X_t(\varphi_t(\gamma(s))) \bigg|_{s=0} = \sum_{j=0}^{2n} A_{ij}(t)\nabla v_j(t) X_t(\varphi_t(x)).
\]

By the definition of \( \mathbf{Rm} \), we also have

\[
\frac{D}{dt}\frac{D}{dt}d\varphi_t(v_i(0)) + \sum_{j=0}^{2n} A_{ij}(t)\mathbf{Rm}(v_j(t), X_t(\varphi_t(x)))X_t(\varphi_t(x))
\]

\[
= \frac{D}{ds}X_t(\varphi_t(\gamma(s))) \bigg|_{s=0} = \sum_{j=0}^{2n} A_{ij}(t) \left( \nabla v_j(t)(\dot{X}_t + \nabla X_t X_t) \right) (\varphi_t(x)).
\]
By (3.1) and (3.3), we have
\begin{equation}
N(t) = A(t)^{-1} \dot{A}(t) + \mathfrak{W}(t).
\end{equation}

By (3.2) and (3.4), we also have
\begin{equation}
A(t)^{-1} \ddot{A}(t) + 2A(t)^{-1} \dot{A}(t) \mathfrak{W}(t)
+ \mathfrak{W}(t) + \mathfrak{W}(t)^2 + R(t) - M(t) = 0.
\end{equation}

If we combine (3.5) and (3.6), then we obtain
\begin{equation}
\dot{N}(t) = -N(t)^2 - N(t) \mathfrak{W}(t) - \mathfrak{W}(t)^T N(t) - R(t) + M(t)
\end{equation}
as claimed.

Before ending this section, let us discuss the relationships between parallel transported frames and the Tanaka connection. In the usual Riemannian case, if \( v_1(t), ..., v_n(t) \) is a parallel orthonormal frame defined along a path \( \gamma \), then one can define the covariant derivative \( \dot{v}(t) \) of a vector field
\[ v(t) = a_1(t)v_1(t) + ... + a_n(t)v_n(t) \]
defined along \( \gamma(t) \) by
\[ \dot{v}(t) = \dot{a}_1(t)v_1(t) + ... + \dot{a}_n(t)v_n(t). \]
It is, of course, well-known that the covariant derivative is closely related to the corresponding Levi-Civita connection.

Similarly, one can define certain covariant derivative corresponding to the above parallel transported frames. More precisely, let
\[ \mathbf{w}((\gamma(t)), v_1(t), ..., v_{2n}(t) \]
be a parallel transported frame defined along a path \( \gamma \) in an almost contact metric manifold. If
\[ v(t) = a_0(t)\mathbf{w}(\gamma(t)) + a_1(t)v_1(t) + ... + a_{2n}(t)v_{2n}(t) \]
is a vector field defined along \( \gamma(t) \), then the covariant derivative \( \frac{D}{dt} \) corresponding to the parallel transported frames of \( v(t) \) along \( \gamma \) is defined by
\[ \frac{D}{dt} v(t) = \dot{a}_0(t)\mathbf{w}(\gamma(t)) + \dot{a}_1(t)v_1(t) + ... + \dot{a}_{2n}(t)v_{2n}(t). \]
Note that the definition of \( \frac{D}{dt} \) is well-defined.

The following lemma gives some basic properties of \( \frac{D}{dt} \) and some of its relationships with the Tanaka connection \( \nabla \). Since it is not needed for the rest of the paper, the proof is omitted.
Recall that Lemma 6.1, we have

Proof. Let us use the notation as in Lemma 3.3 with two constants and let $a(t)$ be a smooth function. Then

1. \( \frac{d}{dt} (c_1 w_1(t) + c_2 w_2(t)) = c_1 \frac{d}{dt} w_1(t) + c_2 \frac{d}{dt} w_2(t), \)
2. \( \frac{d}{dt} a(t) w(t) = \dot{a}(t) w(t) + a(t) \frac{d}{dt} w(t), \)
3. \( \frac{d}{dt} w = 0, \)
4. if both $w_1(t)$ and $w_2(t)$ are contained in $D$, then
   \[ \frac{d}{dt} \langle w_1(t), w_2(t) \rangle = \left( \frac{\tilde{D}}{dt} w_1(t), w_2(t) \right) + \left( w_1(t), \frac{\tilde{D}}{dt} w_2(t) \right), \]
5. if $Y$ is a vector field contained in $D$, then
   \[ \frac{\tilde{D}}{dt} Y(\gamma(t)) = \tilde{\nabla}_{\dot{\gamma}(t)} Y - \frac{1}{2} \langle w, \dot{\gamma}(t) \rangle JY. \]

4. Distributions with Transversal Symmetries

In this section, we assume that the orthogonal complement of the given distribution $D$ is involutive and is spanned by $n - k$ horizontal isometries denoted by $w_1, ..., w_{n-k}$. We will also assume that the vector field $X_t$ in the previous section is the horizontal gradient $\nabla_{hor} f_t$ of a one-parameter family of functions $f_t$ defined on the manifold $M$ and specialize Lemma 3.4 to this case.

Lemma 4.1. Assume that the orthogonal complement $D^\perp$ of the distribution $D$ is involutive and is given by the span of $n - k$ horizontal isometries. If $\varphi_t$ is the flow of a time-dependent vector field $\nabla_{hor} f_t$. Then

\[
\frac{d}{dt} \Delta_{hor} f_t(\varphi_t) \leq \frac{1}{k} (\Delta_{hor} f_t(\varphi_t(x)))^2 + \Delta_{hor} \left( \dot{f}_t + \frac{1}{2} |\nabla_{hor} f|^2 \right) - Rc_{\varphi_t(x)}(\nabla f_t, \nabla f_t) + Rc_{\varphi_t(x)}(\nabla_{hor} f_t, \nabla_{hor} f_t) - 4 \sum_k \langle \nabla_{hor} f_t w_k, \nabla w_k \nabla_{hor} f_t \rangle_{\varphi_t(x)}.
\]

Proof. Let us use the notation as in Lemma 3.3 with $X_t = \nabla_{hor} f_t$. Recall that $u_1(t), ..., u_{n-k}(t), v_1(t), ..., v_k(t)$ is a parallel adapted frame along the path $t \mapsto \varphi_t(x)$, where $\varphi_t$ is the flow of $\nabla_{hor} f_t$. By (1) of Lemma 6.1, we have

\[
\text{tr}(N_{11}(t)) = \sum_i \langle \nabla v_i(t) \nabla_{hor} f_t, v_i(t) \rangle = \Delta_{hor} f_t(\varphi_t(x)).
\]
By (1) of Lemma [6.1] and the symmetry of the Hessian, we also have
\[ \text{tr}(N_{11}(t)^2) = \sum_{i,j} \langle \nabla_{v_i(t)} \nabla_{\text{hor}} f, v_j(t) \rangle \langle \nabla_{v_j(t)} \nabla_{\text{hor}} f, v_i(t) \rangle \]
\[ = \sum_{i,j} \left( \langle \nabla_{v_i(t)} \nabla f, v_j(t) \rangle - \langle \nabla_{v_j(t)} \nabla f, v_i(t) \rangle \right) \cdot \left( \langle \nabla_{v_j(t)} \nabla f, v_i(t) \rangle - \langle \nabla_{v_i(t)} \nabla f, v_j(t) \rangle \right) \]
\[ = \sum_{i,j} \left( \langle \nabla_{v_i(t)} \nabla f, v_j(t) \rangle + \langle \nabla_{v_j(t)} \nabla f, v_i(t) \rangle \right) \cdot \left( \langle \nabla_{v_i(t)} \nabla f, v_j(t) \rangle - \langle \nabla_{v_j(t)} \nabla f, v_i(t) \rangle \right) \]
\[ = \sum_{i,j} \left( \langle \nabla_{v_i(t)} \nabla f, v_j(t) \rangle \right)^2 - \sum_{i,j} \langle \nabla_{v_j(t)} \nabla f, v_i(t) \rangle^2. \]

Therefore, by (11) of Lemma [6.1] and the Cauchy-Schwarz inequality, we have
\[ \text{tr}(N_{11}(t)^2) \geq \frac{1}{k} (\Delta_{\text{hor}} f_t(\varphi_t(x)))^2 - \text{Rc}_{\varphi_t(x)}^{\text{hor}} \langle \nabla_{\text{ver}} f, \nabla_{\text{ver}} f \rangle. \]

Let \( O(t) \) be a family of orthogonal matrices such that
\[ u_i(t) = \sum_j O_{ij}(t) w_j(\varphi_t(x)). \]

It follows from (1) of Lemma [6.1] that
\[ W_{ij}(t) = \langle \dot{u}_i(t), v_j(t) \rangle = \sum_k O_{ik}(t) \langle \nabla_{\nabla_{\text{hor}} f_t} w_k(\varphi_t(x)), v_j(t) \rangle \]
\[ = -\sum_k O_{ik}(t) \langle \nabla_{v_j(t)} w_k(\varphi_t(x)), \nabla_{\text{hor}} f_t(\varphi_t(x)) \rangle \]
\[ = \sum_k O_{ik}(t) \langle w_k(\varphi_t(x)), \nabla_{v_j(t)} \nabla_{\text{hor}} f_t \rangle = \langle \nabla_{v_i(t)} \nabla_{\text{hor}} f_t(u(t)) \rangle. \]

Therefore, we have
\[ W(t)^T = N_{10}(t). \]

Therefore, the \( ij \)-th component of \( W(t)^T N_{01}(t) = N_{10}(t) N_{01}(t) \) is given by
\[ \sum_t \langle \dot{u}_i(t), v_i(t) \rangle \langle \nabla_{u_i(t)} \nabla_{\text{hor}} f_t, v_j(t) \rangle \]
\[ = \sum_{k,l,s} O_{ik}(t) \langle \nabla_{\nabla_{\text{hor}} f_t} w_k(\varphi_t(x)), v_i(t) \rangle O_{ls}(t) \langle \nabla_{w_s} \nabla_{\text{hor}} f_t(\varphi_t(x)), v_j(t) \rangle \]
\[ = \sum_k \langle \nabla_{\nabla_{\text{hor}} f_t} w_k(\varphi_t(x)), v_i(t) \rangle \langle \nabla_{w_k} \nabla_{\text{hor}} f_t(\varphi_t(x)), v_j(t) \rangle \]
we obtain
\( \text{tr}(\mathcal{N}_{10}(t)\mathcal{N}_{01}(t)) = \text{tr}(W(t)^T\mathcal{N}_{01}(t)) \)

(4.4)
\[
= \sum_k \left\langle (\nabla_{\text{hor}f_t}w_k)_{\text{hor}}, (\nabla_w \nabla_{\text{hor}f_t})_{\text{hor}} \right\rangle_{\varphi_t(x)}.
\]

The \( ij \)-th component of \( \mathcal{N}_{10}(t)\mathcal{N}_{10}(t)^T \) is
\[
\sum_{k,s} O_{lk}(t) \left\langle v_i(t), \nabla_{\text{hor}f}w_k \right\rangle O_{ts}(t) \left\langle \nabla_{\text{hor}f}w_s, v_j(t) \right\rangle
= \sum_k \left\langle v_i(t), \nabla_{\text{hor}f}w_k \right\rangle \left\langle \nabla_{\text{hor}f}w_k, v_j(t) \right\rangle.
\]

Therefore,
\[
|\mathcal{N}_{01}(t)|^2 = \text{tr}(\mathcal{N}_{10}(t)\mathcal{N}_{10}(t)^T)
\]

(4.5)
\[
= \sum_k |(\nabla_{\text{hor}f_t}w_k)_{\text{hor}}|^2_{\varphi_t(x)} = \text{Rc}_{\varphi_t(x)}(\nabla_{\text{hor}f_t}w_t, \nabla_{\text{hor}f_t}w_t).
\]

By (1) and (3) of Lemma 6.1, it follows that
\[
\text{tr}(\mathcal{M}_{11}(t)) = \sum_i \left\langle \nabla_v(t)(\nabla_{\text{hor}}\hat{f}_t + \nabla_{\text{hor}f_t}v_i(t)) \right\rangle
= \Delta_{\text{hor}} \left( \hat{f}_t + \frac{1}{2}|\nabla_{\text{hor}f_t}|^2 \right) - 2 \sum_i \left\langle \nabla_v(t)\nabla_{\text{hor}f_t}v_i(t) \right\rangle.
\]

On the other hand, we have, by (8) and (12) of Lemma 6.1
\[
\sum_i \left\langle \nabla_v(t)\nabla_{\text{hor}f_t}\nabla_{\text{ver}f_t}, v_i(t) \right\rangle = \sum_{i,l} \left\langle \nabla_v(t)(w_l f_i \nabla_{\text{hor}f_t}w_l), v_i(t) \right\rangle
= \text{Rc}_{\varphi_t(x)}(\nabla_{f_t}v_t, \nabla_{\text{ver}f_t} + \sum_i \left\langle \nabla_{\text{hor}}(w_{f_i})_{\text{hor}}, \nabla_{\text{hor}f_t}w_i \right\rangle_{\varphi_t(x)}
= \text{Rc}_{\varphi_t(x)}(\nabla_{f_t}v_t, \nabla_{\text{ver}f_t} + \sum_i \left\langle \nabla_{w_l} \nabla_{\text{hor}f_t}w_l, \nabla_{\text{hor}f_t}w_l \right\rangle_{\varphi_t(x)}.
\]

Therefore, by combining this with (4.1), (4.2), (4.3), (4.4), and (4.5), we obtain
\[
\frac{d}{dt} \Delta_{\text{hor}} f_t(\varphi_t) = \frac{d}{dt} \text{tr}(\mathcal{N}_{11}(t))
\]
\[
= -\text{tr}(\mathcal{N}_{11}(t)^2) - 2\text{tr}(\mathcal{N}_{10}(t)\mathcal{N}_{01}(t)) - |\mathcal{N}_{10}(t)|^2 - \text{tr}(R_{11}(t)) + \text{tr}(\mathcal{M}_{11}(t))
\]
\[
\leq -\frac{1}{k} (\Delta_{\text{hor}} f_t(\varphi_t(x)))^2 - \text{Rc}_{\varphi_t(x)}(\nabla_{f_t}v_t, \nabla_{f_t}v_t) + \text{Rc}_{\varphi_t(x)}(\nabla_{\text{hor}f_t}, \nabla_{\text{hor}f_t})
- 4 \sum_k \left\langle (\nabla_{\text{hor}f_t}w_k)_{\text{hor}}, (\nabla_w \nabla_{\text{hor}f_t})_{\text{hor}} \right\rangle_{\varphi_t(x)} + \Delta_{\text{hor}} \left( \hat{f}_t + \frac{1}{2}\nabla_{\text{hor}f_t}|^2 \right).
\]
5. Proof of the main results

This section is devoted to the proofs of the main results. We begin with

**Lemma 5.1.** Assume that the orthogonal complement $D^\perp$ of the distribution $D$ is involutive and is given by the span of $n - k$ horizontal isometries. Then the followings hold:

1. $\frac{d}{dt}(f_t(\varphi_t)) = -\dot{f}_t(\varphi_t) + 2\Delta_{\text{hor}}f_t(\varphi_t) + 2V(\varphi_t) + 2Kf_t(\varphi_t),$
2. $\frac{d}{dt}(\dot{f}_t)(\varphi_t) = \Delta_{\text{hor}}\dot{f}_t(\varphi_t) + K\dot{f}_t(\varphi_t),$
3. $\frac{d}{dt} \left( \frac{1}{2} |\nabla_{\text{ver}}f_t|_{\varphi_t}^2 \right) = \Delta_{\text{hor}} \left( \frac{1}{2} |\nabla_{\text{ver}}f_t|_{\varphi_t}^2 \right)(\varphi_t)
   - \sum_i \langle (\nabla w_i \nabla_{\text{hor}}f_t - \nabla_{\text{hor}}f_t w_i)_{\text{hor}}|_{\varphi_t}^2 + \langle \nabla_{\text{ver}}f_t, \nabla V \rangle_{\varphi_t}.$

**Proof of Lemma 5.1.** The first assertion follows from (2.1). By (2.1), we have

\[
 \frac{d}{dt}(\dot{f}_t(\varphi_t)) = \frac{d}{dt} \left( -\frac{1}{2} |\nabla_{\text{hor}}f_t|_{\varphi_t}^2 + \Delta_{\text{hor}}f_t(\varphi_t) + V(\varphi_t) + Kf_t(\varphi_t) \right)
\]

\[= -\left( \nabla_{\text{hor}}f_t, \nabla_{\text{hor}}\dot{f}_t \right)_{\varphi_t} - \frac{1}{2} \langle \nabla |\nabla_{\text{hor}}f_t|_{\varphi_t}^2, \nabla_{\text{hor}}f_t \rangle_{\varphi_t} + \Delta_{\text{hor}}\dot{f}_t(\varphi_t)
+ \langle \nabla_{\text{hor}}\Delta_{\text{hor}}f_t, \nabla_{\text{hor}}f_t \rangle + \langle \nabla V, \nabla_{\text{hor}}f_t \rangle + K\dot{f}_t(\varphi_t) + K|\nabla_{\text{hor}}f_t|_{\varphi_t}^2
= \Delta_{\text{hor}}\dot{f}_t(\varphi_t) + K\dot{f}_t(\varphi_t).
\]

By (1) and (8) of Lemma 6.1 we have

\[
\langle \nabla_{\text{hor}}(w_i f), \nabla_{\text{hor}}f \rangle = \langle \nabla w_i, \nabla f - \nabla_{\text{hor}}f w_i, \nabla_{\text{hor}}f \rangle
= \left\langle \nabla \left( \frac{1}{2} |\nabla_{\text{hor}}f|_{\varphi_t}^2 \right), w_i \right\rangle.
\]

Therefore, by combining this with (2.1), we have

\[
\frac{d}{dt} \left( \frac{1}{2} |\nabla_{\text{ver}}f_t|_{\varphi_t}^2 \right) = \sum_i w_i f_i(\varphi_t) \frac{d}{dt}(w_i f_i(\varphi_t))
= \sum_i w_i f_i(\varphi_t) \left( w_i \dot{f}_i(\varphi_t) + \langle \nabla_{\text{hor}}f_t, \nabla_{\text{hor}}(w_i f_t) \rangle_{\varphi_t} \right)
= \sum_i w_i f_i(\varphi_t) w_i(\Delta f_t + V + K f_t)(\varphi_t)
= \langle \nabla_{\text{ver}}f_t, \nabla |\nabla_{\text{hor}}f_t + \nabla V \rangle_{\varphi_t} + K|\nabla_{\text{ver}}f|_{\varphi_t}^2.\]
Finally, by (10) of Lemma 6.1, we obtain
\[
\frac{d}{dt} \left( \frac{1}{2} |\nabla_{\text{ver}} f_t|_{\phi_t}^2 \right) = \Delta_{\text{hor}} \left( \frac{1}{2} |\nabla_{\text{ver}} f_t|_{\phi_t}^2 \right) (\phi_t) + K |\nabla_{\text{ver}} f_t|_{\phi_t}^2 \\
- \sum_i |(\nabla_{w_i} \nabla_{\text{hor}} f_t - \nabla_{\nabla_{\text{hor}} f_t w_i})_{\text{hor}}|_{\phi_t}^2 + \langle \nabla_{\text{ver}} f_t, \nabla V \rangle_{\phi_t}.
\]
\[\square\]

**Proof of Theorem 2.1.** By Lemma 4.1 and (12) of Lemma 6.1,
\[
\frac{d}{dt} \Delta_{\text{hor}} f_t (\phi_t) \leq \frac{a_4(t)^2}{k} - \frac{2a_4(t)}{k} \Delta_{\text{hor}} f_t (\phi_t) + \Delta_{\text{hor}} \left( \dot{f}_t + \frac{1}{2} |\nabla_{\text{hor}} f_t|^2 \right) (\phi_t) \\
- \text{Re}_{\phi_t} (\nabla f_t, \nabla f_t) - 3\text{Re}_{\phi_t} (\nabla_{\text{hor}} f_t, \nabla_{\text{hor}} f_t) \\
- 4 \sum_k \langle \nabla_{\nabla_{\text{hor}} f_t w_k}, \nabla_{w_k} \nabla_{\text{hor}} f_t - \nabla_{\nabla_{\text{hor}} f_t w_k} \rangle_{\phi_t}.
\]

By Young’s inequality and (12) of Lemma 6.1, the above inequality becomes
\[
\frac{d}{dt} \Delta_{\text{hor}} f_t (\phi_t) \leq \frac{a_4(t)^2}{k} - \frac{2a_4(t)}{k} \Delta_{\text{hor}} f_t (\phi_t) + \Delta_{\text{hor}} \left( \dot{f}_t + \frac{1}{2} |\nabla_{\text{hor}} f_t|^2 \right) (\phi_t) \\
+ a_2(t) \sum_k |(\nabla_{\nabla_{\text{hor}} f_t w_k} - \nabla_{w_k} \nabla_{\text{hor}} f_t)_{\text{hor}}|_{\phi_t}^2 + \frac{4}{a_2(t)} \text{Re}_{\phi_t} (\nabla_{\text{hor}} f_t, \nabla_{\text{hor}} f_t) \\
- \text{Re}_{\phi_t} (\nabla f_t, \nabla f_t) - 3\text{Re}_{\phi_t} (\nabla_{\text{hor}} f_t, \nabla_{\text{hor}} f_t).
\]

Using the first and the second assumptions,
\[
\frac{d}{dt} \Delta_{\text{hor}} f_t (\phi_t) \leq \frac{a_4(t)^2}{k} - \frac{2a_4(t)}{k} \Delta_{\text{hor}} f_t (\phi_t) + \Delta_{\text{hor}} \left( \dot{f}_t + \frac{1}{2} |\nabla_{\text{hor}} f_t|^2 \right) (\phi_t) \\
+ a_2(t) \sum_k |(\nabla_{\nabla_{\text{hor}} f_t w_k} - \nabla_{w_k} \nabla_{\text{hor}} f_t)_{\text{hor}}|_{\phi_t}^2 \\
+ \frac{4K_3}{a_2(t)} |\nabla_{\text{hor}} f_t|_{\phi_t}^2 - K_1 |\nabla_{\text{hor}} f_t|_{\phi_t}^2 - K_2 |\nabla_{\text{ver}} f_t|_{\phi_t}^2.
\]
Lemma 5.1, we have

\[ \dot{F}_t(x) = \Delta_{\text{hor}} f_t(x) + a_1(t) \dot{f}_t(x) + \frac{a_2(t)}{2} |\nabla_{\text{ver}} f_t(x)|^2 + a_3(t) f_t(x). \]

By Lemma 5.1, we have

\[
\frac{d}{dt} F_t(\varphi_t) \leq \frac{a_4(t)^2}{k} - 2a_4(t) \Delta_{\text{hor}} f_t(\varphi_t) + \Delta_{\text{hor}} \left( \dot{f}_t + \frac{1}{2} |\nabla_{\text{hor}} f_t|^2 \right)(\varphi_t)
\]

\[ + \frac{4K_3}{a_2(t)} |\nabla_{\text{hor}} f_t|^2 - K_1 |\nabla_{\text{hor}} f_t|^2 - K_2 |\nabla_{\text{ver}} f_t|^2 + \dot{a}_1(t) \dot{f}_t(\varphi_t) + \frac{\dot{a}_2(t)}{2} |\nabla_{\text{ver}} f_t|^2.
\]

By Young’s inequality and the definition of $F_t$, we obtain

\[
\frac{d}{dt} F_t(\varphi_t) \leq \frac{a_4(t)^2}{k} + \Delta_{\text{hor}} F_t + \Delta_{\text{hor}} V + 2a_3(t) V(\varphi_t)
\]

\[ + 2 \left( \frac{4K_3}{a_2(t)} - K_1 \right) V + \left( \frac{\dot{a}_2(t)}{2} - K_2 \right) |\nabla_{\text{ver}} f_t|^2
\]

\[ + \left( a_3(t) - K_2 \right) |\nabla_{\text{hor}} f_t|^2
\]

\[ + \left( \dot{a}_1(t) + a_1(t) K - a_3(t) \right) \dot{f}_t(\varphi_t). \]

By Young’s inequality and the definition of $F_t$, we obtain

\[
\frac{d}{dt} F_t(\varphi_t) \leq \frac{a_4(t)^2}{k} + \Delta_{\text{hor}} F_t + \Delta_{\text{hor}} V(\varphi_t) + 2a_3(t) V(\varphi_t)
\]

\[ + 2 \left( \frac{4K_3}{a_2(t)} - K_1 \right) V(\varphi_t) + \left( \frac{\dot{a}_2(t)}{2} - K_2 - 4K_3 \right) |\nabla_{\text{ver}} f_t|^2
\]

\[ + a_2(t) \left( a_2(t) - a_3(t) + K + \frac{8K_3}{a_2(t)} - 2K_1 \right) \frac{K_5}{4K_5} |\nabla_{\text{ver}} f_t|^2
\]

\[ + \left( \dot{a}_1(t) + a_1(t) K - a_3(t) \right) \frac{2K_1 - a_3(t) - 8K_3}{a_2(t)} \dot{f}_t(\varphi_t).
\]
By assumptions (6), (7), and (8), the inequality becomes
\[
\frac{d}{dt} F_t(\varphi_t) \leq \frac{a_4(t)^2}{k} + \Delta_{hor} F_t(\varphi_t) + \Delta_{hor} V(\varphi_t) + K_5|\nabla_{ver} V|^2_{\varphi_t} \\
+ \left( a_3(t) - \frac{2a_4(t)}{k} + K + \frac{8K_3}{a_2(t)} - 2K_1 \right) F_t(\varphi_t) \\
+ 2 \left( a_3(t) + \frac{4K_3}{a_2(t)} - K_1 \right) V(\varphi_t).
\]
By assumptions (3), (4), and (5),
\[
\frac{d}{dt} F_t(\varphi_t) \leq \frac{a_4(t)^2}{k} + \Delta_{hor} F_t(\varphi_t) + K_6 + 2 \left( a_3(t) + \frac{4K_3}{a_2(t)} \right) K_4 \\
+ \left( a_3(t) - \frac{2a_4(t)}{k} + K + \frac{8K_3}{a_2(t)} - 2K_1 \right) F_t(\varphi_t).
\]
Let \( \epsilon > 0 \) and let \( r_\epsilon(\cdot) \) be a solution of
\[
\dot{r}_\epsilon(t) = \frac{a_4(t)^2}{k} + K_6 + 2 \left( a_3(t) + \frac{4K_3}{a_2(t)} \right) K_4 \\
+ \left( a_3(t) - \frac{2a_4(t)}{k} + K + \frac{8K_3}{a_2(t)} - 2K_1 \right) r_\epsilon(t) + \epsilon.
\]
with condition \( r_\epsilon(t) \to \infty \) as \( t \to 0 \).
Let \( t_0 > 0 \) be the first time where there is a point \( x \) in \( M \) satisfying
\( F_{t_0}(\varphi_{t_0}(x)) = r(t_0) \). Then
\[
\dot{r}_\epsilon(t_0) \leq \frac{d}{dt} F_t(\varphi_t) \leq \frac{a_4(t_0)^2}{k} + K_6 + 2 \left( a_3(t_0) + \frac{4K_3}{a_2(t_0)} \right) K_4 \\
+ \left( a_3(t_0) - \frac{2a_4(t_0)}{k} + K + \frac{8K_3}{a_2(t_0)} - 2K_1 \right) r_\epsilon(t_0) < \dot{r}_\epsilon(t_0)
\]
which is a contradiction.

Therefore, \( F_t(x) < r_\epsilon(t) \) for all \( t \geq 0 \) and all \( x \) in \( M \). The result follows from stability of \( r \).

**Proof of Corollary 2.2.** This follows from Theorem 2.1 by setting \( K_1 = K = 0, a_3 \equiv 0, a_1 \equiv c, a_4(t) = \frac{a(c+1)K_3K_1}{ca_2(t)}, a_2(t) = 2(2K_2 - \frac{4K_3}{c}), t \), and \( r(t) = \frac{a(c+1)^2kK_1^2}{(cK_2 - 4K_3)(8K_3 - cK_2)} \).

**Proof of Corollary 2.3.** If we set \( K_1 = K = 0, a_3 \equiv 0, a_1 \equiv c \) is a constant, \( a_2(t) = c_1 \tanh(c_2t), a_4(t) = \frac{4k(c+1)K_1}{ca_2(t)}, \) and \( K_5 = \frac{a_1K_6(8K_3 - cK_2)}{c_2}, K_4 = \frac{K_6c_2}{c_2} \coth(c_2t) + cK_4 \) in Theorem 2.1 Then a computation shows that \( r(t) = \frac{K_6}{c_2} \coth(c_2t) + cK_4 \) and the result follows from Theorem 2.1.
6. Appendix

In this appendix, we provide the detail calculations that we used in the proof of Theorem 2.1.

**Lemma 6.1.** Suppose that the assumptions in Lemma 4.1 hold. Let \(w_1, \ldots, w_{n-k}\) be the horizontal isometries which span \(D^\perp\). Let \(X_1\) and \(X_2\) be vector fields contained in \(D\) and let \(Z\) be a vector field contained in \(D^\perp\). Then the followings hold:

1. \(\langle \nabla_{X_1} Z, X_2 \rangle = - \langle \nabla_{X_2} Z, X_1 \rangle\),
2. \(\nabla_{v_i} v_i\) is horizontal,
3. \(\nabla_{w_i} w_i\) is vertical,
4. \(\nabla_{v_i} w_i\) is horizontal,
5. \(\nabla_{\nabla_{hor} f} \nabla_{hor} f = \frac{1}{2} \nabla_{hor} |\nabla_{hor} f|^2 - 2 (\nabla_{\nabla_{hor} f} \nabla_{ver} f)_{hor}\),
6. \(\sum_j \langle [w_i, v_j] \nabla_{hor} f, v_j \rangle = 0\),
7. \(\sum_j \langle \nabla_{[w_i, v_j]} \nabla_{hor} f, v_j \rangle = - \sum_j \langle \nabla_{v_j} \nabla_{hor} f, [w_i, v_j] \rangle\),
8. \(\nabla_{hor} (w_i f) = (\nabla_{w_i} \nabla_{hor} f - \nabla_{\nabla_{hor} f} w_i)_{hor}\),
9. \(\Delta_{hor} (w_i f) = w_i (\Delta_{hor} f)\),
10. \(\Delta_{hor} \left(\frac{1}{2} \nabla_{ver} f^2\right) = \langle \nabla_{ver} f, \nabla \Delta_{hor} f \rangle + \sum_i \langle (\nabla_{v_i} \nabla_{hor} f - \nabla_{\nabla_{hor} f} w_i)_{hor} \rangle^2\),
11. \(Rc_{hor} (\nabla_{ver} f, \nabla_{ver} f) = \sum_j (\langle \nabla_{v_j} \nabla_{hor} f, v_j \rangle\rangle\),
12. \(Rc_{hor} (\nabla f, w_i) = \sum_j \langle \nabla_{v_j} \nabla_{hor} f w_i, v_j \rangle\),
13. \(\langle \nabla_{v_j} w_i, [w_j, w_i] \rangle \rangle = \langle \nabla_{v_j} w_i, \nabla_{v_j} w_i \rangle\).

**Proof.** By assumption, we have

\[
\langle \nabla_{w_i} X_1, X_2 \rangle + \langle \nabla_{w_i} X_2, X_1 \rangle = w_i \langle X_1, X_2 \rangle = ([w_i, X_1], X_2) + ([w_i, X_2], X_1) + \langle [w_i, X_1], X_2 \rangle + \langle [w_i, X_2], X_1 \rangle.
\]

This gives (1).

It follows from (1) that

\[
\langle \nabla_{v_i} v_i, w_j \rangle = - \langle v_i, \nabla_{w_i} w_j \rangle = 0
\]

which is (2).

Since \(D\) is involutive, we also have

\[
\langle \nabla_{w_i} w_j, v_k \rangle = - \langle w_j, \nabla_{w_i} v_k \rangle = - \langle w_j, \nabla_{v_k} w_i \rangle = - \langle v_k, \nabla_{w_i} w_j \rangle.
\]

This gives (3).

Since

\[
\langle \nabla_{v_j} w_i, w_k \rangle = \langle \nabla_{w_i} v_j, w_k \rangle = - \langle v_j, \nabla_{w_i} w_k \rangle = 0,
\]

(4) holds.
The statement (5) follows from

\[ \left\langle \frac{1}{2} \nabla |\nabla_{\text{hor}} f|^2, X_1 \right\rangle = \left\langle \nabla X_1 \nabla_{\text{hor}} f, \nabla_{\text{hor}} f \right\rangle = \left\langle \nabla X_1 \nabla f, \nabla_{\text{hor}} f \right\rangle - \langle \nabla X_1 \nabla_{\text{ver}} f, \nabla_{\text{hor}} f \rangle = \left\langle \nabla \nabla_{\text{hor}} f \nabla f, X_1 \right\rangle + \left\langle \nabla \nabla_{\text{hor}} f \nabla_{\text{ver}} f, X_1 \right\rangle = \left\langle \nabla \nabla_{\text{hor}} f \nabla_{\text{hor}} f, X_1 \right\rangle + 2 \left\langle \nabla \nabla_{\text{hor}} f \nabla_{\text{ver}} f, X_1 \right\rangle. \]

Since \( w_i \) is a horizontal isometry, \([w_i, v_j] \) is horizontal. Therefore,

\[ \sum_j \left\langle \nabla_{[w_i,v_j]} \nabla f, v_j \right\rangle = \sum_j \left\langle \nabla_{v_j} \nabla f, [w_i, v_j] \right\rangle = \sum_{j,k} \left\langle \nabla_{v_j} \nabla f, v_k \right\rangle \left\langle v_k, [w_i, v_j] \right\rangle = - \sum_{j,k} \left\langle \nabla_{v_j} \nabla f, v_k \right\rangle \left\langle v_j, [w_i, v_k] \right\rangle = 0 \]

which is (6).

It follows from (1) and (6) that

\[ 0 = \sum_j \left\langle \nabla_{[w_i,v_j]} \nabla f, v_j \right\rangle = \sum_j \left\langle \nabla_{v_j} \nabla f, [w_i, v_j] \right\rangle = \sum_j \left\langle \nabla_{v_j} \nabla_{\text{hor}} f, [w_i, v_j] \right\rangle + \left\langle \nabla_{v_j} \nabla_{\text{ver}} f, [w_i, v_j] \right\rangle = \sum_j \left\langle \nabla_{v_j} \nabla_{\text{hor}} f, [w_i, v_j] \right\rangle - \left\langle \nabla_{[w_i,v_j]} \nabla_{\text{ver}} f, v_j \right\rangle. \]

By (6), we also have

\[ 0 = \sum_j \left\langle \nabla_{[w_i,v_j]} \nabla_{\text{hor}} f, v_j \right\rangle + \sum_j \left\langle \nabla_{[w_i,v_j]} \nabla_{\text{ver}} f, v_j \right\rangle. \]

Therefore, (7) follows.

Statement (8) follows from

\[ \left\langle \nabla (w_i f), v_j \right\rangle = \left\langle \nabla_{v_j} \nabla f, w_i \right\rangle + \left\langle \nabla_{\text{hor}} f, \nabla_{v_j} w_i \right\rangle + \left\langle \nabla_{\text{ver}} f, \nabla_{v_j} w_i \right\rangle = \left\langle \nabla w_i \nabla f, v_j \right\rangle + \left\langle \nabla_{\text{hor}} f, \nabla_{v_j} w_i \right\rangle + \left\langle \nabla_{\text{ver}} f, \nabla w_i v_j \right\rangle = \left\langle \nabla w_i \nabla_{\text{hor}} f, v_j \right\rangle - \left\langle \nabla \nabla_{\text{hor}} f w_i, v_j \right\rangle. \]
By (1) and (8), we have
\[
\Delta_{\text{hor}}(w_if) = \sum_j \langle \nabla_{v_j} (\nabla_w \nabla_{\text{hor}} f - \nabla_{\nabla_{\text{hor}} f} w_i), v_j \rangle
\]
\[
= \sum_j \langle \nabla_w \nabla_{v_j} \nabla_{\text{hor}} f, v_j \rangle - \langle \nabla_{\nabla_{\text{hor}} f} \nabla_{v_j} w_i, v_j \rangle
\]
\[
+ \sum_j \langle \nabla_{[v_j, w_i]} \nabla_{\text{hor}} f - \nabla_{[v_j, \nabla_{\text{hor}} f]} w_i, v_j \rangle.
\]

By (1), (3), and (4), the above equation becomes
\[
\Delta_{\text{hor}}(w_if) = \sum_j \langle \nabla_{w_i} \nabla_{v_j} \nabla_{\text{hor}} f, v_j \rangle - \langle \nabla_{\nabla_{\text{hor}} f} \nabla_{v_j} w_i, v_j \rangle
\]
\[
+ \sum_j \langle \nabla_{[v_j, w_i]} \nabla_{\text{hor}} f - \nabla_{[v_j, \nabla_{\text{hor}} f]} w_i, v_j \rangle.
\]

By (1) and (7),
\[
\Delta_{\text{hor}}(w_if) = \sum_j \langle \nabla_{w_i} \nabla_{v_j} \nabla_{\text{hor}} f, v_j \rangle
\]
\[
- \sum_j \langle \nabla_{v_j} \nabla_{\text{hor}} f, [v_j, w_i] \rangle + \langle \nabla_{v_j} w_i, \nabla_{v_j} \nabla_{\text{hor}} f \rangle
\]
\[
= \sum_j \langle \nabla_{w_i} \nabla_{v_j} \nabla_{\text{hor}} f, v_j \rangle + \sum_j \langle \nabla_{v_j} \nabla_{\text{hor}} f, \nabla_{w_i} v_j \rangle.
\]

This, together with (6), gives (9).

By (8) and (9), we have
\[
\Delta_{\text{hor}} \left( \frac{1}{2} |\nabla_{\text{ver}} f|^2 \right) = \sum_{i,j} (v_i w_j f)^2 + \sum_j w_j f \Delta_{\text{hor}}(w_j f)
\]
\[
= \sum_j |\nabla_{\text{hor}}(w_j f)|^2 + \langle \nabla_{\text{ver}} f, \nabla \Delta_{\text{hor}} f \rangle
\]
\[
= \sum_j |(\nabla_{w_i} \nabla_{\text{hor}} f - \nabla_{\nabla_{\text{hor}} f} w_i)_{\text{hor}}|^2 + \langle \nabla_{\text{ver}} f, \nabla \Delta_{\text{hor}} f \rangle
\]
which is (10).

By (3), $\nabla_{\nabla_{\text{ver}} f} \nabla_{\text{ver}} f$ is vertical. Therefore, by (1), we have
\[
Rc_{\text{hor}}(\nabla_{\text{ver}} f, \nabla_{\text{ver}} f) = -\sum_j \langle \nabla_{\nabla_{\text{ver}} f} \nabla_{v_j} \nabla_{\text{ver}} f, v_j \rangle - \sum_j \langle \nabla_{[v_j, \nabla_{\text{ver}} f]} \nabla_{\text{ver}} f, v_j \rangle.
\]
By (1), the above becomes
\[ R_c^{\text{hor}}(\nabla_{\text{ver}} f, \nabla_{\text{ver}} f) = \sum_j \left( \langle \nabla v_j \nabla_{\text{ver}} f, \nabla_{\text{ver}} f \rangle + \langle [v_j, \nabla_{\text{ver}} f], \nabla_{\text{ver}} f \rangle \right) \]
\[ = \sum_j \left( \langle \nabla v_j \nabla_{\text{ver}} f, \nabla_{\text{ver}} f \rangle \right) + |(\nabla v_j \nabla_{\text{ver}} f)_{\text{hor}}|^2. \]

It follows from (4) that \( \nabla_{\text{ver}} f v_j \) is horizontal. Therefore, (11) holds.

By (1) and (3),
\[ R_c^{\text{hor}}(\nabla f, w_i) = \sum_j \langle \nabla v_j \nabla_{\text{hor}} f w_i, v_j \rangle + \langle \nabla v_j w_i, \nabla_{\text{ver}} f v_j \rangle - \langle [v_j, \nabla_{\text{ver}} f] w_i, v_j \rangle \]
\[ = \sum_j \langle \nabla v_j \nabla_{\text{hor}} f w_i, v_j \rangle + \langle \nabla v_j w_i, \nabla_{\text{ver}} f v_j \rangle + \langle \nabla v_j w_i, [v_j, \nabla_{\text{ver}} f] \rangle \]
\[ = \sum_j \langle \nabla v_j \nabla_{\text{hor}} f w_i, v_j \rangle + \langle \nabla v_j w_i, \nabla_{\text{ver}} f \rangle. \]

Statement (12) follows since
\[ \sum_j \langle \nabla v_j w_i, \nabla_{\text{ver}} f \rangle = \sum_{j,k} \langle \nabla v_j w_i, v_k \rangle \langle v_k, \nabla_{\text{ver}} f \rangle = 0 \]
by (1).

By (1) and (3), we have
\[ \langle Rm(v_j, w_i) w_k, v_j \rangle = \langle \nabla v_j w_k, \nabla_{\text{ver}} f v_j \rangle + \langle \nabla v_j w_k, [v_j, w_i] \rangle \]
\[ = \langle \nabla v_j w_i, \nabla_{\text{ver}} f \rangle \]
which is (13).

**References**

[1] B. Andrews: Harnack inequalities for evolving hypersurfaces. Math. Z. 217 (1994), no. 2, 179-197.
[2] D. Bakry, M. Ledoux: A logarithmic Sobolev form of the Li-Yau parabolic inequality. Rev. Mat. Iberoam. 22 (2006), no. 2, 683-702.
[3] F. Baudoin, N. Garofalo: Curvature-dimension inequalities and Ricci lower bounds for sub-Riemannian manifolds with transverse symmetries, arXiv: 1101.3590 (2012).
[4] D.E. Blair: Riemannian geometry of contact and symplectic manifolds. Second edition. Progress in Mathematics, 203. Birkhuser Boston, Inc., Boston, MA, 2010.
[5] H.-D. Cao: On Harnack’s inequalities for the Kähler-Ricci flow. Invent. Math. 109 (1992), no. 2, 247-263.
[6] H.-D. Cao, S.-T. Yau: Gradient estimates, Harnack inequalities and estimates for heat kernels of the sum of squares of vector fields. Math. Z. 211 (1992), no. 3, 485-504.
[7] B. Chow: On Harnack’s inequality and entropy for the Gaussian curvature flow. Comm. Pure Appl. Math. 44 (1991), no. 4, 469-483.
[8] B. Chow: The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature. Comm. Pure Appl. Math. 45 (1992), no. 8, 1003-1014.
[9] R. Hamilton: A matrix Harnack estimate for the heat equation. Comm. Anal. Geom. 1 (1993), no. 1, 113-126.
[10] R. Hamilton: The Harnack estimate for the Ricci flow. J. Differential Geom. 37 (1993), no. 1, 225-243.
[11] R. Hamilton: Harnack estimate for the mean curvature flow. J. Differential Geom. 41 (1995), no. 1, 215-226.
[12] P.W.Y. Lee: On the Li-Yau estimate, arXiv: 1211.5559 (2012).
[13] P. Li, S.-T. Yau: On the parabolic kernel of the Schrodinger operator. Acta Math. 156 (1986), no. 3-4, 153-201.
[14] J. Moser: A Harnack Inequality for Parabolic Differential Equations. Commun. Pure Appl. Math 17 (1964), 101-134.
[15] L. Ni: Monotonicity and Li-Yau-Hamilton inequalities. Surveys in differential geometry. Vol. XII. Geometric flows, 251-301, Surv. Differ. Geom., 12, Int. Press, Somerville, MA, 2008.

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Differential Harnack inequalities on Sasakian manifolds