LONGTIME ROBUSTNESS AND SEMI-UNIFORM COMPACTNESS OF A PULLBACK ATTRACTOR VIA NONAUTONOMOUS PDE

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ABSTRACT. This paper is concerned with the robustness of a pullback attractor as the time tends to infinity. A pullback attractor is called forward (resp. backward) compact if the union over the future (resp. the past) is pre-compact. We prove that the forward (resp. backward) compactness is a necessary and sufficient condition such that a pullback attractor is upper semi-continuous to a compact set at positive (resp. negative) infinity, and also obtain the minimal limit-set. We further prove the lower semi-continuity of the pullback attractor and get the maximal limit-set at infinity. Some criteria for such robustness are established when the evolution process is forward or backward omega-limit compact. Those theoretical criteria are applied to prove semi-uniform compactness and robustness at infinity in pullback dynamics for a Ginzburg-Landau equation with variable coefficients and a forward or backward tempered nonlinearity.

1. Introduction. To mathematically analyze the dynamics of a nonautonomous model, an important concept of a pullback attractor was introduced with notable developments (see [1, 4, 5, 14, 25, 26, 33] and references therein). Although a pullback attractor is time-dependent, many properties (such as compactness, attraction, dimension and semi-continuity) were investigated w.r.t. a single time-segment in monographs (cf. [6, 17]). These segmental properties are just similar as the autonomous case.

In this paper, we are concerned with the robustness of a pullback attractor as the time tends to a finite number or infinity, in particular, we wonder whether a pullback attractor has robustness at infinity, i.e. longtime robustness. This subject specifically focuses on a non-autonomous dynamical system since the attractor is time-independent in an autonomous system.

There are many papers discussing the semi-continuity for a family of attractors, where the object of study was aimed at a family of dynamical systems and caused by external factors such as: varieties of coefficients in equations (cf. [7, 15, 27, 29]),
perturbations of the domain (cf. [2, 3, 19, 23]) and varieties of density for random noise (cf. [11, 13, 18, 24, 34, 35, 37, 40, 42]) etc.

However, in this paper, we consider the semi-continuity of a pullback attractor for a single (rather than a family) dynamical system, where the variation tendency is caused by an intrinsic factor, i.e. the time parameter. This type of semi-continuity seems to be new in literatures.

It is relatively easy to solve the robustness problem at a finite time. Indeed, we prove in Sec.2 that a pullback attractor $A = \{A(t)\}_{t \in \mathbb{R}}$ in a metric space $(X, d)$ is always upper semi-continuous w.r.t. the Hausdorff semi-metric, i.e.

$$d(A(t), A(t_0)) := \sup_{a \in A(t)} \inf_{b \in A(t_0)} d(a, b) \to 0 \text{ as } t \to t_0,$$

and lower semi-continuous, i.e. $\lim_{t \to t_0} d(A(t_0), A(t)) = 0$ for all $t_0 \in \mathbb{R}$.

Main concern of this paper is to consider the robustness at infinity. Unlike the finite time case, the possible limit-set at infinity is unknown. So the first task is to look for this limit-set.

**Question 1.** What is the possible limit-set of a pullback attractor $A(t)$ as $t \to \pm \infty$?

Some new terminologies of upper limit-set $A_U(\pm \infty)$ and lower limit-set $A_L(\pm \infty)$ for a pullback attractor are given by Defs. 3.1, 4.1. We may regard the upper (resp. lower) limit-set as a possible limit-set at infinity in the upper (resp. lower) semi-continuity sense.

However, example 1 in Sec.3 indicates that a pullback attractor may not be semi-continuous to any compact set as $t \to \pm \infty$. So the second task is to find some required conditions.

**Question 2.** What are the conditions to ensure that $A(t)$ is upper semi-continuity to the upper limit-set as $t \to \pm \infty$? i.e.

$$\lim_{t \to -\infty} d(A(t), A_U(-\infty)) = 0, \quad \lim_{t \to +\infty} d(A(t), A_U(\infty)) = 0,$$

and lower semi-continuity to the lower limit-set? i.e.

$$\lim_{t \to -\infty} d(A_L(-\infty), A(t)) = 0, \quad \lim_{t \to +\infty} d(A_L(\infty), A(t)) = 0.$$

A pullback attractor is called forward compact or backward compact (collectively referred to as semi-uniformly compact) if $\cup_{r \geq t} A(r)$ or $\cup_{s \leq t} A(s)$ is pre-compact for each $t \in \mathbb{R}$. We then prove an interesting result that the forward (resp. backward) compactness is just a necessary and sufficient condition such that a pullback attractor is upper semi-continuous to a compact set at positive (resp. negative) infinity (see Theorem 3.4). In this case, the upper limit-set is minimal as a limit-set of a pullback attractor at infinity (see Prop.1).

In Sec.4, we further prove that, under the semi-uniform compactness assumption, a pullback attractor is lower semi-continuous to the lower limit-set at infinity and the lower limit-set is maximal as a limit-set (see Theorem 4.4). So, a pullback attractor is both upper and lower semi-continuous to the same set if and only if both upper and lower limit-sets are identical (see Theorem 4.5).

In Sec.5, we give some criteria, in terms of the evolution process, for semi-uniform compactness of a pullback attractor, which leads to the robustness at infinity. The criteria for backward compactness had been established in recent works (cf. [10, 21]). We establish the criteria for forward compactness in Theorems 5.4 and 5.8. Note that the backward compactness coincides with the pullback dynamics, while the forward compactness does not.
In Sec.6, we consider the application on the non-autonomous Ginzburg-Landau equation with some variable coefficients and a time-dependent nonlinearity. If the nonlinearity is forward tempered, then the GL system has a forward compact pullback attractor, which leads to the semi-continuity result at positive infinity. Correspondingly, the semi-continuity result at negative infinity is established.

2. Continuity of a pullback attractor at finite time. Throughout this paper, \((X, d)\) is a metric space, \(\mathcal{B}\) is the collection consisted of all nonempty bounded subsets in \(X\).

An evolution process on \(X\) means a family of mappings \(S(t, s) : X \to X\) \((t \geq s)\) satisfying
\[
S(s, s) = id_X, \quad S(t, s) = S(t, r)S(r, s) \quad \text{for all } t \geq r \geq s, \tag{4}
\]
and the mapping \(\mathcal{S}_a : [a, +\infty) \times X \mapsto X, (s, x) \mapsto S(s, a)x\) is continuous for all \(a \in \mathbb{R}\).

A time-dependent family of nonempty subsets \(\mathcal{K}(t) (t \in \mathbb{R})\) in \(X\) is said to be a brochette over \(X\). The terminology of a brochette was used in [32, 43], we denote it as \(\{\mathcal{K}(t) : t \in \mathbb{R}\}\) or even \(\mathcal{K}\) for briefness.

**Definition 2.1.** A brochette \(\mathcal{K} = \{\mathcal{K}(t) : t \in \mathbb{R}\}\) over \(X\) is called

(i) compact (resp. bounded) if \(\mathcal{K}(t)\) is compact (resp. bounded) for each \(t \in \mathbb{R}\),

(ii) locally compact if it is compact and \(\cup_{s \in [a, b]} \mathcal{K}(s)\) is pre-compact for any \(a < b\),

(iii) forward compact if both \(\mathcal{K}(t)\) and \(\cup_{r \geq t} \overline{\mathcal{K}(r)}\) are compact for each \(t \in \mathbb{R}\),

(iv) backward compact if it is compact and \(\cup_{s \leq t} \mathcal{K}(s)\) is pre-compact for each \(t \in \mathbb{R}\),

(v) globally compact if it is compact and \(\cup_{s \in \mathbb{R}} \mathcal{K}(s)\) is pre-compact,

(vi) decreasing (resp. increasing) if \(\mathcal{K}(t_1) \supset \mathcal{K}(t_2)\) (resp. \(\mathcal{K}(t_1) \subset \mathcal{K}(t_2)\)) for \(t_1 \leq t_2\).

**Definition 2.2.** A brochette \(\mathcal{A}\) over \(X\) is said to be a pullback attractor for an evolution process \(S(\cdot, \cdot)\) if it is: (a) compact, (b) invariant, i.e. \(S(t, s)\mathcal{A}(s) = \mathcal{A}(t)\) for all \(t \geq s\),

(c) pullback attracting, i.e. for each \(D \in \mathcal{B}, t \in \mathbb{R}\),
\[
\lim_{\tau \to +\infty} d(S(t, t - \tau)D, \mathcal{A}(t)) = 0, \tag{5}
\]

(d) minimal, i.e. \(\mathcal{A}(t) \subset \mathcal{K}(t)\) if \(\mathcal{K}\) is a closed and pullback attracting brochette.

Here, we use the definition of a pullback attractor given by Carvalho et al. [6]. In particular, we need the minimality to ensure the uniqueness.

**Lemma 2.3.** A pullback attractor \(\mathcal{A}\) for an evolution process \(S(\cdot, \cdot)\) must be locally compact.

**Proof.** Let \(a, b \in \mathbb{R}\) with \(a < b\). We need to prove \(\cup_{s \in [a, b]} \mathcal{A}(s)\) is a pre-compact set (actually a compact set). If we define a mapping \(\mathcal{S} : [a, +\infty) \times X \mapsto X, (s, x) \mapsto S(s, a)x\), then \(\mathcal{S}\) is a continuous mapping since the process \(S\) is continuous as assumed in the definition. By the invariance, we know \(S(s, a)\mathcal{A}(a) = \mathcal{A}(s)\) for each \(s \geq a\). Hence, \(\cup_{s \in [a, b]} \mathcal{A}(s) = \mathcal{S}([a, b] \times \mathcal{A}(a)), \) which is a compact set since it is the range of a compact set under a continuous mapping.

By using the local compactness of a pullback attractor, we can give some easy-to-verify criteria for forward or backward compactness.
Lemma 2.4. A pullback attractor $\mathcal{A}$ is forward (resp. backward) compact if and only if $\bigcup_{t \geq t_0} \mathcal{A}(t)$ (resp. $\bigcup_{s \leq t_0} \mathcal{A}(s)$) is pre-compact for some $t_0 \in \mathbb{R}$.

Proof. Let $t \in \mathbb{R}$ be arbitrary. If $t \leq t_0$, then $\bigcup_{s \leq t} \mathcal{A}(s) \subset \bigcup_{s \leq t_0} \mathcal{A}(s)$. Since the latter is pre-compact, so is the former. If $t > t_0$, then we can rewritten the set by $\bigcup_{s \leq t} \mathcal{A}(s) = (\bigcup_{s \leq t_0} \mathcal{A}(s)) \cup (\bigcup_{t_0 < s \leq t} \mathcal{A}(s))$. By Lemma 2.3, $\bigcup_{t_0 < s \leq t} \mathcal{A}(s)$ is pre-compact and so $\bigcup_{s \leq t} \mathcal{A}(s)$ is still pre-compact. It is similar to prove the forward compactness assertion.

The following result indicates that a pullback attractor is always continuous at any finite time.

Theorem 2.5. Kloeden et al. [17]. A pullback attractor $\mathcal{A}(t)$ is always continuous at any finite time, i.e.

$$
\lim_{t \to t_0} \text{dist}_h(\mathcal{A}(t), \mathcal{A}(t_0)) = 0, \quad \forall t_0 \in \mathbb{R},
$$

where the Hausdorff metric $\text{dist}_h(A, B) = \max\{d(A, B), d(B, A)\}$ and $d(A, B)$ is as given in (1).

The proof of Theorem 2.5 requires a result on the continuity of a set-valued mapping, which slightly generalizes the corresponding result in Roxin [28] (or see page 31-32 in Kloeden et al. [17]).

Lemma 2.6. Suppose $f : (a, \infty) \times X \to X$ is continuous and $K$ is a nonempty compact subset. Then the set-valued mapping $t \mapsto f(t, K) := \bigcup_{x \in K} \{f(t, x)\}$ is continuous, i.e.

$$
\lim_{t \to t_0} \text{dist}_h(f(t, K), f(t_0, K)) = 0, \quad \text{for all } t_0 > a.
$$

Proof. We prove the lower semi-continuity: $d(f(t_0, K), f(t, K)) \to 0$ as $t \to t_0$ with $t_0 > a$. If this is not true, then there are $\delta > 0$, $a < t_n \to t_0$ and $x_n \in f(t_0, K)$ such that $d(x_n, f(t_n, K)) \geq \delta$ for all $n \in \mathbb{N}$. Since $f(t_0, K)$ is obviously compact, there is a subsequence (not relabeled) such that $x_n \to x \in f(t_0, K)$. We write $x = f(t_0, y)$ with $y \in K$ and $y_n = f(t_n, y) \in f(t_n, K)$. By the continuity of $f$, $y_n = f(t_n, y) \to f(t_0, y) = x$. Then we have

$$
d(x_n, f(t_n, K)) \leq d(x_n, x) + d(x, y_n) + d(y_n, f(t_n, K)) \to 0,
$$

which gives a contradiction. It is easy to prove the upper semi-continuity.

To prove Theorem 2.5, we let $t_0 \in \mathbb{R}$, $a = t_0 - 1$ and take

$$
f(t, x) = S(t, t_0 - 1)x \quad \text{for } t > t_0 - 1, \quad K = \mathcal{A}(t_0 - 1).
$$

Then, by the invariance of $\mathcal{A}$, we have $f(t, K) = \mathcal{A}(t)$ for all $t > t_0 - 1$. Hence, by Lemma 2.6, $\mathcal{A}(t)$ is continuous to $\mathcal{A}(t_0)$ as $t \to t_0$, which proves Theorem 2.5.

We remark here that Kloeden [17, page 31] had given the following formula:

$$
\text{dist}_h(\mathcal{A}(t), \mathcal{A}(t_0)) = \text{dist}_h(S(t, t_0)\mathcal{A}(t_0), S(t_0, t_0)\mathcal{A}(t_0)) \to 0, \quad \text{as } t \to t_0.
$$

However, this formula only proved the right-continuity (i.e. $t \to t_0^+$) in (6) since $S(t, t_0)$ is well-defined only when $t \geq t_0$. So we add the proof of left-continuity as mentioned above.

3. Longtime robustness of a pullback attractor. From now on, we intend to consider the semi-continuity of a pullback attractor at infinity, which is the main content of this paper.
3.1. Upper limit-sets of a pullback attractor. Motivating from the meanings of the upper semi-continuity, we guess that possible limit-sets at infinity are defined as follows.

**Definition 3.1.** Let \( \mathcal{K} = \{ \mathcal{K}(t) : t \in \mathbb{R} \} \) be a brochette over \( X \). We define the upper limit-set by

\[
\mathcal{K}_U(-\infty) = \cap_{t \in \mathbb{R}} \overline{\cup_{s \leq t} \mathcal{K}(s)}, \quad \mathcal{K}_U(\infty) = \cap_{t \in \mathbb{R}} \overline{\cup_{t \geq r} \mathcal{K}(r)},
\]

where the overline denotes the closure of a set in \( X \).

**Lemma 3.2.** Let \( \mathcal{K} \) be a brochette over \( X \). Then

(i) \( x \in \mathcal{K}_U(-\infty) \) if and only if there are a sequence \( s_n \downarrow -\infty \) and \( x_n \in \mathcal{K}(s_n) \) such that \( x_n \to x \).

(ii) \( y \in \mathcal{K}_U(\infty) \) if and only if there are a sequence \( r_n \uparrow +\infty \) and \( y_n \in \mathcal{K}(r_n) \) such that \( y_n \to y \).

(iii) We have alternative formulas for upper limit-sets.

\[
\mathcal{K}_U(-\infty) = \cap_{n=1}^{\infty} \overline{\cup_{s \leq n} \mathcal{K}(s)}, \quad \mathcal{K}_U(\infty) = \cap_{n=1}^{\infty} \overline{\cup_{t \geq r} \mathcal{K}(r)}.
\]

**Proof.** It is easy to prove (8). We then prove (ii). Let \( y \in \mathcal{K}_U(\infty) \). Then \( y \in \mathcal{K}_U(\infty) \) if and only if there is a sequence \( y_n \in \mathcal{K}(r_n) \) such that \( y_n \to y \). In general, since \( y \in \mathcal{K}_U(\infty) \), we successively choose \( n \to \max\{r_n-1,n\} \) and \( n \to \mathcal{K}(r_n) \) such that \( d(y_n,y) < 1/n \). Then we have \( r_n \to +\infty \) and \( y_n \to y \) with \( y_n \in \mathcal{K}(r_n) \).

Conversely, let \( y_n \in \mathcal{K}(r_n) \) such that \( r_n \to +\infty \) and \( y_n \to y \). Then, for each \( t \in \mathbb{R} \), there is an \( n_t \in \mathbb{N} \) such that \( r_n \geq t \) for all \( n \geq n_t \), which implies \( y_n \in \mathcal{K}(r_n) \subset \mathcal{U}_{t \geq t} \mathcal{K}(r) \) for \( n \geq n_t \). We know \( y \in \mathcal{K}_U(\infty) \) for all \( t \in \mathbb{R} \) and so \( y \in \mathcal{K}_U(\infty) \) as required. It is similar to prove (i). \( \square \)

We meet a trouble that the upper limit-set of a pullback attractor may be empty as shown by the following example.

**Example 1.** We consider a simple non-autonomous ODE: \( u_t + \alpha u = f(t), u(s) = u_0 \in \mathbb{R} \), where \( \alpha > 0 \). The explicit solution defines an evolution process by

\[
S(t, t-\tau)u_0 = e^{-\alpha \tau}u_0 + \int_{-\tau}^{0} e^{\alpha r} f(r + t) dr, \quad \forall t \in \mathbb{R}, \quad \tau \geq 0.
\]

Assume that \( u^*(t) := \int_{-\infty}^{0} e^{\alpha r} f(r + t) dr \) is integrable, then there is a pullback attractor \( \mathcal{A} = \{ u^*(t) \} \).

If we take \( f(t) = e^{-\alpha t/2} \), then it is easy to calculate \( u^*(t) = \frac{2}{\alpha} e^{-\frac{\alpha}{2} t} \). Hence \( \mathcal{U}_{t \leq t} \mathcal{A}(s) = \left[ \frac{2}{\alpha} e^{-\frac{\alpha}{2} s}, +\infty \right) \) and \( \mathcal{U}_{t \geq r} \mathcal{A}(r) = \left[ 0, \frac{2}{\alpha} e^{-\frac{\alpha}{2} r} \right) \). So, \( \mathcal{A}_U(-\infty) = \emptyset \) and \( \mathcal{A}_U(\infty) = \{0\} \neq \emptyset \).

If we take \( f(t) = e^{\alpha t} \), then we have \( u^*(t) = \frac{1}{2\alpha} e^{\alpha t} \). In this case, \( \mathcal{U}_{t \leq t} \mathcal{A}(s) = \left[ \frac{1}{2\alpha} e^{\alpha t}, +\infty \right) \) and \( \mathcal{U}_{t \geq r} \mathcal{A}(r) = \left[ \frac{1}{2\alpha} e^{\alpha r}, +\infty \right) \), which implies that \( \mathcal{A}_U(-\infty) = \{0\} \neq \emptyset \) and \( \mathcal{A}_U(\infty) = \emptyset \).

**Lemma 3.3.** Let \( \mathcal{A} \) be a pullback attractor for an evolution process. Then \( \mathcal{A}_U(-\infty) \) is nonempty compact if \( \mathcal{A} \) is backward compact, and \( \mathcal{A}_U(\infty) \) is nonempty compact if \( \mathcal{A} \) is forward compact. Both \( \mathcal{A}_U(-\infty) \) and \( \mathcal{A}_U(\infty) \) are nonempty compact if \( \mathcal{A} \) is globally compact.
Proof. Let $C_n = \bigcup_{s \leq -n} \mathcal{A}(s)$ $(n \in \mathbb{N})$. Lemma 3.2 implies $\mathcal{A}_U(-\infty) = \cap_{n=1}^{\infty} C_n$. If $\mathcal{A}$ is backward compact, then $\{C_n\}$ is a decreasing sequence of compact subsets. Therefore, by the theorem of nested compact sets, $\cap_{n=1}^{\infty} C_n$ is nonempty compact and so is $\mathcal{A}_U(-\infty)$, which proves the first conclusion.

Suppose $\mathcal{A}$ is forward compact. Then $K_n := \bigcup_{r \geq n} \mathcal{A}(r)$ $(n \in \mathbb{N})$ defines still a decreasing sequence of compact subsets. Then Lemma 3.2 implies that $\mathcal{A}_U(\infty) = \cap_{n=1}^{\infty} K_n$ is nonempty and compact, which proves the second conclusion.

Suppose $\mathcal{A}$ is globally compact. Then $\mathcal{A}$ is both backward and forward compact and so both $\mathcal{A}_U(-\infty)$ and $\mathcal{A}_U(\infty)$ are nonempty compact.

3.2. Upper semi-continuity of a pullback attractor at infinity. We prove an interesting result, which indicates that the upper semi-continuity at infinity is completely determined by the semi-uniform compactness.

**Theorem 3.4.** Let $\mathcal{A}$ be a pullback attractor for an evolution process $S(\cdot, \cdot)$. Then

(i) $\mathcal{A}_U(-\infty)$ is nonempty compact such that $\mathcal{A}(t)$ is upper semi-continuous to $\mathcal{A}_U(-\infty)$, i.e.

$$\lim_{t \to -\infty} d(\mathcal{A}(t), \mathcal{A}_U(-\infty)) = 0$$

if and only if $\mathcal{A}$ is backward compact.

(ii) $\mathcal{A}_U(\infty)$ is nonempty compact such that $\mathcal{A}(t)$ is upper semi-continuous to $\mathcal{A}_U(\infty)$, i.e.

$$\lim_{t \to +\infty} d(\mathcal{A}(t), \mathcal{A}_U(\infty)) = 0$$

if and only if $\mathcal{A}$ is forward compact.

(iii) Both (9) and (10) hold true with compact $\mathcal{A}_U(\pm\infty)$ if and only if $\mathcal{A}$ is globally compact.

**Proof.** (i) Necessity. Suppose $\mathcal{A}_U(-\infty)$ is nonempty compact such that (9) holds true and let $t \in \mathbb{R}$ be fixed. We need to prove the pre-compactness of $\cup_{s \leq t} \mathcal{A}(s)$. Taking a sequence $\{x_n\}$ from this set, we then choose $s_n \leq t$ such that $x_n \in \mathcal{A}(s_n)$. We will prove that the sequence $\{x_n\}$ has a convergent subsequence by dividing into the following two cases.

If $s_0 = \inf_{n \in \mathbb{N}} s_n \neq -\infty$, then, for all $n \in \mathbb{N}$, $s_n \in [s_0, t]$ and so $\{x_n\} \subset \cup_{s_0 \leq s \leq t} \mathcal{A}(s)$. By the local compactness of a pullback attractor (see Lemma 2.3), the latter is pre-compact and so $\{x_n\}$ is pre-compact as required.

If $\inf_{n \in \mathbb{N}} s_n = -\infty$, then, passing to a subsequence, we may assume $s_n \downarrow -\infty$. By the upper semi-continuity assumption (9), we have, as $n \to \infty$,

$$d(x_n, \mathcal{A}_U(-\infty)) \leq d(\mathcal{A}(s_n), \mathcal{A}_U(-\infty)) \to 0.$$  

For each $n \in \mathbb{N}$ we choose a $y_n \in \mathcal{A}_U(-\infty)$ such that $d(x_n, y_n) < d(x_n, \mathcal{A}_U(-\infty)) + 1/n$. By the compactness assumption of $\mathcal{A}_U(-\infty)$, we know the sequence $\{y_n\}$ has a convergent subsequence such that $y_{n_k} \to y$ as $k \to \infty$. Then

$$d(x_{n_k}, y) \leq d(y_{n_k}, y) + d(x_{n_k}, y_{n_k})$$

$$\leq d(y_{n_k}, y) + d(x_{n_k}, \mathcal{A}_U(-\infty)) + 1/n_k,$$

which together with (11) implies that $x_{n_k} \to y$ as required.

Sufficiency. Suppose $\mathcal{A}$ is backward compact. By Lemma 3.3, $\mathcal{A}_U(-\infty)$ is nonempty and compact. We need to prove the upper semi-continuity at $t = -\infty$. 


Let Proposition 1.

Corollary 1. Let \( \mathcal{A} \) be a pullback attractor for an evolution process.

(i) There is a nonempty compact set \( C \) such that \( \lim_{t \to -\infty} d(\mathcal{A}(t), C) = 0 \) if and only if \( \mathcal{A} \) is backward compact.

(ii) There is a nonempty compact set \( C \) such that \( \lim_{t \to +\infty} d(\mathcal{A}(t), C) = 0 \) if and only if \( \mathcal{A} \) is forward compact.

Proof. We need only to prove (ii) since it is similar to prove (i). Suppose \( \mathcal{A} \) is forward compact. Then it follows from Theorem 3.4 that \( \mathcal{A}_U(\infty) \) is a nonempty compact set such that \( \mathcal{A}(t) \) is upper semi-continuous to this compact set as \( t \to +\infty \).

Conversely, suppose \( C \) is a compact set such that \( \lim_{t \to +\infty} d(\mathcal{A}(t), C) = 0 \). Let \( t \in \mathbb{R} \) be fixed and take a sequence \( x_n \in \mathcal{A}(r_n) \) with \( r_n \geq t \). If \( r_0 = \sup_{n \in \mathbb{N}} r_n < +\infty \), then \( \{x_n\} \subset \bigcup_{r \leq r_0} \mathcal{A}(r) \), which is pre-compact from the local compactness given in Lemma 2.3. If \( \sup_{n \in \mathbb{N}} r_n = +\infty \), without lose of generality, we assume \( r_n \uparrow +\infty \). Hence \( d(x_n, C) \leq d(\mathcal{A}(r_n), C) \to 0 \). Since \( C \) is nonempty compact, it is similar as the proof in Theorem 3.4 to prove that \( \{x_n\} \) has a convergent subsequence and so \( \mathcal{A} \) is forward compact.

We have the following minimality of the upper limit-set in the upper semi-continuity sense.

Proposition 1. Let \( \mathcal{A} \) be a pullback attractor for an evolution process.

(i) Suppose \( \mathcal{A} \) is backward compact. Then \( \mathcal{A}_U(\infty) \) is the minimal closed set in the upper semi-continuity sense. More clearly, if \( K \) is a closed set such that \( \lim_{t \to -\infty} d(\mathcal{A}(t), K) = 0 \), then \( \mathcal{A}_U(\infty) \subset K \).

(ii) Suppose \( \mathcal{A} \) is forward compact. Then \( \mathcal{A}_U(\infty) \) is the minimal closed set such that \( \mathcal{A}(t) \) is upper semi-continuous to a closed set as \( t \to +\infty \).

Proof. Let \( x \in \mathcal{A}_U(\infty) \). By Lemma 3.2(i), there are a sequence \( \{s_n\} \) with \( s_n \downarrow -\infty \) and \( x_n \in \mathcal{A}(s_n) \) such that \( x_n \to x \). By the assumption, \( \mathcal{A}(t) \) is upper semi-continuous to the closed set \( K \) as \( t \to -\infty \), we have \( d(\mathcal{A}(s_n), K) \to 0 \) as \( n \to \infty \). Hence

\[
d(x, K) \leq d(x, x_n) + d(x_n, K) \leq d(x, x_n) + d(\mathcal{A}(s_n), K) \to 0,
\]

which means \( d(x, K) = 0 \). Since \( K \) is closed, \( x \in K \) and thus \( \mathcal{A}_U(\infty) \subset K \), which proves (i). It is similar to prove (ii).
4. Lower semi-continuity of a pullback attractor at infinity. It is more difficult to analyze the lower semi-continuity at infinite time. We respectively discuss the lower semi-continuity in two cases: negative or positive infinity.

4.1. Lower semi-continuity of a pullback attractor at negative infinity. We give a suitable definition of the possible limit-set for the lower semi-continuity as \( t \to -\infty \).

Definition 4.1. (a) Let \( \{K_n : n \in \mathbb{N}\} \) be a sequence of subsets in \( X \). We define its limit-set by

\[
\lim_{n \to \infty} K_n = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} K_n. \tag{13}
\]

(b) Let \( \mathcal{K} = \{\mathcal{K}(t) : t \in \mathbb{R}\} \) be a brochette over \( X \). We define the lower limit-set \( \mathcal{K}_L(-\infty) \) by

\[
\mathcal{K}_L(-\infty) = \bigcap_{s_n \downarrow -\infty} \lim_{n \to \infty} \mathcal{K}(s_n), \tag{14}
\]

where we take the interjection of limit-sets over all sequences \( \{s_n\} \) with \( s_n \downarrow -\infty \).

Lemma 4.2. (i) \( x \in \lim_{n \to \infty} K_n \) if and only if there are a subsequence \( \{n_k\} \) of \( \{n\} \) and \( x_k \in K_{n_k} \) such that \( x_k \to x \) as \( k \to \infty \).

(ii) \( y \in \mathcal{K}_L(-\infty) \) if and only if, whenever \( s_n \downarrow -\infty \), there are a subsequence \( \{n_k\} \) of \( \{n\} \) and \( y_k \in \mathcal{K}(s_{n_k}) \) such that \( y_k \to y \).

(iii) \( \mathcal{K}_L(-\infty) \subseteq \mathcal{K}_U(-\infty) = \bigcup_{s_n \downarrow -\infty} \lim_{n \to \infty} \mathcal{K}(s_n) \), where \( \mathcal{K}_U(-\infty) \) is the upper limit-set.

Proof. (i) Let \( x \in \lim_{n \to \infty} K_n \). By (13), \( x \in \bigcup_{n \geq k} K_n \) for all \( k \in \mathbb{N} \). We then take \( x_k \in K_{n_k} \) with \( n_k \geq k \) such that \( d(x, x_k) \leq 1/k \to 0 \), which proves the necessity. The sufficiency is similar to prove.

(ii) The assertion follows from (i) and (14) immediately.

(iii) We need to prove the equality about the upper limit-set. If \( x \in \mathcal{K}_U(-\infty) \), then, by Lemma 3.2, there exist a sequence \( s_n \downarrow -\infty \) and \( x_n \in \mathcal{K}(s_n) \) such \( x_n \to x \). By (i), \( x \in \lim_{n \to \infty} \mathcal{K}(s_n) \). Hence, \( \mathcal{K}_U(-\infty) \subseteq \bigcup_{s_n \downarrow -\infty} \lim_{n \to \infty} \mathcal{K}(s_n) \).

Conversely, if \( x \in \bigcup_{s_n \downarrow -\infty} \lim_{n \to \infty} \mathcal{K}(s_n) \), then there is a sequence \( s_n \downarrow -\infty \) such that \( x \in \lim_{n \to \infty} \mathcal{K}(s_n) \). By (i), there exist a subsequence \( \{n_k\} \) and \( x_k \in \mathcal{K}(s_{n_k}) \) such that \( x_k \to x \). Since \( s_{n_k} \downarrow -\infty \) as \( k \to \infty \), it follows from Lemma 3.2 that \( x \in \mathcal{K}_U(-\infty) \), which implies

\[
\mathcal{K}_U(-\infty) \supseteq \bigcup_{s_n \downarrow -\infty} \lim_{n \to \infty} \mathcal{K}(s_n) \quad \text{and so} \quad \mathcal{K}_U(-\infty) = \bigcup_{s_n \downarrow -\infty} \lim_{n \to \infty} \mathcal{K}(s_n). \tag{15}
\]

Now, both (14) and (15) obviously deduce \( \mathcal{K}_L(-\infty) \subseteq \mathcal{K}_U(-\infty) \). \( \square \)

The following example indicates the inclusion in Lemma 4.2(iii) may be strict.

Example 2. Let \( \mathcal{K}(t) = \{\sin t\} \) for \( t \in \mathbb{R} \). It is easy to prove that \( \mathcal{K} \) is a pullback attractor of an ODE: \( u_t + u = \sin t + \cos t \). On the other hand, \( \cup_{s \in \xi} \mathcal{K}(s) = \cup_{t \geq \xi} \mathcal{K}(t) = [-1, 1] \) for each \( t \in \mathbb{R} \), and by (7)

\[
\mathcal{K}_U(-\infty) = \mathcal{K}_U(\infty) = [-1, 1].
\]

However, we can prove \( \mathcal{K}_L(-\infty) = \emptyset \). Indeed, if we take two sequences by

\[
s_n = -2n\pi + \frac{\pi}{6}, \quad \bar{s}_n = -2n\pi + \frac{\pi}{2}, \quad \text{for } n \in \mathbb{N},
\]
then $s_n \downarrow -\infty$ and $\tilde{s}_n \downarrow -\infty$. But we have

$$\lim_{n \to \infty} \mathcal{K}(s_n) = \{1/2\}, \quad \lim_{n \to \infty} \mathcal{K}(\tilde{s}_n) = \{1\}, \quad \lim_{n \to \infty} \mathcal{K}(s_n) \cap \lim_{n \to \infty} \mathcal{K}(\tilde{s}_n) = \emptyset.$$ 

We now intend to consider the lower limit-set of a pullback attractor.

**Lemma 4.3.** Let $\mathcal{A}$ be a pullback attractor for an evolution process such that $\mathcal{A}$ is backward compact.

(a) For each sequence $s_n \downarrow -\infty$, $\lim_{n \to \infty} \mathcal{A}(s_n)$ is nonempty and compact.

(b) $\mathcal{A}_L(-\infty)$ is compact if it is nonempty.

*Proof.* (a) We denote $C_n := \bigcup_{k \geq n} \mathcal{A}(s_k)$ for all $n \in \mathbb{N}$. Since $\{s_n\}$ is decreasing, we have $C_n \subset \bigcup_{s < s_n} \mathcal{A}(s)$ for each $n \in \mathbb{N}$, then it follows from the backward compactness assumption of $\mathcal{A}$ that $C_n$ is compact. Hence $\{C_n\}$ defines a decreasing sequence of nonempty compact sets, which implies that $\lim_{n \to \infty} \mathcal{A}(s_n) = \bigcap_{n=1}^{\infty} C_n$ is nonempty and compact.

(b) By (a) and (14), $\mathcal{A}_L(-\infty)$ is regarded as the interjection of compact subsets and thus it is compact.

It is possible that $\mathcal{A}_L(-\infty)$ is an empty set even if $\mathcal{A}_U(-\infty)$ is nonempty (see the above example). However, we can give a criterion for $\mathcal{A}_L(-\infty) \neq \emptyset$.

**Proposition 2.** Let $\mathcal{A}$ be a pullback attractor for an evolution process such that $\mathcal{A}$ is backward compact. Then the following statements are equivalent to each other.

(i) $\mathcal{A}_L(-\infty)$ is nonempty.

(ii) there is an $x \in X$ such that $d(x, \mathcal{A}(t)) \to 0$ as $t \to -\infty$.

(iii) there is a nonempty set $B$ such that $d(B, \mathcal{A}(t)) \to 0$ as $t \to -\infty$.

*Proof.* (i)$\Rightarrow$(ii). Let $x \in \mathcal{A}_L(-\infty)$. We need to prove $d(x, \mathcal{A}(t)) \to 0$ as $t \to -\infty$. If this is not true, then there are $\delta > 0$ and a sequence $s_n \downarrow -\infty$ such that $d(x, \mathcal{A}(s_n)) \geq \delta$, for all $n \in \mathbb{N}$.

Since $x \in \mathcal{A}_L(-\infty) \subset \lim_{n \to \infty} \mathcal{A}(s_n)$, it follows from Lemma 4.2(i) that there is a subsequence $\{n_k\}$ of $\{n\}$ and $x_k \in \mathcal{A}(s_{n_k})$ such that $x_k \to x$. Hence, $d(x, \mathcal{A}(s_{n_k})) \leq d(x, x_k) \to 0$, which is a contradiction.

(ii)$\Rightarrow$(iii) is obvious. We need only to prove (iii)$\Rightarrow$(i). Suppose a nonempty set $B$ such that $d(B, \mathcal{A}(t)) \to 0$ as $t \to -\infty$. We take an $x \in B$, then $d(x, \mathcal{A}(t)) \leq d(B, \mathcal{A}(t)) \to 0$ as $t \to -\infty$. Let $\{s_n\}$ be an arbitrary sequence such that $s_n \downarrow -\infty$. Then $\lim_{n \to \infty} d(x, \mathcal{A}(s_n)) = 0$. Hence, for each $n \in \mathbb{N}$, we can choose an $x_n \in \mathcal{A}(s_n)$ such that $d(x, x_n) \leq d(x, \mathcal{A}(s_n)) + \frac{1}{n} \to 0$ as $n \to \infty$.

We then consider the maximal limit-set in the lower semi-continuity sense.

**Theorem 4.4.** Let $\mathcal{A}$ be a pullback attractor for an evolution process such that $\mathcal{A}$ is backward compact and $\mathcal{A}_L(-\infty)$ is nonempty. Then $\mathcal{A}_L(-\infty)$ is compact such that $\mathcal{A}(t)$ is lower semi-continuous to this compact set, that is,

$$\lim_{t \to -\infty} d(\mathcal{A}_L(-\infty), \mathcal{A}(t)) = 0.$$ 

(16)
On the other hand, \( \mathcal{A}_L(-\infty) \) is the maximal limit-set in the lower semi-continuity sense, that is, if a nonempty subset \( B \) is such that \( \lim_{t \to -\infty} d(B, \mathcal{A}(t)) = 0 \), then \( B \subseteq \mathcal{A}_L(-\infty) \).

**Proof.** By Lemma 4.3(b), \( \mathcal{A}_L(-\infty) \) is a compact set. We then prove the lower semi-continuity (16). If this is not true, then there are \( \delta > 0 \) and a sequence \( s_n \downarrow -\infty \) such that

\[
d(\mathcal{A}_L(-\infty), \mathcal{A}(s_n)) \geq 3\delta, \quad \text{for all } n \in \mathbb{N}.
\]

By the definition of Hausdorff semi-metric, for each \( n \in \mathbb{N} \), we can choose \( x_n \in \mathcal{A}_L(-\infty) \) such that

\[
d(x_n, \mathcal{A}(s_n)) \geq d(\mathcal{A}_L(-\infty), \mathcal{A}(s_n)) - \delta \geq 2\delta, \quad \text{for all } n \in \mathbb{N}.
\]

(17)

Since \( \mathcal{A}_L(-\infty) \) is compact, there is a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( x_{n_k} \to x \in \mathcal{A}_L(-\infty) \) as \( k \to \infty \). Without lose of generality, we assume \( d(x_{n_k}, x) < \delta \) for all \( k \in \mathbb{N} \), then by (17)

\[
d(x, \mathcal{A}(s_{n_k})) \geq d(x_{n_k}, \mathcal{A}(s_{n_k})) - d(x_{n_k}, x) \geq \delta, \quad \text{for all } k \in \mathbb{N}.
\]

(18)

Since \( s_{n_k} \downarrow -\infty \), it follows from the definition of \( \mathcal{A}_L(-\infty) \) that \( x \in \mathcal{A}_L(-\infty) \subseteq \lim_{k \to \infty} \mathcal{A}(s_{n_k}) \). By Lemma 4.2(i), there are a subsequence \( \{k_i\} \) of \( \{n_k\} \) and \( y_i \in \mathcal{A}(s_{k_i}) \) such that \( y_i \to x \) as \( i \to \infty \). Hence

\[
d(x, \mathcal{A}(s_{k_i})) \leq d(x, y_i) \to 0, \quad \text{as } i \to \infty,
\]

which contradicts with (18) since \( \{k_i\} \) is a subsequence of \( \{n_k\} \). Therefore (16) holds true.

We then prove the maximality of \( \mathcal{A}_L(-\infty) \) as the limit-set in the lower semi-continuity sense. Indeed, suppose a subset \( B \) in \( X \) satisfies \( \lim_{t \to -\infty} d(B, \mathcal{A}(t)) = 0 \). If \( x \in B \), then

\[
\lim_{t \to -\infty} d(x, \mathcal{A}(t)) \leq \lim_{t \to -\infty} d(B, \mathcal{A}(t)) = 0.
\]

Let \( \{s_n\} \) be an arbitrary sequence such that \( s_n \downarrow -\infty \). Then \( \lim_{n \to \infty} d(x, \mathcal{A}(s_n)) = 0 \). Hence for each \( n \in \mathbb{N} \) we can choose an \( x_n \in \mathcal{A}(s_n) \) such that

\[
d(x, x_n) \leq d(x, \mathcal{A}(s_n)) + \frac{1}{n} \to 0 \quad \text{as } n \to \infty,
\]

which implies that \( x_n \to x \) with \( x_n \in \mathcal{A}(s_n) \). By Lemma 4.2(i), we know \( x \in \lim_{n \to \infty} \mathcal{A}(s_n) \). Since \( \{s_n\} \) be an arbitrary sequence such that \( s_n \downarrow -\infty \), it follows from the definition of \( \mathcal{A}_L(-\infty) \) that \( x \in \mathcal{A}_L(-\infty) \) and so \( B \subseteq \mathcal{A}_L(-\infty) \) as required.

\( \square \)

**Remark.** Under the assumptions of Theorem 4.4, any pullback attracting brochette \( K \) is always lower semi-continuous to \( \mathcal{A}_L(-\infty) \) in view of the minimality of \( \mathcal{A} \).

The following result give a criterion for the continuity of a pullback attractor at negative infinity.

**Theorem 4.5.** Let \( \mathcal{A} \) be a pullback attractor for an evolution process. Then there is a nonempty compact set \( C \) such that

\[
\lim_{t \to -\infty} \text{dist}_h(\mathcal{A}(t), C) = 0 \tag{19}
\]

if and only if \( \mathcal{A} \) is backward compact and \( \mathcal{A}_L(-\infty) = \mathcal{A}_U(-\infty) \). In this case, \( C \) must equal to \( \mathcal{A}_L(-\infty) \).
Proof. Necessity. Suppose (19) holds true. In particular, $\mathcal{A}(t)$ is upper semi-continuous to $C$, i.e. $d(\mathcal{A}(t), C) \to 0$ as $t \to -\infty$. Then it follows from Corollary 1 that $\mathcal{A}$ is backward compact, and by Proposition 1 we further have $\mathcal{A}_U(-\infty) \subset C$.

On the other hand, it follows from (19) that $\mathcal{A}(t)$ is lower semi-continuous to $C$, i.e. $d(C, \mathcal{A}(t)) \to 0$ as $t \to -\infty$. Then, by Proposition 2, $\mathcal{A}_L(-\infty) \neq \emptyset$, and by Theorem 4.4 we have $C \subset \mathcal{A}_L(-\infty)$. In a word, by Lemma 4.2(iii), we have

$$C \subset \mathcal{A}_L(-\infty) \subset \mathcal{A}_U(-\infty) \subset C,$$

which implies that $C = \mathcal{A}_L(-\infty) = \mathcal{A}_U(-\infty)$ as required.

Sufficiency. If $\mathcal{A}$ is backward compact, by Theorem 3.4, $\mathcal{A}_U(-\infty)$ is nonempty and compact such that $d(\mathcal{A}(t), \mathcal{A}_U(-\infty)) \to 0$ as $t \to -\infty$. If we further assume $\mathcal{A}_L(-\infty) = \mathcal{A}_U(-\infty)$, then $\mathcal{A}_L(-\infty)$ is still nonempty, which together with Theorem 4.4 implies that $d(\mathcal{A}_L(-\infty), \mathcal{A}(t)) \to 0$ and so $d(\mathcal{A}_U(-\infty), \mathcal{A}(t)) \to 0$ as $t \to -\infty$. Therefore,

$$\lim_{t \to -\infty} \text{dist}_h(\mathcal{A}(t), \mathcal{A}_U(-\infty)) = 0,$$

which proves (19) if we set $C = \mathcal{A}_U(-\infty)$. \hfill $\square$

Observing the second case in the example given in Sec.3, it is easy to verify $\mathcal{A}_L(-\infty) = \{0\} = \mathcal{A}_U(-\infty)$. In fact, if $\mathcal{A}_U(-\infty)$ is consisted of a single point, we can prove a general result.

Corollary 2. Let $\mathcal{A}$ be a backward compact pullback attractor such that $\mathcal{A}_U(-\infty)$ is a single point. Then we have

$$\lim_{t \to -\infty} \text{dist}_h(\mathcal{A}(t), \mathcal{A}_U(-\infty)) = 0. \quad (20)$$

Proof. We need only to prove that $\mathcal{A}_L(-\infty) = \mathcal{A}_U(-\infty)$ if $\mathcal{A}_U(-\infty) = \{x\}$, which consists of a single point $x \in X$. Indeed, let $\{s_n\}$ be an arbitrary sequence such that $s_n \downarrow -\infty$. By Lemma 4.3(a), $\lim_{n \to -\infty} \mathcal{A}(s_n) \neq \emptyset$. Then there exists at least a $y \in \lim_{n \to -\infty} \mathcal{A}(s_n)$. By Lemma 4.2(i), there are a subsequence $\{n_k\}$ of $\{n\}$ and $y_k \in \mathcal{A}(s_{n_k})$ such that $y_k \to y$. Since $s_{n_k} \downarrow -\infty$, it follows from Lemma 3.2(i) that $y \in \mathcal{A}_U(-\infty)$. But by (22), $\mathcal{A}_U(-\infty)$ is the single point $\{x\}$, we have $x = y \in \lim_{n \to -\infty} \mathcal{A}(s_n)$ for any sequence $\{s_n\}$ with $s_n \downarrow -\infty$. So it follows from (14) that $x \in \mathcal{A}_L(-\infty)$, which means $\mathcal{A}_U(-\infty) \subset \mathcal{A}_L(-\infty)$. By Lemma 4.2(iii), we have $\mathcal{A}_L(-\infty) = \mathcal{A}_U(-\infty)$. Then the needed conclusion follows from Theorem 4.5. \hfill $\square$

4.2. Lower semi-continuity of a pullback attractor at positive infinity. In this subsection, we state the lower semi-continuity result at positive infinity, which completely parallels to the corresponding results in the negative infinity case and so we omit the proof.

For a pullback attractor $\mathcal{A}$, we define its lower limit-set at positive infinity by

$$\mathcal{A}_L(\infty) = \bigcap_{r_n \uparrow +\infty} \lim_{n \to +\infty} \mathcal{A}(r_n), \quad (21)$$

where we take the interjection of limit-sets over all sequence $\{r_n\}$ with $r_n \uparrow +\infty$.

Theorem 4.6. Let $\mathcal{A}$ be a pullback attractor such that $\mathcal{A}$ is forward compact and $\mathcal{A}_L(\infty)$ is nonempty. Then $\mathcal{A}_L(\infty)$ is compact such that

$$\lim_{t \to +\infty} d(\mathcal{A}_L(\infty), \mathcal{A}(t)) = 0. \quad (22)$$
On the other hand, $\mathcal{A}_L(\infty)$ is the maximal set in the lower semi-continuity sense, that is, if a nonempty subset $B$ in $X$ is such that $\lim_{t \to +\infty} d(B, \mathcal{A}(t)) = 0$, then $B \subset \mathcal{A}_L(\infty)$.

**Corollary 3.** Let $\mathcal{A}$ be a pullback attractor for an evolution process.

(i) There is a nonempty compact set $C$ such that $\lim_{t \to +\infty} \text{dist}_h(\mathcal{A}(t), C) = 0$ if and only if $\mathcal{A}$ is forward compact and $\mathcal{A}_L(\infty) = \mathcal{A}_U(\infty)$. In this case, $C$ must equal to $\mathcal{A}_U(\infty)$.

(ii) If $\mathcal{A}$ is forward compact such that $\mathcal{A}_U(\infty)$ is a single point, then we have

$$\lim_{t \to +\infty} \text{dist}_h(\mathcal{A}(t), \mathcal{A}_U(\infty)) = 0.$$  \hspace{1cm} (23)

5. Criteria for semi-uniform compactness of a pullback attractor. In this section, we will establish the criteria for the semi-continuity of a pullback attractor at infinity, these criteria are given in terms of the evolution process. By the corresponding results given in Sec. 3, 4, we need to establish the criteria for the existence of a pullback attractor with backward or forward compactness respectively.

5.1. **Backward compact pullback attractors.** There are a few papers (cf. [10, 21, 39, 41]) discussing the backward compactness of a pullback attractor. We list the needed terminologies and results as follows.

**Definition 5.1.** An evolution process $S(\cdot, \cdot)$ on $X$ is called **backward asymptotically compact** if, for each $t \in \mathbb{R}$, the sequence $\{S(s_n, s_n - \tau_n)x_n\}_{n \in \mathbb{N}}$ is pre-compact whenever $s_n \leq t$, $\tau_n \uparrow +\infty$ and $\{x_n\}$ is bounded in $X$.

An evolution process $S(\cdot, \cdot)$ on $X$ is called **backward omega-limit compact** if

$$\lim_{\tau_0 \to +\infty} \kappa(\bigcup_{\tau \geq \tau_0} \bigcup_{s \leq t} S(s, s - \tau)B) = 0, \forall B \in \mathfrak{B}, \forall t \in \mathbb{R}.$$  \hspace{1cm} (24)

Recall that $\mathfrak{B}$ is all of (deterministic) bounded subsets in $X$ and the Kuratowski measure $\kappa(\cdot)$ of a set is defined by

$$\kappa(A) = \inf\{d > 0 : A \text{ has a finite cover by sets of diameter less than } d\}.$$  

In fact, the above two properties are equivalent to the backward flattening property in a uniformly convex Banach space (see [21]), where the usual flattening property is introduced in [16]. However, we do not pursue the latter property in this paper.

**Definition 5.2.** A brochette $\mathcal{K}(\cdot)$ in $X$ is called **pullback absorbing** for an evolution process $S(\cdot, \cdot)$ if for each $t \in \mathbb{R}$ and $B \in \mathfrak{B}$ there is a $\tau_0 := \tau_0(t, B) > 0$ such that

$$S(t, t - \tau)B \subset \mathcal{K}(t), \quad \forall \tau \geq \tau_0.$$  \hspace{1cm} (25)

The following existence conclusion of a backward compact pullback attractor can be found in [21].

**Theorem 5.3.** Let $S(\cdot, \cdot)$ be a backward omega-limit compact process in a metric space $X$. Then the following statements are equivalent.

(i) $S(\cdot, \cdot)$ has a backward compact pullback attractor $\mathcal{A}$,

(ii) $S(\cdot, \cdot)$ has an increasing, bounded and pullback absorbing brochette $\mathcal{K}$.

In this case, the attractor is given by $\mathcal{A}(t) = \omega(\mathcal{K}(t), t)$, where the omega-limit set is defined by

$$\omega(D, t) = \bigcap_{\tau_0 > 0} \overline{\bigcup_{\tau \geq \tau_0} S(t, t - \tau)D}, \quad \forall t \in \mathbb{R}, \quad D \subset X.$$  \hspace{1cm} (26)
5.2. Criteria for forward compactness of a pullback attractor for a compactly dissipative process. From this subsection, we mainly consider the forward compactness in the pullback dynamics. Since the backward compactness coincides with the pullback dynamics, while the forward compactness does not, it is relatively difficult to deduce the forward compactness in the pullback dynamics.

We start from a simple case for a compact system, which means that the evolution process has a compact and absorbing brochette.

**Theorem 5.4.** An evolution process \( S(\cdot, \cdot) \) has a forward compact pullback attractor \( A \) if and only if it has a pullback attracting brochette \( K \) such that \( K \) is forward compact. In this case, \( A(t) \subset \omega(\bigcup_{s \leq t} K(s), t) \) and it is an equality if \( K \) is backward bounded.

**Proof.** The necessity is obvious since the attractor itself is pullback attracting and forward compact. We need only to prove the sufficiency. For this end, we define a brochette \( A \) by

\[
A(t) = \bigcup_{B \in \mathcal{B}} \omega(B, t), \quad \forall t \in \mathbb{R}, \tag{27}
\]

where, we take the union over all bounded subsets in \( X \). Note that the pullback attracting brochette \( K \) is forward compact and by Definition 2.1 \( K \) is compact. But [6, Theorem 2.12] tells us that there is a pullback attractor if and only if there is a compact attracting brochette, in this case, the brochette \( A \) given by (27) is just a pullback attractor.

We then prove that \( A \) is forward compact. Indeed, since a pullback attractor is the minimal closed attracting brochette (see Def.2.2(d)), we have \( A(t) \subset K(t) \) for all \( t \in \mathbb{R} \). Hence, \( \bigcup_{t \geq t_0} A(t) \subset \bigcup_{t \geq t_0} K(t) \) for all \( t \in \mathbb{R} \). Then the forward compactness of \( A \) follows from the forward compactness of \( K \).

Let now \( t \in \mathbb{R} \) be fixed. Notice that \( A(t - \tau) \subset K(t - \tau) \subset \bigcup_{s \leq t} K(s) \) for all \( \tau \geq 0 \), by the invariance of \( A \), we have

\[
A(t) = S(t, t - \tau)A(t - \tau) \subset S(t, t - \tau)(\bigcup_{s \leq t} K(s)), \quad \forall \tau \geq 0,
\]

which implies that

\[
A(t) \subset \bigcap_{\tau_0 > 0} \bigcup_{\tau \geq \tau_0} S(t, t - \tau)(\bigcup_{s \leq t} K(s)) = \omega(\bigcup_{s \leq t} K(s), t).
\]

On the other hand, if we assume \( K \) is backward bounded, then \( \bigcup_{s \leq t} K(s) \) is a bounded set. By the construction (27), we have \( \omega(\bigcup_{s \leq t} K(s), t) \subset A(t) \) and thus it is an equality.

Since an absorbing brochette must be pullback attracting, we have the following corollary.

**Corollary 4.** An evolution process has a forward compact pullback attractor \( A \) if there is a forward compact absorbing brochette \( K \). In this case, \( A(t) \subset \omega(\bigcup_{s \leq t} K(s), t) \) and it is an equality if \( K \) is backward bounded.

**Remark.** Note that \( K \) may not be backward bounded even if it is forward compact. Then, in the case of forward compactness, we can not deduce a similar formula of \( A \) as given in Theorem 5.3.
5.3. Criteria for forward compactness of a pullback attractor for a bounded dissipative process. In this subsection, we establish a sufficient condition to ensure the existence of a forward compact pullback attractor for a non-compact non-autonomous dynamical system.

**Definition 5.5.** An evolution process \( S(\cdot, \cdot) \) on \( X \) is called forward omega-limit compact if

\[
\lim_{\tau \to +\infty} \kappa(\bigcup_{\tau \geq \tau_0} \bigcup_{r \geq t} S(r, r - \tau)B) = 0, \ \forall B \in \mathcal{B}, \ t \in \mathbb{R}.
\]  

While, the process is called forward asymptotically compact if, for each \( t \in \mathbb{R} \), the sequence \( \{S(r_n, r_n - \tau_n)y_n\}_{n \in \mathbb{N}} \) is pre-compact whenever \( r_n \geq t, \ \tau_n \uparrow +\infty \) and \( \{y_n\} \) is bounded.

We generalize the concept of a omega-limit set to define a forward omega-limit set by

\[
\Omega(D, t) = \bigcap_{\tau_0 > 0} \bigcup_{\tau \geq \tau_0} \bigcup_{r \geq t} S(r, r - \tau)D, \ \forall t \in \mathbb{R}, \ D \subset X.
\]  

**Lemma 5.6.** (i) \( x \in \Omega(D, t) \) if and only if there are \( r_n \geq t, \ \tau_n \uparrow +\infty \) and \( x_n \in D \) such that

\[
y = \lim_{n \to \infty} S(r_n, r_n - \tau_n)x_n.
\]

(ii) For each \( D \subset X \), \( \Omega(D, \cdot) \) is decreasing, i.e. \( \Omega(D, t_1) \supset \Omega(D, t_2) \) if \( t_1 \leq t_2 \).

We also have

\[
\Omega(D, t) \supset \bigcup_{r \geq t} \omega(D, r), \ \forall t \in \mathbb{R}.
\]  

(iii) For each \( t \in \mathbb{R} \), \( \omega(\cdot, t) \) is increasing, i.e. \( \Omega(D_1, t) \subset \Omega(D_2, t) \) if \( D_1 \subset D_2 \).

**Proof.** (i) It is similar as the case of the usual omega-limit set.

(ii) If \( t_1 \leq t_2 \), then \( \bigcup_{r \geq t_1} S(r, r - \tau)D \supset \bigcup_{r \geq t_2} S(r, r - \tau)D \). Hence, by (29), we have \( \Omega(D, t_1) \supset \Omega(D, t_2) \), which proves the decreasing assertion. On the other hand,

\[
\bigcup_{r \geq t} \omega(D, r) = \bigcup_{r \geq t} \bigcap_{\tau_0 > 0} \bigcup_{\tau \geq \tau_0} S(r, r - \tau)D \subset \bigcap_{\tau_0 > 0} \bigcup_{\tau \geq \tau_0} \bigcup_{r \geq t} S(r, r - \tau)D = \Omega(D, t),
\]

which implies (30) in view of the closedness of \( \Omega(D, t) \). The assertion (iii) is obvious from (29).

**Lemma 5.7.** Let an evolution process \( S(\cdot, \cdot) \) be forward asymptotically compact. Then, for each \( B \in \mathcal{B}, \ \Omega(B, \cdot) \) is nonempty and forward compact. Furthermore, the brochette \( \Omega(B, \cdot) \) forward-uniformly pullback attracts \( B \) in the following sense:

\[
\lim_{\tau \to +\infty} \sup_{r \geq t} d(S(r, r - \tau)B, \Omega(B, t)) = 0, \ \forall t \in \mathbb{R}.
\]  

**Proof.** Let \( B \) be a nonempty bounded set and \( t \in \mathbb{R} \). We can take three sequences such that \( r_n \geq t, \ \tau_n \uparrow +\infty \) and \( x_n \in B \). By the forward asymptotic compactness, there are subsequences (not relabeled) such that \( S(r_n, r_n - \tau_n)x_n \to x \). By Lemma 5.6(i), \( x \in \Omega(B, t) \) and so \( \Omega(B, t) \) is nonempty.

We then prove that \( \Omega(B, t) \) is pre-compact and thus compact. Indeed, Let \( \{y_n\} \) be a sequence taken from \( \Omega(B, t) \). By Lemma 5.6(i), for each \( n \in \mathbb{N} \), we can take \( r_n \geq t, \ \tau_n \geq \max\{r_{n-1}, n\} \) and \( x_n \in B \) such that \( d(S(r_n, r_n - \tau_n)x_n, y_n) \leq 1/n \). By the forward asymptotic compactness, there is a subsequence such that \( S(r_{n_k}, r_{n_k} - \tau_{n_k})x_{n_k} \to x \), which implies \( d(y_{n_k}, x) \to 0 \) and so \( \Omega(B, t) \) is compact.
By Lemma 5.6(ii), $\Omega(B, \cdot)$ is decreasing and so $\cup_{r \geq t} \Omega(B, r) = \Omega(B, t)$. Therefore, $\Omega(B, \cdot)$ is forward compact.

Next, we prove the forward-uniform attraction. By the definition of Hausdorff semi-metric, (31) is equivalent to

$$\lim_{\tau \to \infty} d((\cup_{r \geq t} S(r, r - \tau)B, \Omega(B, t)) = 0.$$  (32)

If (32) is not true, then there are $\delta > 0$ and $\tau_n \uparrow +\infty$ such that

$$d((\cup_{r \geq t} S(r, r - \tau_n)B, \Omega(B, t)) \geq 2\delta,$$

for all $n \in \mathbb{N}$.

By the definition of Hausdorff semi-metric, we further choose $r_n \geq t$ and $x_n \in B$ such that

$$d(S(r_n, r_n - \tau_n)x_n, \Omega(B, t)) \geq d((\cup_{r \geq t} S(r, r - \tau_n)B, \Omega(B, t)) - \delta \geq \delta, \ \forall n \in \mathbb{N},$$

which is a contradiction, since the forward asymptotic compactness implies that, passing to a subsequence, $S(r_n, r_n - \tau_n)x_n \to x$ with $x \in \Omega(B, t)$.

**Proposition 3.** Let $S(\cdot, \cdot)$ be an evolution process in a metric space $X$, then the process is forward omega-limit compact if and only if it is forward asymptotically compact.

**Proof.** Necessity. Let $t \in \mathbb{R}$, $B \in \mathcal{B}$ and take three sequences such that $r_n \geq t$, $\tau_n \uparrow +\infty$ and $x_n \in B$. By the forward omega-limit compactness, we have

$$\kappa\{S(r_n, r_n - \tau_n)x_n : n \geq m\} \leq \kappa(\bigcup_{\tau \geq \tau_m} \bigcup_{r \geq t} S(r, r - \tau)B) \to 0 \text{ as } m \to \infty.$$  

Then it follows from [19, Lemma 2.7] that $\{S(r_n, r_n - \tau_n)x_n\}$ has a convergent subsequence.

Sufficiency. Suppose $S(\cdot, \cdot)$ is forward asymptotically compact and $B \in \mathcal{B}$. Then, by Lemma 5.7, the brochette $\Omega(B, \cdot)$ forward-uniformly attracts $B$. Hence (32) holds true and further implies that, for each $\varepsilon > 0$ and $t \in \mathbb{R}$, there is $\tau_\varepsilon > 0$ such that

$$d((\cup_{r \geq t} S(r, r - \tau)B, \Omega(B, t)) < \varepsilon/2 \text{ for all } \tau \geq \tau_\varepsilon.$$  

Hence,

$$\bigcup_{\tau \geq \tau_\varepsilon} \bigcup_{r \geq t} S(r, r - \tau)B \subset \bigcup_{x \in \Omega(B, t)} N_X(x, \varepsilon/2) =: N_X(\Omega(B, t), \varepsilon/2),$$  (33)

where $N_X(\cdot, \varepsilon)$ denotes the $\varepsilon$-neighborhood of a set. Since $\Omega(B, t)$ is compact, its Kuratowski measure is zero, it follows from (33) that

$$\kappa(\bigcup_{\tau \geq \tau_\varepsilon} \bigcup_{r \geq t} S(r, r - \tau)B) \leq \kappa(N_X(\Omega(B, t), \varepsilon/2)) < \varepsilon,$$

which proves that the process is forward omega-limit compact.

**Remark.** It is well-known that the omega-limit compactness is equivalent to the asymptotic compactness under an additional assumption of bounded absorption (see [20, 22]). However, we have proved this equivalence even in the forward case without any extra assumption.

**Theorem 5.8.** Let an evolution process $S(\cdot, \cdot)$ be forward asymptotically compact (or equivalently forward omega-limit compact). Then the process has a forward compact pullback attractor $A$ if there is an absorbing brochette $\mathcal{P}$ such that $\mathcal{P}$ is globally bounded. In this case, $A(t) = \omega(\cup_{s \leq t} \mathcal{P}(s), t)$. 


Proof. Let $\mathcal{K}(t) := \cup_{s \leq t} \mathcal{P}(s)$ for $t \in \mathbb{R}$. Since the absorbing brochette $\mathcal{P}$ is assumed to be globally bounded, we know the brochette $\mathcal{K}$ is increasing, globally bounded and absorbing. Let now

$$\mathcal{A}(t) = \omega(\mathcal{K}(t), t) = \omega(\cup_{s \leq t} \mathcal{P}(s), t).$$

We need to prove that $\mathcal{A}$ is a forward compact pullback attractor by the following four steps.

Step 1. We prove $\mathcal{A}$ is forward compact. Let $t \in \mathbb{R}$ be fixed, we need to prove that $\cup_{r \geq t} \mathcal{A}(r)$ is pre-compact. Indeed, by Lemma 5.6(ii), $\Omega(D, \cdot)$ is decreasing for each fixed $D \subset X$. Then we have

$$\bigcup_{r \geq t} \mathcal{A}(r) = \bigcup_{r \geq t} \omega(K(r), r) \subset \bigcup_{r \geq t} \Omega(K(r), r) \subset \bigcup_{r \geq t} \Omega(K(r), t).$$

By Lemma 5.6(iii), we know, for each $r \geq t$, $\Omega(K(r), t) \subset \Omega(\cup_{r \geq t} K(r), t)$, which implies that

$$\bigcup_{r \geq t} \mathcal{A}(r) \subset \bigcup_{r \geq t} \Omega(K(r), t) \subset \Omega(\bigcup_{r \geq t} K(r), t) =: \Omega(K, t), \quad (34)$$

where $K = \cup_{r \geq t} K(r)$ is a bounded set since $\mathcal{K}$ is forward bounded. By Lemma 5.7, $\Omega(K, t)$ is a compact set. Then, by (34), both $\cup_{r \geq t} \mathcal{A}(r)$ and $\mathcal{A}(t)$ are pre-compact. Note that $\mathcal{A}(t) = \omega(K(t), t)$ is obviously closed, so $\mathcal{A}(t)$ is compact. Therefore, $\mathcal{A}$ is forward compact.

Step 2. We prove that $\mathcal{A}$ pullback attracts each $B \in \mathfrak{B}$ at $t \in \mathbb{R}$. If this is not true, then there are $\delta > 0$, $\tau_n \uparrow +\infty$ and $x_n \in B$ such that

$$d(S(t, t - \tau_n)x_n, \omega(K(t), t)) \geq \delta, \quad \forall n \in \mathbb{N}. \quad (35)$$

By the forward asymptotic compactness and by taking $r_n \equiv t$, we know there is a subsequence (not relabeled) such that $S(t, t - \tau_n)x_n \rightarrow x \in X$. For each fixed $k \in \mathbb{N}$, since $\tau_n \uparrow +\infty$, we can choose a $n_k \in \mathbb{N}$ such that $\tau_{n_k} - k$ is large enough. Since $\mathcal{K}$ is increasing and absorbing, we have

$$y_k := S(t - k, t - k - (\tau_{n_k} - k))x_{n_k} \in K(t - k) \subset K(t).$$

By (4) we have $S(t, t - k)y_k = S(t, t - \tau_{n_k})x_{n_k} \rightarrow x$ as $k \rightarrow \infty$, which yields $x \in \omega(K(t), t)$ and gives a contradiction with (35).

Step 3. We prove the positive invariance, i.e. $S(t, s)\omega(K(s), s) \subset \omega(K(t), t)$ for $t \geq s$. Indeed, let $x \in \omega(K(s), s)$, then there are $\tau_n \uparrow +\infty$ and $x_n \in K(s)$ such that $S(s, s - \tau_n)x_n \rightarrow x$ as $n \rightarrow \infty$. By the continuity of the process, we have

$$S(t, t - (t - s + \tau_n))x_n = S(t, s)S(s, s - \tau_n)x_n \rightarrow S(t, s)x.$$

Since $x_n \in K(s) \subset K(t)$ and $t - s + \tau_n \uparrow +\infty$, it follows $S(t, s)x \in \omega(K(t), t)$.

We then prove the negative invariance: $\omega(K(t), t) \subset S(t, s)\omega(K(s), s)$ for $t \geq s$. Let $x \in \omega(K(t), t)$, then there are $\tau_n \uparrow +\infty$ and $x_n \in K(t)$ such that $S(t, t - \tau_n)x_n \rightarrow x$ as $n \rightarrow \infty$. Since $\tau_n \uparrow +\infty$, there is an $N$ such that $t - \tau_n < s$ for all $n > N$ and so $0 < s - t + \tau_n \uparrow +\infty$ for all $n > N$. By the forward asymptotic compactness, passing to a subsequence, $S(s, t - \tau_n)x_n = S(s, s - (s - t + \tau_n))x_n \rightarrow y \in X$.

For each fixed $k \in \mathbb{N}$, we can choose a subsequence $\tau_{n_k} \uparrow +\infty$ such that $s - k - t + \tau_{n_k}$ is large enough and, by the increasing absorption of $K(\cdot)$,

$$y_k := S(s - k, s - k - (s - k - t + \tau_{n_k}))x_{n_k} \in K(s - k) \subset K(s).$$
By (4), we have, as \( k \to \infty \),
\[
S(s, s - k)y_k = S(s, s - k)S(s - k, t - \tau_n_k)x_{n_k} = S(s, t - \tau_n_k)x_{n_k} \to y,
\]
which proves \( y \in \omega(K(s), s) \). By the continuity of the process,
\[
x = \lim_{k \to \infty} S(t, t - \tau_n_k)x_{n_k} = S(t, s) \lim_{k \to \infty} S(s, t - \tau_n_k)x_{n_k} = S(t, s).
\]
Therefore, \( x \in S(t, s)\omega(K(s), s) \), which proves the negative invariance.

**Step 4.** We prove the minimality. Suppose \( C \) is a closed attracting brochette. We need to prove \( A(t) \subset C(t) \) for all \( t \in \mathbb{R} \). For this end, we let \( B = \bigcup_{s \leq t} A(s) \). For each \( s \leq t \), by the absorption of \( K \), there is a \( \tau_1 := \tau_1(s, D) \) such that \( S(s, s - \tau)K(s) \subset K(s) \) for all \( \tau \geq \tau_1 \), which yields
\[
A(s) = \omega(K(s), s) = \bigcap_{\tau_0 > 0} \bigcup_{\tau \geq \tau_0} S(s, s - \tau)K(s) \subset K(s).
\]
So \( B \subset \bigcup_{s \leq t} K(s) = K(t) \), which is bounded. Then, by the attraction of \( C \) and the invariance of \( A \), we have, as \( \tau \to +\infty \),
\[
d(A(t), C(t)) = d(S(t, t - \tau)A(t - \tau), C(t)) \leq d(S(t, t - \tau)B, C(t)) \to 0.
\]
So \( d(A(t), C(t)) = 0 \) and thus \( A(t) \subset C(t) \) since \( C(t) \) is closed. \( \square \)

**Remark.** Although the forward boundedness of the absorbing brochette \( P \) has deduced the forward compactness of the omega-limit set in Theorem 5.8, we have to assume the backward boundedness of \( P \) to ensure the existence of a pullback attractor. This is different from the backward compact case in Theorem 5.3.

5.4. **Relationship with uniform attractors w.r.t.** \( \mathbb{R} \) or \( \mathbb{R}^+ \). In this subsection, we give some remarks to compare our criteria for semi-uniform compactness of a pullback attractor with that for the existence of uniform attractors w.r.t. \( \mathbb{R} \) or \( \mathbb{R}^+ \), which are introduced by Chepyzhov and Vishik [9, Chapters IV, VII], also cf. [8, 30, 38].

**Definition 5.9.** A time-independent set \( P \) is called \( \mathbb{R} \)-uniformly attracting for a process \( S \) if for each \( B \in \mathcal{B} \),
\[
\lim_{t \to +\infty} \sup_{s \in \mathbb{R}} \text{dist}(S(s + t, s)B, P) = 0. \tag{36}
\]
A compact set \( P \) is said to be a \( \mathbb{R} \)-uniform attractor if it is \( \mathbb{R} \)-uniformly attracting and minimal (among all of compact \( \mathbb{R} \)-uniformly attracting sets).

**Theorem 5.10.** [9, Theorem IV.6.1]. If a process \( S \) has a compact \( \mathbb{R} \)-uniformly attracting set, then it has a \( \mathbb{R} \)-uniform attractor.

Although the \( \mathbb{R} \)-uniform attraction defined by (36) is forward, it is easy to prove that (36) is equivalent to the pullback uniform attraction, i.e.
\[
\lim_{t \to +\infty} \sup_{r \in \mathbb{R}} \text{dist}(S(r, r - t)B, P) = 0, \forall B \in \mathcal{B}.
\]
By using the above equivalence, along with Theorems 5.3, 5.4 and 3.4, one can easily obtain the following result.

**Proposition 4.** If a process \( S \) has a \( \mathbb{R} \)-uniform attractor \( P \), then it has a pullback attractor \( A \) which is both forward and backward compact. So the robustness of the pullback attractor at (positive and negative) infinity holds true with \( A_U(\pm \infty) \subset P \).
However, Example 1 in Sec.3 shows that there does not exist any $\mathbb{R}$-uniform attractor even if there is a forward or backward compact pullback attractor. So the (necessary and sufficient) criteria in Theorem 5.4 is weaker than those given in Theorem 5.10.

We then recall some concepts and existence conclusions for $\mathbb{R}^+$-uniform attractor given by Chepyzhov and Vishik [9, Chapter VII].

**Definition 5.11.** A set $P$ is called $\mathbb{R}^+$-uniformly attracting for a process (actually for a semiprocess $\{U(t, \tau) : t \geq \tau \geq 0\}$) if for each $B \in \mathfrak{B}$,

$$
\lim_{t \to +\infty} \sup_{\tau \geq 0} \text{dist}(U(\tau + t, \tau)B, P) = 0.
$$

(37)

A compact set $P$ is said to be a $\mathbb{R}^+$-uniform attractor if it is $\mathbb{R}^+$-uniformly attracting and minimal (among all of compact $\mathbb{R}^+$-uniformly attracting sets).

**Theorem 5.12.** [9, Corollary VII.1.2]. If a process $S$ has a compact $\mathbb{R}^+$-uniformly attracting set, then it (or the restricted semiprocess on $\mathbb{R}^+$) has a $\mathbb{R}^+$-uniform attractor.

Unlike the $\mathbb{R}$-uniform attractor, the $\mathbb{R}^+$-uniform attraction (37) has not given any information about pullback attraction (i.e. the initial time $\tau \to -\infty$). So this semi-uniform attractor may relate to the forward attractor (cf. [17]), but it is another topic which is completely different from pullback attractors. On the other hand, the semi-uniform forward attraction (37) is completely different from the semi-uniform pullback attraction described by (31).

6. Applications on non-autonomous Ginzburg-Landau equations. We consider the following non-autonomous Ginzburg-Landau equation:

$$
\left\{ 
\begin{array}{l}
\frac{\partial u}{\partial t} = (\lambda + i\alpha(t))\Delta u - (\kappa + i\beta(t))|u|^2u + \gamma u + f(t, x), \quad \text{for } t \geq s, x \in \Omega, \\
u(t, x) = 0, \quad t \geq s, x \in \partial \Omega, \quad \text{and} \quad u(s, x) = u_0, \quad x \in \Omega.
\end{array}
\right.
$$

(38)

where $s \in \mathbb{R}$, $\lambda, \kappa, \gamma > 0$, $\Omega$ is a bounded smooth domain in $\mathbb{R}^n$ with $n = 1, 2$ and the unknown $u$ is a complex-valued function.

If the coefficients $\alpha(\cdot)$ and $\beta(\cdot)$ are constants, then a global attractor was obtained in [31] for $f \equiv 0$ and a random attractor was obtained in [36] if $f$ is replaced by an additive noise.

If both $\alpha(\cdot)$ and $\beta(\cdot)$ are time-dependent coefficients with some special assumptions and $f$ is a random noise, then a pullback random attractor was obtained in [12].

In this paper, we make the following simple assumption for the variable coefficients.

**Hypothesis A.** $\alpha(\cdot) \in C(\mathbb{R}, \mathbb{R})$ and $\beta(\cdot) \in C_b(\mathbb{R}, \mathbb{R})$.

Let $H = L^2(\Omega) = L^2(\Omega) \times L^2(\Omega)$ and $V = H^1_0(\Omega) = H^1_0(\Omega) \times H^1_0(\Omega)$. If $f \in L^1_{\text{loc}}(\mathbb{R}, H)$, one can prove that Eq. (38) is well-posed and the unique solution defines an evolution process $S(t, s) : H \to H$ by $S(t, s)u_0 = u(t, s; u_0)$ for $t \geq s$, $u_0 \in H$, which can be rewritten as

$$
S(t, t - \tau)u_0 = u(t, t - \tau; u_0) \quad \text{for } t \in \mathbb{R}, \quad \tau \geq 0, \quad u_0 \in H.
$$

(39)
6.1. Forward compact attractor and semi-continuity at positive infinity.

We need to give a special assumption for the external force $f$.

**Hypothesis B1.** $f \in L_{\text{loc}}(\mathbb{R}, H)$ is forward tempered in the following sense:

$$F(t) := \sup_{r \geq t} \int_{-\infty}^{r} e^{\gamma(q-r)} \|f(q, \cdot)\|^2 dq < +\infty, \ \forall t \in \mathbb{R}. \quad (40)$$

where the constant $\gamma$ is given in Eq. (38).

**Lemma 6.1.** Let the hypotheses A and B1 are satisfied. Then the solution of Eq.(38) satisfies: for $t \in \mathbb{R}$, $\tau \geq 0$ and $u_{0} \in H$,

$$\sup_{r \geq t} \|u(r, r - \tau; u_{0})\|^2 \leq e^{-\gamma \tau} \|u_{0}\|^2 + c_{1}(1 + F(t)), \quad (41)$$

$$\sup_{r \geq t} \int_{r - \tau}^{r} e^{\gamma(q-r)}(\|u(q, r - \tau; u_{0})\|_{V}^{2} + \|u(q, r - \tau; u_{0})\|_{4}^{2}) dq \leq c_{2} e^{\gamma \tau} \|u_{0}\|^2 + c_{2}(1 + F(t)). \quad (42)$$

where $F(\cdot)$ (given by (40)) is finite and decreasing.

**Proof.** Multiplying Eq.(38) by the conjugate $\bar{u}$, integrating over $\Omega$ and taking the real part, we get

$$\frac{1}{2} \frac{d}{dr} \|u\|^2 + \lambda \|\nabla u\|^2 + \kappa \|u\|_{4}^{2} - \gamma \|u\|^2 = \text{Re} \int_{\Omega} (f(r,x) \bar{u}(r,x)) dx.$$ 

The Young inequality implies

$$\text{Re} \int_{\Omega} (f(r,x) \bar{u}(r,x)) dx \leq \gamma \|u(r)\|^2 + \frac{1}{4\gamma} \|f(r, \cdot)\|^2.$$ 

We then use an obvious inequality $\kappa \|u\|_{4}^{2} - 5\gamma \|u\|^2 \geq -\frac{25\gamma^2}{4\kappa} |\Omega|$ to deduce

$$\frac{d}{dr} \|u\|^2 + \gamma \|u\|^2 + 2\lambda \|\nabla u\|^2 + \kappa \|u\|_{4}^{2} \leq c(1 + \|f(r, \cdot)\|^2).$$

Let $t \in \mathbb{R}$ and $u_{0} \in H$ be fixed. Applying the Gronwall lemma on $[r - \tau, r]$, we have, for all $r \geq t$ and $\tau \geq 0$,

$$\|u(r, r - \tau; u_{0})\|^2 \leq e^{-\gamma \tau} \|u_{0}\|^2 + c \int_{r - \tau}^{r} e^{\gamma(q-r)}(1 + \|f(q)\|^2) dq, \quad (43)$$

$$\int_{r - \tau}^{r} e^{\gamma(q-r)} \|\nabla u(q)\|^2 dq \leq ce^{-\gamma \tau} \|u_{0}\|^2 + c \int_{r - \tau}^{r} e^{\gamma(q-r)}(1 + \|f(q)\|^2) dq, \quad (44)$$

$$\int_{r - \tau}^{r} e^{\gamma(q-r)} \|u(q)\|_{4}^{2} dq \leq ce^{-\gamma \tau} \|u_{0}\|^2 + c \int_{r - \tau}^{r} e^{\gamma(q-r)}(1 + \|f(q)\|^2) dq. \quad (45)$$

By taking the maximums w.r.t. all $r \in [t, +\infty)$, we obtain (41) from (43) and get (42) from both (44) and (45).

We generalize the uniform Gronwall lemma (cf. [31, Lemma III 1.1]) to a non-autonomous version.

**Lemma 6.2.** Let $y, h_{1}, h_{2}$ be nonnegative, locally integrable and $y'$ locally integrable on $\mathbb{R}$ such that

$$\frac{dy}{dq} \leq h_{1}(q)y + h_{2}(q), \quad \text{for } q \geq r - \tau, \quad (46)$$
where \( r \in \mathbb{R} \) and \( \tau > 0 \). Suppose \( \sigma \in (0, \tau) \), then we have
\[
y(r) \leq e^{b_1(r)}(b_2(r) + \frac{b_3(r)}{\sigma}),
\]
where \( b_1(r) := \int_{r-\sigma}^{r} h_1(q) \, dq \), \( b_2(r) := \int_{r-\sigma}^{r} h_2(q) \, dq \) and \( b_3(r) := \int_{r-\sigma}^{r} y(q) \, dq \).

Proof. We multiply (46) by \( \exp(\int_{q}^{r} h_1(\tilde{q}) \, d\tilde{q}) \) and obtain the inequality
\[
\frac{d}{dq}(y(q) \exp(\int_{q}^{r} h_1(\tilde{q}) \, d\tilde{q})) \leq h_2(q) \exp(\int_{q}^{r} h_1(\tilde{q}) \, d\tilde{q}) \quad \text{for all } q \in [r-\sigma, r].
\]
Integrating the above inequality from \( q \in [r-\sigma, r] \) to \( r \), we find for all \( q \in [r-\sigma, r] \),
\[
y(r) \leq y(q) \exp(\int_{r-\sigma}^{r} h_1(\tilde{q}) \, d\tilde{q}) + \int_{r-\sigma}^{r} h_2(\tilde{q}) \, d\tilde{q} \cdot \exp(\int_{r-\sigma}^{r} h_1(\tilde{q}) \, d\tilde{q}).
\]
Then we obtain (47) by integrating the above inequality from \( q = r-\sigma \) to \( q = r \).

**Lemma 6.3.** Let the hypotheses \( A \) and \( B_1 \) be satisfied. Then there are constants \( c_3, c_4 > 0 \) such that, for each ball \( B = N(0, R) \) in \( H \), there is a \( \tau_0 = \tau_0(R) \geq 2 \) such that
\[
\sup_{r \geq 1} \sup_{\tau \geq \tau_0} \sup_{u_0 \in B} \|\nabla u(r, r - \tau; u_0)\|^2 \leq c_3(1 + F(t))e^{c_4(1 + F(t))}, \quad \forall t \in \mathbb{R}.
\]

Proof. Multiplying Eq. (38) by \(-\Delta \tilde{u}\), integrating over \( \Omega \) and taking the real part, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + \lambda \|\Delta u\|^2 - \gamma \|\nabla u\|^2 = \text{Re}((\kappa + i\beta(t))\|u\|^2 u, \Delta u) - \text{Re}(f(t, \cdot), \Delta u).
\]
The Young inequality implies that \( |\text{Re}(f(t, \cdot), \Delta u(t))| \leq \frac{\lambda}{4} \|\Delta u(t)\|^2 + c\|f(t, \cdot)\|^2 \). By the hypothesis \( A \), we let \( M := \sup_{t \in \mathbb{R}} |\beta(t)| < \infty \). Then
\[
\text{Re}((\kappa + i\beta(t))\|u\|^2 u, \Delta u) = -\text{Re}(\kappa + i\beta(t)) \int_{\Omega} (2|u|^2|\nabla u|^2 + u^2\nabla \tilde{u} \cdot \nabla u) dx \\
\leq 3(\kappa + M) \int_{\Omega} |u|^2|\nabla u|^2 dx.
\]
We need to use the following interpolation inequality (cf. [31, (5.28)]).
\[
\|\nabla \varphi\|_4^2 \leq c\|\nabla \varphi\|(\|\varphi\|^2 + \|\Delta \varphi\|^2)^{1/2}, \quad \text{for } \varphi \in V.
\]
Then, the last term in (49) is bounded by
\[
c\|u\|_2^2\|\nabla u\|_4^2 \leq c\|u\|_2^2\|\nabla u\|(\|u\|^2 + \|\Delta u\|^2)^{1/2} \leq \frac{\lambda}{4}(\|\Delta u\|^2 + \|u\|^2) + c\|u\|_2^2 \|\nabla u\|^2.
\]
All above estimates deduce that
\[
\frac{d}{dq}\|\nabla u\|^2 \leq c(1 + \|u\|_4^2\|\nabla u\|^2 + c(\|u\|^2 + \|f(q, \cdot)\|^2)).
\]

Fix \( t \in \mathbb{R} \), we apply Lemma 6.2 on (50) over the interval \([r - \tau, r]\) with \( r \geq t \). For each \( B = N(0, R) \subset H \), we choose \( \tau_0 = \tau_0(R) \geq 2 \) such that \( e^{-\gamma \tau_0 R^2} \leq 1 \). Then it follows from (42) that for all \( r \geq t, \tau \geq \tau_0 \) and \( u_0 \in B \),
\[
b_1(r) := \int_{r-1}^{r} c(1 + \|u(q, r - \tau; u_0)\|^4_{\frac{1}{4}}) \, dq \\
\leq c + ce^\gamma \int_{r-\tau}^{r} e^{\gamma(q-r)}\|u(q, r - \tau; u_0)\|_{\frac{1}{4}}^4 dq \leq c_5(1 + F(t)),
\]
Note that \( \int_{r-1}^r \|f(q, \cdot)\|^2 dq \leq e^r F(t) \). It follows from (41) that
\[
b_2(r) := c \int_{r-1}^r (\|u(q, r - \tau; u_0)\|^2 + \|f(q, \cdot)\|^2) dq \leq c_6(1 + F(t)). \tag{52}
\]
Similarly, by (42), we have
\[
b_3(r) := \int_{r-1}^r \|\nabla u(q, r - \tau; u_0)\|^2 dq \leq c_7(1 + F(t)). \tag{53}
\]
Therefore, the uniform Gronwall inequality (47) implies that for all \( r \geq t, \tau \geq \tau_0 \) and \( u_0 \in B \),
\[
\|\nabla u(r, r - \tau; u_0)\|^2 \leq (c_6 + c_7)(1 + F(t))e^{c_5(1+F(t))},
\]
which proves (48) as required.

\[\square\]

**Theorem 6.4.** Suppose \( \alpha \in C(\mathbb{R}, \mathbb{R}), \beta \in C_0(\mathbb{R}, \mathbb{R}) \) and \( f \in L_{loc}(\mathbb{R}, H) \) is forward tempered in the sense of Hypothesis B1. Then the non-autonomous Ginzburg-Landau equation has a forward compact pullback attractor \( \mathcal{A} = \{ \mathcal{A}(t) : t \in \mathbb{R} \} \) in \( H \) such that the set-mapping
\[
limit_{t \to -\infty} d(\mathcal{A}(t), \mathcal{A}_U(\infty)) = 0, \quad \limit_{t \to +\infty} d(\mathcal{A}_L(\infty), \mathcal{A}(t)) = 0.
\]
Moreover, the upper limit-set \( \mathcal{A}_U(\infty) \) is nonempty compact and minimal in the upper semi-continuity sense, and the lower limit-set \( \mathcal{A}_L(\infty) \) is maximal in the lower semi-continuity sense.

**Proof.** Define a brochette \( \mathcal{K} = \{ \mathcal{K}(t) : t \in \mathbb{R} \} \) over \( V \) (thus over \( H \)) by
\[
\mathcal{K}(t) = \{ u \in V : \|u\|^2 \leq c_3(1 + F(t))e^{c_4(1+F(t))} \}, \quad \forall t \in \mathbb{R}.
\]
By the Sobolev compact embedding, we know \( \mathcal{K}(t) \) is compact in \( H \) for each \( t \in \mathbb{R} \). Since \( t \to F(t) \) is a deceasing function, the mapping \( t \to c_3(1 + F(t))e^{c_4(1+F(t))} \) is still decreasing. Hence \( \mathcal{K} \) is a decreasing brochette and so \( \cup_{r \geq 1} \mathcal{K}(r) = \mathcal{K}(t) \), which is compact. Therefore, \( \mathcal{K} \) is forward compact in \( H \). By Lemma 6.3, \( \mathcal{K} \) pullback absorbs all bounded subsets in \( H \). Then Theorem 5.4 implies that the evolution process given in (39) has a forward compact pullback attractor. The semi-continuity result follows from Theorems 3.4 and 4.6.

\[\square\]

**6.2. Backward compact attractor and semi-continuity at negative infinity.** To discuss the semi-continuity at negative infinity, we need to give another assumption for \( f \).

**Hypothesis B2.** \( f \in L_{loc}(\mathbb{R}, H) \) is backward tempered in the following sense:
\[
\sup_{s \leq t} \int_{-\infty}^s e^{(q-s)}\|f(q, \cdot)\|^2 dq < +\infty, \quad \forall t \in \mathbb{R}. \tag{54}
\]

The following result is parallel with Theorem 6.4 and so we omit the proof.

**Theorem 6.5.** Suppose \( f \in L_{loc}(\mathbb{R}, H) \) is backward tempered. Then the Ginzburg-Landau equation has a backward compact pullback attractor \( \mathcal{A} \) such that
\[
limit_{t \to -\infty} d(\mathcal{A}(t), \mathcal{A}_U(-\infty)) = 0 \quad \text{and} \quad \limit_{t \to -\infty} d(\mathcal{A}_L(-\infty), \mathcal{A}(t)) = 0.
\]
Moreover, \( \mathcal{A}_U(-\infty) \) is nonempty compact and minimal in the upper semi-continuity sense, and \( \mathcal{A}_L(-\infty) \) is maximal in the lower semi-continuity sense.
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