Transferring spherical multipliers on compact symmetric spaces

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Abstract. We prove a two-sided transference theorem between $L^p$ spherical multipliers on the compact symmetric space $U/K$ and $L^p$ multipliers on the vector space $i_p$, where the Lie algebra of $U$ has Cartan decomposition $k \oplus i_p$.

This generalizes the classic theorem transference theorem of deLeeuw relating multipliers on $L^p(T)$ and $L^p(R)$.

1. Introduction

Let $G$ be any non-discrete, locally compact, unimodular group and let $L^p(G)$ denote the space of $p$-integrable functions on $G$ with respect to the Haar measure. A bounded linear operator $T : L^p(G) \to L^p(G)$ is said to be a multiplier (on $L^p(G)$) if $T$ commutes with translations on $G$.

It is well known that a bounded linear map $T$ is a multiplier on $L^p(R)$ precisely when there is a bounded measurable function $m$ on $R$ such that $\hat{T}f = \hat{m}\hat{f}$ for all $f \in L^p \cap L^2(R)$. Similarly, a bounded linear map $T$ is a multiplier on $L^p(T)$ if and only if there is a bounded function $m$ on $Z$ such that $\hat{T}f = \hat{m}\hat{f}$ for all $f \in L^p \cap L^2(T)$. To emphasize the association with $m$, we denote the operator by $T_m$.

In 1965, deLeeuw proved two remarkable facts relating the multipliers of $L^p(R)$ to those of $L^p(T)$.

Theorem 1. Let $m$ be a uniformly continuous function on $R$ and for $\varepsilon > 0$ let $m_\varepsilon$ be its restriction to $Z/\varepsilon \subseteq R$, which we identify with $Z$.

1. If $T_m$ is a multiplier on $L^p(R)$, then $T_{m_\varepsilon}$ is a multiplier on $L^p(T)$ and $\|T_{m_\varepsilon}\|_{p,p} \leq \|T_m\|_{p,p}$, where $\|\cdot\|_{p,p}$ denotes the operator norm.

2. If the operators $T_{m_\varepsilon}$ are uniformly bounded on $L^p(T)$, then $T_m$ is a bounded linear operator on $L^p(R)$ with $\|T_m\|_{p,p} \leq \sup_\varepsilon \|T_{m_\varepsilon}\|_{p,p}$.

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These two elegant theorems became the prototype for a number of “transference” results, where the boundedness of a multiplier operator (for instance, a convolution kernel on a group) may be checked on a different, hopefully simpler, group. Finding analogues in various settings continues to be of interest today, c.f. [1].

This theme was taken up in the context of non-commutative harmonic analysis, first by Coifman and Weiss [2, 3], who proved Marcinkiewicz type results for SU(2) and other Lie groups, replacing the quotient map: \( \mathbb{R} \to \mathbb{T} \) by the mapping \( X \to \exp(X) \) from the Lie algebra to its Lie group. Rubin [16] used similar ideas in the context of \( SO(3) \) and the Euclidean motion group \( M(2) \). The approach was next pursued by Dooley and others, who showed that the notion of a contraction, or a continuous deformation of Lie groups, was the key underlying idea: The Lie group \( G_2 \) is said to be a contraction of the Lie group \( G_1 \) if there is a family \( (\pi_\varepsilon)_{\varepsilon > 0} \) of local diffeomorphisms \( \pi_\varepsilon : G_2 \to G_1 \), which are approximate homomorphisms in the sense that \( \pi_\varepsilon(x) \to e \) as \( \varepsilon \to 0 \) and \( \pi_\varepsilon^{-1}(\pi_\varepsilon(x)\pi_\varepsilon(y)) \to xy \) as \( \varepsilon \to 0 \). This notion generalizes the homomorphism/dilation relationship between \( \mathbb{R} \) and \( \mathbb{T} \). For example, if we have the Cartan decomposition \( u = t \oplus ip \) of a Riemannian compact symmetric pair \( (U, K) \), then the Cartan motion group \( p \rtimes K \) is a contraction of \( U \) by the maps \( \pi_\varepsilon(X, k) = k \exp(\varepsilon X) \). In [7], for example, a version of the second part of deLeeuw’s Theorem was proved in this setting. Further results in this spirit were also given in [5, 8, 11] and [15].

The results of [7] can also be viewed as generalizations of the results of Stanton in [17] where a version of (2) was proven for the transference of spherical multipliers on \( U/K \) to \( \text{Ad}(K) \)-invariant multipliers on \( p \).

Subsequently, in [6] a version of the first part of deLeeuw’s theorem, both for the Cartan motion group contraction and the Coifman-Weiss contraction of \( U \) to \( p \), was given. Unfortunately, the versions of (1) proven in [6] no longer gave an exact converse of the version of (2) from [2, 7, 17] etc. Thus an important open question remains to find a suitable version of the “restriction” for which analogues of both (1) and (2) (simultaneously) hold.

In this article, we will find a suitable version of the restriction for which both directions of deLeeuw’s theorem hold in the case of a contraction of \( U/K \) to \( ip \) for compact symmetric spaces \( U/K \).

2. Harmonic Analysis of Symmetric Spaces

2.1. Symmetric spaces notation. Let \( U \) be a compact, simply connected, semisimple Lie group and suppose \( \theta \) is an involution of \( U \). The set of fixed points under \( \theta \), denoted \( K \), is a compact, connected subgroup of \( U \), and the quotient space, \( U/K \), is known as a compact symmetric space. We will let \( \pi : U \to U/K \) denote the quotient map and given \( x \in U \), we let \( \pi(x) = \pi(x) \pi(K) \) denote the coset \( xK \).

The involution \( \theta \) induces an involution of \( u \), the Lie algebra of \( U \), which we also denote by \( \theta \). Let \( t \) and \( ip \) denote the \( \pm 1 \) eigenspaces of \( \theta \) respectively. The decomposition \( u = t \oplus ip \) is known as the Cartan decomposition.

Let \( g^C \) denote the complexification of \( u \) and let \( g_0 = t \oplus p \). Fix a maximal abelian subspace \( a \) of \( p \) and choose a Cartan subalgebra \( \mathfrak{h} \) of \( g_0 \) containing \( a \). Let \( \mathfrak{h}^C \) denote its complexification and let \( \Sigma \) denote the set of roots of \( g^C \) with respect to \( \mathfrak{h}^C \). Denote by \( \Sigma^+ \) the positive roots and let \( \Phi^+ \) be given by

\[ \Phi^+ = \{ \beta \in \Sigma^+ : \beta|_a \neq 0 \} \]
We write \( a^+ \) for the subset 
\[
   a^+ = \{ H \in a : \alpha(H) > 0 \text{ for all } \alpha \in \Phi^+ \}.
\]
The subsets \( w(\alpha^+ \wedge) \) are disjoint for distinct \( w \in W \), the Weyl group of \( U/K \), and 
\[ a = \bigcup_{w \in W} w(\alpha^+). \]
We will let 
\[
   D = \dim p = \dim U/K \text{ and } r = \dim a.
\]
For notational convenience, we will put 
\[
   p_* := ip \text{ and } a^+_* := -ia^+.
\]
The linear operator \( Ad(k) \) maps \( p_* \to p_* \) whenever \( k \in K \), and a function \( f \) on \( p_* \) is said to be \( Ad(K) \)-invariant if \( f(Ad(k)X) = f(X) \) for all \( X \in p_* \) and \( k \in K \).

Any continuous function defined on \( \pi^p \) has a unique \( Ad(K) \)-invariant extension to \( p_* \).

The notation \( \mu_E \) will denote Haar measure when \( E = U, K, a_* \) or \( p_* \) and will denote a \( U \)-invariant measure on \( U/K \). The measures will be normalized on \( U \) and \( K \), and chosen consistently so that the integration formulas,

\[
   \int_{p_*} f(X) d\mu_{p_*}(X) = \int a^+_* \int_K f(Ad(k)H) \prod_{\alpha \in \Phi^+} \alpha(H) d\mu_K(k) d\mu_{a^+_*}(H)
\]
and
\[
   \int_U f d\mu_U = \int_{U/K} \int_K f(uk) d\mu_K(k) d\mu_{U/K},
\]
hold for continuous functions \( f \) of compact support (on the appropriate domains). In particular, \( \mu_{U/K}(S) = \mu_U(\pi^{-1}(S)) \) for Borel sets \( S \). We often omit the writing of \( \mu_E \) if the underlying space is clear.

As usual, by \( L^p(E) \) we mean the functions defined on \( E \) with \( ||f||_{L^p(E)}^p = \left( \int_E |f(X)|^p d\mu_E(X) \right)^{1/p} < \infty \). Functions on \( U/K \) can be identified with the right \( K \)-invariant functions on \( U \), and these have the same \( L^p \) norm.

2.2. Multipliers on \( L^p(p_*) \). The vector space \( p_* \) can be viewed as a locally compact, abelian group which is self-dual under the killing form \( B(\cdot, \cdot) \). The Fourier transform of \( f \in L^2(p_*) \) is given by

\[
   \hat{f}(Y) = \int_{p_*} f(X) e^{-iB(X,Y)} d\mu_{p_*}(X) \text{ for } Y \in p_*
\]
and the Fourier inversion formula by

\[
   \hat{f}(X) = \int_{p_*} f(Y) e^{iB(X,Y)} d\mu_{p_*}(Y).
\]

A bounded linear operator \( T : L^p(p_*) \to L^p(p_*) \) is called an \( L^p \) multiplier if there is a measurable function \( m \) on \( p_* \) so that for all \( Y \in p_* \) and \( f \in L^2 \cap L^p(p_*) \) we have \( \hat{T}f(Y) = m(Y) \hat{f}(Y) \). Often we write \( T_m \) for \( T \). We denote the operator norm of \( T_m \) by \( ||T_m||_{p,p} \).

Slightly abusing notation, we will also refer to \( m \) as an \( L^p \) multiplier and write \( ||m||_{p,p} \) for the operator norm of \( T_m \).
2.3. Spherical multipliers on $L^p(U/K)$. The left regular representation $\rho$ of $U$, on the Hilbert space $L^2(U/K)$, provides a decomposition of $L^2(U/K)$ into an orthogonal direct sum of invariant subspaces. The irreducible subrepresentations are the class 1 representations of $(U, K)$, those with a one dimensional subspace of $K$-fixed vectors. Let $\Lambda$ be the set of class 1 highest weights. It is known ([T3, p.129]) that these are precisely of the form

$$\lambda = \sum_{j=1}^{r} n_j \sigma_j$$

where $n_j$ are non-negative integers for $j = 1, ..., r$ and $\{\sigma_1, ..., \sigma_r\}$ is a suitable basis for $\mathfrak{a}$ (or more formally, the dual of $\mathfrak{a}$, which we identify with $\mathfrak{a}$).

We will let $\{H_1, ..., H_r\}$ denote the dual basis of $\mathfrak{a}$ with respect to the Killing form $B$, i.e.,

$$B(H_j, X) = \sigma_j(X)$$

for all $X \in \mathfrak{a}$.

With this notation, we have

$$\mathfrak{a}^+ = \langle \sum_{j=1}^{r} n_j H_j : n_j \geq 0 \rangle.$$ 

Given $\lambda$ as above, by $H_\lambda \in \mathfrak{a}$ we mean the element $H_\lambda = \sum_{j=1}^{r} n_j H_j$. Conversely, when $Z = \sum n_j H_j \in \mathfrak{a}^+$, we let $\lambda_Z$ be the weight

$$\lambda_Z := \sum n_j \sigma_j.$$

By $d_\lambda$ we mean the degree of $\lambda \in \Lambda$. Having chosen a $K$-fixed norm-one vector, $v_\lambda$, in the $\lambda$-representation space, we let $\phi_\lambda$ be the spherical function given by $\phi_\lambda(u) = \langle \rho(u)v_\lambda, v_\lambda \rangle$ for $u \in U$. Since $v_\lambda$ is $K$-invariant, we can also view $\phi_\lambda$ as defined on $U/K$ in the natural way.

If $f \in L^2(U/K)$, then we define

$$f * \phi_\lambda(\pi(x)) = \int_U f(\pi(y))\phi_\lambda(y^{-1}x)d\mu_U(y)$$

for $x \in U$.

The Fourier series of $f$ is the formal sum

$$\sum_{\lambda \in \Lambda} d_\lambda f * \phi_\lambda.$$

A bounded linear operator $T : L^p(U/K) \to L^p(U/K)$ is called a spherical multiplier on $L^p(U/K)$ if there is a function $\{m(\lambda)\}_{\lambda \in \Lambda}$ such that for $f \in L^2 \cap L^p(U/K)$, we have

$$T(f) = \sum_{\lambda \in \Lambda} d_\lambda m(\lambda) f * \phi_\lambda.$$ 

As before, we denote this operator by $T_m$ and also refer to the sequence $m$ as a spherical multiplier on $L^p$. We write $\|T_m\|_{p,p}$ or $\|m\|_{p,p}$ for its operator norm.

Spherical multipliers are characterized by the property that they commute with left translation by $U$.

We also view $m = \{m(\lambda)\}_{\lambda \in \Lambda}$ as being defined on the “integer-valued” elements of $\mathfrak{a}^+$ and $\mathfrak{a}^+_+$: If $Z \in \mathfrak{a}^+$ has the form $Z = \sum n_j H_j$ with $n_j \in \mathbb{Z}^+$, we put

$$m(Z) := m(\lambda_Z).$$

Similarly, if $Z = -\sum in_j H_j \in \mathfrak{a}^+_+$ with $n_j \in \mathbb{Z}^+$, then we set $m(Z) := m(\lambda_{-Z})$. 
For more details about symmetric spaces and their harmonic analysis we refer the reader to [12] - [14] for example.

3. Spherical multipliers on $U/K$ transfer to multipliers on $p_*$

3.1. Notation. We let $\exp : u \to U$ denote the exponential map. The notation $\Pi_1$ will be used to denote the exponential map from $p_*$ to $U/K$,

$$\Pi_1(X) = \pi(\exp X) \text{ for } X \in p_*.$$ 

More generally, for $t \geq 1$ we will let $\Pi_t : p_* \to U/K$ be given by

$$\Pi_t(X) = \Pi_1(X/t).$$

We will let $\Omega$ denote a convex neighbourhood of the identity in $p_*$ on which $\Pi_1$ is a diffeomorphism and let $J$ be the Jacobian of $\Pi_1$,

$$J(Y) = \prod_{\alpha \in \Phi^+} \frac{\sin\alpha(iH)}{\alpha(iH)} \text{ where } Y = \text{Ad}(k)H, \ H \in a_+.$$ 

We will also assume that $\Omega$ is chosen suitably small that $J$ is bounded away from 0 and of course, $J$ is bounded by 1.

We have the (change of variable) identity

$$\int_{\Pi_1(\Omega)} f(\pi(\exp(Z)))d\mu_{U/K}(Z) = \int_{\Omega} f(\pi(\exp(Z)))J(Z)d\mu_{p^*}(Z).$$

Given $Z = \sum n_j H_j \in a^+$ and $t > 0$, we put

$$[tZ] = \sum_j [tn_j]H_j$$ 

where $[tn_j]$ denotes the integer part of $tn_j$. As $[tn_j] \geq 0$, $[tZ] \in a^+$ and $\lambda_{[tZ]} = \sum [tn_j]\sigma_j$. We similarly understand $[itZ]$ when $Z \in a^+$.

3.2. A version of deLeeuw’s theorem (2) for spherical multipliers. First, we prove an analogue of the second part of deLeeuw’s Theorem for the pair $U/K, p_*$ that extends Stanton’s Theorem 2.5 of [17].

**Theorem 2.** Let $1 < p < \infty$. Suppose $\{m_t\}_{t > 0}$ is a family of spherical multipliers on $L^p(U/K)$ with $\sup_t \|m_t\|_{p,p} < \infty$. Assume we can define a continuous function $m$ on $a^+_+$ by

$$m(Z) = \lim_{t \to \infty} m_t([tZ]) \text{ for } Z \in a^+.$$ 

Then $m$ extends uniquely to a continuous $\text{Ad}(K)$-invariant function on $p_*$ and the linear map $T_m$ is a multiplier on $L^p(p_*)$ satisfying $\|m\|_{p,p} \leq C \sup_t \|m_t\|_{p,p}$, where $C$ is a constant depending only on $p$.

For the proof we require the following Lemma that is a natural generalization of [17] Prop. 2.4. As it is technical, we will defer its proof until after the conclusion of the proof of the Theorem.

**Lemma 1.** For $Z \in a^+_+, X, Y \in p_*$, and $\lambda_t = \lambda_{[tZ]}$, we have

$$\lim_{t \to \infty} \phi_{\lambda_t}(\exp(Y/t)\exp(X/t)) = \int_K e^{iB(Z,\text{Ad}(k)(X+Y))}dk.$$
Proof of Theorem. Throughout the proof, \( c \) will denote a constant that may change.

First, assume that the family \( \{m_t\} \) satisfies a decay condition, namely, there are constants \( C_1, C_2 \) such that

\[
|m_t([itZ])| \leq C_1 \exp(-C_2 \|Z\|^2)
\]

for all \( Z \in \mathfrak{a}^+_t \) and large \( t \).

The unique extension of \( m \) to a continuous, \( \text{Ad}(K) \)-invariant function on \( \mathfrak{p}_* \) is clear. Thus if we let \( T = T_m \) denote the corresponding linear operator, it will be enough to show that there is a constant \( C \) so that if

\[
I := \int_{\mathfrak{p}_*} T f(X) g(X) dX,
\]

then

\[
|I| \leq C \sup_{t} \|m_t\|_{p,p} \|f\|_{L^p(\mathfrak{p}_*)} \|g\|_{L^q(\mathfrak{p}_*)}
\]

whenever \( f, g \) are \( C^\infty \) functions on \( \mathfrak{p}_* \) with compact support and \( q \) is the conjugate index to \( p \).

Choose \( t \) sufficiently large so that the set \( \mathfrak{t}(t^{-1} \text{supp } f) \cup (t^{-1} \text{supp } g) \) is contained in \( \Omega \) and hence \( \Pi_1 \) is a diffeomorphism there. Define functions \( f_t \) and \( g_t \) on \( \mathfrak{u}/\mathfrak{k} \) by

\[
f_t(\pi(\exp(X))) = f_t(\Pi_1(X)) = f(tX)
\]

and similarly for \( g_t \). These are well defined because of the choice of \( t \).

We will write \( T_t \) for the spherical multiplier corresponding to \( m_t \) and put

\[
I_t = \int_{\mathfrak{u}/\mathfrak{k}} T_t(f_t(y)) g_t(y) d\mathfrak{u}.
\]

Obviously, we have

\[
|I_t| \leq \sup_t \|m_t\|_{p,p} \|f_t\|_{L^p(\mathfrak{u}/\mathfrak{k})} \|g_t\|_{L^q(\mathfrak{u}/\mathfrak{k})}.
\]

The first step is to calculate the \( p \)-norm of \( f_t \). As \( f_t \) is supported on \( \Pi_1(\Omega) \), the change of variables formula gives

\[
\|f_t\|_{L^p(\mathfrak{u}/\mathfrak{k})} = \int_{\mathfrak{u}/\mathfrak{k}} |f_t(\mathfrak{u})|^p d\mathfrak{u} = \int_{\Pi_1(\Omega)} |f_t(\mathfrak{u})|^p d\mathfrak{u}
\]

\[
= \int_{\Omega} |f_t(\Pi_1(\mathfrak{y}))|^p J(\mathfrak{y}) d\mathfrak{y}
\]

\[
= \int_{\mathfrak{p}_*} |f(t\mathfrak{y})|^p J(t^{-1}Y) d\mathfrak{y} = t^{-D} \int_{\mathfrak{p}_*} |f(Y)|^p J(t^{-1}Y) d\mathfrak{y}.
\]

As \( |J(Y)| \leq 1 \) for all \( Y \), we see that \( \|f_t\|_{L^p(\mathfrak{u}/\mathfrak{k})} \leq t^{-D/p} \|f\|_{L^p(\mathfrak{p}_*)} \).

Similarly, \( \|g_t\|_q \leq t^{-D/q} \|g\|_q \), so that

\[
|I_t| \leq \sup_t \|m_t\|_{p,p} t^{-D} \|f\|_p \|g\|_q.
\]

Thus it will be enough to prove that \( \lim_{t \to \infty} t^D I_t = cI \) for some constant \( c \).

Now,

\[
T_t(f_t) = \sum_{\lambda \in \Lambda} d_\lambda m_t(\lambda) f_t * \phi_\lambda,
\]
Combining this identity together with (3.4) and writing $\lambda$ if $\alpha / t$

For $t$ sufficiently large, change of variable arguments and the definitions of $f_t$ and $g_t$ show that $t^D I_t$ equals

(3.4)

Recall that $\lambda \in \Lambda$ has the form $\lambda = \sum_{j=1}^r n_j \sigma_j$ where $n_j \in \mathbb{Z}^+$, so that the sum over $\Lambda$ can be replaced by the sum over $\mathbb{Z}^r$. This gives

\[
\sum_{\lambda \in \Lambda} d_{\lambda t} m_t(\lambda) \phi_{\lambda}(\exp(-Y/t) \exp(X/t)) = \sum_{(n_1, \ldots, n_r) \in \mathbb{Z}^r} d_{\Sigma n_j \sigma_j} m_t(\Sigma n_j \sigma_j) \phi_{\Sigma n_j \sigma_j}(\exp(-Y/t) \exp(X/t))
\]

\[
= \sum_{n \in \mathbb{Z}^r} t^r \int_{\mathbb{R}^+} \cdots \int_{\mathbb{R}^+} m_t(\Sigma [t z_j] \sigma_j) d_{\Sigma [t z_j] \sigma_j} \phi_{\Sigma [t z_j] \sigma_j}(\exp(-Y/t) \exp(X/t)) dz_1 \cdots dz_r
\]

\[
= t^r \int_{\mathbb{R}^+} m_t(\lambda_{itZ}) d_{\lambda_{itZ}}(\lambda)(\exp(-Y/t) \exp(X/t)) dZ
\]

Combining this identity together with (3.4) and writing $\lambda_t$ for $\lambda_{itZ}$ gives

\[
t^D I_t = t^{-D} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} m_t(\lambda_t) d_{\lambda_t} \phi_{\lambda_t}(\exp(-Y/t) \exp(X/t)) f(Y) g(X) J(Y/t) J(X/t) dZ dY dX.
\]

The Weyl dimension formula states that

\[
d_{\lambda_{itZ}} = \prod_{\alpha \in \Phi^+} \frac{\langle \alpha, \lambda_{itZ} \rangle + \delta}{\langle \alpha, \delta \rangle}
\]

where $\delta$ is half the sum of the positive roots. As $\lambda_{itZ}$ is class 1, we have $\langle \alpha, \lambda_{itZ} \rangle = 0$ if $\alpha \not\in \Phi^+$, thus writing $\{itZ\}$ for the ‘fractional’ part of $itZ$ we have

(3.5)

\[
d_{\lambda_{itZ}} = t^{\Phi^+} \prod_{\alpha \in \Phi^+} \frac{\alpha(itZ - \{itZ\}/t + H_{\delta}/t)}{\langle \alpha, \delta \rangle}
\]

hence

\[
\lim_{t \to \infty} \frac{d_{\lambda_{itZ}}}{t^{\Phi^+}} = \prod_{\alpha \in \Phi^+} \frac{\alpha(itZ)}{\langle \alpha, \delta \rangle}.
\]
Moreover, \( D = \dim p_* = \dim a + |\Phi^+| = r + |\Phi^+| \), hence the Lemma implies that for \( Z \in a^+_* \),
\[
\lim_{t \to \infty} t^{r-D} m_t(\lambda|_{U(Z)}) d\lambda|_{U(Z)}(\exp(-Y/t) \exp(X/t)) J(Y/t) J(X/t) \\
= \prod_{\alpha \in \Phi^+} \alpha(iZ) m(Z) \int_K \exp(iB(Z, Ad(k)(X - Y))) dk.
\]

One can see from formula \( 3.5 \) that there is some polynomial in several variables, \( P \), such that \( t^{r-D} d_{\lambda|_{U(Z)}} \leq |P(Z)| \) for all \( t \). Furthermore, \( |\phi_{\alpha}|, |J| \leq 1 \), hence for \( Z \in a^+_* \),
\[
|t^{r-D} m_t(\lambda|_{U(Z)}) d\lambda, \phi_{\lambda}(\exp(-Y/t) \exp(X/t)) J(Y/t) J(X/t)| \\
\leq |P(Z)| C_1 \exp(-C_2 \|Z\|^2),
\]
which is integrable over \( a_* \). Since \( f, g \) are continuous, compactly supported functions, it follows from the Dominated convergence theorem that
\[
(3.6) \quad tD I_t \to \int_{p_*} \int_{p_*} \int_{a^*_+} \prod_{\alpha \in \Phi^+} \alpha(iZ) m(Z) \int_K e^{iB(Ad(k)Z, X - Y)} dk f(Y) g(X) dZ dY dX.
\]

Hence, it only remains to prove that the RHS of \( 3.6 \) is equal to \( cI \) for some suitable constant \( c \).

As \( m \) is \( Ad(\Lambda) \)-invariant and \( \alpha(iZ) \geq 0 \), the integration formula \( 2.1 \) implies
\[
\int_{a^*_+} \int_K m(Z) e^{iB(Ad(k)Z, X - Y)} dk \prod_{\alpha \in \Phi^+} \alpha(iZ) dZ = \int_{p_*} m(Z) e^{iB(Z, X - Y)} dZ.
\]

Thus the RHS of \( 3.6 \) is equal to
\[
(3.7) \quad c \int_{p_*} \int_{p_*} \int_{p_*} m(Z) e^{iB(Z, X - Y)} f(Y) g(X) dZ dY dX
\]
where \( c = \prod_{\alpha \in \Phi^+} (\alpha, \delta)^{-1} \) and Fubini’s theorem is justified by the exponential decay in the function \( m \). The Fourier transform and inversion formulas (see \( 2.3 \), \( 2.4 \)) simplify \( 3.7 \) to
\[
c \int_{p_*} \int_{p_*} m(Z) \hat{f}(Z) e^{iB(Z, X)} g(X) dZ dX = c \int_{p_*} \int_{p_*} \hat{T} f(Z) e^{iB(Z, X)} g(X) dZ dX
\]
\[
= c \int_{p_*} T^* f(X) g(X) dX.
\]
As this equals \( cI \), the proof that \( \|m\|_{p,p} \leq C \sup_t \|m_t\|_{p,p} \) for a suitable constant \( C \) is complete under the additional decay assumption.

In the general case, for each \( \varepsilon > 0 \) and \( t \) large, let \( n_{t,\varepsilon}(\lambda) = \exp(-\varepsilon \|\lambda\|^2 / t^2) \). The rapid decay of the function \( z \to \exp(-\delta \|z\|^2) \) for \( z \in \mathbb{R}^n \) and any \( \delta > 0 \), together with all its derivatives, allows one to use the Hormander-Mihlin style central multiplier theorem for \( L^p(U) \) (c.f., \( 18 \)) to deduce that the functions \( n_{t,\varepsilon} \) are \( L^p \) spherical multipliers on \( U/K \) and, furthermore, that their operator norms are bounded by a constant \( C_1 \) that depends only on \( p \).
It follows that the functions \( m_{t, \epsilon}(\lambda) = m_t(\lambda) \) satisfy \( \|m_{t, \epsilon}(\lambda)\|_{p,p} \leq C_1 \sup_t \|m_t\|_{p,p} \), as well as the decay condition \( 3.3 \). By the first part of the proof,

\[
    m_{\epsilon}(Z) = \lim_{t \to \infty} m_{t, \epsilon}([itZ])
\]

is an \( L^p(p_\ast) \) multiplier with operator norm

\[
    \|m_{\epsilon}\|_{p,p} \leq C \sup_t \|m_{t, \epsilon}\|_{p,p} \leq CC_1 \sup_t \|m_t\|_{p,p}.
\]

Letting \( \epsilon \to 0 \), it follows that \( m \) is also an \( L^p(p_\ast) \) multiplier with norm also bounded by \( CC_1 \sup_t \|m_t\|_{p,p} \). \( \square \)

We turn now to proving Lemma \( 4 \)

**Proof of Lemma.** Put \( g^C = \mathfrak{g}^c \oplus \mathfrak{a}^c \oplus \mathfrak{n}^c \) and let \( \mathcal{P} \) denote the projection onto \( \mathfrak{a}^c \). Let \( G^C \) be the complexification of \( U \) and denote by \( G_0 \) its subgroup with Lie algebra \( \mathfrak{g}_0 \). Then \( G_0 \) has Iwasawa decomposition \( G_0 = KAN \). Let \( \mathcal{H} : G_0 \to \mathfrak{a} \) be given by the rule \( x = k \exp \mathcal{H}(x) n \) and continue it analytically to a neighbourhood of \( c \) in \( G^C \). It is shown in [17] Lemma 2.2, Prop. 2.3 that for \( s \) small and \( Z \) in a suitable neighbourhood of 0 in \( g^C \),

\[
    \mathcal{H}(\exp sZ) = s\mathcal{P}(Z) + O(s^2)
\]

and also that if \( X \in g^C \) has sufficiently small norm, then

\[
    \phi_\lambda(\exp X) = \int_K e^{\lambda(\mathcal{H}(\exp Ad(k)(X)))} dk.
\]

By the Hausdorff-Campbell formula,

\[
    \exp(-Y/t) \exp(X/t) = \exp((X-Y)/t + W_t(X,Y))
\]

where \( \|W_t(X,Y)\| \leq O(1/t^2) \). Putting these facts together and recalling that \( \lambda_t = \lambda_{[itZ]} \), we see that

\[
    \phi_{\lambda_t} \left( \exp \left( \frac{Y}{t} \right) \exp \left( \frac{X}{t} \right) \right) = \int_K \exp \left( \lambda_t(\mathcal{H}(\exp Ad(k)(t^{-1}(X-Y) + W_t(X,Y)))) \right) dk
\]

\[
    = \int_K \exp \left( B([itZ], \mathcal{H}(\exp \frac{1}{t} Ad(k)(X-Y + tW_t(X,Y)))) \right) dk
\]

\[
    = \int_K \exp \left( B([itZ], \frac{1}{t} \mathcal{P}(Ad(k)(X-Y + tW_t) + O(\frac{1}{t^2})) \right) dk.
\]

Writing \( [itZ] = itZ - \{itZ\} \), this becomes

\[
    \int_K \exp \left( iB(Z, \mathcal{P}(Ad(k)(X-Y)) + O(\frac{1}{t^2})) - B([itZ], \frac{1}{t} \mathcal{P}(Ad(k)(X-Y)) + O(\frac{1}{t^2})) \right) dk.
\]

As \( \|\{itZ\}\| \) is bounded (over all \( Z \) and \( t \)),

\[
    \left| B \left( [itZ], \frac{1}{t} \mathcal{P}(Ad(k)(X-Y)) + O(\frac{1}{t^2}) \right) \right| \leq O(\frac{1}{t})
\]

uniformly over \( k \in K \), thus another application of the Dominated convergence theorem implies that as \( t \to \infty \)

\[
    \phi_{\lambda_{[itZ]}} \left( \exp \left( \frac{Y}{t} \right) \exp \left( \frac{X}{t} \right) \right) \to \int_K e^{iB(Z, \mathcal{P}(Ad(k)(X-Y)))} dk = \int_K e^{iB(Z, Ad(k)(X-Y))} dk,
\]

where the final equality was shown in the Proof of Prop 2.4 in [17]. \( \square \)
A special case of the theorem is [17] Thm. 2.5.

**Corollary 1.** Suppose \( m \) is an \( \text{Ad}(K) \)-invariant, continuous, bounded function on \( p_\ast \). For \( t > 0 \), define \( m_t(Z) = m(Z/t) \) for \( Z \in \mathbb{A}_1^+ \) and assume that \( \{m_t\}_{t > 0} \) is a family of spherical multipliers on \( L^p(U/K) \) with uniformly bounded operator norms. Then \( T_m \) is a multiplier on \( L^p(p_\ast) \).

**Proof.** It is enough to note that a continuity argument implies
\[
\lim_{r \to \infty} m_t([tZ]) = \lim_{t} m([tZ]/t) = m(Z) \quad \text{for all } Z \in \mathbb{A}_1^+.
\]
Then call upon the theorem. \( \square \)

**4. Multipliers on \( p_\ast \) transfer to spherical multipliers on \( U/K \)**

In this section we prove an analogue of the first part of deLeeuw’s theorem that is a direct converse of the analogue of the second part that we proved in the previous section, for a restricted class of multipliers.

We continue to assume that \( \Omega \) is a convex neighbourhood of the identity in \( p_\ast \) on which \( \Pi_1 \) is a diffeomorphism and that the Jacobian of \( \Pi_1 \), \( J \), is bounded away from zero. We fix a convex, symmetric neighbourhood of the identity in \( p_\ast \), \( O \subseteq \Omega \), that is relatively compact.

We remind the reader that the notation \( [tZ] \) was defined in (3.2).

**Theorem 3.** Let \( 1 < p < \infty \). Assume \( \xi \in L^1(p_\ast) \) is an \( \text{Ad}(K) \)-invariant function supported on the neighbourhood \( O \). There are constants \( C_1, C_2 > 0 \) and \( L^p \) spherical multipliers on \( U/K \), \( \{m_t(\lambda)\}_{\lambda \in \Lambda} \), such that
\[
\lim_{t \to \infty} m_t(\lambda[tZ]) = \hat{\xi}(Z) \quad \text{for all } Z \in \mathbb{A}_1^+.
\]
and
\[
C_1 \limsup_{t \to \infty} \|m_t\|_{p,p} \leq \|\xi\|_{p,p} \leq C_2 \limsup_{t \to \infty} \|m_t\|_{p,p},
\]
where we view \( \xi \) as a linear operator on \( L^p \) with the action given by convolution.

**Proof.** The approach we take to this proof is motivated by [9] and [10]. Throughout the proof \( c \) will denote a constant that may change. Without loss of generality we will assume \( t \geq 1 \).

Let \( q \) be the conjugate index to \( p \). Given \( F \in L^p(U/K) \) and \( G \in L^q(U/K) \), we define \( F_t \) and \( G_t \) on \( p_\ast \) by
\[
F_t(Z) = \begin{cases} t^{-D/p} F(\Pi_t(Z)) & \text{if } Z \in O \\ 0 & \text{else} \end{cases}
\]
and
\[
G_t(Z) = \begin{cases} t^{-D/q} G(\Pi_t(Z)) & \text{if } Z \in O + O \\ 0 & \text{else} \end{cases}.
\]
Note that \( O/t \subseteq O \), hence, with \( c > 0 \) chosen such that \( J \geq 1/c \) on \( O \), we have
\[
\|F_t\|_{L^p(p_\ast)}^p = t^{-D} \int_O |F(\Pi_t(X))|^p dX 
\leq ct^{-D} \int_O |F(\Pi_t(X))|^p J(t^{-1}X) dX.
\]
Spherical Multipliers

Making the change of variable $X = tW$ and simplifying gives

$$
\|F_t\|_{L^p(\mathbb{P}_s)} \leq c \left( \int_{\mathcal{O}/t} |F(\Pi_t(W))|^p J(W)dW \right)^{1/p}
= c \left( \int_{\Pi_t(\mathcal{O})} |F(\mathbf{x})|^p d\mathbf{x} \right)^{1/p} = c \|F|_{\Pi_t(\mathcal{O})}\|_{L^p(U/K)}.
$$

Similarly, we have

$$
\|G_t\|_{L^q(\mathbb{P}_s)} \leq c \|G|_{\Pi_t(\mathcal{O}+\mathcal{O})}\|_{L^q(U/K)}.
$$

For $f \in L^p(\mathbb{P}_s)$ and $g \in L^q(\mathbb{P}_s)$, we will define the linear action

$$
< f, g >_{p_s} = \int_{\mathbb{P}_s} f(X)g(X)d\mu_p(X)
$$

and similarly for $F, G$ defined on $U/K$. The above computations show that

$$
|< F_t * \xi, G_t >_{p_s}| \leq \|F_t * \xi\|_{L^p(\mathbb{P}_s)} \|G_t\|_{L^q(\mathbb{P}_s)} \leq \|\xi\|_{L^p} \|F_t\|_p \|G_t\|_q
\leq c \|\xi\|_{L^p} \|F|_{\Pi_t(\mathcal{O})}\|_{L^p(U/K)} \|G|_{\Pi_t(\mathcal{O}+\mathcal{O})}\|_{L^q(U/K)}.
$$

This proves we can define a bounded linear operator $V_t : L^p(U/K) \to L^p(U/K)$ by the rule that for each $F \in L^p(U/K)$, the linear function

$$
V_t(F) : L^q(U/K) \to \mathbb{C}
$$

is given by

$$
< V_t(F), G >_{U/K} = < F_t * \xi, G_t >_{p_s} \quad \text{for all } G \in L^q(U/K).
$$

As $V_t$ need not commute with left translation by elements of $U$, we consider the linear map $T_t : L^p(U/K) \to L^p(U/K)$ given by

$$
< T_t F, G >_{U/K} = t^D \int_U < V_t(\rho_y(F)), \rho_y(G) >_{U/K} dy \quad \text{for } G \in L^q(U/K)
$$

where $\rho_y(F)(\mathbf{x}) = F(y^{-1}\mathbf{x}) = F(\pi(y^{-1}x))$ for any $y \in U$. We remark that $\rho_{yz} = \rho_y \rho_z$. An application of Holder’s inequality shows that for any $t$,

$$
|< T_t F, G >| \leq ct^D \int_U \|\xi\|_{L^p} \|\rho_y(F)|_{\Pi_t(\mathcal{O})}\|_{L^p(U/K)} \|\rho_y(G)|_{\Pi_t(\mathcal{O}+\mathcal{O})}\|_{L^q(U/K)} dy
\leq ct^D \|\xi\|_{L^p} \left( \int_U \|\rho_y(F)|_{\Pi_t(\mathcal{O})}\|_p^p dy \right)^{1/p} \left( \int_U \|\rho_y(G)|_{\Pi_t(\mathcal{O}+\mathcal{O})}\|_q^q dy \right)^{1/q}.
$$

Fubini’s theorem gives

$$
\int_U \|\rho_y(F)|_{\Pi_t(\mathcal{O})}\|_p^p dy = \int_{U/K} \int_U \chi_{\Pi_t(\mathcal{O})}(\mathbf{x}) |F(\pi(y^{-1}x))|^p dy d\mathbf{x}
= \int_U \left( \int_U |F(\pi(y^{-1}x))|^p dy \right) \chi_{\Pi_t(\mathcal{O})}(\pi(x)) dx
= \int_U \left( \int_U |F(\pi(y^{-1}))|^p dy \right) \chi_{\Pi_t(\mathcal{O})}(\pi(x)) dx
= \int_U \left( \int_U |F(\pi(y^{-1})|^p dy \right) \chi_{\Pi_t(\mathcal{O})}(\pi(x)) dx.
$$
after replacing $y$ by $xy$. Thus
\[
\int_U \| (\rho_y F) \|_{L^p(U/K)}^p \, dy = \int_U \left( \| F \|_{L^p(U/K)}^p \right) \chi_{\Pi_t(O)} (\pi(x)) \, dx
\]
\[= \mu_{U/K} (\Pi_t(O)) \| F \|_{L^p(U/K)}^p.
\]
Similarly,
\[
\int_U \| \rho_y G \|_{L^q(U+O)}^q \, dy = \mu_{U/K} (\Pi_t(O)) \| G \|_{L^q(U/K)}^q
\]
Since $\Pi_t(O) \subseteq \Pi_t(\Omega) \subseteq U/K$,
\[\mu_{U/K} (\Pi_t(O)) = \int_{U/K} \chi_{\Pi_t(O)} (\pi) d\pi
\]
More change of variables arguments and the fact that $\Pi_1$ is a diffeomorphism on $\Omega$ means that
\[\mu_{U/K} (\Pi_t(O)) = \int_{\Omega} \chi_{\Pi_t(O)} (\Pi_1(Z)) J(Z) \, dZ
\]
\[= t^{-D} \int_{\Omega} \chi_{\Pi_t(O)} (\Pi_1(Z)) J(t^{-1} Z) \, dZ \leq t^{-D} \mu_{U/K}(O)
\]
and this is finite as $O$ is pre-compact. Similarly,
\[\mu_{U/K} (\Pi_t(O + O)) \leq t^{-D} \mu_{U/K}(O + O) < \infty.
\]
Thus
\[\langle T; F, G \rangle_{U/K} \leq c \| \xi \|_{L^p(U/K)} \| F \|_{L^q(U/K)}.
\]
Next, we check that $T_t$ commutes with translation on $U$. Indeed, suppose $F, G$ are continuous functions on $U/K$. For each $v \in U$,
\[\langle T_t (\rho_v F), G \rangle_{U/K} = \int_U V_t (\rho_v F), \rho_y(G) \, dy
\]
\[= t^D \int_U V_t (\rho_v F), \rho_y(G) \, dy
\]
\[= \langle T_t (\rho_v F), G \rangle_{U/K} = \langle \rho_v T_t F, G \rangle_{U/K},
\]
so $T_t (\rho_v F) = \rho_v (T_t F)$ for all $F \in C(U/K)$. A denseness argument implies $T_t$ commutes with $\rho_v$ for all $v \in U$.

These facts establish that for each $t$, $T_t$ is a spherical multiplier satisfying $\| T_t \|_{p,p} \leq c \| \xi \|_{p,p}$.

Now define a multiplier $S_t$ by $S_t(F) = \frac{1}{\mu(O)} T_t (F)$; of course $\| S_t \|_{p,p} \leq c \| \xi \|_{p,p}$ for a new constant $c$. Let $m_t$ be the associated multiplier sequence, i.e., $S_t = S_{m_t}$.

We will next prove that $\lim_{t \to \infty} m_t (\lambda_{t(\xi)}) = \xi(Z)$. The final statement of the theorem, $\| \xi \|_{p,p} \leq C_2 \limsup_{t \to \infty} \| m_t \|_{p,p}$, will then follow from Theorem 2. To do
this, we will find a different formulation of \( T_t \). Applying the definitions of \( T_t, V_t \) and \( \Pi_t \) gives

\[
< T_t F, G >_{U/K} = t^D \int_U < V_t \rho_y(F), \rho_y(G) >_{U/K} dy
\]

\[
= t^D \int_U < (\rho_y F)_t * \xi, (\rho_y G)_t >_{\mathcal{P}_*} dy = t^D \int_U \int_{\mathcal{P}_*} ( (\rho_y F)_t * \xi ) (X) (\rho_y G)_t(X) dX dy
\]

\[
= \int_U \int_{\mathcal{O} + \mathcal{O}} \int_{\mathcal{O}} (\rho_y F)(\Pi_t(W)) \xi(-W + X) (\rho_y G)(\Pi_t(X)) dW dX dy
\]

\[
= \int_U \int_{\mathcal{O} + \mathcal{O}} \int_{\mathcal{O}} F(\pi(y^{-1} \exp W/t)) \xi(-W + X) G(\pi(y) dW dX dY.
\]

After doing the change of variable \( y \rightarrow (\exp X/t)y \) and inversion \( (y \rightarrow y^{-1}) \) we obtain

\[
< T_t F, G >_{U/K} = t^D \int_U \int_{\mathcal{O} + \mathcal{O}} \int_{\mathcal{O}} F(\pi(y \exp(-X/t) \exp(W/t)) \xi(-W + X) dW dX.
\]

As \( F \) is continuous, this proves that

\[
T_t F(\overline{y}) = \int_{\mathcal{O} + \mathcal{O}} \int_{\mathcal{O}} F(\pi(y \exp(-X/t) \exp(W/t)) \xi(-W + X) dW dX.
\]

Changing the order of integration, noting that for a given \( W \in \mathcal{O} \) the integral over the variable \( X \) is limited to \( X \in W + \mathcal{O} \), and then doing the change of variable \( X \rightarrow X + W \) gives

\[
(4.1) \quad T_t F(\overline{y}) = \int_{\mathcal{O}} \int_{\mathcal{O}} F(\pi(y \exp(-(X + W)/t) \exp(W/t)) \xi(X) dX dW.
\]

Recall the Fourier series formula

\[
S_t F = \sum_{\lambda \in \Lambda} d_\lambda m_t(\lambda) F * \phi_\lambda.
\]

For \( Z \in \mathfrak{a}^+_* \), \( \lambda_t = \lambda_{[itZ]} \) and \( F = \phi_\lambda \), we have

\[
S_t \phi_\lambda(\epsilon) = \sum_{\lambda \in \Lambda} d_\lambda m_t(\lambda) \phi_\lambda * \phi_\lambda(\epsilon).
\]

As \( \phi_\lambda * \phi_\sigma(\epsilon) = 0 \) if \( \lambda \neq \sigma \) and equals \( 1/d_{\lambda} \) else, it follows that \( S_t \phi_\lambda(\epsilon) = m_t(\lambda_{[itZ]}) \).

According to Lemma [1]

\[
\lim_{t \rightarrow \infty} \phi_\lambda(\epsilon, \exp(-(X + W)/t) \exp(W/t)) = \mathcal{J}(Z, -X).
\]

where

\[
\mathcal{J}(Z, X) = \int_k e^{i(B(Z, A^d(k)X))} dk.
\]

Since \( |\phi_\lambda| \leq 1 \) and \( \mathcal{O} \) is relatively compact, this identity, together with (4.1) and the fact that \( \xi \) is supported on \( \mathcal{O} \) implies

\[
\lim_{t \rightarrow \infty} S_t \phi_\lambda(\epsilon) = \frac{1}{m_{\mathcal{P}_*, \mathcal{O}}} \int_{\mathcal{O}} \int_{\mathcal{O}} \mathcal{J}(Z, -X) \xi(X) dW dX = \int_{\mathcal{P}_*} \mathcal{J}(Z, -X) \xi(X) dX.
\]
Apply the integration formula (2.1), to get
\[
\lim_{t \to \infty} S_t \phi_{\lambda_t}(e) = \int_{a_+^*} \int_K \xi(Ad(k)H) F(Z, -Ad(k)H) \prod_{\alpha \in \Phi^+} \alpha(H) \, dkdH.
\]
Finally, using $K$-invariance and symmetry properties, we obtain
\[
\lim_{t \to \infty} m_t(\lambda_{itZ}) = \lim_{t \to \infty} S_t \phi_{\lambda_t}(e) = \int_{a_+^*} \xi(H) F(-H, Z) \prod_{\alpha \in \Phi^+} \alpha(H) \, dH
\]
\[
= \int_{a_+^*} \int_K \xi(Ad(k)H) e^{-iB(Z, Ad(k)H)} \prod_{\alpha \in \Phi^+} \alpha(H) \, dkdH
\]
\[
= \int_{a_+^*} e^{-iB(Y, Z)} \xi(Y) \, dY = \hat{\xi}(Z)
\]
for $Z \in a_+^*$, as we desired to show. \qed

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