Competing electric and magnetic excitations
in backward electron scattering
from heavy deformed nuclei

R. Nojarov, Amand Faessler and M. Dingfelder
Institut für Theoretische Physik, Universität Tübingen,
Auf der Morgenstelle 14, D-72076 Tübingen, Germany
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Important $E2$ contributions to the $(e,e')$ cross sections of low-lying orbital $M1$ excitations are found in heavy deformed nuclei, arising from the small energy separation between the two excitations with $I^*K = 2^+1$ and $1^+1$, respectively. They are studied microscopically in QRPA using DWBA. The accompanying $E2$ response is negligible at small momentum transfer $q$ but contributes substantially to the cross sections measured at $\theta = 165^\circ$ for $0.6 < q_{\text{eff}} < 0.9$ fm$^{-1}$ ($40 \leq E_i \leq 70$ MeV) and leads to a very good agreement with experiment. The electric response is of longitudinal $C2$ type for $\theta \leq 175^\circ$ but becomes almost purely transverse $E2$ for larger backward angles. The transverse $E2$ response remains comparable with the $M1$ response for $q_{\text{eff}} > 1.2$ fm$^{-1}$ ($E_i > 100$ MeV) and even dominant for $E_i > 200$ MeV. This happens even at large backward angles $\theta > 175^\circ$, where the $M1$ dominance is limited to the lower $q$ region.

25.30.Dh, 21.60.Jz, 27.70.+q
I. MOTIVATION

Inelastic electron scattering at backward angle $\theta = 180^\circ$ was used by Barber, Peterson et al. 
more than 30 years ago to study nuclear magnetic dipole ($M1$) excitations at the linear accelerator in Stanford. The predominance of $M1$ excitations in backward scattering is expected from qualitative considerations of the $(e,e')$ cross section in the plane wave Born approximation (PWBA). The cross section can be decomposed in this case into longitudinal (Coulomb) and transverse (electric and magnetic) terms, multiplied by corresponding kinematical factors. The electric field of the incoming electron can have both longitudinal and transverse components, while its magnetic field is purely transverse. If the electron rest mass and the nuclear excitation energy can be neglected in comparison with the incident electron energy, which is often the case with low-lying excitations, the longitudinal kinematical factor vanishes for $\theta = 180^\circ$ and only transverse multipoles are excited by the inelastic scattering.

The $(e,e')$ cross section can be related further to the transition probability for photoabsorption through approximate considerations within the "virtual photon" method. One arrives in this way to the rough qualitative estimate for dominant magnetic over electric transverse excitations (both of the same multipolarity). This property is due to the momentum transfer during electron scattering. It stands in contrast with the corresponding relationship for radiative excitation, where no momentum is transferred and the $EL$ excitation is one order of magnitude stronger than the $ML$ excitation ($L$ stands for multipolarity).

A more definite qualitative conclusion about the magnetic dominance at backward angle can be drawn by assuming a purely spin-flip transition. The transverse electric field of the incoming electron is negligibly small in this particular case if the excitation energy $E_x$ is small in comparison with the transferred momentum $q$, i.e., $E_x \ll q\hbar c$. The same result is obtained for a spin-flip transition in the limit of small momentum transfer, $q \rightarrow E_x/\hbar c$. All these considerations are made in the long-wave limit, $qR \ll 1$, where $R$ is the radius of the target nucleus. The momentum transfer has throughout this article the inverse dimension of length ($\text{fm}^{-1}$), i.e., $q/\hbar$ is denoted by $q$.

The peculiar selectivity of the backward $(e,e')$ scattering towards magnetic excitations was used extensively in the last decades to study $M1$ transitions mainly in spherical nuclei. The $M1$ operator can excite there particle-hole states, which are spin-orbit partners. These excitations are therefore often called spin-flip, though the orbital part of the $M1$ operator has also non-vanishing matrix elements. In addition to these predominantly spin transitions, a qualitatively new type of mainly orbital $M1$ excitations with $I^*K = 1^+1$ become possible in deformed nuclei. They are closely related to the so-called scissors mode, predicted by the two-rotor model [8]. It describes isovector rotational oscillations of neutrons against protons, which exist only in deformed nuclei. Their experimental study started a decade ago through backward inelastic electron scattering at the linear accelerator in Darmstadt, see e.g. the review articles [9].

We have calculated within the quasiparticle random-phase approximation (QRPA) the $(e,e')$ form factors for low-lying orbital $M1$ excitations in Titanium [1], rare-earth [2], and actinide [3] nuclei. The present study was motivated firstly by noticing at higher momentum transfer some discrepancies between our theoretical $M1$ form factors and experimental data. These deviations are not present in lighter nuclei but seemed to occur systematically in heavy nuclei. The first diffraction maximum of the experimental $(e,e')$ cross section is usually well described theoretically when the experimental $B(M1)$ value is reproduced in QRPA. In contrast, the $q$-dependence of the theoretical cross section becomes very sensitive to the details of the microscopic wave function at higher momentum transfer $q$.

Our theoretical $M1$ transition density of the $1^+$ excitation at 3.78 MeV in $^{48}$Ti [1] describes well the corresponding experimental form factor [14] not only at low but also up to the highest measured momentum transfer, $q_{\text{eff}} \lesssim 2 \text{ fm}^{-1}$. We were even able to predict the form factor of the strongest $M1$ excitation (at 7.2 MeV) in this nucleus in the same $q$-range before the corresponding experimental data became available [13]. The comparison with experiment can be seen in Fig. 3 of Ref. [7].

On the other hand, our theoretical $M1$ form factors for rare-earth nuclei [2], though lying within the experimental error bars, seem to underestimate systematically the experimental values even at lower momentum transfer (for $q > 0.6 \text{ fm}^{-1}$). In contrast to $^{48}$Ti, high-$q$ experimental data are not available for single $M1$ excitations in heavy deformed nuclei. We were inclined to attribute initially the above discrepancies to the adopted parametrization of the Woods-Saxon potential. The $K^* = 1^+$ excitations in deformed nuclei are weakly collective [14], because the main collectivity is concentrated in the spurious isoscalar rotational state. Thus, the microscopic wave functions of the low-lying orbital $M1$ excitations are very sensitive to the details of the single-particle level scheme around the Fermi surface. In order to improve the agreement with the experimental form factors, we decided to shift in rare-earth nuclei the energy of several single-particle levels close to the nuclear surface [14], a procedure used formerly in microscopic calculations for spherical nuclei. The modified level scheme did not produce, however, a substantial improvement.

Thus, we turn our attention to possible contributions from other multipoles, which should have the property of appearing only in heavy nuclei. We are going to present in the next section preliminary qualitative estimates for the expectation of $E2$ contributions to the $(e,e')$ cross sections at backward angle. The microscopic
calculation of transition densities within our QRPA formalism is described in Sect. III. The theoretical \((e, e')\) cross sections, obtained from these transition densities in the distorted wave Born approximation (DWBA) for \(\theta = 165^\circ\), are compared to experiment in Sect. IV. A qualitative analysis of our DWBA results for larger backward angles is presented in Sect. V using more transparent PWBA estimates. The conclusions are summarized in Sect. VI. The most important ingredients of our formalism can be found in Appendices A-D.

II. PRELIMINARY ESTIMATES

Our search for possible contributions from higher multipoles was stimulated by the experimental study \cite{18} of the \(E2\) transition accompanying the strongest \(M1\) excitation, \(E_x = 3.075\) MeV, in \(156\)Gd. The \(E2\) transition excites the first member (with \(I^\pi K = 2^+1\)) of the rotational band built on the top of the vibrational state with \(I^\pi K = 1^+1\). This is the only accompanying \(E2\) transition studied until now in heavy deformed nuclei. The energy separation between the two excitations (\(M1\) and \(E2\)) corresponds to the rotational energy

\[
\Delta E_{12} = \frac{\left[I(I+1) - K(K+1)\right] h^2}{2 J} = \frac{2 h^2}{J},
\]

where \(J\) is the moment of inertia of the rotational band developed on the intrinsic \(1^+\) state. Since its value is not known, one could use as a rough estimate the ground-state moment of inertia, determined from the energy \(E(2^+_1) = 3 h^2 / J_{g.s.}\) of the first rotational \(2^+\) state in \(156\)Gd \cite{19}. The so obtained energy separation, \(\Delta E_{12} = 59.3\) keV, should be considered as an upper limit, because the rotational bands of excited states have usually larger moments of inertia than the g.s. band \cite{20}. The \(2^+1\) state found in \(156\)Gd \cite{19} lies indeed only 21 keV above its \(1^+1\) band head with \(E_x = 3.075\) MeV. This energy separation is comparable with the energy resolution \(\Delta E_{exp.}\) of the considered \((e, e')\) experiments, 20 keV < \(\Delta E_{exp.}\) < 40 keV, depending on the incident electron energy \cite{21}.

The largest backward angle was \(\theta = 165^\circ\) in these experiments. It corresponds to a ratio between the transverse and longitudinal kinematic factors \cite{1}

\[
\frac{V_t}{V_l} = \frac{1}{2} + \tan^2 \left(\frac{\theta}{2}\right) = 58.
\]

Although the longitudinal Coulomb excitation is strongly damped by this large factor, it could still provide a \(C2\) contribution to the cross section comparable with the \(M1\) contribution. This is due to the fact that the longitudinal \(E2\) transition matrix element, before being reduced by the above factor, is usually much larger than the \(M1\) matrix element. A new \(180^\circ\) electron scattering facility is already in operation in Darmstadt \cite{22}. Though the aperture of such facilities leads to a somewhat smaller effective mean angle of 178.5° \cite{23}, the longitudinal contributions are already negligible at this angle, where the ratio \(\frac{\Delta E_{exp.}}{\Delta E_{12}}\) amounts to 5848. On the other hand, the worsened energy resolution, typically \(\Delta E_{exp.} = 100\) keV at a fully backward angle \cite{23}, makes quite plausible the appearance of transverse \(E2\) contributions to the measured \(M1\) cross section, if both are of comparable magnitude.

The ground state of even-even nuclei corresponds to \(I^\pi K = 0^+0\), so that due to the selection rules each one of the above considered \(1^+1\) and \(2^+1\) states can be excited only by \(M1\) and \(E2\) transitions, respectively, but not by both of them. However, the multipolarity of the excited states studied is usually not known in experiment. Thus, most of the former qualitative estimates for the relative \(CL, EL,\) and \(ML\) contributions to the PWBA cross section consider transitions with the same multipolarity \(L\) in order to find out what kinematics would enhance selectively the \(ML\) excitation.

In contrast, we are interested in the present study in the relative \(E2\) and \(M1\) contributions. They excite two different states, but contribute nevertheless to the experimentally measured cross section only because in heavy deformed nuclei the energy separation of the two states is comparable with the experimental energy resolution. This situation does not occur in lighter even-even nuclei whose small moment of inertia produces a large energy separation between the two states and the two cross sections are easily resolved in experiment. We denote by \(E2\) both longitudinal \(C2\) and transverse \(E2\) components, unless one of them is specified explicitly.

Our study was motivated also by one of the first calculations of \((e, e')\) cross sections in heavy deformed nuclei \cite{21}. The PWBA cross sections for members of the ground-state and \(\beta\)-vibrational bands in \(152\)Sm are obtained in the collective Tassie model assuming an incompressible and irrotational fluid. Although a purely longitudinal (Coulomb) excitation is described by this simple model, the qualitative \(q\)-behaviour of the calculated cross sections supports our expectations of improving the \(M1\) cross sections by taking \(E2\) contributions into account: i) the diffraction maxima of higher multipoles have smaller amplitudes but are broader, ii) the \(I^\pi K = 2^+0\) state of the \(\beta\)-vibrational band has even a larger cross section than the band head (\(0^+0\)), i.e. the magnitude ordering is inverted for the intrinsic excitation with respect to the g.s. band. These qualitative features were confirmed experimentally some 20 years later \cite{23} through \((e, e')\) on \(154\)Gd.

The above arguments about possible appreciable \(E2\) electroexcitations even at backward angles stimulated us to examine more carefully the qualitative estimates, on which the commonly expected \(M1\) dominance in backward electron scattering is based. The relative contribution of the transverse magnetic and electric terms to the \((e, e')\) cross section is given by the approximate PWBA relationship \cite{3}:
\[ \frac{d\sigma(ML)}{d\sigma(EL)} \approx \frac{1}{10} \left( \frac{qhc}{E_x} \right)^2, \]

where we have used the correct ratio from Eq. (30) of Ref. [3]. The factor 10 is a rough estimate for the ratio between the two cross sections at the photon point where they are proportional to the respective reduced probabilities for radiative transitions. Thus, one should expect predominantly magnetic excitations in backward electron scattering, because in most experiments \( E_x \ll qhc \). But the qualitative relationship [4] was derived within the “virtual photon” method valid for \( q \to E_x/hc \). It was stated explicitly [2] that the relationship (3) is not enough to upset in general the dominant magnetic character of backward electron scattering. More detailed considerations will be discussed in Sect. V below.

Without recurring to virtual photons, Willey arrives to a more general qualitative relationship between the two cross sections [22]:

\[ \frac{d\sigma(ML)}{d\sigma(EL)} \approx \frac{1}{(qR)^2}. \]

The magnetic transition is obviously dominant, because [4] is derived in the long-wave limit, \( qR \ll 1 \). However, the long-wave approximation is not fulfilled in the above discussed experiments with heavy nuclei [10–11], where \( q > 0.2 \text{ fm}^{-1}, \ R \approx 6.5 \text{ fm}, \) and \( qR > 1.3 \). With this numerical value one could conclude from [4] that the transverse electric should be comparable with the transverse magnetic electroexcitation, and even dominant for \( q > 0.5 \text{ fm}^{-1}, \) i.e. in the region where we expect important \( E2 \) contributions. Of course, [3] is no more valid for such large values of \( qR \).

The above simple considerations are not able to provide a reliable prediction for the relative interplay of electric and magnetic electroexcitations in backward scattering. They ignore the dynamics contained in the wave functions of the studied nuclear excitations. More definite conclusions can be drawn only from realistic microscopic calculations which are able to provide satisfactory quantitative description of the experimental data. We are using the DWBA because the large charge of heavy nuclei produces a strong Coulomb potential distorting the electromagnetic field of the incoming electron. After our first encouraging results on \(^{156}\text{Gd}\) [27,28], we are going to study here the problem of electric contributions to backward scattering more systematically and to compare our theoretical results with all the available experimental cross sections for single \( M1 \) excitations in rare-earth nuclei.

III. TRANSITION DENSITIES IN QRPA

We solve the QRPA equations of motion for intrinsic excitations with \( K^* = 1^+ \) using a model Hamiltonian [26,30].

\[ \hat{H} = \hat{H}_0 + \hat{H}_{FF} + \hat{H}_{SS}, \]

\[ \hat{F}(m,t) = [\hat{H}_0, \hat{J}(m,t)], \ t = n, p, \ r = \frac{k_1}{k_0}, \]

where the isospin index \( t = n, p \) denotes neutrons or protons. The quasiparticle (q.p.) mean field \( \hat{H}_0 \) [30] is given by an axially-symmetric Woods-Saxon potential [31] plus BCS pairing. Apart from the deformation, all the remaining Woods-Saxon parameters are given for each nucleus by the isospin dependent parametrization of Ref. [22]. It is based on a single set of isospin independent parameters for all nuclei without respect to their mass and charge numbers. The hexadecapole deformation is taken from experiment [32], while the quadrupole one is determined for each nucleus by fitting the microscopically calculated ground state quadrupole moment to the corresponding experimental value [24]. As in most of our previous works, we use an oscillator constant \( \hbar c_0 = 50 \Delta^{-1/3} \text{ MeV} \) and include in the basis (A1) all the oscillator eigenfunctions with principal quantum numbers \((\text{A}2, \mathcal{N} \leq 11)\). The pairing gaps are chosen to reproduce roughly the even-odd mass differences [33].

The next two terms in the model hamiltonian [30] are separable residual interactions of quadrupole and spin type, respectively. The former interaction is constructed with operators of quadrupole type \( \hat{F} \) [30] derived from the deformed part of the q.p. mean field [16,17]. The total angular momentum \( \hat{J}(m,t) \) is symmetrized over the signature \( m = \pm 1 \), see Appendix B. The isoscalar coupling constant \( k_0 \) of \( \hat{H}_{FF} \) is determined microscopically [30] in order to restore the rotational invariance of the model hamiltonian, violated by the deformation. The value \( r = -2 \) is adopted for the ratio [4] between the isovector and the isoscalar coupling constants. This choice, discussed in [22], gives rise in QRPA to a high-lying \( E2 \) strength, whose energy distribution reproduces qualitatively the isovector giant quadrupole resonance observed in \((e,e') \) experiments. We adopt the above rough round value, because of the uncertainties of the scarce experimental data on heavy deformed nuclei. Our attention is focused here on the low-lying orbital \( M1 \) excitations, which are not very sensitive towards the strength of the isovector quadrupole interaction. They are influenced much stronger by the exclusion of spuriousity through restoration of the rotational invariance. They acquire afterwards a well pronounced character of isovector rotational vibrations [16,17].

The last term in (5) is the spin-spin interaction \( \hat{H}_{SS} \) [30]. Its coupling constants,

\[ e^+ A = 200 \text{ MeV}, \quad c^- = -0.5 c^+, \]

are derived from nuclear matter calculations using the Reid soft core interaction in the way explained in [4].

The coupling strengths [4] provide a good description [37] of the experimental spin \( M1 \) strength in rare-earth nuclei, deduced from inelastic proton scattering.

We present here results for all the rare-earth nuclei with experimentally measured \((e,e') \) cross sections of sin-
ingle $1^+$ excitations. They are characterized by a relatively strong $M1$ transition and an excitation energy of about 3 MeV. The energies $E_x$ of the $M1$ excitations, the $B(M1)$ and $B(E2)$ values, and the orbit-to-spin ratio $R_{o.s.}$ of the $M1$ matrix elements are compared with experiment in Table 1. Expressions for $R_{o.s.}$, $M1$ and $E2$ transition probabilities in terms of the RPA amplitudes are given in [4]. The $B(M1)$ values for transitions to $I^+K = 1^+1$ states are calculated with bare orbital and effective spin gyromagnetic factors, $g_s = 0.77g_s^0$. The $B(E2)$ values are analyzed by comparison with the microscopic re-

sponse [41] in two $M_{158}$ MeV with $M_{156}$ MeV. 

Electroexcitations are described by the following one-

electron excitation energy $E_x$ and an excitation energy of about $3\text{ MeV}$ [18].

The experimental situation is more complicated in $^{158}$Gd, where the $(\varepsilon',\varepsilon)$ peak at $3.2\text{ MeV}$ with $E_x(2^+1) = 3.096\text{ MeV}$ [13].

The large ratio $R_{o.s.}$ between the orbital and spin $M1$ matrix elements of each excitation, listed in the last column of Table 1, indicates that these are predominantly orbital transitions. Most of them represent the strongest single $M1$ transition in the corresponding nucleus. They were analyzed by comparison with the microscopic re-

alization [13] of the scissors mode [4], as well as with synthetic rotational phonons [13]. This analysis allows us to interpret them as a weakly collective scissors mode [16,17,12].

Electroexcitations are described by the following one-

body charge and current density operators [27,4]:

$$\hat{\rho}(r) = e \sum_{j=1}^{A} \varepsilon_j \delta(r - r_j),$$  

$$\hat{J}(r) = \hat{J}^C(r) + \hat{J}^S(r),$$  

$$\hat{J}^C(r) = \frac{e}{Me} \sum_{j=1}^{A} \varepsilon_j \hat{P}_j \delta(r - r_j), \quad \hat{P}_j = -i\hbar \nabla_j,$$

$$\hat{J}^S(r) = \nabla \times \hat{\mu}(r),$$  

where $e, M, \varepsilon$ and $g^s$ are the unit charge, the nucleon mass, effective charges and spin gyromagnetic ratios, respectively. $\hat{\rho}(r)$ and $\hat{J}(r)$ are scalar charge density and vector current density operators, respectively. The latter is a sum of the convection current $\hat{j}^C(r)$ and the magnetization (or spin) current $\hat{j}^S(r)$. The summation runs over the coordinates of all the nucleons.

The momentum operator $\hat{p}$ in (10) has to be hermitian symmetrized in order to satisfy the continuity equation [28]: $\hat{P}_{\gamma\nu}^{\gamma\nu} = \frac{1}{2}(\hat{p} + \hat{p}^\dagger)$. However, we do not need this symmetrization in (10) because we work always with signature symmetrized operators, defined below [28]. They possess even higher symmetries, listed in Table 1. Appendix B, including the hermitian property required for the convection current $\hat{J}^C(r)$.

In consistence with our QRPA calculations, we do not use effective charges in (10), but the spin operator is renormalized by a factor of 0.7, included in the spin gyromagnetic ratios:

$$e^n = 0.9, \quad g^n_\nu = 1, \quad g^n_s = 0.77g^s_0, \quad g^n_p = 3.8263, \quad g^n_{p'} = 5.5858.$$  

Thus, only protons contribute to the calculated transition densities of charge and convection currents.

The charge and current density operators $\hat{\rho}(r)$ and $\hat{J}(r)$ can be expanded in spherical harmonics and vector spherical functions, respectively [28]:

$$\hat{\rho}(r) = \sum_{LM} \hat{\rho}_{LM}(r) Y_{LM}^*(\Omega),$$

$$\hat{J}(r) = \sum_{LL'M} \hat{Y}_{LL'M}(\Omega) \hat{J}_{LL'M}(r),$$

The resulting multipole density operators have the form

$$\hat{\rho}_{LL'M}(r) = \int Y_{LM}(\Omega) \hat{\rho}(r) d\Omega,$$

$$\hat{J}_{LL'M}(r) = \int Y_{LL'M}(\Omega) \cdot \hat{J}(r) d\Omega,$$  

where $L' = L, L \pm 1$.  

We are interested only in $M1$ and $E2$ electroexcitations. The former are generated by the transverse $M1$ current density operator $\hat{J}_{1L'M}(r)$. The latter receive longitudinal (Coulomb) contributions from the charge density operator $\hat{\rho}_{2LM}(r)$ and transverse contributions from the $E2$ current density operators $\hat{J}_{2L'M}(r)$, $L' = 1, 3$. Only operators with $M = \pm 1$ contribute to the considered intrinsic excitations with $K^\pi = 1^+$. We use $m$-symmetrized operators in our QRPA formalism [16]. This is necessary to close the algebra of
the commutation relations. Such operators possess also definite symmetries with respect to complex, hermitian, and time conjugation by simply changing their phase under these transformations, see Table I in Appendix B. We are able to ensure in this way the hermiticity of the convection current \( \hat{J} \), to take further symmetries of the microscopic expressions analytically into account, and to reach the minimal set of irreducible terms. The signature \( m = \pm 1 \) corresponds to the intrinsic \( x \)- and \( y \)-axes, which are indistinguishable in axially symmetric nuclei. Thus, final expression for physical observables are independent of \( m \).

There are two main types of operators in the quasiboson formalism: longitudinal and transverse, according on their phase \( \sigma = \pm 1 \) with respect to combined hermitian and time conjugation, considered in Appendix B. Operators of the same type commute with each other in the quasiboson approximation (B9). Non-commuting operators are, e.g., the quadrupole operator \( \hat{Q}(m) \) and the total angular momentum \( \hat{J}(m) \), being of longitudinal and transverse type, respectively. We define here additionally \( m \)-symmetrized multipole charge \( \hat{\rho}^m(r) \) and current \( \hat{J}^m_{LL/1}(m, r) \) density operators using (13):

\[
\hat{J}(m) = \frac{1}{2}[m\hat{J}_z - \hat{J}_y], \quad \hat{J}_z = \hat{J}_x \pm i\hat{J}_y, \\
\hat{Q}(m) = \frac{1}{2}[m\hat{Q}_{21} + \hat{Q}_{21}], \\
\hat{\rho}^m(m, r) = \frac{1}{2}[m\hat{\rho}_{21}(r) + \hat{\rho}_{21}^*(r)], \\
\hat{J}^m_{LL/1}(m, r) = \frac{1}{2}[m\hat{J}_{LL/1}(r) + \hat{J}_{LL/1}^*(r)].
\]

It is seen from their quasiboson representation (B5) that the charge density is of longitudinal (quadrupole) type with \( \sigma = +1 \), while the current densities are of transverse (momentum) type with \( \sigma = -1 \). The definitions (14) ensure that the quasiparticle (q.p.) matrix elements of all transition operators have the same symmetries (B11).

The \((e, e')\) transition densities are the reduced matrix elements (r.m.e.) of the multipole density operators (I3) between the initial and final nuclear states. We describe here in QRPA intrinsic excitations with \( K^\pi = 1^+ \) in axially symmetric even-even nuclei, where \( K \) is the projection of the total angular momentum \( I \), on the intrinsic symmetry axis of the nucleus and \( \pi \) is the parity of the wave function. The states are characterized by \( I^\pi K, \nu \), where \( \nu = 1, 2, ... \) labels the intrinsic QRPA excitations \( \Gamma^\nu_1(m) \) (C1), obtained by solving the QRPA equations of motions.

The transition \( TL \) with multipolarity \( L \) (M1 or E2) takes place between the ground state with \( I^\pi K = 0^+0 \) and an excited state \( \nu \) with \( I^\pi K = L^\pi 1 \). The transition densities (C7), corresponding to the multipole operators (C4), are obtained from (C4). We use the final expression (C9) in terms of q.p. matrix elements (C10) of step-up operators, because the microscopic sum (C9) contains the minimal set of independent terms, not related with each other through any symmetry. Thus, the E2 and M1 transition densities for the \( \nu \)-th QRPA excitation have the form:

\[
\rho^\nu_2(r) = \langle 2^\nu 1, \nu \| \hat{\rho}^2_1(r) \| 0 \rangle = \frac{1}{2} \sum_{0 < \nu < f} \left[ F^\nu_+ \rho_{21}(fi, r) + F^\nu_- \rho_{21}(f'i, r) \right] \]

\[
\hat{J}^\nu_{LL/1}(r) = i\langle L^\nu 1, \nu \| \hat{J}^\nu_{LL} \| 0 \rangle = \frac{i}{2} \sum_{0 < \nu < f} \left[ F^\nu_+ \hat{J}_{LL/1}(fi, r) + F^\nu_- \hat{J}_{LL/1}(f'i, r) \right],
\]

where \( \rho_{21}(fi, r), \hat{J}_{LL/1}(fi, r), \) are the q.p. matrix elements (D3), (D4), (D7), and (D8) of the transition density operators \( \hat{\rho}_{21}(r) \) (longitudinal) and \( \hat{J}_{LL/1}(r) \) (transverse), given by (I3). The transverse transition density (I5a) is decomposed into a sum of convection and spin (or magnetization) parts, according to (B8)-(B10). The information about the considered \( \nu \)-th QRPA excitation is contained in the factors \( F^\nu_\nu(f'i), \) which are linear combinations (C10) of the QRPA amplitudes (C1).

The QRPA transition densities for M1 and E2 excitations of the \( K^\pi = 1^+ \) state in \( ^{156}\text{Gd} \) from Table I are displayed in Fig. 4. This is the strongest low-lying (orbital) M1 transition in \( ^{156}\text{Gd} \). The longitudinal charge transition density \( \rho_2(r) \) has the largest amplitude (note the different ordinate scales), followed by the transverse M1 transition density \( \hat{J}_{11}(r) \). Each transverse density (continuous curve) is a sum of three components, arising from the proton convection (long dashed curve) and spin (dotted curve for protons and short dashed curve for neutrons) currents (B8). In the case of \( \hat{J}_{11}(r) \) and \( \hat{J}_{21}(r) \) the proton convection current is dominant, while the spin currents for protons and neutrons are out-of-phase, as it should be expected for an isovector vibration, and cancel each other to a great extent.

We do not display the transition densities of the other \( 1^+ \) excitations from Table I because they are very similar to those in Fig. 4, apart from the cases of \( ^{156}\text{Gd} \) and \( ^{168}\text{Er} \). Since the latter two cases exhibit also similarities, we show only the transition densities for \( ^{168}\text{Er} \) in Fig. 4. This excitation has a more pronounced volume character because, in contrast to Fig. 4, the transition densities are no more peaked near the nuclear surface. Although the neutrons have a large contribution to the RPA wave function of this excitation, the neutron spin current is very small, as seen from the short-dashed curves in Fig. 4.

IV. INTERPLAY OF E2 AND M1 \((e, e')\) CROSS SECTIONS

We use the QRPA transition densities (I5) for a selected intrinsic excitation \( \Gamma^\nu_1 \) to calculate its \((e, e')\) cross sections with the DWBA code of J. Heisenberg [46]. The M1 excitation is fully specified by the transverse transition density \( \hat{J}^\nu_{11}(r) \), while the E2 excitation has also
a longitudinal (Coulomb) part, \( \rho_{L}^{\ell}(r) \), besides the two transverse components \( J_{2L}^{\ell}(r) \), \( L' = 1, 3 \). The DWBA code uses only \( J_{2L}^{\ell}(r) \) as input and takes the other transverse transition density through the continuity equation into account.

The DWBA \((e,e')\) cross sections of the five \( 1^+ \) excitations from Table II are plotted in Figs. 3 and compared to experiment. Each one of these \( 1^+ \) states corresponds to the strongest low-energy orbital \( M1 \) transition in the respective nucleus. These are all the cases in which experimental \((e,e')\) cross sections of single \( 1^+ \) excitations in rare-earth nuclei have been published until now. The QRPA transition densities for \(^{156}\text{Gd}\) and \(^{168}\text{Er}\), used in the DWBA calculations, are shown in Figs. II and II.

The DWBA cross sections in Figs. II and II correspond to the scattering angle \( \theta = 165^\circ \), at which the experimental data [5, 14, 16] (dots with error bars) were taken. The wave of the incoming electron is distorted by the ground state charge distribution of the target nucleus, described by a two-parameter Fermi formula. We use for \(^{154}\text{Sm}\) and \(^{156}\text{Gd}\) the experimental values [17] of the two parameters, extracted from electron scattering. Since no experimental values are available for the other studied nuclei, the values for the latter nucleus are used also for \(^{154,156}\text{Gd}\), while the experimental values for \(^{168}\text{Er}\) [17] are employed for \(^{168}\text{Er}\).

It is seen from Figs. II and II that the theoretical \( M1 \) cross section of the \( I^+ K = 1^+ 1 \) excitation (dashed curve) underestimates systematically the experimental data, especially at higher momentum transfer \( q_{\text{eff}} > 0.6 \text{ fm}^{-1} \), i.e., \( q > 0.4 \text{ fm}^{-1} \), corresponding to incident electron energy \( E_{\text{e}} > 40 \text{ MeV} \). This peculiarity was discussed already in the Motivation, Sect. I. At small momentum transfer (close to the photon point) the cross section of the accompanying \( E2 \) transition to the \( I^+ K = 2^+ 1 \) state (dotted curve) is two orders of magnitude smaller than the \( M1 \) cross section. Thus, the \( E2 \) transition will not introduce practically any correction to the experimental \( B(M1) \) value, which is extracted by extrapolating the experimental cross section to the photon point \( q = E_{\text{x}} / \hbar c \).

As discussed in Sect. II, the accompanying \( E2 \) response was identified experimentally only in \(^{156}\text{Gd}\) as yet [13]. Its experimental \( B(E2) \) value is reproduced in our calculations together with the corresponding \( B(M1) \) value (Table II). This is an additional proof for the correct magnitudes at low \( q \) of our theoretical \( E2 \) and \( M1 \) DWBA cross sections for \(^{156}\text{Gd}\), displayed on the top plot of Fig. II. The correct transition probabilities provide, moreover, an additional test for our QRPA transition densities \( \rho_{L}^{\ell}(r) \) and \( J_{2L}^{\ell}(r) \) from Fig. II because of the simple relationships [16]:

\[
B(E2) = \left[ \int_{0}^{\infty} \rho_{L}^{\ell}(r)r^{4}dr \right]^{2},
\]

\[
B(M1) = \frac{1}{2} \int_{0}^{\infty} J_{2L}^{\ell}(r)r^{3}dr \] .

(16)

The \( E2 \) and \( M1 \) cross sections become comparable in magnitude in the region \( 0.7 < q_{\text{eff}} < 0.8 \text{ fm}^{-1} \) \((0.5 < q < 0.6 \text{ fm}^{-1}, 50 \leq E_{\text{i}} \leq 60 \text{ MeV})\), as seen from Figs. III and III. The sum of the \( E2 \) and \( M1 \) theoretical cross sections (continuous curve) agrees very well with the available experimental data on \((e,e')\) cross sections of single \( 1^+ \) excitations in rare-earth nuclei. The discrepancies in this \( q\)-region, typical for the \( M1 \) cross section alone, have been removed. It is seen from Figs. III and III that the accompanying \( E2 \) transition provides an important contribution to the measured cross sections for \( 0.6 < q_{\text{eff}} < 0.9 \text{ fm}^{-1} \) \((0.4 < q < 0.7 \text{ fm}^{-1}) \). \( 40 \leq E_{\text{i}} \leq 70 \text{ MeV} \).

Let us consider further in more detail the \( 1^+ \) excitation in \(^{156}\text{Gd}\) as an example. Its theoretical \( M1 \) cross section (dashed line in the top plot of Fig. III) is plotted once more as a continuous curve in Fig. III up to a higher momentum transfer. It is obtained from the QRPA transition density \( J_{2L}^{\ell}(r) \) (continuous line in the corresponding plot of Fig. III). If one neglects the spin current and uses only the convection transition density \( J_{2L}^{\ell}(r) \) (long dashed line in Fig. III), one obtains the DWBA cross section displayed by a dashed curve in Fig. III.

The comparison between the continuous and dashed curves in Fig. III shows that the \( M1 \) excitation has a predominantly convection character, as it should be expected for an orbital transition characterized by the large orbit-to-spin ratio \( R_{\text{os}} \). in Table II. The total (convection plus spin) PWBA cross section (dot-dashed curve in Fig. III) is about one order of magnitude smaller than the corresponding DWBA cross section (continuous curve). Note that the PWBA cross section is plotted, as usual, versus \( q \) (not \( q_{\text{eff}} \)).

It is easier to draw qualitative conclusions in PWBA, where the cross section can be decomposed into a sum of longitudinal and transverse terms [4, 5]:

\[
\frac{d\sigma}{d\Omega}_{\text{PWBA}} = \left( \frac{Ze^2}{E_{\text{i}}} \right)^2 \left\{ V_{I} F_{\ell}^{I}(q)^2 + V_{T} \left[ |F_{\ell}^{M}(q)|^2 + |F_{L}^{M}(q)|^2 \right] \right\},
\]

(17)

\[
V_{I} = \frac{\cos^2(\theta/2)}{4 \sin^4(\theta/2)}, \quad V_{T} = \frac{1 + \sin^2(\theta/2)}{8 \sin^4(\theta/2)},
\]

\[
V_{T} = V_{I}/V_{T} = \frac{1}{2} + t^2(\theta/2).
\]

(18)

\( Z \) and \( e \) are the nuclear proton number and the unit charge, respectively. Note that the squared form factors of [4, 5] have to be multiplied by \( 4\pi/Z^2 \) in order to obtain our definitions [4, 5] for the form factors \( |F_{L}(q)|^2 \) [17], coinciding with those of [12]. Only the term \( F_{I}^{I}(q) \) survives in [17] in the case of \( M1 \) transitions, while the \( E2 \) excitations are specified by \( F_{2}^{I}(q) \) and \( F_{2}^{E}(q) \). The PWBA cross section [17] is obtained in the relativistic limit (neglected electron rest mass) and assuming a small excitation energy in comparison with the momentum transfer, i.e., \( E_{\text{x}} \ll \hbar c \).
It follows from (18) that the transverse contribution to the \(E2\) cross section is enhanced at \(\theta = 165^\circ\) by a kinematical factor \(V_\theta = 58\) with respect to the longitudinal contribution. On the other hand, it is seen from Fig. 1 that the amplitude of the longitudinal transition density \(\rho_2^L(r)\) is one order of magnitude larger than the transverse densities \(\rho_2^T(r)\), \(L' = 1, 3\). The squares of the corresponding form factors will differ roughly by a factor of 100, since the form factors are Fourier transforms of the transition densities. Thus, kinematical and dynamical factors act in opposite directions in backward scattering and almost compensate each other for \(\theta = 165^\circ\).

V. QUALITATIVE ANALYSIS OF BACKWARD SCATTERING

More definite conclusions can be drawn from Fig. 2, where the PWBA cross section of the \(I^*K = 2^+1\) excitation in \(^{156}\)Gd is plotted for \(\theta = 165^\circ, 175^\circ, 178.5^\circ,\) and \(179.5^\circ\). The total \(E2\) cross section (continuous curve) is the sum of the longitudinal \(C2\) (dashed curve) and transverse \(E2\) (dotted curve) cross sections, as given by (14). For small momentum transfer, \(q \approx 0.3\) fm\(^{-1}\), the total \(E2\) cross section at \(165^\circ\) is one order of magnitude smaller than the corresponding PWBA \(M1\) cross section, shown by a dot-dashed curve in Fig. 1. Thus, the \(M1\) dominance over \(E2\) at small \(q\) is preserved in PWBA, although it is more pronounced in DWBA where the difference is two orders of magnitude at smaller \(q\), as seen from the top plot of Fig. 1.

It is seen from Fig. 1 that the Coulomb cross section \(C2\) (dashed curve) provides the main contribution to the \(E2\) cross section at \(\theta = 165^\circ\) up to a high momentum transfer \(q < 1.4\) fm\(^{-1}\). The transverse \(E2\) cross section remains constant in all the four plots because the transverse factor \(V_\theta\) (18) has already reached its maximal value at \(\theta = 165^\circ\) and does not increase any more appreciably with \(\theta\). In contrast, the longitudinal \(C2\) cross section (dashed curve) is reduced further when going to larger backward angles since \(V_\theta\) decreases rapidly. As seen from Fig. 1, the longitudinal and transverse cross sections become comparable at \(175^\circ\).

The total \(E2\) cross section originates exclusively from the transverse current at \(175.5^\circ\) in the whole \(q\)-range studied. As discussed in Sect. II, this is the largest effective angle reachable in experiments with backward \((e, e')\) scattering [23]. Further increase in \(\theta\) (179.5\(^\circ\)) does not produce any change in the \(E2\) cross section. The transverse diffraction minima occur at higher \(q\)-values than the longitudinal ones because the transverse transition densities \(J^\nu_{2L}(r), L' = 1, 3\) are peaked at a smaller radial distance than the charge transition density \(\rho_2^L(r)\), as seen from Fig. 1.

The above qualitative analysis in PWBA indicates that the important \(E2\) contributions to the experimental \(M1\) cross section at intermediate transferred momenta and \(\theta = 165^\circ\), seen in Figs. [11], originate mainly from the Coulomb excitation. Since it is already strongly damped at \(\theta = 178.5^\circ\), as seen from Fig. 5, we plot the DWBA cross sections for this angle in Fig. 6. It is seen from the latter figure that the summed cross section (continuous curve) originates mainly from the \(M1\) excitation (dashed curve) for \(q_{\text{eff}} < 1.2\) fm\(^{-1}\). Since both \(M1\) and \(E2\) cross sections are transverse at this angle, the difference in magnitude arises mainly from the respective transition densities, shown in Fig. 1. The two cross sections are of comparable magnitude for \(1.2 < q_{\text{eff}} < 2\) fm\(^{-1}\). One has to expect an \(E2\) dominance for larger momentum transfer (not shown in Fig. 5) having in view that lower multipoles decrease faster for larger \(q\) [24], as discussed in Sect. II.

Therefore, the analysis of the interplay between \(M1\) and \(E2\) \((e, e')\) responses of the low-lying orbital \(1^+\) excitations leads to the following conclusion. Even at large backward angles, where the strong longitudinal \(C2\) response is switched off by the kinematics, the usually expected \(M1\) dominance at low \(q\) is no more preserved for larger momentum transfer, \(q_{\text{eff}} > 1.2\) fm\(^{-1}\).

We learned recently [18] that the rôle of longitudinal contributions at \(\theta = 180^\circ\) was studied earlier in even-odd nuclei [19]. Their importance was demonstrated for the ground-state rotational band of \(^{181}\)Ta. The multipole mixing is more important in even-odd nuclei since their ground state has a large angular momentum (spin) and many multipoles are allowed by the selection rules. In contrast, even-even nuclei have a zero spin in the ground state and no multipole mixing occurs in transitions to the ground state. Thus, the effect we are studying has a different origin: it is entirely due to the large moment of inertia of heavy nuclei, as discussed in Sect. II.

The DWBA \(C2\) cross sections were calculated in [11] by including the Coulomb and neglecting the transverse \(E2\) contributions. The latter, which are strongly enhanced by the transverse kinematical factor at \(180^\circ\), were calculated separately in PWBA for comparison. They are smaller than the Coulomb DWBA cross sections but, according to the above analysis, they could become larger than the longitudinal contribution if calculated in DWBA. As noted in [14], more precise conclusions could be drawn after including both longitudinal and transverse contributions in DWBA calculations, where they interfere in contrast to the coherent PWBA sum (17). The transitions within a rotational band, studied in [19], are characterized by strong Coulomb excitations because the intrinsic wave function does not change in the final state and the matrix element of the charge density operator is large. In our case of intrinsic excitations with \(K^\pi = 2^+\) the Coulomb excitation is due to the predominantly orbital character of the low-lying \(M1\) transitions, involving a large non spin-flip component in the RPA wave function.

The longitudinal kinematical factor \(V_\theta\) (18) vanishes for \(\theta = 180^\circ\), so that a longitudinal contribution can arise
only if the electron rest mass, neglected in (17), (18), is taken into account. The diverging ratio $V_T$ becomes finite in this case (49): \[ V_T(180^\circ) \approx \left( \frac{qhc}{2m_e^2} \right)^2 \approx \left( \frac{E_i}{m_e^2 c^2} \right)^2, \] (19) where $m_e c^2 = 0.511$ MeV is the electron rest mass. The latter relation in (19) is valid for $E_x \ll E_i$. It is seen from (19) that the longitudinal kinematical factor will be comparable in magnitude with the transverse one ($V_T \lesssim 1$) only for unrealistically small momentum transfer $q \lesssim 0.005$ fm$^{-1}$ or incident electron energy $E_i \leq 2$ MeV.

A large Coulomb $C2$ cross section was found however in (49) for $\theta = 180^\circ$ and $q_{\text{eff}} < 0.7$ fm$^{-1}$. This can be understood in PWBA by assuming a transverse electric excitation of convection type and using the "virtual photon method" (2) for small $q$ to obtain a qualitative estimate for the relative $CL$ and $EL$ cross sections, given in Table XXIV of (3): \[ \frac{d\sigma(\text{CL})}{d\sigma(\text{EL})_{\text{conv}}} \approx \left( \frac{qhc}{E_x} \right)^2 \approx \left( \frac{2m_e c^2}{E_x} \right)^2, \] (20) where also $V_T(180^\circ)$ from (15) was used. Note that the wrong leading $EL$ term $q/M$ was chosen in (20) not because it is spin-flip, as stated in (3), but because it is smaller than the correct leading term $E_x/q$ for low $q$. In contrast to the severe restrictions in (3), it is seen from (20) that for any multipolarity $L$ the Coulomb cross section will be larger than the transverse electric one for excitation energies $E_x \leq 1$ MeV. This is indeed the case in (49), where the studied first two members of the ground state rotational band have very small energies, 0.14 and 0.3 MeV.

The situation is more unfavourable in our case, where the considered $1^+$ excitations from Table 1 have $E_x \approx 3$ MeV. In this case we have still a dominant Coulomb contribution at $165^\circ$ for $q < 1.2$ fm$^{-1}$, but for $\theta > 178^\circ$ the $C2$ cross section is negligible in comparison with the transverse $E2$ cross section, as seen from Fig. 3. This result agrees with the qualitative PWBA prediction (21) for a difference of one order of magnitude at 180$^\circ$. Therefore, our results differ qualitatively from those for the rotational excitations of even-odd nuclei (49), where the attention was focused on a fully backward angle, $\theta = 180^\circ$, so that the main effect found there (large $C2$ contributions) is due to the small but non-negligible electron rest mass (15), (20). This is not the case with the $1^+$ excitations in even-even nuclei, studied in the present work, where the Coulomb contribution becomes negligible already before reaching a full backward angle.

Thus, in our case of low-lying orbital $1^+$ excitations the interplay of transverse electric and magnetic contributions is of prime importance for $\theta > 178^\circ$. We borrow the PWBA estimates in the low-$q$ limit from Table XXIV of (3): \[ \frac{d\sigma(M, L - 1)}{d\sigma(\text{EL})_{\text{conv}}} \approx \left( \frac{\hbar c^2}{E_x R^2 M^2 c^2} \right)^2 \approx E_x^{-2}, \] (21) where $M$ is the nucleon mass. A nuclear radius $R = 6.5$ fm was assumed for the rare-earth region to obtain the latter relationship in (21), where the excitation energy $E_x$ must be supplied in MeV. The spin contribution to the transverse electric cross section is negligible with respect to the convection contribution at low $q$ and the ratio between the $M1$ and $E2$ cross sections is determined by (21). This relationship should be valid for any scattering angle. It applies to both the convection and spin parts of the magnetic excitation ($M, L - 1$).

It is seen from (21) that a dominant $M1$ excitation is predicted in PWBA for small rotational energies, e.g. $E_x \leq 0.3$ MeV (19), but not in our case where $E_x \approx 3$ MeV (Table 1). The PWBA expectation for a dominant $E2$ transition, which should be valid at least for low $q$, is not supported by our microscopic results, shown in Fig. 3: the $M1$ cross section (dashed curve) is one order of magnitude larger than the $E2$ response (dotted curve) for $q_{\text{eff}} < 1.2$ fm$^{-1}$.

The main features of the competing $E2$ and $M1$ responses to inelastic electron scattering, found by the qualitative analysis in this section, are common for all the low-lying $1^+$ excitations from Table 1, studied in this work. Each of them is the strongest orbital $M1$ excitation in the respective nucleus and has the character of a weakly collective scissors mode (10, 17).

VI. SUMMARY AND CONCLUSIONS

We have studied the competition between electric and magnetic excitations with $K^+ = 1^+$ in response to inelastic electron scattering from heavy deformed nuclei. This study, whose initial results were reported briefly in (23), is, to our knowledge, the first theoretical investigation of possible electric contributions to $M1$ cross sections of even-even nuclei, measured experimentally at backward scattering angle. The topic is of some general interest, because the backward electron scattering has established itself since three decades as one of the main tools for the experimental study of nuclear magnetic dipole excitations due to its particular sensitivity towards the magnetic response at backward angles (13).

Electric contributions have been usually neglected at backward angles, apart from a theoretical indication (49) for a possible Coulomb excitation in the ground state rotational band of heavy even-odd nuclei on the example of $^{181}$Ta. The multipole mixing is allowed in even-odd nuclei since they have a non-vanishing angular momentum in the ground state. In contrast, the mixing is forbidden by selection rules for transitions to the ground state of even-even nuclei, which has a zero spin.

However, a rotational band develops in deformed nuclei on the top of each intrinsic excitation with $K^+ = 1^+$. The $M1$ transition with $I^+K = 1^+1$ is accompanied by an $E2$ transition to the first rotational member with $I^+K = 2^+1$ of this band. The energy separation between the two
excitations is small in heavy nuclei with a large moment of inertia and comparable with the experimental energy resolution in electron scattering. Such a mechanism for multipole mixing was not considered previously, although it gains in importance through the systematic occurrence of accompanying \( E2 \) transitions. It stands in contrast to the usual low probability for the random presence of some other intrinsic \( E2 \) excitation in the vicinity of the studied \( M1 \) transition.

The accompanying \( E2 \) transition was identified experimentally only in \(^{156}\text{Gd} \) as yet. We were motivated by this work in the search for a possible explanation for the systematic deviations of the theoretical \( M1 \) form factors from experiment at higher momentum transfer, occurring only in heavy nuclei, but absent in light ones even up to higher transferred momentum.

We describe the \( \pm \) excitations in heavy nuclei within QRPA using a deformed Woods-Saxon potential, BCS pairing, and separable residual interactions. Our formalism was able to provide a good description of \( \pm \) excitations (see the review) in Titanium, rare-earth, and actinide nuclei, studied in \((e,e')\), \((\gamma,\gamma')\), and \((p,p')\) experiments, e.g., \([6,7,38,39]\), reviewed in \([40,41]\).

Using this formalism, we have studied here the competing \( E2 \) and \( M1 \) excitations in all the rare-earth nuclei with experimentally measured \((e,e')\) cross sections. These are the strongest \( M1 \) excitations in each nucleus. They have an orbital nature and correspond to a weakly collective low-energy scissors mode \([42,43]\). The theoretical energies, \( B(M1) \) values, and orbit-to-spin ratios are in agreement with experiment. The only available experimental \( B(E2) \) value \([44]\) is also reproduced. The necessary transition densities are obtained microscopically from the QRPA wave functions within our formalism, developed in a cylindrical basis. The \((e,e')\) cross sections are calculated in DWBA from the transition densities.

The comparison with the experimental \((e,e')\) cross sections, measured at scattering angle \( \theta = 165^\circ \), shows that the previous systematic deviations of the theoretical \( M1 \) cross sections from experiment at higher momentum transfer \( q \) are removed by taking the \( E2 \) response into account. At low \( q \) the theoretical \( E2 \) cross section is two orders of magnitude smaller than the corresponding \( M1 \) cross section. Therefore, the \( E2 \) response does not introduce any corrections to the \( B(M1) \) values derived by extrapolating the experimental cross sections to the photon point (minimal \( q \) value).

The \( E2 \) and \( M1 \) cross sections become comparable in magnitude in the region \( 0.7 < q_{\text{eff}} < 0.8 \text{ fm}^{-1} \) (\( 0.5 < q < 0.6 \text{ fm}^{-1}, 50 \leq E_i \leq 60 \text{ MeV} \)). The accompanying \( E2 \) response provides an important contribution to the measured cross sections for \( 0.6 < q_{\text{eff}} < 0.9 \text{ fm}^{-1} \) (\( 0.4 < q < 0.7 \text{ fm}^{-1}, 40 \leq E_i \leq 70 \text{ MeV} \)). The sum of the \( E2 \) and \( M1 \) theoretical cross sections agrees quite well with the available experimental data. The qualitative behaviour, the improvement, and the achieved agreement with experiment are manifested in a similar way in all the excitations studied and exhibit a well pronounced systematic character. This allows us to conclude that the \( E2 \) response of low-lying orbital \( \pm \) excitations to inelastic electron scattering plays an important rôle in heavy deformed nuclei for higher momentum transfer even at backward scattering angles.

The PWBA analysis of the cross sections has shown that the \( E2 \) excitation is dominated by the Coulomb \( C2 \) response at \( \theta = 165^\circ \) and this qualitative feature is preserved for \( \theta \leq 175^\circ \). In contrast, the longitudinal \( C2 \) response is already negligible for the largest experimentally reachable effective backward angle \( \theta = 178.5^\circ \). It is simply damped by the small longitudinal kinematic factor. The interplay between magnetic and electric responses is determined at this angle (and beyond it until \( 180^\circ \)) by the competing transverse magnetic and electric responses. Both of them receive the main contribution from the proton convection current.

The transverse \( M1 \) response is still dominant at \( 178.5^\circ \) for \( q_{\text{eff}} < 1.2 \text{ fm}^{-1} \), corresponding to incident electron energy \( E_i \leq 100 \text{ MeV} \). The two responses are comparable beyond this energy and the \( E2 \) cross section is expected to be even the dominant one for \( E_i > 200 \text{ MeV} \). Therefore, the commonly expected \( M1 \) dominance in backward scattering does not take place at high momentum transfer even when the full backward angle is reached. This is true at least for the orbital \( M1 \) excitations studied in the present work.

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**APPENDIX A: CYLINDRICAL BASIS**

A large number of formulas for multipole transition densities has been derived in the past using non-trivial relationships from the differential tensor calculus, see, e.g., \([45]\) and references therein. They are very useful for microscopic calculations, but almost all of them are formulated in a spherical basis, whose properties are essential for obtaining the final results. The spherical basis is not suitable for deformed nuclei. We follow the approach of our former work \([44]\) and derive the necessary expressions for deformed basis in cylindrical coordinates \( r = (\rho, z, \varphi) \), where \( \rho \) and \( \varphi \) are the azimuth radius and angle, and \( z \) is the symmetry axis of the nucleus.

We are using a deformed, axially symmetric Woods-Saxon potential \([46]\), whose eigenfunctions are expanded in the basis of the eigenfunctions of the deformed, axially symmetric harmonic oscillator
\[ \Psi_j(r) = \Psi(n_\rho, n_z, \Lambda, \Sigma, r) = \psi(n_\rho, |\Lambda|, \rho) \psi(n_z, z) \chi(\Sigma), \quad (A1) \]

where \( j = (n_\rho, n_z, \Lambda, \Sigma) \) denotes a set of four quantum numbers fully specifying the wave function. \( \chi(\Sigma) \) is a two-component spinor with projection \( \Sigma = \pm 1/2 \) of the spin \( S \) on the quantization z-axis. The principal quantum number \( N \), parity \( \pi \), projection \( \Lambda \) of the orbital angular momentum \( L \) and projection \( K \) of the total angular momentum \( J = L + S \) on the nuclear symmetry axis \( z \) are related in the following way:

\[ N = n_z + 2n_\rho + |\Lambda|, \quad K = \Lambda + \Sigma, \quad \pi = (-1)^N. \quad (A2) \]

The remaining functions in (A1) are expressed through the Hermite and associate Laguerre polynomials \( H_n, L_n^\Lambda \):

\[ \psi(n_\rho, |\Lambda|, \rho) = \psi_\Lambda^\Lambda(\rho) = N^{\Lambda|\Lambda|}_n \sqrt{2n! \rho^{2n}} \exp\left(-\frac{n^2}{2}\right) \frac{L_n^\Lambda(|\Lambda|)}{(|\Lambda|)!} \exp\left(-\frac{\eta^2}{2}ight) \frac{(\rho^2)^n}{n!}, \]

\[ \psi(n_z, z) = \psi(z) = N_{n_z} \sqrt{R_z} \exp\left(-\frac{\xi^2}{2}\right) H_{n_z}(\xi), \]

\[ \psi(\Lambda, \varphi) = \frac{1}{\sqrt{2\pi}} \exp(i\Lambda\varphi), \quad (A3) \]

where shorter notations are introduced for the wave functions with \( j \) denoting either \( n_\rho \) or \( n_z \), to remind that both of them belong to the same set of four quantum numbers specifying the eigenfunction (A1). The norms \( N \) and the coordinate scaling factors \( R \) are given by

\[ N_n^{\Lambda|\Lambda|} = \sqrt{n!/(n_\rho + |\Lambda|)!}, \quad N_{n_z} = (\sqrt{\pi} 2^{n_z} n_z!)^{1/2}, \]

\[ R_\perp = \sqrt{\rho^2 R_\perp^2}, \quad \xi = z R_z, \]

where \( M \) is the nucleon mass and \( \omega_\perp, \omega_z \) are the two frequencies of the axially symmetric harmonic oscillator. The time-reversed basis wave functions (A1) have the form

\[ \hat{\Psi}_j(r) = \hat{T}\Psi_j(r) = (-1)^{1/2-\Sigma}\Psi(n_\rho, n_z, -\Lambda, -\Sigma, r), \quad (A5) \]

Let us introduce short notations \( D \) for the derivatives of (A3) with respect to \( \rho \) and \( z \):

\[ \frac{\partial}{\partial \rho} \psi(n_\rho, |\Lambda|, \rho) = D_\rho^\Lambda(\rho) = \left|\frac{\Lambda}{\rho}\right| \psi(n_\rho, |\Lambda|, \rho), \]

\[ D_\rho^\Lambda(\rho) = -R_\perp \sqrt{\rho^2 \psi(n_\rho - 1, |\Lambda| + 1, \rho)} + \sqrt{\rho^2 + 1} \psi(n_\rho + 1, |\Lambda| + 1, \rho) \]

\[ D_z(z) = \frac{\partial}{\partial z} \psi(n_z, z) = \frac{R_z}{\sqrt{2}} \left[ \sqrt{n_z} \psi(n_z - 1, z) - \sqrt{n_z + 1} \psi(n_z + 1, z) \right], \]

\[ \frac{\partial}{\partial \varphi} \psi(\Lambda, \varphi) = i\Lambda \psi(\Lambda, \varphi). \quad (A6) \]

The eigenfunctions \( \Phi_i \) of the axially symmetric Woods-Saxon potential [31] are linear combinations of (A1) with definite projection \( K \) and parity \( \pi \). Different functions are labelled with \( i \), because there are many orthogonal functions with the same \( K^\pi \). For \( K = K_i \geq 1/2 \):

\[ \Phi_i(K^\pi, r) = \sum_j C_{ij}^\pi \Phi_j(n_\rho, n_z, \Lambda, \Sigma = \frac{1}{2}, K, r) \]

\[ + \sum_{j'} C_{i'j'}^\pi \Phi_j(n'_\rho, n'_z, \Lambda' = 1 + \Lambda, \Sigma' = -\frac{1}{2}, K, r) \quad (A7) \]

where \( C_{ij}, j = (n_\rho, n_z) \), are expansion coefficients found after numerical diagonalization of the hamiltonian in some truncated basis with \( N \leq N_{\text{max}} \). The expansion (A7) is separated explicitly in two terms with constant \( (\Lambda, \Sigma) \), because this property will be used below in the analytical derivation of expressions for transition densities. The time-reversed state corresponding to (A7) is easily obtained from (A3):

\[ \check{\Phi}_i(K^\pi, r) = \sum_j C_{ij}^\pi \check{\Phi}_j(n_\rho, n_z, -\Lambda, \Sigma = -\frac{1}{2}, -K, r) \]

\[ - \sum_{j'} C_{i'j'}^\pi \check{\Phi}_j(n'_\rho, n'_z, -\Lambda - 1, \Sigma' = -\frac{1}{2}, -K, r). \quad (A8) \]

We describe intrinsic excitations with \( \Delta K = |K_f - K_i| = 1 \). Thus, time-reversed states appear only in matrix elements between initial and final states with \( K_i = -1/2, K_f = 1/2 \) and \( \Lambda = 0 \). It is explained in Sect. D how they can be obtained from the expressions (A7), (A8) by some phase changes.

**APPENDIX B: SYMMETRIES OF OPERATORS**

The charge and current density operators (B3) have the following properties under time-reversal \( T \) and hermitian conjugation:

\[ \check{\rho}(r) = \check{\rho}(r), \quad \check{T}\check{\rho}(r)\check{T}^{-1} = \check{\rho}(r), \]

\[ \check{J}(r) = \check{J}(r), \quad \check{T}\check{J}(r)\check{T}^{-1} = (1)^\mu \check{J}_{-\mu}(r), \]

\[ \check{J}_0 = \check{J}_z, \quad \check{J}_\mu = -\frac{\mu}{\sqrt{2}} \left( \check{J}_x + \mu \check{J}_y \right), \quad \mu = \pm 1, \quad (B1) \]

where \( \check{J}_{-\mu}, \mu = 0, \pm 1 \) are the cyclic (tensorial) components of the vector. It follows from (B1) that the multipole density operators (B3) have the properties

\[ \check{\rho}_{LM}(r) = (1)^M \check{\rho}_{L,-M}(r), \]

\[ \check{T}\check{\rho}_{LM}(r)\check{T}^{-1} = (1)^M \check{\rho}_{L,-M}(r), \]

\[ \check{J}_{LL'M}(r) = (1)^{L+L'+M+1} \check{J}_{LL',-M}(r), \]

\[ \check{T}\check{J}_{LL'M}(r)\check{T}^{-1} = (1)^{L+L'+M} \check{J}_{LL',-M}(r). \quad (B2) \]
The \( m \)-symmetrized operators \([\text{14}]\) change only a phase under hermitian and time conjugation. Let us denote with \( c^H \), \( c^T \) the corresponding phases of such an operator \( \hat{B}(m) \):

\[
\hat{B}^\dagger(m) = c^H \hat{B}(m), \quad \hat{T} \hat{B}(m) \hat{T}^{-1} = c^T \hat{B}(m). \tag{B3}
\]

It follows from \([\text{12}]\) and corresponding relationships for \( \hat{Q}(m) \), \( \hat{J}(m) \) that the \( m \)-symmetrized operators \([\text{14}]\) have the properties listed in Table 1.

The spin and orbital angular momentum operators are symmetrized in the same way as \( \hat{J}(m) \) in \([\text{14}]\). It is seen from Table 1 that all transition operators have real matrix elements in our cylindrical basis \([\text{14}], [\text{17}], [\text{18}]\), apart from the current density operators \( \hat{J}_{LL'}(m, r) \), whose matrix elements are imaginary. Operators with \( m = +1(-1) \) are hermitian (antihermitian), apart from the angular momentum and spin operators in the last column of Table 1, which have the opposite property.

All the operators acquire a phase of the type \( c^C c^T = \pm m \). This phase can always be chosen to be \( +m \) by redefining \( \hat{B}(m) \rightarrow \hat{B}(-m) \), if necessary. The convention

\[
c^C c^T = +m \tag{B4}
\]

has already been taken in the definitions \([\text{14}]\) into account, as seen from Table 1. It can be shown that, after adhering to \([\text{B4}]\) and neglecting in QRPA the scattering terms \( \alpha \alpha' \), \( \alpha' \alpha' \), any operator with the properties \([\text{B3}]\) has the following quasiboson representation:

\[
\hat{B}_\sigma(m) = \sum_{i,f > 0} \left\{ b_\sigma(f_i, m)
\times \left[ A^\dagger(if,m) + \sigma m A(if,m) \right]
+ b_\sigma(\tilde{f}_i, m)[\tilde{A}^\dagger(if,m) + \sigma m \tilde{A}(if,m)] \right\}, \tag{B5}
\]

\[
b_\sigma(f_i, m) = \frac{1}{\sqrt{2}} \langle u_f v_i + \sigma u_i v_f | (f | \hat{B}_\sigma(m) | i) \rangle,
\]

\[
b_\sigma(\tilde{f}_i, m) = \frac{1}{\sqrt{2}} \langle u_f v_i + \sigma u_i v_f | (f | \hat{B}_\sigma(m) | i) \rangle, \tag{B6}
\]

\[
\sigma = c^{\text{TH}} = c^T c^H = \pm 1, \tag{B7}
\]

where \( u, v \) are the BCS occupation probabilities and the indices \( (i, f) \) denote the Woods-Saxon eigenfunctions \([\text{17}], [\text{18}]\), used to calculate the single-particle matrix elements in \([\text{B3}]\). The summation runs only over states with \( K > 0 \) because time-reversed states (with \( K < 0 \)) are treated explicitly in \([\text{B3}]\).

The q.p. basis is provided by the \( m \)-symmetrized 2q.p. creation and annihilation operators \([\text{14}]\), appearing in \([\text{B5}]\),

\[
A^\dagger(if,m) = \frac{1}{\sqrt{2}} (\alpha_i^1 \alpha_f^1 - m \alpha_i^1 \alpha_f^1),
\]

\[
\tilde{A}^\dagger(if,m) = \frac{1}{\sqrt{2}} (\alpha_i^1 \alpha_f^1 + m \alpha_i^1 \alpha_f^1). \tag{B8}
\]

Basis operators \([\text{B8}]\) with different \( m \)-values commute and determine the same property of the quasiboson operators \([\text{B7}]\),

\[
[ A(if,m), \ A^\dagger(if,m') ] = 0,

[ \hat{B}_\sigma(m), \ \hat{B}_\sigma'(m) ] = 0,

[ \hat{B}_\sigma(m), \ \hat{B}_\sigma(m') ] = 0, \quad \text{for } m' \neq m,

[ \hat{B}_\sigma(m), \ \hat{B}_\sigma'(m') ] \neq 0, \quad \text{for } \sigma' \neq \sigma. \tag{B9}
\]

Creation and annihilation operators with the same \( \sigma, m \) commute as well, because the hermitian conjugation produces only a phase \([\text{B3}]\). This property is a direct consequence from the fact that the QRPA vacuum \( \Gamma_\mu(m) \) \( \text{[13]} \) is not a vacuum for the quasiboson operators \([\text{B3}]\), i.e., \( \hat{B}_\sigma(m) \) \( \neq 0 \). Only operators with the same \( m \) and different \( \sigma \) (longitudinal and transverse) have a nonvanishing commutator, reflected in the last relation \([\text{B3}]\).

It is seen from Table 1 that the quadrupole and charge density operators are characterized by \( \sigma = 1 \) (longitudinal type), while the spin, angular momentum, and current density operators, corresponding to axial vectors in the \( (x, y) \)-plane, have \( \sigma = -1 \) (transverse type). This fundamental property cannot be changed by redefining the \( m \)-symmetrization in \([\text{14}]\). The q.p. matrix elements of \( \hat{Q}(m) \) and \( \hat{J}(m) \) are given in \([\text{33}]\). Note that we are using here a different definition of \( \hat{J}(m) \) \( \text{[37],[39]} \). The q.p. matrix elements \([\text{B6}]\) of the density operators \([\text{14}]\) have the form:

\[
\rho_2(f_i, m, r) = \frac{1}{\sqrt{2}} U(f, i)(f | \hat{\rho}_2(m, r) | i),
\]

\[
\rho_2(\tilde{f}_i, m, r) = \frac{1}{\sqrt{2}} U(f, i)(\tilde{f} | \hat{\rho}_2(m, r) | i),
\]

\[
U(f, i) = u_f v_i + u_i v_f , \quad V(f, i) = u_f v_i - u_i v_f ,
\]

\[
\hat{J}_{LL'}(f_i, m, r) = -\frac{1}{\sqrt{2}} V(f, i)(f | \hat{\rho}_2(m, r) | i),
\]

\[
\hat{J}_{LL'}(\tilde{f}_i, m, r) = \frac{1}{\sqrt{2}} V(f, i)(\tilde{f} | \hat{\rho}_2(m, r) | i). \tag{B10}
\]

The q.p. matrix elements \([\text{B10}]\) have the same symmetries as those of the remaining operators \([\text{14}]\). Thus, one can write in general for the q.p. matrix elements of any operator \( \hat{B}_\sigma(m) \) \( \text{[14]} \), among those listed in Table 1,

\[
B(f_i, m) = mB(if,m) = mB(\tilde{f}_i, m) = B(\tilde{f}_i, m) = mB(if,m)
\]

\[
= -mB(f_i, m) = -B(\tilde{f}_i, m) \tag{B11}
\]

where the phase \( \sigma \) and the radial dependence of density operators are skipped for simplicity. The definitions of the \( m \)-symmetrized operators \([\text{14}]\) have been specially chosen to ensure the above uniform properties of their matrix elements, valid for both longitudinal (\( \sigma = 1 \)) and transverse (\( \sigma = -1 \)) operators.
APPENDIX C: MULTIPOLe TRANSITIONS

The QRPA phonon creation operators \( \hat{B}_\sigma(m) \) are expanded over the q.p. basis \( \{ \nu \} \),
\[
\Gamma^\nu_\nu(m) = \frac{1}{2} \sum_{i,f>0} \left[ \psi_\nu(f,i,m) A^\dagger(i,f,m) - \phi_\nu(f,i,m) A(i,f,m) \right] + \lambda_\nu(f,i,m) \tilde{A}(i,f,m),
\]
where \( \psi, \lambda \) are forward- and \( \phi, \mu \) backward-going RPA amplitudes, found numerically for each excited state \( \nu \) by solving the RPA equations of motion. The \( m \)-symmetrized wave function is obtained in the laboratory frame after projecting the intrinsic excitation \( \Gamma^\nu_\nu(m) \) on a good angular momentum \( I^\nu \). One obtains with this wave function the reduced matrix element of any transition operator \( \hat{T}^\dagger_L \) (irreducible tensor of rang \( L \)) between the nuclear ground state and the considered \( \nu \)-th RPA excitation with a total angular momentum \( I \), its projection \( K \) on the nuclear symmetry axis, and parity \( \pi \):
\[
\langle I^\pi K = L^+ 1, \nu \parallel \hat{T}^\dagger_L \parallel 0, g.s. \rangle
= \frac{1}{2} \sum_{m=\pm 1} \left[ \langle |\Gamma^\nu_\nu(m)\hat{T}^\dagger_L| \rangle + \langle -1 \rangle^{L+1} m \langle |\Gamma^\nu_\nu(m)\hat{T}^\dagger_L| \rangle \right].
\]

The multipolarity \( L \) of \( \hat{T}^\dagger_L \) in \((C2)\) specifies the corresponding \( m \)-symmetrized operators from \((E)\), i.e., \( L = 1 \) refers to \( \hat{J}_11(m,r), \hat{J}(m), \hat{L}(m) \), and \( \hat{S}(m) \), while \( L = 2 \) means \( \hat{J}_2(m,r), \hat{J}_{2L'}(m,r) \), or \( \hat{Q}(m) \). The necessary cyclic components \( \hat{T}^\dagger_{L \pm 1} \), appearing in \((C2)\), are easily obtained from \((E)\), where all the operators are taken in the quasiboson representation \((B)\):
\[
\hat{T}^\dagger_{LM} = \hat{B}_\sigma(m = -\sigma) + M \hat{B}_\sigma(m = \sigma), \quad \hat{B} \neq \hat{J}, \quad \hat{J}^\dagger_{LM} = \hat{T}^\dagger_{LM}/\sqrt{2}, \quad \text{for} \quad \hat{B}(m) = J(m). \tag{C3}
\]

The matrix elements in \((C2)\), calculated with the help of \((C3)\), have the form:
\[
\langle \Gamma^\nu_\nu(m) \parallel \hat{B}_\sigma(m,r) \rangle = \langle \Gamma^\nu_\nu(m) \parallel \hat{B}_\sigma(m,r) \rangle
= \sum_{i,f>0} \left[ F^\nu_\sigma(f,i,m) b_\sigma(f,i,m,r) + F^\dagger_\sigma(f,i,m) b_\sigma(f,i,m,r) \right],
\]
where \( b_\sigma(f,i,m,r) \) are the q.p. matrix elements of \( \hat{B}_\sigma(m,r) \) \((B)\). They are given by \((B1)\) in the case of transition density operators. The coefficients \( F^\nu_\sigma(f,i,m) \) are linear combinations of the RPA amplitudes \((C1)\):
\[
F^\nu_\sigma(f,i,m) = \psi_\sigma(f,i,m) + \sigma m \phi_\sigma(f,i,m), \quad F^\dagger_\nu_\sigma(f,i,m) = \lambda_\sigma(f,i,m) + \sigma m \mu_\sigma(f,i,m). \tag{C5}
\]

A different notation, \( F_\nu(f,i,m) \equiv F^\nu_\sigma(f,i,m), G_\nu(f,i,m) \equiv F^\dagger_\nu_\sigma(f,i,m) \), was used in our previous works, e.g. \((10,11,12)\). It can be shown that the amplitudes \((C3)\) have exactly the same properties as the q.p. matrix elements \((B1)\). The definitions \((14)\) ensure that these properties are the same for the q.p. matrix elements of all the operators \((14)\). One can show in this way that the intrinsic transition matrix elements \((C2)\) do not depend on the signature \( m \), i.e.,
\[
\langle \Gamma_\nu(m = +1) \parallel \hat{B}_\sigma(m = +1,r) \rangle \quad = \langle \Gamma_\nu(m = -1) \parallel \hat{B}_\sigma(m = -1,r) \rangle. \tag{C6}
\]

Using \((C3)\) and \((C6)\), one obtains for the reduced transition matrix elements of the type \((C2)\):
\[
\langle I^\pi K = L^+ 1, \nu \parallel \hat{T}^\dagger_L \parallel 0, g.s. \rangle = 2 \langle \Gamma_\nu(m) \parallel \hat{B}_\sigma(m,r) \rangle, \tag{C7}
\]
where the intrinsic matrix element in the r.h.s of \((C7)\), given by \((C4)\), \((C5)\), can be calculated either for \( m = 1 \) or \( m = -1 \). It is seen from \((C7)\) that in our \( m \)-symmetrized formalism the QRPA ground state is not only a vacuum for the QRPA annihilation operators \((2)\), \( \Gamma_\nu(m) \rangle = 0 \), but also for the transition operators \( \hat{T}^\nu \) in the laboratory frame. This property is not trivial, because it is not fulfilled for the corresponding transition operators \( B_\sigma(m,r), B^\dagger_\sigma(m,r) \) \((14)\) in the intrinsic frame: both of them have a non-vanishing action on the QRPA vacuum, as it can be seen from \((C4)\) having in view the hermitian properties \((B)\), listed in Table \(I\).

Half of the terms in microscopic sums of the type \((C4)\) are redundant because of the relationships \((B1)\). Thus, we make a transition from \( m \)-symmetrized \((B)\) to the usual step-up operators \( \hat{J}_+ \), \( \hat{L}_+ \), \( \hat{S}_+ \) \((4)\) or tensors with \( M = +1 \) for the remaining operators \((1)\). Let us define with them the following q.p. matrix elements:
\[
\rho_{21}(f,i,r) = U(f,i)\langle \rho_{21}(r)|\hat{J}_+(r)|\rangle, \quad \rho_{21}(f,i,r) = U(f,i)\langle \hat{J}_+(r)|\rho_{21}(r)|\rangle, \quad \rho_{21}(f,i,r) = U(f,i)\langle \hat{J}_+(r)|\hat{J}_+(r)|\rangle.
\]

The corresponding matrix elements of \( \hat{Q}_{21} \) are defined in analogy with \( \rho_{21} \) above, and those of \( \hat{J}_+ \) and \( \hat{S}_+ \) – in analogy with \( \hat{J}_+ \). The symmetries of such step-up and step-down matrix elements can be derived from \((B1)\) using \((4)\). These symmetries allow us to manipulate the microscopic sums in \((C4)\) in order to get rid of the redundant terms. Upon insertion in \((C7)\), one arrives to
the the following final expression for the r.m.e. (C7) in
terms of step-up matrix elements:

\[
\langle L''1, \nu \parallel \hat{T}_I \parallel 0, \text{g.s.} \rangle = \frac{1}{2} \sum_{0<\nu<i} \left[ F_{\nu}^{\nu}(fi)b_{\nu}(fi, r) + F_{\nu}^{\nu}(\tilde{f}i)b_{\nu}(\tilde{f}i, r) \right],
\]

where \(b_{\nu}(fi, r), b_{\nu}(\tilde{f}i, r)\) are the step-up matrix elements (C8) of the multipole transition density operators. The notation \(0<i<f\) indicates that the final state \(f\) has a larger \(K\)-value than the initial state \(i\). The amplitudes of the QRPA phonon wave function are contained in the new step-up factors \(F_{\nu}^{\nu}\) (C9), which are related with the old ones (C5) through:

\[
F_{\nu}^{\nu}(fi) = \sqrt{2} \left[ F_{\nu}^{\nu}(fi, -1) - F_{\nu}^{\nu}(fi, +1) \right],
\]

\[
F_{\nu}^{\nu}(\tilde{f}i) = -\sqrt{2} \left[ F_{\nu}^{\nu}(\tilde{f}i, -1) + F_{\nu}^{\nu}(\tilde{f}i, +1) \right],
\]

i.e., they are obtained from the old factors (C5) by simply replacing q.p. matrix elements of \(m\)-symmetrized operators with those (C8) of the corresponding step-up operators.

APPENDIX D: MATRIX ELEMENTS OF TRANSITION DENSITIES

We are going to derive expressions for the matrix elements (C8) of the transition density operators \(\hat{\rho}_{22}(r)\) and \(\hat{J}_{LL'}(r)\) (13). They are necessary for the calculation of the \(E2\) and \(M1\) transition densities (15a), (15b). The vector scalar product in (13) is expressed by the covariant components \(T_\mu\) of the involved vectors (14):

\[
[Y_{LL'M}(\Omega)]_\mu = (-1)^{\mu}(L', M + \mu, 1, -\mu|LM) \times Y_{LL'M}(\Omega),
\]

\[
Y_{LL'M}(\Omega) \cdot T_1 = \sum_\mu \langle L', M + \mu, 1, -\mu|LM \rangle \times Y_{LL'M}(\Omega)T_{1,\mu},
\]

\[
[Y_{LL'M}(\Omega) \times \nabla \times T_1]_1 = -i\sqrt{2} \sum_{\mu \nu \lambda} \langle L, M + \mu, 1, -\mu|LM \rangle \langle 1, \nu, 1, \lambda|1\mu \rangle \times Y_{LL'M}(\Omega)\nabla_{1,\lambda}T_{1,\lambda},
\]

where the round brackets denote the Clebsch-Gordan coefficients.

The action of \(\phi\)-dependent operators on the basis functions \(\psi(\Lambda, \phi)\) (A1) and spin components on the spinors \(\chi(\Sigma)\) is calculated analytically. Thus, the integration over the azimuth angle \(\phi\) and the spin variables, performed analytically, results in selection rules for the quantum numbers \(\Lambda, \Sigma\) of the involved Woods-Saxon initial and final wave functions \((f, i)\) (A4). Therefore, only the first two basis functions, i.e., the Hermite and Laguerre polynomials from (A1), remain in the final expressions, which have to be integrated over the polar angle \(\theta\). Due to the different selection rules, we treat explicitly the two different terms with \(\Lambda, \Lambda'\) in the Woods-Saxon eigenfunctions (A7), (A8). Let us introduce a shorter notation \(A^{\nu}_{ij}\) for the product of these two functions, belonging to the initial and final Woods-Saxon states (A7), involved in the considered matrix element \((f, i)\). The derivatives of such an expression with respect to \(\rho\) and \(z\) can also be written afterwards in a compact form:

\[
A^{\nu}_{ij}(\Lambda_f, \Lambda_i, \rho, z) = \sum_{kj} C^j_k(\Lambda_f)C^i_j(\Lambda_i) \times \psi^{\Lambda_i}_{\nu}(\rho)\psi^{\Lambda_f}_{\nu}(\rho)\psi(z)\psi_j(z),
\]

\[
B^{ji}(\Lambda_f, \Lambda_i, \rho, z) = \frac{\partial}{\partial \rho_i} A^{ji} - |\Lambda_i| A^{ji},
\]

\[
C^{ji}(\Lambda_f, \Lambda_i, \rho, z) = \frac{\partial}{\partial z_j} A^{ji}(\Lambda_f, \Lambda_i, \rho, z),
\]

\[
R^{ji}(\Lambda_f, \Lambda_i, \rho, z) = \frac{\partial}{\partial \rho} A^{ji} = \sum_{kj} C^j_k(\Lambda_f)C^i_j(\Lambda_i) \times \left[ \psi^{\Lambda_i}_{\nu}(\rho)D^\Lambda_{ji}(\rho) \psi^{\Lambda_f}_{\nu}(\rho) \psi(z)\psi_j(z) + \frac{|\Lambda_f| + |\Lambda_i|}{\rho} A^{ji}(\Lambda_f, \Lambda_i, \rho, z) \right],
\]

\[
Z^{ji}(\Lambda_f, \Lambda_i, \rho, z) = \frac{\partial}{\partial z} A^{ji}(\Lambda_f, \Lambda_i, \rho, z) = \sum_{kj} C^j_k(\Lambda_f)C^i_j(\Lambda_i) \psi^{\Lambda_i}_{\nu}(\rho) \psi^{\Lambda_f}_{\nu}(\rho) \times \left[ \psi(z)D_j(z) + D_j(z)\psi_j(z) \right].
\]

The factors \(C^j_k, C^i_j\) in the above expressions are the expansion coefficients (A7) of the initial and final Woods-Saxon states, respectively. The symbolic notations \(\partial/\partial \rho_i\) and \(\partial/\partial z_j\), appearing in (D2b), indicate that only the basis function belonging to the initial state is subjected to differentiation. It is seen from (D2b), (D2c) that all derivatives are expressed through the elementary derivatives \(D^\Lambda_{ji}(\rho)\) and \(D_j(z)\) (A6) with respect to \(\rho\) and \(z\).

Using the above short notations for derivatives and (D1) for the vector scalar product, one can write the q.p. matrix elements (C8) of the multipole transition densities (15a), (15b) in a more compact form. The matrix...
elements of the charge transition density (15a) are given by:

\[
\rho_{21}(f, i, r) = e\varepsilon U(f, i) \int Y_{21}(\Omega)\Phi^\dagger_f(r)\Phi_i(r)d\Omega
\]

\[
= -e\varepsilon U(f, i) \sqrt{\frac{15}{8\pi}} \int \frac{\rho_z}{r^2} d\Omega
\times \left[ A^{f_i}(\Lambda + 1, \Lambda, \rho, z) + A^{f_i}(\Lambda + 2, \Lambda + 1, \rho, z) \right] d\cos\theta,
\]

\[
\rho_{21}(f, i, r) = -e\varepsilon U(f, i) \sqrt{\frac{15}{8\pi}} \int \frac{\rho_z}{r^2} d\Omega
\times \left[ A^{f_i}(1, 0, \rho, z) - A^{f_i}(0, -1, \rho, z) \right] d\cos\theta,
\]

where \( \Phi(r) \) are the Woods-Saxon wave functions (A7). The integration over the polar angle in (D3) and similar expressions below is limited to the interval \( 0 \leq \theta \leq \pi/2 \) (or \( 0 \leq \cos\theta \leq 1 \)) due to reflection symmetry of the basis with respect to the plane \( z = 0 \).

The matrix elements (C8) of the transverse (current) multipole density operators \( \hat{J}^{LL'}_1(r) \) (13) can be decomposed into convection and spin parts, originating from the convection and magnetization current density operators \( j^C(r) \) (9) and \( j^S(r) \) (10),

\[
\hat{J}^{LL'}_1(f, i, r) = \hat{J}^{LL'}_1^C(f, i, r) + \hat{J}^{LL'}_1^S(f, i, r),
\]

and the same for \( \hat{J}_{LL'}^1(f, i, r) \). All of them are purely imaginary in our basis, see Table II. They are necessary for the calculation of the transverse (current) transition densities (15b). The matrix elements (D4), obtained from (D1) and expressed in nuclear magnetons \( \mu_N = \frac{e\hbar}{2Mc} \approx 0.1052 \text{ e fm} \), have the form:
\[ \mathcal{J}^{C}_{L,L'}(f_i, r) = -2i\varepsilon \mu_N V(f, i) \sum_{\mu} (L', 1 + \mu, 1, -\mu|L1) \int Y_{L',1+\mu}(\Omega) \Phi^*_j(r) \left[ \nabla_1 \Phi_j(r) \right] d\Omega, \tag{D5} \]

\[ \mathcal{J}^{S}_{L,L'}(f_i, r) = -i\mu_N g^S V(f, i) \sqrt{2} \sum_{\mu, \lambda} (L', 1 + \mu, 1, -\mu|L1)(1, \nu, 1, \lambda|1, -\mu) \int Y_{L',1+\mu}(\Omega) \nabla_1 \Phi_j(r) \hat{S}_{1,\lambda} \Phi_j(r) d\Omega. \tag{D6} \]

One obtains from (D5) the convection matrix elements of the transverse \( M \) current with \( LL' = 11 \) and \( E2 \) currents with \( LL' = 21, 23 \):

\[ \mathcal{J}^{C}_{111}(f_i, r) = -i\varepsilon \mu_N V(f, i) \sqrt{\frac{3}{4\pi}} \sum_{\lambda'} \frac{1}{\lambda + \lambda + 1} \int \left\{ \frac{z^2}{r} \left[ B^{fi}(\lambda' + 1, \lambda', \rho, z) + \frac{(|\lambda'| - \lambda)}{\rho} A^{fi}(\lambda' + 1, \lambda', \rho, z) \right] \right. \]
\[ \left. - \frac{\rho}{\rho} C^{fi}(\lambda' + 1, \lambda', \rho, z) \right\} d\cos \theta, \tag{D7a} \]

\[ \mathcal{J}^{C}_{211}(f_i, r) = i\varepsilon \mu_N V(f, i) \sqrt{\frac{3}{4\pi}} \sum_{\lambda'} \frac{1}{\lambda + \lambda + 1} \int \left\{ \frac{z^2}{r} \left[ B^{fi}(\lambda' + 1, \lambda', \rho, z) + \frac{(|\lambda'| - \lambda)}{\rho} A^{fi}(\lambda' + 1, \lambda', \rho, z) \right] \right. \]
\[ \left. + \frac{\rho}{\rho} C^{fi}(\lambda' + 1, \lambda', \rho, z) \right\} d\cos \theta, \tag{D7b} \]

\[ \mathcal{J}^{C}_{231}(f_i, r) = \frac{i\varepsilon \mu_N}{\sqrt{2\pi}} V(f, i) \sum_{\lambda'} \frac{1}{\lambda + \lambda + 1} \int \left\{ \frac{(z^2 - 4\rho^2)}{r^3} \left[ zB^{fi}(\lambda' + 1, \lambda', \rho, z) + \rho C^{fi}(\lambda' + 1, \lambda', \rho, z) \right] \right. \]
\[ \left. - \frac{z\lambda'}{\rho} A^{fi}(\lambda' + 1, \lambda', \rho, z) \right\} d\cos \theta, \tag{D7c} \]

where \( \Lambda = K_i + 1/2 \) in (D7) is determined by the \( K \)-number of the initial state \( K_i \). Let us note that here and below we have always \( K_f = K_i + 1 \). Although \( \Lambda \geq 0 \) in the above two-term sums over \( \lambda' \), the expressions are written in a more general way in order to use them also for the corresponding matrix elements \( \mathcal{J}^{C}_{L,L'}(f_i, r) \), involving a time-reversed initial state. They are obtained from (D7) by putting \( \Lambda = K_i - 1/2 = -1 \) and multiplying the first term (\( \Lambda' = -1 \)) with an overall factor \(-1\), i.e., the two-term sums run over \( \Lambda' = -1, 0 \) in this special case. The same structure was written explicitly in (D3) for \( \rho_{211}(f_i, r) \). The phase \(-1\) originates from the second term of the time-reversed Wood-Saxon wave function (A8).

The spin matrix elements of the transverse \( M \) and \( E2 \) currents, obtained from (D6), have the form:

\[ \mathcal{J}^{S}_{111}(f_i, r) = -i g^S \mu_N V(f, i) \frac{1}{8} \sqrt{\frac{3}{4\pi}} \int \left\{ \frac{z}{r} \left[ R^{fi}(\Lambda + 1, \Lambda, \rho, z) - R^{fi}(\Lambda + 2, \Lambda + 1, \rho, z) \right] \right. \]
\[ \left. + \frac{1}{\rho} \left[ A^{fi}(\Lambda + 1, \Lambda, \rho, z) - A^{fi}(\Lambda + 2, \Lambda + 1, \rho, z) \right] - 2Z^{fi}(\Lambda + 1, \Lambda + 1, \rho, z) \right\} d\cos \theta, \tag{D8a} \]

\[ \mathcal{J}^{S}_{211}(f_i, r) = i g^S \mu_N V(f, i) \frac{1}{8} \sqrt{\frac{3}{4\pi}} \int \left\{ \frac{z}{r} \left[ R^{fi}(\Lambda + 1, \Lambda, \rho, z) - R^{fi}(\Lambda + 2, \Lambda + 1, \rho, z) \right] \right. \]
\[ \left. + \frac{1}{\rho} \left[ A^{fi}(\Lambda + 2, \Lambda + 1, \rho, z) - A^{fi}(\Lambda + 1, \Lambda + 1, \rho, z) + 2Z^{fi}(\Lambda + 1, \Lambda + 1, \rho, z) \right] \right\} d\cos \theta. \tag{D8b} \]
\[ J_{231}^S(f, r) = \frac{ig^S \mu_N}{4\sqrt{2\pi}} V(f, i) \int \left\{ \frac{z}{r} \left[ R^f_i(\Lambda + 1, \Lambda, \rho, z) - R^f_i(\Lambda + 2, \Lambda + 1, \rho, z) \right] \right. \\
+ \left. \frac{(z^2 - 4\rho^2)}{r^3} \left[ \rho \left[ R^f_i(\Lambda + 2, \Lambda, \rho, z) - R^f_i(\Lambda + 1, \Lambda + 1, \rho, z) \right] - 2A^f_i(\Lambda + 2, \Lambda, \rho, z) \right. \\
+ \left. \frac{z}{\rho} \left[ A^f_i(\Lambda + 1, \Lambda, \rho, z) - A^f_i(\Lambda + 2, \Lambda + 1, \rho, z) \right] \right] \right\} d\cos \theta, \] (D8c)

The two terms with \( \Lambda' = \Lambda, \Lambda + 1 \) from (D7) are written explicitly in the expressions (D8) because their symmetry is lost in the spin matrix elements. Nevertheless, one can still use (D8) also for the corresponding matrix elements \( J_{LL'}^S(f, r) \). They are obtained from (D8) by putting \( \Lambda = K_i - 1/2 = -1 \) and multiplying with \(-1\) each functional \( A, R, \) or \( Z \) (D2), which has \(-1\) as a second argument. The radius \( r \) in the r.h.s. of (D7), (D8) is a constant with respect to the integration. Thus, \( \rho \) and \( z \) are not independent integration variables but simple functions of the integration angle \( \theta \), given in (D3).
TABLE I. Comparison with experiment of QRPA excitation energies $E_x$, upwards ($\uparrow$) $M1$ and $E2$ transition probabilities and orbit/spin ratios $R_{o.s.}$ of the $M1$ matrix elements for the strongest orbital $K^\pi = 1^+$ excitation in each rare-earth nucleus with experimentally measured ($e,e'$) cross section of this excitation.

| Nucleus | $E_x$ (MeV) | $B(M1)$ ($\mu_N^2$) | $B(E2)$ ($e^2$fm$^4$) | $R_{o.s.}$ |
|---------|--------------|----------------------|------------------------|------------|
| $^{154}\text{Sm}$ | exp.$^a$ | 3.20 | 0.80(20) | > 0.8 |
|          | th. | 3.16 | 0.87 | 28 | 7.3 |
| $^{154}\text{Gd}$ | exp.$^e$ | 2.94 | 0.85(9) | |
|          | th. | 2.94 | 1.00 | 30 | 6.8 |
| $^{156}\text{Gd}$ | exp.$^f$ | 3.07 | 1.30(20) | 40(6) | > 1.3 |
|          | th. | 2.90 | 1.24 | 42 | 7.8 |
| $^{158}\text{Gd}$ | exp.$^g$ | 3.20 | 0.77(9) | |
|          | th. | 3.48 | 0.89 | 37 | 7.0 |
| $^{168}\text{Er}$ | exp.$^d$ | 3.39 | 0.90(20) | |
|          | th. | 3.35 | 0.88 | 18 | 9.6 |

$^a$ A reference for the experimental $E_x$ and $B(M1)$ values is given in this column.

$^b$ Experimental data are available only for $^{156}\text{Gd}$.

$^c$ Experimental data from $(e,e')$ and $(p,p')$.

$^d$ Ref. [18].

$^e$ Ref. [8].

$^f$ Ref. [8].

$^g$ Ref. [8].

$^h$ Ref. [12].

$^i$ Ref. [33].

$^j$ Ref. [12].

$^k$ Ref. [12].

$^l$ Ref. [12].

TABLE II. Phases $c^{H}$, $c^{T}$, $c^{C}$ of the $m$-symmetrized operators (15) under hermitian and time conjugation, respectively. $c^{C}$ is the phase of the matrix elements in the basis (13), (15, (18).

| Phase | $\hat{Q}(m)$, $\hat{q}_2(m,r)$ | $\hat{J}_{L,L'}(m,r)$ | $\hat{J}(m)$, $\hat{L}(m)$, $\hat{S}(m)$ |
|-------|-----------------|-----------------|-----------------|
| $c^{H}$ | $+m$ | $+m$ | $-m$ |
| $c^{T}$ | $+m$ | $-m$ | $+m$ |
| $c^{C}$ | $+1$ | $-1$ | $+1$ |

FIG. 1. QRPA transition densities (15) for $M1$ ($L = 1$) and $E2$ ($L = 2$) excitations of the $K^\pi = 1^+$ state from Table 1 in $^{154}\text{Gd}$. The total (continuous curve) transverse transition density $J_{1L}^T(r)$ is a sum of the proton convection (long dashed curve) and magnetization (short dashed curve for neutrons and dotted curve for protons) transition densities.

FIG. 2. The same as in Fig. 1 but for $^{168}\text{Er}$.

FIG. 3. DWBA $(e,e')$ cross sections for $M1$ (dashed curves) and $E2$ (dotted curves) excitations, corresponding to the QRPA state with $K^\pi = 1^+$ from Table 1 in $^{154}\text{Sm}$ (top plot) and $^{154}\text{Gd}$ (bottom plot). Scattering angle $\theta = 165^\circ$. For each nucleus the sum of the $M1$ and $E2$ cross sections (continuous curve), plotted versus the effective momentum transfer $q_{ed}$, is compared to experimental data (dots with error bars) from [12] and [28], respectively.

FIG. 4. The same as in Fig. 3 but for $^{156,158}\text{Gd}$ with experimental data from [18,23]. The DWBA cross sections for $^{156}\text{Gd}$ are obtained with the QRPA transition densities from Fig. 1.

FIG. 5. The same as in Fig. 3 but for $^{168}\text{Er}$ with experimental data from [12]. The DWBA cross sections are obtained with the QRPA transition densities from Fig. 2.

FIG. 6. The theoretical $M1$ cross section for $^{156}\text{Gd}$ from the upper plot of Fig. 3 is plotted here as a continuous curve up to a higher momentum transfer $q_{ed}$. Dashed curve: the contribution from the proton convection transition density $J_{1L}^{C}(r)$ (15b), (D3a) alone. These DWBA results are compared also to the total (convection plus spin) $M1$ cross section in PWBA (dot-dashed curve).

FIG. 7. Decomposition of the theoretical PWBA electric cross section (continuous curve: total $E2$) into a sum of its longitudinal $C2$ (dashed curve) and transverse $E2$ (dotted curve) components. They correspond to the $E2$ excitation in $^{156}\text{Gd}$, whose DWBA cross section is shown in the upper plot of Fig. 4. The PWBA cross section is plotted here for scattering angles $\theta = 165^\circ$, $175^\circ$, $178.5^\circ$, and $179.5^\circ$ up to a higher momentum transfer $q_{ed}$.

FIG. 8. The same as in the upper plot of Fig. 3 but for $^{156}\text{Gd}$, but for a scattering angle $\theta = 178.5^\circ$. 

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\[ \frac{d\sigma}{d\Omega} [\text{fm}^2/\text{sr}] \]

- \(156\text{ Gd}\)
- \(158\text{ Gd}\)

- \(M1 + E2\)
- \(M1\)
- \(E2\)
\[ \theta = 178.5^\circ \]

\[ \frac{d\sigma}{d\Omega} \text{ [fm}^2/\text{sr}] \]

\[ 156\text{ Gd} \]

\[ q_{\text{eff}} \text{ [fm}^{-1}] \]