VARIATIONAL CALCULATIONS OF THE $^{12}\text{C}$ NUCLEUS STRUCTURE IN A $3\alpha$ MODEL USING A DEEP POTENTIAL WITH FORBIDDEN STATES

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Abstract

The energy spectrum of the $^{12}\text{C}$ nucleus with $(J^\pi, T) = (0^+, 0)$ and $(2^+, 0)$ is investigated in the framework of the multicluster dynamical model by using a deep $\alpha\alpha$-potential with forbidden states in the S and D waves. A very high sensitivity of the compact ground and first excited $2^+_1$ states energy levels to the description of the two-body forbidden states wave functions has been established. It is shown also that the chosen method of orthogonalizing pseudopotentials yields convergent results for the energies of the excited $(0^+_2, 0)$ and $(0^+_3, 0)$ states of the $^{12}\text{C}$ nucleus with a well developed cluster like structure.

PACS number(s): 21.45.+v; 21.60.Gx; 03.65.Ge

Key words: cluster model, Pauli principle, pseudopotential
1 Introduction

A systematic study of the $^{12}C$ nucleus structure has been successfully performed on the basis of the microscopic three-alpha models [1, 2, 3, 4, 5]. These models reproduce the general features of the low-lying $^{12}C$ spectrum. In the calculations using the resonating group method (RGM) and the orthogonality condition model (OCM) the ground and first excited $2^+\,^1_1$, $2^+_2$, $3^-_1$ and $1^-_1$ resonance states energies of the $^{12}C$ nucleus are overbound by 7-10 MeV, although the energy positions of the $0^+_2$, $2^+_2$, $3^-_1$ and $1^-_1$ resonance states are reproduced fairly well. In the recent work [6], which used the RGM method, the low-lying energy spectrum of this nucleus was described with a good accuracy. This work has shown also that the astrophysical significant $0^+_2$ state of the carbon is a genuine three-alpha resonance in the continuum. This fact is important, since almost all the carbon is produced via capture reaction $\alpha + ^8 Be \rightarrow ^{12} C + \gamma$, which goes through $0^+_2$ resonance of the $^{12}C$ nucleus.

Although the microscopic models are useful tool for the study of the structure of light nuclei, but they can not yield the wave functions in a convenient form for future application in nuclear reactions with these nuclei, such as above capture process. Therefore, it is important to develop macroscopic models. These models should provide main properties of the structure of the nucleus to be investigated.

There are two alternative local potential models, describing the alpha-alpha data. The first potential suggested by Ali and Bodmer (AB) [7] involves a strong repulsive core. The second deep potential model was proposed in [8] by Buck, Friedrich and Wheatley (BFW), which has three forbidden states in the lowest S- and D-waves. Thus, these alternative models differ each from other in describing the Pauli repulsion part of the alpha-alpha interaction. As a result, the local on-shell equivalent potential models give still different wave functions for the $^8 Be$ ground state: while the BFW potential yields a nodal behavior, the AB potential does not describe this microscopically substantiated property.

Macroscopic three-alpha models also are believed to describe well the low-lying states of the $^{12}C$ nucleus due-to large binding energy of $\alpha$ particle. However, in sharp contrast to the microscopic models, the local AB potential gave a very weak binding with $E = -0.60$ [9] for the g.s. energy $E_{exp} = -7.27$ [10] of the $^{12}C$ nucleus as a strong off-shell effect of the repulsive core presented in this potential model. On the other hand, when using the BWF alpha-alpha potential, one has the problem of elimination of forbidden states from relative motion wave function of the system. However, the construction of the many body projector is a very complicated procedure even for the case of the three body system. Therefore, various approaches to the full many-body projector are used.

The approach used in the work [11] is based on the strong truncation of the functional space of relative motion. In spite of the truncation of the relative motion space, the calculated value overbound the ground state binding energy of the $^{12}C$ nucleus by about 5 MeV. The inclusion of full relative motion space yields a stronger overbinding by about 15 MeV [12] with neglecting the Coulomb forces, which is estimated by 5-6 MeV. To overcome the three-alpha overbinding problem the authors of the work [12] proposed the Finite Pauli repulsion model (FPRM) for the $\alpha\alpha$ interaction. However, this model allows admixtures of the forbidden states into the physical relative motion space.

On the other hand, from above mentioned $3\alpha$ macroscopic calculations with orthogonality conditions, one can not see, how accurate the full 3-body projector is approximated by chosen orthogonalization method. Since the antisymmetrization has strong dynamical effects, the $3\alpha$ problem has to be investigated in a model, which allows to examine the convergence of the orthogonalizing procedure. In other words, the dynamical effects of the Pauli principle has to be described properly.
In the present work we investigate the $3\alpha$-problem by using the BFW potential in the framework of the multicluster dynamical model [13] for light nuclei based on a high accuracy variational method which has been used successfully in many structure calculations of various atomic and nuclear systems [14, 15, 16, 17, 18]. For the elimination of forbidden states we use the method of orthogonalizing pseudopotentials (OPP) which allows to work in the full relative motion space. The main feature of our work is the description of the convergence of the orthogonalizing procedure for the ground and lowest excited states of the $^{12}$C nucleus. When using the OPP method one can examine the convergence of the three-body energy as a function of the projecting constant $\lambda$ [16, 19]. We check also the the convergence in respect to the description of the two-body forbidden states fixed by chosen $\alpha\alpha$ potential. We show that the convergence of the orthogonalizing procedure has different character for the compact shell-model like bound states and for the resonance states with a well developed cluster like structure.

In Section 2 we describe the model for the investigation of the $3\alpha$-system structure. The numerical results are presented in Section 3. Discussion and conclusion are given in Section 4 and Section 5, respectively.

## 2 Model

For the description of the lowest states of the $^{12}$C nucleus we use the multicluster $3\alpha$ model with a local $\alpha\alpha$-interaction potential of Buck, Friedrich and Wheatley [8]

$$V(r) = V_0 \exp(-\eta r^2),$$

with $V_0 = -122.6225$ MeV, $\eta = 0.22$ fm$^{-2}$ [11]. This potential describes well the experimental phase shifts of the $\alpha\alpha$-scattering $\delta_L(E)$ with $L = 0, 2, 4$ up to 40 MeV energy. The Coulomb interaction potential in our calculations is taken in the form

$$V_{Coul}(r) = 4e^2 erf(br)/r,$$

where $b=0.75$ fm$^{-1}$, which corresponds to the $\alpha$ particle charge distribution being the Gaussian form with a width of $1/b$. We use a value $\hbar^2/m_\alpha = 10.4469$ MeV fm$^2$ in our calculations.

The potential (1), with including the Coulomb interaction in the form (2) has three nonphysical bound states forbidden by the Pauli principle in each $\alpha\alpha$-subsystem, with the energies $E(0^+_1) = -72.6249149$ MeV, $E(0^+_2) = -25.6174$ MeV, $E(2^+) = -21.9991037$ MeV. The $0^+_1$ state corresponds to the shell configuration $s^8$, while the $0^+_2$ and $2^+$ states correspond to the $s^6p^2$. These forbidden states in our three-body calculations are eliminated by using the method of orthogonalising pseudo-potentials (OPP). This method was developed in the work [20] and successfully employed for the investigations of the structure of nuclei with $A = 6$ and $3N$ system [15, 16, 19].

The variational method on a Gaussian basis used for the calculations of the spectrum of the ground and excited states of the $^{12}$C nucleus has been given in details in Ref.[14]. A high accuracy of the method has been demonstrated in a number of works [14, 15, 16, 17, 18]. A general formalism of the method and analytical expressions of the matrix elements of the overlapping integral, the kinetic energy operator, central, spin-orbital and tensor interaction potentials for the system of three identical fermions with a spin value $1/2$ have been given in Ref.[14]. A corresponding part of these matrix elements can be used for the variational calculations of the $3\alpha$-system. Therefore, we give here only main formulas for the expression of the wave function of the $3\alpha$-system in series of symmetrized Gaussian basis functions.
The three-body Schrödinger equation is solved by using a pseudopotential of the form
\[ \tilde{V}_i(r) = V_i(r) + V_{\text{coul},i}(r) + \sum_f \lambda_f \hat{\Gamma}_i^{(f)}, \] (3)
where \( \lambda_f \) is the projecting constant, \( \hat{\Gamma}_i^{(f)} \) is the projecting operator to the \( f \)-wave forbidden state in the two-body subsystem \((j + k), (i, j, k) = (1, 2, 3)\), and their cyclic permutations.

We note that the method of OPP for the elimination of forbidden states uses the first term of the expansion for the full three-body projector [2]
\[ \hat{P} = \sum_{i=1}^{3} \hat{P}_i - \sum_{i,j=1}^{3} \hat{P}_i \hat{P}_j + \sum_{i,j,k=1}^{3} \hat{P}_i \hat{P}_j \hat{P}_k - \cdots, \] (4)
where
\[ \hat{P}_i = \sum_f \hat{\Gamma}_i^{(f)}, \] (5)
which corresponds exclusively to the two-cluster Pauli forces. We neglect three-cluster (triple) Pauli forces when using this approach. The method of OPP allows us to obtain the solution of the Schrödinger equation at large values of \( \lambda_f \) which is orthogonal to the two-body forbidden states.

Thus, we use next three-body pseudo-hamiltonian in our \( 3\alpha \)-cluster variational calculations:
\[ \tilde{H} = H_0 + \tilde{V}_1 + \tilde{V}_2 + \tilde{V}_3, \] (6)
where \( H_0 \) is the kinetic energy operator of the three \( \alpha \)-particles. The wave function of the \( 3\alpha \)-system is expanded in the series of symmetrized Gaussian functions [14]:
\[ \Psi_{s}^{JM} = \sum_{\gamma_j} c_j^{(\lambda,l)} \varphi_{\gamma_j}^{s}, \] (7)
where \( \varphi_{\gamma_j}^{s} = \varphi_{\gamma_j}(1; 2, 3) + \varphi_{\gamma_j}(2; 3, 1) + \varphi_{\gamma_j}(3; 1, 2), \)
\[ \varphi_{\gamma_j}(k; l, m) = N_j x_k^\lambda y_k^l \exp(-\alpha_{\lambda j} x_k^2 - \beta_{l j} y_k^2) \mathcal{F}_{ML}^{JM}(\vec{x}_k, \vec{y}_k) \] (8)
Here \((k; l, m) = (1; 2, 3), (2; 3, 1), (3; 1, 2), \) \( \gamma = (\lambda, l, J, M) = (\gamma_0, J, M) \); \( \vec{x}_k, \vec{y}_k \) are the normalized Jacobi coordinates in the \( k \)-set:
\[ \bar{x}_k = \sqrt{\frac{\mu}{\hbar}} (\vec{r}_l - \vec{r}_m) \equiv \tau^{-1} \vec{r}_{l,m}; \]
\[ \bar{y}_k = \frac{2\sqrt{\mu}}{\sqrt{3\hbar}} (\frac{\vec{r}_l + \vec{r}_m}{2} - \vec{r}_k) \equiv \tau_{l}^{-1} \vec{r}_k, \] (9)
\( N_j \) is a normalizing multiplier. The nonlinear variational parameters \( \alpha_{\lambda j}, \beta_{l j} \) are chosen as the nodes of the Chebyshev grid:
\[ \alpha_{\lambda j} = \alpha_0 t g(\frac{2j - 1}{2N_{\lambda}}), j = 1, 2, \ldots N_{\lambda}, \]
\[ \beta_{l j} = \beta_0 t g(\frac{2j - 1}{2N_l}, j = 1, 2, \ldots N_l, \] (10)
where \( \alpha_0 \) and \( \beta_0 \) are scale parameters for each \((\lambda l)\) partial component of the complete wave function. When we use the Chebyshev grid the basis frequencies \( \alpha_{\lambda j}, \beta_{l j} \) cover larger and
larger intervals around the scale parameters as the numbers $N_\lambda$ and $N_l$ increase. This allows us to take into account both short-range and long-range correlations of particles. The extraordinary flexibility of the many-particle Gaussian basis makes it possible to describe three- and four-particle configurations that are formed in the ground state and excited states of multicluster systems, and which exhibit an extremely high degree of clustering [14].

The angular part of the Gaussian basis (8) is factorized into the angular component and the internal wave functions $\phi(i)$ of the $\alpha$-particles and has the form:

$$\mathcal{F}_{JM}^{\lambda l}(\vec{x}_k, \vec{y}_k) = \{Y_\lambda(\vec{x}_k) \otimes Y_l(\vec{y}_k)\}J_M\phi(1)\phi(2)\phi(3)$$  \hspace{1cm} (11)

Here the orbital momenta $\lambda$ and $l$ are conjugate to the Jacobi coordinates $\vec{x}_k$ and $\vec{y}_k$, respectively.

We note that the use of the symmetrized Gaussian basis allows us to take into account contributions actually of all partial waves to the binding energy of the system due to overlapping of various three-body channels with the same total orbital momentum $L$ (and total spin $S$ in the case of the $3N$-system). On the other hand, the solution of the Schrödinger equation should be symmetric over the permutation of any two $\alpha$ particles. This requirement is satisfied automatically when using a symmetrized basis for any number of basis functions, whereas in the case of nonsymmetrical basis this requirement is fulfilled approximately and depends on the precision of the expansion.

One of the main advantages of the variational method on a Gaussian basis is the possibility of fully analytical calculation of moments of interaction potentials of some special forms. For example, to calculate the integral

$$G(\alpha, \beta, n) = \int_0^\infty x^n e^{-(\alpha x^2 - \beta x)} dx, \alpha > 0, n \geq 0,$$  \hspace{1cm} (12)

which arises when Yukawa type potentials are used, we suggest more convenient and effective recurrent formulas (compare with Ref.[17]):

$$G(\alpha, \beta, 0) = \frac{\sqrt{\pi}}{2\sqrt{\alpha}} e^{\frac{\beta^2}{4\alpha}} \text{erfc}\left(\frac{\beta}{2\sqrt{\alpha}}\right)$$  \hspace{1cm} (13)

$$G(\alpha, \beta, 1) = \frac{1}{2\alpha} \beta G(\alpha, \beta, 0)$$  \hspace{1cm} (14)

$$G(\alpha, \beta, n + 1) = \frac{n}{2\alpha} G(\alpha, \beta, n - 1) - \frac{\beta}{2\alpha} G(\alpha, \beta, n), n > 0$$  \hspace{1cm} (15)

And for the calculation of moments of the nonpoint Coulomb interaction potential (2)

$$G_{\text{erf}}(\alpha, \beta, n) = \int_0^\infty x^n e^{-(\alpha x^2 - \beta x)} \text{erf}(\beta x) dx, \alpha > 0, n \geq 0,$$  \hspace{1cm} (16)

we obtained also next recurrent formulas:

$$G_{\text{erf}}(\alpha, \beta, 1) = \frac{\beta}{2\alpha \sqrt{\alpha + \beta^2}}$$  \hspace{1cm} (17)

$$G_{\text{erf}}(\alpha, \beta, n) = \frac{n - 1}{2\alpha} G_{\text{erf}}(\alpha, \beta, n - 2) + \frac{\beta}{2\alpha \pi} \frac{\Gamma(n/2)}{(\alpha + \beta^2)^{n/2}}$$  \hspace{1cm} (18)

These formulas allow us to essentially economize computer time when we tabulate the values of $G(\alpha, \beta, n)$ and $G_{\text{erf}}(\alpha, \beta, n)$ for the fixed values of $\alpha$ and $\beta$. We note also that the zero-th moment $G_{\text{erf}}(\alpha, \beta, 0)$ can be estimated numerically. However, in our calculations we need moments with only odd values of $n$. 
3 Numerical results

First we note that a very large symmetrized Gaussian basis is used in our calculations. For the calculations of the energy levels with \( (J^\pi, T) = (0^+, 0) \) we used a basis containing 80 functions in the each of the three-body channels \( (\lambda, l) = \{(0, 0); (2, 2); (4, 4)\} \). We calculated the levels with \( (J^\pi, T) = (2^+, 0) \) on the basis containing 437 Gaussians chosen in main three-body channels \( (\lambda, l) = \{(0, 2); (2, 0); (2, 2); (2, 4); (4, 2); (4, 4)\} \). The results indicate that a further extension of the basis does not have a remarkable influence on the accuracy of the expansion.

The projector on the \( f \)-wave forbidden state in each two-body subsystem has the form:

\[
\hat{\Gamma}_i^{(f)} = \frac{1}{2f+1} \sum_{m_f} \left| \varphi_{f m_f}(\vec{x}_i) \right> < \varphi_{f m_f}(\vec{x}_i') \left| \delta(\vec{y}_i - \vec{y}_i') \right.
\]

with the forbidden state function

\[
\varphi_{f m_f}(\vec{x}_i) = x_i^f \sum_m N_m^{(f)} b_m^{(f)} \exp\left(-\frac{r_i^2}{2r_0^{(f)} m}\right) Y_m^{(f)}(\hat{x}_i).
\]

Here \( r_0 \) is the ”projector radius”, and \( N_m^{(f)} \) is the normalizing multiplier:

\[
N_m^{(f)} = 2^{7/4} \frac{\alpha_m^{(2f+3)/4}}{\pi^{1/4} [2(2f+1)!]^{1/2}}, \quad \alpha_m = \tau^2/(2r_{0m}^2).
\]

In order to check the behaviour of the \( 3\alpha \)-system energy when improving the accuracy of the expansion of the two-body \( \alpha\alpha \) forbidden states wave functions in a series of Gaussian functions we choose several sets of the Gaussian approximations. Corresponding approximate values of the deep forbidden state energies of the \( \alpha\alpha \)-system are shown in the Table 1. In Set 1 the forbidden state wave function of the \( 0^+_1 \)-level of the \( \alpha\alpha \)-system is approximated via \( N=1 \) Gaussian, and wave functions of the forbidden \( 0^+_2 \) - and \( 2^+\)-levels are approximated via \( N=2 \) Gaussians, et al. The values of the forbidden two-body states energies are given in corresponding squares. By comparison of these numbers with the corresponding exact values of the two-body \( 0^+_1, 0^+_2, 2^+\)-forbidden states energies one can conclude about the quality of the approximation for the given number \( N \). We note that the wave function of the \( \alpha\alpha \) forbidden \( 0^+_2\)-state contains a node, therefore it can not be expressed by a single Gaussian.

The spectrum of the energy levels of the \( ^{12}\text{C} \) nucleus with \( (J^\pi, T) = (0^+, 0) \) and \( (2^+, 0) \) for the several variants of the Gaussian expansion for the forbidden state wave functions are shown in Tables 2 and 3. For the sake of convenience we use an identical value of the projecting constant \( \lambda_f \) for all forbidden states. In all three tables the symbol ” \(< P > \)” denotes the total admixture of the forbidden states to the energy of the \( 3\alpha \)-system:

\[
< P > = < \Psi_s | \lambda(\hat{P}_1 + \hat{P}_2 + \hat{P}_3) | \Psi_s > .
\]

In Table 4 we give the experimental values of the lowest energy levels of the \( ^{12}\text{C} \) nucleus and our theoretical estimation comparing with results from the literature. The results of the Ref.[11] were obtained by using the hyperspherical garmornics method and the BFW \( \alpha\alpha \)-potential. Microscopical calculations in the Ref.[6] were performed with an effective MN nucleon-nucleon potential.

From Tables 2 and 3 one can see that we have a reasonable saturation of the energy levels of the \( ^{12}\text{C} \) nucleus when increasing the projecting constant \( \lambda \) to the infinity for a given set of approximation of the two-body forbidden states. However, beginning from the value of
the projecting constant $\lambda \geq 10^4$ MeV the ground and first excited $2_1^+$ states energies of the $^{12}C$ nucleus begin to display a high sensitivity to the description of the two-body forbidden states wave functions. When improving a quality of the approximation of the $\alpha\alpha$-forbidden states in these cases we go to the weakly bound system.

For the other excited states ($0_2^+, 0_3^+, 2_1^+$) of the $^{12}C$ nucleus we have a quite reasonable convergence of the energy with respect to the description of the $\alpha\alpha$-forbidden states. We note also that admixture of the forbidden states in the ground and first excited $2_1^+$ states of the $^{12}C$ nucleus decreases slowly as increasing the projecting constant $\lambda$ to the infinity. In the case of the $3N$ system at $\lambda = 10^5$ MeV the value of the forbidden states admixture in the $^3H$ ground state energy was approximately $10^{-3}$ MeV when using the Moscow $NN$ potential with forbidden states [15, 16].

4 Discussion

First of all we discuss the reason of the strong dependence of the energies of the ground and excited $2_1^+$ states of the $^{12}C$ nucleus on the expansion accuracy of the two-body forbidden states wave functions. We know from the experiment that these states are bound strongly and have consequently a well developed compact shell-model structure [10]. Thus, in the states with $(J^\pi, T) = (0_1^+, 0)$ and $(J^\pi, T) = (2_1^+, 0)$ three $\alpha$-clusters with a large probability overlaps simultaneously each with other. In other words, three-cluster Pauli forces should play an important role in the description of these states due to the requirement of the Pauli principle. However, the method of OPP used here for the elimination of forbidden states in the full three-body system takes into account really two-body Pauli effects.

It is known that the two-body projectors $\hat{P}_i$ and $\hat{P}_j$ do not commute each with other and are not mutually orthogonal [2]. We suggest that the overlapping of these two-body projectors is very strong for the three-alpha system, and hence the energies of the compact ground and excited $2_1^+$ states of the $^{12}C$ nucleus display a high sensitivity to the description of the forbidden states beginning from $\lambda \approx 10^4$ MeV where the next terms in the expansion (11) begin to play a role. And these neglected terms present a mathematical expression of the Pauli triple forces.

We note that the authors of the Ref. [21] also came to the conclusion that three-body Pauli forces should be taken into account when investigating the structure of the $3\alpha$-system. They made this conclusion on the basis of the calculations of the $^{12}C$ nucleus ground state energy in the framework of the Alt-Grassberg-Sandhas equation method by using $\alpha\alpha$-potentials obtained by the RGM and modified by the OCM.

On the other hand, as we noted in the previous section, the energies of the excited levels with $(J^\pi, T) = (0_2^+, 0)$ and $(J^\pi, T) = (0_3^+, 0)$ change slowly from Set 1 ($N \leq 2$) to Set 4 ($N = 7$). Moreover, at $\lambda \approx 10^7 - 10^8$ MeV one can see a good saturation of the energy of these levels with respect to values of the projecting constant $\lambda$. Thus, for the energies of these levels we can write (see Table 2): $E_{\text{theor}}(0_2^+) = 1.551$ MeV and $E_{\text{theor}}(0_3^+) = 4.055$ MeV. It is important to note that the difference of these values $\Delta_{\text{theor}} = 2.50$ MeV is in a good agreement with it’s experimental value $\Delta_{\text{exp}} = 2.65$ MeV. [10]. And this agreement can be seen for all sets 1-4 at large values $\lambda \approx 10^7 - 10^8$ MeV. Thus, the excited levels of the $^{12}C$ nucleus with $(J^\pi, T) = (0_2^+, 0)$ and $(J^\pi, T) = (0_3^+, 0)$ can be described qualitatively in the framework of the method of OPP in contradistinction to the ground and first $2_1^+$ excited states.
5 Conclusions

The energy spectrum of the $^{12}C$ nucleus with $(J^{\pi}, T) = (0^+, 0)$ and $(J^{\pi}, T) = (2^+, 0)$ was calculated in the framework of the multicluster dynamical model using the $\alpha\alpha$-potential of Buck, Friedrich and Wheatley with forbidden states in the $S$ and $D$ waves. For the elimination of forbidden states in the full three-body system the method of orthogonalising pseudo-potentials (OPP) has been used.

It was shown that the energies of the compact $(0^+_1, 0)$ and $(2^+_1, 0)$ states of the $^{12}C$ nucleus with a shell-model like structure display a very high sensitivity to the description of the two-body forbidden states. Therefore, for the energies of these strongly bound states, the chosen model does not give convergent results. We suggest that this situation is due-to neglecting of three-body Pauli forces, which should play an important role in the description of the structure of these strongly bound states.

We obtained also that in our model the resonance states $(0^+_2, 0)$ and $(0^+_3, 0)$ with a well developed cluster-like structure can be described quite well. In contrast to the ground and first excited states, the orthogonalization procedure yields convergent results. However, the energy position of the astrophysically significant $0^+_2$ resonance $E_{exp}(0^+_2) = 0.3796$ MeV is overestimated by about 1.2 MeV.

ACKNOWLEDGEMENTS

Author is thankful to Prof. D.Baye, Prof. A.Mukhamedzhanov, Prof. A.Csoto and Dr.G.Kim for useful discussions, and Dr. G.Ryzhikh and Dr. K.Varga for comparison and discussion of presented results. This work was supported in part by DAAD.

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Table 1: Sets of Gaussian approximations for the forbidden states wave functions of the $\alpha\alpha$ system and corresponding energy values in MeV

|                | $0^+_1$       | $0^+_2$       | $2^+$          |
|----------------|---------------|---------------|----------------|
| exact $E$      | $-72.624915$  | $-25.6174$    | $-21.999104$   |
| $N = 1$: $E$   | $-72.5445$    | $-25.106$     | $-21.676$      |
| $N = 2$: $E$   | $-72.6126$    | $-25.5558$    | $-21.8837$     |
| $N = 3$: $E$   | $-72.6233$    | $-25.6111$    | $-21.9576$     |
| $N = 7$: $E$   | $-72.624905$  | $-25.6173$    | $-21.999098$   |

Table 2: The energy spectrum of the $^{12}C$ nucleus with $(J^{\pi}, T) = (0^+, 0)$ in MeV for the several sets of Gaussian approximations of the two-cluster forbidden states at several values of the projecting constant

| $\lambda$ (MeV) | $10^3$ | $10^4$ | $10^5$ | $10^6$ | $10^7$ | $10^8$ |
|-----------------|--------|--------|--------|--------|--------|--------|
| $N \leq 2$      | $E_1$  | -210.80| -45.194| -18.999| -12.85 | -12.56 |
|                 | $E_2$  | -150.72| -20.87  | 0.274  | 1.210  | 1.39   |
|                 | $E_3$  | -109.37| -15.88  | 1.306  | 2.98   | 3.446  |
| $< P >$         |        | 29.83  | 30.36  | 1.133  | 1.73   | 3.00   |
| $N = 3$         | $E_1$  | -210.71| -43.298| -16.492| -6.366 | -3.35  |
|                 | $E_2$  | -150.05| -17.18  | 0.021  | 1.238  | 1.331  |
|                 | $E_3$  | -109.53| -15.65  | 1.380  | 2.553  | 3.361  |
| $< P >$         |        | 29.93  | 30.36  | 1.133  | 1.73   | 3.00   |
| $N = 4$         | $E_1$  | -210.72| -44.149| -20.15  | -12.85 | -12.56 |
|                 | $E_2$  | -150.28| -17.42  | 0.021  | 1.238  | 1.331  |
|                 | $E_3$  | -109.37| -15.43  | 1.380  | 2.553  | 3.361  |
| $< P >$         |        | 29.92  | 30.36  | 1.133  | 1.73   | 3.00   |
| $N = 7$         | $E_1$  | -210.69| -44.207| -20.15  | -12.85 | -12.56 |
|                 | $E_2$  | -150.32| -17.58  | 0.021  | 1.238  | 1.331  |
|                 | $E_3$  | -109.32| -15.31  | 1.380  | 2.553  | 3.361  |
| $< P >$         |        | 29.95  | 30.36  | 1.133  | 1.73   | 3.00   |

Table 3: The same for the levels with $(J^{\pi}, T) = (2^+, 0)$

| $\lambda$ (MeV) | $10^3$ | $10^4$ | $10^5$ | $10^6$ | $10^7$ | $10^8$ |
|-----------------|--------|--------|--------|--------|--------|--------|
| $N \leq 2$      | $E_1$  | -17.19 | -16.15 | -14.55 | -12.103| -11.176| -11.04 |
|                 | $E_2$  | 0.944  | 0.95   | 0.952  | 0.965  | 1.022  | 1.079  |
| $< P >$         |        | 0.923  | 0.95   | 1.024  | 0.807  | 0.136  | 2.8E-2 |
| $N = 3$         | $E_1$  | -17.213| -15.665| -11.142| -5.953 | -3.950 | -3.80   |
|                 | $E_2$  | 1.029  | 1.032  | 1.034  | 1.047  | 1.141  | 1.19    |
| $< P >$         |        | 1.024  | 0.918  | 2.842  | 1.495  | 5.8E-2 | 8.5E-3  |
| $N = 4$         | $E_1$  | -17.433| -15.725| -8.578 | 1.042  | 1.032  | 1.082  |
|                 | $E_2$  | 1.029  | 1.032  | 1.034  | 1.694  | 2.656  | 3.038  |
| $< P >$         |        | 1.075  | 1.095  | 5.85   | 9.1E-3 | 2.9E-2 | 2.8E-2  |
| $N = 7$         | $E_1$  | -17.361| -15.649| -8.243 | 1.042  | 1.086  | 1.162  |
|                 | $E_2$  | 1.030  | 1.032  | 1.034  | 1.475  | 2.524  | 2.643  |
| $< P >$         |        | 1.062  | 1.127  | 5.974  | 1.2E-2 | 3.1E-2 | 2.7E-2  |
Table 4: Experimental results and theoretical estimations of energies of the ground and lowest excited states of the $^{12}C$ nucleus in MeV

| State | Exp. [10] | BFW $\alpha\alpha$-potential [11] | MN effective $NN$-potential [6] | BFW $\alpha\alpha$-potential with OPP |
|-------|-----------|----------------------------------|---------------------------------|-----------------------------------|
| $0^+_1$ | -7.275    | -12.5                            | -10.43                          | not converged                     |
| $0^+_2$ | 0.3796    | -2.7                             | 0.64                            | 1.55                              |
| $0^+_3$ | 3.0       | –                                | 5.43                            | 4.055                             |
| $2^+_1$ | -2.836    | -10.0                            | -7.63                           | not converged                     |
| $2^+_2$ | 3.89      | -3.7                             | 6.39                            | 2.64                              |