Incorporating the Heisenberg and Pauli principles into the kinetic approach to neutrino oscillations

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Abstract. Neutrinos play an important role in the supernovae explosions. The energy transfer from the core to the outer layers depends on the neutrino flavor evolution, affected by collective neutrino oscillations. The latter are damped by kinematic decoherence, developing in the course of the neutrino propagation. The kinematic decoherence caused by dephasing of many neutrinos is readily accounted for in the kinetic approach to neutrino oscillations. We show that the kinematic decoherence caused by dephasing of momentum modes of the neutrino wave-packets can also be accounted for in the kinetic approach through a choice of the initial conditions consistent with the uncertainty principle. According to the existing estimates, this effect may have a substantial impact on the phenomenology of supernovae neutrinos.

1. Introduction

The wave-particle duality is one of the key concepts of quantum mechanics. The wave properties of the neutrino have been extensively discussed in the literature since the first hints towards the existence of flavor neutrino oscillations. This has led to the emergence of the by now standard quantum-mechanical approach to neutrino oscillations, operating with the neutrino wave function. This approach is automatically consistent with the Heisenberg and Pauli principles. The former implies that the neutrino has a non-zero momentum uncertainty, which leads to damping of the oscillations via the effect of wave packet separation.

The particle properties of the neutrino are manifest in the scattering processes, important in the early universe and core-collapse supernovae. Scattering processes are taken into account by the Boltzmann kinetic equation which, however, does not include the oscillations. In a seminal paper [1] Dolgov extended the kinetic equation to mixed neutrinos by introducing matrices of generalized occupation numbers. The kinetic approach to neutrino oscillations has been further developed by Sigl and Raffelt [2], who included effects non-linear in the neutrino density matrix such as Pauli blocking of neutrino final states or neutrino refraction in a medium of neutrinos. In the context of supernovae these non-linear effects lead to collective neutrino oscillations, which substantially affect the neutrino flavor evolution and therefore the energy transfer from the core to the outer layers. Damping of the collective oscillations due to kinematic decoherence caused by dephasing of many neutrinos is readily taken into account in the kinetic approach. On the other hand, the question of how to take into account kinematic decoherence due to dephasing of momentum modes of the neutrino wave-packets has not been addressed yet.

This question is related to a more general question of how to account for the Heisenberg and Pauli principles in the kinetic approach to neutrino oscillations, addressed in this contribution.
In section 2 we discuss the Heisenberg principle and argue that it can be accounted for through a choice of the initial conditions consistent with this principle. The related effect of wave packet separation is then automatically taken into account. The Pauli principle is addressed in section 3. To compute the expectation value, we need to specify the initial state. Because the Schrödinger and Heisenberg pictures coincide at the production time, the initial state vector in the Heisenberg representation is constructed as

\[ |\psi_i(0, p)\rangle a^\dagger_i(0, p)|0\rangle. \]

(3)

Taking the expectation value of equation (2) with respect to the state vector in equation (3) we obtain the matrix of densities at the production time 4

\[ \rho_{ij}(0, x, p) = \int \frac{d^3p}{(2\pi)^3} \psi_i(0, p) a^\dagger_j(0, p)|0\rangle \rho_{ij}(0, x, p) = \int \frac{d^3p}{(2\pi)^3} \psi_i(0, p + \Delta/2) \psi_j^\dagger(0, p - \Delta/2). \]

(4)

The momentum-space initial wave function is frequently assumed to be a Gaussian,

\[ \psi_i(0, p) \propto \exp\left(-\frac{(p - P_{w})^2}{4\sigma_p^2}\right) e^{-ipx_w}. \]

(5)

where \( \sigma_p \) is the neutrino momentum uncertainty, \( P_{w} \) – its characteristic momentum, and \( x_w \) is the initial position of the wave packet. If the wave function is strongly peaked in the vicinity of the characteristic momentum, \( \sigma_p \ll P_w \), then the matrix of densities is well approximated by

\[ \rho_{ij}(0, x, p) \propto \exp\left(-\frac{(p - P_{w})^2}{2\sigma_p^2}\right) \exp\left(-\frac{(x - x_w)^2}{2\sigma_x^2}\right), \]

(6)

where \( \sigma_x \equiv 1/2\sigma_p \) is the size of the neutrino wave packet in configuration space. This example demonstrates that the matrix of densities is not a classical object and that its coordinate and...
momentum arguments are not coordinate and momentum of the classical neutrino. The condition \( \sigma_p \ll p_w \) ensures smallness of the off-shell effects. In configuration space it translates to \( \sigma_x \gg \lambda_w \), where \( \lambda_w \equiv p_w^{-1} \) is the de Broglie wavelength. This condition is fulfilled for solar neutrinos \(^9\) but may be violated for supernovae neutrinos \(^7\). As is evident from equation (4), the matrix of densities is localized neither in configuration nor in momentum space, i.e. it is consistent with the Heisenberg principle.

It is well known that dephasing of the momentum modes of the neutrino wave packet in the course of neutrino propagation leads to damping of the oscillations even in vacuum. The larger the momentum uncertainty (and, by the Heisenberg principle, the smaller the coordinate uncertainty) the sooner it happens. As this effect is of purely kinematic origin, it is refereed to as kinematic decoherence.

3. The Pauli principle

In core-collapse supernovae the neutrino density is very large and the induced neutrino-potential exceeds the vacuum oscillation frequency. This results in collective neutrino oscillations, which strongly affect neutrino flavor evolution \(^8\) and hence the energy transfer from the core to the outer layers. The state vector of the \( N \)-particle neutrino system is constructed similarly to the single-particle one,

\[
|\psi\rangle = \frac{1}{\sqrt{N!}} \sum_{i_1...i_N} \int \frac{d^3p_1}{(2\pi)^3} \cdots \frac{d^3p_N}{(2\pi)^3} \\
\times \psi_{i_1...i_N}(0, \mathbf{p}_1 \cdots \mathbf{p}_N) \hat{a}_{i_1}^\dagger(0, \mathbf{p}_1) \cdots \hat{a}_{i_N}^\dagger(0, \mathbf{p}_N)|0\rangle,
\]

where \( \psi_{i_1...i_N}(0, \mathbf{p}_1 \cdots \mathbf{p}_N) \) is the \( N \)-particle wave function. The Pauli principle requires the wave function to change its sign under permutations of any two particles. For fermions the ladder operators anticommute and \( \hat{a}_{i_1}^\dagger(t_{ij}, \mathbf{p}_1) \cdots \hat{a}_{i_N}^\dagger(t_{ij}, \mathbf{p}_N)|0\rangle \) changes its sign as well. Therefore, the state vector remains invariant. Proceeding as above, we obtain for the matrix of densities at the production time \(^4\)

\[
\varrho_{ij}(0, \mathbf{x}, \mathbf{p}) = N \int \frac{d^3\Delta}{(2\pi)^3} e^{i\Delta \mathbf{x}} \sum_{l_2...l_N} \int \frac{d^3p_2}{(2\pi)^3} \cdots \frac{d^3p_N}{(2\pi)^3} \\
\times \psi_{l_2...l_N}(0, \mathbf{p} + \Delta/2, \mathbf{p}_2, \ldots, \mathbf{p}_N) \psi_{i_1}^*_{l_2...l_N}(0, \mathbf{p} - \Delta/2, \mathbf{p}_2, \ldots, \mathbf{p}_N). \tag{8}
\]

As the wave function satisfies the Pauli principle, the same applies to the matrix of densities.

To illustrate this expression we approximate the wave function by the Slater determinant,

\[
\psi_{i_1...i_N}(t, \mathbf{p}_1, \ldots, \mathbf{p}_N) = \mathcal{N} \begin{vmatrix} \psi_{1, i_1}(t, \mathbf{p}_1) & \cdots & \psi_{1, i_N}(t, \mathbf{p}_N) \\ \cdots & \cdots & \cdots \\ \psi_{N, i_1}(t, \mathbf{p}_1) & \cdots & \psi_{N, i_N}(t, \mathbf{p}_N) \end{vmatrix}, \tag{9}
\]

where \( \psi_n \) are wave functions of the individual neutrinos. For non-overlapping individual wave functions the normalization factor is given by \( \mathcal{N} = (N!)^{-\frac{1}{2}} \). Neglecting the overlap contributions, small even in supernovae \(^9\), we arrive at \( \varrho_{ij}(0, \mathbf{x}, \mathbf{p}) = \sum_{l=1}^{N} \varrho_{ij}^{(l)}(0, \mathbf{x}, \mathbf{p}) \), where \( \varrho_{ij}^{(l)}(0, \mathbf{x}, \mathbf{p}) \) refer to the individual neutrinos, see equation (4). For Gaussian wave packets \( \varrho_{ij}^{(l)}(0, \mathbf{x}, \mathbf{p}) \) are given by equation (6) with (in general) different \( p_w \) and \( x_w \) for different \( l \).

Akhmedov and Mirizzi argued in \(^7\) that even if all \( p_w \) are equal, the momentum uncertainty inherent to each individual neutrino leads to kinematic decoherence through dephasing of momentum modes of the neutrino wave-packets and results in damping of the collective oscillations. In supernovae the neutrino momentum uncertainty can be as large as 1 MeV, such that neutrinos
decohere after propagating over the distance \( \sim 10 \) km \(^9\). In other words, this effect may have a substantial impact on the collective neutrino oscillations and hence on the energy transfer from the core to the outer layers. Raffelt and Tamborra studied the case of vanishingly small momentum uncertainty and the \( p_w \) distribution reflecting the energy spectrum of the neutrino ensemble \(^10\). Also in this setup the collective oscillations are damped by kinematic decoherence, which now stems from dephasing of many neutrinos. To our knowledge, the combined impact of these two sources of kinematic decoherence has not been studied yet.

4. The Wigner approach
In the preceding sections we have constructed initial conditions for the matrix of densities, consistent with the Heisenberg and Pauli principles, starting from the neutrino wave function in the Schrödinger representation. In principle, from the fact that the Heisenberg equation, used to derive the kinetic equation, produces solutions consistent with the uncertainty and exclusion principles it is evident that the resulting solutions of the kinetic equation are consistent with these fundamental quantum principles also for \( t > 0 \).

However, because the matrix of densities depends on the two canonically conjugate variables, coordinate and momentum, consistency of the kinetic approach with the uncertainty principle is sometimes questioned. It is also argued that the Heisenberg principle can be properly accounted for only in the quantum-mechanical approach to neutrino oscillations, that relies on the Schrödinger equation for the neutrino wave function,

\[
i \partial_t \psi_i(t, \mathbf{x}) = H_{ij}(t, \mathbf{x}) \psi_j(t, \mathbf{x}). \tag{10}\]

As is evident from equation (4), for single neutrinos the quantum-mechanical counterpart of the matrix of densities is the one-particle Wigner function,

\[
\rho_{ij}(t, \mathbf{x}, \mathbf{p}) = \int \frac{d^3\Delta}{(2\pi)^3} e^{i\Delta \mathbf{x}} \psi_i(t, \mathbf{p} + \Delta/2) \psi_j^*(t, \mathbf{p} - \Delta/2). \tag{11}\]

Being a functional of the wave function it satisfies the uncertainty principle at any given time. The evolution equation for the Wigner function can be obtained from the Schrödinger equation. Neglecting small off-shell contributions we obtain (to leading order in the gradients) an evolution equation for the Wigner function whose form matches the form of the collisionless part of equation (1),

\[
\partial_t \rho(t, \mathbf{x}, \mathbf{p}) + \frac{i}{2} \left\{ \partial_\mathbf{p} H(t, \mathbf{x}, \mathbf{p}), \partial_\mathbf{x} \rho(t, \mathbf{x}, \mathbf{p}) \right\} - \frac{1}{2} \left\{ \partial_\mathbf{x} H(t, \mathbf{x}, \mathbf{p}), \partial_\mathbf{p} \rho(t, \mathbf{x}, \mathbf{p}) \right\}
\approx -i \left[H(t, \mathbf{x}, \mathbf{p}), \rho(t, \mathbf{x}, \mathbf{p}) \right]. \tag{12}\]

As both the form of the equations and the form of the initial conditions match, in the collisionless limit the quantum-mechanical and kinetic approaches produce identical results. These considerations are readily generalized to \( N \)-particle systems \(^4\). Based on this observation we conclude that the quantum-mechanical and kinetic approaches to neutrino oscillations are equivalent in the collisionless limit. This confirms that solutions of the kinetic equation account for the Pauli and Heisenberg principles if the initial conditions are consistent with these fundamental quantum principles.

5. Summary
In the present contribution, based on references \(^4\)\(^5\), we show the equivalence of the quantum-mechanical and kinetic approaches to flavor neutrino oscillations in the collisionless limit. This conclusion is based on two observations. First, the form of the evolution equation for the Wigner
function matches the form of the kinetic equation. Second, the initial conditions for the matrix of densities, constructed from the neutrino wave function, match those for the Wigner function. As the form of the evolution and kinetic equations and the form of the respective initial conditions match, the two approaches produce identical solutions. Thus the Heisenberg and Pauli principles can be accounted for also in the kinetic approach to neutrino oscillations if the kinetic equation is supplemented by initial conditions consistent with these fundamental quantum principles.

On the one hand, the equivalence of the two approaches means that the kinetic approach can reproduce all the results obtained in the quantum-mechanical approach until now. On the other hand, it means that many aspects of the quantum-mechanical approach can be transferred to the kinetic approach. This implies in particular, that the neutrino momentum uncertainty is an integral part of the initial conditions for the matrix of densities. As has been extensively discussed in the literature, in supernovae the large neutrino density leads to a sizable neutrino-neutrino potential and results in collective neutrino oscillations, which may substantially impact the energy transfer from the core to the outer layers. The collective oscillations are damped by kinematical decoherence induced by dephasing of many neutrinos, and by dephasing of momentum modes of neutrino wave packets. The latter source of kinematic decoherence is intrinsically related to a finite size of the neutrino wave packet and hence to the uncertainty principle. The reported results provide a means of incorporating the effect of wave packet separation into the kinetic approach to neutrino oscillations and support the approach and conclusions of Akhmedov and Mirizzi.

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