ULRICH BUNDLES ON DOUBLE COVERS OF PROJECTIVE SPACES

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Abstract. In this article, we prove that any smooth projective variety \( X \) which is a double cover of the projective space \( \mathbb{P}^n \) \((n \geq 2)\) admits an Ulrich bundle. When \( n = 2 \), we show that on any such \( X \), there is an Ulrich bundle of rank two.

1. Introduction

Arithmetically Cohen-Macaulay (ACM) vector bundles have been at the center of the study of vector bundles over projective varieties. These are defined to be vector bundles \( E \) on projective varieties \( X \) satisfying
\[
H^i(X, E(l)) = 0 \quad \text{for all} \ l \in \mathbb{Z} \ \text{and} \ 0 < i < \dim X .
\]

It was proved by Horrocks [9] that ACM vector bundles on projective spaces \( \mathbb{P}^n \) are precisely those which split as a direct sum of line bundles. Following this, there has been a lot of interest in understanding ACM bundles over all varieties. Among ACM bundles \( E \), Ulrich bundles are those whose associated module \( \bigoplus_t H^0(X, E(t)) \) has the maximum number of generators.

In his 1984 paper [17], Ulrich asked whether every smooth projective variety carries an Ulrich bundle. Eisenbud and Schreyer [7] drew attention to this question and also asked for the minimal rank of such a bundle whenever it exists. The existence of Ulrich bundles has been established in the case of several varieties, we refer to [1] and [3] and references therein for a complete list of such varieties.

In this note, we consider complex smooth projective varieties \( X \) which admit a degree two morphism \( \pi : X \to \mathbb{P}^n \) \((n \geq 2)\) to the projective space. We discuss the existence of Ulrich bundles over \( X \) with respect to the morphism \( \pi \), or equivalently with respect to the ample and globally generated line bundle \( \pi^* \mathcal{O}_{\mathbb{P}^n}(1) \). Observe that a vector bundle \( E \) on \( X \) is Ulrich with respect to \( \pi \), if \( \pi_* E \) is the trivial vector bundle on \( \mathbb{P}^n \). When \( n = 2 \), i.e. if \( X \) is a double cover of \( \mathbb{P}^2 \), we prove the following result regarding the existence of Ulrich bundles on \( X \) in §1.

Theorem 1.1. A smooth double cover \( X \xrightarrow{\pi} \mathbb{P}^2 \) of the projective plane branched along a smooth curve \( B \) admits a rank two Ulrich bundle.

In [14], the authors investigated the existence of Ulrich line bundles over double covers of \( \mathbb{P}^2 \) and showed that a smooth double cover of \( \mathbb{P}^2 \) branched over a curve of degree \( 2d \) when \( d = 1, 2 \) admits Ulrich line bundles. When \( d \geq 3 \), it was shown that a generic double cover does not admit an Ulrich line bundle, but there are special classes of double covers which do indeed admit Ulrich line bundles. In [15], Sebastian and Tripathi prove that
general plane double covers admit rank two Ulrich bundles using different techniques. We remark that Theorem 1.1 completely answers the question of existence and minimal rank of Ulrich bundles over all smooth double covers of $\mathbb{P}^2$.

The proof of the above theorem involves the analysis of double covers of smooth curves $p : C \to D$ and identifying the conditions on the curve $C$ for it to admit a line bundle whose direct image is trivial, cf. Theorem 4.4. This condition is indeed met when $D$ is a smooth plane curve of degree $d$, and this will be used in the proof of Theorem 1.1.

We remark that, previously, we had attempted to use [10, Theorem A] which also studies line bundles on curves $C$ where $p : C \to D$ is a double cover of smooth curves. However, we realised that the proof of Theorem A in [10] is not quite correct and we provide a counterexample to their result in the Appendix, cf. §7.

In case of double covers of $\mathbb{P}^n$ when $n \geq 3$, we prove the following theorem.

**Theorem 1.2.** Let $X \overset{\pi}{\rightarrow} \mathbb{P}^n$ be a smooth double cover branched over a smooth hypersurface $B$ of degree $2d$. Then $X$ admits an Ulrich bundle.

We give two different proofs of the above theorem in §5. The first approach involves embedding $\mathbb{P}^n$ in a higher dimension projective space $\mathbb{P}^N$ by the $d$-uple embedding and understanding the existence of Ulrich sheaves on double covers of $\mathbb{P}^N$ branched along quadrics. In this regard, we use the well known and seminal result of Herzog, Ulrich and Backelin [8] regarding the existence of Ulrich sheaves on complete intersection varieties $Z$. We reproduce their proof in the special case when $Z$ is quadric hypersurface for the sake of completeness, cf. Theorem 5.2.

In the second approach, we consider appropriate complete intersection subvarieties $Y$ inside $\mathbb{P}^n$ and their double covers $\tilde{Y}$ inside $X$. We prove that the existence of vector bundles on $\tilde{Y}$ whose direct image in $Y$ is trivial give rise to Ulrich bundles on $X$, cf. Theorem 4.1. We then show that such vector bundles indeed exist on $\tilde{Y}$, cf. Lemma 5.3.

We make some observations about the ranks of the Ulrich bundles obtained on the double covers $X$ of $\mathbb{P}^n$ when $n \geq 3$ in §6. Theorem 6.4 gives an upper bound for the minimal rank of Ulrich bundles on double covers $X$ of $\mathbb{P}^n$ when $n \geq 3$. However, computing the minimal rank is still an open question which we are investigating.

## 2. Preliminaries

### 2.1. Ulrich bundles

Here we briefly introduce Ulrich bundles, cf. [11] and [3] for thorough surveys on Ulrich bundles.

**Definition 2.1.** Let $X$ be a smooth projective variety over $\mathbb{C}$ together with a finite morphism $\pi : X \to \mathbb{P}^{\dim X}$ of degree $m$. A vector bundle $E$ of rank $r$ on $X$ is said to be Ulrich with respect to the morphism $\pi$ if $\pi^*E = \mathcal{O}_{\mathbb{P}^{\dim X}}^r$.

If we denote $\pi^*\mathcal{O}_{\mathbb{P}^{\dim X}}(1) = \mathcal{O}_X(1)$, then $E$ is Ulrich on $X$ if and only if $H^i(X, E(-p)) = 0$ for $1 \leq p \leq \dim X$ and for all $i$.

### 2.2. Secant varieties

Given a subvariety $X$ of $\mathbb{P}^n$, we can define the higher order secant varieties of $X$ as follows.
**Definition 2.2.** Let $X$ be a variety in $\mathbb{P}^n$, then the Secant variety of order $(r - 1)$ of $X$ is the closure of the union of all linear subspaces of $\mathbb{P}^n$ spanned by $r$ points in $X$. More precisely,

$$\text{Sec}_{r-1}X = \bigcup_{v_1, v_2, \ldots, v_r \in X} \langle v_1, v_2, \ldots, v_r \rangle \subset \mathbb{P}^n$$

where for $S \subset \mathbb{P}^n$, the notation $\langle S \rangle$ denotes the smallest linear subspace of $\mathbb{P}^n$ containing $S$.

We are interested in the higher order secant varieties of a particular projective variety $X$ which is the set of degree 2 hypersurfaces in $\mathbb{P}^n$ which are reducible into two degree $d$ hypersurfaces, i.e.

$$X = \{ [F] \in \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2d)) \mid F = F_1F_2 \text{ and } F_i \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \}.$$ 

It is known that $X \subset \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2d))$ is a subvariety with dimension $2^{(d+n)} - 2$, cf. [11], [2].

Note that if $[F] \in \text{Sec}_{r-1}X$, then $F$ can be written as

$$F = \Sigma_{i=1}^r F_iG_i \text{ where } F_i, G_i \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) .$$

It is clear that we have a chain of inclusions

$$X \subset \text{Sec}_1X \subset \cdots \subset \text{Sec}_{r-1}X \subset \text{Sec}_rX \subset \cdots \subset \mathbb{P}V .$$

We will see in Lemma 6.1 that there is an integer $m$ such that $\text{Sec}_mX$ covers $\mathbb{P}V$.

### 3. Polynomial expression of a plane curve

Let $F \in \mathbb{C}[x, y, z]$ be a smooth homogeneous polynomial of degree $2d$. Let $[F] \in \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2d))$ denote the corresponding hypersurface of $\mathbb{P}^2$. In [2], the authors show that $[F] \in \text{Sec}_1X$ where $X$ is described in equation (1), i.e. $F$ can be written in the form

$$F = F_1G_1 + F_2G_2 ,$$

where $F_1, G_1, F_2, G_2 \in \mathbb{C}[x, y, z]$ are homogeneous polynomials of degree $d$. We broadly explain their ideas in our context. The following is the key lemma.

**Lemma 3.1.** [2, Lemma 4.1, Lemma 4.3] Let $S = \mathbb{C}[x, y, z]$. The following statements are equivalent.

1. The generic degree $2d$ curve ($F = 0$) in $\mathbb{P}^2$ can be written in the form $F = F_1G_1 + F_2G_2$ where $F_1, F_2, G_1, G_2 \in S$ are of degree $d$.

2. The Secant variety $\text{Sec}_1X$ covers the space of degree $2d$ forms in 3 variables i.e.$$
\text{Sec}_1X = \mathbb{P}(S_{2d}) .$$

3. Let $F_1, F_2, G_1, G_2 \in S$ be generic forms of degree $d$, then

$$H \left( \frac{S}{(F_1, F_2, G_1)} , 2d \right) = 0$$

where $H(\_ , 2d)$ denotes the Hilbert function in degree $2d$ of the ring.
In order to prove that a generic $F \in S_{2d}$ has an expression of the form

$$F = F_1G_1 + F_2G_2$$

where $F_1, F_2, G_1, G_2 \in S_d$, the authors in [2] use condition (3) of the above lemma i.e. they show that the corresponding Hilbert function is zero. In particular, we refer to the proof of [2, Theorem 5.1]. Here they consider the ring

$$(2) \quad A = \frac{\mathbb{C}[x, y, z]}{(F_1, F_2, G_2)}$$

where $F_1, F_2, G_2$ are generic forms of degree $d$. They then show that $H\left(\frac{A}{G_1}, 2d\right) = 0$, where $G_1$ is the image of of a generic degree $d$ form $G_1 \in \mathbb{C}[x, y, z]$ in the ring $A$. Showing $H\left(\frac{A}{G_1}, 2d\right) = 0$ is equivalent to showing that the multiplication by $G_1$ map:

$$m : A_d \to A_{2d}$$

is surjective.

We now prove condition (3) of Lemma 3.1 by showing that the morphism $m$ above is surjective for generic choices of $F_1, F_2, G_1, G_2$, by considering certain extensions of vector bundles in this upcoming subsection.

3.1. Extensions. Let $C$ be a smooth projective curve of genus $g$ with canonical bundle $K_C$. Let $L$ be a line bundle with degree greater than $2g - 2$. We wish to consider extensions,

$$0 \to \mathcal{O}_C \to E \to L \to 0.$$  

These are parametrized by $\text{Ext}^1(L, \mathcal{O}_C) = H^1(L^{-1})$. We denote this vector space (or affine space when applicable) by $V$.

Let $V_s \subset V$ be the subset consisting of elements $\eta \in V$ such that the corresponding extension $E_\eta$ has $H^1(E_\eta) \neq 0$. Dual to our extension, we have

$$0 \to L^{-1} \otimes K_C \to E^\vee \otimes K_C \to K_C \to 0.$$  

Let $M = L^{-1} \otimes K_C$ and $F = E^\vee \otimes K_C$, then $\eta \in V_s \subset V = \text{Ext}^1(K_C, M)$ is just the subset where $H^0(F_\eta) \neq 0$ by Serre duality. Note that $\text{deg } M = -e$ with $e > 0$ by our assumption on the degree of $L$.

**Theorem 3.2.** Let $M$ be a line bundle of degree $-e$, with $e > 0$ and notation as above. Then $V_s \subset V$ is a closed subset of dimension at most $3g - 3$, where for $g = 0$, this is to be understood as $V_s = \emptyset$.

**Proof.** If $g = 0$, the result is obvious since both $M$ and $K_C$ have negative degrees and thus every extension has zero global sections. So, we assume $g > 0$ for the rest of the proof.

Let $\mathbb{P} = \mathbb{P}(H^0(K_C)) = \mathbb{P}^{g-1}$, the set of non-zero sections of $K_C$. Consider $Z \subset \mathbb{P} \times V = X$ consisting of $(\alpha, \eta)$ where $\alpha$ is a non-zero section of $K_C$ and $\alpha$ lifts to $F_\eta$ for the extension

$$0 \to M \to F_\eta \to K_C \to 0.$$
Thus, the image of $Z$ in $V$ under projection is contained in $V_s$. Conversely, if $\eta \in V_s$, we have a non-zero section of $F_\eta$, which by composition gives a non-zero section of $K_C$, since $\deg M < 0$. This says that $V_s$ is precisely the image of $Z$ under the projection to $V$.

On $\mathbb{P}$, we have the universal map $\mathcal{O}_\mathbb{P}(-1) \to H^0(K_C) \otimes \mathcal{O}_\mathbb{P}$ which we pull back to $X$. Similarly, on $V$ we have the universal map $H^0(K_C) \otimes \mathcal{O}_V \to H^1(M) \otimes \mathcal{O}_V$, cupping with $\eta \in V = H^1(M \otimes K_C^{-1})$ which also can be pulled back to $X$. Then we get a composition $\mathcal{O}_X(-1) \to H^1(M) \otimes \mathcal{O}_X$. Then $Z$ is just the points on $X$ where this map is zero and thus $Z$ is a closed subset of $X$ and since $V_s$ is the image under the proper map to $V$, we see that $V_s$ is a closed subset of $V$.

We calculate the dimension of $Z$ by looking at the projection to $\mathbb{P}$. If $(\alpha, \eta) \in Z$, we look at the pull back diagram:

\[
\begin{array}{ccccccccc}
0 & \to & M & \to & G & \to & \mathcal{O}_C & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M & \to & F_\eta & \to & K_C & \to & 0
\end{array}
\]

The section $\alpha$ lifts to $F_\eta$ says the sequence $0 \to M \to G \to \mathcal{O}_C \to 0$ splits and conversely. So the fibre of $\alpha \in \mathbb{P}$ are the extensions in the kernel of the induced map, $V = \text{Ext}^1(K_C, M) \to \text{Ext}^1(\mathcal{O}_C, M)$.

We have an exact sequence

\[
0 \to \mathcal{O}_C \to K_C \to T \to 0
\]

where $T$ is a sheaf of length $2g - 2$. Taking $\text{Hom}(\_, M)$, we get,

\[
0 = H^0(M) \to \text{Ext}^1(T, M) \to \text{Ext}^1(K_C, M) = V \to \text{Ext}^1(\mathcal{O}_C, M) \to .
\]

The kernel of the map $V \to \text{Ext}^1(\mathcal{O}_C, M)$, which is precisely $\text{Ext}^1(T, M)$ is a vector space of dimension $2g - 2$. Thus, $Z \to \mathbb{P}$ is an affine bundle of fiber dimension $2g - 2$ and $\dim Z = 3g - 3$. So, $\dim V_s \leq 3g - 3$. \hfill \Box

**Corollary 3.3.** For a general element $\eta \in V$, the extension

\[
0 \to \mathcal{O}_C \to E_\eta \to L \to 0
\]

where $\deg L > 2g - 2$ has $H^1(E_\eta) = 0$.

**Proof.** We only need to show that $V_s \neq V$. But

\[
\dim V = \deg L + g - 1 > 3g - 3 .
\]

We next have the following definition.

**Definition 3.4.** For a vector bundle $E$, we say a point $P \in C$ is a base point, if the natural map $H^0(E) \otimes \mathcal{O}_C \to E$ is not surjective at $P$, That is, the cokernel has $P$ in its support.

We denote by $V_{bp}$, the set of $\eta \in V = H^1(L^{-1}) = \text{Ext}^1(L, \mathcal{O}_C)$ such that the corresponding extension $E_\eta$ has a base point.
**Theorem 3.5.** Assume that \( \deg L = r > 2g \). Then \( V_{bp} \) is contained in a subvariety of dimension at most \( 3g - 1 \).

**Proof.** We restrict our attention to the open set \( U = V - V_s \), and \( V_{bp} \cap U \), since \( \dim V_s \leq 3g - 3 \) by the previous theorem. Let \( X = C \times U \) and \( p : X \to C \), \( q : X \to U \) the two projections. On \( X \) we have the universal extension,

\[
0 \to \mathcal{O}_X \to E \to p^*L \to 0.
\]

Since for any \( \eta \in U \), one has \( H^1(E\eta) = 0 \), we see that \( q_*E \) is a vector bundle (\( q \) is proper and flat) and so we get a map \( q^*q_*E \to E \). Let \( Z \) be the support of the cokernel of this map, which is a closed subset of \( X \). It is immediate that \( q(Z) = V_{bp} \cap U \) and thus, it is a closed subset of \( U \).

We calculate the dimension of \( Z \) by using the projection \( p \). For a point \( P \in C \), the fibre in \( Z \) over \( p \) is the set of all \( \eta \in U \) such that \( P \) is a base point of \( E\eta \).

A pair \((P, \eta) \in Z \) if and only if \( P \) is a base point of \( E\eta \). However, this says, the image of \( H^0(E\eta) \to H^0(L) \) factors through \( H^0(L(P)) \).

We look at the following pullback diagram.

\[
\begin{array}{ccc}
0 & \to & \mathcal{O}_C \\
\downarrow & & \downarrow \\
0 & \to & E \eta \\
\downarrow & & \downarrow \\
0 & \to & L \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
\]

Note that \( H^0(E\eta) \to H^0(L_P) \) is the zero map where \( L_P \) comes from the exact sequence

\[
0 \to L(-P) \to L \to L_P \to 0.
\]

This gives us \( H^0(F) = H^0(E\eta) \).

If \( e \) is the dimension of the image of \( H^0(E\eta) \to H^0(L) \), one has \( e + g = h^0(L) \), since \( H^1(E\eta) = 0 \). But, by assumption, \( e \) is also the dimension of the image of \( H^0(F) \to H^0(L(-P)) \) and since \( h^0(L(-P)) = h^0(L) - 1 \), we see that \( H^1(F) \neq 0 \).

Let \( W = H^1(L(-P)) = \text{Ext}^1(L(-P), \mathcal{O}_C) \), and \( W_s \subset W \) be defined as earlier. Since \( H^1(F) \neq 0 \), the class corresponding to that extension in \( W \) must in fact be in \( W_s \).

Taking \( \text{Hom}(_, \mathcal{O}_C) \) of the exact sequence,

\[
0 \to L(-P) \to L \to L_P \to 0
\]

we get,

\[
0 \to \text{Ext}^1(L_P, \mathcal{O}_C) \to V \to W \to 0.
\]

Consider \((P, \eta) \in Z \). Then the image of \( \eta \) under the map \( V \to W \) lands in \( W_s \) as seen above. Thereby, such an \( \eta \) is in the inverse image of \( W_s \).

Since \( \deg(L(-P)) = \deg L - 1 > 2g - 2 \), by Theorem 3.2, we get that \( \dim W_s \) is at most \( 3g - 3 \). Also \( \text{Ext}^1(L_P, \mathcal{O}_C) \) is a one dimensional vector space. So the inverse image of \( W_s \) in \( V \) has dimension at most \( 3g - 2 \). This is true for any \( P \in C \) and thus \( \dim Z \leq 3g - 1 \), which implies \( \dim V_{bp} \cap U \leq 3g - 1 \).

**Corollary 3.6.** If \( \deg L > 2g \), a general extension

\[
0 \to \mathcal{O}_C \to E \to L \to 0
\]
has $H^1(E) = 0$ and $E$ is globally generated.

Proof. As before, letting $V = H^1(L^{-1})$, $\dim V = \deg L + g - 1$ by Riemann-Roch and $\deg L + g - 1 > 3g - 1$. On the other hand, $\dim V_{bp} \leq 3g - 1$ and $\dim V_s \leq 3g - 3$. □

Remark 3.7. Both bounds obtained for the line bundles in this section are the best possible in general.

- For example, if we allow $\deg L = 2g - 2$ in Theorem 3.2, then we may take $L = K_C$ and since for any extension $E_\eta$, we have a surjection $H^1(E_\eta) \to H^1(K_C) \neq 0$, we see that all extensions are special, i.e. $V_s = V$.
- Suppose we allow $\deg L = 2g$ in Theorem 3.5, then a general extension $E_\eta$ is non-special i.e. $H^1(E_\eta) = 0$. From the sequence
  
  \[ 0 \to \mathcal{O}_C \to E_\eta \to L \to 0 \]

  we get the surjective map $H^0(L) \to H^1(\mathcal{O}_C)$. Thus the dimension of the image of $H^0(E_\eta) \to H^1(L)$ is equal to $h^0(L) - g = 1$. If $g > 0$, one section can not generate $L$ and thus $E_\eta$ has base points.

Finally, we derive the corollaries we want.

Corollary 3.8. Let $C \subset \mathbb{P}^2$ be a smooth curve of degree $d$. Then we have an extension

\[ 0 \to \mathcal{O}_C \to E \to \mathcal{O}_C(d) \to 0 \]

such that, $H^1(E) = 0$ and $E$ is globally generated.

Proof. By Theorem 3.2 and Theorem 3.5, we just need to compute $\deg \mathcal{O}_C(d)$:

\[ \deg \mathcal{O}_C(d) = d^2 > 2 \frac{(d-1)(d-2)}{2} = 2g(c). \]

□

Corollary 3.9. For any $d > 0$, there exist four homogeneous polynomials in three variables of degree $d$ so that the ideal generated by them contains all degree $2d$ homogeneous polynomials. (This is an open condition, so the conclusion is true for a general set of four degree $d$ polynomials.)

In particular, for a general set of four polynomials $F_1, F_2, G_1, G_2$ the multiplication map

\[ A_d = \frac{\mathbb{C}[x,y,z]}{(F_1,F_2,G_2)_d} \cong \frac{\mathbb{C}[x,y,z]}{(F_1,F_2,G_2)_{2d}} = A_{2d} \]

is surjective.

Proof. Let $C \subset \mathbb{P}^2$ be a smooth curve of degree $d$ and let

\[ 0 \to \mathcal{O}_C \to E \to \mathcal{O}_C(d) \to 0 \]

be an extension such that $H^1(E) = 0$ and $E$ globally generated. Then we can pick three general sections which generate $E$ to get a sequence,

\[ 0 \to \mathcal{O}_C(-d) \to \mathcal{O}_C^3 \to E \to 0. \]

Dualize and twist by $d$ to get a sequence,

\[ 0 \to E^\vee(d) \to \mathcal{O}_C(d)^3 \to \mathcal{O}_C(2d) \to 0. \]
But \( E^\vee(d) \simeq E \) and thus \( H^1(E^\vee(d)) = 0 \). This implies that the map \( H^0(\mathcal{O}_C(d))^3 \to H^0(\mathcal{O}_C(2d)) \) is onto.

In particular, if the curve \( C \) is given by the equation \( (G_1 = 0) \), we get that the following map is onto

\[
m_{F_1,F_2,G_2} : \left( \mathbb{C}[x,y,z] \right) (G_1) \to \mathbb{C}[x,y,z] (G_1)_{2d}.
\]

This means that, given any degree \( 2d \) homogeneous polynomial \( Q(x,y,z) \), there exist degree \( d \) homogeneous polynomials \( f, g, h \in \mathbb{C}[x,y,z] \) such that

\[
Q - fF_1 - gF_2 - hG_2 \in (G_1).
\]

It is now easy to see the equivalence of \( m_{F_1,F_2,G_2} \) being surjective and the following map \( m \) being surjective

\[
m : \left( \mathbb{C}[x,y,z] \right) (F_1,F_2,G_2) \to \mathbb{C}[x,y,z] (F_1,F_2,G_2)_{2d}.
\]

\[\square\]

The above theorem shows that the morphism \( m \) as defined in (3) is indeed surjective and thus \( H\left( \frac{A}{G_1}, 2d \right) = 0 \).

Thus the generic degree \( 2d \) curve \( (F = 0) \) in \( \mathbb{P}^2 \) can be written in the form \( F = F_1G_1 + F_2G_2 \) where \( F_1, F_2, G_1, G_2 \in S \) are of degree \( d \).

**Remark 3.10.** We remark that every degree \( 2d \) form \( F \in S \) can be written in the form \( F = F_1G_1 + F_2G_2 \) where \( F_1, F_2, G_1, G_2 \in S \) are of degree \( d \). The proof of [2, Theorem 5.1] shows that \( \text{Sec} X = \mathbb{P}(S_{2d}) \) where \( X \) is as defined in [11]. [2, Remark 5.4] explains that points of the variety of secant lines lie on a true secant line or on a tangent line to \( X \). However, if \( q \) is a point on the tangent line to \( X \) at a point \( p = [FG] \in X \), then \( q \) can be written in the form \( [\alpha F'G' + \beta F''G'] \) for some forms \( F', G' \) of degree \( d \) and scalars \( \alpha, \beta \), and thus \( q \) lies on the secant line joining \( [F'G'] \) and \( [FG] \).

**Lemma 3.11.** Let \( F \in S \) be smooth of degree \( 2d \). Then \( F \) can be written in the form \( F = F'_1G'_1 + F'_2G'_2 \) where \( F'_i \) and \( G'_i \) have degree \( d \), \( F'_i \) is smooth and \( F'_2 \) and \( G'_2 \) intersect transversely at \( d^2 \) distinct points.

**Proof.** By the previous remark, any \( F \in S_{2d} \) can be written in the form

\[
F = F_1G_2 + F_2G_2 \text{ where } F_i, G_i \in S_d.
\]

Consider the linear system \( \mathbb{P}V \) associated to the vector space \( V \) generated by \( \{F_1, F_2\} \). Any element of \( V \) has the form \( aF_1 + bF_2 \) where \( a, b \in \mathbb{C} \). Hence, a general element of the linear system \( \mathbb{P}V \) looks like \( F_1 + \alpha F_2 \) where \( \alpha \in \mathbb{C} \).

By Bertini’s theorem, a general element of \( \mathbb{P}V \) is non-singular away from the basepoints of the linear system, i.e. the points \( b = \{P \in \mathbb{P}^2 \mid F_1(P) = 0 = F_2(P)\} \).

Let \( P \in b \). Then note that at least one of \( F_1 \) or \( F_2 \) is non-singular at \( P \). Indeed, if both are singular at \( P \), then all the first partials \( F_x, F_y, F_z \) of \( F \) vanish at \( P \). That is, \( F \) itself is singular at \( P \) which is a contradiction. Therefore, there is a dense open subset
$U_1 \subset \mathbb{C}$ such that for $\alpha \in U_1$, $F_1 + \alpha F_2$ is non-singular at all points $P \in b$. For $\alpha \in U_1$ we rewrite $F$ in the form

$$F = F_1 G_1 + F_2 G_2 = (F_1 + \alpha F_2) G_1 + F_2 (G_2 - \alpha G_1) = F_\alpha G_1 + F_2 G_\alpha.$$  

We now need to ensure the transversal intersection of the second component of the summation above.

Consider the linear system generated by $\{F_\alpha, F_2\}$. A general element of the linear system is smooth away from the basepoints of the linear system. By a similar argument as above $F_2 + \beta (F_\alpha)$ is a non-singular element of the linear system whenever $\beta \in U_2$ where $U_2 \subset \mathbb{C}$ is a dense open subset. Now we can rewrite $F$ as

$$F = F_\alpha G_1 + F_2 G_\alpha = F_\alpha (G_1 - \beta G_\alpha) + (F_2 + \beta F_\alpha) G_\alpha = F_\alpha G_\beta + F_\beta G_\alpha,$$

where $F_\alpha$ and $F_\beta$ are smooth.

The linear system defined by $\{G_1, G_2\}$ give the following rational map

$$\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$$

which is a morphism away from the basepoints of the linear system. The fiber whenever it makes sense is defined by $\{G_\alpha = 0\} = \{G_2 - \alpha G_1 = 0\}$. Consider the restriction of $\phi$ to smooth curves $(F_\beta = 0)$ where $\beta \in U_2$.

The fibers of this restriction precisely give the intersection of $F_\beta$ with $G_\alpha$. The differential of this restriction is non-zero since $(F_\beta = 0)$ is smooth and hence, separated. Therefore, the tangent directions of $F_\beta$ and $(G_\alpha)$ are independent for a general $\alpha$ and a general $\beta$ implying that $F_\beta$ and $G_\alpha$ intersect transversally.

\[\square\]

4. Ulrich bundles on double covers of the projective plane

In this section, we use induction on the dimension of the variety to prove the existence of Ulrich bundles on smooth double covers. We start with the following general theorem.

**Theorem 4.1.** Consider $\pi : X \rightarrow Y$ a double covering of smooth projective varieties of dimension $n > 1$ defined over $\mathbb{C}$. Let $\pi_* \mathcal{O}_X \simeq \mathcal{O}_Y \oplus L^{-1}$ where $L \in \text{Pic} Y$ and $\pi$ is branched over a smooth hypersurface $B \in |L^{\otimes 2}|$. Suppose that

1. $H^1(Y, \mathcal{O}_Y) = 0$, and
2. for a smooth and irreducible divisor $D \in |L|$ which is such that $\pi|_D : X_D := X \times_Y D \rightarrow D$ is a double covering, there is a vector bundle $E'$ of rank $r$ on $X_D$ whose direct image in $D$ is trivial.

Then there is a vector bundle $E'$ on $X$ of rank $2r$ whose direct image in $Y$ is trivial.
Proof. We note that for a general divisor \( D \in |L| \), the morphism \( \pi_D : X_D \to D \) is a double cover and we have the following fibered diagram.

\[
\begin{array}{ccc}
X_D & \xrightarrow{i} & X \\
\pi_D \downarrow & & \downarrow \pi \\
D^c & \xrightarrow{\iota} & Y
\end{array}
\]

By hypothesis, there is such a smooth and irreducible \( D \in |L| \) and a vector bundle \( E \) on \( X_D \) such that \( (\pi_D)_* E \simeq \mathcal{O}_{\pi_D}^{\oplus 2r} \). The finite morphism \( \pi_D \) gives the surjection

\[
(\text{4}) \quad \mathcal{O}_{X_D}^{\oplus 2r} \xrightarrow{\iota} (\pi_D)_* E \to \mathcal{O}_{\pi_D}^{\oplus 2r} = 0,
\]

which shows that \( E \) is globally generated. Also,

\[
\dim^0 (X_D, E) = \dim^0 (D, (\pi_D)_* E) = \dim^0 (D, \mathcal{O}_{\pi_D}^{\oplus 2r}) = 2r.
\]

Pushing forward the surjection (4) to \( X \), consider the following composition

\[
\mathcal{O}_{X}^{\oplus 2r} \to \iota_* \mathcal{O}_{X_D}^{\oplus 2r} \to \iota_* E \to 0.
\]

We get the following short exact sequence on \( X \) where \( G \) denotes the kernel of the above surjection:

\[
0 \to G \to \mathcal{O}_{X}^{\oplus 2r} \to \iota_* E \to 0.
\]

Since \( G \) is obtained by the process of elementary transformation of the trivial vector bundle on \( X \) along a divisor \( X_D \subset X \), the sheaf \( G \) is locally free and has rank \( 2r \). We next push forward the above short exact sequence to \( Y \):

\[
0 \to \pi_* G \to \pi_* \mathcal{O}_{X}^{\oplus 2r} \to \pi_* \iota_* E \to 0.
\]

Since \( \pi_* \mathcal{O}_{X}^{\oplus 2r} \simeq (\mathcal{O}_{Y} \oplus L^{-1})^{\oplus 2r} \) and \( \pi_* \iota_* E \simeq j_* \mathcal{O}_{D}^{\oplus 2r} \), we have the following commutative diagram.

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & (L^{-1})^{\oplus 2r} \\
\downarrow & \to & \downarrow \\
\iota_* G & \to & \mathcal{O}_{Y}^{\oplus 2r} \oplus (L^{-1})^{\oplus 2r} \\
\downarrow & \to & \downarrow \\
0 & \to & \pi_* \mathcal{O}_{D}^{\oplus 2r} \\
\downarrow & \to & \downarrow \\
0 & \to & (L^{-1})^{\oplus 2r} \\
\downarrow & \to & \downarrow \\
0 & \to & 0
\end{array}
\]

Hence \( \pi_* G \) is a vector bundle on \( Y \) given by the extension:

\[
0 \to (L^{-1})^{\oplus 2r} \to \pi_* G \to (L^{-1})^{\oplus 2r} \to 0.
\]

However, \( \text{Ext}^1 ((L^{-1})^{\oplus 2r}, (L^{-1})^{\oplus 2r}) \simeq H^1 (Y, \mathcal{O}_{Y}^{\oplus 4r}) = 0 \) by assumption. Hence, we get \( \pi_* G \simeq (L^{-1})^{\oplus 2r} \oplus (L^{-1})^{\oplus 2r} \). Thereby, the required vector bundle is \( G \otimes \pi^* L \), since by projection formula,

\[
\pi_* (G \otimes \pi^* L) \simeq \pi_* G \otimes L \simeq (L^{-1})^{\oplus 4r} \otimes L \simeq \mathcal{O}_{Y}^{\oplus 4r}.
\]
The above theorem will be used to show that smooth double covers of $\mathbb{P}^2$ carry rank two Ulrich bundles. To that end, we examine in the upcoming subsection, double covers $p : C \to D$ of smooth curves and identify when the curve $C$ has line bundles $A$ whose direct image on $D$ is trivial.

4.1. Line bundles on double covers of curves. Let $p : C \to D$ be a double cover of smooth curves and $p_*\mathcal{O}_C = \mathcal{O}_D \oplus L^{-1}$ for a line bundle $L$ on $D$ with degree $L = d$. Let the genus of $C$ and $D$ be $g$ and $h$ respectively. The Riemann-Hurwitz formula gives,

$$2g - 2 = 2(2h - 2) + 2d \text{ i.e.}$$

$$d = g - 2h + 1 \text{ or equivalently } g = d + 2h - 1.$$

Also, we have an element $r \in H^0(L^2)$, unique up to non-zero constants such that $\text{div } r$ is the branch locus. Since $C$ is smooth, $\text{div } r$ is a sum of $2d$ distinct points.

**Lemma 4.2.** Let $p : C \to D$ be a double cover of smooth curves as above. Let $A$ be a degree $e$ line bundle on $C$ with $e \leq d$, which is basepoint free and not composed with $p$ (i.e. $A$ is not a pull back of a line bundle from $D$). Then $h^0(A) = 2$, $e = d$ and $p_* A = \mathcal{O}_D^2$.

**Proof.** Since $A$ is basepoint free, $e \geq 0$. If $e = 0$, then $A = \mathcal{O}_C$ and thus trivially composed with $p$. So assume $e > 0$, again the basepoint free property of $A$ gives $h^0(A) \geq 2$. From Riemann-Roch, we get $\chi(A) = e - g + 1 = e - (d + 2h - 1) + 1 = e - d + 2 - 2h$. Thus, $\chi(p_*A) = e - d + 2 - 2h = e - d + 2(1 - h)$, which gives $\text{deg } p_* A = e - d \leq 0$. If the subsheaf generated by global sections of $p_* A$ had rank two, taking two general sections, we have an inclusion $\mathcal{O}_D^2 \hookrightarrow p_* A$ and since $\text{deg } p_* A \leq 0$ this degree must be zero and so $e = d$ and this map is an isomorphism, thus $p_* A = \mathcal{O}_D^2$ showing that $h^0(C, A) = h^0(D, p_* A) = 2$.

If the rank of the subsheaf generated by global sections is one, taking its saturation, we have a line subbundle $M \subset p_* A$ and $H^0(M) = H^0(p_* A)$. So, one has a map $p^* M \to A$, which can not be zero, since sections of $A$ all come from $M$. But since $A$ is globally generated $p^* M = A$, contradicting our hypothesis that $A$ is not a pull back from $D$. □

The proof of the following lemma is well understood.

**Lemma 4.3.** Let $\psi : Z \to D$ be a finite flat map of degree 2 (where $D$ is a smooth projective curve as before). Then one of the following holds.

1. $Z$ is reduced and irreducible.
2. $Z$ is reduced but not irreducible. Then, $Z = Z_1 \cup Z_2$ where $Z_i$ irreducible and the restriction maps $Z_i \to D$ are isomorphisms.
3. $Z$ is not reduced. Then the map $Z_{\text{red}} \to D$ is an isomorphism.

We now identify conditions for the existence of line bundles $A$ on $C$ as in the above lemma.

**Theorem 4.4.** With notations as above, the following are equivalent.

1. There exists a line bundle $A$ of degree $d$ on $C$ which is basepoint free not composed with $p$. 
(2) There exist elements \(l, m, a \in H^0(L)\) such that the branch locus \(\text{div} \, r\) is given by \(r = lm + a^2\).

**Proof.** First we show that (1) implies (2). Let \(\sigma\) be the involution on \(C\) such that \(C/\langle \sigma \rangle = D\). We will consider the two cases separately, when \(A \neq \sigma^* A\) or \(A = \sigma^* A\).

Assume first that \(A \neq \sigma^* A\). By Lemma 4.2 we have \(h^0(A) = 2\). Taking two sections of \(A\) and taking their conjugate sections in \(\sigma^* A\), we get a morphism \(q : C \to \mathbb{P}^1 \times \mathbb{P}^1\). The involution acts on \(\mathbb{P}^1 \times \mathbb{P}^1\) by switching factors and then the morphism \(q : C \to \mathbb{P}^1 \times \mathbb{P}^1\) is equivariant for the action of \(\sigma\). We get the following commutative diagram when we take quotient by \(\sigma\).

\[
\begin{array}{ccc}
C & \xrightarrow{p} & C/\langle \sigma \rangle = D \\
\downarrow q & & \downarrow q' \\
\mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{p'} & \mathbb{P}^2
\end{array}
\]

Note that

\[
\deg q^* p^* \mathcal{O}_{\mathbb{P}^2}(1) = \deg q^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1) = \deg A \otimes \sigma^* A = 2d.
\]

Hence, \(\deg p^* q^* \mathcal{O}_{\mathbb{P}^2}(1) = 2d\), and since \(p\) is a degree two map, we have \(\deg q^* \mathcal{O}_{\mathbb{P}^2}(1) = d\).

Let us denote \(B := q^* \mathcal{O}_{\mathbb{P}^2}(1)\). We claim that \(B = L\). From the diagram, we see that \(p^* B = A \otimes \sigma^* A\). Computing global sections,

\[
H^0(C, A \otimes \sigma^* A) = H^0(C, p^* B) = H^0(D, B \otimes p_* \mathcal{O}_C) = H^0(D, B) \oplus H^0(D, B \otimes L^{-1}).
\]

If \(B \neq L\), then \(B \otimes L^{-1}\) is a non-trivial degree zero line bundle and so, \(H^0(D, B \otimes L^{-1}) = 0\). That is, all sections of \(A \otimes \sigma^* A\) are invariant under \(\sigma\), which can happen only if the action of \(\sigma\) on \(\mathbb{P}^1 \times \mathbb{P}^1\) is trivial. Hence, we have a contradiction and \(L = B\).

Consider the fibered product \(Z := D \times_{\mathbb{P}^2} (\mathbb{P}^1 \times \mathbb{P}^1)\). We then have natural morphisms \(\xi : Z \to \mathbb{P}^1 \times \mathbb{P}^1\), \(\psi : Z \to D\) and \(\phi : C \to Z\).

\[
\begin{array}{ccc}
C & \xrightarrow{p} & D \\
\downarrow q & & \downarrow q' \\
\mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{p'} & \mathbb{P}^2
\end{array}
\]

\(Z\) is reduced and irreducible. Indeed, we prove this by contradiction by using Lemma 4.3. Suppose \(Z\) is as in case (2) of the Lemma, i.e. \(Z = Z_1 \cup Z_2\). Then, since \(C\) is irreducible, \(\phi : C \to Z\) must map to one of the \(Z_i\)s, say \(Z_1\). Let \(\xi' : Z_1 \to \mathbb{P}^1 \times \mathbb{P}^1\) be the restriction of \(\xi\) to \(Z_1\). So, we have \(A = \phi^* \xi'^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 0)\). But, \(\psi : Z_1 \to D\) is an isomorphism implies \(A\) is a pull back from \(D\), which we have assumed is not the case. Thus, case (2) of Lemma 4.3 is not possible. If we are in case (3) of the Lemma, again, the map \(\phi : C \to Z\) factors through \(Z_{\text{red}}\) and the same argument as above shows that this case is again not possible.
Thereby, $Z$ is reduced and irreducible. Then, $\deg \phi = 1$ and $C$ is just the normalization of $Z$. The morphism $p'$ is branched along a smooth quadric $Q \subset \mathbb{P}^2$. Restricting to the complement, we see that $\psi$ is etale on $D \setminus q'^{-1}(Q)$ and this open set of $Z$ is smooth. Thereby $\phi$ is an isomorphism on $\psi^{-1}(D \setminus q'^{-1}(Q))$, which in turn implies that $p$ is an etale double cover outside $q^{-1}(Q)$. Since $\deg q^*\mathcal{O}_{\mathbb{P}^2}(1) = d$, $q'^{-1}(Q)$ has degree $2d$. Since $p$ is branched along a divisor of degree $2d$ and is contained in $q'^{-1}(Q)$, we see that $q'^{-1}(Q)$ must be precisely the branch locus of $p$. But the branch locus is given by $r \in H^0(L^2)$. Clearly $r = lm + a^2$, by pulling back the equation of the quadric $Q$ to $D$.

Next we consider the case $A = \sigma^*A$. Take a general section $s$ of $A$. If $s$ and $\sigma^*s$ are linearly dependent, $\text{div} s$ is invariant under $\sigma$. Since $s$ was general, we may assume this divisor is a sum of distinct points and none of them is in the ramification locus. But then clearly $\text{div} s = p^*p(\text{div} s)$ and then $A = p^*(\mathcal{O}_D(p(\text{div} s)))$ is the pull back of a line bundle from $D$, contradicting our hypothesis. Thus, $s$ and $\sigma^*s$ are linearly independent, and since $h^0(A) = 2$ by Lemma 4.2 we may assume that $s$ and $\sigma^*s$ generate $H^0(A)$. This gives a map $q : C \to \mathbb{P}^1$ which equivariant for the action of $\sigma$, where $\sigma$ acts on $\mathbb{P}^1$ by switching coordinates. As before, we have a commutative diagram.

\[
\begin{array}{ccc}
C & \xrightarrow{p} & D \\
q \downarrow & & \downarrow q' \\
\mathbb{P}^1 & \xrightarrow{p'} & \mathbb{P}^1
\end{array}
\]

It is immediate that $\deg q' = d$. Let $Z = D \times_{\mathbb{P}^1} \mathbb{P}^1$, and we use the notation as earlier.

\[
\begin{array}{ccc}
C & \xrightarrow{p} & D \\
q \downarrow & & \downarrow q' \\
\mathbb{P}^1 & \xrightarrow{p'} & \mathbb{P}^1
\end{array}
\]

Exactly as in the previous case, $Z$ is reduced irreducible and $\phi : C \to Z$ has degree 1. Since $p'$ is branched along two points, a quadric $Q$ (given by product of two linear forms) as in the previous paragraph, one checks that the map $\psi : Z \to D$ is etale outside $q'^{-1}(Q)$ and so by degree consideration, we see that $q'^{-1}(Q)$ is the branch locus of $p$ and hence $r = q'^{-1}(Q)$ is a section of $L^2$.

As before, we check that $q^*\mathcal{O}_{\mathbb{P}^1}(1) = L$.

\[
H^0(C, A) = H^0(\mathbb{P}^1, q^*p^*\mathcal{O}_{\mathbb{P}^1}(1)) \\
= H^0(p^*q^*\mathcal{O}_{\mathbb{P}^1}(1)) \\
= H^0(D, q^*\mathcal{O}_{\mathbb{P}^1}(1)) \oplus H^0(D, q^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes L^{-1})
\]

If $q^*\mathcal{O}_{\mathbb{P}^1}(1) \neq L$, then $H^0(C, A) = H^0(D, q^*\mathcal{O}_{\mathbb{P}^1}(1))$ and all sections of $A$ are $\sigma$ invariant which is not possible. Hence, $q^*\mathcal{O}_{\mathbb{P}^1}(1) = L$. Thus, $r \in H^0(L^2)$ which is obtained by pulling back $Q$ is given by an equation $r = lm$ for two sections $l, m \in H^0(L)$.
We now prove the converse (2) implies (1). So assume \( r = lm + a^2 \) for \( l, m, a \in H^0(L) \). Since \( \text{div} r \) is smooth, we see that \( l, m, a \) generate \( L \), because if they all vanished at a point, then \( r \) would vanish doubly at that point. Thus, the sections \( l, m, a \) give a morphism \( q : D \to \mathbb{P}^2 \), with \( q^* \mathcal{O}_{\mathbb{P}^2}(1) = L \). By abuse of notation, we think of \( l, m, a \) as sections from \( H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \) and \( r \) as a quadric in \( \mathbb{P}^2 \).

The dimension of the subspace generated by \( l, m, a \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \) cannot be zero and can not be one, since they generate \( L \), a positive degree line bundle. Hence \( h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) = 2 \text{ or } 3 \). If it is three, then \( lm + a^2 \) is a non-singular quadric and if it is two, then we may assume (by abuse of notation) it is \( r = lm \) (any homogeneous polynomial in two variables is product of linear factors). Also, since \( \text{div} r \) on \( D \) is smooth, we see that in the latter case, the point of intersection of \( l, m \) is not in the image of \( D \).

We first look at the case when \( l, m, a \) are linearly independent and so \( r = lm + a^2 \) is a smooth quadric on \( \mathbb{P}^2 \). We have a fiber product diagram, with \( p' : \mathbb{P}^1 \times \mathbb{P}^1 = Z \to \mathbb{P}^2 \), the double cover branched along \( r = 0 \).

\[
\begin{array}{ccc}
C & p \rightarrow & D \\
q' \downarrow & & \downarrow q \\
Z & \rightarrow & \mathbb{P}^2 \\
\end{array}
\]

The two projections \( C \to \mathbb{P}^1 \) give two globally generated line bundles \( A \) and \( B \) on \( C \). Since \( A, B \) are globally generated their degrees are non-negative. In fact the degree is not zero. Indeed, if \( \text{deg} A = 0 \), then the projection corresponding to \( A \) is constant and thus the map \( q' \) factors through \( \{t\} \times \mathbb{P}^1 \) for some \( t \in \mathbb{P}^1 \). But then \( p' \circ q'(C) \) is contained in a line in \( \mathbb{P}^2 \). This means \( q(D) \) is contained in a line, which contradicts our assumption that \( l, m, a \) are linearly independent.

So \( \text{deg} A > 0 \) and \( \text{deg} B > 0 \). Also,

\[
A \otimes B = q'^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1) = q'^* p'^* \mathcal{O}_{\mathbb{P}^2}(1) = p^* q^* \mathcal{O}_{\mathbb{P}^2}(1) .
\]

This says \( \text{deg} A \otimes B = 2d \) and \( A \otimes B \) is a pull back from \( D \). Since both \( A, B \) are of positive degree, we see that at least one of them must have degree at most \( d \).

We claim that neither \( A \) nor \( B \) is a pull back from \( D \). If one of them is, then both are, since \( A \otimes B \) itself is a pullback from \( D \). Let \( A = p^* A' \) and \( B = p^* B' \). We have \( \text{deg} A' \), \( \text{deg} B' \) are both less than \( d \). Then

\[
\begin{align*}
H^0(C, A) = & \ H^0(D, p_* A) \\
= & \ H^0(D, p_* p^* A') \\
= & \ H^0(D, A') \oplus H^0(D, A' \otimes L^{-1}) .
\end{align*}
\]

Since \( \text{deg} (A' \otimes L^{-1}) < 0 \), the last term above is just \( H^0(D, A') \). So, all sections of \( A \) descend to \( A' \) and then these sections generate \( A' \), since they generate \( A \). Applying this to \( B, B' \), we get a morphism \( \phi : D \to \mathbb{P}^1 \times \mathbb{P}^1 \) such that \( \phi \circ p = q' \). By universal property of fiber product, we get a morphism \( D \to C \) which is absurd. Hence, neither \( A \) nor \( B \) is a pull back from \( D \).

If \( \text{deg} A \leq d \), applying Lemma 4.2 we see that \( \text{deg} A = d \) which proves what we need.
Next assume $r = lm$. The proof is similar in this case as well. We have seen that $l, m$
 will generate $L$ and thus we get a morphism $q : D \to \mathbb{P}^1$ with $q^*\mathcal{O}_{\mathbb{P}^1}(1) = L$ and $l = 0$
 defines $0$ and $m = 0$ defines $\infty$. Take the double cover $\mathbb{P}^1 \to \mathbb{P}^1$ simply ramified at zero
 and $\infty$. Then, the pull back is just $C$ and we have a morphism $C \to \mathbb{P}^1$, resulting in the
 following Cartesian diagram.

\[
\begin{array}{ccc}
C & \xrightarrow{p} & D \\
q' \downarrow & & \downarrow q \\
\mathbb{P}^1 & \xrightarrow{\pi'} & \mathbb{P}^1
\end{array}
\]

Again this gives a line bundle $A = q'^*\mathcal{O}_{\mathbb{P}^1}(1)$ of the desired type. We just need to check
is that $A$ is not a pull back from $D$. Clearly $A \neq \mathcal{O}_C$ and as before, if $A = p^*A'$ then
$H^0(A) = H^0(A')$. Thus, we get a morphism $\phi : D \to \mathbb{P}^1$ such that $\phi \circ p = q'$. Then we
get a morphism $D \to C$ by universal property of fiber products, which is absurd. $\square$

4.2. Rank two Ulrich bundles on double covers of the plane. We are now ready
to give a proof of Theorem 1.1.

Proof of Theorem 1.1 Let $\pi : X \to \mathbb{P}^2$ be a smooth double cover and $\deg B = 2d$.
Then by the results in §3 we see that the branch curve $B = (F = 0)$ can be written
as $F = F_1G_1 + F_2G_2$ where $F_i, G_i$ are homogeneous polynomials of degree $d$ and $F_1$
 is smooth, and $F_2$ and $G_2$ intersect transversely at $d^2$ points.

Now let $D = (F_1 = 0)$. Then the branch locus $B$ when restricted to $D$ is given by
$r = (F_2G_2 = 0)$ which is in the form $r = lm + a^2$ as required by Theorem 4.4. Let
$C = X \times_{\mathbb{P}^2} D$. Then $C$ is a double cover of $D$ branched along $\div r$ and we have a
morphism $p : C \to D$. By Theorem 4.4 there is a basepoint free line bundle $A$ on $C$ of
degree $d$ not composed with $p$. Lemma 4.2 tells us that $p_*A = \mathcal{O}_D^2$.

Thus by Theorem 4.1 $X$ admits a rank two Ulrich bundle. $\square$

Theorem 1.1 settles the questions of existence and minimal rank of Ulrich bundles on
all smooth double covers of $\mathbb{P}^2$.

5. Ulrich bundles on double covers of $\mathbb{P}^n$ when $n \geq 3$

In this section we show that any smooth double cover of $\mathbb{P}^n$ admits an Ulrich bundle.
The proof depends on the result of Herzog, Ulrich and Backelin [8] that any complete
intersection variety admits an Ulrich sheaf. We reproduce their proof in the special case
of quadric hypersurfaces. We begin with the following

Lemma 5.1. Let $X$ be an irreducible quadric hypersurface defined by $(Q = 0)$ in $\mathbb{P}^n$
where $n \geq 3$ and $Q \in \mathbb{C}[x_0, x_1, \ldots, x_n]$. If there is a $2^s \times 2^s$ matrix $A$ on $\mathbb{P}^n$
with linear entries such that $A^2 = Q \cdot \text{Id}$, then $X$ admits an Ulrich sheaf of rank $2^{s-1}$.

Proof. Let $F = \mathcal{O}_{\mathbb{P}^n}^2$. Then we have the following short exact sequence on $\mathbb{P}^n$

\[
0 \to F(-1) \xrightarrow{\Delta} F \to G \to 0.
\]
Note that multiplication by $A$ is injective, since multiplication by $A^2$ is. Then consider the following diagram:

```
|   | 'Kernel' |
|---|----------|
| 0 | F(-2)    | F(-1) |
| ∫  | A        | A  |
|    | F        | F    |
|    | F|        | G    |
| 0 | F        | G    |
```

By snake lemma, ‘Kernel’ $\simeq G(-1)$. Hence we have the following short exact sequence on $X$

$$0 \to G(-1) \to F|_X \to G \to 0.$$  

Since $F$ has rank $2^s$ on $X$, the sheaf $G$ on $X$ has rank $2^{s-1}$. We now show that $G$ is Ulrich on $X$. By projection, we have a degree two morphism

$$\pi : X \to \mathbb{P}^{n-1}$$

So, on $\mathbb{P}^{n-1}$ we have a short exact sequence:

$$0 \to \pi_*G(-1) \to \pi_*\mathcal{O}_{\mathbb{P}^{n-1}}^{2^s} \to \pi_*G \to 0.$$  

Notice that since $X$ is a quadric,

$$\pi_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-1).$$

Next, using the exact sequence [5], we see that $H^i(G(*)) = 0$ for $0 < i < \dim X = n - 1$. This says $H^i(\pi_*G(*)) = 0$ for $0 < i < n - 1$, and so by Horrocks’ criterion (cf. [9] or [13, Theorem 2.3.1]), $\pi_*G$ is a direct sum of line bundles. Further, since $n - 1 \geq 2$, we have $\Ext^1(\pi_*G, \pi_*G(-1)) = 0$, and thereby the exact sequence [6] splits:

$$\pi_*G \oplus \pi_*G(-1) \simeq \mathcal{O}_{\mathbb{P}^{n-1}}^{2^s} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{2^s}.$$  

Hence $\pi_*G = \mathcal{O}_{\mathbb{P}^{n-1}}^\alpha \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-1)^\beta$ for some $\alpha$ and $\beta$.

But, by projection formula,

$$\pi_*G(-1) = \pi_*(G \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(-1)|_X) \simeq \pi_*(G \otimes \pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(-1)) \simeq \pi_*G \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(-1).$$

Thereby the exponent $\beta$ must be zero in the expression of $\pi_*G$ since otherwise $\pi_*G(-1)$ will have $\mathcal{O}_{\mathbb{P}^{n-1}}(-2)$ as a summand and cannot be a direct summand of $\mathcal{O}_{\mathbb{P}^{n-1}}^{2^s} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-1)^{2^s}$.

So, $\pi_*G = \mathcal{O}_{\mathbb{P}^{n-1}}^{2^s}$ showing $G$ is Ulrich of rank $2^{s-1}$.  

**Theorem 5.2.** Let $X$ be an irreducible quadric hypersurface defined by $(Q = 0)$ in $\mathbb{P}^n$ where $n \geq 3$ and $Q \in \mathbb{C}[x_0, x_1, \ldots, x_n]$. Then $X$ admits an Ulrich sheaf.

**Proof.** Note that there exists a linear subspace $\mathbb{P}^{n_0} \subset \mathbb{P}^n$ and a smooth quadric in $\mathbb{P}^{n_0}$ such that $X$ is a cone over this smooth quadric. In particular, $Q$ involves only variables from $\mathbb{P}^{n_0}$ and involves all of them. Depending on the parity of $n_0$, we can write

$$Q = l_1m_1 + l_2m_2 + \cdots + l_sm_s,$$
where $l_i$’s form a basis of $H^0(\mathbb{P}^{n_0}, \mathcal{O}_{\mathbb{P}^{n_0}}(1))$ and

1. if $n_0 = 2p$, then $l_s = m_s$ and $s = p + 1$;
2. if $n_0 = 2p + 1$, then $s = p + 1$.

In either case, $s = \lceil \frac{n_0}{2} \rceil + 1$.

We claim that there is an Ulrich sheaf of rank $2s - 1$ on $X$. By Lemma 5.1, it is enough to find a matrix $A$ of size $2s$ on $\mathbb{P}^n$ with linear entries such that $A^2 = Q \cdot \text{Id}$. We construct such an $A$ by induction on $s$. If $Q = l_1m_1$, take

$$A = \begin{bmatrix} 0 & l_1 \\ m_1 & 0 \end{bmatrix}. $$

Suppose that we have constructed a matrix $A_1$ for $Q_1 = l_1m_1 + \cdots + l_pm_p$ with $A_1^2 = Q_1 \cdot \text{Id}$ and if $Q = l_1m_1 + \cdots + l_pm_p + l_{p+1}m_{p+1}$, then, take

$$(7) \quad A = \begin{bmatrix} A_1 & l_{p+1} \text{Id} \\ m_{p+1} \text{Id} & -A_1 \end{bmatrix},$$

which indeed satisfies $A^2 = Q \cdot \text{Id}$. □

We can now prove that any smooth double cover of the projective space admits an Ulrich bundle.

**Proof (I) of Theorem 1.2.** Let $N := (\frac{d+n}{2}) - 1$ and consider the $d$-tuple embedding

$$i : \mathbb{P}^n \hookrightarrow \mathbb{P}^N.$$

Under this embedding $i^* \mathcal{O}_{\mathbb{P}^N}(1) \simeq \mathcal{O}_{\mathbb{P}^n}(d)$ and the morphism

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(2)) \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2d))$$

is surjective since the embedding is projectively normal. Thus there exists a quadric hypersurface $\widetilde{B} \subset \mathbb{P}^N$ such that the degree $2d$ hypersurface $B \subset \mathbb{P}^n$ is the restriction of $\widetilde{B}$, i.e. $B = \widetilde{B} \cap \mathbb{P}^n$. The hypersurface $\widetilde{B}$ may not be smooth. Consider the double cover $\widetilde{X}$ of $\mathbb{P}^N$ branched along $\widetilde{B}$. Then we have the following commutative diagram.

$$\begin{array}{ccc}
\mathbb{P}^n & \xrightarrow{i} & \widetilde{X} \\
\downarrow \pi & & \downarrow \pi' \\
\mathbb{P}^N & \xrightarrow{j} & \mathbb{P}^N \\
\end{array}$$

Note that $\widetilde{X}$ is a quadric hypersurface in $\mathbb{P}^{N+1}$. In fact if the hypersurface $\widetilde{B} \subset \mathbb{P}^N$ is defined by the equation $(F = 0)$ where $F \in \mathbb{C}[Y_0, Y_1, \ldots, Y_N]$ is a quadric, then $\widetilde{X}$ is given by the equation $(T^2 - F = 0)$ where the homogeneous coordinate ring of $\mathbb{P}^{N+1}$ is $\mathbb{C}[Y_0, Y_1, \ldots, Y_N][T]$. By Theorem 5.2, $\widetilde{X}$ admits an Ulrich sheaf, say $E'$. Hence $\pi'_* E'$ is a trivial vector bundle on $\mathbb{P}^N$.

Consider the pullback $i^* E'$ to $X$. This is a sheaf satisfying $\pi_* i^* E' = j^* \pi'_* E'$ [16, Tag 02KE]. Since the direct image of $E'|_X$ under $\pi$ is trivial, $E'|_X$ is Maximal Cohen Macaulay on $X$. But an MCM module over a regular local ring is free, and thereby $E'|_X$ is locally free, and an Ulrich bundle on $X$. □
We can also use Theorem 4.1 to prove that every smooth double cover of $\mathbb{P}^n$ admits an Ulrich bundle. We first need a Lemma, whose proof is very similar to the above Proof (I) of Theorem 1.2.

**Lemma 5.3.** Let $f : C \to D$ be a double cover of smooth projective curves such that the branch locus $B \in |L^2|$ where $L$ is a very ample line bundle on $D$. Assume that the multiplication map

$$H^0(L) \otimes H^0(L) \to H^0(L^2)$$

is surjective. Then there is a vector bundle $E$ on $C$ such that $f_*E$ is trivial.

**Proof.** Let $V = H^0(L)^*$. The very ample line bundle $L$ embeds $D$ in $\mathbb{P}(V)$. Under this embedding $i : D \hookrightarrow \mathbb{P}(V)$, the branch locus $B$ is the restriction of a quadric $\tilde{B} \subset \mathbb{P}(V)$ to $D$ by the surjection (8). Consider the double cover $X$ of $\mathbb{P}(V)$ branched along $\tilde{B}$, then the curve $C$ is simply the fibred product $C = X \times_{\mathbb{P}(V)} D$ and we have

$$\begin{array}{ccc}
C & \longrightarrow & X \\
\downarrow f & & \downarrow f' \\
D & \longrightarrow & \mathbb{P}(V)
\end{array}$$

Since $f' : X \to \mathbb{P}(V)$ is a double cover branched over a quadric, $X$ is a quadric hypersurface in $\mathbb{P}(V \oplus \mathbb{C})$. The rest of the proof is similar to Proof (I) of Theorem 1.2. □

This lemma leads us to an alternate way of constructing Ulrich bundles on double covers of projective spaces.

**Proof (II) of Theorem 1.2**. Choose sections $s_1, s_2, \ldots, s_{n-1} \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ such that $Y_i := Z(s_1) \cap Z(s_2) \cap \cdots \cap Z(s_i)$ is a general irreducible smooth complete intersection inside $\mathbb{P}^n$. Here by general we mean that, if $X_i$ denotes the fibred product $X_i = X \times_{\mathbb{P}^n} Y_i$, then $\pi_i : X_i \to Y_i$ is a double cover. In particular, we have the following commutative diagram.

$$\begin{array}{ccc}
X_{n-1} & \longrightarrow & X_{n-2} \\
\downarrow \pi_{n-1} & & \downarrow \pi_{n-2} \\
Y_{n-1} & \longrightarrow & Y_{n-2}
\end{array}$$

So $Y_{n-1}$ is a smooth complete intersection curve and $\pi_{n-1}$ is a double cover of smooth curves branched along a divisor, say $B_{n-1} \in |\mathcal{O}_{\mathbb{P}^n}(2d)|_{Y_{n-1}}$. The line bundle $L = \mathcal{O}_{\mathbb{P}^n}(d)|_{Y_{n-1}}$ is very ample and satisfies the condition that

$$H^0(L) \otimes H^0(L) \to H^0(L^2)$$

is surjective.

Hence, there is a vector bundle $V$ on $X_{n-1}$ such that $\pi_{n-1}^*V$ is trivial. Further, for $1 \leq i \leq n - 2$, by induction $Y_i$ also satisfies $H^1(Y_i, \mathcal{O}_{Y_i}) = 0$. Thereby, we can apply Theorem 4.1 and construct a vector bundle $E$ on $X$ whose direct image is trivial. □

Consider a smooth double cover $\pi : X \to \mathbb{P}^n$ branched along a smooth hypersurface $B \in |\mathcal{O}_{\mathbb{P}^n}(2d)|$. Then $B$ is defined by a degree $2d$ homogeneous polynomial $F \in \mathbb{C}[x_0, x_1, \ldots, x_n]$. 
Proposition 5.4. Assume that $F$ can be written in the form
$$F = F_1G_1 + F_2G_2 + \cdots + F_nG_n,$$
for $F_i, G_i \in |\mathcal{O}_{\mathbb{P}^n}(d)|$, $i = 1, 2, \ldots, n$, where

- $Y_i = \cap_{j=1}^i Z(F_j)$ for $i = 1, 2, \ldots, n - 1$ define general smooth irreducible complete intersection varieties of type $(d, d, \ldots, d)$.
- $(F_nG_n = 0)$ defines a smooth subvariety of $Y_{n-1}$.

Then $X$ admits an Ulrich bundle of rank $2^{n-1}$.

Proof. Consider the following fibred diagram
$$
\begin{array}{ccc}
X_{n-1} & \xrightarrow{p^{n-1}} & X_{n-2} \\
p^{n-1} & \downarrow & \downarrow p^{n-2} \\
Y_{n-1} & \xrightarrow{p^{n-2}} & Y_{n-2} \\
p^{1} & \downarrow & \downarrow p^{1} \\
\vdots & & \vdots \\
\end{array}
$$

Without loss of generality, one may assume that, $p^i : X_i \to Y_i$ is a double cover. Note that each $p^i : X_i \to Y_i$ is branched along $B \cap Y_i \in |\mathcal{O}_{\mathbb{P}^n}(2d)|_{Y_i}|$.

In particular, the branch locus $B \cap Y_{n-1}$ of $p^{n-1} : X_{n-1} \to Y_{n-1}$ is given by $(F_nG_n = 0)$ since $(F_i = 0)$ for $i = 1, 2, \ldots, n - 1$ on $Y_{n-1}$. Thereby $p^{n-1} : X_{n-1} \to Y_{n-1}$ is a double cover of smooth curves whose branch locus has the form $r = lm + a^2$. By Theorem 1.4 there is a line bundle $A$ on $X_{n-1}$ whose direct image is trivial. Since $H^1(Y_i, \mathcal{O}_{Y_i}) = 0$ for $1 \leq i \leq n - 2$, the repeated application of Theorem 1.4 gives an Ulrich vector bundle $E$ of rank $2^{n-1}$ on $X$. \qed

Both proofs of Theorem 1.2 in this section do not make any assertions on the rank of the Ulrich bundle on double covers of $\mathbb{P}^n$ so obtained. We deal with that in the next section.

6. Estimating the rank of Ulrich bundles on double covers of higher $\mathbb{P}^n$

From the previous sections, we see that the rank of the Ulrich bundle obtained on the double cover depends on the expression of the degree $2d$ branch locus $F$ in the form
$$F = \Sigma_i F_iG_i,$$
where $F_i$ and $G_i$ are degree $d$ homogeneous forms. We now use this to obtain an upper bound for the rank of an Ulrich bundle $E$ on a double cover $\pi : X \to \mathbb{P}^n$ that we constructed in the Proof (1) of Theorem 1.2.

Consider the space $V := H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2d))$ of degree $2d$ polynomials in $n + 1$ variables $x_0, x_1, \ldots, x_n$, and its projectivization $\mathbb{P}V$. We saw in §2.2 that there is a stratification of $\mathbb{P}V$ by the secant varieties $\text{Sec}_r X$, where
$$X = \{[F] \in \mathbb{P}V \mid F = F_1F_2, F_i \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))\}.$$ 

In particular, we have a chain of inclusions
$$X \subset \text{Sec}_1X \subset \cdots \subset \text{Sec}_{r-1}X \subset \text{Sec}_X \subset \cdots \subset \mathbb{P}V$$
and if $[F] \in \text{Sec}_{r-1}X$, then $F$ can be written as $F = \Sigma_{i=1}^r F_iG_i$ where $F_i, G_i \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$. 


Note that any degree $2d$ hypersurface in $\mathbb{P}^n$ can be thought of as a pullback of a quadric from $\mathbb{P}^N$, where $\mathbb{P}^n \hookrightarrow \mathbb{P}^N$ is the $d$-uple embedding. Then, the expression of a quadric as in Theorem 5.2 gives that the above filtration of $\mathbb{P}^N$ by secant varieties is finite, as stated in the lemma below.

**Lemma 6.1.** There is a positive integer $m$ such that $\text{Sec}_m X$ will cover $\mathbb{P}^n V$. In particular, $m = \left[ \frac{N}{2} \right]$ where $N = \binom{d+n}{n} - 1$ and the above chain of inclusions becomes $X \subset \text{Sec}_1 X \subset \cdots \subset \text{Sec}\left[ \frac{N}{2} \right] X = \mathbb{P}^n V$.

As before, let $\pi : X \rightarrow \mathbb{P}^n$ be a smooth double cover branched along a smooth hypersurface $B = (F = 0)$ of degree $2d$.

**Lemma 6.2.** Let $r = \min\{s \mid [F] \in \text{Sec}_s X\}$. Then $X$ admits an Ulrich bundle of rank $2^r$ or $2^{r+1}$.

**Proof.** Since $r$ is the minimum integer such that $F \in \text{Sec}_r X$, there is an irredundant and minimal expression of $F$ in the form
\begin{equation}
F = \sum_{i=1}^{r+1} F_i G_i,
\end{equation}
where $F_i, G_i \in H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(d))$. Let $N := \binom{d+n}{n} - 1$ and consider the $d$-tuple embedding $i : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$.

There is a quadric hypersurface $\widetilde{B} \subset \mathbb{P}^N$ such that the degree $2d$ hypersurface $B \subset \mathbb{P}^n$ is the restriction of $\widetilde{B}$, i.e. $B = \widetilde{B} \cap \mathbb{P}^n$. If $\widetilde{B} = (Q = 0)$, where $Q$ is a degree two polynomial in the homogeneous coordinate ring of $\mathbb{P}^N$, then $Q$ has the expression:

$$Q = \sum_{i=1}^{r+1} l_i m_i,$$

where $l_i, m_i \in H^0(\mathbb{P}^N, O_{\mathbb{P}^N}(1))$ and we have the restrictions $(l_i = 0)|_{\mathbb{P}^n} = (F_i = 0)$ and $(m_i = 0)|_{\mathbb{P}^n} = (G_i = 0)$. This expression is irredundant and minimal, since the expression of $F$ is. Therefore, in the expression of $Q$ as above, we either have

(a) $l_i \neq m_i$ for all $i = 1, 2, \ldots, r+1$, or
(b) $l_i \neq m_i$ for $i = 1, 2, \ldots, r$ and $l_{r+1} = m_{r+1}$.

The double cover $\widetilde{X}$ of $\mathbb{P}^N$ branched along $\widetilde{B}$ is a quadric hypersurface in $\mathbb{P}^{N+1}$ and is given by $(T^2 - \sum_{i=1}^{r+1} l_i m_i = 0)$. We now get the following two cases based on the Proof (I) of Theorem 1.2

(a) Suppose that $Q$ has the form $Q = \sum_{i=1}^{r+1} l_i m_i$ where $l_i \neq m_i$ for all $i$. Then, the equation of $\widetilde{X}$ is $(T^2 - Q = 0)$. Note that the expression of the polynomial $T^2 - Q = T^2 - \sum_{i=1}^{r+1} l_i m_i$ is itself irredundant and minimal (as a sum of products of two linear forms). Hence, the double cover $\widetilde{X}$ and hence $X$ admit an Ulrich bundle of rank $2^{r+1}$.

(b) On the other hand, assume that $Q$ has the form $Q = \sum_{i=1}^{r} l_i m_i + l_{r+1}^2$. Then the equation of $\widetilde{X}$ which is $(T^2 - Q = 0)$ can be rewritten as

$$(T + l_{r+1})(T - l_{r+1}) - \sum_{i=1}^{r} l_i m_i = l_0 m_0 - \sum_{i=1}^{r} l_i m_i = 0.$$ 

Thus, in this case $\widetilde{X}$ and $X$ admit an Ulrich bundle of rank $2^r$.

$\square$
Remark 6.3. In §3 we discussed the expression of any smooth plane degree $2d$ curve $F$ in the form $F = F_1G_1 + F_2G_2$. So any $[F] \in \text{Sec}_X$. Then from the proof of the above Lemma, we see that $r = \min\{s | [F] \in \text{Sec}_X\} = 1$ in our case. Further, since the expression of a general $F$ is of the form as described in part (a) of the proof above, $X$ admits an Ulrich bundle of rank $2^{r+1} = 4$. Hence, the above method does not give the minimal rank Ulrich bundle, which we obtained from the first technique cf. Theorem 1.1.

A direct application of Lemma 6.1 gives us the following

Theorem 6.4. Let $\pi : X \to \mathbb{P}^n$ be a smooth double cover branched along a smooth hypersurface $B$ of degree $2d$. Then $X$ admits an Ulrich bundle of rank $\leq 2^{r+1}$ where $N = \left(\frac{d+n}{n}\right) - 1$.

There is a lower bound on the rank of Ulrich bundles on $\pi : X \to \mathbb{P}^n$ which can be constructed using the above techniques. This is shown in the following lemma and corollary.

Lemma 6.5. Let $F \in \mathbb{C}[x_0, x_1, ..., x_n]$ be a homogeneous polynomial of degree $2d$ which defines a smooth hypersurface in $\mathbb{P}^n$. If $F$ has an expression of the form

$$F = \Sigma_{i=1}^{r} F_iG_i$$

where $F_i, G_i \in \mathbb{C}[x_0, x_1, ..., x_n]$ are homogeneous of degree $d$, then $2r \geq n + 1$.

Proof. Consider the ideal

$$J = (F_1, G_1, F_2, G_2, ..., F_r, G_r) \subset \mathbb{C}[x_0, x_1, ..., x_n].$$

Since $F = \Sigma_{i=1}^{r} F_iG_i$ defines a smooth hypersurface in $\mathbb{P}^n$, the zero set of $J$ in $\mathbb{C}^{n+1}$ is

$$Z(J) = (0, 0, ..0).$$

Denote $m = (x_0, x_1, ..., x_n)$, the irrelevant maximal ideal. Then by Hilbert’s Nullstellensatz,

$$\sqrt{J} = I(Z(J)) = m.$$

This shows that $J$ is $m$-primary and consequently, $m$ is the minimal prime over $J$. Then, by Krull’s principal ideal theorem [6, Theorem 10.2],

$$\text{codim } m = n + 1 \leq 2r.$$

As a direct consequence of this Lemma and Lemma 6.2, we obtain the following

Corollary 6.6. Let $\pi : X \to \mathbb{P}^n$ be a double cover branched along a smooth hypersurface $B = (F = 0)$ of degree $2d$. Then the Ulrich bundle which can be produced by the above methods on $X$ has rank $\geq 2^{r-1}$ where $2r \geq n + 1$. 

7. Appendix

While proving that any smooth double cover of \( \mathbb{P}^2 \) carries a rank two Ulrich bundle, one crucial step we undertook was the analysis of double covers \( p : C \rightarrow D \) of smooth curves in order to identify when there is a line bundle \( A \) on the curve \( C \) whose direct image on \( D \) is trivial.

Our initial approach was to use the following theorem from the paper [10] which seemed appropriate for our situation.

**Theorem 7.1.** [10, Theorem A] Let \( C \) be a curve of genus \( g \) which admits a double covering \( p : C \rightarrow D \) such that the genus of \( D \) is \( h \geq 0 \) and \( g \geq 4h \). Then \( C \) has a basepoint free and complete \( g^1_{g-2h+1} \) not composed with \( p \).

We applied the above Theorem in our context by considering a general degree \( d \) curve \( D \hookrightarrow \mathbb{P}^2 \) and its double cover as follows.

\[
\begin{array}{ccc}
C' & \xrightarrow{i} & X \\
\downarrow{\pi_D} & & \downarrow{\pi} \\
D & \xrightarrow{j} & \mathbb{P}^2
\end{array}
\]

In this case, one can check that \( \text{genus}(C) \geq 4 \text{genus}(D) \). Theorem 7.1 then indicates that there is a complete and basepoint free \( g^1_{g-2h+1} \) not composed with \( p \) on \( C \). Such a line bundle \( A \) has the property that \( p^*A \) is a trivial vector bundle on \( D \), cf. Lemma 4.2. This enabled us to obtain results about the existence of Ulrich bundles on \( X \).

However, as mentioned in the main text, we later identified that the proof of the Theorem 7.1 is not quite correct. In this appendix, we provide a counterexample to Theorem 7.1. We first recall the statement of the Theorem 4.4 which we have proved in §4.

**Theorem 4.4.** Let \( p : C \rightarrow D \) be a double cover of smooth curves and \( p^*\mathcal{O}_C = \mathcal{O}_D \oplus L^{-1} \) for a line bundle \( L \) on \( D \) with degree \( L = d = g - 2h + 1 \) where \( g \) and \( h \) are the genera of \( C \) and \( D \) respectively. Let \( r \in H^0(L^2) \), unique upto non-zero constants be such that \( \text{div} \ r \) is the branch locus.

1. There exists a line bundle \( A \) of degree \( d \) on \( C \) which is basepoint free not composed with \( p \).
2. There exist elements \( l, m, a \in H^0(L) \) such that the branch locus \( \text{div} \ r \) is given by \( r = lm + a^2 \).

We now describe the counterexample to Theorem 7.1 by exhibiting a double cover \( p : C \rightarrow D \) with genus \( (C) \geq 4 \) genus \( (D) \) whose base locus \( \text{div} \ r \) cannot be written in the form \( r = lm + a^2 \).

**The Counterexample.** Let \( D \) be a smooth projective curve of genus 2. Consider the line bundle \( L = \mathcal{O}_D(2K_D + P) \) for any point \( P \in D \). Let \( s, t \in H^0(K_D) \) be sections which form a basis for \( H^0(K_D) \). We can check that \( h^0(2K_D) = 3 \) and so \( s^2, t^2, st \in H^0(2K_D) \) form a basis. We have a natural inclusion \( H^0(2K_D) \subset H^0(L) \), so we can consider \( s^2, t^2, st \in H^0(L) \) and they are linearly independent. Now \( L \) is a line bundle
of degree 5 and $h^0(L) = 4$. Embed $D$ inside $\mathbb{P}^3$ using $H^0(L)$. Since $s^2t^2 = (st)^2$, the curve $D$ is contained in a quadric $Q$ of rank 3. If the homogeneous coordinates on $\mathbb{P}^3$ are $x, y, z, t$, then the quadric is $xy - z^2 = 0$ which by change of coordinates can be written as $x^2 + y^2 + z^2 = 0$. This quadric is singular and irreducible.

Let $V = H^0(L)$, the vector space generated by $x, y, z, t$. It is known that $H^0(L) \otimes H^0(L) \to H^0(L^2)$ is onto [12, Page 55, Corollary]. Thereby we get that

$$S^2V \to H^0(L^2)$$

is onto. Note that $\dim S^2V = 10$ since $\dim V = 4$ and $h^0(L^2) = 9$. So the kernel of (10) is one dimensional. Since $D \subset Q = (x^2 + y^2 + z^2 = 0)$, the kernel is generated by $Q$.

Suppose any element of $r \in H^0(L^2)$ can be written in the form $lm + a^2$ with $l, m, a \in V$. Then starting with an element $r \in S^2V$, there exist $l, m, a \in V$ such that $r - lm - a^2$ goes to zero in $H^0(L^2)$, i.e. $r - lm - a^2 = \alpha Q$ for some $\alpha \in \mathbb{C}$. That is,

$$r - \alpha Q = lm + a^2.$$  

But any element of the form $lm + a^2$ is a singular quadric. For a general $r$, we have $r - \alpha Q$ is non-singular. For instance, if $r = xy + ity + zt$, we can check that for all $\alpha \in \mathbb{C}$, $r - \alpha Q$ is non-singular.

This shows that, a general $r \in H^0(L^2)$ cannot be written in the form $lm + a^2$. Let $p : C \to D$ be the double cover of $D$ branched along such an $r$. If $g = \text{genus}(C)$ and $h = \text{genus}(D)$, the Riemann-Hurwitz gives

$$2g - 2 = 2(2h - 2) + 2d.$$  

But $h = 2$ is our assumption and $d = \deg L = 5$. Then,

$$2g - 2 = 4 + 10 = 14,$$

that is

$$g = 8 \geq 4h.$$  

So we have a double cover of smooth curves $p : C \to D$ with $g \geq 4h$ such that the branch locus $r$ cannot be written in the form $r = lm + a^2$. This is equivalent to saying that there is no line bundle $A$ on $C$ whose direct image is trivial. This gives a counterexample to the statement of Theorem 7.1.

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