CHARACTERISING SURFACE GROUPS BY THEIR VIRTUAL SECOND BETTI NUMBER

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WITH AN APPENDIX BY ISMAEL MORALES

Abstract. Define the virtual second betti number of a finitely generated group $G$ as $vb_2(G) = \sup \{ \dim H_2(H;\mathbb{Q}) | H \leq G \text{ of finite index} \} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. We show that if $G$ is a one-ended word-hyperbolic group obtained as the fundamental group of a graph of free groups with cyclic edge groups then $vb_2(G)$ is finite if and only if $G$ is the fundamental group of a closed surface, in which case $vb_2(G) = 1$. We extend this result to limit groups and prove that the virtual second betti number of a limit group $L$ is finite if and only if $L$ is either free, free abelian or the fundamental group of a closed and connected surface. As an application we give an alternative proof to Wilton’s result which states that surface groups are determined among limit groups (and hyperbolic fundamental groups of graphs of free groups with cyclic edge groups) by their profinite completions.

An appendix to this paper by Morales investigates relations between the virtual second betti number and pro-soluble groups, and explains why the results above imply that Jaikin-Zapirain’s methods for showing that groups in the pro-soluble genera of free and surface groups are residually-$p$ cannot be used for other hyperbolic limit groups.

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1. Introduction

The virtual $i$-th betti number of a group $G$ is defined as the supremum over the set consisting of the $i$-th betti numbers of all finite index subgroups of $G$. We denote the virtual $i$-th betti number of $G$ by $vb_i(G)$ (and the $i$-th betti number of $G$ by $b_i(G)$).

Wilton showed in [44] that a one-ended word-hyperbolic group obtained as the fundamental group of a finite graph of finitely generated free groups with infinite cyclic edge groups contains a surface subgroup, and therefore its virtual second betti number is positive. The aim of this paper is to extend this result and give a full classification of hyperbolic fundamental groups of graphs of free groups with cyclic edge groups by their virtual second
betti number. This classification is remarkably simple: apart from in the obvious cases, the virtual second betti number of a hyperbolic fundamental group of a graph of free groups with cyclic edge groups is always infinite. More precisely, we prove the following:

**Theorem A.** Let $G$ be a word-hyperbolic fundamental group of a finite graph of finitely generated free groups with infinite cyclic edge groups. Then $\text{vb}_2(G) \in \{0, 1, \infty\}$ and

1. $\text{vb}_2(G) = 0$ if and only if $G$ is free,
2. $\text{vb}_2(G) = 1$ if and only if $G \cong \pi_1(\Sigma)$ where $\Sigma$ is a closed, connected surface,
3. $\text{vb}_2(G) = \infty$ otherwise.

Therefore surface groups are characterized among hyperbolic fundamental groups of graphs of free groups with cyclic edge groups by having a positive and finite virtual second betti number. The idea behind the proof is that once a surface subgroup of infinite index appears in $G$, one can employ structural properties of $G$ to replicate it and produce subgroups of $G$ containing many surface subgroups, independent from each other in second homology. The techniques used for producing such subgroups come from the covering theory of graphs of spaces, and the methods are often inspired by previous works of Wilton ([43] and [44]).

Hypercyclic fundamental groups of graphs of free groups with cyclic edge groups also play a special role in the study of limit groups, namely finitely generated, fully residually free groups (for the full definition and a few key properties of limit groups see Subsection 2.2). This class of groups has been extensively studied over the past sixty years, and gained popularity due to its importance in Sela’s proof that all non-abelian free groups have the same first-order theory, answering a longstanding question of Tarski’s (see [39] et seq. as well as [24] et seq.). Limit groups exhibit a cyclic hierarchical structure: they can be built up from easy-to-understand building blocks (namely free and free abelian groups) taking cyclic amalgams and HNN extensions repeatedly. In particular, fundamental groups of graphs of free groups with cyclic edge groups lie (almost) at the bottom of this hierarchical structure of a limit group. This allows us to extend the statement of Theorem A and classify limit groups by their virtual second betti number:

**Theorem B.** Let $L$ be a limit group. $\text{vb}_2(L)$ is equal to

1. $0$ if and only if $L$ is free,
2. $1$ if and only if $L$ is the fundamental group of a closed, connected surface,
3. $\infty$ if and only if $L \cong \mathbb{Z}^d$ (for $d > 2$), and
4. $\infty$ otherwise.

At this point we would like to mention a few previous works on virtual betti numbers of groups: in [8], Bridson and Kochloukova show that the virtual first betti number of a finitely presented nilpotent-by-abelian-by-finite group is finite, and in [25], Kochloukova and Mokari show that the same holds for some abelian-by-polycyclic groups. Agol [1], Venkataramana [41] and Cooper, Long and Reid [13] each show that under some conditions the virtual first betti number of arithmetic hyperbolic 3-manifold groups is infinite. We are not aware of works concerning with virtual second betti numbers of groups. A work that is more closely related to this paper is [9] in which Bridson and Kochloukova show that the second $\ell_2$ betti number of a limit group $L$ is always 0. The definition of $\ell_2$ betti numbers is beyond the scope of this paper, but by Lück’s approximation theorem the second $\ell_2$ betti
number of a limit group \( L \) with coefficients in a field \( K \) is equal to \( \lim_{n \to \infty} \dim \frac{H_2(H_n; K)}{|L:H_n|} \)
where \( (H_n) \) is an exhausting nested sequence of normal subgroups of finite index in \( L \). The proof of Theorem B can easily be modified to produce a nested sequence \( (G_n) \) of finite index subgroups of a given limit group \( L \) satisfying \( \lim_{n \to \infty} \dim \frac{H_2(G_n; K)}{|L:G_n|} > 0 \). It follows that such a sequence \( (G_n) \) can never be exhausting, i.e. \( \bigcap_n G_n \neq \{1\} \).

1.1. An application to profinite rigidity and open questions. Theorems A and B also relate to the study of \textit{profinite rigidity} of groups, namely the study of the extent to which a group \( G \) is determined by its finite quotients. Showing that a residually finite group is \textit{absolutely} profinitely rigid, i.e. it is completely determined by its finite quotients, is notoriously hard. Most finitely generated examples of absolutely profinitely rigid groups are fairly simple, with the exception of examples by Bridson, McReynolds, Reid and Spitler [7, 10] which exhibit interesting geometry. A natural way in which to make the question of absolute profinite rigidity more approachable is to ask whether \( G \) is profinitely rigid \textit{within} a more restrictive class of groups. The existence of surface subgroups of non-free limit groups implies that free groups are profinitely rigid within the class of limit groups [44, Corollary E]. Furthermore, Wilton showed in [45] that the same holds for surface groups. Theorems A and B give an alternative proof to this assertion:

**Corollary C.** Let \( G \) be the fundamental group of a closed, connected surface and let \( \Gamma \) be a limit group or a word-hyperbolic group that splits as a graph of free groups with cyclic edge groups. If \( \hat{G} = \hat{\Gamma} \) then \( G \cong \Gamma \).

Before showing how Corollary C follows from Theorems A and B we collect a few relevant definitions and facts. A group \( G \) is \textit{homologically good} (in the sense of Serre) if the natural map on homology \( H_n(G; M) \to H_n(\hat{G}; M) \) induced by \( G \to \hat{G} \) is an isomorphism for every finite \( G \)-module \( M \) and every \( n \geq 1 \). This notion is usually defined in cohomological terms; we use the homological version because the results of this paper are stated in homological terms. In [15], Grunewald, Jaikin-Zapirain and Zalesskii showed that limit groups are cohomologically good. Minasyan and Zalesskii [31] and later Wilkes and Kropholler [26] proved that hyperbolic virtually special groups, and therefore hyperbolic fundamental groups of graphs of free groups with cyclic edge groups, are cohomologically good. Both these results remain true when rephrased in homological terms.

Denote by \( \text{vb}_{2,p}(G) \) the virtual second betti number of \( G \) with coefficients in \( \mathbb{F}_p \), that is
\[
\text{vb}_{2,p}(G) = \sup\{\dim \frac{H_2(H; \mathbb{F}_p)}{|H|} | H \leq G \text{ of finite index}\}
\]
and clearly if \( G \) is homologically good then \( \text{vb}_{2,p}(\hat{G}) = \text{vb}_{2,p}(G) \). We also remark that the proofs of Theorems A and B remain valid when the homology is calculated with coefficients in \( \mathbb{F}_p \). Corollary C now follows:

**Proof of Corollary C.** By the above discussion,
\[
\text{vb}_{2,p}(\Gamma) = \text{vb}_{2,p}(\hat{\Gamma}) = \text{vb}_{2,p}(\hat{G}) = \text{vb}_{2,p}(G) = 1.
\]
Therefore \( \Gamma \) is the fundamental group of a closed, connected surface. Surface groups are distinguished from each other by their profinite completions, hence \( \Gamma \cong G \). □
Due to the nature of the classification of limit groups and hyperbolic graphs of free groups with cyclic edge groups by their virtual second betti numbers, one cannot deduce further profinite rigidity results by looking at the virtual second betti number (apart from when $G$ is free or free abelian). We would therefore like to introduce another notion which might be of help:

**Definition 1.1.** Let $G$ be a finitely generated group. The second betti spectrum of $G$ is the set $\text{Spectrum}_{b_2}(G) = \{\dim H_2(H; \mathbb{Q}) | H \leq G \text{ of finite index}\}$.

Denote by $\text{Spectrum}_{b_2, p}(G)$ the second betti spectrum of $G$ with coefficients in $\mathbb{F}_p$, and note that if $G$ is a limit group or a hyperbolic graph of free groups with cyclic edge groups, $\text{Spectrum}_{b_2, p}(\hat{G}) = \text{Spectrum}_{b_2, p}(G)$. In other words, the second betti spectrum with coefficients in $\mathbb{F}_p$ is a profinite invariant of $G$. This raises the following question:

**Question 1.2.** Which numbers appear in the second betti spectrum of a given limit group or a given hyperbolic fundamental group of a graph of free groups with cyclic edge groups?

One can even strengthen the notion of a second betti spectrum, and define the filtered second betti spectrum of $G$ to be

$$\mathcal{F}\text{Spectrum}_{b_2}(G) = \left\{ \frac{\dim H_2(H; \mathbb{Q})}{[G : H]} | H \leq G \text{ of finite index} \right\}.$$  

Again, if $G$ is homologically good then the filtered second betti spectrum of $G$ (with coefficients in $\mathbb{F}_p$) is a profinite invariant of $G$.

**Question 1.3.** Which limit groups and hyperbolic fundamental groups of graphs of free groups with cyclic edge groups are characterized by their filtered second betti spectrums?

The proofs of Theorems A and B rely heavily on virtual retractions (see Section 3). Some information is lost in the process of transferring homological data via virtual retractions, so answering these questions should involve methods different to the ones used in this paper.

Lastly, the author would like to mention that this work is part of an ongoing project that aims to understand profinite completions of limit groups and other related groups via their surface subgroups.

The paper is organized as follows: in Section 2.1 we recall the definition of a graph of spaces and discuss properties of limit groups. Section 3 reviews separability properties of groups, and mentions relevant results regarding limit groups and hyperbolic fundamental groups of graphs of free groups with cyclic edge groups. In Section 4 we give a brief survey of the covering theory of graphs of spaces, and Section 5 is devoted to proving Theorems A and B. Appendix A by Ismael Morales relates this work to the pro-soluble and pro-nilpotent genera of hyperbolic limit groups.

**Acknowledgements.** I am grateful to Ismael Morales for many stimulating discussions preceding this work. I would also like to thank Martin Bridson for his generous support and Henry Wilton for an interesting conversation.

2. Preliminaries

For readability, we will often utilize skewed terminology when referring to the main objects of interest in this paper: we will use the term "graph of free groups with cyclic
edges" when we refer to the fundamental group of a finite graph of groups whose vertex groups are all finitely generated free groups and whose edge groups are all infinite cyclic.

2.1. Graphs of spaces. We will often realize graphs of free groups with cyclic edges as graphs of spaces. We therefore begin by defining graphs of spaces and recollecting a few basic facts. For a detailed account of the theory of graphs of spaces we refer the reader to Scott and Wall [36].

Definition 2.1. A graph of spaces $X$ consists of the following data:

- a graph $\Xi$,
- for each vertex $v \in V(\Xi)$ a connected CW-complex $X_v$,
- for each edge $e \in E(\Xi)$ a connected CW-complex $X_e$ and two $\pi_1$-injective maps $\partial^\pm_e : X_e \to X_v^\pm$ (called attaching maps) where $v^+$ and $v^-$ are the endpoints of the edge $e$.

Note that each graph of spaces $X$ has a topological space naturally associated to it, namely its geometric realization. The geometric realization of $X$ is defined as

$$\left( \bigsqcup_{v \in V(\Xi)} X_v \sqcup \bigsqcup_{e \in E(\Xi)} X_e \times [-1,1] \right) / \sim$$

where the equivalence relation $\sim$ identifies $(x, \pm 1) \in X_e \times [-1,1]$ with $\partial^\pm_e(x) \in X_v^\pm$. We will often abuse notation and use $X$ to refer to the geometric realization of $X$.

The fundamental group of $X$ clearly admits a graph of groups decomposition: $\pi_1(X)$ is the fundamental group of the graph of groups with underlying graph $\Xi$ and whose vertex and edge groups are $\{\pi_1(X_v) | v \in V(\Xi)\}$ and $\{\pi_1(X_e) | e \in E(\Xi)\}$ respectively. The edge maps of this graph of groups are given by $(\partial^\pm_e)_* : \pi_1(X_e) \to \pi_1(X_v^\pm)$. We will denote this graph of groups decomposition by $G(X)$ and refer to the vertex and edge groups of $G(X)$ as $G_v$ and $G_e$.

We remark that from this perspective, the fundamental group of a graph of free groups with cyclic edge groups can always be viewed as the fundamental group of a graph of spaces whose vertex spaces are graphs and whose edge spaces are all isomorphic to $S^1$.

2.2. Limit groups and hierarchies. A limit group $L$ is a finitely generated and fully residually free group: for every finite subset $E \subset L$ there is a homomorphism from $L$ to a free group whose restriction to $E$ is injective. Groups with this property have been extensively studied since the 1960s, and were popularized by Sela as they played an integral role in his solution to Tarski’s problem about the first-order theory of non-abelian free groups. Limit groups have a rich structural theory and they are exactly the finitely generated subgroups of fundamental groups of $\omega$-residually free towers. Loosely speaking, an $\omega$-residually free tower is a space obtained by repeatedly attaching simple building blocks to the wedge sum of circles, surfaces and tori. These simple building blocks are of one of two kinds: tori and surfaces with boundary. We will not use this structure of limit groups explicitly, and we refer the reader to Sela’s work [38] for the precise definition of an $\omega$-residually free tower. This structure theorem implies that limit groups admit a hierarchical structure which we will use in the proof of Theorem B.
Definition 2.2. A cyclic hierarchy of a group $G$ is a set $\mathcal{H}(G)$ of subgroups of $G$ obtained by iterating the following procedure, starting with $G$: for $H \in \mathcal{H}$, if $H$ admits a splitting $G(H)$ with (possibly trivial) cyclic vertex groups, add each of these vertex groups to $H$.

A priori the process described in Definition 2.2 above does not necessarily terminate. If this process comes to a halt after finitely many steps (in which case $\mathcal{H}$ is finite), $G$ is said to have a finite hierarchy. The groups appearing at the bottom of the hierarchy, that is groups which do not split over a cyclic subgroup, are called rigid. Sela showed in [39] that if $G$ is a limit group then this process always ends after finitely many steps:

Theorem 2.3. Limit groups have a finite hierarchy. Moreover, rigid groups appearing in the hierarchy of a limit group are free abelian.

Sela also showed that limit groups without $\mathbb{Z}^2$ subgroups are hyperbolic, which implies the following:

Corollary 2.4. If $L$ is a one-ended limit group and $H \in \mathcal{H}(L)$ is a one-ended group with no one-ended groups below it in the hierarchy, then one of the following holds:

1. $H$ is rigid, and therefore free abelian, or
2. $H$ is a hyperbolic group that decomposes as a graph of free groups with cyclic edge groups.

This close relation between limit groups and graphs of free groups with cyclic edge groups will help us deduce Theorem B from Theorem A.

3. Separability properties

Since we study the homology groups of finite index subgroups of a given group, it is natural to make use of separability properties of groups which have close connections with finite index subgroups.

Definition 3.1. A subgroup $H$ of a group $G$ is called separable (in $G$) if it is the intersection of finite index subgroups of $G$, that is $H = \bigcap \{G_0 | H \leq G_0 \text{ and } G_0 \leq G \text{ of finite index}\}$. $G$ is called subgroup separable, or LERF (locally extended residually finite), if every finitely generated subgroup of $G$ is separable in $G$.

Subgroup separability is a strong tool for generating finite index subgroups of a given group, as evident from the following elementary lemma which will prove to be useful later on:

Lemma 3.2. Let $G$ be a subgroup separable group and let $H$ be a finitely generated subgroup of $G$. Then for every finite index subgroup $H_0 \leq H$, there is a finite index subgroup $G_0 \leq G$ such that $G_0 \cap H = H_0$.

Proof. $H_0$ is a finitely generated subgroup of $G$ and therefore separable in $G$. Let $g_1, \ldots, g_n$ be coset representatives of all non-trivial cosets of $H_0$ in $H$, and for every $i \leq n$ let $G_i$ be a finite subgroup of $G$ which contains $H_0$ but doesn’t contain $g_i$. Letting $G_0 = \bigcap_{i=1}^{n} G_i$ we have that $G_0 \cap H = H_0$. □

Another powerful tool for generating finite index subgroups with certain homological properties are virtual retracts.
Definition 3.3. Let $G$ be a group and let $H \leq G$. We say that $G$ virtually retracts onto $H$ if there is a finite index subgroup $G_0$ of $G$ containing $H$ and a retraction $r : G_0 \to H$, that is a homomorphism $r$ such that $r(h) = h$ for every $h \in H$. In this case we say that $H$ is a virtual retract of $G$. If $G$ retracts onto all of its finitely generated subgroups we say that $G$ admits local retractions.

Remark 3.4. If $G$ is a residually finite group that admits local retractions then $G$ is subgroup separable. In fact, if $G$ is residually finite and $H$ is a virtual retract of $G$ then $H$ is separable in $G$ (see [30, Lemma 2.2]). In addition, if $H \leq G$ and $G$ admits local retractions then so does $H$.

The following lemma reduces the problem of showing that $\text{vb}_2(G) = \infty$ to finding finitely generated subgroups of $G$ with an arbitrarily large second betti number, as long as $G$ admits local retractions.

Lemma 3.5. Let $G$ be a finitely generated group and let $H$ be a virtual retract of $G$. Then there is a finite index subgroup $G_0 \leq G$ such that $b_2(G_0) \geq b_2(H)$.

Proof. Let $G_0$ be a finite index subgroup of $G$ which retracts onto $H$; denote the retraction by $r : G_0 \to H$ and denote by $i : H \to G_0$ the inclusion map. Note that $r \circ i : H \to H$ is the identity map. It follows that the same holds for the induced maps on second homology, that is $r_* \circ i_* = \text{Id}_{H_2(H;\mathbb{Q})}$. In particular, $i_* : H_2(H;\mathbb{Q}) \to H_2(G_0;\mathbb{Q})$ is injective and $b_2(G_0) \geq b_2(H)$. \qed

We conclude the discussion by mentioning a few separability properties of hyperbolic fundamental groups of graphs of free groups with cyclic edge groups and limit groups. Wise showed in [46] that hyperbolic graphs of free groups with cyclic edge groups are subgroup separable. Hsu and Wise went on and showed in [19] that such groups are in fact compact special (the definition of a special group is highly technical and is beyond the scope of this paper; we therefore refer the reader to Haglund and Wise’s work [18]). The benefits that follow from being compact special will play a key role in the proof of Theorem A:

Theorem 3.6 (follows from [4, Theorem D]). Let $G$ be the fundamental group of a graph of free groups with cyclic edge groups. If $G$ is hyperbolic then $G$ is locally quasiconvex.

Theorem 3.7 (follows from [18, Corollary 6.7 and Theorem 7.3]). Hyperbolic compact special groups virtually retract onto their quasiconvex subgroups.

Corollary 3.8. Let $G$ be a hyperbolic group which is the fundamental group of a graph of free groups with cyclic edge groups. Then $G$ admits local retractions.

Wilton showed that the same assertion holds for limit groups:

Theorem 3.9 ([42, Theorems A and B]). Limit groups are subgroup separable and admit local retractions.

4. Covers and precovers of graphs of spaces

The (finite index) subgroups of the fundamental group $G$ of a graph of spaces $X$ correspond to (finite-sheeted) coverings of $X$. We will therefore make heavy use of the covering
theory of graphs of spaces as developed by Wise in [46]. We borrow notation and definitions from [43] and refer the reader to Wilton’s work for proofs of the statements appearing in this section.

Let $X$ be a graph of spaces with an underlying graph $\Xi$, vertex spaces $\{X_v\}$, edge spaces $\{X_e\}$ and attaching maps $\partial^\pm_e : X_e \to X_{v^\pm}$. Suppose now that $\widehat{X}$ is a covering of $X$ and note that $\widehat{X}$ inherits a graph of spaces structure from $X$, as follows:

- The vertex spaces of $\widehat{X}$ are the connected components of the preimages of the vertex spaces of $X$ under the covering map; each of these forms a covering space of a vertex space of $X$.
- The edge cylinders of $\widehat{X}$ are the connected components of the preimages of the edge cylinders of $X$ under the covering map; again, each edge space of $\widehat{X}$ is a covering space of an edge space of $X$.
- The underlying graph $\widehat{\Xi}$ of $\widehat{X}$ can be obtained from $\widehat{X}$ by collapsing each vertex space to a point and each edge cylinder to an arc.
- The attaching maps of $\widehat{X}$, also called elevations of the attaching maps of $X$, are the restrictions of pullbacks of the attaching maps and the covering map to the edge spaces of $\widehat{X}$. In more detail, let $\partial^\pm_e : X_e \to X_{v^\pm}$ be an attaching map of $X$ and let $\widehat{X}_{v^\pm}$ be a vertex space of $\widehat{X}$ which lies above $X_{v^\pm}$. Let $\{\widehat{X}_{e_1}, \ldots, \widehat{X}_{e_k}\}$ be the edge spaces of $\widehat{X}$ such that each $\widehat{X}_{e_i}$ is in the preimage of $e \in E(\Xi)$ and incident to $\widehat{v}$. This data fits into the following commuting diagram:

\[
\begin{array}{c}
\bigsqcup_{i=1}^k \widehat{X}_{e_i} \xrightarrow{\bigsqcup_{i=1}^k \widehat{\partial}_{e_i}} \bigsqcup_{i=1}^k \widehat{X}_{v^\pm} \\
\downarrow \quad \downarrow \\
X_v \xrightarrow{\partial^\pm_e} X_v
\end{array}
\]

where the map $\bigsqcup_{i=1}^k \widehat{\partial}_{e_i}$ is the pullback of the attaching map $\partial^\pm_e$ and the covering map. The restriction of this map to each $\widehat{X}_{e_i}$ is an attaching map of $\widehat{X}$.

Remark 4.1. The covering map $\widehat{X} \to X$ induces a graph homomorphism $\widehat{\Xi} \to \Xi$, i.e. a map which sends vertices to vertices, edges to edges and preserves adjacency relations.

Another important notion is the following:

Definition 4.2. The degree of an elevation $\partial^\pm_{e_i}$ is the conjugacy class of $\pi_1(\widehat{X}_{e_i})$ in $\pi_1(X_e)$. In particular, when $X_e$ is a circle, the degree of $\partial^\pm_{e_i}$ coincides with the degree of the covering map $\widehat{X}_{e_i} \to X_e$.

We are now ready to define precovers. Informally, a precover $X'$ of $X$ is a graph of spaces which partially covers $X$: the vertex and edge spaces of $X'$ cover those of $X$ and the degrees of the two ends of each attaching map coincide.

Definition 4.3. Let $X$ be a graph of spaces. Construct another graph of spaces $X'$ in the following manner: let $\{X'_{v'} \to X_v\}$ be a collection of covering maps of vertex spaces of $X$
and for each $e \in E(\Xi)$ let $\{\partial_{e_1}^+\}$ and $\{\partial_{e_1}^-\}$ be subsets of the elevations of $\partial_{e_1}^+$ and $\partial_{e_1}^-$ to $\bigsqcup_{e'} X_{e'}$. Assume further that the degrees of $\partial_{e_1}^+$ and $\partial_{e_1}^-$ coincide. Now let $X'$ be the graph of spaces obtained by gluing the collection $\{X_{e'}\}$ along the attaching maps $\{\partial_{e_1}^\pm\}$. A graph of spaces obtained in this manner is called a \textit{precover} of $X$. Elevations of attaching maps of $X$ to $X'$ which are not attaching maps of $X'$ are called \textit{hanging elevations}.

The key observation regarding precovers is that precovers of $X$ give rise to subgroups of $X$:

\begin{align*}
\text{Lemma 4.4 (\cite[43, Lemmas 15 and 16]{43}).} & \quad \text{If $X'$ is a precover of $X$ then the natural map $X' \to X$ can be extended to a covering map $\hat{X} \to X$. The map $\pi_1(X') \to \pi_1(\hat{X})$ is an isomorphism, and hence the map $\pi_1(X') \to \pi_1(X)$ is injective.} \\
\end{align*}

This lemma, together with Lemma 3.5, implies that if $G$ is the fundamental group of a graph of spaces $X$ and $G$ admits local retractions, then it suffices to construct precovers of $X$ with arbitrarily large second betti numbers to obtain that $\text{vb}_2(G) = \infty$.

5. Precovers with large second homology

In this section we prove theorems $A$ and $B$. The proof of Theorem $A$ is inspired by Wilton’s work on surface subgroups of graphs of free groups with cyclic edges, and in particular borrows ideas from \cite[Lemmas 5.9 and 5.10]{44}. We begin by amassing the ingredients required for the proofs.

The first result that we mention is a generalization of a famous lemma due to Shenitzer which states that an amalgamated product of two free groups along a cyclic group is free if and only if the amalgamating cyclic group is a free factor in one of the free groups. This theorem will enable us to detect locally (that is, by looking at vertex spaces and their adjacent edge spaces) when the precovers that we construct are one-ended (and therefore contain a surface subgroup).

\begin{align*}
\text{Theorem 5.1 (Relative Shenitzer’s Lemma \cite[Theorem 18]{43}).} & \quad \text{Let $G$ be a finitely generated group which is the fundamental group of a graph of groups with infinite cyclic edge groups. Then $G$ is one-ended if and only if every vertex group is freely indecomposable relative to the incident edge groups.} \\
\end{align*}

5.1. Surface subgroups: from one to many. One-endedness is particularly important to us as it will serve as an indicator for when a hyperbolic graph of free groups with cyclic edge groups is not free (and therefore possibly has non-trivial second homology). Wilton showed in \cite{44} that one-ended hyperbolic graphs of free groups with cyclic edge groups contain surface subgroups; since these groups admit local retractions Lemma 3.5 implies that they have a positive virtual second betti number. We record the full statement of Wilton’s theorem below:

\begin{align*}
\text{Theorem 5.2 (\cite[Theorem 6.1]{44}).} & \quad \text{Let $G$ be the fundamental group of a graph of free groups with cyclic edge groups. If $G$ is hyperbolic and one-ended then $G$ contains a surface subgroup.} \\
\end{align*}

The main idea behind the proof of Theorems $A$ and $B$ is that in some instances the existence of a surface subgroup can be promoted to the existence of many surface subgroups.
These surface subgroups will be independent from each other in the second homology of the fundamental group of a precover, producing a subgroup with a large second betti number. This will result in an infinite virtual second betti number. The two lemmas below describe a few of these instances, and we will make repeated use of them in the proofs of Theorems A and B.

**Lemma 5.3.** Suppose that $G$ admits local retractions and splits as a finite and connected graph of spaces $X$ in which all edge groups are infinite cyclic. Suppose further more that there is a vertex space $X_v$ of $X$ whose fundamental group retracts onto a surface subgroup. If at least one of the following holds,

1. the underlying graph $\Xi$ of $X$ is not a tree,
2. there is a vertex group $G_u$ of $\tilde{G}(X)$, different from $G_v$, that is not virtually cyclic,

then $\text{vb}_2(G) = \infty$.

**Proof.** Suppose first that $\Xi$ is not a tree, so there is an edge $e \in E(\Xi)$ with $\Xi - \{e\}$ a connected graph. Let $n \in \mathbb{N}$, take $n$ copies of $X$ and enumerate them $X_1, \ldots, X_n$; let $e_i$ be the copy of $e$ in $E(\tilde{\Xi})$. Remove $X_{e_i} \times (-1, 1)$ from $X_i$ to obtain a precover $X'_i$ of $X_i$ with two hanging elevations $\partial_i^+$. Let $\hat{X}$ be the space obtained by splicing together $X'_1, \ldots, X'_n$, pairing the elevation $\partial_i^+$ with $\partial_{i+1}^-$ (and $\partial_n^+$ with $\partial_1^-$). The resulting space for $n = 3$ appears in Figure 1.

Note that $\hat{X}$ contains $n$ copies of $X_v$ as vertex spaces, and by Lemma 3.5 $b_2(X_v) \geq 1$. We will use Mayer-Vietoris inductively to show that $b_2(\tilde{G}) = \sum_{\tilde{e} \in V(\hat{\Xi})} b_2(\tilde{G}_{\tilde{e}})) \geq n$. $\tilde{G}$ can be obtained by taking an iterated amalgamated product (over infinite cyclic groups) of its different vertex groups, followed by a sequence of HNN extensions. Write $V(\hat{\Xi}) = \{\tilde{v}_1, \ldots, \tilde{v}_m\}$ where each $\tilde{v}_i$ is adjacent to one of $\tilde{v}_1, \ldots, \tilde{v}_{i-1}$ by an edge $\tilde{e}_i$, and let $\tilde{G}_i$ be the iterated amalgamated product of $\tilde{G}_{\tilde{e}_1}, \ldots, \tilde{G}_{\tilde{e}_i}$ (over the groups $\tilde{G}_{\tilde{e}_i}$). Assume that $b_2(\tilde{G}_i) = \sum_{j=1}^i b_2(\tilde{G}_{\tilde{e}_j})$, note that $\tilde{G}_{i+1} = \tilde{G}_i \ast_{\tilde{G}_{\tilde{e}_{i+1}}} \tilde{G}_{\tilde{e}_{i+1}}$ and look at the Mayer-Vietoris exact sequence:

$$
\cdots \longrightarrow H_2(\tilde{G}_{\tilde{e}_{i+1}}; \mathbb{Q}) \longrightarrow H_2(\tilde{G}_i; \mathbb{Q}) \oplus H_2(\tilde{G}_{\tilde{e}_{i+1}}; \mathbb{Q}) \longrightarrow H_2(\tilde{G}_{i+1}; \mathbb{Q}) \longrightarrow \cdots
$$

**Figure 1.** Splicing together three copies of a precover of $X$ along the red edge
Since $H_2(\hat{G};\mathbb{Q})$ is trivial, $H_2(G;\mathbb{Q}) \oplus H_2(G;\mathbb{Q})$ injects into $H_2(\hat{G};\mathbb{Q})$. It follows that $b_2(\hat{G}) = \sum_{i=1}^n b_2(\hat{G}_i)$. A similar argument shows that taking an HNN extension over an infinite cyclic group leaves the second betti number unchanged, which implies that $b_2(\hat{G}) = \sum_{i=1}^n b_2(\hat{G}_i)$. Recall that at least $n$ of the summands come from copies of $G$, and are therefore positive, hence $b_2(\hat{G}) \geq n$ and $\text{vb}_2(\hat{G}) = \infty$.

We still need to treat the case where $\Xi$ is a tree. Let $e \in E(\Xi)$ be the first edge appearing along the path from $u$ to $v$ in $\Xi$. Since $G_u$ is not cyclic, there exists $g \in G_u$ that lies outside of $G_e \leq G_u$. By Remark 3.4, $G_e$ is separable in $G_u$ so there is a finite quotient $q_1 : G_u \rightarrow Q_1$ where $q_1(g) \notin q_1(G_e)$. Let $X_u^1$ be the finite-sheeted covering of $X_u$ that corresponds to $\ker q_1$. Since $q_1(G_e) \subseteq Q_1$, the preimage of $X_e$ in the cover $X_u^1$ has at least two connected components. Since $G_u$ is not virtually cyclic, one may repeat this procedure for one of the elevations of $G_e$ to $G_u$. After $n$ times one obtains a finite cover $X_u^n$ of $X_u$ in which the preimage of $X_e$ has at least $n$ connected components. By Lemma 3.2 there is a finite cover $\hat{X}$ of $X$ and $\hat{u} \in V(\hat{\Xi})$ with $\hat{X}_{\hat{u}} = X_u^n$; note that $\deg(\hat{u}) \geq n$ in $\hat{\Xi}$. Therefore, if $\hat{\Xi}$ is a tree, $\hat{\Xi} = \{\hat{u}\}$ has at least $n$ connected components. Each of these connected components must contain a vertex that lies above $v$, and therefore $\Xi$ contains at least $n$ vertices whose corresponding vertex group has a positive second betti number. As before, employing Mayer-Vietoris we obtain that $b_2(\hat{G}) \geq n$ and therefore $\text{vb}_2(g) = \infty$. Finally, if $\hat{\Xi}$ is a tree, the first part of the proof implies that $\text{vb}_2(\hat{G}) = \infty$ and hence $\text{vb}_2(\hat{G}) = \infty$. \hfill \square

Lemma 5.4. If $G$ splits as a free product where one of the factors has a positive second betti number and the other factor is infinite and residually finite then $\text{vb}_2(G) = \infty$.

Proof. Write $G = G_1 \ast G_2$ where $b_2(G_1) > 0$ and $G_2$ is residually finite. Let $n \in \mathbb{N}$ and since $G_2$ is residually finite it has a subgroup $\hat{G}_2$ of finite index $k \geq n$. Now $G$ has a finite index subgroup $\hat{G}$ that splits as a free product with $k + 1$ factors, one of which is $\hat{G}_2$ and the rest are conjugates of $G_1$. As in the proof of Lemma 5.3, Mayer-Vietoris implies that $b_2(\hat{G}) \geq k \cdot b_2(G_1) \geq n$ and $\text{vb}_2(G) = \infty$. \hfill \square

5.2. Relative JSJ decompositions. Another key component in the proof of Theorem A is JSJ decompositions. Roughly speaking, a (cyclic) JSJ decomposition of a group $G$ serves as a dictionary in which one can "look up" the different ways in which torsion-free hyperbolic groups and by Bowditch for one-ended hyperbolic groups, possibly with torsion (see [6]). The standard modern text covering the theory of JSJ decompositions is Guirardel and Levitt’s book [17] to which we refer the reader for more detail. We will use a relative version of JSJ decompositions which applies to free groups, following Cashen (see [12]).

Before describing the JSJ decomposition of a free group relative to a collection of words, we introduce further notation following [43]. A collection of non-trivial words $\vec{w}$ of a free group $F$ is called a multiword. A peripheral structure on a free group $F$ is a set of pairwise non-conjugate maximal cyclic subgroups of $F$; every multiword $\vec{w}$ in $F$ gives rise to a peripheral structure on $F$ which we denote by $[\vec{w}]$. We will refer to a free group accompanied by a peripheral structure as a pair and denote this data by $(F, [\vec{w}])$. \hfill \square

Remark 5.5. Note that if $G$ is the fundamental group of a graph of free groups with cyclic edge groups then every vertex group $G_v$ of $G$ comes equipped with a peripheral structure
induced by the edge groups corresponding to the edges adjacent to \( v \); we denote this peripheral structure of \( G_v \) by \([w,v]\) and refer to the pair \((G_v,[w,v])\) as the induced pair at \( v \).

Suppose now that \( \hat{G} \) is a finite index subgroup of \( G \), and recall that \( \hat{G} \) inherits a graph of groups decomposition from \( G \). Let \( \hat{v} \) be a vertex of the underlying graph of \( \hat{G} \) which lies above \( v \). In this case, we say that the induced pair at \( \hat{v} \), \((\hat{G}_{\hat{v}},[\hat{w}_v])\), covers \((G_v,[w_v])\). The peripheral structure \([\hat{w}_v]\) on \( \hat{G}_{\hat{v}} \) is called the pullback of \([w_v]\) to \( \hat{G}_{\hat{v}} \). Note that given a pair \((F,[w])\) and a finite index subgroup \( \hat{F} \leq F \), one can always pull back the peripheral structure \([w]\) on \( F \) to \( \hat{F} \) and obtain a pair \((\hat{F},[\hat{w}])\) that covers \((F,[w])\). We continue by describing three types of pairs of particular interest:

**Definition 5.6.** A pair \((F,[w])\) is said to be

- **one-ended** if \( F \) does not split freely relative to the elements of \([w]\), that is, the elements of \([w]\) are hyperbolic in every free splitting of \( F \),
- **rigid** if the elements of \( F \) are hyperbolic in every cyclic splitting of \( F \), and
- **of surface type** if there is a closed surface with boundary \( \Sigma \) and an isomorphism \( F \cong \pi_1\Sigma \) which identifies \([w]\) with the conjugacy classes of the cyclic subgroups of \( \pi_1\Sigma \) that correspond to \( \partial\Sigma \).

**Remark 5.7.** If \((F,[w])\) is of surface type and the corresponding surface \( \Sigma \) is a thrice-punctured sphere then the pair \((F,[w])\) is also rigid; we will refer to pairs of this kind as pairs of surface type and not as rigid pairs.

We can now state the relative JSJ decomposition theorem:

**Theorem 5.8 ([12, Theorem 4.25]).** Suppose that the pair \((F,[w])\) is one-ended. Then \( F \) is the fundamental group of a finite graph of groups with the following properties:

1. each vertex group is either a non-abelian free group or an infinite cyclic group,
2. if \( G_v \) is a non-abelian vertex group then the induced pair \((G_v,[w_v])\) is either rigid or of surface type,
3. the underlying graph of this graph of groups decomposition is bipartite, and every edge adjoins an infinite cyclic vertex group to a non-abelian free vertex group.
4. every element of \([w]\) is conjugate into a unique infinite cyclic vertex group.

The following theorem will also prove to be extremely useful for replicating surface subgroups in a precover:

**Theorem 5.9 ([43, Theorem 8]).** If \((F,[w])\) is rigid then there is a finite index subgroup \( \hat{F} \leq F \) such that for every finite index subgroup \( F' \leq \hat{F} \), the pair \((F',[w'])\) obtained by pulling back \([w]\) to \( F' \) admits the following property: for any component \( w'_i \) of \( w' \), the pair \((F',[w'_i - \{w'_i\}])\) is one-ended.

### 5.3. The proofs of Theorems A and B.

We are finally in a position to prove Theorem A. We divide the proof into two cases, depending on whether the relative JSJ decomposition of some induced pair contains a rigid vertex. For the subsequent two lemmas, \( G \) is a one-ended and word-hyperbolic group that splits as a graph of free groups with cyclic edge groups. \( X \) will be a graph of spaces whose underlying graph coincides with that of the aforementioned splitting of \( G \) and whose vertex spaces are either graphs or closed and connected surfaces with boundary (this will depend on our point of view, and we will
explicitly describe the vertex spaces of $X$ whenever relevant). Recall that we denote the vertex space corresponding to $v \in V(\Xi)$ by $X_v$, and write $G_v$ for $\pi_1 X_v$; fix the same notation for an edge $e \in E(\Xi)$.

**Lemma 5.10.** If there is a vertex $v \in V(\Xi)$ such that the relative JSJ decomposition of the induced pair $(G_v, [w_v])$ at $v$ has a rigid vertex then $\text{vb}_2(G) = \infty$.

**Proof.** We begin by replacing $X$ with a refined graph of spaces, obtained by replacing $X_v$ with a graph of spaces that corresponds to its relative JSJ decomposition; by our assumption, $X$ now contains a vertex space $X_u$ such that the induced pair $(G_u, [w_u])$ at $u$ is rigid. By [46] $G$ is subgroup separable, and by Lemma 3.2 we may replace $X$ with a finite-sheeted cover $\widehat{X}$ that contains a vertex space $\widehat{X}_u$ satisfying the property described in Theorem 5.9.

In a manner similar to [44, Lemma 5.9], we construct a precover of $\widehat{X}$ as follows: write $k = \text{deg}(\widehat{u})$ and let $\widehat{e}_1, \ldots, \widehat{e}_k \in E(\widehat{\Xi})$ be the edges adjacent to $\widehat{u}$. Take $k$ copies of $\widehat{X}_u$ and enumerate them $\widehat{X}_{u1}, \ldots, \widehat{X}_{uk}$; denoting $\pi_1(\widehat{X}_{ui}) = \widehat{G}_i$ and referring to the peripheral structure on $\widehat{G}_i$ induced by the adjacent edges as $\widehat{\omega}_i = \{\widehat{w}_i^1, \ldots, \widehat{w}_i^k\}$, the induced pair at $\widehat{X}_{ui}$ in the resulting precover will be $(\widehat{G}_i, [\widehat{w}_i - \{\widehat{w}_i^i\}])$. Take $k - 1$ copies of each vertex space $X_v$ of $\widehat{X}$ for every $\widehat{v} \neq \widehat{u}$ and enumerate these $\{\widehat{X}_{v1}, \ldots, \widehat{X}_{vk-1}\}$. In order to define the precover $X'$, it suffices to specify which elevations of the attaching maps of $X$ will be attaching maps of $X'$ and to verify that there is a suitable degree-preserving bijection between them. Every elevation of an attaching map to this collection of vertex spaces will be an attaching map of $X'$, except for the elevations that correspond to the edge space of $\widehat{e}_i$ in $\widehat{X}_{ui}$. Note that if $\partial_{e_i}^\pm$ is an attaching map in $X$ then every elevation $\widehat{\partial}_{e_i}^\pm$ to $\widehat{X}$ appears $k - 1$ times as a hanging elevation in the collection of vertex spaces we have just defined. Therefore one can pick a suitable degree-preserving bijection between these hanging elevations and obtain a precover $X'$ of $X$. There are exactly $k$ hanging elevations in this precover, each of them corresponding to the edge space of $\widehat{e}_i$ in $\widehat{X}_{ui}$ for some $1 \leq i \leq k$.

Let $X'_1$ be the connected component of $X'$ containing $\widehat{X}_{u1}$ and note that by Theorem 5.9 the induced pair at every vertex of $X'_1$ is one-ended. Therefore, by Theorem 5.1, $G'_1 = \pi_1 X'_1$ is one-ended and by [44, Theorem 6.1] $G'_1$ contains a surface subgroup $\pi_1 \Sigma$. By Corollary 3.8 we may replace $G'_1$ with a finite index subgroup that retracts onto $\pi_1 \Sigma$.

Note that $X'_1$ contains at least one hanging elevation, corresponding to the edge space of $\widehat{e}_i$ in $\widehat{X}_{ui}$; denote this elevation by $\partial_{i_1}^+$. We consider two distinct cases, depending on whether $X'_1$ contains an additional hanging elevation of suitable degree that may be paired up $\partial_{i_1}^+$ (after adding another cyclic vertex space to $X'_1$).

Suppose first that $X'_1$ contains a suitable hanging elevation $\partial_{i_1}^-$ that may be paired up with $\partial_{i_1}^+$. Let $Y$ be the graph of spaces obtained by attaching another edge cylinder to $X'_1$ along $\partial_{i_1}^\pm$. Regard $Y$ as a graph of spaces with a single vertex space $X'_1$ and a single edge. The underlying graph of $Y$ consists of one vertex and one edge, and in particular it is not a tree. In addition, the fundamental group of $X'_1$ retracts onto its surface subgroup $\pi_1 \Sigma$. By Lemma 5.3 $\text{vb}_2(\pi_1 Y) = \infty$ and therefore by Lemma 3.5 $\text{vb}_2(G) = \infty$. 

Next, treat the second case, and suppose that \( X'_i \) does not contain a hanging elevation that may be paired up with \( \partial_i^+ \). By Lemma 4.4, we can complete \( X'_i \) to a cover \( \overline{X} \) of \( X \). Let \( Y \) be the precover of \( X \) which contains the embedded copy of \( X'_i \) in \( \overline{X} \), together with an extra vertex space that is attached to the vertex space \( \hat{X}_{\bar{w}_i} \) of \( X'_i \) along \( \partial_i^+ \); if this vertex space is cyclic, we attach another vertex space with a free fundamental group to this cyclic vertex. We have that \( \pi_1 Y \) splits as an amalgamated product \( \pi_1 X'_i \ast_C F \) where \( C \) is cyclic and \( F \) is a non-abelian free group. This splitting satisfies condition (2) of Lemma 5.3, and therefore \( \text{vb}_2(\pi_1 Y) = \infty \); as before, this implies that \( \text{vb}_2(G) = \infty \). □

We next deal with the case where none of the relative JSJ decompositions of the induced pairs of \( X \) has a rigid vertex. We will make use of the following lemma which determines the finite-sheeted covering spaces of an orientable surface:

**Lemma 5.11** ([33, Lemma 3.2]). Let \( \Sigma \) be an orientable and connected surface with positive genus and let \( \alpha \geq 1 \). For each boundary component of \( \Sigma \), pick a collection of degrees summing to \( \alpha \). Then there is a connected \( \alpha \)-sheeted covering \( \hat{\Sigma} \to \Sigma \) such that the connected components of the preimage of each boundary component \( \partial \Sigma_i \) of \( \Sigma \) cover \( \partial \hat{\Sigma}_i \) with the prescribed degrees if and only if the number of boundary components of \( \hat{\Sigma} \) has the same parity as \( \alpha \cdot \chi(\Sigma) \).

**Lemma 5.12.** Suppose that for every \( v \in V(\Xi) \) the vertices of the relative JSJ decomposition of the induced pair \( (G_v,[\omega_v]) \) at \( v \) are all of surface type. If \( G \) is not the fundamental group of a closed surface then \( \text{vb}_2(G) = \infty \).

**Proof.** Refine \( X \) by subdividing each edge and putting a cyclic vertex in the middle with vertex group \( \mathbb{Z} \); we also replace each vertex space \( X_v \) with the relative JSJ decomposition of the induced pair at \( X_v \). By assumption, every non-cyclic vertex of \( X \) is a surface with boundary. By [46], \( G \) is subgroup separable and we may replace \( X \) with a finite-sheeted covering \( \hat{X} \) with the following properties: all of the attaching maps at cyclic vertices are isomorphisms, and every non-cyclic vertex space \( \hat{X}_v \) of \( \hat{X} \) is orientable.

Since \( G \) is torsion free, Nielsen realization implies that \( \hat{G} = \pi_1 \hat{X} \) is not a surface group. Therefore, there exists \( \hat{v} \in V(\hat{\Xi}) \) such that \( \deg \hat{v} > 2 \) and \( \pi_1 \hat{X}_{\hat{v}} \) is cyclic. We will use the fact that \( b_2(\pi_1 X) > n \) for a given \( n \). Let \( \hat{v}_1, \hat{v}_2, \hat{v}_3 \) be three distinct edges adjacent to \( \hat{v} \) in \( \hat{\Xi} \) and assume that \( \hat{v}_i \) adjoins \( \hat{v} \) to a vertex \( \hat{v}_i \in \hat{\Xi} \) (it is possible that the vertices \( \hat{v}_i \) to be distinct). For \( 1 \leq i \leq 3 \) let \( \hat{X}_i \) be the precover of \( X \) obtained by removing the edge cylinder \( \hat{X}_{\hat{v}_i} \times (-1,1) \). \( \hat{X}_i \) has two hanging elevations which we denote by \( \hat{\partial}_i^+ : \hat{X}_{\hat{v}_i} \to \hat{X}_{\hat{v}_i} \) and \( \hat{\partial}_i^- : \hat{X}_{\hat{v}_i} \to \hat{X}_{\hat{v}_i} \).

We next construct three precovers \( X'_i \) of \( X \) (and \( \hat{X} \)) with the property that the geometric realization of \( X'_i \) is a connected, orientable surface with two boundary components, each mapped homeomorphically to \( \hat{X}_{\hat{v}_i} \subset \hat{X}_{\hat{v}_i} \) under the natural map \( X'_i \to \hat{X} \).

Take two copies of each surface type vertex space of \( \hat{X}_i \) and take \( \deg(\hat{v}_i) \) copies of each cyclic vertex \( \hat{X}_{\hat{v}_i} \) of \( \hat{X}_i \). We may now pair the hanging elevations in this collection of vertex spaces to obtain a precover \( X'_i \) of \( X \) making sure that

- each non-hanging elevation whose target space is of surface-type in \( \hat{X}_i \) is non-hanging in \( X'_i \),
If $X'_i$ is a cyclic vertex space of $X'_j$ then $\deg(v') = 2$,

- the two copies of $\partial^+_i$ in $X'_i$ are the only hanging elevations in $X'_i$ whose target space

is of surface-type.

Since all of the attaching maps of $X'_i$ are isomorphisms and all of the surface-type vertices
are orientable, $X'_i$ is an orientable surface with two boundary components as desired. If $X'_i$

is not connected, replace it with a connected component that has a non-empty boundary. If

this connected component has a single boundary component, replace it with a 2-sheeted covering
with two boundary components of degree 1; the existence of such a cover is guaranteed by Lemma 5.11. We are finally ready to construct a precover $X'$ of $X$ with a

prescribed second betti number.

Let $n > 2$ and again by Lemma 5.11 there is a finite-sheeted cover of $X'_j$ with $2 \cdot (n - 1)$
boundary components, each of degree one. Name this space $Y$. Take $n - 1$ copies of each
of $X'_j$ and $X'_k$ and label them $X'_1, \ldots, X'_{n-1}, X'_1, \ldots, X'_{n-1}$. Enumerate the boundary
components of $Y$, $S_1, \ldots, S_{2(n-1)}$ and glue $X'_1$ and $X'_k$ to $Y$ along $S_{2i-1} \cup S_{2i}$; the geometric
realization of the resulting precover $X'$ is described in Figure 2. It is now left to show that $b_2(\pi_1 X') = n$. For each $0 \leq i \leq n - 1$ define the following spaces:

- $Z_i = Y \cup X'_1 \cup \cdots \cup X'_{i-1} \cup X'_i \cup X'_{i+1} \cup \cdots \cup X'_{n-1}$ (the space $Z_2$ is the thickened
  surface in Figure 2). Here $Z_0 = Y \cup X'_1 \cup \cdots \cup X'_{n-1}$. Each of the spaces $Z_i$ is a
  closed and connected surface.
- $W_i = Y \cup X'_1 \cup \cdots \cup X'_{i-1} \cup X'_{i+1} \cup \cdots \cup X'_{n-1}$. Each of the spaces $W_i$ is a
  connected surface with boundary and $\partial W_i = S_{2i-1} \cup S_{2i}$ (except for $W_0 = Z_0$).

We have that $\bigcup_{i=0}^{n-1} Z_i = X'$ and $(\bigcup_{i=0}^{m-1} Z_i) \cap Z_m = W_m$ for $1 \leq m \leq n - 1$. This translates to the following decomposition of $\pi_1 X'$ as an iterated amalgamated product in the level of fundamental groups:

$$\pi_1 X' = ((\pi_1 Z_0 * \pi_1 W_1 \pi_1 Z_1) * \pi_1 W_2 \pi_1 Z_2) * \pi_1 W_3 \cdots) * \pi_1 W_{n-1} \pi_1 Z_{n-1})$$

Figure 2. The space $X'$. The surface $Z_2$ is highlighted in bold.
Since the amalgamating subgroups $\pi_1W_i$ are all free and each $\pi_1Z_i$ is a surface group, applying Mayer-Vietoris repeatedly we obtain that $b_2(\pi_1X') = \sum_{i=0}^{n-1} b_2(\pi_1Z_i) = n$. As before, this implies that $\text{vb}_2(G) = \infty$. \hfill \blackslug

Theorem A easily follows from Lemmas 5.10 and 5.12:

*Proof of Theorem A.* First note that if $G$ is free then clearly $\text{vb}_2(G) = 0$, and if $G$ is the fundamental group of a closed, connected surface then $\text{vb}_2(G) = 1$. Assume therefore that $G$ is not a free or a surface group. If $G$ is one-ended then $\text{vb}_2(G) = \infty$ by Lemmas 5.10 and 5.12, so assume further that $G$ is not one-ended. By Grushko’s theorem we may write $G = G_1 \ast \cdots \ast G_n \ast F$ where each $G_i$ is one-ended, $F$ (which might not appear in this decomposition) is free and there are at least 2 factors in this free product. A standard argument using Bass-Serre theory shows that in fact each $G_i$ is the fundamental group of a finite graph of finitely generated free groups with infinite cyclic edge groups: let $T$ be the Bass-Serre tree that corresponds to the cyclic splitting of $G$ and let $T_i$ be a minimal $G_i$-invariant subtree of $T$. Taking the core of the quotient of $T_i$ by $G_i$ we obtain a finite graph of groups decomposition $\mathcal{G}(G_i)$ of $G_i$. Since $G_i$ is freely indecomposable, the edge groups of $\mathcal{G}(G_i)$ are all infinite cyclic; in particular, these edge groups are finitely generated. By Grushko’s theorem $G_i$ is finitely generated which implies that the vertex groups of $\mathcal{G}(G_i)$ are finitely generated. In addition, since the vertex groups of the cyclic splitting of $G$ are free, the vertex groups of $\mathcal{G}(G_i)$ are all free.

If some $G_i$ is not the fundamental group of a closed, connected surface, Lemmas 5.10 and 5.12 imply that $\text{vb}_2(G_i) = \infty$ and by Lemma 3.5 $\text{vb}_2(G) = \infty$. We may therefore suppose that every $G_i$ is a surface group. In particular $G$ splits as a free product where one of the factors has a positive second betti number and the other factor is infinite and residually finite. By Lemma 5.4 $\text{vb}_2(G) = \infty$ as desired. \hfill \blackslug

*Proof of Theorem B.* First note that if $L$ is free then $\text{vb}_2(L) = 0$, if $L$ is a surface group then $\text{vb}_2(L) = 1$ and if $L \cong \mathbb{Z}^d$ then $\text{vb}_2(L) = \binom{d}{2}$. Suppose now that $L$ is not free, surface or free abelian and let $G$ be a one-ended group appearing in the hierarchy of $L$ with no one-ended groups below it. If $G$ is the fundamental group of a graph of free groups with cyclic edge groups, and $G$ is not a surface group, then by Theorem A $\text{vb}_2(G) = \infty$ and since limit groups admit local retractions, $\text{vb}_2(L) = \infty$.

Otherwise, by Corollary 2.2, $G$ is either free abelian or the fundamental group of a surface. Since $L$ is not a surface or a free abelian group, there is a group $G'$ lying above $G$ in the hierarchy. If $G$ is a free factor of $G'$, by Lemma 5.4 $\text{vb}_2(G') = \infty$ and therefore $\text{vb}_2(L) = \infty$. Otherwise, $G'$ contains a free factor $H$ that splits as the fundamental group of a graph of groups with cyclic edge groups; note that $G$ is a vertex group of this splitting $H$. Since $G$ is not a free factor of $G'$ either $H$ is an HNN extension of $G$ (and its underlying graph is not a tree), or $H$ contains an additional vertex that is not virtually cyclic. Therefore we may evoke Lemma 5.3 which shows that $\text{vb}_2(H) = \infty$. $\text{vb}_2(L) = \infty$ follows. \hfill \blackslug
Appendix A. On the soluble genus of limit groups and related questions

By Ismael Morales

Let \( k \geq 0 \) be an integer and \( p \) be a prime. We first recall that the \( k \)-th \( p \)-betti and \( k \)-th betti numbers of a group \( G \) are defined as \( b_{k,p}(G) = \dim_{\mathbb{F}_p} H_k(G; \mathbb{F}_p) \) and \( b_k(G) = \dim_{\mathbb{Q}} H_k(G; \mathbb{Q}) \), respectively. Let \( \mathcal{C} \) be the variety of either finite groups, of finite soluble groups, of finite nilpotent groups or of finite \( p \)-groups. Given a finitely generated group \( G \), we say that it is residually-\( \mathcal{C} \) if, for all \( 1 \neq g \in G \), there exists \( N \triangleleft G \) such that \( g \notin N \) and \( G/N \in \mathcal{C} \). Following the terminology of [16], given two residually-\( \mathcal{C} \) groups \( G \) and \( \Gamma \), we say that \( G \) belongs to the \( \mathcal{C} \)-genus of \( \Gamma \) if they have the same collection of isomorphism types of finite quotients belonging to \( \mathcal{C} \). Equivalently, this is true if there is an abstract isomorphism \( G \overset{\cong}{\rightarrow} \Gamma \) between their pro-\( \mathcal{C} \) completions (by [14]).

It is clear that \( b_1 \) and \( b_{1,p} \) are invariants within any of these genera. Lubotzky proved in [27, Proposition 1.4] that being residually soluble, residually nilpotent or residually-\( p \) is not a profinite property; so these genera may differ in general. On the positive side, Jaikin-Zapirain proved in [20, Theorem 1.1] that if \( G \) is a group in the finite or soluble genus of a finitely generated free or surface group \( S \), then it is residually-\( p \) for all primes \( p \) (and so \( G \) is in the nilpotent and \( p \)-genus of \( S \)).

In this short note, we want to emphasise some particular and strong features of free and surface groups that are used during [20, Theorem 3.3], collected below in Definition A.1.

There are two big classes of groups that include free and surface groups: limit groups and hyperbolic fundamental groups of graphs of free groups with cyclic edge groups. It follows from Theorem A and Theorem B, that free and surface groups are the only such groups satisfying those properties.

When one tries to extend Jaikin-Zapirain’s result to other classes of groups of cohomological dimension at most 2 (such as hyperbolic limit groups), with the same \( L^2 \)-methods, the following three properties arise naturally.

**Definition A.1.** We introduce the following properties of groups \( G \):

- **Property (A):** The virtual second betti number of \( G \) is finite. In other words, the set of integers \( \{ b_2(H) : H \leq G \text{ of finite index} \} \) is bounded.
- **Property (B):** For every finite index subgroup \( H \leq G \), the group \( H_1(H; \mathbb{Z}) \) is torsion-free.
- **Property (C) at the prime \( p \):** The pro-\( p \) completion \( G\hat{p} \) is residually-(torsion-free nilpotent).

Reproducing the methods of [20], one can prove the following theorem (which, for the purpose of this paper, is stated for limit groups).

**Theorem A.2.** Let \( \Gamma \) be a hyperbolic limit group that satisfies properties (A), (B) and (C) at \( p \). Suppose that \( G \) belongs to the soluble or nilpotent genus of \( \Gamma \), then \( G \) is residually-\( p \).

Before moving on to the proof, we make a few comments about the properties (A)–(C).

- The property (A) is proven to characterise free and surface groups among hyperbolic limit groups in this article.
- By Kurosh’s decomposition theorem, property (B) is closed under taking free products and it is satisfied for free abelian groups, free groups and surface groups (in fact, we do not expect to have more examples, see Question A.10). No hyperbolic
three-manifold has this property by [40]. In particular, another possibly surprising consequence is that many RAAGs do not have this property. The reason is that hyperbolic three-manifold groups are virtually compact special (by the work of Agol, Kahn-Markovic, and Wise) and hence it is a virtual retract of some bigger RAAG containing them (by the work of Haglund). On the other side, many parafree groups, such as the one-relator group \(\langle a, b, c | a^2b^2c^3 \rangle\), do not enjoy \((B)\) either.

- The property \((C)\) is true for all primes \(p\) if the lower central quotients \(G/\gamma_nG\) are torsion-free for all \(n \geq 1\). The latter property is ensured for many limit groups. For example, as we review in Propositions 2.16 and 2.17 of [32], property \((C)\) is true for doubles of free groups \(F \ast_{a=a} F\) along words \(a \in F\) with the following property: if \(k\) is the biggest integer such that \(a \in \gamma_kF\), then the image of \(a\) in \(\gamma_kF/\gamma_{k+1}F\) is not a proper power. This class also includes many one-relator parafree groups (see [2], [3] and [22]) and one-relator parasurfaces (see [3]).

We will now prepare the ground for the proof of Theorem A.2.

**Notation A.3.** Given a group \(G\), we denote by \(\widehat{G}\), \(G_S\) and \(G_p\) its pro-finite, pro-soluble and pro-\(p\) completion; respectively. We write \(b_1^{(2)}(G)\) for its first \(L^2\)-betti number. We denote profinite groups in bold letters \(\mathbf{G}, \mathbf{H}, \mathbf{K}, \mathbf{M}, \mathbf{U}\) to distinguish them from abstract groups.

For more background on \(L^2\)-betti numbers, we refer the reader to the books of Lück [29] and Kammeyer [23]. We are mostly interested in computations and estimations of \(b_1^{(2)}(G)\) by means of Lück approximation-type principles in the sense of [28], as the following.

**Proposition A.4.** Let \(G\) be a residually-(locally indicable amenable) group. Then \(b_1^{(2)}(G) \leq b_1(G) - 1\).

**Proof.** Since \(H\) is residually-(locally indicable amenable), then the Hughes-free universal division ring of fractions \(D_{\mathbb{Q}G}\) exists (by [21, Theorem 1.1]), and using Lück’s approximation theorem for rank functions ([21, Theorem 1.2]) one can prove that \(b_1^{(2)}(G) = \dim_{D_{\mathbb{Q}G}} H_1(G; D_{\mathbb{Q}G}) = \dim_{D_{\mathbb{Q}G}} \left( D_{\mathbb{Q}G} \otimes \mathbb{I}_G \right) - 1\). Furthermore, by universality (see [21, Corollary 1.4]), \(\dim_{D_{\mathbb{Q}G}} \left( D_{\mathbb{Q}G} \otimes \mathbb{I}_G \right) \leq \dim_{\mathbb{Q}} \left( \mathbb{Q} \otimes \mathbb{I}_G \right) = b_1(G)\). The conclusion follows. \(\square\)

**Proposition A.5.** Let \(G\) be a hyperbolic limit group. Then \(b_1(G) - b_2(G) = 1 + b_1^{(2)}(G)\).

**Proof.** By [9, Corollary C], \(-\chi(G) = b_1^{(2)}(G)\). On the other side, Sela [39] proved that hyperbolic limit groups have a finite cyclic hierarchy terminating in free groups. So \(G\) has cohomological dimension at most 2 and \(\chi(G) = \sum_{k \geq 0} (-1)^k b_k(G) = 1 - b_1(G) + b_2(G)\). \(\square\)

**Lemma A.6.** Let \(\Gamma\) be a finitely generated non-abelian limit group. Let \(N\) be a non-trivial closed normal subgroup of the pro-soluble completion \(\Gamma_S\). Then there exists a prime \(q\) and an open normal subgroup \(K \leq_N \Gamma_S\) such that \(N/K\) is isomorphic to the direct sum of at least \(b_1(\Gamma)\) copies of \(\mathbb{Z}/q\).
Proof. Since \( N \) is non-trivial, there exists a quotient \( \phi : \Gamma_S \longrightarrow Q \) to a finite soluble group for which the image \( \phi(N) \) is non-trivial. Since non-trivial soluble groups \( Q \) are not perfect (i.e. \([Q, Q]\) is a proper subgroup of \( Q \)), there must exist a prime \( q \), a non-trivial elementary \( q \)-abelian group \( A \) and a surjective map \( \psi : Q \longrightarrow A \) such that \( \psi(\phi(N)) \) non-trivial. In particular, the image of \( N \) in the \( q \)-abelianisation \( \Gamma_S/\Gamma_S^q[\Gamma_S, \Gamma_S] \) is non-trivial. So we can name \( M \) to be the image of \( N \) under the map \( \Gamma_S \longrightarrow \Gamma_{\bar{q}} \). By construction, \( M \) is non-trivial. We claim that \( M/\Phi(M, M) \) has rank at least \( b_1(\Gamma) \).

(a) Suppose that \( M \) has infinite-index. Since \( \Gamma \) is non-abelian limit, then \( \Gamma \) has positive \( q \)-deficiency and, by [34, Theorem 1.1], then the rank \( M/\Phi(M, M) \) is infinite.

(b) If \( M \) has finite index in \( G_{\bar{q}} \), then \( b_1(M) \geq b_1(\Gamma_{\bar{q}}) \geq b_1(\Gamma) \).

Hence, the rank of \( M/\Phi(M) \) is at least \( b_1(\Gamma) \), as we claimed. Furthermore, the closed subgroup \( M \) is normal, so [35, Proposition 2.8.9] says that

\[
\Phi(M) = \bigcap_{M \leq U \leq \Gamma_{\bar{q}}} \Phi(U).
\]

From this, it is clear that there exists an open subgroup \( M \leq U \leq \Gamma_{\bar{q}} \) such that the induced map \( M/\Phi(M) \longrightarrow U/\Phi(U) \) has an image of rank at least \( b_1(\Gamma) \). Thus, \( M/\Phi(U)/\Phi(U) \) is a \( q \)-elementary abelian group of rank at least \( b_1(\Gamma) \), and so is \( MK/K \), where \( K \) is the normal core of the open subgroup \( \Phi(U) \) of \( \Gamma_{\bar{q}} \). \( \square \)

Proof of Theorem A.2. We only prove the statement for the case when \( G \) is in the soluble genus of \( \Gamma \) (the case of the nilpotent genus is completely analogous). Let us fix a prime \( p \) and prove that \( G \) is residually-\( p \). Let \( s = \sup\{b_2(\Lambda) : \Lambda \leq \Gamma \) of finite index\}. From the fact that \( b_1(\Gamma) \geq 1 \), it is clear from Proposition A.5 that for any finite index subgroup \( \Lambda \) of \( \Gamma \), we have that \( b_1(\Lambda) \geq b_1(\Gamma) \) by [11, Proposition 7.5]. We name \( N \) to be the kernel of the map \( G_S \longrightarrow \tilde{G}_S \). We want to prove that \( N = 1 \), since this would prove that the map \( G \longrightarrow G_{\bar{q}} \) is injective and hence that \( G \) is residually-\( p \). We proceed by contradiction. Suppose the opposite. We apply Lemma A.6 to \( N \neq 1 \). It follows that there exists a prime \( q \) (possibly different from the fixed \( p \)) and a normal open \( N \leq G_S \) such that \( NK/K \) is a direct sum of at least \( b_1(G) \) copies of \( \mathbb{Z}/q \).

We name \( T = G \cap NK \) and we denote by \( \tilde{T} \) the image of \( T \) under the surjective map \( G \longrightarrow \tilde{G} \). Notice that \( T/G \cap K \) is a dense subgroup of \( G \). Since \( G = \tilde{G} \). Notice that \( T/G \cap K \) is a dense subgroup of the finite \( q \)-elementary group \( NK/K \) and that \( |G : T| = |G : \tilde{G}| \). So \( T/G \cap K \cong NK/K \). From this, we deduce that the rank of the image of \( G \cap K \) in the \( q \)-abelianisation of \( T \) is at most \( b_1(T) - b_1(G) \). Since \( \tilde{T} \) is a quotient of \( T \) that is also generated by the image of \( G \cap K \), then we derive that \( b_1(\tilde{T}) \leq b_1(T) - b_1(G) \).

We also notice that \( G \) is a subgroup of \( G_{\bar{q}} \cong G_{\bar{q}} \cong \Lambda_{\bar{q}} \cong \Gamma_{\bar{q}} \). Since \( \Gamma \) satisfies property (C), then \( G \) is a subgroup of a residually-(toral-free nilpotent) group and, in particular, it is residually-locally indicable amenable. By Proposition A.4, this implies that \( b_1(\tilde{T}) \geq \)
\[ b_1^{(2)}(\tilde{T}) + 1. \] Hence,
\[ b_1(T) = b_{1,q}(T) \geq b_{1,q}(\tilde{T}) + b_1(G) \geq 1 + b_1^{(2)}(\tilde{T}) + b_1(G) \]
\[ = 1 + |G : T|b_1^{(2)}(\tilde{T}) + b_1(G) \geq 1 + |G : T|b_1^{(2)}(\Gamma) + b_1(\Gamma). \]
Since \( G/T \) is a finite soluble group, there exists a subgroup \( \Lambda \leq \Gamma \) with \( |\Gamma : \Lambda| = |G : T| \) and \( b_1(\Lambda) = b_1(T) \). By the previous inequalities and Proposition A.5, we know that
\[ b_1(\Lambda) \geq 1 + b_1^{(2)}(\Lambda) + b_1(G) \geq b_1(\Lambda) - b_2(\Lambda) + s + 1, \]
which is a contradiction because \( s = \sup \{ b_2(\Lambda) : \Lambda \leq \Gamma \text{ of finite index} \} \).

□

Remark A.7. The same proof of Theorem A.2 works for groups \( \Gamma \) that are residually-p and have \( b_1^{(2)}(\Gamma) > 0 \) as long as one has uniform bounds on the absolute difference \( |b_1(\Lambda) - b_1^{(2)}(\Lambda)| \) for finite-index subgroups \( \Lambda \) (which is equivalent to \( \Gamma \) having finite virtual second betti number if \( \Gamma \) is a limit group of cohomological dimension 2).

Question A.8. Apart from free abelian, free and surface groups, are there more examples of infinitely finitely generated residually finite groups \( G \) for which there exists a constant \( C > 0 \) such that \( |b_1(H) - b_1^{(2)}(H)| \leq C \) for all finite index subgroups \( H \leq G \)?

Both Theorem A and Theorem B suggest the following question.

Question A.9. Let \( G \) be a hyperbolic group of cohomological dimension 2. If \( G \) has finite virtual second betti number, does it follow that \( G \) is a surface group?

Our last question culminates the above discussion about property \( (B) \) and is supported by computational evidence of the authors. Moreover, it was also asked by Ilir Snopce and Pavel Zalesskii in discussions that happened during the workshop “New Trends around Profinite Groups 2022”, held in Varese.

Question A.10. Let \( G \) be a one-ended limit group with property \( (B) \). Does it follow that \( G \) is isomorphic to either a free abelian group or to a surface group?

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