Aspects of Massive Gauge Theories on Three Sphere in Infinite Mass Limit

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Abstract

We study the $S^3$ partition function of three-dimensional supersymmetric $\mathcal{N} = 4 \ U(N)$ SQCD with massive matter multiplets in the infinite mass limit with the so-called Coulomb branch localization. We show that in the infinite mass limit the specific point of the Coulomb branch is chosen and contributes to the partition function dominantly. Therefore we can argue whether each multiplet included in the theory is effectively massless or not in this limit even on $S^3$ and conclude that the partition function becomes that of the effective theory on the specific point of the Coulomb branch in the infinite mass limit. In order to investigate which point of the Coulomb branch is dominant, we use the saddle point approximation in the large $N$ limit because the solution of the saddle point equation can be regarded as a specific point of the Coulomb branch. Then we calculate the partition functions for small rank $N$ and actually confirm that their behaviors in the infinite mass limit are consistent with the conjecture from the results in the large $N$ limit. Our result suggests that the partition function in the mass infinite limit corresponds to that of an interacting superconformal field theory.

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1 Introduction

In three dimension, the Yang-Mills coupling has positive mass dimension. This means that three-dimensional Yang-Mills theories are super-renormalizable. The Yang-Mills term is irrelevant and can not contribute to the infrared physics independently of the gauge group and the matter content. It might be expected that 3d gauge theories flow to the non-trivial infrared fixed point, which depends on its matter content. In fact, $U(N)$ QCD with $N_f \geq N_{\text{crit}}$ massless flavor, where $N_{\text{crit}}$ is some critical value, might flow to an interacting IR fixed point while with $N_f < N_{\text{crit}}$ massless flavors the theory is expected to flow to a gapped phase in the IR [1–3]. The number of the flavors plays an important role to determine the IR structure of the 3d gauge theories. However, it is generally difficult to know the non-perturbative properties of such a theory.

Three-dimensional supersymmetric gauge theories have several interesting features which four-dimensional supersymmetric gauge theories do not have. Especially, we are interested in the fact that there are real parameters: real mass and FI parameters. These are not given by the background chiral superfields. Thus the dynamics triggered by the real parameter deformation is not restricted by the holomorphy. This means that there can occur non-trivial phase transitions. When we give the matter fields infinite mass, the massive matter fields decouple from the theory. Decoupling of the flavors changes IR physics and an interesting phase transition would occur. Moreover, supersymmetric gauge theories are known to have the exactly calculable quantities such as the partition function on compact manifold $M$ by the localization methods in three dimension [20–23]. In this paper, we focus on the round three-sphere partition function, which is given by the matrix type finite dimensional integral. The localization methods admit the real mass and FI terms deformation by weakly gauging a global symmetry and giving the background field which couple its current an expectation value. Then we can approach the non-trivial dynamics triggered by the real mass parameter with the localization methods. In [4–15] the phase structure of the mass deformed gauge theories on $S^3$ is argued.

In this paper, we focus on $\mathcal{N} = 4$ $U(N)$ SQCD with $N_f$ pairs of chiral multiplets in the fundamental and anti-fundamental representation of $U(N)$. These theories are classified in [16] by their low energy properties. The authors define three types of the theories; “good”, “ugly” and “bad” theories. A 3d gauge theory is a good theory if all the monopole operators obey the unitarity bound. In this case, the R-symmetry in the IR is the same as that in the UV. For an $\mathcal{N} = 4$ $U(N)$ SQCD it is a good theory when $2N_f \geq N$. A gauge theory is called “ugly” if the monopole operators satisfy the unitarity bound, however several monopole operators saturate it. This type of theories are likely to flow to an interacting SCFT with R-symmetry visible in the UV with the decoupled free sector consisted of the monopole operators which saturate the unitarity bound. An $\mathcal{N} = 4$ $U(N)$ SQCD is an ugly theory when $N_f = 2N - 1$. In a bad theory, there are the monopole operators with zero or negative R-charge of the R-symmetry.
manifest in the UV. Because the monopole operators violate the unitarity bound of the UV R-symmetry, a bad theory flow to an interacting SCFT whose R-symmetry is not manifest in the UV. An \( \mathcal{N} = 4 \) \( U(N) \) SQCD becomes an bad theory when \( N \leq N_f \leq 2N - 2 \). It is known that whether the \( S^3 \) partition function diverge or not is related with the criterion of "bad" theories. The partition function of a "bad" theory is divergent \([21]\). This might be because the localization methods use the R-symmetry manifest in the UV to put the gauge theory on a compact manifold. Thus we should carefully treat with the number of the flavors.

Our aim of this paper is to study the \( S^3 \) partition function of real mass deformed theories in the infinite mass limit. For example, we consider that we give real mass to enough matter multiplets of "good" theory for the theory to become a "bad" theory after the massive matter fields are decoupled. It could be naively thought that the massive matter multiplets will be decoupled from the theory in that limit. However, a matrix model of a "bad" theory is not well-defined. It is interesting to know what happens to this matrix model in the infinite mass limit. Hence our interest is to know which hypermultiplets become effectively massless or massive in the infinite mass limit on three-sphere. In order to argue decoupling of matter fields, we must choose a vacuum when a theory is put on the flat space. However, there are no choices of vacuum of the theories on three-sphere. Especially, we calculate the sphere partition function with the help of so-called Coulomb branch localization and it is given by the integral over the classical Coulomb branch parameters. Namely, the three-sphere partition function is represented by the integrals of a part of vacua in terms of the theory on flat space. Thus it is not simple to argue whether the massive multiplets will be decoupled when we take infinite mass or not.

For example, we consider that \( U(2) \mathcal{N} = 4 \) SQCD with \( \frac{N_f}{2} \) pairs of hypermultiplets with real mass \( \pm m \). The figure shows the real parts of the two classical Coulomb branch parameter and there are some special points. When we fix a generic point of the Coulomb branch (blue dot), the effective theory is \( U(1) \times U(1) \) with massive matter fields and W-bosons while on a specific point like a green or red point, the effective theory has \( \frac{N_f}{2} \) or \( N_f \) massless hypermultiplets respectively. The origin (black dot) is also special in the sense that the gauge symmetry is enhanced to \( U(2) \). It is non-trivial which points dominantly contribute to the three-sphere partition function in the infinite mass limit because all the points of the Coulomb branch can contribute to it, including generic and above special ones.

To investigate this, we focus on the solution of the saddle point equation because the solution corresponds to a classical Coulomb branch point and in the large \( N \) limit the solution gives a dominant contribution to the sphere partition function. Hence we can argue decoupling of the massive matter fields and also which theory will appear as an effective theory on a point of the

\[ \text{Recent progress of "bad" theories in terms of the geometry of the moduli space of vacua is seen in \([17,19]\).} \]
\[ \text{The infinite mass limit of the matrix model of 3d gauge theories is also considered in \([24,27,30]\) in the context of finding new examples of Seiberg-like dualities \([28,29]\).} \]
\[ \text{The magnetic theory of a "bad" theory in terms of the Seiberg-like duality is considered as a good theory \([27]\).} \]
Figure 1: This figure schematically shows that real parts of the two classical Coulomb branch parameters of $U(2)$ $\mathcal{N} = 4$ SQCD with $N_f/2$ pairs of hypermultiplets with real mass $\pm m$. There are some special points where new massless degrees of freedom appear or gauge symmetry is enhanced to $U(2)$. Here we assume $\sigma_2 \geq \sigma_1$ due to the Weyl symmetry of $U(2)$.

Coulomb branch. We deduce the effective theory from the solution of saddle point equation in the infinite mass limit and confirm that the solution of the saddle point equation of the effective theory coincides with that of the original massive theory in the infinite mass limit.

Investigating the solution of the saddle point equation is just a way to know which point of Coulomb branch gives the dominant contribution to the partition function in the infinite mass limit. Even when we do not take the large $N$ limit, it is expected that there also exists a dominant point of the Coulomb branch and the matrix model becomes a specific effective theory in the infinite mass limit. This is because the mass infinite limit also corresponds to the decompactified limit ($r_{S^3} \to \infty$) and thus the point of the Coulomb branch should be chosen in this limit. We check this in the matrix models for small $N$ and confirm the effective theory is the same as that which we deduced from the calculations in the large $N$ limit. We conclude that this vacuum selection does not need the large $N$ limit, just only the infinite mass limit.

The rest of this paper is organized as follows: In section 2 we review the localization methods and introduce the building blocks of matrix models. In section 3 we solve the saddle point equation of $\mathcal{N} = 4$ SQCD with massless or massive matter fields and investigate the theory which appears in the infinite mass limit. In section 4 we calculate the partition function of finite rank SQCDs and evaluate the leading part in the infinite mass limit. In section 5 we end with a conclusion and some discussion. In appendix A we introduce the techniques of the resolvent methods which we use in this paper to solve the saddle point equation in the large $N$ limit. In appendix B we introduce mixed Chern-Simons terms which must appear in the infinite mass limit as 1-loop effects. We try to interpret what happens in the infinite mass limit in terms of the mixed Chern-Simons terms. In appendix C we discuss the convergent bound of the matrix.

$^{14}$The mass $m$ must appear as the combination $m r_{S^3}$ in the partition function. So we can not distinguish the infinite mass limit and the decompactified limit. In our convention we take $r_{S^3}$ to 1.
model and reconsider the matrix model of the effective theory in the infinite mass limit from
the viewpoint of the convergence bound of it. In appendix D we introduce an example which
becomes ABJM theory in the infinite mass limit while it is just a SQCD when \( m = 0 \).

2 Localization and matrix model

In this paper, we investigate the round three-sphere partition function of gauge theories. It
is given by the finite dimensional integral instead of the path integral with the help of the
localization technique \[20\]–\[23\]. To use the localization technique we put a gauge theory on \( S^3 \)
with preserving supersymmetry and deform the action on \( S^3 \) by a Q-exact term , where Q is a
generator of the supersymmetry. The partition function of the deformed action is independent
of the deformation parameter. Thus we take the parameter to infinite and the path integral
reduces to the finite dimensional matrix integral since the path integral is determined by the
finite dimensional saddle point configuration in the field configuration space. Since the saddle
point approximation is one-loop exact, the action of the matrix model is written by classical
and 1-loop parts as

\[
Z = \frac{1}{N!} \int \left( \prod_{i=1}^{\text{Rank}(G)} \, d\sigma_i \right) |J| Z_{\text{Classical}}(\sigma) Z_{\text{vec,1-loop}}(\sigma) Z_{\text{mat,1-loop}}(\sigma),
\]

where \(|J|\) is the usual Vandermonde determinant and \( Z_{\text{vec,1-loop}} \) and \( Z_{\text{mat,1-loop}} \) are 1-loop parts from
the vector multiplets and matter multiplets respectively. The integral valuable \( \sigma_i \) corresponds
to a eigenvalue of the scalar fields of a \( \mathcal{N} = 2 \) vector multiplet.

2.1 Vector multiplet

We consider here so-called Coulomb branch localization, where the integral valuables correspond
to the eigenvalues of the scalar fields \( \sigma \) of the vector multiplet. We only consider \( U(N) \) gauge
theories in this paper. The Yang-Mills term can not contribute to the partition function since
the Yang-Mills term is Q-exact. On the other hand, the Chern-Simons term can contribute to
it as a classical contribution, but, we do not consider it in this paper. The 1-loop part of the
vector multiplets is given by

\[
Z_{\text{vec,1-loop}}^{\text{vec,1-loop}}(\sigma) = \prod_{i<j}^N \frac{4 \sinh^2 \pi (\sigma_i - \sigma_j)}{\pi^2 (\sigma_i - \sigma_j)^2},
\]

where the denominator cancels against Vandermonde determinant, which appears when we
choose the diagonal gauge of \( \sigma \). When there is \( U(1) \) part of the gauge group, the FI term can
be introduced and contributes to the partition function as a classical term

\[
e^{2\pi i \zeta \sum_{i=1}^N \sigma_i}.
\]
2.2 Matter multiplet

Next we consider the contributions of chiral multiplets. The chiral multiplets can contribute to the partition function through only one-loop parts because its kinetic and superpotential terms are Q-exact. The 1-loop parts of chiral multiplets are determined by the representation both of the gauge group and the flavor symmetry. By weakly gauging a flavor symmetry we can couple its current with a background vector multiplets in a supersymmetric way. Thus we can give the corresponding scalar $\sigma_b$ an expectation value and regard it as a real mass for the chiral multiplets. Moreover we can give the chiral multiplets R-charge $[22, 23]$. However, we do not consider such a deformation in this paper and we consider the chiral multiplets have canonical dimension $\frac{1}{2}$.

The 1-loop part of the chiral multiplets in the representation $R$ of $U(N)$ is given by

$$\prod_{\rho} e^{\ell\left(\frac{1}{2} + i\rho(\sigma)\right)}, \quad (2.4)$$

where $\rho$ is a weight vector of the representation $R$. In [22] the function $\ell(z)$ is defined as

$$\ell(z) = -z \log \left(1 - e^{2\pi iz}\right) + \frac{i}{2} \left(\pi z + \frac{1}{\pi} \text{Li}_2(e^{2\pi iz})\right) - \frac{i\pi}{12}. \quad (2.5)$$

Its remarkable property we will often use is

$$e^{\ell\left(\frac{1}{2} + ix\right)}e^{\ell\left(\frac{1}{2} - ix\right)} = \frac{1}{2 \cosh \pi x}. \quad (2.6)$$

We focus on SQCDs, which are super Yang-Mills theories with $N_f$ pairs of chiral multiplets in the fundamental and anti-fundamental representation of $U(N)$. In this paper, we consider following two mass deformation: In case (i) we give real mass $m$ to $\frac{N_f}{2}$ flavors while we give real mass $-m$ to the remaining $\frac{N_f}{2}$ flavors. This breaks each SU($N_f$) of the flavor symmetry SU($N_f$)$\times$SU($N_f$) to SU($\frac{N_f}{2}$)$\times$SU($\frac{N_f}{2}$). Its total 1-loop part is given by

$$Z_{\text{1-loop}}^{\text{mat}}(\sigma) = \prod_{i=1}^{N_f} e^{\ell\left(\frac{1}{2} + i(\sigma_i + m)\right) + \ell\left(\frac{1}{2} + i(\sigma_i - m)\right) + \ell\left(\frac{1}{2} + i(-\sigma_i + m)\right) + \ell\left(\frac{1}{2} + i(-\sigma_i - m)\right)}$$

$$= \prod_{i=1}^{N_f} \frac{1}{2 \cosh \pi \left(\sigma_i + m\right) \cosh \pi \left(\sigma_i - m\right)} \left(\frac{N_f}{2}\right). \quad (2.7)$$

In case (ii) we give $\frac{N_f}{3}$ flavors real mass $m$ while we give other $\frac{N_f}{3}$ flavors real mass $-m$. Then we keep the remaining $\frac{N_f}{3}$ flavors massless. This real mass assignment breaks each of SU($N_f$)

\[\text{[15]}\] In an $\mathcal{N} = 4$ vector multiplet there exists an adjoint chiral multiplets in terms of the $\mathcal{N} = 2$ language. Then it seems that we must consider the 1-loop part of it. However, because its canonical R-charge is 1, the adjoint chiral multiplet does not contribute to the partition function without axial mass parameter [21][23].

\[\text{[16]}\] where we assume $\frac{N_f}{3}$ is integer.
flavor symmetry of the matter fields to $SU(N_f)\times SU(N_f)\times SU(N_f)$\textsuperscript{†7}. Its total 1-loop part of the chiral multiplets is given by

$$Z_{1\text{-loop}}^{\text{mat}}(\sigma) = \prod_{i=1}^{N} e^{\frac{N_f}{2} (\ell(\frac{1}{2}+i(\sigma_i+m))+\ell(\frac{1}{2}+i(-\sigma_i+m))+\ell(\frac{1}{2}+i(\sigma_i-m))+\ell(\frac{1}{2}+i(-\sigma_i-m))+\ell(\frac{1}{2}+i\sigma_i)+\ell(\frac{1}{2}-i\sigma_i))}$$

$$= \prod_{i=1}^{N} \frac{1}{(2 \cosh \pi (\sigma_i + m) 2 \cosh \pi (\sigma_i - m) 2 \cosh \pi \sigma_i)^{N_f}}. \quad (2.8)$$

3 Large $N$ solution and Coulomb branch point

3.1 SQCD with massless hypermultiplets

In this subsection, we solve the saddle point equation of $U(N)$ SQCD with massless hypermultiplets for latter use. The solution is given as the eigenvalue density function $\rho(x)$, which determine the large $N$ behavior of the theory. Its partition function is written by

$$Z = \frac{1}{N!} \int \prod_{i=1}^{N} d\lambda_j \prod_{i<j} 4 \sinh^2 (\pi (x_i - x_j)) \prod (2 \cosh(\pi x_i))^{N_f}. \quad (3.1)$$

It is difficult to calculate the partition function exactly because there are $N$ dimensional integral. In principle, it’s leading part in the large $N$ limit can be evaluated by the saddle point approximation. The saddle point equation for this theory is given by

$$0 = N_f \tanh(\pi x_i) - 2 \sum_{j \neq i} \coth(\pi(x_i - x_j)). \quad (3.2)$$

We assume that the eigenvalues become dense in the large $N$ limit and we take the continuous limit as following:

$$\frac{i}{N} \to s \in [0, 1], \quad x_i \to x(s), \quad \frac{1}{N} \sum_{i=1}^{N} \to \int ds. \quad (3.3)$$

The planer part of this saddle point equation is rewritten as a singular integral equation\textsuperscript{†8}

$$0 = \xi \tanh(\pi x) - 2 \left( \text{P} \int dy \rho(y) \coth(\pi(x - y)) \right), \quad (3.4)$$

\textsuperscript{†7}We assume $N_f$ is integer.

\textsuperscript{†8} We represent a principal value integral as

$$\text{P} \int dx.$$
where we also took \( N_f \) infinite with \( \xi \equiv \frac{N_f}{N} \) finite and introduced the density function \( \rho(x) \), which determines the large \( N \) behavior of the theory, defined as

\[
\frac{ds}{dx} \equiv \rho(x),
\]

and this means that we regard the values of the eigenvalues \( x \) as the fundamental variable. The density function \( \rho(x) \) counts the number of the eigenvalues which exists between \( x \) and \( x + dx \) and satisfy the normalization condition which depends on how to take the continuous limit.

\[
\int_I dx \rho(x) = 1.
\]

In order to solve the equation (3.5) and obtain the density function \( \rho(x) \), we use the resolvent methods. We give a brief summary of the resolvent methods in appendix A. We take \( e^{2\pi x} \equiv X \), and \( e^{2\pi y} \equiv Y \) and define the resolvent \( \omega(Z) \) and the potential \( V'(x) \) as

\[
\omega(X) = 2 \int_I dy \rho(y) e^{\pi(x-y)} + e^{-\pi(x-y)} = 2 \int_I dy \rho(y) \frac{X + Y}{X - Y} = 2 \left( 1 + \int_C \frac{dY}{\pi} \frac{\rho(Y)}{Y - X} \right),
\]

\[
V'(X) = \frac{X - 1}{X + 1} \xi,
\]

where \( I \) and \( C \) are intervals \([x_{\min}, x_{\max}]\) and \([b, a]\) respectably. The resolvent is determined from the analyticity and the one-cut solution of the resolvent is given by (A.14) as

\[
\omega(X) = \xi \left( \frac{X - 1}{X + 1} - \frac{2 \sqrt{(X - a) \sqrt{(X - \frac{1}{a})}}}{(X + 1) \sqrt{(1 + a)(1 + \frac{1}{a})}} \right) = \omega_0 \left( X; 1; a, \frac{1}{a} \right),
\]

where \( b = \frac{1}{a} \) because of the symmetry of the saddle point equation. We should carefully consider the branch of the square root. For latter convenience we introduced the following function:

\[
\omega_0(X; A; a, b) = \xi \left( \frac{X - A}{X + A} - \frac{2A \sqrt{(X - a) \sqrt{(X - b)}}}{(X + A) \sqrt{(1 + a)(1 + b)}} \right).
\]

The density function \( \rho(X) \) defined on \([\frac{1}{a}, a]\) is given by (A.8) as

\[
\rho(X) = \frac{\xi}{(X + 1) \sqrt{\frac{(X - \frac{1}{a})(a - X)}{(1 + a)(1 + \frac{1}{a})}}}.
\]

The end of the cut \( a \) is determined by the asymptotic behavior of the resolvent \( \omega(X) \) from (3.9). The asymptotic behavior in \( X \to 0 \) is decided by the following equation:

\[
-\frac{2}{\xi} = -1 + \frac{2}{\sqrt{(1 + a)(1 + \frac{1}{a})}}.
\]
The solution is given by

\[ a = \frac{\xi^2 + 4 \xi - 4 + 4\sqrt{(\xi - 1)\xi^2}}{(\xi - 2)^2}, \tag{3.13} \]

where this solution exist only when \( \xi \geq 2 \) since the right-hand-side of (3.12) is always more than \(-1\) as a function of \( a \).

Here we consider the relation between this large \( N \) solution and a point of the classical Coulomb branch. The equation (3.13) means that when we take \( r_{S^3} \) to infinite, \( x_{\min} \) and \( x_{\max} \) become 0 since the radius is recovered as \( x \rightarrow xr_{S^3} \) and \( a = e^{2\pi r_{S^3} x_{\max}} \). Thus the saddle point solution becomes condensed to the origin. Taking the radius to infinite corresponds to considering the theory on the flat space. So this solution corresponds to a point of the Coulomb branch of the theory on the flat space and it is the origin of the classical Coulomb branch. The origin of the Coulomb branch is most singular in the sense that all the massive W-boson on the generic point of Coulomb branch become massless. On this point, the theory at the deep IR of the RG flow expected to be an interacting superconformal field theory. It is expected that the sphere partition function of SQCD with massless hypermultiplets always represents that of a non-trivial SCFT.

The solution exists when \( \xi \geq 2 \). This reflects the bound of the convergence of the matrix model. In this paper, we will add real mass to matter fields while keeping the special flavor symmetry. Even for that case this bound always appears in our analysis.

### 3.1.1 Adding FI term

Here we consider \( U(N) \) gauge theories with \( N_f \) massless hypermultiplets with FI term. Especially, we consider imaginary FI terms in this section for the latter part of this paper, where it appears when we take the infinite mass limit as one-loop effects, which are certain mixed Chern-Simons terms. The density function is almost the same as that in the previous section. However, an FI term breaks the symmetry of the saddle point equation under the simultaneous change of the sign of all eigenvalues \( x_i \rightarrow -x_i \). For this theory the matrix model is written by

\[ Z = \frac{1}{N!} \int \prod_{i=1}^{N} dx_i \frac{e^{i \zeta \sum_i x_i} \prod_{i<j} 4 \sinh^2 (\pi (x_i - x_j))}{\prod_i \left( 2 \cosh \pi (x_i) \right)^{N_f}}, \tag{3.14} \]

where \( \zeta \in \mathbb{R} \) is a imaginary FI parameter in the sense that ordinary FI terms are considered as \( e^{i \pi \sum_i x_i} \). This FI term can be considered as the R-charge of monopole operator since the real part of the monopole operator is \( e^{-2\pi \Delta_m \sigma_i} \), where \( \Delta_m \) is R-charge of the monopole operator \([34, 37]\). Its saddle point equation in the continuous limit is

\[ 0 = \eta + \xi \tanh \pi x_i - 2 \left( \Pi \int dy \rho(y) \coth \pi (x - y) \right), \tag{3.15} \]
where we also take \( N_f \) and \( \zeta \) infinite while keeping 
\[
\xi \equiv \frac{N_f}{N}, \quad \eta \equiv \frac{\zeta}{N}, \tag{3.16}
\]
finite to solve the saddle point equation. We can solve this saddle point equation in the large \( N \) limit by resolvent methods. We define resolvent \( \omega(X) \) and potential \( V'(X) \) for this theory as
\[
\omega(X) \equiv 2 \int dy \rho(y) \frac{X + Y}{X - Y} = 2 \left( 1 + \int \frac{dY}{\pi} \frac{\rho(Y)}{X - Y} \right), \tag{3.17}
\]
\[
V'(X) \equiv \eta + \frac{X - 1}{X + 1} \xi. \tag{3.18}
\]
The resolvent \( \omega(Z) \) is obtained by the same calculation of that which appear in the previous section since a FI term does not change the singular structure of the resolvent,
\[
\omega(X) = \eta + \omega_0(X; A; a, b). \tag{3.19}
\]
The density function is given by the equation (A.8) as
\[
\rho(X) = \xi \left[ \frac{\sqrt{(a - X)(X - b)}}{(X + 1)\sqrt{(1 + a)(1 + b)}} \right], \tag{3.20}
\]
where \( a \) and \( b \) is determined by the equation from the asymptotic behavior of \( \omega(X) \) at \( X = 0, \infty \)
\[
\frac{\eta}{\xi} = \frac{1 - \sqrt{ab}}{\sqrt{(1 + b)(1 + a)}}, \quad 1 - \frac{2}{\xi} = \frac{1 + \sqrt{ab}}{\sqrt{(1 + b)(1 + a)}}. \tag{3.21, 3.22}
\]
Because a FI term breaks the \( \mathbb{Z}_2 \) symmetry under which \( x_i \to -x_i \) in the saddle point equation, the \( a \) and \( b \) do not satisfy the condition \( ab = 1 \). The solution of (3.21) and (3.22) is given by
\[
a = \frac{-4 + 4\xi + \xi^2 - \eta^2 + 4\sqrt{(\xi - 1)(\xi^2 - \eta^2)}}{(-2 + \xi + \eta)^2}, \quad b = \frac{-4 + 4\xi + \xi^2 - \eta^2 - 4\sqrt{(\xi - 1)(\xi^2 - \eta^2)}}{(-2 + \xi + \eta)^2}. \tag{3.23}
\]
From (3.21) and (3.22) we recognize that the solution only exists when
\[
\xi \geq 2 + |\eta|. \tag{3.24}
\]
This condition is same as the condition that the matrix model converges in the large \( N \) limit. In appendix C we will discuss the convergence bound of the matrix model of SQCDs.
3.2 SQCD with massive hypermultiplets

In this subsection, we consider $U(N)$ SQCD with $N_f$ pairs of chiral multiplets with real mass by weakly gauging its flavor symmetry and coupling its current to $\mathcal{N} = 2$ vector multiplets as background fields so that the matrix model is given by

$$Z = \frac{1}{N!} \int \prod_{i=1}^{N} d\lambda_i \frac{\prod_{i<j} 4 \sinh^2 (\pi (x_i - x_j))}{\prod_i (2 \cosh \pi (x_i + m) 2 \cosh \pi (x_i - m))^{\frac{N_f}{2}}} \ .$$  \hspace{1cm} (3.25)

When $m = 0$, this matrix model becomes that of $U(N)$ with $N_f$ massless fundamental hypermultiplets. When we take the infinite mass limit, if the massive matter multiplets decouple, the matrix model is not well-defined. Therefore we investigate what happens to this matrix model of the theory in the infinite mass limit.

The saddle point equation is written as

$$2 \sum_i \coth \pi (x_i - x_j) = \frac{N_f}{2} \left( \tanh \pi (x_i + m) + \tanh \pi (x_i - m) \right), \hspace{1cm} (3.26)$$

and in the continuous limit it becomes

$$4 \left( P \int dy \rho(y) \coth \pi (x - y) \right) = \xi \left( \tanh \pi (x + m) + \tanh \pi (x - m) \right), \hspace{1cm} (3.27)$$

Next we define the resolvent $\omega(X)$ and potential $V'(X)$ as

$$\omega(X) = 4 \int dy \rho(y) \frac{X + Y}{X - Y} = 4 \left( 1 + \int \frac{dY}{\pi} \frac{\rho(Y)}{X - Y} \right), \hspace{1cm} (3.28)$$

$$V'(X) = \xi \left( \frac{X - M^{-1}}{X + M^{-1}} + \frac{X - M}{X + M} \right), \hspace{1cm} (3.29)$$

where $M = e^{2\pi m}$. The resolvent is determined by its analytic properties \textit{(A.14)} as

$$\omega(X) = \omega_0 \left( X; M; a, b \right) + \omega_0 \left( X; M^{-1}; a, b \right). \hspace{1cm} (3.30)$$

The density function is given by \textit{(A.8)} as

$$\rho(X) = \frac{\xi}{2} \left[ \frac{M \sqrt{(a - X)(X - b)}}{(X + M) \sqrt{(M + a)(M + b)}} + \frac{M^{-1} \sqrt{(a - X)(X - b)}}{(X + M^{-1}) \sqrt{(M^{-1} + a)(M^{-1} + b)}} \right]. \hspace{1cm} (3.31)$$

The constant $a$ and $b$ are decided by the symmetry and the asymptotic behavior when $Z = 0,

$$-4 = 2 \xi \left( -1 + \frac{1}{\sqrt{(M + a)(M + \frac{1}{a})}} + \frac{1}{\sqrt{(M^{-1} + a)(M^{-1} + \frac{1}{a})}} \right). \hspace{1cm} (3.32)$$
This equation immediately means that $a$ exists when $\xi \geq 2$. We conclude that this type of mass deformation does not affect the bound of the existence of the solution. Here $a$ is given by

$$a = \frac{2(\xi - 1)(M^2 + 1) + M\xi^2 + 2(M + 1)\sqrt{(\xi - 1)(\xi - 1 + M^2(\xi - 1) + M(\xi^2 - 2\xi + 2))}}{M(\xi - 2)^2}.$$  \tag{3.33}$$

Figure 2: These figure show that the numerical solution (Blue dots) and the analytic solution $\rho(x)$ (Green line). The left one is with parameter $(N, N_f, m)=(100,2000,2)$. The right one is with parameter $(N, N_f, m)=(100,800,0.5)$.

In this theory, when we take the infinite mass limit, it is naively considered that the theory goes to a bad theory and its matrix model diverges. However, this argument is not correct in the following sense: the density function has the peak around $\pm m$ and the eigenvalues gather around these peaks as $m$ becomes large. Thus in the large $N$ limit the partition function of this massive SQCD corresponds to that of the effective theory on the point of the Coulomb branch points where the half of the eigenvalues sit on $+m$ and the others sit on $-m$ like as

$$\sigma = \begin{pmatrix} -m1_{\frac{N}{2} \times \frac{N}{2}} & 0 \\ 0 & m1_{\frac{N}{2} \times \frac{N}{2}} \end{pmatrix}. \tag{3.34}$$

In fact, this argument is confirmed following direction: we assume that the eigenvalues are separated like as

$$x_i = \begin{cases} m - \lambda_i & (i = 1, \ldots \frac{N}{2}), \\ -m - \tilde{\lambda}_i & (i = \frac{N}{2} + 1 \ldots N), \end{cases} \tag{3.35}$$

where we assume that $\lambda_i$ and $\tilde{\lambda}_i$ do not depend on $m$. The saddle point equations (3.26) for the first $\frac{N}{2}$ eigenvalues are written as

$$0 = -2 \sum_{j \neq i} \coth \pi (\lambda_i - \lambda_j) - 2 \sum_j \coth \pi \left( \lambda_i - \tilde{\lambda}_j - 2m \right) + \frac{N_f}{2} \left( \tanh \pi \lambda_i + \tanh \pi \left( \lambda_i - 2m \right) \right).$$
\[ 0 = N \left( \frac{N_f}{2N} - 1 \right) + 2 \sum_{j \neq i}^{N} \coth \pi (\lambda_i - \lambda_j) - \frac{N_f}{2} \tanh \pi \lambda_i, \tag{3.36} \]

where we took the infinite mass limit in the second line and we note that the first term can be interpreted as the gauge-R mixed Chern-Simons term \[31, 33\] induced by integrating out massive gauginos and complex fermions of chiral multiplets. For the latter \( \frac{N_f}{2} \) eigenvalues the saddle point equation in the large mass limit is almost same as (3.36),

\[ 0 = N \left( 1 - \frac{N_f}{2N} \right) + 2 \sum_{j \neq i}^{N} \coth \pi (\tilde{\lambda}_i - \tilde{\lambda}_j) - \frac{N_f}{2} \tanh \pi \tilde{\lambda}_i. \tag{3.37} \]

The equations (3.36) and (3.37) mean that in the infinite mass limit the matrix model (3.25) becomes \[9\]

\[ Z \sim Z_{\text{Massive}}(m) \int d^2 \lambda e^{\pi N \left( \frac{N_f}{2N} - 1 \right) \sum_i \lambda_i} \prod_{i<j} (2 \sinh \pi (\lambda_i - \lambda_j))^2 \prod_i \left( 2 \cosh \pi \lambda_i \right)^{\frac{N_f}{2}} \times \int d^2 \tilde{\lambda} e^{-\pi N \left( \frac{N_f}{2N} - 1 \right) \sum_i \tilde{\lambda}_i} \prod_{i<j} \left( 2 \sinh \pi (\tilde{\lambda}_i - \tilde{\lambda}_j) \right)^2 \prod_i \left( 2 \cosh \pi \tilde{\lambda}_i \right)^{\frac{N_f}{2}}, \tag{3.38} \]

because the saddle point equation is equivalent to (3.36) and (3.37). The factor \( Z_{\text{Massive}}(m) \) is a part of the decoupled free massive degrees of freedom. We can evaluate \( Z_{\text{Massive}} \sim M^{-\frac{N}{2}(N_f - N)} \).

This part can not be read from the saddle point equations. This matrix model is made by the two \( U(\frac{N}{2}) \) with \( \frac{N_f}{2} \) fundamental hypermultiplets FI parameter \( \pm N \left( 1 - \frac{N_f}{2N} \right) \) \[10\] SQCD theories. As we note before, the FI term is induced by one-loop effects as the mixed Chern-Simons term from vector multiplets of gauge and R-symmetry by integrating out the effectively massive fermions. We argue this point in appendix \[13\]. This FI term can not appear when we consider the gauge theories on the flat space.

In fact, we check our claim by comparing the density function of the matrix model of the effective theory (3.38) with that of the matrix model (3.25) in the infinite mass limit. First, we consider the density function of SQCD with massive hypermultiplets (3.31) in the infinite mass limit. We rewrite \( X \) as \( X = MZ \) and assume \( X \) is order \( \mathcal{O}(M^0) \). This procedure corresponds to the simultaneous shift of \( x_i \) by \( m \) and focusing on the peak of the density function around \( +m \). We have to consider the expansion of \( a \) (3.33) around \( m = \infty \). it is given by

\[ a = \alpha M + \mathcal{O}(M^0), \quad \alpha \equiv \frac{4(\xi - 1)}{(\xi - 2)^2}. \tag{3.39} \]

\[ ^{19} \text{The overall factor of the matrix model can not be determined in this procedure.} \]

\[ ^{10} \text{Exactly speaking, the FI parameter is given by } \frac{1}{\pi^2} \left( 1 - \frac{N_f}{2N} \right) \text{ if we recover the radius of } S^3 \text{ because in three-dimensional theory a FI parameter has mass dimension 1.} \]
Thus the density function is expanded around $m = \infty$ as

$$\rho(Z) = \frac{\xi}{2(Z+1)} \sqrt{\frac{Z(\alpha - Z)}{1+\alpha}} + \mathcal{O}(M^{-1}), \quad (3.40)$$

where $Z \in [0, \alpha]$ in the infinite mass limit. Then we compare this with that of the $\lambda$ part of the matrix model (3.38) since $\lambda$ part corresponds to the part concentrated around $m$ of the eigenvalues of the massive SQCD. The solution of its saddle point equation (3.36) is just given by the calculation of section 3.1.1. In this case, $a$ and $b$ are

$$a = \alpha, \quad b = 0, \quad (3.41)$$

and its density function is

$$\rho(Z) = \frac{\xi}{2(Z+1)} \sqrt{\frac{Z(\alpha - Z)}{1+\alpha}}, \quad (3.42)$$

where the additional $\frac{1}{2}$ factor due to the fact that the effective theory has two $U(\frac{N}{2})$ as the gauge group and the normalization condition should be taken as

$$\int \frac{dZ}{2\pi} \rho(Z) = \frac{1}{2}. \quad (3.43)$$

The density function (3.40) and (3.42) are completely equivalent. Next we should consider the part concentrated around $-m$. In this time we must rewrite $X = M^{-1}Z$ in (3.31) and the density function is in this limit

$$\rho(Z) = \frac{\xi}{2(Z+1)} \sqrt{\frac{Z - \frac{1}{\alpha}}{1 + \frac{1}{\alpha}}} + \mathcal{O}(M^{-1}), \quad (3.44)$$

where $Z \in \left[\frac{1}{\alpha}, \infty\right]$. Then we consider the $\tilde{\lambda}$ part of the (3.38). The solution of its saddle point equation is just given by applying the result of 3.1.1 to (3.37) and we obtain

$$a = \infty, \quad b = \frac{1}{\alpha}, \quad (3.45)$$

and its density function is

$$\rho(Z) = \frac{\xi}{2(Z+1)} \sqrt{\frac{Z - \frac{1}{\alpha}}{1 + \frac{1}{\alpha}}}. \quad (3.46)$$

This is the same as (3.44). Therefore we conclude that SQCD with $N_f$ massive hypermultiplets we study here become two SQCDs in the infinite mass limit: $U(\frac{N}{2})$ SQCD with $\frac{N_f}{2}$ massless hypermultiplets and FI term $\tilde{\zeta} = \pm iN(\frac{N_f}{N} - 1)$. This result suggests that if we assume the massive matter fields will be decoupled, the sphere partition function of a massive theory
which would become a bad theory always become that of a specific effective theory. This means that an interacting SCFTs on the specific singular point of the Coulomb branch appears in the infinite mass limit instead of appearing of a bad theory. This result may also suggest that the massive theory can not be used as the UV regularization of the bad theory. In section 4 we will check our claim from the exact calculation of the partition function of finite rank SQCD. It is expected that the partition function can be written as the sector of the decoupled free massive multiplets and that of the effective theory in the infinite mass limit.

3.3 SQCD with massive and massless hypermultiplets

In the previous subsection, all matter fields of the theory are massive. In this subsection, we consider the SQCD theory with both massive and massless matter fields. It is expected that the asymptotic behavior of the partition function in the infinite mass limit depends on the number of the massless matter fields since the sufficient number of matter fields make the matrix model to converge.

We consider U($N$) SQCD with $N_f^3$ pairs of massive hypermultiplets with $\pm m$ and $N_f^3$ massless hyper multiplets. We assume $N_f^3$ is integer. The matrix model is given by

$$Z = \frac{1}{N!} \int \prod_{i=1}^N dx_i \prod_{i<j} \frac{4 \sinh^2(\pi (x_i - x_j))}{2 \cosh \pi (x_i + m) \cosh \pi (x_i - m) \cosh \pi (x_i)}^{N_f^3}. \tag{3.47}$$

The saddle point equation is given by

$$2 \sum_{j \neq i} \coth \pi (x_i - x_j) = \frac{N_f}{3} \left( \tanh \pi (x_i + m) + \tanh \pi (x_i - m) + \tanh \pi x_i \right), \tag{3.48}$$

and we take the continuous limit of this. It is written as

$$6 \left( P \int_\mathcal{C} dy \coth \pi (\lambda_i - \lambda_j) \right) = \xi \left( \tanh \pi (x + m) + \tanh \pi (x - m) + \tanh \pi x \right). \tag{3.49}$$

We define the resolvent $\omega(X)$ and potential $V'(X)$ as

$$\omega(X) = 6 \int dy \rho(y) \frac{X + Y}{X - Y} = 6 \left( 1 + \int \frac{dY}{\pi} \frac{\rho(Y)}{X - Y} \right), \tag{3.50}$$

$$V'(X) = \xi \left( \frac{X - 1}{X + 1} + \frac{X - M}{X + M} + \frac{X - M^{-1}}{X + M^{-1}} \right), \tag{3.51}$$

where $M = e^{2\pi m}$. The resolvent is obtained by (A.14) as

$$\omega(X) = \omega_0(X; 1; a, \frac{1}{a}) + \omega_0(X; M; a, \frac{1}{a}) + \omega_0(X; M^{-1}; a, \frac{1}{a}). \tag{3.52}$$
The cut $\mathcal{C} = [\frac{1}{a}, a]$ is determined by the following asymptotic equation;

$$-\frac{6}{\xi} = -3 + \frac{2}{\sqrt{(1 + a)(1 + \frac{1}{a})}} + \frac{2}{\sqrt{(M + a)(M + \frac{1}{a})}} + \frac{2}{\sqrt{(M^{-1} + a)(M^{-1} + \frac{1}{a})}}. \quad (3.53)$$

Unfortunately there are generally no explicit forms of the solution because this equation corresponds to the octic equation of $a$. However, we can know the solution of it numerically or in the infinite mass limit. The density function of this case is given by (A.8) as

$$\rho(X) = \frac{\xi}{3} \left[ \frac{M\sqrt{(a - X)(X - b)}}{(X + M)\sqrt{(M + a)(M + b)}} + \frac{M^{-1}\sqrt{(a - X)(X - b)}}{(X + M^{-1})\sqrt{(M^{-1} + a)(M^{-1} + b)}} \right. + \left. \frac{\sqrt{(a - X)(X - b)}}{(X + 1)\sqrt{(1 + a)(1 + b)}} \right]. \quad (3.54)$$

Let us consider what happens to the matrix model when the number of the matter fields varies. When $\frac{N_f}{3} \geq 2N$, the matrix model is still well-defined after we take the infinite mass limit and all massive matter fields decouple from the theory. In fact, in this case, the limit takes mass to infinity and the integrals of the matrix model commute $^{111}$. This immediately means that all the massive matter fields decoupled and the remaining theory is $U(N)$ SQCD with $\frac{N_f}{3}$ massless matter fields. This situation is reflected into the equation (3.53). We assume that the solution does not depend on $M$ $^{112}$ when we take mass $m$ to infinity. Then the equation (3.53) becomes the same equation as (3.12) for the case with the number of the flavor $\frac{N_f}{3}$

$$3 - \frac{6}{\xi} = \frac{2}{\sqrt{(1 + a)(1 + \frac{1}{a})}}. \quad (3.56)$$

This means that the solution of (3.53) which does not depend on mass $m$ can exist when $\frac{N_f}{3} \geq 2N$ while the constant solution can not exist in the infinite mass limit when $\frac{N_f}{3} < 2N$. The numerical analysis of (3.53) supports existence of such a solution. Indeed, the density function is the same as that of $U(N)$ gauge theory with $\frac{N_f}{3}$ massless hypermultiplets and this means that all the massive hypermultiplets decouple from the theory since in the infinite mass limit the origin of the Coulomb branch is dominant.

On the other hand, when $\frac{N_f}{3} < 2N$, $a$ is proportional to $M$ in the infinite mass limit and the density function has three peaks at the origin and two-point around $x = \pm m$. We show $^{111}$ In this paper, we focus on just the leading part of the mass infinite limit. Namely, when there exist a finite constant $\alpha$ and $\beta$ so that the relation

$$\lim_{M \to \infty} \left( \int_{-\infty}^{\infty} dx f(x, M)M^\alpha \right) = \int_{-\infty}^{\infty} dx \lim_{M \to \infty} (f(x, M)M^\alpha) = \beta, \quad (3.55)$$

is satisfied for $f(x, M)$, which is a function of $x$ and $M$, we say that the infinite integral commutes with the limit of $M$.

$^{112}$This assumption means that the effectively massless degrees of freedom can not appear.
the behavior of the density function $\rho(x)$ in figure 3. Then we study the effective theory which appears in this situation by analyzing the behavior of the density function when we take the infinite mass limit. First, we have to know how the gauge group $U(N)$ is broken into. From the density function we know that $U(N)$ breaks to three parts. Thus we assume

$$U(N) \to U(N_1) \times U(N_2) \times U(N_3), \quad (N_1 + N_2 + N_3 = N).$$

(3.57)

The rank of each three gauge groups is determined by the ratio of the number of the eigenvalues around each peak. The density function $\rho(x)$ counts the number of the eigenvalues between $x$ and $x + dx$. Therefore we count the number of the eigenvalues which exist around each peak by integrating corresponding density function in the infinite mass limit and determine $N_1$, $N_2$ and $N_3$.

We assume that there exists a solution proportional to $M$. The equation (3.53) becomes in the infinite mass limit

$$1 - \frac{2}{\xi} = \frac{2}{3\sqrt{(1 + \beta)}},$$

(3.58)

where we assume $a = M\beta$. We immediately determine $\beta$ as

$$\beta = \frac{(5\xi - 6)(-\xi + 6)}{9(\xi - 2)^2}.$$  

(3.59)

In order to know the behavior of the density function around $x = m$, we redefine $X$ by order $O(M^0)$ variable $Z$ as $X = MZ$ and take $M \to \infty$. The density function (3.54) becomes in this limit

$$\rho_\pm(Z) = \frac{\xi}{3(Z + 1)} \sqrt{\frac{Z(\beta - Z)}{1 + \beta}},$$

(3.60)

where $Z \in [0, \beta]$. Next we look at the density function around the peak on $x = -m$ by regarding $X$ as $X = M^{-1}Z$ in (3.54). It becomes by the same calculation of (3.54).

$$\rho_\pm(Z) = \frac{\xi}{3(Z + 1)} \sqrt{\frac{Z - \frac{1}{\beta}}{1 + \frac{1}{\beta}},}$$

(3.61)

where $Z \in [\frac{1}{\beta}, \infty]$. The final part is the density function around $x = 0$. To investigate this part of the density function, we assume $X$ is order $O(M^0)$. Then we take the infinite mass limit and the density function becomes

$$\rho_0(Z) = \frac{\xi\sqrt{Z}}{3(Z + 1)},$$

(3.62)

where $Z$ takes value $\in [0, \infty]$. To know $N_1$, $N_2$ and $N_3$ we integrate (3.60), (3.61) and (3.62).
Figure 3: These figure shows the density function $\rho(x)$ (Green line) and numerical one from the saddle point equation (Blue dots). The left and right figure correspond to $(N,N_f,m)=$(200,1000,3) and (200,420,3) respectively.

We obtain

\[
\begin{align*}
\int_0^\beta dY \frac{2\pi Y}{2}\rho_+(Y) &= \frac{6 - \xi}{12}, \\
\int_\frac{1}{2}^\infty dY \frac{2\pi Y}{2}\rho_-(Y) &= \frac{6 - \xi}{12}, \\
\int_0^\infty dY \frac{2\pi Y}{2}\rho_0(Y) &= \frac{\xi}{6}.
\end{align*}
\]

This result means that the gauge group $U(N)$ is broken to as following:

\[
N_1 = \frac{\xi}{6} N, \quad N_2 = N_3 = \frac{6 - \xi}{12} N,
\]

where we assume that $\frac{\xi}{6} N$ and $\frac{6 - \xi}{12} N$ are integer. This means that in the infinite mass limit the theory becomes the effective theory on a point of the Coulomb branch like as

\[
\sigma = \begin{pmatrix}
-m1_{N_2 \times N_2} & 0_{N_1 \times N_1} \\
0_{N_1 \times N_1} & m1_{N_2 \times N_2}
\end{pmatrix}.
\]

We assume that the eigenvalues are separated as

\[
x_i = \begin{cases}
-m - \lambda_1^i, & (i = 1, \ldots, N_2), \\
\lambda_2^i, & (i = N_2 + 1, \ldots N_1 + N_2), \\
m - \lambda_3^i & (i = N_1 + N_2 + 1, \ldots, N).
\end{cases}
\]

With the similar calculation in the previous subsection the saddle point equation is rewritten as the following three parts:

\[
0 = 2N \left( \frac{6 + \xi}{12} - \frac{\xi}{3} \right) + 2 \sum_{j \neq i}^{N_2} \coth \pi (\lambda_1^i - \lambda_1^j) - \frac{N_f}{3} \tanh \pi \lambda_1^i, \quad (i = 1, \ldots N_2),
\]

\[
\left(\begin{array}{c}
0_{N_1 \times N_1} & m1_{N_2 \times N_2} \\
-m1_{N_2 \times N_2} & 0_{N_1 \times N_1}
\end{array}\right)
\]

We assume that the eigenvalues are separated as

\[
x_i = \begin{cases}
-m - \lambda_1^i, & (i = 1, \ldots, N_2), \\
\lambda_2^i, & (i = N_2 + 1, \ldots N_1 + N_2), \\
m - \lambda_3^i & (i = N_1 + N_2 + 1, \ldots, N).
\end{cases}
\]

With the similar calculation in the previous subsection the saddle point equation is rewritten as the following three parts:

\[
0 = 2N \left( \frac{6 + \xi}{12} - \frac{\xi}{3} \right) + 2 \sum_{j \neq i}^{N_2} \coth \pi (\lambda_1^i - \lambda_1^j) - \frac{N_f}{3} \tanh \pi \lambda_1^i, \quad (i = 1, \ldots N_2),
\]

\[
\left(\begin{array}{c}
0_{N_1 \times N_1} & m1_{N_2 \times N_2} \\
-m1_{N_2 \times N_2} & 0_{N_1 \times N_1}
\end{array}\right)
\]

We assume that the eigenvalues are separated as

\[
x_i = \begin{cases}
-m - \lambda_1^i, & (i = 1, \ldots, N_2), \\
\lambda_2^i, & (i = N_2 + 1, \ldots N_1 + N_2), \\
m - \lambda_3^i & (i = N_1 + N_2 + 1, \ldots, N).
\end{cases}
\]

With the similar calculation in the previous subsection the saddle point equation is rewritten as the following three parts:

\[
0 = 2N \left( \frac{6 + \xi}{12} - \frac{\xi}{3} \right) + 2 \sum_{j \neq i}^{N_2} \coth \pi (\lambda_1^i - \lambda_1^j) - \frac{N_f}{3} \tanh \pi \lambda_1^i, \quad (i = 1, \ldots N_2),
\]

\[
\left(\begin{array}{c}
0_{N_1 \times N_1} & m1_{N_2 \times N_2} \\
-m1_{N_2 \times N_2} & 0_{N_1 \times N_1}
\end{array}\right)
\]
\[ 0 = 2 \sum_{j \neq i}^N \coth \pi (\lambda_i^2 - \lambda_j^2) - \frac{N_f}{3} \tanh \pi \lambda_i^2, \quad (i = 1, \ldots N_1) \] (3.70)

\[ 0 = -2N \left( \frac{6 + \xi}{12} - \frac{\xi}{3} \right) + 2 \sum_{j \neq i}^N \coth \pi (\lambda_i^3 - \lambda_j^3) - \frac{N_f}{3} \tanh \pi \lambda_i^3, \quad (i = 1, \ldots N_2). \] (3.71)

These equations mean that the matrix model (3.25) in the infinite mass limit becomes the following matrix model:

\[
Z_{\text{massive}}(m) \int d^N \lambda \frac{e^{2\pi N \left( \frac{\xi - 6 + \xi}{12} \right) \sum_i \lambda_i^7 \prod_{i<j}^N \left( \frac{2 \sinh \pi (\lambda_i^2 - \lambda_j^2)}{\prod_{i<j}^N (2 \cosh \pi \lambda_i^2)} \right)^2 \prod_i^N}}{\prod_i^N \left( 2 \cosh \pi \lambda_i^3 \right)^{\frac{N_f}{3}}} \int d^N \lambda \frac{e^{-2\pi N \left( \frac{\xi - 6 + \xi}{12} \right) \sum_i \lambda_i^7 \prod_{i<j}^N \left( \frac{2 \sinh \pi (\lambda_i^3 - \lambda_j^3)}{\prod_{i<j}^N (2 \cosh \pi \lambda_i^3)} \right)^2 \prod_i^N}}{\prod_i^N \left( 2 \cosh \pi \lambda_i^1 \right)^{\frac{N_f}{3}}}.
\] (3.72)

Two \( U(N_2) \) have FI terms which also come from the gauge-R-symmetry mixed Chern-Simons term we discussed in the previous section and \( U(N_1) \) part has no FI terms since there are pairs of the mixed Chern-Simons terms which have opposite overall sign corresponding to the sign of masses of effectively massive fermions. The decoupled massive free sector can be estimated by \( Z_{\text{massive}}(m) \sim M^{-\frac{N_f N(6 + \xi)}{36}} \).

In fact, we can confirm that each density function obtained in the infinite mass limit is same as each density function obtained from (3.69), (3.70) and (3.71). First, the solution of (3.69) is given by (3.11) and (3.13). We obtain

\[ a = \infty, \quad \rho(Z) = \frac{\xi \sqrt{Z}}{3\pi (Z + 1)}, \] (3.73)

where \( Z \in [0, \infty) \) and we assume \( Z \) is zero since when we scale \( Z = a \tilde{Z}, \rho(\tilde{Z}) \) is \( \mathcal{O}(\frac{1}{a}) \) and only order \( \mathcal{O}(a^0) \) part of \( Z \) can contribute to \( \rho(Z) \). This density function is same as \( \rho_0(Z) \). Next we consider the solution of (3.70). we can obtain its solution with the equations (3.23) as

\[ a = \infty, \quad b = \frac{(\tilde{\xi} - 2)^2}{4(\tilde{\xi} - 1)} = \frac{1}{\beta}, \quad \tilde{\xi} \equiv \frac{4\xi}{6 - \xi}, \] (3.74)

and the density function is given by

\[ \rho(Z) = \frac{6 - \xi}{12} \frac{\tilde{\xi}}{(Z + 1)} \sqrt{\frac{Z - \frac{1}{\beta}}{1 + \frac{1}{\beta}}} = \frac{\xi}{3(Z + 1)} \sqrt{\frac{Z - \frac{1}{\beta}}{1 + \frac{1}{\beta}}}, \] (3.75)

where \( Z \in [rac{1}{\beta}, \infty) \). This corresponds to \( \rho_{-}(X) \). Finally we solve (3.71). Its solution is given by the same way of (3.70) case. The solution is given as

\[ a = \frac{4(\tilde{\xi} - 1)}{(\xi - 2)^2} = \beta, \quad b = 0, \] (3.76)
and

$$\rho(Z) = \frac{6 - \xi}{12} \frac{\tilde{\xi}}{Z + 1} \sqrt{\frac{Z(\beta - Z)}{1 + \beta}} = \frac{\xi}{3(Z + 1)} \sqrt{\frac{Z(\beta - Z)}{1 + \beta}}. \quad (3.77)$$

This density function is same as $\rho_+(Z)$. In the above calculation the normalization condition of each density functions is decided to corresponds to the ratio of the rank of the gauge groups (3.66). We conclude that the matrix model (3.47) becomes the matrix model (3.38) when we take the infinite mass limit. The above result shows that the massive multiplets can not decouple from the theory so that the matrix model of the theory diverges, in other words, the theory can not become a bad theory after the massive matter fields of a good theory are decoupled though the matrix model can become a good theory after decoupling of the massive matter multiplets. The remarkable result is that the gauge group of the effective theory depends on the number of flavors. When $\xi = 2$, the effective theory is three SQCDs whose gauge group $U(\frac{N}{3})$ with $\frac{N_{_{_f}}}{3}$ hypermultiplets and the gauge group is maximally broken between $2 \leq \xi \leq 6$. Then the gauge group recovers to $U(N)$ as $\xi$ increases. We note that the matrix model of the effective theory when $2 \leq \xi \leq 6$ is always convergent.

4 Finite rank SQCD

Especially, in SQCD cases, the matrix model can be actually calculated at least for a sufficiently low rank of the gauge group. In this section, we confirm by the exact results that our argument of the effective theory is true even in finite $N$ cases and what happens in the infinite mass limit the theories which are not covered by our argument in the large $N$ limit.

4.1 With massive hypermultiplets

$U(1)$ SQED

The matrix model of SQED with massive matter fields is given by

$$Z^{N_{_{_f}}} U(1) = \int_{-\infty}^{\infty} dx \frac{1}{(2 \cosh \pi(x + m) 2 \cosh \pi(x - m))^{\frac{N_{_{_f}}}{2}}}.$$ (4.1)

This model is considered in [10] with FI term. In this case, this theory may not become the effective theory which we expected in the previous section since $\frac{N}{2}$ is not integer number. Thus we want to know what happens in this case when $m \to \infty$. The exact result for any $N_{_f}$ is written by the hypergeometric function as [10]

$$Z^{N_{_{_f}}} U(1) = \frac{\Gamma(\frac{N_{_{_f}}}{2})}{2^{\frac{N_{_{_f}}}{2}} \sqrt{2\pi} \Gamma(\frac{N_{_{_f}}}{2} + \frac{1}{2})} \frac{1}{(\cosh 2\pi m + 1)^{\frac{N_{_{_f}}}{2}}} 2F_1 \left(\frac{1}{2}, \frac{1}{2}, \frac{N_{_{_f}}}{2} + \frac{1}{2}; \frac{1}{2}(1 - \cosh 2\pi m)\right). \quad (4.2)$$
The leading part in the infinite mass limit of this partition is given by

\[ Z_{N_f}^{U(1)} \xrightarrow{m \to \infty} \frac{1}{N_f} \frac{\log M}{M^{\frac{N_f}{2}}}, \quad M \equiv e^{2\pi m}. \] (4.3)

This is strange in the sense of our argument so far since there cannot exist \( \log M \) term when a massive theory splits into a decoupled sector and a SCFT sector in the mass infinite limit. So we may not know what the effective theory is in this case from the previous our argument.

**U(2) SQCD**

The partition function for this theory is given by

\[ Z_{N_f}^{U(2)} = \frac{1}{2^N} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{4 \sinh^2(2\pi \xi)}{\cosh(2\pi M (x + m)) \cosh(2\pi M (x - m)) \cosh(2\pi M (y + m)) \cosh(2\pi M (y - m))}, \] (4.4)

and the results for small \( N_f \) are summarized in the following table:

| \( N_f = 4 \) | \( N_f = 6 \) | \( N_f = 8 \) | \( N_f = 10 \) |
|--------------|-------------|-------------|-------------|
| \( \frac{1}{(2\pi)^2 M^2} \) | \( \frac{1}{(4\pi)^2 M^4} \) | \( \frac{1}{(6\pi)^2 M^6} \) | \( \frac{1}{(8\pi)^2 M^8} \) |

Table 1: The leading part of \( Z_{U(2)}^{N_f} \) when \( m \to \infty \).

In fact these results show that in the infinite mass limit the partition function can be interpreted as that of the expected theory from the large \( N \) calculation since the following relation is checked:

\[ Z_{\text{Massive}} Z_{U(1) \times U(1)} = \frac{4 \sinh^2(2\pi m)}{(2 \cosh(2\pi m))^{N_f}} \int_{-\infty}^{\infty} dx \frac{e^{\pi(2 - \frac{N_f}{2})x}}{(2 \cosh(\pi x))^{N_f}} \int_{-\infty}^{\infty} dx \frac{e^{-\pi(2 - \frac{N_f}{2})x}}{(2 \cosh(\pi x))^{N_f}} \xrightarrow{m \to \infty} \frac{1}{((N_f - 2)\pi)^2 M^{N_f - 2}}, \] (4.5)

where the two integrals mean \( U(1) \times U(1) \) theory and the prefactor \( Z_{\text{massive}} \) means decoupled massive free sector whose denominator comes from massive hypermultiplets and numerator comes from the vector multiplets. Because \( \frac{N_f}{2} \) is integer \( U(2) \) can be broken to \( U(1) \times U(1) \). In this case, we see that the partition functions are equal including the overall factor, which can not be checked from the large \( N \) analysis.
U(3) SQCD

The partition function for this case is given by

\[ Z_{U(3)}^{N_f} = \frac{1}{3!} \int_{-\infty}^{\infty} dxdydz \frac{4 \sinh^2 \pi(x - y)4 \sinh^2 \pi(x - z)4 \sinh^2 \pi(y - z)}{(2 \cosh \pi(x \pm m)2 \cosh \pi(y \pm m)2 \cosh \pi(z \pm m))^{N_f}}, \] (4.6)

where we define

\[ 2 \cosh \pi(X \pm Y) \equiv 2 \cosh \pi(X + Y)2 \cosh \pi(X - Y). \] (4.7)

These results for small \( N_f \) are given by the following table:

| \( N_f \) | \( \log M^{\frac{N_f}{3}} \) |
|-----------|-----------------|
| 6         | \( \frac{1}{16\pi^3M^3} \) |
| 8         | \( \frac{1}{144\pi^3M^3} \) |
| 10        | \( \frac{1}{576\pi^3M^{11}} \) |
| 12        | \( \frac{1}{1600\pi^3M^{11}} \) |

Table 2: The leading part of \( Z_{U(3)}^{N_f} \) when \( m \to \infty \)

From these results it may also be impossible that the effective theory is composed by an SCFT and a massive free sector because of the same reason for U(1) case, namely, \( \frac{N_f}{2} \) is not integer.

4.2 With massive and massless hypermultiplets

U(1) SQED

This case is trivial because the limit which take mass to infinity is commutative with the infinite integral. The massive matter fields just decouple from the theory and the remaining theory is SQED with \( \frac{N_f}{d} \) massless hypermultiplets. In fact,

\[ \tilde{Z}_{U(1)}^{N_f} = \int_{-\infty}^{\infty} dx \frac{1}{(2 \cosh \pi x2 \cosh \pi(x + m)2 \cosh \pi(x - m))^{N_f}} \to \left( \frac{1}{M} \right)^{\frac{N_f}{2}} \tilde{Z}_{U(1)}^{N_f} \big|_{m=0}, \]

where \( N = 4 \) U(\( N \)) SQCD with massless flavors part \( \tilde{Z}_{U(N)}^{N_f} \big|_{m=0} \) can be calculated for \( N_f \geq 2N \) as

\[ \tilde{Z}_{U(N)}^{N_f} \big|_{m=0} = \frac{1}{N!} \frac{1}{(2\pi)^N} \prod_{k=0}^{N-1} \frac{\Gamma(k+2) \left( \Gamma\left( \frac{N_f}{2} - N + k + 1 \right) \right)^2}{\Gamma(N_f - N + k + 1)}. \] (4.8)
U(2) SQCD

In what follows we show the exact calculation of the partition function of U(2) SQCD $N_f$ fundamental hypermultiplets,

$$\tilde{Z}_{U(2)}^{N_f} = \frac{1}{2!} \int_{-\infty}^{\infty} dx dy \frac{4 \sinh^2 \pi (x-y)}{(2 \cosh \pi (x \pm m) 2 \cosh (y \pm m) 2 \cosh \pi x 2 \cosh \pi y)^{N_f}}.$$  \hfill (4.9)

The results for small $N_f$ are summarized in following table:

| $N_f$ | 1/4M | (log $M$)$^2$/4$\pi^2$M$^4$ | 1/32M$^6$ | 1/48$\pi^2$M$^8$ |
|-------|------|-----------------|-----------|-----------------|
| 3     |      |                 |           |                 |
| 6     |      |                 |           |                 |
| 9     |      |                 |           |                 |
| 12    |      |                 |           |                 |

Table 3: The leading part of $\tilde{Z}_{U(2)}^{N_f}$ when $m \to \infty$.

When $N_f = 3$ we can deduce the effective theory $U(1) \times U(1)$ is as follows

$$Z_{\text{Massive}} Z_{U(1) \times U(1)} = \frac{4 \sinh^2 2\pi m}{(2 \cosh 2\pi m 2 \cosh \pi m)^2} \left( \int_{-\infty}^{\infty} dx \frac{1}{2 \cosh \pi x} \right)^2 \longrightarrow \frac{1}{4M}.$$ \hfill (4.10)

This means that the effective theory appears when we chose the point of the Coulomb branch

$$\sigma = \left( \begin{array}{c} -m \\ m \end{array} \right),$$ \hfill (4.11)

in the sense of theories on the flat space. Because our expectation can not be applied to this case since $\frac{\xi}{6} N$ and $\frac{12 - \xi}{12} N$ is not integer, it may be expected that a log $M$ term also appears in the infinite mass limit like as the case with only massive matter fields. However, a log $M$ term does not appear in this case and the effective theory is ordinary one. When $N_f = 6$ there appears a log $M$ term. In this case, the effective theory may not be $U(1) \times U(1)$. It is also remarkable that whether a log $M$ appears or not depends on the number of flavors. When $N_f \geq 9$ the infinite mass limit is commutative with the integral. Thus the result is trivial. \[\text{This is consistent with the fact that the $\frac{2N_f}{3}$ massive matter fields with the real mass $m$ are decoupled by choosing the origin of the Coulomb branch since the remaining theory is a good theory. In the infinite mass limit the integral is written as}

$$\tilde{Z}_{U(2)}^{N_f} \longrightarrow \left( \frac{1}{M} \right)^{\frac{2N_f}{3}} Z_{U(2)}^{N_f} |_{m=0}.$$ \hfill (4.12)

\[\text{Exactly speaking, the matrix converges when } 2N - 2 \leq \frac{N_f}{3}. \text{ In the large } N \text{ limit the order one part is neglected.} \]
For this case the partition function is given by

$$
\tilde{Z}_{U(3)}^{N_f} = \frac{1}{3!} \int_{-\infty}^{\infty} \frac{4 \sinh^2 \pi (x - y) 4 \sinh^2 \pi (x - z) 4 \sinh^2 \pi (y - z) dx dy dz}{(2 \cosh \pi x) (2 \cosh \pi y) (2 \cosh \pi z) 2 \cosh \pi (x \pm m) 2 \cosh \pi (y \pm m) 2 \cosh \pi (z \pm m)^{N_f}}.
$$

(4.13)

The results for small $N_f$ are summarized in the following table:

| $N_f = 6$ | $N_f = 9$ | $N_f = 12$ | $N_f = 15$ |
|-----------|-----------|------------|------------|
| $\frac{1}{(2\pi)^4 M^4}$ | $\frac{9}{2^{12} M^8}$ | $\frac{1}{48 \pi^4 M^{12}}$ | $\frac{1}{2^{13} M^{16}}$ |

Table 4: The leading part of $\tilde{Z}_{U(3)}^{N_f}$ when $m \to \infty$.

In $N_f = 6$ case the theory will become the $U(1) \times U(1) \times U(1)$ gauge theory and each gauge group has two massless fundamental hypermultiplets. This theory is expected from the large $N$ analysis when $\xi = 2$ and $\frac{12 - \xi}{6} N$ are integer. In fact its matrix model is given by

$$
Z_{Massive}^{U(1) \times U(1) \times U(1)} = \left( \frac{4 \sinh^2 \pi m}{2 \cosh \pi m} \right)^2 \left( \frac{4 \sinh^2 2 \pi m}{2 \cosh 2 \pi m} \right)^6 \left( \int_{-\infty}^{\infty} dx \frac{1}{(2 \cosh \pi x)^2} \right)^3
$$

$$
\xrightarrow{m \to \infty} \frac{1}{(2\pi)^3 M^4}.
$$

(4.14)

When $N_f = 9$ this means $\xi = 3$, however, $\frac{9}{6} N$ and $\frac{12 - \xi}{12} N$ are not integer. Therefore we can not apply the result from the large $N$ analysis to this case. However, we can guess the effective theory as the $U(1) \times U(1) \times U(1)$ gauge theory and each group has three massless hypermultiplets. As opposed to $N_f = 6$ case, two of three $U(1)$ have a imaginary FI term which comes from one-loop effects. Actually the matrix model is

$$
Z_{Massive}^{U(1) \times U(1) \times U(1)} = \left( \frac{4 \sinh^2 \pi m}{2 \cosh \pi m} \right)^2 \left( \frac{4 \sinh^2 2 \pi m}{2 \cosh 2 \pi m} \right)^6 \left( \int_{-\infty}^{\infty} dx \frac{e^{-2 \pi x}}{(2 \cosh \pi x)^3} \right)^2 \left( \int_{-\infty}^{\infty} dx \frac{1}{(2 \cosh \pi x)^3} \right)
$$

$$
\xrightarrow{m \to \infty} \frac{9}{2^{12} M^8},
$$

(4.15)

and same as $\tilde{Z}_{U(3)}^{N_f = 9}$ in the infinite mass limit. Thus, in $N_f = 6, 9$ case, we conclude that the IR effective theory corresponds to the theory on a non-trivial Coulomb branch point

$$
\sigma = \begin{pmatrix}
  m \\
  0 \\
  -m
\end{pmatrix}.
$$

(4.16)
in the sense of theories in the flat space.

In case that \( N_f = 12 \), a \( \log M \) term appears and we do not have any interpretation of the effective theory. It may be worth emphasizing that a \( \log M \) term will appear when \( \frac{N_f}{3} = 2N - 2 \), \( N_f = 2N - 2 \) is threshold of a bad theory of \( \mathcal{N} = 4 \) U(\( N \)) SQCD with \( N_f \) flavors. When \( N_f \geq 15 \) we can see that the massive multiplets decouple in the infinite mass limit since we can change the order of the limit of mass and the integrals. Actually

\[
\tilde{Z}^{N_f}_{U(3)} \xrightarrow{m \to \infty} \left( \frac{1}{M} \right)^{3N_f/3} \tilde{Z}^{N_f}_{U(3)} \bigg|_{m=0},
\]

and in this case the IR effective theory corresponds to the theory on the trivial Coulomb branch point

\[
\sigma = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

5 Conclusion and Discussion

It is known that the IR dynamics of three-dimensional supersymmetric SQCD theories strongly depends on the number of matter multiplets. Hence it is reasonable that we give infinite mass to matter multiplets to decouple and investigate its effects. We considered the round three-sphere partition function of U(\( N \)) SQCD with massive hypermultiplets and what happens when we take the infinite mass limit with two types of mass deformation: (i) only massive matter fields (ii) massive and massless matter fields. Generally speaking, the vacuum, (in this paper we only consider Coulomb branch), must be chosen to argue whether a matter hypermultiplet is decoupled from the theory or not in the viewpoint of theories on the flat space. On the other hand, since the sphere partition function is written by the integrals over all possible Coulomb branch parameter it seems that we can not argue whether the matter hypermultiplets are decoupled or not. Thus we focused on the large \( N \) limit to fix a point of the Coulomb branch. The partition function is determined by the solution of the saddle point equation and the solution corresponds to a specific point of the Coulomb branch. Therefore we could argue decoupling of matter fields and its effective IR theory by following the solution. Finally, we confirmed that the effective theory on a non-trivial point of the Coulomb branch appears.

In the case (i), if we assume the theory on trivial Coulomb branch and take mass to infinite, all massive matter fields decouple from the theory and its partition function diverges. In fact, this argument is not valid because it is not guaranteed that the limit of mass is commutative with the integrals of the matrix model. We saw that the solution of the saddle point solution has two separated regions; the half one is concentrated on around \( m \) and the other is on \(-m\). This means that in the infinite mass limit the gauge group U(\( N \)) is broken to U(\( \frac{N}{2} \)) \( \times \) U(\( \frac{N}{2} \)) with
massless hypermultiplets and FI terms. Even for finite $N$ cases this picture we obtained from the large $N$ analysis may be true except when $\frac{N}{2}$ is not an integer.

In the case (ii), the behavior of the partition function depends on whether $\frac{Nf}{3} \geq 2N - 2$ or not. When $\frac{Nf}{3} \geq 2N - 2$ the limit of mass is commutable with the integrals and the massive matter fields simply decouple from the theory. This corresponds to the solution of the saddle point solution is concentrated on the origin of the Coulomb branch. In case that $\frac{Nf}{3} < 2N - 2$, we saw the gauge group is broken into three parts and the rank of each gauge group depends on the number of the flavors. For finite $N$ cases, we confirmed that the non-trivial effective theory appears in the infinite mass limit except for a few cases.

We also will provide the model which becomes the ABJM theory in the infinite mass limit with the help of this vacuum selection in the appendix D. This can be regarded as an example which connects the theory whose free energy is proportional to $N^2$ and that whose free energy is proportional to $N^{\frac{3}{2}}$ by a continuous parameter. This is consistent with F-theorem 

Let us comment on F-theorem. In our analysis, it can be considered that the free energies of many theories become divided into two part in the infinite mass limit like as

$$F \rightarrow F_{\text{SCFT}} + F_{\text{Massive}},$$

where $F_{\text{SCFT}}$ is the free energy of interacting SCFT and $F_{\text{Massive}}$ is that of free massive multiplets part. $F_{\text{Massive}}$ is proportional to $m$ and we can counter by a local counter term, which is the Einstein-Hilbert action of $S^3$ [31]. We expect

$$F_{\text{UV}} > F_{\text{SCFT}},$$

where $F_{\text{UV}}$ is the free energy when $m = 0$ since it can be naively considered that the limit $m \rightarrow \infty$ corresponds to a deep IR limit and $m = 0$ corresponds to a UV limit. In fact, this relation is valid at least in our results in section D. We also remark that we encounter the exceptional theories, whose partition function has a log $M$ behavior in the infinite mass limit and can not interpret the theory like as (5.1). It may be interesting to investigate the IR behavior of such a theory and consider F-theorem.

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*14 In this argument, we omit the decoupled massive free sectors.*
A  A brief summary of resolvent methods

Here let us introduce the resolvent methods and the more details of the calculation of the density function in this paper. We follow the argument of the resolvent in [38–40]. First we assume that the eigenvalues become dense in the large $N$ limit and we can take the continuous limit as following:

\[
\frac{i}{N} \to s \in [0, 1], \quad x_i \to x(s), \quad \frac{1}{N} \sum_{i=1}^{N} \to \int ds.
\]  

We introduce the density function $\rho(x)$ defined as

\[
\rho(x) \equiv \frac{ds}{dx},
\]

and impose the normalization condition as

\[
\int_I \rho(x) = 1,
\]

where $I$ is the interval where the density function $\rho(x)$ is defined. In this paper, we consider the following type of the saddle point equation which is given as an singular integral equation:

\[
\alpha \left( P \int_I dy \rho(y) \coth \pi(x - y) \right) = V'(x),
\]

where $\alpha$ is constant. It is convenient for us to take $X = e^{2\pi x}$ and $Y = e^{2\pi y}$ since various techniques developed so far are available. The saddle point equation is written as

\[
\alpha \left( 1 + P \int_C \frac{dY}{\pi} \frac{\rho(Y)}{X - Y} \right) = V'(X).
\]

where $X \in C = [b, a]$. We introduce an auxiliary function $\omega(Z)$ as

\[
\omega(X) \equiv \alpha \left( 1 + \int_C \frac{dY}{\pi} \frac{\rho(Y)}{X - Y} \right).
\]

This function is defined on the complex plane except on $C$, where $\omega(X)$ has a discontinuity when we across the interval $C$. It satisfies the following properties:

\[
\lim_{X \to 0} \omega(X) = -\alpha, \quad \lim_{X \to \infty} \omega(X) = \alpha,
\]

\[
\rho(X) = -\frac{1}{2\alpha i} \lim_{\epsilon \to 0} \left( \omega(X + i\epsilon) - \omega(X - i\epsilon) \right), \quad X \in C,
\]

\[
V'(X) = \frac{1}{2} \lim_{\epsilon \to 0} \left( \omega(X + i\epsilon) + \omega(X - i\epsilon) \right), \quad X \in C.
\]
Here we give a proof of (A.8) and (A.9), which comes from the discontinuity of $\omega(X)$. The following relation is obtained by changing the integral contour:

$$
\int_C \frac{dY}{\pi} \frac{\rho(Y)}{X+\epsilon - Y} = \left( \int_b^{X-\epsilon} + \int_{X+\epsilon}^a \right) \frac{dY}{\pi} \frac{\rho(Y)}{X-Y} + \int_{C_{\epsilon}^-} \frac{dY}{\pi} \frac{\rho(Y)}{X-Y}, \quad (X \in C), \quad (A.10)
$$

where $C_{\epsilon}^-$ is a circle with the radius $\epsilon$ around $Y = X$ in the lower half plane and oriented counterclockwise. By the definition of the principal value integral and the residue theorem, we finally obtain

$$
\lim_{\epsilon \to 0} \omega(X + i\epsilon) = \alpha \left( 1 + P \int_C \frac{dY}{\pi} \frac{\rho(Y)}{X-Y} - i\rho(X) \right). \quad (A.11)
$$

By the same calculation, we also obtain

$$
\lim_{\epsilon \to 0} \omega(X - i\epsilon) = \alpha \left( 1 + P \int_C \frac{dY}{\pi} \frac{\rho(Y)}{X-Y} + i\rho(X) \right). \quad (A.12)
$$

Then the important equations (A.8) and (A.9) are proofed.

From the analyticity the resolvent is given by

$$
\omega(X) = \oint_C \frac{dZ}{2\pi i} \frac{V'(Z)}{X-Z} \frac{\sqrt{(X-a)\sqrt{(X-b)}}}{\sqrt{(Z-a)\sqrt{(Z-b)}}}, \quad (A.13)
$$

where $C$ is a circle which encloses $C$. The density function is determined once the potential $V'(Z)$ is given. We assume that the $n_i$ degree poles $X_{0i}$, ($i = 1, \ldots, n_0$) of $V'(X)$ exist outside of $C$. We deform the integral contour $C$ to infinity and pick the poles $Z = X, \ Z = X_{0i}, \ (i = 1, \ldots, n_0)$. Thus the resolvent is written as

$$
\omega(X) = -\oint_{\infty} \frac{dZ}{2\pi i} \frac{V'(Z)}{X-Z} \frac{\sqrt{(X-a)\sqrt{(X-b)}}}{\sqrt{(Z-a)\sqrt{(Z-b)}}}
\quad = V'(X) - \sum_{i=1}^{n_0} \text{Res} \left( \frac{V'(Z)}{X-Z} \frac{\sqrt{(X-a)\sqrt{(X-b)}}}{\sqrt{(Z-a)\sqrt{(Z-b)}}}, X_{0i} \right). \quad (A.14)
$$

To determine the edge of the cut $C$ we just solve the equation (A.7) with the resolvent $\omega(X)$ we obtained in (A.14).

**B Mixed Chern-Simons terms**

It is known that in three dimension there exist the various Chern-Simons terms, which are consist of not only dynamical gauge fields but also background ones which couple to the current of the global symmetry. These Chern-Simons terms must appear in the mass infinite limit as the 1-loop effects by integrating out the massive fermions charged under the symmetries. Especially,
on $S^3$, we can consider the background vector fields which couples to R-symmetry current and the Chern-Simons terms including the background fields. They are important for us in order to understand the remains after the infinite mass limit \[31–33\]. The Chern-Simons terms which will appear are flavor-R and gauge-R mixed Chern-Simons terms given by

\begin{align}
S_{CS}^{FR} & \sim \frac{k_{FR}}{2\pi} \int_{S^3} \sqrt{g} d^3x \left( \sigma_f + iD_f \right), \\
S_{CS}^{GR} & \sim \frac{k_{GR}}{2\pi} \text{Tr} \int_{S^3} \sqrt{g} d^3x \left( \sigma + iD \right),
\end{align}

where $\sigma_f$ and $D_f$ mean the scalar and auxiliary fields of the background vector superfields respectively. Here we write only the parts of the action which contribute to the action after using localization methods. The induced Chern-Simons levels are given by integrating out a complex fermion $\psi$ as

\begin{align}
\kappa_{FR}^\psi & = \frac{\Delta}{2} \text{sgn}(M_\psi) \sum_f q_{\psi,f}, \\
\kappa_{GR}^\psi & = \frac{\Delta}{2} \text{sgn}(M_\psi) \sum_i q_{\psi,i},
\end{align}

where $q_f$, $q_g$ and $\Delta$ correspond to flavor, gauge and R charges respectively. $M_\psi$ is the effective real mass of the fermions on a point of the Coulomb branch. In this paper, $M_\psi = \sum_f q_f \sigma_f + \sum_i q_i \sigma_i$, where $i$ labels U(1) gauge groups on the Coulomb branch.

We use the terms \[B.1\] and \[B.2\] after using localization technique. These are given by

\begin{align}
e^{S_{CS}^{FR}} & = e^{2\pi k_{FR} \sigma_f}, \\
e^{S_{CS}^{GR}} & = e^{2\pi k_{GR} \sigma},
\end{align}

where the supersymmetric configuration of the background fields and the localization locus were needed:

\begin{align}
D_f = -i\sigma_f, \quad D = -i\sigma, \quad (\text{Other fields}) = 0.
\end{align}

The real mass is given by the expectation value of the background field $\sigma_f = m$. Thus the Flavor-R Chern-Simons terms \[B.1\] induced in the mass infinite limit can corresponds to the contributions from free massive degrees of freedom. The induced gauge-R Chern-Simons term \[B.2\] corresponds to the FI term and the contributions from massive degree of freedom when we shift $\sigma$ by $m$.

As an example, we try to obtain the decoupled free massive sector and the FI terms of the U(2) SQCD case in section \[14\] from the induced Chern-Simons terms \[15\]. We assume that the classical Coulomb branch parameters $(\sigma_1, \sigma_2)$ are written as $(-m - \delta \sigma_1, m - \delta \sigma_2)$. Then
the gauge group $U(2)$ is broken to $U(1)_L \times U(1)_R$. The effective mass and the charges of the massive gauginos and complex fermions of chiral multiplets are summarized in the table $5^{15}$\[16\].

The contributions of the massive gauginos to the induced Chern-Simons terms are following:

\[
\lambda_+ : e^{\pi \Delta_\lambda \text{sgn}(-2m-\delta \sigma_1 + \delta \sigma_2)(-m-\delta \sigma_1 - (-\delta \sigma_2 + m))} \tag{B.8}
\]

\[
\lambda_- : e^{\pi \Delta_\lambda \text{sgn}(2m+\delta \sigma_1 - \delta \sigma_2)(-m-\delta \sigma_1 + (-\delta \sigma_2 + m))} \tag{B.9}
\]

Thus the total contributions of massive gauginos when $m \to \infty$ are

\[
e^{2\pi \Delta_\lambda (2m+\delta \sigma_1 - \delta \sigma_2)}. \tag{B.10}
\]

The first term can be interpreted as the massive free part and second and third term are induced FI terms of $U(1)_L$ and $U(1)_R$.

The contributions of massive matter fermions are summarized as follows:

\[
\psi_{1-} : e^{\pi \Delta_\psi \text{sgn}(-2m-\delta \sigma_1)(-2m-\delta \sigma_1)} \tag{B.11}
\]

\[
\psi_{2+} : e^{\pi \Delta_\psi \text{sgn}(2m-\delta \sigma_2)(2m-\delta \sigma_2)} \tag{B.12}
\]

\[
\tilde{\psi}_{1+} : e^{\pi \Delta_\psi \text{sgn}(2m+\delta \sigma_1)(2m+\delta \sigma_1)} \tag{B.13}
\]

\[
\tilde{\psi}_{2-} : e^{\pi \Delta_\psi \text{sgn}(-2m+\delta \sigma_2)(-2m+\delta \sigma_2)}. \tag{B.14}
\]

Then the total contributions of the massive matter fermions to the induced Chern-Simons term are given by

\[
e^{\pi N_f / 2 \Delta_\psi (8m+2\delta \sigma_1 - 2\delta \sigma_2)}. \tag{B.15}
\]

\[^{15}\text{We would like to thank Masazumi Honda for giving us useful comments and discussion in this point.}\]

\[^{16}\text{An $\mathcal{N} = 4$ gauge theory has a chiral multiplet in adjoint representation of the gauge group. So it seems that we must consider the contributions from the chiral multiplets. However, the canonical R-charge of the chiral multiplet is 1. Then the R-charge of the fermion component is 0 and it does not contribute to the gauge-R mixed Chern-Simons level.}\]
The first term can be interpreted as the contributions from massive free sector and the second and third can be FI terms. Then we conclude that in this case the total contributions from free massive sectors when \( m \to \infty \) are given by
\[
e^{2m\pi(2-N_f)},
\]
and the total induced FI terms are given by
\[
e^{\pi(2-N_f)\delta \sigma_1}, \quad \text{for } U(1)_L,
\]
\[
e^{-\pi(2-N_f)\delta \sigma_2}, \quad \text{for } U(1)_R.
\]
These results are same as those we obtained in the section 4.1 from the calculation of the matrix model. Thus, indeed, the effects which appear in the infinite mass limit in the matrix model can be regarded as the induced mixed Chern-Simons terms. The result can be easily generalized to other theories in this paper. These consequences are not surprising because the 1-loop parts of the vector and chiral multiplets in the matrix model must inherit such 1-loop effects after integrating out massive fermions.

### C Convergence bound of matrix models

In this section, we discuss the convergence of the matrix models. The convergence bound of the matrix model of the SQCD is first discussed in [30]. In [41] it is also pointed out that the convergence bound is undistinguishable with the unitarity bound of the monopole operator in the Veneziano limit.

We consider the convergence of the matrix model which we introduced (3.14)
\[
Z = \frac{1}{N!} \int \prod_{i=1}^{N} dx_i \frac{e^{\pi \zeta \sum x_i} \prod_{i<j} 4 \sinh^2 \left( \pi (x_i - x_j) \right) \prod_i \left( 2 \cosh \pi (x_i) \right)^{N_f}}{\prod_i \left( 2 \cosh \pi (x_i) \right)^{N_f}}.
\]

In order to check whether the integral is convergent or not, it is enough to know the asymptotic behavior of the integrand when we take one of the integral valuables \( |x_i| \to \infty \). Then we focus on \( x_1 \) and study the asymptotic behavior of the integrand. When \( |x_1| \to \infty \), the part of the integrand which is related to the convergence is
\[
e^{\pi |x_1| (\text{sign}(x_1) \zeta + 2(N-1) - N_f)}.
\]
Thus, for the matrix model to converge, the relation
\[
|\zeta| + 2(N-1) - N_f < 0
\]

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must be needed. This threshold corresponds to the condition that the solution of the saddle point equation of (3.14) in the large $N$ limit. We note that the matrix models of the effective theory (3.38) and (3.72) satisfy the above relation and converge. In fact, each matrix model satisfies the bound narrowly. For example, for the case (3.38), the left-hand side of the bound (C.3) is $-2$, which does not depend on any parameter. So the convergence of the matrix model restricts the theory which appears in the infinite mass limit. In fact, in the subsection 3.2, we assume that the solution of the saddle point equation where the gauge group $U(N)$ is broken to $U(N_1) \times U(N_2)$ $(N_1 > N_2)$ is allowed. Then the condition for the convergence of the matrix model corresponding to the effective theory \footnote{Without loss of generality we assume this situation.} is given by

\[ 0 > N_f/2 - 2N_2 + 2(N_1 - 1) - N_f/2 = 2(N_1 - N_2) - 2, \quad (C.4) \]

\[ 0 > \left| N_f/2 - 2N_1 \right| + 2(N_1 - 1) - N_f/2. \quad (C.5) \]

The first line is not satisfied when $N_1 > N_2$. Therefore only $N_1 = N_2$ case is allowed due to the convergence of the matrix model of the effective theory \footnote{We would like to thank Tomoki Nosaka for pointing out and discussing this point.}. By the same argument about the convergence bound, We can also understand the reason why $N_1$, $N_2$ and $N_3$ satisfy the relation (3.66).

\section{ABJM theory as effective theory}

Here we consider the theory whose effective theory in the large mass limit is ABJM theory. Naively speaking, the theory when $m = 0$ corresponds to UV theory in the sense that the energy scale which is decided by the radius of the three-sphere is very bigger than the mass parameter while the theory in the infinite mass limit corresponds to the IR theory in the same sense. The SYM theory we introduce here flows from the SYM theory to the ABJM theory in the above sense. Which effective theories will appear depends on the mass assignment to matter multiplets and the representation of the matter fields. It is possible to anticipate for the ABJM theory to appear as the effective theory in the infinite mass limit by the insight so far.

We consider the $U(2N)$ SYM theory with 2 massive hypermultiplets in the adjoint representation and $2N_f$ massive fundamental hypermultiplets. The matrix model is given by

\[ Z = \frac{1}{(2N)!} \int d^N x \prod_{i<j}^{2N} \frac{4 \sinh^2 \pi (x_i - x_j)}{(2\cosh \pi (x_i - x_j - 2m) 2\cosh \pi (x_i - x_j - 2m))^2} \times \prod_{i}^{2N} \frac{1}{(2\cosh \pi (x_i + m) 2\cosh \pi (x_i - m))^{N_f}}, \quad (D.1) \]
where it is needed to give the adjoint hypermultiplets real mass $\pm 2m$ and the hypermultiplets real mass $\pm m$. Then we assume that the saddle point configuration is splitting, which means the saddle point solution $x_{0i}$ has the following separated region:

$$
\begin{align*}
  x_{0i} &= m + \lambda_i & i & \in 1, \ldots, N, \\
  x_{0i} &= -m + \tilde{\lambda}_i & i & \in 1, \ldots, N
\end{align*}
$$

Under this assumption the free energy $F = -\log Z$ is evaluated with the solution $x_{0i}$ as

$$
F = - \sum_{i>j} \left[ \log 4 \sinh^2 \pi (\lambda_i - \lambda_j) + \log 4 \sinh^2 \pi (\tilde{\lambda}_i - \tilde{\lambda}_j) \right] + 2 \sum_{i,j} \log 2 \cosh \pi (\tilde{\lambda}_i - \lambda_j) \\
+ N_f \sum_i \log 2 \cosh \pi \lambda_i + N_f \sum_i \log 2 \cosh \pi \tilde{\lambda}_i \\
- \sum_{i>j} \left[ \log 4 \sinh^2 \pi (\tilde{\lambda}_i - \lambda_j + 2m) - 2 \log \cosh \pi (\tilde{\lambda}_i - \lambda_j + 4m) \right] \\
+ 2 \sum_{i>j} \log 2 \cosh \pi (\lambda_i - \lambda_j + 2m) \cosh \pi (\lambda_i - \lambda_j - 2m) \\
+ 2 \sum_{i>j} \log 2 \cosh \pi (\tilde{\lambda}_i - \lambda_j + 2m) \cosh \pi (\tilde{\lambda}_i - \lambda_j - 2m) \\
+ N_f \sum_i \log 2 \cosh \pi (\lambda_i + 2m) + N_f \sum_i \log 2 \cosh \pi (\tilde{\lambda}_i - 2m).
\]$$

$$
= - \sum_{i>j} \left[ \log 4 \sinh^2 \pi (\lambda_i - \lambda_j) + \log 4 \sinh^2 \pi (\tilde{\lambda}_i - \tilde{\lambda}_j) \right] + 2 \sum_{i,j} \log 2 \cosh \pi (\tilde{\lambda}_i - \lambda_j) \\
+ N_f \sum_i \log 2 \cosh \pi \lambda_i + N_f \sum_i \log 2 \cosh \pi \tilde{\lambda}_i + \text{(massive part)}. \quad (D.3)
$$

The massive part in the infinite mass limit is given by $N_0 m$ where $N_0$ is some constant. The massless part of the action corresponds to the free energy of the $U(N) \times U(N)$ quiver SYM with two bi-fundamental multiplets and $N_f$ fundamental hypermultiplets multiplets charged under each $U(N)$, which is evaluated with saddle point approximation. The matrix model of this effective theory is given by

$$
Z_{\text{eff}} \sim \int d^N \lambda d^N \tilde{\lambda} \frac{\prod_{i>j} 4 \sinh^2 \pi (\lambda_i - \lambda_j) \sinh^2 \pi (\tilde{\lambda}_i - \tilde{\lambda}_j)}{\prod_{i,j} \left[ 2 \cosh \pi (\lambda_i - \lambda_j) \right] \prod_i \left[ 2 \cosh \pi \lambda_i \cosh \pi \tilde{\lambda}_i \right]^{N_f}}. \quad (D.4)
$$

The massive part is proportional to $N^2 m$ when we consider mass $m$ is much bigger than the typical order of the eigenvalues. We will show that this matrix model is same as square of the matrix model of the ABJM theory with Chern-Simons level $k = N_f$ in the large $N$ limit. We will solve the saddle point equation of (D.4) by following the spirit in [44], where the eigenvalues are proportional to $\sqrt{N}$ in the large $N$ limit with Chern-Simons level $k$ kept finite.
We assume that the saddle point configuration satisfies the condition
\[ \lambda_i = \tilde{\lambda}_i. \] (D.5)

This is plausible in the sense that the action \( S(\lambda, \tilde{\lambda}) \) is invariant under exchanging \( \lambda \) and \( \tilde{\lambda} \). Under this assumption it is enough to consider the saddle point equation of the matrix model
\[
\tilde{Z}_{\text{eff}} = \frac{1}{(2N)!} \int d^N\lambda \frac{\prod_{i>j} 4\sinh^2 \pi(\lambda_i - \lambda_j)}{\prod_{i,j} (2 \cosh \pi (\lambda_i - \lambda_j))^{N_f}} \prod_i (2 \cosh \pi \lambda_i)^{N_f} \equiv \int d^N\lambda e^{-\tilde{S}_{\text{eff}}(\lambda)}. \] (D.6)

For \( N_f = 1 \) this matrix model is known as the mirror dual matrix model of ABJM theory with \( k = 1 \) [21]. This matrix model is studied in [22,23] in the Veneziano limit where \( N_f \) is taken to infinite while \( \frac{N_f}{N} \) is finite. In this paper, we do not use the Veneziano limit. We take \( N \) infinite while keeping \( N_f \) finite.

We evaluate the action is explicitly written by
\[
\tilde{S}_{\text{eff}}(\lambda) = -\sum_{i>j} \log 4 \sinh^2 \pi (\lambda_i - \lambda_j) + \sum_{i,j} \log 2 \cosh \pi (\lambda_i - \lambda_j) + N_f \sum_i \log 2 \cosh \pi \lambda_i. \] (D.7)

Here we take the continuous limit in the large \( N \) limit. We define the continuous parameter \( s \) made by the label of eigenvalue as \( s = \frac{i}{N} + s_b \). The continuous value \( s \) runs from \( s_b \) to \( s_b + 1 \) and \( s_b \) is constant. The eigenvalues are replaced by a function of \( s \) which is monotonically increasing and differentiable. The summation is replaced by the integral as
\[
\sum_i \rightarrow N \int_{s_b}^{s_b+1} ds, \] (D.8)

where we do not introduce the density function. We assume that the ABJM type ansatz of the eigenvalues which are proportional to \( \sqrt{N} \) in the large \( N \) limit like as [44]
\[ \lambda(s) = \sqrt{N}x(s), \] (D.9)

From the above expression the final term of the action (D.7) is evaluated as
\[
N_f \sum_i \log 2 \cosh \pi \sqrt{N} x_i \rightarrow N^\frac{3}{2} N_f \pi \int_{s_b}^{s_b+1} |x(s)|. \] (D.10)

The evaluation of the first and second term of the action is non-trivial since the naive order of these part is \( N^2 \) and reduces to \( N^\frac{3}{2} \). We briefly review the fact by the technique which is developed in [9]. We rewrite the first term as following:
\[
\sum_{i>j} \log 4 \sinh^2 \pi (\lambda_i - \lambda_j)
\]
\[ N^2 \int ds'ds \log 4 \sinh^2 \pi (\sqrt{N}(x - x')) = \frac{N^2}{2} \int ds'ds' \left[ 2\pi \sqrt{N}|x - x'| + \log \left[ 4 \sinh^2 \pi (\sqrt{N}(x - x')) e^{-2\pi \sqrt{N}|x - x'|} \right] \right], \quad \text{(D.11)} \]

\[ \sum_{i,j} \log 2 \cosh \pi (\lambda_i - \lambda_j) \]

\[ \rightarrow N^2 \int ds'ds' \log 2 \cosh \pi (\sqrt{N}(x - x')) \]

\[ = N^2 \int ds'ds' \left[ \pi \sqrt{N}|x - x'| + \log \left[ 2 \cosh \pi (\sqrt{N}(x - x')) e^{-\pi \sqrt{N}|x - x'|} \right] \right], \quad \text{(D.12)} \]

where \( x \) and \( x' \) means \( x(s) \) and \( x(s') \) respectively. The first terms in (D.11) and (D.12) cancel. The second terms in (D.11) and (D.12) is evaluated by the following approximation formula:

\[ \int_{s_0} ds \log (1 \pm e^{-2z(s)}) \sim \frac{1}{\sqrt{N} \dot{x}(s_0)} \int_{C_+} dt \log (1 \pm e^{-2t}) \], \quad \text{(D.13)}

\[ \int_{s_0} ds \log (1 \pm e^{2z(s)}) \sim \frac{1}{\sqrt{N} \dot{x}(s_0)} \int_{C_-} dt \log (1 \pm e^{-2t}) \], \quad \text{(D.14)}

for \( \dot{x}(s)|_{s=s_0} > 0 \) where

\[ z(s) = \sqrt{N}x(s) + v(s), \quad \text{(D.15)} \]

\( x(s_0) = 0 \) and the path \( C_\pm \) is a straight line between \( t = \pm v(s_0) \) and \( t = \sqrt{N} \dot{x}(s_0) \) with \( N \rightarrow \infty \).

In our case \( v(s) = 0 \) and \( u(s) \) is real. The contour \( C_\pm \) is a half line from 0 to \( \infty \). Thus, the remaining parts of the free energy is

\[ N^{\frac{3}{2}} \int \frac{ds'}{\pi \dot{x}(s')} \left( -2 \int_0^{\infty} dt \log (\sinh(t)e^{-t}) + \int_0^{\infty} dt \log (\cosh(t)e^{-t}) \right) \]

\[ = N^{\frac{3}{2}} \int \frac{ds'}{\pi \dot{x}(s')} \left( -2 \int_0^{\infty} dt \log (\frac{\sinh(t)}{\cosh(t)}) \right) \]

\[ = N^{\frac{3}{2}} \int \frac{ds'}{\dot{x}(s')} \frac{1}{4\pi}, \quad \text{(D.16)} \]

where we have assumed \( \dot{x}(s') > 0 \) and there is no singularities in \( t \)-plane for deforming the contour \( C_\pm \). However, there are singularities in the action where a \( \cosh \) factor vanish. We can see that if

\[ -\frac{1}{4} < \text{Im}(v(s)) - \text{Re}(v(s)) \frac{\text{Im}(\dot{x}(s))}{\text{Re}(\dot{x}(s))} < \frac{1}{4}, \quad s \in [s_b, s_b + 1] \quad \text{(D.17)} \]

there is no obstruction for the deformation of the contour. If this is not the case, we can shift \( z_2 \rightarrow z_2 + in/2, \) where \( n \) is an integer, in order to satisfy the condition (D.17). In this case,
the above bound (D.17) is always satisfied since \( \text{Im} (x(s)) = 0 \). Plugging the above expressions into the action (D.7), we obtained the leading part of the free energy as

\[
F[x] = \pi N_f^3 \int_{s_b}^{s_{b+1}} ds \left[ N_f|x(s)| + \frac{1}{4 \dot{x}(s)} \right].
\] (D.18)

The saddle point equation is given by

\[
0 = \frac{\delta F[x]}{\delta x(s)} = N_f \text{sign}(x(s)) + \frac{1}{4} \frac{d}{ds} \left( \frac{1}{(\dot{x}(s))^2} \right),
\] (D.19)

with the boundary condition

\[
\left. \frac{1}{(\dot{x}(s))^2} \right|_{\text{boundary}} = 0.
\] (D.20)

The solution of the equation of (D.19) is potentially discontinuous at the zeros of \( x(s) \) because there a sign function in (D.19). However, we must make \( x(s) \) continuous everywhere in \( (s_b, s_{b+1}) \) since we assumed \( x(s) \) is differentiable. We must take into account with this fact and the boundary condition is explicitly written as

\[
\left. \frac{1}{x(s)} \right|_{s=s_b} = 0, \quad \left. \frac{1}{x(s)} \right|_{s=s_{b+1}} = 0
\] (D.21, D.22)

where these equation come from the edge of the domain of \( x(s) \). Then the solution of the saddle point equation (D.19) is

\[
x(s) = \frac{\text{sign} \left( s - s_b - \frac{1}{2} \right)}{\sqrt{2N_f}} \left[ 1 - \sqrt{1 - 2 \left| s - s_b - \frac{1}{2} \right|} \right],
\] (D.23)

\[
\frac{ds}{dx} \equiv \rho(x) = 2N_f \left( \frac{1}{\sqrt{2N_f}} - |x| \right).
\] (D.24)

It is convenient to take \( \tilde{s} \equiv s - s_b - \frac{1}{2} \) and we rewrite \( \tilde{s} \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \) as \( s \). Consequently, we can evaluate the free energy by plugging this solution into (D.18) as

\[
F = \pi N_f^3 \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tilde{s} \left[ \sqrt{\frac{N_f}{2}} \left( 1 - \sqrt{1 - 2|\tilde{s}|} \right) + \sqrt{\frac{N_f}{8}} \sqrt{1 - 2|\tilde{s}|} \right] = \frac{\pi \sqrt{2N_f}N_f^3}{3}
\] (D.25)

We can see that the number of the fundamental flavor \( N_f \) of the SYM theory corresponds to the Chern-Simons level of the ABJM theory at least in the large \( N \) limit since the free energy

\footnote{There is choice of the over all sign \( \pm \) due to the square root. We choose the sign so that the solution monotonically increases.}
Figure 4: The left figure shows the solution of the saddle point equation of (D.1) with $(N, N_f, m) = (200, 2, 500)$. The horizontal line means the label of the eigenvalues. The right one shows that the density of the eigenvalues with the same parameter. The horizontal line means the degree of the eigenvalues.

Figure 5: The left figure shows the comparison of the numerical solution (Blue dots) of the saddle point equation of (D.1) with the analytic solution (D.23) (Green line). The right one shows that the comparison of the density function from numerical analysis(Blue dots) with the analytic one (Green line). In this figures we focus on the eigenvalues of the first $\frac{N}{2}$ eigenvalues and shifted the valuable $x$ by $m$.

of the ABJM theory with Chern-Simons level $k$ is $\frac{\pi \sqrt{2}}{3} N^\frac{3}{2}$. We show the numerical solution of the saddle point equation of the (D.1) in the large $N$ and large mass region and compare it with the analytic solution (D.23). From the following figure we can see that the solution has the two separated regions and in fact the distance between the two regions is given by $m$. Therefore we conclude that the free energy of (D.1) becomes in the large $N$ and large mass limit with keeping $\sqrt{N} \ll m$

$$ F = 2F_{ABJM}(N, k = N_f) + F_{\text{massive}}, \quad (D.26) $$

where $F_{ABJM}(N, k)$ is the free energy of $U(N)_{k} \times U(N)_{-k}$ ABJM theory and the $F_{\text{massive}}$ is the free energy of the free massive sector, which comes from massive hyper and vector multiplets.
$F_{\text{massive}}$ is proportional to $N^2$ while $F_{\text{ABJM}}$ is proportional to $N^{3/2}$. The free energy of the IR effective theory is $2F_{\text{ABJM}}$ and it is consistent with the "F-theorem" in the sense that the free energy of the UV theory, which means that we take $m = 0$, is proportional to $N^2$ while the deep IR theory, which means that we take $m = \infty$, is proportional to $N^{3/2}$. Thus we conclude that this model is an example which connects a theory whose free energy is proportional to $N^2$ with one whose free energy is proportional to $N^{3/2}$ by a continuous parameter.

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120 In the infinite mass limit, $F_{\text{Massive}}$ is proportional to mass and this term is a scheme dependent term in the three-dimensional theory. Thus we can counter this term by a local counter term $\Lambda \int_{S^3} \sqrt{g}(R + \cdots)$, where $\Lambda$ has mass dimension one.
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