Poisson geometry with a 3-form background

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We study a modification of Poisson geometry by a closed 3-form. Just as for ordinary Poisson structures, these “twisted” Poisson structures are conveniently described as Dirac structures in suitable Courant algebroids. The additive group of 2-forms acts on twisted Poisson structures and permits them to be seen as glued from ordinary Poisson structures defined on local patches. We conclude with remarks on deformation quantization and twisted symplectic groupoids.

The ideas presented in this note grew out of an attempt to understand how Poisson geometry on a manifold is affected by the presence of a closed 3-form “field”. Such forms are playing an important role in contemporary string theory. We refer, for example, to Park as well as to Cornalba and Sciappa and Klimčík and Ströbl. Our aim here is to show that the notions of Courant algebroid and Dirac structure provide a framework in which one can easily carry out computations in Poisson geometry in the presence of a background 3-form. It seems clear that a proper understanding of the global effect of such a 3-form involves gerbes (see for example Brylinski); our work here should at least partially substantiate the claim that Courant algebroids are appropriate infinitesimal objects to associate with gerbes.

Our work was stimulated in part by the many talks at the Workshop on Deformation Quantization and String Theory at Keio University (March, 2001) in which such 3-forms played an essential role. It is essentially an application of some of the ideas contained in a series of letters from Ševera to Weinstein written in 1998. Some of the material in this paper was presented in June, 2001 at the Colloque en l'honneur d’Yvette Kosmann-Schwarzbach at the Institut Henri Poincaré and the Workshop on Poisson Geometry at the Erwin Schrödinger Institute.

Our basic idea is as follows. Poisson structures on a manifold $M$ may be

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identified with certain Dirac structures in the standard Courant algebroid $E_0 = TM \oplus T^* M$; see Courant\(^{6}\) and Liu et al.\(^{12}\) It turns out that a closed 3-form $\phi$ on $M$ may be used to modify the bracket on $E_0$, yielding a new Courant algebroid $E_\phi$. A bivector $\pi$ on $M$ now corresponds to a Dirac structure in $E_\phi$ if and only if it satisfies the equation

$$[\pi, \pi] = \wedge^3 \pi(\phi).$$

Here, and elsewhere in this note, the operator $\tilde{B} : V \to V^*$ is defined for any bilinear form $B$ on a vector space $B$ by $\tilde{B}(\alpha)(\beta) = B(\alpha, \beta)$ for $\alpha$ and $\beta$ in $V$. Given $\phi$, we refer to solutions $\pi$ of (1) as $\phi$-Poisson structures or, if we do not want to specify $\phi$, twisted Poisson structures.

Twisted Poisson structures arose from the study of topological sigma models in the work of Park\(^{14}\), as well as Klimčík and Ströbl\(^{9}\), who called them WZW-Poisson structures. A related notion of quasi-Poisson structure on a manifold with a group action has been introduced by Alekseev et al.\(^{1}\), in connection with the theory of group-valued momentum mappings. Their Jacobi anomaly comes from a trivector on the group rather than a 3-form on the manifold, but it is possible that there is a general notion (perhaps involving Lie algebroid actions) which will incorporate both twisted Poisson and quasi-Poisson structures.

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§1. Courant algebroids and Dirac structures

The original bracket of Courant\(^{6}\) and the Courant algebroid brackets of Liu et al.\(^{12}\) were skew-symmetric. A non-skew-symmetric version of the bracket was also introduced by Liu et al.\(^{12}\), where some of its nice properties were noted. It was then observed by Kosmann-Schwarzbach, Xu, and the first author (all unpublished) that this bracket satisfied the Jacobi identity written in Leibniz form. We will use the non-skew-symmetric Poisson bracket in this paper.

A Courant algebroid over a manifold $M$ is a vector bundle $E \to M$ equipped with a field of nondegenerate symmetric bilinear forms $(\cdot, \cdot)$ on the fibres, an $\mathbb{R}$-bilinear bracket $[\cdot, \cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ on the space of sections of $E$, and a bundle map $\rho : E \to TM$ (the anchor), such that the following properties are satisfied:

1. for any $e_1, e_2, e_3 \in \Gamma(E)$, $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$;
2. for any $e_1, e_2 \in \Gamma(E)$, $\rho(e_1, e_2) = [\rho e_1, \rho e_2]$;
3. for any $e_1, e_2 \in \Gamma(E)$ and $f \in C^\infty(M)$, $[e_1, fe_2] = fe_1[e_2] + (\rho(e_1)f)e_2$;
4. for any $e, h_1, h_2 \in \Gamma(E)$, $\rho(e)(h_1, h_2) = ([e, h_1], h_2) + (h_1, [e, h_2])$;
5. for any $e \in \Gamma(E)$, $[e, e] = D(e, e)$,

where $D : C^\infty(M) \to \Gamma(E)$ is the map defined by $D = \frac{1}{2} \beta^{-1} \rho^* d$, where $\beta$ is the
isomorphism between $E$ and $E^*$ given by the bilinear form. In other words,
\[ (Df, e) = \frac{1}{2} \rho(e) f. \]  

Equivalently, instead of the bracket $\{ \cdot, \cdot \}$, we can use a linear map $e \mapsto Z_e$ which maps sections of $E$ to vector fields on the total space of $E$. The vector field $Z_e$ is a lift of $\rho(e)$ from $M$ to $E$, and the first four axioms just say that the flows of $Z_e$’s preserve the structure of $E$. The bracket $[e_1, e_2]$ is the Lie derivative of $e_2$ by $Z_{e_1}$. This feature of the present non-skew-symmetric bracket makes it particularly convenient.

A Dirac structure in $E$, also called an $E$-Dirac structure on $M$, is a maximal isotropic subbundle $L$ of $E$ whose sections are closed under the bracket, i.e. which is preserved by the flow of $Z_e$ for any $e \in \Gamma(L)$. The properties of a Courant algebroid imply that the restriction of the bracket and anchor to any Dirac structure $L$ form a Lie algebroid structure on $L$.

On any manifold $M$, we have the standard Courant algebroid $E_0 = TM \oplus T^*M$ with bilinear form $((X_1, \xi_1), (X_2, \xi_2)) = \xi_1(X_2) + \xi_2(X_1)$, anchor $\rho(X, \xi) = X$, and bracket
\[ [(X_1, \xi_1), (X_2, \xi_2)] = ([X_1, X_2], \mathcal{L}_{X_1} \xi_2 - i_{X_2} d\xi_1). \]  

Thus, $Z_{(X,0)}$ is the natural lift of $X$ from $M$ to $TM \oplus T^*M$, while $Z_{(0,\xi)}$, at a point $(Y, v) \in TM \oplus T^*M$, is equal to the vertical vector with value $-i_Y d\xi$.

If $\pi$ is a bivector field on $M$, the graph $L_\pi$ of $\tilde{\pi} : T^*M \to TM$ is an $E_0$-Dirac structure if and only if $\pi$ is a Poisson structure, i.e. if $[\pi, \pi] = 0$. The Lie algebroid structure on $L_\pi$ may be transferred by projection to $T^*M$, where it becomes the usual Lie algebroid structure associated to $\pi$.

For a 2-form $\omega$ on $M$, the graph $L_\omega$ of $\tilde{\omega} : TM \to T^*M$ is an $E_0$ Dirac structure if and only if $d\omega = 0$. When the Lie algebroid structure on $L_\omega$ is transferred to $TM$ by projection, it always becomes the standard one.

\section{Twisted Poisson structures}

Now let $\phi$ be a 3-form on $M$. We define a new bracket on $E_0$ by adding the term $\phi(X_1, X_2, \cdot)$ to the right hand side of (3). A simple computation (see the beginning of Section 3 for another argument using less computation) shows that the new bracket together with the original bilinear form and anchor constitute a Courant algebroid structure on $TM \oplus T^*M$ if and only if $d\phi = 0$. We denote this modified Courant algebroid by $E_\phi$.

We call a 2-form $\omega$ on $M$ $\phi$-closed if $L_\omega$ is an $E_\phi$ Dirac structure. This just means that $d\omega = \phi$; hence, $\phi$-closed forms exist only when $\phi$ is exact. Of course, any 2-form becomes “twisted closed” if we allow ourselves to choose $\phi$ “after the fact.”

More interesting are the $\phi$-Poisson structures, i.e. bivector fields $\pi$ for which $L_\pi$ is an $E_\phi$-Dirac structure. It is easily seen that $\pi$ is a $\phi$-Poisson structure if and only if it satisfies (1).

To understand the meaning of (1), we introduce Poisson brackets and hamiltonian vector fields by the usual definitions; i.e. $\{f, g\} = \pi(df, dg)$ and $H_f = \{\cdot, f\}$. 

The usual Jacobi equation then acquires an extra term:

\[ \{\{f, g\}, h\} + \text{c.p.} + \phi(H_f, H_g, H_h) = 0, \quad (4) \]

where “c.p.” means the sum of the two terms obtained from the previous expression by circular permutation of the three variables.

If \( M \) is 3-dimensional, the hamiltonian vector fields span a space of dimension 0 or 2 at each point, so the extra term in the Jacobi identity is zero; i.e. a twisted Poisson structure is a Poisson structure. Nevertheless, as we shall see, the presence of \( \phi \) still has an effect on some of the standard Poisson-geometric constructions. Thus, one should consider \( \phi \) as part of the twisted Poisson structure, even if it does not affect the Jacobi identity.

Twisted Poisson structures lead to Lie algebroid structures on the cotangent bundle, just as ordinary Poisson structures do, since any Dirac structure is a Lie algebroid for the induced Courant bracket; identifying \( L_\pi \) with \( T^*M \) by projection to the second summand of \( E_\phi \) transfers the Lie algebroid structure to the cotangent bundle. The explicit formula for this bracket is:

\[ [\omega_1, \omega_2] = \mathcal{L}_{\tilde{\pi}(\omega_1)}\omega_2 - \mathcal{L}_{\tilde{\pi}(\omega_2)}\omega_1 - d(\pi(\omega_1, \omega_2)) + \phi(\tilde{\pi}(\omega_1), \tilde{\pi}(\omega_2), \cdot). \quad (5) \]

Since this bracket satisfies the Jacobi identity, but the Poisson bracket of functions does not, it should not be surprising that the relation between the two brackets is also altered by the presence of \( \phi \), namely

\[ [df, dg] = d\{f, g\} + \phi(H_f, H_g, \cdot). \quad (6) \]

Similarly, the mapping from functions to their hamiltonian vector fields is no longer an antihomomorphism. Instead, we have

\[ H_{\{f, g\}} + [H_f, H_g] = -\tilde{\pi}(\phi(H_f, H_g, \cdot)). \]

The dual of the Lie algebroid \( L_\pi \) may be identified with the tangent bundle \( TM \). The Lie algebroid cohomology differential is no longer simply \( d_\pi = [\pi, \cdot] \) (which is not of square zero unless \([\pi, \pi] = 0\)), but is now given by the formula:

\[ d_{\pi, \phi} = d_\pi + (\wedge^2 \tilde{\pi} \otimes 1)(\phi). \]

In this formula, \((\wedge^2 \tilde{\pi} \otimes 1)(\phi)\) is a section of the tensor bundle \( \wedge^2 TM \otimes T^*M \). It acts as a degree 1 operator on multivector fields by contraction with the factor in \( T^*M \). This operator is zero on functions, so for \( f \in C^\infty(M) \) we have

\[ d_{\pi, \phi}f = d_\pi f = [\pi, f] = H_f \]

as usual. On the other hand, for a vector field \( X \),

\[ d_{\pi, \phi}X = d_\pi X + (\wedge^2 \tilde{\pi} \otimes 1)(\phi)X, \]

which operates on a pair of 1-forms by

\[ (d_{\pi, \phi}X)(\omega_1, \omega_2) = -(\mathcal{L}_X \pi)(\omega_1, \omega_2) + \phi(\tilde{\pi}_1, \tilde{\pi}_2, X). \]
Since \( d^2_{π, φ} = 0 \), we have for each hamiltonian vector field \( H_f \)
\[
(\mathcal{L}_{H_f} π)(ω_1, ω_2) = φ(\tilde{π}ω_1, \tilde{π}ω_2, H_f),
\]
so the flow of a hamiltonian vector field does not in general preserve \( π \). This is,
of course, evident from the failure of the Jacobi identity. Incidentally, hamiltonian
flows do not in general preserve \( φ \) either, as may be seen easily in the 3-dimensional
case, where \( φ \) is arbitrary.

An important feature of Poisson manifolds is their decomposition into symplectic
leaves. For twisted Poisson manifolds, we have a similar decomposition, into the
orbits of the Lie algebroid \( L_π \) (whose anchor is \( \tilde{π} \), just as in the ordinary case).
Each orbit carries a nondegenerate 2-form, but these forms are only twisted closed;
their differentials equal the pullback of the 3-form; i.e. a twisted Poisson manifold
is decomposed into twisted symplectic leaves.

\[ \text{§3. Gauge transformations associated to 2-forms} \]

For any 2-form \( B \) on \( M \), we define the endomorphism \( τ_B \) of \( TM \oplus T^*M \) by
\[
τ_B(X, ξ) = (X, ξ + \tilde{B}(X)).
\]
A simple computation shows that \( τ_B \) preserves the
symmetric bilinear form and the anchor common to all the \( E_φ \) and that, for any
closed 3-form \( φ \), \( τ_B \) is an isomorphism of Courant bracket structures from \( E_φ \) to
\( E_φ - dB \). This shows that \( τ_B \) is an automorphism of \( E_φ \) if and only if \( B \) is closed,
and that the isomorphism class of \( E_φ \) depends only on the class \([φ]\) in the de Rham
cohomology space \( H^3(M, \mathbb{R}) \). Since any closed 3-form \( φ \) is locally exact, application
of a transformation \( τ_B \) also shows immediately that the bracket for \( E_φ \) does indeed
satisfy the Courant algebroid axioms.

If \( ξ \) is any 1-form, then the gauge transformation \( τ_{−dξ} \) is the time-1 flow of \( Z_{(0, ξ)} \).
This recovers without further computation the fact that \( τ_{−dξ} \) is an automorphism;
since any closed 2-form is locally exact, it also gives another proof of the fact that
gauge transformation by a closed 2-form is an automorphism.

The additive group of closed 2-forms on \( M \) acts on the space of \( φ \)-Dirac structures for each \( φ \),
while if \( φ = dB \), \( τ_B \) maps \( φ \)-Dirac structures to ordinary Dirac structures. We will refer to these operations (whether \( B \) is closed or not) as \textbf{gauge transformations}.
Furthermore, when gauge-equivalent Dirac structures correspond
to (twisted) closed 2-forms or (twisted) Poisson structures, we will say that the forms
or Poisson structures are gauge-equivalent.

Although gauge-equivalence looks like a rather coarse relation (for instance, any
two closed 2-forms are gauge-equivalent as \( E_0 \)-Dirac structures), it is still nontrivial.
First of all, the Lie algebroid structures on gauge-equivalent Dirac structures are
always isomorphic, since the Lie algebroid operations on any \( E \)-Dirac structure are
simply the restriction of the Courant algebroid bracket and anchor on \( E \).

Gauge-equivalence is especially interesting when applied to twisted Poisson structures.
Given a bivector \( π \) and a 2-form \( B \), \( τ_B(L_π) \) is of course a maximal isotropic
subspace of \( TM \oplus T^*M \), but it is not necessarily of the form \( L_{π'} \) for another bivector
\( π' \). If it is, then \( π' = \tilde{π}(1 + B\tilde{π})^{-1} \). (This formula is equivalent to \( π'^{-1} = \tilde{π}^{-1} + B \)
when \( π' \) and \( \tilde{π} \) are invertible.) In fact, the condition for \( τ_B(L_π) \) to correspond to a
bivector is just that the endomorphism $1 + \tilde{B}\pi$ of $TM$ be invertible. By abuse of notation, we will write $\tau_B\pi$ for $\pi'$. For general $\pi$ and $B$, we may think of $\tau_B\pi$ as a bivector with singularities.

Since gauge-equivalent twisted Poisson structures $\pi$ and $\tau_B\pi$ give rise to isomorphic Lie algebroids, they have many features in common. (Note that the particular Lie algebroid structure on $T^*M$ is changed under a gauge transformation of $\pi$, though the structures coming from $\pi$ and $\tau_B\pi$ are isomorphic via the map $1 + \tilde{B}\pi$.) Since the (twisted) Poisson cohomology for $\pi$ is isomorphic to the Lie algebroid cohomology of $L_\pi$, (twisted) Poisson cohomology is gauge invariant. Gauge-equivalent structures have the same Casimir functions. Their decompositions into twisted symplectic leaves are the same, though the 2-forms along the leaves differ by the pullbacks of $B$.

In particular, for $\phi = 0$ and $dB = 0$, the variation from leaf to leaf of the cohomology class of the symplectic structure along the leaves (the fundamental class of Dazord and Delzant) is the same.

It is natural to ask at this point whether every twisted Poisson structure is gauge-equivalent to an ordinary Poisson structure. If $\pi$ is a $\phi$-Poisson structure and $dB = \phi$, then $\tau_{-B}\pi$ will exist and be a Poisson structure if and only if $1 - \tilde{B}\pi$ is invertible. (If it is not, $\tau_{-B}\pi$ still makes sense as a Dirac structure; see Examples 4.1 and 4.2 below.) So the problem is to find a primitive $B$ for $\phi$ such that $1 - \tilde{B}\pi$ is invertible. If $\phi$ is not exact, we have no hope of finding a global primitive $B$; Example 4.1 shows even an exact $\phi$ cannot always be “gauged away.”

Using the Poincaré lemma, it is always possible to find a locally finite covering of $M$ by open subsets $U_\alpha$ carrying primitives $B_\alpha$ for $\phi$ which are small enough so that $1 - \tilde{B_\alpha}\pi$ is invertible. We thus have a family of locally defined ordinary Poisson structures $\pi_\alpha = \tau_{-B_\alpha}\pi$ which are related on overlaps $U_\alpha \cap U_\beta$ by gauge transformations corresponding to the exact Čech 1-cocycle $B_\alpha - B_\beta$ with values in 2-forms. The usual argument due to Weil takes us from here to a real-valued 3-cocycle representing the cohomology class of $\phi$. In this way, our twisted Poisson structures resemble other twisted structures, such as vector bundles, arising from the presence of cohomology classes of degree 3. Although we are now veering very close to the world of gerbes which we have tried to avoid, we will return to this point in the last section.

§4. Examples

The following example shows that, even when $\phi$ is exact, it might not be possible to remove it by a global gauge transformation of twisted Poisson structures. Since our example will be 3-dimensional, we remind the reader that $\phi$ should be considered as part of the structure, even if $\pi$ already satisfies the Jacobi identity.

Example 4.1 On $\mathbb{R}^3$, we consider the Lie-Poisson structure $\pi = x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + c.p.$ as a quasi-Poisson structure with 3-form $\phi = 3\lambda dx_1 \wedge dx_2 \wedge dx_3$, where $\lambda$ is an arbitrary real constant. An obvious primitive for $\phi$ is $B = \lambda(x_1 dx_2 \wedge dx_3 + c.p.)$; when we apply the corresponding gauge transformation, we find that the Poisson structure $\tau_{-B}\pi$ is equal to $\pi/(1 + \lambda r^2)$, where $r^2 = x_1^2 + x_2^2 + x_3^2$. If $\lambda$ is positive, this is fine, but
if $\lambda$ is negative, the structure is singular along the sphere of radius $(-\lambda)^{-1/2}$. The corresponding Dirac structure has this sphere as a “presymplectic leaf” carrying the zero 2-form.

We may ask whether a gauge transformation using another choice of primitive would lead to a globally defined Poisson structure, i.e. whether we can have $1 - \tilde{B}\tilde{\pi}$ invertible. In fact, this is not possible. To see this, we use the usual identifications arising from the orientation and metric on $\mathbb{R}^3$ to replace the skew-symmetric $3 \times 3$ matrix-valued functions $\tilde{B}$ and $\tilde{\pi}$ by 3-dimensional vector fields $B$ and $x$. (The latter is just the identity vector field.) The conditions which must be satisfied by $B$ are $\nabla \cdot B = 3\lambda$ and $x \cdot B(x) \neq -1$. Since $x$ vanishes at the origin, we must have $x \cdot B(x) > -1$. Taking the surface integral of $B$ over a sphere of radius $r$ and applying the divergence theorem, we obtain the inequality $\lambda r^2 > -1$. If $\lambda$ is negative, this means that $\tau_{-B\pi}$ cannot be a nonsingular Poisson structure beyond the sphere of radius $(-\lambda)^{-1/2}$.

The next example was first found (in a more general form) by Klimečk and Ševera; it was rediscovered independently by Alekseev and Ströbl (unpublished).

**Example 4.2** On a Lie group $G$ with bi-invariant metric (not necessarily definite), a natural choice of 3-form is the bi-invariant Cartan form (sometimes called the Chern-Simons form) defined on the Lie algebra by $\phi(u, v, w) = \frac{1}{2}[u, v] \cdot w$. We will use the metric to identify $TG$ with $T^*G$; when $\phi$ is thus interpreted as a bilinear map on vector fields with values in vector fields, its value on a pair of left-invariant and right-invariant vector fields is just half their bracket. For any element $a$ of the Lie algebra $g$, we will denote the corresponding left-invariant and right-invariant vector fields by $a^L$ and $a^R$.

In $E = TG \oplus TG$, we consider the maximal isotropic subbundle $L$ given by the values of the sections $e_a = (a^L - a^R, \frac{1}{2}(a^L + a^R))$ as $a$ ranges over $g$. Computing the ordinary Courant bracket of $e_a$ and $e_b$ gives an expression which differs from $e_{[a,b]}$ by a term in the 1-form component which vanishes if we add $-\phi(e_a, e_b, \cdot)$ to the bracket; i.e. $L$ is an $E_{-\phi}$-Dirac structure.

Since the map $a \mapsto e_a$ from $g$ to sections of $L$ is an isomorphism to each fibre of $L$, it follows that the Lie algebroid $L$ is isomorphic to the action Lie algebroid. Examining the component of $e_a$ in $TG$, we see that the action is just the conjugation action, so the orbits of the Lie algebroid, and hence the twisted presymplectic leaves of the twisted Dirac structure, are the connected components of the conjugacy classes in $G$.

$L$ is a twisted Poisson structure over the subset of $G$ where the vector fields $a^L + a^R$ are linearly independent, i.e. the open, dense subset where $\text{Ad}_g + 1$ is invertible. Over this subset, $L$ is of the form $L_x$, where $\tilde{\pi}(g)$ is the conjugate via left translation by $g$ of the operator $2(\text{Ad}_g - 1)/(\text{Ad}_g + 1)$ on $g$. (Recall that we are identifying tangent and cotangent vectors with the inner product.) As in the previous example, we may also wish to think of $\pi$ as a singular twisted Poisson structure on the whole group; on the conjugacy classes where it blows up, the twisted closed 2-form becomes degenerate.
This example also shows the utility of computing with Dirac structures. One could define \( \pi \) directly in terms of the operator \( 2(\text{Ad}_g - 1)/(\text{Ad}_g + 1) \), but it would be much more complicated to compute \([\pi, \pi]\) than it is to compute the Courant bracket of elements of \( L_\pi \).

We end this example by noting that, if one replaces the operator \( 2(\text{Ad}_g - 1)/(\text{Ad}_g + 1) \) by \( \text{Ad}_g \) itself, one obtains a globally defined bivector field \( \pi_1 \) which is a quasi-Poisson structure in the sense of Alekseev et al. \(^1\); i.e. \([\pi_1, \pi_1]\) is a multiple of the Cartan trivector field associated to the Cartan 3-form by the invariant quasi-algebraic structure on \( C \).

\[ \pi \]

The exactness of \( \partial \) cannot be expected to be the identity; rather it should be an inner automorphism related to a function \( f \), which is a primitive for the closed 1-form \( \theta \). By formality there are isomorphisms \( I_{\alpha \beta} \) between the quantized algebras on the overlap regions. On a triple intersection \( U_\alpha \cap U_\beta \cap U_\gamma \), the product \( I_{\alpha \beta}I_{\beta \gamma}I_{\gamma \alpha} \) cannot be expected to be the identity; rather it should be an inner automorphism related to a function \( f_{\alpha \beta \gamma} \), which is a primitive for the closed 1-form \( \theta_{\alpha \beta} + \theta_{\beta \gamma} + \theta_{\gamma \alpha} \). On quadruple intersections, these functions in turn will give

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\section{Deformation quantization and symplectic groupoids}

This final section is devoted to some speculations.

Poisson structures on \( M \) correspond to infinitesimal deformations of the associative algebra structure on \( C^\infty(M) \); see Bayen et al. \(^2\) and Kontsevich \(^10\). To what algebraic objects do twisted Poisson structures correspond? As already suggested by Park \(^14\) as well as by Cornalba and Schiappa \(^5\), the anomaly in the Jacobi identity for twisted Poisson structures should correspond to an anomaly in associativity for deformations of \( C^\infty(M) \), but the precise algebraic nature of the nonassociativity has yet to be determined. On the other hand, our approach to twisted Poisson structures via gauge transformations suggests a different approach to quantization.

As in the ordinary deformation quantization, we start with a formal \( \phi \)-Poisson structure \( \pi \in \Gamma(\wedge^2 TM)[[\epsilon]] \) such that \( \pi = O(\epsilon) \), i.e.

\[ \pi = \pi^{(1)} \epsilon + \pi^{(2)} \epsilon^2 + \pi^{(3)} \epsilon^3 + \cdots. \]

We choose a cover of \( M \) by open subsets \( U_\alpha \) carrying 2-forms \( B_\alpha \) such that \( dB_\alpha = \phi \), so that \( \tau_{-B_\alpha} \pi = \pi_\alpha \) become (formal) Poisson structures. Assuming further that we have chosen a quasi-isomorphism of Kontsevich for \( M \), we obtain an associative \( \mathbb{R}[[\epsilon]] \)-algebra structure on \( A_\alpha = C^\infty(U_\alpha)[[\epsilon]] \).

On \( U_\alpha \cap U_\beta \), we have \( \pi_\alpha = \pi_{B_\alpha - B_\beta} \pi_\beta \). We may write \( B_\alpha - B_\beta = d\theta_{\alpha \beta} \), where the \( \theta_{\alpha \beta} \) form a 1-cochain of 1-forms which is a cocycle only when considered modulo closed 1-forms. The exactness of \( B_\alpha - B_\beta \) implies that \( \pi_\alpha \) and \( \pi_\beta \) are equivalent Poisson structures, i.e. there is a formal curve of diffeomorphisms carrying \( \pi_\alpha \) to \( \pi_\beta \).

To prove this we construct a time-dependent formal Poisson structure \( \pi(t) \) connecting \( \pi_\alpha \) with \( \pi_\beta \), namely \( \pi(t) = \tau_{(B_\alpha - B_\beta)} \pi_{\alpha} \), and a formal time-dependent vector field \( v(t), v(t) = O(\epsilon) \), such that \( d\pi(t)/dt = \mathcal{L}_{v(t)} \pi(t) \). It is easy: we know that \( \tau_{(B_\alpha - B_\beta)} \) is the flow of \( Z_{(0, \theta_{\alpha \beta})} \), but we know that the flow of \( Z_{(\pi(t)\theta_{\alpha \beta}, \theta_{\alpha \beta})} \) preserves \( L_{\pi(t)} \), so that we can put \( v(t) = -\pi(t)\theta_{\alpha \beta} \). By formality there are isomorphisms \( I_{\alpha \beta} \) between the quantized algebras on the overlap regions. On a triple intersection \( U_\alpha \cap U_\beta \cap U_\gamma \), the product \( I_{\alpha \beta}I_{\beta \gamma}I_{\gamma \alpha} \) cannot be expected to be the identity; rather it should be an inner automorphism related to a function \( f_{\alpha \beta \gamma} \), which is a primitive for the closed 1-form \( \theta_{\alpha \beta} + \theta_{\beta \gamma} + \theta_{\gamma \alpha} \). On quadruple intersections, these functions in turn will give
rise to a Čech 3-cochain which represents the cohomology class $[\phi]$.

The exact nature of the resulting algebraic structure and its relation to nonassociativity are still unclear to us.

Let us remark here (related constructions were considered by Park$^{14}$) that the graded Lie algebra of multivector fields $\Gamma(\wedge TM)[1]$ is twisted by $\phi$ to an $L_\infty$ algebra denoted $\Gamma(\wedge TM)[1]_\phi$, such that formal $\phi$-Poisson structures are the formal solutions of Maurer-Cartan equation. The transformations $\tau_B$ give rise to $L_\infty$-isomorphisms $\Gamma(\wedge TM)[1]_\phi \to \Gamma(\wedge TM)[1]_{\phi-dB}$, hence $\Gamma(\wedge TM)[1]_\phi$ can be seen as glued from the graded Lie algebras $\Gamma(\wedge TU_\alpha)[1]$. It is natural to ask if there is a natural twist of the Hochschild complex that would be quasi-isomorphic to $\Gamma(\wedge TM)[1]_\phi$ (of course, we can define a twist just by using Kontsevich’s quasi-isomorphism).

To describe the $L_\infty$-structure on $\Gamma(\wedge TM)[1]_\phi$, we can use the language (and ideas) of Kontsevich$^{10}$. We define a graded $Q$-manifold $X_\phi$: as a graded supermanifold, $X_\phi = T[1]M \times \mathbb{R}[2]$, the vector field $Q$ is $d + \phi \partial_t$, where $d$ is the deRham differential on $M$ and $t$ is the coordinate on $\mathbb{R}[2]$. Let $j^1 X_\phi$ be the space of one-jets of sections of the projection $X_\phi \to T[1]M$; it is again a graded $Q$-manifold and so is the (infinite-dimensional) space of legendrian submanifolds of $j^1 X_\phi$, denoted $\Lambda_\phi$. Then the formal graded $Q$-manifold corresponding to $\Gamma(\wedge TM)[1]_\phi$ is just the formal neighborhood of a point in $\Lambda_\phi$ — the legendrian submanifold of $j^1 X_\phi$ given by the partially defined section of $X_\phi \to T[1]M$, $m \mapsto (m, 0)$, $m \in M$. The transformations $\tau_B$ act on $X_\phi$ by $(y, t) \mapsto (y, t + B)$; this action is inherited by $\Lambda_\phi$ and $\Gamma(\wedge TM)[1]_\phi$.

Finally, we recall that symplectic groupoids provide a geometric model (Weinstein$^{10}$) as well as a potential means of construction (Cattaneo and Felder$^4$) for deformed algebras. The twisted version of this story appears to go roughly as follows, following the pattern of Kosmann-Schwarzbach$^{11}$, which treated the case where $M$ is a point. Given a $\phi$-Poisson structure $\pi$ on $M$, the $\phi$-Dirac structure $L_\pi$ and the “horizontal” subbundle $TM$ form a Manin pair in $E_\phi$, and so they form a “quasi-Lie bialgebroid.” Assuming that the Lie algebroid $L_\pi$ can be integrated to a groupoid $G$, an extension of the methods of Mackenzie and Xu$^{13}$ should permit the construction of a nondegenerate twisted Poisson structure on $G$ which is compatible with the groupoid structure, making $G$ into a “twisted symplectic groupoid.” (The twisting 3-form would be the difference between the pullbacks of $\phi$ from $X$ to $G$ by the target and source maps.) The relation of this kind of object to deformations remains a mystery.

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