Abstract

Noncommutative tori are among historically the oldest and by now the most developed examples of noncommutative spaces. Noncommutative Yang-Mills theory can be obtained from string theory. This connection led to a cross-fertilization of research in physics and mathematics on Yang-Mills theory on noncommutative tori. One important result stemming from that work is the link between T-duality in string theory and Morita equivalence of associative algebras. In this article we give an overview of the basic results in differential geometry of noncommutative tori. Yang-Mills theory on noncommutative tori, the duality induced by Morita equivalence and its link with the T-duality are discussed. Noncommutative Nahm transform for instantons is introduced.

1 Noncommutative tori

1.1 The algebra of functions

The basic notions of noncommutative differential geometry were introduced and illustrated on the example of a two-dimensional noncommutative torus by A. Connes in [1]. To define an algebra of functions on a $d$-dimensional noncommutative torus consider a set of linear generators $U_n$ labelled by $n \in \mathbb{Z}^d$ - a $d$-dimensional vector with integral entries. The multiplication is defined by the formula

$$U_n U_m = e^{\pi i n_j \theta^{jk} m_k} U_{n+m}$$

(1)

where $\theta^{jk}$ is an antisymmetric $d \times d$ matrix, and summation over repeated indices is assumed. We further extend the multiplication from finite linear combinations to formal infinite series $\sum_n C(n) U_n$ where the coefficients $C(n)$ tend to zero faster than any power of $\|n\|$. The resulting algebra constitutes an algebra of smooth functions on a noncommutative torus and will be denoted as $T^d_\theta$. Sometimes for brevity we will omit the dimension label $d$ in the notation of the algebra. We introduce an involution $\ast$ in $T^d_\theta$ by the rule: $U^*_n = U_{-n}$. The elements $U_n$ are assumed to be unitary with respect to this involution, i.e. $U^*_n U_n = U_{-n} U_n = 1 \equiv U_0$. One can further introduce a norm and take an appropriate completion of the involutive algebra $T^d_\theta$ to obtain a $C^*$-algebra of functions on a noncommutative torus.

For our purposes the norm structure will not be important. A canonically normalized trace on $T^d_\theta$ is introduced by specifying

$$\text{Tr} U_n = \delta_{n,0}.$$  

(2)

1.2 Projective modules

According to general approach to noncommutative geometry finitely generated projective modules over the algebra of functions are natural analogs of vector bundles. Throughout

Noncommutative tori, Yang-Mills and string theory

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this article when speaking of a projective module we will assume a finitely generated left projective module.

A free module $(T^d_\theta)^N$ is equipped with a $T^d_\theta$-valued Hermitian inner product $\langle.,.\rangle_{T^d_\theta}$ defined by the formula

$$\langle (a_1,\ldots,a_N),(b_1,\ldots,b_N)\rangle_{T^d_\theta} = \sum_{i=1}^N a_i^* b_i.$$ 

A projective module $E$ is by definition a direct summand in a free module. Thus it inherits the inner product $\langle.,.\rangle_{T^d_\theta}$. Consider the endomorphisms of module $E$, i.e. linear mappings $E \to E$ commuting with the action of $T^d_\theta$. These endomorphisms form an associative unital algebra denoted $\text{End}_{T^d_\theta}E$. A decomposition $(T^d_\theta)^N = E \oplus E'$ determines an endomorphism $P: (T^d_\theta)^N \to (T^d_\theta)^N$ that projects $(T^d_\theta)^N$ onto $E$. The algebra $\text{End}_{T^d_\theta}E$ can then be identified with a subalgebra in $\text{Mat}_N(T^d_{-\theta})$ - the endomorphisms of free module $(T^d_{-\theta})^N$. The latter one has a canonical trace that is a composition of the matrix trace with the trace specified in [2]. By restriction it gives rise to a canonical trace $\text{Tr}$ on $\text{End}_{T^d_\theta}E$. The same embedding also provides a canonical involution on $E$ by a composition of the matrix transposition and the involution $\ast$ on $T^d_\theta$.

A large class of examples of projective modules over noncommutative tori constitute the so called Heisenberg modules. They are constructed as follows. Let $G$ be a direct sum of $\mathbb{R}^p$ and an abelian finitely generated group, and let $G^\ast$ be its dual group. In the most general situation $G = \mathbb{R}^p \times \mathbb{Z}^d \times F$ where $F$ is a finite group. Then $G^\ast \cong \mathbb{R}^p \times T^d_\theta \times F^\ast$.

Consider a linear space $\mathcal{S}(G)$ of functions on $G$ decreasing at infinity faster than any power. We define operators $U_{(\gamma,\tilde{\gamma})}: \mathcal{S}(G) \to \mathcal{S}(G)$ labelled by a pair $(\gamma,\tilde{\gamma}) \in G \times G^\ast$ acting as follows

$$(U_{(\gamma,\tilde{\gamma})} f)(x) = \tilde{\gamma}(x) f(x + \gamma).$$

One can check that the operators $U_{(\gamma,\tilde{\gamma})}$ satisfy the commutation relations

$$U_{(\gamma,\tilde{\gamma})} U_{(\mu,\tilde{\mu})} = \tilde{\mu}(\gamma) \tilde{\gamma}^{-1}(\mu) U_{(\mu,\tilde{\mu})} U_{(\gamma,\tilde{\gamma})}.$$ 

If $(\gamma,\tilde{\gamma})$ run over a $d$-dimensional discrete subgroup $\Gamma \subset G \times G^\ast$, $\Gamma \cong \mathbb{Z}^d$ then formula (4) defines a module over a $d$-dimensional noncommutative torus $T^d_{\theta}$ with

$$\exp(2\pi i \theta_{ij}) = \tilde{\gamma}_i(\gamma_j) \tilde{\gamma}_j^{-1}(\gamma_i)$$

for a given basis $(\gamma_i,\tilde{\gamma}_i)$ of the lattice $\Gamma$. This module is projective if $\Gamma$ is such that $G \times G^\ast/\Gamma$ is compact. If that is the case then the projective $T^d_{\theta}$-module at hand is called a Heisenberg module and denoted by $E_{\Gamma}$.

Heisenberg modules play a special role. If the matrix $\theta_{ij}$ is irrational in the sense that at least one of its entries is irrational then any projective module over $T^d_{\theta}$ can be represented as a direct sum of Heisenberg modules. In that sense Heisenberg modules can be used as building blocks to construct an arbitrary module.

1.3 Connections

Next we would like to define connections on a projective module over $T^d_{\theta}$. To this end let us first define a Lie algebra of shifts $L_\theta$ acting on $T^d_{\theta}$ by specifying a basis consisting of derivations $\delta_j : T^d_{\theta} \to T^d_{\theta}$, $j = 1,\ldots,d$ satisfying

$$\delta_j(U_n) = 2\pi i n_j U_n.$$
These derivations span a $d$-dimensional abelian Lie algebra that we denote by $L_\theta$.

A connection on a module $E$ over $T^d_\theta$ is a set of operators $\nabla_X : E \to E$, $X \in L_\theta$ depending linearly on $X$ and satisfying

$$[\nabla_X, U_n] = \delta_X(U_n)$$

where $U_n$ are operators $E \to E$ representing the corresponding generators of $T^d_\theta$. In the standard basis this relation reads as

$$[\nabla_j, U_n] = 2\pi i n_j U_n.$$  

(9)

The curvature of connection $\nabla_X$ defined as a commutator $F_{XY} = [\nabla_X, \nabla_Y]$ is an exterior two-form on the adjoint vector space $L^*_\theta$ with values in $\text{End}_{T^d_\theta}E$.

1.4 K-theory. Chern character

The $K$-groups of a noncommutative torus coincide with those for commutative tori:

$$K_0(T^d_\theta) \cong \mathbb{Z}^{2d-1} \cong K_1(T^d_\theta).$$

A Chern character of a projective module $E$ over a noncommutative torus $T^d_\theta$ can be defined as

$$\text{ch}(E) = \text{Tr} \exp \left( \frac{F}{2\pi i} \right) \in \Lambda^{\text{even}}(L^*_\theta)$$

where $F$ is the curvature form of a connection on $E$, $\Lambda^{\text{even}}(L^*_\theta)$ is the even part of the exterior algebra of $L^*_\theta$ and $\text{Tr}$ is the canonical trace on $\text{End}_{T^d_\theta}E$. This mapping gives rise to a noncommutative Chern character

$$\text{ch} : K_0(T^d_\theta) \to \Lambda^{\text{even}}(L^*_\theta).$$

(11)

The component $\text{ch}_0(E) = \text{Tr}1 \equiv \text{dim}(E)$ is called the dimension of module $E$.

A distinctive feature of the noncommutative Chern character is that its image does not consist of integral elements, i.e. there is no lattice in $L^*_\theta$ that generates the image of the Chern character. However there is a different integrality statement that replaces the commutative one. Consider a basis in $L^*_\theta$ in which the derivations corresponding to basis elements satisfy [7]. Denote the exterior forms corresponding to the basis elements by $\alpha^1, \ldots, \alpha^d$. Then an arbitrary element of $\Lambda(L^*_\theta)$ can be represented as a polynomial in anticommuting variables $\alpha^i$. Next let us consider a subset $\Lambda^{\text{even}}(\mathbb{Z}^d) \subset \Lambda^{\text{even}}(L^*_\theta)$ that consists of polynomials in $\alpha^i$ having integer coefficients. It was proved by Elliott that the Chern character is injective and its range on $K_0(T^d_\theta)$ is given by the image of $\Lambda^{\text{even}}(\mathbb{Z}^d)$ under the action of the operator $\exp \left( -\frac{1}{2} \frac{\partial}{\partial \theta^i} \frac{\partial}{\partial \theta^j} \right)$. This fact implies that the K-group $K_0(T^d_\theta)$ can be identified with the additive group $\Lambda^{\text{even}}(\mathbb{Z}^d)$.

A K-theory class $\mu(E) \in \Lambda^{\text{even}}(\mathbb{Z}^d)$ of a module $E$ can be computed from its Chern character by the formula

$$\mu(E) = \exp \left( \frac{1}{2} \frac{\partial}{\partial \alpha^i} \frac{\partial}{\partial \alpha^j} \right) \text{ch}(E).$$

(12)

Note that the anticommuting variables $\alpha^i$ and the derivatives $\frac{\partial}{\partial \alpha^i}$ satisfy the anticommutation relation $\{\alpha^i, \frac{\partial}{\partial \alpha^j}\} = \delta^i_j$.
The coefficients of $\mu(E)$ standing at monomials in $\alpha^i$ are integers to which we will refer as the topological numbers of module $E$. These numbers also can be interpreted as numbers of D-branes of a definite kind although in noncommutative geometry it is difficult to talk about branes as geometrical objects wrapped on torus cycles.

One can show that for noncommutative tori $T^d_\theta$ with irrational matrix $\theta_{ij}$ the set of elements of $K_0(T^d_\theta)$ that represent a projective module (i.e. the positive cone) consists exactly of the elements with a positive dimension. Moreover if $\theta_{ij}$ is irrational any two projective modules which represent the same element of $K_0(T^d_\theta)$ are isomorphic that is the projective modules are essentially specified in this case by their topological numbers.

Complex differential geometry of noncommutative tori and its relation with mirror symmetry is discussed in [17].

2 Yang-Mills theory on noncommutative tori

Let $E$ be a projective module over $T^d_\theta$. We call a Yang-Mills field on $E$ a connection $\nabla_X$ compatible with the Hermitian structure, that is a connection satisfying

$$<\nabla_X \xi, \eta>_{T^d_\theta} + <\xi, \nabla_X \eta>_{T^d_\theta} = \delta_X(<\xi, \eta>_{T^d_\theta})$$

for any two elements $\xi, \eta \in E$. Given a positive-definite metric on the Lie algebra $L_\theta$ we can define a Yang-Mills functional

$$S_{YM}(\nabla_i) = \frac{V}{4g^2_{YM}} g^{ik} g^{jl} \text{Tr}(F_{ij} F_{kl}).$$

Here $g^{ij}$ stands for the metric tensor in the canonical basis, $V = \sqrt{|\text{det} g|}$, $g_{YM}$ is the Yang-Mills coupling constant, $\text{Tr}$ stands for the canonical trace on $\text{End}_{T^d_\theta} E$ discussed above and summation over repeated indices is assumed. Compatibility with the Hermitian structure can be shown to imply the positive definiteness of the functional $S_{YM}$. The extrema of this functional are given by the solutions to the Yang-Mills equations

$$g^{ki}[\nabla_k, F_{ij}] = 0.$$  

A gauge transformation in the noncommutative Yang-Mills theory is specified by a unitary endomorphism $Z \in \text{End}_{T^d_\theta} E$, i.e. an endomorphism satisfying $ZZ^* = Z^*Z = 1$. The corresponding gauge transformation acts on a Yang-Mills field as

$$\nabla_j \mapsto Z \nabla_j Z^*.$$  

The Yang-Mills functional and the Yang-Mills equations are invariant under these transformations.

It is easy to see that Yang-Mills fields whose curvature is a scalar operator, i.e. $[\nabla_i, \nabla_j] = \sigma_{ij} \cdot 1$ with $\sigma_{ij}$ a real number valued tensor, solve the Yang-Mills equations. A characterization of modules admitting a constant curvature connection and a description of the moduli spaces of constant curvature connections (that is the space of such connections modulo gauge transformations) is reviewed in [4]. Another interesting class of solutions to the Yang-Mills equations is instantons (see below).

As in the ordinary field theory one can construct various extensions of the noncommutative Yang-Mills theory by adding other fields. To obtain a supersymmetric extension of
one needs to add a number of endomorphisms $X_I \in \text{End}_{T_\theta} E$ that play the role of bosonic scalar fields in the adjoint representation of the gauge group and a number of odd Grassmann parity endomorphisms $\psi^\alpha_i \in \Pi \text{End}_{T_\theta} E$ endowed with an $SO(d)$-spinor index $\alpha$. The latter ones are analogs of the usual fermionic fields.

In string theory one considers a maximally supersymmetric extension of the Yang-Mills theory (14). In this case the supersymmetric action depends on $10 - d$ bosonic scalars $X_I, I = d, \ldots, 9$ and the fermionic fields can be collected into an $SO(9,1)$ Majorana-Weyl spinor multiplet $\psi^\alpha, \alpha = 1, \ldots, 16$. The maximally supersymmetric Yang-Mills action takes the form

$$S_{\text{SYM}} = \frac{V}{4g^2} \text{Tr} \left( F_{\mu\nu} F^{\mu\nu} + [\nabla_\mu, X_I] [\nabla_\nu, X^I] + [X_I, X_J] [X^I, X^J] \right) - 2\psi^\alpha \sigma^\mu_{\alpha\beta} [\nabla_\mu, \psi^\beta] - 2\psi^\alpha \sigma^I_{\alpha\beta} [X_I, \psi^\beta] \right).$$

(17)

Here the curvature indices $F_{\mu\nu}, \mu, \nu = 0, \ldots, d - 1$ are assumed to be contracted with a Minkowski signature metric, $\sigma^A_{\alpha\beta}$ are blocks of the ten-dimensional $32 \times 32$ Gamma-matrices $\Gamma_A = \left( \begin{array}{cc} 0 & \sigma^A_{\alpha\beta} \\ (\sigma_A)_{\alpha\beta} & 0 \end{array} \right), \quad A = 0, \ldots, 9$.

This action is invariant under two kinds of supersymmetry transformations denoted by $\delta_\epsilon$, $\tilde{\delta}_\epsilon$ and defined as

$$\delta_\epsilon \psi = \frac{1}{2} (\sigma^{jk} F_{jk} \epsilon + \sigma^{Ij} [\nabla_j, X_I] \epsilon + \sigma^{Ij} [X_I, X_J] \epsilon),$$

$$\delta_\epsilon \nabla_j = \epsilon \sigma_j \psi, \quad \delta_\epsilon X_J = \epsilon \sigma_J \psi,$$

$$\tilde{\delta}_\epsilon \psi = \epsilon, \quad \tilde{\delta}_\epsilon \nabla_j = 0, \quad \tilde{\delta}_\epsilon X_J = 0.$$  

(18)

where $\epsilon$ is a constant 16-component Majorana-Weyl spinor. Of particular interest for string theory applications are solutions to the equations of motion corresponding to (17) that are invariant under some of the above supersymmetry transformations. Further discussion can be found in [14].

### 3 Morita equivalence

The role of Morita equivalence as a duality transformation in noncommutative Yang-Mills theory was elucidated by A. Schwarz in [15]. We will adopt a definition of Morita equivalence for noncommutative tori which can be shown to be essentially equivalent to the standard definition of strong Morita equivalence. We will say that two noncommutative tori $T^d_\theta$ and $T^d_\tilde{\theta}$ are Morita equivalent if there exists an $(T^d_\theta, T^d_\tilde{\theta})$-bimodule $Q$ and an $(T^d_\tilde{\theta}, T^d_\theta)$-bimodule $P$ such that

$$Q \otimes_{T^d_\theta} P \cong T^d_\theta, \quad P \otimes_{T^d_\tilde{\theta}} Q \cong T^d_\tilde{\theta}.$$  

(19)

where $T^d_\theta$ on the right hand side is considered as a $(T^d_\theta, T^d_\tilde{\theta})$-bimodule and analogously for $T^d_\tilde{\theta}$. (It is assumed that the isomorphisms are canonical). Given a $T_\theta$-module $E$ one obtains a $T^d_\theta$-module $\hat{E}$ as

$$\hat{E} = P \otimes_{T^d_\tilde{\theta}} E.$$  

(20)

One can show that this mapping is functorial. Moreover the bimodule $Q$ provides us with an inverse mapping $Q \otimes_{T^d_\tilde{\theta}} E \cong E$. 


We further introduce a notion of gauge Morita equivalence (originally called “complete Morita equivalence”) that allows one to transport connections along with the mapping of modules \cite{20}. Let $L$ be a $d$-dimensional commutative Lie algebra. We say that $(T^d_\theta, T^d_\hat{\theta})$ Morita equivalence bimodule $P$ establishes a gauge Morita equivalence if it is endowed with operators $\nabla^P_X, X \in L$ that determine a constant curvature connection simultaneously with respect to $T^d_\theta$ and $T^d_\hat{\theta}$, i.e. satisfy

$$
\nabla^P_X(ea) = (\nabla^P_X e)a + e(\delta_X a),
\nabla^P_X(\hat{a}e) = \hat{a}(\nabla^P_X e) + (\hat{\delta}_X \hat{a})e,
[\nabla^P_X, \nabla^P_Y] = 2\pi i \sigma_{XY} \cdot 1.
$$

(21)

Here $\delta_X$ and $\hat{\delta}_X$ are standard derivations on $T_\theta$ and $T_\hat{\theta}$ respectively. In other words we have two Lie algebra homomorphisms

$$
\delta : L \to L_\theta, \quad \hat{\delta} : L \to L_\hat{\theta}.
$$

(22)

If a pair $(P, \nabla^P_X)$ specifies a gauge $(T_\theta, T_\hat{\theta})$ equivalence bimodule then there exists a correspondence between connections on $E$ and connections on $\hat{E}$. A connection $\hat{\nabla}_X$ on $\hat{E}$ corresponding to a given connection $\nabla_X$ on $E$ is defined as

$$
\nabla_X \mapsto \hat{\nabla}_X = 1 \otimes \nabla_X + \nabla^P_X \otimes 1.
$$

(23)

More precisely, an operator $1 \otimes \nabla_X + \nabla^P_X \otimes 1$ on $P \otimes_\mathbb{C} E$ descends to a connection $\hat{\nabla}_X$ on $\hat{E} = P \otimes_{T_\theta} E$. It is straightforward to check that under this mapping gauge equivalent connections go to gauge equivalent ones

$$
\hat{Z}^! \nabla_X Z = \hat{Z}^! \hat{\nabla}_X \hat{Z}
$$

where $\hat{Z} = 1 \otimes Z$ is the endomorphism of $\hat{E} = P \otimes_{T_\theta} E$ corresponding to $Z \in \text{End}_{T_\theta} E$.

The curvatures of $\nabla_X$ and $\hat{\nabla}_X$ are connected by the formula

$$
F^\nabla_X = \hat{F}^{\hat{\nabla}}_X + 1\sigma_{XY}
$$

(24)

that in particular shows that constant curvature connections go to constant curvature ones.

Since noncommutative tori are labelled by an antisymmetric $d \times d$ matrix $\theta$, gauge Morita equivalence establishes an equivalence relation on the set of such matrices. To describe this equivalence relation consider an action $\theta \mapsto h\theta = \hat{\theta}$ of $SO(d, d|\mathbb{Z})$ on the space of antisymmetric $d \times d$ matrices by the formula

$$
\hat{\theta} = (M\theta + N)(R\theta + S)^{-1}
$$

(25)

where $d \times d$ matrices $M, N, R, S$ are such that the matrix

$$
h = \begin{pmatrix} M & N \\ R & S \end{pmatrix}
$$

(26)

belongs to the group $SO(d, d|\mathbb{Z})$. The above action is defined whenever the matrix $A \equiv R\theta + S$ is invertible. One can prove that two noncommutative tori $T^d_\theta$ and $T^d_\hat{\theta}$ are gauge Morita equivalent if and only if the matrices $\theta$ and $\hat{\theta}$ belong to the same orbit of $SO(d, d|\mathbb{Z})$-action \cite{25}. 

The duality group $SO(d, d|\mathbb{Z})$ also acts on the topological numbers of moduli $\mu \in \Lambda^{\text{even}}(\mathbb{Z}^d)$. This action can be shown to be given by a spinor representation constructed as follows. First note that the operators $a^i = \alpha^i$, $b_i = \partial/\partial \alpha^i$ act on $\Lambda(\mathbb{R}^d)$ and give a representation of the Clifford algebra specified by the metric with signature $(d, d)$. The group $O(d, d|\mathbb{C})$ thus can be regarded as a group of automorphisms acting on the Clifford algebra generated by $a^i$, $b_j$. Denote the latter action by $W_h$ for $h \in O(d, d|\mathbb{C})$. One defines a projective action $V_h$ of $O(d, d|\mathbb{C})$ on $\Lambda(\mathbb{R}^d)$ according to

$$V_h a^i V_h^{-1} = W_h^{-1}(a^i), \quad V_h b_j V_h^{-1} = W_h^{-1}(b_j).$$

This projective action can be restricted to yield a double-valued spinor representation of $SO(d, d|\mathbb{Z})$ on $\Lambda(\mathbb{R}^d)$ by choosing a suitable bilinear form on $\Lambda(\mathbb{R}^d)$. The restriction of this representation to the subgroup $SO(d, d|\mathbb{Z})$ acting on $\Lambda^{\text{even}}(\mathbb{Z}^d)$ gives the action of Morita equivalence on the topological numbers of moduli.

The mapping (23) preserves the Yang-Mills equations of motion (15). Moreover, one can define a modification of the Yang-Mills action functional (14) in such a way that the values of functionals on $\nabla X$ and $\hat{\nabla} X$ coincide up to an appropriate rescaling of coupling constants. The modified action functional has the form

$$S_{YM} = \frac{V}{4g^2} \text{Tr}(F^i_{jk} + \Phi^i_{jk} \cdot 1)(F^i_{jk} + \Phi^i_{jk} \cdot 1) \quad (27)$$

where $\Phi^i_{jk}$ is a number valued tensor that can be thought of as some background field. Adding this term will allow us to compensate for the curvature shift by adopting the transformation rule

$$\Phi_{XY} \mapsto \Phi_{XY} - \sigma_{XY}.$$

Note that the new action functional (27) has the same equations of motion (15) as the original one.

To show that the functional (27) is invariant under gauge Morita equivalence one has to take into account two more effects. Firstly, the values of trace change by a factor $c = \dim(\hat{E})(\dim(E))^{-1}$ as $\text{Tr} \hat{X} = c \text{Tr} X$. Secondly, the identification of $L_\theta$ and $L_{\hat{\theta}}$ is established by means of some linear transformation $A^k_l$ the determinant of which will rescale the volume $V$. Both effects can be absorbed into an appropriate rescaling of the coupling constant.

One can show that the curvature tensor, the metric tensor, the background field $\Phi_{ij}$ and the volume element $V$ transform according to

$$F^\hat{\nabla}_{ij} = A^k_l F^\nabla_{kl} A^l_j + \sigma_{ij}, \quad \hat{g}_{ij} = A^k_l g_{kl} A^l_j,$$

$$\hat{\Phi}_{ij} = A^k_l \Phi_{kl} A^l_j - \sigma_{ij}, \quad \hat{V} = V |\det A| \quad (28)$$

where $A = R \theta + S$ and $\sigma = -RA^i$. The action functional (27) is invariant under the gauge Morita equivalence if the coupling constant transforms according to

$$g^2_{YM} = g^2_{YM}|\det A|^{1/2}. \quad (29)$$

Supersymmetric extensions of Yang-Mills theory on noncommutative tori were shown to arise within string theory essentially in two situations. In the first case one considers compactifications of the (BFSS or IKKT) Matrix model of M-theory [13]. A discussion regarding the connection between T-duality and Morita equivalence in this case can be found
in section 7 of [14]. Noncommutative gauge theories on tori can be also obtained by taking the so-called Seiberg-Witten zero slope limit in the presence of a Neveu-Schwarz $B$-field background [14]. The emergence of noncommutative geometry in this limit is discussed in the article “Noncommutative geometry from string theory” in this volume. Below we give some details on the relation between T-duality and Morita equivalence in this approach. Consider a number of $D_p$-branes wrapped on $T_p$ parameterized by coordinates $x^i \sim x^i + 2\pi r$ with a closed string metric $G_{ij}$ and a $B$-field $B_{ij}$. The $SO(p, p|\mathbb{Z})$ T-duality group is represented by the matrices

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

that act on the matrix

$$E = \frac{r^2}{\alpha'}(G + 2\pi \alpha'B)$$

by a fractional transformation

$$T : E \mapsto (aE + b)(cE + d)^{-1}.$$  \hspace{1cm} (31)

The transformed metric and $B$-field are obtained by taking respectively the symmetric and antisymmetric parts of $E'$. The string coupling constant is transformed as

$$T : g_s \mapsto g_s' = \frac{g_s}{(\det(cE + d))^{1/2}}.$$  \hspace{1cm} (32)

The zero slope limit of Seiberg and Witten is obtained by taking

$$\alpha' \sim \sqrt{\epsilon} \to 0, \quad G_{ij} \sim \epsilon \to 0.$$  \hspace{1cm} (33)

Sending the closed string metric to zero implies that the $B$-field dominates in the open string boundary conditions. In the limit (33) the compactification is parameterized in terms of open string moduli

$$g_{ij} = -(2\pi \alpha')^2 (BG^{-1}B)_{ij}, \quad \theta^{ij} = \frac{1}{2\pi r^2} (B^{-1})^{ij}.$$  \hspace{1cm} (34)

which remain finite. One can demonstrate that $\theta^{ij}$ is a noncommutativity parameter for the torus and the low energy effective theory living on the $Dp$-brane is a noncommutative maximally supersymmetric gauge theory with a coupling constant

$$G_s = g_s \left( \frac{\det g}{\det G} \right)^{1/4}.$$  \hspace{1cm} (35)

From the transformation law (31) it is not hard to derive the transformation rules for the moduli (34) in the limit (33)

$$T : g \mapsto g' = (a + b\theta)g(a + b\theta)^t, \quad T : \theta \mapsto \theta' = (c + d\theta)(a + b\theta)^t.$$  \hspace{1cm} (36)

Furthermore the effective gauge theory becomes a noncommutative Yang-Mills theory [17] with a coupling constant

$$(g_{YM})^{-2} = \frac{(\alpha')^{3-p}}{(2\pi)^{p-2}G_s}.$$
which goes to a finite limit under (33) provided one simultaneously scales $g_s$ with $\epsilon$ as

$$g_s \sim \epsilon^{(3-p+k)/4}$$

where $k$ is the rank of $B_{ij}$. The limiting coupling constant $g_{YM}$ transforms under the T-duality (31), (32) as

$$T : g_{YM} \mapsto g'_{YM} = g_{YM} (\det(a + b\theta))^{1/4}.$$  

(37)

We see that the transformation laws (31) and (37) have the same form as the corresponding transformations in (25), (28), (29) provided one identifies matrix (26) with matrix (30) conjugated by $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The need for conjugation reflects the fact that in the BFSS M(atrix) model in the framework of which the Morita equivalence was originally considered, the natural degrees of freedom are D0 branes versus Dp branes considered in the above discussion of T-duality.

One can further check that the gauge field transformations following from gauge Morita equivalence match with those induced by the T-duality. It is worth stressing that in the absence of a $B$-field background the effective action based on the gauge field curvature squared is not invariant under T-duality.

4 Instantons on noncommutative $T_\theta^4$

Consider a Yang-Mills field $\nabla_X$ on a projective module $E$ over a noncommutative four-torus $T_\theta^4$. Assume that the Lie algebra of shifts $L_\theta$ is equipped with the standard Euclidean metric such that the metric tensor in the basis (7) is given by the identity matrix. The Yang-Mills field $\nabla_i$ is called an instanton if the self-dual part of the corresponding curvature tensor is proportional to the identity operator $F_{+jk} \equiv \frac{1}{2} (F_{jk} + \frac{1}{2} \epsilon_{jkmn} F_{mn}) = i \omega_{jk} \cdot 1 \quad \text{(38)}$

where $\omega_{jk}$ is a constant matrix with real entries. An antiinstanton is defined the same way by replacing the self-dual part with the antiself-dual one.

One can define a noncommutative analog of Nahm transform for instantons [16] that has properties very similar to those of the ordinary (commutative) one. To that end consider a triple $(\mathcal{P}, \nabla_i, \hat{\nabla}_i)$ consisting of a (finite projective) $(T_\theta^4, T_\theta^4)$-bimodule $\mathcal{P}$, $T_\theta^4$-connection $\nabla_i$ and $T_\theta^4$-connection $\hat{\nabla}_i$ that satisfy the following properties. The connection $\nabla_i$ commutes with the $T_\theta$-action on $\mathcal{P}$ and the connection $\hat{\nabla}_i$ with that of $T_\theta$. The commutators $[\nabla_i, \nabla_j]$, $[\hat{\nabla}_i, \hat{\nabla}_j]$, $[\nabla_i, \hat{\nabla}_j]$, $[\nabla_i, \hat{\nabla}_j]$ are proportional to the identity operator

$$[\nabla_i, \nabla_j] = \omega_{ij} \cdot 1, \quad [\hat{\nabla}_i, \hat{\nabla}_j] = \hat{\omega}_{ij} \cdot 1, \quad [\nabla_i, \hat{\nabla}_j] = \sigma_{ij} \cdot 1. \quad \text{(39)}$$

The above conditions mean that $\mathcal{P}$ is a $T_{\theta \oplus (-\theta)}^8$ module and $\nabla_i \oplus \hat{\nabla}_i$ is a constant curvature connection on it. In addition we assume that the tensor $\sigma_{ij}$ is non-degenerate.

For a connection $\nabla^E$ on a right $T_\theta^4$-module $E$ we define a Dirac operator $D = \Gamma^i (\nabla^E_i + \nabla_i)$ acting on the tensor product

$$(E \otimes_{T_\theta^4} \mathcal{P}) \otimes S$$
where $S$ is the $SO(4)$ spinor representation space and $\Gamma^i$ are four-dimensional Dirac gamma-matrices. The space $S$ is $\mathbb{Z}_2$-graded: $S = S^+ \oplus S^-$ and $D$ is an odd operator so that we can consider

$$
D^+ : (E \otimes T_\theta P) \otimes S^+ \to (E \otimes T_\theta P) \otimes S^-, \\
D^- : (E \otimes T_\theta P) \otimes S^- \to (E \otimes T_\theta P) \otimes S^+.
$$

A connection $\nabla^E_i$ on a $T^4_\theta$-module $E$ is called $P$-irreducible if there exists a bounded inverse to the Laplacian

$$
\Delta = \sum_i (\nabla^E_i + \nabla_i)(\nabla^E_i + \nabla_i).
$$

One can show that if $\nabla^E$ is a $P$-irreducible instanton then $\ker D^+ = 0$ and $D^- D^+ = \Delta$. Denote by $\hat{E}$ the closure of the kernel of $D^-$. Since $D^-$ commutes with the $T^4_\theta$ action on $(E \otimes T_\theta P) \otimes S^-$ the space $\hat{E}$ is a right $T^4_\theta$-module. One can prove that this module is finite projective. Let $P : (E \otimes T_\theta P) \otimes S^- \to \hat{E}$ be a Hermitian projector. Denote by $\nabla^{\hat{E}}$ the composition $P \circ \nabla$. One can show that $\nabla^{\hat{E}}$ is a Yang-Mills field on $\hat{E}$.

The noncommutative Nahm transform of a $P$-irreducible instanton connection $\nabla^E$ on $E$ is defined to be the pair $(\hat{E}, \nabla^{\hat{E}})$. One can further show that $\nabla^{\hat{E}}$ is an instanton.

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