Hook-lengths and Pairs of Compositions

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Abstract. The monomial basis for polynomials in N variables is labeled by compositions. To each composition there is associated a hook-length product, which is a product of linear functions of a parameter. The zeroes of this product are related to "critical pairs" of compositions; a concept defined in this paper. This property can be described in an elementary geometric way; for example: consider the two compositions (2,7,8,2,0,0) and (5,1,2,5,3,3), then the respective ranks, permutations of the index set \{1,2,...,6\} sorting the compositions, are (3,2,1,4,5,6) and (1,6,5,2,3,4), and the two vectors of differences (between the compositions and the ranks, respectively) are (-3,6,6,-3,-3,-3) and (2,-4,-4,2,2,2), which are parallel, with ratio -3/2. For a given composition and zero of its hook-length product there is an algorithm for constructing another composition with the parallelism property and which is comparable to it in a certain partial order on compositions, derived from the dominance order. This paper presents the motivation from the theory of nonsymmetric Jack polynomials and the description of the algorithm, as well as the proof of its validity.

1. Introduction

A composition is an element of \( \mathbb{N}^N \) (where \( \mathbb{N}_0 := \{0, 1, 2, 3, \ldots \} \)); a typical composition is \( \alpha = (\alpha_1, \ldots, \alpha_N) \) and the components \( \alpha_i \) are called the parts of \( \alpha \). Compositions have the obvious application of labeling the monomial basis of polynomials in the variables \( x_1, \ldots, x_N \) and they also serve as labels for the nonsymmetric Jack polynomials (a set of homogeneous polynomials which are simultaneous eigenfunctions of a certain parametrized and commuting set \( \{U_i : 1 \leq i \leq N\} \) of difference-differential operators). In this context the ranks of the parts of a composition become significant. The ranks are based on sorting on magnitude and index so that the largest part has rank 1; if a value is repeated then the one with lower index has lower rank. This is made precise in the following (the cardinality of a set \( E \) is denoted by \( \#E \)):

**Definition.** For \( \alpha \in \mathbb{N}^N_0 \) and \( 1 \leq i \leq N \) let \( r(\alpha, i) := \# \{ j : \alpha_j > \alpha_i \} + \# \{ j : 1 \leq j \leq i, \alpha_j = \alpha_i \} \) be the rank function.
A consequence of the definition is that \( r(\alpha, i) < r(\alpha, j) \) is equivalent to \( \alpha_i > \alpha_j \), or \( \alpha_i = \alpha_j \) and \( i < j \). For any \( \alpha \) the function \( i \mapsto r(\alpha, i) \) is one-to-one on \( \{1, 2, \ldots, N\} \). A partition is a composition satisfying \( \alpha_i \geq \alpha_{i+1} \) for all \( i \), equivalently, \( r(\alpha, i) = i \) for all \( i \). For a fixed \( \alpha \in \mathbb{N}_0^N \) the values \( \{r(\alpha, i) : 1 \leq i \leq N\} \) are independent of trailing zeros, that is, if \( \alpha' \in \mathbb{N}_0^M, \alpha'_i = \alpha_i \) for \( 1 \leq i \leq N \) and \( \alpha'_i = 0 \) for \( N < i \leq M \) then \( r(\alpha, i) = r(\alpha', i) \) for \( 1 \leq i \leq N \), and \( r(\alpha', i) = i \) for \( N < i \leq M \). A formal parameter \( \kappa \) appears in the construction of nonsymmetric Jack polynomials; their coefficients are in \( \mathbb{Q}(\kappa) \), a transcendental extension of \( \mathbb{Q} \). The relevant information in a composition label is encoded as the function \( i \mapsto \alpha_i - \kappa r(\alpha, i) \). We will be concerned with situations where a pair \( (\alpha, \beta) \) of compositions has the property that \( \alpha_i - \kappa r(\alpha, i) = \beta_i - \kappa r(\beta, i) \) for all \( i \), when \( \kappa \) is specialized to some negative rational number. This is equivalent to the condition that \( (r(\beta, i) - r(\alpha, i))\kappa + \alpha_i - \beta_i \) is a rational multiple of \( mk + n \) for some fixed \( m, n > 0 \) (or that the vectors \( (\alpha_i - \beta_i)_{i=1}^N \) and \( (r(\alpha, i) - r(\beta, i))_{i=1}^N \) are parallel). For our application an additional condition is imposed on the pair \( (\alpha, \beta) \) which is stated in terms of a partial order on compositions. Let \( S_N \) denote the symmetric group on \( N \) objects, considered as the permutation group of \( \{1, 2, \ldots, N\} \). The action of \( S_N \) on compositions is defined by \( w(\alpha)_i = \alpha_{w^{-1}(i)}, 1 \leq i \leq N \).

**Definition 2.** For a composition \( \alpha \in \mathbb{N}_0^N \) let \( |\alpha| := \sum_{i=1}^N \alpha_i \) and let \( \ell(\alpha) := \max \{j : \alpha_j > 0\} \) be the length of \( \alpha \).

**Definition 3.** For \( \alpha \in \mathbb{N}_0^N \) let \( \alpha^+ \) denote the unique partition such that \( \alpha^+ = w\alpha \) for some \( w \in S_N \). For \( \alpha, \beta \in \mathbb{N}_0^N \) the partial order \( \alpha \triangleright \beta \) (\( \alpha \) dominates \( \beta \)) means that \( \alpha \neq \beta \) and \( \sum_{i=1}^j \alpha_i \geq \sum_{i=1}^j \beta_i \) for \( 1 \leq j \leq N \); and \( \alpha \triangleright \beta \) means that \( |\alpha| = |\beta| \) and either \( \alpha^+ \triangleright \beta^+ \) or \( \alpha^+ = \beta^+ \) and \( \alpha \triangleright \beta \).

For a given \( \alpha \in \mathbb{N}_0^N \) let \( w \) be the inverse function of \( i \mapsto r(\alpha, i) \) then \( r(\alpha, w(j)) = j \) for \( 1 \leq j \leq N \) and \( \alpha = w\alpha^+ \). This permutation appears again in part (iii) of Proposition 3.

**Definition 4.** A pair \( (\alpha, \beta) \) of compositions is a \( (-\frac{n}{m}) \)-critical pair (where \( m, n \geq 1 \)) if \( \alpha \triangleright \beta \) and \( mk + n \) divides \( r(\beta, i) - r(\alpha, i) \) \( \kappa + \alpha_i - \beta_i \) (in \( \mathbb{Q}(\kappa) \)) for each \( i \).

The divisibility property is equivalent to \( (r(\beta, i) - r(\alpha, i)) n = m (\alpha_i - \beta_i) \) for all \( i \). By elementary arguments we show why only negative numbers appear in the critical pairs, and we also find a bound on \( \ell(\beta) \). A simple example shows that \( m = 0 \) is possible: let \( \alpha = (3, 0) \) and \( \beta = (2, 1) \), then both \( \alpha \) and \( \beta \) have ranks \( (1, 2) \).

**Proposition 1.** Suppose \( \alpha, \beta \in \mathbb{N}_0^N, \alpha \triangleright \beta \) and there are integers \( m, n \) such that \( (r(\beta, i) - r(\alpha, i)) \kappa + \alpha_i - \beta_i \) \( / (mk + n) \in \mathbb{Q} \) for \( 1 \leq i \leq N \), then \( mn \geq 0 \) and \( n \neq 0 \).

**Proof.** The case \( n = 0 \) is impossible since that would imply \( \alpha_i - \beta_i = 0 \) for all \( i \), that is, \( \alpha = \beta \). So we assume \( n \geq 1 \) and then show \( m \geq 0 \). Let \( w \) be the inverse function of \( i \mapsto r(\alpha, i) \) (so that \( r(\alpha, w(i)) = i \)). By definition either \( \alpha^+ \triangleright \beta^+ \) or \( \alpha^+ = \beta^+ \) and \( \alpha \triangleright \beta \). Suppose that \( \alpha^+ \triangleright \beta^+ \) and let \( k \geq 1 \) have the property that \( \beta_{w(j)} = \alpha_{w(j)} \) and \( r(\beta, w(j)) = j \) for \( 1 \leq j < k \) and at least one of \( \beta_{w(k)} \neq \alpha_{w(k)} \) and \( r(\beta, w(k)) > k = r(\alpha, w(k)) \) holds. Define \( l \) by \( r(\beta, l) = k \), then by the definition of the dominance order \( > \) we have that \( \alpha_{w(k)} \geq \beta_l \). Also \( \beta_l \geq \beta_{w(k)} \).
because \( r(\beta, w(k)) \geq k \). The case \( \alpha_{w(k)} = \beta_{w(k)} \) and \( r(\beta, k) > k \) (thus \( n = 0 \)) is impossible hence \( \alpha_{w(k)} > \beta_{w(k)} \). If \( r(\beta, k) = k \) then \( m = 0 \) or else \( r(\beta, k) > k \) and \( m > 0 \).

Now suppose \( \alpha^+ = \beta^+ \) and \( \alpha \succ \beta \), and let \( k \geq 1 \) have the property that \( \beta_j = \alpha_j \) for \( 1 \leq j < k \) and \( \alpha_k > \beta_k \) (the existence of \( k \) follows from the definition of \( \alpha \succ \beta \)). Since \( \beta \) is a permutation of \( \alpha \) we have that \( r(\alpha, j) = r(\beta, j) \) for \( 1 \leq j < k \) and \( r(\alpha, k) < r(\beta, k) \). This implies \( m > 0 \).

**Proposition 2.** Suppose that \((\alpha, \beta)\) is a \((-\frac{n}{m})\)-critical pair, for some \( m, n \geq 1 \), then \( \ell(\beta) \leq \ell(\alpha) + |\alpha| \).

**Proof.** First we show that if \( i > \ell(\alpha) \) and \( \beta_i = 0 \) then \( \beta_i = 0 \) for all \( j > i \). By hypothesis \((mn+n)\) divides \( (r(\beta, i) - r(\alpha, i)) \kappa + (\alpha_i - \beta_i) = (r(\beta, i) - i) \kappa \), hence \( r(\beta, i) = i \). This implies that \( 0 \leq \beta_j \leq \beta_i = 0 \) for all \( j > i \). Thus if \( \ell(\beta) > \ell(\alpha) \) then \( \beta_i \geq 1 \) for \( \ell(\alpha) < i \leq \ell(\beta) \). Since \( |\beta| = |\alpha| \) this shows that \( \ell(\beta) - \ell(\alpha) \leq |\alpha| \).

The motivation for the concept of hook-lengths associated with a composition came from the representation theory of the symmetric group, where it appeared in the famous hook-length formula for the degree of an irreducible representation. In Section 2 we will explain the connection with nonsymmetric Jack polynomials. However the following definitions are logically independent of this theory. Suppose \( \alpha \in \mathbb{N}_0^N \) and \( \ell(\alpha) = m \); the (modified for compositions) Ferrers diagram of \( \alpha \) is the set \( \{(i, j) : 1 \leq i \leq m, 0 \leq j \leq \alpha_i \} \). For each node \((i, j)\) with \( 1 \leq j \leq \alpha_i \) there are two special subsets of the Ferrers diagram, the **arm** \( \{(i, l) : l < i, j \leq \alpha_l \leq \alpha_i \} \) and the **leg** \( \{(l, j) : l > i, j \leq \alpha_l \leq \alpha_i \} \cup \{(l, j-1) : l < i, j-1 \leq \alpha_l \leq \alpha_i \} \). The node itself, the arm and the leg make up the **hook**. (Note that for the case of partitions the nodes \((i, 0)\) are omitted from the Ferrers diagram.)

Here is an example: the Ferrers diagram for the composition \( \alpha = (1, 0, 5, 3, 4, 2) \) (where the first part corresponds to the top row) is

```
1 2
1
○ ○ b ○ ○ ○
○ a 2 ○
○ ○ 2 ○ ○
○ 1 2
```

The leg of the node \((4, 1)\), labeled “a”, consists of the nodes labeled “1”, and the leg of the node \((3, 2)\), labeled “b”, consists of the nodes labeled “2”.

The cardinality of the leg is called the leg-length, formalized by the following:

**Definition 5.** For \( \alpha \in \mathbb{N}_0^N, 1 \leq i \leq \ell(\alpha) \) and \( 1 \leq j \leq \alpha_i \) the **leg-length** is

\[
L(\alpha; i, j) := \# \{l : l > i, j \leq \alpha_l \leq \alpha_i \} + \# \{l : l < i, j \leq \alpha_l + 1 \leq \alpha_i \}.
\]

For \( t \in \mathbb{Q}(\kappa) \) the **hook-length** and the hook-length product for \( \alpha \) are given by

\[
h(\alpha, t; i, j) := (\alpha_i - j + t + \kappa L(\alpha; i, j))^{\ell(\alpha) - \alpha_i}
\]

\[
h(\alpha, t) := \prod_{i=1}^{\ell(\alpha)} \prod_{j=1}^{\alpha_i} h(\alpha, t; i, j),
\]


Note that the indices \( \{ i : \alpha_i = 0 \} \) are omitted in the product \( h(\alpha, t) \). (In the present paper \( t \) almost always has the value \( \kappa + 1 \), but \( t = 1 \) and \( t = \kappa \) do occur in some formulae for Jack polynomials.) The results of Knop and Sahi imply that for any node \((i, j) \), \(1 \leq j \leq \alpha_i \), so that \( h(\alpha, \kappa + 1; i, j) = (L(\alpha; i, j) + 1) \kappa + \alpha_i + 1 - j \), there must exist at least one \( \beta \) such that \( (\alpha, \beta) = -(\alpha_i + 1 - j) / (L(\alpha; i, j) + 1) \)-critical. The main purpose of this paper is to construct such a composition \( \beta \) by direct algorithmic means. This forms the content of Section 3. There are examples and discussion of open problems in Section 4.

First we assume that the node is in the largest part, that is, \( r(\alpha, i) = 1 \). The modification for other parts is trivial - one merely ignores all larger parts \( \alpha_k \) (with \( r(\alpha, k) < r(\alpha, i) \)). This will be explained in detail later. We illustrate how the algorithm works on a partition \( \alpha \), thereby avoiding some technical complexity. Choose \( n \leq \alpha_1 \) and suppose \( L(\alpha; 1, \alpha_1 + 1 - n) = m - 1 \) (thus \( h(\alpha, \kappa + 1; 1, \alpha_1 + 1 - n) = m \kappa + n \) ). Then \( \alpha_m > \alpha_1 - n \geq \alpha_{m+1} \). Define a sequence by \( \xi_{mk+i} = \alpha_i - nk \) for \( 1 \leq i \leq m \) and \( k \geq 0 \). Then \( \xi_j \geq \xi_{j+1} \) for all \( j \): indeed if \( j = mk + i \) with \( i < m \) then \( \xi_j - \xi_{j+1} = (\alpha_i - nk) - (\alpha_{i+1} - nk) \geq 0 \) and if \( j = mk \) then \( \xi_{mk} - \xi_{mk+1} = (\alpha_m - n (k - 1)) - (\alpha_{m} - nk) = \alpha_m - (\alpha_1 - n) > 0 \). Also \( \alpha_{m+1} \leq \alpha_1 - n = \xi_{m+1} \). Since the values \( \{ \xi_j \} \) are eventually negative there exists a unique \( T \) such that \( \alpha_{m+x} \leq \xi_{m+x+1} \) for \( 0 < s < T \) and \( \alpha_{m+T} > \xi_{m+T+1} \) (or \( T = 1 \) when \( \alpha_{m+1} > \xi_{m+2} \)). Set \( t := ((T - 1) \bmod m) + 1 \) and \( k := (T - t) / m \) (thus \( T = mk + t \) and \( 1 \leq t \leq m \)). Define \( \beta \in N_0^N \) by

\[
\beta_i := \begin{cases} 
\xi_{(k+1)m+i} = \alpha_i - (k+1)n, & 1 \leq i \leq t \\
\xi_{km+i} = \alpha_i - kn, & t < i \leq m \\
\alpha_i + n, & m + 1 \leq i \leq m + T \\
\alpha_i, & m + T < i.
\end{cases}
\]

In this context, an upper bound on \( N \) is not needed; that is, \( \alpha_i \) is defined for all \( i \geq 1 \) and \( \alpha_i = 0 \) for \( i > \ell(\alpha) \). However one can show that \( l(\beta) \leq \max (l(\alpha), m + T_0) \) where \( T_0 := \sum_{i=0}^m \left[ \frac{m}{i^2} \right] (\lfloor r \rfloor) \) is the largest integer \( \leq r \) and it suffices to take \( N \) as large as this bound. Then \( \alpha_{m+T+n} > \beta_{m+T+1} > \beta_{m+T} > \beta_{m+T-1} = \alpha_{m+T+1} \) and \( r(\beta, m + i) = i \) for \( 1 \leq i \leq T \), \( r(\beta, i) = T + m - t + t = m (k + 1) + t \) for \( 1 \leq i \leq t \), \( r(\beta, i) = mk + i \) for \( t < i \leq m \) and \( r(\beta, i) = i \) for \( i > m + T \). The proof of these facts is a special case of the general result.

The computational scheme can be set up in algorithmic fashion: consider the example \( \alpha = (9, 8, 8, 7, 4, 3, 3, 2, 2) \) with \( h(\alpha, \kappa + 1; 1, 7) = 4 \kappa + 3 \), so \( m = 4 \) and \( n = 3 \). Generate enough of the sequence \( \{ \xi_i \}_{i=1}^N \) to determine the value of \( T \); note that \( \{ \xi_i \}_{i=5}^N = (9 - 3, 8 - 3, 8 - 3, 7 - 3, 9 - 6, 8 - 6, \ldots) \). Comparing the sequences

\[
\alpha = (9, 8, 8, 7, 4, 3, 3, 2, 2, 0, 0, 0, 0, 0, \ldots) \\
\xi = (9, 8, 8, 7, 6, 5, 5, 4, 3, 2, 2, 1, 0, -1, \ldots)
\]

term-by-term we see that \( T = 9 \) \( (\alpha_{4+9} > \xi_{4+10} \land \alpha_{4+s} \leq \xi_{5+s} \) for \( 1 \leq s \leq 8 \)). Finally \( t = 1 \), \( k = 2 \) and the formula produces \( \beta = (0, 2, 2, 1, 7, 6, 6, 5, 5, 3, 3, 3, 3) \); it can be checked that \( (\alpha, \beta) \) is a \( \left( -\frac{4}{3} \right) \)-critical pair.

2. Nonsymmetric Jack polynomials and hook-length products

For \( \alpha \in N_0^N \) the corresponding monomial is \( x^\alpha := \prod_{i=1}^N x_i^{\alpha_i} \) and the degree of \( x^\alpha \) is \( |\alpha| \). For \( 1 \leq i, j \leq N \) and \( i \neq j \) the transposition of \( i \) and \( j \) is denoted by \( (i, j) \) (that is, the permutation \( w \) with \( w(i) = j \), \( w(j) = i \) and \( w(k) = k \) for all \( 1 \leq k < N \) is denoted by \( (i, j) \)).
The action of $S_N$ on coordinates is defined by $(xw)_i = x_{w(i)}$ and is extended to polynomials by $(wp)(x) := p(xw)$ with the effect that $w(x^\alpha) = x^{w\alpha}$.

The operators $U_i$ for $1 \leq i \leq N$ are defined by

$$U_i p(x) := \frac{\partial}{\partial x_i} (xp(x)) + \kappa \sum_{j=1, j \neq i}^{N} \frac{x_i p(x) - x_j p(x(i,j))}{x_i - x_j} - \kappa \sum_{j=1}^{i-1} p(x(i,j)),$$

where $p$ is a polynomial ($\in \mathbb{Q}(\kappa)[x_1, x_2, \ldots, x_N]$). Then (see [2] pp.291-2 for details) $U_i U_j = U_j U_i$ for $1 \leq i, j \leq N$ and there is a crucial triangularity (in the sense of matrices) property: $U_i x^\alpha = \xi_i(\alpha) x^\alpha + q_{\alpha,i}(x)$ where $q_{\alpha,i}(x)$ is a sum of terms of the form $\pm \kappa x^\beta$ with certain $\beta \prec \alpha$ and

$$\xi_i(\alpha) := (N - r(\alpha,i)) \kappa + \alpha_i + 1$$

for $1 \leq i \leq N$ and $\alpha \in \mathbb{N}_0^N$.

The existence of the nonsymmetric Jack polynomials follows from a theorem of elementary linear algebra. Suppose $\{A^{(i)} : 1 \leq i \leq N\}$ is a collection of pairwise commuting lower triangular $M \times M$ matrices over a field. If for each pair $(i, j), 1 \leq i < j \leq M$ there is at least one matrix $A^{(k)}$ such that $A^{(k)} \neq A^{(j)}$, then there exists a unique set of $M$ linearly independent simultaneous (column) eigenvectors for $\{A^{(i)}\}$ with each eigenvector of the form $(0, \ldots, 0, 1, \ldots)^T$. Equivalently, there is a unique lower triangular unipotent matrix $V$ such that $V^{-1} A^{(i)} V$ is diagonal for each $i$. Now apply this result to the action of $\{U_i : 1 \leq i \leq N\}$ on the spaces of homogenous polynomials with the standard basis $\{x^\alpha : \alpha \in \mathbb{N}_0^N, |\alpha| = k\}$, ordered by $\triangleright$, for $k \in \mathbb{N}_0$. It is clear that $\alpha, \beta \in \mathbb{N}_0^N$ and $\alpha \neq \beta$ implies that $\xi_i(\alpha) \neq \xi_i(\beta)$ for any $i$ with $\alpha_i \neq \beta_i$ and generic $\kappa$. Thus for each $\alpha \in \mathbb{N}_0^N$ there is a unique polynomial, called the nonsymmetric Jack polynomial,

$$\zeta_\alpha(x) = x^\alpha + \sum_{\beta \prec \alpha} A_{\beta\alpha} x^\beta,$$

with coefficients $A_{\beta\alpha} \in \mathbb{Q}(\kappa)$ such that

$$U_i \zeta_\alpha = \xi_i(\alpha) \zeta_\alpha, \text{ for } 1 \leq i \leq N.$$

The coefficients, as rational functions of $\kappa$, can have poles only at certain negative rational numbers, which in turn are linked to the critical pairs $(\alpha, \beta)$. This is a sketch of the argument (for a detailed proof see [11]): by the triangularity property there are coefficients $B_{\beta\alpha} \in \mathbb{Q}(\kappa)$ such that $x^\alpha = \zeta_\alpha + \sum_{\beta \prec \alpha} B_{\beta\alpha} \zeta_\beta$ for each $\alpha \in \mathbb{N}_0^N$. Extend the field $\mathbb{Q}(\kappa)$ by adjoining a formal transcendental $\nu$ and consider the operator (on polynomials with coefficients in $\mathbb{Q}(\kappa, \nu)$)

$$T_\alpha := \prod_{\beta \prec \alpha} \frac{\sum_{\nu=1}^{N} \nu^i (U_i - \xi_i(\beta))}{\sum_{\nu=1}^{N} \nu^i (\xi_i(\alpha) - \xi_i(\beta))}.$$

Now apply $T_\alpha$ to the expression for $x^\alpha$ and obtain:

$$\prod_{\beta \prec \alpha} \frac{\sum_{\nu=1}^{N} \nu^i (U_i - \xi_i(\beta))}{\sum_{\nu=1}^{N} \nu^i (\xi_i(\alpha) - \xi_i(\beta))} x^\alpha = \zeta_\alpha + \sum_{\beta \prec \alpha} B_{\beta\alpha} T_\alpha \zeta_\beta = \zeta_\alpha.$$
Since the variable \( \nu \) does not appear in \( \zeta_\alpha \), the denominators of the coefficients \( B_{\beta\alpha} \) must come from reducible factors of the form

\[
\sum_{i=1}^{N} \nu^{i} (\xi_i (\alpha) - \xi_i (\beta)) = \sum_{i=1}^{N} \nu^{i} (r (\beta, i) - r (\alpha, i)) \kappa + \alpha_i - \beta_i
\]

\[
= \left( \sum_{i=1}^{N} c_i \nu^{i} \right) (mk + n)
\]

with \( c_i \in \mathbb{Q} \), \( \beta \prec \alpha \) and \( m, n \geq 1 \). This is exactly the property that \( (\alpha, \beta) \) is a \( \left( -\frac{\nu}{mk} \right) \)-critical pair.

By combinatorial means Knop and Sahi [3] showed that all coefficients of \( h (\alpha, \kappa + 1) \zeta_\alpha^\kappa \) are in \( \mathbb{N}[\kappa] \) (polynomials in \( \kappa \) with nonnegative integer coefficients) and the coefficient of \( x_{k+1} x_{k+2} \ldots x_{k+i-1} \) in \( \zeta_\alpha \) is \( \nu^i / h (\alpha, \kappa + 1) \) for \( k = \ell (\alpha) \) and \( l = |\alpha| \).

We conclude from the above discussion that for any node \((i, j), 1 \leq j \leq \alpha_i \) with \( h (\alpha, \kappa + 1; i, j) = (L (\alpha; i, j) + 1) \kappa + \alpha_i + 1 - j \) there must exist at least one \( \beta \) such that \((\alpha, \beta)\) is \(-\frac{\nu}{mk} \)-critical. The main reason for setting up the machinery of critical pairs is to provide a tool for analyzing the dependence of the poles (as functions of \( \kappa \)) in the coefficients of \( \zeta_\alpha \) on the number of variables. This will be illustrated in the last section.

3. The Construction of Critical Pairs

The main difficulty in extending the method from partitions to compositions is to deal with tied values. Recall that for \( \alpha_i = \alpha_j \) we have \( r (\alpha, i) < r (\alpha, j) \) if and only if \( i < j \). The definition of leg-length \( L \) is more subtle for compositions. The introduction of small deformations in the values makes it possible to use essentially the same method as for partitions. Loosely speaking we use an infinitesimal quantity \( \nu \) which satisfies \( 0 < i \nu < 1 \) for all \( i \geq 1 \), but of course the inequality will only be needed for all \( i \leq N \) for some \( N \geq \max (\ell (\alpha), \ell (\beta)) \). In the sequel we let \( N = \ell (\alpha) + |\alpha| \) which suffices by Proposition 2 and we let \( \nu := \frac{1}{N+1} \).

**Definition 6.** For \( \alpha \in \mathbb{N}^N_0 \) let \( \bar{\alpha} \in \mathbb{Q}^N \) be given by \( \bar{\alpha}_i := \alpha_i - i \nu \) for \( 1 \leq i \leq N \).

**Proposition 3.** For any \( \alpha \in \mathbb{N}^N_0 \) the following hold:

(i) If \( i \neq j \) then \( \bar{\alpha}_i - \bar{\alpha}_j \notin \mathbb{Z} \), in particular, \( \bar{\alpha}_i \neq \bar{\alpha}_j \),

(ii) \( r (\alpha, i) = r (\bar{\alpha}, i) \) for \( 1 \leq i \leq N \),

(iii) there is a unique permutation \( w \) of \( \{1, \ldots, N\} \) such that \( r (\alpha, w (i)) = i \).

**Proof.** To show part (i) suppose \( \alpha_i > \alpha_j \) then \( \bar{\alpha}_i - \bar{\alpha}_j = \alpha_i - \alpha_j - (i - j) \nu \geq 1 - (i - j) \nu > 0 \), or suppose \( \alpha_i = \alpha_j \) and \( i < j \) then \( \bar{\alpha}_i - \bar{\alpha}_j = (j - i) \nu > 0 \). In both cases, \( \bar{\alpha}_i - \bar{\alpha}_j \notin \mathbb{Z} \). This shows that \( r (\bar{\alpha}, i) = \# \{ j : \bar{\alpha}_j > \bar{\alpha}_i \} + 1 \) (extending the definition of the rank function to elements of \( \mathbb{Q}^N \)). Now

\[
\begin{align*}
r (\alpha, i) &= \# \{ j : j > i, \alpha_j > \alpha_i \} + \# \{ j : j < i, \alpha_j \geq \alpha_i \} + 1 \\
&= \# \{ j : j > i, \bar{\alpha}_j > \bar{\alpha}_i \} + \# \{ j : j < i, \bar{\alpha}_j > \bar{\alpha}_i \} + 1 \\
&= r (\bar{\alpha}, i)
\end{align*}
\]

for \( 1 \leq i \leq N \). The fact that the sorting permutation \( w \) is unique follows trivially from part (i). \( \square \)
As noted before, part (iii) implies that \( \alpha_i^+ = \alpha_{w(i)} \), also that \( \alpha_i = \alpha_j \) and \( i < j \) implies \( w(i) < w(j) \) (consider this as the formal proof that \( \beta \mapsto r \) \((\alpha, \beta)\) is one-to-one). Here is the formula for leg-length in terms of \( \tilde{\alpha} \).

**Proposition 4.** For \( \alpha \in \mathbb{N}_0^N \) and for \( 1 \leq i \leq \ell(\alpha) \) so that \((\alpha,\beta)\) with \( k \) one-to-one. Here is the formula for leg-length in terms of \( \tilde{\alpha} \).

This shows that the \( \tilde{\alpha} \) is strictly decreasing and \( \tilde{\alpha}(w(j)) \) for large enough \( \tilde{\alpha} \). There is a unique \( T \) such that \( \tilde{\alpha}(w(m+s)) < \xi_{m+s+1} \) for \( 1 \leq s < T \) and \( \tilde{\alpha}(w(m+T)) > \xi_{m+T+1} \) (the case of equality is ruled out).

**Proof.** The decreasing property has two cases. If \( 1 \leq i < m \) then \( \xi_{mk+i} = \tilde{\alpha}_w(i+n) - nk \), and \( \tilde{\alpha}_w(i) - nk > 0 \). If \( i = m \) then \( \xi_{mk+m} = \tilde{\alpha}_w(m) - nk \) is strictly decreasing, and \( \tilde{\alpha}_w(i) - nk \) for all \( i < j \) is impossible for \( \tilde{\alpha}_w(m+s) = \xi_{m+s+1} \) when \( s \geq 1 \).

Let \( T_0 = \sum_{i=1}^{m} \left\lfloor \alpha_{w(i)/n} \right\rfloor \). We claim \( \xi_{m+T_0+1} = n - 1 \). Set \( \alpha_{w(i)} = nq_i + r_i \) with \( q_i \geq 0 \) and \( 0 \leq r_i \leq n - 1 \) for \( 1 \leq i \leq m \). Then \( q_i \geq 1 \) since \( n \leq \alpha_w(i) \) and \( q_i \geq q_i - 1 \). This follows from the inequalities \( \alpha_w(i) \geq \alpha_w(i) \), that is,

\[
\begin{align*}
nq_i + r_i & \geq nq_i + r_i \geq (n-1)q_i + r_i, \\
r_i - r_i & \geq n(q_i - q_i) \geq r_i - r_i - n,
\end{align*}
\]

but \( 1 - n \leq r_i - r_i \leq n - 1 \) so \( 1 - 2n \leq n(q_i - q_i) \leq n - 1 \). Furthermore \( q_i \geq q_i + 1 \) for \( 1 \leq i < m \) since \( \alpha_w(i) \geq \alpha_w(i+1) \). Thus there exists \( k \) with \( 1 < l \leq m \) so that \( q_i = q_i \) for \( 1 \leq i \leq l \) and \( q_i = q_i - 1 \) for \( l < i \leq m \). Write \( T_0 + m + 1 = mk + i \) with \( k \geq 0 \) and \( 1 \leq i \leq m \). Then \( mk + i = (lq_i + (m-l)(q_i - 1)) + m + 1 = mq_i + l + 1 \) and so \( i = (l+1) \bmod m \). If \( l < m \) then \( k = q_i, i = l + 1 \) and \( \xi_{T_0 + m + 1} = \alpha_{w(i+1)} - nq_i - w(l+1) = r_{i+1} - n - w(l+1) < 1 \). If \( l = m \) then \( k = q_i + 1, i = 1 \) and \( \xi_{T_0 + m + 1} = \alpha_{w(i)} - nq_i - w(l+1) = r_1 - n - w(l+1) < 1 \).
Finally $T_0 \leq \frac{|\alpha|}{\ell}$ and $m + T_0 + 1 \leq \ell (\alpha) + |\alpha| + 1$; also $w(j) = j$ for $j > \ell (\alpha)$ and thus $\tilde{\alpha}_{w(j)} > -1 > \xi_{j+1}$ for all sufficiently large $j \leq N$. The existence and uniqueness of $T$ is now obvious. By part (i) of Proposition 3, $\tilde{\alpha}_{w(m+s)} = \xi_{m+s+1}$ is impossible for $s \geq 1$.

**Definition 7.** With $\alpha$ and $T$ as described above let $t := ((T - 1) \mod m) + 1$, $k := \frac{T + t - m}{t}$ (so that $T = mk + t$ and $1 \leq t \leq m$) and define $\beta \in \mathbb{N}_0^N$ by

$$\beta_{w(i)} := \begin{cases} 
\alpha_{w(i)} - (k + 1) n, & 1 \leq i \leq t \\
\alpha_{w(i)} - kn, & t < i \leq m \\
\alpha_{w(i)} + n, & m + 1 \leq i \leq m + T \\
\alpha_{w(i)}, & m + T < i.
\end{cases}$$

(3.1)

The following is the main result. The notations $\alpha, \beta, \xi, w, m, n, T, t$ continue with the definitions given above. The proof is broken up in several lemmas. It is possible that $m = 1$, in which case one makes the obvious modifications in the following statements.

**Theorem 1.** For $\alpha \in \mathbb{N}_0^N$ the composition $\beta$ in Definition 7 has the property that $mk + n$ divides $(r (\beta, w (i)) - i) \kappa + \alpha_{w(i)} - \beta_{w(i)}$ for all $i$, and $\alpha \triangleright \beta$, that is, $(\alpha, \beta)$ is a $(-\frac{m}{\kappa})$-critical pair.

Before we present the details of the proof we explain how the Theorem can be used as an algorithm. Here is an informal description.

**Algorithm 1.** Start with $\alpha, m, n, N = |\alpha| + \ell (\alpha), \nu$ as described above.

(1) Compute the permutation $w \in S_N$ with the property $r (\alpha, w (i)) = i$ for $1 \leq i \leq N$.

(2) for $i = 1, 2, \ldots, m$ set $\xi_i := \tilde{\alpha}_{w(i)}$, and set $\xi_{m+1} := \xi_1 - n$.

(3) for $s = 2, 3, \ldots$ set $\xi_{m+s} := \xi_s - n$ until $\xi_{m+s} < \tilde{\alpha}_{w(m+s-1)}$.

(4) set $T := s - 1$ (where $s$ is the first value in step 3 for which $\xi_{m+s} < \tilde{\alpha}_{w(m+s-1)}$).

(5) set $t := ((T - 1) \mod m) + 1$, $k := \frac{T + t - m}{t}$.

(6) use Equation (3.1) to compute $\beta$.

Here is an example: let $\alpha = (0, 3, 5, 6, 6, 1)$; then $h (\alpha, \kappa + 1; 4, 4) = 4\kappa + 3$ and $(w (i))_{i=1}^N = (4, 5, 3, 2, 6, 1, 7, 8, \ldots)$. The sequence $(\tilde{\alpha}_{w(i)})_{i=1}^N$ is

$$(6 - 4\nu, 6 - 5\nu, 5 - 3\nu, 3 - 2\nu, 1 - 6\nu, -\nu, -7\nu, -8\nu, -9\nu, -10\nu, -11\nu, \ldots)$$

and the sequence $(\xi_i)_{i=1}^N$ is computed up to the $11^{th}$ term

$$(6 - 4\nu, 6 - 5\nu, 5 - 3\nu, 3 - 2\nu, 3 - 4\nu, 3 - 5\nu, 2 - 3\nu, -4\nu, -4\nu, -5\nu, -1 - 3\nu, \ldots)$$

since $\tilde{\alpha}_{w(4+6)} > \xi_{4+7}$ and $\tilde{\alpha}_{w(4+s)} < \xi_{5+s}$ for $1 \leq s < 6$. So $T = 6, t = 2, k = 1$ and $(\beta_{w(i)})_{i=1}^N = (0, 0, 0, 0, 0, 4, 3, 3, 3, 3, 3, 0, \ldots)$, $\beta = (3, 0, 2, 0, 0, 4, 3, 3, 3, 3)$. Now $(r (\beta, i))_{i=1}^N = (2, 8, 7, 9, 10, 1, 3, 4, 5, 6)$ and $(\alpha, \beta)$ is indeed a $(-\frac{m}{\kappa})$-critical pair.

It is possible that $T = 1$, for example let $\alpha = (2, 6, 5, 2)$ then $h (\alpha, \kappa + 1; 2, 4) = 2\kappa + 3, (\tilde{\alpha}_{w(i)})_{i=1}^N = (6 - 2\nu, 5 - 3\nu, 2 - 4\nu)$ and $\xi = (6 - 2\nu, 5 - 3\nu, 3 - 2\nu, 2 - 3\nu)$ so that $\tilde{\alpha}_{w(2+1)} > \xi_4$; then $\beta = (5, 3, 5, 2)$.

The proof of the theorem is broken up into several lemmas.
LEMMA 2. The following inequalities hold:

(i) \( \bar{\alpha}_{w(m+T+1)} < \xi_{T+m} < T + 1 < \bar{\alpha}_{w(m+T)} + n \) (omit \( \xi_{T+m} \) if \( m = 1 \))
(ii) for \( t < m, \bar{\beta}_{w(m+T)} > \bar{\beta}_{w(t+1)} > \ldots > \bar{\beta}_{w(m)} > \bar{\beta}_{w(1)} > \ldots > \bar{\beta}_{w(t)} > \bar{\beta}_{w(m+T+1)} \)
(iii) for \( t = m, \bar{\beta}_{w(m+T)} > \bar{\beta}_{w(1)} > \ldots > \bar{\beta}_{w(m)} > \bar{\beta}_{w(m+T+1)} \).

PROOF. By construction \( \bar{\alpha}_{w(m+T)} + n > \xi_{m+T+1} + n = (\xi_{T+1} - n) + n > \xi_{T+m} \), by the decreasing property of \( \{\xi_j\} \). If \( T > 1 \) then \( \alpha_{w(m+T-1)} > \bar{\alpha}_{w(m+T+1)} \). If \( T = 1 \) then \( \bar{\alpha}_{w(m+1)} + n > \xi_2 > \alpha_{w(m+1)} > \bar{\alpha}_{w(m+2)} \). This proves part (i). As before \( T = km + t \). By construction,

\[
\bar{\beta}_{w(i)} = \begin{cases} 
\xi_{mk+i}, & 1 \leq i \leq t \\
\xi_{mk+i}, & t < i \leq m \\
\alpha_{w(i)} + n, & m + 1 \leq i \leq m + T \\
\alpha_{w(i)}, & m + T < i.
\end{cases}
\]

Thus the inequality in part (i) shows \( \bar{\beta}_{w(m+T+1)} < \bar{\beta}_{w(t)} < \bar{\beta}_{w(t+1)} < \bar{\beta}_{w(m+T)} \). \( \square \)

LEMMA 3. The following rank values hold:

(i) \( r(\beta, w(m+1)) = i \) for \( 1 \leq i \leq T \),
(ii) \( r(\beta, w(t+i)) = mk + t + i \) for \( 1 \leq i \leq m - t \),
(iii) \( r(\beta, w(i)) = m(k+1) + i \) for \( 1 \leq i \leq t \),
(iv) \( r(\beta, w(i)) = i \) for \( m + T + 1 \leq i \leq N \).

PROOF. By parts (ii) and (iii) of the previous lemma, in decreasing order the \( m \) largest values of \( \beta \) are \( \bar{\beta}_{w(m+1)}, \ldots, \bar{\beta}_{w(m+T)} \), the next \( m - t \) values are \( \bar{\beta}_{w(t+1)}, \ldots, \bar{\beta}_{w(m)} \), the next \( t \) values are \( \bar{\beta}_{w(t)}, \ldots, \bar{\beta}_{w(t)} \), and the remaining are \( \bar{\beta}_{w(m+T+1)}, \ldots \). Note that \( r(\beta, w(t+i)) = T + i = mk + t + i \) in part (ii) and \( r(\beta, w(i)) = T + (m-t) + i = m(k+1) + i \).

LEMMA 4. The composition \( \beta \) satisfies the condition that \( mk + n \) divides \( r(\beta, w(i) - i) + \alpha_{w(i)} - \beta_{w(i)} \) for all \( i \leq N \).

PROOF. Let \( \gamma_i = (r(\beta, w(i)) - i) + \alpha_{w(i)} - \beta_{w(i)} \). For \( 1 \leq i \leq t \), \( \gamma_i = (m(k+1) + i - i) + \alpha_{w(i)} - (\alpha_{w(i)} + (k+1)n) = (k+1)(mk+n) \). For \( t < i \leq m \), \( \gamma_i = m(k+1) + \alpha_{w(i)} - (\alpha_{w(i)} + kn) = k(mk+n) \). For \( m+1 \leq i \leq m + T \), \( \gamma_i = ((i-m) - i) + \alpha_{w(i)} - (\alpha_{w(i)} + n) = -(mk+n) \).

We must show that \( \alpha \triangleright \beta \) to complete the proof of the theorem. For \( 1 \leq i \leq N \) let \( \varepsilon(i) \in \mathbb{N}^N \) denote the standard basis element, that is, \( \varepsilon(i)_j = \delta_{ij} \). The idea is to describe the construction as a sequence of compositions, each of which is produced by adding \( n(\varepsilon(w(m+i)) - \varepsilon(w((i-1) \mod m+1))) \) to the previous one, for \( i = 1, 2, \ldots, T \). The effect of the argument \((i-1) \mod m + 1 \) is to cycle through the values \( w(1), \ldots, w(m) \).

LEMMA 5. Let \( \beta(s) = \alpha + n \sum_{i=1}^s (\varepsilon(w((i-1) \mod m + 1)) - \varepsilon(w(m+i))) \) for \( 0 \leq s \leq T \). Then \( \beta(0) = \alpha, \beta(T) = \beta(s) \triangleright \beta(s+1) \) for \( 0 \leq s < T \).

PROOF. We use Lemma 8.2.3 from [2] p.289. This states that if \( \lambda \) is a partition such that \( 1 \leq n < \lambda_i \) (for some \( i < j \)) then \( \lambda \triangleright (\lambda - n(\varepsilon(i) - \varepsilon(j)))^+ \). As a consequence we have that if \( \gamma \in \mathbb{N}^N \) and \( 1 \leq n < \gamma_i - \gamma_j \) for some \( i, j \) then \( \gamma^+ \triangleright (\gamma - n(\varepsilon(i) - \varepsilon(j)))^+ \), that is, \( \gamma \triangleright (\gamma - n(\varepsilon(i) - \varepsilon(j))) \). Suppose
$s = ml + i < m + T$ with $0 \leq i < m$ and $l \geq 0$. Then $\beta^{(s)}_{w(i)} = \alpha_{w(i)} + n$ for $m + 1 \leq j \leq m + s$, $\beta^{(s)}_{w(j)} = \alpha_{w(j)} - n (l + 1)$ for $1 \leq j \leq i$, $\beta^{(s)}_{w(i)} = \alpha_{w(i)} - nl$ for $i + 1 \leq j \leq m$ and $\beta^{(s)}_{w(m+s+1)} = \alpha_{w(m+s+1)}$. By definition, $\beta^{(s+1)} = \beta^{(s)} - n (\varepsilon (w(i+1)) - \varepsilon (w(m + s + 1)))$. Let $\delta_s$ denote the difference between the two affected values, that is,

$$\delta_s := \beta^{(s)}_{w(i+1)} - \beta^{(s)}_{w(m+s+1)}$$

$$= \alpha_{w(i+1)} - nl - \alpha_{w(m+s+1)}$$

$$= \xi_{ml+i+1} - \alpha_{w(m+s+1)} + (w(s+1) - w(m+s+1)) \nu.$$ 

By the construction of the Algorithm $\xi_{m+s+1} > \alpha_{w(m+s)}$ for $s < T$ and $\alpha_{w(m+s)} > \alpha_{w(m+s+1)}$. Thus $\delta_s - n = \xi_{m+s+1} - \alpha_{w(m+s+1)} + (w(s+1) - w(m+s+1)) \nu > (w(s+1) - w(m+s+1)) \nu > -1$. If $\delta_s - n \geq 1$ then by the above argument we have $\beta^{(s)} > \beta^{(s+1)}$. If $\delta_s - n = 0$ then $\beta^{(s+1)}_{w(m+s+1)} = \beta^{(s)}_{w(i+1)}$, $\beta^{(s+1)}_{w(i+1)} = \beta^{(s)}_{w(i+1)}$ and $w(s+1) < w(m+s+1)$; thus $(\beta^{(s+1)})^+ = (\beta^{(s)})^+$ and $\beta^{(s)} > \beta^{(s+1)}$ (for any composition $\gamma$, if $i < j$ and $\gamma_i > \gamma_j$ then $\gamma > (i,j) \gamma$, that is, $\beta^{(s)} > \beta^{(s+1)}$.

This completes the proof of the theorem when the hook length $mk+n$ is associated with the largest part (either $\alpha_{w(1)} > \alpha_{w(2)}$ or $\alpha_{w(1)} = \alpha_{w(2)}$ and $i \neq 1$ implies $w(1) < w(i)$). Suppose for some $l \geq 1$ that $L(\alpha; w(l+1), \alpha_{w(l+1)} + 1 - n) = m - 1$. Then the algorithm is applied with the arguments of $w$ and the ranks all shifted by $l$; and the largest $l$ parts are not changed. Thus $\beta^{(s)}_{w(i)} := \alpha_{w(i)}$ for $1 \leq i \leq l$ and

$$\beta^{(s)}_{w(l+i)} := \begin{cases} 
\alpha_{w(l+i)} - (k + 1) n, & 1 \leq i \leq t \\
\alpha_{w(l+i)} - kn, & t < i \leq m \\
\alpha_{w(l+i)} + n, & m + 1 \leq i \leq m + T \\
\alpha_{w(l+i)}, & m + T < i,
\end{cases}$$

using the same notations $k, T, t$ as above.

4. Examples and Discussion

The first example is a partition type: $\alpha = (9, 8, 8, 7, 4, 3, 3, 2, 2)$, $n = 3, m = 4, T = 9$ and $\beta = (0, 2, 2, 1, 7, 6, 6, 5, 5, 3, 3, 3, 3)$. The ranks of $\beta$ are $(r(\beta,i))_{i=1}^{13} = (13, 10, 11, 12, 1, 2, 3, 4, 5, 6, 7, 8, 9)$. To illustrate the situation where the node $(i,j)$ (with hook length $h(\alpha, \kappa + 1; i,j) = L(\alpha; i,j) + 1 \kappa + \alpha_i + 1 - j$) is not in the row of rank 1, consider the hook at $(2,5)$ in $\alpha = (9, 8, 8, 5, 4, 4)$, then $L(\alpha; 2, 5) = 2, n = 4$ and $\beta = (9, 4, 4, 5, 8, 8)$ with ranks $(1, 5, 6, 4, 2, 3)$. The next example is a composition $\alpha = (0, 3, 5, 6, 6, 4, 1)$ (with $(r(\alpha,i))_{i=1}^{12} = (7, 5, 3, 1, 2, 4, 6, 4)$) with $n = 3$; then $m = L(\alpha; 4, 4) + 1 = 5, T = 6$ and $\beta = (3, 0, 2, 0, 0, 1, 4, 3, 3, 3, 3, 3)$ (with $(r(\beta,i))_{i=1}^{12} = (2, 10, 8, 11, 12, 9, 1, 3, 4, 5, 6, 7)).$

There is an analogous situation for critical pairs when $\beta$ is the given composition; in this case the expansion of nonsymmetric Jack polynomials in terms of the $p$-basis (see [24] p.298) suggests that the linear factors $(mk+n)$ of $h(\beta, 1)$ lead to $(-\frac{1}{\beta})$-critical pairs $(\alpha, \beta)$. However we consider the uniqueness problem described below as more important for applications, and will not further investigate $h(\beta, 1)$.

A natural question occurs: for a given $\alpha$ and hook-length $mk+n$ does one step of the algorithm produce all possible solutions for $\beta$? It makes sense to consider a sequence of steps because there is a transitive property for critical pairs: if $(\alpha, \beta^{(1)})$
and \((\beta^{(1)}, \beta^{(2)})\) are \((-\frac{n}{m})\)-critical pairs then so is \((\alpha, \beta^{(2)})\) (\(\succ\) being a partial order). If one step sufficed then there would be a uniqueness result of the form: suppose the multiplicity of the linear factor \((m\kappa + n)\) in the hook-length product \(h(\alpha, \kappa + 1)\) is one, then there is a unique \(\beta\) so that \((\alpha, \beta)\) is a \((-\frac{n}{m})\)-critical pair. However, this may fail if \(m, n\) are not relatively prime: for \(\alpha = (6, 3, 1, 1), n = 6, m = 4\) the algorithm produces \(\beta^{(1)} = (0, 3, 1, 1, 6)\); the multiplicity of \((2\kappa + 3)\) is one in both \(h(\alpha, \kappa + 1)\) and \(h(\beta^{(1)}, \kappa + 1)\). Apply the algorithm to \(\beta^{(1)}\) with \(n = 3, m = 2\) to obtain \(\beta^{(2)} = (0, 3, 4, 1, 3)\). The process stops since \(h(\beta^{(2)}, \kappa + 1)\) does not have \((2\kappa + 3)\) as a factor. We conjecture there is uniqueness if \(m, n\) are relatively prime (and the multiplicity is one): a particular case of this was established in [1]. This is crucial because it is used to show that \(\zeta_\alpha\) has no pole at \(\kappa = -\frac{n}{m}\) when the number of variables is less than \(\ell(\beta)\). Here is an example: \(\alpha = (7, 6, 6, 4, 4)\) with \(n = 2\), so that \(m = 3\), then the unique \(\beta = (1, 0, 0, 6, 6, 2, 2, 2, 2, 2, 2, 2)\); the application is to \(\zeta_\alpha\) on \(\mathbb{R}^{10}\) and the uniqueness of \(\beta\) implies that \(\kappa = -\frac{2}{3}\) is not a pole of \(\zeta_\alpha\) (for less than 12 variables). However the uniqueness result uses specific properties of a class of partitions and \(m, n\) are relatively prime. There is a weak uniqueness result concerning the sign changes in the sequence \(\left(\tilde{\alpha}_{w(m+i)} - \xi_{m+i+1}\right)_{i=1}^N\). Recall that \(T\) is chosen so that \(\tilde{\alpha}_{w(m+s)} - \xi_{m+s+1} < 0\) for \(1 \leq s < T\) and \(\tilde{\alpha}_{w(m+T)} - \xi_{m+T+1} > 0\).

Proposition 5. Suppose for some \(s > T\) that \(\tilde{\alpha}_{w(m+s)} - \xi_{m+s+1} > 0\) and \(\tilde{\alpha}_{w(m+s+1)} - \xi_{m+s+2} < 0\), then the hook-length at the node \((w(i), \alpha_{w(i)} + 1 - nl)\) is \(l(m\kappa + n)\) where \(m + s + 1 = ml + i\) and \(1 \leq i \leq m\).

Proof. By hypothesis \(\tilde{\alpha}_{w(ml+i)} > \tilde{\alpha}_{w(i)} - nl > \xi_{m+s+2} > \tilde{\alpha}_{w(ml+i)}\). This implies \(L(\alpha; w(i), \alpha_{w(i)} + 1 - nl) = ml - 1\).

The Proposition implies that if there is only one hook-length divisible by \(m\kappa + n\) in the rows \(w(i)\) for \(1 \leq i \leq m\) then \(\tilde{\alpha}_{w(m+i)} > \xi_{m+i+1}\) for all \(i \geq T\); so there is only one sign-change.

It appears that there can be a considerably larger number of solutions than the multiplicity. Here is an example: \(\alpha = (9, 7, 6, 5, 2)\), the multiplicity of \((2\kappa + 3)\) in \(h(\alpha, \kappa + 1)\) is 4. The relevant hook lengths are \(2\kappa + 3\) at nodes \((1, 7)\) and \((3, 4)\), and \(4\kappa + 6\) at nodes \((1, 4)\) and \((2, 2)\). The algorithm produces \((6, 7, 9, 5, 2)\), \((9, 7, 0, 2, 5, 3, 3)\), \((3, 7, 6, 5, 8)\), \((9, 1, 0, 5, 2, 6, 6)\) respectively for these nodes. But one can continue the process: for example the multiplicity of \((2\kappa + 3)\) in \(h((6, 7, 9, 5, 2), \kappa + 1)\) is 3, and the algorithm (applied to \((6, 7, 9, 5, 2)\)) produces three more solutions for \(\beta\), one being \((6, 7, 3, 5, 8)\). One could speculate that there is a lattice of solutions, ordered by \(\succ\). Finally, one can ask if there are \((-\frac{n}{m})\)-critical pairs without a corresponding hook-length \(m\kappa + n\). It seems doubtful, but we will leave this unanswered.

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