Solve differential equations via a hybrid method between homotopy analytical method and sine cosine optimization algorithm

Mostafa raed najeeb, Omar Saber Qasim and Ahmed Entesar

Department of Mathematics, College of Computer Sciences and Mathematics, University of Mosul, Iraq

E-mail: mostafa.csp115@student uomosul.edu.iq

Abstract. In this paper, ordinary differential equations (ODEs) of two types, linear and non-linear, will be solved by using the homotopy analytical method (HAM), and the sine cosine algorithm (SCA) will be used to modify the parameter h and obtain a better approximate solution than what was in the previous method HAM. The suggested method, HAM-SCA, covers a solution that reveals the reliability and the efficiency correspondingly to the default method HAM by computing the maximum absolute errors (MAE) and mean square error (MSE).

1. Introduction

All-natural, physical, engineering, and other phenomena are expressed by DEs (linear and nonlinear), these equations differential in the method of solution, a few of them can be solved directly, so it was necessary to use approximate methods to address problems and arrive at the solution, as the HAM is one of these methods [1].

HAM is the method that the world Liao Shijun discovered in 1992 [2, 3], he relied on using the homotopic method to obtain a convergent series solution (convergent series) for various types of equations whether linear or non-linear equations, DEs, integral equations, and other types of equations, the HAM method describes a kind of deformation or disparity in mathematical, for example, a circle can be deformed continuously to a square or an ellipse, as well as a cup of coffee, can be deformed continuously into a football [1, 4], in general, the HAM connects objects different has the same properties, one of what distinguishes this method is that it is free of any )small or large( concrete parameter, as well as provides appropriate methods to guarantee convergence in the series of solutions even if the non-linear boundary is complex, and it gives us the privilege in choosing the type of equation for sub-problems, the initial guess, and the basic function.

SCA is one of the swarm algorithms invented by the scientist Sidali Mergalieli, used to solve optimization problems and obtain the best solution, based on the principle of exploration and exploitation as it creates and searches many random solutions depending on the properties of the sine and cosine and switches between them according to a specific condition. [5-7].

We can represent the algorithm in the following form

\[ w_{n+1}^p = w_n^p + d_1 \sin(d_2)(d_3 p_n^p - w_n^p) \tag{1} \]

\[ w_{n+1}^p = w_n^p + d_1 \cos(d_2)(d_3 p_n^p - w_n^p) \tag{2} \]
As $w_n^v$ is the current subject of research, in $n$th dimension at $v$th iteration, and $d_1, d_2, d_3, d_4$ are random numbers, and $p_n$ is the location of the destination point in the $n$th dimension, and $\|\|$ is the absolute value [6]. We can combine the two previous equations as follows.

$$w_{n+1}^v = \begin{cases} 
  w_n^v + d_1 \sin(d_2)|d_3 p_n^v - w_n^v|, & d_4 < 0.5 \\
  w_n^v + d_1 \cos(d_2)|d_3 p_n^v - w_n^v|, & d_4 \geq 0.5 
\end{cases}$$  \hspace{1cm} (3)$$

Where $d_4$ is arbitrary within the closed interval [0,1].

The parameter $d_1$ specifies the next position (in which direction the move will be) by the following law:

$$d_1 = m - t \frac{m}{T}$$  \hspace{1cm} (4)$$

Where $t$ is the value of the current frequency, $T$ is the ultimate number of iterations, and $m$ is a mass constant, which can be either the distance between solutions, the destination or outwards and, the parameter $d2$ assigns how unlikely the movement must be pointed or outward, the $d3$ parameter specifies a random weight for a ($d3 > 1$) random check, or ($d3 < 1$) reduces the focus for the destination effect when specifying a distance.

SCA named after that name for its use of the sine and cosine functions completely in its general form, as it changes and modifies the range of the sine and cosine functions each time to determine a new search site, whether it is in the space or outside the space or if it is between it and another solution, then a value is chosen random $d2$ for a certain period, this mechanism clarifies the true meaning of the processes of exploration and exploitation, and this algorithm makes the correct selection and reconciliation between exploration and exploitation to reach the goal [8,9], which is through the convergence of solutions as shown in the Figure (1).

![Figure 1: Demonstrates the convergence of solutions in SCA](image-url)

2. General Concepts

In this part, will mention some important definitions in this paper

2.1 Definition

The maximum absolute errors MAE is determined by the following formula

$$\|z_{\text{Exact}}(y) - \alpha_i(y)\|_\infty = \max_{i = 1, 2, \ldots} \{z_{\text{Exact}}(y) - \alpha_i(y)\}$$  \hspace{1cm} (5)$$

Here $\alpha_i(y), i = 1, 2, \ldots$ are the successive estimates of the solution $z(y)$ [4].

2.2 Definition

Let's have the vector $x_i^m$ where $i = 1, 2, \ldots$, the MSE is a sum the square of the exact solution Exact $(y_i)$ minus the approximate solution divided by the number of iterations $k$ which is as follows [10]

$$MSE = \frac{1}{k} \sum_{i=1}^{k} (\text{Exact}(y_i) - \theta(y_i))^2$$  \hspace{1cm} (6)$$
2.3 Definition

The MAE is defined by the following formula

$$\frac{1}{N} \sum_{i=1}^{k} |Exact (y_i) - \Phi(y_i)|, \ i = 1,2, ...$$  \hspace{1cm} (7)

Which represents the absolute value of the sum of the product of subtracting the real value from the approximate value divided by the number of iterations \([11]\).

2.4 Adomian Polynomials

Will use the Adomian method to solve the non-linear terms from the ordinary DEs \([12]\). When \(f(u) = u^2\) \([13, 14]\).

Can use the Adomian method to extend the nonlinear term to the \(f(u) = u^2\) document using

\[
x_0 = u_0^2 \\
x_1 = 2u_0u_1 \\
x_2 = 2u_0u_2 + u_1^2 \\
x_3 = 2u_0u_3 + 2u_1u_2 \\
x_4 = 2u_0u_4 + 2u_1u_3 + u_2^2 \\
x_5 = 2u_0u_5 + 2u_1u_4 + 2u_2u_3, \\
\vdots
\]

\[x_n = \sum_{i=0}^{n} u_i u_{n-i}, \ n \geq i, \ n = 0,1,2, \ldots\]

When \(f(u) = uu'\) \([13,14]\)

\[
x_0 = u_0 u_0' \\
x_1 = u_1 u_0'' + u_0 u_1' \\
x_2 = u_2 u_0'' + u_1 u_1' + u_0 u_2' \\
x_3 = u_3 u_0'' + u_2 u_1' + u_1 u_2' + u_0 u_3' \\
x_4 = u_4 u_0'' + u_3 u_1' + u_2 u_2' + u_1 u_3' + u_0 u_4' \\
x_5 = u_5 u_0'' + u_4 u_1' + u_3 u_2' + u_2 u_3' + u_1 u_4' + u_0 u_5' \\
\vdots
\]

\[x_n = \sum_{i=0}^{n} u_i u_{n-i}, \ n \geq i, \ n = 0,1,2, \ldots\]

3. The Basic Ideas of HAM \([1,15,18]\)

Let us have the following linear Des \([17]\).

\[N[z(t)] = 0\]  \hspace{1cm} (8)

Where \(N\) is a non-linear operator, \(t\) represents the independent variable, and \(z\) is an undefined function, by driving the classic homotopy technique, we get the zero orders deformation equation, which is defined below \([18]\).

\[(1 - q) L [\beta(t, q) - z_0(t)] = q h H(t) N[\beta(t, q)]\]  \hspace{1cm} (9)

Where \(q \in [0,1]\) represents the embedding parameter \([19]\), which is called the homotopic parameter, \(L\) represents the auxiliary linear limit that fulfills the following property \(L(z) = 0\), \(\beta(t, q)\) also refers to the initial estimation of the exact solution, and \(h\) is called the convergence control parameter where \(h \neq 0\), and \(H(t)\) is an auxiliary function where \(H(t) \neq 0\), when have \(q = 0\) get

\[\beta(t, 0) = z_0(t)\]  \hspace{1cm} (10)

Which represents the exact solution, and if had \(q = 0\) then get

\[\beta(t, 0) = z(t)\]  \hspace{1cm} (11)

Which represents the real solution to Eq. 8, thus as the parameter of homotopy \(q\) increases how much zero to one, the solution changes continuously from the initial guess \(z_0(t)\) to the exact solution \(z(t)\). This continuous change in solution is called the deformation in homotopy, as it is using the Taylor series for the parameter \(q\).
\[ z_1(t) = \beta(t, q) = z_0(t) + \sum_{j=1}^{+\infty} z_0(t) q^j \]  
\[ \text{Whereas the} \]
\[ z_j(t) = \frac{1}{j!} \frac{\partial^j \beta(t, q)}{\partial q^j} \]  
If \( L, z_0(t), H(t) \) was correctly selected then the Taylor series would look like this
\[ z_1(t) = \beta(t, q) = z_0(t) + \sum_{j=1}^{+\infty} z_0(t) \]  
By deriving the zero-order deformation equation \( j \) times for the parameter \( q \), then substituting for \( q = 0 \) and dividing the equation by \( j! \) get the deformation equation of order \( j \) which is as follows:
\[ L[z_j(t) - \chi_j z_{j-1}(t)] = \hbar H(t) R_{j-1}(z^*_{j-1}(t)), \quad z_j(0) = 0 \]  
Where the vector is known as
\[ z_j^*(t) = \{ z_0(t), \quad z_1(t), \quad z_2(t), \ldots, \quad z_j(t) \} \]  
From Eq. 15 get
\[ R_{j-1}(z^*_{j-1}(t)) = \frac{1}{(j-1)!} \frac{\partial^{j-1} N[\beta(t, q)]}{\partial q^{j-1}}|_{q=0} \]  
\[ \chi_j = 0, \quad j \geq 1 \]
\[ 1, \quad j < 1 \]  

4. The Proposed Method SCA-HAM
The proposed method is based on calculating the optimal value of the HAM parameter \( h \) to figure out linear and non-linear ODEs by combining the HAM method with the SCA method, where the initial calculation is with a formula HAM.
\[ z_j(t) = \chi_{j-1} z_{j-1}(t) + h L^{-1} R_j(z^*_{j-1}(t)) \]  
Using SCA will optimize the parameter \( h \) value in the HAM method to get the best solution.

5. Some Applications of SCA-HAM
Will solve some examples of ordinary linear and nonlinear DEs using HAM and improve the results using SCA.

5.1 Example [16]
\[ y''' + 5y'' + 4y = 0 \]
And the exact solution
\[ z(t) = \frac{4}{3} \cos(t) + \frac{1}{3} \sin(t) - \frac{1}{3} \cos(2t) - \frac{1}{6} \sin(2t) \]
And pick out the initial estimation
\[ z_0(t) = 1 + \frac{1}{6} t^3 \]
Then the linear operator (which represents the general solution)
\[ L[\beta(t; q)] = \frac{\partial^4 \beta(t; q)}{\partial t^4} \]
Then have
\[ L[c_1 + c_2 s + c_3 s^2 + c_4 s^3] \]
The linear operator will look like this
\[ N[\beta(t; q)] = \frac{\partial^4 \beta(t; q)}{t^4} + 5 \frac{\partial^2 \beta(t; q)}{t^2} + 4 \beta(t; q) = 0 \]
And the initial condition
\[ z(0) = 1, z'(0) = 0, z''(0) = 0, \] and \( z(0) = 1 \)
Where
\[ R_j(z^*_{j-1}) = z'''_{j-1} + 5z''_{j-1} + 4z_{j-1} \]
Will work on finding the terms of the series using the following HAM law

\[ z_j(t) = \chi_{j-1} z_{j-1} + h L^{-1} R_j (z_{j-1}) \]

\[ z_1(t) = \frac{1}{6} h t^4 + \frac{1}{22} h t^5 + \frac{1}{1260} h t^7 \]

\[ z_2(t) = \frac{1}{6} h t^4 + \frac{1}{6} h^2 t^4 + \frac{1}{24} h t^5 + \frac{1}{36} h^2 t^6 + \frac{1}{1260} h t^7 + \frac{29}{5040} h^2 t^7 + \frac{1}{2520} h^2 t^8 + \frac{1}{9072} h^2 t^9 + \frac{1}{2494800} h t^{11} \]

\[ z_3(t) = \frac{1}{2043241200} h^3 t^{15} + \frac{1}{25945920} h^3 t^{13} + \frac{1}{7484400} h^3 t^{12} + \frac{1}{12474400} h^2 t^{11} + \frac{1}{997200} h^3 t^{11} + \frac{1}{22680} h^3 t^{10} + \frac{41}{72576} h^3 t^9 + \frac{1}{4536} h^2 t^8 + \frac{1}{1260} h^2 t^8 + \frac{1}{3360} h^3 t^7 + \frac{1}{2520} h^2 t^7 + \frac{3}{280} h^3 t^7 + \frac{1}{1260} h t^7 + \frac{1}{18} h^2 t^6 + \frac{1}{18} h^2 t^6 + \frac{1}{12} h t^5 + \frac{1}{24} h t^5 + \frac{1}{24} h t^5 + \frac{1}{2} h^2 t^4 + \frac{1}{6} h^3 t^4 \]

\[ z_4(t) = \frac{1}{4751761734200} h^4 t^{19} + \frac{1}{27788083200} h^4 t^{18} + \frac{1}{81729648000} h^4 t^{17} + \frac{1}{6810804000} h^3 t^{16} + \frac{504504000}{2492800} h^4 t^{15} + \frac{1}{90810720} h^4 t^{14} + \frac{87}{648640} h^3 t^{13} + \frac{143}{38918880} h^4 t^{13} + \frac{29}{249280} h^4 t^{12} + \frac{41}{3324400} h^4 t^{12} + \frac{83}{324400} h^3 t^{11} + \frac{1}{3628800} h^4 t^{11} + \frac{1}{831600} h^3 t^{11} + \frac{1}{7560} h^4 t^{10} + \frac{1}{840} h^4 t^{10} + \frac{1}{294192} h^3 t^9 + \frac{1}{3024} h^2 t^8 + \frac{11}{8064} h^3 t^8 + \frac{29}{3360} h^4 t^8 + \frac{1}{1120} h^3 t^8 + \frac{1}{7560} h^4 t^8 + \frac{1}{280} h^3 t^7 + \frac{29}{1680} h^2 t^7 + \frac{1}{3260} h^2 t^7 + \frac{79}{5040} h^4 t^7 + \frac{1}{12} h^2 t^6 + \frac{1}{12} h^4 t^6 + \frac{1}{8} h^2 t^5 + \frac{1}{24} h^4 t^5 + \frac{1}{8} h^3 t^5 + \frac{1}{8} h^3 t^5 + \frac{1}{6} h^2 t^4 + \frac{1}{6} h^3 t^4 + \frac{1}{6} h^3 t^4 + \]

So, substitute the value of \( h = -1 \) get

\[ z_1(t) = -\frac{1}{6} t^4 - \frac{1}{24} t^5 - \frac{1}{1260} t^7 \]

\[ z_2(t) = \frac{1}{36} + \frac{1}{1008} t^7 + \frac{1}{2520} t^8 + \frac{1}{9072} t^9 + \frac{1}{7484400} t^{11} \]

\[ z_3(t) = \frac{1}{2043241200} t^{15} - \frac{1}{25945920} t^{13} - \frac{1}{7484400} t^{12} - \frac{1}{133056} t^{11} - \frac{1}{22680} t^{10} - \frac{25}{72576} t^9 - \frac{2016}{5} t^8 \]

\[ z_4(t) = \frac{1}{7451761734200} t^{19} + \frac{1}{27788083200} t^{17} + \frac{1}{81729648000} t^{16} + \frac{1}{544864320} t^{15} + \frac{1}{90810720} t^{14} + \frac{5}{15567552} t^{13} + \frac{1}{399168} t^{12} + \frac{1}{1596672} t^{11} + \frac{1}{36288} t^{10} \]

Hence, the solution of the above series is shown in the following form

\[ z(t) = z_0(t) + z_1(t) + z_2(t) + z_3(t) + z_4(t) + \ldots \]

\[ z(t) \approx 1 + \frac{1}{6} t^3 - \frac{1}{6} t^4 - \frac{1}{1260} t^7 + \frac{1}{1008} t^6 + \frac{1}{36} t^7 + \frac{1}{2520} t^8 + \frac{1}{9072} t^9 + \ldots \]

Now and after using the SCA method to improve the parameter \( h \), got better results by substituting for the value of \( h^{\text{new}} = -0.9776 \). The series solution of HAM-SCA can be written as a form

\[ z^{\text{new}}(t) = z_0^{\text{new}}(t) + z_1^{\text{new}}(t) + z_2^{\text{new}}(t) + z_3^{\text{new}}(t) + z_4^{\text{new}}(t) + \ldots \]

\[ z^{\text{new}}(t) \approx 0.4572607681 \times 10^{-5} t^{15} - 0.3609028549 \times 10^{-7} t^{13} - 0.1248321897 \times 10^{-6} t^{12} - 0.6621571294 \times 10^{-5} t^{11} - 0.0004119462260 t^{10} - 0.000217671603 t^9 - 0.001920860538 t^8 + 0.004159320415 t^7 + 0.0273658885 t^6 - 0.041666919835 t^5 - 0.166667493 t^4 + 0.1666667 t^3 + 1 + \ldots \]

The comparison of results between HAM and HAM-SCA is shown in Table 1 and Figure 2.

**Table 1:** Comparison of MAE and MSE for HAM and HAM-SCA for Example 1.

| Error Criteria | HAM | HAM-SCA |
|----------------|-----|---------|
| MSE            | 2.334360 E − 9 | 9.033345 E − 12 |
| MAE            | 2.035680 E − 5 | 1.555098 E − 6 |
Figure 2: Illustrates the matching process between HAM, HAM-SCA, and exact solution.

5.2 Example [16]

\[ y'' + y'^2 = 0 \]

And the exact solution

\[ z(t) = 1 + \ln(1 + 2t) \]

And pick out the initial estimation

\[ z_0(t) = 1 + 2s \]

And the initial condition

\[ z(0) = 1, \quad z(0) = 2 \]

The linear operator (which represents the general solution)

\[ L[\beta(t; q)] = \frac{\delta^2 \beta(t; q)}{\delta t^2} \]

Then have \[ L[c_1 + c_2 s] = 0 \]

Using Adomian polynomials expand the nonlinear \(y'^2\) a term which will be in the following form

\[ \sum_{i=0}^{j-1} z_i'(t) \cdot z_{j-1-i}'(t) \]

The linear operator will look like this

\[ N[\beta(t; q)] = \frac{\delta^2 \beta(t; q)}{s^2} + \frac{\delta \beta^2(t; q)}{s^2} \]

And

\[ R_j(z_{j-1}^{-}) = z_j''''(t) + \sum_{i=0}^{j-1} z_i'(t) \cdot z_{j-1-i}'(t) \]

Will work on finding the terms of the series using the following (HAM) law

\[
\begin{align*}
    z_j(t) &= \chi_{j-1} z_{j-1} + h L^{-1} R_j(z_{j-1}^-) \\
    z_1(t) &= 2ht^2 \\
    z_2(t) &= 2ht^2 + 2h^2t^2 + \frac{8}{3} h^2 t^3 \\
    z_3(t) &= 4h^3 t^4 + \frac{16}{3} h^2 t^3 + \frac{16}{3} h^3 t^3 + 2ht^2 + 4h^2 t^2 + 2h^3 t^2 \\
    z_4(t) &= \frac{32}{5} h^4 t^5 + 12h^3 t^4 + 12h^4 t^4 + 8h^2 t^3 + 8h^4 t^3 + 16h^3 t^3 + 2ht^2 + 6h^2 t^2 + 2h^4 t^2 \\
    &\vdots
\end{align*}
\]

So, substitute in the value of \(h = -1\) get

\[ z_1(t) = -2t^2 \]
\[ z_2(t) = \frac{8}{3} t^3 \]
\[ z_3(t) = -4t^4 \]
\[ z_4(t) = \frac{32}{5} t^5 \]

Then the series solution of HAM with \( h = -1 \) can be written as a form
\[ z(t) = z_0(t) + z_1(t) + z_2(t) + z_3(t) + z_4(t) + \ldots \]
\[ z(t) \cong 1 + 2t - 2t^2 + \frac{8}{3} t^3 - 4t^4 + \frac{32}{5} t^5 + \ldots \]

Now and after using the SCA method to improve the parameter \( h \), got better results by substituting for the value of \( h_{\text{new}} = -0.4888 \), The series solution of HAM-SCA can be written as a form
\[ z_{\text{new}}(t) \equiv z_{0,\text{new}}(t) + z_{1,\text{new}}(t) + z_{2,\text{new}}(t) + z_{3,\text{new}}(t) + z_{4,\text{new}}(t) + \ldots \]
\[ z_{\text{new}}(t) \equiv 0.365346341626839 t^5 - 1.18356369060168 t^4 + 1.78803919216422 t^3 - 1.86341802881761 t^2 + 2s + 1 + \ldots \]

The comparison of results between HAM and HAM-SCA is shown in Table 2 and Figure 3

| Error Criteria | HAM       | HAM-SCA  |
|---------------|-----------|----------|
| MSE           | 2.056038  | 9.204092 \( E - 6 \) |
| MAE           | 0.758987  | 2.33752 \( E - 3 \) |

Table 2: Comparison of MAE and MSE for HAM and HAM-SCA for Example 2.

Figure 3: Illustrates the matching process between HAM, HAM-SCA, and exact solution.

5.3 Example [16]
\[ y'''' + yy'' - y'^2 = 0 \]

And the exact solution
\[ z(t) = e^t - 1 \]

And pick out the initial estimation
\[ z_0(t) = t + \frac{1}{2} t^2 + \frac{1}{6} t^3 \]

And the initial condition
\[ z(0) = 0 , \quad z'(0) = 1 , \quad z''(0) = 1 , \quad z'''(0) = 1 \]
the linear operator (Which represents the general solution)

\[ L[\beta(t; q)] = \frac{\partial^2 \beta(t; q)}{\partial s^4} \]

Then have \( L[c_1 + c_2 s + c_3 s^2 + c_4 s^3] \)

Using Adomian polynomials expand the nonlinear \( y y'' - y'^2 \) a term which will be in the following form

\[ \sum_{i=0}^{j-1} z_i(t) z_{j-1-i}(t) - \sum_{i=0}^{j-1} z'_i(t) z'_{j-1-i}(t) \]

The linear operator will look like this

\[ N[\beta(t; q)] = \frac{\partial^4 \beta(t; q)}{t^4} + \frac{\partial \beta^2(t; q)}{t^2} \beta(t; q) - \frac{\partial^2 \beta(t; q)}{t^2} \]

And \( R_j(z^*_{j-1}) = z''_{j-1}(t) + \sum_{i=0}^{j-1} z_i(t) z'_{j-1-i}(t) - \sum_{i=0}^{j-1} z'_i(t) z'_{j-1-i}(t) \)

Will work on finding the terms of the series using the following (HAM) law

\[ z_j(t) = \chi_{j-1} z_{j-1} + h L^{-1} R_j(z^*_{j-1}) \]

\[ z_1(t) = -\frac{1}{24} h t^4 - \frac{1}{120} h t^5 - \frac{1}{270} h t^6 - \frac{1}{2520} h t^7 - \frac{1}{20160} h t^8 \]

\[ z_2(t) = h \left(-\frac{1}{24} h t^4 - \frac{1}{120} h t^5 - \frac{1}{270} h t^6 - \frac{1}{2520} h t^7 - \frac{1}{20160} h t^8 \right) + \left(\frac{1482624000}{11404800} h t^{13} - \frac{1}{120} h t^5 - \frac{1}{24} h t^6 - \frac{1}{12} h t^7 + \frac{1}{2520} h t^8 \right) \]

\[ z_3(t) = -\left(\frac{1}{20160} h t^8 - \frac{1}{2520} h t^9 - \frac{1}{12} h t^7 - \frac{1}{120} h t^8 - \frac{1}{270} h t^9 - \frac{1}{24} h t^10 - \frac{1}{720} h t^{11} - \frac{1}{270} h t^{12} - \frac{1}{120} h t^{13} \right) \]

So, substitute in the value of \( h = -1 \) get

\[ z_1(t) = \frac{1}{24} t^4 + \frac{1}{120} t^5 + \frac{1}{270} t^6 + \frac{1}{2520} t^7 + \frac{1}{20160} t^8 \]

\[ z_2(t) = \frac{1}{148262400} t^{13} + \frac{1}{11404800} t^{12} + \frac{1}{19958400} t^{11} + \frac{1}{453600} t^{10} + \frac{1}{362880} t^9 + \frac{1}{24} t^8 + \frac{1}{120} t^7 + \frac{1}{2520} t^6 + \frac{1}{20160} t^5 \]

\[ z_3(t) = \frac{1}{5353114214400} t^{14} + \frac{1}{2964061900800} t^{13} + \frac{1}{10674982800} t^{12} + \frac{1}{32691859200} t^{11} + \frac{1}{40320} t^{10} + \frac{53}{1981324800} t^{9} + \frac{1}{147840} t^8 + \frac{1}{181440} t^7 + \frac{1}{1260} t^6 + \frac{1}{24} t^5 \]

Then the series solution can be written as a form

\[ z(t) = z_0(t) + z_1(t) + z_2(t) + z_3(t) + z_4(t) + \ldots \]

\[ z(t) \approx \frac{1}{24} t^4 + \frac{1}{120} t^5 + \frac{1}{270} t^6 + \frac{1}{2520} t^7 + \frac{1}{20160} t^8 + \ldots \]

When using SCA to improve the parameter \( h \), the best result was \( h = -1 \), which is the same in HAM.
The comparison of results between HAM and HAM-SCA is shown in Table 3 and Figure 4.

**Table 3:** Comparison of MAE and MSE for HAM and HAM-SCA for Example 3

| Error Criteria | HAM       | HAM-SCA   |
|----------------|-----------|-----------|
| MSE            | 2.056038  | 9.204092  $E - 6$ |
| MAE            | 0.758987  | 2.33752  $E - 3$ |

Figure 4: Illustrates the matching process between HAM, HAM-SCA, and exact solution.

**Conclusion**

In this work, a hybrid method SCA-HAM is implemented between the HAM and the SCA. A series of approximate solutions in the HAM has been applied as a fitness function in the SCA to find the optimal h parameter. The outcome of the SCA-HAM method (which contains the optimal parameter (h)) was compared with the HAM through three examples illustrated in Tables (1-3) and Figure (2-4), where the SCA-HAM shows a clear superiority over HAM in discovering approximate solutions through calculating the MSE and MAE.

**References**

[1] A. Sami Bataineh, M. S. M. Noorani, and I. Hashim 2008 Solving systems of ODEs by homotopy analysis method *Communications in Nonlinear Science and Numerical Simulation* **13** 2060-2070

[2] J. Ahmad and S. Rubab 2017 Efficient Homotopy Analysis Method to System of Delay Differential Equations *Annals of the Faculty of Engineering Hunedoara* **15** 133

[3] S.-J. Liao 1992 The proposed homotopy analysis technique for the solution of nonlinear problems Ph. D. Thesis, Shanghai Jiao Tong University Shanghai

[4] L. Shijun 2013 Advances in the Homotopy Analysis Method *World Scientific*

[5] H. Nenavath and R. K. J. Jatoth 2018 Hybridizing sine cosine algorithm with differential evolution for global optimization and object tracking *Applied Soft Computing* **62** 1019-1043

[6] S. Mirjalili 2016 SCA: A Sine Cosine Algorithm for solving optimization problems,” *Knowledge-Based Systems* **96** 120-133

[7] S. Ekiz, P. Erdoğan, and B. Özgür 2017 Solving constrained optimization problems with sine-cosine algorithm *Periodicals of Engineering and Natural Sciences (PEN)* **5**
[8] O. S. Qasim, and Z. Y. J. I. J. o. M 2020 Algamal, Engineering, and M. Sciences, Feature selection using different transfer functions for binary bat algorithm 5 697-706
[9] O. S. Qasim, N. A. Al-Thanoon, Z. Y. J. C. Algamal, and I. L. Systems 2020 Feature selection based on chaotic binary black hole algorithm for data classification 204 104104
[10] Wang, F., Yuan, X., Liew, S. C., & Guo, D. 2013 Wireless MIMO switching: Weighted sum mean square error and sum rate optimization. IEEE transactions on information theory 59(9) 5297-5312
[11] Chai, T., & Draxler, R. R. 2014 Root mean square error (RMSE) or mean absolute error (MAE)?--Arguments against avoiding RMSE in the literature. Geoscientific model development 7(3), 1247-1250.
[12] E. Babolian and J. Biazar 2002 On the order of convergence of Adomian method,” Applied Mathematics and Computation 130 383-387
[13] Wazwaz, A. M. 2000 A new algorithm for calculating Adomian polynomials for nonlinear operators. Applied Mathematics and computation, 111(1) 33-51
[14] Al-Hayani, W., Alzubaidy, L., and Entesar, A. 2017 Solutions of singular IVP’s of Lane–Emden type by homotopy analysis method with genetic algorithm. Appl. Math. Inf. Sci 11(2) 407-416
[15] Entesar, A., Saber, O., and Al-Hayani, W. 2018 Hybridization of Genetic Algorithm with Homotopy Analysis Method for Solving Fractional Partial Differential Equations Eurasian Journal of Science and Engineering 4
[16] A. S. Bataineh, M. S. M. Noorani, and I. Hashim 2009 Homotopy analysis method for singular IVPs of Emden–Fowler type Communications in Nonlinear Science and Numerical Simulation 14 1121-1131
[17] Liao, S. 2003 Beyond perturbation: introduction to the homotopy analysis method. CRC press
[18] He, J. H. 2003 Homotopy perturbation method: a new nonlinear analytical technique Applied Mathematics and computation 135(1) 73-79
[19] Liao, S. 2012 Homotopy analysis method in nonlinear differential equations, 153-165 Beijing: Higher education press.