CONCURRENT $\pi$-VECTOR FIELDS AND ENERGY $\beta$-CHANGE

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Abstract. The present paper deals with an intrinsic investigation of the notion of a concurrent $\pi$-vector field on the pullback bundle of a Finsler manifold $(M, L)$. The effect of the existence of a concurrent $\pi$-vector field on some important special Finsler spaces is studied. An intrinsic investigation of a particular $\beta$-change, namely the energy $\beta$-change $(\widetilde{L}^2(x, y) = L^2(x, y) + B^2(x, y)$ with $B := g(\zeta, \eta)$; $\zeta$ being a concurrent $\pi$-vector field), is established. The relation between the two Barthel connections $\Gamma$ and $\tilde{\Gamma}$, corresponding to this change, is found. This relation, together with the fact that the Cartan and the Barthel connections have the same horizontal and vertical projectors, enable us to study the energy $\beta$-change of the fundamental linear connection in Finsler geometry: the Cartan connection, the Berwald connection, the Chern connection and the Hashiguchi connection. Moreover, the change of their curvature tensors is concluded.

It should be pointed out that the present work is formulated in a prospective modern coordinate-free form.

Keywords: Special Finsler space, Pullback bundle, Energy $\beta$-change, Concurrent $\pi$-vector field, Canonical spray, Barthel connection, Cartan connection, Berwald connection, Chern connection, Hashiguchi connection.

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Introduction

An important aim of Finsler geometry is the construction of a natural geometric framework of variational calculus and the creation of geometric models that are appropriate for dealing with different physical theories, such as general relativity, relativistic optics, particle physics and others. As opposed to Riemannian geometry, the extra degrees of freedom offered by Finsler geometry, due to the dependence of its geometric objects on the directional arguments, make this geometry potentially more suitable for dealing with such physical theories at a deeper level.

Studying Finsler geometry, however, one encounters substantial difficulties trying to seek analogues of classical global, or sometimes even local, results of Riemannian geometry. These difficulties arise mainly from the fact that in Finsler geometry all geometric objects depend not only on positional coordinates, as in Riemannian geometry, but also on directional arguments.

In Riemannian geometry, there is a canonical linear connection on the manifold $M$, whereas in Finsler geometry there is a corresponding canonical linear connection due to E. Cartan. However, this is not a connection on $M$ but is a connection on $T(TM)$, the tangent bundle of $TM$, or on $\pi^{-1}(TM)$, the pullback of the tangent bundle $TM$ by $\pi : TM \rightarrow M$.

The concept of a concurrent vector field in Riemannian geometry had been introduced and studied by K. Yano [14]. On the other hand, the notion of a concurrent vector field in Finsler geometry had been studied locally by S. Tachibana [12], M. Matsumoto and K. Eguchi [8] and others.

In this paper, we study intrinsically the notion of a concurrent $\pi$-vector field on the pullback bundle $\pi^{-1}(TM)$ of a Finsler manifold $(M, L)$. Some properties of concurrent $\pi$-vector fields are discussed. These properties, in turn, play a key role in obtaining other interesting results. The effect of the existence of a concurrent $\pi$-vector field on some important special Finsler spaces is investigated.

The infinitesimal transformations (changes) in Finsler geometry are important, not only in differential geometry, but also in application to other branches of science, especially in the process of geometrization of physical theories [9]. For this reason, we investigate intrinsically a particular $\beta$-change, which will be referred to as an energy $\beta$-change:

$$\tilde{L}^2(x, y) = L^2(x, y) + B^2(x, y),$$

where $(M, L)$ is a Finsler manifold admitting a concurrent $\pi$-vector field $\zeta$ and $B := g(\zeta, \eta)$, $\eta$ being the fundamental $\pi$-vector field. Moreover, the relation between the two Barthel connections $\Gamma$ and $\tilde{\Gamma}$, corresponding to this change, is obtained. This relation, together with the fact that the Cartan and the Barthel connections have the same horizontal and vertical projectors, enable us to study the energy $\beta$-change of the fundamental linear connections on the pullback bundle of a Finsler manifold, namely, the Cartan connection, the Berwald connection, the Chern connection and the Hashiguchi connection. Moreover, the change of their curvature tensors is concluded.

Finally, it should be pointed out that a global formulation of different aspects of Finsler geometry may give more insight into the infrastructure of physical theories and helps better understand the essence of such theories without being trapped into the complications of indices. This is one of the motivations of the present work, where all results obtained are formulated in a prospective modern coordinate-free form.
1. Notation and Preliminaries

In this section, we give a brief account of the basic concepts of the pullback approach to intrinsic Finsler geometry necessary for this work. For more detail, we refer to [1], [3] and [13]. We assume, unless otherwise stated, that all geometric objects treated are of class $C^\infty$. The following notation will be used throughout this paper:

- $M$: a real paracompact differentiable manifold of finite dimension $n$ and of class $C^\infty$.
- $\mathfrak{F}(M)$: the $\mathbb{R}$-algebra of differentiable functions on $M$.
- $\mathfrak{X}(M)$: the $\mathfrak{F}(M)$-module of vector fields on $M$.
- $\pi_M: TM \to M$: the tangent bundle of $M$.
- $\pi^*_M: T^*M \to M$: the cotangent bundle of $M$.
- $\pi: \mathcal{T}M \to M$: the subbundle of nonzero vectors tangent to $M$.

$V(TM)$: the pullback of the cotangent bundle $T^*M$ by $\pi$.
$P: \pi^{-1}(TM) \to \mathcal{T}M$: the pullback of the tangent bundle $TM$ by $\pi$.
$P^*: \pi^{-1}(T^*M) \to T^*M$: the pullback of the cotangent bundle $T^*M$ by $\pi$.
$\mathfrak{X}(\pi(M))$: the $\mathfrak{F}(TM)$-module of differentiable sections of $\pi^{-1}(TM)$.
$\mathfrak{X}^*(\pi(M))$: the $\mathfrak{F}(TM)$-module of differentiable sections of $\pi^{-1}(T^*M)$.

Elements of $\mathfrak{X}(\pi(M))$ will be called $\pi$-vector fields and will be denoted by barred letters $\overline{X}$. Tensor fields on $\pi^{-1}(TM)$ will be called $\pi$-tensor fields. The fundamental $\pi$-vector field is the $\pi$-vector field $\overline{\pi}$ defined by $\overline{\pi}(u) = (u, u)$ for all $u \in \mathcal{T}M$.

We have the following short exact sequence of vector bundles, relating the tangent bundle $T(TM)$ and the pullback bundle $\pi^{-1}(TM)$:

$$0 \to \pi^{-1}(TM) \xrightarrow{\gamma} T(TM) \xrightarrow{\rho} \pi^{-1}(TM) \to 0,$$

where the bundle morphisms $\rho$ and $\gamma$ are defined respectively by $\rho := (\pi_\tau_M, d\pi)$ and $\gamma(u, v) := j_u(v)$, where $j_u$ is the natural isomorphism $j_u: T_{\pi_M(v)}M \to T_u(T_{\pi_M(v)}M)$. The vector 1-form $J$ on $TM$ defined by $J := \gamma \circ \rho$ is called the natural almost tangent structure of $TM$. The vertical vector field $\mathcal{C}$ on $TM$ defined by $\mathcal{C} := \gamma \circ \overline{\pi}$ is called the canonical or Liouville vector field.

Let $D$ be a linear connection (or simply a connection) on the pullback bundle $\pi^{-1}(TM)$. We associate with $D$ the map

$$K: TTM \to \pi^{-1}(TM): X \mapsto D_X\overline{\pi},$$

called the connection (or the deflection) map of $D$. A tangent vector $X \in T_u(TM)$ is said to be horizontal if $K(X) = 0$. The vector space $H_u(TM) = \{X \in T_u(TM): K(X) = 0\}$ of the horizontal vectors at $u \in TM$ is called the horizontal space to $M$ at $u$. The connection $D$ is said to be regular if

$$T_u(TM) = V_u(TM) \oplus H_u(TM), \quad \forall u \in TM. \quad \text{(1.1)}$$

If $M$ is endowed with a regular connection, then the vector bundle maps

$$\gamma: \pi^{-1}(TM) \to V(TM),$$
$$\rho|_{H(TM)}: H(TM) \to \pi^{-1}(TM),$$
$$K|_{V(TM)}: V(TM) \to \pi^{-1}(TM)$$
Lemma 1.1. \[17\] Let \( D \) be a regular connection on \( \pi^{-1}(TM) \) whose \((h)hv\)-torsion tensor \( T \) has the property that \( T(X, Y) = 0 \), then, we have:
\[
(a) \quad [\beta X, \beta Y] = \gamma \hat{R}(X, Y) + \beta(D_{\beta X}Y - D_{\beta Y}X - Q(X, Y)),
\]
\[
(b) \quad [\gamma X, \beta Y] = -\gamma \hat{P}(X, Y) + D_{\beta Y}X + \beta(D_{\gamma X}Y - T(X, Y)),
\]
\[
(c) \quad [\gamma X, \gamma Y] = \gamma(D_{\gamma X}Y - D_{\gamma Y}X + \hat{S}(X, Y)).
\]

The following theorem guarantees the existence and uniqueness of the Cartan connection on the pullback bundle.

Theorem 1.2. \[17\] Let \((M, L)\) be a Finsler manifold and \( g \) the Finsler metric defined by \( L \). There exists a unique regular connection \( \nabla \) on \( \pi^{-1}(TM) \) such that
\[
(a) \quad \nabla \text{ is metric}: \nabla g = 0,
\]
The (h)h-torsion of $\nabla$ vanishes: $Q = 0$.

The (h)hv-torsion $T$ of $\nabla$ satisfies: $g(T(X, \overline{Y}), \overline{Z}) = g(T(X, \overline{Z}), \overline{Y})$.

One can show that the (h)hv-torsion of the Cartan connection is symmetric and has the property that $T(X, \pi) = 0$ for all $X \in \mathfrak{X}(\pi(M))$.

**Definition 1.3.** Let $(M, L)$ be a Finsler manifold and $g$ the Finsler metric defined by $L$. We define:

- $\ell(X) := L^{-1}g(X, \pi)$,
- $h := g - \ell \otimes \ell$ : the angular metric tensor,
- $T(X, \overline{Y}, \overline{Z}) := g(T(X, \overline{Y}), \overline{Z})$ : the Cartan tensor,
- $C(X) := \text{Tr}\{Y \mapsto T(X, Y)\}$ : the contracted torsion,
- $g(C, \overline{X}) := C(X) : C$ is the $\pi$-vector field associated with the $\pi$-form $C$,
- $\text{Ric}^v(X, \overline{Y}) := \text{Tr}\{Z \mapsto S(X, \overline{Z})Y\}$ : the vertical Ricci tensor,
- $g(\text{Ric}^v_0(X), \overline{Y}) := \text{Ric}^v(X, \overline{Y})$ : the vertical Ricci map $\text{Ric}^v_0$,
- $\text{Sc}^v := \text{Tr}\{X \mapsto \text{Ric}^v_0(X)\}$ : the vertical scalar curvature.

Deicke theorem [2] can be formulated globally as follows:

**Lemma 1.4.** Let $(M, L)$ be a Finsler manifold. The following assertions are equivalent:

(a) $(M, L)$ is Riemannian,

(b) The (h)hv-torsion tensor $T$ vanishes,

(c) The $\pi$-form $C$ vanishes.

Concerning the Berwald connection on the pullback bundle, we have

**Theorem 1.5.** [17] Let $(M, L)$ be a Finsler manifold. There exists a unique regular connection $D^\circ$ on $\pi^{-1}(TM)$ such that

(a) $D^\circ_{\rho X}L = 0$,

(b) $D^\circ$ is torsion-free: $T^\circ = 0$,

(c) The (v)hv-torsion tensor $\hat{P}^\circ$ of $D^\circ$ vanishes: $\hat{P}^\circ(X, \overline{Y}) = 0$.

Such a connection is called the Berwald connection associated with the Finsler manifold $(M, L)$.

**Theorem 1.6.** [17] Let $(M, L)$ be a Finsler manifold. The Berwald connection $D^\circ$ is expressed in terms of the Cartan connection $\nabla$ as

$$D^\circ_X Y = \nabla_X Y + \hat{P}(\rho X, \overline{Y}) - T(KX, \overline{Y}), \quad \forall X \in \mathfrak{X}(TM), \overline{Y} \in \mathfrak{X}(\pi(M)).$$

In particular, we have:

(a) $D^\circ_{\pi X} Y = \nabla_{\pi X} Y - T(X, \overline{Y})$,

(b) $D^\circ_{\beta X} Y = \nabla_{\beta X} Y + \hat{P}(\overline{X}, \overline{Y})$. 


We terminate this section by some concepts and results concerning the Klein-Grifone approach to intrinsic Finsler geometry. For more details, we refer to [5], [6] and [7].

A semispray is a vector field $X$ on $TM$, $C^\infty$ on $TM$, $C^1$ on $TM$, such that $\rho \circ X = \eta$. A semispray $X$ which is homogeneous of degree 2 in the directional argument ($[C, X] = X$) is called a spray.

**Proposition 1.7.** [7] Let $(M, L)$ be a Finsler manifold. The vector field $G$ on $TM$ defined by $i_G \Omega = -dE$ is a spray, where $E := \frac{1}{2}L^2$ is the energy function and $\Omega := dd_J E$. Such a spray is called the canonical spray.

A nonlinear connection on $M$ is a vector 1-form $\Gamma$ on $TM$, $C^\infty$ on $TM$, $C^0$ on $TM$, such that $J\Gamma = J$, $\Gamma J = -J$. The horizontal and vertical projectors $h_\Gamma$ and $v_\Gamma$ associated with $\Gamma$ are defined by $h_\Gamma := \frac{1}{2}(I + \Gamma)$ and $v_\Gamma := \frac{1}{2}(I - \Gamma)$. To each nonlinear connection $\Gamma$ there is associated a semispray $S$ defined by $S = h_\Gamma S'$, where $S'$ is an arbitrary semispray. A nonlinear connection $\Gamma$ is homogeneous if $[C, \Gamma] = 0$. The torsion of a nonlinear connection $\Gamma$ is the vector 2-form $t$ on $TM$ defined by $t := \frac{1}{2}[J, \Gamma]$. The curvature of $\Gamma$ is the vector 2-form $R$ on $TM$ defined by $R := -\frac{1}{2}[h_\Gamma, h_\Gamma]$. A nonlinear connection $\Gamma$ is said to be conservative if $d_{h_\Gamma} E = 0$.

**Theorem 1.8.** [6] On a Finsler manifold $(M, L)$, there exists a unique conservative homogeneous nonlinear connection with zero torsion. It is given by:

$$\Gamma = [J, G],$$

where $G$ is the canonical spray.

Such a nonlinear connection is called the canonical connection, the Barthel connection or the Cartan nonlinear connection associated with $(M, L)$.

It should be noted that the semispray associated with the Barthel connection is a spray, which is the canonical spray.

**Proposition 1.9.** [16] Under a change $L \rightarrow \tilde{L}$ of Finsler structures on $M$, the corresponding Barthel connections $\Gamma$ and $\tilde{\Gamma}$ are related by

$$\tilde{\Gamma} = \Gamma - 2L, \quad \text{with } L := \gamma o N \rho.$$  

(1.5)

Moreover, we have $\tilde{h} = h - L$, $\tilde{v} = v + L$. (the definition of $N$ is found in [16]).

## 2. Finsler spaces admitting concurrent $\pi$-vector fields

The notion of a concurrent vector field has been introduced and investigated in Riemannian geometry by K. Yano [14]. Concurrent vector fields have been studied in Finsler geometry by Matsumoto and Eguchi [8], Tachibana [12] and others. These studies were accomplished by the use of local coordinates. In this section, we introduce and investigate intrinsically the notion of a concurrent $\pi$-vector field in Finsler geometry. The properties of concurrent $\pi$-vector fields are obtained.

In what follows $\nabla$ will denote the Cartan connection associated with a Finsler manifold $(M, L)$ and $S$, $P$ and $R$ will denote the three curvature tensors of $\nabla$. 
Definition 2.1. Let \((M, L)\) be a Finsler manifold. A \(\pi\)-vector field \(\zeta \in \mathfrak{X}(\pi(M))\) is called a concurrent \(\pi\)-vector field if it satisfies the following conditions
\[
\nabla_\beta \zeta = -X, \quad \nabla_\gamma \zeta = 0.
\]
In other words, \(\zeta\) is a concurrent \(\pi\)-vector field if \(\nabla_X \zeta = -\rho X\) for all \(X \in \mathfrak{X}(TM)\), or briefly, \(\nabla \zeta = -\rho\).

The following two Lemmas are useful for subsequence use.

Lemma 2.2. Let \((M, L)\) be a Finsler manifold. If \(\zeta \in \mathfrak{X}(\pi(M))\) is a concurrent \(\pi\)-vector field and \(\alpha \in \mathfrak{X}^*(\pi(M))\) is the \(\pi\)-form associated with \(\zeta\) under the duality defined by the metric \(g\):

\[\alpha = i_\zeta g,\]
then the \(\pi\)-form \(\alpha\) has the properties

(a) \((\nabla_\beta X) \alpha(Y) = -g(X, Y)\),
(b) \((\nabla_\gamma X) \alpha(Y) = 0\).

Equivalently, \(\nabla_X \alpha = -i_\rho X g\).

Proof. As \(\nabla_X g = 0\), we get
\[
(\nabla_X \alpha)(Y) = \nabla_X g(\zeta, Y) - g(\zeta, \nabla_X Y) = (\nabla_X g)(\zeta, Y) + g(\nabla_X \zeta, Y) = -g(\rho X, Y). \quad \square
\]

Lemma 2.3. For every \(X, Y \in \mathfrak{X}(TM)\) and \(Z, W \in \mathfrak{X}(\pi(M))\), we have
\[
g(K(X, Y)Z, W) = -g(K(X, Y)W, Z),
\]
where \(K\) is the (classical) curvature of the Cartan connection.

Proof. Follows from Lemma 2.4 of [20] since \(\nabla g = 0\).

Now, we have the following

Proposition 2.4. Let \(\zeta \in \mathfrak{X}(\pi(M))\) be a concurrent \(\pi\)-vector field on \((M, L)\).

For the \(v\)-curvature tensor \(S\), the following relations hold:

(a) \(S(\overline{X}, \overline{Y}) \zeta = 0\), \(S(\overline{X}, \overline{Y}, \overline{Z}, \zeta) = 0\).
(b) \((\nabla_\gamma Z) S(\overline{X}, \overline{Y}, \zeta) = 0\), \((\nabla_\beta Z) S(\overline{X}, \overline{Y}, \zeta) = S(\overline{X}, \overline{Y}) Z\).
(c) \((\nabla_\zeta S)(\overline{X}, \overline{Y}, \zeta) = 0\).

For the \(hv\)-curvature tensor \(P\), the following relations hold:

(d) \(P(\overline{X}, \overline{Y}) \zeta = -T(\overline{Y}, \overline{X})\), \(P(\overline{X}, \overline{Y}, \overline{Z}, \zeta) = T(\overline{X}, \overline{Y}, \overline{Z})\).
(e) \((\nabla_\gamma Z) P(\overline{X}, \overline{Y}, \zeta) = -(\nabla_\gamma Z) T(\overline{Y}, \overline{X})\), \((\nabla_\beta Z) P(\overline{X}, \overline{Y}, \zeta) = -(\nabla_\beta Z) T(\overline{Y}, \overline{X}) + P(\overline{X}, \overline{Y}) Z\).
(f) \((\nabla_\beta \zeta P)(\overline{X}, \overline{Y}, \zeta) = -(\nabla_\beta \zeta T)(\overline{Y}, \overline{X}) - T(\overline{Y}, \overline{X})\).

For the \(h\)-curvature tensor \(R\), the following relations hold:

(g) \(R(\overline{X}, \overline{Y}) \zeta = 0\), \(R(\overline{X}, \overline{Y}, \overline{Z}, \zeta) = 0\).
(h) \((\nabla_\gamma Z) R(\overline{X}, \overline{Y}, \zeta) = 0\), \((\nabla_\beta Z) R(\overline{X}, \overline{Y}, \zeta) = R(\overline{X}, \overline{Y}) Z\).
(i) \((\nabla_{\overline{\gamma}} R)(\overline{X}, \overline{Y}, \overline{\zeta}) = 0\).

**Proof.** The proof follows from the properties of the curvature tensors \(S, P\) and \(R\) investigated in [20] together with Definition 2.1 and Lemma 2.3, taking into account the fact that the (h)h-torsion of the Cartan connection vanishes. \[\square\]

**Corollary 2.5.** Let \(\overline{\zeta} \in \mathfrak{X}(\pi(M))\) be a concurrent \(\pi\)-vector field. For every \(\overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M))\), we have

(a) \(\overline{T}(\overline{X}, \overline{\zeta}) = \overline{T}(\overline{\zeta}, \overline{X}) = 0\),
(b) \(\overline{\hat{P}}(\overline{X}, \overline{\zeta}) = \overline{\hat{P}}(\overline{\zeta}, \overline{X}) = 0\),
(c) \(P(\overline{X}, \overline{\zeta})\overline{Y} = P(\overline{\zeta}, \overline{X})\overline{Y} = 0\).

**Proof.**

(a) The proof follows from Proposition 2.4(d) by setting \(\overline{Z} = \overline{\zeta}\), taking into account the fact that \(g(\overline{T}(\overline{X}, \overline{Y}), \overline{Z}) = g(\overline{T}(\overline{X}, \overline{Z}), \overline{Y})\) and \(g(\overline{P}(\overline{X}, \overline{Y})\overline{Z}, \overline{Z}) = 0\) (lemma 2.3) together with the symmetry of \(T\).

(b) Follows from the identity \(\overline{\hat{P}}(\overline{X}, \overline{Y}) = (\nabla_{\overline{\beta}} T)(\overline{X}, \overline{Y})\) [20], making use of (a) and the fact that \(T(\overline{X}, \overline{\eta}) = 0\).

(c) We have [20]

\[
P(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = g((\nabla_{\overline{\beta}} T)(\overline{Y}, \overline{X}), \overline{W}) - g((\nabla_{\overline{\eta}} T)(\overline{Y}, \overline{X}), \overline{Z}) - g(T(\overline{X}, \overline{W}), \overline{\hat{P}}(\overline{Z}, \overline{Y})) + g(T(\overline{X}, \overline{Z}), \overline{\hat{P}}(\overline{W}, \overline{Y})). \tag{2.2}
\]

From which, by setting \(\overline{Y} = \overline{\zeta}\) (resp. \(\overline{X} = \overline{\zeta}\)) and using (a) and (b) above, the result follows. \[\square\]

**Lemma 2.6.** Let \((M, L)\) be a Finsler manifold and \(D^\circ\) the Berwald connection on \(\pi^{-1}(TM)\). Then, we have

(a) A \(\pi\)-vector field \(\overline{\nabla} \in \mathfrak{X}(\pi(M))\) is independent of the directional argument \(y\) if, and only if, \(D^\circ_{\overline{\gamma}} \overline{\nabla} = 0\) for all \(\overline{X} \in \mathfrak{X}(\pi(M))\),

(b) A scalar (vector) \(\pi\)-form \(\omega\) is independent of the directional argument \(y\) if, and only if, \(D^\circ_{\overline{\gamma}} \omega = 0\) for all \(\overline{X} \in \mathfrak{X}(\pi(M))\).

**Theorem 2.7.** A concurrent \(\pi\)-vector field \(\overline{\zeta}\) and its associated \(\pi\)-form \(\overline{\alpha}\) are independent of the directional argument \(y\).

**Proof.** By Theorem 1.6(a), we have

\[D^\circ_{\overline{\gamma}} \overline{\nabla} = \nabla_{\overline{\gamma}} \overline{\nabla} - T(\overline{X}, \overline{Y}).\]

From which, by setting \(\overline{Y} = \overline{\zeta}\), taking into account (2.1), Corollary 2.5(a) and Lemma 2.6, we conclude that \(\overline{\zeta}\) is independent of the directional argument.

On the other hand, we have from the above relation

\[(D^\circ_{\overline{\gamma}} \overline{\alpha})(\overline{Y}) = (\nabla_{\overline{\gamma}} \overline{\alpha})(\overline{Y}) + g(T(\overline{X}, \overline{Y}), \overline{\zeta}).\]

This, together with Lemma 2.2(b) and Corollary 2.5(a), imply that \(\overline{\alpha}\) is also independent of the directional argument. \[\square\]
3. Special Finsler spaces admitting concurrent $\pi$-vector fields

In this section, we investigate the effect of the existence of a concurrent $\pi$-vector field on some important special Finsler spaces. The intrinsic definitions of the special Finsler spaces treated here are quoted from [18].

For later use, we need the following lemma.

**Lemma 3.1.** Let $(M, L)$ be a Finsler manifold which admits a concurrent $\pi$-vector $\zeta$. Then, we have:

(a) The concurrent $\pi$-vector field $\zeta$ is everywhere non-zero.

(b) The scalar function $B := g(\zeta, \eta)$ is everywhere non-zero.

(c) The $\pi$-vector field $\overline{\zeta} := \zeta - \frac{B}{L^2} \eta$ is everywhere non-zero and is orthogonal to $\eta$.

(d) The $\pi$-vector fields $\overline{\zeta}$ and $\zeta$ satisfy $g(\overline{\zeta}, \overline{\zeta}) = g(\overline{\zeta}, \zeta) \neq 0$.

(e) The angular metric tensor $h$ satisfies $h(\overline{\zeta}, \overline{X}) \neq 0$ for all $\overline{X} \neq \overline{\eta}$.

**Proof.** Property (a) is clear.

(b) If $B := g(\zeta, \eta) = 0$, then

$$0 = (\nabla_\gamma X g)(\zeta, \eta) = \nabla_\gamma X g(\zeta, \eta) - g(\zeta, X) = -g(\zeta, X), \quad \forall \ X \in \mathfrak{X}(\pi(M)),$$

which contradicts (a).

(c) If $\overline{\zeta} = 0$, then $L^2 \zeta - B \eta = 0$. Differentiating covariantly with respect to $\gamma \overline{X}$, we get

$$2g(X, \eta)\zeta - B X - g(X, \zeta)\eta = 0. \quad (3.1)$$

From which,

$$g(X, \zeta) = \frac{B}{L^2} g(X, \eta). \quad (3.2)$$

By (3.1), using (3.2), we obtain

$$0 = 2g(X, \eta)g(Y, \zeta) - B g(X, Y) - g(X, \zeta)g(Y, \eta)$$

$$= 2\frac{B}{L^2} g(Y, \eta)g(X, \eta) - B g(X, Y) - \frac{B}{L^2} g(X, \eta)g(Y, \eta)$$

$$= -B\{g(X, Y) - \frac{1}{L^2} g(Y, \eta)g(X, \eta)\} = -B h(X, Y).$$

From which, since $B \neq 0$, we are led to a contradiction: $h = 0$.

On the other hand, the orthogonality of the two $\pi$-vector fields $\overline{\zeta}$ and $\overline{\eta}$ follows from the identities $g(\overline{\eta}, \overline{\eta}) = L^2$ and $g(\overline{\eta}, \overline{\zeta}) = B$.

(d) Follows from (c).

(e) Suppose that $h(\overline{X}, \overline{\zeta}) = 0$ for all $\overline{X} \neq \overline{\eta} \in \mathfrak{X}(TM)$, then, we have

$$0 = (\nabla_\beta \overline{X} h)(\overline{Y}, \overline{\zeta}) = \nabla_\beta \overline{X} h(\overline{Y}, \overline{\zeta}) - h(\nabla_\beta \overline{X} \overline{Y}, \overline{\zeta}) + h(\overline{X}, \overline{Y}) = h(\overline{X}, \overline{Y}),$$

which contradicts the fact that $h \neq 0$. \qed
Definition 3.2. A Finsler manifold $(M, L)$ is:

(a) Riemannian if the metric tensor $g(x, y)$ is independent of $y$ or, equivalently, if

$$T(\overline{x}, \overline{y}) = 0, \quad \text{for all } \overline{x}, \overline{y} \in \mathfrak{X}(\pi(M)).$$

(b) locally Minkowskian if the metric tensor $g(x, y)$ is independent of $x$ or, equivalently,

$$\nabla^\beta \pi T = 0 \quad \text{and} \quad R = 0.$$

Definition 3.3. A Finsler manifold $(M, L)$ is:

(a) a Berwald manifold if the torsion tensor $T$ is horizontally parallel:

$$\nabla^\beta \pi T = 0.$$

(b) a Landsberg manifold if $\hat{P}(\overline{x}, \overline{y}) = 0$, or equivalently, if

$$\nabla^\beta \eta T = 0 \quad \text{and} \quad \hat{P} = 0.$$

(c) a general Landsberg manifold if the trace of the linear map $\overline{Y} \mapsto \hat{P}(\overline{x}, \overline{Y})$ is identically zero for all $\overline{x} \in \mathfrak{X}(\pi(M))$, or equivalently, if $\nabla^\beta \eta C = 0$.

Now, we have

Theorem 3.4. Let $(M, L)$ be a Finsler manifold which admits a concurrent $\pi$-vector field $\zeta$. Then, the following assertions are equivalent:

(a) $(M, L)$ is a Berwald manifold,

(b) $(M, L)$ is a Landsberg manifold,

(c) $(M, L)$ is a Riemannian manifold.

Proof. The implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) are trivial. Now, we prove the implication (b) $\Rightarrow$ (c). As $(M, L)$ is a Landsberg manifold, $\hat{P} = 0$. Consequently, the hv-curvature $P$ vanishes [20]. Hence, $0 = P(\overline{x}, \overline{Y}, Z, \zeta) = T(\overline{x}, \overline{Y}, \overline{Z})$ by Proposition 2.4(d). The result follows then from Deicke theorem (Lemma 1.4).

Definition 3.5. A Finsler manifold $(M, L)$ is said to be:

(a) $C^h$-recurrent if the (h)hv-torsion tensor $T$ satisfies the condition

$$\nabla_{\beta\pi} T = \lambda_0(\overline{x}) T,$$

where $\lambda_0$ is a $\pi$-form of order one.

(b) $C^e$-recurrent if the (h)hv-torsion tensor $T$ satisfies the condition

$$(\nabla_{\gamma\pi} T)(\overline{Y}, \overline{Z}) = \lambda_0(\overline{x}) T(\overline{Y}, \overline{Z}).$$

(c) $C^0$-recurrent if the (h)hv-torsion tensor $T$ satisfies the condition

$$(D_{\gamma\pi} T)(\overline{Y}, \overline{Z}) = \lambda_0(\overline{x}) T(\overline{Y}, \overline{Z}).$$

Theorem 3.6. Let $(M, L)$ be a Finsler manifold which admits a concurrent $\pi$-vector field $\zeta$ such that $\lambda_0(\zeta) \neq 0$. Then, the following assertions are equivalent:

(a) $(M, L)$ is a $C^h$-recurrent manifold,

(b) $(M, L)$ is a $C^e$-recurrent manifold,

(c) $(M, L)$ is a $C^0$-recurrent manifold,
(d) \((M, L)\) is a Riemannian manifold.

Proof. It is to be noted that (b), (c) and (d) are equivalent despite of the existence of a concurrent \(\pi\)-vector field \([18]\). The implication \((d) \implies (a)\) is trivial. It remains to prove that \((a) \implies (d)\). Setting \(\overline{W} = \overline{\zeta}\) in \((2.2)\), making use of \(\hat{P}(\overline{\zeta}, \overline{X}) = 0 = T(\overline{\zeta}, \overline{X})\) (Corollary \(2.5)\), \(P(\overline{X}, \overline{Y}, \overline{Z}, \overline{\zeta}) = T(\overline{X}, \overline{Y}, \overline{Z})\) (Proposition \(2.4\) and \(g((\nabla_{\overline{\zeta}} T)(\overline{X}, \overline{Y}), \overline{W}) = g((\nabla_{\overline{\zeta}} T)(\overline{X}, \overline{W}), \overline{Y})\) (Proposition \(3.3\) of \([20]\)), we get
\[
\nabla_{\overline{\zeta}} T = 0.
\]

On the other hand, Definition \(3.5\)(a) for \(\overline{X} = \overline{\zeta}\), yields
\[
\nabla_{\overline{\zeta}} T = \lambda_{\circ}(\overline{\zeta}) T.
\]

the above two equations imply that \(T = 0\) and hence \((M, L)\) is Riemannian. \(\square\)

Definition 3.7. A Finsler manifold \((M, L)\) is said to be:

(a) quasi-\(C\)-reducible if \(\dim(M) \geq 3\) and the Cartan tensor \(T\) has the form
\[
T(\overline{X}, \overline{Y}, \overline{Z}) = A(\overline{X}, \overline{Y})C(\overline{Z}) + A(\overline{Y}, \overline{Z})C(\overline{X}) + A(\overline{Z}, \overline{X})C(\overline{Y}),
\]

where \(A\) is a symmetric \(\pi\)-tensor field satisfying \(A(\overline{X}, \overline{\eta}) = 0\).

(b) semi-\(C\)-reducible if \(\dim M \geq 3\) and the Cartan tensor \(T\) has the form
\[
T(\overline{X}, \overline{Y}, \overline{Z}) = \frac{\mu}{n + 1}\{h(\overline{X}, \overline{Y})C(\overline{Z}) + h(\overline{Y}, \overline{Z})C(\overline{X}) + h(\overline{Z}, \overline{X})C(\overline{Y})\} + \frac{\tau}{C^2} C(\overline{X})C(\overline{Y})C(\overline{Z}), \tag{3.3}
\]

where \(C^2 := C(\overline{C}) \neq 0\), \(\mu\) and \(\tau\) are scalar functions satisfying \(\mu + \tau = 1\).

(c) \(C\)-reducible if \(\dim M \geq 3\) and the Cartan tensor \(T\) has the form
\[
T(\overline{X}, \overline{Y}, \overline{Z}) = \frac{1}{n + 1}\{h(\overline{X}, \overline{Y})C(\overline{Z}) + h(\overline{Y}, \overline{Z})C(\overline{X}) + h(\overline{Z}, \overline{X})C(\overline{Y})\}. \tag{3.4}
\]

(d) \(C^2\)-like if \(\dim M \geq 2\) and the Cartan tensor \(T\) has the form
\[
T(\overline{X}, \overline{Y}, \overline{Z}) = \frac{1}{C^2} C(\overline{X})C(\overline{Y})C(\overline{Z}).
\]

Proposition 3.8. If a quasi-\(C\)-reducible Finsler manifold \((M, L)\) \((\dim M \geq 3)\) admits a concurrent \(\pi\)-vector field, then \((M, L)\) is Riemannian, provided that \(A(\overline{\zeta}, \overline{\zeta}) \neq 0\).

Proof. Follows from the defining property of quasi-\(C\)-reducibility by setting \(\overline{X} = \overline{Y} = \overline{\zeta}\) and using the fact that \(C(\overline{\zeta}) = 0\) and \(A(\overline{\zeta}, \overline{\zeta}) \neq 0\). \(\square\)

Theorem 3.9. Let \((M, L)\) be a Finsler manifold \((\dim M \geq 3)\) which admits a concurrent \(\pi\)-vector field \(\overline{\zeta}\), then, we have

(a) A \(C\)-reducible manifold \((M, L)\) is a Riemannian manifold.

(b) A semi-\(C\)-reducible manifold \((M, L)\) is a \(C^2\)-like manifold.
Proof.

(a) Follows from the defining property of $C$-reducibility by setting $X = Y = \zeta$, taking into account Lemma 3.1(e), Lemma 1.3 and $C(\zeta) = 0$.

(b) Let $(M, L)$ be semi-$C$-reducible. Setting $X = Y = \zeta$ and $Z = C$ in (3.3), taking into account Corollary 2.5(a) and $C(\zeta) = 0$, we get

$$\mu h(\zeta, \zeta) C(\zeta) = 0.$$ 

From which, since $h(\zeta, \zeta) \neq 0$ and $C(\zeta) \neq 0$, it follows that $\mu = 0$. Consequently, $(M, L)$ is $C_2$-like.

Definition 3.10. The condition

$$T(X, Y, Z, W) := L(\nabla_\gamma X T)(Y, Z, W) + S_{X,Y,Z,W}(T) = 0$$

(3.5)

will be called the $T$-condition.

The more relaxed condition

$$T_0(X, Y) := L(\nabla_\gamma X C)(Y) + S_{X,Y}(C(Y) = 0$$

(3.6)

will be called the $T_0$-condition.

Theorem 3.11. Let $(M, L)$ be a Finsler manifold which admits a concurrent $\pi$-vector field $\zeta$. Then, the following assertions are equivalent:

(a) $(M, L)$ satisfies the $T$-condition,

(b) $(M, L)$ satisfies the $T_0$-condition,

(c) $(M, L)$ is Riemannian.

Proof.

(a) $\implies$ (c): Follows from (3.5) by setting $W = \zeta$, taking into account that $T(\zeta, X) = T(C, X) = 0$ and $T(C) = B \neq 0$.

(b) $\implies$ (c): Follows from (3.6) by setting $X = \zeta$, taking into account that $C(\zeta) = 0$, $(\nabla_\gamma X C)(Y) = (\nabla_\gamma Y C)(X)$ and $\ell(\zeta) \neq 0$.

The other implications are trivial.

Definition 3.12. A Finsler manifold $(M, L)$ is said to be $S_3$-like if $\dim(M) \geq 4$ and the $v$-curvature tensor $S$ has the form:

$$S(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \frac{S_{e^v}}{(n-1)(n-2)}\{h(\overline{X}, \overline{Z})h(\overline{Y}, \overline{W}) - h(\overline{X}, \overline{W})h(\overline{Y}, \overline{Z})\}.$$  

(3.7)

Theorem 3.13. If an $S_3$-like manifold $(M, L)$ ($\dim M \geq 4$) admits a concurrent $\pi$-vector field $\overline{\zeta}$, then, the $v$-curvature tensor $S$ vanishes.

Proof. Setting $\overline{Z} = \overline{\zeta}$ in (3.7), taking Proposition 2.4 into account, we immediately get

$$\frac{S_{e^v}}{(n-1)(n-2)}\{h(\overline{X}, \overline{\zeta})h(\overline{Y}, \overline{W}) - h(\overline{X}, \overline{W})h(\overline{Y}, \overline{\zeta})\} = 0.$$ 

Taking the trace of the above equation, we have

$$\frac{S_{e^v}}{(n-1)(n-2)}\{(n-1)h(\overline{X}, \overline{\zeta}) - h(\overline{X}, \overline{\zeta})\} = \frac{S_{e^v}}{(n-1)}h(\overline{X}, \overline{\zeta}) = 0$$

From which, since $h(\overline{X}, \overline{\zeta}) \neq 0$ (Lemma 3.1), the vertical scalar curvature $S_{e^v}$ vanishes.

Now, again, from (3.7), the result follows. \qed
**Definition 3.14.** A Finsler manifold \((M, L)\), where \(\text{dim} M \geq 3\), is said to be:

(a) \(P_2\)-like if the hv-curvature tensor \(P\) has the form:

\[
P(X, Y, Z, W) = \omega(Z)T(X, Y, W) - \omega(W)T(X, Y, Z),
\]

where \(\omega\) is a \((1)\) \(\pi\)-form (positively homogeneous of degree 0).

(b) \(P\)-reducible if the \(\pi\)-tensor field \(\hat{P}(X, Y, Z) := g(\hat{P}(X, Y), Z)\) has the form

\[
\hat{P}(X, Y, Z) = \delta(X)h(Y, Z) + \delta(Y)h(X, Z) + \delta(Z)h(X, Y),
\]

where \(\delta\) is the \(\pi\)-form defined by \(\delta(X) = \frac{1}{n+1}(\nabla_{\beta\gamma}(C))(X)\).

**Theorem 3.15.** Let \((M, L)\) be a Finsler manifold \((\text{dim} M \geq 3)\) which admits a concurrent \(\pi\)-vector field \(\zeta\), then, we have

(a) A \(P_2\)-like manifold \((M, L)\) is a Riemannian manifold, provided that \(\omega(\zeta) \neq -1\).

(b) A \(P\)-reducible manifold \((M, L)\) is a Landsberg manifold.

**Proof.**

(a) Setting \(Z = \zeta\) in \((3.8)\), taking into account Proposition 2.4 and Corollary 2.5, we immediately get

\[
(\omega(\zeta) + 1)T(X, Y) = 0.
\]

Hence, the result follows.

(b) Setting \(X = Y = \zeta\) in \((3.9)\) and taking into account that \((\nabla_{\beta\gamma}(C))(\zeta) = 0\), we get

\[
h(\zeta)h(\zeta) = 0, \text{ with } h(\zeta, \zeta) = 0 \quad (\text{Lemma 3.1}).
\]

Consequently, \(\nabla_{\beta\gamma}C = 0\). Hence, again, from Definition 3.14(b), the (v)hv-torsion tensor \(\hat{P} = 0\).

**Definition 3.16.** A Finsler manifold \((M, L)\), where \(\text{dim} M \geq 3\), is said to be:

(a) \(h\)-isotropic if there exists a scalar \(k_o\) such that the horizontal curvature tensor \(R\) has the form

\[
R(X, Y)Z = k_o\{g(X, Z)Y - g(Y, Z)X\}.
\]

(b) of scalar curvature if there exists a scalar function \(k : TM \to \mathbb{R}\) such that

\[
R(\eta, X, \eta, Y) = kL^2h(X, Y),
\]

where \(k\) called the scalar curvature.

**Theorem 3.17.** Let \((M, L)\), \(\text{dim} M \geq 3\), be an \(h\)-isotropic Finsler manifold admitting a concurrent \(\pi\)-vector field \(\zeta\), then we have

(a) The \(h\)-curvature tensor \(R\) of the Cartan connection vanishes.

(b) If \((M, L)\) is a Berwald manifold, then it is a flat Riemannian manifold.

**Proof.**

(a) From Definition 3.16(a), we have

\[
R(X, Y, Z, W) = k_o\{g(X, Z)g(Y, W) - g(Y, Z)g(X, W)\}. \tag{3.10}
\]
Setting $Z = \zeta$ and $X = m$ and noting that $R(X, Y)\zeta = 0$ (Proposition 2.4), we have

$$k_o\{g(m, \zeta)g(Y, W) - g(Y, \zeta)g(m, W)\} = 0.$$  

Taking the trace of this equation, we get

$$k_o(n - 1)g(m, \zeta) = 0.$$  

From which, since $g(m, \zeta) = g(m, m) \neq 0$ (Lemma 3.1) and $\dim M \geq 3$, the scalar $k_o$ vanishes. Now, again, from (3.10), the result follows.

(b) Follows from (a), taking Theorem 3.4 into account.  

Theorem 3.18. Let $(M, L), \dim M \geq 3$, be a Finsler manifold of scalar curvature admitting a concurrent $\pi$-vector field $\zeta$. The following statements hold:

(a) The scalar curvature $k$ vanishes.

(b) The deviation tensor field $H$ ($H(X) := \hat{R}(\eta, X)$) vanishes.

(c) The $(v)h$-torsion tensor $\hat{R}$ of Cartan connection vanishes.

(d) The $h$-curvature tensor $R^\circ$ of Berwald connection vanishes.

(e) The horizontal distribution is completely integrable.

Proof.

(a) Follows from Definition 3.16(b) by setting $Y = \zeta$, taking into account Proposition 2.4(g) and Lemma 3.1(e).

(b) Follows from (a) and Definition 3.16(b).

(c), (d) and (e) Follow from Theorem 4.7 of [20].

4. Energy $\beta$-change and Cartan nonlinear connection (Barthel connection)

In the present and the next sections we consider a perturbation, by a concurrent $\pi$-vector field $\zeta$, of the energy function $E = \frac{1}{2}L^2$ of a Finsler structure $L$.

Let $(M, L)$ be a Finsler manifold. Consider the change

$$\tilde{L}^2(x, y) = L^2(x, y) + B^2(x, y), \text{ with } B := g(\eta, \zeta) = \alpha(\eta), \quad (4.1)$$

$\tilde{L}$ defines a new Finsler structure on $M$. The Finsler structure $\tilde{L}$ is said to be obtained from the Finsler structure $L$ by the $\beta$-change (4.1). The $\beta$-change (4.1) will be referred to as an energy $\beta$-change (as it can be written in the form $\tilde{E} = E + \frac{1}{2}B^2$, where $E$ and $\tilde{E}$ are the energy functions corresponding to the Lagrangians $L$ and $\tilde{L}$ respectively).

The following two lemmas are useful for subsequence use.

Lemma 4.1. The function $B(x, y)$ given by (4.1) has the properties

(a) $B = d_J E(\beta \zeta), \quad d_J B = \alpha \circ \rho, \quad d_h B = -(i_\eta g) \circ \rho.$
(b) \(dd_J B^2(\gamma \bar{X}, \beta \bar{Y}) = 2\alpha(\bar{X})\alpha(\bar{Y}), \quad dd_J B^2(\gamma \bar{X}, \gamma \bar{Y}) = 0.\)

(c) \(dd_J B^2(\beta \bar{X}, \beta \bar{Y}) = 2(\alpha \wedge i_\gamma g)(\bar{X}, \bar{Y}).\)

**Proof.**

(a) From \(g(\bar{\eta}, \bar{\eta}) = 2E^2\) and \(\nabla g = 0\), one can show that
\[d_J E(X) = g(\rho X, \bar{\eta}), \quad \forall X \in \mathfrak{X}(TM).\]

Setting \(X = \beta \bar{\zeta}\), we obtain \(B = d_J E(\beta \bar{\zeta}).\)

On the other hand,
\[d_J B(X) = JX \cdot B = JX \cdot g(\bar{\zeta}, \nabla_J X \bar{\eta}) = g(\bar{\zeta}, \rho X) = \alpha(\rho X).\]

Similarly,
\[d_h B(X) = hX \cdot B = hX \cdot g(\bar{\zeta}, \bar{\eta}) = g(\nabla_{hX} \bar{\zeta}, \bar{\eta}) = -g(\rho X, \bar{\eta}) = -i_\gamma g(\rho X).\]

(b) Making use of (a), we have
\[
\begin{align*}
dd_J B^2(\gamma \bar{X}, \beta \bar{Y}) &= \gamma \bar{X} \cdot d_J B^2(\beta \bar{Y}) - \beta \bar{Y} \cdot d_J B^2(\gamma \bar{X}) - d_J B^2(\gamma \bar{X}, \beta \bar{Y}) \\
&= 2\gamma \bar{X} \cdot (B\gamma \bar{Y} \cdot B) - 2\beta \bar{Y} \cdot (BJ\gamma \bar{X} \cdot B) - 2BJ[\gamma \bar{X}, \beta \bar{Y}] \cdot B \\
&= 2\gamma \bar{X} \cdot (Bg(\bar{\gamma} \bar{Z}, \bar{\zeta})) - 2Bg(\rho[\gamma \bar{X}, \beta \bar{Y}], \bar{\zeta}) \\
&= 2\left\{g(\bar{X}, \bar{\zeta})g(\bar{\gamma} \bar{Z}, \bar{\zeta}) + B\gamma \bar{X} \cdot g(\bar{\gamma} \bar{Z}, \bar{\zeta})\right\} - 2Bg(\nabla_{\gamma \bar{X}} \bar{\gamma} \bar{Y} - T(\bar{\gamma} \bar{X}, \bar{\zeta}), \bar{\zeta}) \\
&= 2\left\{g(\bar{X}, \bar{\zeta})g(\bar{\gamma} \bar{Z}, \bar{\zeta}) + Bg(\nabla_{\gamma \bar{X}} \bar{\gamma} \bar{Y}, \bar{\zeta})\right\} - 2Bg(\nabla_{\gamma \bar{X}} \bar{\gamma} \bar{Y}, \bar{\zeta}) \\
&= 2\alpha(\bar{X})\alpha(\bar{Y}).
\end{align*}
\]

Similarly, \(dd_J B^2(\gamma \bar{X}, \gamma \bar{Y}) = 0.\)

(c) The proof is analogous to that of (b). \(\square\)

**Lemma 4.2.** Let \((M, L)\) be a Finsler manifold. Let \(g\) be the Finsler metric associated with \(L\) and \(\nabla\) the Cartan connection on \(\pi^{-1}(TM)\). Then, the following relation holds
\[g(\bar{X}, \bar{Y}) = \Omega(\gamma \bar{X}, \beta \bar{Y}) \quad \text{for all} \quad \bar{X}, \bar{Y} \in \mathfrak{X}(\pi(M)), \tag{4.2}\]
where \(\Omega := dd_J E\) and \(\beta\) is the horizontal map associated with \(\nabla\).

**Proof.** Using the relations \(d_J E(X) = g(\rho X, \bar{\eta}), \nabla g = 0, T(\rho X, \bar{\eta}) = 0\) and \(g(T(\bar{X}, \bar{Y}), \bar{Z}) = g(T(\bar{X}, \bar{Z}), \bar{Y})\), we get
\[
\begin{align*}
\Omega(\gamma \bar{X}, \beta \bar{Y}) &= \gamma \bar{X} \cdot d_J E(\beta \bar{Y}) - \beta \bar{Y} \cdot d_J E(\gamma \bar{X}) - d_J E(\gamma \bar{X}, \beta \bar{Y}) \\
&= \gamma \bar{X} \cdot g(\bar{Y}, \bar{\eta}) - g(\rho[\gamma \bar{X}, \beta \bar{Y}], \bar{\eta}) \\
&= g(\nabla_{\gamma \bar{X}} \bar{\gamma} \bar{Y}, \bar{\eta}) + g(\bar{Y}, \nabla_{\gamma \bar{X}} \bar{\gamma} \bar{Y}) - g(\rho[\gamma \bar{X}, \beta \bar{Y}], \bar{\eta}) \\
&= g(\bar{X}, \bar{Y}) + g(T(\bar{X}, \bar{Y}), \bar{\eta}) = g(\bar{X}, \bar{Y}),
\end{align*}
\]
which proves the required relation. \(\square\)

The following result gives the relationship between \(g\) and \(\tilde{g}\).
Proposition 4.3. Under the energy $\beta$-change (4.1), the Finsler metrics $g$ and $\tilde{g}$ are related by

$$\tilde{g}(\mathbf{X}, \mathbf{Y}) = g(\mathbf{X}, \mathbf{Y}) + \alpha(\mathbf{X})\alpha(\mathbf{Y}), \quad \text{for all } \mathbf{X}, \mathbf{Y} \in \mathcal{X}(\pi(M)), \quad (4.3)$$

$\alpha$ being the $\pi$-form associated with $\tilde{\zeta}$ under the duality defined by the metric $g$.

Proof. The proof follows by applying the operator $\frac{1}{2}dd_J$ on both sides of (4.1), taking into account (4.2), Proposition 1.9 and Lemma 4.1(b).

In more detail,

$$\tilde{g}(\mathbf{X}, \mathbf{Y}) = dd_J\tilde{E}(\gamma\mathbf{X}, \beta\mathbf{Y}),$$

$$= dd_J\tilde{E}(\gamma\mathbf{X}, \beta\mathbf{Y}) - dd_J\tilde{E}(\gamma\mathbf{X}, L\beta\mathbf{Y}),$$

$$= dd_J\tilde{E}(\gamma\mathbf{X}, \beta\mathbf{Y}) + \frac{1}{2}dd_JB^2(\gamma\mathbf{X}, \beta\mathbf{Y}), \quad \text{as } dd_J\tilde{E}(\gamma\mathbf{X}, \gamma\mathbf{Y}) = 0$$

$$= g(\mathbf{X}, \mathbf{Y}) + \alpha(\mathbf{X})\alpha(\mathbf{Y}).$$

$\square$

Corollary 4.4. Under the energy $\beta$-change (4.1), the exterior 2-forms $\Omega$ and $\tilde{\Omega}$ are related by

$$\tilde{\Omega} = \Omega + \frac{1}{2}dd_JB^2.$$

Theorem 4.5. Let $(M, L)$ and $(M, \tilde{L})$ be two Finsler manifolds related by the energy $\beta$-change (4.1). The associated canonical sprays $G$ and $\tilde{G}$ are related by

$$\tilde{G} = G + \frac{L^2}{1 + p^2}\gamma\tilde{\zeta}; \quad p^2 := g(\tilde{\zeta}, \tilde{\zeta}).$$

Proof. From the above Corollary, we have

$$\tilde{\Omega} = \Omega + \frac{1}{2}dd_JB^2.$$

As the difference between two sprays is a vertical vector field, assume that $\tilde{G} = G + \gamma\tilde{\mu}$, for some $\tilde{\mu} \in \mathcal{X}(\pi(M))$, then we have

$$-d\tilde{E}(X) = i_{\tilde{G}}\tilde{\Omega}(X) = i_{G+\gamma\tilde{\mu}}\tilde{\Omega}(X)$$

$$= i_G\Omega(X) + \frac{1}{2}i_Gdd_JB^2(X) + i_{\gamma\tilde{\mu}}\Omega(X) + \frac{1}{2}i_{\gamma\tilde{\mu}}dd_JB^2(X). \quad (4.4)$$

Now, we compute the terms on the right hand side (using Lemma 4.1 and Lemma 4.2):

$$i_G\Omega(X) = -dE(X).$$

$$i_{\gamma\tilde{\mu}}\Omega(X) = \Omega(\gamma\tilde{\mu}, X) = \Omega(\gamma\tilde{\mu}, \gamma KX + \beta\rho X) = \Omega(\gamma\tilde{\mu}, \beta\rho X) = g(\tilde{\mu}, \rho X).$$

$$\frac{1}{2}i_Gdd_JB^2(X) = \frac{1}{2}\{dd_JB^2(\tilde{\mu}, \beta\rho X) - dd_JB^2(\gamma KX, \beta\tilde{\eta})\}$$

$$= \{g(\tilde{\eta}, \tilde{\zeta})g(\rho X, \tilde{\eta}) - g(\tilde{\eta}, \tilde{\zeta})g(\rho X, \tilde{\zeta})\} - g(KX, \tilde{\zeta})g(\tilde{\eta}, \tilde{\zeta})$$

$$= \{g(\tilde{\eta}, \tilde{\zeta})g(\rho X, \tilde{\eta}) - g(\tilde{\eta}, \tilde{\zeta})g(\rho X, \tilde{\zeta})\} - \{X \cdot g(\tilde{\eta}, \tilde{\zeta}) + g(\tilde{\eta}, \rho X)\}g(\tilde{\eta}, \tilde{\zeta})$$

$$= -L^2g(\tilde{\zeta}, \rho X) - BdB(X).$$

$$\frac{1}{2}i_{\gamma\tilde{\mu}}dd_JB^2(X) = \frac{1}{2}dd_JB^2(\gamma\tilde{\mu}, \beta\rho X + \gamma KX) = \frac{1}{2}dd_JB^2(\gamma\tilde{\mu}, \beta\rho X) = g(\tilde{\zeta}, \tilde{\mu})g(\tilde{\zeta}, \rho X).$$
From these, together with \(d\tilde{E}(X) = dE(X) + BdB(X)\), Equation (4.4) reduces to
\[
g(\bar{\mu}, \rho X) + g(\bar{\zeta}, \bar{\mu})g(\zeta, \rho X) - L^2 g(\zeta, \rho X) = 0. \tag{4.5}
\]
Consequently,
\[
\bar{\mu} = (L^2 - g(\zeta, \bar{\mu}))\bar{\zeta}.
\]
Again, from (4.5), by setting \(X = \beta \bar{\zeta}\), we obtain
\[
g(\zeta, \bar{\mu}) = \frac{L^2 g(\zeta, \bar{\zeta})}{1 + g(\zeta, \zeta)}.
\]
Therefore,
\[
\bar{\mu} = \frac{L^2}{1 + g(\zeta, \zeta)} \bar{\zeta}.
\]
Hence the result.

\textbf{Theorem 4.6.} Let \((M, L)\) and \((M, \tilde{L})\) be two Finsler manifolds related by the energy \(\beta\)-change (4.1). The associated Barthel connections \(\Gamma\) and \(\tilde{\Gamma}\) are related by
\[
\tilde{\Gamma} = \Gamma + 2 \frac{d_J E}{1 + p^2} \otimes \gamma \zeta.
\]

\textit{Proof.} From Theorem 4.5 and the formula [4]
\[
[f X, J] = f[X, J] + df \wedge i_X J - d_J f \otimes X,
\]
we obtain
\[
\tilde{\Gamma} = [J, \tilde{G}] = \left[ J, G + \frac{L^2}{1 + p^2} \gamma \zeta \right] = [J, G] - \left[ \frac{L^2}{1 + p^2} \gamma \zeta, J \right]
\]
\[
= [J, G] - \frac{L^2}{1 + p^2} [\gamma \zeta, J] - d\left( \frac{L^2}{1 + p^2} \right) \wedge i_X J + d_J \left( \frac{L^2}{1 + p^2} \right) \otimes \gamma \zeta.
\]

Now,
\[
d_J \left( \frac{L^2}{1 + p^2} \right) = \frac{2d_J E}{(1 + p^2)}, \quad \text{as } d_J p^2 = 0, \text{ by Theorem 2.7}
\]
whereas,
\[
[\gamma \zeta, J]X = [\gamma \zeta, JX] - J[\gamma \zeta, X]
\]
\[
= \gamma \left\{ \nabla_{\gamma \zeta} \rho X - \nabla_{JX} \zeta \right\} - \gamma \left\{ \nabla_{\gamma \zeta} \rho X - T(\zeta, \rho X) \right\} = 0,
\]
by Lemma 1.1 and Corollary 2.5. Consequently,
\[
\tilde{\Gamma} = \Gamma + \frac{2d_J E}{1 + g(\zeta, \zeta)} \otimes \gamma \zeta,
\]
which proves the result.

\textbf{Remark 4.7.} \textit{Comparing Theorem 4.6 and Proposition 1.9, we find that}
\[
\tilde{\Gamma} = \Gamma - 2L,
\]
\textit{where}
\[
L = -\frac{d_J E}{1 + p^2} \otimes \gamma \zeta. \tag{4.6}
\]
Lemma 4.8. Let $L$ be the 1-form defined by (4.6). The following formulae hold:

(a) $[LX, LY] = \mathcal{U}_{X,Y} \left\{ \frac{g(\rho X, \eta)}{1 + p^2} \gamma \nabla_{\gamma \rho Y, \eta} + g(\rho Y, \zeta) \right\} \gamma \zeta$.

(b) $[hX, LY] = -\frac{g(\rho Y, \eta)}{1 + p^2} [hX, \gamma \zeta] - \frac{g(\nabla_{hX} \rho Y, \eta)}{1 + p^2} \gamma \zeta - \frac{2g(\rho Y, \eta)g(\rho X, \zeta)}{(1 + p^2)^2} \gamma \zeta,$

where $\mathcal{U}_{X,Y} \{A(X, Y)\} = A(X, Y) - A(Y, X)$.

Proof. We prove (a) only; the proof of (b) is similar. Since $L(X) = -\frac{g(\rho X, \eta)}{1 + p^2} \gamma \zeta$, we have

$$[LX, LY] = \left[ \frac{g(\rho X, \eta)}{1 + p^2} \gamma \zeta, \frac{g(\rho Y, \eta)}{1 + p^2} \gamma \zeta \right] = \frac{g(\rho X, \eta)g(\rho Y, \eta)}{(1 + p^2)^2} [\gamma \zeta, \gamma \zeta] + \frac{g(\rho X, \eta)}{1 + p^2} \left\{ \gamma \zeta \cdot \frac{g(\rho Y, \eta)}{1 + p^2} \right\} \gamma \zeta - \frac{g(\rho Y, \eta)}{1 + p^2} \left\{ \gamma \zeta \cdot \frac{g(\rho X, \eta)}{1 + p^2} \right\} \gamma \zeta.$$

The result follows from this relation together with the following expressions

$$[\gamma \zeta, \gamma \zeta] = \gamma \{ \nabla_{\gamma \rho \zeta} \zeta - \nabla_{\gamma \zeta} \zeta \} = 0,$$

$$\gamma \zeta \cdot \frac{g(\rho X, \eta)}{1 + p^2} = \frac{\gamma \zeta \cdot g(\rho X, \eta)}{1 + p^2} = \frac{g(\nabla_{\gamma \rho X, \eta}) + g(\rho X, \zeta)}{1 + p^2},$$

by Theorem 2.1.

Proposition 4.9. Under the energy $\beta$-change (1.1), the curvature tensors $\mathcal{R}$ and $\mathcal{R}$ of the associated Barthel connections $\tilde{\Gamma}$ and $\Gamma$ are related by

$$\tilde{\mathcal{R}}(X, Y) = \mathcal{R}(X, Y) + \mathcal{U}_{X,Y} \{ \mathcal{H}(X, Y) \},$$

where $\mathcal{H}(X, Y) := \frac{g(\rho Y, \eta)}{1 + p^2} JX + \frac{g(\rho X, \eta)g(\rho Y, \zeta)}{(1 + p^2)^2} \gamma \zeta$.

Proof. By the identity $\mathcal{R}(X, Y) = -v[hX, hY]$ (13), we have

$$\tilde{\mathcal{R}}(X, Y) = -\nabla \left[ \tilde{h}X, \tilde{h}Y \right].$$

Using Proposition 1.9 and the fact that $L[LX, LY] = 0$, we get

$$\tilde{\mathcal{R}}(X, Y) = \mathcal{R}(X, Y) - [LX, LY] - L[hX, hY] + \mathcal{U}_{X,Y} \{ v[hX, LY] + L[hX, LY] \}.$$  

The result follows from (4.8) and the following formulae which can be easily computed using Lemma 4.8 and Theorem 4.6

$$[LX, LY] = \mathcal{U}_{X,Y} \left\{ \frac{g(\rho X, \eta)}{1 + p^2} \left\{ g(\nabla_{\gamma \rho Y, \eta}) + g(\rho Y, \zeta) \right\} \gamma \zeta \right\},$$

$$v[hX, LY] = \frac{g(\rho Y, \eta)}{1 + p^2} v[hX, \gamma \zeta] - \frac{2g(\rho X, \zeta)g(\rho Y, \eta)}{(1 + p^2)^2} \gamma \zeta - \frac{g(\nabla_{hX} \rho Y, \eta)}{1 + p^2} \gamma \zeta,$$

$$L[hX, LY] = \frac{g(\rho Y, \eta)}{(1 + p^2)^2} g(\rho [hX, \gamma \zeta], \eta) \gamma \zeta,$$

$$L[hX, hY] = -\frac{g(\rho [hX, hY], \eta)}{1 + p^2} \gamma \zeta.$$ 

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Corollary 4.10. Under the energy $\beta$-change (1.1), the curvature tensors $R$ is invariant if and only if, the vector 2-form $H$ is symmetric.

5. Fundamental connections under an energy $\beta$-change

In this section, we investigate the transformation of the fundamental linear connections of Finsler geometry, as well as their curvature tensors, under the energy $\beta$-change (4.1). We start our investigation with the Cartan connection.

The following lemmas are useful for subsequent use.

Lemma 5.1. [17] Let $(M, L)$ be a Finsler manifold and $g$ the Finsler metric associated with $L$. The Cartan connection $\nabla$ is completely determined by the relations:

(a) $2g(\nabla_{vX}\rho Y, \rho Z) = vX \cdot g(\rho Y, \rho Z) + g(\rho Y, \rho[hZ, vX]) + g(\rho Z, \rho[vX, hY]).$

(b) $2g(\nabla_{hX}\rho Y, \rho Z) = hX \cdot g(\rho Y, \rho Z) + hY \cdot g(\rho Z, \rho X) - hZ \cdot g(\rho X, \rho Y) - g(\rho X, \rho[hY, hZ]) + g(\rho Y, \rho[hX, hZ]) + g(\rho Z, \rho[hX, hY]).$

Since the Cartan connection $\nabla$ has the same horizontal and vertical projectors as the Barthel connection $\Gamma = [J, G]$ [17], we get

Lemma 5.2. Under the energy $\beta$-change (1.1), we have

$$\widetilde{h} = h - L, \quad \widetilde{v} = v + L \quad \text{with} \quad L := - \frac{dIE}{1 + g(\zeta, \zeta)} \otimes \gamma\zeta.$$

Theorem 5.3. Let $(M, L)$ and $(M, \tilde{L})$ be two Finsler manifolds related by the energy $\beta$-change (1.1). Then the associated Cartan connections $\nabla$ and $\tilde{\nabla}$ are related by:

$$\tilde{\nabla}_X\rho Y = \nabla_X\rho Y - \frac{g(\rho X, \rho Y)}{1 + p^2} \zeta, \quad \text{with} \quad p^2 := g(\zeta, \zeta). \quad (5.1)$$

In particular,

(a) $\tilde{\nabla}_{\gamma X}\overline{Y} = \nabla_{\gamma X}\overline{Y},$

(b) $\tilde{\nabla}_{\beta X}\overline{Y} = \nabla_{\beta X}\overline{Y} - U(\overline{X}, \overline{Y}), \quad \text{with} \quad U(\overline{X}, \overline{Y}) = \frac{g(\overline{X}, \overline{Y})}{1 + p^2} \zeta - \frac{g(\overline{X}, \eta)}{1 + p^2} \nabla_{\gamma \overline{Z}}\overline{Y}.$

Proof. Using Lemma 5.1(a), Proposition 4.3 and Lemma 5.2, noting the fact that $\rho[\tilde{h}Z, \tilde{v}X] = \rho[hZ, vX]$ (as $\rho[LZ, vX] = 0$), we get

$$2\tilde{g}(\tilde{\nabla}_{\tilde{v}X}\rho Y, \rho Z) = \{vX \cdot g(\rho Y, \rho Z) + g(\rho Y, \rho[hZ, vX]) + g(\rho Z, \rho[vX, hY])\}$$

$$+ \{LX \cdot g(\rho Y, \rho Z) + g(\rho Y, \rho[hZ, LX]) + g(\rho Z, \rho[LX, hY])\}$$

$$+ \{g(\nabla_{vX}\rho Y, \zeta)g(\rho Z, \zeta) + g(\rho Y, \zeta)g(\nabla_{vX}\rho Z, \zeta) + g(\rho Y, \zeta)g(\nabla_{vX}\rho Z, \zeta) + g(\rho Z, \zeta)g(\nabla_{vX}\rho Y, \zeta)\}$$

$$+ \{g(\nabla_{LX}\rho Y, \zeta)g(\rho Z, \zeta) + g(\rho Y, \zeta)g(\nabla_{LX}\rho Z, \zeta) + g(\rho Z, \zeta)g(\nabla_{LX}\rho Y, \zeta) + g(\rho Y, \zeta)g(\nabla_{LX}\rho Z, \zeta)\}.$$
Since \( T(LX, hY) = -\frac{g(X,Y)}{1+\rho^2} T(\zeta, \rho Y) = 0 \) and \( g(T(vX, hY), \zeta) = -g(T(KX, \zeta), \rho Y) = 0 \), the above relation takes the form

\[
2\tilde{g}(\nabla_{\tilde{\varphi}} X, \rho Z) = 2g(\nabla_{\tilde{\varphi}} Y, \rho Z) + 2g(\nabla_{LX} \rho Y, \rho Z)
\]

From which, setting \( Z = \beta \zeta \), we get

\[
g(\nabla_{\tilde{\varphi}} Y, \zeta) = 0.
\]

Then, Equations (5.5) and (5.6) imply that

\[
\tilde{\nabla}_{\tilde{\varphi}} Y = \nabla_{\varphi} Y.
\]

Similarly, by Lemma 5.1(b), Proposition 4.3 and Lemma 5.2 noting that \( T(\zeta, X) = T(X, \zeta) = 0 \), we get after long but easy calculations

\[
2\tilde{g}(\nabla_{\tilde{\varphi}} X, \rho Z) = 2g(\nabla_{\varphi} Y, \rho Z) + 2g(\nabla_{LX} \rho Y, \rho Z)
\]

Consequently,

\[
g(\nabla_{\tilde{\varphi}} Y, \zeta) = g(\nabla_{\varphi} Y, \zeta) = g(\nabla_{\varphi} Y, \zeta, \rho Z).
\]

From which, setting \( Z = \beta \zeta \), we get

\[
\tilde{g}(\nabla_{\tilde{\varphi}} Y, \zeta) = -\frac{g(\zeta, \rho Y)}{1 + g(\zeta, \zeta)}.
\]

Then, Equations (5.5) and (5.6) imply that

\[
\tilde{\nabla}_{\tilde{\varphi}} Y = \nabla_{\varphi} Y - \frac{g(\rho X, \rho Y)}{1 + g(\zeta, \zeta)} \zeta.
\]

Now, (5.4) follows from (5.4) and (5.7).

As a consequence of Theorem 5.3 and Definition 2.1, we have

**Corollary 5.4.** If a Finsler manifold admits a concurrent \( \pi \)-vector field \( \zeta \), then the vector field \( \zeta \) is no more concurrent with respect to the transformed metric (4.1).

**Corollary 5.5.** Under the energy \( \beta \)-change (4.1), we have:
(a) The maps $\mathcal{Y} \mapsto \nabla_{\mathcal{Y}} \mathcal{Y}$ and $\mathcal{Y} \mapsto \nabla_{\mathcal{Y}} \mathcal{X}$ are invariant.

(b) The (h)hv-torsion $\mathcal{T}$ and the Cartan tensor are invariant.

**Theorem 5.6.** Let $(M, L)$ and $(\tilde{M}, \tilde{L})$ be two Finsler manifolds related by the energy $\beta$-change \([4.1]\). The curvature tensors of the associated Cartan connections $\nabla$ and $\tilde{\nabla}$ are related by:

$$\tilde{\mathcal{K}}(X, Y)\mathcal{Z} = \mathcal{K}(X, Y)\mathcal{Z} + \frac{g(\mathcal{T}(X, Y), \mathcal{Z})}{1 + p^2} \zeta + H(\rho X, \rho Y)\mathcal{Z}, \quad (5.8)$$

where $H$ is the $\pi$-tensor field defined by

$$H(\rho X, \rho Y)\mathcal{Z} = \mathcal{U}_{X,Y} \left\{ \frac{g(\rho X, \mathcal{Z})}{1 + p^2} \rho Y + \frac{g(\rho Y, \mathcal{Z})}{(1 + p^2)^2} \rho \right\}. \quad (5.9)$$

In particular,

(a) $\tilde{\mathcal{S}}(X, Y)\mathcal{Z} = \mathcal{S}(X, Y)\mathcal{Z},$

(b) $\tilde{\mathcal{P}}(X, Y)\mathcal{Z} = \mathcal{P}(X, Y)\mathcal{Z} - \frac{g(\mathcal{T}(\mathcal{Y}, X), \mathcal{Z})}{1 + p^2} \zeta,$

(c) $\tilde{\mathcal{R}}(X, Y)\mathcal{Z} = \mathcal{R}(X, Y)\mathcal{Z} + H(X, Y)\mathcal{Z}.$

**Proof.** By Theorem 5.3, we have:

$$\tilde{\nabla}_X \tilde{\nabla}_Y \mathcal{Z} = \nabla_X \nabla_Y \mathcal{Z} - \frac{g(\rho X, \nabla_Y \mathcal{Z})}{1 + p^2} \zeta + \frac{g(\rho Y, \mathcal{Z})}{1 + p^2} \rho X - \frac{g(\rho Y, \mathcal{Z})}{(1 + p^2)^2} \zeta - \frac{g(\nabla_Y \rho X, \mathcal{Z})}{1 + p^2} \zeta - \frac{g(\rho Y, \nabla_X \mathcal{Z})}{1 + p^2} \zeta,$$

$$\tilde{\nabla}_{[X,Y]} \mathcal{Z} = \nabla_{[X,Y]} \mathcal{Z} - \frac{g(\nabla_X Y, \mathcal{Z})}{1 + p^2} \zeta + \frac{g(\nabla_Y \rho X, \mathcal{Z})}{1 + p^2} \zeta + \frac{g(\mathcal{T}(X, Y), \mathcal{Z})}{1 + p^2} \zeta.$$

The result follows from the above identities and the definition of $\tilde{\mathcal{K}}$. \hfill $\square$

In view of the above theorem, we have

**Corollary 5.7.** The $v$-curvature tensor $\mathcal{S}$ and the (v)hv-torsion tensor $\tilde{\mathcal{P}}$ are invariant under the energy $\beta$-change \([4.1]\).

Now, we turn our attention to the Chern (Rund) connection $D^\diamond$.

**Theorem 5.8.** \([19]\) Let $(M, L)$ be a Finsler manifold. The Chern connection $D^\diamond$ is expressed in terms of the Cartan connection as

$$D^\diamond_X \mathcal{Y} = \nabla_X \mathcal{Y} - T(KX, \mathcal{Y}), \quad \forall X \in \mathcal{X}(TM), \mathcal{Y} \in \mathcal{X}(\pi(M)).$$

In particular, we have:

(a) $D^\diamond_{\gamma X} \mathcal{Y} = \nabla_{\gamma X} \mathcal{Y} - T(\mathcal{X}, \mathcal{Y}).$

(b) $D^\diamond_{\beta X} \mathcal{Y} = \nabla_{\beta X} \mathcal{Y}.$
Lemma 5.9. If \( \zeta \in \mathcal{X}(\pi(M)) \) is a concurrent \( \pi \)-vector field, then we have
\[
D^\circ_{\mathcal{X}\zeta} \zeta = -\mathcal{X}, \quad D^\circ_{\gamma \mathcal{X}} \zeta = 0.
\] (5.10)

Proof. The proof follows from Definition [2.1] and the above theorem, taking into account the fact that \( T(\zeta, \mathcal{X}) = T(\mathcal{X}, \zeta) = 0 \), for all \( \mathcal{X} \in \mathcal{X}(\pi(M)) \).

Using the above Lemma, we get

Proposition 5.10. Let \( \zeta \in \mathcal{X}(\pi(M)) \) be a concurrent \( \pi \)-vector field on \((M, L)\).

The hv-curvature tensor \( P^\circ \) has the properties:

(a) \( P^\circ(\mathcal{X}, \mathcal{Y}) \zeta = P^\circ(\mathcal{X}, \zeta) \mathcal{Y} = 0 \).

(b) \( (D^\circ_{\mathcal{Z}} P^\circ)(\mathcal{X}, \mathcal{Y}, \zeta) = P(\mathcal{X}, \mathcal{Y}) \mathcal{Z} \).

(c) \( (D^\circ_{\mathcal{Z}} P^\circ)(\mathcal{X}, \mathcal{Y}, \zeta) = P^\circ(\mathcal{X}, \mathcal{Y}) \mathcal{Z} \).

(d) \( (D^\circ_{\mathcal{Z}} P^\circ)(\mathcal{X}, \mathcal{Y}, \zeta) = P^\circ(\mathcal{X}, \mathcal{Y}) \mathcal{Z} \).

The h-curvature tensor \( R^\circ \) has the properties:

(e) \( R^\circ(\mathcal{X}, \mathcal{Y}) \zeta = 0 \).

(f) \( (D^\circ_{\mathcal{Z}} R^\circ)(\mathcal{X}, \mathcal{Y}, \zeta) = 0 \).

(g) \( (D^\circ_{\mathcal{Z}} R^\circ)(\mathcal{X}, \mathcal{Y}, \zeta) = R^\circ(\mathcal{X}, \mathcal{Y}) \mathcal{Z} \).

(h) \( (D^\circ_{\mathcal{Z}} R^\circ)(\mathcal{X}, \mathcal{Y}, \zeta) = 0 \).

From Theorems 5.8 and 5.3, taking into account the fact that the (h)hv-torsion \( T \) is invariant under (4.1) and \( T(\zeta, \mathcal{X}) = T(\mathcal{X}, \zeta) = 0 \), we have

Theorem 5.11. Let \((M, L)\) and \((M, \tilde{L})\) be two Finsler manifolds related by the energy \( \beta \)-change (4.1). Then the associated Chern connections \( D^\circ \) and \( \tilde{D}^\circ \) are related by:
\[
\tilde{D}^\circ_{\mathcal{X}\rho Y} = D^\circ_{\mathcal{X}\rho Y} - \frac{g(\rho X, \rho Y)}{1 + p^2} \zeta.
\] (5.11)

In particular,

(a) \( \tilde{D}^\circ_{\mathcal{X}\rho Y} = D^\circ_{\mathcal{X}\rho Y} \),

(b) \( \tilde{D}^\circ_{\mathcal{X}\rho Y} = D^\circ_{\mathcal{X}\rho Y} - U^\circ(\mathcal{X}, \mathcal{Y}), \) with \( U^\circ(\mathcal{X}, \mathcal{Y}) = \frac{g(\mathcal{X}, \mathcal{Y})}{1 + p^2} \zeta - \frac{g(\mathcal{X}, \mathcal{Y})}{1 + p^2} D^\circ_{\mathcal{Z}} \mathcal{Y} \).

Concerning the curvature tensors of the Chern connection, we have

Theorem 5.12. Let \((M, L)\) and \((M, \tilde{L})\) be two Finsler manifolds related by the energy \( \beta \)-change (4.1). The curvature tensors of the associated Chern connections \( D^\circ \) and \( \tilde{D}^\circ \) are related by:
\[
\tilde{K}^\circ(\mathcal{X}, \mathcal{Y}) \mathcal{Z} = K^\circ(\mathcal{X}, \mathcal{Y}) \mathcal{Z} + \frac{g(\mathcal{X}, \mathcal{Y})}{1 + p^2} \zeta + H^\circ(\rho X, \rho Y) \mathcal{Z},
\]
where \( H^\circ \) is the \( \pi \)-tensor field defined by
\[
H^\circ(\rho X, \rho Y) \mathcal{Z} = \mathcal{U}_{\mathcal{X}, \mathcal{Y}} \left\{ \frac{g(\rho X, \mathcal{Z})}{1 + p^2} \rho Y - \frac{g(\rho X, T(\mathcal{X}, \mathcal{Z}))}{1 + p^2} \zeta + \frac{g(\mathcal{Y}, \mathcal{Z}) g(\rho X, \mathcal{Z})}{(1 + p^2)^2} \right\}.
\]

In particular,
(a) \( \tilde{S}^\circ(X, \overline{Y}) \overline{Z} = S^\circ(X, \overline{Y}) \overline{Z} = 0 \),

(b) \( \tilde{P}^\circ(X, \overline{Y}) \overline{Z} = P^\circ(X, \overline{Y}) \overline{Z} - 2 \frac{g(T(\overline{Y}, X), \overline{Z})}{1 + p^2} \zeta \),

(c) \( \tilde{R}^\circ(X, \overline{Y}) \overline{Z} = R^\circ(X, \overline{Y}) \overline{Z} + \mathcal{U}_{X \overline{Y}} \left\{ \frac{g(X, \overline{Z})}{1 + p^2} \overline{Y} + \frac{g(Y, \overline{Z})g(X, \zeta)}{(1 + p^2)^2} \right\} \).

Now, we investigate the effect of the energy \( \beta \)-change \((4.1)\) on the Hashiguchi connection \( D^* \).

**Theorem 5.13.** [19] Let \((M, L)\) be a Finsler manifold. The Hashiguchi connection \( D^* \) is expressed in terms of the Cartan connection \( \nabla \) as

\[
D^*_X \overline{Y} = \nabla_X \overline{Y} + \hat{P}(\rho X, \overline{X}), \quad \forall X \in \mathfrak{X}(TM), \overline{Y} \in \mathfrak{X}(\pi(M)).
\]

In particular, we have:

(a) \( D^*_\gamma \overline{X} \overline{Y} = \nabla^\gamma_X \overline{Y} \).

(b) \( D^*_\beta \overline{X} \overline{Y} = \nabla^\beta_X \overline{Y} + \hat{P}(X, \overline{Y}) \).

As \( \hat{P}(X, \zeta) = 0 \), we get

**Lemma 5.14.** If \( \overline{\zeta} \in \mathfrak{X}(\pi(M)) \) is a concurrent \( \pi \)-vector field, then it satisfies the following properties

\[
D^*_\beta \overline{X} \overline{Y} \overline{\zeta} = 0, \quad D^*_\gamma \overline{X} \overline{\zeta} = 0.
\]  \quad \text{(5.12)}

**Proposition 5.15.** Let \( \overline{\zeta} \in \mathfrak{X}(\pi(M)) \) be a concurrent \( \pi \)-vector field on \((M, L)\).

The v-curvature tensor \( S^* \) satisfies the properties:

(a) \( S^*(X, \overline{Y}) \overline{\zeta} = 0 \).

(b) \( (D^*_{\gamma \overline{X}} S^*) (X, \overline{Y}, \overline{\zeta}) = 0 \).

(c) \( (D^*_{\beta \overline{Z}} S^*) (X, \overline{Y}, \overline{\zeta}) = S^*(X, \overline{Y}) \overline{Z} \).

The hv-curvature tensor \( P^* \) satisfies the properties:

(d) \( P^*(X, \overline{Y}) \overline{\zeta} = -T(\overline{Y}, X) \), \( P^*(X, \overline{\zeta}) \overline{Y} = 0 \).

(e) \( (D^*_{\gamma \overline{Z}} P^*) (X, \overline{Y}, \overline{\zeta}) = -(D^*_{\gamma \overline{Z}} T)(\overline{Y}, X) \).

(f) \( (D^*_{\beta \overline{Z}} P^*) (X, \overline{Y}, \overline{\zeta}) = -(D^*_{\beta \overline{Z}} T)(\overline{Y}, X) + P^*(X, \overline{Y}) \overline{Z} \).

The h-curvature tensor \( R^* \) satisfies the properties:

(g) \( R^*(X, \overline{Y}) \overline{\zeta} = 0 \).

(h) \( (D^*_{\gamma \overline{Z}} R^*) (X, \overline{Y}, \overline{\zeta}) = 0 \).

(i) \( (D^*_{\beta \overline{Z}} R^*) (X, \overline{Y}, \overline{\zeta}) = R^*(X, \overline{Y}) \overline{Z} \).
From Theorem 5.13 and Theorem 5.3, taking into account the fact that the (v)hv-torsion tensor $\tilde{P}$ is invariant and $\tilde{P}(\zeta, X) = \tilde{P}(X, \zeta) = 0$, we have

**Theorem 5.16.** Let $(M, L)$ and $(M, \tilde{L})$ be two Finsler manifolds related by the energy $\beta$-change (4.1). Then the associated Hashiguchi connections $D^* \beta$ and $\tilde{D}^* \beta$ are related by:

$$\tilde{D}^* \beta X \rho Y = D^* X \rho Y - \frac{g(\rho X, \rho Y)}{1 + p^2} \zeta.$$  

(5.13)

In particular,

(a) $\tilde{D}^* X Y = D^* X Y$,

(b) $\tilde{D}^* \beta X Y = D^* \beta X Y - U^*(X, Y)$, with $U^*(X, Y) = \frac{g(X, Y)}{1 + p^2} - \frac{g(X, \zeta)}{1 + p^2} D^* \gamma X$.

**Theorem 5.17.** Let $(M, L)$ and $(M, \tilde{L})$ be two Finsler manifolds related by the energy $\beta$-change (4.1). The curvature tensors of the associated Hashiguchi connections $D^* \beta$ and $\tilde{D}^* \beta$ are related by:

$$\tilde{K}^* (X, Y) Z = K^* (X, Y) Z + \frac{g(T(X, Y), Z)}{1 + p^2} \zeta + H^* (\rho X, \rho Y) Z,$$

where $H^*$ is the $\pi$-tensor field defined by

$$H^* (\rho X, \rho Y) Z = \zeta \left\{ \frac{g(\rho X, Z)}{1 + p^2} \rho Y + \frac{g(\rho Y, Z)g(\rho X, \zeta)}{(1 + p^2)^2} \zeta \right\}.$$

In particular,

(a) $\tilde{S}^*(\bar{X}, \bar{Y}) \bar{Z} = S^*(\bar{X}, \bar{Y}) \bar{Z},$

(b) $\tilde{P}^*(\bar{X}, \bar{Y}) \bar{Z} = P^*(\bar{X}, \bar{Y}) \bar{Z} - \frac{g(T(\bar{Y}, \bar{X}), \bar{Z})}{1 + p^2} \zeta,$

(c) $\tilde{R}^*(\bar{X}, \bar{Y}) \bar{Z} = R^*(\bar{X}, \bar{Y}) \bar{Z} + H^* (\bar{X}, \bar{Y}) \bar{Z}$.

We terminate our study by the fourth fundamental connection in Finsler geometry, namely the Berwald connection $D^\circ$.

By Theorem 1.6, as $P(\bar{X}, \bar{\zeta}) = T(\bar{X}, \bar{\zeta}) = 0$, we have

**Lemma 5.18.** If $\bar{\zeta} \in \mathfrak{X}(\pi(M))$ is a concurrent $\pi$-vector field, then

$$D^\circ \beta \bar{X} \bar{\zeta} = -\bar{X}, \quad D^\circ \gamma \bar{X} \bar{\zeta} = 0.$$  

(5.14)

**Proposition 5.19.** Let $\bar{\zeta} \in \mathfrak{X}(\pi(M))$ be a concurrent $\pi$-vector field on $(M, L)$. The hv-curvature tensor $P^\circ$ has the properties:

(a) $P^\circ(\bar{X}, \bar{Y}) \bar{\zeta} = P^\circ(\bar{X}, \bar{\zeta}) \bar{Y} = 0$.

(b) $(D^\circ_{\gamma} P^\circ)(\bar{X}, \bar{Y}, \bar{\zeta}) = 0$.

(c) $(D^\circ_{\beta} P^\circ)(\bar{X}, \bar{Y}, \bar{\zeta}) = P^\circ(\bar{X}, \bar{Y}) \bar{Z}$.

The h-curvature tensor $R^\circ$ has the properties:


(d) \( R^o(\overline{X}, \overline{Y}) \overline{\zeta} = 0. \)

(e) \( (D^o_\gamma R^o)(\overline{X}, \overline{Y}, \overline{\zeta}) = 0. \)

(f) \( (D^o_\beta R^o)(\overline{X}, \overline{Y}, \overline{\zeta}) = R^o(\overline{X}, \overline{Y}) \overline{Z}. \)

From Theorem 1.6 and Theorem 5.3, taking into account that the torsion tensors \( T \) and \( \hat{P} \) are invariant under (4.1), \( \hat{P}(\overline{\zeta}, \overline{X}) = \hat{P}(\overline{X}, \overline{\zeta}) = 0 \) and \( T(\overline{\zeta}, \overline{X}) = T(\overline{X}, \overline{\zeta}) = 0 \), we obtain

**Theorem 5.20.** Let \((M, L)\) and \((M, \tilde{L})\) be two Finsler manifolds related by the energy \( \beta \)-change (4.1). Then the associated Berwald connections \( D^o \) and \( \tilde{D}^o \) are related by:

\[
\tilde{D}^o_{\gamma X} Y = D^o_{\gamma X} Y - g(\rho X, \rho Y) \frac{1}{1 + p^2} \overline{\zeta}.
\]

In particular,

(a) \( \tilde{D}^o_{\gamma X} \overline{Y} = D^o_{\gamma X} \overline{Y}, \)

(b) \( \tilde{D}^o_{\beta X} \overline{Y} = D^o_{\beta X} \overline{Y} - U^o(\overline{X}, \overline{Y}), \) with \( U^o(\overline{X}, \overline{Y}) = g(\overline{X}, \overline{Y}) \frac{1}{1 + p^2} \overline{\zeta} - g(\overline{X}, \overline{\zeta}) \frac{1}{1 + p^2} D^o_{\gamma X} \overline{Y}. \)

**Theorem 5.21.** Let \((M, L)\) and \((M, \tilde{L})\) be two Finsler manifolds related by the energy \( \beta \)-change (4.1). The curvature tensors of the associated Berwald connections \( D^o \) and \( \tilde{D}^o \) are related by:

\[
\tilde{K}^o(X,Y)Z = K^o(X,Y)Z + \frac{g(T(Y,X),Z)}{1 + p^2} \zeta + H^o(\rho X, \rho Y)Z,
\]

where \( H^o \) is the \( \pi \)-tensor field defined by

\[
H^o(\rho X, \rho Y)Z = \mathfrak{U}_{X,Y} \left\{ \frac{g(\rho X, Z)}{1 + p^2} \rho Y - \frac{g(\rho X, T(Y,X),Z)}{1 + p^2} \zeta + \frac{g(\rho Y, Z)g(\rho X, \zeta)}{(1 + p^2)^2} \zeta \right\}.
\]

In particular,

(a) \( \tilde{S}^o(\overline{X}, \overline{Y})Z = S^o(\overline{X}, \overline{Y})Z = 0, \)

(b) \( \tilde{P}^o(\overline{X}, \overline{Y})Z = P^o(\overline{X}, \overline{Y})Z - 2 \frac{g(T(\overline{Y}, \overline{X}),Z)}{1 + p^2} \zeta, \)

(c) \( \tilde{R}^o(\overline{X}, \overline{Y})Z = R^o(\overline{X}, \overline{Y})Z + \mathfrak{U}_{X,Y} \left\{ \frac{g(\overline{X}, Z)}{1 + p^2} \overline{Y} + \frac{g(\overline{Y}, Z)g(\overline{X}, \zeta)}{(1 + p^2)^2} \zeta \right\}. \)

**Concluding remarks**

On the present work, we have the following comments and remarks:
• The fundamental $\pi$-vector field $\eta$ and a concurrent $\pi$-vector field $\zeta$ have different properties, namely
\[
\nabla_\gamma X \eta = X, \quad \nabla_\gamma X \zeta = 0, \\
\nabla_\beta X \eta = 0, \quad \nabla_\beta X \zeta = -X.
\]
Moreover, an important difference is that $\eta$ is dependent on the directional argument $y$, whereas $\zeta$ is independent of the directional argument $y$ as has been proved.

• Although there are differences between the fundamental $\pi$-vector field $\eta$ and a concurrent $\pi$-vector field $\zeta$, the $(h)hv$-torsion tensor $T$ of $\nabla$ has the common properties:
\[
T(\overline{X}, \eta) = T(\eta, \overline{X}) = 0; \quad T(\overline{X}, \zeta) = T(\zeta, \overline{X}) = 0.
\]
Moreover, the $(v)hv$-torsion tensor $\hat{P}$ of $\nabla$ has the properties:
\[
\hat{P}(\overline{X}, \eta) = \hat{P}(\eta, \overline{X}) = 0; \quad \hat{P}(\overline{X}, \zeta) = \hat{P}(\zeta, \overline{X}) = 0.
\]

• Special Finsler spaces play an important role in Finsler geometry. For this reason, we studied the effect of the existence of a concurrent $\pi$-vector field on some important special Finsler spaces. An interesting result obtained is that many of these special Finsler spaces admitting a concurrent $\pi$-vector field are equivalent to a Riemannian space.

• Randers spaces, which are obtained by the $\beta$-change $\tilde{L}(x, y) = L(x, y) + B(x, y)$, where $B := b_i(x) y^i$, are very important in physical applications [11]. In this paper, we investigate intrinsically an energy $\beta$-change (similar in form to a Randers change),
\[
\tilde{L}^2(x, y) = L^2(x, y) + B^2(x, y),
\]
where $B := g(\overline{\zeta}, \overline{\eta})$, $\overline{\zeta}$ being a concurrent $\pi$-vector field. Under this change, the torsion tensors $T$ and $\hat{P}$ are invariant, $\hat{P}(\overline{X}, \overline{\zeta}) = \hat{P}(\overline{\zeta}, \overline{X}) = 0$ and $T(\overline{\zeta}, \overline{X}) = T(\overline{X}, \overline{\zeta}) = 0$. Consequently, the difference tensors of the fundamental Finsler connections, namely, the Cartan connection, the Berwald connection, the Chern connection and the Hashiguchi connection have not only simple but also similar forms.

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