Implicit parametrizations in shape optimization: boundary observation

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Abstract

We present first a brief review of the existing literature on shape optimization, stressing the recent use of Hamiltonian systems in topology optimization. In the second section, we collect some preliminaries on the implicit parametrization theorem, especially in dimension two, which is a case of interest in shape optimization. The formulation of the problem is also discussed. The approximation via penalization and its differentiability properties are analyzed in Section 3. Next, we investigate the discretization process in Section 4. The last section is devoted to numerical experiments.

Key Words: Hamiltonian systems, implicit parametrizations, shape optimization, optimal control, boundary observation, boundary and topological variations

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1 Introduction

Geometric optimization problems have a very long history (we mention just the Dido’s problem, almost three thousands years old, Kline [12]), but shape optimization problems are a relatively young development of the calculus of variations. There exist already some very good monographs, Pironneau [23], Haslinger and Neittaanmäki [9], Sokolowski and Zolesio [27], Delfour and Zolesio [4], Neittaanmäki, Sprekels and Tiba [19], Bucur and Buttazzo [2], Henrot and Pierre [10], devoted to this subject. In general, just certain
types of boundary variations for the unknown domains, are taken into account. The well
known level set method, [22], [21], [1], [13], investigates topological optimization ques-
tions as well, both from the theoretical and numerical points of view. We underline that
our approach combines boundary and topological variations and is essentially different
from the level set method, although level functions are used (for instance the Hamilton-
Jacobi equation is not necessary here - we just use ordinary differential Hamiltonian
systems, etc.).

A typical example of shape optimization problem, defined on a given family \( \mathcal{O} \) of
bounded domains \( \Omega \in \mathcal{O}, \Omega \subset D \subset \mathbb{R}^d \), looks as follows:

\[
\begin{align*}
\min_{\Omega \in \mathcal{O}} & \int_{\Lambda} j(x, y_{\Omega}(x)) \, dx, \\
-\Delta y_{\Omega} & = f \text{ in } \Omega, \\
y_{\Omega} & = 0 \text{ on } \partial \Omega.
\end{align*}
\]

Other boundary conditions, other differential operators or cost functionals may be as well
considered in (1.1)-(1.3). Supplementary constraints on \( \Omega \) or \( y_{\Omega} \) may be also imposed.

Above, \( \Lambda \) may be \( \Omega \) or some part of \( \Omega \), or it may be \( \partial \Omega \) or some part of \( \partial \Omega \). The
functional \( j(\cdot, \cdot) : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is Carathéodory, \( f \in L^p(D), p > 2 \). The cost may
also depend on \( \nabla y_{\Omega} \) in certain situations. Regularity assumption on \( \Omega \in \mathcal{O} \), other
assumptions, will be imposed in the sequel, when necessity appears.

Shape optimization problems (1.1)-(1.3) have a similar structure with an optimal
control problem, but the minimization parameter is the domain \( \Omega \) itself, where the
problem is defined.

In optimal control theory, boundary observation is an important and realistic case
and this paper is devoted to the study of boundary cost functionals in optimal design
theory. Special cases of this type have been already considered by Pironneau [23],
Haslinger and Neittaanmäki [9], Sokolowski and Zolesio [27].

The recent implicit parametrization approach, using Hamiltonian systems develop-
oped by Tiba [29], [30], Nicolai and Tiba [20] offers a new way of handling effectively
boundary cost integrals and clarifies regularity questions, allowing developments up to
numerical experiments. Related results can be found in Tiba [33], [32], [15], where the
employed methodology is based on the penalization of the Dirichlet problem, but also
uses the representation of the unknown geometry via Hamiltonian systems. The family
of unknown admissible domains is very general and the functional variations introduced
in [17], [18] allow simultaneous topological and boundary variations. This method is
of fixed domain type and avoids drawbacks like remeshing and recomputing the mass
matrix, in each iteration. In fact, in [14], again for Dirichlet boundary conditions and
distributed cost, we have put together all these developments and obtained a com-
plete approximation technique with the potential to solve general shape optimization
problems (general cost functionals, general boundary conditions, various differential op-
erators, including parabolic operators as well, etc.). We continue in this paper with the
case of boundary observation and we show that the new approach, with certain natural modifications and adaptations, gives good results too. Notice that such ideas are also applicable in free boundary problems, for instance for fluid-structure interaction [6], [7]. Other applications are in optimization and optimal control [31].

In the next section, we collect some preliminaries on the implicit parametrization theorem, especially in dimension $d = 2$, which is a case of interest in shape optimization. The formulation of the problem is also discussed. The approximation via penalization and its differentiability properties are analyzed in Section 3. Next, we investigate the discretization process in Section 4. The last section is devoted to numerical experiments.

2 Preliminaries and problem formulation

In this paper, we fix our attention on the problem $(\mathcal{P})$:

$$
\min_{\Omega \in \mathcal{O}} \int_{\partial \Omega} j(x, \nabla y_\Omega(x)) \, d\sigma,
$$

subject to (1.2)-(1.3) and with $j : D \times \mathbb{R}^2 \to \mathbb{R}$, a Carathéodory mapping. The dependence of $j$ on $y_\Omega$ is not necessary here since $y_\Omega = 0$ on $\partial \Omega$. A classical example is the normal derivative $j(x, \nabla y_\Omega(x)) = |\partial y_\Omega / \partial n|^2$.

According to the functional variations approach, introduced in [17], [18], we consider that the family $\mathcal{O}$ of admissible domains given in (2.1), is defined starting from a family of admissible function $\mathcal{F} \subset C(D)$ (where $D$ is a bounded domain in $\mathbb{R}^2$) via the relation:

$$
\Omega = \Omega_g = \text{int} \{ x \in D ; \, g(x) \leq 0 \}, \quad g \in \mathcal{F}.
$$

While relation (2.2) defines a family of open sets (not necessarily connected), by imposing further natural geometric constraints, relation (2.2) defines a family of domains. One example is the selection of the connected component containing $E$

$$
E \subset \Omega, \quad \forall \Omega \in \mathcal{O},
$$

where $E$ is a given subdomain such that $\overline{E} \subset D$. In the formulation (2.2), inclusion (2.3) is expressed as

$$
g(x) \leq 0, \quad \forall x \in \overline{E}.
$$

Another example is the selection of the connected component via $x^0 \in \partial \Omega$, for any $\Omega$ in $\mathcal{O}$. This can be reformulated as

$$
g(x^0) = 0, \quad \forall g \in \mathcal{F}
$$

if $\mathcal{F} \subset C^1(D)$ and satisfies the following conditions (according to [31]):

$$
g(x) > 0, \quad \text{on } \partial D, \quad \nabla g(x) > 0, \quad \text{on } \mathcal{G} = \{ x \in D ; \, g(x) = 0 \}.
$$
This is due to the implicit functions theorem applied to the equation \( g(x) = 0 \), around \( x^0 \) from (2.5). By (2.6), (2.7), we get that \( \Omega_g \cap \partial D = \emptyset \) for any \( g \in \mathcal{F} \) and (2.2) can be equivalently expressed as

\[
\Omega_g = \{ x \in D; \ g(x) < 0 \}, \ g \in \mathcal{F}.
\] (2.8)

Similarly, if we want that a given manifold \( C \subset D \) is contained in \( \partial \Omega_g \) for any \( g \in \mathcal{F} \), then we impose

\[
g(x) = 0, \ x \in C, \ g \in \mathcal{F}.
\] (2.9)

We notice that the family \( \mathcal{F} \) is very large and very flexible in imposing various geometric constraints on the admissible domains \( \mathcal{O} \), via simple conditions on \( \mathcal{F} \). It includes, for instance, multimodal functions of class \( \mathcal{C}^1(D) \) that may have unbounded many extremal points in \( D \). Moreover, the obtained domains \( \Omega_g \) are connected but not simply connected, in general. Consequently, our approach, allows topological optimization and performs, in fact, simultaneous topological and boundary variations, which is a characteristic of functional variations \[17], \[18]. We ask that \( D \subset \mathbb{R}^2 \), which is an important case in shape optimization. This restriction is due to the use of Poincaré-Bendixson type arguments, in some of the following results (see Hirsch, Smale and Devaney \[14\], Ch. 10 or Pontryagin \[24\]).

**Proposition 2.1** (Tiba \[31\]) If \( D \subset \mathbb{R}^2, \mathcal{F} \subset \mathcal{C}^2(D) \) and assumptions (2.6), (2.7) are valid, then \( \mathcal{G} = \{ x \in D; \ g(x) = 0 \} \) is a finite union of disjoint closed curves of class \( \mathcal{C}^2 \), without self intersections, and not intersecting \( \partial D \). They are parametrized by the solution of the Hamiltonian system:

\[
x_1'(t) = -\frac{\partial g}{\partial x_2} (x_1(t), x_2(t)), \ t \in I, \quad (2.10)
\]
\[
x_2'(t) = \frac{\partial g}{\partial x_1} (x_1(t), x_2(t)), \ t \in I, \quad (2.11)
\]
\[
(x_1(0), x_2(0)) = x^0 = (x^0_1, x^0_2) \in D, \quad (2.12)
\]

where some \( x^0 \) is chosen on each component of \( \mathcal{G} \).

Here, the constraint (2.4) is not necessarily valid and \( \Omega_g \) from (2.8) is a finite union of domains, that may be multiplied connected. The existence interval \( I \) from (2.10)-(2.12) may be taken \( I = \mathbb{R} \) or just the corresponding period (the solutions of (2.10)-(2.12) are periodic - this is the consequence of the Poincaré-Bendixson result and hypotheses (2.6), (2.7)). In higher dimension, iterated Hamiltonian systems have to be used and their solution may be just a local one, Tiba \[30\]. This is the case of the implicit parametrization method, a recent extension of the implicit function theorem.

Consider now another mapping \( h \in \mathcal{C}^2(D) \) and satisfying (2.6), (2.7). We define the functional perturbation \( g + \lambda h, \lambda \in \mathbb{R} \) “small”, such that (2.6), (2.7) are still satisfied by \( g + \lambda h \), due to some simple argument based on the Weierstrass theorem.
Proposition 2.2 (Tiba [31]) If $\epsilon > 0$ is small enough, there is $\lambda(\epsilon) > 0$ such that, for $\lambda \in \mathbb{R}$, $|\lambda| < \lambda(\epsilon)$, we have that $G_\lambda$ is included in $V_\epsilon$ and $G_\lambda$ is a finite union of $C^2$ curves.

Here

$$G_\lambda = \{ x \in D; (g + \lambda h)(x) = 0 \},$$
$$V_\epsilon = \{ x \in D; d[x, G] < \epsilon \}$$

with $d[x, G]$ being the distance between a point and $G$. In particular, Proposition 2.2 shows that $G_\lambda \to G$ in the Hausdorff-Pompeiu sense, Neittaanmäki et al. [19], Appendix 3.

Proposition 2.3 (Murea and Tiba [14]) Denote by $T_g$, $T_\lambda$ the periods of the Hamiltonian system (2.10)-(2.12), respectively the perturbed Hamiltonian system. Then $T_\lambda \to T_g$ as $\lambda \to 0$.

Remark 2.1 A discussion of the dependence of the period $T_g$ with respect to certain perturbations can be found in Teschl [28], Ch. 12. In general, the perturbation of a periodic system may not be periodic and the approximation properties have an asymptotic character, Sideris [26]. In [10], we have established that the period $T_g$ has even differentiability properties with respect to functional variations and this will be used in the next Section.

3 Approximation and differentiability

We shall use a variant of the penalization method from Tiba [31], that has good differentiability properties as well. The main new ingredient in this approach is that we penalize directly the cost functional and not the state equation as in [33], [32], [15]. This appears as the application of classical optimization techniques and its advantage is the possibility to extend it to any boundary conditions. We underline that the Hamiltonian handling of the unknown geometries plays an essential role in the formulation below.

The penalized optimization problem is given by

$$\min_{g,u} \int_{I_g} \left[ j(z_g(t), \nabla y(z_g(t))) + \frac{1}{\epsilon} (y(z_g(t)))^2 \right] |z'_g(t)| dt \quad (3.1)$$

$$-\Delta y = f + g^2 u, \quad \text{in } D, \quad (3.2)$$
$$y = 0, \quad \text{on } \partial D, \quad (3.3)$$
$$g(x) \leq 0, \quad \text{on } \overline{E} \subset D, \quad \text{given} \quad (3.4)$$
where \( z_g : I_g \to D, z_g \in (C^1(I_g))^2 \) is the solution of the Hamiltonian system (2.10)-(2.12) associated to \( g \in \mathcal{F} \) and \( I_g = [0, T_g] \) is its period. In case \( \partial \Omega_g \) has several components (their number is finite according to Section 2), then the penalization part in the functional (3.1) has to be understood as a finite sum of terms corresponding to each component. Notice that the corresponding periods and the initial conditions (2.12) can be obtained via standard numerical methods in the examples, see Remark 4.1.

The minimization is performed over \( g \in \mathcal{F} \), satisfying (3.4), (2.6), (2.7) and \( u \) measurable such that \( g^2 u \in L^p(D), p > 2 \). It is possible that the original cost (2.1) (the first term in (3.1)) is defined just on one component of \( \partial \Omega_g \) and this can be singled out by a condition like (2.5) and a corresponding given \( x^0 \notin E \). However the penalization term in (3.1) has to be defined on all the components of \( \partial \Omega_g \) since it controls in fact the Dirichlet condition (1.3). For simplicity, we shall not investigate such details here, related to (3.1).

If \( \partial D \) is in \( C^{1,1} \), then the state \( y \in W^{2,p}(D) \cap H^2_0(D) \), due to (3.2), (3.3). Consequently \( y \in C^1(\overline{D}) \). Then, the cost functionals (2.1), (3.1) make sense since \( \nabla y \) is continuous in \( \overline{D} \) and similar regularity properties are valid on \( \Omega_g \) under the assumptions on \( g \in \mathcal{F} \).

**Proposition 3.1** Let \( j(\cdot, \cdot) \) be a Carathéodory function on \( D \times \mathbb{R}^2 \), bounded by a constant from below. Let \( [y_n^g, u_n^g] \) be a minimizing sequence in the penalized problem (3.1)-(3.4), for some given \( \epsilon > 0 \). Then, on a subsequence denoted by \( n(m) \) the (not necessarily admissible) pairs \( [\Omega_{g_n(m)}, y_n^g(m)] \) give a minimizing cost in (2.1), satisfy (1.2) in \( \Omega_{g_n(m)} \) and (1.3) is fulfilled with a perturbation of order \( \epsilon^{1/2} \) on \( \partial \Omega_{g_n(m)} \).

**Proof.** Let \( [y_{g_n}, g_n] \in W^{2,p}(\Omega_{g_n}) \times \mathcal{F} \) be a minimizing sequence in the problem (2.1), (1.2), (1.3), (3.4) where \( \Omega = \Omega_g \) is defined by (2.8) and \( g \) satisfies \( g^2 u \in L^p(D) \). By Proposition 2.1 \( \partial \Omega_g \) is of class \( C^2 \) and this ensures the regularity for (1.2), (1.3) since \( f \in L^p(D) \).

Take \( \overline{y}_{g_n} \in W^{2,p}(D \setminus \overline{\Omega}_{g_n}) \), not unique, given by the trace theorem such that \( \overline{y}_{g_n} = y_{g_n} \) on \( \partial \Omega_{g_n} \), \( \frac{\partial y_{g_n}}{\partial \mathbf{n}} = \frac{\partial \overline{y}_{g_n}}{\partial \mathbf{n}} \) on \( \partial \Omega_{g_n} \), \( \overline{y}_{g_n} = 0 \) on \( \partial D \). We define an admissible control \( u_{g_n} \) in (3.2) by

\[
 u_{g_n} = -\frac{\Delta \overline{y}_{g_n} + f}{(g_n)^2_+}, \quad \text{in } D \setminus \overline{\Omega}_{g_n}
\]

and zero otherwise. It yields \( (g_n)^2 u_{g_n} \in L^p(D) \) and this control pair is admissible for the problem (3.1)-(3.4). Moreover, the corresponding state \( \overline{y}_{g_n} \) in (3.2)-(3.3) is obtained by concatenation of \( y_{g_n} \) and \( \overline{y}_{g_n} \) and the associated penalization term in (3.1) is null, due to (1.3).

We get the inequality:

\[
\int_{I_{g_n}(m)} j \left( z_{\theta_n^g(m)}, \nabla y_{n}^g(m) (z_{\theta_n^g(m)}) \right) + \frac{1}{\epsilon} \left( y_{n}^g(m) (z_{\theta_n^g(m)}) \right)^2 + \frac{\epsilon}{2} |z_{\theta_n^g(m)}| dt \leq \int_{\partial \Omega_{g_n}} j(x, \nabla y_{g_n}(x)) d\sigma \to \inf(\mathcal{P}),
\]
for \( n(m) \) big enough, due to the minimizing property of the sequence \([y_{n}, g_{n}, u_{n}']\) respectively \([y_{n}, g_{n}, g_{n}']\), By (3.6) we infer

\[
\int_{\partial \Omega} (y'_{n(m)})^2 d\sigma \leq C\epsilon
\]  

with \( C \) a constant independent of \( \epsilon, m \) since \( j \) is bounded below by a constant. Relation (3.7) proves the last statement in the proposition. As \((g_{n(m)}')_+ \) is null in \( \Omega_{g_{n(m)}} \), we see that (1.2) is satisfied here, due to (3.2). The minimizing property with respect to the original cost (2.1) is a clear consequence of (3.6). □

Remark 3.1 In [31], [16] a detailed study of the approximating properties with respect to \( \epsilon \to 0 \), is performed in related problems.

We consider now \([u, g] \in L^p(D) \times \mathcal{F}, p > 2 \) satisfying (3.4), (2.5) together with perturbations \([u + \lambda v, g + \lambda r], \lambda \in \mathbb{R}, v \in L^p(D) \) such that (3.4), (2.5) are satisfied by \( r \in \mathcal{F} \). The state system is, in fact, given by (3.2), (3.3), (2.10)-(2.12) and the corresponding perturbed system has solutions \( y^\lambda, z_{g+\lambda r} \). We study its differentiability properties.

**Proposition 3.2** The system in variations corresponding to (3.2), (3.3), (2.10)-(2.12) is:

\[
-\Delta q = g^2 u + 2g u r, \quad \text{in } D, \\
q = 0, \quad \text{on } \partial D, \\
w_1' = -\nabla \partial_2 g(z_g) \cdot w - \partial_2 r(z_g), \quad \text{in } I_g, \\
w_2' = \nabla \partial_1 g(z_g) \cdot w + \partial_1 r(z_g), \quad \text{in } I_g, \\
w_1(0) = 0, \quad w_2(0) = 0,
\]  

where \( q = \lim_{\lambda \to 0} \frac{y^\lambda - y}{\lambda} \), \( w = [w_1, w_2] = \lim_{\lambda \to 0} \frac{z_{g+\lambda r} - z_g}{\lambda} \) with \( y^\lambda \in W^{2,p}(D) \cap H^1_0(D) \) being the solution of (3.2), (3.3) corresponding to \( g + \lambda r, u + \lambda v \) and “.” is the scalar product in \( \mathbb{R}^2 \). The limits exist in the spaces of \( y, z_g \), respectively.

**Proof.** Subtracting the equations of \( y^\lambda \) (i.e. (3.2), (3.3) with perturbed controls) and \( y \), we get

\[
-\Delta \frac{y^\lambda - y}{\lambda} = \frac{1}{\lambda} \left[ (g + \lambda r)^2 (u + \lambda v) - g^2 u \right], \quad \text{in } D,
\]  

with zero boundary conditions on \( \partial D \). A standard passage to the limit in (3.13), gives (3.8), (3.9).

For (3.10)-(3.12), the argument is similar as in Proposition 6, Tiba [29]. The convergence is in \( C^1(I_g) \) on the whole sequence \( \lambda \to 0 \) due to the uniqueness property for the
linear systems \((3.8) - (3.12)\) and the periodicity of the solutions \(z_g, z_{g+\lambda r}\) by Proposition 2.1. □

We assume now that \(j(x, \cdot)\) is \(C^1(\mathbb{R}^2)\), \(j(x^0, \cdot) \equiv 0\) and \(f \in W^{1,p}(D), \partial D\) is in \(C^{2,1}\). Notice that by imposing \(F \subset C^2(\overline{D})\), we get that \(g_{x}^2 \in W^{1,\infty}(D)\) and \(g_{x}^2 u \in W^{1,p}(D)\) if \(u \in W^{1,p}(D)\).

**Proposition 3.3** Under the above hypotheses, if \(y(x^0) = 0\), then the directional derivative of the penalized cost \((3.1)\) in the direction \([v, r] \in W^{1,p}(D) \times F\) is given by:

\[
\begin{align*}
&\int_{I_g} \nabla_1 j(z_g(t), \nabla y(z_g(t))) \cdot w(t) |z'_g(t)| dt \\
+ &\int_{I_g} \nabla_2 j(z_g(t), \nabla y(z_g(t))) \cdot H(y(z_g(t))) \cdot w(t) |z'_g(t)| dt \\
+ &\int_{I_g} \nabla_2 j(z_g(t), \nabla y(z_g(t))) \cdot \nabla q(z_g(t)) |z'_g(t)| dt \\
+ &\frac{2}{\epsilon} \int_{I_g} y(z_g(t)) [\nabla y(z_g(t)) \cdot w(t) + q(z_g(t))] |z'_g(t)| dt \\
+ &\int_{I_g} \left[ j(z_g(t), \nabla y(z_g(t))) + \frac{1}{\epsilon} (y(z_g(t)))^2 \right] \frac{z'_g(t) \cdot w'(t)}{|z'_g(t)|} dt.
\end{align*}
\]  

(3.14)

The notations are explained in the proof.

**Proof.** We compute

\[
\lim_{\lambda \to 0} \frac{1}{\lambda} \left\{ \int_{I_{g+\lambda r}} j(z_{g+\lambda r}(t), \nabla y^\lambda(z_{g+\lambda r}(t))) + \frac{1}{\epsilon} (y^\lambda(z_{g+\lambda r}(t)))^2 \right\} |z'_{g+\lambda r}(t)| dt \\
- \int_{I_g} \left[ j(z_g(t), \nabla y(z_g(t))) + \frac{1}{\epsilon} (y(z_g(t)))^2 \right] |z'_g(t)| dt,
\]

(3.15)

where we use the notations from Proposition 3.2. The above assumptions on \(F, u, v\) ensure that \(y^\lambda, y \in W^{3,p}(D) \subset C^2(\overline{D})\), Grisvard [5], and \(y^\lambda \to y\) in \(C^2(\overline{D})\), \(z_{g+\lambda r} \to z_g\) in \(C^2(I_g)\).

We study first the term:

\[
\begin{align*}
\frac{1}{\lambda} \int_{I_g}^{T_{g+\lambda r}} j(z_{g+\lambda r}(t), \nabla y^\lambda(z_{g+\lambda r}(t))) + \frac{1}{\epsilon} (y^\lambda(z_{g+\lambda r}(t)))^2 \right\} |z'_{g+\lambda r}(t)| dt \\
= \frac{T_{g+\lambda r} - T_g}{\lambda} \left[ j(z_{g+\lambda r}(\tau), \nabla y^\lambda(z_{g+\lambda r}(\tau))) + \frac{1}{\epsilon} (y^\lambda(z_{g+\lambda r}(\tau)))^2 \right] |z'_{g+\lambda r}(\tau)| \\
\to 0
\end{align*}
\]  

(3.16)

due to the differentiability properties of \(T_g\) with respect to functional variations \(g + \lambda r\) (see [16]) and the convergence properties of \(y^\lambda, z_{g+\lambda r}\) and the regularity assumptions on
In 

\( r, v \rightarrow q \) given by (3.8), (3.9) and by \( B : C^2(D) \rightarrow C^1(I_g)^2 \) the linear continuous operator \( r \rightarrow w \). In these definitions, \( g \in C^2(D) \) and \( u \in W^{1,p}(D) \) are fixed.
Corollary 3.1 The relation (3.14) can be rewritten as:

\[
\begin{align*}
&\int_{I_g} \nabla j(z_g(t), \nabla y(z_g(t))) \cdot Br(z_g(t)) |z'_g(t)| dt \\
+ &\int_{I_g} \nabla 2j(z_g(t), \nabla y(z_g(t))) \cdot H(y(z_g(t))) \cdot Br(z_g(t)) |z'_g(t)| dt \\
+ &\int_{I_g} \nabla 2j(z_g(t), \nabla y(z_g(t))) \cdot \nabla A(r, v)(z_g(t)) |z'_g(t)| dt \\
+ &\frac{2}{\epsilon} \int_{I_g} y(z_g(t)) [\nabla y(z_g(t)) \cdot Br(z_g(t)) + A(r, v)(z_g(t))] |z'_g(t)| dt \\
+ &\int_{I_g} \left[ j(z_g(t), \nabla y(z_g(t))) + \frac{1}{\epsilon} (y(z_g(t)))^2 \right] \frac{z'_g(t)}{|z'_g(t)|} \cdot [-\partial_2 r, \partial_1 r](z_g(t)) dt \\
+ &\int_{I_g} C(t) \cdot Br(z_g(t)) dt.
\end{align*}
\] (3.17)

Here \( C(t) \) is a vector obtained by replacing \( w'(t) \) as expressed in (3.10), (3.11) and separating the part including \([-\partial_2 r, \partial_1 r]\).

Remark 3.2 The regularity hypotheses are natural and necessary when making variations of boundary integrals. The conditions \( j(x^0, \cdot) \equiv 0 \) can be obtained by a translation and \( y(x^0) = 0 \) reflects that \( x^0 \in \mathcal{G} \) and the admissible states in the original shape optimization problem are automatically null on \( \mathcal{G} \). It is possible to remove these two conditions (see [16]), but the relation (3.14) becomes more complex.

4 Finite element discretization

We assume that \( D \) is polygonal and let \( \mathcal{T}_h \) be a triangulation of \( D \) where \( h \) is the size of \( \mathcal{T}_h \). We introduce the linear space

\[
\mathbb{W}_h = \{ \varphi_h \in C(\overline{D}); \varphi_h|_T \in \mathbb{P}_3(T), \forall T \in \mathcal{T}_h \}
\]

where \( \mathbb{P}_3 \) is the piecewise cubic finite element. We use a standard basis of \( \mathbb{W}_h \), \( \{\phi_i\}_{i \in I} \), where \( I = \{1, \ldots, n\} \) and \( \phi_i \) is the hat function associated to the node \( A_i \), see for example [3], [25]. There are ten nodes for the cubic finite element on a triangle.

We can approach \( g \) and \( u \) by the finite element functions \( g_h = \sum_{i \in I} G_i \phi_i \) and \( u_h = \sum_{i \in I} U_i \phi_i \). We introduce the \( \mathbb{R}^n \) vectors \( G = \{G_i\}_{i \in I} \), \( U = \{U_i\}_{i \in I} \) and \( g_h \) can be identified by \( G \), etc. It is possible to use for \( u \) a low order finite element, like piecewise linear \( \mathbb{P}_1 \).

We also set \( \mathbb{V}_h = \{ \varphi_h \in \mathbb{W}_h; \varphi_h = 0 \text{ on } \partial D \} \).
\[ I_0 = \{ i \in I; \ A_i \notin \partial D \} \text{ where } n_0 = \text{card}(I_0) \text{ and the vector} \]
\[
F = (F_i)_{i \in I_0} = \left( \int_D f \phi_i \, dx \right)_{i \in I_0} \in \mathbb{R}^{n_0}.
\]

The discrete weak formulation of (3.2)-(3.3) is: find \( y_h \in \nabla_h \) such that
\[
\int_D \nabla y_h \cdot \nabla \varphi_h \, dx = \int_D (f_h + (g_h)^2 u_h) \varphi_h \, dx, \quad \forall \varphi_h \in \nabla_h.
\] (4.1)

The finite element approximations of \( y \) is \( y_h(x) = \sum_{j \in I_0} Y_j \phi_j(x) \) with \( Y_j = (Y_j)_{1 \leq j \leq n_0} \in \mathbb{R}^{n_0} \) and similarly for \( f, f_h, F \).

Let us define \( K \) the square matrix of order \( n_0 \) by
\[
K = (K_{ij})_{i,j \in I_0}, \quad K_{ij} = \int_D \nabla \phi_j \cdot \nabla \phi_i \, dx
\]
and the \( n_0 \times n \) matrix \( B^1(G) \) defined by
\[
B^1(G) = (B^1_{ij})_{i,j \in I}, \quad B^1_{ij} = \int_D (g_h)^2 \phi_j \phi_i \, dx.
\]

The matrix \( K \) is symmetric, positive definite and the linear system associated to the state system (3.2)-(3.3) is:
\[
KY = F + B^1(G)U. \tag{4.2}
\]

For the time step \( \Delta t > 0 \), the forward Euler scheme can be used:
\[
\begin{align*}
Z^1_{k+1} &= Z^1_k - \Delta t \frac{\partial g_h}{\partial x_2} (Z^1_k, Z^2_k), \tag{4.3} \\
Z^2_{k+1} &= Z^2_k + \Delta t \frac{\partial g_h}{\partial x_1} (Z^1_k, Z^2_k), \tag{4.4} \\
(Z^1_0, Z^2_0) &= (x^0_1, x^0_2). \tag{4.5}
\end{align*}
\]
for \( k = 0, 1, \ldots \), in order to solve numerically the ODE system (2.10)-(2.12). We set \( Z_k = (Z^1_k, Z^2_k) \), in fact, \( Z_k \) is an approximation of \( z_g(t_k) \), where \( t_k = k\Delta t, k \in \mathbb{N} \). When \( Z_m \), for some \( m \in \mathbb{N}^* \) is “close” to \( Z_0 \), we stop the algorithm and we set the computed period \( T_g = t_m \). We have the uniform partition \([t_0, \ldots, t_k, \ldots, t_m]\) of \([0, T_g]\). We denote \( Z = (Z^1, Z^2) \) in \( \mathbb{R}^m \times \mathbb{R}^m \), with \( Z^1 = (Z^1_k)_{1 \leq k \leq m} \) and \( Z^2 = (Z^2_k)_{1 \leq k \leq m} \). One can apply more efficient numerical methods, like explicit Runge-Kutta, however we use (4.3)-(4.5) for the sake of simplicity.

We define the function \( Z : [0, T_g] \to \mathbb{R}^2 \)
\[
Z(t) = \frac{t_{k+1} - t}{\Delta t} Z_k + \frac{t - t_k}{\Delta t} Z_{k+1}, \quad t_k < t \leq t_{k+1}
\]
for \( k = 0, 1, \ldots, m - 1 \). We remark that \( Z \) is derivable on each interval \((t_k, t_{k+1})\) and 
\[
Z'(t) = \frac{1}{\Delta t}(Z_{k+1}^j - Z_k^j, Z_{k+1}^j - Z_k^j) \text{ for } t_k < t \leq t_{k+1}.
\]
We define the \( n \times n \) matrix \( N(Z) \) as follow
\[
N(Z) = \left( \int_0^{T_y} \phi_j(Z(t)) \phi_i(Z(t)) |Z'(t)| \, dt \right)_{i \in I_0, j \in I_0}
\]
and, with this notation, the second term of (3.1) is approached by \( \frac{1}{\epsilon} Y^T N(Z) Y \).

We define the partial derivatives for a piecewise cubic function. If \( g_h(x) = \sum_{i \in I} G_i \phi_i(x) \), we set \( \Pi^1_h G \in \mathbb{R}^n \)
\[
(\Pi^1_h G)_i = \frac{1}{\sum_{j \in J_i} area(T_j)} \sum_{j \in J_i} area(T_j) \partial^1_1 g_h|T_j(A_i)
\]
here \( J_i \) represents the set of index \( j \) such that the node \( A_i \) belongs to the triangle \( T_j \).

In each triangle \( T_j \), the finite element function \( g_h \) is a cubic polynomial function, then \( \partial^1_1 g_h|T_j \) is well defined. In the same way, we construct \( \Pi^2_h G \in \mathbb{R}^n \) for \( \partial_2 \). We have that, \( \Pi^1_h \) and \( \Pi^2_h \) are two square matrices of order \( n \) depending on \( T_h \).

We define 
\[
\partial^1_1 g_h(x) = \sum_{i \in I} (\Pi^1_h G)_i \phi_i(x) \in \mathbb{W}_h
\]
and similarly for \( \partial^1_2 g_h \). Putting \( \nabla^h g_h = (\partial^1_1 g_h, \partial^1_2 g_h) \) and since \( y_h \in \mathbb{W}_h \subset \mathbb{W}_h \), we can also define \( \partial^2_1 y_h \) and \( \partial^2_2 y_h \).

A typical objective function \( j \) depends on the normal derivative \( \frac{\partial y}{\partial n} \). Here, the outward unit normal vector \( n \) of the domain \( \Omega_g \) is approached by
\[
n^h(x) = \frac{1}{\sqrt{(\partial^1_1 g_h(x))^2 + (\partial^1_2 g_h(x))^2}} \nabla^h g_h(x). \tag{4.6}
\]
The first term of (3.1) can be approached by
\[
J_1(G, Z, Y) = \int_0^{T_y} j \left( Z(t), \nabla^h g_h(Z(t)) \right) |Z'(t)| \, dt
\]
and the discrete form of the optimization problem (3.1)-(3.3) is
\[
\min_{G, U \in \mathbb{R}^n} J(G, U) = J_1(G, Z, Y) + \frac{1}{\epsilon} Y^T N(Z) Y \tag{4.7}
\]
subject to (4.2). We remark that, \( Y \) depends on \( G \) and \( U \) from (4.2) and \( Z \) depends on \( G \) from (4.3)-(4.5). For (3.4), we have to impose similar sign conditions on \( G \).

Let \( r_h, v_h \) be in \( \mathbb{W}_h \) and \( R, V \) in \( \mathbb{R}^n \) be the associated vectors. The discrete weak formulation of (3.8)-(3.9) is: find \( q_h \in \mathbb{V}_h \) such that
\[
\int_D \nabla q_h \cdot \nabla \varphi_h \, dx = \int_D \left( (g_h)^2 \frac{\partial v_h}{\partial n} + 2(g_h + u_h r_h) \varphi_h \right) \, dx, \quad \forall \varphi_h \in \mathbb{V}_h. \tag{4.8}
\]
We set $Q \in \mathbb{R}^{n_0}$ the vector associated to $q_h$ and we construct the $n_0 \times n$ matrix $C^1(G, U)$ defined by

$$C^1(G, U) = \left( \int_D 2(g_h) + u_h \phi_j \phi_i \, dx \right)_{i \in I_0, j \in I}.$$ 

The linear system of (4.8) is

$$KQ = B^1(G)V + C^1(G, U)R.$$ 

(4.9)

The term containing $q$ at the fourth line of (3.14) is approached by

$$\frac{2}{\epsilon} Y^T N(Z)Q$$ 

(4.10)

where the matrix $N(Z)$ was defined in the previous subsection.

The numerical integration over the interval $I_g$ is obtained using the right Riemann sum [35]. We set $F^3 = (F^3_i) \in \mathbb{R}^{n_0}$ by

$$F^3_i = \sum_{k=1}^m \Delta t \nabla_2 j (Z(t_k), \nabla^h y_h(Z(t_k))) \cdot \nabla^h \phi_i(Z(t_k)) |Z'(t_k)|$$

for $i \in I_0$. The third line of (3.14) is approached by

$$(F^3)^T Q.$$ 

(4.11)

In order to solve the ODE system (3.10)-(3.12), we use the backward Euler scheme on the partition constructed before:

$$W^1_{k+1} = W^1_k - \Delta t \nabla_h \partial^2 g_h (Z_{k+1}) \cdot (W^1_{k+1}, W^2_{k+1}),$$ 

(4.12)

$$W^2_{k+1} = W^2_k + \Delta t \nabla_h \partial^2 g_h (Z_{k+1}) \cdot (W^1_{k+1}, W^2_{k+1}),$$ 

(4.13)

$$W^1_0 = 0, W^2_0 = 0,$$ 

(4.14)

for $k = 0, \ldots, m - 1$. Contrary to the system (2.10)-(2.12), the system (3.10)-(3.12) is linear in $w$ and we can use without difficulties an implicit method to solve it.

We set $W_k = (W^1_k, W^2_k)$ and $W_k$ is an approximation of $w(t_k)$. We write $W = (W^1, W^2)$ in $\mathbb{R}^m \times \mathbb{R}^m$, with $W^1 = (W^1_k)_{1 \leq k \leq m}$ and $W^2 = (W^2_k)_{1 \leq k \leq m}$. The function $W : [0, T_g] \rightarrow \mathbb{R}^2$ can be constructed in the same way as for $Z$

$$W(t) = \frac{t_{k+1} - t}{\Delta t} W_k + \frac{t - t_k}{\Delta t} W_{k+1}, \quad t_k < t \leq t_{k+1}$$

for $k = 0, 1, \ldots, m - 1$. We have $W(t_k) = W_k$ and $W'(t) = \frac{1}{\Delta t} (W^1_{k+1} - W^1_k, W^2_{k+1} - W^2_k)$ for $t_k < t \leq t_{k+1}$. 

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We denote

\[
\Lambda_1(t) = \nabla_{1,j} (Z(t), \nabla^h y_h(Z(t))) |Z'(t)| \in \mathbb{R}^2
\]
\[
\Lambda_2(t) = \nabla_{2,j} (Z(t), \nabla^h y_h(Z(t))) \cdot H^h (y_h(Z(t))) |Z'(t)| \in \mathbb{R}^2
\]
\[
\Lambda_4(t) = y_h(Z(t)) \nabla^h y_h(Z(t)) |Z'(t)| \in \mathbb{R}^2
\]

and we introduce the vectors:

\[
\tilde{\Lambda}_1 = (\tilde{\Lambda}_1^1, \tilde{\Lambda}_1^2) \in \mathbb{R}^m \times \mathbb{R}^m \text{ with the components } (\Delta t) \Lambda_1(t_k), 1 \leq k \leq m,
\]
\[
\tilde{\Lambda}_2 = (\tilde{\Lambda}_2^1, \tilde{\Lambda}_2^2) \in \mathbb{R}^m \times \mathbb{R}^m \text{ with the components } (\Delta t) \Lambda_2(t_k), 1 \leq k \leq m \text{ and}
\]
\[
\tilde{\Lambda}_4 = (\tilde{\Lambda}_4^1, \tilde{\Lambda}_4^2) \in \mathbb{R}^m \times \mathbb{R}^m \text{ with the components } (\Delta t) \Lambda_4(t_k), 1 \leq k \leq m.
\]

The first, second and the term containing \( w \) at the fourth line of (3.14) are approched by

\[
(\tilde{\Lambda}_1^1)^T W^1 + (\tilde{\Lambda}_1^2)^T W^2 + (\tilde{\Lambda}_2^1)^T W^1 + (\tilde{\Lambda}_2^2)^T W^2 + \frac{2}{\epsilon} (\tilde{\Lambda}_4^1)^T W^1 + (\tilde{\Lambda}_4^2)^T W^2). \tag{4.15}
\]

We also introduce

\[
\Lambda_6(t) = j (Z(t), \nabla^h y_h(Z(t))) \frac{Z'(t)}{|Z'(t)|} \in \mathbb{R}^2
\]
\[
\Lambda_7(t) = (y_h(Z(t)))^2 \frac{Z'(t)}{|Z'(t)|} \in \mathbb{R}^2
\]

and the vectors:

\[
\tilde{\Lambda}_6 = (\tilde{\Lambda}_6^1, \tilde{\Lambda}_6^2) \in \mathbb{R}^m \times \mathbb{R}^m \text{ with the components } \Lambda_6(t_k) - \Lambda_6(t_{k+1}), 1 \leq k \leq m - 1 \text{ and}
\]

the last component \( \Lambda_6(t_m) \)
\[
\tilde{\Lambda}_7 = (\tilde{\Lambda}_7^1, \tilde{\Lambda}_7^2) \in \mathbb{R}^m \times \mathbb{R}^m \text{ with the components } \Lambda_7(t_k) - \Lambda_7(t_{k+1}), 1 \leq k \leq m - 1 \text{ and}
\]

the last component \( \Lambda_7(t_m) \). The last line of (3.14) is approched by

\[
(\tilde{\Lambda}_6^1)^T W^1 + (\tilde{\Lambda}_6^2)^T W^2 + \frac{1}{\epsilon} (\tilde{\Lambda}_7^1)^T W^1 + (\tilde{\Lambda}_7^2)^T W^2). \tag{4.16}
\]

**Proposition 4.1** The discrete version of the relation (3.14) is

\[
dJ_{(G,U)}(R, V) = (\tilde{\Lambda}_1^1)^T W^1 + (\tilde{\Lambda}_1^2)^T W^2 + (\tilde{\Lambda}_2^1)^T W^1 + (\tilde{\Lambda}_2^2)^T W^2 \\
+ (F^3)^T Q + \frac{2}{\epsilon} (\tilde{\Lambda}_4^1)^T W^1 + (\tilde{\Lambda}_4^2)^T W^2) + \frac{2}{\epsilon} Y^T N(Z) Q \\
+ (\tilde{\Lambda}_6^1)^T W^1 + (\tilde{\Lambda}_6^2)^T W^2 + \frac{1}{\epsilon} (\tilde{\Lambda}_7^1)^T W^1 + (\tilde{\Lambda}_7^2)^T W^2). \tag{4.17}
\]

**Proof.** We get (4.17) by adding (4.10), (4.11), (4.15) and (4.16). \( \square \)
We point out that $Q$ depends on $V, R$ and $W$ depends on $R$, but $\Lambda_i, F^3, N(Z)$ as well as $Y$ are independent of $V, R$.

From (4.9), we get
\[
Q = K^{-1} B^1(G)V + K^{-1} C^1(G, U)R \tag{4.18}
\]
and the discrete version of the operator $A$ in the Corollary 3.1 is
\[
(R, V) \in \mathbb{R}^n \times \mathbb{R}^n \rightarrow A^1(R, V) = K^{-1} B^1(G)V + K^{-1} C^1(G, U)R.
\]

Next, we present how $W$ depends on $R$. Let us introduce the square matrices of order 2
\[
A_2(k) = \begin{pmatrix}
-\Delta t \, \partial^h_1 \partial^h_2 g_h(Z_{k+1}) & -\Delta t \, \partial^h_2 \partial^h_1 g_h(Z_{k+1}) \\
\Delta t \, \partial^h_1 \partial^h_1 g_h(Z_{k+1}) & \Delta t \, \partial^h_2 \partial^h_2 g_h(Z_{k+1})
\end{pmatrix},
I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
M_2(k) = (I_2 - \Delta t A_2(k))^{-1}
\]
and the $2 \times n$ matrix
\[
N_2(k) = \begin{pmatrix}
-\Delta t \, \Phi^T(Z_{k+1}) \Pi^2_h \\
\Delta t \, \Phi^T(Z_{k+1}) \Pi^1_h
\end{pmatrix}
\]
where $\Phi(Z_k) = (\phi_i(Z_k))_{i \in I} \in \mathbb{R}^n$. We can rewrite the system (4.12)-(4.13) as
\[
\begin{pmatrix}
W_{k+1}^1 \\
W_{k+1}^2
\end{pmatrix} = M_2(k) \begin{pmatrix}
W_k^1 \\
W_k^2
\end{pmatrix} + M_2(k) N_2(k) R.
\]

We have the following equality
\[
\begin{pmatrix}
W_1^1 \\
W_1^2 \\
\vdots \\
W_m^1 \\
W_m^2
\end{pmatrix} = M_{2m} \times \begin{pmatrix}
N_2(0) \\
N_2(1) \\
\vdots \\
N_2(m-1)
\end{pmatrix} R \tag{4.19}
\]
the right-hand side, $M_{2m}$ is a square matrix of order $2m$ given by
\[
\begin{pmatrix}
M_2(0) & 0 & \cdots & 0 & 0 \\
M_2(1) M_2(0) & M_2(1) & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\Pi_{k=0}^{m-1} M_2(k) & \Pi_{k=1}^{m-1} M_2(k) & \cdots & \Pi_{k=m-2}^{m-1} M_2(k) & M_2(m-1)
\end{pmatrix}
\]
and the second matrix, which contains $N_2$, is of size $2m \times n$. Now, $W$ depends on $R$ by (4.19), we define the linear operator approximation of $B$ from the Corollary 3.1
\[
R \in \mathbb{R}^n \rightarrow W = (W^1, W^2) = (B^2(G, Z) R, B^3(G, Z) R) \in \mathbb{R}^m \times \mathbb{R}^m. \tag{4.20}
\]
We can rewrite (4.17) as
\[
dJ_{(G,U)}(R, V) = (\tilde{\Lambda}^1 + \tilde{\Lambda}^2 + \frac{2}{\epsilon} \tilde{\Lambda}^4) B^2(G, Z) R + \frac{2}{\epsilon} Y^T N(Z) K^{-1} B^1(G) V
\]
\[
+ (F^3 + \frac{2}{\epsilon} Y^T N(Z))^T K^{-1} B^1(G) V
\]
\[
+ \frac{2}{\epsilon} Y^T N(Z) K^{-1} C(G, U) R + (\tilde{\Lambda}^1_0 + \frac{1}{\epsilon} \tilde{\Lambda}^1_7) T B^2(G, Z) R
\]
\[
+ (\tilde{\Lambda}^0 + \frac{1}{\epsilon} \tilde{\Lambda}^3) T B^3(G, Z) R
\] (4.21)

The first four lines of (4.21) represent an approximation of the first four lines of (3.17).

**Descent direction method**

The descent direction method needs at each step a descent direction, i.e. \((R^k, V^k)\) such that \(dJ_{(G^k, U^k)}(R^k, V^k) < 0\) and the next step is defined by
\[
(G^{k+1}, U^{k+1}) = (G^k, U^k) + \lambda(R^k, V^k),
\]
where \(\lambda > 0\) is computed by some line search
\[
\lambda \in \arg \min_{\lambda > 0} J \left( (G^k, U^k) + \lambda(R^k, V^k) \right).
\]
The algorithm stops if \(dJ_{(G^k, U^k)}(R^k, V^k) = 0\) or \(|J(G^{k+1}, U^{k+1}) - J(G^k, U^k)| < tol\) for some prescribed tolerance parameter \(tol\).

**Proposition 4.2** A descent direction for \(J\) at \((G, U)\) is \((R^*, V^*) \in \mathbb{R}^n \times \mathbb{R}^n\) given by
\[
(V^*)^T = -(F^3 + \frac{2}{\epsilon} Y^T N(Z))^T K^{-1} B^1(G)
\]
\[
(R^*)^T = - (\tilde{\Lambda}^1 + \tilde{\Lambda}^2 + \frac{2}{\epsilon} \tilde{\Lambda}^4) T B^2(G, Z)
\]
\[- \frac{2}{\epsilon} Y^T N(Z) K^{-1} B^1(G, V)
\]
\[- (F^3 + \frac{2}{\epsilon} Y^T N(Z))^T K^{-1} C(G, U)
\]
\[- (\tilde{\Lambda}^1_0 + \frac{1}{\epsilon} \tilde{\Lambda}^1_7) T B^2(G, Z)
\]
\[- (\tilde{\Lambda}^0 + \frac{1}{\epsilon} \tilde{\Lambda}^3) T B^3(G, Z).
\]
Proof. We can rewrite (4.21) as \( dJ_{(G,U)}(R, V) = -(V^*)^T V - (R^*)^T R \), then \( dJ_{(G,U)}(R^*, V^*) = -\|V^*\|_{\mathbb{R}^n}^2 - \|R^*\|_{\mathbb{R}^n}^2 \leq 0 \). If the gradient \( dJ_{(G,U)} = (R^*, V^*) \) is non null (non stationary points), the inequality is strict. \( \square \)

Let us introduce a simplified adjoint system: find \( p_h \) in \( \mathbb{V}_h \) such that

\[
\int_D \nabla \varphi_h \cdot \nabla p_h \, dx = \int_0^{T_g} \nabla_2 J(Z(t), y_h(Z(t))) \cdot \nabla^h \varphi_h(Z(t)) |Z'(t)| \, dt + \frac{2}{\epsilon} \int_0^{T_g} y_h(Z(t)) \varphi_h(Z(t)) |Z'(t)| \, dt
\]

(4.22)

\( \forall \varphi_h \in \mathbb{V}_h \) and \( Z(t) \) satisfying (4.3)-(4.5). We have \( p_h = \sum_{i \in I_0} P_i \phi_i \) and \( P = (P)_{i \in I_0} \in \mathbb{R}^{n_0} \).

Proposition 4.3 Given \( g_h, u_h \in \mathbb{W}_h \), let \( y_h \in \mathbb{V}_h \) be the solution of (4.1). For \( r_h = -p_h u_h, v_h = -p_h, \) with \( p_h \in \mathbb{V}_h \) the solution of (4.22), then

\[
\int_0^{T_g} \nabla_2 J(Z(t), y_h(Z(t))) \cdot \nabla^h q_h(Z(t)) |Z'(t)| \, dt + \frac{2}{\epsilon} \int_0^{T_g} y_h(Z(t)) q_h(Z(t)) |Z'(t)| \, dt \leq 0,
\]

(4.23)

where \( q_h \in \mathbb{V}_h \) is the solution of (4.8) depending on \( r_h \) and \( v_h \).

Proof. Putting \( \varphi_h = p_h \) in (4.8) and \( \varphi_h = q_h \) in (4.22), we get

\[
\int_D \left( (g_h)^2 v_h + 2(g_h) u_h r_h \right) p_h \, dx = \int_D \nabla q_h \cdot \nabla p_h \, dx
\]

\[
= \int_0^{T_g} \nabla_2 J(Z(t), y_h(Z(t))) \cdot \nabla^h q_h(Z(t)) |Z'(t)| \, dt + \frac{2}{\epsilon} \int_0^{T_g} y_h(Z(t)) q_h(Z(t)) |Z'(t)| \, dt.
\]

For \( v_h = -p_h \), we have

\[
\int_D (g_h)^2 v_h p_h \, dx = - \int_D (g_h)^2 p_h^2 \, dx \leq 0
\]

and for \( r_h = -p_h u_h \), we have

\[
\int_D 2(g_h) u_h r_h p_h \, dx = - \int_D 2(g_h) (u_h p_h)^2 \, dx \leq 0
\]

since \( (g_h)_+ \geq 0 \) in \( D \). \( \square \)
Remark 4.1 The terms from (3.14), (4.22), (4.23), containing $q$, can be rewritten as integrals over $\partial \Omega_g$. For instance, for (4.22) we obtain:

$$\int_D \nabla \varphi_h \cdot \nabla p_h \, dx = \int_{\partial \Omega_{\Omega_h}} \nabla^2 j(s, y_h(s)) \cdot \nabla^h \varphi_h(s) \, ds$$

$$+ \frac{2}{\epsilon} \int_{\partial \Omega_{\Omega_h}} y_h(s) \varphi_h(s) \, ds.$$

However, the way they are expressed in (3.14), (4.22), (4.23) avoids the use of the unknown geometry and all the elements are easily computable. For instance, $T_g$ is obtained automatically when solving the Hamiltonian system (2.10)-(2.12), while the initial conditions (on each component of $G$) are simply obtained via the equation $g = 0$ and standard routines, together with a simple iterative procedure to generate all of them. See as well [14].

5 Numerical tests

In the numerical examples, we have employed the software FreeFem++, [8].

The functional appearing in the objective function is

$$j(x, \nabla y(x)) = \frac{1}{2} \left( \frac{\partial y}{\partial n}(x) - \delta(x) \right)^2$$

where $\delta \in H^1(D)$ is a given function. It follows that

$$\nabla_1 j(x, \nabla y(x)) = - \left( \frac{\partial y}{\partial n}(x) - \delta(x) \right) \nabla \delta(x)$$

$$\nabla_2 j(x, \nabla y(x)) = \left( \frac{\partial y}{\partial n}(x) - \delta(x) \right) n.$$
Example 1.

a) The computational domain is $D = [-1,1] \times [-1,1]$, the load is $f = -4$ and $\delta = 1$. This problem has the solution $y_e(x_1,x_2) = x_1^2 + x_2^2 - 0.5^2$ defined on the disk of center $(0,0)$ and radius 0.5. The mesh of $D$ has 53290 triangles and 26946 vertices. The penalization parameter is $\epsilon = 10^{-4}$ and the tolerance parameter for the stopping test is $tol = 10^{-6}$. The initial domain is the disk of center $(0.2,0.2)$ and radius 0.5, given by

$$g_0(x_1,x_2) = (x_1 - 0.2)^2 + (x_2 - 0.2)^2 - 0.5^2.$$  

As descent direction, we use $(R^k,V^k)$ given by Proposition 4.3. For $r_h, v_h$ given by Proposition 4.3 and a scaling parameter $\gamma > 0$, then $\gamma r_h$ and $v_h$ also give a descent direction. We take here $\gamma = \frac{1}{\|r_h\|_{\infty}}$.

Figure 1: Example 1a. The zero level sets of the computed optimal $g$, $y$ (top, left), the final state $y$ (top, right), the solution of the system (1.2)-(1.3) in $\Omega_g$ (bottom, left) and in the domain of boundary the zero level sets of $y$ (bottom, right).

Notice that the difference between the two curves (Figure 1 top, left) is due to the fact that the penalization integral is not null at the final step.

The stopping test is obtained for $k = 13$. The objective function (1.3) is 0.072180 for the solution of the elliptic system (1.2)-(1.3) in the domain $\Omega_g$ and 0.077413 for the solution in the domain of boundary the zero level sets of $y$. The initial, intermediate and the final domains are presented in Figure 2 and the corresponding values of the objective function (3.1) are detailed in Table 1.
Figure 2: Example 1a. Initial domain (top, left), for \( k = 4 \) (top, right), for \( k = 8 \) (bottom, left) and the final domain (bottom, right).

| iteration | k=0 | k=4 | k=8 | final |
|-----------|-----|-----|-----|-------|
| \( t_1 \) | 0.268404 | 0.259176 | 0.151267 | 0.093539 |
| \( t_2 \) | 2.38981 | 0.110803 | 0.036932 | 0.002103 |
| \( J \)   | 23898.4 | 1108.29 | 369.471 | 21.133 |

Table 1: Example 1a. The computed objective function (3.1), i.e. \( J = t_1 + \frac{1}{\epsilon} t_2 \), where \( t_1 = \int_{\partial \Omega_0} j(s, \nabla y(s)) \, ds \) and \( t_2 = \int_{\partial \Omega_0} (y(s))^2 \, ds \).
b) We have the same parameters as before, just the initial domain is the disk of center \((0.2, 0.2)\) and radius 0.4 with a circular hole of center \((0.2, 0.2)\) and radius 0.2 with \(g_0(x_1, x_2)\) given by

\[
\max \left((x_1 - 0.2)^2 + (x_2 - 0.2)^2 - 0.4^2, -(x_1 - 0.2)^2 - (x_2 - 0.2)^2 + 0.2^2\right).
\] (5.2)

Figure 3: Example 1b. The zero level sets of the computed optimal \(g, y\) (top, left), the final state \(y\) (top, right), the solution of the problem (1.2)-(1.3) written in \(\Omega_g\) (bottom, left) and in the domain of boundary the zero level sets of \(y\) (bottom, right).

The stopping test is obtained for \(k = 5\). The objective function (1.3) is 0.455005 for the solution of the elliptic system (1.2)-(1.3) written in \(\Omega_g\) and 0.204318 for the solution in the domain of boundary the zero level sets of \(y\). The domain changes its topology, the initial domain is double connected and the final one is simply connected, see Figure 4. The penalization term is here a sum of two integrals as explained after (3.4). The corresponding values of the objective function (3.1) are reported in Table 2.
Figure 4: Example 1b. Initial domain (top, left), intermediate and the final domain (bottom, right).

| iteration | initial | k=2   | k=2   | k=2   | final |
|-----------|---------|-------|-------|-------|-------|
| $t_1$     | 1.57612 | 2.03842 | 2.10354 | 2.12639 | 1.70448 | 0.494585 |
| $t_2$     | 3.70008 | 0.147847 | 0.123044 | 0.106989 | 0.047237 | 0.002589 |
| $J$       | 37002.4 | 1480.51 | 1232.54 | 1072.02 | 474.078 | 26.3898 |

Table 2: Example 1b. The computed objective function (3.1), i.e. $J = t_1 + \frac{1}{\epsilon} t_2$, where $t_1$ and $t_2$ are as before. The columns 4, 5, 6 correspond to intermediate configurations obtained during the line-search after k=2.
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