Minimal triangulations of sphere bundles over the circle

Bhaskar Bagchi and Basudeb Datta

Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Bangalore 560 059, India
Department of Mathematics, Indian Institute of Science, Bangalore 560 012, India

To appear in ‘Journal of Combinatorial Theory, Ser. A’

Abstract

For integers \(d \geq 2\) and \(\varepsilon = 0\) or \(1\), let \(S^{1,d-1}(\varepsilon)\) denote the sphere product \(S^1 \times S^{d-1}\) if \(\varepsilon = 0\) and the twisted sphere product \(S^1 \times -S^{d-1}\) if \(\varepsilon = 1\). The main results of this paper are: (a) if \(d \equiv \varepsilon \pmod{2}\) then \(S^{1,d-1}(\varepsilon)\) has a unique minimal triangulation using \(2d + 3\) vertices, and (b) if \(d \equiv 1 - \varepsilon \pmod{2}\) then \(S^{1,d-1}(\varepsilon)\) has minimal triangulations (not unique) using \(2d + 4\) vertices. In this context, a minimal triangulation of a manifold is a triangulation using the least possible number of vertices. The second result confirms a recent conjecture of Lutz. The first result provides the first known infinite family of closed manifolds (other than spheres) for which the minimal triangulation is unique. Actually, we show that while \(S^{1,d-1}(\varepsilon)\) has at most one \((2d + 3)\)-vertex triangulation (one if \(d \equiv \varepsilon \pmod{2}\), zero otherwise), in sharp contrast, the number of non-isomorphic \((2d + 4)\)-vertex triangulations of these \(d\)-manifolds grows exponentially with \(d\) for either choice of \(\varepsilon\). The result in (a), as well as the minimality part in (b), is a consequence of the following result: (c) for \(d \geq 3\), there is a unique \((2d + 3)\)-vertex simplicial complex which triangulates a non-simply connected closed manifold of dimension \(d\). This amazing simplicial complex was first constructed by Kühnel in 1986. Generalizing a 1987 result of Brehm and Kühnel, we prove that (d) any triangulation of a non-simply connected closed \(d\)-manifold requires at least \(2d + 3\) vertices. The result (c) completely describes the case of equality in (d). The proofs rest on the Lower Bound Theorem for normal pseudomanifolds and on a combinatorial version of Alexander duality.

Mathematics Subject Classification (2000): 57Q15, 57R05.

Keywords: Triangulated manifolds; Stacked spheres; Non-simply connected manifolds.

1 Preliminaries

With a single exception in Section 3, all simplicial complexes considered here are finite. For a simplicial complex \(X\), \(V(X)\) will denote the set of all the vertices of \(X\) and \(|X|\) will denote the geometric carrier of \(X\). One says that \(X\) is a triangulation of the topological space \(|X|\). If \(|X|\) is a manifold then we say that \(X\) is a triangulated manifold. The unique \((d+2)\)-vertex triangulation of the \(d\)-sphere \(S^d\) is denoted by \(S^d_{d+2}\) and is called the standard \(d\)-sphere. The unique \((d+1)\)-vertex triangulation of the \(d\)-ball is denoted by \(B^d_{d+1}\) and is called the standard \(d\)-ball. For \(n \geq 3\), the unique \(n\)-vertex triangulation of the circle \(S^1\) is denoted by \(S^1_n\) and is called the \(n\)-cycle.

---

1 E-mail addresses: bbagchi@isibang.ac.in (B. Bagchi), dattab@math.iisc.ernet.in (B. Datta).
For $i = 1, 2$, the $i$-faces of a simplicial complex $K$ are also called the edges and triangles of $K$, respectively. For a simplicial complex $K$, the graph whose vertices and edges are the vertices and edges of $K$ is called the edge graph (or 1-skeleton) of $K$. Recall that a graph is nothing but a simplicial complex of dimension at most 1. A set of vertices in a graph is called a clique if these vertices are mutually adjacent (i.e., any two of them form an edge). Note that any simplex in a simplicial complex is a clique in its edge graph.

For a simplex $σ$ in a simplicial complex $K$, the number of vertices in $lk_K(σ)$ is called the degree of $σ$ in $K$ and is denoted by $\deg_K(σ)$ (or by $\deg(σ)$). So, the degree of a vertex $v$ in $K$ is the same as the degree of $v$ in the edge graph of $K$.

Recall that for any face $α$ of a complex $X$, its link $lk_X(α)$ is the simplicial complex whose faces are the faces $β$ of $X$ such that $α \cap β = \emptyset$ and $α \cup β ∈ X$. Likewise, the star $st_X(α)$ of the face $α$ has all the maximal faces $γ ⊇ α$ of $X$ as its maximal faces.

A simplicial complex $X$ is called a combinatorial $d$-sphere (respectively, combinatorial $d$-ball) if $|X|$ (with the induced pl structure from $X$) is pl homeomorphic to $S^{d+2}_{d+2}$ (respectively, $B^d_{d+1}$). A simplicial complex $X$ is said to be a combinatorial $d$-manifold if $|X|$ (with the induced pl structure) is a pl $d$-manifold. Equivalently, $X$ is a combinatorial $d$-manifold if all its vertex links are combinatorial spheres or combinatorial balls. In this case, we also say that $X$ is a combinatorial triangulation of $|X|$. A simplicial complex $X$ is a combinatorial manifold without boundary if all its vertex links are combinatorial spheres. A combinatorial manifold will usually mean one without boundary.

A simplicial complex $K$ is called pure if all the maximal faces (facets) of $K$ have the same dimension. For $d ≥ 1$, a $d$-dimensional pure simplicial complex is said to be a weak pseudomanifold if each $(d−1)$-simplex is in exactly two facets. Clearly, any $d$-dimensional weak pseudomanifold has at least $d + 2$ vertices, with equality only for $S^{d+2}_{d+2}$.

For a pure $d$-dimensional simplicial complex $K$, let $Λ(K)$ be the graph whose vertices are the facets of $K$, two such vertices being adjacent in $Λ(K)$ if the corresponding facets intersect in a $(d−1)$-face. If $Λ(K)$ is connected then $K$ is called strongly connected. A strongly connected weak pseudomanifold is called a pseudomanifold. Thus, for a $d$-pseudomanifold $K$, $Λ(K)$ is a connected $(d+1)$-regular graph. This implies that $K$ has no proper subcomplex which is also a $d$-pseudomanifold. (Or else, the facets of such a subcomplex would provide a disconnection of $Λ(X)$.) By convention, $S^0$ is the only 0-pseudomanifold.

A connected $d$-dimensional weak pseudomanifold is said to be a normal pseudomanifold if the links of all the simplices of dimension up to $d−2$ are connected. Clearly, the normal 2-pseudomanifolds are just the connected combinatorial 2-manifolds. But, normal $d$-pseudomanifolds form a broader class than connected combinatorial $d$-manifolds for $d ≥ 3$. In fact, any triangulation of a connected closed manifold is a normal pseudomanifold.

Observe that if $X$ is a normal pseudomanifold then $X$ is a pseudomanifold. (If $Λ(X)$ is not connected then, since $X$ is connected, $Λ(X)$ has two components $G_1$ and $G_2$ and two intersecting facets $σ_1, σ_2$ such that $σ_i ∈ G_i, i = 1, 2$. Choose $σ_1, σ_2$ among all such pairs such that dim($σ_1 ∩ σ_2$) is maximum. Then dim($σ_1 ∩ σ_2$) ≤ $d−2$ and $lk_X(σ_1 ∩ σ_2)$ is not connected, a contradiction.) Notice that all the links of simplices of dimensions up to $d−2$ in a normal $d$-pseudomanifold are normal pseudomanifolds.

Let $X, Y$ be two simplicial complexes with disjoint vertex sets. (Since we identify isomorphic complexes, this is no real restriction on $X, Y$.) Then their join $X * Y$ is the simplicial complex whose simplices are those of $X$ and of $Y$, and the (disjoint) unions of simplices of $X$ with simplices of $Y$. It is easy to see that if $X$ and $Y$ are combinatorial spheres (respectively normal pseudomanifolds) then their join $X * Y$ is a combinatorial sphere (respectively normal pseudomanifold).
By a subdivision of a simplicial complex $K$ we mean a simplicial complex $K'$ together with a homeomorphism from $|K'|$ onto $|K|$ which is facewise linear. Two complexes $K, L$ have isomorphic subdivisions if and only if $|K|$ and $|L|$ are pl homeomorphic. Let $X$ be a pure $d$-dimensional simplicial complex and $\sigma$ be a facet of $X$, then take a symbol $v$ outside $V(X)$ and consider the pure $d$-dimensional simplicial complex $Y$ with vertex set $V(X) \cup \{v\}$ whose facets are facets of $X$ other than $\sigma$ and the $(d+1)$-sets $\tau \cup \{v\}$ where $\tau$ runs over the $(d-1)$-simplices in $\sigma$. Clearly, $Y$ is a subdivision of $X$. The complex $Y$ is called the subdivision obtained from $X$ by starring a new vertex $v$ in the facet $\sigma$.

If $U$ is a non-empty subset of the vertex set $V(X)$ of a simplicial complex $X$ then the simplices of $X$ which are subsets of $U$ form a simplicial complex. This simplicial complex is called the induced subcomplex of $X$ on the vertex set $U$ and is denoted by $X[U]$.

**Definition 1.1.** If $Y$ is an induced subcomplex of a simplicial complex $X$ then the simplicial complement $C(Y, X)$ of $Y$ in $X$ is the induced subcomplex of $X$ with vertex set $V(X) \setminus V(Y)$.

By abuse of notation, for any face $\sigma$ of $X$, the induced subcomplex of $X$ on the complement of $\sigma$ will be denoted by $C(\sigma, X)$.

**Definition 1.2.** Let $\sigma_1, \sigma_2$ be two facets in a pure simplicial complex $X$. Let $\psi : \sigma_1 \rightarrow \sigma_2$ be a bijection. We shall say that $\psi$ is admissible if ($\psi$ is a bijection and) the distance between $x$ and $\psi(x)$ in the edge graph of $X$ is $\geq 3$ for each $x \in \sigma_1$. Notice that if $\sigma_1, \sigma_2$ are from different connected components of $X$ then any bijection between them is admissible. Also note that, in general, for the existence of an admissible map $\psi : \sigma_1 \rightarrow \sigma_2$, the facets $\sigma_1$ and $\sigma_2$ must be disjoint.

**Definition 1.3.** Let $X$ be a weak pseudomanifold with disjoint facets $\sigma_1, \sigma_2$. Let $\psi : \sigma_1 \rightarrow \sigma_2$ be an admissible bijection. Let $X^\psi$ denote the weak pseudomanifold obtained from $X \setminus \{\sigma_1, \sigma_2\}$ by identifying $x$ with $\psi(x)$ for each $x \in \sigma_1$. Then $X^\psi$ is said to be obtained from $X$ by an elementary handle addition. If $X_1, X_2$ are two $d$-dimensional weak pseudomanifolds with disjoint vertex-sets, $\sigma_i$ a facet of $X_i$ ($i = 1, 2$) and $\psi : \sigma_1 \rightarrow \sigma_2$ any bijection, then $(X_1 \cup X_2)^\psi$ is called an elementary connected sum of $X_1$ and $X_2$, and is denoted by $X_1 \#_\psi X_2$ (or simply by $X_1 \# X_2$). Note that the combinatorial type of $X_1 \#_\psi X_2$ depends on the choice of the bijection $\psi$. However, when $X_1, X_2$ are connected triangulated $d$-manifolds, $|X_1 \#_\psi X_2|$ is the topological connected sum of $|X_1|$ and $|X_2|$ (taken with appropriate orientations). Thus, $X_1 \#_\psi X_2$ is a triangulated manifold whenever $X_1, X_2$ are triangulated $d$-manifolds.

**Lemma 1.1.** Let $N$ be a $(d-1)$-dimensional induced subcomplex of a $d$-dimensional simplicial complex $M$. If both $M$ and $N$ are normal pseudomanifolds then

(a) for any vertex $u$ of $N$ and any vertex $v$ of the simplicial complement $C(N, M)$, there is a path $P$ (in $M$) joining $u$ to $v$ such that $u$ is the only vertex in $P \cap N$, and

(b) the simplicial complement $C(N, M)$ has at most two connected components.

**Proof.** Part (a) is trivial if $d = 1$ (in which case, $N = S^0_2$ and $M = S^1_n$). So, assume $d > 1$ and we have the result for smaller dimensions. Clearly, there is a path $P$ (in the edge graph of $M$) joining $u$ to $v$ such that $P = x_1 x_2 \cdots x_k y_1 \cdots y_l$ where $x_1 = u$, $y_l = v$ and $x_i$’s are the only vertices of $P$ from $N$. Choose $k$ to be the smallest possible. We claim that $k = 1$, so that the result follows. If not, then $x_{k-1} \in \text{lk}^M(x_k) \subset \text{lk}^N(x_k)$ and $y_1 \in C(\text{lk}^N(x_k), \text{lk}^M(x_k))$. Then, by induction hypothesis, there is a path $Q$ in $\text{lk}^M(x_k)$ joining $x_{k-1}$ and $y_1$ in which $x_{k-1}$ is the only vertex from $\text{lk}^N(x_k)$. Replacing the part
as above, let \( \tilde{E} \) be the set of all edges of \( M \) with exactly one end in \( S \). Then, for every \( e \in E \), \( \tilde{E} \) meets either \( U^+ \) or \( U^- \) but not both. Put \( E^\pm = \{ e \in E : e \cap U^\pm \neq \emptyset \} \). Then no element of \( E^+ \) is adjacent in \( G \) with any element of \( E^- \). From the previous argument, one sees that each \( x \in A \) is in an edge from \( E^+ \) and in an edge from \( E^- \). Thus, both \( E^+ \) and \( E^- \) are non-empty. So, \( G \) is disconnected. \( \square \)

**Lemma 1.3.** Let \( X \) be a normal d-pseudomanifold with an induced two-sided standard \((d - 1)\)-sphere \( S \). Then there is a d-dimensional weak pseudomanifold \( \tilde{X} \) such that \( X \) is obtained from \( \tilde{X} \) by elementary handle addition. Further,

(a) the connected components of \( \tilde{X} \) are normal d-pseudomanifolds,

(b) \( \tilde{X} \) has at most two connected components,

(c) if \( \tilde{X} \) is not connected, then \( X = Y_1 \# Y_2 \), where \( Y_1, Y_2 \) are the connected components of \( \tilde{X} \), and

(d) if \( C(S,X) \) is connected then \( \tilde{X} \) is connected.

**Proof.** As above, let \( E \) be the set of all edges of \( X \) with exactly one end in \( S \). Let \( E^+ \) and \( E^- \) be the connected components of the graph \( G \) (with vertex-set \( E \)) defined above (cf. Lemma 1.2). Notice that if a facet \( \sigma \) intersects \( V(S) \) then \( \sigma \) contains edges from \( E \), and the graph \( G \) induces a connected subgraph on the set \( E_\sigma = \{ e \in E : e \subseteq \sigma \} \). (Indeed, this
subgraph is the line graph of a complete bipartite graph.) Consequently, either \( E_a \subseteq E^+ \) or \( E_b \subseteq E^- \). Accordingly, we say that the facet \( \sigma \) is positive or negative (relative to \( S \)). If a facet \( \sigma \) of \( X \) does not intersect \( V(S) \) then we shall say that \( \sigma \) is a neutral facet.

Let \( V(S) = W \) and \( V(X) \setminus V(S) = U \). Take two disjoint sets \( W^+ \) and \( W^- \), both disjoint from \( U \), together with two bijections \( f_+: W \to W^+ \) and \( f_-: W \to W^- \). We define a pure simplicial complex \( \tilde{X} \) as follows. The vertex-set of \( \tilde{X} \) is \( U \sqcup W^+ \sqcup W^- \). The facets of \( \tilde{X} \) are: (i) \( W^+ \), \( W^- \), (ii) all the neutral facets of \( X \), (iii) for each positive facet \( \sigma \) of \( X \), the set \( \tilde{\sigma} := (\sigma \cap U) \cup f_+ (\sigma \cap W) \), and (iv) for each negative facet \( \tau \) of \( X \), the set \( \tilde{\tau} := (\tau \cap U) \cup f_- (\tau \cap W) \). Clearly, \( \tilde{X} \) is a weak pseudomanifold. Let \( \psi = f_- \circ f_+^{-1}: W^+ \to W^- \). It is easy to see that \( \psi \) is admissible and \( X = (\tilde{X})^\psi \).

Since the links of faces of dimension up to \( d - 2 \) in \( X \) are connected, it follows that the links of faces of dimension up to \( d - 2 \) in \( \tilde{X} \) are connected. This proves (a).

As \( X \) is connected, choosing two vertices \( f_+ (x_0) \in W^+ \) of \( \tilde{X} \), one sees that each vertex of \( \tilde{X} \) is joined by a path in the edge graph of \( \tilde{X} \) to either \( f_+ (x_0) \) or \( f_- (x_0) \). Hence \( \tilde{X} \) has at most two components. This proves (b). This arguments also shows that when \( \tilde{X} \) is disconnected, \( W^+ \) and \( W^- \) are facets in different components of \( \tilde{X} \). Hence (c) follows.

Observe that \( C(S, X) = C(W^+ \sqcup W^-, \tilde{X}) \). Assume that \( C(S, X) \) is connected. Now, for any \((d - 1)\)-simplex \( \tau \subseteq W^+ \), there is a vertex \( x \) in \( C(S, X) \) such that \( \tau \cup \{x\} \) is a facet of \( \tilde{X} \). So, \( C(S, X) \) and \( W^+ \) are in the same connected component of \( \tilde{X} \). Similarly, \( C(S, X) \) and \( W^- \) are in the same connected component of \( \tilde{X} \). This proves (d).

\( \square \)

**Definition 2.1.** A simplicial complex \( X \) is said to be a **stacked \( d \)-sphere** if there is a finite sequence \( X_0, X_1, \ldots, X_m \) of simplicial complexes such that \( X_0 = S^{d}_{d+2} \), the standard \( d \)-sphere, \( X_m = X \) and \( X_i \) is obtained from \( X_{i-1} \) by starring a new vertex in a facet of \( X_{i-1} \) for \( 1 \leq i \leq m \). Thus an \( n \)-vertex stacked \( d \)-sphere is obtained from the standard \( d \)-sphere by

---

(Extracted content follows text formatting and structure)
(n - d - 2)-fold starring. This implies that every stacked sphere is a combinatorial sphere. Since, for d > 1, each starring increases the number of edges by d + 1, it follows that any n-vertex stacked d-sphere has exactly \((d + 2) + (n - d - 2)(d + 1) = n(d + 1) - \binom{d + 2}{2}\) edges.

In [5], Barnette proved that any n-vertex polytopal d-sphere has at least \(n(d + 1) - \binom{d + 2}{2}\) edges. In [8], Kalai proved this result for triangulated manifolds and also proved that, for \(d \geq 3\), equality holds in this inequality only for stacked spheres. In [15], Tay generalized these results to normal pseudomanifolds to prove:

**Theorem 1.** (Lower Bound Theorem for Normal Pseudomanifolds) For \(d \geq 2\), any n-vertex normal d-pseudomanifold has at least \(n(d + 1) - \binom{d + 2}{2}\) edges. For \(d \geq 3\), equality holds only for stacked spheres.

In [4], we have presented a self-contained combinatorial proof of Theorem 1. Using induction, it is not difficult to prove the next four lemmas (see [4] for complete proofs).

**Lemma 2.1.** Let \(X\) be a normal pseudomanifold of dimension \(d \geq 2\).

(a) If \(X\) is not the standard \(d\)-sphere then any two vertices of degree \(d + 1\) in \(X\) are non-adjacent.

(b) If \(X\) is a stacked sphere then \(X\) has at least two vertices of degree \(d + 1\).

**Lemma 2.2.** Let \(X, Y\) be normal d-pseudomanifolds. Suppose \(Y\) is obtained from \(X\) by starring a new vertex in a facet of \(X\). Then \(Y\) is a stacked sphere if and only if \(X\) is a stacked sphere.

**Lemma 2.3.** The link of a vertex in a stacked sphere is a stacked sphere.

**Lemma 2.4.** Any stacked sphere is uniquely determined by its edge graph.

**Lemma 2.5.** Let \(X_1, X_2\) be normal d-pseudomanifolds. Then \(X_1 \# X_2\) is a stacked d-sphere if and only if both \(X_1, X_2\) are stacked d-spheres.

**Proof.** Induction on the number \(n \geq d + 3\) of vertices in \(X_1 \# X_2\). If \(n = d + 3\) then both \(X_1, X_2\) must be standard d-spheres (hence stacked spheres) and then \(X_1 \# X_2 = S_2^d \ast S_{d+1}^d\) is easily seen to be a stacked sphere. So, assume \(n > d + 3\), so that at least one of \(X_1, X_2\) is not the standard d-sphere. Without loss of generality, say \(X_1\) is not the standard d-sphere. Of course, \(X = X_1 \# X_2\) is not a standard d-sphere. Let \(X\) be obtained from \(X_1 \cup X_2 \setminus \{\sigma_1, \sigma_2\}\) by identifying a facet \(\sigma_1\) of \(X_1\) with a facet \(\sigma_2\) of \(X_2\) by some bijection. Then, \(\sigma_1 = \sigma_2\) is a clique in the edge graph of \(X\), though it is not a facet of \(X\). Notice that a vertex \(x \in V(X_1) \setminus \sigma_1\) is of degree \(d + 1\) in \(X_1\) if and only if it is of degree \(d + 1\) in \(X\). If either \(X_1\) is a stacked sphere or \(X\) is a stacked sphere then, by Lemma 2.1, such a vertex \(x\) exists. Let \(\bar{X}_1\) (respectively, \(\bar{X}_2\)) be obtained from \(X_1\) (respectively, \(X\)) by collapsing this vertex \(x\). Notice that \(\bar{X} = \bar{X}_1 \# \bar{X}_2\). Therefore, by induction hypothesis and Lemma 2.2, we have: \(X\) is a stacked sphere \(\iff\bar{X}\) is a stacked sphere \(\iff\) both \(\bar{X}_1\) and \(\bar{X}_2\) are stacked spheres \(\iff\) both \(X_1\) and \(X_2\) are stacked spheres. \(\square\)

**Definition 2.2.** For \(d \geq 2\), \(\mathcal{K}(d)\) will denote the family of all normal d-pseudomanifolds \(X\) such that the link of each vertex of \(X\) is a stacked \((d - 1)\)-sphere. Since all stacked spheres are combinatorial spheres, it follows that the members of \(\mathcal{K}(d)\) are combinatorial d-manifolds. Notice that, Lemma 2.3 says that all stacked d-spheres belong to the class \(\mathcal{K}(d)\). Also, for \(d \geq 2\), \(K_{2d+3}^d\) and all the simplicial complexes \(K_{2d+4}^d(p)\) constructed in Section 3 are in the class \(\mathcal{K}(d)\) (cf. Proof of Lemma 3.2).
**Lemma 2.6. (Walkup)** Let $X$ be a normal $d$- pseudomanifold and $\psi: \sigma_1 \rightarrow \sigma_2$ be an admissible bijection, where $\sigma_1, \sigma_2$ are facets of $X$. Then $X^\psi \in K(d)$ if and only if $X \in K(d)$.

**Proof.** For a vertex $v$ of $X$, let $\tilde{v}$ denote the corresponding vertex of $X^\psi$. Observe that $lk_{X^\psi}(\tilde{v})$ is isomorphic to $lk_X(v)$ if $v \in V(X) \setminus (\sigma_1 \cup \sigma_2)$ and $lk_{X^\psi}(\tilde{v}) = lk_X(v) \# lk_X(\psi(v))$ if $v \in \sigma_1$. The result now follows from Lemma 2.5. \hfill \Box

**Theorem 2.** For $d \geq 2$, there is a unique $(3d+4)$-vertex stacked $d$-sphere $\mathcal{S} = \mathcal{S}_{3d+4}$ which has a pair of facets with an admissible bijection between them. Further, this pair of facets and the admissible bijection between them is unique up to automorphisms of $\mathcal{S}$.

**Proof.** Uniqueness: Let $V^+$ and $V^-$ be two (disjoint) facets in a $(3d+4)$-vertex stacked $d$-sphere $\mathcal{S}$, and $\psi: V^+ \rightarrow V^-$ be an admissible bijection. Put $V(\mathcal{S}) = U \cup V^+ \cup V^-$. Thus, $\#(U) = d + 2$. Since $\psi$ is admissible, for each $x \in V^+$, none of the $3d+2$ vertices of $\mathcal{S}$ other than $x$ and $\psi(x)$ is adjacent (in the edge graph of $\mathcal{S}$) with both $x$ and $\psi(x)$. Further, $x$ and $\psi(x)$ are non-adjacent. Therefore,

$$\deg(x) + \deg(\psi(x)) \leq 3d + 2, \quad x \in V^+. \tag{1}$$

Also, for $y \in U$, $y$ is adjacent to at most one vertex in the pair $\{x, \psi(x)\}$ for each $x \in V^+$, and these $d + 1$ pairs partition $V(\mathcal{S}) \setminus U$. So, each $y \in U$ has at most $d + 1$ neighbours outside $U$. Since $y$ can have at most $d + 1 = \#(U \setminus \{y\})$ neighbours in $U$, it follows that

$$\deg(y) \leq 2d + 2, \quad y \in U. \tag{2}$$

From (1) and (2), we get by addition,

$$\sum_{x \in V^+} \deg(x) + \sum_{x \in V^+} \deg(\psi(x)) + \sum_{y \in U} \deg(y) \leq (d+1)(3d+2) + (d+2)(2d+2) = (d+1)(5d+6).$$

Now, the left hand side in this inequality is the sum of the degrees of all the vertices of $\mathcal{S}$, which equals twice the number of edges of $\mathcal{S}$. Thus $\mathcal{S}$ has at most $(d+1)(5d+6)/2$ edges. But, as $\mathcal{S}$ is a $(3d+4)$-vertex stacked $d$-sphere and $d \geq 2$, it has exactly $(3d+4)(d+1) - \binom{d+2}{2} = (d+1)(5d+6)/2$ edges. Hence we must have equality in (1) and (2). Thus we have equality throughout the arguments leading to (1) and (2). Therefore we have: (a) $U$ is a $(d+2)$-clique in the edge graph $G$ of $\mathcal{S}$, and (b) for each $y \in U$ and $x \in V^+$, $y$ is adjacent to exactly one of the vertices $x$ and $\psi(x)$. Notice that, since $U$, $V^+$ and $V^-$ are cliques and there is no edge between $V^+$ and $V^-$, it follows that $G$ is completely determined by its (bipartite) subgraph $H$ whose edges are the edges of $G$ between $U$ and $V^+$.

Let $0 \leq m \leq d + 1$.

**Claim.** There exist $x_i^+, 1 \leq i \leq m$, in $V^+$ and $y_i, 1 \leq i \leq m$, in $U$ such that for each $i (1 \leq i \leq m)$, the $i$ vertices $y_1, \ldots, y_i$ are the only vertices from $U$ adjacent to $x_i^+$. Further, there is a stacked $d$-sphere $X(m)$ with vertex-set $V(\mathcal{S}) \setminus \{x_i^+ : 1 \leq i \leq m\}$ whose edge graph is the induced subgraph $G_m$ of $G$ on this vertex set.

We prove the claim by finite induction on $m$. The claim is trivially correct for $m = 0$ (take $X(0) = \mathcal{S}$, $G_0 = G$). So, assume $1 \leq m \leq d + 1$ and the claim is valid for all smaller values of $m$. By Lemma 2.1, $X(m-1)$ has at least two vertices of degree $d + 1$ and they are non-adjacent in $G_{m-1}$. Since each vertex of $U$ has degree $2d + 2$ in $G$, it has degree $\geq 2d + 2 - (m - 1) > d + 1$ in $G_{m-1}$. Since $V^-$ is a clique of $G_{m-1}$, at least one of the degree $d + 1$ vertices of $G_{m-1}$ is in $V^+ \setminus \{x_i^+ : 1 \leq i < m\}$. Let $x_m^+$ be a vertex of degree $d + 1$ in
Remark 2.1. (a) The proof of Theorem 2, in conjunction with the Lower Bound Theorem, actually shows the following. If X is an n-vertex normal d-pseudomanifold with an admissible bijection, then \( n \geq 3d + 4 \), and equality holds only for \( X = S^d_{3d+4} \). (b) If \( \psi \) is the admissible bijection on \( S^d_{3d+4} \), then it is possible to verify directly that \( (S^d_{3d+4})^\psi = K^d_{2d+3} \). This is also immediate from the proof of Theorem 4 below.

3 Some Examples

Recall that for any positive integer \( n \), a partition of \( n \) is a finite weakly increasing sequence of positive integers adding to \( n \). The terms of the sequence are called the parts of the partition. Let’s say that a partition of \( n \) is even (respectively, odd) if it has an even (respectively, odd) number of even parts. Let \( P(n) \) (respectively \( P_0(n) \), respectively \( P_1(n) \)) denote the total number of partitions (respectively even partitions, respectively odd partitions) of \( n \).
To appreciate the construction given below, it is important to understand the growth rate of these number theoretic functions \( P_\varepsilon \), \( \varepsilon = 0, 1 \). Recall that if \( f, g \) are two real valued functions on the set of positive integers, then one says that \( f, g \) are asymptotically equal (in symbols, \( f(n) \sim g(n) \)) if \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1 \). A famous theorem of Hardy and Ramanujan (cf. [12]) says that

\[
P(n) \sim \frac{c_1}{n} e^{\sqrt{\frac{2}{3}}n} \quad \text{as} \quad n \to \infty,
\]

where the absolute constants \( c_1, c_2 \) are given by

\[
c_1 = \frac{1}{4\sqrt{3}}, \quad c_2 = \pi \sqrt{\frac{2}{3}}.
\]

We observe that:

**Lemma 3.1.** \( P_0(n) \sim \frac{c_1}{2n} e^{\sqrt{\frac{2}{3}}n} \), \( P_1(n) \sim \frac{c_1}{2n} e^{\sqrt{\frac{2}{3}}n} \) as \( n \to \infty \).

**Proof.** In view of (3), it suffices to show that \( P_0(n) \sim \frac{1}{2} P(n) \), \( P_1(n) \sim \frac{1}{2} P(n) \) as \( n \to \infty \). Now, \((p_1, \ldots, p_k) \mapsto (1, p_1, \ldots, p_k)\) is a one to one function from the set of even (respectively, odd) partitions of \( n-1 \) to the set of even (respectively, odd) partitions of \( n \). Also, \((p_1, \ldots, p_k) \mapsto (p_1, \ldots, p_{k-1}, p_k+1)\) is a one to one function from the set of even (respectively, odd) partitions of \( n-1 \) to the set of odd (respectively, even) partitions of \( n \). Therefore, \( \min(P_0(n), P_1(n)) \geq \max(P_0(n-1), P_1(n-1)) \). Since \( P_0(n-1) + P_1(n-1) = P(n-1) \), it follows that

\[
P_0(n) \geq \frac{1}{2} P(n-1) \quad \text{and} \quad P_1(n) \geq \frac{1}{2} P(n-1).
\]

But, from (3) it follows that \( P(n-1) \sim P(n) \). Therefore, \( \lim \inf_{n \to \infty} \frac{P_0(n)}{P(n)} \geq \frac{1}{2}, \lim \inf_{n \to \infty} \frac{P_1(n)}{P(n)} \geq \frac{1}{2}. \)

But, \( P_0(n) + P_1(n) = P(n) \). Therefore, \( \lim_{n \to \infty} \frac{P_0(n)}{P(n)} = \frac{1}{2} = \lim_{n \to \infty} \frac{P_1(n)}{P(n)}. \)

**The Construction:** For \( d \geq 2 \), let \( N^{d+1} \) denote the pure \((d+1)\)-dimensional simplicial complex with vertex-set \( \mathbb{Z} \) (the set of all integers) such that the facets of \( N^{d+1} \) are the sets of \( d+2 \) consecutive integers. Then \( N^{d+1} \) is a combinatorial \((d+1)\)-manifold with boundary \( M^d = \partial N^{d+1} \). Now, \( M^d \) is a combinatorial \( d \)-manifold (\( \in K(d) \)) and triangulates \( \mathbb{R} \times S^{d-1} \) (cf. [9]). Clearly, the facets of \( M^d \) are of the form \( \sigma_{n,i} := \{n, n+1, \ldots, n+d+1\} \setminus \{n+i\} \), \( 1 \leq i \leq d, n \in \mathbb{Z} \) (intervals of length \( d+2 \) minus an interior point).

For \( m \geq 1 \), let \( N^{d+1}_{m+d+1} \) (respectively, \( M^d_{m+d+1} \)) denote the induced subcomplex of \( N^{d+1} \) (respectively, \( M^d \)) on \( m+d+1 \) consecutive vertices (without loss of generality we may take \( V(N^{d+1}_{m+d+1}) = V(M^d_{m+d+1}) = \{1, 2, \ldots, m+d+1\} \)). Clearly, \( M^d_{m+d+1} \) triangulates \([0,1] \times S^{d-1}\) and \( \partial M^d_{m+d+1} = S^{d-1}(A_m) \cup S^{d-1}(B_m) \), where \( A_m = \{1, \ldots, d+1\} \) and \( B_m = \{m+1, \ldots, m+d+1\} \).

**Lemma 3.2.** (a) \( \partial N^{d+1}_{m+d+1} \) is a stacked \( d \)-sphere and \( A_m, B_m \) are two of its facets. (b) If \( \psi : B_m \to A_m \) is an admissible bijection then \( X^d_m(\psi) := (\partial N^{d+1}_{m+d+1})^\psi \) is a combinatorial \( d \)-manifold and triangulates \( S^{1,d-1}(\varepsilon) \), where \( \varepsilon = 0 \) if \( X^d_m(\psi) \) is orientable and \( \varepsilon = 1 \) otherwise.

**Proof.** Observe that \( \partial N^{d+1}_{m+2} \) is the standard \( d \)-sphere and for \( m \geq 2 \), \( \partial N^{d+1}_{m+d+1} \) is obtained from \( \partial N^{d+1}_{m+d} \) by starring the new vertex \( m+d+1 \) in the facet \( B_{m-1} = \{m, \ldots, m+d\} \) of \( \partial N^{d+1}_{m+d} \). Thus, \( \partial N^{d+1}_{m+d+1} \) is a stacked \( d \)-sphere. \( A_m \) is a facet of \( \partial N^{d+1}_{i+d+1} \) for all \( i \geq 1 \) and from construction, \( B_m \) is a facet of \( \partial N^{d+1}_{m+d+1} \). This proves (a).
Thus, by Lemma 2.3, $\partial N^{d+1}_{m+d+1}$ is in $\mathcal{K}(d)$. Then, by Lemma 2.6, $X^d_{\psi}(\psi)$ is in the class $\mathcal{K}(d)$. In consequence, $X^d_{\psi}(\psi)$ is a combinatorial $d$-manifold. Since $M^d_{m+d+1}$ triangulates $[0,1] \times S^{d-1}$ and $M^d_{m+d+1} = \partial N^{d+1}_{m+d+1} \setminus \{A_m, B_m\}$, it follows that $X^d_m(\psi) (= \partial N^{d+1}_{m+d+1}(\psi))$ triangulates an $S^{d-1}$-bundle over $S^1$. But, there are only two such bundles: $S^{1,d-1}(\varepsilon)$, $\varepsilon = 0, 1$ (cf. [14, pages 134–135]). This is orientable for $\varepsilon = 0$ and non-orientable for $\varepsilon = 1$. Hence the result. □

Notice that $x \in B_m$ is at a distance $\geq 3$ from $y \in A_m$ (in the edge graph of $\partial N^{d+1}_{m+d+1}$) if and only if $x - y \geq 2d + 3$. Therefore, if $m \leq 2d + 2$, it is easy to see that there is no admissible bijection $\psi: B_m \to A_m$. For $m \geq 2d + 3$ the map $\psi: B_m \to A_m$ defined by $\psi_0(m+i) = i$ is admissible. When $m = 2d+3$, it is the only admissible map and the resulting combinatorial manifold $X^d_{2d+3}(\psi_0)$ is Kühnel’s $K^d_{2d+3}$, triangulating $S^{1,d-1}(\varepsilon)$, $d \equiv \varepsilon (\text{mod } 2)$, whose uniqueness we prove in Section 4 below. For $m \geq 2d + 3$, Kühnel and Lassmann constructed $X^d_m(\psi_0)$ and proved that for $m$ odd $X^d_m(\psi_0)$ is orientable if and only if $d$ is even (cf. [10]). Here we have:

**Lemma 3.3.** Let $m \geq 2d + 3$. If $md$ is even then for any admissible $\psi: B_m \to A_m$, the combinatorial $d$-manifold $X^d_m(\psi)$ is orientable if and only if $\psi \circ \psi_0^{-1}$ is an even permutation. In other words, if $\psi \circ \psi_0^{-1}$ is an even (respectively, odd) permutation then $X^d_m(\psi)$ is a combinatorial triangulation of $S^{1,d-1}(0)$ (respectively, $S^{1,d-1}(1)$).

**Proof.** For $1 \leq k \leq m$, $1 \leq i \leq d$, let $\sigma_{k,i}$ denote the facet $\{k, k+1, \ldots, k+d+1\} \setminus \{k+i\}$ and for $0 \leq i < j \leq d+1$, $(i,j) \neq (0,d+1)$, let $\sigma_{k,i,j}$ denote the $(d-1)$-simplex $\{k, k+1, \ldots, k+d+1\} \setminus \{k+i, k+j\}$ of $M^d_{m+d+1}$. Consider the orientation on $M^d_{m+d+1}$ given by:

$$
+\sigma_{k,i,j} = (-1)^{2d+i+j} k, \ldots, k+i-1, k+i+1, \ldots, k+j-1, k+j+1, \ldots, k+d+1,
+\sigma_{k,i} = (-1)^{2d+i} k, k+1, \ldots, k+i-1, k+i+1, \ldots, k+d+1.
$$

(4)

By an easy computation one sees that the incidence numbers satisfy the following:

$$
[\sigma_{k,i}, \sigma_{k,i,j}] = -1; \quad [\sigma_{k,j}, \sigma_{k,i,j}] = 1 \quad \text{for } 1 \leq i < j \leq d, 1 \leq k \leq m \quad \text{and} \quad [\sigma_{k,i}, \sigma_{k,0,i}] = 1,
\sigma_{k+1,0,i} = [\sigma_{k+1,i,1}, \sigma_{k+1,0,i-1,d+1}] = (-1)^{2d-1} = -1 \quad \text{for } 1 \leq i \leq d, 1 \leq k \leq m.
$$

Thus, (4) gives an orientation on $M^d_{m+d+1}$.

Let $\tilde{\sigma}_{k,i}$ and $\tilde{\sigma}_{k,i,j}$ denote the corresponding simplices in $X^d_m(\psi_0)$. Observe that $\tilde{\sigma}_{k,0,j} = \tilde{\sigma}_{k+1,1,j-1,d+1}$ for $1 \leq k < m$ and $\tilde{\sigma}_{m,0,j} = \tilde{\sigma}_{1,j-1,d+1}$. (The vertex-set of $X^d_m(\psi_0)$ is the set of integers modulo $m$.) Then the above orientation induces an orientation on $X^d_m(\psi_0)$. (This is well defined since $+\sigma_{m,0,j} = (-1)^{md+j}(m+1, \ldots, m+j-1, m+j+1, \ldots, m+d+1) = (-1)^{j+1, \ldots, j+1, \ldots, d+1} = (-1)^{j+1, \ldots, j+1, \ldots, d+1} = +\sigma_{1,j-1,d+1}$. Now, $[\tilde{\sigma}_{m,j}, \tilde{\sigma}_{m,0,j}] = 1; \quad [\tilde{\sigma}_{1,j-1}, \tilde{\sigma}_{m,0,j}] = [\tilde{\sigma}_{1,j-1,d+1}] = -1$. Thus, $[\tilde{\sigma}_{m,j}, \tilde{\sigma}_{m,0,j}] = [-\tilde{\sigma}_{1,j-1,d+1}]$. Therefore, the induced orientation on $X^d_m(\psi_0)$ is coherent. So, $X^d_m(\psi_0)$ is orientable. This implies that $X^d_m(\psi_0)$ triangulates $S^1 \times S^{d-1} = S^{1,d-1}(0)$.

Since $[M^d_{m+d+1}]$ is homeomorphic to $[S^{d-1}_{d+1}(B_m)] \times [0,1]$, we can choose an orientation on $[S^{d-1}_{d+1}(B_m)]$ so that the orientation on $[M^d_{m+d+1}]$ as the product $[S^{d-1}_{d+1}(B_m)] \times [0,1]$ is the same as the orientation given in (4). This also induces an orientation on $[S^{d-1}_{d+1}(A_m)]$. Let $S_B$ (respectively, $S_A$) denote the oriented sphere $[S^{d-1}_{d+1}(B_m)]$ (respectively $[S^{d-1}_{d+1}(A_m)]$) with this orientation. Then, as the boundary of an oriented manifold, $\partial([M^d_{m+d+1}]) = S_A \cup (-S_B)$. [In fact, it is not difficult to see that the orientation defined in (4) on $S^{d-1}_{d+1}(A_m)$ (respectively $S^{d-1}_{d+1}(B_m)$) is the same as the orientation in $S_A$ (respectively $S_B$).]

Let $\psi_0: S_B \to S_A$ be the homeomorphism induced by $\psi_0$. Since $|X^d_m(\psi_0)|$ is orientable, it follows that $\psi_0: S_B \to S_A$ is orientation preserving (cf. [14, pages 134–135]).
Therefore, $\psi \circ \psi_0^{-1}$ is an even (respectively odd) permutation $\Rightarrow |\psi \circ \psi_0^{-1}|: S_A \to S_A$ is orientation preserving (respectively reversing) $\Rightarrow |\psi| = |\psi \circ \psi_0^{-1}| \circ |\psi_0|: S_B \to S_A$ is orientation preserving (respectively reversing) $\Rightarrow |X^d_m(\psi)|$ is orientable (respectively non-orientable). Hence, the result follows from Lemma 3.2. □

Now take $m = 2d + 4$. A bijection $\psi: \{2d + 5, \ldots, 3d + 5\} \to \{1, \ldots, d + 1\}$ is admissible for $\partial N^{d+1}$ if and only if $x - \psi(x) \geq 2d + 3$ for $2d + 5 \leq x \leq 3d + 5$. It turns out that there are $2^d$ distinct admissible choices for $\psi$. But it seems difficult to decide when two admissible choices for $\psi$ yield isomorphic complexes $X^d_{2d+4}(\psi)$. So, we specialize as follows:

Let $p = (p_1, p_2, \ldots, p_k)$ be a partition of $d + 1$. Put $s_0 = 0$ and $s_j = \sum_{i=1}^j p_i$ for $1 \leq j \leq k$. (Thus, in particular, $s_1 = p_1$ and $s_k = d + 1$.) Let $\pi_p$ be the permutation of $\{1, 2, \ldots, d + 1\}$ which is the product of $k$ disjoint cycles $(s_{j-1} + 1, s_{j-1} + 2, \ldots, s_j)$, $1 \leq j \leq k$. Notice that $\pi_p$ is an even (respectively, odd) permutation if $p$ is an even (respectively, odd) partition of $d + 1$. Now, define the bijection $\psi_p: \{2d + 5, 2d + 6, \ldots, 3d + 5\} \to \{1, 2, \ldots, d + 1\}$ by $\psi_p(2d + 4 + i) = \pi_p(i)$, $1 \leq i \leq d + 1$. Since $\pi_p(i) \leq i + 1$ for $1 \leq i \leq d + 1$, it follows that $\psi_p$ is an admissible bijection. Clearly, the corresponding complex $X^d_{2d+4}(\psi_p)$ depends only on the partition $p$ of $d + 1$. We denote it by $K^d_{2d+4}(p)$. Note that $\pi_p = \psi_p \circ \psi_0^{-1}$.

Therefore, by Lemma 3.3, $K^d_{2d+4}(p)$ triangulates $S^{1,d-1}(0)$ (respectively, $S^{1,d-1}(1)$) if $p$ is an even (respectively odd) partition of $d + 1$.

Let $G_p$ denote the non-edge graph of $K^d_{2d+4}(p)$. Its vertex-set is $V(K^d_{2d+4}(p))$, and two distinct vertices $x$, $y$ are adjacent in $G_p$ if $xy$ is not an edge of $K^d_{2d+4}(p)$. It turns out that $G_p$ has a clear description in terms of the partition $p$. For $b \geq 1$, let $K_{b,b}$ denote the unique graph with one vertex of degree $b$ and $b$ vertices of degree one. Also, let $p = (p_1, p_2, \ldots, p_k)$, and put $p_0 = 1$. Then a computation shows that $G_p$ is the disjoint union of $K_{p_0,p_1}, 0 \leq i \leq k$.

Thus, if $p$ and $q$ are distinct partitions of $d + 1$ then $G_p$ and $G_q$ are non-isomorphic (this is where our assumption that $p$, $q$ are weakly increasing sequences comes into play!) and hence $K^d_{2d+4}(p)$ and $K^d_{2d+4}(q)$ are non-isomorphic complexes. Thus we have proved:

**Theorem 3.** For any partition $p$ of $d + 1 \geq 3$, let $\varepsilon = \varepsilon(p) = 0$ if $p$ is even and $= 1$ if $p$ is odd. Then $K^d_{2d+4}(p)$ is a $(2d + 4)$-vertex triangulation of $S^{1,d-1}(\varepsilon)$. Further, distinct partitions $p$ of $d + 1$ correspond to non-isomorphic triangulations of $S^{1,d-1}(\varepsilon)$. In consequence, for $\varepsilon = 0, 1$, there are $(2d + 4)$-vertex combinatorial triangulations of $S^{1,d-1}(\varepsilon)$ and the number of non-isomorphic triangulations is at least $P_{\varepsilon}(d + 1) \sim \frac{\varepsilon_2}{2d} e^{2\sqrt{d}}$.

This theorem provides an affirmative solution of the conjecture (made by Lutz in [11]) that $S^{1,d-1}(1)$ can be triangulated by $2d + 4$ vertices for $d$ even. Notice that each $(2d + 4)$-vertex triangulation of $S^{1,d-1}(\varepsilon)$ constructed here has $d + 2$ non-edges. We conjecture that this is the maximum possible number of non-edges. If this is true then, for $d \equiv 1 - \varepsilon$ (mod 2), our construction yields triangulations of $S^{1,d-1}(\varepsilon)$ with the minimum number of vertices and edges.

**4 Uniqueness of $K^d_{2d+3}$**

Recall from Section 3 that for $d \geq 2$, $K^d_{2d+3}$ is the $(2d + 3)$-vertex combinatorial $d$-manifold constructed by Kühnel in [9]. It triangulates $S^{1,d-1}(\varepsilon)$, where $\varepsilon \in (0, 1)$ is given by $\varepsilon \equiv d$ (mod 2). One description of $K^d_{2d+3}$ is implicit in Section 3. An equivalent (and somewhat simpler) description is as follows. It is the boundary complex of the combinatorial $(d + 1)$-manifold with boundary whose vertices are the vertices of a cycle $S^1_{2d+3}$ of length $2d + 3$,
and facets are the sets of $d + 2$ vertices spanning a path in the cycle. From this picture, it is clear that the dihedral group of order $4d + 6 (= \text{Aut}(S^1_{2d+3}))$ is the full automorphism group of $K_{2d+3}^d$. Here we prove that for $d \geq 3$, up to simplicial isomorphism, $K_{2d+3}^d$ is the unique $(2d + 3)$-vertex non-simply connected triangulated $d$-manifold.

**Lemma 4.1.** (Simplicial Alexander duality) Let $L \subset L'$ be induced subcomplexes of a triangulated $d$-manifold $X$. Let $R \supset R'$ be the simplicial complements in $X$ of $L$ and $L'$ respectively. Then $H_{d-j}(L', L; \mathbb{Z}_2) \cong H_j(R, R'; \mathbb{Z}_2)$ for $0 \leq j \leq d$.

**Proof.** Fix a piecewise linear map $f : |X| \to \mathbb{R}$ such that for all vertices $u$ of $L$, $v$ of $R$ we have $f(u) < f(v)$, and for all vertices $u'$ of $L'$, $v'$ of $R'$ we have $f(u') < f(v')$. Choose $c < c'$ in $\mathbb{R}$ such that $f(u) < c < f(v)$ and $f(u') < c' < f(v')$ for all such $u, v, u', v'$. Define $L = \{x \in |X| : f(x) \leq c\}$, $R = \{x \in |X| : f(x) > c\}$, $L' = \{x \in |X| : f(x) \leq c'\}$, $R' = \{x \in |X| : f(x) > c'\}$. Since $f$ is piecewise linear, it follows that $L, L'$ are compact polyhedra (i.e., geometric carriers of finite simplicial complexes). Also, $(|L'|, |L|)$ (respectively $(|R|, |R'|)$) is a strong deformation retract of $(L', L)$ (respectively $(R, R')$). Hence we have

$$H_{d-j}(L', L; \mathbb{Z}_2) \cong H_{d-j}(L, |L'; \mathbb{Z}_2) \cong H_{d-j}(L', L; \mathbb{Z}_2) \cong H^d_{d-j}(L', L; \mathbb{Z}_2) \cong H_j(R, R'; \mathbb{Z}_2)$$

for $0 \leq j \leq d$.

Here, the fourth isomorphism is because of Alexander duality (cf. [13, Theorem 17, Page 296]). The usual statement of this duality refers to Alexander cohomology, but this agrees with singular cohomology for polyhedral pairs (cf. [13, Corollary 11, Page 291]). Also, Alexander duality applies to orientable closed manifolds, but any closed manifold (such as $|X|$ in our application) is orientable over $\mathbb{Z}_2$. The third isomorphism holds since over a field, homology and cohomology are isomorphic.

**Lemma 4.2.** Let $X$ be a non-simply connected $n$-vertex triangulated manifold of dimension $d \geq 3$. Then $n \geq 2d + 3$. If further, $n = 2d + 3$, then for any facet $\sigma$ of $X$ and any vertex $x$ outside $\sigma$, either the induced subcomplex of $X$ on $V(X) \setminus (\sigma \cup \{x\})$ is an $S^{d-1}_{d+1}$ or the induced subcomplex $\text{lk}_X(x)[\sigma]$ of $\text{lk}_X(x)$ on the vertex set $\sigma$ is disconnected.

**Proof.** Let $\sigma$ be a facet and $C = C(\sigma, X)$ be its simplicial complement. Choose a small (simply connected) neighbourhood $U$ of $|\sigma|$ in $|X|$ such that $U \cap (|X| \setminus |\sigma|)$ is homeomorphic to $S^{d-1} \times (0, 1)$. Now, $|X|$ is non-simply connected, $|X| = U \cup (|X| \setminus |\sigma|)$ and $d \geq 3$. So, by Van Kampen’s theorem, $|X| \setminus |\sigma|$ is non-simply connected. But $|C|$ is a strong deformation retract of $|X| \setminus |\sigma|$. Therefore, $C$ is non-simply connected.

Now fix a facet $\sigma$ of $X$. Choose an ordering $x_1, x_2, \ldots, x_n$ of $V(X)$ so that $\sigma = \{x_1, \ldots, x_{d+1}\}$. For $1 \leq i \leq n$, let $L_i$ (respectively $R_i$) be the induced subcomplex of $X$ on the vertex-set $\{x_1, \ldots, x_i\}$ (respectively $\{x_{i+1}, \ldots, x_n\}$). Then, by Lemma 4.1,

$$H_j(R_i, R_{i+1}) \cong H_{d-j}(L_{i+1}, L_i), \quad \text{for } 0 \leq j \leq d \quad \text{and} \quad 1 \leq i < n. \quad (5)$$

Here the homologies are taken with coefficients in $\mathbb{Z}_2$.

Since $L_1 = \{x_1\}$ is simply connected but $L_n = X$ is not, it follows that there is a (smallest) index $i$ such that $L_i$ is simply connected but $L_{i+1}$ is not. Note that $i \geq d + 1$. Choose this $i$. Since $L_{i+1} = L_i \cup \text{st}_{L_{i+1}}(x_{i+1})$ and $L_i \cap \text{st}_{L_{i+1}}(x_{i+1}) = \text{lk}_{L_{i+1}}(x_{i+1})$, Van Kampen’s theorem implies that $\text{lk}_{L_{i+1}}(x_{i+1})$ is not connected. Hence $H_1(L_{i+1}, L_i) \cong H_1(\text{st}_{L_{i+1}}(x_{i+1}), \text{lk}_{L_{i+1}}(x_{i+1})) \cong \overline{H}_0(\text{lk}_{L_{i+1}}(x_{i+1})) \neq \{0\}$. Thus, there is an index $i \geq d + 1$ such that $H_1(L_{i+1}, L_i) \neq \{0\}$. Hence, from (5), it follows that

$$H_{d-2}(\text{lk}_{R_i}(x_{i+1})) \cong H_{d-1}(R_i, R_{i+1}) \neq \{0\} \quad \text{for some } i \geq d + 1. \quad (6)$$
Notice that we have \( R_{i+1} \subset R_i \subseteq C = C(\sigma, X) \). Since \( H_{d-1}(R_i, R_{i+1}) \neq \{0\} \), \( R_i \) contains at least two \((d - 1)\)-faces. Hence the number of vertices in \( R_i \) is \( \geq d + 1 \).

First suppose \( R_i \) has exactly \( d + 1 \) vertices. Since \( H_{d-2}(\text{lk}_{R_i}(x_{i+1})) \neq \{0\} \) and \( \text{lk}_{R_i}(x_{i+1}) \) has at most \( d \) vertices, it follows that \( \text{lk}_{R_i}(x_{i+1}) = S_{d-2}^d \). Since \( d \geq 3 \), it follows that \( R_i \) is simply connected. As \( C \) is not simply connected, we have \( R_i \subset C \) (proper inclusion). Thus \( n \geq (d + 1) + 1 - (d + 1) = 2d + 3 \). Also, if the number \( n - i \) of vertices in \( R_i \) is \( \geq d + 2 \). Then \( n \geq i + d + 2 \geq 2d + 3 \). This proves the inequality.

Now assume that \( n = 2d + 3 \). Let \( x \notin \sigma \) be a vertex such that \( \text{lk}_{X}(x) \cap L_{d+1} = (\text{st}_{X}(x) \cap L_{d+1}) \) is connected. Choosing the vertex order so that \( x_{d+2} = x \), we get that \( L_{d+2} \) is simply connected (by Van Kampen theorem). Therefore \( i \geq d + 2 \). Hence \( R_i \) has \( \leq n - d - 2 = d + 1 \) vertices. But, \( H_{d-1}(R_i, R_{i+1}) \neq \{0\} \), so that \( R_i \) has \( \geq d + 1 \) vertices. Therefore \( R_i \) has exactly \( d + 1 \) vertices and hence \( i = d + 2 \). Thus, \( H_{d-2}(\text{lk}_{R_{d+2}}(x_{d+3})) \cong H_{d-1}(R_{d+2}, R_{d+3}) \neq \{0\} \). Since \( \text{lk}_{R_{d+2}}(x_{d+3}) \) has at most \( d \) vertices, it follows that \( \text{lk}_{R_{d+2}}(x_{d+3}) = S_{d-2}^d \). Since any vertex of \( R_{d+2} \) may be chosen to be \( x_{d+3} \) in this argument, we get that all the vertex links of \( R_{d+2} \) are isomorphic to \( S_{d-2}^d \). Hence the induced subcomplex \( R_{d+2} \) of \( C \) on the vertex set \( V(X) \setminus (\sigma \cup \{x\}) \) is an \( S_{d-1}^{d-1} \). This proves the lemma. \( \square \)

**Remark 4.1.** For combinatorial manifolds, the inequality in Lemma 4.2 is a theorem due to Brehm and Künnel [6].

**Lemma 4.3.** Let \( X \) be a \((2d + 3)\)-vertex non-simply connected triangulated manifold of dimension \( d \geq 3 \). Then, there is a facet \( \sigma \) of \( X \) such that its simplicial complement \( C(\sigma, X) \) contains an induced \( S_{d+1}^d \).

**Proof.** Suppose the contrary. Then, by Lemma 4.2, for each facet \( \sigma \) of \( X \) and each vertex \( x \notin \sigma \), the induced subcomplex \( \text{lk}_{X}(x)[\sigma] \) of \( \text{lk}_{X}(x) \) on \( \sigma \) is disconnected. If \( \tau \) were a \((d - 2)\)-face of \( X \) of degree 3, say with \( \text{lk}_{X}(\tau) = S_{3}^1(\{x_1, x_2, x_3\}) \), then the induced subcomplex \( \text{lk}_{X}(x_3) \) on the facet \( \tau \cup \{x_1, x_2\} \) would be connected - a contradiction. So, \( X \) has no \((d - 2)\)-face of degree 3. Now, no face \( \gamma \) of \( X \) of dimension \( e \leq d - 2 \) can have \((\text{minimal}) \) degree \( d - e + 1 \). (In other words, the link of \( \gamma \) cannot be a standard sphere.) Or else, any \((d - 2)\)-face \( \tau \supseteq \gamma \) of \( X \) would have degree 3. So, no standard sphere of positive dimension occurs as a link in \( X \).

Now fix a facet \( \sigma \) of \( X \). For each \( x \in \sigma \), there is a unique vertex \( x' \notin \sigma \) such that \((\sigma \setminus \{x\}) \cup \{x'\}\) is a facet. This defines a map \( x \mapsto x' \) from \( \sigma \) to its complement. This map is injective: if we had \( x_1' = y = x_2' \) for \( x_1 \neq x_2 \) then the induced subcomplex of \( \text{lk}_{X}(y) \) on \( \sigma \) would be connected. Also, since \( \text{lk}_{X}(x')[\sigma] \) is disconnected, it follows that \( x \) must be an isolated vertex in \( \text{lk}_{X}(x')[\sigma] \). This implies that \( xx' \) is an edge of \( X \), and \( V(\text{lk}_{X}(x')) \subseteq V(X) \setminus (\sigma \cup \{x'\}) \). Hence \( xx' \) is an edge of degree \( \leq d + 1 \). Therefore, by the observation in the previous paragraph (with \( e = 1 \)), \( \deg_{X}(xx') = d + 1 \). In consequence, \( \text{lk}_{X}(xx') \) is a \((d + 1)\)-vertex normal \((d - 2)\)-pseudo manifold. But all such normal pseudo manifolds are known: we must have \( \text{lk}_{X}(xx') = S_{m+2}^n \ast S_{n+2}^m \) for some \( m, n \geq 0 \) with \( m + n = d - 3 \) (cf. [2]). If \( m > 0 \) or \( n > 0 \) then \( S_{3}^1 \) occurs as a link (of some \((d - 4)\)-simplex) in this sphere and hence it occurs as the link of a \((d - 2)\)-simplex (containing \( xx' \)) in \( X \). Hence, we must have \( m = n = 0 \). Thus \( d = 3 \) and each of the four edges \( xx' (x \in \sigma) \) is of degree 4.

Then \( \text{lk}_{X}(xx') \) is an \( S_{1}^4 = S_{2}^0 \ast S_{2}^0 \) with vertex set \( V(X) \setminus (\sigma \cup \{x'\}) \). In consequence, putting \( C = C(\sigma, X) \), one sees that \( C \) is a 5-vertex non-simply connected simplicial complex (by the proof of Lemma 4.2) such that for at least four of the vertices \( x' \) in \( C \), \( \text{lk}_{C}(x') \supseteq S_{1}^4 \). In consequence, all \( \binom{5}{2} = 10 \) edges occur in \( C \). Since \( C \) is non-simply connected, it follows that \( C \) has at least one missing triangle (induced \( S_{1}^4 \)), say with vertices \( y_1, y_2, y_3 \). At
least two of these vertices (say \(y_1, y_2\)) have \(S_4^1\) in their links. It follows that \(\text{lk}_C(y_1) \supseteq S_0^d(\{y_2, y_3\}) \ast S_0^d(\{y_4, y_5\})\) and \(\text{lk}_C(y_2) \supseteq S_0^d(\{y_1, y_3\}) \ast S_0^d(\{y_4, y_5\})\) where \(y_4, y_5\) are the two other vertices of \(C\). Hence \(C \supseteq C_0 = (S_1^d(\{y_1, y_2, y_3\}) \ast S_0^d(\{y_4, y_5\})) \cup \{y_4y_5\}\). But all 5-vertex simplicial complexes properly containing \(C_0\) and not containing the 2-simplex \(y_1y_2y_3\) are simply connected. So, \(C = C_0\). But, then two of the vertices of \(C\) (viz. \(y_4, y_5\)) have no \(S_1^1\) in their links, a contradiction. This completes the proof.

**Theorem 4.** For \(d \geq 3\), Kühnel’s complex \(K_{2d+3}^d\) is the only non-simply connected \((2d + 3)\)-vertex triangulated manifold of dimension \(d\).

**Proof.** Let \(X\) be a non-simply connected \((2d + 3)\)-vertex triangulated manifold of dimension \(d \geq 3\). By Lemma 4.3, \(X\) must have a facet \(\sigma\) such that \(C(\sigma, X)\) contains an induced subcomplex \(S\) which is an \(S_{d+1}^{d-1}\). Let \(x\) be the unique vertex in \(C(\sigma, X) \setminus S\). If \(xy\) is a non-edge for each \(y \in \sigma\) then \(S_{d+1}^{d-1}\)-pseudomanifold \(\text{lk}_X(x)\) is a subcomplex of the \((d-1)\)-sphere \(S\) and hence \(\text{lk}_X(x) = S\). This implies that \(C(\sigma, X)\) is the combinatorial \(d\)-ball \(\{x\} \ast S\). This is not possible since \(C(\sigma, X)\) is non-simply connected. Thus, \(x\) forms an edge with a vertex in \(\sigma\). This implies that \(C(S, X)\) is connected.

Thus, \(S\) is an induced \(S_{d+1}^{d-1}\) in \(X\), and \(C(S, X)\) is connected. Since \(d \geq 3\), \(S\) is two-sided in \(X\). By Lemma 1.3, we may delete the handle over \(S\) to get a \((3d + 4)\)-vertex normal \(d\)-pseudomanifold \(\tilde{X}\). Since \(X\) has at most \(\binom{2d+3}{2}\) edges, it follows that \(\tilde{X}\) has at most \(\binom{2d+3}{2} + \binom{d+1}{2}\) edges. But \(\binom{2d+3}{2} + \binom{d+1}{2} = (3d + 4)(d + 1) - \binom{d+2}{2}\) is the lower bound on the number of edges of a \((3d + 4)\)-vertex normal \(d\)-pseudomanifold given by the Lower Bound Theorem (cf. Theorem 1). Therefore, \(\tilde{X}\) attains the lower bound, and hence, by Theorem 1, \(\tilde{X}\) is a stacked sphere. Now, Lemma 1.3 implies that \(X = \tilde{X}^\psi\) where \(\psi: \sigma_1 \rightarrow \sigma_2\) is an admissible bijection between two facets of \(\tilde{X}\). Thus, \(\tilde{X}\) is a \((3d + 4)\)-vertex stacked sphere with an admissible bijection \(\psi\). Therefore, by Theorem 2, \(\tilde{X} = S_{3d+4}^d\) and \(\psi\) are uniquely determined, hence so is \(X = \tilde{X}^\psi\). Since \(K_{2d+3}^d\) satisfies the hypothesis, it follows that \(X = K_{2d+3}^d, \quad \square\)

**Corollary 5.** Let \(X\) be an \(n\)-vertex triangulation of an \(S_{d-1}^{d-1}\)-bundle over \(S^1\). If \(d \geq 2\) then \(n \geq 2d + 3\). Further, if \(n = 2d + 3\), then \(X\) is isomorphic to \(K_{2d+3}^d\).

**Proof.** Since an \(S_{d-1}^{d-1}\)-bundle over \(S^1\) is non-simply connected, the result is immediate from Lemma 4.2 and Theorem 4 for \(d \geq 3\). For \(d = 2\), this result is classical. \(\square\)

**Corollary 6.** If \(d \geq 2\), \(\varepsilon \equiv d \pmod{2}\) then \(S_{d-1}^{1,d-1}(\varepsilon)\) has a unique \((2d + 3)\)-vertex triangulation, namely \(K_{2d+3}^d\).

**Proof.** Since \(S_{d-1}^{1,d-1}(\varepsilon)\) (with \(\varepsilon \equiv d \pmod{2}\)) is non-simply connected and is the geometric carrier of \(K_{2d+3}^d\), the result is immediate from Theorem 4 for \(d \geq 3\). For \(d = 2\), this result is classical. \(\square\)

**Corollary 7.** If \(d \geq 2\), \(\varepsilon \not\equiv d \pmod{2}\) then any triangulation of \(S_{d-1}^{1,d-1}(\varepsilon)\) requires at least \(2d + 4\) vertices. Thus, for this manifold, the \((2d + 4)\)-vertex triangulations in Section 3 are vertex minimal.

**Proof.** Since \(S_{d-1}^{1,d-1}(\varepsilon)\) (with \(\varepsilon \not\equiv d \pmod{2}\)) is non-simply connected and \(K_{2d+3}^d\) does not triangulate this space, the result is immediate from Theorem 4 for \(d \geq 3\). For \(d = 2\), this result is classical. \(\square\)
Corollary 8. (Walkup, Altshuler and Steinberg) $K_9^3$ is the unique 9-vertex triangulated 3-manifold which is not a combinatorial 3-sphere. In consequence, every closed 3-manifold other than $S^3$ and $S^1 \times S^2$ requires at least 10 vertices for a triangulation.

Proof. Note that any triangulated 3-manifold is a combinatorial 3-manifold. The result is immediate from Theorem 4, since by the Poincaré-Perelman theorem, the 3-sphere is the only simply connected closed 3-manifold. However, it is not necessary to invoke such a powerful result. Since a simply connected 3-manifold is clearly a homology 3-sphere, and by a result of [3] any homology 3-sphere (other that $S^3$) requires at least 12 vertices, the corollary follows from Theorem 4. \[\square\]

A few days after we posted the first two versions of this paper in the arXiv (arXiv:math.GT/0610829) a similar paper [7] was posted in the arXiv (arXiv:math.CO/0611039) by Chestnut, Sapir and Swartz. In that paper, the authors prove the uniqueness of $K_{2d+3}^d$ in the broader class of homology $d$-manifolds (compared to the class of triangulated $d$-manifolds considered here) but with a much more restrictive topological condition (viz., $\beta_1 \neq 0$ and $\beta_2 = 0$, compared to our hypothesis of non-simply connectedness).

Acknowledgement: The authors thank the anonymous referees for many useful comments which led to substantial improvements in the presentation of this paper. The authors are thankful to Siddhartha Gadgil for useful conversations. The second author was partially supported by DST (Grant: SR/S4/MS-272/05) and by UGC-SAP/DSA-IV.

References

[1] A. Altshuler, L. Steinberg, An enumeration of combinatorial 3-manifolds with nine vertices, Discrete Math. 16 (1976) 91–108.
[2] B. Bagchi, B. Datta, A structure theorem for pseudomanifolds, Discrete Math. 168 (1998) 41–60.
[3] B. Bagchi, B. Datta, Combinatorial triangulations of homology spheres, Discrete Math. 305 (2005) 1–17.
[4] B. Bagchi, B. Datta, Lower bound theorems for pseudomanifolds (preprint).
[5] D. Barnette, A proof of the lower bound conjecture for convex polytopes, Pacific J. Math. 46 (1973), 349–354.
[6] U. Brehm, W. Kühnel, Combinatorial manifolds with few vertices, Topology 26 (1987) 465–473.
[7] J. Chestnut, J. Sapir, E. Swartz, Enumerative properties of triangulations of spherical bundles over $S^1$, European J. Combin. (to appear).
[8] G. Kalai, Rigidity and the lower bound theorem 1, Invent. math. 88 (1987) 125–151.
[9] W. Kühnel, Higher dimensional analogues of Császár’s torus, Results in Mathematics 9 (1986) 95–106.
[10] W. Kühnel, G. Lassmann, Permuted difference cycles and triangulated sphere bundles, Discrete Math. 162 (1996) 215–227.
[11] F. H. Lutz, Triangulated manifolds with few vertices: Combinatorial manifolds, v1 (2005) arXiv:math.CO/0506372.
[12] H. Rademacher, Topics in Analytic Number Theory, Springer-Verlag, New York Heidelberg, 1973.
[13] E. H. Spanier, Algebraic Topology, Springer-Verlag, New York Berlin Heidelberg, 1966.
[14] N. Steenrod, The topology of fibre bundles, Princeton Univ. Press, Princeton, 1951.
[15] T. S. Tay, Lower-bound theorems for pseudomanifolds, Discrete Comput Geom. 13 (1995) 203–216.
[16] D. W. Walkup, The lower bound conjecture for 3- and 4-manifolds, Acta Math. 125 (1970) 75–107.