Concentration Bounds for the Collision Estimator

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Abstract
We prove a strong concentration result about the natural collision estimator, which counts the number of collisions that occur within an iid sample. This estimator is at the heart of algorithms used for uniformity testing and entropy assessment.

While the prior works were limited to only variance, we use elegant techniques of independent interest to bounds higher moments and conclude concentration properties. As an immediate corollary we show that the estimator achieves high-probability guarantee on its own and there is no need for boosting (aka median/majority trick).

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1 Introduction

1.1 Collision Estimation
For many applications, such as key derivation in cryptography [10], property testing [12] and general algorithms [1] it is of interest to estimate the collision probability of a distribution $X$

$$Q \triangleq \sum_x \Pr[X = x]^2$$

(1)

Given a sample $X_1, \ldots, X_n \sim iid$ $X$ one defines the *natural* collision estimator as

$$\hat{Q} \triangleq \frac{1}{n(n-1)} \sum_{i \neq j} 1(X_i = X_j).$$

(2)

which resembles the birthday paradox. In this work we obtain a strong result about its concentration properties, which can be formally stated as follows.

1.2 Our Contribution

▶ Theorem 1 (Tails of Collision Estimator). The estimator (2), after centering, has tails

$$\Pr[|\hat{Q} - Q| > \epsilon] \leq O(1) \exp(-\Omega(\min(\epsilon^2/v^2, \epsilon/b, n\sqrt{\epsilon}))).$$

(3)

where we define

$$v^2 \triangleq \sum_x \Pr[X = x]^2/n^2 + \sum_x \Pr[X = x]^3/n$$

(4)

$$b \triangleq \max_x \Pr[X = x]/n$$

(5)

▶ Remark 2 (Intuition: variance and scale). Best way to understand the concentration bounds in Theorem 1 is to think of $v^2$ as a variance proxy (in fact we have $\text{Var}(\hat{Q}) = O(v^2)$) and of $b$ as as scale parameter. Then the tail of $\exp(-\Omega(\min(\epsilon^2/v^2, \epsilon/b)))$ is typical for so-called sub-gamma distributions [6]. The term with $\epsilon^{1/2}$ appears due to a possible heavy tail behavior: when the moments grow like $d^{2d}$ we get the tail of $e^{-\Omega(\epsilon^{1/2})}$.
1.2.1 Related Work

To the best knowledge of the author, there are no prior works on exact concentration of the collision estimator. The variance of collision estimator itself has been studied extensively in the context of uniformity testing [4, 13, 18, 9, 12], and Renyi entropy estimation [1, 2, 16], but we lack of understanding of higher moments and concentration properties. The techniques used to handle the variance were merely manipulation of algebraic expressions with some combinatorics to carry out term cancellations, which is hard to scale to higher moments. It is also not possible to derive a concentration result by a black-box application of known concentration inequalities: the main problem is that the estimator $\tilde{Q}$ is a quadratic form of correlated inputs, where the inputs are possibly very rare events. Leaving aside the problem of correlation, the right tool to attack the quadratic form would be the Hanson-Wright inequality; however examining the state-of-art variants [20, 5] we find them insufficient in our context (for example, we get very weak scale term $b$). For these reason we resort to direct moment estimates; again there is no directly applicable formula, but at the core of our proof is the sharp moment characterization due to Latala [15].

1.2.2 Outline of Techniques

Our result in Theorem 1 is of interest not only because of its strong quantitative guarantees (which we will see later when discussing applications) but also because of elegant techniques of independent interest, that are used in the proof. We elaborate on that below.

1.2.2.1 Negative Dependence

Negatively dependency of random variables, roughly speaking, captures the property that one of them increases others are more likely to decrease. This property is a very strong form of negative correlation known to imply concentration bounds comparable to those of independent random variables; essentially (in the context of concentration bounds) one works with negatively dependent random variables as if they were independent, which simplifies an analysis to a great extent. Best-known from applications to balls and bin problems, the theory of negative dependence has been summarized in [14] and [11].

In this work we leverage the negative dependence by proving that this property holds for estimator’s contributions (thought as loads) from possible outcomes of the distribution (thought as bins); more precisely negatively dependent are $\tilde{Q}_x \triangleq \sum_{i \neq j} I(X_i = x) I(X_j = x)$ indexed by $x$. This reduces the problem to studying sums of independent random variables distributed as $\tilde{Q}_x$. As a remark we note that this trick can be also used to simplify a bulk of computations in the case of higher-order collisions, studied in higher-order Renyi entropy estimators [2, 1, 16].

1.2.2.2 Subtle Moment Methods

Most of concentration results in TCS papers are obtained by a black-box application of Chernoff-like bounds, and it is not so common to face up a case where these inequalities fail to produce good results. As we point out in this work, collision estimation seems to be such a use case. The problem is that observing every fixed element $x$ in a sample is a rare event with extremely small probability (for example, in the birthday paradox setup we have $\Pr[X = x] = 1/m$ whereas $n = O(\sqrt{m})$). Since we have $\tilde{Q}_x = \sum_{i \neq j} \|X_i = x\| \|X_j = x\|$ which are quadratic forms of these rare events, we may want to apply a variant of Hanson-Wright’s Inequality [20, 5] and then to assemble obtained concentrations of $\tilde{Q}_x$ into a
concentration result for $\sum_x \tilde{Q}_x$ (e.g. by Cramer-Chernoff); unfortunately best known bounds for quadratic forms do not behave well if inputs are very small (they contain distribution-free terms in exponent; in our case we would end up with much weaker $b = O(1/n)$).

This motivates the broader question on what to do when exponential inequalities fail? Our solution is to resort to subtle moment methods that have been studied particularly by Latala [15]. One important contributions of this paper is that we simplify one of his results, showing how to estimate moments (and hence concentration properties) of sums $\sum_x Z_x$ by controlling sum of moments, e.g. by bounding $\sum_x E|Z_x|^d$. This bound is very convenient as individual moments are much easier to compute; we call such conditions Rosenthal-type due to the celebrated result of Rosenthal [19] in same spirit. We note that our technique can be used in other problems where applications of exponential concentration bounds are problematic, for example applied to the problem of missing mass [17].

1.2.2.3 Reduction to Binomial Moments

Armed with the Rosenthal-type concentration bounds we are left with estimating the moments of estimator contributions $\tilde{Q}_x$. Here we apply the tricks that have been proven useful when dealing with quadratic forms: centering, symmetrization and decoupling. Eventually we are able to link moments of $\tilde{Q}_x$ with those of (symmetrized) binomial distributions. More precisely for $p = Pr[X = x], S, S' \sim iid Binom(n, p)$ we obtain the following bound

$$E|\tilde{Q}_x - E\tilde{Q}_x|^d \leq O(E|S - S'|^d) + O(dn^2p^2)^{d/2}E|S - S'|^d$$

This explains the specific form of Theorem 1: consider for simplicity $d = 2$ then the first and second term on the right-hand side contribute respectively $p = Pr[X = x]^2$ and $p = Pr[X = x]^3$; with higher $d$ they contribute respectively $p^{d/2}$ and $p^{3d/2}$.

The crucial step here is to use asymptotically sharp bound on binomial moments (so that we have optimal dependency on $d$). Since they are hard to find in the literature, we prove such bounds using an elementary combinatorial approach along with symmetrization.

1.3 Applications

1.3.1 Application to Uniformity Testing

In uniformity testing one wants to know how close is some unknown distribution to the uniform one, based on a random sample. An appealing idea is to relate the closeness to the collision probability $Q$: for a distribution over $m$ elements the smallest value of $Q$ is $1/m$ which is realized by the uniform distribution $U_m$. Then the closer is $Q$ to $1/m$, the closer is $X$ to $U_m$. Such a test was studied by a number of authors [4, 13, 18, 9, 12] with optimal bounds found in [9]. Remarkably, our concentration bounds imply that such a test achieves high-probability guarantees on its own, as stated in Corollary 3. Prior to our work the test guarantees were quite weak so it was necessary to run multiple tests in parallel.

```python
def l2closeness_to_uniform(X, n, epsilon):
    m ← |dom(X)| /* domain size */
    x[1]...x[n] ← iid X /* get iid samples */
    Q ← #{(i, j) : x[i] = x[j], i ≠ j} /* count collisions */
    Q ← Q/n(m−1) /* normalize */
    if Q > (1 + ϵ)/m:
        return False
    elif Q < (1 + ϵ)/m:
        return True
```
Corollary 3 (Optimal Sublinear Collision Tester). If $X$ is distributed over $m$ elements, with

$$n = O(\log(1/\delta)m^{1/2}/\epsilon)$$

samples the algorithm in Listing 1 distinguishes with probability $\delta$ between a) $\|P_X - U_m\|_2^2 \leq \frac{1}{zm}$ and b) $\|P_X - U_m\|_2^2 \geq \frac{2\epsilon}{m}$, when $1/\sqrt{m} \leq \epsilon \leq 1$.

Remark 4 (Comparison with [9]). The novelty is that we do not require parallel runs to get small error probability $\delta$. The sample size $n$ matches the best bound due to [9] under the mild restriction that $n = O(m)$ (which implies $\epsilon = \Omega(m^{-1/2})$ in Corollary 3) that is the number of samples is at most linear in the alphabet size. Such sublinear algorithms are of practical interest when the alphabet is huge; the restriction $\epsilon = \Omega(m^{-1/2})$ is also sufficient for virtually all cryptographic applications because when $X \in \{0,1\}^d$ we have that $m^{-1/2} = 2^{-d/2}$ corresponds to exponential security guarantees).

1.3.2 Application to Rényi Entropy Estimation

Consider the problem of relative estimation, where $\epsilon := \epsilon Q$. This can be seen as estimation of collision entropy $H_2(X) \triangleq -\log Q$ within an additive error of $\epsilon$ [2]. Our result implies again that the estimator achieves high-probability guarantee on its own, without parallel runs.

Corollary 5 (Collision Estimation). We have

$$\Pr[|\tilde{Q} - Q| > \epsilon Q] \leq O(1) \exp(-\Omega(n\epsilon^2Q^{1/2})), \quad 0 < \epsilon < 1.$$  

Which shows that the estimator requires

$$n = O(\log(1/\delta)Q^{-1/2}/\epsilon^2)$$

samples to achieve relative error of $\epsilon$ and probability guarantee of $1 - \delta$.

Remark 6 (Optimality). In the worst case we have $n = O(m^{1/2}/\epsilon^2)$ when the domain of $X$ has $m$ elements, which matches the lower bounds [2].

Remark 7 (Difference from Uniformity Testing: Phase Transition). The bounds for collision estimation are sharp when $Q = \Omega(1/m)$, however for uniformity testing one considers the different regime of $Q = 1/m \cdot (1 + o(1))$, so the lower bounds [2] no longer apply. Indeed, uniformity testing allows a better dependency on $\epsilon$ that suggested by the general collision estimation. This is an interesting phenomena that could be seen as a "phase transition".

1.4 Organization

In Section 3 we show in detail how to derive results on applications claimed above. The proof of the main result is given in Section 4 and follows the presented outline. In Section 5 we conclude the work.
2 Preliminaries

2.1 Sub-Gamma Distributions

A random variable \( Z \) is sub-gamma with variance factor \( v^2 \) and scale \( b \) when
\[
E \exp(tZ) \leq \exp \left( \frac{v^2 t^2}{2(1 - bt)} \right), \quad \text{when } |t| < 1/b
\]
Such a distribution has gamma-like tails (by the Cramer-Chernoff method [8, 7], see [6])

\begin{proposition}[Sub-Gamma Tails]
If \( Z \) is sub-gamma with variance factor \( v^2 \) and scale \( b \)
\[
\Pr(|Z| > t) \leq 2 \exp \left( -\frac{t^2}{2(v^2 + b \cdot t)} \right), \quad t > 0.
\]
\end{proposition}

The sub-gamma property aggregates when taking sums of independent variables [3]

\begin{proposition}[Sub-Gamma Aggregation]
Let \( Z_i \) be sub-gamma with variance factor \( v_i^2 \) and scale \( b_i \), then \( \sum Z_i \) is sub-gamma with parameters
\[
v^2 = \sum v_i^2 \quad \text{and} \quad b = \max_i b_i.
\]
\end{proposition}

The sub-gamma property can be verified by the moments (see [6], Theorem 2.3)

\begin{proposition}[Sub-Gamma Property via Moments]
Let \( Z \) be centered. If \( Z \) is sub-gamma with variance factor \( v^2 \) and scale \( b \) then
\[
(E|Z|^d)^{1/d} = O(1) \cdot \inf \{ T : \sum x \log E|1 + W_x/T|^d \leq d \}
\]

\end{proposition}

2.2 Rosenthal-type Moment Bounds

The following result bounds the moments of a sum of random variables by controlling moments of individual components.

\begin{lemma}[Sharp Bounds for Moments of Independent Sums [15]]
For \( W_x \) independent
\[
(E \sum x W_x|^d)^{1/d} = \Theta(1) \cdot \inf \{ T : \sum x \log E|1 + W_x/T|^d \leq d \}
\]
holds for any real \( d \geq 1 \).
\end{lemma}

\begin{lemma}[Simplified Latala’s Bound]
Let \( W_x \) be independent and centered, let \( d \) be even and define the function
\[
\phi(u) = \frac{(1+u)^d + (1-u)^d}{2} - 1.
\]
Then
\[
(E \sum x W_x|^d)^{1/d} = O(1) \cdot \inf \{ T : \sum x E\phi(W_x/T) \leq d \}
\]
\end{lemma}

\begin{proof}[Proof of Lemma 12]
By Jensen’s inequality
\[
\sum x \log E|1 + W_x/T|^d \leq m \log (m^{-1} \sum x E|1 + W_x/T|^d)
\]
By the symmetrization trick we can assume that \( W_x \) are symmetric, loosing a factor \( O(1) \) in the upper bound. Expanding the \( d \)-th power and computing moments we obtain
\[
\frac{1}{m} \sum x E|1 + W_x/T|^d = 1 + \frac{1}{m} \sum x E\phi(W_x/T)
\]
and since \( 1 + \frac{1}{m} \sum x E\phi(W_x/T) \leq \exp(\frac{1}{m} \sum x E\phi(W_x/T)) \) we finally obtain
\[
\sum x \log E|1 + W_x/T|^d \leq \sum x E\phi(W_x/T)
\]
which finishes the proof.
\end{proof}
2.3 Growth of Binomial Moments

**Lemma 13** (Symmetrized Binomial Moments). Let $S \sim \text{Binom}(n,p)$ and $S'$ be an independent copy of $S$. Then letting $\sigma^2 = 2p(1-p)$ we have for any even positive $d$

$$
E(S - S')^d \leq O(d^{d/2} \sum_{\ell=1}^{d/2} \binom{n}{\ell} \ell^{d/2} \sigma^2).
$$

**Proof of Lemma 13.** Let $\eta, \eta'$ be independent distributed as $Bern(p)$, we can write $S - S' = \sum_{i=1}^n \eta_i$ where $\eta_i \sim iid \eta - \eta'$. Consider now the multinomial expansion for even $d$

$$
E|S - S'|^d = E(\sum_{i=1}^d \eta_i)^d = \sum_{\ell=1}^{d/2} \sum_{c_1, \ldots, c_d \text{ even, positive}} \binom{d}{\ell} \prod_{k=1}^d \eta_i^{c_k}.
$$

Say that the tuple $(i_1, \ldots, i_d)$ has $\ell$ distinct values which appear with multiplicities $c_1, \ldots, c_k > 0$, $c_1 + \ldots + c_k = d$. Then $E[\prod_{k=1}^d \eta_i^{c_k}] = \prod_{k=1}^d E(\eta - \eta')^{c_k}$. Since $\eta - \eta'$ is symmetric we can consider only the case where all $c_k$ are even and since $|\eta - \eta'| \leq 1$ we have for each $k$

$$
E(\eta - \eta')^{c_k} \leq E(\eta - \eta')^2 = \text{Var}[\eta - \eta'] = 2p(1-p).
$$

The number of such tuples is $\binom{n}{\ell}(\frac{d}{2})_{c_1, \ldots, c_k}$, and therefore

$$
E\left(\sum_{i=1}^d \eta_i\right)^d \leq \sum_{\ell=1}^{d/2} \sum_{c_1, \ldots, c_d \text{ even, positive}} \binom{d}{\ell} \prod_{k=1}^d \eta_i^{c_k} \sigma^{2\ell}.
$$

We are left with the combinatorial problem of determining the sum of even multinomial coefficients. It is known that this quantity equals the moment of a rademacher sum

$$
\sum_{c_1, \ldots, c_d \text{ even, positive}} \binom{d}{c_1, \ldots, c_d} = E(r_1 + \ldots + r_d)^d, \quad r_i \sim iid \pm 1 \text{ w.p. } \frac{1}{2}.
$$

By Khintchine’s Inequality this is at most $O(d!)^{d/2}$ and the proof is finished. 

3 Applications

**Proof of Corollary 3.** Since $\|P_X - U_m\|^2 = \|P_X\|^2 - 1/m = Q - 1/m$, the two cases in Corollary 3 are equivalent to a) $Q \leq \frac{1+\alpha}{m}$ and b) $Q \geq \frac{1+2\alpha}{m}$. Let $\alpha$ be such that $Q = (1+\alpha)/m$, then it suffices to prove that Listing 1 estimates $Q$ estimates within an additive error $\max(\epsilon, \alpha)/2m$ and correctness probability $1 - \delta$. To this end we show that the exponent in Theorem 1 with $\epsilon := \max(\epsilon, \alpha)/2m$ for $n = \Theta(n \log(1/\delta)/\epsilon$ is $\Omega(\log(1/\delta))$. In fact we show

$$
\min \left( \frac{\max(\epsilon, \alpha)^2}{m^2v^2}, \frac{\max(\epsilon, \alpha)}{mb}, \frac{n\sqrt{\max(\epsilon, \alpha)}}{m} \right) = \Omega(\log(1/\delta)), \quad m^{-1/2} \leq \epsilon \leq 1.
$$

Observe that $n\sqrt{\max(\epsilon, \alpha)/m} \geq n\sqrt{\epsilon/m} \geq nm^{-1/2} \epsilon$ because $\epsilon \leq 1$. Moreover we have $nb = \Pr[X = x] \leq \sum \Pr[X = x]^{1/2} \leq \sqrt{(1+\alpha)/m}$, so that $\max(\epsilon, \alpha)/mb = \Omega(1)nm^{-1/2} \max(\epsilon, \alpha)/\sqrt{1+\alpha} \geq \Omega(1)nm^{-1/2} \epsilon$, the inequality is immediate when $\alpha \leq 1$ and
for $\alpha > 1$ follows because $\sqrt{1 + \alpha} = \Omega(1)$ while $\epsilon \leq 1$. In both cases the exponent is at least $\Omega(\log(1/\delta))$, therefore it remains to prove this in the last case

$$\max(\epsilon, \alpha)^2/m^2 \geq \Omega(\log(1/\delta)), \quad m^{-1/2} \leq \epsilon \leq 1.$$ 

We have $\alpha/m = \sum_x (\Pr[X = x] - 1/m)^2$ by the definition of $\alpha$, thus $\sum_x \Pr[X = x]^3 = 1 + 3\alpha + \sum_x (\Pr[X = x] - 1/m)^3$, we can now bound $\sum_x \Pr[X = x]^3 \leq (1 + 3\alpha)/m^2 + (\alpha/m)^{3/2}$. and consequently $\nu^2 \leq (1 + \alpha)/mn^2 + (1 + 3\alpha)/m^2 + (\alpha/m)^{3/2}/n$. Therefore

$$\max(\epsilon, \alpha)^2/m^2 \nu^2 \geq \frac{1}{3} \min \left( n^2 m^{-1} \frac{\max(\epsilon, \alpha)^2}{1 + \alpha}, n^2 m^{-1} \frac{\max(\epsilon, \alpha)^2}{1 + 3\alpha}, \frac{\max(\epsilon, \alpha)^2}{\alpha^{3/2}} \right).$$

Since $\frac{\max(\epsilon, \alpha)^2}{1 + \alpha} = \Omega(\epsilon^2)$, $\frac{\max(\epsilon, \alpha)^2}{1 + 3\alpha} = \Omega(\epsilon^2)$ and $\frac{\max(\epsilon, \alpha)^2}{\alpha^{3/2}} = \Omega(\epsilon^{1/2}) = \Omega(\epsilon)$ for $\epsilon \leq 1$

$$\max(\epsilon, \alpha)^2/m^2 \nu^2 \geq \Omega(1) \min \left( n^2 m^{-1} \epsilon^2, \frac{n^2 \epsilon^2}{\alpha^{3/2}}, \frac{n m^{-1/2} \epsilon^{1/2}}{\alpha^{3/2}} \right).$$

Finally we use the assumption $\epsilon \geq m^{-1/2}$ which gives us

$$\max(\epsilon, \alpha)^2/m^2 \nu^2 \geq \Omega(1) \min((nm^{-1/2})^2, nm^{-1/2} \epsilon) = \Omega(\min(\log^2(1/\delta), \log(1/\delta)))$$

which finishes the proof because $\delta < 1$ and $\log^2(1/\delta) \geq \log(1/\delta)$.

**Proof of Corollary 5.** Observe that we can bound $\sum_x \Pr[X = x]^3 < (\Pr[X = x])^2 \sum_x \Pr[X = x]^2/3$ which implies $\nu^2 \leq Q/n^2 + Q^3/n$ and $b \leq Q^{1/2}/n$. Therefore the exponent in Theorem 1 is at least $\Omega(1) \min(\epsilon^2 Q^2/\nu^2, \epsilon Q/\nu, nQ^{1/2}) = \Omega(1) \min(n^2 \epsilon^2, nQ^{1/2}, \sqrt{nQ})$, and the claim on collision estimation follows because $\epsilon \leq 1$.

## 4 Proof of Main Result

### 4.1 Collision Estimator as Function of Histogram

The first trick is to condition on possible values $x$ in the sample. We have

$$\tilde{Q} = \frac{1}{n(n-1)} \sum_x \sum_{i \neq j} \mathbb{I}(X_i = x) \mathbb{I}(X_j = x) \quad (6)$$

Next we decompose the estimator into the sum of contributions from different $x$

$$\tilde{Q} = \frac{1}{n(n-1)} \sum_x \tilde{Q}_x, \quad \tilde{Q}_x = S^2_x - S_x, \quad S_x = \sum_i \mathbb{I}(X_i = x) \quad (7)$$

which is the relation to the histogram of the sample $X_1, \ldots, X_n$, as $S_x$ is the load of bin $x$. Observe also that $\tilde{Q}_x/n(n-1)$ is, for each $x$, an unbiased estimator for $\Pr[X = x]^2$.

### 4.2 Utilizing Negative Dependence

**Lemma 15 (Contributions from Bins are Negatively Dependent).** Random variables $\{S^2_x - S_x\}_x$, and therefore $Q_x$ (defined in Equation (7)) are negatively dependent.

**Proof.** Observe that for any fixed $i$ the random variables $\mathbb{I}(X_i = x)$, indexed by $x$, are negatively dependent because they are boolean and add up to one (zero-one property, see Lemma 8 in [11]). Since $X_i$ for different $i$ are independent, we obtain that $(\mathbb{I}(X_i = x))_x$ indexed by both $i$ and $x$ are negatively dependent (augmentation property, see Proposition
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7 part 1 in [11]). Observe that \( S_x^2 - S_x = f((I(X_i = x))_i) \) with \( f(u_1, \ldots, u_n) = \sum_{i \neq j} u_iu_j \) increasing in each \( u_i \) when \( u_i \geq 0 \). Applying increasing functions to non-overlapping subsets of negatively dependent variables produces variables that are also negatively dependent (aggregation by monotone functions, see Proposition 7 part 2 in [11]), therefore \( S_x^2 - S_x \) are negatively dependent. Same holds for \( \tilde{Q}_x \) which differ only by a scaling factor.

\[ \boxed{\text{4.3 Concentration in Single Bins}} \]

We will now study the properties of \( \tilde{Q}_x \), for each fixed value of \( x \). For brevity we denote

\[ p = \Pr[X = x] \]

\[ \boxed{\text{4.3.1 Centering}} \]

We start by centering random variables \( I(X_i = x) \). Direct calculations show that

\[ \boxed{\text{Proposition 16 (Estimator Bin Contributions). The centered contribution from bin } x \text{ is}} \]

\[ S_x^2 - S_x - \mathbb{E}[S_x^2 - S_x] = U_2 + 2(n - 1)p \cdot U_1 \]

where \( U_1 \) and \( U_2 \) are zero-mean given by

\[ U_1 = \sum_i \xi_i, \quad U_2 = \sum_{i \neq j} \xi_i\xi_j, \quad \xi_i = I(X_i = x) - \Pr[X = x] \]

\[ \boxed{\text{Proof.}} \]

Let \( Z_i = I(X_i = x) \), then \( S_x^2 - S_x - \mathbb{E}[S_x^2 - S_x] = \sum_{i \neq j} (Z_iZ_j - \mathbb{E}[Z_iZ_j]) \) and \( \mathbb{E}[Z_iZ_j] = p^2 \) when \( i \neq j \). The result follows from the identity \( Z_iZ_j - p^2 = (\xi_i + p)(\xi_j + p) - p^2 = \xi_i\xi_j + p(\xi_i + \xi_j) \), summed over pairs \( i \neq j \).

\[ \boxed{\text{Note 17 (Symmetric Polynomials).}} \]

Observe that \( U_1 \) and \( U_2 \) are the first and second elementary symmetric polynomials in variables \( \xi_i \)

\[ \boxed{\text{4.3.2 Bounding Variance}} \]

\[ \boxed{\text{Corollary 18 (Total Variance of Collision Estimator). The contribution from bin } x \text{ satisfies}} \]

\[ \Var[\tilde{Q}_x] = O(p^2/n^2 + p^3/n^3), \quad p = \Pr[X = x] \]

and the total variance of the collision estimator is

\[ \Var[\tilde{Q}] \leq \sum_x \Var[\tilde{Q}_x] = O\left( \sum_x \Pr[X = x]^2/n^2 + \sum_x \Pr[X = x]^3/n^3 \right) \]

\[ \boxed{\text{Proof.}} \]

By inspection of Equation (8) we see that \( U_1 \) and \( U_2 \) are uncorrelated so that

\[ \Var[\tilde{Q}_x] = \Var[U_1] + \Theta((np)^2)\Var[U_2] \]

Easy inspection shows \( \mathbb{E}[U_1^2] = n(n - 1)p^2 \) and \( \mathbb{E}[U_2^2] = np \); since \( \mathbb{E}[U_2^2] = \mathbb{E}[U_1] = 0 \) this shows \( \Var[\tilde{Q}_x] = \Theta((np)^2 + (np)^3) \). The total variance bound follows because by negative dependence \( \Var[\sum_x \tilde{Q}_x] \leq \sum_x \Var[\tilde{Q}_x] \).
4.3.3 Bounding Moments by Decoupling and Symmetrization

We will bound higher moments of \( U_2 \) and \( U_1 \) in Equation (8) in terms of binomial moments.

The following is a well-known decoupling inequality (cf. Theorem 6.1.1 in [21])

- **Proposition 19 (Decoupling for Quadratic Forms).** Let \( \xi = (\xi_1, \ldots, \xi_n) \) be a random vector with centered independent components, let \( A = a_{i,j} \) be a diagonal-free matrix of shape \( n \times n \). Then for any convex function \( f \)

\[
E(f(\xi^T A \xi)) \leq 4E(f(\xi^T A \xi))
\]

where \( \xi' \) is independent and identically distributed as \( \xi \).

We also need the following standard fact on symmetrization (cf. Lemma 6.1.2 in [21])

- **Proposition 20 (Symmetrization Trick).** Let \( Y, Z \) be independent and \( EZ = 0 \), then \( E(f(Y)) \leq E(f(Y+Z)) \) for any convex \( f \).

- **Note 21.** These are crucial for proving Hanson-Wright’s Lemma.

By combining Proposition 19 and Proposition 20 we obtain that when calculating the moments of \( U_2 \) we can assume (loosing a constant factor) that \( \xi_i \) are symmetric and decoupled.

- **Lemma 22 (Bounding Quadratic Contributions).** For \( \xi_i \sim iid \ Bern(p) - p \) and even \( d \)

\[
E(\sum_{i\neq j} \xi_i \xi_j) \leq 4 \cdot E(\sum_{i\neq j} \eta_i \eta_j) \quad \eta_1, \eta_2, \ldots, \eta_1', \eta_2' \sim iid \eta - \eta', \quad \eta, \eta' \sim Bern(p)
\]

- **Note 23.** The off-diagonal assumption, true in our case, is crucial to apply decoupling.

Similarly we estimate the term \( U_1 \)

- **Lemma 24 (Bounding Linear Contributions).** For \( \xi_i \sim iid \ Bern(p) - p \) and even \( d \geq 2 \)

\[
E(\sum_{i} \xi_i) \leq E(\sum_{i} \eta_i) \quad \eta_1, \eta_2, \ldots \sim iid \eta - \eta', \quad \eta, \eta' \sim Bern(p)
\]

- **Note 25.** For the proof we need only symmetrization.

Finally we reformulate the obtained bounds in terms of binomials

- **Corollary 26 (Binomial Bounds for Linear and Quadratic Contributions).** For \( \xi_i \sim iid \ Bern(p) - p \), even \( d \geq 2 \), and \( S, S' \sim iid \ Bernomial(n, p) \) we have

\[
E(\sum_i \xi_i) \leq E(\sum_i \eta_i) \quad E(\sum_{i\neq j} \xi_i \xi_j) \leq 16E(S-S')^d
\]

**Proof of Corollary 26.** The first bound follows directly as \( \sum_i \eta_i \) is distributed as \( S - S' \). To prove the second inequality, it suffices to show for even \( d \geq 2 \) that

\[
E(\sum_{i\neq j} \eta_i \eta_j) \leq E(\sum_{i\neq j} \eta_i \eta_j + \sum_i \eta_i \eta_i')^d
\]

because then \( E(\sum_{i\neq j} \eta_i + \sum_i \eta_i')^d = E(\sum_i \eta_i)^d (\sum_i \eta_i')^d = E(\sum_i \eta_i)^d \cdot E(\sum_i \eta_i')^d \).

To prove the claim denote \( A = \sum_{i\neq j} \eta_i \eta_j' \), \( B = \sum_i \eta_i \eta_i' \) then we have to prove \( E(A+B)^d \geq EA^d \). Write \( E(A+B)^d = \sum_k E(A^k B^{d-k}) \) and observe that \( E(A^k B^{d-k}) \geq 0 \) which follows by expanding \( A^k \) and \( B^{d-k} \) into sums of products of \( \eta_i, \eta_i' \) and utilizing their symmetry (terms with odd number of repetitions for some \( \eta_i \) or \( \eta_i' \) will have zero expectations, so we are left with non-negative square terms). Thus \( E(A+B)^d \geq EA^d + EB^d \geq EA^d \).
Note 27. An alternative bound avoids combinatorial argument and uses the triangle inequality to establish \((E|\sum_{i\neq j} \eta_i \eta_j|^d)^{1/d} \leq (E|\sum_{i,j} \eta_i \eta'_j|^d)^{1/d} + (E|\sum_{i=j} \eta_i \eta'_j|^d)^{1/d}\). The second term then behaves like a centered binomial with parameters \(n, p^2\).

Proof of Lemma 22. Write the \(d\)-th moment as \(E f(\sum_{i\neq j} \xi_i \xi_j)\) with convex \(f(u) = |u|^d\). By Proposition 19 it is upper bounded by \(4E f(\sum_{i\neq j} \xi_i \xi'_j)\), where \(\xi_i\) and \(\xi'_i\) are indentically distributed and independent. Now look at some chosen \(\xi_i\) and the expectation \(E f(\sum_{i\neq j} \xi_i \xi'_j)\) conditioned on the fixed values of the remaining variables (that is \(\xi_j\) for \(j \neq i\) and \(\xi'_j\) for all \(j\)). By Proposition 20 we get that replacing \(\xi_i\) by \(\eta_i - \eta'_i\) where \(\eta_i, \eta'_i\) are independent copies of \(\xi_i\) gives an upper bound. We repeat this for all \(\xi_i\) and the same for \(\xi'_i\). Note that each time we replace with the distribution \((\eta - p) - (\eta' - p) = \eta - \eta'\) where \(\eta, \eta' \sim \text{iid Bern}(p)\).

Proof of Lemma 24. We replace \(\xi_i\) iteratively as in the proof of Lemma 22.

4.3.4 Auxiliary Function

The bounds in Lemma 13 depends on expressions of form \(a^\ell b^{-\ell}\) that we analyze closer below.

Proposition 28 (Auxiliary Function). The function \(g(\ell) = a^\ell b^{-\ell}\), for any parameters \(a, b > 0\), is maximized at \(\ell = \ell^* = b/W(be/a)\); it increases for \(0 < \ell < \ell^*\) and decreases for \(\ell^* < \ell < +\infty\). Where \(W(\cdot)\) is the main branch of Lambert-W function.

Proof of Proposition 28. The derivative equals

\[
\frac{\partial g}{\partial \ell} = (a/\ell)^\ell \ell^{-1+b}(b - \ell + \ell \log(a/\ell))
\]

and only the last factor can be possibly zero, therefore

\[
\frac{\partial g}{\partial \ell} = 0 \Leftrightarrow u \log u = be/a, \quad u \triangleq e\ell/a
\]

so the zero is at \(u = e^{W(be/a)} = be/aW(b/e/a)\) and the formula for \(\ell^*\) follows. We also conclude that \(g\) is monotone in both intervals \(0 < \ell < \ell^*\) and \(\ell^* < \ell < +\infty\). By the equations above we see that \(\frac{\partial g}{\partial \ell} > 0\) when \(\ell \to 0\) and \(\frac{\partial g}{\partial \ell} < 0\) when \(\ell \to +\infty\), therefore we conclude that \(g(\ell)\) increases for \(0 < \ell < \ell^*\) and decreases when \(\ell^* < \ell < +\infty\). 

![Figure 1](auxiliary_function.png) Auxiliary function \(g(\ell)\) studied in Proposition 28, here with parameters: \(a = 1/5, b = 3\).
Proposition 29 (Supremum of Auxiliary Function). Let $g$ be as in Proposition 28, then

$$\sup\{g(\ell) : 1 \leq \ell \leq b\} \leq a \max(a, b)^{b-1}$$

Proof of Proposition 29. Suppose that $a > b$, then by Proposition 28 we find that $g(\ell)$ is maximized at $\ell^* = b/W(\ell^*)$ for $\ell \in [1, b]$ if $u < e$; then $g(\ell) \leq g(b) = a^b$ for $\ell \in [1, b]$. If $a \leq b$ then $g(\ell) = a \cdot a^{\ell-1} \ell^{b-\ell} \leq a b^{\ell-1} \ell^{b-\ell} \leq a b^{b-1}$ when $\ell \in [1, b]$.

Proof of Proposition 29. Since $W(u) < 1$ when $0 < u < e$, we have $W(\ell/e) < 1$ when $a > b$ so $\ell^* > b$ and the supremum is at $\ell = b$. Suppose now $a/b \geq 1$, then $W(\ell/e) \geq 1$ and

$$\left(a/\ell^*\right)^{1/W} = W(\ell/a)^{1/W(\ell/a)} \cdot (a/b)^{W(\ell/a)} = W(u)^{1/W(u)} \cdot (u/e)^{1/W(u)} , \quad u = \ell/e.$$  

Since $u \geq e$ and $W(u) \geq 1$ we have $W(u)^{1/W(u)} = \Theta(1)$ and $(u/e)^{1/W(u)} = \Theta(1)$ because for $u \geq 1$ we have $W(u) = \Theta(\log u)$. Therefore $a/\ell^* = \Theta(1)$ and

$$g(\ell^*) = \Theta(1)^b (\ell^*)^b = \Theta(b/W(u))^b \leq O(b)^b$$

The result now follows, as the maximum can be either at $\ell = 1$ or $\ell = \ell^*$ or $\ell = b$.

4.3.5 Moments of Bin Contributions

Having estimated $U_1$ and $U_2$ in Proposition 16 we are in position to give formulas that control the moments of $\tilde{Q}_x$. The exact bound is stated below

Corollary 30 (Moments of Bin Contributions). For every $x$ and $p = \Pr[X = x]$ we have

$$\mathbb{E}[\tilde{Q}_x - \mathbb{E}[\tilde{Q}_x]]^d \leq O(d)^{(d/2)\sum_{\ell=1}^{d/2} \binom{n}{\ell} e^{\ell/2} \sigma^{2\ell}} + O(d)^{(d/2)(np)^{d/2} \sum_{\ell=1}^{d/2} \binom{n}{\ell} e^{\ell/2} \sigma^{2\ell}}$$

for $p = \Pr[X = x]$ and $U_1$ and $U_2$ as defined there. By Corollary 26 and Lemma 13

$$\mathbb{E}[\tilde{Q}_x - \mathbb{E}[\tilde{Q}_x]]^d \leq O(d)^{(d/2)\sum_{\ell=1}^{d/2} \binom{n}{\ell} e^{\ell/2} \sigma^{2\ell}}$$

Overestimating $\left(\sum_{\ell=1}^{d/2} \binom{n}{\ell} e^{\ell/2} \sigma^{2\ell}\right)^2 \leq d/2 \cdot \sum_{\ell=1}^{d/2} \left(\binom{n}{\ell} e^{\ell/2} \sigma^{2\ell}\right)^2$ (Jensen’s inequality) and $\binom{n}{\ell} \leq (ne/\ell)^{\ell}$ (the well-known binomial bound) and $\sigma^2 \leq 2p$ we obtain

$$\mathbb{E}[\tilde{Q}_x - \mathbb{E}[\tilde{Q}_x]]^d \leq O(d)^{d/2} \sum_{\ell=1}^{d/2} g(\ell)^2 + O(d)^{d/2} (np)^d \sum_{\ell=1}^{d/2} g(\ell), \quad g(\ell) \triangleq (np)^{\ell} e^{\ell/2 - \ell}.$$  

The result follows now from Proposition 29.
4.4 Assembling Bin Concentrations

Armed with Corollary 30 we are in position to estimate concentration of the sum of bin contributions, and therefore the tails of the estimator. To this end we distinguish between tails heavier and lighter than gamma. For fixed even $d$ we define light and heavy bins as

$$\mathcal{X}^- \triangleq \{ x : n \Pr[X = x] \geq d \}, \quad \mathcal{X}^+ \triangleq \{ x : n \Pr[X = x] < d \}$$

Now Theorem 1 follows directly from the following facts

**Lemma 32** (Concentration of Contributions with Light Tails). We have

$$\Pr[\| \sum_{x \in \mathcal{X}^-} (\tilde{Q}_x - \mathbb{E}\tilde{Q}_x) \| > \epsilon] \leq 2 \exp(-\Omega(\epsilon^2/(v^2 + be))) = 2 \exp(-\Omega(\min(\epsilon^2/v^2, \epsilon/b)))$$

where $v^2 \triangleq n^2 \sum_x \Pr[X = x]^2 + n^3 \sum_x \Pr[X = x]^3$, $b \triangleq n \max_x \Pr[X = x]$.

**Lemma 33** (Concentration of Contributions with Heavy Tails). We have

$$\Pr[\| \sum_{x \in \mathcal{X}^+} (\tilde{Q}_x - \mathbb{E}\tilde{Q}_x) \| > \epsilon] \leq O(1) \exp(-\Omega(\epsilon/\sqrt{d}))$$

The condition $\epsilon = \Omega(v)$ can be ignored because of the term $\epsilon^2/v^2$, this regime gives the trivial bound of $O(1)$. The result follows by $Q = \frac{1}{n(n-1)} \sum_x \tilde{Q}_x$ and changing $\epsilon := n^2 \epsilon$.

**Proof of Lemma 32.** The result follows directly from known aggregation of sub-gamma distributions $\tilde{Q}_x$ which have parameters $v^2 = \sum_x v_x^2$ and $b = \max_x b_x$ as given in Corollary 30. Aggregation is based on calculating moment generating functions, thus it does apply for negatively dependent random variables.

**Proof of Lemma 33.** Observe that, by Corollary 30 applied to $d$ replaced with $k$, we have

$$\mathbb{E}(\tilde{Q}_x - \mathbb{E}\tilde{Q}_x)^k \leq O(k)^{2k-2} n^2 \Pr[X = x]^2, \quad k = 2, 4, \ldots, d, \quad x \in \mathcal{X}^+$$

Thus

$$\mathbb{E}(1 + (\tilde{Q}_x - \mathbb{E}\tilde{Q}_x)/T)^d \leq 1 + n^2 \Pr[X = x]^2/T^2 \sum_{k=2}^d \binom{d}{k} O(k)^{2k-2}/T^{k-2}.$$}

We now apply Lemma 12, since $\phi(u) = \sum_{k=2}^d \binom{d}{k} u^k$ we need to find $T$ such that

$$n^2 \sum_{x \in \mathcal{X}^+} \Pr[X = x]^2/T^2 \sum_{k=2}^d \binom{d}{k} O(k)^{2k-2}/T^{k-2} \leq d$$

This holds for $T = \Theta(v/\sqrt{d} + d^2)$, where $v^2 = n^2 \sum_{x \in \mathcal{X}^+} \Pr[X = x]^2$ with appropriate constants. If this is the case then

$$\mathbb{E}\left[ \sum_{x \in \mathcal{X}^+} (\tilde{Q}_x - \mathbb{E}\tilde{Q}_x)^d \right] \leq O(d^2 + v/\sqrt{d})^d$$
By Markov’s inequality we obtain
\[
\Pr[\left| \sum_{x \in \mathcal{X}^+} (\tilde{Q}_x - E\tilde{Q}_x) \right| > \epsilon] \leq O((d^2/\epsilon + v/\sqrt{d}\epsilon)^d)
\]
We set \(d\) so that \(d^2 = O(\epsilon)\) and \(v/\sqrt{d} = O(\epsilon)\) with appropriate constants, which gives the tail of \(2^{-d}\). Note that \(d\) has to be even and at least 2, thus we the tail is \(2^{-\Omega(\epsilon^{1/2})}\) if \(\epsilon = \Omega(1)\) and \(\epsilon = \Omega(v)\); the first condition can be ignored because the bound is then \(O(1)\).

5 Conclusion
We have derived strong concentration guarantees for the collision estimator, which subsumes variance bounds from previous works. Such concentration bounds can be used for example to eliminate the need for boosting of weak estimators (majority/median tricks).

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