Computing the $L_1$ Geodesic Diameter and Center of a Simple Polygon in Linear Time*

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Abstract. In this paper, we show that the $L_1$ geodesic diameter and center of a simple polygon can be computed in linear time. For the purpose, we focus on revealing basic geometric properties of the $L_1$ geodesic balls, that is, the metric balls with respect to the $L_1$ geodesic distance. More specifically, in this paper we show that any family of $L_1$ geodesic balls in any simple polygon has Helly number two, and the $L_1$ geodesic center consists of midpoints of shortest paths between diametral pairs. These properties are crucial for our linear-time algorithms, and do not hold for the Euclidean case.

1 Introduction

Let $P$ be a simple polygon with $n$ vertices in the plane. The diameter and radius of $P$ with respect to a certain metric $d$ are the most natural and important among several common measures of $P$. The diameter with respect to $d$ is defined to be the maximum distance over all pairs of points in $P$, that is, $\max_{p,q\in P} d(p,q)$, while the radius is defined to be the min-max value $\min_{p\in P} \max_{q\in P} d(p,q)$. Here, the polygon $P$ is considered as a closed and bounded space and thus the diameter and radius of $P$ with respect to $d$ are well defined. A pair of points in $P$ realizing the diameter is called a \textit{diametral pair} and the center is defined to be the set of points $c \in P$ such that $\max_{q\in P} d(c,q)$ is equal to the radius.

One of the most natural metrics on a simple polygon $P$ is induced by the length of the Euclidean shortest paths that stay within $P$, namely, the (Euclidean)
geodesic distance. The problem of computing the diameter and center of a simple polygon with respect to the geodesic distance has been intensively studied in computational geometry since the early 1980s. The diameter problem was first studied by Chazelle [6], where a $O(n^2)$-time algorithm was given. The running time was afterwards improved to $O(n \log n)$ by Suri [20]. Finally, Hershberger and Suri [10] presented a linear-time algorithm based on a fast matrix search technique. Recently, Bae et al. [3] considered the diameter problem for polygons with holes.

The first algorithm for finding the Euclidean geodesic center was given by Asano and Toussaint [2]. Their algorithm runs in $O(n^4 \log n)$-time, and was afterwards reduced to $O(n \log n)$ by Pollack, Sharir, and Rote [16]. Since then, it has been a longstanding open problem whether the geodesic center can be computed in linear time (as also mentioned later by Mitchell [13]).

Another popular metric with a bit different flavor is the link distance, which measures the smallest possible number of links (or turns) of piecewise linear paths. The currently best algorithms that compute the link diameter or center run in $O(n \log n)$ time [7,12,19]. The rectilinear link distance measures the minimum number of links when feasible paths in $P$ are constrained to be rectilinear. It is known that the problem with respect to the rectilinear link distance can be solved in linear time by Nilsson and Schuierer [14,15].

In order to tackle the open problem of computing the Euclidean geodesic center, we investigate another natural metric: the $L_1$ metric. To the best of our knowledge, only a special case where the input polygon is rectilinear has been considered in the literature. This result is by Schuierer [17], where he showed how to compute the $L_1$ geodesic diameter and center of a simple rectilinear polygon in time.

This paper aims to provide a clear and complete exposition on the diameter and center of general simple polygons with respect to the $L_1$ geodesic distance. We first focus on revealing basic geometric properties of the geodesic balls (that is, the metric balls with respect to the $L_1$ geodesic distance). Among other results, we show that any family of $L_1$ geodesic balls has Helly number two (see Theorem 1). This is a critical property that does not hold for the Euclidean geodesic distance, and thus we identify that the main difficulty of the open problem lies there.

We then show that the method of Hershberger and Suri [10] for computing the Euclidean diameter extends to $L_1$ metrics, and that the running time is preserved. However, the algorithms for computing the Euclidean center do not easily extend to rectilinear metrics. Indeed, even though the approach of Pollack et al. [16] can be adapted for the $L_1$ metric, the running time will increase to $O(n \log n)$. On the other hand, the algorithm of Schuierer [17] for the rectilinear simple polygons heavily exploits properties derived from rectilinearity. Thus, its extension to general simple polygons is not straightforward either.

In this paper we use a different approach: using the previously mentioned Helly-type theorem, we show that the $L_1$ geodesic center coincides with the intersection of a finite number of geodesic balls. Afterwards we show how to