On factorized Lax pairs for classical many-body integrable systems

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Abstract

In this paper we study factorization formulae for the Lax matrices of the classical Ruijsenaars-Schneider and Calogero-Moser models. We review the already known results and discuss their possible origins. The first origin comes from the IRF-Vertex relations and the properties of the intertwining matrices. The second origin is based on the Schlesinger transformations generated by modifications of underlying vector bundles. We show that both approaches provide explicit formulae for $M$-matrices of the integrable systems in terms of the intertwining matrices (and/or modification matrices). In the end we discuss the Calogero-Moser models related to classical root systems. The factorization formulae are proposed for a number of special cases.

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1 Introduction

In this paper we deal with the Lax pairs of the Calogero-Moser [12, 29] and Ruijsenaars-Schneider [42] models. More precisely, we study the factorization formulae for the Lax matrices of these models. For the elliptic $\mathfrak{gl}_N$ Ruijsenaars-Schneider model it is of the form

$$L_{\text{RS}}(z) = g^{-1}(z)g(z + \hbar')e^{P/c} \in \text{Mat}(N, \mathbb{C}),$$

(1.1)

where $\hbar'$ and $c$ are constants, $z$ is the spectral parameter, and

$$P = \text{diag}(p_1, ..., p_N) \in \text{Mat}(N, \mathbb{C}), \quad g(z) = g(z, q_1, ..., q_N) \in \text{Mat}(N, \mathbb{C}),$$

(1.2)

where $g(z, q)$ is given by (2.27). The positions of particles $q_i$ and momenta $p_i$ are canonically conjugated $\{p_i, q_j\} = \delta_{ij}$. The form (1.1) was observed in [24] at quantum level. It was used for the proof that the quantum version of the gauge transformed Lax matrix

$$g(z)L_{\text{RS}}(z)g^{-1}(z) = g(z + \hbar')e^{P/c}g^{-1}(z)$$

(1.3)

satisfies the quantum exchange (or RLL) relations with the non-dynamical Baxter-Belavin $R$-matrix [8]. In $N = 2$ case this result reproduces the representation of the quantum Sklyanin algebra [15] through the difference operators [16], and for generic $N$ it provides similar representation for the $\text{GL}_N$ analogue of the Sklyanin algebra [16]. The application to exchange relations establish a link between (1.1) and the IRF-Vertex correspondence [9, 26], which maps dynamical and non-dynamical $R$-matrices into each other. Up to some additional diagonal gauge the matrix $g(z)$ entering (1.1) is the matrix of the intertwining vectors introduced (for the elliptic case) in [9, 26]. It is used for construction of the elliptic analogue of the Drinfeld twist [19]. We will review the above mentioned relations in the next Section. The classical analogue of the IRF-Vertex relations based on (1.1) and the corresponding parameterization of the classical Sklyanin algebra in terms of the Ruijsenaars-Schneider variables (the classical bosonization formulae or the classical representation formulae) are directly obtained from the results of [24]. See [13] for the quasi-classical limit. A general form for such parameterization follows from (1.3) by taking residue at $z = 0$. Namely, the components of the matrix

$$S = S(p, q, \hbar', c) = \text{Res}_{z=0} \left( g(z)L_{\text{RS}}(z)g^{-1}(z) \right) = g(\hbar')e^{P/c}g(0), \quad \check{g}(0) = \text{Res}_{z=0} g^{-1}(z)$$

(1.4)

---

3The form (1.1) is defined up to multiplication of $L(z)$ by a scalar non-dynamical function. In what follow we will fix this freedom as given in (2.26) to match the custom form (2.2).

4The expression (1.3) itself satisfies the classical quadratic exchange relations (2.25) with the classical (non-dynamical) $r$-matrix of the Belavin-Drinfeld [10] type (in the elliptic case).
are the generators of the classical Sklyanin algebra. In (1.4) we also used the property of \( g(z) \) that near \( z = 0 \)
\[
g^{-1}(z) = \frac{1}{z} \check{g}(0) + A + O(z),
\]
i.e. \( g(z) \) is degenerated at \( z = 0 \), and \( \det g(z) \) has the first order zero at \( z = 0 \). Let us also mention that the first example of the classical IRF-Vertex like relation was observed in [48] between the nonlinear Schrödinger equation and the classical Heisenberg magnet.

While in the elliptic case we deal with the Lax representation with spectral parameter, for the trigonometric and rational cases there are Lax representations without spectral parameter. The factorization formulae exist for each of the cases. From the IRF-Vertex relations viewpoint the trigonometric case without spectral parameter is related to \( R \)-matrix structure of the chiral Potts model [7] based on [15, 30], while the trigonometric case with spectral parameter is described by the intertwining matrix of the "non-standard" trigonometric \( R \)-matrix [2] generalizing the 7-vertex \( R \)-matrix [15]. Similarly, the rational case without spectral parameter is related to the \( R \)-matrices of the Cremmer-Gervais type [17, 6], while the rational case with spectral parameter comes from \( GL_N \) generalization [1, 34] of the rational 11-vertex \( R \)-matrix [15]. Factorization formulae for all the cases will be also reviewed in the next Section.

In the non-relativistic limit \( \hbar' = \nu' / c, c \to \infty \) (1.1) provides the Lax matrix of the Calogero-Moser model written in the following form:
\[
L_{\text{CM}}(z) = P + \nu' g^{-1} g'.
\]
where \( g' = \partial_z g(z) \) and \( \nu' \) is the coupling constant. The custom form of the elliptic model is achieved by setting \( \nu' = N \nu \), see (2.32). Similarly to (1.3) the gauge transformed Lax matrix
\[
g(z)L_{\text{CM}}(z)g^{-1}(z) = g(z)P\check{g}^{-1}(z) + \nu' g'(z)g^{-1}(z).
\]
satisfies the classical linear exchange relations (2.20) with the classical (non-dynamical) \( r \)-matrix of the Belavin-Drinfeld type (in the elliptic case). While the residue of (1.3) is the classical representation of the Sklyanin algebra, the residue of (1.7)
\[
S = S(p, q, \nu') = g(0)P\check{g}(0) + \nu' g'(0)\check{g}(0)
\]
is the classical representation of the \( gl_N \) Lie algebra. The Poisson brackets between the matrix elements of \( S \) are the Poisson-Lie brackets on the Lie coalgebra \( gl_N^* \). Moreover, the matrix \( \check{g}(0) \) is of rank one (see (3.23)), and therefore, the \( S \) matrices (1.8) and (1.4) are of rank one too: \( S = \xi(p, q) \otimes \psi(q) \). In the rational case the components of \( \psi \) vector are elementary symmetric functions of the coordinates \( q_i \), while the components of the \( \xi \) vector are canonically conjugated to those of \( \psi \): \( \{\xi_i, \psi_j\} = \delta_{ij} \) (for the non-relativistic case (1.8)). These type variables were used for reformulation of the quantum Calogero-Moser model in terms of the Lie algebra data in [41] and [39].

A general scheme for the classical IRF-Vertex relations was suggested in [32] and is known as the symplectic Hecke correspondence. It unifies a set of integrable models related by gauge transformations of \( g(z, q) \) type. The Lax matrices under consideration are known [29] to be sections of bundles over the base spectral curve \( \Sigma \) with a local coordinate \( z \): \( L(z) \in \Gamma(\text{End}V, \Sigma) \). The underlying vector bundles \( V \) are also related by the action of the gauge transformations, which change the degrees of the bundles by one. It happens due to the special local structure (1.5) of \( g(z, q) \). Its action adds a zero (or a pole) towards a certain direction. Such gauge
transformations are called modifications of bundles \[20, 3\]. In this respect (1.1) is a combination of two modifications \[47\]. The set of models unified by the symplectic Hecke correspondence consists of the Calogero-Moser model (including its spin generalizations), elliptic integrable tops and intermediate models, which are described by partially dynamical \(R\)-matrices \[37, 49\]. The gauge transformation relating (1.6) and (1.7) is then treated as transition from the Calogero-Moser model (with variables \(p_i, q_j\)) to the special elliptic top, where the matrix of dynamical variables \(S(1.8)\) belong to the coadjoint orbit (of \(GL_N\) Lie group) of the minimal dimension, i.e. when \(S\) is of rank one. The relation (1.8) provides explicit change of variables between the systems in this case.

The purpose of the paper is two-fold. The first one is to clarify possible origins of the factorization formulae (1.1) and (1.6). In fact, the factorization is neither necessary nor sufficient for integrability. A natural set up of the problem is as follows. Which \(g(z, q)\) provide the Lax matrices for integrable models? Put it differently, for which \(g(z, q)\) there exist \(M\)-matrix such that the Lax equations

\[
\dot{L}(z) = [L(z), M(z)]
\]

hold true identically in \(z\) and are equivalent to equations of motion of an integrable system defined by the Lax matrix (1.1) or (1.6)? It is easy to verify that a generic matrix \(g(z, q)\) does not provide Lax matrix. Only very special \(g(z, q)\) lead to an integrable system, and the information about integrability of (1.1) or (1.6) is encoded in the form of the matrix \(g(z, q)\). Therefore, it is reasonable to expect that the rest of the data (not only the Lax matrix) is formulated through \(g(z, q)\). We focus on derivation of the \(M\)-matrices for the Calogero-Moser and Ruijsenaars-Schneider models in terms of \(g(z, q)\).

From the above we see that there are two natural possible origins for \(g(z, q)\) with the property that it provides Lax matrix of an integrable model. They come from the algebraic and geometric viewpoints. The algebraic origin is the IRF-Vertex correspondence, i.e. the treatment of the matrix \(g(z, q)\) as an intertwining matrix (in the fundamental representation) entering the Drinfeld twist. The geometric origin is interpretation of \(g(z, q)\) matrix as modification of bundle on the base spectral curve related to the Lax matrix (1.1) or (1.6). Using these two treatments of \(g(z, q)\) we obtain expressions for the \(M\)-matrices of the Ruijsenaars-Schneider (1.1) and Calogero-Moser (1.6) models. Namely, we prove the following

**Theorem 1** The \(M\)-matrix of the Ruijsenaars-Schneider model defined by the Lax matrix (1.1) can be written in terms of the \(g(z, q)\) matrix (2.27) as follows:

\[
M^{RS}(z) = -g^{-1}(z)g'(z)G - F + g^{-1}(z)\frac{d}{dt} g(z),
\]

with

\[
G = \text{tr}_2 \left( O_{12} \frac{\partial^2}{\partial(\hbar)} \dot{g}_2(0) g_2(N\hbar) e^{P_2/c} \right), \quad F = \text{tr}_2 \left( O_{12} \frac{\partial^2}{\partial(\hbar)} A_2 g_2(N\hbar) e^{P_2/c} \right).
\]

where we assume that (in the elliptic case) \(\hbar' = N\hbar\), the matrix \(A\) is the one from the expansion (1.3), and

\[
O_{12} = \sum_{i,j} E_{ii} \otimes E_{ji}.
\]
Theorem 2 The M-matrix of the Calogero-Moser model defined by the Lax matrix (1.6) can be written in terms of the $g(z,q)$ matrix (2.27) as follows:

$$M = g^{-1}(z) \frac{d}{d\tau} g(z) - g^{-1}(z) \frac{d}{dt} g(z)$$

where

$$\text{diag}(q)_\tau - \text{diag}(q)_t = -\frac{1}{N} d, \quad d_i = \sum_{k \neq i} E_1(q_{ik}).$$

The statements of both theorems hold true for trigonometric and rational cases as well. The partial derivative with respect to the moduli $\tau$ should be transformed into the second derivative with respect to the argument (through the heat equation) in these cases. See Section 3.4.

The proof of the first statement (1.10) is based on the algebraic treatment of $g(z,q)$. Following [44] we mention that the IRF-Vertex correspondence provides the following relation between quantum non-dynamical $R$-matrix and the intertwining matrix $g(z,q)$:

$$\frac{1}{N} \tilde{g}_2(0,q) R_{12}^{h}(z) = g_1(z + N\hbar, q) O_{12} g_2^{-1}(N\hbar, q) g_1^{-1}(z, q),$$

where $O_{12}$ is (1.12). Next, we use the $R$-matrix formulation for integrable tops based on the quasiclassical limit of 1-site chain [45]. It was shown in [35] that the Lax equations (1.9) with

$$L^h(S, z) = \frac{1}{N} \text{tr}_2 \left( R_{12}^{h}(z) S_2 \right), \quad M^h(S, z) = -\frac{1}{N} \text{tr}_2 \left( r_{12}(z) S_2 \right),$$

where $r_{12}(z)$ is the classical $r$-matrix ($R_{12}^{h}(z) = 1 \otimes 1 + r_{12}(z) + O(\hbar)$), provide equations of motion for the (relativistic) top model if the quantum unitary $R$-matrix satisfies the associative Yang-Baxter equation. It is verified explicitly using (1.15) that under substitution $S = S(p, q, \hbar, c)$ (1.4) the Lax matrix $L^h(S, z)$ turns into the gauged transformed Ruijsenaars-Schneider one (1.3). Therefore, the $M$-matrix of the Ruijsenaars-Schneider model can be evaluated by the inverse gauge transformation of the $M^h(S(p, q, \hbar, c), z)$. In this way we come to the expression (1.10), which is then verified by direct calculation.

The proof of the second Theorem (1.13) uses the geometric treatment of $g(z,q)$. The non-trivial part of the Lax matrix (1.6) is a $z$-component of the pure gauge connection. To obtain it we need to allow transition from the Lax matrix to the connection along the $z$ coordinate on the base spectral curve. It is exactly the statement of the Painlevé-Calogero correspondence [31]: the Lax pair of the elliptic Calogero-Moser model satisfies not only the Lax equation (1.9) but also the monodromy preserving equations (zero-curvature condition)

$$2\pi i \frac{d}{d\tau} L - \frac{d}{dz} M = [L, M],$$

which lead to the higher Painlevé equations (4.3) with the time variable being the moduli of the elliptic curve $\tau$. Then the Lax matrix (1.6) can be obtained by combining the Schlesinger transformation (the action of the modification of bundle on the connection) and the Painlevé-Calogero correspondence, see [45]. Applying the same procedure to the $M$-matrix we come to the from (1.13).

Another purpose of the paper is to study possible extension of the factorization formulae to the models associated with the root systems of the classical Lie algebras [38, 18, 11, 21, 14].
Some of the constructions discussed above are naturally extended to these cases. For instance, the symplectic Hecke correspondence and underlying modifications of bundles can be defined for $G$-bundles with $G$ being a simple complex Lie group \[36\]. At the same time the intertwining matrix in the elliptic case is known to exist for $A_N$ root system only \[10\]. The question which intertwining vectors generate the factorized Lax pairs deserves further elucidations.

Instead of using (1.1) and/or (1.6) in the rational (and trigonometric) cases without spectral parameter we can rewrite them in a slightly different way using that $g' = C_0 g$ in these cases, where $C_0$ is some constant matrix. This is due to $g$-matrix for the latter cases is of Vandermonde type. Then (1.6) turns into

$$L_{CM}(z) = P + \nu' g^{-1} C_0 g. \quad (1.18)$$

In the last Section we propose factorization formulae of type (1.18) for the rational Calogero models related to root systems $B, C, D$. This study is inspired by possible application to quantum-classical duality \[23\].

## 2 Brief review

### 2.1 Ruijsenaars-Schneider and Calogero-Moser models

The elliptic $gl_N$ Ruijsenaars-Schneider model \[42\] describes $N$ interacting particles on the complex plane with positions $q_k$ and equations of motion

$$\ddot{q}_i = \sum_{k \neq i} \dot{q}_i \dot{q}_k (2E_1(q_{ik}) - E_1(q_{ik} + \hbar) - E_1(q_{ik} - \hbar)), \quad i = 1 \ldots N, \quad (2.1)$$

where $q_{ij} = q_i - q_j$, $E_1(x)$ is the function (A.12) and $\hbar$ is the coupling constant. The model is described by Mat($N, \mathbb{C}$)-valued Lax matrix with the spectral parameter $z$:

$$L_{RS}^{ij} = \phi(z, q_{ij} + \hbar) \prod_{k \neq j} \frac{\vartheta(q_{jk} - \hbar)}{\vartheta(q_{jk})} e^{p_j/c} = \phi(z, q_{ij} + \hbar) \frac{D_j^{-\hbar}}{D_j^0} e^{p_j/c}, \quad D_j^{\eta} = \prod_{k \neq j} \vartheta(q_{jk} + \eta), \quad (2.2)$$

where $c$ is the light speed and $\phi(x, y)$ is the Kronecker function (A.11). The Hamiltonian arises as the trace of (2.2). More precisely,

$$H_{RS} = c \frac{\text{tr}L_{RS}}{\phi(z, \hbar)} = c \sum_{j=1}^{N} \frac{D_j^{-\hbar}}{D_j^0} e^{p_j/c}. \quad (2.3)$$

Then\[5\]

$$\dot{q}_j = \frac{D_j^{-\hbar}}{D_j^0} e^{p_j/c}. \quad (2.4)$$

and the Lax matrix (2.2) acquires the form:

$$L_{RS}^{ij} = \phi(z, q_{ij} + \hbar) \dot{q}_j. \quad (2.5)$$

\[5\]The canonical Poisson brackets are assumed: $\{p_i, q_j\} = \delta_{ij}$.
The definition of the velocities (2.4) is not unique. A family of canonical maps

\[ p_j \rightarrow p_j + c_1 \log \prod_{k \neq j} \frac{\vartheta(q_j - q_k + c_2)}{\vartheta(q_j - q_k - c_2)} , \tag{2.6} \]

with arbitrary constants \(c_{1,2}\) can be used as well. Equations of motion (2.1) (they are independent of (2.6)) can be written in the Lax form

\[ \dot{L}^{RS} \equiv \{ H^{RS}, L^{RS} \} = [L^{RS}, M^{RS}] , \tag{2.7} \]

where the \(M\)-matrix is as follows:

\[ M^{RS}_{ij} = -(1 - \delta_{ij}) \phi(z, q_i - q_j) \dot{q}_j \]

\[ -\delta_{ij} \left( \dot{q}_i (E_1(z) + E_1(\hbar)) + \sum_{k \neq i} \dot{q}_k (E_1(q_{ik} + \hbar) - E_1(q_{ik})) \right) , \tag{2.8} \]

In the non-relativistic limit \(\hbar = \nu/c, c \to \infty\) the Lax pair of the Calogero-Moser model \([12]\) is reproduced \([29]\):

\[ L^{CM}_{ij} = (\dot{q}_i + \nu E_1(z)) \delta_{ij} + \nu (1 - \delta_{ij}) \phi(z, q_{ij}) , \quad \dot{q}_i = p_i - \nu \sum_{k \neq i} E_1(q_{ik}) , \tag{2.9} \]

\[ M^{CM}_{ij} = \nu d_i \delta_{ij} + \nu (1 - \delta_{ij}) f(z, q_{ij}) , \quad d_i = \sum_{k \neq i} E_2(q_{ik}) , \tag{2.10} \]

See the definitions of \(E_2(x)\) and \(f(x, y)\) in \([A.12], [A.19]\). The Hamiltonian

\[ H^{CM} = \sum_{i=1}^{N} \frac{\dot{q}_i^2}{2} - \nu^2 \sum_{i>j}^{N} \varphi(q_i - q_j) , \tag{2.11} \]

where \(\dot{q}_i = \dot{q}_i(p, q)\) (2.9) provides equations of motion

\[ \ddot{q}_i = \nu^2 \sum_{k \neq i} \varphi'(q_{ik}) . \tag{2.12} \]

In trigonometric and rational cases the functions used above are as follows. In the trigonometric limit

\[ \phi(z, q) \to \coth(z) + \coth(q) , \quad E_1(z) \to \coth(z) , \quad D_j^\eta \to \prod_{k \neq j} \sinh(q_{jk} + \eta) , \]

\[ f(z, q) \to -\frac{1}{\sinh^2(q)} , \quad E_2(z) , \varphi(z) \to \frac{1}{\sinh^2(z)} , \tag{2.13} \]

and in the rational limit

\[ \phi(z, q) \to \frac{1}{z} + \frac{1}{q} , \quad E_1(z) \to \frac{1}{z} , \quad D_j^\eta \to \prod_{k \neq j} (q_{jk} + \eta) , \]

\[ f(z, q) \to -\frac{1}{q^2} , \quad E_2(z) , \varphi(z) \to \frac{1}{z^2} . \tag{2.14} \]
2.2 Elliptic integrable tops

The elliptic top \[32\] is the model of the Euler-Arnold type. Dynamical variables are arranged into matrix \(S \in \text{Mat}(N, \mathbb{C})\), and the equations of motion are

\[
\dot{S} = [S, J(S)], \quad S = \sum_{i,j=1}^{N} E_{ij} S_{ij} = \sum_{\alpha \in \mathbb{Z}_N \times \mathbb{Z}_N; \alpha \neq 0} T_{\alpha} S_{\alpha}, \quad (2.15)
\]

\[
J(S) = \sum_{\alpha \neq 0} T_{\alpha} S_{\alpha} J_{\alpha}, \quad J_{\alpha} = -E_{2}(\omega_{\alpha}), \quad \omega_{\alpha} = \frac{\alpha_1 + \alpha_2 \tau}{N}, \quad (2.16)
\]

where \(\{E_{ij}\}\) is the standard matrix basis and \(\{T_{\alpha}\}\) is the one \((A.1)\). The Lax pair is of the form:

\[
L^{\text{top}}(z) = \sum_{\alpha \neq 0} T_{\alpha} S_{\alpha} \varphi_{\alpha}(z, \omega_{\alpha}), \quad M^{\text{top}}(z) = \sum_{\alpha \neq 0} T_{\alpha} S_{\alpha} f_{\alpha}(z, \omega_{\alpha}). \quad (2.17)
\]

The Hamiltonian

\[
H^{\text{top}} = \sum_{\alpha \neq 0} S_{\alpha} S_{-\alpha} E_{2}(\omega_{\alpha}) \quad (2.18)
\]

is evaluated from \(\text{tr}(L^{\text{top}}(z))^2\), and the Poisson structure is the Poisson-Lie one\(^6\)

\[
\{S_1, S_2\} = [S_1, P_{12}], \quad (2.19)
\]

coming from the classical \(r\)-matrix structure

\[
\{L_{1}^{\text{top}}(z), L_{2}^{\text{top}}(w)\} = [L_{1}^{\text{top}}(z) + L_{2}^{\text{top}}(w), r_{12}(z - w)], \quad (2.20)
\]

where \(r_{12}(z - w)\) is the Belavin-Drinfeld \(r\)-matrix (see \((3.3)\)).

The model \((2.15)-(2.18)\) possesses the relativistic extension \[34\] described by equations of motion

\[
\dot{S} = [S, J^{n}(S)], \quad (2.21)
\]

\[
J^{n}(S) = \sum_{\alpha \neq 0} T_{\alpha} S_{\alpha} J^{n}_{\alpha}, \quad J^{n}_{\alpha} = E_{1}(\eta + \omega_{\alpha}) - E_{1}(\omega_{\alpha}) \quad (2.22)
\]

and the Lax pair

\[
L^{n}(z) = \sum_{\alpha} T_{\alpha} S_{\alpha} \varphi_{\alpha}(z, \omega_{\alpha} + \eta), \quad M^{n}(z) = -\sum_{\alpha \neq 0} T_{\alpha} S_{\alpha} \varphi_{\alpha}(z, \omega_{\alpha}). \quad (2.23)
\]

The Hamiltonian appears from \(\text{tr}L^{n}(z)\) as

\[
H^{\text{rel}} = S_0, \quad (2.24)
\]

and the Poisson structure is the \(\text{GL}_N\) generalization of the classical Sklyanin algebra \[15\]. It comes from the quadratic \(r\)-matrix structure

\[
\{L_{1}^{n}(z), L_{2}^{n}(w)\} = [L_{1}^{n}(z)L_{2}^{n}(w), r_{12}(z - w)] \quad (2.25)
\]

with the same \(r\)-matrix as in \((2.20)\). The general (including not only elliptic case) form of the Poisson structure follows from the local expansion of \((2.25)\) near \(z = 0\) and \(w = 0\), see \[34\].

In case when \(N - 1\) eigenvalues of the matrix \(S\) equal to each other the relativistic top is gauge equivalent to the Ruijsenaars-Schneider model \((1.3)\), and the non-relativistic top is gauge equivalent to the Calogero-Moser model \((1.7)\).

\(^6\)\(P_{12}\) is the permutation operator \((A.6)\).
2.3 Factorization formulae

Elliptic Ruijsenaars-Schneider model

The Lax matrix (2.2) is factorized as follows

\[
L_{RS}^{ij} = \frac{\vartheta'(0)}{\vartheta(h)} \sum_k g_{ik}^{-1}(z, q)g_{kj}(z + N h, q) e^{p_j/c}, \tag{2.26}
\]

where

\[
g(z, q) = \Xi(z, q) (D^0)^{-1} \tag{2.27}
\]

with

\[
\Xi_{ij}(z, q) = \vartheta \left[ \frac{1}{2} - \frac{i}{N} \right] \left( z - Nq_j + \sum_{m=1}^{N} q_m \left| N \tau \right) \right), \tag{2.28}
\]

and

\[
D^0_{ij}(z, q) = \delta_{ij}D^0_j = \delta_{ij} \prod_{k \neq j} \vartheta(q_j - q_k). \tag{2.29}
\]

See (A.7)-(A.10) for the definitions of theta-functions with characteristics. The matrix (2.28) was introduced in [26] as the intertwining matrix entering the IRF-Vertex relations (which we review below).

Consider also the Lax matrix

\[
L_{RS'}^{ij} = \phi(z, q_{ij} + h) \frac{D^h_j}{D^h_j} e^{p_j/c}, \quad D^h_j = \prod_{k \neq j} \vartheta(q_j + h), \tag{2.30}
\]

which differs from the one (2.2) by the sign of \(h\) in \(D^h\). The Lax matrices (2.30) and (2.2) are related by the canonical map (2.6) with \(c_1 = c\) and \(c_2 = h\). The one (2.30) is also factorized but in a slightly different way:

\[
L_{RS'}^{ij} = \frac{\vartheta'(0)}{\vartheta(h)} \sum_k (D^h_k)^{-1} \Xi_{ik}^{T}(z + N h, -q) \left( \Xi^T \right)_{kj}^{-1} (z, -q) D^h_j e^{p_j/c}. \tag{2.31}
\]

The latter follows from (2.2) by the transposition (denoted by \(T\)) and changing \(q \rightarrow -q\). Curiously, both factorization (for \(L^{RS}\) and \(L^{RS'}\)) emerge in the framework of the quantum-classical correspondence [23]. They emerge for two possible values of the \(\mathbb{Z}_2\)-grading parameter in the supersymmetric spin chains.

Elliptic Calogero-Moser model

The non-relativistic limit to the Calogero-Moser model is achieved by setting \(h = \nu/c\) and \(c \rightarrow \infty\) in (2.26). This yields

\[
L^{CM} = P + N \nu g^{-1}(z)g'(z), \tag{2.32}
\]

where the non-trivial part can be written explicitly:

\[
(g^{-1}(z)g'(z))_{ij} = \frac{1}{N} \delta_{ij} \left( E_1(z) - \sum_{k \neq i} E_1(q_{ik}) \right) + \frac{1}{N} (1 - \delta_{ij})\phi(z, q_{ij}). \tag{2.33}
\]
Trigonometric Ruijsenaars-Schneider model

The Lax matrix for the trigonometric gl$_N$ Ruijsenaars-Schneider model with spectral parameter is of the form:

$$L_{RS}^{(z)}_{ij} = e^{\hbar(N-2)} \sinh (q_i - q_j + \hbar) + \sinh (Nz) e^{p_j/c} \prod_{k \neq j}^{N} \frac{\sinh (q_j - q_k - \hbar)}{\sinh (q_j - q_k)}.$$  \hspace{1cm} (2.34)

It admits the following factorization formula:

$$L_{RS}^{(z)} = D_0 \tilde{\Xi}^{-1} (z) \tilde{\Xi} (z + \hbar) (D_0)^{-1} e^{P/c},$$  \hspace{1cm} (2.35)

where

$$D_0^{ij} = \delta_{ij} \prod_{k \neq i} (e^{-2q_i} - e^{-2q_k}),$$  \hspace{1cm} (2.36)

$$\tilde{\Xi}_{ij} (z) = \begin{cases} x_j^{i-1}, & i \leq N, \\ x_j^{N-1} + \frac{(-1)^N}{x_j}, & i = N \end{cases}$$

with $x_j = e^{-2q_i + 2z + 2q_i}$. Here $\bar{q} = \frac{1}{N} \sum_{k=1}^{N} q_k$ is the center of mass coordinate.

The Lax matrix for the trigonometric gl$_N$ Ruijsenaars-Schneider model without spectral parameter is of the form:

$$L_{RS}^{(z)} = \frac{\sinh \hbar}{\sinh (q_i - q_j + \hbar)} e^{p_j/c} \prod_{k \neq j}^{N} \frac{\sinh (q_j - q_k - \hbar)}{\sinh (q_j - q_k)}.$$  \hspace{1cm} (2.37)

The factorization is as follows:

$$L_{RS}^{(z)} = D^0(q) \tilde{\Xi}^{-1} (q,z) \tilde{\Xi} (q,z + \hbar) (D^0)^{-1}(q)e^{P/c} =$$

$$= D^0(q) \tilde{\Xi}^{-1} (q,z) Y(h) \tilde{\Xi} (q,z) (D^0)^{-1}(q)e^{P/c},$$  \hspace{1cm} (2.38)

where

$$\tilde{\Xi}_{ij} (z) = \exp \left((2i - 1 - N)(z - q_j)\right),$$  \hspace{1cm} (2.39)

$$\langle D^0 \rangle_{ij} = \delta_{ij} \prod_{k \neq i} \sinh (q_i - q_k)$$  \hspace{1cm} (2.40)

and

$$Y(\lambda)_{ij} = \delta_{ij} \exp \left(-(N + 1 - 2i)\lambda\right).$$  \hspace{1cm} (2.41)

Trigonometric Calogero-Moser model

The Lax matrix of the trigonometric gl$_N$ Calogero-Moser model with spectral parameter is of the following form:

$$L_{ij}^{CM} (z) = \delta_{ij} \left(p_i + \nu(N - 2) + \nu \coth(Nz) - \nu \sum_{k \neq i}^{N} \coth(q_i - q_k)\right) +$$

$$+ \nu(1 - \delta_{ij})(\coth(q_i - q_j) + \coth(Nz)).$$  \hspace{1cm} (2.42)
The factorization formula is as follows:

\[ L^{CM}(z) = P + \nu D^0 \hat{\Xi}^{-1}(\partial_z \hat{\Xi})(D^0)^{-1}, \] (2.43)

where \( \hat{\Xi} \) and \( D^0 \) are defined in (2.36).

The Lax matrix of the trigonometric \( \mathfrak{gl}_N \) Calogero-Moser model without spectral parameter is of the following form:

\[ L^{CM} = \delta_{ij}(p_i - \nu \sum_{k \neq i}^{N} \coth (q_i - q_k)) + (1 - \delta_{ij}) \frac{\nu}{\sinh (q_i - q_j)}. \] (2.44)

Its factorization is as follows:

\[ L^{CM} = P + \nu \tilde{D}^0 \tilde{V}^{-1}(\lambda)\partial_\lambda \tilde{V}(\lambda)(\tilde{D}^0)^{-1} = P + \tilde{D}^0 \tilde{V}^{-1}(\log Y)\tilde{V}(\tilde{D}^0)^{-1}, \] (2.45)

where

\[ (\log Y)_{ij} = \delta_{ij} \nu(2i - 1 - N) \] (2.46)

while \( \tilde{V} \) and \( \tilde{D}^0 \) are those from (2.39) and (2.40).

**Rational Ruijsenaars-Schneider model**

The Lax matrix for the rational \( \mathfrak{gl}_N \) Ruijsenaars-Schneider model with spectral parameter is of the form:

\[ L_{ij}^{RS}(z) = \hbar \left( \frac{1}{q_i - q_j + \hbar} + \frac{1}{Nz} \right) e^{p_j/c} \prod_{k \neq j}^{N} \frac{q_j - q_k - \hbar}{q_j - q_k}. \] (2.47)

It admits the following factorization formula:

\[ L^{RS}(z) = D^0(q)\Xi^{-1}(q, z)\Xi(q, z + \hbar)(D^0)^{-1}(q)e^{P/c}, \] (2.48)

where

\[ (D^0)_{ij}(q) = \delta_{ij} \prod_{k \neq i}^{n} (q_i - q_k), \] (2.49)

\[ \Xi_{ij}(q, z) = (z - q_j + \bar{q})^{\varrho(i)}, \quad \bar{q} = \frac{1}{N} \sum_{k=1}^{N} q_k \]

with

\[ \varrho(i) = \begin{cases} i - 1 \text{ for } 1 \leq i \leq N - 1, \\ N \text{ for } i = N. \end{cases} \] (2.50)

The Lax matrix for the rational \( \mathfrak{gl}_N \) Ruijsenaars-Schneider model without spectral parameter is of the form:

\[ L_{ij}^{RS} = \frac{\hbar e^{p_j/c}}{q_i - q_j + \hbar} \prod_{k \neq j}^{N} \frac{q_j - q_k - \hbar}{q_j - q_k}. \] (2.51)

It admits the following factorization formula:

\[ L^{RS} = D^0(q)V^{-1}(z)V(z + \hbar)(D^0)^{-1}(q)e^{P/c} = \]

\[ = D^0(q)V^{-1}(z)C_h V(z)(D^0)^{-1}(q)e^{P/c}, \] (2.52)
where
\[ V_{ij}(z) = (z - q_j + \bar{q})^{i-1} \] (2.53)
is the Vandermonde matrix, \( D^0(q) \) is (2.49) and
\[
(C_\lambda)_{ij} = \begin{cases} 
\frac{(i-1)!\lambda^{i-j}}{(j-1)!(i-j)!}, & j \leq i, \\
0, & j > i.
\end{cases}
\] (2.54)
The following simplification of (2.52) is also correct:
\[
L_{RS} = D^0(q) V^{-1}(q) C_0 V(q) (D^0)^{-1}(q) e^{P/c},
\] (2.55)
where
\[ V_{ij}(q) = (-q_j)^{i-1}. \] (2.56)

**Rational Calogero-Moser Model**

The Lax matrix of the rational \( gl_N \) Calogero-Moser model with spectral parameter is of the following form:
\[
L_{CM}^{ij}(z) = \delta_{ij}(p_i - \sum_{k \neq i}^N \nu \frac{\nu}{q_i - q_k}) + (1 - \delta_{ij}) \frac{\nu}{q_i - q_j} + \nu \frac{\nu}{Nz}.
\] (2.57)
The factorization formula is as follows:
\[
L^{CM}(z) = P + \nu D^0 \Xi^{-1}(\partial_z \Xi)(D^0)^{-1},
\] (2.58)
where \( \Xi \) and \( D^0 \) are those from (2.49).

The Lax matrix of the rational \( gl_N \) Calogero-Moser model without spectral parameter is of the following form:
\[
L_{CM}^{ij} = \delta_{ij}(p_i - \sum_{k \neq i}^N \nu \frac{\nu}{q_i - q_k}) + (1 - \delta_{ij}) \frac{\nu}{q_i - q_j}.
\] (2.59)
Its factorization is given by
\[
L^{CM}(z) = P + \nu D^0 V^{-1}(z)(\partial_z V)(z)(D^0)^{-1},
\] (2.60)
where \( D^0 \) is defined in (2.49) and \( V(z) \) – in (2.53). Equivalently, one can represent (2.59) in the form:
\[
L^{CM} = P + \nu D^0 V^{-1}(q) C_0 V(q)(D^0)^{-1},
\] (2.61)
with \( V(q) \) (2.56) and
\[
(C_0)_{ij} = \begin{cases} 
 i - 1, & i = j + 1, \\
0, & \text{otherwise}.
\end{cases}
\] (2.62)
3 IRF-Vertex relations

3.1 IRF-Vertex correspondence

First, let us introduce three quantum $R$-matrices.

**Baxter-Belavin (non-dynamical) $R$-matrix** [8] (see also [10]):

\[
R^h_{12}(z) = \sum_\alpha T_\alpha \otimes T_{-\alpha} \varphi_\alpha(z, \omega_\alpha + h), \quad \text{Res}_{z=0} R^h_{12}(z) = NP_{12}.
\] (3.1)

The classical limit (near $\hbar = 0$)

\[
R^h_{12}(z) = \frac{1 \otimes 1}{\hbar} + r_{12}(z) + \hbar m_{12}(z) + O(\hbar^2)
\] (3.2)

provides the classical Belavin-Drinfeld [10] $r$-matrix

\[
r_{12}(z) = 1 \otimes 1 E_1(z) + \sum_{\alpha \neq 0} T_\alpha \otimes T_{-\alpha} \varphi_\alpha(z, \omega_\alpha).
\] (3.3)

The Baxter-Belavin $R$-matrix $R^h_{12}(z_1, z_2) = R^h_{12}(z_1 - z_2)$ (3.1) satisfies the quantum Yang-Baxter equation

\[
R^h_{12}(z_1, z_2) R^h_{13}(z_1, z_3) R^h_{23}(z_2, z_3) = R^h_{23}(z_2, z_3) R^h_{13}(z_1, z_3) R^h_{12}(z_1, z_2)
\] (3.4)

In this Section we will also use notation

\[
R^h_{12}(h, z_1, z_2) = R^h_{12}(h, z_1 - z_2) = \frac{1}{N} R^h_{12} N(z_1 - z_2).
\] (3.5)

**Felder’s (dynamical) $R$-matrix** [22]:

\[
R^F_{12}(h, z_1, z_2|q) = R^F_{12}(h, z_1 - z_2|q) =
\]

\[
= \sum_{i \neq j} E_{ii} \otimes E_{jj} \phi(h, -q_{ij}) + \sum_{i \neq j} E_{ij} \otimes E_{ji} \phi(z_1 - z_2, q_{ij}) + \phi(h, z_1 - z_2) \sum_i E_{ii} \otimes E_{ii}.
\] (3.6)

It is a solution of the quantum dynamical Yang-Baxter equation

\[
R^h_{12}(z_1, z_2|q) R^h_{13}(z_1, z_3|q - h^{(2)}) R^h_{23}(z_2, z_3|q) =
\]

\[
= R^h_{23}(z_2, z_3|q - h^{(1)}) R^h_{13}(z_1, z_3|q) R^h_{12}(z_1, z_2|q - h^{(3)}),
\] (3.7)

where the shifts of dynamical arguments $u$ are performed as follows:

\[
R^h_{12}(z_1, z_2|q + h^{(3)}) = P^h_3 R^h_{12}(z_1, z_2|q) P^h_3^{-1}, \quad P^h_3 = \sum_{k=1}^N 1 \otimes 1 \otimes E_{kk} \exp(\hbar \frac{\partial}{\partial q_k}).
\] (3.8)

**Arutyunov-Chekhov-Frolov (semi-dynamical) $R$-matrix** [4]:

\[
R^{ACF}_{12}(h, z_1, z_2|q) = \sum_{i \neq j} E_{ii} \otimes E_{jj} \phi(h, q_{ij}) + \sum_{i \neq j} E_{ij} \otimes E_{ji} \phi(z_1 - z_2, q_{ij})-
\]

\[
- \sum_{i \neq j} E_{ij} \otimes E_{jj} \phi(z_1 + h, q_{ij}) + \sum_{i \neq j} E_{jj} \otimes E_{ij} \phi(z_2, q_{ij}) +
\]

\[
+ (E_1(h) + E_1(z_1 - z_2) + E_1(z_2) - E_1(z_1 + h)) \sum_i E_{ii} \otimes E_{ii}
\] (3.9)
satisfies the following (semi-dynamical) Yang-Baxter equation:

\[
R^h_{12}(z_1, z_2 | q) R^h_{13}(z_1 - h, z_3 - h | q) R^h_{23}(z_2, z_3 | q) =
\]
\[
= R^h_{23}(z_2 - h, z_3 - h | q) R^h_{13}(z_1, z_3 | q) R^h_{12}(z_1 - h, z_2 - h | q).
\]

**IRF-Vertex correspondence** \([9, 26, 24]\) establishes an explicit relationship between dynamical and non-dynamical \(R\)-matrices \((3.5)\) and \((3.6)\):

\[
g_2(z_2, q) g_1(z_1, q + \hbar^{(2)}) R^p_{12}(h, z_1 - z_2 | q) = R^p_{12}(h, z_1 - z_2) g_1(z_1, q) g_2(z_2, q + \hbar^{(1)}).
\]

For the semi-dynamical \(R\)-matrix \((3.9)\) the following relations hold true \([4, 44]\):

\[
R^p_{12}(h, z_1 - z_2 | q) = g^{-1}_1(z_1, q + \hbar^{(2)}) g_1(z_1 + h, q) R^{ACP}_{12}(h, z_1, z_2 | q) g_2(z_2 + h, q) g_2(z_2, q + \hbar^{(1)}).
\]

Combining \((3.11)\) and \((3.12)\) we get

\[
R^p_{12}(h, z_1 - z_2) = g_1(z_1 + h, q) g_2(z_2, q) R^{ACP}_{12}(h, z_1, z_2 | q) g_2^{-1}(z_2 + h, q) g_1^{-1}(z_1, q).
\]

Following \([44]\) let us rewrite relation \((3.13)\) as

\[
g_2^{-1}(z_2, q) R^p_{12}(h, z_1 - z_2) = g_1(z_1 + h, q) R^{ACP}_{12}(h, z_1, z_2 | q) g_2^{-1}(z_2 + h, q) g_1^{-1}(z_1, q)
\]

and take the residues at \(z_2 = 0\) of both parts:

\[
\tilde{g}_2(0, q) R^p_{12}(h, z) = g_1(z + h, q) \mathcal{O}_{12} g_2^{-1}(h, q) g_1^{-1}(z, q),
\]

where

\[
\tilde{g}(0, q) = \text{Res}_{z=0} g^{-1}(z)
\]

and

\[
\mathcal{O}_{12} = \text{Res}_{z=0} R^{ACP}_{12}(h, z_1, z_2 | q) = \sum_{i,j} E_{ii} \otimes E_{ji}.
\]

Then, for the \(R\)-matrix \((3.11)\) we have\(^7\)

\[
\frac{1}{\mathcal{N}} \tilde{g}_2(0, q) R^h_{12}(z) = g_1(z + Nh, q) \mathcal{O}_{12} g_2^{-1}(Nh, q) g_1^{-1}(z, q).
\]

This formula is \(R\)-matrix analogue of the Lax matrix factorization. We will use it in Section \((3.4)\) for evaluation of the \(M\)-matrix.

### 3.2 Classical IRF-Vertex relations

The classical IRF-Vertex transformations relate the classical dynamical \(r\)-matrix structures of the Ruijsenaars-Schneider (or Calogero-Moser) models with the non-dynamical \(r\)-matrix structures of the relativistic top \((2.25)\) (or the non-relativistic top \((2.20)\), see e.g. \([6, 13]\). At the

\(^7\)Notice that for \(\mathcal{N} = 1\) \((3.18)\) reproduces the definition of the Kronecker function \((A.11)\).
level of the classical Lax matrices the IRF-Vertex transformation is the gauge transformation generated by the matrix $g(z)$:

$$L^\text{top}(z) = g(z)L^\text{CM}(z)g^{-1}(z), \quad (3.19)$$

for the Lax matrices (2.9) and (2.17). Similarly,

$$L^h(z) = g(z)L^\text{RS}(z)g^{-1}(z), \quad (3.20)$$

for the Lax matrices (2.26) and (2.22). Being written as (3.19) and (3.20) these tops are just alternative forms of the Ruijsenaars-Schneider and Calogero-Moser models respectively. However, these are only special cases of the tops corresponding to the rank one matrix $S$. In the general case the dimensions of the phase spaces of the tops are large than those for the spinless many-body systems.

**Structure of the intertwining matrix.** The intertwining matrix $g(z)$ (2.27) satisfies the following properties [24, 26, 40]:

1. The matrix $g(z)$ is degenerated at $z = 0$. See (A.30).
2. The matrix $g(0)$ has one-dimensional kernel in the direction of the vector-column

$$\rho = (1, 1, \ldots, 1)^T \in \mathbb{C}^N \quad (3.21)$$

Consider $g^{-1}(z)$ near $z = 0$:

$$g^{-1}(z) = \frac{1}{z} \hat{g}(0) + A + O(z), \quad \hat{g}(0, q) = \text{Res}_{z=0} g^{-1}(z). \quad (3.22)$$

Then the matrix $\hat{g}(0)$ is of rank one:

$$\hat{g}(0) = \rho \otimes \psi; \quad \psi = (\psi_1(q), \ldots, \psi_N(q)) \in \mathbb{C}^N. \quad (3.23)$$

and

$$\psi = \frac{1}{N} \rho^T \hat{g}(0). \quad (3.24)$$

Indeed, by expanding $g^{-1}(z)g(z) = 1_N$ near $z = 0$ we get $\hat{g}(0)g(0) = 0$. The kernel of $g(0)$ is one-dimensional. Therefore, the kernel of $\hat{g}(0)$ is $N - 1$ dimensional. The latter means that $\hat{g}(0)$ is a product of a vector by covector. On the other hand, $g(0)\hat{g}(0) = 0$. Thus, the vector should lie in the kernel $g(0)$, i.e. it is proportional to $\rho$ (3.21). This gives (3.23).

**Classical bosonization formulae** are the classical analogues of the representation of the Sklyanin algebra generators in terms of the difference operators, i.e. the top’s variables $S_\alpha$ (entering $\text{GL}_N$ classical Sklyanin algebra) are expressed in terms of the Ruijsenaars-Schneider variables. The non-relativistic limit leads to the classical Lie (co)algebra variables expressed in terms of the Calogero-Moser variables. For the explicit change of variables see [46, 24, 13] and [11, 34, 27].

In the above formulae (3.19), (3.20) the top models are of very special type. The matrices $S$ in both cases are of rank one, while in (2.17) and/or (2.22) they are arbitrary. Indeed, the matrices $S$ are residues of the corresponding Lax matrices. Assuming (2.26)- (2.27)

$$L^\text{RS} = \frac{\partial^\prime(0)}{\partial(h)} g^{-1}(z, q)g(z + Nh, q) e^{P/c}, \quad P = \text{diag}(p_1, \ldots, p_N) \quad (3.25)$$

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and (3.20) we get
\[ L^h(z) = \frac{\vartheta'(0)}{\vartheta(h)} g(z + N\hbar, q) e^{P/c} \tilde{g}^{-1}(z, q) . \] (3.26)

Therefore, for the matrix \( S \) of dynamical variables in the relativistic top we have
\[ S = \text{Res}_{z=0} L^h(z) = \frac{\vartheta'(0)}{\vartheta(h)} g(N\hbar) e^{P/c} \tilde{g}(0) \xi \otimes \psi , \] (3.27)

where
\[ \xi = \frac{\vartheta'(0)}{\vartheta(h)} g(N\hbar) e^{P/c} \rho , \quad \psi = \frac{1}{N} \rho^T \tilde{g}(0) . \] (3.28)

The point of the phenomenon is that the Lax matrix (3.26) is expressed through the variables \( S \) (3.27).

The row-vector \( \psi \) can be found in a different way. The residue of the Ruijsenaars-Schneider Lax matrix (2.2) is of the form:
\[ \text{Res}_{z=0} L^{RS}(z) = \rho \otimes \rho^T D^{-h}(D^0)^{-1} e^{P/c} . \] (3.29)

On the other hand, from (3.25) we have
\[ \text{Res}_{z=0} L^{RS}(z) = \frac{\vartheta'(0)}{\vartheta(h)} \tilde{g}(0) g(N\hbar) e^{P/c} = \frac{\vartheta'(0)}{\vartheta(h)} \rho \otimes \psi g(N\hbar) e^{P/c} . \] (3.30)

By comparing (3.29) and (3.30) we come to
\[ \psi = \frac{\vartheta'(0)}{\vartheta(h)} \rho^T D^{-h}(D^0)^{-1} g^{-1}(N\hbar) . \] (3.31)

Notice that by the definition (3.23) \( \psi \) is independent of \( \hbar \). Therefore, the parameter \( \hbar \) in the r.h.s. of (3.31) is arbitrary. Tending it to zero we reproduce (3.24). Plugging (3.31) into (3.27) we get
\[ S = g(N\hbar) e^{P/c} \rho \otimes \rho^T D^{-h}(D^0)^{-1} g^{-1}(N\hbar) . \] (3.32)

In the non-relativistic limit the generators of the Poisson-Lie structure appear. By setting \( \hbar = \nu/c \) and taking the limit \( c \to \infty \) in (3.27) we obtain:
\[ S = g(0) P \tilde{g}(0) + N \nu g'(0) \tilde{g}(0) = \mu \otimes \psi , \quad \mu = (g(0) P + N \nu g'(0)) \rho . \] (3.33)

The Poisson-Lie brackets for \( S \) follow from the canonical brackets between components of \( \mu \) and \( \psi \): \( \{ \mu_i, \psi_j \} \propto \delta_{ij} \). This lead to a natural quantization \( \hat{\mu}_i \propto \partial/\partial \psi_i \). Such coordinates were used in [41] and [39] for reformulation of the quantum Calogero-Moser model.

**Modifications of bundles.** The IRF-Vertex intertwining matrix can be treated as modification of bundle [47] [32]. It is no coincidence that the vector \( \rho \) (3.21) enters the residue of the Lax matrix (3.29). In fact, dealing with the singular gauge transformation (degenerated at point \( z = 0 \)) we must impose condition for an eigenvector (\( \rho \)) of the residue of the Lax matrix under transformation to lie in the kernel of the gauge transformation at point \( z = 0 \): \( \rho \in \text{Ker} g(0) \). This condition comes from the requirement not to produce the second order pole at \( z = 0 \) when performing conjugation by the matrix \( g(z) \). We explain it below. Here, for the Lax matrix...
it is easy to see by expansion of the r.h.s. of (3.20) near \( z = 0 \). The vanishing of the second order pole is equivalent to
\[
g(0) \text{Res}_{z=0} L^{RS}(z) \tilde{g}(0) = 0. \tag{3.34}
\]
It is fulfilled due to
\[
\text{Res}_{z=0} L^{RS}(z) \rho = \lambda_0 \rho, \quad \lambda_0 = \sum_k \dot{q}_k = H^{RS}/c \quad \text{and} \quad g(0) \rho = 0.
\]

The Lax matrices with spectral parameter \( z \) can be viewed as sections of bundles over the base curve \( \Sigma \) with local coordinate \( z \). In our case \( \Sigma \) is the elliptic curve with moduli \( \tau \).

The Lax matrices are fixed by their residues and quasi-periodic behavior on the lattice \( \mathbb{Z} + \tau \mathbb{Z} \). The latter means that they are sections of \( \text{End}(V) \)-bundles for some holomorphic vector bundles \( V \). For the Lax matrices of the Calogero-Moser (2.9) and the elliptic top (2.17) models using (A.14) we have
\[
L_{CM}(z + 1) = L_{CM}(z), \quad L_{CM}(z + \tau) = e^{-2\pi i \text{diag}(q)} L_{CM}(z) e^{2\pi i \text{diag}(q)}, \tag{3.35}
\]
where
\[
\text{diag}(q) = \text{diag}(q_1, q_2, \ldots, q_N) \in \text{Mat}(N, \mathbb{C})
\]
is the diagonal matrix built of coordinates of particles, and
\[
L^{\text{top}}(z + 1) = Q^{-1} L^{\text{top}}(z) Q, \quad L^{\text{top}}(z + \tau) = \Lambda^{-1} L^{\text{top}}(z) \Lambda, \tag{3.37}
\]
where \( Q, \Lambda \) are the matrices (A.2). In the relativistic case an additional factor \( \exp(-2\pi i \hbar) \) appears for the shift of \( z \) by \( \tau \). It can be removed by dividing the Lax matrix by function \( \phi(z, \hbar) \).

The IRF-Vertex transformation acts as gauge transformation, which changes the quasi-periodic properties from (3.35) to (3.37). In fact, this condition almost fixes the matrix \( g(z) \) (2.27). More precisely, it fixes the \( \Xi(z) \) part of \( g(z) \), while the \( D^0 \) factor comes from the discussed above requirement for the vector \( \rho \) to belong to the kernel of \( g(0) \).

The map between two bundles, which is an isomorphism everywhere except a point, where it has one-dimensional kernel is known as the modification of (the initial) bundle \([20, 8]\). In our case it is performed at point \( z = 0 \) in the direction \( \rho \). Locally the modification is described as follows. Let us choose the basis in of sections in a way that the residue \( L_{-1} \) at \( z = 0 \) of the initial Lax matrix \( L(z) \in \Gamma(\text{End}(V)) \) is of the form
\[
L_{-1} = \begin{pmatrix}
\lambda & *_{1 \times (N-1)} \\
0_{(N-1) \times 1} & *_{(N-1) \times (N-1)}
\end{pmatrix}.
\tag{3.38}
\]
Then its eigenvector is \( v = (1, 0, \ldots, 0)^T \): \( L_{-1} v = \lambda v \). The modification towards this direction is given by
\[
g(z) = \begin{pmatrix}
z & 0_{1 \times (N-1)} \\
0_{(N-1) \times 1} & 1_{(N-1) \times (N-1)}
\end{pmatrix}.
\tag{3.39}
\]
In this case \( g(0) v = 0 \) and \( \text{Res}_{z=0} g^{-1}(z) = v \otimes v^T - \text{rank one matrix} \). We also have
\[
g(z) L_{-1} g^{-1}(z) = \begin{pmatrix}
\lambda & z *_{1 \times (N-1)} \\
0_{(N-1) \times 1} & *_{(N-1) \times (N-1)}
\end{pmatrix}.
\tag{3.40}
\]
This demonstrates that the second order pole does not appear. Notice also that the transformation \( (3.39) \) adds the zero at \( z = 0 \) to the section of the \( \det V \). This results in changing degree of the initial vector bundle \( V \) by one. So that the Calogero-Moser model correspond to \( \deg V = 0 \), while the elliptic top model – to \( \deg V = 1 \). The vector bundles over elliptic curves were classified in [5]. In the Hitchin approach [25] to elliptic integrable systems the moduli space of the underlying vector bundles play the role of the configuration space of the integrable system. Its dimension is equal to \( \text{g.c.d.}(\text{rk}(V), \deg(V)) \). This could be understood as follows. For the \( \deg(V) = k \) bundle the quasi-periodic properties of the Lax matrix are

\[
L^{\text{top}}(z + 1) = Q^{-1}L^{\text{top}}(z)Q, \quad L^{\text{top}}(z + \tau) = \Lambda^{-k}L^{\text{top}}(z)\Lambda^k. \quad (3.41)
\]

If \( \text{g.c.d.}(N, k) = m > 1 \) then there exist a matrix \( X \) parameterized by \( m \) variables \( (q_i) \) with the property \( [Q, X] = [\Lambda^k, X] = 0 \), so that the boundary conditions \( (3.41) \) become degenerated. This degeneracy can be eliminated by redefinition of \( (3.41) \) as

\[
L^{\text{top}}(z + 1) = Q^{-1}L^{\text{top}}(z)Q, \quad L^{\text{top}}(z + \tau) = X^{-1}\Lambda^{-k}L^{\text{top}}(z)\Lambda^kX. \quad (3.42)
\]

By reexpressing \( X \) through \( m \) variables of \( q_i \) type we get a model representing an intermediate case between the many-body and the tops systems [37]. Thus, starting with \( k = 0 \) and increasing the degree of \( V \) by modifications provides a family of gauge equivalent integrable Hitchin type systems including the (spin) Calogero-Moser model and the elliptic top. This scheme was called the symplectic Hecke correspondence [32, 49], and it is naturally generalized to the case when the structure group of the principle bundle (associated with the vector bundle \( V \)) is an arbitrary complex simple Lie group [36].

### 3.3 Factorization of the Lax matrix

To proceed we need the \( R \)-matrix formulation for the tops models [34]. The Lax pair of the relativistic top model (2.23) can be written in terms of the Belavin’s \( R \)-matrix (3.1)-(3.3) as follows\(^{8}\):

\[
L^h(S, z) = \frac{1}{N} \text{tr}_2 R^h_{12}(z)S_2, \quad M^h(S, z) = -\frac{1}{N} \text{tr}_2 (r^h_{12}(z)S_2). \quad (3.43)
\]

The factor \( 1/N \) comes from \([A.5]\). In fact, the formulae \( (3.43) \) are valid for a wider class of integrable tops, which appear when the underlying \( R \)-matrix satisfies the associative Yang-Baxter equation together with appropriate classical limit and skew-symmetry and/or unitarity conditions [35].

Multiply both sides of \( (3.18) \) by \( g_2(Nh)e^{P_{2/c}q_2}/\vartheta(h) \) from the left:

\[
\frac{\vartheta(0)}{N\vartheta(h)} g_2(Nh)e^{P_{2/c}q_2}(0, q) R^h_{12}(z) = \frac{\vartheta(0)}{\vartheta(h)} g_2(Nh)e^{P_{2/c}g_1(z + Nh,q)}O_{12}g_2^{-1}(Nh,q)g_1^{-1}(z,q).
\]

The trace over the second space provides in the first space the Lax matrix \( (3.43) \) with \( S = S(p, q) \) (3.27):

\[
L^h_1(S(p, q), z) = \text{tr}_2 \left( \frac{\vartheta(0)}{\vartheta(h)} e^{P_{2/c}g_1(z + Nh,q)}O_{12}g_1^{-1}(z,q) \right). \quad (3.44)
\]

\(^{8}\)The general idea is similar to the quasi-classical description presented originally in [45].

\(^{9}\)Here the \( M \)-matrix differs from the one in (2.23) by the term proportional to identity matrix (it is cancelled out from the Lax equations).
Taking into account that for any matrix \( T = \sum_{i,j} E_{ij} T_{ij} \in \text{Mat}(N, \mathbb{C}) \)

\[
\text{tr}_2 (O_{12} T_2) = \sum_i E_{ii} \sum_j T_{ij}
\]

we come to the factorized form of the Lax matrix:

\[
L^h(S(p, q), z) = \frac{\vartheta'(0)}{\vartheta(h)} g(z + Nh, q) e^{\frac{P}{\hbar}} g^{-1}(z, q).
\]

The inverse gauge transformation provides (3.25):

\[
L^{RS}(z) = g^{-1}(z, q) L^h(S(p, q), z) g(z, q) = \frac{\vartheta'(0)}{\vartheta(h)} g^{-1}(z, q) g(z + Nh, q) e^{\frac{P}{\hbar}}.
\]

### 3.4 Ruijsenaars-Schneider \( M \)-matrix in terms of \( g(z) \)

Let us compute the Ruijsenaars-Schneider \( M \)-matrix using representation (3.43). Consider expansion of the identity (3.18) near \( \hbar \). Using (3.2) and (3.22) we get in the \( \hbar^{-1} \) order:

\[
\hat{g}_2(0) = g_1(z) O_{12} g_1^{-1}(z) \hat{g}_2(0).
\]

It holds true by the following reason. Due to (3.23) \( \hat{g}_{km}(0) = \psi_m \). Then the r.h.s. of (3.48) acquires the form:

\[
g_1(z) O_{12} g_1^{-1}(z) \hat{g}_2(0) = \sum_{i,j,k,l,m} g_{ik}(z) g_{kj}(z) E_{ij} \otimes E_{lm} \hat{g}_{km}(0) = \sum_{i,j,l,m} E_{ij} \delta_{ij} \otimes E_{lm} \psi_m.
\]

The latter is equal to \( \hat{g}_2(0) \). For the \( \hbar^0 \) order of the expansion of (3.18) we have:

\[
\frac{1}{N} \hat{g}_2(0, q) r_{12}(z) = g'_1(z) O_{12} \hat{g}_2(0) g_1^{-1}(z) + g_1(z) O_{12} A_2 g_1^{-1}(z)
\]

with matrix \( A \) defined in (3.52). As in the previous paragraph let us multiply both sides of (3.50) by \( g_2(Nh) e^{\frac{P_2}{\hbar}} \vartheta(0)/\vartheta(h) \) from the left and compute the trace over the second space. This provides

\[
-M^h(S(p, q), z) = \frac{1}{N} \text{tr}_2(r_{12}(z) S_2(p, q)) = g'_1(z) G_1 g_1^{-1}(z) + g_1(z) F_1 g_1^{-1}(z),
\]

where

\[
G_1 = \text{tr}_2 \left( O_{12} \frac{\vartheta'(0)}{\vartheta(h)} \hat{g}_2(0) g_2(Nh) e^{\frac{P_2}{\hbar}} \right)
\]

and

\[
F_1 = \text{tr}_2 \left( O_{12} \frac{\vartheta'(0)}{\vartheta(h)} A_2 g_2(Nh) e^{\frac{P_2}{\hbar}} \right).
\]

From (3.43) and the inverse gauge transformation (3.47) for the \( M \)-matrix

\[
M^{RS'}(z) = g^{-1}(z, q) M^h(S(p, q), z) g(z, q) + g^{-1}(z, q) \hat{g}(z, q)
\]

we get

\[
-M^{RS'}(z) = g^{-1}(z) g'(z) G + F - g^{-1}(z) \hat{g}(z),
\]
where
\[ \dot{g}(z) = g'(z) \left( -N \text{diag}(\dot{q}) + 1_{N \times N} \sum_k \dot{q}_k \right) - g(z) \dot{D}^0 (D^0)^{-1}. \] (3.56)
with \( \text{diag}(\dot{q}) \) being the diagonal matrix of the velocities \( (2.4) \) defined as in \( (3.36) \).

**Proposition 3.1** The matrix \( M^{RS'}(z) \) in \( (3.55) \) coincides with the Ruijsenaars-Schneider \( M \)-matrix \( (2.8) \) up to unimportant term proportional to identity matrix.

**Proof:** Consider expansion of the Ruijsenaars-Schneider Lax matrix near \( z = 0 \)
\[ L^{RS} = \frac{1}{z} L^{RS}_{-1} + L^{RS}_0 + O(z) \] (3.57)
in two ways. First, from the definition \( (2.5) \):
\[ L^{RS}_{-1} = \rho \otimes \rho^T \text{diag}(\dot{q}), \] (3.58)
\[ (L^{RS}_0)_{ij} \overset{A.22}{=} E_1 (q_{ij} + \hbar \dot{q}_j), \] (3.59)
where \( \rho \) is the vector-column \( (3.21) \). The second way to get \( (3.57) \) is to use \( (3.47) \):
\[ L^{RS}_{-1} = \frac{\vartheta'(0)}{\vartheta(\hbar)} \dot{g}(0) g(\hbar) e^{P/c}, \] (3.60)
\[ L^{RS}_0 = \frac{\vartheta'(0)}{\vartheta(\hbar)} \dot{g}(0) g'(\hbar) e^{P/c} + \frac{\vartheta'(0)}{\vartheta(\hbar)} A g(\hbar) e^{P/c}. \] (3.61)

Then, from \( (3.52) \) and \( (3.60) \)
\[ G_1 = \text{tr}_2 \left( \mathcal{O}_{12}(L^{RS}_{-1})_2 \right) \overset{3.58}{=} \text{tr}_2 \left( \mathcal{O}_{12}(\rho \otimes \rho^T)_2 \text{diag}(\dot{q})_2 \right) \overset{3.45}{=} 1_{N \times N} \sum_k \dot{q}_k. \] (3.62)

Plugging this into \( (3.55) \) we obtain
\[ -M^{RS'}(z) = Ng^{-1}(z)g'(z) \text{diag}(\dot{q}) + F + \dot{D}^0 (D^0)^{-1}. \] (3.63)

Notice that the last two terms are diagonal, so that the non-diagonal part of \( M^{RS'}(z) \) is defined by the first term only. The coincidence of the non-diagonal parts of \( M^{RS'}(z) \) and \( (2.8) \) is due to identity \( (2.33) \), which comes from the non-relativistic limit of \( (3.47) \).

To complete the proof let us compute the matrix \( F \) \( (3.53) \). For this purpose substitute the matrix \( L^{RS}_0 \) \( (3.59) \) into \( (3.61) \)
\[ \frac{\vartheta'(0)}{\vartheta(\hbar)} A g(\hbar) e^{P/c} = L^{RS}_0 - \frac{\vartheta'(0)}{\vartheta(\hbar)} \dot{g}(0) g'(\hbar) e^{P/c} \] (3.64)
and compute the last term in the r.h.s. by differentiating the identity \( (3.60) \) with respect to \( \hbar \):
\[ \partial_\hbar L^{RS}_{-1} = -E_1(\hbar) L^{RS}_{-1} + N \frac{\vartheta'(0)}{\vartheta(\hbar)} \dot{g}(0) g'(\hbar) e^{P/c}. \] (3.65)
From (3.64) and (3.65) we find
\[
\frac{\vartheta'(0)}{\vartheta(h)} A g(N h) e^{P/c} = L_{0}^{RS} - \frac{1}{N} \partial_{h} L_{-1}^{RS} - \frac{1}{N} E_{1}(h) L_{-1}^{RS}.
\] (3.66)

Plugging here (3.58)-(3.59) yields
\[
(\frac{\vartheta'(0)}{\vartheta(h)} A g(N h) e^{P/c})_{st} = (L_{0}^{RS})_{st} + \frac{1}{N} q_{i} \sum_{k \neq l} E_{1}(q_{ik} - h).
\] (3.67)

Then, for the matrix $F$ (3.53) using (3.45) we obtain
\[
F_{ij} = \delta_{ij} \left( \sum_{k} (L_{0}^{RS})_{ik} + \frac{1}{N} \sum_{k \neq l} q_{i} E_{1}(q_{ik} - h) \right).
\] (3.68)

We now turn back to (3.63) and compute the diagonal part of its r.h.s. The input (to $ii$-th diagonal element) of the first term $(Ng^{-1}(z)g'(z) \text{diag}(\dot{q}))$ is evaluated from (2.33):
\[
\dot{q}_{i} E_{1}(z) - \dot{q}_{i} \sum_{k \neq i} E_{1}(q_{ik}).
\] (3.69)

The input of the $F$ matrix term comes from (3.68) and (3.59):
\[
\dot{q}_{i} E_{1}(h) + \sum_{k \neq i} \dot{q}_{k} E_{1}(q_{ik} + h) + \frac{1}{N} \sum_{k,l \neq i} \dot{q}_{i} E_{1}(q_{ik} - h).
\] (3.70)

At last, the input of the $\dot{D}^{0}(D^{0})^{-1}$ term is equal to
\[
\sum_{k \neq i} (\dot{q}_{i} - \dot{q}_{k}) E_{1}(q_{ik}).
\] (3.71)

Summing up (3.69)-(3.71) we reproduce $-M_{ij}^{RS}$ (2.8) except the last term from (3.70) which is independent of $i$. It is proportional to the identity matrix, and it has no affects on the Lax equations. ■

The $M$-matrix for rational Ruijsenaars-Schneider system has the form:
\[
M_{ij}^{RS} = -(1 - \delta_{ij}) \left( \frac{1}{q_{i} - q_{j}} + \frac{1}{Nz} \right) \dot{q}_{j} - \delta_{ij} \left( \frac{1}{Nz} + \frac{1}{h} \right) + \sum_{k \neq i} q_{k} \left( \frac{1}{q_{i} - q_{k} + h} - \frac{1}{q_{i} - q_{k}} \right).
\] (3.72)

The $M$-matrix without spectral parameter can be obtained by sending $z \to \infty$.

**Example 3.1** The $M$-matrix with spectral parameter for rational Ruijsenaars-Schneider system, up to some unimportant terms proportional to identity matrix, can be written in the following form:
\[
M^{RS}(z) = -g^{-1}(z)g'(z)\text{diag}(\dot{q}) - F - \dot{D}^{0}(D^{0})^{-1},
\] (3.73)
where
\[ g = \Xi(D^0)^{-1}, \]  
\[ F_{ij} = \delta_{ij} \sum_{k=1}^{N} \frac{\dot{q}_k}{q_i - q_k + \hbar}. \]  
(3.74)

The matrices \( \Xi, D^0 \) were defined in \((2.49)\).

Example 3.2 The \( M \)-matrix without spectral parameter for rational Ruijsenaars-Schneider system, up to some unimportant terms proportional to identity matrix, can be written in the following form:
\[ M_{RS} = -g^{-1}(z)g'(z)\text{diag}(\dot{q}) - F - \dot{D}^0(D^0)^{-1}, \]  
where
\[ g = V(D^0)^{-1}, \]  
\[ F_{ij} = \delta_{ij} \sum_{k=1}^{N} \frac{\dot{q}_k}{q_i - q_k + \hbar}. \]  
(3.75)

The matrix \( V \) was defined in \((2.53)\) and \( D^0 \) in \((2.49)\).

The \( M \)-matrix for trigonometric Ruijsenaars-Schneider system with spectral parameter has the following form:
\[ M_{ij}^{RS}(z) = -(1 - \delta_{ij})(\coth(q_i - q_j) + \coth(Nz))\dot{q}_j - \]  
\[ -\delta_{ij} \left( \dot{q}_i (\coth(Nz) + \coth(\hbar)) + \sum_{k\neq i}^{N} \dot{q}_k (\coth(q_i - q_k + \hbar) - \coth(q_i - q_k)) \right). \]  
(3.77)

The corresponding \( M \)-matrix without spectral parameter:
\[ M_{ij}^{RS} = -(1 - \delta_{ij}) \frac{\dot{q}_j}{\sinh(q_i - q_j)} - \]  
\[ -\delta_{ij} \left( \dot{q}_i \coth(\hbar) + \sum_{k\neq i}^{N} (\dot{q}_k (\coth(q_i - q_k + \hbar) - \coth(q_i - q_k)) \right). \]  
(3.78)

Example 3.3 The \( M \)-matrix with spectral parameter for the trigonometric Ruijsenaars-Schneider system, up to some unimportant terms proportional to identity matrix, can be written in form:
\[ M^{RS}(z) = -g^{-1}(z)g'(z)\text{diag}(\dot{q}) - F - \dot{D}^0(D^0)^{-1}, \]  
where
\[ g = \hat{\Xi}(D^0)^{-1}, \]  
\[ F_{ij} = \delta_{ij} \sum_{k=1}^{N} \coth(q_i - q_k + \hbar)\dot{q}_k. \]  
(3.80)

The matrices \( \hat{\Xi} \) and \( D^0 \) were defined in \((2.36)\).
Example 3.4  The M-matrix without spectral parameter for the trigonometric Ruijsenaars-Schneider system, up to some unimportant terms proportional to identity matrix, can be written in form:

\[ M^{RS} = -g^{-1}(z)g'(z)\text{diag}(\dot{q}) - F - \dot{D}^{0}(D^{0})^{-1}, \]

where

\[ g = \tilde{V}(\tilde{D}^{0})^{-1}, \]

\[ F_{ij} = \delta_{ij} \sum_{k=1}^{N} \coth(q_i - q_k + \hbar) \dot{q}_k. \]

The matrices \( \tilde{V} \) and \( \tilde{D}^{0} \) were defined in (2.39) and (2.40).

4 Schlesinger transformation

In this Section we will show that the Lax pair of the Calogero-Moser model (2.9)-(2.10) is naturally obtained from the Schlesinger transformation generated by the intertwining matrix (2.27)-(2.29).

The Schlesinger transformation [43, 3] is a (singular in the local coordinate \( z \)) gauge transformation

\[ A(z) \rightarrow hA(z)h^{-1} - \partial_z hh^{-1} \]

of (the \( z \) component of) a connection, which changes its residues. For example, in the simplest case the scalar connection on \( \mathbb{C}P^1 \) \( A(z) = \partial_z + \nu_0/z \), where \( \nu_0 \) is a constant, is transformed via (4.1) with \( h = z \) as \( \nu_0 \rightarrow \nu_0 - 1 \). Similarly, on the elliptic curve the scalar connection \( A(z) = \partial_z + \nu_0 E_1(z) \), where \( E_1(z) \) is (A.12) is transformed via (4.1) with \( h = \vartheta(z) \) as \( \nu_0 \rightarrow \nu_0 - 1 \) as well. As we know from (2.33) the non-trivial part (corresponding to the non-zero coupling constant \( \nu \)) of the Lax matrix (2.32) has form of a pure gauge connection along the coordinate \( z \) on the elliptic curve. We are going to treat it as a result of the Schlesinger like transformation. To make sense of a connection along the spectral parameter \( z \) we should proceed to the monodromy preserving equations.

Classical Painlevé-Calogero correspondence. As is known from [31] the Lax pair (2.9)-(2.10) satisfies not only the Lax equation \( \dot{L} = [L, M] \) but also the zero curvature condition

\[ 2\pi i \frac{d}{d\tau} L - \frac{d}{dz} M = [L, M]. \]

More precisely, the \( M \)-matrix (2.10) should be shifted by the identity matrix multiplied by \( \partial_{\tau} \log \vartheta(z) \): \( M \rightarrow M + 1_N \partial_{\tau} \log \vartheta(z) \) in order to compensate \( 2\pi i \partial_{\tau} E_1(z) \) coming from the first term in the l.h.s. of (4.2). Then (4.2) is equivalent (identically in \( z \)) to the higher Painlevé equations

\[ (2\pi i)^2 \frac{d^2 q_i}{d\tau^2} = \nu^2 \sum_{k \neq i} \vartheta'(q_{ik}). \]

This system of equations is treated as non-autonomous version of the Calogero-Moser equations of motion (2.12) in the sense that the elliptic moduli \( \tau \) (entering the r.h.s. of (4.3) explicitly) plays the role of the time variable. Technically, equivalence of (4.2) and (4.3) is similar
to derivation of the Lax equations for the Calogero-Moser model together with the usage of $2\pi i \partial_{\tau} L = \frac{d}{dz} M$. The latter follows from the heat equations (A.25)-(A.26).\footnote{Let us remark that the property $2\pi i \partial_{\tau} L = \frac{d}{dz} M$ is gauge dependent, so that the gauge choice including $D^0$ matrix is important here.}

Another important argument is that all models connected by the symplectic Hecke correspondence satisfy the property of the Painlevé-Calogero correspondence as well \cite{[33]}. So that the gauge transformed Lax pair again satisfies not only the Lax equation but also the zero-curvature condition (4.2) if the gauge transformation is given by the modification of the underlying bundle.

Then we may perform the following procedure. Consider the Lax matrix of the Calogero-Moser model with the coupling constant $\nu_0$:

$$L_0 = P + N\nu_0 g^{-1} g'. \quad (4.4)$$

At first, perform the gauge transformation generated by $g$-matrix. Secondly, transform the Lax matrix into the connection by adding $\partial_z$. Thirdly, perform the inverse gauge transformation generated by $g^{-1}$-matrix. At last, reduce the connection to the Lax matrix. The validity of the second and the last steps is guaranteed by the Painlevé-Calogero correspondence. Schematically, the procedure is as follows:

$$L_0 \rightarrow gL_0 g^{-1} \rightarrow \partial_z + gL_0 g^{-1} \rightarrow \partial_z + L_0 + g^{-1} g' \rightarrow L_0 + g^{-1} g' = P + (N\nu_0 + 1)g^{-1} g'. \quad (4.5)$$

As a result we get the same Lax matrix with the coupling constant shifted as $\nu_0 \rightarrow \nu_0 + 1/N$.

Calogero-Moser $M$-matrix in terms of $g(z)$. The described above procedure is a way to get the non-trivial part of the Calogero-Moser Lax matrix in the form of the pure gauge connection. Let us repeat all the steps to get the $M$-matrix. For convenience let us set $\nu_0 = 0$. Then the initial $M$-matrix equals zero since it corresponds to the free model. The analogue of (4.5) is as follows:

$$M_0 = 0 \rightarrow -\dot{g} g^{-1} \rightarrow 2\pi i \partial_{\tau} \dot{g} g^{-1} \rightarrow 2\pi i \partial_{\tau} - g^{-1} \dot{g} g^{-1} \frac{d}{d\tau} g \rightarrow M \quad (4.6)$$

where

$$M = g^{-1} \frac{d}{d\tau} g - g^{-1} \frac{d}{dt} g. \quad (4.7)$$

Both derivatives are the full derivatives, i.e. they include differentiation with respect to explicit and implicit dependencies on these variables. The implicit one is contained in $q_i(t)$ or $q_i(\tau)$. The relation between momenta and velocities comes from the Hamiltonian equations with the Hamiltonian function being computed from $(1/2)\text{tr}L^2$. Notice that at the first and the second stages of (4.5) we have $p_i = \dot{q}_i$, while on the last two stages an additional terms appear coming from the diagonal part of the $g^{-1} g'$:

$$\text{diag}(q)_t = P, \quad \text{diag}(q)_\tau = P - \frac{1}{N} d. \quad (4.8)$$

where

$$d_i = \sum_{k \neq i} E_1(q_{ik}). \quad (4.9)$$

So that

$$\text{diag}(q)_\tau - \text{diag}(q)_t = -\frac{1}{N} d. \quad (4.10)$$
The latter relation explains how to compute $M$-matrix via (4.7).

Introduce notations:

$$Ng^{-1}g' = l(z),$$  \hspace{1cm} (4.11)

$$l_{ii}(z) = E_1(z) - \sum_{k \neq i} E_1(q_{ik}) = E_1(z) - d_i.$$  \hspace{1cm} (4.12)

From (4.11) we also have

$$Ng^{-1}g'' = \partial_z l(z) + \frac{1}{N} I^2(z).$$  \hspace{1cm} (4.13)

**Proposition 4.1** The matrix $M(z)$ (4.7) with the relation (4.10) coincides with the Calogero-Moser $M$-matrix (2.10) up to unimportant terms proportional to the identity matrix.

**Proof:**

From the explicit form of $g$ (2.27) and (4.10) we get

$$M = \frac{N}{2} g^{-1}g'' - 2\pi i D^{-1} \partial_r D - D^{-1} \dot{D} \big|_{q_i = -d_i/N} - Ng^{-1}g'(\text{diag}(q)_r - \text{diag}(q)_t) =$$

$$= \frac{N}{2} g^{-1}g'' - 2\pi i D^{-1} \partial_r D + \frac{1}{N} D^{-1} \dot{D} \big|_{q_i = d_i} + g^{-1}g'd.$$  \hspace{1cm} (4.14)

Non-diagonal part:

$$(Ng^{-1}g'')_{ij} = \partial_z l_{ij} + \frac{1}{N} l_{ij}(l_{ii} + l_{jj}) + \frac{1}{N} \sum_{k \neq i,j} l_{ik} l_{kj}. \hspace{1cm} (4.15)$$

Using (4.15) and

$$\partial_z l_{ij} \overset{(A.19)}{=} \phi(z, q_{ij})(E_1(z + q_{ij}) - E_1(z)) \hspace{1cm} (4.16)$$

together with

$$l_{ik} l_{kj} \overset{(A.17)}{=} \phi(z, q_{ij})(E_1(z) + E_1(q_{ik}) + E_1(q_{kj}) - E_1(z + q_{ij})) \hspace{1cm} (4.17)$$

we get

$$\frac{N}{2} g^{-1}g''_{ij} = \frac{1}{N} f(z, q_{ij}) - \frac{1}{N} l_{ij} d_j,$$  \hspace{1cm} (4.18)

which means that for $i \neq j$ the statement of the Proposition indeed holds true.

Diagonal part:

The inputs coming from (4.14) are as follows. From (4.13) using (A.18) we find

$$\frac{1}{2} (Ng^{-1}g'')_{ii} = \frac{1}{2N} \left( E_1^2(z) - E_2(z) \right) + \frac{1}{2N} d_i^2 - \frac{1}{2N} \sum_{k \neq i} E_2(q_{ik}) - \frac{1}{N} E_1(z) d_i.$$  \hspace{1cm} (4.19)

Next,

$$-2\pi i D^{-1} \partial_r D_i = -\frac{1}{2} \sum_{k \neq i} \partial_r \log \vartheta(q_{ik}) = -\frac{1}{2} \sum_{k \neq i} E_1^2(q_{ik}) + \frac{1}{2} \sum_{k \neq i} E_2(q_{ik}).$$  \hspace{1cm} (4.20)
Next,
\[
\frac{1}{N} \sum_{k \neq i} (d_i - d_k) E_1(q_{ik}) = \frac{1}{N} \sum_{k \neq i}(d_i - d_k) E_1(q_{ik}).
\]

Finally,
\[
(g^{-1} g'd)_{ii} = \frac{1}{N} E_1(z) d_i - \frac{1}{N} d_i^2.
\]

Summarizing (4.19)-(4.22) for the diagonal part of (4.14) we get
\[
M_{ii} = \frac{1}{N} \partial_r \log \vartheta(z) + \frac{N-1}{2N} \sum_{k \neq i} E_2(q_{ik}) + \frac{1}{2N} d_i^2 - \frac{1}{2} \sum_{k \neq i} E_1^2(q_{ik}) - \frac{1}{N} \sum_{k \neq i} d_k E_1(q_{ik}).
\]

Introduce notation
\[
\sum'' = \sum_{k,l \neq i,i \neq k,l}.
\]

Since
\[
\sum_{k \neq i} d_k E_1(q_{ik}) = - \sum_{k \neq i} E_1^2(q_{ik}) + \sum'' E_1(q_{ik}) E_1(q_{kl})
\]
and
\[
d_i^2 = \sum_{k \neq i} E_1^2(q_{ik}) + \sum'' E_1(q_{ik}) E_1(q_{il})
\]
the expression (4.23) acquires the form:
\[
M_{ii} = \frac{1}{N} \partial_r \log \vartheta(z) + \frac{N-1}{2N} \sum_{k \neq i} E_2(q_{ik}) - \frac{N - 3}{2N} \sum_{k \neq i} E_1^2(q_{ik}) + \frac{1}{2N} \sum'' E_1(q_{ik}) E_1(q_{il}) - \frac{1}{N} \sum'' E_1(q_{ik}) E_1(q_{kl}).
\]

Consider the following sums
\[
\Delta_i = \sum'' (E_1(q_{ik}) + E_1(q_{kl}) + E_1(q_{ki}))^2.
\]

Due to
\[
\sum'' E_1^2(q_{kl}) = \sum_{k,l \neq i} E_1^2(q_{kl}) - 2 \sum_{k \neq i} E_1^2(q_{ik})
\]
and
\[
\sum'' E_1^2(q_{ik}) = (N - 2) \sum_{k \neq i} E_1^2(q_{ik})
\]
we have
\[
\Delta_i =
\]
\[
= \sum_{k,l\neq i} E_1^2(q_{kl}) + 2(N - 3) \sum_{k \neq i} E_1^2(q_{ik}) + 4 \sum'' E_1(q_{ik}) E_1(q_{kl}) - 2 \sum'' E_1(q_{ik}) E_1(q_{il}).
\]
Then expression (4.27) is simplified as follows:

\[
M_{ii} = \frac{1}{N} \partial_r \log \vartheta(z) + \frac{1}{4N} \sum_{k,l: k \neq l} E_1^2(q_{kl}) + \frac{N - 1}{2N} \sum_{k \neq i} E_2(q_{ik}) - \frac{1}{4N} \Delta_i. \tag{4.32}
\]

Notice that the first and the second terms are independent of index \(i\). They provide the term proportional to the identity matrix. The sum \(\Delta_i\) (4.28) can be written in a different way using (A.27). Plugging for each term of the sum (4.28) the r.h.s. of (A.27) we obtain:

\[
\Delta_i = (N - 1)(N - 2) \vartheta'''(0) \vartheta'(0) + 2(N - 3) \sum_{k \neq i} E_2(q_{ik}) + \sum_{k,l: k \neq l} E_2(q_{kl}) \tag{4.33}
\]

Then for the diagonal part of the \(M\)-matrix (4.32) we get

\[
M_{ii} = \frac{1}{N} \partial_r \log \vartheta(z) - \frac{(N - 1)(N - 2)}{4N} \frac{\vartheta''(0)}{\vartheta'(0)} + \frac{1}{4N} \sum_{k,l: k \neq l} (E_1^2(q_{kl}) - E_2(q_{ik})) + \\
+ \frac{1}{N} \sum_{k \neq i} E_2(q_{ik}). \tag{4.34}
\]

All terms in the upper line of (4.34) are independent of index \(i\), and the lower line is the diagonal part of (2.10) with \(\nu = 1/N\). ■

Examples.

**Example 4.1** The \(M\)-matrix of the rational Calogero-Moser model

\[
M_{ij} = \delta_{ij} \left( \sum_{k \neq i}^N \frac{\nu}{(q_i - q_k)^2} \right) - (1 - \delta_{ij}) \frac{\nu}{(q_i - q_j)^2} \tag{4.35}
\]

up to sum unimportant terms proportional to the identity matrix can be written as follows:

\[
M = \nu \left( \frac{1}{2} g^{-1} g'' + g^{-1} g' d + (D^0)^{-1} D^0 \big|_{q_i = d_i} \right), \tag{4.36}
\]

where

\[
g = \Xi(D^0)^{-1},
\]

\[
d_{ij} = \delta_{ij} d_i = \delta_{ij} \sum_{k \neq i}^N \frac{1}{q_i - q_k}. \tag{4.37}
\]

The matrices \(\Xi, D^0\) were defined in (2.49).

**Example 4.2** The \(M\)-matrix of the rational Calogero-Moser model

\[
M_{ij} = \delta_{ij} \left( \sum_{k \neq i}^N \frac{\nu}{(q_i - q_k)^2} \right) - (1 - \delta_{ij}) \frac{\nu}{(q_i - q_j)^2} \tag{4.38}
\]
up to sum unimportant terms proportional to the identity matrix can be written as follows:

\[
M = \nu \left( \frac{1}{2} g^{-1} g'' + g^{-1} g' d + (D^0)^{-1} \dot{D}^0|_{q_i = d_i} \right),
\]

(4.39)

where

\[
g = V(D^0)^{-1},
\]

(4.40)

\[
d_{ij} = \delta_{ij}d_i = \delta_{ij} \sum_{k \neq i}^{N} \frac{1}{q_i - q_k}.
\]

(4.43)

The matrix \(V\) was defined in (2.53) and \(D^0\) in (2.49).

**Example 4.3** The \(M\)-matrix of the trigonometric Calogero-Moser model

\[
M_{ij} = \delta_{ij} \left( \sum_{k \neq i}^{N} \frac{\nu}{\sinh^2(q_i - q_k)} \right) - \nu(1 - \delta_{ij}) \frac{1}{\sinh^2(q_i - q_j)}
\]

(4.41)

up to some unimportant terms proportional to the identity matrix can be written in form:

\[
M = \nu \left( \frac{1}{2} g^{-1} g'' + g^{-1} g' d + (\tilde{D}^0)^{-1} \dot{\tilde{D}}^0|_{q_i = d_i} \right),
\]

(4.42)

where

\[
g = \tilde{\Xi}(\tilde{D}^0)^{-1},
\]

(4.43)

\[
d_{ij} = \delta_{ij}d_i = \delta_{ij} \left( \sum_{k \neq i}^{N} \coth(q_i - q_k) \right) - (N - 2).
\]

The matrices \(\tilde{\Xi}\) and \(\tilde{D}^0\) were defined in (2.36).

**Example 4.4** The \(M\)-matrix of the trigonometric Calogero-Moser model

\[
M_{ij} = \delta_{ij} \left( \sum_{k \neq i}^{N} \frac{\nu}{\sinh^2(q_i - q_k)} \right) - \nu(1 - \delta_{ij}) \frac{\coth(q_i - q_j)}{\sinh(q_i - q_j)}
\]

(4.44)

up to some unimportant terms proportional to the identity matrix can be written in form:

\[
M = \nu \left( \frac{1}{2} g^{-1} g'' + g^{-1} g' d + (\tilde{D}^0)^{-1} \dot{\tilde{D}}^0|_{q_i = d_i} \right),
\]

(4.45)

where

\[
g = \tilde{V}(\tilde{D}^0)^{-1},
\]

(4.46)

\[
d_{ij} = \delta_{ij}d_i = \delta_{ij} \sum_{k \neq i}^{N} \coth(q_i - q_k).
\]

The matrices \(\tilde{V}\) and \(\tilde{D}^0\) were defined in (2.39)-(2.40).
5 Classical root systems

In this Section we propose factorization formulae for the rational Calogero-Moser systems associated with classical root systems $D_N$, $C_N$, $B_N$. As was mentioned in the Introduction, in case when there is no spectral parameter the factorization of the $A_N$ Lax matrix takes the form (1.18). It is due to the fact that

$$V'(z) = C_0 V(z),$$

where $V(z)$ is the Vandermonde matrix (2.33) and

$$(C_0)_{ij} = \begin{cases} i-1, & i = j + 1, \\ 0, & \text{otherwise} \end{cases}.$$ (5.2)

Below we suggest analogues of (1.18) for the models related to $D_N$, $C_N$, $B_N$ root systems. The proofs are given in the Appendix B.

5.1 Calogero-Moser model associated with classical root systems

The $B_{CN}$ model is described by the following Hamiltonian:

$$H_{cm} = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - \left( \sum_{i=1}^{N} \frac{m_2^2}{(q_i - q_j)^2} + \frac{m_2^2}{(q_i + q_j)^2} \right) + \sum_{i=1}^{N} \frac{m_4^2}{(2q_i)^2} + \sum_{i=1}^{N} \frac{m_1^2}{q_i^2}. \quad (5.3)$$

Its Lax matrix is of $(2N+1) \times (2N+1)$ size [38, 18]:

$$L^{CM}(m_1, m_2, m_4) = \begin{pmatrix} A & B & C \\ -B & -A & -C \\ -C^T & C^T & 0 \end{pmatrix} \quad (5.4)$$

where the blocks $A$, $B$ are $N \times N$ matrices and $C$ is $N$-dimensional column-vector:

$$A_{ij} = \delta_{ij} \left( p_i - \frac{\sqrt{2}m_4}{2q_i} - \frac{\sqrt{2}m_1}{q_i} - m_2 \sum_{k \neq i}^{N} \left( \frac{1}{q_i - q_k} + \frac{1}{q_i + q_k} \right) \right) + (1 - \delta_{ij}) \frac{m_2}{q_i - q_j},$$

$$B_{ij} = (1 - \delta_{ij}) \frac{m_2}{q_i + q_j} + \delta_{ij} \frac{\sqrt{2}m_4}{2q_i},$$

$$C_i = \frac{m_1}{q_i}.$$ (5.5)

The system is integrable if $m_1(m_1^2 - 2m_2^2 + \sqrt{2}m_2m_4) = 0$. It reduces to the classical root systems $D_N$, $C_N$ and $B_N$ by following choice of the coupling constants:

$$D_N : \quad m_1 = 0, \ m_4 = 0,$$

$$C_N : \quad m_1 = 0,$$ (5.6)

$$B_N : \quad m_4 = 0, \ m_1^2 = 2m_2^2.$$ Notice that for $C_N$ and $D_N$ cases the Lax matrix is effectively of dimension $2N$, therefore we will consider such matrices as Mat$(2N, \mathbb{C})$-valued.
5.2 Factorization formulae for classical root systems

Factorization for \(C_n\) and \(D_N\) root systems

Introduce the following notations for \(2N \times 2N\) matrices:

\[
D^0_{ij} = \delta_{ij} \begin{cases} 
2q_i \prod_{k \neq i}^N ((q_i - q_k)(q_i + q_k)) , & i \leq N , \\
-2q_{i-N} \prod_{k \neq i-N}^N ((q_{i-N} - q_k)(q_{i-N} + q_k)) , & N+1 \leq i \leq 2N , 
\end{cases}
\]

(5.7)

\[
V_{ij} = \begin{cases} 
q_j^{i-1}, & j \leq N , \\
(-q_{j-N})^{i-1}, & N+1 \leq j \leq 2N 
\end{cases}
\]

(5.8)

and

\[
\tilde{C}_{ij} = \begin{cases} 
1, & i = j + 1 , \ i \ -\ \text{even} \\
0, & \text{otherwise} 
\end{cases}
\]

(5.9)

The Lax matrices (5.4) for the \(C_N\) and \(D_N\) cases (5.6) admit the following factorization formula:

\[
L^{CM}(m_2, m_4, 0) = P - D^0 V^{-1}(m_2 C_0 - (m_2 - \sqrt{2}m_4)\tilde{C})V(D^0)^{-1},
\]

(5.10)

where \(C_0\) is the one (5.2) but of the size \(2N \times 2N\), and

\[
P_{ij} = \delta_{ij} \begin{cases} 
p_i, & i \leq N , \\
-p_{i-N} , & N+1 \leq i \leq 2N 
\end{cases}
\]

(5.11)

For the choice \(m_4 = 0\) (5.10) reproduces the Lax matrix for \(D_N\) root system, otherwise we get the \(C_N\) case.

Factorization for \(B_N\) root system

Let us introduce the notations for \((2N + 1) \times (2N + 1)\) matrices:

\[
D^0_{ij} = \delta_{ij} \begin{cases} 
\sqrt{2}q_i^2 \prod_{k \neq i}^N ((q_i - q_k)(q_i + q_k)) , & i \leq N , \\
\sqrt{2}q_{i-N}^2 \prod_{k \neq i-N}^N ((q_{i-N} - q_k)(q_{i-N} + q_k)) , & N+1 \leq i \leq 2N , \\
\prod_{k=1}^N (-q_k^2) , & i = 2N + 1 
\end{cases}
\]

(5.12)

\[
V_{ij} = \begin{cases} 
q_j^{i-1}, & j \leq N , \\
(-q_{j-N})^{i-1}, & N+1 \leq j \leq 2N \\
\delta_{i,1}, & j = 2N + 1 
\end{cases}
\]

(5.13)

and

\[
\tilde{C}_{ij} = \begin{cases} 
1, & i = j + 1 , \ i \ -\ \text{even} \\
0, & \text{otherwise} \ ; \ i, j = 1, \ldots, 2N + 1 
\end{cases}
\]

(5.14)

The Lax matrix (5.4) for the \(B_N\) case (5.6) admits the following factorization formula:

\[
L^{CM}(m_2, 0, \sqrt{2}m_2) = P - m_2 D^0 V^{-1}(C_0 + \tilde{C})V(D^0)^{-1},
\]

(5.15)
where $C_0$ is the one (5.2) but of the size $(2N + 1) \times (2N + 1)$, and

$$P_{ij} = \delta_{ij} \begin{cases} p_i, & i \leq N, \\ -p_{i-N}, & N + 1 \leq i \leq 2N, \\ 0, & i = 2N + 1. \end{cases}$$ (5.16)

6 Appendix A

Finite dimensional representation of Heisenberg group. Instead of the standard basis in $\text{Mat}_N$ the following one is widely used in elliptic $R$-matrices:

$$T_a = T_{a_1a_2} = \exp \left( \frac{\pi i}{N} a_1a_2 \right) Q^{a_1}_1 \Lambda^{a_2}, \quad a = (a_1, a_2) \in \mathbb{Z}_N \times \mathbb{Z}_N,$$ (A.1)

where

$$Q_{kl} = \delta_{kl} \exp \left( \frac{2\pi i}{N} k \right), \quad \Lambda_{kl} = \delta_{k-l+1=0 \mod N}, \quad Q^N = \Lambda^N = 1_{N \times N}.$$ (A.2)

These are the generators of the finite dimensional representation of Heisenberg group

$$\Lambda^{a_2} Q^{a_1} = \exp \left( \frac{2\pi i}{N} a_1a_2 \right) Q^{a_1} \Lambda^{a_2}. \quad \text{(A.3)}$$

Then for the product of basis matrices we have

$$T_a T_\beta = \kappa_{\alpha, \beta} T_{\alpha + \beta}, \quad \kappa_{\alpha, \beta} = \exp \left( \frac{\pi i}{N} (\beta_1 a_2 - \beta_2 a_1) \right),$$ (A.4)

where $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$. Therefore

$$\text{tr}(T_a T_\beta) = N \delta_{\alpha, -\beta}, \quad \text{(A.5)}$$

The permutation operator takes the following form in this basis:

$$P_{12} = \sum_{i,j=1}^N E_{ij} \otimes E_{ji} = \frac{1}{N} \sum_{a \in \mathbb{Z}_N \times \mathbb{Z}_N} T_a \otimes T_{-a}, \quad \text{(A.6)}$$

where $E_{ij}$ is the standard basis in $\text{Mat}_N$.

Theta functions. The Riemann theta-functions with characteristics

$$\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z \mid \tau) = \sum_{j \in \mathbb{Z}} \exp \left( 2\pi i (j + a)^2 \frac{\tau}{2} + 2\pi i (j + a)(z + b) \right), \quad a, b \in \frac{1}{N} \mathbb{Z}. \quad \text{(A.7)}$$

are defined on elliptic curve $\Sigma_\tau = \mathbb{C}^2/(\mathbb{Z} \oplus \tau \mathbb{Z})$ with moduli $\tau$ (Im$\tau > 0$). They behave on the lattice $\mathbb{Z} \oplus \tau \mathbb{Z}$ as follows:

$$\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z + 1 \mid \tau) = \exp(2\pi i a) \theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z \mid \tau), \quad \text{(A.8)}$$

$$\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z + a' \tau \mid \tau) = \exp \left( -2\pi i a'^2 \frac{\tau}{2} - 2\pi i a' (z + b) \right) \theta \left[ \begin{array}{c} a + a' \\ b \end{array} \right] (z \mid \tau). \quad \text{(A.9)}$$

We also use a shorthand notation for the odd theta function

$$\vartheta(z \mid \tau) \equiv \vartheta(z) \equiv \theta \left[ \begin{array}{c} 1/2 \\ 1/2 \end{array} \right] (z \mid \tau). \quad \text{(A.10)}$$
The following set of functions numerated by $a \wp$ where Genus one Fay trisecant identity is as follows:

Kronecker and Eisenstein functions. The Kronecker function is defined in terms of \(A.10\):

$$
\phi(\eta, z) = \frac{\vartheta'(0) \vartheta(z + q)}{\vartheta(z) \vartheta(q)}
$$

(A.11)

The first Eisenstein and the second Eisenstein functions $E_1(z) = \frac{\vartheta'(z)}{\vartheta(z)}$, $E_2(z) = -\partial_z E_1(z) = \varphi(z) - \frac{1}{3} \vartheta''(0)$, (A.12)

where $\varphi(z)$ is the Weierstrass $\varphi$-function. The function $E_2(z)$ is double-periodic on the lattice $\mathbb{C}/\mathbb{Z} + \tau \mathbb{Z}$, while for the first Eisenstein and the Kronecker functions we have:

$$
E_1(z + 1) = E_1(z), \quad E_1(z + \tau) = E_1(z) - 2\pi i,
$$

(A.13)

$$
\phi(z + 1, w) = \phi(z, w), \quad \phi(z + \tau, w) = e^{-2\pi iw} \phi(z, w).
$$

(A.14)

The following set of functions numerated by $a = (a_1, a_2) \in \mathbb{Z}_N \times \mathbb{Z}_N$ (as in (A.1)) is also used:

$$
\varphi_a(z, \omega_a) = \exp(2\pi i \frac{a_2}{N} z) \phi(z, \omega_a + \hbar), \quad \omega_a = \frac{a_1 + a_2 \tau}{N}.
$$

(A.15)

Genus one Fay trisecant identity is as follows:

$$
\phi(h, z)\phi(\eta, w) = \phi(h - \eta, z)\phi(\eta, z + w) + \phi(\eta - h, w)\phi(h, z + w)
$$

(A.16)

Its degenerations:

$$
\phi(\eta, z)\phi(\eta, w) = \phi(\eta, z + w)(E_1(\eta) + E_1(z) + E_1(w) - E_1(z + w + \eta)),
$$

(A.17)

$$
\phi(h, z)\phi(h, -z) = \varphi(h) - \varphi(z) = E_2(h) - E_2(z).
$$

(A.18)

For the derivative of the Kronecker function with respect to the second argument we keep notation

$$
f(z, q) = \partial_q \phi(z, q) = \phi(z, q)(E_1(z + q) - E_1(q)).
$$

(A.19)

It satisfies identities:

$$
\phi(z, q)f(z, -q) - f(z, q)\phi(z, -q) = \varphi'(q),
$$

(A.20)

$$
\phi(z, q_{ab})f(z, q_{bc}) - f(z, q_{ab})\phi(z, q_{bc}) = \phi(z, q_{ac})(\varphi(q_{ab}) - \varphi(q_{bc})).
$$

(A.21)

Due to the local expansion near $z = 0$

$$
\phi(z, q) = \frac{1}{z} + E_1(q) + \frac{1}{2} \left( E_1^2(q) - \varphi(q) \right) + O(z^2)
$$

(A.22)

we also have

$$
f(0, q) = -E_2(q).
$$

(A.23)

Heat equation. For the theta functions (A.7) the following relation holds

$$
4\pi i \partial_\tau \theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z \mid \tau) = \partial_2^2 \theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z \mid \tau).
$$

(A.24)

In particular, it is true for $\vartheta(z)$ (A.10). Then using the definitions (A.11)-(A.12) we can get

$$
2\pi i \partial_\tau \phi(z, q) = \partial_2 \partial_q \phi(z, q)
$$

(A.25)

and

$$
2\pi i \partial_\tau \log \vartheta(z) = \frac{1}{2} (E_1^2(z) - E_2(z)).
$$

(A.26)
Identities.

\[(E_1(x) + E_1(y) + E_1(-x - y))^2 = \varphi(x) + \varphi(y) + \varphi(x + y) = \]

\[= E_2(x) + E_2(y) + E_2(x + y) + \frac{\varphi''(0)}{\varphi'(0)} \quad \text{(A.12)} \]

\[\frac{\varphi(h)}{\varphi'(0)} \sum_k g_{ik}(z, q) \phi(z, q_{kj} + h) = g_{ij}(z + Nh, q) \prod_{m \neq j} \frac{\varphi(q_{mj})}{\varphi(q_{mj} + h)}. \quad \text{(A.28)} \]

For

\[X_{ij}(x_j) = \vartheta \left[ \frac{1}{2} - \frac{i}{N} \right] (Nx_j | N\tau) \quad \text{(A.29)} \]

we have

\[\det X = C_N(\tau) \vartheta \left( \sum_{k=1}^{N} x_k \right) \prod_{i<j} \vartheta(x_j - x_i), \quad C_N(\tau) = \frac{(-1)^{N-1}}{(i\eta(\tau))^{1/2}}, \quad \text{(A.30)} \]

where \(\eta(\tau)\) is the Dedekind eta-function:

\[\eta(\tau) = e^{\frac{\pi i}{12}} \prod_{k=1}^{\infty} (1 - e^{2\pi i k}). \quad \text{(A.31)} \]

Then for the matrix (2.28)

\[\det \Xi(z, q) = C_N(\tau) \vartheta(z) \prod_{i<j} \vartheta(q_i - q_j). \quad \text{(A.32)} \]

7 Appendix B

Proof of formula (5.10). To prove (5.10) we need to show that

\[J = D^0 V^{-1} \tilde{C} V (D^0)^{-1} = \left( \begin{array}{cc} \delta_{ij} & \delta_{ij} \\ \frac{2q_i}{\delta_{ij}} & \frac{2q_i}{\delta_{ij}} \end{array} \right). \quad \text{(B.1)} \]

The proof of (B.1) is a direct evaluation, which uses explicit form of the inverse Vandermonde matrix:

\[J_{ij} = (D^0 V^{-1} \tilde{C} V (D^0)^{-1})_{ij} = D^0_{i\alpha} V^{-1}_{\alpha\beta} \tilde{C}_{\beta\gamma} V_{\gamma\nu} (D^0)^{-1}_{\nu j} = \]

\[= \sum_{\gamma-odd} D^0_{i\gamma} V^{-1}_{\gamma+1} V_{\gamma,j} = \frac{D^0_{i\gamma}}{q_j D^0_{j\gamma}} \sum_{\gamma-even} V^{-1}_{\gamma} V_{\gamma,j}. \quad \text{(B.2)} \]

To see how the matrix \(J_{ij}\) changes under substitutions \(i \rightarrow i + N\) and \(j \rightarrow j + N\) consider the changes of its factors: \(D^0_j \rightarrow -D^0_j\), \((D^0)^{-1}_i \rightarrow -(D^0)^{-1}_i\), \(q_i \rightarrow -q_i\), \(V_{\gamma,j} \rightarrow -V_{\gamma,j}\), \(V^{-1}_i \rightarrow -V^{-1}_i\). The penultimate relation holds true because since the summation goes over odd \(\gamma\). Therefore, \(J_{ij}\) does not change the sign under the substitution \(i \rightarrow i + N\), and \(J_{ij}\) changes the sign under \(j \rightarrow j + N\). Thus the matrix \(J\) has the form:

\[J = \left( \begin{array}{cc} \tilde{J} & -\tilde{J} \\ \tilde{J} & -\tilde{J} \end{array} \right). \quad \text{(B.3)} \]
Further, we will consider $1 \leq i, j \leq N$, since this is sufficient to determine matrix $J$.

$$
\sum_{\gamma \text{even}} V_{\gamma i}^{-1} V_{\gamma j} = \sum_{\gamma \text{odd}} V_{i, \gamma + 1}^{-1} V_{\gamma + 1, j} =
$$

$$=
\sum_{\gamma \text{odd}} \frac{q_j}{q_i} \sum_{\gamma \text{odd}} q_j^{-1} \frac{1}{(\gamma - 1)!} \partial^{(\gamma - 1)}_{\rho} \left[ \frac{\prod_{s \neq i} (\rho - q_s) (\rho + q_s)}{2q_i \prod_{s \neq i} (q_i - q_s) (q_i + q_s)} \right]_{\rho=0}
$$

$$= \frac{q_j}{q_i} \sum_{\gamma \text{odd}} q_j^{-1} \frac{1}{(\gamma - 1)!} \partial^{(\gamma - 1)}_{\rho} \left[ \frac{\prod_{s \neq i} (\rho - q_s) (\rho + q_s)}{2 \prod_{s \neq i} (q_i - q_s) (q_i + q_s)} \right]_{\rho=0}
$$

Using this relation we obtain:

$$
\sum_{\gamma \text{even}} V_{\gamma i}^{-1} V_{\gamma j} = \delta_{ij} \frac{1}{2} \quad \text{(B.5)}
$$

and, therefore

$$
\tilde{J}_{ij} = \frac{D^0_i}{q_j D^0_j} \frac{\delta_{ij}}{2} = \frac{\delta_{ij}}{2q_j} \quad \text{(B.6)}
$$

In this way the formula (5.10) is proved.

**Proof of formula (5.15):** First, determine the structure of matrix:

$$
G_{ij} = (D^0 V^{-1} (C + \bar{C}) V (D^0)^{-1})_{ij} = \frac{D^0_i}{D^0_j} \sum_{\gamma=1}^{2N} V_{i, \gamma + 1}^{-1} (C + \bar{C})_{\gamma + 1, j} V_{\gamma j} \quad \text{(B.7)}
$$

Consider $1 \leq i, j \leq N$. Let us find out how its matrix elements change under the substitutions $i \rightarrow i + N$ and $j \rightarrow j + N$:

$$
D^0_i \rightarrow D^0_i, \quad D^0_j \rightarrow D^0_j, \quad V_{\gamma j} \rightarrow (-1)^{-1} V_{\gamma j}, \quad V_{i, \gamma + 1}^{-1} \rightarrow (-1)^{\gamma} V_{i, \gamma + 1}^{-1} \quad \text{(B.8)}
$$

Therefore, $G_{ij} \rightarrow -G_{ij}$. Similar properties are valid for $1 \leq i \leq N, \ N + 1 \leq j \leq 2N$ and the substitutions $i \rightarrow i + N, \ j \rightarrow j - N$. Thus, we get $G_{ij} \rightarrow -G_{ij}$. Then consider the case $i = j = 2N + 1$:

$$
G_{2N+1,2N+1} = \sum_{\gamma=1}^{2N} V_{2N+1, \gamma + 1}^{-1} (C + \bar{C})_{\gamma + 1, 2N+1} V_{\gamma, 2N+1}^{-1} = V_{2N+1,2}^{-1} (C + \bar{C})_{2,1} = 0 \quad \text{(B.9)}
$$
and the case $j = 2N + 1, 1 \leq i \leq N$:

$$G_{i,2N+1} = \frac{D^0_i}{D^0_{2N+1}} \sum_{\gamma=1}^{2N} \frac{V_{i,\gamma+1}(C + \bar{C})_{\gamma+1,\gamma} V_{\gamma,2N+1}}{V_{2}^{-1} = 2 \frac{D^0_i}{D^0_{2N+1}}} = 2 - \frac{D^0_i}{D^0_{2N+1}} \partial_{\rho} \left[ \frac{\rho(\rho + q_i) \prod_{s \neq i} (\rho^2 - q_s^2)}{2q_i^2 \prod_{s \neq i} (q_i^2 - q_s^2)} \right] \bigg|_{\rho=0} = - \frac{D^0_i}{D^0_{2N+1}} \frac{1}{q_i^2} \prod_{s \neq i} (q_i^2 - q_s^2) = - \sqrt{2}.$$  

(B.10)

Similarly, for $N + 1 \leq i \leq 2N$ we get:

$$G_{i,2N+1} = \frac{\sqrt{2}}{q_{i-N}}.$$  

(B.11)

Calculate $G_{2N+1,j}$ for $1 \leq j \leq N$:

$$G_{2N+1,j} = \frac{D^0_i}{D^0_{2N+1}} \sum_{\gamma=1}^{2N} \frac{V_{2N+1,\gamma+1}(C + \bar{C})_{\gamma+1,\gamma} V_{\gamma,j}}{V_{2}^{-1} = 2 \frac{D^0_i}{D^0_{2N+1}}} = \frac{D^0_i}{D^0_{2N+1}} \sum_{\gamma=1}^{2N} q_{j-1} \frac{1}{(\gamma - 1)!} \partial_{\rho}^{(\gamma)} \left[ \prod_{s \neq j}^{N} (\rho^2 - q_s^2) \right] \bigg|_{\rho=0} = \frac{D^0_i}{D^0_{2N+1}} \sum_{\gamma=1}^{2N} q_{j-1} \frac{1}{(\gamma - 1)!} \left[ \frac{\gamma}{(\gamma - 2)!} \partial_{\rho}^{(\gamma-2)} \left[ \prod_{s \neq j}^{N} (\rho^2 - q_s^2) \right] \right] \bigg|_{\rho=0} - \frac{q_i^2}{(\gamma - 1)!} \partial_{\rho}^{(\gamma)} \left[ \prod_{s \neq j}^{N} (\rho^2 - q_s^2) \right] \bigg|_{\rho=0} = \frac{1}{D^0_i} \sum_{\gamma=1}^{2N} q_{j-1} \frac{\gamma}{(\gamma - 2)!} \partial_{\rho}^{(\gamma-2)} \left[ \prod_{s \neq j}^{N} (\rho^2 - q_s^2) \right] \bigg|_{\rho=0} - \frac{q_i^2}{(\gamma - 1)!} \partial_{\rho}^{(\gamma)} \left[ \prod_{s \neq j}^{N} (\rho^2 - q_s^2) \right] \bigg|_{\rho=0} = \frac{1}{D^0_i} \sum_{\gamma=1}^{2N} q_{j-1} \frac{\gamma}{(\gamma - 2)!} \left[ \frac{\gamma}{(\gamma - 2)!} \partial_{\rho}^{(\gamma-2)} \left[ \prod_{s \neq j}^{N} (\rho^2 - q_s^2) \right] \right] \bigg|_{\rho=0} - \frac{q_i^2}{(\gamma - 1)!} \partial_{\rho}^{(\gamma)} \left[ \prod_{s \neq j}^{N} (\rho^2 - q_s^2) \right] \bigg|_{\rho=0}.

(B.12)
Therefore,

\[ G_{2N+1,j} = \frac{\sqrt{2}}{q_j} \left( \sum_{\gamma \text{ even}} q_j^{\gamma-2} \frac{\gamma}{(\gamma-2)!} \partial^{(\gamma-2)}_\rho \left[ \frac{(\rho + q_j) \prod_{s \neq j} (\rho^2 - q_s^2)}{2q_j \prod_{s \neq j} (q_j^2 - q_s^2)} \right] \right|_{\rho=0} \]

(B.13)

\[ -\frac{\sqrt{2}}{q_j} \left( \sum_{\gamma \text{ even}} q_j^{\gamma} \frac{\gamma}{\gamma!} \partial^{(\gamma)}_\rho \left[ \frac{2q_j \prod_{s \neq j} (q_j^2 - q_s^2)}{\prod_{s \neq j} (q_j^2 + 1)} \right] \right|_{\rho=0} = \]

(B.14)

\[ = \frac{2\sqrt{2}}{q_j} \sum_{\gamma \text{ even}} (V^D)_{j,\gamma-1} V^D_{\gamma-1,j} - 2 \sqrt{2} \delta_{jj} = \frac{\sqrt{2}}{q_j} \]

In the last equalities the result from the proof of formula (5.10) was used. Under the substitution \( j \rightarrow j + N: D^0_j \rightarrow D^0_j, V_{ij} \rightarrow (-1)^{\gamma-1} V_{ij} \). Taking into account that the sum goes over even \( \gamma \) we obtain \( G_{2N+1,j} \rightarrow -G_{2N+1,j} \). Thus, we proved that the Lax matrix is of the form:

\[ L(m_2, 0, \sqrt{2}m_2) = \left( \begin{array}{ccc} A & B & C \\ -B & -A & -C \\ -C^T & C^T & 0 \end{array} \right), \quad C_i = \frac{\sqrt{2}m_2}{q_i} \]

(B.14)

Then, for the last two blocks \((1 \leq i \leq N, N + 1 \leq j \leq 2N)\) we have:

\[ B_{i,j-N} = -m_2 D^0_i \sum_{\gamma=1}^{2N} V_{i,\gamma+1}(C + \tilde{C})_{\gamma+1,\gamma} V_{\gamma,j} = \]

(B.15)

\[ = -m_2 \frac{D^0_i}{D^0_j} \sum_{\gamma=1}^{2N} (-q_j-N)^{\gamma-1}(C + \tilde{C})_{\gamma+1,\gamma} \frac{1}{\gamma!} \partial^{(\gamma-1)}_\rho \left[ \frac{(\rho + q_i) \prod_{s \neq i} (\rho^2 - q_s^2)}{2q_i \prod_{s \neq i} (q_i^2 - q_s^2)} \right] \right|_{\rho=0} \]

(B.15)

\[ = m_2 \frac{q_i}{q_j-N} \frac{D^0_i}{D^0_j} \frac{1}{q_i} \sum_{\gamma=1}^{2N} (-q_j-N)^{\gamma-1}(C + \tilde{C} - 2)_{\gamma+1,\gamma} \frac{1}{(\gamma-1)!} \times \]

\[ \times \partial^{(\gamma-1)}_\rho \left[ \frac{(\rho + q_i) \prod_{s \neq i} (\rho^2 - q_s^2)}{2q_i \prod_{s \neq i} (q_i^2 - q_s^2)} \right] \right|_{\rho=0} + 2 \sum_{\gamma=1}^{2N} (V^D)_{\gamma-1} (V^D)^{-1} \]
\[ -m_2 D^0 D^0 \sum_{\gamma=2}^{2N} (q_j - N) \gamma^{-2} (C_{\gamma+1,\gamma} + \tilde{C}_{\gamma+1,\gamma} - 2) (V^D)^{-1}_{i\gamma} = \]
\[ = -m_2 D^0 D^0 \sum_{\gamma=1}^{2N-1} (q_j - N) \gamma^{-1} (C_{\gamma+1,\gamma} - \tilde{C}_{\gamma+1,\gamma}) (V^D)^{-1}_{i,\gamma+1} = B^D_{ij}. \]

Notice that the term \( 2 \cdot \sum_{1 \leq \gamma \leq 2N} (V^D)^{-1}_{\gamma j} V^D_{i\gamma} = 2\delta_{ij} \) vanishes since \( i \neq j \) in our case. This gives the block \( B \) from the Lax matrix in \( D_N \) case. The last one block corresponding to \( 1 \leq i \leq N, 1 \leq j \leq N \) is evaluated in a similar way, apart from the term
\[ 2 \sum_{1 \leq \gamma \leq 2N} V^D_{i\gamma} (V^D)^{-1}_{\gamma j} = 2\delta_{ij} \] (B.16)

which does not vanish. Therefore, we get
\[ A_{ij} = A^D_{ij} - m_2 \frac{2\delta_{ij}}{q_i}. \] (B.17)

This finishes the proof of (5.15).

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