Dynamics of phantom model with O(N) symmetry in loop quantum cosmology

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Abstract

Many astrophysical data show that the expansion of our universe is accelerating. In this paper, we study the model of phantom with O(N) symmetry in background of loop quantum cosmology (LQC). We investigate the phase-space stability of the corresponding autonomous system and find no stable node but only 2 saddle points in the field of real numbers. The dynamics is similar to the single-field phantom model in LQC\textsuperscript{1}. The effect of O(N) symmetry just influence the detail of the universe’s evolution. This is a sharp contrast with the result in general relativity, in which the dynamics of scalar fields models with O(N) symmetry are quite different from the single-field models\textsuperscript{2,3,4}.

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1 Introduction

Recently, many astrophysical data, such as WMAP\cite{5}, type Ia supernovae\cite{6} and large scale galaxy surveys\cite{7}, show that our universe is almost flat, and currently undergoing a period of accelerating expansion which is driven by a yet unknown dark energy. About 70\% of the energy density in our universe is dark energy nowadays. There are many candidates of dark energy such as the cosmological constant\cite{8}, quintessence\cite{9, 10, 11, 12, 13, 2}, phantom\cite{14, 15, 16, 3}, quintom\cite{17, 4} etc. The equation-of-state parameter $w = p/\rho$ plays an important role in these models. The parameter of quintessence and phantom lies in the range $-\frac{1}{3} > w > -1$ and $w < -1$ respectively. And for quintom $w$ could cross -1.

Li et al. generalized the quintessence and phantom models to fields possess O($N$) symmetry\cite{2, 3}. Setare et al. extended the generalization to the quintom model\cite{4}. In the above papers they showed that the behaviors of the dynamic system with O($N$) symmetry are quite different from the single-field models.

Many works on dark energy have been done in the framework of Einstein’s classical general relativity(GR). However, most physicists believe that the gravity should be quantized in a ultimate theory. Thus, we should consider the quantum effects in the evolution of our universe. Loop quantum gravity(LQG) is a outstanding candidate of quantum gravity theory. LQG is a background independent and non-perturbative theory. At the quantum level, the classical spacetime continuum is replaced by a discrete quantum geometry and the operators corresponding to geometrical quantities have discrete eigenvalues. Many topics on cosmology are discussed within the framework of LQG in various literature in which it is known as Loop Quantum Cosmology (LQC)\cite{1, 18, 19, 20, 21}. Recently, Samart et al.\cite{1} have studied the phantom dynamics in LQC. The work of quintom and hessence model in loop quantum cosmology have been done by Wei et al.\cite{19}. They found the behaviors is different from the ones in standard FRW, such as the avoidance of future singularities.

In this paper, we investigate the evolution of universe dominated by phantom with O($N$) symmetry. The phase-space stability of the corresponding autonomous system has been discussed. We find there’s no stable attractor but only 2 saddle points, this result is similar to the single-field phantom in LQC. In order to research the evolution of our model, we numerically study the system in exponential potential, and draw 2 graph to display the evolution of Hubble parameter $H$ and energy density $\rho$ to cosmic time $t$. The figures reveal that in LQC even for a universe dominated by phantom, the Hubble parameter $H$ may turn negative and oscillate. Thus, the universe will enter a oscillatory regime. The relation between $\rho$ and $H$ is also shown in a graph.
2 Loop quantum cosmology

Loop quantum cosmology (for review, see Refs. [22, 23]) can describe the homogeneous and isotropic spacetimes by effective modified Friedmann equation for flat universe. The quantum effects could be reflected by adding a correction term $\rho/\rho_c$ into the equation. In Einstein’s general relativity, the standard Friedmann equation which describes a flat universe is:

$$H^2 = \frac{\kappa^2}{3} \rho$$  \hspace{1cm} (2.1)

Instead of Eq. (2.1), the modified Friedmann equation for a flat universe in LQC is given by [24, 18]

$$H^2 = \frac{\kappa^2}{3} \rho \left(1 - \frac{\rho}{\rho_c}\right)$$  \hspace{1cm} (2.2)

As in GR, $H = \dot{a}/a$ is Hubble parameter, $\rho$ is the total energy density, and a dot denotes the derivative with respect to cosmic time $t$. $\rho_c$ is critical loop quantum density:

$$\rho_c = \frac{\sqrt{3}}{16\pi^2 \gamma^3 G^2 h},$$  \hspace{1cm} (2.3)

Where $\gamma$ is the dimensionless Barbero-Immirzi parameter and is suggested that $\gamma \simeq 0.2375$ by the black hole thermodynamics in LQG [25]. The modified Friedmann equation provides an effective description for LQC which very well approximates the underlying discrete quantum dynamics. Differentiating Eq. (2.2) and use the conservation law

$$\dot{\rho} + 3H (\rho + p) = 0$$  \hspace{1cm} (2.4)

We obtain the effective modified Raychaudhuri equation

$$\dot{H} = -\frac{\kappa^2}{2} (\rho + p) \left(1 - 2\frac{\rho}{\rho_c}\right)$$  \hspace{1cm} (2.5)

where $p$ is the total pressure. In LQC the accelerating condition is

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 > 0$$  \hspace{1cm} (2.6)

The dynamics of our universe could be analyzed through Eq. (2.2), (2.4), (2.5) and the equation of the fields which dominate the universe.
3 O(N) Phantom

We consider the flat Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t) \left( dx^2 + dy^2 + dz^2 \right)$$ (3.7)

The Lagrangian density for the Phantom with O(N) symmetry is given by [3]

$$L_{\Phi} = -\frac{1}{2} g^{\mu\nu} (\partial_\mu \Phi^a)(\partial_\nu \Phi^a) - V(|\Phi^a|)$$ (3.8)

where $\Phi^a$ is the component of the scalar field, $a = 1, 2, \cdots, N$. In order to impose the O(N) symmetries, following [3], we write it in the form

$$\Phi^1 = R(t) \cos \varphi_1(t)$$
$$\Phi^2 = R(t) \sin \varphi_1(t) \cos \varphi_2(t)$$
$$\Phi^3 = R(t) \sin \varphi_1(t) \sin \varphi_2(t) \cos \varphi_3(t)$$

$$\cdots$$

$$\Phi^{N-1} = R(t) \sin \varphi_1(t) \cdots \sin \varphi_{N-2}(t) \cos \varphi_{N-1}(t)$$
$$\Phi^N = R(t) \sin \varphi_1(t) \cdots \sin \varphi_{N-2}(t) \sin \varphi_{N-1}(t)$$ (3.9)

Thus, we have $|\Phi^a| = R$ and assume that the potential $V(|\Phi^a|)$ depends only on $R$. The radial equation of scalar fields is

$$\ddot{R} + 3H \dot{R} - \frac{\Omega^2}{a^6 R^3} - \frac{\partial V(R)}{\partial R} = 0$$ (3.10)

The “angular components” contribute an effective term $\frac{\Omega^2}{a^6 R^3}$ to the system’s dynamics. The energy density $\rho$ and the pressure $p$ of the O(N) phantom are given as

$$\rho = -\frac{1}{2} \left( \dot{R}^2 + \frac{\Omega^2}{a^6 R^2} \right) + V(R)$$ (3.11)

and

$$p = -\frac{1}{2} \left( \dot{R}^2 + \frac{\Omega^2}{a^6 R^2} \right) - V(R)$$ (3.12)

Hence, the equation-of-state for the O(N) phantom is

$$w = -\frac{1}{2} \left( \dot{R}^2 + \frac{\Omega^2}{a^6 R^2} \right) - V(R) \frac{-\frac{1}{2} \left( \dot{R}^2 + \frac{\Omega^2}{a^6 R^2} \right) + V(R)}{\dot{R}^2 + \frac{\Omega^2}{a^6 R^2} + V(R)}$$ (3.13)

Different from GR, the accelerating condition for universe in LQC is not $w < -1/3$ but Eq.(2.16).
4 EVOLUTION AND STABILITY OF THE MODEL

In this section we discuss the dynamics of universe dominated by phantom with $O(N)$ symmetry in LQC. In order to get a possible evolution of the model, we choose exponential potential

$$ V = V_0 e^{-\lambda R}, \quad (4.14) $$

The evolution of universe is governed by Eq.(2.2),(2.5),(3.10). To investigate the stability of the model, we introduce the following dimensionless variables

$$ x = \frac{\kappa \dot{R}}{\sqrt{6H}}, \quad y = \frac{\kappa \sqrt{V(R)}}{\sqrt{3H}}, \quad z = \frac{\kappa}{\sqrt{6H a^3 R}}, \quad m = \frac{\rho}{\rho_c}, \quad \xi = \frac{1}{\kappa R}, \quad N = \ln a \quad (4.15) $$

Using these variables, Eq.(2.2),(2.5),(3.10) can be written as the following autonomous system:

$$ \begin{align*}
\frac{dx}{dN} &= -3x + \sqrt{6} \xi z^2 - \sqrt{\frac{3}{2}} \lambda y^2 - (3x^3 + 3xz^2)(1 - 2m) \\
\frac{dy}{dN} &= -\sqrt{\frac{3}{2}} xy \lambda - (3x^2 y + 3yz^2)(1 - 2m) \\
\frac{dz}{dN} &= -3z - \sqrt{6} xz \xi - (3x^2 z + 3z^3)(1 - 2m) \\
\frac{dm}{dN} &= -3m(1 + \frac{-x^2 - y^2 - z^2}{-x^2 + y^2 - z^2}) \\
\frac{d\xi}{dN} &= -\sqrt{6} x \xi^2
\end{align*} \quad (4.17) $$

In the following, it is straightforward to analyze the critical points as well as their stability. We can get critical points by setting the right hand of the equation(4.17) to zero. As presented in table 1 in the field of real numbers there are only 2 real critical points: $A$ and $B$. We expand the variables around the critical point $A$ and $B$ in the form $x = x_c + \delta x, y = y_c + \delta y, z = z_c + \delta z, m = m_c + \delta m, \xi = \xi_c + \delta \xi$, where $\delta x, \delta y, \delta z, \delta m, \delta \xi$ are perturbations of the variables near the critical points and we consider them forming a column vector denoted as $U$. Substituting the above expression into (4.17) and neglecting higher order terms in the perturbations, we can obtain the equations for the perturbations up to first order as:

$$ U' = M \cdot U, \quad (4.18) $$

where the prime denotes differentiation with respect to $N$. $M$ is a $5 \times 5$ matrix formed by the coefficients of the perturbation equations (4.18). The matrix for the critical point
Critical point \( \frac{x}{y} \frac{z}{m} \frac{\xi}{c} \)

| Critical point \( \frac{x}{y} \frac{z}{m} \frac{\xi}{c} \) | \( \frac{x}{y} \frac{z}{m} \frac{\xi}{c} \) |
|---------------------|---------------------|
| A \( \frac{-\lambda}{\sqrt{6}} \) \( \frac{-6-\lambda^2}{\sqrt{6}} \) \( 0 \) \( 0 \) \( 0 \) |
| B \( \frac{\lambda}{\sqrt{6}} \) \( \frac{-6+\lambda^2}{\sqrt{6}} \) \( 0 \) \( 0 \) \( 0 \) |

Table 1: The real critical points of the autonomous system (4.17).

| Label | Eigenvalues | Stability | \( w \) | Acceleration |
|-------|-------------|-----------|---------|--------------|
| A | \(-\lambda^2, \frac{1}{3}(-6 - \lambda^2), \frac{1}{2}(-6 - \lambda^2), \lambda^2, 0\) | saddle point | \(-1 - \frac{\lambda^2}{3}\) | Yes |
| B | \(-\lambda^2, \frac{1}{3}(-6 - \lambda^2), \frac{1}{2}(-6 - \lambda^2), \lambda^2, 0\) | saddle point | \(-1 - \frac{\lambda^2}{3}\) | Yes |

Table 2: The eigenvalues of the matrix \( M \) and the properties of critical points in the autonomous system (4.17).

\( (x_c, y_c, z_c, m_c, \xi_c) \) is given by

\[
M = \begin{pmatrix}
\frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} & \frac{\partial x'}{\partial m} & \frac{\partial x'}{\partial \xi} \\
\frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} & \frac{\partial y'}{\partial m} & \frac{\partial y'}{\partial \xi} \\
\frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} & \frac{\partial z'}{\partial m} & \frac{\partial z'}{\partial \xi} \\
\frac{\partial m'}{\partial x} & \frac{\partial m'}{\partial y} & \frac{\partial m'}{\partial z} & \frac{\partial m'}{\partial m} & \frac{\partial m'}{\partial \xi} \\
\frac{\partial \xi'}{\partial x} & \frac{\partial \xi'}{\partial y} & \frac{\partial \xi'}{\partial z} & \frac{\partial \xi'}{\partial m} & \frac{\partial \xi'}{\partial \xi}
\end{pmatrix}
\] \quad (4.19)

The eigenvalue of (4.19) determine the type and stability of the critical points. We present them in table 2. As shown, both critical A and B are saddle points, it follows immediately that the phase trajectory is very sensitive to initial conditions given to the system. This result is similar to the single-field phantom in LQC[1]. It is interesting to contrast the influence of \( O(N) \) symmetry in GR and LQC. Several works[2][3][4] shows that the dynamics of quintessence, phantom and quintom with \( O(N) \) symmetry in GR is quite different from the single-field model.

In the following, we study the possible evolution of our model numerically. In order to do this, we set the value of some parameters: \( \lambda = -0.1, \Omega = 1, \kappa = \sqrt{8\pi}, \rho_c = 1.5, V_0 = 1.2, \) and draw 3 graphs: Fig[1] Fig[2] Fig[3]. Fig[1] shows the evolution of Hubble parameter \( H \) with respect to cosmic time \( t \). From Fig[1] we found \( H \) may take maximum value while \( \rho = \rho_c/2 \), and then decline even to negative interval. After \( H \) taking minimum value, it would bounce and then oscillate. Thus, the accelerating expansion of our universe would end in finite time and undergoes contraction from halting(\( H = 0 \)). Then the universe enters an oscillating stage. Fig[2] shows the evolution of cosmic total...
energy density $\rho$, it oscillate while $t$ goes by. In fact, from Fig.3 we can get some insight into the evolution of the model. Fig.3 reveals the relation between $H$ and $\rho$. As time goes by, the trajectory makes a closed region, and the oscillation of $H$ and $\rho$ are restricted in a certain interval.

Figure 1: The evolution of $H$ with respect to $t$ for $\lambda = -0.1$, $\Omega = 1$, $\kappa = \sqrt{8\pi}$, $\rho_c = 1.5$, $V_0 = 1.2$.

Figure 2: The evolution of cosmic total energy density $\rho$ with respect to $t$ for $\lambda = -0.1$, $\Omega = 1$, $\kappa = \sqrt{8\pi}$, $\rho_c = 1.5$, $V_0 = 1.2$.

5 Conclusions

In this paper, we investigate the dynamics of phantom model with $O(N)$ symmetry in background of loop quantum cosmology. We discuss its stability by analyzing the autonomous system Eq.(4.17). It shows that there is no attractor in the system, but
only 2 saddle points. This result is similar to the single-field model in LQC([1]). The effect of O(N) symmetry just influence the detail of the universe’s evolution. It is a sharp contrast with the result in GR, in which the dynamics of scalar fields models with O(N) symmetry are quite different from the single-field models. Finally, to show some insight into our model, we draw 3 graphs by Numerical analysis. The possible oscillation of $H$ and $\rho$ are revealed in Fig.1. and Fig.2. Fig.3 shows that as time goes by, $H$ and $\rho$ are restricted in a certain interval, and the trajectory makes a closed region.

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References

[1] Daris Samart and Burin Gumjudpai, Phys.Rev.D76:043514,2007. arXiv:0704.3414.
[2] X. Zh. Li, J. G. Hao and D. J. Liu, Class. Quant. Grav. 19, 6049 (2002) arXiv:astro-ph/0107171.
[3] X. Li and J. Hao, Phys. Rev. D 69, 107303 (2004) arXiv:hep-th/0303093.
[4] M. R. Setare and E. N. Saridakis, JCAP 0809:026,2008 arXiv:0809.0114.
[5] E. Komatsu et al. [WMAP Collaboration], Astrophys. J. Suppl. 180, 330 (2009) [arXiv:0803.0547 [astro-ph]].

[6] M. Hicken et al., Astrophys. J. 700, 1097 (2009) [arXiv:0901.4804 [astro-ph.CO]].

[7] R. Scranton et al. [SDSS Collaboration], [arXiv:astro-ph/0307335].

[8] P. J. Peebles and B. Ratra, Rev. Mod. Phys. 75, 559 (2003) [arXiv:astro-ph/0207347].

[9] Caidwell.R.R, Dave.R and Steinhardt.P.J, Phys.Rev.Lett.80 (1998) 1582

[10] C. J. Gao and Y. G. Shen. (2002). Phys.Lett.B541,1

[11] C. J. Gao and Y. G. Shen. (2002). Chin.Phys.Lett.19,1396

[12] P. J. E. Peebles, B. Ratra, Rev. Mod. Phys. 75(2003)599 ; T. Padmanabhan, Phys. Rept. 380(2003)235

[13] R. R. Caldwell, R. Dave and P. J. Steinhardt, Phys. Rev. Lett. 80(1998)1582

[14] J.-An Gu and W. -Y. P. Hwang. (2001) Phys.Lett. B517, 1

[15] Caidwell.R.R, Phys.Lett. B 545 (2002) 23.

[16] Z. Y. Sun and Y. G. Shen. Gen.Rel.Grav. 37(2005) 243

[17] B. Feng, X. L. Wang and X. M. Zhang, Phys. Lett. B 607, 35 (2005) [astro-ph/0404224].

[18] Xin Zhang and Yi Ling, JCAP0708:012,2007. [arXiv:0705.2656]

[19] Hao Wei and Shuang Nan Zhang, Phys.Rev.D76:063005,2007. [arXiv:0705.4002]

[20] Xiangyun Fu, Hongwei Yu and Puxun Wu, PhysRevD.78:063001. [arXiv:0808.1382]

[21] Songbai Chen, Bin Wang and Jiliang Jing, [arXiv:0808.3482]

[22] M. Bojowald, Living Rev. Rel. 8, 11 (2005) [gr-qc/0601085]; M. Bojowald, [gr-qc/0505057].

[23] A. Ashtekar, M. Bojowald and J. Lewandowski, Adv. Theor. Math. Phys. 7, 233 (2003)

[24] A. Ashtekar, AIP Conf. Proc. 861, 3 (2006) [gr-qc/0605011]. [gr-qc/0304074]; A. Ashtekar, [gr-qc/0702030].
[25] A. Ashtekar, J. Baez, A. Corichi and K. Krasnov, Phys. Rev. Lett. 80, 904 (1998) [gr-qc/9710007]; M. Domagala and J. Lewandowski, Class. Quant. Grav. 21, 5233 (2004) [gr-qc/0407051]; K. A. Meissner, Class. Quant. Grav. 21, 5245 (2004) [gr-qc/0407052].

[26] A. Cohen, D. Kaplan and A. Nelson, hep-th/9803132 Phys. Rev. Lett. 82 (1999) 4971.

[27] S.D.H. Hsu, hep-th/0403052.

[28] M. Li, hep-th/0403127

[29] Yungui Gong, Phys.Rev.D 70(2004) 064029

[30] A.Zee,Phys.Rev.Lett. 42(1979)417

[31] Frank S.Accetta, David J.Zoller and Michael S.Turner, Phys.Rev.D 31(1985) 3046

[32] Y. G. Shen Chin. Phys. Lett 12(1995) 509

[33] W. Fischler and L. Susskind, hep-th/9806039 R. Bousso, JHEP 9907 (1999) 004.