REAL ANALYTIC SOLUTIONS TO THE WILLMORE FLOW

YUANZHEN SHAO

Abstract. In this paper, a regularity result for the Willmore flow is presented. It is established by means of a truncated translation technique in conjunction with the Implicit Function Theorem.

1. Introduction

The Willmore flow consists in looking for an oriented, closed, compact moving hypersurface \( \Gamma(t) \) immersed in \( \mathbb{R}^3 \) evolving subject to the law
\[
\begin{aligned}
V(t) &= -\Delta_{\Gamma(t)} H_{\Gamma(t)} - 2H_{\Gamma(t)} (H_{\Gamma(t)}^2 - K_{\Gamma(t)}), \\
\Gamma(0) &= \Gamma_0.
\end{aligned}
\] (1.1)

Here \( V(t) \) denotes the velocity in the normal direction of \( \Gamma \) at time \( t \) and \( \Delta_{\Gamma(t)} \) stands for the Laplace-Beltrami operator, while \( H_{\Gamma(t)} \) is the normalized mean curvature of \( \Gamma(t) \). Finally, \( K_{\Gamma(t)} \) denotes the Gaussian curvature.

The equilibria of (1.1) appear as the critical points of the Willmore functional, or sometimes called the Willmore energy. For a smooth immersion \( f : \Gamma \to \mathbb{R}^3 \) of a closed oriented two-dimensional manifold \( \Gamma \), the Willmore functional is defined as:
\[
W(f) = \int_{f(\Gamma)} H_{f(\Gamma)}^2 d\sigma,
\] (1.2)

where \( d\sigma \) is the area element on \( f(\Gamma) \) with respect to the Euclidean metric in \( \mathbb{R}^3 \).

The critical surfaces of this functional, called the Willmore surfaces, satisfy the equation:
\[
\Delta_{f(\Gamma)} H_{f(\Gamma)} + 2H_{f(\Gamma)}^2 - 2H_{f(\Gamma)} K_{f(\Gamma)} = 0.
\] (1.3)

The reader may consult [27, Section 7.4] for a brief historical account and a proof of this variational formula. The proof therein is derived by computing the critical points of all normal variations of the hypersurface \( f(\Gamma) \).

A generalization of the Willmore functional (1.2) in higher dimensions is studied by B.-Y. Chen [4]. He extends (1.2) for smooth immersions \( f : \Gamma \to \mathbb{R}^{m+1} \) of the \( m \)-dimensional closed oriented manifold \( \Gamma \) into \( \mathbb{R}^{m+1} \):
\[
W(f) = \int_{f(\Gamma)} H_{f(\Gamma)}^m d\sigma
\]
with $d\sigma$ standing for the volume element with respect to the Euclidean metric in $\mathbb{R}^{m+1}$. The critical points of this functional are now of the form:

$$\Delta_{f(\Gamma)} f(\Gamma) = 0.$$ 

Here $R_{f(\Gamma)}$ denotes the scalar curvature. We may observe that $R_{f(\Gamma)} = 2K_{f(\Gamma)}$ when $m = 2$, so this Euler-Lagrange equation agrees with (1.3) in the two-dimensional case. However, this generalization has the drawback that the corresponding Willmore functional is no longer conformally invariant except when $m = 2$.

The Willmore problem has been studied by many authors, among them T.J. Willmore, W. Blaschke, B.-Y. Chen, J.L. Weiner, P. Li, S.-T. Yau, R. Bryant, R. Kusner, L. Simon, U.F. Mayer, G. Simonett, M. Bauer, E. Kuwert, R. Schätzle, U. Pinkall, I. Sterling, M.U. Schmidt, C.M. Fernando, and N. André. See for example [2, 3, 4, 9, 11, 12, 13, 14, 15, 17, 18, 19, 20, 22, 24, 25, 26, 27]. It is well-known that the Willmore functional is bounded below by $4\pi$ with equality only for the round sphere. Then the famous Willmore conjecture due to T.J. Willmore asserts that for any immersed 2-dimensional torus into $\mathbb{R}^3$ we have $W(f) \geq 2\pi^2$, and it suggests that the 2-dimensional Clifford torus achieves the minimum of the Willmore functional amongst all immersed tori in $\mathbb{R}^3$. In 1982, P. Li and S.-T. Yau [15] show that any immersion with $W(f) < 8\pi$ must in fact be an embedding. In other words, it will suffice to estimate $W(f)$ for embeddings. A classification of all Willmore immersions $f : S^2 \to \mathbb{R}^3$ is obtained by R.L. Bryant [3]. The possible values of

$$W(f) = \int_{f(S^2)} H^2_f d\sigma$$

are $4n\pi$ with $n = 1$, or $n \geq 4$ and $n$ even, or $n \geq 9$ and $n$ odd. Existence and regularity for embedded tori in the Willmore conjecture has been proven by L. Simon [24], and later this result is generalized by M. Bauer, E. Kuwert [2] for an extension of the conjecture by R. Kusner [11] to higher genus cases. An existence, uniqueness and regularity result on the Willmore flow is presented by G. Simonett [25]. It is proven therein that the Willmore flow admits a unique smooth solution. Moreover, this solution exists globally when it is initially close enough to spheres in the $C^{2+\alpha}$-topology and is exponentially attracted by spheres. In [17], U.F Mayer and G. Simonett prove that the Willmore flow can drive embedded surfaces to a self-intersection in a finite time interval. Moreover, numerical simulations in [18] indicate that the Willmore flow can develop true singularities (topological changes) in finite time. E. Kuwert and R. Schätzle [12] show that the smooth solutions are global as long as the initial Willmore energy is sufficiently small. Later, the same authors improve this result in [14] by finding an explicit optimal bound for the restriction on the initial energy, that is, if the smooth immersion $f_0 : \Gamma \to \mathbb{R}^3$ satisfies $W(f_0) \leq 8\pi$, then the solution with initial data $f_0$ exists smoothly for all time and converges to a round sphere. Recently, in a breakthrough paper, C.M. Fernando and N. André [9] prove the Willmore conjecture for surfaces of arbitrary genus $g \geq 1$, i.e., $W(f) \geq 2\pi^2$ for all embedded $\Gamma$ with genus $g \geq 1$, and the equality holds iff $\Gamma$ is conformal to the Clifford torus.
**Assumptions:** Throughout this paper, we always assume that \((M, g)\) is a compact, closed, immersed, oriented, real analytic hypersurface in \(\mathbb{R}^3\) endowed with the Euclidean metric \(g\). We may find for \(M\) a normalized atlas \((O_\kappa, \varphi_\kappa)_{\kappa \in \Lambda}\), where an atlas is said to be normalized if \(\varphi_\kappa(O_\kappa) = \mathbb{B}^2\) for all \(\kappa \in \Lambda\). Here \(\mathbb{B}^2\) is the open unit ball centered at the origin in \(\mathbb{R}^2\). Put \(\psi_\kappa = \varphi_\kappa^{-1}\).

A family \((\pi_\kappa)_{\kappa \in \Lambda}\) is called a localization system subordinate to \((O_\kappa, \varphi_\kappa)_{\kappa \in \Lambda}\) if:

1. \(\pi_\kappa \in D(O_\kappa, [0, 1])\) and \((\pi^2_\kappa)_{\kappa \in \Lambda}\) is a partition of unity subordinate to the open cover \((O_\kappa)_{\kappa \in \Lambda}\).

2. Any \(\pi_\kappa\) and \(\pi_\eta\) satisfying \(\text{supp}(\pi_\kappa) \cap \text{supp}(\pi_\eta) \neq \emptyset\) have their supports located within the same local chart.

For any manifold satisfying the above assumptions, there exists a localization system. See [1, Lemma 3.2] for a proof.

**Notations:** Throughout this paper, \(\mathbb{N}_0\) stands for the set of all natural numbers including 0. For any time interval \(I, \dot{I}\) always denotes the interior of \(I\).

Fix \(0 < \alpha < 1\). Let \(\gamma \in (0, 1]\) and \(E_0 := h^\alpha(M), E_1 := h^{4+\alpha}(M)\). For notational brevity, we simply write \(\mathfrak{g}(O, \mathbb{R})\) and \(\mathfrak{g}(M, \mathbb{R})\) as \(\mathfrak{g}(O)\) and \(\mathfrak{g}(M)\), where \(O\) is an open subset of \(\mathbb{R}^2\) and \(\mathfrak{g}\) stands for any of the function spaces in this paper.

In the sequel, we always denote \((E_0, E_1)\) by \(E_\gamma\), where \((\cdot, \cdot)_\gamma\) is the continuous interpolation method. See [16, Definition 1.2.2] for a definition. Note that the continuous interpolation method \((\cdot, \cdot)_0^{0, \infty}\) coincides with \((\cdot, \cdot)_0\) in the suggested reference. In particular, we set \((E_0, E_1)_{1} := E_1\).

For some fixed interval \(I = [0, T]\) and some Banach space \(E\), we define

\[
BUC_{1-\gamma}(I, E) := \{ u \in C(\dot{I}, E); [t \mapsto t^{1-\gamma} u] \in BUC(\dot{I}, E), \lim_{t \to 0^+} t^{1-\gamma} ||u|| = 0 \},
\]

\[
\|u\|_{C_{1-\gamma}} := \sup_{t \in I} t^{1-\gamma} \|u(t)\|_E,
\]

and

\[
BUC_{1-\gamma}^1(I, E) := \{ u \in C^1(\dot{I}, E); u, \dot{u} \in BUC_{1-\gamma}(I, E) \}.
\]

In particular, we put

\[
BUC_0(I, E) := BUC(I, E) \quad \text{and} \quad BUC^1_0(I, E) := BUC^1(I, E).
\]

In addition, if \(I = [0, T]\) is a half open interval, then

\[
C_{1-\gamma}(I, E) := \{ v \in C(\dot{I}, E); v \in BUC_{1-\gamma}([0, t], E), \ t < T \},
\]

\[
C_{1-\gamma}^1(I, E) := \{ v \in C^1(\dot{I}, E); v, \dot{v} \in C_{1-\gamma}(I, E) \}.
\]

We equip these two spaces with the natural Fréchet topology induced by the topology of \(BUC_{1-\gamma}(0, T], E)\) and \(BUC_{1-\gamma}^1(0, T], E)\), respectively.

Last but not least, we set

\[
E_0(I) := C(I, E_0) \quad \text{and} \quad E_1(I) := C(I, E_1) \cap C^1(I, E_0).
\]
It will be shown in this paper that the Willmore flow (1.1) admits a real analytic solution jointly in time and space. Our motivation for a real analytic solution is mainly stimulated by the following facts: a compact closed real analytic manifold cannot have a "flat part", and real analyticity in time implies that the hypersurface should move permanently in the interval of existence.

**Theorem 1.1.** Let $0 < \alpha < 1$. Suppose that $\Gamma_0$ is a compact closed immersed oriented hypersurface in $\mathbb{R}^3$ belonging to the class $h^{2+\alpha}$. Then the Willmore flow (1.1) has a unique local solution $\Gamma = \{\Gamma(t) : t \in [0, T]\}$ for some $T > 0$. Moreover, 

$$
\mathcal{M} := \bigcup_{t \in (0,T)} (\{t\} \times \Gamma(t))
$$

is a real analytic submanifold in $\mathbb{R}^4$. In particular, each manifold $\Gamma(t)$ is real analytic for $t \in (0,T)$.

For any open subset $O \subset \mathbb{R}^2$, the little Hölder space $h^s(O)$ of order $s > 0$ with $s \not\in \mathbb{N}$ is the closure of $BUC^\infty(O)$ in $BUC^s(O)$. Here $BUC^s(O)$ is the Banach space of all bounded and uniformly Hölder continuous functions. The little Hölder space $h^s(M)$ on $M$ is defined in terms of a smooth atlas, that is, a function $u$ belongs to $h^s(M)$ iff $\psi_\kappa^* \pi_\kappa u \in h^s(\mathbb{R}^2)$, for each $\kappa \in \Lambda$.

### 2. Parameterization over a Reference Manifold

In equation (1.1), if we fix an initial hypersurface $\Gamma_0$ belonging to the class $h^{2+\alpha}$, then by the discussion in [21, Section 4] we can find a real analytic compact closed embedded oriented hypersurface $M$, a function $\rho_0 \in h^{2+\alpha}(M)$ and a parameterization

$$
\Psi_{\rho_0} : M \to \mathbb{R}^3, \quad \Psi_{\rho_0}(p) := p + \rho_0(p) \nu_M(p)
$$

such that $\Gamma_0 = \text{im}(\Psi_{\rho_0})$. Here $\nu_M(p)$ denotes the unit normal with respect to a chosen orientation of $M$ at $p$, and $\rho_0 : M \to (-a, a)$ is a real-valued function on $M$, where $a$ is a sufficiently small positive number depending on the inner and outer ball condition of $M$. The reader may consult [21, Section 4.1] for the precise bound of $a$. Thus $\Gamma_0$ lies in the $a$-tubular neighborhood of $M$. In fact, it will suffice to assume $\Gamma_0$ to be a $C^2$-manifold for the existence of such a parameterization and a real analytic reference manifold. See [21] Section 4] for a detailed proof.

Analogously, if $\Gamma(t)$ is $C^1$-close enough to $M$, then we can find a function $\rho : [0, T) \times M \to (-a, a)$ for some $T > 0$ and a parameterization

$$
\Psi_{\rho} : [0, T) \times M \to \mathbb{R}^3, \quad \Psi_{\rho}(t, p) := p + \rho(t,p) \nu_M(p)
$$

such that $\Gamma(t) = \text{im}(\Psi_{\rho}(t, \cdot))$ for every $t \in [0, T)$. It is worthwhile to mention that $\Psi_{\rho}$ admits an extension on $\mathbb{R}^3$, called Hanzawa transform, which was first introduced by E.I. Hanzawa in [10].

For any fixed $t$, I do not distinguish between $\rho(t, \cdot)$ and $\rho(t, \psi_\kappa(\cdot))$ in each local coordinate $(O_\kappa, \varphi_\kappa)$ and abbreviate $\Psi_{\rho}(t, \cdot)$ to be $\Psi_{\rho} := \Psi_{\rho}(t, \cdot)$. In addition, the hypersurface $\Gamma(t)$ will be simply written as $\Gamma_{\rho}$ as long as the choice of $t$ is of no
importance in the context, or \( \rho \) is independent of \( t \).

We put

\[
\mathcal{U} := \{ \rho \in h^{2+\alpha}(M) : \|\rho\|_\infty < a \}.
\]

Here \( \|\rho\|_\infty := \sup_{p \in M} |\rho(p)| \). For any \( \rho \in \mathcal{U} \), \( \text{im}(\Psi_\rho) \) constitutes a \( h^{2+\alpha} \)-hypersurface \( \Gamma_\rho \). In this case, \( \Psi_\rho \) defines a \( h^{2+\alpha} \)-diffeomorphism from \( M \) onto \( \Gamma_\rho \).

Here and in the following, it is understood that the Einstein summation convention is employed and all the summations run from 1 to 2 for all repeated indices.

In [21], J. Prüss and G. Simonett derive global expressions for many geometric objects of \( \Gamma_\rho \) in terms of the function \( \rho \). I will use some results therein to translate equation (1.1) into a differential equation in \( \rho \). By [21] formula (23), (28), we have the following explicit expressions for the components of the first fundamental form and the normal vector of \( \Gamma_\rho \):

\[
g^{ij}_\rho = g_{ij} - 2\rho l^i_{ij} + \rho^2 l^i_l j_r + \partial_i \rho \partial_j \rho,
\]

and

\[
\nu_t = \beta(\rho)(\nu_M - a(\rho)).
\]

In (2.1), the \( l^i_j \)'s are the components of the Weingarten tensor \( L_M \) of \( M \) with respect to \( g \), i.e., \( L_M = l^i_j \tau^i \otimes \tau_j \), where \( \{ \tau_i = \partial_i \} \) forms a basis of \( T_p M \) at \( p \in M \) and \( \{ \tau^i \} \) is the dual basis to \( \{ \tau_i \} \), i.e., \( (\tau^i | \tau_j) = \delta_i^j \). The extension of \( L_M \) into \( \mathbb{R}^3 \), by identifying it to be zero in the normal direction, is denoted by \( L^E_M \), namely, \( L^E_M = l^i_j \tau^i \otimes \tau_j + 0 \cdot \nu_M \otimes \nu_M \). It is a simple matter to check that

\[
\tau^i_i = (I - \rho L^E_M) \tau_i + \nu_M \partial_i \rho
\]

forms the standard basis of \( T_{\Psi_\rho(p)} \Gamma_\rho \). In addition, the \( l^i_j \)'s are the components of the second fundamental form \( \tilde{L}_M \) of the metric \( g \). Finally, \( g^{ij}_\rho = (\tau^i_j | \tau^j_i) \) are the components of the first fundamental form of the Euclidean metric \( g_T \) on \( \Gamma_\rho \). We set \( G^\Gamma(\rho) = (g^{ij}_\rho)_{ij} \) and \( G^{-1}_\Gamma(\rho) \) for its inverse.

In (2.2), the terms \( a(\rho) \) and \( \beta(\rho) \) read as

\[
a(\rho) = (I - \rho L^E_M)^{-1} \nabla_M \rho \quad \text{and} \quad \beta(\rho) = [1 + |a(\rho)|^2]^{-1/2}.
\]

Here \( \nabla_M \) is the surface gradient on \( M \).

For sufficiently small \( a > 0 \), \( (I - \rho L^E_M) \) is invertible. One can check that

\[
I - \rho L^E_M = (\delta^j_i - \rho l^j_l ) \tau^i \otimes \tau_j + \nu_M \otimes \nu_M.
\]

Thus

\[
(I - \rho L^E_M)^{-1} = r^j_i(\rho) \tau^i \otimes \tau_j + \nu_M \otimes \nu_M,
\]

where \( R_\rho = (r^j_i(\rho))_{ij} = [(\delta^j_i - \rho l^j_l )_{ij}]^{-1} \). By Cramer’s rule, all the entries of \( R_\rho \) possess the expression

\[
r^j_i(\rho) = \frac{P^j_i(\rho)}{Q^j_l(\rho)}
\]

in every local chart, where \( P^j_l \) and \( Q^j_l \) are polynomials in \( \rho \) with real analytic coefficients and \( Q^j_l \neq 0 \).
Substituting \((I - \rho L^E_M)^{-1}\) by (2.3), we get

\[
|a(\rho)|^2 = (r_i^j(\rho)\partial_j \rho \tau^i |r_k^l(\rho)\partial_l \rho \tau^k) = g^{ik} r_i^j(\rho) r_k^l(\rho) \partial_j \rho \partial_l \rho.
\]

Then

\[
\beta(\rho) = [1 + |a(\rho)|^2]^{-1/2} = [1 + g^{ik} r_i^j(\rho) r_k^l(\rho) \partial_j \rho \partial_l \rho]^{-1/2}.
\]

Note that in every local chart

\[
\beta^2(\rho) = \frac{P^3(\rho)}{Q^3(\rho, \partial_j \rho)},
\]

where \(P^3(\rho)\) is a polynomial in \(\rho\) with real analytic coefficients and \(Q^3(\rho, \partial_j \rho) \neq 0\) is a polynomial in \(\rho\) and its first order derivatives with real analytic coefficients.

The normal velocity can be expressed as

\[
V(t) = (\partial_t \Psi_M |\nu_T) = (\rho \nu_M |\nu_T) = \beta(\rho) \rho_t.
\]

Therefore, the first line of equation (1.1) is equivalent to

\[
\rho_t = -\frac{1}{\beta(\rho)} [\Psi^*_p \Delta_{\Gamma^p} H_{\Gamma^p} + 2 \Psi^*_p H_{\Gamma^p} (H^2_{\Gamma^p} - K_{\Gamma^p})].
\]

Next we shall calculate the Gaussian curvature \(K_{\Gamma^p}\) in terms of \(\rho\). For simplicity, we write \(K_{\rho}\) instead of \(\Psi^*_p K_{\Gamma^p}\).

Because

\[
\partial_j \tau_i = \Gamma^k_{ij} \tau_k + l_{ij} \nu_M \quad \text{and} \quad \partial_j \tau^i = -\Gamma^k_{jk} \tau^k + l^i_{jk} \nu_M,
\]

we may readily compute

\[
\partial_j L^E_M = \partial_j l^i_{jk} \tau^j \otimes \tau_k - \Gamma^i_{jk} l^j_{hk} \tau^i \otimes \tau_k + l^i_{jk} l^j_{hk} \nu_M \otimes \tau_k + l_{jk} l^i_{jk} \tau^i \otimes \nu_M. \quad (2.5)
\]

Denote by \(L^i = (l^i_{jk})_{ij}\) the second fundamental form of \(\Gamma^p\) with respect to \(g_T\). Then by (2.2) and (2.3), we can compute its components \(l^i_{ij}\) as follows:

\[
l^i_{ij} = -(\tau^i |\partial_j \nu_T)
\]

\[
= -(\rho L^E_M)_{\tau_i} + \nu_M \partial_i \rho |(\partial_j \nu_M - \partial_j a(\rho))) - (r_i^j |\partial_j \beta |\nu_T)
\]

\[
= \beta |l_{ij} + \rho (L^E_M)_{\tau_i} |\partial_j \nu_M + \tau_i |\partial_j (\nabla_M \rho)) + (\rho L^E_M)_{\tau_i} |\partial_j [(I - \rho L^E_M)^{-1}]|\nabla_M \rho) + \partial_i \rho (\nu_M (I - \rho L^E_M)^{-1})|\partial_j (\nabla_M \rho))
\]

\[
= \beta |l_{ij} + \rho l_{jk} \tau^k |\partial_j \nu_M + (\tau_i |\partial_j (\nabla_M \rho)) + (\rho L^E_M)_{\tau_i} |\partial_j [(I - \rho L^E_M)^{-1}]|\nabla_M \rho) + \partial_i \rho (\nu_M (I - \rho L^E_M)^{-1})|\partial_j (\nabla_M \rho))
\]

\[
= \beta |l_{ij} - l_{ik} l^k_{\rho} + \partial_i \rho l^k_{\rho} + \partial_k \rho l^i_{\rho} |(\partial_j l^k_{\rho} + \Gamma^k_{ij} l^i_{\rho} - \Gamma^k_{ik} l^i_{\rho}) \partial_l \rho
\]

\[
+ l^i_{jk} (\rho) l^j_{hk} \partial_i \rho |\partial_l \rho + l^i_{jk} (\rho) l^j_{hk} \rho \partial_i \rho |\partial_l \rho + l^i_{jk} \partial_i \rho \partial_l \rho |\partial_k \rho).
\]

We have used (2.5) and the following facts in the above computation:

- \((\nu_M |\partial_j \nu_M) = 0\).
- \((\tau^i |\nu_T) = 0\).
\[ \partial_j \nu_M = - l_{ij} \tau^i. \]

\[ (I - \rho L_M^\xi)^{-1} \nu_M = \nu_M. \]

\[ \partial_j \alpha(\rho) = (I - \rho L_M^\xi)^{-1} \partial_j (\nabla_M \rho) + \partial_j [(I - \rho L_M^\xi)^{-1}] \nabla_M \rho. \]

\[ \partial_j [(I - \rho L_M^\xi)^{-1}] = (I - \rho L_M^\xi)^{-1} \partial_j (\rho L_M^\xi) (I - \rho L_M^\xi)^{-1}. \]

Therefore, \( \det(L^F) \) can be expressed in every local chart as

\[ \det(L^F) = \beta^2(\rho) \frac{P^F(\rho, \partial_j \rho, \partial_{ij} \rho)}{Q^F(\rho)}. \]

Here \( P^F(\rho, \partial_j \rho, \partial_{ij} \rho) \) is a polynomial in \( \rho \) and its derivatives up to second order with real analytic coefficients. Moreover, \( Q^F(\rho) \) is a polynomial in \( \rho \) with real analytic coefficients. In particular, we have \( Q^F \neq 0. \)

In full view of the above computations, within every local chart \( K_\rho = \det[G_r^{-1}(\rho) L^F] \) can be expressed locally as

\[ K_\rho = \beta^2(\rho) \frac{P^F(\rho, \partial_j \rho, \partial_{ij} \rho)}{\det(G^F(\rho)) Q^F(\rho)} \quad (2.6) \]

As a straightforward conclusion of the above computation, we obtain an explicit expression for \( H_\rho := \Psi_\rho H_{\Gamma^\rho} : \)

\[ 2H_\rho = g_{ij}^F |_{ij} \]

\[ = \beta(\rho) g_{ij}^F \left[ l_{ij} - l_{ik} l_{kj}^F \rho + \partial_{ij} \rho - \Gamma_{ij}^k \partial_k \rho + r_{ik}^F (\partial_j l_{jk}^F) \right. \]

\[ + r_{ik}^F (\partial_j l_{jk}^F + \Gamma_{jh}^k l_{ij}^F - \Gamma_{ij}^h l_{jk}^F) \partial_h \rho + \left. r_{ik}^F (l_{ij}^F l_{jk}^F \partial_h \rho + l_{ik}^F l_{kj}^F \partial_h \rho) \right]. \]

(2.7)

The reader may also find a different global expression for \( H_\rho \) in [21, formula (32)]. We can decompose \( H_\rho \) into \( H_\rho = P_1(\rho) + F_1(\rho) \):

\[ F_1(\rho) = \frac{\beta(\rho)}{2} g_{ij}^F (l_{ij} - l_{ik} l_{kj}^F \rho) = \frac{\beta(\rho)}{2} \text{Tr}[G_r^{-1}(\rho)(L^M - \rho L^M L^M)], \]

where \( \text{Tr}(\cdot) \) denotes the trace operator, and

\[ P_1(\rho) = \frac{\beta(\rho)}{2} \left\{ g_{ij}^F \partial_{ij} \sum_{k} g_{ij}^F (l_{ik}^F l_{kj}^F \partial_h \rho - \Gamma_{ij}^k \partial_k \rho) \right\} \]

\[ + g_{ij}^F \left[ r_{ik}^F (l_{ij}^F l_{jk}^F \partial_h \rho + r_{ik}^F (\partial_j l_{jk}^F + \Gamma_{jh}^k l_{ij}^F - \Gamma_{ij}^h l_{jk}^F) \partial_h \rho + r_{ik}^F (l_{ij}^F l_{jk}^F \partial_h \rho + l_{ik}^F l_{kj}^F \partial_h \rho)) \right]. \]

in every local chart. Note that \( \text{Tr}[G_r^{-1}(\rho)L^M] \) changes like \( H_M \) under transition maps and thus is invariant. Analogously, we can check that \( F_1 \) is a well-defined global operator. Hence so is \( P_1(\rho) \).

In addition, it is a well-known fact that \( \Psi_\rho^* \Delta_{\Gamma^\rho} = \Delta_\rho \Psi_\rho^* \), where \( \Delta_{\Gamma^\rho} \) and \( \Delta_\rho \) are the Laplace-Beltrami operators on \( (\Gamma_\rho, g_{\Gamma}) \) and \( (M, \sigma(\rho)) \), respectively. Here \( \sigma(\rho) := \Psi_\rho^* g_{\Gamma} \) stands for the pull-back metric of \( g_{\Gamma} \) on \( M \) by \( \Psi_\rho \). Then in every local chart, the Laplace-Beltrami operator \( \Delta_\rho \) can be expressed as

\[ \Delta_\rho = \sigma^{jk}(\rho)(\partial_j \partial_k - \gamma_{jk}^i(\rho) \partial_i). \]

(2.8)

Here \( \sigma^{jk}(\rho) \) are the components of the induced metric \( \sigma^*(\rho) \) of \( \sigma(\rho) \) on the cotangent bundle. Note that \( \sigma^{jk}(\rho) \) involves the derivatives of \( \rho \) merely up to order one. \( \gamma_{jk}^i(\rho) \)
are the corresponding Christoffel symbols of \( \sigma(\rho) \), which contain the derivatives of \( \rho \) up to second order.

There exists a global operator \( R(\rho) \in \mathcal{L}(h^{3+\alpha}(M), E_0) \) such that \( R(\cdot) \) is well defined on \( \mathcal{O} \) and:

\[
R(\rho) = \frac{1}{2\beta(\rho)} \Delta_\rho [\beta(\rho) \text{Tr}(G^{-1}_\Gamma(\rho)L^M)] - \frac{\rho}{2\beta(\rho)} \Delta_\rho [\beta(\rho) \text{Tr}(G^{-1}_\Gamma(\rho)L^ML^M)].
\]

We set

\[
P(\rho) := \frac{1}{\beta(\rho)} \Delta_\rho F_1 + R(\rho), \quad \rho \in \mathcal{O},
\]

\[
F(\rho) := -\frac{1}{\beta(\rho)} \Delta_\rho F_1(\rho) + R(\rho)\rho - \frac{2}{\beta(\rho)} H_\rho (H_\rho^2 - K_\rho), \quad \rho \in \mathcal{O} \cap h^{3+\alpha}(M).
\]

Note that third order derivatives of \( \rho \) do not appear in \( F(\rho) \). Hence it is actually well-defined on \( \mathcal{O} \). Based on the above discussion, these two maps enjoy the following smoothness properties:

\[
P \in C^\infty(\mathcal{O}, \mathcal{L}(E_1, E_0)) \quad \text{and} \quad F \in C^\infty(\mathcal{O}, E_0).
\]

**Definition 2.1.** Let \( l \in \mathbb{N}_0 \). A linear operator \( A : D(M) \rightarrow C(M) \) is called a differential operator of order \( l \) with continuous coefficients on \( M \) if for any \( u \in D(M) \) it holds that

\[
\psi^*(\kappa)(Au) = A_{\kappa}(\psi^*_\kappa u)
\]

for every local chart \( (O_\kappa, \varphi_\kappa) \) and some differential operator \( A_{\kappa} = \sum_{|\alpha| \leq l} a^\kappa_\alpha \partial^\alpha \) with \( a^\kappa_\alpha \in C(\mathbb{B}^2) \) defined on \( \mathbb{B}^2 \), and at least one of the \( A_{\kappa} \)'s is of order \( l \). In particular, when \( l = 0 \), \( Au = au \) for some \( a \in C(M) \).

By the above definition, \( P(\rho) \) is a fourth order differential operator with continuous coefficients on \( M \) for each \( \rho \in \mathcal{O} \). In every local chart \( (O_\kappa, \varphi_\kappa) \), the principal part of the local expression of \( P(\rho) \) can be written as

\[
P^\kappa_{\rho} := \sigma(\rho) g^*_{ijkl} \partial_{ijkl}.
\]

Given \( \xi \in T^*M \), we estimate the symbol of \( P^\kappa_{\rho} \) as follows.

\[
P^\kappa_{\rho}(\xi) = \sigma(\rho)(\xi, \xi) g^*_{ijkl}(\xi, \xi) \geq c|\xi|^4
\]

for some \( c > 0 \), and \( g^*_{ijkl} \) denotes the induced metric of \( g_\rho \) on the cotangent bundle of \( M \). Hence, \( P(\rho) \) is a uniformly elliptic fourth order operator acting on functions over \( M \) for each \( \rho \in \mathcal{O} \). By \cite{23} Theorem 4.3, \( P(\rho) \in \mathcal{H}(E_1, E_0) \), namely that \(-P(\rho)\) generates an analytic semigroup on \( E_0 \) with \( D(-P(\rho)) = E_1 \), \( \rho \in \mathcal{O} \).

Now the Willmore flow \( \{1.1\} \) can be rewritten as:

\[
\begin{aligned}
\rho_t + P(\rho)\rho &= F(\rho), \\
\rho(0) &= \rho_0,
\end{aligned}
\]

where \( \rho_0 \in \mathcal{O} \). A different characterization of the problem can be found in \cite{6, 7, 25}.
Applying [5, Theorem 4.1], the existence and regularity result in [25] can be restated as:

**Theorem 2.2.** [25, Theorem 1.1] Suppose that \( \rho_0 \in h^{2+\alpha}(M) \). Then equation (2.9) has a unique solution \( \rho \) such that
\[
\rho \in C^1_\frac{1}{2}(J(\rho_0), E_0) \cap C^1_\frac{1}{2}(J(\rho_0), E_1) \cap C(J(\rho_0), h^{2+\alpha}(M)) \cap C^{\frac{1}{2}-\beta_0}(J(\rho_0), E_{\beta_0})
\]
for any \( \beta_0 \in [0, \frac{1}{2}] \). Moreover, each hypersurface \( \Gamma(t) \) is of class \( C^\infty \) for \( t \in J(\rho) \).

### 3. Parameter-Dependent Diffeomorphisms

The main purpose of the last two sections is to show that the classical solution obtained in Theorem 2.2 is in fact real analytic jointly in time and space. To this end, I will construct a family of parameter-dependent diffeomorphisms acting on functions over \( M \) first. Because the construction applies to manifolds of arbitrary dimensions, in this section we assume that \( M \) is a \( m \)-dimensional manifold with the properties imposed in Section 1.

For a given point \( p \in M \), we choose a normalized atlas \( (O_\kappa, \varphi_\kappa)_\kappa \in \Lambda \) for \( M \) such that \( \varphi_1(p) = 0 \in \mathbb{R}^m \). Choose several open subsets \( B_i \) in \( \mathbb{B}^m \), the open unit ball centered at the origin in \( \mathbb{R}^m \), in such a manner that:
- \( B_i := \mathbb{B}^m(0, i\varepsilon_0) \), for \( i = 1, 2, 3 \) and some \( \varepsilon_0 > 0 \).
- \( B_3 \subset \subset B_4 \subset \subset \mathbb{B}^m \).

Next, I further pick two cut-off functions on \( \mathbb{B}^m \):
- \( \chi \in D(B_2, [0, 1]) \) such that \( \chi|_{\mathbb{B}_1} \equiv 1 \). We write \( \chi_\kappa = \varphi_\kappa^* \chi \).
- \( \zeta \in D(B_4, [0, 1]) \) such that \( \zeta|_{\mathbb{B}_3} \equiv 1 \). We write \( \zeta_\kappa = \varphi_\kappa^* \zeta \).

We define a rescaled translation on \( \mathbb{B}^m \) for any \( \mu \in \mathbb{B}(0, r) \subset \mathbb{R}^m \) with \( r \) sufficiently small:
\[
\theta_\mu(x) := x + \chi(x) \mu, \quad x \in \mathbb{B}^m.
\]

This localization technique in Euclidean spaces was first introduced in [8] by J. Escher, J. Prüss and G. Simonett to establish regularity for solutions to parabolic and elliptic equations.

Given a function \( v \in L_{1,loc}(\mathbb{B}^m) \), its pull-back and push-forward induced by \( \theta_\mu \) are defined as:
\[
\theta_\mu^* v := v \circ \theta_\mu \quad \text{and} \quad \theta_\mu v := v \circ \theta_\mu^{-1}.
\]

The diffeomorphism \( \theta_\mu \) induces a transformation \( \Theta_\mu \) on \( M \) by:
\[
\Theta_\mu(q) = \begin{cases} 
\psi_1(\theta_\mu(\varphi_1(q))) & q \in O_1, \\
q & q \notin O_1.
\end{cases}
\]
It can be shown that $\Theta_\mu \in \text{Diff}^\infty(M)$ for $\mu \in B(0,r)$ with sufficiently small $r > 0$. See [23] for details.

For any $u \in L^1_{\text{loc}}(M)$, we can define its pull-back and push-forward induced by $\Theta_\mu$ analogously as:

$$\Theta^*_\mu u := u \circ \Theta_\mu \quad \text{and} \quad \Theta^{\mu} u := u \circ \Theta_\mu^{-1}. $$

We may find an explicit global expression for the transformation $\Theta^*_\mu$ on $M$,

$$\Theta^*_\mu u = \varphi^*_\mu \theta^*_\mu \psi^*_\mu (\zeta_1 u) + (1 - \zeta_1) u. $$

Here and in the following it is understood that a partially defined and compactly supported function is automatically extended over the whole base manifold by identifying it to be zero outside its original domain.

Likewise, we can express $\Theta^{\mu}$ as

$$\Theta^{\mu} u = \varphi^{\mu} \theta^{\mu} \psi^{\mu} (\zeta_1 u) + (1 - \zeta_1) u. $$

Let $I = [0, T], T > 0$. Assuming that $J \subset (0, T)$ is an open interval and $t_0 \in J$ is a fixed point, we choose $\varepsilon_0$ to be so small that $B(t_0, 3\varepsilon_0) \subset J$. Next we pick another auxiliary function

$$\xi \in D(B(t_0, 2\varepsilon_0), [0, 1]) \quad \text{with} \quad \xi|_{B(t_0, \varepsilon_0)} \equiv 1. $$

The above construction now engenders a parameter-dependent transformation in terms of the time variable:

$$\varrho_\lambda(t) := t + \xi(t) \lambda, \quad \text{for any} \ t \in I \text{ and } \lambda \in \mathbb{R}. $$

Now we are in a situation to define a family of parameter-dependent transformations on $I \times M$. Given a function $u : I \times M \to \mathbb{R}$, we set

$$u_{\lambda, \mu}(t, \cdot) := \Theta^*_{\lambda, \mu} u(t, \cdot) := T^*_\mu(t) \varrho^*_\lambda u(t, \cdot), $$

where $T^*_\mu(t) = \Theta^*_{\xi(t)\mu}$ and $(\lambda, \mu) \in B(0,r)$.

It is important to note that $u_{\lambda, \mu}(0, \cdot) = u(0, \cdot)$ for any $(\lambda, \mu) \in B(0,r)$ and any function $u$.

The importance of this family of parameter-dependent diffeomorphisms lies in the following theorems. Their proofs as well as additional properties of this technique can be found in [23].

**Theorem 3.1.** Let $k \in \mathbb{N}_0 \cup \{\infty, \omega\}$. Suppose that $u \in C(I \times M)$. Then we have that $u \in C^k(I \times M)$ iff for any $(t_0, p) \in I \times M$, there exists $r = r(t_0, p) > 0$ and a corresponding family of parameter-dependent diffeomorphisms $\Theta^*_{\lambda, \mu}$ such that

$$[\lambda, \mu] \mapsto \Theta^*_{\lambda, \mu} u \in C^k(B(0, r), C(I \times M)). $$

Here $\omega$ is the symbol for real analyticity.
Proposition 3.2. Suppose that \( u \in E_1(I) \). Then \( u_{\lambda, \mu} \in E_1(I) \), and
\[
\partial_t [u_{\lambda, \mu}] = (1 + \xi^t) \Theta_{\lambda, \mu}^* u_t + B_{\lambda, \mu}(u_{\lambda, \mu}),
\]
where
\[
[(\lambda, \mu) \mapsto B_{\lambda, \mu}] \in C^\omega(\mathbb{B}(0, r), C(I, \mathcal{L}(E_1, E_0))).
\]
Furthermore, \( B_{\lambda, 0} = 0 \).

Proposition 3.3. Let \( s \in [0, h] \) and \( l \in \mathbb{N}_0 \). Suppose that \( A \) is a differential operator of order \( l \) with continuous coefficients on \( M \) satisfying \( a_\alpha^k \in BC^h(\mathbb{B}^m) \) and \( a_1^1 \in BC^h(\mathbb{B}^m) \cap C^\omega(\mathcal{O}) \) for some open subset \( \mathcal{O} \) such that \( B_3 \subset \subset \mathcal{O} \subset \subset \mathbb{B}^m \). Then
\[
[\mu \mapsto T_\mu A T_\mu^{-1}] \in C^\omega(\mathbb{B}(0, r), C(I, \mathcal{L}(h^{s+l}(M), h^s(M)))).
\]

Proposition 3.4. Let \( s \geq 0 \). Suppose that \( u \in C^\omega(\psi_1(\mathcal{O})) \cap h^s(M) \), where \( \mathcal{O} \) is defined in Proposition 3.3. Then
\[
[\mu \mapsto T_\mu u] \in C^\omega(\mathbb{B}(0, r), C(I, h^s(M))).
\]

4. Real Analyticity

By setting \( G(\rho) := P(\rho)\rho - F(\rho) \), we may rewrite equation (2.9) as
\[
\begin{cases}
\rho_t + G(\rho) = 0, \\
(\rho(0) = \rho_0).
\end{cases}
\]

Theorem 4.1. Let \( 0 < \alpha < 1 \). Suppose that \( \rho_0 \in h^{2+\alpha}(M) \). Then equation (4.1) has a unique local solution \( \rho \) in the interval of maximal existence \( \tilde{J}(\rho_0) \) such that \( \rho \in C^\omega(\tilde{J}(\rho_0) \times M) \).

Proof. I will indicate herein all the key steps of the proof. More details can be found in [23].

For any \((t_0, p) \in \tilde{J}(\rho_0) \times M\) and sufficiently small \( r > 0 \), a family of parameter-dependent diffeomorphisms \( \Theta_{\lambda, \mu}^* \) can be defined for \((\lambda, \mu) \in \mathbb{B}(0, r) \). Henceforth, we always use the notation \( \rho \) exclusively for the solution to (2.9) and hence to (4.1). Set \( u := \rho_{\lambda, \mu} \). Then as a consequence of Proposition 3.2, \( u \) satisfies the equation
\[
\begin{align*}
u_t &= \partial_t [\rho_{\lambda, \mu}] = (1 + \xi^t) \Theta_{\lambda, \mu}^* \rho_t + B_{\lambda, \mu}(u) \\
&= -(1 + \xi^t) \Theta_{\lambda, \mu}^* G(\rho) + B_{\lambda, \mu}(u) \\
&= -(1 + \xi^t) T_\mu G(\rho_\lambda^* \rho) + B_{\lambda, \mu}(u) \\
&= -(1 + \xi^t) T_\mu G(T_\mu^{-1} u) + B_{\lambda, \mu}(u) := -H_{\lambda, \mu}(u).
\end{align*}
\]
Pick $I: [\varepsilon, T] \subset J(\rho_0)$ such that $t_0 \in I$ and $B(t_0, 3\varepsilon_0) \subset I$. Then we define $E_0(I)$ and $E_1(I)$ as in Section 1 by moving the initial point from 0 to $\varepsilon$. Set $E_0^a(I) := \{ v \in E_1(I) : \| v \|_\infty < a \}$, where $\| v \|_\infty := \sup_{(t, q) \in I \times M} |v(t, q)|$. For $\mathcal{A} \in \mathcal{H}(E_1, E_0)$, we say that $(E_0(I), E_1(I))$ is a pair of maximal regularity of $\mathcal{A}$, if

$$\left( \frac{d}{dt} + \mathcal{A}, \gamma_\varepsilon \right) \in \text{Isom}(E_1(I), E_0(I) \times E_1),$$

where $\gamma_\varepsilon$ is the evaluation map at $\varepsilon$, i.e., $\gamma_\varepsilon(u) = u(\varepsilon)$. Next we define

$$\Phi: E_0^a(I) \times \mathbb{B}(0, r) \rightarrow E_0(I) \times E_1 \text{ as } \Phi(v, (\lambda, \mu)) \mapsto \left( v_t + H_{\lambda, \mu}(v) \right).$$

Note that $\Phi(\rho_{\lambda, \mu}, (\lambda, \mu)) = \left( \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right)$ for any $(\lambda, \mu) \in \mathbb{B}(0, r)$.

(i) My first goal is to prove that $\Phi(\rho_{\lambda, \mu}, (\lambda, \mu)) = \left( \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right)$ for any $(\lambda, \mu) \in \mathbb{B}(0, r)$.

By Proposition 3.2, $B_{\lambda, \mu} \in C^\omega(\mathbb{B}(0, r), C(I, \mathcal{L}(E_1, E_0)))$. We define a bilinear and continuous map:

$$f: C(I, \mathcal{L}(E_1, E_0)) \times E_1(I) \rightarrow E_0(I), \quad (T(t), u(t)) \mapsto T(t)(u(t)).$$

Hence $\{(v, (\lambda, \mu)) \mapsto f(B_{\lambda, \mu}, v) = B_{\lambda, \mu}(v) \} \in C^\omega(E_0^a(I) \times \mathbb{B}(0, r), E_0(I))$.

On the other hand, let $\pi = \sum_{\eta \in \mathcal{C}(1)} \pi_\eta$, where

$$\mathcal{C}(1) := \{ \eta \in \Lambda : \text{supp}(\pi_\eta) \cap \text{supp}(\pi_1) \neq \emptyset \}.$$ We decompose $G$ into

$$G = \pi G + \sum_{\eta \notin \mathcal{C}(1)} \pi_\eta G.$$ According to our construction of $\Theta_\mu^*$ and of the localization system, we may assume that $\pi|_0 \equiv 1$, where $\Theta$ is defined in Proposition 3.3 with $m = 2$. See [11 Lemma 3.2] for details.

Taking into account (2.6), (2.7), and (2.8), in every local chart $(O_\kappa, \varphi_\kappa)$ and for any $v \in E_0^a(I)$, $G(v)$ can be expressed as

$$\beta^{2h(v)}P^G(v, \ldots, \partial_{ijkl}v) / \det(G^T(v))^{s_1} \det([\sigma(v)])^{s_2} Q^G(v),$$

where $h, s_1, s_2 \in \mathbb{N}$. $[\sigma(v)]$ is the matrix representation of the metric $\sigma(v)$. Here $\sigma(v)$ is defined in a similar manner to $\sigma(\rho)$ with $\rho$ replaced by $v$. Analogously, $G^T(v)$ is defined in a similar way to $G^T(\rho)$. Meanwhile, $P^G$ is a polynomial in $v$ and its derivatives up to fourth order with real analytic coefficients, and $Q^G$ is a polynomial in $v$ with real analytic coefficients. In particular, $\det([\sigma(v)])$ only involves first order derivatives of $v$. 


Therefore, \( \pi_G(v) \) can be decomposed globally into
\[
\frac{P_0 + P_1^1 v \cdots P_k^1 v + \cdots + P_n^1 v}{Q_0 + Q_1^1 v \cdots Q_k^1 v + \cdots + Q_n^1 v}
\]
where \( P_0, Q_0 \in C^\infty(M) \cap C^\omega(\psi_1(O)) \) and \( P_i^j \)'s are linear differential operators with continuous coefficients on \( M \) up to fourth order, and the \( Q_i^j \)'s are linear differential operators of order at most one with continuous coefficients on \( M \). Their coefficients in every local chart satisfy that \( a_{ij}^k \in BC^\infty(B^2) \) and \( a_{ij}^1 \in BC^\infty(B^2) \cap C^\omega(O) \). By Proposition 3.3, we deduce that
\[
[\mu \mapsto (T_\mu P_0, T_\mu Q_0)] \in C^\omega(B(0, r), C(I, E_1) \times C(I, E_1)).
\]
Analogously, it follows from Proposition 3.3 that
\[
[\mu \mapsto T_\mu P_i^j, T_\mu^{-1}] \in C^\omega(B(0, r), C(I, L(E_1, E_0)))
\]
and
\[
[\mu \mapsto T_\mu Q_i^j, T_\mu^{-1}] \in C^\omega(B(0, r), C(I, L(E_1, h^{3+a}(M))))
\]
Combining the above discussion with point-wise multiplication theorems on Riemannian manifolds, we infer that
\[
[\mu \mapsto T_\mu (\pi G) T_\mu^{-1} v] \in C^\omega(E_0(I) \times B(0, r), E_0(I)).
\]
Applying these arguments repeatedly to the other terms \( \pi^3_n G \), we conclude that
\[
\Phi \in C^\omega(E_0(I) \times B(0, r), E_0(I) \times E_1).
\]
(ii) Next we look at the Fréchet derivative of \( \Phi \) in the first component:
\[
D_1 \Phi(v, (\lambda, \mu))w = \left( w_t + (1 + \xi(\lambda)) T_\mu DG(T_\mu^{-1} v) T_\mu^{-1} w - B_{\lambda, \mu}(w) \right)_{\gamma_v w}
\]
Thus
\[
D_1 \Phi(\rho, (0, 0))w = \left( w_t + DG(\rho)w \right)_{\gamma_v w}.
\]
Observe that \( DG(\rho) \) is a fourth order linear differential operator whose coefficients satisfy \( a_{ij}^k \in E_0 \). The principal part of \( DG(\rho) \) in every local chart coincides with that of \( P(\rho) \), that is, \( P_0^0(\rho) \). By the discussion in Section 2, we know that \( DG(\rho(t, \cdot)) \) is a uniformly elliptic operator for every fixed \( t \geq 0 \). As a consequence of [23, Theorem 4.5, Proposition 4.7], it follows that \( (E_0(I), E_1(I)) \) is a pair of maximal regularity for \( DG(\rho(t, \cdot)) \).
We set \( \mathcal{A}(t) = DG(\rho(t, \cdot)) \). It follows that
\[
\frac{d}{dt} + A(s, \gamma_v) \in \text{Isom}(E_1(I), E_0(I) \times E_1), \quad \text{for every } s \in I.
\]
By [5, Lemma 2.8(a)], we have
\[
\frac{d}{dt} + A(\cdot, \gamma_v) \in \text{Isom}(E_1(I), E_0(I) \times E_1).
\]
Now we are in a position to apply the Implicit Function Theorem. It follows right away that there exists an open neighborhood, say \( B(0, r_0) \subset B(0, r) \), such that
\[
[(\lambda, \mu) \mapsto \rho_{\lambda, \mu}] \in C^\omega(B(0, r_0), E_1(I)).
\]
As a consequence of Theorem 3.1, we deduce that $\rho \in C^\omega(J(\rho_0) \times M)$. This completes the proof.

Proof of Theorem 1.1. For each $(t_0, q) \in \mathcal{M} = \bigcup_{t \in J(\rho_0)} \{t\} \times \Gamma(t)$, there exists a $p \in M$ such that $\Psi_\rho(t_0, p) = q$. Here $\Gamma(t) = \text{im}(\Psi_\rho(t, \cdot))$. Theorem 4.1 states that there exists a local patch $(\mathcal{O}_\kappa, \varphi_\kappa)$ such that $p \in \mathcal{O}_\kappa$ and $\rho \circ \psi_\kappa$ is real analytic in $J(\rho_0) \times \mathbb{B}^2$. Therefore, we conclude that

$$\left[(t, x) \mapsto (t, \psi_\kappa(x) + \rho(t, \psi_\kappa(x)))\nu_M(\psi_\kappa(x))\right] \in C^\omega(J(\rho_0) \times \mathbb{B}^2, \mathcal{M}).$$

This proves the assertion of Theorem 1.1.



References

[1] H. Amann, Function spaces on singular manifolds. Math. Nachr., 1-40 (2012) / DOI 10.1002/mana.201100157. arXiv:1106.2033.

[2] M. Bauer, E. Kuwert, Existence of minimizing Willmore surfaces of prescribed genus. Int. Math. Res. Not. 2003, no. 10, 553-576.

[3] R.L. Bryant, A duality theorem for Willmore surfaces. J. Differential Geom. 20 (1984), no. 1, 23-53.

[4] B.-Y. Chen, On a variational problem on hypersurfaces. J. London Math. Soc. (2) 6 (1973), 321-325.

[5] P. Clément, G. Simonett, Maximal regularity in continuous interpolation spaces and quasi-linear parabolic equations. J. Evol. Equ. 1 (2001), no. 1, 39-67.

[6] J. Escher, G. Simonett, A center manifold analysis for the Mullins-Sekerka Model. J. Differential Equations 143 (1998), no. 2, 267-292.

[7] J. Escher, U.F. Mayer, G. Simonett, The surface diffusion flow for immersed hypersurfaces. SIAM J. Math. Anal. 29 (1998), no. 6, 1419-1433.

[8] J. Escher, J. Prüss, G. Simonett, A new approach to the regularity of solutions for parabolic equations. Evolution equations, 167-190, Lecture Notes in Pure and Appl. Math., 234, Dekker, New York, 2003.

[9] C.M. Fernando, N. André Neves, Min-Max theory and the Willmore conjecture. arXiv:1202.6036.

[10] E.I. Hanzawa, Classical solution of the Stefan problem. Tôhoku Math. Jour. 33 (1981), 297-335.

[11] R. Kusner, Comparison surfaces for the Willmore problem. Pacific J. Math. 138 (1989), no. 2, 317-345.

[12] E. Kuwert, R. Schätzle, The Willmore flow with small initial energy. J. Differential Geom. 57 (2001), no. 3, 499-441.

[13] E. Kuwert, R. Schätzle, Gradient flow for the Willmore functional. Comm. Anal. Geom. 10 (2002), no. 2, 307-339.

[14] E. Kuwert, R. Schätzle, Removability of point singularities of Willmore surfaces. Ann. of Math. (2) 160 (2004), no. 1, 315-357.

[15] P. Li, S.-T. Yau, A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces. Invent. Math. 69 (1982), no. 2, 269-291.

[16] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems. Birkhäuser Verlag, Basel, 1995.

[17] U.F. Mayer, G. Simonett, Self-intersections for Willmore flow. Evolution equations: applications to physics, industry, life sciences and economics (Levico Terme, 2000), 341-348, Progr. Nonlinear Differential Equations Appl., 55, Birkhäuser, Basel, 2003.

[18] U.F. Mayer, G. Simonett, A numerical scheme for axisymmetric solutions of curvature-driven free boundary problems, with applications to the Willmore flow. Interfaces Free Bound. 4 (2002), no. 1, 89-109.
[19] U. Pinkall, *Hopf tori in $S^3*$. Invent. Math. 81 (1985), no. 2, 379-386.
[20] U. Pinkall, I. Sterling, *Willmore surfaces*. Math. Intelligencer 9 (1987), no. 2, 38-43.
[21] J. Prüss, G. Simonett, *On the manifold of closed hypersurfaces in $\mathbb{R}^n**. arXiv.
[22] M.U. Schmidt, *A proof of the Willmore conjecture*. arXiv:math/0203224.
[23] Y. Shao, *A family of parameter-dependent diffeomorphisms acting on function spaces over a Riemannian manifold and applications to geometric flows*. In preparation.
[24] L. Simon, *Existence of surfaces minimizing the Willmore functional*. Comm. Anal. Geom. 1 (1993), no. 2, 281-326.
[25] G. Simonett, *The Willmore flow near spheres*. Differential Integral Equations 14 (2001), no. 8, 1005-1014.
[26] J.L. Weiner, *On a problem of Chen, Willmore, et al*. Indiana Univ. Math. J. 27 (1978), no. 1, 19-35.
[27] T.J. Willmore, *Riemannian Geometry*. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.

Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA

E-mail address: yuanzhen.shao@vanderbilt.edu