LEVEL SPACINGS AND NODAL SETS AT INFINITY FOR RADIAL PERTURBATIONS OF THE HARMONIC OSCILLATOR

THOMAS BECK AND BORIS HANIN

ABSTRACT. We study properties of the nodal sets of high frequency eigenfunctions and quasimodes for radial perturbations of the Harmonic Oscillator. In particular, we consider nodal sets on spheres of large radius (in the classically forbidden region) for quasimodes with energies lying in intervals around a fixed energy $E$. For well chosen intervals we show that these nodal sets exhibit quantitatively different behavior compared to those of the unperturbed Harmonic Oscillator. These energy intervals are defined via a careful analysis of the eigenvalue spacings for the perturbed operator, based on analytic perturbation theory and linearization formulas for Laguerre polynomials.

1. Introduction

In this article we obtain new information about nodal (i.e. zero) sets of high frequency eigenfunctions and eigenvalue spacings for semi-classical Schrödinger operators that are small radial perturbations of the isotropic Harmonic Oscillator:

$$P_h(\varepsilon) := H_0 + \varepsilon h V(|x|^2), \quad H_0 := -\frac{h^2}{2} \Delta_{R^d} + \frac{|x|^2}{2}, \quad d \geq 2.$$ (1)

See [4] for the assumptions we place on $V$. Although much is known about nodal sets of eigenfunctions of the Laplacian on a compact manifold, comparatively little has been proved about nodal sets of eigenfunctions of Schrödinger operators $-\frac{h^2}{2} \Delta + V(x)$, even on $\mathbb{R}^d$. When $V(x) \to +\infty$ as $|x| \to \infty$, such operators have a discrete spectrum and a complete eigenbasis for $L^2(\mathbb{R}^d, dx)$. Fixing an energy $E > 0$, and letting $h \to 0$, any energy $E$ eigenfunction $\psi_{h,E}$ of $-\frac{h^2}{2} \Delta + V(x)$ is rapidly oscillating (with frequency $h^{-1}$) in the classically allowed region

$$A_E := \{ x \in \mathbb{R}^d \mid V(x) \leq E \}$$

and exponentially decaying in the classically forbidden region

$$F_E := \{ x \in \mathbb{R}^d \mid V(x) > E \}.$$ The nodal set of $\psi_{h,E}$ undergoes a qualitative change as it crosses from $A_E$ to $F_E$. This transition is illustrated in Figure 1.

In $A_E$, the eigenfunction $\psi_{h,E}$ behaves much like an eigenfunction of the Laplacian. For instance, if $V$ is real analytic, then Jin [19] proved that for any bounded open $B \subseteq A_E$ there exists $c, C > 0$ such that

$$c h^{-1} \text{vol}(B) \leq \mathcal{H}^{d-1}(\{ \psi_{h,E} = 0 \} \cap B) \leq C h^{-1} \text{vol}(B).$$ (2)
Here and throughout, $\mathcal{H}^k$ denotes $k-$dimensional Hausdorff measure. The same estimates were proved for compact real analytic Riemannian manifolds by Donnelly-Fefferman \[7\] for eigenfunctions of the Laplacian. Except when $d = 1$, when $\psi_{h,E}$ has no zeros in $\mathcal{F}_E$, much less is known about the nodal set of $\psi_{h,E}$ in $\mathcal{F}_E$. Jin established that his upper bound in \[2\] continues to hold in the forbidden region.

Aside from this, we are aware of only several strands of prior work on the subject. The oldest are the articles of Hoffman-Ostenhof \[15, 16\] and Hoffman-Ostenhoff-Swetina \[17, 18\] that study nodal for potentials that vanish at infinity. They show that the nodal set of an eigenfunctions on the sphere at infinity looks locally like the nodal set of a Hermite polynomial. There is also the paper of Canzani-Toth \[6\] about the persistence of forbidden hypersurfaces in nodal sets of Schrödinger eigenfunctions on a compact manifold and the articles of Bérard-Helfer \[2, 3, 4\] on nodal domains for eigenfunctions of the harmonic oscillator and similar operators (mainly in the allowed region). Finally, we mention the articles of Hanin-Zelditch-Zhou \[10, 11\], which study the typical size of the nodal set in $\mathcal{F}_E$ and near the caustic $\partial \mathcal{A}_E = \{|x|^2 = 2E\}$ for random fixed energy eigenfunctions of $\text{HO}_h$. We also refer the reader to the interesting heuristic physics paper of Bies-Heller \[5\].

In particular, in \[10\] it is shown that for every bounded $B \subseteq \mathcal{F}_E$ there exists $C > 0$ depending only on the minimum and maximum distance from a point in $B$ to $\mathcal{A}_E$ so that

$$E \left[ \mathcal{H}^{d-1} (\{\psi_{h,E} = 0\} \cap B) \right] = Ch^{-1/2} \text{vol}(B) (1 + O(h)) .$$

(3)
While the typical nodal density for $\psi_{h,E}$ in $F_E$ is therefore $h^{-1/2}$, there are no matching deterministic upper and lower bounds. Indeed, for every bounded open $B \subseteq F_E$

$$\inf_{\psi_{h,E} \in \ker(H_0 - E)} \mathcal{H}^{d-1} \left( \{ \psi_{h,E} = 0 \} \cap B \right) = 0$$

$$\sup_{\psi_{h,E} \in \ker(H_0 - E)} \mathcal{H}^{d-1} \left( \{ \psi_{h,E} = 0 \} \cap B \right) = C h^{-1} \text{vol}(B).$$

The infimum is attained when $\psi_{h,E}$ is the unique radial eigenfunction of $H_0$ with given energy $E$, which has no nodal set whatsoever in $F_E$, and the supremum is attained when $\psi_{h,E}$ is any of the purely angular eigenfunctions, which are eigenfunctions of the Laplacian on $S^{d-1}$ of frequency $\approx h^{-1}$.

The difference in the exponents in the various estimates above raises the question of what happens to the nodal sets of eigenfunctions for other Schrödinger operators. We take up this question in the present article for the small radial perturbations $P_{\epsilon}(1)$ of the harmonic oscillator. We are concerned primarily with the behavior of nodal sets on the sphere at infinity for eigenfunctions of $P_{\epsilon}(1)$ with approximately the same energy. Our main results in this direction are Theorems 3 and 4, which establish upper and lower bounds on the size of the nodal set of both eigenfunctions and certain quasi-modes near a fixed energy $E$.

Since $P_{\epsilon}(1)$ is rotationally symmetric for all $\epsilon$, its eigenfunctions can be obtained by separating variables (see (5) and (12)). The radial parts of these separation of variables eigenfunctions are deformations in $\epsilon$ of the Laguerre functions (25), while the angular parts are the eigenfunctions of the Laplacian on the round sphere $S^{d-1}$. At a fixed energy $E$, all such products have the same rate of growth at infinity when $\epsilon = 0$ (see (27) in §4.1), and hence spherical harmonics of many different angular momenta may contribute to the nodal set of eigenfunctions at infinity.

However, for $\epsilon \neq 0$, the energies $E_{\ell,n}^V(\epsilon)$, defined in (5), for different angular momenta $\ell$ will no longer be the same (Theorem 1). Hence, since the rate of growth at infinity of the radial eigenfunctions is an increasing function of $E_{\ell,n}^V(\epsilon)$ (Proposition 2), we see that the nodal sets at infinity of energy $\approx E$ eigenfunctions and quasimodes for $P_{\epsilon}(1)$ depend on the level spacings of the perturbed energies $E_{\ell,n}^V(\epsilon)$ for various angular momenta $\ell$. We obtain precise information on these level spacings, for what we call slowly-varying potentials $V$, in Theorem 4 which is our main technical result.

2. Statement of Results

Theorem 1 concerns the eigenvalue spacings for $P_{\epsilon}(1)$. It holds for $V$ that satisfy

$$V \in C^\infty(\mathbb{R}_+, \mathbb{R}), \quad \limsup_{|x| \to \infty} |x|^{-\eta} V(|x|^2) \leq C, \quad V(0) = V'(0) = 0$$

for some $\eta > 0$ and are slowly varying in the sense of Definition 1 below. The last assumption in (4) is only a matter of convenience since $V(0)$ (resp. $V'(0)$) can be absorbed as shifts (resp. scalings) of the spectrum of $P_{\epsilon}(0) = H_0$. 

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Since $P_h(\varepsilon)$ is rotationally symmetric for all $\varepsilon$ its spectrum can be decomposed as a union (with multiplicity):

$$
\text{Spec } (P_h(\varepsilon)) = \bigcup_{\ell \geq 0} \text{Spec } (P_{h,\ell}(\varepsilon)),
$$

where $P_{h,\ell}(\varepsilon) := P_h(\varepsilon)|_{L^2_\ell}$ is the restriction of $P_h(\varepsilon)$ to functions with fixed angular momentum:

$$
L^2_\ell = L^2(\mathbb{R}^+, r^{d-1}dr) \otimes \ker (\Delta_{S^{d-1}} + \ell (\ell + d - 2)), \quad L^2(\mathbb{R}^d, dx) = \bigoplus_{\ell \geq 0} L^2_\ell.
$$

In the previous line, $\Delta_{S^{d-1}}$ is the Laplacian for the round metric on $S^{d-1}$, whose spectrum is $\{ -\ell (\ell + d - 2) \}_{\ell \geq 0}$.

The spectrum

$$
\text{Spec } (P_{h,\ell}(\varepsilon)) = \{ E^V_{\ell,n}(\varepsilon) \}_{n \geq \ell, n \equiv \ell (\mod 2)}, \quad E^V_{\ell,n}(0) = h \left( n + \frac{d}{2} \right)
$$

of the radial operator for each angular momentum $\ell$ is simple for small $\varepsilon$ since it is an analytic perturbation of the simple spectrum

$$
\text{Spec } (P_{h,\ell}(0)) = \{ h (n + d/2) \}_{n \geq \ell, n \equiv \ell (\mod 2)}.
$$

Let us fix $E > 0$ and define

$$
h_n := \frac{E}{n + \frac{d}{2}}, \quad n \in \mathbb{N}.
$$

At $\varepsilon = 0$, the spectra of the radial oscillators $P_{h_n,\ell}(0)$ overlap and contain the same energy

$$
E = E^V_{\ell,n}(0)
$$

for all $\ell \leq n$ congruent to $n$ modulo 2. However, for small $\varepsilon$ and generic $V$, we expect the spectra of $P_{h_n,\ell}(\varepsilon)$ will be disjoint for various $\ell$. Although we do not have a proof of this fact, Theorem 1 implies that the eigenspaces of $P_{h_n,\ell}(\varepsilon)$ will have bounded multiplicity uniformly in $n, \ell$ (see (11)). Theorem 1 concerns the relative positions of the perturbations $E^V_{\ell,n}(\varepsilon)$ of $E$ as a function of $\ell$.

**Definition 1.** Let $E, \delta > 0$. We say a potential $V \in C^\infty(\mathbb{R}^+, \mathbb{R})$ is $\delta$-slowly varying in the allowed region for energy $2E$ if it satisfies (4), the condition $\|V\|_{L^\infty} \leq 1$, and

$$
\frac{\delta^2}{2} \leq |V''(0)| \leq \delta^2 \quad \text{and} \quad \sup_{r \in [0, \sqrt{4E}]} \left| \frac{V^{(k)}(r)}{k!} \right| \leq \delta^k
$$

for all $k \geq 3$.

**Theorem 1.** There exist constants $C_1, C_2 > 0$ with the following property. Suppose $E > 0$, $\delta \in (0, (C_1 E)^{-1})$ and $V$ is $\delta$-slowly varying in the allowed region for energy $2E$. Then, for all

$$
n \text{ s.t. } h_n < 1, \quad \varepsilon \in [0, 1/5), \quad \ell \leq n, \ell \equiv n (\mod 2) \quad (8)
$$

we have

$$
E^V_{\ell,n}(\varepsilon) = E + \varepsilon h_n V''(0) \left( \frac{E}{2} - \frac{d}{4} \right) \left[ 3 + \frac{\ell^2}{n^2} (-1 + S(\ell, n, \varepsilon)) + T(n, \varepsilon) \right] + O(h_n^\infty). \quad (9)
$$
Proposition 2. Fix sense. The energy \( E \) of \( \psi \) is eigenfunction equation \( \Delta \delta = 1/\delta \). Then, there exists a finite constant \( C \) such that

\[
|A_n| \leq C_N h_n^\gamma
\]

for all \( n \geq 1 \).

Remark 1. Here and throughout, a quantity \( A_n \) is \( O(h_n^\infty) \) if, for each \( \gamma \geq 1 \), there exists a constant \( C_\gamma \) such that

\[
|A_n| \leq C_\gamma h_n^\gamma
\]

Theorem 1 shows that \( E_{\ell,n}(\varepsilon) - E \) is essentially a monotone function of \( \ell \) if \( \delta \) and \( \varepsilon \) are sufficiently small. More precisely, if \( \max\{|\delta,\varepsilon| < (2C_1)^{-1} \) then

\[
\ell' > \frac{\ell}{1 - 2C_2 \max\{|\delta,\varepsilon|}
\Rightarrow \quad \text{sgn}(\Delta_{\ell,n}(\varepsilon)) \left( E_{\ell,n}(\varepsilon) - E_{\ell',n}(\varepsilon) \right) > 0.
\]

2.1. Nodal Sets of Eigenfunctions for \( \text{P}_n(\varepsilon) \). In this section, we state our results on nodal sets. We define \( U_{\ell,n}(\varepsilon) \) to be the span of the eigenfunctions of \( \text{P}_n(\varepsilon) \) of energy \( E_{\ell,n}(\varepsilon) \). The vector spaces \( U_{\ell,n}(\varepsilon) \) have multiplicity bounded independent of \( n, \ell \) (see (11)). Setting \( x \in \mathbb{R}^d \mapsto (r, \omega) \) to be the polar decomposition, \( U_{\ell,n}(\varepsilon) \) is spanned by functions of the form

\[
v_{\ell,n,m}(\varepsilon, x) = \psi_{\ell,n}(\varepsilon, r)Y_{m}^\ell(\omega), \quad 1 \leq m \leq D_d, \ell,
\]

where \( D_d,\ell = \dim(\text{ker}(\Delta_{S^{d-1}} + (\ell + 2d - 1))) \) and the spherical harmonics \( Y_m^\ell(\omega) \) are an ONB for the \( -\ell(\ell + d - 2) \) eigenspace of the Laplacian on \( S^{d-1} \):

\[
\text{ker}(\Delta_{S^{d-1}} + (\ell + 2d - 1)) = \text{Span}\{Y_m^\ell, m = 1, \ldots, D_d,\ell\}.
\]

The function \( \psi_{\ell,n}(\varepsilon, r) \) is the unique, tempered, \( L^2(r^d dr) \)-normalized solution to the eigenfunction equation

\[
\text{P}_n(\varepsilon)\psi_{\ell,n}(\varepsilon) - E_{\ell,n}(\varepsilon)\psi_{\ell,n} = 0.
\]

The energy \( E_{\ell,n}(\varepsilon) \) controls the rate of growth of \( v_{\ell,n,m}(\varepsilon, x) \) for large \( |x| \) in the following sense.

Proposition 2. Fix \( \eta > 0 \), and let \( V \in C^\infty(\mathbb{R}_+; \mathbb{R}) \), such that

\[
\lim_{r \to \infty} r^n V(r^2) < \infty.
\]

Then, there exists a finite constant \( C_{\ell,n,\varepsilon} \neq 0 \) such that

\[
\lim_{r \to \infty} \frac{d^j}{dr^j} \left( r^N e^{\frac{r^2}{2N}} \psi_{\ell,n}(\varepsilon, r) \right) = C_{\ell,n,\varepsilon} \cdot \delta_{0,j},
\]

where \( N = \frac{1}{h_n} E_{\ell,n}(\varepsilon) - \frac{d}{4} \).

Proposition 2 is essentially a classical result (see §§3.1-3.4 in [8]). We give a brief derivation in §3.1. Our next result concerns the nodal sets of eigenfunctions of \( \text{P}_n(\varepsilon) \) whose eigenvalue \( E_{\ell,n}(\varepsilon) \) nearly extremizes the distance to \( E = E_{\ell,n}(0) \), and hence, by Theorem 1, are close to

\[
E_{0,n}(\varepsilon) = E + \left( \frac{E}{2} - \frac{h_n d}{4} \right)^2 \varepsilon h_n V''(0)(3 + O(\max\{\varepsilon, \delta\})).
\]
when \( h_n, \varepsilon \) and \( \delta \) are small. Define
\[
I_{\varepsilon, \gamma, h_n} = [E^V_{0,n}(\varepsilon) - h_n^{1+2\gamma}, E^V_{0,n}(\varepsilon) + h_n^{1+2\gamma}],
\]
for some \( 0 \leq \gamma \leq 1 \), and consider the span of the corresponding eigenfunctions
\[
V_{\varepsilon, \gamma, h_n} = \text{span}\{U_{\ell,n}(\varepsilon) : E^V_{\ell,n}(\varepsilon) \in I_{\varepsilon, \gamma, h_n}\}.
\]
The following concerns the nodal sets of functions in \( V_{\varepsilon, \gamma, h_n} \).

**Theorem 3.** Under the assumptions of Theorem 1, there exists \( \varepsilon^*, \delta^*, h^* > 0 \) such that for every \( \delta < \delta^* \), \( \varepsilon \in [0, \varepsilon^*] \), \( h_n < h^* \), and \( \gamma \in [0,1] \), we have
\[
\inf_{v \in V_{\varepsilon, \gamma, h_n}} \limsup_{R \to \infty} \mathcal{H}^{d-2}(\{v = 0\} \cap S^{d-1}_R) = 0,
\]
and there exist absolute constants \( c, C > 0 \) such that
\[
ch_n^{-1+\gamma} \leq \sup_{v \in V_{\varepsilon, \gamma, h_n}} \limsup_{R \to \infty} \mathcal{H}^{d-2}(\{v = 0\} \cap S^{d-1}_R) \leq Ch_n^{-1+\gamma}.
\]
Here \( S^{d-1}_R \) is the \( d-1 \)-dimensional sphere of radius \( R \) centred at the origin, and \( \mathcal{H}^{d-2} \) is the Haar (probability) measure on \( S^{d-1}_R \).

In the case where the energies \( E^V_{\ell,n}(\varepsilon) \) are distinct, we can conclude additional properties of the nodal sets.

**Theorem 4.** Under the assumptions of Theorem 1, and the additional assumption that the energies \( E^V_{\ell,n}(\varepsilon) \) are distinct, we have the following: For each \( v \in V_{\varepsilon, \gamma, h_n} \),
\[
\lim_{R \to \infty} \mathcal{H}^{d-2}(\{v(x) = 0\} \cap S^{d-1}_R)
\]
exists, and there exists absolute constants \( c, C > 0 \) such that for every \( v \in V_{\varepsilon, \gamma, h_n} \) in the complement of a co-dimension 1 subspace, we have
\[
\lim_{R \to \infty} \mathcal{H}^{d-2}(\{v(x) = 0\} \cap S^{d-1}_R) = 0 \quad \text{if } V''(0) > 0,
\]
\[
ch_n^{-1+\gamma} \leq \lim_{R \to \infty} \mathcal{H}^{d-2}(\{v(x) = 0\} \cap S^{d-1}_R) \leq Ch_n^{-1+\gamma} \quad \text{if } V''(0) < 0.
\]

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3. **Proof of Theorems 3 and 4**

We now explain how to derive Theorems 3 and 4 from Theorem 1 and Proposition 2. We then prove Proposition 2 in §3.1 below and Theorem 1 in §5. Our derivation relies on several well-known properties of the spherical harmonics \( Y^\ell_m(\omega) \). The first is an estimate on the measure of the total nodal set: Since for each \( 1 \leq m \leq D_{d,\ell} \),
$Y_m^\ell(\omega)$ is an eigenfunction on the round sphere $S^{d-1}$ with eigenvalue $-\ell(\ell + d - 2)$, the Donnelly-Fefferman bounds [7] show that there exist constants $c,C > 0$ such that

$$c\ell \leq \mathcal{H}^{d-2}\left( \left\{ \omega \in S^{d-1} : \sum_{1 \leq m \leq D_{d,\ell}} a_m Y_m^\ell(\omega) = 0 \right\} \right) \leq C\ell. \quad (16)$$

In particular, since $E_{0,n}^V(\varepsilon) \in I_{\varepsilon,\gamma,h_n}$ by construction for all $0 \leq \gamma \leq 1$, this immediately gives the first statement of Theorem 3. Moreover, by a simple application of the Crofton formula (see for example Theorem [9]), the upper bound in (16) holds also for (non-identically zero) linear combinations of spherical harmonics up to frequency $\ell$:

$$\mathcal{H}^{d-2}\left( \left\{ \omega \in S^{d-1} : \sum_{s \leq \ell, 1 \leq m \leq D_{d,\ell}} a_{s,m} Y_m^s(\omega) = 0 \right\} \right) \leq C\ell. \quad (17)$$

By the approximate monotonicity of the energies $E_{\ell,n}^V(\varepsilon)$ from Theorem 1, provided $\varepsilon,\delta,h_n$ are sufficiently small, the largest value of $\ell$ for which $E_{\ell,n}^V(\varepsilon) \in I_{\varepsilon,\gamma,h_n}$ is bounded above and below by a constant multiplied by $h_n^{-1+\gamma}$. The second statement in Theorem 3 then follows from the lower bound in (16) and the upper bound in (17).

To prove Theorem 4 we have the extra assumption that the energies $E_{\ell,n}^V(\varepsilon)$ are distinct. In this case, by Proposition 2 the radial part of the eigenfunctions, $\psi_{\ell,n}(\varepsilon,r)$, grows at different rates as $r \to \infty$ for different values of $\ell$. Given $\psi(x) \in V_{\varepsilon,\gamma,h_n}$, we can write

$$v(x) = \sum_{\ell \in J_{\varepsilon,\gamma,h_n}} \sum_{1 \leq m \leq D_{d,\ell}} a_{\ell,m} \psi_{\ell,n}(\varepsilon,r) Y_m^\ell(\omega).$$

Among the values of $\ell$ for which $a_{\ell,m} \neq 0$ for some $m$, let $\ell^* \in J_{\varepsilon,\gamma,h_n}$ correspond to the largest energy $E_{\ell,n}^V(\varepsilon)$. Then, by Proposition 2 the function $r^{N^*} e^{2\pi m} v(r,\omega)$ converges in $C^\infty(S^{n-1})$ to

$$C_{\ell^*,n,\varepsilon} \sum_{1 \leq m \leq D_{d,\ell^*}} a_{\ell^*,m} Y_m^{\ell^*}(\omega) \quad (18)$$

as $r \to \infty$, where $N^* = \frac{1}{m_n} E_{\ell^*,n}^V(\varepsilon) - \frac{d}{2}$. Moreover, the function in (18) has co-dimension 2 singular set in $S^{d-1}$ (see e.g. [12] [13]), and so by Corollary 2 in [1] we have the convergence of the nodal set measures,

$$\lim_{R \to \infty} \mathcal{H}_R^{d-2}\left( \{ v(x) = 0 \} \cap S^{d-1}_R \right) = \mathcal{H}^{d-2}\left( \left\{ \omega \in S^{d-1} : \sum_{1 \leq m \leq D_{d,\ell^*}} a_{\ell^*,m} Y_m^{\ell^*}(\omega) = 0 \right\} \right). \quad (19)$$

By Theorem 1 provided $\varepsilon,\delta,h_n$ are sufficiently small, for almost every $v \in V_{\varepsilon,\gamma,h_n}$, $\ell^*$ is equal to 0 if $V''(0) > 0$, while $\ell^* = O(h_n^{-1+\gamma})$ if $V''(0) < 0$. Thus, (19) together with the Hausdorff measure estimates in (16) imply Theorem 4.
3.1. Proof of Proposition 2. Suppose that \( u_{\ell,n}(\varepsilon, r) \) satisfies the equation

\[
\left( -\frac{h_n^2}{2} \frac{d}{dr^2} + \frac{d - 1}{r} \frac{d}{dr} - \frac{\ell(\ell + d - 2)}{r^2} \right) + \frac{r^2}{2} + \varepsilon h_n V(r^2) - E_{\ell,n}(\varepsilon) \right) u_{\ell,n}(\varepsilon, r) = 0
\]

for \( r \) sufficiently large. Setting \( t = \frac{r^2}{2h_n} \), and \( z_{\ell,n}(\varepsilon, t) = t^{-N/2} e^t u_{\ell,n}(\varepsilon, r) \), with \( N = \frac{1}{h_n} E_{\ell,n}(\varepsilon) - \frac{d}{2} \), this equation becomes

\[
\left( \frac{d^2}{dt^2} + 2 \left( -1 + \frac{N/2 + d/4}{t} \right) \frac{d}{dt} + \frac{F_{\ell,n}(\varepsilon, t)}{t^2} \right) z_{\ell,n}(\varepsilon, t) = 0.
\]

Here the function \( F_{\ell,n}(\varepsilon, t) \) is given by

\[
F_{\ell,n}(\varepsilon, t) = -\varepsilon t V(2h_n t) - \frac{\ell(\ell + d - 2)}{4} + \frac{N}{2} (N/2 - 1) + \frac{Nd}{4},
\]

and so by the assumption on \( V(t) \) from (15),

\[
|F_{\ell,n}(\varepsilon, t)| \leq C_{\ell,n,\varepsilon} t^{1-n/2},
\]

for a constant \( C_{\ell,n,\varepsilon} \) for large \( t \). Then, for fixed \( \ell, n, \varepsilon \), Erdelyi [8] (with \( \omega = -1, \rho = -N/2 - d/4 \)) gives a solution \( z_{\ell,n}^{(1)}(\varepsilon, t) \) to (21) for \( t \geq t_0 \), under the assumption on \( F_{\ell,n}(\varepsilon, t) \) in (22), such that

\[
\limsup_{t \to \infty} t^q \left| z_{\ell,n}^{(1)}(\varepsilon, t) - 1 \right| < \infty.
\]

(Note that in [8], equation (7), the assumption placed on \( F_{\ell,n}(\varepsilon, t) \) is that it is bounded in \( x \), but the same proof works for the sub-linear growth from (22).)

Going back to the original function \( u_{\ell,n}(\varepsilon, r) \), we obtain a solution \( u_{\ell,n}^{(1)}(\varepsilon, r) \) to (20) for \( r \geq r_0 \), which is non-zero, and satisfies

\[
\lim_{t \to \infty} t^{N/2} e^t u_{\ell,n}^{(1)}(\varepsilon, \sqrt{2h_n} t) = 1, \quad \lim_{t \to \infty} \frac{d^j}{dt^j} \left( t^{N/2} e^t u_{\ell,n}^{(1)}(\varepsilon, \sqrt{2h_n} t) \right) = 0
\]

for \( N = \frac{1}{h_n} E_{\ell,n}(\varepsilon) - \frac{d}{2}, j \geq 1 \). To obtain another solution to (20) for large \( r \), we first set \( u_{\ell,n}^{(2)}(\varepsilon, r) = r^{-d/2} u_{\ell,n}^{(1)}(\varepsilon, r) \), to remove the coefficient of \( \frac{d}{dr} \) in (20). Then defining

\[
u_{\ell,n}^{(2)}(\varepsilon, r) = r^{-d/2} u_{\ell,n}^{(2)}(\varepsilon, r),
\]

where

\[
w_{\ell,n}^{(2)}(\varepsilon, r) = w_{\ell,n}^{(1)}(\varepsilon, r) \int_{r_0}^{r} w_{\ell,n}^{(1)}(\varepsilon, s)^{-2} ds,
\]

gives the other linearly independent solution to (20) for \( r \geq r_0 \). Since \( u_{\ell,n}^{(2)}(\varepsilon, r) \) grows exponentially as \( r \) tends to infinity, it is not \( L^2(r^{d-1} dr) \)-normalisable, and so our eigenfunction \( \psi_{\ell,n}(\varepsilon, r) \) must be proportional to \( u_{\ell,n}^{(1)}(\varepsilon, r) \) for \( r \geq r_0 \). The proposition then follows from the estimates in (24). □
4. Background to Proof of Theorem

4.1. Spectral Theory of $\text{HO}_\hbar$. The spectrum of the isotropic harmonic oscillator $\text{HO}_\hbar$ is

$$\text{Spec}(\text{HO}_\hbar) = \{\hbar (n + d/2)\}_{n \in \mathbb{N}}.$$ 

In this article, we will use repeatedly properties of the radial eigenfunctions of $\text{HO}_\hbar$, which we now recall. Recall from [13] the spectrum of the Laplacian on $S^{d-1}$ and the corresponding real-valued eigenfunctions $\{Y^\ell_\omega, m = 1, \ldots, D_{d,\ell}\}$.

A standard calculation shows that an ONB for $\ker(\text{HO}_\hbar - \hbar (n + d/2))$ is given by

$$\psi_{\hbar, \ell, n}(r) \cdot Y^\ell_\omega(\omega), \quad 0 \leq \ell \leq n, \ell \equiv n \pmod{2}, \quad m = 1, \ldots, D_{d,\ell},$$

where $x \in \mathbb{R}^d \mapsto (r, \omega)$ is the polar decomposition and

$$\psi_{\hbar, \ell, n}(r) = \hbar^{-\frac{\ell - d}{2}} N_{\ell, n} \cdot r^\ell e^{-r^2/2\hbar} L^{(\alpha)}_{\ell, n'}(r^2/\hbar), \quad N^{2}_{\ell, n} = \frac{2 \cdot \Gamma\left(\frac{n+\ell}{2} + 1\right)}{\Gamma\left(\frac{n+\ell+d}{2}\right)}.$$

In the above, we have set

$$n' = n - \frac{\ell}{2}, \quad \alpha = \ell + \frac{d - 2}{2},$$

and denoted by $L^{(\alpha)}_k$ the generalized Laguerre polynomials. We often fix $E > 0$ and define $\hbar = \hbar_n$ to be a function of $n$ and $E$ as in [6]. In this case, we abbreviate

$$\psi_{\ell, n} := \psi_{\hbar, \ell, n}.$$ 

As explained in the introduction, the energy $E$ determines a classically forbidden region $F_E = \{r^2 > 2E\}$, where the fixed energy eigenfunctions $\psi_{\hbar_n, \ell, m}$ for $m \approx n$ are uniformly exponentially small. More precisely, for any $\varepsilon > 0$ there exists $C > 0$ such that

$$\sup_{0 \leq \ell \leq m, |m-n| < \frac{\hbar}{2}, \ell \equiv m \pmod{2}, r \in [\sqrt{4E}, \infty)} \left| e^{(1-\varepsilon)r^2/2\hbar} \psi_{\hbar_n, \ell, m}(r) \right| \leq C.$$ 

(26)

Since $\lim_{x \to \infty} x^{-k} L^{(\alpha)}_k(x) = (-1)^k k!$, for each fixed $n$, the radial eigenfunctions $\psi_{\ell, n}$ differ at infinity only by a constant:

$$0 < \lim_{r \to \infty} \frac{\psi_{\ell, n}(r)}{h_n^{-\frac{\ell - d}{2}} \cdot r^\ell e^{-r^2/2\hbar}} = N_{\ell, n} < \infty \quad \forall 0 \leq \ell \leq n, \ell \equiv n \pmod{2}.$$ 

(27)

4.2. Linearization Formulas for Laguerre Functions. In order to perform perturbative calculations about $\text{Spec}(P_\hbar(\varepsilon))$, we will need a convenient expression for

$$J_{a,b,k}^{\alpha} := \int_0^\infty e^{-\rho} \rho^{a+k} L^{(\alpha)}_a(\rho) L^{(\alpha)}_b(\rho) d\rho,$$

where as in [25] $L^{(\alpha)}_a$, $L^{(\alpha)}_b$ are the generalized Laguerre polynomials.
Proposition 5 (Special Case of [20] Eqn. (2.5)). For any \( a, b, k \in \mathbb{N} \) and every \( \alpha > -1 \), we have
\[
J_{a,b,k}^\alpha = (\alpha + 1)_k \left( \frac{k}{a-b} \right) \left[ \frac{(a \vee b + 1)|a-b|}{(a \vee b + \alpha + 1)|a-b|} \right]^{1/2} \binom{3 \Gamma}{2} \left[ -k^{-1} \frac{k+1}{a-b+1} \alpha + 1 ; 1 \right].
\]
(28)

We have written \( a \vee b \) for the minimum of \( a, b \),
\[
(x)_q := \frac{\Gamma(x + q)}{\Gamma(x)}
\]
for the Pochhammer symbol and
\[
\binom{3 \Gamma}{2} \left[ a \ a \ a \ b \ b \ b \ ; 1 \right] = \sum_{q=0}^\infty \frac{(a_1)_q (a_2)_q (a_3)_q}{(b_1)_q (b_2)_q}
\]
for a hypergeometric function. In the case of (28), the sum terminates at \( q = a \vee b \) since \( (x)_q \) vanishes for \( x = -1, -2, \ldots \). The expression in (28) differs slightly from the one in [20] because our Laguerre functions are \( L^2 \)-normalized while the ones in [20] are not.

4.3. Analytic Perturbation Theory. We recall in this section several results from analytic perturbation theory. These results are classical, and we mainly follow the notes [21] of M. Taylor. Suppose that \( H \) is an unbounded self-adjoint operator on a Hilbert space \( \mathcal{H} \) with discrete spectrum \( \{ \lambda_j \}_{j=0}^\infty \) and corresponding eigenfunctions
\[
\mathcal{H}u_j = \lambda_j u_j.
\]
Suppose further that \( W \) is a bounded self-adjoint operator on \( \mathcal{H} \). Consider some \( \lambda = \lambda_n \in \text{Spec}(H) \), and write \( u = u_n \) for the corresponding eigenfunction. Then for all \( \varepsilon \) sufficiently small the operator
\[
H(\varepsilon) := H + \varepsilon W
\]
has a simple eigenvalue \( \lambda(\varepsilon) \) with
\[
(H + \varepsilon W) u(\varepsilon) = \lambda(\varepsilon) u(\varepsilon).
\]
Both \( \lambda(\varepsilon) \) and \( u(\varepsilon) \) are analytic in \( \varepsilon \). Explicitly, write
\[
\lambda(\varepsilon) = \lambda + \varepsilon \sum_{k \geq 0} \varepsilon^k \mu_k, \quad u(\varepsilon) = u + \varepsilon \sum_{k \geq 0} \varepsilon^k v_k,
\]
and impose the normalization
\[
(u(\varepsilon) - u) \perp u.
\]
We have the following recursive formulas for \( \mu_k, v_k \) for each \( k \geq 0 \)
\[
\begin{cases}
  w_k = \sum_{m=0}^{k-1} \mu_{k-m-1} v_m, & w_0 = 0 \\
  v_k = (H - \lambda)^{-1} \left[ \Pi^+_u (Wv_{k-1}) + w_k \right], & v_{-1} = u \\
  \mu_k = \langle Wv_{k-1}, u \rangle
\end{cases}
\]
(29)
The operator $\Pi_u^\perp$ is the projection onto the orthogonal complement of $u$. Using this recursion and integration by parts, we have for any $X \in \mathcal{H}$

$$\langle Wv_k, X \rangle = -\langle Wv_{k-1}, G(X) \rangle, \quad G = (H - \lambda)^{-1} \circ \Pi_u^\perp \circ W. \quad (30)$$

Writing $u = u_n$ and using (30), we find for $k \geq 0$

$$\mu_k = (-1)^k \langle Wu, G^{(k)}(u) \rangle. \quad (31)$$

Using the definition of $G$ we obtain

$$\mu_0 = \langle Wu, u \rangle$$

$$\mu_k = \sum_{m_1, \ldots, m_k \neq n} \langle Wu, u_{m_1} \rangle \prod_{i=1}^k \frac{\langle Wu_{m_i}, u_{m_{i+1}} \rangle}{\lambda_n - \lambda_{m_i}}, \quad k \geq 1, \quad (32)$$

with the convention that $u_{m_{k+1}} = u$. We will also need the following simple estimate.

**Lemma 6.** Suppose that $H$ not only has simple spectrum but also that the spacing between any two consecutive eigenvalues is bounded below by $\eta > 0$. Then, if $\|W\|_{L^\infty} \leq \eta$, 

$$\sup_{n \in \mathbb{N}} |\lambda_n(\epsilon) - \lambda_n(0)| < \eta/4. \quad (33)$$

**Proof.** Let $\mu_k$ given by (31). Since

$$\|G\|_{\mathcal{H} \to \mathcal{H}} \leq \eta^{-1} \|W\|_{L^\infty} \leq 1,$$

we conclude

$$|\mu_k| \leq 1.$$ 

Thus, for $\epsilon \in [0, \frac{1}{5}]$ we can write

$$\lambda_n(\epsilon) - \lambda_n(0) = \epsilon \sum_{k=0}^\infty \epsilon^k \mu_k,$$

and, in particular

$$|\lambda_n(\epsilon) - \lambda_n(0)| \leq \eta \cdot \frac{\epsilon}{1 - \epsilon} \leq \frac{\eta}{4}. \quad \square$$

5. **Proof of Theorem 1**

Throughout this section, we fix $E > 0$ and use the convention

$$\hbar = \hbar_n = \frac{E}{n + \frac{d}{2}}$$

as in §1. The proof of Theorem 1 consists of three steps, which we describe below.
5.1. **Step 1.** The first step is to replace both $V(r^2)$ and $E_{\ell,n}(\epsilon)$ by an $h$–dependent Taylor series around $r = 0$ and $\epsilon = 0$, respectively. More precisely, for each $K \in \mathbb{N}$, define

$$V_K(r) := \sum_{k=0}^{K} \frac{V^{(k)}(0)}{k!} r^{2k}.$$  

**Proposition 7.** There exists a constant $C_1 > 0$ with the following property. For all $E > 0$, any $\delta \in (0,(C_1E)^{-1})$, each $\delta$–slowly varying potential $V$, and every $n, \epsilon$ such that $h_n < 1$ and $\epsilon \in [0, 1/5]$, we have

$$\sup_{\ell \leq n, \ell \equiv n \pmod{2}} \left| E_{\ell,n}(\epsilon) - \sum_{j=0}^{J} \frac{(h_n \epsilon)^j}{j!} \frac{d^j}{d\epsilon^j} \right|_{\epsilon = 0} E_{\ell,n}(\epsilon) = O(h_n^{\infty})$$  

provided $K = K(n)$ and $J = J(n)$ satisfy

$$\limsup_{n \to \infty} \frac{K(n)}{\log n} = \limsup_{n \to \infty} \frac{J(n)}{\log n} = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{K(n)J(n)}{n} = 0. \quad (35)$$

The approximation $[34]$ is the source of the $O(h_n^{\infty})$ error in $[9]$. The function $E_{\ell,n}^{V_K}(\epsilon)$ whose jets appear in Proposition $[7]$ is formally defined in the same way as $E_{\ell,n}^{V}(\epsilon)$. However, note that $V_K$ is not a bounded operator on $L^2([0, \infty), r^{d-1}dr)$. It therefore does not strictly follow from the discussion in $[4.3]$ that these jets are well-defined. Nonetheless, we simply define these jets by $\mu_{\ell,n}^{V_K}(0) = E$ and for $j \geq 1$

$$\mu_{\ell,n}^{V_K}(j) := \frac{1}{j!} \frac{d^j}{d\epsilon^j} \left|_{\epsilon = 0} E_{\ell,n}^{V_K}(\epsilon) = (-1)^{j-1} \left\langle V_K \psi_{h_n,\ell,n}, G_{\ell,K}^{(j-1)}(\psi_{h_n,\ell,n}) \right\rangle \right. \quad (36)$$

where the inner product is in $L^2([0, \infty), r^{d-1}dr)$ and

$$G_{\ell,K} := (P_h - E_{\ell,n}^{V}(0))^{-1} \circ \Pi_{\psi_{h_n,\ell,n}} \circ V_K.$$  

The inner products on the right hand side of $[36]$ are finite provided $K(n)$ satisfies $[35]$ by the Agmon estimates $[26]$. We prove Proposition $7$ in $[7]$.

5.2. **Step 2.** The second step in the proof of Theorem $[1]$ is to write the derivatives of $E_{\ell,n}(\epsilon)$ at $\epsilon = 0$ that appear in Proposition $[7]$ in terms of hypergeometric functions and obtain their asymptotics. Unwinding the definition of $G_K$, using $[36]$, and recalling that the spectrum of $HO_h = P_h(0)$ has level spacings $h$, we may write $\mu_{\ell,n}^{V_K}(j)$ as

$$h_n^{1-j} \sum_{m_1, \ldots, m_{j-1} \neq n \atop |m_i - m_{i+1}| \leq 2K \atop |n - m_i| \leq 2K} \left( V_K \psi_{h_n,\ell,n}, \psi_{h_n,\ell,m_1} \right)^{j-1} \prod_{i=1}^{j-1} \frac{V_K \psi_{h_n,\ell,m_i}, \psi_{h_n,\ell,m_{i+1}}}{m_i - n} \quad (37)$$

with the convention that $m_{j+1} = n$. The restriction that $|m_i - m_{i+1}| \leq 2K$ comes from the binomial coefficient in $[42]$ below.
To state our next result, we augment the notation in §1.1 and write for each \( n \geq \ell \), \( \ell \equiv n (\text{mod } 2) \) and all \( s, t \geq \ell \) with \( s, t \equiv \ell (\text{mod } 2) \)

\[
s' := \frac{s - \ell}{2}, \quad t' := \frac{t - \ell}{2}, \quad \alpha := \ell + \frac{d - 2}{2}, \quad s' \vee t' := \min\{s', t'\}.
\]  

(38)

For \( n \in \mathbb{N} \), we will be interested in the values of \( s, t \), and \( \ell \) in the set

\[
U_{s,t,\ell}^{n} = \left\{ (s, t, \ell) \in \mathbb{N}^3 : s, t, \ell \equiv n (\text{mod } 2), \ell \leq n, |s - n| < \frac{n}{2}, |t - n| < \frac{n}{2} \right\}.
\]  

(39)

For each \( K, s, t \in \mathbb{N} \), we recall our assumptions \( V(0) = V'(0) = 0 \) and write

\[
\langle V_K \psi_{h_{n, \ell}, s} | \psi_{h_{n, \ell}, t} \rangle = \sum_{k=2}^{K} \frac{V(k)(0)}{k!} h_n^k A_{k, s, t, \ell}.
\]  

(40)

The following Proposition is proved in §6.

**Proposition 8.** There exist constants \( C_1, C_2 > 0 \) with the following property. For any \( E > 0 \), if \( \delta \in (0, (C_1 E)^{-1}) \) and \( V \) is a \( \delta \)-slowly varying potential in the allowed region for energy \( 2E \) (Definition 7), then for each

\[
k_0 \geq 2, \quad n \text{ s.t. } h_n < 1, \quad (s, t, \ell) \in U_{s,t,\ell}^{n},
\]

we have

\[
\left| \sum_{k \geq k_0} \frac{V(k)(0)}{k!} h_n^k A_{k, s, t, \ell} - T(n, s, t) \right| \leq C_2 \frac{1 + \ell^2}{(s \vee t)^2} e^{-|s - t| / (E \delta^{k_0})}.
\]  

(41)

Here, as usual \( K = K(n) \) satisfies (35), and \( T(n, s, t) \) is \( \ell \)-independent and satisfies

\[
\sup_{h_n < h_n^*; \quad (s, t, \ell) \in U_{s,t,\ell}^{n}} e^{-|s - t| / (E \delta^{k_0})} |T(n, s, t)| \leq (C_2 E \delta)^{k_0}.
\]

**Remark 2.** We will only use Proposition 8 for \( k_0 = 2, 3 \). Also, we will obtain the following exact formula for \( A_{k, s, t, \ell} \):

\[
A_{k, s, t, \ell} = (\alpha + 1)_k \left( \frac{k}{|s' - t'|} \right)^{1/2} - \frac{(s' \vee t' + 1)|s' - t'|}{(s' \vee t' + \alpha + 1)|s' - t'|} \right)^{1/2} 3F_2\left[\begin{array}{c} -k, k + 1 - (s' \vee t') \end{array} \mid \begin{array}{c} \alpha + 1, 1 \end{array}\right],
\]  

(42)

where the notation is from (38). In particular, \( A_{k, s, t, \ell} = 0 \) whenever \( |s' - t'| > k \).

5.3. **Step 3.** The final step in the proof of Theorem 1 is to observe that combining Proposition 8 with the expression for \( \frac{1}{J} \frac{d}{d\epsilon} \bigg|_{\epsilon=0} E_{K,n}^{V}(\epsilon) \) from (37) and (40), we obtain the following estimates.

**Proposition 9.** There exist constants \( C_1, C_2 > 0 \) with the following property. Fix \( E > 0 \) and \( \delta \in (0, (C_1 E)^{-1}) \) and a \( \delta \)-slowly varying potential \( V \). For \( n \in \mathbb{N} \), and \( K = K(n) \) and \( J = J(n) \) satisfying (35), we have

\[
\sum_{j=2}^{J} \frac{(h_n \epsilon)^j}{j!} \frac{d}{d\epsilon} \bigg|_{\epsilon=0} E_{K,n}^{V}(\epsilon) = T_K(n, \epsilon) + \frac{\ell^2}{n^2} S_K(\ell, n, \epsilon).
\]
Moreover, in the notation of Proposition 8, we have
\[ \max\{|T_K(n, \varepsilon)|, |S_K(\ell, n, \varepsilon)|\} \leq C_2 h_n \varepsilon^2 (E \delta)^2. \]

Moreover, in the notation of Proposition 8, we have
\[ \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} E_{\ell,n}^{V_n}(\varepsilon) = \langle V_n \psi_{n,\ell,n}, \psi_{n,\ell,n} \rangle = \left( h_n^2 \frac{V''(0)}{2} A_{2,n,\ell,n} + Y(n) + \frac{\ell^2}{n^2} X(\ell, n) \right), \]
with
\[ \max\{|X(\ell, n)|, |Y(n)|\} \leq C_2 (E \delta)^3 \]
for \( \ell, n, \varepsilon \) satisfying (43).

The proof of Theorem 1 is complete once we choose \( C_1 \) to be the maximum of the \( C_i \)'s that are provided by Propositions 7 and 9, use that
\[ h^2 A_{2,n,\ell,n} = 6 \left( \frac{h_n n}{2} \right)^2 \left( 1 - \frac{1}{3} \cdot \frac{\ell^2}{n^2} + \frac{2 - d - \ell}{3n} \cdot \frac{d}{n} + \frac{d(d + 2)}{6 n^2} \right), \]
and substitute the estimates from Proposition 9 into (34).

6. Proof of Proposition 8

Let us first derive (40) and (42). Recall from (25) that, as a function of the radial variable \( r = |x| \), the radial eigenfunctions of the unperturbed operator (\( \varepsilon = 0 \)) are
\[ \psi_{n,\ell,s}(r) = h_n^{\frac{\ell - d}{2}} N_{s,\ell,d} \cdot r^\ell e^{-r^2/2h} L_\ell^{(s,\ell)}(r^2/h_n), \quad N_{s,\ell,d}^2 = \frac{2 \cdot \Gamma(\frac{s-\ell+1}{2})}{\Gamma\left(\frac{s+\ell+d}{2}\right)}, \]
where \( L_k^{(\alpha)} \) are the generalized Laguerre polynomials. Hence, for \( \alpha = \ell + \frac{d-2}{2} \), we have
\[ \langle V_n \psi_{n,\ell,s}, \psi_{n,\ell,t} \rangle = \frac{N_{s,\ell,d} N_{t,\ell,d}}{2} \int_0^\infty V_n(\sqrt{h_n} \rho) \rho^\alpha e^{-\rho} L_\ell^{(s,\ell)}(\rho) L_t^{(s,\ell)}(\rho) d\rho \]
\[ = \frac{N_{s,\ell,d} N_{t,\ell,d}}{2} \sum_{k=0}^K \frac{h_n^{k} V^{(k)}(0)}{k!} \int_0^\infty \rho^{\alpha+k} e^{-\rho} L_\ell^{(s,\ell)}(\rho) L_t^{(s,\ell)}(\rho) d\rho. \]
Writing
\[ \frac{N_{s,\ell,d} N_{t,\ell,d}}{2} = \left[ \frac{(s' \lor t') + 1)}{(s' \lor t') + 1)} \right]^{1/2} \cdot \frac{\Gamma((s' \lor t') + 1)}{\Gamma((s' \lor t') + 1)} \]
and using equation (2.5) in [20], then proves (40) and (42). Next, we will show that for all \( n \in \mathbb{N} \) and \( (s, t, \ell) \in U_{s,t,\ell}^n \), \( A_{k,s,t,\ell} \) has the expansion
\[ h_n^k A_{k,s,t,\ell} = \left( \frac{h_n}{2} \right)^k \left( T_1(k, s, t) + \frac{\ell^2}{(s \lor t)^2} T_2(k, s, t, \ell) \right). \]
Here for some $C_1 > 0$, we have

$$\sup_{s, t \in \mathbb{N}, |s-n|, |t-n| \leq \frac{n}{2}} |T_1(k, s, t)| \leq C \cdot C_1^k, \quad \sup_{s, t, \ell \in \mathbb{N}, |s-n|, |t-n| \leq \frac{n}{2}, \ell \leq n, \ell \equiv n \mod 2} |T_2(k, s, t, \ell)| \leq C \cdot C_1^k. \quad (46)$$

Note that for $s \lor t \leq 3n/2$, we have that $\frac{h_\alpha(s \lor t)}{2} \leq E$. Moreover, by (42), $A_{k, s, t, \ell}$ is equal to zero when $|s' - t'| = |s - t| > k$. Hence, the term $e^{-|s-t|}$ appearing in (41) is bounded by $e^{-2k}$ and can be absorbed into the constant $C_1$ in (45) and (46). Thus, these estimates, together with Definition 1 of a $\delta$–slowly varying potential allow us to sum over $k$ to establish (41) and complete the proof of Proposition 8.

To obtain the estimates in (45) and (46), we need two lemmas, in which we abbreviate $N = s \lor t, \beta = |s' - t'|$. In particular, this means that $|N - n| \leq \frac{n}{2}$. Since $A_{k, s, t, \ell} = 0$ for $|s' - t'| > k$, we can and will restrict to the case where $0 \leq \beta \leq k \leq K(n) \ll n$.

**Lemma 10.** There exists $C_2 > 0$ such that for every $0 \leq \beta \leq k, 0 \leq \ell \leq N, \ell \equiv N \mod 2$,

$$\left| \alpha \frac{(N-\ell + 1)\beta}{(N-\ell + \alpha + 1)\beta} - \left[ 1 - 2\beta \cdot \frac{\ell}{N} + S(\beta, N) \right] \right| \leq \frac{1 + \ell^2}{N^2} \cdot C_2^\beta,$$

where $S(\beta, N)$ is $\ell$–independent and satisfies

$$|S(\beta, N)| \leq \frac{C_2\beta}{N}.$$

**Lemma 11.** There exists $C_3 > 0$ such that for every $0 \leq \beta \leq k$ and each $0 \leq \ell \leq N, \ell \equiv N \mod 2$, we have

$$\left| (\alpha + 1)_{2k} F_2 \left[ \frac{-k, k + 1, -N'}{\beta, 1, \alpha + 1}; 1 \right] - \frac{(2k)!}{k!(\beta + 1)k} \left( \frac{N}{2} \right)^k \left( 1 + \beta \cdot \frac{\ell}{N} + T(\beta, k, N) \right) \right| \leq \frac{k(1 + \ell^2)}{N^2} C_3^k$$

where $T(\beta, k, N)$ is $\ell$–independent and satisfies

$$\sup_{0 \leq \beta \leq k} |T(\beta, k, N)| \leq C_3 \cdot \frac{k^2}{N}.$$

We will prove these lemmas in §§6.1-6.2 below. Assuming them for the moment, we prove (45) and (46) (which were used to complete the proof of Proposition 8). Using
Lemmas 10 and 11 and the expansion for $A_{k,s,t,\ell}$ from (42) we find that we have

$$A_{k,s,t,\ell} = \left( \frac{k}{\beta} \right) \left( \frac{(N' + 1)_{\beta}}{(N' + \alpha + 1)_{\beta}} \right)^{1/2} \left( \alpha + 1 \right)_{\kappa} \frac{3}{F_2} \left[ -k, k + 1, -N' \beta + 1, \alpha + 1 ; 1 \right]$$

$$= \left( \frac{k}{\beta} \right) \cdot \left( \frac{N}{2} \right)^{k} \cdot \left( \frac{(2k)!}{k!(\beta + 1)_{k}} \right) \cdot \left[ 1 + \beta \cdot \frac{\ell}{N} + T(\beta, k, N) \right] + O \left( \frac{1 + \ell^2}{N^2} \cdot C^2_3 \right)$$

$$\cdot \left( 1 - 2\beta \cdot \frac{\ell}{N} + S(\beta, N) + O \left( \frac{1 + \ell^2}{N^2} \cdot C^3_3 \right) \right)^{1/2}$$

$$= \left( \frac{k}{\beta} \right) \cdot \left( \frac{N}{2} \right)^{k} \left( \frac{(2k)!}{k!(\beta + 1)_{k}} \right) \cdot \left[ 1 + T(\beta, k, N) \right] \cdot \left[ 1 + S(\beta, N) \right]^{1/2} + O \left( \frac{1 + \ell^2}{N^2} \cdot (C_3^2)^k \right),$$

with $S(\beta, N) = O \left( \frac{\beta}{N} \right)$, $T(\beta, k, N) = O \left( \frac{k^2}{N} \right)$. Since

$$k! \cdot \frac{(2k)!}{k!(\beta + 1)_{k}} = \left( \frac{k}{\beta} \right) \cdot \left( \frac{2k}{\beta} \right) \cdot \frac{k!}{(\beta + 1)_{k}} = O(8^k),$$

and $A_{k,s,t,\ell} = 0$ unless $|s' - t'| \leq k$, we obtain (45) and (46). □

6.1. **Proof of Lemma 10.** We want to estimate

$$h_\beta \left( \frac{\ell}{N} \right) := \frac{(N - \ell + 1)_{\beta}}{(N - \ell + \alpha + 1)_{\beta}} = \frac{g_\beta \left( \frac{\ell}{N} \right)}{f_\beta \left( \frac{\ell}{N} \right)},$$

where

$$g_\beta(x) = \prod_{j=1}^{\beta} \left( 1 - x + \frac{2j}{N} \right), \quad \text{and} \quad f_\beta(x) = \prod_{j=0}^{\beta-1} \left( 1 + x + \frac{d+2j}{N} \right).$$

We have the estimates for $|x| \leq 1$,

$$g_\beta(x) = \prod_{j=1}^{\beta} \left( 1 - x + \frac{2j}{N} \right) \leq \left( 2 + \frac{2\beta}{N} \right)^\beta = O \left( 2^{\beta} e^{\beta^2/N} \right)$$

$$g_\beta'(x) = \sum_{j=1}^{\beta} \prod_{j \neq j_1} \left( 1 - x + \frac{2j}{N} \right) = O \left( 2^{\beta} \beta \cdot e^{\beta^2/N} \right)$$

$$g_\beta''(x) = \sum_{1 \leq j_1 < j_2 \leq \beta} \prod_{j_1 < j_2} \left( 1 - x + \frac{2j}{N} \right) = O \left( 2^{\beta} \beta^2 \cdot e^{\beta^2/N} \right),$$

and we have the analogous estimates for the function $f_\beta(x)$. By Taylor’s Theorem,

$$h_\beta \left( \frac{\ell}{N} \right) = h_\beta(0) + \frac{\ell}{N} h_\beta'(0) + \frac{\ell^2}{N^2} O \left( \| h''_\beta \|_{L^\infty([0,\ell/N])} \right),$$

and by the estimates above, $h_\beta(0) = 1 + O(\beta/N)$ and $h'_\beta(0) = -2\beta + O(\beta/N)$. Since

$$h''_\beta(x) = \frac{f_\beta(x)^2 \left[ g_\beta''(x) f_\beta(x) - f_\beta''(x) g_\beta(x) \right] - 2 f_\beta(x) f'_\beta(x) \left[ g_\beta'(x) f_\beta(x) - f'_\beta(x) g_\beta(x) \right]}{f_\beta(x)^4},$$
and \( f_\beta(x) \geq 1 \), the estimates above also imply that for any \( C_2 > 8 \),
\[
\sup_{x \in [0, \ell/N]} |h''_\beta(x)| \leq C_2^\beta.
\]
as required. \( \square \)

6.2. **Proof of Lemma 11.** By definition,
\[
(\alpha + 1)_{k-3}F_2 \left[ \begin{array}{c} -k & k + 1 & -N' \\ \beta + 1 & \alpha + 1 \\ \end{array} ; 1 \right] = \sum_{q=0}^{k-2} \frac{1 (-k)(k+1)q}{(q+1)! (\beta+1)_q (\alpha+1)_q}.
\]
Let us check that there exists \( C > 0 \) so that
\[
\sum_{q=0}^{k-2} a_{q,\beta} b_q = O \left( \frac{1 + \ell^2}{N^2} \cdot (CN)^k \right),
\]
where the implied constant is independent of \( \beta, \ell, k, N \). Note that \( |a_{q,\beta}| \leq |a_{q,0}| \). Hence, it is sufficient to establish (47) for \( \beta = 0 \). Define

\[
f(q) := \frac{a_{q+1,0}}{a_{q,0}} = \frac{(k-q)(k+q+1)}{(q+1)^2}.
\]
We have
\[
f'(x) = -\frac{2(k-x)(k+x+1)}{(x+1)^3} - \frac{2x+1}{(x+1)^2} < 0, \quad \forall x \in [0, k],
\]
and hence
\[
\sup_{q=0,\ldots,k-2} |a_{q,0}| = |a_{q^*,0}|, \quad q^* := \max \{ q \mid f(q) \geq 1 \}.
\]
The equation \( f(x) = 1 \) is
\[
(k-x)(k+x+1) = (x+1)^2,
\]
which has a unique positive solution \( \eta k \) with \( \eta \in [1/2, 1] \). Using Stirling’s approximation, we find there exists \( C > 0 \) so that
\[
\sup_{q=0,\ldots,k-2} |a_{q,0}| = O \left( \frac{(k(1+\eta))!}{((k\eta)!)^2 (k(1-\eta))!} \right) = O \left( \frac{1}{k^k C^k} \right).
\]
Hence, to prove (47), it remains to establish the estimate
\[
|b_q| = O \left( (1 + \ell^2)N^{k-2}2^{k-2} \right).
\]
To do this, write
\[
|b_q| = \left( \frac{N}{2} \right)^k \prod_{j=0}^{q-1} \left( 1 - \frac{\ell}{N} - \frac{2j}{N} \right) \prod_{j=q+1}^{k} \left( \frac{\ell + \frac{d-2}{2} + j}{N} \right).
\]
Since $q \leq k - 2$, we have
\[
f_1(\ell/N) = \frac{\left(\frac{\ell}{N} + \frac{d-2}{2} + k\right) \left(\frac{\ell}{N} + \frac{d-2}{2} + k - 1\right)}{N^2} \prod_{j=q+1}^{k-2} \left(\frac{\ell}{N} + \frac{d-2}{2} + j\right).
\]
Next, since $\ell \leq N$ and $k \leq N/2$, we have
\[
\prod_{j=q+1}^{k-2} \left(\frac{\ell}{N} + \frac{d-2}{2} + j\right) \leq 2^{k-q-2}.
\]
Observing that
\[
\frac{\left(\frac{\ell}{N} + \frac{d-2}{2} + k\right) \left(\frac{\ell}{N} + \frac{d-2}{2} + k - 1\right)}{N^2} = O\left(\frac{k^2(1 + \ell^2)}{N^2}\right)
\]
confirms (48) and completes the proof of (47). For the remaining two terms, we write
\[
a_{k,\beta} b_k + a_{k-1,\beta} b_{k-1} = \frac{1}{k!} \cdot \frac{(-1)^k(2k)!}{(\beta + 1)_k} \left(-N\right)_k + \frac{1}{(k-1)!} \cdot \frac{(-1)^{k-1}(2k-1)!}{(\beta + 1)_{k-1}} \left(-N\right)_{k-1} \alpha + k
\]
\[
= \frac{(2k)!}{k!(\beta + 1)_k} \left(\frac{N}{2}\right)^k \tilde{g}\left(\frac{\ell}{N}\right) \left(1 + \frac{\ell}{N} (\beta + k - 1) + T(\beta,k,N)\right).
\]
Here $T(\beta,k,N) = O\left(\frac{k^2}{N}\right)$ and is independent of $\ell$, and
\[
\tilde{g}(x) := \prod_{j=0}^{k-2} \left(1 - x - \frac{2j}{N}\right).
\]
We have
\[
\tilde{g}(\ell/N) = \tilde{g}(0) + \frac{\ell}{N} \tilde{g}'(0) + O\left(\frac{\ell^2}{N^2} \sup_{x \in [0,\ell/N]} |\tilde{g}''(x)|\right),
\]
where
\[
\tilde{g}(0) = 1 + O(k/N), \quad \tilde{g}'(0) = (1 - k) \left(1 + O(k/N)\right),
\]
and
\[
\sup_{x \in [0,\ell/N]} |\tilde{g}''(x)| = \sup_{x \in [0,\ell/N]} \left|\sum_{0 \leq j_1 < j_2 \leq k-2 j \neq j_1, j_2} \prod_{0 \leq j_1 < j_2 \leq k-2 j \neq j_1, j_2} \left(1 - \frac{\ell}{N} - \frac{2j}{N}\right)\right| = O(k^2).
\]
Putting this all together, we obtain,
\[
(\alpha + 1)_{k+3} F_2 \left[\begin{array}{c}
-k, k + 1, -N' \\
\beta + 1, \alpha + 1
\end{array}\right] 1
\]
\[
= \left(\frac{N}{2}\right)^k \left\{ (2k)! \left(1 + \beta \cdot \frac{\ell}{N} + T(\beta,k,N)\right) + O\left(\frac{(1 + \ell^2)}{N^2} c_\beta\right) \right\}
\]
as required. \qed
7. Proof of Proposition 7

To prove Proposition 7, we begin with the following result, which allows us to replace \( E_{\ell,n}(\varepsilon) \) by a finite number (depending on \( n \)) of its jets at \( \varepsilon = 0 \).

**Proposition 12.** For any \( V \in L^\infty(\mathbb{R}_+) \) with \( \| V \|_{L^\infty} = 1 \), \( n \) such that \( h_n < 1 \), and any \( J = J(n) \) satisfying \( \liminf_{n \to \infty} J(n)/\log n = \infty \), we have

\[
\sup_{\ell \leq n, \ell \equiv n (\mod 2) \atop \varepsilon \in [0,1/5]} \left| E_{\ell,n}(\varepsilon) - \sum_{j=0}^{J} \frac{(h_n\varepsilon)^j}{j!} \frac{d^j}{d\varepsilon^j} E_{\ell,n}(\varepsilon) \right| = O(h_n^\infty). \tag{49}
\]

**Proof.** Applying Lemma 6 with \( W(r) = \varepsilon h_n V(r^2) \), we find that for every \( \varepsilon \in [0,1/5] \)

\[
\sup_{\ell \leq n, \ell \equiv n (\mod 2) \atop \varepsilon \in [0,1/5]} d \left( E_{\ell,n}^V(\varepsilon), \text{Spec}(P_{h_n,\ell}(\varepsilon)) \setminus \{ E_{\ell,n}^V(\varepsilon) \} \right) > \frac{h_n}{2}, \tag{50}
\]

where \( d(x,A) \) denotes the distance from a point \( x \) to a set \( A \). As explained in §4.3, we have

\[
\frac{d^j}{d\varepsilon^j} E_{\ell,n}^V(\varepsilon) = (-1)^{j-1} \left( V\psi_{h_n,\ell,n}(\varepsilon), [G_{\ell}(\varepsilon)]^{(j-1)}(\psi_{h_n,\ell,n}) \right),
\]

where

\[
G_{\ell}(\varepsilon) = (P_{h_n,\ell} - E_{\ell,n}^V(\varepsilon))^{-1} \circ \Pi_{\psi_{h_n,\ell,n}} \circ V,
\]

and \( V \) denotes multiplication by the function \( V(r^2) \). Hence, using (50) and that \( \| V \|_{L^\infty} \leq 1 \), we find

\[
\sup_{\ell \leq n, \ell \equiv n (\mod 2) \atop \varepsilon \in [0,1/5]} \| G_{\ell}(\varepsilon) \| \leq \frac{2}{h_n} \Rightarrow \sup_{\ell \leq n, \ell \equiv n (\mod 2) \atop \varepsilon \in [0,1/5]} \left| \frac{d^j}{d\varepsilon^j} E_{\ell,n}^V(\varepsilon) \right| \leq \left( \frac{2}{h_n} \right)^j. \tag{52}
\]

Applying Taylor’s theorem then gives

\[
\sup_{\ell \leq n, \ell \equiv n (\mod 2) \atop \varepsilon \in [0,1/5]} \left| E_{\ell,n}(\varepsilon) - \sum_{j=0}^{J} \frac{(h_n\varepsilon)^j}{j!} \frac{d^j}{d\varepsilon^j} E_{\ell,n}(\varepsilon) \right| = O \left( (2^J (J+1)!)^{-1} \right) = O(h_n^\infty)
\]

since \((J+1)! \geq e^{-\log h_n}^2\).

To complete the proof of Proposition 7, it remains to check that, provided \( h_n < 1 \) and \( J(n), K(n) \) satisfy (35), we have

\[
\sup_{\ell \leq n, \ell \equiv n (\mod 2) \atop 0 \leq j \leq J} h_n^j \left| \frac{d^j}{d\varepsilon^j} \right| \frac{d^j}{d\varepsilon^j} E_{\ell,n}^V(\varepsilon) - \frac{d^j}{d\varepsilon^j} E_{\ell,n}^V(\varepsilon) \right| = O(h_n^\infty). \tag{53}
\]

To prove this estimate, we again use

\[
\frac{d^j}{d\varepsilon^j} E_{\ell,n}^V(\varepsilon) = (-1)^{j-1} \left( V\psi_{h_n,\ell,n}, G_{\ell}^{(j-1)}(\psi_{h_n,\ell,n}) \right),
\]

where

\[
G_{\ell} = G_{\ell}(0) = (P_{h_n,\ell}(0) - E)^{-1} \circ \Pi_{\psi_{h_n,\ell,n}} \circ V.
\]
Setting for each $K \geq 1$, 
\[ G_{\ell,K} := (P_{\psi_{n,\ell}}(0) - E)^{-1} \circ \Pi_{\psi_{n,\ell}} \circ V_K, \]
we have 
\[ \frac{d^j}{d\varepsilon^j} \bigg|_{\varepsilon=0} E_{\psi_{n,\ell}}^V(\varepsilon) = (-1)^{j-1} \left\langle V_K \psi_{n,\ell}, G_{\ell,K}^{(j-1)}(\psi_{n,\ell}) \right\rangle \]
and (53) reduces to showing that for each $j \leq J$ and every $\ell \leq n$, $\ell \equiv n \pmod{2}$
\[ \left| h_n^j \left( \left\langle V \psi_{n,\ell}, G_{\ell}^{(j-1)}(\psi_{n,\ell}) \right\rangle - \left\langle V_K \psi_{n,\ell}, G_{\ell,K}^{(j-1)}(\psi_{n,\ell}) \right\rangle \right) \right| = O(h_n^\infty), \tag{54} \]
with the implied constant independent of $j, \ell, n$. We will establish (54) by induction with the help of the following lemma.

**Lemma 13.** Suppose $h_n < 1$ and $J = J(n)$, $K = K(n)$ satisfy (35). Then, there exists a constant $C_1 > 0$ so that if $V$ is a $\delta$–slowly varying potential for the energy $E$, with $\delta \in (0, (C_1 E)^{-1})$,
\[ \sup_{|m-n| \leq \frac{\delta}{2}} \frac{\left| \langle V - V_K \rangle \psi_{n,\ell,m}, X \rangle \right|}{\|X\|} = O \left( h_n^\infty \right). \tag{55} \]

**Proof.** Let $\chi(r)$ be an auxiliary cut-off function that equals 1 for $r \leq \sqrt{4E}$ and 0 otherwise. Then, since $V$ is bounded, by the exponential decay (26) of $\psi_{n,\ell,m}$
\[ \sup_{|m-n| \leq \frac{\delta}{2}} \frac{\left| \left\langle (1 - \chi) V \psi_{n,\ell,m}, X \right\rangle \right|}{\|X\|} = O \left( h_n^\infty \right). \tag{56} \]

Using the definition (7) that $V$ is $\delta$–slowly varying on the support of $\chi$ and the assumption $\limsup_{n \to \infty} K(n)/\log n = \infty$, we have, for all $\delta E$ sufficiently small
\[ \sup_{|m-n| \leq \frac{n}{2}} \frac{\left| \langle \chi \left( V - V_K \right) \psi_{n,\ell,m}, X \rangle \right|}{\|X\|} = O \left( h_n^\infty \right). \tag{57} \]

Finally, again using the exponential decay (26) of $\psi_{n,\ell,m}$ and $\liminf_{n \to \infty} K(n)/n = 0$, we obtain
\[ \sup_{|m-n| \leq \frac{n}{2}} \frac{\left| \left\langle (1 - \chi) V_K \psi_{n,\ell,m}, X \right\rangle \right|}{\|X\|} = O \left( h_n^\infty \right), \tag{58} \]

which completes the proof. \qed

To prove (54) by induction, note that Lemma 13 is precisely the base case $j = 1$. Next, suppose we have already shown (54) for some $j \geq 1$. Then, using Lemma 13 and the norm estimate from (52), we have
\[ h_n^j \left( \left\langle V \psi_{n,\ell,n}, G_{\ell}^{(j)}(\psi_{n,\ell,n}) \right\rangle \right) = h_n^j \left( \left\langle V_K \psi_{n,\ell,n}, G_{\ell,K}^{(j)}(\psi_{n,\ell,n}) \right\rangle + O(h_n^\infty) \right). \tag{59} \]

The adjoint of $G_{\ell}$ is 
\[ G_{\ell}^* = V \circ (P_{\psi_{n,\ell}} - E)^{-1} \circ \Pi_{\psi_{n,\ell}}^1. \]
and hence
\[ G_\ell^* (V_K \psi_{h_n, \ell, n}) = h_n^{-1} \sum_{m \text{ s.t. } |m-n| \leq 2K, m \neq n} \frac{\langle V_K \psi_{h_n, \ell, m}, \psi_{h_n, \ell, n} \rangle}{m-n} \psi_{h_n, \ell, m}. \]

The sum in the previous line is truncated to $|m - n| \leq 2K$ since by Proposition 8, the numerator vanishes unless $|m - n| \leq 2K$. To complete the proof, we write
\[
\begin{align*}
\langle V_{h_n, \ell, n}, G_\ell^{(j)} \rangle &= \langle V_{h_n, \ell, m}, G_\ell^{(j)} \rangle + O(h_\infty) \\
&= \sum_{m \text{ s.t. } |m-n| \leq 2K, m \neq n} \frac{\langle V_K \psi_{h_n, \ell, m}, \psi_{h_n, \ell, n} \rangle}{m-n} \psi_{h_n, \ell, m} \\
&= \sum_{m \text{ s.t. } |m-n| \leq 2K, m \neq n} \frac{\langle V_K \psi_{h_n, \ell, m}, \psi_{h_n, \ell, n} \rangle}{m-n} \psi_{h_n, \ell, m} + O(h_\infty) \\
&= h_n^2 \langle V_{h_n, \ell, n}, G_\ell^{(j)} \rangle + O(h_\infty),
\end{align*}
\]

where in the second-to-last line we used the inductive hypothesis and the fact that $\limsup_{n \to \infty} K(n) = 0$.

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(T. Beck) Department of Mathematics, MIT, Cambridge, United States.
E-mail address: tdbeck@mit.edu

(B. Hanin) Department of Mathematics, Texas A&M, College Station, United States.
E-mail address: bhanin@math.tamu.edu