Exact correlators in the Gaussian Hermitian matrix model

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Abstract

We present the $W_{1+\infty}$ constraints for the Gaussian Hermitian matrix model, where the constructed constraint operators yield the $W_{1+\infty}$ $n$-algebra. For the Virasoro constraints, we note that the constraint operators give the null 3-algebra. With the help of our Virasoro constraints, we derive a new effective formula for correlators in the Gaussian Hermitian matrix model.

Keywords: Conformal and $W$ Symmetry, Matrix Models, $n$-algebra

1 Introduction

The various constraints for matrix models have attracted remarkable attention, such as Virasoro/$W$-constraints \cite{1}-\cite{5} and Ding-Iohara-Miki constraints \cite{6,7}. Due to the Bagger-Lambert-Gustavsson (BLG) theory of M2-branes \cite{8,9}, $n$-algebra and its applications have aroused much interest \cite{10}-\cite{17}. In the context of matrix models, usually the Virasoro/$W$-constraint operators do not yield the closed $n$-algebra. Whether there exist such kind constraint operators leading to the closed $n$-algebra has recently been investigated for the (elliptic) Hermitian one-matrix models. By inserting the special multi-variable realizations of the $W_{1+\infty}$ algebra under the integral, it was found that the derived constraint operators for the Hermitian one-matrix model may yield the closed $W_{1+\infty}$ ($n$)-algebras \cite{18}. For the case of the elliptic matrix model, one can obtain the constraint operators associating with the $q$-operators \cite{19,20}. The situation is different from that of the Hermitian one-matrix model, since the derived constraint operators do not yield the closed algebra. However, it was shown that the ($n$)-commutators of the constraint operators are compatible with the desired generalized $q$-$W_{\infty}$ ($n$)-algebras once we act on the partition function \cite{20}.

The partition functions of various matrix models can be obtained by acting on elementary functions with exponents of the given operators. For the Gaussian Hermitian matrix model, its

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partition function is generated by the operator $\hat{W}_-$. This operator is also the constraint operator for the Hermitian one-matrix model which associates with the Lassalle operator and the potential of the $A_{N-1}$-Calogero model. The correlators in the Gaussian Hermitian matrix model have been well investigated. A compact formula for correlators has been given by finite sums over Young diagrams of a given size, which involve also the well known characters of symmetric group. Moreover, the $2m$-fold Gaussian correlators of rank $r$ tensors have been given by $r$-linear combinations of dimensions with the Young diagrams of size $m$. In this letter, we reinvestigate the Gaussian Hermitian matrix model and present its Virasoro/$W$-constraints. We intend to further explore the properties of the constraints and derive a new formula for correlators in this matrix model.

2 $W_{1+\infty}$ constraints for the Gaussian Hermitian matrix model

Let us consider the Gaussian Hermitian matrix model

$$Z_G = \int_{N \times N} d\phi \exp\left(-\text{tr}\phi^2/2 + \sum_{k=0}^{\infty} t_k \text{tr}\phi^k\right) = e^{Nt_0} (1 + C_{i_1}(N) t_{i_1} + \frac{1}{2!} C_{i_1i_2}(N) t_{i_1} t_{i_2} + \frac{1}{3!} C_{i_1i_2i_3}(N) t_{i_1} t_{i_2} t_{i_3} + \cdots),$$

(1)

where the coefficients $C_{i_1\cdots i_l}(N)$ are the so-called $l$-point correlators, which are given by the Gaussian integrals

$$C_{i_1\cdots i_l}(N) = \langle \text{tr}\phi^{i_1} \cdots \text{tr}\phi^{i_l} \rangle = \int_{N \times N} d\phi \text{tr}\phi^{i_1} \cdots \text{tr}\phi^{i_l} \exp(-\frac{1}{2} \text{tr}\phi^2).$$

(2)

Due to the reflection symmetry of the action $\text{tr}\phi^2$, when $i_1 + \cdots + i_l$ is odd, we have $C_{i_1\cdots i_l}(N) = 0$.

The partition function of the Gaussian model can also be expressed as

$$Z_G = \sum_{s=0}^{\infty} Z^{(s)}_G = e^{\hat{W}_-/2} e^{Nt_0},$$

(3)

where

$$Z^{(s)}_G = e^{Nt_0} \sum_{l=0}^{\infty} \sum_{i_1 + \cdots + i_l = s} \langle \text{tr}\phi^{i_1} \cdots \text{tr}\phi^{i_l} \rangle \frac{t_{i_1} \cdots t_{i_l}}{l!},$$

(4)

and the operator $\hat{W}_-$ is given by

$$\hat{W}_- = \sum_{j_1, j_2 = 0}^{\infty} (j_1 j_2 t_{j_1} t_{j_2} \frac{\partial}{\partial t_{j_1+j_2-2}} + (j_1 + j_2 + 2) t_{j_1+j_2+2} \frac{\partial}{\partial t_{j_1}} \frac{\partial}{\partial t_{j_2}}).$$

(5)
It indicates that the partition function (1) can indeed be generated by the operator $\hat{W}_{-2}$.

The action of the operator $\hat{W}_{-2}$ on $Z_G^{(s)}$ leads to increase the grading in the following sense:

$$\hat{W}_{-2}Z_G^{(s)} = (s + 2)Z_G^{(s+2)}.$$  \hfill (6)

The operator preserving the grading is given by (21)

$$\hat{D} = \sum_{j=0}^{\infty} j t_j \frac{\partial}{\partial t_j},$$  \hfill (7)

which acting on $Z_G^{(s)}$ gives

$$\hat{D}Z_G^{(s)} = sZ_G^{(s)}.$$  \hfill (8)

The commutation relation between $\hat{D}$ and $\hat{W}_{-2}$ is

$$[\hat{D}, \hat{W}_{-2}] = 2\hat{W}_{-2}.\)  \hfill (9)

Note that the actions of $\hat{D}$ and $\hat{W}_{-2}$ on $Z_G$ give

$$\hat{D}Z_G = \hat{W}_{-2}Z_G.$$  \hfill (10)

For the operators $\frac{\partial}{\partial t_2}$ and $\hat{D}$, there is the similar commutation relation as (9)

$$[\hat{D}, \frac{\partial}{\partial t_2}] = -2\frac{\partial}{\partial t_2}.$$  \hfill (11)

The actions of $\frac{\partial}{\partial t_2}$ and $\hat{D}$ on $Z_G$ give

$$\frac{\partial}{\partial t_2}Z_G = (\hat{D} + N^2)Z_G.$$  \hfill (12)

By means of (11) and (12), it is easy to show that

$$\frac{\partial}{\partial t_2}Z_G^{(s)} = (s - 2 + N^2)Z_G^{(s-2)}.\)  \hfill (13)

In contrast with the operator $\hat{W}_{-2}$, we see that the operator $\frac{\partial}{\partial t_2}$ decreases the grading in the sense (13).

Let us introduce the operators

$$W^r_m = (-\frac{1}{2})^{r-1}(\hat{W}_{-2})^m(\hat{W}_{-2} - \hat{D})^{r-1}, \ m, r \in \mathbb{N}, r \geq 2,$$  \hfill (14)

which obviously satisfy

$$W^r_mZ_G = 0.$$  \hfill (15)
The remarkable property is that these constraint operators yield

\[ [W_{m_1}^{r_1}, W_{m_2}^{r_2}] = \left( \sum_{k=0}^{r_1-1} C_{r_1-1}^k m_2^k - \sum_{k=0}^{r_2-1} C_{r_2-1}^k m_1^k \right) W_{m_1+m_2}^{r_1+r_2-1-k}, \]  

(16)

and \( n \)-algebra

\[ [W_{m_1}^{r_1}, W_{m_2}^{r_2}, \ldots, W_{m_n}^{r_n}] := \epsilon_{i_1 \ldots i_n}^{j_1 \ldots j_n} W_{m_1}^{r_{i_1}} W_{m_2}^{r_{i_2}} \cdots W_{m_n}^{r_{i_n}} = \epsilon_{i_1 \ldots i_n}^{j_1 \ldots j_n} \left( \sum_{\alpha_i=0}^{\beta_i} m_{i_1}^{\alpha_1} \cdots m_{i_n}^{\alpha_n} \right) W_{m_1}^{r_{i_1}+\cdots+r_{i_n}-(n-1)-\alpha_1-\cdots-\alpha_n}, \]

(17)

where \( C_r^k = \frac{r(r-1) \cdots (r-k+1)}{k!} \), \( \beta_k = \left\{ \begin{array}{ll} r_{i_1} - 1, & k = 1, \\ \sum_{j=1}^{k} r_{i_j} - k - \sum_{i=1}^{k-1} \alpha_i, & 2 \leq k \leq n - 1, \end{array} \right. \)

and \( \epsilon_{i_1 \ldots i_n}^{j_1 \ldots j_n} \) is given by

\[ \epsilon_{i_1 \ldots i_p}^{j_1 \ldots j_p} = \det \left( \begin{array}{ccc} \delta_{j_1}^{i_1} & \cdots & \delta_{j_p}^{i_1} \\ \vdots & \ddots & \vdots \\ \delta_{j_1}^{i_p} & \cdots & \delta_{j_p}^{i_p} \end{array} \right). \]

It is noted that (16) and (17) completely match with the \( W_{1+\infty} \) \((n-)\)-algebra presented in Ref. [18]. The \( W_{1+\infty} \) \((n-)\)-algebra (17) with \( n \) even is a generalized Lie algebra (or higher order Lie algebra), which satisfies the generalized Jacobi identity

\[ \epsilon_{1 \ldots 2n-1}^{i_1 \ldots i_{2n-1}} [\left[ A_{i_1}, A_{i_2}, \ldots, A_{i_{n}}, A_{i_{n+1}}, \ldots, A_{i_{2n-1}} \right] = 0. \]

(18)

For the constraint operators \( W_m^{r_n} \) (14) with fixed \( r = n + 1 \), by taking the appropriate scaling transformations, it is not difficult to show that these operators constitute the subalgebras

\[ [W_{m_1}^{n+1}, W_{m_2}^{n+1}, \ldots, W_{m_{2n}}^{n+1}] = \prod_{1 \leq j < k \leq 2n} (m_k - m_j) W_{m_1+m_2+\cdots+m_{2n}}^{n+1}, \]

(19)

and

\[ [W_{m_1}^{n+1}, \ldots, W_{m_{2n+1}}^{n+1}] = 0. \]

(20)

For the \( W_{1+\infty} \) constraints (15), it is noted that the constraint operators (14) contain the operators increasing and preserving the grading. Let us now introduce the following operators in terms of the operators decreasing and preserving the grading:

\[ \tilde{W}_m^r = (-\frac{1}{2})^{r-1} \frac{\partial}{\partial t_2} m (\hat{D} - \frac{\partial}{\partial t_2} + N^2)^{r-1}, m, r \in \mathbb{N}, r \geq 2. \]

(21)
Straightforward calculation shows that they also yield the $W_{1+\infty}$ algebra and $n$-algebra. By carrying out the action of the operators on the partition function of the Gaussian model, it gives another $W_{1+\infty}$ constraints

$$\tilde{W}_m^T Z_G = 0.$$  

(22)

### 3 Correlators in the Gaussian Hermitian matrix model

Let us first recall the correlators in the Gaussian Hermitian matrix model. Harer and Zagier presented a generating function for exact (all-genera) 1-point correlators in the Gaussian Hermitian matrix model [22, 23],

$$C_i = \text{coefficient of } x^i \lambda^N \text{ in } \frac{1}{\lambda - (1 - \lambda) - (1 + \lambda)x^2},$$  

(23)

where $i$ is even. By using Toda integrability of the model, Morozov and Shakirov derived the 2-point generalization of the Harer-Zagier 1-point function [26],

$$\frac{C_{(2k+1)(2m+1)}}{(2k+1)!!(2m+1)!!} = \text{coefficient of } x^{2k+1} y^{2m+1} \lambda^N \text{ in } \frac{\lambda}{(\lambda - 1)^{3/2}} \frac{\arctan\left(\frac{xy\sqrt{\lambda - 1}}{\sqrt{\lambda - 1 + (1 + \lambda)(x^2 + y^2)}}\right)}{\sqrt{\lambda - 1 + (1 + \lambda)(x^2 + y^2)}}.$$  

(24)

However, it should be noted that it is difficult to give the higher correlators in this way. Recently Mironov and Morozov presented a compact formula for correlators by finite sums over Young diagrams of a given size [27],

$$C_{i_1 i_2 \cdots i_l}(N) \equiv O_\Lambda = \sum_{R \vdash |\Lambda|} \frac{1}{d_R} \chi_R\{t_n = \frac{1}{2} \delta_{n,2}\} \cdot D_R(N) \cdot \psi_R(\Lambda),$$  

(25)

where $\Lambda = \{i_1 \geq i_2 \geq \cdots \geq i_l > 0\}$ and $R$ are the Young diagrams of the given size $\sum_k i_k$, and $D_R(N)$, $\chi_R\{t\}$, $\psi_R(\Lambda)$ and $d_R$ are respectively the dimension of representation $R$ for the linear group $GL(N)$, the linear character (Schur polynomial), the symmetric group character and the dimension of representation $R$ of the symmetric group $S_{|R|}$ divided by $|R|!$. Furthermore, a representation of the correlators in terms of permutations is given by [28]

$$O_\sigma = \sum_{R \vdash m} \varphi_R(\{2^m\}) \cdot D_R(N) \cdot \psi_R(\sigma),$$  

(26)
where $\varphi_R(2^m)$ are the symmetric group characters.

Let us turn to consider the Virasoro constraints in (22)
\[ \tilde{W}_l^2 Z_G = 0, \ l \in \mathbb{N}. \] (27)
The constraint operators yield the Witt algebra
\[ [\tilde{W}_l^2, \tilde{W}_{l_1}^2] = (l_2 - l_1)\tilde{W}_{l_1+l_2}^2, \] (28)
and null 3-algebra
\[ [\tilde{W}_l^2, \tilde{W}_{l_1}^2, \tilde{W}_{l_2}^2] = 0. \] (29)

When $l \neq 0$, by using the expression (21) to calculate left-hand side of (27), we obtain
\[
\sum_{i=1}^{\infty} it_i (C_2 \ldots 2 i (N) + \sum_{i_1=1}^{\infty} C_2 \ldots 2 i i_1 (N)t_{i_1} + \frac{1}{2!} \sum_{i_1,i_2=1}^{\infty} C_2 \ldots 2 i i_1 i_2 (N)t_{i_1}t_{i_2} + \cdots )
\]
\[
+(N^2 + 2l)(C_2 \ldots 2 (N) + \sum_{i_1=1}^{\infty} C_2 \ldots 2 i_1 (N)t_{i_1} + \frac{1}{2!} \sum_{i_1,i_2=1}^{\infty} C_2 \ldots 2 i_1 i_2 (N)t_{i_1}t_{i_2} + \cdots )
\]
\[
-(C_2 \ldots 2 (N) + \sum_{i_1=1}^{\infty} C_2 \ldots 2 i_1 (N)t_{i_1} + \frac{1}{2!} \sum_{i_1,i_2=1}^{\infty} C_2 \ldots 2 i_1 i_2 (N)t_{i_1}t_{i_2} + \cdots ) = 0. \] (30)

From the fact that the constant term in the left-hand side of (30) should be zero, we have
\[ C_2 \ldots 2 (N) = (N^2 + 2l)C_2 \ldots 2 (N). \] (31)

Taking the special constraint operator $\tilde{W}_0^2$ in (27), it is easy to obtain
\[ C_2 (N) = N^2. \] (32)

Thus from (31), we obtain
\[ C_2 \ldots 2 (N) = \prod_{j=0}^{l-1} (N^2 + 2j). \] (33)

By collecting the coefficients of $t_{i_1}t_{i_2} \cdots t_{i_k}$ in (30) and setting to zero, we have
\[
C_{\ldots l} \ldots 2 i_1 \ldots i_k (N) = (N^2 + i_1 + \cdots + i_k + 2l)C_{\ldots l} \ldots 2 i_1 \ldots i_k (N)
\]
\[
= \prod_{j=0}^{l} (N^2 + i_1 + \cdots + i_k + 2j)C_{i_1 \cdots i_k} (N), \ l \in \mathbb{N}. \] (34)
Let us take the constraint operator $W^2_0$ in (15), i.e.,

$$(\hat{W}_{-2} - \hat{D})Z_G = 0.$$ \hfill (35)

After a straightforward calculation of the left-hand side of (35), we obtain

$$\sum_{j_1, j_2 = 1}^{\infty} (j_1 + j_2 + 2) t_{j_1 + j_2 + 1} + \sum_{i_1 = 1}^{\infty} C_{j_1 j_2 i_1} (N) t_{i_1} + \frac{1}{2!} \sum_{i_1, i_2 = 1}^{\infty} C_{j_1 j_2 i_1 i_2} (N) t_{i_1} t_{i_2} + \cdots$$

$$+ t^2_1 N (1 + \sum_{i_1 = 1}^{\infty} C_{i_1} (N) t_{i_1} + \frac{1}{2!} \sum_{i_1, i_2 = 1}^{\infty} C_{i_1 i_2} (N) t_{i_1} t_{i_2} + \frac{1}{3!} \sum_{i_1, i_2, i_3 = 1}^{\infty} C_{i_1 i_2 i_3} (N) t_{i_1} t_{i_2} t_{i_3} + \cdots)$$

$$+ 2 t^2_2 N^2 (1 + \sum_{i_1 = 1}^{\infty} C_{i_1} (N) t_{i_1} + \frac{1}{2!} \sum_{i_1, i_2 = 1}^{\infty} C_{i_1 i_2} (N) t_{i_1} t_{i_2} + \frac{1}{3!} \sum_{i_1, i_2, i_3 = 1}^{\infty} C_{i_1 i_2 i_3} (N) t_{i_1} t_{i_2} t_{i_3} + \cdots)$$

$$+ \sum_{j_1, j_2 = 1}^{\infty} j_1 j_2 t_{j_1 + j_2} (N) + \sum_{i_1 = 1}^{\infty} C_{1 + j_2 - 1, i_1} (N) t_{i_1} + \frac{1}{2!} \sum_{i_1, i_2 = 1}^{\infty} C_{1 + j_2 - 2, i_1, i_2} (N) t_{i_1} t_{i_2} + \cdots$$

$$+ \cdots + 2 \sum_{j_1 = 1}^{\infty} (j_1 + 2) t_{j_1 + 2} N (C_{1 j_1} (N) + \sum_{i_1 = 1}^{\infty} C_{1 j_1 i_1} (N) t_{i_1} + \frac{1}{2!} \sum_{i_1, i_2 = 1}^{\infty} C_{1 j_1 i_1 i_2} (N) t_{i_1} t_{i_2} + \cdots)$$

$$- \sum_{j_1 = 1}^{\infty} j_1 t_{j_1} (N) + \sum_{i_1 = 1}^{\infty} C_{1 i_1} (N) t_{i_1} + \frac{1}{2!} \sum_{i_1, i_2 = 1}^{\infty} C_{1 i_1 i_2} (N) t_{i_1} t_{i_2} + \cdots = 0.$$ \hfill (36)

By collecting the coefficients of $t^2_1$ and setting to zero, we obtain

$$C_{1,1} (N) = N.$$ \hfill (37)

Similarly, for the case of the coefficients of $t^l_1$ with $l$ even, we have

$$C_{1, \ldots, 1} (N) = (l - 1) NC_{1, \ldots, 1} (N).$$ \hfill (38)

Substituting (37) into the recursive relation (35), we obtain

$$C_{1, \ldots, 1} (N) = (l - 1)!! N^l, \ for \ l \ even.$$ \hfill (39)

Motivated by the exact $l$-point correlators $C_{1, \ldots, 1} (N)$ and $C_{2, \ldots, 2} (N)$, we now proceed to derive the general $l$-point correlators $C_{i_1 \ldots i_l} (N)$. Let us consider the Virasoro constraints in (15)

$$W^2 m Z_G = 0.$$ \hfill (40)

The constraint operators $W^2 m$ also yield the Witt algebra (28) and null 3-algebra (29). By means of (9) and (10), we may rewrite (40) as

$$(\hat{W}_{-2})^{m+1} Z_G = \prod_{j=0}^{m} (\hat{D} - 2j) Z_G.$$ \hfill (41)
Let us focus on the coefficients of \( t_1 t_2 \cdots t_i \) with \( \sum_{j=1}^l i_j = 2(m+1) \) on the both sides of (41). Note that the form of \((\hat{W}_2)^{m+1}\) appears to become more complicated very rapidly as one proceeds to higher power. We may formally express the \((m+1)\)-th power of \(\hat{W}_2\) as

\[
(\hat{W}_2)^{m+1} = \sum_{k,l=1}^{2(m+1)} \sum_{j_1,j_2,\ldots,j_k=0}^{\infty} \sum_{i_1,i_2,\ldots,i_l=\rho} \frac{\partial}{\partial t_{j_1}} \cdots \frac{\partial}{\partial t_{j_k}},
\]

where \(\rho = \sum_{n=1}^k j_n + 2(m+1)\) and \(P_{j_1,j_2,\ldots,j_k}^{i_1,i_2,\ldots,i_l}\) are polynomials in \(i_\alpha, \alpha = 1,\cdots,l\) and \(j_\beta, \beta = 1,\cdots,k\).

When \(j_\beta = 0\) for \(\beta = 1,\cdots,k\) in (42), the corresponding terms acting on \(Z_G\) give the coefficients of \(t_1 t_2 \cdots t_i\) with \(\sum_{j=1}^l i_j = 2(m+1)\) on the left-hand side of (41)

\[
\sum_{k=1}^{2(m+1)} \sum_{\sigma} P_{0,\ldots,0}^{\sigma(i_1),\sigma(i_2),\ldots,\sigma(i_l)} N^k e^{Nt_0},
\]

where \(\sigma\) denotes all distinct permutations of \((i_1, i_2, \cdots, i_l)\). By means of (8), the right-hand side of (41) becomes

\[
\prod_{j=0}^m (\hat{D} - 2j)Z_G = \sum_{s=0}^{m} \prod_{j=0}^s (s - 2j)Z_G^{(s)}
= e^{Nt_0} \sum_{s=0}^{\infty} \sum_{i_1+i_2+\cdots+i_l=s} \frac{1}{l!} \prod_{j=0}^m (s - 2j)C_{i_1 \cdots i_l}(N) t_1 \cdots t_i.
\]

From (44), we obtain that the coefficients of \(t_1 t_2 \cdots t_i\) with \(\sum_{j=1}^l i_j = 2(m+1)\) on the right-hand side of (41) are

\[
e^{Nt_0} \sum_{\sigma} \frac{2^{m+1}(m+1)!}{l!} C_{\sigma(i_1),\ldots,\sigma(i_l)}(N) = \frac{2^{m+1}(m+1)!}{l!} \lambda_{(i_1 \cdots i_l)} e^{Nt_0} C_{i_1 \cdots i_l}(N),
\]

where we denote by \(\lambda_{(i_1 \cdots i_l)}\) the number of distinct permutations of \((i_1, i_2, \cdots, i_l)\).

By equating (43) and (45), we obtain the \(l\)-point correlators \(C_{i_1 \cdots i_l}(N)\)

\[
C_{i_1 \cdots i_l}(N) = \frac{l!}{2^{m+1}(m+1)!} \lambda_{(i_1 \cdots i_l)} \sum_{k=1}^{2(m+1)} \sum_{\sigma} P_{0,\ldots,0}^{\sigma(i_1),\sigma(i_2),\ldots,\sigma(i_l)} N^k,
\]

where \(\sum_{j=1}^l i_j\) is even and \(m = \frac{1}{2} \sum_{j=1}^l i_j - 1\).
When particularized to the 1-point correlators in (46), we have

\[ C_i(N) = \frac{1}{\sqrt{2^i (\frac{i}{2})!}} \sum_{k=1}^{i} P_{0,\ldots,0}^{i,k} N^k. \]  

Comparing (46) with (25) and (26), we see that (46) is different from the other two expressions. Hence (46) is a new formula for correlators, where the operators \((\hat{W}_{-2})^{m+1}\) play an crucial role to determinate the polynomials in \(i_{\alpha}, \alpha = 1, \cdots, l\) in the correlators.

For clarity of calculation, let us consider the m = 1 case in (46), i.e., \(\sum_{j=1}^{l} i_j = 4\). From the expression

\[
(\hat{W}_{-2})^2 = \sum_{i_1,i_2=0}^{\infty} \sum_{i_1+i_2=i_3+i_4+4} (i_3 + i_4 + 2) \hat{t}_{i_1} \hat{t}_{i_2} \frac{\partial}{\partial t_{i_3}} \frac{\partial}{\partial t_{i_4}} + 2 \sum_{i_1,i_2=0}^{\infty} i_1 i_2 \hat{t}_{i_1+i_2+2} \frac{\partial}{\partial t_{i_1+i_2-2}} \\
+ 2 \sum_{i_1,i_2,i_3=0}^{\infty} (i_1 + i_2 - 2) \hat{t}_{i_1} \hat{t}_{i_2} \hat{t}_{i_3} \frac{\partial}{\partial t_{i_1+i_2+i_3-4}} \frac{\partial}{\partial t_{i_3}} \frac{\partial}{\partial t_{i_4}} + 2 \sum_{i_1,i_2,i_3,i_4=0}^{\infty} \hat{t}_{i_1} \hat{t}_{i_2} \hat{t}_{i_3} \hat{t}_{i_4} \frac{\partial}{\partial t_{i_1+i_2-2}} \frac{\partial}{\partial t_{i_3+i_4-2}} \frac{\partial}{\partial t_{i_4}} \\
+ 4 \sum_{i_1,i_2,i_3=0}^{\infty} i_2 \hat{t}_{i_1+i_2} \hat{t}_{i_3} \frac{\partial}{\partial t_{i_1}} \frac{\partial}{\partial t_{i_2+i_3-2}} + 2 \sum_{i_1,i_3,i_4=0}^{\infty} (i_3 + i_4 + 2) \hat{t}_{i_1+i_3+i_4+4} \frac{\partial}{\partial t_{i_1+i_3+i_4}} \frac{\partial}{\partial t_{i_4}},
\]

(48)

where \(\hat{t}_j = j t_j\), we have

\[ P_0^4 = 8, \quad P_{0,0,0}^4 = 16, \quad P_{0,0}^{1,3} = 6, \quad P_{0,0}^{3,1} = 18, \quad P_{0,0}^{2,2} = 8, \]

\[ P_{0,0,0,0}^{2,2} = 4, \quad P_{0,0,0}^{1,1,1} = 1, \quad P_{0,0}^{1,2,1} = P_{0,0}^{2,1,1} = P_{0,0}^{1,1,2} = 4. \]

(49)

Substituting (49) into (46), we obtain

\[
C_4(N) = \frac{1}{2^4 \cdot 2! \cdot \lambda(4)} (P_0^4 N + P_{0,0,0}^4 N^3) = 2N^3 + N,
\]

\[
C_{1,3}(N) = \frac{2!}{2^2 \cdot 2! \cdot \lambda(1,3)} (P_{0,0}^{1,3} + P_{0,0}^{3,1}) N^2 = 3N^2,
\]

\[
C_{2,2}(N) = \frac{2!}{2^2 \cdot 2! \cdot \lambda(2,2)} (P_{0,0}^{2,2} N^2 + P_{0,0,0,0}^{2,2} N^4) = N^4 + 2N^2,
\]

\[
C_{1,1,2}(N) = \frac{3!}{2^2 \cdot 2! \cdot \lambda(1,1,2)} [(P_{0,0}^{1,2,1} + P_{0,0,0}^{2,1,1}) N + P_{0,0,0,0}^{1,1,2} N^3] = N^3 + 2N,
\]

\[
C_{1,1,1,1}(N) = \frac{4!}{2^4 \cdot 2! \cdot \lambda(1,1,1,1)} P_{0,0}^{1,1,1,1} N^2 = 3N^2,
\]

(50)

where \(\lambda(4) = 1, \lambda(1,3) = 2, \lambda(2,2) = 1, \lambda(1,1,2) = 3\) and \(\lambda(1,1,1,1) = 1\).
4 Summary

It is known that the partition function of the Gaussian Hermitian matrix model can be obtained by acting on an elementary function with exponent of the operator $\hat{W}_{-2}$. This operator increases the grading in the sense (6). Based on the operators $\hat{W}_{-2}$ and $\hat{D}$ preserving the grading, we have constructed the $W_{1+\infty}$ constraints (15) for the Gaussian model, where the constraint operators yield not only the $W_{1+\infty}$ algebra, but also the closed $W_{1+\infty} n$-algebra. In contrast with the operator $\hat{W}_{-2}$, we observed that the operator $\frac{\partial}{\partial t}$ decreases the grading in the sense (13). Another $W_{1+\infty}$ constraints (22) for the Gaussian model have been presented in terms of the operators $\frac{\partial}{\partial t}$ and $\hat{D}$, where the constraint operators also constitute the closed $W_{1+\infty} (n)$-algebras. When particularized to the Virasoro constraints in (15) and (22), respectively, the corresponding constraint operators give the null 3-algebra.

Based on the Virasoro constraints (27), we have presented the exact correlators $C_{2\cdots 2}(N)$ (33). However, it appears to be impossible to obtain arbitrary correlators from (27). With the help of another Virasoro constraints (40), we have derived a new formula (46) for correlators in the Gaussian Hermitian matrix model. Our results confirm that the constraint operators which lead to the higher algebraic structures provide new insight into the matrix models.

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