Classification of Conformal Representations Induced from the Maximal Cuspidal Parabolic

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Abstract

In the present paper we continue the project of systematic construction of invariant differential operators on the example of representations of the conformal algebra induced from the maximal cuspidal parabolic.

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1 Introduction

Invariant differential operators play very important role in the description of physical symmetries. In a recent paper [1] we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the parabolic subgroups and subalgebras from which the necessary representations are induced. Thus we have set the stage for study of different non-compact groups and induction from different parabolics.

In the present paper we focus on the algebra \( \text{so}(4,2) \) and representations induced from the maximal cuspidal parabolic.

This paper is a sequel of [1] and we refer to it and [2] for motivations and extensive list of literature on the subject.

2 Preliminaries

Let \( G \) be a semisimple non-compact Lie group, and \( K \) a maximal compact subgroup of \( G \). Then we have an Iwasawa decomposition \( G = KA_0N_0 \), where \( A_0 \) is abelian simply connected vector subgroup of \( G \), \( N_0 \) is a nilpotent simply connected subgroup of \( G \) preserved by the action of \( A_0 \). Further, let \( M_0 \) be the centralizer of \( A_0 \) in \( K \). Then the subgroup \( P_0 = M_0A_0N_0 \) is a minimal parabolic subgroup of \( G \). A parabolic subgroup \( P = MAN \) is any subgroup of \( G \) which contains a minimal parabolic subgroup.

The importance of the parabolic subgroups comes from the fact that the representations induced from them generate all (admissible) irreducible representations of \( G \) [3, 4]. Actually, induction from the cuspidal parabolic subgroups is enough for this classification result.

For our purposes here we restrict to maximal cuspidal parabolic subgroups \( P \), so that rank \( A = 1 \).

Let \( \nu \) be a (non-unitary) character of \( A \), \( \nu \in A^* \), let \( \mu \) fix an discrete series representation \( D^\mu \) of \( M \) on a vector space \( V_\mu \), or a limit thereof.

We consider induced representation \( \chi = \text{Ind}^G_P(\mu \otimes \nu \otimes 1) \). These belong to the family of elementary representations of \( G \) [5]. (They are called generalized principal series representations (or limits thereof) in mathematical literature [6].) Their spaces of functions are:

\[
C_\chi = \{ \mathcal{F} \in C^\infty(G, V_\mu) \mid \mathcal{F}(g\text{man}) = e^{-\nu(H)} \cdot D^\mu(m^{-1})\mathcal{F}(g) \} \quad (2.1)
\]

where \( a = \exp(H) \in A \), \( H \in A \), \( m \in M \), \( n \in N \). The representation action is the left regular action:

\[
(T^\chi(g)\mathcal{F})(g') = \mathcal{F}(g^{-1}g') \ , \ g, g' \in G . \quad (2.2)
\]

Note that for our considerations it is enough to use the infinitesimal left action:

\[
(X_L\mathcal{F})(g) = \left. \frac{d}{dt} \mathcal{F}(\exp(-tX)g) \right|_{t=0} \quad (2.3)
\]
An important ingredient in our considerations are the highest weight representations of $G$ associated to the ER $\chi$. These can be realized as (factor-modules of) Verma modules $V^\Lambda$ over $G^C$, where $\Lambda \in (H^C)^*$, $H^C$ is a Cartan subalgebra of $G^C$, the weight $\Lambda = \Lambda(\chi)$ is determined uniquely from $\chi$ [7]. We recall that the Verma module is explicitly given by $V^\Lambda = U(G^C) \otimes v_0$, where we employ the triangular decomposition of $G^C$: $G^C = G^C_+ \oplus H^C \oplus G^C_-$, $G^C_+$, $G^C_-$ being the raising, lowering generators of $G^C$.

To employ this HWM structure we shall use the right action of $G^C$ by the standard formula:

$$ (X_R \mathcal{F})(g) \doteq \frac{d}{dt} \mathcal{F}(g \exp(tX))|_{t=0} \quad (2.4) $$

where $\mathcal{F} \in C_\chi$, $g \in G$, first $X \in G$, then we use complex linear extension to extend this action to $G^C$. Note that this action takes $\mathcal{F}$ out of $C_\chi$ for some $X$ but that is exactly why it is used for the construction of the intertwining differential operators.

Further we need to introduce $C$-valued realization $\tilde{C}_\chi$ of the space $C_\chi$ by the formula:

$$ \varphi(g) \equiv \langle u_0, \mathcal{F}(g) \rangle \quad (2.5) $$

where $\langle , \rangle$ is the $M$-invariant scalar product in $V_\mu$, $u_0$ is the highest weight vector in the discrete series representation space $V_\mu$, or the limit thereof.

On these functions the left/right action of $G^C$ is defined by:

$$ (X_{L/R} \varphi)(g) \equiv \langle u_0, (X_{L/R} \mathcal{F})(g) \rangle \quad (2.6) $$

Generically, Verma modules are irreducible, but for the construction of intertwining differential operators we need the reducible ones. There is a simple criterion [8]: the Verma module $V^\Lambda$ is reducible if holds

$$ (\Lambda + \rho, \beta^\vee) = m \quad , \quad \beta^\vee \equiv 2\beta/(\beta, \beta) \quad (2.7) $$

where $\beta \in \Delta^+$ (the positive roots of $(G^C, H^C)$), $m \in \mathbb{N}$, $(\cdot, \cdot)$ is the bilinear product in $(H^C)^*$, $\rho$ is half the sum of the positive roots.

Whenever the above is fulfilled there exists [9] in $V^\Lambda$ a submodule which is also a Verma module with shifted weight: $V^{\Lambda-m\beta}$. This submodule is generated by a unique vector $v^s \in V^\Lambda$, called singular vector, such that $v^s \notin C v_0$ and it has the properties of the highest weight vector of $V^{\Lambda-m\beta}$:

$$ X v^s = (\Lambda - m\beta)(X) \cdot v^s \quad , \quad X \in H^C $$

$$ X v^s = 0 \quad , \quad X \in G^C_+ $$

The above situation will be depicted as follows:

$$ V^\Lambda \longrightarrow V^{\Lambda-m\beta} \quad (2.9) $$

the arrow points to the submodule [7].

The singular vector is expressed via [7]

$$ v^s_{m,\beta} = \mathcal{P}_{m,\beta} v_0 $$
where $P_{m,\beta}(G^C)$ is a polynomial in the universal enveloping algebra $U(G^C)$.

Then there exists [7] an intertwining differential operator of order $m = m_{\beta}$:

$$D_{m,\beta} : \tilde{\mathcal{C}}_{\chi(\Lambda)} \longrightarrow \tilde{\mathcal{C}}_{\chi(\Lambda-m_{\beta})}$$  \ (2.10)

given explicitly by:

$$D_{m,\beta} = P_{m,\beta}((G^C)^R)$$  \ (2.11)

where $(G^C)^R$ denotes the right action of the elements of $G^C$ on the functions $\varphi$.

Thus, in each such situation we have an invariant differential equation of order $m = m_{\beta}$:

$$D_{m,\beta} \varphi = \varphi', \quad \varphi \in \tilde{\mathcal{C}}_{\chi(\Lambda)}, \quad \varphi' \in \tilde{\mathcal{C}}_{\chi(\Lambda-m_{\beta})}$$  \ (2.12)

One main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called multiplets [7, 10]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible ERs and the lines between the vertices correspond to intertwining operators. The explicit parametrization of the multiplets and of their ERs is important for understanding of the situation.

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators, as we shall demonstrate.

3 Multiplets of $so(4,2)$ using maximal cuspidal parabolic

Let $G = so(n,2)$, $n > 2$. It has three nonconjugate parabolic subalgebras:

$$\mathcal{P} = M \oplus A \oplus N$$
$$\mathcal{P}_0 = so(n-2) \oplus A_0 \oplus N_0$$
\[ \dim A_0 = 2, \quad \dim N_0 = 2(n-1) \]
$$\mathcal{P}_1 = so(n-2) \oplus sl(2, \mathbb{R}) \oplus A_1 \oplus N_1$$
\[ \dim A_1 = 1, \quad \dim N_1 = 2n-3 \]
$$\mathcal{P}_2 = so(n-1,1) \oplus A_2 \oplus N_2$$
\[ \dim A_2 = 1, \quad \dim N_2 = n \]

where $\mathcal{P}_0$ is the minimal parabolic, $\mathcal{P}_1$ is maximal cuspidal, $\mathcal{P}_2$ is maximal noncuspidal. Usually it is the latter that is used, but in the present paper we study induction from the maximal cuspidal parabolic subalgebra $\mathcal{P}_1$.

3.1 The $so(4,2)$ main multiplets using maximal cuspidal parabolic

For $so(4,2)$ the maximal cuspidal parabolic is:

$$\mathcal{P}_1 = so(2) \oplus sl(2, \mathbb{R}) \oplus A_1 \oplus N_1$$  \ (3.14)
The signatures of the ERs are

\[ \chi_1 = \{ n', k, \epsilon, \nu' \} \]  

where \( n' \in \mathbb{Z} \) is a character of \( SO(2) \), \( \nu' \in \mathbb{C} \) is a character of \( A_1 \), \( k, \epsilon \) fix a discrete series representation of \( SL(2, \mathbb{R}) \), \( k \in \mathbb{N} \), \( \epsilon = \pm 1 \), or a limit thereof when \( k = 0 \).

The relation with the \( sl(4) \) Dynkin labels is as follows [11]:

\[ m_1 = \frac{1}{2}(k - \nu' + n') \, , \, m_2 = -k \, , \, m_3 = \frac{1}{2}(k - \nu' - n') \]  

The main multiplet of reducible ERs contains 12 members which we parametrize as part of the main \( sl(4) \) multiplet with 24 members. Thus, the 12-plet has the following signatures:

\[
\begin{align*}
\Lambda_2 &= (m_{12}, -m_2, m_{23}) = \{ n' = m_1 - m_3, k = m_2, \nu' = -m_{13} \} \\
\Lambda_{12} &= (m_2, -m_{12}, m_{13}) = \{ -m_1 - m_3, m_{12}, -m_{23} \} \\
\Lambda_{32} &= (m_{13}, -m_{23}, m_2) = \{ m_1 + m_3, m_{23}, -m_{12} \} \\
\Lambda_{121} &= (-m_2, -m_1, m_{13}) = \{ -m_{13} - m_2, m_1, -m_3 \} \\
\Lambda_{132} &= (m_{23}, -m_{13}, m_1) = \{ m_3 - m_1, m_{13}, -m_2 \} \\
\Lambda_{232} &= (m_{13}, -m_3, -m_2) = \{ m_{13} + m_2, m_3, -m_1 \} \\
\Lambda_{1232} &= (m_{23}, -m_3, -m_1) = \{ m_{13} + m_2, m_3, m_1 \} \\
\Lambda_{1321} &= (-m_{23}, -m_1, m_1) = \{ -m_{13} - m_2, m_1, m_3 \} \\
\Lambda_{2132} &= (m_3, -m_{13}, m_1) = \{ m_3 - m_1, m_{13}, m_2 \} \\
\Lambda_{12132} &= (m_3, -m_{23}, -m_1) = \{ m_1 + m_3, m_{23}, m_1 \} \\
\Lambda_{21321} &= (-m_3, -m_{12}, -m_1) = \{ -m_1 - m_3, m_{12}, m_{23} \} \\
\Lambda_{121321} &= (-m_3, -m_2, -m_1) = \{ m_1 - m_3, m_2, m_{13} \}
\end{align*}
\]

where

\[ \Lambda_{i_1i_2...i_t} = \sigma_{i_t} \cdots \sigma_{i_2} \sigma_{i_1} \Lambda_0 \]

\( \sigma_j \), \( j = 1, 2, 3 \), are the three simple \( sl(4) \) reflections, \( \Lambda_0 \) is the ER, or rather the corresponding Verma module with dominant highest weight with signature \( (m_1, m_2, m_3) \), \( m_k \in \mathbb{N} \), which module is fixing the \( sl(4) \) 24-plet. We have given the signature in both the \( sl(4) \) signature notation \((\cdot, \cdot, \cdot)\) and in the \( P_1 \)-induced notation (3.15).

We would like to follow connections with ERs induced from the maximal noncuspidal parabolic \( P_2 \). Thus, below we shall replace the \( m_k \) notation with equivalent one:

\[ (p, \nu, n) = (m_1, m_2, m_3) \]

Thus, we give the same 12-plet in \( p, \nu, n \) parametrization and adding the Harish-Chandra (HC) parameters [12] for the three nonsimple roots in the order \( \alpha_{12}, \alpha_{23}, \alpha_{13} \). Thus, in the 4-th, 5-th, 6-th, place the parameters are : \( m_{12} = m_1 + m_2, m_{23} = m_2 + m_3, m_{13} = m_1 + m_2 + m_3 \), and of course they are redundant but some representation theoretic statements are formulated
easier in their terms. In particular, the \(K\)-noncompact HC parameters are:
\(m_2, m_{12}, m_{23}, m_{13}\).

Thus, the 12-plet is given now as:
\[
\begin{align*}
\Lambda^0_0 &= \Lambda_0 = (p + \nu, -\nu, n + \nu; p, n, p + \nu + n) = \chi^\nu_{p,n} \\
\Lambda^1_a &= \Lambda_{12} = (\nu, -p - \nu, p + \nu + n; -p, n, n + \nu) = \chi^{-\nu}_{p,n} \\
\Lambda^1_b &= \Lambda_{32} = (p + \nu + n, -n - \nu, \nu; p, -n, p + \nu) = \chi^{\nu}_{p,n} \\
\Lambda^2_c &= \Lambda_{121} = (-\nu, -p, p + \nu + n; -p - \nu, n + \nu, n) \\
\Lambda^2_d &= \Lambda_{132} = (n + \nu, -(p + \nu + n), p + \nu; -p, -n, \nu) = \chi^{\nu}_{p,n} \\
\Lambda^3_e &= \Lambda_{232} = (p + \nu + n, -n - \nu, \nu; p - \nu, n - \nu, p) \\
\Lambda^3_f &= \Lambda_{1232} = (n + \nu, -n, -p - \nu; \nu, -p - \nu - n, -p) \\
\Lambda^4_g &= \Lambda_{2132} = (n, -p - \nu + n, p; -p - \nu, -\nu - n, -\nu) = \chi^{+\nu}_{p,n} \\
\Lambda^4_h &= \Lambda_{12132} = (-n - \nu, -p, p + \nu; -p - \nu - n, -\nu, -\nu - n, ) \\
\Lambda^5_i &= \Lambda_{121321} = (-n, -\nu - n, -p - \nu, -p - \nu - n) \\
\end{align*}
\]

where we have also indicated the five cases \(\chi^{(\nu,\nu)}_{p,n}\) coinciding by signatures with ERs induced from the maximal noncuspidal parabolic. The notations \(\Lambda^\pm_0, \Lambda^\pm_a, \) etc, are used in Fig.1, where we present this 12-plet. The arrows denote both the intertwining differential operators between ERs and embeddings between Verma modules. Notation at the arrows denote the representation parameter and the root, e.g., \(\nu_1\) denotes singular vector (embedding) of weight \(\nu \alpha_1, p_{12}\) denotes singular vector (embedding) of weight \(\alpha_1 \alpha_2\). Note that only the \(M\)-non-compact roots are involved [7]. Thus, the \(M\)-compact root \(\alpha_2\) is not relevant for the intertwining differential operators.

Some remarks on the ERs content: As in the general situation the ERs \(\Lambda^+_d = \chi^{\nu}_{p,n}\) in (3.18) contain holomorphic discrete series representations when the discrete parameter \(\epsilon = 1\), and the antiholomorphic discrete series representations when the discrete parameter \(\epsilon = -1\). (The criterion for holomorphicity is that the \(K\)-compact HC parameters are positive, while the \(K\)-noncompact HC parameters are negative [6].)

In the ERs \(\Lambda^-_a = \chi^{\nu}_{p,n}\) and \(\Lambda^-_b = \chi^+_{p,n}\) are contained the massless representations with conformal weight \(d = 1 + j\) with spin \(j \geq 1\), where \(j = (p + 1)/2\) and \(j = (n + 1)/2\), resp. (the three with lower spin are considered below).

Our diagrams account also for the Knapp-Stein (KS) [13] integral operator relevant for \(P_1\) induction. It acts on the signatures as the highest \(sl(4)\) root \(\alpha_{13}, [11,14]\). Thus, on the \(P_1\)-signature it acts by changing the sign of the last entry in the \(\{\}\) -notation (\(\nu\)). On the figure the KS operators intertwine the ERs symmetric w.r.t. the dashed line. Of course, the KS from \(\Lambda^+_c, \Lambda^+_d, \Lambda^+_e\) to \(\Lambda^-_c, \Lambda^-_d, \Lambda^-_e\), resp., degenerated to differential operators of degrees \(n, \nu, p\), resp., as shown on Fig.1. The KS operators from \(\Lambda^+_c, \Lambda^+_d, \Lambda^+_e\) to \(\Lambda^-_c, \Lambda^-_d, \Lambda^-_e\), resp., remain integral operators.

The same remarks about the KS integral operators will be true verbatim for all further Figures below and will not be repeated.
3.2 Reduced multiplets

We have several types of reduced multiplets.

3.2.1 Symmetrically reduced multiplets

- The first case is a septuplet depending on two parameters, and the signatures may be obtained by setting formally \(\nu = 0\) in (3.17):

\[
\begin{align*}
\Lambda_2 &= (p, 0, n) = \{n' = p - n, k = 0, \nu' = -p - n\} = \chi_{p0n}^- = 2\chi_{pn}^-
\Lambda_{12} &= (0, -p, p + n) = \{-p - n, p, -n\} = \chi_{p0n}^+
\Lambda_{32} &= (p + n, -n, 0) = \{p + n, n, -p\} = \chi_{p0n}^+
\Lambda_{132} &= (n, -p - n, p) = \{n - p, p + n, 0\} = \chi_{p0n}^+ = \chi_{pn}^+ = 2\chi_{pn}^+
\Lambda_{1232} &= (n, -n, -p) = \{p + n, n, p\}
\Lambda_{1321} &= (-n, -p, p) = \{-p - n, p, n\}
\Lambda_{121321} &= (-n, 0, -p) = \{p - n, 0, p + n\}
\end{align*}
\]

(3.19)

or adding the HC parameters:

\[
\begin{align*}
\Lambda_0^- &= \Lambda_2 = (p, 0, n; p, n, p + n) = 2\chi_{pn}^-
\Lambda_a^- &= \Lambda_{12} = (0, -p, p + n; -p, n, n)
\Lambda_b^- &= \Lambda_{32} = (p + n, -n, 0; p, -n, p)
\Lambda_c &= \Lambda_{132} = (n, -p - n, p; -p, -n, 0) = 2\chi_{pn}^+
\Lambda_b^+ &= \Lambda_{132} = (n, -n, -p; 0, -n - p, -p)
\Lambda_a^+ &= \Lambda_{3212} = (-n, -p, p; -n - p, 0, -n)
\Lambda_0^+ &= \Lambda_{213213} = (-n, 0, -p; -n, -p, -n - p)
\end{align*}
\]

(3.20)

The first and fourth entries \(2\chi_{pn}^\pm\) form a doublet w.r.t. induction from the maximal noncuspidal parabolic.

The first entry \(2\chi_{pn}^-\) is induced from a limit of \(sl(2, \mathbb{R})\) discrete series. For \(p = n = 1\) it contains the scalar massless representation with conformal weight \(d = 1\).

The fourth entry \(2\chi_{pn}^+\) contains limits of holomorphic/antiholomorphic discrete series (for \(\epsilon = \pm 1\)).

This septuplet is shown on Fig.2.

3.2.2 Asymmetrically reduced multiplets

- The second case is also septuplet depending on two parameters, the signatures may be obtained by setting formally \(p = 0\) in (3.17):

\[
\begin{align*}
\Lambda_{0n}^- &= \Lambda_2 = (\nu, -\nu, n + \nu; 0, n, \nu + n) = \chi_{0\nu n}^- = \chi_{0\nu n}^+ = 1\chi_{\nu n}^-
\Lambda_b^- &= \Lambda_{32} = (\nu + n, -n - \nu, \nu; 0, -n, \nu) = \chi_{0\nu n}^+ = \chi_{0\nu n}^- = 1\chi_{\nu n}^+
\Lambda_d^- &= \Lambda_{212} = (-\nu, 0, \nu + n; -\nu, n + \nu, n)
\Lambda_c &= \Lambda_{232} = (\nu + n, -n, -\nu; \nu, -n - \nu, 0)
\Lambda_d^+ &= \Lambda_{3212} = (-n - \nu, 0, \nu; -\nu - n, \nu, -n)
\Lambda_b^+ &= \Lambda_{2132} = (n, -n + \nu, 0; -\nu, -\nu - n, -\nu) = \chi_{0\nu n}^+
\Lambda_{0n}^+ &= \Lambda_{23212} = (-n, -\nu, 0; -\nu - n, -\nu, -\nu - n)
\end{align*}
\]

(3.21)
The first two entries $\chi_{\nu n}^{\pm}$ form a doublet w.r.t. induction from the maximal noncuspidal parabolic. The ER $\chi_{11}^{\pm}$ contains one (of the two) spin 1/2 massless representations with conformal weight $d = 3/2$.

The third and fifth entries are induced from limits of discrete series.

This septuplet is shown on Fig.3n.

The conjugate septuplet may be obtained by setting formally

$$\Lambda_{0p} = \Lambda_2 = (p + \nu, -\nu, \nu; 0, 0, p + \nu) = \chi_{0\nu 0}^{r-} = \chi_{0\nu 0}^{m+} = 3\chi_{\nu 0}^{p-}$$
$$\Lambda_a = \Lambda_{12} = (-\nu, -p - \nu, p + \nu; -p, 0, \nu) = \chi_{p\nu 0}^{r-} = \chi_{p\nu 0}^{r+} = 3\chi_{p\nu 0}^{p+}$$
$$\Lambda_\nu = \Lambda_{232} = (p + \nu, 0, -\nu; p + \nu, -\nu, p)$$
$$\Lambda_{d} = \Lambda_{212} = (-\nu, -p - \nu, p + \nu; -p - \nu, \nu, 0)$$
$$\Lambda_{e} = \Lambda_{232} = (\nu, 0, -\nu - p - \nu, -\nu - p - \nu, \nu) = \chi_{p\nu 0}^{p+}$$
$$\Lambda_0^+ = \Lambda_{213213} = (0, -\nu, -p; -\nu, -p - \nu, -p - \nu)$$

(3.22)

The interpretation of the members is exactly as for the last case, e.g., the first two entries $3\chi_{p\nu}^{\pm}$ form a doublet w.r.t. induction from the maximal noncuspidal parabolic. The ER $3\chi_{11}^{\pm}$ contains the other spin 1/2 massless representation with conformal weight $d = 3/2$.

This septuplet is shown on Fig.3p.

### 3.2.3 Symmetrically doubly reduced multiplets

- The next case is a quartet depending on one parameter with signatures may be obtained by setting formally $p = n = 0$ in (3.17):

$$\Lambda_0^- = \Lambda_2 = (\nu, -\nu, \nu; 0, 0, 0) = \chi_{00\nu}^{r-} = \chi_{00\nu}^{m-} = \chi_{00\nu}^{r+} = \chi_{00\nu}^{p+}$$
$$\Lambda_{d} = \Lambda_{212} = (-\nu, 0, 0, -\nu, 0)$$
$$\Lambda_{e} = \Lambda_{232} = (\nu, 0, -\nu, -\nu, 0)$$
$$\Lambda_0^+ = \Lambda_{2132} = (0, -\nu, 0, -\nu, 0) = \chi_{00\nu}^{p+}$$

(3.23)

The first entry is a singlet w.r.t. induction from the maximal noncuspidal parabolic. For $\nu = 1$ it is a Lorentz scalar positive energy representation with conformal weight $d = 2$, i.e., above the unitarity threshold $d = 1$, but below the limit of holomorphic discrete series $d = 3$. The second and third entries are induced from limits of discrete series.

This quartet is shown on Fig.4.

### 3.2.4 Asymmetrically doubly reduced multiplets

- The next two conjugate cases are triplets depending on one parameter.

The first be obtained by setting formally $p = \nu = 0$ in (3.17):

$$\Lambda_{00\nu}^- = \Lambda_2 = (0, 0, 0; 0, 0, 0) = \chi_{000}^{r-} = \chi_{000}^{m-}$$
$$\Lambda_6 = \Lambda_{32} = (n, -n, 0; 0, -n, 0) = \chi_{00n}^{m+} = \chi_{00n}^{r+} = \chi_{00n}^{p+}$$
$$\Lambda_0^- = \Lambda_{3212} = (-n, 0, 0; -n, 0, -n)$$

(3.24)
The first and third entries are induced from limits of discrete series.

- The conjugate case:

\[
\Lambda_{0\nu} = \Lambda_{2} = (p, 0, 0; p, 0, p) = \chi_{p00}^{+} = \chi_{p00}^{-}
\]
\[
\Lambda_{a} = \Lambda_{12} = (0, -p, p; -p, 0, 0) = \chi_{p00}^{+} = \chi_{p00}^{-} = \chi_{p00}^{+}
\]
\[
\Lambda_{0\nu}^{+} = \Lambda_{12321} = (0, 0, -p; 0, -p, -p)
\]

(3.25)

The two conjugate triplets are shown on Fig. 5p,5n.

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Fig. 1. Main multiplets for $su(2, 2)$

using maximal cuspidal parabolic
Fig. 2. Symmetrically reduced multiplets for $su(2, 2)$ using maximal cuspidal parabolic

Fig. 3p,3n. Asymmetrically reduced multiplets for $su(2, 2)$ using maximal cuspidal parabolic
Fig. 4. Symmetrically doubly reduced multiplets for \( su(2, 2) \) using maximal cuspidal parabolic

Fig. 5p,5n. Asymmetrically doubly reduced multiplets for \( su(2, 2) \) using maximal cuspidal parabolic