An asymptotic expansion of the Casorati determinant and its application to discrete integrable systems

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Abstract. The Hankel determinant appears in the representation of solutions to several integrable systems. Asymptotic expansion of the Hankel determinant thus plays a key role for investigating asymptotic analysis of such integrable system. In this paper, an asymptotic expansion formula of a certain Casorati determinant is presented as an extension of the Hankel case. It is also shown that an application of it to an asymptotic analysis of the discrete hungry Lotka-Volterra system, which is one of basic models in mathematical biology.

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1. Introduction

The Toda equation and the Lotka-Volterra (LV) system are the basic integrable systems which describe the current-voltage in an electric circuit and a prey-predator relationship of distinct species, respectively. The discrete Toda equation \([7]\) is a time-discretization of the Toda equation, and is known to be just equal to the recursion formula of the qd algorithm for computing eigenvalues of symmetric tridiagonal matrix \([11, 6]\) and singular values of bidiagonal one \([9]\). The discrete LV (dLV) system \([13]\), which is a time-discretization of the LV system, also has an interesting application to computing for bidiagonal singular values \([9]\). The solutions to both the discrete Toda equation and the dLV system are expressed by using the Hankel determinant, for bidiagonal singular values \([10]\). The discrete LV (dLV) system \([13]\), which is a qd algorithm for computing eigenvalues of symmetric tridiagonal matrix \([11, 6]\) and of the Toda equation, and is known to be just equal to the recursion formula of the

\[
H_0^{(n)} := 1,
\]

\[
H_j^{(n)} := \begin{vmatrix}
    a^{(n)} & a^{(n+1)} & \cdots & a^{(n+j-1)} \\
    a^{(n+1)} & a^{(n+2)} & \cdots & a^{(n+j)} \\
    \vdots & \vdots & \ddots & \vdots \\
    a^{(n+j-1)} & a^{(n+j)} & \cdots & a^{(n+2j-2)}
\end{vmatrix}, \quad j = 1, 2, \ldots, \tag{1}
\]

where \(j\) and \(n\) correspond the discrete spatial variable and the discrete time one, respectively \([13]\). Here, the formal power series \(f(z) = \sum_{n=0}^{\infty} a^{(n)} z^n\) associated with \(H_j^{(n)}\) is assumed to be analytic at \(z = 0\) and meromorphic in the disk \(D = \{z | |z| < \sigma\}\). The finite or infinite number of poles \(u_1^{-1}, u_2^{-1}, \ldots\) of \(f(z)\) are numbered such that \(0 < |u_1^{-1}| < |u_2^{-1}| < \cdots < \sigma\). Then, there exists a nonzero constant \(c_j\) independent of \(n\) such that, for \(\sigma\) satisfying \(|u_j| > \sigma > |u_{j+1}|\),

\[
H_j^{(n)} = c_j (u_1 u_2 \cdots u_j)^n \left(1 + O \left(\frac{\sigma}{|u_j|}\right)^n\right), \tag{2}
\]

as \(n \to \infty\) \([6]\). The asymptotic expansion \((2)\) of the Hankel determinant \((1)\) as \(n \to \infty\) enables us to analyze the discrete Toda equation and the dLV system asymptotically as in \([11, 6]\) and in \([9]\), respectively.

A generalization of the Hankel determinant \(H_j^{(n)}\) is the determinant of a nonsymmetric square matrix of order \(j\),

\[
C_{k,0}^{(n)} := 1,
\]

\[
C_{k,j}^{(n)} := \begin{vmatrix}
    a_k^{(n)} & a_{k+1}^{(n)} & \cdots & a_{k+j-1}^{(n)} \\
    a_{k+1}^{(n+1)} & a_{k+1}^{(n+1)} & \cdots & a_{k+j-1}^{(n+1)} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_k^{(n+j-1)} & a_{k+1}^{(n+j-1)} & \cdots & a_{k+j-1}^{(n+j-1)}
\end{vmatrix}, \quad k = 0, 1, \ldots, \quad j = 1, 2, \ldots, \tag{3}
\]

which is called the Casoratian determinant, or the Casoratian. According to \([15]\), the Casoratian is a useful determinant in the theory of difference equations, which plays the role similar to the Wronskian in the theory of differential equations, and it has some applications to difference equations in mathematical physics. No one feels something
wrong that the formal power series \( f_k(z) = \sum_{n=0}^{\infty} a_k^{(n)} z^n \) is associated with the Casorati determinant \( C_{k,j}^{(n)} \) for each \( k \). The formal power series \( f_0(z), f_1(z), \ldots \) differ from \( f(z) \) in that not only the subscripts but also superscripts appear in the coefficients.

The main purpose of this paper is to present an asymptotic expansion of \( C_{k,j}^{(n)} \) as \( n \to \infty \). As an application of it, we also give an asymptotic analysis for the discrete hungry LV (dhLV) system, which is a generalization of the dLV system. The dhLV system is a time-discretization of the hungry LV (hLV) system \([1, 8]\) which grasps more complicated prey-predator relation, and is shown in \([2, 3, 16]\) to enable us to give the \( LR \) and the sifted \( LR \) transformations for computing eigenvalues of a banded totally nonnegative matrix whose all minors are nonnegative.

This paper is organized as follows. In Section 2, we first observe that the entries in \( C_{k,j}^{(n)} \) can be expressed by using poles of \( f_k(z) \). We next give an asymptotic expansion of the Casorati determinant in terms of poles of \( f_k(z) \) as \( n \to \infty \) by expanding the theorem on analyticity for Hankel determinant in \([6]\). With the help of the resulting theorem, in Section 3, we also clarify asymptotic behaviors of the solution to the dhLV system. Finally, we give concluding remark in Section 4.

### 2. An asymptotic expansion of the Casorati determinant

In this section, we first give an expression of the entries of the Casorati determinant \( C_{k,j}^{(n)} \) in terms of the poles of the formal power series \( f_k(z) \) associated with \( C_{k,j}^{(n)} \). Referring to the theorem on analyticity for Hankel determinant in \([6]\), we next present an asymptotic expansion of the Casorati determinant \( C_{k,j}^{(n)} \) as \( n \to \infty \) by using the poles of \( f_k(z) \). We also describe an asymptotic expansion of \( C_{k,j}^{(n)} \) as \( n \to \infty \) under some restriction on the poles of \( f_k(z) \).

Let \( f_k(z) = \sum_{n=0}^{\infty} a_k^{(n)} z^n \), which is the formal power series associated with \( C_{k,j}^{(n)} \) for \( k = 0, 1, \ldots \), be analytic at \( z = 0 \) and meromorphic in the disk \( D = \{ z | |z| < \sigma \} \). Moreover, let \( r_{1,k}^{-1}, r_{2,k}^{-1}, \ldots \), denote the poles of \( f_k(z) \) such that \( |r_{1,k}^{-1}| < |r_{2,k}^{-1}| < \ldots < \sigma \). By extracting the principal parts in \( f_k(z) \), we derive

\[
 f_k(z) = \frac{\alpha_{1,k}}{r_{1,k}^{-1} - z} + \frac{\alpha_{2,k}}{r_{2,k}^{-1} - z} + \cdots + \frac{\alpha_{j,k}}{r_{j,k}^{-1} - z} + \sum_{n=0}^{\infty} b_k^{(n)} z^n, \tag{4}
\]

where \( \alpha_{1,k}, \alpha_{2,k}, \ldots, \alpha_{j,k} \) are some nonzero constants and \( b_k^{(n)} \), which contains the terms with respect to \( r_{j+1,k}^{-1}, r_{j+2,k}^{-1}, \ldots \), satisfies

\[
 |b_k^{(n)}| \leq \mu_k r_k^{-n} \tag{5}
\]

for some nonzero positive constants \( \mu_k \) and \( \rho_k \) with \( |r_{j+1,k}| < \rho_k < |r_{j,k}| \). The proof of \([5]\) is given in \([6]\) through the Cauchy coefficient estimate. We here give a lemma for an expression of \( a_k^{(n)} \) appearing in \( f_k(z) = \sum_{n=0}^{\infty} a_k^{(n)} z^n \).

**Lemma 1.** Let us assume that the poles \( r_{1,k}^{-1}, r_{2,k}^{-1}, \ldots, r_{j,k}^{-1} \) of \( f_k(z) \) are not multiple.
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Then \( a_k^{(n)} \) is expressed by using \( r_{1,k}, r_{2,k}, \ldots, r_{j,k} \) as

\[
a_k^{(n)} = \sum_{\ell=1}^{j} c_{\ell,k} r_{\ell,k}^{n+k+1} + b_k^{(n)},
\]

where \( c_{1,k}, c_{2,k}, \ldots, c_{j,k} \) are some nonzero constants.

**Proof.** The key point is the replacement \( \alpha_{1,k} = c_{1,k} r_{1,k}^k, \alpha_{2,k} = c_{2,k} r_{2,k}^k, \ldots, \alpha_{j,k} = c_{j,k} r_{j,k}^k \) in (4), namely,

\[
f_k(z) = \frac{c_{1,k} r_{1,k}^k}{r_{1,k}^k - z} + \frac{c_{2,k} r_{2,k}^k}{r_{2,k}^k - z} + \ldots + \frac{c_{j,k} r_{j,k}^k}{r_{j,k}^k - z} + \sum_{n=0}^{\infty} b_k^{(n)} z^n.
\]

Since each \( c_{\ell,k} r_{\ell,k}^k/(r_{\ell,k}^k - z) \) in (7) can be regarded as the summation of geometric series, we get

\[
f_k(z) = \sum_{n=0}^{\infty} \left( \sum_{\ell=1}^{j} c_{\ell,k} r_{\ell,k}^{n+k+1} \left( n \right) \right) + b_k^{(n)} z^n,
\]

which implies (6). \( \square \)

Along the line similar to an asymptotic expansion as \( n \to \infty \) of the Hankel determinant \( H_k^{(n)} \) in [6], we have the following theorem for that of the Casorati determinant \( C_{k,j}^{(n)} \) in (3).

**Theorem 1.** Let us assume that the poles \( r_{1,k}^{-1}, r_{2,k}^{-1}, \ldots, r_{j,k}^{-1} \) of \( f_k(z) \) are not multiple. Then there exists some constant \( c_{\kappa_1,\kappa_2,\ldots,\kappa_j} \) independently of \( n \) such that, as \( n \to \infty \),

\[
C_{k,j}^{(n)} = \sum_{\kappa_1,\kappa_2,\ldots,\kappa_j=1,2,\ldots} \left[ c_{\kappa_1,\kappa_2,\ldots,\kappa_j} \left( r_{\kappa_1,k} r_{\kappa_2,k+1} \ldots r_{\kappa_j,k+j-1} \right)^n \right]
\]

\[
\left( 1 + \sum_{\ell=1}^{j} O \left( \left( \frac{\rho_{k+\ell-1}}{|r_{\kappa_{\ell,k},k+\ell-1}|} \right)^n \right) \right)
\]

where \( \rho_{k+\ell-1} \) is some constant such that \( r_{j+1,k+\ell-1} < \rho_{k+\ell-1} < |r_{j,k+\ell-1}| \).

**Proof.** By applying Lemma [1] and the addition formula of determinant to the Casorati determinant \( C_{k,j}^{(n)} \), we derive

\[
C_{k,j}^{(n)} = \sum_{\kappa_1,\kappa_2,\ldots,\kappa_j=1,2,\ldots} D_{k,\kappa_1,\kappa_2,\ldots,\kappa_j}^{(n)} + \sum_{\kappa_1,\kappa_2,\ldots,\kappa_j=1,2,\ldots,j} \hat{D}_{k,\kappa,\kappa_2,\ldots,\kappa_j}^{(n)}
\]

(9)
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where in the 1st summation

\[
D^{(n)}_{k_1,k_2,\ldots,k_j} := \begin{vmatrix}
C_{k_1,k_1+k+1} & C_{k_1,k_2+k+2} & \cdots & C_{k_1,k_2,j+1} \\
C_{k_2,k_1+k+1} & C_{k_2,k_2+k+2} & \cdots & C_{k_2,k_2,j+1} \\
\vdots & \vdots & \ddots & \vdots \\
C_{k_1,k_1+k+2} & C_{k_1,k_2+k+3} & \cdots & C_{k_1,k_2,j+1} \\
\end{vmatrix},
\]

and \( \hat{D}^{(n)}_{k_1,k_2,\ldots,k_j} \) in the 2nd summation denotes a determinant of the same form as \( D^{(n)}_{k_1,k_2,\ldots,\kappa_j} \) except that at least one of the \( \kappa_j \) columns are replaced with \( b^{(n+1)},\ldots,b^{(n+j-1)} \). By evaluating the 1st summation in (9), we get

\[
\sum_{\kappa_1,\kappa_2,\ldots,\kappa_j=1,2,\ldots,j} D^{(n)}_{k_1,k_2,\ldots,\kappa_j} = \sum_{\kappa_1,\kappa_2,\ldots,\kappa_j=1,2,\ldots,j} c_{\kappa_1,\kappa_2,\ldots,\kappa_j} \left( r_{\kappa_1,k} r_{\kappa_2,k+1} \cdots r_{\kappa_j,k+j-1} \right)^n \tag{10}
\]

where \( c_{\kappa_1,\kappa_2,\ldots,\kappa_j} \) is a constant

\[
c_{\kappa_1,\kappa_2,\ldots,\kappa_j} = \begin{vmatrix}
C_{\kappa_1,k_1+1,k_1} & C_{\kappa_1,k_2+1,k_2} & \cdots & C_{\kappa_1,k_2,j+1} \\
C_{\kappa_2,k_1+1,k_2} & C_{\kappa_2,k_2+1,k_2} & \cdots & C_{\kappa_2,k_2,j+1} \\
\vdots & \vdots & \ddots & \vdots \\
C_{\kappa_1,k_1+1,k_2} & C_{\kappa_1,k_2+1,k_2} & \cdots & C_{\kappa_1,k_2,j+1} \\
\end{vmatrix}.
\tag{11}
\]

In order to estimate the 2nd summation in (9), we consider the case where 1st column is replaced with \( b_1 \). For example, it follows that

\[
\sum_{\kappa_1,\kappa_2,\ldots,\kappa_j=1,2,\ldots,j} \sum_{\ell=1}^{j} \left( r_{\kappa_1,k} r_{\kappa_2,k+1} \cdots r_{\kappa_\ell+1,k+\ell-1} r_{\kappa_\ell+1,k+\ell} \cdots r_{\kappa_j,k+j-1} \right)^n = O((\rho_k r_{\kappa_2,k+1} \cdots r_{\kappa_j,k+j-1})^n).
\tag{11}
\]

It is easy to check all the permutations for \( \kappa_1,\kappa_2,\ldots,\kappa_j \) in the 2nd summation. Thus, we can rewrite the second summation as

\[
\sum_{\kappa_1,\kappa_2,\ldots,\kappa_j=1,2,\ldots,j} \sum_{\ell=1}^{j} O(r_{\kappa_1,k} r_{\kappa_2,k+1} \cdots r_{\kappa_\ell+1,k+\ell-1} r_{\kappa_\ell+1,k+\ell} \cdots r_{\kappa_j,k+j-1}).
\tag{12}
\]

Therefore, by taking account that \( r_{1,k} > r_{2,k} > \cdots > r_{j,k} > \rho_k > |r_{j+1,k}| \), from (10)–(12) we get (8).

Hereinafter, let us consider the restricted case where \( r_{1,k} = r_1, r_{2,k} = r_2, \ldots, r_{j,k} = r_j \) in \( f_k(z) \). This restriction admits the relationship of \( a^{(n)}_0, a^{(n)}_1, \ldots \), for example, appearing in the next section concerning an asymptotic analysis for dhLV system as \( n \to \infty \). Then, by the replacement of \( r_{\ell,k} \) with \( r_{\ell} \) in (5), we easily get

\[
a^{(n)}_k = \sum_{\ell=1}^{j} c_{\ell,k} r_{\ell,k}^{n+\ell+1} + b^{(n)}_k.
\tag{13}
\]

As a specialization of Theorem 1, we thus derive the following theorem for an asymptotic expansion of the Casorati determinant \( C^{(n)}_{k,j} \) with the restricted \( a^{(n)}_k \) as \( n \to \infty \).
Theorem 2. Let us assume that the poles $r_1^{-1}, r_2^{-1}, \ldots, r_j^{-1}$ of $f_k(z)$ are not multiple. Then there exists some constant $c_{k,j} \neq 0$ independently of $n$ such that, for $|r_{j+1}| < \rho_k < |r_j|$, as $n \to \infty$,

$$C_{k,j}^{(n)} = c_{k,j}(r_1 r_2 \ldots r_j)^n \left(1 + \sum_{\ell=1}^{j} O \left( \frac{\rho_{k+\ell-1}}{|r_j|} \right) \right). \quad (14)$$

Proof. The replacement $r_{1,k} = r_1, r_{2,k} = r_2, \ldots, r_{j,k} = r_j$ in (11) gives

$$c_{\kappa_1, \kappa_2, \ldots, \kappa_j} = \begin{vmatrix} c_{\kappa_1,k} r_{\kappa_1}^{k+1} & c_{\kappa_2,k} r_{\kappa_2}^{k+1} & \cdots & c_{\kappa_j,k} r_{\kappa_j}^{k+1} \\ c_{\kappa_1,k} r_{\kappa_1}^{k+2} & c_{\kappa_2,k} r_{\kappa_2}^{k+2} & \cdots & c_{\kappa_j,k} r_{\kappa_j}^{k+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{\kappa_1,k} r_{\kappa_1}^{k+j} & c_{\kappa_2,k} r_{\kappa_2}^{k+j} & \cdots & c_{\kappa_j,k} r_{\kappa_j}^{k+j} \end{vmatrix}. \quad (15)$$

So, we simplify (10) as

$$\sum_{\kappa_1, \kappa_2, \ldots, \kappa_j = 1, 2, \ldots, j} D_{k, \kappa_1, \kappa_2, \ldots, \kappa_j}^{(n)} \quad (16)$$

$$= c_{\kappa_1,k} c_{\kappa_2,k+1} \cdots c_{\kappa_j,k+j-1} (r_1 r_2 \ldots r_j)^n \times \sum_{\kappa_1, \kappa_2, \ldots, \kappa_j = 1, 2, \ldots, j} \begin{vmatrix} p_{\kappa_1}^{k+1} & p_{\kappa_2}^{k+1} & \cdots & p_{\kappa_j}^{k+1} \\ p_{\kappa_1}^{k+2} & p_{\kappa_2}^{k+2} & \cdots & p_{\kappa_j}^{k+2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{\kappa_1}^{k+j} & p_{\kappa_2}^{k+j} & \cdots & p_{\kappa_j}^{k+j} \end{vmatrix} \quad (17)$$

It is of significance to note that there exists a constant $\rho_k$, which is not equal to one in Theorem 1 such that $|r_{j+1}| < \rho_k < |r_j|$. This is because $\rho_k$ and $\rho_{k+1}$ does not always satisfy $\rho_k = \rho_{k+1}$ even if $r_{1,k} = r_1, r_{2,k} = r_2, \ldots, r_{j,k} = r_j$ in Theorem 1. Thus, (12) becomes

$$\sum_{\ell=1}^{j} O \left( (r_1 r_2 \ldots r_{j-1} r_{k+\ell-1})^n \right). \quad (18)$$

Therefore, from (16) and (18) we have (14). \qed

Theorem 1 is expected to be useful for asymptotic analysis of dynamical system whose solution are expressed in terms of the Casorati determinant $C_{k,j}^{(n)}$. Theorem 2 also covers an asymptotic expansion of the Hankel determinant.

3. Asymptotic analysis for the discrete hungry Lotka-Volterra system

In this section, we first explain that the solution to the dhLV system is written in terms of the Casorati determinant By using Theorem 2, we next clarify an asymptotic behavior of the dhLV variables as $n \to \infty$.

The hLV system is known as one of the mathematical prey-predator models which is an extension of the LV system. The hLV system differs from the simple LV system in
that more than one food and predator exists for each species. A skillful discretization of the hLV system enable us to give the dhLV system with hungry degree $M$,

$$
\frac{u_k^{(n+1)}}{u_k^{(n)}} = \prod_{j=1}^{M} \frac{\delta^{(n)} + u_{k+j}^{(n)}}{\delta^{(n+1)} + u_{k-j}^{(n)}},
$$

\[ k = 0, 1, \ldots, (M + 1)m - M - 1, \quad n = 0, 1, \ldots, \] (19)

$$
u_{M}^{(n)} := 0, u_{M+1}^{(n)} := 0, \ldots, u_{-1}^{(n)} := 0,
$$

$$
u_{(M+1)m-M}^{(n)} := 0, u_{(M+1)m+1}^{(n)} := 0, \ldots, u_{(M+1)m-1}^{(n)} := 0,
$$

where $u_k^{(n)}$ and $\delta^{(n)}$ denote the number of the $k$th species and the discretization parameter at the discrete time $n$, respectively. The dhLV system (19) with $M = 1$ coincides with the simple LV system. It is shown in [14] that the dhLV system through considering the three-term recurrence $T_{k+1}(x) = xT_k(x) - u_k^{(n)}T_{k-2}(x)$. Similarly, it is easy to get the arbitrary integer case of $M$ through considering the three-term recurrence $T_{k+1}(x) = xT_k(x) - u_k^{(n)}T_{k-M}(x)$. At a first glance, the dhLV system (19) seems to differ from in [2] from the viewpoint of the position of discrete parameter $\delta^{(n)}$, but they are essentially equivalent to each other.

Let us introduce the auxiliary variable

$$
u_k^{(n)} := u_{k-M}^{(n)} \prod_{j=1}^{M} (\delta^{(n)} + u_{k-M-j}^{(n)}),
$$

\[ k = M, M + 1, \ldots, (M + 1)m - 1. \] (20)

Then, $\nu_k^{(n)}$ can be expressed according to [14] as

$$
u_k^{(n)} = \frac{\tau_{k+1}^{(n)}\tau_{k-M}^{(n)}}{\tau_k^{(n)}\tau_{k-M+1}^{(n)}}, \quad k = M, M + 1 \ldots, (M + 1)m - 1,
$$

by using the tau-function $\tau_k^{(n)}$ with the following determinant representation in the cases where $k = j(M + 1)$ and $k = i + j(M + 1)$,

$$
\tau_0^{(n)} := 1, \quad \tau_{j(M+1)}^{(n)} := \left| \begin{array}{cccc}
\tau_{0,M}^{(n)} & \tau_{1,M}^{(n)} & \cdots & \tau_{j-1,M}^{(n)} \\
\tau_{M,M}^{(n)} & \tau_{1,M}^{(n)} & \cdots & \tau_{j-1,M}^{(n)} \\
\vdots & \vdots & \ddots & \vdots \\
\tau_{(j-1)M,M}^{(n)} & \tau_{(j-1)M+1,M}^{(n)} & \cdots & \tau_{(j-1)(M+1)-1,M}^{(n)} \\
\tau_{jM,i-1}^{(n)} & \tau_{jM+1,i-1}^{(n)} & \cdots & \tau_{j(M+1)-1,i-1}^{(n)}
\end{array} \right|,
$$

$$
\tau_{i+j(M+1)}^{(n)} := \left| \begin{array}{cccc}
\tau_{0,M}^{(n)} & \tau_{1,M}^{(n)} & \cdots & \tau_{j-1,M}^{(n)} \\
\tau_{M,M}^{(n)} & \tau_{1,M}^{(n)} & \cdots & \tau_{j-1,M}^{(n)} \\
\vdots & \vdots & \ddots & \vdots \\
\tau_{(j-1)M,M}^{(n)} & \tau_{(j-1)M+1,M}^{(n)} & \cdots & \tau_{(j-1)(M+1)-1,M}^{(n)} \\
\tau_{jM,i-1}^{(n)} & \tau_{jM+1,i-1}^{(n)} & \cdots & \tau_{j(M+1)-1,i-1}^{(n)}
\end{array} \right|,
$$

\[ i = 1, 2, \ldots, M, \]
where
\[
\tau^{(n)}_{\ell,s} := \begin{pmatrix}
\alpha^{(n)}_{\ell} & \alpha^{(n)}_{\ell+1} & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
\alpha^{(n)}_{\ell+s} & \cdots \\
\end{pmatrix} \in \mathbb{R}^{(s+1) \times (s+1)},
\]
with the relationship concerning the evolution from \(n\) to \(n+1\),
\[
a^{(n+1)}_{k} = a^{(n)}_{k} - (\delta^{(n)})^{M+1} a^{(n)}_{k}.
\] (22)

It is easy to check that (22) admits the assumption \(r_{1,k} = r_{1}, r_{2,k} = r_{2}, \ldots, r_{j,k} = r_{j}\) in \(f_{k}(z)\). The 1st, the 2nd, \ldots, the \((j-1)\)th raw and column blocks in \(\tau^{(n)}_{1+j(M+1)}\) are \(M\)-by-\(M\) matrices, but the \(j\)th raw and column blocks in it are \((i-1)\)-by-\((i-1)\) matrices. The following lemma gives the representation of \(v_{k}^{(n)}\) in terms of \(C_{k,j}^{(n)}\) appearing in Section 1.

**Lemma 2.** The auxiliary variable \(v_{k}^{(n)}\) is expressed as
\[
v_{i+j(M+1)}^{(n)} = \frac{C^{(n)}_{i,j+1} C^{(n)}_{i+1,j-1}}{C^{(n)}_{i,j} C^{(n)}_{i+1,j}},
\] (23)
i = 0, 1, \ldots, \(M - 1\), \quad j = 1, 2, \ldots, \(m - 1\),
\[
v_{M+j(M+1)}^{(n)} = \frac{C^{(n)}_{M,j+1} C^{(n)}_{0,j}}{C^{(n)}_{M,j} C^{(n)}_{0,j+1}},
\] (24)
j = 0, 1, \ldots, \(m - 1\).

**Proof.** Let us introduce a new determinant of a square matrix of order \(j\),
\[
G_{i,0}^{(n)} := 1,
\]
\[
G_{i,j}^{(n)} := \begin{vmatrix}
\alpha_{i}^{(n)} & \alpha_{i+1}^{(n)} & \cdots & \alpha_{i+j-1}^{(n)} \\
\alpha_{i+M}^{(n)} & \alpha_{i+M+1}^{(n)} & \cdots & \alpha_{i+M+j-1}^{(n)} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{i+M(j-1)}^{(n)} & \alpha_{i+M(j-1)+1}^{(n)} & \cdots & \alpha_{i+(M+1)(j-1)}^{(n)}
\end{vmatrix},
\] (25)
j = 1, 2, \ldots.

We give an explanation that \(\tau_{j(M+1)}^{(n)}\) can be transformed into the block diagonal determinant with respect to \(G_{0,j}^{(n)}, G_{1,j}^{(n)}, \ldots, G_{M,j}^{(n)}\). By interchanging the 2nd, the 3rd, \ldots the \(j\)th rows and columns with the \([1 + (M + 1)]\)th, the \([1 + 2(M + 1)]\)th, \ldots, the \([1 + (j - 1)(M + 1)]\)th ones in \(\tau_{j(M+1)}^{(n)}\), we observe that the same form of \(G_{0,j}^{(n)}\) appears in the 1st diagonal block of \(\tau_{j(M+1)}^{(n)}\). The entries in the 1st, the 2nd, \ldots, the \(j\)th rows and columns in \(\tau_{j(M+1)}^{(n)}\) are simultaneously all 0, except for those in the diagonal block part. The permutations similar to the above makes the forms of \(G_{1,j}^{(n)}, G_{2,j}^{(n)}, \ldots, G_{M,j}^{(n)}\) as the 2nd, the 3rd, \ldots, the \((M + 1)\)th blocks in \(\tau_{j(M+1)}^{(n)}\). Thus, \(\tau_{j(M+1)}^{(n)}\) can be expressed in terms of \(G_{0,j}^{(n)}, G_{1,j}^{(n)}, \ldots, G_{M,j}^{(n)}\) as
\[
\tau_{j(M+1)}^{(n)} = \prod_{\ell=0}^{M} G_{\ell,j}^{(n)}.
\] (26)
Similarly, $\tau_{i+j(M+1)}^{(n)}$ can be transformed into the determinant of the block diagonal matrix whose $M + 1$ blocks are the notices in $G_{0,j+1}^{(n)}, G_{1,j+1}^{(n)}, \ldots, G_{i-1,j+1}^{(n)}$ and $G_{i,j}^{(n)}, G_{i+1,j}^{(n)}, \ldots, G_{M,j}^{(n)}$. Thus, it follows that

$$\tau_{i+j(M+1)}^{(n)} = \left( \prod_{\ell=0}^{i-1} G_{\ell,j+1}^{(n)} \right) \left( \prod_{\ell=i}^{M} G_{\ell,j}^{(n)} \right). \quad (27)$$

The cases where $k = i + j(M + 1)$ and $k = M + j(M + 1)$ in (21) become

$$v_{i+j(M+1)}^{(n)} = \frac{\tau_{i+j(M+1)+1}^{(n)} - \tau_{i+(j-1)(M+1)+1}^{(n)}}{\tau_{i+j(M+1)}^{(n)} - \tau_{i+(j-1)(M+1)+2}^{(n)}},$$

$$v_{M+j(M+1)}^{(n)} = \frac{\tau_{j+1(M+1)}^{(n)} - \tau_{j(M+1)+1}^{(n)}}{\tau_{M+j(M+1)}^{(n)} - \tau_{j(M+1)+1}^{(n)}}. \quad (28)$$

Consequently, by combining them with (26) and (27), we get

$$v_{i+j(M+1)}^{(n)} = \frac{G_{i,j+1}^{(n)} G_{i+1,j-1}^{(n)}}{G_{i,j}^{(n)} G_{i+1,j}^{(n)}}, \quad i = 0, 1, \ldots, M - 1, \quad (28)$$

$$v_{M+j(M+1)}^{(n)} = \frac{G_{M,j+1}^{(n)} G_{0,j+1}^{(n)}}{G_{M,j}^{(n)} G_{0,j+1}^{(n)}}. \quad (29)$$

The entries in the $j$th row of $G_{i,j}^{(n)}$ are rewritten as the linear combination

$$a_{i+M(j-1)+\ell}^{(n+1)} = a_{i+M(j-2)+\ell}^{(n+1)} + (\delta^{(n)})_{M+1} a_{i+M(j-2)+\ell}^{(n+1)} \quad \text{for} \quad \ell = 0, 1, \ldots, j - 1. \quad \text{By multiplying} \quad (j-1)th \quad \text{by} \quad (\delta^{(n)})_{M+1} \quad \text{and by adding it to the} \quad jth, \quad \text{we get the row} \quad (a_{i+M(j-2)+1}^{(n+1)}, \ldots, a_{i+M(j-2)+1}^{(n+1)} \quad \text{as the new} \quad jth. \quad \text{Similarly, for the} \quad (j-1)th, \quad \text{the} \quad (j-2)th, \ldots, \quad \text{the} \quad 2nd \quad \text{rows, it follows that}$$

$$G_{i,j}^{(n)} = \begin{bmatrix}
    a_{i}^{(n+1)} & a_{i+1}^{(n+1)} & \cdots & a_{i+j-1}^{(n+1)} \\
    a_{i}^{(n+1)} & a_{i+1}^{(n+1)} & \cdots & a_{i+j-1}^{(n+1)} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{i+M(j-2)-1}^{(n+1)} & a_{i+M(j-2)+1}^{(n+1)} & \cdots & a_{i+M(j-2)+1}^{(n+1)} \\
\end{bmatrix}. \quad (30)$$

It is here worth noting that the subscript $M$ can be regarded as be transformed into the superscript 1. Thus, $G_{i,j}^{(n)}$ in (23) after all is equal to the Casorati determinant $C_{i,j}^{(n)}$ in (3). Therefore, by taking account of it in (28) and (29), we have (23) and (24). \hfill \square

Lemma [2] with Theorem [2] leads to the following theorem for an asymptotic behavior of $v_{k}^{(n)}$ as $n \to \infty$.

**Theorem 3.** The auxiliary variables $v_{j(M+1)}^{(n)}, v_{1+j(M+1)}^{(n)}, \ldots, v_{M-1+j(M+1)}^{(n)}$ converge to 0, and $v_{M+j(M+1)}^{(n)}$ goes to some constant $\bar{c}_{j}$ as $n \to \infty$. 
Proof. From Theorem 2 and Lemma 2 we derive, as \( n \to \infty \),

\[
v^{(n)}_{i+j(M+1)} = \frac{c_{0,j+1}c_{i+1,j-1}}{c_{0,j}c_{i+1,j}} \left( \frac{r_{j+1}}{r_j} \right)^n \times \left( 1 + \sum_{\ell=1}^{j+1} O \left( \left( \frac{\rho_{\ell-1}}{|r_{j+1}|} \right)^n \right) \right) \left( 1 + \sum_{\ell=1}^{j} O \left( \left( \frac{\rho_{\ell-1}}{|r_j|} \right)^n \right) \right),
\]

Thus, by taking account that \( |r_j| > |r_{j+1}| \), we see that \( v^{(n)}_{j(M+1)} \to 0 \), \( v^{(n)}_{1+j(M+1)} \to 0 \), \ldots, \( v^{(n)}_{M-1+j(M+1)} \to 0 \) as \( n \to \infty \). Similarly, it follows that

\[
v^{(n)}_{M+j(M+1)} = \frac{c_{M,j+1}c_{0,j}}{c_{M,j}c_{0,j+1}} \left( 1 + \sum_{\ell=1}^{j+1} O \left( \left( \frac{\rho_{M+\ell-1}}{|r_{j+1}|} \right)^n \right) \right) \left( 1 + \sum_{\ell=1}^{j} O \left( \left( \frac{\rho_{\ell-1}}{|r_j|} \right)^n \right) \right),
\]

which implies that \( v_{M+j(M+1)} \to \hat{c}_j := c_{M,j+1}c_{0,j}/(c_{M,j}c_{0,j+1}) \) as \( n \to \infty \).

By recalling the relationship of the dhLV variable \( u^{(n)}_k \) to the auxiliary variable \( v^{(n)}_k \) in (20), we have the following theorem concerning an asymptotic convergence of \( u^{(n)}_k \) as \( n \to \infty \).

**Theorem 4.** The dhLV variable \( u^{(n)}_{j(M+1)} \) converges to some nonzero constant \( \hat{c}_j \), and \( u^{(n)}_{1+j(M+1)-M}, u^{(n)}_{2+j(M+1)-M}, \ldots, u^{(n)}_{M+j(M+1)} \) go to 0 as \( n \to \infty \), provided that the limit of \( \delta^{(n)} \) as \( n \to \infty \) exists.

**Proof.** The proof is given by induction for \( j \). Without loss of generality, let us assume that \( \lim_{n \to \infty} \delta^{(n)} = \bar{\delta} \) where \( \bar{\delta} \) denotes some constant. From (20), it holds that

\[
u^{(n)}_k = \frac{v^{(n)}_{k+M}}{\prod_{\ell=1}^{M} (\delta^{(n)} + u^{(n)}_{k-\ell})}.
\]

By taking the limit as \( n \to \infty \) of the both hand side in (30) with \( k = 0 \) and by using \( v^{(n)}_M \to \hat{c}_0 \) as \( n \to \infty \), we get

\[
\lim_{n \to \infty} u^{(n)}_0 = \bar{c}_0,
\]

where \( \bar{c}_0 = \hat{c}_0/\delta^M \). By taking account of Theorem 3 with (31) in the case where \( k = 1, 2, \ldots, M \) in (30), we successively check that \( u^{(n)}_1 \to 0, u^{(n)}_2 \to 0, \ldots, u^{(n)}_M \to 0 \) as \( n \to \infty \).
Let us assume that $u_{j(M+1)}^{(n)} \to \bar{c}_j$ and $u_{1+j(M+1)}^{(n)} \to 0$, $u_{2+j(M+1)}^{(n)} \to 0, \ldots, u_{M+j(M+1)}^{(n)} \to 0$ as $n \to \infty$. Equation (30) with $k = (j+1)(M+1)$ becomes

$$u_{(j+1)(M+1)}^{(n)} = \frac{v_{M+(j+1)(M+1)}^{(n)}}{\prod_{\ell=1}^M (\delta^{(n)} + u_{(j+1)(M+1)-\ell}^{(n)})}.$$  

(32)

It is obvious that denominator on the right hand side of (32) converges to $\delta^M$ as $n \to \infty$ under the assumption. By combining it with $v_{M+(j+1)(M+1)}^{(n)} \to \hat{c}_j$ as $n \to \infty$, we observe that $u_{(j+1)(M+1)}^{(n)} \to \bar{c}_j = \hat{c}_j / \delta^M$ as $n \to \infty$. Moreover, it follows that

$$\lim_{n \to \infty} u_{i+(j+1)(M+1)+1}^{(n)} = \lim_{n \to \infty} \frac{v_{i+(j+2)(M+1)}^{(n)}}{\prod_{\ell=1}^M (\delta^{(n)} + u_{i+(j+1)(M+1)-\ell}^{(n)})} = 0,$$

$$i = 0, 1, \ldots, M - 1,$$

(33)

since $\prod_{\ell=1}^M (\delta^{(n)} + u_{i+(j+1)(M+1)-\ell}^{(n)}) \to \delta^{M-1}(\delta + \bar{c}_j)$ and $v_{i+(j+2)(M+1)}^{(n)} \to 0$ as $n \to \infty$.

The convergence theorem concerning the dhLV system (19) in [2] is restricted in the case where the dhLV variable $u_k^{(n)}$ is positive and the discretization parameter $\delta^{(n)}$ is fixed positive for every $n$. Theorem 4 clams that the $[j(M+1)]$th species survives and the $[1+j(M+1)]$th, the $[2+j(M+1)]$th, $\ldots$, the $[M+j(M+1)]$th species vanish as $n \to \infty$ even in the case where $\delta^{(n)}$ is variable negative for every $n$. Though the case of negative $u_k^{(n)}$ is not longer recognized as a biological model, we also realize that the convergence is not different from the positive case.

4. Concluding remark

In this paper, we associate a formal power series $f_k(z) = \sum_{n=0}^{\infty} a_k^{(n)} z^n$ with the Casorati determinant, and then give an asymptotic expansion of the Casorati determinant as $n \to \infty$ in Theorem 2. As an application of Theorem 2, we also clarify an asymptotic behavior of the dhLV variables as $n \to \infty$ in Theorem 4.

Theorem 2 will contribute to asymptotic analysis for another discrete integrable systems. An example is the discrete hungry Toda (dhToda) equation which is derived from the numbered box and ball system through inverse ultra-discretization [12]. A special solution to the dhToda equation is shown in [12] to be written by using the Hankel determinant. The solution with Casorati determinant is expected as more generalized solution, since the dhToda equation has a relationship of variables to the dhLV system whose solution is given in the Casorati determinant [4]. The Casorati determinant directly appear in, for example, the solution to the discrete Darboux-Pöschl-Teller equation which is a discretization of the dynamical system concerning a special class of potentials for 1-dimensional Schrödinger equation [5].
It is proved in [3] that the dhLV system (19) with fixed positive $\delta^{(n)}$ is associated with the LR transformation for a totally nonnegative matrix. The paper [16] also suggests that the dhLV system (19) with variable negative $\delta^{(n)}$ generates the sifted LR transformation. Theorem 4 will be useful for investigating the convergence of the shifted LR transformation based on the dhLV system (19) in the variable negative case of $\delta^{(n)}$.

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