Stochastic Resonance in Maps and Coupled Map Lattices.

Prashant M. Gade
Jawaharlal Nehru Centre for Advanced Scientific Research
Jakkur, Bangalore-560064, INDIA

Renuka Rai and Harjinder Singh
Department of Chemistry,
Panjab University, Chandigarh-160014, INDIA

We demonstrate the phenomenon of stochastic resonance (SR) for discrete-time dynamical systems. We investigate various systems that are not necessarily bistable, but do have two well defined states, switching between which is aided by external noise which can be additive or multiplicative. Thus we find it to be a fairly generic phenomenon. In these systems, we investigate kinetic aspects like hysteresis which reflect the nonlinear and dissipative nature of the response of the system to the external field. We also explore spatially extended systems with additive or parametric noise and find that they differ qualitatively.

I. INTRODUCTION

A seemingly counterintuitive scenario that a weak signal can be enhanced by addition of noise was proposed by Benzi and coworkers in connection with the glaciation cycle of the earth. Since then stochastic resonance (SR) has been employed in explaining various phenomena (see e.g. [2, 3]) and has been studied extensively experimentally as well as theoretically [4, 5]. The major emphasis of the studies has been on the original model by Benzi et al. [1] in which the stochastic system in consideration has two stable fixed points in absence of noise and the driving force, though few other models have also received attention [6, 7]. The essence of the phenomenon is that even a weak periodic signal which is undetectable in absence of noise can force a bistable system to switch between its two states, periodically, in presence of an optimal noise. Often one calculates the power spectrum of the output signal of the system filtered through a two-level filter. The ratio of output power in the frequency of the signal with the background noise, also called as signal to noise ratio (SNR) is a relevant quantifier here. In fact, nonmonotonic behavior of SNR with the noise intensity has become a ‘fingerprint’ of this phenomenon.

Additive noise has been the focus of most of the studies. The multiplicative noise, which is not equivalent to additive noise in the presence of a periodic field [1] and thus in principle can show qualitatively different behavior, has not been studied much [12]. Multiplicative noise occurs in a variety of physical phenomena [13, 14, 15] and is certainly important. Also of importance are the kinetic aspects of the phenomenon which reflect the way a system responds to the signal. These aspects have been mainly investigated by measuring the phase shift between the signal and the response [16, 17]. Hysteresis which reflects phase shift and also the extent of losses in a periodically driven system, is an equally important quantifier [17, 18]. However, studies of usual hysteresis behavior in presence of noise have hardly been attempted. One notable exception has been a numerical work by Mahato and Shenoy [17]. However, they define the hysteresis loop in a different way than is usually done. Thus, as they state, we cannot expect their results to be carried over to the usual case even qualitatively. Recently there have been few interesting studies on SR in spatially extended systems [20] (see also [21, 22]). The reasons for the recent surge of interest in analysis of spatiotemporal systems are not far to seek. These systems are important from the point of view of potential applications that range from coupled nonlinear devices and signal processing to neurophysiology and merit further attention.

We feel that SR is a generic feature of two-state systems neither state of which needs to be a stable fixed point. Any system with two well defined and well separated states switching between which can be aided by noise, can possibly show stochastic resonance in the presence of a weak signal. We will illustrate this with systems switching between two chaotic attractors, a chaotic attractor and a fixed point, and will also present the standard model of two stable fixed points. It is easy and computationally inexpensive to construct such cases using discrete-time systems like maps and we will be using maps for demonstration. We have used both additive and multiplicative noise in our simulations. In many physical and chemical systems noise is generated internally (see e.g. [13]) and such systems have been observed to show stochastic resonance. As we have pointed out, the multiplicative noise in the presence of a periodic force is not equivalent to additive noise coupled with a periodic signal. Interestingly, in the case of a single map, both show a similar qualitative behavior as far as SR is concerned, i.e., in both cases SNR shows nonmonotonic behavior of response as a function of noise and has a peak at some value of noise.

In this dissipative and nonlinear system, the response to external field is likely to be delayed and nonlinear. The simplest way to gauge it is to study hysteresis in these
systems which will reflect both losses as well as delay in the response \[17\]. This will be information additional to the one given by simple signal to noise ratio, e.g., \(SNR\) does not reflect the phase shift in the response.

Finally we investigate spatially extended systems where local dynamics is governed by these maps. Such systems, popularly known as coupled map lattices (CML) \[23\] have gained considerable attention in recent times due to their computational simplicity, and ability to reproduce qualitative features in various phenomena. To name a few interesting applications one can point out the modelling of phase ordering dynamics \[23\], spatiotemporal intermittency \[25\], spiral waves \[26\] etc.. However, we are not aware of any attempt to observe SR in these systems. Here we see a qualitative difference between the systems subjected to an additive noise and parametric noise.

The section II and III we will define our models and present their analysis. In section IV we will discuss the hysteresis and effect of noise. In section V we will define the spatially extended systems in a way popularly known as coupled map lattice and discuss results in it. Finally in section V we conclude and discuss questions that we are interested in.

II. THE MODELS

Let us consider the following maps in absence of any periodic or noisy drive.

\[
f_S(x) = S \tanh(x) \quad x \in (-\infty, \infty) \quad S > 0 \tag{1}
\]

\[
g_a(x) = \exp(a|x|_{\text{mod1}}) \quad x \in (-1, 1) \tag{2}
\]

\[
h_r(x) = r|x|_{\text{mod1}} \quad x \in (0, \frac{1}{2}]
\]

\[
h_r(x) = r(1 - x)|_{\text{mod1}} \quad x \in \left[\frac{1}{2}, 1\right]
\]

\[
h_r(x) = -h_r(-x) \quad x \in (-1, 0) \tag{3}
\]

a) The map \(f_S(x)\) which has a hamiltonian symmetry \((f_S(-x) = -f_S(x))\) and two stable fixed points symmetric around \(x = 0\), is clearly analogous to the original model of bistable potential \[1\]. In this case also, the basin of attraction of positive fixed point \(x_S^+\) is \((0, \infty)\) while that of the negative fixed point that is symmetrically placed at \(-x_S^-\) is \((-\infty, 0)\). Thus the system has two stable fixed point attractors any of which is reached depending on initial conditions. (See Fig. 1(a)) \[23\]

b) The map \(g_a(x)\) shows a chaotic behavior if \(a > 0\) or has a stable fixed point 0 as attractor if \(a < 0\). Thus the map has a chaotic or fixed point attractor depending on value of \(a\). (Fig. 1(b) shows the map \(g_p(x)\) and \(g_q(x)\) for \(p > 0\) and \(q < 0\).)

c) The map \(h_r\) also has a hamiltonian symmetry like map \(f_S(x)\). It has a fixed point attractor at \(x = 0\) for \(|r| < 1\). However for \(|r| > 1\), it has an interesting behavior. In this range, it shows two chaotic attractors symmetric around 0. The map is such that for any initial condition \(x_0 \in [0, 1]\), \(h^T(x_0) \in [0, 1]\) for all times \(T\). In fact, for \(1 < r < 2\) the attractor on the positive side does not span an entire unit interval but is in the interval \([a_0, a_1]\) = \([h^T(x), h^T(x)]\) for \(x \in (0, 1)\), while for \(x \in (-1, 0)\), the values assymptotically span interval \([-a_1, -a_0]\). (For \(r \geq 2\), \(a_0 = 0\), \(a_1 = 1\) This is a system with two symmetric chaotic attractors any of which is reached depending on initial conditions. The attractors are separated by 2 \(a_0\). (See Fig. 1(c))

III. SR IN MAPS

The above maps have the desired property of having two different attractors between which the system can switch when aided by noise. We investigate the following systems.

\[
a) x(t+1) = f_S(x(t)) + z\cos(2\pi\omega t) + \eta_t \tag{4}
\]

\[
a1) x(t+1) = f_S(x(t)) + z\cos(2\pi\omega t) \tag{5}
\]

\[
b) x(t+1) = (g_a(x) + z\cos(2\pi\omega t)|x(t) - p_0) + p_0|_{\text{mod1}} \tag{6}
\]

\[
c) x(t+1) = (h_r(x(t)) + z\cos(2\pi\omega t) + \eta_t)|_{\text{mod1}} \tag{7}
\]

\[
c1) x(t+1) = (h_r(x(t)) + z\cos(2\pi\omega t))|_{\text{mod1}} \tag{8}
\]

where \(\eta\) is delta correlated random number with variance \(D\). Except in case of eq.\[4\], where \(\eta\) has a gaussian distribution, we have a uniform distribution for \(\eta\). The maps \(f_S(x)\) and \(h_r(x)\) have hamiltonian symmetry, i.e., \(f_S(-x) = -f_S(x)\) and \(h_r(-x) = -h_r(x)\). They have symmetric attractors on positive and negative side. Since we are interested in inter-state switching we neglect the intra-state fluctuations in cases of eqs. \[4\] \[5\] \[6\] \[7\] \[8\] while analysing the output. The output is analysed in the usual manner, i.e. one takes the fourier transform of the time series thus generated and averages the power over various phases and also initial conditions. The \(SNR\) is defined as the ratio of the intensity of \(\delta\)-spike in the power spectrum at the frequency \(\Omega = 2\pi\omega\) to the height of the smooth fluctuational background \(Q^0(\Omega)\) at the same frequency \(\Omega\) then

\[
SNR = \log_{10} \frac{\text{Total power in the frequency } \Omega}{Q^0(\Omega)} \tag{9}
\]

Variations in this definition do not change the results qualitatively.

a) This is the simplest system which mimics well studied bistable potential model \[3\] of Benzi and coworkers. Here the system toggles between the positive and the negative fixed points of the map \(f_S\), \((x_S^+\) and \(-x_S^-\)\). The system is defined over the entire range \((-\infty, \infty)\). We apply gaussian noise. It is seen that as the noise intensity increases the spectral strength of the signal also increases, but this happens at the expense of noise, thus
increasing signal to noise ratio. This happens since when the
signal is at its peak, the little noise aids the system to
flip from the basin of attraction of one fixed point to
other. For large noise, the flips can occur almost all the
time, the regularity is reduced and SNR decreases again.
This behavior is shown in Fig. 2a) which shows SNR as
a function of noise intensity $D$ for the system defined by
\[ H = \frac{1}{2} \sum_{j=1}^{N} M_j \sum_{k=1}^{N} \delta_{j,k} \alpha + \beta \frac{\partial M}{\partial x} \]

a) In this map we also have a possibility of parametric
noise as in eq. [4]. The value of noise changes the position
and the stability of the fixed point. For large enough
noise the fixed point can come arbitrarily close to zero,
which coupled with the periodic signal can cause flips
which can be very regular at optimal noise level. Thus
SNR shows as standard SR behavior as a function of $D$
(See Fig. 2b).

b) Now we explore the possibility of competition be-
tween a fixed point attractor and a chaotic attractor
switching between which is aided by noise. (We have
added a small constant $p_0$ in the eq. [4] unlike eq. [2]
for numerical reasons. The position of the fixed point
now changes to $p_0$. For $p_0 = 0$, trajectory that comes
close to zero within numerical precision will stay there.
This change does alter the description of the map qual-
itatively.) At the minimum value of the drive, it is likely
that $a_t = a + \eta_t + z \cos(2\pi \omega t) < 0$ and the system will be
attracted to the fixed point. While small noise will aid
this repetitive attraction towards fixed point, very large
noise is likely to reduce it. As a result, we see a non-
monotonic response of SNR to noise intensity. In some
sense, this system mimics excitable dynamics, where SR
has been observed. [10] (though the excited state in this
case is not a chaotic state.) Fig. 2c) shows SNR as
a function of noise intensity $D$ for this system.

This system is like a random walk when viewed on
a logarithmic scale. Neglecting the modulo factor, the
variable value at time $n + 1$ goes as $ln(x_{n+1}) = a_n +
a_{n-1} + \ldots + a_0 + ln(x_0)$ where $a_t = a + \eta_t + z \cos(2\pi \omega t)$.
Thus it is like a random walk with initial position $ln(x_0)$
and displacement $a_i$ at $i$th time step. The value of $x_t$
is bounded from above by unity due to modulo condition
and $x_t$ does not tend to zero asymptotically since $a > 0$.
Due to $a > 0$, this is a case of a biased 1-d random walk
bounded from above. Thus this system is comparable with
stochastic resonance seen in random walk. [21] Fig. 3 shows a schematic diagram for the above description.

c) This is another interesting but unexplored possibility.
Here we have two chaotic attractors switching be-
tween which is aided by noise and a periodic signal. We
would like to point out that apart from the nature of the
attractors, this system is very similar to system defined
by eq.[4] as far as the dynamics of interstate switching is
concerned. Let us consider system defined in eq.[7]. As
noted above, the two attractors are well separated in ab-
sence of noise and periodic signal and the system stays
in either of them depending on initial conditions. How-
ever, in the presence of noise, the system can ‘leak’ out of
the attractor. This ‘leaking’, i.e. switching from positive
attractor to negative and vice versa is likely to occur at
most opportune times, i.e., at the minimum and the
maximum of the signal. Thus one may expect stochastic
resonance here which is indeed the case (See Fig. 2d).

c1) Here one could have a parametric noise as an alter-
ative to additive noise. The parametric noise can change
the value of $a_0$ which controls the distance between two
attractors thus aiding the switching. Fig. 2e shows SNR
as a function of noise intensity $D$ in these systems. One
can see a clear non-monotonicity in response.

IV. HYSTERESIS

Now we discuss the kinetic aspects of this phenomenon.
Hysteresis is a kinetic phenomenon which is the signature
of the response of the system to external field sweep. In
general, due to the frictional losses, system is not able to
follow the external signal exactly. There is an accumu-
lated strain after which it responds to the signal. The
quantity that reflects these losses is the area of the hys-
teresis loop. The most familiar example is the behaviour
of magnetization $M$ as a function of alternating external
magnetic field $H$. We have a two-state system in our ex-
amples and we are analyzing the signal filtered through
two state filter. This makes it easy to define an analogue
of magnetization. We define

$$ m(t) = \frac{1}{N} \sum_{j=1}^{N} sgn(x(jT + t)) $$

where $T$ is the period of the applied periodic force and
t = 1, 2, . . . , T/2. It is clear that $m(t)$ is just difference
between number of times the value of the variable $x$
is greater than zero and that it is less than zero, at times
t modulo $T$ where $1 \leq t < T/2$. We normalize $m(t)$
properly so that $r(t) = m(t)/M$ is confined between 1
and -1. ($M$ is the maximum value that $m(t)$ takes. By
symmetry, we would expect it to be the same on positive
and negative side.) This gives one of the branches of
hysteresis loop. The other branch can be constructed by
symmetry or computed by

$$ m(T/2 - t) = \frac{1}{N} \sum_{j=1}^{N} sgn(x(jT + T/2 + t)) $$

for $t = 1, 2, \ldots , T/2$ and $r(t) = m(t)/M$. Here one can
have two extreme cases. If the response is exactly in tune
with the field, the magnetization will be zero at zero value
of the periodic force (no remnant magnetization) and the
hysteresis area will be zero. On the contrary, if the re-
sponse is so late that only at the end of the half-cycle, the
flips start occuring, loop will have maximum area. Thus
more delayed the response is, higher is the area of hysteresis loop and hence is the popular notion of hysteresis area giving indication of losses in the system. One would a priori think that as the noise intensity $D$ increases, which is an equivalent of increasing temperature, the hopping between the different states will be facilitated and thus the area of the loop will decrease at larger noise values. This is exactly what our observation is! We have plotted the area of hysteresis loop for the system defined by eq.4 in Fig 4b. Here, we see a clear decrease in hysteresis loop area with increase in noise intensity. As noise increases, the frictional losses are reduced since the system will not stay in metastable state for long. A sudden fluctuation will force it to respond to the signal and there will be little memory or remnant magnetization in the system. Apart from the area of the hysteresis loop, the shape of the hysteresis is also an interesting object to investigate. Though for a noise higher than some critical value, the maxima and minima in magnetization start occurring at the maxima and minima of the field, for smaller noise they occur at different times. (See, Fig. 4a, where we have plotted the hysteresis loop for four different values of noise intensity.) We can see that the response is delayed from the field by a finite phase shift. This phase shift reduces with increasing noise. The results indicate that in the limit of small noise the phase shift is of the order of quarter of the total period, $-\pi/2$ which is expected for a two-state analysis that does not take in account intra-well fluctuations.

V. SR IN COUPLED MAP LATTICES

Let us discuss cooperative phenomena possible in the spatially extended versions of this system. Spatially discretized periodically forced time dependent Ginzburg-Landau equation [21], as well as one dimensional array of coupled bistable oscillators [22] have been studied before. Major result in the work by Lindner and coworkers [22] has been that the largest value of SR is higher for a given oscillator of the coupled system as compared to uncoupled one. However, the maximum of SR does not occur at the same value of noise intensity. Not only the best value of SR is higher for the coupled case, the value of SR at a given value of noise intensity $D$ is better for the coupled system as compared to uncoupled one. A simple interpretation that can be offered for the above observation is that even when an oscillator misses the interstate switching, the nearby oscillators may not, thus forcing the individual oscillator to switch. This cooperative behavior should induce enhanced regularity in the switching of the oscillators and increase SR for a given oscillator. The simple interpretation which can be true for any inter-state switching mechanism should also work for the systems we are studying.

We define the following spatially extended system. We will follow a evolution scheme popularly known as coupled map lattice. Let us consider a linear array of length $N$. At each lattice point $i$ ($i = 1, \ldots, N$) we attach a variable $x_i(t)$ at time $t$. The time evolution of $x_i(t)$ is described by

$$x_i(t + 1) = (1 - \epsilon) F(x_i(t)) + \frac{\epsilon}{2} (F(x_{i-1}(t)) + F(x_{i+1}(t))$$

(10)

for $2 \leq i \leq N - 1$ and

$$x_1(t + 1) = (1 - \epsilon) F(x_1(t)) + \epsilon F(x_2(t))$$

$$x_N(t + 1) = (1 - \epsilon) F(x_N(t)) + \epsilon F(x_{N-1}(t))$$

(11)

Where $F$ denotes some time evolving map. Of course, we will be in particular interested in single maps that show SR. (e.g eq.1).

We have used the maps defined by eqs. [23][24] and 8 as function $F$ in the above equation and have made a detailed study of SR at various values of coupling $\epsilon$ and noise intensity $D$ for $N = 8$. Following Lindner et al [24] we have looked at the response of the middle oscillator. Fig. 5a and 5b) show the best value of SR as a function of coupling for maps [23][24] i.e. the tanh map with additive and parametric noise while Figs. 5c) and 5d) show the same for maps defined by eq. [23] and eq.[24] i.e., chaotic map with additive and multiplicative noise respectively. We have the following observations.

a) In all these cases, the SR of the middle oscillator as a function of noise for a given value of coupling is non-monotonic and shows a peak at some optimal value as in single map case. The optimal value of noise need not be the same as one for uncoupled case. b) If the single system is perturbed with additive noise the coupling between such systems always enhances the SR, i.e., best SR for the coupled system is better than the one for a single oscillator case. c) If the single system is perturbed by parametric noise, the coupling between such systems does not enhance SR much. In fact for a large value of coupling, there is a clear decrease in SR.

While observations a) and b) are in tune with the studies by Lindner et al [24] in coupled bistable systems, the case with parametric noise has not been studied before.

We feel that the reason for this qualitative difference is following. When two systems with different parameters are coupled, i.e., a system with high Kramer’s rate is coupled to the one with low Kramer’s rate, the Kramer’s rate for the coupled system is like that of the slower system [23]. For higher coupling, this effect is more pronounced. Thus, instead of an induced switching, one could have a slowed down switching in presence of coupling. In this context, we would like to point out a rather curious outcome of our numerical investigation that the best SR has the least value around $\epsilon = 2/3$ in the case of parametric noise. This is the value at which all the
maps in the neighbourhood have equal weight in eq. (10) 
\(1 - \epsilon = \epsilon/2 = 1/3\).

VI. DISCUSSION

Due to computational simplicity of the system defined above, and its ability to produce key features, the system defined above holds promise for carrying out work in various directions with relative ease in future. Here we would like to point out that using a coupled map like formulation, Oono and Puri have been able to get a 
quantitative agreement in modeling phase ordering dynamics of Ising-type systems [24]. Miller and Huse have also found that chaotic coupled maps with hamiltonian symmetry show a phase transition with static and dynamic critical exponents consistent to the Ising class [31,32]. On the other hand, Ising system have been reported to show SR in 1-d and 2-d [33,34]. As we have pointed out, recently there have been studies on coupled bistable oscillators in 1-d which shows SR [21]. Although one does not expect all the detailed behavior in one model to carry over to the other, all these things do point towards a broader universality between coupled bistable oscillators, coupled maps and Ising systems. The investigations on these lines should be useful in understanding SR in spatially extended systems which are relatively unexplored but clearly important from various points of view. Given the wide variety of physical situations that Ising model is able to simulate, this similarity should not come as a surprise. In fact, the similarity between coupled bistable oscillators and Ising type systems has been pointed out before [33]. While studies of the Ising system itself will be useful in analytic investigations, investigations in the coupled map type systems will be computationally more efficient. Studies in globally coupled maps [22] and detection of noise induced transitions in maps [26] could be carried out in these systems. Though our preliminary numerical investigations do not indicate any scaling as in [22, the behavior of \(SNR\) as a function of coupling \(\epsilon\) and number of maps \(N\) could to be studied further and these investigations are in progress. One could also investigate SR in 2-dimensional coupled map lattice since higher dimensional spatially extended systems are not investigated. One more important question that needs to be addressed is the effect of multiplicative noise and disorder in SR in spatially extended systems.

Authors have enjoyed discussions with Prof. N. Kummar(RRI). RR would like to thank UGC for financial support and RRI for hospitality while HS would like to thank the Indian Academy of Sci. for the support for the visit to JNCASR where the work was carried out.

Figure Captions

Fig. 1 The maps \(f_S\), \(g_s\) and \(h_s\) are shown in as defined in eqs. (1), (2) and (3) are depicted in a) b) and c)
respectively.

Fig. 2 The \(SNR\) as a function of \(D\) for systems defined by eqs. (4) and (5) are shown in figures a), b), c), d) and e) respectively. \(\omega = 1/8\), \(S = 2\), \(z = .5\) for a) and b), \(\omega = 1/8\), \(r = 1.4\), \(a = 12\) for d) and e), and \(\omega = 1/32\), \(p_0 = .01\), \(z = .1\) for c). Noise is gaussian for case of unbounded map defined by eq. (3) while is uniform on an unit interval in other cases.

Fig. 3 This figure shows the schematic diagram of how the evolution under eq. (2) resembles a 1-d random walk on a logarithmic scale with a boundary condition at \(ln(x) = 0\) coming due to modulo condition on the right hand side while the evolution is unbounded on the left hand side.

Fig. 4 a)Hysteresis curve for \(D = 0.2\), \(D = 1\), \(D = 1.4\) and \(D = 5.2\) for the map defined by eq. (4) for \(S = 2\), \(z = 0.5\) and \(T = 1/\omega = 32\). One can clearly see that for low \(D\) maxima and minima of magnetization are not in tune with the drive. b) Area of hysteresis loop as a function of noise intensity \(D\) in this case. One can clearly see a decrease at larger noise values.

Fig. 5 Best value of \(SNR\) as a function of coupling \(\epsilon\) for maps defined by eqs. (6) and (7) are shown in a), b), c) and d). It is clear that while for additive noise the best \(SNR\) is enhanced, it is not so for parametric noise.

[1] R. Benzi, A. Sutera and A. Vulpiani, J. Phys. A 14, L453 (1981), R. Benzi et al, Tellus 34,10 (1982).
[2] K. Wiesenfeld and F. Moss, Nature 373, 33(1995), A. R. Bulsara and L. Gammatoni, Physics Today, 39 (Mar. 1996).
[3] See, for example, Proceedings of the NATO ARW Stochastic Resonance in Physics and Biology, edited by F. Moss, A. Bulsara and M. F. Shlesinger [J. Stat. Phys. 70.1 (1993)]
[4] B. McNamara and K. Wiesenfeld, Phys. Rev. E 39, 4854 (1989).
[5] S. Fauve and H. Leslort, Phys. Lett. 97 A, 5 (1983).
[6] B. McNamara, K. Wiesenfeld and R. Roy, Phys. Rev. Lett., 60, 2626 (1988).
[7] Recently a non-dynamical model of this phenomenon was proposed. See Z. Gingl, L. B. Kiss and F. Moss, Europhys. Lett., bf 29, 191 (1995).
[8] There has been a study on SR in monostable well [4]. There have also been studies on systems with excitable dynamics [11]. The two states of this system are a stable fixed point and an excited state. The other state is not a stationary state and the system returns back to the fixed point after certain refractory period.
N. G. Stocks et al, J. Phys. A, 26, L385 (1993).

K. Wiesenfeld et al, Phys. Rev. Lett. 72, 2125 (1994).

H. Risken, The Fokker-Planck Equation, (Springer, Berlin, 1984).

L. Gammaitoni et al, Phys. Rev. E, 49, 4878 (1994).

D. S. Leonard and L. E. Reichl, Phys. Rev. E 49, 1734 (1994).

R. Graham and A. Schenzle, Phys. Rev. A 25, 1731 (1982).

L. Gammaitoni et al, Phys. Lett. 158 A, 449 (1991). L. Gammaitoni and F. Marchesoni, Phys. Rev. Lett., 70, 874 (1993).

M. Dykman et al, Phys. Rev. Lett. 70, 874 (1993). P. Jung and P. Hanggi, Z. Phys. B 90, 255 (1993). M. Morillo and J. Gómez-Ordóñez, Phys. Rev. Lett. 71, 9 (1993).

L. Gammatoni et al, Phys. Lett. 158 A, 449 (1991). L. Gammatoni and F. Marchesoni, Phys. Rev. Lett., 70, 874 (1993).

D. S. Leonard, Phys. Rev. A 46, 6742 (1992).

A. Neiman, Phys. Rev. E 49, 3484 (1994).

It is difficult to get reliable statistics at lower noise intensities, since there are fewer flips.

J. Miller and D. A. Huse, Phys. Rev. A 48, 2528 (1993).

J. J. Brey and A. Pandos, Phys. Lett. 216A, 240 (1996).

Z. Néda, Phys. Rev. E 51, 5315 (1993).

L. Kiss et al, J. Stat. Phys., 70, 451 (1993). M. Inchiosa and A. R. Bulsara, Phys. Rev. E 52, 327 (1995). P. Jung and G. Mayer-Kress, Phys. Rev. Lett., 74, 2130 (1995).

H. Gang, H. Haken and X. Fagen, Phys. Rev. Lett. 77, 1925 (1996).
Fig. 1a
Fig. 1b

\[ g_a(x) \]

\[-a_1 \quad -a_0 \quad 0 \quad a_0 \quad a_1 \]

\[-1 \quad -0.5 \quad 0 \quad 0.5 \quad 1 \]
Fig. 1c

\[ h_p(x) \]

\[ h_q(x) \]

\[ h_p(x) \]
Fig. 2a

SNR vs. $D$
Fig. 2b

![Graph showing the relationship between SNR and D](image-url)
Fig. 2c

SNR

$D$

0 0.5 1 1.5 2 2.5 3

0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9
Fig. 2e

SNR

\( D \)
Fig. 3
\begin{figure}[h]
  \centering
  \includegraphics[width=\textwidth]{fig4a.png}
  \caption{Graph showing the function $m(t)$ for different values of $D$.}
  \label{fig:4a}
\end{figure}

$D = 5.6$
$D = 1.4$
$D = 1$
$D = 0.2$

$\pi$
Fig. 4b

Area

\[
\begin{align*}
2\pi & \\
3\pi/2 & \\
\pi & \\
\pi/2 & \\
\end{align*}
\]

\[
\begin{align*}
0.1 & \\
1 & \\
10 & \\
\end{align*}
\]

\[D\]
Fig. 5a

Best SNR

\[
\begin{array}{c}
\text{Best SNR} \\
\epsilon
\end{array}
\]
Fig. 5c

Best SNR vs. $\epsilon$
Fig. 5d

Best SNR

$\epsilon$

0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1