Abstract
The paper surveys some concepts of signed graph coloring.

1 Introduction and definitions

This paper surveys concepts of colorings of signed graphs. The majority of them are natural extension/generalizations of vertex coloring and the chromatic number of unsigned graphs. However, it turns out that there are coloring concepts which are equivalent for unsigned graphs but they are not equivalent for signed graphs in general. Consequently, there are several versions of a chromatic number of a signed graph. We give a brief overview of the various concept and relate some of them to each other. We use standard terminology of graph theory. However, we will first give some necessary definitions, some of which are less standard.

We consider finite graphs $G$ with vertex set $V(G)$ and edge set $E(G)$. An edge $e$ consists of two half-edges each of which is incident to a vertex. If the half-edges of $e$ are incident to vertices $v$ and $w$, then the half-edges are denoted by $h_e(v)$ and $h_e(w)$. If $v = w$, then $e$ is a loop. We say that $e$ is incident to $v$ and $w$ or that $v$ and $w$ are the end vertices of $e$ and we will also use the term $vw$ to denote $e$.

Let $G$ be a graph and $C$ be a set. A mapping $c : V(G) \to C$ is a coloring of $G$, if $uv \in E(G)$ implies that $c(u) \neq c(v)$. A coloring $c$ is a $k$-coloring, if $|\{c(v) : v \in V(G)\}| \leq k$. The chromatic number of $G$ is the minimum $k$ for which there is a $k$-coloring of $G$.

Let $H(G)$ be the set of half-edges of $G$, and $H_G(v)$ be the set of half-edges which are incident to $v$. The cardinality of $H_G(v)$ is the degree of $v$ which is denoted by $d_G(v)$.

The maximum degree of a vertex of $G$ is denoted by $\Delta(G)$, and the minimum degree of a vertex of $G$ is denoted by $\delta(G)$. A graph $G$ is $k$-regular, if $d_G(v) = k$ for all $v \in V(G)$. Let $X \subseteq V(G)$ be a set of vertices. The subgraph of $G$ induced by $X$ is denoted by $G[X]$, and the set of edges with precisely one end in $X$ is denoted by $\partial(X)$. A circuit is a connected
2-regular graph. For \( k \geq 1 \), a circuit of length \( k \) is denoted by \( C_k \), where \( C_1 \) is a loop, and \( C_2 \) consists of two vertices and two edges between them.

Let \( G \) be a graph and \( \tau : H(G) \to \{±1\} \) be a function. We define a signed graph \((G, \sigma)\) to be a graph \( G \) together a function \( \sigma : E(G) \to \{±1\} \) with \( \sigma(e) = \tau(h_e(v))\tau(h_e(w)) \). The function \( \sigma \) is called a signature of \( G \), and the set of negative edges of \((G, \sigma)\) is denoted by \( N_\sigma \). The set \( E(G) - N_\sigma \) is the set of positive edges, which is also denoted by \( E^+_\sigma(G) \). A circuit or a path is positive if the product of the signs of its edges is positive, otherwise it will be negative. A subgraph \((H, \sigma|_{E(H)})\) of \((G, \sigma)\) is balanced if all circuits in \((H, \sigma|_{E(H)})\) are positive, otherwise it is unbalanced. If \( \sigma(e) = 1 \) for all \( e \in E(G) \), then \( \sigma \) is the all positive signature and it is denoted by \( 1 \), and if \( \sigma(e) = -1 \) for all \( e \in E(G) \), then \( \sigma \) is the all negative signature and it is denoted by \(-1\).

**Theorem 1.1** (15). A signed graph \((G, \sigma)\) is balanced if and only if \( V(G) \) can be partitioned into two sets \( A \) and \( B \) (possibly empty) such that all edges of \( E(G[A]) \cup E(G[B]) \) are positive and all edges of \( \partial(A) \) are negative.

A resigning of a signed graph \((G, \sigma)\) at a vertex \( v \) defines a graph \((G, \sigma')\) with \( \tau'(h) = -\tau(h) \) if \( h \in H_G(v) \), \( \tau'(h) = \tau(h) \) otherwise, and \( \sigma'(e) = \tau'(h_e(v))\tau'(h_e(w)) \) for each \( vw \in E(G) \). Note that a resigning at \( v \) changes the sign of each edge incident to \( v \) but the sign of a loop incident to \( v \) is unchanged. We say that \((G, \sigma_1)\) and \((G, \sigma_2)\) are equivalent if they can be obtained from each other by a sequence of resignings. We also say that \( \sigma_1 \) and \( \sigma_2 \) are equivalent signatures of \( G \); this is denoted by \( \sigma_1 \sim \sigma_2 \). This equivalence relation is well defined, since we have the following fundamental statement.

**Theorem 1.2** (48). Two signed graphs \((G, \sigma)\) and \((G, \sigma')\) are equivalent if and only if they have the same set of negative circuits.

If \( A \subseteq V(G) \), then resigning at every vertex of \( A \) changes the sign of every edge of \( \partial(A) \) and the signs of all other edges are unchanged. Hence, it follows with Theorem 1.1

**Corollary 1.3.** A signed graph \((G, \sigma)\) is balanced if and only if it is equivalent to \((G, 1)\).

We define a signed graph \((G, \sigma)\) to be antibalanced if it is equivalent to \((G, -1)\). Clearly, \((G, \sigma)\) is antibalanced if and only if the sign product of every even circuit is 1 and it is -1 for every odd circuit. Note, that a balanced bipartite graph is also antibalanced. The underlying unsigned graph of \((G, \sigma)\) is denoted by \( G \).

Zaslavsky proposed in 48 that there is a canonical form to any given resigning class on a graph \( G \) with respect to a maximal forest on \( G \):

**Proposition 1.4** (48). Let \( G \) be a graph and \( T \) a maximal forest. Each equivalence class of the set of signed graphs on \( G \) has a unique representative whose edges are positive on \( T \).

Hence, the number of non-equivalent signatures on any given finite, loopless graph is easy to determine.
Proposition 1.5 ([34]). If $G$ is a loopless graph with $m$ edges, $n$ vertices and $c$ components, then there are $2^{(m-n+c)}$ non-equivalent signatures on $G$.

In particular, by Proposition 1.5, there is only one such class on any given forest. By Theorem 1.2, each signature $\sigma$ defines an equivalence class on the set of all signed graphs which have $G$ as the underlying unsigned graph. To avoid overloading papers technically, many authors do not use different notations for the equivalence class and for a representative of this class. In most cases, this does not cause any problems if characteristics of signed graphs are studied, which are invariant under resigning. We will follow this approach in this paper.

As far as we know, Cartwright and Harary [6] were the first to consider the question of signed graph coloring. In Section 2, we shortly introduce their concept from 1968, which seems to be motivated by Theorem 1.1. In the 1980s, Zaslavsky was the first who considered the natural constraints for a coloring $c$ of a signed graph $(G, \sigma)$, that $c(v) \neq \sigma(e)c(w)$ for each edge $e = vw$, and that the colors can be inverted under resigning, i.e. equivalent signed graphs have the same chromatic number. In order to guarantee these properties of a coloring, Zaslavsky [47] used the set $\{-k, \ldots, 0, \ldots, k\}$ of $2k+1$ “signed colors” and studied the interplay between colorings and zero-free colorings through the chromatic polynomial. Section 3 summarizes some results of his pioneering work on signed graph coloring.

In Section 4 we display the approaches of Mácajová, Raspaud, and Škoviera [32] and Kang and Steffen [25], which both coincide with ordinary colorings on unsigned graphs.

Section 4.6 generalizes some approaches of the previous section by considering permutations on the edges instead of signatures. This work is mainly driven by question on coloring planar signed graphs.

An unsigned graph $G$ has a $k$-coloring if and only if there is an homomorphism from $G$ into the the complete graph on $k$ vertices. This approach is generalized to signed graphs by Naserasr, Rollová and Sopena [34]. In Section 5 we display the basic results for this part of study of signed graph coloring.

The study of vertex colorings of unsigned graphs can be reduced to simple graphs. This is not the case in the context of signed graphs. We will use the following definition occasionally. For a loopless graph $G$ let $\pm G$ be the signed multigraph obtained from $G$ by replacing each edge by two edges, one positive and one negative.

2 A first approach by Cartwright and Harary

In 1968, Cartwright and Harary [6] gave the following definition of a coloring of signed graphs.

Definition 2.1 ([6]). An $k$-coloring of a signed graph $(G, \sigma)$ is a partition of $V(G)$ into $k$ subsets (called color sets) such that for every edge $e$ with endpoints $v$ and $w$:
(i) if $\sigma(e) = -1$, then $v$ and $w$ are in different color sets,

(ii) if $\sigma(e) = 1$, then $v$ and $w$ are in the same color set.

We say a signed graph has a coloring if it has an $k$-coloring for some $k > 0$.

**Theorem 2.2** (6). The following statements are equivalent for a signed graph $(G, \sigma)$:

(i) $(G, \sigma)$ has a coloring.

(ii) $(G, \sigma)$ has no negative edge joining two vertices of a positive component.

(iii) $(G, \sigma)$ has no circuit with exactly one negative edge.

Hence, a signed complete graph has a coloring if and only if it has no triangle with exactly one negative edge. Cartwright and Harary (6) observed that $(G, \sigma)$ has a 2-coloring if and only if $(G, \sigma)$ is balanced. For the all-negative signed graph $(G, -1)$, Theorem 2.2 implies a classical result of König (29), that a graph is bipartite if and only if it does not contain an odd circuit. Cartwright and Harary studied further variants of these kind of colorings.

Bezhad, Chartrand (2) gave a definition of a signed line graph of a signed graph and extended this coloring concept to edge-colorings of signed graphs.

## 3 The fundamental approach by Zaslavsky

Zaslavsky’s papers (46, 47, 48, 49, 50) in the early 1980s can be considered as the pioneering work on signed graph colorings. His approaches opened the door for further research on this topic. The natural constraints for a coloring $c$ of a signed graph $(G, \sigma)$ are

(i) that $c(v) \neq \sigma(e)c(w)$, for each edge $e = vw$, and

(ii) that equivalent signed graphs should have the same chromatic number.

The second condition implies that colors have to be changed under resigning. In order to guarantee these properties of a coloring, Zaslavsky (50) defined a coloring with $k$ colors or with $2k + 1$ “signed colors” of a signed graph $(G, \sigma)$ as a mapping from the vertex set of $G$ to the set $\{-k, \ldots, 0, \ldots, k\}$. Let $c$ be a coloring of $(G, \sigma)$. It is easy to see that if $(G, \sigma')$ is obtained from $(G, \sigma)$ by resigning at a vertex $u$, then $c'$ with $c'(v) = c(v)$ if $v \neq u$ and $c'(u) = -c(u)$ is a coloring of $(G, \sigma')$.

A coloring is zero-free if it does not use the color 0. The exceptional role of the color 0 is due to the fact, that it is self-inverse. That is, if $c$ is a coloring of a signed graph $(G, \sigma)$, which uses the color 0, then $G[c^{-1}(0)]$ is an independent set in $G$ while for $t \neq 0$, $G[c^{-1}(t)]$ may contain negative edges. Zaslavsky (47, 50) defined the chromatic polynomial $\chi_G(\lambda)$ to be the function for odd positive numbers $\lambda = 2k + 1$ whose value is
equal to the number of proper signed colorings with $k$ colors. For even positive numbers $\lambda = 2k$, he defined the balanced chromatic polynomial $\chi^b_G(\lambda)$ whose value is the number of zero-free proper signed colorings of $(G, \sigma)$ with $k$ colors. Consequently, his definition for the chromatic number $\gamma_G(G, \sigma)$ of a signed graph $(G, \sigma)$ is the smallest $k \geq 0$ for which $\chi_G(2k + 1) > 0$, and the zero-free chromatic number $\gamma^*(G, \sigma)$ is the smallest number $k$ for which $\chi^*_G(2k) > 0$. Clearly, if $(G, \sigma)$ has a $k$-coloring, then it has a zero-free $(k + 1)$-coloring.

Major parts of Zalavsky’s work on signed graph coloring are devoted to the interplay between colorings and zero-free colorings through the chromatic polynomial. Due to the large number of highly interesting results in this field we refer the interested reader to the original papers [46, 47, 48, 49, 50] in this respect and focus on results on (upper) bounds for this chromatic numbers of signed graphs.

**Theorem 3.1 (50).** The zero-free chromatic number of a signed graph $(G, \sigma)$ is equal to the minimum number of antibalanced sets into which $V(G)$ can be partitioned, and to $\min(\chi(G|E^+_\sigma(G))): \sigma' \sim \sigma$.

**Corollary 3.2.** Let $G$ be a graph. If $\chi(G)$ is even, then $\gamma(G, 1) = \frac{1}{2}\chi(G) = \gamma^*(G, 1)$. If $\chi(G)$ is odd, then $\gamma(G, 1) = \frac{1}{2}(\chi(G) - 1)$ and $\gamma^*(G, 1) = \frac{1}{2}(\chi(G) + 1)$.

This gives us a first set of general upper bounds for the zero-free chromatic number. In fact, from these and the above mentioned properties one can derive the following classification.

**Corollary 3.3 (50).** Let $(G, \sigma)$ have no loops and $|V(G)| = n$. Then $\gamma^*(G, \sigma) = n$ if $(G, \sigma) = \pm K_n$; $\gamma^*(G, \sigma) = n - 1$ if $(G, \sigma) = \pm K_n \setminus E'$ where $E' \subseteq E(\pm K_n)$ is either a non-empty set of edges at one vertex of $\pm K_n$ or an unbalanced triangle; and otherwise $\gamma^*(G, \sigma) \leq n - 2$, and $\gamma^*(G, \sigma) = 1$ if and only if $(G, \sigma)$ is antibalanced.

An obvious lower bound for the zero-free chromatic number of $(G, \sigma)$ is $\lceil \frac{1}{2}\chi(G) \rceil$. Furthermore, upper bounds for the zero-free chromatic number in terms of the order of a graph are given.

**Theorem 3.4 (50).** Let $(G, \sigma)$ be a simple signed graph. If $|V(G)| = n$, then $\gamma^*(G, \sigma) \leq \lceil \frac{n}{2} \rceil$, with equality precisely when $(G, \sigma) = (K_n, 1)$ or $n$ is even and $(G, \sigma)$ contains a $(K_{n-1}, 1)$ or $n = 4$ and $(G, \sigma)$ is an unbalanced circuit on 4 vertices or $n = 6$ and $(G, \sigma)$ is equivalent to $(K_6, \sigma^*)$, where $N_{\sigma^*}$ is the edge set of a circuit of length 5.

Also, $\gamma^*(G, \sigma) \geq 1$, with equality precisely when $(G, \sigma)$ is equivalent to $(G, -1)$.

A signed graph is orientable embeddable into a surface $S$ if it is embeddable into $S$ and a closed walk preserves orientation if and only if its sign product is 1. Indeed, the later condition is equivalent to a closed walk reverses orientation if and only if its sign product is $-1$. 


Theorem 3.5 \([52]\). Let \((G, \sigma)\) be a signed graph without positive loops. If \((G, \sigma)\) is orientable embeddable into the projective plane or into the Klein bottle, then \(\gamma(G, \sigma) \leq 2\) and \(\gamma^*(G, \sigma) \leq 3\). For both surfaces there are signed graphs where equality holds.

4 Modifications of signed coloring

This section considers concepts for coloring signed graphs which satisfy conditions (i) and (ii) of the previous section, and additionally, (iii) the corresponding chromatic numbers of the all-positive graph \((G, 1)\) are equal to the chromatic number of \(G\).

4.1 The approach of Mácajová, Raspaud, and Škoviera

Mácajová, Raspaud, and Škoviera \([32]\) modified Zaslavsky’s definition of signed graph coloring. For each \(n \geq 1\), a set \(M_n \subseteq \mathbb{Z}\) of colors is defined as
\[
M_n = \{\pm 1, \pm 2, \ldots, \pm k\}
\]
if \(n = 2k\), and \(M_n = \{0, \pm 1, \pm 2, \ldots, \pm k\}\) if \(n = 2k + 1\). An \(n\)-coloring is a coloring of a signed graph \((G, \sigma)\) is a mapping \(c : V(G) \rightarrow M_n\), such that \(c(v) \neq \sigma(e)c(w)\), for each edge \(e = vw\). The smallest number \(n\) such that \((G, \sigma)\) admits an \(n\)-coloring is the signed chromatic number of \((G, \sigma)\) and it is denoted by \(\chi_\pm((G, \sigma))\). This chromatic number has the advantage that, in the case of a balanced signed graph, it coincides with the chromatic number of its underlying unsigned graph and can therefore be regarded as a natural extension.

Of course, there is a direct relationship between \(\chi_\pm\) and the pair \(\gamma\) and \(\gamma^*\). The relationship can be expressed as
\[
\chi_\pm((G, \sigma)) = \gamma((G, \sigma)) + \gamma^*((G, \sigma))
\]
for every signed graph \((G, \sigma)\). Therefore, it is obvious that \(\chi_\pm((G, \sigma))\) is invariant under resigning.

The new defined signed chromatic number \(\chi_\pm\) has different, but often similar bounds as the zero-free chromatic number \(\gamma^*\) Zaslavsky studied in \([50]\). For example a very simple bound using the chromatic number of the underlying graph is given in the following theorem. One only needs to see that any unsigned coloring of an underlying graph is also a coloring of any corresponding signed graph. Schweser and Stiebitz showed that the class of signed expansions provides examples for equality in the bound above.

Theorem 4.1 \([38]\). If \(G\) is a graph, then \(\chi_\pm(\pm G) = 2\chi(G) - 1\).

Since every simple signed graph \((G, \sigma)\) is a subgraph of \(\pm G\), it follows that \(2\chi(G) - 1\) is an upper bound for the signed chromatic number of a signed graph \((G, \sigma)\). However, Mácajová, Raspaud and Škoviera proved that the bound is attained by a simple signed graph.

Theorem 4.2 \([32]\). For every loopless signed graph \((G, \sigma)\) we have \(\chi_\pm((G, \sigma)) \leq 2\chi(G) - 1\). Furthermore, this bound is sharp.
Another bound can be regarded as an extension of the well known characterization of bipartite graphs.

**Lemma 4.3** ([32]). A signed graph \((G, \sigma)\) is antibalanced if and only if \(\chi_{\pm}((G, \sigma)) \leq 2\).

Important for the study of choosability of signed graphs, a topic which will be introduced later in this section, is the notion of degeneracy. A graph \(G\) is called \(k\)-degenerate if every subgraph of \(G\) has a vertex of degree at most \(k\). This property can also be used for another bound, using the fact that if a graph is \(k\)-degenerate, then there is an ordering \(v_1, v_2, \ldots, v_n\) of its vertices such that for every \(1 < i \leq n\) the vertex \(v_i\) has at most \(k\) neighbors in \(\{v_1, \ldots, v_{i-1}\}\) and greedy coloring.

**Lemma 4.4** ([32]). If a signed graph \((G, \sigma)\) is \(k\)-degenerate, then \(\chi_{\pm}((G, \sigma)) \leq k + 1\).

We recall that the vertex arboricity \(a(G)\) of a graph \(G\), is the minimum number of subsets into which \(V(G)\) can be partitioned so that each set induces a forest. Similarly, the edge arboricity of a graph \(G\), denoted by \(a'(G)\), is the minimum number of forests into which its edges can be partitioned. An acyclic coloring (of an unsigned graph) is a coloring in which every two color classes induce a forest and \(\chi_{a}(G)\) denotes the acyclic chromatic number of a graph \(G\). With these notions, one can describe a set of bounds specifically for simple signed graphs.

**Proposition 4.5** ([32]). Let \((G, \sigma)\) be a simple signed graph. Then the following statements hold.

(i) If \(G\) is \(K_4\)-minor-free, then \(\chi_{\pm}((G, \sigma)) \leq 3\).

(ii) If \(G\) is the union of two forests (i.e. \(a'((G, \sigma)) \leq 2\)), then \(\chi_{\pm}((G, \sigma)) \leq 4\).

(iii) If \(a(G) \leq k\), then \(\chi_{\pm}((G, \sigma)) \leq 2k\).

(iv) \(\chi_{a}'((G, \sigma)) \leq \chi_{a}(G)\).

The most fundamental result in [32] is a signed version of the famous theorem of Brooks, which states a relationship between the chromatic number of a connected graph and its maximum degree. The two extremal cases in the original theorem, the complete graph and the odd circuit, carry over into the version for signed graphs as the balanced signed graph and the balanced odd circuit. However, it is interesting that for signed graphs there is a third extremal case, the even unbalanced circuit.

**Theorem 4.6** ([32]). Let \((G, \sigma)\) be a simple connected signed graph. If \((G, \sigma)\) is not a balanced complete graph, a balanced odd circuit, or an unbalanced even circuit, then \(\chi_{\pm}((G, \sigma)) \leq A(G)\).

It was proved in [12] that depth-first search and greedy coloring can be used to find a proper coloring of connected signed graphs \((G, \sigma)\) using at most \(A(G)\) colors, provided \((G, \sigma)\) is different from the above mentioned extremal cases. Another important theorem that has its pendant in unsigned graph theory is the five color theorem for signed planar graphs.
Theorem 4.7 ([32]). Let \((G, \sigma)\) be a simple signed planar graph, then \(\chi_{\pm}((G, \sigma)) \leq 5\). Furthermore,

(i) if \(G\) is triangle-free, then \(\chi_{\pm}((G, \sigma)) \leq 4\), and
(ii) if \(G\) has girth at least 5, then \(\chi_{\pm}((G, \sigma)) \leq 3\).

Note, that it has been conjectured in [32] that the famous 4-color theorem does also hold for simple signed planar graphs. However, this conjecture is disproved, see Section 4.6.

Theorem 4.7 approximates Steinberg’s Theorem, which states that every triangle-free planar graph is 3-colorable. In this context, Erdős raised the question (see problem 9.2 in [32]) whether there exists a constant \(k\) such that every planar graph without cycles of length from 4 to \(k\) is 3-colorable? The question was studied by Hu and Li in [17] for signed graphs.

Theorem 4.8 ([17]). Let \((G, \sigma)\) be a signed planar graph. If \(G\) does not contain a circuit of length \(k\) for all \(k \in \{4, 5, 6, 7, 8\}\), then \(\chi_{\pm}((G, \sigma)) \leq 3\).

4.2 Circular colorings of signed graphs

In [23], Kang and Steffen extended the concept of \((k, d)\)-coloring to signed graphs and thus introduced a new approach to coloring of signed graphs. For unsigned graphs, the concept of the circular chromatic number had been introduced under the name the star chromatic number by Vince [42] in 1988 as the infimum over all rational numbers \(\frac{r}{t}\) so that there is a mapping \(\varphi\) from \(V(G)\) to the cyclic group of integers modulo \(n\), \(\mathbb{Z}/n\mathbb{Z}\) (or just \(\mathbb{Z}_n\)) with the property that if \(u\) and \(v\) are adjacent vertices in \(G\), then \(\varphi(u)\) and \(\varphi(v)\) are at distance of at least \(k\) in \(\mathbb{Z}_n\). For signed graphs, this concept has to be modified to take into account the signs of the edges.

For \(x \in \mathbb{R}\) and a positive real number \(r\), we denote by \([x]_r\), the remainder of \(x\) divided by \(r\), and define \([x]_r = \min([x], [-x]_r)\). Thus, \([x]_r \in [0, r)\) and \([x]_r = |-x]_r\). For \(a, b \in \mathbb{Z}\) and an integer \(k \geq 2\), \(|a - b|_k\) can be regarded as the distance of \(x\) and \(y\) in \(\mathbb{Z}_k\). For two positive integers \(k\) and \(d\) with \(k \geq 2d\), a \((k, d)\)-coloring of a signed graph \((G, \sigma)\) is a mapping \(c : V(G) \rightarrow \mathbb{Z}_k\) with the property, that \(d \leq |c(e) - \sigma(e)c(w)|_k\) for each edge \(e = vw\). The circular chromatic number \(\chi_c((G, \sigma))\) of a signed graph \((G, \sigma)\) is the infimum over all \(\frac{r}{t}\) so that \((G, \sigma)\) has a \((k, d)\)-coloring. The minimum \(k\) such that \((G, \sigma)\) has a \((k, 1)\)-coloring is called the chromatic number of \((G, \sigma)\), and it is denoted by \(\chi((G, \sigma))\). In the context of integer colorings, a \((k, 1)\)-coloring will also be denoted as a \(\mathbb{Z}_k\)-coloring or a modular \(k\)-coloring.

Similar to \(\chi_\pm\), we have \(\chi((G, 1)) = \chi(G)\) and \(\chi_c((G, 1)) = \chi_c(G)\). Therefore, this concept naturally generalizes circular coloring of unsigned graphs to signed graphs. Like for other definitions of signed graph coloring, the circular chromatic number is unchanged under resigning classes of signed graphs. Given a signed graph \((G, \sigma)\) with a \((k, d)\)-coloring \(c\),
a resigning of \((G, \sigma)\) at \(u \in V(G)\) together with a change of \(c(u)\) to \(-c(u)\) results in an equivalent graph \((G, \sigma')\) with \((k, d)\)-coloring \(c'\).

**Proposition 4.9** ([25]). Let \(k\) and \(d\) be positive integers, \((G, \sigma)\) be a signed graph and \(c\) be a \((k, d)\)-coloring of \((G, \sigma)\). If \((G, \sigma)\) and \((G, \sigma')\) are equivalent, then there exists a \((k, d)\)-coloring \(c'\) of \((G, \sigma')\). Therefore, \(\chi_c((G, \sigma)) = \chi_c((G, \sigma'))\).

Note, that if \((G, \sigma)\) has a \((k, d)\)-coloring \(c\), then there is an equivalent graph \((G, \sigma')\) with corresponding \((k, d)\)-coloring \(c'\) such that \(c'(v) \in \{0, 1, 2, \ldots, \left\lfloor \frac{d}{k} \right\rfloor\}\) for every \(v \in V(G)\). This is easy to see if one resigns \((G, \sigma)\) in every vertex where \(c(v) > \left\lfloor \frac{d}{k} \right\rfloor\).

In [25] it is shown that the number of used colors in a smallest \((k, d)\)-coloring can be bounded by a function of the order of the graph. Hence, as in the case of unsigned graphs, the circular chromatic number is a minimum. Therefore, if \(\chi_c((G, \sigma)) = \frac{k}{d}\), then \((G, \sigma)\) has a \((k, d)\)-coloring.

**Theorem 4.10** ([25]). Let \((G, \sigma)\) be a signed graph on \(n\) vertices, then \(\chi_c((G, \sigma)) = \min \{\frac{k}{d} : (G, \sigma)\) has a \((k, d)\)-coloring and \(k \leq 4n\}\).

From circular graph coloring on unsigned graphs, it is known that \([\chi_c(G)] = \chi(G)\), [42]. There is a similar result for the case of signed graphs as well.

**Theorem 4.11** ([25]). Let \((G, \sigma)\) be a signed graph. Then \(\chi((G, \sigma)) - 1 \leq \chi_c((G, \sigma)) \leq \chi((G, \sigma))\).

Note that, unlike for unsigned graphs, it is actually possible that the difference between \(\chi_c\) and \(\chi\) equals 1. Actually, this case is equivalent to the existence of a different coloring.

**Theorem 4.12** ([25]). Let \((G, \sigma)\) be a signed graph with \(\chi((G, \sigma)) = t + 1\) for a positive integer \(t\). Then, \(\chi_c((G, \sigma)) = t\) if and only if \((G, \sigma)\) has a \((2t, 2)\)-coloring.

Also, there is a simple construct of a graph whose all-positive subgraph is a complete \(l\)-partite graph that allows to prove the existence of the above mentioned property.

**Theorem 4.13** ([23, 25]). For every \(k \geq 2\), there exists a signed graph \((G, \sigma)\) with \(\chi((G, \sigma)) - 1 = \chi_c((G, \sigma)) = k\).

If, however, the difference between \(\chi\) and \(\chi_c\) is not 1, then this lower bound on \(\chi_c\) can be further improved. Recalling Theorem [11], we know that the value of \(\chi_c\) can be stated as \(\frac{p}{q}\) with \(p\) and \(q\) being coprime integers and \(p \leq 4n\), thus, if \(\chi_c\) and \(\chi\) differ, their difference has to be at least \(\frac{1}{2}\) and from this inequality we can derive the following theorem.

**Theorem 4.14** ([25]). Let \((G, \sigma)\) be a signed graph on \(n\) vertices. If \(\chi((G, \sigma)) - 1 \neq \chi_c((G, \sigma))\), then \((\chi((G, \sigma)) - 1)\left(1 + \frac{1}{4n}\right) \leq \chi_c((G, \sigma)) \leq \chi((G, \sigma)) - \chi_c((G, \sigma)) < 1 - \frac{1}{4n}$.\)
4.3 Circular r-colorings of signed graphs

On unsigned graphs, there is a coloring equivalent to \((k,d)\)-coloring, called circular coloring, that was introduced by Zhu in [53]. On signed graphs, this definition has to be modified in a similar way.

Let \((G,\sigma)\) be a signed graph and \(r \geq 1\) be a real number. A circular-\(r\)-coloring of \((G,\sigma)\) is a function \(f : V(G) \rightarrow [0,r)\) such that for any edge \(e = uv\) of \((G,\sigma)\), if \(\sigma(e) = 1\), then \(1 \leq |f(u) - f(v)| \leq r - 1\), and if \(\sigma(e) = -1\), then \(1 \leq |f(u) + f(v) - r| \leq r - 1\). This definition can be equivalently stated if one identifies 0 and \(r\) on the interval \([0,r]\), thus obtaining a circle with perimeter \(r\), denoted by \(S'\). Now the colors are points on \(S'\) and the distance between two points \(a\) and \(b\) on \(S'\) is the length of the shorter arc of \(S'\) connecting \(a\) and \(b\), which can be described as \(|a - b|_r\). If we now define the inverse of \(a \in S'\) to be \(r - a\), then we can describe a circular \(r\)-coloring of \((G,\sigma)\) as a function \(f : V(G) \rightarrow [0,r)\) such that for any edge \(e = uv\) of \((G,\sigma)\), if \(\sigma(e) = 1\), then \(1 \leq |f(u) - f(v)|_r\), and if \(\sigma(e) = -1\), then \(1 \leq |f(u) + f(v) - r|_r\). This definition also enables the easy conversion under resigning. If \(f\) is a circular \(r\)-coloring of \((G,\sigma)\) and \((G,\sigma')\) can be obtained by resigning \((G,\sigma)\) at \(v \in V(G)\), then \(f'\) defined as \(f\) on \(V(G) \setminus \{v\}\) and \(f'(v) = r - f(v)\) is a circular \(r\)-coloring of \((G,\sigma')\). Hence, for every circular \(r\)-coloring on a signed graph, there is a circular \(r\)-coloring on a resigning equivalent graph that only uses colors in the interval \([0,\frac{r}{2}]\).

Like on unsigned graphs, there is an equivalence of \((k,d)\)-colorings to \(r\)-coloring on signed graphs.

Theorem 4.15 ([25]). Let \(k\) and \(d\) be positive integers with \(2d \leq k\). A signed graph \((G,\sigma)\) has a \((2k,2d)\)-coloring if and only if \((G,\sigma)\) has a circular \(\frac{k}{2}\)-coloring.

Theorem 4.16 ([25]). Let \((G,\sigma)\) be a signed graph. Then \(\chi_c((G,\sigma)) = \min\{r : (G,\sigma)\text{ has a circular }r\text{-coloring}\}\).

4.4 Relations between coloring parameters and the chromatic spectrum of a graph

The chromatic number \(\chi((G,\sigma))\) is different from the modified signed chromatic number \(\chi_\pm(G,\sigma)\) by Múčajová, Raspaud and Škoviera, as the following theorem shows.

Proposition 4.17 ([25]). If \((G,\sigma)\) is a signed graph, then \(\chi_\pm((G,\sigma)) - 1 \leq \chi((G,\sigma)) \leq \chi_\pm((G,\sigma)) + 1\).

The following proposition classifies some easy examples showing that these bounds are tight.

Proposition 4.18 ([25]). Let \((G,\sigma)\) be a connected signed graph with \(|V(G)| \geq 3\).

(i) If \((G,\sigma)\) is antibalanced and not bipartite, then \(\chi_\pm((G,\sigma)) = 2\) and \(\chi((G,\sigma)) = 3\).
Lemma 4.20 \([\ref{24}]\). The following statements hold.

(ii) If \((G, \sigma)\) is bipartite and not antibalanced, then \(\chi_{\pm}((G, \sigma)) = 3\) and \(\chi((G, \sigma)) = 2\).

Obviously, the signed chromatic number of a signed graph depends not only on the structure of the underlying graph, but also on the signature. For example, every all-negative signed graph can be colored using only the color 1. It is therefore important to study how much the signed chromatic number of a given signed graph can be changed by replacing the corresponding signature.

The set \(\Sigma_{\chi}(G) = \{\chi((G, \sigma)) : \sigma\) is a signature on \(G\}\) is the chromatic spectrum of \(G\), and let \(M_{\chi} = \max \Sigma_{\chi}(G)\) and \(m_{\chi} = \min \Sigma_{\chi}(G)\). Analogously, \(\Sigma_{\chi_{\pm}} = \{\chi_{\pm}((G, \sigma)) : \sigma\) is a signature on \(G\}\) is the chromatic spectrum of \(G\) with respect to \(\chi_{\pm}\) and \(m_{\chi_{\pm}}(G)\) and \(M_{\chi_{\pm}}(G)\) denote the minimum and maximum of this set, respectively.

Proposition 4.19 \([\ref{24}]\). Let \(G\) be a nonempty graph. The following statements hold.

(i) \(\Sigma_{\chi}(G) = \{1\} \iff m_{\chi} = 1 \iff E(G) = \emptyset\).

(ii) If \(E(G) \neq \emptyset\), then \(\Sigma_{\chi}(G) = \{2\}\) if and only if \(m_{\chi} = 2\) if and only if \(G\) is bipartite.

(iii) If \(G\) is not bipartite, then \(m_{\chi} = 3\), and

(iv) \(\Sigma_{\chi_{\pm}}(G) = \{1\}\) if and only if \(E(G) = \emptyset\).

(v) If \(E(G) \neq \emptyset\), then \(m_{\chi_{\pm}} = 2\).

The third statement is obtained by using an all-negative signature on \(G\) and coloring every vertex 1 in \(\mathbb{Z}_3\). Using a similar method on \(G - H\), with \(H\) being an induced subgraph of \(G\), the following statement is obtained.

Lemma 4.20 \([\ref{24}]\). Let \(k \geq 3\) be an integer. If \(H\) is an induced subgraph of a graph \(G\) with \(k \in \Sigma_{\chi}(H)\) \((k \in \Sigma_{\chi_{\pm}}(H))\), then \(k \in \Sigma_{\chi}(G)\) \((k \in \Sigma_{\chi_{\pm}}(G))\).

It turns out that the chromatic spectrum of \(G\) with respect to these two coloring parameters is an interval of integers.

Theorem 4.21 \([\ref{24}]\). If \(G\) is a graph, then \(\Sigma_{\chi}(G) = \{k : k \in \mathbb{N} \text{ and } m_{\chi}(G) \leq k \leq M_{\chi}(G)\}\) and \(\Sigma_{\chi_{\pm}}(G) = \{k : k \in \mathbb{N} \text{ and } m_{\chi_{\pm}}(G) \leq k \leq M_{\chi_{\pm}}(G)\}\).

We close this section with relating \(\chi_{\pm}\) and \(\chi_{c}\) to each other.

Proposition 4.22. Let \((G, \sigma)\) be a signed graph and \(k \in \mathbb{N}\). If \((G, \sigma)\) has a 2\(k\)-coloring, then \((G, \sigma)\) has a \((4k, 2)\)-coloring and a circular 2\(k\)-coloring. In particular, if \(\chi_{\pm}((G, \sigma)) = 2k\), then \(\chi_c((G, \sigma)) \leq \chi_{\pm}((G, \sigma))\).

Proof. Let \(\phi : V(G) \to \{-1, \ldots, \pm k\}\) be a 2\(k\)-coloring. By possibly resigning we can assume that \(\phi(v)\) is positive for every \(v \in V(G)\). Let \(\phi' : V(G) \to \mathbb{Z}_{4k}\) with \(\phi'(v) = 2\phi(v) - 1\). It is easy to see that \(\phi'\) is a \((4k, 2)\)-coloring of \((G, \sigma)\). By Theorem 4.15 \([\ref{15}]\), \((G, \sigma)\) has a circular 2\(k\)-coloring. The second part of the statement follows directly from these facts. \(\square\)
4.5 Choosability on signed graphs

Both definitions of the previous two subsections (\(k\)-colorings and \(\mathbb{Z}_k\)-colorings) are used to study list colorings and choosability on signed graphs. In the context of signed graph coloring, another important adoption from the theory of unsigned graphs is the notion of list-colorings and choosability on signed graphs. For unsigned graphs, these notions had been introduced by Erdős, Rubin and Taylor in [10] and many results on these parameters can be transformed into similar ones for signed graphs. The coloring number of a graph \(G\), denoted by \(\text{col}(G)\), is the maximum ranging over the minimum degree of all subgraphs of \(G\) plus 1. Therefore, a graph with coloring number at most \(k + 1\) is also \(k\)-degenerate. Note that the coloring number is unchanged under any signature assignment to \(G\).

Let \((G, \sigma)\) be a signed graph, \(k \geq 0\) be an integer and \(f : V(G) \rightarrow \mathbb{N}_0\) a function. A list-assignment \(L\) of \((G, \sigma)\) is a function that maps every vertex \(v\) of \(G\) to a nonempty set (list) of colors \(L(v) \subseteq \mathbb{Z}\). More specific, if \(|L(v)| = f(v)\) for every \(v \in V(G)\) we call \(L\) an \(f\)-assignment and if \(|L(v)| = k\) for every \(v \in V(G)\) we call it a \(k\)-assignment. An \(L\)-coloring of \((G, \sigma)\) is a proper coloring \(\phi\) of \((G, \sigma)\) such that for all \(v \in V(G)\) the coloring \(\phi(v) \in L(v)\). If \((G, \sigma)\) admits an \(L\)-coloring, then \((G, \sigma)\) is said to be \(L\)-colorable or, more generally, list-colorable. Resulting from this, if \((G, \sigma)\) is \(L\)-colorable for every \(f\)-assignment \(L\) of \((G, \sigma)\), it is called \(f\)-list-colorable and similarly \((G, \sigma)\) is called \(k\)-list-colorable or \(k\)-choosable if it is \(L\)-colorable for every \(k\)-assignment \(L\). The signed list-chromatic number or signed choice number of \((G, \sigma)\) is the smallest integer \(k \geq 0\) such that \((G, \sigma)\) is \(k\)-choosable. We denote it as \(\chi^L_k((G, \sigma))\).

The proposition that every \((d - 1)\)-degenerate graph is \(d\)-choosable can easily be generalized via induction on the vertex set of a signed graph.

**Theorem 4.23** ([20]). Let \((G, \sigma)\) be a signed graph. If \(G\) is \((d - 1)\)-degenerate, then \((G, \sigma)\) is \(d\)-choosable.

Since a signed coloring of \((G, \sigma)\) with color set \(M_k\) can be regarded as an \(L\)-coloring for the \(k\)-assignment \(L\) with \(L(v) = M_k\) for every \(v \in V(G)\), it is easy to see that \(\chi_L((G, \sigma)) \leq \chi^L_k((G, \sigma))\). In fact, \(\chi^L_k\) can be incorporated in a chain of inequalities that is an extension of the Brooks type formula stated earlier.

**Proposition 4.24** ([38]). Every signed graph \((G, \sigma)\) satisfies

\[\chi_L((G, \sigma)) \leq \chi^L_k((G, \sigma)) \leq \text{col}((G, \sigma)) \leq \Delta(G) + 1.\]

Since \(\chi_L((G, \sigma))\) is invariant under resigning, it makes sense that this also holds true for the signed choice number and some sort of resigning defined for a list-assignment. In [20] there is such a definition.

Let \((G, \sigma)\) be a signed graph, \(L\) be a list-assignment of \((G, \sigma)\), and \(\phi\) be an \(L\)-coloring of \((G, \sigma)\). Let \(X \subseteq V(G)\). We say \(L'\) is obtained from \(L\) by a resign at \(X\) if

\[L'(v) = \{-\alpha : \alpha \in L(v)\}, \text{ if } v \in X, \text{ and } L'(v) = L(v), \text{ if } v \in V(G) \setminus X.\]
With this definition we easily get

**Proposition 4.25** \([20]\). Let \((G, \sigma)\) be a signed graph, \(L\) be a list-assignment of \(G\) and \(\phi\) be an \(L\)-coloring of \((G, \sigma)\). If \(\sigma', L'\) and \(\phi'\) are obtained from \(\sigma, L\) and \(\phi\) by resigning at a subset of \(V(G)\), then \(\phi'\) is an \(L'\)-coloring of \((G, \sigma')\). Furthermore, two resigning equivalent signed graphs have the same signed choice number.

Therefore, Zaslavsky’s proposition in \([48]\) that it is possible to write signed graph theory in terms of resigning classes still seems convincing. A signed graph \((G, \sigma)\) is called degree-choosable if \((G, \sigma)\) is \(f\)-list-colorable for the degree function \(f(v) = d_G(v)\) for all \(v \in V(G)\). Not every signed graph is degree-choosable but every signed graph \((G, \sigma)\) is \(f\)-list-colorable if \(f(v) = d_G(v) + 1\) for every \(v \in V(G)\). In fact, Schweser and Stiebitz showed an extension of a result of \([10]\) regarding the characterization of degree-choosable unsigned graphs for which we will recall the notion of a block of \((G, \sigma)\), a maximal connected subgraph of \((G, \sigma)\) that has no separating vertex. Also, we call a signed graph \((G, \sigma)\) a brick if \((G, \sigma)\) is a balanced complete graph, a balanced odd circuit, an unbalanced even circuit, a signed extension of \(K_n\) for an integer \(n \geq 2\), or a signed extension of \(C_n\) for an odd integer \(n \geq 3\). One can see that this class of signed graphs is an extension of the extremal cases for Brooks’ theorem for signed graphs.

**Theorem 4.26** \([38]\). Let \((G, \sigma)\) be a connected signed graph. Then \((G, \sigma)\) is not degree-choosable if and only if each block of \((G, \sigma)\) is a brick.

The following corollary is a Brooks’ type theorem for the list-chromatic number of signed graphs.

**Corollary 4.27** \([38]\). Let \((G, \sigma)\) be a connected signed graph. If \((G, \sigma)\) is not a brick, then

\[
\chi'_\pm((G, \sigma)) \leq \Delta(G).
\]

There are also generalizations for results in the theory of choosability on planar graphs to the case of signed planar graphs. In particular, Jin, Kang and Steffen were able to extend some major theorems for the case of planar signed graphs. The next theorem for example is an extension of Thomassen’s work in \([40]\) and its proof uses the same method.

**Theorem 4.28** \([20]\). Every signed planar graph is 5-choosable.

Also extendable is a result of Voigt in \([43]\) for the existence of not 4-choosable planar unsigned graphs. Interestingly, there are signed graphs with this property, but their underlying unsigned graphs are 4-choosable.

**Theorem 4.29** \([20]\). There exists a signed planar graph \((G, \sigma)\) that is not 4-choosable but \(G\) is 4-choosable.

With Theorem \([12]\) one can characterize a class of planar graphs that are 4-choosable.
Theorem 4.30 \([\text{[20]}]\). Let \((G, \sigma)\) be a signed planar graph. For each \(k \in \{3, 4, 5, 6\}\), if \((G, \sigma)\) has no \(k\)-circuits, then \((G, \sigma)\) is 4-choosable.

Furthermore, the proof of Thomassen in \([\text{[41]}]\) regarding the 3-choosability of every planar graph of girth at least 5 also works for signed graphs.

Theorem 4.31 \([\text{[20]}]\). Every signed simple planar graph with neither 3-circuit nor 4-circuit is 3-choosable.

Recently, Kim and Yu \([\text{[28]}]\) proved that every planar graph with no 4-circuits adjacent to 3-circuits is 4-choosable.

With the theory of list-colorability of signed graphs, naturally the notion of list-critical signed graphs emerges. Let \((G, \sigma)\) be a signed graph and let \(L\) be a list assignment of \((G, \sigma)\). The signed graph \((G, \sigma)\) is called \(L\)-critical if \((G, \sigma)\) is not \(L\)-colorable, but every proper subgraph of \((G, \sigma)\) is. Particularly, if \(L\) is a \((k - 1)\)-list-assignment, we call \((G, \sigma)\) \(k\)-list-critical. Also, a signed graph \((G, \sigma)\) is called \(k\)-critical if \(\chi_+(\sigma(G)) = k\) and for every proper subgraph \((H, \sigma')\) of \((G, \sigma)\), \(\chi_+(\sigma(H)) \leq k - 1\). With the same argument as before, we see that every \(k\)-critical signed graph is \(k\)-list-critical. A signed graph \((G, \sigma)\) is called \(k\)-choice-critical if \(\chi_+^c(G, \sigma) = k\) and for every signed proper subgraph \((H, \sigma')\) of \((G, \sigma)\) we have \(\chi_+^c(G, \sigma) \leq k - 1\). Again, we get the result that every \(k\)-choice-critical signed graph is \(k\)-list-critical.

Especially the class of 3-critical signed graphs can be easily characterized.

Lemma 4.32 \([\text{[38]}]\). A signed graph is 3-critical if and only if it is a balanced odd circuit or an unbalanced even circuit.

The following lemma states some of the basic properties of list-critical signed graphs.

Lemma 4.33 \([\text{[38]}]\). Let \((G, \sigma)\) be an \(L\)-critical signed graph for a list-assignment \(L\) of \((G, \sigma)\). Let \(H = \{v : d_G(v) > |L(v)|\}\) and \(F = V(G) \setminus H\) with \(\emptyset \neq X \subseteq F\). The following statements hold true.

(i) Every block of \((G, \sigma)\) : \(X\) is a brick.

(ii) If \(L\) is a list-assignment with \(k \geq 1\), then \(H \neq \emptyset\) or \((G, \sigma)\) is a brick. Furthermore, if \((G, \sigma)\) contains a \(K_k\), then \((G, \sigma)\) is a balanced complete graph of order \(k\).

This lemma implies, that for a \(k\)-list-critical signed graph \((G, \sigma)\) the minimum degree \(\delta(G)\) is at least \(k - 1\) and so we have a lower bound for the number of edges in a \(k\)-list-critical signed graph of the form \(|E((G, \sigma))| \geq \frac{1}{2}(k - 1)|V(G)|\). This bound can be improved further for certain signed simple graphs.

Theorem 4.34 \([\text{[38]}]\). Let \((G, \sigma)\) be a \(k\)-list-critical signed simple graph with \(k \geq 4\) that is not a balanced complete graph of order \(4\), then

\[
2|E((G, \sigma))| \geq \left(k - 1 + \frac{k - 3}{k^2 - 3}\right)|V(G)|.
\]
In a recent paper, Zhu [55] extended his study on signed graph coloring which lead to a refinement of the concept of choosability such that the two extremal cases are \( k \)-choosability and \( k \)-colorability and in between there are gradually changing concepts of coloring which depend on the possible partitions of the integer \( k \). In [55] it is shown that several kinds of generalized signed graph coloring can be expressed in terms of a refined choosability concept.

### 4.6 Coloring generalized signed graphs

Jin et al. generalize the concepts of sections 4.1 and 4.2 in [19, 21]. To describe their concept we need some definitions. In this context, a graph is considered as a symmetric digraph, where each edge \( vw \) is replaced by two opposite arcs \( e = (v, w) \) and \( e^{-1} = (w, v) \). Let \( S \) be an inverse closed set of permutations of positive integers. An \( S \)-signature of \( G \) is a mapping \( \sigma : E(G) \to S \) such that \( \sigma(e) = \sigma(e)^{-1} \) for every arc \( e \), and \((G, \sigma)\) is called an \( S \)-signed graph. Let \( [k] = \{1, 2, \ldots, k\} \). A \( k \)-coloring of \((G, \sigma)\) is a mapping \( c : V(G) \to [k] \) such that \( \sigma(e)(c(v)) \neq c(w) \) for each arc \( e = (v, w) \). The graph \( G \) is \( S \)-\( k \)-colorable if \((G, \sigma)\) is \( k \)-colorable for every \( S \)-signature of \( G \).

The image of an integer \( a \not \in [k] \) is irrelevant in an \( S - k \)-coloring of a graph \( G \). Analogously, if \( a \in [k] \) and \( \pi(a) \not \in [k] \), then \( \pi(a) \) is irrelevant. In this sense, in [19, 21] only permutations which are bijections between subsets of \([k]\) are considered.

Let \( S_k \) be the set of permutations of \([k]\), \( id \) be the identity and let \( \pi = (1, 2)(3, 4) \ldots ((2q-1), 2q) \), where \( q = \lfloor \frac{k}{2} \rfloor \) and \( \pi' = (1, 2)(3, 4) \ldots ((2q'-1), 2q') \), where \( q' = \lfloor \frac{k}{2} \rfloor - 1 \). An \( k \)-coloring of a signed graph \((G, \sigma)\) is a mapping \( c : V(G) \to [k] \) such that \( c(v) \neq c(w) \) if \( vw \) is a positive edge and \( \pi(c(v)) \neq c(w) \) if \( vw \) is a negative edge. A modular \( k \)-coloring of \((G, \sigma)\) is defined analogously with \( \pi' \) instead of \( \pi \). Note that in this context, the \( S \)-chromatic numbers of \( G \) with respect to modular colorings (Section 4.2) and \( n \)-colorings (Section 4.1) are \( M_k(G) \) and \( M_{k_2}(G) \), respectively.

The coloring of \( S \)-signed graphs generalizes some well known notions of colorings.

**Proposition 4.35 ([19]).** Let \( S \) be a subset of \( S_k \).

- If \( S = \{id\} \), then \( S - k \)-coloring is equivalent to conventional \( k \)-coloring.
- If \( S = \{id, (1, 2)(3, 4) \ldots ((2q-1), 2q) \} \) and \( q = \lfloor \frac{k}{2} \rfloor \), then \( S - k \)-coloring is equivalent to \( k \)-coloring.
- If \( S = \{id, (1, 2)(3, 4) \ldots ((2q'-1), 2q') \} \) and \( q' = \lfloor \frac{k}{2} \rfloor - 1 \), then \( S - k \)-coloring is equivalent to modular \( k \)-coloring.
- If \( S = S_k \), then \( S - k \)-coloring is equivalent to \( DP \)-\( k \)-coloring, as defined in [22].
- If \( S = < (1, 2, \ldots, k) > \) is the cyclic group generated by the permutation \((1, 2, \ldots, k)\), then \( S - k \)-coloring is equivalent to \( \mathbb{Z}_k \)-coloring, as defined by Jaeger, Linial, Payan and Tarsi [18].
Jin et al. [21] give also an equivalent formulation of a $k$-coloring of a gain graph $(G, \phi)$ with gain group $\Gamma$, see [51], in terms of a $S-k'$-coloring of $(G, \sigma)$.

The papers [19, 21, 26, 54, 56] focus on planar graphs. Two subsets $S$ and $S'$ of $S_k$ are conjugate if there is a permutation $\pi$ in $S_k$ such that $S' = \{ \pi \sigma \pi^{-1} : \sigma \in S \}$. A subset $S$ of $S_4$ is good if every planar graph is $S$-4-colorable, and it is bad if it is not good. Hence, by the 4-Color-Theorem, $\{id\}$ is good. Mácajová, Raspaud, and Škoviera [32] conjectured that $\{id, (1, 2)(3, 4)\}$ and Kang and Steffen conjectured that $\{id, (1, 2)\}$ is good. Therefore, a natural question is whether there are subsets of $S_4$ which are good.

Jin et al. [21] answer the question completely for subsets $S$ of $S_4$ as long as $S$ contains $id$, which is somehow the canonical label for a positive edge. They summarize: Král et al. [30] showed that the set $\{id, (1234), (13)(24), (1432)\}$ as well as the set $\{id, (12)(34), (13)(24), (14)(23)\}$ are not good. In a first version of [22] Jin et al. excluded $\{id, (123), (12, (123)), (12)(13), (12)(34), (13), (12)(34), (13)(24)\}$. Zhu [55] shows that $\{id, (12)\}$ is bad, and therefore, he disproved the conjecture of Kang and Steffen. Kardoš and Narboni [26] disproved the conjecture of Mácajová, Raspaud, and Škoviera by showing that $\{id, (12), (34)\}$ is bad. The remaining two cases for $S \in \{id, (123), (12, (123))\}$ are shown to be bad in the revised version [21] of [22] by Jin et al., which completes the proof of the following theorem.

**Theorem 4.36.** A subset $S$ of $S_4$ is good if and only if $S = \{id\}$.

Jiang et al. study in [19] the question whether Grötzsch’s Theorem can be generalized to signed graphs. A non-empty subset $S$ of $S_3$ is TFP-good, if every triangle-free planar graph is $S$-3-colorable. Grötzsch Theorem says that $\{id\}$ is TFP-good. They proved that an inverse closed subset of $S_3$ not isomorphic to $\{id, (12)\}$ is TFP-good if and only if $S = \{id\}$. Hence, the only remaining open case for this question is whether $\{id, (12)\}$ is TFP-good.

Using the relation between signed graph coloring and DP-colorings, Kim and Ozeki [27] gave a structural characterization of graphs that do not admit a DP-coloring which generalizes Theorem 4.26.

Further concept of graph parameters which are closely related to coloring are extended to signed graphs. For instance, Wang et al. study the Alon-Tarsi number of signed graphs in [45] and Lajou [31] the achromatic number of signed graphs.

## 5 Signed graph coloring via signed homomorphisms according to Naserasr, Rollová and Sopena

In [14] B. Guenin introduced the notion of homomorphisms on signed graphs. This concept can be used to define a chromatic number on signed graphs that is different from those which we have discussed so far. In the following, we consider graphs to be simple and loopless except when explicitly stated otherwise.
Homomorphism theory on signed graphs, in conformity with Zaslavsky’s earlier statement, can be discussed in terms of resigning classes of signed graphs. We will follow the definitions stated in [34] and define homomorphisms on signed graphs as homomorphisms on resigning classes of signed graphs. For the sake of simplicity, we will call both \((G, \sigma)\) and its corresponding resigning class \([G, \sigma]\) signed graphs. The difference can always be spotted by looking at the braces.

Given two signed graphs \([G, \sigma_1]\) and \([H, \sigma_2]\), we say that there is a homomorphism of \([G, \sigma_1]\) to \([H, \sigma_2]\) if there is a representation \([G, \sigma'_1]\) of \([G, \sigma_1]\) and a representation \((H, \sigma'_2)\) of \([H, \sigma_2]\) together with a vertex-mapping \(\phi : V(G) \rightarrow V(H)\) such that if \(xy \in E(G)\), then \(\phi(x)\phi(y) \in E(H)\) and \(\phi(x)\phi(y)\) has the same sign as \(xy\). In other words, \(\phi\) preserves signed adjacency. We will state the existence of a homomorphism of \([G, \sigma_1]\) to \([H, \sigma_2]\) as \([G, \sigma_1] \rightarrow [H, \sigma_2]\).

Let \(\phi\) be a homomorphism of \([G, \sigma_1]\) to \([H, \sigma_2]\) using the representations \((G, \sigma'_1)\) of \([G, \sigma_1]\) and \((H, \sigma'_2)\) of \([H, \sigma_2]\) and let \(X \subseteq V(H)\) be the resigning set that forms \((H, \sigma'_2)\) from \((H, \sigma_2)\). Let \((G, \sigma''_1)\) be the signed graph we receive if we resign \((G, \sigma'_1)\) at \(\phi^{-1}(X) \subseteq V(G)\). Then, \(\phi\) is also a homomorphism of \([G, \sigma_1]\) to \([H, \sigma_2]\) using the representations \((G, \sigma''_1)\) and \((H, \sigma_2)\).

An automorphism of \([G, \sigma]\) is a homomorphism of the signed graph to itself that is bijective on the vertex set \(V(G)\) and the induced edge-mapping is surjective. If, for each pair \(x, y\) of vertices of \([G, \sigma]\), there exists an automorphism \(\rho\) of \([G, \sigma]\) such that \(\rho(x) = y\), \([G, \sigma]\) is called vertex-transitive. Similarly, if for each pair \(e_1 = xy\) and \(e_2 = uv\) of edges of \([G, \sigma]\), there exists an automorphism \(\rho\) of \([G, \sigma]\) such that \(\{\rho(x), \rho(y)\} = \{u, v\}\), we call \([G, \sigma]\) edge-transitive. An unbalanced circuit is an example of a signed graph that is both vertex-transitive and edge-transitive since it has representatives with only one negative edge which can be moved around by resigning one of its endpoints. Thus, there are automorphisms that induce a ”rotation” on the circuit. We say that a signed graph \([G, \sigma]\) is isomorphic to \([H, \sigma']\) if there is a homomorphism of \([G, \sigma]\) to \([H, \sigma']\) that is bijective on the vertex set and the induced edge-mapping is bijective as well.

A core of a signed graph \([G, \sigma]\) is a minimal subgraph of \([G, \sigma]\) to which \([G, \sigma]\) admits a homomorphism. A signed core is a signed graph that admits no homomorphism to a proper subgraph of itself, equivalently if every homomorphism of \([G, \sigma]\) to \([G, \sigma]\) is an automorphism, then \([G, \sigma]\) is a core. The following lemma shows that the core of a signed graph is well-defined.

**Lemma 5.1 ([34])**. Let \([G, \sigma]\) be a signed graph. The core of \([G, \sigma]\) is unique up to isomorphy of signed graphs.

Since, in the context of homomorphisms, the signature of the image graph is of no concern, the binary relation of the existence of a homomorphism on signed graphs is associative.
Lemma 5.2 ([34]). Let \( G, \sigma \) and \( H, \sigma \) be signed graphs. Then the relation \( [G, \sigma] \rightarrow [H, \sigma] \) is associative.

Hence, the relation \( [G, \sigma] \rightarrow [H, \sigma] \) is a quasi-order on the class of all signed graphs which is a poset on the class of all signed cores. Naserasr, Rollová and Sopena ([34]) call this order the homomorphism order of signed graphs and say that \( [H, \sigma] \) bounds \( [G, \sigma] \), or that \( [G, \sigma] \) is smaller than \( [H, \sigma] \) instead of writing \( [G, \sigma] \rightarrow [H, \sigma] \).

Furthermore, they extend the notion to classes of graphs, so if \( C \) is a class of signed graphs, they say that \( [H, \sigma] \) bounds \( C \) if \( [H, \sigma] \) bounds every member of \( C \).

Now, as in the case of homomorphisms of unsigned graphs, the smallest order of a signed graph which bounds \( [G, \sigma] \) is called the signed chromatic number of \( [G, \sigma] \), denoted by \( \chi_{\text{Hom}}([G, \sigma]) \). Analogously, the notion of signed graph coloring can be defined in the following manner. A proper coloring of a signed graph \( [G, \sigma] \) is an assignment of colors to the vertices of \( G \) such that adjacent vertices do not receive the same color and there is a representation \( (G, \sigma') \) of \( [G, \sigma] \) such that for every pair of colors \( a, b \) every edge with one endpoint colored \( a \) and the other colored \( b \) has the same sign in \( (G, \sigma') \). The signed chromatic number \( \chi_{\text{Hom}}([G, \sigma]) \) of \( [G, \sigma] \) can then be regarded as the minimum number of colors needed for a proper coloring of \( [G, \sigma] \).

Note, that not all the different representatives \( (G, \sigma') \) of \( [G, \sigma] \) necessary admit a homomorphism to the bounding graph of \( [G, \sigma] \) (since we know from above that the domain graph is not free in its representation). Therefore, the signed chromatic number \( \chi_{\text{Hom}}([G, \sigma]) \) may also be expressed as a minimum over all representations of \( [G, \sigma] \): Let \( \text{Ord}(G, \sigma) \) be the smallest order of a signed graph to which \( (G, \sigma) \) admits a homomorphism. Then \( \chi_{\text{Hom}}([G, \sigma]) = \min \{\text{Ord}(G, \sigma) : (G, \sigma) \text{ is a representative of } [G, \sigma]\} \).

Note, that for an unbalanced circuit of length 4 we have \( \chi_\pm((C_4, \sigma)) = 3, \chi((C_4, \sigma)) = 2 \) and \( \chi_{\text{Hom}}([C_4, \sigma]) = 4 \). However, Zaslavsky’s approach of signed colors on signed graphs can be formulated in terms of homomorphisms. Let \( k \geq 1 \) be an integer and \( \pm K_{k+1} \) be the signed multigraph with vertex set \( \{0, \ldots, k\} \) and one positive and one negative edge between each pair of distinct vertices, as well a negative loop on every vertex except 0. Furthermore, let \( \pm K_k' \) be the graph obtained from \( \pm K_{k+1} \) by deleting the vertex 0 and its incident edges.

Lemma 5.3 ([4]). Let \( [G, \sigma] \) be a signed graph. Then \( [G, \sigma] \) admits a proper \( k \)-coloring (as defined by Zaslavsky) if and only if \( [G, \sigma] \rightarrow \pm K_{k+1} \). Furthermore, \( [G, \sigma] \) admits a proper zero-free \( k \)-coloring (as defined by Zaslavsky) if and only if \( [G, \sigma] \rightarrow \pm K_k' \).

The question of the existence of a homomorphism between two signed graphs is essential for this definition of signed graph coloring. In this concern, the signed chromatic number itself provides a first test for the possibility of existence of a homomorphism of \( [G, \sigma_1] \) to \( [H, \sigma_2] \).

Lemma 5.4 ([34]). Let \( [G, \sigma_1] \) and \( [H, \sigma_2] \) be signed graphs. If \( [G, \sigma_1] \rightarrow [H, \sigma_2] \), then \( \chi_{\text{Hom}}([G, \sigma_1]) \leq \chi_{\text{Hom}}([H, \sigma_2]) \).
This lemma follows from the associativity of the relation of \([G,\sigma_1] \rightarrow [H,\sigma_2]\). There is a set of lemmas in [34] that provide such tests, they are called “no homomorphism lemmas”.

In general, the problem \(s - \text{Hom}(H,\sigma)\): "Does a signed graph \([G,\sigma']\) admit a homomorphism to \([H,\sigma]\)?” is not easy to solve and its complexity has been studied e.g. in [4, 5, 13].

5.1 Minor construction

The construction of minors of signed graphs, as introduced in the context of signed graph homomorphism in [34], differs from the one Zaslavsky proposed in [48]. A minor of a signed graph \([G,\sigma]\) is a signed graph \([H,\sigma']\) obtained from \([G,\sigma]\) by a sequence of four operations: Deleting vertices, deleting edges, contracting positive edges and resigning. While a negative edge can not directly be contracted, one can resign the graph by one of its endpoints and then contract it. Since in this chapter we generally consider signed graphs to be simple, every contraction in this process that results in a multiple edge should be followed by the removal of all of these edges but one. Note that there is no rule concerning the choice of the remaining edge, so it can be chosen freely. Using this rule also prevents the creation of loops in a minor.

The set of unbalanced circuits determines a signed graph, so it is worthwhile to consider that the contraction of positive edges does not change the balance of a circuit, hence we get the following lemma.

**Lemma 5.5 ([34])**. Let \([H,\sigma']\) be a minor of a signed graph \([G,\sigma]\) that is obtained only by the operation of contracting positive edges, then the image (of the mapping induced by minor-construction) of an unbalanced circuit of \([G,\sigma]\) is an unbalanced circuit in \([H,\sigma']\).

**Corollary 5.6 ([34])**. Let \([H,\sigma']\) be a minor of a signed graph \([G,1]\). Then \([H,\sigma'] = [H,1]\).

This follows immediately from the fact that minor-construction, following the rules above, does not create circuits.

5.2 Signed Cliques

Another important extension to signed graphs is the notion of cliques. In [34] the following generalization for signed graphs was introduced. Let \([G,\sigma]\) be a signed graph, \([G,\sigma]\) is a signed clique, or short S-clique, if its signed chromatic number equals the number of its vertices. One may equivalently call a signed graph an S-clique if its homomorphic images are all isomorphic to itself. Recall that in unsigned graph theory, a clique is any complete graph, so an all-positive graph is an S-clique if and only if its underlying graph is a clique. Thus, this definition is a natural extension. Note that every signed complete graph \([K_n,\sigma]\) is an S-clique, but the converse is not true. A useful tool for determining whether a signed graph is an S-clique or not is the following lemma.
Lemma 5.7 ([34]). A signed graph \([G,\sigma]\) is an S-clique if and only if for each pair \(u\) and \(v\) of vertices of \(G\) either \(uv\) is an edge in \(G\) or \(u\) and \(v\) are vertices of an unbalanced circuit of length 4.

The following corollary follows from this lemma immediately.

Corollary 5.8 ([34]). An S-clique cannot have a cut-vertex.

An interesting example of an S-clique that is not a signed complete graph is the signed complete bipartite graph \([K_{n,n},\sigma_M]\), where \(X = \{x_1, \ldots, x_n\}\) and \(Y = \{y_1, \ldots, y_n\}\) is the bipartition of its vertices and \(N_{\sigma_M}\) is the matching \(\{x_1y_1, \ldots, x Ny_n\}\). It is easy to see that \([K_{n,n},\sigma_M]\) is an S-clique for \(n \geq 3\) if one checks for the existence of an unbalanced circuit of length 4 on every pair of vertices of the same partition. Note that, since the core of a signed graph is unique up to isomorphism, every S-clique is a core.

Following the definition of signed cliques, there are two natural definitions of the signed clique number of a signed graph. The absolute S-clique number of \([G,\sigma]\), denoted by \(\omega_{sa}[G,\sigma]\), equals the order of the largest subgraph \([H,\sigma']\) of \([G,\sigma]\) that is an S-clique itself. The relative S-clique number of \([G,\sigma]\), denoted by \(\omega_{sr}[G,\sigma]\), is the order of the largest subgraph \([H,\sigma']\) of \([G,\sigma]\) such that in every homomorphic image \(\phi[G,\sigma]\) the order of the induced subgraph \(\phi[H,\sigma']\) equals that of \([H,\sigma']\). Again it is easy to see that these definitions equal their unsigned counterpart in the case of an all-positive graph. It was also proved in [34] that these definitions are independent of resigning, so we can continue to consider resigning classes instead of actual signed graphs. However, the difference between the absolute S-clique number and the relative S-clique number of a signed graph can be arbitrarily large. These numbers also follow the homomorphism order of signed graphs, which gives rise to another “no homomorphism lemma”.

Lemma 5.9 ([34]). Let \([G,\sigma_1]\) and \([H,\sigma_2]\) be two signed graphs. If \([G,\sigma_1] \rightarrow [H,\sigma_2]\), then \(\omega_{sa}[G,\sigma_1] \leq \omega_{sa}[H,\sigma_2]\) and \(\omega_{sr}[G,\sigma_1] \leq \omega_{sr}[H,\sigma_2]\).

Also, from their definition arises the following relationship between these two S-clique numbers and the signed chromatic number.

Theorem 5.10 ([34]). Let \([G,\sigma]\) be a signed graph. Then \(\omega_{sa}[G,\sigma] \leq \omega_{sr}[G,\sigma] \leq \chi_{Hom}([G,\sigma]).\)

The signed complete bipartite graph \([K_{n,n},\sigma_M]\) is an example for the large differences that can occur between the signed chromatic number of a signed graph and the chromatic number of its underlying graph. It is bipartite and therefore, it is 2-colorable, but \(\chi_{Hom}([K_{n,n},\sigma_M]) = 2n\). Furthermore, we have the following statement.

Proposition 5.11 ([34]). For every graph \(G\) and every signature \(\sigma\) of \(G\), \(\chi_{Hom}([G,\sigma]) \geq \chi(G)\). Furthermore the difference \(\chi_{Hom}([G,\sigma]) - \chi(G)\) can be arbitrarily large.

In [34] it is shown that the problem of determining the relative or absolute S-clique number of a signed graph is NP-hard. However, in [6] the relative S-clique number is determined for graphs of some families of planar and outerplanar signed graphs.
Now that the basic definitions necessary for signed graph coloring in the context of homomorphisms are introduced, we continue by stating some of their elemental properties. We know that there is (up to resigning) only one signed graph on any given tree and its core is always the $[K_2, 1]$. Therefore we get the following proposition regarding the S-clique numbers on a signed tree.

**Proposition 5.12** ([34]). Let $[G, \sigma]$ be a signed graph. If the underlying graph $G$ is a tree, then $\omega_{sa}(G, \sigma) = \omega_{sr}(G, \sigma) = \chi_{\text{Hom}}([G, \sigma]) = 2$.

Furthermore, the set of 2-colorable signed graphs can be completely classified by two properties.

**Theorem 5.13** ([34]). A signed graph $[G, \sigma]$ is 2-colorable if and only if $G$ is bipartite and $[G, \sigma]$ is balanced.

For other values than 2 however, it is difficult to compute the signed chromatic number. In [34] it is shown that the problem “Is $\chi_{\text{Hom}}([G, \sigma]) \leq k$?” is computable in polynomial-time for $k = 1, 2$ and it is NP-complete for $k \geq 3$.

Motivated by Brooks’ Theorem there is a first approach towards the study of the relation between the signed chromatic number and the maximum degree of a signed graph, providing an upper and lower bound.

**Theorem 5.14** ([8]). For every signed graph $[G, \sigma]$ with $\Delta(G) \geq 3$: $2^{\frac{\Delta(G)}{2}\Delta(G) - 1} \leq \chi_{\text{Hom}}([G, \sigma]) \leq (\Delta(G) - 1)^2(2^{\Delta(G) - 1}) + 2$.

There are further upper bounds for the chromatic number in terms of homomorphisms to some target graphs which are studied by Ochem and Pinlou in [36] and by Naserasr, Rollová and Sopena in [33]. The latter three authors studied minor closed families of signed graphs and achieved the following results on the chromatic number.

**Theorem 5.15** ([34]). Let $G$ be a $K_4$-minor-free graph. If $[G, \sigma]$ is a signed graph, then $\chi_{\text{Hom}}([G, \sigma]) \leq 5$, and this bound is tight.

This theorem implies that in particular the chromatic number of an outerplanar signed graph $[G, \sigma]$ is at most 5. It is shown in [34] that the bound is tight for this class as well.

In [1], there is a proof of the fact that every $m$-edge-colored graph whose underlying graph has an acyclic chromatic number of at most $k$ admits a homomorphism to an $m$-edge-colored graph of order at most $km^{k-1}$. This result was generalized to colored mixed graphs in [35] and there is a version of this in [34] for the case of signed graphs as well and thus giving a bound for the class of signed graphs whose underlying graphs are acyclically $k$-colorable.

**Theorem 5.16** ([36]). Let $[G, \sigma]$ be a signed graph. If $G$ is acyclically $k$-colorable, then $\chi_{\text{Hom}}([G, \sigma]) \leq k2^{k-2}$. 
Ochem and Pinlou note that the bound of Theorem 5.16 is tight, which was shown in [11]. Theorem 5.16 can be used to give a more general rule regarding the upper bounds of the chromatic number of some classes of signed graphs. Recall that a \( k \)-tree is a graph that can be constructed from the complete graph \( K_k \) by repeatedly adding vertices in such a way that each added vertex is joined to \( k \) vertices that already form an \( k \)-clique. A subgraph of a \( k \)-tree is called a partial \( k \)-tree. Since all \( k \)-trees are acyclically \((k+1)\)-colorable, following the last theorem we get:

**Corollary 5.17.** Let \( G \) be a partial \( k \)-tree and let \( \sigma \) be any signature on \( G \). Then \( \chi_{\text{Hom}}([G,\sigma]) \leq (k+1)2^{k-1} \).

In [44], it was proved that \( K_4 \)-minor-free graphs are exactly partial 2-trees and for these the above formula only gives an upper bound of 8 instead of the earlier mentioned bound of 5, so its bounds are generally not tight. There is also an upper bound on the signed chromatic number of the class of planar signed graphs that can be obtained by using the bound on the acyclic chromatic number of planar graphs and techniques similar to the ones applied in [37] and [1], equivalently from Theorem 5.16 and the fact that every planar graph is acyclically 5-colorable (see [3]). The following theorem seems to be proved parallel in [34] and [36]. In [34] it is proved for 48 instead of 40, but as remarked in [34] using Theorem 5.16 yields the following the statement.

**Theorem 5.18** ([34, 36]). Let \([G,\sigma]\) be a planar signed graph. Then\( \chi_{\text{Hom}}([G,\sigma]) \leq 40 \). Also, there is a planar \( S \)-clique of order 8 and a planar signed graph with signed chromatic number 10.

Let \( \mathcal{P}_g \) denote the class of planar signed graphs with girth at least \( g \) and similarly \( \mathcal{O}_g \) the class of outerplanar signed graphs.

**Theorem 5.19** ([36]). Let \([G,\sigma]\) be a planar signed graph.

(i) If \([G,\sigma]\) \( \in \mathcal{O}_4 \), then \( \chi_{\text{Hom}}([G,\sigma]) \leq 4 \)

(ii) If \([G,\sigma]\) \( \in \mathcal{P}_4 \), then \( \chi_{\text{Hom}}([G,\sigma]) \leq 25 \)

(iii) If \([G,\sigma]\) \( \in \mathcal{P}_5 \), then \( \chi_{\text{Hom}}([G,\sigma]) \leq 10 \)

(iv) If \([G,\sigma]\) \( \in \mathcal{P}_6 \), then \( \chi_{\text{Hom}}([G,\sigma]) \leq 6 \)

An interesting fact is that if the maximum signed chromatic number of any planar signed graph were \( k \), then this would imply the existence of a signed graph of order \( k \) to which every signed planar graph admits a homomorphism. There are important results for other properties of signed planar graphs as well. For example the maximum order of a planar \( S \)-clique and therefore the maximum of the absolute \( S \)-clique number of planar graphs.

**Theorem 5.20** ([34]). The maximum order of a planar \( S \)-clique is 8.
The proof of this theorem heavily relies on Lemma 5.7, indicating it as a useful tool regarding S-cliques. Furthermore, the relations between unsigned graphs and signed bipartite graphs can be used to restate Hadwiger’s conjecture and in the following, leads to possibilities of a strengthening of Hadwiger’s conjecture for the class of even signed graphs. This topic is extensively studied in [34].

6 Final remarks and some conjectures

A definition of a “chromatic number” for signed graphs strongly depends on properties of the colors, as those of the “signed colors” in the definitions of Zaslavsky and Máčajová et al. or on the permutations which are associated to the edges as in the case of generalized signed graphs. Since every element of an additive abelian group has an inverse element, the condition \( c(v) \neq \sigma vw c(w) \) is equal to the condition \( c(v) \neq c(w) \) if the color \( c(v) \) is self-inverse; i.e. \( \sigma \) is the identity on \( c(v) \). The self-inverse elements play a crucial role in such colorings, since the color classes which are induced by these elements are independent sets. Hence, the following statement is true.

**Proposition 6.1.** Let \( G \) be a graph and \( \chi(G) = k \). If \( C \) is a set of \( k \) pairwise different self-inverse colors (e.g. of \( \mathbb{Z}_{2}^n \ (k \leq 2^n) \)), then every \( k \)-coloring of \( G \) with colors from \( C \) is a \( k \)-coloring of \( (G, \sigma) \), for every signature \( \sigma \) of \( G \). In particular, the chromatic number of \( (G, \sigma) \) with respect to colorings with colors of \( C \) is equal to the chromatic number of \( G \) for every signature \( \sigma \).

A graph \( G \) together with a function \( f : V(G) \rightarrow \{ \pm 1 \} \) is a marked graph. This naturally induces a signature on \( G \), where an edge is positive if its two vertices have the same mark, and it is negative otherwise. Harary and Kabell [16] noticed that the signed graph where the signature is obtained from a marking of the vertices is balanced. This fact implies that the edge-chromatic number of a signed graph \( (G, \sigma) \) is equal to edge-chromatic number of the unsigned graph \( G \), which was noticed by Schweser and Stiebitz in [38].

Despite of the the early approached of Cartwright and Harary [6], almost all concepts for coloring of signed graphs are natural generalization of the corresponding concepts for unsigned graphs. The next problem is only of interest for signed graphs.

**Problem 6.2.** What is the complexity of the following decision problem: Let \( k \) and \( t \) be integers and \( (G, \sigma) \) be a signed graph and \( \chi(G) = k \). Is \( \chi((G, \sigma)) \leq t \) \( (\chi_{\pm}((G, \sigma)) \leq t) \)?

It is easy to figure out the trivial cases. For the chromatic number of a signed graph we have that the problem is trivial if \( k \in \{1, 2\} \) or \( t \geq \min\{\Delta(G), 2k - 2\} \). Brewster et al. [4] proved that it can be decided in polynomial time whether two signed graphs are equivalent.

Let \( K_{S} \) be the largest complete graph such that \( \pm K_{S} \) has an orientable embedding into \( S \).
Conjecture 6.3. Let $S$ be a surface and $n_S$ be the largest order of a complete graph that embeds into $S$. If $S$ is not the projective plane, then $\gamma(G, \sigma) \leq \gamma(\pm K_S)$ and $\gamma^*(G, \sigma) \leq \gamma^*(\pm K_S)$, and there are graphs where equality holds.

Theorem 3.5 is a first result towards a proof of this conjecture.

In [24] it is shown that the chromatic spectrum is an interval of integers for $n$- and for modular $n$-colorings. Let $(G, \sigma)$ be an $S$-signed graph and $\chi_S(G, \sigma)$ be the minimum $k$ such that $(G, \sigma)$ has an $S$-$k$-coloring.

**Problem 6.4.** Let $S$ be an inverse closed set of permutations of $S_k$. Is it true that the set $\{\chi(\sigma(G)) : \sigma \text{ is a } S\text{-signature of } G\}$ is an interval of integers?

In the context of coloring planar graphs, the following questions might be of interest. The first one was formulated for $n$-colorings of signed graphs (see Section 4.1) by Kardoš and Narboni [26].

**Problem 6.5.** Let $S \subset S_4$ and $id \in S$. What is the smallest order of a non-$S$-4-colorable planar graph?

Conjecture 6.6. Every triangle-free signed graph is signed 3-colorable.

Theorem 4.11 says that the difference between the circular chromatic number and the chromatic number can be 1.

**Problem 6.7.** Is it true that every planar signed graph has circular chromatic number 4?

While seemingly every of the studied approaches of signed graph coloring shares many properties with their conventional counterparts, the research done in the field of the homomorphism related definition seems particular fruitful concerning results that are applicable in ordinary graph theory.

Remarkably, while there is a strong connection between the signed colorings and the circular coloring, visible through the proximity of their respective chromatic numbers, a similar connection to the homomorphism-based chromatic number is not yet apparent.

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