GENERALISED SEIBERG-WITTEN EQUATIONS AND ALMOST-HERMITIAN GEOMETRY

VARUN THAKRE

International Centre for Theoretical Sciences (ICTS-TIFR),
Bengaluru, India

Abstract. The article gives an interpretation of the 4-dimensional generalised Seiberg-Witten equations in terms of almost-complex geometry on the underlying 4-manifold.

1. Introduction

The aim of this article is to study the relationship between a generalisation of the Seiberg-Witten equations and the almost-complex geometry of the underlying 4-manifold. Let $X$ be a 3 or a 4-dimensional, oriented, Riemannian manifold. A spinor bundle over $X$ is a vector bundle with typical fibre $\mathbb{H}$, the vector space of quaternions. In dimension 3, Taubes [1] observed that one can replace the spinor representation with a hyperKähler manifold $(M,g,I_1,I_2,I_3)$ admitting a certain action of the group $Sp(1) \cong Spin(3)$. Spinors can then replaced by sections of the associated bundle with a typical fibre $M$, called generalised spinors. The interplay of the $Sp(1)$-action with the quaternionic structure on $M$ allows one to define the Clifford multiplication. Composing Clifford multiplication with the covariant derivative defines the generalised Dirac operator. Additionally, using a twisting principal $G$-bundle one obtains a twisted Dirac operator for every connection on the $G$-bundle. This idea was extended to dimension 4 by Pidstrygach [2]. The second component of the Seiberg-Witten equations is the quadratic map. This is nothing but the hyperKähler moment map for the $U(1)$-action on $\mathbb{H}$. Therefore by replacing the quadratic map with a hyperKähler moment map for the $G$ action, one obtains the generalised Seiberg-Witten equations.

Many well-known gauge-theoretic equations occur as special cases of the generalised equations. For instance, choosing $G = PU(2)$ and $M = \mathbb{H} \otimes \mathbb{C}^2$, one...
obtains the \( PU(2) \)-monopole equations. The monopoles are conjectured to establish an equivalence between Donaldson invariants and Seiberg-Witten invariants \([3]\). The lesser known Vafa-Witten equations have recently gained attention for their connection with five-dimensional gauge theory, which is being used to study Khovanov homology and Fukaya-Seidel categories \([4, 5]\). The equations are obtained by choosing \( M = \mathbb{H} \otimes g \). In another recent development, \( Pin(2) \)-monopole equations were used by Manolescu \([6]\) to disprove the triangulation conjecture.

In this article, we will be concerned with the case when \( G = U(1) \). Broadly speaking, the article is divided into two parts. In the first part, we consider a class of target hyperKähler manifolds that are obtained via Swann’s construction \([7]\). Starting with a quaternionic Kähler manifold \( N \) of positive scalar curvature, Swann constructs a fibration over \( N \) whose total space carries a hyperKähler structure. The total space of the bundle is a Riemann cone over a 3-Sasakian and therefore admits a natural action of \( \mathbb{R}^+ \). In this setting, one may talk about “weighted spinors”.

Let \( X \) be a 4-dimensional Riemann manifold and \( \pi_1 : P_{CO(4)} \to X \) be the conformal bundle of frames for a fixed metric \( g_X \). Let \( \pi : Q \to X \) denote the reduction to the conformal \( Spin^c(4) \) group. Let \( M \) be the total space of a Swann bundle over a quaternionic Kähler manifold of positive scalar curvature.

\[ \textbf{Theorem 1.1.} \] Let \( f \) be a smooth, real-valued function on \( X \). Consider the metric \( g_X' := e^{2f} g_X \) in the conformal class \( [g_X] \). Denote by \( \varphi' \) and \( \varphi \) the Levi-Civita connections associated to \( g_X \) and \( g_X' \) respectively and let \( A_{\varphi} \) and \( A_{\varphi'} \) be the corresponding lifts to \( Q \). Then the generalised Dirac operators \( D_{A_{\varphi}} \) and \( D_{A_{\varphi'}} \) are related as:

\[
D_{A_{\varphi'}}(B u) = B \left( T e^{-b/2 \pi i f} D_{A_{\varphi}}(e^{b/2 \pi i f} u) \right)
\]

(1)

where, \( B \) is the lift of the automorphism \( B : P_{CO(4)} \to P_{CO(4)} \), given by

\[ p \mapsto e^{-f} p, \]

\( u \in C^\infty(Q, M)^{Spin(4)} \) is a spinor.

In the second part, we use Theorem 1.1 to show that away from a singular set, the generalised Seiberg-Witten equations can be interpreted in terms of almost-complex geometry of the underlying 4-manifold, as equations for a compatible almost-complex structure and a real-valued function which is related to a conformal factor. Recall that on a Riemannian 4-manifold \( (X, g_X) \), the compatible almost-complex structures on \( X \) are parametrized by sections of the twistor bundle \( \mathcal{Z} \), which is a sphere bundle in \( \Lambda^+ \). Thus the almost-complex structures can be thought of as self-dual, 2-forms \( \Omega \) with \( |\Omega| = 1 \). An almost-complex structure gives a splitting of \( \Lambda^+ \) into the direct sum of the trivial bundle spanned by \( \Omega \) and its orthogonal complement \( \overline{K} \), where \( K \) is a complex line bundle. Since \( |\Omega| = 1 \), its covariant derivative is a section of \( T^*X \otimes_R \overline{K} \). Using the almost-complex structure,
we get the isomorphism
\[ T^*X \otimes_{\mathbb{R}} K \cong T^*X \otimes_{\mathbb{C}} K \oplus T^*X \otimes_{\mathbb{C}} K. \]
Moreover, the wedge product gives a complex, bi-linear map
\[ T^*X \times T^*X \longrightarrow \Lambda^2 T^*X = K. \]
using which, we can identify \( TX \cong T^*X \otimes_{\mathbb{C}} K \). Thus \( \nabla \Omega \) has two components: the first component in \( T^*X \otimes_{\mathbb{C}} K \) is the Nijenhuis tensor and the second one in \( TX \) is \( d\Omega \). Let \( \langle \cdot, \cdot \rangle \) denote the obvious \( K \)-valued pairing between \( TX \) and \( T^*X \otimes K \).

**Theorem 1.2.** Fix a metric \( g_X \) on \( X \) and let \( [g_X] \) be its conformal class. Let \( M \) be the total space of a Swann bundle obtained as a quotient of a flat-hyperKähler manifold. Then, there exists a 1-1 correspondence between the following:

- pairs consisting of a metric \( g'_X \in [g_X] \) and a solution \((u, A)\) to the generalised Seiberg-Witten equations, such that the image of \( u \) does not contain a fixed point of the \( U(1) \) action on \( M \)
- pairs consisting of a metric \( g''_X \in [g_X] \) and a self-dual 2-form \( \Omega \) satisfying
  \[ (\nabla^* \nabla \Omega)^{\perp} + \langle d\Omega, N_{\Omega} \rangle = 0, \quad \frac{3}{2} |N_{\Omega}|^2 + \frac{1}{2} |d\Omega|^2 + s_X(g''_X) < 0 \]
  where \( s_X(g''_X) \) denotes the scalar curvature with respect to the metric \( g''_X \).

Theorem 1.2 is a generalization of the one obtained by Donaldson [11] for the Seiberg-Witten equations, which serves as the author’s motivation for this work.

The first equation in the second bullet of Theorem 1.2 is nothing but a perturbation of Euler-Lagrange equation of the energy functional
\[ \int_X |\nabla \Omega|^2. \]  
(2)
The functional was studied by Wood [8]. Critical points of the functional correspond to a choice of “optimal” almost-complex structures, amongst all possible almost-complex structures on \( X \).

Another criterion to single out the “best” almost-complex structures amongst all - suggested by Calabi and Gluck [9] - is to consider those sections \( \Omega \) whose image in the twistor space \( Z \) is of minimal volume. This raises an interesting question: *When does a critical point of (2) define a minimal isometric embedding of \( X \) in \( Z \)?* The question has addressed by studied by Davidov, Haq and Mushkarov in [10]. It would be interesting to ask if and when the solutions to generalised Seiberg-Witten equations define such a minimal isometric embedding.

2. Acknowledgement

The author wishes to thank Prof. Clifford Taubes for pointing out a crucial mistake in the earlier version of the article. A major part of this article is based on the author’s doctoral dissertation. The author is grateful to his supervisor Prof. V. Pidstrygach for his unwavering support and encouragement.
3. Preliminaries

3.1. HyperKähler manifolds. A 4n-dimensional Riemannian manifold \((M, g_M)\) is hyperKähler if it admits a triple of almost-complex structures \(I_i \in \text{End}(TM)\) \(i = 1, 2, 3\), which are covariantly constant with respect to the Levi-Civita connection and satisfy quaternionic relations \(I_i I_j = \delta_{ij} I_k\). The tangent space over each point of \(M\) has a quaternionic structure, and therefore the dimension of \(M\) is \(4n\), where \(n\) is an integer.

Let \(\text{Sp}(1)\) denote the group of unit quaternions and \(\mathfrak{sp}(1)\) its Lie algebra. It is often convenient to think of the complex structures as covariantly constant endomorphisms of \(TM\) with values in \(\mathfrak{sp}(1)^* = (\text{Im}(\mathbb{H}))^*\).

\[ I \in \Gamma(M, \text{End}(TM) \otimes \mathfrak{sp}(1)^*), \quad I_\xi := \xi_1 I_1 + \xi_2 I_2 + \xi_3 I_3, \quad \xi \in \mathfrak{sp}(1) \]  

Note that if \(\xi \in S^2 \subset \mathfrak{sp}(1)\), then \(I_\xi\) is again a complex structure. In other words, \(M\) has an entire family of Kähler structures parametrized by \(S^2\). The associated Kähler 2-forms can be thought of as a single \(\mathfrak{sp}(1)^*\)-valued 2-form, defined as

\[ \Omega \in \Lambda^2 M \otimes \mathfrak{sp}(1)^*, \quad \Omega_\xi(\cdot, \cdot) = g_M(I_\xi(\cdot), \cdot) \]

Definition 1. An isometric action of \(\text{Sp}(1)\) on \(M\) is said to be \textit{permuting} if the induced action on the 2-sphere of complex structures is the standard rotating action of \(SO(3) = \text{Sp}(1)/\pm 1\) on \(S^2\):

\[ dq I_\xi dq^{-1} = I_{q\xi\bar{q}}, \quad \text{for} \quad q \in \text{Sp}(1), \quad \xi \in \mathfrak{sp}(1), \quad ||\xi||^2 = 1 \]

Definition 2. An isometric action of a Lie group \(G\) on \(M\) is called \textit{tri-holomorphic}, if it preserves the hyperKähler structure

\[ \eta_\ast I_i = I_i \eta_\ast, \quad i = 1, 2, 3, \quad \eta \in \mathfrak{g} \]

where \(\mathfrak{g}\) is the Lie algebra of \(G\). In particular, \(G\) fixes the 2-sphere of complex structures on \(M\). Additionally, suppose that the action is \textit{tri-Hamiltonian}. This means that the action is Hamiltonian with respect to each \(\Omega_i\). Then the three moment maps can be combined together to define a single map \(\mu : M \rightarrow \mathfrak{sp}(1)^*\), which satisfies

\[ d(\mu, e_i \otimes \eta) = \iota_{K^M_\eta} \Omega_i, \quad \eta \in \mathfrak{g}, \quad \xi_i \in \mathfrak{sp}(1) \]

where \(K^M_\eta\) denotes the fundamental vector-field due to \(\eta\). We call \(\mu\) a \textit{hyperKähler moment map}.

Definition 3. A \textit{hyperKähler potential} is a smooth function \(f : M \rightarrow \mathbb{R}^+\) which is simultaneously a Kähler potential for all the three complex structures \(I_1, I_2, I_3\).

Let us consider a few examples:

Example 1. Let \(M = \mathbb{H}^n\) and consider the \(\text{Sp}(1)\)-action on \(\mathbb{H}^n\) by

\[ \text{Sp}(1) \times \mathbb{H}^n \ni (q, h) \mapsto q h \]
The action is a permuting action. On the other hand, consider an action of $U(1)$, given by

$$U(1) \times \mathbb{H}^n \ni (z, h) \mapsto zh$$

The action commutes with the $Sp(1)$-action and is tri-Hamiltonian, with a moment map

$$\mu(h) = \frac{1}{2} h \{h, \}.$$  

**Example 2 (Swann bundles [7]).** A quaternionic Kähler manifold is a 4n-dimensional manifold whose holonomy is contained in $Sp(n)Sp(1) := (Sp(n) \times Sp(1))/\pm 1$. Let $N$ be a quaternionic Kähler manifold of positive scalar curvature and $F$ be the $Sp(n)Sp(1)$ reduction of the frame bundle $P_{SO(4n)}$ of $N$. Then $S(N) := F/Sp(n)$ is a principal $SO(3)$-bundle, which is the frame bundle of the three-dimensional vector sub-bundle of skew symmetric endomorphisms of $TN$. The $Sp(1)$-action, by left multiplication, descends to an isometric action of $SO(3)$ on $\mathbb{H}^*/\mathbb{Z}_2$. Swann bundle over $N$ is the principal $\mathbb{H}^*/\mathbb{Z}_2$ bundle over $N$

$$U(N) := S(N) \times_{SO(3)} (\mathbb{H}^*/\mathbb{Z}_2) \longrightarrow N$$

**Theorem 3.1.** [7] The manifold $U(N)$ is a hyperKähler manifold with a free, permuting action of $SO(3)$ and admits a hyperKähler potential given by $\rho_0 = \frac{1}{2} r^2$. The vector field $X_0 = I_{\xi} K_\xi^M$ is independent of $\xi \in \mathfrak{sp}(1)$ and grad $\rho_0 = X_0$. Moreover, if a Lie group $G$ acts on $N$, preserving the quaternionic Kähler structure, then the action can be lifted to a tri-Hamiltonian action of $G$ on $U(N)$.

The Riemannian metric on the total space $U(N)$ is given by $g_{U(N)} = g_{\mathbb{H}^*/\mathbb{Z}_2} + r^2 g_N$ where $r$ is the radial co-ordinate on $\mathbb{H}^*/\mathbb{Z}_2$ and $g_{\mathbb{H}^*/\mathbb{Z}_2}$ is the quotient metric obtained from $\mathbb{H}$. Alternatively, one can write

$$U(N) = (0, \infty) \times S(N)$$

with metric $g_{U(N)} = dr^2 + r^2 (g_N + g_{\mathbb{R}P^3})$, where $g_{\mathbb{R}P^3}$ is the quotient metric on $\mathbb{R}P^3$ derived from its double cover $S^3$. Thus, $U(N)$ is a metric cone over $S(N)$. The manifold $U(N)$ is equipped with a natural left action of $\mathbb{H}^* \cong \mathbb{R}^+ \times Sp(1)$

$$((\lambda, q)(r, s)) \mapsto (\lambda \cdot r, q \cdot s).$$

(4)

3.2. **Target hyperKähler manifold.** Consider a permuting action of $Sp(1)$ on $M$ and let $G$ be a compact Lie group whose action on $M$ commutes with the $Sp(1)$-action and is tri-holomorphic. Let $\varepsilon \in G$ be a central element of order 2. Then $(-1, \varepsilon) \in Sp(1) \times G$ generates a normal subgroup of order 2, which we denote by $\pm 1$. Assume that $\pm 1$ acts trivially on $M$ so that the action of $Sp(1) \times G$ descends to an action of $Sp(1) \times_{\pm 1} G =: Spin^G(3)$. Such an action is said to be permuted action of $Spin^G(3)$.

An action of $Spin^G(4) := (Sp(1)_+ \times Sp(1)_-) \times_{\pm 1} G$ is said to be permuting if the action is induced by a permuting action of $Spin^G(3)$ via the homomorphism $\rho : Spin^G(4) \longrightarrow Spin^G(4)/Sp(1)_- \cong Spin^G(3)$. Note that in this case $Sp(1)_-$ acts trivially.
Keeping further exposition in mind, we will henceforth focus on the case when \( G = U(1) \), so that \( \text{Spin}^{G}(4) = \text{Spin}^{c}(4) \).

### 3.3. Generalised Dirac operator

Fix \( M \) to be a hyperKähler manifold with permuting action of \( \text{Spin}^{c}(4) \). Fix a \( \text{Spin}^{c} \)-structure \( \pi : Q \to X \) and denote by \( \pi_{SO} : Q \to P_{SO(4)} \) the projection to the frame bundle. The Levi-Civita connection \( \varphi \) on \( P_{SO(4)} \) and a connection \( A \) on the principal \( U(1) \)-bundle \( P_{U(1)} := Q/\text{Spin}(4) \to X \) together determine a unique \( \text{Spin}^{c} \)-connection on \( Q \). Let \( \mathcal{A} \) denote the space of all connections on \( Q \), which are the lifts of the Levi-Civita connection. We define the space of \textit{generalised spinors} to be the space of smooth, equivariant maps

\[
S := C^{\infty}(Q,M)^{\text{Spin}^{c}} \cong \Gamma(X,Q \times_{\text{Spin}^{c}} M).
\]

Covariant derivative of a spinor \( u \in S \), with respect to \( A \in \mathcal{A} \) is defined to be

\[
D_{A} : C^{\infty}(Q,M)^{\text{Spin}^{c}} \to \text{Hom}(TQ,TM)_{\text{hor}}, \quad D_{A}u = du + K_{A}^{M}|_{u}
\]

where \( K_{A}^{M}|_{u} : TQ \to u^{*}TM \) is a vector bundle homomorphism \( K_{A}^{M}|_{u}(v) = K_{A(v)}^{M}|_{u(p)} \) for \( v \in T_{p}Q \). Alternatively, one can view the covariant derivative as

\[
D_{A} : C^{\infty}(Q,M)^{\text{Spin}^{c}} \to C^{\infty}(Q,(\mathbb{R}^{4})^{*} \otimes TM)^{\text{Spin}^{c}} \quad (D_{A}u(q),w) = du(q)(\tilde{w})
\]

where, \( w \in \mathbb{R}^{4} \), \( \tilde{w} \) denotes the horizontal lift of \( \pi_{SO}(q)(w) \in T_{\pi(q)}X \).

**Clifford multiplication.** The second ingredient we need to define the Dirac operator is Clifford multiplication. From (3), we can construct an action of \( \mathcal{C}_{4}^{l} \cong \mathcal{C}_{3} \) on \( TM \) by

\[
\mathbb{R}^{3} \cong \mathfrak{m}(\mathbb{H}) \to \text{End}(TM), \quad h \mapsto I_{h}
\]

The map extends to a \( \text{Spin}^{c} \)-equivariant map \( \mathcal{C}_{3} \to \text{End}(TM) \). Thus \( TM \) is naturally a \( \mathcal{C}_{4}^{l} \) module. Now consider \( TM := \mathcal{C}_{4} \otimes \mathcal{C}_{4}^{l} E \), where \( E = (TM,I_{1}) \).

Since \( TM \) is a \( \mathcal{C}_{4}^{l} \)-module, we get a \( \mathbb{Z}_{2} \)-graded \( \mathcal{C}_{4} \)-module

\[
\widetilde{TM} = W^{+} \oplus W^{-}, \quad W^{+} = \mathcal{C}_{4}^{1} \otimes \mathcal{C}_{4}^{l} E, \quad W^{-} = \mathcal{C}_{4}^{1} \otimes \mathcal{C}_{4}^{l} E.
\]

More precisely, \( W^{+} \) is the \( \text{Spin}^{c} \)-equivariant bundle \( TM \) with an action induced by \( \rho \), whereas \( W^{-} \) is the \( \text{Spin}^{c} \)-equivariant vector bundle \( TM \) equipped the left-action:

\[
[q_{+}, q_{-}, g] \cdot w_{-} = I_{q_{-}}I_{q_{+}}dg_{+}dg w_{-}.
\]

We identify \( \mathbb{R}^{4} \) with \( \mathbb{H} \) by mapping the standard, oriented basis \( (e_{1}, e_{2}, e_{3}, e_{4}) \) of \( \mathbb{R}^{4} \), to \((1, i, j, k)\). The \( \text{Spin}^{c} \)-action on \( \mathbb{H} \) is given by \([q_{+}, q_{-}, g] \cdot h = q_{-}hq_{+}^{*}\). Clifford multiplication is the \( \text{Spin}^{c}(4) \)-equivariant map

\[
\bullet : (\mathbb{R}^{4})^{*} \cong \mathbb{H} \to \text{End}(W^{+} \oplus W^{-}), \quad g_{\mathbb{R}^{4}}(h) \mapsto \begin{bmatrix} 0 & -I_{h} \\ I_{h} & 0 \end{bmatrix}
\]

\[\text{The subscript hor implies that } D_{A}u \text{ vanishes on vertical vector fields.}\]
Since \( h \cdot h = -g_{\mathbb{R}^4}(h, h) \cdot id_{W^+ \oplus W^-} \), by universality property, the map \( \cdot \) extends to a map of algebras \( \cdot : \mathcal{C}l_4 \to \text{End}(W^+ \oplus W^-) \). Composing \( \cdot \) with the covariant derivative, we get the \textit{generalised Dirac operator}:

\[
\mathcal{D}_A u \in C^\infty(Q, u^* W^-)^{Spin^c}, \quad \mathcal{D}_A u = \sum_{i=0}^3 e_i \cdot \mathcal{D}_A u(\tilde{e}_i)
\]

where the latter expression follows from equation (6).

\textbf{Generalised Seiberg-Witten equations.} Henceforth, unless mentioned otherwise, we assume that the permuting action of \( Spin^c(4) \) on \( M \) is such that the action of \( U(1) \)-action is tri-Hamiltonian. Let \( \mu \) denote the hyperKähler moment map for the \( U(1) \)-action. The \textit{generalised Seiberg-Witten equations} for a pair \((u, A) \in \mathcal{S} \times \mathcal{A}\), in dimension 4, are

\[
\begin{cases}
\mathcal{D}_A u = 0 \\
F_A^+ - \mu \circ u = 0
\end{cases}
\]

where \( F_A \in \text{Map}(Q, \Lambda^2(\mathbb{R}^4)^*) \) denotes the curvature of \( A \).

4. \textbf{Conformal transformation of generalised Dirac operator}

In this section we fix \( M = U(N) \) for some quaternionic Kähler manifold \( N \) of positive scalar curvature. We show that under the conformal change of metric on \( X \), the space of harmonic, generalised spinors remains invariant. For more details on ideas used in this section, we refer the interested reader to [12].

Let \( X \) be a 4-manifold and fix a metric \( g_X \) on \( X \). Let \([g_X]\) denote its conformal class. We denote by \( \pi_1 : P_{\mathcal{C}l(4)} \to X \) the bundle of all conformal frames on \((X, [g_X])\). This is a reduction of the frame bundle of \( X \) to the conformal group. Let \( \theta : P_{\mathcal{C}l(4)} \to \mathbb{R}^4 \) denote the canonical one-form

\[
\theta_p(v) = p^{-1}((\pi_1)_*(v)), \quad p \in P_{\mathcal{C}l(4)}, \quad v \in T_p P_{\mathcal{C}l(4)}.
\]

A metric on \( X \) is a section \( g_X \in \Gamma(X, S^2(T^* X)) \), which can viewed as an equivariant map in \( C^\infty(P_{\mathcal{C}l(4)}, S^2(\mathbb{R}^4)^*)^{\mathcal{C}l(4)} \)

\[
\pi_1^* g_X(\cdot, \cdot) = g_{\mathbb{R}^4}(\theta_p(\cdot), \theta_p(\cdot)).
\]

For a smooth, real-valued function \( f \) on \( X \), consider the metric \( g_X' = e^{2(\pi_1 f)} g_X \) in the conformal class of \( g_X \). The metrics \( g_X \) and \( g_X' \) determine two isomorphic \( SO(4) \) bundles:

\[
P_{SO(4)} = \{ p \in P_{\mathcal{C}l(4)} \mid g_{\mathbb{R}^4}(\theta_p, \theta_p) = \pi_1^* g_X(\cdot, \cdot) \}
\]

\[
P_{SO(4)}' = \{ p \in P_{\mathcal{C}l(4)} \mid g_{\mathbb{R}^4}(\theta_p, \theta_p) = e^{2(\pi_1 f)} \pi_1^* g_X(\cdot, \cdot) \}
\]

where, \( g_{\mathbb{R}^4}(\cdot, \cdot) \) is the standard metric on \( \mathbb{R}^4 \). Let \( \varphi \) be a connection on \( P_{\mathcal{C}l(4)} \). Then \( \varphi + \theta \) define a 1-form with values in \( \mathfrak{so}(4) \oplus \mathbb{R}^4 \). We can extend the bracket on the Lie algebra \( \mathfrak{so}(4) \) to \( \mathfrak{so}(4) \oplus \mathbb{R}^4 \) as

\[
[A, x] = -[x, A] = Ax, \quad [x, y] = 0, \quad \text{for } x, y \in \mathbb{R}^4 \text{ and } A \in \mathfrak{so}(4).
\]
Therefore, we need to find a one form $\beta$ what needs to be added to $\phi$.

Note that $\pi \phi$ where $b \zeta$ from is measured by

$$\text{Lie}(\text{Aut}(\zeta))$$

is given by $\zeta \cdot g = -g(\zeta, \cdot) - g(\cdot, \zeta)$. That is the reason we have a negative sign in the second line. If $\zeta \in \mathbb{R} = \text{Lie}(\mathbb{R}^+) \subset \text{Lie}(\text{Aut}(\mathbb{R}^4))$, then $\zeta \cdot g = -2g(\zeta, \cdot)$.

Therefore $(d + \phi + \pi \phi_1)(\epsilon^{2(\pi \phi_1)} g x) = 0$. But the torsion of the connection $\phi + \pi \phi_1 df$ is non-zero, since

$$(d + \phi + \pi \phi_1 df)\theta = \pi \phi_1 df \cdot \theta.$$  

We now have a $g_\phi$-compatible connection on $X$. The difference between $\phi'$ and $\phi + \pi \phi_1 df$ is called \textit{contorsion form}. In other words, a contorsion form is exactly what needs to be added to $\phi + \pi \phi_1 df$ to make it torsion-free and thus equal to $\phi'$.

Therefore, we need to find a one form $\beta \in \Omega^1(P_{\text{SO}(4)}, (\mathbb{R}^4)^* \otimes \mathfrak{so}(4))^{\text{SO}(4)}$ so that

$$(d + \phi + \pi \phi_1 df + \beta)\theta = 0.$$  

Note that $\pi \phi_1 df$ is an element in $(\mathbb{R}^4)^* \subset (\mathbb{R}^4)^* \otimes \mathfrak{so}(4)$. 

This defines an affine Lie algebra which is best identified with the frame bundle of $\mathbb{R}^4$. The failure of the 1-form $\phi + \theta$ to conform with the associated Maurer-Cartan form is measured by

$$d(\phi + \theta) + [\phi + \theta, \phi + \theta] = \mathcal{R}(\phi) + T(\phi)$$

where

$$\mathcal{R}(\phi) = d\phi + \frac{1}{2}[\phi, \phi], \quad T(\phi) = d\theta + [\phi, \theta].$$

and the Lie brackets are carried out simultaneously with wedging of 1-forms. These are horizontal-valued 2-forms on the conformal frame bundle, which are nothing but the curvature and the torsion tensors, respectively.

Suppose that $\phi$ is a connection on $P_{\text{CO}(4)}$ such that:

$$(d + \phi)g_x = 0, \quad (d + \phi)\theta = 0. \quad (10)$$

Then $\phi$ is just the Levi-Civita connection for the metric $g_x$. Let $\phi'$ denote the Levi-Civita connection for the metric $g'_x$. The difference of the 2-connections is a horizontal 1-form on $P_{\text{CO}(4)}$ and therefore can be written as contraction of $\theta$ with an equivariant function $\xi \in \text{Hom}(\mathbb{R}^4, \mathfrak{co}(4)) \cong (\mathbb{R}^4)^* \otimes \mathfrak{co}(4)$. More precisely,

$$(\theta_p, \xi)(Y) = (\theta_p(Y), \xi), \quad Y \in T_p P_{\text{CO}(4)}$$

which is a horizontal 1-form on $P_{\text{CO}(4)}$. Therefore we may write

$$\phi' - \phi = (\theta, \xi) \quad \text{for some } \xi \in (\mathbb{R}^4)^* \otimes \mathfrak{co}(4). \quad (11)$$

Throughout, we will suppress the pairing with $\theta$ and simply write $\phi' - \phi = \xi$. The covariant derivative of the metric $g'_x$ with respect to $\phi$ is

$$(d + \phi)(g'_x) = -e^{2(\pi \phi_1)} 2(\pi \phi_1 df) g_x.$$  

**Remark 1.** The left action of $\text{Aut}(\mathbb{R}^4) \subset S^2(\mathbb{R}^4)^*$ is given by

$$S^2(\mathbb{R}^4)^* \ni g_x \mapsto b \cdot g_x(\cdot, \cdot) := g_x(b^{-1}, b^{-1}),$$

where $b \in \text{Aut}(\mathbb{R}^4)$. Therefore the action of an element $\zeta$ in the Lie algebra $	ext{Lie}(\text{Aut}(\mathbb{R}^4))$ is given by $\zeta \cdot g = -g(\zeta, \cdot) - g(\cdot, \zeta)$. That is the reason we have a negative sign in the second line. If $\zeta \in \mathbb{R} = \text{Lie}(\mathbb{R}^+) \subset \text{Lie}(\text{Aut}(\mathbb{R}^4))$, then $\zeta \cdot g = -2g(\zeta, \cdot)$. 

Therefore $(d + \phi + \pi \phi_1 df)(e^{2(\pi \phi_1)} g x) = 0$. But the torsion of the connection $\phi + \pi \phi_1 df$ is non-zero, since

$$(d + \phi + \pi \phi_1 df)\theta = \pi \phi_1 df \cdot \theta.$$  

We now have a $g_\phi$-compatible connection on $X$. The difference between $\phi'$ and $\phi + \pi \phi_1 df$ is called \textit{contorsion form}. In other words, a contorsion form is exactly what needs to be added to $\phi + \pi \phi_1 df$ to make it torsion-free and thus equal to $\phi'$. Therefore, we need to find a one form $\beta \in \Omega^1(P_{\text{SO}(4)}, (\mathbb{R}^4)^* \otimes \mathfrak{so}(4))^{\text{SO}(4)}$ so that

$$(d + \phi + \pi \phi_1 df + \beta)\theta = 0.$$  

Note that $\pi \phi_1 df$ is an element in $(\mathbb{R}^4)^* \subset (\mathbb{R}^4)^* \otimes \mathfrak{so}(4)$.
For a given connection on \( P_{CO(4)} \), let \( D_p \) denote the horizontal subspace at a point \( p \in P_{CO(4)} \), with respect to the connection. Then, the torsion tensor is a map from \( \Lambda^2 D_p \cong \Lambda^2 \mathbb{R}^4 \overset{\delta}{\to} \mathbb{R}^4 \). Therefore, for any two connections \( \alpha \) and \( \beta \), the difference between the torsions, at a point \( p \), is given by:

\[
T(\alpha)_p(x \wedge y) - T(\beta)_p(x \wedge y) = \frac{1}{2}(\xi_p(x)y - \xi_p(y)x), \quad x, y \in \mathbb{R}^4,
\]

where \( \xi = \alpha - \beta \). In terms of the \( CO(4) \)-equivariant homomorphism:

\[
\delta : (\mathbb{R}^4)^* \otimes \mathfrak{so}(4) \mapsto (\mathbb{R}^4)^* \otimes (\mathbb{R}^4)^* \otimes \mathbb{R}^4 \mapsto \Lambda^2(\mathbb{R}^4)^* \otimes \mathbb{R}^4 \cong \Lambda^2(\mathbb{R}^4)^* \otimes (\mathbb{R}^4)^*
\]

where, the first map is the inclusion and the second one is the anti-symmetrization, we may write

\[
T(\alpha) - T(\beta) = -\delta \xi.
\]

For simplicity, we shall use the abbreviation \( \Lambda^k \) for the space \( \Lambda^k(\mathbb{R}^4)^* \).

Recall that there is a natural isomorphism \( \mathfrak{so}(n) \cong \Lambda^2 \) obtained by associating the skew-symmetric endomorphism, to a pair of vectors \( v, w \in \mathbb{R}^n \),

\[
v \wedge w = \langle v, w \rangle - \langle w, v \rangle v.
\]

(12)

Now \( \delta|_{\mathfrak{so}(4)} : \Lambda^1 \otimes \Lambda^2 \mapsto \Lambda^2 \otimes \Lambda^1 \) maps the difference of two connections to the difference of their torsions and is an isomorphism. Indeed, this can be seen as follows: let \( a_{ijk} \in \Lambda^1 \otimes \Lambda^2 \) denote the difference of Christoffel symbols of the two connections. Then, \( \delta(a_{ijk}) = \frac{1}{2}(a_{ijk} - a_{jik}) \). It is easily seen that if \( a_{ijk} \in \ker(\delta) \), then \( a_{ijk} = 0 \) and hence \( \delta|_{\mathfrak{so}(4)} \) is an isomorphism. If one of the connections is torsion-free, then the inverse of \( \delta|_{\mathfrak{so}(4)} \) gives the contorsion form.

Recall that a point \( p \in P_{CO(4)} \) can be viewed as a linear isomorphism

\[
p : \mathbb{R}^4 \overset{\cong}{\to} T_{\pi(p)} X
\]

which is equivariant under the right action of \( CO(4) \). Using this, define

\[
f_i(p) = \pi_1^* df(p(\widetilde{e}_i)),
\]

where, \( \widetilde{e}_i \in \mathbb{R}^4 \) is the standard basis element of \( \mathbb{R}^4 \) and \( p(\widetilde{e}_i) \) is the horizontal lift of \( p(e_i) \) with respect to \( \varphi \). The 1-form \( \pi_1^* df \) on \( P_{CO(4)} \) can be written as \( \sum_i f_i e^i \).

We can also view this as a 1-form with values in \( \mathfrak{so}(4) \), by writing

\[
\pi_1^* df = \sum_{i,j} f_i e^i \otimes e^j \otimes e_j \in (\mathbb{R}^4)^* \otimes (\mathbb{R}^4)^* \otimes \mathbb{R}^4.
\]

Using the isomorphism \( \mathbb{R}^4 \cong (\mathbb{R}^4)^* \), we can write \( \pi_1^* df = \sum_{i,j} f_i e^i \otimes e^j \otimes e^j \).

Therefore,

\[
\delta|_{\mathfrak{so}(4)}^{-1}(\delta(\pi_1^* df)) = -\sum_{i,j} f_i e^i \otimes (e^i \wedge e^j).
\]

This is indeed the contorsion form, for

\[
T \left( \varphi + \pi_1^* df - \delta|_{\mathfrak{so}(4)}^{-1}(\delta(\pi_1^* df)) - \varphi' \right) = -\delta(\pi_1^* df) - \delta(-\pi_1^* df) = 0.
\]
We can now write the Levi-Civita connection $\varphi'$ explicitly as
\[ \varphi' = \varphi + \pi^* f \, df + \sum_{i,j} f_i e^j \otimes (e^i \wedge e^j). \]

For simplicity, let $\alpha = \pi^* f \, df + \sum_{i,j} f_i e^j \otimes (e^i \wedge e^j)$.

**Proposition 4.1. ([13] Prop. 6.2, Chap. I)** The adjoint representation induces the Lie algebra isomorphism $\zeta : \text{spin}(n) \to \mathfrak{so}(n)$ is given by:
\[ \zeta(e_i e_j) = 2e_i \wedge e_j, \]
where, $\{e_i e_j\}_{i<j}$ are the basis elements of $\text{spin}(n)$. Consequently for $v, w \in \mathbb{R}^n$,
\[ \zeta^{-1}(v \wedge w) = \frac{1}{2}[v, w]. \]

Therefore for $e_i \wedge e_j \in \Lambda^2 \mathbb{R}^n$, $\zeta^{-1}(e_i \wedge e_j) = \frac{1}{2}(e_i e_j - e_j e_i)$. Under this isomorphism coupled with the identification $\mathbb{R}^4 \cong (\mathbb{R}^4)^*$,
\[ \pi^* f \, df + \sum_{i,j} f_i e^j \otimes (e^i \wedge e^j) \mapsto \pi^* f \, df + \frac{1}{4} \sum_{i,j} f_i e^j \otimes (e^i e^j - e^j e^i), \]
and denoted again by $\alpha$. Recall from example 2 that for a quaternionic Kähler manifold $N$ of strictly positive scalar curvature, the Swann bundle over $N$ can be written as a Riemann cone over a 3-Sasakian and therefore carries a natural action of $\mathbb{R}^+$.

Let $u \in C^\infty(Q, M)^{C\text{Spin}^c}$ be a spinor and $A$ be a fixed connection on the principal $U(1)$-bundle. Denote by $A_\varphi$ and $A_{\varphi'}$, the respective lifts of the Levi-Civita connections $\varphi$ and $\varphi'$ to $Q$. Then, from (4)
\[ D_{A_{\varphi'}} u = D_{A_\varphi} u + K^M_{\alpha} \mid_u \in C^\infty(Q, (\mathbb{R}^4)^* \otimes u^* TM)^{C\text{Spin}^c}. \tag{13} \]

Recall that $U(N)$ admits a hyperkähler potential $\rho_0$ and $\lambda_0 = \text{grad} \rho_0$. For $\lambda \in \mathbb{R} \setminus \{0\}$,
\[ \rho_0(e^{\lambda} x) = \frac{1}{2} g^M(X_0 \mid_{e^\lambda x}, X_0 \mid_{e^\lambda x}) = \frac{1}{2} e^{2\lambda} g^M(X_0 \mid_x, X_0 \mid_x) = e^{2\lambda} \rho_0(x). \]

So
\[ \frac{d}{dt} \rho_0(e^{2t\lambda} x)_{t=0} = d\rho_0(\frac{d}{dt}(e^{2t\lambda} x)) = 2d\rho_0(K^M_{\lambda}^{M,R^+}) \mid_x = g^M(X_0 \mid_x, K^M_{\lambda}^{M,R^+} \mid_x). \]

But
\[ \frac{d}{dt} \rho_0(e^{2t\lambda} x)_{t=0} = \frac{d}{dt}(e^{2t\lambda}) \rho_0(x) = 2\lambda \rho_0(x) = g_M(X_0 \mid_x, X_0 \mid_x). \]

This gives
\[ \lambda g_M(X_0 \mid_x, X_0 \mid_x) = \lambda g_M(X_0 \mid_x, K^M_{\lambda}^{M,R^+} \mid_x) \]
for every $x \in M$, which, in turn implies $K^M_{\lambda}^{M,R^+} = \lambda X_0$.

We are now in a position to give the proof of Theorem 1.1. First, we need the following Lemma:
Lemma 4.2. For $f \in C^\infty(X, \mathbb{R})$, we have

$$D_A(e^{-\pi f}u) = Te^{-\pi f} D_A u - \pi_i df \cdot \mathcal{X}_0 \circ u.$$  \hfill (14)

**Proof.** Let $p \in Q$ and $v \in T_p Q$. Let $\gamma : [0, 1] \rightarrow Q$ be a curve in $Q$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Evaluating the covariant derivative of $e^{-\pi f}u$ for $v$:

$$D_A(e^{-\pi f}u)(v) = T(e^{-\pi f}u)(v) + K^M_M(e^{-\pi f(p)}u(p))$$

The first term is

$$T(e^{-\pi f}u)(v) = \frac{d}{dt}(e^{-\pi f(t)}u(\gamma(t)))|_{t=0}$$

$$= \frac{d}{dt}(e^{-\pi f(t)}u(\gamma(t)))|_{t=0}$$

$$= Te^{-\pi f(p)}T u(v) + K^M_M(e^{-\pi f(p)})u(p)$$

and the second term is

$$K^M_M(e^{-\pi f(p)}u(p)) = Te^{-\pi f(p)} K^M_M|u(p)$$

In conclusion,

$$D_A(e^{-\pi f}u) = Te^{-\pi f} D_A u - d(\pi_i f) \otimes \mathcal{X}_0 \circ u$$

Applying Clifford multiplication, proves the Lemma. \hfill \Box

**Proof of Theorem 1.1.** Let $\bullet$ denote the Clifford multiplication w.r.t the metric $g_X$. Then w.r.t the metric $e^{2\pi f}g_X$, the Clifford multiplication is given by $\bullet' = Te^{-\pi f}\bullet$. Substituting for $a$ in (13) and applying the Clifford multiplication we get:

$$D_{A_{\bullet'}} u = Te^{-\pi f}(D_{A_{\bullet'}} u + \pi_i df \bullet \mathcal{X}_0 \circ u + \frac{1}{4} \sum_{i<j} f_i e^j \cdot K^M_M(e_i e^j - e^j e^i)|u)$$

Note that using the identification $(\mathbb{R}^4)^* \cong \mathbb{H}$, the element $(e_i e^j - e^j e^i)$ belongs to the Lie algebra $\mathfrak{sp}(1) \cong \mathfrak{im}(\mathbb{H})$ and has norm 1. Now recall from example 2 the vector field $\mathcal{X}_0 = -I_{\xi} K^M_M$ is independent of $\xi \in \mathfrak{sp}(1)$. In particular for $|\xi| = 1$, we have $I_{\xi} \mathcal{X}_0 = K^M_M$. Therefore,

$$K^M_M(e_i e^j - e^j e^i)|u = I(e_i e^j - e^j e^i)\mathcal{X}_0 \circ u = (e_i e^j - e^j e^i) \bullet \mathcal{X}_0 \circ u.$$ 

Substituting

$$D_{A_{\bullet'}} u = Te^{-\pi f}(D_{A_{\bullet'}} u + \pi_i df \bullet \mathcal{X}_0 \circ u + \frac{1}{4} \sum_{i<j} f_i e^j \bullet (e^i e^j - e^j e^i) \bullet \mathcal{X}_0 \circ u)$$

$$= Te^{-\pi f}(D_{A_{\bullet'}} u + \pi_i df \bullet \mathcal{X}_0 \circ u + \frac{1}{4}(4 \sum_i f_i e^i - 2 \sum_{i,j} f_i e^j \delta_{i,j})$$

$$+ 4 \sum_i f_i e^i) \bullet \mathcal{X}_0 \circ u)$$
\[ = T e^{-\pi i f} \left( DA \varphi u + \pi^*_1 df \bullet X_0 \circ u + \frac{3}{2} \pi^*_1 df \bullet X_0 \circ u \right). \]

Now observe that
\[
DA (e^{-\pi i f} u) = Te^{-\pi i f} \left( Te^{-\pi i f} DA \varphi u + Te^{-\pi i f} \frac{3}{2} \pi^*_1 df \bullet X_0 \circ u \right) = Te^{-\pi i f} \left( Te^{-\frac{3}{2} \pi i f} DA \varphi \left( e^{\frac{3}{2} \pi i f} u \right) \right).
\]

Thus, we conclude
\[
DA (\mathcal{D} u) = \mathcal{D} \left( T e^{-\frac{3}{2} \pi i f} DA \varphi \left( e^{\frac{3}{2} \pi i f} u \right) \right).
\]

(15)

\[
\square
\]

5. Almost Hermitian geometry and generalised Seiberg-Witten

In this section, we will restrict to those Swann bundles, whose hyperKähler structure can be obtained via hyperKähler reduction of a flat hyperKähler manifold.

Many interesting examples of hyperKähler manifolds in literature are obtained by starting with a flat hyperKähler manifold and then taking the quotient by a linear action of a group. The list includes moduli space of Bogomolny monopoles, co-adjoint orbits of semi-simple Lie groups, moduli space of framed instantons on \( S^4 \), moduli space of framed \( SU(r) \)-instantons on \( \mathbb{R}^4 \) of charge \( k \), etc.

Let \( V \) be a Hermitian vector space and \( H := V \oplus V^* \). Then \( H \) is a flat-hyperKähler manifold. Define a left action of \( U(1) \) on \( H \) by
\[
z \cdot (v, w) = (z \cdot v, z \cdot w)
\]

The action is hyper-Hamiltonian and the real and complex moment maps for the action are given by
\[
\mu_R (v, w) = \frac{1}{2} (\|v\|^2 - \|w\|^2), \quad \mu_C (v, w) = \langle v, w \rangle
\]

Suppose that \( H \) is acted upon by another Lie group \( G \) and the action is hyper-Hamiltonian. Assume also that the action commutes with the \( U(1) \)-action. If zero is a regular value of the \( G \)-moment map \( \mu_G : H \longrightarrow \mathfrak{sp}(1)^* \otimes \mathfrak{g}^* \). Then, \( U(1) \) preserves the zero set and descends to a hyper-Hamiltonian action on the quotient \( M := \mu_G^{-1}(0)/G \).

**Example 1.** Take the flat space \( \mathbb{H}^n = \mathbb{C}^n \oplus j \mathbb{C}^n \). With respect to the complex structure \( i \) and \( V = \mathbb{C}^n \), this can be written as \( V \oplus V^* \). Let \( G \subset Sp(n) \) be a sub-group of \( Sp(n) \). Then the action of \( G \) is hyper-Hamiltonian with moment map \( \mu_g : \mathbb{H}^n \longrightarrow \mathfrak{sp}(1)^* \otimes \mathfrak{g}^* \). We take this moment map to be \( \langle \mu_g (q), \eta \rangle = \frac{1}{2} q \cdot \eta \). Then the zero set of the moment map is invariant under the action of \( \mathbb{H}^* \) on \( \mathbb{H}^n \), given by right multiplication by the conjugate. Therefore the quotient \( \mu_g^{-1}(0)/G \) is a Swann bundle over a quaternionic Kähler manifold of positive curvature, obtained via quaternionic Kähler reduction of \( \mathbb{H}P^n \) by \( G \). On the other hand, given a compact, semi-simple Lie group \( G \) such that all the simple factors of the Lie algebra are simple, there exists a number \( m \) and a sub-group \( H \subset Sp(m) \) such that the
orbit of nil-potent elements of $g^C$ under the adjoint action of $G^C$, can be obtained as a quotient of $\mathbb{H}^m$ by $H$ [15].

**Example 2.** Let $M_{n,r}(\mathbb{K})$ denote a matrix with $n$ rows and $r$ columns from $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Let

$$H = M_{n,n}(\mathbb{H}) \oplus M_{n,r}(\mathbb{H}) \cong M_{n,n} (\mathbb{C}) \oplus M_{n,n} (\mathbb{C}) \oplus M_{r,n} (\mathbb{C}).$$

Let $V = M_{n,n}(\mathbb{C}) \oplus M_{r,n}(\mathbb{C})$. Then $H = V \oplus V^*$. Consider the action of $U(n)$ on $H$

$$g \cdot (b_1, b_2, c, d) \mapsto (g^{-1} b_1 g, g^{-1} b_2 g, g a, b g^{-1}).$$

The action commutes with the $U(1)$ action

$$z \cdot (b_1, b_2, c, d) \mapsto (z b_1, z^{-1} b_2, z g, z^{-1} b).$$

Let $\mu_{u(n)}$ denote the $U(n)$-moment map and let $\mathcal{M}_0 (r,n) := \mu_{u(n)}^{-1}(0)/U(n)$. Then $\mathcal{M}_0 (r,n)$ carries a hyperKähler structure, outside of a singular locus [16]. Donaldson and Kronheimer [17] proved that there is a bijective correspondence between $\mathcal{M}_0 (r,n)$ and the moduli space of framed ideal instantons on $\mathbb{R}^4$. Maciocia [18] gives an explicit construction of hyperKähler potential on the moduli space of framed $SU(r)$-instantons of charge $k$ over $\mathbb{R}^4$, which we again denote by $\mathcal{M}_0 (k,r)$. Therefore, $\mathcal{M}_0 (k,r)$ can be obtained via Swann's construction from a quaternionic Kähler manifold of positive scalar curvature [7].

5.1. **Modified Seiberg-Witten equations.** We restrict the generalised Seiberg-Witten equations to the case where the target hyperKähler manifold $M$ is obtained via hyperKähler reduction of $H$. To begin with, note that $H$ carries a permuting action of $Sp(1)$. Indeed, writing $(v,w) \in H$ as $v + \pi^4 j$ the $Sp(1)$-action is given by multiplication by conjugate on the right. This action is permuting left-action. Suppose that $H$ carries a tri-Hamiltonian action of $G \times U(1)$, that commutes with the $Sp(1)$-action. Define the group

$$\hat{G} := Spin(4) \times \pm 1 \ (U(1) \times G)$$

where $\pm 1$ is the order 2-subgroup of $Spin(4)$ ($U(1) \times G$) generated by $\{-1, -1, \epsilon\}$, where $\epsilon \in G$ is an element of order two in the centralizer of $G$ and $\epsilon \neq 1$.

Assume that $\hat{G}$ action on $H$ is permuting and $\mu_{\hat{g}}^{-1}(0)/G = M$. Since the actions commute, $\mu_{\hat{g}}^{-1}(0)$ is preserved by $Spin^c(4)$.

For simplicity, denote by $P := \mu_{\hat{g}}^{-1}(0)$ the $Spin^c$-equivariant principal $G$-bundle over $M$. Consider the following diagram

$$
\begin{array}{ccc}
\mathbb{H}^n & \xrightarrow{\pi_1} & Q \\
\pi_2 & & \pi \\
\downarrow & & \downarrow \\
\pi & & M \\
\downarrow & & \downarrow \\
X & & M
\end{array}
$$

Given a smooth, equivariant map $\hat{u} : P_{\hat{G}} \to H$, such that $\mu_{\hat{g}} \circ \hat{u} = 0$, define $u : Q \to M$ by $u(q) = \pi_2 (\hat{u}(p))$, $q \in Q$, $p \in \pi_1^{-1}(q)$. Clearly then, diagram commutes. On the other hand, given a smooth spinor $u : Q \to M$, it defines a principal $G$-bundle over $Q$, via pull-back of $P$ and canonically defines $\hat{u}$, making the diagram commutative.

In summary,
Lemma 5.1. There is a bijective correspondence between
\[ \{ u \in C^\infty(Q,M)^{\text{Spin}^c} \} \leftrightarrow \{ \tilde{u} \in C^\infty(P_{\tilde{G}},H_{\tilde{G}}) \mid \mu^e \circ \tilde{u} = 0 \} \]

Fix a connection \( A \) on \( Q \). This is uniquely determined by the Levi-Civita connection on \( X \) and a connection \( b \) on \( P_{U(1)} \).

The principal bundle \( P \rightarrow M \) is a Riemannian submersion and therefore carries a canonical connection \( a \), defined by: \( K^P,G|_P(v) = -\text{proj}^{\text{im}K^P,G}(v) \), \( v \in T_P P \). This is just the projection to the vertical sub-bundle. The pull-back of this connection by \( \tilde{u} \), along with the connection \( A \) on \( Q \), uniquely determine a connection \( A \) on \( P_{\tilde{G}} \) (see [2])
\[ A = \pi^*A \oplus A_\theta \in \Lambda^1(P_{\tilde{G}}, \tilde{g})^G \] (18)
where \( A_\theta = \tilde{u}^*a - (\pi^*A,_{\text{Spin}^c}\tilde{u}^*a) \).

We can define a Dirac operator acting on maps \( \tilde{u} \), twisted by the connection \( A \).

Proposition 5.2. Then, there is a 1-1 correspondence between
\[ \{ (\tilde{u}, A) \mid \mathcal{D}_A \tilde{u} = 0, \mu^e \circ \tilde{u} = 0 \} \quad \text{and} \quad \{ (u, A) \mid \mathcal{D}_A u = 0 \} \] (19)
Whenever \( \mathcal{D}_A \tilde{u} = 0, \mu^e \circ \tilde{u} = 0, \) and \( \text{proj}_b A = A_\theta \) as in (18) and therefore, \( A \) is uniquely determined by a \( U(1) \)-connection \( b \).

Proof. For \( h \in P \) such that \( \mu^e(h) = 0 \), define \( \mathcal{H}_h := \ker d\mu^e(h) \cap (\text{im} K^P,G) \perp. \) This is just the horizontal subspace over \( h \) with respect to the canonical connection \( a \) on \( P \).

We will prove the proposition in two steps. In what follows, we shall denote the \( G \) and \( \text{Spin}^c \)-components of \( A \) by \( A_\theta \) and \( A \) respectively.

Step 1: In the first step we will prove that \( I_\xi D_A \tilde{u}(v) \in \mathcal{H}_{\tilde{u}} \) for every \( \xi \in \mathfrak{sp}(1) \) and \( v \in \mathcal{H}_A \subset TP_{\tilde{G}} \). Indeed, if \( \mu^e \circ \tilde{u} = 0 \), then \( d\mu^e(\tilde{u})(p) \). Also, \( K^P,G|_{\tilde{u}} \in \ker d\mu^e(\tilde{u})(p) \) and \( K^{P,\text{Spin}^c}_{A}|_{\tilde{u}} \in \ker d\mu^e(\tilde{u})(p) \). Therefore, \( D_A \tilde{u}(v) \in \ker d\mu^e(\tilde{u})(p) \). Consequently
\[ 0 = \langle d\mu^e(D_A \tilde{u}(v)), \xi \otimes \eta \rangle = \langle I_\xi K^P,G|_{\tilde{u}(p)}, D_A \tilde{u}(v) \rangle = -\langle K^P,G|_{\tilde{u}(p)}, I_\xi D_A \tilde{u}(v) \rangle \]
for \( \xi, \eta \in \mathfrak{sp}(1) \) and so \( I_\xi D_A \tilde{u}(v) \in (\text{im} K^P,G)^\perp \) for all \( \xi \in \mathfrak{sp}(1) \). Also, for \( \xi', \xi'' \in \mathfrak{sp}(1) \),
\[ (d\mu_G(I_\xi D_A \tilde{u}(v)), \xi' \otimes \eta) = (d\mu_G(D_A \tilde{u}(v)), [\xi, \xi'] \otimes \eta) = 0 \]
which implies \( I_\xi D_A \tilde{u}(v) \in \ker d\mu_G(\tilde{u}(p)) \) for all \( \xi \in \mathfrak{sp}(1) \). Thus, \( I_\xi D_A \tilde{u}(v) \in \mathcal{H}_{\tilde{u}} \).

Step 2: In this step, we prove the equivalence (19). If \( D_A \tilde{u} = 0 \), then from (8), we have
\[ 0 = D_A \tilde{u}(\tilde{e}_0) - \sum_{i=1}^3 I_i D_A \tilde{u}(\tilde{e}_i) \]
From Step 1, $D_A \hat{u}(\hat{e}_i) \in \mathcal{H}_G$. It follows that $D_A \hat{u}(\hat{e}_i) \in \mathcal{H}_G$ for all $i = 1, 2, 3$. Consequently, for any $v \in \mathcal{H}_A$, $\text{proj}^{imK^{G,G}} D_A \hat{u}(v) = 0$ and we get $K^{P,G}_{\mathcal{A}_u(v)} = -\text{proj}^{imK^{G,G}} d\hat{u}(v)$. In other words, the $\mathfrak{g}$-connection component of $A$ is just the pull-back of the canonical connection on $P$.

Since the diagram commutes, $d\pi_2(D_A \hat{u}) = D_A u$. Also, as $D_A \hat{u}(\hat{e}_i) \in \mathcal{H}_G$ for all $i = 0, 1, 2, 3$, we have $t^*I_i = \pi_2^*\hat{I}_i$ and so,

$$0 = d\pi_2(D_A \hat{u}) = d\pi_2 \left(D_A \hat{u}(\hat{e}_0) - \sum_{i=1}^{3} t^*I_i \ D_A \hat{u}(\hat{e}_i)\right) = D_A u$$

Thus, $D_A \hat{u} = 0$ implies $D_A u = 0$. On the other hand if $K^{P,G}_{\mathcal{A}_u(v)} = -\text{proj}^{imK^{G,G}} d\hat{u}(v)$ then $D_A \hat{u} \in \mathcal{H}_G$ and so $d\pi_2(D_A \hat{u}) = D_A u$. Therefore, if $D_A u = 0$, it implies that $D_A \hat{u} \in \text{im} K^{P,G}$. But since,

$$D_A \hat{u} = D_A \hat{u}(\hat{e}_0) - \sum_{i=1}^{3} \pi_2^*\hat{I}_i \ D_A \hat{u}(\hat{e}_i) \in \mathcal{H}_G$$

it follows that $D_A \hat{u} \in (\text{im} K^{P,G})^\perp$ and so $D_A \hat{u} = 0$. This proves the statement.

With this observation, it is now easy to construct a “lift” of the equations:

**Proposition 5.3.** Fix a connection $b$ on $P_{U(1)}$. There is a 1-1 correspondence between the following systems of equations

$$\begin{cases}
D_A \hat{u} = 0 \\
F^+_b - \mu \circ \hat{u} = 0 \quad \text{and} \quad D_A u = 0 \\
\mu_g \circ \hat{u} = 0
\end{cases}$$

where $\mu : \mathcal{H} \to i\mathbb{R}$ denotes the moment map for $U(1)$-action on $\mathcal{H}$.

Since the tri-Hamiltonian action of $U(1)$ descends to $M$, we denote the $U(1)$-moment map by $\mu$ itself.

The above correspondence was independently obtained by Pidstrygach [19].

### 5.2. Almost-complex geometry and generalised Seiberg-Witten

In this subsection, we give a proof of Theorem 1.2. It exploits the equivalence (20) and Theorem 1.1. Firstly, note that the generalised Seiberg-Witten are not conformally invariant. On the other hand, from Theorem 1.1, we know that the generalised spinors are conformally invariant. It follows that there is 1-1 correspondence between the solutions $(\hat{u}', \Lambda')$ of the system (20) with respect to the metric $g'_x \in [g_x]$, such that image of $\hat{u}$ does not contain a fixed point of the $U(1)$-action on $\mathcal{H}$, and the triples $(g''_x, \hat{u}'', \Lambda'')$ such that $||\mu \circ \hat{u}''|| = 1$ and

$$\begin{cases}
D_{\Lambda''} \hat{u}'' = 0 \\
F^+_b - \lambda \mu \circ \hat{u}'' = 0 \\
\mu_g \circ \hat{u}'' = 0
\end{cases}$$
for some strictly positive function \( \lambda \). To see the correspondence, first note that \( \tilde{u}' \) is nowhere vanishing. We now choose \( g''_X = |\tilde{u}'|^2/3 g'_X \). Suppose we are given a triple \((g''_X, \tilde{u}, A)\) satisfying (21) with \(|\mu \circ \tilde{u}| = 1\). This essentially translates to saying \(|\tilde{u}| = 1\) so that \( \tilde{u} \) is non-vanishing. Then \( \Omega = \Phi(\mu \circ \tilde{u}) \) is a non-degenerate, self-dual 2-form on \( X \) and defines an almost-complex structure on \( X \).

**Lemma 5.4.** Given a nowhere vanishing spinor \( \tilde{u} \) and a connection \( \Lambda_0 \), there exists a unique 1-form \( a \) on \( X \) such that \( D_A \tilde{u} = 0 \), where \( A = \Lambda_0 + a \).

**Proof.** Observe that \( D_A \tilde{u} = D_{\Lambda_0} \tilde{u} + a \bullet \tilde{u} \). The statement for the lemma follows from the fact that Clifford multiplication \( \bullet : (\mathbb{R}^4)^* \times H \to H \) is just rightmultiplication by conjugate. Therefore, if \( h_1, h_2 \in H \), \( h_1 \neq 0 \), then choosing \( a = \frac{h_2}{h_1} \) satisfies \( a \bullet h_1 = h_2 \).

Given this lemma, the Dirac equation can be used to eliminate \( A \), since \( A \) is determined by \( \Lambda_0 \) and \( \tilde{u} \). Therefore \( a = a(\tilde{u}) = a(\Omega) \).

Now \( F^+_A = F^+_{\Lambda_0} + d^+ a(\Omega) \) and so, the second equation in (20) can be written as \( d^+ a(\Omega) = -F^+_{\Lambda_0} + \lambda \Omega \). Using the decomposition \( X^+ = \mathbb{R} \Omega \oplus \mathbb{K} \) we get two conditions

\[
\langle d^+ a(\Omega), \Omega \rangle > 0, \quad d^+_{0,2} a(\Omega) + F^{0,2}_{\Lambda_0} = 0
\]

We see now that the generalised Seiberg-Witten equations, can be expressed in terms of an almost-complex structure. The Theorem 1.2 is a consequence of the following lemma

**Lemma 5.5.** In the situation described above, the following formulae hold

- \( d^+_{0,2} a(\Omega) + F^{0,2} = (\nabla^* \nabla \Omega) + \langle d\Omega, N_{\Omega} \rangle \)
- \( d^+ a(\Omega) = -\left( \frac{3}{2} |N_{\Omega}|^2 + \frac{1}{2} |d\Omega|^2 + s_X(g''_X) \right) \)

where \( s_X(g''_X) \) is the scalar curvature of \( X \) with respect to the metric \( g''_X \).

Let \( B : H \times H \to \mathfrak{sp}(1) \) denote the symmetric bi-linear 2-form associated to the moment map and \( B \) denote the induced map on \( T^*X \otimes \Lambda^+ \). Then, \( \Omega = B(\tilde{u}, \tilde{u}) \).

This gives

\[
\nabla^* \nabla \Omega = 2 \left( B(D_A^* D_A \tilde{u}, \tilde{u}) - \tilde{B}(D_A \tilde{u}, D_A \tilde{u}) \right)
\]

Applying the Weitzenböck formula

\[
D^*_A D_A \tilde{u} = D^*_A D_A \tilde{u} + \frac{s_X(g''_X)}{4} \tilde{u} + F^+_{\tilde{A}g} \bullet \tilde{u} + F^+_{\tilde{A}g} \bullet \tilde{u}
\]

(22) gives

\[
\nabla^* \nabla \Omega = -\frac{s_X(g''_X)}{2} \Omega - B(F^+_g \bullet \tilde{u}, \tilde{u}) - B(F^+_g \bullet \tilde{u}, \tilde{u}) - 2 \tilde{B}(D_A \tilde{u}, D_A \tilde{u})
\]

We claim that the term \( B(F^+_g \bullet \tilde{u}, \tilde{u}) \) vanishes. This follows from the following Lemma:
Lemma 5.6. Assume that \( \mu_\mathfrak{g}(h) = 0 \) and let \( \xi \in \mathfrak{sp}(1) \) and \( \eta \in \mathfrak{g} \). Then
\[
B(\tilde{u}, \eta \tilde{u} \xi) = 0
\]

Proof. For \( v \in V \), \( w \in V^* \), let \( h = v + w^\dagger j \), where \( \dagger \) denotes the conjugate transpose. Then the moment map (17) can be written as
\[
\mu(h) = \frac{1}{2} h^\dagger i h \quad \text{and so} \quad B(h_1, h_2) = \frac{1}{4}(h_1^\dagger i h_2 + h_2^\dagger i h_1).
\]
Taking \( h_1 = h \) and \( h_2 = \eta \tilde{u} \xi \), we get \( B(h, \eta \tilde{u} \xi) = \frac{1}{4}(h^\dagger i \eta h \xi + \xi h^\dagger \eta^\dagger i h) \).
Therefore, for any \( \beta \in \mathfrak{sp}(1) \),
\[
\langle B(h, \eta h \xi), \beta \rangle_{\mathfrak{sp}(1)} = -\Re \left( \frac{1}{4}(\beta h^* i \eta h \xi) \right) - \Re \left( \frac{1}{4}(\beta \xi \eta^* i h) \right)
\]
\[
= \frac{1}{4} \Re \left( (i h \beta) (\eta h \xi) \right) - \frac{1}{4} \Re \left( (\eta h \xi \beta) (i h) \right)
\]
\[
= \frac{1}{4} \langle I_\beta K_i^H|h, I_\xi K_i^H|h \rangle_H - \frac{1}{4} \langle I_\beta I_\xi I_\eta K_i^H|h, K_i^H|h \rangle_H
\]
\[
= \frac{1}{2} \langle I_\xi K_i^H|h, I_\beta K_i^H|h \rangle_H = \langle d\mu_H, \beta \xi \eta \rangle (K_i^H|h) = 0
\]
Since \( \mu_\mathfrak{g}(h) = 0 \) and \( U(1) \) action commutes with that of \( G \), \( U(1) \) preserves \( \mu_\mathfrak{g}^{-1}(0) \), \( K_i^H|h \in \ker d\mu_\mathfrak{g} \). The last equality therefore holds for all \( \beta \in \mathfrak{sp}(1) \). In other words, \( B(\tilde{u}, \eta \tilde{u} \xi) = 0 \).
\[
\Box
\]

Now note that \( F^\dagger_\mathfrak{A}_\mathfrak{g} \cdot \tilde{u} = \sum_{i=1}^{3} \langle F^\dagger_\mathfrak{A}_\mathfrak{g}, \eta_i \rangle_{\mathfrak{sp}(1)} \tilde{u}_i \). It now follows from the above Lemma that the term \( B(F^\dagger_\mathfrak{A}_\mathfrak{g} \cdot \tilde{u}, \tilde{u}) \) vanishes. Therefore,
\[
\nabla^* \nabla \Omega = - \left( \frac{s_x(g''_x)}{2} + \lambda \right) \Omega - 2 \tilde{B}(D_\lambda \tilde{u}, D_\lambda \tilde{u})
\]
(23)

Proof of Lemma (5.5). Observe that since \( |\Omega| = 1 \),
\[
0 = \Delta |\Omega| = 2 \langle \nabla^* \nabla \Omega, \Omega \rangle - 2 |\nabla \Omega|^2
\]
Using (23), we get
\[
2\lambda = -s_x(g''_x) - 2 |\nabla \Omega|^2 = 2 \left( \tilde{B}(D_\lambda \tilde{u}, D_\lambda \tilde{u}), \Omega \right)
\]
(24)

Also, it follows from (23) that \( (\nabla^* \nabla \Omega)^{\perp \alpha} + \tilde{B}(D_\lambda \tilde{u}, D_\lambda \tilde{u})^{\perp \alpha} = 0 \). To complete out proof, we merely need to show that
\[
(\nabla^* \nabla \Omega)^{\perp \alpha} + \tilde{B}(D_\lambda \tilde{u}, D_\lambda \tilde{u})^{\perp \alpha} = 0, \quad \left( \tilde{B}(D_\lambda \tilde{u}, D_\lambda \tilde{u}), \Omega \right) = \frac{1}{4} \left( |N_\Omega|^2 - |d\Omega|^2 \right).
\]
In order to do this, it suffices to restrict to the standard model when \( X = \mathbb{R}^4 \) and the connection \( \Lambda \) is trivial. The key issue here is to identify the kernel of the
Clifford multiplication with the map $\tilde{B}$. Let $s_1, s_2, \cdots, s_{2n}$ denote the basis for the spinors and write $\tilde{u}$ as

$$\tilde{u} : \mathbb{R}^4 \to \mathbb{H}, \quad \tilde{u} = \sum_{i=1}^{n} f_i s_i + \sum_{i=n+1}^{2n} g_{i-n} s_i$$

where $f_i, g_i \in C^\infty(\mathbb{R}^4, \mathbb{C})$.

Let $f$ and $g$ denote the vectors

$$f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}, \quad g = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}.$$

Then

$$B(\tilde{u}, \tilde{u}) = \left( \frac{|f|^2 - |g|^2}{2} \right) \Omega_0 + \text{Re}(f \overline{g}) \Omega_1 + \text{Im}(f \overline{g}) \Omega_2$$

where,

$$\Omega_0 = dx_0 \, dx_1 + dx_2 \, dx_3, \quad \Omega_1 = dx_0 \, dx_2 + dx_3 \, dx_1, \quad \Omega_3 = dx_0 \, dx_3 + dx_1 \, dx_2.$$

We compute at the origin. Assume that $g_i = 0$ and $|f| = 1$ at the origin. Then, the unit spinor defines the standard complex structure on $\mathbb{R}^4$. This allows us to use the complex co-ordinates

$$z = x_0 + ix_1, \quad w = x_2 + ix_3$$

where $x_0, x_1, x_2, x_3$ are the standard co-ordinates on $\mathbb{R}^4$. The Dirac equation now reads

$$f_z = g_w, \quad f_w = -g_z \quad (25)$$

The condition that $|\tilde{u}| = 1$ implies that the derivative of $f$ is purely imaginary at the origin. Indeed this can be seen as follows: assume, without loss of generality, that at the origin $f(0) = (r_0, r_1, \cdots, r_n)$, where each $r_i$ is a real number. Identify $\mathbb{H}^n$ with $\mathbb{C}^{2n}$ let $\langle \cdot, \cdot \rangle$ denote the standard Hermitian metric on $\mathbb{C}^{2n}$. Since $|\tilde{u}| = 1$, we have

$$0 = \frac{\partial}{\partial z} \langle \tilde{u}, \tilde{u} \rangle = \text{Re} \left( \frac{\partial \tilde{u}}{\partial z} \cdot \tilde{u} \right).$$

At the origin, $g = 0$ and $f(0) = (r_0, r_1, \cdots, r_n)$, which gives, at origin

$$0 = \frac{\partial}{\partial z} \langle f, f \rangle = \text{Re} \left( \frac{\partial f}{\partial z}, f \right) = \left( \text{Re} \frac{\partial f}{\partial z}, f \right)$$

which implies that $\text{Re} \frac{\partial f}{\partial z} = 0$. In other words, $\frac{\partial f}{\partial z}$ is purely imaginary. Similarly, $\frac{\partial f}{\partial x_0}$, $\frac{\partial f}{\partial x_1}$ and $\frac{\partial f}{\partial x_2}$ are purely imaginary. Therefore

$$f_z = -\overline{f}_w \quad \text{and} \quad f_w = -\overline{f}_z$$

The component of $\tilde{B}(D\tilde{u}, D\tilde{u})$ along $\Omega_0$ is

$$\frac{1}{2} \sum_{l=0}^{3} \left| \frac{\partial f}{\partial x_l} \right|^2 - \left| \frac{\partial g}{\partial x_l} \right|^2.$$
\[
\frac{1}{8} \left( |f_z|^2 + |f_{z\bar{v}}|^2 + |f_{w\bar{v}}|^2 + |f_{w\bar{v}}|^2 - |g_z|^2 + |g_{w\bar{v}}|^2 + |g_{w\bar{v}}|^2 \right)
\]

Using the identities (25) and (5.2), we get
\[
\left< \tilde{B}(D\tilde{u}, D\tilde{u}), \frac{1}{2} \Omega_0 \right> = \frac{1}{16} \left( |g_z|^2 + |g_{w\bar{v}}|^2 \right) - \frac{1}{16} \left( |g_{w\bar{v}}|^2 + |g_{w\bar{v}}|^2 \right)
\]

The space orthogonal to \( \Omega_0 \) is spanned by \( \Omega_c = \bar{\sigma} \cdot \bar{\omega} \) and therefore the component of \( B(D\tilde{u}, D\tilde{u}) \) orthogonal to \( \Omega_0 \) is
\[
(B(D\tilde{u}, D\tilde{u}))^{\perp \Omega_0} = \sum_{l=0}^{3} \left[ \left( \frac{\partial f}{\partial \xi_l} \right)^\dagger \frac{\partial g}{\partial \xi_l} \right] \Omega_c
\]
\[
= \frac{1}{4} \left( f_\xi^\dagger g_z + f_{\bar{v}}^\dagger g_{w\bar{v}} + f_{w\bar{v}}^\dagger g_{w\bar{v}} + f_{w\bar{v}}^\dagger g_{w\bar{v}} \right) \Omega_c = \frac{1}{4} (g_{w\bar{v}}^\dagger g_z + g_{w\bar{v}}^\dagger g_{w\bar{v}}) \Omega_c
\]
where the \( \dagger \) denotes the conjugate transpose. Now \( \Omega \) is a section of the twistor bundle and therefore its covariant derivative at the origin is given by the derivative of \( f^\dagger g \) which is nothing but the derivative of \( g \). The holomorphic part \( (g_z, g_{w\bar{v}}) \) corresponds to the Nijenhuis tensor \( N_\Omega \) whereas the anti-holomorphic component \( (g_{\bar{v}}, g_{w\bar{v}}) \) corresponds to \( d\Omega \).

Recall that there is a natural \( \mathcal{K} \)-valued pairing between \( TX \) and \( T^* X \otimes \mathcal{K} \). Applying this to \( d\Omega \) and \( N_\Omega \), the pairing corresponds to \( (g_{w\bar{v}}^\dagger g_z + g_{w\bar{v}}^\dagger g_{w\bar{v}}) \Omega_c \). Therefore,
\[
(B(D\tilde{u}, D\tilde{u}))^{\perp \Omega_0} = \frac{1}{4} \times 4 \langle d\Omega, N_\Omega \rangle = \langle d\Omega, N_\Omega \rangle
\]
\[
\left< \tilde{B}(D\tilde{u}, D\tilde{u}), \frac{1}{2} \Omega_0 \right> = \frac{1}{16} \times 4 \left( |N_\Omega|^2 - |d\Omega|^2 \right) = \frac{1}{4} \left( |N_\Omega|^2 - |d\Omega|^2 \right)
\]

Substituting in equation (23), we have
\[
\nabla^* \nabla \Omega = -\left( \frac{s_\lambda(g_{w\bar{v}}^\dagger)}{2} + \lambda \right) \Omega + \frac{1}{2} \left( |d\Omega|^2 - |N_\Omega|^2 \right) \Omega - 2 \langle d\Omega, N_\Omega \rangle
\]

The statement follows from eq. (29) and eq. (24).

\[\square\]

6. Some Remarks

- Given an action of \( U(1) \), as in (16), the only fixed point of the action is the origin. Therefore, all that we need is that \( \tilde{u} \) be non-vanishing. If one considers the associated vector bundle, with a typical fibre \( H \), then the rank of the vector bundle is strictly greater than the dimension of \( X \). So, it is reasonable to expect that the condition that a non-vanishing spinor is a solution to (20) is satisfied often in this case.

- In the case of usual Seiberg-Witten equations, Donaldson remarks that for a fixed metric, the Seiberg-Witten equations can be written in terms of \( \Omega \) as
\[
\nabla^* \nabla \Omega = -\left( \frac{s_\lambda}{2} + |\Omega|^2 \right) \Omega - 2(d\Omega + *d|\Omega|, N_\Omega) + \frac{1}{2} \left( \frac{|d\Omega|^2}{|\Omega|^2} - |N_\Omega|^2 \right) \Omega
\]
We conjecture that using similar techniques, one can show that the generalised Seiberg-Witten equations can be written in terms of Ω, exactly as (30). In fact, it may be possible to generalize this approach to the non-Abelian case, where Ω is a non-degenerate, self-dual 2-form, but with values in a Lie algebra. The tensor NΩ can be generalised to a suitable vector-valued tensor. In such a case, one ends up with a system of elliptic equations in terms of self-dual 2-forms.

• Under mild restrictions, it can be shown that dΩ = 0 and hence such a solution defines a symplectic structure on X. It can then be inferred from (29) that Ω is an extremum of the functional \( \int_X |∇Ω| \), i.e, \( (∇^*∇Ω)\perp = 0 \). In other words, we get a harmonic almost-complex structure.

It would be interesting to extend the results of this article to the infinite-dimensional setting. This has applications in higher-dimensional gauge theory, particularly in Spin(7)-instanton theory. The author intends to pursue his studies in the above direction.

**Appendix A. Vector bundles and connections**

Let \( π_E : E \to X \) be a vector bundle. Then consider \( Tπ_E : TE \to TX \). Then \( V_E \subset \ker(Tπ_E) \subset TE \) is called the vertical sub-bundle. A connection on E is a choice of a smooth horizontal sub-bundle \( H_E \) such that \( TE = H_E \oplus V_E \). Denote by proj\( v \) and proj\( H_E \) the projections to the vertical and the horizontal sub-bundles respectively. A connection on E is said to be linear if proj\( V_E \) is linear w.r.t \( Tπ_E \).

**Vertical lift.** Consider the pull-back bundle \( E \times_M E \). The map
\[
vl_E : E \times_M E \to V_E, \quad (v, w) \mapsto \frac{d}{dt}(v + tw)|_{t=0}
\]
is an isomorphism. We call this the vertical lift.

**Connector.** A connector is a smooth map \( K : TE \to E \) that satisfies \( K \circ vl_E = \text{proj}_2 : E \times_M E \to E \) and is a vector bundle homomorphism for both the vector bundle structures on \( E \); i.e \( Tπ_E : TE \to TX \) and \( π : TE \to E \).

Given a linear connection \( Φ : TE \to V_E \), its connector \( K^Φ \) is given by the composition
\[
\begin{align*}
TE \xrightarrow{Φ} V_E & \xrightarrow{(vl_E)^{-1}} E \times_M E \xrightarrow{\text{proj}_2} E
\end{align*}
\]

A connector on E induces a covariant derivative
\[
\Gamma(X, T^*X \otimes E) \ni \nabla^Φ_v(s) = K^Φ(Ts(v)) \quad \text{for} \quad v \in TX, \quad s \in Γ(X, E)
\]

**Proposition A.1.** [20, Theorem 42.1] Let X be a compact manifold. Then the space \( \text{Map}(P, M)^G \) is a Frechét manifold modelled on topological vector spaces:
\[
T_u\text{Map}(P, M)^G = Γ(P, u^*TM)^G
\]
The covariant derivative can be interpreted as a section of the infinite-dimensional Fréchet vector-bundle:

\[ D_A : \text{Hom}(\mathcal{H}_A, TM)^G \to \text{Map}(P, M)^G \]

Note that

\[ T\text{Hom}(\mathcal{H}_A, TM)^G = \text{Hom}(\mathcal{H}_A, TTM)^G, \quad \nabla \text{Hom}(\mathcal{H}_A, TM)^G = \text{Hom}(\mathcal{H}_A, \nabla TM)^G \]

Given a connector \( \psi : TTM \to TM \), it induces a connector \( \Psi \) on \( \text{Hom}(\mathcal{H}_A, TM)^G \). Using this, one can compute the linearization of \( \nabla^\Psi D_A \).

**Lemma A.2.** [14, Section 2.4] The linearization \( \nabla^\Psi D_A \) of the covariant derivative coincides with the first-order differential operator

\[ \nabla^{A,\psi} : C^\infty(P, TM)^G \to \text{Hom}(TP, TM)^G_{\text{hor}}, \quad v \mapsto \psi \circ T\nu \circ \text{proj}_{\mathcal{H}_A} \]  

where \( \text{proj}_{\mathcal{H}_A} \) denotes the projection the horizontal bundle defined by \( A \).

**References**

[1] C. H. Taubes, “Nonlinear Generalizations of a 3-manifold’s Dirac operator,” in Trends in mathematical physics (Knoxville, TN, 1998), AMS/IP Stud. Adv. Math., vol. 13, pp. 475–486, Amer. Math. Soc., Providence, RI, 1999.

[2] V. Y. Pidstrygach, “HyperKähler manifolds and Seiberg-Witten equations,” Proc. Steklov Inst. Math., pp. 249–262, 2004.

[3] V. Pidstrygach and A. Tyurin, “Localisation of the Donaldson’s invariants along Seiberg-Witten classes.” arxiv, 1995. Pre-print.

[4] A. Haydys, “Fukaya-Seidel category and gauge theory,” J. Symplectic Geom., vol. 13, pp. 151–207, 2015.

[5] E. Witten, “Fivebranes and Knots.” arxiv, 2011. Pre-print.

[6] C. Manolescu, “Pin(2)-Equivariant Seiberg-Witten Floer homology and the Triangulation Conjecture,” J. Amer. Math. Soc., vol. 29, pp. 147–176, 2016.

[7] A. Swann, “HyperKähler and Quaternionic Kähler geometry,” Math. Ann., vol. 3, pp. 421–450, 1991.

[8] J. Wood, “Harmonic almost complex structures,” Compos. Math., vol. 99, pp. 183–212, 1995.

[9] E. Calabi and H. Gluck, “What are the best almost-complex structures on the 6-sphere?,” Proc. Sym. Pure Math., vol. 54, pp. 99–106, 1993.

[10] J. Davidov, A. U. Haq, and O. Mushkarov, “Almost complex structures that are harmonic maps.” arxiv, 2015. Pre-print.

[11] S. K. Donaldson, “The Seiberg-Witten Equations and Almost-Hermitian Geometry,” Contemp. Math., vol. 288, pp. 32–38, 2001.

[12] S. Salamon, *Riemannian geometry and holonomy groups*, vol. 201 of Pitman Research notes in Mathematics. Longman Scientific & Technical, 1989.

[13] B. Lawson and M. Michelsohn, *Spin Geometry*, 38, Princeton University Press, 1989.

[14] H. Schumacher, “Generalized Seiberg-Witten equations: Swann bundles and \( L^\infty \)-estimates,” Master’s thesis, Mathematisches Institut, Georg-August-Universität, Göttingen, http://www.uni-math.gwdg.de/preprint/mg.2010.02.pdf, 2010.
[15] P. Kobak and A. Swann, “Classical nilpotent orbits as hyperkähler quotients,” *Int. J. Math.*, vol. 07, pp. 193–210, 1996.

[16] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček., “Hyperkähler metric and Super-symmetry,” *Communications in Mathematical Physics*, vol. 108, no. 4, pp. 535–589, 1987.

[17] S. Donaldson and P. Kronheimer, *The geometry of four-manifolds*. Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1990.

[18] A. Maciocia, “Metrics on the Moduli Spaces of Instantons Over Euclidean 4-Space,” *Commun. Math. Phys*, vol. 135, pp. 467–482, 1991.

[19] V. Pidstrygach, “Bogomolny-Monopole und 4-Mannigfaltigkeiten.” Seminar Talk, Universität Bielefeld, 2006.

[20] A. Kriegl and P. Michor, *The Convenient Setting of Global Analysis*, vol. 53 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.