Gap of the First Two Eigenvalues of the Schrödinger Operator with Nonconvex Potential

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Dedicated to Manfredo do Carmo on his 80th Birthday

In this essay, I will extend my previous work [9] on operators whose potential is nonconvex. In particular, the results given here can be applied to the double well potential. I define an invariant associated to the potential in §4. It defines a distance between the point where $\frac{u_2}{u_1}$ achieves its maximum, $\sup u_2$, to the point where $\frac{u_2}{u_1} = \varepsilon \sup u_1$. Here $u_i$ are the eigenfunctions of the Schrödinger operator. I will show how the gap $\lambda_2 - \lambda_1$ can be estimated from below in terms of this distance. Theorem 6.1 is the main theorem of this essay. It is a very interesting problem to locate the maximum of $\frac{u_2}{u_1}$ and its zeroes. The upper bound of $\lambda_2 - \lambda_1$ depends on the choice of a good trial function, and I shall come back to this question in the future.

This line of research on gradient estimates started from my work on bounded harmonic functions [8] and the method was used by Peter Li [2] and Li-Yau [3] for the Laplacian of a manifold. Li-Yau [3] also applied it to the Schrödinger operator where a distance function similar to the one used here was introduced.

The Li-Yau type distance function was also used by Perelman in his famous work [6].

In the Li-Yau’s approach of estimating the first eigenvalue of the Laplacian, it was conjectured by Li-Yau and proved by Zhong-Yang [10] that if $d$ is the diameter of a manifold with nonnegative Ricci curvature, then $\lambda_1 d^2$ has an universal lower bound which is achieved when the manifold is a circle.

Convex domain and convex potential can be considered as an analogue of manifold with
nonnegative curvature. In Singer-Wong-Yau-Yau [7], we improved on the log concavity result of Brascamp-Lieb [1] and proved that $(\lambda_2 - \lambda_1)d^2$ has a universal lower bound. It is natural for us to expect that the interval will give this optimal lower estimate.

I would like to dedicate this work to my friend Manfredo do Carmo whose works on minimal surfaces are very original and influential.

§1 Generalized log concavity of the first eigenfunction

In [7, 9], I used method of continuity to generalize the log concavity result of Brascamp and Lieb [1] when the potential is convex. I generalize it further in this section.

Let $u_1$ be the first eigenfunction of the operator $-\Delta + V$ on a domain $\Omega_1$ with zero boundary valued data. Let $\varphi = -\log u_1$. Then we have the following theorem:

**Theorem 1.1** The Hessian of $\varphi$ has eigenvalue greater than $g(x)$ where

$$g(x) = \sup\{f(x)\mid f \text{ is a bounded smooth function defined on } \Omega \text{ so that the lowest eigenvalue of the Hessian of } V \text{ plus } \Delta f \text{ is greater than } f^2\}$$

**Proof** Differenting the equation

$$\Delta \varphi = |\nabla \varphi|^2 - V + \lambda_1$$  \hspace{1cm} (1.1)

we obtain

$$\Delta(\varphi_{ii} - f) = \sum \varphi_{ji}^2 - V_{ii} - \Delta f$$  \hspace{1cm} (1.2)

where $\varphi_{ji} = \frac{\partial^2 u}{\partial x_j \partial x_i}$ and $V_{ii} = \frac{\partial^2 V}{\partial x_i^2}$.

Minimizing $\varphi_{ii} - f$, we obtain

$$\varphi_{ji} = 0 \quad \text{for} \quad j \neq i$$  \hspace{1cm} (1.3)

$$\Delta(\varphi_{ii} - f) \geq 0$$  \hspace{1cm} (1.4)
Hence at such a point,

$$\varphi_{ii}^2 \geq V_{ii} + \Delta f$$

(1.5)

Note that the continuity argument was introduced by me and discussed in [7]. (I had lectured on this in 1979 as was noted in [4].) It can be applied in the following way.

Replace the potential $V$ by $V_t = ||x||^2 + tV$. When $t = 0$, the theorem is obviously true. We have to prove the theorem for all $t > 0$.

Suppose the theorem is true for $t < t_0$. Then at $t = t_0$, $\varphi_{ii} \geq f$ and there is a function $f$, depending on $t$, so that $(V_t)_{ii} + \Delta f > f^2$ and at some point, $\varphi_{ii} = f$. By (1.5), we obtain

$$f^2 \geq (V_t)_{ii} + \Delta f$$

(1.6)

This contradicts the choice of $f$.

Hence $t$ can be arbitrarily large and we conclude that Theorem 1.1 holds.

**Remark 1.1** The argument of Theorem 1.1 can be generalized to manifolds with negative curvature.

§2 Gradient estimate of first eigenfunction

Let $\Omega$ be a convex subdominant. Then we can choose a smooth nonnegative function $\rho$ with compact support.

Let

$$G = \rho^2(V + \alpha)^{-1} \Delta \varphi$$

(2.1)

where $\alpha$ is a constant to be chosen later. Then according to (1.1), we find

$$\nabla G = G(2\nabla \log \rho - \nabla \log(V + \alpha)) + \rho^2(V + \alpha)^{-1}(2\nabla \varphi \cdot \nabla \nabla \varphi - \nabla V)$$

(2.2)
\[ \Delta G = \nabla G(2\nabla \log \rho - \nabla \log (V + \alpha)) + G(2\Delta \log \rho - \Delta \log (V + \alpha)) \\
+2\nabla G \cdot \nabla \log \rho - 4|\nabla \log \rho|^2 G - \nabla G \cdot \nabla \log (V + \alpha) \\
+4G \nabla \log (V + \alpha) \cdot \nabla \log \rho - G|\nabla \log (V + \alpha)|^2 \\
+\rho^2 (V + \alpha)^{-1} \left[ 2|\nabla \nabla \varphi|^2 + 4\nabla \varphi \cdot \nabla \nabla \varphi \cdot \nabla \varphi - 2\nabla \varphi \cdot \nabla V - \Delta V \right] \\
= \nabla G(4\nabla \log \rho - 2\nabla \log (V + \alpha) + 2\nabla \varphi) \\
+G(2\Delta \log \rho - \Delta \log (V + \alpha) - 4|\nabla \log \rho|^2 - |\nabla \log (V + \alpha)|^2 \\
-4\nabla \log \rho \cdot \nabla \varphi + 2\nabla \log (V + \alpha) \cdot \nabla \log \rho) \\
+\rho^2(V + \alpha)^{-\frac{1}{2}} (|\nabla \nabla \varphi|^2 - \Delta V) \tag{2.3} \]

By (1.1)

\[ |\nabla \nabla \varphi|^2 \geq \frac{1}{n}|\Delta \varphi|^2 \tag{2.4} \]

Hence at the point where $G$ achieves its maximum,

\[ 0 \geq \frac{2}{n}G^2 - \rho^4(V + \alpha)^{-2} \Delta V \\
+\rho^2 (V + \alpha)^{-1} G[2\Delta \log \rho - \Delta \log (V + \alpha) - 4|\nabla \log \rho|^2 \\
-|\nabla \log (V + \alpha)|^2 - 4 \log \rho \cdot \nabla \varphi + 2\nabla \log (V + \alpha) \cdot \nabla \log \rho] \tag{2.5} \]

Note that

\[ \rho^2(V + \alpha)^{-1} |\nabla \varphi|^2 \leq \rho^2 (V + \alpha)^{-1} (|\nabla \varphi|^2 - V - \lambda_1) + \rho^2 (V + \alpha)^{-1} (V - \lambda_1) \]

\[ = G + \rho^2 (V + \alpha)^{-1} (V - \lambda_1) \tag{2.6} \]

Hence

\[ \frac{2}{n}G^2 \leq \rho^4(V + \alpha)^{-2} \Delta V \\
-\rho^2 (V + \alpha)^{-1} G[2\Delta \log \rho - \Delta \log (V + \alpha) - 4|\nabla \log \rho|^2 \\
-|\nabla \log (V + \alpha)|^2 - 2\nabla \log (V + \alpha) \cdot \nabla \log \rho] \tag{2.7} \]

\[ +4\rho(V + \alpha)^{-\frac{1}{2}} \]

Therefore,
either
\[ G \leq \frac{4}{n^2} \rho^2 (V + \alpha)^{-1} \left| \nabla \log \rho \right|^2 \] (2.8)

or
\[ G \leq 3\sqrt{\frac{n}{2}} \rho^2 (V + \alpha)^{-1} \sqrt{(\Delta V)_+} \] (2.9)

or
\[ G \leq \frac{3n}{2} \rho^2 (V + \alpha)^{-1} \left[ -2\Delta \log \rho + 4|\nabla \log \rho|^2 + \Delta \log (V + \alpha) \right. \\
+ |\nabla \log (V + \alpha)|^2 - 2\nabla \log (V + \alpha) \cdot \nabla \log \rho + 4(V - \lambda_1)^{\frac{3}{2}} \left. \right] \] (2.10)

**Theorem 2.1** Let \( u_1 \) be a positive solution of \((\Delta + V)u_1 = \lambda_1 u_1 \) and \( \varphi = -\log u_1 \). Then
\[
\rho^2 (V + \alpha)^{-1} \left| \nabla \varphi \right|^2 - V + \lambda_1 \leq 10n \rho^2 (V + \alpha)^{-1} \left( |\nabla \log \rho|^2 + |\Delta \log \rho| \right) + \frac{3n}{2} \rho^2 (V + \alpha)^{-1} \left\{ (\Delta \log (V + \alpha))_+ + 2|\nabla \log (V + \alpha)|^2 + 4(V - \lambda_1)^{\frac{3}{2}} \right\} \] (2.11)

In particular,
\[
\rho^2 (V + \alpha)^{-1} |\nabla \varphi|^2 \leq \sup V - \lambda_1 \left( \frac{\sup V}{\sup V + \alpha} + 10n^2 \sup (V + \alpha)^{-1} \rho^2 \left( |\nabla \log \rho|^2 + |\Delta \log \rho| \right) + 3n \sup [(V + \alpha)^{-2} ((\Delta V)_+) + (V + \alpha)^{-3} |\nabla V|^2] \right. \\
\left. + 6n \sup (V + \alpha)^{-\frac{1}{2}} \left( \frac{\sup V - \lambda_1}{\sup V + \alpha} \right)^{\frac{1}{2}} \right) \]

§3 Gradient estimate for \( \frac{u_2}{u_1} \)

Let \( u_2 \) be the second eigenfunction of \(-\Delta + V \) on \( \Omega \). Let \( u = \frac{u_2}{u_1} \).

Then
\[ \Delta u = -(\lambda_2 - \lambda_1)u + 2\nabla \varphi \cdot \nabla u \] (3.1)
Let $c > \sup_{\Omega} \frac{u_2}{u_1}$ and $\psi = -\ln(c - u)$.

Then

$$\Delta \psi = (\lambda_2 - \lambda_1)(1 - ce^\psi) + 2\nabla \varphi \nabla \psi + |\nabla \psi|^2. \quad (3.2)$$

Let

$$F = \rho^2(V + \alpha)^{-1} \left[|\nabla \psi|^2 + (\lambda_2 - \lambda_1)(1 - ce^\psi) \right] \quad (3.3)$$

when $\alpha > 0$ is a constant to be chosen.

Then

$$\nabla F = 2F \nabla \log \rho - F \nabla \log(V + \alpha) + \rho^2(V + \alpha)^{-1} \left[2\nabla \psi \nabla \nabla \psi - c(\lambda_2 - \lambda_1)e^\psi \nabla \psi \right] \quad (3.4)$$
\[ \Delta F = \nabla F(2\nabla \log \rho - \nabla \log(V + \alpha)) + F(2\Delta \log \rho - \Delta \log(V + \alpha)) \\
+ 4\rho(V + \alpha)^{-1}\nabla \rho \cdot \nabla \psi \cdot \nabla \nabla \psi - 2\rho^2(V + \alpha)^{-2} \nabla \nabla \psi \cdot \nabla \psi \\
+ 2\rho^2(V + \alpha)^{-1} \nabla \nabla \psi \cdot \nabla(\Delta \psi) \\
- 2c\rho(V + \alpha)(\lambda_2 - \lambda_1)e^\psi \nabla \rho \cdot \nabla \psi + c\rho^2(V + \alpha)^{-2}(\lambda_2 - \lambda_1)e^\psi \nabla \nabla \psi \\
- c\rho^2(V + \alpha)^{-1}(\lambda_2 - \lambda_1)e^\psi |\nabla \psi|^2 \\
- c\rho^2(V + \alpha)^{-1}(\lambda_2 - \lambda_1)e^\psi \left[ |\nabla \psi|^2 + 2\nabla \varphi \cdot \nabla \psi + (\lambda_2 - \lambda_1)(1 - ce^\psi) \right] \\
= \nabla F(2\nabla \log \rho - \nabla \log(V + \alpha)) + F(2\Delta \log \rho - \Delta \log(V + \alpha)) \\
+ 2\nabla F \cdot \nabla \log \rho - 4F|\nabla \log \rho|^2 + 2F\nabla \log(V + \alpha) \cdot \nabla \log \rho \\
+ 2c\rho^2(V + \alpha)^{-1}(\lambda_2 - \lambda_1)e^\psi \nabla \nabla \psi \cdot \nabla \log \rho - (V + \alpha)^{-1} \nabla F \cdot \nabla \nabla \psi \\
+ 2F\nabla \log \rho \cdot \nabla \log(V + \alpha) - F|\nabla \log(V + \alpha)|^2 \\
- c\rho^2(V + \alpha)^{-2}(\lambda_2 - \lambda_1)e^\psi \nabla \nabla \psi \cdot \nabla \nabla \psi + 2\rho^2(V + \alpha)^{-1} |\nabla \nabla \psi|^2 \\
+ 2\rho^2(V + \alpha)^{-1}(2\nabla \psi \cdot \nabla \nabla \psi \cdot \nabla \psi + 2\nabla \psi \cdot \nabla \nabla \psi \cdot \nabla \psi) \\
\text{ (3.5)} \\
- 2c\rho(V + \alpha)(\lambda_2 - \lambda_1)e^\psi \nabla \nabla \psi \cdot \nabla \psi + c\rho^2(V + \alpha)^{-2}(\lambda_2 - \lambda_1)e^\psi \nabla \nabla \psi \\
- c\rho^2(V + \alpha)^{-1}(\lambda_2 - \lambda_1)e^\psi |\nabla \psi|^2 \\
- c\rho^2(V + \alpha)^{-1}(\lambda_2 - \lambda_1)e^\psi \left[ |\nabla \psi|^2 + 2\nabla \varphi \cdot \nabla \psi + (\lambda_2 - \lambda_1)(1 - ce^\psi) \right] \\
= \nabla F(4\nabla \log \rho - 2\nabla \log(V + \alpha)) + F(2\Delta \log \rho - \Delta \log(V + \alpha) - 4|\nabla \log \rho|^2 \\
\text{ (3.5)} \\
+ 4\nabla \log(V + \alpha) \cdot \nabla \log \rho - |\nabla \log(V + \alpha)|^2 \\
+ 2\rho^2(V + \alpha)^{-1} |\nabla \nabla \psi|^2 + 2\nabla F \cdot \nabla \psi - 2F\nabla \log \rho \cdot \nabla \psi \\
- 2F\nabla \log(V + \alpha) \cdot \nabla \psi + 4\rho^2(V + \alpha)^{-1} \nabla \psi \cdot \nabla \nabla \psi \cdot \nabla \psi + 2\nabla F \cdot \nabla \varphi \\
- 2F\nabla \log \rho \cdot \nabla \varphi + 2F\nabla \log(V + \alpha) \nabla \varphi + 2c\rho^2(V + \alpha)^{-1}(\lambda_2 - \lambda_1)e^\psi \nabla \psi \cdot \nabla \varphi \\
- c\rho^2(V + \alpha)^{-1}(\lambda_2 - \lambda_1)e^\psi |\nabla \psi|^2 \\
- c\rho^2(V + \alpha)^{-1}(\lambda_2 - \lambda_1)e^\psi \left[ |\nabla \psi|^2 + 2\nabla \varphi \cdot \nabla \psi + (\lambda_2 - \lambda_1)(1 - ce^\psi) \right] \\
= \nabla F(4\nabla \log \rho - 2\nabla \log(V + \alpha) + 2\nabla \psi + 2\nabla \varphi) \\
+ F(2\Delta \log \rho - \Delta \log(V + \alpha) - 4|\nabla \log \rho|^2 \\
\text{ (3.5)} \\
+ 4\nabla \log(V + \alpha) \cdot \nabla \log \rho - |\nabla \log(V + \alpha)|^2 - 2\nabla \log \rho \cdot \nabla \psi \\
- 2\nabla \log(V + \alpha) \cdot \nabla \psi - 2\nabla \log \rho \cdot \nabla \varphi + 2\nabla \log(V + \alpha) \cdot \nabla \varphi \\
+ 2\rho^2(V + \alpha)^{-1} |\nabla \nabla \psi|^2 + 4\rho^2(V + \alpha)^{-1} \nabla \psi \cdot \nabla \nabla \psi \cdot \nabla \psi \\
- 2c\rho^2(V + \alpha)^{-1}(\lambda_2 - \lambda_1)e^\psi |\nabla \psi|^2 - c(\lambda_2 - \lambda_1)^2 \rho^2 (V + \alpha)^{-1} e^\psi (1 - ce^\psi) \\
\text{ (3.5)
Now
\[ |\nabla \nabla \psi|^2 \geq \frac{1}{n} (\Delta \psi)^2 \]  \hspace{1cm} (3.6)

and
\[ \Delta \psi - 2 \nabla \psi \cdot \nabla \varphi = \rho^{-2} (V + \alpha) F \]  \hspace{1cm} (3.7)

Hence
\[ \rho^2 (V + \alpha)^{-1} |\nabla \nabla \psi|^2 \geq \rho^2 (V + \alpha)^{-1} \left[ (V + \alpha)^2 \rho^{-4} F^2 + 4 \rho^{-2} (V + \alpha) F \nabla \psi \cdot \nabla \varphi + 4 (\nabla \psi \cdot \nabla \varphi)^2 \right] \]  \hspace{1cm} (3.8)

When \( F \) achieves its maximum, \( \nabla F = 0 \) and \( \Delta F \leq 0 \). Therefore,
\[ 0 \geq F (2 \Delta \log \rho - \Delta \log (V + \alpha) - 4 |\nabla \log \rho|^2 + 4 \nabla \log (V + \alpha) \cdot \nabla \log \rho \\
- |\nabla \log (V + \alpha)|^2 - 2 \nabla \log \rho \cdot \nabla \psi - 2 \nabla \log (V + \alpha) \cdot \nabla \psi \\
- 2 \nabla \log \rho \cdot \nabla \varphi + 2 \nabla \log (V + \alpha) \cdot \nabla \varphi \\
+ \frac{2}{n} \rho^{-2} (V + \alpha) F^2 + \frac{8}{n} F \nabla \psi \cdot \nabla \varphi + \frac{8}{n} \rho^2 (V + \alpha)^{-1} (\nabla \psi \cdot \nabla \varphi)^2 \\
+ 4 \rho^2 (V + \alpha)^{-1} \nabla \psi \cdot \nabla \nabla \varphi \cdot \nabla \psi \\
- 2 c \rho^2 (V + \alpha)^{-1} (\lambda_2 - \lambda_1) e^\psi \left[ |\nabla \psi|^2 + (\lambda_2 - \lambda_1) (1 - c e^\psi) \right] \\
+ c (\lambda_2 - \lambda_1)^2 \rho^2 (V + \alpha)^{-1} e^\psi (1 - c e^\psi) \]  \hspace{1cm} (3.9)

Note that
\[ \rho^2 (V + \alpha)^{-1} \nabla \psi \cdot \nabla \nabla \varphi \cdot \nabla \psi \geq \inf \left( \rho^2 (V + \alpha)^{-1} g(x) \right) F \]  \hspace{1cm} (3.10)

where \( g \) is defined in Theorem 1.1.

Hence
\[ 0 \geq \frac{2}{n} F^2 + 4 \inf \left( \rho^2 (V + \alpha)^{-1} g(x) \right) F \\
- F \left( \rho^2 (V + \alpha)^{-1} |\nabla \psi|^2 \right) \left[ \frac{8}{n} \left( \rho^2 (V + \alpha)^{-1} |\nabla \varphi|^2 \right)^\frac{1}{2} + 2 \left( \rho^2 (V + \alpha)^{-1} |\nabla \log \rho|^2 \right) \right] \\
+ 2 \left( \rho^2 (V + \alpha)^{-1} |\nabla \log (V + \alpha)|^2 \right)^\frac{1}{2} \\
+ \rho^2 (V + \alpha)^{-1} F \left[ 2 \Delta \log \rho - 7 |\nabla \log \rho|^2 - 2 |\nabla \varphi|^2 - \Delta \log (V + \alpha) - |\nabla \log (V + \alpha)|^2 \right] \\
- 2 c \rho^2 (V + \alpha)^{-1} (\lambda_2 - \lambda_1) e^\psi F + c (\lambda_2 - \lambda_1)^2 \rho^2 (V + \alpha)^{-2} e^\psi (1 - c e^\psi) \]  \hspace{1cm} (3.11)
By definition of $F$, either
\[
\rho^2 (V + \alpha)^{-1} |\nabla \psi|^2 \leq 2F \tag{3.12}
\]
or
\[
F \leq (\lambda_2 - \lambda_1)(V + \alpha)^{-1} (1 - ce^\psi) \tag{3.13}
\]

Let us assume (3.12) first. In that case, either
\[
F^2 \leq \frac{3n}{2} \left[ \frac{8}{n} \left( \rho^2 (V + \alpha)^{-1} |\nabla \varphi|^2 \right)^{\frac{1}{2}} + 2 \left( \rho^2 (V + \alpha)^{-1} |\nabla \log \rho|^2 \right)^{\frac{1}{2}} \right] + 2 \left( \rho^2 (V + \alpha)^{-1} |\nabla \log(V + \alpha)|^2 \right)^{\frac{1}{2}} \tag{3.14}
\]
or
\[
F + 6n \inf \left( \rho^2 (V + \alpha)^{-1} g(x) \right) \leq \frac{3n}{2} \rho^2 (V + \alpha)^{-1} \left[ 2\Delta \log \rho + 7|\nabla \log \rho|^2 + \Delta \log(V + \alpha) + |\nabla \log(V + \alpha)|^2 + 2|\nabla \varphi|^2 + 2c(\lambda_2 - \lambda_1)e^\psi \right] \tag{3.15}
\]

or
\[
F^2 \leq \frac{3n}{2} c(\lambda_2 - \lambda_1)^2 \rho^2 (V + \alpha)^{-2} e^\psi |(1 - ce^\psi)| \tag{3.16}
\]

From this, we conclude:

**Theorem 3.1** Let $\Omega$ be a domain and $\rho$ be a smooth function with compact support in $\Omega$. Let $u_i$ be smooth function satisfying the equation $(-\Delta + V)u_i = \lambda_i u_i$ so that $u_1 > 0$ and $\varphi = -\log u_1$ satisfies the conclusion of Theorem 1.1. Let $c$ be a constant so that $c > \sup u$ where $u = \frac{u_2}{u_1}$. Let $\psi = -\log(c - u)$. Then one of the following inequalities hold:

(1)
\[
\rho^2(V + \alpha)^{-1} \left[ |\nabla \psi|^2 + (\lambda_2 - \lambda_1)(1 - ce^\psi) \right] \leq (\lambda_2 - \lambda_1) \sup(V + \alpha)^{-1}(1 - ce^\psi) \tag{3.17}
\]
\[(2)\]
\[
\rho^2 (V + \alpha)^{-1} \left[ |\nabla \psi|^2 + (\lambda_2 - \lambda_1)(1 - ce^\psi) \right]
\leq 144 \left( \frac{\sup V - \lambda_1}{\sup V + \alpha} \right) + 20n^2 \frac{\|D\log \rho\|}{\sup V + \alpha} \left[ |\nabla \log \rho|^2 + |\Delta \log \rho| \right]
+ 4n \sup \left( (V + \alpha)^{-2} (\Delta V)_+ + (V + \alpha)^{-3} |\nabla V|^2 \right)
+ 6n \sup (V + \alpha)^{-\frac{1}{2}} \left( \frac{\sup V - \lambda_1}{\sup V + \alpha} \right)^{\frac{1}{2}}
\]

\[(3)\]
\[
\rho^2 \left[ (V + \alpha)^{-1} |\nabla \psi|^2 + (\lambda_2 - \lambda_1)(1 - ce^\psi) \right] + 6n \inf \left( \rho^2 (V + \alpha)^{-1} g(x) \right)
\leq 20n^2 \sup (V + \alpha)^{-1} \left[ |\nabla \rho|^2 + \rho |\Delta \rho| \right]
+ 10n \sup \rho^2 \left( (V + \alpha)^{-2} (\Delta V)_+ + (V + \alpha)^{-3} |\nabla V|^2 \right)
+ 10 \left( \frac{\sup V - \lambda_1}{\sup V + \alpha} \right) + 6n \sup (V + \alpha)^{-\frac{1}{2}} \left( \frac{\sup V - \lambda_1}{\sup V + \alpha} \right)^{\frac{1}{2}}
+ 3nc(\lambda_2 - \lambda_1) \sup \rho^2 (V + \alpha)^{-1} e^\psi
\]

\[(4)\]
\[
\rho^2 (V + \alpha)^{-1} \left( |\nabla \psi|^2 + (\lambda_2 - \lambda_1)(1 - ce^\psi) \right)
\leq \sqrt{3nc} (\lambda_2 - \lambda_1) \sup \rho^2 (V + \alpha)^{-1} \left[ e^\psi (1 - ce^\psi) \right]^{\frac{1}{2}}
\]

§4 Estimate of the gap \(\lambda_2 - \lambda_1\) in terms of the potential

Let \(u = \frac{u_2}{u_1}\) be defined as in §3. We assume that it is bounded on \(\Omega\) and is zero somewhere in \(\Omega\).

Assume that for some \(\delta > 0\), \(u(x_0) = \sup u\) and \(u(x_1) = \delta \sup u\). Then for each smooth function \(\rho\) with compact support and constant \(\alpha \geq 0\), we can define
\[
L(\rho, \alpha, \delta) = \inf_x \int_0^1 \left( \rho^{-1} \sqrt{V + \alpha} \right) (x(t)) |\dot{x}| dt
\]

where \(x\) is any path in \(\Omega\) with \(x(0) = x_0\) and \(x(1) = x_1\).
Now we can define

\[ L_\delta = \inf_{\rho, \alpha} L(\rho, \alpha, \delta) \left\{ 20n^2 \sup (V + \alpha)^{-1} \rho^2 \left( |\Delta \rho| + |\nabla \rho|^2 \right) + 10n \sup \rho^2 (V + \alpha)^{-3} \left( (\Delta V)_+ + |\nabla V|^2 \right) + 10 \frac{\sup V - \lambda_1}{\sup V + \alpha} \right\} \]

\[ + 6n \sup (V + \alpha)^{-\frac{1}{2}} \left( \frac{\sup V - \lambda_1}{\sup V + \alpha} \right)^{\frac{1}{2}} - 6n \inf \rho^2 (V + \alpha)^{-1} g(x) \right\} \]

where \( \alpha > 0 \) is a constant and \( \rho \) is any smooth function with compact support in \( \Omega \).

Based on Theorem 3.1, we conclude that

**Theorem 4.1**

\[ \left| \log \frac{1}{\delta} \right| \leq L_\delta + (\lambda_2 - \lambda_1) \left( \frac{1}{\delta} + \inf_{\rho, \alpha} \left\{ L(\rho, \alpha, \delta) \sup \frac{\rho^2 (V + \alpha)^{-1}}{\delta} \right\} \right) \]

In particular if for some \( \delta \), \( \log \frac{1 - \delta}{\delta} - L_\delta > 0 \), there is a lower estimate of \( \lambda_2 - \lambda_1 \), in terms of \( L_\delta \) and \( \inf_{\rho, \alpha} \left\{ L(\rho, \alpha, \delta) \sup \frac{\rho^2 (V + \alpha)^{-1}}{\delta} \right\} \).

**§5 Oscillation of the function \( \frac{u_2}{u_1} \)**

Note that in §4, we do not need to assume \( u_i \) satisfies any boundary coordinates on \( \Omega \).

If we assume \( u_i = 0 \) on \( \partial \Omega \), \( u_1 > 0 \) and

\[ \int_{\Omega} u_i^2 = 1 \quad (5.1) \]
\[ \int_{\Omega} u_1 u_2 = 0 \quad (5.2) \]

we find

\[ \int_{\Omega} u^2 u_i^2 = 0 \quad (5.3) \]
\[ \int_{\Omega} u u_i^2 = 0 \quad (5.4) \]
The eigenfunction equations give
\[
\int_{\Omega} |\nabla u_i|^2 + \int_{\Omega} V u_i^2 = \lambda_i \tag{5.5}
\]
Note that assuming (5.1), (5.2) can also be written as
\[
\int_{\Omega} (u_1 + u_2)^2 = 2 \tag{5.6}
\]
or
\[
\int_{\Omega} (u_1 - u_2)^2 = 2 \tag{5.7}
\]
Let
\[
\Omega_t = \{ x \in \Omega | d(x_1 \partial \Omega) \geq t \} \tag{5.8}
\]
We are interested in the behavior of \( \int_{\Omega_t} u_i^2 \) and \( \int_{\Omega_t} u_1 u_2 \). Hence we compute
\[
\frac{d^2}{dt^2} \int_{\Omega_t} u_i^2 = \frac{d}{dt} \int_{\Omega_t} u_i^2 = \int_{\partial \Omega_t} u_i^2 H_t + 2 \int_{\partial \Omega_t} u_i \frac{\partial u_i}{\partial \nu} = \int_{\partial \Omega_t} u_i^2 H_t + \int_{\Omega_t} \Delta (u_i^2) \tag{5.9}
\]
where \( H_t \) is the mean curvature of \( \partial \Omega_t \), measured by the outward normal.
But
\[
\int_{\Omega_t} \Delta (u_i)^2 = 2 \int_{\Omega_t} |\nabla u_i|^2 + 2 \int_{\Omega_t} u_i \Delta u_i = 2 \int_{\Omega_t} (|\nabla u_i|^2 + V u_i^2) - 2 \lambda_i \int_{\Omega_t} u_i^2 = 2 \inf(V - \lambda_i) \tag{5.10}
\]
Since \( u_i = 0 \) on \( \partial \Omega \),
\[
\frac{d}{dt} \int_{\Omega_t} u_i^2 = 0 \tag{5.11}
\]
where \( t = 0 \).

We conclude that

\[
\int_{\Omega_t} u_i^2 = \int_0^t \frac{d}{dt} \left( \int_{\Omega_t} u_i^2 \right) + \int_{\Omega_t} u_i^2 \\
= \int_0^t \left( \int_0^s \frac{d^2}{ds^2} \left( \int_{\Omega_s} u_i^2 \right) \right) + \int_{\Omega_t} u_i^2
\]

\[
\geq \inf(V - \lambda_i) t^2 + 1
\]

Similarly,

\[
\frac{d^2}{dt^2} \int_{\Omega_t} (u_1 + u_2)^2 \\
\geq 2 \int_{\Omega_t} (\nabla u_1 + \nabla u_2)^2 + 2 \int_{\Omega_t} (u_1 + u_2)V(u_1 + u_2) - 2 \int_{\Omega_t} (u_1 + u_2)(\lambda_1 u_1 + \lambda_2 u_2)
\]

\[
\geq 2 \int_{\Omega_t} \left( V - \frac{\lambda_1 + \lambda_2}{2} \right) (u_1 + u_2)^2 + (\lambda_2 - \lambda_1) \int_{\Omega_t} u_1^2 + (\lambda_1 - \lambda_2) \int_{\Omega_t} u_2^2
\]

\[
\geq 2 \inf \left( V - \frac{\lambda_1 + \lambda_2}{2} \right) + (\lambda_1 - \lambda_2)
\]

\[
\int_{\Omega_t} (u_1 + u_2)^2 \geq 2 + \left[ \inf \left( V - \frac{\lambda_1 + \lambda_2}{2} \right) + \frac{\lambda_1 - \lambda_2}{2} \right] t^2 \]

\[
= 2 + \left( \inf \Omega V - \lambda_2 \right) t^2
\]

\[
\int_{\Omega_t} (u_1 - u_2)^2 \geq 2 + \left[ \inf \left( V - \frac{\lambda_1 + \lambda_2}{2} \right) + \frac{\lambda_1 - \lambda_2}{2} \right] t^2
\]

\[
= 2 + \left( \inf \Omega V - \lambda_2 \right) t^2
\]

From (5.14), we obtain

\[
\int_{\Omega_t} u_1 u_2 \geq \frac{1}{2} \left[ \inf \Omega (V - \lambda_2) \right] t^2
\]

From (5.15), we obtain

\[
\int_{\Omega_t} u_1 u_2 \leq - \frac{1}{2} \left[ \inf \Omega (V - \lambda_2) \right] t^2
\]
Let \( u = \frac{u_2}{u_1} \). Then from (5.12),

\[
\int_{\Omega_t} u^2 u_1^2 = \int_{\Omega_t} u_2^2
\geq 1 + \inf_{\Omega_t} (V - \lambda_2) t^2
\]

(5.18)

Hence

\[
\sup_{\Omega_t} u^2 \geq 1 + \inf_{\Omega_t} (V - \lambda_2) t^2
\]

(5.19)

On the other hand,

\[
\inf_{\Omega_t} |u| \leq \left| \int_{\Omega_t} u u_1^2 \right|
\leq \frac{t^2}{2} \left[ - \inf_{\Omega} (V - \lambda_2) \right]
\]

(5.20)

Combining (5.19) and (5.20), we conclude

\[
\inf_{\Omega_t} |u| \leq \frac{t^2 \left[ - \inf_{\Omega} (V - \lambda_2) \right]}{2 + 2 \inf_{\Omega} (V - \lambda_1) t^2}
\]

(5.21)

In the next section, we shall apply the estimates in §4.

### §6 Distance function and the estimate of the gap

For simplicity, we shall assume that \( \Omega \) to be convex in this section. We also assume that

\[
|\nabla V| \leq c_\alpha (V + \alpha)^{\frac{3}{2}}
\]

(6.1)

\[
|\Delta V| \leq c_\alpha (V + \alpha)^{3}
\]

(6.2)

We introduce

\[
d_\alpha(x_0, x_1) = \inf \int_{0}^{1} \sqrt{\bar{V} + \alpha(x(t))} \ |\dot{x}| \ dt
\]

(6.3)

where the infimum is taken over all paths \( x : [0, 1] \to \Omega \) joining \( x_0 \) to \( x_1 \).
If
\[ u(x_0) = \sup u \]  
(6.4)
and
\[ u(x_1) = \delta \sup u \]  
(6.5)

We define \( L_\Omega(\delta) \) to be \( d(x_0, x_1) \). It is of course dominated by
\[ L(\Omega) = \sup_{\tilde{x}_0, \tilde{x}_1 \in \Omega} d(\tilde{x}_0, \tilde{x}_1) \]  
(6.6)

Using the terminology of §4 and §5, we set \( \rho \) to be function of \( t \) so that \( \rho = 0 \) on \( \partial \Omega \) and \( \rho = 1 \) when \( t \geq (\inf_{\partial \Omega} V)^{-\frac{1}{2}} \).

We can assume
\[ \rho^2 \left( |\nabla \log \rho|^2 + |\Delta \log \rho| \right) \leq 3 \left( \inf_{\partial \Omega} V \right) \]  
(6.7)

Note that
\[
\frac{1}{V + \alpha} - \frac{1}{\inf_{\partial \Omega} V + \alpha} \leq \frac{|\nabla V|}{(V + \alpha)^2} t \leq \frac{c_\alpha t}{(V + \alpha)^{\frac{5}{2}}} \leq \frac{c_\alpha t}{(\inf_{\partial \Omega} V + \alpha)^{\frac{5}{2}}} + \frac{1}{2} c^2 t^2
\]  
(6.8)

Hence,
\[
\sup (V + \alpha)^{-1} \rho^2 (\Delta \rho + |\nabla \rho|^2) \leq \frac{3(\inf V)}{(\inf_{\partial \Omega} V + \alpha)} + \frac{c_\alpha (\inf_{\partial \Omega} V)^{\frac{5}{2}}}{(\inf_{\partial \Omega} V + \alpha)^{\frac{5}{2}}} + \frac{3c^2}{2}
\]  
(6.9)

From (5.21), we obtain
\[
\frac{\inf_{\partial \Omega} |u|}{\sup_{\partial \Omega} |u|} \leq \frac{\inf_{\overline{\Omega}} (V - \lambda_2)}{2 \inf_{\partial \Omega} V + 2 \inf_{\overline{\Omega}} (V - \lambda_1)}
\]  
(6.10)

Now assume
\[ |\inf_{\overline{\Omega}} (V - \lambda_2)| \leq \varepsilon \inf_{\partial \Omega} V \]  
(6.11)
Then
\[
\inf_{\Omega_t} |u| \leq \frac{\varepsilon}{2(1 - \varepsilon)}
\]  
(6.12)

According to Theorem 4.1, we have proved

**Theorem 6.1** Assume (6.1), (6.2) and (6.11) and \( t = (\inf_{\partial \Omega} V)^{-\frac{1}{2}} \)

\[
\left| \log \left( \frac{\varepsilon}{2(1 - \varepsilon)} \right) \right| \leq \tilde{c}_\alpha L(\Omega_t) + \frac{2(\lambda_2 - \lambda_1)}{\varepsilon} \left[ 1 + L(\Omega_t) \sup_{\Omega} (V + \alpha)^{-1} \right]
\]  
(6.13)

Here \( \tilde{c}_\alpha \) depends on \( c_\alpha \) and \( -\inf((V + \alpha)^{-1} g(x)) \).  

Note that when \( V \) grows fast

\[
\left| \log \left( \frac{\varepsilon}{2(1 - \varepsilon)} \right) \right| > \tilde{c}_\alpha L(\Omega_+)
\]

and we have a lower bound for \( \lambda_2 - \lambda_1 \) from (6.13).

If we know the location of the points to achieve \( \inf_{\Omega_t}(u) = u(x_0) \) and \( \sup_{\Omega_t} |u| = u(x_1) \), then we can replace \( L(\Omega_t) \) by \( d_\alpha(x_0, x_1) \). This can be applied when we have the double well potential.

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