RELATION BETWEEN INTERSECTION HOMOLOGY AND HOMOTOPY GROUPS

DAVID CHATAUR, MARTINTXO SARALEGI-ARANGUREN, AND DANIEL TANRÊ

Abstract. As Goresky and MacPherson intersection homology is not the homology of a space, there is no preferred candidate for intersection homotopy groups. Here, they are defined as the homotopy groups of a simplicial set which P. Gajer associates to a couple \((X, \mathbf{p})\) of a filtered space and a perversity. We first establish some basic properties for the intersection fundamental groups, as a Van Kampen theorem.

For general intersection homotopy groups on Siebenmann CS sets, we prove a Hurewicz theorem between them and the Goresky and MacPherson intersection homology. If the CS set and its intrinsic stratification have the same regular part, we establish the topological invariance of the \(\mathbf{p}\)-intersection homotopy groups. Several examples justify the hypotheses made in the statements. Finally, intersection homotopy groups also coincide with the homotopy groups of the topological space itself, for the top perversity on a connected, normal Thom-Mather space.

Introduction

Poincaré duality is an extraordinary property of manifolds. Trivial examples show that this feature disappears when the space in consideration presents some singularities, even in the case of an amalgamation of manifolds of different dimensions as in a complex of manifolds of Whitney ([35]). Using an extra parameter \(\mathbf{p}\) called perversity, Goresky and MacPherson [17, 18] define new homologies depending on the choice of a perversity and recreate a Poincaré duality between some spaces with singularities, as the pseudomanifolds.

Many invariants and structures of differential or algebraic topology also found their place in intersection homology or cohomology: Morse theory, characteristic classes, Hodge theory, bordism, existence of cup products, foliations,... (See [12, Chapter 10] for a documented list.) From a homotopical point of view, the first observation is that intersection homology is not a homotopy invariant. On the other hand, F. Quinn ([27]) has given a presentation of some filtered spaces which allows their study with homotopical tools. Here, what we have in mind is a notion of intersection homotopy groups which could be related to intersection homotopy, more or less as the homotopy and homology groups of a space are. In this direction, the first question is: “Let \(X\) be a given filtered space and \(\mathbf{p}\) be a perversity. Does there exist a topological space \(Y\) whose ordinary homology groups are the \(\mathbf{p}\)-intersection homology groups of \(X\)?”

As quoted by N. Habbegger in the introduction of [20], his study of Thom operations for intersection homology “destroys the hope of calculating intersection homology as the ordinary homology of a suitable space, in general, since the Thom operations are natural.”

Nevertheless the well-established properties of the homology of a space encourage a search for substitutes, i.e. the search for spaces \(\mathcal{P}X\) associated to a pseudomanifold \(X\) and a perversity \(\mathbf{p}\), and which is “close” to \(X\) in a direction to be specified. This is the approach of M. Banagl and we send the reader to [2] for an explicit description of this procedure and its properties; we will not use them in this work.

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Briefly, the idea guiding Banagl’s construction is the replacement of the singular links by a truncation of their Moore space decomposition. In general, the homology of $\mathcal{I}^pX$ is not the Goresky and MacPherson intersection homology but satisfies a generalized Poincaré duality when $X$ is closed and oriented. The difficulty lies in the construction of the spaces $\mathcal{I}^pX$. Their existence is established for pseudomanifolds with isolated singularities and some cases of depth two pseudomanifolds in [2, 3, 11]. In [1], their construction is carried out for arbitrary depth but with trivial link bundles, which covers the case of toric varieties.

Our approach is of a different spirit. We keep the Goresky and MacPherson intersection homology and introduce intersection homotopy groups as homotopy groups of a space defined by Gajer in [14, 15]. To explain that, it is better to come back at the beginning of the story. Given a filtered space $X$ and a perversity $\mathfrak{p}$, a selection is made among the singular simplexes as follows. A simplex $\sigma: \Delta \to X$ is $\mathfrak{p}$-allowable if, for each singular stratum $S$, the set $\sigma^{-1}S$ verifies

$$\dim \sigma^{-1}S \leq \dim \Delta - \text{codim } S + \mathfrak{p}(S).$$

(0.1)

A $\mathfrak{p}$-allowable chain is a linear combination of $\mathfrak{p}$-allowable simplexes. These definitions present major shortcomings.

i) If $\xi \in \mathcal{I}^pX$, its boundary $\partial \xi$ is not necessarily $\mathfrak{p}$-allowable.

ii) If $\sigma$ is a $\mathfrak{p}$-allowable simplex and $\partial i$ is a face operator, the simplex $\partial i \sigma$ is not necessarily $\mathfrak{p}$-allowable.

If we are interested by homology, we do a more restrictive choice at the level of chains: a singular chain $\xi$ of $\mathfrak{p}$-intersection if $\xi$ and its boundary $\partial \xi$ are $\mathfrak{p}$-allowable ([17]). The homology of the complex of singular chains of $\mathfrak{p}$-intersection, $C^p_\ast(X; G)$, is the $\mathfrak{p}$-intersection homology $H^p_\ast(X; G)$, with coefficients in an $R$-module $G$ over a Dedekind ring $R$.

If we are interested by simplicial sets, we act on the simplexes themselves: a simplex $\sigma$ is $\mathfrak{p}$-full if $\sigma$ and all its iterated faces are $\mathfrak{p}$-allowable. This is the approach of [14]: Gajer gets a Kan simplicial set that we denote by $\mathfrak{G}_pX$ and called the Gajer space. The $\mathfrak{p}$-intersection homotopy groups are now defined as the homotopy groups of $\mathfrak{G}_pX$ and denoted by $\pi^{G}_p(X)$. What we do, in the present paper, is the study of these homotopy groups and their relation with intersection homology. For the dimension of $\sigma^{-1}S$ in (0.1), we are dealing with the polyhedral dimension of $\mathfrak{G}_pX$ (see Definition 1.11), revisited in [9]. In [14], Gajer falsely believed that the homology of $\mathfrak{G}_pX$ is the $\mathfrak{p}$-intersection homology of $X$. So in [14], some properties as the existence of a Mayer-Vietoris sequence for $H_\ast(\mathfrak{G}_pX)$ with open subsets of $X$ appeared as a consequence of well-known properties of intersection homology. Also we work with topological filtered spaces and not PL-ones. For these reasons, in Section 2, we provide the proofs of basic properties, with references to their first occurrence if we find one.

It’s time to clarify what we mean by perversity. In this work, a perversity is a map from the set of strata of a filtered space with values in $\mathbb{Z} \cup \{\pm \infty\}$, taking the value 0 on the regular strata. In particular, any map $f: \mathbb{N} \to \mathbb{Z}$ such that $f(0) = 0$ defines a perversity $\mathfrak{p}$ by $\mathfrak{p}(S) = f(\text{codim } S)$. Such perversities are said codimensional. Among them are the original GM-perversities ([17]) of Goresky and MacPherson, see Definition 1.8, and the top perversity defined by $\mathfrak{t}(i) = i - 2$, for $i \geq 2$.

In the case of a filtered space $X$ and a perversity $\mathfrak{p}$, we establish a Van Kampen theorem for $\pi^p_\ast(X)$ in Theorem 4.1. If $X$ is a connected normal Thom-Mather space with a finite number of strata and $\mathfrak{t}$ is the top perversity, we deduce, in Corollary 4.2, the isomorphisms, $\pi^p_j(X) \cong \pi_j(X)$, for any $j$. This result enhances the well-known homology result of intersection homology, $H_\ast(X) \cong H_\ast(X)$.

Our main results concern the locally conical filtered spaces, introduced by Siebenmann in [33], and called CS sets, see Definition 1.4. One of their first delicate peculiarities is that links of points in the same stratum are not necessarily homeomorphic ([12, Example 2.3.6]). We know that these links have isomorphic intersection homology groups, and we prove here that they also have isomorphic intersection homotopy groups, cf. Proposition 3.9.
In Example 4.3, we point out that the double suspension of the Poincaré sphere is a counter-example to the topological invariance of $\mathcal{P}$-intersection fundamental groups. In this example, some singular points of the CS set $X$ become regular in the intrinsic stratification $X^*$, introduced by King in [23]. The next result (see Theorem 4.6 and Theorem 5.1) shows that, except this generic case, the perversity homotopy groups are topological invariants.

**Theorem A.** Let $X$ be a CS set without stratum of codimension 1, of intrinsic stratification $\nu: X \to X^*$ and $\mathcal{P}$ be a Goresky and MacPherson perversity. We suppose that the regular parts of $X$ and $X^*$ coincide. Then the map $\nu$ induces isomorphisms, $\pi_0\mathcal{P}(X,x) \cong \pi_0\mathcal{P}(X^*,x)$, for any regular point $x$ and any $j$.

We also establish a $\mathcal{P}$-intersection analog of the Hurewicz theorem (Theorem 6.1) between intersection homotopy groups and Goresky and MacPherson intersection homology groups. Example 6.8 justifies the hypotheses put on the links. We call $\mathcal{P}$-intersection Hurewicz homomorphism,

$$h^\mathcal{P}_j: \pi_j\mathcal{P}(X,x) \to H^\mathcal{P}_j(X;\mathbb{Z}),$$

the composition of the Hurewicz map for $\mathcal{P}_X$ with the homomorphism $H_\nu(\mathcal{P}_X;\mathbb{Z}) \to H^\mathcal{P}_j(X;\mathbb{Z})$ coming from the inclusion between the corresponding chain complexes.

**Theorem B.** Let $X$ be a CS set and $\mathcal{P}$ be a perversity such that $\pi_0\mathcal{P}(X) = 0$.

i) For any regular point $x$, the $\mathcal{P}$-intersection Hurewicz map $h^\mathcal{P}_j: \pi_j\mathcal{P}(X,x) \to H^\mathcal{P}_j(X;\mathbb{Z})$ induces an isomorphism between the abelianisation of $\pi_j\mathcal{P}(X,x)$ and $H^\mathcal{P}_j(X;\mathbb{Z})$.

ii) Let $k \geq 2$. We suppose $\pi_j\mathcal{P}(X) = \pi_j\mathcal{P}(L) = 0$ for every link $L$ of $X$, and each $j \leq k - 1$. Then, the intersection Hurewicz homomorphism $h_j^\mathcal{P}: \pi_j\mathcal{P}(X,x_0) \to H^\mathcal{P}_j(X;\mathbb{Z})$ is an isomorphism for $j \leq k$ and a surjection for $j = k + 1$.

Some natural questions on $\mathcal{P}$-intersection homotopy groups follow from these results. Let us list some of them.

- In the case of a cone on a compact space, $\mathcal{C}X$, of apex $v$, with the conical stratification, $\{v\} \subset \mathcal{C}X$, one has (Proposition 3.5)

$$\mathcal{P}_X(\mathcal{C}X) = P_k\text{Sing}X,$$

where $P_k$ denotes the $k$th space of the Postnikov decomposition of $\text{Sing}X$ and $k = D\mathcal{P}(v)$. In this particular case, the Gajer space appears as an Eckmann-Hilton dual ([22, Section 4.H]) of Banagl’s intersection space, $\mathcal{P}X$: the (co)truncation of the Moore decomposition is replaced by the truncation of the Postnikov tower. In [8], we come back to this point of view and extend this relation to more general spaces than cones.

- By using an adapted version of the PL forms of D. Sullivan, we present in [5] a notion of perverse minimal model. In a future work, we will connect the indecomposables of this model with the intersection homotopy groups. This will bring a new notion of $\mathcal{P}$-formality for a fixed GM-perversity $\mathcal{P}$.

- Let’s extend this list with: “Can we relate the perverse Eilenberg-MacLane spaces introduced in [10] with the intersection homotopy groups?” and “Is there a relation to perverse sheaves?”

- Let us conclude with a question about small resolutions, which has its source in [18, Section 6.2]. Recall that a small resolution of a complex algebraic variety is an algebraic map, $f: V \to X$, which is a resolution of singularities of $X$ such that, for all $r > 0$,

$$\text{codim}_C \{x \in X \mid \text{dim}_C f^{-1}(x) \geq r\} > 2r.$$

In [18], the authors prove that a small resolution $f$ induces isomorphisms, $f_*: H_*(V) \cong H^\mathcal{P}_*(X)$, between the (ordinary) homology of $V$ and the intersection homology of $X$ for the middle perversity $\mathcal{P}$. When they do exist, small resolutions are not necessarily unique and the previous result implies that the homology groups $H_*(V)$ do not depend on the small resolution of $X$. Thus we ask if there are similar results for the intersection homotopy groups. For instance, under some connectedness conditions, can we find an
integer $k$ such that the homotopy groups $\pi_j(V)$ do not depend on the small resolution $V$, for $j \leq k$. More precisely, we make the following conjecture.

**Conjecture.** Let $f : V \to X$ be a small resolution. Suppose that $V$ and $X$ are simply connected and that there exists an integer $k$ such that $\pi_j(L) = 0$ for all links and all $j < k$. Then, we have an isomorphism

$$f_* : \pi_j(V) \to \pi_j^{\text{tr}}(X), \quad \text{for all} \quad j \leq k.$$  

In the case of isolated singular points of a local complete intersection of complex dimension $n$, this conjecture implies the existence of isomorphisms, $\pi_j(V) \cong \pi_j^{\text{tr}}(X)$, for any $j \leq n - 1$ and any small resolution $V$ of $X$.

**Notation and convention.** The word “space” means compactly generated topological space. When we work on CS sets that are locally path-connected spaces, we use “connected” and “path-connected” interchangeably. In the text, the letter $R$ denotes a Dedekind ring and $G$ an $R$-module. The singular chain complex of a simplicial set, $X$, is denoted by $C_*(X; G)$, $C_*(X; \mathcal{E})$ or $C_*(X)$ if there is no ambiguity.

We denote by $\Delta[\ell]$ the simplicial set whose $k$-simplexes are the $(k+1)$-uple of integers $(j_0, \ldots, j_k)$ with $0 \leq j_0 \leq \cdots \leq j_k \leq \ell$, and by $d_i : \Delta[\ell]_k \to \Delta[\ell]_{k-1}$, $s_i : \Delta[\ell]_k \to \Delta[\ell]_{k+1}$, its faces and degeneracies, for $i \in \{0, \ldots, k\}$. Its geometric realization is the subspace of $\mathbb{R}^{\ell+1}$ defined by $\Delta^\ell = \{(t_0, \ldots, t_\ell) \mid \sum_{i=0}^{\ell} t_i = 1, t_i \geq 0\}$. We denote by $\Delta' = \Delta^\ell \setminus \partial \Delta^\ell$ the interior of $\Delta^\ell$. The $k$th horn, $\Lambda[\ell, k]$, of $\Delta[\ell]$ is obtained from the boundary $\partial \Delta[\ell]$ by removing the $k$th face. Its geometric realization is denoted by $\Lambda^\ell_k$.

The family $\Delta^e$ is a cosimplicial space with cofaces and codegeneracies, $d^i : \Delta^{e-1} \to \Delta^{e}$ and $s^i : \Delta^{e+1} \to \Delta^{e}$, for $i \in \{0, \ldots, \ell\}$. The image $d^i \Delta^{e-1}$ is called the $i$-face of $\Delta^{e}$ and denoted by $\partial_i \Delta^{e}$. If $\sigma : \Delta^{e} \to X$ is a singular simplex the precomposition of $\sigma$ with $d^i$ is denoted by $\partial_i \sigma = \sigma \circ d^i$. The domain of $\sigma$ is denoted by $\Delta^{e}$ or $\Delta_\sigma$, or simply $\Delta$, depending on the parameter concerned at this place.

Our program is carried out in Sections 1-6 below, whose headings are self-explanatory.

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1. **Background on filtered spaces and intersection homology**

The singular spaces considered by Goresky and MacPherson in their pioneer works ([17, 18]), are *pseudomanifolds*. They allow a Poincaré duality through intersection homology, based on a parameter called perversity. The singular spaces presented in this section are general filtered spaces and the CS sets of Siebenmann ([33]) well adapted to proofs by induction. We also make use of perversities defined on the poset of strata.
1.1. Filtered spaces and CS sets. We denote by Top the category of compactly generated spaces (henceforth called “space”) with arrows the continuous maps. Let us emphasize that our definition of compact space includes the Hausdorff property. We now enter in the filtered world.

**Definition 1.1.** A filtered space of formal dimension $n$ is a non-empty space $X$ with a filtration,

$$X_{-1} = \emptyset \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{n-2} \subseteq X_{n-1} \subseteq X_n = X,$$

by closed subsets. The subspaces $X_i$ are called skeleta of the filtration and the non-empty components of $X^i = X_i \setminus X_{i-1}$ are the strata of $X$. The set of strata is denoted by $S_X$ (or $S$ if there is no ambiguity).

The subspace $\Sigma_X = X_{n-1}$ (or $\Sigma$) is called the singular subspace. Its complementary, $X \setminus \Sigma_X$, is the regular subspace and the strata in $X \setminus \Sigma_X$ are called regular. The formal dimension of a stratum $S \subseteq X^i$ is $\dim S = i$; its formal codimension is $\text{codim} S = n - i$. (The formal dimension is not necessarily related to any topological notion of dimension.) The filtered space is of locally finite stratification if every point has a neighborhood that intersects only a finite number of strata.

A stratified space is a filtered space such that the closure of any stratum is a union of strata of lower dimension. The set of strata of a stratified space is a poset, $(S, \prec)$, for the relation $S_0 \prec S_1$ if $S_0 \subset \overset{\smile}{S}_1$. The depth of $X$ is the greatest integer $m$ for which it exists a chain of strata, $S_0 \prec S_1 \prec \cdots \prec S_m$. We denote it by $\text{depth} X = m$.

Let $X$ be a filtered space of formal dimension $n$. An open subset $U \subset X$ is a filtered space for the induced filtration given by $U_i = U \cap X_i$. The product $Y \times X$ with a space $Y$ is a filtered space for the product filtration defined by $(Y \times X)_i = Y \times X_i$. If $X$ is compact, the open cone $\overset{\smile}{x} X = X \times \{0\} / X \times \{0\}$ is endowed with the conical filtration defined by $(\overset{\smile}{x} X)_i = \overset{\smile}{x} X_{i-1}$, $0 \leq i \leq n+1$. By convention, $\overset{\smile}{x} \emptyset = \{v\}$, where $v = [-1, 0]$ is the apex of the cone.

**Definition 1.2.** A stratified map $f : X \to Y$ between two filtered spaces is a continuous map such that for each stratum $S$ of $X$ there exists a stratum $S^f$ of $Y$ with $f(S) \subset S^f$ and $\text{codim} S^f \leq \text{codim} S$. A stratified homeomorphism is a homeomorphism such that $f$ and $f^{-1}$ are stratified maps. We denote this relation by $X \simeq_s Y$.

In this text, we will frequently encounter stratified maps $f$ which are homeomorphisms but for which the reverse application $f^{-1}$ is not stratified. We call them homeomorphisms and stratified maps. It is fundamental not to confuse them with stratified homeomorphisms.

**Definition 1.3.** Let $X, Y$ be filtered spaces, we endow the cylinder $X \times [0, 1]$ with the product filtration, $(X \times [0, 1])_i = X_i \times [0, 1]$. Two stratified maps, $f, g : X \to Y$, are stratified homotopic if there exists a stratified map $H : X \times [0, 1] \to Y$ such that $H = H(-, 0)$ and $g = H(-, 1)$. We denote this relation by $f \simeq_s g$.

Two filtered spaces, $X$ and $Y$, are stratified homotopy equivalent if there exist stratified maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq_s \text{id}_X$, $f \circ g \simeq_s \text{id}_Y$ and if $\text{codim} S = \text{codim} S^f$, $\text{codim} T = \text{codim} T^g$, for any strata $S$ of $X$ and $T$ of $Y$. We denote this relation by $X \simeq_s Y$ and the maps $f$ and $g$ are called stratified homotopy equivalences.

A stratified homotopy equivalence, $f : X \to Y$, induces a bijection between the two sets of strata, $S_X \cong S_Y$, and verifies $S^{g \circ f} = S$, $T^{f \circ g} = T$ see [12, Remark 2.9.11]. Let us end these recalls with the CS sets, introduced by Siebenmann in [33].

**Definition 1.4.** A CS set of dimension $n$ is a filtered space $X$ of dimension $n$, whose $i$-dimensional strata are $i$-dimensional topological manifolds for each $i$, and such that for each point $x \in X_i \setminus X_{i-1}$, $i \neq n$, there exist an open neighborhood $V$ of $x$ in $X$, endowed with the induced filtration, an open neighborhood $U$ of $x$ in $X_i \setminus X_{i-1}$, a compact filtered space, $L$, of dimension $n - i - 1$, and a stratified homeomorphism $\varphi : U \times \overset{\smile}{x} L \to V$ such that

$$\varphi(U \times \overset{\smile}{x} L_i) = V \cap X_{i+j+1}, \quad (1.1)$$
for each \( j \in \{0, \ldots, n - i - 1\} \). If \( v \) is the apex of the cone \( \hat{c}L \), the homeomorphism \( \varphi \) is also required to verify \( \varphi(u, v) = u \), for each \( u \in U \). The pair \((V, \varphi)\) is a conical chart of \( x \) and the filtered space \( L \) is a link of \( x \). The CS set \( X \) is called normal if its links are connected.

From the definition, we see that a CS set has a topological dimension. Moreover, in a CS set, the notions of formal and topological dimension coincide.

**Remark 1.5.** If \( X \) is a manifold with a structure of CS set, then any link of \( X \) is a homology sphere. So, if the CS set \( X \) has no codimensional 1 strata, it is normal. Let us see that.

Given a link \( L \) we have a stratified homeomorphism \( \varphi: \mathbb{R}^k \times \hat{c}L \to V \), where \( V \) is an open subset of \( X \). Without loss of generality, we can suppose that \( V \) is homeomorphic to an open subset of \( \mathbb{R}^n \). So, we have a topological embedding \( \psi: \mathbb{R}^k \times \hat{c}L = \hat{c}(S^{k-1} \ast L) \to \mathbb{R}^n = cS^{n-1} \) preserving the apexes. Following Stalling’s invertible cobordism, we get a homeomorphism \( \hat{c}(S^{k-1} \ast L) = cS^{n-1} \) preserving the apexes ([23, Proposition 1]) and thus \( L \) is a homology sphere.

There can be some differences in the various definitions of CS sets in the literature. In Definition 1.4, the links of singular strata are supposed to be non-empty, thus the open subset \( X_n \setminus X_{n-1} \) is dense. This implies that, for each link \( L \), the regular part, \( L \setminus L \), is dense in \( L \). The links of a stratum are not uniquely defined but if one of them is connected, so all of them are too ([12, Remark 2.6.2]). A CS set is a stratified space ([5, Appendix A2]) and its strata are locally closed and form a locally finite family ([12, Lemma 2.3.8]).

**Remark 1.6.** Proceeding as in [12, Lemma 2.6.3], one can prove that a normal CS set \( X \) is connected if, and only if, its regular part, \( X \setminus \Sigma \), is connected. We can also prove that \( X \) is normal if, and only if, the regular part \( L \setminus \Sigma_L \) of any link \( L \) is connected.

1.2. Perversities. Intersection homology of Goresky and MacPherson is defined from a parameter, called perversity. The original ones introduced in [17] depend only on the codimension of the strata. More general perversities ([24, 12, 30, 31]) are defined on the set of strata. In [4], a blown-up cohomology is defined in this setting, to establish a Poincaré duality for pseudomanifolds with a cup product in [6, 32]. Let us recall their definitions.

**Definition 1.7.** A perversity on a filtered space, \( X \), is a map \( \overline{\varphi}: S_X \to \mathbb{Z} = \mathbb{Z} \cup \{\pm \infty\} \) taking the value 0 on the regular strata. The pair \((X, \overline{\varphi})\) is called a perverse space, or a perverse CS set if \( X \) is a CS set.

A constant perversity \( \underline{k} \), with \( k \in \mathbb{Z} \), is defined by \( \underline{k}(S) = k \) for any singular stratum \( S \). The top perversity \( \overline{T} \) is defined by \( \overline{T}(S) = \text{codim} S - 2 \), if \( S \) is a singular stratum. Given a perversity \( \overline{\varphi} \) on \( X \), the complementary perversity on \( X \), \( D\overline{\varphi} \), is characterized by \( D\overline{\varphi} + \overline{\varphi} = \overline{T} \).

Any map \( f: \mathbb{N} \to \mathbb{Z} \) such that \( f(0) = 0 \) defines a perversity \( \overline{\varphi} \) by \( \overline{\varphi}(S) = f(\text{codim} S) \). Such perversity is called codimensional. In general, we denote by the same letter the perversity \( \overline{\varphi} \) and the map \( f \). Among the codimensional perversities we find the original perversities of [17].

**Definition 1.8.** A Goresky and MacPherson perversity (or GM-perversity) is a map \( \overline{\varphi}: \{2, 3, \ldots\} \to \mathbb{N} \) such that \( \overline{\varphi}(2) = 0 \) and \( \overline{\varphi}(i) \leq \overline{\varphi}(i + 1) \leq \overline{\varphi}(i) + 1 \) for all \( i \geq 2 \).

When using a GM-perversity, the filtered spaces under consideration have no 1-codimensional strata. If \( \overline{\varphi} \) is a GM-perversity, so is its complementary.

**Definition 1.9.** Let \( f: X \to Y \) be a stratified map and \( \overline{\varphi} \) be a perversity on \( Y \). The pullback perversity of \( \overline{\varphi} \) by \( f \) is the perversity \( f^*\overline{\varphi} \) on \( X \) defined by \( f^*\overline{\varphi}(S) = \overline{\varphi}(f(\text{top}(S))) \) for any stratum \( S \) of \( X \). In the case of the canonical injection of an open subset endowed with the induced filtration, \( \iota: U \to Y \), we still denote by \( \overline{\varphi} \) the perversity \( \iota^*\overline{\varphi} \) and call it the induced perversity.

The product with a topological space, \( X \times M \), is endowed with the pull-back perversity of \( \overline{\varphi} \) by the canonical projection \( X \times M \to X \), also denoted by \( \overline{\varphi} \). If \( X \) is compact, a perversity \( \overline{\varphi} \) on the open cone \( \hat{c}X \) induces a perversity on \( X \), also denoted by \( \overline{\varphi} \) and defined by \( \overline{\varphi}(S) = \overline{\varphi}(S \times [0, 1]) \).
Remark 1.10. A perversity $\overline{\mathcal{p}}$ defined on a CS set, $X$, induces a perversity on any link $L$. Let $\varphi : \mathbb{R}^k \times cL \to V$ be a conical chart of $x \in S \in X$. Then, $V \setminus S$ is an open subset of $X$ and is given with the induced perversity still denoted by $\overline{\mathcal{p}}$. The strata of $V \setminus S$ are the products $\mathbb{R}^k \times [0,1] \times T$, where $T$ is a stratum of $L$. We set $\overline{\mathcal{p}}(T) = \overline{\mathcal{p}}(\mathbb{R}^k \times [0,1] \times T)$ which defines a perversity on $L$. There is also a relation between the perversity on $S$ itself and the conical chart by setting $\overline{\mathcal{p}}(v) = \overline{\mathcal{p}}(S)$.

1.3. Intersection homology. Let $(X, \overline{\mathcal{p}})$ be an $n$-dimensional perverse space. The starting point in intersection homology is the use of the perversity for making a selection among singular simplexes.

Before stating it, we specify the notion of dimension we are using for subspaces of a Euclidean simplex. We employ the polyhedra containing it (see [29] for the used properties on polyhedra).

Definition 1.11. A subspace $A \subset \Delta$ of a Euclidean simplex is of polyhedral dimension less than or equal to $\ell$ if $A$ is included in a polyhedron $Q$ with $\dim Q \leq \ell$. This definition verifies

$$\dim(A_1 \cup A_2) = \max(\dim A_1, \dim A_2).$$

We choose this definition and do a selection among singular simplexes. As it appears below, the allowability condition is usually most conveniently expressed in terms of the complementary perversity, $D\overline{\mathcal{p}}$, rather than $\overline{\mathcal{p}}$.

Definition 1.12. Let $(X, \overline{\mathcal{p}})$ be a perverse space. A simplex $\sigma : \Delta \to X$ is $\overline{\mathcal{p}}$-allowable if, for each singular stratum $S$, the set $\sigma^{-1}S$ verifies

$$\dim \sigma^{-1}S \leq \Delta - \codim S + \overline{\mathcal{p}}(S) = \Delta - 2 - D\overline{\mathcal{p}}(S),$$

with the convention $\dim \emptyset = -\infty$. A singular chain $\xi$ is $\overline{\mathcal{p}}$-allowable if it can be written as a linear combination of $\overline{\mathcal{p}}$-allowable simplexes.

In Remark 1.14, we illustrate this definition in low dimensions with a perversity such that $D\overline{\mathcal{p}} \geq 0$, which is the case of the GM-perversities.

As explained in the introduction, we need a more restrictive choice than $\overline{\mathcal{p}}$-allowability for obtaining a chain complex.

Definition 1.13. A singular chain $\xi$ is of $\overline{\mathcal{p}}$-intersection if $\xi$ and its boundary $\partial \xi$ are $\overline{\mathcal{p}}$-allowable. We denote by $C^\bullet(X; G)$ the complex of singular chains of $\overline{\mathcal{p}}$-intersection and by $H^\bullet(X; G)$ its homology, called $\overline{\mathcal{p}}$-intersection homology of $X$ with coefficients in a module $G$ over a Dedekind ring $R$.

If $f : (X, \overline{\mathcal{p}}) \to (Y, \overline{\mathcal{p}})$ is a stratified map between two perverse spaces with $f^*D\overline{\mathcal{p}} \leq D\overline{\mathcal{p}}$, the association $\sigma \mapsto f \circ \sigma$ sends a $\overline{\mathcal{p}}$-allowable simplex on a $\overline{\mathcal{p}}$-allowable simplex ([9, Proposition 1]) and defines a chain map $f_* : C^\bullet(X) \to C^\bullet(Y)$.

Remark 1.14. In the original definition of King ([23]), the dimension chosen for a subspace of a Euclidean simplex comes from the dimension of the skeleta containing it. In [9], we show that the two choices of dimension give isomorphic intersection homology for CS sets. The proof needs a Mayer-Vietoris exact sequence and therefore a small chains replacement. As usual, this property comes from a subdivision process but, with the choice of the polyhedra dimension in Definition 1.11, that is a subtle issue.

Let us take basic examples with a perversity $\overline{\mathcal{p}}$, such that $D\overline{\mathcal{p}} \geq 0$. A 0-simplex and a 1-simplex are $\overline{\mathcal{p}}$-allowable if, and only if, $\sigma^{-1}S = \emptyset$ for any singular stratum $S$; this means that they must lie in the regular part. For a 2-simplex, the situation is different: the set $\sigma^{-1}S$ can be a finite subset. Let us consider a 2-simplex $\Delta^2$ with one singular point, $v$, in its interior, which is thus $\overline{\mathcal{p}}$-allowable. Now we want to subdivide $\Delta^2$ while keeping the $\overline{\mathcal{p}}$-allowability condition for the small 2-simplexes of the subdivision. To achieve this, the point $v$ must not belong to any 1-simplex of the subdivision.

Therefore, the subdivision of a $\overline{\mathcal{p}}$-allowable simplex $\Delta^1$, that we need, cannot be the barycentric subdivision but a subdivision, called pseudobarycentric subdivision, which verifies the following properties:
• a diameter of the new simplexes strictly smaller than that of the initial simplex,
• the preservation of $\mathcal{P}$-allowability,
• a construction by induction, taking the cone with vertex a suitably chosen interior point $b$ in $\Delta^\ell$ and base the pseudobarycentric subdivision of the boundary of $\Delta^\ell$. The point $b$ is called a pseudobarycentre of $\Delta^\ell$.

In [9, Proposition 6], we show that such subdivisions exist for any $\mathcal{P}$-allowable simplex. In this work we only use the properties recalled above.

2. Gajer spaces

Let $(X, \mathcal{P})$ be an $n$-dimensional perverse space. We emphasize that in the definition of intersection homology (see Section 1.3), we do not consider simplexes but chains, $\xi$, and we ask for the allowability of $\xi$ and its boundary $\partial\xi$. Now we want to construct a simplicial set, thus the requirements concern the simplexes and all their faces.

**Definition 2.1.** Let $(X, \mathcal{P})$ be a perverse space. A simplex $\sigma: \Delta^\ell \to X$ is $\mathcal{P}$-full if $\sigma$ and all its faces, $\partial_1 \sigma, \ldots, \partial_n \sigma$, are $\mathcal{P}$-allowable.

**Remark 2.2.** Similarly, a simplex $\sigma: \Delta^\ell \to X$ is $\mathcal{P}$-full if, for all faces $F$ of $\Delta$ and all singular stratum $S$, one has $\dim((\sigma^{-1} S \cap F)) \leq \dim F - D_{\mathcal{P}}(S) - 2$.

We continue with basic properties. Let us begin with the following result, crucial for the definition of $\mathcal{P}$-intersection homotopy groups.

**Proposition 2.3.** ([14, Page 946]). Let $(X, \mathcal{P})$ be a perverse space. Then the set of $\mathcal{P}$-full simplexes is a simplicial set verifying the Kan condition. We denote it by $\mathcal{G}_F X$ and call it the Gajer $\mathcal{P}$-space associated to $X$.

**Proof.** Let $\sigma: \Delta^\ell \to X$ be a $\mathcal{P}$-full simplex and $S$ be a singular stratum. By definition, the face $\partial_k \sigma$ is $\mathcal{P}$-full for any face of $\sigma$. Let $s^i: \Delta^{\ell+1} \to \Delta^\ell$ be a codegeneracy and $d^j: \Delta^\ell \to \Delta^{\ell+1}$ be a coface. The commutator rules express $\sigma \circ s^i \circ d^j$ as a composition $\sigma \circ d^j \circ s^i$. By induction on the dimension $\ell$, we deduce that $\sigma \circ s^i$ is $\mathcal{P}$-full. Thus the set of $\mathcal{P}$-full simplexes is a simplicial subset of the singular set $\text{Sing} X$. By using the adjunction between the functor $\text{Sing}$ and the realization functor, the Kan extension condition is equivalent to the construction of a $\mathcal{P}$-full simplex $\tau$ making commutative the following diagram,

\[
\begin{array}{ccc}
\Lambda_k & \xrightarrow{\Psi} & X, \\
\downarrow \sigma & & \\
\Delta^\ell & \xrightarrow{\tau} & \\
\end{array}
\]

for any $0 \leq k \leq \ell$. If $\rho_k: \Delta^\ell \to \Lambda_k$ denotes the radial projection from the barycentre of the face $\partial_k \Delta^\ell$, we set $\tau = (\Psi \circ \rho_k)^{-1}$. For any singular stratum $S$, we have

\[
\dim((\Psi \circ \rho_k)^{-1} S) \leq \dim((\Psi)^{-1} S) + 1 \leq (\ell - 1 - D_{\mathcal{P}}(S) - 2) + 1 = \ell - D_{\mathcal{P}}(S) - 2,
\]

and the $\mathcal{P}$-allowability condition of $\tau$ is satisfied. As $\rho_k$ is an affine map, the same argument works for any face of $\Delta^\ell$.

The Gajer space depends on a filtration $\mathcal{F}$ on the topological space $X$ and we should denote it by $\mathcal{G}_F(X, \mathcal{F})$. In fact, we do not mention explicitly the filtration and simply write $\mathcal{G}_F X$.

**Proposition 2.4.** ([14, Example 1]) Let $(X, \mathcal{P})$ be a perverse space and $Y$ be a topological space. The product $Y \times X$ is equipped with the product filtration and the product perversity, also denoted by $\mathcal{P}$. Then, the canonical projections, $p_Y: Y \times X \to Y$ and $p_X: Y \times X \to X$, induce an isomorphism

\[\mathcal{G}_\mathcal{P}(Y \times X) \cong \text{Sing} Y \times \mathcal{G}_\mathcal{P} X.\]
Proof. Let \( \sigma = (\sigma_Y, \sigma_X) : \Delta^1 \to Y \times X \) be a simplex. By definition, \( \sigma \) is \( \mathfrak{p} \)-full if, and only if, \( \sigma_X \) is \( \mathfrak{p} \)-full also. The projections therefore induce an isomorphism \( \sG_{\mathfrak{p}}(Y \times X) \cong \text{Sing} Y \times \sG_{\mathfrak{p}}X \).

The following statement concerns the functoriality of the association \( (X, \mathfrak{p}) \mapsto \sG_{\mathfrak{p}}X \) and its compatibility with homotopy.

**Proposition 2.5.** Let \( f : (X, \mathfrak{p}) \to (Y, \mathfrak{q}) \) be a stratified map between perverse spaces such that \( f^* D\mathfrak{q} \leq D\mathfrak{p} \). Then, \( f \) induces a map \( \sG_{\mathfrak{p}}f : \sG_{\mathfrak{p}}X \to \sG_{\mathfrak{q}}Y \). Moreover, if \( \varphi : (X \times [0,1], \mathfrak{p}) \to (Y, \mathfrak{q}) \) is a stratified homotopy between two stratified maps \( f, g : (X, \mathfrak{p}) \to (Y, \mathfrak{q}) \) with \( f^* D\mathfrak{q} \leq D\mathfrak{p} \), then we also have \( g^* D\mathfrak{q} \leq D\mathfrak{p} \) and the simplicial maps, \( \sG_{\mathfrak{p}}f \) and \( \sG_{\mathfrak{p}}g \), are homotopic.

Proof. As we have already noticed, the association \( \sigma \mapsto f \circ \sigma \) sends a \( \mathfrak{p} \)-allowable simplex on a \( \mathfrak{p} \)-allowable simplex. Let \( S \) be a stratum of \( X \). Recall that the perversity on \( X \times [0,1] \) is still denoted by \( \mathfrak{p} \) and defined by \( \mathfrak{p}(S \times [0,1]) = \mathfrak{p}(S) \). From the stratification of the product \( X \times [0,1] \), we have \( S^f = (S \times [0,1])^\varphi = S^\varphi \). Thus, \( f^* D\mathfrak{q}(S) = D\mathfrak{q}(S^f) = D\mathfrak{q}(S \times [0,1])^\varphi = \varphi^* D\mathfrak{q}(S \times [0,1]) \) and, in the same way, \( g^* D\mathfrak{q}(S) = \varphi^* D\mathfrak{q}(S \times [0,1]) \). Thus the three conditions \( f^* D\mathfrak{q} \leq D\mathfrak{p} \), \( g^* D\mathfrak{q} \leq D\mathfrak{p} \) are equivalent and verified by hypothesis. The associations \( \sigma \mapsto f \circ \sigma, g \circ \sigma, \varphi \circ \sigma \) being simplicial maps, the maps \( f, g \) and the homotopy \( \varphi \) induce simplicial maps \( \sG_{\mathfrak{p}}f, \sG_{\mathfrak{p}}g, \sG_{\mathfrak{p}}\varphi : \sG_{\mathfrak{p}}X \to \sG_{\mathfrak{q}}Y \) and \( \sG_{\mathfrak{p}}f, \sG_{\mathfrak{p}}g, \sG_{\mathfrak{p}}\varphi : \sG_{\mathfrak{p}}(X \times [0,1]) \to \sG_{\mathfrak{q}}Y \) using \( \text{Proposition 2.4} \). We get the desired homotopy as the composition

\[
\sG_{\mathfrak{p}}X \times \Delta[1] \to \sG_{\mathfrak{p}}X \times \text{Sing}(\Delta^1) \to \sG_{\mathfrak{p}}(X \times \Delta^1) \xrightarrow{f \circ \sigma} \sG_{\mathfrak{q}}Y.
\]

In particular, Definition 1.3 and Proposition 2.5 imply the following result.

**Corollary 2.6.** Let \( f : (X, \mathfrak{p}) \to (Y, \mathfrak{q}) \) be a stratified homotopy equivalence with \( D\mathfrak{q} = f^* D\mathfrak{q} \). Then \( \sG_{\mathfrak{p}}f : \sG_{\mathfrak{p}}X \to \sG_{\mathfrak{q}}Y \) is a homotopy equivalence.

Proof. Let \( S \) be a stratum of \( X \). By hypothesis, there exists a stratified map \( g : (Y, \mathfrak{q}) \to (X, \mathfrak{p}) \) and a stratified homotopy \( \varphi \) between \( g \circ f \) and \( \text{id}_X \). From the stratification of the product \( X \times [0,1] \), we deduce \( S^{g \circ f} = (S \times [0,1])^\varphi = S^\varphi = S \). Similarly, for any stratum \( T \) of \( Y \), we have \( T^f = T \). These equalities and the hypothesis \( \sG_{\mathfrak{p}}f \) and \( \sG_{\mathfrak{p}}g \) give \( g^* D\mathfrak{q} = D\mathfrak{p} \). With Proposition 2.5, we get two simplicial maps, \( \sG_{\mathfrak{p}}f : \sG_{\mathfrak{p}}X \to \sG_{\mathfrak{q}}Y \) and \( \sG_{\mathfrak{p}}g : \sG_{\mathfrak{q}}Y \to \sG_{\mathfrak{p}}X \) and homotopies between their compositions and the identity maps.

Let us look at the particular case of isolated singular points.

**Proposition 2.7.** Let \((X, \mathfrak{p})\) be a perverse space with an isolated singularity \( x \). Set \( \ell = D\mathfrak{p}(\{x\}) + 1 \). Then, there is an isomorphism between the \( \ell \)-skeleta of \( \sG_{\mathfrak{p}}X \) and \( \sG_{\mathfrak{p}}(X\setminus\{x\}) \).

Proof. The \( \mathfrak{p} \)-allowability condition of a simplex \( \sigma : \Delta^1 \to X \) for a singular stratum \( \{x\} \) is:

\[
\dim \sigma^{-1}x \leq j - D\mathfrak{p}(\{x\}) - 2. \tag{2.1}
\]

Thus, for any \( j < D\mathfrak{p}(\{x\}) + 1 \), the inequality (2.1) gives \( \dim \sigma^{-1}x \leq -1 \). This implies \( \sigma^{-1}x = \emptyset \) and an isomorphism of the \( j \)-skeleta \( (\sG_{\mathfrak{p}}X)^j \cong (\sG_{\mathfrak{p}}(X\setminus\{x\}))^j \).

Let \( K \) be a simplicial set and \( \Pi K \) be its fundamental groupoid. Let us recall that a local coefficient system (of abelian groups) on \( K \) is a contravariant functor \( \mathcal{E} : \Pi K \to \text{Ab} \), with values in the category of abelian groups, see [13, Appendix I] or [16, Page 340]. Such functor generates a chain complex \( C_\ell(K; \mathcal{E}) \) whose homology is called the homology of \( K \) with coefficients in the local system \( \mathcal{E} \). The elements of \( C_j(K; \mathcal{E}) \) are chains \( \xi = \sum_{i \in I} a_i \sigma_i \) with \( \sigma_i \in K_j \) and \( a_i \in \mathcal{E}(\sigma_i(0)) \). The differential of \( a \sigma \), with \( \sigma \in K_j \) and \( a \in \mathcal{E}(\sigma(0)) \), is given by

\[
\partial(a \sigma) = \sum_{i=1}^j (-1)^i a \partial_i \sigma + \mathcal{E}(\sigma([01]))^{-1}(a) \partial_j \sigma.
\]
Isomorphisms between homotopy groups of simplicial sets (or spaces) can be detected from the existence of isomorphisms between fundamental groups and isomorphisms of homology groups for any local coefficient system ([26, Proposition II.3.4]).

Lemma 2.8. For a simplicial map $f : K \to K'$, the following conditions are equivalent.

1. The map $f$ is a weak equivalence.
2. The map $f$ induces isomorphisms, $\pi_0(K) \cong \pi_0(K')$, $\pi_1(K, x) \cong \pi_1(K', f(x))$ for any $x \in K_0$, $H_j(K'; \mathcal{E}) \cong H_j(K; f^*\mathcal{E})$, for any local coefficient system $\mathcal{E}$ on $K'$ and any $j$.

We also use the next well-known slight modification.

Lemma 2.9. Let $f : K \to K'$ be a simplicial map between Kan simplicial sets whose induced maps verify the following properties: $\pi_0(K) \cong \pi_0(K')$, $\pi_1(K, x) \cong \pi_1(K', f(x))$ for any $x \in K_0$, $H_j(K; f^*\mathcal{E}) \cong H_j(K'; \mathcal{E})$, for any local coefficient system $\mathcal{E}$ on $K'$, any $j \leq \ell$ and $H_{\ell+1}(K; f^*\mathcal{E}) \to H_{\ell+1}(K'; \mathcal{E})$ is a surjection. Then, the map induced by $f$ between the homotopy groups is an isomorphism for $j \leq \ell$ and a surjection for $j = \ell + 1$.

Proof. Let $\tilde{K} \to K$ be the universal cover of $K$. The key observation in the proof given by Quillen ([26]) is the degeneracy of the Serre spectral sequence, $H_j(K, p^*\mathbb{Z}) \cong H_j(\tilde{K}; \mathbb{Z})$, for coefficients $\mathbb{Z}$ on $\tilde{K}$ and the induced local coefficient system $p^*\mathbb{Z}$ on the homology of the fibres. The result follows from the classic Whitehead theorem applied to the universal covers. □

Proposition 2.10. Let $X$ be a compact filtered space and $\mathcal{E}X$ be the open cone, of apex $v$, with the conic filtration. Let $\mathcal{P}$ be a perversity on $\mathcal{E}X$, we also denote by $\mathcal{P}$ the perversity induced on $X$. Then, for any local coefficient system $\mathcal{E}$ on $\mathcal{E}X$, we have

$$H_j(\mathcal{P}(\mathcal{E}X), \mathcal{P}X; \mathcal{E}) = 0 \quad \text{for any} \quad j \leq D(\mathcal{P}(v)) + 1.$$ (2.2)

Proof. We apply Proposition 2.7 to $\mathcal{E}X$, its apex $v$ and $\ell = D(\mathcal{P}(v)) + 1$. The equality (2.2) follows from the isomorphism between the $\ell$-skeleta, $(\mathcal{P}(\mathcal{E}X))^\ell \cong (\mathcal{P}(\mathcal{E}X\backslash\{v\}))^\ell$, Proposition 2.4 and the long exact homology sequence of the pair $(\mathcal{P}(\mathcal{E}X), \mathcal{P}X)$. □

For sake of simplicity let us take $\mathbb{Z}$ as coefficients. The determination done in Proposition 2.10 for the homology of $\mathcal{P}(\mathcal{E}X)$ gives a result of different spirit than the $\mathcal{P}$-intersection homology of a cone, $H^\mathcal{P}_j(\mathcal{E}X)$. If $j \leq D(\mathcal{P}(v))$, this last homology verifies $H^\mathcal{P}_j(\mathcal{E}X) \cong H^\mathcal{P}_j(X)$, which is similar to (2.2). But we also have $H^\mathcal{P}_j(\mathcal{E}X) = 0$ if $j > D(\mathcal{P}(v))$, see [9, Proposition 3] or [7, Proposition 5.2]. This last condition is not true for the homology of the Gajer space on a cone.

Before giving concrete examples, let’s analyze the difference between the elements of $C^\mathcal{P}(Y)$ and those of $C_r(\mathcal{E}Y)$ for a perverse space $(Y, \mathcal{P})$. Let $\xi = \sum_i r_i \sigma_i$ be a singular chain on $Y$. For having $\xi \in C_r(\mathcal{E}Y)$, the allowability condition has to be satisfied for all the faces of all $\sigma_i$. In contrast, in $C^\mathcal{P}(Y)$ there is no requirement on the faces that cancel out in the boundary $\partial \xi$. As already observed in [15], the homology of $\mathcal{E}Y$ is not isomorphic to the $\mathcal{P}$-intersection homology. To see that, let $Y = \mathcal{E}(S^2 \times S^3)$ with the perversity determined by the value $\mathcal{P}(v) = D(\mathcal{P}(v)) = 2$ on the apex. Using the determination (Proposition 3.5) of the homotopy groups of the Gajer space of a cone, the only non-zero intersection homotopy group is $\pi_2(\mathcal{E}Y) = \mathbb{Z}$. Therefore, the space $\mathcal{E}Y$ is the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$. The homology of $\mathcal{E}Y$ does not fit with the $\mathcal{P}$-intersection homology groups of $Y$ which are zero in degrees strictly greater than 2. We continue with other examples of this feature which do not need the use of $\mathcal{P}$-intersection homotopy groups.

Example 2.11. Let $L$ be an oriented $\ell$-dimensional connected compact manifold verifying $\pi_\ell(L) = 0$ (as the torus for $\ell = 2$), endowed with the GM-perversity $\mathcal{P}$. Below, we prove:

$$H^\mathcal{P}_\ell(\mathcal{E}L; \mathbb{Z}) = 0 \neq H_\ell(L; \mathbb{Z}) = H_\ell(\mathcal{E}(\mathcal{P}(\mathcal{E}L); \mathbb{Z}).$$
Let us see that. The dual perversity of $\overline{0}$ being $D\overline{0} = \overline{t - 1}$, we deduce from the cone formula for intersection homology $H^\ell_!(\mathbb{L}; \mathbb{Z}) = 0$. The second part of the claim holds if we prove that the natural inclusion induces an isomorphism $H_\ell(L; \mathbb{Z}) \to H_\ell(\mathfrak{I}(\mathbb{L}); \mathbb{Z})$. This map is an epimorphism since, at the level of chain complexes, we have (see Proposition 2.7) $C^\ell_{\leq 1}(L \times [0, 1]; \mathbb{Z}) = C^\ell_{\leq 1}(\mathfrak{I}(\mathbb{L}); \mathbb{Z})$.

Let $[\alpha] \in H_\ell(L; \mathbb{Z})$ be the orientation class of $L$. We reason by the absurd and suppose $[\alpha] = 0$ in $H_\ell(\mathfrak{I}(\mathbb{L}); \mathbb{Z})$. The claim will be proven if we get a contradiction. By hypothesis, there exists

$$\gamma = \sum_{i \in I} n_i \sigma_i \in C_{\ell + 1}(\mathfrak{I}(\mathbb{L}); \mathbb{Z}),$$

with $\alpha = \partial \gamma$. Let $v$ be the apex of $\mathbb{L}$. The allowability condition of Definition 1.12 implies that, for each simplex $\sigma_i : \Delta^\ell \to \mathbb{L}$, the set $\sigma_i^{-1}(v)$ is finite and included in the interior of $\Delta^\ell$. Thus, for each $i \in I$, the restriction $\bar{\sigma}_i : \partial \Delta^\ell \to \mathbb{L}$ of $\sigma_i$ takes value in $\mathbb{L}\{v\}$. This restriction defines a homotopy class in $\pi_\ell(\mathbb{L}\{v\}) = \pi_\ell(\mathbb{L}) = 0$. So, there exists a continuous map $\tau_i : \Delta^\ell \to \mathbb{L}\{v\}$ extending $\bar{\sigma}_i$. We obtain $\tau_i \in C_{\ell + 1}(\mathbb{L}\{v\}; \mathbb{Z})$ with $\partial \tau_i = \partial \sigma_i$. Finally,

$$[\alpha] = \left[ \sum_{i \in I} n_i \partial \sigma_i \right] = \left[ \sum_{i \in I} n_i \partial \tau_i \right] = \left[ \partial \sum_{i \in I} n_i \tau_i \right] = 0$$

in $H_\ell(\mathbb{L}\{v\}; \mathbb{Z}) = H_\ell(L; \mathbb{Z})$. We get a contradiction since $[\alpha]$ is the orientation class.

Evidently, as the homology of $\mathfrak{I}(X)$ is the homology of a space, it possesses Mayer-Vietoris sequences. But the Mayer-Vietoris sequence that we are considering below is related to an open covering of the space $X$. The existence of such sequence is a consequence of a theorem of $U$-small chains that we first establish.

**Proposition 2.12.** Let $(X, \mathfrak{p})$ be a perverse space with a locally finite stratification, $U$ be an open covering of $X$ and $E$ be a local coefficient system on $\mathfrak{I}(X)$. We denote $\mathfrak{I}(U)X$ the simplicial subset whose simplexes are the $\mathfrak{p}$-full simplexes included in an element of $U$. Then the following properties are verified.

1) There exists a chain map $sd : C_\ast(\mathfrak{I}(U)X; E) \to C_\ast(\mathfrak{I}(X); E)$ such that, for each $\mathfrak{p}$-full simplex $\sigma$, there exists $r \in \mathbb{N}$ with $sd^r \sigma \in C_\ast(\mathfrak{I}(U)X; E)$.

2) The inclusion $i_\ast : C_\ast(\mathfrak{I}(U)X; E) \to C_\ast(\mathfrak{I}(X); E)$ induces an isomorphism in homology.

**Proof.** Let $\sigma : \Delta^\ell \to X$ be a $\mathfrak{p}$-full simplex. The construction comes from a process of subdivision applied to $\sigma$. Here we have to adapt it to allowable simplexes and to the presence of local coefficients. For local coefficients, the adaptation is classic: if $F$ and $K$ are faces of a simplex $\Delta$, with the join $\sigma_F \ast \sigma_K$ defined and $a \in E(\sigma_F(0), \sigma_K(0))$, one sets $\sigma_F \ast (a \sigma_K) = E(\sigma_F(0), \sigma_K(0))^{-1}(a) \sigma_F \ast \sigma_K$. The adaptation to the compatibility with perversities is obtained by replacing the classic barycentric subdivision with the pseudobarycentric subdivision recalled in Remark 1.14. As in the classical topological case, this gives a chain map $sd : C_\ast(\mathfrak{I}(U)X; E) \to C_\ast(\mathfrak{I}(X); E)$ satisfying the first point of the statement, see [9, Proposition 7] in the case of ordinary coefficients.

The proof of 2) uses a morphism $T : C_\ast(X; E) \to C_\ast(X; E)$ verifying $id - sd = T \circ \partial + \partial \circ T$ and sending a $\mathfrak{p}$-allowable simplex on a $\mathfrak{p}$-allowable chain. Similarly to that of the previous subdivision, the construction of the map $T$ is an adaptation of the classical case, with an induction from the pseudobarycentric subdivision. If $\sigma$ is a $\mathfrak{p}$-full simplex, we obtain a chain $T(\sigma) \in C_\ast(\mathfrak{I}(X); E)$, as in [9, Proposition 6] for the case of ordinary coefficients.

The following two properties arise from Proposition 2.12, as in the classic case (see [28, Chapter 6]).

**Theorem 2.13.** Let $(X, \mathfrak{p})$ be a perverse space with a locally finite stratification, $\{U, V\}$ be an open covering of $X$ and $E$ be a local coefficient system on $\mathfrak{I}(X)$. Then, there exists a Mayer-Vietoris long exact sequence,

$$\cdots \to H_\ast(\mathfrak{I}(U \cap V); E) \to H_\ast(\mathfrak{I}(U); E) \oplus H_\ast(\mathfrak{I}(V); E) \to H_\ast(\mathfrak{I}(X); E) \to H_{\ast - 1}(\mathfrak{I}(U \cap V); E) \to \cdots$$
Theorem 2.14. Let \((X, \mathcal{P})\) be a perverse space, \(E\) be a local coefficient system on \(\mathcal{P}_X\) and \(Z \subset A\) be two subspaces of \(X\) such that \(Z\) is included in the interior of \(A\). Then there is an isomorphism
\[ H_*(\mathcal{P}_X \setminus Z, \mathcal{P}_A \setminus Z; E) \cong H_*(\mathcal{P}_X, \mathcal{P}_A; E). \]

3. Intersection homotopy groups

A pointed perverse space is a triple \((X, \mathcal{P}, x_0)\) where \((X, \mathcal{P})\) is a perverse space and \(x_0 \in X \setminus \Sigma_X\) is a regular point. We also denote \(x_0\) the simplicial subset of \(\text{Sing} X\) generated by \(x_0\): \(\Delta^0 \to X\).

Definition 3.1. ([14]) Let \((X, \mathcal{P}, x_0)\) be a pointed perverse space. The \(\mathcal{P}\)-intersection homotopy groups (or perverse homotopy groups) are the homotopy groups of the simplicial set \(\mathcal{P}_X\),
\[ \pi^\mathcal{P}_\ell(X, x_0) = \pi_\ell(\mathcal{P}_X, x_0). \]

As \(\mathcal{P}_X\) is Kan, an element of \(\pi^\mathcal{P}_\ell(X, x_0)\) is given by a \(\mathcal{P}\)-full simplex, \(\sigma: \Delta^\ell \to X\), with \(\sigma(\partial \Delta^\ell) = \{x_0\}\). Two such simplexes, \(\sigma_0\) and \(\sigma_1\), are equivalent if there exists a \(\mathcal{P}\)-full simplex \(\Phi: \Delta^{\ell+1} \to X\) verifying \(\partial_i \Phi = x_0\) if \(i \geq 2\) and \(\partial_i \Phi = \sigma_i\) for \(i = 0, 1\). We continue with the set \(\pi^\mathcal{P}_\ell(X)\) of the connected components of \(\mathcal{P}_X\).

Definition 3.2. A perverse space, \((X, \mathcal{P})\), is said \(\mathcal{P}\)-connected if \(\pi^\mathcal{P}_0(X) = 0\).

If \((X, \mathcal{P})\) is a \(\mathcal{P}\)-connected perverse space, we do not need to specify the basepoint of the homotopy groups and sometimes we will write \(\pi^\mathcal{P}_\ell(X)\) instead of \(\pi^\mathcal{P}_\ell(X, x_0)\).

Remark 3.3. Since each regular point of \(X\) is a \(\mathcal{P}\)-full simplex, we have the natural map \(\pi_0(X \setminus \Sigma_X) \to \pi^\mathcal{P}_0(X)\). If the perversity \(\mathcal{P}\) verifies \(\mathcal{P} \leq 7\), then the \(\mathcal{P}\)-full simplexes \(\sigma: \Delta^\ell \to X\), for \(\ell = 0, 1\), do not meet the singular part of \(X\). So, \(\pi^\mathcal{P}_\ell(X) = \pi_\ell(X \setminus \Sigma_X)\). On the other hand, if \(\mathcal{P} \geq 7 + 2\) then the \(\mathcal{P}\)-allowability condition is always fulfilled by the 0 and 1 simplexes. So \(\pi^\mathcal{P}_\ell(X) = \pi_\ell(X)\).

If \((X, \mathcal{P})\) is a perverse CS set, the set \(\pi^\mathcal{P}_0(X)\) is related with the family of connected components of \(X \setminus \Sigma_X\) in a more specific way.

Proposition 3.4. Let \((X, \mathcal{P})\) be a perverse CS set. For any point \(x \in X\), there exists a path, \(\beta: [0, 1] \to X\), with \(\beta(0) = x\), \(\beta([0, 1]) \subset X \setminus \Sigma\). Thus the canonical inclusion induces a surjection \(\pi_0(X \setminus \Sigma) \to \pi^\mathcal{P}_0(X)\).

Proof. If \(x\) is regular, we choose the constant path. If not, we consider a conical chart, \(\varphi: \mathbb{R}^k \times \epsilon L \to U\) with \(\varphi(0, v) = x\). We join \(x\) to a regular point with the path, \(t \mapsto (0, [z, t])\). Here, we have written \(v = [z, 0]\) where \(z\) is a regular point of the link \(L\), which exists since \(L \setminus \Sigma_L \neq \emptyset\). \(\square\)

The first concrete computation of \(\mathcal{P}\)-intersection homotopy groups concerns the basic example of the cone on a filtered space. The following statement is the Eckmann-Hilton dual of the intersection homology of cones ([17, Section 6] or [7, Proposition 5.2]).

Proposition 3.5. [14, Example 2] Let \((X, x_0)\) be a compact filtered space, pointed by a regular point \(x_0\), and \(\varepsilon X\) be the open cone of apex \(\varepsilon\), with the conic filtration and pointed by \(y_0 = [x_0, 1/2]\). Let \(\mathcal{P}\) be a perversity on \(\varepsilon X\), we denote also by \(\mathcal{P}\) the perversity induced on \(X\). Then, the \(\mathcal{P}\)-intersection homotopy groups of \(\varepsilon X\) are given by
\[ \pi^\mathcal{P}_\ell(\varepsilon X, y_0) = \begin{cases} \pi^\mathcal{P}_\ell(X, x_0) & \text{if } \ell \leq D\mathcal{P}(\varepsilon), \\ 0 & \text{if } \ell > D\mathcal{P}(\varepsilon). \end{cases} \]

Proof. For \(\ell \leq D\mathcal{P}(\varepsilon)\), the statement is a consequence of Propositions 2.7 and 2.4. Consider now a \(\mathcal{P}\)-full simplex \(\sigma = [\sigma_0, \sigma_1]: (\Delta^\ell, \partial \Delta^\ell) \to (\varepsilon X, y_0)\) with \(\ell \geq D\mathcal{P}(\varepsilon) + 1\). For \(\ell > 0\), the result will be established if we define a \(\mathcal{P}\)-full simplex, \(\Phi: \Delta^{\ell+1} \to \varepsilon X\) such that \(\partial_i \Phi = y_0\) if \(i \geq 1\) and \(\partial_0 \Phi = \sigma\). These requirements define \(\Phi|_{\partial \Delta^{\ell+1}}\). To extend it to the whole simplex, we consider the barycentre \(b\) of \(\Delta^{\ell+1}\) and set \(\Phi(b) = \varepsilon\).

We extend \(\Phi\) linearly to the rest of \(\Delta^{\ell+1}\). To check the allowability condition for \(\Phi\), we distinguish according to the two possible types of singular strata.
(i) The stratum is the vertex $v$ of the cone.
- If $\sigma^{-1}v \neq \emptyset$, then we have $\dim \Phi^{-1}v = \dim \mathfrak{c}_b\sigma^{-1}v \leq 1 + \dim \mathfrak{c}_b\sigma^{-1}(v) - 1$. By connectedness of $X$, we also get
- If $\sigma^{-1}v = \emptyset$, then $\Phi^{-1}v = b$. The restriction $\ell \geq D\mathfrak{c}(v) + 1$ implies $\dim \Phi^{-1}(v) = 0 \leq 1 + \ell - D\mathfrak{c}(v) - 2$.

(ii) The stratum is the product $S \times [0, 1]$ where $S$ is a singular stratum of $X$.
- If $\sigma^{-1}(S \times [0, 1]) \neq \emptyset$, then we have $\dim \Phi^{-1}(S \times [0, 1]) = \{x + (1 - \ell)b \mid [\sigma_0(x), \ell\sigma_1(x)] \in S \times [0, 1]\} = \sigma^{-1}(S \times [0, 1]) \times [0, 1]$. We deduce $\dim \Phi^{-1}(S \times [0, 1]) = \dim \mathfrak{c}_b\sigma^{-1}(S \times [0, 1]) + 1$ and the same argument than above applies here.
- If $\sigma^{-1}(S \times [0, 1]) = \emptyset$, then $\Phi^{-1}(S \times [0, 1]) = \emptyset$.

For $\ell = 0$ the situation is slightly different. We have two $\mathfrak{p}$-full simplexes $y_1, y_2 : \Delta^0 \to \mathfrak{c}X$ and we need to find a $\mathfrak{p}$-full simplex, $\Phi : \Delta^1 \to \mathfrak{c}X$ such that $\partial_1 \Phi = y_1$ and $\partial_0 \Phi = y_2$. We leave the adaptation to the reader.

This determination implies the following properties. The first one is a consequence of the homotopy exact long sequence of a pair.

**Corollary 3.6.** With the hypotheses of Proposition 3.5, the relative $\mathfrak{p}$-intersection homotopy groups of the pair $(\mathfrak{c}X, X)$ verifies $\pi_\ell^\mathfrak{p}(\mathfrak{c}X, X) = 0$, for any $\ell \leq D\mathfrak{c}(v) + 1$.

**Corollary 3.7.** With the hypotheses of Proposition 3.5 for a topological space $X$ with the trivial filtration, the simplicial set $\mathcal{H}(\mathfrak{c}X)$ is a $D\mathfrak{c}(v)$-Postnikov stage of the simplicial set $\text{Sing} X$.

**Corollary 3.8.** Let $\mathfrak{c}X$ be the cone on a simply connected, compact space $X$, filtered by its apex $\{v\} \subset \mathfrak{c}X$, and let $\mathfrak{p}$ be a perversity given by $\mathfrak{p}(v)$. If $(\Lambda Z, d)$ is the Sullivan minimal model of $X$, then the cdga $(\Lambda Z \leq D\mathfrak{c}(v), d)$ is the minimal model of $\mathcal{H}(\mathfrak{c}X)$.

**Proof.** This property comes from the relation between Sullivan minimal models and Postnikov towers, see [34].

In opposition with the PL situation, in a CS set $X$, one can encounter links of the same point that are not homeomorphic ([12, Example 2.3.6]). In the case of intersection homology groups, this phenomenon does not matter since all these links have the same intersection homology groups. We prove below that the links of points in the same stratum have isomorphic $\mathfrak{p}$-intersection homotopy groups, for any perversity $\mathfrak{p}$ on the total space. We analyze this property again in Remark 4.4. (Recall that a perversity defined on $X$ induces a perversity on the conical charts thus on the links, see Remark 1.10.)

**Proposition 3.9.** Let $(X, \mathfrak{p})$ be a normal perverse CS set, we also denote by $\mathfrak{p}$ the perversities induced on each link. Let $S$ be a singular stratum, $x_1, x_2$ be two points of $S$ and $L_1, L_2$ be respective links of $x_1$ and $x_2$. Then, for any $\ell \geq 1$, there is a group isomorphism,

$$\pi_\ell^\mathfrak{p}(L_1, x_1) \cong \pi_\ell^\mathfrak{p}(L_2, x_2).$$

**Proof.** By connectedness of $S$, we can suppose that $x_1 = x_2 = x$. In the proof of the analogous result for the intersection homology (see [12, Lemma 5.3.13]), one constructs open subsets, linked by canonical inclusions as follows,

$$V'_1 \xrightarrow{f} V_2 \xrightarrow{g} V_1 \xrightarrow{h} V'_2,$$

with $V'_2$ stratified homotopy equivalent to $L_2$, $V'_1$ stratified homotopy equivalent to $L_1$ and such that $gf$ and $hg$ are stratified homotopy equivalences. As a stratified homotopy equivalence induces an isomorphism between intersection homotopy groups (Corollary 2.6), the compositions $hg$ and $gf$ induce isomorphisms between intersection homotopy groups. As $L_1$ and $L_2$ are connected, the map $g$ induces an isomorphism $\pi_\ell(L_1) \cong \pi_\ell(L_2)$.

By taking the perversity $\mathfrak{p} = \infty$, the previous result implies $\pi_\ell(L_1 \setminus \Sigma L_1) \cong \pi_\ell(L_2 \setminus \Sigma L_2)$. With $\mathfrak{p} = -\infty$, we also get $\pi_\ell(L_1) \cong \pi_\ell(L_2)$. 


4. Intersection fundamental group

Let \((X, \mathcal{P})\) be a perverse space, with \(\mathcal{P} \leq \mathcal{I}\). Recall that any \(\mathcal{P}\)-allowable simplex, \(\sigma: \Delta^\ell \to X\), verifies \(\dim \sigma^{-1}S \leq \ell - \text{codim } S + \mathcal{P}(S) \leq \ell - 2\), for any singular stratum \(S\). Thus, the vertices and the edges of a \(\mathcal{P}\)-full simplex stay in the regular part. This implies

\[
\text{the surjectivity of } \pi_1(X \setminus \Sigma X, x_0) \to \pi_1^\mathcal{P}(X, x_0) \text{ for } x_0 \in X \setminus \Sigma X.
\]  

(4.1)

We begin with a Van Kampen theorem if \(\mathcal{P} \leq \mathcal{I}\). The proof consists in the observation that the classic demonstration can be done in accordance with the \(\mathcal{P}\)-allowability conditions. For that, we use [22, Section 1.2] as guideline.

Let \((X, \mathcal{P})\) be a perverse space with \(\mathcal{P} \leq \mathcal{I}\). \([U_0, U_1]\) be an open cover of \(X\). The open subsets \(U_0, U_1, U_0 \cap U_1\) are supposed to be path-connected and we endow them with the induced filtration and the induced perversity. We denote by \(j_k: \pi_1^\mathcal{P}(U_0 \cap U_1, x_0) \to \pi_1^\mathcal{P}(U_k, x_0)\) and \(i_k: \pi_1^\mathcal{P}(U_k, x_0) \to \pi_1^\mathcal{P}(X, x_0)\) the homomorphisms induced by the canonical inclusions for \(k = 0, 1\). The homomorphisms \(i_k\) extend to a homomorphism defined on the free product of groups,

\[
\Phi: \pi_1^\mathcal{P}(U_0, x_0) \ast \pi_1^\mathcal{P}(U_1, x_0) \to \pi_1^\mathcal{P}(X, x_0).
\]

The Van Kampen theorem consists of a determination of the kernel and the image of \(\Phi\).

**Theorem 4.1.** The map \(\Phi\) is surjective and its kernel is the normal subgroup generated by all elements of the form \(j_0(\omega) \ast j_1(\omega)^{-1}\).

**Proof.** Let \(\alpha\) be a \(\mathcal{P}\)-allowable loop in \(X\). We already noted that the inclusion map induces a surjective homomorphism \(\pi_1(X \setminus \Sigma X, x_0) \to \pi_1^\mathcal{P}(X, x_0)\). Thus, the \(\mathcal{P}\)-homotopy class of \(\alpha\) is the image of the homotopy class of a loop \(\beta\) in \(X \setminus \Sigma X\). From the compactness of \([0, 1]\), we obtain a decomposition \([\alpha] = \Phi([\gamma_0] \ast \cdots \ast [\gamma_\ell])\) with the support of each \(\gamma_j\) in one of the two subsets \(U_0, U_1\). By using the path-connectedness of \(U_0, U_1, U_0 \cap U_1\), we can suppose that each \(\gamma_i\) is a loop based in \(x_0\). This first step gives the surjectivity of \(\Phi\).

The critical point of the proof is the determination of the kernel of \(\Phi\). We consider two factorizations as above, \(\Phi([\gamma_0] \ast \cdots \ast [\gamma_\ell]) = \Phi([\gamma'_0] \ast \cdots \ast [\gamma'_\ell])\). Thus, the two loops \(\gamma_0 \cdots \gamma_\ell\) and \(\gamma'_0 \cdots \gamma'_\ell\) are \(\mathcal{P}\)-homotopic and there exists a full \(\mathcal{P}\)-homotopy \(F: [0, 1] \times [0, 1] \to X\) between them. Unlike the situation of paths, the support of the homotopy is not necessarily in \(X \setminus \Sigma X\); i.e. the subset \(F^{-1}(\Sigma X)\) is not necessarily empty. However the \(\mathcal{P}\)-allowability condition gives us a control on it and we know that \(F^{-1}(\Sigma X)\) consists in a finite number of points located in the interior of the square. Without loss of generality, by using an appropriate subdivision, we can suppose that \(F^{-1}(\Sigma X)\) is reduced to one point \(v\) in the interior of \([0, 1] \times [0, 1]\). As in the surjective part of the proof, we use the compactness of \([0, 1] \times [0, 1]\) to get a subdivision of it so that each small square lies either in \(U_0\) or in \(U_1\). Here, we can arrange the subdivision in such a way that the point \(v\) does not belong to any of the boundaries of the small squares. This implies that each small square is a full \(\mathcal{P}\)-homotopy in \(U_0\) or \(U_1\). The rest of the proof consists to the use of the path-connectedness of \(U_0, U_1, U_0 \cap U_1\) to transform each small square in a \(\mathcal{P}\)-homotopy between loops based in \(x_0\). This is done with the introduction of paths from \(x_0\) to the corners of the small squares. As this can be done in \(X \setminus \Sigma X\), each of the small squares remains a full \(\mathcal{P}\)-homotopy. This gives the conclusion, as in the classic topological situation.

**Corollary 4.2.** Let \(X\) be a connected normal Thom-Mather space ([25]) with a finite number of strata. Then, for any \(j \geq 0\), there is an isomorphism

\[
\pi_j(X) \cong \pi_j(X).
\]

**Proof.** From Remark 1.6, we deduce the connectivity of \(X \setminus \Sigma\) and we can omit the reference to the basepoint. Let \(S\) be one of the minimal strata. There is a locally trivial fibration \(E \to S\), playing the role of a tubular neighborhood. From [14, Theorem 2.3], we deduce a long exact sequence
... \rightarrow \pi_1^E(cL) \rightarrow \pi_1^E(E) \rightarrow \pi_1^E(S) \rightarrow \ldots. The space \( S \) has only one stratum so \( \pi_1^E(S) = \pi_1(S) \). From Proposition 3.5, we know \( \pi_1^E(cL) = 0 \). Thus we deduce \( \pi_1^E(E) = \pi_1(S) \). Now we do an induction on the depth and the number of minimal strata. The inductive step is reduced to an open cover of \( E \) given by \( E \backslash S \). The hypotheses of connectivity are satisfied: for \( E \), it is a consequence of the connectivity of \( S \) and \( cL \), for \( E \backslash S \) it comes from \( X \backslash \Sigma \) connected, \( X \backslash \Sigma \subset X \backslash S \) and \( X \backslash \Sigma = X \). Finally, \( E \backslash S \) is the total space of a locally trivial fiber bundle, of basis \( S \) and fiber \( L \times [0,1] \), thus \( E \backslash S \) is connected since \( L \) and \( S \) are so. The intersection fundamental group of \( E \) is computed as above. On the two open subsets, \( X \backslash S \), \( (X \backslash S) \cap E = E \backslash S \), we have the induction hypothesis. Thus, from Theorem 4.1, we deduce \( \pi_1^E(X) \cong \pi_1(X) \).

With Lemma 2.8, we can now replace homotopy groups by homology groups in a local system of coefficients, \( \mathcal{E} \). We use a theorem of King ([23, Section 3]) detailed in [7, Theorem 5.1], applied to the natural transformation \( \psi_X : H_*(\mathcal{F}X; \mathcal{E}) \rightarrow H_*(X; \mathcal{E}) \). This theorem says that such transformation is an isomorphism if its restrictions to the open subsets of \( CS \) sets verify the following properties.

i) There are Mayer-Vietoris sequences and \( \psi \) induces a commutative diagram between them.

ii) If \( (U_\alpha) \) is an increasing sequence of open subsets and \( \psi_{U_\alpha} \) is an isomorphism for each \( \alpha \), then \( \psi_{\cup U_\alpha} \) is an isomorphism.

iii) If \( L \) is a compact filtered space such that \( X \) has an open subset \( U \) stratified homeomorphic to \( \mathbb{R}^i \times cL \) and if \( \psi_U \) is an isomorphism for \( U' = \mathbb{R}^i \times (cL \backslash \{v\}) \) (where \( v \) is the apex) then so is \( \psi_U \).

iv) If \( U \) is an open subset of \( X \) contained within a single stratum and homeomorphic to an Euclidean space, then \( \psi_U \) is an isomorphism.

As \( H_*(\mathcal{F}X; \mathcal{E}) \) admits Mayer-Vietoris sequences (Theorem 2.13), the point iii) is the only one which deserves attention. Looking to its conclusion, we notice from Propositions 2.4 and 3.5, that \( \pi_1(\mathcal{F}(\mathbb{R} \times cL)) \cong \pi_1(\mathcal{F}cL) \) is trivial except in degree 0. The hypothesis of normality gives \( \pi_0(\mathcal{F}(\mathbb{R} \times cL)) \cong \pi_0(\mathbb{R} \times cL) = 0 \) for any \( i \), the result follows from the Whitehead theorem.

We are interested in the topological invariance of the intersection fundamental group of \( CS \) sets for GM-perversities. The following example shows that this property is not satisfied in general.

**Example 4.3.** The suspension \( \Sigma M \) of a topological space \( M \) being the main ingredient of the example, let us begin with two basic properties.

1) If \( \Sigma M \) is homeomorphic to a sphere \( S^n \), then \( M \) has the homotopy type of \( S^{n-1} \). (Just remove the two vertices of \( \Sigma M \) to get a homeomorphism \( M \times ]-1,1[ \cong S^{n-1} \times ]-1,1[ \).)

2) If \( \Sigma M \) is homeomorphic to a sphere \( S^n \) and \( M \) is a manifold, from 1) and the theorems of Smale-Freedman-Donaldson-Perelman (also called “Poincaré conjecture”), then \( M \) is a sphere.

Let \( P \) be the Poincaré sphere, a manifold of dimension three with the homology of the 3-sphere and such that \( \pi_1(P, v) \neq 1 \), with \( v \in P \). From the previous recalls, we deduce that the suspension \( \Sigma P \) is not homeomorphic to \( S^4 \) and has no structure of manifold, but has the homotopy type of \( S^4 \). Let \( v_1, v_2 \) be the two new apexes of \( \Sigma P \), constituting the singular set of \( \Sigma P \). We choose a GM-perversity \( \mathfrak{p} \) with \( D\mathfrak{p}(4) = 1 \). By definition, two \( \mathfrak{p} \)-allowable simplexes, \( \xi : (\Delta^4, \partial \Delta^4) \rightarrow (\Sigma P, v) \) and \( \eta : (\Delta^2, \partial \Delta^2) \rightarrow \Sigma P \), verify \( \dim \xi^{-1}(v_i) \leq \dim \Delta - 2 \) and \( \dim \eta^{-1}(v_i) \leq -1 \). Thus, we have \( \pi_1^\mathfrak{p}(\Sigma P, v) = \pi_1(P, v) \).

We now consider the double suspension of \( P \) and denote by \( w_1, w_2 \) the two new apexes. We filter \( \Sigma^2 P \) by \( \{w_1, w_2\} \subset \Sigma \{v_1, v_2\} \subset \Sigma^2 P \). We choose a GM-perversity such that \( D\mathfrak{p}(4) = D\mathfrak{p}(5) = 1 \). A similar computation gives \( \pi_1^\mathfrak{p}(\Sigma^2 P, v) = \pi_1^\mathfrak{p}(\Sigma P, v) = \pi_1(P, v) \neq 1 \). As \( \Sigma^2 P \) is homeomorphic to the sphere \( S^5 \), this example shows that the intersection fundamental group is not a topological invariant in general. This situation does not appear in the PL case since the homeomorphism between \( \Sigma^2 P \) and \( S^5 \) is not a PL map. In fact, in [14, Theorem 2.7] Gajer shows that the \( \mathfrak{p} \)-intersection homotopy groups of a polyhedron is a PL invariant.
Remark 4.4. The overlapping of Proposition 3.9 and Example 4.3 calls out. In the first reference, we prove that links of a point in a CS set have isomorphic intersection homotopy groups for any perversity.

In the second one, we use the example of two CS set structures on a particular suspension space, with an ad’hoc perversity, to highlight the failure of a general topological invariance of intersection homotopy groups.

Let us first make a few reminders about the properties of links in CS sets. We consider two CS set structures on the same topological space $X$ and a point $x \in X$. Let $N = (\xi L, \nu)$ and $N' = (\xi L', \nu')$ be two conical charts of $x$ in the respective CS set structures. From a result of Stallings on conical neighborhoods (see [12, Corollary 2.10.2]), there exists a relative homeomorphism between $N$ and $N'$. (This does not imply the existence of a homeomorphism between the links!)

In the case of a fixed structure of CS set on a topological space $X$, the invariance of the intersection homotopy groups of the links, established in Proposition 3.9, follows from the existence of stratified homotopy equivalences between charts and links.

Let us consider two homeomorphic spaces with CS set structures, and a GM-perversity $\mathfrak{p}$. The topological invariance means that the corresponding $\mathfrak{p}$-intersection homotopy groups are isomorphic. That is, we now have two different CS set structures on the “same” topological space and we ask whether the intersection homotopy groups are isomorphic. Our test for this topological invariance is the double suspension $\Sigma^2P$. We endow $\Sigma^2P$ with the filtration $\{u_1, u_2\} \subset \Sigma\{v_1, v_2\} \subset \Sigma^2P$ of Example 4.3. The singular set is the circle $\gamma = \Sigma\{v_1, v_2\}$. Let $\varphi: \Sigma^2P \to S^5$ be the Edwards homeomorphism. We get two different structures of CS set on the topological space $S^5$: a “trivial” one comes from the manifold structure of $S^5$ and a second one is the image by $\varphi$ of the previous chosen filtration on $\Sigma^2P$. As we only consider these two structures in this remark, we call “non-trivial” the second one. As said above, conical charts in $u_1$ (or $u_2$) are homeomorphic, so $\mathbb{R}^0 \times \varepsilon S^4 \cong \mathbb{R}^0 \times \varepsilon \Sigma P$. We emphasize that this is only a topological result without mention of filtrations. This is a crucial point in the analysis of links made below.

Set $u = \varphi(u_1) \in S^5$. From the manifold structure, the point $u$ admits an open neighborhood homeomorphic to $\varepsilon S^4$. We endow $S^4$ with the filtration induced by the non-trivial filtration of $S^5$. At the request of a referee, we study whether the sphere $S^4$, equipped with this filtration, can be a link of $u$ in the non-trivial CS set structure of $S^5$. We prove:

Claim: For the non-trivial filtration of $S^5$, $S^4$ is not a link of $u$.

Let us suppose that $S^4$ is such a link; i.e., $S^4$ and $\Sigma P$ are two links of $u$. The claim will be inferred from the existence of a contradiction. We choose a perversity $\mathfrak{p}$ such that $D\mathfrak{p}(4) = D\mathfrak{p}(5) = 1$. Having only one CS set structure on $S^5$ (the non-trivial one), we can apply Proposition 3.9 and get $\pi_1^\mathfrak{p}(S^4) \cong \pi_1^\mathfrak{p}(\Sigma P)$. From a computation made in Example 4.3, we deduce

$$\pi_1^\mathfrak{p}(S^4) \cong \pi_1^\mathfrak{p}(\Sigma P) \neq 1. \quad (4.2)$$

To obtain a contradiction, we directly determine this fundamental intersection group. Let us first note that $\varphi(\gamma) \cap S^4$ is a set of isolated points, possibly empty. By definition, two $\mathfrak{p}$-allowable simplexes, $\xi: \Delta^1 \to S^4$ and $\eta: \Delta^2 \to S^4$, verify $\dim \xi^{-1}\varphi(\gamma) \leq \dim \Delta - 2 - D\mathfrak{p}(4) = -2$ and $\dim \eta^{-1}\varphi(\gamma) \leq -1$. Therefore $\pi_1^\mathfrak{p}(S^4) \cong \pi_1(S^4 \setminus (\varphi(\gamma) \cap S^4))$. From (1.1) in the definition of a CS set, we deduce that, for each $x \in \varphi(\gamma) \cap S^4$, the segment $[x, u]$ is included in $\varphi(\gamma)$. The map $\varphi$ being an embedding, the intersection $\varphi(\gamma) \cap S^4$ is a set of at most two points. In any of the three possible cases, we have $\pi_1(S^4 \setminus \varphi(\gamma) \cap S^4) = 1$ and a contradiction with (4.2).

In conclusion, the obstruction to $S^4$ being a link is the non-triviality of the intersection homotopy group $\pi_1^\mathfrak{p}(\Sigma P)$. Let us also point out that the topological embedding $\gamma: S^4 \to S^5$ is wild and far from a smooth one, as shown for instance in [21].

In Example 4.3, some singular points of $\Sigma^2P$ become regular in $S^5$. The next result proves that is the only obstruction to having a topological invariant. Recall that a coarsening of a filtered space $X$ is
a filtered space $X$ on the same topological space, such that each stratum of $Y$ is a union of strata of $X$. The identity induces a stratified map, $\nu: X \to Y$.

**Definition 4.5.** Let $\nu: X \to Y$ be a coarsening of a filtered space $X$. An exceptional stratum is a singular stratum of $X$ which is included in a regular stratum of $Y$. A source stratum of a stratum $T$ of $Y$ is a stratum $S$ of $X$ included in $T$ and of the same dimension.

**Theorem 4.6.** Let $\nu: X \to Y$ be a coarsening of CS sets and $\overline{\nu}$ be a codimensional perversity such that $\overline{\nu} \leq T$. If there is no exceptional stratum, then the map $\nu$ induces isomorphisms, $\pi_0^\nu(X) \cong \pi_0^\nu(Y)$ and $\pi_1^\nu(X, x_0) \cong \pi_1^\nu(Y, x_0)$, for any regular point $x_0$.

Any GM-perversity satisfies the required hypothesis on $\overline{\nu}$. The condition imposed on the strata means that the two singular sets are identical and we denote it by $\Sigma = \Sigma_X = \Sigma_Y$.

**Proof.** The first assertion comes from Remark 3.3 and the surjectivity of $\nu_*: \pi_0^\nu(X, x_0) \to \pi_0^\nu(Y, x_0)$ from (4.1). For the injectivity of $\nu_*$, we consider a $\overline{\nu}$-allowable loop $\xi: [0, 1] \to X$ and suppose that there exists a $\overline{\nu}$-allowable simplex $\eta: \Delta^2 \to Y$, whose boundary is $\xi$ on one edge and the constant loop on $x_0$ on the other two. We have to construct a $\overline{\nu}$-allowable simplex $\eta': \Delta^2 \to X$ such that $\partial \eta = \partial \eta'$. From the $\overline{\nu}$-allowability condition on $\eta$, we know that the subset $\eta^{-1}\Sigma = \{ \eta^{-1}T \mid T \text{ is a singular stratum of } Y \text{ with } L \overline{\nu}(\text{codim } T) = 0 \}$ is finite. If $\eta^{-1}\Sigma = \emptyset$, we choose $\eta' = \eta$. Otherwise, by subdivision, we can suppose that $\eta^{-1}\Sigma = \{ p \}$ in $\Delta^2$ and that the image of $\eta$ is included in a conical chart $U$. Let us denote by $\varphi: \mathbb{R}^m \times \varepsilon L \to U$ this conical chart with $\varphi(0, v) = \eta(p)$, where $v$ is the apex of the cone. The point $\eta(p)$ belongs to a singular stratum $S$ of $X$ and to the singular stratum $S''$ of $Y$. By the previous choices, we have $L \overline{\nu}(\text{codim } S'') = 0$. The family of source strata of $S''$ being an open dense subset of $S''$, we can find a source stratum $Q$ of $S''$ and a point $x = \varphi(u_0, v) \in U$ as close as we want to $\eta(p)$. The map $\varphi^{-1} \circ \eta: \partial \Delta^2 \to \mathbb{R}^m \times \varepsilon L$ can be written

$$\varphi^{-1}(\eta(y)) = (f_1(y), [f_2(y), f_3(y)])$$

Its image is in $L \setminus \Sigma_L$ and the map $f_3$ does not vanish. Let us denote by $\hat{\nu}$ the barycentre of $\Delta^2$. We define $\eta': \Delta^2 \to U$ by

$$\eta'(ty + (1-t)\hat{\nu}) = \varphi((1-t)u_0 + t f_1(x), [f_2(x), t f_3(x)])$$

By construction, we have $\eta'^{-1}\Sigma = \eta^{-1}Q = \{ b \}$ and for $t = 1$ we get $\partial \eta = \partial \eta'$. As $Q$ is a stratum of $S''$, it has the same codimension, and we have

$$\dim \eta'^{-1}\Sigma = \dim \eta^{-1}Q = \dim \{ b \} = 2 - \text{codim } Q - 2 = \dim \eta - \text{codim } Q - 2.$$ 

This gives the $\overline{\nu}$-allowability of $\eta$ in the CS set $X$, as required.

Among the coarsenings of a CS set $X$, there is the intrinsic coarsest CS set, $\nu: X \to X^*$, whose properties are recalled below. Theorem 4.6 applied to this coarsest stratification implies that the intersection fundamental group for a codimensional perversity is a topological invariant if no singular stratum of $X$ becomes regular in $X^*$. Example 4.3 shows the necessity of this restriction.

**Recall 4.7. (Intrinsic CS set)** Let $X$ be a CS set. For the construction of a new structure of CS set on the same topological space in [23], King utilizes an equivalence relation he credits Dennis Sullivan with: two points $x_0$ and $x_1$ are equivalent if there exist neighborhoods $U_i$ of $x_i$ with a homeomorphism $(U_0, x_0) \cong (U_1, x_1)$. The equivalence classes are union of strata and the CS set $X$ has a new filtration by choosing $X^*_j$ as the union of equivalence classes which only contains components of strata of dimension less than or equal to $j$. This defines a CS set denoted by $X^*$ and called the intrinsic CS set associated to $X$. The identity map is a stratified map denoted by $\nu: X \to X^*$. By construction, a stratum $T$ of $X^*$ is a locally finite union of strata of $X$. The set of source strata (Definition 4.5) of $T$ is a dense open subset of $T$. 


The CS set $X^*$ is a coarsening of $X$. We can observe that a singular stratum of $X$ can be included in a singular stratum or in a regular stratum of $X^*$ but a regular stratum of $X$ stays regular in $X^*$. Thus the singular set $\Sigma_{X^*} = X \backslash X_{n-1}^*$ is a closed subspace of $\Sigma_X$. The CS sets $X$ and $X^*$ have the same underlying topological space; if there is no ambiguity, we denote it by $X$. If $U$ is an open subset of $X$, we also denote by $U$ the CS set with the structure induced by $X$ and by $U^*$ the CS set with the structure induced by $X^*$. Let us recall from [23] how are the structures of $X$ and $X^*$ in the neighborhood of a point. In the following diagram,

\[
\begin{array}{ccc}
\mathbb{R}^k \times \hat{W} & \xrightarrow{\varphi} & V \\
\downarrow h & & \downarrow f \\
\mathbb{R}^m \times \hat{L} & \xrightarrow{\psi} & V^*,
\end{array}
\]

the map $\varphi$ is a conical chart for $X$ and $\psi$ a conical chart for $X^*$. The maps $\varphi$ and $\psi$ are stratified homeomorphisms but $f$ and $h$ are only homeomorphisms and stratified maps. The space $W$ is a link of $X$ and $L$ is a link of $X^*$. We have $m \geq k$. (In the case of a regular point, the link is the empty set.) Without loss of generality, we can suppose $h(0, w) = (0, v)$, where $v$ and $w$ are the apexes of the cones. We denote $s = \dim W$, $t = \dim L$ and deduce $s \geq t$ from $s + k = m + t$. The map $h$ also verifies $h(\mathbb{R}^k \times \{v\}) = B \times \{v\}$, with $B$ closed, and $h^{-1}(\mathbb{R}^m \times \{v\}) = \mathbb{R}^k \times \hat{A}$. By writing $\mathbb{R}^m \cong \mathbb{R}^k \times \hat{A}$ as $\mathbb{R}^{s-k-1} \cong \mathbb{R}^{s-k-1} \times \hat{A} \cong \mathbb{R}^{s-k-1} \times A$, we see that $A$ is a homology sphere of dimension $(m - k - 1)$. In particular, $W$ itself is an $(m - k - 1)$-dimensional homological sphere when $L = \emptyset$.

**Definition 4.8.** Let $\overline{p}$ be a GM-perversity. The **cleaving point** of $\overline{p}$ is the number $\ell_{\overline{p}} \in \mathbb{N}$ defined by

\[\ell_{\overline{p}} = \sup \{ k \in \mathbb{N} | \overline{p}(k) = \overline{t}(k) \} = \sup \{ k \in \mathbb{N} | D\overline{p}(k) = 0 \} .\]

In the general case of a filtered space, we show that the cleaving point reduces the determination of the $\overline{p}$-intersection fundamental groups to the case of the top perversity.

**Proposition 4.9.** Let $\overline{p}$ be a GM-perversity, $X$ be an $n$-dimensional filtered space without stratum of codimension 1 and $x_0$ be a regular point of $X$. Then, we have

\[\pi_1^\overline{p}(X, x_0) = \pi_1^\overline{p}(X \backslash X_{n-\ell_{\overline{p}}}, x_0).\]

**Proof.** In the proof of Theorem 4.6, we have already noticed that the 1-skeleton of $\mathcal{S}_{\overline{p}} X$ is isomorphic to the 1-skeleton of $\text{Sing}(X \backslash \Sigma_X)$ and that a 2-simplex, $\eta: \Delta^2 \to X$ is $\overline{p}$-full if, and only if, $\dim \eta^{-1} S \leq 2 - D\overline{p} (\text{codim} S) = 2 - (2 - D\overline{p} (\text{codim} S))$ for any singular stratum $S$. Since $D\overline{p} (\text{codim} S) \geq 0$, we distinguish two cases.

- If $D\overline{p} (\text{codim} S) \geq 1$, which corresponds to $\text{codim} S \geq \ell_{\overline{p}} + 1$ or $\text{dim} S \leq n - \ell_{\overline{p}} - 1$, then we have $\dim \eta^{-1} S \leq -D\overline{p} (\text{codim} S) \leq -1$. This implies $\eta^{-1} S = \emptyset$, which is equivalent to $\eta: \Delta^2 \to X \backslash X_{n-\ell_{\overline{p}}}$, with the restriction on $S$.

- If $D\overline{p} (\text{codim} S) = 0$, which corresponds to $\text{codim} S \leq \ell_{\overline{p}}$ or $\text{dim} S \geq n - \ell_{\overline{p}}$ then we have $\dim \eta^{-1} S \leq -D\overline{p} (\text{codim} S) = 0$ which is equivalent to $\dim \eta^{-1} S \leq 2 - D\overline{p} (\text{codim} S) - 2$, with the restriction on $S$.

In short, the $\overline{p}$-full 2-simplexes are exactly the simplexes $\eta: \Delta^2 \to X \backslash X_{n-\ell_{\overline{p}}}$ such that $\dim \eta^{-1} S \leq 2 - D\overline{p} (\text{codim} S) - 2$ for any stratum $S$ of $X \backslash X_{n-\ell_{\overline{p}}}$. We have proven

\[\mathcal{S}_{\overline{p}}(X) = \mathcal{S}_{\overline{p}}(X \backslash X_{n-\ell_{\overline{p}}})\]

and, therefore, $\pi_1^\overline{p}(X, x_0) = \pi_1^\overline{p}(X \backslash X_{n-\ell_{\overline{p}}}, x_0)$. \(\square\)

Under the hypothesis of Corollary 4.2, we can reduce the determination of $\overline{p}$-intersection fundamental groups to that of classic fundamental groups with $\pi_1^\overline{p}(X, x_0) = \pi_1(X \backslash X_{n-\ell_{\overline{p}}}, x_0)$.
5. Topological invariance of intersection homotopy groups

In this section, we study the topological invariance of perverse homotopy groups and prove it when the regular parts of $X$ and of its intrinsic filtration coincide. Let us notice that this is a condition on the topological space $X$ itself: it means that there is no regular point equivalent to a singular point for the relation of King ([23]) quoted in Recall 4.7. We have already proven that the perverse fundamental group is a topological invariant under the same hypothesis (Theorem 4.6) and that there are counterexamples in the general case (Example 4.3).

Theorem 5.1. Let $\overline{\mathcal{P}}$ be a GM-perversity and $X$ be a CS set without stratum of codimension 1. We denote by $X^*$ the associated CS set with the intrinsic filtration. If there is no exceptional stratum, then the map $\nu : X \to X^*$ induces an isomorphism $\pi_i^\mathcal{P}(X, x_0) \cong \pi_i^\mathcal{P}(X^*, x_0)$, for any $j$ and any regular point $x_0$.

The proof begins with a local version in a conical chart.

Proposition 5.2. Let $\overline{\mathcal{P}}$ be a GM-perversity, $X$ be a CS set without stratum of codimension 1. We denote by $X^*$ the associated CS set with the intrinsic filtration. Let $S$ be a singular stratum of $X$ and $V$ be a conical chart of $x \in S$. We suppose that the stratum $S$ is included in a singular stratum of $X^*$ and that there are isomorphisms, $\pi_0^\mathcal{P} \nu : \pi_0^\mathcal{P}(V) \cong \pi_0^\mathcal{P}(V^*)$ and $\pi_j^\mathcal{P} \nu : \pi_j^\mathcal{P}(V, x_0) \cong \pi_j^\mathcal{P}(V^*, x_0)$, for any regular point $x_0$. If the map

$$\pi_j^\mathcal{P} \nu : \pi_j^\mathcal{P}(V, x_0) \to \pi_j^\mathcal{P}(V^*, x_0), \quad (5.1)$$

is an isomorphism for any $j$, then the next map is also an isomorphism,

$$\pi_0^\mathcal{P} \nu : \pi_0^\mathcal{P}(V, x_0) \to \pi_0^\mathcal{P}(V^*, x_0). \quad (5.2)$$

Proof. We can suppose that $V \cong \mathbb{R}^k \times \mathbb{C}W$ is a conical chart, with $\mathbb{W}$ the apex of the cone $\mathbb{C}W$. Thus, there is a homeomorphism $S \cap V \cong \mathbb{R}^k \times \{w\}$. We take the description given in Recall 4.7. We have $V^* \cong \mathbb{R}^m \times \mathbb{C}L$ with $\mathbb{W}$ the apex of the cone $\mathbb{C}L$ and $m \geq k$. If we denote by $B \times \{v\}$ the image of $\mathbb{R}^k \times \{w\}$ under the previous homeomorphism, we also have $(V^* \backslash S) \cong \mathbb{R}^m \times \mathbb{C}L \backslash (B \times \{v\})$. We set $s = \dim W$ and $t = \dim L$; they verify $s \geq t$ since $s + k = m + t$. The hypothesis on the stratum $S$ means $s \geq 1$ and $t \geq 1$. For the rest of the proof, we shorten the notation as follows,

$$V^* \cong P \cong \mathbb{R}^m \times \mathbb{C}L, \quad V^* \backslash S \cong Q = (\mathbb{R}^m \times \mathbb{C}L) \backslash (B \times \{v\}), \quad R = (\mathbb{R}^m \times \mathbb{C}L) \backslash (\mathbb{R}^m \times \{w\}),$$

$$V \cong P' = \mathbb{R}^k \times \mathbb{C}W, \quad V \backslash S \cong Q' = (\mathbb{R}^k \times \mathbb{C}W) \backslash (\mathbb{R}^k \times \{w\}).$$

By using Lemmas 2.8, 2.9 and the hypotheses on the perverse fundamental groups, the existence of the isomorphisms (5.1) and (5.2) between perverse homotopy groups is equivalent to the existence of isomorphisms in homology with local coefficients. Let us consider a local coefficient system, $\mathcal{E}$, on $\mathcal{P}^\mathcal{P}(P)$. We abuse notation and still denote by $\mathcal{E}$ the local coefficient systems on $\mathcal{P}^\mathcal{P}(P^*)$, $\mathcal{P}^\mathcal{P}(Q)$, $\mathcal{P}^\mathcal{P}(Q')$, $\mathcal{P}^\mathcal{P}(R)$, obtained by pullback along $\nu$ or by induction on a simplicial subset. Under these conditions, we know there exist homology exact sequences of pairs (or triples) and excisions (Theorem 2.14). If $D\mathcal{P}(s + 1) = 0$, as $\mathcal{P}^\mathcal{P}(V)$ and $\mathcal{P}^\mathcal{P}(V^*)$ are simply connected, we choose the constant system with coefficients in $Z$. We do not mention explicitly the coefficient $\mathcal{E}$ or $Z$ in the rest of the proof. We proceed in two steps.

- Let $i \leq D\mathcal{P}(s + 1)$. We set $Z = B \times (\mathbb{C}L \backslash \{v\})$. As $Z \subset R \subset Q$ and the closure of $Z$ being included in the interior of $Q$, we can use an excision (Theorem 2.14) and get

$$H_i(\mathcal{P}^\mathcal{P}(Q), \mathcal{P}^\mathcal{P}(R)) \cong H_i(\mathcal{P}^\mathcal{P}(Q, Z), \mathcal{P}^\mathcal{P}(R, Z)).$$

Taking their image by the restriction (see (4.3)) of $h^{-1}$ gives stratified homeomorphisms $Q, Z \cong ((\mathbb{R}^k \times \mathbb{C}A) \backslash (\mathbb{R}^k \times \{w\})) \times \mathbb{C}L$ and $R^i/Z \cong ((\mathbb{R}^k \times \mathbb{C}A) \backslash (\mathbb{R}^k \times \{w\})) \times (\mathbb{C}L \backslash \{v\})$, where $w$ is the apex of $\mathbb{C}A$. By using Proposition 2.4 and the fact that $A$ is a homology sphere of dimension $m - 1 - k$, we deduce for
any i,
\[ H_i(\mathcal{G}_P, \mathcal{G}_R) \cong H_i(\mathcal{G}(A \times \mathcal{L}), \mathcal{G}(A \times (\mathcal{L}\setminus\{v\}))) \]
\[ \cong H_i(\text{Sing } A \times \mathcal{G}(\mathcal{L}), \text{Sing } A \times \mathcal{G}(\mathcal{L}\setminus\{v\})) \]
\[ \cong H_i(\mathcal{G}(\mathcal{L}), \mathcal{G}(\mathcal{L}\setminus\{v\})) \oplus H_{i-m+1+k}(\mathcal{G}(\mathcal{L}), \mathcal{G}(\mathcal{L}\setminus\{v\})) \]
\[ \cong H_i(\mathcal{G}_P, \mathcal{G}_R) \oplus H_{i-m+1+k}(\mathcal{G}(\mathcal{L}), \mathcal{G}(\mathcal{L}\setminus\{v\})). \] (5.3)

In particular, this isomorphism implies that the canonical injection induces a surjective map \( H_i(\mathcal{G}_P, \mathcal{G}_R) \to H_i(\mathcal{G}_P, \mathcal{G}_R) \). By incorporating this information in the homology long exact sequence of the triple \((\mathcal{G}_P, \mathcal{G}_Q, \mathcal{G}_R)\), we obtain short exact sequences, for any i,
\[ 0 \to H_{i+1}(\mathcal{G}_P, \mathcal{G}_Q) \to H_i(\mathcal{G}_P, \mathcal{G}_R) \to H_i(\mathcal{G}_P, \mathcal{G}_R) \to 0. \] (5.4)

Recall \( t \leq s \). As \( D\mathcal{P} \) is a GM-perversity, we have the inequality \( D\mathcal{P}(s+1) \leq (s-t) + D\mathcal{P}(t+1) \). By using it and \( i \leq D\mathcal{P}(s+1) \), we get \( i - m + 1 + k \leq D\mathcal{P}(s+1) + t - s + 1 \leq D\mathcal{P}(t+1) + 1 \). Thus, from Proposition 2.10, the isomorphism (5.3) becomes
\[ H_i(\mathcal{G}_P, \mathcal{G}_R) \cong H_i(\mathcal{G}_P, \mathcal{G}_R) \] for \( i \leq D\mathcal{P}(s+1) \).

From (5.4), we deduce
\[ H_i(\mathcal{G}_P, \mathcal{G}_Q) = 0 \] for \( i \leq D\mathcal{P}(s+1) + 1. \) (5.5)

Let us write \( h_j \), for \( j = 1, 2, 3 \), the homomorphisms induced by the homeomorphism \( h \) of the diagram (4.3), between the homology long exact sequences of the pairs \((\mathcal{G}_P, \mathcal{G}_Q)\) and \((\mathcal{G}_P, \mathcal{G}_R)\),
\[ \cdots \to H_{i+1}(\mathcal{G}_P, \mathcal{G}_Q) \to H_i(\mathcal{G}_P, \mathcal{G}_Q) \to H_i(\mathcal{G}_P, \mathcal{G}_R) \to H_i(\mathcal{G}_P, \mathcal{G}_R) \to \cdots \]
\[ \cdots \to H_{i+1}(\mathcal{G}_P, \mathcal{G}_Q) \to H_i(\mathcal{G}_P, \mathcal{G}_Q) \to H_i(\mathcal{G}_P, \mathcal{G}_R) \to H_i(\mathcal{G}_P, \mathcal{G}_R) \to \cdots \]

By hypothesis, the map \( (h_1)_i \) is an isomorphism for any \( i \). For \( i \leq D\mathcal{P}(s+1) + 1 \), we know that \( H_i(\mathcal{G}_P, \mathcal{G}_Q) = 0 \) from Proposition 2.10 and \( H_i(\mathcal{G}_P, \mathcal{G}_R) = 0 \) from (5.5). This implies, with the five lemma, that \( (h_2)_i \), is an isomorphism for any \( i \leq D\mathcal{P}(s+1) \) and a surjection for \( i = D\mathcal{P}(s+1) + 1 \). We can apply Lemma 2.9 to \( \mathcal{G}_P' \to \mathcal{G}_P \) and deduce
\[ \pi_i(\mathcal{G}_P', v) \cong \pi_i(\mathcal{G}_P, h(v)) \] for \( i \leq D\mathcal{P}(s+1) \) and each basepoint \( v \).

• From \( D\mathcal{P}(s+1) + 1 \geq D\mathcal{P}(t+1) + 1 \), Propositions 2.4 and 3.5 imply \( \pi_i(\mathcal{G}_P', v) = \pi_i(\mathcal{G}_P, v) = 0 \), for all \( i \geq D\mathcal{P}(s+1) + 1 \) and each basepoint \( v \).

Combining the two cases, we get the expected result (5.2). \( \square \)

Proof of Theorem 5.1. All the CS sets that appear in the rest of the proof are supposed to be without exceptional stratum. From Theorem 4.6, we know that \( \nu \) induces an isomorphism between the fundamental groups, \( \pi_1^J(X, x_0) \cong \pi_1^X(X^*, x_0) \) for any regular point \( x_0 \in X \). From Remark 3.3 and the hypothesis, we have \( \pi_0^J(X) = \pi_0(X/\Sigma X) \cong \pi_0(X/\Sigma X^*) \cong \pi_0^X(X^*) \). Let \( U \) be an open subset of \( X \), as the regular parts of \( U \) and \( U^* \) coincide, we also get \( \pi_1^J(U) \cong \pi_1^X(U^*) \) and \( \pi_1^J(U, x_0) \cong \pi_1^X(U^*, x_0) \) for any regular point \( x_0 \in U \).

We repeat an argument of King in [23, Section 3] (see also [12, Theorem 5.1]) using an induction on the depth of \( X \). The result is obviously true for CS sets of depth 0. We say that the CS set \( X \) has the property \( W \) if the map \( \pi_1^J: \pi_1(X, x_0) \to \pi_1(X^*, x_0) \) is an isomorphism for any \( j \) and any regular point \( x_0 \). We consider the following properties.

**P(\ell):** The CS sets of depth \( \leq \ell \) have property \( W \).

**Q(\ell):** The CS sets of the form \( M \times \mathcal{L} \mathcal{W} \), with \( M \) a trivially filtered manifold and \( W \) a compact filtered space of depth \( \leq \ell \), have property \( W \).
The CS sets of the form $\mathbb{R}^k \times \mathcal{C}W$, with $\mathbb{R}^k$ trivially filtered and $W$ a compact filtered space of dimension $\leq \ell$, have property $W$.

We will show, $P(\ell) \Rightarrow R(\ell)$, $R(\ell) \Rightarrow Q(\ell)$ and $P(\ell) \land Q(\ell) \Rightarrow P(\ell + 1)$, which gives the proof.

$P(\ell) \Rightarrow Q(\ell)$. We know ([23, Lemma 2]) that there is a coarsening $Z$ of $\mathcal{C}W$ such that $(M \times \mathcal{C}W)^* \cong M \times Z$ and $(\mathbb{R}^k \times \mathcal{C}W)^* \cong \mathbb{R}^k \times Z$. From Proposition 2.4, we deduce that $\mathcal{G}(\mathbb{R}^k \times \mathcal{C}W)^*$ is of the homotopy type of $\mathcal{G}Z$ and that $\mathcal{G}(\mathbb{R}^k \times \mathcal{C}W)$ is of the homotopy type of $\mathcal{G}(\mathcal{C}W)$. The conclusion comes from a new application of Proposition 2.4 to $M \times \mathcal{C}W$ and $M \times Z$.

$P(\ell) \Rightarrow R(\ell)$. This is Proposition 5.2.

$P(\ell) \land Q(\ell) \Rightarrow P(\ell + 1)$. If $U_i$ is an increasing sequence of open subsets of $X$ such that $\pi^*_i(U_i) \cong \pi^*_i(U_i^*)$, for any $i$, then the classical argument of compactness (see [19, Proposition 15.9]) gives an isomorphism $\pi^*_i(\bigcup U_i) \cong \pi^*_i((\bigcup U_i)^*)$. With the properties on fundamental perverse groups and connected components recalled at the beginning of this proof, we can use Lemma 2.8 and are reduced to prove that $\mathcal{G}(\mathcal{G}X \to \mathcal{G}X^*)$ induces an isomorphism in homology for any local coefficient system, $\mathcal{E}$, on $\mathcal{G}X^*$. As we have established a Mayer-Vietoris exact sequence with coefficients in $\mathcal{E}$ in Theorem 2.13, the presentation using Zorn’s Lemma, made by G.Friedman in [12, Page 257], can be reproduced verbatim here. □

6. Intersection Hurewicz theorem

Let $(X, \mathcal{P})$ be a perverse space and $x_0 \in X$ be a regular point. Any $\mathcal{P}$-full simplex of $X$ being a chain of $\mathcal{P}$-intersection, we have a canonical map, $J^\mathcal{P}_i: H_i(\mathcal{G}X; \mathbb{Z}) \to H_i^\mathcal{P}(X; \mathbb{Z})$, induced by the inclusion of chain complexes, $C_i(\mathcal{G}X; \mathbb{Z}) \to C_i^\mathcal{P}(X; \mathbb{Z})$. By definition, we call $\mathcal{P}$-intersection Hurewicz homomorphism, the composition

$$h^\mathcal{P}_i: \pi^\mathcal{P}_i(X, x_0) \to H_i^\mathcal{P}(X; \mathbb{Z})$$

(6.1) of $J^\mathcal{P}_i$ with the Hurewicz homomorphism, $h_\#(\mathcal{G}X): \pi_\#(\mathcal{G}X, x_0) \to H_\#(\mathcal{G}X, x_0; \mathbb{Z})$, of $\mathcal{G}X$. This section is devoted to the proof of the following result.

Theorem 6.1. Let $(X, \mathcal{P})$ be a $\mathcal{P}$-connected perverse CS set. Then the following properties are verified.

1) The intersection Hurewicz homomorphism, $h^\mathcal{P}_i: \pi^\mathcal{P}_i(X, x_0) \to H_i^\mathcal{P}(X; \mathbb{Z})$, is isomorphic to the abelianisation $\pi^\mathcal{P}_i(X, x_0) \to \pi^\mathcal{P}_i(X, x_0)^{ab}$, for any regular point $x_0$ of $X$.

2) Let $k \geq 2$. We suppose $\pi^\mathcal{P}_j(X) = \pi^\mathcal{P}_j(L) = 0$ for every link $L$ of $X$, and each $j \leq k - 1$. Then, the intersection Hurewicz homomorphism $h^\mathcal{P}_j: \pi^\mathcal{P}_j(X, x_0) \to H_j^\mathcal{P}(X; \mathbb{Z})$ is an isomorphism for $j \leq k$ and a surjection for $j = k + 1$.

We already know that the $\mathcal{P}$-intersection homology of a filtered space, $X$, can be different from the homology of $\mathcal{G}X$, see Example 2.11. Taking in account the (classic) Hurewicz theorem of $\mathcal{G}X$, Theorem 6.1 above is proven if we establish a convenient relationship between the homologies $H_\#(\mathcal{G}X; \mathbb{Z})$ and $H_j^\mathcal{P}(X; \mathbb{Z})$. As the proof requires different techniques, we split it into two parts. Let us begin with the two first homologies.

Theorem 6.2. Let $(X, \mathcal{P})$ be a perverse space. The inclusion of chain complexes induces isomorphisms in homology, in degrees 0 and 1,

(a) $J^\mathcal{P}_0: H_0(\mathcal{G}X; \mathbb{Z}) \cong H_0^\mathcal{P}(X; \mathbb{Z})$,
(b) $J^\mathcal{P}_1: H_1(\mathcal{G}X; \mathbb{Z}) \cong H_1^\mathcal{P}(X; \mathbb{Z})$.

In general, $J^\mathcal{P}_i$ is not an isomorphism, as shows the torus in Example 2.11. Let us emphasize that we are dealing with a general perversity $\mathcal{P}$, without any restriction on it. (For instance, the global argument is simpler with the hypothesis $\mathcal{P} \leq \mathcal{T}$.) We begin with a first technical lemma.
Lemma 6.3. Let \((X, \mathcal{P})\) be a perverse space. Let \(\sigma_0, \sigma_1 : [0, 1] \to X\) be two \(\mathcal{P}\)-allowable simplexes with \(\sigma_0(1) = \sigma_1(0)\) and \(\sigma_0(0), \sigma_1(1) \in C^0_0(X)\). We consider the simplex \(\sigma_0 * \sigma_1 : [0, 1] \to X\) defined by
\[\sigma_0 * \sigma_1(t) = \begin{cases} \sigma_0(2t), & \text{if } t \leq 1/2, \\ \sigma_1(2t - 1), & \text{if } t \geq 1/2. \end{cases}\]
Then, there exists a \(\mathcal{P}\)-intersection 2-chain \(\tau\) verifying
\[\partial \tau = \sigma_0 + \sigma_1 - \sigma_0 * \sigma_1.\] (6.2)

Proof. This is a well-known argument, the delicate point is the verification of the \(\mathcal{P}\)-allowability of the auxiliary simplexes and of their boundary. First, the 1-simplex \(\overline{\sigma}_0\) defined by \(\overline{\sigma}_0(t) = \sigma_0(1 - t)\) is \(\mathcal{P}\)-allowable since \(\sigma_0\) is so. The 1-simplex \(\sigma_0 * \sigma_1\) is also \(\mathcal{P}\)-allowable since, for each singular stratum \(S\), we have:
\[\dim(\sigma_0 * \sigma_1)^{-1}S \leq \max(\dim \sigma_0^{-1}S, \dim \sigma_1^{-1}S) \leq 1 - D \mathcal{P}(S) - 2.\]
For any \(j \in \mathbb{N}\), we consider \(j\)-simplexes, constant on \(\sigma_0(0), \varepsilon_j : \Delta^j \to X\). They are \(\mathcal{P}\)-allowable if, for each singular stratum \(S\) with \(\sigma_0(0) \in S\), we have \(\dim \varepsilon_j^{-1}S \leq j - D \mathcal{P}(S) - 2\), which is a consequence of \(\sigma_0(0) \in C^0_0(X)\). Since \(\varepsilon_j\) is 0 or \(\varepsilon_{j-1}\), the simplexes \(\varepsilon_j\) are of \(\mathcal{P}\)-intersection.

Let \(a_0, a_1, a_2\) be the vertices of \(\Delta^2\). The 2-simplex, \(\beta : \Delta^2 \to X\), defined by \(\beta(t_0a_0 + t_1a_1 + t_2a_2) = \sigma_0(t_1)\) is \(\mathcal{P}\)-allowable since, for each singular stratum \(S\), we have
\[\dim \beta^{-1}S \leq 1 + \dim \sigma_0^{-1}S \leq 2 - D \mathcal{P}(S) - 2.\]
A direct computation gives \(\partial \beta = \sigma_0 - \varepsilon_1 + \overline{\sigma}_0\) which is a \(\mathcal{P}\)-allowable chain. Let \(f : \Delta^2 \to [0, 1]\) be the linear map such that \(f(a_0) = 0\), \(f(a_1) = 0\) and \(f(a_2) = 1/2\). The 2-simplex, \(\tau : \Delta^2 \to X\), defined by \(\tau = (\sigma_0 * \sigma_1) \circ f\), is also \(\mathcal{P}\)-allowable since, for each singular stratum \(S\), we have
\[\dim \tau^{-1}S = \dim f^{-1}((\sigma_0 * \sigma_1)^{-1}S) \leq 1 + \dim(\sigma_0 * \sigma_1)^{-1}S \leq 2 - D \mathcal{P}(S) - 2.\]
A computation gives \(\partial \tau = \overline{\sigma}_0 - \sigma_1 + \sigma_0 * \sigma_1\) which is a \(\mathcal{P}\)-allowable chain. In summary, the 2-chain \(\gamma = \varepsilon_2 + \beta - \tau\) is of \(\mathcal{P}\)-intersection and verifies
\[\partial \gamma = \partial \varepsilon_2 + \partial \beta - \partial \tau = \varepsilon_1 + (\sigma_0 - \varepsilon_1 + \overline{\sigma}_0) - \overline{\sigma}_0 + \sigma_1 - \sigma_0 * \sigma_1 = \sigma_0 + \sigma_1 - \sigma_0 * \sigma_1.\]

Proof of Theorem 6.2. The equality \(C^0_0(X) = C_0(\mathcal{P}X)\) gives the surjectivity of \(\mathcal{J}^0\). Let us study the injectivity of \(\mathcal{J}_1^0\) and the surjectivity of \(\mathcal{J}_2^0\). We get the claim if for any chain \(\eta \in C^0(X)\) of this type, with \(\partial \eta \in C_0(\mathcal{P}X) = C^0_0(X)\), we find \(\alpha \in C_1(\mathcal{P}X)\) with \(\eta = \alpha - \partial C^0_0(X)\). This chain can be uniquely written \(\eta = \sum_{i \in I} n_i \sigma_i\) with \(n_i = \pm 1\).

As \(\eta \in C^1_1(X)\), the chain \(\eta\) and its boundary \(\partial \eta\) are \(\mathcal{P}\)-allowable. There can exist simplexes \(\sigma_i\) with a non-\(\mathcal{P}\)-allowable face but this fact must then be eliminated in the boundary \(\partial \eta\). Unfortunately such a simplex is not accepted in a Gajer space. We must therefore substitute them by simplexes with \(\mathcal{P}\)-allowable faces. Using a subdivision, we can suppose that each \(\sigma_i\), \(i \in I\), has at most one vertex which is not \(\mathcal{P}\)-allowable. Thus, we can write
\[\eta = \sum_{i \in I_0} n_i \sigma_i + \sum_{i \in I_1} n_i \sigma_i = \eta_0 + \eta_1,\]
where \(\sigma_i(0), \sigma_i(1) \in C^0_0(X)\) if \(i \in I_0\) and exactly one of these two points is \(\mathcal{P}\)-allowable if \(i \in I_1\). By definition, we have \(\eta_0 \in C_1(\mathcal{P}X)\). Let \(k \in I_1\) and suppose for simplicity that \(\sigma_k(1) \notin C^0_0(X)\). Since \(\partial \eta_1 \in C^0_0(X)\), there exists \(j \in I_1\) with \(\sigma_k(1) = \sigma_j(0)\) and \(n_k = n_j\). The chain \(n_1\) is a sum of chains \(n_k \sigma_k + n_j \sigma_j\) of this type. Following Lemma 6.3, we have \(\sigma_k + \sigma_j - \sigma_k * \sigma_j \in \partial C_2^0(X)\) and \(\sigma_k * \sigma_j \in C_1(\mathcal{P}X)\).

Let's get to the difficult point: the map \(\mathcal{J}_1^0\) is a monomorphism. Given \(\eta \in C^0_0(X)\) with \(\partial \eta \in C_1(\mathcal{P}X)\), we have to find \(\alpha \in C_2(\mathcal{P}X)\) with \(\partial \eta = \partial \alpha\). Let's start from a \(\mathcal{P}\)-allowable 2-simplex \(\Delta^2 \to X\), and
apply to it the pseudobarycentric subdivision recalled in Remark 1.14. By construction, all the simplexes containing the pseudobarycentre of $\Delta^2$ are $\mathfrak{P}$-allowable. This means that all the 1-faces of the new “small” simplexes are $\mathfrak{P}$-allowable, except at most one. Also, all the vertices are $\mathfrak{P}$-allowable except at most two. Thus, any chain $\eta \in C^2_\mathfrak{P}(X)$ can be uniquely written
\[ \eta = \sum_{i \in J_1} n_i \sigma_i + \sum_{i \in J_2} n_i \sigma_i, \]
with $n_i = \pm 1$, all the 1-faces of the $\sigma_i$ with $i \in J_1$ are $\mathfrak{P}$-allowable and the $\sigma_i$ with $i \in J_2$ have exactly two $\mathfrak{P}$-allowable 1-faces. We proceed in two steps, beginning with the 1-faces.

- **First step:** Cancelation of the bad 1-faces of $J_2$. Let $\sigma_k : \langle a_0, a_1, a_2 \rangle \to X$ with $k \in J_2$. Suppose that the restriction $\tau$ of $\sigma_k$ to $\langle a_0, a_1 \rangle$ is the bad face. So, there exists $\sigma_j : \langle a_0, a_1, a_3 \rangle \to X$, $j \in J_2$, in such a way that $\tau$ does not appear in the boundary of $n_k \sigma_k + n_j \sigma_j$. We proceed to a pseudobarycentric subdivision of $\sigma_k$, of pseudobarycentre $b$, see Remark 1.14. By construction, the restrictions to $\langle a_0, b \rangle$ and $\langle a_1, b \rangle$ are $\mathfrak{P}$-allowable. Let us define two new simplexes $\sigma'_k : \Delta^2 \to X$ and $\sigma'_j : \Delta^2 \to X$ as indicated in the figure below.

![Diagram showing the cancelation of bad 1-faces](image)

So, by construction, the 2-simplexes $\sigma'_k, \sigma'_j$ and all their 1-faces are $\mathfrak{P}$-allowable and we also have $\partial (n_k \sigma'_k + n_j \sigma'_j) = \partial(n_k \sigma_k + n_j \sigma_j)$. The 2-chain
\[ \eta' = \eta - (n_k \sigma_k + n_j \sigma_j) + (n_k \sigma'_k + n_j \sigma'_j) \in C^2_\mathfrak{P}(X) \]
verifies $\partial \eta' = \partial \eta$ and the cardinal of the corresponding subset $J'_2$ is strictly smaller than that of $J_2$. With iterations, we get a 2-chain $\alpha = \sum_{i \in K} n_i \sigma_i$, with $n_i = \pm 1$ and all the faces of the $\sigma_i$ $\mathfrak{P}$-allowable except possibly two vertices.

- **Second step:** the vertices. We start with the previous $\alpha$. Proceeding as before to a pseudobarycentric subdivision, we can suppose that all the faces are $\mathfrak{P}$-allowable, except possibly one vertex. Therefore, we decompose
\[ \alpha = \sum_{i \in K_1} n_i \sigma_i + \sum_{i \in K_2} n_i \sigma_i = \alpha_1 + \alpha_2 \]
where all the faces of the $\sigma_i$ with $i \in K_1$ are $\mathfrak{P}$-allowable and all the faces of the $\sigma_i$ with $i \in K_2$ are $\mathfrak{P}$-allowable except one vertex. Let $i \in K_2$ and suppose that $\sigma_i(a_0)$ is the only not $\mathfrak{P}$-allowable vertex. Recall $\partial \alpha = \partial \eta \in C_1(\mathfrak{P}X)$. Thus $\sigma_i(a_0)$ does not belong to $\partial \alpha$ and there exists $k \in K_2$ such that the restrictions of $\sigma_i$ and $\sigma_k$ to the face $\langle a_0, a_1 \rangle$ verifies
\[ n_i \sigma_i(\langle a_0, a_1 \rangle) + n_k \sigma_k(\langle a_0, a_1 \rangle) = 0. \]
(We can have a cancellation with other faces of $\sigma_k$ but it is the same pattern with a sign adjustment.)

By repeating this process, we construct a chain with $\{i_0, \ldots, i_m\} \subset K_2$ and $\partial \beta \in C_1(\mathfrak{P}X)$. This chain appears as a map still denoted by $\beta : K_\beta \to X$ from a polyhedron $K_\beta$ as in the figure beside. The
letters $b_i$ are a notation for the vertices of $\Delta^2 = \langle a_0, a_1, a_2 \rangle$.

The illustrated situation corresponds to an annulation of the vertex $\sigma_{i_k}(a_0)$ after an iteration of 4 steps. The case of 0 iteration corresponds to a disk. In general, we get a similar figure after a finite number $m$ of steps.

The chain $\alpha$ is the sum of $\alpha_1 \in C_2(\mathcal{F}_pX)$ and some chains of type (6.3). So, we get the claim if we find $\beta' \in C_2(\mathcal{F}_pX)$ with $\partial \beta = \partial \beta'$. This new chain $\beta'$ is obtained from $\beta$ slightly moving the vertex $a_0$ in $a'_0$, keeping the $b_i$ unchanged, and corresponding to a new triangulation of the polyhedron $K_\beta$. We denote by $\sigma'_i$ the simplex corresponding to $\sigma_i$, in this new triangulation. We need to prove that each $\sigma'_{i_k}$ is a $\mathcal{F}$-full simplex for a convenient choice of $a'_0$. Since $\partial \eta \in C_1(\mathcal{F}_pX)$ then it suffices to prove that all the faces of the simplexes $\sigma'_{i_k}$ meeting $\beta(a'_0)$ are $\mathcal{F}$-allowable. For that, we adapt to the case of polyhedra the proof of [9, Proposition 6] made for simplexes. We distinguish the three possible dimensions.

1) The 2-dimensional faces. The simplexes $\sigma'_{i_k}$ are $\mathcal{F}$-allowable for any choice of $a'_0$, since we have, for any singular stratum,

$$\max\{\dim \sigma'_{i_k} \ S \mid 0 \leq k \leq m\} = \dim \beta^{-1} S = \max\{\dim \sigma_{i_k}^{-1} S \mid 0 \leq k \leq m\}\}
\leq 2 - \mathcal{D}(S) - 2,$$

Note that $\beta^{-1} S = \emptyset$ for each singular stratum $S \in S_X$ with $\mathcal{D}(S) > 0$.

Before studying the 0 and 1-dimensional faces, we establish some notation. Since the simplexes of $\beta$, as well as their faces, are $\mathcal{F}$-allowable, we get the following properties (cf. Definition 1.12).

- The subset $D = \{\beta^{-1} S \mid D_p(S) = 0\}$ is a finite subset included in $K_\beta = K_\beta \setminus \partial K_\beta$.
- The subset $E = \{\beta^{-1} S \mid D_p(S) = -1\}$ is included in $K_\beta$ and its dimension is smaller than 1. We choose a polyhedron $P$ such that $E \subset P$ and $\dim P \leq 1$.

2) The 0-dimensional faces. All vertices, except $a'_0$, belong to the boundary and are therefore $\mathcal{F}$-allowable. The restriction of $\beta$ to $\langle a'_0, b_i \rangle$ is $\mathcal{F}$-allowable if $a'_0 \notin D \cup P$. By dimensional reasons, the open set $O = K_\beta \setminus (D \cup P)$ is dense in $K_\beta$. If we choose $a'_0$ in this subset (which is not empty!) we get the $\mathcal{F}$-allowability of all the 0-dimensional faces of the simplexes $\sigma'_{i_k}$.

3) The 1-dimensional faces. The one-dimensional faces appearing in $\partial \beta$ are $\mathcal{F}$-allowable. Thus, we are reduced to the 1-simplexes obtained by the restriction of $\beta$ to $\Delta^1 = \langle a'_0, b_i \rangle$, with $0 \leq i \leq m$. They are $\mathcal{F}$-allowable if the following conditions hold, for any $i$,

$$\left\{ \begin{array}{ll} (i) \ D \cap \langle a'_0, b_i \rangle = \emptyset, \\
(ii) \ P \cap \langle a'_0, b_i \rangle \text{ is a finite set.} \end{array} \right. \quad (6.4)$$

Since $D$ is a finite set, the family of points $a'_0 \in K_\beta$ verifying (i) are a dense open subset $O'$ of $K_\beta$. Notice that $O \cap O'$ is also a dense open subset of $K_\beta$. Applying the general position principle ([29, Section 5.34]), we find $a'_0 \in O \cap O'$ verifying (ii) for any $i$. This gives (6.4).

We have proven that all faces of the simplexes $\sigma'_{i_k}$ are $\mathcal{F}$-allowable. We obtain $\beta' \in C_2(\mathcal{F}_pX)$ and $\partial \beta = \partial \beta'$ which ends the proof.\qed
We continue our study of the relationship between the homologies $H_*(\mathcal{F}_p X)$ and $H^T_k(X)$ by looking to higher degrees. As the links are supposed to be $\mathcal{P}$-connected, we can suppress the mention of the basepoint in the rest of this section.

**Theorem 6.4.** Let $(X, \mathcal{P})$ be a perverse CS set and $k \geq 2$ be an integer. We suppose $\pi^T_j(L) = 0$ for each link $L$ of $X$ and each $j \leq k - 1$. Then, the inclusion of chain complexes, $C_*(\mathcal{F}_p X; \mathbb{Z}) \hookrightarrow C^T_*(X; \mathbb{Z})$, induces

(i) an isomorphism: $j^T_j: H_j(\mathcal{F}_p X; \mathbb{Z}) \xrightarrow{\cong} H^T_j(X; \mathbb{Z})$, for $j \leq k$, and

(ii) an epimorphism $\varphi^T_{k+1}: H_{k+1}(\mathcal{F}_p X; \mathbb{Z}) \twoheadrightarrow H^T_{k+1}(X; \mathbb{Z})$.

Before the proof, we introduce a property adapted to conification and Mayer-Vietoris sequences.

**Definition 6.5.** Let $(X, \mathcal{P})$ be a perverse space and $k \in \mathbb{Z}$. We say that $X$ verifies the property $P_k$, written $P_k(X)$, if the inclusion, $C_*(\mathcal{F}_p X; \mathbb{Z}) \hookrightarrow C^T_*(X; \mathbb{Z})$, of chain complexes induces

(i) the isomorphism $j^T_j: H_j(\mathcal{F}_p X; \mathbb{Z}) \xrightarrow{\cong} H^T_j(X; \mathbb{Z})$, for $j \leq k$, and

(ii) the epimorphism $\varphi^T_{k+1}: H_{k+1}(\mathcal{F}_p X; \mathbb{Z}) \twoheadrightarrow H^T_{k+1}(X; \mathbb{Z})$.

Notice that Theorem 6.2 gives $P_0(X)$.

**Proposition 6.6.** Let $L$ be a compact filtered space and $\mathcal{E} L$ be the open cone, with the conic filtration. Let $\mathcal{P}$ be a perversity on $\mathcal{E} L$, we also denote by $\mathcal{P}$ the perversity induced on $L$. Let $k \geq 2$ be an integer. We suppose $\pi^T_j(L) = 0$ for each $j \leq k - 1$. Then, the property $P_k(L)$ implies $P_k(\mathcal{E} L)$.

In other words, we prove that $P_k(\mathcal{E} L \setminus \{v\})$ implies $P_k(\mathcal{E} L)$, cf. Proposition 2.4 and [9, Corollary 1].

**Proof.** Let $v$ be the apex of the cone $\mathcal{E} L$, $\mathcal{P}(v) = p$ and $D_{\mathcal{P}}(v) = q$.

(i) With Theorem 6.2, we take $2 \leq j \leq k$. We have $C_{\leq q+1}(\mathcal{F}_p \mathcal{E} L) = C_{\leq q+1}(\mathcal{F}_p (\mathcal{E} L \setminus \{v\}))$ and $C_{\leq q+1}(\mathcal{E} L) = C_{\leq q+1}(\mathcal{E} L \setminus \{v\})$. If $j \leq q$, then the map $j^T_{\mathcal{F}, \mathcal{P}}: H_j(\mathcal{F}_p \mathcal{E} L) \to H^T_j(\mathcal{E} L)$ is isomorphic to the map, $j_{\mathcal{F}, \mathcal{P}}: H_j(\mathcal{F}_p (\mathcal{E} L \setminus \{v\})) \to H^T_j(\mathcal{E} L \setminus \{v\})$ induced by the inclusion. This is an isomorphism with the hypothesis $P_k(\mathcal{E} L \setminus \{v\})$.

Suppose $q + 1 \leq j$. From $q \leq k - 1$ and Proposition 3.5, we deduce $\pi^T_j(\mathcal{E} L) = 0$. The (classic) Hurewicz theorem implies $H_*(\mathcal{E} L) = 0$ and the map $j^T_{\mathcal{F}, \mathcal{P}}$ is a monomorphism. From [9, Proposition 3], we have $H^T_j(\mathcal{E} L) = 0$ and the map $j^T_{\mathcal{F}, \mathcal{P}}$ is an epimorphism.

(ii) We have to prove that $\varphi^T_{k+1, \mathcal{E} L}: H_{k+1}(\mathcal{F}_p \mathcal{E} L) \to H^T_{k+1}(\mathcal{E} L)$ is an epimorphism.

If $k + 1 \geq q + 1$ then $H^T_{k+1}(\mathcal{E} L) = 0$ (cf. [9, Proposition 3] and $k + 1 \neq 0$) and therefore $\varphi^T_{k+1, \mathcal{E} L}$ is an epimorphism. Let us suppose $k + 1 \leq q$. Consider the commutative diagram,

$$
\begin{array}{ccc}
\pi^T_{k+1}(L) & \xrightarrow{J_2} & \pi^T_{k+1}(\mathcal{E} L) \\
\downarrow J_{k+1, \mathcal{E} L} \quad & & \quad \downarrow \varphi^T_{k+1, \mathcal{E} L} \\
H_{k+1}(\mathcal{F}_p L) & \xrightarrow{\varphi^T_{k+1, \mathcal{E} L}} & H^T_{k+1}(L) \\
\end{array}
$$

where the maps $J_*$ are induced by the inclusion $L \to \mathcal{E} L$ and $h_{*, *}$ are the (classic) Hurewicz homomorphisms. We know that $J_1$ is an isomorphism ([9, Proposition 3] and $k + 1 \leq q$) and that $\varphi^T_{k+1, \mathcal{E} L}$ is an epimorphism grants to $P_k$(L). From $\pi^T_j(\mathcal{E} L) = 0$ for $\ell \leq k - 1$ and the classic Hurewicz theorem ($k \geq 2$) we obtain the surjectivity of $h_{k+1, \mathcal{E} L}$. We deduce the surjectivity of $\varphi^T_{k+1, \mathcal{E} L}$.

**Proposition 6.7.** Let $(X, \mathcal{P})$ be a perverse space and $k \in \mathbb{N}$. For any open covering, $(U, V)$, of $X$, we have

$$
P_k(U), P_k(V), P_k(U \cap V) \implies P_k(X).
$$
It suffices to apply the Five Lemma to the morphism $\mathcal{J}_j^P$:

$$
\begin{align*}
&H_j((\mathcal{P}(U \cap V)) \oplus H_j((\mathcal{P}(U))) = H_j((\mathcal{P}(V))) \quad H_j((\mathcal{P}(U \cap V))) = H_j((\mathcal{P}(U))) \oplus H_j((\mathcal{P}(V))) \\
&H_j((\mathcal{P}(U \cap V)) \oplus H_j((\mathcal{P}(U))) = H_j((\mathcal{P}(V))) \quad H_j((\mathcal{P}(U \cap V))) = H_j((\mathcal{P}(U))) \oplus H_j((\mathcal{P}(V)))
\end{align*}
$$

between the Mayer-Vietoris exact sequences (cf. Theorem 2.13 and [9, Theorem 1]).

**Proof of Theorem 6.4.** We proceed by induction on the depth of the CS sets. If the depth is null, then $\mathcal{J}_j^P$ is an isomorphism for each $j \in \mathbb{N}$ since $H_j((\mathcal{P}(X)) = H_j((\mathcal{P}(X)) = H_j((\mathcal{P}(X))$.

Let $\mathcal{U}$ be a conical chart. We define a cube as a product $[a_1, b_1] \times \cdots \times [a_m, b_m] \subset U$, with $a_\bullet, b_\bullet \in \mathbb{Q}$, and denote by $C$ the family of cubes. The truncated cone $\hat{c}_iL$ is the quotient $\hat{c}_iL = L \times [0, 1/L] \times \{0\}$. The following family,

$$
\mathcal{U} = \{\varphi(C \times c_iL) \mid C \in \mathcal{C}, 0 < t < 1, t \in \mathbb{Q}\} \cup \{\varphi(C \times L \times [a, b]) \mid C \in \mathcal{C}, 0 < a < b \leq 1, a, b \in \mathbb{Q}\}
$$

is a countable open basis of $V$ closed by finite intersections. By induction hypothesis we have $P_k(L)$. From it and Proposition 6.6, we deduce $P_k(\varphi(C \times c_iL))$ since $C$ is homeomorphic to $\mathbb{R}^n$ and $\varphi$ is a stratified homeomorphism. Similarly, from $P_k(L)$, we deduce $P_k(\varphi(C \times L \times [a, b]))$. In summary, we have proven $P_k(W)$ for any $W \in \mathcal{U}$.

Let $U_1$ be the family of finite unions of elements of $\mathcal{U}$. We consider $U = \bigcup_{i=1}^p U_i \in U_1$. By induction on $p$, from $(U_1 \cup \cdots \cup U_{p+1}) \cap U_p = (U_1 \cap U_{p+1}) \cup \cdots \cup (U_{p-1} \cap U_p)$, we deduce $P_k((U_1 \cup \cdots \cup U_{p-1}) \cap U_p)$. Proposition 6.7 implies $P_k(U)$ for each $U \in U_1$. Notice that $U_1$ is a countable open basis of $V$ closed by finite intersections.

Let $U_2$ be the family of numerable unions of elements of $\mathcal{U}$. We consider $U = \bigcup_{i \in \mathbb{N}} U_i \in U_2$. By setting $V_i = V_{i-1} \cup U_i$, we get $U = \bigcup_{i \in \mathbb{N}} V_i$ where $(V_i)_i$ is an increasing sequence of open subsets in $U_1$. A classic argument for homotopy theories with compact supports gives

$$
H_*((\mathcal{P}(U)) = \lim_{i \to } H_*((\mathcal{P}(V_i)) \quad H_*((\mathcal{P}(U)) = \lim_{i \to } H_*((\mathcal{P}(V_i))
$$

From $P_k(V_i)$ for all $i \in \mathbb{N}$, we deduce $P_k(U)$ for each $U \in U_2$. We get the claim since $U_2$ is the topology of $V$.

**Second Step: we prove $P_k(X)$.** We consider the family $V$ of open subsets of $X$ with $P_k(V)$. We order $V$ by inclusion. If $(V_i)_i \in I$ is an increasing family of elements of $V$, an argument as above implies $P_k(\bigcup_{i \in I} V_i)$. Therefore, by Zorn’s lemma, the family $V$ has a maximal element $W$. We are reduce to show $W = X$. Let us suppose the existence of $x \in X \setminus W$. Let $\varphi : U \times \hat{c}_iL \to V$ be a conical chart of $x$. The first step gives $P_k(V)$ and $P_k(W \cap V)$. We also have $P_k(W)$ by choice of $W$. From Proposition 6.7, we deduce $P_k(W \cup V)$. The maximality of $W$ implies $W = W \cup V$ and $x \in W$. From this contradiction, we deduce $W = X$.

**Proof of Theorem 6.1.** The two properties are consequences of the Hurewicz theorem for $\mathcal{P}(X)$ and results on the map $H_j((\mathcal{P}(X); \mathbb{Z}) \to H_j((\mathcal{P}(X); \mathbb{Z})$. Property 1) is a consequence of Theorem 6.2 and Property 2) is covered by Theorem 6.4.

**Example 6.8.** We construct a CS set $X$ with $\pi_1^j(X) = 0$, for $j \leq 2$, (i.e., $k = 3$ in Theorem 6.4) and $\mathcal{J}_j^P$ not surjective. Thus the hypotheses on the links cannot be removed in Theorem 6.1.2).

We start with the Hopf principal bundle $S^3 \to S^7 \to \mathbb{R}^7$ and quotient it by the $\mathbb{Z}_2$-action to obtain the bundle $\mathbb{R}P^3 \to \mathbb{R}P^7 \to \mathbb{R}P^3$. Let $c\mathbb{R}P^3 = \mathbb{R}P^3 \times [0, 1]/\mathbb{R}P^3 \times \{0\}$ be the closed cone. The total space of the associated cone bundle of $\tau$, $c\mathbb{R}P^3 \to \tau' \to \mathbb{R}P^7$, is a CS set with $\mathbb{R}P^7$ as only singular stratum. Finally, we consider the double mapping cylinder of $\tau'$ and $\mathbb{R}P^7 \to *$:

$$X = T \cup_{\beta = \mathbb{R}P^7} c\mathbb{R}P^7.$$
This is a CS set with two singular strata, $S^4$ and the apex $v$ of the cone. Their links are $\mathbb{R}P^3$ and $\mathbb{R}P^7$, respectively. We use the perversity $\mathfrak{p}$ defined by $D_\mathfrak{p}(S^4) = 0$ and $D_\mathfrak{p}(v) = 2$. The hypotheses of Theorem 6.1.2) are not satisfied since $\pi_7(\mathbb{R}P^7) = \pi_7(\mathbb{R}P^3) = \mathbb{Z}_2 \neq 0$.

Let $0 < j < 2$. For the determination of $\pi_j^\mathfrak{p}(X)$, we observe that the $\mathfrak{p}$-allowability condition (1.3) gives $\pi_j^\mathfrak{p}(X) = \pi_j^\mathfrak{p}(X \setminus \{v\})$. As $X \setminus \{v\}$ is stratified homotopy equivalent to $T$, we get $\pi_j^\mathfrak{p}(X) = \pi_j^\mathfrak{p}(T)$. From the homotopy long exact sequence of the bundle $\tau^\mathfrak{p}$ (\cite[Theorem 2.2]{14}), we deduce $\pi_j^\mathfrak{p}(T) = \pi_j^\mathfrak{p}(\mathcal{E}\mathbb{R}P^3) = 0$.

Therefore, the condition “$\pi_j^\mathfrak{p}(X) = 0$ for $0 < j < 2$” is satisfied.

We claim that $h_4^\mathfrak{p}: \pi_4^\mathfrak{p}(X) \to H_4^\mathfrak{p}(X; \mathbb{Z})$ is an isomorphism and $h_2^\mathfrak{p}: \pi_2^\mathfrak{p}(X) \to H_2^\mathfrak{p}(X; \mathbb{Z})$ is not an epimorphism, which is equivalent to: $T^\mathfrak{p}$ is an isomorphism and $J^\mathfrak{p}_T$ is not surjective. We prove that claim by using the Mayer-Vietoris exact sequences ([7, Proposition 4.1] and Theorem 2.13) associated to the covering $\{ \{v\}, X \setminus S^4 \}$. Let us notice the existence of stratified homotopy equivalences, $X \setminus \{v\} \simeq_T T$, $X \setminus S^4 \simeq \mathcal{E}\mathbb{R}P$ and $(X \setminus \{v\}) \cap (X \setminus S^4) \simeq \mathbb{R}P$. We compute the various homologies.

- In $T$, the perversity $\mathfrak{p}$ is the top perversity $7$. From [7, Proposition 5.4], Lemma 2.8 and Corollary 4.2, we get $H^\mathfrak{p}_j(T) = H_4^\mathfrak{p}(T) = H_4(S^4)$, and $H_2(\mathcal{E}\mathbb{R}P) = H_2(T) = H_2(S^4)$.

- From [7, Proposition 5.2], we have $H^\mathfrak{p}_j(\mathcal{E}\mathbb{R}P) = 0$ if $j > 2$ and $H^\mathfrak{p}_j(\mathcal{E}\mathbb{R}P) = H_j(\mathbb{R}P) = \mathbb{Z}_2$. Proposition 3.5 implies that $\mathcal{E}\mathbb{R}P(\mathcal{E}\mathbb{R}P)$ is the Eilenberg-MacLane space $K(\mathbb{Z}_2,1) = \mathbb{R}P^\infty$.

Thus, in low degrees, the Mayer-Vietoris sequences reduce to

$$
\begin{array}{ccccccccc}
0 & \to & Z & \to & H_4(\mathcal{E}\mathbb{R}P X) & \to & Z_2 & \to & Z_2 & \to & H_4(\mathcal{E}\mathbb{R}P X) & \to & 0 \\
 & & & \downarrow{J_T} & & & \downarrow{\text{inclusion}} & & & \downarrow{\text{inclusion}} & & & \downarrow{J_T} & \\
0 & \to & Z & \to & H_2^\mathfrak{p}(X) & \to & Z_2 & \to & 0 & \to & H_2^\mathfrak{p}(X) & \to & 0.
\end{array}
$$

The horizontal map $Z_2 \to Z_2$ being an isomorphism, the map $J^\mathfrak{p}_T$ is an isomorphism and $J^\mathfrak{p}_T$ is not surjective.

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LAMPA, Université de Picardie Jules Verne, 33, rue Saint-Leu, 80039 Amiens Cedex 1, France
*Email address*: David.Chataur@u-picardie.fr

LABORATOIRE DE MATHEMATIQUES DE LENS, EA 2462, Université d’Artois, SP18, rue Jean Souvraz, 62307 Lens Cedex, France
*Email address*: martin.saraleguiaranquren@univ-artois.fr

DéPARTEMENT DE MATHEMATIQUES, UMR-CNRS 8524, Université de Lille, 59655 Villeneuve d’Ascq Cedex, France
*Email address*: Daniel.Tanre@univ-lille.fr