Some aspects of Skyrme–Chern–Simons densities

D H Tchrakian

School of Theoretical Physics, Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland
Department of Computer Science, National University of Ireland Maynooth, Maynooth, Ireland
E-mail: tigran@stp.dias.ie

Received 6 December 2021, revised 7 April 2022
Accepted for publication 20 April 2022
Published 26 May 2022

Abstract
The gauge transformation properties of the Skyrme–Chern–Simons (SCS) densities is studied. Two types of SCS actions are identified, type I in which the gauge group is smaller than the largest possible one, and type II which are gauged with the largest allowed gauge group. Type I SCS feature only one power of the gauge connection and no curvature, while type II feature both the gauge connection and the curvature. The abelian type I SCS turn out to be explicitly gauge invariant while non-abelian type I and all type II SCS are gauge invariant only up to a total divergence term, and hence lead to gauge covariant equations of motion. SCS actions are the gauged Skyrmion analogues of the usual Chern–Simons (CS) actions, except that unlike the CS which are defined only in odd dimensions, the SCS are defined also in even dimensions. Some areas of application in the construction of solitons are pointed out.

Keywords: SCS, feature, aspects, Skyrme, Chern–Simons, densities, gauged Skyrme systems, Chern–Simons actions, anomaly related actions

1. Introduction
The role of the Chern–Simons (CS) action in soliton physics was recognised a long time ago in the construction of electrically-charged spinning vortices of $SO(2)$ gauged Higgs model [1–3] and $O(3)$ Skyrme model [4–6] in $2 + 1$ dimensions, and, $SO(2)$ gauged $O(5)$ Skyrme model [7] in $4 + 1$ dimensions. In the case of the gauged Skyrme solitons [6, 7], an energy lower bound departing from the topological ‘baryon number’ due to the gauge field was applied. This new ‘deformed baryon number’ was first proposed in [10] and applied many times since, and most recently is elaborated in appendix B of [8]. The gauging prescription for the $O(D + 1)$ Skyrme
scalar on $\mathbb{R}^D$ proposed in [8] (and references therein) is effected by the $SO(N), (2 \leq N \leq D)$ connection gauging $N$ components of the $D + 1$ component Skyrmic scalar.

Further to the construction of solitons of abelian gauged Skyrmion models in $2 + 1$ and $4 + 1$ dimensions, remarkable dynamical effects resulting from the CS action were discovered in $2 + 1$ dimensions in [12–14], and in $4 + 1$ dimensions, in reference [15] fields encoding the CS action in a specific way. They are (a) the non-standard dependence of the mass/energy $E$ on the electric charge $Q_e$ and the angular momentum $J$, whose slope can be negative (as well as positive) in some regions of the parameter space, and (b) the evolution of the ‘topological charge’ (baryon number) away from its integer value prior to gauging, due to its deformation resulting from the gauging.

Thus in ‘all odd dimensions’ where the CS density is defined, namely through the examples of $2 + 1$ and $4 + 1$ cited in the previous paragraph, we learn that the $E$ vs $Q_e$ and $E$ vs $J$ slopes can be negative as a result of the CS dynamics. Now it is a longstanding problem to explain why the electrically neutral neutron described by the Skyrmion is heavier than the electrically charged proton which is presumably described by the abelian gauged Skyrmion. The latter should display a negative $E$ vs $Q_e$ slope. But in $3 + 1$ dimensions there is no CS term defined, so the Skyrmion of the abelian gauged $O(4)$ sigma model cannot be influenced by this mechanism to produce a negative $E$ vs $Q_e$ slope. It would be desirable for this purpose to avail of a CS like density in even dimensions. Such an ‘anomaly associated’ density with the appropriate parity reversal properties, defined in terms of the abelian field interacting with the Skyrmie scalar in $3 + 1$ dimensions, was displayed in reference [16], albeit not in the context of the $E$ vs $Q_e$ slope question. The SCS actions in $3 + 1$ dimensions can be seen as a possible alternative to the ‘anomaly associated’ action of reference [16].

It is precisely to fill this gap that the Skyrme–Chern–Simons (SCS) densities, which are defined in both even and odd dimensions, were proposed in reference [8]. The present note is intended to clarify and elucidate aspects of these SCS densities, principally to demonstrate that the variational equations of these actions are gauge covariant\(^1\). In this note two types of SCS densities are identified, typeI and typeII. TypeI SCS densities are those which are gauged with the single gauge group $SO(N)$, where $N < d + 1$, $SO(d + 1)$ being the largest gauge group a SCS density in $d$ dimensions can be gauged with. The typical feature of typeI is that only a single connection $A_\mu$ appears and no curvature $F_{\mu\nu}$ is displayed. A technical feature of typeI SCS is that they can be expressed in closed form in all dimensions $d$, for each $SO(N)$. TypeII SCS densities on the other hand are those pertaining to the largest allowed gauge group in $d$ dimensions, $SO(d + 1)$, as well as those gauged with the direct product of all possible subgroups $SO(d + 1)$. The important distinction between typeI and typeII SCS is that the former display one power of the connection $A_\mu$ only, while the later display both connection and curvature $(A_\mu, F_{\mu\nu})$ for $SO(d + 1)$, or as the case may be, for each of the (direct product) subgroups therein. In this sense typeII SCS are the germane analogues of the usual CS terms which are expressed in terms of $(A_\mu, F_{\mu\nu})$, except that unlike the CS which defined in only odd $d$, the typeII SCS are defined in all $d$.

In section 2, a revision of the definition of the usual CS densities is given, followed by a brief presentation of the generic SCS densities. The typeI SCS densities in arbitrary $d$ dimensions for gauge groups $SO(2)$ and $SO(3)$ are presented in section 3, one abelian and one non-abelian example, the smallest of $SO(N)$ ($N < d + 1$) gauge groups. Section 4 deals in principle with

\(^1\)It could be mentioned in passing that in addition to the SCS actions, the so-called Higgs–Chern–Simons (HCS) densities are proposed in [8], lead to gauge covariant variational equations. This aspect is obvious in the case of HCS densities, since in even dimensions these are manifestly gauge invariant and in odd dimensions they consist of a Higgs dependant part which is gauge invariant, plus, the usual CS density in the given (odd) dimension.
2. Chern–Simons (CS) and Skyrme–Chern–Simons (SCS)

In the first subsection, a brief review of the usual CS densities is presented with the purpose of putting into context the definition of the SCS densities proposed in [8], which is summarised in the subsequent subsection below.

2.1. Brief review of Chern–Simons (CS)

The starting point in the definition of the CS density in \(d\) dimensions is the Chern–Pontryagin (CP) density in \((d + 1)\), even, dimensions

\[
\varrho \equiv \Omega^{(d+1)}_{\text{CP}} = \varepsilon_{i_1i_2\ldots i_{d+1}} \text{Tr} F_{i_1i_2} F_{i_3i_4} \ldots F_{i_{d}i_{d+1}},
\]

which happens to be the topological charge density stabilising \(SO(\pm (d + 1))\) ‘instantons’ in all even dimensions (see e.g., [17]). The density (2.1) is known to be a total divergence.

The usual CS density \(\Omega^{(d)}_{\text{CS}}\) in \((d)\) dimensions is extracted from the (total divergence) CP density in \((d + 1)\) as

\[
\Omega^{(d+1)}_{\text{CP}} = \partial_i \Omega^{(d+1)}_{\text{CP},i}, \quad i = 1, 2, \ldots, d + 1
\]

as the \((d + 1)\)th component of \(\Omega^{(d+1)}_{\text{CP}}\) in (2.2),

\[
\Omega^{(d)}_{\text{CS}} \overset{\text{def}}{=} \Omega^{(d+1)}_{\text{CP},d+1},
\]

and since \(\Omega^{(d+1)}_{\text{CP}}\) is a ‘curl’ defined in terms of the totally antisymmetric tensor \(\varepsilon^{i_1i_2\ldots i_{d+1}}\), fixing one component, say \(i = d + 1\), is tantamount to a descent by one dimension, such that \(\Omega^{(d)}_{\text{CS}}\) defined by (2.3) is a scalar in \(d\) dimensions. Thus, the CS density (2.3) expressed in terms of the gauge connection \(A_\mu\) and the curvature \(F_{\mu\nu}\) is defined in a \(d\) dimensional space with coordinates \(x_\mu\)

\[
\Omega^{(d)}_{\text{CS}} = \Omega^{(d)}_{\text{CS}}[A_\mu, F_{\mu\nu}], \quad \mu = 1, 2, \ldots, d, \quad d \text{ odd}.
\]

Most remarkably, CS densities and are explicitly gauge variant, as implied by the notation used in (2.4). It is important to stress that theories endowed with dynamical CS terms in the Lagrangian are defined on spacetimes with Minkowskian signature. Since the CS term is independent of the metric tensor, the resulting stress tensor does not feature it and the static Hamiltonian (and hence energy) is gauge invariant as it should be\(^2\).

Of course, the CP densities and the resulting CS densities, can be defined in terms of both abelian and non-abelian gauge connections and curvatures. The context of the present notes

\(^2\)Should one employ a CS density on a space with Euclidean signature, with the CS density appearing in the static Hamiltonian itself, then the energy would not be gauge invariant. Hamiltonians of this type have been considered in the literature, e.g., in [18]. CS densities on Euclidean spaces, defined in terms of the composite connection of a sigma model, find application as the topological charge densities of Hopf solitons.
is the construction of soliton solutions\(^3\), rather than the study of topologically massive field theories as in [19, 20]. In this respect, the choice of gauge group any given dimension must be made with due regard to regularity, and the models chosen must be consistent with the Derrick scaling requirement for the finiteness of energy. Accordingly, in all but \(2 + 1\) dimensions, our considerations are restricted to non-abelian gauge fields.

2.1.1. Gauge transformation of CS. Consider the transformation of \(\Omega^{(d+1)}_i\) in (2.2) under the infinitesimal gauge transformation \(g(x_i)\)

\[
\Omega_i^{(d+1)} \overset{g}{\rightarrow} \Omega_i^{(d+1)} + \delta \Omega_i^{(d+1)}.
\]  

(2.5)

Since \(\varrho = \partial_i \Omega_i^{(d+1)}\) is gauge invariant, i.e., \(\delta \varrho = 0\), it follows that

\[
\partial_i (\delta \Omega_i^{(d+1)}) = 0,
\]  

(2.6)

which allows to express \(\delta \Omega_i^{(d+1)}\) formally as

\[
\delta \Omega_i^{(d+1)} = \varepsilon_{ijk_{k_{2}}...k_{d-1}} \partial_j V_{k_{1}k_{2}...k_{d-1}},
\]  

(2.7)

where \(V_{k_{1}k_{2}...k_{d-1}}\) is a totally antisymmetric tensor defined in terms of the connection and the curvature fields.

From the definition (2.3) of the CS density, (2.7) implies the following transformation

\[
\delta \Omega^{(d)}_{CS} = \delta \Omega^{(d+1)}_{CS} + \varepsilon_{\mu\nu_{1}\nu_{2}...\nu_{d-1}} \partial_{\mu} V_{\nu_{1}\nu_{2}...\nu_{d-1}},
\]  

(2.8)

which is clearly defined on the space with the \(d\)-dimensional coordinates \(x_{\mu}\).

It follows from (2.8) that under an infinitesimal gauge transformation \(g(x_i)\), the CS density \(\Omega^{(d)}_{CS}\) transforms as

\[
\Omega^{(d)}_{CS} \overset{g}{\rightarrow} \Omega^{(d)}_{CS} + \varepsilon_{\mu\nu_{1}\nu_{2}...\nu_{d-1}} \partial_{\mu} V_{\nu_{1}\nu_{2}...\nu_{d-1}},
\]  

(2.9)

meaning that the CS density is gauge invariant up to a total divergence. One concludes that the action of \(\Omega^{(d)}_{CS}\), namely its volume integral

\[
\int d^d x \, \Omega^{(d)}_{CS} \overset{g}{\rightarrow} \int d^d x \, \Omega^{(d)}_{CS}
\]

remains invariant under the action of \(g\), resulting in the Euler–Lagrange equations (2.13) and (2.14) being gauge covariant.

This statement can be given concrete expression in terms of examples in dimensions \(d = 3, 5, 7\), which can be extended to all odd dimensional spacetimes systematically.

The CS densities \(\Omega^{(d)}_{CS}\), defined by (2.3), for \(d = 3, 5, 7\), are

\[
\Omega^{(3)}_{CS} = \varepsilon_{\lambda\mu\nu} \text{ Tr } A_{\lambda} \left[ F_{\mu\nu} - \frac{2}{3} A_{\mu} A_{\nu} \right],
\]  

(2.10)

\[
\Omega^{(5)}_{CS} = \varepsilon_{\lambda\mu\rho\sigma} \text{ Tr } A_{\lambda} \left[ F_{\mu\rho} F_{\rho\sigma} - F_{\mu\sigma} A_{\rho} A_{\sigma} + \frac{2}{5} A_{\mu} A_{\rho} A_{\sigma} A_{\rho} A_{\sigma} \right],
\]  

(2.11)

\(^3\)The term soliton solutions here is used rather loosely, implying only the construction of regular and finite energy solutions, without insisting on topological stability in general.
Ω^{(7)}_{CS} = \varepsilon_{\lambda\mu\rho\sigma\tau\kappa} \text{Tr} A_\lambda \left[ F_{\mu\nu} F_{\rho\sigma} A_\tau A_\kappa - \frac{4}{5} F_{\mu\nu} A_\rho A_\sigma A_\tau A_\kappa - \frac{2}{5} F_{\mu\nu} A_\rho F_{\pi\sigma} A_\tau A_\kappa \right. \\
+ \left. \frac{4}{5} F_{\mu\nu} A_\rho A_\sigma A_\tau A_\kappa - \frac{8}{35} A_\mu A_\rho A_\sigma A_\tau A_\kappa \right].
(2.12)

Restricting to orthogonal groups, one notes that the CS term in \(d\) dimensions features the product of \(d-1\) powers of the (algebra valued) gauge field/connection in front of the trace, which would vanish if the gauge group is not larger than SO\((d-1)\). In that case, the YM connection would describe only a ‘magnetic’ component, with the ‘electric’ component necessary for the nonvanishing of the CS density would be absent. As in [21], the most convenient choice is SO\((d+1)\). Since \(d+1\) is always even, the representation of SO\((d+2)\) are the chiral representation in terms of (Dirac) spin matrices. This completes the definition of the usual non-abelian CS densities in \(d\) spacetimes.

From (2.10)–(2.12), it is clear that the CS density is explicitly gauge variant. Their Euler–Lagrange equations w.r.t. the variation of \(A_\lambda\) are nonetheless gauge invariant

\[ \delta A_\lambda \Omega^{(2)}_{CS} = \varepsilon_{\lambda\mu} F_{\mu\nu}, \]
\[ \delta A_\lambda \Omega^{(3)}_{CS} = \varepsilon_{\lambda\mu\nu} F_{\mu\rho} F_{\nu\sigma}, \]
\[ \delta A_\lambda \Omega^{(4)}_{CS} = \varepsilon_{\lambda\mu\nu\rho\sigma} F_{\mu\rho} F_{\nu\sigma} F_{\lambda\kappa}. \]

This remarkable property of CS densities can be understood by noting that, while these appear as explicitly gauge-variant densities, these actions are actually gauge invariant up to a surface term. To see this one subjects them to transformation under the action of an element, \(g\) of the (non-abelian) gauge group.

The transformations for the two examples (2.10) and (2.11) are explicitly

\[ \Omega^{(3)}_{CS} \rightarrow \Omega^{(3)}_{CS} - \frac{2}{3} \varepsilon_{\lambda\mu\nu} \text{Tr} \alpha_\lambda \alpha_\mu \alpha_\nu - 2 \varepsilon_{\lambda\mu\nu} \partial_\lambda \alpha_\mu A_\nu, \]
\[ \Omega^{(5)}_{CS} \rightarrow \Omega^{(5)}_{CS} - \frac{2}{5} \varepsilon_{\lambda\mu\nu\rho\sigma} \text{Tr} \alpha_\lambda \alpha_\mu \alpha_\nu \alpha_\rho \alpha_\sigma + \frac{2}{3} \varepsilon_{\lambda\mu\nu\rho\sigma} \partial_\lambda \alpha_\mu \left( F_{\rho\sigma} - \frac{1}{2} A_\rho A_\sigma \right) \\
+ \left( F_{\rho\sigma} - \frac{1}{2} A_\rho A_\sigma \right) A_\nu - \frac{1}{2} A_\nu \alpha_\rho A_\sigma - \alpha_\nu \alpha_\rho A_\sigma, \]

where \(\alpha_\mu = \partial_\mu gg^{-1}\), as distinct from the algebra valued quantity \(\beta_\mu = g^{-1}\partial_\mu g\) that appears as the inhomogeneous term in the gauge transformation of the non-abelian connection. (2.16) and (2.17) are the explicit versions of (2.9).

As seen from (2.16) and (2.17), the gauge variation of \(\Omega^{(d)}_{CS}\) consists of a term which is explicitly a total divergence, and, another term

\[ \omega^{(d)} = \frac{2}{d} \varepsilon_{\mu_1\mu_2...\mu_d} \text{Tr} \alpha_{\mu_1} \alpha_{\mu_2} \cdots \alpha_{\mu_d}, \]

which in the appropriate parametrisation can be cast in the form of a winding number density.
2.2. Brief definition of Skyrme–Chern–Simons (SCS)

This subsection is intended to provide a self-contained definition of SCS densities, which are discussed at length in [8].

The $O(d+2)$ sigma system is defined in terms of the Skyrme scalar $\phi^a$, $a = 1, 2, \ldots, d+2$, subject to $|\phi^a|^2 = 1$. The SCS density $\Omega^{(d+1)}_{\text{CS}}$ in $d$ dimensions is defined in terms of the Skyrme scalar $\phi^a$.

The definition of the SCS density in $d$ dimensions follows exactly analogously to the step (2.2) $\rightarrow$ (2.3) by the one-step descent of the CP density $\Omega^{(d+1)}_{\text{CP}}$ to the CS density $\Omega^{(d+1)}_{\text{CS}}$. This is done by carrying out such a descent of a density in $d + 1$ dimensions that might be denoted as $\Omega^{(d+1)}_{\text{SCP}}$, down to the SCS density

\[ \Omega^{(d+1)}_{\text{SCP}} = p a_i \partial_i \Omega^{(d+1)} \]

and

\[ \Omega^{(d+1)}_{\text{CS}} = \Omega^{(d+1)}_{i d + 1} \] (2.20)

Just as the CP density $\Omega^{(d+1)}_{\text{CP}}$ in (2.2) is both gauge invariant and total divergence, so too is $\Omega^{(d+1)}_{\text{SCP}}$ in (2.19), and this is expressed symbolically in terms of the vector valued density $\partial_a \phi$.

The definition of the SCS density in (2.19) is gauge invariant by construction and $\Omega^{(d+1)}_{\text{CS}}$ is purely symbolic and is meant to underline the analogy with the usual CP density. Taking this nomenclature literally, as Skyrme–Chern–Pontryagin, is misleading. It is the density presenting the lower bound on the ‘energy’ of the gauged Skyrme system, and is deformation of the (topological) winding number density (2.21).

4 With solitons in mind, this is taken to be a $d$ dimensional Minkowski space, but the choice of this signature is not important in principle.

5 The subscript on $\Omega^{(d+1)}_{\text{SCP}}$ is purely symbolic and is meant to underline the analogy with the usual CP density. Taking this nomenclature literally, as Skyrme–Chern–Pontryagin, is misleading. It is the density presenting the lower bound on the ‘energy’ of the gauged Skyrme system, and is deformation of the (topological) winding number density (2.21).
The gauge group calculations are carried out directly, using the Leibnitz rule and the tensor identities.

Collecting the gauge invariant pieces \( g_G \) and \( W \) in (2.23), and separately, the individually gauge variant pieces \( g_0 \) and \( \omega_i \), one has two equivalent definitions of a density

\[
\rho = g_G + W[F, D\phi] \quad \text{(2.24)}
\]

\[
= g_0 + \partial_i \Omega_i^{(d+1)}[A, \phi], \quad \text{(2.25)}
\]

which is adopted as the definition for the density \( \rho = \Omega^{(d+1)}_{\text{SCP}} \) presenting a lower bound on the ‘energy’ in the same way as does the usual CP density.

The two equivalent definitions (2.24) and (2.25) of \( \rho \) are, as required, both gauge invariant and (essentially) total divergence. The quantity \( \rho \) is the deformation of the of \( g_0 \), namely of the (topological) ‘baryon number’ when the gauge field is switched on.

Noting that \( g_0 \) is essentially total divergence\(^6\) and hence in a constraint compliant parametrisation it is explicitly total divergence, say

\[
g_0 = \partial_i \omega_i^{(d+1)}, \quad \text{(2.26)}
\]

(2.25) can be expressed (explicitly) as the total divergence \( \partial_i \hat{\Omega}_i^{(d+1)} \) in (2.19).

Thus, the expression (2.25) for \( \rho \) can be written as

\[
\rho = \rho_0 + \partial_i \hat{\Omega}_i^{(d+1)} \equiv \Omega^{(d+1)}_{\text{SCP}}, \quad \text{(2.27)}
\]

from which follows immediately, the definition of the SCS density (2.20)

\[
\Omega^{(d)}_{\text{SCS}} = \hat{\Omega}_i^{(d+1)} = \omega_i^{(d+1)} + \Omega^{(d+1)}_{\text{SCP}} \equiv \Omega^{(d)} \quad \text{(2.28)}
\]

where we have denoted \( \omega_i^{(d+1)} = \omega_i^{(d)}(x_i) \) and \( \Omega^{(d+1)}_{\text{SCP}} = \Omega^{(d)}(x_i) \). The density \( \omega^{(d)} \) in (2.29) is the Wess–Zumino (WZ) term.

2.2.1. The \( W \) term in SCP definition (2.23) and (2.24). In defining the SCS density (2.29) and (2.30) in \( d \) dimensions, only the definition (2.25) was employed, and not the variant (2.24). It is nonetheless reasonable for the sake of being self-contained, to illustrate the provenance of the gauge invariant term \( W \) that appears in the crucial relation (2.23) which splits\(^7\), into the two terms \( \partial_i \Omega_i^{(d+1)} \) and \( W \) leading to the two equivalent definitions (2.24) and (2.25) for the SCP \( \rho \). This splitting occurs for the \( SO(2) \) SCP in two dimensions \([9]\), albeit in a somewhat different

\(^6\)The meaning of the phrase essentially total divergence used here is, that this quantity is not explicitly total divergence, but rather that the resulting Euler–Lagrange equations are trivial as in the case of an explicitly total divergence density. Throughout, the term total divergence is used as a synonym for explicitly total divergence.

\(^7\)It may be relevant to mention that such a definition for a gauge invariant and total divergence density like \( \rho \) in (2.24) and (2.25) can be made for a \( SO(d) \) gauged system of \( d \)-tuplet Higgs field, though in that case that is not necessary since the ‘energy’ density of such monopoles are is bounded by the Higgs–Chern–Pontryagin (HCP) density defined in \([17]\). Such lower bounds as (2.24) and (2.25) for Higgs systems were discussed in \([11]\), where again this splitting becomes unique in the \( d = 2 \) case only, and also the lower bound is saturated when the usual abelian Higgs model is chosen. In all \( d \geq 3 \), this splitting has some freedom and the relevant lower bound is not saturated. Moreover these lower bounds are always higher than the HCP lower bounds \([11]\).
approach to here. The reference [9] was the template for extending this construction to \( d = 3 \) for \( SO(3) \) and to \( d = 4 \) for \( SO(4) \), in \([10]\), and subsequently to smaller gauge groups in each case (see [8] and references therein).

The term \( W \) in the SCP density (2.24) is involved in stating Bogomol’nyi-like ‘energy lower bounds’, which is saturated only for the \( SO(2) \) case in \( d + 1 = 2 \) dimensions presented in [9]. As well as this, it turns out that in dimensions \( d + 1 \geq 3 \) the splitting in (2.23) is not always unique. To demonstrate these features, it helps to consider the \( W \) terms for the special examples considered in sections 3.1, 3.2 and 4.1 below.

These examples pertain to the type I \( SO(2) \) and \( SO(3) \) SCP densities (3.41) and (3.52) respectively, both in \( d + 1 \) dimensions, and, to the type II \( SO(4) \) SCP in \( d + 1 = 4 \), given by \( \Omega_i^{(d+1)} \) in (4.79) below. The \( W \) terms for these examples are listed as

\[
W = \frac{1}{2} \varepsilon_{ijk_{\ell}k_{\ell-1}-\beta} \varepsilon^{A_1A_2-A_d} F_{ij} D_{k_1} \phi^{A_1} D_{k_2} \phi^{A_2} \ldots D_{k_{d-1}} \phi^{A_{d-1}} F_{n}^{A_d}, \tag{2.31}
\]

\[
W = \frac{1}{2} \varepsilon_{ijk_{\ell}k_{\ell-1}-\beta} \varepsilon^{A_1A_2-A_{d-1}} F_{ij}^{\alpha} \phi^{\alpha} D_{k_1} \phi^{A_1} D_{k_2} \phi^{A_2} \ldots D_{k_{d-1}} \phi^{A_{d-1}}, \tag{2.32}
\]

\[
W = 3! \varepsilon_{ijkl} \varepsilon^{abcd} \phi^a \left\{ \frac{1}{24} \left( \phi^b \phi^c \phi^d \phi^{[ijkl]} + \frac{1}{2} \phi^{[abc]} \phi^{d]} \phi^{[ijkl]} \right) \right\}, \tag{2.33}
\]

where in (2.31) and (2.32) the indices \( A_1, A_2, \ldots \) label the ungauged components of the Skyrme scalar \( \phi^a \).

On close inspection, it is clear that a gauge-invariant and total-divergence term can be extracted from (2.31) and from the second term in (2.33), while no such term can be isolated in (2.32). It can be noted that this extraction can be carried out for \( SO(n) \) for even \( n \) gauge groups, with the exception of \( SO(2) \) gauging of the \( O(3) \) sigma model [9]. Thus in cases such as (2.31) and (2.33), the total-divergence term extracted from the term \( W \) appearing in (2.24), can be transferred to (2.25) and incorporated in \( \delta \Omega_i \), redefining the latter.

This is the freedom present in the splitting carried out in (2.23), which can be gainfully employed in casting the resulting Bogomol’nyi-like inequalities in a useful form. Clearly, this choice influences also the definition of the corresponding SCS densities, and in all examples encountered it leads also to an optimal choice for the latter.

### 2.2.2. Gauge transformation of SCS

Consider the transformation of \( \hat{\Omega}_i^{(d+1)} \), appearing in the definition of the SCP density \( \varrho = \partial i \hat{\Omega}_i^{(d+1)} \) in (2.19) (or (2.28)), under the infinitesimal gauge transformation \( \varrho(x_i) \)

\[
\delta \hat{\Omega}_i^{(d+1)} = \frac{\partial_i \varrho}{\varrho} \hat{\Omega}_i^{(d+1)} + \delta_i \hat{\Omega}_i^{(d+1)}. \tag{2.34}
\]

Since \( \varrho \) is gauge invariant, i.e., \( \partial \varrho = 0 \), it follows that

\[
\partial_i (\delta \hat{\Omega}_i^{(d+1)}) = 0, \tag{2.35}
\]

which allows to express \( \delta \hat{\Omega}_i^{(d+1)} \) formally as

\[
\delta \hat{\Omega}_i^{(d+1)} = \varepsilon_i \delta_{k_{\ell}k_{\ell-1}} \partial_i V_{k_1k_2k_{d-1}}. \tag{2.36}
\]

where \( V_{k_1k_2k_{d-1}} \) is a totally antisymmetric tensor defined in terms of the connection and curvature fields, as well as the Skyrme scalar.

From the definition of the SCS density (2.29) and (2.36) implies the following transformation

\[
\hat{\Omega}_i^{(d+1)} = \hat{\Omega}_i^{(d+1)} + \delta \hat{\Omega}_i^{(d+1)}.
\]
\[ \delta \Omega^{(d)}_{\text{SCS}} = \delta \hat{\Omega}^{(d+1)}_{\text{SCP}} = \varepsilon_{i(d+1)\mu_1\mu_2...\mu_{d-1}} \partial_{\mu_i} V_{\mu_1\mu_2...\mu_{d-1}} \] (2.37)

which is clearly defined on the space with the \( d \)-dimensional coordinates \( x_\mu \).

It follows from (2.37) that under an infinitesimal gauge transformation \( g(x_\mu) \), the SCS density \( \Omega^{(d)}_{\text{SCS}} \) transforms as

\[ \Omega^{(d)}_{\text{SCS}} \to \Omega^{(d)}_{\text{SCS}} + \varepsilon_{i\mu_1\mu_2...\mu_{d-1}} \partial_{\mu_i} V_{\mu_1\mu_2...\mu_{d-1}}, \] (2.38)

meaning that the SCS density is gauge invariant up to a total divergence, as a result, the Euler–Lagrange equations are gauge covariant.

3. **Type I SO(2) and SO(3) SCS in \( d \) dimensions**

The largest gauge group of a SCS density in \( d \) dimensions is \( SO(d+1) \), namely the gauge group of the SCP density (2.19) of the \( O(d+2) \) Skyrmion system in \( d+1 \) dimensions. Type I SCS are those which are gauged with \( SO(N) \), with \( N < d+1 \). What is distinctive with the type I SCS is that they can be expressed in a uniform format for a given \( N \) in any dimension \( d \).

This format is typified by the linear dependence of \( \Omega^{(d)} \) in the definition of the SCS (2.30), on the gauge connection \( A_\mu \), and the absence there of the gauge curvature \( F_{\mu\nu} \).

In the next two subsections, only the \( SO(2) \) and \( SO(3) \) cases are analysed, restricting attention to one abelian and one non-abelian case.

3.1 **SO(2) gauged SCS, in \( d \) dimensions**

Consider the \( O(d+2) \) sigma model in \( d+1 \) dimensions, gauged with \( SO(2) \). As per the prescription described above, we start with the SCP density in \( d+1 \) dimensions, and after the descent by one step arrive at the SCS density in \( d \) dimensions. Here only two components of the \( d+2 \) component \( O(d+2) \) Skyrmion scalar \( \phi^a = (\phi^\alpha, \phi^A) \) are gauged with \( SO(2) \) according to the gauging prescription

\[ D_\alpha \phi^a = \partial_\alpha \phi^\alpha + A_\alpha (\varepsilon \phi)^\alpha, \quad \alpha = 1, 2, \quad (\varepsilon \phi)^\alpha = \varepsilon^{\alpha\beta} \phi^\beta \] (3.39)

\[ D_\alpha \phi^A = \partial_\alpha \phi^A, \quad A = 1, 2, \ldots, d, \text{ or } A = 3, 4, \ldots, d+2. \] (3.40)

Examples of \( SO(2) \) gauged SCP densities \( \partial \hat{\Omega}^{(d+1)}_2 \), (2.27), in various dimensions are listed in reference [8], from which follows the SCP in \( d+1 \) dimensions

\[ \partial \hat{\Omega}^{(d+1)}_2 = \partial_i (\omega_i^{(d+1)} + \Omega_i^{(d+1)}) = \partial_i \left[ \omega_i^{(d+1)} + \varepsilon_{ijk_2...k_{d-1}} \varepsilon^{A_1A_2...A_d} A_j \partial_{k_1} \phi^{A_1} \partial_{k_2} \phi^{A_2} \ldots \partial_{k_{d-1}} \phi^{A_{d-1}} \phi^{A_d} \right]. \] (3.41)

by induction.

From (3.41) follows via the one-step descent (2.19) and (2.20) or (2.27)–(2.30), fixing \( i = d+1 \), the SCS density (2.30) in \( d \) dimensions

\[ \Omega^{(d)}_{\text{SCS}} = \omega^{(d)} + \Omega^{(d)} \]

in which \( \Omega^{(d)} \) is given by

\[ \Omega^{(d)} = \varepsilon_{\nu\mu_1\mu_2...\mu_d} \varepsilon^{A_1A_2...A_d} A_\nu \partial_{\mu_1} \phi^{A_1} \partial_{\mu_2} \phi^{A_2} \ldots \partial_{\mu_{d-1}} \phi^{A_{d-1}} \phi^{A_d}. \] (3.42)
To evaluate the WZ term $\omega^{(d)}$ in (2.30) one must first evaluate $\varrho_0$ defined by (2.21) in $d+1$ dimensions, cast it in total divergence form $\partial_\nu \omega^{(d+1)}$, and then perform the one-step descent (2.19) and (2.20) or (2.27)–(2.30), fixing $i = d + 1$. For this, it is necessary to employ a parametrisation which is compliant with the constraint $|\phi^\alpha|^2 = (|\phi^\alpha|^2 + |\phi^A|^2) = 1$,

$$\phi^\alpha = \sin f \, n^\alpha, \quad \phi^A = \cos f \, n^A; \quad \alpha = 1, 2; \quad A = 1, 2, \ldots, d,$$  

where $n^\alpha$ and $n^A$ are vector valued functions of unit length. The two-component unit vector $n^\alpha = (\cos \psi, \sin \psi)$ being parametrised by the angular coordinate $\psi$ on $S^1$, and the $(d-1)$-component unit vector $n^A$ by the coordinates on $S^{d-2}$.

Substituting (3.43) in (2.21), and noting that $\varepsilon^{\alpha\beta} n^\alpha \partial_{\nu+1} n^\beta = \partial_{\nu+1} \psi$,

$$\varrho_0 = - \varepsilon_{ijk_1-k_{d-1}} (\partial_\nu \cos f) (\varepsilon^{\alpha\beta} n^\alpha \partial_{\nu} n^\beta) \times (\varepsilon^{A_1A_2 \ldots A_d} \partial_{\nu_1} n^{A_1} \partial_{\nu_2} n^{A_2} \ldots \partial_{\nu_{d-1}} n^{A_{d-1}} n^{A_d})$$

$$= - \partial_\nu \left[ (\cos f) \varepsilon_{ij} (\partial_\nu \psi) \times (\varepsilon_{ijk_1-k_{d-1}} \varepsilon^{A_1A_2 \ldots A_d} \partial_{\nu_1} n^{A_1} \partial_{\nu_2} n^{A_2} \ldots \partial_{\nu_{d-1}} n^{A_{d-1}} n^{A_d}) \right] \partial_\nu \psi. \tag{3.44}$$

from which follows the WZ term

$$\omega^{(d)} = - \varepsilon_{\nu_1\nu_2 \ldots \nu_{d-1}} \cos f$$

$$\times (\varepsilon^{A_1A_2 \ldots A_d} \partial_{\nu_1} n^{A_1} \partial_{\nu_2} n^{A_2} \ldots \partial_{\nu_{d-1}} n^{A_{d-1}} n^{A_d}) \partial_\nu \psi. \tag{3.45}$$

Next, we evaluate $\Omega^{(d)}$ in the parametrisation (3.43) by substituting the latter in (3.42), yielding

$$\Omega^{(d)} = \varepsilon_{\nu_1\nu_2 \ldots \nu_{d-1}} \cos f (\varepsilon^{A_1A_2 \ldots A_d} \partial_{\nu_1} n^{A_1} \partial_{\nu_2} n^{A_2} \ldots \partial_{\nu_{d-1}} n^{A_{d-1}} n^{A_d}) A_\nu, \tag{3.46}$$

and adding (3.45) to (3.46) we end up with the SCS density (2.30) in $d$ dimensions

$$\Omega^{(d)} = \varepsilon_{\nu_1\nu_2 \ldots \nu_{d-1}} \cos f (\varepsilon^{A_1A_2 \ldots A_d} \partial_{\nu_1} n^{A_1} \partial_{\nu_2} n^{A_2} \ldots \partial_{\nu_{d-1}} n^{A_{d-1}} n^{A_d}) (A_\nu - \partial_\nu \Lambda), \tag{3.47}$$

Since according to the gauging prescription (3.39) and (3.40) the scalar function $f$ and the vector function $n^\alpha$ are inert under gauge transformations, it follows that the SCS density (3.47) is invariant under the abelian gauge transformation

$$A_\nu \rightarrow A_\nu + \partial_\nu \Lambda \tag{3.48}$$

with $\psi$ in (3.47) compensating for $\Lambda$ in (3.48).

The remarkable feature here is the fact that the abelian SCS action (3.47) is explicitly gauge invariant, unlike the usual CS term which is seen in subsection 2.1.1 to be gauge invariant up to total divergence only. It appears that in this case, the gauge transformation of the abelian connection is compensated completely by that of the Skyrme scalar.

3.2. SO(3) gauged SCSs in $d$ dimensions

The gauging prescription for the $O(d + 2)$ sigma model in $d + 1$ dimensions is given by the definition of the covariant derivatives

$$D_i \phi^\alpha = \partial_i \phi^\alpha + A_i^{\alpha\beta} \phi^\beta, \quad \alpha = 1, 2, 3 \tag{3.49}$$

$$D_i \phi^A = \partial_i \phi^A, \quad A = 1, 2, \ldots, d - 1, \text{ or } A = 4, 5, \ldots, d + 2, \tag{3.50}$$

where in (3.49), $A_i^{\alpha\beta} = \varepsilon^{\alpha\beta\gamma} A_i^\gamma$. 

10
Examples of $SO(3)$ gauged SCP densities $\partial_i \Omega_i^{(d+1)}$, (2.27), in various dimensions are listed in reference [8], from which follows the SCP in $d + 1$ dimensions

$$\partial_i \Omega_i^{(d+1)} = \partial_i (\omega_i^{(d+1)} + \Omega_i^{(d+1)})$$

(3.51)

$$= \partial_i \left( \omega_i^{(d+1)} + \varepsilon_{ijkl} \partial_{jk}^1 \phi^A \partial_{kl}^2 \phi^B \ldots \partial_{k_1}^1 \phi^A_{k_2} \right)$$

(3.52)

by induction.

From (3.52) follows via the one-step descent (2.19) and (2.20) or (2.27)–(2.30), fixing $i = d + 1$, the SCS density (2.30) in $d$ dimensions

$$\Omega_S^{(d)} = \omega^{(d)} + \Omega^{(d)}$$

in which $\Omega^{(d)}$ is given by

$$\Omega^{(d)} = \varepsilon_{ijkl} \omega^{(d)} \varepsilon_{ijkl} \varepsilon^{A1A2...A_d-1} A_\nu^A \phi_\mu^A \partial_{\mu_1}^A \partial_{\mu_2}^A \partial_{\mu_3}^A \ldots \partial_{\nu_d}^A \phi^{A_{d-1}}.$$  

(3.53)

As in the previous case with $SO(2)$ gauging, to evaluate the WZ term $\omega^{(d)}$ in (2.30) we must evaluate the winding number density $\phi_\theta$ defined by (2.21) in $d + 1$ and cast it in total divergence form $\partial_i \omega^{(d+1)}$. After that the one-step descent (2.19) and (2.20) (or (2.27)–(2.30)) can be performed, fixing $i = d + 1$. As was done in (3.43) above, this is achieved by employing a constraint compliant parametrisation which in this case is formally that as (3.43), namely

$$\phi^\alpha = \sin f n^\alpha, \quad \phi^A = \cos f n^A; \quad \alpha = 1, 2, 3; \quad A = 1, 2, \ldots, d - 1,$$

(3.54)

with $n^\alpha$ and $n^A$ being 2 and $(d - 1)$ component unit vectors respectively, $n^\alpha = n^\alpha (\chi, \psi)$ being parametrised by the polar and azimuthal coordinate on $S^2$, and $n^A$ by the angular coordinates on $S^{d-2}$.

Substituting (3.54) in (2.21) yields,

$$\phi_\theta = \frac{1}{4} \varepsilon_{ijkl} \omega^{(d)} \varepsilon_{ijkl} \varepsilon^{A1A2...A_d-1} A_\nu^A \phi_\mu^A \partial_{\mu_1}^A \partial_{\mu_2}^A \partial_{\mu_3}^A \ldots \partial_{\nu_d}^A \phi^{A_{d-1}}$$

(3.55)

in which the symbol $\Phi^{(d)}$ is

$$\Phi^{(d)} = \frac{1}{d + 1} \left( f - \frac{1}{d + 1} \sin (d + 1) f \right), \quad \text{for odd } d,$$

(3.56)

$$= \sin^{d+1} f, \quad \text{for even } d.$$

(3.57)

It follows from (3.55) that the WZ term $\omega^{(d)}$ is

$$\omega^{(d)} = \frac{1}{4} \Phi^{(d)} \varepsilon_{ijkl} \omega^{(d)} \varepsilon_{ijkl} \varepsilon^{A1A2...A_d-1} A_\nu^A \phi_\mu^A \partial_{\mu_1}^A \partial_{\mu_2}^A \partial_{\mu_3}^A \ldots \partial_{\nu_d}^A \phi^{A_{d-1}}.$$ 

(3.58)
We next evaluate $\Omega^{(d)}$ given by (3.53) in the parametrisation (3.54)

$$\Omega^{(d)} = -\varepsilon_{\nu_1 \nu_2 \mu_1 \mu_2 \ldots \mu_d} (A_{\nu_1}^\alpha \n^\alpha \partial_\nu \Phi^{(d)})$$

$$\cdot \varepsilon^{A_1 A_2 \ldots A_d-1} \partial_{\mu_1} n^{A_1} \partial_{\mu_2} n^{A_2} \ldots \partial_{\mu_d-2} n^{A_{d-2}} n^{A_{d-1}},$$

(3.59)

and finally adding (3.58) and (3.59), we have the SCS density

$$\Omega^{(d)}_{\text{SCS}} = \frac{1}{4} \varepsilon_{\nu_1 \nu_2} (4 \partial_{\nu_1} \Phi^{(d)} A_{\nu_2}^\alpha n^\alpha + \Phi^{(d)} \varepsilon^{\alpha \beta \gamma} n^\alpha \partial_{\nu_1} n^\beta \partial_{\nu_2} n^\gamma),$$

(3.60)

### 3.2.1 Gauge dependence.

It is useful to express (3.60) as

$$\Omega^{(d)}_{\text{SCS}} = \frac{1}{2} \Xi_{\nu_1 \nu_2} \left( 4 \partial_{\nu_1} \Phi^{(d)} A_{\nu_2}^\alpha n^\alpha + \Phi^{(d)} \varepsilon^{\alpha \beta \gamma} n^\alpha \partial_{\nu_1} n^\beta \partial_{\nu_2} n^\gamma \right),$$

(3.61)

where the prefactor

$$\Xi_{\nu_1 \nu_2} = \varepsilon_{\nu_1 \nu_2 \mu_1 \mu_2 \ldots \mu_d} \varepsilon^{A_1 A_2 \ldots A_d-1} \partial_{\mu_1} n^{A_1} \partial_{\mu_2} n^{A_2} \ldots \partial_{\mu_d-2} n^{A_{d-2}} n^{A_{d-1}}$$

(3.62)

is gauge invariant.

Instead of working with the real parametrisation of the $SO(3)$ gauge group, it is convenient to work with the $SU(2)$ parametrisation

$$g = \cos \frac{\Lambda}{2} + i \bar{\sigma} \cdot \bar{m} \sin \frac{\Lambda}{2}, \quad \text{with} \quad |\bar{m}|^2 = 1$$

(3.63)

such that the vector $n^\alpha$ and the connection $A_\mu^\alpha$ are now expressed as $SU(2)$ algebra elements

$$n = n^\alpha \sigma^\alpha \equiv \bar{n} \cdot \bar{\sigma}, \quad A_\mu^\alpha = A_\mu^\alpha \sigma^\alpha \equiv \bar{A}_\mu \cdot \bar{\sigma}$$

(3.64)

and they transform under the $SU(2)$ gauge group element $g$ as

$$n \overset{g}{\rightarrow} g^{-1} n g$$

(3.65)

$$A_\mu \overset{g}{\rightarrow} g^{-1} A_\mu g + i g^{-1} \partial_\mu g.$$  

(3.66)

In the parametrisation (3.64) the action (3.61) is expressed as

$$\Omega^{(d)}_{\text{SCS}} = \frac{1}{2} \Xi_{\mu \nu} \left( 4 \partial_\mu \Phi^{(d)} \text{Tr}(n A_\nu) - i \Phi^{(d)} \text{Tr}(n \partial_\mu n \partial_\nu n) \right).$$

(3.67)

Under infinitesimal transformation with $\cos \Lambda \approx 1$ and $\sin \Lambda \approx \Lambda$ in (3.63), $g$ reduces to

$$g = \mathbb{I} + \frac{i}{2} \Lambda \bar{n} \cdot \bar{\sigma}$$

(3.68)

under which $n$ transforms, up to first order in $\Lambda$, as

$$n \overset{g}{\rightarrow} \bar{n} \cdot \bar{\sigma} \overset{g}{\rightarrow} \bar{n} \cdot \bar{\sigma} - \bar{n} \times (\Lambda \bar{m}) \cdot \bar{\sigma},$$

(3.69)

and hence the term $\text{Tr}(n A_\nu)$ in (3.67) as
\[ \text{Tr}(n A_\nu) \xrightarrow{\xi} \text{Tr}(n A_\nu) + i \text{Tr}(n \partial_\mu g^{-1}) \]
\[ = \text{Tr}(n A_\nu) - \vec{n} \cdot \partial_\nu (\Lambda \vec{m}) \]  
(3.70)
\[ = 2 \vec{n} \cdot A_\mu - \vec{n} \cdot \partial_\mu (\Lambda \vec{m}), \]  
(3.71)
and the term \( \Xi_{\mu\nu} \text{Tr}(n \partial_\mu n \partial_\nu n) \) in (3.67), as
\[ \Xi_{\mu\nu} \text{Tr}(n \partial_\mu n \partial_\nu n) \xrightarrow{\xi} \varepsilon_{\mu\nu} \text{Tr}(g^{-1} n g) \partial_\mu (g^{-1} n g) \partial_\nu (g^{-1} n g) \]
\[ = 2i \varepsilon_{\mu\nu} \left[ \vec{n} \cdot (\partial_\mu \vec{n} \times \partial_\nu \vec{n}) - 2 \partial_\mu \vec{n} \cdot \partial_\nu (\Lambda \vec{m}) \right]. \]  
(3.73)
The result is
\[ \Omega^{(d)}_{\text{SCS}} \xrightarrow{\xi} \Omega^{(d)}_{\text{SCS}} - 2 \Xi_{\mu\nu} \left[ \partial_\mu \Phi^{(d)} \vec{n} \cdot \partial_\nu (\Lambda \vec{m}) + \partial_\nu \Phi^{(d)} \vec{n} \cdot \partial_\mu (\Lambda \vec{m}) \right] \]
\[ = \Omega^{(d)}_{\text{SCS}} - 2 \Xi_{\mu\nu} \partial_\mu \left[ \Phi^{(d)} \vec{n} \cdot \partial_\nu (\Lambda \vec{m}) \right], \]  
(3.75)
and since \( \Xi_{\mu\nu} \) is by definition (3.62) antisymmetric in \( \mu\nu \),
\[ \Omega^{(d)}_{\text{SCS}} \xrightarrow{\xi} \Omega^{(d)}_{\text{SCS}} - 2 \partial_\mu \left[ \Xi_{\mu\nu} \Phi^{(d)} \vec{n} \cdot \partial_\nu (\Lambda \vec{m}) \right], \]  
(3.76)
i.e., that the SCS density is gauge invariant up to a total divergence term like the usual CS density which is shown in subsection 2.1.1.

4. Type II SCS in \( d = 3 \) dimensions

Type II SCS terms are the germane analogues of the CS densities, in that unlike type I SCS they feature both the gauge curvature \( F_{\mu\nu} \) and the gauge connection \( A_\mu \). Unlike their type I counterparts however, the construction of type II SCS densities does not lend itself to simple uniform formats in all dimensions \( d \), and, they are defined only for non-abelian gauge groups. Indeed, in [8], which is our source for SCS densities, the only example given is that of the \( \text{SO}(4) \) gauged SCS in \( d = 3 \) dimensions.

What was not presented in reference [8] is a type II SCS in \( d = 3 \), pertaining to gauging with a subgroup\(^8\) of \( \text{SO}(4) \). Below in section 4.1 is presented the type II \( \text{SO}(4) \) SCS in \( d = 3 \), and the type II \( \text{SO}(2) \times \text{SO}(2) \) SCS in \( d = 3 \) section 4.2.

4.1. \( \text{SO}(4) \) gauged type II SCS in \( d = 3 \)

This example is given in [8] but it is repeated here, to make the presentation of the \( \text{SO}(2) \times \text{SO}(2) \) SCS in section 4.2 self-contained. The gauge transformation property of the type II \( \text{SO}(4) \) SCS is not carried out here, since that analysis is very cumbersome.

The covariant derivative of the \( O(5) \) Skyrme scalar \((\phi^a, \phi^5)\), \( a = 1, 2, 3, 4 \) is
\[ D_\mu \phi^a = \partial_\mu \phi^a + A_\mu^a \phi^b, \quad a = 1, 2, 3, 4 \]  
(4.77)
\[ D_\mu \phi^5 = \partial_\mu \phi^5. \]  
(4.78)

\(^8\) Clearly this does not include the group contraction \( \text{SO}(4) \rightarrow \text{SO}(3) \), since that results in the type I SCS analysed in section 3.2 above, for \( d = 3 \).
To define the SCS density, what is needed are the definitions of $\omega^{(3+1)}$ and $\Omega^{(3+1)}$ in (2.27), from which follows the definition of the SCS density by (2.20). The quantity $\Omega^{(3+1)}$ in (2.27) in this case is,

$$\Omega^{(3+1)} = \omega^{(3+1)} + \Omega^{(3+1)}_0,$$

where the quantity $\Omega^{(3+1)}_0$ in $3 + 1$ dimensions, appearing in [7, 8, 12], is

$$\Omega^{(3+1)}_0 \equiv 3! \varepsilon_{ijkl} \varepsilon^{abcd} \phi^5 \left\{ \frac{1}{2} F_{ik}^{ab} \phi^e D_j \phi^b + \partial_j \left[ A_{ik}^{ab} \phi^e \left( \partial_k \phi^d + \frac{1}{2} A_{ik} \phi^d \right) \right] \right\} + \frac{1}{4} \left( 1 - \frac{1}{3} \phi^5 \right)^2 \left[ \partial_j A_{ik}^{ab} + \frac{2}{3} (A_{ik} A_{ij})^{cd} \right], \quad (4.79)$$

from which follows by the one-step descent, the quantity $\Omega^{(3)}$ in (2.30)

$$\Omega^{(3)} = 3! \varepsilon_{ijkl} \varepsilon^{abcd} \phi^5 \left\{ -\frac{1}{2} F_{ik}^{cd} \phi^e D_j \phi^b + \partial_j \left[ A_{ik}^{ab} \phi^e \left( \partial_k \phi^d + \frac{1}{2} A_{ik} \phi^d \right) \right] \right\} + \frac{1}{4} \left( 1 - \frac{1}{3} \phi^5 \right)^2 \left[ \partial_j A_{ik}^{cd} + \frac{2}{3} (A_{ik} A_{ij})^{cd} \right], \quad (4.80)$$

This density is clearly gauge variant, seen for example from the term multiplying $(1 - \frac{1}{3} \phi^5)$ in the second line, which is the (Euler, rather than the Pontryagin) CS density for the $SO(4)$ gauge field in $d = 3$.

To evaluate the term $\omega^{(3)}$ in (2.30) however, it is necessary to employ the constraint compliant parametrisation

$$\phi^a = \sin f \, n^a, \quad \phi^5 = \cos f, \quad |n^a|^2 = 1 \quad (4.81)$$

of the Skyrme scalar $(\phi^a, \phi^5)$.

To this end, one evaluates $\varrho_0$ in (2.25) defined by (2.21), yielding

$$\varrho_0 = -4 \, \varepsilon_{ijkl} \varepsilon^{abcd} \partial_i \left( \phi^5 - \frac{1}{3} \phi^5 \right) \cdot n^a \partial_j n^b \partial_k n^c \partial_l n^d$$

$$= -4 \, \varepsilon_{ijkl} \varepsilon^{abcd} \partial_i \left[ \left( \phi^5 - \frac{1}{3} \phi^5 \right) n^a \partial_j n^b \partial_k n^c \partial_l n^d \right] \quad \text{def} \quad \partial_i \omega^{(3+1)} \quad (4.82)$$

whence we have

$$\omega^{(3)} \equiv \omega^{(3+1)} \equiv 4 \, \varepsilon_{ijkl} \varepsilon^{abcd} \left( \phi^5 - \frac{1}{3} \phi^5 \right) n^a \partial_j n^b \partial_k n^c \partial_l n^d. \quad (4.83)$$

Adding (4.83) and (4.80)

$$\Omega^{(3)}_{SCS} = \omega^{(3)} + \Omega^{(3)} \quad (4.84)$$

for $SO(4)$ gauging.

The WZ term $\omega^{(3)}$ in (4.84) is given in the constraint-compliant parametrisation (4.81), but this is not done for the term $\Omega^{(3)}$ here.
4.2. SO(2) × SO(2) gauged type II SCS in \( d = 3 \)

The task is as above the calculation of \( \Omega^{(3)}_{SCS} = \omega^{(3)} + \Omega^{(3)} \), (2.30). It is useful to start with the calculation of \( \Omega^{(3)} \) for this case by subjecting (4.80) to the gauge group contraction

\[
SO(4) \rightarrow SO(2) \times SO(2)
\]

on the gauge connection \( A_{\mu}^{ab} = (A_{\mu}^{\alpha\beta}, A_{\mu}^{AB}, A_{\mu}^{A}) \)

\[
A_{\mu}^{\alpha\beta} = A_{\mu} \varepsilon^{\alpha\beta}, \quad A_{\mu}^{AB} = B_{\mu} \varepsilon^{AB}, \quad A_{\mu}^{A} = 0, \quad (4.85)
\]

where \( A_{\mu} \) and \( B_{\mu} \) are the connections of the two (distinct) abelian subgroups of \( SO(4) \).

We denote the corresponding components of the abelian curvature by

\[
F_{\mu\nu}^{\alpha\beta} = \varepsilon_{\alpha\beta} F_{\mu\nu}, \quad F^{AB}_{\mu\nu} = \varepsilon_{AB} G_{\mu\nu}, \quad F^{A}_{\mu\nu} = 0, \quad (4.86)
\]

where

\[
F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \quad \text{and} \quad G_{\mu\nu} = \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu}
\]

in an obvious notation.

The corresponding \( SO(4) \) covariant derivatives contract to

\[
D_{\mu} \phi^{\alpha} = \partial_{\mu} \phi^{\alpha} + A_{\mu} (\varepsilon \phi)^{\alpha}, \quad \alpha = 1, 2, \quad (\varepsilon \phi)^{\alpha} = \varepsilon^{\alpha\beta} \phi^{\beta} \quad (4.87)
\]

\[
D_{\mu} \phi^{A} = \partial_{\mu} \phi^{A} + B_{\mu} (\varepsilon \phi)^{A}, \quad A = 3, 4, \quad (\varepsilon \phi)^{A} = \varepsilon^{AB} \phi^{B} \quad (4.88)
\]

\[
D_{\mu} \phi^{5} = \partial_{\mu} \phi^{5}. \quad (4.89)
\]

Substituting (4.85), (4.86) and (3.39)–(4.89) in (4.80), we obtain \( \Omega^{(3)} \) for the gauge group \( SO(2) \times SO(2) \)

\[
\Omega^{(3)} = 3! \epsilon^{4\mu\nu\lambda} \phi^{5} \left\{ \frac{1}{3} (\phi^{5})^{2} (\partial_{\lambda} F_{\mu\nu} + B_{\lambda} F_{\mu\nu}) - A_{\lambda} B_{\mu} \partial_{\lambda} (|\phi^{\alpha}|^{2} - |\phi^{A}|^{2}) \right. \\
\left. - 2 [A_{\lambda} (\varepsilon \partial_{\mu} \phi)^{A} \partial_{\nu} \phi^{A} + B_{\lambda} (\varepsilon \partial_{\mu} \phi)_{B} \partial_{\nu} \phi^{B}] \right\} . \quad (4.90)
\]

Next, calculate \( \omega^{(3)} \) in (2.30) to complete the construction of the SCS density \( \Omega^{(3)}_{SCS} \). This can be done only after expressing \( \varrho_{0} = \partial \omega^{(3+1)}_{SCS} \) in constraint compliant parametrisation of the \( O(5) \) Skyrme scalar \( (\phi^{a}, \phi^{5}) \), with \( \phi^{a} = (\phi^{a}, \phi^{A}), \alpha = 1, 2; A = 3, 4 \), which is

\[
\phi^{\alpha} = \sin f \sin g n_{(1)}^{\alpha}, \quad \phi^{A} = \sin f \cos g n_{(2)}^{A}, \quad \phi^{5} = \cos f \quad (4.91)
\]

\[
n_{(1)}^{\alpha} = \begin{pmatrix} \cos \psi \\ \sin \psi \end{pmatrix}, \quad n_{(2)}^{A} = \begin{pmatrix} \cos \chi \\ \sin \chi \end{pmatrix}. \quad (4.92)
\]

The result is

\[
\varrho_{0} = -3! \cdot 2 \epsilon_{ijkl} \left[ \partial_{i} \left( \cos f - \frac{1}{3} \cos^{3} f \right) \right] (\partial_{j} \sin^{2} g) \partial_{k} \psi \partial_{l} \chi, \quad (4.93)
\]

from which one can choose e.g., to extract the partial derivative \( \partial_{i} \), resulting in \( \omega^{(3)} \defeq \omega^{(3+1)}_{SCS} \)

\[
\omega^{(3)} = -3! \cdot 2 \epsilon_{ijkl} \left( \cos f - \frac{1}{3} \cos^{3} f \right) (\partial_{k} \sin^{2} g) \partial_{i} \psi \partial_{l} \chi. \quad (4.94)
\]
Finally, the SCS density for the gauge group $SO(2) \times SO(2)$ expressed entirely in the parametrisation (4.91) and (4.92), is

$$\Omega_{SCS}^{(3)} = \omega^{(3)} + \Omega^{(3)}$$

$$\frac{1}{3!} \Omega_{SCS}^{(3)} = \varepsilon^{\mu \nu \lambda} \left\{ -2 \left( \cos f - \frac{1}{3} \cos^3 f \right) \left( \partial_{\lambda} \sin^2 g \right) \partial_{\mu} \psi \partial_{\nu} \chi 
\right.
\left. + \frac{1}{3} \cos^3 f \left( A_{\lambda} G_{\mu \nu} + B_{\lambda} F_{\mu \nu} \right) - A_{\mu} B_{\nu} \cos f \left[ \partial_{\lambda} \left( \sin^2 f \sin^2 g \right) - \partial_{\lambda} \left( \sin^2 f \cos^2 g \right) \right] 
\right.
\left. + 2 \left[ A_{\lambda} \partial_{\mu} \left( \sin^2 f \sin^2 g \right) \partial_{\nu} \chi + B_{\lambda} \partial_{\mu} \left( \sin^2 f \cos^2 g \right) \partial_{\nu} \psi \right] \right\}. \quad (4.95)$$

Setting $B_{\mu} = 0$ in (4.95) results in the type I SCS density (3.47) for $d = 3$. The action (4.95) is manifestly gauge variant. The question is, as a CS like density, is it like the latter gauge invariant up to total divergence? A straightforward way to test this is to show that the resulting Euler–Lagrange equations are gauge covariant.

The variational equations w.r.t. $A_{\tau}$ resp. $B_{\tau}$ are

$$\varepsilon^{\tau \mu \nu} \cos f \left[ \frac{1}{3} \cos^2 f G_{\mu \nu} - \partial_{\nu} \left( \sin^2 f \cos^2 g \right) \right] = 0, \quad (4.96)$$

$$\varepsilon^{\tau \mu \nu} \cos f \left[ \frac{1}{3} \cos^2 f F_{\mu \nu} - \partial_{\nu} \left( \sin^2 f \sin^2 g \right) \right] = 0, \quad (4.97)$$

and the variational equations w.r.t. $\psi$ resp. $\chi$ are

$$\varepsilon^{\tau \mu \nu} \left[ \frac{1}{2} G_{\mu \nu} \cos f \partial_{\tau} \left( \sin^2 f \sin^2 g \right) + \partial_{\nu} f \sin^3 f \partial_{\lambda} \left( \sin^2 g \right) \right] = 0, \quad (4.98)$$

$$\varepsilon^{\tau \mu \nu} \left[ \frac{1}{2} F_{\mu \nu} \cos f \partial_{\tau} \left( \sin^2 f \cos^2 g \right) - \partial_{\nu} f \sin^3 f \partial_{\lambda} \left( \cos^2 g \right) \right] = 0, \quad (4.99)$$

which are manifestly gauge invariant. The variational equations w.r.t. $f$ and $g$ are also gauge invariant but they are too cumbersome to display here.

It may be relevant to comment on the degenerate model resulting from replacing $B_{\mu} = A_{\mu}$ in (4.90). That model features only the abelian field $(A_{\mu}, F_{\mu \nu})$. The variational equation w.r.t. $A_{\nu}$ will be gauge invariant only if one sets the functions $\psi = \chi$, which is absurd as these angles parameterise the components $\phi^A$ and $\phi^B$ of the $O(5)$ Skyrme scalar, which are independent degrees of freedom, i.e., that the submodel of (4.95) with $B_{\mu} = A_{\mu}$ leads to gauge variant equations of motion.

5. Summary and outlook

In this note, aspects of the SCS densities proposed in reference [8] are elaborated on, with the main emphasis being on the gauge dependance of the SCS. It is shown that the SCS actions in $d$ spacetime dimensions, as in the case of the usual CS densities, are gauge invariant up
to total divergence and hence that their Euler–Lagrange equations are gauge invariant. The importance of verifying gauge invariance of the equations of any given model is, that in the concrete application of such actions, this is a necessary check of the correctness of the model at hand.

In section 3, type I SCS in $d$ dimensions are presented. In subsection 3.1 the case with gauge group $SO(2)$, and in subsection 3.2 with gauge group $SO(3)$ respectively. Type I SCS in $d$ space-time dimensions, are those gauged with $SO(N)$ with $N < d + 1$, $SO(d + 1)$ being the largest gauge group allowed.

Concrete analyses are carried out in two examples, $SO(2)$ and $SO(3)$, one abelian and one non-abelian. It is found that $SO(2)$ type I SCS are explicitly gauge invariant as they stand, and hence the resulting equations of motion are automatically gauge covariant. By contrast, $SO(3)$ type I SCS is gauge invariant only up to a total divergence, and hence their equations of motion are gauge invariant as described in section 2.2.2. (One can surmise that this property for $SO(3)$ holds in all non-abelian gauge groups as well.) Thus, like the usual CS (in odd dimensions) type I SCS (in all dimensions) lead to gauge covariant equations of motion. A distinguishing feature of type I SCS is that the density $\Omega^{(d)}$ in (2.30) features only one power of the gauge connection $A_{\mu}$, and no curvature term $F_{\mu\nu}$. This sets a limitation on the application of type I SCS in the context of static fields, since for static fields the term $\omega^{(d)}$ in (2.30) vanishes, while the term $\Omega^{(d)}$ now displays only the temporal component $A_0$ of the gauge connection. But we know from the results of references [12–15] that the important new features resulting from CS dynamics hinge on the interrelation of the electric and the magnetic fields $A_0$ and $A_i$. Of course in odd dimensions, when the usual CS term is present, the type I SCS will have a quantitative effect.

In section 4, type II SCS in $d = 3$ dimensions are presented. Type II SCS are those gauged with the largest allowed gauge group $SO(d + 1)$ or with some direct product of subgroups of $SO(d + 1)$. In these notes attention is restricted to dimension $d = 3$, thus to the gauge group $SO(4)$ and its subgroup $SO(2) \times SO(2)$. In subsection 4.1 gauge group $SO(4)$ is considered, and in subsection 4.2 gauge group $SO(2) \times SO(2)$ is analysed concretely.

Type II SCS feature both connection and curvature ($A_{\mu}, F_{\mu\nu}$) so that in the static limit both electric and magnetic fields ($A_0, A_i$) will persist. This is the important aspect distinguishing type II SCS from type I. It is the type II SCS that promise to reproduce the special effects of CS dynamics observed in references [12–15] in odd dimensions. In this sense, it is the type II SCS that are the germane extensions of the usual CS densities, with the added all important feature that they are defined in both odd and even dimensions.

In the absence of a CS action in even dimensional spacetime, the SCS action which is defined in all dimensions is potentially important. Of special importance is the physical Minkowskian $3 + 1$ dimensional theory. In that case such an action is also the ‘anomaly related density’ appearing in reference [16] for the $U(1)$ gauged Skyrme theory. Specifically, the latter is akin to the SCS II in that it displays both the $SO(2)$ connection $A_{\mu}$ and the curvature $F_{\mu\nu}$. A significant difference between the SCS action $\Omega_{\text{SCS}}^{(d=4)}$ defined by (2.29) and (2.30) proposed here, and the ‘anomaly related term’ appearing in [16], is that in the latter the Skyrme scalar employed in its construction is the $O(4)$ sigma model field that supports the (topologically stable) Skyrmion, while by contrast the Skyrme scalar employed in the construction of the SCS $\Omega_{\text{SCS}}^{(d=4)}$ is the $O(6)$ sigma model field which in $3 + 1$ dimensions does not support a soliton$^{10}$.

$^{10}$This is simply because the SCS in $3 + 1$ dimensions is descended, much in the same spirit proposed earlier in reference [22], from a SCP density in five dimensions which is defined in terms of the $O(6)$ sigma model scalar.
Generally, the SCS can be employed to investigate the effects of CS-like dynamics in both even and odd dimensions. In this context, the case of $3 + 1$ dimensions is special since in that case the ‘anomaly related term’ of [16] is also a candidate for this role. Moreover, the latter is defined in terms of the $O(4)$ Skyrme scalar unlike its SCS counterpart that is defined in terms of the $O(6)$ Skyrme scalar.

It would be interesting to compare the potential roles of the CS-like densities in [16] and the SCS action $\Omega_{\text{SCS}}^{(d=4)}$ proposed here. In both cases the topologically stable Skyrmion stabilised by the baryon number is deformed first by the abelian gauge field, after which the respective CS-like action further influences the dynamics. In the first [16] case no new (scalar) field is involved while in the second [8] case the $O(6)$ Skyrme scalar enters.

The largest group with which the $O(6)$ scalar can be can be gauged is $SO(5)$. The $O(4)$ Skyrme scalar describing the usual $SO(2)$ gauged Skyrmion, can interact with the $O(6)$ Skyrme scalar describing the SCS density only through the $SO(2)$ gauge field. Thus the $O(6)$ scalar must be gauged with the subgroup $SO(2 \times SO(3))$ of $SO(5)$, in which the $SO(3)$ gauge sector can play the role of an ‘auxiliary gauge field’

12. Concrete investigations of gauged Skyrmions influenced by SCS dynamics are under active consideration at present.

Acknowledgments

My thanks for invaluable support go to Valery Rubakov. I am grateful to Francisco Navarro-Lerida (FNL) and Eugen Radu for their past collaboration in this area. Thanks to ER for help in preparing this report, and to FNL for raising the question which instigated the analysis carried out here. My thanks go to the Referee of J. Phys. A, for generous and constructive comments and suggestions.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

ORCID iDs

D H Tchrakian © https://orcid.org/0000-0001-7892-0948

References

[1] Paul S K and Khare A 1986 Charged vortices in abelian Higgs model with Chern–Simons term Phys. Lett. B 174 420

Paul S K and Khare A 1986 Phys. Lett. B 177 453 (erratum)

[2] Hong J, Kim Y and Pac P Y 1990 On the multivortex solutions of the abelian Chern–Simons–Higgs theory Phys. Rev. Lett. 64 2230

The action in [16] follows from the results derived in reference [22], which is specific to four dimensions. We are not aware of higher dimensional versions of this result in the literature.

12 There is also the academic possibility of starting with the $SO(3)$ gauged $O(4)$ Skyrme model given in reference [23] which is also endowed with an ‘energy lower bound’. This $SO(3)$ gauged Skyrmion is then deformed further by the $SO(3 \times SO(2))$ gauged SCS action, so that the $O(4)$ and $O(6)$ scalars see each other via the $SO(3)$ gauge field. In this case it is the $SO(2)$ field which plays the role of ‘auxiliary gauge field’.
[3] Jackiw R and Weinberg E J 1990 Selfdual Chern–Simons vortices Phys. Rev. Lett. 64 2234
[4] Ghosh P K and Ghosh S K 1996 Topological and nontopological solitons in a gauged $O(3)$ sigma model with Chern–Simons term Phys. Lett. B 366 199
[5] Kimm K, Lee K-M and Lee T 1996 Anyonic Bogomolnyi solitons in a gauged $O(3)$ sigma model Phys. Rev. D 53 4436
[6] Arthur K, Tchrakian D H and Yang Y 1996 Topological and nontopological selfdual Chern–Simons solitons in a gauged $O(3)$ sigma model Phys. Rev. D 54 5245
[7] Navarro-Lérida F, Radu E and Tchrakian D H 2020 SO(2) gauged Skyrmions in $4 + 1$ dimensions Phys. Rev. D 101 125014
[8] Tchrakian D H 2015 Higgs– and Skyrmé–Chern–Simons densities in all dimensions J. Phys. A: Math. Theor. 48 375401
[9] Schroers B J 1995 Bogomolny solitons in a gauged $O(3)$ sigma model Phys. Lett. B 356 291
[10] Tchrakian D H 1997 Topologically stable lumps in $SO(d)$ gauged $O(d + 1)$ sigma models in $d$ dimensions: $d = 2, 3$ Lett. Math. Phys. 40 191–201
[11] Tchrakian D H 2002 Winding number versus Chern–Pontryagin charge (arXiv:hep-th/0204040)
[12] Navarro-Lérida F, Radu E and Tchrakian D H 2017 Effect of Chern–Simons dynamics on the energy of electrically charged and spinning vortices Phys. Rev. D 95 085016
[13] Navarro-Lérida F and Tchrakian D H 2019 Vortices of SO(2) gauged Skyrmions in $2 + 1$ dimensions Phys. Rev. D 99 045007
[14] Navarro-Lérida F, Radu E and Tchrakian D H 2019 On the topological charge of SO(2) gauged Skyrmions in $2 + 1$ and $3 + 1$ dimensions Phys. Lett. B 791 287–92
[15] Navarro-Lérida F, Radu E and Tchrakian D H 2021 On the effects of the Chern–Simons term in an abelian gauged Skyrme model in $d = 4 + 1$ dimensions Phys. Lett. B 814 136083
[16] Callan C G Jr and Witten E 1984 Monopole catalysis of Skyrmion decay Nucl. Phys. B 239 161–76
[17] Tchrakian D H 2011 Notes on Yang–Mills–Higgs monopoles and Dyons on $R^d$, and Chern–Simons–Higgs solitons on $R^{d+2}$: dimensional reduction of Chern–Pontryagin densities J. Phys. A: Math. Theor. 44 343001
[18] Rubakov V A and Tavkhelidze A N 1985 Stable anomalous states of superdense matter in gauge theories Phys. Lett. B 165 109
[19] Deser S, Jackiw R and Templeton S 1982 Phys. Rev. Lett. 48 975
[20] Deser S, Jackiw R and Templeton S 1982 Topologically massive gauge theories Ann. Phys. 140 372
Deser S, Jackiw R and Templeton S 1988 Ann. Phys. 185 406 (erratum)
Deser S, Jackiw R and Templeton S 2000 Ann. Phys. 281 409
[21] Brihaye Y, Radu E and Tchrakian D H 2010 AdS$_5$ solutions in Einstein–Yang–Mills–Chern–Simons theory Phys. Rev. D 81 064005
[22] Witten E 1983 Global aspects of current algebra Nucl. Phys. B 223 422–32
[23] Arthur K and Tchrakian D H 1996 Phys. Lett. B 378 187–93