Arnold diffusion of charged particles in ABC magnetic fields

Alejandro Luque∗
Instituto de Ciencias Matemáticas
Consejo Superior de Investigaciones Científicas
28049 Madrid (Spain).

Daniel Peralta-Salas†
Instituto de Ciencias Matemáticas
Consejo Superior de Investigaciones Científicas
28049 Madrid (Spain).

15th September 2015

Abstract

We prove the existence of diffusing solutions in the motion of a charged particle in the presence of an ABC magnetic field. The equations of motion are modeled by a 3DOF Hamiltonian system depending on two parameters. For small values of these parameters, we obtain a normally hyperbolic invariant manifold and we apply the so-called geometric methods for a priori unstable systems developed by A. Delshams, R. de la Llave, and T.M. Seara. We characterize explicitly sufficient conditions for the existence of a transition chain of invariant tori having heteroclinic connections, thus obtaining global instability (Arnold diffusion). We also check the obtained conditions in a computer assisted proof. ABC magnetic fields are the simplest force-free type solutions of the magnetohydrodynamics equations with periodic boundary conditions, so our results are of potential interest in the study of the motion of plasma charged particles in a tokamak.

Keywords: Motion of charges in magnetic fields, Hamiltonian dynamical systems, Arnold diffusion, global instability, heteroclinic connections.

∗luque@icmat.es
†dperalta@icmat.es
1 Introduction

The study of the motion of a charged particle in a magnetic field is a classical subject in several areas of physics, such as condensed matter theory, accelerator physics, magnetobiology and plasma physics. The equation of motion of a (non-relativistic) unit-mass, unit-charge particle at the position \( q \in \mathbb{R}^3 \) in the presence of a magnetic field \( B \) is given by the Newton-Lorentz law

\[
\dot{q} = \dot{q} \times B(q),
\]

where the dot over \( q \) denotes, as usual, the time derivative, and \( \times \) stands for the standard vector product in \( \mathbb{R}^3 \).

An important observation is that Eq. (1) can be written equivalently in a Hamiltonian way whenever there is a globally defined vector potential \( A \) such that \( B = \nabla \times A \). If this is the case, the Hamiltonian function is

\[
H(q, p) = \frac{1}{2} (p - A(q))^2.
\]

In this paper we are interested in the motion of charges in ABC magnetic fields. These fields arise in the theory of magnetic dynamos (see [29] and references therein) and were introduced independently by Arnold [2] and Childress [10] in the 1960’s. The well-known family of ABC magnetic fields depends on three real parameters, \( A, B \) and \( C \), and reads in Cartesian coordinates \( q = (x, y, z) \) as

\[
B_{ABC} = (A \sin z + C \cos y, B \sin x + A \cos z, C \sin y + B \cos x).
\] (2)

ABC magnetic fields are stationary solutions of the magnetohydrodynamics equations of force-free type, thus implying that the field exerts no force on the current distribution generating it. Indeed, it is straightforward to check that \( B_{ABC} \) is divergence-free and force-free because \( \nabla \times B_{ABC} = B_{ABC} \), and so the ABC field admits the globally defined vector potential \( A_{ABC} = B_{ABC} \). ABC magnetic fields are minimizers of the energy functional \( \int B^2 \) acting on the space of divergence-free fields of fixed helicity.

Since the dependence of the ABC magnetic field and its vector potential with the variables \( x, y, z \) is \( 2\pi \)-periodic, it is customary to consider that these fields are defined in the 3-torus \( \mathbb{T}^3 = \mathbb{R}^3/(2\pi \mathbb{Z})^3 \) so that \( (x, y, z) \in \mathbb{T}^3 \). By rescaling and reordering the space variables and the time, all the non-trivial cases can be reduced to \( A = 1 \geq B \geq C \geq 0 \), so we shall assume it in what follows. The Newton-Lorentz equation of motion (1) for the ABC magnetic field can be described as a 3DOF Hamiltonian system defined in the phase space \( \mathbb{T}^3 \times \mathbb{T}^3 \supset (x, y, z, p_x, p_y, p_z) \) by the Hamiltonian function:

\[
H = \frac{1}{2} (p_x - C \cos y - \sin z)^2 + \frac{1}{2} (p_y - B \sin x - \cos z)^2 + \frac{1}{2} (p_z - B \cos x - C \sin y)^2.
\] (3)

Force free fields are very important in applications and model diverse physical systems, as stellar atmospheres [7], the solar corona [24] and relaxed states of toroidal plasmas [40]. Moreover, the motion of a charge in an ABC field can be interpreted as a model for the motion of plasma charged particles in a tokamak. A wide examination of system (1) was recently presented in [36], proving the existence of confinement regions of charges near some magnetic lines and also that the problem gives rise to non-integrability and chaotic motions. In this study we go one step further and obtain global instability, i.e. Arnold diffusion.

Characterizing global instabilities in Hamiltonian systems is a relevant problem that has called the attention of mathematicians, physicists and engineers. For example, in the context of beam physics, designers of accelerators or plasma confinement devices are interested in the characterization of these instabilities in order to avoid them as much as possible (e.g. in the confinement of hot plasmas for fusion power generation, diffusion is a very relevant phenomenon because of the harmful plasma-wall interaction). Global instability deals with the question of whether Hamiltonian perturbations of a regular integrable system accumulate over time, giving rise to a long term effect, or whether they average out. This problem was first formulated by Arnold. Indeed, in the celebrated paper [11], Arnold constructed a concrete example, suitably and cleverly chosen, such that some trajectories can jump around KAM tori thus obtaining diffusion (after [11] this problem is known as Arnold diffusion). These diffusing orbits were constructed...
using a mechanism of transition chains. It consists in obtaining heteroclinic intersections between the stable and unstable manifolds of a sequence of whiskered invariant tori.

In the last decades there has been a significant advance in the understanding of diffusion and, following [9], the studies are classified in two different groups: the a priori unstable case and the a priori stable case. Arnold diffusion in a priori unstable systems (where the unperturbed system has hyperbolic properties of some kind) has been approached using geometric methods in [16, 18, 19, 25], the separatrix map in [42, 43], topological methods in [26, 27] and variational methods in [3, 11]. A combination of topological and geometric methods has been recently presented in [28]. The more difficult case of a priori stable systems (where the unperturbed system is foliated by Lagrangian invariant tori) is less understood, but significant advances have been presented along the last few years in [4, 8, 32, 33, 34, 35, 38, 47].

Our aim in this paper is to prove the existence of Arnold diffusion in the dynamics of a charged particle in an ABC magnetic field, which is modeled by the Hamiltonian system (3). If $B = C = 0$, we obtain an integrable Hamiltonian system $H_0$ having a normally hyperbolic invariant manifold (NHIM) $\Lambda_0$ foliated by whiskered invariant tori (see details in Section 2.1). Then, the problem considered in this paper falls into the a priori unstable setting. It is worth mentioning that one of the main difficulties in the study of a priori unstable systems was the so-called large gap problem (see [49]). This problem arises because a generic perturbation of size $\varepsilon$ creates gaps at most of size $\sqrt{\varepsilon}$ between the persisting primary KAM tori and, in principle, only orbits separated an amount $\varepsilon$ could be connected by heteroclinic connections between invariant tori. This issue was solved in the previously mentioned references, using different tools for the study of Arnold diffusion. We observe that recent mechanisms of diffusion have been proposed in order to avoid big gaps using very little information of the dynamics restricted to the NHIM (see [6, 21]). Here, we follow the geometric methods developed in [16, 18] in order to prove the existence of Arnold diffusion in the Hamiltonian (3) for small values of $B$ and $C$. Concretely, we prove the following theorem, which establishes sufficient conditions for the existence of a transition chain between whiskered invariant tori, thus producing large unstable motions in the perturbed system:

**Main Theorem (informal statement).** Let us consider the Hamiltonian (3) with $B = \varepsilon \hat{B} \neq 0$ and $C = \varepsilon \hat{C} \neq 0$, and a non-empty set $\mathcal{I} = [a_1, b_1] \times [a_2, b_2]$ for given (positive) values of $a_i, b_i$. Then, under some explicit non-degeneracy and transversality conditions, if $|\varepsilon|$ is small enough, the ABC system exhibits Arnold diffusion in $\mathcal{I}$, i.e. there exists a trajectory of (3) connecting two arbitrary values of $(p_x, p_y)$ in the interior of $\mathcal{I}$.

A precise statement of this theorem is given in Theorem 2.2 (Section 2), after a detailed discussion of the unperturbed ABC system. Moreover, we implement the non-degeneracy and transversality conditions included in the Main Theorem in a computer assisted proof (CAP) in Section 6. As a consequence, we obtain an open set of initial conditions in phase space where we can construct a transition chain. For example, we obtain the following result which serves as an illustration:

**Corollary 1.1.** Let us consider Hamiltonian (3) with $\hat{B} = 10$ and $\hat{C} = 0.1$. Then, the non-degeneracy and transversality conditions of the Main Theorem hold in the set $\mathcal{I} = [0.1, 0.9] \times [0.5, 0.9]$. Therefore, for $|\varepsilon|$ small enough, there exists a trajectory of (3) connecting two arbitrary values of $(p_x, p_y)$ in $(0.1, 0.9) \times (0.5, 0.9)$.

We remark that the choice $\hat{B} = 10$ and $\hat{C} = 0.1$ is arbitrary. Analogous results can be obtained for any other choice of parameters. The computational cost to verify the hypotheses for a fixed set $\mathcal{I}$ increases when the difference between $\hat{B}$ and $\hat{C}$ is reduced. It is worth mentioning that if we take “narrow” sets of the form $\mathcal{I} = [a_1, a_1 + \delta] \times [a_2, b_2]$ or $\mathcal{I} = [a_1, b_1] \times [a_2, a_2 + \delta]$, with $\delta$ small, then the computational cost of the CAP is reduced significantly. In this case, it is also possible to check the conditions for open sets of parameters $\hat{B}$ and $\hat{C}$. We have produced analogous results to Corollary 1.1 and we have not found obstructions to diffusion in any case.

To the best of our knowledge, the Main Theorem and Corollary 1.1 are the first rigorous results on the existence of diffusing orbits in the motion of charges in magnetic fields, even though physicists have been aware of this phenomenon for a long time (cf. [45, 46]) and the effect is sometimes known as drift motion in the physics literature. Of course, we want to mention other significant problems where Arnold diffusion have been characterized. In particular,
we can find remarkable contributions in the context of celestial mechanics: diffusion along mean motion resonances in the restricted planar three-body problem [22]; instability mechanism in a special configuration of the 5-body problem [39, 48]; transition chains of invariant tori around the point $L_2$ in the elliptic three body problem as a perturbation of the circular problem [12], improved recently in [6]; instability around the point $L_1$ in the circular spatial restricted three-body problem, focusing on homoclinic trajectories [13]; instability in the elliptic restricted problem close to the parabolic orbits of the Kepler problem between the comet and the Sun [21]. We observe that some parts of the arguments in [6, 12, 13, 22] are non-rigorous, but are strongly backed by convincing numerical computations. It is also worth mentioning the example discussed in [20], where the geometric mechanism for diffusion introduced in [19] is illustrated in a representative model. The model simplifies some of the hypotheses, thus saving a significant amount of computations, so they can present the geometric mechanism of diffusion in a clear understandable way. In the system (5) studied in this paper, some of these simplifications cannot be used and we must perform some ad hoc analysis and specific computations. The reader interested in numerical studies is referred to [30].

The mechanisms governing Arnold diffusion are very complex and there are still many questions to answer and many aspects to understand. As is posed in [27], it is relevant to detect, combine, and compare different mechanisms of diffusion displayed by concrete systems. In this way, Hamiltonian (5) can be an ideal framework to apply and compare different approaches and methods in the literature (e.g. topological methods, variational techniques, use of multiple scattering maps, etc). On the one hand, the ABC system is complicated enough to contain all the difficulties that are present in a general a priori unstable problem. On the other hand, the ABC system is explicit and simple enough to perform analytic computations. Moreover, it is a problem that appears in a natural way in physics.

The proof of the Main Theorem consists in combining the internal dynamics on the NHIM with its outer (asymptotic) dynamics, which is modeled by the scattering map [17]. The procedure is divided in the following steps:

**Characterization of the NHIM:** The first step is to characterize the perturbed NHIM $\Lambda_\varepsilon$ and its stable and unstable manifolds (we summarize some basic concepts in Section 3.1). We pay special attention to describe explicitly the geometric procedure that allows us to parameterize the NHIM in a natural way, thus obtaining a suitable symplectic structure on the NHIM (see Section 3.2). The construction presented has special interest since we give explicit formulas to use the deformation theory introduced in [17]. To this end, we have to compute perturbatively a symplectic frame associated to the manifold. Explicit computations for the ABC system are detailed in Section 3.3.

**Invariant tori on the NHIM:** To study the inner dynamics on the NHIM, where the so-called big gaps are present, we perform averaging theory of the vector field restricted to the manifold. After choosing a suitable parameterization in the previous step, we follow [16, 18] *mutatis mutandis* in Section 3.4.1. Explicit computations for the ABC system are detailed in Section 3.4.2. In Proposition 3.9 we obtain an approximation of the level sets that characterize the invariant objects inside the NHIM. In particular, we find a set of whiskered invariant tori (primary and secondary) covering $\Lambda_\varepsilon$ except for a set of measure $O(\varepsilon^{3/2})$.

**Scattering map:** In Section 4 we describe the outer dynamics associated to our problem. For the sake of completeness, in Section 4.1 we summarize the construction of the Melnikov potential that characterizes the intersections of the stable and unstable manifolds associated to the NHIM (cf. [41]). In Section 4.2 we compute the scattering map for the ABC system.

**Combination of inner and outer dynamics:** The combination of both dynamics, obtaining explicit transversality conditions for the existence of diffusion, is performed in Section 5. We remark that, since the unperturbed scattering map has a so-called phase shift, there is an additional term in the transversality conditions that is not present in [16, 18]. In the domain where the conditions are satisfied, we construct a sequence $\{T_i\}_{i=1}^\infty$ of whiskered tori satisfying $W^u_{T_i} \cap W^s_{T_{i+1}}$, that is, we construct a transition chain along $\Lambda_\varepsilon$.

We remark again that the hypotheses in the Main Theorem are explicit and involve a series of standard, but cumbersome, computations. First, we evaluate some integrals that depend on $(p_x, p_y)$ as parameters. We solve a one-dimensional non-linear equation that depends on these integrals. We approximate the derivatives with respect to
parameters of the previous solution. Finally, we evaluate several complicated formulas that depend on the previous objects. In Section 6 we rigorously perform these computations with the help of a computer.

2 Setting of the problem and statement of the main theorem

In this Section we present a detailed description of the geometry of our problem and state a precise version of the Main Theorem. More precisely, in Section 2.1 we fully describe the motion of the unperturbed Hamiltonian system (Eq. (3) with $B = C = 0$), thus characterizing a normally hyperbolic invariant manifold with coincident stable and unstable invariant manifolds. Then, in Section 2.2 we provide explicit sufficient conditions for the existence of Arnold diffusion in the perturbed problem (Eq. (3) with $B = \varepsilon \hat{B}, C = \varepsilon \hat{C}$).

2.1 Geometric features of the unperturbed problem

For $B = C = 0$, the ABC magnetic field has the simple expression

$$B_{ABC} = (\sin z, \cos z, 0),$$

which implies that the field is linear on each toroidal surface $z = z_0$, periodic or quasi-periodic depending on the value of $\tan z_0$. Concerning the equations of motion, the Hamiltonian function in Eq. (3) is given by

$$H_0 = \frac{1}{2}(p_x - \sin z)^2 + \frac{1}{2}(p_y - \cos z)^2 + \frac{1}{2}p_z^2. \quad (4)$$

The system of ODEs associated to (4) is

$$\dot{x} = p_x - \sin z, \quad \dot{p}_x = 0,$$
$$\dot{y} = p_y - \cos z, \quad \dot{p}_y = 0,$$
$$\dot{z} = p_z, \quad \dot{p}_z = p_x \cos z - p_y \sin z,$$

so $p_x$ and $p_y$ are constants of the motion. There is no loss of generality in taking positive values of $p_x$ and $p_y$, so we shall assume it throughout the paper. In addition, we observe that the system $(z, p_z)$ is pendulum-like and has an effective potential

$$V(z) := -p_x \sin z - p_y \cos z.$$

Notice that this system has a hyperbolic equilibrium at the point

$$z^* := \arctan \frac{p_x}{p_y} + \pi, \quad p_z^* = 0,$$

and, since $p_x > 0$ and $p_y > 0$, we have the identities

$$\sin z^* = \frac{-p_x}{\sqrt{p_x^2 + p_y^2}}, \quad \cos z^* = \frac{-p_y}{\sqrt{p_x^2 + p_y^2}}.$$

We denote the positive eigenvalue of the linearized equation at the hyperbolic equilibrium as

$$\lambda := (p_x^2 + p_y^2)^{1/4}, \quad (5)$$

which allows us to write the constants of the motion as $p_x = \lambda^2 \sin \alpha$, and $p_y = \lambda^2 \cos \alpha$, with $\alpha = \arctan(p_x/p_y) \in (0, \pi/2)$. With this notation, the pendulum-like equation in the variables $(z, p_z)$ reads as

$$\ddot{z} = p_x \cos z - p_y \sin z = \lambda^2 \sin(\alpha - z),$$
thus obtaining that there is a homoclinic orbit connecting the equilibrium point given by
\[ z^0(t) = 4 \arctan e^{\lambda t} + z^*, \quad p_z^0(t) = \frac{2\lambda}{\cosh(\lambda t)}. \]

It is straightforward to check that \( z^0(t) \to z^* \) and \( p_z^0(t) \to p_z^* = 0 \), exponentially with exponent \( \lambda \), as \( t \to \pm \infty \). There is a second homoclinic trajectory connecting the equilibrium point given by \( z^0(t) = -z^0(t) + 2\alpha \) and \( p_z^0(t) = -p_z^0(t) \), but it will not be used in what follows.

The previous computations show that the Hamiltonian system \( H_0 \) has a 4-dimensional normally hyperbolic invariant manifold
\[ \Lambda_0 := \{(q, p) \in T^3 \times \mathbb{R}^3 : z = z^*, p_z = p_z^* \}, \]
which is foliated by 2-dimensional invariant tori \( \mathcal{T}_{p_x, p_y} \) obtained by fixing \( p_x \) and \( p_y \), i.e. \( \Lambda_0 = \bigcup_{p_x, p_y} \mathcal{T}_{p_x, p_y} \). A direct computation shows that the dynamics on each invariant torus \( \mathcal{T}_{p_x, p_y} \) is linear with frequency vector \( \omega = (\omega_1, \omega_2) \) given by
\[ \omega_1 := p_x - \sin(z^*) = p_x(1 + (p_x^2 + p_y^2)^{-1/2}), \]
\[ \omega_2 := p_y - \cos(z^*) = p_y(1 + (p_x^2 + p_y^2)^{-1/2}). \]

The stable and unstable manifolds of \( \Lambda_0 \) are 5-dimensional invariant sets defined by
\[ W^s(\Lambda_0) = W^u(\Lambda_0) = \{(q, p) \in T^3 \times \mathbb{R}^3 : z = z^0(\tau), p_z = p_z^0(\tau), \tau \in \mathbb{R} \}, \]
so the set \( W^s(\Lambda_0) \) (or \( W^u(\Lambda_0) \)) is the union of the stable (unstable) manifolds of the invariant tori \( \mathcal{T}_{p_x, p_y} \), i.e.
\[ W^s(\Lambda_0) = \bigcup_{p_x, p_y} W^s(\mathcal{T}_{p_x, p_y}) = W^u(\Lambda_0) = \bigcup_{p_x, p_y} W^u(\mathcal{T}_{p_x, p_y}). \]

In order to work with the invariant torus \( \mathcal{T}_{p_x, p_y} \) and its whiskers \( W^s(\mathcal{T}_{p_x, p_y}) = W^u(\mathcal{T}_{p_x, p_y}) \), we introduce appropriate parameterizations. Indeed, \( \mathcal{T}_{p_x, p_y} \subset \Lambda_0 \) can be parameterized as
\[ u^* \equiv u^*(x, y) = (x, y, z^*, p_x, p_y, p_z^*), \]
where \( p_x \) and \( p_y \) are fixed and \( (x, y) \in T^2 \). Moreover, the stable manifold \( W^s(\mathcal{T}_{p_x, p_y}) \) is given by the set of points of the form
\[ u^0 \equiv u^0(\tau, x, y) = (x + F_1(\tau), y + F_2(\tau), z^0(\tau), p_x, p_y, p_z^0(\tau)), \]
where \( \tau \in \mathbb{R}, (x, y) \in T^2 \), the functions \( z^0 \) and \( p_z^0 \) are defined in Eq. (6), and
\[ F_1(\tau) := \sin(z^*)\tau - \int_0^\tau \sin(z^0(\sigma))d\sigma, \quad F_2(\tau) := \cos(z^*)\tau - \int_0^\tau \cos(z^0(\sigma))d\sigma. \]
Finally, we introduce some notation that will be useful in Section 4. If \( \phi_t^0 \) is the flow of the Hamiltonian system \( H_0 \) and we consider points \( u^* \in \Lambda_0 \) and \( u^0 \in W^s(\Lambda_0) = W^u(\Lambda_0) \), then
\[ \phi_t^0(u^*) = (x + \omega_1 t, y + \omega_2 t, z^*, p_x, p_y, p_z^*), \]
\[ \phi_t^0(u^0) = (x + F_1(\tau + t) + \omega_1 t, y + F_2(\tau + t) + \omega_2 t, z^0(\tau + t), p_x, p_y, p_z^0(\tau + t)), \]

We observe that the functions \( F_1 \) and \( F_2 \) depend on the constants \( p_x \) and \( p_y \) through \( z^* \) and \( z^0 \), but we omit this dependence in order to avoid cumbersome notation. After straightforward computations we obtain the following explicit formulas
\[ F_1(\tau) = \left( \frac{2(\tanh(\lambda \tau) - 1)}{\lambda} + \frac{2}{\lambda} \right) \sin z^* - \left( \frac{2\sech(\lambda \tau)}{\lambda} - \frac{2}{\lambda} \right) \cos z^*, \]
\[ F_2(\tau) = \left( \frac{2(\tanh(\lambda \tau) - 1)}{\lambda} + \frac{2}{\lambda} \right) \cos z^* + \left( \frac{2\sech(\lambda \tau)}{\lambda} - \frac{2}{\lambda} \right) \sin z^*, \]
where the constant $\lambda$ is defined in Eq. (5). These functions allow us to compute the phase shift of any trajectory when traveling along $W^s(T_{p_x,p_y})$. Indeed, the phase-shift is defined by the limits

$$
x_+ := \lim_{t \to \infty} F_1(\tau + t), \quad x_- := \lim_{t \to -\infty} F_1(\tau + t),
$$

$$
y_+ := \lim_{t \to \infty} F_2(\tau + t), \quad y_- := \lim_{t \to -\infty} F_2(\tau + t),
$$

which can be explicitly computed and do not depend on $\tau$, that is

$$
x_\pm = 2 \frac{(\mp p_x - p_y)}{(p_x^2 + p_y^2)^{3/4}}, \quad y_\pm = 2 \frac{(p_x \mp p_y)}{(p_x^2 + p_y^2)^{3/4}}. \quad (13)
$$

Observe that the limits $x_+$ and $x_-$ are different, which means that any point in the homoclinic orbit approaches different points of the same invariant torus if we consider the limit in the future and in the past. This is the reason why the terminology phase-shift is used for this phenomenon, see e.g. [5, 15, 22]. As we will show in Section 5 this phase-shift contributes to the expression involved in the transversality conditions used to obtain diffusion.

Remark 2.1. It is interesting to note that the invariant tori $T_{p_x,p_y}$ project onto the toroidal magnetic surfaces $z = z^*$ of the unperturbed ABC magnetic field in the configuration space $T^3$. Moreover, the magnetic field on each surface is linear, i.e. $B_{ABC}|_{z=z^*} = (\sin z^*, \cos z^*, 0)$, and the trajectories follow the magnetic lines. Let us observe that the slope of the magnetic lines $\tan z^*$ coincides with the ratio of the frequencies $\omega_1/\omega_2$, cf. Eqs. (7) and (8).

2.2 Main Theorem: diffusion along a NHIM

Let us consider the following Hamiltonian for the ABC system

$$
H = \frac{1}{2} (p_x - \sin z - \varepsilon \hat{C} \cos y)^2 + \frac{1}{2} (p_y - \cos z - \varepsilon \hat{B} \sin x)^2 + \frac{1}{2} (p_z - \varepsilon \hat{C} \sin y - \varepsilon \hat{B} \cos x)^2, \quad (14)
$$

where we have introduced a scaling $B = \varepsilon \hat{B}$ and $C = \varepsilon \hat{C}$. The following result states sufficient conditions for the existence of diffusing orbits:

Theorem 2.2. Consider the Hamiltonian (14) of the ABC system with $\hat{B} \geq \hat{C} \neq 0$. Assume that the following hypotheses hold:

A1. Considering the notation introduced in Section 2.1 we define the functions $M^0_i \equiv M^0_i(p_x,p_y)$ as

$$
M^0_1 := \hat{B} \int_{-\infty}^{\infty} \left( (p_y - \cos z^*) \sin(x_\pm + \omega_1 \sigma) - (p_y - \cos z^0) \sin(F_1 + \omega_1 \sigma) - p_z^0 \cos(F_1 + \omega_1 \sigma) \right) d\sigma,
$$

$$
M^0_2 := \hat{C} \int_{-\infty}^{\infty} \left( (p_x - \sin z^*) \cos(y_\pm + \omega_2 \sigma) - (p_x - \sin z^0) \cos(F_2 + \omega_2 \sigma) - p_z^0 \sin(F_2 + \omega_2 \sigma) \right) d\sigma,
$$

$$
M^0_3 := \hat{B} \int_{-\infty}^{\infty} \left( (p_y - \cos z^*) \cos(x_\pm + \omega_1 \sigma) - (p_y - \cos z^0) \cos(F_1 + \omega_1 \sigma) + p_z^0 \sin(F_1 + \omega_1 \sigma) \right) d\sigma,
$$

$$
M^0_4 := \hat{C} \int_{-\infty}^{\infty} \left( (p_x - \sin z^0) \sin(F_2 + \omega_2 \sigma) - (p_x - \sin z^*) \sin(y_\pm + \omega_2 \sigma) - p_z^0 \cos(F_2 + \omega_2 \sigma) \right) d\sigma,
$$

with $F_1 \equiv F_1(\sigma)$ and $F_2 \equiv F_2(\sigma)$, and where the notation $x_\pm$ (resp. $y_\pm$) means that we take $x_-$ (resp. $y_-$) when we integrate in the interval $(-\infty, 0)$, and $x_+$ (resp. $y_+$) when we integrate in the interval $(0, \infty)$. We assume that there exists a non-empty set $\mathcal{I} = [a_1, b_1] \times [a_2, b_2]$, for positive values of $a_1, b_1$, such that $M^0_1$ and $M^0_3$ do not vanish simultaneously, and the same for $M^0_2$ and $M^0_4$, provided that $(p_x,p_y) \in \mathcal{I}$.

A2. Assume that for any value $(p_x,p_y) \in \mathcal{I}$ there exists a non-empty domain $\mathcal{J}_{p_x,p_y} \subset T^2$ with the property that

$$
\mathcal{D} := \bigcup_{(p_x,p_y) \in \mathcal{I}} \mathcal{J}_{p_x,p_y} \times \{(p_x,p_y)\} \subset T^2 \times \mathcal{I}
$$
is a domain, and when \((x, y, p_x, p_y) \in \mathcal{D}\) there is a unique critical point \(\tau^* \equiv \tau^*(x, y, p_x, p_y)\) of the map
\[
\tau \mapsto M_1^0 \cos(x - \omega_1 \tau) + M_2^0 \cos(y - \omega_2 \tau) + M_3^0 \sin(x - \omega_1 \tau) + M_4^0 \sin(y - \omega_2 \tau),
\]
which defines a smooth function on \(\mathcal{D}\).

Assume that we can chose a constant \(L > 0\) such that for every \((x, y, p_x, p_y) \in \mathcal{D}\) we have
\[
\begin{cases}
\Delta_1 \Delta_3 - \Delta_2^2 \neq 0, & \text{if } |p_x - p_y| \geq L, \\
\Delta_1 \Delta_4 - \Delta_2 \Delta_3 \neq 0, & \text{if } |p_x - p_y| \leq L,
\end{cases}
\]
where \(\{\Delta_i\}_{i=1,2,3}\) and \(\{\hat{\Delta}_i\}_{i=1,2,3,4}\) are certain explicit functions depending on \((x, y, p_x, p_y)\) that are defined in Section 5, cf. Eqs. (94)–(96) and (98)–(101).

Then, given two pairs \((y^0_x, y^0_y) \in \mathcal{I}\) and \((p^1_x, p^1_y) \in \mathcal{I}\) and given \(\delta > 0\), there exists \(\epsilon^* = \epsilon^*(\delta, I)\) such that if \(0 < \epsilon < \epsilon^*\) then there is a trajectory \((x(t), y(t), z(t), p_x(t), p_y(t), p_z(t))\) of the system (14) satisfying
\[
\begin{align*}
\text{dist} \left( (p^0_x, p^0_y), (p_x(0), p_y(0)) \right) & \leq \delta, \\
\text{dist} \left( (p^1_x, p^1_y), (p_x(T), p_y(T)) \right) & \leq \delta.
\end{align*}
\]
for some \(T > 0\).

We would like to emphasize that the above hypotheses are given in a very explicit way. To evaluate all the functions involved in the statement of Theorem 2.2, we only need to compute the coefficients \(\{M_i^0\}_{i=1,2,3,4}\) in Hypothesis A1, together with the partial derivatives \(\tau_x^*, \tau_y^*, \tau_{xx}^*, \tau_{yy}^*, \tau_{xy}^*\) of the critical point in Hypothesis A2. As was sketched in the introduction, the proof of Theorem 2.2 consists in combining the internal dynamics on the NHIM with its outer (asymptotic) dynamics. Details are presented and discussed in Sections 3 and 5. Finally, in Section 6 we show that the hypotheses of the theorem can be rigorously checked in a computer assisted proof.

Remark 2.3. Since the invariant tori \(T_{p_x, p_y}\) correspond to the toroidal magnetic surfaces of the unperturbed ABC magnetic field, c.f. Remark 2.1, Theorem 2.2 implies the existence of drift motions connecting any two magnetic surfaces (compatible with the set \(I\)) for the perturbed ABC system. This diffusion of charged particles is a very harmful phenomenon for the confinement of hot plasmas for fusion power generation, as explained in the introduction.

3 Inner dynamics of the normally hyperbolic invariant manifold

The study of normally hyperbolic invariant manifolds is a very classical (and important) topic and it has been extensively considered in the literature. Most of the results that we use in this section are standard and can be found in [23, 37]. Our purpose here is to present a basic overview, notation and perturbative formulas that we require to study the perturbation of the normally hyperbolic invariant manifold introduced in Section 2.1.

We recall that our goal is to study the Hamiltonian (14) for small values of \(\epsilon\). Hence, we write \(H = H_\epsilon\) perturbatively as follows
\[
H_\epsilon = H_0 + \epsilon H_1 + \epsilon^2 H_2,
\]
where
\[
\begin{align*}
H_0 &= \frac{1}{2}(p_x - \sin z)^2 + \frac{1}{2}(p_y - \cos z)^2 + \frac{1}{2}p_z^2, \\
H_1 &= -\dot{C} \cos y(p_x - \sin z) + \dot{B} \sin x(p_y - \cos z) - p_z(\dot{C} \sin y + \dot{B} \cos x), \\
H_2 &= \frac{\dot{C}^2}{2} + \frac{\dot{B}^2}{2} + \dot{B}\dot{C} \cos x \sin y.
\end{align*}
\]
The unperturbed Hamiltonian $H_0$ was studied in Section 2.1, where we characterized the corresponding NHIM $\Lambda_0$. Now, we are interested in characterizing the perturbed invariant manifold $\Lambda_\varepsilon$ together with the restricted dynamics on it (mainly the existence and approximation of invariant tori). To this end, we will follow closely the methodology introduced in the papers [16, 17] [18].

Let us remark that the Hamiltonian (16) is real-analytic. This will imply that all the objects obtained in this section will be of class $C^r$, with arbitrarily large $r$ (this follows from Fenichel rate conditions) so that we can omit all the discussions concerning regularity. This will simplify many technical issues, for example when applying averaging and KAM theory. The interested reader is referred to [16, 18] for details on regularity.

3.1 Normally hyperbolic invariant manifolds and perturbative setting

Let $M$ be a smooth finite dimensional manifold and let us consider a flow $\phi_t$, of class $C^r$ with $r \geq 1$, acting on $M$.

**Definition 3.1.** Let $\Lambda \subset M$ be a submanifold invariant under the flow, i.e., $\phi_t(\Lambda) = \Lambda$. We say that $\Lambda$ is a normally hyperbolic invariant manifold (NHIM), if there exist a constant $c > 0$, expansion rates $0 < \mu < \lambda$, and a splitting for every $x \in \Lambda$

$$T_x M = E^s_x \oplus E^u_x \oplus T_x \Lambda,$$

characterized as follows

$$v \in E^s_x \iff |D\phi_t(x)v| \leq ce^{-\lambda t}|v|, \quad t \geq 0,$$

$$v \in E^u_x \iff |D\phi_t(x)v| \leq ce^{-\mu t}|v|, \quad t \leq 0,$$

$$v \in T_x \Lambda \iff |D\phi_t(x)v| \leq c e^{|t|}|v|, \quad t \in \mathbb{R}.$$

The classical theory of NHIMs guarantees that if $\Lambda$ is normally hyperbolic, then it is persistent under small perturbations. Moreover, if the system depends smoothly on parameters, the manifolds —they may not be unique— can be chosen to depend smoothly on parameters. NHIMs are robust under perturbations, so we do not require a symplectic structure on $M$ and $\phi_t$. Nevertheless, the problem considered in this paper is endowed with a symplectic structure and hence we will be interested in characterizing a symplectic structure on the perturbed NHIM.

In order to apply the geometric mechanism for a priori unstable systems (c.f. [16] [18]) we must compute explicitly some expansions in $\varepsilon$ of the NHIM associated to the Hamiltonian (16). Notice that in our case we can model the NHIM by means of the canonical manifold $N = T^2 \times \mathbb{R}^2$ (see Section 2.1), that is, we look for a parameterization $P_\varepsilon : N \to M$, with $P_\varepsilon(N) = \Lambda_\varepsilon$, characterized by the invariance equation

$$X_\varepsilon \circ P_\varepsilon = DP_\varepsilon R_\varepsilon$$

where $R_\varepsilon$ is a vector field on $N$ and $X_\varepsilon$ is the Hamiltonian vector field associated to $H_\varepsilon$. Using the expansions

$$X_\varepsilon = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \ldots,$$

$$P_\varepsilon = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \ldots,$$

$$R_\varepsilon = R_0 + \varepsilon R_1 + \varepsilon^2 R_2 + \ldots,$$

and equating terms in the expansion of $\varepsilon$ of the invariance equation (22), we obtain (this approach was used in [16])

0th order: \hspace{1cm} $X_0 \circ P_0 = DP_0 R_0$, \hspace{1cm} (23)

1st order: \hspace{1cm} $(DX_0 \circ P_0)P_1 + X_1 \circ P_0 = DP_0 R_1 + DP_1 R_0$, \hspace{1cm} (24)

2nd order: \hspace{1cm} $(DX_0 \circ P_0)P_2 + \frac{1}{2} (D^2 X_0 \circ P_0) P_1^2 + (DX_1 \circ P_0)P_1 + X_2 \circ P_0$

\hspace{1cm} $= DP_0 R_2 + DP_1 R_1 + DP_2 R_0$, \hspace{1cm} (25)

$n$th order: \hspace{1cm} $(DX_0 \circ P_0)P_n - DP_n R_0 - DP_0 R_n = -X_n \circ P_0 + S_n$, \hspace{1cm} (26)
where $S_n$ is a polynomial in $X_0, \ldots, X_{n-1}$, their derivatives, $P_0, \ldots, P_{n-1}$, their derivatives, and $R_0, \ldots, R_{n-1}$.

Clearly (see the discussion in Section 2.1) Eq. (23) has the solution

$$
P_0(x, y, px, py) = (x, y, z^*, px, py, p^*_x)
$$

$$
R_0(x, y, px, py) = \omega_1(px, py)\partial_x + \omega_2(px, py)\partial_y
$$

where $\omega_1$ and $\omega_2$ are given by (7) and (8), respectively. In this case, since the unperturbed internal field $R_0$ does not depend on the angular variables $(x, y)$, the equations of the form (26) lead to simple cohomological equations in a suitable frame. Hence, these equations can be solved explicitly using Fourier expansions. It is worth mentioning that there are more general theories that allow us to solve equations of the form (26) even if the motion on the base is not quasi-periodic.

As will be discussed in subsequent sections, the solution of equations (23), (24), (25), and (26) is not uniquely determined. We will use this freedom in order to obtain certain symplectic properties. More specifically, we follow the ideas in [17] to maintain the canonical symplectic structure on $\Lambda_e$, so that we can easily characterize and manipulate the Hamiltonian associated to the restricted vector field $R_\varepsilon$.

### 3.2 Symplectic properties of NHIMs of Hamiltonian systems

Let $M$ be a symplectic manifold with symplectic form $\omega$, represented by a matrix-valued function $\Omega$, and let us assume that a $C^r$ Hamiltonian $H_0$, with $r \geq 2$, has a NHIM $\Lambda_0$ parameterized by $P_0 : N \to M$. Then, it is well known (c.f. [23, 37]) that for every perturbed Hamiltonian $H_\varepsilon$ of class $C^r$ there exists a NHIM $\Lambda_\varepsilon$ parameterized by $P_\varepsilon$ of class $C^{r-1}$. Moreover, $\Lambda_\varepsilon$ is $O(\varepsilon)$-close to $\Lambda_0$ in the $C^{r-2}$ sense. Here and in what follows, when we say that a map depending on parameters is of class $C^r$, we shall mean that it is of class $C^r$ in all variables including the parameters.

Given a family of Hamiltonians having a family of NHIMs $\Lambda_\varepsilon = P_\varepsilon(N)$, with $P_\varepsilon : N \to M$, we consider the maps $R_\varepsilon : N \to TN$ corresponding to the vector fields restricted to the NHIMs. The maps $P_\varepsilon$ and $R_\varepsilon$ are related by the invariance equation (22).

It is well known that the solutions of (22) are not uniquely defined, since we have the possibility of choosing different coordinates in the reference manifold $N$. It is natural to use this freedom to satisfy certain properties, like asking $P_\varepsilon$ to be a graph or asking $R_\varepsilon$ to be as simple as possible. In this paper, we are interested in choosing the solution that preserves the Hamiltonian structure of the problem, that is, we want that

$$
\frac{d}{d\varepsilon}(P_\varepsilon^*\omega) = 0.
$$

(27)

The fact that this can be achieved was proved in [17]. In this paper, since we need to perform some explicit computations, we have to give some additional details on the procedure presented in [17]. The aim of this section is to explain the explicit computations required to handle a particular problem.

A natural way to obtain (27) is to use deformation theory. Let us recall some standard definitions. Given two connected manifolds $M$ and $N$, and given a family $f_\varepsilon : N \to M$ such that $(x, \varepsilon) \mapsto f_\varepsilon(x)$ is $C^1$ in all its arguments, we define the infinitesimal deformation of $f_\varepsilon$ as the vector field $F_\varepsilon$ that satisfies

$$
\frac{d}{d\varepsilon}f_\varepsilon = F_\varepsilon \circ f_\varepsilon,
$$

and we observe that $F_\varepsilon = (\frac{d}{d\varepsilon}f_\varepsilon) \circ f_\varepsilon^{-1}$ is defined on $f_\varepsilon(N) \subset M$.

Let $P_\varepsilon$ be the infinitesimal deformation of the family $P_\varepsilon$ with initial condition $P_0$. It is clear that $P_\varepsilon : \Lambda_\varepsilon \to TM$, so we can consider the projections of $P_\varepsilon$ according to the splitting (20). Then we have the following result [17]:

**Proposition 3.2.** Let us consider a family of parameterizations $P_\varepsilon : N \to M$ with $\Lambda_\varepsilon = P_\varepsilon(N)$. Assume that the infinitesimal deformation $P_\varepsilon$ satisfies that the projection on the space $T_x\Lambda_\varepsilon$ vanishes for every $x \in \Lambda_\varepsilon$. Then, the symplectic form $P_\varepsilon^*\omega_{*,\varepsilon}$ is independent of $\varepsilon$, where $\omega_{*,\varepsilon}$ is the original form $\omega$ expressed in a basis of the splitting (20).
Similarly, we say that a frame 

\[ \Omega_{n+1} = \begin{pmatrix} \Omega_n^0 & O_{2n \times 2} \\ O_{2 \times 2n} & \Omega_1^0 \end{pmatrix}, \quad \Omega_n^0 = \begin{pmatrix} O_n & -I_n \\ I_n & O_n \end{pmatrix} \]

where from now on we use the notation \( O_{n \times m}, I_{n \times m} \equiv O_{n \times n} \), and \( I_n \equiv I_{n \times n} \), for the zero and identity matrices, respectively. Moreover, we denote by \( M_{m \times n} \) the space of \( m \times n \)-matrices with real coefficients.

**Definition 3.4.** Given a parameterization \( P_0 : N \to M \) of a NHIM, with \( N = \mathbb{T}^n \times \mathbb{R}^n \) and \( M = \mathbb{T} \times \mathbb{R} \), we say that \( P_0 \) is compatible with the symplectic form \( \omega \) if

\[ DP_0(u, p_u) \top \Omega_{n+1} DP_0(u, p_u) = \Omega_n^0. \]

Similarly, we say that a frame

\[ \mathcal{F} : N \times \mathbb{R}^{2n+2} \to T_{P_0(N)} M \]

\[ (u, p_u, \xi) \mapsto (P_0(u, p_u), C_0(u, p_u) \xi) \]

with \( C_0 : N \to M_{(2n+2) \times (2n+2)} \), is symplectic if

\[ C_0(u, p_u) \top \Omega_{n+1} C_0(u, p_u) = \Omega_{n+1}. \]
Let us also introduce some notation regarding derivatives that will be useful in computations. Given a vector field\( R \) on a NHIM, and given a function\( \xi : N \to \mathbb{R} \), we denote the Lie derivative of\( \xi \) with respect to\( R \) as follows
\[
L_R(\xi) = D\xi R = \sum_{i=1}^{n} \frac{\partial \xi}{\partial u_i} R_i + \sum_{i=1}^{n} \frac{\partial \xi}{\partial p_{u,i}} R_{n+i}.
\]
Moreover, given a parameterization\( P : N \to M \), and vector fields\( X \) and\( R \) on\( M \) and\( N \), respectively, we introduce the operator
\[
\mathcal{R}_{P,X,R}(\xi) = DX \circ P\xi - L_R(\xi),
\]
acting on functions\( \xi : N \to \mathbb{R} \). We extend the notation in (29) and (30) component-wise for matrix functions\( \xi : N \to M_{m \times n} \). In order to simplify the notation, we will write\( \mathcal{R}_0 \equiv \mathcal{R}_{P_0,X_0,R_0} \).

Given a parameterization\( P_0 : N \to M \) of a NHIM, with\( N = \mathbb{T}^n \times \mathbb{R}^n \) and\( M = N \times \mathbb{T} \times \mathbb{R} \), we can take derivatives at both sides of the invariance equation\( X_0 \circ P_0 = D P_0 R_0 \) thus obtaining
\[
DX_0 \circ P_0 D P_0 = D(D P_0 R_0) = L_{R_0}(D P_0) + D P_0 D R_0.
\]
This means that the tangent vectors of\( P_0(N) \) partially characterize the action of the operator\( \mathcal{R}_0 \) in (30) as
\[
\mathcal{R}_0(D P_0) = D P_0 D R_0.
\]
Since\( P_0(N) \) is normally hyperbolic, there exist maps\( W_0 : N \to M_{(2n+2) \times 2} \) parameterizing the normal bundle of\( P_0(N) \), and\( \Gamma_0 : N \to M_{2 \times 2} \) such that
\[
\mathcal{R}_0(W_0) = W_0 \Gamma_0.
\]
From now on, we assume that\( \Gamma_0 \) is diagonal, and due to the Hamiltonian structure we can write
\[
\Gamma_0 = \begin{pmatrix} \lambda_0 & 0 \\ 0 & -\lambda_0 \end{pmatrix}.
\]
Moreover, if we assume that\( P_0 \) is compatible with the symplectic form\( \omega \), then it turns out that the matrix\( W_0 \) can be scaled in such a way that the juxtaposed matrix\( C_0 := (D P_0 W_0) \in M_{(2n+2) \times (2n+2)} \) defines a symplectic frame as in Definition 3.4

The operator\( \mathcal{R}_0 \) introduced above appears in the perturbative equations (23)–(26) obtained in Section 3.1. The following lemma approaches the study of these equations using the previously constructed frame. It is worth mentioning that the fact the frame\( \mathcal{C} \) is assumed to be symplectic is not really necessary. Nevertheless, it simplifies some computations (for example the computation of the inverse\( C_0^{-1} \)).

**Lemma 3.5.** Assume that\( P_0 : N \to M \) satisfies\( X_0 \circ P_0 = D P_0 R_0 \), with\( N = \mathbb{T}^n \times \mathbb{R}^n \) and\( M = N \times \mathbb{T} \times \mathbb{R} \). Given a map\( \eta : N \to \mathbb{R}^{2n+2} \), we consider the following equation
\[
DX_0 \circ P_0 \dot{\xi} - D\xi R_0 - D P_0 \rho = \eta
\]
for the unknowns\( \xi : N \to \mathbb{R}^{2n+2} \) and\( \rho : N \to \mathbb{R}^{2n} \). Then, using the symplectic frame\( \mathcal{C} \) associated to the matrix\( C_0 = (D P_0 W_0) \) constructed above, it turns out that Eq. (31) leads to
\[
- L_{R_0}(\dot{\xi}^C) + D R_0 \dot{\xi}^C = \dot{\eta}^C + \rho
\]
\[
- L_{R_0}(\dot{\xi}^H) + \Gamma_0 \dot{\xi}^H = \dot{\eta}^H
\]
where
\[
\xi = C_0 \dot{\xi} = D P_0 \dot{\xi}^C + W_0 \dot{\xi}^H \quad \text{and} \quad \dot{\eta} = \begin{pmatrix} \dot{\eta}^C \\ \dot{\eta}^H \end{pmatrix} = -\Omega_{n+1} C_0^T \Omega_{n+1} \eta,
\]
with\( \eta^C : N \to \mathbb{R}^{2n} \), \( \dot{\xi}^C : N \to \mathbb{R}^{2n} \), \( \eta^H : N \to \mathbb{R}^{2} \) and \( \dot{\xi}^H : N \to \mathbb{R}^{2} \).
Proof. Let us observe that the fact that $C$ is chosen to be symplectic allows us to compute the inverse of $C_0$ as follows

$$C_0^{-1} = \Omega_{n+1}^{-1} C_0^T \Omega_{n+1} = -\Omega_{n+1} C_0^T \Omega_{n+1}.$$ 

We also notice that the action of $R_0$ on the matrix $C_0 \hat{\xi}$ takes the form

$$R_0(C_0 \hat{\xi}) = R_0(C_0) \hat{\xi} - C_0 L_{R_0}(\xi),$$

and that

$$C_0^{-1} R_0(C_0) = -\Omega_{n+1} C_0^T \Omega_{n+1} (DX_0 \circ \rho_0 C_0 - L_{R_0}(C_0))$$

$$= -\Omega_{n+1} \begin{pmatrix} DP_0^T \Omega_{n+1} DP_0 DR_0 & DP_0^T \Omega_{n+1} W_0 \Gamma_0 \\ W_0^T \Omega_{n+1} DP_0 DR_0 & W_0^T \Omega_{n+1} W_0 \Gamma_0 \end{pmatrix}$$

$$= -\Omega_{n+1} \Omega_{n+1} \begin{pmatrix} DR_0 & O_{2n \times 2} \\ O_{2n \times 2} & \Gamma_0 \end{pmatrix} = \begin{pmatrix} DR_0 & O_{2n \times 2} \\ O_{2n \times 2} & \Gamma_0 \end{pmatrix}. $$

Introducing $\xi = C_0 \hat{\xi} = DP_0 \hat{\xi}^C + W_0 \hat{\xi}^H$ into Eq. (31), we obtain

$$\begin{pmatrix} DR_0 & O_{2n \times 2} \\ O_{2n \times 2} & \Gamma_0 \end{pmatrix} \begin{pmatrix} \xi^C \\ \xi^H \end{pmatrix} = \begin{pmatrix} L_{R_0}(\hat{\xi}^C) \\ L_{R_0}(\hat{\xi}^H) \end{pmatrix} - C_0^{-1} DP_0 \rho = C_0^{-1} \eta.$$

Then, we observe that

$$-C_0^{-1} DP_0 \rho = \Omega_{n+1} C_0^T \Omega_{n+1} DP_0 \rho = \Omega_{n+1} \begin{pmatrix} DP_0^T \Omega_{n+1} DP_0 \\ W_0^T \Omega_{n+1} DP_0 \end{pmatrix} \rho = \begin{pmatrix} \rho \end{pmatrix}.$$ 

Finally, using the symplectic structure, we introduce $\hat{\eta}^C$ and $\hat{\eta}^H$ as in the statement of the lemma, thus ending up with the equations (32) and (33).

It is standard to check that the solution of Eq. (33) is unique. In our particular case (see computations in Section 3.3), it turns out that $R_0$ produces an integrable quasi-periodic motion in $N$, and hence, we can solve (33) using Fourier series. In particular, if we have a function $\beta : N \to \mathbb{R}$ expressed in Fourier series as

$$\beta(u, p_u) = \sum_{k \in \mathbb{Z}^n} (\beta_k^{\cos}(p_u) \cos(k \cdot u) + \beta_k^{\sin}(p_u) \sin(k \cdot u)), $$

with $\beta_k^{\sin} \equiv 0$, then it turns out that the solution $\xi$ of the equation $\lambda \xi - L_{R_0}(\xi) = \beta$ is given by

$$\xi(u, p_u) = \sum_{k \in \mathbb{Z}^n} (\xi_k^{\cos}(p_u) \cos(k \cdot u) + \xi_k^{\sin}(p_u) \sin(k \cdot u)), $$

with

$$\xi_k^{\cos} = \frac{\beta_k^{\cos} \lambda + \omega \cdot \beta_k^{\sin}}{\lambda^2 + (\omega \cdot k)^2}, \quad \xi_k^{\sin} = \frac{\beta_k^{\sin} \lambda - \omega \cdot \beta_k^{\cos}}{\lambda^2 + (\omega \cdot k)^2}. $$

In case that $R_0$ takes a more general form, Eq. (33) can be solved using the asymptotic properties of the cocycle.

As was mentioned in Section 3.1, the solution of Eq. (32) is not unique. A simple choice consists in taking

$$\hat{\xi}^C = O_{2n \times 1}, \quad \rho = -\hat{\eta}^C,$$

but, in general, this solution will not determine a parameterization which is compatible with the symplectic structure of the problem. The final goal of this section is to compute the deformation of the symplectic frame $C$ with respect to the perturbation parameter and to combine Proposition 3.2 and Lemma 3.5 in order to obtain the canonical symplectic structure in the deformed NHIM.
Assume that \( P_\varepsilon : N \to M \), with \( N = \mathbb{T}^n \times \mathbb{R}^n \) and \( M = N \times \mathbb{T} \times \mathbb{R} \), is a family of parameterizations satisfying \( X_\varepsilon \circ P_\varepsilon = D P_\varepsilon R_\varepsilon \), where \( X_\varepsilon \) is a family of Hamiltonian vector fields with the symplectic form \( \omega \) given by (28). Let us consider \( P_\varepsilon \), the infinitesimal deformation of the family \( P_\varepsilon \) with initial condition \( P_0 \). A simple computation shows that
\[
\frac{dP_\varepsilon}{d\varepsilon} = P_0 + 2P_2\varepsilon + 3P_3\varepsilon^2 + \ldots = P_0 \circ P_0 + (D P_0 \circ P_0 P_1 + P_1 \circ P_0)\varepsilon + \ldots ,
\]
thus obtaining
\[
\begin{align*}
0\text{th order:} & \quad P_1 = P_0 \circ P_0 \\
1\text{st order:} & \quad 2P_2 = P_1 \circ P_0 + DP_0 \circ P_0 P_1 \\
n\text{th order:} & \quad nP_n = P_n \circ P_0 + S_n
\end{align*}
\]
where \( S_n \) is an explicit expression depending recursively on the previously computed objects.

Let us consider the first order correction determined by Eq. (24). We apply Lemma 3.5 with
\[
\xi = P_1, \quad \rho = R_1, \quad \eta = -X_1 \circ P_0
\]
and we consider the unique solution of Eqs. (32) and (33) satisfying Eq. (35). In Eq. (36) we observe that \( P_1 \) is proportional to \( P_0 \). Hence, it turns out that the deformation \( P_0 \) vanishes on the central directions. By Proposition 3.2, we conclude that the reduced vector field \( R_1 \) is a Hamiltonian vector field with respect to the form \( \Omega_\varepsilon^0 \).

The second order correction is not so simple. On the one hand, we observe that \( P_2 \) and \( P_3 \) are no longer proportional. On the other hand, we have to consider Proposition 3.2 on the deformed symplectic frame. Let us assume that we have computed \( P_1, R_1 \), and also the first order correction of the symplectic frame, that is, \( C_\varepsilon = C_0 + \varepsilon C_1 + \mathcal{O}(\varepsilon^2) \). Then, we express the infinitesimal deformation \( P_\varepsilon \) on the frame \( C_\varepsilon \) perturbatively as
\[
C_\varepsilon^{-1} P_\varepsilon (P_\varepsilon) = C_0^{-1} P_0 + \varepsilon (C_0^{-1} DP_0 \circ P_0 P_1 + C_0^{-1} P_1 \circ P_0 - C_0^{-1} C_1 C_0^{-1} P_0 \circ P_0) + \mathcal{O}(\varepsilon^2).
\]

By construction, it is clear that
\[
C_0^{-1} P_0 \circ P_0 = C_0^{-1} P_0 = C_0^{-1} C_0 \xi_1 = \xi_1 = \begin{pmatrix} O_{2n \times 1} \\ \xi_1^H \end{pmatrix}.
\]

We ask the same condition for the \( \varepsilon \)-order terms, thus obtaining that
\[
C_0^{-1} DP_0 \circ P_0 P_1 + C_0^{-1} P_1 \circ P_0 - C_0^{-1} C_1 C_0^{-1} P_0 \circ P_0 = \begin{pmatrix} O_{2n \times 1} \\ \xi_1^H / \xi_1 \end{pmatrix},
\]
for certain \( \zeta : N \to \mathbb{R}^2 \) whose expression is not important for us. Then, we use again that \( P_0 \circ P_0 = P_1 = C_0 \xi_1 \), we replace \( P_1 \circ P_0 \) using (37), and we write \( P_2 = C_0 \xi_2 \), thus obtaining the condition
\[
2 \xi_2 - C_0^{-1} C_1 \begin{pmatrix} O_{2n \times 1} \\ \xi_1^H / \xi_1 \end{pmatrix} = \begin{pmatrix} O_{2n \times 1} \\ \xi_1 \end{pmatrix},
\]
(39)

that determines the first \( 2n \) components \( \xi_2^C \) of \( \xi_2 \). Therefore, we can solve the second order correction of the invariance equation, given by (25), using Lemma 3.5 with
\[
\xi = P_2, \quad \rho = R_2, \quad \eta = -X_2 \circ P_0 + DP_1 R_1 - \frac{1}{2} D^2 X_0 \circ P_0 P_2 + DX_1 \circ P_1
\]
and choosing the unique solution obtained by fixing \( \xi_2^C \) satisfying Eq. (39). Then, the corresponding correction of the reduced vector field,
\[
R_2 = DR_0 \xi_2^C + L_R_0 (\xi_2^C) - \eta^C,
\]
(40)
is a Hamiltonian vector field with respect to the form $\Omega_0$. Finally, we need to give a simple recipe to compute the first order correction $C_1$ of the symplectic frame. The construction is analogous up to any order, but this is enough for our purposes. We will construct the frame taking $C_1 = (DP_1 \ W_1)$, where $W_1$ is computed as follows. On the one hand, we assume that we have computed $P_\varepsilon = P_0 + \varepsilon P_1 + O(\varepsilon^2)$ so that we have (the computation is direct)

$$R_{P_\varepsilon, X_\varepsilon, R_\varepsilon}(DP_0 + \varepsilon DP_1) = (DP_0 + \varepsilon DP_1)(DR_0 + \varepsilon DR_1) + O(\varepsilon^2),$$

where we recall that $R_{P_\varepsilon, X_\varepsilon, R_\varepsilon}$ is given by Eq. (30). On the other hand, we look for $W_1$ and $\Gamma_1$ is such a way that the action of $R_{P_\varepsilon, X_\varepsilon, R_\varepsilon}$ on the matrix $W_0 + \varepsilon W_1$ is given by

$$R_{P_\varepsilon, X_\varepsilon, R_\varepsilon}(W_0 + \varepsilon W_1) = (W_0 + \varepsilon W_1)(\Gamma_0 + \varepsilon \Gamma_1) + O(\varepsilon^2).$$

We observe that this condition is satisfied if

$$(DX_0 \circ P_0)W_1 - LR_0(W_1) - W_0 \Gamma_1 - W_1 \Gamma_0 = S_1,$$  \hspace{1cm} (41)

where

$$S_1 := LR_1(W_0) - DX_1 \circ P_0 W_0 - D^2X_0 \circ P_0 P_1 \otimes W_0.$$

Again, the solutions of Eq. (41) are obtained by considering the action of the unperturbed operator $R_0$. In the following result, analogous to Lemma 3.5, we study the above equation.

**Lemma 3.6.** Assume that $P_0 : N \rightarrow M$ satisfies $X_0 \circ P_0 = DP_0 R_0$, with $N = \mathbb{T}^n \times \mathbb{R}^n$ and $M = N \times \mathbb{T} \times \mathbb{R}$. Assume that the pair $P_1$ and $R_1$ is a solution of equation (24), that is, we have

$$(X_0 + \varepsilon X_1) \circ (P_0 + \varepsilon P_1) = (DP_0 + \varepsilon DP_1)(R_0 + \varepsilon R_1) + O(\varepsilon^2).$$

Then, using the symplectic frame $\mathcal{C}$ associated to the matrix $C_0 = (DP_0 \ W_0)$, it turns out that equation (41) leads to

$$-LR_0(\hat{W}_1^C) + DR_0 \hat{W}_1^C - \hat{W}_1^C \Gamma_0 = \hat{S}_1^C,$$

$$-LR_0(\hat{W}_1^H) + \Gamma_0 \hat{W}_1^H - \hat{W}_1^H \Gamma_0 = \hat{S}_1^H - \Gamma_1,$$  \hspace{1cm} (42, 43)

where

$$W_1 = C_0 \hat{W}_1 = DP_0 \hat{W}_1^C + W_0 \hat{W}_1^H \text{ and } \begin{pmatrix} \hat{S}_1^C \\ \hat{S}_1^H \end{pmatrix} = -\Omega_{n+1}C_0^T \Omega_{n+1} S_1.$$  \hspace{1cm} (44)

**Proof.** We recall that the frame $\mathcal{C}$ satisfies

$$R_0(C_0) = DX_0 \circ P_0 C_0 - D(C_0) R_0 = C_0 \begin{pmatrix} DR_0 & O_{2n \times 2} \\ O_{2n \times n} \Gamma_0 \end{pmatrix}.$$  \hspace{1cm} (45)

Then, we compute the action of $R_0$ on $W_1 = C_0 \hat{W}_1$ as follows

$$R_0(C_0 \hat{W}_1) = C_0 \begin{pmatrix} DR_0 & O_{2n \times 2} \\ O_{2n \times n} \Gamma_0 \end{pmatrix} \hat{W}_1 - C_0 LR_0(\hat{W}_1),$$

and we introduce this expression into (41), thus obtaining

$$C_0 \begin{pmatrix} DR_0 & O_{2n \times 2} \\ O_{2n \times n} \Gamma_0 \end{pmatrix} \hat{W}_1 - C_0 LR_0(\hat{W}_1) - W_0 \Gamma_1 - C_0 \hat{W}_1 \Gamma_0 = S_1.$$  \hspace{1cm} (46)

Using the symplectic properties of the frame, we multiply both sides by $C_0^{-1} = -\Omega_{n+1}C_0^T \Omega_{n+1}$ and we end up with

$$\begin{pmatrix} DR_0 & O_{2n \times 2} \\ O_{2n \times n} \Gamma_0 \end{pmatrix} \hat{W}_1 - D(\hat{W}_1) R_0 + \Omega_{n+1} C_0^T \Omega_{n+1} W_0 \Gamma_1 - \hat{W}_1 \Gamma_0 = -\Omega_{n+1} C_0^T \Omega_{n+1} S_1.$$
Finally, we observe that
\[ \Omega_{n+1} C_0^T \Omega_{n+1} W_0 = \Omega_{n+1} \left( DP_0^T \Omega_{n+1} W_0 \right) = \left( O_{2n \times 2} I_2 \right), \]
and using the notation in (44) we obtain the equations (42) and (43).

Finally, we discuss the solution of equations (42) and (43). On the one hand, we observe that equation (42) is similar to equation (33) in the sense that it can be solved using Fourier series, obtaining a unique solution. On the other hand, we observe that the diagonal part of the left hand side of equation (43) is resonant. We can avoid this resonance by selecting \( \Gamma_1 \). To this end, we consider the particular choice
\[ \Gamma_1 = \text{diag}(S_1^H), \]
where \( \langle \cdot \rangle \) stands for the average with respect to the variables \( u \in \mathbb{T}^n \). Obviously, this choice preserves the diagonal character of the matrix \( \Gamma_1 = \Gamma_0 + \varepsilon \Gamma_1 + \ldots \).

### 3.3 Perturbative computation of the NHIM of the ABC system

The goal of this section is to compute the NHIM \( \Lambda_\varepsilon \) associated to the ABC system in the perturbative setting given by Eqs. (16)–(19). We follow the notation and methodology described in Section 3.2.

First, it is convenient to reorder the phase-space coordinates as \( (x, y, p_x, p_y, z, p_z) \) rather than \( (x, y, z, p_x, p_y, p_z) \). In analogy with Section 3.2, we have \( (u, p_u) = (x, y, p_x, p_y) \in \mathbb{T} = \mathbb{T}^2 \times \mathbb{R}^2 \) and \( (v, p_v) = (z, p_z) \in \mathbb{T} \times \mathbb{R} \). Then, we consider the symplectic form \( \omega \), and its matrix representation \( \Omega_3 \), given by
\[ \omega = dp_x \wedge dx + dp_y \wedge dy + dp_z \wedge dz, \quad \Omega_3 = \begin{pmatrix} \Omega_0^0 & O_{4 \times 2} \\ O_{2 \times 4} & \Omega_0^1 \end{pmatrix}. \]

With the above notation, we have \( X_\varepsilon = \Omega_3^{-1} DH_\varepsilon^\top = -\Omega_3 DH_\varepsilon^\top \).

We start with the explicit characterization of the unperturbed problem, giving rise to the expressions
\[ X_0 = \begin{pmatrix} p_x - \sin z \\ p_y - \cos z \\ 0 \\ 0 \\ p_z \\ p_x \cos z - p_y \sin z \end{pmatrix}, \quad P_0 = \begin{pmatrix} x \\ y \\ p_x \\ p_y \\ z^* = \arctan(p_x/p_y) + \pi \\ p_z^* = 0 \end{pmatrix}, \]
and the corresponding derivatives
\[ DX_0 \circ P_0 = \begin{pmatrix} 0 & 0 & 1 & 0 & -\cos z^* & 0 \\ 0 & 0 & 0 & 1 & \sin z^* & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \cos z^* & -\sin z^* & \lambda^2 & 0 \end{pmatrix}, \quad DP_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & p_y \lambda^{-4} & -p_z \lambda^{-4} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]
where we recall that \( \lambda = (p_x^2 + p_y^2)^{1/4} \), \( \sin z^* = -p_x/\lambda^2 \), and \( \cos z^* = -p_y/\lambda^2 \) (see computations in Section 2.1). Notice that the parameterization \( P_0 \) given above is compatible with the form \( \omega \) according to Definition 3.4.

To obtain the unperturbed symplectic frame we take the columns of \( DP_0 \) and we complement them with the eigenvectors of \( DX_0 \circ P_0 \) of eigenvalues \( \lambda \) and \( -\lambda \), that we suitable scale in order to obtain a symplectic frame.
Specifically, we take

\[
C_0 = \begin{pmatrix}
1 & 0 & 0 & 0 & \frac{(\sqrt{2}/2)p_y\lambda^{-2}}{2} & \frac{(\sqrt{2}/2)p_y\lambda^{-2}}{2} \\
0 & 1 & 0 & 0 & -\frac{(\sqrt{2}/2)p_x\lambda^{-2}}{2} & -\frac{(\sqrt{2}/2)p_x\lambda^{-2}}{2} \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\lambda^{-4}}{2}
\end{pmatrix},
\]

and we left as an exercise to the reader to check that \(C_0(x)^T \Omega_3 C_0(x) = \Omega_3\), where \(\Omega_3\) is the matrix of the canonical symplectic form. The inverse of \(C_0\) is given by

\[
C_0^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \frac{p_y\lambda^{-4}}{2} \\
0 & 1 & 0 & 0 & 0 & \frac{p_x\lambda^{-4}}{2} \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda
\end{pmatrix},
\]

and it turns out that this frame allows us to reduce \(DX_0 \circ P_0\) as follows

\[
C_0^{-1}DX_0 \circ P_0 C_0 = \begin{pmatrix}
0 & 0 & \frac{p_y^2\lambda^{-6} + 1}{2} & -\frac{p_x p_y \lambda^{-6}}{2} & 0 & 0 \\
0 & 0 & -\frac{p_x p_y \lambda^{-6}}{2} & \frac{p_y^2\lambda^{-6} + 1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & -\lambda
\end{pmatrix}.
\]

Notice that this expression corresponds to \(C_0^{-1}R_0(C_0)\) since in this case \(C_0^{-1}D(C_0)R_0 = 0\). Finally, the reduced vector field is given by \(R_0 = \omega_1 \partial_x + \omega_2 \partial_y\), where we recall that \(\omega_1 = \dot{p}_x (1 + \lambda^{-2})\), and \(\omega_2 = \dot{p}_y (1 + \lambda^{-2})\).

To obtain the corrections \(P_1\) and \(R_1\) of the parameterization and the reduced vector field, respectively, we consider the equation

\[
(DX_0 \circ P_0)P_1 - DP_0 R_1 - DP_1 R_0 = \eta
\]

where

\[
\eta = -X_1 \circ P_0 = \begin{pmatrix}
\dot{C} \cos y \\
\dot{B} \sin x \\
-\dot{B} \omega_2 \cos x \\
\dot{C} \omega_1 \sin y \\
\dot{C} \cos z \ast \cos y - \dot{B} \sin z \ast \sin x
\end{pmatrix}.
\]

Using Lemma 3.3, with \(P_1 = C_0 \hat{\xi}\) and \(R_1 = \rho\), we obtain the equivalent system of equations

\[
\begin{align*}
(p_y^2\lambda^{-6} + 1)\dot{\xi}_4 - p_x p_y \lambda^{-6} \dot{\xi}_4 - L_{R_0} (\dot{\xi}_1) &= \dot{\eta}_1 + \rho_1, \\
-p_x p_y \lambda^{-6} \dot{\xi}_3 + (p_x^2\lambda^{-6} + 1)\dot{\xi}_4 - L_{R_0} (\dot{\xi}_2) &= \dot{\eta}_2 + \rho_2, \\
L_{R_0} (\dot{\xi}_3) &= \dot{\eta}_3 + \rho_3, \\
L_{R_0} (\dot{\xi}_4) &= \dot{\eta}_4 + \rho_4, \\
\lambda \dot{\xi}_5 - L_{R_0} (\dot{\xi}_5) &= \dot{\eta}_5, \\
-\lambda \dot{\xi}_6 - L_{R_0} (\dot{\xi}_6) &= \dot{\eta}_6.
\end{align*}
\]
where \( L_{R_0}(\hat{\xi}_i) = \omega_1 \partial_x \hat{\xi}_i + \omega_2 \partial_y \hat{\xi}_i \), and

\[
\dot{\eta} = C_0^{-1} \eta = \begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \\ \dot{\eta}_4 \\ \dot{\eta}_5 \\ \dot{\eta}_6 \end{pmatrix} = \begin{pmatrix} A_1 \cos y + A_2 \sin x \\ A_3 \cos y + A_4 \sin x \\ A_5 \cos x \\ A_6 \sin y \\ A_7 \cos x + A_8 \cos y + A_9 \sin x + A_{10} \sin y \\ A_{11} \cos x + A_{12} \cos y + A_{13} \sin x + A_{14} \sin y \end{pmatrix}.
\]

The coefficients \( A_i, i = 1, \ldots, 14 \), are functions depending on the action variables \( p_x, p_y \), given by

\[ \begin{align*}
A_1 &= \hat{C} (1 + p_y^2 \lambda^{-6}) \\
A_2 &= - \hat{B} p_y \lambda^{-6} \\
A_3 &= - \hat{C} p_x p_y \lambda^{-6} \\
A_4 &= \hat{B} (1 + p_x^2 \lambda^{-6}) \\
A_5 &= - \hat{B} \omega_2 \\
A_6 &= \hat{C} \omega_1 \\
A_7 &= \sqrt{\frac{2}{2} (\lambda^6 + \lambda^2 p_y^2 + p_y^2)} \hat{B} \lambda^{-11/2} \\
A_8 &= - \sqrt{\frac{2}{2} \hat{C} p_y \lambda^{-5/2}} \\
A_9 &= \sqrt{\frac{2}{2} \hat{B} p_x \lambda^{-5/2}} \\
A_{10} &= \sqrt{\frac{2}{2} (\lambda^6 + \lambda^2 p_x^2 + p_x^2) \hat{C} \lambda^{-11/2}} \\
A_{11} &= - \sqrt{\frac{2}{2} (\lambda^6 + \lambda^2 p_y^2 + p_y^2) \hat{B} \lambda^{-11/2}} \\
A_{12} &= - \sqrt{\frac{2}{2} \hat{C} p_y \lambda^{-5/2}} \\
A_{13} &= \sqrt{\frac{2}{2} \hat{B} p_x \lambda^{-5/2}} \\
A_{14} &= - \sqrt{\frac{2}{2} (\lambda^6 + \lambda^2 p_x^2 + p_x^2) \hat{C} \lambda^{-11/2}}
\end{align*} \tag{51} \]

The solution of Eqs. (49) and (50), using Fourier series, is obtained using (34)

\[
\hat{\xi}_5 = B_1 \cos x + B_2 \cos y + B_3 \sin x + B_4 \sin y, \\
\hat{\xi}_6 = B_1 \cos x - B_2 \cos y - B_3 \sin x + B_4 \sin y,
\]

where the coefficients \( B_i \) have the following expressions:

\[
B_1 = \frac{A_7 \lambda + \omega_1 A_9}{\lambda^2 + \omega_1^2}, \quad B_2 = \frac{A_8 \lambda + \omega_2 A_{10}}{\lambda^2 + \omega_2^2}, \quad B_3 = \frac{A_9 \lambda - \omega_1 A_7}{\lambda^2 + \omega_1^2}, \quad B_4 = \frac{A_{10} \lambda - \omega_2 A_8}{\lambda^2 + \omega_2^2},
\]

which are functions depending on the action variables \( p_x, p_y \). The solution of Eqs. (45)–(48) is given by \( \hat{\xi}_1 = \hat{\xi}_2 = \hat{\xi}_3 = \hat{\xi}_4 = 0 \) and

\[
R_1 = \rho = \begin{pmatrix} -A_1 \cos y - A_2 \sin x \\ -A_3 \cos y - A_4 \sin x \\ -A_5 \cos x \\ -A_6 \sin y \end{pmatrix} \tag{52}
\]

By construction, the vector field \( R_1 \) in (52) is Hamiltonian with respect to the symplectic form \( dp_x \land dx + dp_y \land dy \) (c.f. Section 3.2).

The specific computations regarding \( C_1, R_2 \) and \( P_2 \) are omitted, since they will not be used in what follows. The only thing that we need to know in the next section is which resonances appear in the averaging process of the Hamiltonian corresponding to \( R_2 \). We remark that in our problem, it turns out that \( R_2 \) is a trigonometric polynomial of degree 2. This claim follows from the the construction explained in Section 3.2 and the fact that we know the degrees of the functions \( X_0, X_1, X_2, P_0, P_1 \) and \( C_0 \) that appear in Eqs. (39) and (40).

### 3.4 Invariant tori on the NHIM

From the computations presented in Section 3.3 we obtain that the dynamics reduced to the perturbed NHIM is given by the Hamiltonian system:

\[
r_\varepsilon(x, y, p_x, p_y) = r_0(p_x, p_y) + r_1(x, y, p_x, p_y) \varepsilon + r_2(x, y, p_x, p_y) \varepsilon^2 + O(\varepsilon^3). \tag{53}
\]
The Hamiltonian functions $r_i$ satisfy $R_i = -\Omega_i^0 \mathbf{D}r_i^T$, where $R_i$ is the reduced vector field on the NHIM computed in Section 3.3. Specifically, we have

\[
\begin{align*}
    r_0(p_x, p_y) &= \frac{p_x^2 + p_y^2}{2} + \sqrt{p_x^2 + p_y^2}, \\
    r_1(x, y, p_x, p_y) &= A_5 \sin x - A_6 \cos y, \\
    r_2(x, y, p_x, p_y) &= A_{15} + A_{16} \cos x + A_{17} \cos y + A_{18} \sin x + A_{19} \sin y \\
    &+ A_{20} \cos(2x) + A_{21} \cos(2y) + A_{22} \cos(x + y) + A_{23} \cos(x - y) \\
    &+ A_{24} \sin(2x) + A_{25} \sin(2y) + A_{26} \sin(x + y) + A_{27} \sin(x - y),
\end{align*}
\]

where $A_5$ and $A_6$ are given in Eqs. (51) and the remaining coefficients are certain explicit functions of $(p_x, p_y)$ whose explicit expressions are not important in the computations performed later. Here we are denoting as $(x, y, p_x, p_y)$, with abuse of notation, the reduced variables on the perturbed NHIM, but they are not the same as the coordinate variables in the phase space $\mathbb{T}^3 \times \mathbb{R}^3$. However, at first order in $\varepsilon$ the parameterization is a graph (see Eq. (36)), and hence, the reduced variables and the coordinate variables only differ in terms of order $\varepsilon^2$.

### 3.4.1 The global averaging method

The task now is to characterize invariant tori on the perturbed NHIM. The idea introduced in [16] consists in performing several steps of averaging in a global way on the whole NHIM. To this end, a normal form procedure is applied but, when we are close to a given resonance, the resonant normal form is defined by evaluating the corresponding coefficient on the resonant manifold (see also [18]). It is worth mentioning that since the problems considered in [16, 18] are non-autonomous, a suitable projection is the so-called projection along the $k$-direction. In our case, due to the fact that the studied Hamiltonian is autonomous, the orthogonal projection is more appropriate to perform computations.

Although we are interested in the ABC system, the discussion of this section is presented in a general setting. This allows us to use a more convenient notation and, moreover, we think that it will help the reader to link with the exposition in [16, 18] and to consult the details that we omit in our discussion.

Let us consider a Hamiltonian system on $N = \mathbb{T}^n \times \mathbb{R}^n$ of the form

\[
h(u, p_u) = h_0(p_u) + \sum_{i=1}^{m} \varepsilon^i h_i(u, p_u) + O(\varepsilon^{m+1}),
\]

where every $h_i$ is written in Fourier series as

\[
h_i(u, p_u) = \sum_{k \in \mathcal{Z}_i} \left( h_{i, k}^\cos(p_u) \cos(k \cdot u) + h_{i, k}^\sin(p_u) \sin(k \cdot u) \right),
\]

where $\mathcal{Z}_i$ is the support of the Fourier series, which is assumed to be a finite set. For the sake of consistency we take $h_{i, 0}^{\sin} \equiv 0$.

The averaging of Eq. (54) consists in performing (recursively) a suitable change of variables in such a way that we obtain a new Hamiltonian system depending on the variables $u \in \mathbb{T}^n$ in a simple way. Setting $(h)_0(u, p_u) := h(u, p_u)$, let us assume that we have performed $m - 1 \geq 0$ steps of averaging, so we have

\[
(h)_{m-1}(u, p_u) = h_0(p_u) + \sum_{i=1}^{m-1} \varepsilon^i h_i(u, p_u) + \varepsilon^m h_m(u, p_u) + O(\varepsilon^{m+1}).
\]

Then, given a Hamiltonian system $\varepsilon^m g_m$ with time-1 flow $\phi^{\varepsilon_m}$, we introduce the new Hamiltonian

\[
(h)_m(u, p_u) = (h)_{m-1} \circ \phi^{\varepsilon_m}(u, p_u)
\]

\[
= h_0(p_u) + \sum_{i=1}^{m-1} \varepsilon^i h_i(u, p_u) + \varepsilon^m \left( h_m(u, p_u) + \{h_0, g_m\}(u, p_u) \right) + O(\varepsilon^{m+1}),
\]
and we ask it to be as simple as possible by taking

\[ h_m(u, p_u) + \{ h_0, g_m \}(u, p_u) = \bar{h}_m(u, p_u). \]

Here \( \{ \cdot, \cdot \} \) is the Poisson bracket, defined as

\[ \{ f, g \} = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial u_i} \frac{\partial g}{\partial p_{u,i}} - \frac{\partial f}{\partial p_{u,i}} \frac{\partial g}{\partial u_i} \right). \]

Using an expansion as in (55), we obtain the following set of equations for the Fourier coefficients:

\[
\begin{align*}
(\omega \cdot k)\bar{h}_{m,k}^{\cos}(p_u) &= \bar{h}_{m,k}^{\sin}(p_u) - \tilde{h}_{m,k}^{\sin}(p_u), \\
-(\omega \cdot k)\bar{g}_{m,k}^{\sin}(p_u) &= \bar{h}_{m,k}^{\cos}(p_u) - \tilde{h}_{m,k}^{\cos}(p_u),
\end{align*}
\]

for every \( k \in \mathbb{Z}^n \setminus \{0\} \), and we take \( \bar{h}_{m,0}^{\cos} = \bar{h}_{m,0}^{\sin} \) and \( \tilde{h}_{m,0}^{\sin} = 0 \), so that \( \bar{g}_{m,0}^{\cos} \) and \( \tilde{g}_{m,0}^{\sin} \) can take any value. In these equations \( \omega \equiv \bar{\omega}(p_u) := \frac{\partial \bar{h}}{\partial p_u} \).

**Definition 3.7.** Given a Hamiltonian \( h : \mathcal{I} \subset \mathbb{R}^n \to \mathbb{R} \), for each \( k \in \mathbb{Z}^n \setminus \{0\} \) we define the resonant set

\[ R_k := \{ p_u \in \mathcal{I} : \bar{\omega}(p_u) \cdot k = 0 \}. \]

Let us assume in what follows that the function \( \bar{\omega}(p_u) \cdot k \) has no critical points on \( R_k \), so that the resonant manifolds are smooth surfaces (a condition that depends only on the unperturbed problem and that is certainly satisfied by the ABC system). Then, it makes perfect sense to introduce some additional definitions and notation. Indeed, given a resonant set \( R_k \) and a small constant \( L > 0 \), we denote the tubular neighborhood of \( R_k \) of radius \( L \) (measured with the Euclidean metric) as \( \text{Tub}(R_k, L) \). Moreover, for every resonant set we introduce the orthogonal projection \( \Pi_k : \text{Tub}(R_k, L) \subset \mathbb{R}^n \to R_k \). Finally, given a resonant set \( R_k \) we denote by \( \text{dist}(p_u, R_k) \) the Euclidean distance of the point \( p_u \) to the manifold \( R_k \).

Notice that \( R_k = R_{m \ell k} \) for any \( m \in \mathbb{Z} \). Then, given two sets \( R_k \) and \( R_\ell \), we have, generically, the following trichotomy:

- They are the same manifold: \( R_k = R_\ell \), i.e, \( k = m \ell \) for some \( m \in \mathbb{Z} \).
- They do not intersect: \( R_k \cap R_\ell = \emptyset \).
- They intersect transversely in a manifold of codimension two without boundary.

It is worth mentioning that the third case does not play an important role in our problem. Indeed, for the ABC system, resonant sets are 1-dimensional manifolds and their intersections define sets of dimension zero. The case of higher dimensions has been discussed recently in [18] proving that the existence of multiple resonances is not a limitation to prove diffusion.

If there is a finite number of resonant sets, it is clear that we can choose a constant \( L > 0 \) small enough such that for every pair \( k, \ell \in \mathbb{Z}^n \) we have either \( R_k = R_\ell \) or \( R_k \cap \text{Tub}(R_\ell, L) = \emptyset \). Then, following [18], we construct a solution of Eq. (56) in a global way, that is, for all values \( p_u \in \mathcal{I} \). Of course, we only want to modify the Fourier coefficients in the support of the series, that is, if \( k \notin \mathbb{Z} \) we take \( \tilde{h}_{m,k}^{\cos}(p_u) = \tilde{h}_{m,k}^{\sin}(p_u) = 0 \), and hence \( \bar{g}_{m,k}^{\cos}(p_u) = \bar{g}_{m,k}^{\sin}(p_u) = 0 \). Then, if \( k \in \mathbb{Z} \), we take

\[
\begin{align*}
\bar{h}_{m,k}^{\cos}(p_u) &= h_{m,k}^{\cos}(\Pi_k(p_u)) \psi \left( \frac{1}{L} \text{dist}(p_u, R_k) \right), \\
\bar{h}_{m,k}^{\sin}(p_u) &= h_{m,k}^{\sin}(\Pi_k(p_u)) \psi \left( \frac{1}{L} \text{dist}(p_u, R_k) \right),
\end{align*}
\]

where \( \psi : \mathbb{R} \to \mathbb{R} \) is a fixed \( C^\infty \) function such that \( \psi(t) = 1 \), if \( t \in [-1, 1] \), and \( \psi(t) = 0 \), if \( t \notin [-2, 2] \). The Fourier coefficients of the Hamiltonian \( g_m \) are obtained from Eq. (56), passing to the limit when \( p_u \) tends to \( R_k \). For details, we refer to Lemma 8.8 in [16] and to Lemma 10 in [18]. With this choice we distinguish two different zones:
• **Non-resonant region:** If \( p_u \notin \text{Tub}(R_k, 2L) \), we have \( \bar{h}_{m,k}^{\cos}(p_u) = 0 = \bar{h}_{m,k}^{\sin}(p_u) \).

• **Resonant region:** If \( p_u \in \text{Tub}(R_k, L) \), we have \( \bar{h}_{m,k}^{\cos}(p_u) = h_{m,k}(\Pi_k(p_u)) \), and \( \bar{h}_{m,k}^{\sin}(p_u) = h_{m,k}(\Pi_k(p_u)) \).

**Remark 3.8.** The choice of \( L \) is arbitrary. This implies that we do not need to study the regions at a distance between \( L \) and \( 2L \) of the resonant set \( R_k \).

### 3.4.2 Adapted coordinates on the averaged system

Let us now apply the averaging procedure described in Section 3.4.1 to the reduced Hamiltonian (53). In this case, resonant sets are expressed as

\[
R_k = \{(p_x, p_y) \in \mathcal{I} : \omega_1 k_1 + \omega_2 k_2 = 0\}
\]

where \( k = (k_1, k_2) \), the set \( \mathcal{I} \subset \{p_x > 0\} \times \{p_y > 0\} \), and the frequency \( \omega = (\omega_1, \omega_2) \) is given by Eqs. (7) and (8). Then, it is clear that there are no resonances associated to the averaging of order \(|k| \leq 1\), since \( p_x \neq 0 \) and \( p_y \neq 0 \), and so we have \( \omega_1 \neq 0 \) and \( \omega_2 \neq 0 \). For the same reason, in the averaging of order \(|k| \leq 2\), we must deal only with the set \( \omega_1 - \omega_2 = 0 \), that corresponds to the straight line \( p_x = p_y \). The orthogonal projection associated to this particular resonance, that we simply write as \( \Pi \), has the following explicit expression:

\[
\Pi(p_x, p_y) = \left( \frac{p_x + p_y}{2}, \frac{p_x + p_y}{2} \right). \tag{57}
\]

• **Non-resonant region:** we can eliminate all the terms in \( r_1(u, p_u) \) and \( r_2(u, p_u) \), so the second order averaged system is given by

\[
\langle r_\varepsilon \rangle_2(x, y, p_x, p_y) = r_0(p_x, p_y) + \mathcal{O}(\varepsilon^3).
\]

Neglecting the \( \mathcal{O}(\varepsilon^3) \) terms, we obtain an integrable unperturbed system. The invariant tori of this unperturbed system are given by the level sets

\[
\begin{align*}
p_x &= e_1, \\
p_y &= e_2.
\end{align*} \tag{58}
\]

When the perturbation \( \mathcal{O}(\varepsilon^3) \) is taken into account, KAM theorem guarantees that most of these invariant tori persist for the perturbed system, covering the non-resonant region except for a set of measure of order \( \mathcal{O}(\varepsilon^{3/2}) \). We remark again that, since our problem is real analytic, we do not need to care about the technical difficulties regarding regularity in the KAM theorem.

• **Resonant region:** we consider the projection (57), and we obtain that the second order averaged reduced Hamiltonian is given by

\[
\langle r_\varepsilon \rangle_2(x, y, p_x, p_y) = r_0(p_x, p_y) + \varepsilon^2 \left( A_{23}(\Pi(p_x, p_y)) \cos(x - y) + A_{27}(\Pi(p_x, p_y)) \sin(x - y) \right) + \mathcal{O}(\varepsilon^3).
\]

Then, it is natural to perform a canonical change of variables in order to introduce a resonant angle:

\[
\begin{align*}
\theta_1 &= x, \\
\theta_2 &= x - y,
\end{align*}
\]

thus obtaining the Hamiltonian

\[
\langle r_\varepsilon \rangle_2(\theta_1, \theta_2, I_1, I_2) = r_0(I_1 + I_2, -I_2) + \varepsilon^2 \left( A_{23} \left( \frac{I_1}{2}, \frac{I_2}{2} \right) \cos(\theta_2) + A_{27} \left( \frac{I_1}{2}, \frac{I_2}{2} \right) \sin(\theta_2) \right) + \mathcal{O}(\varepsilon^3). \tag{59}
\]
The next step is to perform a Taylor expansion around the resonance. It is clear that the resonance \( p_x = p_y \) is equivalent to \( I_2 = -I_1/2 \). Hence, we consider the expansion \( I_2 = -I_1/2 + \delta \) and we write the unperturbed Hamiltonian as follows

\[
 r_0\left(\frac{I_2}{2} + \delta, \frac{I_1}{2} - \delta\right) = r_0\left(\frac{I_2}{2}, \frac{I_1}{2}\right) + \frac{1}{2} \left( \frac{\partial^2 r_0}{\partial p_x^2} - 2 \frac{\partial^2 r_0}{\partial p_x \partial p_y} + \frac{\partial^2 r_0}{\partial p_y^2} \right) \left(\frac{I_2}{2}, \frac{I_1}{2}\right) \delta^2 + \mathcal{O}(\delta^3),
\]

where we have used that \( \omega_1 = \omega_2 \) at \( \delta = 0 \). Moreover, using the specific expression of \( r_0 \), it turns out that we can write the Hamiltonian \( \mathcal{H}_2 \) as

\[
 \langle r_\epsilon \rangle_2(\theta_1, \theta_2, I_1, -\frac{I_1}{2} + \delta) = \frac{I_1^2}{4} + \frac{I_1}{\sqrt{2}} + (1 + \sqrt{2}I_1^{-1})\delta^2 + \varepsilon^2 \left( A_{23}\left(\frac{I_1}{2}, \frac{I_1}{2}\right) \cos(\theta_2) + A_{27}\left(\frac{I_1}{2}, \frac{I_1}{2}\right) \sin(\theta_2) \right), \tag{60}
\]

modulo terms of order \( \mathcal{O}(\varepsilon^3, \delta^3) \). This corresponds to a pendulum-like Hamiltonian system in the variables \( (\theta_2, \delta) \) depending on the variable \( I_1 \). In other words, we observe that \( I_1 \) is an integral of motion of the truncated Hamiltonian \( \mathcal{H}_2 \) and the motion of the variables \( (\theta_2, \delta) \) is described by the system

\[
 \dot{\theta}_2 = 2(1 + \sqrt{2}I_1^{-1})\delta, \\
 \dot{\delta} = \varepsilon^2 \left( A_{23}\left(\frac{I_1}{2}, \frac{I_1}{2}\right) \sin(\theta_2) - A_{27}\left(\frac{I_1}{2}, \frac{I_1}{2}\right) \cos(\theta_2) \right).
\]

The above system has a hyperbolic equilibrium point at \( (\theta_2, \delta) = (\theta_2^*(I_1), 0) \), and we denote by \( H^* \) the energy of this point. Then, the level sets of \( I_1 \) and \( \langle r_\epsilon \rangle_2 \) are characterized as follows

\[
 \frac{I_1^2}{4} + \frac{I_1}{\sqrt{2}} + (1 + \sqrt{2}I_1^{-1})\delta^2 + \varepsilon^2 \left( A_{23}\left(\frac{I_1}{2}, \frac{I_1}{2}\right) \cos(\theta_2) + A_{27}\left(\frac{I_1}{2}, \frac{I_1}{2}\right) \sin(\theta_2) \right) + \mathcal{O}(\varepsilon^3, \delta^3) = e_2. \tag{61}
\]

We observe that these level sets have different topology depending if \( e_2 > H^* \) (two primary tori), if \( e_2 = H^* \) (two whiskered tori with coincident whiskers) or if \( e_2 < H^* \) (two secondary tori). Again, applying the KAM theorem to consider the effect of the perturbation, we obtain that many of the invariant tori in the previous picture persist, covering the resonant region except for a set of measure \( \mathcal{O}(\varepsilon^{3/2}) \). We refer to \cite{16,18,19} for full details on the application of the KAM theorem close to the separatrix. As before, we do not discuss here the specific technical details since they are covered by the fact that our Hamiltonian is real-analytic.

We have obtained an approximation of the level sets that characterize the invariant objects inside the NHIM. Such level sets are not written in terms of the original variables of the problem but in the averaged variables. In the following result we translate the previous construction into the coordinate variables in phase space.

**Proposition 3.9.** Let us consider the original Hamiltonian system

\[
 H_\varepsilon = H_0 + \varepsilon H_1 + \varepsilon^2 H_2,
\]

where \( H_0, H_1 \) and \( H_2 \) are given by Eqs. \( \text{[17]} - \text{[19]} \). Then, the invariant tori inside the NHIM are characterized by the level sets

\[
 p_x - \varepsilon \hat{B}_{\omega_1} \sin x + \mathcal{O}(\varepsilon^2) = e_1, \\
 p_y - \varepsilon \hat{C}_{\omega_2} \cos y + \mathcal{O}(\varepsilon^2) = e_2, \tag{62}
\]

in the non-resonant region, and

\[
 \frac{e_1^2}{4} + \frac{e_1}{\sqrt{2}} + (1 + \sqrt{2}e_1^{-1}) \left( \frac{e_1}{2} - p_y + \varepsilon \hat{C}_{\omega_2} \cos y \right)^2 + \mathcal{O}(\varepsilon^2) = e_2, \tag{63}
\]

in the resonant region. Recall that the frequencies \( \omega_1 \) and \( \omega_2 \) are defined in Eqs. \( \text{[7]} \) and \( \text{[8]} \).
Proof. We only have to undo the different changes of variables in the averaging construction previously explained. In particular we recall that we defined
\[ \langle r_\varepsilon \rangle_1(u,p_u) = r_\varepsilon \circ \phi^{g_1}(u,p_u), \]
where \( \phi^{g_1} \) is the time-1 flow of a Hamiltonian \( \varepsilon g_1 \) satisfying \( r_1 + \{ r_0, g_1 \} = 0 \), with \( r_1 = A_5 \sin x - A_6 \cos y \). The expressions of \( A_5 \) and \( A_6 \) are given in Eqs. (51). Since there are no resonances involved, we can solve the previous equation taking
\[ g_1(u,p_u) = G_1 \cos x + G_2 \sin y, \]
with \( G_1 = \tilde{\dot{B}} \frac{\omega_2}{\omega_1} \) and \( G_2 = -\tilde{\dot{C}} \frac{\omega_1}{\omega_2} \). This means that we have to invert the change of variables
\[ x \mapsto x + \varepsilon \partial_p x g_1 + O(\varepsilon^2), \]
\[ y \mapsto y + \varepsilon \partial_p y g_1 + O(\varepsilon^2), \]
\[ p_x \mapsto p_x - \varepsilon \partial_x g_1 + O(\varepsilon^2), \]
\[ p_y \mapsto p_y - \varepsilon \partial_y g_1 + O(\varepsilon^2), \]
that lead to the averaged system. As explained at the beginning of Section 3.4 the reduced variables are the same as the coordinate variables up to terms of order \( \varepsilon^2 \), so we can safely assume that the variables \( (x,y,p_x,p_y) \) in Eq. (64) are the phase space coordinates.

Let us first consider the non-resonant region. The unperturbed invariant tori of the averaged system are given by the level sets \( p_x = e_1 \) and \( p_y = e_2 \), cf. Eq. (58). Inverting the change of variables (64) we obtain that the surviving invariant tori satisfy the expression in Eq. (62).

In the resonant region, we obtained that the unperturbed invariant tori of the averaged system are given by the level sets in Eq. (61). Following [18], we first replace the expression \( I_1 = e_1 \) into the second expression in (61), thus obtaining the equivalent system
\[ \frac{e_1^2}{4} + \frac{e_1}{\sqrt{2}} + (1 + \sqrt{2} e_1^{-1}) \delta^2 + O(\varepsilon^2) = e_2. \]

This choice will simplify subsequent computations. Then, we recall the definition of the variable \( \delta \) and we invert the change of variables \( (p_x,p_y) \mapsto (I_1,I_2) \), thus obtaining
\[ \delta = I_2 + \frac{I_1}{2} = -p_y + \frac{e_1}{2}. \]
Then, inverting the change of variables (64), we obtain Eq. (63).

We would like to remark that the terms of order \( \varepsilon \) in Eqs. (62) and (63) will be important in the computations of Section 5. These terms are not necessary in [16, 18, 19], due to the fact that the unperturbed outer dynamics is the identity and hence there is no phase-shift.

4 Outer dynamics of the NHIM

In this section we consider the outer dynamics of the NHIM for the perturbed system. This dynamics is modelled by the so-called scattering map of a normally hyperbolic invariant manifold with intersecting stable and unstable invariant sets along a homoclinic manifold. This remarkable tool was introduced in [15] to study Arnold diffusion in the context of periodic perturbations of geodesic flows in \( \mathbb{T}^2 \), and it was crucial for applications in [16, 18, 20, 26]. The paper [17] contains a complete description of the geometric properties of the scattering map, together with a systematic development of perturbative formulas for its computation.

In Section 4.1 we recall the construction of the so-called Melnikov potential, which was introduced in [15], in the setting considered in this paper. In Section 4.2 we present a brief definition of the scattering map and we obtain its first order approximation for the case of the ABC system.
4.1 The Poincaré-Melnikov function

As was discussed in Section 3, for small values of \( \varepsilon \) there exists a perturbed NHIM, denoted by \( \Lambda_\varepsilon \), together with local invariant manifolds \( W_{loc}^s(\Lambda_\varepsilon) \) and \( W_{loc}^u(\Lambda_\varepsilon) \). These manifolds are \( \mathcal{O}(\varepsilon) \)-close to \( \Lambda \) and \( W^s(\Lambda) = W^u(\Lambda) \), respectively. As usual, we globalize the invariant manifolds as \( W^s(\Lambda_\varepsilon) = \bigcup_{t<0} \phi_t^\varepsilon(W_{loc}^s(\Lambda_\varepsilon)), W^u(\Lambda_\varepsilon) = \bigcup_{t>0} \phi_t^\varepsilon(W_{loc}^u(\Lambda_\varepsilon)) \), where \( \phi_t^\varepsilon \) is the flow of the perturbed Hamiltonian \( H_\varepsilon \). The intersections of the stable and unstable invariant manifolds are given by the following proposition. All along this section we use the notation introduced in Section 2.1 for the unperturbed problem.

**Proposition 4.1.** Let us consider an analytic Hamiltonian system of the form \( H_\varepsilon(q,p) = H_0(q,p) + \varepsilon h(q,p,\varepsilon) \), having a NHIM \( \Lambda_\varepsilon \), where the unperturbed Hamiltonian \( H_0 \) is given by \( \mathcal{L}_0 \). The homoclinic intersections of the invariant manifolds \( W^s(\Lambda_\varepsilon) \) and \( W^u(\Lambda_\varepsilon) \) are described, at first order in \( \varepsilon \), by the critical points of the Poincaré function (also known as Melnikov potential):

\[
L(\tau,x,y,p_x,p_y) = \int_{-\infty}^{\infty} h(\phi_\sigma^0(u^0),0) - h(\phi_\sigma^0(u^0 + u_\pm),0) \, d\sigma,
\]

where \( \phi_\sigma^0 \) is the time-\( \sigma \) flow of the unperturbed Hamiltonian \( H_0 \). In particular, \( \phi_\sigma^0(u^0 + u_\pm) \) is given by Eqs. (10) and (13), and \( \phi_\sigma^0(u^0) \) is given by Eq. (11). Recall that the compact notation \( u_\pm \) means that we take \( u_+ \) for \( \sigma > 0 \) and \( u_- \) for \( \sigma < 0 \).

We observe that this expression of \( L(\tau,x,y,p_x,p_y) \) differs from the one used in [16, 18, 19] by the fact that it depends on the phase-shift. A Melnikov potential of this type is given in Proposition 3 of [41] and analogous expressions can be found in [14, 15]. We invite the reader to compare Proposition 4.1 with Theorem 32 in [17] that is stated in a more general setting. For the sake of completeness, we present here a complete proof of this proposition that may be of valuable help for the general reader. The arguments, which we adapt from [16], are standard in Melnikov theory and well known to experts.

Let us consider the function

\[
\mathcal{P}(x,y,z,p_x,p_y) := \frac{\mu^2}{2} - \lambda^2 (\cos(z - \alpha) + 1),
\]

which is a first integral of the Hamiltonian system defined by \( H_0 \), where \( \alpha = \arctan(p_x/p_y) \). This function is used to estimate the distance between the invariant manifolds associated to the NHIM (see Lemma 4.2 below). Indeed, at every point \( u^0 = u^0(\tau,x,y) \in W^s(\Lambda_0) = W^u(\Lambda_0) \), given by (9), we have

\[
\mathcal{P}(u^0) = \frac{2\lambda^2}{\cosh^2(\lambda \tau)} - \lambda^2 \left( \cos(4 \arctan e^{\lambda \tau} + \pi) + 1 \right)
\]

\[
= \lambda^2 \left( \frac{8 e^{2\lambda \tau}}{(1 + e^{2\lambda \tau})^2} \cos(4 \arctan e^{\lambda \tau}) - 1 \right) = 0.
\]

Then, for every point \( u^0 \in W^s(\Lambda_0) = W^u(\Lambda_0) \) we consider the straight line \( \Sigma \), transversal to \( W^s(\Lambda_0) = W^u(\Lambda_0) \), given by

\[
\Sigma = \Sigma(u^0) = \{ u^0 + \mu \nabla(z,p_x)\mathcal{P}(u^0) : \mu \in \mathbb{R} \},
\]

where we are using the notation \( \nabla(z,p_x)\mathcal{P} := (0,\frac{\partial \mathcal{P}}{\partial z},0,0,\frac{\partial \mathcal{P}}{\partial p_x}) \). We denote by \( u^s = \Sigma(u^0) \cap W^s(\Lambda_\varepsilon) \) and \( u^u = \Sigma(u^0) \cap W^u(\Lambda_\varepsilon) \) the intersections of the line \( \Sigma \) with the stable and unstable manifolds of \( \Lambda_\varepsilon \), respectively. Then, there exist constants \( \mu^s \in \mathbb{R} \) and \( \mu^u \in \mathbb{R} \) such that these intersections are given by

\[
u^s = \left( x + F_1(\tau), y + F_2(\tau), z^0(\tau) + \mu^s \partial_z \mathcal{P}^0, p_x, p_y, p_z^0(\tau) + \mu^s \partial_{p_z} \mathcal{P}^0 \right),
\]

\[
u^u = \left( x + F_1(\tau), y + F_2(\tau), z^0(\tau) + \mu^u \partial_z \mathcal{P}^0, p_x, p_y, p_z^0(\tau) + \mu^u \partial_{p_z} \mathcal{P}^0 \right),
\]
where \( \partial_z \mathcal{P}^0 := \frac{\partial}{\partial z} (u^0(\tau, x, y)) \) and \( \partial_p \mathcal{P}^0 = \frac{\partial}{\partial p} (u^0(\tau, x, y)) \). Then, the following result states the relationship between \( \mathcal{P}(u) \) and the intersections of \( W^s(\Lambda_\varepsilon) \) and \( W^u(\Lambda_\varepsilon) \) for \( \varepsilon \neq 0 \):

**Lemma 4.2.** For each fixed \( u^0 \), the homoclinic intersections of the stable and unstable manifolds are characterized by

\[
\begin{align*}
    u^s &= u^u & \iff & \mu^s = \mu^u & \iff & \mathcal{P}(u^s) = \mathcal{P}(u^u).
\end{align*}
\]

**Proof.** It is clear that these implications hold from the left to the right. The converse follows from the fact that the function

\[
f(\mu) := \mathcal{P}(u^0 + \mu \nabla_{(x, p)} \mathcal{P}(u^0)) = \left( \frac{p^0(\tau) + \mu \partial_p \mathcal{P}^0}{2} \right) - \lambda^2 \cos(z^0(\tau) + \mu \partial_p \mathcal{P}^0 - \alpha) - \lambda^2,
\]

has no critical points if \( \mu \) is small enough. Indeed, an easy computation shows that the derivative

\[
f'(0) = \frac{4\lambda^2}{\cosh^2(\lambda \tau)} + \lambda^4 \sin^2(4 \arctan e^{\lambda \tau} + \pi)
\]

does not vanish in the region \( \{ p_x > 0, p_y > 0 \} \) because \( \lambda = (p_x^2 + p_y^2)^{1/4} \). \( \square \)

Before proving Proposition 4.1, we summarize some basic asymptotic properties of the flows \( \phi^0_t \) and \( \phi^\varepsilon_t \). We recall that the dynamics of the unperturbed problem has a phase-shift \( u_\pm = (x_\pm, y_\pm, 0, 0, 0) \), where \( x_\pm \) and \( y_\pm \) are given by Eq. (13). The trajectories on the invariant torus \( \mathcal{T}_{p_x, p_y} \) and the trajectories on the whiskers \( W^s(\mathcal{T}_{p_x, p_y}) = W^u(\mathcal{T}_{p_x, p_y}) \) converge exponentially to each other, with rate \( \lambda \) as \( t \to \pm \infty \). More precisely

\[
\begin{align*}
    |\phi^0_t(u^0) - \phi^0_t(u^* + u_+)| &\leq C_1 e^{-\lambda t}, \quad t \geq 0, \\
    |\phi^0_t(u^0) - \phi^0_t(u^* + u_-)| &\leq C_1 e^{-\lambda |t|}, \quad t \leq 0,
\end{align*}
\]

for some constant \( C_1 > 0 \). These estimates are obtained using the explicit expressions computed in Section 2.1 and they just reflect the normal hyperbolicity of the NHIM. Analogous expressions hold for the perturbed system. In this case, given \( u^s \in W^s(\Lambda_\varepsilon) \) and \( u^u \in W^u(\Lambda_\varepsilon) \), there exist points on the NHIM, \( u^s_\pm, u^u_\pm, u^u_{\pm} \in \Lambda_\varepsilon \), that are \( \varepsilon \)-close to their unperturbed counterparts. These points satisfy

\[
\begin{align*}
    |\phi^\varepsilon_t(u^s) - \phi^\varepsilon_t(u^s_\pm + u^u_\pm)| &\leq C_2 e^{-\lambda_\varepsilon t}, \quad t \geq 0, \\
    |\phi^\varepsilon_t(u^u) - \phi^\varepsilon_t(u^u_{\pm} + u^u_{\pm})| &\leq C_2 e^{-\lambda_\varepsilon |t|}, \quad t \leq 0,
\end{align*}
\]

for some constant \( C_2 > 0 \), where \( \lambda_\varepsilon = \lambda + \mathcal{O}(\varepsilon) \). We also need to recall some estimates that allow us to compare the perturbed and the unperturbed flows. The following estimates, which hold for all \( t \in \mathbb{R} \), are standard and immediate to obtain (for certain positive constants \( C_3, C_4, C_5 \) and \( K \)):

\[
\begin{align*}
    |\phi^\varepsilon_t(u^s) - \phi^0_t(u^0)| &\leq C_3 |u^s - u^0| e^{K \varepsilon |t|} \leq C_5 \varepsilon e^{K \varepsilon |t|}, \\
    |\phi^\varepsilon_t(u^u) - \phi^0_t(u^0)| &\leq C_3 |u^u - u^0| e^{K \varepsilon |t|} \leq C_5 \varepsilon e^{K \varepsilon |t|}, \\
    |\phi^\varepsilon_t(u^s_{\pm} + u^u_{\pm}) - \phi^0_t(u^* + u_+)| &\leq C_4 |u^s_{\pm} + u^u_{\pm}| e^{K \varepsilon |t|} \leq C_5 \varepsilon e^{K \varepsilon |t|}, \\
    |\phi^\varepsilon_t(u^u_{\pm} + u^u_{\pm}) - \phi^0_t(u^* + u_-)| &\leq C_4 |u^u_{\pm} + u^u_{\pm}| e^{K \varepsilon |t|} \leq C_5 \varepsilon e^{K \varepsilon |t|}.
\end{align*}
\]

These expressions state that for \( \varepsilon > 0 \) there may be unstable motions inside the NHIM and we cannot have a global control on the dynamics for all time. Nevertheless, we have the bounds

\[
C_5 \varepsilon e^{K \varepsilon |t|} \leq C_7 e^{\rho_1 |t|}, \quad \text{for} \ |t| \leq C_6 \log(1/\varepsilon),
\]

with \( C_6 > 0, C_7 > 0 \) and \( 0 < \rho_1 < 1 \), which is enough for our purposes.
Proof of Proposition 4.7. To monitor the evolution of the function $P$, given by Eq. (67), along the perturbed flow, we use the formula

$$\frac{d}{dt}(P(\phi_t^\varepsilon(u))) = \{P, H^\varepsilon\}(\phi_t^\varepsilon(u)) = \varepsilon\{P, h\}(\phi_t^\varepsilon(u)),$$

where we have used that $\{P, H_0\} = 0$. Integrating this equation we obtain

$$P(\phi_t^\varepsilon(u)) = P(\phi_{t_1}^\varepsilon(u)) + \varepsilon \int_{t_1}^{t_2} \{P, h\}(\phi_s^\varepsilon(u))ds. \quad (72)$$

Using (72) with $t_2 = 0$, $t_1 = \infty$ and $u = u^\varepsilon$, we have

$$P(u^\varepsilon) = P(\phi_{t_1}^\varepsilon(u^\varepsilon)) - \varepsilon \int_0^\infty \{P, h\}(\phi_s^\varepsilon(u^\varepsilon))ds, \quad (73)$$

and using (72) with $t_2 = 0$, $t_1 = \infty$ and $u = u^{ss} + u_+^\varepsilon$, we get

$$P(u^{ss} + u_+^\varepsilon) = P(\phi_{t_1}^\varepsilon(u^{ss} + u_+^\varepsilon)) - \varepsilon \int_0^\infty \{P, h\}(\phi_s^\varepsilon(u^{ss} + u_+^\varepsilon))ds.$$

Subtracting these expressions we obtain

$$P(u^\varepsilon) - P(u^{ss} + u_+^\varepsilon) = P(\phi_{t_1}^\varepsilon(u^\varepsilon)) - P(\phi_{t_1}^\varepsilon(u^{ss} + u_+^\varepsilon))$$

$$- \varepsilon \int_0^\infty \{P, h\}(\phi_s^\varepsilon(u^\varepsilon)) - \{P, h\}(\phi_s^\varepsilon(u^{ss} + u_+^\varepsilon))ds. \quad (74)$$

Now, we observe that $P(u^{ss} + u_+^\varepsilon) = P(u^* + u_+^\varepsilon + O(\varepsilon^2)) = O(\varepsilon^2)$, since both $P$ and $\nabla P$ vanish on the unperturbed NHIM $\Lambda_0$. Moreover, from the asymptotic properties (69) it follows that

$$|P(\phi_{t_1}^\varepsilon(u^\varepsilon)) - P(\phi_{t_1}^\varepsilon(u^{ss} + u_+^\varepsilon))| \leq C_8|\phi_{t_1}^\varepsilon(u^\varepsilon) - \phi_{t_1}^\varepsilon(u^{ss} + u_+^\varepsilon)| \rightarrow 0.$$

To study the integral term, we recall that we cannot control the dynamics on the NHIM for all time, so we consider

$$\int_{C_6 \log(1/\varepsilon)}^\infty \left( \{P, h\}(\phi_s^\varepsilon(u^\varepsilon)) - \{P, h\}(\phi_s^\varepsilon(u^{ss} + u_+^\varepsilon)) \right)ds$$

$$\leq \int_{C_6 \log(1/\varepsilon)}^\infty C_9|\phi_s^\varepsilon(u^\varepsilon) - \phi_s^\varepsilon(u^{ss} + u_+^\varepsilon)|ds$$

$$\leq C_9C_2 \int_{C_6 \log(1/\varepsilon)}^\infty e^{-\lambda_\varepsilon \sigma}d\sigma = \frac{C_9C_2}{\lambda_\varepsilon} e^{-\lambda_\varepsilon C_6 \log(1/\varepsilon)} = O(\varepsilon^{\rho_2})$$

for certain constant $\rho_2 > 0$. Notice that we have used Eq. (69) to derive the second inequality. Accordingly, these estimates and Eq. (74) imply that

$$P(u^\varepsilon) = -\varepsilon \int_0^{C_6 \log(1/\varepsilon)} \left( \{P, h\}(\phi_s^\varepsilon(u^\varepsilon)) - \{P, h\}(\phi_s^\varepsilon(u^{ss} + u_+^\varepsilon)) \right)ds + O(\varepsilon^2) + O(\varepsilon^{1+\rho_2}).$$

Now, we can control the quantities $\phi_s^\varepsilon(u^\varepsilon) - \phi_s^0(0^\varepsilon)$ and $\phi_s^\varepsilon(u^{ss} + u_+^\varepsilon) - \phi_s^0(u^* + u_+)$ using Eqs. (70) and (71), so we write

$$P(u^\varepsilon) = -\varepsilon \int_0^{C_6 \log(1/\varepsilon)} \left( \{P, h\}(\phi_s^0(u^\varepsilon)) - \{P, h\}(\phi_s^0(u^* + u_+)) \right)ds + I + O(\varepsilon^2) + O(\varepsilon^{1+\rho_2}),$$

where

$$I \leq \varepsilon C_9 \int_0^{C_6 \log(1/\varepsilon)} \left( |\phi_s^\varepsilon(u^\varepsilon) - \phi_s^0(u^\varepsilon)| + |\phi_s^\varepsilon(u^{ss} + u_+^\varepsilon) - \phi_s^0(u^* + u_+)| \right)ds \leq 2C_9C_7C_6\varepsilon^{1+\rho_1} \log(1/\varepsilon) = O(\varepsilon^{1+\rho_1})$$

References
for certain constant $0 < \rho_3 < 1$. We conclude that
\[
\mathcal{P}(u^\delta) = -\varepsilon \int_0^\infty \left( \{\mathcal{P}, h\}(\phi^0_\sigma (u^\delta)) - \{\mathcal{P}, h\}(\phi^0_\sigma (u^\delta + u_+)) \right) d\sigma + O(\varepsilon^{1+\rho}),
\]
for some constant $\rho > 0$. Here we have used the bound
\[
\int_C \log(1/\varepsilon) \left( \{\mathcal{P}, h\}(\phi^0_\sigma (u^\delta)) - \{\mathcal{P}, h\}(\phi^0_\sigma (u^\delta + u_+)) \right) d\sigma = O(\varepsilon^{\rho_3}).
\]
Finally, obtaining a similar formula for $\mathcal{P}(u^u)$ and subtracting, we obtain
\[
\mathcal{P}(u^u) - \mathcal{P}(u^\delta) = \varepsilon \int_{-\infty}^\infty \left( \{\mathcal{P}, h\}(\phi^0_\sigma (u^u)) - \{\mathcal{P}, h\}(\phi^0_\sigma (u^\delta + u_+)) \right) d\sigma + O(\varepsilon^{1+\rho}).
\]
Recalling that the unperturbed flow $\phi^0_\sigma$ satisfies Eq. (11), we can write
\[
\frac{\partial}{\partial \tau} (h(\phi^0_\sigma (u^\delta))) = \partial_x h^0 \dot{F}_1(\tau + \sigma) + \partial_y h^0 \dot{F}_2(\tau + \sigma) + \partial_z h^0 z^0(\tau + \sigma) + \partial_p h^0 \dot{p}(\tau + \sigma),
\]
where we are using the notation $\partial \varepsilon := \frac{\partial h}{\partial \varepsilon}(\phi^0_\sigma (u^0))$. Now we observe that $\dot{F}_1(\tau) = -\sin(\alpha) - \sin(z^0(\tau))$, and $\dot{F}_2(\tau) = -\cos(\alpha) - \cos(z^0(\tau))$, so we obtain
\[
\frac{\partial}{\partial \tau} (h(\phi^0_\sigma (u^\delta))) = -[\sin(\alpha) + \sin(z^0(\tau + \sigma))\partial_x h^0 - [\cos(\alpha) + \cos(z^0(\tau + \sigma))\partial_y h^0 + p_z(\tau + \sigma)\partial_z h^0 + (p_x \cos(z^0(\tau + \sigma)) - p_y \sin(z^0(\tau + \sigma)))\partial_p h^0.
\]
Using the definition of $\mathcal{P}$ in (67), we end up with
\[
\frac{\partial}{\partial \tau} (h(\phi^0_\sigma (u^\delta))) = -\{\mathcal{P}, h\}(\phi^0_\sigma (u^\delta)).
\]
Hence, the expression (75) is equivalent to
\[
\mathcal{P}(u^u) - \mathcal{P}(u^\delta) = -\varepsilon \frac{\partial}{\partial \tau} L(\tau, x, y, p_x, p_y) + O(\varepsilon^{1+\rho}),
\]
where we have considered the expansion $h(u, \varepsilon) = h(u, 0) + O(\varepsilon)$ and used the definition of $L$ in Eq. (60). By Lemma 4.2 homoclinic intersections are characterized by the condition $\mathcal{P}(u^\delta) = \mathcal{P}(u^u)$. Therefore, we conclude that the existence of homoclinic intersections is given, at first order perturbation theory, by the zeros of a directional derivative of the Poincaré function $L(\tau, x, y, p_x, p_y)$, as we wanted to prove. \hfill \Box

Let us consider the Hamiltonian of the ABC system, written as $H_\varepsilon = H_0 + \varepsilon H_1 + \varepsilon^2 H_2$, where $H_0$, $H_1$ and $H_2$ are given by Eqs. (17) - (19). Then, using expressions (10) and (11), the Poincaré function $L$ has the form
\[
L(\tau, x, y, p_x, p_y) = M_1 \cos x + M_2 \cos y + M_3 \sin x + M_4 \sin y,
\]
where the coefficients $M_i \equiv M_i(\tau, p_x, p_y)$ are given by the integrals
\[
M_1 := B \int_{-\infty}^\infty \left( p_y - \cos z^* \sin(x_\pm + \omega_1 \sigma) - (p_y - \cos z^0) \sin(F_1 + \omega_1 \sigma) - p_z^0 \cos(F_1 + \omega_1 \sigma) \right) d\sigma,
\]
\[
M_2 := C \int_{-\infty}^\infty \left( p_x - \sin z^* \cos(y_\pm + \omega_2 \sigma) - (p_x - \sin z^0) \cos(F_2 + \omega_2 \sigma) - p_z^0 \sin(F_2 + \omega_2 \sigma) \right) d\sigma,
\]
\[
M_3 := B \int_{-\infty}^\infty \left( p_y - \cos z^* \cos(x_\pm + \omega_1 \sigma) - (p_y - \cos z^0) \cos(F_1 + \omega_1 \sigma) + p_z^0 \sin(F_1 + \omega_1 \sigma) \right) d\sigma,
\]
\[
M_4 := C \int_{-\infty}^\infty \left( p_x - \sin z^* \sin(F_2 + \omega_2 \sigma) - (p_x - \sin z^0) \sin(y_\pm + \omega_2 \sigma) - p_z^0 \cos(F_2 + \omega_2 \sigma) \right) d\sigma.
\]
Here $F_1 = F_1(\tau + \sigma)$ and $F_2 = F_2(\tau + \sigma)$ are given by Eq. (12), and $z^0 = z^0(\tau + \sigma)$ and $p_z^0 = p_z^0(\tau + \sigma)$ are given by Eq. (6).
4.2 The Scattering map

It is convenient to introduce the notation

\[ \mathcal{L}(x - \omega_1 \tau, y - \omega_2 \tau, p_x, p_y) := L(0, x - \omega_1 \tau, y - \omega_2 \tau, p_x, p_y). \]  

(80)

Since the properties of the unperturbed flow imply that

\[ L(0, x - \omega_1 \tau, y - \omega_2 \tau, p_x, p_y) = L(\tau, x, y, p_x, p_y), \]

we can consider the critical points of the function

\[ \tau \mapsto \mathcal{L}(x - \omega_1 \tau, y - \omega_2 \tau, p_x, p_y) \]

(81)

in order to study the homoclinic intersections.

Then, we introduce the domain \( \mathcal{D} \subset \mathbb{T}^2 \times \mathcal{I} \subset \mathbb{T}^2 \times \mathbb{R}^2 \) in hypothesis \( A_2 \) of Theorem 2.2, such that for each \((x, y, p_x, p_y)\) in \( \mathcal{D} \), there exists a unique critical point \( \tau^* = \tau^*(x, y, p_x, p_y) \) of the map (81) defining a smooth function on \( \mathcal{D} \). This implies that the points

\[ (x, y, z^0(\tau^*), p_x, p_y, \tau^*_0) + \mathcal{O}(\varepsilon) \in \mathcal{W}^s(\Lambda_{\varepsilon}) \cap \mathcal{W}^u(\Lambda_{\varepsilon}) \]

define a manifold \( \Gamma_{\varepsilon} \), called homoclinic manifold. The scattering map associated to \( \Gamma_{\varepsilon} \) is defined in a domain \( \mathcal{D}_{\varepsilon,b} \subset \mathcal{D} \) in the following way (see [16, 17]):

\[ s_{\varepsilon} : \mathcal{D}_{\varepsilon,b} \subset \mathcal{D} \quad \text{u}_b \quad \mathcal{D}_{\varepsilon,f} \subset \mathbb{T}^2 \times \mathbb{R}^2, \]

(82)

with \( \text{u}_f = s_{\varepsilon}(\text{u}_b) \) if and only if there exists \( u \in \Gamma_{\varepsilon} \) such that

\[ |\phi_\varepsilon^t(u) - \phi_\varepsilon^t(P_\varepsilon(u_f))| \rightarrow 0, \quad t \rightarrow \infty, \]

and

\[ |\phi_\varepsilon^t(u) - \phi_\varepsilon^t(P_\varepsilon(u_b))| \rightarrow 0, \quad t \rightarrow -\infty, \]

where \( P_\varepsilon \) is the parameterization of the perturbed NHIM \( \Gamma_{\varepsilon} \) introduced in Section 3.1. Since the parameterizing variables \((x, y, p_x, p_y)\) and the phase space variables coincide up to order \( \varepsilon^2 \), we can safely assume that they are the same. The sets \( \mathcal{D}_{\varepsilon,b} \) and \( \mathcal{D}_{\varepsilon,f} \) are defined as:

\[ \mathcal{D}_{\varepsilon,b} := \bigcup_{u \in \Gamma_{\varepsilon}} \{ \text{u}_b \}, \quad \mathcal{D}_{\varepsilon,f} := \bigcup_{u \in \Gamma_{\varepsilon}} \{ \text{u}_f \}. \]

The scattering map relates the past asymptotic trajectory of any orbit in the homoclinic manifold to its future asymptotic behavior.

The scattering map (82) is exact symplectic (see [17]) and it is given by the time-1 flow of the Hamiltonian function

\[ S_{\varepsilon} = S_0 + \varepsilon S_1 + \mathcal{O}(\varepsilon^2) \]

where \( S_0 \) corresponds to the unperturbed outer dynamics, and \( S_1 \) is given by the Poincaré function (80) evaluated at \( \tau = \tau^* \). Notice that the unperturbed scattering map for the ABC system satisfies

\[ \text{u}_f = \text{u}_b + u_+ - u_-, \quad u_\pm = (x_\pm, y_\pm, 0, 0, 0), \]

where \( x_\pm, y_\pm \) are given by Eq. (13). Hence, we obtain the following expressions for \( S_0 \) and \( S_1 \):

\[ S_0 = -8(p_x^2 + p_y^2)^{1/4}, \quad S_1(x, y, p_x, p_y) = \mathcal{L}(x - \omega_1 \tau^*, y - \omega_2 \tau^*, p_x, p_y), \]
where $\tau^* = \tau^*(x, y, p_x, p_y)$ is the critical point of the function (81) that has the following expression for the ABC system:

$$\tau \mapsto M_1^0 \cos(x - \omega_1 \tau) + M_2^0 \cos(y - \omega_2 \tau) + M_3^0 \sin(x - \omega_1 \tau) + M_4^0 \sin(y - \omega_2 \tau),$$

where $M_i^0 := M_i(0, p_x, p_y)$ are obtained evaluating the integrals (76), (77), (78), and (79). Finally, we discuss some conditions that allows us to justify that there exists a domain $D$ where the above construction is well posed for the ABC system. We fix a value of $(p_x, p_y)$ and notice that the function

$$(x, y) \mapsto \mathcal{L}(x, y) = M_1^0 \cos(x) + M_2^0 \cos(y) + M_3^0 \sin(x) + M_4^0 \sin(y),$$

has four critical points $(x_c, y_c)$ given by

$$x_c = \arctan \frac{M_3^0}{M_1^0}, \quad y_c = \arctan \frac{M_4^0}{M_2^0}. \quad (83)$$

It is easy to check that these critical points are nondegenerate provided that $M_1^0$ and $M_3^0$ do not vanish simultaneously, and the same for $M_2^0$ and $M_4^0$. This implies, in particular, that $\hat{B} \neq 0$ and $\hat{C} \neq 0$, as required in the statement of Theorem 2.2. Hence, we observe that we are in the same situation considered in [20], where the existence of $\tau^*$ was justified in detail using the tangential intersection of straight lines in the direction $(\omega_1, \omega_2)$ with the regular level curves of $\mathcal{L}(x, y)$, which are periodic curves which fill out a region bounded by the level curves containing the saddle points.

5 Combination of inner and outer dynamics

In this Section we conclude the proof of Theorem 2.2 and we give explicit formulas for the condition (15). To this end, we combine the inner and outer dynamics. In Proposition 3.9 we showed that the invariant tori (both primary and secondary) of the ABC system are given by the level sets of a couple of functions. This couple defines an $\mathbb{R}^2$-valued map that will be denoted as $F_\varepsilon$ all along this section. The scattering map described in Section 4.2 transports the level sets of $F_\varepsilon$ onto the level sets of $F_\varepsilon \circ s_\varepsilon$. Then, following [15,16], it turns out that (c.f. Lemma 10.4 in [16]) given two manifolds $\Sigma_1, \Sigma_2 \subset \Lambda_\varepsilon$ that are invariant under the inner dynamics, if $\Sigma_1$ intersects transversally $s_\varepsilon(\Sigma_2)$ in $\Lambda_\varepsilon$, then $W^s_{\Sigma_1} \cap W^u_{\Sigma_2}$. This is the main ingredient to create heteroclinic intersections between the KAM tori in $\Lambda_\varepsilon$.

To characterize the action of the scattering map on the level sets of a given function, we follow the computations in [19] (which are also used in [18,20]). Given a function $F = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \ldots$ we can approximate $F \circ s_\varepsilon$ as

$$F \circ s_\varepsilon = F + \{F, S_\varepsilon\} + O(\varepsilon^2) = F + \{F_0 + \varepsilon F_1, S_0 + \varepsilon S_1\} + O(\varepsilon^2),$$

$$= F + \{F_0, S_0\} + \varepsilon(\{F_1, S_0\} + \{F_0, S_1\}) + O(\varepsilon^2). \quad (84)$$

It is worth mentioning that this expression does not correspond with the expression obtained in [18,19,20], due to the presence of a phase-shift in the unperturbed problem. We observe that $F_0$ and $S_0$ depend only on the momenta, so we have $\{F_0, S_0\} = 0$.

Let us consider a function $F : T^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ that defines the level sets

$$F_1 = F_{1,0} + \varepsilon F_{1,1} + O(\varepsilon^2) = e_1, \quad (85)$$

$$F_2 = F_{2,0} + \varepsilon F_{2,1} + O(\varepsilon^2) = e_2, \quad (86)$$

and the transformation of these level sets by means of the scattering map

$$F_1 \circ s_\varepsilon = F_{1,0} + \varepsilon F_{1,1} + \varepsilon(\{F_{1,1}, S_0\} + \{F_{1,0}, S_1\}) + O(\varepsilon^2) = e_1', \quad (87)$$

$$F_2 \circ s_\varepsilon = F_{2,0} + \varepsilon F_{2,1} + \varepsilon(\{F_{2,1}, S_0\} + \{F_{2,0}, S_1\}) + O(\varepsilon^2) = e_2', \quad (88)$$
for certain $\epsilon_1', \epsilon_2'$. We will use Eqs. (85) – (88) to determine transversal intersections between these level sets. Indeed, if we subtract these expressions, we have

$$
\{F_{1,1}, S_0\} + \{F_{1,0}, S_1\} + \mathcal{O}(\epsilon) = \frac{\epsilon_1' - \epsilon_1}{\epsilon},
$$

(89)

$$
\{F_{2,1}, S_0\} + \{F_{2,0}, S_1\} + \mathcal{O}(\epsilon) = \frac{\epsilon_2' - \epsilon_2}{\epsilon}.
$$

(90)

Then, if we use Eqs. (85) and (86) to write $p_x = p_x(x, y, e_1, e_2)$ and $p_y = p_y(x, y, e_1, e_2)$, and we introduce these expressions into Eqs. (89) and (90), it turns out that we will have intersection as long as $\frac{\epsilon_1' - \epsilon_1}{\epsilon}$ and $\frac{\epsilon_2' - \epsilon_2}{\epsilon}$ are small enough, close to the non-degenerate solutions of

$$
\{F_{1,1}, S_0\} + \{F_{1,0}, S_1\} = 0,
$$

(91)

$$
\{F_{2,1}, S_0\} + \{F_{2,0}, S_1\} = 0.
$$

(92)

The non-degeneracy condition, which implies that the intersection is transversal, reads as

$$
\det \left( \frac{\partial}{\partial x} \left( \{F_{1,1}, S_0\} + \{F_{1,0}, S_1\}\right), \frac{\partial}{\partial y} \left( \{F_{1,1}, S_0\} + \{F_{1,0}, S_1\}\right) \right) \neq 0
$$

(93)

for each point at the intersection of the level sets of $F$. Let us remark that the condition (93) is evaluated by fixing $p_x$ and $p_y$ by means of $F_{1,1} = S_0$ and $F_{1,0} = S_1$, where we have neglected the $\mathcal{O}(\epsilon)$ terms.

**Remark 5.1.** It is easy to check that, in the non-resonant case, the matrix in Eq. (93) is symmetric. This is a consequence of the geometric structure of the problem, since the functions $F_{1,1}$ and $F_{2,1}$ are obtained by means of the change of variables (64).

Finally, let us express the condition (93) in a explicit way for the case of the ABC system. We consider separately the non-resonant and the resonant zones:

- **Non-resonant region:** From Proposition 3.3 it follows that we have to consider the level sets (62). In order to check the condition (93) we introduce $F_1 = F_{1,0} + \epsilon F_{1,1} + \mathcal{O}(\epsilon^2)$ and $F_2 = F_{2,0} + \epsilon F_{2,1} + \mathcal{O}(\epsilon^2)$, where

$$
F_{1,0} := p_x,
$$

$$
F_{1,1} := - \dot{B}_1 \omega_1 \sin x,
$$

$$
F_{2,0} := p_y,
$$

$$
F_{2,1} := - \dot{C}_2 \omega_2 \cos y.
$$

A direct computation shows that

$$
\{F_{1,1}, S_0\} = 4 \dot{B}_1 \omega_1 p_x (p_x^2 + p_y^2)^{-3/4} \cos x,
$$

$$
\{F_{1,0}, S_1\} = - M_3 \omega_1 \left( 1 - \omega_1 \frac{\partial x^*}{\partial x} \right) \cos(x - \omega_1 \tau^*) + M_3 \omega_2 \frac{\partial x^*}{\partial x} \cos(y - \omega_2 \tau^*) + M_4 \omega_1 \frac{\partial x^*}{\partial x} \sin(x - \omega_1 \tau^*) - M_4 \omega_2 \frac{\partial x^*}{\partial x} \sin(y - \omega_2 \tau^*),
$$

$$
\{F_{2,1}, S_0\} = - 4 \dot{C}_2 \omega_2 p_y (p_x^2 + p_y^2)^{-3/4} \sin y,
$$

$$
\{F_{2,0}, S_1\} = M_3 \omega_1 \frac{\partial x^*}{\partial y} \cos(x - \omega_1 \tau^*) - M_4 \omega_1 \left( 1 - \omega_1 \frac{\partial x^*}{\partial y} \right) \cos(y - \omega_2 \tau^*) - M_3 \omega_1 \frac{\partial x^*}{\partial y} \sin(x - \omega_1 \tau^*) + M_4 \omega_2 \left( 1 - \omega_2 \frac{\partial x^*}{\partial y} \right) \sin(y - \omega_2 \tau^*),
$$
and a straightforward but cumbersome computation allows us to compute the $2 \times 2$ matrix in Eq. (93), which reads as

$$
\begin{pmatrix}
\Delta_1 & \Delta_2 \\
\Delta_2 & \Delta_3
\end{pmatrix},
$$

where the coefficients have the expressions

$$
\Delta_1 := \frac{\partial}{\partial x}\left(\{F_{1,1}, S_0\} + \{F_{1,0}, S_1\}\right)
= -4 \hat{B} \frac{\partial p_x}{\partial \omega} (p_x^2 + p_y^2)^{-3/4} \sin x + \left[M^0_3 \omega_1 \tau^*_x + M^0_1 (1 - \omega_1 \tau^*_x) \right] \cos(x - \omega_1 \tau^*)
+ \left[M^0_3 \omega_2 \tau^*_x + M^0_2 (1 - \omega_2 \tau^*_x) \right] \cos(y - \omega_2 \tau^*)
+ \left[-M^0_1 \omega_1 \tau^*_y + M^0_3 (1 - \omega_1 \tau^*_y) \right] \sin(x - \omega_1 \tau^*)
+ \left[-M^0_2 \omega_2 \tau^*_y + M^0_4 (1 - \omega_2 \tau^*_y) \right] \sin(y - \omega_2 \tau^*),
$$

$$
\Delta_2 := \frac{\partial}{\partial y}\left(\{F_{1,1}, S_0\} + \{F_{1,0}, S_1\}\right)
= \frac{\partial}{\partial x}\left(\{F_{2,1}, S_0\} + \{F_{2,0}, S_1\}\right)
= \left[M^0_3 \omega_1 \tau^*_y - M^0_1 \omega_1 \tau^*_y (1 - \omega_1 \tau^*) \right] \cos(x - \omega_1 \tau^*)
+ \left[M^0_3 \omega_2 \tau^*_y - M^0_2 \omega_2 \tau^*_y (1 - \omega_2 \tau^*_y) \right] \cos(x - \omega_1 \tau^*)
+ \left[-M^0_1 \omega_1 \tau^*_y - M^0_3 \omega_1 \tau^*_y (1 - \omega_1 \tau^*) \right] \cos(x - \omega_1 \tau^*)
+ \left[-M^0_2 \omega_2 \tau^*_y - M^0_4 \omega_2 \tau^*_y (1 - \omega_2 \tau^*_y) \right] \sin(y - \omega_2 \tau^*),
$$

$$
\Delta_3 := \frac{\partial}{\partial y}\left(\{F_{1,2}, S_0\} + \{F_{2,1}, S_0\}\right)
= -4 \hat{C} \frac{\partial p_y}{\partial \omega} (p_x^2 + p_y^2)^{-3/4} \cos y + \left[M^0_3 \omega_1 \tau^*_y + M^0_1 (1 - \omega_1 \tau^*_y) \right] \cos(x - \omega_1 \tau^*)
+ \left[M^0_3 \omega_2 \tau^*_y + M^0_2 (1 - \omega_2 \tau^*_y) \right] \cos(y - \omega_2 \tau^*)
+ \left[-M^0_1 \omega_1 \tau^*_y + M^0_3 (1 - \omega_1 \tau^*_y) \right] \cos(x - \omega_1 \tau^*)
+ \left[-M^0_2 \omega_2 \tau^*_y + M^0_4 (1 - \omega_2 \tau^*_y) \right] \sin(y - \omega_2 \tau^*).
$$

Here the subscripts in $\tau^*$ denote, as usual, partial differentiation. Then, the transversality condition in the non-resonant region, using the functions $\Delta_i$, reads as

$$
\Delta_1 \Delta_3 - \Delta_2^2 \neq 0.
$$

**Resonant region:** From Proposition 3.9 it follows that we have to consider the level sets $\Delta_2^2$. In order to check the condition in Eq. (93) we introduce $F_1 = F_{1,0} + \varepsilon F_{1,1} + O(\varepsilon^2)$, and $F_2 = F_{2,0} + \varepsilon F_{2,1} + O(\varepsilon^2)$, where

$$
F_{1,0} := p_x + p_y,
F_{1,1} := -\hat{B} \frac{\partial p_x}{\partial \omega} \sin x - \hat{C} \frac{\partial p_y}{\partial \omega} \cos y,
F_{2,0} := \frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{\sqrt{e}} + (1 + \sqrt{2} e^{-1}) \left(\frac{\varepsilon^2}{4} - p_y\right)^2,
F_{2,1} := 2 \hat{C} \frac{\partial p_y}{\partial \omega} (1 + \sqrt{2} e^{-1}) \left(\frac{\varepsilon^2}{4} - p_y\right) \cos y.
$$

Then, the matrix in Eq. (93) has the form

$$
\begin{pmatrix}
\hat{\Delta}_1 & \hat{\Delta}_2 \\
\hat{\Delta}_2 & \hat{\Delta}_3
\end{pmatrix},
$$

with

$$
\hat{\Delta}_1 := \frac{\partial}{\partial x}\left(\{F_{1,1}, S_0\} + \{F_{1,0}, S_1\}\right)
= -4 \hat{B} \frac{\partial p_x}{\partial \omega} (p_x^2 + p_y^2)^{-3/4} \sin x
+ \left[M^0_3 \omega_1 (\tau^*_x + \tau^*_y) + M^0_1 (1 - \omega_1 (\tau^*_x + \tau^*_y)) \right] \cos(x - \omega_1 \tau^*)
+ \left[M^0_3 \omega_2 (\tau^*_x + \tau^*_y) - M^0_0 \omega_2 \tau^*_x (1 - \omega_2 (\tau^*_x + \tau^*_y)) \right] \cos(y - \omega_2 \tau^*)
+ \left[-M^0_1 \omega_1 (\tau^*_x + \tau^*_y) + M^0_3 (1 - \omega_1 (\tau^*_x + \tau^*_y)) \right] \sin(x - \omega_1 \tau^*)
+ \left[-M^0_2 \omega_2 (\tau^*_x + \tau^*_y) - M^0_4 \omega_2 \tau^*_x (1 - \omega_2 (\tau^*_x + \tau^*_y)) \right] \sin(y - \omega_2 \tau^*),
$$

$$
\hat{\Delta}_2 := \frac{\partial}{\partial y}\left(\{F_{1,1}, S_0\} + \{F_{1,0}, S_1\}\right)
= \frac{\partial}{\partial x}\left(\{F_{2,1}, S_0\} + \{F_{2,0}, S_1\}\right)
= \left[M^0_3 \omega_1 \tau^*_x - M^0_1 \omega_1 \tau^*_x (1 - \omega_1 \tau^*) \right] \cos(x - \omega_1 \tau^*)
+ \left[M^0_3 \omega_2 \tau^*_x - M^0_2 \omega_2 \tau^*_x (1 - \omega_2 \tau^*_x) \right] \cos(x - \omega_1 \tau^*)
+ \left[-M^0_1 \omega_1 \tau^*_x - M^0_3 \omega_1 \tau^*_x (1 - \omega_1 \tau^*) \right] \cos(x - \omega_1 \tau^*)
+ \left[-M^0_2 \omega_2 \tau^*_x - M^0_4 \omega_2 \tau^*_x (1 - \omega_2 \tau^*_x) \right] \sin(y - \omega_2 \tau^*),
$$

$$
\hat{\Delta}_3 := \frac{\partial}{\partial y}\left(\{F_{1,2}, S_0\} + \{F_{2,1}, S_0\}\right)
= -4 \hat{C} \frac{\partial p_y}{\partial \omega} (p_x^2 + p_y^2)^{-3/4} \cos y
+ \left[M^0_3 \omega_1 \tau^*_y - M^0_1 \omega_1 \tau^*_y (1 - \omega_1 \tau^*) \right] \cos(x - \omega_1 \tau^*)
+ \left[M^0_3 \omega_2 \tau^*_y - M^0_2 \omega_2 \tau^*_y (1 - \omega_2 \tau^*_y) \right] \cos(x - \omega_1 \tau^*)
+ \left[-M^0_1 \omega_1 \tau^*_y - M^0_3 \omega_1 \tau^*_y (1 - \omega_1 \tau^*) \right] \cos(x - \omega_1 \tau^*)
+ \left[-M^0_2 \omega_2 \tau^*_y - M^0_4 \omega_2 \tau^*_y (1 - \omega_2 \tau^*_y) \right] \sin(y - \omega_2 \tau^*).
\[ \hat{\Delta}_2 := \frac{\partial}{\partial y} (\{F_{1,1}, S_0\} + \{F_{1,0}, S_1\}) = -4\hat{C}_{\omega x} p_y (p_x^2 + p_y^2)^{-3/4} \cos y \\
+ [M_3^0 \omega_1 (\tau_{xy} + \tau_{yy}) - M_1^0 \omega_1 \tau_{xy}^* (1 - \omega_1 \tau_{xy})] \cos (x - \omega_1 \tau^*) \\
+ [M_4^0 \omega_2 (\tau_{xy} + \tau_{yy}) + M_2^0 (1 - \omega_2 \tau_{yy})] \cos (y - \omega_2 \tau^*) \\
+ [-M_1^0 (\tau_{xy} + \tau_{yy})^* - M_2^0 \omega_1 \tau_{yy}^* (1 - \omega_1 \tau_{yy})] \sin (x - \omega_1 \tau^*) \\
+ [-M_2^0 \omega_2 (\tau_{xy} + \tau_{yy}) + M_4^0 (1 - \omega_2 \tau_{yy})] \sin (y - \omega_2 \tau^*), \]
\[ \hat{\Delta}_3 := \frac{\partial}{\partial x} (\{F_{2,1}, S_0\} + \{F_{2,0}, S_1\}) = \gamma [M_3^0 \omega_1 \tau_{xy}^* + M_1^0 \omega_1 \tau_{xy}^* (1 - \omega_1 \tau_{xy})] \cos (x - \omega_1 \tau^*) \\
+ \gamma [M_1^0 \omega_2 \tau_{xy}^* + M_3^0 \omega_2 \tau_{xy}^* (1 - \omega_2 \tau_{xy})] \cos (y - \omega_2 \tau^*) \\
+ \gamma [M_2^0 \omega_2 \tau_{xy}^* + M_4^0 \omega_2 \tau_{xy}^* (1 - \omega_2 \tau_{xy})] \sin (y - \omega_2 \tau^*), \]
\[ \hat{\Delta}_4 := \frac{\partial}{\partial y} (\{F_{2,1}, S_0\} + \{F_{2,0}, S_1\}) = 8\hat{C}_{y} \left( p_x^2 + p_y^2 \right)^{-3/4} \omega_2 (1 + \frac{\sqrt{\pi}}{\sqrt[4]{\omega_2}}) \left( p_x - p_y \right) \cos y \\
+ \gamma [-M_2^0 \omega_1 \tau_{yy}^* - M_1^0 (\omega_1 \tau_{yy}^*)^2] \cos (x - \omega_1 \tau^*) \\
+ \gamma [-M_1^0 \omega_2 \tau_{yy}^* - M_2^0 (1 - \omega_2 \tau_{yy}^*)^2] \cos (y - \omega_2 \tau^*) \\
+ \gamma [M_2^0 \omega_2 \tau_{yy}^* - M_4^0 (1 - \omega_2 \tau_{yy}^*)^2] \sin (y - \omega_2 \tau^*). \]

Here we are using the notation \( \gamma := \frac{2(1 + \sqrt{2})}{\sqrt{\pi}} \left( p_x - p_y \right) \). Then, the transversality condition in the resonant region takes the form

\[ \hat{\Delta}_1 \hat{\Delta}_4 - \hat{\Delta}_2 \hat{\Delta}_3 \neq 0. \]  

Putting together the information gathered on the inner and the outer dynamics, we can construct chains of invariant tori giving rise to large motions in the action space. Assume that \( D \) is the domain introduced in Section 4.2. It is obvious that we can safely assume, by shrinking \( D \) if necessary, that the domain \( D_{x,y} \) of the scattering map (82) coincides with \( D \). Then, we assume that we can choose a constant \( L \) such that such that for every \((x, y, p_x, p_y) \in D \) we have

\[
\begin{align*}
\Delta_1 \Delta_4 - \Delta_2 \Delta_3 &\neq 0, & &\text{if } |p_x - p_y| \geq L, \\
\Delta_1 \Delta_4 - \Delta_2 \Delta_3 &\neq 0, & &\text{if } |p_x - p_y| \leq L.
\end{align*}
\]

This condition is precisely Eq. (15) in Theorem 2.2. It then follows that we can find a sequence \( \{T_i\}_{i=0}^{\infty} \) of tori which are at a distance \( O(\varepsilon) \) from each other and that satisfy \( s_\varepsilon(T_i) \cap T_{i+1} \). By applying Lemma 10.4 in [16], it turns out that these tori satisfy \( W^u_\varepsilon(T_i) \cap W^s_{\varepsilon}(T_{i+1}) \), that is, they form a transition chain. The claim of Theorem 2.2 (Arnold diffusion) then follows from the general theory presented in [16] [13].

6 Rigorous verification of the hypotheses of the main theorem

In this section we illustrate the effective verification of the hypotheses of Theorem 2.2 thus obtaining Arnold diffusion in the ABC Hamiltonian system [13]. We rigorously evaluate the involved functions and obtain rigorous bounds for the critical points with the help of the computer. Our approach is as simple as possible, in the sense that we do not pretend to present a fast and efficient methodology to study large regions of the phase-space systematically. Our interest here is to convince the general reader that the hypotheses of Theorem 2.2 can be rigorously checked with the help of a computer.

In rigorous computations, real numbers are substituted by intervals whose extremal are computer representable real numbers. That is, when implementing interval operations in a computer, the result of an operation with intervals is an interval that includes the result. The reader can consult the recent introductory book [14] on rigorous computations. All the computations presented in this section have been performed using FILIB [31] that uses double precision arithmetics.
Rigorous bounds of the Melnikov coefficients are obtained in Section 6.1. In Section 6.2, we control the critical point $\tau^*$ and its derivatives with respect to the angles $(x, y)$. Finally, in Section 6.3, we present a direct application of the previous ideas and we describe the implementation details giving rise to Corollary 1.1.

6.1 On the evaluation of the Melnikov coefficients

Given certain values of $p_x$ and $p_y$ (represented using interval arithmetics), we are interested in the rigorous evaluation of the Melnikov coefficients $M_1, M_2, M_3$ and $M_4$ in Eqs. (76–79). Let us recall that we are particularly interested in the values $M_i^0 = M_i(0, p_x, p_y)$ in order to check the hypotheses of Theorem 2.2. If we denote by $f_i(\sigma)$ the function that we have to integrate to evaluate $M_i^0$, and we introduce the notation $f_{i,+}(\sigma) = f_i(\sigma)$ for $\sigma > 0$ and $f_{i,-}(\sigma) = f_i(\sigma)$ for $\sigma < 0$, we can write the expressions for $M_i^0$ as follows

$$M_i^0 = \int_{-\infty}^{\infty} f_i(\sigma) d\sigma = \int_{-\infty}^{-a} f_{i,-}(\sigma) d\sigma + \int_{-a}^{0} f_{i,-}(\sigma) d\sigma + \int_{0}^{a} f_{i,+}(\sigma) d\sigma + \int_{a}^{\infty} f_{i,+}(\sigma) d\sigma,$$

where $a > 0$ is a constant that will be fixed later. The integrals at infinity (called tails from now on) will be bounded using the asymptotic properties discussed in Section 2.1. Of course, one can obtain general formulas for the tails in terms of a uniform control on the Hamiltonian and the Lyapunov exponent. However, in this case, we present specific formulas for the ABC system giving rise to sharper estimates of the tails. This allows us to keep the modulus of the tails under a prefixed tolerance using a small value of $a$.

**Lemma 6.1.** The following bounds hold for the ABC system:

$$\left| M_i^0 - \int_{-a}^{0} f_{i,-}(\sigma) d\sigma - \int_{0}^{a} f_{i,+}(\sigma) d\sigma \right| \leq \Sigma_i,$$

where $\Sigma_i$ are given by

$$\Sigma_1 = \Sigma_3 := \hat{B} \frac{4}{\lambda^2} |p_y - \cos z^*| \left(\log (1 + e^{-2\lambda a}) \sin z^* - 2 \left(\arctan \left(\frac{e^{-\lambda a} - 1}{e^{-\lambda a} + 1}\right) + \frac{\pi}{4}\right) \cos z^*\right) + \hat{B} \Sigma_0,$$

$$\Sigma_2 = \Sigma_4 := \hat{C} \frac{4}{\lambda^2} |p_x - \sin z^*| \left(\log (1 + e^{-2\lambda a}) \cos z^* + 2 \left(\arctan \left(\frac{e^{-\lambda a} - 1}{e^{-\lambda a} + 1}\right) + \frac{\pi}{4}\right) \sin z^*\right) + \hat{C} \Sigma_0,$$

where $\Sigma_0 := 8 e^{-\lambda a} \frac{\lambda}{\lambda^2} + \frac{8}{3} e^{-3\lambda a} + 4 \hat{B} \arctan(\sinh(\lambda a)) - \pi$.

**Proof.** We will only consider the case of $M_1^0$ because the control of the tails of $M_2^0, M_3^0$ and $M_4^0$ is completely analogous. We first split the function $f_{1,+}$ into three terms as

$$f_{1,+}(\sigma) = f_{1,+}^{(1)} + f_{1,+}^{(2)} + f_{1,+}^{(3)}.$$

In this splitting each term is given by

$$f_{1,+}^{(1)} := \hat{B} (p_y - \cos z^*) (\sin(x_+ + \omega_1 \sigma) - \sin(F_1(\sigma) + \omega_1 \sigma)),$$

$$f_{1,+}^{(2)} := \hat{B} \sin(F_1(\sigma) + \omega_1 \sigma) (\cos z_0(\sigma) - \cos z^*),$$

$$f_{1,+}^{(3)} := -\hat{B} p_2(\sigma) \cos(F_1(\sigma) + \omega_1 \sigma).$$

Then, a straightforward computation shows that

$$\left| \int_{-a}^{\infty} f_{1,+}^{(1)}(\sigma) d\sigma \right| \leq \hat{B} |p_y - \cos z^*| \left| \int_{-a}^{\infty} (F_1(\sigma) - x_+) d\sigma \right|,$$

$$\left| \int_{-a}^{\infty} f_{1,+}^{(2)}(\sigma) d\sigma \right| \leq \hat{B} \left| \int_{-a}^{\infty} (4 \arctan e^{\lambda \sigma} - 2\pi) d\sigma \right|,$$

$$\left| \int_{-a}^{\infty} f_{1,+}^{(3)}(\sigma) d\sigma \right| \leq \hat{B} \left| \int_{-a}^{\infty} \frac{2\lambda}{\cosh(\lambda \sigma)} d\sigma \right|.$$
By the asymptotic properties discussed in Section 2.1 we know that these three integrals are convergent. Next, we
give some explicit expressions to control the above integrals. To this end, we use the expression of the primitives of
the functions that we are integrating. First, we introduce
\[ g(\sigma) := \int (F_1(\sigma) - x_+ d\sigma = \frac{2}{\lambda^2} \log(\cosh(\lambda \sigma)) \sin z^* - \left( \frac{4}{\lambda^2} \arctan(\tanh(\frac{\lambda \sigma}{2})) - \frac{2\sigma}{\lambda} \right) \cos z^* - x_+ \sigma, \]
which allows us to control Eq. (103) in terms of the expression \( g(\infty) - g(a) \). However, the direct evaluation of this
expression with a computer presents a huge rounding error. A more suitable formula is obtained using the limit
\[ \lim_{\sigma \to \infty} \log(\cosh(\lambda \sigma)) = \lim_{\sigma \to \infty} \log\left( \frac{e^{e^{-\lambda a} + \lambda a} - e^{e^{-\lambda a}}}{2} \right) = \lim_{\sigma \to \infty} (\lambda \sigma - \log 2), \]
which allows us to control the term (103) as follows
\[ \left| \int_a^\infty f_{1,+}^{(1)}(\sigma)d\sigma \right| \leq B \frac{2}{\lambda^2} |p_y - \cos z^*| \log(1 + e^{-2\lambda a}) \sin z^* - 2 \left( \arctan \left( \frac{e^{-\lambda a} - 1}{e^{-\lambda a} + 1} \right) + \frac{\pi}{4} \right) \cos z^*. \]
Using Taylor series, it is easy to check that the term (104) is bounded as
\[ \left| \int_a^{\infty} (4 \arctan e^{\lambda \sigma} - 2\pi)d\sigma \right| \leq 4 \frac{e^{-\lambda a}}{\lambda} + \frac{4}{3} \frac{e^{-3\lambda a}}{\lambda} , \]
and that the term (105) is estimated as
\[ \left| \int_a^{\infty} \frac{2\lambda}{\cosh(\lambda \sigma)} d\sigma \right| \leq 2 |\arctan(\sinh(\lambda a)) - \pi| . \]
Analogously, we can estimate the term
\[ \int_{-\infty}^{-a} f_{1,-}(\sigma)d\sigma , \]
thus proving the lemma.

We use Lemma 6.1 to rigorously control the Melnikov coefficients. Specifically, we evaluate directly the obtained
expressions of \( \Sigma_i \) using interval arithmetics. The integrals \( \int_{-\infty}^{0} f_{i,-} \) and \( \int_{0}^{a} f_{i,+} \) are controlled using Simpson’s rule
with rigorous bounds on the error, which are obtained using explicit formulas for the 4th-order derivatives of the
functions \( f_i(\sigma) \) computed with a symbolic manipulator.

Next, we illustrate the rigorous evaluation of the Hypothesis A_1 of Theorem 2.2. Obviously, it is enough to
consider the coefficients \( M^0_i \) for the parameters \( \hat{B} = \hat{C} = 1 \) and store conveniently the obtained values. If we are
interested in other values of \( \hat{B} \) and \( \hat{C} \) we simply have to scale the previously computed values. In Table 1 we present
some rigorous enclosures of the coefficients \( M_i^0 \) corresponding to \( \hat{B} = \hat{C} = 1 \), \( p_x \in [0.4, 0.4001] \) and different
interval values of \( p_y \). To control the tails we use Lemma 6.1 with \( a = 20 \) and to enclose the finite integrals we use
Simpson’s rule with 130 subintervals. This implementation parameters are enough to guarantee that the coefficients
\( M_i^0 \) do not vanish for a non-empty set \( I \) of momenta. We can obtain a similar result for a much larger domain \( I \) by
systematically performing this computation.

Finally, let us illustrate how to check Hypothesis A_2 of Theorem 2.2. It turns out that, for \( p_x \in [0.4, 0.4001] \) and
all the intervals \( p_y \) in Table 1 the four critical points of the map \( (x, y) \mapsto L(x, y) \) in Eq. (83) are non-degenerate and
given by two maxima and two saddle points, so that we can use the arguments in [20] to see that there is a unique
smooth critical point \( \tau^* \) of the function \( \tau \mapsto L(x - \omega_1 \tau, y - \omega_2 \tau, p_x, p_y) \). In the following section we discuss the
rigorous enclosure of \( \tau^* \) and its derivatives.
6.2 On the evaluation of the critical points

We discuss a simple methodology to rigorously enclose the critical points of the function

\[
\tau \mapsto L(x - \omega_1 \tau, y - \omega_2 \tau, p_x, p_y) = M_1^0 \cos(x - \omega_1 \tau) + M_2^0 \cos(y - \omega_2 \tau) + M_3^0 \sin(x - \omega_1 \tau) + M_4^0 \sin(y - \omega_2 \tau),
\]

(106)

where \((x, y, p_x, p_y) \in \mathbb{T}^2 \times I\), and the coefficients \(M_i^0 = M_i(0, p_x, p_y)\) are given by Eqs. (76)–(79). The critical points \(\tau^* = \tau^*(x, y, p_x, p_y)\) are characterized by the zeros of the function

\[
Q(\tau) := \omega_1 \left( - M_1^0 \sin(x - \omega_1 \tau) + M_3^0 \cos(x - \omega_1 \tau) \right) + \omega_2 \left( - M_2^0 \sin(y - \omega_2 \tau) + M_4^0 \cos(y - \omega_2 \tau) \right),
\]

which, of course, depends on the variables \((x, y, p_x, p_y)\). There are several techniques in computer-assisted proofs that allow us to study solutions of nonlinear equations as above, e.g. the interval Newton method [44]. However, in the following discussion we choose to use the simplest possible method with the aim of convincing a reader that is not familiar with computer-assisted methods. More advanced techniques will give the possibility to validate large regions of phase-space with a reduced computational cost. This is not the aim in this article, since the required techniques are not related with the ideas that we want to highlight and they would require to provide a larger amount of computational and implementation details. To enclose \(\tau^*\) we proceed using a bisection-like procedure:

- Given \((x, y, p_x, p_y)\), that may be numbers or interval values, we enclose \(M_i^0 = M_i(0, p_x, p_y)\) following Section 6.1.
- Given an integer \(N > 1\), we consider an increasing sequence of interval values \(\tau_i \in [(i - 1)/N, i/N]\). Then we compute \(Q(\tau_i)\). While \(0 \notin Q(\tau_i)\) we increase the index \(i\), until we obtain an interval such that \(0 \in Q(\tau_i)\). This gives a lower estimate for \(\tau^*\).
- After the previous computations, we continue the process of increasing the index \(i\) and computing \(Q(\tau_i)\). When we reach an interval such that \(0 \notin Q(\tau_i)\) then we have obtained an upper estimate for \(\tau^*\).

As a result of the above procedure, we obtain an interval enclosure of the critical point \(\tau^*\). Then, the derivatives of \(\tau^*\) with respect to \((x, y)\) are computed using the following equations:

\[
\tau^*_{\alpha} = -\frac{Q_{\alpha}(\tau^*)}{Q_{\tau}(\tau^*)}, \quad \tau^*_{\alpha\beta} = -\frac{Q_{\alpha\beta}(\tau^*) + Q_{\alpha\tau}(\tau^*)\tau^*_{\beta} + (Q_{\beta\tau}(\tau^*)\tau^*_{\alpha})}{Q_{\tau}(\tau^*)},
\]

where the subscripts denote, as usual, partial differentiation, and \(\alpha\) and \(\beta\) can be chosen to be \(x\) or \(y\). For the case of the ABC system, we have

\[
Q_x(\tau) = -M_1^0 \cos(x - \omega_1 \tau)\omega_1 - M_3^0 \sin(x - \omega_1 \tau)\omega_1,
\]

\[
Q_y(\tau) = -M_2^0 \cos(y - \omega_2 \tau)\omega_2 - M_4^0 \sin(y - \omega_2 \tau)\omega_2,
\]

| \(p_y\) | \(M_1^0\) | \(M_2^0\) | \(M_3^0\) | \(M_4^0\) |
|---|---|---|---|---|
| [0.1, 0.1001] | [−5.1237, −4.9193] | [−11.314, −10.972] | [−1.282, −1.0754] | [1.4611, 1.807] |
| [0.2, 0.2001] | [−5.3464, −5.1279] | [−9.185, −8.8068] | [−2.2718, −2.0471] | [−3.0694, −2.7531] |
| [0.3, 0.3001] | [−5.911, −5.6582] | [−6.5333, −6.2423] | [−3.557, −3.0967] | [−5.5463, −5.2556] |
| [0.4, 0.4001] | [−6.4914, −6.2559] | [−4.4201, −4.1843] | [−4.4214, −4.1829] | [−6.4917, −6.2557] |
| [0.5, 0.5001] | [−7.0975, −6.7721] | [−2.9704, −2.7501] | [−5.2892, −5.0544] | [−6.7487, −6.5249] |
| [0.6, 0.6001] | [−7.4463, −7.2257] | [−1.988, −1.7805] | [−6.8843, −5.6657] | [−6.701, −6.4904] |
| [0.7, 0.7001] | [−7.8251, −7.6315] | [−1.3078, −1.129] | [−6.2495, −6.0596] | [−6.5182, −6.3291] |
| [0.8, 0.8001] | [−8.1753, −7.9784] | [−0.85053, −0.66663] | [−6.4659, −6.2811] | [−6.2954, −6.0987] |
| [0.9, 0.9001] | [−8.4879, −8.2889] | [−0.53277, −0.3423] | [−6.575, −6.3929] | [−6.0643, −5.8485] |

Table 1: We show rigorous enclosures of the coefficients \(M_i^0\), for the values \(B = C = 1\) and \(p_x \in [0.4, 0.4001]\).
\[ Q_\tau(\tau) = M_1^0 \cos(x - \omega_1 \tau)\omega_1^2 + M_3^0 \sin(x - \omega_1 \tau)\omega_1^2 + M_2^0 \cos(y - \omega_2 \tau)\omega_2^2 + M_4^0 \sin(y - \omega_2 \tau)\omega_2^2, \]
\[ Q_{xx}(\tau) = M_1^0 \sin(x - \omega_1 \tau)\omega_1 + M_3^0 \cos(x - \omega_1 \tau)\omega_1, \]
\[ Q_{yy}(\tau) = M_2^0 \sin(y - \omega_2 \tau)\omega_2 - M_4^0 \cos(y - \omega_2 \tau)\omega_2, \]
\[ Q_{xt}(\tau) = - M_1^0 \sin(x - \omega_1 \tau)\omega_1^2 - M_3^0 \cos(x - \omega_1 \tau)\omega_1^2, \]
\[ Q_{yt}(\tau) = - M_2^0 \sin(y - \omega_2 \tau)\omega_2^2 - M_4^0 \cos(y - \omega_2 \tau)\omega_2^2, \]
\[ Q_{tt}(\tau) = M_1^0 \sin(x - \omega_1 \tau)\omega_1^3 - M_3^0 \cos(x - \omega_1 \tau)\omega_1^3 + M_2^0 \sin(y - \omega_2 \tau)\omega_2^3 - M_4^0 \cos(y - \omega_2 \tau)\omega_2^3. \]

In Table 2 we illustrate these computations taking \( \hat{B} = \hat{C} = 1 \) and considering the values of \( p_x \) and \( p_y \) used in Table 1. In all the cases, we fix the angles as \( x = y = 0 \), and we compute the critical point \( \tau^* \) using \( N = 100 \).

| \( p_y \) | \( \tau^* \) | \( \tau_y^* \) |
|---|---|---|
| 0.1, 0.1001 | [2.34, 2.43] | [0.74112, 0.91778] | [-0.41917, -0.3401] |
| 0.2, 0.2001 | [2.71, 2.81] | [0.47369, 0.73294] | [0.29594, 0.4514] |
| 0.3, 0.3001 | [2.38, 2.45] | [0.43611, 0.54436] | [0.41137, 0.51753] |
| 0.4, 0.4001 | [2.1, 2.16] | [0.40377, 0.50552] | [0.40346, 0.5051] |
| 0.5, 0.5001 | [1.9, 1.96] | [0.37234, 0.4717] | [0.39116, 0.50717] |
| 0.6, 0.6001 | [1.75, 1.81] | [0.33403, 0.42893] | [0.38246, 0.51976] |
| 0.7, 0.7001 | [1.61, 1.67] | [0.27007, 0.36171] | [0.38473, 0.54755] |
| 0.8, 0.8001 | [1.48, 1.55] | [0.1825, 0.2837] | [0.37799, 0.60038] |
| 0.9, 0.9001 | [1.36, 1.44] | [0.09254, 0.18709] | [0.37117, 0.65404] |

Table 2: We show rigorous enclosures of the critical point \( \tau^* \) and its derivatives with respect to \( (x, y) \) for the values \( B = C = 1 \), \( p_x \in [0.4, 0.4001] \), and \( x = y = 0 \).

### 6.3 On the verification of the transversality conditions

With the rigorous estimates obtained in Sections 6.1 and 6.2, let us now explain how to use interval arithmetics to check the transversality condition in Hypothesis \( A_3 \) of Theorem 2.2. This consists in enclosing the functions \( \{\Delta_i\}_{i=1,2,3} \) and \( \{\tilde{\Delta}_i\}_{i=1,2,3,4} \) given by Eqs. (94)–(101). In order to check the condition in the resonant region, we observe that \( \Delta_3 \) and \( \Delta_4 \) are proportional to \( p_x - p_y \) so that they tend to zero when we approach the resonance. For this reason, we eliminate the factor \( p_x - p_y \) in the computation of the expression \( \Delta_1 \Delta_4 - \Delta_2 \Delta_3 \). This allows us to check that the condition holds in any tubular neighborhood that is close enough to the resonance.

An important observation is that we have the freedom of choosing the angles \( (x, y) \) to evaluate the critical point \( \tau^* \). In fact, there is no optimal way to choose the angles \( (x, y) \) in order to verify the transversality conditions. The reason is that optimal values selected numerically may fail to fulfill such conditions when rigorous interval operations are used. This is because the enclosed value of \( Q_x(\tau^*) \) may be very close to zero (or even contain this point), thus producing a large enclosure in the evaluation of the conditions. Our experience in this problem is that choosing random values of \( (x, y) \), until we reach a suitable pair, is the simplest and fast strategy.

For example, in Table 3 we present some rigorous enclosures of the transversality conditions in Hypothesis \( A_3 \) of Theorem 2.2 corresponding to \( \hat{B} = \hat{C} = 1 \), \( p_x \in [0.4, 0.40001] \), and different interval values of \( p_y \). To control the tails we use Lemma 6.1 with \( a = 20 \) and to enclose the finite integrals we use Simpson's rule with 300 subintervals. To obtain the critical point and its derivatives, we use the approach described in Section 6.2 with \( N = 300 \). This
implementation parameters are enough to guarantee that the functions $\Delta_1 \Delta_3 - \Delta_2^2$ and $\hat{\Delta}_1 \hat{\Delta}_4 - \hat{\Delta}_2 \hat{\Delta}_3$ do not vanish for a non-empty set $\mathcal{I}$ of momenta. By computing simultaneously the condition in the non-resonant region (4th column of Table 3) and in the resonant region (5th column of Table 3), it is clear that we can select a number $L > 0$ that allows us to obtain diffusing orbits crossing the resonance.

| $p_y$  | $x$     | $y$     | $\Delta_1 \Delta_3 - \Delta_2^2$ | $\hat{\Delta}_1 \hat{\Delta}_4 - \hat{\Delta}_2 \hat{\Delta}_3$ |
|-------|---------|---------|---------------------------------|---------------------------------|
| 0.1, 0.100001 | 5.2923 | 0.93117 | $[-20.8399, -3.5905]$ | $[2.7401, 101.72]$ |
| 0.2, 0.200001 | 2.7665 | 0.55732 | $[8.3643, 23.588]$ | $[-68.148, -6.1705]$ |
| 0.3, 0.300001 | 1.1969 | 0.37322 | $[26.796, 48.326]$ | $[-135.08, -61.886]$ |
| 0.4, 0.400001 | 3.7869 | 4.19530 | $[11.818, 28.517]$ | $[-73.026, -30.082]$ |
| 0.5, 0.500001 | 4.9160 | 6.1701 | $[-13.819, -6.4592]$ | $[18.387, 34.8]$ |
| 0.6, 0.600001 | 1.7342 | 2.53840 | $[-17.72, -3.2241]$ | $[8.7055, 41.964]$ |
| 0.7, 0.700001 | 4.7928 | 5.74470 | $[-16.15, -6.0621]$ | $[13.441, 36.899]$ |
| 0.8, 0.800001 | 5.0542 | 6.19650 | $[-20.256, -6.1962]$ | $[12.115, 45.704]$ |
| 0.9, 0.900001 | 1.5249 | 4.18960 | $[-6.1939, -0.93573]$ | $[0.38193, 12.601]$ |

Table 3: We show rigorous enclosures of the transversality conditions of Theorem 2.2 for the values $\hat{B} = \hat{C} = 1$, $p_x \in [0.4, 0.40001]$, for different interval values of $p_y$. The critical point $\tau^*$ and its derivatives are evaluated at the points $(x, y)$ shown in the 2nd and the 3rd columns.

Finally, we describe the implementation parameters of the CAP that lead to the result stated in Corollary 1.1. We take $\hat{B} = 10$ and $\hat{C} = 0.1$ and we divide the set $\mathcal{I} = [0.1, 0.9] \times [0.5, 0.9]$ in subsets of size $10^{-4} \times 10^{-4}$. For every subset, we use Lemma 6.1 with $\alpha = 20$ and to enclose the finite integrals we use Simpson’s rule with 130 subintervals. To obtain the critical point and its derivatives, we use the approach described in Section 6.2 with $\hat{B} = 100$. For all these sets we obtain that the function $\Delta_1 \Delta_3 - \Delta_2^2$ does not vanish in $\mathcal{I}$, and the function $\hat{\Delta}_1 \hat{\Delta}_4 - \hat{\Delta}_2 \hat{\Delta}_3$ only vanishes on the resonant line $p_x = p_y$.

Acknowledgements

The authors are very grateful to A. Delshams, M. Guardia, A. Haro, G. Huguet, R. de la Llave, and T.M. Seara for useful discussions and suggestions. We especially want to thank T.M. Seara for her patience and kindness answering several questions on the papers [16][17][18]. The authors are supported by the ERC Starting Grant 335079. This work is supported in part by the ICMAT–Severo Ochoa grant SEV-2011-0087 and the grants MTM2012-3254 (A.L.) and 2014SGR1145 (A.L.).

References

[1] V.I. Arnold, Instability of dynamical systems with several degrees of freedom. Sov. Math. Dokl. 5 (1964) 581–585.
[2] V.I. Arnold, Sur la topologie des écoulements stationnaires des fluides parfaits. C. R. Acad. Sci. Paris 261 (1965) 17–20.
[3] P. Bernard, The dynamics of pseudographs in convex Hamiltonian systems. J. Amer. Math. Soc. 21 (2008) 615–669.
[4] P. Bernard, V. Kaloshin and K. Zhang, Arnold diffusion in arbitrary degrees of freedom and crumpled 3-dimensional normally hyperbolic invariant cylinders. Preprint.
[5] S. Bolotin and D. Treschev, Unbounded growth of energy in nonautonomous Hamiltonian systems. Nonlinearity 12 (1999) 365–388.
[6] M. Capinski, M. Gidea and R. de la Llave, Arnold diffusion in the elliptic circular restricted three body problem. Preprint.
[7] S. Chandrasekhar and L. Woltjer, On force-free magnetic fields. Proc. Natl. Acad. Sci. 44 (1958) 285–289.

[8] C.Q. Cheng, Arnold diffusion in nearly integrable hamiltonian systems. Preprint.

[9] L. Chierchia and G. Gallavotti, Drift and diffusion in phase space. Ann. Inst. H. Poincaré Phys. Théor. 60 (1994) 1–144.

[10] S. Childress, New solutions of the kinematic dynamo problem. J. Math. Phys. 11 (1970) 3063–3076.

[11] C.Q. Cheng and J. Yan, Existence of diffusion orbits in a priori unstable Hamiltonian systems. J. Differential Geom. 67 (2004) 457–517.

[12] M. Capinski and P. Zgliczynski, Transition Tori in the Planar Restricted Elliptic Three Body Problem. Nonlinearity 24 (2011) 1395–1432.

[13] A. Delshams, M. Gidea and P. Roldán. Arnold’s mechanism of diffusion in the spatial circular restricted three-body problem: A semi-numerical argument. Preprint.

[14] A. Delshams and P. Gutiérrez, Splitting potential and Poincaré-Melnikov method for whiskered tori in Hamiltonian systems. J. Nonlinear Sci. 10 (2000) 433–476.

[15] A. Delshams, R. de la Llave and T.M. Seara, A geometric approach to the existence of orbits with unbounded energy in generic periodic perturbations by a potential of generic geodesic flows of $\mathbb{T}^2$. Comm. Math. Phys. 209 (2000) 353–392.

[16] A. Delshams, R. de la Llave and T.M. Seara, A geometric mechanism for diffusion in hamiltonian systems overcoming the large gap problem: heuristics and rigorous verification on a model. Mem. Amer. Math. Soc. 2006.

[17] A. Delshams, R. de la Llave and T.M. Seara, Geometric properties of the scattering map of a normally hyperbolic invariant manifold. Adv. Math. 217 (2008) 1096–1153.

[18] A. Delshams, R. de la Llave and T.M. Seara, Instability of high dimensional Hamiltonian systems: multiple resonances do not impede diffusion. Preprint.

[19] A. Delshams and G. Huguet, Geography of resonances and Arnold diffusion in a priori unstable Hamiltonian systems. Nonlinearity 22 (2009) 1997–2077.

[20] A. Delshams and G. Huguet, A geometric mechanism of diffusion: Rigorous verification in a priori unstable Hamiltonian systems. J. Differential Equations 250 (2011) 2601–2623.

[21] A. Delshams, V. Kaloshin, A. de la Rosa and T.M. Seara. Global instability in the elliptic restricted three body problem. Preprint.

[22] J. Féjoz, M. Guardia, V. Kaloshin and P. Roldán, Diffusion along mean motion resonance in the restricted planar three-body problem. J. Eur. Math. Soc. in press.

[23] N. Fenichel, Persistence and smoothness of invariant manifolds for flows. Indiana Univ. Math. J. 21 (1971/1972) 193–226.

[24] N. Flyer, B. Fornberg, S. Thomas and B.C. Low, Magnetic field confinement in the solar corona I: force-free magnetic fields. Astrophys. J. 606 (2004) 1210–1222.

[25] V. Gelfreich and D. Turaev, Unbounded energy growth in Hamiltonian systems with a slowly varying parameter. Comm. Math. Phys. 283 (2008) 769–794.
[26] M. Gidea and R. de la Llave, Topological methods in the instability problem of Hamiltonian systems. Discrete Contin. Dyn. Syst. A 14 (2006) 295–328.

[27] M. Gidea and C. Robinson, Diffusion along transition chains of invariant tori and Aubry-Mather sets. Ergodic Theor. & Dynam. Sys. 33 (2013) 1401–1449.

[28] M. Gidea, R. de la Llave and T.M. Seara, A general mechanism of diffusion in Hamiltonian systems: qualitative results. Preprint.

[29] A.D. Gilbert, Magnetic field evolution in steady chaotic flows. Phil. Trans. R. Soc. Lond. A 339 (1992) 627–656.

[30] M. Guzzo, E. Lega and C. Froeschlé, A numerical study of Arnold diffusion in a priori unstable systems. Comm. Math. Phys. 290 (2009) 557–576.

[31] W. Hofschuster and W. Kraemer, A Fast Public Domain Interval Library in ANSI C. Proceedings of the 15th IMACS World Congress on Scientific Computation 2 (1997) 395–400.

[32] V. Kaloshin and M. Saprykina, An example of a nearly integrable Hamiltonian system with a trajectory dense in a set of maximal Hausdorff dimension. Comm. Math. Phys. 315 (2012) 643–697.

[33] V. Kaloshin and K. Zhang, A strong form of Arnold diffusion for two and a half degrees of freedom. Preprint.

[34] V. Kaloshin and K. Zhang, Partial averaging and dynamics of the dominant Hamiltonian, with applications to Arnold diffusion. Preprint.

[35] V. Kaloshin and K. Zhang, A strong form of Arnold diffusion for three and a half degrees of freedom. Preprint.

[36] A. Luque and D. Peralta-Salas, Motion of charged particles in ABC magnetic fields. SIAM J. Appl. Dyn. Syst. 12 (2013) 1889–1947.

[37] M.W. Hirsch, C.C. Pugh and M. Shub, Invariant Manifolds. Lecture Notes in Math. 583, Springer-Verlag, Berlin, 1977.

[38] J.N. Mather, Arnold diffusion I. Announcement of results. J. Math. Sci. 124 (2004) 5275–5289.

[39] R. Moeckel, Transition tori in the five-body problem. J. Differential Equations 129 (1996) 290–314.

[40] J.B. Taylor, Relaxation and magnetic reconnection in plasmas. Rev. Mod. Phys. 58 (1986) 741–763.

[41] D. Treschev, Multidimensional symplectic separatrix maps. J. Nonlinear Sci. 12 (2002) 27–58.

[42] D. Treschev, Evolution of slow variables in a priori unstable Hamiltonian systems. Nonlinearity 17 (2004) 1803–1841.

[43] D. Treschev, Arnold diffusion far from strong resonances in multidimensional a priori unstable Hamiltonian systems. Nonlinearity 9 (2012) 2717–2757.

[44] W. Tucker. Validated Numerics: A Short Introduction to Rigorous Computations. Princeton University Press, Princeton, 2011.

[45] H.P. Warren, A. Bhattacharjee and M.E. Mauel, On Arnold diffusion in a perturbed magnetic dipole field. Geophys. Res. Lett. 19 (1992) 941–944.

[46] G.M. Zaslavskii, M.Y. Zakharov, R.Z. Sagdeev, D.A. Usikov and A.A. Chernikov, Stochastic web and diffusion of particles in a magnetic field. Sov. Phys. JETP 64 (1986) 294–303.

[47] K. Zhang, Speed of Arnold diffusion for analytic Hamiltonian systems. Invent. Math. 186 (2011) 255–290.

[48] Y. Zheng, Arnold diffusion for a priori unstable systems and a five-body problem. Preprint.