TWO-DIMENSIONAL MODULI SPACES OF RANK 2 HIGGS BUNDLES
OVER $\mathbb{C}P^1$ WITH ONE IRREGULAR SINGULAR POINT

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ABSTRACT. We give a complete description of the two-dimensional moduli spaces of stable Higgs bundles of rank 2 over $\mathbb{C}P^1$ with one irregular singular point, having a regular leading-order term, and endowed with a generic compatible parabolic structure such that the parabolic degree of the Higgs bundle is 0. Our method relies on elliptic fibrations of the rational elliptic surface, an equivalence of categories between irregular Higgs bundles and some sheaves on a ruled surface, and an analysis of stability conditions.

1. INTRODUCTION

In this article we consider 2 complex dimensional moduli spaces of singular Higgs bundles over $\mathbb{C}P^1$ with irregular singularities. It is known [5] that if one fixes finitely many points on a curve $C$ and suitable polar parts for a Higgs bundle near those points, then one gets a holomorphic symplectic moduli space of Higgs bundles over $C$ with the given asymptotic behaviour at the singularities. In some cases these spaces turn out to be of complex dimension 2. Our aim in this article is to give a complete description of the two-dimensional holomorphic symplectic moduli spaces of rank 2 Higgs bundles over $\mathbb{C}P^1$ having a unique pole of order 4 as singularity, and regular leading-order term. One needs to distinguish two cases, depending on whether the leading-order term is a regular semi-simple endomorphism (untwisted case), or has non-vanishing nilpotent part (twisted case). As we will see, the corresponding fiber at infinity of the Hitchin fibration is $\tilde{E}_7$ in the untwisted case and $\tilde{E}_8$ in the twisted case. The corresponding de Rham moduli spaces of irregular connections are related to the Painlevé II (untwisted case) and Painlevé I (twisted case) equations. The polar part of irregular Higgs bundles depends on some complex parameters

(1) $a_\pm, b_\pm, c_\pm, \lambda_\pm \in \mathbb{C}, \quad a_+ \neq a_-$

in the untwisted case and

(2) $b_{-8}, \ldots, b_{-3} \in \mathbb{C}, \quad b_{-7} \neq 0$

in the twisted case, see Subsection 2.3.

In the following statements we let $\mathcal{M}$ be a moduli space of rank 2, parabolic degree 0 stable parabolic irregular Higgs bundles over $\mathbb{C}P^1$ with a unique pole of order 4 with a regular leading-order term and fixed parameters (1) or (2). For details, see Subsection 2.3. Observe that by the genericity assumption on the parabolic structure, the degree of the underlying vector bundle is necessarily equal to $-1$. It is expected that moduli spaces of irregular Higgs bundles with fixed polar parts underlie completely integrable systems with Abelian varieties as generic fibers. If $\dim_{\mathbb{C}}(\mathcal{M}) = 2$ this would then imply that $\mathcal{M}$ is an elliptic fibration over a curve. Our results below will confirm this expectation, with one singular fiber of type $\tilde{E}_7$ (untwisted case) or $\tilde{E}_8$ (twisted case). On the other hand, there are several possibilities for the other singular fibers [15].

In [16], a general equivalence of categories between irregular Higgs bundles and some pure 1-dimensional rank one sheaves on a ruled surface was shown to hold, assuming that the leading order term of the Higgs field is semi-simple. We will use this equivalence to prove our first result, giving a complete description of these further singular fibers in the
untwisted case in terms of the parameters of (1). (For the definition of various types of singular fibers see [11] or Section 3.)

**Theorem 1.1.** Assume that the polar part of the Higgs bundles is untwisted. Then $\mathcal{M}$ is birational to the complement of the fiber at infinity (of type $\tilde{E}_7$) in an elliptic fibration of the rational elliptic surface such that the set of other singular fibers of the fibration is:

1. a type III fiber if $\Delta = 0$ and $\lambda_+ = 0$;
2. a type II and an $I_1$ fiber if $\Delta = 0$ and $\lambda_+ \neq 0$;
3. an $I_2$ and an $I_1$ fiber if $\Delta \neq 0$ and $\lambda_+ = 0$;
4. and three $I_1$ fibers otherwise,

where

$$\Delta = \left( (b_6 - b_+) - 4(a_+ - a_+) (c_- - c_+) \right)^3 - 432 (a_+ - a_+) ^4 \lambda_+^2.$$  

Notice that according to [15, Proposition 4.2] this is a complete list of the possible singular fibers of elliptic fibrations on the rational elliptic surface without multiple fibers and having a singular fiber of type $\tilde{E}_7$. The proof of Theorem 1.1 is given in Sections 4 and 5, where an explicit description of the Hitchin fibers corresponding to the reducible singular curves in the fibration is given. In Section 5, we also work out the stability analysis in the case of rank 2 irregular Higgs bundles in degree 0 case; strictly speaking we do not need this analysis to prove the theorem, nevertheless we found it interesting enough to include it.

Similarly to Theorem 1.1, the next theorem provides a complete description of the singular fibers of the fibration in the twisted case, in terms of the parameters (2).

**Theorem 1.2.** Assume that the polar part of the Higgs bundles is twisted. Then $\mathcal{M}$ is birational to the complement of the fiber at infinity (of type $\tilde{E}_8$) in an elliptic fibration of the rational elliptic surface such that the set of other singular fibers of the fibration is:

1. type II fiber if $D = 0$;
2. and two type $I_1$ fibers otherwise,

where

$$D = \left( b_6^2 + 4b_- \right)^2 - 24b_- \lambda_+ (b_6 b_- + 2b_-^3).$$

Notice again that according to [15, Section 4.1] this is a complete list of the possible singular fibers of elliptic fibrations without multiple fibers and having a singular fiber of type $\tilde{E}_8$. We prove Theorem 1.2 in Section 6.

Now let us give an outline of the paper. In Section 2, we fix our notations and provide some well-known background material used later. Namely, in Subsection 2.1, we introduce the Hirzebruch surface of index 2. In Subsection 2.2, we gather some results about the compactified Jacobian of singular curves, with special emphasis on the compactifications of curves of type $I_1$ and $I_1$ and Oda–Seshadri stability in the case of a curve of type $I_2$.

In Subsection 2.3, we discuss rank 2 irregular Higgs bundles on $\mathcal{O} \mathbb{P}^1$ in the untwisted and twisted case, compatible parabolic structures and stability, and in Subsection 2.4, we link the previous two subsections by presenting spectral data of irregular Higgs bundles and the irregular Hitchin map.

In Section 3, we give a detailed analysis of elliptic fibrations on the rational elliptic surface with one singular fiber of type $\tilde{E}_7$ or of type $\tilde{E}_8$.

In Section 4, we first construct the rational surface $Y$ governing the moduli space $\mathcal{M}$ in the untwisted case. Quoting the general categorical equivalence of [16], we then achieve the proof of Theorem 1.1 up to the stability analysis of irregular Higgs bundles with reducible spectral curve. This latter, in turn, is carried out in Section 5. The analysis of the case of a type $I_2$ curve proceeds along the lines of Section 4 of Schaub’s paper [14].

We start Section 5 by some straightforward computations expressing the coefficients of the Puiseux-expansion of the eigenvalues of the Higgs field in terms of the parameters (2). We then go on to construct the rational surface $Y$ governing the moduli space $\mathcal{M}$ in the twisted case. Next, in Proposition 6.4 we give an analogue of the general categorical
equivalence of [16] between twisted irregular Higgs bundles and some pure 1-dimensional
rank one sheaves on \( Y \). This then allows us to prove Theorem [1.2].

Let us make a few remarks on related literature. In the paper [13], spaces of initial
conditions for Painlevé equations are studied using rational surfaces and root systems. In
particular, in Appendix B loc. cit. configurations of curves similar to ours appear. In [9]
the singular fiber of the Hitchin map corresponding to a singular spectral curve of type \( A_k \)
determined. Our Section 5 is reminiscent to (special cases of) their results. The work
[8] (in particular, Section 9 thereof) undertakes the analysis of wall-crossing phenomena
related to Hitchin systems with irregular singularities. Finally, let us mention that we hope
to treat the 2-dimensional moduli spaces of rank 2 irregular Higgs bundles over \( \mathbb{C}P^1 \)
with several marked points in the future.

2. PREPARATORY MATERIAL

We denote by \( O \) and \( K \) the sheaf of regular functions and the canonical sheaf respectively. We identify holomorphic line bundles over \( \mathbb{C}P^1 \) with their sheaves of sections. We
equally let \( O(1) \) stand for the ample line bundle and for \( n \in \mathbb{Z} \) set \( K(n) = K \otimes O(n) \).

2.1. The second Hirzebruch surface and the basic birational map. Throughout the
paper we will consider the Hirzebruch surface

\[
X = \mathbb{P}(K(4) \otimes O),
\]

the fiberwise projectivization of the rank 2 holomorphic line bundle \( K(4) \otimes O \) over \( \mathbb{C}P^1 \).
Given that the line bundle \( K(4) \) is isomorphic to \( O(2) \), we get that \( X \) is biholomorphic to
the Hirzebruch surface of index 2. The surface \( X \) naturally fibers over \( \mathbb{C}P^1 \) with fibers
isomorphic to \( \mathbb{C}P^1 \):

\[
p : X \rightarrow \mathbb{C}P^1.
\]

This morphism is sometimes called the ruling. We denote its generic fibre by \( F \) and the
homology class of \( F \) by \( [F] \in H_2(X; \mathbb{Z}) \).

It is known that \( X \) admits two further remarkable closed curves denoted by \( C_0, C_\infty \)
and called the 0-section and section at infinity, respectively. Both \( C_0 \) and \( C_\infty \) are sections
of \( p \), in particular they are biholomorphic to \( \mathbb{C}P^1 \). Specifically, if we let \( 0 \) stand for the
0-section of \( K(4) \) and \( 1 \) stand for the constant section equal to 1 of \( O \) then

\[
C_0 = \{[0_q : 1_q] \mid q \in \mathbb{C}P^1\},
\]

where the subscripts \( q \) mean evaluation of the given sections at \( q \), and as usual \( [\cdot : \cdot] \) denote
projective coordinates. Locally, the section at infinity can be defined similarly, however it
is not possible to pick a single section of \( K(4) \) because any such section vanishes at two
points of \( \mathbb{C}P^1 \). So, letting \( \kappa \) stand for a local non-vanishing section of \( K(4) \) on some open
set \( U \subset \mathbb{C}P^1 \), we define

\[
C_\infty \cap p^{-1}(U) = \{[\kappa_q : 0_q] \mid q \in U\}
\]

where \( 0 \) stands for the 0-section of \( O \). It can be checked that if \( V \) is another open subset of
\( \mathbb{C}P^1 \) with a non-vanishing section \( \mu \) then these definitions of \( C_\infty \) agree on \( U \cap V \), hence
these formulas give a well-defined curve. We denote the homology classes defined by these
sections by \([C_0], [C_\infty]\).

The second homology \( H_2(X; \mathbb{Z}) \) is generated by the classes of any two of the above
three curves, the relation between them being

\[
[C_\infty] = [C_0] - 2[F].
\]

The intersection pairing is given by the formulas

\[
[C_\infty]^2 = -2, \quad [C_0]^2 = 2, \quad [F]^2 = 0, \quad [C_\infty] : [C_0] = 0, \quad [C_\infty] : [F] = [C_0] : [F] = 1.
\]
As it is well-known, $X$ is birational to $\mathbb{C}P^2$ by the morphisms
\[
\begin{array}{c}
\tilde{X} \\
\downarrow \omega
\end{array}
\quad\begin{array}{c}
X \\
\downarrow \omega
\end{array}
\quad\begin{array}{c}
\mathbb{C}P^2
\end{array}
\]
where $\tilde{X} \to X$ is the blow-up of a point $(\kappa_q : 1_q) \in X \setminus C_\infty$ for any $q \in U \subset \mathbb{C}P^1$ and local section $\kappa \in H^0(U; K(4))$, and $\tilde{X} \to \mathbb{C}P^2$ is the blow-up of two infinitely close points on $\mathbb{C}P^2$. For sake of concreteness, we may take the locus of this reduced point to be $(0 : 0 : 1)$. The proper transform of the fiber $F_q$ of the map $p$ of Equation (3) over $q \in \mathbb{C}P^1$ is the exceptional divisor of the second blow-up of $\mathbb{C}P^2$. On the other hand, the proper (which in this case is the same as the total) transform of $C_\infty$ in $\tilde{X}$ is equal to the proper transform of the exceptional divisor of the first blow-up of $\mathbb{C}P^2$ under the second blow-up. Throughout the paper we will use $\omega$ to go back and forth between $X$ and $\mathbb{C}P^2$.

2.2. Elliptic fibrations and their relative compactified Picard schemes. In this section we summarize some facts concerning families of curves that we will need in the paper.

Let $B$ be a scheme over $\mathbb{C}$ and $X \to B$ be a flat projective map of relative dimension 1. For a geometric point $b$ of $B$ we call the fiber at $b$ the base change of $X$ under the inclusion map $b \to B$, and we denote the fiber at $b$ by $X_b$. Throughout this section we assume that for each geometric point $b$ of $B$ the fiber $X_b$ is reduced. We furthermore assume that each singular fiber is of the following types:

1. a simple nodal rational curve $I_1$;
2. two smooth rational curves meeting transversely in two distinct points $I_2$;
3. a cuspidal rational curve $II$.

(Again, for the definition of the various singularities appearing in elliptic fibrations see [11] or Section 3.) In this situation there exists a relative compactified Picard scheme $\text{Pic}^{\mathbf{G}_m|B}$ parametrizing torsion-free sheaves $S$ of $\mathcal{O}_{X_b}$-modules of rank 1. It naturally decomposes according to the (total) degree $\delta$ of $S$ as
\[
\text{Pic}^{\mathbf{G}_m|B} = \bigoplus_{\delta \in \mathbb{Z}} \text{Pic}^{\delta|B}
\]
where the degree is defined by
\[
\deg(S) = \chi(S) - \chi(\mathcal{O}_{X_b})
\]
with $\chi$ standing for Poincaré characteristic. For types $I_1$ and $II$ the scheme $\text{Pic}^{\mathbf{G}_m|B}$ was constructed by [7]. The $I_2$ case is a particular case of [12]; we will come back to this case in Subsection 2.2.1. In order to introduce the ideas to be used later in various other situations, let us give here the description of (5) in the cases $I_1$ and $II$ according to [12, Section 13] and [6, Chapter 4]. Our argument can be made more precise using generalised parabolic line bundles on the normalization introduced by [4].

**Proposition 2.1.** (Oda–Seshadri [12], Altman–Kleiman [2])

1. Let $X_b$ be a curve of type $I_1$. Then for any $\delta \in \mathbb{Z}$ the scheme $\text{Pic}^{\delta|B}$ is isomorphic to a curve of type $I_1$.
2. Let $X_b$ be a curve of type $II$. Then for any $\delta \in \mathbb{Z}$ the scheme $\text{Pic}^{\delta|B}$ is isomorphic to a curve of type $II$.

**Proof.** We only treat part (1). Let $\pi : \tilde{X}_b \to X_b$ stand for the normalization of $X_b$. Then $\tilde{X}_b$ is a smooth rational curve. Let us denote by $x_0 \in X_b$ the only singular point and by $0, \infty \in \tilde{X}_b$ its preimages under the map $\pi$. Then a
degree 0 line bundle on $X_b$ is the same thing as a line bundle $L$ of degree 0 on $\tilde{X}_b$ endowed with an isomorphism

$$L_0 \cong L_\infty,$$

where $L_p$ denotes the fiber of $L$ over $p \in \tilde{X}_b$. Now there is just one degree 0 holomorphic line bundle on $\tilde{X}_b$, namely $L = O_{\tilde{X}_b}$, so the data above reduces to just the identification of the fibers. This in turn can be described by the image $\lambda \in \mathbb{C}^* \subset L_\infty$ of $1 \in L_0$. Intrinsically $\lambda$ can be understood as an element of the projective line

$$\mathbb{P}(L_0 \oplus L_\infty).$$

Let us denote by $L(\lambda)$ the degree 0 line bundle on $X_b$ obtained by the above identification of the fibers; clearly, for $\lambda' \neq \lambda$ the line bundle $L(\lambda')$ is not isomorphic to $L(\lambda)$. To sum up, the universal line bundle on $X_b$ is given by

$$L(\cdot) \to \mathbb{C}^* \times X_b \subset \mathbb{P}(L_0 \oplus L_\infty) \times X_b.$$

Our aim is to find the limit of $L(\lambda)$ as $\lambda \to 0$ or $\infty$ in $\mathbb{P}(L_0 \oplus L_\infty)$. In the case $\lambda = 0$ the limit consists of a line bundle on $\tilde{X}_b$ with an identification of the fiber $L_0$ to 0 $\in L_\infty$; said differently, there is a short exact sequence

$$0 \to L(0) \to \pi_* L \to L_0 \to 0,$$

hence $L(0) = \pi_* \mathcal{O}_{\tilde{X}_b}(-\{0\})$. Similarly, the limit $\lambda \to \infty$ fits into the short exact sequence

$$0 \to L(\infty) \to \pi_* L \to L_\infty \to 0,$$

hence $L(\infty) = \pi_* \mathcal{O}_{\tilde{X}_b}(-\{\infty\})$. As $\tilde{X}_b$ is of genus 0, the bundles $\mathcal{O}_{\tilde{X}_b}(-\{0\})$ and $\mathcal{O}_{\tilde{X}_b}(-\{\infty\})$ are isomorphic to each other, therefore so are their direct images by $\pi$. The statement in the case of $I_1$ now follows.

As for part [2], see [2] Theorem 18. \end{proof}

### 2.2.1. Oda–Seshadri stability for $I_2$ curves.

In this subsection we continue the summary of known results concerning compactified Picard schemes. For families with singular fibers $I_n$ for $n \geq 2$ (and more generally, for reduced curves with only simple nodes as singular points) the compactifications of the Picard scheme were studied in [12]. In this case, the degree of the restriction of $\mathcal{S}$ to each component of $X_b$ needs to be centered about some values. Let us restrict our attention to the case $n = 2$ and denote by $X_+, X_-$ the irreducible components of $X_b$. These are smooth curves of genus 0, attached at two points. We may assume for ease of notations that the common points are 0, $\infty \in X_\pm$ so that 0 $\in X_+$ is identified with 0 $\in X_-$ and $\infty \in X_+$ is identified with $\infty \in X_-$. We will also denote by 0 and $\infty$ the point of $X_b$ obtained by the above identification. The curve

$$\tilde{X}_b = X_+ \amalg X_-$$

is called the normalization of $X_b$. There is an obvious map

$$\sigma : \tilde{X}_b \to X_b.$$

It turns out that in order to get a moduli scheme one needs to impose a further condition of stability on the sheaves $\mathcal{S}$ that we wish to parametrize. This stability condition depends on some parameters $(\phi_+, \phi_-) \in \mathbb{R}^2$ satisfying

$$\phi_+ + \phi_- = 0.$$

For a torsion-free coherent sheaf $\mathcal{S}$ of $\mathcal{O}_{X_b}$-modules of rank 1 let us set

$$\mathcal{L}(\mathcal{S}) = \sigma^* \mathcal{S}/\mathcal{T}or^{\mathcal{O}_{\tilde{X}_b}}(\sigma^* \mathcal{S})$$

with $\mathcal{T}or^{\mathcal{O}_{\tilde{X}_b}}(\sigma^* \mathcal{S})$ denoting the torsion part of the $\mathcal{O}_{\tilde{X}_b}$-module $\sigma^* \mathcal{S}$, and for $i \in \{ \pm \}$ define

$$\delta_i = \deg(\mathcal{L}(\mathcal{S})|_{X_i}),$$
where deg stands for the degree with respect to the standard polarization on $X_i$. Notice that for any $i$ there exists a canonical morphism $S \to \mathcal{L}(S)|_{X_i}$ from the composition

$$S \to \sigma^* S \to \mathcal{L}(S) \to \mathcal{L}(S)|_{X_i}.$$  

(10)

Setting

$$J(S) = \{ j \in \{0, \infty \} : \text{ $S$ is locally free near } j \},$$

we have a short exact sequence of coherent sheaves

$$0 \to S \to \mathcal{L}(S)|_{X_i} \oplus \mathcal{L}(S)|_{X_i} \to \oplus_{j \in J(S)} C \to 0,$$

hence

$$\chi(S) + |J(S)| = \chi(\mathcal{L}(S)|_{X_i}) + \chi(\mathcal{L}(S)|_{X_i}).$$

(12)

Applying this formula to $S = O_{X_i}$ we get

$$\chi(0_{X_i}) + 2 = \chi(0_{X_i}) + \chi(0_{X_i}).$$

(13)

Now subtracting (13) from (12) and taking into account definitions (6) and (9), we infer

$$\deg(S) = \delta_i + \delta_j + 2 - |J(S)|.$$  

(14)

The construction of Oda and Seshadri uses the dual graph $\Gamma = (V, E)$ associated to $X_i$; by definition, $V = \{X_i, X_\perp\} = \{+, -\}$ is the set of all connected components of the normalization $\tilde{X}_b$, $E = \{0, \infty\}$ is the set of all double points of $X_b$, and an edge $j$ is adjacent to a vertex $i$ if and only if the double point corresponding to $j$ lies on the connected component corresponding to $i$. For $i \in \{\pm\}$ Oda and Seshadri define the value

$$d(J - J(S))_i,$$

as the number of edges $j \in \{0, \infty\}$ such that $i$ is one of the end-points of $j$ and $S$ is not locally free at $j$. As both $i = \pm$ are end-points of both edges $j \in \{0, \infty\}$, it is obvious from this definition that the quantity $d(J - J(S))_i$ does not depend on $i \in \{\pm\}$, and we have the equality

$$d(J - J(S))_i = |J - J(S)| = 2 - |J(S)|.$$  

Furthermore, for any non-trivial subset $I' \subset \{\pm\}$, Oda and Seshadri set $I'' = \{\pm\} - I'$ and denote by

$$\langle \delta_J(S), v(I''), \delta_J(S), v(I'') \rangle$$

the number of edges $j \in \{0, \infty\}$ such that $S$ is locally free near $j$ and has one end-point in $I'$ and the other one in $I''$. As any non-trivial $I' \subset \{\pm\}$ is necessarily of the form $I' = \{i\}$ for some $i \in \{\pm\}$ and every edge has both vertices $i$ as end-point, clearly the last condition on the edges is vacuous. Hence (15) simply gives the number of edges such that $S$ is locally free near $j$, said differently we find

$$\langle \delta_J(S), v(I''), \delta_J(S), v(I'') \rangle = |J(S)|.$$  

With these preliminaries Oda and Seshadri call $S$ $\phi$-semistable if for both $i \in \{\pm\}$ the inequalities

$$\delta_i + \frac{1}{2}d(J - J(S))_i - \phi_i \leq \frac{\langle \delta_J(S), v(I - \{i\}), \delta_J(S), v(I - \{i\}) \rangle}{2}$$

are fulfilled, and $\phi$-stable if the corresponding strict inequalities hold. Plugging the formulas found above into this inequality we find that in the case of an $I_2$ curve $X_0$ the semi-stability condition reads as

$$\delta_i - \phi_i \leq |J(S)| - 1,$$

(16)

and stability is defined by the corresponding strict inequality. Taken into account the equality of (14), this may be equivalently rewritten as

$$\delta - 1 < \delta_i - \phi_i \leq |J(S)| - 1.$$
The compactified Picard scheme
\[ \text{Pic}^\delta \phi_{X_b} \]
of degree \( \delta \in \mathbb{Z} \) is then defined as the scheme parametrizing \( \phi \)-stable torsion-free sheaves of degree \( \delta \) over \( X_b \). More precisely, Oda and Seshadri define the Picard functor of \( \phi \)-stable torsion-free sheaves and they show that it is representable by a scheme.

2.3. Irregular Higgs bundles. We study rank 2 irregular Higgs bundles \( (E, \theta) \) defined over \( \mathbb{C}P^1 \), where \( E \) is a rank 2 vector bundle and \( \theta \) is a meromorphic section of \( \mathcal{E}nd(E) \otimes K \) called the Higgs field. We set \( \deg(E) = d \).

We will limit ourselves to the case where \( \theta \) has a single pole \( q \) of order 4:
\[ \theta : E \to E \otimes K(4 \cdot \{q\}). \]
Up to an isomorphism of \( \mathbb{C}P^1 \) we may fix the point \( q = 0 \in \mathbb{C} \) and use the local coordinate \( z_1 \) vanishing at \( q \). Then over \( \mathbb{C} \) the line bundle \( K(4 \cdot \{q\}) \) admits the trivializing section
\[ \kappa = \frac{dz_1}{z_1^4}. \]
In some trivialization of \( E \) near \( q \) we have
\[ \theta = \sum_{n \geq -4} A_n z_1^n \otimes dz_1, \]
where \( A_n \in \text{gl}(2, \mathbb{C}) \). This may be rewritten as
\[ \theta = \sum_{n \geq 0} A_{n-4} z_1^n \otimes \kappa \]
Introduce its characteristic polynomial in the indeterminate \( \zeta \in H^0(X, K(4 \cdot \{q\})) \) as
\[ \chi_\theta(\zeta) = \det(\zeta I_E - \theta) = \zeta^2 + \zeta F(z_1) + G(z_1), \]
for some
\[ F \in H^0(\mathbb{C}P^1, K(4 \cdot \{q\})), \quad G \in H^0(\mathbb{C}P^1, K(4 \cdot \{q\}) \otimes 2). \]
Said differently, \( F \) is a meromorphic differential and \( G \) is a meromorphic quadratic differential. Now, over \( \mathbb{C} \) the line bundle \( K(4 \cdot \{q\}) \) has the trivialization \( \kappa \), and this induces a trivialization \( \kappa^2 \) of \( K(4 \cdot \{q\}) \otimes 2 \). Let us set \( \vartheta = \sum_{n \geq 0} A_n z_1^n \) so that we have
\[ \theta = \vartheta \otimes \kappa \]
and introduce \( z_2 \) by \( \zeta = z_2 \otimes \kappa \). If we now factor \( \kappa \) in (19), then the characteristic polynomial may be rewritten as
\[ \chi_\theta(z_2) = \det(z_2 I_E - \vartheta) = z_2^2 + z_2 f(z_1) + g(z_1), \]
with
\[ F = f \kappa, \quad G = g \kappa^2. \]
Now, as \( K(4 \cdot \{q\}) \cong \mathcal{O}(2) \), the coefficients \( f \) and \( g \) are polynomials in \( z_1 \) of degree 2 and 4, respectively.

In this section we explain how to fix the polar parts of \( \theta \) depending on whether its leading-order term is regular semi-simple (the so-called untwisted case) or has a non-trivial nilpotent part (twisted case).
2.3.1. Untwisted case. In this case we will fix scalars \( a_\pm \in \mathbb{C} \) with \( a_+ \neq a_- \) and assume that the leading-order term of \( \theta \) (i.e., the coefficient \( A_{-4} \) of \( z_1^{-4} \) in its Laurent series) is semi-simple with eigenvalues \( a_\pm \). Then there exists a polynomial gauge transformation in the indeterminate \( z_1 \) that transforms \( \theta \) into the form

\[
\theta = \left[ z_1^{-4} \begin{pmatrix} a_+ + b_+ z_1 + c_+ z_1^2 + \lambda_+ z_1^3 \\ 0 \\ a_- + b_- z_1 + c_- z_1^2 + \lambda_- z_1^3 \end{pmatrix} + \cdots \right] \otimes dz_1
\]

in some local trivialization of \( \mathcal{E} \) near \( q \) where the dots stand for higher-order matrices in \( z_1 \). Indeed, up to applying a constant base change we may assume that \( A_{-4} \) is diagonal. Furthermore the action of

\[
\gamma(z_1) = 1 + \gamma_n z_1^n
\]

on (18) is

\[
\gamma(z_1) \theta(z_1) \gamma(z_1)^{-1} = \left( A_{-4} z_1^{-4} + \cdots + A_{n-5} z_1^{n-5} + (A_{n-4} - \text{ad}_{A_{-4}}(g_n)) z_1^{n-4} + O(z_1^{n-3}) \right) \otimes dz_1,
\]

and since the image of \( \text{ad}_{A_{-4}} \) is the subspace of off-diagonal matrices we can successively apply such gauge transformations with \( n = 1, 2 \) and 3 to cancel the off-diagonal terms of \( A_{-3} \), then those of \( A_{-2} \) and finally those of \( A_{-1} \).

The matrices appearing in (20) are called the polar part of \( \theta \) at the singularity. From now on we assume that the constants \( a_\pm, b_\pm, c_\pm, \lambda_\pm \in \mathbb{C} \) appearing in (20) are fixed. A necessary condition for the existence of Higgs bundles with this polar part is given by the residue theorem which states that

\[
\lambda_+ + \lambda_- = 0.
\]

We therefore assume that the parameters are fixed so that this equality holds.

A generic parabolic structure compatible with \( (\mathcal{E}, \theta) \) is a choice

\[
(\alpha_+^q, \alpha_-^q) \in [0, 1)^2
\]

of two distinct numbers for the singular point \( q \in \mathcal{P}_{\text{red}} \); the scalars \( \alpha_q^\pm \) are called parabolic weights. Essentially, \( \alpha_q^\pm \) are associated to the \( \lambda_\pm \) in the above polar parts at \( q \), and they correspond to the flag

\[
\mathcal{E}_q \supset L_q^+ \supset \{0\}
\]

invariant under the polar part of \( \theta \). Notice that this flag is uniquely determined by the polar part, so the only choice for the parabolic structure is that of the weights \( \alpha_q^\pm \). A Higgs subbundle of \( (\mathcal{E}, \theta) \) is a pair \( (\mathcal{F}, \theta|_{\mathcal{F}}) \) with \( \mathcal{F} \) a holomorphic subbundle of \( \mathcal{E} \) such that

\[
\theta|_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes K(\mathcal{P}).
\]

One immediately sees that if this is the case then the fiber \( \mathcal{F}_q \) of \( \mathcal{F} \) at \( q \) must be one of the eigenlines \( L_q^\pm \). In particular, if \( (\mathcal{E}, \theta) \) is endowed with a compatible parabolic structure then any Higgs subbundle \( (\mathcal{F}, \theta|_{\mathcal{F}}) \) inherits a parabolic structure from \( (\mathcal{E}, \theta) \) in a natural way: according to whether \( \mathcal{F}_q = L_q^\pm \) we set

\[
\alpha_q(\mathcal{F}) = \alpha_q^\pm
\]

to be the parabolic weight of \( (\mathcal{F}, \theta|_{\mathcal{F}}) \) at \( q \in \mathcal{P}_{\text{red}} \). We then define

\[
\text{par-deg}(\mathcal{E}) = \deg(\mathcal{E}) + (\alpha_+^q + \alpha_-^q)
\]

and

\[
\text{par-deg}(\mathcal{F}) = \deg(\mathcal{F}) + \alpha_q(\mathcal{F}).
\]

We say that \( (\mathcal{E}, \theta) \) is \( \tilde{\alpha} \)-semistable if and only if for all Higgs subbundles \( (\mathcal{F}, \theta|_{\mathcal{F}}) \) we have

\[
\text{par-deg}(\mathcal{F}) \leq \frac{\text{par-deg}(\mathcal{E})}{2}
\]

and \( \tilde{\alpha} \)-stable if strict inequality holds. Observe that if \( \text{par-deg}(\mathcal{E}) = 0 \) then these conditions simplify to

\[
\text{par-deg}(\mathcal{F}) \leq 0.
\]
(respectively $<$. If $(\mathcal{F}, \theta|\mathcal{F})$ is a Higgs subbundle of $(\mathcal{E}, \theta)$ then $\theta$ also induces a morphism on the quotient vector bundle
\[ Q = \mathcal{E}/\mathcal{F}, \]
and we denote the resulting Higgs field by
\[ \theta|_Q : Q \to Q \otimes K(P). \]
In this situation we say that $(Q, \theta|_Q)$ is a quotient Higgs bundle of $(\mathcal{E}, \theta)$. Furthermore, if $(\mathcal{E}, \theta)$ is endowed with a compatible parabolic structure then it induces a parabolic structure on $Q$: if $\alpha_q(\mathcal{F}) = \alpha_q^+$ then we simply set
\[ \alpha_q(Q) = \alpha_q^+. \]
Just as above, we set
\[ \text{par-deg}(Q) = \deg(Q) + \alpha_q(Q). \]
By additivity of the degree, we have an equivalent definition of $\tilde{\alpha}$-stability in terms of quotients: namely, $(\mathcal{E}, \theta)$ is $\alpha$-semistable if and only if for any quotient Higgs bundle $(Q, \theta|_Q)$ we have
\[ \text{par-deg}(Q) \geq \frac{\text{par-deg}(\mathcal{E})}{2} \]
and $\alpha$-stable if strict inequality holds. Again, if $\text{par-deg}(\mathcal{E}) = 0$ then these conditions simplify to
\[ \text{par-deg}(Q) \geq 0 \]
(respectively $>$).

We will be interested in the moduli spaces
\[ \mathcal{M}^{(s)*} = \mathcal{M}^{(s)*}(\mathbb{C}P^1, q, a_q, b_q, c_q, \lambda_q, \alpha_q^+) \]
of $\alpha$-stable (resp. $\alpha$-semi-stable) irregular Higgs bundles on $\mathbb{C}P^1$ of 0 parabolic degree with the polar parts at $q$ as prescribed in (20), up to gauge equivalence. The general construction of moduli spaces $\mathcal{M}^s$ parametrizing isomorphism classes of stable objects was given in [5] using gauge theoretic methods. In particular, it is proved that if semi-stability is equivalent to stability and the adjoint orbits of the residues are closed, then the moduli space $\mathcal{M}^s$ is a complete hyper-Kähler manifold. On the other hand, in order to consider moduli spaces $\mathcal{M}^{ss}$ parametrizing equivalence classes of semi-stable objects one needs to slightly relax the notion of equivalence. Namely, to any strictly semi-stable object $(\mathcal{E}, \theta)$ it is possible to find a Jordan–Hölder filtration
\[ 0 \subset (\mathcal{E}_1, \theta_1) \subset (\mathcal{E}, \theta) \]
(in our case necessity of length 2) such that both $(\mathcal{E}_1, \theta_1)$ and $(\mathcal{E}_2, \theta_2)$ are stable (where $\mathcal{E}_2 = \mathcal{E}/\mathcal{E}_1$ and $\theta_2$ is the Higgs field on $\mathcal{E}_2$ induced by $\theta$). We then call
\[ (\mathcal{E}_1, \theta_1) \oplus (\mathcal{E}_2, \theta_2) \]
the associated graded irregular Higgs bundle of $(\mathcal{E}, \theta)$ and we call $(\mathcal{E}, \theta)$ and $(\mathcal{E}', \theta')$ S-equivalent if their associated graded irregular Higgs bundles agree. This definition reduces to isomorphism in the case of stable irregular Higgs bundles. We expect that there exists a quasi-projective smooth coarse moduli scheme $\mathcal{M}^{ss}$ parametrizing S-equivalence classes of semi-stable irregular Higgs bundles.
2.3.2. Twisted case. We now consider the case where $A_{-4}$ has non-trivial nilpotent part. In a convenient trivialization we then have

$$A_{-4} = \begin{pmatrix} b_{-8} & 1 \\ 0 & b_{-8} \end{pmatrix}$$

for some $b_{-8} \in \mathbb{C}$ (the labelling will shortly become clear). Observe that then $\text{im}(\text{ad}_{A_{-4}})$ is spanned by the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ Using the same argument as in the twisted case it follows that there exists a polynomial gauge transformation $\gamma(z)$ that transforms $\theta$ into the form

$$\theta = \left( \begin{pmatrix} b_{-8} & 1 \\ 0 & b_{-8} \end{pmatrix} z^{-4} + \begin{pmatrix} 0 & 0 \\ b_{-7} & b_{-6} \end{pmatrix} z^{-3} + \begin{pmatrix} 0 & 0 \\ b_{-5} & b_{-4} \end{pmatrix} z^{-2} + \begin{pmatrix} 0 & 0 \\ b_{-3} & b_{-2} \end{pmatrix} z^{-1} + O(1) \right) \otimes dz.\]$$

Observe that by virtue of the residue theorem this time we have

$$b_{-2} = 0.$$

On the other hand, notice that if $b_{-7} = 0$ in the above matrix then $A_{-4}$ can be diagonalized using the meromorphic gauge transformation

$$\gamma(z) = 1 + \begin{pmatrix} 0 & -b_{-1}^2 \\ 0 & 0 \end{pmatrix} z^{-1}$$

unless $b_{-6}$ also vanishes. Since in this section we are interested in the case where $A_{-4}$ is not diagonalizable (even by meromorphic gauge transformations), from now on we therefore assume that

$$b_{-7} \neq 0.$$ and that the constants $b_{-8}, \ldots, b_{-3} \in \mathbb{C}$ appearing in (24) are fixed.

This time the data of the parabolic structure compatible with $\theta$ is trivial, i.e. is the trivial flag

$$E_q \supset \{0\}$$

with an arbitrary weight $\alpha_q$. Indeed, as the rank of $E$ is 2, the only other possibility would be a full flag as in the untwisted case; however, then the graded pieces of the polar parts would be of dimension 1, and we could not get nilpotent graded polar parts.

Again, we will be interested in the moduli spaces

$$\mathcal{M}^{ss}(C^{P1}, q, b_{-8}, \ldots, b_{-3}, \alpha_q)$$

of S-equivalence classes of (semi-)stable irregular Higgs bundles on $CP^1$ with polar part at $q$ with respect to some trivialization as prescribed in (24). We will see that in this case the weight $\alpha_q$ actually plays no role. The existence of a moduli space parametrizing isomorphism classes of stable objects should follow from [5], and we again expect that there should exist a quasi-projective smooth coarse moduli scheme $\mathcal{M}^{ss}$ parametrizing S-equivalence classes of semi-stable objects.

2.4. Spectral data of irregular Higgs bundles and the irregular Hitchin map. A categorical equivalence between the groupoid of irregular Higgs bundles with semi-simple polar part and the relative Picard functor of a Hilbert scheme of curves on a certain multiple blow-up $Y$ of the Hirzebruch surface $X$ from Subsection 2.1 was described in [16]. We will refer to this equivalence as the refined Beauville–Narasimhan–Ramanan (BNR-) correspondence. The sheaf associated to an irregular Higgs bundle by this correspondence is called its spectral sheaf, usually denoted by $S$. The general formula relating the degrees appearing in the two setups is

$$\delta = d + \frac{1}{2} r(r - 1) \deg(K(4)) = d + 2,$$
where \( d = \deg(\mathcal{E}) \) and \( \delta \) denotes the degree of \( \mathcal{S} \) defined in (6). (Recall that in the latter formula \( X_0 \) denotes the support of \( \mathcal{S} \).) We refer the reader to [10] for the general correspondence; in Section 4 we will spell it out explicitly in the untwisted case. In the twisted case we prove an analogous result in Section 6. We expect that such a result should hold in general, and not only in the particular case we are treating here.

A closely related concept is the irregular Hitchin map. Namely, to an irregular Higgs bundle one may associate the support \( \Sigma \) of \( \mathcal{S} \), called the spectral curve. This curve belongs to a complete linear system \(|D|\) of curves on \( Y \) determined by the map \( Y \to X \). It follows from the properties of the equivalence that the natural map

\[
(\mathcal{E}, \theta) \mapsto \Sigma
\]

obtained by composing the refined BNR-correspondence above and the forgetful functor mapping a sheaf to its support, actually takes values in an affine subspace \(|D|_0 \subset |D|\). Therefore, the above association gives rise to a map

\[
H : \mathcal{M}^{ss} \to |D|_0.
\]

We will refer to \( H \) as the irregular Hitchin map, as it is a straightforward analogue of the map defined in [10]. It follows from [5] that for generic choices of the singularity parameters (namely, assuming that the adjoint orbits of the residues are closed), the irregular Dolbeault moduli spaces are complete holomorphic-symplectic smooth manifolds. Based on this fact and the above analogy, it is therefore natural to expect that \( H \) is a proper map which endows \( \mathcal{M}^{ss} \) with the structure of an algebraically completely integrable system.

### 3. Elliptic Fibrations on Rational Elliptic Surfaces

In this section we will study singular fibers of elliptic fibrations on rational elliptic surfaces. As 4-manifolds, these surfaces are diffeomorphic to the 9-fold blow-up \( \mathbb{CP}^2 \# 9\mathbb{CP}^2 \) of the complex projective plane \( \mathbb{CP}^2 \). The potential singular fibers are classified by Kodaira [11]. Here we will concentrate only on those fibrations which contain singular fibers of types \( \tilde{E}_8 \) and \( \tilde{E}_7 \). (For the plumbing description of these singular fibers see Figure 1)

![Figure 1](image)

**Figure 1.** Plumbings of singular fibers of types (a) \( \tilde{E}_8 \) and (b) \( \tilde{E}_7 \) (integers next to vertices indicate the multiplicities of the corresponding homology classes in the fiber). All self-intersections are equal to \(-2\).

The usual way to construct an elliptic fibration on the rational elliptic surface is by giving a pencil of cubic curves in \( \mathbb{CP}^2 \) (with the additional property that the pencil contains at least one smooth cubic) and then blowing up the basepoints of the pencil. In turn, the pencil can be given by specifying two degree-3 homogeneous polynomials \( p_0 \) and \( p_1 \) in three variables and considering the curves \( C(p_t) \) corresponding to the polynomials \( p_t = t_0 p_0 + t_1 p_1 \) for \( t = [t_0 : t_1] \in \mathbb{CP}^1 \). The pencil will not contain smooth curves if \( p_0 \) and \( p_1 \) admit common singular points, hence usually this case is avoided.

Recall that the singular fiber in an elliptic fibration with a single node is called \( I_1 \) (or a fishtail fiber), the fiber with a cusp singularity (which can be modeled by the cone on the trefoil knot \( \tilde{I}_{2,3} \), or can be given by the local equation \( y^2 = x^3 \)) is a cusp fiber (also denoted by \( I_1 \)). A singular fiber with two rational curves intersecting each other in two distinct points (and having self-intersection \(-2\)) form an \( I_2 \) fiber. If the two rational curves
are tangent to each other (still with self-intersection \(-2\)) then we have a type \(III\) fiber. (There are further singular fibers in the Kodaira list, but we will not meet them in our subsequent arguments.)

The determination of the type of all singular fibers in an elliptic fibration specified by two cubic polynomials \(p_0, p_1\) can be a rather tedious problem. By choosing specific polynomials, the existence of two singular fibers is quite transparent, but the identification of the further ones usually requires further computations.

3.1. The case of singular fibers of type \(\tilde{E}_8\). Suppose first that we have an elliptic fibration on \(\mathbb{C}P^2 \# 9\mathbb{C}P^2\) with a singular fiber of type \(\tilde{E}_8\). We will also assume that the fibration comes from blowing up a pencil, hence it admits a section. This section then necessarily intersects the \(\tilde{E}_8\)-fiber in the unique curve with multiplicity 1. Consider a generic fiber \(C\) of the fibration, and blow down the section and then consecutively the next six curves of the \(\tilde{E}_8\)-fiber. The image of \(C\) (now of self-intersection 7) will intersect two curves \(E_1, E_2\) (both of self-intersection \((-1))\) from the fiber, one of which (say \(E_2\)) is further intersected by the leaf \(E_3\) of the \(\tilde{E}_8\) fiber, and is of multiplicity 2. (We point out that, as it is obvious from the construction, the two curves \(E_1, E_2\) intersect at the same point, cf. the left diagram of Figure 2).

There is a choice in continuing the blow-down process. If we blow down \(E_1\), then we get a configuration of curves in the second Hirzebruch surface, where the image of \(E_2\) is a fiber, \(E_3\) is the section at infinity, and \(C\) blows down to a multisection, intersecting the generic fiber twice and being tangent to \(E_2\). On the other hand, blowing down \(E_2\) first, and then \(E_3\), the curve \(C\) blows down to a cubic curve \(C_0\) in \(\mathbb{C}P^2\), and the image of \(E_1\) will be a projective line, triply tangent to \(C_0\) (at one of its inflection points). The two results are related by the birational morphism \(\omega\) of Equation (4).

In conclusion,

**Theorem 3.1.** Any elliptic fibration on \(\mathbb{C}P^2 \# 9\mathbb{C}P^2\) with a section and with a singular fiber of type \(\tilde{E}_8\) can be blown up from a pencil defined by either

1. the union of the infinity section (with multiplicity 2) with a fiber (with multiplicity 4) in the second Hirzebruch surface, and with a double section which is tangent to the chosen fiber, or
2. a cubic curve in \(\mathbb{C}P^2\), with a triple tangent line (at one of the inflection points of the cubic), the latter with multiplicity three.

3.2. The case of singular fibers of type \(\tilde{E}_7\). Next we would like to analyze pencils resulting in fibrations with singular fibers of type \(\tilde{E}_7\). Assume therefore that the fibration on \(\mathbb{C}P^2 \# 9\mathbb{C}P^2\) contains such a singular fiber, and that the fibration results from a pencil, hence it also admits a section. Indeed, since the pencil should have at least two basepoints (otherwise the fibration has a singular fiber which contains a chain of 8 curves with self-intersection \((-2)\), which is impossible next to a fiber of type \(\tilde{E}_7\)), we can assume that
there are two sections, intersecting the type $E_7$ singular fibers in the two $(-2)$-curves with multiplicity 1. As before, let $C$ be a regular fiber of the fibration.

After 7 blow-downs (by blowing down the two sections and two, respectively three curves from the two long arms of the $E_7$-fiber) we get a configuration of 4 curves: the image of the fiber $C$, two $(-1)$-curves (called $E_1$ and $E_2$) intersecting it in two distinct points (and each other) and a $(-2)$-curve $E_3$ intersecting $E_2$ only, cf. the right diagram of Figure 2. As in the case of an $E_8$-fiber, we have a choice in performing the next blow-down. If we blow down $E_1$, we get a configuration again in the second Hirzebruch surface, while if we blow down $E_2$ (and then $E_3$), we get a configuration in $CP^2$. Consequently we get

**Theorem 3.2.** Any elliptic fibration on $CP^2 \# 9\overline{CP^2}$ with two sections and with a singular fiber of type $E_7$ can be blown up from a pencil defined by either

1. the union of the infinity section (with multiplicity 2) with a fiber (with multiplicity 4) in the second Hirzebruch surface, and with a double section which intersects the distinguished fiber in two distinct points, or
2. a cubic in $CP^2$, with a tangent line which intersects the cubic in one further point, the latter with multiplicity three.

Assume now that the elliptic fibration contains (besides the type $E_7$-fiber) a further singular fiber which is either of type $I_2$ or of type $III$. By further inspecting the blow-down process, now choosing the curve $C'$ to be a singular fiber of type $I_2$ or $III$ we get:

**Proposition 3.3.** If an elliptic fibration with a fiber of type $E_7$ and two sections contains a further singular fiber either of type $I_2$ or of type $III$, then the pencil of curves resulting from the repeated blow-down in the second Hirzebruch surface contains a double section which is the union of two sections of the ruling of the surface.

The same argument (now by blowing down the configuration to $CP^2$) shows that the pencil in $CP^2$ can be chosen to be generated by a projective line $\ell$ (with multiplicity three, just as before) and another curve, which has two components, a line $\ell_1$ and a quadric $q$, where $\ell$ intersects $\ell_1$ in one point $P$, while $\ell$ is tangent to the quadric $q$ (in a point distinct from $P$). The pencil gives rise to a fibration which has (besides a type $E_7$ fiber) an $I_2$ fiber if $\ell_1$ intersects $q$ in two distinct points, and a type $III$ fiber if $\ell_1$ is tangent to $q$.

4. UNTWISTED CASE

We start by describing a certain blow-up $Y$ of the surface $\overline{X}$ whose geometry governs $\mathcal{M}$. Namely, a local trivialization of $K(4) \cong K \otimes O(4\cdot(q_1))$ near $q_1 = 0$ is given by $z_1^4dz_1$, so the expressions $z_1^{-4}(a_+ + b_+z_1 + c_+z_1^2 + \lambda_+z_1^3)dz_1$ specify non-reduced subschemes of dimension 0 and length 4 in $\overline{X}$. We define $Y$ as the blow-up of $\overline{X}$ along these subschemes.

In concrete terms, as in Section 2, we set $q = 0$, $U = \mathbb{C} = CP^1 \setminus \{\infty\}$, $\kappa = z_1^4dz_1$, and parametrize $p^{-1}(U) \setminus C_\infty$ by coordinates $(z_1, z_2) \in \mathbb{C}^2$ as follows: we let the point of $X$ corresponding to these parameters be

$$[z_2\kappa z_1 : 1z_1].$$

We may assume that $\overline{X}$ is the blow-up of $X$ in the point $[a_+\kappa_0 : 1z_1]$.

Alternatively, over $p^{-1}(U)$ the surface $\overline{X}$ is defined by

$$(z_1z_2 - (z_2 - a_+)z'_2) \in \mathbb{C}^2 \times CP^1$$

where $[z_1' : z_2'] \in CP^1$ are homogeneous coordinates corresponding to the direction of tangent vectors at $z_1 = 0, z_2 = a_+$. We denote this blow-up by

$$\sigma_{1+} : X_{1+} = \overline{X} \to X$$
and its exceptional divisor by
\[ E_{1+} = \{ z_1 = 0, z_2 = a_+, [z'_1 : z'_2] \}. \]

Next we blow up \( \widetilde{X} \) in the point
\[ [z'_1 : z'_2] = [1 : b_+] \in E_{1+}. \]

For this purpose, we introduce the local chart \( U_{1+} \) of \( \widetilde{X} \) given by \( z'_1 \neq 0 \). Here we may normalize \( z'_1 = 1 \), and so a local coordinate chart of \( U_{1+} \) is given by \( z_1, z_2' \). The blow-up
\[ \sigma_{2+} : X_{2+} \to X_{1+} \]

we consider is then the blow-up of the point with coordinates \( z_1 = 0, z_2' = b_+ \). Similarly to the above, we denote the exceptional divisor of \( \sigma_{2+} \) by \( E_{2+} \), and we get canonical coordinates \([z''_1 : z''_2]\) parametrizing \( E_{2+} \) starting from the coordinates \( z_1, z'_2 \). We then blow up the point
\[ [z''_1 : z''_2] = [1 : c_+] \in E_{2+} \]

and call the corresponding birational map
\[ \sigma_{3+} : X_{3+} \to X_{2+}. \]

Finally, just as above we get canonical coordinates \([z'''_1 : z'''_2]\) on the exceptional divisor \( E_{3+} \) of \( \sigma_{3+} \), and we define the blow-up
\[ \sigma_{4+} : X_{4+} \to X_{3+} \]

of the point with coordinates
\[ [z'''_1 : z'''_2] = [1 : \lambda_+] \in E_{3+}. \]

We then let \( X_{0-} = X_{1+} \) and carry out a similar procedure for the length 4 non-reduced subschemes corresponding to the expression \( z_1^{-1}(a_- + b_- z_1 + c_- z_2^2 + \lambda_+ z_1 \lambda_-)^dz_1 \). We denote the birational maps and their exceptional divisors by
\[ \sigma_{i-} : X_{i-} \to X_{(i-1)-} \]

and \( E_{i-} \) for \( 1 \leq i \leq 4 \). By an abuse of notation, we will continue to denote the proper transforms of \( E_{i+} \) and \( E_{i-} \) along the subsequent maps \( \sigma_{j+} \) and \( \sigma_{j-} \) by the same symbols. The surface of interest to us is
\[(27) \quad Y = X_{4-} \to X.\]

Clearly then there is a diagram

\[
\begin{array}{c}
\text{Y} \\
\downarrow \\
X \\
\downarrow \\
\mathbb{CP}^2
\end{array}
\]

where the left-hand map is a blow-up of \( X \) in 8 points and the right-hand map is a blow-up of \( \mathbb{CP}^2 \) in 9 point. In particular, as a smooth 4-manifold \( Y \) is diffeomorphic to \( \mathbb{CP}^2 \# 9 \mathbb{CP}^1 \). By an abuse of notation, we will denote the composition of \( \widetilde{X} \to X \) with \( p : X \to \mathbb{CP}^1 \) by \( \widetilde{X} \to \mathbb{CP}^1 \) and also the composition of \( Y \to X \) with \( p : X \to \mathbb{CP}^1 \) by \( p : Y \to \mathbb{CP}^1 \).

It follows from [16 Theorem 4.3] that irregular rank 2 Higgs bundles on \( \mathbb{CP}^1 \) with a pole of order 4 of the local form \((20)\) are in one-to-one correspondence with data of the form \((\Sigma, S)\)

where \( \Sigma \) is a holomorphic curve in \( Y \) satisfying certain properties and \( S \) is a torsion-free sheaf of \( \mathcal{O}_\Sigma \)-modules of some given degree \( \delta \). In concrete terms, we require the following conditions to hold:

1. \( \Sigma \) is disjoint from the proper transform of \( C_\infty \) in \( Y \);
(2) \( p : \tilde{\Sigma} \to \mathbb{C}P^1 \) is a double ramified cover;
(3) \( \tilde{\Sigma} \) intersects the exceptional divisors \( E_{4\pm} \) in one point each, away from their “points at infinity” \( \{ z^{(i)}_1 : z^{(i)}_2 \} = [0 : 1] \in E_{4\pm} \).

In particular, the last two conditions imply that \( \tilde{\Sigma} \) intersects neither the proper transform \( \tilde{F}_0 \) of the fiber \( F_0 \) in \( Y \) nor the exceptional divisors \( E_{1\pm} \) with \( 1 \leq i \leq 3 \).

**Lemma 4.1.** Let \( \tilde{\Sigma} \) be a holomorphic curve in \( Y \) fulfilling the above conditions. Then the restriction of the birational map \( Y \to \mathbb{C}P^2 \) establishes a biholomorphism between \( \tilde{\Sigma} \) and a cubic curve in \( \mathbb{C}P^2 \). In particular, \( \tilde{\Sigma} \) is of arithmetic genus 1.

**Proof.** Under the map \( Y \to \mathbb{C}P^2 \) the generic fibers of \( p : Y \to \mathbb{C}P^1 \) get mapped to curves of self-intersection number 1, i.e. to lines \( \ell \) in \( \mathbb{C}P^2 \) passing through \( [0 : 0 : 1] \). Thus the image of a curve \( \Sigma \) is a curve in \( \mathbb{C}P^2 \) intersecting the generic such line \( \ell \) in two points distinct from \([0 : 0 : 1]\) (corresponding to the intersection points of \( \Sigma \) with the generic fibre of \( Y \)). Furthermore it is easy to see that the point \([0 : 0 : 1]\) is a base point of such curves \( \tilde{\Sigma} \), but blowing it up once is sufficient to separate them. In different terms, \( \tilde{\Sigma} \) intersects the generic line \( \ell \) passing through \([0 : 0 : 1]\) in 3 points (counted with multiplicity). By the conditions, no component of \( \tilde{\Sigma} \) gets contracted to a point and moreover no two points of \( \tilde{\Sigma} \) get identified. We infer that the restriction is one to one. \( \square \)

**Proof of Theorem 3.2.** We first show that \( \mathcal{M} \) is a relative compactified Picard scheme of torsion-free sheaves of total degree 1 over an elliptic fibration with singular fibers as described in the Theorem. For this purpose, let \( \tilde{F}_0 \) denote the proper transform under the map \( (27) \) of the fibre \( F_0 \) of \( p \) over \( 0 \) and recall again our convention that \( E_{j\pm} \) stands for the proper transform in \( Y \) of the exceptional divisor of the blow-up \( \sigma_{j\pm} \). According to Theorem 3.2 in any elliptic fibration on \( Y \) with one \( E_7 \) singular fiber this latter must be given by

\[
2C_{\infty} + 4\tilde{F}_0 + 3(E_{1+} + E_{1-}) + 2(E_{2+} + E_{2-}) + (E_{3+} + E_{3-}).
\]

Next we will start identifying the singular fibers of the resulting elliptic fibration. In the untwisted case, in (20) we gave a local form for \( \theta \). We also wrote \( \chi_{\theta}(z_2) \) as the characteristic polynomial of \( \theta \) in the trivialization given by \( \kappa \) and \( \kappa^2 \), see (27) cf. also Subsection 2.3. This gives

\[
\chi_{\theta}(z_2, z_1) = z_2^3 + f(z_1)z_2 + g(z_1).
\]

Consider the polynomials \( f \) and \( g \) given by the following equations:

\[
f(z_1) = -(p_2z_1^2 + p_1z_1 + p_0),
\]

\[
g(z_1) = -(q_4z_1^4 + q_3z_1^3 + q_2z_1^2 + q_1z_1 + q_0),
\]

where all coefficients are elements of \( \mathbb{C} \).

The roots of the characteristic polynomial in \( z_2 \) have expansions with respect to \( z_1 \). The first several terms of the expansion are the same as the diagonal elements of the matrix in (30). More precisely, the series of the “negative” root of \( \chi_{\theta}(z_2) \) up to third order is equal to \( a_- + b_-z_1 + c_-z_1^2 + \lambda_zz_1^3 + O(z_1^4) \) and the “positive” root up to third order is equal to \( a_+ + b_+z_1 + c_+z_1^2 + \lambda_+z_1^3 + O(z_1^4) \). From these equations we get the following expressions:

\[
f(z_1) = -(a_- + a_+) - (b_- + b_+)z_1 - (c_- + c_+)z_1^2,
\]

\[
g(z_1) = a_-a_+ + (a_2b_1 + a_1b_2)z_1 + (a_+c_- + a_-c_+ + b_-b_+)z_1^2 + (a_-\lambda_+ + a_+\lambda_- + b_+c_- + b_-c_+)z_1^3 - q_4z_1^4.
\]

According to the residue theorem (24) we know that \( \lambda_+ + \lambda_- = 0 \), hence we can eliminate \( \lambda_+ = -\lambda_- \). Furthermore, we will denote \( q_4 \) by \( t \). Hence we get a pencil
Finally, we substitute \((31)\)

We express \((35)\) and \((36)\) from the third and the previous equations:

\[
\chi_{\theta,E7}(z_2, z_1, t) = z_2^2 + (-a_- a_+ + (-b_- b_+) z_1 + (-c_- c_+) z_1^2) z_2 + (a_- a_+ + (a_+ b_- + a_- b_+) z_1 + (a_+ c_- + a_- c_+ + b_- b_+) z_1^2 + ((a_- a_+) \lambda_+ + b_+ c_- + b_- c_+) z_1^3 - t z_1^4 = 0
\]

where the degree of the polynomial is 2 in the variable \(z_2\) and 4 in \(z_1\).

According to Theorem 3.2, the pencil \(\chi_{\theta,E7}\) gives rise to an elliptic fibration in \(\mathbb{C}P^2 \# 9\mathbb{C}P^2\) with a singular fiber of type \(E_7\).

In identifying the singular fibers in the pencil, we look for triples \((z_2, z_1, t)\) such that \((z_2, z_1)\) fits the curve with parameter \(t\), and the partial derivates below vanish:

\[
\frac{\partial \chi_{\theta,E7}(z_2, z_1, t)}{\partial z_2} = 0,
\]

\[
\frac{\partial \chi_{\theta,E7}(z_2, z_1, t)}{\partial z_1} = 0.
\]

The second equation is very simple, we can express \(z_2\) from it as:

\[
z_2 = \frac{1}{2} (a_- + a_+ + (b_- + b_+) z_1 + (c_- + c_+) z_1^2)
\]

We express \(t\) from the third and the previous equations:

\[
t(z_1) = -\frac{1}{8 z_1^3} \left( (a_- - a_+) (b_- - b_+) + (2 (a_- - a_+) (c_- - c_+) + (b_- - b_+)^2) z_1 + 3 ((b_- - b_+) (c_- - c_+) - 2 (a_- - a_+) \lambda_+) z_1^2 + 2 (c_- + c_+) z_1^3 \right)
\]

Finally, we substitute \(z_2\) and \(t\) in the first equation and we get a polynomial in \(z_1\):

\[
0 = (2 (a_- - a_+) + (b_- - b_+) z_1) (a_- - a_+ + (b_- - b_+) z_1 + (c_- - c_+) z_1^2) + 2 (a_- - a_+) \lambda_+ z_1^3.
\]

Since this is a degree-3 polynomial, generally we get three distinct roots, and this corresponds to the fact that there are at most three singular fibers in the fibration.

The problem becomes more convenient to study on another chart, given by the following coordinates and trivialization of \(K (4 \cdot (q))\): take

\[
w_1 = \frac{1}{z_1} \quad \text{and} \quad k_2 = dw_1.
\]

The conversion from \(k_2\) to \(k_2^2\) is the following:

\[
k = \frac{dz_1}{z_1^2} = -w_1^2 dw_1 = -w_1^2 k_2.
\]

Then \(F = f_2 k_2\) and \(G = g_2 k_2^2\) in (19) with

\[
f_2(w_1) = p_0 w_1^2 + p_1 w_1 + p_2
\]

\[
g_2(w_1) = - (g_0 w_1^4 + q_1 w_1^3 + q_2 w_1^2 + q_3 w_1 + q_4).
\]

We describe the pencil in the new coordinates as:

\[
\chi_{\theta,E7}(z_2, w_1, t) = z_2^2 + f_2(w_1) z_2 + g_2(w_1, t) = z_2^2 + ((a_- + a_+) w_1^2 + (b_- + b_+) w_1 + c_- + c_+) z_2 + (a_- a_+ w_1^3 + (a_+ b_- + a_- b_+) w_1^2 + (a_+ c_- + a_- c_+ + b_- b_+) w_1 + (a_- - a_+) \lambda_+ + b_+ c_- + b_- c_+) w_1 - t = 0
\]
To identify the singular fibers, we once again compute the partial derivatives:

\[
\begin{align*}
\chi_{\delta, E^7}(z_2, w_1, t) &= 0, \\
\frac{\partial \chi_{\delta, E^7}(z_2, w_1, t)}{\partial z_2} &= 0, \\
\frac{\partial \chi_{\delta, E^7}(z_2, w_1, t)}{\partial w_1} &= 0.
\end{align*}
\]

We express \(z_2\) and \(t\) from the first and second equations by \(w_1\):

\[
z_2(w_1) = -\frac{1}{2} \left( (a_1 + a_2) w_1^2 + (b_1 + b_2) w_1 + c_1 + c_2 \right),
\]

\[
t(w_1) = -\frac{1}{4} \left( (a_- - a_+)^2 w_1^4 + 2 (a_- - a_+) (b_- - b_+) w_1^3 + \right.
\]

\[
+ \left( 2 (a_- - a_+) (c_- - c_+) + (b_- - b_+)^2 \right) w_1^2 +
\]

\[
+ \left( 2 (b_- - b_+) (c_- - c_+) - 4 (a_- - a_+) \lambda_s \right) w_1 + (c_- + c_+)^2 \right).
\]

(37)

Now, we substitute the resulting expression into the third equation and get

\[
0 = (2 (a_- - a_+) w_1 + b_- - b_+) \left( (a_- - a_+) w_1 + b_- - b_+ \right) w_1 + c_- - c_+ +
\]

\[
+ 2 (a_- - a_+) \lambda_s.
\]

(38)

The cubic polynomial of (32) with variable \(z_1\) has one or two roots if and only if its discriminant vanishes. The polynomial in (38) with variable \(w_1\) behaves similarly, since the discriminants of the two polynomials are the same:

\[
\Delta = (a_- - a_+)^2 \left( (b_- - b_+) - 4 (a_- - a_+) (c_- - c_+) \right)^3 - 432 (a_- - a_+)^4 \lambda_s^2.
\]

With the choice \(a_- = a_+\) the configuration reduces to the case of a type \(E_8\) singular fiber (to be treated in Section 6), therefore we can assume that \(a_- \neq a_+\). Let \(\Delta\) denote \(\Delta / (a_- - a_+)^2\).

By the classification of singular fibers in elliptic fibrations [13], the singular fibers besides the \(E_7\)-fiber and with at most two singular points are: either a \(II\) fiber or a cusp fiber (\(II\)) together with a fishtail fiber (\(I_1\)). The two cases are distinguished by the number of singular fibers.

The cubic in (32) has one root if and only if the discriminant \(\Delta\) vanishes, and the derivative of (32) with respect to \(z_1\) has one root, hence the discriminant of the latter quadratic equation also vanishes. This means that \(\Delta' = 0\) with

\[
\Delta' = 9 (a_- - a_+) (b_- - b_+) (2 (a_- - a_+) \lambda_s - (b_- - b_+) (c_- - c_+)) +
\]

\[
+ \left( 2 (a_- - a_+) (c_- - c_+) + (b_- - b_+)^2 \right)^2.
\]

In the \((z_2, w_1)\) chart the cubic of (38) has one root if and only if \(\Delta\) and the discriminant \(\Delta'_2\) of the derivative of (38) with respect to \(w_1\) vanishes, where

\[
\Delta'_2 = 3 (a_- - a_+)^2 \left( (b_- - b_+) - 4 (a_- - a_+) (c_- - c_+) \right).
\]

When (32) has one root then the pencil has one singular curve in the corresponding chart. It is not necessarily a curve of type \(III\), since the pencil might have a singular curve with singularity at infinity (hence not contained by the chart). In our case, however, \(\Delta = \Delta'_2 = 0\) implies that the pencil has a singularity of type \(III\) on any chart, because the \(z_1 = 0\) line \((w_1 = \infty)\) has no singularity by Condition \(\tilde{E}\) preceding Lemma [13].

It is easy to see \(\Delta = \Delta'_2 = 0\) is equivalent to \(\Delta = \lambda_s = 0\), hence we get part (1) of Theorem [13].

At the same time we verified part (2) as well, since \(\Delta = 0\) and \(\Delta'_2 \neq 0\) is equivalent to \(\Delta = 0\) and \(\lambda_s \neq 0\), furthermore equation (38) has two distinct roots. The corresponding two singularities give two singular fibers, with singularities of types \(II\) and \(I_1\).
Remark 4.2. Solve the equations $\Delta = 0$ and $\Delta' = 0$ simultaneously for the variables $(a_\cdot - a_\ast)$ and $\lambda_\ast$ on the chart given by the coordinates $(z_2, z_1)$. If $b_\cdot \neq b_\ast$ and $c_\cdot \neq c_\ast$, we have solutions:

\[
(a_\cdot - a_\ast)_1 = \frac{(b_\cdot - b_\ast)^2}{4(c_\cdot - c_\ast)}, \quad (\lambda_\ast)_1 = 0,
\]

\[
(a_\cdot - a_\ast)_2 = -\frac{(b_\cdot - b_\ast)^2}{4(c_\cdot - c_\ast)}, \quad (\lambda_\ast)_2 = -\frac{(c_\cdot - c_\ast)^2}{b_\cdot - b_\ast}.
\]

The first solution provides a curve of type III, but the second will be different. Indeed, by substituting $(a_\cdot - a_\ast)_2$ and $(\lambda_\ast)_2$ to the equation of (38), and solving it, we get two distinct roots:

\[
(w_1)_1 = 0, \quad (w_1)_2 = \frac{3(c_\cdot - c_\ast)}{b_\cdot - b_\ast}.
\]

The number of roots shows that there are two singular fibers, which are of type II and I\(_1\), where one of the two curves has a singularity at $z_1 = \infty$ in the $(z_2, z_1)$ chart.

Finally, $b_\cdot = b_\ast$ implies $c_\cdot = c_\ast$ because $\Delta' = 0$; and conversely, $c_\cdot = c_\ast$ implies $b_\cdot = b_\ast$. Furthermore, $\lambda_\ast = 0$ since $\Delta = 0$. In this case, the singularity of type III is at $w_1 = 0$ in the $(z_2, w_1)$ chart, and at $z_1 = \infty$ in the $(z_2, z_1)$ chart.

Remark 4.3. Note that according to Proposition [3], the fibration has the fiber of type III or $I_2 + I_1$ if and only if the pencil contains a double section which is the union of two sections, and this last condition is easily seen to be equivalent to $\lambda_\ast = 0$.

Now, we consider the $\Delta \neq 0$ case, where we have three singularities.

Lemma 4.4. $\Delta \neq 0$ and $\lambda_\ast = 0$ holds if and only if the pencil has $I_2$ and $I_1$ singularities.

Proof. The first direction is simple, because the equation of (32) can be easily solved and the three $z_1$ values can be substituted into (31). We get two distinct $t$ values, which correspond to one curve with two singularities and another curve with one singularity.

The three $z_1$ values are:

\[
(z_1)_1 = \frac{2(a_\cdot - a_\ast)}{b_\cdot - b_\ast},
\]

\[
(z_2, z_3) = z_2 = b_\cdot - b_\ast \pm \frac{\sqrt{(b_\cdot - b_\ast)^2 - 4(a_\cdot - a_\ast)(c_\cdot - c_\ast)}}{2(c_\cdot - c_\ast)}.
\]

If $b_\cdot = b_\ast$ then the pencil contains a curve with an $I_1$ singularity at $z_1 = \infty$ on the $(z_2, z_1)$ chart (actually this appear on the $(z_2, w_1)$ chart with $w_1 = 0$), and if $c_\cdot = c_\ast$ then the pencil contains a curve with an $I_2$ singularity, and one of the singular points of $I_2$ is at $z_1 = \infty$. Moreover, if $b_\cdot = b_\ast$ and $c_\cdot = c_\ast$ then the two singular fibers (one of type $I_2$ and the other of type $I_1$) degenerate to a type III singular fiber, since $\Delta$ vanish. These singularities appear on the $(z_2, w_1)$ chart.

Let us see now the converse direction. We investigate this case in the $s_2$ trivialization, since all singularities appear in the $(z_2, w_1)$ chart, and (37) is a polynomial in contrast to (31). If the pencil contains an $I_2$ and an $I_1$ curve then the equation of (38) has three distinct roots. Let us denote these roots by $y_1, y_2, y_3$. Two roots (say $y_1$ and $y_2$) provide singularities on the same curve, that is, $t(y_1) = t(y_2)$. Equivalently

\[
0 = t(y_1) - t(y_2) = 4(a_\cdot - a_\ast)\lambda_\ast - ((a_\cdot - a_\ast)(y_1 + y_2) + b_\cdot - b_\ast) \cdot \left(\frac{(a_\cdot - a_\ast)(y_1^2 + y_2^2) + (b_\cdot - b_\ast)(y_1 + y_2) + 2(c_\cdot - c_\ast)}{4} \right),
\]

where we simplify with $\frac{1}{4}(y_1 - y_2)$. 

Obviously, the three distinct roots provide two values for \( t \) if and only if
\[
0 = (t(y_1) - t(y_2))(t(y_2) - t(y_3))(t(y_3) - t(y_1)).
\]

This expression is a symmetric polynomial in \( y_1, y_2, y_3 \), hence can be written as a polynomial of the elementary symmetric polynomials \( s_1 = y_1 + y_2 + y_3, s_2 = y_1y_2 + y_2y_3 + y_3y_1 \) and \( s_3 = y_1y_2y_3 \). The above vanishing condition would yield a long expression, but \( s_1, s_2, s_3 \) can be determined from the coefficient of equation (38) by Vieta’s formulas. The relations between the symmetric polynomials and the coefficients then provide
\[
s_1 = -\frac{3 (b_- - b_+)}{2 (a_- - a_+)},
\]
\[
s_2 = -\frac{3 (b_- - b_+)}{2 (a_- - a_+)},
\]
\[
s_3 = \frac{2 (a_- - a_+)}{2 (a_- - a_+)} \lambda_+ - (b_- - b_+) (c_- - c_+) 2 (a_- - a_+)^2.
\]

After simplifications, we get a condition for fibration to contain an \( I_2 \) curve:
\[
\lambda_+ \left( \left( (b_- - b_+)^2 - 4 (a_- - a_+) (c_- - c_+) \right) 3 - 432 (a_- - a_+)^4 \lambda_+^2 \right) = 0.
\]

Now \( \Delta \) is in the nominator, and \( \lambda_+ \) is a multiplication factor. Since the pencil has three singularities, we have that \( \Delta \neq 0 \), consequently \( \lambda_+ = 0 \) concluding the proof of Lemma 4.4. \( \square \)

In order to complete the proof of Theorem 1.1 we need to examine the case when \( \Delta \neq 0 \) and \( \lambda_+ \neq 0 \). In this case there is a single possibility for the singular fibers: there are three \( I_1 \) fibers in the fibration. We summarize the cases in Table 1.

| \( \Delta \) | \( \lambda_+ \) | Type |
|---|---|---|
| 0 | 0 | \( I_1 \) |
| 0 | 0 | \( I_2 + I_1 \) |
| 0 | 0 | 3\( I_1 \) |

Table 1. The type of singular curves in untwisted case

Now, observe that the fibration obtained from the pencil have a section (actually, even two sections). It then follows that over the locus \( C^{\text{unr}} \) parametrizing smooth curves in \( Y \) with the given properties, the relative Abel–Jacobi map gives an algebraic isomorphism between the fibration and its relative Picard scheme. Therefore, in order to conclude the proof of the Theorem, one merely needs to specify the singular fibres of \( H \). In the cases of curves of types \( I_1 \), \( II \) this was carried out in Proposition 2.1. As for curves of types \( I_2 \) and \( III \), the analysis is carried out in the next section. \( \square \)

5. Stability analysis in the untwisted case

In cases \( 2 \) and \( 3 \) of Theorem 1.1 the singular fibers of the elliptic pencil (except for the type \( E_7 \) fiber at infinity) are integral (i.e. irreducible and reduced), so the Hitchin fiber of the moduli space corresponding to the singular fibers is just the usual compactified Picard scheme of degree \( \delta \). In the other cases however we need to determine the Hitchin fibres of \( M \) corresponding to the reducible singular fibres of the pencil.
5.1. Stability analysis in the case \( E_7 + I_2 + I_3 \). We use the results and notations of Subsection 2.2.1. We let \( b \in B \) denote the point whose preimage in the pencil is the singular fibre of type \( I_2 \). We assume that

\[ E = p_*(S) \]

for some torsion-free sheaf \( S \) of \( O_{X_b} \)-modules of rank 1 and use the definitions (9). By assumption we have

\[ 0 = \text{par-deg}(E) = \deg(E) + \alpha_0^+ + \alpha_0^- \tag{39} \]

Then in view of (14) and (26) the above formula may be rewritten as

\[ 0 = (\delta_+ + \alpha_0^+) + (\delta_- + \alpha_0^-) - |J(S)|. \tag{40} \]

For any non-trivial Higgs subbundle \((\mathcal{F}, \theta|_{\mathcal{F}})\) of \((E, \theta)\) the scheme

\[ (\theta|_{\mathcal{F}} - \lambda) \subset X \]

is a sub-scheme of \( X_b \) that is a one-to-one cover of \( \mathbb{C}P^1 \). Clearly the same also holds for non-trivial quotient Higgs bundles \((\mathcal{Q}, \theta|_{\mathcal{Q}})\). On the other hand, for any \( i \in \{ \pm \} \) the functor \( p_* \) applied to the morphism (10) gives rise to a quotient Higgs bundle \((\mathcal{Q}_i, \theta|_{\mathcal{Q}_i})\). Again by (26) the degree of this quotient is given by \( \delta_i \) so its parabolic degree is

\[ \delta_i + \alpha_0^i. \]

It is easy to see that these are the only quotient Higgs bundles of \((E, \theta)\), because the support of the spectral sheaf of any such quotient is a component of \( X_b \), and there are exactly two such components. We infer that \((E, \theta)\) is \( \vec{\alpha} \)-stable if and only if the two inequalities

\[ \delta_i + \alpha_0^i > 0 \]

for \( i \in \{ \pm \} \) hold. Taken into account the formula (40) these inequalities are also equivalent to

\[ 0 < \delta_i + \alpha_0^i < |J(S)| \tag{41} \]

for \( i \in \{ \pm \} \). Let us point out that this can only have a solution if \( |J(S)| \in \{ 1, 2 \} \). Now, setting

\[ \phi_i = 1 - \alpha_0^i, \]

we see that the stability condition (41) transforms into (16), which is the Oda–Seshadri stability condition for the values \((\phi_+, \phi_-)\). (Notice however that the equality

\[ \phi_- + \phi_+ = 0 \]

holds if and only if

\[ \alpha_0^+ + \alpha_0^- = 2, \]

which is incompatible with our assumption that \( 0 \leq \alpha_0^+ < 1 \).

Let us explicitly write down the corresponding Hitchin fibers. For simplicity let us set

\[ \alpha^i = \alpha_0^i. \]

Since

\[ \alpha^+ + \alpha^- = -\deg(E) \]

is an integer and \( \alpha^+, \alpha^- \in [0, 1) \), it follows that

- either we have \( \deg(E) = -1 \) and

\[ \alpha^+ + \alpha^- = 1 \tag{42} \]

- or we have \( \deg(E) = 0 \) and

\[ \alpha^+ = 0 = \alpha^- \tag{43} \].
5.1.1. Case of degree $-1$. Assume that $d = \deg(E) = -1$. By virtue of (26) this amounts to $\delta = \deg(S) = 1$. Let us first study the sheaves with $|J(S)| = 2$, i.e. invertible sheaves on $X_b$. Assumptions (43) and $\alpha_i \in [0, 1)$ imply that $\alpha_i \in (0, 1)$, therefore, by condition (41) we have either

$$\delta_+ = 0, \delta_- = 1$$

or

$$\delta_+ = 1, \delta_- = 0.$$  

Let us introduce the notation

$$\mathcal{L}_i = \mathcal{L}(S)|_{X_i}$$

and fix one of the two conditions on the degrees spelled out above. Then, as $X_b$ are rational curves, the isomorphism class of $\mathcal{L}_b$ is completely determined. Moreover, according to (11), $S$ is obtained by first identifying the fibers $(\mathcal{L}_i)_0$ and $(\mathcal{L}_i)_0$ by an isomorphism, then identifying the fibers $(\mathcal{L}_i)_\infty$ and $(\mathcal{L}_i)_\infty$ by an isomorphism. The possible identifications between these pairs of lines are parametrized by $\mathbb{C}^* \times \mathbb{C}^*$. Indeed, for trivializations $\sigma_i$ over open affine subsets of $X_i$, we have

$$\sigma_+(0) = \lambda_0 \sigma_-(0), \quad \sigma_+(\infty) = \lambda_\infty \sigma_-(\infty)$$

for some

$$(\lambda_0, \lambda_\infty) \in \mathbb{C}^* \times \mathbb{C}^* \subset \mathbb{C}^2.$$  

However, we may act on this space of identifications by constant automorphisms of one of the bundles $\mathcal{L}_i$ (say $\mathcal{L}_+$) without changing the isomorphism class of the sheaf $S$ obtained by the identifications. Constant automorphisms are isomorphic to $\mathbb{C}^*$ and $t \in \mathbb{C}^*$ obviously acts by

$$t(\lambda_0, \lambda_\infty) = (t\lambda_0, t\lambda_\infty).$$

Therefore, we are left with a parameter space

$$\text{Pic}^{\delta_+\delta_-} = \mathbb{C}^* \times \mathbb{C}^* / \mathbb{C}^* = \mathbb{C}^* \subset \mathbb{C}P^1$$

for such invertible sheaves. It is easy to see that these sheaves are all non-isomorphic. This implies that the universal line bundle on $X_b$ of bidegree $(\delta_+, \delta_-)$ is given by

$$L^{\delta_+\delta_-}(-) \rightarrow \text{Pic}^{\delta_+\delta_-} \times X_b = \mathbb{C}^* \times X_b.$$  

Now let us consider the case of sheaves $S$ with $|J(S)| = 1$. These sheaves are locally free in a neighbourhood of exactly one of the two points $\{0, \infty\}$. Clearly, if $S$ is locally free near $0$ and not locally free near $\infty$ then $S$ cannot be isomorphic to a sheaf $S'$ that is locally free near $\infty$ and not locally free near $0$. Thus there exist at least 2 points in

$$\overline{\text{Pic}^{\delta_+\delta_-} \times (\text{Pic}^{0,1} \cup \text{Pic}^{1,0})}.$$  

Our aim is to show that there exist exactly 2 points in this complement. Indeed, we first observe that if $|J(S)| = 1$ then (41) only allows for

$$\delta_+ = 0 = \delta_-.$$  

As $X_b$ are rational curves, the isomorphism class of line bundles of degree 0 on $X_b$ is unique, they are given by $\mathcal{L}_b = \mathcal{O}_{X_b}$. Now, assume that $S$ is locally free near $0$. Then $S$ is obtained by identifying the fibers $(\mathcal{L}_i)_0$ and $(\mathcal{L}_i)_0$ by a linear isomorphism. The choices for such an isomorphism are parametrized by $\mathbb{C}^*$. However, we again get isomorphic sheaves if we apply a constant automorphism to one of $\mathcal{L}_i$. It follows that there exists a single stable sheaf $S_0$ that is locally free near $0$ but not locally free near $\infty$. Similarly, there exists a unique stable sheaf $S_\infty$ that is locally free near $\infty$ but not locally free near $0$.  

Finally, we show that both $S_0$ and $S_\infty$ are in the closure of both $\text{Pic}^{0,1}$ and $\text{Pic}^{1,0}$ in $\overline{\text{Pic}^{\delta_+\delta_-}}$. The argument closely follows the one in the proof of Proposition (24). Let us for instance work in the chart $\lambda_\infty = 1$ of $\mathbb{C}P^1$, and fix one of the two conditions on the degrees spelled out above, say $(1, 0)$. We will consider the limit $L^{1,0}(0)$ of the line bundles
$L^{1,0}(\lambda)$ as $\lambda = \lambda_0 \to 0$. Let us denote the two preimages of $0 \in X_b$ by $0_+ \in X_+, 0_- \in X_-$ respectively. For $\lambda_0 = 0$ we get

$$\sigma_+(0_+) = 0, \sigma_-(0_-),$$

hence

$$\mathcal{T}_{\mathcal{O} X_+.\mathcal{O}}(\pi^* L^{1,0}(0)) \cong \mathbb{C}_{0_+},$$

is generated by $\sigma_-(0_-)$, and

$$\mathcal{T}_{\mathcal{O} X_+.\mathcal{O}}(\pi^* L^{1,0}(0)) \cong 0.$$

At the points $\infty_+, L^{1,0}(0)$ is locally free. We infer that the line bundle $\mathcal{L}_+(0)$ of $\mathbb{C}_+$ over $X_+$ associated to $L^{1,0}(0)$ fits into the short exact sequence

$$0 \to \mathcal{L}_+(0) \to \mathcal{O}_{X_+}(1) \to \mathbb{C}_{0_+} \to 0,$$

and that $\mathcal{L}_-(0) = \mathcal{O}_{X_-}$; in other words, these line bundles are both of degree 0. As we have already shown, $\mathcal{S}_\infty$ is up to isomorphism the unique sheaf of bidegree $(0,0)$ which is locally free near $\infty$ but not locally free near 0. We infer that

$$L^{1,0}(0) = \mathcal{S}_\infty.$$

A similar argument for $L^{0,1}$ over the affine chart $\lambda_\infty = 1$ now shows that the limit of $L^{0,1}(\lambda)$ as $\lambda \to 0$ is a sheaf of bidegree $(-1,1)$, locally free near $\infty$ but not locally free near 0. Let us denote by $X_0$ the partial normalization of $X_b$ at the point $0 \in X_b$. By the uniqueness of $\mathcal{S}_\infty$ we see that

$$L^{0,1}(0) \cong \mathcal{S}_\infty \otimes_{\mathcal{O}_{X_0}} \mathcal{O}_{X_0}(-\{0_+\} + \{0_+\}).$$

However, as the arithmetic genus of $X_0$ is 0, the latter sheaf is trivial. Hence, $L^{0,1}(0)$ is also isomorphic to $\mathcal{S}_\infty$.

The case of $\mathcal{S}_0$ can then be obtained by exchanging the roles of 0 and $\infty$.

We infer from the discussion above that the moduli space has the structure of an elliptic fibration near the point $b \in B$ corresponding to the singular fiber. Furthermore, it is easy to check (using the fact that the parabolic weights are non-zero) that in this case semi-stability is equivalent to stability. Therefore, by the moduli space is complete. It then follows that the fiber of the Hitchin map $H$ over $b$ is either a smooth elliptic curve or one of the singular fibers on Kodaira’s list. As we have shown above, this fiber is homeomorphic to two copies of $\mathbb{C}P^1$ attached at two different points. In particular, the fiber is singular, and as the only fiber on Kodaira’s list homeomorphic to two copies of $\mathbb{C}P^1$ attached at two points is $I_2$, we conclude that $H^{-1}(b)$ is a type $I_2$ curve.

5.1.2. Case of degree 0. The analysis is similar to the case of degree $-1$, hence we only give the outline. In the case $|J(S)| = 2$ of invertible sheaves, we obtain

$$\delta_+ = 1 = \delta_-, \delta_- = 2 - \delta_+, \delta_+ \in \{0,1,2\},$$

and if $|J(S)| = 1$ then no $(\delta_+, \delta_-)$ solves $\mathbb{H}$. We infer that stable sheaves are parametrized by $\mathbb{C}^\times$. Let us now consider strictly semi-stable sheaves. Then, the solutions in the case $|J(S)| = 2$ are

$$\delta_+ \in \{0,1,2\},$$

with $\delta_- = 2 - \delta_+$. The parameter space consists of 3 copies of $\mathbb{C}^\times$. The solutions $(\delta_+, \delta_-)$ with $|J(S)| = 1$ are

$$(0,1), (1,0),$$

each being parametrized by a point. The point corresponding to bidegree $(0,1)$ is both a limit point of the $\mathbb{C}^\times$ parametrizing invertible sheaves of bidegree $(0,2)$ and the one parametrizing invertible sheaves of bidegree $(1,1)$. Similarly, the point corresponding to bidegree $(1,0)$ is both a limit point of the $\mathbb{C}^\times$ parametrizing invertible sheaves of bidegree $(2,0)$ and the one parametrizing invertible sheaves of bidegree $(1,1)$. All the semi-stable solutions are parametrized by a copy of $\mathbb{C}P^1$ with two copies of $\mathbb{C}$ attached to it at two
different points of $\mathbb{C}P^1$. In contrast with the case of degree $-1$, this time there do exist strictly semi-stable Higgs bundles, and in addition the parabolic weights are not all distinct. Hence, we cannot use a completeness argument to determine the algebraic type of the singular fiber.

5.2. **Stability analysis in the case** $\tilde{E}_7 + III$. We now let $b \in B$ be the point whose preimage in the pencil is the singular fibre of type $III$. We again have \([39]\).

We assume that 
\[ E = p_*(S) \]
and use the definitions of \([2]\). The curve $X_b$ has a single singular point $x$ which is a tacnode (an $A_3$-singularity). It is known that there exists a fractional ideal 
\[ O_{X_b,x} \subseteq I \subseteq \tilde{O}_{X_b,x} \]
of $O_{X_b,x}$ such that 
\[ S_x \cong I. \]
The length of $S$ at $x$ is by definition 
\[ l(S) = \dim_C(I/O_{X_b,x}), \]
and we have the inequalities 
\[ 0 \leq l(S) \leq \dim_C(O_{X_b,x}/O_{X_b,x}) = 2. \]
Now there exists a short exact sequence of sheaves 
\[ 0 \to S \to \mathcal{L}(S)|_{X_b} \oplus \mathcal{L}(S)|_{X_b} \to \mathbb{C}^{2-l(S)} \to 0, \]
hence 
\[ \chi(S) + 2 - l(S) = \chi(S)|_{X_b} + \chi(S)|_{X_b}. \]
Applying this to $O_{X_b}$ in the place of $S$ we get 
\[ \chi(O_{X_b}) + 2 = \chi(O_{X_b}) + \chi(O_{X_b}). \]
Subtracting the second formula from the first we infer 
\[ \delta - l(S) = \delta_+ + \delta_-, \]
with $\delta, \delta_+, \delta_-$ the degrees of $S, \mathcal{L}(S)|_{X_b}$ and $\mathcal{L}(S)|_{X_b}$, respectively. Using this formula and \([26]\) we can rewrite \([39]\) as 
\[ (45) \quad 0 = \delta_+ + \delta_- + l(S) - 2 + \alpha_+ + \alpha_. \]
The canonical morphisms \([10]\) give quotient irregular parabolic Higgs bundles $E_i$ of $E$ of rank 1 and degree 
\[ d_i = \delta_i \]
for $i \in \{ \pm \}$. Furthermore, these are again the only non-trivial Higgs quotient bundles of $E$.

The parabolic weight associated to $E_i$ is $\alpha_i$, so the parabolic degree of $E_i$ is 
\[ \text{par-deg}(E_i) = \delta_i + \alpha_i. \]
It follows that the parabolic stability of $(E, \theta)$ is equivalent to the inequalities 
\[ 0 < \delta_i + \alpha_i \]
for $i \in \{ \pm \}$. Taken \([45]\) into account, this is equivalent to 
\[ (46) \quad \delta_+ + \alpha_+ + 2l(S) - 2 < \delta_- + \alpha_- + l(S) < \delta_+ + \alpha_+ + 2. \]
This time this inequality immediately implies that there exist no stable Higgs bundles with spectral sheaf $S$ of length 2.

We again set 
\[ \alpha_i = \alpha_i \]
and we need to distinguish two cases:
5.2.1. Case of degree $-1$. Let us first treat the case of (47). Assume first $l(S) = 0$, i.e. $S$ is an invertible sheaf on $X_b$. Then, independently of the values of $\alpha^\pm$ satisfying (47), condition (46) implies either

$$\delta_+ = 0, \delta_- = 1$$

or

$$\delta_+ = 1, \delta_- = 0.$$

Therefore, such sheaves are parameterized by $\mathbb{C} \amalg \mathbb{C}$, as it readily follows from the long exact sequence associated to

$$0 \to \mathcal{O}_{X_b} \to \mathcal{O}_{\tilde{X}_b} \to \mathcal{O}_{\tilde{X}_b,x} \to 0$$

using the fact that $\tilde{X}_b$ has two connected components.

If, on the other hand, we have $l(S) = 1$ then the only solution is

$$\delta_+ = 0 = \delta_-,$$

again independently of the values of $\alpha^\pm$. This latter sheaf is in the closure of both components $\mathbb{C}$ parameterizing invertible sheaves. We infer that up to homeomorphism, the Hitchin fiber over the point $b$ is parameterized by two copies of $\mathbb{C}P^1$ attached at one point.

As the generic fiber of the Hitchin-fibration is an elliptic curve and the moduli space is complete by [5], the fiber over $b$ must be again one of the fibers of Kodaira’s list. However, the only singular fiber on the list that is homeomorphic to two copies of $\mathbb{C}P^1$ glued at one point is the fiber of type $III$. Therefore, the Hitchin fiber $H^{-1}(b)$ is a singular curve of type $III$.

5.2.2. Case of degree $0$. Let us now study the case of (48): in this case, by virtue of (45) we have

$$2 - l(S) = \delta_+ + \delta_-.$$

If $l(S) = 0$ then we readily see that the only solution to equation (46) is

$$\delta_+ = 1 = \delta_-,$$

and just as above one can show that such sheaves are parameterized by $\mathbb{C}$.

On the other hand, if $l(S) = 1$ then (46) has no solutions; however, if we relax the inequalities in (46) to not necessarily strict ones, then there exist two solutions:

$$\delta_+ = 0, \delta_- = 1$$

and

$$\delta_+ = 1, \delta_- = 0.$$

The sheaves with these properties are parameterized by one point in each of the two cases.

Let us first analyze the case (49): in this case, the destabilizing quotient of $(E, \theta)$ is $E_-$; indeed, we have

$$0 = \delta_+ = \deg(E_-) = \deg(E) = \deg(E) = \deg(E),$$

since the parabolic weights vanish. The destabilizing Higgs subbundle of $E$ is

$$\ker(E \to E_-),$$

which is a lower elementary transformation of $E_-;$$

$$\ker(E \to E_-) = E_-(\{-t\}),$$
where \( t \in \mathbb{C}P^1 \) is the image under \( p \) of the singular point of \( X_b \). Indeed, we have
\[
\deg(\mathcal{E}(\{t\})) = \deg(\mathcal{E}_-) - 1 = 0,
\]
and \( \mathcal{E}(\{t\}) \) is preserved by \( \theta \) simply because the image by \( \theta \) of vanishing sections of \( \mathcal{E} \) at \( t \) also vanish at \( t \), in particular, they belong to \( \mathcal{E}_- \). The Jordan–Hölder filtration of \( (\mathcal{E}, \theta) \) is therefore given by
\[
\mathcal{E}_- \subset \mathcal{E},
\]
with associated graded
\[
\mathcal{E}_- \oplus \mathcal{E}_+ \text{ endowed with the action}
\]
\[
\begin{pmatrix}
\theta_- & 0 \\
0 & \theta_+
\end{pmatrix},
\]
where \( \theta_\pm \) are the morphisms induced by \( \theta \) on the two direct summands. According to (51), the vector bundle underlying this graded Higgs bundle is isomorphic to the trivial bundle of rank 2 over \( \mathbb{C}P^1 \). Moreover, the action of \( \theta_\pm \) in the above matrix clearly have spectral curves \( X_\pm \) respectively.

The case of (49) can be treated in a very similar manner, except that one needs to exchange the roles of \( \mathcal{E}_- \) and \( \mathcal{E}_+ \). It then follows that the destabilizing Higgs subbundle of \( \mathcal{E} \) is
\[
\mathcal{E}_+ \oplus \mathcal{E}_- \oplus \mathcal{E}_+ \text{ (52)},
\]
and that the graded Higgs bundle associated to the Jordan–Hölder filtration is
\[
\mathcal{E}_- \oplus \mathcal{E}_+ \oplus \mathcal{E}_+ \text{ (52)},
\]
the trivial vector bundle of rank 2 over \( \mathbb{C}P^1 \), with Higgs field given by the formula (52).

To sum up, in the degree 0 case the Hitchin fiber over the point \( b \) is homeomorphic to the compactification of \( \mathcal{C} \) (corresponding to invertible sheaves) by a unique point (corresponding to sheaves of length 1). However, the parabolic weights are equal and there exist strictly semi-stable Higgs bundles, so we cannot use a completeness argument to determine algebraically the special fiber of the Hitchin map.

6. Twisted Case

In this section we determine a certain blow-up \( Y \) of \( \tilde{X} \) depending on the parameters appearing in (2) with the property that certain sheaves on \( Y \) are in one-to-one correspondence with Higgs bundles of the local form (24). We need two preliminary lemmas.

**Lemma 6.1.** Let \( \theta \) be a Higgs field of the local form (24). Let us denote by \( \zeta dz \) the eigenvalues of \( \theta \); \( \zeta \) is a ramified bi-valued meromorphic function of \( z_1 \).

1. Assume that \( b_7 \neq 0 \). Then, for \(-8 \leq n \leq -3\) the coefficients of the Puiseux expansion
\[
\zeta = \sum_{n=-8}^{\infty} a_n z_1^{\frac{n}{7}}.
\]

Admit expressions
\[
a_n = a_n(b_-8, \sqrt[7]{b_7}, b_-6, \ldots, b_0) \in \mathbb{C}[b_-8, b_-6, b_-7, \ldots, b_0]
\]
in the parameters \( b_n \), and \( a_-7 \neq 0 \).
Vice versa, if \( \theta \) is of the local form (24) and \( a_{-7} \neq 0 \) then the parameters \( b_{-8}, \ldots, b_{-3} \) admit polynomial expressions

\[
 b_n = b_n(a_{-8}, \ldots, a_n) \in \mathbb{C}[a_{-8}, \ldots, a_n]
\]

in function of the Puiseux coefficients of \( \zeta \), and \( b_{-7} \neq 0 \).

**Proof.** This is a straightforward computation. Specifically, we have

\[
a_{-8} = b_{-8},
\]

for \( n \in \{-6, -4\} \) we have

\[
a_n = \frac{b_n}{2},
\]

and the coefficients with odd indices are given by

\[
a_{-7} = \sqrt{b_{-7}},
\]

\[
a_{-5} = \frac{1}{8\sqrt{b_{-7}}}(b_{-6}^2 + 4b_{-5}),
\]

\[
a_{-3} = \frac{1}{8\sqrt{b_{-7}}}(2b_{-4}b_{-6} + 4b_{-3}) - \frac{1}{128b_{-7}\sqrt{b_{-7}}}(b_{-6}^2 + 4b_{-5})^2,
\]

(the square root of \( b_{-7} \) depending on the choice of square root of \( z \) in the Puiseux series).

The inverse transformations are given by

\[
b_{-7} = a_{-7}^2,
\]

\[
b_{-5} = 2a_{-5}a_{-7} - a_{-6}^2
\]

\[
b_{-3} = 2a_{-3}a_{-7} - 2a_{-4}a_{-6} + a_{-5}^2.
\]

\[\Box\]

In the lemma below we follow the conventions and notations introduced in Sections 2 and 4. In particular, in view of the definition of the affine coordinate system \((z_1, z_2)\) near \( p^{-1}(q) \setminus C^\infty \) and Lemma 6.1, the equation of the spectral curve of a Higgs field of the local form (24) reads as

\[
z_2 = \sum_{n=0}^{\infty} a_{n-8} z_1^n.
\]

**Lemma 6.2.** Assume the above Puiseux expansion holds.

1. If \( a_{-7} \neq 0 \), then for \( 2 \leq n \leq 6 \) there exist polynomials

\[
d_n = d_n(a_{-7}, \ldots, a_{n-9}) \in \mathbb{C}[a_{-7}^\pm, a_{-6}, \ldots, a_{n-9}]
\]

such that we have the Taylor series

\[
z_1 = d_2(z_2 - a_{-8})^2 + \cdots + d_6(z_2 - a_{-8})^6 + O((z_2 - a_{-8})^7).
\]

Moreover, \( d_2 \neq 0 \).

2. Conversely, the value \( a_{n-9} \) is a polynomial in \( d_2^{3/2}, d_3, \ldots, d_n \), and \( a_{-7} \neq 0 \).

**Proof.** By assumption we have

\[
\frac{z_2 - a_{-8}}{a_{-7}} = \sum_{n=1}^{\infty} \frac{a_{n-8}}{a_{-7}} \frac{z_1^n}{z_1^7} = z_1^\frac{4}{7} + O(z_1).
\]
Let geometry governs coordinates step. We assume that recursively find the point on the exceptional divisor that we blow up in the following step. We define affine coordinates as the blow-up of the point. In concrete terms, on the affine chart we find so on a curve \(E\), \(\sigma_1 : \tilde{X} \to X\) is the blow-up of \(X\) in the point
\[
[a_{-8}^0 : 1_0].
\]
Let \(E_1 \subset \tilde{X}\) denote the corresponding exceptional divisor, see Figure 2. Observe that the coordinates \([z'_1 : z'_2]\) on \(E_1\) now satisfy
\[
\frac{z'_1}{z'_2} = \frac{z_1}{z_2 - a_{-8}}.
\]
so on a curve \(\Sigma\), having the expansion of \((54)\), we have
\[
\frac{z'_1}{z'_2} = -\sum_{n=2}^{\infty} d_n(z_2 - a_{-8})^{n-1}.
\]
We define
\[
\sigma_2 : X_2 \to \tilde{X}
\]
as the blow-up of the point
\[
[z'_1 : z'_2] = [0 : 1] \in E_1.
\]
In concrete terms, on the affine chart \(V_1 = \{z'_2 \neq 0\} \subset \tilde{X}\) we normalize \(z'_2 = 1\) and in the affine coordinates \((z'_1, z'_2)\) on \(V_1\) we consider
\[
\{(z'_1, z_2) \mid (z'_1, z'_2) \in V \times \mathbb{C}P^1, z'_2 z'_1 - z'_1 (z_2 - a_{-8}) = 0\}.
\]

\[\text{Lemma 6.3. Assume that Lemmas 6.1 and 6.2 hold.}\]

1. If \(b_{-7} \neq 0\), then for \(2 \leq n \leq 6\) there exist polynomials
\[
d_n = d_n(b_{-7}, \ldots, b_{-9}) \in \mathbb{C}[b_{-7}, b_{-6}, \ldots, b_{-9}]
\]
such that we have the Taylor series
\[
z_1 = d_2(z_2 - b_{-8})^2 + \cdots + d_6(z_2 - b_{-8})^6 + O((z_2 - b_{-8})^7).
\]
Moreover, \(d_2 \neq 0\).

2. Converse, the value \(b_{-9}\) is a polynomial in \(d_2^2, d_3, \ldots, d_6\) and \(b_{-7} \neq 0\). 

\[\text{Proof. The lemma directly follows from the previous two lemmas.}\]

We now proceed to construct the surface \(Y\) with a birational morphism to \(\tilde{X}\) whose geometry governs \(\mathcal{M}\). The idea is similar to the untwisted case: we use the above expansions to recursively find the point on the exceptional divisor that we blow up in the following step. We assume that \(\sigma_1 : \tilde{X} \to X\) is the blow-up of \(X\) in the point
\[
[0 : 1].
\]
With these definitions, over $V_2 = \{z_2' \neq 0\} \subset X_2$ on a curve $\Sigma$ having the expansion of $\Sigma_2$, we have

$$\frac{z_1''}{z_2'} = \frac{z_1'}{z_2 - a_{-8}} = \frac{z_1 z_2'}{(z_2 - a_{-8})^2} = \frac{z_1}{(z_2 - a_{-8})^2} = \sum_{n=2}^{\infty} d_n (z_2 - a_{-8})^{n-2}$$

(recall we have set $z_2' = 1$).

From this point on, the pattern of the construction of $Y$ is clear and similar to the construction in the untwisted case. Namely, for $3 \leq n \leq 8$ we successively consider the blow-up

$$\sigma_n : X_n \rightarrow X_{n-1}$$

of the point

$$[z_1^{n-1} : z_2^{n-1}] = [d_n : 1] \in E_{n+1}$$

and denote by $E_{n+2}$ the exceptional divisor of $\sigma_n$. We set

$$Y = X_8,$$

and define

$$\sigma = \sigma_8 \circ \cdots \circ \sigma_1 : Y \rightarrow X.$$

**Proposition 6.4.** There exists an equivalence of categories between the groupoids of

1. Higgs bundles on $\mathbb{C}P^1$ with one singular point $q = 0$ and local form given by $[24]$ with $b_{-7} \neq 0$,

2. pure sheaves of dimension 1 and rank 1 on $Y$ supported on a curve $\Sigma$ which is disjoint from $E_1, \ldots, E_3$ and intersects $E_4$ with algebraic multiplicity 1.

**Proof.** Let $(E, \theta)$ be a Higgs-field as in part (1). Consider its spectral sheaf

$$S_0 = \text{coker} \left( p^* (E) \otimes \Theta_C (-4 \cdot \{0\}) \right),$$

where $\Theta_C (-4 \cdot \{0\})$ is the dual bundle of $K_C (4 \cdot \{0\})$, and $\xi \in H^0 (Z, \mathcal{O}_Z (1))$, $\zeta \in H^0 (Z, p^* (K_C (4 \cdot \{0\}_C) \otimes \mathcal{O}_Z (1)))$ are the canonical sections. Let us denote by $\Sigma_0$ the support of $S_0$. Assume that $\Sigma_0$ is integral (i.e. irreducible and reduced). Then, by [3], we have

- $\Sigma_0$ is disjoint from $C_\infty$,
- $p$ is finite over $\Sigma_0$,
- $S_0$ is torsion-free on $\Sigma_0$,
- $p_* S_0 = E$,
- the direct image of multiplication by $\zeta$ on $S_0$ induces $\theta$.

Conversely, any sheaf $S_0$ satisfying the first three of these properties is the spectral sheaf of an irregular Higgs bundle $(E, \theta)$. The integrality requirement on $\Sigma_0$ was later lifted in [14].

The idea of the proof is to use the properties of proper transform functor of coherent sheaves under the blow-up introduced in [1]. Namely, for any smooth surface $W$ and a point $w \in W$, let us denote by $\tau : \overline{W} \rightarrow W$ the blow-up of $w$ and by $E$ the exceptional divisor. Now, given any coherent sheaf $\mathcal{F}$ of $\mathcal{O}_W$-modules we set

$$\mathcal{F}^E := \mathcal{T}_{\mathcal{O}_W}^C (\tau^* \mathcal{F}, \mathcal{O}_W (E)_E)$$
and
\[ \mathcal{F}^* = \tau^* \mathcal{F}/\mathcal{F}^E. \]

With these notations, we have the following result.

**Lemma 6.5** (Lemma 5.12 [1]). Suppose that the homological dimension of \( \mathcal{F} \) at \( x \) satisfies \( \text{dh}(\mathcal{F}_x) = 1 \).

1. If \( \mathcal{F}_x \) is torsion, then \( \text{dh}(\mathcal{F}_y^*) = 1 \) for any \( y \in E \).
2. We have \( R^0(\sigma_*) = \mathcal{F}^* \) and \( R^i(\tau_*) = 0 \) for all \( i > 0 \).
3. If \( \mathcal{F} \) is pure of dimension \( 1 \) then \( E \notin \text{supp}(\mathcal{F}^*) \).

The definition of \( \mathcal{S}_0 \) makes it clear that it is a torsion module, of homological dimension \( 1 \). As the surface \( X \) is regular, according to the Auslander–Buchsbaum formula we also get that \( \mathcal{S}_0 \) is pure of dimension 1. Let us write
\[ \mathcal{S}_1 = (\mathcal{S}_0)^{\tau_1}. \]

Then part (1) of the Lemma applied to \( W = X, w \in X \) the point given by \( \Sigma_{10} \) and \( \mathcal{F} = \mathcal{S}_0 \) implies that \( \mathcal{S}_1 \) is also of homological dimension 1, and as above we also get that it is pure of dimension 1. Furthermore, part (2) of the lemma implies that
\[ R^0(\sigma_*) (\mathcal{S}_1) = \mathcal{S}_0. \]

We recursively define for all \( n \in \{2, \ldots, 8\} \) the coherent sheaf
\[ \mathcal{S}_n = (\mathcal{S}_{n-1})^{\tau_n} \]
on \( X_n \). Recursive application of part (1) of the lemma then implies that \( \mathcal{S}_n \) is of homological dimension 1 and pure of dimension 1, and by part (2) it satisfies
\[ R^0(\sigma_*) (\mathcal{S}_n) = \mathcal{S}_{n-1}. \]

Let us set \( \mathcal{S} = \mathcal{S}_8 \). It then follows that using the map of \( \Sigma_{10} \) we have
\[ R^0(\sigma_*) (\mathcal{S}) = \mathcal{S}_0. \]

Using the properties of \( \mathcal{S}_0 \) we then get that
\[ R^0(\rho \circ \sigma)_* (\mathcal{S}) = \mathcal{E}. \]

We now show that
\[ \mathcal{E} \mapsto \mathcal{S} = \mathcal{S}_8 \]
gives a map from the set of objects of (1) to the set of objects of (2). Indeed, purity follows from Lemma 6.5 as observed above. The rank of \( \mathcal{S} \) is equal to 1 because of (57), given that the rank of \( \mathcal{E} \) is 2 and that \( \rho \circ \sigma_{\Sigma} \) is a double cover of \( \mathbb{P}^1 \). Finally, by part (3) of the lemma, the exceptional divisors \( E_1, \ldots, E_{10} \) are not contained in \( \Sigma \). Moreover, according to part (1) of Lemma 6.1 and part (1) of Lemma 6.2 for each \( n \) the center of the blow-up \( \sigma_n \) is the only intersection point of the proper transform of \( \Sigma_0 \) in \( X_{n-1} \) with the exceptional divisor \( E_{n+1} \). This implies the statement about intersections.

Conversely, suppose that a sheaf \( \mathcal{S} \) fulfilling the properties of (2) is given. Then, we define a holomorphic vector bundle \( \mathcal{E} \) by \( \mathcal{E}_1 \), and we define a Higgs field \( \theta \) as the direct image of multiplication by \( \zeta d\zeta \) on \( \mathcal{E}_0 = R^0(\sigma_*) (\mathcal{S}) \). If the curve \( \Sigma \) is disjoint from \( E_1, \ldots, E_6 \) and intersects \( E_{10} \) with algebraic multiplicity 1, then the expansion of its image \( \sigma(\Sigma) \) near \( q \) is given by \( \Sigma_{10} \). By virtue of part (3) of Lemma 6.2 this implies the converse expansion \( \Sigma_{10} \) with \( a_{-\gamma} \neq 0 \). Then, according to part (2) of Lemma 6.1 the coefficients in the form \( \Sigma_{24} \) are as required. This then gives the inverse map of (58) on objects.

Now, let us consider the map on morphisms. Recall that an isomorphism \( (\mathcal{E}_1, \theta_1) \cong (\mathcal{E}_2, \theta_2) \) amounts to an isomorphism of vector bundles
\[ \Psi : E_1 \rightarrow E_2 \]
such that
\[ \theta_2 \circ \Psi = (\Psi \otimes I_K) \circ \theta_1, \]
where \( I_K \) stands for the identity map of the canonical bundle \( K_{\mathbb{P}^1} \). Therefore, if \((\mathcal{E}_1, \theta_1)\) and \((\mathcal{E}_2, \theta_2)\) are isomorphic, then we have a diagram
\[
\begin{array}{cccc}
0 & \longrightarrow & p^* \mathcal{E}_1 & \xrightarrow{\xi \otimes p^* \theta_1 \mathcal{E}_1} & \mathcal{S}_0(\mathcal{E}_1, \theta_1) \otimes p^*(K(4 \cdot \{0\})) & \longrightarrow & 0 \\
0 & \longrightarrow & p^* \mathcal{E}_2 & \xrightarrow{\xi \otimes p^* \theta_2 \mathcal{E}_2} & \mathcal{S}_0(\mathcal{E}_2, \theta_2) \otimes p^*(K(4 \cdot \{0\})) & \longrightarrow & 0
\end{array}
\]

It follows from this diagram that there exists a morphism of sheaves of \( \mathcal{O}_K \)-modules
\[
\mathcal{S}_0(\mathcal{E}_1, \theta_1) \leftrightarrow \mathcal{S}_0(\mathcal{E}_2, \theta_2),
\]
which is an isomorphism with inverse induced by \( \Psi^{-1} \) in the same way. This isomorphism in turn induces isomorphisms
\[
\mathcal{S}_b(\mathcal{E}_1, \theta_1) \leftrightarrow \mathcal{S}_b(\mathcal{E}_2, \theta_2)
\]

by functoriality of the proper transform operation. On the other hand, such an isomorphism of spectral sheaves gives an isomorphism of Higgs bundles by functoriality of the direct image functor. This finishes the proof of Proposition 6.4

**Proof of Theorem 1.2** According to Proposition 6.4, describing the moduli space of irregular Higgs bundles with local form given by (24) is equivalent to describing the relative Picard scheme of degree 1 torsion-free sheaves on curves satisfying the properties listed in part (2) of the Proposition.

As in the untwisted case, we write the characteristic polynomial of \( \theta \) in the trivialization given by \( \kappa \) and \( \kappa^2 \) of (23). The polynomials \( f \) and \( g \) are given in (29) and (30), and the characteristic polynomial is:
\[
\chi_{\theta, ES}(z_2, z_1) = z_2^2 - (p_2 z_1^2 + p_1 z_1 + p_0) z_2 - (q_4 z_1^4 + q_3 z_1^3 + q_2 z_1^2 + q_1 z_1 + q_0).
\]

Now the roots of \( \chi_{\theta, ES}(z_2, 0) \) in \( z_2 \) are equal, because the curve intersects the \( z_1 = 0 \) line in one point. This requirement is satisfied if the discriminant of \( \chi_{\theta, ES} \) vanishes at \( z_1 = 0 \), that is,
\[
p_0^2 + 4q_0 = 0.
\]

After this simplification, we consider the expansions of the roots of \( \chi_{\theta, ES}(z_2, z_1) \) with respect to \( z_1 \). It is enough to consider the positive root, because the expansions of the two roots differ in a negative sign in certain terms. The expansion is:
\[
z_2 \left[ \frac{p_0}{2} + \frac{1}{2} \sqrt{2p_0 p_1 + 4q_1} z_1 + \frac{p_1}{4} z_1 + \frac{p_1^2 + 2p_0 p_2 + 4q_2}{4 \sqrt{2p_0 p_1 + 4q_1}} - \left( \frac{p_1^2 + 2p_0 p_2 + 4q_2}{16 (p_0 p_1 + 2q_1)^2} \right)^{1/2} \right].
\]

We write the local form of \( \theta \) in the twisted case as in (24). We described the matrix eigenvalues in Lemma 6.1. by the Puiseux expansion, with coefficients \( a_{i \nu} \).

These two expansions are the same, hence by comparing the coefficients we get the following:
\[
\chi_{\theta, ES}(z_2, z_1, t) = -b_{-6} z_2^2 + (b_{-4} z_2^2 - b_{-6} z_1 - 2b_{-8}) z_2 - t z_1^4 - b_{-3} z_1^3 + (b_{-8} b_{-6} - b_{-7}) z_1 + b_{-8},
\]
where (as in the untwisted case) we denote \( q_4 \) by \( t \), and the degree of the polynomial is 2 in the variable \( z_2 \) and 4 in \( z_1 \). Therefore, we get a pencil parametrized by \( t \) with baselocus \((0, b_{-8})\) in \( \mathbb{C}^2 \).
According to the Theorem 3.1 $\chi_{\theta,E_8}$ provides an elliptic fibration in $\mathbb{C}P^2 \# 9\mathbb{C}P^2$ with a singular fiber of type $E_8$, and the pencil determines the types of further singular fibers in the elliptic fibration. In the following we will identify the types of these further singular fibers in terms of the defining constants of the pencil.

Since the $z_1 = 0$ line (being of multiplicity 4) contains no singular point of any curve in the pencil, the problem becomes more convenient to study on the $(z_2, w_1)$ chart (which, therefore, will contain all singularities). We will use the trivialization given by $\kappa_2$ and $\kappa_2'$ according to (33) and (34) on that chart. The polynomials $f_2$ and $g_2$ are given in (35) and (36), and the characteristic polynomial is:

$$\chi_{\theta,E_8}(z_2, w_1, t) = z_2^2 + f_2(w_1)z_2 + g_2(w_1, t) =$$

$$z_2^2 + (p_0w_1^2 + p_1w_1 + p_2)z_2 + \frac{1}{4}p_0^2w_1 - q_1w_1^3 - q_2w_1^2 - q_3w_1 - t =$$

$$z_2^2 + (2b_{-s}w_1^2 + b_{-6}w_1 + b_{-4})z_2 + b_{-8}w_1^4 + (b_{-7}b_{-6} - b_{-7})w_1^3 +$$

$$+ (b_{-8}b_{-4} - b_{-s})w_1^2 - b_{-3}w_1 - t.$$

In identifying the further singular fibers in the fibration, we look for triples $(z_2, w_1, t)$ such that $(z_2, w_1)$ fits the curve with parameter $t$, and the partial derivates below vanish:

$$\frac{\partial \chi_{\theta,E_8}(z_2, w_1, t)}{\partial z_2} = 0,$$

$$\frac{\partial \chi_{\theta,E_8}(z_2, w_1, t)}{\partial w_1} = 0.$$

Notice that the second and third equations do not involve $t$, hence we can solve this system for $z_2$ and $w_1$. Indeed, by solving the second equations for the variable $z_2$ we get

$$z_2 = -\frac{1}{2}(2b_{-s}w_1 + b_{-6})w_1 - \frac{b_{-4}}{2}.$$ 

We substitute the resulting expression to the third equation, leading to

(59) $$0 = 6b_{-7}w_1^2 + (b_{-6}^2 + 4b_{-5})w_1 + b_{-4}b_{-4} + 2b_{-3}.$$

This polynomial is quadratic in $w_1$ and has one root if and only if the discriminant

$$D = (b_{-6}^2 + 4b_{-5})^2 - 24b_{-7}(b_{-6}b_{-4} + 2b_{-3})$$

vanishes. In this case the pencil has a single further singular fiber, which has a cusp singularity. If $D \neq 0$ then the fibration has two $I_1$ singular fibers.

**Remark 6.6.** If $b_{-6}b_{-4} + 2b_{-3} = 0$ but $b_{-6}^2 + 4b_{-5} \neq 0$, then the polynomial in (59) has two roots: $(w_1)_1 = 0$ and $(w_1)_2 \neq 0$. Geometrically this means that the pencil has two singular curves, each with a fishtail singularity, and one of the singular points is not visible on the $(z_2, z_1)$ (corresponding to the value $z_1 = \infty$).

Similarly, if both expressions above are zero, then $D = 0$, hence the pencil has a curve with a type $II$ singularity, with the singular point having $z_1 = \infty$, again invisible on the $(z_2, z_1)$ chart. These two cases justify our use of the $(z_2, w_1)$ chart in our computations.

The fibration obtained from the pencil has a section, so just as in the proof of Theorem 1.1 we may apply the relative Abel–Jacobi map to identify the fibration and its relative Picard scheme over the locus of smooth curves. Thus it is sufficient to describe the singular fibers of $H$. By Proposition 2.7 these latter are as stated in Theorem 1.2 concluding the proof. □
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