Optimal $L^2$-Extensions on Tube Domains and a Simple Proof of Prékopa’s Theorem

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Received: 1 May 2021 / Accepted: 10 August 2021 / Published online: 10 December 2021
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Abstract
We prove the optimal $L^2$-extension theorem of Ohsawa–Takegoshi type on a tube domain. As an application, we give a simple proof of Prékopa’s theorem.

Keywords Prékopa’s theorem · $L^2$-extension · Convexity · Minimal extension property

Mathematics Subject Classification 32U05 · 52A39

1 Introduction
Prékopa’s theorem [12], which can be seen as a generalization of the Brunn–Minkowski theorem, plays an important role in convex geometry. The theorem asserts that if $\varphi : \mathbb{R} \times \mathbb{R}_x^n \rightarrow \mathbb{R}$ is a convex function, the function $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$e^{-\Phi(t)} := \int_{\mathbb{R}_x^n} e^{-\varphi(t,x)} d\lambda(x)$$

is also convex.

Replacing $\mathbb{R}$ by $\mathbb{C}$ and convex functions by plurisubharmonic functions, we can consider a version of Prékopa’s theorem in the complex setting. Unfortunately, it is known that this complex Prékopa problem does not hold in general (see [10]). However, Berndtsson [1, Theorem 1.3, 2] proved that if a plurisubharmonic function $\varphi : D \times (V + \sqrt{-1} \mathbb{R}_x^n) \subset \mathbb{C}_t \times \mathbb{C}_x^n \rightarrow \mathbb{R} \cup \{-\infty\}$ is independent of $\text{Im}(z)$, the function $\Phi$ on $D$ defined by
$e^{-\Phi(\tau)} := \int_V e^{-\psi_{\tau, \Re(z)}} d\lambda(\Re(z))$

is plurisubharmonic as well, where $V \subset \mathbb{R}^n$ is a convex domain and $V + \sqrt{-1}\mathbb{T}^n := \{z = x + \sqrt{-1}y \in \mathbb{C}^n \mid x \in V\}$ is a tube domain. The above assumption of $\varphi$ is appropriate in the following sense. If $\varphi$ is a convex function on $V$, the associated function $\hat{\varphi}(z) := \varphi(x)$ is plurisubharmonic on $V + \sqrt{-1}\mathbb{T}^n$. Conversely, if $\hat{\varphi}$ is plurisubharmonic on $V + \sqrt{-1}\mathbb{T}^n$ and independent of $\Im(z)$, the well-defined function $\varphi(x) := \hat{\varphi}(x + \sqrt{-1}\mathbb{T}^n)$ is convex on $V$. This simple observation allows us to study the convexity of functions via complex analytic methods. For the Prékopa theorem and the complex Prékopa theorem, one main tool to prove them is the $L^2$-estimate of the optimal $L^2$-extension theorem due to [2,8], initially proved by Ohsawa and Takegoshi [11] for some constant, not necessarily optimal.

In this article, we give a proof of Prékopa’s theorem by using $L^2$-extension theorems without any regularity assumption or direct computation of curvature. In order to give the proof, we prove the following optimal $L^2$-extension theorem.

**Theorem 1.1** Let $D \subset \mathbb{C}_r$ be a domain, $V$ be a bounded convex domain in $\mathbb{R}^n_x$ and $V_x + \sqrt{-1}\mathbb{T}^n_x \subset \mathbb{C}^n_x$ be a tube domain. Assume that $\varphi(\tau, z)$ is a plurisubharmonic function on $D \times (V + \sqrt{-1}\mathbb{T}^n)$, which is independent of $y = \Im(z)$. Then, for any point $a \in D$ and any $r > 0$ such that $\int_V e^{-\psi(a, x)} d\lambda(x) < +\infty$ and $\Delta(a; r) = \{|\tau-a| < r\} \subset D$, there exists a holomorphic function $f$ on $\Delta(a; r)$ satisfying $f(a) = 1$ and

$$\int_{\Delta(a; r) \times V} |f(\tau)|^2 e^{-\psi(\tau, x)} d\lambda(\tau, x) \leq \pi r^2 \int_V e^{-\psi(a, x)} d\lambda(x).$$

This is a version of the optimal $L^2$-extension theorem due to [2,8], initially proved by Ohsawa and Takegoshi [11] for some constant, not necessarily optimal.

The proof of Theorem 1.1 is a little bit complex. On the other hand, if we regard the optimal $L^2$-extension theorem above as a fact, we can give a quite simple proof of Prékopa’s theorem. A key notion is the minimal extension property or the optimal $L^2$-extension property, which is introduced in [9] or [4,5], respectively.

## 2 Optimal $L^2$-Extensions and Minimal Extension Property

In this article, we let $\lambda_n$ denote the standard Lebesgue measure on $\mathbb{R}^n$ and omit $n$. First, we introduce the optimal $L^2$-extension theorem in the following form.

**Theorem 2.1** ([2,8]) Let $D$ be a bounded pseudoconvex domain with $D \subset \mathbb{C}^{n-1} \times \{|z_n| < r\}$ for $r > 0$. We also let $\varphi$ be a plurisubharmonic function on $D$ and $H := \Omega \cap \{z_n = 0\}$. Then for any holomorphic function $f$ on $H$ with $\int_H |f(z')|^2 e^{-\psi(z', 0)} d\lambda(z') < +\infty$, there exists a holomorphic function $F$ on $D$ satisfying $F|_H = f$ and

$$\frac{1}{\pi r^2} \int_D |F(z', z_n)|^2 e^{-\psi(z', z_n)} d\lambda(z', z_n) \leq \int_H |f(z')|^2 e^{-\psi(z', 0)} d\lambda(z'),$$
where \((z') = (z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1}\).

Then we introduce the notion of the minimal extension property and the optimal \(L^2\)-extension property (hereafter, we will use the former term).

**Definition 2.2** (minimal extension property [9], the optimal \(L^2\)-extension property [4,5]) Let \(\varphi : D \to \mathbb{R} \cup \{-\infty\}\) be an upper semi-continuous function on a domain \(D \subset \mathbb{C}\). We say that \(\varphi\) satisfies minimal extension property if for any \(a \in D\) with \(\varphi(a) \neq -\infty\) and for any \(r > 0\) satisfying \(\Delta(a; r) \subset D\), there exists a holomorphic function on \(\Delta(a; r)\) such that \(f(a) = 1\) and

\[
\frac{1}{\pi r^2} \int_{\Delta(a; r)} |f|^2 e^{-\varphi} \, d\lambda \leq e^{-\varphi(a)}.
\]

Note that the minimal extension property can be defined for an \(n\)-dimensional domain. In this paper, we only consider the case \(n = 1\). If \(\varphi\) is plurisubharmonic, due to Theorem 2.1, \(\varphi\) satisfies the above minimal extension property. As a converse, it is known that the following result holds.

**Theorem 2.3** ([4, Theorem 1.4], cf. [5,8,9]) Keep the notation above. If an upper semi-continuous function \(\varphi\) satisfies the minimal extension property, \(\varphi\) is plurisubharmonic.

This type of idea was initially observed by Guan and Zhou in [8]. For the sake of completeness, we give the proof.

**Proof** It is enough to show that \(\varphi\) satisfies the mean value inequality at any point \(a \in D\) with \(\varphi(a) > -\infty\). Take any \(r > 0\) satisfying \(\Delta(a; r) \subset D\). Thanks to the assumption, we can take a holomorphic function \(f\) on \(\Delta(a; r)\) satisfying \(f(a) = 1\) and

\[
\frac{1}{\pi r^2} \int_{\Delta(a; r)} |f|^2 e^{-\varphi} \, d\lambda \leq e^{-\varphi(a)}.
\]

Taking logarithms and using Jensen’s inequality, we have

\[
-\varphi(a) \geq \log \left( \frac{1}{\pi r^2} \int_{\Delta(a; r)} |f|^2 e^{-\varphi} \, d\lambda \right)
\geq \frac{1}{\pi r^2} \int_{\Delta(a; r)} \log |f|^2 \, d\lambda - \frac{1}{\pi r^2} \int_{\Delta(a; r)} \varphi \, d\lambda.
\]

Since \(\log |f|^2\) is plurisubharmonic and \(f(a) = 1\), we obtain

\[
\frac{1}{\pi r^2} \int_{\Delta(a; r)} \varphi \, d\lambda \geq \varphi(a).
\]

\(\square\)
3 Optimal $L^2$-Extension Theorems on Tube Domains

In this section, we prove Theorem 1.1. Ohsawa–Takegoshi type $L^2$-extension theorems usually require the boundedness of domains. To extend holomorphic functions on unbounded domains such as tube domains, we take a functional analytic approach. The proof is inspired by the method in [1]. Throughout the proof, we simply write $y$ instead of some $y_j$ (for example, $\frac{\partial}{\partial y}$). We also say that a function $f$ is holomorphic on a non-open set $K$ if $f$ is holomorphic on some open neighborhood $U$ of $K \subset U$.

**Proof of Theorem 1.1.** The proof is divided into three steps.

(Step 1) Construct holomorphic functions on each bounded domain.

Let $B_R \subset \mathbb{R}^n$ denote $B_R := \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n \mid |y|^2 := |y_1|^2 + \cdots + |y_n|^2 < R^2\}$ for $R > 0$. Consider a constant function 1 on $\{a\} \times (V + \sqrt{-1}\mathbb{B}_R)$. Then, due to Theorem 2.1, we get a holomorphic function $f_R$ on $\Delta(a; r) \times (V + \sqrt{-1}\mathbb{B}_R)$ satisfying $f_R|_{\{a\} \times (V + \sqrt{-1}\mathbb{B}_R)} \equiv 1$ and

$$\int_{\Delta(a; r) \times (V + \sqrt{-1}\mathbb{B}_R)} |f_R|^2 e^{-\varphi(\tau, x, y)} d\lambda(\tau, x, y) \leq \pi r^2 \int_{(V + \sqrt{-1}\mathbb{B}_R)} e^{-\varphi(a, x, y)} d\lambda(x, y)$$

(3.1)

$$\leq \pi r^2 (\sigma_n R^n) \int_V e^{-\varphi(a, x)} d\lambda(x)$$

(3.2)

for each $R > 0$ since $\varphi$ is independent of $y$. Here, $\sigma_n$ is the volume of the unit ball in $\mathbb{R}^n$. Roughly speaking, we would like to consider the limit $\lim_{R \to +\infty} f_R / \sqrt{\sigma_n R^n}$. To do this procedure precisely, we take a convolution of $f_R$ with bump functions.

(Step 2) Take a convolution and estimate $L^2$ norms.

Define a bump function $\chi_R(y)$ on $\mathbb{R}^n$ as follows: $0 \leq \chi_R \leq 1$, $\chi_R$ is smooth and has compact support in $\{|y| < R - \sqrt{R}\}$, $\chi_R|_{\{|y| < R - 2\sqrt{R}\}} \equiv 1$ and $|\nabla \chi_R| \leq C / \sqrt{R}$ for some positive constant $C > 0$. We also let $a_R := \int_{\mathbb{R}^n} \chi_R(w) d\lambda(w)$. Take a convolution of $f_R$ with $\chi_R/a_R$

$$\tilde{f}_R(\tau, x, y) := \frac{1}{a_R} \int_{\mathbb{R}^n} f_R(\tau, x, y - w) \chi_R(w) d\lambda(w)$$

$$= \frac{1}{a_R} \int_{\mathbb{R}^n} \chi_R(y - w) f_R(\tau, x, w) d\lambda(w).$$

Here, we regard $f_R \equiv 0$ on $\Delta(a; r) \times (V + \sqrt{-1}\mathbb{B}_R)$ and take the convolution on $\mathbb{R}^n$.

Note that $f_R(\tau, x, w) \equiv 0$ and $\chi_R(w) \equiv 0$ if $w \in \mathbb{R}^n \setminus \overline{B_R}$. Then we have that

$$|\tilde{f}_R(\tau, x, y)|^2 = \frac{1}{a_R^2} \left| \int_{B_R} \chi_R(y - w) f_R(\tau, x, w) d\lambda(w) \right|^2$$

(3.3)

$$\leq \frac{1}{a_R^2} \int_{B_R} |\chi_R(y - w)|^2 d\lambda(w) \int_{B_R} |f_R(\tau, x, w)|^2 d\lambda(w)$$

(3.4)
\[ \leq \frac{\sigma_n R^n}{a_R^2} \int_{B_R} |f_R(\tau, x, w)|^2 d\lambda(w) \quad (3.5) \]

for \((\tau, x, y) \in \Delta(a; r) \times (V + \sqrt{-1}I^n)\), and

\[
\int_{\Delta(a; r) \times V} |\tilde{f}_R(\tau, x, y)|^2 e^{-\phi(\tau, x)} d\lambda(\tau, x)
\leq \frac{\sigma_n R^n}{a_R^2} \int_{\Delta(a; r) \times (V + \sqrt{-1}B_R)} |f_R(\tau, x, w)|^2 e^{-\phi(\tau, x, w)} d\lambda(\tau, x, w) \quad (3.6)
\]

\[
\leq \left( \frac{R}{R - 2\sqrt{R}} \right)^{2n} \pi r^2 \int_V e^{-\phi(a, x)} d\lambda(x) \quad (3.7)
\]

\[
\leq \left( \frac{R}{R - 2\sqrt{R}} \right)^{2n} \pi r^2 \int_V e^{-\phi(a, x)} d\lambda(x) \quad (3.8)
\]

due to (3.2). Here we use the fact that \(\sigma_n (R - 2\sqrt{R})^{2n} \leq a_R\). Note that \(\{R/(R - 2\sqrt{R})\}_{R>0}\) is decreasing and has an upper bound independent of \(R\). For instance, if \(R \geq 100\), we can estimate

\[
\int_{\Delta(a; r) \times V} |\tilde{f}_R(\tau, x, y)|^2 e^{-\phi(\tau, x)} d\lambda(\tau, x) \leq \left( \frac{5}{4} \right)^{2n} \pi r^2 \int_V e^{-\phi(a, x)} d\lambda(x). \quad (3.9)
\]

We also obtain

\[
\frac{\partial \tilde{f}_R}{\partial y}(\tau, x, y) = \frac{1}{a_R} \int_{\mathbb{R}^n} \frac{\partial \chi_R}{\partial y}(y - w) f_R(\tau, x, w) d\lambda(w),
\]

and

\[
\left| \frac{\partial \tilde{f}_R}{\partial y}(\tau, x, y) \right|^2 \leq \frac{1}{a_R^2} \int_{B_R} \left| \frac{\partial \chi_R}{\partial y}(y - w) \right|^2 d\lambda(w) \int_{B_R} |f_R(\tau, x, w)|^2 d\lambda(w)
\leq \frac{\sigma_n R^n}{a_R^2} \left( \frac{C}{\sqrt{R}} \right)^2 \int_{B_R} |f_R(\tau, x, w)|^2 d\lambda(w).
\]

Repeating the argument above, we get

\[
\int_{\Delta(a; r) \times V} \left| \frac{\partial \tilde{f}_R}{\partial y}(\tau, x, y) \right|^2 e^{-\phi(\tau, x)} d\lambda(\tau, x)
\leq \left( \frac{R}{R - 2\sqrt{R}} \right)^{2n} \left( \frac{C}{\sqrt{R}} \right)^2 (\pi r^2) \int_V e^{-\phi(a, x)} d\lambda(x) \quad (3.10)
\]

and \(R/(R - 2\sqrt{R})^{2n} \leq (5/4)^{2n}\) when \(R \geq 100\) as well. Since \(\chi_R\) has compact support in \(|w| < R - \sqrt{R}\) for \(y \in \{|y| < \sqrt{R}\}\), we may assume that \(|y - w| < R\) when we consider the integration.
\[
\int_{\mathbb{R}^n} f_R(\tau, x, y - w) \chi_R(w) d\lambda(w) = \int_{B_{R^{-1/2}}} f_R(\tau, x, y - w) \chi_R(w) d\lambda(w).
\]

On \(|y| < \sqrt{R}/2\), we have

\[
\begin{align*}
\frac{\partial\tilde{f}_R}{\partial x}(\tau, x, y) &= \frac{1}{a_R} \int_{B_{R^{-1/2}}} \frac{\partial f_R}{\partial x}(\tau, x, y - w) \chi_R(w) d\lambda(w), \\
\frac{\partial\tilde{f}_R}{\partial y}(\tau, x, y) &= \frac{1}{a_R} \int_{B_{R^{-1/2}}} \frac{\partial f_R}{\partial y}(\tau, x, y - w) \chi_R(w) d\lambda(w), \\
\frac{\partial\tilde{f}_R}{\partial t}(\tau, x, y) &= \frac{1}{a_R} \int_{B_{R^{-1/2}}} \frac{\partial f_R}{\partial t}(\tau, x, y - w) \chi_R(w) d\lambda(w), \\
\frac{\partial\tilde{f}_R}{\partial s}(\tau, x, y) &= \frac{1}{a_R} \int_{B_{R^{-1/2}}} \frac{\partial f_R}{\partial s}(\tau, x, y - w) \chi_R(w) d\lambda(w),
\end{align*}
\]

where \(\tau = t + \sqrt{-1}s\). Then we see that \(\partial/\partial \bar{z}\) and \(\partial/\partial z\) commute with the integral as well, which implies that \(\tilde{f}_R\) is holomorphic on \(\Delta(a; r) \times (V + \sqrt{-1}\mathbb{R}^n)\).

(Step 3) Take the limit \(R \to +\infty\).

We fix a monotonically increasing sequence \(\{R_j\}_{j \in \mathbb{N}}\) of positive numbers such that \(R_1\) is sufficiently large, \(R_{j+1} > R_j\) and \(\lim_j R_j = +\infty\).

We also take an exhaustion by compact sets \(\{K_i\}_{i \in \mathbb{N}}\) in \(\Delta(a; r) \times (V + \sqrt{-1}\mathbb{R}^n)\) such that \((K_1)^c \neq \emptyset\), \(K_i \subset (K_{i+1})^c\) and \(\cup_i K_i = \Delta(a; r) \times (V + \sqrt{-1}\mathbb{R}^n)\). For each \(K_i\), we can obtain compact subsets \(L_{i_1} \subset \Delta(a; r), L_{i_2} \subset V\) and \(L_{i_3} \subset \mathbb{R}^n\) satisfying \(K_i \subset (L_{i_1} \times L_{i_2} \times L_{i_3})^c\).

First, we consider \(L^2\)-estimates on \(K_1\). It follows that there exists \(R_{n_1}\) such that \(L_{n_1} \subset \{|y| < \sqrt{R_{n_1}}/2\}\), that is, for every \(j \geq n_1\), \(\tilde{f}_j\) is holomorphic on \(L_{1_1} \times L_{1_2} \times L_{1_3}\).

Note that thanks to (3.8) and (3.9), we have

\[
\sup_{y \in L_{1_3}} ||\tilde{f}_{R_j}(. , y)||_{L^2_{\varphi}} \leq C < +\infty, \tag{3.11}
\]

where \(||\tilde{f}_{R_j}(. , y)||_{L^2_{\varphi}} = \int_{\Delta(a; r) \times V} |\tilde{f}_{R_j}(\tau, x, y)|^2 e^{-\varphi(\tau, x)} d\lambda(\tau, x)\) and \(C\) is a positive constant, which is independent of \(R_j, K_i\) and \(L_{i_1}, L_{i_2}, L_{i_3}\). For \(i = 1\), \(\{\sup_{y \in L_{1_3}} ||\tilde{f}_{R_{j_1}}(. , y)||_{L^2_{\varphi}}\}_{j_1}\) is a bounded sequence. Hence, there exists a convergent subsequence \(\{\sup_{y \in L_{1_3}} ||\tilde{f}_{R_{j_{1,k}}}(. , y)||_{L^2_{\varphi}}\}_{j_{1,k}}\). We may assume that \(j_{1,k} \geq n_1\). Since \(\varphi\) is locally bounded above, there is a positive constant \(C_1\) such that \(\varphi \leq C_1\), that is, \(e^{-\varphi} \geq e^{-C_1}\) on \(L_{1_1} \times L_{1_2} \times L_{1_3}\). Then we have that

\[
\sup_{y \in L_{1_3}} ||\tilde{f}_{R_{j_{1,k}}}(. , y) - \tilde{f}_{R_{j_{1,k}}}(. , y)||_{L^2_{\varphi}} \geq \sup_{y \in L_{1_3}} \int_{L_{1_1} \times L_{1_2}} |\tilde{f}_{R_{j_{1,k}}}(\tau, x, y) - \tilde{f}_{R_{j_{1,k}}}(\tau, x, y)|^2 e^{-\varphi(\tau, x)} d\lambda(\tau, x)
\]
for some positive constant $C_{K_1, L_{1_1}, L_{1_2}, L_{1_3}} > 0$ since $\tilde{f}_{R_{j_1,k}}$ and $\tilde{f}_{R_{j_1,\ell}}$ are holomorphic on $L_{1_1} \times L_{1_2} \times L_{1_3}$. Then $\{\tilde{f}_{R_{j_1,k}}\}_k$ forms a Cauchy sequence in the space of continuous functions on $K_1$ with the sup norm. Hence, there exists a function $f_{K_1, \infty}$ on $K_1$ such that $\{\tilde{f}_{R_{j_1,k}}\}_k$ uniformly converges to $f_{K_1, \infty}$ on $K_1$. Here $f_{K_1, \infty}$ is holomorphic in $K_1^0$.

Next, we consider the $L^2$-estimates on $K_2$. Repeating the above argument, we can get a convergent subsequence $\{\tilde{f}_{R_{j_2,k}}\}_k$ of $\{\tilde{f}_{R_{j_1,k}}\}_k$ and a function $f_{K_2, \infty}$. Since $\{\tilde{f}_{R_{j_2,k}}\}_k$ is also uniformly converging to $f_{K_2, \infty}$, it holds that $f_{K_2, \infty}|_{K_1} = f_{K_1, \infty}$.

By using the diagonal argument, we can finally conclude that there exists a holomorphic function $f_\infty$ on $\Delta(a; r) \times (V + \sqrt{-1}\mathbb{R}^n)$ such that $\{\tilde{f}_{R_{j_k,k}}\}_k$ uniformly converges to $f_\infty$ on every compact set.

Then $\frac{\partial \tilde{f}_{R_{j_k,k}}}{\partial y}$ also uniformly converges to $\frac{\partial f_\infty}{\partial y}$ on every compact set. Fix any point $(\tau_0, x_0, y_0) \in \Delta(a; r) \times (V + \sqrt{-1}\mathbb{R}^n)$ and take $K_n \ni (\tau_0, x_0, y_0)$. By (3.10), we have that

$$e^{-C'} \int_{L_{n_1} \times L_{n_2}} \left| \frac{\partial \tilde{f}_{R_{j_k,k}}}{\partial y}(\tau, x, y_0) \right|^2 d\lambda(\tau, x)$$

$$\leq \int_{L_{n_1} \times L_{n_2}} \left| \frac{\partial \tilde{f}_{R_{j_k,k}}}{\partial y}(\tau, x, y_0) \right|^2 e^{-\psi(\tau, x)} d\lambda(\tau, x)$$

$$\leq \left( \frac{R_{j_{k_0}, k_0}}{R_{j_{k_0}, k_0} - 2 \sqrt{R_{j_{k_0}, k_0}}} \right)^{2n} \left( \frac{C}{\sqrt{R_{j_{k_0}, k_0}}} \right)^2 (\pi r^2) \int_V e^{-\psi(a, x)} d\lambda(x) < +\infty$$

for $k \geq k_0$. Then we obtain

$$\int_{L_{n_1} \times L_{n_2}} \left| \frac{\partial f_\infty}{\partial y}(\tau, x, y_0) \right|^2 d\lambda(\tau, x) \leq \frac{C''}{R_{j_{k_0}, k_0}} \left( \frac{R_{j_{k_0}, k_0}}{R_{j_{k_0}, k_0} - 2 \sqrt{R_{j_{k_0}, k_0}}} \right)^{2n}$$

$$\int_V e^{-\psi(a, x)} d\lambda(x) < +\infty$$
for $C'' > 0$. Letting $k_0 \to \infty$, we get $\frac{\partial f_\infty}{\partial y}(\tau, x, y_0) = 0$ on $L_{n_1} \times L_{n_2}$, that is, $\frac{\partial f_\infty}{\partial y}(\tau_0, x_0, y_0) = 0$. Since $(\tau_0, x_0, y_0)$ is arbitrary, $f_\infty$ is a holomorphic function independent of $y$. Hence, $f_\infty$ is independent of $z = x + \sqrt{-1}y$. Then we define the well-defined holomorphic map $f : \Delta(a; r) \to \mathbb{C}$ by $f(\tau) := f_\infty(\tau, x, y)$. For $j \in \mathbb{N}$, by (3.8), we get

$$
\int_{\Delta(a;r) \times V} |\tilde{f}_{R_{jk,k}}(\tau, x, y)|^2 e^{-\varphi(\tau, x)} d\lambda(\tau, x) 
\leq \left( \frac{R_{jk,k}}{R_{jk,k} - 2\sqrt{R_{jk,k}}} \right)^{2n} \pi r^2 \int_V e^{-\varphi(a,x)} d\lambda(x).
$$

Taking the limit $k \to \infty$, thanks to Fatou’s lemma, we have that

$$
\int_{\Delta(a;r) \times V} |f_\infty(\tau, x, y)|^2 e^{-\varphi(\tau, x)} d\lambda(\tau, x) \leq \pi r^2 \int_V e^{-\varphi(a,x)} d\lambda(x),
$$

that is,

$$
\int_{\Delta(a;r) \times V} |f(\tau)|^2 e^{-\varphi(\tau, x)} d\lambda(\tau, x) \leq \pi r^2 \int_V e^{-\varphi(a,x)} d\lambda(x).
$$

We also have that

$$
\tilde{f}_{R_{jk,k}}(a, x, 0) = \frac{1}{a_{R_{jk,k}}} \int_{[a]} f_{R_{jk,k}}(a, x, -w) \cdot \chi_{R_{jk,k}}(w) d\lambda(w) = \frac{1}{a_{R_{jk,k}}} \int_{B_{R_{jk,k}}} 1 \cdot \chi_{R_{jk,k}}(w) d\lambda(w) = 1.
$$

Then we see that $f(a) = f_\infty(a, x, 0) = \lim_{k \to \infty} \tilde{f}_{R_{jk,k}}(a, x, 0) = 1$, which completes the proof.

\[ \square \]

**Remark 3.1** The constant in the $L^2$-extension of $\tilde{f}_R$ is not optimal and changes for each $R > 0$ (see (3.8)). However, by taking the limit $R \to \infty$, we can estimate the $L^2$-norm of $f_\infty$ with the optimal constant.

### 4 A Simple Proof of Prékopa’s Theorem

In this section, applying Theorem 1.1, we give a simple proof of Prékopa’s theorem. The proof based on a non-optimal $L^2$-extension theorem for the “complex” version of Prékopa’s theorem also appeared in [6,7]. Our main purposes are to establish the optimal $L^2$-extension theorem on tube domains and to give a proof of Prékopa’s theorem in the “real” setting directly.
First, we consider the following case.

**Theorem 4.1** Let $V$ be a convex domain in $\mathbb{R}^n$ and $\varphi$ be a convex function on $\mathbb{R}_t \times V_x$. Assume that $V$ is bounded and

$$e^{-\Phi(t)} := \int_V e^{-\varphi(t,x)} \, d\lambda(x) < +\infty$$

for each $t \in \mathbb{R}$. Then $\Phi$ is convex.

**Proof** We consider the following tube domains $\mathbb{R}_t + \sqrt{-1} \mathbb{R}_s$ and $V_x + \sqrt{-1} \mathbb{R}_y$, and set $\tau = t + \sqrt{-1}s$, $z = x + \sqrt{-1}y$. We also let $\hat{\Phi}(\tau) := \Phi(t)$ be a function on $(\mathbb{R} + \sqrt{-1} \mathbb{R})_\tau$ and $\hat{\varphi}(\tau, z) := \varphi(t, x)$ be a function on $(\mathbb{R} + \sqrt{-1} \mathbb{R})_\tau \times (V + \sqrt{-1} \mathbb{R}^n)_z$.

Then it clearly holds that $\hat{\varphi}$ is a plurisubharmonic function and

$$e^{-\hat{\Phi}(\tau)} = \int_V e^{-\hat{\varphi}(\tau, z)} \, d\lambda(x).$$

It is enough to show that $\hat{\Phi}$ is plurisubharmonic. Note that $\hat{\Phi}$ is independent of $s$ and $\hat{\varphi}$ is independent of $s$ and $y$. We only need to show that $\hat{\Phi}$ satisfies the minimal extension property since $\hat{\Phi}$ is upper semi-continuous thanks to Fatou’s lemma (cf. Theorem 2.3).

Take a point $a \in \mathbb{R} + \sqrt{-1} \mathbb{R}$ and $r > 0$. Then, by Theorem 1.1, there exists a holomorphic function $f$ on $\Delta(a; r)$ satisfying $f(a) = 1$ and

$$\int_{\Delta(a; r) \times V} |f(\tau)|^2 e^{-\hat{\Phi}(\tau, x)} \, d\lambda(\tau, x) \leq \pi r^2 \int_V e^{-\hat{\varphi}(a, x)} \, d\lambda(x) < +\infty,$$

that is,

$$\frac{1}{\pi r^2} \int_{\Delta(a; r)} |f(\tau)|^2 e^{-\hat{\Phi}(\tau)} \, d\lambda(\tau) \leq e^{-\hat{\Phi}(a)},$$

which completes the proof. \(\square\)

**Remark 4.2** The above type proof can be applied to the complex Prékopa theorem as well.

If $V$ is an unbounded convex domain such as $\mathbb{R}^n$, we need to take a convex exhaustion. We only show the proof in the case that $V = \mathbb{R}^n$ without loss of generality.

**Theorem 4.3** Keep the notation above. Set $V = \mathbb{R}^n$. Suppose that

$$e^{-\Phi(t)} := \int_{\mathbb{R}^n} e^{-\varphi(t,x)} \, d\lambda(x) < +\infty.$$

Then $\Phi$ is convex.
Proof Let \( B_j := \{ |x| < j \} \subset \mathbb{R}^n \) for \( j \in \mathbb{N} \). We define
\[
e^{-\Phi_j(t)} := \int_{B_j} e^{-\varphi(t,x)} \, d\lambda(x) < +\infty.
\]
Then we know that \( \Phi_j \) is convex. It holds that \( \Phi_j \) is decreasing to \( \Phi \). Then \( \Phi \) is convex as well. \( \square \)

Acknowledgements The author would like to thank Bo Berndtsson for reading and commenting on a draft version. He is also grateful to the anonymous referee for careful reading and pointing out a gap in the proof of the main theorem.

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