Chaotification via Higher-order Nonlinear Schrödinger Equations for Secured Communication

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Higher-order nonlinear Schrödinger (HNLS) equation which can be used to describe the propagation of short light pulses in the optical fibers, is studied in this paper. Using the phase plane analysis, HNLS equation is reduced into the equivalent dynamical system, periodicity of such system is obtained with the phase projections and power spectra given. By means of the time-delay feedback method, with the original dynamical system rewritten, we construct a single-input single-output system, and propose a chaotic system based on the chaotification of HNLS. Numerical studies have been conducted on such system. Chaotic motions with different time delays are displayed. Power spectra of such chaotic motions are calculated. Lyapunov exponents are given to corroborate that those motions are indeed chaotic.

Keywords: Anti-control of chaos; Higher-order nonlinear Schrödinger equation; Chaotic Motion; Time delay

Theoretical and experimental studies on the solitons have revealed that the propagation of optical solitons can be used to carry data at a high bit rate\cite{1}. Especially, a source of ultrashort optical pulses, which can be used in the high bit rate and long-distance optical communication systems, has attracted people’s attention\cite{2} Techniques have been proposed, including the measurement of ultrafast processes\cite{3} optoelectronic terahertz time domain spectroscopy\cite{5} and optoelectronic sampling\cite{6}.

With the consideration of higher-order effects, including the third-order dispersion (TOD), self-steepening (SS) and stimulated Raman scattering (SRS) in a dispersion-shift fiber, the following higher-order nonlinear Schrödinger (HNLS) equation reads as\cite{7}:

\[
\begin{align*}
    u_x - \frac{i}{2} u_{tt} - iN |u|^2 u + \frac{1}{6} L_H u_{ttt} + SN^2 (|u|^2 u)_t - \tau_R N^2 u|u|^2 = 0,
\end{align*}
\]

where \( u \), a complex function of the normalized propagation distance \( x \) and normalized time \( t \), refers to the normalized amplitude of the electromagnetic field, the subscripts denote the partial derivatives, \( N \) and \( L_H \) are respective for the nonlinearity and dispersion, \( S \) and \( \tau_R \) account for, respectively, the SS and SRS,

\[
N^2 = \frac{\gamma P_0 T_0^2}{|\beta_2|^2}, \quad L_H = \frac{\beta_3}{T_0 |\beta_2|}, \quad S = \frac{1}{\omega_0 T_0}, \quad \tau_R = \frac{T_R}{T_0},
\]

\( P_0 \) is the peak power of the incident pulse, \( \omega_0 \) is the carrier frequency, \( T_R \) represents the Raman resonant time constant, \( \beta_2 \) is the group-velocity dispersion parameter which is responsible for the pulse broadening, \( \beta_3 \) is the TOD coefficient, \( T_0 \) is the half-width of the input pulse, and \( \gamma \) is the fiber nonlinearity coefficient\cite{7,22}.

Eq. (1) describe the soliton-effect pulse compression of ultrashort solitons in optical fiber\cite{11,22}. Studies on Eq. (1) include the numerical calculations via the split-step Fourier method\cite{11,22}, bright soliton solutions, dynamics of the bright solitons with the random initial perturbations, dark soliton solutions\cite{13} stochastic soliton solutions\cite{15} and rogue wave solutions.

Chaos, which may occur in a deterministic nonlinear system, is a sustained and disorderly-looking long-term evolution that satisfies certain criteria. Chaotic motions have been found in some perturbed equations and such dynamical systems as the Lorenz system\cite{16} and Chua’s circuit\cite{17}. With the increasing demands on chaos in such areas as the secure communication and information security, the task of making a non-chaotic system chaotic, called the anti-control of chaos, or chaotification, has appeared\cite{16,18}.

Such methods as the time-delay feedback\cite{19,20} topological conjugate\cite{14} and impulsive control\cite{21} have been developed to chaotify the dynamical systems. A system with time-delay is inherently infinitely dimensional, so it can produce the bifurcation and chaos, even a first-order system\cite{19,20}. Particularly, chaotic motions have been found in some time-delay equations due to their associated differential equations\cite{17,19,20}. Therefore, the time-delay feedback method has been thought as a straightforward one to chaotify a non-chaotic system.

To our knowledge, little work has been completed on Eq. (1) with respect to the chaos. Motivated by the applications of Eq. (1) in the optical fibers, its potential applications in the secured optical communications will be the focus in this paper. Using the time-delay feedback method, we aim to propose a new chaotic system with the time-delay introduced into Eq. (1), and find whether some chaotic motions can be observed. Furthermore, when the time-delay is fixed, we want to investigate the possible soliton solutions of such chaotic system, and whether such time-delay perturbation can affect the soliton propagation.
Methods have been developed to study the nonlinear evolution equations for their integrability and periodicity\(^\text{33}\), such as the variational approximation\(^\text{30}\), phase-plane analysis\(^\text{32,33,42,43}\) and perturbation method based on the inverse scattering transform\(^\text{35,36}\). Among them, the phase-plane analysis will be used next.

Setting \(u(x,t) = \psi(\xi)e^{i\vartheta}\) with \(\xi = a_1x - b_1t\) and \(\vartheta = a_2x - b_2t\), and substituting them into Eq. (1), we have

\[
\psi_{\xi\xi\xi} + r_1\psi_{\xi\xi} + r_2\psi_{\xi} + r_3\psi + r_4\psi^3 + r_5\psi^2\psi_{\xi} = 0,
\]

(3)

with

\[
r_1 = \frac{3iL_Hb_2 - 3i}{L_Hb_1}, \quad r_2 = -\frac{6a_1 - 6b_1b_2 + 3L_Hb_1b_2^2}{L_Hb_1^3},
\]

(4)

\[
r_3 = -\frac{6ia_2 - 3ib_2^2 - ib_3^2L_H}{L_Hb_1^2}, \quad r_4 = \frac{6iN^2 + 6ib_2SN^2}{L_Hb_1^3},
\]

(5)

\[
r_5 = \frac{18b_1SN^2 - 12\tau_RN^2b_1}{L_Hb_1^3},
\]

(6)

where \(\psi\) is a real function, \(a_j\)'s and \(b_j\)'s \((j = 1, 2)\) are all the real constants. Thus, Eq. (3) can be rewritten as a three-dimensional planar dynamic system \((X \equiv \psi, Y \equiv \psi_{\xi}, Z \equiv \psi_{\xi\xi})\),

\[
\begin{align*}
X_\xi &= Y, \\
Y_\xi &= Z, \\
Z_\xi &= -r_1Z - r_2Y - r_3X - r_4X^3 - r_5X^2Y.
\end{align*}
\]

(7)

Phase projections for System (7) when \(r_2 = 0\) and \(r_2 \neq 0\) are shown in Figs. 1(a) and 1(b), respectively. Power spectra for the solutions of System (7) in the two cases are displayed in Figs. 2(a) and 2(b).

When \(r_2 = 0\), i.e., with the group-velocity dispersion parameter \(\beta_2\), TOD coefficient \(\beta_3\) and half-width of the input pulse \(T_0\) satisfying that \(\beta_3 = \frac{(2b_1b_2 - 2a_2)b_3}{b_1^3}\), we can see a closed curve, which means that there exists a center point, as shown in Fig. 1(a). But a focal point can be found in System (7) when \(r_2 \neq 0\), as displayed in Fig. 1(b). Their respective power spectra for the solutions of System (7) are calculated via the spectral analysis in Figs. 2(a) and 2(b), and the periodicity of \(Z\) is verified because of the single frequency. Hereby, in Figs. 2, the abscissa represents the frequency of \(Z\) under the certain conditions, where the frequency is the number of occurrences of a repeating event per unit time and it is the reciprocal of the periodicity\(^\text{43}\) and the ordinate \(fft(Z)\) represents the fast Fourier transform (FFT) of \(Z\).\(^\text{27}\)

System (7) can be rewritten as

\[
\begin{pmatrix}
X \\
Y \\
Z
\end{pmatrix}_{\xi} = \begin{pmatrix}
Y \\
Z \\
-r_1Z - r_2Y - r_3X - r_4X^3 - r_5X^2Y
\end{pmatrix},
\]

(8)
FIG. 2: (b) The same as 1(a) but $r_2 = 0.2$.

FIG. 3: (a) Power spectrum for the solutions of System (7) which correspond with Fig. 1(a).

with $\mathbf{x}_1 = (0, 0, 0)^T$, $\mathbf{x}_2 = \left(\sqrt{-\frac{r_2}{r_1}}, 0, 0\right)^T$ and $\mathbf{x}_3 = \left(-\sqrt{-\frac{r_2}{r_1}}, 0, 0\right)^T$ being its equilibrium points, $T$ means the vector transpose. Without loss of generality, we choose $r_2$, the coefficient of damped term, as the parameter which can be affected by the perturbation via the time-delay feedback method.

To chaotify Eq. (1), according to the time-delay feedback method\textsuperscript{25,39}, we construct a single-input single-output nonlinear system as follows:

\begin{align}
\dot{\mathbf{x}}_\xi &= \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\delta(\xi), \\
y &= h(\mathbf{x}),
\end{align}

where $\mathbf{x}$ and $y$ label the input and output, respectively, $\dot{\mathbf{x}}_\xi = d\mathbf{x}/d\xi$, $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ are both real vector functions, $\delta(\xi)$ corresponds to a system parameter perturbation, and $h(\mathbf{x})$ is a smooth real function and refers to the output. In the case of System (8), $\mathbf{x} = (X, Y, Z)^T$, $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ can be expressed as

\begin{align}
\mathbf{f}(\mathbf{x}) &= \begin{pmatrix} Y \\ Z \\ -r_1 Z - r_2 Y - r_3 X - r_4 X^3 - r_5 X^2 Y \end{pmatrix}, \\
\mathbf{g}(\mathbf{x}) &= \begin{pmatrix} 0 \\ 0 \\ Y \end{pmatrix}.
\end{align}

Thus, System (9) can be rewritten as

\begin{align}
\dot{\mathbf{x}}_\xi &= \begin{pmatrix} Y \\ Z \\ -r_1 Z - r_2 Y - r_3 X - r_4 X^3 - r_5 X^2 Y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ Y \end{pmatrix} \delta(\xi), \\
y &= h(\mathbf{x}),
\end{align}
where \( h(x) \) can be determined based on \( \delta(\xi) \).

Based on Expressions (11), we have

\[
\text{ad}_f g(x) = \begin{pmatrix} 0 \\ -Y \\ r_1 Y + Z \end{pmatrix}, \quad \text{ad}_f^2 g(x) = \begin{pmatrix} Y \\ -r_1 Y - Z \\ (r_1^2 - r_2)Y - r_3 X - r_4 X^3 - r_5 X^2 Y \end{pmatrix},
\]

where \( \text{ad}_f g(x) \) is the Lie bracket of the two smooth vector functions \( f(x) \) and \( g(x) \). Based on the conclusions in Refs. [26, 39], the relative degree of System (9) is three, i.e., the dimension of \( x \). Hereby, the definition of “relative degree” can be seen in Refs. [26, 39], and “Lie bracket” \( \text{ad}_f g(x) \) and \( \text{ad}_f^2 g(x) \) can be obtained as [26, 39].

\[
\text{ad}_f g(x) = \frac{\partial f(x)}{\partial x} g(x) - f(x) \frac{\partial g(x)}{\partial x},
\]

\[
\text{ad}_f^2 g(x) = \text{ad}_f [\text{ad}_f g(x)] = \frac{\partial f(x)}{\partial x} [\text{ad}_f g(x)] - f(x) \frac{\partial \text{ad}_f g(x)}{\partial x}.
\]

Via the time-delay feedback method [26, 39], we know that \( h(x) \) should satisfy

\[
\frac{\partial h(x)}{\partial x} [g(x), \text{ad}_f g(x), \text{ad}_f^2 g(x)] = 0,
\]

i.e.,

\[
\frac{\partial h(x)}{\partial x} g(x) = \frac{\partial h}{\partial Z} Y = 0,
\]

\[
\frac{\partial h(x)}{\partial x} \text{ad}_f g(x) = \frac{\partial h}{\partial Y} (-Y) + \frac{\partial h}{\partial Z} (r_1 Y + Z) = 0,
\]

so that we have \( h(x) = X \). In line with Refs. [26, 39], \( \delta(\xi) \) can be expressed as

\[
\delta(\xi) = \varsigma \sin[\sigma X(\xi - \tau)],
\]

with \( \varsigma \) and \( \sigma \) being both the real constants, \( \tau \) referring to the time delay, and \( \xi \) given in Sec. 2.

As a generalization of this part, the chaotification of Eq. (1) can be given as

\[
X_\xi = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_\xi = \begin{pmatrix} Y \\ Z \\ -r_1 Z - (r_2 + \delta)Y - r_3 X - r_4 X^3 - r_5 X^2 Y \end{pmatrix},
\]

with \( \delta = \delta(\xi) = \varsigma \sin[\sigma X(\xi - \tau)] \) being the time-delay perturbation on \( r_2 \).

To study the possible chaotic motions of System (17), we will investigate the phase projections of \( Z \) in Figs. 3. Note that the difference between Figs. 3(a) and 3(b) roots in the different values of the time delay \( \tau \). Then, we calculate
FIG. 5: (a) Phase projection of System (17) with $r_1=1.5$, $r_2=5$, $r_3=-1$, $r_4=-5$, $r_5=1.75$, $\zeta=1$, $\sigma=0.5$, $\tau=10$.

FIG. 6: (b) Phase projection of System (17) with $r_1=1.5$, $r_2=5$, $r_3=-1$, $r_4=-5$, $r_5=1.75$, $\zeta=1$, $\sigma=0.5$, $\tau=26$.

FIG. 7: (a) Power spectrum for the solutions of System (17) which correspond with Fig. 3(a).
FIG. 8: (b) Power spectrum for the solutions of System (17) which correspond with Fig. 4(a).

FIG. 9: (a) Lyapunov exponents when the time-delay $\tau$ is the bifurcation parameter, while $r_1=1.5$, $r_2=2$, $r_3=-1$, $r_4=5$, $r_5=1.75$, $\varsigma=1$ and $\sigma=0.5$.

their respective power spectra in Figs. 4, which can be used for people to corroborate that the motions in Figs. 3 are indeed chaotic.

Comparing Figs. 3-4 with Figs. 1-2, we can see that there exits no periodicity for System (17), and the frequency of System (7) has been broken. As defined in Refs. [46,47], developed chaos occurs when the solutions ignore the driver periods and represent a random sequence of uncorrelated shocks. Thus, developed chaos occurs in Figs. 3.

To verify that the behaviors in Figs. 3 are indeed chaotic from another point of view, we will investigate the Lyapunov exponents of the time delay $\tau$ and parameter $r_2$, as displayed in Figs. 5.

Dynamic behaviors of System (17) varying with $\tau$ can be found in Fig. 5(a). We can see that System (17) can give rise to the periodic motions when $2.5 \leq \tau \leq 15$ or $26.5 \leq \tau \leq 28$ with other parameters fixed, while System (17) may turn into the chaos when $15 < \tau < 26.5$ or $28 < \tau < 30$. Dynamic behaviors of System (17) varying with $r_2$ are shown in Fig. 5(b). Note that System (17) turns into the chaos when $2.5 < r_2 < 11$, and System (17) is periodic when $11 \leq r_2 \leq 30$. As expected, the Lyapunov exponents in Figs. 5 are consistent with the chaotic motions shown in Figs. 3. It also shows that Eq. (1) can be chaotified via the time-delay method when the parameters are in certain ranges.

In this paper, we have focused ourselves on a HNLS equation [i.e., Eq. (1)] in the secured optical communication, which can be used to describe the propagation of short light pulses in the optical fibers. With the time-delay perturbation introduced, we have proposed a chaotic system [i.e., System (17)], which has been studied numerically with the phase projections, power spectra and Lyapunov exponents. Furthermore, to study the potential applications of Eq. (1) in the secured optical communication from another point of view, we have investigated the soliton solutions
of System (17) when the time delay is fixed. The main results can be summarized as below:

- Using the phase plane analysis, we have given the periodicity of Eq. (1), as seen in Figs. 1 and 2. Effect of damped coefficient $r_2$ has been obtained, i.e., center point has been shown in Fig. 1(a) if $r_2 = 0$, or focal point in Fig. 2(a).
- Making use of the time-delay feedback method, we have chaotified Eq. (1), and a chaotic system [i.e., System (17)] has been constructed with the time-delay perturbation introduced into Eq. (1).
- Phase projections of System (17) with different time delays have been given in Figs. 3(a) and 3(b), and their respective power spectra have been displayed in Figs. 4(a) and 4(b).
- Lyapunov exponents of the time-delay $\tau$ and parameter $r_2$ have been shown in Figs. 5(a) and 5(b), respectively. We have found that when $r_2$ is fixed, System (17) is periodic when $2.5 \leq \tau \leq 15$ or $26.5 \leq \tau \leq 28$, while it may turn into the chaos when $15 < \tau < 26.5$ or $28 < \tau < 30$. When $\tau$ is fixed, System (17) is chaotic when $2.5 < r_2 < 11$, and it is periodic when $11 \leq r_2 \leq 30$.

Acknowledgments The authors acknowledge *** for the discussions during the works.

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1 H. Toda, Y. Furukawa, T. Kinoshita, Y. Kodama and A. Hasegawa, *IEEE Photon. Technol. Lett.* **9**, 1415 (1997); F. K. Abdullaev and J. Garnier, *J. Phys. Rev. E* **72**, 035603 (2005).
2 G. P. Agrawal, *Nonlinear Fiber Optics* (Acad., California, 2002).
3 A. Hasegawa, *Optical Solitons in Fibers* (Springer, Berlin, 1989).
4 D. H. Auston and M. C. Nuss, *IEEE J. Quantum Electron.* **24**, 184 (1988).
5 B. B. Hu and M. C. Nuss, *Opt. Lett.* **20**, 1716 (1995).
6 T. Nagatsuma, M. Yaita, M. Shinagawa, K. Kato, A. Kozen, K. Iwatsuki and K. Suzuki, *Electron. Lett.* **30**, 814 (1994).
7 A. Biswas, M. Fessak, S. Johnson, S. Beartice, D. Milovic, Z. Jovanoski, R. Kohl and F. Majid, *Opt. Laser Technol.* **44**, 263 (2012); D. Y. Tang, B. Zhao, D. Y. Shen, C. Lu, W. S. Man and H. Y. Tam, *Phys. Rev. A* **66**, 033806 (2002).
8 D. Rand, I. Glesk, C. S. Bres, D. A. Nolan, X. Chen, J. Koh, J. W. Fleischer, K. Steiglitz and P. R. Prucnal, *Phys. Rev. Lett.* **98**, 053902 (2007); Q. Tian, L. Wu, J. F. Zhang, B. A. Malomed, D. Mihalache and W. M. Liu, *Phys. Rev. E* **83**, 016602 (2011).
9 Garnett, P. Williams, *Chaos Theory Tamed* (Joseph Henry, Washington D. C., 1997).
10 Morris W. Hirsch, Stephen Smale and Robert L. Devaney, *Differential Equations, Dynamical Systems, and an Introduction to Chaos* (Elsevier, New York, 2004).
11 M. I. Weinstein, *SIAM J. Math. Anal.* **16**, 472 (1985); T. Ueda and W. L. Kath, *Phys. Rev. A* **42**, 563 (1990).
12 T. Kanna, M. Lakshmanan, P. Tchofo Dinda, and Nail Akhmediev, *Phys. Rev. E* **73**, 026604 (2006).
13 R. Radhakrishnan, M. Lakshmanan and J. Hietarinta, *Phys. Rev. E* **56**, 2213 (1997); T. Kanna and M. Lakshmanan, *Phys. Rev. Lett.* **86**, 5043 (2001); M. Vijayayajanthy, T. Kanna and M. Lakshmanan, *Phys. Rev. A* **77**, 013820 (2008).
14 Y. Ohta and J. K. Yan, *Proc. R. Soc. A* **468**, 1716 (2012).
15 X. Yu, *Int. J. Syst. Sci.* **27**, 355 (1995).
16 G. Chen, *Controlling Chaos and Bifurcations in Engineering Systems* (Chemical Rubber Company, New York, 1999).
17 G. Chen and X. Dong, *From Chaos to Order: Perspectives, Methodologies, and Applications* (World Sci., Singapore, 1998).
18 A. Vanecek and S. Celikovsky, *Control Systems: From Linear Analysis to Synthesis of Chaos* (Prentice-Hall, London, 1996).
19 P. Celka, *Physica D* **104**, 127 (1997).
20 M. Gotz, U. Feldmann and W. Schwarz, *IEEE Trans. Circ. Syst.* **40**, 854 (1993).
21 J. N. Lu, Q. Wu and J. H. Lü, *Phys. Lett. A* **305**, 365 (2002).
22 Arman Kiani-B, Kial Fallahi, Naser Pariz and Henry Leung, *Commun. Nonl. Sci. Numer. Simulat.* **14**, 863 (2009).
23 G. Kolumban, M. P. Kennedy and LO. Chua, *IEEE Trans. Circ. Syst.* **44**, 927 (1997).
24 G. Heidari-Bateni and C. D. McGillem, *IEEE Trans. Commun.* **42**, 154 (1994).
25 M. Lakshmanan and K. Murali, *Chaos in Nonlinear Oscillators: Controlling and Synchronization* (World Scientific, Singapore, 1996).
26 T. Zhao, G. R. Chen and Q. Yang, *Chaos* **14**, 662 (2004); X. S. Yang and Y. Tang, *Chaos Solitons Fract.* **19**, 841 (2004); X. F. Wang, G. R. Chen and K. F. Man, *IEEE Trans. Circ. Syst.* **48**, 641 (2001b); X. F. Wang, G. R. Chen and X. Yu, *Chaos* **10**, 771 (2000).
27 G. D. Bergland, *IEEE Spectrum* **6**, 228 (1969); P. Stoica and R. L. Moses, *Introduction to Spectral Analysis* (Prentice Hall, New Jersey, 1997).
28 M. C. Mackey and L. Class, *Science* **197**, 287 (1977); J. D. Farmer, *Physica D* **4**, 336 (1982); K. Ikeda and K. Matsumoto, *Physica D* **29**, 223 (1987).
29 O. Diekmann, S. A. Gils, S. M. Lunel and H. O. Walther, *Delay Equations: Functional, Complex, and Nonlinear Analysis* (Springer, Berlin, 1995).
30 D. Anderson, M. Lisak and T. Reichel, *J. Opt. Soc. Am. B* **5**, 207 (1988).
31 E. Infeld and G. Rolands, *Nonlinear Waves, Soliton and Chaos* (Cambridge Univ. Press, Cambridge, 1990).
32 A. Kumar and A. K. Sarna, *Opt. Commun.* **234**, 427 (2004); S. K. Adhikari, *Phys. Rev. E* **71**, 16611 (2005).
33 V. Skarka and N. B. Aleksic, *Phys. Rev. Lett.* **96**, 013903 (2006); S. K. Adhikari and B. A. Malomed, *Phys. Rev. A* **76**, 043626 (2007).
34 J. Fujioka, E. Cortès, R. Pérez-Pascual, R. F. Rodríguez, A. Espinosa and B. A. Malomed, *Chaos* **21**, 033120 (2011).
35 Y. S. Kivshar and B. A. Malomed, *Rev. Mod. Phys.* **61**, 763 (1989).
36 N. C. Panoiu, D. Mihalache, D. Mazilu, L. C. Crasovan, I. V. Melnikov and F. Lederer, *Chaos* **10**, 625 (2000).
37 T. Zhou, G. Chen and Q. Yang, *Chaos* **14**, 662 (2004).
38 L. Yang, Z. R. Liu and G. Chen, *Int. J. Bifur. Chaos* **12**, 1121 (2002).
39 H. Nijmeijer and A. Schaft, *Nonlinear Dynamical Control Systems* (Springer, New York, 1990); A. Isidori, *Nonlinear Control Systems* (Springer, Berlin, 1995); A. Isidori, *Nonlinear Control Systems II* (Springer, Berlin, 1999).
40 I. S. Amiria, A. Afroozeh, I. N. Nawi, M. A. Jalil, A. Mohamad, J. Ali and P. P. Yupapind, *Procedia Engineering*, **8**, 417 (2011).
41 I. S. Amiria, A. Afroozeh, I. N. Nawi, M. A. Jalil, A. Mohamad, J. Ali and P. P. Yupapind, *Procedia Engineering*, **8**, 360 (2011).
42 J. Yu, W. J. Zhang and X. M. Gao, *Chaos Solitons Fract.* **33**, 1307 (2007).
43 E. Infeld and G. Rolands, *Nonlinear Waves, Soliton and Chaos* (Cambridge Univ., Cambridge, 1990).
44 H. J. Cao, J. M. Seoane and A. F. Sanjuán, *Chaos Soliton. Fract.* **34**, 197 (2007).
45 B. Knobnob, S. Mitatha, K. Dejhan, S. Chaiyasoonthorn and P. P. Yupapind, *Optick* **121**, 1743 (2010).
46 N. M. Ryskin and V. N. Titov, *Tech. Phys.* **48**, 1170 (2011).
47 C. C. Lalescu, C. Meneveau and G. L. Eyink, *Phys. Rev. Lett.* **110**, 084102 (2013).