Neutrino Yukawa textures within type-I see-saw

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Abstract

The arbitrariness of Yukawa couplings can be reduced by the imposition of some flavor symmetries and/or by the realization of texture zeros. We review neutrino Yukawa textures with zeros within the framework of the type-I seesaw with three heavy right chiral neutrinos and in the basis where the latter and the charged leptons are mass diagonal. An assumed non-vanishing mass of every ultralight neutrino and the observed non-decoupling of any neutrino generation allow a maximum of four zeros in the Yukawa coupling matrix $Y_{\nu}$ in family space. There are seventy two such textures. We show that the requirement of an exact $\mu\tau$ symmetry, coupled with the observational constraints, reduces these seventy two allowed textures to only four corresponding to just two different forms of the light neutrino mass matrix $M_{\nu A}/M_{\nu B}$, resulting in an inverted/normal mass ordering. The effect of each of these on measurable quantities can be described, apart from an overall factor of the neutrino mass scale, in terms of two real parameters and a phase angle all of which are within very constrained ranges. The masses and Majorana phases of ultralight neutrinos are predicted within definite ranges with $3\sigma$ laboratory and cosmological observational inputs. The rate for $0\nu\beta\beta$ decay, though generally below the reach of planned experiments, could approach it in some parameteric regions. Within the same framework, we also study Yukawa textures with a fewer number of zeros, but with exact $\mu\tau$ symmetry. We further formulate the detailed scheme of the explicit breaking of $\mu\tau$ symmetry in terms of three small parameters for allowed four zero textures. The observed sizable mixing between the first and third generations of neutrinos is shown to follow for a suitable choice of these symmetry breaking parameters.

Invited review, to appear in a special issue of Advances in High Energy Physics (AHEP) on neutrinos
7.1 Introduction

The impressive experimental progress from neutrino oscillation studies [1]-[5] and the sharpening [6, 7] of the cosmological upper bound on the neutrino mass sum have underscored two fundamental but distinct puzzles. 1) Why are the observed neutrinos so ultralight, i.e. with masses in the sub-eV range? 2) Why is the three neutrino mixing pattern of two large and one small (but measurable) angles so different from the sequentially small CKM mixing angles of quarks? There is a widespread feeling that the former is due to some kind of a see-saw mechanism[8]-[14] yielding ultralight Majorana neutrinos. It is our contention that the latter has to do do with zeros in neutrino Yukawa textures plus a broken $\mu\tau$ symmetry. Let us start with the simplest scheme of three weakly interacting flavored ultralight neutrinos discarding any possible light sterile ones mixing with them. We hold that there should be a fundamental principle behind a massless particle, as with gauge invariance and the photon. Since no such principle is identifiable with any single neutrino, we take each to have a nonzero mass. Though there are other types of proposed see-saw mechanisms, such as type-II [11, 12], type-III [13], inverse see-saw [14, 15] etc, in a minimalist approach we stick to the original type-I with three heavy right chiral electroweak singlet neutrinos denoted by the column vector $N_R$.

We next turn to the issue of texture zeros. By a texture we mean a configuration of a Yukawa coupling matrix with some vanishing elements. Texture zeros have a long history in the quark sector where four zero Yukawa textures [16]-[18] have had distinguished success in fitting the known quark masses and CKM parameters. The problem is simpler there since the Dirac quark mass matrix of a given charge, which is the corresponding Yukawa coupling matrix times the Higgs VEV, contains all information about physical quark masses. In the case of see-saw induced ultralight Majorana neutrinos,
the elements of the Dirac mass matrix $M_D$ do not carry all information about physical neutrino masses. The latter are contained in the elements of the complex symmetric Majorana neutrino mass matrix $M_\nu$ which is related to $M_D$ through the standard see-saw formula. There have been initial as well as continuing efforts [19]-[23] to assume the vanishing of certain elements in $M_\nu$. But, we strongly feel that an occurrence of zeros must be linked to some fundamental symmetry [24]-[26] or suppression mechanism [27] inherent in the Lagrangian itself. It seems more natural then to postulate the occurrence of such zeros in some elements of the neutrino Yukawa coupling matrix (equivalently $M_D$) which appears in the Lagrangian [28]-[36]. There are ways [37]-[39] to ensure the stability of such zeros under quantum corrections in type-I seesaw models.

An important point in the context of texture zeros is that of Weak Basis dependence. Both $M_D$ and $M_\nu$ change [40] under general (and different) unitary transformations of the left and right chiral fermion fields. In consequence, any Yukawa texture is basis dependent. It is further known that those fermions, which do not couple mutually in the Lagrangian, can be simultaneously put into a mass diagonal form by suitable basis transformations. Without loss of generality, we can therefore choose a Weak Basis in which the charged lepton fields $\ell$ and the very heavy right chiral neutrino fields $N_R$ are mass diagonal with real masses. The question arises as to how a flavor model, corresponding to a given set of texture zeros in such a basis, would be recognized in a different basis. It has been shown [40] that the vanishing of certain Weak Basis invariants would be a hallmark of those zeros. This is also related to the linkage of CP violation at low energies, probed in short or long baseline experiments, and at high energies, as relevant to leptogenesis. Though that linkage is a major motivation for postulating Yukawa texture zeros [28]-[30], it is outside the scope of the present review.

In this article we focus on the role of texture zeros, occurring in $M_D$, in understanding the observed pattern of neutrino masses and mixing angles. More generally, we show how they affect key aspects of low energy neutrino phenomenology. Four is shown to be the maximum number of such zeros allowed within our framework [28]. We classify all possible four zero textures, seventy two in total [28]. Then we introduce $\mu\tau$ symmetry [31], [41]-[66] as an invariance under the interchange of flavors $\mu$ (2) and $\tau$ (3) in the neutrino sector which is motivated by an automatic prediction of vanishing (maximal) mixing between the first (second) and third generations of neutrinos. This symmetry reduces the preceding seventy two textures to four which lead to only two distinct forms of $M_\nu$ whose phenomenological consequences are worked out [30]-[32]. Three zero textures with $\mu\tau$ symmetry are also shown to have similar consequences, while textures with a lesser number of zeros have little predictivity [33]. We then discuss the general explicit breaking of $\mu\tau$ symmetry in terms of three small parameters and show, within the lowest order of perturbation in those parameters, that the observed small mixing of first and third generations of neutrinos can be explained within our framework [33].

In Section 7.2 we set up our formalism. Section 7.3 contains the classification of all four zero textures and a discussion of $\mu\tau$ symmetry. Section 7.4 addresses the consequent phenomenological implications. In Section 7.5 we discuss the realization of other $\mu\tau$ symmetric texture zeros. Section 7.6 contains a general discussion of explicit $\mu\tau$ symmetry breaking and how that fits observation. Finally, in Section 7.7 we summarize our conclusions.

### 7.2 Framework and Formalism

The relevant mass terms in our starting Lagrangian are

$$-\mathcal{L}^m = \bar{\nu}_L^C M_D N_R^0 + \frac{1}{2} \bar{N}_R^0 C M_R N_R^0 + \bar{\ell}_L^C M_l l_R^0 + h.c.$$
where we have used the general definition of a conjugate fermion field \( \psi^C = \gamma_0 C \psi^* \) (\( C \) being the charge conjugation matrix) and the identity

\[
\bar{\psi}_L m \psi'_R = \psi'^C_T m \psi^C_R.
\]

(7.2)

Here \( M_R, M_D \) and \( M_l \) respectively denote the right chiral complex symmetric Majorana mass matrix, the neutrino Dirac mass matrix and the charged lepton mass matrix in a three dimensional family space. The superscripts '0' identifies the corresponding fields as flavor eigenstate ones. The complex symmetric \( 6 \times 6 \) neutrino mass matrix in the second line of (7.1) is denoted \( \mathcal{M} \), i.e.

\[
\mathcal{M} = \begin{pmatrix}
0 & M_D \\
M_D^T & M_R
\end{pmatrix}.
\]

(7.3)

The energy scale of \( M_R \) is taken to be very high (\( > 10^9 \) GeV), as compared with the electroweak scale \( v \simeq 246 \) GeV.

The complete diagonalization of \( \mathcal{M} \) leads to

\[
\mathcal{V}^\dagger \mathcal{M} \mathcal{V} = \text{diag}(d, D),
\]

(7.4)

where \( \mathcal{V} \) is a \( 6 \times 6 \) unitary matrix

\[
\mathcal{V} = \begin{pmatrix}
K & G \\
S & T
\end{pmatrix}
\]

(7.5)

with \( 3 \times 3 \) blocks \( K, G, S \) and \( T \). In (7.4) \( d \) and \( D \) are three dimensional diagonal mass matrices, each with ultralight and heavy real positive entries respectively:

\[
d = \text{diag}(m_1, m_2, m_3), \quad m_{1,2,3} < 1 \text{ eV},
\]

(7.6)

\[
D = \text{diag}(M_1, M_2, M_3), \quad M_{1,2,3} > 10^9 \text{ GeV}.
\]

(7.7)

Charged current interactions can then be written in terms of the semi-weak coupling strength \( g \) as well as the respective ultralight neutrino and heavy neutrino fields \( \nu_i \) and \( N_i \):

\[
\mathcal{L}^{\text{cc}} = -\frac{g}{\sqrt{2}} \left( \bar{\nu}_L \gamma_\mu K_{ij} \nu_{jL} + \bar{\nu}_L \gamma_\mu G_{ij} N_{jL} \right) W^\mu.
\]

(7.8)

In an excellent approximation, the ultralight neutrino masses and mixing angles can now be obtained from

\[
M_\nu \simeq - M_D M_R^{-1} M_D^T,
\]

(7.9)

\[
U^\dagger M_\nu U^* = d.
\]

(7.10)
Eq. (7.9) is the well-known see-saw formula. We also choose to define the matrix $h = M \nu M^\dagger \nu$ and have

$$U^\dagger h U = d^2.$$  (7.11)

In (7.10), $U$ is the Pontecorvo, Maki, Nakagawa, Sakata (PMNS) matrix admitting the standard parametrization

$$U_{\text{PMNS}} = \begin{pmatrix}
  c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i \delta_D} \\
  -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i \delta_D} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i \delta_D} & s_{23} c_{13} \\
  s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i \delta_D} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i \delta_D} & c_{23} c_{13}
\end{pmatrix}\begin{pmatrix}
  e^{i \alpha_{M_1}/2} & 0 & 0 \\
  0 & e^{i \alpha_{M_2}/2} & 0 \\
  0 & 0 & 1
\end{pmatrix}$$  (7.12)

with $c_{ij} = \cos \theta_{ij}$, $s_{ij} = \sin \theta_{ij}$, and $\delta_D (\alpha_{M_1}, \alpha_{M_2})$ being the yet unknown Dirac (Majorana) phase(s).

We note for the sake of completeness that the unitary transformation between the column of mass eigenstate of left chiral neutrino fields $\nu_L$ and the corresponding flavor eigenstate $\nu^0_L$ is

$$\nu^0_L = U \nu_L.$$  (7.13)

The additional approximate relations to keep in mind are those between $U$ and the submatrices $K$, $G$ of $V$ and $M_D$ of $M$:

$$K_{ij} = U_{ij} + O(v^2/M_R^2),$$  (7.14)

$$M_k G_{jk} = (M_D)_{jk} + O(v^3/M_R^2), \quad k \text{ not summed.}$$  (7.15)

Needless to add, we always neglect terms of order $v^2/M_R^2$.

As mentioned earlier, without loss of generality, we can choose the Weak Basis in which $M_l$ and $M_R$ are

$$M_l = \text{diag} (m_e, m_\mu, m_\tau),$$  (7.16)

$$M_R = \text{diag} (M_1, M_2, M_3),$$  (7.17)

with real positive entries. All CP-violating phases, stemming from $M$, are contained in the Dirac mass matrix $M_D$ in this Weak Basis. As a consequence of (7.6, 7.7) and (7.9, 7.10), $M_D$ can be written in the Casas-Ibarra form [67]

$$M_D = i U \sqrt{d} R \sqrt{M_R},$$  (7.18)

where $R$ is a complex orthogonal matrix: $R^T R = R R^T = I$. An important comment on $M_D$, following from (7.9), is that our condition of no massless neutrino, i.e. $\det M_\nu \neq 0$, implies that $\det M_D \neq 0$. This means that textures of $M_D$ with one vanishing row or column or with a quartet of zeros (i.e. zeros in $ij$, $lk$, $ik$ and $lj$ elements with $i \neq l$ and $k \neq j$ and $l = 1, 2$ or 3) are inadmissible since they make $\det M_D$ vanish. Furthermore, in our Weak Basis, for any nonzero entry in $M_D$ with all other elements in its row or column being zero, $M_\nu$ from (7.9) develops a block diagonal form that is incompatible with the observed simultaneous mixing of three neutrinos. The same logic holds for any block diagonal texture of $M_D$. Indeed, if any row in a texture of $M_D$ is orthogonal, element by element, to both the others, one neutrino family decouples and therefore makes such a texture inadmissible. These arguments have been shown [28] to be sufficient to rule out all textures in $M_D$ with more than four zeros. Four is then the maximum permitted number of zeros in a neutrino Yukawa texture.
7.3 Classification of four zero textures and the role of $\mu\tau$ symmetry

In this section we provide the classification of all possible four zero neutrino Yukawa textures and forms of the surviving textures, since these details were not given in [28, 68]. There are $9C_4 = 126$ possible four zero neutrino Yukawa textures which can be classified into four classes [68]. In making this classification, we rule out the orthogonality between any two rows or columns by some artificial cancellation; orthogonality is to be ensured in terms of a vanishing product, element by element. We can now enumerate four cases.

(i) $\det M_D \neq 0$ and one family of neutrinos decouples: 9 textures.

For each texture of $M_D$ here, one row is orthogonal to the other rows. It follows that, in the neutrino mass matrix $M_\nu$ in our chosen basis with a diagonal $M_R$, one neutrino family always decouples. So, though all neutrinos are massive here, these textures are to be discarded.

(ii) $\det M_D = 0$ and one family of neutrinos decouples: 18 textures

Here each texture has a vanishing row and there are six such textures for every such row. Such a row generates a vanishing mass eigenvalue and the corresponding family decouples. Hence this class is also excluded.

(iii) $\det M_D = 0$ and no family decouples: 27 textures [68]

Each of 18 textures in this class has a vanishing column and each of the remaining 9 has a quartet of zeros, leading to a vanishing $\det M_\nu$. So, this class is rejected.

(iv) $\det M_D \neq 0$ and no family decouples: 72 textures

These remaining textures are allowed by the criteria we have set up.

The retained textures are sub-divided into two categories $A$ and $B$. We wish to elaborate on this categorization [28]. Let us consistently use the complex parameters $a_k$, $b_k$ and $c_k$ for elements in $M_D$ belonging to the $k$th column and the first, second and third rows respectively. The two categories then are as follows.

Category $A$:

Here every texture has two mutually orthogonal rows ($i$, $j$ say, with $i \neq j$) and the corresponding derived $M_\nu$ has $(M_\nu)_{ij} = 0$. Thus there are 54 such textures divided into three sub-categories, each containing 18 textures: (A1) those with orthogonal rows 1 and 2 which generate $(M_\nu)_{12} = (M_\nu)_{21} = 0$; (A2) those with orthogonal rows 2 and 3 which generate $(M_\nu)_{23} = (M_\nu)_{32} = 0$; (A3) those with orthogonal rows 1 and 3 which generate $(M_\nu)_{13} = (M_\nu)_{31} = 0$. The explicit form of each of the 54 textures in category $A$ within the three sub-categories is shown in Table 7.1.

Category $B$:

There are 18 textures in this category. Each has two orthogonal columns, while no pair of rows is orthogonal. Invariably, then, it turns out that one row ($i$, say) has two zeroes and the other two rows (say $k$, $l \neq i$) have one zero each. It is now a consequence of (7.9) that, in the derived neutrino mass-matrix $M_\nu$, we have the relation

$$\det \text{cofactor } [(M_\nu)_{kl}] = 0. \quad (7.19)$$

Once again, one can make three subcategories with six entries each. $B1$ has two zeros in the first row and one zero in each of the other two rows. $B2$ has two zeros in the second row and one zero in each of the other two rows. $B3$ has two zeros in the third row and one zero in each of the other two rows. All 18 textures of Category $B$ are shown in Table 7.2 within the three subcategories.
Neutrino Physics

Neutrino Yukawa textures within type-I see-saw

We now raise the question of $\mu \tau$ symmetry which we had explained in the Introduction. This symmetry is evidently invalid for the charged lepton mass terms. However, for elements in the Dirac mass matrix $M_D$ of neutrinos, it immediately implies the relations

\[
(M_D)_{12} = (M_D)_{13}, \quad (M_D)_{21} = (M_D)_{31}, \\
(M_D)_{23} = (M_D)_{32}, \quad (M_D)_{22} = (M_D)_{33}.
\]

Moreover, for the masses of the very heavy right-chiral neutrinos, we have

\[
(M_R)_{22} = (M_R)_{33},
\]

a result which is transparent as $M_2 = M_3$ in our chosen basis. On account of (7.9) and (7.20) as well

| Category A1 (Orthogonality between rows 1 and 2) |
|-----------------------------------------------|
| $0 \ a_2 \ 0$ | $0 \ a_2 \ 0$ | $0 \ 0 \ a_3$ | $0 \ 0 \ a_3$ | $a_1 \ 0 \ 0$ | $a_1 \ 0 \ 0$ |
| $b_1 \ 0 \ b_3$ | $b_1 \ 0 \ b_3$ | $b_1 \ b_2 \ 0$ | $b_1 \ b_2 \ 0$ | $b_1 \ 0 \ b_3$ | $b_1 \ 0 \ b_3$ |
| $0 \ c_2 \ c_3$ | $c_1 \ c_2 \ 0$ | $0 \ 0 \ c_3$ | $c_1 \ 0 \ c_3$ | $0 \ b_2 \ b_3$ | $0 \ b_2 \ b_3$ |
| $0 \ a_2 \ 0$ | $0 \ a_2 \ 0$ | $0 \ 0 \ a_3$ | $0 \ 0 \ a_3$ | $a_1 \ 0 \ 0$ | $a_1 \ 0 \ 0$ |
| $0 \ 0 \ b_3$ | $b_1 \ 0 \ b_3$ | $b_1 \ 0 \ b_3$ | $0 \ 0 \ b_3$ | $0 \ 0 \ b_3$ | $0 \ 0 \ b_3$ |
| $c_1 \ c_2 \ c_3$ | $c_1 \ c_2 \ c_3$ | $c_1 \ c_2 \ c_3$ | $c_1 \ c_2 \ c_3$ | $c_1 \ c_2 \ c_3$ | $c_1 \ c_2 \ c_3$ |

| Category A2 (Orthogonality between rows 2 and 3) |
|-----------------------------------------------|
| $0 \ a_2 \ a_3$ | $a_1 \ a_2 \ 0$ | $0 \ a_2 \ a_3$ | $a_1 \ 0 \ a_3$ | $a_1 \ 0 \ a_3$ | $a_1 \ a_2 \ 0$ |
| $0 \ b_2 \ 0$ | $0 \ b_2 \ 0$ | $0 \ 0 \ b_3$ | $b_1 \ 0 \ b_3$ | $b_1 \ 0 \ b_3$ | $b_1 \ 0 \ b_3$ |
| $c_1 \ 0 \ c_3$ | $c_1 \ 0 \ c_3$ | $c_1 \ c_2 \ 0$ | $c_1 \ c_2 \ 0$ | $0 \ c_2 \ c_3$ | $0 \ c_2 \ c_3$ |
| $a_1 \ a_2 \ a_3$ | $a_1 \ a_2 \ a_1$ | $a_1 \ a_2 \ a_3$ | $a_1 \ a_2 \ a_3$ | $a_1 \ a_2 \ a_3$ | $a_1 \ a_2 \ a_3$ |
| $0 \ b_2 \ 0$ | $0 \ b_2 \ 0$ | $0 \ 0 \ b_3$ | $b_1 \ 0 \ b_3$ | $b_1 \ 0 \ b_3$ | $b_1 \ 0 \ b_3$ |
| $0 \ 0 \ c_3$ | $c_1 \ 0 \ c_3$ | $0 \ c_2 \ c_3$ | $c_1 \ 0 \ c_3$ | $0 \ c_2 \ c_3$ | $0 \ c_2 \ c_3$ |

| Category A3 (Orthogonality between rows 1 and 3) |
|-----------------------------------------------|
| $0 \ a_2 \ 0$ | $0 \ a_2 \ 0$ | $0 \ 0 \ a_3$ | $0 \ 0 \ a_3$ | $a_1 \ 0 \ 0$ | $b_1 \ 0 \ b_3$ |
| $0 \ b_2 \ b_3$ | $b_1 \ b_2 \ 0$ | $0 \ b_2 \ b_3$ | $b_1 \ 0 \ b_3$ | $b_1 \ 0 \ b_3$ | $b_1 \ 0 \ b_3$ |
| $c_1 \ 0 \ c_3$ | $c_1 \ 0 \ c_3$ | $c_1 \ c_2 \ 0$ | $c_1 \ c_2 \ 0$ | $0 \ c_2 \ c_3$ | $0 \ c_2 \ c_3$ |
| $0 \ a_2 \ 0$ | $0 \ a_2 \ 0$ | $0 \ 0 \ a_3$ | $0 \ 0 \ a_3$ | $a_1 \ 0 \ 0$ | $a_1 \ 0 \ 0$ |
| $b_1 \ b_2 \ b_3$ | $b_1 \ b_2 \ b_3$ | $b_1 \ b_2 \ b_3$ | $b_1 \ b_2 \ b_3$ | $b_1 \ b_2 \ b_3$ | $b_1 \ b_2 \ b_3$ |
| $0 \ 0 \ c_3$ | $c_1 \ 0 \ c_3$ | $0 \ 0 \ c_3$ | $c_1 \ 0 \ c_3$ | $0 \ 0 \ c_3$ | $0 \ 0 \ c_3$ |
| $a_1 \ 0 \ a_3$ | $a_1 \ 0 \ a_3$ | $a_1 \ 0 \ a_3$ | $a_1 \ 0 \ a_3$ | $a_1 \ 0 \ a_3$ | $a_1 \ 0 \ a_3$ |
| $0 \ b_2 \ b_3$ | $b_1 \ b_2 \ 0$ | $b_1 \ 0 \ b_3$ | $b_1 \ 0 \ b_3$ | $b_1 \ 0 \ b_3$ | $b_1 \ 0 \ b_3$ |
| $0 \ c_2 \ 0$ | $0 \ c_2 \ 0$ | $0 \ 0 \ c_3$ | $c_1 \ 0 \ c_3$ | $c_1 \ 0 \ c_3$ | $c_1 \ 0 \ c_3$ |

Table 7.1: Four zero Yukawa textures of $M_D$ in Category A with subcategories A1, A2, and A3
Neutrino Physics

Neutrino Yukawa textures within type-I see-saw

Table 7.2: Four zero Yukawa textures of $M_D$ in Category $B$ with subcategories $B_1$, $B_2$, and $B_3$

| Category | First row with two zeros, a pair of orthogonal columns, non-orthogonal rows |
|----------|-----------------------------------------------------------------------------|
| $B_1$    | $\begin{pmatrix} 0 & a_2 & 0 \\ 0 & b_1 & b_3 \\ c_1 & c_2 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & a_2 & 0 \\ b_1 & b_2 & 0 \\ 0 & c_2 & c_3 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & a_3 \\ b_1 & 0 & b_3 \\ c_1 & 0 & c_3 \end{pmatrix}$, $\begin{pmatrix} a_1 & 0 & 0 \\ b_1 & 0 & b_3 \\ c_1 & c_2 & 0 \end{pmatrix}$, $\begin{pmatrix} a_1 & 0 & 0 \\ b_1 & b_2 & 0 \\ c_1 & 0 & c_3 \end{pmatrix}$ |

| Category | Second row with two zeros, a pair of orthogonal columns, non-orthogonal rows |
|----------|-------------------------------------------------------------------------------|
| $B_2$    | $\begin{pmatrix} 0 & a_2 & a_3 \\ 0 & b_2 & 0 \\ c_1 & c_2 & 0 \end{pmatrix}$, $\begin{pmatrix} a_1 & a_2 & 0 \\ 0 & b_2 & 0 \\ 0 & c_2 & c_3 \end{pmatrix}$, $\begin{pmatrix} 0 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ c_1 & 0 & c_3 \end{pmatrix}$, $\begin{pmatrix} a_1 & 0 & a_3 \\ b_1 & 0 & 0 \\ c_1 & c_2 & 0 \end{pmatrix}$, $\begin{pmatrix} a_1 & a_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}$ |

| Category | Third row with two zeros, a pair of orthogonal columns, non-orthogonal rows |
|----------|------------------------------------------------------------------------------|
| $B_3$    | $\begin{pmatrix} a_1 & a_2 & 0 \\ 0 & b_2 & b_3 \\ 0 & c_2 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & a_2 & a_3 \\ b_1 & b_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$, $\begin{pmatrix} a_1 & 0 & a_3 \\ b_1 & 0 & b_3 \\ c_1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{pmatrix}$ |

as (7.21), one is immediately led to the following relations among elements of the complex symmetric ultralight neutrino Majorana mass matrix $M_\nu$:

$$(M_\nu)_{12} = (M_\nu)_{13} \quad (M_\nu)_{22} = (M_\nu)_{33}. \quad (7.22)$$

We take these as statements of a custodial $\mu\tau$ symmetry in the ultralight neutrino sector. One can now invert (7.10) and explore the consequences of (7.22) in the parametrization of (7.12). An immediate consequence is the fixing of the two mixing angles pertaining to the third flavor: $\theta_{23} = \pi/4$, $\theta_{13} = 0$. Since the measured former angle is compatible with $45^\circ$ within errors and the latter has been found to be small ($\simeq 9^\circ$), the occurrence of at least a broken $\mu\tau$ symmetry in nature is a reasonable supposition that we adhere to. An interesting footnote to this discussion is the issue of tribimaximal mixing [69, 70] which subsumes $\mu\tau$ symmetry but posits the additional relation

$$(M_\nu)_{11} + (M_\nu)_{13} = (M_\nu)_{22} + (M_\nu)_{23}, \quad (7.23)$$

leading to a fixation of the remaining mixing angle $\theta_{12} = \sin^{-1}(1/\sqrt{3}) \simeq 35.26^\circ$. However, we shall not make use of (7.23).

An immediate consequence of the imposition of $\mu\tau$ symmetry, via (7.20), is the drastic reduction of the seventy two allowed four zero textures of $M_D$ to only four [31]. This is seen just by inspection. The allowed $\mu\tau$ symmetric textures are the following, each involving only three complex parameters.

**Category A**

$$M_{DA1} = \begin{pmatrix} a_1 & a_2 & a_2 \\ 0 & 0 & b_1 \\ 0 & b_1 & 0 \end{pmatrix}, \quad M_{DA2} = \begin{pmatrix} a_1 & a_2 & a_2 \\ 0 & b_1 & 0 \\ 0 & 0 & b_1 \end{pmatrix}, \quad (7.24)$$

**Category B**

$$M_{DB1} = \begin{pmatrix} a_1 & 0 & 0 \\ b_1 & 0 & b_2 \\ b_1 & b_2 & 0 \end{pmatrix}, \quad M_{DB2} = \begin{pmatrix} a_1 & 0 & 0 \\ b_1 & b_2 & 0 \\ b_1 & 0 & b_2 \end{pmatrix}. \quad (7.25)$$
It may be noted that, in either category, any texture can be obtained from the other by the interchange of rows 2 and 3 or columns 2 and 3. Because of \(\mu\tau\) symmetry, this means that the physical content of the two textures in each category is the same. Indeed, by use of (7.9), we obtain the same \(M_\nu\) for either. Thus we have just two allowed ultralight neutrino Majorana mass matrices

\[
M_{\nu A} = -\begin{pmatrix}
\frac{a_1^2}{M_1} + 2\frac{a_2^2}{M_2} & a_2b_1/M_2 & a_2b_1/M_2 \\
a_2b_1/M_2 & b_1^2/M_2 & 0 \\
a_2b_1/M_2 & 0 & b_1^2/M_2
\end{pmatrix}
\]

and

\[
M_{\nu B} = -\begin{pmatrix}
a_1^2/M_1 & a_1b_1/M_1 & a_1b_1/M_1 \\
a_1b_1/M_1 & b_1^2/M_1 + b_2^2/M_2 & b_1^2/M_1 \\
a_1b_1/M_1 & b_1^2/M_1 & b_1^2/M_1 + b_2^2/M_2
\end{pmatrix}
\] (7.26)

for categories \(A\) and \(B\) respectively.

### 7.4 Phenomenology with \(\mu\tau\) symmetric four zero Yukawa textures

Given \(\mu\tau\) symmetry, one automatically obtains that \(\theta_{23} = \pi/4\) and \(\theta_{13} = 0\). The current 3\(\sigma\) limits on these are 38.6\(^o\) < \(\theta_{23}\) < 53.1\(^o\) and 7.0\(^o\) < \(\theta_{13}\) < 10.9\(^o\) [1]. We shall later consider a small breaking of \(\mu\tau\) symmetry. But, for the moment, let us assume the latter to be the exact. The other mass and mixing parameters in the ultralight neutrino sector are kept free. Their experimentally allowed 3\(\sigma\) ranges will be used to constrain the non-zero elements of \(M_{\nu A}\) and \(M_{\nu B}\) in Table 7.3. We define \(\Delta_{ij}^2 = m_i^2 - m_j^2\) where \(i, j (= 1, 2, 3)\) refer to the mass eigenstate neutrinos. It will now be convenient to reparametrize the elements of \(M_{\nu A}\) and \(M_{\nu B}\) in (7.26) and (7.27) respectively in the way given in Table 7.3. Here \(k_1, k_2, l_1, l_2\) are real and positive quantities while \(\alpha, \alpha', \beta, \beta'\) are phases. However, the phases \(\alpha'\) and \(\beta'\) can be absorbed in the definition of the first family neutrino field \(\nu_e\) for \(M_{\nu A}\) and \(M_{\nu B}\) respectively and therefore are not physical. Moreover, the overall phase in \(m_{A,B}\) can also be absorbed by a further redefinition of all flavor eigenstate neutrino fields. So we can treat \(m_{A,B}\) as real for further discussions. In addition, we have defined in Table 7.3 sets of derived real quantities \(X_{1-4}^{A,B}\) which will be related to various observables.

| Category \(A\) | Category \(B\) |
|----------------|----------------|
| \(M_{\nu A}\) | \(M_{\nu B}\) |
| \(m_A = -\frac{b_1^2}{M_2}\), \(k_1e^{\alpha + \alpha'} = \frac{a_1}{b_1}\sqrt{\frac{M_1}{M_2}}\), \(k_2e^{\alpha'} = \frac{a_2}{b_2}\), \(\alpha = \arg a_{12}^A\) | \(m_B = -\frac{b_2^2}{M_2}\), \(l_1e^{\beta + \beta'} = \frac{a_1}{b_1}\sqrt{\frac{M_1}{M_2}}\), \(l_2e^{\beta'} = \frac{b_1}{b_2}\sqrt{\frac{M_1}{M_2}}\), \(\beta = \arg a_{12}^B\) |
| \(X_1^A = 2\sqrt{2}k_2[(1 + 2k_2^2)^2 + k_1^2 + 2k_1^2(1 + 2k_2^2)\cos 2\alpha]^{1/2}\) | \(X_1^B = 2\sqrt{2}l_2[(l_2^2 + 2l_2^2)^2 + 1 + 2(l_2^2 + 2l_2^2)\cos 2\beta]^{1/2}\) |
| \(X_2^A = 1 - k_1^2 - 4k_2^2 - 4k_1k_2^2\cos 2\alpha\) | \(X_2^B = 1 + 4l_2^2\cos 2\beta + 4l_2^2 - l_2^4\) |
| \(X_3^A = 1 - 4k_1^2 - k_1^2 - 4k_1k_2^2\cos 2\alpha - 4k_2^2\) | \(X_3^B = 1 - (l_2^2 + 2l_2^2)^2 - 4l_2^2\cos 2\beta\) |
| \(X_4^A = k_1^4 + 4k_1^2 + 4k_1^2k_2^2\cos 2\alpha\) | \(X_4^B = l_2^4\) |

Table 7.3: Reparametrized quantities and relevant functions in Category \(A\) and Category \(B\)
Neutrino Yukawa textures within type-I see-saw

| Quantity | Experimental 3σ range |
|----------|------------------------|
| $\Delta^2_{21}$ | $7.12 \times 10^{-3} \ eV^2 < \Delta^2_{21} < 8.20 \times 10^{-3} \ eV^2$ |
| $\Delta^2_{32} < 0$ | $-2.76 \times 10^{-3} \ eV^2 < \Delta^2_{32} < -2.22 \times 10^{-3} \ eV^2$ |
| $\Delta^2_{32} > 0$ | $2.18 \times 10^{-3} \ eV^2 < \Delta^2_{32} < 2.70 \times 10^{-3} \ eV^2$ |
| $\theta_{12}$ | $37.46^\circ < \theta_{12} < 31.30^\circ$ |

Table 7.4: Input experimental values [1]

With the reparametrization given in Table 7.3, $M_{\nu A}$ and $M_{\nu B}$ assume the simple forms

$$M_{\nu A} = m_{A} \begin{pmatrix} k_1 e^{2i\alpha} + 2k_2 & k_2 & k_2 \\ k_2 & 1 & 0 \\ k_2 & 0 & 1 \end{pmatrix}, \quad M_{\nu B} = m_{B} \begin{pmatrix} l_1^2 & l_1 l_2 e^{i\beta} & l_1 l_2 e^{i\beta} \\ l_1 l_2 e^{i\beta} & l_2^2 e^{2i\beta} + 1 & l_2^2 e^{2i\beta} \\ l_1 l_2 e^{i\beta} & l_2^2 e^{2i\beta} + 1 \end{pmatrix}. \quad (7.28)$$

These lead us, through the diagonalization of the matrix $h_{\nu}$ of (7.11), to the relations

$$\Delta^2_{21} = m^2 X, \quad \Delta^2_{32} = \frac{m^2}{2} (X_3 - X), \quad \tan 2\theta_{12} = \frac{X_1}{X_2}, \quad (7.29)$$

where $m = m_{A,B}$ for category $A, B$ and

$$X = \sqrt{X_1^2 + X_2^2}. \quad (7.30)$$

One can further make use of (7.10) to calculate [32] the ultralight masses $m_{1,2,3}$ in terms $\Delta^2_{21}$ and also the Majorana phases $\alpha_{M_1}, \alpha_{M_2}$, cf. (7.12), in terms of $m_1, m_2, m_3$. The former are given by

$$m_{1,2} = \left| \Delta^2_{21} \left( 2 - \frac{X_3 + X}{2X} \right) \right|^{1/2}, \quad m_3 = \left| \Delta^2_{21}/X \right|^{1/2} \quad (7.31)$$

and the latter by

$$\cos(\alpha_{M_1} - \arg Z) = \frac{|Z|^2 m_3^2 + m_2^2 \sin^4 \theta_{12} - m_2^2 \cos^4 \theta_{12}}{2m_1 m_2 \sin^2 \theta_{12}|Z|},$$

$$\cos(\alpha_{M_2} - \arg Z) = \frac{|Z|^2 m_2^2 + m_3^2 \cos^4 \theta_{12} - m_3^2 \sin^4 \theta_{12}}{2m_1 m_2 \cos^2 \theta_{12}|Z|}. \quad (7.32)$$

Here $Z = [(M_{\nu})_{22} + (M_{\nu})_{23}][(M_{\nu})_{22} - (M_{\nu})_{23}]^{-1}$. The last quantity of physical interest that we calculate in this section is the effective mass $m_{\beta\beta} = |(M_{\nu})_{11}|$ appearing in the transition amplitude for the yet unobserved neutrinoless nuclear double beta decay. That is given by

$$m_{\beta\beta} = |\Delta^2_{21} X_4 X^{-1}|^{1/2}, \quad (7.33)$$

with $X_4$ as given in Table 7.3.

Feeding the experimental $3\sigma$ ranges from Table 7.4, we find that only the inverted mass ordering $\Delta^2_{32} < 0$ is allowed in Category $A$ while only the normal mass ordering $\Delta^2_{32} > 0$ is permitted for Category $B$. Moreover, in the corresponding $k_1 - k_2/l_1 - l_2$ parameter plane [31], very constrained domains are allowed, as shown in Fig 7.1. The phases $\alpha, \beta$ are also severely restricted in magnitude,
Neutrino Physics

Neutrino Yukawa textures within type-I see-saw

Figure 7.1: Variation of $k_1$ and $k_2$ in category A and of $l_1$ and $l_2$ in category B with $\mu\tau$ symmetry over the $3\sigma$ allowed ranges of $\Delta_{21}^{\alpha}$, $|\Delta_{22}^{\alpha}|$ and $\theta_{12}$ [31].

specifically $89.0^0 \leq |\alpha| < 90^0$ and $87.0^0 \leq |\beta| < 90^0$. These allow just a very limited region in the $X_3 - X$ plane, leading to $3\sigma$ lower and upper bounds on the neutrino mass sum $m_1 + m_2 + m_3$, namely [0.156, 0.5] eV/[0.074, 0.132] eV for an inverted/normal mass-ordering [32]. It may be recalled that there is already a lower bound of 0.05 eV on the said sum from atmospheric neutrino data. Furthermore, the general consensus [7] on the least model dependent cosmological upper bound on it is 0.5 eV.

Turning to the individual neutrino masses $m_1/eV$, $m_2/eV$ and $m_3/eV$ respectively, we obtain by use of (7.31) the respective $3\sigma$ intervals [0.0452, 0.1682], [0.0457, 0.1684], [0.077, 0.1632] for Category A and [0.0110, 0.0335], [0.0144, 0.0345], [0.0485, 0.0638] for Category B. However, there are correlated constraints among these masses. These are shown in the left-most panel of Fig 7.2. Given these allowed intervals and correlated constraints, it is not possible right now to distinguish between the hierarchical and quasi-degenerate possibilities. But a future reduction of these ranges and domains could pin this down. We next come to the Majorana phases $\alpha_{M_1}$, $\alpha_{M_2}$. One can ab initio restrict them to the interval $-\pi$ to $\pi$ and utilize (7.32) as well as the expressions for $Z$, $m_1$, $m_2$, $m_3$ and $\tan 2\theta_{12}$ in terms of the basic parameters $(k_1, k_2, \alpha)/(l_1, l_2$ and $\beta)$, depending on the category. The further application of the phenomenologically acceptable ranges of these parameters, as given above, leads to the allowed $3\sigma$ intervals $-98.0^0 \leq \alpha_{M_1} \leq 20.0^0$, $9.2^0 \leq \alpha_{M_2} \leq 36.4^0$ for Category A and $-88.6^0 \leq \alpha_{M_1} \leq 7.97^0$, $90.7^0 \leq \alpha_{M_2} \leq 128.8^0$ for Category B. Allowed values of $\alpha_{M_1}$ and $\alpha_{M_2}$ are shown in the middle panel of Fig 7.2.

Another quantity to be considered in this section is the double $\beta$–decay effective mass $m_{\beta\beta}$, cf. (7.33). The currently accepted upper bound [71] on it is $m_{\beta\beta} < 0.35$ eV. In comparison, our allowed values $m_{\beta\beta}$ vs the neutrino mass sum $\sum_i m_i$ are shown in the rightmost panels of Fig 7.2. More absolutely, we can say that $0.038 \leq m_{\beta\beta}/eV \leq 0.161$ for Category A and $0.003 \leq m_{\beta\beta}/eV \leq 0.0186$ for Category B. The region near the upper bound in Category A may be accessible in forthcoming experiments.

11
An interesting question pertains to the consequences of the effect of $\mu\tau$ symmetry on couplings between the heavy right chiral and the ultralight left chiral neutrinos. The corresponding neutral gauge boson induced interactions are down by factor $O(v^2/M^2_R)$. On the other hand, the Higgs boson induced interactions affect leptogenesis modes and have been discussed in detail in [30]. Since leptogenesis is outside the scope of the present article, we do not go into those discussions here.

### 7.5 Realization of other texture zeros with $\mu\tau$ symmetry

Though four is the maximum number of allowed neutrino Yukawa texture zeros, we examine other textures with a lesser number of zeros for completeness [33]. Let us work in the same Weak Basis of real diagonal $M_l$ and $M_R$. We wish to study only those textures that are compatible with $\mu\tau$ symmetry which we believe to be approximately valid in the real world. The $\mu\tau$ symmetric forms of $M_D$ and $M_R$ now are

$$M_D = \begin{pmatrix} a & b & b \\ c & d & e \\ c & e & d \end{pmatrix}, \quad M_R = \text{diag} (M_1, M_2, M_2)$$

(7.34)

with $a, b, c, d, e$ as complex numbers.
Three zero textures
We first identify possible three zero textures which are compatible with (7.34). Apart from $a$, the other four complex parameters in $M_D$ come in pairs. So, for any texture with an odd number of zeros, $a$ must vanish. For three zero textures the remaining two zeros can be arranged in $^4C_1 = 4$ ways. So, the four allowed three zero $\mu\tau$ symmetric textures of $M_D$ are:
\[
\begin{pmatrix}
0 & b & b \\
c & 0 & e \\
c & e & 0
\end{pmatrix},
\begin{pmatrix}
0 & b & b \\
c & d & 0 \\
c & 0 & d
\end{pmatrix},
\begin{pmatrix}
0 & 0 & b \\
c & 0 & d \\
c & d & e
\end{pmatrix},
\begin{pmatrix}
0 & 0 & b \\
c & d & 0 \\
c & e & d
\end{pmatrix}.
\tag{7.35}
\]
The last two textures have one vanishing row and one vanishing column respectively. These can be discarded with our requirement of no massless neutrino, i.e. $\det M_\nu \neq 0$, leaving only the first two textures as acceptable. In general, there can be $^9C_3 = 84$ three zero textures. The conditions of (1) $\mu\tau$ symmetry, (2) the non-zero value of $\det M_\nu$ and (3) the non-decoupling of any neutrino generation reduce this number to only two.

The first two textures of (7.35) have only three complex parameters each and we can just use $b, c$ and $d$ for both allowed textures of $M_D$:
\[
\begin{pmatrix}
0 & b & b \\
c & 0 & d \\
c & d & 0
\end{pmatrix},
\begin{pmatrix}
0 & b & b \\
c & d & 0 \\
c & 0 & d
\end{pmatrix}.
\tag{7.36}
\]
Using the see-saw formula, we obtain an identical form of $M_\nu$ for both textures in (7.36), namely
\[
M_\nu = -\left(\begin{array}{ccc}
\frac{2k_1^2}{M_2^2} & \frac{b d}{M_2^2} & \frac{b d}{M_2^2} \\
\frac{b d}{M_2^2} & \frac{c^2 M_1}{M_2^2} + \frac{d^2 M_2}{M_2^2} & \frac{c^2 M_1}{M_2^2} \\
\frac{b d}{M_2^2} & \frac{c^2 M_1}{M_2^2} + \frac{d^2 M_2}{M_2^2} & \frac{c^2 M_1}{M_2^2} + \frac{d^2 M_2}{M_2^2}
\end{array}\right),
\tag{7.37}
\]
Eq. (7.37) can be written in the following form under a further reparametrization
\[
M_\nu = m_0 \begin{pmatrix}
2k_1^2 e^{i2\alpha_1} \\
k_1 e^{i\alpha_1} k_1 e^{i\alpha_1} k_2 e^{i\alpha_2} \\
k_1 e^{i\alpha_1} k_2 e^{i\alpha_2} + 1
\end{pmatrix},
\tag{7.38}
\]
with $m_0 = -\frac{d}{M_2}$, $k_1 e^{i\alpha_1} = \frac{b}{d}$ and $k_2 e^{i\alpha_2} = \frac{c}{d} \sqrt{\frac{M_2}{M_1}}$. From $M_\nu$, we can remove the phase $\alpha_1$ and any phase $\theta_m$ in $m_0$ by rotating $M_\nu$ with the diagonal phase matrix $\text{diag} \left( e^{-i\alpha_1}, 1, 1 \right) e^{-i\theta_m/2}$. Thus, $M_\nu$ has three real parameters, namely $m_0$, $k_1$ and $k_2$ and only one phase $\alpha_2$. The interesting point to be noted is that the number of independent parameters in $M_\nu$ for $\mu\tau$ symmetric three zero Yukawa textures is the same as that for $\mu\tau$ symmetric four zero textures. We then have the same phenomenological expressions as in (7.29)-(7.33), but now with changed definitions of $X_{1,2,3,4}$, namely
\[
X_1 = 2\sqrt{2} k_1 \sqrt{(1 + 2k_1^2)^2 + 4k_1^4 + 4k_2^2(1 + 2k_1^2) \cos 2\alpha_2},
\]
\[
X_2 = 4k_1^4 + 1 + 4k_2^2 \cos 2\alpha_2 - 4k_1^4,
\]
\[
X_3 = 1 - 4k_1^4 - 4k_1^2 - 4k_2^2 - 4k_2^2 \cos 2\alpha_2,
\]
\[
X_4 = 4k_1^2.
\tag{7.39}
\]
Two zero textures

Again, looking at the $\mu\tau$ symmetric form of $M_D$ in (7.34), we can conclude that, for any even zero textures, $a \neq 0$. Two zeros can be fitted to each of the remaining four pairs of parameters in four ways. The four possible two zero textures of $M_D$ are:

\[
\begin{pmatrix}
  a & b & b \\
  c & 0 & d \\
  c & d & 0
\end{pmatrix}, \quad \begin{pmatrix}
  a & b & b \\
  c & d & 0 \\
  c & 0 & d
\end{pmatrix},
\]

\[
\begin{pmatrix}
  a & 0 & 0 \\
  c & b & d \\
  c & d & b
\end{pmatrix}, \quad \begin{pmatrix}
  a & b & b \\
  0 & c & d \\
  0 & d & c
\end{pmatrix}
\]

(7.40)

and are all allowed. So, the number of allowed $\mu\tau$ symmetric two zero textures is the same as that of similar four zero textures. We need only the four parameters $a, b, c$ and $d$ to write down all four textures. The latter lead to three allowed forms of $M_\nu$; the first two such textures yield one form and the remaining two lead to two forms of $M_\nu$. These are respectively given by

\[
M_\nu = -\begin{pmatrix}
\frac{a^2}{M_1} + \frac{2b^2}{M_2} & \frac{ac}{M_1} + \frac{bd}{M_2} & \frac{ac}{M_1} + \frac{bd}{M_2} \\
\frac{ac}{M_1} + \frac{bd}{M_2} & \frac{c^2}{M_1} + \frac{d^2}{M_2} & \frac{c^2}{M_1} + \frac{d^2}{M_2} \\
\frac{ac}{M_1} + \frac{bd}{M_2} & \frac{c^2}{M_1} + \frac{d^2}{M_2} & \frac{e^2}{M_1} + \frac{d^2}{M_2}
\end{pmatrix},
\]

(7.41)

\[
M_\nu = -\begin{pmatrix}
\frac{a^2}{M_1} & \frac{bc}{M_1} + \frac{d^2}{M_2} & \frac{bc}{M_1} + \frac{d^2}{M_2} \\
\frac{bc}{M_1} + \frac{d^2}{M_2} & \frac{c^2}{M_1} + \frac{d^2}{M_2} & \frac{c^2}{M_1} + \frac{d^2}{M_2} \\
\frac{bc}{M_1} + \frac{d^2}{M_2} & \frac{c^2}{M_1} + \frac{d^2}{M_2} & \frac{e^2}{M_1} + \frac{d^2}{M_2}
\end{pmatrix},
\]

(7.42)

\[
M_\nu = -\begin{pmatrix}
\frac{a^2}{M_1} & \frac{bc}{M_1} + \frac{d^2}{M_2} & \frac{bc}{M_1} + \frac{d^2}{M_2} \\
\frac{bc}{M_1} + \frac{d^2}{M_2} & \frac{c^2}{M_1} + \frac{d^2}{M_2} & \frac{c^2}{M_1} + \frac{d^2}{M_2} \\
\frac{bc}{M_1} + \frac{d^2}{M_2} & \frac{c^2}{M_1} + \frac{d^2}{M_2} & \frac{e^2}{M_1} + \frac{d^2}{M_2}
\end{pmatrix}.
\]

(7.43)

Compared to four and three zero Yukawa textures, there are more independent parameters here. Apart from the overall mass scale, there will be three moduli and two irremovable phases. It is easier to fit the neutrino data with such a larger number of parameters and we do not discuss it any further.

One zero

One zero textures represent the most trivial case among the $\mu\tau$ symmetric neutrino Yukawa texture zeros. This is since, as an odd zero texture, it must have $a = 0$ in (7.34). The single allowed texture of $M_D$ is

\[
\begin{pmatrix}
  0 & b & b \\
  c & d & e \\
  c & e & d
\end{pmatrix}
\]

(7.44)

and yields the following form of $M_\nu$:

\[
M_\nu = -\begin{pmatrix}
\frac{bc}{M_2} + \frac{bd}{M_2} & \frac{be}{M_2} + \frac{bd}{M_2} & \frac{be}{M_2} + \frac{bd}{M_2} \\
\frac{bc}{M_2} + \frac{bd}{M_2} & \frac{c^2}{M_1} + \frac{d^2}{M_2} & \frac{c^2}{M_1} + \frac{d^2}{M_2} \\
\frac{bc}{M_2} + \frac{bd}{M_2} & \frac{c^2}{M_1} + \frac{d^2}{M_2} & \frac{e^2}{M_1} + \frac{d^2}{M_2}
\end{pmatrix}.
\]

(7.45)

Like the two zero textures, this allowed one zero texture has six parameters: one overall real mass scale, three moduli and two phases. These can easily fit the extant neutrino data.
7.6 The breaking of $\mu\tau$ symmetry

As mentioned in previous sections, the results $\theta_{13} = 0$ and $\theta_{23} = \pi/4$ are consequences of the custodial $\mu\tau$ symmetry in $M_\nu$. But present neutrino data from T2K, DOUBLE CHOOZ, RENO and DAYA BAY experiments rule out $\theta_{13} = 0$ by $5\sigma$. So, the breaking of $\mu\tau$ symmetry is an inevitable need in order to generate a nonzero value of $\theta_{13}$. In addition, a departure of $\theta_{23} = \pi/4$ would arise from the same breaking. Another interesting consequence of a nonzero $\theta_{13}$ would be the observability of a CKM-type of CP violation in the lepton sector. Our previous expressions for $\Delta_{21}^2$, $\Delta_{32}^2$ and $\tan 2\theta_{12}$ will be modified if $\mu\tau$ symmetry is broken.

This symmetry can be broken explicitly or spontaneously or dynamically as with the Renormalization Group evolution of Lagrangian parameters. Spontaneous breakdown generally requires the presence of extra scalars and needs a model with them. We do not go into a discussion of such models here. On the other hand, RG effects on neutrino mass can be incorporated with the methodology presented in Ref.[72, 73] in terms of the $\tau$ lepton mass arising through the running of the Yukawa coupling strength from the GUT scale $\Lambda$ to the weak scale $\lambda$. The effect is characterized by the parameter $\Delta_\tau$ which has the 1-loop expression

$$\Delta_\tau \simeq \frac{m_\tau^2}{8\pi^2 v^2} (\tan^2 \beta + 1) \ln \left( \frac{\Lambda}{\lambda} \right),$$

(7.46)

where $\tan \beta$ is the ratio of the VEVs of the up-type and down-type neutral Higgs fields in the MSSM and $v^2$ is twice the sum of their squares. Even for a large $\tan \beta$, ($\tan \beta \simeq 60$) $\Delta_\tau$ is $O(10^{-3})$ and cannot generate a $\theta_{13}$ of the order of $9^\circ$.

We therefore turn to an explicit breaking of $\mu\tau$ symmetry in $M_D$. That can be realized as the following way:

**Category A**

$$M_{DA1} = \begin{pmatrix} a_1 & a_2 & a_2(1 - \epsilon_1 e^{i\theta}) \\ 0 & 0 & b_1(1 - \epsilon_2 e^{i\phi}) \\ 0 & b_1 & 0 \end{pmatrix}, \quad M_{DA2} = \begin{pmatrix} a_1 & a_2 & a_2(1 - \epsilon_1 e^{i\theta}) \\ 0 & b_1(1 - \epsilon_2 e^{i\phi}) & 0 \\ 0 & 0 & b_1 \end{pmatrix},$$

(7.47)

**Category B**

$$M_{DB1} = \begin{pmatrix} a_1 & 0 & 0 \\ b_1(1 - \epsilon_1 e^{i\theta}) & 0 & b_2(1 - \epsilon_2 e^{i\phi}) \\ b_1 & b_2 & 0 \end{pmatrix}, \quad M_{DB2} = \begin{pmatrix} a_1 & b_1(1 - \epsilon_1 e^{i\theta}) & b_2(1 - \epsilon_2 e^{i\phi}) \\ 0 & b_1 & 0 \\ 0 & 0 & b_2 \end{pmatrix}.$$

(7.48)

Furthermore,

$$M_R = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_2(1 - \epsilon_3) \end{pmatrix},$$

(7.49)

where $\epsilon_1 e^{i\theta}$, $\epsilon_2 e^{i\phi}$ are complex $\mu\tau$ symmetry breaking parameters ($\epsilon_{1,2}$ real) and $\epsilon_3$ is a real $\mu\tau$ symmetry breaking parameter in $M_R$. For these modified $M_D$ and $M_R$, we have the following $M_\nu$'s:
Neutrino Physics

A

Category

H

where elements of

Category

B

Appendix 7.9. The detailed diagonalization and expressions for mass differences and mixing angles are given in \( \epsilon \) of \( \Delta^2 \mu_{12}, \Delta^2 \mu_{13} \) and \( \Delta^2 \nu_{BR} \), cf. Appendix. An appropriate choice of symmetry breaking parameters, i.e. \( \epsilon_{1,2,3} \approx 0.1 \), and slightly shifted parameter spaces for \( k_1, k_2, \alpha \) in category A and \( l_1, l_2, \beta \) in category B are needed. A nonzero \( CP \) violating effect can be effected through the Jarlskog invariant \( J_{CP} \):

\[
J_{CP} = \sqrt{\sum_{i=1}^{3} \sum_{j=1}^{3} H_{ij} H_{ij}}
\]

The detailed diagonalization and expressions for mass differences and mixing angles are given in Appendix 7.9.

A nonzero \( \theta_{13} \) arises after \( \mu \tau \) symmetry breaking. The value \( \theta_{13} \approx 9^\circ \) is possible for 3\( \sigma \) variations of \( (\Delta^2_{31})^{e_{1,2,3}}, (\Delta^2_{32})^{e_{1,2,3}}, (\Delta^2_{21})^{e_{1,2,3}} \), cf. Appendix. An appropriate choice of symmetry breaking parameters, i.e. \( \epsilon_{1,2,3} \approx 0.1 \), and slightly shifted parameter spaces for \( k_1, k_2, \alpha \) in category A and \( l_1, l_2, \beta \) in category B are needed. A nonzero \( CP \) violating effect can be effected through the Jarlskog invariant \( J_{CP} \):

\[
J_{CP} = \sqrt{\sum_{i=1}^{3} \sum_{j=1}^{3} H_{ij} H_{ij}}
\]

where elements of \( H \) and mass squared differences \( (\Delta^2_{31})^{e_{1,2,3}}, (\Delta^2_{32})^{e_{1,2,3}} \) are given in Appendix 7.9. N.B. \( (\Delta^2_{31})^{e_{1,2,3}} = (\Delta^2_{31})^{e_{1,2,3}} + (\Delta^2_{32})^{e_{1,2,3}} \). A detailed treatment of explicitly broken \( \mu \tau \) symmetric four zero and three zero textures is given in [33].
7.7 Concluding summary

We have reviewed neutrino Yukawa textures with zeros within the type-I seesaw with three heavy right chiral neutrinos and in the basis where the latter and the charged leptons are mass diagonal. The conditions of a non-vanishing mass of every ultralight neutrinos and of the non-decoupling of any neutrino generation allow a maximum of four zeros in the neutrino Yukawa coupling matrix $Y_\nu$. There are seventy two such textures. We show that the requirement of an exact $\mu \tau$ symmetry, coupled with observational constraints, reduces the seventy two allowed textures in such a $Y_\nu$ to only four corresponding to just two different forms of the light neutrino mass matrix $M_{\nu A}/M_{\nu B}$, resulting in an inverted/normal mass ordering. Apart from an overall mass scale, $M_\nu$ for every category has two real parameters and a irremovable phase. These parameters $k_1$, $k_2$ and $|\alpha|$ for Category A and $l_1$, $l_2$ and $|\beta|$ for Category B get highly restricted, given the 3σ ranges of measured neutrino mass squared differences and mixing angles. Neutrino masses and Majorana phases are predicted within definite ranges with 3σ laboratory and cosmological inputs. The predicted respective masses $m_1/eV$, $m_2/eV$, $m_3/eV$ are [0.0452, 0.1682], [0.0457, 0.1684], [0.077, 0.1632] for Category A and [0.0110, 0.0335], [0.0144, 0.0345], [0.0485, 0.0638] for Category B. Corresponding intervals of the Majorana phases are $-98.0^\circ \leq \alpha_{M_1} \leq 20.0^\circ$, $9.2^\circ \leq \alpha_{M_2} \leq 36.4^\circ$ for Category A and $-88.6^\circ \leq \alpha_{M_3} \leq 7.97^\circ$, $90.7^\circ \leq \alpha_{M_4} \leq 128.8^\circ$ for Category B. In addition, we predict the range of the mass scale associated with $0\nu\beta\beta$ decay, most of which is well below the reach of planned experiments. We have also studied Yukawa textures with a lesser number of zeros, but with exact $\mu \tau$ symmetry. Finally we have formulated the detailed scheme of three parameter explicit breaking of $\mu \tau$ symmetry for allowed four zero textures. A value of $\theta_{13} \approx 9^\circ$ can be arranged for a suitable choice of small values of these symmetry breaking parameters.

7.8 Acknowledgments

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7.9 Appendix: Expressions for measurable quantities

We can write the general form of a broken $\mu \tau$ symmetric $M_\nu$ in the following way [33]:

$$M_\nu = m \left[ \begin{pmatrix} P & Q & Q \\ Q & R & S \\ S & R & S \end{pmatrix} - \epsilon_1 e^{i \theta} \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{pmatrix} - \epsilon_2 e^{i \phi} \begin{pmatrix} y_1 & y_2 & y_3 \\ y_2 & y_4 & y_5 \\ y_3 & y_5 & y_6 \end{pmatrix} - \epsilon_3 \begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_4 & z_5 \\ z_3 & z_5 & z_6 \end{pmatrix} \right] \right].$$

(7.53)

The explicit expressions of $P$, $Q$, $R$, $S$, $x_{1-6}$, $y_{1-6}$ and $z_{1-6}$ for four forms of neutrino mass matrices after $\mu \tau$ symmetry breaking are given in Table 7.5.

We can now have

$$H = M_\nu M_\nu^\dagger = m^2 \left[ \begin{pmatrix} |P|^2 + 2|Q|^2 & PQ^* + Q(R^* + S^*) & PQ^* + Q(R + S) \\ P^*Q + Q^*(R + S) & |Q|^2 + |R|^2 + |S|^2 & |Q|^2 + RS^* + R^*S \\ P^*S + Q^*(R + S) & |Q|^2 + RS^* + R^*S & |Q|^2 + |R|^2 + |S|^2 \end{pmatrix} \right]$$
Neutrino Physics

Neutrino Yukawa textures within type-I see-saw

| Quantity | Category A1 | Category A2 | Category B1 | Category B2 |
|----------|-------------|-------------|-------------|-------------|
| $P$      | $k_1 e^{2i\alpha} + 2k_2^2$ | $k_1 e^{2i\alpha} + 2k_2^2$ | $l_1^2$ | $l_1^2$ |
| $Q$      | $k_2$      | $k_2$      | $l_1 l_2 e^{i\beta}$ | $l_1 l_2 e^{i\beta}$ |
| $R$      | 1          | 1          | $l_2^2 e^{2i\beta} + 1$ | $l_2^2 e^{2i\beta} + 1$ |
| $S$      | 0          | 0          | $l_2^2 e^{2i\beta}$ | $l_2^2 e^{2i\beta}$ |
| $x_1$    | $2k_2^2$  | $2k_2^2$  | 0          | 0          |
| $x_2$    | $k_2$     | 0          | $l_1 l_2 e^{i\beta}$ | $l_1 l_2 e^{i\beta}$ |
| $x_3$    | 0          | $k_2$     | 0          | 0          |
| $x_4$    | 0          | 0          | $2l_2^2 e^{2i\beta}$ | $2l_2^2 e^{2i\beta}$ |
| $x_5$    | 0          | 0          | $l_2^2 e^{2i\beta}$ | $l_2^2 e^{2i\beta}$ |
| $x_6$    | 0          | 0          | 0          | 0          |
| $y_1$    | 0          | 0          | 0          | 0          |
| $y_2$    | $k_2$     | $k_2$     | 0          | 0          |
| $y_3$    | 0          | 0          | 0          | 0          |
| $y_4$    | 2          | 2          | 2          | 2          |
| $y_5$    | 0          | 0          | 0          | 0          |
| $y_6$    | 0          | 0          | 0          | 0          |
| $z_1$    | $-k_2^2$  | $-k_2^2$  | 0          | 0          |
| $z_2$    | $-k_2$    | 0          | 0          | 0          |
| $z_3$    | 0          | $-k_2$    | 0          | 0          |
| $z_4$    | $-1$      | 0          | $-1$       | 0          |
| $z_5$    | 0          | 0          | 0          | 0          |
| $z_6$    | 0          | $-1$      | 0          | $-1$       |

Table 7.5: Expressions for $P$, $Q$, $R$, $S$, $x_{1-6}$, $y_{1-6}$ and $z_{1-6}$ for categories A1, A2, B1 and B4 from Equations (7.50) and (7.51)

$$ - \epsilon_1 \begin{pmatrix} u_1 & u_2^* & u_3^* \\ u_2 & u_4 & u_5 \\ u_3 & u_5 & u_6 \end{pmatrix} - \epsilon_2 \begin{pmatrix} v_1 & v_2^* & v_3^* \\ v_2 & v_4 & v_5 \\ v_3 & v_5 & v_6 \end{pmatrix} - \epsilon_3 \begin{pmatrix} w_1 & w_2^* & w_3^* \\ w_2 & w_4 & w_5 \\ w_3 & w_5 & w_6 \end{pmatrix} , $$  \hspace{1cm} (7.54)

Here

$$ u_1 = \left[ P^* x_1 + Q^* (x_2 + x_3) \right] e^{i\theta} + \left[ P x_1^* + Q (x_2^* + x_3^*) \right] e^{-i\theta} , $$

$$ u_2 = \left[ P^* x_2 + Q^* (x_4 + x_5) \right] e^{i\theta} + \left[ Q x_1^* + R x_2^* + S x_3^* \right] e^{-i\theta} , $$

$$ u_3 = \left[ P^* x_3 + Q^* (x_5 + x_6) \right] e^{i\theta} + \left[ Q x_1^* + S x_2^* + R x_3^* \right] e^{-i\theta} , $$

$$ u_4 = \left[ Q^* x_2 + R^* x_4 + S^* x_5 \right] e^{i\theta} + \left[ Q x_2^* + S x_3^* + R x_4^* \right] e^{-i\theta} , $$

$$ u_5 = \left[ Q^* x_3 + R^* x_5 + S^* x_6 \right] e^{i\theta} + \left[ Q x_3^* + S x_4^* + R x_5^* \right] e^{-i\theta} , $$

$$ u_6 = \left[ Q^* x_3 + S^* x_5 + R^* x_6 \right] e^{i\theta} + \left[ Q x_3^* + S x_6^* + R x_6^* \right] e^{-i\theta} . $$  \hspace{1cm} (7.55)

Note that $v_i$ and $w_i$ have similar functional forms as $u_i$. If we write $u_i$ in the following way

$$ u_i = f_i(x_1, x_2, ..., x_6, \theta) , $$  \hspace{1cm} (7.56)

then $v_i$ and $w_i$ will be

$$ v_i = f_i(y_1, y_2, ..., y_6, \phi) , $$
\[ w_i = f_i(z_1, z_2, ..., z_6, 0). \]  

(7.57)

The diagonalization of \( H \) yields diag.(\( m_1^2, m_2^2, m_3^2 \)) and also expressions for five relevant measurable quantities. The latter are: \( \Delta^2_{31}, \Delta^2_{32}, \theta_{12}, \theta_{23} \) and \( \theta_{13} \). We will associate superscripts \( \epsilon_{1,2,3} \) with all of these five quantities to distinguish them from their unperturbed expressions. The relevant functions for these physical quantities are given below.

\[
\begin{align*}
U_1 &= \frac{1}{2} \left[ -2c_{12}u_1 + \sqrt{2}c_{12}s_{12} \{(u_2 + u_3)e^{-i\psi} + (u_2^* + u_3^*)e^{i\psi}\} - s_{12}^2(u_4 + u_6 + u_5 + u_5^*) \right], \\
U_2 &= \frac{1}{2} \left[ -\sqrt{2}c_{12}(u_2 + u_3)e^{-i\psi} + \sqrt{2}s_{12}^2(u_2^* + u_3^*)e^{i\psi} + c_{12}s_{12}(u_4 + u_6 - 2u_1 + u_5 + u_5^*) \right], \\
U_3 &= \frac{1}{2} \left[ \sqrt{2}c_{12}(u_2 - u_3)e^{-i\psi} + s_{12}(u_6 - u_4 + u_5 - u_5^*) \right], \\
U_4 &= \frac{1}{2} \left[ -2s_{12}u_2 - \sqrt{2}c_{12}s_{12} \{(u_2 + u_3)e^{-i\psi} + (u_2^* + u_3^*)e^{i\psi}\} - c_{12}^2(u_4 + u_6 + u_5 + u_5^*) \right], \\
U_5 &= \frac{1}{2} \left[ \sqrt{2}s_{12}(u_2 - u_3)e^{-i\psi} - c_{12}(u_6 - u_4 + u_5 - u_5^*) \right], \\
U_6 &= \frac{1}{2} [u_5 + u_5^* - u_4 - u_6],
\end{align*}
\]

(7.58)

where \( \psi = \text{arg } [P^*Q + Q^*(R + S)] \), \( c_{12} = \cos \theta_{12} \), \( s_{12} = \sin \theta_{12} \), \( \theta_{12} \) being the unperturbed mixing angle in (7.29). There are also \( V_i \) and \( W_i, i = 1 - 6 \) which have similar functional forms as \( U_i \). If we write \( U_i \) as

\[ U_i = F_i(u_1, u_2, ..., u_6), \]

(7.59)

then \( V_i \) and \( W_i \) will be

\[
\begin{align*}
V_i &= F_i(v_1, v_2, ..., v_6), \\
W_i &= F_i(w_1, w_2, ..., w_6).
\end{align*}
\]

(7.60)

The final results with three \( \mu\tau \) symmetry breaking parameters are

\[
\begin{align*}
(\Delta^2_{21})^{\epsilon_{1,2,3}} &= \Delta^2_{21} + m^2 \{(U_4 - U_1)\epsilon_1 + (V_4 - V_1)\epsilon_2 + (W_4 - W_1)\epsilon_3\}, \\
(\Delta^2_{32})^{\epsilon_{1,2,3}} &= \Delta^2_{32} + m^2 \{(U_6 - U_4)\epsilon_1 + (V_6 - V_4)\epsilon_2 + (W_6 - W_4)\epsilon_3\}, \\
(\sin \theta_{12})^{\epsilon_{1,2,3}} &= \left| s_{12} + c_{12}m^2 \left\{ \frac{U_2^*\epsilon_1 + V_2^*\epsilon_2 + W_2^*\epsilon_3}{\Delta^2_{21}} \right\} \right|, \\
(\sin \theta_{23})^{\epsilon_{1,2,3}} &= \left| \frac{1}{\sqrt{2}} + \frac{s_{12}m^2}{\sqrt{2}} \left\{ \frac{U_3^*\epsilon_1 + V_3^*\epsilon_2 + W_3^*\epsilon_3}{\Delta^2_{21} + \Delta^2_{32}} \right\} - \frac{c_{12}m^2}{\sqrt{2}} \left\{ \frac{U_5^*\epsilon_1 + V_5^*\epsilon_2 + W_5^*\epsilon_3}{\Delta^2_{32}} \right\} \right|, \\
(\sin \theta_{13})^{\epsilon_{1,2,3}} &= \left| c_{12}m^2 \left\{ \frac{U_1^*\epsilon_1 + V_1^*\epsilon_2 + W_1^*\epsilon_3}{\Delta^2_{21} + \Delta^2_{32}} \right\} + s_{12}m^2 \left\{ \frac{U_5^*\epsilon_1 + V_5^*\epsilon_2 + W_5^*\epsilon_3}{\Delta^2_{32}} \right\} \right|.
\end{align*}
\]

(7.61)
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