A priori bounds for positive solutions of subcritical elliptic equations

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Abstract We provide a-priori $L^\infty$ bounds for positive solutions to a class of subcritical elliptic problems in bounded $C^2$ domains. Our analysis widens the known ranges of subcritical nonlinearities for which positive solutions are a-priori bounded.

Keywords A priori estimates · Positive solutions · Subcritical nonlinearity · Moving planes method · Kelvin transform

Mathematics Subject Classification 35B45 · 35B33 · 35B09 · 35J60

1 Introduction

We provide a-priori $L^\infty(\Omega)$ bounds for classical positive solutions to the boundary value problem:

\[-\Delta u = f(u), \quad \text{in} \quad \Omega, \quad u = 0, \quad \text{on} \quad \partial\Omega, \tag{1.1}\]

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded $C^2$ domain, and $f$ is a subcritical nonlinearity. For simplicity we assume $N > 2$, but our techniques fit well to the case $N = 2$.

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We prove the existence of a priori bounds when \( f(s) = s^{\frac{N+2}{N-2}}/\ln(s + 2)^{\alpha} \), with \( \alpha > 2/(N - 2) \), see Corollary 2.2.

The exponent \( 2^* - 1 = \frac{N+2}{N-2} \) of a nonlinearity \( f(s) = s^{2^*-1} \) is critical from the viewpoint of Sobolev embedding. Observe that \( 2^* = \frac{2N}{N-2} \), and the embedding \( H^1(\Omega) \) in \( L^{2^*}(\Omega) \) is not compact. Pohozaev proved that problem (1.1) does not have a solution if \( \Omega \) is star shaped, see [17], and Bahri and Coron proved that problem (1.1) has a solution if \( \Omega \) has non trivial topology in a certain sens, see [2].

If \( \lim_{s \to \infty} \frac{f(s)}{s^{2^*-1}} = +\infty \), the problem is of supercritical nature. Consider \( f(s) = M(1 + s)^{\beta} \), for \( \beta > \frac{N+2}{N-2} \), \( M \in \mathbb{R} \), Joseph and Lundgren for balls in \( \mathbb{R}^N \), \( N \geq 3 \), provide sufficient conditions guaranteeing that (1.1) has an unbounded sequence of positive solutions, see [13, Theorem 2].

If \( \lim_{s \to \infty} \frac{f(s)}{s^{2^*-1}} = 0 \), the problem is subcritical. Consider \( f(s) = s^{2^*-1-\varepsilon} \) for \( \varepsilon > 0 \). It is well known that problem (1.1) has a solution \( u_\varepsilon \), see Lions [15] and references therein. Atkinson and Pelletier for balls in \( \mathbb{R}^3 \), and Han for non-spherical domains, proved that there exists \( x_0 \in \Omega \) and a sequence \( u_\varepsilon \) such that \( \lim_{\varepsilon \to 0} u_\varepsilon = 0 \) in \( C^1(\Omega \setminus \{x_0\}) \) and \( \lim_{\varepsilon \to 0} |\nabla u_\varepsilon|^2 = C \delta_{x_0} \) in the sense of distributions, where \( \delta \) is the Dirac distribution, and \( C \) depends on \( N \) and on the best Sobolev constant in \( \mathbb{R}^N \), see [1, 12].

A-priori bounds in the \( L^\infty \)-norm of positive solutions provided a great deal of information on the existence of solutions when it is combined with degree theory for compact maps, and it is a longstanding open problem. The supercritical radial case suggest that the growth condition \( |f(s)| < M(1 + |s|^{\frac{N+2}{N-2}}) \) is a necessary condition for the existence of a priori bounds for all positive solutions to (1.1). In [19], Turner provided sufficient conditions for the existence of a priori bounds for (1.1) for regions in \( \mathbb{R}^2 \). If \( |f(s)| \leq M(1 + |s|^\beta) \) for some \( \beta < \frac{N+2}{N-2} \), Nussbaum proves the existence of a priori estimates for positive radial solutions to (1.1) in the subcritical radial case, see [16]. Subsequently, in [4] Brezis and Turner extend the results to general bounded regions in \( \mathbb{R}^N \) under the the restriction \( 1 < \beta < \frac{N+1}{N-2} \). Later, Gidas and Spruck in [9] as well de Figueiredo et al. in [8] extend those result to the subcritical case. The results in [9] depend heavily on the blow up method which requires \( f \) to be essentially of the form \( f(x,s) = h(x)s^p \) with \( p \in (1, \frac{N+2}{N-2}) \) and \( h(x) \) continuous and strictly positive. In [8] the nonlinearity \( f \) is assumed to satisfy

\[
\liminf_{s \to +\infty} \frac{\theta F(s) - sf(s)}{s^2 f(s)^{2/N}} \geq 0, \quad \text{for some } \theta \in [0, 2^*),
\]

where \( F(s) = \int_0^s f(t) \, dt \). They conjecture that this condition is not necessary, but it is essential in proving their results. It can be seen that for \( f_1(s) = s^{2^*-1}/\ln(s + 2)^{\alpha} \) with \( \alpha > 0 \)

\[
\liminf_{s \to +\infty} \frac{\theta F_1(s) - sf_1(s)}{s^2 f_1(s)^{2/N}} = -\infty, \quad \text{for any } \theta \in [0, 2^*),
\]

where \( F_1(s) = \int_0^s f_1(t) \, dt \), see Remark 2.3.
Our analysis substantially extends previous results, widen the known ranges of subcritical nonlinearities for which positive solutions are a priori bounded and also applies to non-convex domains. Our main result is:

**Theorem 1.1** Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^2$ boundary. Assume that the nonlinearity $f$ is locally Lipschitzian and satisfies the following conditions

(H1) $\frac{f(s)}{s^{2^* - 1}}$ is nonincreasing for any $s > 0$.

(H2) There exists a constant $C_1 > 0$ such that $\limsup_{s \to \infty} \frac{\max_{[0,s]} f}{f(s)} \leq C_1$.

(H3) There exists a constant $C_2 > 0$ and a non-increasing function $H : \mathbb{R}^+ \to \mathbb{R}^+$ such that

(H3.1) $\liminf_{s \to +\infty} \frac{2NF(s) - (N - 2)sf(s)}{sf(s)H(s)} \geq C_2 > 0,$

and

(H3.2) $\lim_{s \to +\infty} \frac{f(s)}{s^{2^* - 1} [H(s)]^{\frac{2}{N - 2}}} = 0.$

(H4) $\liminf_{s \to +\infty} \frac{f(s)}{s} > \lambda_1$, where $\lambda_1$ is the first eigenvalue of $-\Delta$ acting on $H^1_0(\Omega)$.

Then, there exists a uniform constant $C$, depending only on $\Omega$ and $f$, such that for every $u > 0$, classical solution to (1.1),

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

If the domain $\Omega$ is convex, we have the following result:

**Theorem 1.2** Assume that $\Omega \subset \mathbb{R}^N$ is a bounded, convex domain with $C^2$ boundary. Assume that the nonlinearity $f$ is locally Lipschitzian, satisfies (H2)–(H4), and also the following conditions:

(H1)' There exists a constant $C_0 > 0$ such that $\liminf_{s \to \infty} \frac{\min_{[s/2,s]} f}{f(s)} \geq C_0$.

Then, there exists a uniform constant $C$, depending only on $\Omega$ and $f$, such that for every $u > 0$, classical solution to (1.1),

$$\|u\|_{L^\infty(\Omega)} \leq C.$$
Hypothesis (H3) looks rather cumbersome. However we prove that functions such as $f_1(s) = s^{2^*-1}/\ln(s+2)^\alpha$ satisfy our hypotheses for $\alpha > 2/(N-2)$ with $H(s) = 1/\ln(s+2)$, see Corollary 2.2, but not those of [8] or [9], see Remark 2.3.

Our proofs of Theorems 1.1 and 1.2, as in [8], use moving plane arguments, the Kelvin transform, and a Pohozaev identity, see [17]. These ideas are well known but we combine them in a slightly different way.

The moving planes method was used by Serrin in [18]. Gidas et al. in [10], using this moving planes method and the Hopf Lemma, prove symmetry of positive solutions of elliptic equations vanishing on the boundary. See also Castro-Shivaji [7], where symmetry of nonnegative solutions is established for $f(0) < 0$. In [10] the authors also characterized regions inside $\Omega$, next to the convex part of the boundary, where a positive solution cannot have critical points. Those regions, called maximal caps, depend only on the local convexity of $\Omega$, and are independent of $f$ and $u$, see the Appendix A for a precise definition. This non-existence of critical points in a maximal cap, is due to any positive solution is strictly increasing in the normal direction, inside of a maximal cap. This is a key point to reach local a priori bounds in a neighborhood of the boundary.

The arguments split into two ways, depending on the convexity of the domain. The reason is the following one. If $\Omega$ is convex, and the nonlinearity $f$ satisfies (H4), then any positive solution is a priori bounded in a neighborhood of the boundary, more precisely, there exists a constant $C$ depending only on $\Omega$ and $f$ but not on $u$, such that

$$\max_{\Omega \setminus \Omega_\delta} u \leq C,$$

where $\Omega_\delta := \{ x \in \Omega : d(x, \partial \Omega) > \delta \}$, see [8].

If $\Omega$ is a general bounded domain, not necessarily convex, the argument on the a priori bounds in a neighborhood of the boundary relies on the Kelvin transform, see (B.2) for a definition. In that case, if the nonlinearity $f$ satisfies (H1) and (H4), then any positive solution is a priori bounded in a neighborhood of the boundary, in other words, conclusion (1.2) is reached, see [8].

We include these Theorems in Appendix A and B in order to clarify which hypothesis are needed in the convex case and in the non-convex case respectively. The starting point in the proof of Theorems 1.1 and 1.2 are a priori bounds in a neighborhood of the boundary, Theorems B.3 and A.3 respectively.

In [5,6] we study the associated bifurcation problem for a nonlinearity $f(\lambda, s) = \lambda s + g(s)$ with $g$ subcritical. We provide sufficient conditions guaranteeing that either for any $\lambda < \lambda_1$ there exists at least a positive solution, or for any continuum $(\lambda, u_\lambda)$ of positive solution, there exists a $\lambda^* < 0$ such that $\lambda^* < \lambda < \lambda_1$ and

$$\|\nabla u_\lambda\|_{L^2(\Omega)} \to \infty, \quad \text{as} \quad \lambda \to \lambda^*,$$

see [6, Theorem 2]. In case $\Omega$ is convex, for any $\lambda < \lambda_1$ there exists at least a positive solution, see [5, Theorem 1.2].

This paper is organized in the following way. In Sect. 2 we prove our main results on a-priori bounds, see Theorems 1.1 and 1.2. We collect results on the a priori bounds
in a neighborhood of the boundary in two Appendices, one for the convex case and another for the general case.

2 Proof of Theorems 1.1 and 1.2

Let us start this section with the following Remark.

Remark 2.1 By hypothesis, \( H : \mathbb{R}^+ \to \mathbb{R}^+ \) is a non-increasing function, therefore \( 0 \leq \lim_{s \to \infty} H(s) < \infty \).

Taking into account hypothesis (H3.2) we also conclude that \( \lim_{s \to +\infty} \frac{f(s)}{s^{2^*-1}} = 0. \)

Next, we prove Theorem 1.2.

Proof of Theorem 1.2 From hypothesis (H3.1), there exists a constant \( C_3 > 0 \) and a non-increasing function \( H \) such that

\[
2N F(s) - (N - 2) sf(s) \geq \frac{C_2}{2} H(s) sf(s), \quad \text{for any } s > C_3. \tag{2.1}
\]

Applying this inequality to any positive solution, and integrating on \( \Omega \) we obtain that

\[
2N \int_\Omega F(u) \, dx - (N - 2) \int_\Omega uf(u) \, dx \geq \frac{C_2}{2} \int_\Omega uf(u) H(u) \, dx - C_4, \tag{2.2}
\]

for some constant \( C_4 \) independent on \( u \). From now on, throughout this proof \( C \) denotes several constants independent of \( u \).

From a slight modification of Pohozaev identity, see [8, Lemma 1.1] and [17], if \( y \in \mathbb{R}^N \) is a fixed vector, then any positive solution \( u \) of (1.1) satisfies

\[
\int_{\partial\Omega} (x - y) \cdot n(x) |\nabla u|^2 \, dS = 2N \int_\Omega F(u) \, dx - (N - 2) \int_\Omega uf(u) \, dx. \tag{2.3}
\]

From (1.2) and de Giorgi–Nash type Theorems, see [14, Theorem 14.1]

\[
\|u\|_{C^{0,\alpha}(\Omega_{\delta/8} \setminus \Omega_{2\delta/8})} \leq C, \quad \text{for any } \alpha \in (0, 1),
\]

where \( \Omega_\tau := \{ x \in \Omega : d(x, \partial \Omega) > \tau \} \).

From Schauder interior estimates, see [11, Theorem 6.2]

\[
\|u\|_{C^{1,\alpha}(\Omega_{\delta/4} \setminus \Omega_{3\delta/4})} \leq C.
\]

Finally, combining \( L^p \) estimates with Schauder boundary estimates, see [3,11]

\[
\|u\|_{W^{2,p}(\Omega_{\delta/2})} \leq C, \quad \text{for any } p \in (1, \infty).
\]
Consequently, there exists two constants $C, \delta > 0$ independent of $u$ such that

$$
\|u\|_{C^{1,\alpha}(\Omega\setminus\Omega_{\delta})} \leq C, \quad \text{for any } \alpha \in (0, 1).
$$

This, (2.3) and (2.2) yield

$$
\int_{\Omega} uf(u) H(u) \, dx \leq C,
$$

for some constant $C$ independent of $u$. Next we prove that also

$$
\int_{\Omega} u|f(u)| H(u) \, dx \leq C.
$$

From hypothesis (H4), there exists a constant $C$ such that if $s > C$ then $f(s) > 0$. Splitting the above integral in the set $S = \{x \in \Omega : |u| \leq C\}$ and its complementary $\Omega \setminus S$, since from (2.5) $\int_{\Omega \setminus S} uf(u) H(u) \, dx \leq C$, then (2.6) holds.

From hypothesis (H3.2), $\lim_{s \to +\infty} \frac{|f(s)|^{\frac{1}{2-\frac{1}{q}}}}{s^{\frac{N}{N+2}}} = 0$. Multiplying numerator and denominator by $|f(s)|H(s)^{\frac{N}{N+2}}$, we can assert that there exists a constant $C$ such that

$$
|f(s)|^{1+\frac{1}{2-\frac{1}{q}}}[H(s)]^{\frac{N}{N+2}} \leq s|f(s)|H(s) + C, \quad \text{for any } s > 0.
$$

Applying this inequality to any positive solution, integrating on $\Omega$, and using (2.6) we obtain that

$$
\int_{\Omega} |f(u)|^{1+\frac{1}{2-\frac{1}{q}}} H(u)^{\frac{N}{N+2}} \, dx \leq C.
$$

Consequently, since $H$ is non-increasing,

$$
\int_{\Omega} |f(u(x))|^{q} \, dx \leq C \frac{1}{H(\|u\|_{\infty})^{\frac{N}{N+2}}} \int_{\Omega} |f(u(x))|^{1+\frac{1}{2-\frac{1}{q}}} H(u)^{\frac{N}{N+2}} \, dx
$$

$$
\leq C \frac{|f(u(\cdot))|_{q-\frac{1}{2-\frac{1}{q}}}}{H(\|u\|_{\infty})^{\frac{N}{N+2}}},
$$

for any $q > N/2$.

Therefore, from elliptic regularity, see [11, Lemma 9.17]

$$
\|u\|_{W^{2,q}(\Omega)} \leq C \|\Delta u\|_{L^{q}(\Omega)} \leq C \frac{\|f(u(\cdot))\|_{q-\frac{1}{2-\frac{1}{q}}}}{[H(\|u\|_{\infty})]^{\frac{N}{N+2q}}}. 
$$

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Let us restrict \( q \in (N/2, N) \). From Sobolev embeddings, for \( 1/q^* = 1/q - 1/N \) with \( q^* > N \) we can write

\[
\|u\|_{W^{1,q^*}(\Omega)} \leq C \|u\|_{W^{2,q}(\Omega)} \leq C \frac{\|f(u(\cdot))\|_\infty}{[H(\|u\|_\infty)]^{N+2/q}}.
\] (2.10)

From Morrey’s Theorem, (see [3, Theorem 9.12 and Corollary 9.14]), there exists a constant \( C \) only dependent on \( \Omega \), \( q \) and \( N \) such that

\[
|u(x_1) - u(x_2)| \leq C|x_1 - x_2|^{1-N/q^*}\|u\|_{W^{1,q^*}(\Omega)}, \quad \forall x_1, x_2 \in \Omega.
\] (2.11)

Therefore, for all \( x \in B(x_1, R) \subset \Omega \)

\[
|u(x) - u(x_1)| \leq CR^{2-N/q^*}\|u\|_{W^{2,q}(\Omega)}.
\] (2.12)

From now on, we shall argue by contradiction. Let \( \{u_k\}_k \) be a sequence of classical positive solutions to (1.1) and assume that

\[
\lim_{k \to \infty} \|u_k\| = +\infty, \quad \text{where} \quad \|u_k\| := \|u_k\|_\infty.
\] (2.13)

Let \( C, \delta > 0 \) be as in (1.2). Let \( x_k \in \overline{\Omega_\delta} \) be such that

\[
u_k(x_k) = \max_{\Omega_\delta} u_k = \max_{\Omega} u_k.
\]

By taking a subsequence if needed, we may assume that there exists \( x_0 \in \overline{\Omega_\delta} \) such that

\[
\lim_{k \to \infty} x_k = x_0 \in \overline{\Omega_\delta}, \quad \text{and} \quad d_0 := \text{dist}(x_0, \partial\Omega) \geq \delta > 0.
\] (2.14)

Let us choose \( R_k \) such that \( B_k = B(x_k, R_k) \subset \Omega \), and

\[
u_k(x) \geq \frac{1}{2}\|u_k\| \quad \text{for any} \quad x \in B(x_k, R_k).
\]

and there exists \( y_k \in \partial B(x_k, R_k) \) such that

\[
u_k(y_k) = \frac{1}{2}\|u_k\|.
\] (2.15)

Let us denote by

\[
m_k := \min_{\|u_k\|/2, \|u_k\|} f, \quad M_k := \max_{[0,\|u_k\|]} f.
\]

Therefore, we obtain

\[
m_k \leq f(u_k(x)) \quad \text{if} \quad x \in B_k, \quad f(u_k(x)) \leq M_k \quad \forall x \in \Omega.
\] (2.16)
Then, reasoning as in (2.8), we obtain

$$\int_{\Omega} |f(u_k)|^q \, dx \leq C \frac{M_k^{\frac{q-1}{q}-\frac{1}{(2^q-1)q}}}{H(\|u_k\|)^N}.$$  \hspace{1cm} (2.17)

From elliptic regularity, see (2.9), we deduce

$$\|u_k\|_{W^{2,q}(\Omega)} \leq C \frac{M_k^{\frac{1}{2}-\frac{1}{(2^q-1)q}}}{[H(\|u_k\|)]^{\frac{N}{N+2q}}}.$$  \hspace{1cm} (2.18)

Therefore, from Morrey’s Theorem, see (2.12), for any \(x \in B(x_k, R_k)\)

$$|u_k(x) - u_k(x_k)| \leq C (R_k)^{2-\frac{N}{q}} \frac{M_k^{\frac{1}{2}-\frac{1}{(2^q-1)q}}}{[H(\|u_k\|)]^{\frac{N}{N+2q}}}.$$  \hspace{1cm} (2.19)

Particularizing \(x = y_k\) in the above inequality and from (2.15) we obtain

$$C (R_k)^{2-\frac{N}{q}} \frac{M_k^{\frac{1}{2}-\frac{1}{(2^q-1)q}}}{[H(\|u_k\|)]^{\frac{N}{N+2q}}} \geq |u_k(y_k) - u_k(x_k)| = \frac{1}{2} \|u_k\|,$$  \hspace{1cm} (2.20)

which implies

$$(R_k)^{2-\frac{N}{q}} \geq \frac{1}{2C} \frac{\|u_k\| [H(\|u_k\|)]^{\frac{N}{N+2q}}}{M_k^{\frac{1}{2}-\frac{1}{(2^q-1)q}}}.$$  \hspace{1cm} (2.21)

or equivalently

$$R_k \geq \left(\frac{1}{2C} \frac{\|u_k\| [H(\|u_k\|)]^{\frac{N}{N+2q}}}{M_k^{\frac{1}{2}-\frac{1}{(2^q-1)q}}}\right)^{1/(2-\frac{N}{q})}.$$  \hspace{1cm} (2.22)

Consequently, taking into account (2.16), and that \(H\) is non-increasing

$$\int_{B(x_k, R_k)} u_k |f(u_k)| H(u_k) \, dx \geq \frac{1}{2} \|u_k\| H(\|u_k\|) m_k \omega (R_k)^N,$$

where \(\omega = \omega_N\) is the volume of the unit ball in \(\mathbb{R}^N\).

Due to \(B(x_k, R_k) \subset \Omega\), substituting inequality (2.22), and rearranging terms, we obtain

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\[
\int_{\Omega} u_k |f(u_k)| H(u_k) \, dx \geq \frac{1}{2} \|u_k\| H(\|u_k\|) m_k \omega \left( \frac{1}{2C} \frac{\|u_k\| [H(\|u_k\|)]^{\frac{N}{N+2q}}}{M_k^{1-\frac{1}{q} - \frac{2}{(2^*-1)q}}} \right)^{\frac{1}{N+2q}} \\
= C m_k \left( \frac{\|u_k\| [H(\|u_k\|)]^{\frac{N}{N+2q}}}{M_k^{1-\frac{1}{q} - \frac{2}{(2^*-1)q}}} \right) \frac{1}{N+2q} \\
= C m_k \left( \frac{\|u_k\|^{\frac{1+\frac{2}{N} - \frac{1}{q}}{\|H(\|u_k\|)\|}^{\frac{2}{(N+2q)}}} M_k^{1-\frac{1}{q} - \frac{2}{(2^*-1)q}}} \right) \frac{1}{N+2q} \\
= C \frac{m_k}{M_k} \left( \frac{\|u_k\|^{\frac{1+\frac{2}{N} - \frac{1}{q}}{\|H(\|u_k\|)\|}^{\frac{2}{(N+2q)}}} M_k^{1-\frac{1}{q} - \frac{2}{(2^*-1)q}}} \right) \frac{1}{N+2q} \\
\]

At this moment, let us observe that from hypothesis (H1)' and (H2)

\[
m_k \geq C, \quad \text{for all } k \text{ big enough.} \tag{2.23}
\]

Hence, taking again into account hypothesis (H2), and rearranging exponents, we can assert that

\[
\int_{\Omega} u_k |f(u_k)| H(u_k) \, dx \geq C \left( \frac{\|u_k\|^{\frac{1+\frac{2}{N} - \frac{1}{q}}{\|H(\|u_k\|)\|}^{\frac{2}{(N+2q)}}} M_k^{1-\frac{1}{q} - \frac{2}{(2^*-1)q}}} \right) \frac{1}{N+2q} \\
\geq C \left( \frac{\|u_k\|^{\frac{1+\frac{2}{N} - \frac{1}{q}}{\|H(\|u_k\|)\|}^{\frac{2}{(N+2q)}}} [f(\|u_k\|)]^{\frac{N}{N+2q}} M_k^{1-\frac{1}{q} - \frac{2}{(2^*-1)q}}} \right) \frac{1}{N+2q} \\
\geq C \left( \frac{\|u_k\|^{(N+2)}[\frac{1}{N} - \frac{1}{(N+2q)}][H(\|u_k\|)]^{\frac{2}{(N+2q)}} M_k^{1-\frac{1}{q} - \frac{2}{(2^*-1)q}}} [f(\|u_k\|)]^{(N-2)} \left( \frac{1}{N} - \frac{1}{(N+2q)} \right) \right) \frac{1}{N+2q} \\
\]

Finally, from hypothesis (H3.2) we deduce

\[
\int_{\Omega} u_k |f(u_k)| H(u_k) \, dx \geq C \left( \frac{\|u_k\|^{\frac{2^*-1}{2^*-2}} [H(\|u_k\|)]^{\frac{2}{2^*-2}} f(\|u_k\|)}{\|u_k\|^{(N-2)\left( \frac{1}{N} - \frac{1}{(N+2q)} \right)}} \right)^{\frac{(N-2)}{N-2} \left( \frac{1}{N} - \frac{1}{(N+2q)} \right) \frac{2^*-1}{2^*-2}} \\
\rightarrow \infty \text{ as } k \rightarrow \infty
\]

which contradicts (2.6), ending the proof. \qed
Next, we prove Theorem 1.1:

**Proof of Theorem 1.1** Clearly hypotheses (H1) implies hypotheses (H1)’.

For non-convex domains, we use the Kelvin transform to get the a-priori bounds in a neighborhood of the boundary. Let us observe that we need additionally hypothesis (H1), see Theorem B.3. All the other arguments work exactly in the same way as in the above proof. \(\square\)

**Corollary 2.2** Assume that \(\Omega \subset \mathbb{R}^N\) is a bounded domain with \(C^2\) boundary.

Let us consider any \(u > 0\), classical solution to

\[
\begin{cases}
-\Delta u = \frac{u^{2^*-1}}{\ln(2+u)^\alpha}, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

with \(\alpha > 2/(N-2)\).

Then, there exists a uniform constant \(C\), depending only on \(\Omega\) and \(f\), such that for every

\[\|u\|_{L^\infty(\Omega)} \leq C.\]

**Proof** We will prove that \(f(s) = s^{2^*-1}/\ln(s+2)^\alpha\) with \(\alpha > 2/(N-2)\) satisfies our hypotheses for \(H(s) = 1/\ln(s+2)\). Hypothesis (H1) and (H2) hold trivially. Let us prove (H3).

(H3.1) From definition, and integrating by parts

\[
F(t) = \int_0^t \frac{s^{2^*-1}}{\ln(2+s)^\alpha} \, ds
\]

\[
= \frac{1}{2^*} \frac{t^{2^*}}{\ln(2+t)^\alpha} + \alpha \int_0^t \frac{1}{\ln(2+s)} \left( \frac{s^{2^*}}{2+s} \right)^{\alpha+1} \, ds
\]

(2.25)

Therefore, using also l’Hôpital rule, and simplifying we can write

\[
\lim_{t \to +\infty} \frac{2^* F(t) - t f(t)}{t f(t) H(t)} = \lim_{t \to +\infty} \frac{\alpha \int_0^t \frac{1}{\ln(2+s)} \left( \frac{s^{2^*}}{2+s} \right)^{\alpha+1} \, ds}{\frac{t^{2^*}}{\ln(2+t)^{\alpha+1}}}
\]

\[
= \lim_{t \to +\infty} \frac{\alpha \left( \frac{1}{\ln(2+t)} \right)^{\alpha+1} \frac{t^{2^*}}{2+t}}{2^* \frac{\ln(2+t)^{\alpha+1}}{t^{2^*}} - (1 + \alpha) \left( \frac{1}{\ln(2+t)} \right)^{\alpha+2} \frac{t^{2^*}}{2+t}} = \frac{\alpha}{2^*} > 0,
\]

(2.26)

and so (H3.1) holds.
(H3.2) From definition of \( f \) and \( H \),
\[
\lim_{t \to +\infty} \frac{f(t)}{t^{2^*-1} [H(t)]^{\frac{2}{N-2}}} = \lim_{t \to +\infty} \frac{\ln(2 + t)}{\sqrt[2^*-1]{(N-2)^2}} = 0,
\]
for any \( \alpha > \frac{2}{(N-2)} \). \( \square \)

**Remark 2.3** Keeping the notation of the above Corollary, we will observe that
\[
L := \lim_{t \to +\infty} \frac{\theta F(t) - tf(t)}{t^2 f(t)^{2/N}} = -\infty, \quad \text{for any } \theta \in [0, 2^*].
\]

Set
\[
L_1 = \lim_{t \to +\infty} \frac{F(t)}{tf(t)H(t)}, \quad L_2 = \lim_{t \to +\infty} \frac{2^* F(t) - tf(t)}{tf(t)H(t)}, \quad L_3 = \lim_{t \to +\infty} \frac{tf(t)H(t)}{t^2 f(t)^{2/N}}
\]
then \( L = [(\theta - 2^*)L_1 + L_2]L_3 \).

From (2.25) and (2.26) we can write
\[
L_1 = \lim_{t \to +\infty} \frac{t^{2^*}}{\ln(2 + t)^\alpha} + \alpha \int_0^t \frac{s^{2^*}}{\ln(2 + s)^{\alpha+1}} ds
\]
\[
= \lim_{t \to +\infty} \left( \ln(2 + t) + \frac{\alpha}{2^*} \right) = +\infty.
\]
Therefore \( \lim_{t \to +\infty} (\theta - 2^*)L_1 = -\infty \), for any \( \theta \in [0, 2^*]. \) From (2.26) we know that \( L_2 = \frac{\alpha}{2^*} \). Finally, from definition we obtain \( L_3 = +\infty \). Consequently \( L = -\infty \).

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**Appendix A: A-priori bounds in a neighborhood of the boundary: the convex case**

In this Appendix, we collect some well known results on the moving planes method, see Proposition A.1. Next, we state results concerning a-priori bounds in a neighborhood of the boundary for the convex case: Theorem A.3. All those results are essentially well known, see [8], we include them here in order to clarify which hypotheses are used in the convex case and which in the non-convex case, see Theorem B.3 in Appendix B.

We will be moving planes in the \( x_1 \)-direction to fix ideas. Let us first define some concepts and notations.

- The moving plane is defined in the following way: \( T_\lambda := \{ x \in \mathbb{R}^N : x_1 = \lambda \} \),
\[ \Sigma_\lambda := \{ x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} \cap \Omega : x_1 < \lambda \}, \]
- the cap:
- the reflected point: \( x^\lambda := (2\lambda - x_1, x') \),
- the reflected cap: \( \Sigma'_\lambda := \{ x^\lambda : x \in \Sigma_\lambda \} \), see Fig. 1a.
- the minimum value for \( \lambda \) or starting value: \( \lambda_0 := \min \{ x_1 : x \in \overline{\Omega} \} \),
- the maximum value for \( \lambda \): \( \lambda^* := \max \{ \lambda : \Sigma'_\mu \subset \overline{\Omega} \text{ for all } \mu \leq \lambda \} \),
- the maximal cap: \( \Sigma := \Sigma_{\lambda^*} \).

We will need the moving plane method for a nonlinearity \( f = f(x, u) \).

**Proposition A.1** Suppose \( u \in C^2(\overline{\Omega}) \) is a positive solution of
\[
- \Delta u = f(x, u), \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial \Omega. \tag{A.1}
\]
Assume \( f = f(x, s) \) and its first derivative \( f_s \) are continuous, for \((x, s) \in \overline{\Omega} \times \mathbb{R} \).
Assume that
\[
f(x^\lambda, s) \geq f(x, s) \text{ for all } x \in \Sigma, \quad \text{for all } s > 0. \tag{A.2}
\]
Then for any \( \lambda \in (\lambda_0, \lambda^*) \) the following conclusion holds
\[
\text{(C) } u(x) < u(x^\lambda) \text{ and } \frac{\partial u}{\partial x_1}(x) > 0 \text{ for all } x \in \Sigma_\lambda.
\]
Furthermore, if \( \frac{\partial u}{\partial x_1}(x) = 0 \) at some point in \( \Omega \cap T_{\lambda^*} \), then \( u \) is symmetric with respect to the plane \( T_{\lambda^*} \), and \( \Omega = \Sigma \cup \Sigma' \cup (T_{\lambda^*} \cap \Omega) \).

**Proof** It is a Corollary of Theorem 2.1’ in [10]. \( \square \)

**Remark A.2** Set \( x_0 \in \partial \Omega \cap T_{\lambda_0} \), see Fig. 1a. Let us observe that by definition of \( \lambda_0 \), \( T_{\lambda_0} \) is the tangent plane to the graph of the boundary at \( x_0 \), and the inward normal at \( x_0 \), is \( n_i(x_0) = e_1 \). The above Theorem says that the partial derivative following the direction given by the inward normal at the tangency point is strictly positive in the whole maximal cap. Consequently, there are no critical points in the maximal cap.

![Fig. 1](image-url)
Now, we can apply the above result in any direction. First, let us fix the notation for a general $v \in \mathbb{R}^N$ with $|v| = 1$. We set

- the moving plane defined as: $T_\lambda(v) = \{x \in \mathbb{R}^N : x \cdot v = \lambda\}$,
- the cap: $\Sigma_\lambda(v) = \{x \in \Omega : x \cdot v < \lambda\}$,
- the reflected point: $x^\lambda(v) = x + 2(\lambda - x \cdot v)v$,
- the reflected cap: $\Sigma^\lambda(v) = \{x^\lambda : x \in \Sigma_\lambda(v)\}$, see Fig. 1b, for $v = -e_1$,
- the minimum value of $\lambda$: $\lambda_0(v) = \min\{x \cdot v : x \in \overline{\Omega}\}$,
- the maximum value of $\lambda$: $\lambda^*(v) = \max\{\lambda : \Sigma^\lambda(v) \subset \overline{\Omega} \text{ for all } \mu \leq \lambda\}$,
- and the maximal cap: $\Sigma(v) = \Sigma^\lambda(v)$, see Fig. 1c, for $v = -e_1$.

Set $x_0 \in \partial \Omega \cap T_{\lambda_0}$. The above Proposition says that the partial derivative following the direction given by the inward normal, $n_i(x_0)$, at the tangency point $x_0$, is strictly positive in the whole maximal cap $\Sigma = \Sigma(n_i(x_0))$, consequently the function $g(t) := u(x_0 + t n_i(x_0))$ is non-decreasing for $t \in [0, t_0]$ for some $t_0 = t_0(x_0) > 0$.

Now consider a neighborhood of $x_0$, denoted by $B_{\delta_0}(x_0)$. We can observe that for any $x \in \partial \Omega \cap B_{\delta_0}(x_0) \cap \Sigma$, also the function $g(t) := u(x + n_i(x_0))$ is non-decreasing for $t \in [0, t_0]$ for some $t_0 = t_0(x_0, x) > 0$. By choosing points $x$ such that $\operatorname{dist}(x, T_{\lambda}(n_i(x_0))) > \delta$, we see that the function $g(t) := u(x + t n_i(x_0))$ is non-decreasing for $t \in [0, \delta]$ for any $x \in \partial \Omega \cap \Sigma(n_i(x_0)) : \operatorname{dist}(x, T_{\lambda}(n_i(x_0))) > \delta$.

Now, let us move to a different cap, in a neighborhood of $x_0$. We can apply this idea, to their corresponding maximal caps $\Sigma$, with their corresponding vectors $v$. Then, choosing points in the intersection of the maximal caps, such that $\operatorname{dist}(x, T_{\lambda}(v)) > \delta$, also the function $g(t) := u(x + t v)$ is increasing for $t \in [0, \delta]$.

From now, the arguments split into two ways, depending on the convexity of the domain. If $\Omega$ is convex, we observe that, reasoning as in [8], any positive solution $u$ is locally increasing in the maximal cap following directions close to the normal direction, which provides $L^\infty$ bounds locally in a neighborhood of the boundary. This is the statement of the following Theorem.

**Theorem A.3** Assume that $\Omega \subset \mathbb{R}^N$ is a bounded, convex domain with $C^2$ boundary. Assume that the nonlinearity $f$ satisfies (H4).

If $u \in C^2(\overline{\Omega})$ satisfies (1.1) and $u > 0$ in $\Omega$, then there exists a constant $\delta > 0$ depending only on $\Omega$ and not on $f$ or $u$, and a constant $C$ depending only on $\Omega$ and $f$ but not on $u$, such that

$$\max_{\Omega \setminus \Omega_{\delta}} u \leq C$$

(A.3)

where $\Omega_{\delta} := \{x \in \Omega : d(x, \partial \Omega) > \delta\}$.

**Proof** See Step 2 in the proof of [8, Theorem 1.1]. \qed

**Appendix B: A-priori bounds in a neighborhood of the boundary: the general case**

Finally, we treat the general case, applying the moving plane method on the Kelvin transform, see below for a precise definition. First, we fix regions where a Kelvin transform of the solution has no critical points, see Theorem B.2. This result will
imply a priori bounds in a neighborhood of the boundary for any solution of the elliptic equation, see Theorem B.3.

Let us recall that every $C^2$ domain $\Omega$ satisfies the following condition, known as the uniform exterior sphere condition,

(P) there exists a $\rho > 0$ such that for every $x \in \partial \Omega$ there exists a ball $B = B_\rho(y) \subset \mathbb{R}^N \setminus \Omega$ such that $\partial B \cap \partial \Omega = x$.

Let $x_0 \in \partial \Omega$, and let $\overline{B}$ be the closure of a ball intersecting $\overline{\Omega}$ only at the point $x_0$. Let us suppose $x_0 = (1, 0, \ldots, 0)$, and $B$ is the unit ball with center at the origin. The inversion mapping

$$x \to h(x) = \frac{x}{|x|^2}, \quad (B.1)$$

is an homeomorphism from $\mathbb{R}^N \setminus \{0\}$ into itself; and observe that $h(h(x)) = x$. We perform an inversion from $\Omega$ into the unit ball $B$, in terms of the inversion map $h | \Omega$, see Fig. 2a.

Let $\tilde{\Omega} = h(\Omega)$ denote the image through the inversion map into the ball $B$. For any $x_0 \in \partial \Omega$, let $\tilde{n}_i(x_0)$ be the normal inward at $x_0$ in the transformed domain $\tilde{\Omega}$, and let $\tilde{\Sigma} = \tilde{\Sigma}(\tilde{n}_i(x_0))$ be its maximal cap, see Fig. 2b.

The following Lemma B.1 states a well known geometrical result, for any boundary point of a $C^2$ domain, the maximal cap in the transformed domain is nonempty. This result could seem surprising in presence of highly oscillatory boundaries. For example, suppose the boundary of $\Omega$ includes $\Gamma_2 = \{(x, f(x)) : f(x) := 1 + x^5 \sin(\frac{1}{x}), x \in [-0.01, 0.01]\}$, to visualize the scale, see in Fig. 3b $\{(x, x^5 \sin(\frac{1}{x})), x \in [-0.01, 0.01]\}$. Let $h(\Gamma_2)$ be the image through the inversion map into the unit ball $B$, and let $\Gamma_3$ be the arc of the boundary $\partial B$ given by $\Gamma_3 = \{(x, g(x)) : g(x) := \sqrt{1 - x^2}, x \in [-0.01, 0.01]\}$, see Fig. 3c. At this scale, the oscillations are not appreciable. We plot in Fig. 3d the derivative of the ‘vertical’ distance between the boundary $\Gamma_2$ and the ball, concretely we plot $f'(x) - g'(x)$ for $x \in [-0.01, 0.01]$. We plot in Fig. 3e the second derivative of the ‘vertical’ distance between the boundary and the ball, which is $f''(x) - g''(x)$ for $x \in [-5 \cdot 10^{-4}, 5 \cdot 10^{-4}]$. Let us observe that this
second derivative is strictly positive, and that $f''(0) - g''(0) = 1$. Consequently, the first derivative is strictly increasing, and therefore the ‘vertical’ distance $f(x) - g(x)$ does not oscillate.

Moreover, let us consider the image through the inversion map of the straight line $y = 1$, i.e. $h(x, 1) = h((x, 1), x \in [-0.01, 0.01])$. In Fig. 3f and g we plot the second coordinate of the difference $h(F_2) - h(x, 1)$ where $h(x, 1)$ is the image of the straight line $y = 1$, g a zoom of the same graphic and h second coordinate of the difference $h(F_2) - h(F_3)$

In Fig. 3a we draw the inversion of the boundary into the unit ball at an inflexion point; more precisely we set $\Gamma_1 := \{(x, f(x)) : f(x) = \frac{x^3}{2} + 1, x \in [-\pi/4, \pi/4]\}$, which has an inflexion point at $x = 0$.

The following Lemma states the local convexity of the transformed domain.

**Lemma B.1** If $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^2$ boundary, then for any $x_0 \in \partial \Omega$, there exists a maximal cap in the transformed domain $\Sigma = \tilde{\Sigma}(\tilde{n}_i(x_0))$ non empty.
Let $u$ solve (1.1). The Kelvin transform of $u$ at the point $x_0 \in \partial \Omega$ is defined in the transformed domain $\tilde{\Omega} := h(\Omega)$ by

$$
v(y) := \left( \frac{1}{|y|} \right)^{N-2} u(h(y)) = \left( \frac{1}{|y|} \right)^{N-2} u \left( \frac{y}{|y|^2} \right), \quad \text{for} \quad y \in \tilde{\Omega}.
$$

(B.2)

Next, we fix regions where a Kelvin transform of the solution has no critical points. This is the statement of the following Theorem.

**Theorem B.2** Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^2$ boundary. Assume that the nonlinearity $f$ satisfies (H1).

If $u \in C^2(\tilde{\Omega})$ satisfies (1.1) and $u > 0$ in $\Omega$, then for any $x_0 \in \partial \Omega$ its maximal cap in the transformed domain $\tilde{\Sigma}$ is nonempty, and its Kelvin transform $v$, defined by (B.2), has no critical point in the maximal cap $\tilde{\Sigma}$.

Consequently, for any $x_0 \in \partial \Omega$, there exists a $\delta > 0$ only dependent of $\Omega$ and $x_0$, and independent of $f$ and $u$ such that its Kelvin transform $v$ has no critical point in the set $B_\delta(x_0) \cap h(\Omega)$.

Finally, we observe that, reasoning on the Kelvin transform, the Kelvin transform of $u$ at $x_0 \in \partial \Omega$ is locally increasing in the maximal cap of the transformed domain, which provides $L^\infty$ bounds for the Kelvin transform locally. By a compactification process, we then translate this into $L^\infty$ bounds in a neighborhood of the boundary for any solution of the elliptic equation. This is the statement of the following Theorem.

**Theorem B.3** Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^2$ boundary. Assume that the nonlinearity $f$ satisfies (H1) and (H4).

If $u \in C^2(\tilde{\Omega})$ satisfies (1.1) and $u > 0$ in $\Omega$, then there exists a constant $\delta > 0$ depending only on $\Omega$ and not on $f$ or $u$, and a constants $C$ depending only on $\Omega$ and $f$ but not on $u$, such that

$$
\max_{\Omega \setminus \Omega_\delta} u \leq C
$$

(B.3)

where $\Omega_\delta := \{ x \in \Omega : d(x, \partial \Omega) > \delta \}$.

*Proof* See the proof of [8, Theorem 1.2].

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