Condensate of charged Bose disks with numerous holes in a uniform magnetic field

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(November 21, 2018)

Abstract

A stack of disks with numerous holes composed of a non-interacting charged Bose gas is modeled as a low dimensional disk. The Bose condensation of the net-like disk system in a uniform magnetic field is studied. Calculation of the condensate fraction of the net-like disk system showed that there still exists a non-zero condensate fraction at low temperature and in weak field.

PACS numbers: 05.30.Jp, 03.75.Fi, 61.43.Hv
It is known that two fermions can be coupled to form a pair, which behaves like a spinless boson. Many bosonic pairs form a kind of charged Bose gas, and it has also been known that the condensation of the Bose system could be a reliable candidate for superfluidity [1,2]. It has also been reported that a dominant contribution to the superfluid density of liquid helium-4 in films and porous media originates from a geometric structure, such as the non-integer dimensionality of the samples [3]. Condensate density plays a key role in the superfluid density. In this note we will discuss a condensation of an odd structure of matter, a system of charged bose disks (CBD).

We assume that the distance between any two fermions of a pair is large enough to neglect the Coulomb interaction. Then, we propose a stack of non-interacting low dimensional charged Bose disks with numerous holes in a uniform magnetic field. In reality, it is a very thin net with negligible thickness and numerous holes. Fig. 1 is a simple example of the CBD. Therefore, the net could be modeled into a low dimensional disk which has a dimension between 1 and 2 in fractal point of view [?,5]. The dimensionality represents the effects in the measure of disorderness in terms of the connectivity and complexity of the system [6,7].

Generally, magnetic field hardly penetrates superconductor, but can pass through the net-like disk without any difficulty. Therefore, the low dimensional CBD may not be antiferromagtic. This is an advantage of the low dimensional CBD. It has been known that an ideal charged Bose gas in two dimensions (2D) cannot be condensed under a magnetic field because of the one dimensional (1D) character of particle motion within the lowest Landau level [8–10]. On the other hand, if we pile up the net-like low dimensional disks in parallel, this gives an extra dimension to the perpendicular axis. The whole dimension of the stack then becomes between 2 and 3, which is large enough to create a nonzero superfluid density.

We apply a simple statistical approach for the $D$ non-integer dimensions. The theory we use begins from the non-interacting Bose gas in disk dimensions. It is uniform in disk directions. This is then extended to a charged system such as the bipolaronic method for the condensate density [1,2]. The partition function of the system is given by
\[
\ln Q_D(z, T) = -\int_0^\infty d\epsilon \rho_D(\epsilon) \ln(1 - ze^{-\epsilon/T}) - \ln(1 - z),
\]
(1)

where \( z \) is the fugacity defined by \( z = e^{-|\mu|/T} \), and \( \epsilon_p = \frac{p^2}{2m} \) is taken for the neutral system. We set \( \hbar = c = k_B = 1 \) for convenience, and unit volume is assumed.

The term \( \rho_D(\epsilon) \) is the \( D \)-dimensional density of states and plays a key role in our analysis. For a neutral and uniform system it is given as \([3,6,7]\)

\[
\rho_D(\epsilon) = a_D \epsilon^{\frac{D}{2} - 1},
\]
(2)

where \( a_D \) is a \( D \)-dimensional coefficient given by \( a_D = \frac{\Gamma \left( \frac{D}{2} \right)}{(\frac{m^*}{2\pi})^{\frac{D}{2}}} \). Here, \( \Gamma \) is the Gamma function and \( m^* \) is the effective mass of a pair.

The average number of particles is obtained from Eq. (1)

\[
n = \frac{z}{z} \frac{\partial \ln Q}{\partial z} = \int_0^\infty d\epsilon \frac{\rho_D(\epsilon)}{z^{-1}(e^{\epsilon/T} - 1)} + n_0,
\]
(3)

where \( n_0 = n_{p=0} \). Next, substituting the \( D \)-dimensional density of states from Eq. (2) into Eq. (3), the condensate fraction is obtained as \([3,7,11]\)

\[
\frac{n_0}{n} = 1 - v \int_0^\infty d\epsilon \frac{\rho_D(\epsilon)}{z^{-1}(e^{\epsilon/T} - 1)}
= 1 - \int_0^\infty d\epsilon \frac{\epsilon^{\frac{D}{2} - 1}}{e^{\epsilon/T} - 1} \left( \int_0^\infty d\epsilon \frac{\epsilon^{\frac{D}{2} - 1}}{e^{\epsilon/T} - 1} \right)^{-1}_{T_c}
= 1 - \left\{ \frac{T}{T_c(D)} \right\}^{\frac{D}{2}},
\]
(4)

where

\[
T_c(D) = \frac{2\pi}{m^* v^{\frac{D}{2}}} \frac{1}{\zeta \left( \frac{D}{2} \right)}.
\]
(5)

Here, \( v \) is the volume density and \( \zeta \) is the Riemann-Zeta function. The \( z = 1 \) limit is taken for the condensation. Note that \( \int_0^\infty dx x^{\frac{D}{2} - 1}/(e^x - 1) = \Gamma \left( \frac{D}{2} \right) \zeta \left( \frac{D}{2} \right) \).

The critical temperature in Eq. (5), \( T_c \), corresponds to the BEC transition temperature. It is rewritten as a function of \( T_c^b \) for the bulk (\( D = 3 \)) as
\[
T_c(D) = \alpha(D)T_c^0,
\]
where \(\alpha(D) = 1.897/\zeta\left(\frac{D}{2}\right)^{2/D}\). Note that \(\zeta(\frac{3}{2})^2 = 1.897\). It can be readily shown that Eq. (6) satisfies both the ideal thin limit \((D = 2)\) and the bulk limit \((D = 3)\). Note also that the transition would not occur for the 2D limit since \(T_c \sim \left|\frac{D}{2} - 1\right|\) as \(D\) approaches to 2 [12].

The CBD in a uniform magnetic field is now extended by using this new \(D\)-dimensional density of states, \(\rho_D(\epsilon, \omega_H)\). Our \(D\)-dimensional system, \(2 < D < 3\), is composed of \((D - 1)\)-dimensional net-like planes and an additional dimension that is parallel to the magnetic field. A uniform magnetic field is applied perpendicular to the direction of the disks. The new density of states in \(D\)-dimensions is derived from the Landau quantization law [1,13]:

\[
\epsilon_n + \epsilon_{pz} = \left(n + \frac{1}{2}\right)\omega_H + \frac{p_z^2}{2m^*},
\]

where \(\omega_H = 2eH/m^*\) is the cyclotron frequency.

The \((D - 1)\)-dimensional degeneracy is

\[
\rho_{D-1}(\epsilon)\omega_H = \frac{1}{\Gamma\left(\frac{D-1}{2}\right)} \left(\frac{m^*}{2\pi}\right)^{\frac{D-3}{2}} \frac{D-1}{\epsilon^{\frac{D-3}{2}}\omega_H},
\]

Applying the Landau quantization energy to the density of states of the CBD, we obtain \(\rho_D(\epsilon, \omega_H)\) as

\[
\rho_D(\epsilon, \omega_H) = \rho_{D-1}(\epsilon)\omega_H \sum_{n=0}^{\infty} \sum_{p_z} \delta(\epsilon - \epsilon_n - \epsilon_{pz}).
\]

\[
= \rho_{D-1}(\epsilon)\omega_H \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \delta(\epsilon - \epsilon_n - \epsilon_{pz})
\]

\[
= \frac{1}{\sqrt{\pi} \Gamma\left(\frac{D-1}{2}\right)} \left(\frac{m^*}{2\pi}\right)^{\frac{D-3}{2}} \frac{D-1}{\epsilon^{\frac{D-3}{2}}\omega_H} \sum_{n=0}^{\infty} \frac{1}{\sqrt{\epsilon - (n + \frac{1}{2})\omega_H}}.
\]

Certainly, this \(\rho_D\) of the net-like system is different from the one of uniform system in Eq. (2).

Therefore, the condensate fraction of the net-like system is obtained as

\[
\frac{n_0}{n}(T, \omega_H) = 1 - \int_0^\infty d\epsilon \frac{\rho_D(\epsilon, \omega_H)}{e^{\epsilon/T} - 1} \left[\int_0^\infty d\epsilon \frac{\rho_D(\epsilon, \omega_H)}{e^{\epsilon/T} - 1}\right]^{-1}_{T_c}
\]

\[
= 1 - \frac{\int_0^\infty d\epsilon \frac{\rho_{D-1}^2}{e^{\epsilon/T} - 1} \sum_{n=0}^{\infty} \frac{1}{\sqrt{\epsilon - (n + \frac{1}{2})\omega_H}}}{\int_0^\infty d\epsilon \frac{\rho_{D-1}^2}{e^{\epsilon/T} - 1} \sum_{n=0}^{\infty} \frac{1}{\sqrt{\epsilon - (n + \frac{1}{2})\omega_H}}}.
\]
We may use Eq. (6) for $T_c$, and introduce the dimensionless variables $t$ and $y$ instead of $T$ and $\omega_H$ as $t = \frac{T}{T_c}$ and $y = \frac{\omega_H}{T_c}$. Then, the condensate fraction is expressed in the following simple form

$$n_0(t, y) = 1 - \left(\frac{t}{\alpha}\right)^{\frac{D-1}{2}} \frac{A(y/t)}{A(y/\alpha)}. \quad (10)$$

$A(y)$ is defined by

$$A(y) = \int_{x_0}^{\infty} dx \frac{x^{D-3}}{e^x - 1} \sum_{n=0}^{\infty} \frac{1}{\sqrt{x - (n + \frac{1}{2})y}}, \quad (11)$$

where $x_0 = (n + \frac{1}{2})y$. $A(y)$ itself can diverge, but $n_0/n$ in Eq. (10) does not.

The condensate density of the charged Boson model should be different from Eq. (4) as $D \to 3$ limit, and contain additional factor of $A(y)$ which gives the effect of the field over that of the neutral model in Eq. (4). From the equation above, we can easily check that the condensate fraction goes to zero at $t > \alpha$ and $y > 1$ region. On the other hand, we also know that there still exists non-zero condensate fraction at low temperature and low frequency range, but it quickly drops to zero as the field or temperature is increase.

It is plotted in FIG. 2, at various disk dimension and frequencies as a function of normalized temperature $t$. The graphs in Fig. 2 are irregular since the summation of $n$ is contained in the integral, but the overall shape of the drop is clear. We see that the higher dimension gives a little bit more condensate fraction.

We calculated the density of states and condensate fraction of the net-like CBD in the uniform magnetic field, and find that the field strongly effects the condensate fraction of the CBD. The condensate fraction drops to zero quickly as the temperature and field increase, but it still exists at the temperature of $t < \alpha$ and in the field of $y < 1$. We think that the numerous holes of the CBD help the condensate fraction to keep non-zero at the range.

We send special thank to A. S. Alexandrov, C. K. Kim, and P. Pfeifer for fruitful advice.
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FIGURES

FIG. 1. A net-like model of charged Bose disks with numerous holes.

FIG. 2. The condensate fraction of the charged Bose disks in a uniform magnetic field. The solid line is for $y = 10$, the dashed line is for $y = 4$, and the dotted line is for $y = 1$. (a) When $D = 2.4$ (the dimension of a disk is 1.4). (b) When $D = 2.8$ (the dimension of a disk is 1.8).
disk dim=1.4
(b)

\[ \eta_g/n \]

\[ T/T_c^b \]

disk dim = 1.8