THE CALOGERO-SUTHERLAND MODEL AND
POLYNOMIALS WITH PRESCRIBED
SYMmetry

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The Schrödinger operators with exchange terms for certain Calogero-Sutherland quantum many body systems have eigenfunctions which factor into the symmetric ground state and a multivariable polynomial. The polynomial can be chosen to have a prescribed symmetry (i.e. be symmetric or antisymmetric) with respect to the interchange of some specified variables. For four particular Calogero-Sutherland systems we construct an eigenoperator for these polynomials which separates the eigenvalues and establishes orthogonality. In two of the cases this involves identifying new operators which commute with the corresponding Schrödinger operators. In each case we express a particular class of the polynomials with prescribed symmetry in a factored form involving the corresponding symmetric polynomials.

1 Introduction

The Schrödinger operator

\[ H^{(C)} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \beta(\beta/2 - 1) \left( \frac{\pi}{L} \right)^2 \sum_{1 \leq j < k \leq N} \frac{1}{\sin^2 \pi(x_j - x_k)/L}, \quad 0 \leq x_j \leq L \]  (1.1)

describes quantum particles on a line of length $L$ interacting through a $1/r^2$ pair potential with periodic boundary conditions, or equivalently quantum particles on a circle of circumference length $L$ (hence the superscript $(C)$) with the pair potential proportional to the inverse square of the chord length. It is one of a number of quantum many body systems in one dimension which are of the Calogero-Sutherland type, meaning that the ground state (i.e. eigenstate with the smallest eigenvalue $E_0$) is of the form $e^{-\beta W/2}$ with $W$ consisting of one and two body terms only. Explicitly, for (1.1) we have

\[ W = W^{(C)} := -\sum_{1 \leq j < k \leq N} \log |e^{2\pi i x_k/L} - e^{2\pi i x_j/L}|. \]  (1.2)
In studying the integrability properties of (1.1) it is useful \[1\] to generalize the Schrödinger operator to include the exchange operator \(M_{jk}\) for coordinates \(x_j\) and \(x_k\):

\[
H^{(C,Ex)} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \beta \left(\frac{\pi}{L}\right)^2 \sum_{1 \leq j < k \leq N} \frac{(\beta/2 - M_{jk})}{\sin^2 \left(\pi (x_j - x_k)/L\right)},
\]

(1.3)

When acting on functions symmetric in \(x_1, \ldots, x_N\), (1.3) reduces to (1.1). Conjugation with the ground state of (1.1) gives the transformed operator

\[
\tilde{H}^{(C,Ex)} := \left(\frac{L}{2\pi}\right)^2 e^{\beta W(C)/2} (H^{(C,Ex)} - E_0^{(C)}) e^{-\beta W(C)/2}
\]

\[
= \sum_{j=1}^{N} \left( z_j \frac{\partial}{\partial z_j} \right)^2 + \frac{N - 1}{\alpha} \sum_{j=1}^{N} z_j \frac{\partial}{\partial z_j} + \frac{2}{\alpha} \sum_{1 \leq j < k \leq N} \frac{z_j z_k}{z_j - z_k}
\]

\times \left[ \left( \frac{\partial}{\partial z_j} - \frac{\partial}{\partial z_k} \right) - \frac{1 - M_{jk}}{z_j - z_k} \right]
\]

(1.4)

where

\[
z_j := e^{2\pi i x_j/L}, \quad \alpha := 2/\beta.
\]

(1.5)

This operator has non-symmetric eigenfunctions of the form

\[
E_\eta(z, \alpha) = z^\eta + \sum_{\nu < \eta} b_{\nu\eta} z^\nu.
\]

(1.6)

\((z^\eta\) will be referred to as the leading term), where \(\eta\) and \(\nu\) are \(N\)-tuples of non-negative integers and the \(b_{\nu\eta}\) are coefficients. To define the partial order <, let \(\kappa\) be a partition and \(P\) be the (unique) permutation of minimal length such that

\[
z^\eta := z_1^{\eta_1} \cdots z_N^{\eta_N} = z_{P(1)}^{\kappa_1} \cdots z_{P(N)}^{\kappa_N},
\]

(1.7)

Equivalently, let \(\kappa_1 = \eta_{P(1)}\) and similarly define the partition \(\mu\) and permutation \(Q\) such that \(\mu_j = \nu_{Q(j)}\). The partial order < is defined by the statement that \(\nu < \eta\) if \(\mu < \kappa\) (dominance ordering) or, in the cases \(\mu = \kappa\), if the first non-vanishing difference \(P(j) - Q(j)\) is positive. An equivalent specification in this later case is that the last nonvanishing difference of \(\eta - \nu\) is negative \([2]\). The eigenfunctions \(E_\eta(z, \alpha)\) are referred to as non-symmetric Jack polynomials \([3, 4]\). The eigenvalue of (1.4) corresponding to the eigenfunction \(E_\eta\) is given by \([5]\)

\[
\epsilon_\eta = \sum_{j=1}^{N} \kappa_j^2 + \frac{1}{\alpha} (N + 1 - 2j) \kappa_j
\]

(1.8)

and is thus independent of the permutation \(P\) relating \(\eta\) to \(\kappa\). Note that this implies the linear combination

\[
\sum_{P=1}^{N!} a_P E_{\pi^{-1}P}(z, \alpha)
\]

(1.9)

is also an eigenfunction of (1.4).

The exchange operator generalization can also be applied to other Schrödinger operators of the Calogero-Sutherland type, in particular when the underlying root system is
respectively, where the operator \( S \) is given by 

\[
H^{(H,Ex)} := -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{\beta^2}{4} \sum_{j=1}^{N} x_j^2 + \beta \sum_{1 \leq j < k \leq N} \frac{(\beta/2 - M_{jk})}{(x_j - x_k)^2} \tag{1.10}
\]

\[
H^{(L,Ex)} := -\sum_{j=1}^{N} \frac{\partial^2}{\partial \phi_j^2} + \frac{\beta a'}{2} \sum_{j=1}^{N} \frac{\beta a'/2 - S_j}{\sin^2 \phi_j} + \frac{\beta b'/2 - S_j}{\cos^2 \phi_j} + \beta \sum_{1 \leq j < k \leq N} \frac{\beta/2 - M_{jk}}{\sin^2(\phi_j - \phi_k)} + \frac{\beta/2 - S_j S_k M_{jk}}{\sin^2(\phi_j + \phi_k)} \tag{1.11}
\]

\[
H^{(J,Ex)} := -\sum_{j=1}^{N} \frac{\partial^2}{\partial \phi_j^2} + \frac{\beta a'/2 - S_j}{\sin^2 \phi_j} + \frac{\beta b'/2 - S_j}{\cos^2 \phi_j} + \beta \sum_{1 \leq j < k \leq N} \frac{\beta/2 - M_{jk}}{\sin^2(\phi_j - \phi_k)} + \frac{\beta/2 - S_j S_k M_{jk}}{\sin^2(\phi_j + \phi_k)} \tag{1.12}
\]

These operators are given by \([1, 6]\) and \([7]\). Also, the transformation analogous to (1.4) gives

The symmetric ground state eigenfunctions of (1.10) and (1.11) are of the form \( e^{-\beta W} \) with \( W \) given by

\[
W^{(H)} := \frac{1}{2} \sum_{j=1}^{N} x_j^2 - \sum_{1 \leq j < k \leq N} \log |x_k - x_j| \tag{1.13}
\]

\[
W^{(L)} := \frac{1}{2} \sum_{j=1}^{N} x_j^2 - \frac{a'}{2} \sum_{j=1}^{N} \log x_j^2 - \sum_{1 \leq j < k \leq N} \log |x_k^2 - x_j^2|, \tag{1.14}
\]

\[
W^{(J)} := -\frac{a'}{2} \sum_{j=1}^{N} \log \sin^2 \phi_j - \frac{b'}{2} \sum_{j=1}^{N} \log \cos^2 \phi_j - \sum_{1 \leq j < k \leq N} \log |\sin^2 \phi_j - \sin^2 \phi_k|. \tag{1.15}
\]

Also, the transformation analogous to (1.4) gives

\[
\tilde{H}^{(H,Ex)} := -\frac{2}{\beta} e^{\beta W^{(H)}/2} (H^{(H,Ex)} - E_0^{(H)}) e^{-\beta W^{(H)}/2}
\]

\[
= \sum_{j=1}^{N} \left( \frac{\partial^2}{\partial y_j^2} - 2 y_j \frac{\partial}{\partial y_j} \right) + \frac{2}{\alpha} \sum_{j < k} \frac{1}{y_j - y_k} \left[ \left( \frac{\partial}{\partial y_j} - \frac{\partial}{\partial y_k} \right) - \frac{1 - M_{jk}}{y_j - y_k} \right] \tag{1.16}
\]

\[
\tilde{H}^{(L,Ex)} := -\frac{1}{2\beta} e^{\beta W^{(L)}/2} (H^{(L,Ex)} - E_0^{(L)}) e^{-\beta W^{(L)}/2}
\]

\[
= \sum_{j=1}^{N} \left( y_j \frac{\partial^2}{\partial y_j^2} + (a + 1 - y_j) \frac{\partial}{\partial y_j} \right) + \frac{1}{\alpha} \sum_{j < k} \frac{1}{y_j - y_k} \left[ 2 \left( y_j \frac{\partial}{\partial y_j} - y_k \frac{\partial}{\partial y_k} \right) - \frac{y_j + y_k (1 - M_{jk})}{y_j - y_k} \right] \tag{1.17}
\]

\( A_N \) or \( B_N \) and there is an external potential or if the underlying root system is \( BC_N \).
\[ \tilde{H}^{(J,Ex)} := -\frac{1}{4} e^{\beta W(J)/2} (H^{(J,Ex)} - E_0^{(J)}) e^{-\beta W(L)/2} \]

\[ = \sum_{j=1}^{N} \left( z_j \frac{\partial}{\partial z_j} \right)^2 + \left( (a + 1/2) \frac{z_j + 1}{z_j - 1} + (b + 1/2) \frac{z_j - 1}{z_j + 1} + \frac{2(N-1)}{\alpha} \right) z_j \frac{\partial}{\partial z_j} \]

\[ + \frac{2}{\alpha} \sum_{1 \leq j < k \leq N} \frac{z_j z_k}{z_j - z_k} \left[ \left( \frac{\partial}{\partial z_j} - \frac{\partial}{\partial z_k} \right) - \frac{1 - M_{jk}}{z_j - z_k} \right] \]

\[ + \frac{2}{\alpha} \sum_{1 \leq j < k \leq N} \frac{1}{z_j z_k - 1} \left[ (z_j \frac{\partial}{\partial z_j} - z_k \frac{\partial}{\partial z_k}) - \frac{z_j z_k (1 - M_{jk})}{z_j z_k - 1} \right] \]

(1.18)

where

\[ a := (a' \beta - 1)/2, \quad b := (b' \beta - 1)/2. \]

To obtain (1.16) we have made the change of variables \( y_j = \sqrt{\beta/2} x_j \), while to obtain (1.17) we have made the change of variables \( y_j = \beta x_j^2/2 \) and imposed the restriction to eigenfunctions which are even in \( x_j \), and to obtain (1.18) we have used the variable \( z_j = e^{2i\phi_j} \) and imposed the restriction to eigenfunctions unchanged by the mapping \( z_j \mapsto 1/z_j \).

Our objective is to initiate a study into properties of the polynomial eigenfunctions of the operators (1.16)-(1.18), and to supplement some of the results of [8, 9] on the polynomial eigenfunctions of (1.4) with a prescribed symmetry (i.e. eigenfunctions which are either symmetric or anti-symmetric with respect to the interchange of specified variables). In Section 2 we consider (1.4). We revise the construction of an eigenoperator for the symmetric polynomial eigenfunctions (the Jack polynomials) which separates the eigenvalues, and how it can be used to establish orthogonality. This construction is then generalized to provide an eigenoperator for the Jack polynomials with prescribed symmetry, which is used to establish orthogonality.

In Sections 3 and 4 we introduce non-symmetric generalized Hermite and Laguerre polynomials as eigenfunctions of (1.16) and (1.17) respectively. Exponential operator formulas are given relating these polynomials to the non-symmetric Jack polynomials. New commuting operators are identified which are used to prove the orthogonality of these polynomials. Generalized Hermite and Laguerre polynomials with prescribed symmetry are defined, and the commuting operators are used to define an eigenoperator which separates the eigenvalues and establishes orthogonality.

In Section 5 we begin by revising known commuting operators which decompose the operator (1.18). Non-symmetric generalized Jacobi polynomials, and Jacobi polynomials with prescribed symmetry are defined as eigenfunctions of these operators, and orthogonality is established.

The final subsection of Section 2-5 is devoted to establishing a formula expressing a particular class of the polynomials with prescribed symmetry in a factored form involving the corresponding symmetric polynomials.

2 Eigenfunctions of \( \tilde{H}(C,Ex) \)
2.1 Revision

The operator $\tilde{H}^{(C,Ex)}$ allows a factorization in terms of so-called Cherednik operators $\hat{D}_j$ [10]. These operators are given in terms of the Dunkl operator

$$T_j := \frac{\partial}{\partial z_j} + \frac{1}{\alpha} \sum_{k \neq j} \frac{1}{z_j - z_k} (1 - M_{jk})$$

(2.1)

for the root system $A_{N-1}$ by

$$\hat{D}_j := z_j T_j + \frac{1}{\alpha} \sum_{k=1}^{j-1} M_{jk}$$

$$= z_j \frac{\partial}{\partial z_j} + \frac{1}{\alpha} \left( \sum_{l<j} \frac{z_l}{z_j - z_l} (1 - M_{lj}) + \sum_{l>j} \frac{z_j}{z_j - z_l} (1 - M_{lj}) \right) + \frac{(j-1)}{\alpha}.$$  

(2.2)

They mutually commute so that

$$[\hat{D}_j, \hat{D}_k] = 0.$$  

(2.3)

This can be checked from the fact that the Dunkl operators commute:

$$[T_j, T_k] = 0.$$  

(2.4)

The non-symmetric Jack polynomials are simultaneous eigenfunctions of the $\hat{D}_j$ for each $j = 1, \ldots, N$, and the corresponding eigenvalues are

$$e_{j,n} = \eta_j + \frac{1}{\alpha} \left( -\sum_{l<j} h(\eta_l - \eta_j) + \sum_{l>j} h(\eta_j - \eta_l) \right) + \frac{(j-1)}{\alpha}.$$  

(2.5)

with

$$h(x) = \begin{cases} 1, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$  

(2.6)

We remark that some authors [11, 12] define non-symmetric Jack polynomials as eigenfunctions of a variant of the Cherednik operators (2.2):

$$\xi_j := \alpha y_j T_j + \sum_{k=j+1}^{N} M_{jk} + (1 - N)$$

$$= \alpha \left[ y_j \frac{\partial}{\partial y_j} + \frac{1}{\alpha} \left( \sum_{l<j} \frac{y_j}{y_j - y_l} (1 - M_{lj}) + \sum_{l>j} \frac{y_l}{y_j - y_l} (1 - M_{lj}) \right) - \frac{(j-1)}{\alpha} \right].$$  

(2.7)

(the choice of notation $y_1, \ldots, y_N$ for the coordinates here is for later convenience). The operators $\xi_j$ have the same polynomial eigenfunctions as $\hat{D}_j$ except that $z_j$ is replaced by $y_{N+1-j}$ $(j = 1, \ldots, N)$ and $\eta_j$ is replaced by $\eta_{N+1-j}$. Following [12, 11] the corresponding eigenvalues are to be denoted $\bar{\eta}_j$ and are explicitly given by

$$\bar{\eta}_j = \alpha \eta_j - \left( \sum_{l<j} h(\eta_l + 1 - \eta_j) + \sum_{l>j} h(\eta_l - \eta_j) \right).$$  

(2.8)
Returning to the relationship to $\tilde{H}^{(C,Ex)}$, a direct calculation shows 4

$$\tilde{H}^{(C,Ex)} = \sum_{j=1}^{N} \left( \hat{D}_j - \frac{N-1}{2\alpha} \right)^2 - \left( \frac{L}{2\pi} \right)^2 E_0^{(C)}. \tag{2.9}$$

Let us revise 4 how this decomposition can be used to prove that the symmetric polynomial eigenfunctions of (1.4) (i.e. the symmetric Jack polynomials) are orthogonal with respect to the inner product

$$\langle f|g \rangle^{(C)} := \int_0^L dx_1 \ldots \int_0^L dx_N \prod_{1 \leq j < k \leq N} |z_k - z_j|^{2/\alpha} f(z_1^*, \ldots, z_N^*) g(z_1, \ldots, z_N), \tag{2.10}$$

where $z_j = e^{2\pi i x_j/L}$ and $^*$ denotes complex conjugate. First we check directly from the definitions (2.2) and (2.10) that

$$\langle f|\hat{D}_j g \rangle^{(C)} = \langle \hat{D}_j f|g \rangle^{(C)} \tag{2.11}$$

which says that the Cherednik operators are self-adjoint with respect to the inner product (2.10). In performing this check we use the facts that

$$\prod_{1 \leq j < k \leq N} |z_k - z_j|^{2/\alpha} = \psi_0^* \psi_0, \quad \text{where} \quad \psi_0 = \prod_{j=1}^{N} z_j^{-(N-1)/\alpha} \prod_{1 \leq j < k \leq N} (z_k - z_j)^{1/\alpha}$$

and

$$\psi_0 \hat{D}_j \psi_0^{-1} = z_j \frac{\partial}{\partial z_j} - \frac{1}{\alpha} \left( \sum_{l \neq j} \frac{z_l}{z_j - z_l} M_{lj} + \sum_{l > j} \frac{z_j}{z_j - z_l} M_{lj} \right) + \frac{(N-1)}{2\alpha}.$$

Note that $|\psi_0|^2 = e^{-\beta W^{(C)}}$, where $W^{(C)}$ is given by (2.2), and is thus the square of the ground state wave function for (1.1). Next, we compare the eigenvalues $\{e_{j,\eta}\}$ to $\{e_{j,\eta'}\}$, where $\eta'$ is obtained from the $N$-tuple $\eta$ by interchanging $\eta_i$ and $\eta_i'$.

**Lemma 2.1** We have

$$e_{i,\eta'} = e_{i',\eta}, \quad e_{i',\eta'} = e_{i,\eta} \quad \text{and} \quad e_{j,\eta'} = e_{j,\eta}, \quad (j \neq i', i).$$

**Proof** These equations are verified directly from (2.3).

**Remark** The result analogous to Lemma 2.1 applies for the eigenvalues (2.8).

From Lemma 2.1 we see that $\{e_{j,\eta}\}_{j=1, \ldots, N}$ with $\eta = P^{-1}\kappa$ is independent of the permutation $P$. Choosing the permutation $P(j) = N + 1 - j$ ($j = 1, \ldots, N$) shows that $\{e_{j,\eta}\}_{j=1, \ldots, N} = \{\kappa_{N+1-j} + (j-1)/\alpha\}_{j=1, \ldots, N}$. This allows an eigenoperator of the $E_{P^{-1}\kappa}$ to be constructed for which the eigenvalues are independent of $P$:

$$\left(1 + u(\hat{D}_1 - (N-1)/2\alpha)\right) \ldots \left(1 + u(\hat{D}_N - (N-1)/2\alpha)\right) E_{P^{-1}\kappa}$$

$$= \prod_{j=1}^{N} \left(1 + u(\kappa_j + (N+1-2j)/2\alpha)\right) E_{P^{-1}\kappa} \tag{2.12}$$
(the constants $-(N - 1)/2\alpha$ are not essential and could have been omitted). Note that
$$
\kappa_1 - \frac{1}{\alpha} > \kappa_2 - \frac{2}{\alpha} > \ldots > \kappa_N - \frac{N}{\alpha}
$$
(2.13)
so the eigenvalues for different partitions $\kappa$ are distinct.

Consider now the symmetric Jack polynomials $J^{(a)}_\kappa$. They can be characterized (up to normalization, which for definiteness we will take to be that adopted by Stanley [13]) as the polynomial eigenfunctions of (1.4) with leading term $m_\kappa$. From the fact that the non-symmetric Jack polynomials $E_{P-1,\kappa}$ are simultaneous eigenfunctions of all the $\hat{D}_j$ with leading term $z^{P-1,\kappa}$ and the triangular structure (1.6) we must have
$$
J^{(a)}_\kappa(z) = \sum_P a_{P-1,\kappa} E_{P-1,\kappa}(z, \alpha)
$$
(2.14)
for some coefficients $a_{P-1,\kappa}$ (these coefficients are given explicitly in [11]). It follows that the symmetric Jack polynomial satisfies the eigenvalue equation (2.12). Since by (2.11) the operator in (2.12) is self-adjoint with respect to the inner product (2.10) and by (2.13) the eigenvalues are distinct, this implies that the symmetric Jack polynomials are orthogonal with respect to (2.10).

With respect to the eigenvalue equation (2.12) with $E_{P-1,\kappa}$ replaced by $J^{(a)}_\kappa$, we remark that Macdonald [14] has constructed an operator $D_N(X; \alpha)$ such that
$$
D_N(X; \alpha) J^{(a)}_\kappa = \prod_{i=1}^N (X + N - i + \alpha \kappa_i) J^{(a)}_\kappa.
$$
(2.15)
Since $\{J^{(a)}_\kappa\}$ is a basis for symmetric functions, it follows by comparison with (2.12) that when acting on symmetric functions [13, 16]
$$
\prod_{j=1}^N (X + \alpha \hat{D}_j) = D_N(X; \alpha).
$$
(2.16)

Another way of establishing the orthogonality of the symmetric Jack polynomials is to use the expansion (2.14) together with the fact that the non-symmetric Jack polynomials form an orthogonal set with respect to (2.10). This later fact can be established by first noting that
$$
\prod_{i=1}^N (1 + u_i \hat{D}_i)
$$
(2.17)
is an eigenoperator of each $E_\eta$ which separates the eigenvalues. The result now follows after using the fact that (2.17) is self-adjoint.

### 2.2 Jack polynomials with prescribed symmetry

As noted above, the fact that the non-symmetric Jack polynomials $E_\eta$ are simultaneous eigenfunctions of $\hat{D}_1, \ldots, \hat{D}_N$ implies that the $E_\eta$ are eigenfunctions of (1.4). Since (1.4) is symmetric in $z_1, \ldots, z_N$ it follows that $E_\eta$ with the variables $z_1, \ldots, z_N$ permuted is also an eigenfunction of (1.4) with the same eigenvalue. Thus, for any permutation $P$, since
the leading order term of $J^\alpha_\kappa(z)$ is proportional to the monomial symmetric function (i.e. the symmetrization of $z^{P^{-1}\kappa}$) we must have

$$J^\alpha_\kappa(z) = A_{P^{-1}\kappa} \, \text{Sym} \left( E_{P^{-1}\kappa}(z, \alpha) \right).$$  \hfill (2.18)

For the case $P^{-1}\kappa = \kappa$, $\text{Sym} \left( E_{P^{-1}\kappa}(z, \alpha) \right)$ has leading term $m_\kappa$, so $A_{P^{-1}\kappa} = v_\kappa$ where $v_\kappa$ is defined and given explicitly in ref. [3]. Eigenfunctions can be constructed in an analogous way which are symmetric with respect to the interchange of certain sets of variables and antisymmetric with respect to the interchange of other sets of variables. We will refer to such polynomials as having a prescribed symmetry. To facilitate a discussion of this situation, let us rewrite the coordinates \( \{ \gamma \}_{j=1, \ldots, N_\gamma} \) as

\[
\left( \bigcup_{\alpha=1}^{q} \{ w^{(\alpha)}_j \}_{j=1, \ldots, N^{(w)}_\alpha} \right) \left( \bigcup_{\gamma=1}^{p} \{ z^{(\gamma)}_j \}_{j=1, \ldots, N^{(z)}_\gamma} \right)
\]

taken in order so that $w^{(1)}_1 = z_1, \ldots, z^{(p)}_N = z_N$ and $N = \sum_{\alpha=1}^{q} N^{(w)}_\alpha + \sum_{\gamma=1}^{p} N^{(z)}_\gamma$. We seek polynomial eigenfunctions of \( (1.4) \), $S_{P^{-1}\kappa}(z, \alpha)$ say, which are symmetric in \( \{ w^{(\mu)}_j \}_{j=1, \ldots, N^{(w)}_\mu} \) and antisymmetric in \( \{ z^{(\gamma)}_j \}_{j=1, \ldots, N^{(z)}_\gamma} \). We have

$$S_{P^{-1}\kappa}(z, \alpha) = \mathcal{O} \left( E_{P^{-1}\kappa}(z, \alpha) \right)$$  \hfill (2.19)

where $\mathcal{O}$ denotes the operation of symmetrization in \( \{ w^{(1)}_j \}_{j=1, \ldots, N^{(w)}_1} \), antisymmetrization in \( \{ z^{(\gamma)}_j \}_{j=1, \ldots, N^{(z)}_\gamma} \) and normalization such that the coefficient of $z^{P^{-1}\kappa}$ is unity. Due to the operation $\mathcal{O}$ the label $P^{-1}\kappa$ in $S_{P^{-1}\kappa}$ can be replaced by $q + p$ partitions $(\rho, \mu) := (\rho^{(1)}, \ldots, \rho^{(q)}, \mu^{(1)}, \ldots, \mu^{(p)})$ where $\rho^{(\mu)}$ consists of $N^{(w)}_\alpha$ parts ($\alpha = 1, \ldots, q$) and $\mu^{(\gamma)}$ consists of $N^{(z)}_\gamma$ parts. For the $N$-tuple $\eta = P^{-1}\kappa$ any rearrangements of

\[
\{ \eta_j \}_{j=1, \ldots, N^{(w)}_1}, \{ \eta_{N^{(w)}_1+j} \}_{j=1, \ldots, N^{(z)}_2}, \ldots, \{ \eta_{N^{(w)}+\sum_{\gamma=1}^{p-1} N^{(z)}_\gamma+j} \}_{j=1, \ldots, N^{(z)}_p},
\]

where $N^{(w)} := \sum_{\mu=1}^{q} N^{(w)}_\mu$, give the same partitions $(\rho, \mu)$ and thus the same polynomial with prescribed symmetry.

A feature of the polynomials $E_\eta(z, \alpha)$ is that if $\eta_i = \eta_{i+1}$ then $E_\eta$ is symmetric in $z_i$ and $z_{i+1}$ (see (2.21) below). It follows that if two parts of any $\mu^{(\gamma)}$ are equal the polynomial $S_{(\rho, \mu)}$ vanishes identically due to the antisymmetrization procedure in its construction. Thus each $\mu^{(\gamma)}$ must be restricted to distinct parts.

Now we know from [12, Lemma 2.4 and Proposition 4.3] that with $s_i := M_{i,i+1}$ and $\delta_i := \eta_i - \eta_{i+1}$,

\[
s_i E_\eta = \begin{cases} 
\frac{1}{\delta_i} E_\eta + (1 - \frac{1}{\delta_i}) E_{s_i \eta}, & \eta_i > \eta_{i+1} \\
E_\eta, & \eta_i = \eta_{i+1} \\
\frac{1}{\delta_i} E_\eta + E_{s_i \eta}, & \eta_i < \eta_{i+1}
\end{cases}
\]

(2.21)

(here $E_\eta$ refers to the eigenfunctions of (2.7), which as noted below (2.7) are related to the non-symmetric Jacks defined as eigenfunctions of (2.2) by relabelling). Also, each permutation can be written as a product of the elementary transpositions $s_i$. Therefore, we conclude that (2.19) can be rewritten as

$$S_{P^{-1}\kappa}(z, \alpha) = \sum_{\text{rearrangements}} b_{Q^{-1}\kappa} E_{Q^{-1}\kappa}(z, \alpha)$$  \hfill (2.22)
where the sum is over rearrangements $Q^{-1} \kappa$ of $P^{-1} \kappa$ obtained by permuting within the sets (2.20).

Now two distinct sequences of partitions $(\rho, \mu)$ and $(\hat{\rho}, \hat{\mu})$ as defined below (2.19) cannot have any rearrangements of (2.20) in common, as they wouldn’t then be distinct. Hence the expansion (2.22) for $S_{(\rho, \mu)}$ and $S_{(\hat{\rho}, \hat{\mu})}$ does not contain any common $E_q$. It follows immediately from the orthogonality of $\{E_q\}$ with respect to (2.11) that $\{S_{(\rho, \mu)}\}$ are also orthogonal with respect to (2.11).

An alternative way to deduce the orthogonality is to note that the operator

$$
\prod_{\mu=1}^{q} \prod_{j=1}^{N_{\mu}^{(w)}} \left(1 + u_{\mu} \hat{D}_{\sum_{i=1}^{\mu-1} N_{i}^{(w)} + j}\right) \prod_{\gamma=1}^{p} \prod_{j=1}^{N_{\gamma}^{(z)}} \left(1 + v_{\gamma} \hat{D}_{\sum_{i=1}^{\gamma-1} N_{i}^{(z)} + j}\right)
$$

(2.23)

is an eigenoperator of $\{S_{(\rho, \mu)}\}$. To see this, note that this is an eigenoperator of the non-symmetric Jacks, and from Lemma 2.1 the corresponding eigenvalue is independent of the particular rearrangements (2.20). Furthermore, the eigenvalues of (2.23) corresponding to $S_{(\rho, \mu)}(z, \alpha)$ and $S_{(\hat{\rho}, \hat{\mu})}(z, \alpha)$ are distinct whenever $\rho \mu$ and $\hat{\rho} \hat{\mu}$ are distinct. Since, by (2.11), (2.23) is self-adjoint with respect to (2.11), the fact that the eigenvalues are distinct implies orthogonality of these functions with respect to the inner product (2.10).

In the case $q = 0, p = 2, |\mu^{(1)}| + |\mu^{(2)}| = 1, 2$ or $3$, explicit formulas for $\tilde{S}_{\mu^{(1)} \mu^{(2)}}(z, \alpha)$ have been given in ref. [17], where the $\tilde{S}$ are eigenfunctions of (1.4) which are symmetric in $\{w_{j}^{(1)}\}$ and $\{w_{j}^{(2)}\}$. However, in general $\tilde{S}$ does not correspond to $S$ as $\tilde{S}$ does not satisfy (2.19), and the $\tilde{S}$ are not orthogonal.

### 2.3 Some special Jack polynomials with prescribed symmetry

In some previous works [8, 9] we have conjectured a formula for certain Jack polynomials with prescribed symmetry in terms of difference products and the symmetric Jack polynomial. In the present notation the conjecture in [8, 9] applies to $S_{(\rho, \kappa)}(z, \alpha)$ with $q = 1$ and

$$(\rho, \kappa) = (\rho_1, \rho_2, \ldots, \rho_{N_0}, N_1 - 1, N_1 - 2, \ldots, 1, N_2 - 1, N_2 - 2, \ldots, 1, N_p - 1, N_p - 2, \ldots, 1)$$

(2.24)

(here we have written $N_{1}^{(w)} =: N_0$, $N_{l}^{(z)} =: N_l$ to be consistent with refs. [8, 9] and it is assumed $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_{N_1} \geq 0$). The conjecture states that for $(\rho, \mu)$ given by (2.24)

$$
S_{(\rho, \mu)}(z, \alpha) = A_{(\rho, \mu)} \prod_{\gamma=1}^{p} \prod_{1 \leq j < k \leq N_{\gamma}} (z_{k}^{(\gamma)} - z_{j}^{(\gamma)}) J_{\rho}^{(p+\alpha)}(w_1, \ldots, w_{N_0}),
$$

(2.25)

where $A_{(\rho, \mu)}$ is some normalization, provided

$$
\rho_1 \leq \min(N_1, \ldots, N_p)
$$

(2.26)

(A stronger conjecture was also given in [8, 9] which replaces $\rho_1$ in (2.24) by $\rho_1 - 1$, however we do not consider that extension here.)

This conjecture can be verified directly by showing that the r.h.s. of (2.25) is an eigenfunction of (1.4) (an abbreviated version of the required calculation was given in
We begin by rewriting the variables $z_1, \ldots, z_N$ in \((1.4)\) as \(\{w_j\}_{j=1,\ldots,N_0}\) and \(\{z_j^{(\gamma)}\}_{j=1,\ldots,N_\gamma}\) \((\gamma = 1, \ldots, p)\). In terms of these variables, when acting on functions symmetric in \(\{w_j\}\) and anti-symmetric in \(\{z_j^{(\gamma)}\}\) we have

\[
\tilde{H}^{(C,Ex)} = \tilde{H}^{(C,w)} + \tilde{H}^{(C,z)} + \tilde{H}^{(C,wz)}
\]  

where

\[
\tilde{H}^{(C,w)} = \sum_{j=1}^{N_0} \left( w_j \frac{\partial}{\partial w_j} \right)^2 + \frac{N - 1}{\alpha} \sum_{j=1}^{N_0} w_j \frac{\partial}{\partial w_j} + \frac{2}{\alpha} \sum_{j \neq k} w_j w_k \frac{\partial}{\partial w_j}
\]

\[
\tilde{H}^{(C,z)} = \sum_{\gamma=1}^{p} \sum_{j=1}^{N_\gamma} \left( z_j^{(\gamma)} \frac{\partial}{\partial z_j^{(\gamma)}} \right)^2 + \frac{N - 1}{\alpha} \sum_{\gamma=1}^{p} \sum_{j=1}^{N_\gamma} z_j^{(\gamma)} \frac{\partial}{\partial z_j^{(\gamma)}}
\]

\[
+ \frac{2}{\alpha} \sum_{\gamma < \nu} \sum_{j=1}^{N_\gamma} \sum_{k=1}^{N_\nu} \left[ \left( \frac{\partial}{\partial z_j^{(\gamma)}} - \frac{\partial}{\partial z_k^{(\nu)}} \right) \frac{1 - M(z_j^{(\gamma)}, z_k^{(\nu)})}{z_j^{(\gamma)} - z_k^{(\nu)}} \right]
\]

\[
+ \frac{2}{\alpha} \sum_{\gamma=1}^{p} \sum_{j<k} \left[ \left( \frac{\partial}{\partial z_j^{(\gamma)}} - \frac{\partial}{\partial z_k^{(\gamma)}} \right) \frac{1}{z_j^{(\gamma)} - z_k^{(\gamma)}} \right]
\]

\[
\tilde{H}^{(C,wz)} = \frac{2}{\alpha} \sum_{\gamma=1}^{p} \sum_{j=1}^{N_\gamma} \sum_{k=1}^{N_\nu} \left( z_j^{(\gamma)} w_k \frac{\partial}{\partial w_k} \right) \left[ \left( \frac{\partial}{\partial z_j^{(\gamma)}} - \frac{\partial}{\partial w_k} \right) \frac{1 - M(z_j^{(\gamma)}, w_k)}{z_j^{(\gamma)} - w_k} \right]
\]

Here we have used the notation \(M(x, y)\) to denote the operator which exchanges the coordinates \(x\) and \(y\). We seek the action of these operators on

\[
\prod_{\gamma=1}^{p} \Delta(z^{(\gamma)}) \prod_{i=1}^{N_0} w_i^{k_i}, \quad \Delta(z^{(\gamma)}) := \prod_{1 \leq j < k \leq N_\gamma} (z_k^{(\gamma)} - z_j^{(\gamma)}).
\]  

For this purpose we require the following result \([3]\), which can be verified directly.

**Lemma 2.2** Let

\[
A(y_j, y_k) := \frac{y_j y_k}{y_j - y_k} \left[ \left( \frac{\partial}{\partial y_j} - \frac{\partial}{\partial y_k} \right) - \frac{1 - M(y_j, y_k)}{y_j - y_k} \right]
\]

For \(\lambda_j \geq \lambda_k\) we have

\[
A(y_j, y_k) y_j^{\lambda_j} y_k^{\lambda_k} = -\lambda_k y_j^{\lambda_j} y_k^{\lambda_k} + \begin{cases} \sum_{l=1}^{\lambda_j - \lambda_k - 1} (\lambda_j - \lambda_k - l) y_j^{\lambda_j - l} y_k^{\lambda_k + l}, & \lambda_j - \lambda_k \geq 2 \\ 0, & \text{otherwise}. \end{cases}
\]

This lemma will first be used to determine the action of \(H^{(C,z)}\) on \((2.31)\).

**Lemma 2.3** Let \(P_\gamma\) be a permutation of \(\{1, 2, \ldots, N_\gamma\}\), and let

\[
\Phi(z) := \prod_{\gamma=1}^{p} \prod_{j=1}^{N_\gamma} (z_j^{(\gamma)})^{P_\gamma(i)-1}.
\]

\[
\Phi(z) := \prod_{\gamma=1}^{p} \prod_{j=1}^{N_\gamma} (z_j^{(\gamma)})^{P_\gamma(i)-1}.
\]
We have
\[ \tilde{H}^{(C, z)} \Phi(z) = \delta^{(C)} \Phi(z) + \Omega(z). \]  
(2.34)
where \( \delta^{(C)} \) is independent of the permutations \( P_\gamma \) and \( \Omega(z) \) is a polynomial such that the exponent of each monomial has at least one repeated part. Hence
\[ \tilde{H}^{(z)} \prod_{\gamma=1}^{p} \Delta(z^{(\gamma)}) = \delta^{(C)} \prod_{\gamma=1}^{p} \Delta(z^{(\gamma)}). \]  
(2.35)

**Proof.** The fact that \( \delta^{(C)} \) is independent of the permutations \( P_\gamma \) follows from the eigenvalue \( (2.8) \) being independent of the permutation, while each monomial having at least one repeated part is a consequence of the fact that the exponents in \( \Phi(z) \) for each set of variables \( \{z_j\}_{j=1}^{N_\gamma} \) consists of the consecutive integers 0, 1, \ldots, \( N_\gamma - 1 \), and the action of the operator \( A(z^{(\gamma)}, z^{(\mu)}) \) noted in Lemma 2.2. Anti-symmetrizing both sides of (2.34) in the variables \( \{z_j\}_{j=1}^{N_\gamma} \) (\( \gamma = 1, \ldots, p \)) the polynomial \( \Omega(z) \) therefore gives zero contribution, and the result (2.35) follows from the Vandermonde determinant formula
\[ \sum_{P=1}^{N!} \varepsilon(P) \prod_{l=1}^{N} z_l^{P(l)-1} = \Delta(z), \]  
(2.36)
where \( \varepsilon(P) \) denotes the parity of the permutation \( P \).

The crucial point in establishing (2.25) is the action of \( H^{(C, wz)} \) on (2.31).

**Lemma 2.4** Let \( F(w, z) = m_{\kappa}(w) \prod_{j=1}^{p} \Delta(z^{(\gamma)}) \), where \( \kappa \) is a partition consisting of \( N_0 \) parts with the largest part \( \kappa_1 \) restricted by \( \kappa_1 \leq \min(N_1, \ldots, N_\gamma) \) and \( m_{\kappa}(w) \) denotes the monomial symmetric function with exponent \( \kappa \). We have
\[ \tilde{H}^{(C, wz)} F(w, z) = \left( \frac{\sum_{j=1}^{N_\gamma} w_j^{\partial} \partial w_j}{\alpha} \right)^2 - \frac{2}{\alpha} (N^{(z)} - p/2) \sum_{j=1}^{N_\gamma} w_j \partial w_j \right) F(w, z). \]  
(2.37)

**Proof.** Consider first the action of \( A(z^{(\gamma)}, w_k) \) on \( \Delta(z^{(\gamma)})w_k^\lambda \). Expanding \( \Delta(z^{(\gamma)}) \) into terms of the form (2.38) we see from the argument of the proof of Lemma 2.3 that for 0 \( \leq \lambda \leq N_\gamma \) only the first term on the r.h.s. of (2.32) for the action of \( A(z^{(\gamma)}, w_k) \) on \( z_j^{(\gamma)}w_k^\lambda \) contributes, and thus
\[ \sum_{j=1}^{N_\gamma} A(z_j^{(\gamma)}, w_k) \Delta(z^{(\gamma)})w_k^\lambda = \left( - \sum_{j=1}^{N_\gamma} \min(\lambda, N_\gamma - j) \right) \Delta(z^{(\gamma)})w_k^\lambda. \]

But for 0 \( \leq \lambda \leq N_\gamma \) a straightforward calculation gives
\[ - \sum_{j=1}^{N_\gamma} \min(\lambda, N_\gamma - j) = \frac{1}{2} \lambda^2 - (N_\gamma - \frac{1}{2}) \lambda. \]

Thus
\[ \sum_{k=1}^{N_0} \sum_{j=1}^{N_\gamma} A(z_j^{(\gamma)}, w_k) \Delta(z^{(\gamma)})w_1^{\kappa Q(1)} \ldots w_{N_0}^{\kappa Q(N_0)} = \left( \frac{1}{2} |\kappa|^2 - (N_\gamma - \frac{1}{2}) |\kappa| \right) w_1^{\kappa Q(1)} \ldots w_{N_0}^{\kappa Q(N_0)}, \]
where $|\kappa| := \sum_{j=1}^{N_0} \kappa_j$, $|\kappa^2| := \sum_{j=1}^{N_0} \kappa_j^2$, independent of the permutation $Q$. Summing over $\gamma$ and comparison with the definition of $\tilde{H}^{(C,wz)}$ shows that

$$\tilde{H}^{(C,wz)} F(w, z) = \frac{2}{\alpha} \left( \frac{p}{2} |\kappa^2| - \frac{p}{2} |\kappa| \right) F(w, z).$$

This equation remains valid with $H^{(C,wz)}$ replaced by the operator on the r.h.s. of (2.37), thus verifying the validity of (2.37).

Substituting the results of Lemmas 2.3 and 2.4 in (2.27), assuming the inequality in Lemma 2.4, we have

$$\tilde{H}^{(C,Ex)} \prod_{\gamma=1}^{p} \Delta(z^{(\gamma)}) m_\kappa(w) =$$

$$\left( \delta^{(C)} + \frac{N - 1 - 2N(z)}{\alpha} \sum_{j=1}^{N_0} w_j \frac{\partial}{\partial w_j} + \left( 1 + \frac{p}{\alpha} \right) \left( \sum_{j=1}^{N_0} \left( \frac{w_j \partial}{\partial w_j} \right)^2 \right) \right) \Delta(z^{(\gamma)}) m_\kappa(w)$$

(2.38)

The first two terms on the r.h.s. of (2.38) are eigenoperators of any homogeneous polynomial in $w$, while the terms in the square brackets form the eigenoperator defining the symmetric Jack polynomial $J^{(p+\alpha)}(w)$ (recall (1.4) with $M_{jk} = 1$). Thus by forming an appropriate linear combination of $m_\kappa(w)$ in (2.38) (with $|\kappa| = |\rho|$ and $\min(N_1, \ldots, N_p) \geq \rho_1 \geq \kappa_1$) we see that indeed (2.25) is an eigenfunction of (1.4), as required.

We remark that the above derivation shows that if we replace $J^{(p+\alpha)}(w)$ in (2.25) by $E_{\rho}(w, \alpha + p)$, then the resulting function is also an eigenfunction of (1.4). This is consistent with the construction (2.19) of $S_{(\rho,\mu)}$. In fact this latter eigenfunction must result from antisymmetrizing $E_{(\rho,\mu)}$, with $(\rho, \mu)$ defined by (2.24), in the variables $\{z^{(\gamma)}\}_{j=1,\ldots,N_\gamma}$. Thus, with this operation defined by $A$ and assuming the inequality (2.26), we see from the structure (1.6) and the fact that antisymmetrization of a monomial with equal exponents vanishes that

$$AE_{(\rho,\mu)} = w^\rho A z^\mu + \sum_{\nu < \rho} c_{\nu\rho} w^\nu A z^\mu$$

$$= \prod_{\gamma=1}^{p} \Delta(z^{(\gamma)}) \left( w^\rho + \sum_{\nu < \rho} c_{\nu\rho} w^\nu \right),$$

(2.39)

for some constants $c_{\nu\rho}$. But $AE_{(\rho,\mu)}$ must be an eigenfunction of (1.4), and the above working gives that the function of $w$ must satisfy a eigenvalue equation in which the eigenoperator is again of the form (1.4), which we know has a unique solution of the form required in (2.39).

In the case $q = 0$, $p = 1$ we can also provide a formula for $S_{(\rho,\mu)} =: S_\mu$ in terms of the symmetric Jack polynomial:

$$S_\mu(z, \alpha) = \Delta(z) \frac{1}{\nu_{\kappa}} (\alpha/(1+\alpha)) J^{(\alpha/(1+\alpha))}_\kappa(z)$$

(2.40)

where

$$\kappa := (\mu_1 - N + 1, \mu_2 - N + 2, \ldots, \mu_N)$$

(2.41)
and $v_\kappa$ is as in (2.18). Note from (2.19) that (2.40) is equivalent to the statement that
\[ \mathcal{A} E_\mu(z, \alpha) = \Delta(z) \frac{1}{v_\kappa(\alpha/(1 + \alpha))} J^{(\alpha/(1 + \alpha))}_\kappa(z), \] (2.42)

where $\mathcal{A}$ denotes antisymmetrization in all variables. The easiest way to verify (2.40) is to try for eigenfunctions of (1.3) of the form
\[ |\Delta(z)|^{1/\alpha} \Delta(z) f \]
where $f$ is symmetric. A straightforward calculation shows that $f$ must be an eigenfunction of (1.4) with $M_{jk} = 1$ and $2/\alpha$ replaced by $2/\alpha + 1$. But the unique symmetric eigenfunction of this equation with leading term $m_\kappa J^{(\alpha/(1 + \alpha))}_\kappa(z)$ is $J^{(\alpha/(1 + \alpha))}_\kappa(z)$. The relationship between $\kappa$ and $\mu$, and thus the result follows from (2.19).

3 Eigenfunctions of $\tilde{H}(H,Ex)$

3.1 The non-symmetric generalized Hermite polynomials

The operator (1.16) has unique polynomial eigenfunctions of the form
\[ y^\eta + \sum_{|\nu|<|\eta|} c_{\eta \nu} y^\nu \] (3.1)
with corresponding eigenvalue $-2|\eta|$. By adding together an appropriate linear combination of these eigenfunctions, we can construct the unique eigenfunction of the form
\[ E^{(H)}_\eta(y, \alpha) := E_\eta(y, \alpha) + \sum_{|\nu|<|\eta|} c'_{\eta \nu} E_\nu(y; \alpha) \] (3.2)
again with eigenvalue $-2|\eta|$. We will refer to the $E^{(H)}_\eta(y, \alpha)$ as the non-symmetric generalized Hermite polynomials (they are related to the symmetric generalized Hermite polynomials defined in [7] by an equation analogous to (2.14); see eq. (3.16) below). In fact by adopting a method due to Sogo [18], an exponential operator formula can be obtained expressing $E^{(H)}_\eta(y, \alpha)$ in terms of $E_\eta(y, \alpha)$, which is the analogue of the formula due to Lassalle [19] (see eq. (3.16) below) expressing the symmetric generalized Hermite polynomials in terms of the symmetric Jack polynomials.

To obtain this formula, we write the eigenvalue equation for the $E^{(H)}_\eta(y, \alpha)$ in the form
\[ (A + \tilde{D}_0) E^{(H)}_\eta(y, \alpha) = 0 \] (3.3)
where
\[ A := -2 \sum_{j=1}^N y_j \frac{\partial}{\partial y_j} + 2|\eta|, \quad \tilde{D}_0 := \sum_{j=1}^N \frac{\partial^2}{\partial y_j^2} + \frac{2}{\alpha} \sum_{j<k} \frac{1}{y_j - y_k} \left[ \left( \frac{\partial}{\partial y_j} - \frac{\partial}{\partial y_k} \right) - \frac{1 - M_{jk}}{y_j - y_k} \right] \] (3.4)
Note that (3.3) only specifies $E^{(H)}_\eta$ uniquely after the specification (3.2). Since $E_\eta(y, \alpha)$ is homogeneous of degree $|\eta|$ we also have $A E_\eta(y, \alpha) = 0$ which can be equated with (3.3) and the resulting equation rearranged to give
\[ E^{(H)}_\eta(y, \alpha) = \left( 1 - (A + \tilde{D}_0)^{-1} \tilde{D}_0 \right) E_\eta(y, \alpha) \] (3.5)
Next we make use of the operator identity
\[(A + \tilde{D}_0)^{-1}\tilde{D}_0 = A^{-1}\tilde{D}_0 - (A^{-1}\tilde{D}_0)^2 + (A^{-1}\tilde{D}_0)^3 + \ldots,\] (3.6)

We note that after \(p\) applications of \(\tilde{D}_0\), \(E_\eta(y, \alpha)\) is a homogeneous polynomial of degree \(|\eta| - 2p\) so we have
\[
A^{-1}(\tilde{D}_0)^p E_\eta(y, \alpha) = \frac{1}{4p}(\tilde{D}_0)^p E_\eta(y, \alpha).
\]
Using this in (3.5) gives
\[
E_{\eta_H}(y, \alpha) = \left(1 - \frac{1}{4} \tilde{D}_0 + \frac{1}{4^2} \frac{1}{2!}(\tilde{D}_0)^2 - \frac{1}{4^3} \frac{1}{3!}(\tilde{D}_0)^3 + \ldots\right) E_\eta(y, \alpha)
= \exp(-\tilde{D}_0/4) E_\eta(y, \alpha),
\] (3.7)
which is consistent with (3.2) and is thus the sought exponential operator formula. Note that the series in (3.7) terminates after the \(|\eta|/2\) application of \(\tilde{D}_0\).

To proceed further we note that the operator \(\tilde{D}_0\) can be written in terms of the Dunkl operator (2.1) with the \(z_i\) replaced by \(y_i\). We have (20)
\[
\tilde{D}_0 = \sum_{j=1}^{N} T_j^2.
\] (3.8)

We will use (3.7) and (3.8) to verify that the \(E_{\eta_H}\) are simultaneous eigenfunctions of a set of operators more basic than \(\tilde{H}^{(H,Ex)}\), which play an analogous role to the Cherednik operators in the theory of the non-symmetric Jack polynomials.

**Proposition 3.1** The non-symmetric generalized Hermite polynomials \(E^{(H)}_{\eta_H}\) are eigenfunctions of the operators
\[
h_i := \xi_i - \frac{\alpha}{2} T_i^2, \quad (i = 1, \ldots, N)
\] (3.9)
with corresponding eigenvalue \(\bar{\eta}_i\). Here \(\xi_i\) is the Cherednik operator (2.7) and \(\bar{\eta}_i\), which is given explicitly by (2.8), is defined as the eigenvalue in the eigenvalue equation
\[
\xi_i E_\eta = \bar{\eta}_i E_\eta.
\] (3.10)

**Remarks**
(i) In [7, Prop. 3.2 with \(j = 1\)] we noted that \(D_1^1 + \frac{1}{4}[D_1^1, D_0]\), where
\[
D_1^1 := \alpha \sum_{j=1}^{N} y_j \frac{\partial}{\partial y_j} + \frac{N(N - 1)}{2}, \quad D_0 := \sum_{j=1}^{N} \frac{\partial^2}{\partial y_j^2} + \frac{2}{\alpha} \sum_{j \neq k} \frac{1}{y_j - y_k} \frac{\partial}{\partial y_j},
\] (3.11)
is an eigenoperator for the symmetric generalized Hermite polynomials \(H_\kappa(y; \alpha)\). Our construction of (3.9) was motivated by this result, (3.8) and the fact that
\[
\sum_{i=1}^{N} (\xi_i + (N - 1)) = D_1^1.
\] (3.12)
We could replace $\xi_i$ in (3.9) by $\hat{D}_{N+1-i}$ (recall the remark below (2.7)). Our use of $\xi_i$ has been influenced by [12, 11].

In further preparation for proving Proposition 3.1 we will evaluate the commutator $[\xi_j, \hat{D}_0]$. Due to (3.8), we should first consider the commutator $[\xi_j, T_i]$.

**Lemma 3.1** We have

$$
\begin{align*}
[\xi_j, T_i] &= T_i M_{ij}, \quad i < j \\
[\xi_j, T_i] &= T_j M_{ij}, \quad i > j \\
[\xi_j, T_j] &= -\alpha T_j - \sum_{p < j} M_{jp} T_j - \sum_{p > j} T_j M_{jp}.
\end{align*}
$$

**Proof** These formulas are verified by straightforward calculation using the formula (2.7) relating $\xi_j$ and $T_j$, the commutator formula (2.4), and the additional easily verified commutator identities

$$
[T_i, y_i] = 1 + \frac{1}{\alpha} \sum_{p \neq i} M_{ip}, \quad [T_i, y_j] = -\frac{1}{\alpha} M_{ij}, \quad i \neq j \quad [T_i, M_{jk}] = 0, \quad i \neq j, k.
$$

Now we can evaluate the commutator $[\xi_i, \hat{D}_0]$.

**Lemma 3.2** We have

$$
[\xi_i, \hat{D}_0] = -2\alpha T_i^2.
$$

**Proof** Using (3.8) we have

$$
[\xi_i, \hat{D}_0] = \sum_{j=1}^N [\xi_i, T_j^2] = \sum_{j=1}^N \left( [\xi_i, T_j] T_j + T_j [\xi_i, T_j] \right).
$$

The result follows after splitting the sum up into parts $j < i$, $j = i$ and $j > i$, then using Lemma 3.1 and the facts that

$$
M_{ij} T_j = T_i M_{ij}, \quad M_{ij} T_k = T_k M_{ij} \quad (k \neq i, j).
$$

With this preparation we can now provide the verification of the claim of Proposition 3.1.

**Proof of Proposition 3.1** Using (3.7) and (3.10) we have

$$
\xi_i \left( e^{\hat{D}_0/4} E_\eta^{(H)} \right) = \bar{\eta}_k \left( e^{\hat{D}_0/4} E_\eta^{(H)} \right).
$$

But according to the Baker-Campbell-Hausdorff formula

$$
\begin{align*}
\xi_i \left( e^{\hat{D}_0/4} E_\eta^{(H)} \right) &= e^{\hat{D}_0/4} \left( \xi_i + \frac{1}{4} [\xi_i, \hat{D}_0] + \frac{1}{4!} \frac{1}{4^2} [[\xi_i, \hat{D}_0], \hat{D}_0] + \ldots \right) E_\eta^{(H)} \\
&= e^{\hat{D}_0/4} \left( \xi_i - \frac{\alpha}{2} T_i^2 \right) E_\eta^{(H)}.
\end{align*}
$$
where to obtain the last line we have used the fact that since \([\xi_i, \tilde{D}_0] = -2\alpha T_i^2\) (by Lemma 3.2) and \(\tilde{D}_0 = \sum_{j=1}^{N} T_j^2\) (eq. (3.8)), the higher order commutators vanish due to (2.4). Equating the r.h.s. of (3.14) with the r.h.s. of (3.13) gives the desired eigenvalue equation.

**Remark** Since \(\{E_{\eta}^{(H)}\}\) form a basis for analytic functions it follows from Proposition 3.1 that \(\{h_i\}\) mutually commute. This fact can also be checked directly using, Proposition 3.1, Lemma 3.1 and (2.3) and (2.4).

From Proposition 3.1, (3.8), (3.12) and (1.16) we have that

\[
\sum_{i=1}^{N} h_i = -\frac{\alpha}{2} \left( \tilde{H}^{(H,Ex)} + N(N - 1)/\alpha \right). \tag{3.15}
\]

Also, by forming the sum (2.14) in (3.7) we have

\[
\sum_p a_{p-1,\kappa} E_{p-1,\kappa}^{(H)}(y;\alpha) = \exp \left(-\frac{1}{4} \tilde{D}_0\right) J_\kappa^{(\alpha)}(y) = 2^{-|\kappa|} J_\kappa^{(\alpha)}(1 N) H_\kappa(y;\alpha) \tag{3.16}
\]

where, after noting that \(\tilde{D}_0 = D_0\) as defined in (3.11) when acting on symmetric functions, the last equality is the exponential operator formula of Lassalle (see ref. [7, eq. (3.21)]). Now from the remark below Lemma 2.1 we have that \(\{\bar{\eta}_j\}_{j=1,...,N} = \{\alpha \kappa_j - (j - 1)\}_{j=1,...,N}\) independent of the permutation relating \(\eta\) to the partition \(\kappa\). Using this fact, (3.15), (3.16) and Proposition 3.1 we see by following the argument of the last two paragraphs of Section 2.1 that

\[
\prod_{j=1}^{N} (1 + u h_i) \tag{3.17}
\]

is an eigenoperator of the symmetric generalized Hermite polynomials \(H_\kappa(y;\alpha)\) (an operator with this property equivalent to (3.17) has recently been identified by Kakei [21]) with corresponding eigenvalue

\[
\prod_{j=1}^{N} \left( 1 + u (\alpha \kappa_j - (j - 1)) \right). \tag{3.18}
\]

Note that the inequalities (2.13) imply that the eigenvalues are distinct.

We remark that in ref. [7, Prop. 3.2] an operator \(\tilde{H}_j^{(H)}\) was constructed such that

\[
\left( \sum_{j=0}^{N} X^{N-j} \tilde{H}_j^{(H)} \right) H_\kappa(y;\alpha) = \prod_{i=1}^{N} (X + N - i + \alpha \kappa_i) H_\kappa(y;\alpha). \tag{3.19}
\]

Comparison with the eigenvalue (3.18) corresponding to the operator (3.17) shows that when acting on symmetric functions

\[
\prod_{i=1}^{N} (X + N + h_i) = \sum_{j=0}^{N} X^{N-j} \tilde{H}_j^{(H)} \tag{3.20}
\]

The eigenoperator (3.17) can be used to establish that \(\{H_\kappa(y;\alpha)\}\) are orthogonal with respect to the inner product

\[
\langle f|g \rangle^{(H)} := \prod_{i=1}^{N} \int_{-\infty}^{\infty} dy_i e^{-y_i^2} \prod_{1 \leq j < k \leq N} |y_k - y_j|^{2/\alpha} fg \tag{3.21}
\]
(note that the weight function in (3.21) is equal to $e^{-\beta W(H)}$, where $W(H)$ is given by (1.13), and is thus the square of the ground state wave function of (1.10)). This is an immediate consequence of the fact that the eigenvalues (3.18) are distinct and (3.17) is self-adjoint with respect to (3.21) (recall the analogous argument in Section 2.1). The latter result follows from the $h_i$ being self-adjoint with respect to (3.21), which is to be established in the subsequent lemma. The orthogonality has previously been established in refs. [7, 21], but the details here are different.

**Lemma 3.3** We have

$$\langle f|T_i g\rangle^{(H)} = \langle (2y_i - T_i)f|g\rangle^{(H)}$$

(3.22)

and thus

$$\langle f|h_i g\rangle^{(H)} = \langle h_i f|g\rangle^{(H)}$$

(3.23)

**Proof** The result (3.22) is given in [22, lemma 3.7]. It is derived using integration by parts. Using (3.22) and the first equation in (2.7) we find

$$\langle f|\xi_i g\rangle^{(H)} = \langle (\xi_i + \alpha(2y_i^2 - T_iy_j - y_jT_i)f|g\rangle^{(H)}.$$

(3.24)

But from Proposition 3.1 $h_i = \xi_i - (\alpha/2)T_i^2$. Noting from (3.22) that

$$\langle f|T_i^2 g\rangle^{(H)} = \langle (2y_i - T_i)^2f|g\rangle^{(H)}$$

(3.25)

the result (3.23) follows by subtracting $\alpha/2$ times (3.25) from (3.24).

Analogous to the theory of the symmetric Jack polynomials revised in Section 2.1, the orthogonality of the symmetric generalized Hermite polynomials can be established from the formula (3.16) and the fact that the non-symmetric generalized Hermite polynomials are orthogonal with respect to (3.21). This latter fact is established by noting that (2.17) with $\hat{D}_i$ replaced by $h_i$ is self-adjoint with respect to (3.21) and is an eigenoperator of each $E_{\eta}^{(H)}$ which separates the eigenvalues.

### 3.2 Generalized Hermite polynomials with prescribed symmetry

Let $S_{(\rho,\mu)}(y,\alpha)$ denote a Jack polynomial with prescribed symmetry. By following the working which led to (3.7) we can construct a polynomial eigenfunction of (1.10) according to

$$S_{(\rho,\mu)}^{(H)}(y,\alpha) = \exp(-\hat{D}_0/4)S_{(\rho,\mu)}(y,\alpha).$$

(3.26)

Note that $S_{(\rho,\mu)}^{(H)}(y,\alpha)$ has the same symmetry properties as $S_{(\rho,\mu)}(y,\alpha)$. We will refer to $\{S_{(\rho,\mu)}^{(H)}(y,\alpha)\}$ as the generalized Hermite polynomials with prescribed symmetry. Due to the expansion (2.22), and the formula (3.7) we see from (3.26) that

$$S_{(\rho,\mu)}^{(H)}(y,\alpha) = \sum_{\text{rearrangements}} b_{Q^{-1}\kappa} E_{Q^{-1}\kappa}^{(H)}(y,\alpha)$$

(3.27)

From this formula we can deduce that the operator (2.23) with each operator $\hat{D}_j$ replaced by $h_j$ is an eigenoperator of $S_{(\rho,\mu)}^{(H)}$, and the corresponding eigenvalues are distinct.
for distinct members of \( \{ S_{\rho, \mu}^{(H)} \} \). This implies \( \{ S_{\rho, \mu}^{(H)} \} \) is an orthogonal set with respect to the inner product (3.21). This fact can also be deduced from (3.27) and the orthogonality of \( \{ E_{\eta}^{(H)} \} \).

### 3.3 Some special generalized Hermite polynomials with prescribed symmetry

The analogue of the conjecture (2.25) in the Hermite case is that for \( \eta =: P \kappa \) given by (2.24)

\[
S_{\eta}^{(H)}(y, \alpha) = A_{\eta}^{(H)} \prod_{\gamma = 1}^{N_0} \prod_{1 \leq j < k \leq N_0} (y_j^{(\gamma)} - y_k^{(\gamma)}) H_{\rho}(\sqrt{\frac{\alpha}{\alpha + p}} x_1, \ldots, \sqrt{\frac{\alpha}{\alpha + p}} x_{N_0}; p + \alpha), \tag{3.28}
\]

provided the inequality (2.26) is satisfied. To verify this conjecture, our task is to show (3.28) is an eigenfunction of (1.16). To do this, we proceed as in (2.27) and write (1.16) when acting on functions symmetric in \( \{ y_j^{(\gamma)} \}_{j = 1, \ldots, N_0} \) and antisymmetric in \( \{ y_j^{(\gamma)} \}_{j = 1, \ldots, N_0} \) (\( \gamma = 1, \ldots, p \)) as the sum of three terms:

\[
\tilde{H}^{(H, Ex)} = \tilde{H}^{(H, x)} + \tilde{H}^{(H, y)} + \tilde{H}^{(H, xy)} \tag{3.29}
\]

where

\[
\tilde{H}^{(H, x)} = \sum_{j = 1}^{N_0} \left( \frac{\partial^2}{\partial x_j^2} - 2x_j \frac{\partial}{\partial x_j} \right) + 2 \alpha \sum_{j \neq k} \frac{1}{x_j - x_k} \frac{\partial}{\partial x_k} \tag{3.30}
\]

\[
\tilde{H}^{(H, xy)} = \frac{2}{\alpha} \sum_{\gamma = 1}^{p} \sum_{j = 1}^{N_0} \sum_{k = 1}^{N_0} \frac{1}{y_j^{(\gamma)} - x_k} \left[ \frac{\partial}{\partial y_j^{(\gamma)}} - \frac{\partial}{\partial x_k} \right] \left( 1 - M(y_j^{(\gamma)}, x_k) \right) \tag{3.31}
\]

and \( \tilde{H}^{(H, y)} \) is given by (1.16) with \( y_1, y_2, \ldots, y_N \) replaced by \( y_1^{(1)}, y_2^{(1)}, \ldots, y_p^{(p)} \).

The fundamental operator in establishing that (3.28) is an eigenfunction of (1.16) is \((y_j y_k)^{-1} A(y_j, y_k)\), where \( A(y_j, y_k) \), along with its action on \( y_j^{(\lambda)} y_k^{(\lambda)} \) is specified in Lemma 2.2. Using this action, we can repeat the argument of the proof of Lemma 2.3 to conclude that \( \prod_{\gamma = 1}^{p} \Delta(y^{(\gamma)}) \) is an eigenfunction of \( \tilde{H}^{(H, y)} \). This action can also be used to establish the analogue of Lemma 2.4.

**Lemma 3.4** Let \( F(x, y) = m_\kappa(x) \prod_{\gamma = 1}^{p} \Delta(y^{(\gamma)}) \), where \( \kappa \) is a partition consisting of \( N_0 \) parts with the largest part \( \kappa_1 \) restricted by \( \kappa_1 \leq \min(N_1, \ldots, N_\gamma) \). We have

\[
\tilde{H}^{(H, xy)} F(x, y) = \left( \frac{p}{\alpha} \right) \sum_{j = 1}^{N_0} \frac{\partial^2}{\partial x_j^2} F(x, y). \tag{3.32}
\]

**Proof** Proceeding as in the proof of Lemma 2.4 we see that for \( 0 \leq \lambda \leq N_\gamma \) only the \( l = 1 \) term on the r.h.s. of (2.32) contributes to the action of \((y_j^{(\gamma)} x_k)^{-1} A(y_j^{(\gamma)}, x_k)\) on \((y_j^{(\gamma)})^{\lambda'} x_k^{\lambda}\), and this requires \( \lambda - \lambda' \geq 2 \). Thus

\[
\sum_{j = 1}^{N_\gamma} (y_j^{(\gamma)} x_k)^{-1} A(y_j^{(\gamma)}, x_k) \Delta(y^{(\gamma)}) x_k^{\lambda} = \left( \frac{1}{\lambda - 2} \right) \sum_{j = 0}^{\lambda - 2} (\lambda - j - 1) \Delta(y^{(\gamma)}) x_k^{\lambda},
\]

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which implies
\[ \tilde{H}^{(H,xy)} F(x, y) = \frac{p}{\alpha} (|\kappa|^2 - |\kappa|) F(x, y), \]
and the result follows.

From Lemma 3.4 and the fact that \( \tilde{H}^{(H,y)} \) is an eigenoperator for \( \prod_{\gamma=1}^p \Delta(y^{(\gamma)}) \) with eigenvalue \( \delta^{(H)} \) say we have, assuming the inequality in Lemma 3.4,
\[ \tilde{H}^{(H,E_x)} \prod_{\gamma=1}^p \Delta(y^{(\gamma)}) m_\kappa(x) = \left( \delta^{(H)} + (1 + \frac{p}{\alpha}) \left[ \sum_{j=1}^{N_0} \frac{\partial^2}{\partial x_j^2} - \frac{2\alpha}{\alpha + p} \frac{1}{x_j} \frac{\partial}{\partial x_j} \right] \right) \prod_{\gamma=1}^p \Delta(y^{(\gamma)}) m_\kappa(x). \]
The fact that (3.28) is an eigenfunction of \( \tilde{H}^{(H,E_x)} \) follows from this equation since the operator in square brackets is the defining eigenoperator for \( H_\kappa((\alpha/(\alpha + p))^{1/2}x; \alpha) \).

As another explicit evaluation of a class of \( S_{(\rho,\mu)}^{(H)} \), in the case \( p = 1, q = 0 \) we have the analogue of (2.40):
\[ S_{\mu}^{(H)}(y, \alpha) = \Delta(y) \frac{2^{-|\kappa|} C_\kappa^{(\alpha)}(1^N)}{v_\kappa(\alpha/(1 + \alpha))} H_\kappa(y; \alpha/(1 + \alpha)), \]
where the prefactors of \( H_\kappa \) are chosen so that the coefficient of \( m_\kappa \) in this factor is unity (see ref. [7]). The derivation of this result is analogous to the derivation of (2.40), and so will be omitted.

### 4 Eigenfunctions of \( \tilde{H}^{(L,E_x)} \)

#### 4.1 The non-symmetric generalized Laguerre polynomials

Analogous to the situation with \( \tilde{H}^{(H,E_x)} \), the operator (1.17) has unique polynomial eigenfunctions of the form (3.1) with corresponding eigenvalue \(-|\eta|\). An appropriate linear combination of these eigenfunctions gives unique eigenfunctions of the form
\[ E_\eta^{(L)}(y, \alpha) := E_\eta(y, \alpha) + \sum_{|\nu|<|\eta|} c_{\eta \nu} E_\nu(y; \alpha) \]
which also have eigenvalue \(-|\eta|\). As well as depending on the parameter \( \alpha \) they also depend on the parameter \( a \) in (1.17), however for notational convenience we have suppressed this dependence in (1.1). The \( E_\eta^{(L)}(y, \alpha) \) will be referred to as the non-symmetric generalized Laguerre polynomials (we recall from ref. [7] that the symmetric generalized Laguerre polynomials \( L_\kappa^{(\alpha)}(y; \alpha) \) are the polynomial eigenfunctions of (1.17) with leading term proportional to the symmetric Jack polynomial \( J_\kappa^{(\alpha)}(y) \).

By repeating the working which led to (3.7), starting with the eigenvalue equation for \( E_\eta^{(L)}(y, \alpha) \), we obtain the exponential operator formula
\[ E_\eta^{(L)}(y, \alpha) = \exp \left( - \left( D_1 + (a + 1) \sum_{j=1}^N \frac{\partial}{\partial y_j} \right) \right) E_\eta(y, \alpha) \]
where

\[ \tilde{D}_1 := \sum_{j=1}^{N} y_j \frac{\partial^2}{\partial y_j^2} + \frac{1}{\alpha} \sum_{j<k} \frac{1}{y_j - y_k} \left[ 2 \left( y_j \frac{\partial}{\partial y_j} - y_k \frac{\partial}{\partial y_k} \right) - \frac{y_j + y_k (1 - M_{jk})}{y_j - y_k} \right]. \]  

\[ (4.3) \]

It is convenient to introduce new variables \( x_j^2 =: y_j \). It follows from \[23\], first eq. pg. 125 that

\[ \tilde{D}_1 + (a + 1) \sum_{j=1}^{N} \frac{\partial}{\partial y_j} \bigg|_{y_j \mapsto x_j^2} = \frac{1}{4} \sum_{i=1}^{N} (T_i^{(B)})^2 \]  

\[ (4.4) \]

where \( T_i^{(B)} \) is the Dunkl operator for the root system \( B_N \):

\[ T_i^{(B)} := \frac{\partial}{\partial x_i} + \frac{1}{\alpha} \sum_{p \neq i} \left( \frac{1 - M_{ip}}{x_i - x_p} + \frac{1 - S_i S_p M_{ip}}{x_i + x_p} \right) + \frac{a + 1/2}{x_i} (1 - S_i) \]  

\[ (4.5) \]

(as in \[1.11\] \( S_j \) is the operator which replaces the coordinate \( x_j \) by \( -x_j \)). The similarity between \( (4.2) \) with the substitution \( (4.4) \), and \( (3.7) \) with the substitution \( (3.8) \) suggest we define operators \( l_i \), say, analogous to \( (3.9) \):

\[ l_i := \hat{\xi}_i - \frac{\alpha}{4} (T_i^{(B)})^2 \]  

\[ (4.6) \]

where \( \hat{\xi}_i \) is the Cherednik operator \[2.7\] with the change of variables \( y_j = x_j^2 \) (a literal analogy would have \( \alpha/4 \) replaced by \( \alpha/2 \) in \( (4.6) \); the reason for modifying this is connected with the remark accompanying Lemma 4.2 below). We want to show that the \( l_i \) are eigenoperators for the \( E_n^{(L)}(x^2, \alpha) \). To do this we require the analogue of Lemma 3.2, and this in turn requires some preliminary results.

**Lemma 4.1** When acting on functions even in \( x_1, \ldots, x_N \)

\[ \frac{1}{4} (T_i^{(B)})^2 = x_i^2 \tilde{T}_i^2 + (a + 1) \tilde{T}_i + \frac{1}{\alpha} \sum_{p \neq i} M_{ip} \tilde{T}_i \]  

\[ (4.7) \]

where \( \tilde{T}_i \) is the \( A \)-type Dunkl operator \[2.1\] with the change of variables \( z_j = x_j^2 \) (\( j = 1, \ldots, N \)):

\[ \tilde{T}_i = \frac{1}{2x_i} \frac{\partial}{\partial x_i} + \frac{1}{\alpha} \sum_{p \neq i} \frac{1 - M_{ip}}{x_i^2 - x_p^2}. \]  

\[ (4.8) \]

**Proof** Let \( f \) be even in \( x_1, \ldots, x_N \). Then from \( (4.5) \) and \( (4.8) \) we see that

\[ T_i^{(B)} f = 2x_i \tilde{T}_i f. \]

Now \( x_i \tilde{T}_i f \) is odd in \( x_i \), and from the definition \( (1.5) \) we see that when acting on a function which is odd in \( x_i \),

\[ T_i^{(B)} = 2x_i \tilde{T}_i + \frac{2a + 1}{x_i} + \frac{1}{\alpha} \sum_{p \neq i} \frac{1 + S_p}{x_i + x_p} M_{ip}. \]
Thus when acting on $f$

$$(T_i^{(B)})^2 = 4(x_i \hat{T}_i)^2 + (4a + 2)\hat{T}_i + \frac{4}{\alpha} \sum_{p \neq i} \frac{1}{x_i + x_p} M_{ip} x_i \hat{T}_i. \quad (4.9)$$

To simplify further, note that

$$(x_i \hat{T}_i)^2 = x_i([\hat{T}_i, x_i] + x_i \hat{T}_i) \hat{T}_i \quad (4.10)$$

and evaluate the commutator:

$$[\hat{T}_i, x_i] = \frac{1}{2x_i} + \frac{1}{\alpha} \sum_{p \neq i} \frac{1}{x_i + x_p} M_{ip}. \quad (4.11)$$

The stated result (4.7) follows by substituting (4.11) in (4.10), substituting the result in (4.9) and simplifying.

The analogue of Lemma 3.1 is given by the following result.

**Lemma 4.2** Let $B_i := \frac{1}{4}(T_i^{(B)})^2$ and let

$$\hat{\xi}_j := \alpha x_j^2 \hat{T}_j + (1 - N) + \sum_{p > j} M_{jp} \quad (4.12)$$

be the Cherednik operator (2.7) with the substitution $y_j = x_j^2$ ($j = 1, \ldots, N$). We have

$$[\hat{\xi}_j, B_i] = B_i M_{ij}, \quad i < j$$

$$[\hat{\xi}_j, B_i] = B_j M_{ji}, \quad i > j$$

$$[\hat{\xi}_j, B_j] = -\alpha B_j - \sum_{p < j} M_{jp} B_j - \sum_{p > j} B_j M_{jp}. \quad (4.13)$$

**Remark** Although the commutators in Lemma 3.1 and Lemma 4.2 have the same structure, note that in Lemma 3.1 the operator $T_i$ occurs while in Lemma 4.2 it is the operator $(T_i^{(B)})^2$ which occurs. Moreover, it seems that there is no simple formula for $[\hat{\xi}_j, T_i^{(B)}]$. 

**Proof** First consider the case $i < j$. From Lemma 4.1 we have

$$[\hat{\xi}_j, B_i] = [\hat{\xi}_j, x_i^2 \hat{T}_i^2 + (a + 1)\hat{T}_i + \frac{1}{\alpha} \sum_{p \neq i} M_{ip} \hat{T}_i]$$

$$= [\hat{\xi}_j, x_i^2 \hat{T}_i^2 + x_i^2 ([\hat{T}_i, \hat{T}_i] \hat{T}_i + \hat{T}_i [\hat{T}_i, \hat{T}_i])]$$

$$+ (a + 1) [\hat{\xi}_j, \hat{T}_i] + \frac{1}{\alpha} [\hat{\xi}_j, \sum_{p \neq i} M_{ip} \hat{T}_i] + \frac{1}{\alpha} \sum_{p \neq i} M_{ip} [\hat{\xi}_j, \hat{T}_i]. \quad (4.13)$$

Now, for $i < j$ a direct calculation gives $[\hat{\xi}_j, x_i^2] = -x_j^2 M_{ij}$, while Lemma 3.1 gives $[\hat{\xi}_j, \hat{T}_i] = \hat{T}_i M_{ij}$. To evaluate the second last commutator in (4.13) we substitute (4.12), and note that

$$[x_j^2 \hat{T}_j, \sum_{p \neq i} M_{ip}] = (x_j^2 \hat{T}_j - x_i^2 \hat{T}_i) M_{ij}, \quad \text{and} \quad [\sum_{q > j} M_{jq}, \sum_{p \neq i} M_{ip}] = 0,$$
where to obtain the latter result the formula $M_{ij}M_{jp} = M_{jq}M_{pi} = M_{qi}M_{ij}$ has been used. Substituting these results in (1.13) gives the stated result in the case $i < j$. The cases $i > j$ and $i = j$ are very similar, although the latter case requires more manipulation to simplify the final expression.

Using Lemma 4.2 the required analogue of Lemma 3.2 follows.

**Lemma 4.3** We have

$$[\hat{\xi}_j, \sum_{i=1}^{N} (T_{i}^{(B)})^2] = -\alpha (T_{j}^{(B)})^2.$$  

We can use Lemma 4.3 in the same way as Lemma 3.2 was used in the proof of Proposition 3.1 to prove the analogue of Proposition 3.1 in the Laguerre case.

**Proposition 4.1** The non-symmetric Laguerre polynomials $E_{\eta}^{(L)}(x^2, \alpha)$ are simultaneous eigenfunctions of the operators $l_i$ (1.6) with corresponding eigenvalue $\bar{\eta}_i$.

**Remark** Since $\{E_{\eta}^{(L)}\}$ form a basis for analytic functions, it follows from Proposition 4.1 that $\{l_i\}$ mutually commute, a fact that can also be derived directly by using Lemma 4.2 and the fact that the $T_i^{(B)}$ commute, as do the operators $\hat{\xi}_i$.

From (3.12) and (4.4) we see from (4.6) and (1.17) that

$$\sum_{i=1}^{N} l_i = -\alpha \tilde{H}^{(L, Ex)} \big|_{y_j \to x_j^2} - N(N - 1)/2.$$  

(4.14)

Also, analogous to (3.10), forming the sum (2.14) in (1.2) we have

$$\sum_{p} a_{p-1, \kappa} E_{p-1, \kappa}^{(L)}(y; \alpha) = \exp \left( - \left( \tilde{D}_1 + (a + 1) \sum_{j=1}^{N} \frac{\partial}{\partial y_j} \right) \right) J_{\kappa}^{(a)}(y)$$

$$= (-1)^{a+1} |\kappa|! J_{\kappa}^{(a)}(1^N) L_{\kappa}^{a}(y; \alpha)$$  

(4.15)

where the last equality follows from [7, eq. (4.39)]. From (4.14), analogous to (3.17), we have that

$$\prod_{j=1}^{N} (1 + ul_j)$$  

(4.16)

is an eigenoperator of the symmetric generalized Laguerre polynomials $L_{\kappa}^{a}(x^2; \alpha)$ with corresponding eigenvalue (3.18).

In ref. [4, Prop. 4.5] an operator $\tilde{H}_j^{(L)}$ was constructed such that (3.19) holds with $\tilde{H}_j^{(H)}$ replaced by $\tilde{H}_j^{(L)}$ and $H_{\kappa}(y; \alpha)$ replaced by $L_{\kappa}^{a}(y; \alpha)$. It follows that (3.20) holds with $h_j$ replaced by $l_j$ and $\tilde{H}_j^{(H)}$ replaced by $\tilde{H}_j^{(L)}$.

We know from [4] (see also [24]) that $\{L_{\kappa}^{a}\}$ are orthogonal with respect to the inner product

$$\langle f | g \rangle_{(L)} := 2^N \prod_{l=1}^{N} \int_{-\infty}^{\infty} dx_l e^{-x_l^2} |x_l|^{2a+1} \prod_{1 \leq j < k \leq N} |x_k^2 - x_j^2|^{2/\alpha}$$

$$\times f(x_1^2, \ldots, x_N^2) g(x_1^2, \ldots, x_N^2)$$  

(4.17)
Note that the weight function is proportional to $e^{-\beta W(L)}$, where $W(L)$ is given by (1.14), and is thus proportional to the square of the symmetric ground state wave function of (1.17). The orthogonality can be deduced in the present setting by first checking (see subsequent lemma) that the $l_i$, and thus the eigenoperator (4.16), are self-adjoint with respect to (4.17) and recalling that the eigenvalues of (4.16) are distinct.

**Lemma 4.4** We have

$$\langle f | \hat{T}_i g \rangle^{(L)} = \langle (1 - \frac{a}{x_i^2} - \hat{T}_i) f | g \rangle^{(L)}$$

and

$$\langle f | l_i g \rangle^{(L)} = \langle l_i f | g \rangle^{(L)}.$$  (4.19)

**Proof** The result (4.18) follows from the explicit formula (4.8) and integration by parts. From (4.18), (4.11) and (4.12) we find

$$\langle f | \hat{\xi}_i g \rangle^{(L)} = \langle (\hat{\xi}_i + \alpha(x_i^2 - a - 2x_i^2 \hat{T}_i - 1 - \frac{1}{\alpha} \sum_{p \neq i} M_{ip}) f | g \rangle^{(L)}.$$  (4.20)

Also, analogous to (3.22) we have

$$\langle f | T_i^{(B)} g \rangle^{(L)} = \langle (2x_i - T_i^{(B)}) f | g \rangle^{(L)}$$

and thus

$$\langle f | (T_i^{(B)})^2 g \rangle^{(L)} = \langle (4x_i^2 - 2x_i T_i^{(B)} - 2T_i^{(B)} x_i + (T_i^{(B)})^2) f | g \rangle^{(L)}.$$  (4.22)

But from the proof of Lemma 4.1 we know that when acting on functions even in $x_1^2, \ldots, x_N^2$,

$$x_i T_i^{(B)} = 2x_i^2 \hat{T}_i$$

and from the working in the same proof we can compute that

$$T_i^{(B)} x_i = 2x_i^2 \hat{T}_i + 2(a + 1) + \frac{2}{\alpha} \sum_{p \neq i} M_{ip}.$$  

Substituting these formulas in (4.22) and subtracting $\alpha/4$ times the result from (4.20) gives the required result (4.19).

Note that the operator (2.17) with the $\hat{D}_j$ replaced by $l_j$ is a self-adjoint (with respect to (4.17)) eigenoperator of the $E_{\eta}^{(L)}(x^2, \alpha)$ which separates the eigenvalues. It follows that $\{E_{\eta}^{(L)}\}$ is an orthogonal set with respect to the inner product (4.17). The orthogonality of $\{L_{\eta}^{(a)}\}$ also follows from this fact and the expansion (4.15).

### 4.2 Generalized Laguerre polynomials with prescribed symmetry

The theory here is analogous to that for the generalized Hermite polynomials with prescribed symmetry. Generalized Laguerre polynomials with prescribed symmetry, $S_{(\rho,\mu)}^{(L)}(y, \alpha)$ say, are defined as the eigenfunctions of (1.17) given by the exponential operator formula

$$S_{(\rho,\mu)}^{(L)}(y, \alpha) = \exp \left( - (\tilde{D}_1 + (a + 1) \sum_{j=1}^{N} \frac{\partial}{\partial y_j}) \right) S_{(\rho,\mu)}(y, \alpha).$$  (4.23)
The operator (2.23) with each $\hat{D}_j$ replaced by $l_j$ is an eigenoperator for each $S_{(\rho,\mu)}^{(L)}(y,\alpha)$ and the eigenvalues are distinct for distinct members of $\{S_{(\rho,\mu)}^{(L)}\}$. From this we see that $\{S_{(\rho,\mu)}^{(L)}\}$ is an orthogonal set with respect to the inner product (4.17).

4.3 Some special generalized Laguerre polynomials with prescribed symmetry

For $\eta =: P^{-1}\kappa$ given by (2.24) and the inequality (2.23) satisfied the analogue of the conjecture (2.23) in the Laguerre case states

$$S_{\eta}^{(L)}(y,\alpha) = A_{\eta}^{(L)} \prod_{\gamma=1}^{p} \prod_{1 \leq j < k \leq N_{\gamma}} (y_{\gamma}^{(j)} - y_{\gamma}^{(k)}) L_{\rho}^{\alpha/(p+\alpha)} \left( \frac{\alpha}{\alpha + p} x_1, \ldots, \frac{\alpha}{\alpha + p} x_{N_{\gamma}} ; p + \alpha \right),$$

(4.24)

Here we will verify this statement by showing that the r.h.s. of (4.24) is an eigenfunction of (4.25). Now, when acting on functions symmetric in $\{x_j\}_{j=1,...,N_0}$ and antisymmetric in $\{y_{\gamma}^{(j)}\}_{j=1,...,N_{\gamma}}$ ($\gamma = 1, \ldots, p$) we have

$$\tilde{H}^{(L,x)} = \tilde{H}^{(L,y)} + \tilde{H}^{(L,xy)}$$

(4.25)

where

$$\tilde{H}^{(L,x)} = \sum_{j=1}^{N_0} \left( x_j \frac{\partial^2}{\partial x_j^2} + (a + 1 - x_j) \frac{\partial}{\partial x_j} \right) + \frac{2}{\alpha} \sum_{j \neq k} \frac{x_j}{x_j - x_k} \frac{\partial}{\partial x_j}$$

(4.26)

and $\tilde{H}^{(L,y)}$ is given by (1.17) with $y_1, y_2, \ldots, y_N$ replaced by $y_1^{(1)}, y_2^{(2)}, \ldots, y_N^{(p)}$.

To compute the action of these operators we require the action of the operator which occurs in the summand of (4.27) on monomials. A direct calculation gives the following result.

**Lemma 4.4** Let

$$A_{jk}^{(L)} := \frac{1}{y_j - y_k} \left[ 2 \left( y_j \frac{\partial}{\partial y_j} - y_k \frac{\partial}{\partial y_k} \right) - \frac{y_j + y_k}{y_j - y_k} (1 - M_{jk}) \right].$$

For $\lambda_j \geq \lambda_k$ we have

$$A_{jk}^{(L)} y_j^{\lambda_j} y_k^{\lambda_k} = \begin{cases} \sum_{l=1}^{\lambda_j - \lambda_k} \left( 2(\lambda_j - \lambda_k - l) + 1 \right) y_j^{\lambda_j - l} y_k^{\lambda_k - l + 1}, & \lambda_j - \lambda_k \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Using Lemma 4.4 the argument of the proof of Lemma 2.3 shows that

$$\prod_{\gamma=1}^{p} \Delta(y_{\gamma}^{(\gamma)})$$

is an eigenfunction of $\tilde{H}^{(L,y)}$. This lemma is also used to determine the action of $\tilde{H}^{(L,xy)}$. The strategy is the same as in the proof of Lemma 2.4, so the details will be omitted.
Lemma 4.5  Let $F(x, y)$ be as in Lemma 3.4. We have

$$\tilde{H}^{(L, xy)}F(x, y) = \frac{p}{\alpha} \sum_{j=1}^{N_0} \left( x_j \frac{\partial^2}{\partial x_j^2} + \frac{\partial}{\partial x_j} \right) F(x, y).$$

From the above working we see that, assuming the inequality in Lemma 3.4,

$$\tilde{H}^{(L, Ex)} \prod_{\gamma=1}^p \Delta(y^{(\gamma)}) m_\kappa(x) = \left( \delta^{(L)} + \frac{(1 + \frac{p}{\alpha})}{2} \sum_{j=1}^{N_0} x_j \frac{\partial^2}{\partial x_j^2} + \left( a\alpha/(p + \alpha) + 1 - x_j\alpha/(p + \alpha) \right) \frac{\partial}{\partial x_j} \right) \prod_{\gamma=1}^p \Delta(y^{(\gamma)}) m_\kappa(x).$$

where $\delta^{(L)}$ is the eigenvalue for the action of $\tilde{H}^{(L, y)}$. Since the operator in square brackets is the defining eigenoperator for $L_\kappa^{a\alpha/(p + \alpha)}(\alpha x/(p + \alpha); \alpha)$, it follows that (1.24) is an eigenfunction of $\tilde{H}^{(L, Ex)}$ as required.

In the case $p = 1, q = 0$ we have the analogue of (2.40) and (3.33):

$$S^{(L)}_\mu(y, \alpha) = \Delta(y) \frac{(-1)^{|\kappa|/|\kappa|} C'_{\kappa} \alpha/2}{v_\kappa \alpha/(1 + \alpha)} L_\kappa^{a\alpha/2}(\alpha/(1 + \alpha)), \quad (4.28)$$

which is derived according to the same method.

5 Eigenfunctions of $\tilde{H}^{(J, Ex)}$

5.1 Decomposition

The operators, $\hat{D}^{(J)}_j$ say, which provide a decomposition of $\tilde{H}^{(J, Ex)}$ analogous to the decomposition (2.9) of $\tilde{H}^{(C, Ex)}$ have been given by Bernard et al. [25] and Hikami [26]. We have

$$\hat{D}^{(J)}_j = \hat{D}_j - \frac{1}{\alpha} \sum_{k=1}^N \frac{1 - S_j S_k M_{jk}}{1 - z_j z_k} - (a + \frac{1}{2}) \frac{1 - S_j}{1 - z_j} - (b + \frac{1}{2}) \frac{1 - S_j}{1 + z_j} + \frac{1}{2} (a + b + 1) \quad (5.1)$$

and

$$\tilde{H}^{(J, Ex)} = \sum_{j=1}^N \left( \hat{D}^{(J)}_j \right)^2 - \frac{1}{4} E_0^{(J)}. \quad (5.2)$$

We want to investigate the eigenfunctions of $\hat{D}^{(J)}_j$ and relate them to the eigenfunctions of $\tilde{H}^{(J, Ex)}$. To begin, we note that $\hat{D}^{(J)}_j$ is self-adjoint with respect to the inner product

$$\langle f | g \rangle^{(J)} := \int_0^{\pi/2} d\phi_1 \cdots \int_0^{\pi/2} d\phi_N |\psi_0^{(J)}|^2 f(z_1^*, \ldots, z_N^*) g(z_1, \ldots, z_N) \quad (5.3)$$

where $z_j = e^{2i\phi_j}$ and

$$\psi_0^{(J)} := \prod_{j=1}^N z_j^{-N-1/2} (z_j - 1)^{a+1/2} (z_j + 1)^{b+1/2} \prod_{j<k} (z_j - z_k)^{1/2} (1 - z_j z_k)^{1/2} \quad (5.4)$$
Note that $|\psi_0^{(J)}|^2$ is proportional to $e^{-\beta W^{(J)}}$ where $W^{(J)}$ is given by (1.13), and is thus proportional to the square of the symmetric ground state wave function of $H^{(J,Ex)}$. The self-adjointness is easily checked upon using the operator identity

$$
\psi_0^{(J)} \hat{D}_{j}^{(J)} (\psi_0^{(J)})^{-1} = z_j \frac{\partial}{\partial z_j} - \frac{1}{\alpha} \left( \sum_{l<j} \frac{z_l}{z_j-z_l} M_{lj} + \sum_{l>j} \frac{z_j}{z_j-z_l} M_{lj} \right) + \frac{1}{\alpha} \sum_{k=1}^{N} \frac{S_j S_k M_{jk}}{1-z_j z_k} + (a+1/2) \frac{S_j}{1-z_j} + (b+1/2) \frac{S_j}{1+z_j}. \quad (5.5)
$$

Also, we remarked in a previous study [7] that $\tilde{H}^{(J,Ex)}$ has a complete set of symmetric polynomial eigenfunctions $G_{\kappa}^{(a,b)}(y;\alpha)$ (the generalized Jacobi polynomials) where $y = \sin^2 \phi = -(z+1/z-2)/4$.

To specify the eigenvalues and eigenfunctions of (5.1), let $\eta$ be an $N$-tuple of non-negative integers as in (1.7), and define the partial order as below (1.7). Let $\epsilon$ be an $N$-tuple with each entry $+1$ or $-1$, and define $\epsilon\eta$ as the $N$-tuple formed from $\epsilon$ and $\eta$ by multiplication of the respective parts. A direct calculation shows

$$
\hat{D}_{j}^{(J)} z^{\epsilon\eta} = e_{j,\epsilon\eta} z^{\epsilon\eta} + \sum_{\eta' < \eta} \epsilon' c_{\epsilon\eta,\epsilon'\eta'} z^{\epsilon'\eta'} \quad (5.6)
$$

where

$$
e_{j,\epsilon\eta} = \epsilon_j \eta_j + \frac{1}{\alpha} \left( - \sum_{l<j} h(\epsilon_l \eta_l - \epsilon_j \eta_j) + \sum_{l>j} h(\epsilon_j \eta_l - \epsilon_l \eta_l) + \sum_{k=1}^{N} h(-\epsilon_j \eta_j - \epsilon_k \eta_k) \right) + \frac{1}{\alpha} (j-1) - (a+b+1) \left( h(-\epsilon_j \eta_j) - \frac{1}{2} \right) \quad (5.7)
$$

with $h(x)$ defined by (2.6). It follows that $\hat{D}_{j}^{(J)}$ has a complete set of Laurent polynomial eigenfunctions such that when the highest weight monomial is $z^{\epsilon\eta}$ the corresponding eigenvalue is $e_{j,\epsilon\eta}$. The fact that each operator $\hat{D}_{j}^{(J)}$ has a unique Laurent polynomial eigenfunction with leading monomial $z^{\epsilon\eta}$ and that the operators $\{\hat{D}_{j}^{(J)}\}$ mutually commute imply each eigenfunction is simultaneously an eigenfunction of all the operators $\hat{D}_{1}^{(J)}, \ldots, \hat{D}_{N}^{(J)}$.

Next we note that the eigenvalues $e_{j,\epsilon\eta}$ and $e_{j,\epsilon\eta}_{|_{\epsilon_p \to -\epsilon_p}}$ are simply related.

**Lemma 5.1** We have

$$
e_{j,\epsilon\eta}_{|_{\epsilon_p \to -\epsilon_p}} = e_{j,\epsilon\eta} \quad (j \neq p), \quad e_{j,\epsilon\eta}_{|_{\epsilon_p \to -\epsilon_p}} = -e_{j,\epsilon\eta} \quad (j = p).
$$

**Proof** This follows directly from (5.7).

A consequence of Lemma 5.1 is that $(\hat{D}_{j}^{(J)})^2$ permits Laurent polynomial eigenfunctions, $E_{\eta}^{(J)}(y)$ say $(y = -(z+1/z-2)/4)$, with leading term $y^q$ and corresponding eigenvalue $(e_{j,\epsilon\eta})^2 = (e_{j,\epsilon})^2$. Now $e_{j,\epsilon\eta} = e_{j,\epsilon} + (a+b+1)/2$ where $e_{j,\epsilon}$ is given by (2.5). From Lemma 2.1 we thus have that $\{(e_{j,\epsilon\eta})^2\}$ is independent of the permutation in the equation $\eta = P\kappa$. 
By following the argument used in the last two paragraphs of Section 2.1 we conclude that

\[(1 + u(\hat{D}_1^{(J)})^2) \ldots (1 + u(\hat{D}_N^{(J)})^2) \tag{5.8}\]
is an eigenoperator for the symmetric (Laurent) polynomial eigenfunctions of (5.2) with leading term proportional to \(y^n\). As remarked below (5.3), these polynomials are the generalized Jacobi polynomials \(G^{(a,b)}_{\kappa}(y; \alpha)\). The corresponding eigenvalue is

\[
\prod_{j=1}^{N} \left(1 + u\left(\kappa_j + (N - j)/\alpha + (a + b + 1)/2\right)^2\right). \tag{5.9}\]

Now since each operator \(\hat{D}_j^{(J)}\) is self-adjoint with respect to the inner product (5.3), the operator (5.8) is also self-adjoint with respect to this inner product. Furthermore, from the inequalities (2.13) each eigenvalue (5.9) is distinct. This immediately implies (as has been proved before [27, 28, in the latter reference an operator equivalent to (5.8) when restricted to acting on symmetric functions of \(y\) is also constructed] that the generalized Jacobi polynomials \(\{G^{(a,b)}_{\kappa}(y; \alpha)\}\) are orthogonal with respect to the inner product (5.3).

Note that the operator (2.17) with \(\hat{D}_j^{(J)}\) replaced by \((\hat{D}_j^{(J)})^2\) can be used to show that \(\{E^{(J)}_{\eta}\}\) is orthogonal with respect to (5.3).

### 5.2 Generalized Jacobi polynomials with prescribed symmetry

From the decomposition (5.2) and the fact that \((\hat{D}_j^{(J)})^2\) and \(\tilde{H}^{(J, \text{Ex})}\) both have unique polynomial eigenfunctions with leading term \(y^n\), we conclude that \(E^{(J)}_{\eta}\) can be specified as the eigenfunction of (5.2) with leading term \(y^n\). To pursue this characterization, it is convenient to introduce the variable \(y = \sin^2 \phi\) in (1.12) and repeat the computation of (1.18). This gives

\[
-\tilde{H}^{(J, \text{Ex})} = \sum_{j=1}^{N} y_j (1 - y_j) \frac{\partial^2}{\partial y_j^2} + (a + 1) \sum_{j=1}^{N} \frac{\partial}{\partial y_j} - (a + b + 2 + \frac{2}{\alpha}(N - 1)) \sum_{j=1}^{N} y_j \frac{\partial}{\partial y_j} + \frac{1}{\alpha} \sum_{j \neq k} \frac{1}{y_j - y_k} \left(2y_j (1 - y_k) \frac{\partial}{\partial y_j} - y_j (1 - y_k) y_j - y_k (1 - M_{jk})\right)

:= U + V \tag{5.10}\]

where

\[
U := -\tilde{H}^{(C, \text{Ex})} - (a + b + 1 + (N - 1)/\alpha) \sum_{j=1}^{N} y_j \frac{\partial}{\partial y_j}, \quad V = \tilde{H}^{(L, \text{Ex})} + \sum_{j=1}^{N} y_j \frac{\partial}{\partial y_j}. \tag{5.11}\]

Following ref. [18] we note that the operator \(U\) is an eigenoperator for the non-symmetric Jack polynomials \(E^{(J)}_{\eta}\) while the operator \(V\) reduces by one the degree of a homogeneous polynomial (c.f. (1.17)). Using these facts and noting \(E^{(J)}_{\eta}(y, \alpha)\) has leading term \(y^n\), we see by proceeding as in the derivation of (3.5) that

\[
E^{(J)}_{\eta}(y, \alpha) = \left(1 - (\hat{U} + V)^{-1} V\right) E^{(J)}_{\eta}(y, \alpha). \tag{5.12}\]
where with $\eta = P^{-1}\kappa$,
\[
\hat{U} = U + \sum_{j=1}^{N} \left( \kappa_j^2 + 2\kappa_j(N - j)/\alpha + \kappa_j(a + b + 1) \right).
\] (5.13)

Note that if this operator is expanded according to (3.6) the series terminates after $|\eta|$ applications of $V$.

The generalized Jacobi polynomials with prescribed symmetry, $S_{(\rho,\mu)}^{(J)}(y, \alpha)$ say, are defined as the eigenfunctions of (5.2) given by the formula
\[
S_{(\rho,\mu)}^{(J)}(y, \alpha) = \left( 1 - (\hat{U} + V)^{-1}V \right) S_{(\rho,\mu)}^{(J)}(y, \alpha).
\] (5.14)

(For any polynomial $p$ which is an eigenfunction of $\hat{U}$, $(1 - (\hat{U} + V)^{-1}V)p$ is an eigenfunction of (5.2)). From (5.14) and (2.22) it follows that
\[
S_{(\rho,\mu)}^{(J)}(y, \alpha) = \sum_{\text{rearrangements}} b_{Q-1}\kappa E_{Q-1}\kappa(y, \alpha),
\] (5.15)

which in turn implies (2.23) is an eigenoperator for $\{S_{(\rho,\mu)}^{(J)}\}$ with $\hat{D}_j$ therein replaced by $(\hat{D}_j^{(J)})^2$. This operator is self-adjoint with respect to (5.3) and separates the eigenvalues, so we conclude that $\{S_{(\rho,\mu)}^{(J)}\}$ is an orthogonal set with respect to the inner product (5.3), with an appropriate change of variables (see ref. [7, eq. 2.17]).

### 5.3 Some special generalized Jacobi polynomials with prescribed symmetry

Here we want to establish the analogue of (2.25) in the Jacobi case:
\[
S_{\eta}^{(J)}(y, \alpha) = A_{\eta}^{(J)} \prod_{\gamma=1}^{p} \prod_{1 \leq j < k \leq N_{\gamma}} \left( y_{j}^{(\gamma)} - y_{j}^{(\gamma)} \right) C_{\rho}^{(u,v)}(x_1, \ldots, x_{N_0}; p + \alpha),
\] (5.16)

where $A_{\eta}^{(J)}$ is some normalization, $\eta$ is given by (2.24), $\rho_1$ satisfies (2.20) and
\[
u := \frac{\alpha}{\alpha + p} \left( \frac{p}{\alpha} + a + 1 \right) - 1, \quad \nu := \frac{\alpha}{\alpha + p} \left( \frac{p}{\alpha} + b + 1 \right) - 1.
\] (5.17)

Now set $N = N_0 + \sum_{\gamma=1}^{p} N_{\gamma}$ and denote the variables $y_1, \ldots, y_N$ as $\{x_j\}_{j=1,\ldots,N_0}$ and $\{y_j^{(\gamma)}\}_{j=1,\ldots,N_{\gamma}} (\gamma = 1, \ldots, p)$. Analogous to (2.27), for $\hat{H}_{(J,Ex)}^{(J,x)}$ acting on functions symmetric in $\{x_j\}$ and anti-symmetric in $\{y_j^{(\gamma)}\}$ we make the decomposition
\[
\hat{H}_{(J,Ex)}^{(J,x)} = \hat{H}_{(J,x)}^{(J)} + \hat{H}_{(J,y)}^{(J)} + \hat{H}_{(J,xy)}^{(J)}
\] (5.18)

where
\[
\hat{H}_{(J,x)}^{(J)} = \sum_{j=1}^{N_0} \left( x_j(1 - x_j) \frac{\partial^2}{\partial x_j^2} + (a + 1) \frac{\partial}{\partial x_j} - (a + b + 2 + \frac{2}{\alpha}(N - 1))x_j \frac{\partial}{\partial x_j} \right)
\] + \frac{2}{\alpha} \sum_{j \neq k} x_j(1 - x_k) \frac{\partial}{\partial x_j} (5.19)
\[
\tilde{H}^{(J,xy)} = \frac{1}{\alpha} \sum_{\gamma=1}^{p} \sum_{j=1}^{N_y} \sum_{k=1}^{N_x} \frac{1}{y_j^{(\gamma)} - x_k} \left[ 2y_j^{(\gamma)}(1 - x_k) \frac{\partial}{\partial y_j^{(\gamma)}} - 2x_k(1 - y_j^{(\gamma)}) \frac{\partial}{\partial x_k} - \frac{y_j^{(\gamma)} + x_k - 2y_j^{(\gamma)} x_k}{y_j^{(\gamma)} - x_k} (1 - M(y_j^{(\gamma)}, x_k)) \right].
\]
(5.20)

and \(\tilde{H}^{(J,y)}\) is given by (1.18) with \(y_1, y_2, \ldots, y_N\) replaced by \(y_1^{(1)}, y_2^{(1)}, \ldots, y_N^{(p)}\). In fact we have that \(\tilde{H}^{(J,y)}\) is a linear combination of \(\tilde{H}^{(C,y)}\) and \(\tilde{H}^{(L,y)}\) and hence is an eigenoperator of \(\prod_{\gamma=1}^{p} \Delta(y^{(\gamma)})\) with eigenvalue \(\delta^{(J)}\) say. Also, from (2.30) and (4.27) we have

\[
\tilde{H}^{(J,xy)} = -\tilde{H}^{(C,xy)} + \tilde{H}^{(L,xy)}
\]

so when acting on \(F(x,y)\) as defined in Lemma 2.4 we see from Lemmas 2.4 and 4.5 that

\[
\tilde{H}^{(J,xy)} = \frac{p}{\alpha} N_0 \sum_{j=1}^{N_y} x_j (1 - x_j) \frac{\partial^2}{\partial x_j^2} + \frac{2}{\alpha} (N^{(z)} - p) \sum_{j=1}^{N_y} x_j \frac{\partial}{\partial x_j} + \frac{p}{\alpha} \sum_{j=1}^{N_y} \frac{\partial}{\partial x_j}.
\]

Hence, for \(\tilde{H}^{(J,Ex)}\) acting on \(F(x,y)\) we have

\[
\tilde{H}^{(J,Ex)} = \delta^{(J)} + \frac{p + \alpha}{\alpha} \mathcal{H}
\]

where \(\mathcal{H}\) is the operator (5.10) with \(N = N_0\), the variables \(y_1, \ldots, y_N\) replaced by \(x_1, \ldots, x_N\), \(M_{jk} = 1\), \(a = u\), \(b = v\) and \(\alpha\) replaced by \(\alpha + p\). The operator \(\mathcal{H}\) is the defining eigenoperator for the generalized Jacobi polynomial in (5.16), thus establishing the validity of that equation.

In the case \(p = 1, q = 0\) we have the analogue of (2.40) and thus another explicit formula:

\[
S_{\mu}^{(J)}(y, \alpha) = \Delta(y) \frac{1}{v_{\kappa \kappa}(\alpha/(1 + \alpha))} G_{\kappa}^{(a,b)}(y; \alpha/(1 + \alpha)).
\]
(5.21)

This result is derived from (1.12) in the same manner as (2.40) is derived from (1.3).

6 Conclusion

The Schrödinger operators (1.4) and (1.16)-(1.18) admit polynomial eigenfunctions of the form given on the r.h.s. of (1.6). The most basic of these polynomials are the non-symmetric eigenfunctions of (1.4), which are referred to as the non-symmetric Jack polynomials. In (3.7), (4.2), (5.12) we give operator formulas which transform the non-symmetric Jack polynomials to the non-symmetric polynomial eigenfunctions of (1.16)-(1.18). The operators (1.4) and (1.16)-(1.18) also admit bases of fully symmetric polynomial eigenfunctions, and polynomial eigenfunctions with a prescribed symmetry. We have established orthogonality of these sets of eigenfunctions with respect to inner products defined as multidimensional integrals, with the corresponding symmetric ground state wave function as the weight function. For the fully symmetric polynomials the orthogonality has previously been established, but for the polynomials with prescribed symmetry this result is new.
We expect the polynomials with prescribed symmetry to be relevant to the calculation of correlation functions in the Calogero-Sutherland model with spin (see e.g. ref. [5] and references therein). For this purpose we require normalization formulas, and an expansion formula expressing the power sum in terms of the Jack polynomials with prescribed symmetry. Regarding the former, we point out that for the special Jack polynomials with prescribed symmetry of Section 2.3, a conjecture for the normalization has been made in ref. [8].

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References

[1] A. P. Polychronakos. Exchange operator formalism for integrable systems of particles. *Phys. Rev. Lett.*, 69:703–705, 1992.

[2] K. Takemura and D. Uglov. Level-0 action of $U_q(\hat{sl}_n)$ on the $q$-deformed Fock spaces. *q-alg/9607031*.

[3] E. M. Opdam. Harmonic analysis for certain representations of graded Hecke algebras. *Acta Math.*, 175:75–121, 1995.

[4] D. Bernard, M. Gaudin, F. D. M. Haldane, and V. Pasquier. Yang-Baxter equation in long-range interacting systems. *J. Phys. A*, 26:5219–5236, 1993.

[5] Y. Kato and Y. Kuramoto. Exact solution of the Sutherland model with arbitrary internal symmetry. *Phys. Rev. Lett.*, 74:1222–1225, 1995.

[6] T. Yamamoto. Multicomponent Calogero model of $B_N$-type confined in harmonic potential. *cond-mat/9508012*.

[7] T. H. Baker and P. J. Forrester. The Calogero–Sutherland model and generalized classical polynomials. *solv-int/9608004*.

[8] P. J. Forrester. Jack polynomials and the multi-component Calogero-Sutherland model. *Int. Jour. Mod. Phys.*, B 10:427–441, 1996.

[9] T. H. Baker and P. J. Forrester. Generalized weight functions and the Macdonald polynomials. *q-alg/9603005*.

[10] I. Cherednik. A unification of the Knizhnik–Zamolodchikov and Dunkl operators via affine Hecke algebras. *Inv. Math.*, 106:411–432, 1991.

[11] S. Sahi. A new scalar product for nonsymmetric Jack polynomials. *q-alg/9608013*.

[12] F. Knop and S. Sahi. A recursion and combinatorial formula for Jack polynomials. Rutgers Uni. preprint.
[13] R. P. Stanley. Some combinatorial properties of Jack symmetric functions. *Adv. in Math.*, 77:76–115, 1989.

[14] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford University Press, Oxford, 2nd edition, 1995.

[15] A. N. Kirillov and M. Noumi. Affine Hecke algebras and raising operators for Macdonald polynomials. [q-alg/9605004].

[16] A. N. Kirillov and M. Noumi. $q$-difference raising operators for Macdonald polynomials and the integrality of transition coefficients. [q-alg/9605005].

[17] H. Awata, Y. Matsuo, and T. Yamamoto. Collective field description of spin Calogero–Sutherland models. *J. Phys. A*, 16:3089–3098, 1996.

[18] K. Sogo. A simple derivation of multivariable Hermite and Legendre polynomials. Kitasato Uni. preprint.

[19] M. Lassalle. Generalized Hermite polynomials: a short survey. Unpublished manuscript.

[20] C. F. Dunkl. Differential-difference operators associated to reflection groups. *Trans. Amer. Math. Soc.*, 311:167–183, 1989.

[21] S. Kakei. Common algebraic structure for the Calogero-Sutherland models. [q-alg/9608009].

[22] C. F. Dunkl. Integral kernels with reflection group invariance. *Canad. J. Math.*, 43:1213–1227, 1991.

[23] C. F. Dunkl. Hankel transforms associated to finite reflection groups. In D. St. Richards, editor, *Contemp. Math.*, volume 138, pages 123–138, 1993.

[24] M. Lassalle. Polynômes de Laguerre généralisés. *C. R. Acad. Sci. Paris, t. Séri es I*, 312:725–728, 1991.

[25] D. Bernard, V. Pasquier, and D. Serban. Exact solution of long-range interacting spin chains with boundaries. *Europhy. Lett.*, 30:301–306, 1995.

[26] K. Hikami. Dunkl operator formalism for quantum many–body problems associated with classical root systems. *J. Phys. Soc. Japan*, 65:394–401, 1996.

[27] R. J. Beerends and E. M. Opdam. Certain hypergeometric series related to the root system BC. *Trans. Amer. Math. Soc.*, 339:581–609, 1993.

[28] J. F. Van Diejen. Commuting difference operators with polynomial eigenfunctions. *Compositio Math.*, 95:183–233, 1995.