Abstract. We introduce, and partially resolve, a conjecture that brings a three-centuries-old derangements phenomenon and its much younger two-decades-old analogue under the same umbrella. Our tools blend combinatorics and analysis in a medley incorporating Inclusion-Exclusion and Tannery’s theorem.

Such permutations are often called derangements, and counting them is a traditional preoccupation of Combinatorics texts.

C.D. Godsil, 1993 [15]

1. INTRODUCTION. The functions in the opening quote are permutations of the set \{1, \ldots, n\} having no fixed points. They are perhaps the first nontrivial combinatorial objects ever counted, having been introduced by Rémond de Montmort in his famous 1708 treatise [32] and determined in its 1713 second edition. Incidentally, Sir Isaac Newton owned a copy of the latter work, a print from which appears as the frontispiece of Bollobás’ monograph [5]. So by now it might be surprising to learn that anybody has anything new to share about derangements.

“Everybody” knows the beautiful appearance of Euler’s number $e$ in their enumeration. As a reminder, the number $d_n$ of derangements of \{1, \ldots, n\} is the closest integer to $n!/e$; equivalently,

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$  

These relations imply that of all the permutations of \{1, \ldots, n\}, about a fraction $1/e$ of them are derangements.

It’s fair to say that most mathematics students see the entrance of $e$ into the discussion of $d_n$ as unexpected, but it is actually quite natural. If we think of assembling a permutation $\pi$ of \{1, \ldots, n\} by letting each $i$ in this domain decide its image, then for the resulting $\pi$ to be a derangement, each $i$ has $n-1$ choices for $\pi(i)$ because it can’t map to itself. If the $n$ decisions had no interdependence (hold it—we know that’s too big an “If”), then the overall proportion of assembled permutations that are deranged would be

$$\left(\frac{n-1}{n}\right)^n = \left(1 - \frac{1}{n}\right)^n \to \frac{1}{e} \quad \text{as } n \to \infty.$$ 

Of course, the requirement that $\pi$ be surjective—in addition merely to moving each domain element $i$—enters dependence into those $n$ decisions. But, especially when $n$ is large, the error in the heuristic thinking is not large, so rigor can be salvaged. Indeed, other recent MONTHLY authors [2] have accomplished this exact feat in an enlightening analysis of proximate probability measures. Though we shall not pursue this angle, the heuristic did play an essential role in formulating Conjecture 1 below.
We shall approach derangements through a graph-theoretic lens. First notice that each permutation \( \pi \) of \( \{1, \ldots, n \} \) can be viewed as a perfect matching in the complete bipartite graph \( K_{n,n} \) (with each \( i \) in the domain matched to its image \( \pi(i) \) across the bipartition). If \( \pi \) happens to be a derangement, then, as a perfect matching, it includes none of the edges of the particular perfect matching \( M = \{11', 22', \ldots, nn'\} \) in \( K_{n,n} \) and thus is a perfect matching in \( K_{n,n} - M \). Conversely, every perfect matching in the latter graph arises from a derangement.

Following Godsil [15], denote the number of perfect matchings in a graph \( G \) by \( \text{pm}(G) \). The preceding remarks give \( \text{pm}(K_{n,n}) = n! \) and \( \text{pm}(K_{n,n} - M) = d_n \), and since any two perfect matchings of \( K_{n,n} \) are isomorphic, the \( M \) in this last identity can be any fixed perfect matching of this graph. These observations, together with (1), show that

\[
\frac{\text{pm}(K_{n,n} - M)}{\text{pm}(K_{n,n})} = \frac{d_n}{n!} \to e^{-1} \quad (\text{as } n \to \infty).
\]  

(2)

Determining \( d_n \) is the “Hat-Check Problem” (see, e.g., [36]), which, as noted in the first paragraph, is more than 300 years old. Its advanced age heightens the surprise when one learns—perhaps right now—that it took almost this long for the following analogue to attract study.

**Kindergartner Problem**

An even number \( 2n \) of kindergartners buddy up for a field trip; at the end of the day, their inexperienced chaperone asks them to buddy up again arbitrarily. What’s the chance that no original buddy pairs persist?

If the children are the vertices of the complete graph \( K_{2n} \), then the original pairing is a perfect matching \( M \) in \( K_{2n} \), and the problem asks about the ratio of \( \text{pm}(K_{2n} - M) \) to \( \text{pm}(K_{2n}) \). With \( (\cdot)!! \) denoting the double factorial function, the latter count is \( \text{pm}(K_{2n}) = (2n - 1)!! \). The matchings counted by \( \text{pm}(K_{2n} - M) \) are called deranged matchings; for comparison with (2), we also use \( D_n \) to denote their count. Brawner [7] conjectured the following analogue of (2):

\[
\frac{\text{pm}(K_{2n} - M)}{\text{pm}(K_{2n})} = \frac{D_n}{(2n - 1)!!} \to e^{-1/2} \quad (\text{as } n \to \infty).
\]  

(3)

Published in 2000, this conjecture arose from an inquiry by the United States Tennis Association concerning the tournament draw for the 1996 U.S. Open. (See [7] for the full story.) The kindergartner-formulation is due to the second author of the present article and first appeared in [26]; see also [27]. The conjecture was proved by Margolius [24] and again by the second author in this Monthly [22].

In their original formulations, these problems weren’t framed to display their similarity so transparently as in (2) and (3). Now that they are, hope springs to put them under one umbrella.

Besides both containing \( 2n \) vertices, the graphs \( K_{n,n} \) and \( K_{2n} \) share a deeper connection, namely being examples of “balanced complete \( r \)-partite” graphs. We spell this out. For a positive integer \( r \), an \( r\text{-partite} \) graph \( G \) is one whose vertex set \( V \) can be partitioned into \( r \) classes, \( V = V_1 \cup \cdots \cup V_r \), so that no edge of \( G \) has both ends in the same partition class. If, moreover, \( G \) contains every conceivable edge \( xy \) with \( x \in V_i \) and \( y \in V_j \) (whenever \( i \neq j \)), then \( G \) is complete \( r\text{-partite} \). Finally, such a graph is \textit{balanced} when each class \( V_i \) contains the same number of vertices. Following
[8], we use $K_{r \times 2n/r}$ to denote the $2n$-vertex balanced complete $r$-partite graph. Thus $K_{n,n} = K_{1 \times n}$ and $K_{2n} = K_{2n \times 1}$.

The latter observation and the enticing similarity between the phenomena (2) and (3) led the second author to apply the heuristic sketched above to general balanced complete $r$-partite graphs. This resulted in the following “linking conjecture,” in which we use standard asymptotic tilde-notation.4

**Conjecture 1 [21].** If $r = r(n) \geq 2$ is integer-valued and divides $2n$, and $M$ is a perfect matching in $K_{r \times 2n/r}$, then

$$\frac{\text{pm}(K_{r \times 2n/r} - M)}{\text{pm}(K_{r \times 2n/r})} \sim e^{-r/(2r-2)} \ (\text{as } n \to \infty).$$

The solutions to the Hat-Check and Kindergartner Problems confirm Conjecture 1 for $r = 2$ and $r = 2n$, respectively. In this article, we give further supporting evidence. Section 3 verifies Conjecture 1 for $r = 3$ and shows that it holds for constant $r \geq 4$ when restricted to certain “nice” perfect matchings $M$. Section 4 settles the conjecture when $r(n)$ is linear in $n$, i.e., when the classes of $K_{r \times 2n/r}$ are of constant size. Up to, but not including, Section 5, our presentation is self-contained. In that short section, we show how a deep result of McLeod [25] allows us to extend the resolution of Conjecture 1 to cases of $r(n)$ dropping to fractional powers $\Omega(n^\delta)$ (for fixed positive $\delta < 1$).2 Thus we leave open essentially just the cases of polylogarithmic $r(n)$ (beyond the general constant $r \geq 4$ noted above).

**Background and methods.** As noted in the introductory paragraph, the study of derangements is generally believed to have been initiated by Montmort [32], who in 1708 described a probabilistic game of coincidences. He analyzed his “Jeu du Treize” by an iterative procedure in 1713, when he also included a solution by his friend Nicolas Bernoulli [1687–1759] (we mention his lifespan to identify the member of this extended mathematical family). Bernoulli’s solution—addressed more recently in [12]—is a classical application of the Principle of Inclusion-Exclusion, which appears as Theorem 2 below.

Other approaches leading to (1) abound. For example, one can deduce combinatorially the recurrence relation $d_n = (n-1)(d_{n-1} + d_{n-2})$ (as did Euler), then solve it iteratively, or by substitution, or by incorporating generating functions. Likewise for the recurrence $d_n = nd_{n-1} + (-1)^n$. Another proof applies the Binomial Inversion Theorem to the identity $n! = \sum_{k=0}^{n} \binom{n}{k} d_k$ (see, e.g., [16, Chapter 5]). Two others derive from the relation $d_n = \int_0^\infty (t-1)^n e^{-t} dt$, which follows from a result of Joni and Rota [19] and Godsil [14] (because $(t-1)^n$ is the “rook polynomial” of a single perfect matching of $n$ edges in an otherwise empty graph—see [15] or, e.g., [13] or [20]). An analogous integral formula for counting perfect matchings in general graphs provides the basis for Lemma 7 below.

Though we’re not attempting to be encyclopedic, the preceding paragraphs give a reasonably complete overview of the basic techniques leading to (1). For more on the math, we refer the reader to the references already cited and to [35] or [36]. Derangements fall in the realm of “permutations with restricted position,” a branch of combinatorics originally explicated by Riordan [33]; this reference also provides some related history. We found the thesis [3] and article [17] similarly helpful in this regard.

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1 Real sequences $(a_n)$ and $(b_n)$ satisfy $a_n \sim b_n$ (as $n \to \infty$) exactly when $\lim_{n \to \infty} (a_n/b_n) = 1$.

2 Positive real sequences $(a_n)$ and $(b_n)$ satisfy $a_n = \Omega(b_n)$, or $a_n \in \Omega(b_n)$, exactly when $\liminf_{n \to \infty} (a_n/b_n) > 0$. 
The latter article presents a generalization of the Hat-Check Problem different from ours. Others have done this, too. David [11] views the problem through the lens of counting certain elements of the symmetric group on \( n \) symbols. In his generalization and treatment, the Poisson probability distribution—and hence powers of \( e \) not unlike ours—appear(s). Penrice [31] considers a ground set \( S \) partitioned into equal-sized parts and determines the limiting probability that a random permutation of \( S \) maps every element of \( S \) outside its partition class. Powers of \( e \) arise in this generalization as well.

Clark [10] introduces a generalization with quite a different character. “Living” on a simple graph \( G = (V, E) \), his graph derangements are bijections \( f \) of \( V \) to itself for which each \( v \in V \) is adjacent to its image \( f(v) \). Graph permutations are defined likewise, except vertices are also allowed to remain fixed. Clark recovers ordinary derangements (resp. permutations) when \( G = K_n \). Aside from establishing some basic properties of this version of derangement, his definition inspired at least a couple of recent studies: [1] and [9]. These explore the ratios of derangement- to permutation-counts in various graphs and digraphs.

Like the published literature, the Internet is replete with information about derangements. For example, the websites [18] and [28] provide detailed historical accounts of Montmort. Not surprisingly, there are also separate Wikipedia® pages on Montmort and derangements. And the numbers \( d_n \), \( D_n \), respectively, of derangements and deranged matchings appear as sequences A000166 [29], A053871 [30] in The On-Line Encyclopedia of Integer Sequences®.

One can observe in many of these earlier sources a shared theme but differing substance from the present paper. These feel tantalizingly close to our work here, but mathematically they stand separately.

The Principle of Inclusion-Exclusion (PIE) furnishes our primary combinatorial tool while Tannery’s theorem proves fruitful on the analysis front. We discuss these results in Section 2, but see [36] and [23], respectively, for thorough treatments. For any omitted graph theory terminology, almost any source will suffice (e.g., [6]).

2. PRELIMINARIES. This section records the main tools just mentioned, starting with PIE. Of the various ways to formulate this principle, the most convenient here is to count the elements in the complement of a finite union of finite sets in terms of those sets’ various intersections. So the initial data includes a finite index set \( I \) and, for each \( t \in I \), a subset \( A_t \) of a finite “universal” set \( U \). With this backdrop, we then have the following

**Theorem 2 (Principle of Inclusion-Exclusion).** The number of elements of \( U \) not contained in any \( A_t \) is given by

\[
\left| \left( \bigcup_{t \in I} A_t \right)^C \right| = \sum_{S \subseteq I} (-1)^{|S|} \left| \bigcap_{t \in S} A_t \right| .
\]

Our other main tool is a useful, though not widely known, fact from analysis. It’s a special case of a famous result—Lebesgue’s Dominated Convergence theorem (see, e.g., [23])—but, especially for those mathematicians working in the “discrete world,” it deserves its own limelight. For the sake of completeness, and maybe a bit of advertising, we include a short proof. The result concerns the legality of interchanging the order of the limit and summation operations when the terms themselves are not assumed to be constants but are converging. Here the initial data includes a sequence \((f_i(\cdot))_{i \geq 1}\) of real-valued functions on \( \mathbb{N} \) together with two real-valued sequences \((f_i)\), \( (g_i)\).
Lemma 3 (Tannery’s theorem). If, for each $i \geq 1$, we have $\lim_{n \to \infty} f_i(n) = f_i$ and $|f_i(n)| \leq M_i$ for all $n \in \mathbb{N}$, and furthermore if $\sum_{i=1}^{\infty} M_i < \infty$, then for any integer-valued sequence $(s_n)$ with $s_n \to \infty$, we have

$$\lim_{n \to \infty} \sum_{i=1}^{s_n} f_i(n) = \sum_{i=1}^{\infty} f_i. \quad (4)$$

This result has various formulations in the literature; we chose the conclusion (4) because it’s the most natural for our applications.

Proof. Observe that $|f_i(n)| \leq M_i$ implies that $|f_i| \leq M_i$. Therefore, by the comparison test, $\sum |f_i|$ converges, and so must $\sum f_i$.

Fix $\varepsilon > 0$. The convergence of $\sum M_i$ implies that there is an integer $N = N(\varepsilon)$ such that

$$\sum_{i=N+1}^{\infty} M_i < \frac{\varepsilon}{4}.$$ 

Now pick $N' = N'(\varepsilon)$ such that whenever $n > N'$, we have $s_n > N$ and

$$|f_i(n) - f_i| < \frac{\varepsilon}{2N}$$

for every $i \in \{1, 2, \ldots, N\}$.

Therefore, if $n > N'$, then

$$\left| \sum_{i=1}^{s_n} f_i(n) - \sum_{i=1}^{\infty} f_i \right| = \left| \sum_{i=1}^{N} (f_i(n) - f_i) + \sum_{i=N+1}^{s_n} f_i(n) - \sum_{i=N+1}^{\infty} f_i \right| \leq \sum_{i=1}^{N} |f_i(n) - f_i| + \sum_{i=N+1}^{s_n} |f_i(n)| + \sum_{i=N+1}^{\infty} |f_i| < N \frac{\varepsilon}{2N} + 2 \sum_{i=N+1}^{\infty} M_i < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$ 

In the proofs of Theorems 5 and 6 (Section 3), we shall need a slightly more general version of Tannery’s theorem: one that disregards the order of the summation index. We close this section with a statement and proof of the generalization. To this end, let us recall that a sequence $(X_n)$ of sets with $X_1 \subset X_2 \subset \cdots$ is monotone increasing and in this event, we have $\lim_{n \to \infty} X_n = \bigcup_{n=1}^{\infty} X_n$. Furthermore, for a countable set $X$, the sum $\sum_{x \in X} f(x)$ is defined only when there is a bijection $g : \mathbb{N} \to X$ such that the series $\sum_{n=1}^{\infty} f(g(n))$ converges absolutely.
Corollary 4 (Tannery’s theorem—unordered index version). Let \((I_n)\) be a monotone increasing sequence of finite subsets of a countable set \(I\) such that \(\lim_{n \to \infty} I_n = I\). If, for each \(t \in I\), we have \(\lim_{n \to \infty} f_t(n) = f_t\) and \(|f_t(n)| \leq M_t\) for all \(n \in \mathbb{N}\), and furthermore if \(\sum_{t \in I} M_t < \infty\), then

\[
\lim_{n \to \infty} \sum_{t \in I_n} f_t(n) = \sum_{t \in I} f_t.
\]

Proof. By the hypotheses on \((I_n)\), there is an enumeration of \(I\) respecting the monotonicity of \((I_n)\), i.e., a bijection \(g : \mathbb{N} \to I\) such that \(\{g(1), g(2), \ldots, g(|I_n|)\} = I_n\) for all \(n \in \mathbb{N}\). Consequently, each \(f_{g(i)}(n)\) converges to \(f_{g(i)}\) as \(n \to \infty\), each \(|f_{g(i)}(n)| \leq M_{g(i)}\) for all \(n \in \mathbb{N}\), and \(\sum M_{g(i)} < \infty\). Therefore we may apply Lemma 3 with \(s_n = |I_n|\) to obtain

\[
\lim_{n \to \infty} \sum_{t \in I_n} f_t(n) = \lim_{n \to \infty} \sum_{i=1}^{|I_n|} f_{g(i)}(n) = \sum_{i=1}^\infty f_{g(i)} = \sum_{t \in I} f_t. \tag*{\blacksquare}
\]

3. INSTANCES OF CONSTANT \(r(n)\). The proof of Conjecture 1 when \(r = r(n) = 3\) sheds lots of light on the cases when \(r\) takes any constant value (dividing \(2n\)), but shadows remain. In this section, we first present that proof (Theorem 5) and then indicate how it lifts to certain other cases of constant \(r\) (Theorem 6).

The hypothesis of \(r\) dividing \(2n\) when \(r = 3\) implies that \(n\) is restricted to values such that \(2n = 6m\) for some positive integer \(m\). In this case, the balanced complete \(r\)-partite graph \(K_{r \times 2n/r} = K_{3 \times 6m/3}\) is the complete tripartite graph \(K_{2m,2m,2m}\). As this notation is not yet too unwieldy, we adopt it for establishing this case of Conjecture 1. (But we draw the line before turning to \(K_{(r-1)m,(r-1)m,\ldots,(r-1)m}\).)

Theorem 5. For any given perfect matching \(M\) in the complete tripartite graph \(K_{2m,2m,2m}\), we have

\[
\lim_{m \to \infty} \frac{\text{pm}(K_{2m,2m,2m} - M)}{\text{pm}(K_{2m,2m,2m})} = e^{-3/4}.
\]

Proof. Let us begin by counting the number of perfect matchings in \(K_{2m,2m,2m}\), whose partition classes we’ll denote by \(X, Y, Z\). Observe that \(M\) has \(3m\) total edges and each partition class is incident to exactly \(2m\) of these edges (as illustrated in Figure 1). Due to the tripartiteness, this implies that in \(M\), there are exactly \(m\) edges between each pair of classes. We imagine building \(M\) step-by-step. Its edges effectively partition the \(2m\) vertices of \(Y\) into two sets, one of size \(m\) (matched with vertices of \(X\)) and one of size \(m\) (matched with vertices of \(Z\)). This can be done in \(\binom{2m}{m}\) ways. The classes \(X\) and \(Z\) are partitioned in a similar fashion. With these three partitions established, there are \(m!\) ways to match the vertices of \(Y\) and \(X\), \(m!\) ways to match those of \(Y\) and \(Z\), and \(m!\) ways to match those of \(X\) and \(Z\) to finish building \(M\). Thus

\[
\text{pm}(K_{2m,2m,2m}) = \binom{2m}{m}^3 (m!)^3. \tag{5}
\]

Now fixing a particular \(M\), we’ll use inclusion-exclusion (Theorem 2) to count the number of perfect matchings in \(K_{2m,2m,2m} - M\). For an edge \(e \in M\), let \(A_e\) be the
family of perfect matchings in $K_{2m,2m,2m}$ that use edge $e$. Thus, for a subset $L \subseteq M$, the intersection $\bigcap_{e \in L} A_e$ is the family of perfect matchings in $K_{2m,2m,2m}$ that include all of the edges of $L$. By Theorem 2, the number of perfect matchings in $K_{2m,2m,2m}$ that do not use any edge of $M$ is

$$\text{pm}(K_{2m,2m,2m} - M) = \sum_{L \subseteq M} (-1)^{|L|} \left| \bigcap_{e \in L} A_e \right| .$$

It’s helpful to expand this sum by specifying the edges of $M$ between pairs of partition classes. Write $M = M_{XY} \cup M_{YZ} \cup M_{XZ}$, where $M_{AB}$ denotes those $M$-edges between classes $A$ and $B$. Thus we have

$$\text{pm}(K_{2m,2m,2m} - M) =$$

$$= \sum_{I \subseteq M_{XY}} \sum_{J \subseteq M_{YZ}} \sum_{K \subseteq M_{XZ}} (-1)^{|I|+|J|+|K|} \left| \bigcap_{e \in I} A_e \cap \bigcap_{f \in J} A_f \cap \bigcap_{g \in K} A_g \right| .$$

A term in this sum counts the number of perfect matchings in $K_{2m,2m,2m}$ that include all edges in $L = I \cup J \cup K$; see Figure 2, where the ellipses capture $L$.

To evaluate these terms, we count the number of ways to build a perfect matching $M'$ that includes the edges of $I \cup J \cup K$. Figure 3 depicts the various parameters in the following discussion. (Therein, if, e.g., $m$ were 17, then the values $i = 11$, $j = 9$, and $k = 7$ would arrange for the numbers of displayed unmatched vertices of $X$, $Y$, and $Z$.) First consider the class $Y$, which has $|I| + |J|$ vertices already matched by edges of $I \cup J$. Since $M'$ must have exactly $m$ edges between each pair of classes, its edges partition the remaining unmatched $2m - |I| - |J|$ vertices of $Y$ into two sets, one of size $m - |I|$ (matched with vertices of $X$) and one of size $m - |J|$ (matched with vertices of $Z$). This can be done in $\binom{2m - |I| - |J|}{m - |I|}$ ways. The classes $Z$ and $X$ are
partitioned in a similar fashion. Once these partitions are set, there are \((m - |I|)!)\ ways to match the as-yet unmatched vertices of \(Y\) and \(X\), \((m - |J|)!)\ ways to match those vertices of \(Z\) and \(Y\), and \((m - |K|)!)\ ways to match those vertices of \(X\) and \(Z\) to finish building \(M'\). Thus a term

\[
\left| \bigcap_{e \in I} A_e \cap \bigcap_{f \in J} A_f \cap \bigcap_{g \in K} A_g \right|
\]

is equal to

\[
\left( 2m - |I| - |J| \right) \left( 2m - |J| - |K| \right) \left( 2m - |K| - |I| \right) \cdot (m - |I|)! (m - |J|)! (m - |K|)!.
\]

Observe that the expression above depends only on the cardinalities of \(I\), \(J\), and \(K\). Therefore we can reindex the sum on the right side of (6) to express \(\text{pm}(K_{2m,2m,2m - M})\) as

\[
\sum_{0 \leq i,j,k \leq m} (-1)^{i+j+k} \binom{m}{i} \binom{m}{j} \binom{m}{k} \binom{2m-i-j}{m-i} \binom{2m-j-k}{m-j} \cdot \binom{2m-k-i}{m-k} (m-i)! (m-j)! (m-k)!
\]

Dividing this sum by \(\text{pm}(K_{2m,2m,2m}) = \left( \frac{2m}{m} \right)^3 (m!)^3\) and simplifying gives

\[
\frac{\text{pm}(K_{2m,2m,2m - M})}{\text{pm}(K_{2m,2m,2m})} = \sum_{0 \leq i,j,k \leq m} (-1)^{i+j+k} \frac{2m-i-j}{m-i} \frac{2m-j-k}{m-j} \frac{2m-k-i}{m-k} \frac{i! j! k!}{(2m)_m (2m)_m (2m)_m}.
\]
To determine the limiting value of this ratio, we aim to apply Corollary 4. Any pair \(a, b \in \{i, j, k\}\) satisfies

\[
\lim_{m \to \infty} \frac{\binom{2m-a-b}{m-a}}{\binom{2m}{m}} = \lim_{m \to \infty} \frac{\prod_{\ell=0}^{a-1} (1 - \ell/m)}{\prod_{\ell=0}^{b-1} (1 - \ell/m)} \frac{\prod_{\ell=0}^{a+b-1} (2 - \ell/m)}{\prod_{\ell=0}^{a+b-1} (2 - \ell/m)} = \frac{1}{2^{a+b}}.
\]

This implies that for all \(i, j, k\), we have

\[
\lim_{m \to \infty} \frac{(-1)^{i+j+k} \binom{2m-i-j}{m-i} \binom{2m-j-k}{m-j} \binom{2m-k-i}{m-k}}{i! j! k!} = \frac{(-1)^{i+j+k} 1}{i! j! k!} \frac{1}{2^{i+j+k}}.
\]

Moreover, the absolute value of the expression inside the limit in (8) is at most \(\frac{1}{i! j! k!}\), and

\[
\sum_{0 \leq i,j,k \leq \infty} \frac{1}{i! j! k!} = \lim_{m \to \infty} \sum_{0 \leq i,j,k \leq m} \frac{1}{i! j! k!} = \lim_{m \to \infty} \left( \sum_{i=0}^{m} \frac{1}{i!} \sum_{j=0}^{m} \frac{1}{j!} \sum_{k=0}^{m} \frac{1}{k!} \right) = e^3 < \infty.
\]
The three assertions starting at (8) (and ending at \( \infty \)) establish the hypotheses of Corollary 4, and therefore

\[
\lim_{m \to \infty} \frac{\text{pm}(K_{2m,2m,2m} - M)}{\text{pm}(K_{2m,2m,2m})} =
\]

\[
= \lim_{m \to \infty} \sum_{0 \leq i,j,k \leq m} \frac{(-1)^{i+j+k}}{i!j!k!} \frac{(2m-i-j)}{m-i} \frac{(2m-j-k)}{m-j} \frac{(2m-k-i)}{m-k}
\]

\[
= \lim_{m \to \infty} \sum_{0 \leq i,j,k \leq m} \frac{(-1)^{i+j+k}}{i!j!k!} \frac{1}{4^{i+j+k}}
\]

\[
= \lim_{m \to \infty} \left( \sum_{i=0}^{m} \frac{(-1)^i}{i!4^i} \sum_{j=0}^{m} \frac{(-1)^j}{j!4^j} \sum_{k=0}^{m} \frac{(-1)^k}{k!4^k} \right)
\]

\[
= e^{-1/4} e^{-1/4} e^{-1/4}
\]

\[
= e^{-3/4}.
\]

As noted at the start of the preceding proof, a perfect matching in a balanced complete tripartite graph always has the same number of edges between each pair of classes. Unfortunately, this property is not shared by a balanced complete \( r \)-partite graph for \( r \geq 4 \). Indeed, consider the complete 4-partite graph with classes \( V_1, V_2, V_3, V_4 \), each of size \( m \). For example, there is a perfect matching with \( m \) edges between \( V_1 \) and \( V_2 \) and \( m \) between \( V_3 \) and \( V_4 \) and consequently no edges between \( V_1 \) and \( V_3 \) (nor between \( V_2 \) and \( V_4 \), etc.).

We’ve arrived at the edge of our Conjecture 1 shadows. In order to generalize Theorem 5—and hence confirm Conjecture 1 for other constants \( r \geq 4 \)—we need to restrict ourselves to perfect matchings with the same number of edges between each pair of classes. We call such perfect matchings balanced. Observe that a complete \( r \)-partite graph \( K_{r \times 2n/\ell} \) has a balanced perfect matching \( M \) if and only if each class is of size \( (r-1)m \) for some positive integer \( m \). In this case, \( M \) has exactly \( m \) edges between each pair of classes, and \( (r-1)m = 2n/\ell \) (so that \( |M| = n = \binom{r}{3} m \)).

We proceed to develop a generalization of Theorem 5 to the graphs \( K_{r \times 2n/\ell} = K_{r \times (r-1)m} \) with constant \( r \geq 3 \) and restricting our counts to balanced perfect matchings. As the details are cumbersome—and don’t offer further illumination—we sketch a proof mirroring that of Theorem 5. We use \( \text{bpm}(G) \) to denote the number of balanced perfect matchings in a graph \( G \).

There are \( \binom{r}{3} \) pairs of classes in \( K_{r \times (r-1)m} \), so following the argument that establishes (5) gives the total number of balanced perfect matchings as

\[
\text{bpm}(K_{r \times (r-1)m}) = \left( \frac{(r-1)m)!}{(m!)^{r-1}} \right)^r \binom{r}{3}.
\]

(9)

Now suppose that \( M \) is a fixed balanced perfect matching. As in the proof of Theorem 5, we use inclusion-exclusion, here to determine \( \text{bpm}(K_{r \times (r-1)m} - M) \). To this end, we count the number of balanced perfect matchings that include edges \( L \subseteq M \) as in (7). In the resulting sum (below), the indices \( x_{ij} \) play the role of \( i, j, k \) in the earlier proof; there, \( (i, j, k) \) would be \( (x_{12}, x_{23}, x_{13}) \). If we sum over all settings of these \( \binom{r}{3} \) indices taking values \( x_{ij} \in \{0, 1, \ldots, m\} \), the analogue expression of (7), which
here counts $\text{bpm}(K_{r \times (r-1)m} - M)$, is
\[
\sum_{X_m} (-1)^{\sum_i x_i} \prod_{i < j} \left( \frac{m}{x_{ij}} \right)^r \frac{(r - 1)m - \sum_{j \neq i} x_{ij}!}{\prod_{j \neq i} (m - x_{ij})!} \prod_{i < j} (m - x_{ij})! \tag{10}
\]
Dividing (10) by (9), simplifying factors, and then applying Corollary 4 gives the following analogue of the final calculation in the proof of Theorem 5:
\[
\lim_{m \to \infty} \frac{\text{bpm}(K_{r \times (r-1)m} - M)}{\text{bpm}(K_{r \times (r-1)m})} = \\
= \lim_{m \to \infty} \sum_{X_m} (-1)^{\sum_i x_i} \prod_{i < j} x_{ij} \left( \frac{1}{r - 1} \right)^{2 \sum_{j < i} x_{ij}} \\
= \lim_{m \to \infty} \prod_{1 \leq i < j \leq r} \sum_{0 \leq x_{ij} \leq m} (-1)^{x_{ij}} (x_{ij})! \left( \frac{1}{(r - 1)^2} \right)^{x_{ij}} \\
= \prod_{1 \leq i < j \leq r} e^{-1/(r-1)^2} \\
= e^{-r/(2r-2)}. \tag{11}
\]

This establishes another special case of Conjecture 1.

**Theorem 6.** If $r \geq 3$ is fixed, and $M$ is a perfect matching in $K_{r \times (r-1)m} = K_{r \times 2n/r}$, then
\[
\lim_{m \to \infty} \frac{\text{bpm}(K_{r \times (r-1)m} - M)}{\text{bpm}(K_{r \times (r-1)m})} = e^{-r/(2r-2)}. \tag{11}
\]

Notice that our invocation of Tannery’s theorem (Corollary 4) in passing to (11) necessitates $r$ being fixed in the statement of Theorem 6. If $r$ were allowed to grow with $n$, then the dependence on $r$ in (11) of the factors $e^{-1/(r-1)^2}$ would be a dependence on $n$ (hence on $m$), and taking the limit leading to (11) would not be justified by Corollary 4 (whose limiting summands are independent of the growing parameter, in this case $m$).

4. **CASES OF LINEAR $r(n)$**. The preceding section progresses toward Conjecture 1 when the number $r$ of partition classes of $K_{r \times 2n/r}$ is constant relative to the total number of vertices $(2n)$. It’s also natural to consider the other extreme, namely when the number of vertices per class is a constant, say, $c \geq 1$. In this case, counting vertices gives $2n = cr$, or $r = 2n/c$. Conversely, consider any case of $r = r(n)$ being a linear function of $n$, say $r = Cn$. Since $r$ must divide $2n$, we have $C \leq 2$, and since the graphs $K_{r \times 2n/r}$ by definition have equal class sizes, we see that the number of vertices per class is $2n/Cn$, a positive constant. (Note, e.g., that if $C$ happens to be $1/17$, then we’re considering 34 vertices per class, and $n \to \infty$ through multiples of 17.) In this section, we prove Conjecture 1 for these cases of linear $r(n)$.

We shall invoke the following identity, which follows from results both of Godsil [14] and of Zaslavsky [37]. The statement mentions the complement $\overline{G}$ of a graph
\(G\), which shares \(G\)'s vertex set and has the complementary edge set \(E(K_{2n}) - E(G)\); it also involves a counting function \(\mu_k(\cdot)\), which gives the number of \(k\)-matchings in its argument, i.e., matchings comprised of exactly \(k\) edges. For the sake of completeness, we include a short proof.

**Lemma 7 [14,37].** The number of perfect matchings in a graph \(G\) on \(2n\) vertices is

\[
\text{pm}(G) = \sum_{k=0}^{n} (-1)^k \mu_k(\overline{G})(2n - 2k - 1)!!.
\]  

**Proof.** With the union of the edge sets of \(G\) and \(\overline{G}\) being \(E(K_{2n})\), it’s useful to consider the set \(M\) of all perfect matchings of \(K_{2n}\). Using inclusion-exclusion (Theorem 2), we count those \(M \in M\) that contain no edge of \(\overline{G}\). So for \(e \in E(\overline{G})\), let \(A_e\) denote the subset of \(M\) each member of which contains \(e\). Then PIE gives

\[
\text{pm}(G) = \left| \left( \bigcup_{e \in E(\overline{G})} A_e \right)^C \right| = \sum_{S \subseteq E(\overline{G})} (-1)^{|S|} \left| \bigcap_{e \in S} A_e \right|.
\]

If an index \(S\) of this sum contains \(k\) edges, then it contributes a nonzero term exactly when \(S\) is a \(k\)-matching in \(\overline{G}\), and there are \(\mu_k(\overline{G})\) of these. Since each such \(S\) spans \(2k\) vertices, the remaining \(2n - 2k\) vertices induce a complete subgraph \(K_{2n - 2k}\) of the parent \(K_{2n}\). Thus there are \((2n - 2k - 1)!!\) ways to complete a perfect matching starting from such an \(S\), and this is the value of the corresponding (unsigned) term. Therefore, if we sum instead over the possible sizes \(k\) of \(S\), we obtain the identity (12).

The expression in (3) establishes the limiting proportion of perfect matchings in \(K_{2n}\) with a perfect matching removed to the total number in \(K_{2n}\). Our next result generalizes this by accounting for the removal of the edges of a \(d\)-regular graph from \(K_{2n}\) (a perfect matching being the \(d = 1\) case). The perhaps unwieldy expression for the vertex degrees arises because it arranges for the complementary (“removed”) graphs to be \(d\)-regular. We view this result mainly as a lemma supporting Corollary 9 below, but it may also be of independent interest.

**Theorem 8.** If \(d\) is a fixed nonnegative integer and \((G_{2n})\) is a sequence of \((2n - d - 1)\)-regular \(2n\)-vertex graphs, then

\[
\lim_{n \to \infty} \frac{\text{pm}(G_{2n})}{\text{pm}(K_{2n})} = e^{-d/2}.
\]  

**Proof.** For convenience, put \(G = G_{2n}\) as noted, the complement \(\overline{G}\) is \(d\)-regular. Applying Lemma 7 to \(G\) and using \(\text{pm}(K_{2n}) = (2n - 1)!!\) shows that the left fraction in (13) is

\[
\frac{\text{pm}(G_{2n})}{\text{pm}(K_{2n})} = \sum_{k=0}^{n} \frac{(-1)^k \mu_k(\overline{G})(2n - 2k - 1)!!}{(2n - 1)!!},
\]

whose limiting value yields to Lemma 3. Toward that goal, we proceed to establish the lemma’s three hypotheses.
The graph $G$ contains $dn$ edges; thus the number $\mu_k(G)$ of its $k$-matchings satisfies the following (crude) upper bound:

$$\mu_k(G) \leq \binom{dn}{k} \leq \frac{d^k}{k!} n^k. \quad (14)$$

On the other hand, we may pick the edges of a $k$-matching one-by-one. There are $dn$ choices for the first edge $e_0$; since each edge is incident to $2d - 2$ others, there are $dn - 2d + 1 > dn - 2d$ choices for the second edge (which also cannot be $e_0$). Continuing in this way and accounting for overlaps among the ruled-out edges—and for the $k!$ ways of arriving at the same $k$-matching—we find that

$$\mu_k(G) \geq \frac{1}{k!} \prod_{i=0}^{k-1} (dn - 2di) \geq \frac{d^k}{k!} \frac{k}{(n - 2(k - 1))^k}. \quad (15)$$

Combining (14) and (15) gives

$$\mu_k(G) \sim \frac{d^k}{k!} n^k \quad \text{(as } n \to \infty \text{ with } k \text{ constant}).$$

Under the same asymptotic condition, we observe that

$$\frac{(2n - 2k - 1)!!}{(2n - 1)!!} = \prod_{i=0}^{k-1} \frac{1}{(2n - 1) - 2i} \sim \frac{1}{(2n)^k}. \quad (16)$$

As $d$ is also constant, these asymptotic relations imply that

$$\lim_{n \to \infty} \frac{(-1)^k \mu_k(G)(2n - 2k - 1)!!}{(2n - 1)!!} = \frac{(-1)^k}{k!} \left( \frac{d}{2} \right)^k. \quad (16)$$

A glance at (14) and a little algebra gives an estimate of the fraction inside the limit:

$$\left| \frac{(-1)^k \mu_k(G)(2n - 2k - 1)!!}{(2n - 1)!!} \right| \leq \frac{d^k}{k!}. \quad (17)$$

Moreover,

$$\sum_{k=0}^{\infty} \frac{d^k}{k!} = e^d < \infty. \quad (18)$$

With the hypotheses (16), (17), and (18) established, we’re ready to invoke Lemma 3:

$$\lim_{n \to \infty} \frac{\text{pm}(G_{2n})}{\text{pm}(K_{2n})} = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{(-1)^k \mu_k(G)(2n - 2k - 1)!!}{(2n - 1)!!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{d}{2} \right)^k = e^{-d/2}. \quad \blacksquare$$
Theorem 8 provides just the tool we need to resolve Conjecture 1 when \( r(n) \) is a linear function of \( n \). For a constant integer \( c \geq 1 \), let us set \( r = \frac{2n}{c} \) as at the start of this section (recalling that \( c \) is then the number of vertices in each partition class of \( K_{r,\frac{2n}{r}} \)). This host graph is thus \( K_{r,c} \), which is \((2n - c)\)-regular, and if we remove a perfect matching \( M \), then the resulting graph is \((2n - c - 1)\)-regular. Therefore Theorem 8 gives

\[
\lim_{n \to \infty} \frac{\text{pm}(K_{r,c} - M)}{\text{pm}(K_{r,c})} = \lim_{n \to \infty} \frac{\text{pm}(K_{r,c} - M)/\text{pm}(K_{2n})}{\text{pm}(K_{r,c})/\text{pm}(K_{2n})} = \frac{e^{-c/2}}{e^{-(c-1)/2}} = e^{-1/2}.
\]

(Technically, we twice applied Theorem 8 to sequences containing the graphs in question as subsequences.) Thus, when \( r \) is linear in \( n \), we have proved the following partial resolution of Conjecture 1.

**Corollary 9.** If \( c \) is a fixed positive integer, then

\[
\lim_{n \to \infty} \frac{\text{pm}(K_{(2n/c)\times c} - M)}{\text{pm}(K_{(2n/c)\times c})} = e^{-1/2}.
\]

**Added in proof:** Having seen Conjecture 1 presented at a conference in 2017, Spiro and Surya took interest and recently produced a manuscript \([34]\) proposing a full proof. At the time of this writing, their manuscript is under review.

5. **REFINEMENT.** As hinted in the preceding section, Lemma 7 reformulates an integral counting identity due to Godsil \([14]\). A deeper analysis of that formula by McLeod \([25]\) led her to a strong generalization of a theorem of Bollobás \([4]\) from the 1980s. One of McLeod’s results takes us substantially further than Corollary 9. A version of \([25, \text{Theorem 17}]\) suitable for our purposes follows. (Again, we relegate the required asymptotic notation to the footnotes.\(^3\))

**Theorem 10** \([25]\). For any given \( \delta > 0 \), if \( 2 \leq d \leq O(n^{1-\delta}) \) and \( G \) is a \((2n - d - 1)\)-regular \( 2n \)-vertex graph, then

\[
\text{pm}(G) = \frac{(2n)!}{2^n n!} \left(1 - \frac{d}{2n}\right)^n \cdot e^{o(1)}.
\]

Notice the refinement of Theorem 8: the leading fraction here is \((2n - 1)!! = \text{pm}(K_{2n})\) while the binomial power is asymptotic to \(e^{-d/2}\).

Theorem 10 shows that for integers \( c = c(n) \) in the range \( 2 \leq c \leq O(n^{1-\delta}) \), we have

\[
\frac{\text{pm}(K_{(2n/c)\times c} - M)}{\text{pm}(K_{(2n/c)\times c})} = \frac{(1 - c/2n)^n}{(1 - (c - 1)/2n)^n} \cdot e^{o(1)} = \left(1 - \frac{1}{2n - c + 1}\right)^n \cdot e^{o(1)}.
\]

And since

\[
\left(1 - \frac{1}{2n - c + 1}\right)^n \sim \exp\left(-\frac{n}{2n - c + 1}\right) \to e^{-1/2} \quad \text{(as } n \to \infty),
\]

our partial resolution of Conjecture 1 is that much closer:

\(^3\)Positive real sequences \((a_n)\) and \((b_n)\) satisfy \(a_n = O(b_n)\), or \(a_n \in O(b_n)\), exactly when \(\limsup_{n \to \infty} (a_n/b_n) < \infty\); alternately, \(a_n = O(b_n)\) if and only if \(b_n = \Omega(a_n)\) (cf. footnote 2). Finally, \(a_n = o(b_n)\) means that \(\lim_{n \to \infty} (a_n/b_n) = 0\).
Corollary 11. If \( c = c(n) \) is a positive integer with \( c \leq O(n^{1-\delta}) \) for a fixed \( \delta > 0 \), then

\[
\lim_{n \to \infty} \frac{\text{pm}(K_{(2n/c) \times c} - M)}{\text{pm}(K_{(2n/c) \times c})} = e^{-1/2}.
\]

(Of course, the case \( c = 1 \) is already known from Corollary 9.)

6. CONCLUDING REMARKS. As mentioned in the Introduction, Conjecture 1 had its genesis in a heuristic but not rigorous argument. As such, its appeal stemmed from the goal of giving the Hat-Check and Kindergartner phenomena a common explanation. Though compelling in itself, this goal leaned on limited evidence. The new results presented here—Theorems 5, 6, and Corollaries 9, 11—add substance to that evidence. Beyond its \( r = 2 \) and \( r = 2n \) cases (the standard derangement and deranged matching instances), Conjecture 1 is now settled for \( r = 3 \), for balanced cases of constant \( r \geq 4 \), and for \( r(n) \in \Omega(n^{\delta}) \) dividing \( 2n \) (for any fixed \( \delta > 0 \)). Although this leaves open a gap—\( \Omega(1) < r(n) < O(n^{\delta}) \)—we feel some satisfaction in having established these bookends.

Aside from perhaps inspiring further progress toward Conjecture 1, our investigations invite some other avenues of inquiry. We can view our parent graphs \( K_{r \times 2n/r} \) as complete graphs \( K_{2n} \) with the edge sets of \( r \) vertex-disjoint copies of smaller complete subgraphs \( K_{2n/r} \) having been removed. So one variation on our theme could be to study what happens if instead of these \( E(K_{2n/r}) \)-removals, we remove the edge sets of triangles (\( K_3 \)'s) or other regular structures. Our Theorem 8 could provide a tool here because it specifically addresses graphs \( G_{2n} \) obtained by the removal from \( K_{2n} \) of the edges of a regular subgraph. Another variation could focus the counting not on perfect matchings—which are 1-regular subgraphs—but on regular subgraphs of degree exceeding 1. We have not attempted to establish a result analogous to Theorem 8 for this situation.

However, we can share a sharper estimate than one we used in the proof of that theorem. The following improvement of (14) could conceivably be helpful in investigations along the lines alluded above:

\[
\mu_k(G) \leq \frac{2n(2n - 2)(2n - 4) \cdots (2n - 2(k - 1))d^k}{2^k k!} = \binom{n}{k} d^k.
\]

We omit the proof.

Turning the focus back to perfect matching enumeration, we should mention that the graphs \( K_{r \times 2n/r} \) are awfully close to the so-called Turán graphs, which are also complete \( r \)-partite graphs—not necessarily balanced, but with the partition classes as close in size as possible (so pairwise all within a count of one). When \( r \) divides \( 2n \)—i.e., when the graphs \( K_{r \times 2n/r} \) are defined—the two graph classes coincide. But even when \( r \) fails to divide \( 2n \), we believe that a variation of Conjecture 1 for Turán graphs should hold. Let it be Conjecture 2.

Prove and conjecture!

Pál Erdős [1913–1996] (attrib.)

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4Turán graphs \( T_{n,r} \) feature prominently in the important subfield of extremal graph theory, one of the many areas championed by Erdős; see, e.g., [6].
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ORCID
P. Mark Kayll http://orcid.org/0000-0003-0643-5008

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DANIEL JOHNSTON earned a bachelor’s degree from St. John’s College in Santa Fe, NM and a doctorate from Western Michigan University. Currently, he is a Harold L. Dorwart Visiting Assistant Professor at Trinity College in Hartford, CT. He enjoys working in graph theory and explaining to his coauthors why the Detroit Pistons are going to be good next year.

Department of Mathematics, Trinity College, Hartford CT 06106
daniel.johnston@trincoll.edu

MARK KAYLL earned a math degree in the ’80s (Simon Fraser University) and another in the ’90s (Rutgers). Though generally enamored with the subject, he caught a combinatorics bug in 1983 during SFU’s Math 243. Still infected and now into his 60s, Mark remains enthusiastic about the state of our discipline and its stewardship by the younger generations—including his coauthors here. He appreciates a good basketball debate. And he likes banjos.

Department of Mathematical Sciences, University of Montana, Missoula MT 59812
mark.kayll@umontana.edu

CORY PALMER earned a B.A. from Berkeley and a Ph.D. from Central European University. He discovered a love for combinatorics while studying abroad in the Budapest Semesters in Mathematics program in 2002. Outside of mathematics, Cory’s interests include lifting weights and talking about lifting weights (and of course, basketball).

Department of Mathematical Sciences, University of Montana, Missoula MT 59812
cory.palmer@umontana.edu