Coarse-grained distributions and superstatistics

Pierre-Henri Chavanis

October 27, 2018

Laboratoire de Physique Théorique, Université Paul Sabatier, 118, route de Narbonne, 31062 Toulouse, France
E-mail: chavanis@irsamc.ups-tlse.fr

Abstract

We show an interesting connection between non-standard (non Boltzmannian) distribution functions arising in the theory of violent relaxation for collisionless stellar systems (Lynden-Bell 1967) and the notion of superstatistics recently introduced by Beck & Cohen (2003). The common link between these two theories is the emergence of coarse-grained distributions arising out of fine-grained distributions. The coarse-grained distribution functions are written as a superposition of Boltzmann factors weighted by a non-universal function. Even more general distributions can arise in case of incomplete violent relaxation (non-ergodicity). They are stable stationary solutions of the Vlasov equation. We also discuss analogies and differences between the statistical equilibrium state of a multi-components self-gravitating system and the metaequilibrium (or quasi-equilibrium) states of a collisionless stellar system. Finally, we stress the important distinction between entropies, generalized entropies, relative entropies and $H$-functions. We discuss applications of these ideas in two-dimensional turbulence and for other systems with long-range interactions.

1 Introduction

Recently, several researchers have questioned the “universality” of the Boltzmann distribution in physics. This problem goes back to Einstein himself who did not accept Boltzmann’s principle $S = k \ln W$ on a general scope because he argued that the statistics of a system ($W$) should follow from its dynamics and cannot have a universal expression [1, 2]. In 1988, Tsallis introduced a generalized form of entropy in an attempt to describe complex systems [3]. This was the starting point for several generalizations of thermodynamics, statistical mechanics and kinetic theories (see, e.g., [4]). A lot of experimental and numerical studies (in an impressive number of domains of physics) has then shown that complex systems exhibit non-standard distributions and that, in many cases, they can be fitted by Tsallis $q$-distributions [5]. However, there also exists physical systems (like those that we shall consider here) that are described neither by Boltzmann nor by Tsallis distributions.

An important question is to understand why non-standard distributions and generalized entropies emerge in a system. We have argued that non-standard distributions arise when microscopic constraints are in action [6]. They sometimes appear as “hidden constraints” inaccessible to the observer. For “simple systems”, the energetically accessible microstates are
equiprobable and a standard combinatorial analysis leads to the Boltzmann entropy. Then, the equilibrium distribution (most probable macrostate) maximizes the Boltzmann entropy at fixed macroscopic constraints (mass, energy,...). For “complex systems”, the a priori accessible microstates are not equiprobable, some being even forbidden, contrary to what is postulated in ordinary statistical mechanics. The non-equiprobability of microstates can be due to microscopic constraints (of various origin) that affect the dynamics. In certain cases, the microscopic constraints can be dealt with by using a generalized form of entropy. In principle, this entropy \( S = \ln W' \) should be obtained from a counting analysis by assuming that the microstates which satisfy the macroscopic constraints and the microscopic constraints are equiprobable. An example of microscopic constraints is provided by the Pauli exclusion principle in quantum mechanics which prevents two fermions with the same spin to occupy the same site in phase space. Because of this constraint, the Boltzmann entropy is replaced by the Fermi-Dirac entropy which puts a bound \( f(x, v) \leq \eta_0 \) on the maximum value of the distribution function. In this example, the exclusion principle is explained by quantum mechanics so it has a fundamental origin. Another example is when the particles are subject to an excluded volume constraint. In simplest models (e.g., a lattice model), this is accounted for by introducing a Fermi-Dirac type entropy in physical space which puts a bound \( \rho(x) \leq \sigma_0 \) on the maximum value of the spatial density. These entropies can be obtained from a combinatorial analysis which carefully takes into account the fact that two particles cannot be in the same microcell in phase space or in physical space. More generally, we can imagine other situations where some microscopic constraints (not necessarily of fundamental origin) act on the system and lead to non-standard forms of distribution functions and entropies.

Non-Boltzmannian distributions can also emerge when the system does not mix well (for some reason) so that the evolution is non-ergodic. In that case, the system does not sample the a priori energetically accessible phase space uniformly and prefers some regions more than others. The effectively accessible phase space can have a complicated geometrical structure. In many cases, we do not know the nature of the microscopic constraints perturbing the dynamics, so that they act as “hidden constraints” inaccessible to the observer. We just see their effect indirectly because they lead to non-standard distributions. The fact that we do not know these microscopic constraints implies an indetermination in the selection of the entropy functional. For example, the Tsallis entropies \([3]\) can be relevant for a certain type of non-ergodic behaviour when the phase space has a fractal or multifractal structure. This is appropriate in particular for porous media and in the case of weak chaos. In Tsallis generalized thermodynamics, the complexity of mixing is encapsulated in a single parameter \( q \) which indexes the entropies and characterizes the degree of mixing \( (q = 1 \) if the evolution is ergodic). In some cases, it is possible to determine the parameter \( q \) directly from the microscopic dynamics. In more complicated situations, it has to be adjusted to the situation by a fit. It would be interesting to obtain Tsallis form of entropy directly from a counting analysis by assuming that the energetically accessible microstates are equiprobable on a fractal phase space. In that case, Tsallis entropy could be viewed as an entropy on a fractal. One interesting aspect of Tsallis entropy is that it exhibits mathematical properties very close to those possessed by the Boltzmann entropy. Therefore, it represents the most natural extension of the Boltzmann entropy to the case of “complex” systems. However, Tsallis entropy is not expected to describe all types of complex systems. Depending on the constraints acting on the underlying dynamics, there exists situations in which the observed distribution differs from a \( q \)-distribution. In that case, we must consider more general forms of entropy \( S = - \int C(f) dx dv \) where \( C(f) \) is a convex function \([6]\).

Several microscopic models have been constructed to show how non-standard distributions and generalized entropies can emerge in a system. By introducing a kinetic interaction principle (KIP), Kaniadakis \([7]\) has obtained a generalized form of Boltzmann and Fokker-Planck
equations that lead to a wide class of distribution functions at equilibrium. These generalized equations arise when the expression of the transition probabilities is more general than usually considered. This can take into account quantum statistics or non-ideal effects (e.g. excluded volume) that are ignored in the standard derivation of the Boltzmann and Fokker-Planck equations. On the other hand, Borland [8] and Chavanis [6] have introduced generalized stochastic processes and generalized Fokker-Planck equations in which the diffusion coefficient and the friction/drift terms explicitly depend on the concentration of particles. The dynamics of particles described by these stochastic processes has a complex (non-ergodic) phase space structure. These equations lead to non-standard distributions at equilibrium and they are associated with generalized free energy functionals which play the role of Lyapunov functions. Generalized Fokker-Planck equations have also been studied by Frank [9]. In fact, as discussed in Chavanis [6], it is possible to generalize the usual kinetic equations (Boltzmann, Landau, Kramers, Smoluchowski,...) in such a way that they satisfy a H-theorem for an arbitrary form of entropy. Boltzmann, Fermi-Dirac, Bose-Einstein and Tsallis entropies are just special cases of this general formalism. As indicated previously, the generalization of standard kinetic models can be viewed as a heuristic attempt to take into account “hidden constraints” in complex systems. What we are doing, essentially, is to develop an effective thermodynamical formalism (E.T.F.) to accommodate from our lack of complete information on the microscopic dynamics of a complex system.

In a different context, Beck & Cohen [10] have shown how non-standard distributions can arise in a system if an external variable (e.g. the temperature) is allowed to fluctuate. The probability of energy $E$ is then given by a Laplace transform $P(E) = \int_0^{+\infty} f(\beta)e^{-\beta E}d\beta$ where $f(\beta)$ is the distribution of fluctuations that must be regarded as given. When $f(\beta)$ is strongly peaked around a temperature $\beta_0$, the Boltzmann distribution $P(E) = \frac{1}{Z}e^{-\beta_0 E}$ is recovered. Beck & Cohen gave particular examples of non-standard distributions $P(E)$ arising from this formalism and Tsallis & Souza [11] constructed the generalized entropies associated with these non-standard distributions.

At the same time (ignoring the works of Beck & Cohen and Tsallis & Souza), we revived the concept of violent relaxation introduced by Lynden-Bell [12] for collisionless stellar systems described by the Vlasov-Poisson system and we showed how this theory predicts metaequilibrium states characterized by non-standard distribution functions [6] [13]. Assuming complete relaxation (ergodicity), the coarse-grained distribution function (DF) is given by $\bar{f}(\epsilon) = \frac{1}{Z} \int_{-\infty}^{+\infty} \chi(\eta)\eta e^{-\eta(\beta\epsilon)}d\eta$ where the function $\chi(\eta)$ accounts for the conservation of the Casimir integrals and is determined by the initial conditions. In this context, the Casimir integrals play the role of “hidden constraints” because they are not accessible at the coarse-grained scale (which is the scale of observation). Due to the Liouville theorem in $\mu$-space, they can give rise to an effective “exclusion principle” similar to the Pauli principle in quantum mechanics [12] [13]. In particular, the coarse-grained distribution is bounded by the maximum value of the initial (fine-grained) distribution: $\bar{f}(x, v, t) \leq \max_{x,v}\{f(x, v, t = 0)\}$. We gave particular examples of non-standard distributions $\bar{f}(\epsilon)$ arising from this formalism, with emphasis on the Fermi-Dirac distribution [14], and we introduced the notion of “generalized entropies” $S[\bar{f}] = -\int C(\bar{f})d\mathbf{x}d\mathbf{v}$ (in $\bar{f}$-space) associated with these coarse-grained distributions. The same ideas apply in two-dimensional (2D) turbulence where the coarse-grained vorticity is given by $\bar{\omega}(\psi) = \frac{1}{Z(\psi)} \int_{-\infty}^{+\infty} \chi(\sigma)\sigma e^{-\sigma(\beta\epsilon+\alpha)}d\sigma$ [15] [16] [17]. In the case of geophysical flows that are forced at small scale, Ellis et al. [18] interpret $\chi(\sigma)$ as a prior vorticity distribution encoding the statistics of forcing while for freely evolving flows $\chi(\sigma)$ is determined from the initial conditions by the Casimirs. In the point of view of Ellis et al. [18], further discussed in Chavanis [19], the function $\chi(\sigma)$ must be regarded as given and it directly determines the form of generalized entropy $S[\bar{\omega}] = -\int C(\bar{\omega})d\mathbf{x}$ (in $\bar{\omega}$-space) associated with the coarse-grained vorticity field. The
small-scale forcing, encapsulated in the function $\chi(\sigma)$, can be viewed as a “hidden constraint” which affects the structure of the coarse-grained vorticity.

The object of this paper is to emphasize the similarity between the Beck-Cohen superstatistics and the coarse-grained distributions arising in theories of violent relaxation. The point of superstatistics is that experimentally or numerically observed distributions are in general coarse-grained distributions which arise as averages of finer-grained distributions. Therefore, Lynden-Bell’s statistics is a sort of superstatistics. This connection has not been noted previously and we think that it deserves to be pointed out in detail. Furthermore, the notion of generalized entropies that we gave in [6] in the context of the theory of violent relaxation is similar to that given by Tsallis & Souza [11] in relation with the Beck-Cohen superstatistics.

The paper is organized as follows. We first start to emphasize the distinction between the statistical equilibrium state of a $N$-stars system described by the Hamilton equations and the metaequilibrium states of a collisionless stellar system described by the Vlasov equation. To stress the analogies and the differences, we consider a stellar system with a distribution of mass. The statistical equilibrium state is described in Sec. 2 and the theory of violent relaxation is discussed in Sec. 3. The similarities (and differences) between coarse-grained distribution functions and superstatistics is shown in Sec. 3.4. We introduce the notion of generalized entropy $S[\mathcal{F}]$ associated with the coarse-grained distributions in Sec. 3.3. We show that the generalized entropies associated with the coarse-grained DF predicted by Lynden-Bell can never be the Tsallis functional $S_q[\mathcal{F}] = -\frac{1}{q-1} \int \left( \frac{\mathcal{F}^q}{q} - \mathcal{F} \right) d\mathbf{r} d\mathbf{v}$ because Lynden-Bell’s distribution is defined for all energies $\epsilon$ while Tsallis $q$-distribution (with $q > 1$) has a compact support (the distribution function drops to zero at a finite energy). Then, in Sec. 3.5 we insist on the notion of incomplete violent relaxation and on the limitations of Lynden-Bell’s statistical prediction. As the fluctuations weaken as the system approaches equilibrium, it can be trapped in a stationary solution of the Vlasov equation which is not the most mixed state. We interpret Tsallis functional $S_q[\mathcal{F}]$ as a particular $H$-function in the sense of Tremaine, Hénon & Lynden-Bell [20], not as an entropy. We show that the proper form of Tsallis entropy in the context of violent relaxation is a functional $S_q[\rho] = -\frac{1}{q-1} \int \left( \frac{\rho^q}{q} - \rho \right) d\eta d\mathbf{r} d\mathbf{v}$ of the fine-grained distribution $\rho(\mathbf{r}, \mathbf{v}, \eta)$. The maximization of $S_q[\rho]$ at fixed mass, energy and Casimirs is a condition of thermodynamical stability (in a generalized sense). By contrast, the maximization of a $H$-function (e.g., the Tsallis $H$-function) at fixed mass and energy is a condition of nonlinear dynamical stability for a steady state of the Vlasov-Poisson system of the form $\mathcal{F} = \mathcal{F}(\epsilon)$ with $\mathcal{F}(\epsilon) < 0$ [6, 13, 21]. The $H$-functions can be used to construct a wide class of stable models of galaxies which can be an alternative to Lynden-Bell’s prediction in case of incomplete relaxation. Another alternative is to develop a dynamical theory of violent relaxation [16, 22] in order to understand what limits mixing. In that case, non-ergodicity is explained as a decay of the fluctuations of the gravitational field driving the relaxation, not by a complex structure of phase space. Generalized entropies like $S_q[\rho]$ or $C(\rho)$ are not necessary in that approach. Finally, in Sec. 4 we discuss these ideas in the context of 2D turbulence and show that the notions of prior vorticity distributions and relative entropies introduced by Ellis et al. [18] make the analogies with superstatistics much closer than for freely evolving systems.

2 Statistical equilibrium state of a multi-components stellar system

We wish to determine the statistical equilibrium state of a stellar system made of stars with different mass $m_i$. This Hamiltonian system is described by the microcanonical ensemble where the
energy $E$ and the particle numbers $N_i$ (for each species) are fixed. A thermal equilibrium state is established due to the development of stellar encounters which randomize the distribution of particles (“collisional” mixing). Mathematically, this statistical equilibrium state is obtained when the infinite time limit $t \to +\infty$ is taken before the thermodynamic limit $N \to +\infty$ defined in [23, 24]. This statistical approach is adapted to the case of globular clusters whose age is of the same order as the Chandrasekhar relaxation time $t_{\text{relax}} \sim (N/\ln N)t_D$ [25]. We shall determine the most probable distribution of stars at statistical equilibrium by using a combinatorial analysis, assuming that all accessible microstates (with given $E$ and $M_i = N_i m_i$) are equiprobable. To that purpose, we divide the $\mu$-space $\{\mathbf{r}, \mathbf{v}\}$ into a very large number of microcells with size $\hbar$. We do not put any exclusion, so that a microcell can be occupied by an arbitrary number of particles. We shall now group these microcells into macrocells each of which contains many microcells but remains nevertheless small compared to the phase-space extension of the whole system. We call $\nu$ the number of microcells in a macrocell. Consider the configuration $\{n_{ij}\}$ where $n_{ij}$ is the number of particles of species $j$ in the macrocell $i$. Using the standard combinatorial procedure introduced by Boltzmann, the probability of the state $\{n_{ij}\}$, i.e. the number of microstates corresponding to the macrostate $\{n_{ij}\}$, is given by

$$W(\{n_{ij}\}) = \prod_{i,j} \frac{N_j^{n_{ij}}}{n_{ij}!}.$$  

This is the Maxwell-Boltzmann statistics. As is customary, we define the entropy of the state $\{n_{ij}\}$ by

$$S(\{n_{ij}\}) = \ln W(\{n_{ij}\}).$$  

It is convenient here to return to a representation in terms of the distribution function giving the phase-space density of species $j$ in the $i$-th macrocell: $f_{ij} = f_j(\mathbf{r}_i, \mathbf{v}_i) = n_{ij} m_j / \nu \hbar^3$. Using the Stirling formula $\ln n! = n \ln n - n$, we have

$$\ln W(\{n_{ij}\}) = -\sum_{i,j} n_{ij} \ln n_{ij} = -\sum_{i,j} \nu \hbar^3 \frac{f_{ij}}{m_j} \ln \frac{f_{ij}}{m_j}.$$  

Passing to the continuum limit $\nu \to 0$, we obtain the usual expression of the Boltzmann entropy for different types of particles

$$S_B = -\sum_i \int \frac{f_i}{m_i} \ln \frac{f_i}{m_i} d^3\mathbf{r} d^3\mathbf{v},$$  

up to some unimportant additive constant. This is the expression used by Lynden-Bell & Wood [26] in their thermodynamical description of “collisional” stellar systems (globular clusters). Assuming ergodicity, the statistical equilibrium state, corresponding to the most probable distribution of particles, is obtained by maximizing the Boltzmann entropy (4) while conserving the mass of each species

$$M_i = \int f_i d^3\mathbf{r} d^3\mathbf{v},$$  

and the total energy

$$E = \frac{1}{2} \int f v^2 d^3\mathbf{r} d^3\mathbf{v} + \frac{1}{2} \int \rho \Phi d^3\mathbf{r}.$$  

5
where \( f(r, v) = \sum_i f_i(r, v) \) is the total distribution function and \( \rho = \int f d^3v \) the total density. The gravitational potential is determined by the Poisson equation

\[
\Delta \Phi = 4\pi G \rho.
\]

Introducing Lagrange multipliers and writing the variational principle in the form

\[
\delta S_B - \beta \delta E - \sum_i \alpha_i \delta M_i = 0,
\]

we get

\[
f_i = A_i e^{-\beta m_i (v^2/2 + \Phi)}.
\]

The total distribution function is therefore given by

\[
f = \sum_i A_i e^{-\beta m_i (v^2/2 + \Phi)}.
\]

It is a superposition of Maxwell-Boltzmann distributions with equal temperature \( k_B T = 1/\beta \) and different mass \( m_i \). According to the theorem of equipartition of energy, the mean squared velocity of species \( i \) decreases with mass such that

\[
\langle v^2 \rangle_i = \frac{\int e^{-\beta m_i (v^2/2 + \Phi)} v^2 d^3v}{\int e^{-\beta m_i (v^2/2 + \Phi)} d^3v} = \frac{3 k_B T}{m_i}.
\]

Therefore, heavy particles have less velocity dispersion to resist gravitational attraction so they preferentially orbit in the inner region of the system. This leads to mass segregation. The effect of mass segregation can also be appreciated by writing the distribution function (9) in the form

\[
f_i(\epsilon) = C_{ij} [f_j(\epsilon)]^{m_i/m_j},
\]

where \( C_{ij} = A_i/A_j^{m_i/m_j} \) is a constant independent on the individual energy \( \epsilon = v^2/2 + \Phi \). On the other hand, developing a kinetic theory for a multi-components self-gravitating system, one obtains the multi-species Landau equation

\[
\frac{\partial f_i}{\partial t} + v \cdot \nabla f_i + F \cdot \nabla f_i = \frac{\partial}{\partial \nu} \sum_j \int K^{\mu \nu} \left( m_j f_j \frac{\partial f_i}{\partial \nu^\prime} - m_i f_i \frac{\partial f_j}{\partial \nu^\prime} \right) d^3v',
\]

\[
K^{\mu \nu} = 2\pi G^2 \frac{1}{u} \ln \Lambda \left( \delta^{\mu \nu} - \frac{u^\mu u^\nu}{u^2} \right),
\]

where \( u = v - v' \) is the relative velocity of the particles involved in an encounter, \( \ln \Lambda = \int_0^{+\infty} dk/k \) is the Coulomb factor (regularized with appropriate cut-offs) and we have set \( f_j = f_j(r, v', t) \) assuming that the collisions can be treated as local (see Kandrup [27] for a critical discussion of this approximation and formal generalizations). The Landau-Poisson system conserves the total mass of each species of particles and the total energy of the system. It also increases the Boltzmann entropy \( S_B \) monotonically: \( \dot{S}_B \geq 0 \) (H-theorem). The linearly dynamically stable stationary solutions of the Landau-Poisson system are determined by the mean-field Maxwell-Boltzmann distributions \( f \) which are local maxima of the Boltzmann entropy at fixed \( E, N_i \), so they correspond to statistical equilibrium states. We emphasize that the Boltzmann distribution is the only stationary solution of the Landau equation. The problems linked with the absence of strict statistical equilibrium state in self-gravitating systems and the notion of long-lived metastable states are discussed in [24].
3 Violent relaxation of collisionless stellar systems

3.1 The Vlasov-Poisson system

We shall now contrast the statistical equilibrium state of “collisional” stellar systems (globular clusters) to the metaequilibrium, or quasi-equilibrium, states of “collisionless” stellar systems (elliptical galaxies). The distinction between collisional and collisionless dynamics is just a question of timescales. The age of elliptical galaxies is by many orders of magnitude smaller than the Chandrasekhar relaxation time \[25\] so that their evolution is governed by the Vlasov-Poisson system

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0,
\]

(15)

\[
\Delta \Phi = 4\pi G \int f d^3 \mathbf{v},
\]

(16)

where \( \mathbf{F} = -\nabla \Phi \) is the force by unit of mass experienced by a particle. Mathematically, the Vlasov equation is obtained when the \( N \to +\infty \) limit is taken before the \( t \to +\infty \) limit. Indeed, the collision term in Eq. (13) scales as \( 1/N \) in a proper thermodynamic limit \[23\] so that it vanishes for \( N \to +\infty \). The Vlasov equation, or collisionless Boltzmann equation, simply states that, in the absence of encounters, the distribution function \( f \) is conserved by the flow in phase space. This can be written \( df/dt = 0 \) by using the advective derivative. The Vlasov equation can also be obtained from the \( N \)-body Liouville equation by making a mean-field approximation, i.e. the \( N \)-body distribution factors out in a product of \( N \) one-body distributions. We note that the individual mass \( m_i \) of the stars does not appear in the Vlasov equation. Therefore, in the collisionless regime, the evolution of the total distribution function does not depend on how many species of particles exist in the system (unlike the Landau equation). This implies that the collisionless dynamics does not lead to a segregation by mass contrary to the collisional dynamics. It is easy to show that the Vlasov equation conserves the total mass \( M \) and the total energy \( E \) of the system. Furthermore, the Vlasov equation conserves an infinite number of invariants called the Casimir integrals. They are defined by \( I_h = \int h(f) d^3 \mathbf{r} d^3 \mathbf{v} \) for any continuous function \( h(f) \). The conservation of the Casimirs is equivalent to the conservation of the moments of the distribution function denoted

\[
M_n = \int f^n d^3 \mathbf{r} d^3 \mathbf{v}.
\]

The Vlasov-Poisson system also conserves angular momentum and impulse but these constraints will not be considered here. Finally, the Vlasov equation admits an infinite number of stationary solutions whose general form is given by the Jeans theorem \[25\].

3.2 The metaequilibrium state

The Vlasov-Poisson system develops very complex filaments as a result of a mixing process in phase space (collisionless mixing). In this sense, the fine-grained distribution function \( f(\mathbf{r}, \mathbf{v}, t) \) will never reach a stationary state but will rather produce intermingled filaments at smaller and smaller scales. However, if we introduce a coarse-graining procedure, the coarse-grained distribution function \( \overline{f}(\mathbf{r}, \mathbf{v}, t) \) will reach a metaequilibrium state \( \overline{f}(\mathbf{r}, \mathbf{v}) \) on a very short timescale, of the order of the dynamical time \( t_D \). This is because the evolution continues at scales smaller
than the scale of observation (coarse-grained). This process is known as “phase mixing” and “violent relaxation” (or collisionless relaxation) \[25\]. Lynden-Bell \[12\] has tried to predict the metaequilibrium state achieved by the system in terms of statistical mechanics. This approach is of course quite distinct from the statistical mechanics of the \(N\)-body system (exposed in Sec. 2) which describes the statistical equilibrium state reached by a discrete \(N\)-body Hamiltonian system for \(t \to +\infty\). In Lynden-Bell’s approach, we make the statistical mechanics of a field, the distribution function \(f(r, v, t)\) whose evolution is governed by the Vlasov-Poisson system, while in Sec. 2 we made the statistical mechanics of a system of point particles described by Hamilton equations. In the following, we shall summarize the theory of Lynden-Bell and make the connection with the notion of superstatistics.

Let \(f_0(r, v)\) denote the initial (fine-grained) distribution function. We discretize \(f_0(r, v)\) in a series of levels \(\eta\) on which \(f_0(r, v) \simeq \eta\) is approximately constant. Thus, the levels \(\{\eta\}\) represent all the values taken by the fine-grained distribution function. If the initial condition is unstable, the distribution function \(f(r, v, t)\) will be stirred in phase space (phase mixing) but will conserve its values \(\eta\) and the corresponding hypervolumes \(\gamma(\eta) = \int \delta(f(r, v, t) - \eta)d^3r d^3v\) as a property of the Vlasov equation (this is equivalent to the conservation of the Casimirs).

Let us introduce the probability density \(\rho(r, v, \eta)\) of finding the level of phase density \(\eta\) in a small neighborhood of the position \(r, v\) in phase space. This probability density can be viewed as the local area proportion occupied by the phase level \(\eta\) and it must satisfy at each point the normalization condition

\[
\int \rho(r, v, \eta)d\eta = 1. \tag{18}
\]

The locally averaged (coarse-grained) distribution function is then expressed in terms of the probability density as

\[
\bar{f}(r, v) = \int \rho(r, v, \eta)\eta d\eta, \tag{19}
\]

and the associated (macroscopic) gravitational potential satisfies

\[
\Delta \Phi = 4\pi G \int \bar{f}d^3v. \tag{20}
\]

Since the gravitational potential is expressed by space integrals of the density, it smoothes out the fluctuations of the distribution function, supposed at very fine scale, so \(\Phi\) has negligible fluctuations (we thus drop the bar on \(\Phi\)). The conserved quantities of the Vlasov equation can be decomposed in two groups. The mass and energy will be called robust integrals because they are conserved by the coarse-grained distribution function: \(\bar{M} = M\) and \(\bar{E} = E\). Hence

\[
M = \int \bar{f}d^3r d^3v, \tag{21}
\]

\[
E = \int \frac{1}{2}\bar{v}^2d^3r d^3v + \frac{1}{2} \int \bar{f}\Phi d^3r d^3v. \tag{22}
\]

As discussed above, the gravitational potential can be considered as smooth, so we have expressed the energy in terms of the coarse-grained fields \(\bar{f}\) and \(\Phi\) neglecting the internal energy of the fluctuations \(\bar{f}\Phi\). Therefore, the mass and the energy can be calculated at any time of
the evolution from the coarse-grained field \( \mathcal{F} \). By contrast, the moments \( M_n \) with \( n > 2 \) will be called \textit{fragile integrals} because they are altered on the coarse-grained scale since \( \mathcal{F}^n \neq \mathcal{F}^n \). Therefore, only the moments of the fine-grained field \( M_n^{f.g} = M_n[f] = \int \mathcal{F}^n d^3r d^3v \) are conserved, i.e.

\[
(23) \quad M_n^{f.g} = \int \rho(r, v, \eta) \eta^n d^3r d^3v d\eta.
\]

The moments of the coarse-grained field \( M_n^{c.g}[\mathcal{F}] = \int \mathcal{F}^n d^3r d^3v \) are not conserved along the evolution since \( M_n[f] \neq M_n[\mathcal{F}] \). In a sense, the moments \( M_n^{f.g} \) are “hidden constraints” because they are expressed in terms of the fine-grained distribution \( \rho(r, v, \eta) \) and they cannot be measured from the coarse-grained field. They can be only computed from the initial conditions before the system has mixed or from the fine-grained field. Since in many cases we do not know the initial conditions nor the fine-grained field, they often appear as “hidden”. Note that instead of conserving the fine-grained moments, we can equivalently conserve the total hypervolume \( \gamma(\eta) = \int \rho d^3r d^3v \) of each level \( \eta \).

After a complex evolution, we may expect the system to be in the most probable, i.e. most mixed state, consistent with all the constraints imposed by the dynamics (see, however, Sec. 3.5). We define the mixing entropy as the logarithm of the number of microscopic configurations associated with the same macroscopic state characterized by the probability density \( \rho(r, v, \eta) \). To get this number, we divide the macrocells \( (r, r + dr; v, v + dv) \) into \( \nu \) microcells of size \( h \) and denote by \( \eta_{ij} \) the number of microcells occupied by the level \( \eta \) in the \( i \)-th macrocell. Note that a microcell can be occupied only by one level \( \eta \). This is due to the fact that we make the statistical mechanics of a continuous field \( f(r, v, t) \) instead of point mass stars as in Sec. 2. Therefore, we cannot “compress” that field, unlike point-wise particles. A simple combinatorial analysis indicates that the number of microstates associated with the macrostate \( \{ \eta_{ij} \} \) is

\[
(24) \quad W(\{ \eta_{ij} \}) = \prod_j N_j! \prod_i \frac{\nu!}{n_{ij}!},
\]

where \( N_j = \sum_i n_{ij} \) is the total number of microcells occupied by \( \eta_j \) (this is a conserved quantity equivalent to \( \gamma(\eta) \)). We have to add the normalization condition \( \sum_j n_{ij} = \nu \), equivalent to Eq. (18) which prevents overlapping of different levels (we note that we treat here the level \( \eta = 0 \) on the same footing as the others). This constraint plays a role similar to the Pauli exclusion principle in quantum mechanics. Morphologically, the Lynden-Bell statistics (24) corresponds to a 4\textsuperscript{th} type of statistics since the particles are distinguishable but subject to an exclusion principle [12]. There is no such exclusion for the statistical equilibrium of point mass stars since they are free a priori to approach each other, so we can put several particles in the same microcell.

Taking the logarithm of \( W \) and using the Stirling formula, we get

\[
(25) \quad \ln W(\{ \eta_{ij} \}) = - \sum_{i,j} n_{ij} \ln n_{ij} = - \sum_{i,j} \nu h^3 \rho_{ij} \ln \rho_{ij}
\]

where \( \rho_{ij} = \rho(r_i, v_i, \eta_j) = n_{ij}/\nu h^3 \). Passing to the continuum limit \( \nu \to 0 \), we obtain the Lynden-Bell mixing entropy

\[
(26) \quad S_{L.B.}[\rho] = - \int \rho(r, v, \eta) \ln \rho(r, v, \eta) d^3r d^3v d\eta.
\]

Note that the Lynden-Bell entropy can be interpreted as the Boltzmann entropy for a distribution of levels \( \eta \) (including \( \eta = 0 \)). Equation (26) is sometimes called a \textit{collisionless entropy} to
emphasize the distinction with the collisional entropy \[^{(4)}\] of Sec. 2. Assuming ergodicity or “efficient mixing” (which may not be realized in practice, see Sec. 3.5), the statistical equilibrium state is obtained by maximizing \( S[\rho] \) while conserving mass \( M \), energy \( E \) and all the Casimirs (or moments \( M_n \)). We need also to account for the local normalization condition \[^{(18)}\]. This problem is treated by introducing Lagrange multipliers, so that the first variations satisfy
\[
\delta S - \beta \delta E - \sum_{n \geq 1} \alpha_n \delta M_n - \int \zeta(r, v) \delta \left( \int \rho(r, v, \eta) d\eta \right) d^3r d^3v = 0,
\]
where \( \beta \) is the inverse temperature and \( \alpha_n \) the “chemical potential” associated with \( M_n \). The resulting optimal probability density is a Gibbs state which has the form
\[
\rho(r, v, \eta) = \frac{1}{Z} \chi(\eta) e^{-(\beta \epsilon + \alpha) \eta},
\]
where \( \epsilon = \frac{v^2}{2} + \Phi \) is the energy of a star by unit of mass. In writing Eq. \[^{(28)}\], we have distinguished the Lagrange multipliers \( \alpha \) and \( \beta \) associated with the robust integrals \( M \) and \( E \) from the Lagrange multipliers \( \alpha_{n>1} \), associated with the conservation of the fragile moments \( M_{n>1} = \int \rho \eta^n d\eta d^3r d^3v \), which have been regrouped in the function \( \chi(\eta) \equiv \exp(-\sum_{n>1} \alpha_n \eta^n) \). This distinction will make sense in the following. Under this form, we see that the equilibrium distribution of phase levels is a product of a universal Boltzmann factor \( e^{-(\beta \epsilon + \alpha) \eta} \) by a non-universal function \( \chi(\eta) \) depending on the initial conditions. The partition function \( Z \) is determined by the local normalization condition \( \int \rho d\eta = 1 \) leading to
\[
Z = \int \chi(\eta) e^{-\eta (\beta \epsilon + \alpha)} d\eta.
\]
Finally, the equilibrium coarse-grained DF defined by \( \overline{f} = \int \rho \eta d\eta \) can be written
\[
\overline{f} = \frac{\int \chi(\eta) \eta e^{-\eta (\beta \epsilon + \alpha)} d\eta}{\int \chi(\eta) e^{-\eta (\beta \epsilon + \alpha)} d\eta},
\]
or, equivalently,
\[
\overline{f} = -\frac{1}{\beta} \frac{\partial \ln Z}{\partial \epsilon} = F(\beta \epsilon + \alpha) = \overline{f}(\epsilon).
\]
It is straightforward to check that this coarse-grained distribution depending only on the energy \( \epsilon \) is a stationary solution of the Vlasov equation \[^{(25)}\]. Thus, for a given initial condition, the statistical theory of Lynden-Bell selects a particular stationary solution of the Vlasov equation (most mixed) among all possible ones (an infinity!). Incidentally, the fact that the coarse-grained DF should be a stationary solution of the Vlasov equation is not obvious; this depends on the definition of coarse-graining, see \[^{(28)}\]. Specifically, the equilibrium state is obtained by solving the differential equation
\[
\Delta \Phi = 4\pi G \int f_{\alpha_n, \beta} \left( \frac{v^2}{2} + \Phi \right) d^3v,
\]
and relating the Lagrange multipliers \( \alpha_n, \beta \) to the constraints \( M_n, E \). We note that the coarse-grained distribution function \( \overline{f}(\epsilon) \) can take a wide diversity of forms depending on the function \( \chi(\eta) \) determined by the fragile moments (“hidden constraints”). Some examples will be given in Sec. \[^{(34)}\]. In the present context, the function \( \chi(\eta) \) is determined from the constraints \( a \)
indeed, we have to solve the full problem in order to get the expression of \(\chi(\eta)\). In this sense, the constraints associated with the conservation of the fine-grained moments are treated microcanonically. We emphasize that the function \(\bar{f}(\epsilon)\) depends on the detail of the initial conditions unlike in ordinary statistical mechanics where only the mass \(M\) and the energy \(E\) matter. Here, we need to know the value of the fine-grained moments \(M_n^{f,g}\) which are accessible only in the initial condition (or from the fine-grained field) since the observed moments are altered for \(t > 0\) by the coarse-graining as the system undergoes a mixing process \((M_n^{c,g} \neq M_n^{f,g})\). This makes the practical prediction of \(\bar{f}(\epsilon)\) very complicated, or even impossible, since we often do not know the initial conditions in detail (e.g., for the formation of elliptical galaxies). In addition, in many cases, we cannot be sure that the initial condition is not already mixed (coarse-grained). If it has a fine-grained structure, this would change a priori the prediction of the metaequilibrium state.

We note that the coarse-grained DF predicted by Lynden-Bell depends only on the individual energy \(\epsilon\) of the stars. According to the Jeans theorem [25], such distribution functions form just a particular class of stationary solutions of the Vlasov equation, corresponding to spherical stellar systems (they even correspond to a sub-class of spherical systems whose general distribution function depends on energy \(\epsilon\) and angular momentum \(r \times v\)). From this simple fact, it is clear that the statistical theory of violent relaxation is not able to account for the triaxial structure of elliptical galaxies. More general stationary solutions of the Vlasov equation can arise in case of incomplete violent relaxation and they differ from Lynden-Bell’s prediction (see Sec. 3.5). We also note that \(\bar{f}(\epsilon)\) is a monotonically decreasing function of energy. Indeed, from Eqs. (28) and (31), it is easy to establish that

\[
\bar{f}'(\epsilon) = -\beta f^2, \quad f^2 = \int \rho(\eta - f)^2 d\eta > 0,
\]

where \(f^2\) is the centered local variance of the distribution \(\rho(r,v,\eta)\). Therefore, \(\bar{f}(\epsilon) \leq 0\) since \(\beta \geq 0\) is required to make the velocity profile normalizable. Finally, the coarse-grained distribution function satisfies \(\bar{f}(r,v) \leq f_0^{\text{max}}\) where \(f_0^{\text{max}}\) is the maximum value of the initial (fine-grained) distribution function. This inequality can be obtained from Eq. (30) by taking the limit \(\epsilon \to -\infty\) for which \(\bar{f}(\epsilon) \to \eta_{\text{max}} = f_0^{\text{max}}\) and using the fact that \(f(\epsilon)\) is a decreasing function. Of course, the inequality \(0 \leq \bar{f} \leq f_0^{\text{max}}\) is clear from physical considerations since the coarse-grained distribution function locally averages over the fine-grained levels. Since the fine-grained distribution function is conserved by the Vlasov equation, the coarse-grained distribution function is always intermediate between the minimum and the maximum values of \(f_0\). Finally, we note that Lynden-Bell’s distribution (30) does not lead to a segregation by mass since the individual mass of the particles does not appear in the Vlasov equation on which the whole theory is based; however, it leads to a segregation by phase levels \(\eta\).

If the initial DF takes only two values \(f_0 = 0\) and \(f_0 = \eta_0\), the statistical prediction of Lynden-Bell for the metaequilibrium state is

\[
\bar{f} = \frac{\eta_0}{1 + e^{\eta_0(\beta\epsilon + \alpha)}},
\]

which is similar to the Fermi-Dirac distribution [12, 14]. This has to be contrasted from the statistical equilibrium state (for \(t \to +\infty\)) of the one component self-gravitating gas which is the Maxwell-Boltzmann distribution

\[
f = Ae^{-\beta m\epsilon}.
\]

In the dilute limit of Lynden-Bell’s theory \(\bar{f} \ll \eta_0\) (which may be a good approximation for elliptical galaxies, see [12]), the DF (34) becomes

\[
\bar{f} = A'e^{-\beta \eta_0 \epsilon}.
\]
This is similar to the statistical equilibrium state (35) of the $N$-stars system. Therefore, in this approximation, collisional and collisionless relaxation lead to similar distribution functions (the Maxwell-Boltzmann distribution) but with a completely different interpretation, corresponding to very different timescales. To emphasize the difference, note in particular the bar on $f$ in Eq. (36) and the fact that the mass of the individual stars $m$ in (35) is replaced by the value $\eta_0$ of the fine-grained distribution function.

### 3.3 Generalized entropies

We have seen that the most probable local distribution of phase levels $\rho(r,v,\eta)$ maximizes the mixing entropy (26) while conserving mass, energy and all the fine-grained moments. This functional of $\rho$ is the proper form of Boltzmann entropy in the context of violent relaxation. It is obtained by a combinatorial analysis taking into account the specificities of the collisionless evolution. We shall now show that the most probable coarse-grained distribution function $\bar{f}(r,v)$ (which is the function directly accessible to the observations) maximizes a certain functional $S[\bar{f}]$ at fixed mass $M$ and energy $E$. This functional of $\bar{f}$ will be called a “generalized entropy” (in a sense different to that given by Tsallis). It is non-universal and depends on the initial conditions. It is determined indirectly by the statistical theory of Lynden-Bell and cannot be obtained from a combinatorial analysis, unlike $S[\rho]$. Such generalized (non-Boltzmannian) functionals arise because they encapsulate the influence of fine-grained constraints (Casimirs) that are not accessible on the coarse-grained scale. They play the role of “hidden constraints” in our general interpretation of non-standard entropies. We note that the entropic functionals $S[\rho]$ and $S[\bar{f}]$ are defined on two different spaces. The $\rho$-space is the relevant one to make the statistical mechanics of violent relaxation [12, 16]. The $\bar{f}$-space is a sort of projection of the $\rho$-space in the space of directly observable (coarse-grained) distributions.

Since the coarse-grained distribution function $\bar{f}(\epsilon)$ predicted by the statistical theory of Lynden-Bell depends only on the individual energy and is monotonically decreasing, it extremizes a functional of the form

$$S[\bar{f}] = -\int C(\bar{f}) d^3r d^3v,$$

at fixed mass $M$ and energy $E$, where $C(\bar{f})$ is a convex function, i.e. $C'' > 0$. Indeed, introducing Lagrange multipliers and writing the variational principle as

$$\delta S - \beta \delta E - \alpha \delta M = 0,$$

we find that

$$C'(\bar{f}) = -\beta \epsilon - \alpha.$$

Since $C'$ is a monotonically increasing function of $\bar{f}$, we can inverse this relation to obtain

$$\bar{f} = F(\beta \epsilon + \alpha) = \bar{f}(\epsilon),$$

where

$$F(x) = (C')^{-1}(-x).$$

From the identity

$$\bar{f}(\epsilon) = -\beta / C''(\bar{f}),$$

(42)
resulting from Eq. (39), \( f(\epsilon) \) is a monotonically decreasing function of energy (if \( \beta > 0 \)). Thus, Eq. (31) is compatible with Eq. (40) provided that we use the identification (41). Therefore, for any function \( F(x) \) determined by the function \( \chi(\eta) \) in the statistical theory, we can associate to the metaequilibrium state (31) a generalized entropy (37) where \( C(f) \) is given by Eq. (41) or equivalently by

\[
C(f) = -\int f F^{-1}(x) dx.
\]

It can be shown furthermore that the coarse-grained distribution (31) maximizes this generalized entropy at fixed energy \( E \) and mass \( M \) (robust constraints) \(^1\). We note that \( C(f) \) is a non-universal function which depends on the initial conditions. Indeed, it is determined by the function \( \chi(\eta) \) which depends indirectly on the initial conditions through the complicated procedures discussed in Sec. 3.2. In general, \( S[f] \) is not the Boltzmann functional

\[
S_B[f] = -\int f \ln f d^3r d^3v
\]

(except in the dilute limit of the theory) due to fine-grained constraints (Casimirs) that modify the form of entropy that we would naively expect. This is why the metaequilibrium state is described by non-standard distributions (even for an assumed ergodic evolution). The existence of “hidden constraints” (here the Casimir invariants that are not accessible on the coarse-grained scale) is the physical reason for the occurrence of non-standard distributions and “generalized entropies” in our problem. In fact, the distribution is standard (Boltzmann-Gibbs) at the level of the local distribution of fluctuations \( \rho(r, v, \eta) \) (\( \rho \)-space) and non-standard at the level of the macroscopic coarse-grained field \( f(r, v) \) (\( f \)-space).

We emphasize that the generalized entropies, which are maximized by the coarse-grained distributions, are phenomenological in nature. The point here is that generalized entropies arise because we want to phenomenologically extend the maximum entropy principle at the level of coarse-grained distributions.

### 3.4 Connection with superstatistics

We would like now to point out some connections between coarse-grained distribution functions and superstatistics. Setting \( E \equiv \beta \epsilon + \alpha \), we can rewrite the “partition function” (29) in the form

\[
Z(E) = \int_0^{+\infty} \chi(\eta)e^{-\eta E} d\eta.
\]

This is the Laplace transform of \( \chi(\eta) \). Therefore, the partition function \( Z(E) = \hat{\chi}(E) \) can be used as a generating function for constructing the moments of the fine-grained distribution \(^2\). The coarse-grained distribution is given by

\[
\overline{f}(E) = \frac{1}{Z(E)} \int_0^{+\infty} \chi(\eta)\eta e^{-\eta E} d\eta.
\]

We note that the Lynden-Bell statistics has a form similar to the superstatistics \( P(E) = \int_0^{+\infty} f(\beta)e^{-\beta E} d\beta \) of Beck & Cohen \(^3\) provided that we identify the distribution of temperature \( f(\beta) \) to the distribution of phase levels \( \chi(\eta) \). Formally, the distribution \( P(E) \) is expressed as a Laplace transform like the partition function \( Z(E) \). However, physically, one should focus on the coarse-grained distribution \(^4\) as being the superstatistics in the present context rather

\(^1\) This implies that \( f \) is dynamically stable (nonlinearly) via the Vlasov-Poisson system. Our discussion implicitly assumes that the system is confined within a box so as to avoid the infinite mass problem (Sec. 3.5).
than the partition function (44). These coarse-grained distributions do not exactly have the form considered by Beck and Cohen, but this is a minor point. Superstatistics is an idea foremost, not a proposition for a fixed form of average distribution. The real point is that the coarse-grained distributions do arise as averages (of some sort) of fine-grained distributions of Boltzmann’s type and so are superstatistics.

Due to these formal and physical analogies, we can transpose the results of Beck & Cohen [10] to the context of violent relaxation. However, in the present case, the physical distribution is given by

\[ f(E) = -\frac{\partial \ln Z}{\partial E}, \]

instead of \( P(E) \). Therefore, for the same \( f(\beta) \) and \( \chi(\eta) \), the distributions \( P(E) \) and \( f(E) \) will differ because of this logarithmic derivative. In addition, we must require that the distribution \( f(E) \) is integrable, i.e. the spatial density \( \rho = \int f d\nu \) must exist. We note finally that the generalized entropy associated to the coarse-grained distribution \( f(E) \) is determined by the relation

\[ C'(\bar{f}) = -E, \]

where the function \( \bar{f} = f(E) \) is specified by Eq. (46) depending on \( \chi(\eta) \). Therefore, \( C(\bar{f}) \) is obtained by inverting the relation \( \bar{f} = -(\ln Z)'(E) \) and integrating the resulting expression. In mathematical terms, we get the nice formula defining the generalized entropy

\[ C(\bar{f}) = -\int \bar{f} [\ln \hat{\chi}]^{-1}(-x)dx. \]

Interestingly, the notion of generalized entropy that we gave in the context of violent relaxation in [6] is similar to the one given independently by Tsallis & Souza [11] in the context of superstatistics and by Almeida in the context of generalized thermodynamics [31]. Let us now consider particular examples similar to those given by Beck & Cohen [10]. These examples are given essentially to illustrate the fact that different forms of non-standard distributions can emerge on the coarse-grained scale. We do not claim that they have any particular physical meaning (except (ii)). Furthermore, many other examples of distributions and generalized entropies could be constructed.

(i) Uniform distribution: We take \( \chi(\eta) = 1/b \) for \( 0 \leq \eta \leq b \) and \( \chi = 0 \) otherwise. Then

\[ Z(E) = \frac{1}{bE}(1 - e^{-bE}), \]

and

\[ \bar{f}(E) = \frac{1}{E} + \frac{b}{1 - e^{bE}}. \]

This distribution satisfies \( \bar{f}(E) \to b \) for \( E \to -\infty \), \( \bar{f}(0) = b/2 \) and \( \bar{f}(E) \sim E^{-1} \) for \( E \to +\infty \).

Since \( \bar{f} \sim v^{-2} \) for \( v \to +\infty \), the density \( \rho = \int f d\nu \) exists only in \( d = 1 \) dimension.

(ii) 2-levels distribution: We take \( \chi(\eta) = \frac{1}{2} \delta(\eta) + \frac{1}{2} \delta(\eta - b) \). Then

\[ Z(E) = \frac{1}{2}(1 + e^{-bE}), \]

and

\[ \bar{f}(E) = \frac{b}{1 + e^{bE}}. \]
This is similar to the Fermi-Dirac distribution \[12, 14\]. We have $\mathbf{\overline{f}}(E) \rightarrow b$ for $E \rightarrow -\infty$, $\mathbf{\overline{f}}(0) = b/2$ and $\mathbf{\overline{f}}(E) \sim e^{-bE}$ for $E \rightarrow +\infty$. Since $\mathbf{f} \sim e^{-bE}$ for $v \rightarrow +\infty$, the density $\rho = \int f dv$ exists in any dimension. Inverting the relation (52), we get

$$-E = \frac{1}{b} \left[ \ln \mathbf{\overline{f}} - \ln(b - \mathbf{\overline{f}}) \right] = C'(\mathbf{\overline{f}}).$$

After integration, we obtain

$$S[\mathbf{\overline{f}}] = -\int \left\{ \frac{\mathbf{\overline{f}}}{b} \ln \frac{\mathbf{\overline{f}}}{b} + \left( 1 - \frac{\mathbf{\overline{f}}}{b} \right) \ln \left( 1 - \frac{\mathbf{\overline{f}}}{b} \right) \right\} d^3r d^3v,$$

which is similar to the Fermi-Dirac entropy. Note that for this two-levels distribution, the generalized entropy (54) in $f$-space coincides with the mixing entropy (26) in $\rho$-space since $\rho(r, v, \eta) = (\mathbf{\overline{f}}/b)\delta(\eta - b) + \left( 1 - \mathbf{\overline{f}}/b \right)\delta(\eta)$. This is because the distribution of phase levels $\rho(r, v, \eta) = p(r, v)\delta(\eta - b) + p'(r, v)\delta(\eta)$ can be expressed in terms of the coarse-grained distribution function $\mathbf{\overline{f}}(r, v) = p(r, v)b$, using the normalization condition $p + p' = 1$. This is the only case where we have the equivalence between the mixing entropy $S[\rho]$ and the generalized entropy $S[f]$. The fact that the ‘averaged’ Shannon entropy (26) and the generalized entropy (37) are different in general has also been noted by Beck \[32\] in a different context.

(iii) Gamma distribution: We take

$$\chi(\eta) = \frac{1}{b \Gamma(c)} \left( \frac{\eta}{b} \right)^{c-1} e^{-\eta/b}$$

with $c > 0$ and $b > 0$. Note that the case $c = 1$ corresponds to the exponential distribution while $b \rightarrow +\infty$ corresponds to a power law. Then

$$Z(E) = (1 + bE)^{-c}.$$

As noted by Beck & Cohen \[10\], this is similar to Tsallis $q$-distribution (with $q < 1$). However, in our context, the physical distribution is

$$\mathbf{\overline{f}}(E) = \frac{cb}{1 + bE}.$$

It is defined only for $E > -1/b$. Furthermore, $\mathbf{\overline{f}}(E) \sim cE^{-1}$ for $E \rightarrow +\infty$ so that the spatial density exists only in $d = 1$ dimension. Inverting the relation (57), we get

$$-E = \frac{1}{b} \left( 1 - \frac{cb}{\mathbf{\overline{f}}} \right).$$

After integration, we obtain

$$C(\mathbf{\overline{f}}) = \frac{\mathbf{\overline{f}}}{b} - c \ln \frac{\mathbf{\overline{f}}}{b}.$$

Note that the first term can be absorbed in the Lagrange multiplier $\alpha$ associated with the mass conservation so that the relevant generalized entropy is

$$S[\mathbf{\overline{f}}] = \int \ln \mathbf{\overline{f}} d^3r d^3v.$$
It could be called the log-entropy. Note that when \( E = v^2/2 \), the corresponding distribution function \( \ln f^q - f \) is the Lorentzian. In a sense, the log-entropy can be viewed as a continuation of Tsallis entropy for \( q = 0 \) (see Eq. (70)). This suggests to introducing the modified functional

\[
S_q[f] = -\frac{1}{q-1} \int \left( \frac{1}{q} f^q - f \right) d^3r d^3v, \tag{61}
\]

which has properties similar to the Tsallis functional for \( q \neq 0 \) and which reduces to Eq. (60), leading to the distribution \( \ln f^q - f \) for \( q \to 0 \), where \( K \) is a constant. Taking the variations \( \delta S_q \) at fixed mass and energy leads to

\[
f = (1 - (q - 1)E)^{1/(q-1)} \quad \text{which passes to the limit for } q \to 0, \tag{70}
\]

(iv) Gaussian distribution: We take

\[
\chi(\eta) = 2 \left( \frac{\gamma}{\pi} \right)^{1/2} e^{-\gamma \eta^2}, \tag{62}
\]

with \( \gamma > 0 \). Then,

\[
Z(E) = e^{E^2/4\gamma} \text{erfc} \left( \frac{E}{2\sqrt{\gamma}} \right). \tag{63}
\]

The corresponding coarse-grained distribution can be written

\[
\bar{f}(E) = \frac{1}{\sqrt{\gamma}} H \left( \frac{E}{2\sqrt{\gamma}} \right), \quad H(x) = x \left\{ \frac{1}{\sqrt{\pi} xe^{x^2} \text{erfc}(x)} - 1 \right\}. \tag{64}
\]

This distribution satisfies \( \bar{f}(E) \sim -\frac{E}{\gamma} \) for \( E \to -\infty \) and \( \bar{f}(E) \sim E^{-1} \) for \( E \to +\infty \). The density exists only in \( d = 1 \) dimension.

In the examples considered above, only the Fermi-Dirac distribution function is relevant for self-gravitating systems since the density \( \rho = \int f dv \) is not defined for the others in \( d = 3 \) dimensions. However, these examples may still be of interest in physics because the theory of violent relaxation is valid for other systems with long-range interactions described by the Vlasov equation [23]. The foregoing distributions may thus be relevant for one-dimensional systems. They can also be relevant in 2D turbulence (see Sec. 4) where the energy \( \epsilon = v^2/2 + \Phi(r) \) is replaced by the stream function \( \psi(r) \), so that there is no condition of normalization equivalent to \( \int f dv < \infty \).

The \( E^{-1} \) behaviour of \( \bar{f}(E) \) for \( E \to +\infty \) arises because we have assumed that the function \( \chi(\eta) \) is regular at \( \eta = 0 \). In fact, the level \( \eta = 0 \) plays a particular role in the theory because it corresponds to the “vacuum” which has a very large phase space extension and which can mix with the non-zero levels. Therefore, we expect that \( \chi(\eta) \to \chi_0 \delta(\eta) \) for \( \eta \to 0 \). As a consequence, the level \( \eta = 0 \) should be treated specifically, and a more physical form of partition function, which isolates the contribution of \( \eta = 0 \), would be

\[
Z(E) = 1 + \int_{a}^{+\infty} \chi(\eta) e^{-\eta E} d\eta, \tag{65}
\]

where \( a \geq 0 \). Note that we can take \( \chi_0 = 1 \) without restriction of generality so that the value of \( \gamma(0) \), which is infinite, never appears in the theory. If we reconsider example (i) with now \( \chi = 1/(b-a) \) for \( a \leq \eta \leq b \) and \( \chi = 0 \) otherwise, we get

\[
\bar{f}(E) = \frac{e^{-aE} - e^{-bE} + E(\alpha e^{-aE} - e^{-bE})}{E[\alpha(b-a) + e^{-aE} - e^{-bE}]}. \tag{66}
\]
If \( a \neq 0 \) (gap), the DF decreases as \( \overline{f} \sim a/(b - a)E^{-1}e^{-aE} \) for \( E \to +\infty \) and if \( a = 0 \) (no gap) as \( \overline{f} \sim (1/b)E^{-2} \). The density profile \( \rho = \int f \, dv \) is now well-defined in \( d = 3 \). If we reconsider example (ii) with (65), we get

\[
(67) \quad \overline{f}(E) = \frac{cb}{(1 + bE)((1 + bE)^c + 1)},
\]

which decreases as \( \overline{f} \sim cb^{-c}E^{-(c+1)} \). We think that the particularity of the level \( \eta = 0 \) is an important point that deserves further consideration.

### 3.5 Incomplete relaxation, Tsallis entropies and \( H \)-functions

The statistical approach presented previously rests on the assumption that the collisionless mixing is efficient so that the ergodic hypothesis which sustains the statistical theory is fulfilled. In reality, this is not the case. It has been understood since the beginning [12] that violent relaxation is incomplete so that the mixing entropy (26) is not maximized in the whole phase space and real stellar systems are not described by Lynden-Bell’s statistics. In fact, for stellar systems, violent relaxation cannot be complete because there is no maximum entropy state in an unbounded domain. The generalized isothermal distribution functions (30) predicted by Lynden-Bell, when coupled to the Poisson equation, yield density profiles whose mass is infinite (the density decreases as \( r^{-2} \) at large distances). But this mathematical problem is rather independent from the physical reason why violent relaxation is incomplete. Physically, real stellar systems tend towards the maximum entropy state during violent relaxation but cannot attain it because the gravitational potential variations die away before the relaxation process is complete. Thus, for dynamical reasons, the system will not explore the whole phase space ergodically as discussed in [16, 22]. However, since the Vlasov equation admits an infinite number of stationary solutions, the coarse-grained distribution \( \overline{f} \) can be trapped in one of them and remain frozen in that state until collisional effects come into play (on longer timescales). This steady solution is not, in general, the most mixed state (it is only partially mixed) so it differs from Lynden-Bell’s statistical prediction. The concept of incomplete violent relaxation explains why galaxies are more confined than predicted by statistical mechanics (the density profile of elliptical galaxies decreases as \( r^{-4} \) instead of \( r^{-2} \)).

In order to quantify the importance of mixing, Tremaine, Hénon & Lynden-Bell [20] have introduced the notion of \( H \)-functions. They are defined by

\[
(68) \quad H[\overline{f}] = -\int C(\overline{f})d^3r d^3v,
\]

where \( C \) is any convex function. It can be shown that the \( H \)-functions \( H[\overline{f}] \) calculated with the coarse-grained distribution function increase during violent relaxation in the sense that \( H[\overline{f}(r, v, t)] \geq H[\overline{f}(r, v, 0)] \) for \( t > 0 \) where it is assumed that, initially, the system is not mixed so that \( \overline{f}(r, v, 0) = f(r, v, 0) \). This is similar to the \( H \)-theorem in kinetic theory. However, contrary to the Boltzmann equation, the Vlasov equation does not single out a unique functional (the above inequality is true for all \( H \)-functions) and the time evolution of the \( H \)-functions is not necessarily monotonic (nothing is implied concerning the relative values of \( H(t) \) and \( H(t') \) for \( t, t' > 0 \)). Yet, this observation suggests a notion of generalized selective decay principle: among all invariants of the collisionless dynamics, the \( H \)-functions (fragile constraints) tend to increase (\( -H \) decrease) on the coarse-grained scale while the mass and the energy (robust constraints) are approximately conserved. According to this phenomenological principle, we might expect (see however the last paragraph of this section) that the metaequilibrium state
reached by the system as a result of incomplete violent relaxation will maximize a certain \( H \)-function (non-universal) at fixed mass and energy. Repeating the calculations of Sec. 3.3 with \( H[\mathcal{F}] \) instead of \( S[\mathcal{F}] \), the extremization of a \( H \)-function at fixed \( E \) and \( M \) determines a distribution function \( \mathcal{F} = \mathcal{F}(\epsilon) \) with \( \mathcal{F}'(\epsilon) < 0 \) which is a stationary solution of the Vlasov equation (recall that our argument applies to the coarse-grained distribution). Moreover, if the DF maximizes the \( H \)-function at fixed \( E \) and \( M \), then it is nonlinearly dynamically stable with respect to the Vlasov-Poisson system \([20, 13, 21]\). In general, the \( H \)-function \( H^*[\mathcal{F}] \) that is effectively maximized by the system as a result of incomplete violent relaxation (if any) is difficult to predict \([6]\). It depends on the initial conditions (due to the Casimirs) and on the efficiency of mixing. If mixing is complete (as may be the case for systems others than gravitational ones), the \( H \)-function that is maximized at equilibrium is the generalized entropy \([18]\), hence \( H^*[\mathcal{F}] = S[\mathcal{F}] \), and the stationary distribution function is the Lynden-Bell distribution \([31]\). If mixing is incomplete, \( H^*[\mathcal{F}] \) and \( \mathcal{F}(\epsilon) \) can take forms that are not compatible with the expressions \([18]\) and \([31]\) derived in the statistical approach.

In the context of incomplete violent relaxation, the Tsallis functional

\[
S_q[\mathcal{F}] = -\frac{1}{q-1} \int (\mathcal{F}^q - \mathcal{F}) d^3r d^3v,
\]

is a particular \( H \)-function whose maximization at fixed mass and energy leads to distribution functions of the form

\[
\mathcal{F}(\mathbf{r}, \mathbf{v}) = \left[ \mu - \frac{\beta(q-1)}{q} \epsilon \right]^{1/(q-1)}.
\]

These distribution functions characterize stellar polytropes \([25]\). They are particular stationary solutions of the Vlasov equation. For \( q > 1 \), the polytropic distribution functions have a compact support (they vanish at a maximum energy \( \epsilon_{\text{max}} \)) unlike the Lynden-Bell distribution functions \([31]\) whose tails extend to infinity. Stellar polytropes with index \( n \leq 5 \) (where \( n = 3/2 + 1/(q-1) \)) describe confined structures with finite mass, unlike isothermal stellar systems. They have been studied for a long time in astrophysics as simple mathematical models of stellar systems. Unfortunately, pure polytropic distributions do not provide a good model of incomplete violent relaxation for elliptical galaxies \([25]\). An improved model is a composite model that is isothermal in the core (justified by Lynden-Bell’s theory of violent relaxation) and polytropic in the halo (due to incomplete relaxation) with an index \( n = 4 \) \([33, 13]\). Since the maximization principle determining the nonlinear dynamical stability of a collisionless stellar system (maximization of a \( H \)-function at fixed mass and energy) is similar to the maximization principle determining the thermodynamical stability of a collisional stellar system (maximization of the Boltzmann entropy at fixed mass and energy) we can use a thermodynamical analogy and develop an effective thermodynamical formalism (E.T.F.) to analyze the nonlinear dynamical stability of collisionless stellar systems \([6, 13, 21]\). We emphasize, however, that the maximization of a \( H \)-function at fixed mass and energy is a condition of nonlinear dynamical stability for the Vlasov equation, not a condition of thermodynamical stability. Therefore, this thermodynamical analogy is purely formal. In particular, in the context of violent relaxation, Tsallis functional \( S_q[\mathcal{F}] \) is a particular \( H \)-function, not an entropy.

\footnote{During mixing \( D\mathcal{F}/Dt \neq 0 \) and the \( H \)-functions \( H[\mathcal{F}] \) increase. Once it has mixed \( D\mathcal{F}/Dt = 0 \) so that \( \dot{H}[\mathcal{F}] = 0 \). Since \( \mathcal{F}(\mathbf{r}, \mathbf{v}, t) \) has been brought to a maximum \( \mathcal{F}_0(\mathbf{r}, \mathbf{v}) \) of a certain \( H \)-function and since \( H[\mathcal{F}] \) is conserved (after mixing), then \( \mathcal{F}_0 \) is a nonlinearly dynamically stable steady state of the Vlasov equation.}
we would need to replace the Lynden-Bell entropy \([20]\) by the \(q\)-entropy

\[
S_q[\rho] = -\frac{1}{q-1} \int (\rho^q(\mathbf{r}, \mathbf{v}, \eta) - \rho(\mathbf{r}, \mathbf{v}, \eta)) d^3\mathbf{r} d^3\mathbf{v} d\eta,
\]

as argued in \([31]\). The generalized mixing entropy \(S_q[\rho]\), which is a functional of the probability \(\rho(\mathbf{r}, \mathbf{v}, \eta)\), would be the proper form of \(q\)-entropy in that context, taking into account the specificities of the collisionless dynamics. For \(q \to 1\), it returns the Lynden-Bell entropy \([20]\). For \(q \neq 1\), it could take into account incomplete mixing and non-ergodicity. In that context, the \(q\) parameter could be interpreted as a measure of mixing and Tsallis entropy could be interpreted as a functional attempting to take into account non-ergodicity in the process of incomplete violent relaxation. Maximizing \(S_q[\rho]\) at fixed mass, energy and Casimirs, we obtain a \(q\)-generalization of the Gibbs state \([28]\). This maximization principle is a condition of thermodynamical stability (in Tsallis generalized sense) in the context of violent relaxation. Then, we can obtain a \(q\)-generalization of the equilibrium coarse-grained distribution function \([31]\) in a fashion similar to that of Sec. \([32]\) after introducing proper averaging procedures (e.g., \(q\)-expectation values). For appropriate values of \(q\), these distribution functions will have finite mass contrary to Lynden-Bell’s distribution. We shall not try, however, to develop this generalized formalism in more detail here. Note that in the case of two levels \(f \in \{0, \eta_0\}\), and in the dilute limit of the theory \(\bar{f} \ll \eta_0\), \(S_q[\rho]\) can be written in terms of the coarse-grained distribution \(\bar{f} = \rho \eta_0\) in the form \(S_q[\bar{f}] = -\frac{1}{q-1} \int [(\bar{f}/\eta_0)^q - (\bar{f}/\eta_0)] d^3\mathbf{r} d^3\mathbf{v}\). In this particular limit, Tsallis functional \(S_q[\bar{f}]\) could be interpreted as a generalized entropy (not just a \(H\)-function). Therefore, Tsallis functional \(S_q[\rho]\) expressed in terms of \(\rho(\mathbf{r}, \mathbf{v}, \eta)\) is a generalized entropy while Tsallis functional \(S_q[\bar{f}]\) expressed in terms of \(\bar{f}(\mathbf{r}, \mathbf{v})\) is either a \(H\)-function (dynamics) or a particular case of entropy \(S_q[\rho]\) (thermodynamics) for two levels in the dilute limit. However, it is not clear why complicated effects of non-ergodicity (incomplete mixing) could be encapsulated in a simple functional such as \([71]\). Indeed, other functionals of the form \(S = -\int C(\rho)d\eta d\mathbf{r} d\mathbf{v}\) where \(C\) is convex could be considered as well. As discussed above, the observations of galaxies do not support the prediction of non-extensive thermodynamics obtained by maximizing Tsallis \(q\)-entropy \([71]\). Furthermore, it is not clear whether the idea of changing the form of entropy in case of incomplete relaxation is the most relevant. An alternative approach developed in \([16, 22]\) is to keep the Lynden-Bell entropy \([20]\) unchanged but describe the dynamical evolution of \(\rho(\mathbf{r}, \mathbf{v}, t)\) by a relaxation equation of the form

\[
\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho - \nabla \Phi \cdot \frac{\partial \rho}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \left\{ D(\mathbf{r}, \mathbf{v}, t) \left[ \frac{\partial \rho}{\partial \mathbf{v}} + \beta(t)(\eta - \bar{f}) \rho \mathbf{v} \right] \right\},
\]

with a diffusion coefficient \(D(\mathbf{r}, \mathbf{v}, t)\) going to zero for large time (as the variations of the gravitational potential \(\Phi\) decay) and in regions of phase-space where the fluctuations \(\delta \Phi\) are not strong enough to provide efficient mixing. The vanishing of the diffusion coefficient can “freeze” the system in a subdomain of phase space and account for incomplete relaxation and non-ergodicity. In general, the resulting state, although incompletely mixed, is not a \(q\)-distribution. This approach in interesting because it is not based on a generalized entropy, so there is no free parameter like \(q\) or \(C(\rho)\). However, it demands to solve a dynamical equation \([72]\) to predict the equilibrium state. The idea is that, in case of incomplete relaxation (non-ergodicity), the prediction of the equilibrium state is impossible without considering the dynamics.

We would like to emphasize again the distinction between entropies and \(H\)-functions. An entropy is a quantity which is proportional to the logarithm of the disorder, where the disorder is equal to the number of microstates consistent with a given macrostate. This is how the Lynden-Bell entropy \([20]\) has been defined. Tsallis entropy \([71]\) could be considered as a
generalization of this definition in the case where the phase-space has a complex structure so that the evolution is non-ergodic. In each case, the entropy is a functional of the probability \( \rho(\mathbf{r}, \mathbf{v}, \eta) \) and the maximization of these entropies at fixed mass, energy and Casimirs is a condition of thermodynamical stability. The \( H \)-functions do not have a statistical origin. They are just arbitrary functionals of the coarse-grained distribution \( \bar{f}(\mathbf{r}, \mathbf{v}, t) \) of the form \( \text{(68)} \). They are useful to characterize the degree of mixing of a collisionless stellar system \[20\]. Furthermore, their maximization at fixed mass and energy provides a condition of nonlinear dynamical stability with respect to the Vlasov equation. Finally, the “generalized entropies” \[37\] defined in Sec. \[3.3\] can be regarded as entropies which are proportional to the logarithm of the number of microstates consistent both with a given macrostate and with the constraints imposed by the Vlasov equation (Casimirs). Their functional form depends on the initial condition. They are defined on a projection space (\( \bar{f} \)-space) where a macrostate is defined by the specification of \( \bar{f}(\mathbf{r}, \mathbf{v}) \) instead of \( \rho(\mathbf{r}, \mathbf{v}, \eta) \).

Finally, we note that the maximization of the Lynden-Bell entropy \[20\], of the Tsallis entropy \[71\] or of a \( H \)-function \[68\] leads to a distribution function of the form \( \bar{f} = \bar{f}(\epsilon) \) with \( \bar{f}(\epsilon) < 0 \) depending only on the energy. These DF can only describe spherical stellar systems (and even a sub-class of them) \[25\]. In reality, stellar systems are not spherical and their distribution functions are not function of the energy alone. Indeed, according to the Jeans theorem \[25\], there exists more general stationary solutions of the Vlasov equation which depend on other integrals of motion. This indicates that the structure of the final state of a collisionless stellar system depends on its dynamical evolution in a complicated manner. An important problem in astrophysics is therefore to find the form of distribution function appropriate to real galaxies. Simple concepts based on entropies and \( H \)-functions are not sufficient to understand the structure of galaxies. This is particularly deceptive. However, conceptually, the theory of violent relaxation is important to explain how a collisionless stellar system reaches a steady state. This is due to phase mixing in phase space. The coarse-grained DF \( \bar{f}(\mathbf{r}, \mathbf{v}, t) \) reaches a steady state \( \bar{f}(\mathbf{r}, \mathbf{v}) \) in a few dynamical times while the fine-grained distribution function \( f(\mathbf{r}, \mathbf{v}, t) \) develops filaments at smaller and smaller scales and is never steady (presumably). Since this mixing process is very complex, the resulting structure \( \bar{f}(\mathbf{r}, \mathbf{v}) \) should be extremely robust and should be therefore a nonlinearly dynamically stable stationary solution of the Vlasov equation. Thus, the theory of incomplete violent relaxation explains how collisionless stellar systems can be trapped in nonlinearly dynamically stable stationary solutions of the Vlasov equation on the coarse-grained scale.

4 Two-dimensional turbulence

4.1 Statistical mechanics of 2D vortices

The same ideas apply in 2D turbulence to understand the formation of coherent structures (jets and vortices) in large-scale flows. The analogy between stellar systems and 2D vortices is discussed in Chavanis \[17\]. A statistical theory of point vortices has been first developed by Onsager \[35\] and Joyce & Montgomery \[36\]. This theory predicts the statistical equilibrium state of a point vortex gas, reached for \( t \to +\infty \) after a “collisional” relaxation, assuming ergodicity. The most probable vorticity profile is given by

\[
\omega(\mathbf{r}) = -\Delta \psi = \sum_i A_i e^{-\beta \gamma_i \psi},
\]

which is similar to the statistical distribution \[10\] of a multi-components system of stars (note that the vorticity is proportional to the density of point vortices). A kinetic theory of point
vortices has been developed by Dubin & O’Neil \[37\] and Chavanis \[38, 17\]. The collision term of the derived kinetic equation, which is the counterpart of the Landau equation \(13\), cancels out when the profile of angular momentum is monotonic so that this equation (valid to order \(1/N\)) does not relax towards the statistical equilibrium state. This implies that the relaxation time scale (if there is ever relaxation) is larger than \(Nt_D\).

In the limit \(N \to +\infty\), the evolution of the system is described by the 2D Euler equation \(100\) which is the counterpart of the Vlasov equation \(15\). The statistical mechanics of continuous vorticity fields described by the 2D Euler equation has been developed by Miller and Robert & Sommeria \[15\]. This is similar to the theory of violent relaxation of Lynden-Bell \[12, 16\]. In that context, we speak of “inviscid relaxation” or “chaotic mixing”. The mixing entropy is

\[
S[\rho] = - \int \rho(\mathbf{r}, \sigma) \ln \rho(\mathbf{r}, \sigma) d^2r d\sigma,
\]

and the Gibbs state reads

\[
\rho(\mathbf{r}, \sigma) = \frac{1}{Z} \chi(\sigma) e^{-\sigma(\beta \psi + \alpha)},
\]

with notations similar to those of Sec. 3.2 (here, \(\sigma\) labels the vorticity levels). The density probability \(\rho(\mathbf{r}, \sigma)\) gives the local distribution of vorticity at statistical equilibrium. It maximizes the mixing entropy \(74\) at fixed energy \(E = \frac{1}{2} \int \mathcal{C}(\omega) d^2r\), circulation \(\Gamma = \int \mathcal{C} d^2r\) (robust constraints) and Casimir constraints or fine-grained moments \(\Gamma_{n>1} = \int \mathcal{C}^n d^2r = \int \rho \sigma^n d\sigma d^2r\) (fragile constraints). The partition function can be written

\[
Z = \int_{-\infty}^{+\infty} \chi(\sigma) e^{-\sigma(\beta \psi + \alpha)} d\sigma,
\]

and the most probable coarse-grained vorticity \(\overline{\omega} = \int \rho \sigma d\sigma\) is related to the stream function by a relation of the form

\[
\overline{\omega} = -\frac{1}{\beta} \frac{1}{\partial \ln Z/\partial \psi} = F(\beta \psi + \alpha) = f(\psi).
\]

This is a steady state of the 2D Euler equation where \(f\) is monotonic (since \(f'(\psi) = -\beta \omega_2\) with \(\omega_2 = \overline{\omega}^2 - \overline{\omega}^2 \geq 0\), it is increasing at negative temperatures and decreasing at positive temperatures). Note that the vorticity levels \(\sigma\) can take positive and negative values contrary to the case of self-gravitating systems for which \(\eta \geq 0\). Note also that \(\omega\) is a vorticity field not a distribution of particles, unlike \(f\) in astrophysics (only in the point vortex model can we interprete \(\omega\) as a distribution of particles since it is related to the density of point vortices). The most probable coarse-grained vorticity \(77\) maximizes a generalized entropy

\[
S[\overline{\omega}] = - \int C(\overline{\omega}) d^2r,
\]

at fixed circulation and energy. Indeed, this optimization problem leads to a relation of the form

\[
C'(\overline{\omega}) = -\beta \psi - \alpha,
\]

which can be identified with Eq. \(77\) with \(f'(\psi) = -\beta/C''(\overline{\omega})\). This identification relates the function \(C(\overline{\omega})\) to the function \(F(x)\) whose form depends on \(\chi(\sigma)\) through Eqs. \(76\) and \(77\). Explicitly, we have

\[
C(\overline{\omega}) = -\int \overline{\omega} F^{-1}(x) dx.
\]
We can also introduce a notion of generalized selective decay principle in 2D turbulence: among all inviscid invariants of the 2D Euler equation, the $H$-functions (fragile constraints) $H[\omega] = -\int C(\omega)d^2\mathbf{r}$ increase ($-H$ decrease) on the coarse-grained scale or in the presence of a small viscosity (Appendix A) while the energy $E[\omega]$ and the circulation $\Gamma[\omega]$ (robust constraints) are approximately conserved. Therefore, the metaequilibrium state resulting from violent relaxation is expected to maximize a certain $H$-function (non-universal) at fixed energy and circulation. This generalizes the usual selective decay principle of 2D turbulence which considers the minimization of enstrophy $\Gamma_2 = \int \omega^2 d^2\mathbf{r}$ at fixed energy and circulation. In our approach, minus the enstrophy $-\Gamma_2[\omega] = -\int \omega^2 d^2\mathbf{r}$ and the Tsallis functionals $S_q[\omega] = -\frac{1}{q-1} \int (\omega^q - \omega)d^2\mathbf{r}$ are particular $H$-functions (note that the enstrophy $\Gamma_2$ is a particular case of Tsallis functional with $q = 2$).

The extremization of a $H$-function at fixed energy and circulation leads to a stationary solution of the 2D Euler equation of the form $\omega = f(\psi)$ where $f$ is a monotonic function specified by the convex function $C(\omega)$. Furthermore, as shown in [18], the condition of maximum provides a refined criterion of nonlinear dynamical stability for the 2D Euler-Poisson system (the physical interpretation of this criterion applying to the coarse-grained vorticity is the same as in the remark of Sec. 3.5). Note that contrary to the Vlasov equation, the relation $\omega = f(\psi)$ is the general form of stationary solution of the 2D Euler equation (for systems with no special symmetries). Therefore, in 2D hydrodynamics, any nonlinearly dynamically stable stationary solution of the 2D Euler equation maximizing a $H$-function at fixed circulation and energy (and, possibly, angular momentum and impulse) contrary to the case of the Vlasov equation in astrophysics where a more general class of steady solutions exists due to the Jeans theorem.

Finally, the Tsallis entropy in the context of the 2D Euler equation is a functional of the vorticity distribution $\rho(\mathbf{r}, \sigma)$ of the form $S_q[\rho] = -\frac{1}{q-1} \int (\rho^q - \rho)d\sigma d^2\mathbf{r}$ generalizing the mixing entropy [17] [34]. This functional could be an attempt to take into account non-ergodicity in the process of violent relaxation of 2D turbulent flows. However, other functionals could be considered as well, and Tsallis entropy does not provide a correct description of non-ergodicity in all observed cases. This means that the type of mixing in 2D turbulence (and stellar dynamics) is more complex than the one (multi-fractal) described by the Tsallis functional [3]. Non-ergodicity (incomplete relaxation) can be taken into account dynamically by using relaxation equations with a space dependent diffusion coefficient related to the fluctuations [39] [16].

### 4.2 Prior vorticity distribution

The statistical approach of Miller-Robert-Sommeria applies to flows that are strictly described by the 2D Euler equation. In this point of view, one must conserve the value of all the Casimir invariants (or vorticity moments). This leads to the expression [17] for the most probable distribution of vorticity, where the function $\chi(\sigma)$ is determined by the initial conditions through the value of the Casimir integrals (this is precisely the Lagrange multiplier associated to these constraints). However, in geophysics, there exists situations in which the flow is continuously forced at small-scales so that the conservation of the Casimirs is destroyed. Ellis et al. [18] have proposed to take into account these situations by fixing the function $\chi(\sigma)$ instead of the Casimirs. Physically, this prior vorticity distribution can be viewed as a global distribution of vorticity imposed by a small-scale forcing. It can be due to convection and 3D effects like in the atmosphere of Jupiter. Its specific form has to be adapted to the situation. Then, two-dimensional turbulence organizes this global distribution of vorticity into large-scale coherent structures. These organized states result from a balance between entropic and energetic effects: the system tends to mix but complete mixing, which would result in a uniform distribution, is prevented by the energy constraint. The most probable local distribution of vorticity is now
obtained by maximizing a relative entropy conditioned by the prior distribution

\[ S_\chi[\rho] = -\int \rho(r,\sigma) \ln \left[ \frac{\rho(r,\sigma)}{\chi(\sigma)} \right] d^2r d\sigma, \]

at fixed circulation and energy (no other constraints). The conservation of the Casimirs has been replaced by the specification of a prior distribution \( \chi(\sigma) \). As shown in Chavanis [19], the relative entropy (81) can be seen as a Legendre transform \( S_\chi = S - \sum_{n>1} \alpha_n \Gamma_{n>1}^{f,g} \) of the mixing entropy (74) when the constraints associated with the conservation of the vorticity moments (Casimirs) are treated canonically. Indeed, the approach of Ellis et al. [18] amounts to fixing the conjugate variables \( \alpha_{n>1} \) instead of the fine-grained moments \( \Gamma_{n>1}^{f,g} \). If we view the vorticity levels as species of particles, this is equivalent to fixing the chemical potentials instead of the total number of particles in each species. This assumes that the 2D system is in contact with a sort of “reservoir”. The forcing and dissipation break the conservation of the Casimirs and impose instead a distribution of vorticity. By contrast, the robust constraints (circulation and energy) are still treated microcanonically. The maximization of \( S_\chi \) at fixed \( E, \Gamma \) again leads to the distribution (75) but with a different interpretation. In the present context, the statistical equilibrium state results from an interplay between 3D effects (the non-universal small-scale homogeneous forcing encapsulated in the prior \( \chi(\sigma) \)) and 2D effects (the universal Gibbs factor \( e^{-\sigma(\alpha+\beta\psi)} \) giving rise to inhomogeneous large-scale structures). The statistical distribution is the product of these two effects. The partition function and the most probable coarse-grained vorticity field are still given by Eqs. (76) and (77). However, in this new approach, the function \( F(x) \) is fixed directly by the prior vorticity distribution \( \chi(\sigma) \) while in the approach of Miller-Robert-Sommeria, it has to be related a posteriori to the initial conditions in a complicated way.

The approach of Ellis et al. [18] is very close to the notion of superstatistics since it considers that the fluctuations of vorticity \( \chi(\sigma) \) are given a priori by an external process, which is also the case for the fluctuations of temperature \( f(\beta) \) in the Beck-Cohen superstatistics. Therefore, the \( \omega - \psi \) relationship and the generalized entropy \( S[\omega] \) are directly determined by the prior vorticity distribution \( \chi(\sigma) \) through the formula

\[ C(\omega) = -\int [-(\ln \chi(\sigma))^\prime - (\omega, \chi)] d\sigma, \]

where \( \chi(E) = \int_{-\infty}^{\infty} \chi(\sigma) e^{-\sigma E} d\sigma, \) according to Eqs. (76), (77) and (78). This makes the generalized entropy \( S[\omega] \) an intrinsic quantity. In the present context, it is determined by the small-scale forcing (through the prior \( \chi \)) while in the approach of Miller-Robert-Sommeria it depends on the initial conditions (through the Casimirs). Furthermore, in the present context, \( S[\omega] \) really has the status of an entropy in the sense of the large deviation theory. Indeed, Ellis et al. [18] show that the probability of the coarse-grained vorticity field \( \omega(r) \) at statistical equilibrium can be written in the form of the Cramer formula

\[ P[\omega] \sim e^{nS[\omega]}, \]

where \( n \) is the number of sites of the underlying lattice introduced in their mathematical analysis. Therefore, the most probable vorticity field \( \omega \) maximizes \( S[\omega] \) at fixed circulation and energy. This maximization principle also provides a refined condition of nonlinear dynamical stability with respect to the 2D Euler-Poisson system [18].
4.3 Example of generalized entropy

Let us consider, for illustration, the prior vorticity distribution $\chi(\sigma)$ introduced by Ellis et al.\[18\] in their model of jovian vortices. It corresponds to a de-centered Gamma distribution

$$\chi(\sigma) = \frac{1}{|\sigma|} R \left[ \frac{1}{\epsilon} \left( \sigma + \frac{1}{\epsilon} \right) ; \frac{1}{\epsilon^2} \right],$$

where $R(z; a) = \Gamma(a)^{-1} z^{a-1} e^{-z}$ for $z \geq 0$ and $R = 0$ otherwise. The scaling of $\chi(\sigma)$ is chosen such that $\langle \sigma \rangle = 0$, $\text{var}(\sigma) = 1$ and $\text{skew}(\sigma) = 2\epsilon$. This distribution is a variant of Gamma distribution considered by Beck & Cohen \[10\]. Setting $E \equiv \beta \psi + \alpha$, we get

$$(85) \quad Z(E) = \hat{\chi}(E) = \frac{e^{E/\epsilon}}{(1 + \epsilon E)^{1/\epsilon^2}},$$

and

$$(86) \quad \omega(E) = -(\ln Z)'(E) = \frac{-E}{1 + \epsilon E}.$$ 

Inversing the relation (86), we obtain

$$(87) \quad -E = \frac{\omega}{1 + \epsilon \omega} = C' (\omega).$$

After integration, we obtain the generalized entropy

$$(88) \quad C(\omega) = \frac{1}{\epsilon} \left[ \omega - \frac{1}{\epsilon} \ln(1 + \epsilon \omega) \right].$$

This form of entropy can also be obtained from the techniques of the large deviation theory as discussed in \[18\]. Our approach, leading to the general formula (82), is a simple alternative to obtain the generalized entropy $C(\omega)$ associated to the prior vorticity distribution $\chi(\sigma)$. On the other hand, for a Gaussian prior distribution

$$(89) \quad \chi(\sigma) = e^{-\sigma^2 / 2},$$

we get

$$(90) \quad Z(E) = \sqrt{2\pi} e^{E^2 / 2}, \quad \omega(E) = -E, \quad C(\omega) = \frac{1}{2} \omega^2.$$ 

Therefore, the $\omega - \psi$ relationship is linear and the generalized entropy $S[\omega] = -\frac{1}{2} \int \omega^2 d^2 r$ is minus the enstrophy. It also corresponds to the limit of Eq. (88) for $\epsilon \to 0$. Other examples of prior vorticity distributions are collected in \[6\]. An example which has not been given previously is when $\chi(E)$ is of the Tsallis form

$$(91) \quad \chi(\sigma) = \left( 1 - \frac{1}{2} \sigma^2 \right)^p, \quad |\sigma| \leq \sqrt{2p}.$$ 

For $p \to +\infty$, we recover the Gaussian distribution (89). For the distribution (91), we get

$$(92) \quad Z(E) = 2^{(3+2p)/4} p^{(5-2p)/4} \sqrt{\pi} \Gamma(p) |E|^{-1/2-p} I_{1/2+p} (\sqrt{2p} |E|).$$
4.4 Generalized Fokker-Planck equations

In the context of freely evolving 2D turbulence, a thermodynamical parametrization of the 2D Euler equation has been proposed by Robert & Sommeria \[39\] in terms of relaxation equations based on a maximum entropy production principle (MEPP). These equations conserve all the Casimirs, increase the mixing entropy \((74)\) and relax towards the Gibbs state \((75)\). In the situations considered by Ellis et al. \[18\] where the system is forced at small scale, we have proposed in \[19\] an alternative parametrization of the 2D Euler equation. In that case, we have seen that only the energy and the circulation (robust constraints) are conserved. The conservation of the Casimirs is replaced by the specification of a prior vorticity distribution \(\chi(\sigma)\) encoding the small-scale forcing. This fixes a form of generalized entropy \((78)\) through the formula \((82)\). In that case, we have proposed to describe the large-scale evolution of the flow on the coarse-grained scale by a relaxation equation which conserves energy and circulation and increases the generalized entropy \((78)\) until the equilibrium state \((77)\) is reached. This can be obtained by using a generalized Maximum Entropy Production Principle. The resulting relaxation equation, introduced in \[6\], has the form of a generalized Fokker-Planck equation

\[
\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nabla \cdot \left\{ D \left[ \nabla \omega + \frac{\beta(t)}{C''(\omega)} \nabla \psi \right] \right\},
\]

and

\[
\beta(t) = -\frac{\int D \nabla \omega \cdot \nabla \psi d^2 \mathbf{r}}{\int D \frac{(\nabla \psi)^2}{C''(\omega)} d^2 \mathbf{r}},
\]

where the evolution of the Lagrange multiplier \(\beta(t)\) accounts for the conservation of energy. Furthermore, the diffusion coefficient can be obtained from a kinetic model leading to \(D = K \epsilon^2 / \sqrt{C''(\omega)}\) where \(\epsilon\) is the resolution scale and \(K\) is a constant of order unity \[19\]. In these equations, the function \(C(\omega)\) is fixed by the prior distribution \(\chi(\sigma)\). These equations are expected to be valid close to the equilibrium state in the spirit of Onsager’s linear thermodynamics. However, they may offer a useful parametrization of 2D flows even if we are far from equilibrium. Alternatively, according to the refined nonlinear dynamical stability criterion of Ellis et al. \[18\] these relaxation equations can be used as powerful numerical algorithms to compute arbitrary nonlinearly dynamically stable stationary solutions of the 2D Euler-Poisson system. These ideas are further discussed in \[19\] in relation with geophysical flows. We note that forced 2D turbulence provides a physical situation of interest in which a rigorous notion of generalized thermodynamics and generalized kinetics emerges. In our formalism, all the complexity of the system is encapsulated in a prior distribution \(\chi(\sigma)\). We can then determine the generalized entropy \(S[\omega]\) by using formula \((82)\) and substitute the result in the relaxation equation \((93)\) to obtain the dynamical evolution of the coarse-grained flow. The problem now amounts to finding the relevant prior \(\chi(\sigma)\). Of course, this depends on the situation contemplated. Furthermore, for a given situation, it is likely that a whole “class” of priors (or generalized entropies) will sensibly give the same results. In practice, one has to proceed by trying and errors to find the relevant “class of equivalence” adapted to the situation considered \[6\].

As discussed previously, the prior \(\chi(\sigma)\) encodes the small-scale forcing. It is due, e.g., to convection (in the jovian atmosphere) or any other complicated process specific to the situation contemplated. It is not our goal here to develop a precise model of convection to determine a relevant form for \(\chi(\sigma)\). We shall rather remain at a phenomenological level and propose to describe the generation of vorticity fluctuations by general stochastic processes. Since the generating process must include a forcing and a dissipation, we consider a generalized Langevin
equation of the form introduced in [6]:

\[
\frac{d\sigma}{dt} = -\xi\sigma + \sqrt{2D\chi\left[\frac{\mathcal{C}(\chi)}{\chi}\right]}\eta(t),
\]

where \(\eta(t)\) is a white noise and \(\mathcal{C}(\chi)\) a convex function of the global distribution of vorticity. The corresponding (generalized) Fokker-Planck equation is

\[
\frac{\partial\chi}{\partial t} = \frac{\partial}{\partial\sigma}\left[D\chi\mathcal{C}''(\chi)\frac{\partial\chi}{\partial\sigma} + \xi\chi\sigma\right].
\]

Its stationary solution determines the prior vorticity distribution \(\chi(\sigma)\) through the relation

\[
\mathcal{C}'(\chi) = -b\frac{\sigma^2}{2} - a,
\]

where \(b = \xi/D\) is a sort of inverse temperature. For example, when the coefficients of dissipation and forcing are constant, corresponding to \(\mathcal{C}(\chi) = \chi\ln\chi\) and leading to standard stochastic processes, the prior distribution is the Gaussian \(N\) leading to a generalized entropy having the form of minus the enstrophy \(S[\omega] = -\Gamma_2[\omega]\) and to a linear \(\omega - \psi\) relationship at equilibrium. However, our formalism allows to treat more general situations. Furthermore, in the preceding discussion, we have implicitly assumed that the prior relaxes more rapidly to its equilibrium value than the coarse-grained vorticity field, so that, in Eq. (93), the generalized entropy \(C(\omega)\) is calculated from \(\chi(\sigma) = \chi(\sigma, +\infty)\). This is probably a relevant approximation. Otherwise, we need to couple the two equations (93) and (96) and determine, at each time, the function \(C(\omega, t)\) from the prior \(\chi(\sigma, t)\), using formula (82).

5 Conclusion

In this paper, we have discussed some analogies between coarse-grained distribution functions characterizing statistical equilibrium states of collisionless stellar systems or inviscid 2D flows and the notion of superstatistics introduced by Beck & Cohen (2003). In particular, we have shown that the coarse-grained distribution functions arising in theories of violent relaxation can be viewed as forms of superstatistics (albeit different from the Beck-Cohen superstatistics). Although the concept of violent relaxation has been introduced by Lynden-Bell (1967) long ago, it remains largely unknown in the statistical mechanics community and this is why we have exposed this theory in some detail here. Non-standard distributions arise on the coarse-grained scale because they are expressed as averages of fine-grained distributions. The observed (coarse-grained) distribution function appears to be a superposition of Boltzmann’s factors weighted by a non-universal function \(\chi(\eta)\) or \(\chi(\sigma)\). To each coarse-grained distribution, we can associate a generalized entropy. For freely evolving systems, the functions \(\chi(\eta)\) or \(\chi(\sigma)\) and the generalized entropies \(S[\mathcal{F}]\) or \(S[\mathcal{\omega}]\) depend on the initial conditions. Alternatively, in certain occasions, it may be justified to regard the function \(\chi\) as imposed by some external processes. This prior distribution then directly determines the generalized entropy. This approach is particularly relevant in the case of geophysical flows that are forced at small scales [18, 19]. It may also be valid in the case of dark matter models in astrophysics where a small-scale forcing can alter the conservation of the Casimirs and impose instead a distribution of fluctuations. In these cases, the relaxation of the coarse-grained field can be described by generalized Fokker-Planck equations where the entropy is determined by the prior \(\chi\) [6, 19]. Alternatively, these relaxation
equations can be used as numerical algorithms to construct arbitrary non-linearly dynamically stable stationary solutions of the Vlasov and Euler equations specified by a convex function $C$.

We have also discussed the two successive equilibrium states achieved by a stellar system. In a first regime, the evolution is collisionless and the system reaches a metaequilibrium state as a result of violent relaxation. This is a non-linearly dynamically stable stationary solution of the Vlasov-Poisson system. On longer timescales, stellar encounters ("collisions") drive the system towards the statistical equilibrium state described by the Boltzmann distribution (when the escape of stars and the gravothermal catastrophe are prevented). The metaequilibrium state (collisionless regime) and the statistical equilibrium state (collisional regime) correspond to quite different processes. They can be written as a superposition of Boltzmann factors for each species of particles (collisional equilibrium) or for the different phase levels (collisionless equilibrium).

In fact, violent relaxation is incomplete in general. A famous example of incomplete relaxation in 2D turbulence is provided by the plasma experiment of Huang & Driscoll \[40\]. In this experiment, the metaequilibrium state resulting from violent relaxation has the form of a self-confined vortex surrounded by un-mixed flow. This strong confinement is in contradiction with the statistical mechanics of Miller-Robert-Sommeria \[15\] which leads to un-restricted vorticity profiles. As discussed in Brands \[23\], the observed confinement is due to incomplete relaxation and lack of mixing/ergodicity. The system has evolved to a stationary solution of the 2D Euler equation which is not the most mixed state. Now, any non-linearly dynamically stable stationary solution of the 2D Euler equation maximizes a $H$-function $S[q]$ at fixed circulation and energy. In the special case considered by Huang & Driscoll, this $H$-function turns out to be related to the enstrophy functional $\Gamma_q[\psi]$, which is a particular form of the Tsallis $H$-function $S_q[\psi]$ with $q = 2$. This “dynamical interpretation” based on $H$-functions is different from the “generalized thermodynamical interpretation” of Boghosian \[11\] where $S_q[\psi]$ is viewed as a Tsallis q-entropy. Since (in our sense) the Tsallis functional $S_q[\psi]$ is a $H$-function, not an entropy, the use of $q$-expectation values is irrelevant in this dynamical context. If we want to apply Tsallis thermodynamics in the context of the 2D Euler equation, we need to introduce an entropy $S_q[\rho]$ which is a functional of the probability density $\rho(r, \sigma)$. However, in that case, the agreement with the plasma experiment fails as shown in \[34\]. Therefore, the experimental result of Huang & Driscoll cannot in fact be explained by Tsallis generalized thermodynamics when the full constraints of the Euler equation are accounted for. The fact that the $\omega - \psi$ relationship resembles a $q$-distribution (in $\psi$-space) is coincidental. This is a particular solution of the Euler equation resulting from incomplete violent relaxation. Since the 2D Euler equation admits an infinity of stationary solutions, there are many other examples of incomplete violent relaxation in 2D turbulence (and stellar dynamics) where the system settles in a steady state that is not described by the Tsallis distribution (in $\psi$-space or in $\rho$-space). The situation described by Huang & Driscoll in which an $\omega - \psi$ relationship resembling a $q$-distribution emerges is fortuitous and not generic.

In this paper, we have tried to distinguish different notions of entropy that arise in the theory of violent relaxation. The mixing entropy \[28\] is the fundamental entropy of the theory. It can be obtained by a combinatorial analysis and its maximization at fixed mass/circulation, energy and Casimir invariants determines the most probable distribution of fine-grained levels $\rho(r, v, \eta)$ through the Gibbs state \[28\], assuming ergodicity (complete mixing). The generalized mixing entropy \[71\] is the appropriate Tsallis generalization of \[28\] in the context of violent relaxation. It can be seen as an attempt to take into account non-ergodic effects and describe them in terms of a single parameter $q$. All the machinery of non-extensive thermodynamics (q-expectation values,...) could be developed in that framework, working with $\rho(r, v, \eta)$ instead of $f(r, v)$. We might also consider other generalizations of entropy.
\[ S = - \int C(\rho) d^3r d^3v d\eta \] where \( C \) is convex. The status of such generalizations is still in debate for the moment because it is not clear whether non-ergodic effects can be encapsulated in a simple functional. One must rather accept that the final state of the system is unpredictable in case of incomplete violent relaxation. The relative entropy \((81)\) is the Legendre transform of the mixing entropy \((74)\) conditioned by a prior vorticity distribution \(\chi(\sigma)\) in the sense of \([18, 19]\). This description can be relevant for 2D turbulent flows that are forced at small-scales. Its maximization at fixed circulation and energy (no other invariants) determines the most probable distribution of fine-grained levels \(\rho(r, \sigma)\) through the Gibbs state \((75)\) conditioned by an imposed global distribution \(\chi(\sigma)\). The generalized entropy \((37)-(43)\) or \((78)-(80)\) is the functional that the most probable coarse-grained distribution \(f(r, v)\) or \(\omega(r)\) given by \((30)-(77)\) maximizes at fixed energy and mass/circulation. For freely evolving systems, it depends on the initial conditions. For forced systems, it is determined by the prior vorticity distribution \(\chi(\sigma)\) through the formula \((82)\). The H-functions \((68)\) are arbitrary functionals (not entropies) of the coarse-grained field. They increase during mixing and their maximization at fixed mass/circulation and energy determines a nonlinearly dynamically stable stationary solution of the Vlasov/Euler equation with a monotonic relationship \(f = f(\epsilon)\) or \(\omega = \omega(\psi)\). These stationary solutions can result from complete or incomplete violent relaxation (in that case, \(f\) and \(\omega\) must be regarded as the coarse-grained fields). When mixing is complete, the H-function that is maximized at equilibrium is the generalized entropy \((37)-(43)\) or \((78)-(80)\). When mixing is incomplete, the H-functions and the coarse-grained distributions can take forms that are not consistent with the statistical theory. For example, Tsallis functional \((69)\) is a particular H-function associated with stellar polytropes and polytropic vortices. They form simple families of stationary solutions of the Vlasov and 2D Euler equations. They sometimes arise as a result of incomplete violent relaxation due to the combined effect of Casimir constraints and non-ergodicity \([10, 34]\). The maximization of a H-function at fixed mass/circulation and energy is a condition of nonlinear dynamical stability. We can develop a thermodynamical analogy and an effective thermodynamical formalism to study the nonlinear dynamical stability of the system, but the notion of “generalized thermodynamics” is essentially effective in that context \([6, 21]\).

In conclusion, a striking property of systems with long-range interactions is the rapid emergence of coherent structures: galaxies in astrophysics, vortices and jets in 2D turbulence, quasi-equilibrium states in the HMF model... Since these metaequilibrium states are not described by the Boltzmann distribution, some authors have proposed to replace the Boltzmann entropy \(S_B[f]\) by the Tsallis entropy \(S_q[f]\), invoking that the system is non-extensive so that standard statistical mechanics is not applicable \([41, 12, 43]\). However, this approach ignores the importance of the Vlasov equation and the concept of violent relaxation introduced by Lynden-Bell \([12]\). The description of coherent structures in Vlasov systems is complicated but it can be explained in terms of “classical” principles without invoking a generalized thermodynamics \([17]\). Our discussion indicates that there are two independent reasons why the quasi-equilibrium states that form as a result of violent relaxation are non-Boltzmannian. This is due, on the one hand, to the existence of fine-grained constraints (the Casimirs) which depend on the initial conditions and, on the other hand, to incomplete relaxation (non-ergodicity, partial mixing). Even in case of ergodicity (complete mixing), we can have a wide diversity of non-standard distributions depending on the initial conditions. They are given by Eq. \((30)\) according to the statistical theory of Lynden-Bell. They are sorts of superstistics. Moreover, if the system does not mix efficiently, the Lynden-Bell prediction breaks down and even more general distributions can be observed. They are stable stationary solutions of the Vlasov equation on the coarse-grained scale. The prediction of the metaequilibrium state in case of incomplete relaxation is extremely complicated, if not impossible. One possibility is to change the form of entropy. However, the metaequilibrium state cannot apparently be described by a universal
functional such as the Tsallis functional, even if it is extended to the form (71) so as to take into account the specificities of the collisionless evolution (Casimir constraints). An alternative approach is to keep the Lynden-Bell entropy but develop a dynamical theory of violent relaxation as initiated in [16, 22] to understand what prevents complete mixing. In that case, we have to solve a dynamical equation with a non-constant diffusion coefficient related to the fluctuations. The $H$-functions can also be useful to construct stable models of galaxies (and 2D vortices) in order to reproduce observed phenomena. In some specific situations, some $H$-functions (belonging to the same “class of equivalence”) may be more appropriate than others to describe the system, so that a phenomenological notion of “effective generalized thermodynamics” (in $f$-space or $\omega$-space) can be developed to deal with complex systems in a simple and practical way [6]. In that point of view, the relevant functional should be found by trying and errors.

A $H$-functions for the 2D Euler equation

We briefly recall, and adapt to the case of the 2D Euler equation, the notion of $H$-functions introduced by Tremaine et al. [20] for the Vlasov equation. These concepts have not been introduced in 2D turbulence. A $H$-function is a functional of the coarse-grained vorticity of the form

\begin{equation}
H = - \int C(\omega) d^2 r,
\end{equation}

where $C$ is a convex function. We assume that the initial condition at $t = 0$ has been prepared without small-scale structure so that the fine-grained and coarse-grained vorticity fields are equal: $\omega(r,0) = \omega(r,0)$. For $t > 0$, the system will mix in a complicated manner and develop intermingled filaments so that these two fields will not be equal anymore. We have

\begin{equation}
H(t) - H(0) = \int \{ C(\omega(r,0)) - C(\omega(r,t)) \} d^2 r
\end{equation}

The fine-grained vorticity is solution of the 2D Euler equation

\begin{equation}
\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = 0,
\end{equation}

where $\mathbf{u}(r,t)$ is an incompressible velocity field. Thus

\begin{equation}
\frac{d}{dt} \int C(\omega) d^2 r = \int C'(\omega) \frac{\partial \omega}{\partial t} d^2 r = - \int C'(\omega) \mathbf{u} \cdot \nabla \omega d^2 r = - \int \mathbf{u} \cdot \nabla C(\omega) d^2 r = 0.
\end{equation}

This shows that the $H$-function $H[\omega]$ calculated with the fine-grained vorticity is independent on time (it is a particular Casimir) so Eq. (99) becomes

\begin{equation}
H(t) - H(0) = \int \{ C(\omega(r,t)) - C(\omega(r,t)) \} d^2 r.
\end{equation}

Now, a macrocell is divided into $\nu$ microcells of size $h = \Delta/\nu$. We call $\omega_i$ the value of the vorticity in a microcell. The contribution of a macrocell to $H(t) - H(0)$ is

\begin{equation}
\Delta \left\{ \frac{1}{\nu} \sum_i C(\omega_i) - C \left( \frac{1}{\nu} \sum_i \omega_i \right) \right\}
\end{equation}
which is positive since $C$ is convex. Therefore, the $H$-functions calculated with the coarse-grained vorticity $H[\overline{\omega}]$ increase in the sense that $H(t) \geq H(0)$ for any $t \geq 0$. Note, however, that nothing is said concerning the relative value of $H(t)$ and $H(t')$ for $t, t' > 0$ so that the increase is not necessarily monotonic.

In 2D hydrodynamics, the viscosity has an effect similar to coarse-graining. Indeed, considering the Navier-Stokes equation

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nu \Delta \omega,$$

with $\nu > 0$, we get

$$\dot{H} = -\frac{d}{dt} \int C(\omega) \, d^2r = -\int C'(\omega) \frac{\partial \omega}{\partial t} \, d^2r = -\nu \int C''(\omega) \Delta \omega \, d^2r$$

$$= \nu \int \nabla C'(\omega) \cdot \nabla \omega \, d^2r = \nu \int C''(\omega) (\nabla \omega)^2 \, d^2r \geq 0.$$  

(105)

In that case, the increase of $H$ is monotonic.

References

[1] A. Pais, *Subtle is the Lord*, Oxford University Press, New York (1982).
[2] E.G.D. Cohen, Physica A 305, 19 (2002).
[3] C. Tsallis, J. Stat. Phys. 52, 479 (1988).
[4] Special issue of Physica A 340, Issue 1-3, edited by G. Kaniadakis and M. Lissia (2004).
[5] See [http://tsallis.cat.cbpf.br/biblio.htm](http://tsallis.cat.cbpf.br/biblio.htm) for a regularly updated bibliography on the subject.
[6] P.H. Chavanis, Phys. Rev. E 68, 036108 (2003); P.H. Chavanis, Physica A 332, 89 (2004); P.H. Chavanis, Banach Center Publ. 66, 79 (2004); P.H. Chavanis, Physica A 340, 57 (2004); P.H. Chavanis, P. Laurenço, M. Lemou, Physica A 341, 145 (2004).
[7] G. Kaniadakis, Physica A 296, 405 (2001).
[8] L. Borland, Phys. Rev. E 57, 6634 (1998).
[9] T.D. Frank, Physics Lett. A 290, 93 (2001).
[10] C. Beck & E.G.D. Cohen, Physica A 322, 267 (2003).
[11] C. Tsallis & A. Souza, Phys. Rev. E 67, 026106 (2003).
[12] D. Lynden-Bell, Mon. Not. R. Astr. Soc. 136, 101 (1967).
[13] P.H. Chavanis, Astron. Astrophys. 401, 15 (2003).
[14] P.H. Chavanis & J. Sommeria, Mon. Not. R. Astr. Soc. 296, 569 (1998).
[15] J. Miller, Phys. Rev. Lett. 65, 2137 (1990); R. Robert & J. Sommeria, J. Fluid Mech. 229, 291 (1991).
[16] P.H. Chavanis, J. Sommeria & R. Robert, Astrophys. J. 471, 385 (1996).
[17] P.H. Chavanis, in *Dynamics and thermodynamics of systems with long range interactions*, edited by Dauxois, T, Ruffo, S., Arimondo, E. and Wilkens, M. Lecture Notes in Physics, Springer (2002) cond-mat/0212223.

[18] R. Ellis, K. Haven & B. Turkington, Nonlinearity 15, 239 (2002).

[19] P.H. Chavanis, Physica D 200, 257 (2005).

[20] S. Tremaine, M. Hénon & D. Lynden-Bell, Mon. Not. R. Astron. Soc. 219, 285 (1986).

[21] P.H. Chavanis & C. Sire, cond-mat/0409569

[22] P.H. Chavanis, Mon. Not. R. Astr. Soc. 300, 981 (1998).

[23] P.H. Chavanis, cond-mat/0409641

[24] P.H. Chavanis, A&A 432, 117 (2005).

[25] J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton Series in Astrophysics, 1987).

[26] D. Lynden-Bell and R. Wood, Mon. Not. R. Astron. Soc. 138, 495 (1968).

[27] H.E. Kandrup, Astrophys. J. 244, 316 (1981).

[28] P.H. Chavanis and F. Bouchet, Astron. Astrophys. 430, 771 (2005)

[29] M. Le Bellac, *Quantum and Statistical Field Theory* (Clarendon Press, Oxford, 1991).

[30] H.W. Capel and R.A. Pasmanter, Phys. Fluids 12, 2514 (2000).

[31] M.P. Almeida, Physica A 300, 424 (2001).

[32] C. Beck, Physica A 342, 139 (2004).

[33] J. Hjorth & J. Madsen, Mon. Not. R. Astr. Soc. 265, 237 (1993).

[34] H. Brands, P.H. Chavanis, J. Sommeria and R. Pasmanter, Phys. Fluids 11, 3465 (1999).

[35] L. Onsager, Nuovo Cimento Suppl. 6, 279 (1949).

[36] G. Joyce and D. Montgomery, J. Plasma Phys. 10, 107 (1973).

[37] D. Dubin and T.M. O’Neil, Phys. Rev. Lett. 60, 1286 (1988); D. Dubin, Phys. Plasmas 10, 1338 (2003)

[38] P.H. Chavanis, Phys. Rev. E 58, R1199 (1998); P.H. Chavanis, Phys. Rev. E 64, 026309 (2001).

[39] R. Robert and J. Sommeria, Phys. Rev. Lett. 69, 2776 (1992); R. Robert and C. Rosier, J. Stat. Phys. 86, 481 (1997).

[40] X.P. Huang and C.F. Driscoll, Phys. Rev. Lett. 72, 2187 (1994).

[41] B. Boghosian, Phys. Rev. E 53, 4754 (1996).

[42] A. Taruya & M. Sakagami, Physica A 322, 285 (2003).

[43] V. Latora, A. Rapisarda & C. Tsallis, Physica A, 305, 129 (2002)