Highly noise resistant multipartite quantum correlations

Wiesław Laskowski,1 Tamás Vértesi,2,3 and Marcin Wieśniak1

1Institute of Theoretical Physics and Astrophysics, University of Gdański, PL-80-952 Gdańsk, Poland
2Institute for Nuclear Research, Hungarian Academy of Sciences, H-4001 Debrecen, P.O. Box 51, Hungary
3Département de Physique Théorique, Université de Genève, 1211 Genève, Switzerland

We analyze robustness of correlations of the N-qubit GHZ and Dicke states against white noise admixture. For sufficiently large N, the Dicke states (for any number of excitations) lead to more robust violation of local realism, than the GHZ states (e.g. for N = 9 for the W state). We also identify states that are the most resistant to white noise. Surprisingly, it turns out that these states are partially product. As a by-product, we obtain a simple three-setting Bell inequality which is violated by any pure entangled state.

PACS numbers: 03.65.Ud

I. INTRODUCTION

Local realism was a model introduced by Einstein, Podolsky, and Rosen[1] in an attempt to reach agreement between predictions of quantum mechanics with our classical intuition. However, in a simple argument Bell has demonstrated[2] that this attempt must necessarily fail. Not only has the contradiction between the assumptions of local realism and quantum mechanics deep fundamental meaning, it also lies at the very heart of more efficient solutions to certain specific communication tasks[3].

It is then natural to confront these two contradicting theories in an experiment. For the Bell theorem, many systems have been studied, but quantum interferometry is the most common field of implementation. However, any experiment will suffer from some imperfections, which could be both of a technical nature (to be removed by more effort of the experimentalist) and intrinsically irreversible. They degrade the quality of quantum correlations, making them more similar to those reproducible by local operations and classical communication. This fact is reflected in the theoretical description by mixing the desired state with a certain noise. The simplest noise model is the so-called white noise. It has a uniform structure and degrades all correlations of the state by a constant factor, v, which corresponds to Michelson’s interference visibility.

Despite lack of a physical motivation in some cases, robustness of Bell inequality violation against white noise is often considered a benchmark of the strength of non-classicality of states. In this contribution we conduct a numerical investigation of multiquant states that are the most robust against this imperfection, in the framework of two measurements per side. We find that the most noise is tolerated in protocols similar to those described in Ref. [4], where a group of observers simply projects on fixed local states. More surprisingly, the most robust states are products of GHZ and pure one-qubit states.

Lately, considerable research effort has been dedicated to the study of noise robustness of nonlocal multipartite entangled states (see e.g., [5,6]). The present study can be considered as part of this research program.

II. NOISE PROPERTIES OF THE GHZ AND DICKE STATES

At the beginning we analyze how correlations of two prominent families of states are resistant to white noise admixture. The states which we consider are: the N-qubit GHZ state[7]:

$$|\text{GHZ}_N\rangle = \frac{1}{\sqrt{2}}(|0\ldots0\rangle + |1\ldots1\rangle)$$ (1)

and the N-qubit Dicke states (with e excitations)[8]:

$$|\text{D}_N^e\rangle = \frac{1}{\sqrt{\binom{N}{e}}} \sum_{\pi} |\pi(0\ldots01\ldots10\ldots0)\rangle,$$ (2)

where $\pi$ denotes a permutation of e ones and $(N-e)$ zeros in the ket, and $\binom{N}{e}$ gives a number of such permutations. The special case of $e = 1$ corresponds to N-qubit W states. Our task is to find, for a given state $\rho$, the critical value $\nu_{\text{crit}}$ of the parameter $\nu$ in the mixture:

$$\rho(\nu) = \nu \rho + \frac{1-\nu}{2^N} \mathbb{I}.$$ (3)

If $\nu > \nu_{\text{crit}}$, there does not exist any local realistic model describing quantum correlations of experimental events, i.e. the state violates a certain Bell inequality. The parameter $\nu_{\text{crit}}$ is usually called “critical visibility”.

A. N-qubit GHZ state

The GHZ states exhibit maximal violation of the Bell inequalities[9,10] for experiments with two alternative measurements setting per party. A noisy N-qubit GHZ state, given by:

$$\rho_{\text{GHZ}}(\nu) = \nu |\text{GHZ}_N\rangle \langle \text{GHZ}_N| + \frac{1-\nu}{2^N} \mathbb{I}$$ (4)

leads to the critical visibility:

$$\nu_{\text{crit}}^{\text{GHZ}} = \frac{1}{2(N-1)/2}.$$ (5)
Actually, the set of inequalities in [10] comprises all tight two-setting correlation-type Bell inequalities, hence the above threshold [13] for the visibilities defines the lowest one among all such correlations.

**B. N-qubit Dicke state**

For the case of the noisy Dicke state:

\[
\rho_{D_N}(v) = v_{GHZ} |D_N^e\rangle \langle D_N^e| + \frac{1-v}{2N} I, \quad (6)
\]

we introduce a new Bell inequality \( \langle C_N \rangle \leq 2 \) for \( N \geq 3 \) in the following iterative form:

\[
C_N = (1-A_1^{(N)})C_{N-1} + 2A_1^{(N)}, \quad (7)
\]

where \( C_2 = A_1^{(1)} A_2^{(2)} + A_1^{(1)} A_2^{(2)} + A_1^{(2)} A_2^{(1)} - A_2^{(1)} A_2^{(2)} \) is the CHSH expression [11] and \( A_1^{(j)} \) denotes a dichotomic observable measured by \( j \)th observer when he/she chooses \( j \)th measurement setting. Note that \( (N-2) \) observers perform only a single measurement \( A_1^{(k)} \) \( (k = 3, \ldots, N) \). The Bell expression (7) can be explicitly written as:

\[
C_N = C_2 + (1-C_2) \left( \sum_{i=3}^{N} A_i^{(i)} - \sum_{3 \leq i<j \leq N} A_i^{(i)} A_j^{(j)} \right) + \sum_{3 \leq i<j<k \leq N} A_i^{(i)} A_j^{(j)} A_k^{(k)} - \cdots + (-1)^{N+1} A_1^{(3)} \cdots A_1^{(N)}. \quad (8)
\]

We also mention that this inequality reduces to the one appeared in [12] for \( N = 3 \).

In order to find the quantum value of [13] let us choose measurement settings for the last \( (N-2) \) observers \( A_1^{(i)} = -\sigma_z \) \( (i = 3, \ldots, N) \). Note that due to the permutational symmetry of the Dicke states the correlations are the same for any particular set of subsystems. The quantum value of the inequality [13] can read:

\[
\langle C_N \rangle_{D_N} = \langle C_2 \rangle_{D_N} + (1-\langle C_2 \rangle_{D_N}) \times \sum_{k=1}^{N} (-1)^{2k+1} \binom{N}{k} T_{z^{0\ldots0}z^{0\ldots0}}^{N-k}_{k}, \quad (9)
\]

where the expectation value of the CHSH operator is equal to \( \langle C_2 \rangle_{D_N} = 2\sqrt{2} \cdot \frac{\langle e \rangle}{\langle v \rangle} v \) and

\[
T_{z^{0\ldots0}z^{0\ldots0}}^{N-k}_{k} = \frac{v}{\langle e \rangle} \sum_{j=0}^{[e/2]} \left[ \binom{n-k}{e-2j} \binom{k}{2j} - \binom{n-k}{e-(2j+1)} \binom{k}{2j+1} \right]. \quad (10)
\]

After calculations we get:

\[
\langle C_N \rangle_{D_N} = v \left( 2 + \frac{2^N (\sqrt{2} - 1)}{\langle e \rangle} \right). \quad (11)
\]

**TABLE I. The critical visibilities for the GHZ and W states for \( N \leq 10 \) and two measurement settings per observer. If \( v > v_{crit} \), there does not exist any local realistic model describing quantum probabilities of experimental events. For \( N \geq 9 \) the W states lead to lower critical visibility than the GHZ states.**

| \( N \) | \( v^\text{GHZ}_{crit} \) | \( v^\text{W}_{crit} \) |
|---|---|---|
| 3 | 0.5000 | 0.6442 |
| 4 | 0.3536 | 0.5294 |
| 5 | 0.2500 | 0.4018 |
| 6 | 0.1768 | 0.2774 |
| 7 | 0.1250 | 0.1736 |
| 8 | 0.0884 | 0.1034 |
| 9 | 0.0625 | 0.0578 |
| 10 | 0.0442 | 0.0313 |

Comparing Eqs. (5) and (12) we can conclude that for any type of the Dicke state \( D_N^e \) (for any arbitrary number of excitations \( e \)), there is a critical number of particles \( N_{e}^{crit} \) such that \( v_{crit} \) for \( D_{N_{e}^{crit}}^e \) is lower then critical visibility for the \( N \)-qubit GHZ state. For example: for \( e = 1 \), \( N_{e}^{crit} = 11 \); for \( e = 2 \), \( N_{e}^{crit} = 19 \); for \( e = 5 \), \( N_{e}^{crit} = 44 \); for \( e = 10 \), \( N_{e}^{crit} = 88 \). It means that for \( N > N_{e}^{crit} \) the Dicke states \( D_N^e \) becomes more robust against white noise admixture than the GHZ states in terms of violation of local realism. The value \( N_{e}^{crit} = 11 \) for \( e = 1 \) was the first time calculated in [4] with some projective method. These results are in good agreement with the recent ones [9] where the most robust Dicke states subject to losses are found to correspond to only few excitations \( e \).

**C. Numerical method**

We also use the numerical method (see e.g. [13]) based on linear programming to find critical visibilities for the GHZ and W states up to 10 qubits in experiments with two alternative measurement settings per side. The method does not need any knowledge at all of the forms of Bell inequalities. However, the results obtained by this method are equivalent to the analysis of the full set of (probabilistic) Bell inequalities formulated for a given experimental situation. The results are presented in Table I.

Analyzing the critical values one can see that the W state is more resistant to white noise than the GHZ state already for \( N = 9 \). This result is stronger than the one obtained by means of the inequality [12] or the projective method [4].
III. PARTIALLY PRODUCT STATES ARE HIGHLY RESISTANT TO NOISE

We also apply the linear programming method to analyze noise resistance of \(k\)-product states of \(N\) qubits in the following form:

\[
|\psi_{N}^{k-\text{prod}}\rangle = \big| \underbrace{0 \ldots 0}_{1 \ldots k} \big| \text{GHZ}_{N-k} \rangle. \tag{13}
\]

The results for two measurement settings per party are presented in Tab. \ref{tab:critical-visibility}. Surprisingly, the obtained critical parameters \(v_{\text{crit}}\) are very low and lower then the corresponding ones for the \(N\)-qubit GHZ state. The surprise is due to the fact that the \(N\)-qubit GHZ state is genuinely \(N\)-partite entangled and maximizes many entanglement conditions and measures \[14\].

Moreover, for \(2 \leq N \leq 6\) we identify states that lead to the lowest critical visibility. This was possible after including an optimization over all pure states in the numerical method. The optimal states are: for \(N = 2\), \(|\text{GHZ}_2\rangle\); for \(N = 3\), \(|\text{GHZ}_3\rangle\); for \(N = 4\), \(|0\rangle_1|\text{GHZ}_3\rangle\); for \(N = 5\), \(|00\rangle_{12}|\text{GHZ}_3\rangle\); and for \(N = 6\), \(|000\rangle_{123}|\text{GHZ}_3\rangle\). These states are the most robust states against white noise admixture.

A. Inequalities

We present two families of the Bell-type inequalities, which recover the results presented in Table \ref{tab:critical-visibility} for the states: \(|\psi_{N}^{(N-2)\text{-prod}}\rangle = |0\ldots0\rangle_{1\ldots N-2}\text{GHZ}_{2}\rangle\) and \(|\psi_{N}^{(N-3)\text{-prod}}\rangle = |0\ldots0\rangle_{1\ldots N-3}\text{GHZ}_{3}\rangle\).

The first inequality, optimal for the state \(\psi_{N}^{(N-2)\text{-prod}}\rangle\), has the same form as the inequality given in Eq. \[4\]. Taking again \(A^{(i)} = -\sigma_z\) for \(2 < i \leq N\), we obtain \(\langle A^{(i)} \rangle_{\psi_{N}^{(N-2)\text{-prod}}} = -1\). The expectation value of \(\langle C_2 \rangle_{\psi_{N}^{(N-2)\text{-prod}}} = 2\sqrt{2}\). Then the critical visibility is given by:

\[
v_{\text{crit}}^{(N-2)\text{-prod}} = \frac{1}{1 + (\sqrt{2} - 1)2^{N-2}} \tag{14}
\]

and for \(N > N_{\text{crit}} = \lceil \log_2(12 + 8\sqrt{2}) \rceil = 5\) is lower than for the GHZ state.

The second inequality, optimal for the state \(\psi_{N}^{(N-3)\text{-prod}}\rangle\) has a form \(\langle M_N \rangle \leq 2\) for \(N \geq 4\), where

\[
C_N = (1 - A_1^{(N)})M_{N-1} + 2A_1^{(N)} \tag{15}
\]

and \(M_3 = -A_1^{(1)}A_1^{(2)}A_1^{(3)} + A_1^{(1)}A_2^{(2)}A_2^{(3)} + A_2^{(1)}A_2^{(2)}A_1^{(3)} + A_2^{(1)}A_1^{(2)}A_2^{(3)}\) is the Mermin expression \[9\]. The observers choose again their measurements as \(A^{(i)} = -\sigma_z\) for \(3 < i \leq N\). We obtain \(\langle A^{(i)} \rangle_{\psi_{N}^{(N-3)\text{-prod}}} = -1\) and the expectation value of \(\langle C_3 \rangle_{\psi_{N}^{(N-3)\text{-prod}}} = 4\). Therefore, the critical visibility is equal to:

\[
v_{\text{crit}}^{(N-3)\text{-prod}} = \frac{8}{8 + 2^N}, \tag{16}
\]

and is lower than the corresponding value for the GHZ state for \(N \geq 4\).

IV. A SINGLE BELL INEQUALITY VIOLATED BY ANY PURE ENTANGLED STATE

Our construction is based on the inequality \[8\]. It is not difficult to see that this inequality is equivalent to the following one:

\[
CH^{(1,2)} = \prod_{k=3}^{N} p(A_{k}^{(k)}) \leq 0, \tag{17}
\]

where \(CH^{(i,j)} = p(A_{i}^{(i)}, A_{j}^{(j)}) + p(A_{i}^{(i)}, A_{j}^{(j)}) + p(A_{i}^{(i)}, A_{j}^{(j)}) - p(A_{i}^{(i)}, A_{j}^{(j)}) - p(A_{j}^{(i)}) - p(A_{j}^{(j)})\) is the Clauser-Horne expression \[10\]. \(p(A_{i}^{(i)})\) denotes the probability of obtaining a “1” result by the \(i\)-th observer choosing observable \(A_{k}\) and \(p(A_{i}^{(i)}, A_{j}^{(j)})\) is the probability of
obtaining “1” by ith and jth observers choosing \(A_i\) and \(A_j\) observable, respectively.

From the above Bell inequality (17), we construct the following symmetrized one (for any \(N \geq 3\)):

\[
\sum_{1 \leq i < j \leq N} CH^{(i,j)} \prod_{k=1 \atop k \neq i,j}^{N} p(A^{(k)}_1) \leq 0. \tag{18}
\]

This inequality involves three binary outcome settings \((A_1, A_2, A_3)\) for each party. For the simplest case of three observables \((N = 3)\), it looks as follows:

\[
CH^{(1,2)} p(A^{(1)}_1) + CH^{(1,3)} p(A^{(2)}_1) + CH^{(2,3)} p(A^{(3)}_1) \leq 0. \tag{19}
\]

In order to demonstrate that the single N-party Bell inequality (18) is violated by any N-party pure entangled states, we make use of the result of Popescu and Rohrlich [16] (see also [17]). They showed that for any N-party pure entangled state there exist \(N - 2\) local projections which leave the remaining two systems (say, systems \(i\) and \(j\)) in a pure entangled state. Since any pair of pure entangled states violate the Clauser-Horne (CH) inequality [18], it implies that inequality (17) is violated by this state (where the successful projections are associated with outcomes “1”). The fact that \(i\) and \(j\) can be any two systems out of the \(N\) systems is captured by the symmetrized inequality (18). Indeed, let the parties \(i\) and \(j\) output 0 for their first measurement (i.e. they carry out a degenerate measurement resulting in \(p(A^{(i)}_1) = 0\) and \(p(A^{(j)}_1) = 0\), whereas the successful projections on the rest of the parties are associated with outcome “1”. In this case the only term which survives in (18) is the one of (17), the other terms giving zero contribution.

Though there exists a more economical Bell inequality in terms of number of settings (it consists of only two settings per party) which is violated by any pure entangled states [19], we believe the present proof based on our new inequality (18) is simpler.

V. CONCLUSIONS

We have analyzed quantum states, which offer the highest Bell inequality violation robustness against an admixture of the white noise. This robustness was studied in a specific experimental scenario, with two measurement settings per side. Up to \(N = 8\) we find that the optimal states are products of three-qubit GHZ states and \((N - 3)\) pure single qubit states. The corresponding inequalities represent a protocol, in which \((N - 3)\) observers attempt to locally project the states, while the remaining three conduct a GHZ experiment. This result challenges common, but contradictory beliefs: that entanglement distributed among many parties is more strongly non-classical, and that the robustness against white noise is a proper measure of quantumness.

ACKNOWLEDGMENTS

This work is a part of the FNP TEAM project cofinanced by the EU Regional Development Fund. W.L. and M.W. are supported by the Polish Ministry of Science and Higher Education Grant No. IdP2011 000361. T.V. acknowledges financial support from a János Bolyai Grant of the Hungarian Academy of Sciences, the Hungarian National Research Fund OTKA (K111734), and SEFRI (COST action MP1006).

[1] A. Einstein, B. Podolski, N. Rosen, Phys. Rev. 47, 777 (1935).
[2] J. S. Bell, Physics (Long Island, N. Y. C.) 1, 195 (1964).
[3] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, S. Wehner, Rev. Mod. Phys. 86, 419 (2014).
[4] A. Sen, U. Sen, M. Wieśniak, D. Kaszlikowski, M. Żukowski, Phys. Rev. A, 86, 062306 (2003).
[5] R. Chaves, A. Acín, L. Aloita, D. Cavalcanti, Phys. Rev. A 89, 042106 (2014).
[6] A. Sohbi, I. Zaquine, E. Diamanti, D. Markham, arXiv:1411.4489 (2014); T.J. Barnea, G. Pütz, J. Bohr Brask, N. Brunner, N. Gisin, Y.-C. Liang, arXiv:1412.5953 (2014).
[7] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in Bells Theorem, Quantum Theory, and Conceptions of the Universe, edited by M. Kafatos (Kluwer Academic, Dordrecht 1989), p. 69.
[8] R. H. Dicke, Phys. Rev. 93, 99 (1954).
[9] N. D. Mermin, Phys. Rev. Lett. 65, 1838 (1990).
[10] H. Weinfurter and M. Żukowski, Phys. Rev. A 64, 010102 (2001); R.F. Werner and M.M. Wolf, ibid. 64, 032112 (2001); M. Żukowski and C. Brukner, Phys. Rev. Lett. 88, 210401 (2002).
[11] J.F. Clauser, M.A. Horne, A. Shimony, R.A. Holt, Phys. Rev. Lett. 23, 880 (1969).
[12] N. Brunner and T. Vértesi, Phys. Rev. A 86, 042113 (2012).
[13] J. Gruca, W. Laskowski, M. Żukowski, N. Kiesel, W. Wieczorek, C. Schmid and H. Weinfurter, Phys. Rev. A 82, 012118 (2010); J. Gondzio, J. A. Gr鲁ka, J.A. Julian Hall, W. Laskowski, M. Žukowski, J. Comp. App. Math. 263, 392 (2014).
[14] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).
[15] J. Clauser, M. Horne, Phys. Rev. D 10, 526 (1974).
[16] S. Popescu and D. Rohrlich, Phys. Lett. A 166, 293 (1992).
[17] D. Cavalcanti, M. L. Almeida, V. Scarani, A. Acín, Nat. Commun. 2, 184 (2011).
[18] N. Gisin, Phys. Lett. A 154, 201 (1991); N. Gisin and A. Peres, Phys. Lett. A 162, 15 (1992).
[19] S. Yu, Q. Chen, C. Zhang, C. H. Lai, C. H. Oh, Phys. Rev. Lett. 109, 120402 (2012).