The Chronological Operator Algebra
and
Formal Solutions of Differential Equations

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Abstract
The aim of this paper is twofold. First, we obtain the explicit exact formal solutions of differential equations of different types in the form with Dyson chronological operator exponents. This allows us to deal directly with the solutions to the equations rather than the equations themselves. Second, we consider in detail the algebraic properties of chronological operators, yielding an extensive family of operator identities. The main advantage of the approach is to handle the formal solutions at least as well as ordinary functions. We examine from a general standpoint linear and non-linear ODEs of any order, systems of ODEs, linear operator ODEs, linear PDEs and systems of linear PDEs for one unknown function. The methods and techniques involved are demonstrated on examples from important differential equations of mathematical physics.

1 Introduction
There are several universal approaches to solving DE. The most promising of them are Lie symmetry methods [1], [2] and advanced methods for the linear equations which are based on differential Galois theory [3].

The aim of this contribution is to show examples of the possibilities for solving differential equations with the method based on Dyson’s operator solution representation in the form of time-ordered exponentials [4] (see also [5], [6]). It is well-known that Dyson’s use of time-ordering is the fundamental conceptual tool in quantum field theory. This tool has now become a natural part of many branches of physics and is even used in parts of engineering.

The method allows us to find the explicit exact operator solutions of many problems if we succeed in reformulating a given problem into a first-order linear one. The next step of the approach is directed to transforming the operator solution into a more practical, calculable or useful expression. The use of operator methods is efficient in manipulating time-ordered expressions with help of exponential identities like the Baker-Campbell-Hausdorff formulae [7], [8]. Our second aim in this paper is to give general methods for obtaining such identities.
We have in mind that at every stage the chronological exponential can be expanded in formal operator power series neglecting all the convergence problems that can appear with, for example, analytical functions.

As a result, this method allows us to deal directly with the solutions to the equations rather than the equations themselves.

The main advantages of the method are in its compactness, clarity and simplicity. It is also essential that we can handle the operator expressions in manner similar to ordinary functions.

It is clear that there are some correlations between the operator method and other common methods, especially the Lie approach.

We choose example equations for the operator method from important differential equations from many branches of science. The main aspects of the method are quite easily generalized for more complicated equations.

As the paper is addressed to practical problems in many branches of physics and engineering for which solutions are not yet at hand and to avoid complications, we choose a simple (and non-rigorous in places) manner of exposition in that we can always verify the solutions by substitution into the initial equations. Analytical restrictions are obvious from the context as a rule.

2 Definitions and notations

An operator \( A \) is defined as a mapping of a function \( f \) (from a ring \( R \)) into a function \( Af \) (from the same ring \( R \)). An operator \( A \) is linear if it maps any two functions \( \phi \) and \( \psi \) in such a way that

\[
A(\phi + \psi) = A\phi + A\psi.
\]

Operators will be written in the bold font seen above to avoid any ambiguities with the exception being the usual differential operator \( \frac{\partial}{\partial x} \) notation. In every case throughout this paper, we shall be dealing exclusively with operators that are linear.

The operator \( \Delta \) is derivative if

\[
\Delta(A + B)f = \Delta Af + \Delta Bf,
\]

\[
\Delta Af = (\Delta A)f + A\Delta f
\]

for any \( f \) (from a ring \( R \)). It follows from this definition that the inner derivative of an operator \( A \) is

\[
(\Delta A) = \Delta A - A\Delta = [\Delta, A],
\]

where \([\Delta, A]\) is the conventional notation of the two operator commutator. In this paper we will consider only differentiable operators \( A \) in sense that the commutator \([\Delta, A]\) exists.
If $\Delta = \partial \partial x$ then such a definition of the inner derivative of the differentiable operator $A(t)$ is in accord with the classical analytic definition

$$\frac{\partial A(x)}{\partial x} = \lim_{\epsilon \to 0} \frac{A(x + \epsilon) - A(x)}{\epsilon}.$$ 

We will find need for an operator-valued function $F(A)$ which is an operator itself. Only functions which can be described by power series are considered in this work.

Important examples for this paper are the following types of linear operators:

$$L_1 = f_0 + f_1 \frac{\partial}{\partial x_1} + \ldots + f_n \frac{\partial}{\partial x_n},$$

(1)

where $f_i$ are arbitrary functions of $t, x_1, \ldots, x_n$, $L_1$ is obviously a derivative and

$$L_2 = F(t, x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}).$$

Though $L_2$ is a generalization of $L_1$, nevertheless $L_2$ is not a derivative in general, as e.g., $(\frac{\partial}{\partial x})^2 = \frac{\partial^2}{\partial x^2}$ does not satisfy the above definition of derivative.

3 The linear first-order differential equations and basic identities for chronological operator exponents

3.1 Dyson’s form of the solution of first-order linear differential equation

Let us start with the first-order linear differential equation with respect to $t$ for $u(t, \bar{\rho}) = u(t)$:

$$\frac{\partial u(t)}{\partial t} = L(t)u(t)$$

(2)

with the initial condition

$$u(t)|_{t=a} = v,$$

(3)

where $\bar{\rho}$ is a set of parameters say $x_1, \ldots, x_n$, $L(t) = L(t, \bar{\rho})$ is a linear operator which does not depend on $\frac{\partial}{\partial t}$ explicitly, and $v$ is some function of $\bar{\rho}$.

This equation can be solved by the following iteration scheme

$$u_0(t) = v,$$

$$\frac{\partial u_n(t)}{\partial t} = L(t)u_{n-1}(t),$$

which for $t > a$ leads to

$$u(t) = E(t, a; L(t))v,$$

(4)
where operator $E$ has the following series expansion ($I$ is the identity operator)

$$E(t, a; L(t)) = I + \int_a^t d\tau \, L(\tau) + \cdots + \int_a^{\tau_1} d\tau_1 \int_a^{\tau_2} d\tau_2 \cdots \int_a^{\tau_{n-1}} d\tau_n \, L(\tau_1) \, L(\tau_2) \cdots L(\tau_n) + \cdots \tag{5}$$

The solution (4) (taking into account (5)) has the following properties:

(a). Operator $E$ satisfies the operator differential equation

$$\frac{\partial E}{\partial t} = L(t)E \tag{6}$$

with initial condition

$$E|_{t=a} = I, \tag{7}$$

or the equivalent integral equation

$$E(t) = I + \int_a^t d\tau \, L(\tau)E(\tau).$$

(b). All operators $L$ in the series expansion of $E$ are ordered in sense that any operator term $L(\tau_1) \, L(\tau_2) \cdots L(\tau_n)$ corresponds to the requirement that $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n$. The ordering here is very essential as operators $L(\tau_i)$ and $L(\tau_j)$ do not commute in general.

The property (a) suggests that the operator $E$ is a kind of $\exp\{\int_a^t d\tau \, L(\tau)\}$ but expansion of the exponent here does not lead to the required operator ordering (b), as for example, the third term of the $E$ expansion

$$\int_a^{\tau_1} d\tau_1 \int_a^{\tau_2} d\tau_2 \, L(\tau_1) \, L(\tau_2)$$

differs from the third term of the exponent expansion

$$\frac{1}{2} \int_a^t d\tau_1 \int_a^t d\tau_2 \, L(\tau_1) \, L(\tau_2).$$

The next trick was proposed by Freeman J. Dyson [4] (see also [5], [6]). Let us demonstrate it on the third term of the $E$ expansion. By changing notation of integration variables and integration order we obtain

$$\int_a^{\tau_1} d\tau_1 \int_a^{\tau_2} d\tau_2 \, L(\tau_1) \, L(\tau_2) = \int_a^{\tau_1} d\tau_1 \int_a^{\tau_2} d\tau_2 \, L(\tau_2) \, L(\tau_1),$$

so

$$\int_a^{\tau_1} d\tau_1 \int_a^{\tau_2} d\tau_2 \, L(\tau_1) \, L(\tau_2) = \frac{1}{2} \int_a^t d\tau_1 \{ \int_a^{\tau_1} d\tau_2 \, L(\tau_1) \, L(\tau_2) + \int_{\tau_1}^t d\tau_2 \, L(\tau_2) \, L(\tau_1) \}.$$
If we now introduce a T-operator defined as

$$ T\{L(\tau_1)L(\tau_2)\} = \begin{cases} L(\tau_1)L(\tau_2) & \text{if } \tau_1 \geq \tau_2; \\ L(\tau_2)L(\tau_1) & \text{if } \tau_1 < \tau_2, \end{cases} $$

then n-th term of the right-hand side of (5) can be transformed to

$$ \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \ldots \int_a^{\tau_{n-1}} d\tau_n \ L(\tau_1) \ L(\tau_2) \ldots L(\tau_n) = $$

$$ \frac{1}{n!} \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \ldots \int_a^{\tau_n} d\tau_n \ T\{L(\tau_1) \ L(\tau_2) \ldots L(\tau_n)\}. $$

Removing the T-operator outside of the integral signs means we can express

$$ E(t,a;L(t)) = T\{1 + \int_a^t d\tau \ L(\tau) + $$

$$ \cdots + \frac{1}{n!} \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \ldots \int_a^{\tau_n} d\tau_n \ L(\tau_1) \ L(\tau_2) \ldots L(\tau_n) + \ldots \} $$

or in the final form with the chronological operator exponential

$$ E(t,a;L(t)) = T \exp\{\int_a^t d\tau \ L(\tau)\}, \quad (8) $$

where we have in mind that (8) represents the series expansion and that

$$ T \{L(\tau_1) \ L(\tau_2) \ldots L(\tau_n)\} = L(\tau_{a_1}) \ L(\tau_{a_2}) \ldots L(\tau_{a_m}). $$

$$ \tau_{a_1} \geq \tau_{a_2} \geq \cdots \geq \tau_{a_m} $$

The expression (8) is the formal solution of the operator differential equation (6) with initial condition (7). The formal solution of the problem (2), (3) is obviously

$$ u(t) = T \exp\{\int_a^t d\tau \ L(\tau)\} v. \quad (9) $$

The main advantage of solutions in the chronological exponential form is the explicit dependence of all parameters of the problem and the relatively clear way to obtaining approximate non-formal solutions through series expansion

$$ T \exp\{\int_a^t d\tau \ L(\tau)\} = 1 + \int_a^t d\tau \ L(\tau) + $$

$$ \cdots + \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \ldots \int_a^{\tau_{n-1}} d\tau_n \ L(\tau_1) \ L(\tau_2) \ldots L(\tau_n) + $$

$$ \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \ldots \int_a^{\tau_{n+1}} d\tau_{n+1} \ L(\tau_1) \ L(\tau_2) \ldots L(\tau_{n+1}) T \exp\{\int_a^{\tau_{n+1}} d\xi \ L(\xi)\}. $$
Analogously, the solution of the problem (2), (3) for $t < a$ is similar to (9), but instead of the operator $T$ it is necessary to replace it with the operator $T_0$, which represents a product of $L$’s, in reverse order
\[ T_0 \{ L(\tau_1) \ L(\tau_2) \ldots L(\tau_n) \} = L(\tau_{\alpha_1}) \ L(\tau_{\alpha_2}) \ldots L(\tau_{\alpha_m}). \]
\[ \tau_{\alpha_1} \leq \tau_{\alpha_2} \leq \cdots \leq \tau_{\alpha_m} \]

The problem (2), (3) is deterministic in the sense that if we start from an initial condition at $t = a$ we can find a unique $u(t, \vec{\rho})$ for any $t$. If we now consider equation (4) with a new initial condition $u(t, \vec{\rho})|_{t=b} = v(\vec{\rho})$ and recalculate $u(t, \vec{\rho})$ for $t \geq b$ we obtain (for $a \leq b \leq t$)
\[ u(t) = T \exp \{ \int_a^t d\tau \ L(\tau) \} v = T \exp \{ \int_b^t d\tau \ L(\tau) \} T \exp \{ \int_a^b d\tau \ L(\tau) \} v \]
for any $v$. The operators $T$ here and everywhere in this paper act on operators of their exponentials only. That is, for $a \leq b \leq t$ we have the following operator identity
\[ T \exp \{ \int_a^t d\tau \ L(\tau) \} = T \exp \{ \int_b^t d\tau \ L(\tau) \} T \exp \{ \int_a^b d\tau \ L(\tau) \}. \] (10)

Analogously, if on the second stage we recalculate $u(t, \vec{\rho})$ from $t$ to $a$ backward, we obtain that
\[ v = T_0 \exp \{ \int_t^a d\tau \ L(\tau) \} u(t) = T_0 \exp \{ -\int_a^t d\tau \ L(\tau) \} T \exp \{ \int_a^t d\tau \ L(\tau) \} v \]
for any $v$. Therefore
\[ T_0 \exp \{ -\int_a^t d\tau \ L(\tau) \} T \exp \{ \int_a^t d\tau \ L(\tau) \} = 1. \] (11)

So the operator
\[ T_0 \exp \{ -\int_a^t d\tau \ L(\tau) \} \]
is inverse to
\[ T \exp \{ \int_a^t d\tau \ L(\tau) \} \]
and these operators commute.

It can be easily seen that the operator
\[ E^{-1}(t) = T_0 \exp \{ \int_a^t d\tau \ L(\tau) \} \] (12)
is the solution of the following operator differential equation
\[ \frac{\partial E^{-1}(t)}{\partial t} = E^{-1}(t)L(t) \] (13)
with an initial condition
\[ E(t)|_{t=a} = 1 \] (14)
and that
\[ E^{-1}(t) = 1 + \int_a^t d\tau E^{-1}(\tau)L(\tau) \] (15)
which leads to the following series expansion
\[ E^{-1}(t) = T_0 \exp \left\{ \int_a^t d\tau L(\tau) \right\} = 1 + \int_a^t d\tau L(\tau) + \cdots + \int_a^{\tau_1} d\tau_1 \int_a^{\tau_2} d\tau_2 \cdots \int_a^{\tau_n} d\tau_n L(\tau_n) L(\tau_{n-1}) \cdots L(\tau_1) + \int_a^t d\tau_1 \int_a^{\tau_2} d\tau_2 \cdots \int_a^{\tau_{n+1}} d\tau_{n+1} T_0 \exp \left\{ \int_a^{\tau_{n+1}} d\xi L(\xi) \right\} L(\tau_{n+1}) L(\tau_n) \cdots L(\tau_1). \] (16)

It follows from properties (13)-(16) (or from (10)) that for \( a \leq b \leq t \)
\[ T_0 \exp \left\{ \int_a^t d\tau L(\tau) \right\} = T_0 \exp \left\{ \int_a^b d\tau L(\tau) \right\} T_0 \exp \left\{ \int_b^t d\tau L(\tau) \right\}. \] (17)

With property (6) at hand one can easily verify that the solution of the
inhomogeneous linear differential equation
\[ \frac{\partial u(t)}{\partial t} = L(t)u(t) + \phi(t) \quad (u(t)|_{t=a} = v) \]
has the following operator form \( t > a \)
\[ u(t) = T \exp \left\{ \int_a^t d\tau L(\tau) \right\} v + \int_a^t d\tau T \exp \left\{ \int_\tau^t d\xi L(\xi) \right\} \phi(\tau). \] (18)

The operator technique in the form given above is immediately suitable only for solving the linear problems. As it will be demonstrated later, an analogous approach is fruitful for more complicated problems. However, before touching such problems we need to consider the algebra of chronological operator exponentials in more details.

3.2 Basic transformation identities for chronological operator exponentials

For any linear operator \( L(t) \) (not necessarily invertible) we can form the chronological operator exponential \( \text{(here and later on } t > a) \)
\[ T \exp \left\{ \int_a^t d\tau L(\tau) \right\}, \]
which is always invertible and differentiable with respect to \( t \).
Suppose we have an arbitrary linear invertible differentiable operator \( A(t) \). From the obvious identity

\[
\frac{\partial A(t)}{\partial t} = \left( \frac{\partial A(t)}{\partial t} A^{-1}(t) \right) A(t) \quad (A(t)|_{t=a} = A(a))
\]

we obtain

\[
A(t) = T \exp \left\{ \int_a^t d\tau \frac{\partial A(\tau)}{\partial \tau} A^{-1}(\tau) \right\} A(a). \tag{19}
\]

It follows that the chronological operator exponential can represent the \( t \)-dependence of any linear invertible differentiable operator, and each such operator corresponds to the following linear first-order differential equation

\[
\frac{\partial u(t)}{\partial t} = \left( \frac{\partial A(t)}{\partial t} A^{-1}(t) \right) u(t) \quad (u(t)|_{t=a} = v)
\]

with the solution

\[
u(t) = A(t) A^{-1}(a) v.
\]

As the product of any two of such operators is the linear invertible differentiable operator (i.e., they form group) then the set of all chronological operators forms group too.

If we invert \( A^{-1}(t) \) by \( B(t) \) we get

\[
A^{-1}(t) = A^{-1}(a) T_0 \exp \left\{ - \int_a^t d\tau \frac{\partial A(\tau)}{\partial \tau} A^{-1}(\tau) \right\}
\]

and replace \( A^{-1}(t) \) by \( B(t) \) we get

\[
B(t) = B(a) T_0 \exp \left\{ - \int_a^t d\tau \frac{\partial B^{-1}(\tau)}{\partial \tau} B(\tau) \right\} = B(a) T_0 \exp \left\{ \int_a^t d\tau B^{-1}(\tau) \frac{\partial B(\tau)}{\partial \tau} \right\}
\]

and conclude that any linear invertible differentiable operator can be represented as an inverse ordered operator exponential too. Hence, the two main forms of opposite ordered operator exponents can be expressed through each other (see \( 19 \) and \( 20 \) below).

Let us consider the following operator

\[
K(t) = b(t) T \exp \left\{ \int_a^t d\tau A(\tau) \right\},
\]

where \( b(t) \) is a linear invertible differentiable operator and \( A(t) \) is a linear one.

Differentiating \( K \) with respect to \( t \) we have

\[
\frac{\partial K(t)}{\partial t} = \left( \frac{\partial b(t)}{\partial t} b^{-1}(t) + b(t) A(t) b^{-1}(t) \right) K(t) \tag{20}
\]
and
\[ K(t)|_{t=a} = b(a). \] (21)

As it was shown in the previous section, the operator differential equation with an initial condition has a solution in the following form
\[ K(t) = T \exp\{ \int_a^t d\tau \left[ \frac{\partial b(\tau)}{\partial \tau} b^{-1}(\tau) + b(\tau) A(\tau) b^{-1}(\tau) \right] \} b(a), \]
so we obtain the important identity
\[ b(t) T \exp\{ \int_a^t d\tau A(\tau) \} = T \exp\{ \int_a^t d\tau \left[ \frac{\partial b(\tau)}{\partial \tau} b^{-1}(\tau) + b(\tau) A(\tau) b^{-1}(\tau) \right] \} b(a). \] (22)

Now if
\[ b(t) = T \exp\{ \int_a^t d\tau B(\tau) \}, \]
then from (22) we obtain one of the basic chronological operator identities
\[ T \exp\{ \int_a^t d\tau B(\tau) \} T \exp\{ \int_a^t d\tau A(\tau) \} = T \exp\{ \int_a^t d\tau \left[ \frac{\partial b(\tau)}{\partial \tau} b^{-1}(\tau) + b(\tau) A(\tau) b^{-1}(\tau) \right] \} \times \]
\[ T \exp\{ \int_a^t d\tau T_0 \exp\{ - \int_a^\tau d\xi B(\xi) \} C(\tau) T \exp\{ \int_a^\tau d\xi B(\xi) \} \}. \] (23)

If we denote
\[ C(\tau) = T \exp\{ \int_a^\tau d\xi B(\xi) \} A(\tau) T_0 \exp\{ - \int_a^\tau d\xi B(\xi) \} \]
and solve it with respect to \( A(\tau) \)
\[ A(\tau) = T_0 \exp\{ - \int_a^\tau d\xi B(\xi) \} C(\tau) T \exp\{ \int_a^\tau d\xi B(\xi) \}, \]
then it follows immediately from (23) that
\[ T \exp\{ \int_a^t d\tau [B(\tau) + C(\tau)] \} = T \exp\{ \int_a^t d\tau B(\tau) \} \times \]
\[ T \exp\{ \int_a^t d\tau T_0 \exp\{ - \int_a^\tau d\xi B(\xi) \} C(\tau) T \exp\{ \int_a^\tau d\xi B(\xi) \} \}. \] (24)

Identities (23) and (24) are generalizations of the well-known Baker-Campbell-Hausdorff (BCH) and Zassenhaus formulae for \( t \)-dependent operators in the sense that the classical BCH formula merges two exponential operators into a single one and Zassenhaus formula splits an exponential operator into a product of exponential operators.

At first sight the identities (23) and (24) represent a circle of some kind. To demonstrate the value of these identities we first consider their classical expansions and later (in Subsection 5.2 below) we show that in many cases of practical important that involve (23) and (24) operator expressions can be obtained exactly by a more direct way without using the expansions like (24), (24).
3.3 Baker-Campbell-Hausdorff and Zassenhaus formulae for chronological operator exponents

Let us consider the following construction

\[ K(a) = T \exp \left\{ \int_a^t d\tau \, B(\tau) \right\} A(t) T_0 \exp \left\{ - \int_a^t d\tau \, B(\tau) \right\}. \tag{25} \]

If we differentiate \( K(a) \) with respect to \( a \) we obviously obtain the following expression

\[ \frac{\partial K(a)}{\partial a} = T \exp \left\{ \int_a^t d\tau \, B(\tau) \right\} \left\{ -B(a)A(t) + A(t)B(a) \right\} T_0 \exp \left\{ - \int_a^t d\tau \, B(\tau) \right\}, \]

where \( K(a) \big|_{a=t} = A(t) \).

If we use the commutator notation \( A(t)B(a) - B(a)A(t) = [A(t), B(a)] \) we get after integration that

\[ K(a) = A(t) + \int_a^t d\tau \; T \exp \left\{ \int_\tau^t d\xi \, B(\xi) \right\} [A(t), B(\tau)] T_0 \exp \left\{ - \int_\tau^t d\xi \, B(\xi) \right\} \]

or

\[ T \exp \left\{ \int_a^t d\tau \, B(\tau) \right\} A(t) T_0 \exp \left\{ - \int_a^t d\tau \, B(\tau) \right\} = A(t) - \int_a^t d\tau \; T \exp \left\{ \int_\tau^t d\xi \, B(\xi) \right\} [A(t), B(\tau)] T_0 \exp \left\{ - \int_\tau^t d\xi \, B(\xi) \right\}. \tag{26} \]

This is an analog of so-called integral BCH formula. Iterations of \( 26 \) leads to the following expansion

\[ T \exp \left\{ \int_a^t d\tau \, B(\tau) \right\} A(t) T_0 \exp \left\{ - \int_a^t d\tau \, B(\tau) \right\} = A(t) - \int_a^t d\tau \; T \exp \left\{ \int_\tau^t d\xi \, B(\xi) \right\} [A(t), B(\tau)] T_0 \exp \left\{ - \int_\tau^t d\xi \, B(\xi) \right\} \]

\[ + (-1)^n \int_a^t d\tau_1 \int_{\tau_1}^t d\tau_2 \cdots \int_{\tau_{n-1}}^t d\tau_n \; T \exp \left\{ \int_{\tau_n}^t d\xi \, B(\xi) \right\} \times \]

\[ [[...[A(t), B(\tau_1)], B(\tau_2)]...], B(\tau_n)] T_0 \exp \left\{ - \int_{\tau_n}^t d\xi \, B(\xi) \right\}. \tag{27} \]
If we now substitute (27) into (23) we obtain the more conventional form of BCH formula

\[
T \exp \left\{ \int_a^t d\tau \ B(\tau) \right\} T \exp \left\{ \int_a^t d\tau \ A(\tau) \right\} = 
\]

\[
T \exp \left\{ \int_a^t d\tau \ \{ A(\tau) + B(\tau) - \int_a^\tau d\tau_1 [A(\tau), B(\tau_1)] \right\} + 
\]

\[
\int_a^\tau d\tau_1 \int_a^\tau d\tau_2 \left[ [A(\tau), B(\tau_1)], B(\tau_2) \right] - \ldots 
\]

\[
+ (-1)^n \int_a^\tau d\tau_1 \int_a^\tau \ldots \int_a^\tau d\tau_n T \exp \left\{ \int_a^\tau d\xi \ B(\xi) \right\} \times 
\]

\[
[[[A(\tau), B(\tau_1)], B(\tau_2)], \ldots, B(\tau_n)] \right\} \backslash T_0 \exp \left\{ - \int_a^\tau d\xi \ B(\xi) \right\} \right\}. \quad (28)
\]

The importance of the last BCH formula (28) lies not in the details of the formula, but in the fact that there is one, and the fact that it gives the product of two operators

\[
T \exp \left\{ \int_a^t d\tau \ B(\tau) \right\} T \exp \left\{ \int_a^t d\tau \ A(\tau) \right\} 
\]

as one operator in terms of A and B, brackets of A and B, brackets of brackets, etc.

Analogously, starting from

\[
K(\lambda) = T_0 \exp \left\{ - \int_a^\lambda d\tau \ B(\tau) \right\} A(t) T \exp \left\{ \int_a^\lambda d\tau \ B(\tau) \right\} \quad (29)
\]

and differentiating K(\lambda) with respect to \lambda we obtain the following expression

\[
\frac{\partial K(\lambda)}{\partial \lambda} = T_0 \exp \left\{ - \int_a^\lambda d\tau \ B(\tau) \right\} [-B(\lambda)A(t) + A(t)B(\lambda)] T \exp \left\{ \int_a^\lambda d\tau \ B(\tau) \right\},
\]

where

\[
K(\lambda)|_{\lambda=a} = A(t).
\]

After integration we get that

\[
K(\lambda) = A(t) + \int_a^\lambda d\tau T_0 \exp \left\{ - \int_a^\tau d\xi \ B(\xi) \right\} [A(t), B(\tau)] T \exp \left\{ \int_a^\tau d\xi \ B(\xi) \right\}
\]

for any \lambda. By setting \lambda = t we arrive to

\[
T_0 \exp \left\{ - \int_a^t d\tau \ B(\tau) \right\} A(t) T \exp \left\{ \int_a^t d\tau \ B(\tau) \right\} = 
\]

\[
A(t) + \int_a^t d\tau T_0 \exp \left\{ - \int_a^\tau d\xi \ B(\xi) \right\} [A(t), B(\tau)] T \exp \left\{ \int_a^\tau d\xi \ B(\xi) \right\}. \quad (30)
\]
This is a mirror-like twin brother of the integral BCH formula. We note here that (30) is the corollary of (26) if we have in mind the properties of the chronological operators (10), (11) and (17). Iterations of (30) lead to the following expansion (compare with (27))

\[
\begin{align*}
T_0 \exp \left\{ -\int_a^t d\tau \, B(\tau) \right\} A(t) & \exp \left\{ \int_a^t d\tau \, B(\tau) \right\} = \\
&= A(t) + \int_a^t d\tau \left[ A(t), B(\tau) \right] + \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \left[ \left[ A(t), B(\tau_1) \right], B(\tau_2) \right] + \\
&\quad \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \ldots \int_a^{\tau_{n-1}} d\tau_n \exp \left\{ -\int_a^{\tau_n} d\xi \, B(\xi) \right\} \times \\
&\quad \left[ \ldots \left[ A(t), B(\tau_1) \right], B(\tau_2) \right], \ldots, B(\tau_n) \right] T \exp \left\{ \int_a^{\tau_n} d\xi \, B(\xi) \right\}.
\end{align*}
\]

(31)

If we now substitute (30) into (24) we obtain

\[
\begin{align*}
T \exp \left\{ \int_a^t d\tau \left[ B(\tau) + C(\tau) \right] \right\} &= T \exp \left\{ \int_a^t d\tau \, B(\tau) \right\} T \exp \left\{ \int_a^t d\tau \, C(\tau) \right\} + \\
&\quad \int_a^t d\tau_1 T_0 \exp \left\{ -\int_a^{\tau_1} d\xi \, B(\xi) \right\} \left[ C(\tau), B(\tau_1) \right] T \exp \left\{ \int_a^{\tau_1} d\xi \, B(\xi) \right\} \right] \times
\end{align*}
\]

(32)

and with the help of the original identity (24) we further obtain

\[
\begin{align*}
T \exp \left\{ \int_a^t d\tau \left[ B(\tau) + C(\tau) \right] \right\} &= T \exp \left\{ \int_a^t d\tau \, B(\tau) \right\} T \exp \left\{ \int_a^t d\tau \, C(\tau) \right\} \times \\
&\quad T \exp \left\{ \int_a^t d\tau \right\} \int_a^{\tau} d\tau_1 T_0 \exp \left\{ -\int_a^{\tau_1} d\xi \, C(\xi) \right\} T_0 \exp \left\{ -\int_a^{\tau_1} d\xi \, B(\xi) \right\} \times \\
&\quad \left[ C(\tau), B(\tau_1) \right] T \exp \left\{ \int_a^{\tau_1} d\xi \, B(\xi) \right\} T \exp \left\{ \int_a^{\tau_1} d\xi \, C(\xi) \right\} \
\end{align*}
\]

(31)

Iterations of (31) with the help of (23) and (24) lead to more conventional form (but noticeably bulky than (28), so we do not represent it here) of a generalized Zassenhaus formula, which nevertheless tells us that the operator

\[
T \exp \left\{ \int_a^t d\tau \left[ B(\tau) + C(\tau) \right] \right\}
\]

can be expressed as product of operators in terms of A and B, brackets of A and B, brackets of brackets, etc.
3.4 Relationship between opposite ordered chronological exponents and generalized linear operator differential equations

Let us consider now

$$T \exp \{ \int_a^t d\tau [A(\tau) - A(\tau)] \} = 1 = T \exp \{ \int_a^t d\tau A(\tau) \} \times$$

$$T \exp \{ - \int_a^t d\tau T_0 \exp \{ - \int_a^t d\xi A(\xi) \} A(\tau) T \exp \{ \int_a^t d\xi A(\xi) \} \},$$

so we deduce at once that

$$T_0 \exp \{ - \int_a^t d\tau A(\tau) \} =$$

$$T \exp \{ - \int_a^t d\tau T_0 \exp \{ - \int_a^t d\xi A(\xi) \} A(\tau) T \exp \{ \int_a^t d\xi A(\xi) \} \} \quad (33)$$

and further

$$T \exp \{ \int_a^t d\tau A(\tau) \} =$$

$$T_0 \exp \{ \int_a^t d\tau T_0 \exp \{ - \int_a^t d\xi A(\xi) \} A(\tau) T \exp \{ \int_a^t d\xi A(\xi) \} \}. \quad (34)$$

These identities allow us to solve the following equation

$$T_0 \exp \{ \int_a^t d\xi A(\xi) \} A(t) T \exp \{ - \int_a^t d\xi A(\xi) \} = B(t) \quad (35)$$

with respect to $A(t)$. The solution of $(35)$ is as follows

$$A(t) = T_0 \exp \{ - \int_a^t d\xi B(\xi) \} B(t) T \exp \{ \int_a^t d\xi B(\xi) \}. \quad (36)$$

One can prove this by substitution of $(36)$ into $(35)$. Really, in view of $(33)$ and
we have the following chain starting from (35) and then substituting (36)

\[ T_0 \exp \left( \int_a^t d\xi \ A(\xi) \right) A(t) T \exp \left\{ - \int_a^t d\xi \ A(\xi) \right\} = \]

\[ = T_0 \exp \left( \int_a^t d\tau \ T_0 \exp \left( - \int_a^\tau d\xi \ B(\xi) \right) B(\tau) T \exp \left( \int_a^\tau d\xi \ B(\xi) \right) \right) \times \]

\[ T_0 \exp \left\{ - \int_a^t d\xi \ B(\xi) \right\} B(t) T \exp \left( \int_a^t d\xi \ B(\xi) \right) \times \]

\[ T \exp \left\{ - \int_a^t d\tau \ T_0 \exp \left( - \int_a^\tau d\xi \ B(\xi) \right) B(\tau) T \exp \left( \int_a^\tau d\xi \ B(\xi) \right) \right\} = \]

\[ = T_0 \exp \left( \int_a^t d\tau \ B(\tau) \right) T_0 \exp \left\{ - \int_a^t d\tau \ B(\tau) \right\} B(t) \times \]

\[ T \exp \left\{ - \int_a^t d\tau \ B(\tau) \right\} T_0 \exp \left\{ - \int_a^t d\tau \ B(\tau) \right\} = B(t). \]

Let us consider the following operator

\[ K(t) = T_0 \exp \left( \int_a^t d\tau \ A(\tau) \right) B(t) T \exp \left( \int_a^t d\tau \ C(\tau) \right). \] (37)

Differentiating \( K(t) \) with respect to \( t \) we have

\[ \frac{\partial K(t)}{\partial t} = \]

\[ = T_0 \exp \left( \int_a^t d\tau \ A(\tau) \right) [A(t)B(t) + \frac{\partial B(t)}{\partial t} + B(t)C(t)] T \exp \left( \int_a^t d\tau \ C(\tau) \right) \]

and obtain the following differential expression

\[ \frac{\partial K(t)}{\partial t} = a(t)K(t) + K(t)c(t) + b(t), \] (38)

where

\[ a(t) = T_0 \exp \left( \int_a^t d\tau A(\tau) \right) A(t) T \exp \left\{ - \int_a^t d\tau A(\tau) \right\}, \] (39)

\[ c(t) = T_0 \exp \left( - \int_a^t d\tau C(\tau) \right) C(t) T \exp \left( \int_a^t d\tau C(\tau) \right), \] (40)

\[ b(t) = T_0 \exp \left( \int_a^t d\tau A(\tau) \right) \frac{\partial B(t)}{\partial t} T \exp \left\{ - \int_a^t d\tau C(\tau) \right\}, \] (41)

and moreover

\[ K(t)_{|_{t=a}} = B(a). \] (42)

We can solve (39) and (40) with respect to \( A(t) \) and \( C(t) \) respectively

\[ A(t) = T_0 \exp \left\{ - \int_a^t d\tau a(\tau) \right\} a(t) T \exp \left( \int_a^t d\tau a(\tau) \right), \] (43)

\[ C(t) = T_0 \exp \left( \int_a^t d\tau C(\tau) \right) C(t) T \exp \left\{ - \int_a^t d\tau C(\tau) \right\}, \] (44)
\[ C(t) = -T_0 \exp \{ \int_a^t d\tau \, c(\tau) \} \, c(t) \, T \exp \left\{ -\int_a^t d\tau \, c(\tau) \right\}. \] (44)

From (41) we have
\[ \frac{\partial B(t)}{\partial t} = T \exp \left\{ -\int_a^t d\tau \, A(\tau) \right\} \, b(t) \, T_0 \exp \left\{ -\int_a^t d\tau \, C(\tau) \right\} \]

and with (42) it leads to
\[ B(t) = K(a) + \int_a^t d\tau T_0 \exp \left\{ -\int_a^\tau d\xi \, a(\xi) \right\} \, b(\tau) \, T_0 \exp \left\{ -\int_a^\tau d\xi \, c(\xi) \right\} \]

and substituting (43) and (44) we have, taking into account identities (33) and (34), that
\[ B(t) = K(a) + \int_a^t d\tau \, T_0 \exp \left\{ -\int_a^\tau d\xi \, a(\xi) \right\} \, b(\tau) \, T \exp \left\{ -\int_a^\tau d\xi \, c(\xi) \right\} \]

If we now express \( A, B \) and \( C \) in (37) via \( a, b \) and \( c \) then it follows that the formal solution of the operator differential equation (38) with the initial condition \( K(t)|_{t=a} = K(a) \), where \( a, b \) and \( c \) are any linear (not necessarily invertible) operators, which do not depend on \( \frac{\partial}{\partial t} \) explicitly, has the following form
\[ K(t) = T_0 \exp \left\{ \int_a^t d\tau \, T_0 \exp \left\{ -\int_a^\tau d\tau_1 \, a(\tau_1) \right\} \, a(\tau) \, T \exp \left\{ \int_a^\tau d\tau_1 a(\tau_1) \right\} \right\} \times \]
\[ \left[ K(a) + \int_a^t d\tau T_0 \exp \left\{ -\int_a^\tau d\xi \, a(\xi) \right\} \, b(\tau) \, T \exp \left\{ -\int_a^\tau d\xi \, c(\xi) \right\} \right] \times \]
\[ T \exp \left\{ -\int_a^t d\tau T_0 \exp \left\{ \int_a^\tau d\tau_1 c(\tau_1) \right\} \, c(\tau) \, T \exp \left\{ -\int_a^\tau d\tau_1 c(\tau_1) \right\} \right\} \]

By using identities (33) and (34) we arrive finally to
\[ K(t) = T \exp \left\{ \int_a^t d\tau a(\tau) \right\} \times \]
\[ \left[ K(a) + \int_a^t d\tau T_0 \exp \left\{ -\int_a^\tau d\xi a(\xi) \right\} \, b(\tau) \, T \exp \left\{ -\int_a^\tau d\xi c(\xi) \right\} \right] \times \]
\[ T_0 \exp \left\{ \int_a^t d\tau c(\tau) \right\}, \] (45)

as the formal solution of (38), which can be verified directly by substitution of (45) into (38).

The equation (38) is a generalization of equations (6) and (13). As we can easily see, their solutions (8) and (12) are particular cases of (45).
3.5 Differentiation of the chronological exponential with respect to a parameter

In the previous sections we considered some analytical properties of chronological operator exponentials mainly under differentiation on \( t \). It is very important from an analytical point of view to consider differentiation of operator exponentials with respect to some parameter, say \( \alpha \).

Let us consider the following operator

\[
\frac{\partial}{\partial \alpha} T \exp \left\{ \int_a^t d\tau A(\tau, \alpha) \right\}
\]

and denote here

\[
a(t, \alpha) = \frac{\partial A(t, \alpha)}{\partial \alpha} = \left[ \frac{\partial}{\partial \alpha}, A(t, \alpha) \right].
\]

From the identity (26) we have

\[
T \exp \left\{ \int_a^t d\tau A(\tau, \alpha) \right\} \frac{\partial}{\partial \alpha} T_0 \exp \left\{ -\int_a^t d\tau A(\tau, \alpha) \right\} = \frac{\partial}{\partial \alpha} - \int_a^t d\tau T \exp \left\{ \int_a^\tau d\xi A(\xi, \alpha) \right\} a(\tau, \alpha) T_0 \exp \left\{ -\int_a^\tau d\xi A(\xi, \alpha) \right\},
\]

so

\[
\frac{\partial}{\partial \alpha} T_0 \exp \left\{ -\int_a^t d\tau A(\tau, \alpha) \right\} = T_0 \exp \left\{ -\int_a^t d\tau A(\tau, \alpha) \right\} \left[ \frac{\partial}{\partial \alpha} - \int_a^t d\tau T \exp \left\{ \int_a^\tau d\xi A(\xi, \alpha) \right\} a(\tau, \alpha) T_0 \exp \left\{ -\int_a^\tau d\xi A(\xi, \alpha) \right\} \right].
\]

Analogously from identity (30) we have

\[
T_0 \exp \left\{ -\int_a^t d\tau A(\tau, \alpha) \right\} \frac{\partial}{\partial \alpha} T \exp \left\{ \int_a^t d\tau A(\tau, \alpha) \right\} = \frac{\partial}{\partial \alpha} + \int_a^t d\tau T_0 \exp \left\{ -\int_a^\tau d\xi A(\xi, \alpha) \right\} a(\tau, \alpha) T \exp \left\{ \int_a^\tau d\xi A(\xi, \alpha) \right\},
\]

which leads to

\[
\frac{\partial}{\partial \alpha} T \exp \left\{ \int_a^t d\tau A(\tau, \alpha) \right\} = T \exp \left\{ \int_a^t d\tau A(\tau, \alpha) \right\} \frac{\partial}{\partial \alpha} + \int_a^t d\tau T_0 \exp \left\{ -\int_a^\tau d\xi A(\xi, \alpha) \right\} a(\tau, \alpha) T \exp \left\{ \int_a^\tau d\xi A(\xi, \alpha) \right\}. \tag{46}
\]

3.6 Shift operators

To transform formal solutions into ordinary expressions we have to express explicitly the action of the corresponding linear operators. Every solved equation
gives us an example of definite action of a given linear operator. Thus, the list of “good” operators is nonempty. Our goal here is to study the properties of a certain class of operators that we will refer to as shift operators.

The Taylor expansion for a sufficiently arbitrary function $\Phi(x)$ can be expressed as

$$
\Phi(x + \alpha) = \sum_{k=0}^{\infty} \frac{1}{k!} \alpha^k \frac{d^k x}{d^k x} \Phi(x) = \exp\{\alpha \frac{d}{dx}\} \Phi(x).
$$

Here $\alpha$ can be either a constant or a function of arguments which do not include $x$ (in the second case $\frac{d}{dx}$ is replaced by $\frac{\partial}{\partial x}$). If we read this expression from the right side, we can obtain the result of the action of an exponential form of a shift operator $\exp\{\alpha \frac{d}{dx}\}$ on a function $\Phi(x)$ (see also (50) and (51) below):

$$
\exp\{\alpha \frac{d}{dx}\} \Phi(x) = \Phi(\exp\{\alpha \frac{d}{dx}\} x) = \Phi(x + \alpha).
$$

Let us consider shift operators with a change of variable $y = \psi(x)$. It is obvious, that

$$
\exp\{\alpha(\tau) \frac{\partial}{\partial x}\} \Phi(y) = \Phi(\exp\{\alpha(\tau) \frac{\partial}{\partial \psi(x)}\} \Phi(y)) = \Phi(\psi(x) + \alpha(\tau)).
$$

When $\psi(x)$ is fixed function, the set of shift operators with different parameters $\alpha(\tau)$ forms an Abelian group. Shift operators with different $\psi(x)$ do not commute.

Simple examples of shift operators are well-known. We give here examples which are somewhat exotic

$$
\exp\{\ln(\alpha) x \ln(x) \frac{\partial}{\partial x}\} \Phi(x) = \Phi(x^\alpha)
$$

and

$$
\exp\{\frac{c}{2} \frac{\partial}{\partial x}\} \exp\{\ln(2) x \ln(x) \frac{\partial}{\partial x}\} \exp\{(b - \frac{c^2}{4}) \frac{\partial}{\partial x}\} \Phi(x) = \Phi(x^2 + cx + b).
$$

These examples demonstrate that combinations of shift operators may produce non-trivial changes of the variable $x$ in the function $\Phi(x)$.

The following useful identities for shift operators are known

$$
\exp\{a(\tau) x^\alpha \frac{\partial}{\partial x}\} x^\beta \frac{\partial}{\partial x} \exp\{-a(\tau) x^\alpha \frac{\partial}{\partial x}\} = \exp\{a(\tau) x^\alpha \frac{\partial}{\partial x}\} x^\beta - \alpha \exp\{-a(\tau) x^\alpha \frac{\partial}{\partial x}\} x^\alpha \frac{\partial}{\partial x} = \exp\{a(\tau) x^\alpha \frac{\partial}{\partial x}\} x^\beta - \alpha \exp\{-a(\tau) x^\alpha \frac{\partial}{\partial x}\} x^\alpha \frac{\partial}{\partial x}.
$$
\[
\begin{align*}
\exp\{(\beta - 1)a(\tau)\}x^{\beta} \frac{\partial}{\partial x} & \quad \text{if } \alpha = 1; \\
x^{\alpha} \frac{\partial}{\partial x} & \quad \text{if } \alpha = \beta; \\
\left[x^{\beta(1-\alpha)} + (1-\alpha)a(\tau)x^{\alpha(1-\alpha)}\right]^{\frac{\beta-\alpha}{\beta-\alpha+1}} \frac{\partial}{\partial x} & \quad \text{if } \alpha \neq 1 \text{ and } \alpha \neq \beta.
\end{align*}
\]

(48)

Alas, in this last type of transformation the “polynomial” nature of the left-hand side of the identities in general does not survive in the right-hand side.

### 3.7 Chronological operator homomorphism

As we will see later, the chronological operators with the derivative operator \(\Delta(t)\) (see definition in Section 2) in the exponential which we will here for short denote as

\[ E = T \exp\left\{ \int_a^t d\tau \Delta(\tau) \right\}, \]

play a very important role.

Let \(B(t) = b(t)\) be a function that is the operator of multiplication on the \(b(t)\). Then, as can be seen from the definition of the derivative operator, the commutator \([b(t), \Delta(\tau)]\) is a function too.

Now consider the following construction

\[ K = E b(t) E^{-1}. \]

As far as the commutator \([b(t), \Delta(\tau)]\) is a function, then all repeated commutators are functions too. So with help of BCH formula (28) we conclude that operator \(K\) is a function and

\[ K = (E b(t)). \] (49)

An analogous conclusion holds for \(E^{-1} b(t) E\).

Let us consider further the following obvious chain

\[ E b_1 b_2 \ldots b_n = E b_1 E^{-1} E b_2 E^{-1} \ldots E b_n = \]

\[ = (E b_1) (E b_2) \ldots (E b_n). \]

With what is observed here and the fact that the chronological operator is linear, leads to conclusion that for any function \(F(b_1, b_2, \ldots, b_n)\), which can be expanded in power series with respect to \(b_1, b_2, \ldots, b_n\), we can obtain the following nice property:

\[ E F(b_1, b_2, \ldots, b_n) = F((E b_1), (E b_2), \ldots, (E b_n)). \] (50)

Analogously

\[ E^{-1} F(b_1, b_2, \ldots, b_n) = F((E^{-1} b_1), (E^{-1} b_2), \ldots, (E^{-1} b_n)). \] (51)
3.8 Some additional useful identities

By taking inverses of both sides of (23) we find the similar identity for the product of inverse ordered operators

\[ T_0 \exp\{\int_a^t d\tau \ B(\tau)\} \ T_0 \exp\{\int_a^t d\tau \ A(\tau)\} = T_0 \exp\{\int_a^t d\tau T_0 \exp\{\int_\tau^t d\xi \ B(\xi)\} A(\tau) T_0 \exp\{\int_\tau^t d\xi \ B(\xi)\}\}. \]

Furthermore, the analog of (24) is

\[ T_0 \exp\{\int_a^t d\tau \ [B(\tau) + C(\tau)]\} = T_0 \exp\{\int_a^t d\tau T_0 \exp\{\int_\tau^t d\xi \ B(\xi)\} C(\tau) T_0 \exp\{\int_\tau^t d\xi \ B(\xi)\}\} \times T_0 \exp\{\int_a^t d\tau B(\tau)\}. \] (52)

If in (22) we substitute \( b(t) = T_0 \exp\{\int_a^t d\tau B(\tau)\} \), then from (22) we obtain the following operator identity

\[ T_0 \exp\{\int_a^t d\tau B(\tau)\} \ T \exp\{\int_a^t d\tau A(\tau)\} = T \exp\{\int_a^t d\tau T_0 \exp\{\int_\tau^t d\xi B(\xi)\} [A(\tau) + B(\tau)] \ T \exp\{\int_\tau^t d\xi B(\xi)\}\}, \]

and by inverting it and making the change of operators \( A \rightarrow -B \) and \( B \rightarrow -A \) we obtain another form

\[ T_0 \exp\{\int_a^t d\tau B(\tau)\} \ T \exp\{\int_a^t d\tau A(\tau)\} = T_0 \exp\{-\int_a^t d\tau T_0 \exp\{-\int_a^t d\xi A(\xi)\} [A(\tau) + B(\tau)] \ T \exp\{\int_\tau^t d\xi A(\xi)\}\}. \]

Since from (24) we have that

\[ T \exp\{\int_a^t d\tau [B(\tau) + C(\tau)]\} = T \exp\{\int_a^t d\tau B(\tau)\} \times T \exp\{\int_a^t d\tau [B(\tau) + C(\tau)]\} T \exp\{\int_a^t d\tau B(\tau)\} \times T \exp\{\int_a^t d\tau C(\tau)\} \times T \exp\{\int_a^t d\tau B(\tau)\} \times T \exp\{\int_a^t d\tau [B(\tau) + C(\tau)]\} \ T \exp\{\int_a^t d\tau C(\xi)\} B(\tau) \ T \exp\{\int_\tau^t d\xi C(\xi)\}\}. \]
then we come to a third form
\[
\begin{align*}
T_0 \exp\{- \int_a^t d\tau C(\tau)\} & T \exp\{\int_a^t d\tau B(\tau)\} = \\
T & \exp\{\int_a^t d\tau T_0 \exp\{- \int_a^\tau d\xi C(\xi)\} B(\tau) T \exp\{\int_a^\tau d\xi C(\xi)\}\} \times \\
T_0 & \exp\{- \int_a^t d\tau T_0 \exp\{- \int_a^\tau d\xi B(\xi)\} C(\tau) T \exp\{\int_a^\tau d\xi B(\xi)\}\}.
\end{align*}
\]

Let us now consider the invertible operator \(B\) which does not depend on \(t\). Then from (22) we have
\[
B T \exp\{\int_a^t d\tau A(\tau)\} = T \exp\{\int_a^t d\tau B A(\tau) B^{-1}\} B.
\]
or
\[
T \exp\{\int_a^t d\tau A(\tau)\} B = B T \exp\{\int_a^t d\tau B^{-1} A(\tau) B\}. \tag{53}
\]

From the expression
\[
K(t) = B T \exp\{\int_a^t d\tau A(\tau) B\}
\]
we obtain by differentiation that
\[
\frac{\partial K(t)}{\partial t} = B A(t) B T \exp\{\int_a^t d\tau A(\tau) B\} = B A(t) K(t),
\]
and
\[K(t)|_{t=a} = B.\]

Solving the above operator differential equation we find that for any linear (not necessarily invertible) “\(t\)-independent” operator \(B\)
\[
B T \exp\{\int_a^t d\tau A(\tau) B\} = T \exp\{\int_a^t d\tau B A(\tau) B\}. 
\]

Analogously,
\[
B T_0 \exp\{\int_a^t d\tau A(\tau) B\} = T_0 \exp\{\int_a^t d\tau B A(\tau) B\}. 
\]

3.9 Example 1. Linear first-order PDE

The general linear first-order partial differential equation for \(u(t, \vec{\rho})\)
\[
\frac{\partial u(t, \vec{\rho})}{\partial t} = \phi(t, \vec{\rho}) + f_0(t, \vec{\rho}) u(t, \vec{\rho}) + f_1(t, \vec{\rho}) \frac{\partial u(t, \vec{\rho})}{\partial x_1} + \cdots + f_n(t, \vec{\rho}) \frac{\partial u(t, \vec{\rho})}{\partial x_n}, \tag{54}
\]
where we abbreviate \( \vec{\rho} = x_1, \ldots, x_n \), with initial condition
\[ u(t, \vec{\rho})|_{t=a} = v(\vec{\rho}) \]
is fitted outright to be solved by the operator method. Its formal solution follows from (18):
\[
\begin{align*}
  u(t, \vec{\rho}) &= T \exp \left\{ \int_a^t d\tau \left[ f_0(\tau, \vec{\rho}) + f_1(\tau, \vec{\rho}) \frac{\partial}{\partial x_1} + \cdots + f_n(\tau, \vec{\rho}) \frac{\partial}{\partial x_n} \right] \right\} v(\vec{\rho}) + \\
  &+ \int_a^t d\tau T \exp \left\{ \int_\tau^t d\xi \left[ f_0(\xi, \vec{\rho}) + f_1(\xi, \vec{\rho}) \frac{\partial}{\partial x_1} + \cdots + f_n(\xi, \vec{\rho}) \frac{\partial}{\partial x_n} \right] \right\} \phi(\tau, \vec{\rho}).
\end{align*}
\]

We indicate that the above general solution can be expressed in terms of some particular solutions of homogeneous first-order PDE:
\[ \frac{\partial \tilde{u}(t, \vec{\rho})}{\partial t} = f_1(t, \vec{\rho}) \frac{\partial \tilde{u}(t, \vec{\rho})}{\partial x_1} + \cdots + f_n(t, \vec{\rho}) \frac{\partial \tilde{u}(t, \vec{\rho})}{\partial x_n}. \] (55)

If we introduce the following notation
\[ \zeta_i(t, \vec{\rho}) = T \exp \left\{ \int_a^t d\tau \left[ f_1(\tau, \vec{\rho}) \frac{\partial}{\partial x_1} + \cdots + f_n(\tau, \vec{\rho}) \frac{\partial}{\partial x_n} \right] \right\} x_i, \quad (i = 1, \ldots, n) \]
we can consider it as the fundamental set of particular solutions of (55) with initial conditions \( \zeta_i(t, \vec{\rho})|_{t=a} = x_i \). Since the operator in the exponential is the derivative, then taking into account (50), the solution of homogeneous first-order PDE (55) may be expressed via \( \zeta_i \) as
\[ \tilde{u}(t, \vec{\rho}) = v(\zeta_1(t, \vec{\rho}), \ldots, \zeta_n(t, \vec{\rho})). \]

The last expression is the general solution of the homogeneous first-order PDE (55) if \( v(\vec{\rho}) \) is an arbitrary function.

If we further denote
\[ Z_i(t, \tau, \vec{\rho}) = T \exp \left\{ \int_\tau^t d\tau \left[ f_1(\tau, \vec{\rho}) \frac{\partial}{\partial x_1} + \cdots + f_n(\tau, \vec{\rho}) \frac{\partial}{\partial x_n} \right] \right\} x_i, \quad (i = 1, \ldots, n), \]
we can find by using (10) that
\[ \zeta_i(t, \vec{\rho}) = T \exp \left\{ \int_\tau^t d\tau \left[ f_1(\tau, \vec{\rho}) \frac{\partial}{\partial x_1} + \cdots + f_n(\tau, \vec{\rho}) \frac{\partial}{\partial x_n} \right] \right\} \zeta_i(\tau, \vec{\rho}) ; \]
or
\[ \zeta_i(t, \vec{\rho}) = \zeta_i(\tau, Z_1(t, \tau, \vec{\rho}), \ldots, Z_n(t, \tau, \vec{\rho})) , \]
so \( Z_i(t, \tau, \vec{\rho}) \) are solutions of the following system of algebraic equations
\[ \zeta_i(\tau, Z_1, \ldots, Z_n) = \zeta_i(t, \vec{\rho}), \quad (i = 1, \ldots, n). \] (56)
Let us take up in passing a nice property of considered type of functions which we will need later. Denoting

$$b_i(t, \vec{\rho}) = T_0 \exp\{-\int_a^t \left[ f_1(\tau, \vec{\rho}) \frac{\partial}{\partial x_1} + \cdots + f_n(\tau, \vec{\rho}) \frac{\partial}{\partial x_n} \right] \} x_i, \quad (i = 1, \ldots, n) \quad (57)$$

then from (11) and (51) we have

$$T_0 \exp\{-\int_a^t \left[ f_1(\tau, \vec{\rho}) \frac{\partial}{\partial x_1} + \cdots + f_n(\tau, \vec{\rho}) \frac{\partial}{\partial x_n} \right] \} \zeta_i(t, \vec{\rho}) = \zeta_i(t, b_1(t, \vec{\rho}), \ldots, b_n(t, \vec{\rho})) = x_i \quad (58)$$

and analogously

$$b_i(t, \zeta_1(t, \vec{\rho}), \ldots, \zeta_n(t, \vec{\rho})) = x_i, \quad (i = 1, \ldots, n) \quad (59)$$

That is \( \zeta_i \) and \( b_i \) are bundled by algebraic systems (58) and (59).

Moreover, it is obvious that

$$\frac{\partial b_i(t, \vec{\rho})}{\partial t} = -T_0 \exp\{-\int_a^t \left[ f_1(\tau, \vec{\rho}) \frac{\partial}{\partial x_1} + \cdots + f_n(\tau, \vec{\rho}) \frac{\partial}{\partial x_n} \right] \} f_i(\tau, \vec{\rho}),$$

and from (51) it follows that

$$\frac{\partial b_i(t, \vec{\rho})}{\partial t} = -f_i(t, b_1(t, \vec{\rho}), \ldots, b_n(t, \vec{\rho})), \quad (i = 1, \ldots, n) \quad (60)$$

The expressions (57) are solutions of the system of \( n \) first-order non-linear ODEs (60).

As by virtue of (24) and (50)

$$T \exp\{\int_a^t d\tau \left[ f_0(\tau, \vec{\rho}) + f_1(\tau, \vec{\rho}) \frac{\partial}{\partial x_1} + \cdots + f_n(\tau, \vec{\rho}) \frac{\partial}{\partial x_n} \right] \} = \frac{\partial}{\partial t} \begin{cases} \end{cases}$$

$$T \exp\{\int_a^t d\tau \left[ f_1(\tau, \vec{\rho}) \frac{\partial}{\partial x_1} + \cdots + f_n(\tau, \vec{\rho}) \frac{\partial}{\partial x_n} \right] \} \times$$

$$\exp\{\int_a^t d\tau \left( T_0 \exp\{\int_a^\tau d\xi \left[ f_1(\xi, \vec{\rho}) \frac{\partial}{\partial x_1} + \cdots + f_n(\xi, \vec{\rho}) \frac{\partial}{\partial x_n} \right] \} f_0(\tau, \vec{\rho}) \} = \exp\{\int_a^t d\tau f_0(\tau, Z_1(t, \vec{\rho}), \ldots, Z_n(t, \vec{\rho}) \} \times$$

$$T \exp\{\int_a^t d\tau \left[ f_1(\tau, \vec{\rho}) \frac{\partial}{\partial x_1} + \cdots + f_n(\tau, \vec{\rho}) \frac{\partial}{\partial x_n} \right] \}.$$ 

If we now return to the general linear first-order PDE (54), we can rewrite its operator solution via \( \zeta_i \) (and \( Z_i \) which are expressed through \( \zeta_i \) by system (56)) into the following form

$$u(t, \vec{\rho}) = v(\zeta_1(t, \vec{\rho}), \ldots, \zeta_n(t, \vec{\rho})) \exp\{\int_a^t d\tau f_0(\tau, Z_1(t, \vec{\rho}), \ldots, Z_n(t, \vec{\rho}) \} +$$

$$\int_a^t d\tau \phi(t, Z_1(t, \vec{\rho}), \ldots, Z_n(t, \vec{\rho})) \exp\{\int_a^t d\xi f_0(\xi, Z_1(t, \xi, \vec{\rho}), \ldots, Z_n(t, \xi, \vec{\rho}) \}.$$
3.10 Example 2. Linear parabolic differential equation

The linear parabolic differential equation

$$\frac{\partial u(t, \vec{\rho})}{\partial t} = f_0(t, \vec{\rho}) u(t, \vec{\rho}) + f_1(t, \vec{\rho}) \frac{\partial^2 u(t, \vec{\rho})}{\partial x_1^2} + \cdots + f_n(t, \vec{\rho}) \frac{\partial^2 u(t, \vec{\rho})}{\partial x_n^2},$$

where $\vec{\rho} = x_1, ..., x_n$, with an initial condition

$$u(t, \vec{\rho})|_{t=0} = v(\vec{\rho}),$$

has the following formal solution

$$u(t, \vec{\rho}) = T \exp \left\{ \int_0^t d\tau \left[ f_0(\tau, \vec{\rho}) + f_1(\tau, \vec{\rho}) \frac{\partial^2}{\partial x_1^2} + \cdots + f_n(\tau, \vec{\rho}) \frac{\partial^2}{\partial x_n^2} \right] \right\} v(\vec{\rho}).$$

It is interesting to note that it is generally accepted that the parabolic equation is a second order PDE, but from the angle of the operator method considered here this equation is first order if the initial condition is specified for $t$ variable.

In the simplest case when all $f_i$ are constants $f_i = k_i$ then it is easy to see with (53) that

$$T \exp \left\{ \int_0^t d\tau \left[ k_0 + k_1 \frac{\partial^2}{\partial x_1^2} + \cdots + k_n \frac{\partial^2}{\partial x_n^2} \right] \right\} \exp \left\{ i(\sigma_1 x_1 + \cdots + \sigma_n x_n) \right\} = \exp \left\{ i(\sigma_1 x_1 + \cdots + \sigma_n x_n) \right\} \exp \left\{ t \left[ k_0 - k_1 \sigma_1^2 - \cdots - k_n \sigma_n^2 \right] \right\},$$

so we can obtain well-known non-operator solutions of such a parabolic equation with help of a Fourier transformation of the initial condition.

4 Non-linear first-order ODE’s

4.1 The operator solution for first-order non-linear ODE

An ordinary non-linear first-order differential equation

$$\frac{du(t)}{dt} = f(t, u(t)) \tag{61}$$

on account of its non-linearity is not immediately suited for application of the above considered operator method. Nevertheless there are many possibilities to convert the problem \(\text{(61)}\) to a linear one and in Subsection 3.9 we have obtained by way of the operator method a solution for such an equation. Here we consider this problem in more detail.

Most linearization procedures are concerned with the introduction of spaces of larger dimensions. Here we demonstrate some of them which are almost generally applicable.
If we introduce a new function

$$S(t, \omega) = e^{\omega u(t)},$$

from (61) one can easily derive the following first-order (with respect to $t$) linear differential equation for $S(t, \omega)$ equivalent to (61)

$$\frac{\partial S(t, \omega)}{\partial t} = \omega f(t, \frac{\partial}{\partial \omega}) S(t, \omega), \quad S(a, \omega) = e^{\omega u(a)},$$

from which the formal solution follows immediately from (9) in the form

$$S(t, \omega) = T \exp \{ \int_a^t d\tau \omega f(\tau, \frac{\partial}{\partial \omega}) \} e^{\omega u(a)}.$$

So the solution of equation (61) is:

$$u(t) = \left. \frac{\partial S(t, \omega)}{\partial \omega} \right|_{\omega=0}.$$  (65)

Direct substitution of the solution (65) reduces the equation (61) to an identity.

The theorem for the uniqueness of a solution of a differential equation shows that if we have an exact solution of a problem in several forms, all the forms can be transformed to each other. So the main problem that remains in this approach is how to transform the operator solution to that form which can be considered most useful.

### 4.2 Solution for first-order ODE in form with derivative operator

The main result of this subsection is another form (67) of the formal solution for non-linear first-order ODE. We can prove the solution correctness by its direct substitution into ODE (61), but it is desirable to outline the way which leads to such form.

Let us start from an operator solution of the equation (63) in the form (64) but let us rewrite it (not only) for convenience in the following notation

$$S(t, c, \omega) = T \exp \{ \int_a^t d\tau \omega f(\tau, \frac{\partial}{\partial \omega}) \} e^{\omega c},$$

where $c = u(t)|_{t=a}$. With the help of identity (53) we find

$$S(t, c, \omega) = e^{\omega c} T \exp \{ \int_a^t d\tau \omega e^{-\omega c} f(\tau, \frac{\partial}{\partial \omega}) e^{\omega c} \} \cdot 1.$$  

CHB-expansion gives

$$e^{-\omega c} f(\tau, \frac{\partial}{\partial \omega}) e^{\omega c} = f(\tau, \frac{\partial}{\partial \omega}) + c[f(\tau, \frac{\partial}{\partial \omega}), \omega] + \frac{c^2}{2} [[f(\tau, \frac{\partial}{\partial \omega}), \omega], \omega] + \ldots$$  (66)
Since it is easily proven by induction that

\[
\left[ \frac{\partial^m}{\partial \omega^m}, \omega \right] = m \frac{\partial^{m-1}}{\partial \omega^{m-1}},
\]

then from expansion of \( f(x) \) into power series and backwards summation it follows that

\[
\left[ f\left( \frac{\partial}{\partial \omega} \right), \omega \right] = f'\left( \frac{\partial}{\partial \omega} \right) \quad \text{(} f'(x) = \frac{\partial f(x)}{\partial x} \text{)},
\]

therefore we can conclude with taking into account (66) and the property of the shift operator (47) that

\[
e^{-\omega c} f(\tau, \frac{\partial}{\partial \omega}) e^{\omega c} = f(\tau, \frac{\partial}{\partial \omega}) + c f'(\tau, \frac{\partial}{\partial \omega}) + \frac{c^2}{2} f''(\tau, \frac{\partial}{\partial \omega}) + \cdots =
\]

\[
f(\tau, c + \frac{\partial}{\partial \omega}) = \exp\left( \frac{\partial}{\partial \omega} \frac{\partial}{\partial c} \right) f(\tau, c) \exp\left( -\frac{\partial}{\partial \omega} \frac{\partial}{\partial c} \right).
\]

Then

\[
S(t, c, \omega) = e^{\omega c} \exp\left( \frac{\partial}{\partial \omega} \frac{\partial}{\partial c} \right) T \exp\left( \int_a^t d\tau \left( \omega - \frac{\partial}{\partial c} \right) f(\tau, c) \right) \exp\left( -\frac{\partial}{\partial \omega} \frac{\partial}{\partial c} \right) \cdot 1 =
\]

\[
e^{\omega c} \exp\left( \frac{\partial}{\partial \omega} \frac{\partial}{\partial c} \right) T \exp\left( \int_a^t d\tau \left[ \omega f(\tau, c) - \frac{\partial f(\tau, c)}{\partial c} - f(\tau, c) \frac{\partial}{\partial c} \right] \right) \cdot 1.
\]

By expanding the chronological exponential of the derivative operator \( f(\tau, c) \frac{\partial}{\partial c} \) with the help of identity (24) and properties (49), (50) we obtain that

\[
S(t, c, \omega) = e^{\omega c} \exp\left( \frac{\partial}{\partial \omega} \frac{\partial}{\partial c} \right) T \exp\left( -\int_a^t d\tau f(\tau, c) \frac{\partial}{\partial c} \right) \times
\]

\[
\exp\left( -\int_a^t d\tau g(\tau, c) \right) \exp(\omega \int_a^t d\tau G(\tau, c)),
\]

where

\[
g(\tau, c) = T_0 \exp\left( \int_a^\tau d\xi f(\xi, c) \frac{\partial}{\partial c} \right) \frac{\partial f(\tau, c)}{\partial c}
\]

and

\[
G(\tau, c) = T_0 \exp\left( \int_a^\tau d\xi f(\xi, c) \frac{\partial}{\partial c} \right) f(\tau, c).
\]

If we differentiate the last expression with respect to \( \omega \) we find that

\[
\frac{\partial S(t, c, \omega)}{\partial \omega} = c S(t, c, \omega) + e^{\omega c} \exp\left( \frac{\partial}{\partial \omega} \frac{\partial}{\partial c} \right) T \exp\left( -\int_a^t d\tau f(\tau, c) \frac{\partial}{\partial c} \right) \times
\]

\[
\exp\left( -\int_a^t d\tau g(\tau, c) \right) \exp(\omega \int_a^t d\tau G(\tau, c)) \int_a^t d\tau G(\tau, c).
\]
If in the second item we recover the initial operator form

\[
\frac{\partial S(t, c, \omega)}{\partial \omega} = c S(t, c, \omega) + T \exp\left\{ \int_a^t d\tau \omega f(\tau, \frac{\partial}{\partial \omega}) \right\} e^{\omega c} \int_a^t d\tau G(\tau, c),
\]

and note that the chronological operator commutes with any function which does not depend on \( \omega \), we arrive at

\[
\frac{\partial S(t, c, \omega)}{\partial \omega} = [c + \int_a^t d\tau G(\tau, c)] S(t, c, \omega)
\]

and inasmuch as \( S(t, c, \omega)|_{\omega=0} \equiv 1 \), then

\[
u(t, c) = \frac{\partial S(t, c, \omega)}{\partial \omega}|_{\omega=0} = c + \int_a^t d\tau G(\tau, c),
\]

hence it follows that

\[
u(t, c) = c + \int_a^t d\tau T_0 \exp\left\{ \int_a^\tau d\xi f(\xi, c) \frac{\partial}{\partial c} \right\} f(\tau, c)
\]

or finally

\[
u(t, c) = T_0 \exp\left\{ \int_a^t d\tau f(\tau, c) \frac{\partial}{\partial c} \right\} c. \tag{67}
\]

The expression (67) is the general solution of the equation (61) if we will consider \( c \) as an arbitrary constant.

Above derivation is not rigorous, of course, so it is very important to verify our conclusion by direct substitution of the obtained solution into equation (61). After differentiation of (67) and using property (50) we can be sure that (67) is really the formal solution of the equation (61).

So we have obtained a solution for a non-linear first-order equation in a more convenient form with the derivative operator.

Let us finish this Subsection with a brief remark about the well-known interconnection between first-order ODEs and linear first-order PDEs.

As we have seen in Example 1 above (see Subsection 3.9) the function

\[
\zeta(t, c) = T \exp\{- \int_a^t d\tau f(\tau, c) \frac{\partial}{\partial c} \} c \tag{68}
\]

satisfies the equation

\[
\frac{\partial \zeta(t, c)}{\partial t} + f(t, c) \frac{\partial \zeta(t, c)}{\partial c} = 0.
\]

If we now act on both sides of (68) by the operator

\[
T_0 \exp\{ \int_a^t d\tau f(\tau, c) \frac{\partial}{\partial c} \},
\]

26
we get
\[ \zeta(t, u(t, c)) = c \] (69)
and
\[ u(t, \zeta(t, c)) = c \]
– the well-known equations expressing the relationship between solutions of first-order ODEs and the corresponding linear first-order PDEs.

In some practical cases it is easier to obtain explicit expression of \( \zeta(t, c) \) for given ODE, for example, by an integrating factor method. (69) then is the implicit form of the solution for the given ODE. Which form of solution of types (67) or (69) is more analyzable depends on the object under investigation for a specific problem.

4.3 Example 3. Bernoulli ODE

The formal solution of a Bernoulli ODE
\[ \frac{du}{dt} = a(t)u^\alpha + b(t)u \]
in the form of (67) is as follows
\[ u(t, c) = T_0 \exp\left\{ \int_0^t d\tau \left[ a(\tau)c^\alpha + b(\tau) \right] \frac{\partial}{\partial c} c \right\}. \]
Expanding the above chronological exponential with the help of identity (52)
\[ u(t, c) = T_0 \exp\left\{ \int_0^t d\tau \left[ a(\tau)c^\alpha + b(\tau) \right] \frac{\partial}{\partial c} c \right\} \times \exp\left\{ \int_0^t d\tau b(\tau)c \frac{\partial}{\partial c} c \right\} \]
and with (48) gives us the solution with only shift operators
\[ u(t, c) = \exp\left\{ \int_0^t d\tau a(\tau) \left[ (\alpha - 1) \int_0^\tau d\xi b(\xi) \right] c^\alpha \frac{\partial}{\partial c} c \right\} \times \exp\left\{ \int_0^t d\tau b(\tau)c \frac{\partial}{\partial c} c \right\}. \]
Executing shift operations we arrive to a classical form of the solution for Bernoulli ODE \((\alpha \neq 1)\)
\[ u(t, c) = [c^{(1-\alpha)}+(1-\alpha) \int_0^t d\tau a(\tau) \exp\left\{ (\alpha - 1) \int_0^\tau d\xi b(\xi) \right\}]^{1-\alpha} \exp\left\{ \int_0^t d\tau b(\tau) \right\}. \]

The technique being used here is applicable not only for Bernoulli ODEs, but it is successful under concatenation of some circumstances, when the original chronological exponential is decomposed into the chain of shift operators on its own without solving any auxiliary differential equations.
4.4 The “integral-free” form of solution for non-linear first-order ODE

Let us consider the following operator chain

\[
\exp\{(t - a) [f(s, c) \frac{\partial}{\partial c} + \frac{\partial}{\partial s}] \}
\]

\[
T_0 \exp\{\int_a^t d\tau e^{(t-a)\frac{\partial}{\partial c}} f(s, c) \frac{\partial}{\partial c} e^{(t-a)\frac{\partial}{\partial s}}\} \exp\{(t - a) \frac{\partial}{\partial s}\}
\]

\[
T_0 \exp\{\int_a^t d\tau f(s + \tau - a, c) \frac{\partial}{\partial c}\} \exp\{(t - a) \frac{\partial}{\partial s}\}.
\]

So

\[
T_0 \exp\{\int_a^t d\tau f(s + \tau - a, c) \frac{\partial}{\partial c}\} c = \exp\{(t - a) [f(s, c) \frac{\partial}{\partial c} + \frac{\partial}{\partial s}]\} c
\]

and if we denote

\[
U(t, c, s) = \exp\{(t - a) [f(s, c) \frac{\partial}{\partial c} + \frac{\partial}{\partial s}]\} c
\]

we obtain from the preceding expression that the general solution of ODE (61) is

\[
u(t, c) = U(t, c, s)|_{s=a}
\]

or

\[
u(t, c) = [\exp\{(t - a) [f(s, c) \frac{\partial}{\partial c} + \frac{\partial}{\partial s}]\} c]|_{s=a}.
\]

The last form of the operator solution has some interesting features in the sense that it does not contain any integration and even ordering operator \(T\), which may be useful for calculating approximate expressions of the solution \(u(t, c)\).

The function \(U(t, c, s)\) obviously satisfies the following PDE

\[
\frac{\partial U(t, c, s)}{\partial t} - f(s, c) \frac{\partial U(t, c, s)}{\partial c} - \frac{\partial U(t, c, s)}{\partial s} = 0, \quad (U(t, c, s)|_{t=a} = c).
\]

4.5 The solution with an arbitrary function

Let us now consider \(f(t, c) = g(t, c) + h(t, c)\) so we can find the following form of the general solution of equation (61)

\[
u(t, c) = T_0 \exp\{\int_a^t d\tau f(\tau, c) \frac{\partial}{\partial c}\} c = T_0 \exp\{\int_a^t d\tau [g(\tau, c) + h(\tau, c)] \frac{\partial}{\partial c}\} c =
\]

\[
T_0 \exp\{\int_a^t d\tau [T_0 \exp\{\int_a^\tau d\xi h(\xi, c) \frac{\partial}{\partial c}\} g(\tau, c)] \times
\]

\[
\exp\{-\int_a^\tau d\xi [T_0 \exp\{\int_a^\xi d\zeta h(\zeta, c) \frac{\partial}{\partial c}\} \frac{\partial h(\xi, c)}{\partial c}] \frac{\partial}{\partial c}\} \times
\]

\[
T_0 \exp\{\int_a^t d\tau h(\tau, c) \frac{\partial}{\partial c}\} c
\]
or

\[
    u(t, c) = T_0 \exp\left\{ \int_a^t d\tau \left[ T_0 \exp\left\{ \int_a^\tau d\xi h(\xi, c) \frac{\partial}{\partial c} \right\} \left[ f(\tau, c) - h(\tau, c) \right] \right] \right\} \times
\]

\[
    \exp\left\{ -\int_a^\tau d\xi \left[ T_0 \exp\left\{ \int_a^\xi d\zeta h(\zeta, c) \frac{\partial}{\partial c} \right\} \right] \frac{\partial h(\xi, c)}{\partial c} \right\} \times
\]

\[
    T_0 \exp\left\{ \int_a^t d\tau h(\tau, c) \frac{\partial}{\partial c} \right\} c,
\]

where we can consider \( h(t, c) \) as an arbitrary differentiable function.

Supposing it is known that

\[
    z(t, c) = T_0 \exp\left\{ \int_a^t d\tau h(\tau, c) \frac{\partial}{\partial c} \right\} c,
\]

then with the help of property (50) we obtain

\[
    u(t, c) = T_0 \exp\left\{ \int_a^t d\tau f(\tau, z(\tau, c)) - \frac{\partial z(\tau, c)}{\partial \tau} \frac{\partial}{\partial c} \right\} z(t, c). \tag{70}
\]

As far as \( h(t, c) \) is an arbitrary function, the preceding expression is valid for any differentiable function \( z(t, c) \). The particular solution of equation (61) with initial condition \( u(t, c)|_{t=a} = c \) is obtained when \( z(t, c)|_{t=a} = c \).

Rewriting (70) as

\[
    u(t, c) = z(t, T_0 \exp\left\{ \int_a^t d\tau f(\tau, z(\tau, c)) - \frac{\partial z(\tau, c)}{\partial \tau} \frac{\partial}{\partial c} \right\} c) \tag{71}
\]

by wisely choosing the form of the function \( z(t, c) \) we may reduce the initial problem to

\[
    \tilde{u}(t, c) = T_0 \exp\left\{ \int_a^t d\tau f(\tau, z(\tau, c)) - \frac{\partial z(\tau, c)}{\partial \tau} \frac{\partial}{\partial c} \right\} c,
\]

which may be a simpler one by losing some troublesome singularities of the initial problem.

5 The systems of non-linear first order ODEs

5.1 The formal solution of the system of non-linear first order ODEs

To find the formal solution of the system of non-linear first order ODEs

\[
    \frac{du_i(t)}{dt} = f_i(t, u_1(t), \ldots, u_n(t)), \quad (i = 1, \ldots, n) \tag{72}
\]
we, of course, could introduce an auxiliary function like (62) and pass through a chain of unwieldy expressions but now we are ready to assert that the general solution of the system (72) has the following operator form

$$u_i(t, \vec{c}) = T_0 \exp\left\{ \int_a^t d\tau \left[ f_1(\tau, \vec{c}) \frac{\partial}{\partial c_1} + \cdots + f_n(\tau, \vec{c}) \frac{\partial}{\partial c_n} \right]\right\} c_i,$$  

(73)

where we abbreviate $\vec{c} = c_1, \ldots, c_n$ and $c_i$ are a set of arbitrary constants.

Differentiating (73) with respect to $t$ and with (51) we have

$$\frac{\partial u_i}{\partial t} = T_0 \exp\left\{ \int_a^t d\tau \left[ f_1(\tau, \vec{c}) \frac{\partial}{\partial c_1} + \cdots + f_n(\tau, \vec{c}) \frac{\partial}{\partial c_n} \right]\right\} f_i(t, \vec{c}) = f_i(t, u_1, \ldots, u_n).$$

If we denote here that

$$\zeta(\phi)(t, \vec{c}) = T \exp\{- \int_a^t d\tau \left[ f_1(\tau, \vec{c}) \frac{\partial}{\partial c_1} + \cdots + f_n(\tau, \vec{c}) \frac{\partial}{\partial c_n} \right]\} \phi(\vec{c}),$$

it is easy to see that $\zeta$ for any function $\phi$ satisfies a PDE similar to (55):

$$\frac{\partial \zeta}{\partial t} + f_1(t, \vec{c}) \frac{\partial \zeta}{\partial c_1} + \cdots + f_n(t, \vec{c}) \frac{\partial \zeta}{\partial c_n} = 0.$$

So if we know the $n$ fundamental solutions of this equation, namely

$$\zeta_i(t, \vec{c}) = T \exp\{- \int_a^t d\tau \left[ f_1(\tau, \vec{c}) \frac{\partial}{\partial c_1} + \cdots + f_n(\tau, \vec{c}) \frac{\partial}{\partial c_n} \right]\} c_i,$$

then

$$\zeta(t, \vec{c}) = \phi(\zeta_1(t, \vec{c}), \ldots, \zeta_n(t, \vec{c}))$$

and we can find $u_i$ as the solution of the algebraic system

$$\zeta_i(t, u_1, \ldots, u_n) = c_i, \quad (i = 1, \ldots, n).$$

Analogous to the first order ODE, we can rewrite the solution (73) in an “integral-free” form as follows

$$u_i(t, \vec{c}) = \left[ \exp\left\{ - \int_a^t \left[ f_1(s, \vec{c}) \frac{\partial}{\partial c_1} + \cdots + f_n(s, \vec{c}) \frac{\partial}{\partial c_n} + \frac{\partial}{\partial s} \right]\right\} c_i \right]_{s=a}.$$

5.2 The direct calculation of BCH type expressions when the involved operators are derivatives

As we have seen, the operators similar to (29) when operators $A(t)$ and $B(t)$ are derivatives similar to (1), play a key role in our approach. And one of the principal points here is the calculation of the following expressions

$$K_i(t, \vec{x}) = T_0 \exp\{- \int_a^t d\tau \sum_{j=1}^m h_j(\tau, \vec{x}) \frac{\partial}{\partial x_j}\} \zeta_i = T_0 \exp\{- \int_a^t d\tau \sum_{j=1}^m h_j(\tau, \vec{x}) \frac{\partial}{\partial x_j}\} \int_a^t \frac{d\tau}{\partial x_i}.$$

(74)
where \( \vec{x} = x_1, \ldots, x_m \), \( h_j(t, \vec{x}) \) are ordinary functions, and we suppose that for a given \( h_j(t, \vec{x}) \) we are able to calculate the fundamental set of of functions

\[
z_i(t, \vec{x}) = T_0 \exp \left\{ - \int_a^t \frac{d\tau}{m} \sum_{j=1}^m h_j(\tau, \vec{x}) \frac{\partial}{\partial x_j} \right\} x_i.
\]

We have calculated above such expressions for \( m = 1 \), but if \( m > 1 \) we need to follow by a different way.

Analogous to (29), we find that

\[
\frac{\partial K_i(t, \vec{x})}{\partial t} = T_0 \exp \left\{ - \int_a^t \frac{d\tau}{m} \sum_{j=1}^m h_j(\tau, \vec{x}) \frac{\partial}{\partial x_j} \right\} \times
\]

\[
T \exp \left\{ \int_a^t \frac{d\tau}{m} \sum_{j=1}^m h_j(\tau, \vec{x}) \frac{\partial}{\partial x_j} \right\}.
\]

The use of (61) leads to the following system of linear operator ODEs

\[
\frac{\partial K_i(t, \vec{x})}{\partial t} = \sum_{j=1}^m g_{ij}(t, \vec{x}) K_j(t, \vec{x}) , \quad (i = 1, \ldots, m),
\]

where

\[
g_{ij}(t, \vec{x}) = T_0 \exp \left\{ - \int_a^t \frac{d\tau}{m} \sum_{j=1}^m h_j(\tau, \vec{x}) \frac{\partial}{\partial x_j} \right\} \frac{\partial h_j(t, \vec{x})}{\partial x_i}
\]

with initial conditions

\[
K_i(t, \vec{x})|_{t=a} = \frac{\partial}{\partial x_i}.
\]

We can write down the formal solution of the system (65) as

\[
K_i(t, \vec{x}) = [T_0 \exp \left\{ \int_a^t \sum_{i=1}^m \sum_{j=1}^m g_{ij}(\tau, \vec{x}) c_j \frac{\partial}{\partial c_i} \right\} |_{c_i=0}] c_i = \frac{\partial}{\partial x_k}.
\]

Since the system (65) is linear, then its solutions depend on initial conditions linearly, i.e.

\[
K_i(t, \vec{x}) = \sum_{k=1}^m p_{ik}(t, \vec{x}) \frac{\partial}{\partial x_k},
\]

where

\[
p_{ik}(t, \vec{x}) = [T_0 \exp \left\{ \int_a^t \sum_{i=1}^m \sum_{j=1}^m g_{ij}(\tau, \vec{x}) c_j \frac{\partial}{\partial c_i} \right\} |_{c_k=1, c_{i\neq k}=0}]
\]

This conclusion is a direct consequence of the fact that the set of operators \( \sum_{j=1}^m h_j(t, \vec{x}) \frac{\partial}{\partial x_j} \) forms a Lie algebra.
So in cases under consideration we can exactly calculate the operators $K_i(t, \vec{x})$ without resorting to BCH expansions.

The usage of such calculations in principle allows us to factor the chronological exponential, e.g. in (73), into a product of relatively simple factors (with a sequence order assigned in advance)

$$T_0 \exp\left\{ \int_a^t d\tau \sum_{j=1}^n f_j(\tau, \vec{c}) \frac{\partial}{\partial c_j} \right\} = \prod_{i=1}^n T_0 \exp\left\{ \int_a^t d\tau g_i(\tau, \vec{c}) \frac{\partial}{\partial c_i} \right\}. $$

Unfortunately this way leads to complicated non-linear PDEs for functions $g_i(t, \vec{c})$.

6 The non-linear $n$th order ODE

For the non-linear $n$th order ODE

$$\frac{d^n u(t)}{dt^n} = f(t, u(t), \frac{du(t)}{dt}, \ldots, \frac{d^{n-1} u(t)}{dt^{n-1}}) \tag{77}$$

we can apply the results of the previous Section as long as the $n$th order ODE can be represented by a system of $n$ first order ODEs:

$$\frac{du(t)}{dt} = u_1(t);$$

$$\ldots$$

$$\frac{du_n(t)}{dt} = u_{n+1}(t);$$

$$\ldots$$

$$\frac{du_{n-1}(t)}{dt} = f(t, u(t), u_1(t), \ldots, u_{n-1}(t)).$$

So the formal solution of ODE (77) can be expressed in the following form (here $\vec{c} = c_1, \ldots, c_n$)

$$u(t, \vec{c}) = T_0 \exp\left\{ \int_a^t d\tau \left[ f(\tau, \vec{c}) \frac{\partial}{\partial c_n} + \cdots + c_{i+1} \frac{\partial}{\partial c_i} + \cdots + c_2 \frac{\partial}{\partial c_1} \right] \right\} c_1, \tag{78}$$

and it is obvious that (here $m = 1, \ldots, n-1$)

$$\frac{\partial^m u(t, \vec{c})}{\partial t^m} = T_0 \exp\left\{ \int_a^t d\tau \left[ f(\tau, \vec{c}) \frac{\partial}{\partial c_n} + \cdots + c_{i+1} \frac{\partial}{\partial c_i} + \cdots + c_2 \frac{\partial}{\partial c_1} \right] \right\} c_{m+1}. $$

If we consider $c_1, \ldots, c_n$ as arbitrary constants, the expression (78) is the general solution of the equation (77).

As we have indicated in Example 1 (see Subsection 3.9), the functions

$$\zeta_i(t, \vec{c}) = T \exp\left\{ - \int_a^t d\tau \left[ f(\tau, \vec{c}) \frac{\partial}{\partial c_n} + \cdots + c_{i+1} \frac{\partial}{\partial c_i} + \cdots + c_2 \frac{\partial}{\partial c_1} \right] \right\} c_i,$$

32
obey the differential equation
\[
\frac{\partial \zeta_i}{\partial t} + f(s, \vec{c}) \frac{\partial \zeta_i}{\partial c_n} + \cdots + c_{i+1} \frac{\partial \zeta_i}{\partial c_i} + \cdots + c_2 \frac{\partial \zeta_i}{\partial c_1} = 0 \quad (\zeta_i|_{t=a} = c_i)
\] (79)
and
\[
\zeta_i(t, u, \frac{du}{dt}, \ldots, \frac{d^{n-1}u}{dt^{n-1}}) = c_i, \quad (i = 1, \ldots, n).
\]

So if we know some, say \(k \leq n\), independent solutions of (79), then we can eliminate \(k\) unknowns from the system
\[
\zeta_i(t, u, \frac{du}{dt}, \ldots, \frac{d^{n-1}u}{dt^{n-1}}) = c_i, \quad (i = 1, \ldots, k).
\]
and as a result reduce the order of initial problem from \(n\) to \((n - k)\).

In the “integral-free” form
\[
u(t, \vec{c}) = \exp\{(t-a) [f(s, \vec{c}) \frac{\partial}{\partial c_n} + \cdots + c_{i+1} \frac{\partial}{\partial c_i} + \cdots + c_2 \frac{\partial}{\partial c_1} + \frac{\partial}{\partial s}] \} c_1 |_{s=a},
\]
the auxiliary function
\[
U(t, \vec{c}, s) = \exp\{(t-a) [f(s, \vec{c}) \frac{\partial}{\partial c_n} + \cdots + c_{i+1} \frac{\partial}{\partial c_i} + \cdots + c_2 \frac{\partial}{\partial c_1} + \frac{\partial}{\partial s}] \} c_1
\]
satisfies the following PDE
\[
\frac{\partial U}{\partial t} = f(s, \vec{c}) \frac{\partial U}{\partial c_n} + \cdots + c_{i+1} \frac{\partial U}{\partial c_i} + \cdots + c_2 \frac{\partial U}{\partial c_1} + \frac{\partial U}{\partial s}, \quad (U|_{t=a} = c_1)
\]
and
\[
u(t, \vec{c}) = U(t, \vec{c}, s)_{|_{s=a}}.
\]

Since here we have a larger number of degrees of freedom than in the case \(n = 1\), we can obtain many equivalent forms of the formal solutions like (71) using the way described in the previous section.

7 Helmholtz equation

Here we consider the essential features of the operator method for linear partial differential equations on example of formal solution of Helmholtz equation under different formulations of the boundary conditions. The solutions of the Helmholtz equation represent the (spatial part of) solutions of the wave equation.

Let us consider the Helmholtz equation for an inhomogeneous medium, which has the following form
\[
\frac{\partial^2 u(x, y, z)}{\partial x^2} = -[\Delta_2 + \varepsilon(x, y, z)]u(x, y, z) + q(x, y, z)
\] (80)
with boundary conditions on \( x = a \)

\[
\begin{aligned}
  u(x, y, z)|_{x=a} &= \alpha(y, z), \\
  \frac{\partial u(x, y, z)}{\partial x}|_{x=a} &= \beta(y, z),
\end{aligned}
\]  

(81)

where \( \Delta_2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \), \( \varepsilon(x, y, z) \) is a function, which takes into account the wave speed dependence on the space point, and \( q(x, y, z) \) is a wave source.

If we introduce the function

\[
S(x, y, z; h, p) = hu(x, y, z) + p \frac{\partial u(x, y, z)}{\partial x},
\]

where \( h \) and \( p \) are auxiliary real parameters, then equation (80) can be transformed into the following form

\[
\frac{\partial S(x, y, z; h, p)}{\partial x} = [h \frac{\partial}{\partial p} - \Delta_2 + \varepsilon(x, y, z)] p \frac{\partial}{\partial h} S(x, y, z; h, p) + pq(x, y, z)
\]

with boundary condition on \( x = a \)

\[
S(x, y, z; h, p)|_{x=a} = h \alpha(y, z) + p \beta(y, z).
\]

Hence the solution of the Helmholtz equation (80) under boundary conditions (81) is \( (x \geq a) \)

\[
\begin{aligned}
  u(x, y, z) &= \frac{\partial}{\partial h} T \exp \left\{ \int_a^x d\tau \left[ h \frac{\partial}{\partial p} - \Delta_2 + \varepsilon(\tau, y, z) \right] p \frac{\partial}{\partial h} \right\} \left[ h \alpha(y, z) + p \beta(y, z) \right] + \\
  \frac{\partial}{\partial h} \int_a^x d\tau T \exp \left\{ \int_{\tau}^x d\xi \left[ h \frac{\partial}{\partial p} - \Delta_2 + \varepsilon(\xi, y, z) \right] p \frac{\partial}{\partial h} \right\} p q(\tau, y, z)
\end{aligned}
\]

(82)

It is easy to see that \( u(x, y, z) \) in (82) does not depend on auxiliary parameters \( h \) and \( p \).

There are boundary value problems for Helmholtz equation (which are more profound from a physical point of view) when one puts certain requirements on the solution behavior at infinity (at \( r = (x^2 + y^2 + z^2)^{1/2} \to \infty \)). Since here the co-ordinate \( r \) is selected by boundary conditions it is expedient to solve this problem in spherical co-ordinates with the initial supposition that \( u \) and its first derivative on \( r \) at \( r = a \) are known. At \( a \to \infty \), \( u \) and its derivative in a medium with \( Im \varepsilon \geq 0 \) have to diminish rapidly enough (there are not wave sources at infinity), hence the first item of the expression of type (82) goes to zero. Therefore for \( u \) to satisfy the radiation conditions at infinity in a boundless inhomogeneous medium we can find the following operator expression \( (Im \varepsilon \geq 0) \)

\[
\begin{aligned}
  u(\vec{r}) &= -\frac{\partial}{\partial h} \int_r^\infty d\zeta T_0 \exp \left\{ \int_\zeta^r d\xi \left[ h \frac{\partial}{\partial p} - r^2 \Delta + \xi^2 \varepsilon(\frac{\xi}{r} \vec{\xi}) \right] p \frac{\partial}{\partial h} \right\} p \zeta^2 q(\frac{\zeta}{r} \vec{r})
\end{aligned}
\]

(83)

where vector \( \vec{r} = (x, y, z) \) and \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \) is the ordinary Laplace operator.
The Green function for a boundless inhomogeneous medium follows from (83) at $q(r) = \delta(\vec{r} - \vec{r}_0)$. By using the well-known method of images one can find from (83) the Green function say for half-space $x > 0$ and to get a solution of a Helmholtz equation (80) one has to fulfil the radiation conditions at infinity and a boundary condition on a plane for example.

8 The system of linear first-order PDEs

Let us consider the systems of linear PDEs, which are first-order with respect to $x$ and $y$, using as an example the following two PDEs for one function $u(x, y, \vec{\rho}) = u$:

$$\frac{\partial u}{\partial x} = A(x, y)u, \tag{84}$$

$$\frac{\partial u}{\partial y} = B(x, y)u,$$

where $\vec{\rho}$ is a set of parameters say $z_1, ..., z_n$, $A(x, y) = A(x, y, \vec{\rho})$ and $B(x, y) = B(x, y, \vec{\rho})$ are linear operators, which do not depend on $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ explicitly.

We can solve the first of them by obtaining

$$u(x, y) = T \exp\{ \int_a^x d\tau A(\tau, y) \} \phi(y, \vec{\rho}),$$

where $\phi(y, \vec{\rho})$ is a yet unknown function. Substituting now this solution into the second equation of the system we have

$$\frac{\partial}{\partial y} T \exp\{ \int_a^x d\tau A(\tau, y) \} \phi(y, \vec{\rho}) = B(x, y)T \exp\{ \int_a^x d\tau A(\tau, y) \} \phi(y, \vec{\rho})$$

and with the help of (46) we obtain the differential expression:

$$\frac{\partial \phi(y, \vec{\rho})}{\partial y} = \{ T_0 \exp\{ - \int_a^x d\tau A(\tau, y) \} B(x, y)T \exp\{ \int_a^x d\tau A(\tau, y) \} - \int_a^x d\tau T_0 \exp\{ - \int_a^{\tau} d\xi A(\xi, y) \} \frac{\partial A(\tau, y)}{\partial y} T \exp\{ \int_a^{\tau} d\xi A(\xi, y) \} \} \phi(y, \vec{\rho}), \tag{85}$$

which have the form of differential equation for an unknown $\phi(y, \vec{\rho})$ if and only if the right-hand side of (85) does not depend on $x$, that is when its derivative with respect to $x$ is equal to zero, which leads to the well-known consistency condition

$$[A(x, y), B(x, y)] + \frac{\partial A(x, y)}{\partial y} - \frac{\partial B(x, y)}{\partial x} = 0. \tag{86}$$
We can solve with the help of Eq. (45) this operator equation with respect to $B(x,y)$, that is rewrite the consistency condition as

$$B(x,y) = T \exp \left\{ \int_a^x d\tau A(\tau,y) \right\} \times \left[ B(a,y) + \int_a^x d\tau \ T_0 \exp \left\{ - \int_a^\tau d\xi A(\xi,y) \right\} \frac{\partial A(\tau,y)}{\partial y} T \exp \left\{ - \int_a^\tau d\xi A(\xi,y) \right\} \right] \times T_0 \exp \left\{ - \int_a^x d\tau A(\tau,y) \right\},$$  \hspace{1cm} (87)

where $B(a,y) = B(x,y)|_{x=a}$ is an arbitrary linear operator (which does not depend on $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ explicitly).

The formal solution of the equation (85) is

$$\phi(y,\vec{\rho}) = T \exp \left\{ \int_b^y d\zeta T_0 \exp \left\{ - \int_a^x d\tau A(\tau,\zeta) \right\} B(x,\zeta) T \exp \left\{ \int_a^x d\tau A(\tau,\zeta) \right\} - \int_a^x d\tau T_0 \exp \left\{ - \int_a^\tau d\xi A(\xi,\zeta) \right\} \frac{\partial A(\tau,\zeta)}{\partial \zeta} T \exp \left\{ \int_a^\tau d\xi A(\xi,\zeta) \right\} \right\} c(\vec{\rho}),$$  \hspace{1cm} (88)

where $c(\vec{\rho})$ is an arbitrary function. Substituting now (87) into (88) we receive

$$\phi(y,\vec{\rho}) = T \exp \left\{ \int_b^y d\zeta B(a,\zeta) c(\vec{\rho}) \right\}$$

and finally the desired solution of considered system (84) if (86) holds is

$$u(x,y,\vec{\rho}) = T \exp \left\{ \int_a^x d\tau A(\tau,y,\vec{\rho}) \right\} T \exp \left\{ \int_b^y d\zeta B(a,\zeta,\vec{\rho}) c(\vec{\rho}) \right\},$$  \hspace{1cm} (89)

where $c(\vec{\rho})$ has an obvious definition as $c(\vec{\rho}) = u(x,y,\vec{\rho})|_{x=a,y=b}$.

Since here $A(x,y,\vec{\rho})$ and $B(x,y,\vec{\rho})$ are operators, then (89) can represent solutions of non-trivial systems of PDEs.

9 Conclusions

We have presented some ways in solving DEs by the chronological operator method. Besides linear first-order DEs and systems of such DEs, we have obtained operator solutions for linear and non-linear ODEs of arbitrary order.

It is easy to note that the obtained solutions contain differential operators with respect to arbitrary constants, which represent initial conditions of the problem. For more complicated problems, e.g. for non-linear PDEs, the formal solutions will contain variational differential operators. Some examples we have touched on can be found in [9] and [10].

In conclusion, we believe that we succeeded in demonstrating the fact that operator forms of DE solutions can be handled analytically no worse than ordinary functions. In some cases its transformation properties are more comfortable than, e.g., for some special functions.
Acknowledgments

The author would like to thank Reece Heineke for a careful reading of this paper and some actual suggestions.

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