Permutation decoding of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes

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Abstract

An alternative permutation decoding method is described which can be used for any binary systematic encoding scheme, regardless whether the code is linear or not. Thus, the method can be applied to some important codes such as $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, which are binary and, in general, nonlinear codes in the usual sense. For this, it is proved that these codes allow a systematic encoding scheme. As a particular example, this permutation decoding method is applied to some Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear codes.

1 Introduction

We denote by $\mathbb{F}^n$ the set of all binary vectors of length $n$ and by $\text{wt}(v)$ the (Hamming) weight of any vector $v \in \mathbb{F}^n$, that is, the number of its nonzero coordinates. The (Hamming) distance between two vectors $u, v \in \mathbb{F}^n$ is defined as $d(u, v) = \text{wt}(u + v)$. Given a binary code of length $n$, $C \subseteq \mathbb{F}^n$, we denote by $d_C$ its minimum distance, that is, the minimum distance between any pair of different codewords in $C$. We say that $C$ is a $t$-error-correcting code, where $t = \lfloor (d_C - 1)/2 \rfloor$.

For a vector $v \in \mathbb{F}^n$ and a set $I \subseteq \{1, \ldots, n\}$, $|I| = k$, we define $v_I \in \mathbb{F}^k$ as the vector $v$ restricted to the $I$ coordinates. For example, if $I = \{1, \ldots, k\}$ and $v = (v_1, \ldots, v_n)$, then $v_I = (v_1, \ldots, v_k)$. If $C$ is a binary code of length $n$, then $C_I = \{v_I : v \in C\}$.

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If $C$ has size $|C| = 2^k$, then $C$ is a systematic code if there is a set $I \subseteq \{1, \ldots, n\}$ of $k$ coordinate positions such that $|C_I| = 2^k$. In other words, $C_I$ is $\mathbb{F}^k$. Such a set $I$ is also referred to as a set of systematic coordinates or an information set. Given a systematic code of size $|C| = 2^k$ with information set $I$, a systematic encoding for $I$ is a one-to-one map $f : \mathbb{F}^k \to \mathbb{F}^n$, such that for any information vector $a \in \mathbb{F}^k$, the corresponding codeword $f(a)$ satisfies that $f(a)_I = a$.

Let us consider the group of permutations on $n$ symbols, $S_n$, acting on $\mathbb{F}^n$ by permuting the coordinates of each vector. That is, for every $v = (v_1, \ldots, v_n) \in \mathbb{F}^n$ and $\pi \in S_n$, $\pi(v_1, \ldots, v_n) = (v_{\pi^{-1}(1)}, \ldots, v_{\pi^{-1}(n)})$. Then, for any binary code $C$, we denote by $\text{PAut}(C)$ its permutation automorphism group, i.e., $\text{PAut}(C) = \{ \pi \in S_n : \pi(C) = C \}$. Moreover, a binary code $C'$ is said to be permutation equivalent to $C$ if there exists $\pi \in S_n$ such that $\pi(C) = C'$.

Not every binary code of size $2^k$ is systematic, but every binary linear code is systematic. Indeed, if $C \subseteq \mathbb{F}^n$ is a binary linear code of dimension $k$, it is permutation equivalent to a code with generator and parity check matrices:

$$G = \begin{pmatrix} \text{Id} & A \end{pmatrix}, \quad H = \begin{pmatrix} A^T & \text{Id} \end{pmatrix}, \quad (1)$$

where $\text{Id}$ denotes the identity matrix and $A^T$ is the transpose of $A$. In general, for any information set $I$, we say that a generator (resp. parity check) matrix is in standard form if the columns in the positions inside (resp. outside of) $I$ are the columns of $\text{Id}$. Then the map $f : \mathbb{F}^k \to \mathbb{F}^n$ given by

$$f(v) = v G, \quad (2)$$

for any $v \in \mathbb{F}^k$, is clearly a systematic encoding.

Permutation decoding was introduced in [11] and [8]. A description of the standard method for linear codes can be found in [9, p.513]. Given a $t$-error-correcting linear code $C \subseteq \mathbb{F}^n$ with fixed information set $I$, we consider $y = x + e$ the received vector, where $x \in C$ and $e$ is the error vector. We assume that $y$ has less than $t + 1$ errors, that is, $\text{wt}(e) \leq t$. The idea of permutation decoding is to use the elements of $\text{PAut}(C)$ in order to move the non-zero coordinates of $e$ out of $I$. So, on the one hand the method is based on the existence of some special subsets $S \subseteq \text{PAut}(C)$, called PD-sets, verifying that for any vector $e \in \mathbb{F}^n$, with $\text{wt}(e) \leq t$, there is an element $\pi \in S$ such that $\text{wt}(\pi(e)_I) = 0$. On the other hand, the main tool of this decoding algorithm is the following theorem which gives us a necessary and sufficient condition for a received vector $y \in \mathbb{F}^n$ having its systematic coordinates correct.
Theorem 1.1 ([9]) Let $C$ be a $t$-error-correcting linear code with information set $I$ and parity check matrix $H$ in standard form. Let $y = x + e$, where $x \in C$ and $e$ verifies that $\text{wt}(e) \leq t$. Then

$$\text{wt}(H_y^T) = \text{wt}(H_e^T) \leq t \iff \text{wt}(e_I) = 0.$$  

Then, let $C \subseteq \mathbb{F}^n$ be a linear code with information set $I$ and parity check matrix $H$ in standard form. Assume that we have found a PD-set for the information set $I$, $S \subseteq \text{PAut}(C)$, and denote by $y = x + e$ the received vector, where $x \in C$ and $e$ is the error vector. Then the permutation decoding algorithm works as follows:

1. If $\text{wt}(H_y^T) \leq t$, then the systematic coordinates of $y$ are correct and we can recover $x$ from (2).

2. Else, we search $\pi \in S$ such that $\text{wt}(H\pi(y)^T) \leq t$. If there is no such $\pi$, we conclude that more than $t$ errors have occurred.

3. If we have successfully found $\pi$, we take $x'$ the unique codeword such that $x'_I = \pi(y)_I$. Then the decoded vector is $\pi^{-1}(x')$.

In this paper, we show that $\mathbb{Z}_2\mathbb{Z}_4$-linear codes are also systematic. Moreover, we give a systematic encoding method. However, for $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, Theorem 1.1 holds just in some obvious cases, not in general. Nevertheless, we give an alternative method for permutation decoding which does not need (3). Such method does not use the syndrome $H_y^T$ to check whether the systematic coordinates are correct or not. Therefore, the method can be used for general $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, of course, assuming that we know an appropriate PD-set.

The paper is organized as follows. In Section 2 we show that any $\mathbb{Z}_2\mathbb{Z}_4$-linear code is systematic. Moreover, we give an information set and a systematic encoding for that information set. We also see under which conditions the standard permutation decoding method works for $\mathbb{Z}_2\mathbb{Z}_4$-linear codes. In Section 3 we present the alternative permutation decoding method. Such method does not use the syndrome of a received vector in order to check whether the systematic coordinates are correct or not. Hence, it is not important if Theorem 1.1 holds or not. We use the new method for the particular case of Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear codes. These are, in general, nonlinear codes in the binary sense, but they have high error-correcting capability.
2 Systematic encoding for $\mathbb{Z}_2\mathbb{Z}_4$-linear codes

For every pair $n_1, n_2$ of nonnegative integers, define the componentwise Gray map $\Phi : \mathbb{Z}_{2}^{n_1} \times \mathbb{Z}_{4}^{n_2} \rightarrow \mathbb{F}_{2^{n_1+2n_2}}$ as

$$\Phi(x, y) = (x, \phi(y_1), \ldots, \phi(y_{n_2}))$$

for all $x \in \mathbb{Z}_{2}^{n_1}$ and $y = (y_1, \ldots, y_{n_2}) \in \mathbb{Z}_{4}^{n_2}$,

where $\phi : \mathbb{Z}_4 \rightarrow \mathbb{Z}_2^2$ is the usual Gray map, that is,

$\phi(0) = (0, 0), \phi(1) = (0, 1), \phi(2) = (1, 1), \phi(3) = (1, 0)$.

The Lee weights over the elements in $\mathbb{Z}_4$ are defined as $\omega_L(0) = 0, \omega_L(1) = \omega_L(3) = 1, \omega_L(2) = 2$. Then the Lee weight of a vector $u = (u_1, \ldots, u_{n_1+n_2}) \in \mathbb{Z}_{2}^{n_1} \times \mathbb{Z}_{4}^{n_2}$ is defined as $\omega_L(u) = \text{wt}(u_1, \ldots, u_{n_1}) + \sum_{i=1}^{n_2} \omega_L(u_{n_1+i})$. Finally, the Lee distance between two vectors $u, v \in \mathbb{Z}_{2}^{n_1} \times \mathbb{Z}_{4}^{n_2}$ is $d_L(u, v) = \omega_L(u-v)$.

The Gray map is an isometry which transforms Lee distances to Hamming distances.

Let $C$ be a $\mathbb{Z}_2\mathbb{Z}_4$-linear code of type $(\alpha, \beta; \gamma, \delta; \kappa)$, length $n = \alpha + 2\beta$ and size $|C| = 2^\kappa = 2^{\gamma+2\delta}$ [2]. As usual, denote by $C$ the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-additive code, i.e. $C = \Phi^{-1}(C)$. If $C$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$, it is permutation equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with generator matrix as follows [2]:

$$G = \begin{pmatrix}
I_\kappa & T_b & 2T_2 & 0 & 0 \\
0 & 0 & 2T_1 & 2I_{\gamma-\kappa} & 0 \\
0 & S_b & R & I_\delta
\end{pmatrix},$$

where $T_b, S_b$ are matrices over $\mathbb{Z}_2$; $T_1, T_2, R$ are matrices over $\mathbb{Z}_4$ with all entries in $\{0, 1\} \subset \mathbb{Z}_4$; and $S_q$ is a matrix over $\mathbb{Z}_4$. We say that $G$ is a matrix in standard form for a $\mathbb{Z}_2\mathbb{Z}_4$-additive code.

Given two vectors $u, v \in \mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta}$, the inner product is defined as in [2]:

$$\langle u, v \rangle = 2(\sum_{i=1}^{\alpha} u_i v_i) + \sum_{j=\alpha+1}^{\alpha+\beta} u_j v_j \in \mathbb{Z}_4,$$

where the computations are made taking the zeros and ones in the $\alpha$ binary coordinates as quaternary zeros and ones, respectively. The additive dual code of $C$, denoted by $C^\perp$, is then defined in the standard way:

$$C^\perp = \{ y \in \mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta} | \langle x, y \rangle = 0, \text{ for all } x \in C \}.$$
It is also shown in [2] that if $C$ has a generator matrix in standard form (4), then $C^\perp$ can be generated by the matrix:

$$
H = \begin{pmatrix}
T^t_b & I_{\alpha-\kappa} & 0 & 0 & 2S^t_b \\
0 & 0 & 2I_{\gamma-\kappa} & 2R^t \\
T^t_2 & 0 & I_{\beta+\kappa-\gamma-\delta} & T^t_1 & -(S_q + RT_1)
\end{pmatrix},
$$

which also represents a parity check matrix for $C$. Moreover, $C^\perp$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \bar{\gamma}, \bar{\delta}; \bar{\kappa})$, where

$$
\bar{\gamma} = \alpha + \gamma - 2\kappa,
\bar{\delta} = \beta - \gamma - \delta + \kappa,
\bar{\kappa} = \alpha - \kappa.
$$

There are some cases where the systematic encoding of $C$ is clear. First, when $C$ is linear, we can apply simply the standard systematic encoding for linear codes by considering the generator matrix $G$ of $C$ as in (1). It is clear that $C$ is linear, for example, when $\beta = 0$ and also when $\delta = 0$. In general, if $G$ is the generator matrix of a $C$ as in (4) and $\{u_i\}_{i=1}^\gamma$ and $\{v_j\}_{j=1}^\delta$ are the row vectors of order two and order four in $G$, respectively, then $C$ is a binary linear code if and only if $2v_j \ast v_k \in C$, for all $j, k$ satisfying $1 \leq j < k \leq \delta$, where $\ast$ is the component-wise product (see [5]).

The second case is when $\gamma = \kappa$. If we consider the generator and the parity check matrix of a code $C$ with $\gamma = \kappa$, then we obtain

$$
G = \begin{pmatrix}
I_k & T_b & 2T_2 & 0 \\
0 & S_b & S_q & I_\delta
\end{pmatrix},
\quad H = \begin{pmatrix}
T^t_b & I_{\alpha-\kappa} & 0 & 2S^t_b \\
T^t_2 & 0 & I_{\beta+\kappa-\gamma-\delta} & -S_q
\end{pmatrix}.
$$

It is clear that for any information vector $(u, v) \in \mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$, we have that $(u, v)G = (u, z, v)$ for some $z \in \mathbb{Z}_2^{\alpha-\gamma} \times \mathbb{Z}_4^{\beta-\delta}$ and, therefore, the set $I = \{1, \ldots, \kappa, \alpha + \beta - \delta + 1, \ldots, \alpha + \beta\}$ is a set of systematic coordinates. Hence, we have the following systematic encoding:

$$
f(a) = \Phi\left(\Phi^{-1}(a)G\right), \quad \forall a \in \mathbb{F}_k.
$$

Even though in those cases the systematic encoding is clear, we can not use the same method to $\mathbb{Z}_2\mathbb{Z}_4$-linear codes in general. We are going to define a method that use the $\mathbb{Z}_2\mathbb{Z}_4$-linearity of the code and can be used for all values of $\alpha, \beta, \gamma, \delta$ and $\kappa$.

Let us consider $C$ a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with a generator matrix in standard form (4). For each quaternary coordinate
α+i, with i ∈ {1, . . . , β}, we denote by φ1(α+i) and φ2(α+i) the corresponding pair of binary coordinates in {1, . . . , α+2β}, that is, φ1(α+i) = α+2i−1 and φ2(α+i) = α+2i. We define the following sets of coordinate positions in \{1, . . . , α+2β\}:

- \(J_1 = \{1, . . . , \kappa\}\), \(|J_1| = \kappa\).
- \(J_2 = \{j_1, \ldots, j_{\gamma-\kappa}\}\), where \(j_i = \varphi_1(\alpha + \beta + \kappa - \gamma - \delta + i)\), \(|J_2| = \gamma - \kappa\).
- \(J_3 = \varphi_1(\alpha + \beta - \delta + 1), \varphi_2(\alpha + \beta - \delta + 1), \ldots, \varphi_1(\alpha + \beta), \varphi_2(\alpha + \beta)\)\), \(|J_3| = 2\delta\).

We are going to show that \(J = J_1 \cup J_2 \cup J_3\) is a set of systematic coordinates for the \(\mathbb{Z}_2\mathbb{Z}_4\)-linear code \(C\). We shall refer to \(J\) as the standard information set or standard set of systematic coordinates.

Given an information vector \(a = (a_1, \ldots, a_{\gamma+2\delta}) \in \mathbb{F}^{\gamma+2\delta}\), we consider the representation \(a = (b, c, d)\), where \(b = (a_1, \ldots, a_\kappa)\), \(c = (c_1, \ldots, c_{\gamma-\kappa}) = (a_{\kappa+1}, \ldots, a_\gamma)\) and \(d = (a_{\gamma+1}, \ldots, a_{\gamma+2\delta})\). Note that \(\Phi^{-1}(a) = (b, c, \Phi^{-1}(d)) \in \mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta\). Consider the codeword \(x = \Phi^{-1}(a)\mathcal{G} \in \mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta\). For each \(i \in \{1, . . . , \gamma - \kappa\}\), define

\[
\eta_i = \begin{cases} 
0 & \text{if } x_{j_i} = c_i, \\
1 & \text{otherwise}, 
\end{cases}
\]

where, following the notation given above, \(J_2 = \{j_1, \ldots, j_{\gamma-\kappa}\}\). Let \(\eta = (\eta_1, . . . , \eta_{\gamma-\kappa})\). Then, we consider the bijection \(\sigma : \mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta \rightarrow \mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta\) given by \(\sigma(\Phi^{-1}(a)) = \sigma(b, c, \Phi^{-1}(d)) = (b, c+\eta, \Phi^{-1}(d))\). It is straightforward that the codeword \(\sigma(\Phi^{-1}(a))\mathcal{G}\) verifies that

\[
\left(\Phi \left(\sigma(\Phi^{-1}(a))\mathcal{G}\right)\right)_J = (b, c, d) = a.
\]

Since \(|J| = \kappa + \gamma - \kappa + 2\delta = \gamma + 2\delta\), we conclude that \(J\) is a set of systematic coordinates. Therefore we have proved the following theorem.

**Theorem 2.1** If \(C\) is a \(\mathbb{Z}_2\mathbb{Z}_4\)-linear code of type \((\alpha, \beta; \gamma, \delta; \kappa)\), then \(C\) is a systematic code. Moreover, if we assume that the generator matrix of \(C = \Phi^{-1}(C)\) is in standard form \(4\), then \(J = J_1 \cup J_2 \cup J_3\) is a set of systematic coordinates for \(C\).

Note that in the case \(\gamma = \kappa\) we have \(a = (b, d)\), so \(\eta\) is the all-zero vector and hence \(\sigma\) is the identity map. Therefore, as a result, for \(\gamma = \kappa\) we obtain the systematic encoding function given in \(8\).
Corollary 2.2 Let $C$ be a $\mathbb{Z}_2\mathbb{Z}_4$-linear code of length $n$, size $|C| = 2^k$ and such that $C = \Phi^{-1}(C)$ has generator matrix in standard form (7). Then, the function $f : \mathbb{F}^k \rightarrow \mathbb{F}^n$ defined as

$$f(a) = \Phi\left(\sigma\left(\Phi^{-1}(a)\right)G\right), \quad \forall a \in \mathbb{F}^k$$

(10)

is a systematic encoding for $C$ and the information set $J$.

The following example shows that the set of systematic coordinates is not unique, in general.

Example 2.3 Consider the $\mathbb{Z}_2\mathbb{Z}_4$-additive code $C$ generated by

$$G = \begin{pmatrix} 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$ 

Let $C = \Phi(C)$ be the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code in $\mathbb{F}^8$. A set of systematic coordinates is $\{2, 4, 6, 8\}$. However, the standard set of systematic coordinates would be $\{1, 5, 7, 8\}$.

Note that this encoding method requires, in some cases, two products by the generator matrix. However, this is not a meaningful change of complexity order.

3 An alternative permutation decoding algorithm

In this section we are going to see that the usual permutation decoding algorithm can be applied to $\mathbb{Z}_2\mathbb{Z}_4$-linear codes just in a few cases. This is because, even if we find a PD-set, Theorem 1.1 can not be used in general. We shall present an alternative permutation decoding algorithm where Theorem 3.2 replaces Theorem 1.1.

Let $C$ be a $t$-error correcting $\mathbb{Z}_2\mathbb{Z}_4$-linear code with information set $J$. Let $C = \Phi^{-1}(C)$ be the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. On the one hand, we have seen that if $C$ is linear then the usual systematic encoding can be applied considering the matrices as in (11). So Theorem 1.1 works.

On the other hand, if $\gamma = \kappa$ then we have seen that we can assume that $C$ has a parity check matrix $H$ containing the identity matrix (see (7)). Then, denote the received vector $y = x + e \in \mathbb{F}^{\alpha+2\beta}$, where $x \in C$ and $e$ is the error.
Proposition 3.1 Let $C$ be a $t$-error-correcting $\mathbb{Z}_2\mathbb{Z}_4$-additive code of length $n$, type $(\alpha, \beta; \gamma, \delta; \kappa)$ and parity check matrix $H$, such that $C = \Phi(C)$ is a binary nonlinear code. Then, $C$ satisfies (11) if and only if $\gamma = \kappa$.

Proof:

The case $\gamma = \kappa$ have been discussed above. So assume $\gamma > \kappa$.

Denote by $e_i$ the binary vector of length $n$ which has weight one and its nonzero coordinate is at position $i$ ($1 \leq i \leq n$). Define the three binary coordinate sets:

1. $L_1 = \{\kappa + 1, \cdots, \alpha\}$,
2. $L_2 = \{\varphi_1(\alpha+1), \varphi_2(\alpha+1), \ldots, \varphi_1(\alpha+\beta+\kappa-\gamma-\delta), \varphi_2(\alpha+\beta+\kappa-\gamma-\delta)\}$,
3. $L_3 = \{j_1, \ldots, j_{\gamma-\kappa}\}$, where $j_i = \varphi_2(\alpha + \beta + \kappa - \gamma - \delta + i)$.

We have that $L = L_1 \cup L_2 \cup L_3 = \{1, \ldots, n\} \setminus J$. Consider an error vector $e \in \mathbb{F}^n$ such that $\text{wt}(e) = t$, $\text{wt}(e_J) = 0$ and $\text{wt}(e_{L_3}) \neq 0$. By the definition of $H$, (11), it is easy to check that for $k_1, \ldots, k_r \in L_3$ we have that

$$\omega_L(\Phi^{-1}(y)) \geq 2t,$$

where

$$\omega_L(\Phi^{-1}(y)) = \omega_L(\Phi^{-1}(x)) \leq t \iff \omega_L(e_J) = 0. \quad (11)$$

The case $\omega_L(e_J) = 0$, where $x \in C$ and $\text{wt}(e) \leq t$.

Theorem 3.2 Let $C$ be a binary systematic $t$-error-correcting code of length $n$. Let $I$ be a set of systematic coordinates and let $f$ be a systematic encoding for $I$. Suppose that $y = x + e$ is a received vector, where $x \in C$ and $\text{wt}(e) \leq t$. Then, the systematic coordinates of $y$ are correct, i.e. $y_I = x_I$, if and only if $\text{wt}(y + f(y_I)) \leq t$. 

\[ \square \]
Proof: If \( \text{wt} (y + f(y_I)) \leq t \), then \( f(y_I) \) is the closest codeword to \( y \), that is, \( f(y_I) = x \). Hence the systematic coordinates are the same \( y_I = x_I \).

If \( x_I = y_I \), then \( \text{wt} (y + f(y_I)) = \text{wt}(y + x) = \text{wt}(e) \leq t \). \( \square \)

Now, let us consider \( C \) a \( \mathbb{Z}_2 \mathbb{Z}_4 \)-linear code with information set \( I \). Assume that \( S \subseteq \text{PAut}(C) \) is a PD-set for \( I \) and \( y \) is a received vector. As an alternative method with respect to the algorithm described in Section \[1\] we can use the following decoding process:

1. If \( \text{wt} (y + f(y_I)) \leq t \), then \( x = f(y_I) \) is the decoded vector and \( y_I \) is the information vector.

2. Else, we search \( \pi \in S \) such that \( \text{wt} (\pi(y) + f(\pi(y)_I)) \leq t \). If there is no such \( \pi \), we conclude that more than \( t \) errors have occurred.

3. If we have successfully found \( \pi \), then the decoded vector is \( x = \pi^{-1}(f(\pi(y)_I)) \).

Note that \( \text{wt} (\pi(y) + f(\pi(y)_I)) \leq t \) implies that \( f(\pi(y)_I) \) is the closest codeword to \( \pi(y) \). Therefore, the closest codeword to \( y \) is \( \pi^{-1}(f(\pi(y)_I)) \).

Example 3.3 Consider the \( \mathbb{Z}_2 \mathbb{Z}_4 \)-additive code \( C \) with generator and parity check matrices:

\[
\mathcal{G} = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix}.
\]

The corresponding \( \mathbb{Z}_2 \mathbb{Z}_4 \)-linear code \( C = \Phi(C) \) is a 1-error-correcting code of type \((0,4;0,2;0)\) (i.e., \( C \) is a \( \mathbb{Z}_4 \)-linear code). In fact, \( C \) is a Hadamard \( \mathbb{Z}_4 \)-linear code [11]. Let \( \vartheta = (1,3,5,7)(2,4,6,8) \). It is straightforward to check that \( \vartheta \in \text{PAut}(C) \) (note that \( C \) is a quaternary cyclic code) [10]. Moreover, \( S = \{ \text{id}, \vartheta, \vartheta^2 \} \) is a PD-set for the standard information set \( I = \{5,6,7,8\} \).

Since \( \gamma = \kappa \), we can use the systematic encoding \( f \) defined in (8).

For example, let \( a = (0,1,0,1) \in \mathbb{F}_4 \) be an information vector. Then

\[
x = f(a) = \Phi(\Phi^{-1}(a)\mathcal{G}) = \Phi((1,1)\mathcal{G}) = \Phi(1,1,1,1) = (0,1,0,1,0,1,0,1).
\]

Suppose now that the received vector is \( y = x + e \), where \( e = (0,0,0,0,0,0,0,1) \). The syndrome of \( y \) is

\[
\Phi(\mathcal{H}\Phi^{-1}(y)^T) = \Phi(\mathcal{H}(1,1,1,0)^T) = \Phi((2,3)^T) = (1,1,1,0)^T,
\]

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which has weight $3 > t = 1$. However, considering the vector $z = \vartheta(y) = (0, 0, 0, 1, 0, 1, 0, 1)$, we have that the syndrome is

$$
\Phi \left( H \Phi^{-1}(z)^T \right) = \Phi((3, 0)^T) = (1, 0, 0, 0)^T,
$$

which has weight $1 \leq t = 1$. Therefore, the systematic coordinates of $z$ have no errors. Hence, we decode $y$ as

$$
\vartheta^{-1} \left( \Phi(\Phi^{-1}(z)G) \right) = \vartheta^{-1} \left( \Phi((1, 1)G) \right) = \vartheta^{-1}(\Phi(1, 1, 1)) = (0, 1, 0, 1, 0, 1, 0, 1) = x,
$$

and the information vector is $x_I = (0, 1, 0, 1)$.

**Example 3.4** Consider the $\mathbb{Z}_2\mathbb{Z}_4$-additive code $C$ with generator matrix:

$$
G = \begin{pmatrix}
2 & 2 & 0 & 0 & 0 & 0 \\
3 & 2 & 1 & 2 & 3 & 0 \\
2 & 3 & 0 & 3 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

The corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code $C = \Phi(C)$ is a 3-error-correcting code of type $(0, 8; 1, 2; 0)$ (i.e., $C$ is a $\mathbb{Z}_4$-linear code). In fact, $C$ is also a Hadamard $\mathbb{Z}_4$-linear code [7]. We know that $\langle \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \rangle \subseteq \text{PAut}(C)$ [11], where

\begin{align*}
\vartheta_1 &= (1, 5)(2, 6)(3, 11)(4, 12)(9, 13)(10, 14)(7, 15)(8, 16), \\
\vartheta_2 &= (1, 3, 5, 11)(2, 4, 6, 12)(9, 7, 13, 15)(10, 8, 14, 16), \\
\vartheta_3 &= (9, 13)(10, 14)(7, 15)(8, 16), \\
\vartheta_4 &= (1, 9)(2, 10)(5, 13)(6, 14).
\end{align*}

Moreover, it is easy to check using the MAGMA software package [4] that we can take the elements in the subgroup $S = \langle \vartheta_1, \vartheta_2, \vartheta_4 \rangle$ as a PD-set for the information set $I = \{11, 13, 14, 15, 16\}$. In this case, we can not use the standard permutation decoding, since $\gamma \neq \kappa$. However, we can still perform a permutation decoding using the alternative method presented in this section.

For example, let $a = (1, 1, 1, 1, 1) \in \mathbb{F}_5$ be an information vector. Using the systematic encoding given by (10), the corresponding codeword is

\begin{align*}
x &= f(a) = \Phi(\sigma(\Phi^{-1}(a))G) = \Phi((1 + \eta_1, 2, 2)G) \\
&= \Phi(2, 2, 2, 2, 2, 2, 2) = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1),
\end{align*}

where $\eta = (\eta_1) = (1)$. Suppose now that the received vector is $y = x + e$, where $e = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 1)$. By considering the standard information set, the information coordinates of $y$ are $y_I = (1, 0, 1, 0, 0)$ and

$$
f(y_I) = \Phi(\sigma(\Phi^{-1}(y_I))G) = (0, 1, 0, 0, 1, 0, 1, 1, 1, 0, 1, 0, 1, 0, 1, 0, 0),
$$
so \( \text{wt}(y + f(y_1)) = 5 > t = 3 \). However, considering the vector \( z = \vartheta_1(y) = (1, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 1) \), we have that \( z_1 = (1, 1, 1, 1) \) and
\[
f(z_1) = \Phi(\sigma(\Phi^{-1}(z_1))\mathcal{G}) = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1),
\]
so \( \text{wt}(z + f(z_1)) = 3 \leq t = 3 \). Therefore, the systematic coordinates of \( z \) have no errors. Hence, we decode \( y \) as \( \vartheta_1^{-1}(f(z_1)) = x \) and the information vector is \( x_1 = (1, 1, 1, 1) \).

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