SPECTRAL ANALYSIS AND DIRICHLET FORMS ON BARLOW-EVANS FRACTALS

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Abstract. We show that if a Barlow-Evans Markov process on a vermiculated space is symmetric, then one can study the spectral properties of the corresponding Laplacian using projective limits. For some examples, such as the Laakso spaces and a Sierpinski Pâte à Choux, one can develop a complete spectral theory, including the eigenfunction expansions that are analogous to Fourier series. In addition, we construct connected fractal spaces isospectral to the fractal strings of Lapidus and van Frankenhuijsen.

1. Introduction

We study symmetric regular Dirichlet forms [12] on the fractal-like spaces $F_\infty$ constructed in [10]. Barlow and Evans in [10] describe a construction for a new, interesting class of state spaces for Markov processes utilizing projective limits. We furthermore assume that the base spaces are Dirichlet metric measure spaces, that is, metric measure spaces equipped with Dirichlet forms. We show that in this case one can develop a complete spectral theory of the associated Laplace operators, including formulas for spectral projections, utilizing the tools of Dirichlet form theory. The characterization of the spectra of the Laplacians presented here is a generalization of those obtained previously by the first author for Laakso spaces in [21, 23].

Given a measure space on which one has a Laplacian it is natural to study the spectrum. As the measure space becomes more complicated this task can become very difficult. On fractal spaces such as the Sierpinski gasket and carpet this problem has been extensively studied [25, 7, 8, 16, 17]. For finitely ramified self-similar highly symmetric fractals a complete spectral analysis is possible although rather complicated, see [2, 3] and references therein. Moreover, it is possible to extend this kind of spectral analysis to finitely ramified fractafolds, that is to metric measure spaces that have local charts from open sets of a reference fractal. This is one way of obtaining new examples from old, including isospectral fractafolds, see [25, 26]. The projective limit construction provides yet another way of controllably obtaining new measure spaces and in this paper we examine how the spectral data transfers to the limit.

The main goal of this paper is an understanding of the spectrum of a class of Laplacians. We have found that working in terms of the associated Dirichlet form to be more straightforward. This is particularly noticeable in Definition 2.4. Where the domain of a Dirichlet form is much easier to write down than the domain of the corresponding Laplacian.

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As we discuss in the final section of this paper, the projective limit construction can produce many connected fractals which are isospectral to a given fractal string (see \cite{20} and references therein). Which makes it possible to ask which fractal strings then correspond to connected fractals on which there are heat kernel estimates for a diffusion process. However, answering this question is beyond the scope of this paper. Determining heat kernel estimates on fractal spaces has a long tradition (see for instance, \cite{11, 12, 13} and an extensive review in \cite{14}). For example Laakso spaces have Gaussian heat kernel estimates while Sierpinski gasket-like fractals have sub-Gaussian estimates often depending on geometric conditions.

One note of caution, our analysis of fractals defined as projective limits is entirely intrinsic on abstractly defined objects. Even in the simplest examples, Laakso spaces, the limit space is not bi-Lipschitz embeddable in any finite dimensional Euclidean space.

We begin with definitions in Section 2. In Sections 3 and 4 we provide the background on projective systems of measure spaces along with the limiting procedure for the Laplacians on each approximating measure space. Section 5 contains the main results of the paper which give a decomposition of the systems of measure spaces along with the limiting procedure for the Laplacians on each approximating bi-Lipschitz embeddable in any finite dimensional Euclidean space. One note of caution, our analysis of fractals defined as projective limits is entirely intrinsic on abstractly defined objects. Even in the simplest examples, Laakso spaces, the limit space is not bi-Lipschitz embeddable in any finite dimensional Euclidean space.

We begin with definitions in Section 2. In Sections 3 and 4 we provide the background on projective systems of measure spaces along with the limiting procedure for the Laplacians on each approximating measure space. Section 5 contains the main results of the paper which give a decomposition of the spectrum of the Laplacian on the limit space. Then in Section 6 we describe three classes of examples of spaces that can be constructed with this method, and outline the computation of the spectrum in each case.

### 2. Definitions

The following definitions are based on those given in \cite{10} and references therein.

**Definition 2.1.** Let $F_0$ be a locally compact second-countable Hausdorff space with a $\sigma$-finite Borel measure $\mu_{F_0}$. In addition we assume there is a sequence of compact second-countable Hausdorff spaces $G_i$ for $i \geq 0$ with Borel probability measures $\mu_{G_i}$.

Inductively define a sequence of locally compact topological measure spaces and maps between them as follows (refer to Figure 1). Suppose $i > 0$, $F_{i-1}$ is defined, (is a locally compact second-countable Hausdorff space,) and $B_i \subset F_{i-1}$ is closed. Set $$F_i = (F_{i-1} \setminus B_i) \times G_i \cup B_i$$ and $$\pi_i(x, g) = \begin{cases} (x, g) & \text{if } x \in F_{i-1} \setminus B_{i-1} \\ x & \text{if } x \in B_{i-1}. \end{cases}$$ The space $F_i$ is topologized by the map $\pi_i$, which means that a subset of $F_i$ is open if and only if its $\pi_i$ preimage is open.

The map $\psi_i$ is the natural projection from $F_{i-1} \times G_i \to F_{i-1}$ and define $\phi_i = \psi_i \circ \pi_i^{-1} : F_i \to F_{i-1}$. Alternatively $\phi_i$ can be defined by

$$\phi_i(x, g) = x \text{ if } x \in F_{i-1} \setminus B_{i-1}$$
$$\phi_i(x) = x \text{ if } x \in B_{i-1}.$$}

**Definition 2.2.** The sequence of spaces and associated maps \{ $F_i$, $G_i$, $\phi_i$, $\pi_i$, $\psi_i$ \} will be called a Barlow-Evans sequence.

For the rest of the paper a space $F_i$ will be assumed to be a member of a Barlow-Evans sequence with all the associated maps assumed. For any $M = 0, 1, \ldots, \infty$ we shall denote the $L^2$ norm on functions over $F_i$ by $\| \cdot \|_{M}$. These norms should not be confused with the $L^p$ norms which are used in this paper.

If $f$ is a function on $F_i$ then $\pi_i^* f$ is a function on $F_{i-1} \times G_i$ defined by $$\pi_i^* f = f \circ \pi_i.$$ Similarly for $\psi_i^*$, $\phi_i^*$ and also set $\phi_{ij}^* = \phi_{i}^* \circ \cdots \circ \phi_{j+1}^*$ taking functions on $F_j$ to functions on $F_i$.

**Proposition 2.1.** For all $i \geq 1$

$$f \in C_0[F_i] \text{ if and only if } \pi_i^* f \in C_0[F_{i-1} \times G_i].$$
Proof. Recall that $F_i$ has the induced topology from $F_{i-1} \times G_i$ by $\pi_i$. That is $U \subset F_i$ is open if and only if $\pi_i^{-1}(U) \subset F_{i-1} \times G_i$ is open. Let $V \subset \mathbb{R}$, then $f^{-1}(V) \subset F_i$ is open if and only if $\pi_i^{-1}(f^{-1}(V)) \subset F_{i-1} \times G_i$ is open. But $\pi_i^{-1}(f^{-1}(V)) = (\pi_i^*f)^{-1}(V)$ so as $V$ ranges over all open sets of $\mathbb{R}$ we have that $f$ and $\pi_i^*f$ are both either continuous or discontinuous. \hfill $\square$

**Definition 2.3.** Given $\mu_{F_0}$ inductively define measures $\mu_{F_i}$ on $F_i$ for $i \geq 1$ by

$$\int_{F_i} f \, d\mu_{F_i} = \int_{F_{i-1} \times G_i} \pi_i^* f \, d(\mu_{F_{i-1}} \times \mu_{G_i}).$$

The measure $\mu_{F_i}$ is defined on the Borel $\sigma$–algebra generated by the above defined topology on $F_i$.

Note that if $\mu_{F_0}$ is a finite measure with mass $|\mu_{F_0}|$ then all $\mu_{F_i}$ have the same total mass. Thus we have $L^2(F_i, \mu_i)$ for all $i$ on which we define Dirichlet forms.

**Definition 2.4.** If a Dirichlet form $((\mathcal{E}_{i-1}, \mathcal{F}_{i-1})$ on $F_{i-1}$ is regular then define $((\mathcal{E}_i, \mathcal{F}_i)$ by

$$\mathcal{E}_i(f, f) = \int_{G_i} \mathcal{E}_{i-1}(\pi_i^* f, \pi_i^* f) \, d\mu_{G_i},$$

and $\mathcal{F}_i$ to be the closure in the $|| \cdot ||^2 + \mathcal{E}_i(\cdot, \cdot)$ metric of $\mathfrak{F}_i = \{ f \in L^2(F_i, \mu_{F_i}) | \pi_i^* f(x, g) = \sum_{k=1}^n f_k(x) h_k(g), \ f_k \in \mathcal{F}_{i-1}, \ h_k \in C(G_i) \}$.\hfill $\square$

Note that $\pi_i^* f$ is a function in two variables, one along $F_{i-1}$ and another along $G_i$. So this definition can be read as applying $\mathcal{E}_{i-1}$ to $\pi_i^* f$ for each element of $G_i$ and then integrating over $G_i$. Before examining the properties of $((\mathcal{E}_i, \mathcal{F}_i)$ we make sure it is well defined.

**Lemma 2.1.** For $f \in \mathfrak{F}_i$ $\mathcal{E}_{i-1}(\pi_i^* f, \pi_i^* f)(g)$ is $\mu_{G_i}$-measurable.

**Proof.** Let $\pi_i^* f = f_1(x) h_1(g)$ where $f_1 \in \mathcal{E}_{i-1}$ and $h_1 \in C(G_i)$. Then

$$\mathcal{E}_{i-1}(\pi_i^* f, \pi_i^* f)(g) = \mathcal{E}_{i-1}(f_1(x) h_1(g), f_1(x) h_1(g)) = h_1^2(g) \mathcal{E}_{i-1}(f_1, f_1)$$

which is a continuous function of $g \in G_i$ and hence Borel measurable. This extends to finite linear combinations naturally. \hfill $\square$

**Lemma 2.2.** If $\mathcal{E}_{i-1}$ is regular then for $i \geq 1$ $\mathfrak{F}_i \cap C_0[F_i]$ is a dense subset of $C_0[F_i]$.\hfill $\square$
Proof. Fix \( f \in C_0[F_i] \) and \( \epsilon > 0 \). Let \( X = \pi_i(\psi_i^{-1} \circ \psi_i(\text{supp}(\pi_i^*f))) \subset F_i \). Then \( X \) is compact and \( f = 0 \) on \( \partial X \). Then on \( X \), the collection of elements of \( \mathfrak{F}_i \) restricted to \( X \) is closed under products and contains constant functions and separates points of \( F_i \). By the Stone-Weierstrass Theorem \( \mathfrak{F}_i \) restricted to \( X \) is dense in \( C[X] \) so there exists a finite combination of elements of \( \mathfrak{F}_i \) that approximate \( f \) uniformly within an error of \( \epsilon \) on \( X \) such that the supports of these functions are contained in the support of \( f \). This finite sum of elements of \( \mathfrak{F}_i \) restricted to \( X \) all vanish in a neighborhood of \( \partial X \) so as functions they can be continued to all of \( F_i \) by setting them equal to zero on \( X^\complement \).

Theorem 2.1. If \( E_0 \) is a regular Dirichlet form, then \( E_i \) are also regular Dirichlet forms for all \( i \geq 0 \). Moreover, if \( E_0 \) is local then \( E_i \) is local as well.

Proof. We proceed by induction. The base case is given as an assumption so it suffices to show that if \( E_{i-1} \) is a local regular Dirichlet form then so is \( E_i \). By Lemma 2.1 \( E_i \) is well defined. The locality, linearity, non-negativity, and Markovian property of \( E_i \) follow directly from the definition and the fact that \( \mu_{G_i} \) is a positive probability measure.

The form \( E_i \) must be shown to be closed for it to be a Dirichlet form. To be closed \( \mathcal{F}_i \) must not only closed but complete under the \( \| \cdot \|_i + E_i(\cdot, \cdot) \) metric. Assume that \( u_n \in \mathfrak{F}_i \) be Cauchy and that \( ||u_n||_i \to 0 \). This is sufficient since \( \mathcal{F}_i \) is the closure of this set. If \( u_n \) are Cauchy in the \( \| \cdot \|_i + E_i(\cdot, \cdot) \) metric then it is Cauchy in \( || \cdot \|_i \). By Definition 2.3 this can be interpreted as \( \pi_i^*u_n(x, g) \) is \( L^2 \) convergent in the \( x \)-variable thus there is an almost everywhere convergent subsequence \( u_{n_k} = u_k \) so that \( ||\pi_i^*u_k(\cdot, g)||_{i-1} \to 0 \) for almost every \( g \in G_i \). It remains to show that \( f_p(g) = E_{i-1}(\pi_i^*u_p(\cdot, g)) \) is Cauchy for some subsequence of \( u_k = u_p \).

In order to see that \( f_p(g) \) is Cauchy for almost every \( g \) for some subsequence \( u_{k_p} = u_p \) recall that \( u_n \) is Cauchy in the \( \| \cdot \|_i + E_i(\cdot, \cdot) \) metric so \( E_i(u_n, u_n) \) is a Cauchy sequence of positive numbers. In Definition 2.3 the form \( E_i \) is defined as the integral over \( G_i \) of \( f_n(g) \) so is a \( L^1(G_i) \) convergence sequence. This means that along the subsequence \( u_p, f_p(g) \) converges for almost every \( g \). Since \( \pi_i^*u_p(\cdot, g) \to 0 \) then \( f_p(g) \to 0 \) by the fact that \( E_{i-1} \) is assumed to be closed by the induction hypothesis. Which is by 12 sufficient for \( E_i \) to be a closed form on \( \mathcal{F}_i \).

By Lemma 2.2 \( \mathfrak{F}_i \cap C_0(F_i) \) is uniformly dense in \( C_0(F_i) \). By construction of \( (E_i, F_i) \) \( \mathfrak{F}_i \) is a dense subset of \( \mathcal{F}_i \) in the \( \| \cdot \|_i + E_i(\cdot, \cdot) \) metric. Hence \( E_i \) is regular if \( E_{i-1} \) is regular.

Let \( u, v \in E_i \) have disjoint support. Then \( \pi_i^*u \) and \( \pi_i^*v \) have disjoint support in \( F_{i-1} \times G_i \). Since \( E_{i-1} \) is by assumption local then for each \( g \in G_i \) \( \pi_i^*u(\cdot, g) \) and \( \pi_i^*v(\cdot, g) \) have disjoint support and thus \( E_{i-1}(\pi_i^*u, \pi_i^*v)(g) = 0 \) for all \( g \in G_i \). Thus taking the integral over \( G_i \) we get that \( E_i(u, v) = 0 \). Thus if \( E_{i-1} \) is local then \( E_i \) is local.

Corollary 2.1. The domains of the Dirichlet forms \( E_i \) are nested. That is

\[ \phi_i^* F_{i-1} \subset F_i. \]

Proposition 2.2. If \( f \in F_i \) then \( \pi_i^*f(\cdot, g) \in F_{i-1} \) for almost every \( g \in G_i \) and \( E_{i-1}(\pi_i^*f)(g) \in L^1(G_i) \).

Proof. This holds if

\[ f \in \{ f \in L^2(F_i, \mu_{F_i}) | \pi_i^*f(x, g) = \sum_{k=1}^{n} f_k(x)h_k(g), \}

Where \( f_k \in F_{i-1}, h_k \in C(G_i) \). Thus it only remains to show that these properties are preserved under the \( \| \cdot \|_i + E_i(\cdot, \cdot) \) norm. The continuation of these properties follow from the proof of Theorem 2.1 in particular the proof of the closability of \( E_i \).

3. Projective Limits

The construction that is considered in this paper is a means of constructing state spaces for the symmetric diffusions via projective limit. Effectively this process takes compatible sequences of topological spaces and taking their limit. Barlow and Evans 10 considered this construction as a way to produce exotic state spaces for Markov processes. Then 21 specialized Barlow and Evans work to Laakso spaces 18.
Figure 3. The projective system of spaces and the maps between them.

Definition 3.1. A projective system of measure spaces is a collection of measure spaces \((F_i, \mu_{F_i})\) with projections \(\phi_i: F_i \to F_{i-1}\) such that for a \(\mu_{F_{i-1}}\)-measurable set \(A\) that \(\mu_{F_i}(\phi_i^{-1}(A)) = \phi_i^* \mu_{F_i}(A) = \mu_{F_{i-1}}(A)\). Then a projective limit can be defined as \(\lim_{\rightarrow} F_i \subset \prod_i F_i\) as the set of sequences \(x_i \in \prod_i F_i\) that have the property \(\phi_i(x_i) = x_{i-1}\). Then the canonical projections \(\Phi_j: \prod_i F_i \to F_j\) can be restricted to \(\lim_{\rightarrow} F_i\) and have the consistency property:

\[
\phi_i \circ \Phi_i = \phi_{i-1}, \quad i \geq 1.
\]

Proposition 3.1 \((\text{11})\). There exists a unique measure on \(\lim_{\rightarrow} F_i\) denoted \(\mu_{F_\infty}\) if the masses of \(\mu_{F_i}\) are uniformly bounded. Then \(\mu_{F_\infty}\) satisfies

\[
(1) \quad \mu_{F_i}(A) = \mu_{F_\infty}(\phi_i^{-1}(A))
\]

for all \(A\) that are \(\mu_{F_i}\)-measurable.

Corollary 3.1. If \(\mu_{F_0}\) is \(\sigma\)-finite then there exists a unique \(\mu_{F_\infty}\) on \(F_\infty\) which satisfies \((\text{7})\).

Proof. Since \(\mu_{F_i}\) is \(\sigma\)-finite there exists a partition of \(F_0\) such that each element of the partition has finite measure. Apply Proposition 3.1 on each member of the partition and then take \(\mu_{F_\infty}\) to be their sum.

We shall often have probability measures on \(F_i\) so that it will be possible to consider directly the limit measure space \((\lim_{\rightarrow} F_i, \mu_{F_\infty})\) without the worry of this corollary. Note that as maps from the Borel functions on \(F_i\) to the Borel functions on \(F_\infty\) that \(\Phi_i^*\) are \(\mathbb{R}\)-linear maps.

Proposition 3.2. Let \(\text{clos}_{\text{uniform}}\) represent the closure operator in the uniform norm then

\[
\text{C}_0(F_\infty) = \text{clos}_{\text{uniform}} \left\{ \bigcup_{i=0}^{\infty} \Phi_i^* \text{C}_0(F_i) \right\}.
\]

4. Projections and Laplacians

Having constructed Dirichlet forms on the approximating \(F_i\) we now turn to a Dirichlet form over \(F_\infty\). Recall that the \(L^2(F_M, \mu_{F_M})\) norm by \(\| \cdot \|_M\) for \(M = 0, 1, 2, \ldots, \infty\). The existence of projective limits of Dirichlet spaces \((L^2\text{ spaces equipped with a Dirichlet form and its domain})\) is briefly discussed in \((\text{11})\). We develop the existence for the sake of the accompanying notation which is then used to describe the decompositions in Theorem 4.3. The decompositions rely on the specific structure of the equivalence relations used in defining the projective system and are not a general feature of projective systems of Dirichlet spaces.

Definition 4.1. Given a Barlow-Evans sequence let \(E_\infty\) be the quadratic form on \(\lim_{\rightarrow} F_i\) defined by

\[
E_\infty(\Phi_i^* u, \Phi_i^* u) = E_i(u, u)
\]
For all $u \in \mathcal{F}_i$, the domain of $\mathcal{E}_\infty$ is

$$\mathcal{F}_\infty = \text{clos}\left\{ \bigcup \Phi_i^* \mathcal{F}_i \right\}$$

where the closure is in the $\mathcal{E}_\infty + \| \cdot \|_\infty$ metric.

**Theorem 4.1.** If $(\mathcal{E}_0, \mathcal{F}_0)$ is a regular Dirichlet form then the pair $(\mathcal{E}_\infty, \mathcal{F}_\infty)$ is a regular Dirichlet form. Furthermore, if $\mathcal{E}_0$ is local then $\mathcal{E}_\infty$ is local as well.

**Proof.** Linearity, the Markovian property, and positivity follow immediately from the definition. What remains to check is that $\mathcal{E}_\infty$ is closed and regular. The proof that $\mathcal{E}_\infty$ is a closed form follows the same type of argument as used in Theorem 2.1. From that theorem we have that all the $\mathcal{E}_i$ are closed. Using the same criterion, we assume that $u_n \in \bigcup \Phi_i^* \mathcal{F}_i$ is Cauchy in the metric given by $\mathcal{E}_\infty(\cdot, \cdot) + \| \cdot \|_\infty$ and that $\| u_n \|_\infty \to 0$. Then we need to show that $\mathcal{E}_\infty(u_n) \to 0$. This will show that $(\mathcal{E}_\infty, \bigcup \Phi_i^* \mathcal{F}_i)$ is a closable quadratic form and the closure will be $(\mathcal{E}_\infty, \mathcal{F}_\infty)$ as defined above. Let $\epsilon > 0$ then there exists $N \geq 0$ such that $\| u_n \|_\infty < \epsilon$ for $n \geq N$. Then there exists $M$ large enough such that $u_n \in \mathcal{F}_M$ for $n \leq M$ and since $\mathcal{E}_M(u_M) < \epsilon$. Such a finite $M$ exists because all of the $\mathcal{E}_i$ are closed and the first $M$ of the $u_n$ form the beginning of a Cauchy sequence in $\mathcal{E}_M(\cdot, \cdot) + \| \cdot \|_M$. This $M$ depends only on $\epsilon$ so by taking a sequence $\epsilon_p \to 0$ there is a subsequence $u_{n_p}$ along which $\mathcal{E}_\infty(u_{n_p}) \to 0$. Since this was a Cauchy sequence by assumption we have $\mathcal{E}_\infty(u_{n_p}) \to 0$.

The next claim is that $\mathcal{E}_\infty$ is regular. This requires a statement of what the continuous functions on $\lim F_i$ are. The continuous functions on the limit space are those functions that are in the uniform closure of functions of the form $\Phi_i^* f$ for some $i \geq 0$ and $f \in C(F_i)$. This is from the topology given by the inverse limit construction on the limit space. Elements of $\mathcal{F}_\infty$ can be used to approximate elements of $C(F_i)$ by the regularity of $\mathcal{E}_i$ any continuous function on $\lim F_i$ can be approximated by a sequence in $\mathcal{F}_\infty$ by taking a diagonal sequence. Similarly one can take a diagonal sequence of continuous functions approximating elements of $\mathcal{F}_\infty$.

Finally, we claim that if the $\mathcal{E}_i$ are local then $\mathcal{E}_\infty$ is local as well. Let $f, g \in \mathcal{F}_\infty$ with disjoint compact support. Then $f$ and $g$ are approximable in $\mathcal{E}_\infty$ by $f_n$ and $g_n$ such that for each $n$ $f_n$ and $g_n$ have disjoint compact support. Since $\mathcal{E}_\infty$ is a closed form

$$\mathcal{E}_\infty(f, g) = \lim_{n \to \infty} \mathcal{E}_\infty(\Phi_i^* f_n, \Phi_i^* g_n) = \lim_{n \to \infty} \mathcal{E}_n(f_n, g_n) = 0.$$ 

$$\square$$

**Theorem 4.2.** If $\Delta_i$ is the Laplacian generated by $\mathcal{E}_i$ and $\Phi_j : \lim F_i \to F_j$ the continuous projection from the projective limit construction. Then

$$\Phi_{i-1}^* \text{Dom}(\Delta_{i-1}) \subset \Phi_i^* \text{Dom}(\Delta_i) \forall i \geq 0.$$ 

**Proof.** For a general Dirichlet form $g \in \text{Dom}(\Delta)$ if and only if there exists $f \in L^2$ such that for all $v \in \mathcal{F}$ that

$$\mathcal{E}(g, v) = \langle f, v \rangle_{L^2}.$$ 

It is sufficient to check that if $u \in \text{Dom}(\Delta_{i-1})$ then $\phi_i^* u \in \text{Dom}(\Delta_i)$. We shall use Proposition 2.2 to ensure that $\pi_i^* v(\cdot, g) \in \mathcal{F}_{i-1}$ for a.e. $g$. Let $u \in \text{Dom}(\Delta_{i-1})$ and $v \in \mathcal{F}_i$. 

\[ E_i(\phi^*_i u, v) = \int_{G_i} E_{i-1}(\pi^*_i \phi^*_i u, \pi^*_i v) \, d\mu_G, \]

\[ = \int_{G_i} E_{i-1}(u, \pi^*_i v) \, d\mu_G, \]

\[ = \int_{G_i} \int_{F_{i-1}} (\Delta_{i-1} u)(\pi^*_i v) \, d\mu_{F_i} \, d\mu_G, \]

\[ = \int_{F_{i-1} \times G_i} (\Delta_{i-1} u)(\pi^*_i v) \, d\mu_{F_{i-1} \times G_i}, \]

\[ = \int_{F_i} (\phi^*_i u) v \, d\mu_{F_i} \]

Thus \( \phi^*_i(\Delta_{i-1} u) = \Delta_i \phi^*_i u. \) \( \square \)

**Definition 4.2.** For \( i \geq 1 \), given a Borel measurable \( f : F_i \to \mathbb{R} \) define the projections \( P_i \) and \( \tilde{P}_i \) by

\[(P_i f)(x) = \phi^*_i \left( \int_{G_i} (\pi^*_i f)(x, g) \, d\mu_G \right)\]

and

\[(\tilde{P}_i f)(x) = \int_{G_i} (\pi^*_i f)(x, g) \, d\mu_G.\]

These projections can be defined with domains \( C_0(F_i), L^2(F_i) \), or any subsets of these. The domain will be made clear in each context.

The integral in this definition maps a function on \( F_{i-1} \times G_i \) to a function on \( F_{i-1} \) so that \( P_i \) takes functions on \( F_i \) and returns another function on \( F_{i-1} \). Note that for \( x \in B_i \), that \( P_i f(x) = f(x) \). Because \( \pi^*_i f(x, g) \) is constant over all the values of \( g \) if \( x \in B_i \). However the \( \tilde{P}_i \) can be composed to project down several levels. Let \( \Pi_i \) be the left inverse of \( \Phi^*_i \), then the families \( P_i \) and \( \Pi_i \) satisfy the following relation for \( f \in L^2(F_i) \)

\[ \Pi_{i-1} \circ \Phi^*_i(f) = \tilde{P}_i(f). \]

**Lemma 4.1.** The generator of \( E_\infty \) denoted \( \Delta_\infty \) is the weak limit of \( \Phi^*_i \Delta_i \Pi_i \) where

\[ \text{Dom}(\Phi^*_i \Delta_i \Pi_i) = \Phi^*_i \text{Dom}(\Delta_i). \]

**Proof.** First \( \Delta_\infty \) is the unique maximal self-adjoint operator on \( L^2(F_\infty) \) such that for all \( f \in \text{Dom}(\Delta_\infty) \subset F_\infty \) and \( g \in F_\infty \) that

\[ \langle \Delta_\infty f, g \rangle = E_\infty(f, g). \]

For \( f \in \text{Dom}(\Delta_i) \) and \( g \in \mathcal{F}_i \)

\[ E_\infty(\Phi^*_i f, \Phi^*_i g) = E_i(f, g) \]

\[ = \langle \Delta_i f, g \rangle_{L^2(F_i)} \]

\[ = \langle \Phi^*_i \Delta_i f, \Phi^*_i g \rangle_{L^2(F_\infty)}. \]

This equality holds for all \( g' \in \mathcal{F}_i \) because \( \Pi_i g' \in \mathcal{F}_i \). Also there exists \( f' \in \mathcal{F}_\infty \) such that \( \Pi_i f' = f \), namely \( \Phi^*_i f \). Hence this equality can be rewritten as

\[ E_\infty(f', g') = \langle \Phi^*_i \Delta_i \Pi_i f', g' \rangle_{L^2(F_\infty)}. \]

Since \( \Pi \) is an orthogonal projection, \( \Delta_i \) is self-adjoint, and \( \Phi^*_i \) is an inclusion map, \( \Phi^*_i \Delta_i \Pi_i \) is a self-adjoint operator with domain \( \Phi^*_i \text{Dom}(\Delta_i) \). Notice that \( \Phi^*_i \Delta_i \Pi_i \) possesses the defining property of
\( \Delta_\infty \) on \( \Phi^* \text{Dom}(\Delta_i) \) so \( \Phi^* \text{Dom}(\Delta_i) \subset \text{Dom}(\Delta_\infty) \). We now have
\[
\langle \Delta_\infty f, g \rangle_{L^2(F_\infty)} = \mathcal{E}_i(f, g)
= \lim_{i \to \infty} \mathcal{E}_i(\Pi_i f, \Pi_i g)
= \lim_{i \to \infty} \langle \Delta_i \Pi_i f, \Pi_i g \rangle_{L^2(F_i)}
= \lim_{i \to \infty} \langle \Phi_i^* \Delta_i \Pi_i f, \Phi_i^* \Pi_i g \rangle_{L^2(F_i)}
\]
Where \( f \in \text{Dom}(\Delta_\infty) \) and \( g \in \mathcal{F}_\infty \) where \( \Pi_i : \text{Dom}(\Delta_\infty) \to \text{Dom}(\Delta_i) \) which follows from Lemma 4.4. Since \( \Phi_i^* \Pi_i g \to g \in \mathcal{F}_\infty \) with \( i \) going to infinity we get that \( \Delta_\infty = w - \lim_{i \to \infty} \Phi_i^* \Delta_i \Pi_i \).

\[\square\]

**Definition 4.3.** Set the following notation:
\[
\ker(\mathcal{P}_i|_{C(F_i)}) = \mathcal{E}_i
\ker(\mathcal{P}_i|_{L^2(F_i)}) = \mathcal{L}_i
\ker(\mathcal{P}_i|_{\mathcal{F}_i}) = \mathcal{F}_i
\]

**Lemma 4.2.** Let \( \mathcal{P}_i \) be defined on \( C_0(F_i) \). Then
\[
C_0(F_i) = \phi_i^*(C_0(F_{i-1})) \oplus \mathcal{E}_i.
\]
Moreover, \( h \in \mathcal{E}_i \) if and only if \( h(x, g) \) satisfies
\[
\int_{G_i} (\pi_i^* h)(x, g) \, d\mu_{G_i} = 0 \quad \forall x \in F_{i-1}.
\]

**Proof.** First note that any characterization of the kernel of \( \mathcal{P}_i \) is a statement about the pull back of functions from \( C_0(F_i) \) to \( C_0(F_{i-1} \times G_i) \). On \( F_{i-1} \times G_i \) one can distinguish two closed sets of functions. The first that are constant over \( G_i \) and the second those that have mean zero over \( G_i \) for every \( x \in F_{i-1} \). These two sets of functions are \( \psi_i^*(C_0(F_{i-1})) \) and \( \pi_i^* \mathcal{E}_i \) respectively and both are subsets of \( \pi_i^* C_0(F_i) \) and only have the constantly zero function in common. Since they are both in \( \pi_i^* C_0(F_i) \) consider their images in \( C_0(F_i) \), these are now the two summands in the statement of the Lemma.

Take \( f \in C_0(F_i) \), we want to write \( f \) as the sum of an element in \( \phi_i^* C_0(F_{i-1}) \) and an element in \( \mathcal{E}_i \). Write \( f = \mathcal{P}_i(f) + (f - \mathcal{P}_i(f)) \). Then \( \mathcal{P}_i(f) \in \phi_i^*(C_0(F_{i-1})) \) since \( \mathcal{P}_i \) is the projection onto precisely those functions. Then we need to check that \( \pi_i^* (f - \mathcal{P}_i(f)) \) has mean zero over \( G_i \) for every \( x \in F_{i-1} \).
\[
\int_{G_i} \pi_i^* (f - \mathcal{P}_i(f))(x, g) \, d\mu_{G_i} = \int_{G_i} \pi_i^* f \, d\mu_{G_i} - \int_{G_i} \pi_i^* \mathcal{P}_i f \, d\mu_{G_i}
= \int_{G_i} \pi_i^* f \, d\mu_{G_i}
- \int_{G_i} \pi_i^* \phi_i^* \int_{G_i} \pi_i^* f \, d\mu_{G_i} d\mu_{G_i},
\]
In (2) the integrand of the outer integral has no \( g \) dependence the integral only serves to push it's integrand back into a function on \( F_{i-1} \). Because Figure [i] is a commutative diagram this integration composed with \( \pi_i^* \phi_i^* \) is the identity operator. Thus
\[
\int_{G_i} \pi_i^* (f - \mathcal{P}_i(f))(x, g) \, d\mu_{G_i} = \int_{G_i} \pi_i^* f \, d\mu_{G_i} - \int_{G_i} \pi_i^* f \, d\mu_{G_i} = 0.
\]

\[\square\]

**Lemma 4.3.** Let \( \mathcal{P}_i \) be defined on \( L^2(F_i) \). Then
\[
L^2(F_i) = \phi_i^*(L^2(F_{i-1})) \oplus \mathcal{L}_i.
\]
Elements of \( \mathcal{L}_i \) have the corresponding property as elements of \( \mathcal{E}_i \) for almost every \( x \in F_{i-1} \).
Proof. This Lemma follows the same ideas as the previous however the technicalities are reduced because we are working with a Hilbert space. If $P_i$ is the projection onto $\phi_i^*(L^2(F_ι−1))$ then the claimed decomposition is just the orthogonal decomposition. By definition $P_i$ is a projection so it only remains to show that it is only $\phi_i^*(L^2(F_ι−1))$. Take any element $f ∈ \phi_i^*(L^2(F_ι−1))$ and calculate $P_i(f)$. This is a short calculation which immediately shows that $f = P_i(f)$ in $L^2(F_ι)$. \qed

Lemma 4.4. Let $P_i$ be defined on $F_i$. Then

$$F_i = \phi_i^*(F_ι−1) ⊕ F_i'.$$

Elements of $F_i'$ have the corresponding property as elements of $L_i$. Moreover the core, $C(F_i) \cap F_i$ of the Dirichlet form $(E_i, F_i)$ has the same decomposition.

Proof. The decomposition of $F_i$ is a consequence of Lemma 4.3 and the characterization of the elements of $\ker P_i$ follows similarly to the previous lemmas. The novel claim is the statement concerning the core of $F_i$. The core of $F_i$ that we chose is $C(F_i) \cap F_i$. That this is a core is a consequence of the regularity result in Theorem 2.1. Lemma 4.2 gives this kind of decomposition of $C(F_i)$ but it still needs to be checked that if $f ∈ C(F_i) \cap F_i$ is decomposed according to Lemma 4.2 that the two components are again in $\phi_i^*(F_ι−1)$ and $F_i'$ respectively.

Let $f ∈ C(F_i) \cap F_i$ and decompose into $f_{i−1} ∈ C(F_i) \cap F_i$ and $f_c ∈ F_i'$. We will show that $f_{i−1} = \phi_i^*(F_ι−1)$ and $f_c ∈ F_i'$. Because $\phi_i^*(F_ι−1)$ is closed in $F_i$ and $E_ι−1$ is assumed to be regular it is enough to check elements of $F_i \cap C(F_i)$ for these properties. Let $f ∈ F_i \cap C(F_i)$ then $(π_i f)(x, g) = \sum_n f_j(x) h_j(g)$ where the $f_j$ and $h_j$ are all continuous. The action of $P_i$ on $f$ is $P_i f = \phi_i (\sum_n f_j(x) \int_G h_j(g) (d\mu_G(g))) = f_{i−1}$. Which is in $\phi_i^*(F_ι−1)$ by the definition of $F_i$. Set $f_c = f - f_{i−1}$, this is in $F_i' \cap C(F_i) \subset E_i$. \qed

Definition 4.4. Let $D_ι = \Phi_ι^0\text{Dom}(\Delta_ι)$. The inductively define $D_i$ by:

$$D_i = \Phi_i^0\text{Dom}(\Delta_i) \cap \bigcap_{j=0}^{i−1} D_j'.$$

Where the orthogonal compliment is taken in $L^2(F_i)$. This implies that $\Phi_i^\dagger \text{Dom}(\Delta_i) = \bigoplus_{j=0}^i D_j'$.

Theorem 4.3.

$$L^2(F_∞, μ_∞) = \text{clos}_{L^2(F_∞, μ_∞)} (\Phi_0^*L^2(F_0, μ_0) \oplus (⊕_ι=1^∞ \Phi_i^*L_i)$$

$$C(F_∞) = \text{clos}_{\text{unif}} (\Phi_0^*C(F_0) \oplus (⊕_ι=1^∞ \Phi_i^*E_i))$$

$$F_∞ = \text{clos}_{\text{unif}} (\Phi_0^*F_0 \oplus (⊕_i=1^∞ \Phi_i^*F_i'))$$

Proof. By definition $L^2(F_∞, μ_∞)$ is the completion of $∪_ι=1^∞ \Phi_i^*L^2(F_ι, μ_ι)$, what is new is the direct sum decomposition. Let $f ∈ L^2(F_ι, μ_ι)$ then notice that $f = (P_ι f) + \phi_ι^*(P_ι f) ∈ L_ι + \phi_ι^*L^2(F_ι)$. In general for $f ∈ L^2(F_2)$ we would have

$$f = (f - P_2 f) + \phi_2^*(P_2 f - P_1 P_2 f) + \phi_2^*\phi_1^*(P_1 P_2 f) ∈ L_2 + \phi_2^*L_1 + \phi_2^*\phi_1^*L^2(F_0).$$

Continuing by this method we have the direct sum expansion for $L^2(F_ι)$ for any $i ≥ 1$. The $L^2(F_∞)$ limits of these expansions must then be all of $L^2(F_∞)$. The same argument works for $C(F_∞)$ and $F_∞$. \qed

The domain of $\Delta_∞$ can be decomposed into the direct sum of $D_i$, or as $\text{Dom}(\Delta_∞) \cap L_i$ or as $\text{Dom}(\Delta_∞) \cap F_i'$. \qed

Lemma 4.5. The three direct sum decompositions of $\text{Dom}(\Delta_∞)$ agree.

Proof. Because $F_∞ \subset L^2(F_∞)$ we know that $F_i' \subset L_i$. This together with the fact that $\text{Dom}(\Delta_∞) \subset F_∞$ implies that $\text{Dom}(\Delta_∞) \cap L_i = \text{Dom}(\Delta_∞) \cap F_i'$. What remains to be shown is that when the closure of the direct sum is taken all of $\text{Dom}(\Delta_∞)$ is obtained as limits of elements of $\text{Dom}(\Delta_∞) \cap F_i'$. Suppose there exists a $g ∈ \text{Dom}(\Delta_∞)$ such that for any sequence $g_n ∈ F_0 \oplus [⊕_i=0^{∞} \text{Dom}(\Delta_∞) \cap F_i')$ there exists $ε > 0$ such that $\|g - g_n\|_{L^2} + \|\Delta_∞(g - g_n)\|_{L^2} > ε$. That is $g ∈ \text{Dom}(\Delta_∞)$ but not in the
Theorem 5.1. The spectrum of \( g \) the left hand side goes to one. So the \( \sigma(\Delta) \) that \( \Pi D \) for these spaces corresponding to the \( B \) closure of \( \oplus_{i=0}^{\infty} (\text{Dom}(\Delta_\infty) \cap \mathcal{F}_i') \). Then choose \( g_n \) such that \( g_n - g_{n-1} \in \mathcal{F}_n' \) and \( g_n \to g \) in \( \mathcal{F}_\infty \) and in \( L^2(F_\infty) \). Hence \( \|\Delta_\infty(g - g_n)\|_{L^2} > \epsilon \). So
\[
e^2 < \|\Delta_\infty(g - g_n)\|_{L^2}^2 \leq \sqrt{\epsilon}(g - g_n)\|\Delta_\infty(g - g_n)\| + \sqrt{\epsilon}(\Delta_\infty(g - g_n)\|g - g_n\|)} \leq \sqrt{\epsilon}(g - g_n).
\]

As \( n \to \infty \) the right hand side goes to zero by the choice of \( g_n \). However since \( \|\Delta_\infty(g - g_n)\|_{L^2} > \epsilon \) the left hand side goes to one. So the \( g \) did not exist. \( \square \)

5. Main Results

Theorem 5.1. The spectrum of \( \Delta_\infty \) is given by:
\[
\sigma(\Delta_\infty) = \bigcup_{i=0}^{\infty} \sigma(\Delta_i) = \bigcup_{i=0}^{\infty} \sigma(\Delta_\infty|\Omega_i)
\]

Proof. Let \( z \in \sigma(\Delta_n) \). Then by Lemma 4.1 \( (\Delta_n - z) \) is not invertible on \( \text{Dom}(\Delta_n) \). Since \( (\Delta_\infty - z) \) agrees with \( (\Delta_n - z) \) on \( \Phi^*_n \text{Dom}(\Delta_n) \) we have that \( (\Delta_\infty - z) \) is not invertible. Hence \( \sigma(\Delta_n) \subset \sigma(\Delta_\infty) \) for all \( n \geq 0 \). This implies that \( \sigma(\Delta_n) = \sigma(\Delta_\infty|\Omega_n) \) where \( \Omega_n = \bigoplus_{i=0}^{n} \Omega_i' \) and \( \sigma(\Delta_\infty|\Omega_n) = \bigcup_{i=0}^{n} \sigma(\Delta_\infty|\Omega_i') \).

Suppose that \( z \in \sigma(\Delta_\infty) \) and \( z \notin \bigcup_{i=0}^{\infty} \sigma(\Delta_i) \). Define \( B_z : L^2(F_\infty) \to \text{Dom}(\Delta_\infty) \) by
\[
B_z = w - \lim_{i \to \infty} \Phi^*_i(\Delta_i - z)^{-1} \Pi_i.
\]

We claim that \( B_z \) is the inverse of \( (\Delta_\infty - z) \) contradicting the assumption that \( z \notin \sigma(\Delta_\infty) \). Recall that \( \Pi_i \) is a bounded linear and hence continuous operator. Let \( f \in \bigcup_{i=0}^{\infty} \Phi^*_i \text{Dom}(\Delta_i) \), then
\[
B_z(\Delta_\infty - z)f = \lim_{n \to \infty} \Phi^*_n(\Delta_n - z)^{-1} \Pi_n \lim_{m \to \infty} \Phi^*_m(\Delta_m - z) \Pi_m f
\]
\[
= \lim_{n \to \infty} \Phi^*_n(\Delta_n - z)^{-1} \Pi_n \Phi^*_i(\Delta_i - z) \Pi_i f
\]
\[
= \Phi^*_i(\Delta_i - z)^{-1}(\Delta_i - z) \Pi_i f
\]
\[
= f.
\]

For large enough \( m \lim_{m \to \infty} \Phi^*_m(\Delta_m - z) \Pi_m f \) stabilizes to \( \Phi^*_i(\Delta_i - z) \Pi_i f \) then as \( n \) grows it will also stabilize for \( n \geq M \). Then since \( \Delta_\infty \) is a closed operator the claim extends to \( \text{Dom}(\Delta_\infty) \). Finally by the decompositions in Lemmas 4.3 and 4.4 the last limit equals \( f \). The same calculation can be used to show that \( (\Delta_\infty - z)B_z = \Pi_i \). Thus there exists no \( z \in \sigma(\Delta_\infty) \) but not in \( \bigcup_{i=0}^{\infty} \sigma(\Delta_i) \). \( \square \)

In the standard theory of self-adjoint operators lie the spectral resolutions of self-adjoint operators [19]. These spectral resolutions are orthogonal projection valued measures over \( \mathbb{R} \) supported on the spectrum of the operator they are representing. For \( \Delta_\infty \) let \( E_\lambda \) be the spectral resolution. Then
\[
\Delta_\infty f = \int_{\sigma(\Delta_\infty)} \lambda dE_\lambda f.
\]

Note that for each \( \lambda \in \mathbb{R} \), \( E_\lambda : L^2(F_\infty) \to \text{Dom}(\Delta_\infty) \) where for \( f \notin \text{Dom}(\Delta_\infty) \) the integral fails to converge. We also have the orthogonal projections \( P_i \) out of \( \text{Dom}(\Delta_\infty) \).

From the previous discussion the following statement follows immediately.

Theorem 5.2. Let \( E_\lambda \) be a spectral projection operator for \( \Delta_\infty \). Then for all \( \lambda \in \mathbb{R} \) and \( i \in \mathbb{N} \)
\[
\mathcal{D}_i' \cap E_\lambda(\text{Dom}(\Delta_\infty)) = E_\lambda \mathcal{D}_i'.
\]

Similar statements could be made for \( L^2(F_\infty) \), \( \mathcal{F}_\infty \), however we have not developed the notation for these spaces corresponding to the \( \mathcal{D}_i' \) notation.
Theorem 5.3. Suppose that $\mathcal{E}_0$ is a local regular Dirichlet form. If $G_i$ have bounded, finite cardinality, $F_i$ is compact, $F_i \setminus B_i$ consists of disjoint open sets whose diameters are monotonically decreasing to zero in $i$ and the cardinality of $B_i$ is finite and increasing then $\sigma(\Delta_i)$ are all discrete and $\sigma(\Delta_\infty) = \bigcup_{n=0}^{\infty} \sigma(\Delta_n)$.

Proof. By the finiteness of the $G_i$ we have that $F_\infty = F_0 \times K$ where $K$ is a Cantor set and the equivalence relation $\sim$ is the union of all the $\sim_i$. Which implies that $F_\infty$ is compact. If it can be shown that $\min i$ increases without bound then the union will be nowhere dense and no new eigenvalues will be generated from the closure.

By the assumptions on the cardinality of $B_i$ and that $F_i \setminus B_i$ each eigenfunction of $\Delta_i$ is decomposable into eigenfunctions of $\Delta_{i-1}$ on each connected component of $F_i \setminus B_i$ which is isomorphic to an open subset of $F_{i-1}$. Let $f \in \mathcal{D}_i$ with eigenvalue $\lambda_i = \min \sigma(\Delta_i|_{\mathcal{D}_i})$. Then $f$ restricted to a single component of $F_i \setminus B_i$ is locally an eigenfunction of $\Delta_{i-1}$ with eigenvalue $\lambda_i$ and Dirichlet boundary conditions at $B_i$. If $\Delta_{i-1}$ has a fundamental solution which is strictly positive then $\lambda_i$ must be larger than $\lambda_{i-1}$ by the assumption on the sizes of the open components of $F_i \setminus B_i$. \qed

6. Examples

6.1. The Laakso fractal and the heat kernel estimates. Laakso spaces were initially introduced in [18] using the Cartesian product of a unit interval and a number of Cantor sets. In [23, 21] it was shown that they could also be constructed using the projective limit construction presented originally in [10] and reiterated above. Take $F_0 = [0, 1]$, the unit interval. Let $G_i = G = \{0, 1\}$. Choose a sequence $\{j_i\}_{i=1}^\infty$ where $j_i \in \{j, j+1\}$ for some fixed integer greater than one. Define

$$d_N = \prod_{j=1}^N j_i \quad L_N = \left\{ \frac{i}{d_N} \right\}_{i=1}^{d_N-1}.$$

Then set $B_n = \phi_{n,0}(L_n \setminus L_{n-1})$. The sets $L_N$ describe the location of what the quotient maps $\pi_i$ collapse and $d_N$ the separation of the new identifications from any old identifications. A Laakso space will be denoted by $L$.

If $\mathcal{E}_0$ is taken to be the simplest Dirichlet form on the unit interval, namely $\mathcal{E}_0(u, v) = \int_0^1 \frac{du}{dx} \frac{dv}{dx} dx$ with the Sobolev space $H^{1,2}([0, 1])$ as $\mathcal{F}_0$, then there is a limiting Dirichlet form, $\mathcal{E}_\infty$, on $L$ which has a generator $\Delta_\infty$. The analysis of $\Delta_\infty$’s spectrum is the topic of [21] and several chapters in [23]. Using the arguments involved in the proofs of Theorems 6.1 and 6.2 the following results are known.

Theorem 6.1 ([21]). Let $L$ be a Laakso space with sequence $\{j_i\}$. The spectrum of $\Delta_\infty$ on this Laakso space is

$$\sigma(A) = \bigcup_{n=0}^{\infty} \bigcup_{k=1}^{\infty} \{k^2 \pi^2 d_n^2\} \cup \bigcup_{n=2}^{\infty} \bigcup_{k=1}^{\infty} \{k^2 \pi^2 4d_n^2\} \cup \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{\infty} \{(2k+1)^2 \pi^2 4d_n^2\}.$$

6.2. Sierpinski Pâte à Choux. The name of this example was suggested by Jean Bellisard who commented that such a space would evoke the memory of puff pastry in the reader. Denote by $SG$ the standard Sierpinski gasket constructed as the limit of the iterated function system $T_l(x) = \frac{1}{2}(x-q_l)+q_l$ for $l = 0, 1, 2$ where $q_0 = (0, 0)$, $q_1 = (1, 0)$, and $q_2 = (\frac{1}{2}, \sqrt{3}/2)$. Let $F_0 = SG$, $G_i = G = \{0, 1\}$ and $B_i = V_i \setminus V_{i-1}$. We use the convention that $V_n$ consists the images of the points $q_i$ under all the $n$-fold compositions of the contractions $T_j$.

Lemma 6.1. The limit space $F_\infty$ is an infinitely ramified fractal with Hausdorff dimension $d_h = 1 + d_H(SG) = \frac{\log(6)}{\log(2)}$ with respect to the geodesic metric.

Proof. The cell structure on $F_\infty$ induced by the cell structures on $SG$ and on the Cantor set have boundaries that are themselves Cantor sets. Hence $F_\infty$ is infinitely ramified. Since $i^{-1}(F_\infty) \subset SG \times K$ the Hausdorff dimension is at most $\frac{\log(6)}{\log(2)}$. By the same argument as in [21] it is at least $\frac{\log(6)}{\log(2)}$. \qed
In light of Theorem 5.3 it would be possible to write out explicitly the spectrum on $F_\infty$ as we did with the Laakso spaces. In particular, it is possible but somewhat involved to write the spectrum in a closed form. The reader can find solution to a similar problem in [23]. We note that, in the limit, the Sierpinski Pâte à Choux is not a Sierpinski fractafold, but the approximations $F_i$ are fractafolds. However, despite the fact that these fractafolds are very complicated, the spectrum of the Laplacian on $F_i$ can be found inductively using methods presented in this paper. In particular, the spectrum of each Laplacian $\Delta_i$ is a union of the spectrum of a large collection of disjoint fractafolds (with Dirichlet boundary conditions). These fractafolds are rescaled copies of two kinds of finite fractafolds, and therefore the spectrum can be found using the methods of [23], and the standard rescaling by $5^n$. This is very similar to how the spectrum is found in the case of the Laakso spaces, described above. Finally, we can comment that the Laakso spaces are built using intervals, which are one-dimensional analogs of the Sierpinski gasket. Therefore, in a sense, the Sierpinski Pâte à Choux is a direct analog of the Laakso spaces. Combining the approaches of [15, 23, 21, 25, 20] one can study all the eigenfunctions and eigenprojections, which will be subject of subsequent work.

6.3. Connected fractal spaces isospectral to the fractal strings of Lapidus and van Frankenhuijsen. Fractal strings are given a comprehensive treatment in [20], in particular in relation to spectral zeta functions, and we will only give a brief description here. We show that our construction can yield connected fractal spaces with Laplacians isospectral to the standard Laplacians on fractal strings. This implies, in particular, that there are symmetric irreducible diffusion processes whose generators are Laplacians with prescribed spectrum, as in the theory of fractal strings developed in [20].

A fractal string is an open subset of $\mathbb{R}$, usually assumed to be a bounded subset, or at least that the lengths of intervals are bounded and tend to zero. Therefore it is a union of countably many finite intervals of lengths $l_i$. We will suppose that the intervals are indexed so that the lengths form a non-increasing sequence. By indexing the fractal string with $l_i$ and $m_i$, lengths and multiplicities we can assume that $l_i$ is strictly decreasing. The Laplacian that we consider on a $I$ is the usual Laplacian on an interval with Dirichlet boundary conditions on all the intervals. The eigenvalues of this Laplacian are all of the form

$$\lambda_{i,k} = \frac{\pi^2 k^2}{l_i^2}$$

with multiplicity $m_i$. What choices of $F_0$, $B_i$, and $G_i$ can be made to create a connected fractal with the same spectrum as a given fractal string? As the desire is to “stitch” the disjoint intervals together there is no unique canonical method.

To begin with, we let $F_0 = [0, l_1]$, $B_1 = \{0, 1\}$ and $G_1 = \{1, 2, \ldots, m_1\}$. Then $F_1$ will be $m_1$ copies of the unit interval with left end points identified and right end points identified. A particular implication of this step is that $F_0 = F_1$ if and only if $m_1 = 1$. We impose zero boundary conditions at the endpoints, and therefore the spectrum of the Laplacian on $F_1$ is the spectrum on $F_0 = [0, l_1]$ repeated, in the sense of multiplicity, $m_1$ times. For the next step $G_2 = \{1, 2, \ldots, m_2 + 1\}$, and we choose

$$B_2 = ([0, 1] \times ([1 - l_2, 1] \times \{1\} \cup [0, 1] \times (G_2 \setminus \{1\}))/ \sim_1$$

where $\sim_1$ is the equivalence relation on $F_0 \times G_1$ determined by $B_1$. This implies that the spectrum on $F_2$ is the union of the spectrum on $F_1$ and the spectrum on $[0, l_2]$ repeated, in the sense of multiplicity, $m_2$ times. For $i = 3$ we take

$$B_3 = [1 - l_3, 1] \times \{(1, 1)\} \cup [0, 1] \times (G_1 \times G_2 \setminus \{(1, 1)\}).$$

Then $B_n$ is constructed inductively in the same manner. This construction is in a sense a non-self-similar version of the nested fractal construction. It is also somewhat similar to construction of some of the so called diamond fractals, see [11, 22].

In this setting one cannot employ Theorem 4.3 directly. However it is easy to replace the condition on the cardinality of $B_i$ with our specific choice of $B_i$, and the same result can be shown. Namely,
the spectrum of the Laplacian $\Delta$ is given by the union of the spectra of $\Delta_n$ and that $D_i$ are functions with eigenvalues $\lambda_{i,k}$ with multiplicity $m_i$.

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