Dynamics of SEIR model with delay effects - latent period and recovery period

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Abstract. A compartmental SEIR epidemic model with constant latent and recovery periods as delay effects is proposed in this paper. Incidence rate applied in the model is non-monotonic. Basic reproduction number for the model is calculated and the dynamical behavior is analyzed by studying the local & global asymptotic stability of disease free and endemic equilibriums of this model. Numerical simulation of this model is given to illustrate the results.

Key words. latent period, recovery period, non-monotonic incidence rate, basic reproduction number, stability.

1. Introduction

Epidemiological models other than the simple SI, SIR, SEIR, etc are essential to study for investigation of role of maturation period, recovery period, immunity period and reaction times in disease transmission. Time delays are included into the epidemic models to reveal the real dynamical behavior of models which is dependent on the past history of systems [1]. In mathematical epidemiology, the transmission of the individuals from one compartment of disease to another at time t depends not only on the state of time t but also on the previous time [2]. Inclusion of time delay in the model formulation makes the epidemic models more realistic by providing exposure on biological and ecological processes.

During latent period, individuals are infected by the disease but not infectious. In this period of time infected individuals are not able to transmit the disease to susceptible individuals as the disease is not sufficiently active to transmit. After completion of latent period when the individuals are infectious, they are able to transmit the disease. Many authors added delay effects in the epidemic models and analyzed dynamical behavior of the models. Cooke [3] incorporated delay effect in the epidemic model to model the latent period. Zhin Ma and Jia Li [2] explained that in the models with time delay, the dynamic behavior of the model at the time t depends also on the state at time t-τ, where τ is a fixed constant. For example the number of the infective at time t also depends on the number of infective t-τ where τ is latent period. They described SIS model with a constant period of recovery as delay and incorporated constant latent period and constant course of infection as delay effect in an SEIR model. Abdelilah Kaddar et al [4] compared the SIR and SEIR epidemic models with time delay and a saturated incidence rate. A Bernoussi et al [5] analyzed a SIRI epidemiological model with time delay and relapse which is applied to the evaluation of the spread of disease in a given population. Khekare et al [6] proposes a machine to maintain cleanliness in water to avoid diseases. Khekare et al [7] implemented emergency system to tackle real time emergencies. Khekare et al [8] proposes a model which provides the accurate result in quick time using unique identifier key.
2. Model Formulation

We propose an SEIR model (2.1) with constant latent period and constant recovery period and non-monotonic incidence rate $\frac{\lambda S(t)I(t)}{1 + al}$. We take the parameter ‘$a$’ as measure of the inhibitory effect as a result of human behavior when there are large number of infectives in the population. Let $P_0$ and $P^*$ denote disease free and endemic equilibriums, respectively.

\[
\begin{align*}
\frac{dS}{dt} &= \beta[K - S(t)] - \frac{\lambda S(t)I(t)}{1 + al} \\
\frac{dE}{dt} &= \frac{\lambda S(t)I(t)}{1 + al} - \frac{\lambda S(t - \tau)I(t - \tau)e^{-\beta \tau}}{1 + al} - \beta E \\
\frac{dl}{dt} &= \frac{\lambda S(t - \tau)I(t - \tau)e^{-\beta \tau}}{1 + al} - \frac{\lambda S(t - \tau - \delta)I(t - \tau - \delta)e^{-\beta(t + \delta)e^{-\mu \delta}}}{1 + al} - (\beta + \mu)I \\
\frac{dR}{dt} &= \frac{\lambda S(t - \tau - \delta)I(t - \tau - \delta)e^{-\beta(t + \delta)e^{-\mu \delta}}}{1 + al} - \beta R
\end{align*}
\]  

(2.1)

The notations used in this model for the compartments and parameters are listed below:

- $t$ denotes time
- $S(t)$ denotes the count of susceptible individuals
- $E(t)$ denotes the count of exposed individuals
- $I(t)$ denotes the count of infected individuals
- $R(t)$ denotes the number of recovered individuals
- $\alpha = \beta K$ is the birth rate of population
- $\beta$ is the natural death rate of population
- $\lambda$ is the infection rate
- $\mu$ is death rate due to infection
- $\tau$ is latent period
- $\delta$ is recovery period

We suppose that $K = \frac{\alpha}{\beta}$ is capability of the environment to hold population taking into consideration that the population will not grow at $N = K$ and decreases at $N > K$. Here $N$ is the total population.

Here, $N = S(t) + E(t) + I(t) + R(t)$

Now we find the equilibrium points, let

\[
\begin{align*}
\beta[K - S] - \frac{\lambda S I}{1 + al} &= 0 \\
\frac{\lambda S I}{1 + al} - \frac{\lambda S I e^{-\beta}}{1 + al} - \beta E &= 0 \\
\frac{\lambda S I e^{-\beta}}{1 + al} - \frac{\lambda S I e^{-\beta(t + \delta)e^{-\mu \delta}}}{1 + al} - (\beta + \mu)I &= 0
\end{align*}
\]  

(2.2) (2.3) (2.4)
\[
\frac{\lambda S I e^{-\beta (t+\delta)} e^{-\mu \delta}}{1 + a I} - \beta R = 0
\] (2.5)

Then the disease free equilibrium of the system (2.1) is \( P_0 = (K, 0, 0, 0) \).

From (2.4), we have
\[
S = \frac{(\beta + \mu)(1 + a I)}{\lambda e^{-\beta t} (1 - e^{-\beta + \mu \delta})}
\] (2.6)

From (2.2) & (2.6), we get,
\[
\beta K e^{-\beta t} (1 - e^{-\beta + \mu \delta}) - \beta (\beta + \mu)(1 + a I) - \lambda (\beta + \mu) I = 0
\]
\[
\beta(\beta + \mu) a I + \lambda (\beta + \mu) I - \beta K e^{-\beta t} (1 - e^{-\beta + \mu \delta}) + \beta (\beta + \mu) = 0
\]
\[
(\beta a + \lambda)(\beta + \mu) I - \beta (\beta + \mu) \frac{\lambda K e^{-\beta t} (1 - e^{-\beta + \mu \delta})}{(\beta + \mu)} - 1 = 0
\]
\[
R_0 = \frac{\lambda K e^{-\beta t} (1 - e^{-\beta + \mu \delta})}{(\beta + \mu)}
\] (2.7)

Where \( R_0 \) denotes basic reproduction number [9].

\[
S^* = \frac{(\beta + \mu)(1 + a I^*)}{\lambda e^{-\beta t} (1 - e^{-\beta + \mu \delta})}
\] (2.8)
\[
E^* = \frac{\lambda (\beta + \mu)(1 - e^{-\beta t}) I^*}{\beta(1 + a I^*)}
\]
\[
E^* = \frac{(\beta + \mu)(1 - e^{-\beta t}) I^*}{\beta e^{-\beta t} (1 - e^{-\beta + \mu \delta})}
\] (2.9)
\[
I^* = \frac{\beta(R_0 - 1)}{\beta a + \lambda}
\] (2.10)

and
\[
R^* = \frac{\lambda e^{-\beta t} e^{-\beta t} e^{-\beta + \mu \delta} S^* I^*}{\beta(1 + a I^*)}
\]
\[
R^* = \frac{(\beta + \mu)e^{-\beta + \mu \delta} I^*}{\beta(1 - e^{-\beta + \mu \delta})}
\] (2.11)

From (2.10) we have

(i) No positive equilibrium exists for \( R_0 < 1 \).

(ii) An unique positive equilibrium \( P^* = (S^*, E^*, I^*, R^*) \), termed as endemic equilibrium, exists at \( R_0 > 1 \).

3. Stability Analysis of the Equilibrium Points
We proceed further to discuss the disease dynamics, by considering the following reduced system.

Let

\[ \frac{dS}{dt} = \beta(K - S) - \frac{\lambda SI}{1 + al} \equiv P(S, I) \]

(2.12)

\[ \frac{dI}{dt} = \frac{\lambda SI e^{-\beta}}{1 + al} - \frac{\lambda SI e^{-\beta(e I + \delta)} e^{-\mu I}}{1 + al} - (\beta + \mu)I \equiv Q(S, I) \]

where \((S, I) \in \Gamma = \{(S, I) : S > 0, I > 0, S + I \leq K\}\)

Linearising the system (2.12), we get

\[ J = \begin{bmatrix}
-\beta - \frac{\lambda I}{1 + al} & -\lambda S \\
\lambda e^{-\beta I}(1 - e^{-(\beta + \mu)I}) & 1 - \lambda Ke^{-\beta I}(1 - e^{-(\beta + \mu)I}) - (\beta + \mu)
\end{bmatrix} \]

\[ l_1 = -\beta - \frac{\lambda I}{1 + al} \]

\[ l_2 = \lambda Ke^{-\beta I}(1 - e^{-(\beta + \mu)I}) - (\beta + \mu) \]

The Jacobian matrix \(J(P_0)\) has two eigen values \(l_1 = -\beta\) and \(l_2 = \lambda Ke^{-\beta I}(1 - e^{-(\beta + \mu)I}) - (\beta + \mu)\). We have \(l_1 < 0\) and if \(R_0 < 1\) then \(l_2 < 0\). Since at \(R_0 < 1\) both the eigen values are negative, local asymptotically stability of \(P_0\) exists. Eigen value \(l_2\) is positive at \(R_0\) greater than 1 hence \(P_0\) is unstable [10, 11].

Hence, there exists equilibrium \(P_0\) in the region \(\Gamma\) for \(R_0 < 1\) for which the system (2.12) is not a closed orbit. Also region \(\Gamma\) is positively invariant for system (2.12) so that all orbits of the system started inside the region \(\Gamma\) cannot go out of \(\Gamma\) as \(t \to \infty\). Thus \(P_0\) is globally asymptotically stable in \(\Gamma\) for \(R_0\) less than 1 i.e. the persistence of epidemic cannot possible and eventually disease dies out [12].

Theorem 3.2: Global asymptotic stability of unique positive equilibrium \(P^*\) exists at \(R_0 > 1\) for system (2.12).

Proof: Jacobian matrix of system (2.12) at \(P^*\) is as follows:

\[ J(P^*) = \begin{bmatrix}
-\beta - \frac{\lambda I^*}{1 + al^*} & -\frac{\lambda S^*}{(1 + al^*)^2} \\
\lambda e^{-\beta I^*}(1 - e^{-(\beta + \mu)I^*}) & \lambda e^{-\beta I^*}(1 - e^{-(\beta + \mu)I^*})S^* - (\beta + \mu)
\end{bmatrix} \]
We have

\[ trf(P^*) = -\beta - \frac{\lambda I^*}{1 + a_I^*} + \frac{\lambda e^{-\beta \tau}(1 - e^{-(\beta + \mu)\delta})S^*}{(1 + a_I^*)^2} - (\beta + \mu) \]

\[ = -\beta - \frac{\lambda I^*}{1 + a_I^*} + (\beta + \mu) \left[ \frac{1}{1 + a_I^*} - 1 \right] \]

\[ = -\beta - \frac{(\lambda + a(\beta + \mu))I^*}{1 + a_I^*} < 0 \]

\[ detf(P^*) = -\left( \beta + \frac{\lambda I^*}{1 + a_I^*} \right) \left( \frac{\lambda e^{-\beta \tau}(1 - e^{-(\beta + \mu)\delta})S^*}{(1 + a_I^*)^2} - (\beta + \mu) \right) \]

\[ + \left( \frac{\lambda S^*}{(1 + a_I^*)^2} \right) \left( \frac{\lambda e^{-\beta \tau}(1 - e^{-(\beta + \mu)\delta})I^*}{1 + a_I^*} \right) \]

\[ = -\left( \beta + \frac{\lambda I^*}{1 + a_I^*} \right) (\beta + \mu) \left[ \frac{1}{1 + a_I^*} - 1 \right] + \lambda (\beta + \mu)I^* \]

\[ = (\beta + \mu) \left[ a \left( \beta + \frac{\lambda I^*}{1 + a_I^*} \right) + \lambda \right] I^* > 0 \]

Since region \( \Gamma \) is a positively invariant for system (2.12), there is no closed orbit of system (2.12) in the interior of \( \Gamma \) is the sufficient condition to show the global stability of endemic equilibrium \( P^* \).

Let \( D(S, I) = \frac{1 + a_I}{I} \) is Dulac function.

\[ DP = \frac{1 + a_I}{I} (\beta k + S) - \lambda S \]

\[ \frac{D(DP)}{DS} = -\frac{\beta (1 + a_I)}{I} - \lambda \]

\[ DQ = \lambda S e^{-\beta \tau} - \lambda S e^{-\beta (\tau + \delta)} e^{-\mu \delta} - (\beta + \mu)(1 + a_I) \]

\[ \frac{D(DQ)}{DI} = -a(\beta + \mu) \]

\[ \frac{\partial(DP)}{\partial S} + \frac{\partial(DQ)}{\partial I} = -\frac{\beta (1 + a_I)}{I} - \lambda - a(\beta + \mu) < 0, \quad \text{for} \ (S, I) \in \text{int} \ \Gamma \]

This proves that no closed orbit is there in interior of \( \Gamma \) of system (2.12). It proves global asymptotically stability of \( P^* \) in region \( \Gamma \) when \( R_0 > 1 \) by the qualitative theory of planar differential systems. Hence the disease persists in the population and become endemic at the equilibrium \( P^* \) [13].

4. Numerical Simulation

We analyze model (2.1) numerically to validate theoretical results using MATLAB and Simulink. Fig. 4.1 shows that disease becomes endemic for \( R_0 > 1 \). We can observe from fig. 4.2 that number of infected individuals \( I^* \) increases with decrease in latent period \( \tau \) and fig. 4.3 shows that \( I^* \) decreases with decrease in recovery period \( \delta \). If the inhibitory effect ‘\( a^\prime \) of host population for the disease increases through awareness about precautions and proper treatment methods then there is remarkable decrease in the value of \( I^* \) which is shown in fig. 4.4.
Fig. 4.1 Here $S(0) = 4, I(0) = 2, \alpha = 1.2, \beta = 0.06, \lambda = 0.1, \mu = 0.005$, $\tau = 7, \delta = 10, R_0 = 9.66$.

Fig. 4.2 Dependence of $I^*$ on latent period $\tau$ (tau)

Fig. 4.3 Dependence of $I^*$ on recovery period $\delta$ (delta)

Fig. 4.4 Dependence of $I^*$ on $a$
We have simulated behaviour of I* with respect to latent period τ, recovery period δ and parameter a in fig 4.2, fig 4.3 and fig. 4.4 respectively which is not shown in [1, 2, 3, 4, 5].

5. Concluding Remarks
In this paper, we proved global asymptotic stability of the disease free equilibrium for $R_0 \leq 1$ and it is unstable for $R_0 > 1$. If $R_0$ exceeds 1 then global asymptotic stability of the endemic equilibrium exists. This means that the disease fades out for $R_0 \leq 1$ and when the value of $R_0$ exceeds 1, disease persists in the population.

We also see that the latent period and recovery period influence basic reproduction number $R_0$. It is clear from the expressions for $R_0$ that $R_0$ decreases as latent period increases and recovery period decreases. Also it is apparent from the same expressions that $R_0$ decreases for decrease in infection rate $\lambda$. So the spread of the disease can be controlled by taking steps to keep the infection rate minimum. In the model (2.1), the non-monotonic incidence rate is applied to make the model more realistic. Thus the model analysis done in this paper helps to predict the disease mechanism and to make the public health policies precisely and accordingly in the next outbreak.

6. References
[1] Hattaf K, Lashari A, Louartassi Y and Yousfi N, 2013, A delayed sir epidemic model with general incidence rate, Electronic Journal of Qualitative Theory of Differential Equations, 3, 1-9.
[2] Ma Z and Li J, 2009, Dynamical modeling and analysis of epidemics, World Scientific Publishing Co. Pte. Ltd., Singapore.
[3] Cooke K, 1992, Book review of: Retarded dynamical systems: stability and characteristic systems, by G. Stepan, Bull. Amer. Math. Soc., 26, 175-179.
[4] Khaddar A, Abta A, Alaoui H, 2011, A comparision of delayed SIR and SEIR epidemic models, Nonlinear Analysis: Modelling and Control, 16, 181-190.
[5] Bernoussi A, Kaddar A and Asserda A, 2014, Global stability of a delayed SIRI epidemic model with nonlinear incidence, International Journal of Engineering Mathematics, 1-6.
[6] Khekare G S, Dhanre U T, Dhanre G T and Yede S S, 2019, Design of Optimized and Innovative Remotely Operated Machine for Water Surface Garbage Assortment, International Journal of Computer Sciences and Engineering, 7, 113-117.
[7] Khekare G S, 2014, Design of emergency system for intelligent traffic system using VANET, International Conference on Information Communication and Embedded Systems (ICICES2014), Chennai, 1-7.
[8] Khekare G S and Verma P, 2020, Design of Automatic Key Finder for Search Engine Optimization in Internet of Everything, 2020 IEEE 1st International Conference for Convergence in Engineering (ICCE), Kolkata, 464-468.
[9] Hethcote H, 2008, The basic epidemiology models: models, expressions for $R_0$, parameter estimation, and applications.
[10] Perko L, 1996 Differential equations and dynamical systems, Springer-Verlag, New York.
[11] Bellman R, 2008, Stability theory of differential equations, Dover Publications Inc., New York.
[12] Allman E and Rhodes J, 2003, Mathematical models in biology, an introduction, University Press, Cambridge.
[13] Brauer F., Chavez C.C., Mubayi A. and Towers S., 2016, Some models for epidemics of vector-transmitted diseases, Infectious Disease Modelling, Vol. 1, 79-87.