Idempotent and Pure Gamma Subacts of Multiplication Gamma Acts

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Abstract

The purpose of this article is to investigate pure gamma subacts of multiplication gamma acts and some related concepts. Some characterizations of pure gamma subacts are given. For this reason, we introduce the concept of idempotent gamma subacts and study some of its properties. Also, we discussed the relation among the pure, multiplication and idempotent gamma acts. Moreover, some result of pure gamma subacts of multiplication gamma acts are considered. Finally, we prove that the product of two pure gamma subacts is also pure.

Introduction

Let R be a commutative ring and M an left R-module. Ribenboim [1] called the submodule N a pure submodule of M if rM ∩ N = rN for each r ∈ R. Anderson and Fuller [2] defined N to be pure in M if IN = N ∩ IM for every ideal I of R. After that, in 1988 Z. A. El-Bast and P. F. Smith [3], introduced the concept of multiplication modules (Let M be an R-module. Then M is a multiplication provided for each submodule N of M, there exists an ideal I of R such that N = IM) and they studied various properties about it. The concept of gamma semigroup was introduced by M.K. Sen [4], as a generalization of semigroup and they studied several notions of semigroups have been extended to gamma semigroups. Recently, M.S. Abbas and Abdulqader Faris [5], introduced the notion of gamma act over Γ-semigroup and they studied some of their basic properties.

Our work is to introduce pure gamma subacts of multiplication gamma acts. In Section 2, we review some basic notions and properties of gamma semigroup, and gamma acts. Also, we give some results and properties of such gamma subacts which are need in our work. In Section 3, the concepts of idempotent and pure gamma subacts was introduced. Also, we give some characterizations and properties of such gamma subacts. For example we prove that if the gamma subact and gamma ideal are idempotent then so is their product. The relation between gamma subact and its residual is studied. Several results about this concepts are studied. Throughout this paper, our definition of gamma purity will be as a generalization to Anderson and Fuller [2], and S will be denote a commutative gamma semigroup with identity.
Basic Concepts

In this section we review some basic definitions and notions of gamma semigroup and gamma act which are need in our work.

Definition 2.1. [4] Let $S$ and $\Gamma$ be nonempty sets, $S$ is called a gamma semigroup (denoted by $\Gamma$-semigroup) if there is a mapping: $S \times \Gamma \times S \rightarrow S$ written by $(s_1, \alpha, s_2) \mapsto s_1\alpha s_2$ which is satisfying the condition $(s_1\alpha s_2)\beta s_3 = s_1(\alpha(s_2\beta s_3))$ for all $s_1, s_2, s_3 \in S$ and $\alpha, \beta \in \Gamma$. A $\Gamma$-semigroup $S$, is called commutative if $st = ts$ for all $s, t \in S$ and $\alpha \in \Gamma$.

Definition 2.2. [4] Let $S$ be a $\Gamma$-semigroup. An element $a$ in $S$ is said to be left (right )identity of $S$ if $a\alpha s = s$ ($s\alpha a = s$) for all $s \in S$ and $\alpha \in \Gamma$. An element $a$ of a $\Gamma$-semigroup $S$ is said to be a identity if it is both a left and right identity of $S$. A $\Gamma$-semigroup $S$ with identity is called a $\Gamma$-monoid. The identity of a $\Gamma$-semigroup (if exists) is denoted by 1.

Definition 2.3. [4] Let $S$ be a $\Gamma$-semigroup. A nonempty subset $A$ of $S$ is called Left (right) $\Gamma$-ideal if $S\Gamma A \subseteq N$ ($A\Gamma S \subseteq N$) such that $S\Gamma A := \{ s\alpha a \mid a \in A, \alpha \in \Gamma \text{ and } s \in S \}$, where $\Gamma$-ideal means Left and right $\Gamma$-ideal.

Definition 2.4. [4] An $\Gamma$-ideal $A$ of a $\Gamma$-semigroup $S$ is said to be a maximal $\Gamma$-ideal if $A$ is a proper $\Gamma$-ideal of $S$ and is not properly contained in any proper $\Gamma$-ideal of $S$.

Definition 2.5. [4] Let $S$ be a $\Gamma$-semigroup. An element $s$ in $S$ is called $\alpha$-idempotent if $s\alpha s = s$ for some $\alpha \in \Gamma$. If all elements of $S$ are $\alpha$–idempotent, then $S$ is said to be an idempotent. An element $s$ in $S$ is called an idempotent if $s\alpha s = s$ for all $\alpha \in \Gamma$. If all elements of $S$ are idempotent, then $S$ is said to be a strongly idempotent. A $\Gamma$-ideal $A$ of a $\Gamma$-semigroup $S$ is said to be globally idempotent (gl-idempotent for short) if $A\Gamma A = A$.

Proposition 2.6. [6] Let $S$ be a $\Gamma$-semigroup. If $A$ and $B$ are globally idempotent $\Gamma$-ideals of $S$, then $A\Gamma B$ is gl-idempotent $\Gamma$-ideal.
We introduce the following definition

**Definition 2.7.** A $\Gamma$-semigroup $S$, is said to be completely gl-idealpotent if every $\Gamma$-ideal of $S$, is gl-idealpotent.

The concepts of acts over semigroups is generalized to the following in [2].

**Definition 2.8.**[5] Let $S$ be $\Gamma$-semigroup. A nonempty set $M$ is called left gamma act over $S$ (denoted by $S_\Gamma$-act) if there is a mapping $S \times \Gamma \times M \rightarrow M$ written $(s, \alpha, m)$ by $s\alpha m$, such that the following condition is satisfied

$$(s_1\alpha_1\beta_1)m = s_1\alpha(s_2\beta_1m)$$

for all $s_1, s_2 \in S$, $\alpha, \beta \in \Gamma$ and $m \in M$. Similarly one can define a right gamma acts. From now on " $S_\Gamma$-act" means "left $S_\Gamma$-act". An $S_\Gamma$-act $M$ is called unitary if $S$ is a $\Gamma$-monoid and $1\alpha m = m$ for all $m \in M$ and $\alpha \in \Gamma$.

**Definition 2.9.**[5] Let $M$ be an $S_\Gamma$-act. An element $\theta \in M$ is called a zero of $M$ if $s\alpha \theta = \theta$ for all $s \in S$ and $\alpha \in \Gamma$. If $S$ is a $\Gamma$-semigroup with zero then $0\alpha m = \theta$.

**Definition 2.10.**[5] Let $M$ be an $S_\Gamma$-act, A nonempty subset $N$ of $M$ is called $S_\Gamma$-subact if $S\Gamma N \subseteq N$, Where $S\Gamma N = \{ s\alpha n \mid s \in S, \alpha \in \Gamma \text{ and } n \in N \}$. For $S_\Gamma$-subact $N$ of $M$, $[N : M]$ is defined as : $[N : M] = \{ s \in S \mid s\alpha m = m \text{ for all } \alpha \in \Gamma \text{ and } m \in M \}$.

Let $\{ N_i \mid i \in I \}$ be an arbitrary collection of $S_\Gamma$-subacts of $M$. Then, $\bigcup_{i \in I} N_i$ is a $S_\Gamma$-subact of $M$.and if $\bigcap_{i \in I} N_i$ is nonempty, then $\bigcap_{i \in I} N_i$ is a $S_\Gamma$-subact of $M$.Also, if $N$ and $L$ are $S_\Gamma$-subacts of $M$ and $A$, $B$ are nonempty subsets of $M$. Then,

1. if $A \subseteq B$ implies that $[N : B] \subseteq [N : A]$.
2. $[N \cap L : A] = [N : A] \cap [L : A]$. [5]

For $S_\Gamma$-subact $N$ of $S_\Gamma$-act $M$, it’s clear to show that $[N : M]$ is a $\Gamma$-ideal of a $\Gamma$-semigroup $S$.

**Definition 2.11.**[5] Let $M$ be an $S_\Gamma$-act. Then $M$ is a simple $S_\Gamma$-act, if it contain no gamma subact other than $M$. A $\Gamma$-semigroup $S$ is said to be simple if $S$ is $S_\Gamma$-act.

**Definition 2.12.**[5] Let $M$ and $N$ be two $S_\Gamma$-acts. A mapping $f : M \rightarrow N$ is called $S_\Gamma$-homomorphism if $f(s\alpha m) = s\alpha f(m)$. for all $s \in S$, $\alpha \in \Gamma$ and $m \in M$. 


Definition 2.13. [7] An $S_{Γ}$-act $M$ is called a multiplication if for every $S_{Γ}$-subact $N$ of $M$, there exists a $Γ$-ideal $A$ of $S$, such that $N = AΓM$. $Γ$-ideal $A$ of $S$, is multiplication if $A$ is $S_{Γ}$-subact of $S_{Γ}$-act $S$.

Clearly, $M$ is multiplication $S_{Γ}$-act if and only if $N = [N:M]ΓM$ for every $S_{Γ}$-subact $N$ of $M$.

Examples and Remarks 2.14.

i. A $Γ$-monoid $S$, is called multiplication if all its $Γ$-ideals are multiplication.

ii. If $S$ is a completely gl-idempotent $Γ$-semigroup, then is a multiplication.

iii. Cyclic $S_{Γ}$-acts are multiplication. [7]

iv. The nonempty intersection of two multiplication $S_{Γ}$-acts is multiplication. [7]

v. A union of two multiplication $S_{Γ}$-acts not necessary that being multiplication $S_{Γ}$-act. [7]

In [7], if $M$ is a $S_{Γ}$-act and $P$ a maximal $Γ$-ideal of $S$, then we define:

$T_p(M) = \{ m \in M : m = pam \text{ for some } p \in P \text{ and } α \in Γ \}$. Clearly $T_p(M)$ is an $S_{Γ}$-subact of $M$.

We say that $M$ is $P$-cyclic provided there exist $q \in S\Gamma P$ such that $q\Gamma M \subseteq S\Gamma m$, for all $m \in M$. As a generalization of $T_p(M)$, we can define $\overline{T}_p(M) = \{ m \in M : m = p\Gamma m \text{, for some } p \in P \}$, it's clear that $T_p(M) \subseteq \overline{T}_p(M)$. We proved that, if $S$ is a $Γ$-monoid, then $M$ is a multiplication $S_{Γ}$-act if and only if for every maximal $Γ$-ideal $P$ of $S$, either $M = T_p(M)$ or $M$ is $P$-cyclic. Thus, we have the following corollary.

Corollary 2.15. [7] Let $S$ be $Γ$-monoid. Then, $M$ is a multiplication $S_{Γ}$-act if and only if $A\Gamma M$ is a multiplication $S_{Γ}$-act for all multiplication $Γ$-ideal $A$ of $S$.

Recall that an $S_{Γ}$-act $M$ is faithful (globally faithful (gl-faithful for short)), if the equality $s\alpha m = t\alpha m$, $(s\Gamma m = t\Gamma m)$ implies that $s = t$ for all $m \in M$ and $α \in Γ$. It's clear that every gl-faithful $S_{Γ}$-act $M$ is faithful. [7]

Theorem 2.16. [7] Let $S$ be a $Γ$-monoid and $M$ a faithful $S_{Γ}$-act. Then $M$ is a multiplication if and only if:

i. $\bigcap_{i \in I} (A_i \Gamma M) = (\bigcap_{i \in I} A_i) \Gamma M$ for any nonempty collection of $Γ$-ideals $A_i$ $(i \in I)$ of $S$, with $(\bigcap_{i \in I} A_i) \neq \emptyset$, and

ii. For any $S_{Γ}$-subact $N$ of $M$ and $Γ$-ideal $A$ of $S$ such that $N \subseteq A\Gamma M$ there exists an $Γ$-ideal $B$ with $B \subseteq A$ and $N \subseteq B\Gamma M$. 


Now, we give some Lemmas which are need in our work.

**Lemma 2.17.** Let $S$ be a $\Gamma$-monoid and $K, N$ $S_\Gamma$-subacts of $S_\Gamma$-act $M$. If $[K \cap N:M] = S$, then $[(K:M) \cap (N:M)] \subseteq [K:M] \cap [N:M]$. Let $a, b \in [K:M] \cap [N:M]$ and $\alpha \in \Gamma$. Then, $a\alpha b \in [K:M] \cap [N:M]$. For the other inclusion, $(K:M) \cap (N:M) = (K:M) \cap (N:M)$.

**Proof.** Let $a, b \in [K:M] \cap [N:M]$ where $a, b \in [K:M] \cap [N:M]$ and $\alpha \in \Gamma$. Thus, $a\alpha b \in [K:M] \cap [N:M]$ and $a\alpha b \in [N:M] \cap [N:M]$ and hence, $a\alpha b \in [K:M] \cap [N:M]$. For the other inclusion, $(K:M) \cap (N:M) = (K:M) \cap (N:M)$.

Therefore, $(K:M) \cap (N:M) = (K:M) \cap (N:M)$.

**Lemma 2.18.** Let $S$ be a $\Gamma$-semigroup and $A, B$ are $\Gamma$-ideals of $S$. If $A \Gamma M = B \Gamma M$, then $A \cap \Gamma B \Gamma M = B \cap \Gamma A \Gamma M$. Where $A \cap \Gamma B \Gamma M = B \cap \Gamma A \Gamma M$.

**Proof:** We will prove by induction on $k$. If, $k=1$, then it's clear. Suppose it's true when $k = n$, and we show that for $n = k + 1$. Thus, $A^{n+1} \Gamma M = A^n \Gamma (A \Gamma M) = A^n \Gamma (B \Gamma M) = B \Gamma (A^n \Gamma M) = B \Gamma (B^n \Gamma M) = B \Gamma (B \Gamma M) = B^{n+1} \Gamma M$.

**Lemma 2.19.** Let $A$ be a $\Gamma$-ideal of $\Gamma$-monoid $S$, and $M$ an $S_\Gamma$-act. If $M$ is faithful multiplication $\Gamma$-ideal of $S$, then $A = [A \Gamma M : M]$.

**Proof:** Let $a \in A$. Then, $a \alpha m \in A \Gamma M$ for all $\alpha \in \Gamma$ and $m \in M$. So, $a \Gamma M \subseteq A \Gamma M$. Hence, $a \in [A \Gamma M : M]$. Conversely, suppose $s \in [A \Gamma M : M]$ then $s \Gamma M \subseteq A \Gamma M$. By faithfulness of $M$, $s \in A$. Therefore, $A = [A \Gamma M : M]$.

The proof of the following two Lemmas are immediate

**Lemma 2.20.** Let $\{ N_i, i \in I \}$ be a nonempty collection of $S_\Gamma$-subacts of an $S_\Gamma$-act $M$, and $A$ be a $\Gamma$-ideal of $S$. Then $\Gamma (\bigcup_{i \in I} N_i) = \bigcup_{i \in I} (A \Gamma N_i)$.
Lemma 2.21. Let $S$ be a $\Gamma$-semigroup ,and $A$ be $\Gamma$-ideal of $S$. Then for any collection of $\Gamma$-ideals $\{ B_i : i \in I \}$ of $S$, we have:
1. $[A : \bigcup_{i \in I} B_i] = \bigcap_{i \in I} [A : B_i]$
2. $[\bigcap_{i \in I} B_i : A] = \bigcap_{i \in I} [B_i : A]$.

Lemma 2.22. Let $S$ be a $\Gamma$-monoid and $M$ an $S\Gamma$-act. Then $K$ is a multiplication $S\Gamma$-subact of $M$ if and only if $N\cap K = [N : K]\Gamma K$ for every $S\Gamma$-subact $N$ of $M$.

**Proof.** $(\Rightarrow)$ Let $x \in N \cap K$, then $x \in K$, since $K$ is a multiplication $S\Gamma$-subact of $M$, there is an $\Gamma$-ideal $A$ in $S$ such that, $S\Gamma x = A\Gamma K \subseteq N$. It follows that $A \subseteq [N:K]$ and hence $A\Gamma K \subseteq [N:K]\Gamma K$. Therefore , $x \in [N : K]\Gamma K$. Now , we show the other inclusion. Let $y \in [N:K]\Gamma K$. Then $y= sak$ for some $s \in [N : K]$, $s\in \Gamma$ and $k \in K$. So, $s\Gamma K \subseteq N$ thus $y = sak \in N$. Also, since $[N:K]\Gamma K \subseteq S\Gamma K \subseteq K$ then $y \in K$. We conclude that $y \in N\cap K$. Hence , $N\cap K = [N:K]\Gamma K$.

$(\Leftarrow)$ Let $L$ be a $S\Gamma$-subact of $K$ . Then by hypothesis , $L= L\cap K = [L : K]\Gamma K$.

**Idempotent and Pure Gamma Subacts of Multiplication Gamma Acts.**

In this section we introduce the definitions of idempotent and pure gamma subacts and we give several characterizations and properties of such gamma subacts.

**Definition 3.1.** An $S\Gamma$-subact $N$ of an $S\Gamma$-act $M$ is said to be idempotent if $N = [N : M]\Gamma N$.

It’s clear that the trivial $S\Gamma$-subacts are idempotent .For $M=S=\Gamma=\mathbb{Z}$ and $N=2\mathbb{Z}$. Then, $[N : M]= 2\mathbb{Z}$. So, $[N : M]\Gamma N = 2\mathbb{Z} = N$.

Now , we study some properties of idempotent gamma subacts.

**Proposition 3.2.** Let $A$ be an $\Gamma$-ideal of a $\Gamma$-semigroup $S$ and $K , N$ $S\Gamma$-subacts of $S\Gamma$-act $M$.

1. If $K$ and $N$ are idempotent then, so is $K \cup N$.
2. Let $M$ be a faithful multiplication and $K , N$ idempotent in $M$. If $S = [K\cap N : M]$ then $K \cap N$ is idempotent in $M$.
3. If $K$ is idempotent in $N$ and $N$ is idempotent in $M$ then $K$ is idempotent in $M$.
4. Let $M$ be a faithful multiplication $S\Gamma$-act. Then $[N : M]$ is gl-idempotent $\Gamma$-ideal of $S$ if and only if $N$ is idempotent.
Proof. (1) By Lemma 2.20, we have \([K \cup N: M] \Gamma (K \cup N) = ([K \cup N: M] \Gamma K) \cup ([K \cup N: M] \Gamma N) \supseteq (\Gamma N \cap \Gamma K) \cup ([N: M] \Gamma N) = K \cup N\). Also, \([K \cup N: M] \Gamma (K \cup N) \subseteq K \cup N\). Hence, \(K \cup N\) is idempotent in \(M\).

(2) By Theorem 2.16 and Lemma 2.17, we have,
\[\begin{align*}
[K \cap N: M] \Gamma (K \cap N) &= ([K: M] \cap [N: M]) \Gamma (K \cap N) \\
&= ([K: M] \cap [N: M]) \Gamma ([K: M] \cap [N: M]) \\
&= ([K: M] \cap [N: M]) \Gamma ((K \cap N) \cap ([K: M] \cap [N: M])) \\
&= ([K: M] \cap [N: M]) \Gamma (K \cap N) \\
&= K \cap N.
\end{align*}\]

(3) By hypothesis, \(K = [K: N] \Gamma K\) and \(N = [N: M] \Gamma N\). It follows that \([K: N] \Gamma N = [K: N] \Gamma ([N: M] \Gamma N) \subseteq [K: N] \Gamma N\). Thus, \([K: N] \Gamma N = [K: N] \Gamma N\). Also, \(K = [K: N] \Gamma K \subseteq [K: N] \Gamma N = [K: M] \Gamma N \subseteq [K: M] \Gamma M \subseteq K\). So, \(K = [K: M] \Gamma M\). Hence, \([K: N] \Gamma N = [K: M] \Gamma N \subseteq [K: M] \Gamma M\). Therefore, \(K = [K: N] \Gamma N = [K: M] \Gamma N\). Finally, \(K = [K: N] \Gamma K = [K: N] \Gamma ([K: M] \Gamma N) = [K: M] \Gamma K\).

(4) \((\Rightarrow)\) If \([N: M] \Gamma\) is an \(S\)-idempotent \(\Gamma\)-ideal of \(S\), then \(N = [N: M] \Gamma M = ([N: M] \Gamma [N: M]) \Gamma M\). Also, \(N = [N: M] \Gamma N\), and hence \(N\) is idempotent in \(M\).

\((\Leftarrow)\) By hypothesis, \([N: M] \Gamma M = [N: M] \Gamma N\). Thus, \([N: M] \Gamma M = [N: M] \Gamma N\) \(\Gamma ([N: M] \Gamma M) = ([N: M] \Gamma ([N: M] \Gamma M)) \Gamma M\). By faithfulness of \(M\), we have \([N: M] = [N: M] \Gamma [N: M] \Gamma M\). Therefore, \([N: M]\) is a \(S\)-idempotent.

Proposition 3.3. Let \(M\) be a multiplication \(S\)-act. If \(A\) is a \(S\)-idempotent \(\Gamma\)-ideal of \(S\) and \(N\) is idempotent \(S\)-subact of \(M\), then \(A \Gamma N\) is an idempotent \(S\)-subact of \(M\).

Proof: Since \(M\) is a multiplication. Then \(A \Gamma N = [A \Gamma N: M] \Gamma M\). Also, \(A \Gamma N = A \Gamma ([N: M] \Gamma M) = (A \Gamma [N: M]) \Gamma M\). By assumptions, \(A \Gamma N = (A \Gamma A) \Gamma ([N: M] \Gamma N) = (A \Gamma [N: M]) \Gamma A \Gamma N = (A \Gamma [N: M]) \Gamma ([A \Gamma N: M]) \Gamma M = [A \Gamma N: M] \Gamma (A \Gamma N)\).
Corollary 3.4. Let \( M \) be a multiplication \( S_\Gamma \)-act. If \( A \) is gl-idempotent \( \Gamma \)-ideal of \( S \), then \( A \Gamma M \) is an idempotent \( S_\Gamma \)-subact of \( M \). The converse is true if \( M \) is faithful \( S_\Gamma \)-act.

**Proof:** It's clear by Proposition 3.3. Conversely, since \( A \Gamma M \) is an idempotent then by Lemma 2.19, \( A \Gamma M = [A \Gamma M : M] \Gamma (A \Gamma M) = A \Gamma [M : M] \Gamma (A \Gamma M) = (A \Gamma S) \Gamma (A \Gamma M) = (A \Gamma A) \Gamma M \). By faithfulness of \( M \), \( A = A \Gamma A \).

**Proposition 3.5.** Let \( M \) be a multiplication \( S_\Gamma \)-act and \( \{N_i, i \in I\} \) a nonempty collection of \( S_\Gamma \)-subacts of \( M \) such that \( M = \bigcup_{i \in I} N_i \). Then \( N_i \) is idempotent for all \( i \in I \).

**Proof:** Let \( i \in I \), by Lemma 2.20 we get \( N_i = [N_i : M] \Gamma M = [N_i : M] \Gamma (U_{i \neq j} N_j) = ([N_i : M] \Gamma N_i) \cup ([N_i : M] \Gamma (U_{i \neq j} N_j)) \). Since \( M \) is multiplication then \( [N_i : M] \Gamma (U_{i \neq j} N_j) = [N_i : M] \Gamma ([U_{i \neq j} N_j : M] \Gamma M) = [U_{i \neq j} N_j : M] \Gamma ([N_i : M] \Gamma M) = [U_{i \neq j} N_j : M] \Gamma N_i \subseteq N_i \). It follows that \( [N_i : M] \Gamma (U_{i \neq j} N_j) \subseteq N_i \cap (U_{i \neq j} N_j) = 0 \). We have \( N_i = [N_i : M] \Gamma N_i \) and hence \( N_i \) is idempotent.

Now, we introduce the definition of pure gamma subact and we discussed some of basic properties.

**Definition 3.6.** An \( S_\Gamma \)-subact \( N \) of \( S_\Gamma \)-act \( M \) is called pure, if \( A \Gamma N = N \cap A \Gamma M \) for each \( \Gamma \)-ideal \( A \) of \( S \). A \( \Gamma \)-ideal \( A \) of \( S \), is pure if \( A \) is \( S_\Gamma \)-subact of \( S_\Gamma \)-act \( S \).

**Example 3.7.** Let \( S = \{x, y, z\}, \Gamma = \{\alpha, \beta, y\} \) and \( M = S \). Define a binary operation in \( M \) as shown in the following table:

|   | x | y | z |
|---|---|---|---|
| x | x | x | x |
| y | x | x | x |
| z | x | y | z |

Then \( M \) is an \( S_\Gamma \)-act under the mapping \( S \times \Gamma \times M \rightarrow M \) which defined by \( x \alpha y = xy \) for all \( x, y \in M \). Here, \( N_1 = \{x, y\} \) and \( N_2 = \{x, z\} \) are \( S_\Gamma \)-subacts of \( M \). Clearly, \( N_1 \) is a pure \( S_\Gamma \)-subact, but \( N_2 \) is not pure \( S_\Gamma \)-subact of \( M \).
**Proposition 3.8.** Pure $S_\Gamma$-subacts of multiplication $S_\Gamma$-act are multiplication and idempotent.

**Proof.** Let $N$ be a pure $S_\Gamma$-subact of multiplication $S_\Gamma$-act $M$, and $K$ an $S_\Gamma$-subact of $N$. Thus $K = A \Gamma M$ for some $\Gamma$-ideal $A$ of $S$. But $N$ is pure, so we have $K = K \cap N = A \Gamma M \cap N = A \Gamma N$. Hence, $N$ is a multiplication. Now, we show that $N$ is idempotent. Since $N = [N:M] \Gamma M$, and $N$ is pure then $[N:M] \Gamma N = N \cap [N:M] \Gamma M = N$. Therefore, $N = [N:M] \Gamma N$, hence $N$ is idempotent in $M$.

**Proposition 3.9.** If each of $K$ and $N$ are pure $S_\Gamma$-subacts of $M$ then so is $K \cup N$.

**Proof:** Let $A$ be an $\Gamma$-ideal of $S$. Since $K$ and $N$ are pure in $M$, then $A \Gamma K = K \cap A \Gamma M$ and $A \Gamma N = N \cap A \Gamma M$. So, by Lemma 2.20, $(K \cup N) \cap A \Gamma M = (K \cap A \Gamma M) \cup (N \cap A \Gamma M) = A \Gamma K \cup A \Gamma N = A \Gamma (K \cup N)$, and hence $K \cup N$ is pure $S_\Gamma$-subact of $M$.

**Proposition 3.10.** Let $S$ be a $\Gamma$-semigroup, and $K$, $N$ $S_\Gamma$-subacts of $S_\Gamma$-act $M$. If $K$ is pure $S_\Gamma$-subact of $N$ and $N$ is pure of $M$, then $K$ is pure of $M$.

**Proof:** Let $A$ be a $\Gamma$-ideal of $S$. By hypothesis $A \Gamma K = K \cap A \Gamma N = K \cap (N \cap A \Gamma M) = K \cap A \Gamma M$. Therefore, $K$ is pure.

An $S_\Gamma$-act $M$ is said to be pure-multiplication if for each pure $S_\Gamma$-subact $N$ of $M$ there exist, a $\Gamma$-ideal $A$ of $S$ such that $N = A \Gamma M$. An $S_\Gamma$-act $M$ is said pure-simple, if it contain no pure $S_\Gamma$-subact other than $M$. An $S_\Gamma$-act $M$ is said to regular if all its $S_\Gamma$-subacts are pure. A $\Gamma$-semigroups is called regular if a $S$ is regular $S_\Gamma$-act.

It's clear that every pure-simple $S_\Gamma$-act is pure multiplication. But the convers is not true in general as the following example:

**Example 3.11.** Let $S = \{ i, 0, -i \}$ and $S = \Gamma = M$. Then $M$ is $S_\Gamma$-act under the multiplication over complex numbers. Here $N_1 = \{0\}$ and $N_2 = M$, are the only $S_\Gamma$-subacts of $M$. It's clear that $N_i$, $i=1,2$ are pure $S_\Gamma$-subacts of $M$, and there is $\Gamma$-ideal $A$ of $S$, such that $N_i = A \Gamma M$ for $i=1,2$. Therefore, $M$ is pure multiplication $S_\Gamma$-act but not simple. Also, $M$ is regular.
Proposition 3.12. Let $M$ be an $S_\Gamma$–act and $\{N_i, i \in I\}$ a nonempty collection of $S_\Gamma$–subacts of $M$. If $M = \bigcup_{i \in I} N_i$ then $N_i$ is a pure $S_\Gamma$–subact of $M$ for all $i \in I$.

Proof: Let $A$ be a $\Gamma$-ideal of $S$. By Lemma 2.20, $A \Gamma M = A \Gamma \left( \bigcup_{i \in I} N_i \right) = \bigcup_{i \in I} (A \Gamma N_i)$. Now, let $j \in I$ then, $A \Gamma M \cap N_j = \bigcup_{i \in I} (A \Gamma N_i) \cap N_j = (A \Gamma N_j \cap N_j) \cup \left( \bigcup_{i \neq j} A \Gamma N_i \cap N_j \right) = A \Gamma N_j \cup \emptyset = A \Gamma N_j$. Hence, $N_i$ is a pure $S_\Gamma$–subacts of $M$ for all $i \in I$.

In the following Theorem we give a relation between pure gamma subacts, multiplication gamma acts and idempotent gamma subacts.

Theorem 3.13. Let $S$ be a $\Gamma$-monoid and $N$ a $S_\Gamma$-subact of faithful multiplication $S_\Gamma$-act $M$.

The following statement are equivalent

i. $N$ is a pure $S_\Gamma$-subact of $M$.

ii. $N$ is multiplication and idempotent in $M$.

iii. $A \Gamma [N:M] = A \cap [N:M]$ for every $\Gamma$-ideal $A$ of $S$.

Proof. (i)$\implies$(ii) It’s clear by Proposition 3.8.

(ii)$\implies$(iii) Let $K$ be a $S_\Gamma$-subact of $M$. Then $[K:N] \Gamma N = [K:N] \Gamma ([N:M] \Gamma N) = ([K:N] \Gamma [N:M]) \Gamma N \subseteq [K: M] \Gamma N$. Since $[K: M] \Gamma N \subseteq [K:N] \Gamma N$, So, $[K:N] \Gamma N = [K:M] \Gamma N$. Since $N$ is multiplication, then by Lemmas 2.22 and 2.19, for every $\Gamma$-ideal $A$ of $S$, $A \Gamma M \cap N = [A \Gamma M \cap N] \Gamma N = A \Gamma N$. Thus, $A \Gamma N = A \Gamma M \cap N$. So, $(A \Gamma [N:M]) \Gamma M = A \Gamma N \cap A \Gamma M \cap N = A \Gamma M \cap [N:M] \Gamma M$. Therefore, $A \Gamma [N:M] \Gamma M = (A \Gamma [N:M]) \Gamma M$ and by Theorem 2.16, $(A \cap [N:M]) \Gamma M = (A \Gamma [N:M]) \Gamma M$. By faithfulness of $M$, $A \Gamma [N:M] = A \cap [N:M]$.

(iii)$\implies$(i) Let $A$ be an $\Gamma$-ideal of $S$. By Theorem 2.16, and hypothesis $N \cap A \Gamma M = [N:M] \Gamma M \cap A \Gamma M = ([N:M] \cap A) \Gamma M = ([N:M] \Gamma A) \Gamma M = (A \Gamma [N:M]) \Gamma M = A \Gamma [N:M] \Gamma M = A \Gamma N$.

Therefore, $N$ is a pure $S_\Gamma$-subact of $M$.

As a special case of Theorem 3.13, the following corollary gives a characterization of pure $\Gamma$-ideal in $\Gamma$-monoids.

Corollary 3.14. If $S$ is a faithful and $A$ $\Gamma$-ideal of $S$. Then the following statement are equivalent:

i. $A$ is a pure $\Gamma$-ideal of $S$. 


ii. A is multiplication and gl-idempotent in S.

iii. \( B \cap [A:S] \) for every \( \Gamma \)-ideal B of S.

**Corollary 3.15.** Let S be a \( \Gamma \)-monoid and M a gl-faithful multiplication \( S_\Gamma \)-act .

i. If N (resp. A) is a pure \( S_\Gamma \)-subact (resp. \( \Gamma \)-ideal) of M (resp. S) then \( \Lambda \Gamma N \) is a pure \( S_\Gamma \)-subact of M.

ii. The nonempty intersection of two pure \( S_\Gamma \)-subacts of M is pure \( S_\Gamma \)-subact of M.

**Proof.** (i) Let N (A) be a pure \( S_\Gamma \)-subact (\( \Gamma \)-ideal) of M (S). Then by Theorem 3.13, N and (A) is a multiplication and idempotent (gl-idempotent). By Corollary 2.15, and Proposition 3.3, \( \Lambda \Gamma N \) is multiplication an idempotent. By Theorem 3.13, \( \Lambda \Gamma N \) is pure \( S_\Gamma \)-subact of M.

(ii) Let \( N_1 \) and \( N_2 \) be a pure \( S_\Gamma \)-subacts of M, with \( N_1 \cap N_2 \neq \emptyset \) . We will show that \( N_1 \cap N_2 \) is an idempotent. Let \( x \in N_1 \cap N_2 \) , then \( x \in N_1 \) and \( x \in N_2 \) . By Theorem 3.13 , \( N_1 \) and \( N_2 \) are idempotent \( S_\Gamma \)-subacts, then \( N_1 = [N_1 : M] \Gamma N_1 \) and \( N_2 = [N_2 : M] \Gamma N_2 \) . Thus, there is \( y_1 \in [N_1 : M] \) and \( y_2 \in [N_2 : M] \) such that \( x = y_1 \alpha \) and \( x = y_2 \beta \) for some \( \alpha , \beta \in \Gamma \) . Also, since \( y_1 \in [N_1 : M] \) and \( y_2 \in [N_2 : M] \) then \( y_1 \Gamma y_2 \subseteq [N_1 : M] \Gamma [N_2 : M] \subseteq [N_1 : M] \cap [N_2 : M] = [(N_1 \cap N_2) : M] \) . Now, \( x = y_1 \alpha (y_2 \beta x) = (y_1 \alpha y_2) \beta x \in [(N_1 \cap N_2) : M] \Gamma (N_1 \cap N_2) \) . Thus, \( N_1 \cap N_2 \) is idempotent and by Remark 1.19 (1), \( N_1 \cap N_2 \) is a multiplication. So, by Theorem 3.13, \( N_1 \cap N_2 \) is a pure \( S_\Gamma \)-subact of M.

**Proposition 3.16.** Let S be a \( \Gamma \)-monoid and N a pure \( S_\Gamma \)-subact of gl-faithful multiplication \( S_\Gamma \)-act M. Then every \( S_\Gamma \)-subact of N is pure.

**Proof:** By Theorem 3.13, N is idempotent, and multiplication. Let K be a \( S_\Gamma \)-subact of \( N \) . Then \( K = [K : N] \Gamma N \). By Corollary 3.15 , K is a pure \( S_\Gamma \)-subact of M.

**Theorem 3.17.** Let S be a \( \Gamma \)-monoid and M a gl-faithful multiplication \( S_\Gamma \)-act. Then :

i. An \( S_\Gamma \)-subact N of M is pure if and only if \([N : M]\) is a pure \( \Gamma \)-ideal of S .

ii. An \( \Gamma \)-ideal A of S is pure if and only if \( \Lambda \Gamma M \) is a pure \( S_\Gamma \)-subact of M.
**Proof:** (i) $\Rightarrow$ Let $A$ be a $\Gamma$-ideal of $S$, and $N$ be a pure $S_\Gamma$-subact of $M$. Then ,by Theorem 3.13, $A \Gamma [N:M] = A \cap [N:M]$. It follows that, $[N:M]$ is pure $\Gamma$-ideal of $S$.

$(\Leftarrow)$ Let $N$ be an $S_\Gamma$-subact of $M$ such that $[N : M]$ is a pure $\Gamma$-ideal of $S$. Then, corollary 3.14 implies that $[N : M]$ is multiplication and gl-idempotent. Since, $N = [N : M] \Gamma M$, then $N$ is multiplication and by Proposition 3.2(4), $N$ is idempotent. Hence, $N$ is pure.

(ii) $\Rightarrow$ Let $B$ be a $\Gamma$-ideal of $S$. Then, by Theorem 2.16, $A \Gamma M \cap B \Gamma M = (A \cap B) \Gamma M = (A \Gamma B) \Gamma M = A \Gamma (B \Gamma M) = B \Gamma (A \Gamma M)$. Hence, $A \Gamma M$ is a pure $S_\Gamma$-subact of $M$.

$(\Leftarrow)$ Let $A$ be an $\Gamma$-ideal of $S$, such that $A \Gamma M$ is a pure $S_\Gamma$-subact of $M$. By Lemma 2.19, $A = [A \Gamma M : M]$, by (1) $A$ is a pure $\Gamma$-ideal of $S$.

In following Proposition, we give a characterization of completely gl-idempotent $\Gamma$-monoids in terms of pure $\Gamma$-ideals.

**Corollary 3.18** Let $S$ be a $\Gamma$-monoid. Then $S$ is completely gl-idempotent if and only if $S$ is regular.

**Proof:** $(\Rightarrow)$ Let $A$ be a $\Gamma$-ideal of $S$. Then $A$ is gl-idempotent and $A$ is multiplication. Hence, by Corollary 3.14, $A$ is pure.

$(\Leftarrow)$ Let $B$ be a pure $\Gamma$-ideal of $S$. By Theorem 3.13, $B$ is gl-idempotent. Hence, $S$ is completely gl-idempotent.

An $S_\Gamma$-subact $N$ of $S_\Gamma$-act $M$ is said to be meet principal if $A \Gamma N \cap K = (A \cap [K : N]) \Gamma N$, for all $\Gamma$-ideal $A$ of $S$ and $S_\Gamma$-subact $K$ of $M$.

**Theorem 3.19.**[6] Let $S$ be a $\Gamma$-monoid. If $M$ is meet principal $S_\Gamma$-act then $M$ is a multiplication. The converse is true, if $M$ is faithful.

It's easy to show that if $A$ and $B$ are $\Gamma$-ideals of $S$, then so is $A \Gamma B$.

**Corollary 3.20.**[6] Let $S$ be a $\Gamma$-monoid. If $A$ and $B$ are faithful multiplication $\Gamma$-ideals of $S$, then $A \Gamma B$ is a multiplication $\Gamma$-ideal of $S$. 

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Proposition 3.21. If $S$ is a faithful and $A$ and $B$ are pure $\Gamma$-ideals of $S$, then $A \Gamma B$ is a pure $\Gamma$-ideal of $S$.

**Proof:** By Corollary 3.20, and Proposition 2.6, $A \Gamma B$ is multiplication and gl-idempotent $\Gamma$-ideal of $S$. So, by Corollary 3.14, $A \Gamma B$ is pure.

Recall that [6], If $N_1$, $N_2$ are $S_\Gamma$-subacts of a multiplication $S_\Gamma$-act $M$ such that $N_1 = A \Gamma M$ and $N_2 = B \Gamma M$ for some $\Gamma$-ideals $A$ and $B$ of $S$, then the product of $N_1$ and $N_2$ is denoted by $N_1 * N_2$ is defined by $N_1 * N_2 = (A \Gamma B) \Gamma M$. Where, $N_1 * N_2$ is an $S_\Gamma$-subact of $M$.

Proposition 3.22. Let $S$ be a $\Gamma$-monoid and $M$ a multiplication $S_\Gamma$-act. If $N_1$ and $N_2$ are pure $S_\Gamma$-subacts of $M$, then $N_1 * N_2$ is a pure $S_\Gamma$-subact of $M$.

**Proof.** Without loss the generality every $\Gamma$-ideals of $S$ is a faithful. Let $N_1$ and $N_2$ be a pure $S_\Gamma$-subacts of $M$. Since $M$ is multiplication then there is $\Gamma$-ideals $A$ and $B$ of $S$, such that $N_1 = A \Gamma M$ and $N_2 = B \Gamma M$. By Theorem3.17, $A$ and $B$ are pure $\Gamma$-ideals and by Proposition 3.21, $A \Gamma B$ is pure $\Gamma$-ideal. Thus, $(A \Gamma B) \Gamma M$ is pure $S_\Gamma$-subact of $M$ (By Theorem3.17). Hence, $N_1 * N_2$ is a pure.

References

[1] Ribenboim P., (1972) *Algebraic Numbers* (New York: Wiley–Interscience).

[2] Anderson F. W., and Fuller K. R., (1974) *Rings and Categories of Modules* (New York:Springer -Verlag).

[3] Smith P., and El-Bast Z., (1988) Multiplication modules *Comm. In Algebra* 16 pp.755-779.

[4] Sen M., and Saha K., (1986) On $\Gamma$-semigroups *Bull. Calcutta Math*. 78 pp. 180–186.

[5] Abbas M., and Faris A.,( 2016) Gamma Acts *Inter. J. of Advan. Resea*. 4 pp. 1592-1601.

[6] Abbas M., and Adnan S., (2019) The product of Gamma Subacts of Multiplication Gamma Act *Int. Sci. Conf. of the Univ. of Bab.* (University of Babylon).

[7] Abbas M., and Adnan S., (2019) Multiplication Gamma Act submitted to *International Electronic Journal of Algebra*. 