Noncompact $\mathbb{CP}^N$ as a phase space of superintegrable systems

Erik Khastyan,‡ Armen Nersessian,†,‡,§ and Hovhannes Shmavonyan

1Yerevan Physics Institute, 2 Alikhanian Brothers St., Yerevan 0036 Armenia
2Yerevan State University, 1 Alex Manoogian St., Yerevan, 0025, Armenia
3Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Russia

We propose the description of superintegrable models with dynamical $so(1,2)$ symmetry, and of the generic superintegrable deformations of oscillator and Coulomb systems in terms of higher-dimensional Klein model (the non-compact analog of complex projective space) playing the role of phase space. We present the expressions of the constants of motion of these systems via Killing potentials defining the $su(N,1)$ isometries of the Kähler structure.

I. INTRODUCTION

The symplectic manifolds are the even-dimensional manifolds equipped with closed non-degenerate two-form, which yields the non-degenerate Poisson brackets (see, e.g. [1]). In accordance with Darboux theorem any symplectic structure can locally be presented in the canonical form corresponding to the canonical Poisson brackets. Furthermore, any cotangent bundle of Riemann manifold can be equipped with the globally defined canonical symplectic structure. Hence, for the Hamiltonian description of systems of particles moving on the Riemann space we can restrict ourselves by the canonical symplectic structure (and canonical Poisson brackets). The non-canonical Poisson brackets are usually used for the description of more sophisticated systems, say, various modifications of tops, (iso)spin dynamics, etc. An important class of symplectic manifolds are Kähler manifolds, which are the Hermitian manifolds whose imaginary parts define the symplectic structures. Kähler manifolds are highly common objects in almost all areas of theoretical physics, especially in the supergravity and string theory. However, they usually appear as configuration spaces of the particles and fields. Only in a limited number of physical problems they appear as phase spaces, mostly for the description of various versions of Hall effect, including its higher-dimensional generalizations (see, e.g. [2,3] and refs therein and to them). Respectively, the number of the known nontrivial (super)integrable systems with Kähler phase space is very restricted, and their study is on the margin of the theory of integrable systems. This is especially surprising, given that quantization of the systems with Kähler phase space has been in the focus of modern geometry since the invention of the concept of geometric quantization. An exceptional integrable model with Kähler phase space extensively studied nowadays is the compactified (1,1)-symmetric Kähler model [4], but even this system is mostly studied in canonical coordinates. On the other hand, relation of the (existing) integrable systems and of its constants of motion with the isometries of Kähler manifold considered as a phase space can be useful to understand the geometry of the system, and could be an important step for the quantization in non-canonical coordinates.

A very simple example of such system is one-dimensional conformal mechanics formulated in terms of the Klein model of Lobachevsky plane (“noncompact complex projective plane”) played the role of phase space [5]. Such description, besides elegance, allows to immediately construct its $N = 2k$ superconformal extension associated with $su(1,1|k)$ superalgebra. Similar formulation of the higher-dimensional systems was given in [6,7] with the aim to geometrize (and generalize to higher dimensions) the so-called ”holomorphic factorization approach” [8] to the (two-dimensional) superintegrable oscillator- and Coulomb-like systems invented in [9] and extended to the spheres and hyperboloids in [10]. It was based on the separation of ”radial” part of the system (spanned by the generators of $so(1,2)$ dynamical symmetry algebra), from the ”angular” part, given by the Casimir of $so(1,2)$ [11]. As a result, the integrable generalizations of conformal mechanics, and of the (Euclidian, spherical and hyperbolical) oscillator and Coulomb systems were formulated on the non-Kähler phase space $\mathbb{CP}^1 \times \mathcal{M}$, with $\mathbb{CP}^1$ being a Lobachevsky plane (“noncompact complex projective plane”) parameterized by the radial coordinate and momentum, and with $\mathcal{M}$ being the phase space of ”angular” part of conformal mechanics. So, the higher-dimensional extension of the approach suggested in [5] led to the loss of $su(1|1)$-symmetric Kähler structure of the phase space. Nevertheless, a few interesting observations were made there, particularly the unified formulation, in terms of complex phase space coordinates, of the hidden symmetry generators
II. PRELIMINARY

The symplectic manifold \((M, \omega)\) is the even-dimensional manifold equipped with closed non-degenerate two-form

\[
\omega = \frac{1}{2} \omega_{ij}(x) dx^i \wedge dx^j : d\omega = 0, \quad \det \omega_{ij} \neq 0. \tag{1}
\]

This two-form defines the non-degenerate Poisson brackets

\[
\{f, g\} = \omega^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}, \quad \text{with} \quad \omega^{ij} \omega_{jk} = \delta^i_k. \tag{2}
\]

Kähler manifold is the manifold with Hermitian metrics \(ds^2 = g_{ab} dz^a d\bar{z}^b\) whose imaginary part defines the symplectic structure

\[
\omega_M = i g_{ab} dz^a \wedge d\bar{z}^b, \quad d\omega_M = 0 \Rightarrow g_{ab} dz^a d\bar{z}^b = \frac{\partial^2 \mathcal{K}}{\partial z^a \partial \bar{z}^b} dz^a d\bar{z}^b, \tag{3}
\]

where \(\mathcal{K}(z, \bar{z})\) is a real function (Kähler potential) defined up to holomorphic and antiholomorphic functions: \(\mathcal{K}(z, \bar{z}) \rightarrow \mathcal{K}(z, \bar{z}) + \mathcal{U}(z) + \mathcal{U}(\bar{z}).\)

Hence, Kähler manifold can be equipped with the Poisson brackets

\[
\{f, g\} = i g^{ab} \left( \frac{\partial f}{\partial z^a} \frac{\partial g}{\partial \bar{z}^b} - \frac{\partial f}{\partial \bar{z}^a} \frac{\partial g}{\partial z^b} \right), \quad g^{ab} g_{bc} = \delta^a_c. \tag{4}
\]

Therefore, the isometries of Kähler structure should preserve both complex and symplectic structures, i.e. they are defined by the holomorphic Hamiltonian vector fields,

\[
V_\mu = \{h_\mu, \cdot\}_M = V^a_\mu(z) \frac{\partial}{\partial z^a} + \bar{V}^a_\mu(\bar{z}) \frac{\partial}{\partial \bar{z}^a}, \quad V^a_\mu = i g^{bc} \partial_c h_\mu(z, \bar{z}). \tag{5}
\]

The real function \(h_\mu(z, \bar{z})\) (sometimes called Killing potential) obeys the equation

\[
\frac{\partial^2 h_\mu}{\partial z^a \partial \bar{z}^b} - \Gamma^c_{ab} \frac{\partial h_\mu}{\partial z^c} = 0, \tag{6}
\]
with $\Gamma_{ab} = g^{cd} \partial_a g_{bd}$ being the non-vanishing components of the Christoffel symbols.

The most known examples of nontrivial Kähler manifolds are the $N$-dimensional complex projective space $\mathbf{CP}^N$ and its non-compact analog which we will further denote as $\widetilde{\mathbf{CP}}^N$. They can be equipped with the $su(N + 1)$-invariant (for the compact case) and the $su(N, 1)$ invariant (for the non-compact case) Kähler metrics, known as the Fubini-Study ones. These metrics and respective Kähler potentials are defined by the expressions (with the upper sign corresponding to $\mathbf{CP}^N$, and the lower sign to $\widetilde{\mathbf{CP}}^N$) \(^1\)

$$g_{ab} dz^a d\bar{z}^b = \frac{g d\bar{z} \bar{d}z}{1 \pm \bar{z} \bar{z}}, \quad h_a = g \frac{2 z^a}{1 \pm z \bar{z}}, \quad \bar{h}_a = g \frac{2 \bar{z}^a}{1 \pm z \bar{z}}. \quad (9)$$

These generators form the $su(N + 1)$ algebra for the upper sign, and the $su(N, 1)$ for the lower one (the generators $h_{ab}$ form $u(N)$ algebra):

$$\{h_a, h_b\} = 0, \quad \{h_a, \bar{h}_b\} = -4i h_{ab}, \quad \{h_a, h_{bc}\} = \mp i (\delta_{ac} h_b + \delta_{bc} h_a), \quad \{h_{ab}, h_{cd}\} = \pm i (\delta_{ad} h_{cb} - \delta_{dc} h_{ab}). \quad (10)$$

Poincaré and Klein models of the Lobachevsky plane

The one-dimensional noncompact complex projective space $\mathbb{CP}^1$ is the Lobachevsky plane (upper sheet of two-sheet hyperboloid) proper. Its Fubini-Study metrics results in the $su(1, 1) = so(2, 1)$-invariant Kähler metric parameterized by the unit disc of two-dimensional plane, which is known as Poincaré model [13]

$$ds^2 = \frac{g d\bar{z} \bar{d}z}{(1 - z \bar{z})^2}, \quad K = -g \log(1 - z \bar{z}), \quad |z| < 1. \quad (11)$$

In this particular case the Killing potentials read

$$h = g \frac{1 + \bar{z} \bar{z}}{1 - z \bar{z}}, \quad h_+ = g \frac{2 \bar{z}}{1 - z \bar{z}}, \quad h_- = g \frac{2 z}{1 - z \bar{z}}; \quad \{h_+, h_-\} = -4i h, \quad \{h_\pm, h\} = \mp 2i h_\pm. \quad (12)$$

Performing the transformation

$$z = \frac{1 - i \bar{w}}{1 + i \bar{w}} \quad (13)$$

we arrive at the so-called Klein model parameterized by upper two-dimensional half-plane [13]

$$ds^2 = \frac{g dwd\bar{w}}{|w \bar{w} - i |^2}, \quad K = -g \log |i(w - \bar{w})|, \quad \text{Im } w < 0. \quad (14)$$

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\(^1\) Through this section we use the notation $z \bar{z} \equiv \sum_{c=1}^N z^c \bar{z}^c, \quad z d\bar{z} \equiv \sum_{c=1}^N z^c d\bar{z}^c$ etc.
The Poisson brackets corresponding to this structure are defined by the relation
\[ \{ w, \bar{w} \} = -\frac{i}{g}(w - \bar{w})^2, \] (15)
while the Killing potentials read
\[ h = g \frac{w\bar{w} + 1}{i(w - \bar{w})}, \quad h_+ = g \frac{(1 + iw)(1 + i\bar{w})}{i(w - \bar{w})}, \quad h_- = g \frac{(1 - iw)(1 - i\bar{w})}{i(w - \bar{w})}. \] (16)
Instead of these generators it is more convenient to use their linear combinations
\[ H_0 = g \frac{w\bar{w}}{i(w - \bar{w})}, \quad K_0 = g \frac{1}{i(w - \bar{w})}, \quad D_0 = g \frac{\bar{w} + w}{i(w - \bar{w})}. \] (17)
\[ \{ K_0, H_0 \} = D_0, \quad \{ D_0, H_0 \} = 2H_0, \quad \{ K_0, D_0 \} = 2K_0. \] (18)
Introducing the canonical phase space variables \((p, x)\)
\[ w = \frac{p}{x} - i \frac{g}{x^2} : \quad \{ x, p \} = 1, \] (19)
we can represent the Killing potentials in the standard form of the generators of one-dimensional conformal mechanics
\[ H_0 = \frac{p^2}{2} + \frac{g^2}{2x^2}, \quad K_0 = \frac{x^2}{2}, \quad D_0 = px. \] (20)

### III. NONCOMPACT COMPLEX PROJECTIVE SPACE: KLEIN MODEL

Let us construct \(N\)-dimensional analog of the Klein model from the Fubini-Study structure of noncompact complex projective space \(\tilde{\mathbb{C}}\mathbb{P}^N\) given by the expressions (7) with a lower sign. For this purpose we perform the transformation
\[ z^N = \frac{1 - iw}{1 + iw}, \quad z^\alpha = \sqrt{\frac{2}{1 + iw}} \bar{z}^\alpha, \] (21)
which yields the following expressions for the Kähler structure and potential (here and further instead of \(\bar{z}^\alpha\) we use the former notation \(z^\alpha\))
\[ ds^2 = g \frac{[dw + iz^\alpha dz^\alpha][d\bar{w} - iz^\beta d\bar{z}^\beta]}{[i(w - \bar{w}) - z^\gamma \bar{z}^\gamma]^2} + \frac{g dz^\alpha d\bar{z}^\alpha}{i(w - \bar{w}) - z^\gamma \bar{z}^\gamma}, \] (22)
\[ \mathcal{K} = -g \log |i(w - \bar{w}) - z^\gamma \bar{z}^\gamma|, \quad \alpha, \beta, \gamma = 1, \ldots, N - 1, \] (23)
with the following range of validity of the coordinates \(w, z^\alpha\)
\[ \text{Im } w < 0, \quad \sum_{\alpha=1}^{N-1} z^\alpha \bar{z}^\alpha < -2 \text{ Im } w. \] (24)
The respective Poisson brackets are defined by the relations
\[ \{ w, \bar{w} \} = -A(w - \bar{w}), \quad \{ w, z^\alpha \} = A \bar{z}^\alpha, \quad \{ z^\alpha, \bar{z}^\beta \} = iA \delta^\beta^\alpha, \] (25)
where
\[ A := \frac{i(w - \bar{w}) - z^\gamma \bar{z}^\gamma}{g}. \] (26)
The Killing potentials of the Kähler structure are defined by the expressions

\[ h_{N\bar{N}} = \frac{w\bar{w} + 1}{A}, \quad h_{\alpha\bar{N}} = \frac{1}{\sqrt{2}} \bar{z}^\alpha (1 - iw), \quad h_{\alpha\bar{\beta}} = \frac{\bar{z}^\alpha z^\beta + \frac{1}{2} \delta_{\alpha\beta} (1 + iw)(1 - i\bar{w})}{A} \]  

(27)

\[ h_N = \frac{(1 + iw)(1 + i\bar{w})}{A}, \quad h_\alpha = \sqrt{2} \bar{z}^\alpha (1 + iw). \]  

(28)

These potentials form \( su(N,1) \) algebra, which in the given notation reads the same as in \([10]\) with a lower sign and \( a = N, \alpha \). Below we will refer to this representation as the \( N \)-dimensional Klein model.

For our purposes, instead of Killing potentials \((27),(28)\) it is more convenient to use the following ones

\[ H = \frac{w\bar{w}}{A}, \quad K = \frac{1}{A}, \quad D = \frac{w + i\bar{w}}{A}, \quad H_{\alpha\bar{N}} = \frac{\bar{z}^\alpha w}{A}, \quad H_\alpha = \frac{\bar{z}^\alpha}{A}, \quad H_{\alpha\bar{\beta}} = \frac{\bar{z}^\alpha z^\beta}{A}. \]  

(29)

Certainly, these functions are not independent, for there are many obvious relations between them, e.g.

\[ H = \frac{1}{2} \sum_{\alpha = 1}^{N-1} \frac{H_{\alpha\bar{N}}H_{\alpha\bar{\beta}}}{H_{\alpha\bar{\alpha}}}, \quad H_{\alpha\bar{\beta}} = \frac{H_\alpha H_{\beta\bar{\alpha}}}{K}, \quad \text{etc.} \]  

(30)

In these terms the \( su(1, N) \) algebra relations read

\[ \{H, K\} = -D, \quad \{H, D\} = -2H, \quad \{K, D\} = 2K, \]  

(31)

\[ \{H, H_\alpha\} = -H_{\alpha\bar{N}}, \quad \{H, H_{\alpha\bar{N}}\} = \{H, H_{\alpha\bar{\beta}}\} = 0, \]  

(32)

\[ \{K, H_{\alpha\bar{N}}\} = H_\alpha, \quad \{K, H_\alpha\} = \{K, H_{\alpha\bar{\beta}}\} = 0, \]  

(33)

\[ \{D, H_\alpha\} = -H_{\alpha\bar{N}}, \quad \{D, H_{\alpha\bar{N}}\} = H_{\alpha\bar{N}}, \quad \{D, H_{\alpha\bar{\beta}}\} = 0, \]  

(34)

\[ \{H_\alpha, H_{\beta\bar{\alpha}}\} = \{H_\alpha, H_{\beta\bar{\beta}}\} = 0, \]  

(35)

\[ \{H_\alpha, H_{\beta\bar{\beta}}\} = -iK \delta_{\alpha\bar{\beta}}, \quad \{H_{\alpha\bar{N}}, H_{\beta\bar{\beta}}\} = -iH \delta_{\alpha\bar{\beta}}, \quad \{H_{\alpha\bar{\beta}}, H_{\beta\bar{\gamma}}\} = i(H_{\alpha\bar{\beta}} \delta_{\gamma\bar{\beta}} - H_{\gamma\bar{\beta}} \delta_{\alpha\bar{\gamma}}), \]  

(36)

\[ \{H_\alpha, H_{\beta\bar{\gamma}}\} = H_{\alpha\bar{\gamma}} + \frac{1}{2} \left( g + \sum_{\gamma} H_{\gamma\gamma} - iD \right) \delta_{\alpha\bar{\gamma}}, \]  

(37)

\[ \{H_\alpha, H_{\beta\bar{\gamma}}\} = -iH \delta_{\alpha\bar{\gamma}}, \quad \{H_{\alpha\bar{N}}, H_{\beta\bar{\gamma}}\} = -iH_{\beta\bar{N}} \delta_{\alpha\bar{\gamma}}. \]  

(38)

So, the generators \( H, K, D \) define the conformal algebra \( su(1,1) = so(1,2) \), and the generators \( H_{\alpha\bar{\beta}} \) define the algebra \( u(N - 1) \).

It is seen that

- the Hamiltonian \( H \) has two sets of constants of motion \( H_{N\alpha} \) and \( H_{\alpha\bar{\beta}} \) (see \([32]\)), therefore it defines superintegrable system;
- the Hamiltonian \( K \) has two sets of constants of motion as well, \( H_\alpha \) and \( H_{\alpha\bar{\beta}} \) (see \([33]\)). Thus, it defines the superintegrable system as well;
- the triples \( (H, H_{N\alpha}, H_{\alpha\bar{\beta}}) \) and \( (K, H_\alpha, H_{\alpha\bar{\beta}}) \) transform into each other within discrete transformation

\[ (w, z^\alpha) \to (-\frac{1}{w}, \frac{z^\alpha}{w}) \Rightarrow D \to -D, \quad \{H, H_{N\alpha}, H_{\alpha\bar{\beta}}\} \to \{K, -H_\alpha, H_{\alpha\bar{\beta}}\}, \quad (K, H_\alpha, H_{\alpha\bar{\beta}}) \to (H, -H_{N\alpha}, H_{\alpha\bar{\beta}}). \]  

(39)

Adding to the Hamiltonian \( H \) the appropriate function of \( K \), we get the superintegrable oscillator- and Coulomb-like systems.

\[ \text{Oscillator-like Hamiltonian} \]

We define the oscillator-like Hamiltonian by the expression (cf.\([20]\))

\[ H_{\text{osc}} = H + \omega^2 K \]  

(40)
and introduce the following generators

\[ A_\alpha = H_{\alpha N} + \omega H_\alpha, \quad B_\alpha = H_{\alpha N} - \omega H_\alpha : \quad \begin{cases} \{H_{\text{osc}}, A_\alpha\} = -\omega A_\alpha \\ \{H_{\text{osc}}, B_\alpha\} = \omega B_\alpha \end{cases}. \]  \hspace{1cm} (41)

These generators and their complex conjugates form the following algebra

\[ \{A_\alpha, A_\beta\} = -i(H_{\text{osc}} - \omega(g + \sum_{\gamma=1}^{N-1} H_{\gamma N}))\delta_{\alpha\beta} + 2i\omega H_{\alpha\beta}, \]

\[ \{B_\alpha, B_\beta\} = -i(H_{\text{osc}} + \omega(g + \sum_{\gamma=1}^{N-1} H_{\gamma N}))\delta_{\alpha\beta} - 2i\omega H_{\alpha\beta}, \]

\[ \{A_\alpha, B_\beta\} = -i\delta_{\alpha\beta}(H_{\text{osc}} - 2\omega^2 K + i\omega D), \hspace{1cm} (44) \]

with their Poisson brackets with \( H_{\alpha\beta} \) reading

\[ \{A_\alpha, H_{\beta\gamma}\} = -i\delta_{\alpha\gamma} A_\beta, \quad \{B_\alpha, H_{\beta\gamma}\} = -i\delta_{\alpha\gamma} B_\beta \hspace{1cm} (45) \]

Then we immediately deduce that the Hamiltonian \( [40] \) besides \( H_{\alpha\beta} \), has the additional constants of motion which provide the system by the maximal superintegrability property

\[ M_{\alpha\beta} = A_\alpha B_\beta = H_{\alpha N} H_{\beta N} + \omega^2 H_\alpha H_\beta + \omega(H_\alpha H_{\beta N} - H_{\alpha N} H_\beta) = \frac{z^\alpha \bar{z}^\beta}{A^2}(\omega^2 + \omega^2): \quad \{H_{\text{osc}}, M_{\alpha\beta}\} = 0. \hspace{1cm} (46) \]

For sure, these constants of motion are functionally dependent, so that among them one can choose the \( N - 1 \) integrals which guarantee superintegrability of the system, e.g. \( P_\alpha \equiv P_{\alpha\alpha} \) only, like in \[ 3 \]. The generators \[ 40 \] and the \( su(N) \) generators \( H_{\alpha\beta} \) form the following symmetry algebra

\[ \{H_{\alpha\beta}, M_{\gamma\delta}\} = i\delta_{\alpha\gamma} M_{\beta\delta} + i\delta_{\beta\delta} M_{\alpha\gamma}, \quad \{M_{\alpha\beta}, M_{\gamma\delta}\} = 0, \hspace{1cm} (47) \]

**Coulomb-like Hamiltonian**

We define the Coulomb-like Hamiltonian with the additional constants of motion which provide the system by the maximal superintegrability property as follows (cf. \[ 20 \])

\[ H_{\text{Coul}} = H - \frac{\gamma}{\sqrt{2K}}, \quad R_\alpha = H_{\alpha N} + \gamma \frac{H_\alpha}{(g + \sum_{\gamma=1}^{N-1} H_{\gamma N})\sqrt{2K}}: \quad \{H_{\text{Coul}}, R_\alpha\} = \{H_{\text{Coul}}, H_{\alpha\beta}\} = 0. \hspace{1cm} (48) \]

The whole symmetry algebra is as follows

\[ \{R_\alpha, R_\beta\} = -i\delta_{\alpha\beta} \left( H_{\text{Coul}} - \frac{\gamma^2}{2(g + \sum_{\gamma=1}^{N-1} H_{\gamma N})} + \frac{\gamma^2 H_{\alpha\beta}}{2(g + \sum_{\gamma=1}^{N-1} H_{\gamma N})^3} \right): \quad \{H_{\alpha\beta}, R_\gamma\} = i\delta_{\alpha\gamma} R_\beta, \quad \{R_\alpha, R_\beta\} = 0. \hspace{1cm} (49) \]

To clarify the origin of these models it is convenient to transit to the canonical coordinates.

**IV. CANONICAL COORDINATES**

For the introduction of the canonical coordinates we transit from the complex coordinates to the real ones

\[ w = x + iy, \quad z^\alpha = q_\alpha e^{i\varphi_\alpha}, \quad \text{where} \quad y < 0, \quad q_\alpha \geq 0, \quad \varphi_\alpha \in [0, 2\pi), \quad q^2 := \sum_{\alpha=1}^{N-1} q_\alpha^2 < -2y. \hspace{1cm} (50) \]

Then we write down the symplectic/Kähler one-form and identify it with the canonical one

\[ \mathcal{A} = -\frac{g}{2} dw + d\bar{w} - i(\bar{z}^\alpha dz^\alpha - \bar{z}^\alpha dz^\alpha) \equiv p_\alpha dx + \pi_\alpha d\varphi_\alpha. \hspace{1cm} (51) \]
This yields the following expressions for the canonical coordinates and momenta,

\[ p_x = \frac{1}{2y+q^2}, \quad \pi_\alpha = -g \frac{q_\alpha}{2y+q^2} \quad \Leftrightarrow \quad q_\alpha = \sqrt{-\frac{\pi_\alpha}{p_x}}, \quad y = \frac{\pi + g}{2p_x}, \quad \text{with} \quad \pi := \sum_{\alpha=1}^{N-1} \pi_\alpha \quad (52) \]

Thus, the complex coordinates are expressed via canonical ones as follows

\[ w = x + i \frac{\pi + g}{2p_x}, \quad z^\alpha = \sqrt{-\frac{\pi_\alpha}{p_x}} e^{i\varphi_\alpha}. \quad (53) \]

For the complete analogy with one-dimensional case \([5]\) we perform further canonical transformation \((x,p_x) \rightarrow (p_r/r,-r^2/2)\) and re-write the above expression in a more convenient form

\[ w = \frac{p_r}{r} - i \frac{\pi + g}{r^2}, \quad z^\alpha = \sqrt{2\pi_\alpha} e^{i\varphi_\alpha}, \quad \text{with} \quad r > 0, \quad \pi_\alpha \geq 0, \quad \varphi_\alpha \in [0,2\pi). \quad (54) \]

and

\[ A = \frac{i(\bar{w} - w) - z^\gamma \bar{z}^\gamma}{g} = \frac{2}{r^2}. \quad (55) \]

In these terms the generators of conformal algebra \([31]\) take the form of conformal mechanics with separated ”radial” and ”angular” parts (cf. \([11]\)),

\[ H = \frac{p_r^2}{2} + \frac{I}{r^2}, \quad K = \frac{r_2}{2}, \quad D = p_r r, \quad (56) \]

where the angular part of Hamiltonian is given by the expression

\[ I = \frac{1}{2} \left( \sum_{\alpha=1}^{N-1} \pi_\alpha + g \right)^2. \quad (57) \]

The rest generators of \(su(1,N)\) algebra read

\[ H_{\alpha N} = \sqrt{2\pi_\alpha} \left( \frac{p_r}{2} - i \frac{\pi + g}{2r} \right) e^{-i\varphi_\alpha}, \quad H_\alpha = r \sqrt{\pi_\alpha} e^{-i\varphi_\alpha}, \quad H_{\alpha\beta} = \sqrt{\pi_\alpha \pi_\beta} e^{-i(\varphi_\alpha - \varphi_\beta)}, \quad (58) \]

with the basic Poisson brackets \(\{r,p\} = 1\) and \(\{\varphi_\alpha, \pi_\alpha\} = 1\).

In these coordinates the oscillator- and Coulomb-like Hamiltonians \([40,48]\) take the form,

\[ H_{\text{osc}} = \frac{p_r^2}{2} + \frac{I}{r^2} + \frac{\omega^2 r^2}{2}, \quad H_{\text{Coul}} = \frac{p_r^2}{2} + \frac{I}{r^2} - \frac{\gamma}{r}. \quad (59) \]

with \(I\) given by \([57]\).

The generic conformal mechanics with the angular part \(I_{\text{gen}}(\pi, \varphi)\) can be defined via \(su(1,N)\) generators by the expression

\[ H_{\text{gen}} = H + \frac{I_{\text{gen}}(\sqrt{K} H_\alpha, \sqrt{K} H_\beta) - (\sum_{\gamma=1}^{N-1} H_{\gamma\gamma} + g)^2}{2K}. \quad (60) \]

However, we are mostly interested in the study of integrable and superintegrable systems. Thus, we have to restrict ourselves by the particular cases of angular Hamiltonians.

**Superintegrable Systems**

In accordance with Liouville theorem, the integrability of the system with \(2N\)-dimensional phase space means the existence \(N\) functionally independent involutive integrals \(F_1 = H, \ldots, F_N : \{F_a, F_b\} = 0\). This yields the existence of the so-called action-angle variables \((I_a(F), \Phi_a)\):

\[ H = H(I), \quad \{I_a, \Phi_b\} = \delta_{ab}, \quad \{I_a, I_b\} = \{\Phi_a, \Phi_b\} = 0, \quad \Phi_a \in [0,2\pi), \quad a, b = 1, \ldots, N. \quad (61) \]
The system becomes maximally superintegrable when the Hamiltonian is expressed via action variables as follows

\[ H = H \left( \sum_{a=1}^{N} n_a I_a \right), \quad n_a \in \mathcal{N} \]  

(62)

where \( n_a \) are integers (or rational numbers). Indeed, in that case the system possesses the additional (non-involutive) integrals \( I_{ab} = \cos(n_a \Phi_b - n_b \Phi_a) \), among them \( N - 1 \) integrals are functionally independent.

Now, let us suppose that \( \pi_\alpha, \varphi_\alpha \) are related with the action-angle variables \((I_\alpha, \Phi_\alpha)\) of some \((N - 1)\)-dimensional angular mechanics by the relations

\[ \pi_\alpha = n_\alpha I_\alpha, \quad \varphi_\alpha = \frac{\Phi_\alpha}{n_\alpha}, \quad \text{where} \quad n_\alpha \in \mathcal{N}. \]

(63)

Upon this identification the angular Hamiltonian \((57)\) takes a form

\[ \mathcal{I} = \frac{1}{2} \left( \sum_{\alpha=1}^{N-1} n_\alpha I_\alpha + g \right)^2, \quad \text{with} \quad n_\alpha \in \mathcal{N}, \]

(64)

This is precisely the class of angular Hamiltonians which provides the superintegrable generalizations of the conformal mechanics, and of the oscillator and Coulomb systems on the \( N \)-dimensional Euclidean spaces \([12]\).

Though the relations \((31)-(38)\) hold upon this identification, the generators \( H_\alpha, H_{\alpha\beta}, H_{\alpha\bar{\beta}} \) become locally defined, \( \varphi_\alpha \in [0, 2\pi/n_\alpha] \), so they fail to be constants of motion. However, taking their relevant powers we get the globally defined generators which form the nonlinear algebra

\[ \tilde{H}_\alpha := (H_\alpha)^{n_\alpha} = d_\alpha(I)r^{n_\alpha}e^{-r\Phi_\alpha}, \]

(65)

\[ \tilde{H}_{\alpha\bar{\beta}} := (H_{\alpha\bar{\beta}})^{n_\alpha n_\bar{\beta}} = d_{\alpha\bar{\beta}}(I)r^{n_\alpha n_\bar{\beta}}e^{-r\Phi_\alpha - n_\alpha \Phi_\beta}, \]

(67)

where

\[ d_\alpha(I) = \left( \frac{n_\alpha I_\alpha}{2} \right)^{n_\alpha/2}, \quad d_{\alpha\beta}(I) = \left( \frac{n_\alpha I_\alpha}{2} \right)^{n_\beta/2}, \quad d_{\alpha\bar{\beta}}(I) = (n_\alpha n_\beta I_\alpha I_{\bar{\beta}})^{n_\alpha n_\bar{\beta}/2}. \]

(68)

Thus, we get

\[ \{H, \tilde{H}_{\alpha\bar{\beta}}\} = \{H, \tilde{H}_{\alpha\bar{\beta}}\} = 0, \quad \{K, \tilde{H}_\alpha\} = \{K, \tilde{H}_{\alpha\beta}\} = 0, \]

(69)

where \( H, K \) are defined by \((56)\) and \((64)\). For sure, the functions \((68)\), being dependent on action variables only, do not affect the commutativity of the additional integrals with the Hamiltonian.

In a similar way we construct the constant of motion of the oscillator- and Coulomb-like systems given by \((40), (64)\) and \((48), (64)\), respectively.

For the oscillator-like system \((40)\) the integrals take the form

\[ \tilde{M}_{\alpha\beta} := (A_\alpha B_{\beta})^{n_\alpha n_\beta} = \frac{1}{2} d_{\alpha\beta}(I) e^{-r(n_\beta \Phi_\alpha - n_\alpha \Phi_\beta)} \left( r_p + \sum_{\gamma=1}^{N-1} \frac{n_\gamma I_\gamma + g}{r} \right)^2, \]

(70)

with \( A_\alpha, B_\beta \) given by \((41), (63)\).

For the Coulomb-like system \((48)\) the integrals take the form

\[ \tilde{R}_\alpha = (R_\alpha)^{n_\alpha} = d_\alpha(I) e^{-r\Phi_\alpha} \left( r_p + \sum_{\gamma=1}^{N-1} \frac{r_\gamma}{n_\gamma I_\gamma + g} \right)^{n_\alpha}, \]

(71)

There are a few interesting simple systems whose angular parts are given by \((64)\) with \( g \neq 0 \), among them are,
• "charge-monopole" system (and respective systems with oscillator/Coulomb potentials),

\[ \mathcal{H} = \sum_{a=1}^{3} \frac{p_a^2}{2} + \frac{s^2}{2x^2}, \quad \{p_a, x^b\} = \delta_{ab}, \quad \{p_a, p_b\} = \frac{\varepsilon_{abc}x^c}{y^2}, \quad \{x^a, x^b\} = 0. \]  

(72)

Its angular part is defined by the (two-dimensional) spherical Landau problem, with the following Hamiltonian (see, e.g. [13], where one can find the expressions for action-angle variables for the angular part)

\[ I = \frac{1}{2} \left( I_1 + I_2 + |s| \right)^2, \quad I_{1,2} \in [0, \infty), \]

(73)

with \( s \) being the monopole number.

• Smorodinsky-Winternitz system

\[ H_{SW} = \sum_{a=1}^{N} \left( \frac{p_a^2}{2} + \frac{g_a^2}{2x_a^2} + \frac{\omega^2 x_a^2}{2} \right). \]

(74)

The angular Hamiltonian of this system is given by the expression (64) with (see, e.g. [16])

\[ k_\alpha = 2\omega, \quad g = \sum_{a=1}^{N-1} |g_a|. \]

(75)

For sure, this system could be viewed as a trivial case of rational Calogero model, which also belongs to the class of systems above.

• Rational Calogero model associated with Coxeter root system \( \mathcal{R} \subset \mathbb{R}^N \),

\[ \mathcal{H}_{Cal} = \sum_{a=1}^{N} \frac{p_a^2}{2} + \sum_{\alpha \in \mathcal{R}^+} \frac{g_{\alpha}^2 (\alpha \cdot \alpha)}{2(\alpha \cdot \alpha)^2}, \quad \{p_a, x_b\} = \delta_{ab} \]

(76)

where \( g_{\alpha} \geq 0 \) is a Weyl-invariant multiplicity function on the set of roots \( \mathcal{R}^+ \).

The spectrum of the angular part of quantum rational Calogero model was found in [19]. Taking its classical limit, one can get the expression of the angular (part of) generalized rational Calogero model in terms of action variables \( \{H_{\alpha N}, H_{\alpha \beta}\} = \{H, H_{\alpha \beta}\} = 0 \). The first condition leads to the vanishing of Kähler structure and Poisson brackets, while the absorption of constant \( g \) by the action variables immediately yields the change of the range of validity of the action variables. However, a minor complication allows to involve in our picture the generic superintegrable conformal mechanics, oscillator and Coulomb systems as well.

Using the expressions of the constants of motion presented in [2], we can immediately write down the constants of motions of those systems written in terms of Killing potentials.

• Conformal mechanics

\[ \mathcal{H} = H - \frac{g(g + 2 \sum_{\gamma=1}^{N-1} H_{\gamma \gamma})}{4K}, \quad \mathcal{H}_{\alpha N} := H_{\alpha N} + \frac{g \varepsilon^\alpha}{2} = H_{\alpha N} + ig \frac{H_{\alpha}}{2K} : \{\mathcal{H}, H_{\alpha N}\} = \{\mathcal{H}, H_{\alpha \beta}\} = 0. \]

(77)

• Oscillator-like system

\[ \mathcal{H}_{osc} = H_{osc} - \frac{g(g + 2 \sum_{\gamma=1}^{N-1} H_{\gamma \gamma})}{4K}, \quad \{\mathcal{H}_{osc}, H_{\alpha \beta}\} = \{\mathcal{H}_{osc}, A_{\alpha} B_{\beta}\} = 0 \]

(78)

where

\[ A_{\alpha} = H_{\alpha N} + ig \frac{H_{\alpha}}{2K} + i\omega H_{\alpha}, \quad B_{\alpha} = H_{\alpha N} + ig \frac{H_{\alpha}}{2K} - i\omega H_{\alpha}. \]

(79)
• Coulomb-like system

\[ \mathcal{H}_{\text{Coul}} = \mathcal{H}_{\text{Coul}} - \frac{g(g + 2 \sum_{\gamma=1}^{N} H_{\gamma \bar{\gamma}})}{4K}, \quad \{\mathcal{H}_{\text{Coul}}, R_{\alpha}\} = 0 \]  

(80)

where

\[ R_{\alpha} = R_{\alpha} + ig \frac{H_{\alpha}}{\sqrt{2K}} \left( \frac{1}{\sqrt{2K}} + \frac{\gamma}{(g + \sum_{\bar{\gamma}=1}^{N} H_{\gamma \bar{\gamma}}) \sum_{\gamma=1}^{N} H_{\gamma \bar{\gamma}}} \right) \]  

(81)

The transition to the action-angle variables is obvious.

Hence, we have shown how to describe the superintegrable deformations of oscillator and Coulomb systems in terms of noncompact complex projective spaces \( \tilde{\mathbb{C}P}^N \).

V. CONCLUDING REMARKS

In this paper we have shown that the superintegrable generalizations of conformal mechanics, oscillator and Coulomb systems can be naturally described in terms of the noncompact complex projective space considered as a phase space. This observation yields some interesting directions for further studies, among them

• the construction of the \( \mathcal{N} = 2k \) superconformal mechanics associated with \( su(1,N|k) \) superalgebra. For this purpose one should consider phase superspace equipped with the Kähler structure with the potential

\[ K = -g \log(i(w - \bar{w}) - z^\alpha \bar{z}^\alpha - \eta_A \bar{\eta}_A), \quad A = 1, \ldots, k, \]  

(82)

where \( \eta_A \) are Grassmann variables. This should be the direct generalization of the one-dimensional system considered in [5]. We expect that it will be possible to construct, in a similar way, the \( \mathcal{N} = 2k \) supersymmetric extensions of the considered oscillator- and (repulsive) Coulomb-like systems as well, in particular, the superextension of Smorodinsky-Winternitz system.

• Performing the transformation to the higher-dimensional Poincaré model via [21], we expect to present the considered models in the Ruijsenaars-Schneider-like form and in this way to find, some superintegrable cases of the Ruijsenaars-Schneider systems, as well as their supersymmetric/superconformal extensions.

• describing the superintegrable deformations of the free particle on the spheres/hyperboloids, and the spherical/hyperbolic oscillators, in a similar way. For this purpose we expect to consider the "c-deformation" of the Kähler structure of the Klein model, in the spirit of the so-called "c-deformation approach" developed in [8].

• constructing spin-extensions of the above models, choosing the noncompact analogs of complex Grassmanians as phase spaces.

We plan to address these problems soon.

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