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Novel generalized Fourier representations and phase transforms

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ABSTRACT

The Fourier representations (FRs) are indispensable mathematical formulations for modeling and analysis of physical phenomena and engineering systems. This study presents a new set of generalized Fourier representations (GFRs) and phase transforms (PTs). The PTs are special cases of the GFRs and true generalizations of the Hilbert transforms. In particular, the Fourier transform based kernel of the PT is derived and its various properties are discussed. The time derivative and integral, including fractional order, of a signal are obtained using the GFR. It is demonstrated that the general class of time-invariant and time-variant filtering operations, analog and digital modulations can be obtained from the proposed GFR. A narrowband Fourier representation for the time-frequency analysis of a signal is also presented using the GFR. A discrete cosine transform based implementation, to avoid end artifacts due to discontinuities present in the both ends of a signal, is proposed. A fractional-delay in a discrete-time signal using the FR is introduced. The fast Fourier transform implementation of all the proposed representations is developed. Moreover, using the analytic wavelet transform, a wavelet phase transform (WPT) is proposed to obtain a desired phase-shift in a signal under-analysis. A wavelet quadrature transform (WQT) is also presented which is a special case of the WPT with a phase-shift of $\pi/2$ radians. Thus, a wavelet analytic signal representation is derived from the WQT. Theoretical analysis and numerical experiments are conducted to evaluate effectiveness of the proposed methods.

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1. Introduction

The Fourier representation (FR) of a signal is the most important mathematical formulation for modeling and analysis of physical phenomena, engineering systems and signals in numerous applications. It has been used to obtain solution of problems in almost all fields of mathematics, science, engineering and technology. It is the fundamental of signal processing, analysis, information extraction and interpretation. There are many variants of the FR such as continuous-time Fourier series (FS), Fourier transform (FT), Fourier sine transform (FST) and Fourier cosine transform (FCT), discrete-time FT (DTFT), discrete-time Fourier series (DTFS), discrete FT (DFT), discrete sine transform (DST) and discrete cosine transform (DCT) [1,2]. All these are the orthogonal transforms which can be efficiently computed using the Cooley–Tukey fast FT (FFT) algorithm [3]. Recently, many studies [4–6] have been performed using the Fourier theory, and many applications including signal decomposition and time-frequency analysis of a nonlinear and nonstationary time-series have been proposed.

The DCT was proposed in the seminal paper [1] for image processing based pattern recognition and Wiener filtering. The modified DCT (MDCT) [7] is based on the DCT of overlapping data which uses the concept of time-domain aliasing cancellation [8]. The DCT and MDCT are widely-used due to decorrelation and energy compaction properties in many applications like image (e.g., JPEG), video (e.g., Motion JPEG, MPEG, Daala, digital video, Theora) and audio (e.g., MP3, WMA, AC-3, AAC, Vorbis, ATRAC) compression, electrocardiogram data analysis [11], and for numerical solution of partial differential equations by spectral methods. There are eight types of DCTs and eight types of DSTs depending upon the symmetry about a data point and the boundary conditions.

The Hilbert transform (HT) [75–79] is an inevitable tool which has been studied and used in numerous applications such as quadrature amplitude modulation, analytic signal representation, time-frequency analysis, signal processing and system identification, signal and speech demodulation, and image processing [53–62]. The Fourier theory based quadrature method was proposed by Gabor [10] in 1946 as a practical approach for obtaining the HT and Gabor analytic signal (GAS) representation of a signal. The GAS has been extensively used in communication engineering, physics, time-frequency-energy (TFE) representation, and signal analysis. The TFE representation of a signal is obtained using the concept of instantaneous frequency (IF) [12–27,46] which is an...
important parameter in many applications. Recently, using eight
types of DCTs and eight types of DSTs, sixteen types of Fourier
quadrature transforms (FQTs) and corresponding Fourier-Singh
analytic signal (FSAS) representations are introduced for a nonlinear
and non-stationary time-series analysis [4]. The FQTs and FSAS
representations are alternatives to the HT and GAS representation,
respectively. The HT and FQTs are $\pi/2$ phase shifters. However,
there is no general method to provide a desired phase shift to
signal under analysis. This study presents a phase transform (PT),
which is based on the proposed generalized Fourier representation
(GFR) of a signal, to obtain the desired phase shift and time-delay.
This work also discusses the various special cases of the GFR,
namely the Fourier representation, PT, time-delay including frac-
tional delay of discrete-time signals, time derivative and integral
including fractional order, amplitude modulation (AM), frequency
modulation (FM) and other digital modulation schemes.

There are numerous applications of wavelet transform (WT)
[31–34] which uses a wavelet function to analyze the signals,
and when analyzing signal is analytic then corresponding WT is
known as the analytic wavelet transform (AWT) [36–38]. An ana-
lytic signal representation of a real-valued and finite energy signal
using the AWT is presented in [40–42], where authors obtained
the analytic wavelet function (AWF) with its real part being an even function (Theorem 2 of [40]) and thus the imaginary part of the AWF is an odd function. The advantages of the AWT in both precision and antinoise perfor-
ance are also demonstrated. This study eliminates the restriction
(that the real part of an AWF has to be an even function to ob-
tain the AWT) and proposes two representations of the wavelet
quadrature transform (WQ) and corresponding AWT using any
AWF. This work also proposes two representations of the wavelet
phase transform (WPT) and shows that the proposed WQT is a
special case of the WPT where phase-shift is $\pi/2$ radians.

There are many generalizations of the FT such as Laplace trans-
form, Gabor transform [80], fractional FT (FrFT) [81,82], acoustic
scattering and Schrödinger’s equation based FT [83], short-time FT
(STFT), wavelet transform [84], quaternion FT [85], chirplet trans-
form [86], S transform [87], and de Branges theory based FT [88].
Most of these transforms are based on the modification in the
forward FT integral (analysis equation) and the original signal
is recovered from the corresponding inverse transforms. On the other
hand, the proposed GFRs are based on the inverse transform integ-
als (synthesis equations), e.g., the inverse FT and inverse wavelet
transform.

The main contributions of this study are summarized as follows:

1. Introduction of the GFRs which are completely based on the
Fourier representations of a signal.
2. Introduction of the phase transforms (PTs) using Fourier repre-
sentations (i.e., FS, FT, DFT, FCT, FST, DSTs and DCTs) which
can be functions of both the frequency and time (hereafter, unless
and until stated, PT means constant or frequency-and-time in-
dependent PT). The proposed PT is a special case of the GFR,
and a true generalization of the FT. The desired phase-shifts
and time-delays can be introduced to a signal under analysis
using the PT. In particular, the Fourier transform based
kernel of the PT is derived and its various properties are discussed.
It is shown that the HT is a special case of the PT when phase-
shift is $\pi/2$ radians. An extension of the one-denominational
PT for the two-dimensional image signals is also provided in
Appendix C, which can be easily further extended for multi-
dimensional signals.
3. The time derivative and time integral, including fractional or-
er, of a signal can be obtained using the GFR. The DCT based
implementation is presented to avoid end artifacts due to dis-
continuities present at the both ends of a signal.
4. Introduction of the fractional delay in a discrete-time signal
using the Fourier representation.
5. It is demonstrated that the zero-phase, linear and nonlinear
phase filters such as low-pass, high-pass, band-pass, and
band-stop or band-reject filters, which can be time-invariant
or time-varying, are special cases of the proposed GFR.
6. Contrary to the perception in the literature [59], it is demon-
strated that the continuous-time both aperiodic (19) and peri-
odic (94) Hilbert kernels possess zero rather than pole (singu-
larity/infinitly) at the origin.
7. Using the proposed GFR, the narrowband Fourier represen-
tation (NBFR) is obtained for the time-frequency representation
and analysis of a signal.
8. The FFT implementations of all the above proposed represen-
tations are developed.
9. Using the AWT, the WPT is introduced to obtain a desired
phase-shift in a signal under-analysis, and two representations
of the WQT are derived. Thus, the WAS representation is ob-
tained using the WQT which is a special case of the WPT
where phase-shift is $\pi/2$ radians.

Moreover, using the PT, It is observed that (i) a constant phase shift
(e.g., HT as $\pi/2$ phase shift) in a signal corresponds to variable
time-delays in all the harmonics, (ii) to obtain a constant time-
delay in a signal, one needs to provide variable phase-shifts in all
the harmonics, (iii) a constant phase-shift is same as the constant
time-delay only for a single frequency sinusoid.

All the acronyms and symbols used in this work are summa-
ized in Appendix A. Rest of the study is organized as follows: The
GFR and its various special cases are presented using Fourier series
in Section 2. The PT using FT is presented in Section 3.1. The PT
using Fourier sine and cosine transforms is presented in Section 3.2.
The WPT and WQF using AWT are presented in Section 3.3. The
implementations of the GFR using the DFT and DCT are presented
in Section 3.4 and Section 3.5, respectively. Simulation results
and discussions are presented in Section 4. Section 5 presents conclu-
sion and future scope of the study.

2. The generalized Fourier representation

This section proposes the GFR, presents its various special cases,
and provides convergence of the GFR.

2.1. The GFR using Fourier series representation

Let $x_t(t)$ be a real valued periodic signal (i.e., $x_t(t + T) =
x_t(t), \forall t$) which follows the Dirichlet conditions. The Fourier series
expansion of $x_t(t)$ is given by

$$x_t(t) = a_0 + \sum_{k=1}^{\infty} [a_k \cos(k \omega_0 t) + b_k \sin(k \omega_0 t)],$$

where

$$\omega_0 = \frac{2\pi}{T} = 2\pi f_0 \text{ rad/s},$$

with

$$a_0 = \frac{1}{T} \int_{t_1}^{t_1+T} x_t(t) \, dt; \quad a_k = \frac{2}{T} \int_{t_1}^{t_1+T} x_t(t) \cos(k \omega_0 t) \, dt$$

and

$$b_k = \frac{2}{T} \int_{t_1}^{t_1+T} x_t(t) \sin(k \omega_0 t) \, dt \text{ for } k = 1, 2, \ldots, \infty.$$ Using $a_k = X_k \cos(\phi_k), b_k = -X_k \sin(\phi_k)$, where $X_k = \sqrt{a_k^2 + b_k^2}$, $|X_k| = \sqrt{a_k^2 + b_k^2}$ and $\phi_k = \tan^{-1}(-b_k/a_k)$, i.e., $|X_k|e^{j\phi_k} = a_k - jb_k$. $X_0 = a_0$, one can write

$$x_t(t) = a_0 + \sum_{k=1}^{\infty} |X_k| \cos(k \omega_0 t + \phi_k).$$

Using the Fourier series representation (2), the GFR is hereby proposed as

$$x_I(t, \tau_k(t), \alpha_k(t)) = a_0 H_0(t) \cos(\alpha_0(t))$$

$$+ \sum_{k=1}^{\infty} H_k(t) |X_k| \cos(k \omega_0 t + \phi_k - \alpha_k(t)), \quad (3)$$

where $0 \leq H_k(t) \leq M < \infty$ and $0 \leq \alpha_k(t) < 2\pi$ (for $k = 0, 1, 2, \ldots, \infty$) are introduced as amplitude and phase scaling/modulating functions of both frequency ($k$) and time ($t$). Now, the various cases of the GFR are presented as follows:

**Case 1:** The GFR (3) is the Fourier series representation of a signal when $H_k(t) = 1$ and $\alpha_k(t) = 0$, for all $t$ and $k = 0, 1, 2, \ldots, \infty$.

**Case 2:** Using the GFR (3) with $H_k(t) = 1$, the four types of phase transforms are hereby proposed as follows:

(i) Frequency- and time-dependent (FDTD) PT as

$$x_I(t, \alpha_k(t)) = a_0 \cos(\alpha_0(t)) + \sum_{k=1}^{\infty} |X_k| \cos(k \omega_0 t + \phi_k - \alpha_k(t)), \quad (4)$$

where introduced phase, $\alpha_k(t)$, is a function of time and frequency.

(ii) Frequency-dependent and time-independent (FDTD) PT as

$$x_I(t, \alpha(t)) = a_0 \cos(\alpha(t)) + \sum_{k=1}^{\infty} |X_k| \cos(k \omega_0 t + \phi_k - \alpha(t)), \quad (5)$$

which can be written as

$$x_I(t, \alpha(t)) = x_I(t, \cos(\alpha(t))) + x_I(t, \pi/2 \sin(\alpha(t))), \quad (6)$$

where $x_I(t, \pi/2)$ is the HT, a special case of the PT (8) with constant phase shift of $\alpha = \pi/2$ radians (or 90°), which is defined as

$$x_I(t, \pi/2) = \sum_{k=1}^{\infty} |X_k| \sin(k \omega_0 t + \phi_k). \quad (7)$$

where $x_I(t, \alpha(t))$ is the HT, a special case of the PT (8) with constant phase shift of $\alpha = \pi/2$ radians (or 90°), which is defined as

Using (2) and (9), one can write the analytic signal (AS) representation as

$$z_I(t) = x_I(t) + j \dot{x}_I(t)$$

$$= a_0 + \sum_{k=1}^{\infty} X_k \left[ \cos(k \omega_0 t + \phi_k) + j \sin(k \omega_0 t + \phi_k) \right]. \quad (10)$$

The kernel of the proposed PT for the periodic signals, $\delta_I(t, \alpha) = \cos(\alpha) \delta_I(t) + \sin(\alpha) \delta_I(t, \pi/2)$, is derived in Example 1. One can easily obtain an arbitrary constant PT of a periodic signal using the periodic convolution with this kernel as, $x_I(t, \alpha) = x_I(t) * \delta_I(t, \alpha) = \cos(\alpha) x_I(t) + \sin(\alpha) x_I(t, \pi/2)$.  

**Case 3:** A time-delay of the periodic signal $x_I(t)$ can be defined as

$$x_I(t - t_k) = a_0 + \sum_{k=1}^{\infty} |X_k| \cos(k \omega_0 t + \phi_k - k \omega_0 t_k). \quad (11)$$

From (7), (9) and (11), we observe that (a) the HT is a constant phase shifter which introduces variable time-delays in all the harmonics, i.e., $t_k = t_0 / (2k \omega_0)$; (b) to obtain a constant time-delay (say, $t_k = t_0$) in a signal, one need to provide variable phase shifts in all the harmonics, i.e., $\alpha_k = k \omega_0 t_0$; (c) a constant phase shift is same as constant time-delay only for a single frequency sinusoid (say $k = 1$ and hence $\alpha_k = \omega_0 t_0$); (d) variable phase shifts are same as variable time-delays only for a zero mean ($\alpha_0 = 0$) signal with $\alpha_k = \omega_0 t_0, \forall k$. It is interesting to observe that the phase shift in a constant signal (7) is presented as, $\alpha_0 \cos(\alpha_0)$, which is valid because (i) for the HT, it is zero, (ii) for the phase shift of $\pi$, it is multiplied by minus one, and (iii) there is no change in its value, if phase shift is zero. However, time-delay operation in a constant signal [e.g., $a_0$ in (11)] does not change its value.

**Case 4:** It is observed that the $\mu$-th order time derivative of a periodic signal is a special case of the GFR (3) when $H_0(t) = H_\mu(t) = 1$ for $0 < \mu < 1$, $H_\mu(t) = (k \omega_0)^\mu$, $\alpha_0(t) = 0$ and $\alpha_k(t) = -\mu \pi/2, \forall k, t$. Thus, from (3), we obtain the $\mu$-th order fractional time derivative of the signal $x_I(t)$ as

$$D^\mu [x_I(t)] = \frac{\alpha_0 t^{-\mu}}{\Gamma(1 - \mu)}$$

$$+ \sum_{k=1}^{\infty} (k \omega_0)^{\mu} |X_k| \cos(k \omega_0 t + \phi_k + \mu \pi/2), \quad \mu \geq 0,$$

where $\Gamma(1 - \mu)$ is a gamma function.

**Case 5:** It is observed that the $\nu$-th order time integral of a periodic signal is a special case of the GFR (3) when $H_0(t) = H^-\nu(t) = 1$ for $0 < \nu < 1$, $H^-\nu(t) = (k \omega_0)^{-\nu}$, $\alpha_0(t) = 0$ and $\alpha_k(t) = \nu \pi/2, \forall k, t$. Thus, from (3), we obtain the $\nu$-th order fractional time integral of the signal $x_I(t)$ as

$$D^-\nu [x_I(t)]$$

$$= \frac{\alpha_0 t^{\nu}}{\Gamma(1 + \nu)}$$

$$+ \sum_{k=1}^{\infty} (k \omega_0)^{-\nu} |X_k| \cos(k \omega_0 t + \phi_k - \nu \pi/2), \quad \nu \geq 0.$$

**Case 6:** Using the proposed GFR (3), we can obtain an amplitude modulated (AM) signal with an arbitrary but fixed value of $k$ (say $k = 1$), carrier frequency $\omega_c = \omega_0$, $\alpha_1(t) = 0$, $H_0(t) = 0$, and $H_1(t) = (A_m + m(t)) \geq 0$, where $m(t)$ is a message signal whose maximum frequency $\omega_m < \omega_c$. From (3), one can also obtain an angle modulated (which includes both the frequency and phase modulation) signal with a fixed value of $k$ (say $k = 1$, carrier frequency $\omega_c = \omega_0$, $H_0(t) = 0$, $H_1(t) = 1$, and $\alpha_1(t) = m(t)$).

Similarly, it can be shown that the digital modulation schemes such as phase-shift keying, frequency-shift keying, amplitude-shift keying and quadrature amplitude modulation are special cases of the GFR (3).

**Case 7:** The GFR (3) represents different types of filtering operations to the signal $x_I(t)$, e.g., the zero-phase filtering operations: (i) low-pass filtering if $H_0(t) = 1$; $H_k(t) = 1$, for $1 \leq k < K$ and $H_k(t) = 0$, for $k > K$; (ii) high-pass filtering if $H_0(t) = 0$; $H_k(t) = 0$, for $< K$; $H_k(t) = 1$, for $k > K$; (iii) band-pass filtering if $H_0(t) = 0$; $H_k(t) = 1$, for $K_1 \leq k < K_2$ and $H_k(t) = 0$, otherwise; (iv) band-stop filtering if $H_0(t) = 1$; $H_k(t) = 0$, for $K_1 \leq k < K_2$ and $H_k(t) = 1$, otherwise;
\( \alpha_k(t) = 0 \). Other than these zero-phase filtering, there are countless possibilities of linear and nonlinear phase filtering operations by properly selecting the \( H_k(t) \) and \( \alpha_k(t) \) based on the applications and requirements.

Moreover, the class of time-varying filters are also possible when \( H_k(t) \) or \( \alpha_k(t) \), or both are functions of time and frequency. For example, the Fourier decomposition method (FDM) proposed in the seminal work [5] maps the FR (2) of possibly infinite constant amplitude-and-frequency components into a set of finite number of amplitude-and-frequency modulated (AM-FM) Fourier intrinsic band functions (FIBFs) as, \( x(T) = a_0 + \sum_{n=1}^{N} \left[ H_n(t) x_n(t) \cos(\phi_n(t)) \right] \), which is a special case of the proposed GFR (3) and can be used to obtain the time-varying filtering (TVF) as

\[
y_T(t) = a_0 + \sum_{m=1}^{M} [H_m(t) x_m(t) \cos(\phi_m(t))].
\]  

(14)

This kind of TVF can be used in many applications such as effective noise removal from the gravitational waves which is presented in the Example 2.

2.2. Convergence of the GFR

We are, generally, interested in the following three types of convergence of the sequence of partial sums

\[
s_{TN}(t) = \sum_{k=0}^{N} X_k(t) \cos(\omega_k t) + \phi_k(t),
\]  

(15)

(i) \( s_{TN} \rightarrow x_T \) pointwise on a set \( \{t_1, t_1 + T\} \) if \( s_{TN}(t) \rightarrow x_T(t) \) as \( N \rightarrow \infty \) for all \( t \in \{t_1, t_1 + T\} \); (ii) \( s_{TN} \rightarrow x_T \) uniformly on a set \( \{t_1, t_1 + T\} \) if \( \sup|s_{TN}(t) - x_T(t)| \rightarrow 0 \) as \( N \rightarrow \infty \) for \( t \in \{t_1, t_1 + T\} \); and (iii) \( s_{TN} \rightarrow x_T \) in 2 norm on a set \( \{t_1, t_1 + T\} \) if \( \int_{t_1}^{t_1+T} (s_{TN}(t) - x_T(t))^2 \, dt \rightarrow 0 \) as \( N \rightarrow \infty \) for \( t \in \{t_1, t_1 + T\} \). All these convergences are well defined in the literature [63–67] and depend on the type of convergence and continuity of the function \( x_T(t) \) which can be decomposed in terms of sine and cosine basis functions using the Fourier representation (2). The Fourier series convergence theorems are summarized as follows:

**Theorem 1.** (Dirichlet 1824) Let \( x_T \) be a T-periodic, piecewise continuous function with piecewise-continuous first-derivative. Then \( s_{TN} \rightarrow x_T \) pointwise on \( \mathbb{R} \) as \( N \rightarrow \infty \), where \( x_T(t) = [x_T(t+) + x_T(t-)]/2 \).

**Theorem 2.** Let \( x_T \) be a T-periodic, continuous function with piecewise-continuous first-derivative. Then \( s_{TN} \rightarrow x_T \) uniformly on \( \mathbb{R} \) as \( N \rightarrow \infty \).

**Theorem 3.** Let \( x_T \) be a T-periodic, piecewise continuous function with piecewise-continuous first-derivative. Then \( s_{TN} \rightarrow x_T \) in \( L^2 \) norm as \( N \rightarrow \infty \).

The convergence of the proposed GFR (3) can be considered similar to the FR (2), provided the phase and amplitude scaling functions are properly selected, as follows: (i) If the FR (2) converges uniformly \( i.e., \sum_{k=-N}^{N} |X_k| < \infty \), as \( \sum_{k=-N}^{N} |H_k(t)| |X_k| \cos(\omega_k t) + \phi_k(t) \rightarrow \sum_{k=-N}^{N} |H_k(t)| |X_k| \), then for the uniform convergence of (3), \( \sum_{k=-N}^{N} |H_k(t)| |X_k| \) must be finite for each \( t \), i.e., \( \sum_{k=-N}^{N} H_k(t) X_k \) \( \leq M \sum_{k=-N}^{N} |X_k| \) \( \leq \alpha(t) \); (ii) If the FR (2) converges in \( L^2 \) norm \( i.e., \sum_{k=-N}^{N} |X_k| \rightarrow \infty \), then for the \( L^2 \) norm convergence of (3), \( \sum_{k=-N}^{N} |H_k(t)| |X_k|^2 \) \( \leq M^2 \sum_{k=-N}^{N} |X_k|^2 \) \( \leq \alpha(t) \); and (iii) If the FR (2) converges pointwise, then the GFR (3) would converge pointwise if \( H_k(t) \) and \( \alpha_k(t) \) are bounded, T-periodic, continuous \( i.e., \) piecewise continuous functions (because the uniform convergence implies both the pointwise and \( L^2 \) convergences). Therefore, for all the three cases, as long as \( H_k(t) \) and \( \alpha_k(t) \) are bounded, T-periodic, continuous \( i.e., \) piecewise continuous first-derivatives, the proposed GFR (3) would converge to a new desired function.

3. Phase transforms using the FT, FCT and FST

This section presents the GFR and PT to achieve a desired phase-shift in a signal using the FT, FCT, and FST. Moreover, the FFT implementations of the various cases of the proposed GFR are also presented.

3.1. PT using the Fourier transform

The FT and inverse FT (IFT) pairs of a signal, \( x(t) \) which satisfies the Dirichlet conditions, are defined as

\[
X(\omega) = \int_{-\infty}^{\infty} x(t) \exp(-j\omega t) \, dt, \quad -\infty < \omega < \infty,
\]

(16)

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \exp(j\omega t) \, d\omega, \quad -\infty < t < \infty,
\]

subject to the existence of the integrals, and these pairs can be denoted by \( x(t) = X(f) \), where \( \omega = 2\pi f \). From the definitions of FT and IFT (16), one can observe that \( X(0) = \int_{-\infty}^{\infty} x(t) \, dt = 0 \), and \( x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \, d\omega = 0 \), provided that \( x(t) \) and \( X(\omega) \) are zero-mean functions, respectively. Moreover, an odd function is a zero-mean function, however, reverse is not always true.

First, we consider subtle details of the zero-mean function, \( x_1(t) = \exp(-a|t|) \operatorname{sgn}(t) \), \( a > 0 \), where sign function is defined [9] as

\[
\operatorname{sgn}(t) = \begin{cases} 
1, & t > 0 \\
0, & t = 0 \\
-1, & t < 0 
\end{cases}
\]

(17)

The FT of \( x_1(t) \) is given by \( X_1(\omega) = \frac{-2a}{\omega^2 + a^2} \), \( \forall a > 0 \), from which we observe that \( X_1(f) = 0 \) at \( f = 0 \), \( \forall a \in \mathbb{R} \) (as \( x_1(t) \) is an odd function, therefore, \( X_1(0) = \int_{-\infty}^{\infty} \exp(-a|t|) \operatorname{sgn}(t) \, dt = 0 \), \( \forall a \in \mathbb{R} \); however, if \( a < 0 \), \( X_1(f) \) does not exist for \( f \neq 0 \). It is pertinent to notice that, \( \lim_{a \to 0} X_1(f) = \frac{1}{\pi|f|}, \lim_{f \to 0} \frac{1}{\pi|f|} = j\infty \), and \( \lim_{f \to 0} \frac{1}{\pi|f|} = j\infty \) which implies that the \( \lim_{f \to 0} \frac{1}{\pi|f|} \) does not exist. The FT of a zero-mean or an odd function is always constrained to be zero at the origin, and thus, we write the FT \( S(f) \) of the zero-mean sign function, \( s(t) = \operatorname{sgn}(t) = \lim_{a \to 0} x_1(t) \), as

\[
\lim_{a \to 0} X_1(f) = S(f) = \begin{cases} 
0, & f = 0 \\
\frac{1}{\pi|f|}, & f \neq 0 
\end{cases}
\]

(18)

Using the duality principle of the FT, i.e., if \( x(t) = X(f) \) then \( X(t) = x(-f) \), one can obtain, \( J S(t) = j \operatorname{sgn}(f) = -j \operatorname{sgn}(f) \). We denote \( J S(t) = h(t) \), and thus obtain

\[
H(f) = -j \operatorname{sgn}(f), \quad \text{and } h(t) = \begin{cases} 
0, & t = 0 \\
\frac{1}{\pi|t|}, & t \neq 0 
\end{cases}
\]

(19)

Now, we compute the function \( h(t) \) from \( H(f) \) using the IFT (16) as:

\[
h(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \sin(\omega t) \, d\omega,
\]

(20)
where, one can clearly observe that, \( h(0) = 0 \), which is also required by the definition (19). Further, we consider a limiting case of the following integral, with \( \sigma > 0 \), and obtain the solution of integration (20) as
\[
\frac{1}{\pi} \int_{0}^{\infty} e^{-\sigma \omega} \sin(\omega t) \, d\omega = \frac{1}{\pi \sigma^2 + t^2}
\]
\[
\Rightarrow h(t) = \lim_{\nu \to 0} \frac{1}{\pi \sigma^2 + t^2} = \begin{cases} 0, & t = 0, \\ \frac{1}{\pi}, & t \neq 0. \end{cases}
\]

(21)

The function, \( \frac{1}{\nu} \), is the well-known HT kernel which is singular at the origin, \( t = 0 \). In the sense of limit, it is not defined at the origin, because, \( \lim_{t \to 0} \frac{1}{\nu} = -\infty \) and \( \lim_{t \to 0} \frac{1}{\nu} = \infty \), therefore, \( \lim_{t \to 0} \frac{1}{\nu} \) does not exist. Thus, we have presented a trivial but important modification to the HT kernel in (19) by defining it at the origin, and obtained its integral form in (20). One can observe that if two or more functions are equal, almost everywhere, except on a set of points where Lebesgue measure (e.g., length, area, or volume) is zero, then the FT of these functions is same. Therefore the FT of the original HT kernel \( \frac{1}{\nu} \) and the modified HT kernel \( h(t) \) (19) is same. These definitions, (19), (20) and (21), of the HT kernel are further supported by the kernel of the discrete-time HT \([5,47,48,49,50]\), defined as \( h[n] = \frac{1}{\nu} \cos(n\pi) \), which is also zero at the origin (i.e., \( h[0] = 0 \)), and it can be obtained by using the discrete counter part of (20), i.e., \( h[n] = \frac{1}{\nu} \sum \sin(2\pi n) \, d\omega \).

For a real-valued signal \( x(t) \), the Gabor analytic signal is defined as
\[
z(t) = x(t) + jx(t) = \frac{1}{\pi} \int_{0}^{\infty} X(\omega) \exp(j\omega t) \, d\omega, \quad -\infty < t < \infty,
\]
(22)

where the real part is the original signal and the imaginary part is the HT of the real part. Thus, the signal \( x(t) \) and its HT \( x(t) \) can be written as
\[
x(t) = \frac{1}{\pi} \int_{0}^{\infty} |X(\omega)| \cos(\omega t + \phi(\omega)) \, d\omega, \quad -\infty < t < \infty,
\]
(23)
\[
\hat{x}(t) = \frac{1}{\pi} \int_{0}^{\infty} X(\omega) \sin(\omega t + \phi(\omega)) \, d\omega, \quad -\infty < t < \infty,
\]
where \( X(\omega) = X_{R}(\omega) + jX_{I}(\omega) = |X(\omega)| \exp(j\phi(\omega)) \). \( \phi(\omega) = \tan^{-1}(X_{I}(\omega)/X_{R}(\omega)) \). The GFR, corresponding to (3), is present here as
\[
x(t, \mathcal{H}(\omega, t), \alpha(\omega, t)) = \frac{1}{\pi} \int_{0}^{\infty} \mathcal{H}(\omega, t) |X(\omega)| \cos(\omega t + \phi(\omega) - \alpha(\omega, t)) \, d\omega.
\]
(24)

The four PTs [corresponding to (4), (5), (7) and (8)] of the signal \( x(t) \) are here defined as
\[
x(t, \alpha(\omega, t)) = \frac{1}{\pi} \int_{0}^{\infty} |X(\omega)| \cos(\omega t + \phi(\omega) - \alpha(\omega, t)) \, d\omega, \quad \text{FDTD-PT},
\]
\[
x(t, \alpha(\omega, t)) = \frac{1}{\pi} \int_{0}^{\infty} |X(\omega)| \cos(\omega t + \phi(\omega) - \alpha(\omega, t)) \, d\omega, \quad \text{FITF-PT},
\]
(25)
\[
x(t, \alpha(\omega, t)) = \frac{1}{\pi} \int_{0}^{\infty} |X(\omega)| \cos(\omega t + \phi(\omega) - \alpha(\omega, t)) \, d\omega, \quad \text{FDTI-PT},
\]
\[
x(t, \alpha(\omega, t)) = \frac{1}{\pi} \int_{0}^{\infty} |X(\omega)| \cos(\omega t + \phi(\omega) - \alpha(\omega, t)) \, d\omega, \quad \text{FITI-PT}.
\]

The PT, \( x(t, \alpha(\omega)) \) defined in (25), is the real part of the PT of analytic signal hereby defined as
\[
z(t, \alpha(\omega)) = \frac{1}{\pi} \int_{0}^{\infty} X(\omega) \exp(j(\omega t - \alpha(\omega))) \, d\omega,
\]
and, therefore, the transfer function is given by
\[
H(\alpha(\omega)) = \begin{cases} e^{-j\alpha(\omega)}, & \omega \geq 0, \\ e^{j\alpha(\omega)}, & \omega < 0. \end{cases}
\]
(27)

For example, to obtain a time-delayed signal \( x(t - t_0) \) from the input signal \( x(t) \), one can set \( \alpha(\omega) = \omega t_0 \), and therefore from (25) and (16), \( x(t, \alpha(\omega)) = x(t - t_0) \Rightarrow \frac{1}{\pi} \int_{0}^{\infty} X(\omega) \exp(j\omega t_0) \times \exp(-j\omega t_0) \, d\omega = \frac{1}{\pi} \int_{0}^{\infty} |X(\omega)| \cos(\omega t) \exp(j\omega t - \omega t_0) \, d\omega.

Thus, we have defined a general phase shifter in (25) which is a generalization of the IFT as well as the HT because it is (i) IFT if \( \alpha(\omega) = 0 \), (ii) HT if \( \alpha(\omega) = \pi/2 \). The impulse response of the PT, designated as PT kernel, from (25) with \( \alpha(\omega) = \alpha \), is hereby derived as
\[
h(t, \alpha) = \cos(\alpha) \delta(t) + \sin(\alpha) \delta(t, \pi/2)
\]
(28)
where \( \delta(t) \) is the Dirac delta function, \( h(t, \pi/2) \) = \( \delta(t) \) = \( \delta(t, \pi/2) \) is the HT kernel as defined in (19) and (20), \( h(t, \alpha) = \delta(t, \alpha) \) and \( \delta(t, 0) = \delta(t) \). The direct derivation of the kernel of the PT (28) the from FITF-PT (25) is elementary. Here, an indirect proof of the PT kernel is presented as follows: one can easily show [using (16) or (22)] and setting \( x(t) = \delta(t) \) \( \Leftrightarrow \) \( X(\omega) = 1 \) and \( \phi(\omega) = 0 \) that, \( \delta(t) = \frac{1}{\pi} \int_{0}^{\infty} \cos(\omega t) \, d\omega = \frac{1}{\pi} \int_{0}^{\infty} \cos(\omega t) \, d\omega \), and \( h(t) = \frac{1}{\pi} \int_{0}^{\infty} |\cos(\omega t)| \, d\omega \). Using these facts, from (25) we obtain, \( h(t, \alpha) = \frac{1}{\pi} \int_{0}^{\infty} |\cos(\omega t - \alpha) \, d\omega = \frac{1}{\pi} \int_{0}^{\infty} |\cos(\omega t + \alpha) \, d\omega + \sin(\alpha) \, \sin(\alpha) \, d\omega \), and thus (28). The kernel of the PT (28) is well-defined for all values of \( \alpha \) [e.g., \( h(t, \pi/4) = \frac{1}{\pi} \delta(t) + \frac{1}{\pi} \delta(t) \), for all time including at the origin, only due to the modification presented in the HT kernel (19) by defining it zero at the origin; otherwise, mathematically it would be invalid to add the delta function with the Hilbert kernel which is so far not defined at the origin in the literature.

We can obtain the kernel of an analytic signal and compute the phase difference between the delta function \( \delta(t) \) and HT kernel \( h(t) = \delta(t, \pi/2) \) (19), as
\[
z_\delta(t) = \delta(t) + jh(t) = a_\delta(t)e^{j\phi_\delta(t)}, \quad \text{where} \quad a_\delta(t) = \delta(t) + |h(t)|,
\]
and
\[
\phi_\delta(t) = \tan^{-1}\left( \frac{h(t)}{\delta(t)} \right) = \frac{\pi}{2} \text{sgn}(t) = \begin{cases} \pi/2, & t > 0, \\ 0, & t = 0, \\ -\pi/2, & t < 0. \end{cases}
\]
(29)

Fig. 1 shows plots of the delta function \( \delta(t) \) (top-left), (top-right) HT kernel \( h(t) \) (19); amplitude \( a_\delta(t) \) (bottom-left), and phase \( \phi_\delta(t) \) (bottom-right) of the kernel of the analytic signal (29) which is well-defined for all time including at the origin due to the modification presented in the HT kernel (19). It is observed that if the HT kernel (19) is not defined or has singularity at the origin, then phase \( \phi_\delta(t) \) would be undefined at the origin in (29). The kernel
of the analytic signal can also be written as $z_2(t) = δ(t) + jh(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \exp(\omega t) d\omega$ or $z_2(t) = \lim_{\sigma \to 0^+} \frac{1}{\pi} \int_{-\infty}^{\infty} \exp(-\omega \sigma - j \omega t) d\omega = \lim_{\sigma \to 0^+} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sigma}{\sigma^2 + \omega^2} d\omega$, which implies (i) $δ(t) = \lim_{\sigma \to 0^+} \frac{1}{\pi} \left( \frac{\sigma}{\sigma^2 + t^2} \right)$ and $h(t) = 0$, for $t = 0$, (ii) $δ(t) = 0$ and $h(t) = \frac{1}{\pi t}$, for $t \neq 0$.

The arbitrary but fixed phase shifter defined in (28) is a linear time-invariant (LTI) system, thus its output can be written as the convolution of input with impulse response, i.e.,

$$x(t,\alpha) = H_0 \{ x(t) \} = x(t) * h(t,\alpha) = \cos(\alpha) x(t) + \sin(\alpha) \hat{x}(t),$$

(30)

or $x(t,\alpha) = \cos(\alpha) x(t,0) + \sin(\alpha) x(t, \pi/2)$, where $x(t,0) = x(t)$ and $x(t, \pi/2) = \hat{x}(t)$ as defined in (23). Clearly, the PT of a time-domain signal $x(t)$ is another time-domain and phase shifted signal $x(t,\alpha)$. There are some obvious properties of the PT (30) which follow directly from the definition such as $H_\alpha = \cos(\alpha) I + \sin(\alpha) H$, where $I$ and $H$ are the identity and HT operators, respectively, i.e., $I \langle x(t) \rangle = x(t)$ and $H \langle x(t) \rangle = \hat{x}(t) = x(t, \pi/2)$; inverse PT $H_\alpha^{-1} = H_{-\alpha}$ (or $H^{-1}_\alpha H_{\alpha} = H_{\alpha} H^{-1}_\alpha = I$); $H_{\alpha m} = H_{\alpha}$, $H_{\alpha} H_{\alpha} = H_{\alpha 1} H_{\alpha 2} = H_{\alpha 1 + \alpha 2}$. Therefore, $H_{\alpha m} \langle x(t) \rangle = x(t, m\alpha)$, $H_{\alpha} \langle x(t,\alpha) \rangle = H_{\alpha} \langle x(t,\alpha_1) \rangle = x(t,\alpha_1 + \alpha_2)$, and $x(t,\alpha_1 + \alpha_2) = x(t, \alpha_1) \delta(t, \alpha_2) - x(t, \alpha_2) \delta(t, \alpha_1)$. If $\alpha_1 + \alpha_2 = 2\pi m, \forall m \in \mathbb{Z}$.

Now, we explore and present the some basic properties of the proposed PT (30) as follows:

1. **Linearity:** The PT is a linear operator, i.e., $H_\alpha \{ a_1 x_1(t) + a_2 x_2(t) \} = a_1 H_\alpha \{ x_1(t) \} + a_2 H_\alpha \{ x_2(t) \}$ for arbitrary scalars $a_1$ and $a_2$, functions $x_1(t)$ and $x_2(t)$.

2. **The PT of a constant signal:** For any constant $c$, $H_\alpha \{ c \} = c \cos(\alpha)$.

3. **Time-shifting and time-dilation:** If $x(t,\alpha)$ is the PT of $x(t)$, i.e., $H_\alpha \{ x(t) \} = x(t,\alpha)$, then $H_\alpha \{ x(t - t_0) \} = x(t - t_0, \alpha)$ and $H_\alpha \{ x(at) \} = \cos(\alpha) x(at) + \sin(\alpha) \text{sgn}(a) \tilde{x}(at)$, where $a \neq 0$.

4. **Relation with the Fourier transform:** We obtain the Fourier transform of the kernel of the PT (30) as, $H(f, \alpha) = \cos(\alpha) + \sin(\alpha) (-j \text{sgn}(f))$, which can be written as

$$H(f, \alpha) = \begin{cases} e^{-j\alpha}, & f > 0, \\ \cos(\alpha), & f = 0, \\ e^{j\alpha}, & f < 0. \end{cases}$$

(31)

thus the PT provides, $-\alpha$ and $+\alpha$ phase shifts to the positive and negative frequencies, respectively, and when, $\alpha = \pi/2$, it becomes the HT. Further, if $X(f)$ is the Fourier transform $x(t)$, then the Fourier transform of $x(t,\alpha)$ is $X(f,\alpha) = X(f) H(f,\alpha)$.

5. **Orthogonality:** If $x(t)$ is a real-valued energy signal (i.e., $E = \langle x(t), x(t) \rangle$), then the inner product of $x(t)$ with $x(t,\alpha)$ is given by

$$\langle x(t), x(t,\alpha) \rangle = \cos(\alpha) \langle x(t), x(t) \rangle + \sin(\alpha) \langle x(t), \tilde{x}(t) \rangle$$

$$\implies \cos(\alpha) = \frac{\langle x(t), x(t,\alpha) \rangle}{\langle x(t), x(t) \rangle},$$

(32)

as $\langle x(t), \tilde{x}(t) \rangle = 0$, and thus they are orthogonal only for the phase shifts $\alpha = \pi/2 (2m + 1), m \in \mathbb{Z}$.

6. **Energy:** If $x(t)$ is a real-valued energy signal, then $x(t,\alpha)$ is also a real-valued energy signal, and its energy ($E_{\alpha}$) is computed by the inner product of $x(t,\alpha)$ with itself as

$$E_{\alpha} = \langle x(t,\alpha), x(t,\alpha) \rangle$$

$$= \cos^2(\alpha) \langle x(t), x(t) \rangle + \sin^2(\alpha) \langle \tilde{x}(t), \tilde{x}(t) \rangle.$$

(33)

For a zero mean signal, energy is preserved in the HT, i.e., $\langle x(t), x(t) \rangle = \langle \tilde{x}(t), \tilde{x}(t) \rangle$, so the energy is preserved in the proposed PT.

7. **Time-derivative:** The PT of the derivative of signal is the derivative of the PT, i.e.,

$$H_\alpha \{ \frac{d}{dt} x(t) \} = \frac{d}{dt} H_\alpha \{ x(t) \}.$$
8. **PT of product of low-pass and high-pass signals:** Let \( x_1(t) \) be a low-pass signal such that its FT \( X_1(f) = 0 \) for \( |f| > f_0 \) and let \( x_2(t) \) be a high-pass signal with \( X_2(f) = 0 \) for \( |f| < f_0 \). Then, \( PT H_o(x_1(t)x_2(t)) = X_1(t)H_o(x_2(t)) = x_1(t)x_2(t, \alpha) \). One can easily show it using the property of HT (i.e., Bedrosian theorem [29,30]) as \( H_o(x_1(t)x_2(t)) = X_1(t)X_2(t) \). Thus, to obtain the \( PT \) of the product of the low-pass signal and high-pass signal, only the high-pass signal needs to be phase shifted.

9. **PT of a analytic signal:** From (26) and (30), we obtain the \( PT \) of analytic signal as \( H_o(z(t)) = x(t, \alpha) + j\hat{x}(t, \alpha) = x(t, \alpha)e^{-j\alpha} = \{x(t) + j\hat{x}(t)\}e^{-j\alpha} \). which gives \( H_o(z(t)) = \cos(\alpha)x(t) + \sin(\alpha)\hat{x}(t) + j[-\sin(\alpha)x(t) + \cos(\alpha)\hat{x}(t)] = x(t, \alpha) + j\hat{x}(t, \alpha + \pi/2) \). Thus, \( x(t, \alpha) \) can be computed by considering the real part of the \( H_o(z(t)) \).

It is to be noted that, in the literature, a generalized Hilbert transform to obtain a phase shift of \( \alpha \) radians is well-defined only in the frequency-domain, e.g., author in study [28] defined \( H(f, \alpha) \) as

\[
H(f, \alpha) = \begin{cases} 
  e^{-j\alpha}, & f > 0, \\
  0, & f = 0, \\
  e^{j\alpha}, & f < 0,
\end{cases}
\]  

(34)

which is same as (31) for \( f \neq 0 \) and differs at \( f = 0 \). However, the time-domain description of (34) is not available so far in the literature. In this paper, we have presented detailed study, both in the time and frequency domains, along with various properties.

Using the proposed GFR (24), the narrowband inverse Fourier transform (NB-IFT) or NBFR \( x_{\hat{o}}(\omega, \lambda) \) is hereby defined for the time-frequency representation and analysis of the signal \( x(t) \), a special case of (24) with \( \mathcal{H}(\omega, t) = \mathcal{H}(\omega - \lambda) \) and \( \alpha(\omega, t) = 0 \), as

\[
x_{\mathcal{H}}(t, \lambda) = \frac{1}{\pi} \int_{0}^{\infty} \mathcal{H}(\omega - \lambda)X(\omega) |\cos(\omega t + \phi(\omega))| d\omega.
\]  

(35)

This is a counter part of the STFT which is obtained using the forward Fourier transform of the time-windowed signal \( x(t)g(t - \tau) \) as

\[
x_{\mathcal{F}}(\omega, \tau) = \int_{-\infty}^{\infty} x(t)g(t - \tau) \exp(-j\omega t) dt.
\]  

(36)

**Observation 3.1 (a):** The proposed PT (30) is a solution of the following partial differential equation

\[
\frac{\partial^2}{\partial x^2}x(t, \alpha) + x(t, \alpha) = 0,
\]  

(37)

which can be considered as (a) the initial value problem with conditions \( x(t, 0) = x(t) \) and \( \frac{\partial x(t, 0)}{\partial \alpha} = \dot{x}(t) \), or (b) the boundary value problem with conditions \( x(t, 0) = x(t) \) and \( \alpha = \pm x(t) \) or \( x(t, \pm \pi/2) = \pm x(t) \).

**Observation 3.1 (b):** Here, we consider three examples of the PT property 8, which can be used to obtain a unique PT and thus the HT of 2D and higher dimensional signals, as follows. First, we consider a signal with frequencies \( \omega_1, \omega_2 \geq 0 \) as \( x(t) = \cos(\omega_1 t) \cos(\omega_2 t) = \frac{1}{2}[\cos(\omega_1 t + \omega_2 t) + \cos(\omega_1 t - \omega_2 t)] = \frac{1}{2}[\cos(\omega_1 t + \omega_2 t) + \cos(\omega_2 t - \omega_1 t)] \). We obtain the PT of signal \( x(t) \) as (i) \( x(t, \alpha) = \frac{1}{2}[\cos(\omega_1 t + \omega_2 t - \alpha) + \cos(\omega_1 t - \omega_2 t + \alpha)] = \cos(\omega_1 t - \alpha) \cos(\omega_2 t) \), if \( \omega_1 > \omega_2 \), and (ii) \( x(t, \alpha) = \frac{1}{2}[\cos(\omega_1 t + \omega_2 t - \alpha) + \cos(\omega_2 t - \omega_1 t + \alpha)] = \cos(\omega_1 t + \alpha) \cos(\omega_2 t - \omega_1 t - \alpha) \), if \( \omega_2 > \omega_1 \). Next, we consider \( x(t) = \sin(\omega_1 t) \sin(\omega_2 t) \), then we obtain the PT as (i) \( x(t, \alpha) = \sin(\omega_1 t - \alpha) \sin(\omega_2 t) \), if \( \omega_1 > \omega_2 \), and (ii) \( x(t, \alpha) = \sin(\omega_1 t - \alpha) \sin(\omega_2 t + \alpha) \), if \( \omega_2 > \omega_1 \). Finally, we consider \( x(t) = \sin(\omega_1 t) \cos(\omega_2 t) \), then we obtain the PT as (i) \( x(t, \alpha) = \sin(\omega_1 t - \alpha) \cos(\omega_2 t) \), if \( \omega_1 > \omega_2 \), and (ii) \( x(t, \alpha) = \sin(\omega_1 t) \cos(\omega_2 t - \alpha) \), if \( \omega_2 > \omega_1 \). Thus, one can observe that to obtain a unique PT, in the phase argument of sin and cos, before introducing a phase \( \alpha \), one must consider \( (\omega_1 - \omega_2)t \) or \( (\omega_2 - \omega_1)t \) depending upon whether \( \omega_1 > \omega_2 \) or \( \omega_2 > \omega_1 \), respectively.

**Observation 3.1 (c):** If the considered signal \( x(t) \) is a complex valued function then the GFR, corresponding to the GFR of real valued function (24), can be written as

\[
x(t, H(\omega, t), \alpha(\omega, t)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega, t)X(\omega)|\exp(j\omega t + \phi(\omega) + \alpha(\omega, t))|d\omega
\]

\[
+ \frac{1}{2\pi} \int_{0}^{\infty} H(\omega, t)X(\omega)|\exp(j\omega t + \phi(\omega) - \alpha(\omega, t))|d\omega,
\]

(38)

and thus the PTs corresponding to (25) can be easily obtained by setting \( H(\omega, t) = 1, \forall t, \omega \).

3.2. **PT using the Fourier sine and cosine transforms**

The FCT and inverse FCT (IFCT) pairs of a signal are defined as

\[
X_c(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} x(t) \cos(\omega t) dt, \quad \omega \geq 0
\]  

(39)

\[
x(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} X_c(\omega) \cos(\omega t) d\omega, \quad t \geq 0,
\]

subject to the existence of the integrals, i.e., \( x(t) \) is absolutely integrable \( (\int_{0}^{\infty} |x(t)| dt < \infty) \) and its derivative \( x'(t) \) is piece-wise continuous in each bounded subinterval of \([0, \infty)\).

The Fourier cosine quadrature transform (FCQT) \( \tilde{x}_c(t) \) of the signal \( x(t) \) is defined [4] as

\[
\tilde{x}_c(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} X_c(\omega) \sin(\omega t) d\omega,
\]

(40)

where

\[
\tilde{x}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \tilde{x}_c(t) \sin(\omega t) dt,
\]

(41)

and

\[
\tilde{x}_c(\omega) = \begin{cases} 
  0, & \omega = 0, \\
  X_c(\omega), & \omega > 0,
\end{cases}
\]  

(42)

The FSAS, using the signal \( x(t) \) and its FCQT \( \tilde{x}_c(t) \), is defined [4] as

\[
\tilde{x}_c(t) = x(t) + j\tilde{x}_c(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} X_c(\omega) \exp(j\omega t) d\omega
\]

(43)

where the real part is the original signal and the imaginary part is the FQFT of the real part.

The PT \( x(t, \alpha(\omega, t)) \) of signal \( x(t) \) using the FCT is hereby defined as

\[
x(t, \alpha(t, \omega)) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} X_c(\omega) \cos(\omega t - \alpha(t, \omega)) d\omega,
\]

(44)
where \(\alpha(t, \omega)\) is the time and frequency dependent phase shift. If the phase shift is an arbitrary constant, i.e., \(\alpha(t, \omega) = \alpha\), then (44) can be written
\[
x(t, \alpha) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} X_\alpha(\omega) \cos(\omega t - \alpha) \, d\omega
\]
\[
= \cos(\alpha) x(t) + \sin(\alpha) \tilde{x}_\alpha(t),
\]
where \(\alpha\) is constrained to its principal value \([0, 2\pi)\) or \([-\pi, \pi)\) when it represents the wrapped phase, and \(\alpha \in \mathbb{R}\) in the case of unwrapped phase. Thus, we have defined the general phase shifter of signal which is a generalization of the IFST as well as FCQT because it is (i) IFCT if \(\alpha = 0\), and (ii) FCQT if \(\alpha = \pi/2\).

The FST and inverse FST (IFST) pairs of a signal are defined as
\[
X_\alpha(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} x(t) \sin(\omega t) \, dt,
\]
subject to the existence of the integrals. The Fourier sine quadrature transform (FSQT) \(\tilde{x}_\alpha(t)\) using the FST of signal \(x(t)\) is defined [4] as
\[
\tilde{x}_\alpha(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} X_\alpha(\omega) \cos(\omega t) \, d\omega,
\]
where one can observe that the both representations, defined as FST of \(x(t)\) in (46) and FCT of \(\tilde{x}_\alpha(t)\) in (48), are same for all the frequencies, i.e., \(X_\alpha(\omega) = X_\omega(\omega)\). The FSAS, using the signal and its FSQT, is defined [4] as
\[
\tilde{x}_\alpha(t) = \tilde{x}_\alpha(t) + jx(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} X_\alpha(\omega) \exp(j\omega t) \, d\omega,
\]
where the imaginary part is the original signal and the real part is the FQT of the imaginary part.

The PT \(x(t, \alpha(t, \omega))\) of signal \(x(t)\) using the FST is hereby defined as
\[
x(t, \alpha(t, \omega)) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} X_\alpha(\omega) \sin(\omega t - \alpha(t, \omega)) \, d\omega.
\]
If \(\alpha(t, \omega) = \alpha\), then (50) can be written as
\[
x(t, \alpha) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} X_\alpha(\omega) \sin(\omega t - \alpha) \, d\omega
\]
\[
= \cos(\alpha) x(t) - \sin(\alpha) \tilde{x}_\alpha(t),
\]
where \(\alpha \in [0, 2\pi]\) is the phase shift introduced in the signal \(x(t)\). Thus, we have defined the general phase shifter which is a generalization of the IFST as well as FSQT because it is (i) IFCT if \(\alpha = 0\), and (ii) FCQT if \(\alpha = \pi/2\).

The FQTs, presented in (40) and (47), are different from the HT (23) by the definition itself. The proposed Fourier–Singh phase transform (FSPT) representations, defined in (44) and (50), are the effective phase shifters which can be used in various applications such as envelop detection, IF estimation, time-frequency-energy representation and analysis of nonlinear and nonstationary data.

So far, the FST, FT, FCT and FST have been used to define PT. Moreover, the STFT can also be used to define the PT for obtaining a desired phase shift in a signal under consideration. The next section presents the other form of PT using the continuous-time AWT.

3.3. PT using the continuous analytic wavelet transform

In this subsection, the continuous AWT is used to define the WPT of a signal, and it is shown that the WQT is a special case of the WPT when phase-shift is \(\pi/2\) radians. The continuous wavelet transform (CWT) of a signal \(x(t) \in L^2(\mathbb{R})\) is defined as [35–37]
\[
X_\psi(s, \tau) \equiv (x(t), \psi_s, \tau(t)) = \int_{-\infty}^{\infty} x(t) \psi^*_s(t - \tau) \, dt,
\]
where at*erkst denotes the complex conjugate operation. The CWT (52) can be represented in the Fourier domain, using the Parseval’s relation \(\int_{-\infty}^{\infty} x(t)^* y(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) Y(\omega) \, d\omega\), as
\[
X_\psi(s, \tau) = \sqrt{\frac{|s|}{2\pi}} \int_{-\infty}^{\infty} X(\omega) \psi^*(s\omega) e^{j\omega t} \, d\omega,
\]
where \(\psi_s, \tau(t) = \frac{1}{\sqrt{|s|}} \psi \left( \frac{t - \tau}{s} \right) = \sqrt{\frac{|s|}{2\pi}} \psi^* \omega e^{-j\omega t}\) with scaling \((s \in \mathbb{R}, s \neq 0)\) and translation \((\tau \in \mathbb{R})\) parameters is a family of daughter wavelets, and \(\psi(t) \in L^2(\mathbb{R})\) is a mother wavelet function which has finite energy and zero mean. The wavelet \(\psi(t) = \Psi(\omega)\) satisfies the admissibility condition [38]
\[
C_\psi = \int_{-\infty}^{\infty} \frac{|\Psi(\omega)|^2}{|\omega|} \, d\omega < \infty,
\]
and ensures reconstruction of the original signal \(x(t)\) using the inverse CWT defined as [35–37]
\[
x(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_\psi(s, \tau) \psi_s, \tau(t) \, ds \, \frac{1}{s^2} \, dt.
\]

If \(\psi(t)\) is an analytic wavelet, then \(\Psi(\omega) = 0\) for \(\omega < 0\), and (52) represents the AWT [36,37], moreover (53) and (54) can be re-presented as
\[
X_\psi(s, \tau) = \sqrt{\frac{|s|}{2\pi}} \int_{0}^{\infty} X(\omega) \Psi^*(s\omega) e^{j\omega t} \, d\omega,
\]
\[
C_\psi = \int_{0}^{\infty} \frac{|\Psi(\omega)|^2}{|\omega|} \, d\omega < \infty.
\]

Now, using the analytic wavelet, we consider the real and imaginary parts of the AWT separately as follows: let \(\psi(t) = \psi_s, \tau(t) + j\psi_\tau, \tau(t)\) be an analytic wavelet, where the imaginary part is the HT or FQT of the real part, \(\psi(t) = 2U(\omega) \Psi_s, \tau(t)\) \(U(\omega)\) is the unit step function, and \(\psi_{s, \tau}(t) = \psi_s, \tau(t) + j\psi_\tau, \tau(t)\). Therefore, the AWT (52) for a real-valued energy signal \(x(t)\) can be written as \(X_\psi(s, \tau) = X_{\psi}(s, \tau) - jX_{\psi}(s, \tau)\), where \(X_{\psi}(s, \tau) = \langle x(t), \psi_{s, \tau}(t) \rangle\) and \(X_{\psi}(s, \tau) = \langle x(t), \psi_{s, \tau}(t) \rangle\). Thus, we obtain
\[ x(t) = \frac{2}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_{\tau}(s, \tau) \psi_{\tau,s}(t) \, dt \, \frac{1}{s^2} \, ds, \]  
\[ x(t) = \frac{2}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_i\psi(s, \tau) \psi_{\tau,s}(t) \, dt \, \frac{1}{s^2} \, ds, \]  
and define the following two representations of the WQT as

\[ x_w(t) = \frac{2}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_{\tau}(s, \tau) \psi_{\tau,s}(t) \, dt \, \frac{1}{s^2} \, ds, \]  
\[ x_w(t) = \frac{2}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_i\psi(s, \tau) \psi_{\tau,s}(t) \, dt \, \frac{1}{s^2} \, ds, \]

where \( x_w(t) = x(t, \tau/2) \) and the subscript \( w \) indicates that the phase-shift has been obtained using the wavelet transform. Moreover, using (57) and (58), the original signal \( x(t) \) (55) can be written as

\[ x(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ X_{\tau}(s, \tau) \psi_{\tau,s}(t) + X_i\psi(s, \tau) \psi_{\tau,s}(t) \right] \, dt \, \frac{1}{s^2} \, ds, \]

\[ + j \left[ X_{\tau}(s, \tau) \psi_{\tau,s}(t) - X_i\psi(s, \tau) \psi_{\tau,s}(t) \right] \, dt \, \frac{1}{s^2} \, ds, \]

\[ = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ X_{\tau}(s, \tau) \psi_{\tau,s}(t) + X_i\psi(s, \tau) \psi_{\tau,s}(t) \right] \, dt \, \frac{1}{s^2} \, ds, \]

Thus, using the two signal representations (57) and the two quadrature components (58), the following four representations of the WAF are here defined as

\[ z_w(t) = x(t) + j x_w(t) = \frac{2}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_{\tau}(s, \tau) \psi_{\tau,s}(t) \, dt \, \frac{1}{s^2} \, ds, \]

\[ z_w(t) = x(t) + j x_w(t) = \frac{2}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_i\psi(s, \tau) \psi_{\tau,s}(t) \, dt \, \frac{1}{s^2} \, ds, \]

\[ z_w(t) = x(t) + j x_w(t) = \frac{2}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} j X_{\tau}(s, \tau) \psi_{\tau,s}(t) \, dt \, \frac{1}{s^2} \, ds, \]

\[ z_w(t) = x(t) + j x_w(t) = \frac{2}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} j X_i\psi(s, \tau) \psi_{\tau,s}(t) \, dt \, \frac{1}{s^2} \, ds. \]

The desired phase in the AWT (56) is introduced as

\[ X_{\tau}(s, \tau, \alpha(\omega)) = \sqrt{\frac{1}{2\pi}} \int_0^{\infty} \Psi^*(s \alpha(\omega)) e^{j\omega \tau} e^{-j\alpha(\omega)} \, d\omega. \]

If the introduced phase in the AWT (61) is independent of frequency, i.e., \( \alpha(\omega) = \alpha \), then we can write \( X_{\tau}(s, \tau, \alpha) = X_{\tau}(s, \tau) e^{-j\alpha} = [\cos(\alpha)X_{\tau}(s, \tau) - \sin(\alpha) - j[\cos(\alpha)X_{\tau}(s, \tau) + \sin(\alpha)X_{\tau}(s, \tau)] \), and we obtain the following two representations of an arbitrary constant WPT of the signal \( x(t) \) as

\[ x(t, \alpha) = \frac{2}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_{\tau}(s, \tau, \alpha) \psi_{\tau,s}(t) \, dt \, \frac{1}{s^2} \, ds, \]

\[ x(t, \alpha) = \frac{2}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_i\psi(s, \tau, \alpha) \psi_{\tau,s}(t) \, dt \, \frac{1}{s^2} \, ds. \]

The WPT can also be obtained from the WAF (60) by considering the real part of the following representation

\[ z_w(t, \alpha) = [x(t) + j x_w(t)] e^{-j\alpha} = x(t, \alpha) + j x_w(t, \alpha), \]

where \( x_w(t, \alpha) = [\cos(\alpha)X_{\tau}(t) - \sin(\alpha)x(t)], \) and the WPT

\[ x(t, \alpha) = \cos(\alpha)X(t) + \sin(\alpha)x(t). \]

3.3.1. A single-integral representation of the WPT

The CWT is a redundant transform which reveals in-depth structural characteristics of the signal, and due to redundancy there may exist many ways to define the inverse. A single-integral can be used for the inverse CWT, if the signal \( x(t) \) and wavelet \( \psi(t) \) are satisfying the following two conditions [39]: (1) the signal \( x(t) \) is a real-valued function, and (2) either the wavelet \( \psi(t) \) is an even function which has a real-valued FT, or the wavelet is an analytic wavelet which has FT that supports only for the positive frequencies, i.e., \( \psi(\omega) \neq 0 \) for \( \omega < 0 \).

If \( x(t) \) and \( y(t) \) are two finite energy signals, \( \psi_1(t) \) and \( \psi_2(t) \) are two wavelet functions which satisfy the two-wavelet admissibility condition, the following equality holds:

\[ C_{\psi_1, \psi_2}(x(t), y(t)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_{\psi_1}(s, \tau) X_{\psi_2}^*(s, \tau) \, dt \, \frac{1}{s^2} \, ds, \]

where \( X_{\psi_1}(s, \tau) = \langle x(t), \psi_1(t, \tau) \rangle, \) \( X_{\psi_2}(s, \tau) = \langle y(t), \psi_2(t, \tau) \rangle \) and

\[ C_{\psi_1, \psi_2} = \int_{-\infty}^{\infty} \left| \Psi_{\psi_1}^*(\omega) \Psi_{\psi_2}(\omega) \right| \frac{1}{|\omega|} \, d\omega. \]

Therefore, the two-wavelet admissibility condition is defined as

\[ \int_{-\infty}^{\infty} \left| \Psi_{\psi_1}^*(\omega) \Psi_{\psi_2}(\omega) \right| \frac{1}{|\omega|} \, d\omega < \infty. \]

The main idea of a single-integral inverse CWT is that the admissibility condition of two-wavelet (67) can be satisfied even if either one of the wavelets is not admissible, and it can be further simplified by allowing one of the signals and wavelets to be distributions. Thus, by considering the analytic wavelet \( \psi_1(t) \), real-valued signal \( x(t) \), both signal \( y(t) \) and wavelet \( \psi_2(t) \) to be the delta function, one can obtain the WAS representation using the following single-integral inverse CWT as

\[ z(t) = x(t) + j x_w(t) = \frac{2}{C_{\psi_1}} \int_0^{\infty} X_{\psi_1}(t, s) \frac{1}{s} \, ds, \]

where the imaginary part \( x_w(t) \) is the WQT of signal \( x(t) \). Therefore, if \( X_{\psi_1}(s, \tau, \alpha) = X_{\psi_1}(s, \tau) e^{-j\alpha} \) as defined in (61), then we can define

\[ z(t, \alpha) = x(t, \alpha) + j x_w(t, \alpha) = \frac{2}{C_{\psi_1}} \int_0^{\infty} X_{\psi_1}(t, s, \alpha) \frac{1}{s} \, ds. \]
where $x(t, \alpha)$ is the proposed WPT which introduces a desired phase-shift $\alpha$ in the signal $x(t)$ using AWT, $x(t, 0) = x(t)$, $x(t, \pi/2) = x_w(t)$ and $x_w(t, \alpha) = x(t, \alpha + \pi/2)$.

**Observation 3.3**: We can represent a real signal as, $x(t) = \alpha(t) \cos(\phi(t))$, and obtain its quadrature component, $x_q(t) = \alpha(t) \sin(\phi(t))$, using any of the AS representations (10), (22), (43), (49), (60) and (68), and obtain the following AM-FM transform, which is a special case of the proposed GFR (24), as

$$x(t, \mathcal{H}(t), \alpha(t)) = \mathcal{H}(t) \alpha(t) \cos(\phi(t) - \alpha(t)),$$

$$= \mathcal{H}(t) \alpha(t) \cos(\phi(t)) \cos(\alpha(t))$$

$$+ \mathcal{H}(t) \alpha(t) \sin(\phi(t)) \sin(\alpha(t)),$$

$$= \mathcal{H}(t) \alpha(t) \cos(\alpha(t)) + \mathcal{H}(t) x_q(t) \sin(\alpha(t)),$$

where $\mathcal{H}(t)$ and $\alpha(t)$ are the functions of time. If $\mathcal{H}(t) = 1$ then, like FDTI-PT and FDTD-PT (25), (70) is the FITD-PT, and thus time-varying filtering (14) can be written as

$$x_T(t, \mathcal{H}_m(t), \alpha_m(t)) = a_0 + \sum_{m=1}^{M} [\mathcal{H}_m(t) x_q(t) \cos(\phi_m(t) - \alpha_m(t))].$$

(71)

### 3.4. Implementation of the GFR using DFT

The DFT and inverse DFT (IDFT) pairs, of a signal $x[n]$ of length $N$, are defined as

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \exp(-j2\pi kn/N), \quad 0 \leq k \leq N - 1,$$

$$x[n] = \sum_{k=0}^{N-1} X[k] \exp(j2\pi kn/N), \quad 0 \leq n \leq N - 1.$$  

(72)

The DFT and IDFT are computed efficiently using the FFT algorithm. Unless otherwise mentioned, we consider $x[n]$ as a real valued signal. We obtain the following kernel of a discrete-time PT corresponding to the kernel of the continuous-time PT (28) as

$$h[n, \alpha] = \cos(\alpha) \delta[n] + \sin(\alpha) h[n],$$

(73)

where $h[n, \alpha] = [h[0, \alpha], \ldots, h[N-1, \alpha]]$, and $h[n, \pi/2] = h[n] = \delta[n] = [\delta[0, \pi/2]]$ is the discrete-time HT kernel. Thus, a constant phase shift in the signal $x[n]$ is represented as

$$x[n, \alpha] = \cos(\alpha) x[n] + \sin(\alpha) \bar{x}[n].$$

(74)

where $x[n] = x[n, 0]$ and $\bar{x}[n] = x[n, \pi/2]$. The PT defined in (7) and (25) can be computed using the FFT by considering the real part of the following analytic signal

$$\bar{z}[n, \alpha_k] = x[n, \alpha_k] + j x_q[n, \alpha_k + \pi/2] = \text{IFFT} [X[k] \mathcal{H}[k]],$$

(75)

where $\bar{z}[n, \alpha_k] = \text{Re}[z[n, \alpha_k]]$, and $\mathcal{H}[k]$ is defined as

$$\mathcal{H}[k] = \begin{cases} \cos(\alpha_k), & k = 0, \\ 2 \exp(-j \alpha_k), & 1 \leq k \leq N/2 - 1, \\ 0, & (N/2 + 1) \leq k \leq N - 1, \end{cases}$$

(66)

if $N$ is an even number, and

$$\mathcal{H}[k] = \begin{cases} \cos(\alpha_k), & k = 0, \\ 2 \exp(-j \alpha_k), & 1 \leq k \leq (N - 1)/2, \\ 0, & ((N - 1)/2 + 1) \leq k \leq N - 1, \end{cases}$$

(77)

if $N$ is an odd number. For a constant phase shift, the imaginary part is the HT of the real part in (75), and if $\alpha_k = 0, \forall k$, the real part is the original signal $x[k]$. In (76) and (77), we can remove the multiplication factor of 2 and define $\mathcal{H}[k] = \exp(-j \alpha_k)$ for positive frequencies $(1 \leq k \leq N/2 - 1)$, and $\mathcal{H}[k] = \exp(j \alpha_k)$ for negative frequencies $(N/2 + 1) \leq k \leq N - 1$.

**Observation 3.3.1**: The phase shift of a sinusoidal signal does not change its amplitude (or energy), except for the lowest (i.e., DC) and highest frequency components. For example, if $x[n] = c + \cos(\pi n)$, then $x[n, \alpha] = \cos(\alpha) (c + \cos(\pi n))$, and for $\alpha = \pi/2$, it becomes zero which cannot be recovered by further phase shift. Therefore, to overcome this issue, we can define the phase shift of a constant signal and highest frequency component as $z[n, \alpha] = (c + \cos(\pi n)) e^{-j \alpha} = e^{-j \alpha} + e^{j(\pi - \alpha)}$, which preserves the energy in these cases as well. Moreover, in the case of $\pi/2$ phase shift, the complete signal is getting transferred to the imaginary signal, and one can recover the original signal by further phase shift of $(\alpha + 2\pi m)$, and considering $x[n, \alpha] = \text{Re}[z[n, \alpha]]$. This is also consistent with the complex plane representation, where multiplication of $+j$ with a complex number $(z = a + jb)$ introduces $+\pi/2$ phase shift (e.g., $jz = ja - b$) without any change in the amplitude. Therefore, one can use $H[k] = e^{-j \mu}$ for $k = 0, N/2$ in (76) and (77).

We observe that the delayed signal $x[n - n_0]$ can be obtained using the IDFT (72) as

$$x[n - n_0] = X[0] + \sum_{k=1}^{N-1} X[k] \exp(j2\pi k(n - n_0)/N),$$

(78)

which is valid only for an integer value of $n_0$, because the complex conjugate of $\exp\left(\frac{j2\pi k n_0}{N}\right)$ is $\exp\left(-\frac{j2\pi k (N-n_0)}{N}\right)$ only for some integer value of $n_0$, and it is not valid for a fractional delay. Therefore, in order to obtain both the integer and fractional delays $n_k \in \mathbb{R}$ in the signal $x[n]$, corresponding to (11) which can be computed by (75), we define $H[k]$ as

$$H[k] = \begin{cases} 1, & k = 0, \\ 2 \exp(-j 2\pi k n_0/N), & 1 \leq k \leq N/2 - 1, \\ \exp(-j \pi n_k), & k = N/2, \\ 0, & (N/2 + 1) \leq k \leq N - 1, \end{cases}$$

(79)

where $N$ is an even number (similarly it can also be defined when $N$ is an odd number), and for some constant delay $n_0$ one can set $n_k = n_0, \forall k$.

The $\mu$-th order derivative of the signal $x[n]$, denoted as $x_\mu[n]$, corresponding to (12), can be estimated by

$$x_{\mu}[n] = a_0 \left(\frac{n/N}{\Gamma(1 - \mu)} \right) + \text{Re}[\text{IFFT} \{X[k] H[k]\}]$$

(80)

using $H[k]$ which is defined as

$$H[k] = \begin{cases} 0, & k = 0, \\ 2(2\pi k/N)^\mu \exp(j \mu \pi/2), & 1 \leq k \leq N/2 - 1, \\ (\pi)^\mu \exp(j \mu \pi/2), & k = N/2, \\ 0, & (N/2 + 1) \leq k \leq N - 1, \end{cases}$$

(81)

where mean-value $a_0 = \sum_{n=0}^{N-1} x[n] / N$, $\mu \geq 0$, and $N$ is an even number (similarly it can also be defined when $N$ is an odd number); and when $\mu < 0$ then it is the $\mu$-th order integral of a signal as defined in (13).

The HT of discrete signal $x[n]$ can be obtained by the inverse DTFT as

$$\hat{x}[\xi] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{sgn}(\Omega) X(\Omega) \exp(j \Omega n) d\Omega,$$

(82)
where $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] \exp(-j\Omega n)$ is the DFT of signal $x[n]$. Practically, the HT (82) is implemented by discretization in frequency with the DFT and inverse DFT using the FFT algorithm. This approach provides an error convergence that is polynomially decreasing with the size of the DFT points. To obtain better error convergence, there are many HT approximation methods such as (i) the HT based on the sinc basis functions expansion [47–50], and (ii) the HT approximation based on an expansion in rational eigenfunctions of the HT operator [89]. An approximation of the HT in terms of the sinc functions can be easily obtained as follows. The HT (82) can be written using the convolution operation as $\hat{x}[n] = x[n] * h[n]$, and thus

$$\hat{x}[n] = \sum_{m=-\infty}^{\infty} x[m] \left( \frac{1 - \cos(\pi (n - m))}{\pi (n - m)} \right),$$

where $\text{sinc}(x) = \sin(\pi x)/\pi x$. If the signal is time limited with length $N$, i.e., $x[n]$ is non-zero for $n \in [0, N-1]$ and zero otherwise, then (83) can be written as

$$\hat{x}[n] = \frac{\pi}{2} \sum_{m=0}^{N-1} x[m](n-m) \text{sinc}^2 \left( \frac{n-m}{2} \right).$$

(84)

Because an accurate estimation of the PT depends on the accuracy of the HT, therefore these HT approximation methods can be used in (74) to compute the PT of a signal.

3.5. Implementation of the GFR using DCTs/DSTs

Let $x[n]$ be a finite energy signal of a length $N$. The DCT type-2 (DCT-2) of the signal $x[n]$ is defined as [1]

$$X_{\text{c2}}[k] = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} x[n] \cos \left( \frac{\pi k(2n+1)}{2N} \right), \quad 0 \leq k \leq N-1,$$

(85)

and the inverse DCT (IDCT) is obtained by

$$x[n] = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} X_{\text{c2}}[k] \cos \left( \frac{\pi k(2n+1)}{2N} \right), \quad 0 \leq n \leq N-1,$$

(86)

where normalization factors $\sigma_k = \sqrt{\frac{2}{N}}$ for $k = 0$, and $\sigma_k = 1$ for $k \neq 0$. If the consecutive samples of the sequence $x[n]$ are correlated, then DCT concentrates energy in a few $X_{\text{c2}}[k]$ and decorrelates them. The DCT basis sequences, $\cos \left( \frac{2k \pi n}{2N} \right)$, which are a class of discrete Chebyshev polynomials [1], form an orthogonal set as inner product $\cos \left( \frac{2k \pi n}{2N} \right), \cos \left( \frac{2m \pi n}{2N} \right) = 0$ for $k \neq m$.

The discrete Fourier cosine quadrature transform (FCQT), $\tilde{x}_{\text{c2}}[n]$, of the signal $x[n]$ is defined as [4]

$$\tilde{x}_{\text{c2}}[n] = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} \sigma_k X_{\text{c2}}[k] \sin \left( \frac{\pi k(2n+1)}{2N} \right), \quad 0 \leq n \leq N-1,$$

(87)

where $X_{\text{c2}}[k]$ is the DCT-2 as defined in (85). The PT using the DCT-2 is hereby defined as

$$x[n, \alpha(k)] = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} \sigma_k X_{\text{c2}}[k] \cos \left( \frac{\pi k(2n+1)}{2N} - \frac{\pi \alpha(k)}{2} \right), \quad 0 \leq n \leq N-1,$$

(88)

and if $\alpha(k) = \alpha$, then (88) can be written as

$$x[n, \alpha] = \cos(\alpha) x[n] + \sin(\alpha) \tilde{x}_{\text{c2}}[n].$$

(89)

Thus, using the PT (88), variable fractional time-delays in the signal $x[n]$ can be introduced as $x[n - n_k] = x[n, \alpha(k)]$, where $\alpha(k) = \pi n_k/N$, $n_k \in \mathbb{R}$, and for a constant fractional time-delay $n_k = n_0, \forall k$.

Considering eight-types of the DCTs and eight-types of the DSTs, sixteen-types of the Fourier quadrature transforms (FQTs) and corresponding Fourier-Singh analytic signal (FSAS) representations are well defined [4]. Therefore, sixteen-types of the PTs can be derived from these FQTs and FSAS representations. So far, we have presented the PT, in (88) and (89), using only the DCT-2 and corresponding FQT. Accordingly, we next consider and define all sixteen-types of the PTs as follows. Using the standard DCTs and DSTs transform matrices, presented in Appendix B (97) and (98), the following sixteen FQTs (i.e., eight FCQTs, $\tilde{x}_{\text{c2}}$, and eight FSQTs, $\tilde{x}_{\text{c2}}$, and corresponding sixteen FSAS representations (FSASRs), $\tilde{x}_{\text{c2}}$, and $\tilde{x}_{\text{c2}}$, for $i = 1, 2, \ldots, 8$, are defined in [4] as

$$x_i = C_i x; \quad x = C_i^T x_i; \quad \text{(DCTs and IDCTs)}$$

$$\tilde{x}_i = S_i^T \tilde{x}_i = \tilde{S}_i^T C_i x; \quad \tilde{x}_i = x + j\tilde{x}_i; \quad \text{(FCQTs and FSASRs)}$$

$$x_i = S_i x; \quad x = S_i^T x_i; \quad \text{(DSTs and IDSTs)}$$

$$\tilde{x}_i = \tilde{C}_i^T x_i = \tilde{C}_i^T S_i x; \quad \tilde{x}_i = \tilde{x}_i + j\tilde{x}_i; \quad \text{(FCFTs and FSASRs)},$$

(90)

where $x_i = [X_i[0] \ X_i[1] \ldots X_i[N-1]]^T$ and $x_i = [X_i[0] \ X_i[1] \ldots X_i[N-1]]^T$ are the DCT and DST of $i$-th type of the signal $x = [x[0] \ x[1] \ldots x[N-1]]^T$, respectively. Hence, the linear transformations of $x$ into $\tilde{x}_i$, and $x$ into $\tilde{x}_i$, are defined with the transformation matrices $S_i^T C_i$ and $\tilde{C}_i^T S_i$, respectively. Thus, the following sixteen PTs, using the sixteen types of FQTs and FSAS representations, are defined as

$$x_i(\alpha) = \text{Re}[e^{-j\alpha} \tilde{x}_i] = \cos(\alpha)x + \sin(\alpha)\tilde{x}_i, \quad 1 \leq i \leq 8,$$

(91)

$$x_i(\alpha) = \text{Im}[e^{-j\alpha} \tilde{x}_i] = \cos(\alpha)x - \sin(\alpha)\tilde{x}_i, \quad 1 \leq i \leq 8.$$
\[ \delta T(t, \alpha) = \cos(\alpha) \frac{1}{T} + 2 \sum_{k=1}^{\infty} \cos(2\pi k f_0 t - \alpha), \quad -\infty < t < \infty, \]
\[ \delta T(t, \alpha) = \cos(\alpha) \delta T(t) + \sin(\alpha) \delta T(t, \pi/2), \quad -\infty < t < \infty, \]
\[ \delta T(t, \pi/2) = 2 \frac{1}{T} \sum_{k=1}^{\infty} \sin(2\pi k f_0 t) = \begin{cases} 0, & t = \pm \frac{m}{f_0}, m \in \mathbb{Z}, \\ \frac{1}{2} \cot(\pi f_0 t), & t \neq \pm \frac{m}{f_0}, \end{cases} \]
\[ \delta T(t, \pi/2) = \lim_{N \to \infty} \frac{2}{T} \sum_{k=1}^{N} \sin(2\pi k f_0 t). \]

where \[ \delta T(t, \pi/2) \] is the periodic HT of the impulse train and defined as

\[ \delta T(t, \pi/2) = \lim_{N \to \infty} \frac{2}{T} \sum_{k=1}^{N} \sin(2\pi k f_0 t). \]

By considering the sum of the series, \[ \frac{1}{T} \sum_{k=1}^{\infty} \exp(-j2\pi f_0 t k) = \exp(-j2\pi f_0 t), \]
for \( \sigma > 0 \) and then taking the imaginary part of the sum as limiting case, one can obtain (95) as

\[ \lim_{\sigma \to 0} \frac{1}{T} \sum_{k=1}^{\infty} \exp(-j2\pi f_0 t k) = \frac{1}{2} \cot(\pi f_0 t). \]

We can easily obtain the HT kernel (19) for the non-periodic signals from the periodic one (95), as expected, by contemplating the limits, \[ \lim_{\sigma \to 0} \lim_{T \to \infty} f_{00} = 0 \text{ as follows:} \]
\[ \lim_{T \to \infty} \frac{1}{2} \cot(\pi f_0 t) = \lim_{T \to \infty} \frac{1}{2} \cot(\pi f_0 t) \]
which can be written as \[ \lim_{f_{00} \to 0} \frac{1}{2} \cot(\pi f_0 t) \]
\[ \lim_{\sigma \to 0} \frac{1}{2} \cot(\pi f_0 t) \]
\[ \lim_{T \to \infty} \frac{1}{T} \sum_{k=1}^{N} \sin(2\pi k f_0 t) = \frac{1}{T}, \]
\[ \delta T(t, \pi/2) = \lim_{T \to \infty} \frac{1}{T} \sum_{k=1}^{N} \sin(2\pi k f_0 t) \]
\[ \delta T(t, \pi/2) = \lim_{N \to \infty} \frac{2}{T} \sum_{k=1}^{N} \sin(2\pi k f_0 t). \]

It is pertinent to note, contrary to perception in the literature, that the Hilbert kernels, presented in (19), (20), and (95), are well-defined and possess zero rather than pole at the origin, which is consistent with (i) the Hilbert kernel in discrete-time domain, and (ii) the definition of an odd function which is zero at the origin, if it is defined or limit exists at the origin.

**Example 2:** Next, in this example, we consider the Gravitational wave event GW150914 data and perform analysis, i.e., noise removal, IF and the Hilbert spectrum (HS) or TFE estimation using the proposed methodology. Albert Einstein predicted the Gravitational waves (GWs) in 1916 which are ripples in the space-time continuum and travel outward at the speed of light from a source of origin. A binary black hole merger event, nearly 1.3 billion light years away, generated the GW event GW150914 [43], marks one of the most important scientific inventions in the history of human life. Analysis of the GWs reveals information about source or cosmic-event that produces ripples in the space-time continuum.

Using the proposed time-varying filtering (TVF) (14), we consider the noise removal, IF estimation and HS representation of the GW150914 data, recorded by the laser interferometer Gravitational-wave observatory (LIGO) with sampling rate \( F_s = 16384 \) Hz, which is publicly available for downloading at [44].

Frequency of this GW150914 signal sweeps upwards from 35 Hz to 250 Hz and amplitude-strain increases to peak GW strain of \( 1.0 \times 10^{-21} \) [43]. As the IF of a GW signal can be used to estimate many parameters such as primary mass and secondary mass, luminosity distance, total mass, chirp mass, separation, effective spin and velocity of binary black hole merger [43], therefore, noise removal and an accurate IF estimation from the GW time-series is of paramount importance.

Fig. 2(a) presents the GW150914 H1 strain time-series (top figure), recorded at the LIGO Hanford, that is heavily corrupted with noise. The Fourier spectrum, shown in the bottom of Fig. 2(a), of this GW time-series is not able to reveal the non-stationarity (i.e., upwards sweep or increase in the frequency and amplitude with time) inherently present in the signal. The TFE representation of the GW150914 H1 time-series without decomposition using the FDM is shown in Fig. 2(b) which clearly shows the signal frequency is increasing with time, however, due to noise there are lots of spurious fluctuations in the time-frequency plane. The GW time-series is decomposed using the FDM into a set of six FIFBs [FIFB1 (25–50 Hz), FIFB2 (50–100), FIFB3 (100–200), FIFB4 (200–400), FIFB5 (400–800), FIFB6 (800–1912) Hz] and a low frequency component (LFC) of band 0–25 Hz which are presented in Fig. 2(c). In Fig. 2(d), the top five FGB-F5 components are obtained by multiplying the time-domain Gaussian window function with corresponding FIFBs (FIFB1-FIFB5), and the bottom graph presents the reconstructed GW waveform which is obtained by the addition of the five FGB-F5 components. In reconstruction of the GW waveform, the LFC and highest frequency FIFB6 have not been considered as they represent out of band noise components present in the GW150914 H1 time-series. The following time-domain Gaussian windows have been used for the TVF

\[ H_1(t) = \begin{cases} 1, & 0 \leq t < \mu_1, \\ \exp(-t^2/2\sigma_1^2), & \mu_1 \leq t \leq T, \end{cases} \]
\[ H_m(t) = \exp(-t^2/2\sigma_m^2), \quad 0 \leq t \leq T \quad \text{and} \quad m = 2, 3, 4, \]
\[ H_5(t) = \begin{cases} \exp(-t^2/2\sigma_5^2), & 0 \leq t < \mu_5, \\ 1, & \mu_5 \leq t \leq T, \end{cases} \]
where \( \mu_m \) is time corresponding to the peak value of FIFB and \( \sigma_n^2 \) is variance of the corresponding FIFB denoted by \( X_{\alpha}(t) \) in (14).

Fig. 3 shows the further analysis, comparison, residue and HS estimation of the GW150914 H1 strain time-series using the proposed TVF (Example 2): (a) the GW150914 H1 strain time-series (top red dotted line), the proposed reconstructed waveform (top blue solid line), and an estimated difference between them, i.e., residue waveform (bottom); (b) the numerical relativity (NR) time-series [43] (top red dash-dot line), the reconstructed waveform (top blue solid line), and difference between the reconstructed and NR time-series (bottom); the HS estimates of the reconstructed and NR time-series are shown in (c) and (d), respectively. The exactly same analysis of the event GW150914 L1 strain time-series, recorded at LIGO Livingston, using the proposed TVF is also presented in Fig. 4.
Fig. 2. The event GW150914 H1 strain time-series, recorded at LIGO Hanford, analysis using the proposed TVF (Example 2): (a) The GW H1 strain time-series (top figure) [43], and its Fourier spectrum (bottom figure), (b) the Hilbert spectrum (TFE) without any decomposition, (c) Decomposition of the GW time-series into a set of FIBFs (FIBF1-FIBF6) and low frequency component (LFC), (d) the proposed TVF produces top five components, FG1-FG5, by multiplication of the FIBF1-FIBF6 with corresponding Gaussian time-windows, and the reconstructed GW (bottom) by sum of FG1-FG5. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Fig. 3. The event GW150914 H1 strain time-series analysis using the proposed TVF (Example 2): (a) the GW150914 H1 strain time-series (top red-dashed line), the reconstructed time-series (top blue solid line), and difference between the original and reconstructed (i.e., residue) signal (bottom), (b) Reconstructed time-series (top blue solid line) and numerical relativity (NR) time-series (top red dashed line), and difference between the reconstructed and NR time-series (bottom), (c) HS (TFE) estimates of the reconstructed time-series, and (d) HS (TFE) estimates of the NR time-series.
The two detectors at Hanford and Livingston are about 3000 km apart which corresponds to the maximum time-delay of 10 ms and phase-shift of $\pi$ radians. The time-delay and phase-shift estimation using the cross-correlation between the Hanford and Livingston waveforms are shown in Fig. 5: (a) the cross-correlation between the H1 and L1 NR signals, (b) the cross-correlation between the reconstructed H1 and L1 NR signals. The cross-correlations, in the both cases, are almost symmetric with respect to negative peak. The estimated time-delays are 7.5 ms and 7.4 ms, which are very close and differs only by 0.1 ms, from the NR signals and reconstructed signals, respectively, and there is a phase-shift of $\pi$ radians. These time-delay and phase-shift are due to relative positioning of the detectors and propagation delay of the observed GW signals at the two observatories. As the GWs are propagating at the speed of light, the time-delay of 7.5 ms corresponds to an effective distance of $\approx 2250$ km between two observatories, which presents the first detection in Livingston and followed by Hanford, and suggests that the GW150914 event originated from the direction of Southern Hemisphere [45]. The negative peaks in the cross-correlations (in both cases) and very beginning of the signals in reconstructed wave-forms in Fig. 5(b) are the evidences of the phase shift of $\pi$ radians which is expected in between the observed GWs due to the relative positioning of the two detectors. Fig. 5(c) presents the H1 reconstructed GW advanced by 7.5 ms and the L1 reconstructed GW shifted by $\pi$ radians (top), and corresponding symmetric cross-correlation (bottom). Fig. 5(d) shows the H1 reconstructed GW, L1 reconstructed GW delayed by 7.5 ms and shifted by $\pi$ radians (top), and corresponding cross-correlation (bottom) which is symmetric around the origin with positive peak. The presented example clearly demonstrates the efficacy of the proposed TVF for the analysis of real-life non-stationary signals, data and other time-series.

**Example 3:** In this example, the phase shift analysis is considered for a Gaussian function, $x(t) = e^{-t^2}$. $0 \leq t < 5$ with sampling frequency $F_s = 1000$ Hz, which is shown in Fig. 6, where in the direction of arrow (a) Phase in the range of $[0, \pi]$ radians is increasing in step of $\pi/20$ radians, first plot is original Gaussian function and last one corresponds to $\pi$ radians phase-shift, plot corresponding to the tip of arrow is the HT of original signal, i.e., $\pi/2$ radian phase shift, (b) Phase in the range of $[\pi, 2\pi]$ radians is increasing in step of $\pi/20$ radians, first plot corresponds to $\pi$ radians phase-shift and the last one is the original Gaussian function obtained with $2\pi$ phase shift, plot corresponding to the tip of arrow is the HT of original signal with minus sign, i.e., $3\pi/2$ radians phase shift, (c) Phase shift in the range of $[0, 2\pi]$ using the DFT which is obtained by combining (a) and (b); (d) Phase shift in the range of $[0, 2\pi]$ using the DCT. It is observed that there is no difference, in phase shift obtained by the DFT [30] and DCT [45] approaches, for a set of signals which represent same underlying periodic extension that inherently present in the DFT (N-sample periodicity) and DCT (2N-sample periodicity with even symmetry) representations.

**Example 4:** Fig. 7 presents the phase shift analysis of a sine function, $x(t) = \sin(2\pi t)$, with $0 \leq t < 1$ and $F_s = 1000$ Hz, where phase is increasing in step of $\pi/10$ radians; (a) using the DFT and (c) using the DCT with phase in the range of $[0, \pi]$, in the direction of arrow, first plot is original sine function and last one corresponds to $\pi$ phase-shift, plot corresponding to the tip of arrow is the $\pi/2$ phase shift; (b) using the DFT and (d) using the DCT with phase in the range of $[\pi, 2\pi]$, in the direction of arrow, first plot is $\pi$ phase shifted sine wave and last one corresponds...
Fig. 5. Time-delay and phase-shift estimation using the cross-correlation between Hanford and Livingston waveforms (Example 2): (a) the cross-correlation between the H1 and L1 NR signals, (b) the cross-correlation between the reconstructed H1 and L1 NR signals. The cross-correlations, in the both cases, are almost symmetric with respect to negative peak. The estimated time-shift of 7.5 ms (7.4 ms) and phase-shift of \( \pi \) radians are due to relative positioning of the detectors and propagation delay of the observed GW signals at the two observatories, (c) H1 reconstructed GW advanced by 7.5 ms and L1 reconstructed GW shifted by \( \pi \) radians (top), and corresponding cross-correlation (bottom), (d) H1 reconstructed GW, L1 reconstructed GW delayed by 7.5 ms and shifted by \( \pi \) radians (top), and corresponding cross-correlation (bottom).

Fig. 6. Phase shift analysis of a Gaussian function of Example 3: (a) Phase in the range of \([0, \pi]\) is increasing in a step of \(\pi/20\), in the direction of arrow, first plot is original Gaussian function and last one corresponds to \(\pi\) phase-shift, plot corresponding to the tip of arrow is the HT of original signal, i.e., \(\pi/2\) phase shift, (b) Phase in the range of \([\pi, 2\pi]\) is increasing in a step of \(\pi/20\), in the direction of arrow, first plot corresponds to \(\pi\) phase-shift and last one is the original Gaussian function obtained by \(2\pi\) phase shift, plot corresponding to the tip of arrow is the HT of original signal with minus sign, i.e., \(3\pi/2\) phase shift. Phase shift in the range of \([0, 2\pi]\) using the DFT (c), and using the DCT (d).
to $2\pi$ phase-shift, plot corresponding to the tip of arrow is $3\pi/2$ phase-shift; (e) using the DFT which is obtained by superimposing (a) and (b); and (f) using the DCT which is obtained by superimposing (c) and (d). We observe clear differences, in phase shift obtained by the DFT (30) and DCT (45) approaches, for a set of signals which yield one periodic signal for the DFT ($N$-sample periodicity) representation and another periodic signal for the DCT ($2N$-sample periodicity with even symmetry) representation.

**Example 5:** This example considers a Gaussian function $x(t) = e^{-(t-5)^2}$, with $0 \leq t < 10$, $F_s = 1/T = 10$ Hz, and thus $x[n] = e^{-(n-5)^2}$. We computed true delayed signal as $x(t - t_0) = e^{-(t-t_0-5)^2}$, and delayed signal $x[n - n_0]$ using the proposed method (79) with fractional delay $t_0 = n_0 T$ sec, where $n_0 = 0.7$. Fig. 8 shows the fractional delay estimation of this Gaussian function: (upper) the original Gaussian function and its delayed version obtained theoretically using the expression $x(t - t_0) = e^{-(t-t_0-5)^2}$, (middle) the original Gaussian function and its delayed version obtained by the proposed method (79), and (lower) an estimated error, which is really small in the order of $10^{-11}$, by taking the difference between truly delayed signal and delayed signal obtained by the proposed method.

We consider a cos function $x(t) = \cos(\pi t)$, with $0 \leq t < 100$, $F_s = 1/T = 1$ Hz, and thus $x[n] = \cos(\pi n)$. We computed true delayed signal as $x(t - t_0) = \cos(\pi (t - t_0))$, and delayed signal $x[n - n_0]$ using the proposed method (79) with the fractional delay $t_0 = n_0 T$ sec, where $n_0 = 0.7$. Fig. 9 shows the fractional delay estimation of this cos function: (upper) the original cos function and its delayed version obtained theoretically using the expression $x(t - t_0) = \cos(\pi (t - t_0))$, (middle) the original cos function and its delayed version obtained by the proposed method (79), and (lower) an estimated error, which is really small in the order of
Fig. 8. Fractional delay analysis of a Gaussian function of Example 5: (upper) the original Gaussian function and its delayed version obtained theoretically, (middle) the original Gaussian function and its delayed version obtained by the proposed method, and (lower) an error estimated by taking the difference between truly delayed signal and delayed signal obtained by the proposed method.

Fig. 9. Fractional delay analysis of cos(\pi t) function of Example 5: (upper) the original cos function and its delayed version obtained theoretically, (middle) the original cos function and its delayed version obtained by the proposed method, and (lower) an error estimated by taking the difference between truly delayed signal and delayed signal obtained by the proposed method.

Fig. 10. Estimation of the fractional derivative of order 0.0, 0.25, 0.5, 0.75 and 1.0 of the sine function of Example 6 by the proposed method.
Fig. 11. Estimation of the fractional integral of order 0.0, 0.25, 0.5, 0.75 and 1.0 of the sine function of Example 6 by the proposed method.

Fig. 12. Phase shift analysis of the cosine function of Example 7 where phase is increasing in step of $\pi / 10$: using the proposed WPT with the Morse wavelet (a) top plot has phase in the range of [0, $\pi$], in the direction of arrow, first plot is the original cosine function and last one corresponds to $\pi$ phase-shift, plot corresponding to the tip of arrow is $\pi / 2$ phase shift; (b) bottom plot has phase in the range of [$\pi$, $2\pi$], in the direction of arrow, first plot is $\pi$ phase shifted cosine function and last one corresponds to $2\pi$ phase-shift, plot corresponding to the tip of arrow is $3\pi / 2$ phase shift.

$10^{-14}$, by taking the difference between truly delayed signal and delayed signal obtained by the proposed method.

**Example 6:** This example presents the fractional order derivative (FOD) and fractional order integral (FOI) of a sine function, $x(t) = \sin(2\pi t)$, with $0 \leq t < 10$ and $F_s = 1000$ Hz. Fig. 10 and Fig. 11 present the FOD and FOI, respectively, of the sine wave where fractional order $\mu \in \{0.0, 0.25, 0.5, 0.75, 1.0\}$; FOD and FOI are estimated using the proposed method (80).

**Example 7:** In this example, a cosine wave is considered and desired phase shift is obtained using the proposed WPT and WQT. Fig. 12 presents the phase shift analysis of a cosine function, $x(t) = \cos(2\pi t)$, with $0 \leq t < 5$ and $F_s = 1000$ Hz, where phase is increasing in step of $\pi / 10$ radians using the proposed WPT with the Morse wavelet: (a) top plot with phase in the range of [0, $\pi$], in the direction of arrow, first plot is the original cosine function and last one corresponds to $\pi$ phase-shift, plot corresponding to the tip of arrow is the $\pi / 2$ phase shift, i.e., the WQT; (b) bottom plot has phase in the range of [$\pi$, $2\pi$], in the direction of arrow, first plot is $\pi$ phase shifted cosine wave and last one corresponds to $2\pi$ phase-shift, plot corresponding to the tip of arrow is $3\pi / 2$ phase-shift in the original signal.

In this study, we have considered many examples and demonstrated the efficacy of the proposed methodologies. However, it may be noted that Fourier methods provide better error convergence when function is smooth, error increases when there are discontinuities in the considered function, and error also changes with respect to the size of the DFT implementation [51,52]. More specifically, error for smooth functions reduces exponentially in the $L^2$ and $L^\infty$ norms. However, the most significant issue is that non-smooth functions suffer from spurious high frequency oscillations near discontinuities, error convergence is polynomial and, in the worst case, error converges slower than the linear function. In particular, the $L^2$-norm of the error for a discontinuous function converges at a sub-linear rate and the $L^\infty$-norm does not converge at all [68] due to manifestation of the Gibbs phenomenon [69]. Because of the maximal efficacy and efficiency of Fourier methods, fortunately many studies such as filtering [70], reprojection [71–73] and mollification [68,74] have been proposed to resolve the Gibbs phenomenon in the non-smooth scenario and recover the exponential error convergence which makes Fourier methods very useful. Thus, these recovery methods and the proposed one may be combined to obtain better error convergence in the case of non-smooth functions.
5. Conclusion and future scope

This work introduced the generalized Fourier representation (GFR), which is completely based on the Fourier representation of a signal, and presented seven special cases of the GFR, namely the Fourier representation, phase transform (PT), time-delay including fractional delay of discrete-time signals, time derivative and integral including fractional order, analog and digital modulations, and filtering operations. The most important and fundamental contribution of this study is the PT which is a special case of the GFR and a true generalization of the Hilbert transform. Using the proposed PT, the desired phase-shift and time-delay can be obtained in a signal under analysis. The kernel of the PT is derived to obtain any constant phase shift, the various properties of the PT are discussed, and it is shown that the HT is a special case of the PT when phase-shift is \( \pi/2 \) radians. An extension of the one-dimensionalal PT is also provided for the two-dimensional image signals, which can easily be extended for higher dimensional signals. Using the PT, it is demonstrated that (i) a constant phase shift (e.g., \( \pi/2 \) phase shift) in a signal corresponds to variable time-delays in all the harmonics, (ii) a frequency dependent phase shift in all the harmonics of the Fourier representation can be used to obtain a constant time-delay in a signal, (iii) a constant phase shift is same as the constant time-delay only for a single frequency sinusoid. Contrary to perception in the literature, it is demonstrated that the kernel of the Hilbert transform in continuous time-domain has a zero rather than a pole at the origin. The narrowband Fourier representation, for the time-frequency representation and analysis, of a signal is also obtained using the proposed GFR.

The time derivative and time integral, including fractional order, of a signal are obtained using the GFR. The DCT based implementation is proposed to avoid end artifacts due to discontinuities present in the both ends of a signal. A new method is proposed to obtain a fractional delay in a discrete-time signal using the Fourier representations, i.e., DFT, DISTS and DCTS. The DFT implementations of all the proposed representations are also developed. Using the analytic wavelet transform (AWT), the wavelet phase transform (WPT) is proposed to introduce a desired phase-shift in a signal under-analysis, and two representations of wavelet quadrature transform (WQT) are presented as special cases of the WPT where phase-shift is \( \pi/2 \) radians.

In this study, we have considered the limited number of examples for error estimations, noise removal, real-time data analysis and processing using the proposed methods. The future directions of research would be to consider and apply the proposed methods in various applications such as biomedical signals electrocardiogram and electroencephalogram processing, seismic data analysis, image processing, audio and other sound signal representation and noise removal. Other interesting future scopes of the study are (1) to perform a detailed mathematical (e.g., pointwise and \( L^p \) norm, \( 1 < p < \infty \)) convergence analysis of the proposed GFRs, (2) to consider and study error convergence in non-smooth functions with the filtering and mollification methods, (3) to investigate and define the generalized fractional Fourier representations using the fractional Fourier transform (FrFT) and inverse FrFT, and explore its various special cases.

Ethics statement

This study did not involve any active collection of human data.

Data accessibility statement

Gravitational wave data is publicly available, and no other data is used in this study which cannot be generated by MATLAB/Python or any other programming language.

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Declaration of competing interest

The author declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Acronyms and symbols

The list of acronyms, symbols and parameters used in this study is presented in Table 1 and Table 2, respectively.

Appendix B. Transform matrices of DCTS and DISTS

The standard sixteen DCTS and DISTS transform matrices (\( C_i \) and \( S_i \) for \( i = 1, 2, \ldots, 8 \), with their \( nk \)-th element, denoted by \( (C_i)_{nk} \) and \( (S_i)_{nk} \), respectively) are defined in [2] as

\[
(C_1)_{nk} = a y_n \gamma_k \cos \left( \frac{nk \pi}{N-1} \right),
\]

\[
(C_2)_{nk} = b \sigma_k \cos \left( \frac{n + \frac{1}{2}}{N} \frac{k \pi}{N} \right),
\]

\[
(C_3)_{nk} = b \sigma_n \cos \left( \frac{k + \frac{1}{2}}{N} \frac{n \pi}{N} \right),
\]

\[
(C_4)_{nk} = b \cos \left( \frac{n + \frac{1}{2}}{N} \frac{k + \frac{1}{2}}{N} \pi \right),
\]

\[
(C_5)_{nk} = c \sigma_k \sigma_n \cos \left( \frac{nk \pi}{N} \right),
\]

\[
(C_6)_{nk} = c \sigma_n \sigma_k \cos \left( \frac{N - 1}{N} \frac{k 2 \pi}{N} \right),
\]

\[
(C_7)_{nk} = c \sigma_n \sigma_k \cos \left( \frac{N - 1}{N} \frac{n 2 \pi}{N} \right),
\]

\[
(C_8)_{nk} = d \cos \left( \frac{n + \frac{1}{2}}{N} \frac{k + \frac{1}{2}}{N} \frac{2 \pi}{2N} \right),
\]

\[
(S_1)_{nk} = b \sin \left( \frac{nk \pi}{N} \right),
\]

\[
(S_2)_{nk} = b \delta_k \sin \left( \frac{n + \frac{1}{2}}{N} \frac{k + \frac{1}{2}}{N} \right),
\]

\[
(S_3)_{nk} = b \delta_n \sin \left( \frac{k + \frac{1}{2}}{N} \frac{n + \frac{1}{2}}{N} \right),
\]

\[
(S_4)_{nk} = b \sin \left( \frac{n + \frac{1}{2}}{N} \frac{k + \frac{1}{2}}{N} \right),
\]

\[
(S_5)_{nk} = c \sin \left( \frac{nk \pi}{N} \right),
\]

\[
(S_6)_{nk} = c \sin \left( \frac{N - 1}{N} \frac{k \pi}{N} \right),
\]

\[
(S_7)_{nk} = c \sin \left( \frac{N - 1}{N} \frac{n \pi}{N} \right),
\]

\[
(S_8)_{nk} = c \sin \left( \frac{N - 1}{N} \frac{k \pi}{N} \right).
\]
Table 1
List of acronyms.

| Abbreviation | Full Form |
|--------------|-----------|
| FR           | Fourier representation |
| FT/FSPT      | Phase transform/Fourier–Singh phase transform |
| AWT          | Analytic wavelet transform |
| FS           | Fourier series |
| SST          | Fourier sine transform |
| DTFT         | Discrete-time Fourier transform |
| DFT          | Discrete Fourier transform |
| STFT         | Short-time Fourier transform |
| DCT          | Discrete cosine transform |
| MODCT        | Modified DCT |
| TFE          | Time-frequency-energy |
| FQT          | Fourier quadrature transform |
| AM           | Amplitude modulation |
| WT           | Wavelet transform |
| AWF          | Analytic wavelet function |
| WQT          | Wavelet quadrature transform |
| AS           | Analytic signal |
| LPC          | Low frequency component |
| FDM          | Fourier decomposition method |
| TVF          | Time-varying filtering |
| IFT          | Inverse FT |
| FCQT         | Fourier cosine quadrature transform |
| FSQT         | Fourier sine quadrature transform |
| IDFT         | Inverse DFT |
| FSAARs       | FSAS representations |
| GW           | Gravitational wave |
| FOD          | Fractional order derivative |
| HS           | Hilbert spectrum |

Table 2
List of symbols and parameters.

| Symbol | Description |
|--------|-------------|
| $x_T(t)$ | A periodic signal with period $T$ |
| $a_k, b_k$ | $k$-th coefficients in FS representation |
| $\phi_k$ | $k$-th phase of complex coefficient |
| $\psi_k(t)$ | $k$-th phase scaling function |
| $\mathcal{H}(\omega, t)$ | Amplitude scaling function |
| $\text{sgn}(t)$ | Sign function |
| $h(t)$ | Kernel of Hilbert transform |
| $z(t)$ | Analytic signal |
| $x_q(t, \lambda)$ | NBF of a function $x(t)$ |
| $\psi(t)$ | Mother wavelet function |
| $\psi(t)$ | Kernel of analytic signal |
| $E_q$ | Energy of a signal $x(t)$ |
| $X_q(t)$ | CWT of a signal $x(t)$ |
| $x(t)$ | Scaling and translation parameters |
| $x_\text{WQFT}(t)$ | Wavelet of a signal $x(t)$ |
| $X[k]$ | DFT of a signal $x[n]$ |
| $X_\text{DCT-2}[k]$ | DCT-2 of a signal $x[n]$ |
| $\mathbf{C}_i$ | $i$-th DCT matrix |
| $\mathbf{S}_i$ | $i$-th DST matrix |

\[
\begin{align*}
\mathbf{(S}_6\mathbf{)}_{nk} & = c \sin \left[ \frac{n + \frac{1}{2}}{2} \right] k \left( n + \frac{1}{2} \right) \frac{2\pi}{2N - 1}, \\
\mathbf{(S}_7\mathbf{)}_{nk} & = c \sin \left[ \frac{k + \frac{1}{2}}{2} \right] \frac{(n + 1)2\pi}{2N - 1}, \\
\mathbf{(S}_8\mathbf{)}_{nk} & = c e_{n}\epsilon_{k} \sin \left[ \frac{(n + 1)}{2} \right] k \left( n + \frac{1}{2} \right) \frac{2\pi}{2N - 1}. 
\end{align*}
\]

where $a = \sqrt{\pi}, b = \sqrt{\pi}, c = \frac{1}{2\sqrt{2N - 1}},$ and $d = \frac{1}{\sqrt{2N - 1}}$; normalization factors are unity except for $\gamma_n = \gamma_k = \frac{1}{\sqrt{2}}$ for $n = k = 0$ or $N - 1$, $\sigma_n = \sigma_k = \frac{1}{2\sqrt{2}}$ for $n = k = 0$, and $\epsilon_n = \epsilon_k = \frac{1}{\sqrt{2}}$ for $n = k = N - 1$; $0 \leq n, k \leq N - 1$ for all the $N$-th-order DCTs/DSTs except for the $(N - 1)$-th-order DST-1 and DST-5 where $1 \leq n, k \leq N - 1$. Inverses of the DCTs and DSTs are computed by transpose relation (as they are unitary transform) $\mathbf{C}^{-1}_i = \mathbf{C}^T_i$ and $\mathbf{S}^{-1}_i = \mathbf{S}^T_i$, respectively. The elements of other sixteen transform matrices, using (97), are defined in [4] as follows:

\[
\begin{align*}
\mathbf{(S}_1\mathbf{)}_{nk} & = a_k \gamma_n \sin \left[ \frac{nk\pi}{N - 1} \right], \\
\mathbf{(S}_2\mathbf{)}_{nk} & = b\sigma_n \sin \left[ \frac{(n + 1)2\pi}{2N - 1} \right], \\
\mathbf{(S}_3\mathbf{)}_{nk} & = b\sigma_n \sin \left[ \frac{k + \frac{1}{2}}{2N - 1} \right], \\
\mathbf{(S}_4\mathbf{)}_{nk} & = b\cos \left[ \frac{(n + 1)2\pi}{2N - 1} \right], \\
\mathbf{(S}_5\mathbf{)}_{nk} & = c\sigma_n \cos \sin \left[ \frac{nk\pi}{2N - 1} \right], \\
\mathbf{(S}_6\mathbf{)}_{nk} & = c\epsilon_n \epsilon_k \sin \left[ \frac{(n + 1)2\pi}{2N - 1} \right].
\end{align*}
\]
Appendix C. Multidimensional PT

In this appendix, we consider the PT of a 2D signal (e.g., image) which can be easily extended for multidimensional signals. Let \( g(x, y) \) be a non-periodic and real valued function, then the 2D-FT is defined as

\[
G(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{-j(\omega_1 x + \omega_2 y)} \, dx \, dy,
\]

and the inverse 2D-FT is defined as

\[
g(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\omega_1, \omega_2) e^{j(\omega_1 x + \omega_2 y)} \, d\omega_1 \, d\omega_2.
\]

The 2D PT transfer function corresponding to 1D counterpart (27) can be written as

\[
H(\alpha(\omega_1, \omega_2)) = \begin{cases} e^{-j\alpha(\omega_1, \omega_2)}, & 0 \leq \omega_1 < \infty, -\infty < \omega_2 < \infty \\ e^{j\alpha(\omega_1, \omega_2)}, & -\infty < \omega_1 \leq 0, -\infty < \omega_2 < \infty \end{cases},
\]

where the 2D analytic signal (2D-AS) is defined by considering the first and fourth quadrants of the 2D-FT plane as [6]

\[
z(x, y, \alpha(\omega_1, \omega_2)) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\omega_1, \omega_2) e^{j(\omega_1 x + \omega_2 y)} e^{-j\alpha(\omega_1, \omega_2)} \, d\omega_1 \, d\omega_2,
\]

and if \( \alpha(\omega_1, \omega_2) = \alpha \), then \( z(x, y, \alpha) = z(x, y)e^{-j\alpha} \), therefore 2D counter part of 1D PT (30) can be defined as

\[
g(x, y, \alpha) = \text{Re}[z(x, y, \alpha)] = \cos(\alpha)g(x, y) + \sin(\alpha)\hat{g}(x, y),
\]

where \( \hat{g}(x, y) \) is the HT of \( g(x, y) \). The 2D-PT can be computed by considering the real part of the 2D-PT of 2D-AS which we defined as

\[
z(x, y) = g(x, y) + j\hat{g}(x, y)
\]

where \( \hat{g}(x, y) \) is the HT of \( g(x, y) \). The 2D-PT can be computed by considering the real part of the 2D-PT of 2D-AS which we defined as

\[
\hat{g}(x, y) \text{ is the HT of } g(x, y).
\]

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\[
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where \( \hat{g}(x, y) \) is the HT of \( g(x, y) \). The 2D-PT can be computed by considering the real part of the 2D-PT of 2D-AS which we defined as
C.1. Phase transform of an image signal

The 2D discrete-time Fourier transform (2D-DFT) and inverse 2D-DFT, for a non-periodic and real-valued signal \( g[m, n] \), are defined as

\[
G(\Omega_1, \Omega_2) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} g[m, n] e^{-j\Omega_1 m - \Omega_2 n} \]

\[
g[m, n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} G(\Omega_1, \Omega_2) e^{j\Omega_1 m + \Omega_2 n} \, d\Omega_1 \, d\Omega_2. \tag{111}
\]

Let \( g[m, n] = \delta[m, n] \), then \( G(\Omega_1, \Omega_2) = 1 \) and from (111), we can write

\[
\delta[m, n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \exp(j\Omega_1 m + \Omega_2 n) \, d\Omega_1 \, d\Omega_2, \tag{112}
\]

which can be further simplified as

\[
\delta[m, n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cos(\Omega_1 m + \Omega_2 n) \, d\Omega_1 \, d\Omega_2, \tag{113}
\]

because the imaginary part of (112) is zero, i.e.,

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sin(\Omega_1 m + \Omega_2 n) \, d\Omega_1 \, d\Omega_2 = 0.
\]

Now, from Observation 3.1 which is based on the Bedrosian theorem, we conclude that the 2D Hilbert transform (2D-HT) as a true extension of 1D-HT can be derived by considering the regions of 2D-DFT as shown in Fig. 13, thus we hereby define the kernel of the 2D-PT in frequency domain as

\[
\Phi(\Omega_1, \Omega_2, \alpha) = \begin{cases} e^{-j\alpha}, & \Omega_1 + \Omega_2 \geq 0, \\ e^{j\alpha}, & \Omega_1 + \Omega_2 \leq 0. \end{cases} \tag{114}
\]

Using (114), we write (113) as

\[
\delta[m, n] = \frac{1}{\pi} \int_{-\pi}^{\pi} \left[ \int_{-\Omega_2}^{\Omega_2} \cos(\Omega_1 m + \Omega_2 n) \, d\Omega_2 \right] \, d\Omega_1, \tag{115}
\]

and define the HT of (115) as

\[
h[m, n] = \delta[m, n, \pi/2] = \frac{1}{\pi} \int_{-\pi}^{\pi} \left[ \int_{-\Omega_2}^{\Omega_2} \sin(\Omega_1 m + \Omega_2 n) \, d\Omega_2 \right] \, d\Omega_1, \tag{116}
\]

and obtain

\[
h[m, n] = \begin{cases} 0, & m = 0, n = 0, \\ \delta[n-m], & m = n, m \neq 0, n \neq 0, \\ \frac{-\cos(\pi m)}{\pi}, & m = 0, n \neq 0, \\ \frac{-\cos(\pi n)}{\pi}, & m \neq 0, n = 0, \\ 0, & m \neq n \neq 0. \end{cases} \tag{117}
\]

Moreover, the kernel of the 2D PT can be written as

\[
\delta[m, n, \alpha] = \cos(\alpha) \delta[m, n] + \sin(\alpha) \delta[m, n, \pi/2]. \tag{118}
\]

Fig. 13. Phase transform regions with four quadrants (I, II, III and IV) in Fourier domain, II quadrant is divided in two parts (a) \( \Omega_2 > |\Omega_1| \) and (b) \( |\Omega_1| > \Omega_2 \); IV quadrant is divided in two parts (a) \( \Omega_1 > |\Omega_2| \) and (b) \( |\Omega_2| > \Omega_1 \); complete region can be divided in two parts by line \( \Omega_1 + \Omega_2 = 0 \);

and thus

\[
g[m, n, \alpha] = \cos(\alpha) g[m, n] + \sin(\alpha) g[m, n, \pi/2], \tag{119}
\]

where \( \delta[m, n, \alpha] = \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{-\Omega_2}^{\Omega_2} \cos((\Omega_1 m + \Omega_2 n) - \alpha) \, d\Omega_2 \, d\Omega_1 \) is the 2D-PT of unit impulse sequence \( \delta[m, n] \); \( g[m, n, \alpha] = \delta[m, n, \pi/2] * g[m, n] \) is the 2D-PT of signal \( g[m, n] \) and (119) is the 2D-HT of signal \( g[m, n] \) when \( \alpha = \pi/2 \). Using (111) and above discussion, we obtain the 2D analytic signal (2D-AS) as

\[
z[m, n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\Omega_2}^{\Omega_2} [G(\Omega_1, \Omega_2) \exp(j\Omega_1 m + \Omega_2 n)] \, d\Omega_1 \, d\Omega_2, \tag{120}
\]

and its HT as

\[
z[m, n, \alpha] = z[m, n] e^{-j\alpha} = g[m, n, \alpha] + jg[m, n, \alpha + \pi/2]. \tag{121}
\]

Similar to (29), one can obtain the kernel of 2D discrete-time analytic signal and compute the phase difference between \( \delta[m, n] \) and its HT \( h[m, n] \) (117) as

\[
\phi_{h,m}[m, n] = \tan^{-1} \left( \frac{h[m, n]}{\delta[m, n]} \right) = \begin{cases} 0, & m = 0, n = 0, \\ \pi/2, & m = n, m > 0, n > 0, \\ -\pi/2, & m = n, m < 0, n < 0, \\ -\pi/2, & (m > 0 \text{ even}) \text{ or } (m < 0 \text{ odd}), n = 0, \\ \pi/2, & (m > 0 \text{ odd}) \text{ or } (m < 0 \text{ even}), n = 0, \\ -\pi/2, & m = 0, (n > 0 \text{ even}) \text{ or } (n < 0 \text{ odd}), \\ \pi/2, & m = 0, (n > 0 \text{ odd}) \text{ or } (n < 0 \text{ even}). \end{cases} \tag{122}
\]

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