Mass matrices in $SU(5) \times Q_6$ SUSY-FUT’s

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Abstract. In the context of two-loop Finite Supersymmetric Theories with gauge group $SU(5)$, we present parametric solutions to the finiteness conditions using $Q_6$ as a family symmetry group. Assuming an MSSM scenario with just one pair of light Higgs doublets, we obtain mass matrices at the GUT scale for this class of theories.

1. Introduction
Since the recent discovery of a Higgs-like state in the LHC with mass around 125 GeV [1, 2], it is worth looking more closely at beyond the Standard Model theories which predict a mass for the Higgs particle around this value. N=1 Supersymmetric Finite Unified Theories (FUT’s) are theories that are characterized by the fact that, at least at two-loops, UV divergences are absent and can even be made finite to all-loops [3–7]. With $SU(5)$ as a gauge group, a FUT model was proposed in which, restricted to one generation, a top mass at $\sim 172$ GeV and a Higgs mass at $\sim 122–126$ GeV were predicted, in good agreement with experimental data [8–10] (for a recent review of finite theories and their phenomenological implications see ref. [11]). In this paper, we analyse $SU(5)$ FUTs with the discrete group $Q_6$ as family symmetry group, and present the resulting mass matrices. A study with different family symmetry group can be found in [12]. Section 2 is devoted to general aspects of finite theories and the form that the finiteness conditions take in $SU(5)$. In section 3 we very briefly review the $Q_6$ properties to proceed with the assignment of the fields with this symmetry group. Then, in Section 4 we present the boundary conditions that the finiteness conditions impose on the MSSM, ending with some concluding remarks in section 5.

2. SUSY-FUT’s in $SU(5)$
For a SUSY theory with a gauge simple group $G$ the absence of UV divergences at two-loop level is guaranteed if it satisfies two requirements [13]. First the Yukawa couplings and the gauge couplings are related as
\[ \sum_{ij} Y_{ij} Y^{*ij} = g^2 \delta^i_k, \] (1)
where $Y^{*ij} = Y^*_{ij}$. The second condition is that the assignment of matter content satisfies the following relation
\[ \sum_i T(R_i) - 3C_2(G) = 0, \] (2)
where $T(R_i)$ is the Dynkin-index associated with the $i$-field that it is in the $R_i$ representation of the gauge group $G$. The remarkable feature of these Finiteness Conditions (FC), is that if one finds a non-degenerate and isolated solution to eq.(1), then finiteness to all order in perturbation theory can be achieved [3, 4].

For $G = SU(5)$, a solution to eq. (2) is obtained with the following matter content [14, 15]:

\[ 5 : X_a, \quad 5 : \bar{X}_a, \quad 24 : \Sigma, \quad 10 : 10_i, \quad 5 : 5_i. \quad (a = 1, 2, 3, 4; \quad i = 1, 2, 3) \]  

(3)

The 10 and 5 fields are interpreted as the fields where the three families of quarks and leptons are. The most general $SU(5)$ Yukawa superpotential with this matter content is [16]

\[ f_y = \frac{1}{2} Y_{1010X} 1010X + Y_{105X} 105X + Y_{X\bar{X}\Sigma} X\Sigma\bar{X} + \frac{1}{6} Y_{\Sigma^3} \Sigma^3 + \frac{1}{2} Y_{\bar{5}510} \bar{5}510 + \frac{1}{2} Y_{\bar{X}\bar{X}10} \bar{X}\bar{X}10 + Y_{X\Sigma\Sigma} X\Sigma\Sigma, \]  

(4)

where generation and gauge indices are omitted. The last three terms would generate renormalizable terms that violate leptonic and baryonic number, so we make $Y_{\bar{5}510} = Y_{\bar{X}\bar{X}10} = Y_{X\Sigma\Sigma} = 0$, we can achieve this either as a result of the family group assignment we will discuss below or because of an extra Abelian symmetry that has been imposed. We denote the family part of the Yukawas as $f_{ij\alpha}, g_{ij\alpha}, h_{ab}$ and $p$ for the first four terms in equation (4), respectively. The general finiteness conditions (FC) reduce to [17]

\[ X^b_a = 3 f_{ij\alpha} f^{ijb} + \frac{24}{5} h_{ac} h^{bc} = \frac{24}{5} g^2 \delta^b_a \]  

(5)

\[ \bar{5}_i^b = 4 g_{kia} g^{kja} = \frac{24}{5} g^2 \delta^b_i \]  

(6)

\[ X^b_a = 4 g_{kja} g^{ijb} + \frac{24}{5} h_{ca} h^{cb} = \frac{24}{5} g^2 \delta^b_a \]  

(7)

\[ 10^b_i = 3 f_{ik\alpha} f^{ikb} + 2 g_{ik\alpha} g^{i\alpha} = \frac{36}{5} g^2 \delta^b_i \]  

(8)

\[ \Sigma = h_{ab} h^{ab} + \frac{21}{5} p^2 p = \frac{36}{5} g^2, \]  

(9)

In principle there are 55 quadratic equations with 89 complex unknowns taking into account the three families. Thus family symmetries become mandatory, but before, we can obtain some useful conditions on the solutions. From eqs. (7) and (8) we have:

\[ \sum_{a}^{4} X^a_a - \sum_{i}^{3} \bar{5}_i^b = \sum_{a,b}^{4} |h_{ab}|^2 = g^2, \]  

(10)

so from this last equation and eq. (9) one can see that

\[ |p|^2 = \frac{15}{4} g^2. \]  

(11)

If only $m$ ($m \leq 4$) fields of $X_a$ are coupled in the 105X term, (i.e., $g_{kib} = 0$ unless $b = b^1, \cdots, b^m$), then, taking the sum $X^b_a$ over $b$ ($b = b^1, \cdots, b^m$) minus the sum $\bar{5}_i^b$ over $I$ ($i = 1, 2, 3$), implies the following relation

\[ \sum_{b}^{b^1,\cdots,b^m} X^b_a - \sum_{i}^{3} \bar{5}_i = \sum_{b}^{b^1,\cdots,b^m} \sum_{c}^{4} |h_{cb}|^2 = (m - 3)g^2. \]  

(12)
Similarly for the 1010X sector; \( f_{ij} = 0 \) unless \( a = a^1, \ldots, a^n, (n \leq 4) \), then

\[
\sum_a X^a_i = (\sum_i 1^{ij} - \frac{1}{2} \sum_i 5^i_j) = \sum_a \sum_c |h_{ac}|^2 = (n - 3)g^2,
\]

so we need at least three fields \( X' \) (or \( X' \)'s) to be coupled to the 105 (1010) terms. If \( m = 3 \) \((n = 3)\) the only non-vanishing elements of \( h_{ab} \) are those of the column \( b \neq \{b^1, b^2, b^3\} \) \((a \neq \{a^1, a^2, a^3\})\). The converse is also true, if the only non-vanishing elements of \( h_{ab} \) are just of one column(row), then \( m = 3(n = 3) \). When \( (m = n = 3) \), \( h_{ab} \) is reduced to having only one non-vanishing element.

3. \( Q_6 \) symmetry group assignment

Since all (Bi)Dihedral groups have only one and two dimensional irreducible representations (irreps), their study becomes important to explain the observed hierarchy of masses, where we have one heavy plus two light generations. The \( Q_6 \) group has four one-dimensional irreps (denoted as \( 1, 1', 1'', 1''' \)) and two dimension two irresp (denoted as \( 2, 2' \)). In tables 1 and 2 the tensor products of the two-dimensional vectors are presented showing explicitly their tensor matrices, also it is shown the assignment necessary for other fields to couple to these tensors in order to form a \( Q_6 \) invariant term.

\[
\begin{array}{c|c|c|c|c|}
\tau_0 & (2 \otimes 2)_1' & (2' \otimes 2')_1'' & (2' \otimes 2')_1''' & (2 \otimes 2')_1'''' \\
\hline
(2 \otimes 2)_1' & 1 & 1 & 1 & 1 \\
(2' \otimes 2')_1'' & 1 & 1 & 1 & 1 \\
(2' \otimes 2')_1''' & 1 & 1 & 1 & 1 \\
(2 \otimes 2')_1'''' & 1 & 1 & 1 & 1 \\
\end{array}
\]

Table 1. The first row represents the tensor matrices that appear in the tensor products of two doublets that form one-dimensional irreps, which are shown in the second row. The third row tells us which one irrep forms a \( Q_6 \) invariant when multiplied with a tensor of the second row.

\[
\begin{array}{c|c|c|c|c|}
(2 \otimes 2')_2 & (2 \otimes 2')_2' & (2 \otimes 2')_2'' & (2 \otimes 2')_2''' & (2 \otimes 2')_2'''' \\
\hline
(2 \otimes 2')_2 & -\tau_1 & \tau_3 & \tau_3 & \tau_0 \\
(2 \otimes 2')_2' & \tau_3 & -\tau_1 & \tau_0 & \tau_3 \\
(2 \otimes 2')_2'' & \tau_0 & \tau_3 & -\tau_1 & \tau_0 \\
(2 \otimes 2')_2''' & \tau_3 & \tau_0 & \tau_3 & -\tau_1 \\
\end{array}
\]

Table 2. Matrices of the two-dimensional part of the tensor product of two doublets. The doublet assigned needed to form an invariant term is also shown.

If the \( 10_i \) \((i = 1, 2, 3)\) fields are all in one dimensional irreps, we can form invariant tensors with \( X_a, (a = 1, 2, 3, 4) \) (in the sector 1010X), only if some of these are also in one-dimensional irreps of \( Q_6 \). Since the main motivation of introducing non-Abelian groups as \( Q_6 \) is to relate matrix elements for the Yukawa elements, we will take \( 10_1 \) and \( 10_2 \) to be a doublet and also we will assign at least one doublet structure to each of the remaining fields, \( 5_i, X_a \) and \( \bar{X}_a \). Thus,

\[
10 \equiv \begin{pmatrix} 10_1 \\ 10_2 \end{pmatrix}, \quad 5 \equiv \begin{pmatrix} 5_1 \\ 5_2 \end{pmatrix}, \quad X \equiv \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad \bar{X} \equiv \begin{pmatrix} \bar{X}_1 \\ \bar{X}_2 \end{pmatrix},
\]

are doublets of \( Q_6 \), whereas \( 10_3 \) and \( 5_3 \) have to be in one of the four one-dimensional irreps. In order to work both sectors at the same time, we define

\[
\Psi^A = 10(10), \quad \Psi^B = 10(5), \quad \Psi^A_3 = 10(10_3) \quad \Psi^C = X(\bar{X}), \quad \Psi^C_2 = X_2(\bar{X}_2), \quad \Psi^B_3 = 10_3(5_3), \quad \Psi^C_3 = X_3(\bar{X}_3), \quad \Psi^C_4 = X_4(\bar{X}_4).
\]
the field before (inside of) the parenthesis applies to the 1010X (105X) sector. We will further assume the same assignment in both sectors, with doublet structure \((2' \times 2' \times 2')\) and \((2 \times 2 \times 2')\) for \(\Psi^A \times \Psi^B \times \Psi^C\). If \(\Psi^C_3\) and \(\Psi^C_4\) are in a one-dimensional irrep then the most general Yukawa part of the superpotential is of the form

\[
\Psi^A \Psi^B (a \Psi^C_3 + a' \Psi^C_4) + b \Psi^A \Psi^B \Psi^C + c \Psi^A \Psi^B \Psi^C + d \Psi^A \Psi^B \Psi^C + \Psi^A_3 \Psi^B_3 (e \Psi^C_3 + e' \Psi^C_4),
\]

where the tensor \(\tau\) matrices are not shown explicitly. We examine for simplicity the case where \(a' = e' = 0\), then one has the situation where \(n = m = 3\), so the solution for \(h_{ab}\) needs to be

\[
h_{ab} = \text{diag}\{0, 0, 0, g\}.
\]

In this case the finiteness conditions read

\[
|b_{f,g}|^2 + |c_{f,g}|^2 + 2|d_{f,g}|^2 = (l_{f,g})^2, \quad 2|a_{f,g}|^2 + |e_{f,g}|^2 = (l_{f,g})^2,
\]

\[
|a_{f,g}|^2 + |c_{f,g}|^2 + 2|d_{f,g}|^2 = (l_{g})^2, \quad 2|b_{f,g}|^2 + |e_{f,g}|^2 = (l_{g})^2,
\]

\[
3(|a_{f}|^2 + |b_{f}|^2 + |c_{f}|^2 + 2|d_{f}|^2) + 2(|b_{g}|^2 + |c_{g}|^2 + 2|d_{g}|^2) = 3(l_{f})^2 + 2(l_{g})^2,
\]

\[
3(2|c_{f}|^2 + |e_{f}|^2) + 2(2|c_{g}|^2 + |e_{g}|^2) = 3(l_{f})^2 + 2(l_{g})^2.
\]

with

\[
(l_{f})^2 = \frac{8}{5} g^2, \quad (l_{g})^2 = \frac{6}{5} g^2,
\]

where the \(f\) and \(g\) couplings correspond to the 1010X and 1050X sectors, respectively. All the non-diagonal conditions are identically zero. The most general solution to this set of quadratic equations is

\[
|a_{f,g}| = |b_{f,g}| = |c_{f,g}| = \frac{1}{\sqrt{2}} (l_{f,g}) \sin(\theta_{f,g}),
\]

\[
|e_{f,g}| = \sqrt{2} |d_{f,g}| = (l_{f,g}) \cos(\theta_{f,g}),
\]

where \(\theta_{g}\) and \(\theta_{f}\) are arbitrary free parameters. By looking at tables 1 and 2, we can find assignments that forbid \((a, b, c)\) or \((e, d)\), in these cases we can achieve all-loop finiteness.

4. Boundary conditions in the MSSM

Taking an \(SU(5)\) spontaneous symmetry breaking of the form \(\langle \Sigma \rangle = w \text{diag}\{-2, -2, -2, +3, +3\}\), we assume that we are left with the MSSM matter content, with only one pair of light Higgses. Thus the FC become boundary conditions at \(M_{GUT}\). We transform the \(X_a\) and \(\bar{X}_a\) sectors by applying an \(U_X\) and \(U_{\bar{X}}\) unitary transformation, respectively [17,18]. Denoting the doublet part of \(SU(2)\) of \(X_a\) (\(\bar{X}_a\)) by \(H_a\) (\(\bar{H}_a\)), and taking the fourth component in this new basis as the lightest, one has that the Yukawa couplings take the form

\[
\left(\sum_i f_{ijk} v_k\right), \quad \left(\sum_i g_{ijk} \bar{v}_k\right)
\]

with

\[
v_i = \langle H_i\rangle = (U_X^*)_{ik} v_k, \quad \bar{v}_i = \langle \bar{H}_i\rangle = (U_{\bar{X}})_{ik} \bar{v}_k,
\]

where \(v_u\) and \(v_d\) are the vacuum expectation values of the up and down Higgs fields, in an obvious notation. We now show explicitly the resulting mass matrices:

\[
\begin{pmatrix}
-dw_1 + aw_3 & dw_2 & bw_1 \\
dw_2 & dw_1 + aw_3 & bw_2 \\
bw_1 & bw_2 & cw_3
\end{pmatrix}, \quad (2' \times 2' \times 2'), \quad \Psi^A_3 = \Psi^B_3 = \Psi^C_3 = 1,
\]

\[
4
\]
and
\[
\begin{pmatrix}
  dv_2 + aw_3 & -2dv_1 & bw_1 \\
  -2dv_1 & -dv_2 + aw_3 & bw_2 \\
  bw_1 & bw_2 & ew_3
\end{pmatrix}, 
(2 \times 2 \times 2'),
\Psi_3^A = \Psi_3^B = 1'', \Psi_3^C = 1', \tag{27}
\]

where \(w_i\) represents \(v_i\) or \(\bar{v}_i\), since we are assuming the same structure for both sectors. This form of the mass matrices appears also in the MSSM-\(Q_6\) model of ref. [19], as well as in an \(S_3\) invariant extension of the SM with three Higgs doublets [20], but with the difference that the mass matrices obtained here are at the GUT scale. The case where \(a'\) and \(e'\) are different from zero in eq. (16) has been worked out, with the result that the FC are always overdetermined. In principle it is still possible to find a different class of solutions if one considers the three fields \(\bar{5}_i\) in a one-dimensional irrep.

5. Conclusions
In this article we have obtained the boundary conditions at the \(M_{GUT}\) scale for the MSSM mass matrices that come from the finiteness conditions and \(Q_6\) as a family symmetry group. The assignments that we have made of the fields in the \(Q_6\) irreps, which consist of taking at least one doublet of \(Q_6\) for each of the matter fields, allow only two-loop finite solutions (although there are all-loop finite solutions for this class of field assignments, corresponding to \(\theta_f = 0\) and \(\theta_g = \pi/2\) values (or vice versa), which generate mass matrices that are too simple to be considered in a realistic MSSM scenario). To study the consequences of these results in low energy physics one has to run the mass matrices using the renormalization group equations to the electroweak scale.

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