The leading twist light-cone distribution amplitudes for the S-wave and P-wave quarkonia and their applications in single quarkonium exclusive productions

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Abstract

In this paper, we calculate twist-2 light-cone distribution amplitudes (LCDAs) of the S-wave and P-wave quarkonia (namely \(^1S_0\) state \(\eta_Q\), \(^3S_1\) state \(J/\psi(\Upsilon)\), \(^1P_1\) state \(h_Q\) and \(^3P_J\) states \(\chi_{QJ}\) with \(J = 0, 1, 2\)) to the next-leading order of the strong coupling \(\alpha_s\) and leading order of the non-relativistic expansion parameter \(v\). We apply these LCDAs to some single quarkonium exclusive production at large center-of-mass energy (say, \(\sqrt{s}\)), such as \(\gamma^* \rightarrow \eta_Q\gamma, \chi_{QJ}\gamma (J = 0, 1, 2)\), \(Z \rightarrow \eta_Q\gamma, \chi_{QJ}\gamma (J = 0, 1, 2)\), \(J/\psi(\Upsilon)\gamma, h_Q\gamma\) and \(h \rightarrow J/\psi\gamma\), by adopting the collinear factorization. The asymptotic behaviors of those processes obtained in NRQCD factorization are reproduced.

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I. INTRODUCTION

One of main fields for precision examination of the perturbative Quantum Chromodynamics (QCD) is the study of the hard exclusive processes with the large momentum transfer involved. The collinear factorization has been a well-established calculation framework for more than three decades\{1, 2\}. In this framework, the amplitudes of many hard exclusive processes can be expressed as convolutions of the perturbatively calculable hard kernels and the universal the light-cone distribution amplitudes (LCDAs), in which the short-distance and long-distance contributions are clearly factorized. For instance, the electromagnetic form-factor of $\gamma^*\gamma \to \pi^0$ at large momentum transfer can be expressed as

$$F(Q^2) = f_{\pi} \int_0^1 dx T_H(x; Q^2, \mu) \phi_\pi(x; \mu) + O(\Lambda_{QCD}^2/Q^2),$$

(1)

where hard kernel $T_H(x; Q^2, \mu)$ contains the short-distance dynamics, while the LCDA of pion $f_{\pi} \phi_\pi(x; \mu)$ is a purely non-perturbative object parametrizing the universal hadronization effects around the light-like distance. The LCDAs for light hadrons are not perturbatively calculable, one has to extract their informations from the experiments, or calculate or constrain them by various non-perturbative methods, such as QCD sum rules, Lattice simulations. However, the dependence of these LCDAs on the renormalization scale $\mu$ are perturbatively calculable. For instance, renormalization scale dependence of the twist-2 LCDA of pion is governed by the celebrated Efremov-Radyushkin-Brodsky-Lepage (EFBL) equation \{3, 4\}

$$\frac{d}{d \ln \mu^2} f_{\pi} \phi_\pi(x; \mu) = \frac{\alpha_s}{2\pi} C_F \int_0^1 V_0(x, y) f_{\pi} \phi_\pi(y; \mu),$$

(2)

where $V_0(x, y)$ is the so-called Brosky-Lepage kernel.

For the quarkonium involved exclusive processes, if the momentum transfer square is much greater than the mass square of the quarkonium, the collinear factorization can be invoked as well \{5, 6\}. Many phenomenological applications along this line have been made for exclusive hard production of charmonium \{7–13\}, exclusive charmonium production in $B$ meson decays \{14–17\}, etc. All of these applications requires the understanding of the LCDAs for quarkonia.

Different from the LCDAs for the light mesons, one believes that the LCDAs for quarkonia can be calculated perturbatively, due to the nature of quarkonium as a non-relativistic bound state of heavy quark and anti-quark. The standard theoretical tool to deal with the heavy quark bound state system is the non-relativistic QCD (NRQCD) factorization \{18, 19\}, in which all information of hadronization of quarkonium is encoded in the NRQCD matrix elements. Thus, there must be connections between the LCDAs of quarkonia and NRQCD matrix elements. For examples, in \{20–22\}, the authors try to constrain their models for the LCDAs of quarkonia by relating the moments of LCDAs with the local NRQCD matrix elements; in \{23, 24\}, the authors calculated the leading twist LCDAs of the S-wave quarkonia within the NRQCD framework, and express the LCDAs in form of the product of perturbatively calculable distribution part and lowest order NRQCD matrix-element.

Especially, the attempts in \{23, 24\} open a way to connect the predictions of hard quarkonium exclusive productions within the collinear factorization directly to those made within the NRQCD factorization (for examples, the many theoretical calculations based on NRQCD factorizations \{25–38\}, triggered by the recent experimental measurements of charmonium
exclusive productions at $B$-factories \cite{33,41}. In particular, in \cite{42,43}, the authors have shown that the collinear factorization indeed can reproduce the exact asymptotic behavior of NRQCD predictions at the leading logarithms (LL) and next-to-leading order (NLO) of the strong coupling $\alpha_s$, respectively, for a certain class of the quarkonium exclusive productions, if one employs the leading twist LCDAs calculated in \cite{24}, and the ERBL equations can be used to resum the large logarithms appearing the NRQCD factorization calculations for the exclusive quarkonium productions, and such resummation cannot be done within the NRQCD factorization.

As a successive work of \cite{42,43}, in this paper, we calculate ten leading twist LCDAs for the S-wave and P-wave quarkonia, namely $^{1}S^0$, $^{3}S^1$, $^{1}P^1$ and $^{3}P^J_J$ ($J = 0, 1, 2$) states, to the NLO of $\alpha_s$ and leading order of non-relativistic parameter $v$, by adopting methods developed in \cite{23,24}. For three LCDAs of S-wave quarkonia, we get slightly different results from those obtained in \cite{23}, and confirm the results of LCDA for $^{1}S^0$ state given in \cite{24}.

The seven leading twist LCDAs of P-wave quarkonia at NLO are totally new. All of these leading twist LCDAs at NLO do obey the ERBL equations, and can be applied to various quarkonium involved hard exclusive processes.

This paper is organized as follows: in Sect.\ref{sect:definitions} we give the definitions of the leading twist LCDAs for the S-wave and P-wave quarkonia, in terms of the matrix-elements of a certain class of non-local QCD operators, and their tree-level forms at the leading order of $v$; in Sect.\ref{sect:results} we present our main results of this paper, the LCDAs at the NLO of $\alpha_s$ and leading order of $v$; in Sect.\ref{sect:applications} as applications and non-trivial examinations of our results, we calculate the $\gamma^* \rightarrow \eta_Q\gamma, \chi_Q\gamma, Z \rightarrow \eta_Q\gamma, \chi_Q\gamma, J/\psi\gamma, h_Q\gamma$ and $h \rightarrow J/\psi\gamma$ within the collinear factorization, by using the LCDAs we calculate, and show how we can reproduce the asymptotic behavior of the NLO NRQCD predictions for those processes exactly; finally, we summarize our work in Sect.\ref{sect:summary}.

II. THE DEFINITIONS OF LCDAS FOR QUARKONIA

A. Notations

We adopt the following notations for the decompositions of momentums: the momentum of quarkonium $H$ is $P^\mu \equiv m_H v^\mu$ with $v^2 = 1$, and a 4-vector $a^\mu$ can be decomposed as $a^\mu = v \cdot a v^\mu + a^T_\perp$ where $v \cdot a_\perp \equiv 0$. We also use the same notation $v$ for the non-relativistic expansion parameter, which is typical size of the relative velocity of quark and anti-quark inside a quarkonium. In the context, one should not confuse these two. We also introduce two light-like vectors $n^\mu_\pm$ such that $n^2_\pm = 0$ and $n_+ n_- = 2$, and any 4-vector $a^\mu$ can be decomposed as $a^\mu = n_+ a n^\mu_+/2 + n_- a n^\mu_-/2 + a_\perp^\mu$ with $n_+ a_\perp \equiv 0$. For conveniences, we set $v^\mu = (n_+ v n^\mu_+ + n_- v n^\mu_-)/2$ (apparently $n_+ v n_- = 1$).

B. Definitions of the LCDAs

The leading twist, i.e. twist-2, LCDAs for the S-wave and P-wave quarkonia are defined of the proper gauge-invariant non-local quark bilinear operators as

$$J[\Gamma](\omega) \equiv (\bar{Q}W_c)(\omega n_+/2)\eta_+ \Gamma(W_c^\dagger Q)(-\omega n_+/2),$$

(3)
where $Q$ is the heavy quark field in QCD, the Wilson-line

$$W_c(x) = \text{P exp} \left( i g_s \int_{-\infty}^{0} ds n_+ A(x + sn_+) \right),$$

is a path-ordered exponential with the path along the $n_+$ direction, and $A_\mu(x) \equiv A_\mu^a(x)T^a$ ($T^a$ are the generators of SU(3) group in the fundamental representation).

The ten non-vanishing twist-2 LCDAs of the S-wave and P-wave quarkonia are defined as

$$\langle H^{(1)S_0, P}|J[\gamma_5](\omega)|0 \rangle = -if_{Pn_+}P \int_0^1 dx e^{i\omega n_+P(x-1/2)} \hat{\phi}_P(x; m, \mu),$$

$$\langle H^{(2)S_1, P, \varepsilon^*}|J[\gamma_1](\omega)|0 \rangle = -if_\nu m_\nu n_+ \varepsilon \int_0^1 dx e^{i\omega n_+P(x-1/2)} \hat{\phi}_V(x; m, \mu),$$

$$\langle H^{(3)S_1, P, \varepsilon^*}|J[\gamma_0^*](\omega)|0 \rangle = -if_{Pn_+}P^a \int_0^1 dx e^{i\omega n_+P(x-1/2)} \hat{\phi}_V^a(x; m, \mu),$$

$$\langle H^{(1)P_1, P, \varepsilon^*}|J[\gamma_5](\omega)|0 \rangle = if_1 m_1 A n_+ \varepsilon \int_0^1 dx e^{i\omega n_+P(x-1/2)} \hat{\phi}_{1A}(x; m, \mu),$$

$$\langle H^{(1)P_1, P, \varepsilon^*}|J[\gamma_0^*](\omega)|0 \rangle = if_{Pn_+}P^a \int_0^1 dx e^{i\omega n_+P(x-1/2)} \hat{\phi}_{1A}^a(x; m, \mu),$$

$$\langle H^{(3)P_0, P}|J[\gamma_1](\omega)|0 \rangle = f_{Sn_+}P \int_0^1 dx e^{i\omega n_+P(x-1/2)} \hat{\phi}_S(x; m, \mu),$$

$$\langle H^{(3)P_1, P, \varepsilon^*}|J[\gamma_5](\omega)|0 \rangle = if_{3A} m_3 A n_+ \varepsilon \int_0^1 dx e^{i\omega n_+P(x-1/2)} \hat{\phi}_{3A}(x; m, \mu),$$

$$\langle H^{(3)P_1, P, \varepsilon^*}|J[\gamma_0^*](\omega)|0 \rangle = if_{Pn_+}P^a \int_0^1 dx e^{i\omega n_+P(x-1/2)} \hat{\phi}_{3A}^a(x; m, \mu),$$

$$\langle H^{(3)P_2, P, \varepsilon^*}|J[\gamma_5](\omega)|0 \rangle = f_T m_2 T n_+ \varepsilon \int_0^1 dx e^{i\omega n_+P(x-1/2)} \hat{\phi}_{T}(x; m, \mu),$$

$$\langle H^{(3)P_2, P, \varepsilon^*}|J[\gamma_0^*](\omega)|0 \rangle = f_T m_2 T n_+ \varepsilon \int_0^1 dx e^{i\omega n_+P(x-1/2)} \hat{\phi}_{T}^a(x; m, \mu),$$

where $f$, $\varepsilon^*$ and $\hat{\phi}(x)$ are decay constants, polarization vectors/tensors, and twist-2 LCDAs of corresponding quarkonia, respectively. $x$ denotes the light-cone fractions, and $\mu$ is the renormalization scale. In whole of this paper, we will also adopt the notation $x \equiv 1 - x$ for any light-cone fraction $x \in [0, 1]$.

Due to the discrete $C$, $P$, and $T$ symmetries, one can check that, when $\omega \to 0$, we have

$$\int_0^1 dx \hat{\phi}_{1A,T}(x) = \int_0^1 dx \hat{\phi}_{3A,T}(x) = \int_0^1 dx \hat{\phi}_S(x) = 0,$$

and corresponding integrals of the rest LCDAs do not vanish. Thus, we set the normalization conditions for the LCDAs as following

$$\int_0^1 dx \hat{\phi}_{P}(x) = \int_0^1 dx \hat{\phi}_{V}(x) = \int_0^1 dx \hat{\phi}_{V}^a(x) = \int_0^1 dx \hat{\phi}_{1A}(x) = \int_0^1 dx \hat{\phi}_{3A}(x) = 1,$$

$$\int_0^1 dx \hat{\phi}_{1A,T}(x)(2x - 1) = \int_0^1 dx \hat{\phi}_{3A,T}(x)(2x - 1) = \int_0^1 dx \hat{\phi}_S(x)(2x - 1) = 1.$$

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1 Here we follow the definitions of the LCDAs for P-wave mesons in series papers by K.C. Yang et al., by setting $z = \omega n_+/2$, and $p^\mu = n_+ P n^\mu / 2$. Thus $p \cdot z \equiv n_+ P \omega / 2$. 

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Then, some decay constants defined above can be related to the following matrix-elements of local operators

$$\langle H(1s_0, P)|Q\gamma^\mu\gamma_5|0\rangle = -if_P P^\mu,$$

$$\langle H(3s_1, P, \varepsilon^*)|Q\gamma^\mu Q|0\rangle = -if_V m_V \varepsilon^* \varepsilon^\mu,$$

$$\langle H(3s_1, P, \varepsilon^*)|\bar{Q}\sigma^{\mu
u}Q|0\rangle = if_V^\dagger (\mu)(P^\mu \varepsilon^\nu - P^\nu \varepsilon^\mu),$$

$$\langle H(1P, P, \varepsilon^*)|Q\gamma^\mu\gamma_5 Q|0\rangle = -if_{1A}(\mu)(P^\mu \varepsilon^\nu - P^\nu \varepsilon^\mu),$$

$$\langle H(3P, P, \varepsilon^*)|Q\gamma^\mu\gamma_5 Q|0\rangle = -if_{3A}(\mu)m_{3A} \varepsilon^* \varepsilon^\mu. \quad (22)$$

In practical calculations, it is convenient to use the Fourier transformed form of the non-local operator defined in Eq. (3)

$$Q[\Gamma](x) = \left[\langle \bar{Q}W_c(x)|\omega_{n_+/2}\Gamma(W_c^\dagger Q)(-\omega_{n_+/2})\right]_{F.T.}$$

$$\equiv \int \frac{d\omega}{2\pi} e^{-i(x-1/2)\omega n_+P} \langle \bar{Q}W_c(\omega_{n_+/2})\Gamma(W_c^\dagger Q)(-\omega_{n_+/2}), \quad (23)$$

which are invariant under the re-parametrization $n_+ \to \alpha n_+$ and $n_- \to \alpha^{-1} n_-$. We have

$$\langle H(1s_1, P)|Q[\gamma_5](x)|0\rangle = -if_P \hat{\phi}_P(x), \quad (24)$$

$$\langle H(3s_1, P, \varepsilon^*)|Q[\gamma_5](x)|0\rangle = -if_V m_V n_+ \varepsilon^\mu \hat{\phi}_V(x), \quad (25)$$

$$\langle H(3s_1, P, \varepsilon^*)|Q[\gamma_5^\alpha](x)|0\rangle = if_{1A}^\dagger \varepsilon^{*\alpha} \hat{\phi}_{1A}(x), \quad (26)$$

$$\langle H(1P, P, \varepsilon^*)|Q[\gamma_5](x)|0\rangle = -if_{1A} \varepsilon^\mu \hat{\phi}_{1A}(x), \quad (27)$$

$$\langle H(1P, P, \varepsilon^*)|Q[\gamma_5^\alpha](x)|0\rangle = -if_{1A} \varepsilon^{*\alpha} \hat{\phi}_{1A}(x), \quad (28)$$

$$\langle H(3P, P, \varepsilon^*)|Q[\gamma_5](x)|0\rangle = f_s \hat{\phi}_S(x), \quad (29)$$

$$\langle H(3P, P, \varepsilon^*)|Q[\gamma_5^\alpha](x)|0\rangle = -if_{3A} \varepsilon^{*\alpha} \hat{\phi}_{3A}(x), \quad (30)$$

$$\langle H(3P, P, \varepsilon^*)|Q[\gamma_5^\alpha](x)|0\rangle = -if_{3A} \varepsilon^{*\alpha} \hat{\phi}_{3A}(x), \quad (31)$$

$$\langle H(3P, P, \varepsilon^*)|Q[\gamma_5](x)|0\rangle = \frac{m_t^2 n_+ \alpha n_+ \beta \varepsilon^{*\alpha\beta}}{(n_+P)^2} \hat{\phi}_T(x), \quad (32)$$

$$\langle H(3P, P, \varepsilon^*)|Q[\gamma_5^\alpha](x)|0\rangle = \frac{m_t^2 n_+ \alpha n_+ \beta \varepsilon^{*\alpha\beta}}{(n_+P)^2} \hat{\phi}_T(x). \quad (33)$$

Here we suppress the dependence of all quantities on the renormalization scale $\mu$.

**C. NRQCD factorization for the LCDAs**

Since quarkonia are non-relativistic bound states of heavy quark and anti quark, all of the LCDAs of quarkonia can be factorized into products of perturbatively calculable distribution parts and non-perturbative NRQCD matrix elements, as what done in [23, 24]. This means that, schematically, at operator level, we have the matching equation

$$Q[\Gamma](x, \mu) \simeq \sum_{n=0}^{\infty} C_n^\mu(x, \mu) O_{\Gamma, n}^{NRQCD}, \quad (34)$$
where \( n \) denotes the order of \( v \)-expansion, \( C^n(x, \mu) \) is the short-distance coefficient as a distribution over the light-cone fraction \( x \), and \( O^\text{NRQCD}_{(n)} \) is the relevant NRQCD operator which scales \( O(v^n) \) in the NRQCD power counting. Thus, the LCDAs of quarkonia can be expressed as

\[
\langle H|Q|\Gamma(x, \mu)|0\rangle \simeq \sum_{n=0}^\infty C^n(x, \mu) \langle H|O^\text{NRQCD}_{(n)}|0\rangle . \tag{35}
\]

At the lowest order of \( v \), the matrix elements of the following relevant NRQCD effective operators will be involved in our calculation:

\[
\begin{align*}
O^{(1)S_0} & \equiv \bar{\psi}_v \gamma_5 \chi_v , \\
O^{(3)S_1} & \equiv \bar{\psi}_v \gamma_\mu \chi_v , \\
O^{(1)P_1} & \equiv \bar{\psi}_v \left[ \left( \frac{i}{2} \right) \overleftrightarrow{\partial}_\mu \gamma_5 \right] \chi_v , \\
O^{(3)P_0} & \equiv \bar{\psi}_v \left[ \left( -\frac{1}{\sqrt{3}} \right) \overleftrightarrow{\partial}_\mu \right] \chi_v , \\
O^{(3)P_1} & \equiv \bar{\psi}_v \left( \frac{1}{2\sqrt{2}} \gamma_\mu \right) \overleftrightarrow{\partial}_\mu \chi_v , \\
O^{(3)P_2} & \equiv \bar{\psi}_v \left[ \left( -\frac{i}{2} \right) \overleftrightarrow{\partial}_\mu \right] \chi_v .
\end{align*}
\tag{36}
\]

Here we use the four-component notations as in \([44]\) for the Non-Relativistic QCD effective Lagrangian,

\[
L^\text{LO}_{\text{NRQCD}} = \bar{\psi}_v \left( iv \cdot D - \frac{(iD^\mu)(iD^\nu)}{2m} \right) \psi_v + \bar{\chi}_v \left( iv \cdot D + \frac{(iD^\mu)(iD^\nu)}{2m} \right) \chi_v , \tag{37}
\]

where \( m \) is the pole mass of the heavy quark, \( \psi_v \) and \( \chi_v \) are the effective fields of the heavy-quark and anti-heavy-quark, respectively, satisfying \( \psi \psi_v = \psi_v \) and \( \gamma \chi_v = -\chi_v \). \( D^\mu = \partial^\mu - ig_s A^\mu \) is the covariant derivative, and \( \gamma^\mu = \gamma^\mu - \gamma_\mu \gamma - \gamma^\mu - 2ig_s A^\mu \). And \( a^{(\mu\nu)}_\tau = (a^{\mu}_\tau b^{\nu}_\tau + a^{\nu}_\tau b^{\mu}_\tau)/2 - a^{\mu}_\tau b^{\nu}_\tau (g^{\mu\nu} - \eta^{\mu\nu})/(d - 1) \) with \( d = 4 \) means the symmetric 3-D traceless part of rank-2 tensor \( a^{\mu}_\tau b^{\nu}_\tau \).

At tree-level, within the color-singlet model, we have

\[
\left\{ \begin{align*}
\langle \eta_q | O^{(1)S_0} | 0 \rangle &= \langle O^{(1)S_0} \rangle , \\
\langle \Psi \gamma | O^{(3)S_1} | 0 \rangle &= \varepsilon^{*\mu} \langle O^{(1)S_0} \rangle , \\
\langle \eta_h | O^{(1)P_1} | 0 \rangle &= \varepsilon^{*\mu} \langle O^{(3)P_0} \rangle , \\
\langle \chi_Q | O^{(3)P_0} | 0 \rangle &= \langle O^{(3)P_0} \rangle , \\
\langle \chi_Q | O^{(3)P_1} | 0 \rangle &= \frac{1}{2 - d} \varepsilon^{*\mu} \left( g^{\mu\nu} - \frac{p_{\mu}p_{\nu}}{m_{Q}^2} \right) - \varepsilon^{*\nu} \left( g^{\mu\nu} - \frac{p_{\mu}p_{\nu}}{m_{Q}^2} \right) \langle O^{(3)P_0} \rangle , \\
\langle \chi_Q | O^{(3)P_2} | 0 \rangle &= \varepsilon^{*\mu\nu} \langle O^{(3)P_0} \rangle , \tag{38}
\end{align*} \right.\]

\[\text{\textsuperscript{2} Here we have used the spin symmetry of heavy quark system to relate the various matrix elements of S-wave operators and P-wave operators.}\]
with
\[
\langle O(1S_0) \rangle = \sqrt{2N_c} \sqrt{2M_{\text{eq}}} \left( \frac{1}{4\pi} \right) R_{10}(0),
\]
(39)
\[
\langle O(3P_0) \rangle = \sqrt{2N_c} \sqrt{2M_{\chi_{Qp}}} (-i) \left( \frac{3}{4\pi} \right) R_{21}'(0),
\]
(40)
in the color-singlet model at the leading order of \( \alpha_s \) and \( v \)-expansions. Here the spin symmetry of the leading non-relativistic interactions has been used. \( R_{nl}(r) \) denotes the radial Schrödinger wave function of the quarkonium with radial quantum number \( n \) and orbital angular momentum \( l \), and the prime denotes a derivative with the respect of \( r \).

### D. Tree-level matching

The short distance coefficient \( C_n^G(x, \mu) \) can be extracted, most conveniently, through matching the matrix-elements between the vacuum and state of a colorless pair of free heavy quark and anti quark with non-relativistic relative motion. In this subsection, we illustrate how to do the matching at tree level. The generalization to the NLO calculation is straightforward.

We start with the heavy quark and anti-quark pair with the momenta
\[
p_1^\mu = mv^\mu + q^\mu, \quad p_2^\mu = mv^\mu + \bar{q}^\mu, \quad p_{1,2}^2 = m^2,
\]
(41)
where the residual momenta \( q \) and \( \bar{q} \) in the rest frame of heavy quark pair scale like \( q^0 \sim \bar{q}^0 \sim mv^2, \quad \bar{q} = -\bar{q} \) and \( |\bar{q}| = |q| \sim mv \), where \( v << 1 \). The total momentum of heavy quark pair
\[
P^\mu = p_1^\mu + p_2^\mu = m_H v^\mu, \quad m_H \equiv \sqrt{P^2} \approx 2m + O(v^2).
\]
(42)
The on-shell spinors of quark and anti-quark can be expanded in \( v \) as
\[
u(p_1) \approx \left( 1 + \frac{\bar{q}}{2m} \right) u_v(p_1), \quad u_v(p_1) \equiv \frac{1 + \bar{q}}{2} u(p_1),
\]
(43)
\[
u(p_2) \approx \left( 1 - \frac{\bar{q}}{2m} \right) v_v(p_2), \quad v_v(p_2) \equiv \frac{1 - \bar{q}}{2} v(p_2).
\]
(44)
Thus, at tree-level, we have
\[
\langle Q^a(p_1) \bar{Q}^b(p_2) | Q[\Gamma](x) | 0 \rangle
\]

\[
= \delta^{ab} \int \frac{d\omega}{2\pi} e^{-i(x-1/2)\omega n + i\omega n + \bar{q} \bar{\omega}} \bar{u}(p_1) \gamma^\mu \Gamma v(p_2)
\]

\[
= \delta^{ab} \frac{\delta}{n+P} \left( x - 1/2 - \frac{n+\bar{q}}{n+P} \right) \bar{u}(p_1) \gamma^\mu \Gamma v(p_2)
\]

\[
= \frac{\delta^{ab}}{n+P} \left( \delta(x - 1/2) - \delta'(x - 1/2) \frac{n+\bar{q}}{n+P} \right) \bar{u}_v(p_1) \gamma^\mu \Gamma v_v(p_2)
\]

\[
+ \delta(x - 1/2) \frac{1}{2m} \bar{u}_v(p_1) \left\{ \bar{q}, \gamma^\mu \Gamma \right\} v_v(p_2) + O(v^2),
\]
(45)
where we define $\bar{q} \equiv (q - \bar{q})/2$, and $a, b$ are color indices for the quark and anti-quark.

For illustration, when $\Gamma = \gamma_5$, we have

\[
\bar{u}_v(p_1)\gamma_5 v_v(p_2) = n_+ v \bar{u}_v(p_1) \gamma_5 v_v(p_2) \sim n_+ v \langle Q \bar{Q} | O(1) S_0 \rangle | 0 \rangle,
\]

\[
\frac{1}{2m} \bar{u}_v(p_1) \{ \gamma, \gamma_5 \} v_v(p_2) = \frac{n_+ v}{2m} \bar{u}_v(p_1) \{ \gamma, \gamma_5 \} \gamma_5 v_v(p_2) \sim \frac{\sqrt{2}}{m} \langle Q \bar{Q} | n_+ \mu O^\mu (\gamma P_1) \rangle | 0 \rangle,
\]

\[
\frac{n_+ q}{n_+ P} \bar{u}_v(p_1) \gamma_5 v_v(p_2) = \frac{n_+ v}{n_+ P} \bar{u}_v(p_1) \gamma_5 v_v(p_2) \sim \frac{n_+ v}{n_+ P} \langle Q \bar{Q} | n_+ \mu O^\mu (1 P_1) \rangle | 0 \rangle.
\]

Thus,

\[
\langle Q[\gamma_5](x) \rangle = \delta(x - 1/2) \left( \frac{n_+ v}{n_+ P} \langle O(1) S_0 \rangle + \frac{\sqrt{2} n_+ \mu}{m n_+ P} \langle O^\mu (3 P_1) \rangle \right) - \frac{\delta'(x - 1/2)}{n_+ P} \frac{n_+ v n_+ \mu}{n_+ P} \langle O^\mu (1 P_1) \rangle + O(v^2).
\]

With the normalization conditions for the LCDAs set by (16,17), we have

\[
\hat{\phi}^{(0)}_P(x) = \hat{\phi}^{(0)}_{3A}(x) = \delta(x - 1/2), \quad \hat{\phi}^{(0)}_{1A}(x) = -\delta'(x - 1/2)/2,
\]

and

\[
f^{(0)}_P = \frac{i}{m_P} \langle O(1) S_0 \rangle, \quad f^{(0)}_{1A} = \frac{2}{m_{1A}} \langle O^3 P_0 \rangle, \quad f^{(0)}_{3A} = \frac{i \sqrt{2}}{m_{3A}} \langle O^3 P_0 \rangle,
\]

where the superscript $(0)$ denotes the quantity at the leading order of $\alpha_s$. Note we have used the fact that $n_+ v/n_+ P = 1/m_H$.

Similarly, one can get

\[
\hat{\phi}^{(0)}_{V}(x) = \hat{\phi}^{(0)}_{T}(x) = \hat{\phi}^{(0)}_{1A}(x) = \delta(x - 1/2),
\]

\[
\hat{\phi}^{(0)}_{S}(x) = \hat{\phi}^{(0)}_{3A}(x) = \hat{\phi}^{(0)}_{T}(x) = \hat{\phi}^{(0)}_{1A}(x) = -\delta'(x - 1/2)/2,
\]

and

\[
f^{(0)}_{V} = f^{(0)}_{T} = \frac{i}{m_V} \langle O(1) S_0 \rangle, \quad f^{(0)}_{1A} = -\frac{i}{m_{1A}} \langle O^3 P_0 \rangle,
\]

\[
f^{(0)}_{S} = -\frac{2}{\sqrt{3} m^2} \langle O^2 P_0 \rangle, \quad f^{(0)}_{3A} = -\frac{i \sqrt{2}}{m_{3A}} \langle O^3 P_0 \rangle,
\]

\[
f^{(0)}_{T} = -f^{(0)}_{T} = -\frac{2}{m^2} \langle O^3 P_0 \rangle.
\]

III. THE CALCULATIONS OF THE LCDAS AT NLO

A. Matching procedure by method of threshold expansion

To extract the short-distance coefficients $C^\mu_\Gamma(x, \mu)$ at NLO of $\alpha_s$ through the matching equation (55), we have to calculate one-loop corrections to the matrix elements of both $Q[\Gamma](x, \mu)$ and $O^\mu_{\Gamma,n}$ in general matching procedure as what done in (23).
However, in this work, we will adopt the method of threshold expansion \[49\] to simplify the matching procedure so that we do not need to calculate the one-loop corrections to the matrix elements of effective operators $O^{\text{NRQCD}}_{\Gamma, n}$. This is equivalent to what done in \[24\].

At one-loop level, the bare matrix element of $Q[\Gamma](x)$ is written as \(^3\)

\[
\langle Q^a(p_1) \bar{Q}^b(p_2) | Q[\Gamma](x) | 0 \rangle_{\text{bare}} = \delta^{ab} \delta \left( x - \frac{n_{+} p_1}{n_{+} P} \right) \frac{\bar{u}(p_1) \gamma_{\pm} \Gamma v(p_2)}{n_{+} P} + \frac{\alpha_s}{4\pi} C_F \delta^{ab} \int [dk] \frac{\bar{u}(p_1) \gamma_{\mu}(\gamma_{\rho} - \gamma_{\rho} P/2 + m) \gamma_{\rho} \Gamma(\gamma_{\rho} + P/2 + m) \gamma_{\mu} v(p_2)}{n_{+} P[(k - q)^2][(k + P/2)^2 - m^2][(k - P/2)^2 - m^2]} \delta \left( x - \frac{1}{2} - \frac{n_{+} k}{n_{+} P} \right)
\]

where $+i\epsilon$ prescription for the propagators are understood, $C_F = \frac{N_c^2 - 1}{2N_c}$ with $N_c = 3$ is rank-2 Casimir in the fundamental representation of SU(3) group, and

\[
[dk] \equiv \frac{(4\pi)^2}{i} \left( \frac{e^{\gamma_E} \mu^2}{4\pi} \right)^\varepsilon \frac{d^d k}{(2\pi)^d}
\]

with $d = 4 - 2\varepsilon$ and $\gamma_E = 0.5772...$ being the Euler constant.

In the following calculations, we will use the extra dimension $\varepsilon$ to regulate both of the ultraviolet and infrared divergences, and we will treat $\gamma_{5}$ in naive dimensional regularization (NDR) scheme, i.e. $\{\gamma_{5}, \gamma_{\mu}\} = 0$.

Considering that $\Gamma$ in $Q[\Gamma](x)$ either commutes or anti-commutes with $\gamma_{\pm}$, the above can be simplified to

\[
\langle Q^a(p_1) \bar{Q}^b(p_2) | Q[\Gamma](x) | 0 \rangle_{\text{bare}} = \delta^{ab} \delta \left( x - \frac{n_{+} p_1}{n_{+} P} \right) \frac{\bar{u}(p_1) \gamma_{\pm} \Gamma v(p_2)}{n_{+} P} + \frac{\alpha_s}{4\pi} C_F \delta^{ab} \int [dk] \frac{2n_{+}(k + P/2) \delta \left( x - \frac{1}{2} - \frac{n_{+} k}{n_{+} P} \right) - \delta \left( x - \frac{n_{+} p_1}{n_{+} P} \right)}{n_{+} P[(k - q)^2][(k + P/2)^2 - m^2][(k - P/2)^2 - m^2]} \bar{u}(p_1) \gamma_{\pm} \Gamma v(p_2)
\]

\[
- \frac{\alpha_s}{4\pi} C_F \delta^{ab} \int [dk] \frac{2n_{+}(k - P/2) \delta \left( x - \frac{1}{2} - \frac{n_{+} k}{n_{+} P} \right) - \delta \left( x - \frac{n_{+} p_1}{n_{+} P} \right)}{n_{+} P[(k - q)^2][(k - P/2)^2 - m^2]} \bar{u}(p_1) \gamma_{\pm} \Gamma v(p_2) . \quad (58)
\]

We will expand the loop integrals in small parameter $\nu \sim |\bar{q}|/m$ by the threshold expansion technique developed in \[48\]. The most important momentum regions are hard region (where loop momentum $k^{\mu} \sim m$), soft region (where $k^{\mu} \sim m\nu$), potential region (where $k^{\mu} \sim m(v^2, \bar{v})$), ultra-soft region (where $k^{\mu} \sim m\nu^2$). The contributions from the low-energy

\(^3\) Her we set the momentum of gluon in the loop as $k - \bar{q}$ as in \[49\]. And note that $p_{1,2} = P/2 \pm \bar{q}$, the momenta of quark and anti-quark propagator will be $k + P/2$ and $P/2 - k$, respectively.
regions, i.e. (ultra)-soft and potential regions, are reproduced by the one-loop corrections to the matrix elements of effective operators in matching equation (33). Thus, to get the NLO part of the short distance coefficient \( C_{\Gamma, n}(x, \mu) \), we only need to calculate the contributions from the hard region.

After tedious expansions of various spin structures and loop integrals in \( \nu \), the hard part of the bared matrix element up to \( \mathcal{O}(\nu) \) is

\[
\langle Q^a(p_1) \bar{Q}^b(p_2) | Q[\Gamma](x) | 0 \rangle^{\text{bare}}_{\text{hard}} = \frac{\delta^{ab} u_v(p_1) \bar{u}_v(p_2) \Gamma v_v(p_2)}{n_+ P} \left\{ \delta \left( x - \frac{1}{2} \right) \right. \\
\left. + \frac{\alpha_s}{4\pi} C_F \int [dk] \frac{\delta \left( x - \frac{1}{2} - \frac{n_+ k}{n_+ P} \right) \left( -4m^2 + c_{\Gamma, \nu}^2 k^2 / (d - 2) - c_{\Gamma, \nu} (n_+ k/n_+ P)^2 \right)}{[k^2 + i\epsilon][k^2 + P \cdot k + i\epsilon][k^2 - P \cdot k + i\epsilon]} \right\}
\]

\[
- \frac{\alpha_s}{4\pi} C_F \int [dk] \frac{\delta \left( x - \frac{1}{2} - \frac{n_+ k}{n_+ P} \right) - \delta \left( x - \frac{1}{2} \right) \left( \frac{2n_+(k + P/2)}{[k^2 + P \cdot k + i\epsilon]} + \frac{2n_+(k - P/2)}{[k^2 - P \cdot k + i\epsilon]} \right)}{[n_+ k][k^2 + i\epsilon][k^2 + P \cdot k + i\epsilon][k^2 - P \cdot k + i\epsilon]} \\
+ \frac{\delta^{ab} u_v(p_1) \bar{u}_v(p_2) \left\{ \frac{\delta \left( x - \frac{1}{2} \right)}{2m n_+ P} \right. \\
\left. + \frac{\alpha_s}{4\pi} C_F \int [dk] \frac{\delta \left( x - \frac{1}{2} - \frac{n_+ k}{n_+ P} \right) \left( c_{\Gamma, \nu}^2 k^2 / (d - 2) + c_{\Gamma, \nu} (n_+ k/n_+ P)^2 \right)}{[k^2 + i\epsilon][k^2 + P \cdot k + i\epsilon][k^2 - P \cdot k + i\epsilon]} \right\} \\
- \frac{\alpha_s}{4\pi} C_F \int [dk] \frac{\delta \left( x - \frac{1}{2} - \frac{n_+ k}{n_+ P} \right) - \delta \left( x - \frac{1}{2} \right) \left( \frac{2n_+(k + P/2)}{[k^2 + P \cdot k + i\epsilon]} + \frac{2n_+(k - P/2)}{[k^2 - P \cdot k + i\epsilon]} \right)}{[n_+ k][k^2 + i\epsilon][k^2 + P \cdot k + i\epsilon][k^2 - P \cdot k + i\epsilon]} \\
+ \frac{\delta^{ab} n_+ \bar{q} u_v(p_1) \bar{u}_v(p_2) \left\{ \delta \left( x - \frac{1}{2} \right) \right. \\
\left. + \frac{\alpha_s}{4\pi} C_F \int [dk] \left( -4m^2 + c_{\Gamma, \nu}^2 k^2 / (d - 2) - c_{\Gamma, \nu} (n_+ k/n_+ P)^2 \right) \frac{n_+ P \left( n_+ - \frac{n_+ k}{n_+ P} \right)}{[k^2 + i\epsilon][k^2 + P \cdot k + i\epsilon][k^2 - P \cdot k + i\epsilon]} \right\} \\
\left. - \frac{\alpha_s}{4\pi} C_F \int [dk] \frac{\delta \left( x - \frac{1}{2} - \frac{n_+ k}{n_+ P} \right) + 4c_{\Gamma, \nu} m n_+ k/n_+ P}{[k^2 + i\epsilon][k^2 + P \cdot k + i\epsilon][k^2 - P \cdot k + i\epsilon]} \right\} \\
+ \frac{\alpha_s}{4\pi} C_F \int [dk] \frac{8mc_{\Gamma, \nu} (n_+ k/n_+ P) k^2 / (d - 2) \delta \left( x - \frac{1}{2} - \frac{n_+ k}{n_+ P} \right)}{[k^2 + i\epsilon][k^2 + P \cdot k + i\epsilon][k^2 - P \cdot k + i\epsilon]} \\
\right. \\
\left. - \frac{\alpha_s}{4\pi} C_F \int [dk] \frac{\delta \left( x - \frac{1}{2} \right) \left( \frac{2n_+(k + P/2)}{[k^2 + P \cdot k + i\epsilon]} + \frac{2n_+(k - P/2)}{[k^2 - P \cdot k + i\epsilon]} \right)}{[n_+ k][k^2 + i\epsilon][k^2 + P \cdot k + i\epsilon][k^2 - P \cdot k + i\epsilon]} \\
\right. \\
\left. - \frac{\alpha_s}{4\pi} C_F \int [dk] \frac{\delta \left( x - \frac{1}{2} - \frac{n_+ k}{n_+ P} \right) - \delta \left( x - \frac{1}{2} \right) \left( \frac{2n_+(k + P/2)}{[k^2 + P \cdot k + i\epsilon]} + \frac{2n_+(k - P/2)}{[k^2 - P \cdot k + i\epsilon]} \right)}{[n_+ k][k^2 + i\epsilon][k^2 + P \cdot k + i\epsilon][k^2 - P \cdot k + i\epsilon]} \\
\right. \\
\left. \times \left( \frac{n_+ P \left( n_+ - \frac{n_+ k}{n_+ P} \right)}{[k^2 + i\epsilon]} + \frac{n_+ P}{[n_+ k]} \right) \right\}\]
\[
+ \frac{\delta^{ab} \tilde{u}_v(p_1) [\hat{g}, \hat{\gamma}_+ \Gamma] u_v(p_2)}{2m} \frac{\alpha_s}{n^2 P} \frac{C_F}{4\pi} \int \frac{[dk]}{[k^2 + i\varepsilon][k^2 + P \cdot k + i\varepsilon][k^2 - P \cdot k + i\varepsilon]} \frac{\delta(x - \frac{1}{2} - \frac{n^+ k}{n^+ P})}{k^2 + i\varepsilon} \frac{1}{[k^2 - P \cdot k + i\varepsilon]}
\times \left\{ 4m^2 \frac{n^+ k}{n^2 v} \left( -2 + c_{\delta+\Gamma} - 2c_{\delta+\Gamma} \frac{k_1^2}{(d - 2)k_2} \right) \right\} + O(v^2),
\]

where \( c_{\delta+\Gamma} \) is defined as \( \gamma^\mu \delta^\nu \Gamma \gamma_\mu \equiv c_{\delta+\Gamma} \delta^\nu \Gamma \) so that

\[
c_{\delta+\Gamma} = \begin{cases} 
2 - d, & \Gamma = 1, \\
2 - d, & \Gamma = \gamma_5, \\
2 - d, & \Gamma = \gamma_5^\alpha, \\
2 - d, & \Gamma = \gamma_5^\alpha \gamma_5,
\end{cases}
\]

with \( d = 4 - 2\varepsilon \).

The hard part of the renormalized matrix-element is

\[
\langle Q^a(p_1) \bar{Q}^b(p_2) | Q[\Gamma](x) | 0 \rangle_{\text{hard}}^{\text{ren}} = Z_2^{\text{os}} \int_0^1 dy Z_{\delta+\Gamma}(x, y) \langle Q^a(p_1) \bar{Q}^b(p_2) | Q[\Gamma](y) | 0 \rangle_{\text{hard}}^{\text{bare}},
\]

where the on-shell renormalization constant for the heavy quark

\[
Z_2^{\text{os}} = 1 - \frac{\alpha_s}{4\pi} C_F \left( \frac{3}{\varepsilon} + 3 \ln \frac{\mu^2}{m^2} + 4 \right),
\]

and the renormalization kernel for the operator \( Q[\Gamma](x) \) in \( \overline{\text{MS}} \) scheme

\[
Z_{\delta+\Gamma}(x, y) = Z_{\delta+\Gamma}(x, y) = \delta(x - y) - \frac{\alpha_s}{4\pi} C_F \frac{2}{\varepsilon} V_0(x, y),
\]

\[
Z_{\delta+\Gamma_5}(x, y) = Z_{\delta+\Gamma_5}(x, y) = \delta(x - y) - \frac{\alpha_s}{4\pi} C_F \frac{2}{\varepsilon} V_1(x, y),
\]

with the Brodsky-Lepage kernel being

\[
V_0(x, y) = \left[ \frac{1 - x}{1 - y} \left( 1 + \frac{1}{x - y} \right) \theta(x - y) + \frac{x}{y} \left( 1 + \frac{1}{y - x} \right) \theta(y - x) \right],
\]

\[
V_1(x, y) = V_0(x, y) - \left[ \frac{1 - x}{1 - y} \theta(x - y) + \frac{x}{y} \theta(y - x) \right].
\]

Thus, schematically, the final matching equation up to \( O(v) \) goes to

\[
\langle Q^a(p_1) \bar{Q}^b(p_2) | Q[\Gamma](x) | 0 \rangle_{\text{hard}}^{\text{ren}} = \sum_{n=0,1} C_{\Gamma,n}(x, \mu) \langle Q^a(p_1) \bar{Q}^b(p_2) | Q_n^{\text{NRQCD}} | 0 \rangle_{\text{NRQCD}} + O(v^2).
\]

B. Final results for LCDAs of quarkonia

With the loop integrals given in Appendix A we get the short-distance coefficients \( C_{\Gamma,n}(x, \mu) \), we reach the final results for the LCDAs at the NLO of \( \alpha_s \) and leading order of
\( v, \) which are

\[
\hat{\phi}_P(x; \mu) = \delta(x - 1/2) + \frac{\alpha_s(\mu)}{4\pi} C_F \left\{ 4 \left[ \left( \ln \frac{\mu^2}{m^2(1-2x)^2} - 1 \right) \left( 1 + \frac{1}{1/2 - x} \right) x \theta(1-2x) \right]_+ \\
+ \left[ \frac{16x\bar{x}}{(1-2x)^2} \theta(1-2x) \right]_{++} + (x \leftrightarrow \bar{x} \right\},
\]

(67)

\[
\hat{\phi}_{V}^\parallel(x; \mu) = \hat{\phi}_P(x; \mu) - \frac{\alpha_s(\mu)}{4\pi} C_F \left[ 8x\theta(1-2x) + 8\bar{x}\theta(2x-1) \right]_+,
\]

(68)

\[
\hat{\phi}_{V}^\perp(x; \mu) = \hat{\phi}_P(x; \mu) - \frac{\alpha_s(\mu)}{4\pi} C_F \left[ \left( \ln \frac{\mu^2}{m^2(1-2x)^2} - 1 \right) (4x\theta(1-2x) + 4\bar{x}\theta(2x-1)) \right]_+,
\]

with

\[
f_P = \left\{ 1 + \frac{\alpha_s(\mu)}{4\pi} C_F (-6) \right\} \frac{i}{m_P} \langle \mathcal{O}(1S_0) \rangle,
\]

(69)

\[
f_V = \left\{ 1 + \frac{\alpha_s(\mu)}{4\pi} C_F (-8) \right\} \frac{i}{m_V} \langle \mathcal{O}(1S_0) \rangle,
\]

(70)

\[
f_{V}^\perp = \left\{ 1 + \frac{\alpha_s(\mu)}{4\pi} C_F \left( -\ln \frac{\mu^2}{m^2} - 8 \right) \right\} \frac{i}{m_V} \langle \mathcal{O}(1S_0) \rangle,
\]

(71)

for S-wave quarkonia, and

\[
\hat{\phi}_{1A}^\parallel(x; \mu) = -\delta'(x - 1/2)/2
\]

\[
+ \frac{\alpha_s(\mu)}{4\pi} C_F \left\{ - \left[ \ln \frac{\mu^2}{m^2(1-2x)^2} - 3 \right] \left( \ln \frac{\mu^2}{m^2(1-2x)^2} - 1 \right) \left( 5 - 8x + 4x^2 \right) \theta(1-2x) \right]_+
\]

\[
- \left[ \frac{8x(7 - 4x)\theta(1-2x)}{(1-2x)^2} \right]_{++} - \left[ \frac{8x\theta(1-2x)}{(1-2x)^3} \right]_{+++} - (x \leftrightarrow \bar{x}) \right\},
\]

(72)

\[
\hat{\phi}_{1A}^\perp(x; \mu) = \hat{\phi}_V^\perp(x; \mu) - \frac{\alpha_s(\mu)}{4\pi} C_F \left[ \frac{8x\bar{x}\theta(2x-1)}{(1-2x)^2} + \frac{8x\bar{x}\theta(2x-1)}{(1-2x)^2} \right]_{++},
\]

(73)

\[
\hat{\phi}_S(x; \mu) = -\delta'(x - 1/2)/2
\]

\[
+ \frac{\alpha_s(\mu)}{4\pi} C_F \left\{ - \left[ \ln \frac{\mu^2}{m^2(1-2x)^2} - 3 \right] \left( \ln \frac{\mu^2}{m^2(1-2x)^2} - 1 \right) \left( 5 - 8x + 4x^2 \right) \theta(1-2x) \right]_+
\]

\[
- \left[ \frac{8x(7 - 8x)\theta(1-2x)}{(1-2x)^2} \right]_{++} - \left[ \frac{8x\theta(1-2x)}{(1-2x)^3} \right]_{+++} - (x \leftrightarrow \bar{x}) \right\},
\]

(74)

\[
\hat{\phi}_{3A}^\parallel(x; \mu) = \hat{\phi}_V^\parallel(x; \mu) - \frac{\alpha_s(\mu)}{4\pi} C_F \left[ \frac{8x\bar{x}\theta(2x-1)}{(1-2x)^2} + \frac{8x\bar{x}\theta(2x-1)}{(1-2x)^2} \right]_{++},
\]

(75)

\[
\hat{\phi}_{3A}^\perp(x; \mu) = -\delta'(x - 1/2)/2
\]

\[
+ \frac{\alpha_s(\mu)}{4\pi} C_F \left\{ - \left[ \ln \frac{\mu^2}{m^2(1-2x)^2} - 3 \right] \left( \ln \frac{\mu^2}{m^2(1-2x)^2} - 1 \right) \left( 5 - 8x + 4x^2 \right) \theta(1-2x) \right]_+
\]

\[
- \left[ \frac{8x(7 - 8x)\theta(1-2x)}{(1-2x)^2} \right]_{++} - \left[ \frac{8x\theta(1-2x)}{(1-2x)^3} \right]_{+++} - (x \leftrightarrow \bar{x}) \right\},
\]

(76)

\[
\hat{\phi}_T(x; \mu) = -\delta'(x - 1/2)/2
\]
for P-wave quarkonia. Here we define $+++$, $++$ and $+-$-functions as

\[
\hat{\phi}_T(x; \mu) = -\delta'(x - 1/2)/2
\]

\[
\hat{\phi}_{TM}(x) = \frac{\alpha_s(\mu)}{4\pi} C_F \left\{ - \left( \ln \frac{\mu^2}{m^2 (1 - 2x)^2} - 4 \right) \frac{4x(5 - 8x + 4x^2)\theta(1 - 2x)}{(1 - 2x)^2} \right\}_{++} + \frac{\alpha_s(\mu)}{4\pi} C_F \left\{ - \left( \ln \frac{\mu^2}{m^2 (1 - 2x)^2} + 2 \right) \frac{16x\bar{x}\theta(1 - 2x)}{(1 - 2x)^2} \right\}_{++} + \frac{32x\bar{x}(1 - 2x)}{1 - 2x}_{++} - \frac{8x\theta(1 - 2x)}{(1 - 2x)^3} - (x \leftrightarrow \bar{x}) \right\},
\]

with

\[
f_{1A} = \left\{ 1 - \frac{\alpha_s(\mu)}{4\pi} C_F \left( \frac{8}{3} \ln \frac{\mu^2}{m^2} + \frac{76}{9} \right) \right\} \frac{2i}{m_{1A}^2} \langle \mathcal{O}^3 P_0 \rangle,
\]

\[
f_{1A}^L = \left\{ 1 + \frac{\alpha_s(\mu)}{4\pi} C_F \left( - \ln \frac{\mu^2}{m^2} - 4 \right) \right\} \frac{2i}{m_{1A}^2} \langle \mathcal{O}^3 P_0 \rangle,
\]

\[
f_S = \left\{ 1 - \frac{\alpha_s(\mu)}{4\pi} C_F \left( \frac{8}{3} \ln \frac{\mu^2}{m^2} - \frac{2}{9} \right) \right\} \frac{2i}{\sqrt{3}m_S^2} \langle \mathcal{O}^3 P_0 \rangle,
\]

\[
f_{3A} = \left\{ 1 + \frac{\alpha_s(\mu)}{4\pi} C_F \left( -4 \right) \right\} \frac{\sqrt{2i}}{mm_{3A}} \langle \mathcal{O}^3 P_0 \rangle,
\]

\[
f_{3A}^L = \left\{ 1 - \frac{\alpha_s(\mu)}{4\pi} C_F \left( 3 \ln \frac{\mu^2}{m^2} + 6 \right) \right\} \frac{-\sqrt{2i}}{m_{3A}^2} \langle \mathcal{O}^3 P_0 \rangle,
\]

\[
f_T = \left\{ 1 - \frac{\alpha_s(\mu)}{4\pi} C_F \left( \frac{8}{3} \ln \frac{\mu^2}{m^2} + \frac{88}{9} \right) \right\} \frac{2i}{m_T^2} \langle \mathcal{O}^3 P_0 \rangle,
\]

\[
f_T^L = \left\{ 1 - \frac{\alpha_s(\mu)}{4\pi} C_F \left( 3 \ln \frac{\mu^2}{m^2} + 10 \right) \right\} \frac{2i}{m_T^2} \langle \mathcal{O}^3 P_0 \rangle,
\]

for P-wave quarkonia. Here we define $+++$, $++$ and $+-$-functions as

\[
\int_0^1 dx [f(x)]_{+++} g(x) = \int_0^1 dx f(x)(g(x) - g(1/2) - g'(1/2)(x - 1/2) - \frac{g''(1/2)}{2}(x - 1/2)^2),
\]

\[
\int_0^1 dx [f(x)]_{++} g(x) = \int_0^1 dx f(x)(g(x) - g(1/2) - g'(1/2)(x - 1/2)),
\]

\[
\int_0^1 dx [f(x)]_{+-} g(x) = \int_0^1 dx f(x)(g(x) - g(1/2)).
\]

One can check that our results for $\hat{\phi}_M(x; \mu)$ preserve the normalizations in (16,17), and $f_M \phi_M(x; \mu)$ satisfy the Brodsky-Lepage equations

\[
\mu^2 \frac{d}{d\mu^2} (f_M \phi_M(x)) = \frac{\alpha_s(\mu^2)}{2\pi} C_F \int_0^1 dy V_M(x,y) (f_M \phi_M(y)).
\]
For the decay constants which can be defined by the local QCD currents, such as \( f_P, f_V, f_3^V, f_{1A} \) and \( f_{3A} \), we find that our results at NLO of \( \alpha_s \) agree with those in literature \[50\]. The decay constants, such as \( f_{1A}^+, f_S, f_{3A}^+, f_T \), and \( f_T^+ \), are actually the first Gegenbauer moment of the corresponding LCDAS, which satisfy the RGE that they should obey \[3, 4\].

\[
\frac{d}{d \ln \mu^2} (f_S, f_{3A}, f_T) = -\frac{\alpha_s}{4\pi} C_F \left( \frac{8}{3} \right) (f_S, f_{3A}, f_T),
\]

\[
\frac{d}{d \ln \mu^2} (f_{1A}^+, f_T^+) = -\frac{\alpha_s}{4\pi} C_F (f_{1A}^+, f_T^+).
\]

We also compare our results for the LCDAs of S-wave quarkonia with those in \[23, 24\]. In \[23\], the authors give all three leading twist LCDAs for S-wave quarkonia, but we find that their results do not lead to correct decay constants at NLO of \( \alpha_s \) after integration over the light fraction. In \[24\], only \( f_P \phi_P(x) \) is calculated, and we find that our results agree.

### C. Some related quantities

In the practical applications of the leading twist LCDAs, since the lowest order hard kernels \( T_H(x) \) for many hard exclusive processes are in form of \( 1/x \) or \( 1/\bar{x} \), the inverse moment of the LCDAs are crucial for final amplitudes.

We define

\[
R_\Gamma \equiv \frac{f_\Gamma}{f_\Gamma^{(0)}}, \quad \Gamma = P, V, V_\perp, 1A_\parallel, 1A_\perp, S, 3A_\parallel, 3A_\perp, T_\parallel, T_\perp.
\]

We have

\[
R_P \int_0^1 dx \frac{\hat{\phi}_P(x, \mu)}{x} = 2 + \frac{\alpha_s}{4\pi} C_F \left( (6 - 4 \ln 2) \ln \frac{\mu^2}{m^2} + 4 \ln 2 - \frac{2\pi^2}{3} \right),
\]

\[
R_V \int_0^1 dx \frac{\hat{\phi}_V(x, \mu)}{x} = 2 + \frac{\alpha_s}{4\pi} C_F \left( (6 - 4 \ln 2) \ln \frac{\mu^2}{m^2} - 4 \ln 2 - \frac{2\pi^2}{3} \right),
\]

\[
R_{3A} \int_0^1 dx \frac{\hat{\phi}_{3A}^+(x, \mu)}{x} = 2 + \frac{\alpha_s}{4\pi} C_F \left( (6 - 4 \ln 2) \ln \frac{\mu^2}{m^2} - 4 \ln 2 + 4 - \frac{2\pi^2}{3} \right),
\]

\[
R_{1A} \int_0^1 dx \frac{\hat{\phi}_{1A}^+(x, \mu)}{x} = 2 + \frac{\alpha_s}{4\pi} C_F \left( (6 - 4 \ln 2) \ln \frac{\mu^2}{m^2} + 8 \ln 2 - \frac{4\pi^2}{3} \right),
\]

\[
R_T \int_0^1 dx \frac{\hat{\phi}_T^+(x, \mu)}{x} = -2 + \frac{\alpha_s}{4\pi} C_F \left( (6 - 4 \ln 2) \ln \frac{\mu^2}{m^2} + 4 \ln 2 + \frac{2\pi^2}{3} \right),
\]

\[
R_S \int_0^1 dx \frac{\hat{\phi}_S(x, \mu)}{x} = -2 + \frac{\alpha_s}{4\pi} C_F \left( (6 - 4 \ln 2) \ln \frac{\mu^2}{m^2} - 20 \ln 2 - 12 + \frac{2\pi^2}{3} \right),
\]

\[
R_{1A} \int_0^1 dx \frac{\hat{\phi}_{1A}^+(x, \mu)}{x} = -2 + \frac{\alpha_s}{4\pi} C_F \left( (6 - 4 \ln 2) \ln \frac{\mu^2}{m^2} + 4 \ln 2 - 4 + \frac{2\pi^2}{3} \right),
\]

\[
R_{1A} \int_0^1 dx \frac{\hat{\phi}_{1A}^+(x, \mu)}{x} = -2 + \frac{\alpha_s}{4\pi} C_F \left( (6 - 4 \ln 2) \ln \frac{\mu^2}{m^2} + 4 \ln 2 - 4 + \frac{2\pi^2}{3} \right),
\]
$$R_{3A}^{\perp} = \int_0^1 dx \phi_{3A}^T(x, \mu) \frac{1}{x} = -2 + \frac{\alpha_s}{4\pi} C_F \left(2 \ln \frac{\mu^2}{m^2} - 8 \ln 2 - 8\right),$$  \hspace{1cm} (101)  

$$R_T^{\perp} = \int_0^1 dx \phi_T^T(x, \mu) \frac{1}{x} = -2 + \frac{\alpha_s}{4\pi} C_F \left(2 \ln \frac{\mu^2}{m^2} - 40 \ln 2 + 40\right).$$  \hspace{1cm} (102)  

IV. APPLICATIONS

In this section, we will apply our results for the LCDAs of quarkonia to calculate the hard exclusive processes $\gamma^* \rightarrow \eta Q\gamma, \chi_Qj\gamma, Z \rightarrow \eta Q\gamma, \chi_Qj\gamma, J/\psi(\Upsilon)\gamma, h_Q\gamma$ and $h \rightarrow J/\psi\gamma$ within the collinear factorization\(^4\). We also compare our results with the asymptotic behavior of the corresponding predictions in the NRQCD factorization. These comparisons can be regarded as a non-trivial test of our results.

A. $\gamma^* \rightarrow \eta Q\gamma, \chi_Qj\gamma$ in the collinear factorization

For the hard exclusive process $\gamma^*(Q, \varepsilon_{\gamma^*}) \rightarrow H(p)\gamma(p', \varepsilon_\gamma)$ with the momenta in the light-cone coordinates

$$p'^\mu = n_- p_n^\mu, \quad \mu = n_+ p_n^\mu + n_- p_n^\mu,$$ \hspace{1cm} (103)

and the polarization vectors $\varepsilon_{\gamma^*}$ and $\varepsilon_\gamma$ for the virtual and real photon, respectively, when $m_H^2/Q^2 \ll 1$ ($Q^2 = (p + p')^2$), we expect the light-cone factorization formula for the transition amplitude

$$i\mathcal{M}(\gamma^*(Q, \varepsilon_{\gamma^*}) \rightarrow H(p)\gamma(p', \varepsilon_\gamma)) = -i e^2 e_Q^2 \varepsilon_{\gamma^*} \varepsilon_\gamma^* \int_0^1 dy \left(\frac{i}{2} \epsilon_{\mu\nu} T_H(y; Q^2, \mu) (H(p)|Q[\gamma_5](y; \mu)|0) + \frac{g_4^{\mu\nu}}{2} T_H(y; Q^2, \mu) (H(p)|Q[1](y; \mu)|0) \right) + \mathcal{O}(m_H^2/Q^2).$$ \hspace{1cm} (104)

Here $e$ is the elementary electric charge, $e_Q$ the fractional electric charge of the quark $Q$ inside of meson $H$, $\epsilon_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} n_- n_+ n^\rho n^\sigma/2$ where $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Cevita tensor with $\epsilon_{0123} = +1$, $T_H^{\mu\nu}(y; Q^2, \mu)$ are the perturbatively calculable hard kernels, the matrix-elements of $Q[\gamma_5](y; \mu)$ and $Q[1](y; \mu)$ are eventually the appropriate leading-twist LCDAs of meson $H$.

In [51], the hard kernels have been obtained at the NLO of $\alpha_s$ which are

$$T_H^P(x; Q^2, \mu) = \frac{m_H^2}{\bar{x}} \left(1 + \frac{\alpha_s}{4\pi} C_F \left[-(3 + 2 \ln \bar{x}) \ln \frac{\mu^2}{Q^2} + \ln^2 \bar{x} - \frac{\mu^2}{\bar{x}} \ln \bar{x} \right] \right) + \mathcal{O}(\alpha_s^2).$$ \hspace{1cm} (105)

\(^4\) In [42], the authors have considered the $\gamma^* \rightarrow \eta h\gamma$ and $h \rightarrow \Upsilon\gamma$ in the collinear factorization at the leading logarithm level, and used ERBL equations to resum the large logarithms. The applications in this paper are kind of extension of the work in [42] at the NLO of $\alpha_s$, but we shall not consider the resummation here.
$$T_H^V(y; Q^2, \mu) = \frac{1}{x} \left\{ 1 + \frac{\alpha_s}{4\pi} C_F \left[ -\frac{2}{x} \ln(\frac{x}{2}) + \frac{4}{3} \right] \right\}$$

Straightforwardly, we apply the LCDAs of quarkonia obtained in the previous section, we have the NLO amplitudes

$$i\mathcal{M}(\gamma^*(Q, \varepsilon_{\gamma^*} \rightarrow \eta_{Q}(p)\gamma(p', \varepsilon_{\gamma})) = \frac{i}{2} e^2 e_Q e_{\mu} \varepsilon_{\gamma^*} \varepsilon_{\gamma} \mathcal{F}(S) \int_0^1 dy T_H^P(y; Q^2, \mu) f_{\phi_p}(y; \mu)$$

$$= -2 e^2 e_Q e_{\mu} \varepsilon_{\gamma^*} \varepsilon_{\gamma} \mathcal{F}(S) \int_0^1 dy T_H^P(y; Q^2, \mu) f_{\phi_p}(y; \mu)$$

$$i\mathcal{M}(\gamma^*(Q, \varepsilon_{\gamma^*} \rightarrow \chi_{Q1}(p, \varepsilon_{\gamma}) \gamma(p', \varepsilon_{\gamma})) = \frac{i}{2} e^2 e_Q e_{\mu} \varepsilon_{\gamma^*} \varepsilon_{\gamma} \mathcal{F}(S) \int_0^1 dy T_H^P(y; Q^2, \mu) f_{\phi_p}(y; \mu)$$

$$= -2 e^2 e_Q e_{\mu} \varepsilon_{\gamma^*} \varepsilon_{\gamma} \mathcal{F}(S) \int_0^1 dy T_H^P(y; Q^2, \mu) f_{\phi_p}(y; \mu)$$

$$i\mathcal{M}(\gamma^*(Q, \varepsilon_{\gamma^*} \rightarrow \chi_{Q2}(p, \varepsilon_{\gamma}) \gamma(p', \varepsilon_{\gamma})) = -i e^2 e_Q e_{\mu} \varepsilon_{\gamma^*} \varepsilon_{\gamma} \mathcal{F}(S) \int_0^1 dy T_H^P(y; Q^2, \mu) f_{\phi_p}(y; \mu)$$

$$= -8 e^2 e_Q e_{\mu} \varepsilon_{\gamma^*} \varepsilon_{\gamma} \mathcal{F}(S) \int_0^1 dy T_H^P(y; Q^2, \mu) f_{\phi_p}(y; \mu)$$

with $L \equiv \ln \frac{Q^2 - i\epsilon}{m^2}$. By squaring the amplitudes, one can easily reproduce the asymptotic behavior of the ratio between the NLO and tree-level cross-sections of $e^+e^- \rightarrow \eta_c\gamma, \chi_{cJ}\gamma (J = 0, 1, 2)$ in [32].

### B. $Z \rightarrow \eta_{Q\gamma}, \chi_{Q\gamma}, J/\psi(\Upsilon)\gamma, h_{Q\gamma}$ in the light-cone factorization frame

The $Z$ boson interacts with quark-anti-quark pair through the tree-level weak interaction as

$$i\mathcal{L}_{ZQQ} = \frac{g}{4\cos\theta_W} Q_{\gamma}^\mu (g_V - g_A \gamma_5) Q_{Z\mu},$$

where $g$ is the weak coupling in $SU(2)_L \times U(1)_Y$ electro-weak gauge theory, $\theta_W$ the Weinberg angle, $g_V = 1 - 8 \sin^2\theta_W / 3$ and $g_A = 1$ for the up-type quark, and $g_V = -1 + 4 \sin^2\theta_W / 3$ and $g_A = -1$ for the down-type quark.

Thus, through the vectorial interaction, $Z$ can decay to $\eta_{Q\gamma}, \chi_{Q\gamma}$ as $\gamma^*$, the corresponding decay amplitudes in the light-cone framework are just similar to $\gamma^* \rightarrow H\gamma$ by replacing the prefactor $e^2 e_Q^2$ with $g g_V e e_Q / (4 \cos\theta_W)$, $\varepsilon_{\gamma^*}$ with the polarization vector of $Z$ boson $\varepsilon_Z$,
and $Q^2$ with $m_Z^2$. Through the axial-vectorial interaction, $Z$ can decay to $J/\psi(\Upsilon)\gamma$, $h_Q\gamma$ as well. The corresponding factorization formula can be reached similarly, i.e.

$$i\mathcal{M}(Z(Q, \varepsilon_Z) \rightarrow H(p)\gamma(p', \varepsilon_\gamma)) = i \frac{g_{A}eQ}{4\cos\theta_W}\varepsilon_Z\varepsilon_{\gamma \nu}\int_0^1 dy \left( \frac{l}{2}\varepsilon_{\mu \nu}T_H^p(y; m_Z^2, \mu)\langle H(p)|Q[1](y; \mu)|0\rangle \right.$$ 

$$+ g_{A}^{\parallel}T_H^0(y; m_Z^2, \mu)\langle H(p)|Q[\gamma](y; \mu)|0\rangle) + O(m_H^2/m_Z^2), \quad (112)$$

where $H = J/\psi(\Upsilon)$ or $h_Q$, and the hard-kernels $T_H$ are identical to those for $\gamma^* \rightarrow H\gamma$ but with the Lorenz structures exchanged.

Straightforwardly, we have the NLO amplitudes

$$i\mathcal{M}(Z(Q, \varepsilon_Z) \rightarrow J/\psi(\Upsilon)(p, \varepsilon)\gamma(p', \varepsilon_\gamma)) = -i \frac{g_{A}eQ}{8\cos\theta_W}\varepsilon_Z\varepsilon_{\gamma \nu}f_{J/\psi(\Upsilon)}\int_0^1 dy T_H^p(y; Q^2, \mu)\hat{\phi}_V(y; \mu)$$

$$= -i \frac{g_{A}eQ}{2\cos\theta_W}\varepsilon_Z\varepsilon_{\gamma \nu}m_{J/\psi(\Upsilon)}\left\{ 1 + \frac{\alpha_s}{4\pi} C_F \left( 3 - 2\ln 2 \right) L + \ln^2 2 - 2\ln 2 - 9 - \frac{\pi^2}{3} \right\}, \quad (113)$$

$$i\mathcal{M}(Z(Q, \varepsilon_Z) \rightarrow h_Q(p, \varepsilon)\gamma(p', \varepsilon_\gamma)) = \frac{g_{A}eQ}{8\cos\theta_W}\varepsilon_Z\varepsilon_{\gamma \nu}f_{1A}\int_0^1 dy T_H^p(y; Q^2, \mu)\hat{\phi}_{1A}(y; \mu)$$

$$= -i \frac{g_{A}eQ}{2\cos\theta_W}\varepsilon_Z\varepsilon_{\gamma \nu}\left\{ 1 + \frac{\alpha_s}{4\pi} C_F \left( 1 - 2\ln 2 \right) L + \ln^2 2 - 3\ln 2 - 4 - \frac{\pi^2}{3} \right\}, \quad (114)$$

with $L \equiv \ln \frac{-m_Z^2}{m^2}$. By squaring the amplitudes, one should easily reproduce the asymptotic behavior of the ratio between the NLO and tree-level cross-sections of $e^+e^- \rightarrow J/\psi\gamma$, $h_Q\gamma$ at $Z^0$-pole. In [52, 53], Chen et al give the asymptotic ratios between the NLO and LO cross section are

$$r^{[3S_1]} \equiv \frac{\sigma^{NLO}(e^+e^- \rightarrow Z^0 \rightarrow J/\psi\gamma)}{\sigma^{LO}(e^+e^- \rightarrow Z^0 \rightarrow J/\psi\gamma)} = \frac{\alpha_s}{2\pi} C_F \left( 3 - 2\ln 2 \right) \ln \frac{m_Z^2}{m_c^2} + \frac{\ln^2 2 - 2\ln 2 - 5 - \frac{\pi^2}{3}}{3}, \quad (115)$$

$$r^{[1P_1]} \equiv \frac{\sigma^{NLO}(e^+e^- \rightarrow Z^0 \rightarrow h_Q\gamma)}{\sigma^{LO}(e^+e^- \rightarrow Z^0 \rightarrow h_Q\gamma)} = \frac{\alpha_s}{2\pi} C_F \left( 1 - 2\ln 2 \right) \ln \frac{m_Z^2}{m_c^2} + 3\ln 2 - 4 - \frac{\pi^2}{3}. \quad (116)$$

Their results agree with ours for $^1P_1$ case, but differ from ours for $^3S_1$ case, by a constant term $(-4)$ at $O(\alpha_s)$, which may be related to the different scheme to treat $\gamma_5$ in loop calculation.

C. $h \rightarrow J/\psi\gamma$ in the light-cone factorization frame

The higgs boson $h$ in the Standard Model interacts with quark-anti-quark pair through the Yukawa interaction

$$i\mathcal{L}_{hQQ} = \frac{i\mu_Q}{\sqrt{2}}\bar{Q}Qh. \quad (117)$$
Here $y_Q \equiv -\sqrt{2 m/v}$ is the Yukawa coupling where $v = 246$ GeV is the vacuum expectation value of the Higgs field, and $m$ is the current mass of quark $Q$ in $\overline{\text{MS}}$ scheme. The corresponding factorization formula for $h \to J/\psi \gamma$ is

$$i \mathcal{M}(h(Q) \to J/\psi(p, \varepsilon_\psi)\gamma(p', \varepsilon_\gamma)) = -i \frac{m_e e c}{2 v} \varepsilon^*_\psi \varepsilon_\gamma \int_0^1 dy T_H(y; m_h^2, \mu)(\gamma(p, \varepsilon_\psi)Q[\gamma'_\mu](y; \mu)|0),$$

(118)

where $\varepsilon_\psi$ is the polarization vector of $J/\psi$, and the hard-kernels $T_H$ can be calculated perturbatively. The NLO hard-kernel is

$$T_H(x; m_h^2, \mu) = \frac{1}{\bar{x}} \left\{ 1 + \frac{\alpha_s}{4\pi} C_F \left[ - \left( 3 + 2 \frac{\ln \bar{x}}{x} \right) \ln \frac{\mu^2}{-m_h^2 - i\epsilon} + \frac{\ln^2 \bar{x}}{x} - 7 \right] \right\},$$

(119)

with the mass of higgs in the Standard Model $m_h \simeq 125$ GeV.

Straightforwardly, we have the NLO amplitudes

$$i \mathcal{M}(h(Q) \to J/\psi(p, \varepsilon_\psi)\gamma(p', \varepsilon_\gamma))$$

$$= -i \frac{m_e e c}{2 v} \varepsilon^*_\psi \varepsilon_\gamma \left\{ \alpha_s \pi \right\} \left( \frac{1}{m_{J/\psi}} \left[ 4 \ln 2 \ln \frac{-m_h^2 - i\epsilon}{m_c^2} - 2 \ln^2 2 - 4 \ln 2 + 7 + \frac{2 \pi^2}{3} \right] \right\} ,$$

(120)

where $m_c$ is the pole mass of charm quark.

Thirty years ago, Shifman et al have calculated $h \to J/\psi \gamma$ to NLO of $\alpha_s$ in color singlet model which is equivalent to the NRQCD calculation [54]. The NLO prediction for $H \to J/\Psi \gamma$, that we quote from Eq.(22) in [54], is written as

$$i \mathcal{M}(h \to J/\Psi \gamma) = i \mathcal{M}_{tr}(h \to J/\Psi \gamma) \left[ 1 - \frac{\alpha_s(m_h^2)C_F}{2\pi} a(\kappa) \right] ,$$

(121)

where

$$a(\kappa) = 4 - \frac{\pi^2}{12(1-\kappa)} - \frac{F(1-2\kappa)}{2(1-\kappa)} + \frac{\kappa - 1}{1-2\kappa} + \frac{2\kappa(\kappa - 2)}{(1-\kappa)^2} \left[ \phi(\kappa) + F(1) - F(-1) \right]$$

$$+ \left( 4 \frac{4}{\kappa} + \frac{8}{1-\kappa} \right) \sqrt{\frac{\kappa}{1-\kappa}} \arctan \sqrt{\frac{\kappa}{1-\kappa}} + \left( \frac{4}{1-\kappa} + 2 + \frac{\kappa}{(2\kappa - 1)^2} \right) \ln(2 - 2\kappa) ,$$

(122)

with $\kappa = m_h^2 / (4m_c^2) + i\epsilon$ and

$$\phi(x) = \int_0^1 \frac{dy}{y - 1/(2x)} \ln \frac{1 - 4y(1 - y)x}{2y(1 - x)} , \quad F(x) = \int_0^x \frac{dy}{y} \ln(1 + y) = -Li_2(-x) .$$

The asymptotic behavior of $a(\kappa)$ at $\kappa \to \infty$ is

$$a(\kappa) = \frac{1}{2} \left[ 4 \ln 2 \ln \frac{-m_h^2 - i\epsilon}{m_c^2} - 2 \ln^2 2 - 4 \ln 2 + \frac{2 \pi^2}{3} + 7 \right] + O(m_c^2/m_h^2) ,$$

(124)

which coincides with Eq.(120).
V. SUMMARY

In this paper, we calculate ten leading twist LCDAs for the S-wave and P-wave quarkonia to the NLO of $\alpha_s$ and leading order of $v$. We demonstrate that applications of these LCDAs in some single quarkonium exclusive processes can lead to correct asymptotic behavior of relevant NRQCD results. This confirms again the conclusion in [43] that there is a tight connection between the collinear factorization method and NRQCD factorization method for a certain class of quarkonium exclusive productions. And also as in [42], together with the ERBL equation, the collinear factorization method can be used to resum the large logarithms in NRQCD calculations. However, as discussed in [42, 55], the so-called "endpoint logarithms" in helicity-flipped exclusive processes, leads to the breakdown of the collinear factorization. Such "endpoint logarithms" seem to be process-dependent, and how to resum them remains unknown.

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Note added: After this work was finished, we were noticed by the authors of a series of paper [56, 57], that they calculated the S-wave and P-wave heavy quarkonium fragmentation functions (FFs) from a heavy quark pair, of which the FFs from a color-singlet heavy quark pair are related to the LCDAs we calculated in this paper. We are very grateful to Y.Q. Ma, J.W. Qiu and H. Zhang (the authors of [56, 57]) for enormous communications and efforts on cross-checking. After correcting some typos and mistakes in original manuscripts, we get completely consistent results.

Appendix A: Some useful integrals

Here we list some loop integrals which are useful for the NLO computation of LCDAs for quarkonia in Sect.III.

\[
\int [dk] \frac{f(n+k)}{[k^2+i\epsilon][k^2+P\cdot k+i\epsilon][k^2-P\cdot k+i\epsilon]} = \left(\frac{-1}{m^2}\right)^n \left(\frac{4\pi\tilde{\mu}^2}{m^2}\right)^\varepsilon \frac{\Gamma(n+\varepsilon)}{\Gamma(n+1)} \int_0^1 dy f((y-1/2)n+P) \left(\frac{(2y)^n\theta(2y-1)}{(1-2y)^{2n+2\varepsilon}} + \frac{(2y)^n\theta(1-2y)}{(1-2y)^{2n+2\varepsilon}}\right),
\]

(A1)

\[
\int [dk] \frac{f(n+k)k^\mu_1 k^\nu_2}{[k^2+i\epsilon][k^2+P\cdot k+i\epsilon][k^2-P\cdot k+i\epsilon]} = \left(\frac{4\pi\tilde{\mu}^2}{m^2}\right)^\varepsilon \frac{\Gamma(\varepsilon)}{2} \int_0^1 dy f((y-1/2)n+P) \left(\frac{2y\theta(2y-1)}{(1-2y)^{2\varepsilon}} + \frac{2y\theta(1-2y)}{(1-2y)^{2\varepsilon}}\right),
\]

(A2)
\[
\int [dk] \frac{f(n_p + k) k_{\perp}^\mu k_{\perp}^\nu}{[k^2 + i\epsilon]^2[k^2 + P \cdot k + i\epsilon][k^2 - P \cdot k + i\epsilon]} \\
= -\frac{1}{m^2} \left( \frac{4\pi \mu^2}{m^2} \right)^\varepsilon \Gamma(1 + \varepsilon) \frac{g_{\perp}^{\mu\nu}}{2} \int_0^1 dy f ((y - 1/2)n + P) \left( \frac{2y^2\theta(2y - 1)}{(1 - 2y)^{2+2\varepsilon}} + \frac{2y^2\theta(1 - 2y)}{(1 - 2y)^{2+2\varepsilon}} \right), \tag{A3}
\]

\[
\int [dk] \frac{f(n_p + k) (n_{-k} - n_{-k}/(n_{+v})^2)n_+ q}{[k^2 + i\epsilon]^2[k^2 + P \cdot k + i\epsilon][k^2 - P \cdot k + i\epsilon]} \\
= \frac{n_+ q}{m^2 n_+ P} \left( \frac{4\pi \mu^2}{m^2} \right)^\varepsilon \Gamma(1 + \varepsilon) \frac{g_{\perp}^{\mu\nu}}{2} \int_0^1 dy f ((y - 1/2)n + P) \\
\times \left( \frac{4y(1 + 2\varepsilon y)\theta(2y - 1)}{(1 - 2y)^{3+2\varepsilon}} + \frac{4y(1 + 2\varepsilon y)\theta(1 - 2y)}{(1 - 2y)^{3+2\varepsilon}} \right), \tag{A4}
\]

\[
\int [dk] \frac{f(n_p + k) k_{\perp}^\mu k_{\perp}^\nu (n_{-k} - n_{-k}/(n_{+v})^2)n_+ q}{[k^2 + i\epsilon]^2[k^2 + P \cdot k + i\epsilon][k^2 - P \cdot k + i\epsilon]} \\
= \frac{n_+ q}{n_+ P} \left( \frac{4\pi \mu^2}{m^2} \right)^\varepsilon \Gamma(1 + \varepsilon) \frac{g_{\perp}^{\mu\nu}}{2} \int_0^1 dy f ((y - 1/2)n + P) \\
\times \left( \frac{2y(1 - 2y - 2\varepsilon y)\theta(2y - 1)}{(1 - 2y)^{1+2\varepsilon}} - \frac{2y(1 - 2y + 2\varepsilon y)\theta(1 - 2y)}{(1 - 2y)^{1+2\varepsilon}} \right), \tag{A5}
\]

\[
\int [dk] \frac{\delta(x - 1/2 - n_{+k}/n_{+P}) - \delta(x - 1/2)}{[n_{+k}][k^2 + i\epsilon]} \left( \frac{n_+(k + P/2)}{[k^2 + P \cdot k + i\epsilon]} + \frac{n_+(k - P/2)}{[k^2 - P \cdot k + i\epsilon]} \right) \\
= -\left( \frac{4\pi \mu^2}{m^2} \right)^\varepsilon \Gamma(1 + \varepsilon) \frac{4x\theta(1 - 2x)}{(1 - 2x)^{1+2\varepsilon}} + \frac{4x\theta(1 - 2x)}{(2x - 1)^{1+2\varepsilon}} + \frac{n_+(k + P/2)}{[k^2 + P \cdot k + i\epsilon]} + \frac{n_+(k - P/2)}{[k^2 - P \cdot k + i\epsilon]} \right), \tag{A6}
\]

\[
\int [dk] \frac{\delta(x - 1/2 - n_{+k}/n_{+P}) - \delta(x - 1/2)}{[n_{+k}][k^2 + i\epsilon]} \left( \frac{n_+(k + P/2)}{[k^2 + P \cdot k + i\epsilon]} + \frac{n_+(k - P/2)}{[k^2 - P \cdot k + i\epsilon]} \right) \\
= \frac{n_+ q}{n_+ P} \left( \frac{4\pi \mu^2}{m^2} \right)^\varepsilon \Gamma(1 + \varepsilon) \frac{4x\theta(1 - 2x)}{(1 - 2x)^{1+2\varepsilon}} + \frac{4x\theta(1 - 2x)}{(2x - 1)^{1+2\varepsilon}} \right), \tag{A7}
\]

\[
\int [dk] \frac{\delta(x - 1/2 - n_{+k}/n_{+P}) - \delta(x - 1/2)}{[n_{+k}][k^2 + i\epsilon]} \left( \frac{n_{-k} - n_{-k}/(n_{+v})^2}{[n_{+k}][k^2 + i\epsilon]} \right) \\
\times \left( \frac{n_+(k + P/2)}{[k^2 + P \cdot k + i\epsilon]} + \frac{n_+(k - P/2)}{[k^2 - P \cdot k + i\epsilon]} \right) \\
= \frac{n_+ q}{n_+ P} \left( \frac{4\pi \mu^2}{m^2} \right)^\varepsilon \Gamma(1 + \varepsilon) \frac{8x(1 - 2x + 4\varepsilon x)\theta(2x - 1)}{(1 - 2x)^{2+2\varepsilon}} - \frac{8x(1 - 2x + 4\varepsilon x)\theta(1 - 2x)}{(1 - 2x)^{2+2\varepsilon}} \right). \tag{A8}
\]

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