Gravity in Brans-Dicke theory with Born-Infeld scalar field and the Pioneer anomaly

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Abstract

In this paper we discuss a model which can be considered as a generalization of the well-known scalar-tensor Brans-Dicke theory. This model possesses an interesting feature: due to Born-Infeld type non-linearity of the scalar field the properties of the interaction between two test bodies depend significantly on their masses. It is shown that the model can be interesting in view of the Pioneer 10, 11 spacecraft anomaly.

1 Introduction and setup

One of the most known scalar-tensor theories of gravity is the Brans-Dicke theory \[1, 2\]. It describes the scalar field non-minimally coupled to gravity with the action

\[
S = \int d^4x \sqrt{-g} \left[ \varphi R - \tilde{\omega} \frac{g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi}{\varphi} + L_{\text{matter}} \right],
\]

where \(\tilde{\omega}\) is the Brans-Dicke parameter and \(L_{\text{matter}}\) is the Lagrangian of matter. In the limit \(\tilde{\omega} \to \infty\) the theory goes to the standard General Relativity. This theory is very well examined, the present days gravitational experiments set stringent limits on possible values of the Brans-Dicke parameter \(\tilde{\omega}\) \[3, 4\].

In this paper we consider a generalization of this theory based on the use of the Born-Infeld scalar field. This field itself was widely discussed in the literature, see, for example, \[5\]–[11] and references therein. We will show that such a highly non-linear covariant theory possesses very interesting features and, in principle, it can account for the anomalous acceleration of Pioneer 10 and Pioneer 11 spacecraft \[12, 13\], leaving the planets of the Solar System devoid of such extra constant acceleration which is excluded by observations \[13\].

To this end let us consider the following four-dimensional action describing Born-Infeld scalar field non-minimally coupled to gravity

\[
S = \int d^4x \sqrt{-g} \left[ \varphi R + f \sqrt{1 - \frac{\omega g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi}{f}} - f + L_{\text{matter}} \right],
\]

with \(f > 0\). The action of the scalar field has a non-standard form, but in the limit \(f \to \infty\)

\[
f \sqrt{1 - \frac{\omega g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi}{f}} \to f - \frac{\omega g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi}{2}
\]

and (2) transforms into the well-know action of the Brans-Dicke theory \[1\] with the Brans-Dicke parameter \(\tilde{\omega} = \frac{\omega}{2}\). The theory with action (2) can be considered as a generalized Brans-Dicke theory. The extra term \(-f\) in (2) is added to preserve the Minkowski background metric in the vacuum state.
The vacuum expectation value of the field $\varphi$ is supposed to be $\varphi_{\text{vac}} = M_{Pl}^2$. Let us represent the scalar and the gravitational fields as

$$\varphi = M_{Pl}^2 + \frac{M_{Pl}}{\sqrt{\omega}} \phi,$$

$$g_{\mu \nu} = \eta_{\mu \nu} + 1 \frac{h_{\mu \nu}}{M_{Pl}}$$

where $\varphi_{\text{vac}} = 0$, $g_{\mu \nu}^{\text{vac}} = \eta_{\mu \nu}$, $\eta_{\mu \nu} = \text{diag}(-1, 1, 1, 1)$ is the flat Minkowski metric, and expand action \([2]\) into series with respect to $\phi$ and $h_{\mu \nu}$. The "gravitational" part of the action

$$\int d^4x \sqrt{-g} \varphi R$$

can be represented as

$$\int d^4 x \left[ M_{Pl} L^{(1)}[h_{\mu \nu}] + \frac{1}{\sqrt{\omega}} \phi L^{(1)}[h_{\mu \nu}] + \left( 1 + \frac{1}{M_{Pl} \sqrt{\omega}} \phi \right) L^{(2)}[h_{\mu \nu}] + \ldots \right],$$

where $L^{(1)}[h_{\mu \nu}]$ is linear in $h_{\mu \nu}$, $L^{(2)}[h_{\mu \nu}]$ is quadratic in $h_{\mu \nu}$ and so on. The Born-Infeld part of the action

$$\int d^4 x \sqrt{-g} \phi \sqrt{1 - \frac{\eta_{\mu \nu} \partial_{\mu} \phi \partial_{\nu} \phi}{f}}$$

can be rewritten as

$$\int d^4 x \sqrt{-g} \phi \sqrt{1 - \frac{1}{M_{Pl} \sqrt{\omega}} \phi + \ldots},$$

where we have omitted the expansion of $\sqrt{-g}$. Here and below indices are raised by $\eta^{\mu \nu}$.

Now let us discuss formulas \([5]\) and \([6]\). First, the term $L^{(1)}[h_{\mu \nu}]$ is simply $L^{(1)}[h_{\mu \nu}] = \partial^\mu \partial^\nu h_{\mu \nu} - \partial^\mu h^\nu_{\nu}$ (where $h = h^\nu_{\nu}$), which is a total derivative. Thus, the term $M_{Pl} L^{(1)}[h_{\mu \nu}]$ vanishes from the action. Second, since we suppose to work in the Newtonian approximation, we can drop the term $\frac{1}{M_{Pl}} h^{\mu \nu}$ (as well as higher corrections in $h_{\mu \nu}$) in comparison with $\eta^{\mu \nu}$. For these reasons we can also drop the terms $L^{(n)}[h_{\mu \nu}]$ for $n > 2$. As for the term $\frac{1}{M_{Pl} \sqrt{\omega}} \phi$, it is not evident that it is much smaller than unity. Nevertheless, let us suppose that $\frac{1}{M_{Pl} \sqrt{\omega}} \phi \ll 1$ and drop the term $\frac{1}{M_{Pl} \sqrt{\omega}} \phi$ and the subsequent terms in \([6]\), as well as the cubic term $\frac{1}{M_{Pl} \sqrt{\omega}} \phi L^{(2)}[h_{\mu \nu}]$. Below we will show that condition $\frac{1}{M_{Pl} \sqrt{\omega}} \phi \ll 1$ indeed holds. Note that we are not able to drop the quadratic term $\frac{\phi}{\sqrt{\omega}} L^{(1)}[h_{\mu \nu}]$ because it ensures the interaction of Born-Infeld scalar field with matter.

Thus we get

$$S_{\text{eff}} = \int d^4 x \left( \frac{\phi}{\sqrt{\omega}} L^{(1)}[h_{\mu \nu}] + L^{(2)}[h_{\mu \nu}] + \int \sqrt{1 - \frac{1}{M_{Pl} \sqrt{\omega}} \phi + \frac{1}{2 M_{Pl}} h^{\mu \nu} t_{\mu \nu}} \right),$$

where $t_{\mu \nu}$ is the energy-momentum tensor of matter and

$$L^{(2)}[h_{\mu \nu}] = L_{FP}[h_{\mu \nu}] =$$

$$-\frac{1}{4} \left[ \partial_{\rho} h_{\mu \nu} \partial^\rho h^{\mu \nu} - \partial_{\rho} h h^\rho_{\rho} + 2 \partial_{\rho} h^{\mu \nu} \partial_{\sigma} h - 2 \partial_{\rho} h^{\mu \nu} \partial^\rho h_{\mu \nu} \right]$$
is the standard Fierz-Pauli Lagrangian. It is convenient to diagonalize action (7) with the help
of the standard redefinition
\[ h_{\mu\nu} = b_{\mu\nu} - \frac{1}{\sqrt{\omega}} \eta_{\mu\nu} \phi. \] (9)
After some algebra we get
\[ L_{\text{eff}} = L_{\text{FP}}[b_{\mu\nu}] + f \sqrt{1 - \frac{1}{f} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi} - \frac{3}{2 \omega} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2 M_{\text{Pl}}} b^{\mu\nu} t_{\mu\nu} - \frac{1}{2 M_{\text{Pl}} \sqrt{\omega}} \phi t, \] (10)
where \( t = \eta^{\mu\nu} t_{\mu\nu}. \) The extra term \(-\frac{3}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi\) in (7) has appeared in the action after diagonalization.

The Born-Infeld part of action (10) has the standard form of the Dirac-Born-Infeld (DBI) scalar field action, though the standard DBI Lagrangian has a different origin. It is necessary to note that we take \( f > 0 \) (like in [8]), contrary to the case \( f < 0 \), which is often discussed in the literature (see, for example, [5]–[7], [10]).

It should be also noted that we neglected the term \(-f\) (see action (2)) while obtaining Lagrangian (10). We will discuss this issue in the next section.

Lagrangian (10) allows one to examine the stability of the model at least above the Minkowski background. To this end we consider \( t_{\mu\nu} = 0 \) and suppose that \( \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \ll f \). Expanding the square root in (10) up to the linear term we get a quadratic Lagrangian
\[ L_2 = L_{\text{FP}}[b_{\mu\nu}] - \frac{\omega + 3}{2 \omega} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \] (11)
For \( \omega \gg 1 \) (this case will be discussed below) the kinetic term of the scalar field \( \phi \) has the proper sign, which leads to the absence of ghosts in the theory. Indeed, for \( \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \ll f \) Lagrangian (10) describes the standard Brans-Dicke theory in the Newtonian approximation, which is known to be stable. Higher corrections to (11) lead to an infinite tower of self-interaction and interaction terms. Nevertheless, the solution which will be discussed below corresponds to a deep non-linear regime of the Born-Infeld part of the model where perturbation theory does not work. It has appeared to be very difficult (maybe even impossible) to examine the perturbations around this solution analytically, one should make numerical analysis. Thus, the question about the stability of the solution presented below has no definite answer yet.

2 Equations of motion and extra anomalous force

The equations of motion following from Lagrangian (10) take the form
\[ \Box b_{\mu\nu} = -\frac{1}{M_{\text{Pl}}} \left( t_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} t \right), \] (12)
\[ \partial_\nu \left( \frac{\eta^{\mu\nu} \partial_\mu \phi}{\sqrt{1 - \frac{1}{f} \eta^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi}} + \frac{3}{\omega} \eta^{\mu\nu} \partial_\mu \phi \right) = \frac{1}{2 M_{\text{Pl}} \sqrt{\omega}} t, \] (13)
where \( \Box = \eta^{\mu\nu} \partial_\mu \partial_\nu. \) We used de Donder gauge condition \( \partial^\mu b_{\mu\nu} - \frac{1}{2} \partial_\nu b = 0 \) while obtaining (12).
We will be interested in the additional interaction between two bodies caused by the scalar field $\phi$. As can be seen from initial action (2), ordinary matter interacts only with the metric, in this sense the weak equivalence principle is fulfilled. Thus the gravitational force acting on a test body can be easily obtained by considering geodesic motion, and in the Newtonian limit we get the well-known formula

$$\ddot{x} = \frac{1}{2M_{Pl}} \nabla h_{00}.$$  

If one considers a non-point-like source, this formula transforms into a formula describing the force acting on the center of mass of the test body

$$m \ddot{R} = \frac{1}{2M_{Pl}} \int_V dV \rho(\vec{x}) \nabla h_{00},$$

where $\rho(\vec{x})$ is the density of the body of volume $V$ such that

$$\int_V \rho(\vec{x}) dV = m$$

and $\vec{R}$ is the vector pointing to the center of mass of the body. Using (9) we get the standard Newtonian force (the contribution of $b_{\mu\nu}$), and an anomalous force

$$\vec{F}_{anom} = \frac{1}{2M_{Pl} \sqrt{\omega}} \int_V dV \rho(\vec{x}) \nabla \phi$$

(the contribution of $\phi$). We will be interested in this anomalous extra force.

Now let us turn to Eq. (13) and consider the static case of spherically symmetric bodies. We suppose that the energy-momentum tensors of the test bodies have the form

$$t^{1,2}_{00} = \rho_{1,2}(\vec{x}), \quad t^{1,2}_{ij} = 0.$$  

Due to the spherical symmetry

$$\rho_2(\vec{x}) = \rho(r), \quad r \leq r_*,$$

$$\rho_2(\vec{x}) = 0, \quad r > r_*,$$

where $r_*$ is the radius of the second body.

Let us denote $\eta^{ij} \partial_j \phi = \nabla \phi = \vec{\phi}$. Then Eq. (13) for the case of two test bodies takes the form

$$\text{div} \left( \frac{\vec{\phi}}{\sqrt{1 - \frac{1}{4} (\vec{\phi} \cdot \vec{\phi})}} + \frac{3 \vec{\omega}}{\omega} \right) = 4\pi \left( -\frac{1}{8\pi \sqrt{\omega} M_{Pl}} \rho_1(\vec{x}) - \frac{1}{8\pi \sqrt{\omega} M_{Pl}} \rho_2(\vec{x}) \right).$$

Now we are ready to examine the force acting on the test body in such a system. The coordinate system that will be used for calculations is presented in Fig. 1. The force will be calculated for the second body (the right body in Fig. 1). The force will be calculated for the second body (the right body in Fig. 1).

The solution to Eq. (19) inside the second body has the form

$$\frac{\vec{\phi}}{\sqrt{1 - \frac{1}{4} (\vec{\phi} \cdot \vec{\phi})}} + \frac{3 \vec{\omega}}{\omega} = -\frac{1}{8\pi \sqrt{\omega} M_{Pl}} \left( \frac{M_1}{l^3} + \frac{m(r) \vec{r}}{r^3} \right),$$

where $l$ is the scale length, $M_1$ is the mass of the first body, $m(r)$ is the mass of the second body.
where \( \vec{l} = \vec{R} + \vec{r} \), \( r = \sqrt{\vec{r}^2} \), \( M \) is the mass of the first body,

\[
m(r) = 4\pi \int_0^r \vec{r}^2 \rho(\vec{r}) d\vec{r}
\]

for \( r < r_* \) and \( m(r)_{r \geq r_*} = m \) (see Fig. 1).

It is convenient to represent the parameter \( f \) as

\[
f = \frac{M^2}{r_M^4 (8\pi)^2 \omega M_{Pl}^2},
\]

(21)

where \( r_M \) is a parameter depending on \( M \).

We suppose that \( \omega \gg 1 \). Since \( \frac{1}{\sqrt{1 - \frac{4}{3} f(\vec{\phi})}} > 1 \) and \( 3/\omega \ll 1 \), we can look for a solution to Eq. (20) using the perturbative approach. After some algebra one can get

\[
\vec{\phi} = \frac{\sqrt{f} \left( \frac{M \vec{l}}{t^3} + \frac{m(r) \vec{r}}{r^3} \right)}{\sqrt{\frac{M^2}{r_M^4} + \left( \frac{M \vec{l}}{t^3} + \frac{m(r) \vec{r}}{r^3} \right)^2}} \left[ 1 - \frac{3}{\omega} \left( \frac{M^2}{r_M^2} + \left( \frac{M \vec{l}}{t^3} + \frac{m(r) \vec{r}}{r^3} \right)^2 \right)^{\frac{3}{2}} \right].
\]

(22)

For the objects which will be discussed in the next section of the paper the correction in (22) \( \sim 3/\omega \) appears to be at least \( \sim 10^{-3} \) or even smaller. Thus we can drop this correction and use

\[
\vec{\phi} = - \left( \frac{M \vec{l}}{t^3} + \frac{m(r) \vec{r}}{r^3} \right) \frac{\sqrt{f}}{\sqrt{\frac{M^2}{r_M^4} + \left( \frac{M \vec{l}}{t^3} + \frac{m(r) \vec{r}}{r^3} \right)^2}}.
\]

(23)

For \( r_M \gg R \) (23) transforms into

\[
\vec{\phi} \approx - \frac{\sqrt{f} \left( \frac{M \vec{l}}{t^3} + \frac{m(r) \vec{r}}{r^3} \right)}{\sqrt{\left( \frac{M \vec{l}}{t^3} + \frac{m(r) \vec{r}}{r^3} \right)^2}}.
\]

(24)

We can estimate the maximal value of the field \( \phi \) itself. The approximate size of the non-linearity zone is \( r_M \), and \( \phi \) can be estimated as

\[
\phi \sim \sqrt{(\vec{\phi} \vec{r})} r_M = \sqrt{f} r_M = \frac{M}{8\pi \sqrt{\omega M_{Pl} r_M}}
\]
and thus

$$\frac{\phi}{\sqrt{\omega M_{Pl}}} \sim \frac{M}{8\pi M^3_{Pl} r_M}.$$ 

A more accurate analysis based on the use of the solutions inside and outside the non-linearity zone (the latter behaves as $\sim 1/L$, where $L \gg r_M$ is a characteristic distance from both bodies), provides an analogous estimate (up to the factor of the order of unity). For the parameters, which will be considered in the next section, $\frac{\phi}{\sqrt{\omega M_{Pl}}} \ll 1$ and the corresponding terms in (5) indeed can be omitted.

Now let us estimate the effects that could be produced by the omitted term $-f$ of action (2) (see previous section). In the non-linearity zone $\sqrt{1 - \frac{1}{f}(\vec{\phi} \vec{\phi})} \ll 1$ and the term $-f$ is not compensated. It indicates that the background metric in the non-linearity zone is not the flat Minkowski metric, but a de Sitter-like background metric leading to a local expansion. For the observer, say, on the first body it looks like a repulsive force acting on the second body. This force has the form

$$\vec{F}_{rep} \sim \frac{m f}{M^3_{Pl}} \vec{R},$$

which can be easily obtained by considering de Sitter metric in the static form [2]. Using (21) we get

$$|\vec{F}_{rep}| \sim \left( \frac{M}{4\pi M^3_{Pl} r_M} \right) \left( \frac{R}{r_M} \right) \frac{1}{16\pi M^2_{Pl} \omega r_M^2} m.$$ (26)

For the values of the parameters that will be used below this force appears to be much smaller than the forces caused by the fields $b_{\mu \nu}$ and $\phi$ obeying (12) and (13) respectively. Thus, for our purposes we can use the Minkowski background metric instead of a de Sitter background metric. We would like to note that analogous estimates can be obtained if we retain the term $-f$ in the action and get slightly modified equations for the fields $b_{\mu \nu}$ and $\phi$.

3 Specific examples

Now we turn to the effects which can be produced by the DBI scalar field in our Solar System. Let us suppose that

$$M = M_\odot, \quad r_M \approx 100 \text{AU}, \quad \omega \approx 700,$$

which means that $f \approx 2 \cdot 10^{-44} \text{GeV}^4$ (our ”reduced” Planck mass $M_{Pl} \approx \frac{1.2 \cdot 10^{19} \text{GeV}}{\sqrt{16\pi}} \approx 1.7 \cdot 10^{18} \text{GeV}$).

In what follows we will consider two cases:

1. A light body with the mass $m$, for which $\frac{m}{r^2} \sim \frac{m(r)}{r^2} \ll \frac{M}{R^2}$, $r_s \ll R \ll r_M$ (for example, a spacecraft like Pioneer 10, 11 with $m \sim 300 \text{kg}$, $r_s \sim 1 \text{m}$). In this case in the leading order

$$\vec{\phi} = -\left( \frac{M \vec{I}}{f^3} + \frac{m(r) \vec{R}}{r^3} \right) \frac{\sqrt{f}}{\sqrt{\left( \frac{M \vec{I}}{f^3} + \frac{m(r) \vec{R}}{r^3} \right)^2}} \approx$$

$$\approx -\left( \frac{M \vec{I}}{f^3} + \frac{m(r) \vec{R}}{r^3} \right) \frac{\sqrt{f}}{\sqrt{\left( \frac{M \vec{I}}{f^3} + \frac{m(r) \vec{R}}{r^3} \right)^2}} = -\sqrt{f} \left( \frac{\vec{R}}{R} + \frac{m(r)R^2}{M r^3} \right).$$ (27)
It is worth mentioning that there is no such static solution for the case \( f < 0 \). Indeed, if \( f < 0 \) then \( r_M^4 < 0 \) and we get negative values under the square root (see Eq. (23)), which is the consequence of the existence of a horizon at a finite distance (see, for example, [9], where solutions with horizons in DBI scalar field theory are discussed). That is why we chose the case \( f > 0 \).

Substituting the latter formula into (16) and integrating over the volume of the body leads to (we use the fact that \( \int \vec{r} d\Omega = 0 \), where \( \Omega \) is the solid angle)

\[
\vec{F}_{anom} = -\frac{1}{16\pi M_{Pl}^2} \frac{M}{\omega r_M^2} m. \tag{28}
\]

It is necessary to note that in principle

\[
\int \frac{\vec{r}}{\sqrt{(\frac{Ml}{l^3} + \frac{m(r)r^2}{r^3})^2}} d\Omega \neq 0. \tag{29}
\]

But we can neglect possible corrections because anyway

\[
\left| \frac{Ml}{l^3} \right| \gg \left| \frac{m(r)r^2}{r^3} \right|, \tag{28}
\]

see (27).

Formula (28) is written in the system of units \( \hbar = c = 1 \). The replacement \( \frac{1}{16\pi M_{Pl}^2} \rightarrow G \) allows one to pass to the SI units, which results in the acceleration towards the Sun in the SI units

\[
a_{anom} = \frac{GM}{\omega r_M^2} \approx 8.7 \cdot 10^{-10} \text{m/s}^2. \tag{30}
\]

We note that this acceleration does not depend on the distance from the Sun, which is exactly the situation with the Pioneer 10 and Pioneer 11 spacecraft [12, 13].

As for the bodies on the surface of the Earth, we can carry out analogous calculations taking \( M = M_\oplus \) (in this case \( r_M \) also changes). Our ideal test bodies with density profile (17), (18) on the surface of the Earth also possess an additional acceleration \( a_{anom} \) towards the center of the planet. It is evident that this acceleration can be neglected in comparison with \( g \approx 9.8 \text{m/s}^2 \) for Earth-based gravitational experiments.

2. Heavy bodies with the mass \( m \) (planets), \( \frac{M}{l^2} \ll \frac{m(r)}{r^2} \), \( r_* \ll R \ll r_M \). In this case we should carry out calculations more precisely because

\[
\left| \frac{Ml}{l^3} \right| \ll \left| \frac{m(r)r^2}{r^3} \right|, \tag{28}
\]

and possible corrections due to (29) can be quite large. Correspondingly, we should take

\[
\sqrt{\left( \frac{Ml}{l^3} + \frac{m(r)r^2}{r^3} \right)^2} \approx \frac{m(r)}{r^2} \sqrt{1 + 2 \frac{Mr(r)R}{m(r)R^3}} \approx \frac{m(r)}{r^2} \left( 1 + \frac{Mr(r)R}{m(r)R^3} \right) \tag{31}
\]
and thus
\[
- \frac{m(r)\vec{r}}{r^3} \frac{\sqrt{f}}{\sqrt{\left(\frac{Ml}{l^3} + \frac{m(r)r^3}{r^3}\right)^2}} \approx -\sqrt{f} \left(\frac{\vec{r}}{r} - \frac{Ml^2}{m(r)R^2} \frac{\vec{R}}{R} \left(\frac{\vec{R}}{R}\right)\right).
\]

Finally we obtain
\[
\vec{\phi} = -\left(\frac{Ml}{l^3} + \frac{m(r)r^3}{r^3}\right) \frac{\sqrt{f}}{\sqrt{\left(\frac{Ml}{l^3} + \frac{m(r)r^3}{r^3}\right)^2}} \approx -\sqrt{f} \left(\frac{\vec{r}}{r} + \frac{Ml^2}{m(r)R^2} \frac{\vec{R}}{R} \left(\frac{\vec{R}}{R}\right)\right).
\]

Substituting the latter formula into (16) and integrating over the volume of the body leads to
\[
\vec{F}_{\text{anom}} = -\frac{1}{16\pi M F^2 Pi^3} \frac{8\pi M}{3\omega r^3 m} \left(\int_0^{r^*} \frac{\rho(r)}{m(r)} r^4 dr\right) \frac{M m \bar{R}}{R^3},
\]
in the SI units
\[
\vec{F}_{\text{anom}} = -G_{\text{eff}} \frac{M m \bar{R}}{R^3}
\]
where
\[
G_{\text{eff}} = G \frac{8\pi M}{3\omega r^3 m} \int_0^{r^*} \frac{\rho(r)}{m(r)} r^4 dr.
\]

One can see that the extra force acting on a heavy body \(\sim 1/R^2\). Such a behavior is inherent to the ordinary Brans-Dicke theory and we can replace the original potential \(\phi\) in (16) by an effective potential \(\sim 1/R\). The effective Brans-Dicke parameter can be easily extracted from (36):
\[
\frac{1}{2\omega_{BD} + 3} = \frac{8\pi M}{3\omega r^3 m} \int_0^{r^*} \frac{\rho(r)}{m(r)} r^4 dr;
\]
\[
\omega_{BD} \approx \frac{3\omega r^3 m}{16\pi M} \frac{m \bar{R}^2}{\int_0^{r^*} \frac{\rho(r)}{m(r)} r^4 dr}.
\]

A significant difference from the original Brans-Dicke theory is that \(\omega_{BD}\) depends on the mass \(m\), i.e. it is different for different planets. To estimate \(\omega_{BD}\) for different planets we suppose that \(\rho(r) = \frac{3m}{4\pi r^2} = \text{const}\). In this case
\[
\omega_{BD} \approx \frac{\omega m r^3}{2M r^3} = \frac{a_{ff}}{2\Delta_{anom}},
\]
where \(a_{ff}\) is the free fall acceleration on the surface of a body (a planet). For example,
\[
\omega_{BD,\text{Mercury}} \approx 2.1 \times 10^9,
\]
\[
\omega_{BD,\text{Jupiter}} \approx 1.4 \times 10^{10}.
\]

Such large values of the Brans-Dicke parameter do not contradict the experimental bounds \(\omega_{BD} > 3500\) obtained in the Solar System gravitational tests [3, 4] (we would like to note...
that the bounds on the $\sim \vec{R}/R^3$ extra force differ from the bounds on the $\sim \vec{R}/R$ extra force).

For the other limiting case $\rho(r) = \frac{m}{2\pi r^2}$

$$\omega_{BD} \approx \frac{3a_f}{4a_{anom}}. \quad (40)$$

One should note that in the case of a heavy body with $\rho(r) = \text{const}$ there exists a region $r < \hat{r}$ such that $\frac{M}{R^2} \approx \frac{m(r)}{r^2}$. For this region one should carry out the calculations described in item 1 (the case of a light body). But even for Mercury (if we suppose $\rho(r) = \text{const}$) $\frac{\hat{r}}{r_s} \approx 1.8 \cdot 10^{-2}$, and the constant extra acceleration appears to be

$$a_{extra} \approx \frac{\hat{r}^3}{r_s^3}a_{anom} \approx 6 \cdot 10^{-6}a_{anom},$$

which does not contradict the existing experimental restriction on a possible extra constant acceleration of the planet [14] (which should be much smaller than that of the Pioneers 10, 11 spacecraft). It is easy to check that for the other planets of the Solar System $a_{extra}$ also does not exceed the experimentally allowed limits [14]. If one takes $\rho \sim \frac{1}{r}$ this region is absent and thus $a_{extra} = 0$. Of course, our density profiles for the planets are an idealization, but more realistic profiles should lead to the values of $a_{extra}$ which lie somewhere between the limiting values obtained above. The same is valid for the values of the effective Brans-Dicke parameter $\omega_{BD}$.

The physical difference between the two cases can be easily explained. In the first case vectors $\vec{\phi}$ are directed approximately parallel to the vector $\vec{R}$ at any point of the second body, whereas in the second case these vectors are approximately parallel to the radius-vectors $\vec{r}$, which leads to the result discussed above.

Thus we have shown that in principle it is possible to construct a covariant theory which "distinguishes" light and heavy test bodies with respect to an external gravitational field of a source. We would also like to note that the value of the parameter $f$ which is necessary to reproduce the anomalous acceleration of the Pioneer spacecraft was chosen to be $f \approx 2 \cdot 10^{-44} \text{GeV}^4$. This value is quite close to the vacuum energy density $\sim 10^{-47} \text{GeV}^4$, responsible for the accelerating expansion of the Universe. In this connection it is very interesting to examine possible cosmological manifestations of the model described by action (2). This issue calls for a more detailed and thorough investigation.

Of course we can not argue that the Pioneer anomaly is indeed caused by the existence of such a DBI scalar field. Moreover, recently it was shown that a part of the anomalous acceleration can be explained by the thermal recoil force effect [15]. Nevertheless the model presented in this paper possesses quite interesting features, does not contradict experimental data at least in the Newtonian limit and seems to be worth an additional examination.

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