A Solution to the Cosmological Constant Problem

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Abstract

The fluctuations of the vacuum energy are treated as a non-equilibrium process and a stochastic model for the cosmological constant is presented, which yields a natural explanation for the smallness or zero value of the constant in the present epoch and its large value in an era of inflation in the early universe.
It is widely considered that the problem of the cosmological constant has not been satisfactorily explained. The problem is this: the vacuum fluctuations of particle fields add up to a contribution which is much larger than the observational limits by many orders of magnitude. Attempts using quantum cosmology, based on e.g., Euclidean path integrals and wormholes, have been criticized for various technical reasons, which question it as a possible solution to the problem \[1\]. Unbroken supersymmetry can “protect” the constant $\Lambda$ from becoming non-zero, but supersymmetry is badly broken in the real universe at an energy $> 1$ TeV. No other symmetry is known to exist in Nature that guarantees that $\Lambda$ remains zero. Recently, a solution using the Jordan, Brans-Dicke theory of gravitation was proposed in which the cosmological “constant” is a smooth function of time \[2\].

It is believed by many that the problem would be solved within a physically consistent theory of quantum gravity. Since such a theory does not presently exist, we shall propose a possible solution to the problem based on treating the vacuum energy as a stochastic system. We describe this system by a phenomenological equation for the cosmological constant $\Lambda$, which appears in Einstein’s field equations:

$$ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad (1) $$

The cosmological constant enters through the vacuum energy density:

$$ T_{V\mu\nu} = -\rho_V g_{\mu\nu} = -\frac{\Lambda V}{8\pi G} g_{\mu\nu}. \quad (2) $$

Today, $\Lambda$ has the incredibly small value, $\Lambda < 10^{-46}$ GeV$^4$, whereas generic inflation models require that $\Lambda$ has a large value during the inflationary epoch. This is the source of the cosmological constant problem.

Let us treat the vacuum energy as a fluctuating environment and consider $\Lambda$ as a variable characterizing the state of this system, which is determined by the equation \[3–5\]:

$$ \dot{\Lambda} = \alpha \Lambda - \Lambda^2, \quad (3) $$

where $\dot{\Lambda} = d\Lambda/dt$, $\alpha$ is a parameter which corresponds to the difference between the rate of
creation and annihilation of particles in the vacuum. The second term is a self-restriction term which limits the growth of $\Lambda$. The solution of (3) is given by

$$\Lambda(t) = \Lambda(0) \exp(\alpha t)\{1 + \Lambda(0)((\exp(\alpha t) - 1)/\alpha)\}^{-1}.$$  

(4)

At $\alpha = 0$, there is only one stable stationary state solution $\bar{\Lambda} = 0$, and at this point the solution becomes unstable and bifurcates into a new branch of steady stable solutions, $\bar{\Lambda} = \alpha$. At $\alpha = 0$ the system undergoes a second-order phase transition.

Let us consider the situation in which the vacuum fluctuations are rapid compared with $\tau_{\text{macro}} = \alpha^{-1}$, which defines the macroscopic scale of time evolution. We shall assume that the parameter $\alpha$ can be written as $\alpha_t = \alpha + \sigma \xi_t$, in which $\alpha$ is the average value, $\xi_t$ is Gaussian noise and $\sigma$ measures the intensity of the vacuum fluctuations. Then, we write Eq.(3) as

$$d\Lambda_t = (\alpha \Lambda_t - \Lambda_t^2)dt + \sigma \Lambda_t dW_t = f(\Lambda_t) + \sigma g(\Lambda_t)dW_t,$$

(5)

where $dW_t$ is a Wiener process. We shall use the Ito integral to describe the diffusion process, although the Stratonovich integral would predict the same qualitative results [6].

The probability density $p(x, t)$ satisfies the Fokker-Planck equation:

$$\partial_t p(x, t) = -\partial_x[(\alpha x - x^2)p(x, t)] + \frac{\sigma^2}{2}\partial_{xx}(x^2p(x, t)).$$  

(6)

The diffusion process is restricted to the positive real half line and 0 and $\infty$ are intrinsic boundaries, because $g(0) = 0$ and $f(\infty) = -\infty$. The probability of the diffusion process reaching infinity as $t \to \infty$ is zero, since infinity is a natural boundary. Moreover, zero is a natural boundary if $\alpha > \sigma^2/2$ [5], so neither boundary is accessible and no boundary conditions need be imposed on the Fokker-Planck equation. For $\alpha < \sigma^2/2$, it can be shown that zero is an attracting boundary.

The stationary-state solution for the probability density, $p_s(x)$, of Eq.(6) is given by

$$p_s(x) = N_s x^{(2\alpha/\sigma^2) - 2} \exp\left(-\frac{2x}{\sigma^2}\right).$$  

(7)
The normalization constant \( N \) is
\[
N^{-1} = \left[ \left( \frac{2}{\sigma^2} \right)^{2(\alpha/\sigma^2)-1} \right]^{-1} \Gamma\left( \frac{2\alpha}{\sigma^2} - 1 \right). \tag{8}
\]
If \( p(x, t) \) is integrable between 0 and \( \infty \), then a stationary state solution exists when \( \alpha > \sigma^2/2 \). If it does not exist, then the probability density will be concentrated at zero, i.e., \( p_s(\Lambda) = \delta(\Lambda) \) for \( \alpha < \sigma^2/2 \).

The extrema of \( p_s \) play the role of order parameters for non-equilibrium phase transitions, and they are determined by the equation:
\[
\Lambda_m^2 - (\alpha - \sigma^2)\Lambda_m = 0. \tag{9}
\]
They are given by \( \Lambda_{m1} = 0 \) and \( \Lambda_{m2} = \alpha - \sigma^2 \) (\( \alpha > \sigma^2 \)). The maximum \( \Lambda_{m2} \) always exists, while the maximum \( \Lambda_{m1} \) exists for \( 0 < \alpha < \sigma^2 \). Two transition points exist: one at \( \alpha = \sigma^2/2 \) and one at \( \alpha = \sigma^2 \). We therefore have the following situation: (a) If \( \alpha < \sigma^2/2 \), then zero is a stable stationary point for \( \Lambda \). (b) The point \( \alpha = \sigma^2/2 \) is a transition point, since \( \Lambda = 0 \) becomes unstable and a new stationary probability density is produced. (c) The stationary density becomes divergent at \( \Lambda = 0 \) when \( \sigma^2/2 < \alpha < \sigma^2 \). Although zero is no longer a stable stationary point, it remains the most probable value. (d) For \( \alpha = \sigma^2 \), the probability density \( p_s(\Lambda) \) undergoes a transition, in which \( \Lambda \) can take on large values by increasing the intensity of the fluctuations, even though the average state of the vacuum is kept constant.

For \( 0 < \alpha < \sigma^2/2 \), the vacuum fluctuations dominate over the growth or decline of \( \Lambda \), although the value zero is still the most probable value for \( \Lambda \), since the distribution function has a vertical slope at \( \Lambda = 0 \). Because we are using a continuous variable, \( \Lambda \) never reaches the boundary zero in a finite time. This model breaks down when \( \Lambda \) is vanishingly small and the probability of having \( \Lambda = 0 \) is defined for \( 0 < \epsilon << 1 \) and \( 0 < \alpha < \sigma^2/2 \) by [3]:
\[
\lim_{t \to \infty} \int_0^\epsilon \rho(x, t)dx. \tag{10}
\]

When \( \alpha > \sigma^2/2 \), the growth of \( \Lambda \) dominates the influence of the vacuum fluctuations, and in the neighborhood of zero the probability of \( \Lambda = 0 \) drops to zero. At \( \alpha = \sigma^2/2 \) real
growth of $\Lambda$ becomes possible corresponding to a change from a degenerate random variable for steady-state behavior to a stochastic variable; the boundary at $\alpha = 0$ switches from attracting to natural. The transition point $\alpha = \sigma^2$ corresponds to a qualitative change in the stochastic variable $\Lambda$ with no change in the nature of the boundary. The probability of $\Lambda = 0$ drops abruptly to zero. This phenomenon is inherently nonlinear.

We now see the following scenario emerging from our model. In the inflation era, the intensity of vacuum fluctuations is large and $\alpha > \sigma^2$, causing a second-order phase transition and a maximum in $\Lambda$ not near zero. This corresponds to the large vacuum energy needed to drive inflation [7]. As the universe expands the intensity of vacuum fluctuations decreases and for $0 < \alpha < \sigma^2/2$ or $\sigma^2/2 < \alpha < \sigma^2$ the probability density is largest when $\Lambda$ is non-vanishing and small, which can lead to a current value of $\Lambda_0$ that can be used to fit the observational data. If the stationary probability density $p_s$ does not exist for $\alpha < \sigma^2/2$, then $\Lambda = 0$ is a stationary point; the drift and diffusion vanish simultaneously for $\Lambda = 0$ and $p_s(\Lambda) = \delta(\Lambda)$. This corresponds to the case when $\Lambda$ is vanishingly small.

Thus, our model provides a natural explanation, in terms of non-equilibrium stochastic processes in an expanding universe, for the behavior of $\Lambda$ needed to fit observational data and still be consistent with inflationary models.

According to general relativity, the equation that governs the expansion factor $R(t)$ is given by

$$H^2 \equiv \left( \frac{\dot{R}}{R} \right) = \frac{8\pi G}{3} \rho_M + \frac{\Lambda}{3} - \frac{k}{R^2},$$

where $\rho_M$ is the mass density; $k = -1, 0, +1$ and $H$ is the Hubble constant, whose observable value at present time $t_0$ is denoted by $H_0$. We define

$$\Omega_{\text{tot}} \equiv \Omega_M + \Omega_\Lambda = 1 - \Omega_k.$$  \hspace{1cm} (12)

Many observational tests can constrain $\Omega_\Lambda$, but the gravitational lensing method provides the most direct constraint with the result: $\Omega_{0, \Lambda} < 0.75$ [3]. A recent analysis of the cosmological data showed that for $\Omega_\Lambda = 0.65 \pm 0.1, \Omega_M = 1 - \Omega_\Lambda$ and a small tilt: $0.8 < n < 1.2$,
models exist which are consistent with the available data and an inflationary spatially flat universe [9].

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