NEW RELIABILITY SCORE FOR COMPONENT STRENGTH USING KULLBACK-LEIBLER DIVERGENCE

The reliability of technical systems is one of the most important research subjects in the point reached by modern science. In many recent studies, this problem is solved by evaluating the operation performance of determined one or more components operating under stress. At this point, $R=P(X<Y)$ is taken as a basis. Here, $X$ is the stress applied on the operating component and $Y$ is the strength of the component. In this study we propose a new method by using Kullback-Leibler divergence for computing the reliability of the component under stress-strength model. The superiority of the proposed method is that when the component durability is equal to applied stress Kullback-Leibler divergence is equal to zero. In addition to that when more than one stresses exists at the same time the formed function can include all stresses at the same time. When $R$ is used, this is not possible because of stresses are evaluated separately. As Kullback-Leibler divergence is calculated depending on time, the strength of the component is evaluated within a dynamic structure.

Keywords: reliability, stress-strength model, multistate system model, Kullback-Leibler divergence.

1. Introduction

All technical systems have been designed to perform their intended tasks in a specific ambient. Some systems can perform their tasks in a variety of distinctive levels. A system that can have a finite set of tasks in a specific ambient. Some systems can perform their functions up to complete failure. The quality of the system is completely determined by components. Generally multi-state system is consisted of components that also can be multi-state. The performance rates of components can also vary as a result of their deterioration or in consequence of variable environmental conditions. Components failure can lead to the degradation of the entire multi-state system performance.

The performance rates of the components can range from perfect functioning to complete failure. The quality of the system is completely determined by components.

In some cases, the status of the system depends on the effect of several stresses which cause degradation. The system may not fail fully, but can degrade and there may exist several states of the system. This situation corresponds to multistate systems. For an excellent review of multistate system we refer to Andrzejczak [1]. Indeed, a binary system is the simplest case of a multi-state system having two distinguished states; perfect functioning and completely failure. In a binary system, the definition domains of the states of the system and its components are $\{0,1\}$.

Multi-state systems have been found to be more flexible tool than binary systems for modeling engineering systems. In literature, much attention has been paid to multi-state system modeling. El-Neweihi et al. [14] provided axioms extending the standard notion of a coherent system to the new notion of a multistate coherent system. For such systems they obtained deterministic and probabilistic properties for system performance which are analogous to well-known results for coherent system reliability. Hudson and Kapur [19] presented some models and their applications, in terms of reliability analyses, to situations where the system and all its components have a multiple states. Ebrahimi [11] proposed two types of multistate coherent system and presented various properties related to them. Brunella and Kapur [7] studied a series of reliability measures and expanded their definitions to be consisted with binary, multistate and continuum models. Kuo and Zuo [22] focused on multistate system reliability models and introduced several special multistate system reliability models. Eryilmaz [15] studied mean residual and mean past lifetime concepts for multistate systems. Also, for more details about multi-state system model one can see Andrzejczak [2] and [3].

For reliability analysis, stress-strength models are of special importance. In the simplest terms, stress-strength model can be described as an assessment of the reliability of the component in terms of $X$ and $Y$ random variables where $X$ is the random “stress” experienced by the...
component and \( Y \) is the random “strength” of the component available to overcome the stress. From this simplified explanation, the reliability of the component is the probability that the component is strong enough to overcome the stress applied on it. Then the reliability of the system is defined as:

\[
P(X < Y) = \int_0^\infty F(x) dG(x),
\]

where \( F(x) \) and \( G(x) \) are distribution functions of \( X \) and \( Y \), respectively. Also, for \( x < 0, F(x) = G(x) = 0 \).

Extensive works have been done for the reliability of the component and its estimation under different choices for stress and strength distributions. Chandra and Owen [8] studied the estimation of the reliability of a component where component is subject to several stresses whereas its strength is a single random variable. Awad and Gharafila [4] used a simulation study which compares minimum variance unbiased estimator, the maximum likelihood estimator and bayes estimator for \( R \) when \( X \) and \( Y \) are two independent but not identically distributed Burr random variables. Kotz et al. [20] presented comprehensive information about all methods and results on the stress-strength model. Nadarajah and Kotz [24] calculated distributions. Eryılmaz and İşçioglu [16] studied multi-state systems

\[
T = \inf \{ t : t \geq 0, X(t) \leq Y(t) \},
\]

where \( X(t) \) and \( Y(t) \) denote the stress that the system is experiencing at time \( t \) and strength at time \( t \), respectively. For a specific time period \((0, t_0)\), the reliability of a stress-strength system, \( R(t_0) \), which is defined as the probability of surviving at time \( t_0 \), follows from (2) that:

\[
R(t_0) = P(T > t_0)
\]

Reliability of a stress-strength system is a function of time. This function has been studied in several papers. Ebrahimi [12] investigated this dynamic model on condition that the stress of the system \( Y(t) \) is decreasing in time. Whitmore [28], Ebrahimi and Ramalllingam [13], Basu and Lingham [6] considered the problem of estimating the reliability of a system when both \( X(t) \) and \( Y(t) \) are assumed to be independent Brownian motion processes.

In this paper, inspired by the idea of Kullback-Leibler (KL) divergence, we aim to propose a new method for computing the reliability of the component under stress-strength model. The proposed method provides a simple way for determining the component operation performance under more than one stresses depending on time.

The rest of this paper is organized as four sections. Section 2 gives some information and properties about KL divergence. In Section 3, we explain the proposed method for evaluation of the component’s performance under the stress-strength model. Section 4 contains some examples to show the usefulness of the proposed method for different marginal lifetime distributions of the stress and strength random variables. In Section 5, we summarize what we have done in the article and give some conclusions.

2. KL divergence and properties

The KL divergence (or relative entropy) which introduced by Kullback and Leibler [21], measures the distance between the distributions of random variables. If the densities \( p(x) \) and \( q(x) \) of \( P \) and \( Q \), respectively, exist with respect to Lebesgue measure, the KL divergence \( D_{KL}(P \parallel Q) \) of \( Q \) from \( P \) is defined as:

\[
D_{KL}(P \parallel Q) = \int_S p(x) \log \frac{p(x)}{q(x)} \, dx,
\]

where \( S \) is the support set of \( p(x) \). Note that, \( D_{KL}(P \parallel Q) \) is finite only if \( P \) is absolutely continuous with respect to \( Q \) and \( +\infty \) otherwise. Importantly, the KL divergence remains non-negative and is known as Gibbs’ inequality and is zero if and only if \( P=Q \) i.e., for any two distributions \( P \) and \( Q \):

\[
D_{KL}(P \parallel Q) \geq 0.
\]

Note that it is not a symmetrical quantity, that is to say:

\[
D_{KL}(P \parallel Q) \neq D_{KL}(Q \parallel P).
\]

In information theory, machine learning and statistics, the KL divergence plays an important role. The applications of its can be found in many areas. In literature, much attention has been paid to it. Hall [18] examined Discrimination Information or KL loss in the context of nonparametric kernel density estimation. Also, he showed that its asymptotic properties are profoundly influenced by tail properties of the kernel and of the unknown density. Dahlhaus [9] calculated the asymptotic KL information divergence of two locally stationary sequences and the limit of the Fisher information matrix. Do [10] proposed a fast algorithm to approximate the KL distance between two hidden Markov models. Rached et al. [25] provided an explicit computable expression for the KL divergence rate between two arbitrary time-invariant finite-alphabet Markov sources. Wang et al. [27] proposed a universal divergence estimator for absolutely continuous distributions \( P \) and \( Q \) based on independent and identically distributed samples generated from each source. In Markov-switching regression models, Smith et al. [26] used KL divergence between the true and candidate models to select the number of states and variables simultaneously. Lee and Park [23] considered estimation of the KL divergence between the true density and a selected parametric model.

3. Proposed method

In this section, we introduce a new approach for determining the component operation performance where component is subject to \( X_1(t), X_2(t), \ldots, X_n(t) \) stresses, whereas its strength, \( Y(t) \), is a single random process. Let us initially assume that the stresses are independent random processes having continuous cumulative distribution functions \( F_i(x) = P(X_i(t) \leq x), i = 1, 2, \ldots, n \) and the stress has the marginal distribution function \( G(x) = P(Y(t) \leq x) \).

In our method, we first form the KL divergence \( D_{KL}(\parallel Y(t) \parallel X_1(t), X_2(t), \ldots, X_n(t) \parallel Y(t)) \) by using (3) for \( \xi = 1, 2, \ldots, n \).

After this, we calculate the \( D_{KL}(\parallel Y(t) \parallel X_1(t), X_2(t), \ldots, X_n(t) \parallel Y(t)) \) for selected values of the parameters of marginal lifetime distributions of the stress and
strength random variables. Using these values the operation level of the component, depending on the number of stresses, can be defined as follows:

\[
\begin{align*}
\text{The level of the component } n, & \quad t < t_0 \\
n-1, & \quad t_0 \leq t < t_{n-1} \\
n-2, & \quad t_{n-1} \leq t < t_{n-2} \\
& \quad \vdots \\
2, & \quad t_2 \leq t < t_1 \\
1, & \quad t_1 \leq t \\
0, & \quad \text{otherwise}
\end{align*}
\]

where \( t_i \) denotes the time when \( D(t) \) is equal to zero, \( i=1,2,\ldots,n \).

Also using \( D(t) \) and \( t_i \) values we can define the following equations:

\[
\xi_{i,v} = \begin{cases} 
D(t)_{kl(v)} & \text{if } t_{v+1} < t \leq t_v \\
0, & \text{otherwise}
\end{cases}
\]

where \( v=1,2,\ldots,n-1 \) and:

\[
\xi_{i,v} = \begin{cases} 
D(t)_{kl(v)} & \text{if } t \leq t_n \\
0, & \text{otherwise}
\end{cases}
\]

Now with the help of the above equations, the new reliability score \( Y(t) \) for the component can be expressed as follows:

\[
Y(t) = \sum_{j=1}^{n} \chi_j \left( j - 1 + \frac{\xi_j}{u_j} \right)
\]

(4)

where:

\[
\chi_j = \begin{cases} 
1, & \xi_j > 0 \\
0, & \xi_j = 0
\end{cases}
\]

and \( u_j = \sup D(t)_{kl(v)} \).

In probabilistic design it is common to use parametric statistical models to compute the reliability obtained from stress-strength interference theory. In the following section we apply our method to a Weibull distributional example.

### 4. A Weibull distributional example

In this section, we apply the proposed method to find the component operation performance. Assume that the component is subject to \( X_i(t), X_2(t) \) and \( X_3(t) \) stresses, which remain fixed over time, whereas its strength, \( Y(t) \), is a single random variable, which is stochastically decreasing in time.

A Weibull process is a useful model for events that are changing over time. Here, let \( G \) be a Weibull cumulative distribution function and its shape parameter \( \beta > 0 \) is constant with aging time, while its scale parameter \( \alpha \) decreases over time.

Then, its cumulative distribution function can be written as:

\[
G(t) = 1 - \exp \left( -\frac{t}{\alpha(t)} \right)^\beta, \quad x > 0.
\]

Similarly, assume that \( X_1, X_2 \) and \( X_3 \) are Weibull random variables with cumulative distribution functions:

\[
F_i(t) = 1 - \exp \left( -\frac{t}{\theta_i} \right)^\beta, \quad x > 0,
\]

where \( \beta > 0 \) is the shape parameter, \( \theta_i > 0 \) is the scale parameter of the distributions and \( i=1,2,3 \). Also both \( \beta \) and \( \theta_i \) are constant with aging time.

For computing the operation performance of a component at first we must form KL divergence \( K_{KL(l)}(l) \) of \( X_i \) from \( Y(t) \) for \( l=1,2,3 \). The KL divergence (3) can also be written for \( X_i(l=1,2,3) \) and \( Y(t) \) as:

\[
D_{KL(l)}(l) = H(y,x_l) - H(y)
\]

(7)

where:

\[
H(y,x_l) = \int g_l(x) \log \frac{1}{g_l(x)} dx
\]

(8)

and:

\[
H(y) = \int g_l(x) \log \frac{1}{g_l(x)} dx
\]

(9)

Here, \( H(y) \) is the differential entropy of a continuous random variable \( Y(t) \) with density \( g_l(x) \). Let \( Y(t), X_1, X_2 \) and \( X_3 \) are independent. Now, using probability density functions of (5) and (6) in (8), we have:

\[
H(y,x_l) = -\int_0^\infty \frac{\beta}{\alpha(t)} \left( \frac{t}{\alpha(t)} \right)^\beta \left( \frac{x}{\alpha(t)} \right)^\beta \log \frac{\beta \left( \frac{t}{\alpha(t)} \right)^\beta \left( \frac{x}{\alpha(t)} \right)^\beta}{\theta_i} \left( \frac{t}{\alpha(t)} \right)^\beta \left( \frac{x}{\alpha(t)} \right)^\beta \log \frac{\beta \left( \frac{t}{\alpha(t)} \right)^\beta \left( \frac{x}{\alpha(t)} \right)^\beta}{\theta_i} \left( \frac{t}{\alpha(t)} \right)^\beta \left( \frac{x}{\alpha(t)} \right)^\beta} dx
\]

(10)

By making the substitution \( u = \left( \frac{x}{\alpha(t)} \right)^\beta \) in (10) and then using following integral:

\[
\int_0^\infty e^u \log t dt = -C,
\]

(11)
where $C=0.577215$ is the Euler’s constant (Eq. 8.367.4 in Gradshteyn and Ryzhik, [17]), $H(y,s_i)$ can be obtained as:

$$H(y,s_i) = -\log\left(\beta\right) + \left(\beta - 1\right)\left(\frac{C}{\beta} - \log\alpha(t)\right) + \left(\frac{\alpha(t)}{\frac{\beta}{\theta}}\right)^\beta. \quad (12)$$

Similarly, using probability density function of (5) in (9), we have:

$$H(y) = -\log\left(\frac{\beta}{\alpha(t)}\right) + \left(\beta - 1\right)\left(\frac{C}{\beta} - \log\alpha(t)\right) + 1 \quad (13)$$

where suitable transformations and simplifications have been applied and also (11) used.

Table 1. Numerical values obtained from Equation (15) for $\beta=0.9, \theta_1=0.01$, $\theta_2=0.02$ and $\theta_3=0.05$

| $t$  | $D^{(l)}_{KL(1)}$ | $D^{(l)}_{KL(2)}$ | $D^{(l)}_{KL(3)}$ |
|------|-------------------|-------------------|-------------------|
| 5    | 11.126            | 4.870             | 1.234             |
| 10   | 4.870             | 1.808             | 0.242             |
| 15   | 2.807             | 0.871             | 0.036             |
| 20   | 1.808             | 0.456             | 0.018             |
| 25   | 1.234             | 0.242             | 0.059             |
| 30   | 0.871             | 0.123             | 0.097             |
| 35   | 0.627             | 0.057             | 0.107             |
| 40   | 0.456             | 0.021             | 0.159             |
| 45   | 0.333             | 0.004             | 0.211             |
| 50   | 0.242             | 0.0               | 0.263             |
| 55   | 0.174             | 0.003             | 0.312             |
| 60   | 0.123             | 0.012             | 0.360             |
| 65   | 0.085             | 0.025             | 0.406             |
| 70   | 0.057             | 0.041             | 0.451             |
| 75   | 0.036             | 0.059             | 0.493             |
| 80   | 0.021             | 0.078             | 0.534             |
| 85   | 0.011             | 0.097             | 0.574             |
| 90   | 0.004             | 0.118             | 0.611             |
| 95   | 0.001             | 0.138             | 0.648             |
| 100  | 0.0               | 0.159             | 0.683             |

Now using (12) and (13) in (7) we have:

$$D^{(l)}_{KL(l)} = \log\left(\frac{\theta_l}{\alpha(t)}\right) + \left(\frac{\alpha(t)}{\frac{\theta_l}{\theta}}\right)^\beta - 1. \quad (14)$$

Because of $\alpha(t)$ decreases over time, in (14), let $\alpha(t)=1/t$ then we have:

$$D^{(l)}_{KL(l)} = \log\left(\frac{\theta_l}{1/t}\right) + \left(\frac{1}{\frac{\theta_l}{\theta}}\right)^\beta - 1, \quad (15)$$

where $l=1,2,3$.

Clearly, when values in Table 1 used, we have:

The level of the component

$$\left\{ \begin{array}{ll} 3, & t < t_5 \\ 2, & t_5 \leq t < t_2 \\ 1, & t_2 \leq t < t_1 \\ 0, & t_1 \leq t \end{array} \right.$$ 

where $t_1=100, t_2=50$ and $t_5=20$. Then, for $n=3$ in (4), we have:

$$Y(t) = \chi^3_{1} + \chi^3_{2} + \chi^3_{3}.$$ 

and:

$$\chi^3_{1} = \left\{ \begin{array}{ll} 1, & \chi^3_{1} > 0 \\ 0, & \chi^3_{1} = 0 \end{array} \right., \quad \chi^3_{2} = \left\{ \begin{array}{ll} 1, & \chi^3_{2} > 0 \\ 0, & \chi^3_{2} = 0 \end{array} \right., \quad \chi^3_{3} = \left\{ \begin{array}{ll} 1, & \chi^3_{3} > 0 \\ 0, & \chi^3_{3} = 0 \end{array} \right..$$

Finally, using (16) for $t=5,10,...,100$ we can obtain the new reliability score presented in Table 2 for the component under stress-strength setup.

It can be observed from numerical values in Table 1 how stresses affect the performance of the component that operates under different parameters. When the component starts working, its strength is greater than all stresses. However, because the component’s strength is decreasing depending on the selected time, as the uptime increases at first the KL divergence $D^{(l)}_{KL(l)}$ decreases to near zero. In this period, the strength of the component will begin to move to the good position declined from a perfect position. From the moment that $D^{(l)}_{KL(l)}=0$, the component will pass to the good working period from a perfect working period, the KL divergence $D^{(l)}_{KL(l)}$ is not considered and instead of the KL divergence $D^{(l)}_{KL(l)}$ is taken into account. The KL divergence $D^{(l)}_{KL(l)}$ will be reduced again depending on the time. From the moment it is equal to zero, the component will pass to the average working period from a good working period, the KL divergence $D^{(l)}_{KL(l)}$ is not considered and instead of the KL divergence $D^{(l)}_{KL(l)}$ is taken into account. The KL divergence $D^{(l)}_{KL(l)}$ will be reduced again depending on the time. From the moment it is equal to zero, the operation of the component will end and because the component’s durability remains weak in three stresses the component will be impaired.
Table 2. New reliability score for the component when $\beta=0.9$, $\theta_1=0.01$, $\theta_2=0.02$ and $\theta_3=0.05$

| $t$ | $Y(t)$ | $t$ | $Y(t)$ |
|-----|--------|-----|--------|
| 5   | 3.0017 | 55  | 0.0156 |
| 10  | 2.1961 | 60  | 0.0110 |
| 15  | 2.0291 | 65  | 0.0076 |
| 20  | 2.0045 | 70  | 0.0051 |
| 25  | 1.0496 | 75  | 0.0032 |
| 30  | 1.0252 | 80  | 0.0018 |
| 35  | 1.0117 | 85  | 0.0009 |
| 40  | 1.0043 | 90  | 0.0003 |
| 45  | 1.0008 | 95  | 0.00008 |
| 50  | 1.0000 | 100 | 0.0000 |

5. Conclusion

In the study, it is theoretically assumed that a component operates under $n$ different stresses and when the component’s strength remains weak in all stresses the component is fail. Here, for reliability evaluation we provide a new approach for obtaining the component operation performance. The proposed method described here is a simple and can clearly show the chance of component operation performance depending on time while under all stresses. The evaluation of the component operation performance naturally depends on the probability distributions of stresses and selection of probability distribution of component strength. The method used in the study does not originally depend on probability distribution. Reliability function is a parametric method, but the reliability score proposed from this aspect is nonparametric method for the component. When different effect functions are used instead of probability functions of stress and strength, the recommended method can be easily used.

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