Spinor techniques for massive fermions with arbitrary polarization

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Abstract

We present a new variant of the spinor techniques for calculating the amplitudes of processes involving massive fermions with arbitrary polarization. It is relatively simple and leads to basic spinor products. Our procedure is not more complex than CALCUL spinor techniques for massless fermions. We obtain spinor Chisholm identities for massive fermions. As an illustration, expressions are given for the amplitudes of electron-positron annihilation into fermions-pairs for several polarizations.

1 Introduction

Studies of high-energy processes with polarization are of fundamental importance in understanding structure of matter. For example, The European Muon Collaboration (EMC) and SLAC experiments on deep inelastic scattering of longitudinally polarized muons on longitudinally polarized target have catalyzed an extraordinary outburst of theoretical activity. With increasing energy of colliders, processes involving many final-state particles (2 → 3, 2 → 4, 2 → 5, . . .) are an important part of the present collider’s physics.

The above-listed directions of high-energy physics have a point of intersection. The calculation of cross sections for these processes is difficult if you used conventional approach. This approach is to square the Feynman amplitude and therefore it is very inconvenient to implement if the number of Feynman diagrams and the number of final state particles are large.

An alternative approach is to compute the Feynman amplitudes symbolically or numerically. The idea of calculating Feynman amplitudes is as old as the conventional approach. For example, covariant method of calculating amplitudes have developed more thirty years ago [1]-[3].

Many different methods are proposed for calculating the Feynman diagrams. A particular spinor technique for massless external fermions was introduced by the CALCUL collaboration [4]-[6]. Generalizations of CALCUL approach to the massive fermion case exist [5]-[7], but only for specific choice of the spin projection. We call this choice KS-spin projection or simple KS-states (these fermion states often are called helicity states, but it is not correct).

A few methods of analytical calculations of reactions with massive fermions are convenient for computer symbolic calculations. Except the above mentioned method of group CALCUL, it is important to mention methods proposed in Refs. [7],[8]. So in Ref. [7] a compact formalism of evaluating matrix elements based on the insertion in spinor lines of a complete set of states build up of unphysical spinors was proposed, that has allowed to create a high speed program of evaluations of Feynman amplitudes.
The aim of this paper is to present a spinor techniques method for calculating the amplitudes of processes involving massive fermions with arbitrary polarizations. This approach lead to expressions of Feynman amplitudes in terms of spinor products. As an illustration we apply our method to compute amplitudes and cross sections of the electron-positron annihilation into fermions-pairs for various usual spin projection of fermion states (helicity, KS-states, spin z-projection of the fermion in its rest frame).

2 Spinor techniques for massless fermions

In this section we briefly recall the spinor techniques of Refs. [4]-[5], but with small modifications. Let us introduce the orthonormal four-vector basis in Minkowski space

\[ n_0 = (1, 0, 0, 0), n_1 = (0, 1, 0, 0), n_2 = (0, 0, 1, 0), n_3 = (0, 0, 0, 1). \]

They satisfy the completeness relation:

\[ n_\mu^0 \cdot n_\nu^0 - n_\mu^1 \cdot n_\nu^1 - n_\mu^2 \cdot n_\nu^2 - n_\mu^3 \cdot n_\nu^3 = g^{\mu\nu} \]

by means of which an arbitrary 4-vector \( p \) can be written as:

\[ p = p \cdot n_0 \cdot n_0 - p \cdot n_1 \cdot n_1 - p \cdot n_2 \cdot n_2 - p \cdot n_3 \cdot n_3. \]

Using (1) we can define light-like vectors

\[ b_0 = n_0 - n_3, b_3 = n_0 + n_3, b_\lambda = n_1 + \lambda n_2, \quad \lambda = \pm 1 \]

with following properties:

\[ b_0 \cdot b_\lambda = 0, \quad b_3 \cdot b_\lambda = 0, \quad b_0 \cdot b_3 = 2, \quad b_+ \cdot b_- = -2, \]

\[ \frac{1}{2} \left( b_0^\lambda \cdot b_\nu^0 + b_3^\lambda \cdot b_\nu^3 - b_+^\lambda \cdot b_-^\nu - b_-^\lambda \cdot b_+^\nu \right) = g^{\mu\nu}. \]

Next we define basic spinor \( U_\lambda(b_0) \) by specifying the corresponding projection operator and phase condition:

\[ U_\lambda(b_0) U_\lambda(b_0) = \omega_\lambda \gamma_0, \]

\[ \frac{\lambda}{2} \gamma_\lambda U_{-\lambda}(b_0) = U_\lambda(b_0) \]

with matrix \( \omega_\lambda = \frac{1}{2} (1 + \lambda \gamma_5) \). Our \( b_0 \) corresponds to the vector called \( k_0 = (1, 1, 0, 0) \) in Ref. [5] and instead of \( k_1 = (0, 0, 1, 0) \) we use \( b_\lambda \).

The CALCUL spinor techniques for calculating processes with massless external fermions involve the following operations:

1 step: The arbitrary massless spinor \( U_\lambda(p) \) of momentum \( p \) and helicity \( \lambda \) is defined in terms of basic spinor

\[ U_\lambda(p) = \frac{\gamma^\mu}{\sqrt{2p \cdot b_0}} U_{-\lambda}(b_0), \]

where \( p \neq const b_0 \).

2 step: Using the spinor Chisholm identity

\[ \gamma^\mu \{ U_\lambda(p) \gamma_\mu U_\lambda(k) \} = 2 U_\lambda(k) U_\lambda(p) + 2 U_{-\lambda}(p) U_{-\lambda}(k) \]

\[ ^1 \text{These are covariant four vectors} \]
Figure 1: Feynman diagrams for the process $e^+e^- \rightarrow f \bar{f}$

and the equation for any real four-vector $p$ with $p^2 = 0$

$$
\not{p} = \sum_{\lambda} U_{\lambda}(p) \overline{U}_{\lambda}(p) \tag{11}
$$

we can reduce the amplitudes of processes with massless fermions to expressions involving spinor products (or inner products)

$$
s_{\lambda}(p, k) \equiv \overline{U}_{\lambda}(p) U_{-\lambda}(k) = -s_{\lambda}(k, p). \tag{12}
$$

The remaining possible spinor products vanish due to helicity conservation or are reduced to $s_{\lambda}$ with the help of the relation

$$
V_{\lambda}(p) = U_{-\lambda}(p),
$$

where $V_{\lambda}(p)$ is the bispinor of an antifermion.

As an illustration we present the amplitudes for $e^+e^- \rightarrow f \bar{f}$, where $f$ is fermion ($f \neq e$). The Feynman diagrams for this process are show in Fig.1. Using standard rules the amplitude can be written as

$$
T(\lambda_1, \lambda_2; \nu_1, \nu_2) = \frac{4\pi\alpha}{s} [T_\gamma(\lambda_1, \lambda_2; \nu_1, \nu_2) + T_{Z^0}(\lambda_1, \lambda_2; \nu_1, \nu_2)], \tag{13}
$$

where

$$
T_\gamma(\lambda_1, \lambda_2; \nu_1, \nu_2) = Q_f \overline{V}_{\lambda_2}(p_2, s_{p_2}) \gamma_\mu U_{\lambda_1}(p_1, s_{p_1}) \overline{U}_{\nu_1}(k_1, s_{k_1}) \gamma^\mu V_{\nu_2}(k_2, s_{k_2}), \tag{14}
$$

$$
T_{Z^0}(\lambda_1, \lambda_2; \nu_1, \nu_2) = R_z \left( g^{\mu\nu} - q^\mu q^\nu / M_Z^2 \right)
\overline{V}_{\lambda_2}(p_2, s_{p_2}) \gamma_\nu \left( q^\nu - g^\nu_{5} \right) U_{\lambda_1}(p_1, s_{p_1}) \overline{U}_{\nu_1}(k_1, s_{k_1}) \gamma_\mu \left( q^\mu - g^\mu_{5} \right) V_{\nu_2}(k_2, s_{k_2}), \tag{15}
$$

with $s = q^2 = (p_1 + p_2)^2$, $R_z = (G_F M_Z^2 s) / (2\sqrt{2}\pi\alpha (s - M_Z^2))$. The $g^\nu_{5}, g^\mu_{5}$ are fermion couplings and $\alpha = e^2/(4\pi)$, $G_F$ are the fine-structure and the Fermi constant respectively, $Q_f$ is the $f$
charge in units $e$. The notation $s_p$ for bispinors indicates that fermion with momentum $p$ has fixed polarization vector $s_p$. Using (10), the nonzero amplitude in massless case can be written in terms of the spinor products (12):

$$T (\lambda, -\lambda; \nu, -\nu) = \frac{8\pi \alpha}{s} \left( Q_f + R_z (g_v^\nu - \lambda g^\nu_0) \left( g^\nu_v - \nu g^\nu_0 \right) \right)$$

$$[\delta_{\lambda,\nu} s_\lambda (k_1, p_2) s_{-\lambda} (p_1, k_2) + \delta_{\lambda,\nu} s_\lambda (p_2, k_2) s_{-\lambda} (k_1, p_1)] .$$

Thus, in the evaluation of the Feynman diagrams with the help of spinor techniques the spinor products are important `building' blocks.

The spinor product (12) due to equations (7)-(8) reduces to the trace

$$s_\lambda (p, k) = \lambda \frac{T r (\omega_2 b_0 k b_\lambda k b_\lambda b_0) - 2}{\sqrt{b_0 \cdot p} \sqrt{b_0 \cdot k}} \left( \frac{\lambda [p \cdot b_0 k \cdot b_\lambda - k \cdot b_0 p \cdot b_\lambda - i \epsilon (b_0, b_\lambda, p, k)}{\sqrt{b_0 \cdot p} \sqrt{b_0 \cdot k}} \frac{\lambda [p \cdot b_0 k \cdot b_\lambda - k \cdot b_0 p \cdot b_\lambda]}{\sqrt{b_0 \cdot p} \sqrt{b_0 \cdot k}} \right).$$

From Eqs.(11),(13) for a real light-like vector $p$ it follows that

$$|p \cdot b_+| = |p \cdot b_-| = \sqrt{p \cdot b_0 p \cdot b_3} .$$

If we define

$$p \cdot b_0 = p^0 - p^\tau \equiv p^-, \quad p \cdot b_3 = p^0 + p^\tau \equiv p^+, \quad p \cdot b_\lambda = \sqrt{p^+ p^-} \exp (i \lambda \varphi_p) ,$$

then it follows that

$$s_\lambda (p, k) = \lambda \left( \sqrt{p^- k^+} \exp (i \lambda \varphi_k) - \sqrt{p^+ k^-} \exp (i \lambda \varphi_p) \right).$$

The spinor product (22) of two four-vectors is not more complicated than the dot Minkowski product $p.k$. It is important, that definition of the spinor product (22) is valid for any real 4-vectors, including also $b_0$ and $b_3$.

There are several possibilities to calculate the amplitude (13). We can construct an orthonormal basis (11) with the help of the physical vectors $p_1, p_2, k$

$$n_0 = \frac{p_1 + p_2}{\sqrt{2}p_1 \cdot p_2}, \quad n_3 = \frac{p_2 - p_1}{\sqrt{2}p_1 \cdot p_2}, \quad n^1_i = \epsilon (\mu, n_0, n_3, n_2), \quad n^2_i = \epsilon (\mu, p_1, p_2, k_1) \frac{\sqrt{2}p_1 \cdot p_2 p_1 \cdot k_1 p_2 \cdot k_1}{\sqrt{2}p_1 \cdot p_2 p_1 \cdot k_1 p_2 \cdot k_1}$$

and obtain an analytic expression for the amplitude in terms of dot products.

Also, using (22), we can write the amplitude in terms of the components of the momenta. Finally, we have the possibility of calculating numerically the amplitude with the help of Eqs.(11), (22).

3 Spinor techniques for massive fermions

Our procedure will be similar to that presented in the above section. Let us consider bispinors, which are related to the basic spinor (see appendix of Ref.[3]) by

$$U_\lambda (p, s_p) = \frac{\tau^\lambda_\nu (p, s_p)}{\sqrt{b_0 \cdot (p + m_p s_p)}} U_{-\lambda} (b_0) ,$$

$$\epsilon (b_0, b_\lambda, p, k) = \epsilon (\mu, \rho) b_0^\mu b_\lambda^\rho p^\rho k^\sigma, \quad \epsilon^{0123} = -1$$
\[ V_\lambda(p, s_p) = \frac{\tau^\lambda_v(p, s_p)}{\sqrt{b_0 \cdot (p + m_p s_p)}} U_\lambda(b_0), \] (25)

where the projection operators \( \tau^\lambda_u(p, s_p), \tau^\lambda_v(p, s_p) \) are

\[ \tau^\lambda_u(p, s_p) = \frac{1}{2} (\not{p} + m_p) (1 + \lambda \gamma_5 \not{s}_p), \] (26)
\[ \tau^\lambda_v(p, s_p) = \frac{1}{2} (\not{p} - m_p) (1 + \lambda \gamma_5 \not{s}_p). \] (27)

We obtain

\[ \not{p} U_\lambda(p, s_p) = m_p U_\lambda(p, s_p), \quad \not{p} V_\lambda(p, s_p) = -m_p V_\lambda(p, s_p), \]
\[ \gamma_5 \not{s}_p U_\lambda(p, s_p) = \lambda U_\lambda(p, s_p), \gamma_5 \not{s}_p V_\lambda(p, s_p) = \lambda V_\lambda(p, s_p) \] (28)

i.e. the bispinors \( U_\lambda(p, s_p) \) and \( V_\lambda(p, s_p) \) satisfy Dirac equation and spin condition for massive fermion and antifermion. We also found, that the bispinors of fermion and antifermion (24)-(25) are related by

\[ V_\lambda(p, s_p) = -\lambda \gamma_5 U_{-\lambda}(p, s_p), \quad \not{v}_\lambda(p, s_p) = \not{v}_{-\lambda}(p, s_p) \lambda \gamma_5. \] (29)

Let us analyze how many spinor products there are for massive fermions. Obviously, the case of massive fermions is more difficult in comparison with the massless one. In the general case for calculating the matrix elements sixteen spinor products are necessary while in the massless case there are only two. However, it is possible to achieve essential simplification.

We notice that the spinor products are not all linearly independent. Using relations (29) we can show, that only eight products are linearly independent. We define these basic spinor products for massive fermions as

\[ \not{v}_\lambda(p, s_p) U_{-\lambda}(k, s_k), \quad \not{v}_\lambda(p, s_p) U_\lambda(k, s_k), \quad \not{v}_{-\lambda}(p, s_p) U_\lambda(k, s_k), \quad \not{v}_{-\lambda}(p, s_p) U_{-\lambda}(k, s_k). \] (30)

As one can see below, the basic spinor products can be calculated with the help of two functions.

The matrix \( \gamma_5 \) is the operator of a spin for massless bispinor i.e.

\[ \gamma_5 U_\lambda(b_0) = \lambda U_\lambda(b_0). \] (31)

Then the definition of bispinors (24)-(27) can be rewritten as

\[ U_\lambda(p, s_p) = \frac{\chi(p, s_p, +1)}{2 \sqrt{b_0 \cdot (p + m_p s_p)}} U_{-\lambda}(b_0), \] (32)
\[ V_\lambda(p, s_p) = \frac{\chi(p, s_p, -1)}{2 \sqrt{b_0 \cdot (p + m_p s_p)}} U_\lambda(b_0), \] (33)

with the function

\[ \chi(p, s_p, a) = (\not{p} + a m_p) (1 + a \not{s}_p). \] (34)

Let us introduce the following functions:

\[ \nu(p, k, s_p, s_k, \lambda, a) = \frac{\lambda \text{Tr} \left( \omega_{-\lambda} \not{y}_0 \chi^i(p, s_p, a) \chi(k, s_k, +1) \not{y}_\lambda \right)}{8 \sqrt{b_0 \cdot (p + m_p s_p)} \sqrt{b_0 \cdot (k + m_k s_k)}}. \] (35)
\[ w(p, k, s_p, s_k, \lambda, a) \equiv \frac{1}{4} Tr \left( \omega_{\lambda} \gamma_0 \chi^\dagger (p, s_p, a) \chi (k, s_k, +1) \right). \]  

All the basic spinor products are reduced then to functions (35)-(37): 

\[
\begin{align*}
U_\lambda (p, s_p) U_{-\lambda} (k, s_k) &= v(p, k, s_p, s_k, \lambda, a = +1), \\
U_\lambda (p, s_p) U_{\lambda} (k, s_k) &= w(p, k, s_p, s_k, \lambda, a = +1), \\
\nV_\lambda (p, s_p) U_{\lambda} (k, s_k) &= v(p, k, s_p, s_k, -\lambda, a = -1), \\
\nV_\lambda (p, s_p) U_{-\lambda} (k, s_k) &= w(p, k, s_p, s_k, -\lambda, a = -1). 
\end{align*}
\]  

As for the massless fermions, the spinor products (37) can be calculated through the components of vectors \( p, k, s_p, s_k \). Certainly, the analytical expressions (37) are more complex on a comparison with the appropriate formulas (14), (24).

An important role in the transformation of the matrix elements to basis spinor products is played by the spinor identities Chisholm of type (10). For a proof of relation (10) the Chisholm trace identity was used in Ref. [3]:

\[ \gamma^\mu Tr(\gamma_\mu \zeta_1 \cdots \zeta_{2n+1}) = 2 (\zeta_1 \cdots \zeta_{2n+1} + \zeta_{2n+1} \cdots \zeta_1). \]  

As an expression of the type \( U_\lambda (p, s_p) \gamma_\mu U_{-\lambda} (k, s_k) \) is reduced to trace, using the formulas (24)-(25), (38) it is possible to obtain appropriate spinor identities for massive fermions. In this case we have four basic identities (compared to one in the massless case):

\[
\begin{align*}
\gamma^\mu \left\{ U_\lambda (p, s_p) \gamma_\mu U_{-\lambda} (k, s_k) \right\} &= U_\lambda (k, s_k) U_\lambda (p, s_p) + U_{-\lambda} (p, s_p) U_{-\lambda} (k, s_k) + \\
&+ V_{-\lambda} (k, s_k) V_{-\lambda} (p, s_p) + V_{\lambda} (p, s_p) V_{\lambda} (k, s_k), \\
\gamma^\mu \left\{ U_\lambda (p, s_p) \gamma_\mu U_{-\lambda} (k, s_k) \right\} &= U_{-\lambda} (k, s_k) U_\lambda (p, s_p) - U_{-\lambda} (p, s_p) U_{-\lambda} (k, s_k) + \\
&+ V_{-\lambda} (p, s_p) V_{-\lambda} (k, s_k) - V_{\lambda} (k, s_k) V_{-\lambda} (p, s_p), \\
\gamma^\mu \left\{ V_\lambda (p, s_p) \gamma_\mu U_{-\lambda} (k, s_k) \right\} &= U_{-\lambda} (k, s_k) V_\lambda (p, s_p) + V_{-\lambda} (p, s_p) U_{-\lambda} (k, s_k) + \\
&+ V_{\lambda} (k, s_k) U_{-\lambda} (p, s_p) + U_{\lambda} (p, s_p) V_{\lambda} (k, s_k), \\
\gamma^\mu \left\{ V_\lambda (p, s_p) \gamma_\mu U_{-\lambda} (k, s_k) \right\} &= U_{-\lambda} (k, s_k) V_\lambda (p, s_p) - U_{\lambda} (p, s_p) V_{-\lambda} (k, s_k) - \\
&- V_{-\lambda} (k, s_k) U_{-\lambda} (p, s_p) - V_{\lambda} (p, s_p) V_{-\lambda} (k, s_k). 
\end{align*}
\]  

The remaining possible combinations are reduced to the above ones with the help of relations (29). For massless fermions the relations (39) and (41) are identical (to within the replacement \( \lambda \rightarrow -\lambda \)) and pass into (10), and Eqs. (10) and (42) both sides are zero.

Using (39)-(42), and Dirac equation we can write the analytical expression of a matrix element (13) in terms of spinor products. We represent the amplitude by \( T(\lambda, -\lambda; \lambda, -\lambda) \) for massive fermions with arbitrary polarizations as an example of our type of spinor techniques:

\[
T(\lambda, -\lambda; \lambda, -\lambda) = \frac{4\pi \alpha}{s} \left\{ (Q_f + R_z (g^a_\alpha g^f_a + g^c_\alpha g^f_c)) \right\} \\
\left[ v(k_1, p_2, \lambda, -1) v(p_1, k_2, -\lambda, 1) + v(k_1, p_2, \lambda, 1) v(p_1, k_2, -\lambda, 1) \right] - \\
\lambda R_z \left( g^a_\alpha g^c_a + g^c_\alpha g^a_c \right) \left[ v(k_1, p_2, \lambda, 1) v(p_1, k_2, -\lambda, -1) + v(k_1, p_2, \lambda, -1) v(p_1, k_2, -\lambda, 1) \right] - \\
\lambda R_z \left( g^a_\alpha g^c_a - g^c_\alpha g^a_c \right) \left[ w(k_1, p_1, \lambda, -1) w(p_2, k_2, -\lambda, 1) - w(k_1, p_1, \lambda, 1) w(p_2, k_2, -\lambda, 1) \right] - \\
\lambda R_z \left( g^a_\alpha g^c_a - g^c_\alpha g^a_c \right) \left[ w(k_1, p_1, \lambda, -1) w(p_2, k_2, -\lambda, -1) - w(k_1, p_1, \lambda, 1) w(p_2, k_2, -\lambda, 1) \right] - \\
4 R_z g^a_\alpha g^c_\alpha m_k m_p w(k_1, k_2, \lambda, 1) w(p_2, p_1, \lambda, 1) / M_Z^2 \right\},
\]  

(43)
where we use the notation \( v(p_1, k_2, \lambda, 1) \equiv v(p, k, s_p, s_k, \lambda, 1) \) (and a similar one for \( w \)). Analogous expressions can be written for other spin configurations of fermions. The amplitude with massive fermions looks more complicated than the massless one \((16)\), but we will see below, that the amplitude \((17)\) has a simple form.

## 4 Spinor products

Let us calculate the functions \( v(p, k, s_p, s_k, \lambda, a) \) and \( w(p, k, s_p, s_k, \lambda, a) \) for several spin projection of the fermion states.

The polarization vector \( s_p \) of a fermion can be expressed through the momentum of the fermion as

\[
s_p = \frac{q \cdot p - m^2 p}{m \sqrt{(q \cdot p)^2 - m^2 q^2}}.
\]

where \( q_p \) is an arbitrary vector \((q_p \neq \text{const } p)\). It is easy see, that \( s_p \) satisfies to standard conditions:

\[
s^2_p = -1, \quad p \cdot s_p = 0.
\]

Choosing

\[
q_p = n_0 = (1, 0, 0, 0)
\]

we find in this case that the state of polarization of a fermion is the **helicity state**.

Taking

\[
q_p = b_0 = n_0 - n_3
\]

then the polarization vector \((44)\) can be written as

\[
s_p = \frac{p}{m_p} - m_p \frac{b_0}{p \cdot b_0}.
\]

Let us call the fermion states with this choice of the spin quantization vector the **KS states** \((\text{see [5]}\)) As we see, helicity states and KS states are different in the general case.

Finally, if in the rest frame of the fermion \( s_p = n_3 \) (axis of spin projection is \( z \)-axis) we have the fermion polarized states, which will be called the **\( z \) states**.

The spinor products can be rewritten as functions

\[
v(p, k, s_p, s_k, \lambda, a) \equiv v(p, k, q_p, q_k, \lambda, a),
\]

\[
w(p, k, s_p, s_k, \lambda, a) \equiv w(p, k, q_p, q_k, \lambda, a).
\]

For the calculation of the amplitudes it is often assumed that \( q_p = q_k = \ldots = q \) i.e. all fermions have the same polarized states (helicity, KS, \( z \) or another possible state). The spinor products \( v \) and \( w \) are denoted

\[
v_{\text{hel}}(p, k, \lambda, a), w_{\text{hel}}(p, k, \lambda, a), \quad v_{\text{KS}}(p, k, \lambda, a), w_{\text{KS}}(p, k, \lambda, a), \quad v_z(p, k, \lambda, a), w_z(p, k, \lambda, a)
\]

for helicity, KS and \( z \) states respectively.

Using \((35)-(36)\) we express the functions \( v \) and \( w \) through the components of the vectors. For the vectors \( p = (p^0, p^x, p^y, p^z) \) and \( k = (k^0, k^x, k^y, k^z) \) we obtain:

\[
v_{\text{hel}}(p, k, \lambda, a) = \frac{\lambda \left( a m_p m_k - (k^0 + |k|)(p^0 + |p|) \right)}{\sqrt{4 |k| |p| (k^0 + |k|)(p^0 + |p|)}}
\]

\[
\left\{ e^{i \lambda \varphi_k} \sqrt{(|k| + k^z)(|p| - p^z)} - e^{i \lambda \varphi_p} \sqrt{(|k| - k^z)(|p| + p^z)} \right\},
\]

\[
\left( e^{i \lambda \varphi_k} \sqrt{(|k| + k^z)(|p| - p^z)} - e^{i \lambda \varphi_p} \sqrt{(|k| - k^z)(|p| + p^z)} \right),
\]

\[
(51)
\]
\[ w_{hel}(p, k, \lambda, a) = \frac{(am_p (k^0 + |\vec{p}|) + m_k (p^0 + |\vec{p}|))}{\sqrt{4 |\vec{k}| |\vec{p}| (k^0 + |\vec{k}|) (p^0 + |\vec{p}|)}} \]

\[ \left\{ \sqrt{(|\vec{k}| - k^z) (|\vec{p}| - p^z) + e^{i\lambda \varphi_k - \varphi_p} \sqrt{(|\vec{k}| + k^z) (|\vec{p}| + p^z)} \right\}, \] (52)

\[ v_{KS}(p, k, \lambda, a) = \lambda \left\{ e^{i\lambda \varphi_p} \sqrt{p^0 - p^z} \sqrt{(k^0 + k^z) - m_k^2 / (k^0 - k^z)} - e^{i\lambda \varphi_k} \sqrt{k^0 - k^z} \sqrt{(p^0 + p^z) - m_k^2 / (p^0 - p^z)} \right\}, \] (53)

\[ w_{KS}(p, k, \lambda, a) = am_p \sqrt{(k^0 - k^z) / (p^0 - p^z)} + m_k \sqrt{(p^0 - p^z) / (k^0 - k^z)}, \] (54)

\[ v_z(p, k, \lambda, a) = \lambda \left\{ e^{i\lambda \varphi_p} \sqrt{|\vec{p}|^2 - (p^z)^2} \left[ (a - 1) (k^0 + m_k) + (a + 1) k^z \right] - e^{i\lambda \varphi_k} \sqrt{|\vec{k}|^2 - (k^z)^2} \left[ (a - 1) (p^0 + m_p) + (a + 1) p^z \right] \right\}, \] (55)

\[ w_z(p, k, \lambda, a) = \frac{1}{\sqrt{4 (k^0 + m_k) (p^0 + m_p)}} \left\{ \left[ (a + 1) (k^0 + m_k) + (1 - a) k^z \right] (p^0 + m_p) + \left[ (a - 1) (k^0 + m_k) - (1 + a) k^z \right] p^z - \lambda e^{i\lambda \varphi_p - \varphi_k} \sqrt{(|\vec{k}|^2 - (k^z)^2) (|\vec{p}|^2 - (p^z)^2)} (a + 1) \right\}. \] (56)

As one can see, the analytical expressions (51)-(56) looks a little bit more complicated in the corresponding relation (22). In the massless limit, the mass term in (51)-(54) can be dropped. As a result, for the functions \( v_{hel} \) and \( v_{KS} \) we obtain the expression (22). Since all \( z \) states have only one spin projection (\( z \)-axis in rest frame), the massless limit is absent for these polarized states in general.

Let us consider possible unstable situations, which can arise in numerical calculations. There are the denominators in (51)-(56), hence the ambiguity of type 0/0 can appear in the calculations. Obviously we have not problems with helicity for massive and massless fermions and \( z \), KS states for massive fermions. If one chooses the vector \( k = (k^0, 0, 0, k^0) \) (the massless fermion moving along \( z \)-axis), then the expressions for the KS states (53)-(54) contain the ambiguity of type 0/0. In this situation for \( v_{KS} \) and \( w_{KS} \) we have the rule in the massless limit:

1 step. To take the mass of the fermion equal to zero.

2 step. Only after 1 step we calculate the spinor products through the components of the vectors.

If another spin projection \( s_k \) is interest of we can decompose the any bispinor \( U_{\lambda} (p, s_p) \) in terms of the another bispinor \( U_{\nu} (k, s_k) \) with the help of spinor products, using completeness relation of the bispinors (24)-(24):

\[ \sum_{\nu} \frac{U_{\nu} (k, s_k) \bar{U}_{\nu} (k, s_k) - V_{\nu} (k, s_k) \bar{V}_{\nu} (k, s_k)}{2m_k} = 1. \] (57)
As an example we present the matrix of decomposition for KS and helicity states through the components of the momenta:

\[
\frac{U_\lambda(p, hel) U_\nu(p, KS)}{2m_p} = \delta_{\lambda,\nu} \sqrt{ \frac{(p^0 + |\vec{p}|) (|\vec{p}| + p^z)}{2 |\vec{p}| (p^0 - p^z)} } - \lambda \delta_{\lambda,-\nu} e^{i \lambda \phi_p} \sqrt{ \frac{(p^0 - |\vec{p}|) (|\vec{p}| + p^z)}{2 |\vec{p}| (p^0 - p^z)} } .
\] (58)

The other important building block is the element of the current \( J^\mu \) defined as follows

\[
J^\mu_\nu(p, k, s_p, s_k; \lambda, \nu) \equiv \frac{\partial \lambda(p, s_p)}{\gamma^\mu U_\nu(k, s_k) ,} \quad J^\nu_\nu(p, k, s_p, s_k; \lambda, \nu) \equiv \nabla_\lambda(p, s_p) \gamma^\nu U_\nu(k, s_k) .
\] (59)

Using the completeness identity (6) we obtain that the current \( J \) of type (53) can be written

\[
J^\mu(p, k, s_p, s_k; \lambda, \nu) = \frac{1}{2} (b_0 \cdot J \cdot b^\mu_0 + b_3 \cdot J \cdot b^\mu_3 - b_+ \cdot J \cdot b^\mu_- - b_- \cdot J \cdot b^\mu_+) \quad (60)
\]

It is easy to verify that the terms \( b_0 \cdot J \) and other are reduced then to spinor products \( \nu \) and \( \bar{\nu} \). Using the analytical expressions (51)-(54) we obtain coefficients of decomposition (60) through the components of the vectors (see these coefficients for helicity states in appendix A).

We now have all the tools necessary to express any Feynman diagrams with arbitrary fermion polarizations in terms of the spinor products or the components of the momenta.

## 5 Application

As an illustration of our method we present the helicity and KS amplitudes for process \( e^+e^- \rightarrow f\bar{f} \). Should be noted, that the spinor techniques with functions (53)-(54) (KS states) are used for the calculating of amplitudes of \( e^+e^- \rightarrow f\bar{f} + n\gamma \) with massless electron and positron [9].

Also we take the massless \( e^+ \) and \( e^- \). For completeness we have included expressions of helicity amplitude (13) with all massive fermion in appendix B.

We will work in the center of momentum system with the initial particles moving along \( z \)-axis. In this frame the momenta take the form

\[
p_1 = \frac{\sqrt{s}}{2}(1, 0, 0, 1) \quad \text{for } e^- ,
\]
\[
p_2 = \frac{\sqrt{s}}{2}(1, 0, 0, -1) \quad \text{for } e^+ ,
\]
\[
k_1 = \frac{\sqrt{s}}{2} (1, \beta_k \sin \theta, 0, \beta_k \cos \theta) \quad \text{for } f ,
\]
\[
k_2 = \frac{\sqrt{s}}{2} (1, -\beta_k \sin \theta, 0, -\beta_k \cos \theta) \quad \text{for } \bar{f} ,
\] (61)

where \( s = (p_1 + p_2)^2, \beta_k = \sqrt{1 - 4m_k^2/s} \).

Using (13), (21)-(24) it is simple obtain the helicity and KS amplitudes:

\[
T_{hel}(\lambda, -\lambda, \nu_1, \nu_2) = \frac{4\pi\alpha}{s} \left( \delta_{\nu_1,-\nu_2} \left( -s \right) \left( 1 + \lambda \nu_1 \cos \theta \right) \left[ Q_f + R_z \left( g_v^e - \lambda g_a^e \right) \left( g_v^f - \nu_1 \beta_k g_a^f \right) \right] \right) + \delta_{\nu_1,\nu_2} \left( 2 \lambda \sqrt{s} m_k \sin \theta \left[ Q_f + R_z g_v^f \left( g_v^e - \lambda g_a^e \right) \right] \right) ,
\] (62)

\[
T_{KS}(\lambda, -\lambda, \nu_1, \nu_2) = \frac{4\pi\alpha}{s} \left( \sqrt{s} \beta_k \sin \theta \left( 1 + \lambda \nu_1 \beta_k \cos \theta \right) \right) \left[ Q_f + R_z \left( g_v^e - \lambda g_a^e \right) \left( g_v^f - \nu_1 g_a^f \right) \right] + \delta_{\nu_1,\nu_2} \left( 2 (\lambda + \nu_1) m_k \left[ Q_f + R_z \left( g_v^e - \lambda g_a^e \right) \right) \right) ,
\] (63)
The helicity amplitude (62) coincide with the amplitude, which was obtained in Ref. [10] up to the phase factor.

In Eqs. (62) and (63) \( \nu_1, \nu_2 \) correspond to the values of the helicity and KS fermion’s polarization respectively. As one can see, the helicity and KS amplitudes have the different angular dependence for same values \( \nu_1, \nu_2 \) . In Fig.2 we represent the angular cross sections \( d\sigma_{\text{KS}}^{z,1,-1,1,-1}(z)/dz \) and \( d\sigma_{\text{hel}}^{RL,RL}(z)/dz \) \( (z = \cos\theta) \) of the process \( e^+e^- \rightarrow t\bar{t} \) with \( \sqrt{s} = 0.5 \text{ TeV} \).

From our example we see, that it is incorrect to call the KS states helicity ones (see, [7]). It is to verify that the unpolarized cross sections, which can obtained from the helicity and KS amplitudes are coincide.

6 Concluding Remarks

We have presented a new spinor techniques method for calculating the amplitudes of processes involving massive fermions with arbitrary polarizations. The method allows one obtain to a Feynman amplitude in terms of spinor products.

We determined , that all possible spinor product can be calculated with the help of the two functions only. We have received the analytical expressions of spinor products for vectors of polarizations of fermions which frequently in the physical appendieces (helicity, KS and z-state). Are discussed possible unstable situations, which can arise in the numerical calculations.

Our procedure is not more complex than CALCUL spinor techniques for massless fermions. The method can be adopted to both analytical and numerical computations. The our variant of spinor techniques is convenient for a realization on the computer, as massless variant. As an example, all procedure of reduction of Feynman diagram to spinor products are constructed with the help of the simple rule-based program in Mathematica.

We obtained the spinor Chisholm identities for massive fermions with arbitrary polarization.

As an illustration, the expressions are given for the amplitudes of electron-positron annihilation into fermions-pairs for helicity and KS-states.
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8 Appendix A

We can write the Feynman amplitudes in terms of the current object (59). For example the matrix element (14) can be rewritten in terms of the dot products of the $J^\mu$

$$T_\gamma (\lambda_1, \lambda_2; \nu_1, \nu_2) = Q_f (\nu_1 \nu_2) \{ J_\nu (p_2, p_1, s_{p_2}, s_{p_1}; \lambda_2, \lambda_1) . J_\nu (k_1, k_2, s_{k_1}, s_{k_2}; -\nu_1, -\nu_2) \}. \quad (64)$$

Therefore, in the calculations the currents are important building blocks.

For massless fermions we can obtain any process in terms of the dot products $J_\nu . J_u$ and the four-vectors of reaction. This formalism ($E$-vector formalism) are used in Ref. [5]. But the terms, which not reduce to the these dot products are exist, if the all fermions are massive.

Using (5), the dot product $b_0 \cdot J$ reduces to the spinor products:

$$b_0 \cdot J_u (p, k, s_p, s_k; \lambda, \nu) \equiv \overline{U}_\lambda (p, s_p) U_\nu (k, s_k) = \sum_p \{ \overline{U}_\lambda (p, s_p) U_\rho (b_0) \} \{ \overline{U}_\rho (b_0) U_\nu (k, s_k) \}. \quad (65)$$

Thus we obtain analytical expressions of the current coefficients through the components of the vectors. For helicity states we have:

$$b_0 \cdot J^{\text{hel}}_u (p, k, \lambda, -\lambda) = \lambda \left[ \sqrt{(p^0 + |\vec{p}|)} \left( k^0 + |\vec{k}| \right) \sqrt{(1 + \beta_z(k))(1 - \beta_z(p))} e^{i\lambda \varphi_k} - \sqrt{(p^0 - |\vec{p}|)} \left( k^0 + |\vec{k}| \right) \sqrt{(1 - \beta_z(k))(1 + \beta_z(p))} e^{i\lambda \varphi_p} \right], \quad (66)$$

$$b_3 \cdot J^{\text{hel}}_u (p, k, \lambda, -\lambda) = \lambda \left[ \sqrt{(p^0 - |\vec{p}|)} \left( k^0 - |\vec{k}| \right) \sqrt{(1 - \beta_z(k))(1 - \beta_z(p))} e^{i\lambda \varphi_k} - \sqrt{(p^0 + |\vec{p}|)} \left( k^0 - |\vec{k}| \right) \sqrt{(1 + \beta_z(k))(1 - \beta_z(p))} e^{i\lambda \varphi_p} \right], \quad (68)$$

$$b_\nu \cdot J^{\text{hel}}_u (p, k, \lambda, -\lambda) = \left[ \sqrt{(p^0 - |\vec{p}|)} \left( k^0 + |\vec{k}| \right) - \sqrt{(p^0 + |\vec{p}|)} \left( k^0 - |\vec{k}| \right) \frac{\delta_{\lambda,\nu} \sqrt{(1 - \beta_z(k))(1 - \beta_z(p))} - \sqrt{(1 + \beta_z(k))(1 + \beta_z(p))} (e^{i(\varphi_k + \varphi_p)} \delta_{\lambda,1} \delta_{\nu,1} - e^{-i(\varphi_k + \varphi_p)} \delta_{\lambda,-1} \delta_{\nu,-1})}{\sqrt{(1 - \beta_z(k))(1 - \beta_z(p))} + \sqrt{(1 + \beta_z(k))(1 + \beta_z(p))}} \right]. \quad (70)$$
\[ b_\nu \cdot J_u^{\text{hel}}(p, k, \lambda, \lambda) = \left[ \sqrt{(p^0 + |\vec{p}|)}(k^0 + |\vec{k}|) - \sqrt{(p^0 - |\vec{p}|)}(k^0 - |\vec{k}|) \right] \]

\[
\{ \sqrt{(1 + \beta_z(k)})(1 - \beta_z(p)) \left[ \delta_{\lambda_1 \delta_{\nu_1} - e^{-i\varphi_k} + \delta_{\lambda_1 \delta_{v_1}} e^{i\varphi_k} \right] \\
\sqrt{(1 - \beta_z(k)})(1 + \beta_z(p)) \left[ \delta_{\lambda_1 \delta_{\nu_1} e^{i\varphi_k} + \delta_{\lambda_1 \delta_{v_1}} e^{-i\varphi_k} \right] \}, \] (71)

where \( \beta_z(k) = k^2/|\vec{k}| \) and in Eqs.(70)-(71) \( \nu = \pm 1 \).

9 Appendix B

In the center of momentum system with the initial particles moving along \( z \)-axis, momenta of massive fermions take form:

\[
p_1 = \frac{\sqrt{s}}{2}(1, 0, 0, \beta_p) - \text{ for } e^-, \\
p_2 = \frac{\sqrt{s}}{2}(1, 0, 0, -\beta_p) - \text{ for } e^+, \\
k_1 = \frac{\sqrt{s}}{2}(1, \beta_k \sin \theta, 0, \beta_k \cos \theta) - \text{ for } f, \\
k_2 = \frac{\sqrt{s}}{2}(1, -\beta_k \sin \theta, 0, -\beta_k \cos \theta) - \text{ for } \bar{f}, \] (72)

where \( \beta_p = \sqrt{1 - 4m_p^2/s} \). Using (43) and (51)-(52) we compute the helicity amplitudes:

\[
T^{\text{hel}}(\lambda, -\lambda, \nu_1, \nu_2) = \frac{4\pi\alpha}{\sqrt{s}} \left( -\delta_{\nu_1, -\nu_2} \sqrt{s} (1 + \lambda \nu_1 \cos \theta) \right) \left[ Q_f + R_z \left( g_v^e - \lambda \beta_p g_a^e \right) \left( g_v^f - \nu_1 \beta_k g_a^f \right) \right] \delta_{\nu_1, \nu_2} 2\lambda m_k \sin \theta \left[ \frac{Q_f + R_z g_v^f (g_v^e - \lambda \beta_p g_a^e)}{Q_f - R_z \left( g_v^e g_v^f + \nu_1 \lambda \beta_p g_a^f \left( s/M_Z^2 - 1 \right) \cos \theta \right)} \right] \] (73)

\[
T^{\text{hel}}(\lambda, \lambda, \nu_1, \nu_2) = \frac{4\pi\alpha}{s} \left( 4m_k m_p \delta_{\nu_1, \nu_2} \left[ Q_f - R_z \left( g_v^e g_v^f + \nu_1 \lambda \beta_p g_a^f \left( s/M_Z^2 - 1 \right) \cos \theta \right) \right] \right) -\delta_{\nu_1, -\nu_2} m_p \sqrt{s} g_a^e R_z \sin \theta \left[ \beta_k g_v^f + \nu_1 g_a^f \left( 4m_k^2/M_Z^2 - 1 \right) \right]. \] (74)

These expressions have a compact form and in the massless limit for \( e^+ \) and \( e^- \) pass into (62).

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Figure captions

Fig. 1 Feynman diagrams for the process $e^+e^- \rightarrow f\bar{f}$

Fig. 2 The angular cross section for the helicity and KS polarized states