Rigidity phenomena on lower $N$-weighted Ricci curvature bounds with $\varepsilon$-range for non-symmetric Laplacian

Kazuhiro Kuwae* and Yohei Sakurai†

Abstract

Lu-Minguzzi-Ohta [10] have introduced the notion of a lower $N$-weighted Ricci curvature bound with $\varepsilon$-range, and derived several comparison geometric estimates from a Laplacian comparison theorem for weighted Laplacian. The aim of this paper is to investigate various rigidity phenomena for the equality case of their comparison geometric results. We will obtain rigidity results concerning the Laplacian comparison theorem, diameter comparisons, and volume comparisons. We also generalize their works for non-symmetric Laplacian induced from vector field potential.

Keywords: $N$-weighted Ricci curvature, Laplacian comparison theorem, Bonnet-Myers theorem, Cheng maximal diameter theorem, Bishop-Gromov volume comparison theorem, Ambrose theorem

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1 Introduction

1.1 Weighted Ricci curvature and comparison geometry

Let $(M, g, f)$ denote an $n$-dimensional weighted Riemannian manifold, namely, $(M, g)$ is an $n$-dimensional complete Riemannian manifold, and $f \in C^\infty(M)$. For $N \in ]-\infty, +\infty]$, the $N$-weighted Ricci curvature is defined as follows ( [2, 7]):

$$\text{Ric}_f^N := \text{Ric}_g + \nabla^2 f - \frac{df \otimes df}{N - n},$$

where when $N = +\infty$, the last term is interpreted as the limit 0, and when $N = n$, we only consider a constant function $f$, and set $\text{Ric}_f^n := \text{Ric}_g$.

It is well-known that lower weighted Ricci curvature bounds lead us various comparison geometric results. In the traditional case of $N \in [n, +\infty]$, under a curvature condition

$$\text{Ric}_f^N \geq Kg$$

(1.1)

*Department of Applied Mathematics, Fukuoka University, Fukuoka 814-0180, Japan (kuwae@fukuoka-u.ac.jp). Supported in part by JSPS Grant-in-Aid for Scientific Research (KAKENHI) 17H02846 and by fund (No.:185001) from the Central Research Institute of Fukuoka University.

†Department of Mathematics, Saitama University, 255 Shimo-Otubo, Sakura-ku, Saitama-City, Saitama, 338-8570, Japan (ysakurai@rimath.saitama-u.ac.jp). Supported in part by JSPS Grant-in-Aid for Scientific Research on Innovative Areas “Discrete Geometric Analysis for Materials Design” 17H06460.
for $K \in \mathbb{R}$, such results have been obtained by [9], [15], [17], and so on. On the other hand, recently, comparison geometry has begun to be developed in the complementary case of $N \in ]-\infty, n[$ (see e.g., [5], [6], [8], [10], [11], [12], [13], [14], [16], [18], [19]). Wylie-Yeroshkin [19] have introduced a curvature condition

$$Ric_f^1 \geq (n-1)ke^{-\frac{4f}{n-1}}g$$

(1.2)

for $\kappa \in \mathbb{R}$, and presented an optimal Laplacian comparison theorem, Bonnet-Myers theorem, Bishop-Gromov volume comparison theorem. After that the first named author and Li [5] have extended their condition to

$$Ric_f^N \geq (n-N)ke^{-\frac{4f}{n-N}}g$$

(1.3)

with $N \in ]-\infty, 1[$, and also done their comparison theorems.

Lu-Minguzzi-Ohta [10] have introduced a new curvature condition that interpolates the conditions (1.1) with $K = (N-1)\kappa$, (1.2) and (1.3). For $N \in ]-\infty, 1] \cup [n, +\infty[$, they have considered the notion of the $\varepsilon$-range:

$$\varepsilon = 0 \text{ for } N = 1, \quad \varepsilon \in ]-\sqrt{\varepsilon_0}, \sqrt{\varepsilon_0}[, \text{ for } N \neq 1, n, \quad \varepsilon \in \mathbb{R} \text{ for } N = n,$$  

(1.4)

where

$$\varepsilon_0 := \frac{N-1}{N-n}.$$  

Here if $N = +\infty$, then we interpret $\varepsilon_0$ as the limit 1. In this range, they have proposed a curvature condition

$$Ric_f^N \geq c^{-1}\kappa e^{-\frac{4(1-\varepsilon)}{n-1}}g$$

(1.5)

for $\kappa \in \mathbb{R}$, where $c = c_{N,\varepsilon} \in ]0, 1]$ is a positive constant defined by

$$c := \frac{1}{n-1} \left(1 - \varepsilon^2 \frac{N-n}{N-1}\right)$$

(1.6)

if $N \neq 1$, and $c := (n-1)^{-1}$ if $N = 1$. When $N \in [n, +\infty[$ and $\varepsilon = 1$ with $c = (N-1)^{-1}$, the curvature condition (1.5) is reduced to (1.1) with $K = (N-1)\kappa$. Also, when $N = 1$ and $\varepsilon = 0$ with $c = (n-1)^{-1}$, it covers (1.2), and when $N \in ]-\infty, 1]$ and $\varepsilon = \varepsilon_0$ with $c = (n-N)^{-1}$, it does (1.3). Under the condition (1.5), they first proved a Laplacian comparison theorem for the distance function, and derived a diameter bound of Bonnet-Myers type, and a volume bound of Bishop-Gromov type under density bounds. Notice that they have worked on Finsler setting beyond weighted Riemannian setting.

### 1.2 Setting

The purpose of this paper is to examine rigidity phenomena on the equality case of comparison theorems under the curvature condition (1.5). We work on a general setting such that a weighted manifold $(M, g, V)$ has a vector field potential $V$ beyond the gradient
case of $V = \nabla f$. Such a weighted manifold is equipped with canonical weighted Laplacian and weighted Ricci curvature. The *weighted Laplacian* is defined by

$$\Delta_V := \Delta - g(V, \nabla \cdot),$$

and the *$N$-weighted Ricci curvature* is done as follows:

$$\text{Ric}^N_V := \text{Ric}_g + \frac{1}{2} \mathcal{L}_V g - \frac{V^* \otimes V^*}{N - n}.$$

Here $\mathcal{L}_V g$ is the Lie derivative of $g$ with respect to $V$, and $V^*$ denotes its dual 1-form. If $N = n$, then we always assume that $V$ vanishes such that $\text{Ric}^N_V = \text{Ric}_g$. In the gradient case of $V = \nabla f$, it coincides with $\text{Ric}^N_f$.

We now describe our setting. We always fix a point $p \in M$, and also $N \in [n, +\infty]$ and $\varepsilon \in \mathbb{R}$ in the range (1.4). We consider an arbitrary positive constant $c_p > 0$ such that

$$c_p = e^{-\frac{2(1-\varepsilon)f(p)}{n-1}}$$

in the gradient case of $V = \nabla f$. We further define two lower semi continuous functions $f_{V,p}, s_{V,p} : M \to \mathbb{R}$ by

$$f_{V,p}(x) := \inf_{\gamma} \int_0^{d(p,x)} g(V, \dot{\gamma}(\xi)) \, d\xi, \quad s_{V,p}(x) := c_p \inf_{\gamma} \int_0^{d(p,x)} e^{-\frac{2(1-\varepsilon)f_{V,p}(\gamma(\xi))}{n-1}} \, d\xi,$$

where the infimum is taken over all unit speed minimal geodesics $\gamma : [0, d(p, x)] \to M$ from $p$ to $x$. In the gradient case of $V = \nabla f$, we see $f_{V,p}(x) = f(x) - f(p)$. Furthermore, $s_{V,p}$ is called the *re-parametrized distance* from $p$ (cf. [19]). For a continuous function $\kappa : [0, +\infty) \to \mathbb{R}$, we also define a function $\kappa_{V,p} : M \to \mathbb{R}$ by

$$\kappa_{V,p} := \kappa \circ s_{V,p}.$$

Our setting is as follows:

$$\text{Ric}^N_V \geq c^{-1} c_p^2 \kappa_{V,p} e^{-\frac{4(1-\varepsilon)f_{V,p}}{n-1}} g,$$

where $c$ is defined as (1.6). In the gradient case of $V = \nabla f$, and the case where $\kappa$ is constant, this is reduced to (1.5). Under the condition (1.8) with non-gradient potential, for instance, Bakry-Qian [3] have obtained a Laplacian comparison theorem and a volume comparison theorem of Bishop-Gromov type for invariant measures when $N \in [n, +\infty]$ and $\varepsilon = 1$ with $c = (N - 1)^{-1}$, Kuwada [4] has given a diameter comparison theorem of Bonnet-Myers type and a maximal diameter theorem of Cheng type when $N \in [n, +\infty]$ and $\varepsilon = 1$ and $\kappa$ is a positive constant, Wylie [18] has proven a splitting theorem of Cheeger-Gromoll type when $N \in [n, +\infty]$, and $\kappa = 0$, and the first named author and Shukuri [6] have studied various comparison geometric properties when $N \in [-\infty, 1]$ and $\varepsilon = \varepsilon_0$ with $c = (n - N)^{-1}$. We also aim to develop such comparison geometry.
In Section 2, we will produce a Laplacian comparison theorem, and its rigidity properties for the equality case, which is a key ingredient of the proof of our rigidity theorems (see Theorem 2.3 and Lemma 2.8). Our Laplacian comparison is a generalization of the one that has been obtained by Lu-Minguzzi-Ohta [10] in the gradient case. In [10], they have derived it from the so-called Bishop inequality, which is based on an algebraic calculation. To analyze the rigidity phenomena in more detail, inspired by the original work of Wylie-Yeroshkin [19], we will deduce it from the Bochner formula.

In Section 3, we study diameter comparison theorems of Bonnet-Myers type, and maximal diameter theorems of Cheng type for the equality cases. We will obtain two maximal diameter theorems. The first one is a generalization of the one that has been proved by Wylie-Yeroshkin [19] under the curvature condition (1.2) concerning the diameter of a conformally deformed Riemannian metric (see Theorem 3.3 and Corollary 3.4). The second one is a new result even in the setting of Wylie-Yeroshkin [19]. We will assume not only the curvature bound but also a density bound in the gradient case. We characterize the equality case of a diameter comparison by standard sphere with constant density (see Theorem 3.6 and Corollary 3.7).

In Section 4, we investigate volume comparisons. We first deduce an absolute comparison of Bishop type, and a relative volume comparison of Bishop-Gromov type concerning a weighted volume of sub-level sets of the re-parametrized distance that have been obtained in the setting of Wylie-Yeroshkin [19] (see Propositions 4.4 and 4.6). We also establish a rigidity theorem for the equality case of them (see Theorem 4.8).

In Section 5, we examine some compactness properties in our setting. First, we consider the notion of $\varepsilon$-completeness from which we derive a compactness property (see Proposition 5.2). We further show a theorem of Ambrose type (see Theorem 5.3).

2 Laplacian

2.1 Riccati inequality

Let $d_p : M \to \mathbb{R}$ be the distance function from $p$ defined as $d_p(x) := d(p, x)$. We denote by $U_p M$ the unit tangent sphere at $p$. We define a function $\tau : U_p M \to [0, +\infty]$ as

$$\tau(v) := \sup\{t > 0 \mid d_p(\gamma_v(t)) = t\},$$

where $\gamma_v : [0, +\infty] \to M$ is the unit speed geodesic with $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$.

We start from the following Riccati inequality that has been already shown by Lu-Minguzzi-Ohta [10] in the gradient case of $V = \nabla f$ via algebraic calculation (see the proof of [10] Proposition 3.5). Inspired by the work of Wylie-Yeroshkin [19], we give its proof by using Bochner formula (cf. [19, Lemma 4.1], [6, Lemma 3.1]).

Lemma 2.1 For all $t \in [0, \tau(v)]$ we have

$$\left(e^{\frac{2(1-\varepsilon)f_V}{n-1} \Delta_V d_p}(\gamma_v(t))\right)' \leq -e^{\frac{2(1-\varepsilon)f_V}{n-1} \Delta_V d_p}(\gamma_v(t)) - c e^{-\frac{2(1-\varepsilon)f_V}{n-1} \Delta_V d_p}(\gamma_v(t)) \left(e^{\frac{2(1-\varepsilon)f_V}{n-1} \Delta_V d_p}(\gamma_v(t))\right)^2. \tag{2.1}$$
Then we see

Remark 2.2

When \( \epsilon \to 0 \) the Jacobi equation \( \psi''(s) + \kappa(s) \psi(s) = 0 \) with \( \psi(0) = 0 \) and \( \psi'(0) = 1 \). We set

\[
C_\kappa := \inf\{ s > 0 \mid s_\kappa(s) = 0 \}, \quad \cot_\kappa(s) := \frac{s_\kappa'(s)}{s_\kappa(s)}, \quad H_\kappa(s) := e^{-1} \cot_\kappa(s).
\]

Note that \( \cot_\kappa(s) \) is a unique solution to the following Riccati equation:

\[
\psi'(s) = -\kappa(s) - \psi(s)^2, \quad \lim_{s \to 0} \psi(s) = 1, \quad \lim_{s \to C_\kappa} (s - C_\kappa) \psi(s) = 1
\]

under \( C_\kappa < +\infty \). We also notice that \( H_\kappa \) is decreasing in the case where \( \kappa \) is non-negative, and \( H_\kappa \) is strictly decreasing in the case where \( \kappa \) is a positive function, or \( \kappa \) is a constant function (cf. \([6\text{, Lemma 7.1}])\).

Let us define functions \( s_{V,v} : [0, +\infty] \to [0, s_{V,v}(+\infty)] \) and \( \tau_V : U_p M \to [0, +\infty] \) by

\[
s_{V,v}(t) := c_p \int_0^t e^{-\frac{2(1-\epsilon)f_v(t)}{n-1}} d\xi, \quad \tau_V(v) := s_{V,v}(\tau(v)).
\]
Let \( t_{V,v} : [0, s_{V,v}(+\infty)] \to [0, +\infty] \) stand for the inverse function of \( s_{V,v} \). We now derive our Laplacian comparison from the Riccati inequality (2.1), which has been obtained by Lu-Minguzzi-Ohta \cite{10} via Bishop inequality in the gradient case of \( V = \nabla f \) (see \cite{10} Proposition 3.5, Remark 3.10):

**Theorem 2.3** Assume \( \text{Ric}_V^N(\gamma_v(t)) \geq c_p^{-1} \kappa(s_{V,v}(t)) e^{-\frac{4(1-c)\text{Ric}_V^N(\gamma_v(t))}{n-1}} \) for all \( t \in [0, \tau(v)] \). Then for all \( t \in [0, \tau(v)] \) with \( s_{V,v}(t) \in [0, \min\{\tau_V(v), C_\kappa\}] \), we have

\[
\Delta_V d_p(\gamma_v(t)) \leq c_p H_\kappa(s_{V,v}(t)) e^{-\frac{2(1-c)\text{Ric}_V^N(\gamma_v(t))}{n-1}}. \tag{2.5}
\]

**Proof.** We define two functions \( F_v : [0, \tau(v)] \to \mathbb{R} \) and \( \hat{F}_v : [0, \tau_V(v)] \to \mathbb{R} \) by

\[
F_v := \left( c_p^{-1} e^{\frac{2(1-c)\text{Ric}_V^N(\gamma_v(t))}{n-1}} \Delta_V d_p \right) \circ \gamma_v, \quad \hat{F}_v := F_v \circ t_{V,v}.
\]

From (2.1) and the curvature assumption, for all \( s \in [0, \tau_V(v)] \),

\[
\hat{F}_v'(s) = F_v'(t_{V,v}(s)) e^{\frac{2(1-c)\text{Ric}_V^N(\gamma_v(t_{V,v}(s)))}{n-1}} c_p^{-1}
\]

\[
\leq - \text{Ric}_V^N(\gamma_v(t_{V,v}(s))) e^{-\frac{4(1-c)\text{Ric}_V^N(\gamma_v(t_{V,v}(s)))}{n-1}} c_p^{-2} - c F_v'(t_{V,v}(s))
\]

\[
\leq -c^{-1} \kappa(s) - c \hat{F}_v^2(s).
\]

The Riccati equation (2.3) implies that for all \( s \in [0, \min\{\tau_V(v), C_\kappa\}] \),

\[
\hat{F}_v'(s) - H_\kappa'(s) \leq -c \left( \hat{F}_v^2(s) - H_\kappa^2(s) \right). \tag{2.6}
\]

Let us consider a function \( G_{\kappa,v} : [0, \min\{\tau_V(v), C_\kappa\}] \to \mathbb{R} \) by

\[
G_{\kappa,v} := s_\kappa^2 (\hat{F}_v - H_\kappa).
\]

From (2.6) it follows that

\[
G_{\kappa,v}'(s) = 2 s_\kappa' s_\kappa (\hat{F}_v - H_\kappa) + s_\kappa^2 (\hat{F}_v' - H_\kappa')
\]

\[
\leq 2 s_\kappa s_\kappa' (\hat{F}_v - H_\kappa) - c s_\kappa^2 (\hat{F}_v^2 - H_\kappa^2)
\]

\[
= -c s_\kappa^2 (\hat{F}_v - H_\kappa)^2 \leq 0.
\]

Since we see \( G_{\kappa,v}(s) \to 0 \) as \( s \to 0 \) by (2.3), the function \( G_{\kappa,v} \) is non-positive; in particular, \( \hat{F}_v \leq H_\kappa \) holds on \( [0, \min\{\tau_V(v), C_\kappa\}] \). This proves (2.5). \[ \square \]

**Remark 2.4** We assume that the equality in (2.5) holds at \( t_0 \). Then \( G_{\kappa,v}(s_{V,v}(t_0)) = 0 \). From \( G_{\kappa,v} \leq 0 \) it follows that \( G_{\kappa,v}(s) = 0 \) on \( [0, s_{V,v}(t_0)] \); in particular, the equality in (2.5) holds on \( [0, t_0] \).

**Remark 2.5** In the gradient case of \( V = \nabla f \), this Laplacian comparison theorem has been obtained by Wylie-Yeroshkin \cite{19} under (1.2), the first named author and Li \cite{5} under (1.3), and Lu-Minguzzi-Ohta \cite{10} under (1.5) (see \cite{19} Theorem 4.4, \cite{5} Theorem 2.4, and \cite{10} Remark 3.10). In the non-gradient case, it has been done by Bakry-Qian \cite{3} in the case of \( N \in \{ n, +\infty \} \) and \( \varepsilon = 1 \), and the first named author and Shukuri \cite{6} in the case of \( N \in \{ -\infty, 1 \} \) and \( \varepsilon = \varepsilon_0 \) (see \cite{3} Theorem 4.2, and \cite{6} Theorem 2.5).
Theorem 2.3 leads us to the following:

**Lemma 2.6** Assume $C_\kappa < +\infty$, and $\text{Ric}_N^N(\dot{\gamma}_v(t)) \geq c^{-1}c_\kappa^2(s_{V,v}(t)) e^{4(1-\varepsilon)(f_{V,p}(\gamma_v(t))) \over n-1}$ for all $t \in [0, \tau_v]$. Then we have

$$\tau_V(v) \leq C_\kappa. \quad (2.7)$$

**Proof.** The proof is by contradiction. Assume $\tau_V(v) > C_\kappa$. In this case, $\tau(v) > t_{V,v}(C_\kappa)$. In virtue of (2.5),

$$\Delta_V d_p(\gamma_v(t)) \leq c_p H(\kappa(s_{V,v}(t))) e^{2(1-\varepsilon)(f_{V,p}(\gamma_v(t))) n-1}$$

for every $t \in [0, t_{V,v}(C_\kappa)]$; in particular, $\Delta_V d_p(\gamma_v(t)) \to -\infty$ as $t \to t_{V,v}(C_\kappa)$ by (2.3). This contradicts with the smoothness of $d_p \circ \gamma_v$ on $[0, \tau(v)]$, and hence (2.7). \(\square\)

**Remark 2.7** Due to Lemma 2.6, one can drop the restriction $s_{V,v}(t) \in [0, \min\{\tau_V(v), C_\kappa\}]$ in Theorem 2.3.

### 2.3 Rigidity of Laplacian Comparison

We next investigate the equality case of the Laplacian comparison theorem.

**Lemma 2.8** Under the same setting as in Theorem 2.3, assume that the equality in (2.5) holds at $t_0 \in [0, \tau(v)]$. Choose an orthonormal basis $\{e_{v,i}\}_{i=1}^n$ of $T_pM$ with $e_{v,n} = v$. Let $\{Y_{v,i}\}_{i=1}^{n-1}$ and $\{E_{v,i}\}_{i=1}^{n-1}$ be the Jacobi fields and parallel vector fields along $\gamma_v$ with $Y_{v,i}(0) = 0_p$, $Y_{v,i}'(0) = e_{v,i}$ and $E_{v,i}(0) = e_{v,i}$, respectively. Then the following properties hold on $[0, t_0]$:

(i) If $N = n$, then

$$V \equiv 0, \quad Y_{v,i}(t) = s_{2n}(t) E_{v,i}(t);$$

(ii) if $N = 1$,

$$\varepsilon = 0, \quad Y_{v,i}(t) = c_p^{-1} \exp(\frac{f_{V,p}(\gamma_v(t)))}{n-1}) s_{2n}(s_{V,v}(t)) E_{v,i}(t);$$

(iii) if $N \neq 1, 0$,

$$\varepsilon = 0, \quad g(V, \dot{\gamma}_v(t)) \equiv 0, \quad Y_{v,i}(t) = s_{2n}(t) E_{v,i}(t).$$

**Proof.** If $N = n$, then $V \equiv 0$ by definition and the rigidity of Jacobi fields under $\varepsilon = 1$ is well-known. For general $\varepsilon \in \mathbb{R}$, its proof can be similarly done (see the proof for (iii) below). If $N = 1$, then the desired assertion has been proved by Kuwae-Shukuri [6] (see [6] Lemma 3.2, and also [19] Lemma 4.3). We may assume $N \neq 1, n$. 

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We first show $\varepsilon = 0$ by contradiction. We suppose $\varepsilon \neq 0$. Then in view of Remarks 2.2 and 2.4, $h_{V,v}(t)$ is equal to

$$-\varepsilon^{-1} \frac{N - 1}{N - n} f_v'(t) = c_p H_{\kappa}(s_{V,v}(t)) e^{-\frac{2(1-\varepsilon) f_v(t)}{n-1}},$$

on $[0, t_0]$, where $h_{V,v} := (\Delta_v d_p \circ \gamma_v$ and $f_v := f_{V,p} \circ \gamma_v$. This is a contradiction since the left hand side converges as $t \to 0$, but the right hand side does not. We now possess $\varepsilon = 0$. Remark 2.2 says $g(V, \gamma_v(t)) \equiv 0$; moreover,

$$\text{Ric}_V(\dot{\gamma}_v(t)) \geq \text{Ric}_V^N(\dot{\gamma}_v(t)) \geq (n - 1) c_p^2 \kappa(s_{V,v}(t)) e^{-\frac{2 f_{V,p}(\gamma_v(t))}{n-1}},$$

$\Delta_v d_p(\gamma_v(t)) = c_p H_{\kappa}(s_{V,v}(t)) e^{-\frac{2 f_{V,p}(t)}{n-1}},$

and hence the equality for $N = 1$ occurs. By the rigidity of Jacobi fields for $N = 1$, and $g(V, \gamma_v(t)) \equiv 0$, we conclude

$$Y_{r,i}(t) = c_p^{-1} s_{\kappa}(c_p t) E_{r,i}(t) = s_{\kappa}^2(t) E_{r,i}(t).$$

Thus we complete the proof. \qed

2.4 LAPLACIAN COMPARISON WITH BOUNDED DENSITY

In this last subsection, we focus on the gradient case of $V = \nabla f$. We show comparison results under a bound for $f$. As stated above, we choose $c_p$ as (1.7).

**Lemma 2.9** Let $V = \nabla f$. We assume

$$C_{\kappa} < +\infty, \quad \text{Ric}_f(\dot{\gamma}_v(t)) \geq c^{-1} \kappa(s_{\nabla f,v}(t)) e^{-\frac{4(1-\varepsilon) f(\gamma_v(t))}{n-1}}, \quad (1 - \varepsilon) f \circ \gamma_v \leq (n - 1) \delta$$

on $[0, \tau(v)]$ for $\delta \in \mathbb{R}$. Then

$$\tau(v) \leq C_{\kappa e^{-4\delta}}.$$

**Proof.** The upper bound for $f$ implies $e^{-2\delta} \tau(v) \leq \tau_{\nabla f}(v)$. By Lemma 2.6 and $C_{\kappa e^{-4\delta}} = e^{2\delta} C_{\kappa}$, we complete the proof. \qed

**Remark 2.10** In the gradient case of $V = \nabla f$, a similar upper bound has been shown by Lu-Minguzzi-Ohta \[10\] (see \[10\], Theorem 3.6]).

We write $\Delta_f := \Delta_{\nabla f}$. In view of Lemma 2.9, we have the following:

**Lemma 2.11** Let $V = \nabla f$. We assume

$$\text{Ric}_f(\dot{\gamma}_v(t)) \geq c^{-1} \kappa(s_{\nabla f,v}(t)) e^{-\frac{4(1-\varepsilon) f(\gamma_v(t))}{n-1}}, \quad (1 - \varepsilon) f \circ \gamma_v \leq (n - 1) \delta$$

on $[0, \tau(v)]$ for $\delta \in \mathbb{R}$. We further assume that $H_{\kappa}$ is decreasing. Then for all $t \in [0, \tau(v)]$,

$$\Delta_f d_p(\gamma_v(t)) \leq e^{2\delta} H_{\kappa e^{-4\delta}}(t) e^{-\frac{2(1-\varepsilon) f(\gamma_v(t))}{n-1}}. \quad (2.8)$$
Proof. From the upper boundedness of $f$, we deduce $s_{\nabla f,v}(t) \geq e^{-2\delta t}$ for every $t \in [0, \tau(v)]$. Now, the assumption for $H_\kappa$ and (2.5) imply
\[
\Delta_f d_p(\gamma_v(t)) \leq H_\kappa(s_{\nabla f,v}(t)) e^{-\frac{2(1-\varepsilon)f(\gamma_v(t))}{n-1}} \leq H_\kappa(e^{-2\delta t}) e^{-\frac{2(1-\varepsilon)f(\gamma_v(1))}{n-1}}. \tag{2.9}
\]
The right hand side is equal to that of (2.8). □

Remark 2.12 Assume that the equality in (2.8) holds at $t_0 \in [0, \tau(v)]$. Then the equalities in (2.9) also hold. The equality in (2.5) holds (see Lemma 2.9). Moreover, if $H_\kappa$ is strictly decreasing, then $s_{\nabla f,v}(t_0) = e^{-2\delta t_0}$ and hence $(1-\varepsilon)f \circ \gamma_v = (n-1)\delta$ on $[0, t_0]$.

Remark 2.13 In the gradient case of $V = \nabla f$, a similar estimate has been shown by Lu-Minguzzi-Ohta [10] (see [10, Theorem 3.9]). Here they further assumed a lower bound of $(1-\varepsilon)f \circ \gamma_v$, and obtained an estimate that does not depend on $f$. Moreover, they have concluded a volume estimate under the same setting (see [10, Theorem 3.11]).

3 Diameter

3.1 Diameter comparison theorem

We present the following comparison of Bonnet-Myers type:

Proposition 3.1 We assume $C_\kappa < +\infty$, and also assume $\text{Ric}_N^f \geq c - \kappa V, p e^{-\frac{4(1-\varepsilon)f(V,p)}{n-1}} g$. Then we have
\[
\sup_{x \in M} s_{V, p}(x) \leq C_\kappa.
\]

Proof. Fix $x \in M$, and a unit speed minimal geodesic $\gamma : [0, d(p, x)] \to M$ from $p$ to $x$. According to Lemma 2.6 we see
\[
s_{V, p}(x) \leq c_p \int_0^{d_p(x)} e^{-\frac{2(1-\varepsilon)f(V, \gamma(\xi))}{n-1}} d\xi \leq \tau_V(v) \leq C_\kappa,
\]
where $v := \dot{\gamma}(0)$. We conclude the desired assertion. □

By the same method, we can deduce the following from Lemma 2.9

Proposition 3.2 Let $V = \nabla f$. We assume
\[
C_\kappa < +\infty, \quad \text{Ric}_f^N \geq c - \kappa V, p e^{-\frac{4(1-\varepsilon)f}{n-1}} g, \quad (1-\varepsilon)f \leq (n-1)\delta
\]
for $\delta \in \mathbb{R}$. Then
\[
\sup_{x \in M} d_p(x) \leq C_{\kappa e^{-4\delta}}.
\]
In particular, $M$ is compact.
3.2 Maximal diameter theorem

We now establish a maximal diameter theorem for the equality case of Proposition 3.1. We consider a conformally deformed Riemannian metric (with singularity) defined by

\[ g_{V,p} := c_p^2 e^{-\frac{4(1-\epsilon) f_{V,p}}{n-1}} g, \]

and its distance function

\[ d_{g_{V,p}}(x, y) = c_p \inf_{\sigma} \int_0^l e^{-\frac{2(1-\epsilon) f_{V,p}(\sigma(\xi))}{n-1}} g(\dot{\sigma}(\xi), \dot{\sigma}(\xi))^{1/2} d\xi, \]

where the infimum is taken over all piecewise smooth curves \( \sigma : [0, l] \to M \) with \( \sigma(0) = x \) and \( \sigma(l) = y \). Note that this satisfies the triangle inequality, and \( d_{g_{V,p}}(p, x) \leq s_{V,p}(x) \) for all \( x \in M \). The following is one of our main theorems:

**Theorem 3.3** We assume \( C_\kappa < +\infty \), and also assume \( \text{Ric}_N^V \geq c^{-1} c_p^2 \kappa_{V,p} e^{-\frac{4(1-\epsilon) f_{V,p}}{n-1}} g \). Then we have

\[ \sup_{x \in M} d_{g_{V,p}}(p, x) \leq C_\kappa. \]  

(3.1)

Moreover, we further assume that \( \kappa(s) = \kappa(C_\kappa - s) \) for all \( s \in [0, C_\kappa] \), and \( \kappa \) is positive. If there exists \( q \in M \) with

\[ \text{Ric}_N^V \geq c^{-1} c_q^2 \kappa_{V,q} e^{-\frac{4(1-\epsilon) f_{V,q}}{n-1}} g, \quad c_q e^{-\frac{2(1-\epsilon) f_{V,q}}{n-1}} = c_p e^{-\frac{2(1-\epsilon) f_{V,p}}{n-1}} \]  

(3.2)

such that

\[ d_{g_{V,p}}(p, q) = C_\kappa, \]

then

\[ d_p + d_q \equiv d(p, q) \]

on \( M \), and by identifying \( U_{pM} \) with the \((n-1)\)-dimensional unit sphere \((S^{n-1}, g_{S^{n-1}})\), we have the following rigidity properties:

(i) If \( N = n \), then \( V \equiv 0 \), and \( g = dt^2 + s_{p_2}(t) g_{S^{n-1}} \);

(ii) if \( N = 1 \), then \( \epsilon = 0 \), and

\[ g = dt^2 + c_p^{-2} \exp \left( \frac{2f_{V,p}(\gamma_v(t))}{n-1} \right) s_{V,v}^2(s_{V,v}(t)) g_{S^{n-1}}; \]

(iii) if \( N \neq 1, n \), then \( \epsilon = 0 \), \( V \) is orthogonal to \( \nabla d_p \) on \( M \setminus \{p, q\} \) and vanishes at \( \{p, q\} \),

and \( g = dt^2 + s_{p_2}(t) g_{S^{n-1}} \).

**Proof.** The inequality (3.1) is a direct consequence of Proposition 3.1. Let us prove the rigidity part. Set

\[ \Omega_{p,q} := \{ x \in M \mid \{p, q\} \setminus \{p, q\} = d_p(x) + d_q(x) = d(p, q) \}. \]

(3.3)
The interior of a unit speed minimal geodesic from $p$ to $q$ lies in $\Omega_{p,q}$, and hence $\Omega_{p,q}$ is a non-empty closed subset of $M \setminus \{p, q\}$.

We show that $\Omega_{p,q}$ is open. Fix $x \in \Omega_{p,q}$. Note that $x$ does not belong to the cut locus of $p$ and $q$. We take a sufficiently small domain $\Omega \subset M$ containing $x$ on which $d_p$ and $d_q$ are smooth. We apply Theorem 2.3 to them with the help of the first assumption in (3.2). By using the second one in (3.2) again, the equality in (2.5) holds on $\text{Ric}$. Then we have

$$\Delta_V(d_p + d_q)(y) \leq c_p H_\kappa(s_{V,p}(y)) e^{-\frac{2(1-\varepsilon)/V,p(y)}{n-1}} + c_q H_\kappa(s_{V,q}(y)) e^{-\frac{2(1-\varepsilon)/V,q(y)}{n-1}}$$

$$= c^{-1} c_p \left( \cot_\kappa(s_{V,p}(y)) + \cot_\kappa(s_{V,q}(y)) \right) e^{-\frac{2(1-\varepsilon)/V,p(y)}{n-1}}.$$

The second one in (3.2) implies $g_{V,p} = g_{V,q}$. The triangle inequality for $d_{g_{V,p}}$ leads to

$$s_{V,p}(y) + s_{V,q}(y) \geq d_{g_{V,p}}(p,y) + d_{g_{V,q}}(q,y) = d_{g_{V,p}}(p,y) + d_{g_{V,q}}(q,y) \geq d_{g_{V,p}}(p,q) = C_\kappa.$$
(iii) if $N \neq 1, n$, then $\varepsilon = 0$, $f$ is constant, and $M$ is isometric to a sphere with constant curvature $\kappa e^{-\frac{4f}{n-1}}$.

**Proof.** Almost all parts are the direct consequence of Theorem 3.3. Actually, the assumption (3.2) is always satisfied in this setting. We only need to verify the radial property of $f$ in the case of $N = 1$. This has been proved by Wylie-Yeroshkin (see [19 Proposition 4.15]). Thus we arrive at the desired conclusion. □

**Remark 3.5** In the gradient case of $V = \nabla f$, Corollary 3.4 has been obtained by Wylie-Yeroshkin [19] under (1.2), and the first named author and Shukuri [6] under (1.3) (see [19, Theorem 4.16], and [6, Corollary 2.22]). In the non-gradient case, Kuwada [4] has proven Theorem 3.3 in the case where $N \in [n, +\infty[$, $\varepsilon = 1$ and $\kappa$ is constant.

### 3.3 Maximal Diameter Theorem with Bounded Density

We next investigate rigidity phenomena for the equality case of Proposition 3.2, which is new even in the setting of Wylie-Yeroshkin [19].

**Theorem 3.6** Let $V = \nabla f$. We assume

$$C_\kappa < +\infty, \quad \text{Ric}^N_f \geq e^{-1} \kappa \nabla f, e^{-\frac{4(1-\varepsilon)}{n-1}} (1 - \varepsilon)f \leq (n-1)\delta$$

for $\delta \in \mathbb{R}$. Then

$$\sup_{x \in M} d_p(x) \leq C_{\kappa e^{-\delta}}. \quad (3.4)$$

Moreover, we further assume that $\kappa(s) = \kappa(C_\kappa - s)$ for all $s \in [0, C_\kappa]$, and $\kappa$ is positive. If there exists $q \in M$ with

$$\text{Ric}^N_f \geq e^{-1} \kappa \nabla f, e^{-\frac{4(1-\varepsilon)}{n-1}} g$$

such that

$$d_p(q) = C_{\kappa e^{-\delta}},$$

then

$$d_p + d_q \equiv C_{\kappa e^{-\delta}} \quad (3.5)$$

on $M$, and the following rigidity properties hold:

(i) If $N = n$, then $(1 - \varepsilon)f \equiv (n-1)\delta$, and $g = dt^2 + s_{\kappa e^{-\delta}}(t) g^{n-1}$;

(ii) if $N \neq n$, then $\varepsilon = 0$, $f \equiv (n-1)\delta$, and $g = dt^2 + s_{\kappa e^{-\delta}}(t) g^{n-1}$.

**Proof.** The inequality (3.4) was proved in Proposition 3.2. We will prove the rigidity part by using Lemma 2.11 instead of Theorem 2.3 along the lines of the proof of Theorem 3.3. Define a non-empty closed subset $\Omega_{p,q}$ of $M \setminus \{p, q\}$ as (3.3). We show the openness of $\Omega_{p,q}$. For a fixed $x \in \Omega_{p,q}$, take a domain $\Omega \subset M$ containing $x$ on which $d_p$ and $d_q$ are smooth. Due to Lemma 2.11 for each $y \in \Omega$,

$$\Delta_f(d_p + d_q)(y) \leq e^{2\delta} \left( H_{\kappa e^{-\delta}}(d_p(x)) e^{-\frac{2(1-\varepsilon)f(y)}{n-1}} + H_{\kappa e^{-\delta}}(d_q(y)) e^{-\frac{2(1-\varepsilon)f(y)}{n-1}} \right)$$

$$= e^{-1} e^{2\delta} \left( \cot_{\kappa e^{-\delta}}(d_p(x)) + \cot_{\kappa e^{-\delta}}(d_q(y)) \right) e^{-\frac{2(1-\varepsilon)f(y)}{n-1}},$$
where we notice that \( H_\kappa \) is strictly decreasing by the positivity of \( \kappa \). Since it holds that \( s_\kappa(s) = s_\kappa(C_\kappa - s) \), we have
\[
\cot_{\kappa e^{-4\delta}}(d_q(y)) \leq \cot_{\kappa e^{-4\delta}}(C_{\kappa e^{-4\delta}} - d_p(y)) = -\cot_{\kappa e^{-4\delta}}(d_p(y)),
\]
and obtain \( \Delta f(d_p + d_q)(y) \leq 0 \). According to the strong maximum principle, we arrive at the openness of \( \Omega_{p,q} \).

From the same argument as in the proof of Theorem 3.3, one can conclude (3.5). Now, the equality in (2.8) holds on \( M \setminus \{p, q\} \) (see Remark 2.12). We can apply Lemma 2.8 to our situation, and \( (1 - \varepsilon)f \equiv (n - 1)\delta \). This completes the proof. \( \square \)

For constant \( \kappa \), we have the following:

**Corollary 3.7** Let \( V = \nabla f \), and let \( \kappa \) be a positive constant. We assume
\[
\text{Ric}_V \geq c^{-1} \kappa e^{-\frac{4(1-\varepsilon)f}{n-1}}, \quad (1 - \varepsilon)f \leq (n - 1)\delta
\]
for \( \delta \in \mathbb{R} \). Then
\[
\sup_{x \in M} d_p(x) \leq C_{\kappa e^{-4\delta}}.
\]
Moreover, if there exists \( q \in M \) such that \( d_p(q) = C_{\kappa e^{-4\delta}} \).

then the following rigidity properties hold:

(i) If \( N = n \), then \( (1 - \varepsilon)f \equiv (n - 1)\delta \), and \( M \) is isometric to a sphere of constant curvature \( \kappa e^{-4\delta} \);

(ii) if \( N \neq n \), then \( \varepsilon = 0 \), \( f \equiv (n - 1)\delta \), and \( M \) is isometric to a sphere of constant curvature \( \kappa e^{-4\delta} \).

4 Volume

4.1 Volume Elements

For \( t \in ]0, \tau(v)[ \), and for the volume element \( \theta(t, v) \) of the \( t \)-level surface of \( d_p \) at \( \gamma_v(t) \),
\[
\theta_V(t, v) := e^{-f_{V,p}(\gamma_v(t))} \theta(t, v), \quad \hat{\theta}_V(s, v) := \theta_V(t_{V,v}(s), v),
\]
where \( t_{V,v} \) is the inverse function of \( s_{V,v} \), defined as (2.4). We first show:

**Lemma 4.1** Assume that \( \text{Ric}_V^V(\gamma_v(t)) \geq c^{-1} \kappa e^{-\frac{4(1-\varepsilon)f}{n-1}} \) for all \( t \in ]0, \tau(v)[ \). Then for all \( s_1, s_2 \in ]0, \tau_v(v)[ \) with \( s_1 \leq s_2 \)
\[
\frac{\hat{\theta}_V(s_2, v)}{\hat{\theta}_V(s_1, v)} \leq \frac{s_1^{1/\kappa}(s_2)}{s_1^{1/\kappa}(s_1)}.
\]
Moreover, if \( c = (n - 1)^{-1} \), then for all \( s \in [0, \tau_v(v)[ \) we have
\[
\hat{\theta}_V(s, v) \leq s_{\kappa}^{n-1}(s).
\]
Proof. Let us use the inequality (2.5). For all \( s \in [0, \tau_V(v)] \) we see

\[
\frac{d}{ds} \log \frac{\hat{\theta}_V(s, v)}{g_{\kappa}(s)} = \left( c_p^{-1} e^{\frac{2(1-\epsilon)/V_p}{n-1}} \Delta_V d_p \right) (\gamma_c(t_{V,v}(s))) - H_k(s) \leq 0.
\]

This implies (4.1). If \( c = (n-1)^{-1} \), then we see \( \hat{\theta}_V(s, v)/g_{\kappa} \to 1 \) as \( s \to 0 \) by (2.3). Hence we arrive at (4.2).

\[\square\]

Remark 4.2 Assume that the equality in (4.2) holds at \( s_0 \in [0, \tau_V(v)] \). Then the equality in (4.2) holds on \([0, s_0]\); in particular, the equality in (2.5) holds on \([0, t_{V,v}(s_0)]\) (see Lemma 2.8).

Remark 4.3 We have \( c = (n-1)^{-1} \) if and only if either (1) \( N = n \); or (2) \( N = 1 \); or (3) \( N \neq 1, n \) and \( \varepsilon = 0 \) (cf. Lemma 2.8).

4.2 Volume comparison theorem

For \( r > 0 \), we define

\[
B_{V,r}(p) := \{ x \in M \mid s_{V,p}(x) < r \},
\]

and also define measures

\[
\mu_{V,p} := e^{-f_{V,p}} v_g, \quad \nu_{V,p} := e^{-\frac{2(1-\epsilon)/V_p}{n-1}} \mu_{V,p},
\]

where \( v_g \) is the Riemannian volume measure. By straightforward argument, one can verify

\[
\nu_{V,p}(B_{V,r}(p)) = \int_{U_p M} \int_0^r \tilde{\theta}_V(s, v) \, ds \, dv,
\]

where

\[
\tilde{\theta}_V(s, v) := \begin{cases} \hat{\theta}_V(s, v) & \text{if } s < \tau_V(v), \\ 0 & \text{if } s \geq \tau_V(v). \end{cases}
\]

We also set

\[
S_\kappa(r) := \int_0^r g_{\kappa}^{1/c} \, ds,
\]

where

\[
\tilde{g}_\kappa(s) := \begin{cases} g_\kappa(s) & \text{if } s < C_\kappa, \\ 0 & \text{if } s \geq C_\kappa. \end{cases}
\]

We first present the following absolute comparison theorem of Bishop type:

Proposition 4.4 We assume \( \text{Ric}_V^N \geq c^{-1} c_p^2 \kappa_{V,p} e^{-\frac{2(1-\epsilon)/V_p}{n-1}} g \), and \( c = (n-1)^{-1} \). Then for all \( r > 0 \) we have

\[
\nu_{V,p}(B_{V,r}(p)) \leq \omega_{n-1} S_\kappa(r),
\]

where \( \omega_{n-1} \) is the volume of the \((n-1)\)-dimensional unit sphere. In particular,

\[
\lim_{r \to +\infty} \frac{\nu_{V,p}(B_{V,r}(p))}{S_\kappa(r)} \leq \omega_{n-1}.
\]
Proof. By (4.2) in Lemma 4.1, for all $s \geq 0$ and $v \in U_p M$

$$\bar{\theta}_V(s,v) \leq \bar{s}^n_{\kappa}(s).$$

(4.5)

Integrating it over $[0,r[$ with respect to $s$, and (4.3) complete the proof. \qed

Remark 4.5 Assume that the equality in (4.4) holds. Then the equality in (4.5) holds for all $s \in [0,r]$ and $v \in U_p M$. We have $\tau_V(v) \geq \min\{r,C_{\kappa}\}$ for all $v \in U_p M$, and the equality in (4.2) holds for all $s \in [0,\min\{r,C_{\kappa}\}]$ and $v \in U_p M$ (see Remark 4.2).

We also prove the following relative comparison of Bishop-Gromov type:

Proposition 4.6 We assume $\text{Ric}_V^N \geq c^{-1}c_p^2 \kappa_{V,p} e^{-\frac{4(1-\varepsilon)\nu_{V,p}}{n-1}} g$. Then for all $r,R > 0$ with $r \leq R$ we have

$$\frac{\nu_{V,p}(B_{V,R}(p))}{\nu_{V,p}(B_{V,r}(p))} \leq \frac{\mathcal{S}_n(R)}{\mathcal{S}_n(r)}.$$

Proof. By using (4.1), for all $s_1,s_2 > 0$ with $s_1 \leq s_2$, and $v \in U_p M$

$$\bar{\theta}_V(s_2,v) \bar{s}^{1/\varepsilon}_{\kappa}(s_1) \leq \bar{\theta}_V(s_1,v) \bar{s}^{1/\varepsilon}_{\kappa}(s_2).$$

Let us integrate the both sides over $]0,r[\] with respect to $s_1$, and over $]r,R[\] with respect to $s_2$. We obtain

$$\int_r^R \bar{\theta}_V(s_2,v) \, ds_2 \leq \frac{\mathcal{S}_n(R) - \mathcal{S}_n(r)}{\mathcal{S}_n(r)}.$$

The formula (4.3) yields

$$\frac{\nu_{V,p}(B_{V,R}(p))}{\nu_{V,p}(B_{V,r}(p))} = 1 + \frac{\int_{U_p M} \int_r^R \bar{\theta}_V(s_2,v) \, ds_2 \, dv}{\int_{U_p M} \int_0^r \bar{\theta}_V(s_1,v) \, ds_1 \, dv} \leq \frac{\mathcal{S}_n(R)}{\mathcal{S}_n(r)}.$$

We complete the proof of Proposition 4.6. \qed

Remark 4.7 In the gradient case of $V = \nabla f$, similar volume comparison theorems have been studied by Wylie-Yeroshkin [19] under (1.2), and by the first named author and Li [5] under (1.3) (see [19, Corollary 4.6], [5, Theorem 2.10]). In the non-gradient case, it has been done by the first named author and Shukuri [6] in the case of $N \in ]-\infty,1]$ and $\varepsilon = \varepsilon_0$ (see [6, Theorem 2.14]).

4.3 RIGIDITY OF VOLUME COMPARISON

We investigate the equality cases of volume comparisons (cf. [19, Theorem 4.17]).

Theorem 4.8 We assume $\text{Ric}_V^N \geq c^{-1}c_p^2 \kappa_{V,p} e^{-\frac{4(1-\varepsilon)\nu_{V,p}}{n-1}} g$, and $c = (n-1)^{-1}$. If

$$\lim_{r \to +\infty} \frac{\nu_{V,p}(B_{V,r}(p))}{\mathcal{S}_n(r)} \geq \omega_{n-1},$$

(4.6)

and if $C_{\kappa} = +\infty$, then $M$ is diffeomorphic to $\mathbb{R}^n$, and the following properties hold:
(i) If $N = n$, then $V \equiv 0$, and $g = dt^2 + s_{\gamma_p}(t) g_{S^{n-1}}$;

(ii) if $N = 1$, then $\varepsilon = 0$, and

$$g = dt^2 + c_p^{-2} \exp \left( \frac{2f_{V,p}(\gamma_v(t))}{n - 1} \right) s^2_v(s_{V,v}(t)) g_{S^{n-1}};$$

(iii) if $N \neq 1, n$, then $\varepsilon = 0$, $V$ is orthogonal to $\nabla d_p$ on $M \setminus \{p\}$ and vanishes at $\{p\}$, and $g = dt^2 + s_{\gamma_p}(t) g_{S^{n-1}}$.

**Proof.** Due to Propositions 4.4 and 4.6, the assumption $(4.6)$ tells us that the equality in $(4.4)$ holds for all $r > 0$. From $C_\kappa = +\infty$ we derive $\tau_v(t) = +\infty$ for all $v \in U_p M$ (see Remark 4.5); in particular, $\tau_v(t) = +\infty$, and $M$ is diffeomorphic to $\mathbb{R}^n$. Now, the equality in $(2.5)$ holds on $M \setminus \{p\}$ (see Remark 4.2). In virtue of Lemma 2.8, we complete the proof of Theorem 4.8. □

**Remark 4.9** The authors do not know whether a similar result holds when $C_\kappa < +\infty$. In this case, under the same setting as in Theorem 4.8, we see $\tau_v(t) = C_\kappa$ for all $v \in U_p M$. Since $\tau_v(t)$ can be either finite or infinite, it seems to be difficult to conclude any rigidity results.

### 4.4 Radial Case

Here we consider the case where $f_{V,p}$ is radial with respect to $p$. In this case, $s_{V,v}(t)$ does not depend on $v$, and we can write it as $s_V(t)$. In particular, Lemma 4.10 can be rewritten as follows:

**Lemma 4.10** Let $f_{V,p}$ be radial with respect to $p$. We assume

$$\text{Ric}^N_N(\dot{\gamma}_v(t)) \geq c^{-1}c_p^{\frac{n}{2}} \kappa(s_V(t)) e^{-\frac{4(1-\varepsilon)|f_{V,p}(\gamma_v(t))|}{n-1}}$$

for all $t \in [0, \tau_v]$. Then for all $t_1, t_2 \in [0, \tau_v]$ with $t_1 \leq t_2$

$$\frac{\theta_V(t_2, v)}{\theta_V(t_1, v)} \leq \frac{s_\kappa^{1/c}(s_V(t_2))}{s_\kappa^{1/c}(s_V(t_1))}.$$

Moreover, if $c = (n - 1)^{-1}$, then for all $s \in [0, \tau_v]$ we have

$$\theta_V(t, v) \leq s_\kappa^{n-1}(s_V(t)).$$

For $r > 0$, we set

$$B_r(p) := \{ x \in M \mid d_p(x) < r \} , \quad S_{\kappa,V}(r) := \int_0^r s_\kappa^{1/c}(s_V(t)) \, dt.$$

Having Lemma 4.10 at hand, we can prove the following assertions along the lines of the proof of the statements in the previous subsections. The proof is left to the readers.
Proposition 4.11 Let \( f_{V,p} \) be radial with respect to \( p \). If \( \text{Ric}^N_V \geq c^{-1} c_p^2 \kappa_{V,p} e^{-\frac{4(1-\epsilon)f_{V,p}}{n-1}} g \) and \( c = (n-1)^{-1} \), then for all \( r > 0 \) we have
\[
\mu_{V,p}(B_r(p)) \leq \omega_{n-1} S_{\kappa,V}(r).
\]
In particular,
\[
\lim_{r \to +\infty} \frac{\mu_{V,p}(B_r(p))}{S_{\kappa,V}(r)} \leq \omega_{n-1}.
\]

Proposition 4.12 Let \( f_{V,p} \) be radial with respect to \( p \). If \( \text{Ric}^N_V \geq c^{-1} c_p^2 \kappa_{V,p} e^{-\frac{4(1-\epsilon)f_{V,p}}{n-1}} g \), then for all \( r, R > 0 \) with \( r \leq R \) we have
\[
\mu_{V,p}(B_R(p)) \leq \frac{S_{\kappa,V}(R)}{S_{\kappa,V}(r)} \mu_{V,p}(B_r(p)).
\]

Theorem 4.13 Let \( f_{V,p} \) be radial with respect to \( p \). Assume \( \text{Ric}^N_V \geq c^{-1} c_p^2 \kappa_{V,p} e^{-\frac{4(1-\epsilon)f_{V,p}}{n-1}} g \), and \( c = (n-1)^{-1} \). If we have
\[
\lim_{r \to +\infty} \frac{\mu_{V,p}(B_r(p))}{S_{\kappa,V}(r)} \geq \omega_{n-1},
\]
and if \( C_\kappa = +\infty \), then \( M \) is diffeomorphic to \( \mathbb{R}^n \), and the following properties hold:

(i) If \( N = n \), then \( V \equiv 0 \), and \( g = dt^2 + s_{c_p^2}(t) g_{S_{n-1}} \);

(ii) if \( N = 1 \), then \( \epsilon = 0 \), and
\[
g = dt^2 + c_p^{-2} \exp \left( \frac{2f_{V,p}(\gamma_v(t))}{n-1} \right) s_{\kappa}(s_V(t)) g_{S_{n-1}};
\]

(iii) if \( N \neq 1, n \), then \( \epsilon = 0 \), \( V \) is orthogonal to \( \nabla d_p \) on \( M \setminus \{p\} \) and vanishes at \( \{p\} \), and \( g = dt^2 + s_{c_p^2}(t) g_{S_{n-1}} \).

5 Compactness

5.1 \( \epsilon \)-completeness

We stated that \( M \) is compact under the setting of Proposition 3.2. We first discuss the compactness under that of Proposition 3.1. We say that \( (M, g, V) \) is \( \epsilon \)-complete at \( p \) if
\[
\lim_{r \to +\infty} \inf_{\gamma} \int_0^r e^{-\frac{2(1-\epsilon)f_{V,p}(\gamma(t))}{n-1}} \, dt = +\infty,
\]
where the infimum is taken over all unit speed minimal geodesics \( \gamma : [0, r] \to M \) with \( \gamma(0) = p \) (cf. [19, Proposition 3.4], [5, Definition 2.1], [6, Definition 2.1], [10, Definition 3.2]). Note that in the gradient case of \( V = \nabla f \), the assumption \( (1-\epsilon)f \leq (n-1)\delta \) for \( \delta \in \mathbb{R} \) in Proposition 3.2 implies the \( \epsilon \)-completeness. We also see the following:
Lemma 5.1 Suppose that \((M, g, V)\) is \(\varepsilon\)-complete at \(p\). Then, for any sequence \(\{q_i\}\) in \(M\) such that \(\text{d}_p(q_i) \to +\infty\) as \(i \to +\infty\), we have \(s_p(q_i) \to +\infty\).

**Proof.** The proof is similar to that of [19, Proposition 3.4]. We omit it. \(\Box\)

Proposition 3.1 together with Lemma 5.1 tells us the following:

**Proposition 5.2** Let \(C_\kappa < +\infty\), and assume that \(\mathrm{Ric}^N_V \geq c^{-1} c^2 p^2 \kappa_{V,p} e^{-\frac{4(1-\varepsilon)}{n-1}} g\), and \((M, g, V)\) is \(\varepsilon\)-complete at \(p\). Then \(M\) is compact.

5.2 **Ambrose Type Theorem**

One can further generalize Proposition 5.2 in the case where \(\kappa\) is a constant function. Let us prove the following Ambrose type theorem (cf. [1]):

**Theorem 5.3** Assume that \((M, g, V)\) is \(\varepsilon\)-complete at \(p\). Suppose additionally that for every unit speed geodesic \(\gamma : [0, +\infty[ \to M\) with \(\gamma(0) = p\), we have

\[
\int_1^{+\infty} e^{\frac{2(1-\varepsilon) f_V(\gamma(t))}{n-1}} \mathrm{Ric}^N_V(\dot{\gamma}(t)) \, dt = +\infty. \tag{5.1}
\]

Then \(M\) is compact.

**Proof.** Suppose that \(M\) is non-compact. Then there exists a unit speed minimal geodesic \(\gamma : [0, +\infty[ \to M\) with \(\gamma(0) = p\). We set

\[
f(t) := f_V(p)(\gamma(t)), \quad \lambda(t) := e^{\frac{2(1-\varepsilon) f(t)}{n-1}} \Delta_V d_p(\gamma(t)).
\]

Note that \(\lambda(t)\) is smooth along \(\gamma\). Lemma 2.1 leads us to

\[
\lambda(r) - \lambda(1) + c \int_1^r e^{\frac{2(1-\varepsilon) f(t)}{n-1}} \lambda(t)^2 \, dt \leq - \int_1^r e^{\frac{2(1-\varepsilon) f(t)}{n-1}} \mathrm{Ric}^N_V(\dot{\gamma}(t)) \, dt.
\]

From the assumption (5.1),

\[
\lim_{r \to +\infty} \left( \lambda(r) + c \int_1^r e^{\frac{2(1-\varepsilon) f(t)}{n-1}} \lambda(t)^2 \, dt \right) = -\infty. \tag{5.2}
\]

In particular, \(\lim_{r \to +\infty} \lambda(r) = -\infty\).

Next we prove that there exists a finite number \(R > 0\) such that \(\lim_{r \to R} \lambda(r) = -\infty\), which contradicts the smoothness of \(\lambda(r)\). By (5.2), given \(C > c^{-1}\) there exists \(r_0 > 1\) such that

\[
-\lambda(r_0) - c \int_1^{r_0} e^{\frac{2(1-\varepsilon) f(t)}{n-1}} \lambda(t)^2 \, dt \geq cC > 1.
\]

By (5.1), there exists \(r_1 \in ]r_0, +\infty[\) such that \(\int_{r_0}^{r_1} e^{\frac{2(1-\varepsilon) f(t)}{n-1}} \mathrm{Ric}^N_V(\dot{\gamma}(t)) \, dt \geq 0\) for all \(r \geq r_1\). Let \(\psi(r)\) be the function defined by

\[
\psi(r) := -\lambda(r) - c \int_1^r e^{\frac{2(1-\varepsilon) f(t)}{n-1}} \lambda(t)^2 \, dt - \int_1^r e^{\frac{2(1-\varepsilon) f(t)}{n-1}} \mathrm{Ric}^N_V(\dot{\gamma}(t)) \, dt.
\]
By Lemma 2.1 we see $\psi'(r) \geq 0$. Hence $\psi(r) \geq \psi(r_0)$ for $r \geq r_1 > r_0$. This implies that

$$-\lambda(r) - c \int_1^r e^{-\frac{2(1-\varepsilon)f(t)}{n-1}} \lambda(t)^2 dt \geq cC > 1$$

(5.3)

holds for all $r \geq r_1$. Let us consider the sequence $\{r_\ell\}$ defined inductively by

$$\int_{r_\ell}^{r_{\ell+1}} e^{-\frac{2(1-\varepsilon)f(t)}{n-1}} dt = c^{-1} (cC)^{-\ell+1} \quad \text{for} \quad \ell \geq 1.$$

Let $R$ be the increasing limit of $\{r_\ell\}$. Then

$$\int_{r_1}^R e^{-\frac{2(1-\varepsilon)f(t)}{n-1}} dt = \frac{C}{cC - 1} < +\infty.$$

In view of the $\varepsilon$-completeness of $(M, g, V)$ at $p$, we obtain $R < +\infty$. Finally we claim that for given $\ell \geq 1$, $-\lambda(r) \geq (cC)\ell$ for all $r \geq r_\ell$. This is true for $\ell = 1$ by (5.3). Suppose that $-\lambda(r) \geq (cC)\ell$ for all $r \geq r_\ell$ and fix $r \geq r_{\ell+1}$. Then using inequality (5.3) again,

$$-\lambda(r) \geq cC + c \int_1^{r_\ell} e^{-\frac{2(1-\varepsilon)f(t)}{n-1}} \lambda(t)^2 dt + c \int_{r_\ell}^r e^{-\frac{2(1-\varepsilon)f(t)}{n-1}} \lambda(t)^2 dt \geq c \int_{r_\ell}^{r_{\ell+1}} e^{-\frac{2(1-\varepsilon)f(t)}{n-1}} \lambda(t)^2 dt \geq c (cC)^{2\ell} \frac{c^{-\ell}}{C^{\ell-1}} = (cC)^{\ell+1}.$$

Hence we prove the claim. In particular, $\lim_{r \to R} \lambda(r) = -\infty$, which is the desired contradiction. We complete the proof. \(\square\)

It is trivial that the condition $\text{Ric}^N \geq e^{-c_p^2 \kappa_{V,p}} e^{-\frac{4(1-\varepsilon)f_{V,p}}{n-1}} g$ implies (5.1) in the case where $\kappa$ is a positive constant and $\varepsilon$-completeness at $p$ holds.

**Remark 5.4** In the gradient case of $V = \nabla f$, the first named author and Li \cite{5} have proved Theorem 5.3 under the curvature condition (1.23) (see \cite[Theorem 2.12]{3}). Furthermore, the first named author and Shukuri \cite{6} have extended it to the non-gradient case (see \cite[Theorem 2.17]{6}).

**References**

[1] W. Ambrose, A theorem of Myers, Duke Math. J. 24 (1957), no. 3, 345–348.

[2] D. Bakry and M. Émery, Diffusion hypercontractives, in: Sém. Prob. XIX, in: Lecture Notes in Math., vol. 1123, Springer-Verlag, Berlin/New York, 1985, pp. 177–206.

[3] D. Bakry and Z.-M. Qian, Volume comparison theorems without Jacobi fields, Current trends in potential theory, 115–122, Theta Ser. Adv. Math., 4, Theta, Bucharest, 2005. Available at http://www.lsp.ups-tlse.fr/Bakry.

[4] K. Kuwada, A probabilistic approach to the maximal diameter theorem, Math. Nachr. 286 (2013), no. 4, 374–378.
[5] K. Kuwae and X.-D. Li, *New Laplacian comparison theorem and its applications to diffusion processes on Riemannian manifolds*, preprint 2019, to appear in Bull. Lond. Math. Soc., Available from arXiv:2001.00444.

[6] K. Kuwae and T. Shukuri, *Laplacian comparison theorem on Riemannian manifolds with modified $m$-Bakry-Émery Ricci lower bounds for $m \leq 1$*, preprint 2020, to appear in Tohoku Math. J..

[7] A. Lichnerowicz, *Variétés riemanniennes à tenseur C non négatif*. (French) C. R. Acad. Sci. Paris Sér. A-B 271 (1970), A650–A653.

[8] A. Lim, *The splitting theorem and topology of noncompact spaces with nonnegative $N$-Bakry Émery Ricci curvature*, Proc. Amer. Math. Soc. 149 (2021), no. 8, 3515–3529.

[9] J. Lott, *Some geometric properties of the Bakry-Émery Ricci tensor*, Comment. Math. Helv. 78 (2003), no. 4, 865–883.

[10] Y. Lu, E. Minguzzi and S. Ohta, *Comparison theorems on weighted Finsler manifolds and spacetimes with ε-range*, preprint (2020), Available from arXiv:2007.00219.

[11] C. H. Mai, *Rigidity for the isoperimetric inequality of negative effective dimension on weighted Riemannian manifolds*, Geom. Dedicata 202 (2019), 213–232.

[12] ——–, *On Riemannian manifolds with positive weighted Ricci curvature of negative effective dimension*, Kyushu J. Math. 73 (2019), no. 1, 205–218.

[13] E. Milman, *Beyond traditional curvature-dimension I: new model spaces for isoperimetric and concentration inequalities in negative dimension*, Trans. Amer. Math. Soc. 369 (2017), no. 5, 3605–3637.

[14] S. Ohta, *$(K,N)$-convexity and the curvature-dimension condition for negative $N$*, J. Geom. Anal. 26 (2016), 2067–2096.

[15] Z. Qian, *Estimates for weighted volumes and applications*, Quart. J. Math. Oxford Ser. (2) 48 (1997), 235–242.

[16] Y. Sakurai, *Comparison geometry of manifolds with boundary under a lower weighted Ricci curvature bound*, Canad. J. Math. 72 (2020), no. 1, 243–280.

[17] G. Wei and W. Wylie, *Comparison geometry for the Bakry-Émery Ricci tensor*, J. Differential Geom. 83 (2009), 377–405.

[18] W. Wylie, *A warped product version of the Cheeger–Gromoll splitting theorem*, Trans. Amer. Math. Soc. 369 (2017), no. 9, 6661–6681.

[19] W. Wylie and D. Yeroshkin, *On the geometry of Riemannian manifolds with density*, preprint 2016, Available from arXiv:1602.08000.