On the density of primes in arithmetic progression having a prescribed primitive root

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Abstract
Let $g \in \mathbb{Q}$ be not $-1$ or a square. Let $P_g$ denote the set of primes $p$ such that $g$ is a primitive root mod $p$. Let $1 \leq a \leq f$, $(a,f) = 1$. Under the Generalized Riemann Hypothesis (GRH) it can be shown that the set of primes $p \in P_g$ with $p \equiv a \pmod{f}$ has a natural density. In this note this density is explicitly evaluated. This generalizes a classical result of Hooley.

1 Introduction

Let $G$ be the set of non-zero rational numbers $g$ such that $g \neq -1$ and $g$ is not a square of a rational number. For arbitrary $g \in \mathbb{Q}^*$ let $P_g$ denote the set of primes $p$ such that $g$ is a primitive root mod $p$. Clearly a necessary condition for $P_g$ to be infinite is that $g \in G$. That this is also a sufficient condition was conjectured in 1927 by Emil Artin. There is no $g \in G$ for which this has been proved, however, Heath-Brown in a classical paper established a result which implies, for example, that there are at most two primes $q$ for which $P_q$ is finite. In 1967 Hooley established Artin’s primitive root conjecture under the assumption of the Generalized Riemann Hypothesis (GRH). Moreover, he showed that under that assumption the set $P_g$ has a natural density, which he evaluated (his result is Theorem 2 below with $a = f = 1$ and $g \in G \cap \mathbb{Z}$). It turns out that this density is equal to a positive rational number times the Artin constant $A$, with

$$A = \prod_p \left(1 - \frac{1}{p(p-1)}\right) \approx 0.3739558136192288054728.$$ 

(Throughout this note the notation $p$ is used to indicate rational primes.)

In connection with his study of Euclidian number fields, Lenstra considered the distribution over arithmetic progressions of the primes in $P_g$. Let $P_{a,f,g}$ denote the set of primes $p$ such that $g$ is a primitive root mod $p$ and $p \equiv a \pmod{f}$. From Lenstra’s work it follows that, under GRH, $P_{a,f,g}$ has a natural density.

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Theorem 1 [3]. Put $\zeta_m = e^{2\pi i/m}$. Let $1 \leq a \leq f$, $(a, f) = 1$. Let $\sigma_a$ be the automorphism of $\mathbb{Q}(\zeta_f)$ determined by $\sigma_a(\zeta_f) = \zeta_f^a$. Let $c_a(n)$ be 1 if the restriction of $\sigma_a$ to the field $\mathbb{Q}(\zeta_f) \cap \mathbb{Q}(\zeta_n, g^{1/n})$ is the identity and $c_a(n) = 0$ otherwise. Put

$$\delta(a, f, g) = \sum_{n=1}^{\infty} \frac{\mu(n)c_a(n)}{[\mathbb{Q}(\zeta_f, \zeta_n, g^{1/n}) : \mathbb{Q}]}.$$ 

Then, assuming GRH, we have

$$\pi_g(x; f, a) = \delta(a, f, g) \frac{x}{\log x} + O\left(\frac{x \log \log x}{\log^2 x}\right),$$

where $\pi_g(x; f, a)$ denotes the number of primes $p \leq x$ that are in $P_{a,f,g}$.

In view of the apparent arithmetical complexity of Lenstra’s formula, the following relatively simple expression for $\delta(a, f, g)$ may come as a bit of a surprise.

Theorem 2 Let $g \in G$ and let $h$ be the largest integer such that $g$ is an $h$-th power. Let $\Delta$ denote the discriminant of the quadratic field $\mathbb{Q}(\sqrt{g})$. Let $1 \leq a \leq f$, $(a, f) = 1$. Let $b = \Delta/(f, \Delta)$. If $b$ is odd, put $\gamma = (-1)^{(b-1)/2}(f, \Delta)$. Put

$$A(a, f, h) = \prod_{p|(a-1, f)} (1 - \frac{1}{p}) \prod_{\stackrel{p|f}{p|h}} (1 - \frac{1}{p-1}) \prod_{\stackrel{p|f}{p|h}} \left(1 - \frac{1}{p(p-1)}\right)$$

if $(a - 1, f, h) = 1$ and $A(a, f, h) = 0$ otherwise. Then

$$\delta(a, f, g) = \frac{A(a, f, h)}{\varphi(f)} \left(1 + \frac{2\gamma}{a} \prod_{p|h, p|\beta} (p-2) \prod_{p|h, p|\beta} (p^2 - p - 1)\right).$$

Here $(\gamma/a)$ denotes the Kronecker symbol.

In case $b$ is even, the Kronecker symbol $(\gamma/a)$ is not defined, then however $\mu(2|b|) = 0$ and the term involving $\mu(2|b|)$ is taken to be zero. Note that $g$ can be uniquely written as $g = g_1g_2^2$, with $g_1$ a squarefree integer and $g_2 \in \mathbb{Q}$. Then $\Delta = g_1$ if $g_1 \equiv 1(\bmod 4)$ and $\Delta = 4g_1$ otherwise. We see that $b$ is odd if and only if $g_1 \equiv 1(\bmod 4)$ or $g_1 \equiv 2(\bmod 4)$ and $8|f$ or $g_1 \equiv 3(\bmod 4)$ and $4|f$. On using the properties of the Kronecker symbol mentioned in Section 2 and quadratic reciprocity for the Jacobi symbol, Theorem 2 can be formulated in terms of $g_1$. (The odd part of $0 \neq m \in \mathbb{Z}$ is $m/2^e$ with $2^e|m$ and $2^{e+1} \nmid m$.)

Theorem 3 Let $a, f, g, h$ and $A(a, f, h)$ be as in Theorem 2. Let $\bar{g}_1$ and $\bar{a}$ denote the odd parts of $g_1$ and $a$, respectively. Let $\beta = g_1/(g_1, f)$. We have

$$\delta(a, f, g) = \frac{A(a, f, h)}{\varphi(f)} \left(1 - \frac{a}{(f, g_1)} \prod_{p|\beta, p|h} (p-2) \prod_{p|\beta, p|h} (p^2 - p - 1)\right)$$

in case $g_1 \equiv 1(\bmod 4)$ or $g_1 \equiv 2(\bmod 4)$ and $8|f$ or $g_1 \equiv 3(\bmod 4)$ and $4|f$ and

$$\delta(a, f, g) = \frac{A(a, f, h)}{\varphi(f)},$$

otherwise. Here $(./.)$ denotes the Jacobi symbol with the stipulation that $(a/2) = (-1)^{(a^2-1)/8}$.
From Theorem 2 various known results can be rather easily deduced. That will be the subject of Section 5. The remaining sections are devoted to proving Theorem 2.

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2 Some facts from algebraic number theory

Although Theorem 1 suggests differently, the problem of evaluating $\delta(a, f, g)$ really only involves cyclotomic fields, quadratic fields and their composita. In this section some facts concerning these fields relevant for the proof of Theorem 2 are discussed. We start by recalling some properties of the Kronecker symbol, a rarely covered topic in books on number theory (in contrast to the Jacobi symbol).

The Kronecker symbol $(c/d)$ is defined for $c \in \mathbb{Z}$, $c \equiv 0 \pmod{4}$ or $c \equiv 1 \pmod{4}$, $c$ not a square, and $d \geq 1$ an integer. If $(c, d) > 1$, then $(c/d) = 0$. If $(c, d) = 1$, then $(c/d) = \pm 1$. If $d_1, d_2 \geq 1$, then $(c/d_1 d_2) = (c/d_1)(c/d_2)$. If $c$ is odd, then $(c/2) = \text{Jacobi symbol } (2/|c|)$. If $d_1 \equiv d_2 \pmod{|c|}$, then $(c/d_1) = (c/d_2)$.

The following lemma allows one to determine all quadratic subfields of a given cyclotomic field (for a proof see e.g. [12], p. 163).

Lemma 1 Let $\mathbb{Q}(\sqrt{d})$ be a quadratic field of discriminant $\Delta_d$. Then the smallest cyclotomic field containing $\mathbb{Q}(\sqrt{d})$ is $\mathbb{Q}(\zeta_{|\Delta_d|})$.

Consider the cyclotomic field $\mathbb{Q}(\zeta_f)$. There are $\varphi(f)$ distinct automorphisms given by $\sigma_a(\zeta_f) = \zeta_a^q$, with $1 \leq a \leq f$ and $(a, f) = 1$. We need to know when the restriction of such an automorphism to a given quadratic subfield of $\mathbb{Q}(\zeta_f)$ is the identity. In this direction we have

Lemma 2 Let $\mathbb{Q}(\sqrt{d}) \subseteq \mathbb{Q}(\zeta_f)$ be a quadratic field of discriminant $\Delta_d$. We have $\sigma_a|_{\mathbb{Q}(\sqrt{d})} = \text{id}$ if and only if $(\Delta_d/a) = 1$, where $(./.)$ denotes the Kronecker symbol.

Proof. We have

$$\sigma_a(\sqrt{d}) = \left( \frac{\mathbb{Q}(\zeta_f)/\mathbb{Q}}{a} \right) \sqrt{d} = \left( \frac{\mathbb{Q}(\sqrt{d})/\mathbb{Q}}{a} \right) \sqrt{d} = (\frac{\Delta_d}{a}) \sqrt{d},$$

where the latter reciprocity symbol is the Kronecker symbol and the other two are Artin symbols. \hfill \Box

Remark. It is also possible to prove Lemma 2 using quadratic reciprocity and properties of Gauss sums.

The next result can be proved by a trivial generalization of an argument given by Hooley [2], pp. 213-214.

Lemma 3 Let $g \in G$ and let $h$ be the largest positive integer such that $g$ is an $h$-th power. Let $\Delta$ denote the discriminant of $\mathbb{Q}(\sqrt{g})$. Let $k$ and $r$ be natural
numbers such that \( k|r \) and \( k \) is squarefree. Put \( k_1 = k/(k,h) \) and \( n(k,r) = \lfloor \mathbb{Q}(\zeta_r, \sqrt[1/k]{f}) : \mathbb{Q} \rfloor \). Then

i) if \( k \) is odd, \( n(k,r) = k_1 \varphi(r) \);

ii) if \( k \) is even and \( \Delta \nmid r \), \( n(k,r) = k_1 \varphi(r) \);

iii) if \( k \) is even and \( \Delta | r \), \( n(k,r) = k_1 \varphi(r)/2 \).

Notice that \( h \) is odd, since by assumption \( g \) is not a square. The next lemma together with Lemma \( 3 \) allows one to compute \( c_a(n) \).

**Lemma 4** Let \( g \in G \) and let \( \Delta \) denote the discriminant of \( \mathbb{Q}(\sqrt{\gamma}) \). Let \( n \geq 1 \) be squarefree and \( f \geq 1 \) be arbitrary. Put \( b = \Delta/(f, \Delta) \). If \( b \) is odd, then

\[
\mathbb{Q}(\zeta_f) \cap \mathbb{Q}(\zeta_n, \sqrt[n]{f}) = \mathbb{Q}(\zeta_{f,n}, \sqrt{(-1)^{(b-1)/2}(f, \Delta)})
\]

if \( n \) is even, \( \Delta \nmid n \) and \( \Delta | \text{lcm}(f,n) \) and

\[
\mathbb{Q}(\zeta_f) \cap \mathbb{Q}(\zeta_n, \sqrt[n]{f}) = \mathbb{Q}(\zeta_{f,n})
\]

otherwise. If \( b \) is even, then \( \mathbb{Q}(\zeta_f) \cap \mathbb{Q}(\zeta_n, \sqrt[n]{f}) = \mathbb{Q}(\zeta_{f,n}) \).

**Proof.** On noting that \( \varphi((f,n)) \varphi(\text{lcm}(f,n)) = \varphi(f) \varphi(n) \) and that

\[
[\mathbb{Q}(\zeta_f, \zeta_n, \sqrt[n]{f}) : \mathbb{Q}] = \frac{[\mathbb{Q}(\zeta_f) : \mathbb{Q}(\zeta_{f,n})][\mathbb{Q}(\zeta_n, \sqrt[n]{f}) : \mathbb{Q}]}{[\mathbb{Q}(\zeta_f) \cap \mathbb{Q}(\zeta_n, \sqrt[n]{f}) : \mathbb{Q}(\zeta_{f,n})]}
\]

it easily follows, using Lemma \( 3 \) that

\[
[\mathbb{Q}(\zeta_f) \cap \mathbb{Q}(\zeta_n, \sqrt[n]{f}) : \mathbb{Q}(\zeta_{f,n})] = 2 \quad (1)
\]

if \( n \) is even, \( \Delta \nmid n \) and \( \Delta | \text{lcm}(f,n) \) and \( \mathbb{Q}(\zeta_f) \cap \mathbb{Q}(\zeta_n, \sqrt[n]{f}) = \mathbb{Q}(\zeta_{f,n}) \) otherwise. In case \( b \) is even, we see that \( \Delta \nmid \text{lcm}(f,n) \) on using that \( n \) is squarefree and we are done. It remains to deal with the case where \( b \) is odd, \( n \) is even, \( \Delta \nmid n \) and \( \Delta | \text{lcm}(f,n) \). The latter divisibility condition implies that \( \Delta | fn \) and hence \( b|n \). Put \( \gamma = (-1)^{(b-1)/2}(f, \Delta) \). Using Lemma \( 1 \) and the fact that \( b|n \), it follows that \( \sqrt{\Delta/\gamma} = \sqrt{(-1)^{(b-1)/2}b} \in \mathbb{Q}(\zeta_n) \). We now distinguish cases according to the residue mod 4 of the odd part of \( \Delta \), respectively the odd part of \( (f, \Delta) \). In each case we check, using Lemma \( 1 \) that \( \sqrt{\gamma} \in \mathbb{Q}(\zeta_f) \). It follows that \( \sqrt{\gamma} \notin \mathbb{Q}(\zeta_{f,n}) \), for otherwise from \( \sqrt{\Delta/\gamma}, \sqrt{\gamma} \in \mathbb{Q}(\zeta_n) \) it would follow that \( \sqrt{\Delta} \in \mathbb{Q}(\zeta_n) \), contradicting our assumption that \( \Delta \nmid n \). Using \( \sqrt{\Delta/\gamma} \in \mathbb{Q}(\zeta_n) \), \( \sqrt{\gamma} \in \mathbb{Q}(\zeta_f) \), \( \sqrt{\gamma} \notin \mathbb{Q}(\zeta_{f,n}) \) and \( 1 \), the result is completed. \( \square \)
3 Euler products

In this section we prove some results that will help us to write down the Euler product of the sums encountered in the proof of the main result (Theorem 2).

**Proposition 1** Let $f, h \geq 1$ be integers. Then the function $w : \mathbb{N} \to \mathbb{N}$ defined by

$$w(k) = \frac{k \varphi(lcm(k,f))}{(k,h)\varphi(f)}$$

is multiplicative. Furthermore,

i) if $p \nmid h$ and $p \nmid f$, then $w(p) = p(p - 1)$

ii) if $p \nmid h$ and $p | f$, then $w(p) = p$

iii) if $p | h$ and $p \nmid f$, then $w(p) = p - 1$

iv) if $p | h$ and $p | f$, then $w(p) = 1$

v) if $h$ is odd, then $w(2) = 2$.

This will help us to prove the following lemma.

**Lemma 5** Let $f, h \geq 1$ be integers with $1 \leq a \leq f$, $(a,f) = 1$ and $h$ odd. Let $\Delta$ be a discriminant of a quadratic number field. Let $b = \Delta/(f,\Delta)$. Put

$$S(b) = \sum_{n=1}^{\infty} \frac{\mu(n)}{w(n)},$$

Let $S_2(b)$ denote the same sum as $S(b)$ but with the restriction that $2 | n$. Then $S(b) = 0$ if $b$ is even and

$$S(b) = \frac{\mu(b)A(a,f,h)}{\prod_{p \mid b}(w(p)-1)}$$

otherwise. Furthermore, $S_2(b) = -S(b)$.

**Proof.** If $b$ is even, then the summation in $S(b)$ runs over non squarefree $n$ only and hence $S(b) = 0$. Next assume that $b$ is odd. We have

$$S(b) = \sum_{n=1}^{\infty} \frac{\mu(n)}{w(n)} = \sum_{d \mid (a-1,f)} \sum_{n=1 \mod (f,n)}^{\infty} \frac{\mu(n)}{w(n)}$$

$$= \sum_{d \mid (a-1,f)} \frac{\mu(d)}{w(d)} \sum_{n=1 \mod (f,n)}^{\infty} \frac{\mu(n)}{w(n)} = \frac{\mu([b])}{w([b])} \sum_{d \mid (a-1,f)} \frac{\mu(d)}{w(d)} \sum_{n=1 \mod (b,n)}^{\infty} \frac{\mu(n)}{w(n)}.$$
Now by assumption $\Delta$ is a discriminant and $b$ is odd. This implies that $(f, b) = 1$ and $b$ is squarefree. Thus

\[
S(b) = \frac{\mu(|b|)}{w(|b|)} \prod_{p|(a-1, f)} \left(1 - \frac{1}{w(p)}\right) \prod_{p \nmid f, b} \left(1 - \frac{1}{w(p)}\right)
\]

\[
= \frac{\mu(|b|)}{w(|b|)} \prod_{p|(a-1, f)} \left(1 - \frac{1}{w(p)}\right) \prod_{p \mid f} \left(1 - \frac{1}{w(p)}\right) \prod_{p \nmid b} \left(1 - \frac{1}{w(p)}\right)^{-1}
\]

\[
= \mu(|b|) A(a, f, h) \prod_{p|b} (w(p) - 1)
\]

where we used that $w(p) > 1$ for $p|b$ and

\[
\prod_{p|(a-1, f)} \left(1 - \frac{1}{w(p)}\right) \prod_{p \mid f} \left(1 - \frac{1}{w(p)}\right) = A(a, f, h),
\]

an identity immediately obtained on invoking Proposition 1.

The latter part of the assertion follows easily on using that $w(2) = 2$. $\blacksquare$

### 4 Proof of the main result

**Proof of Theorem 2.**

Suppose that $b$ is odd. Note that the discriminant of $\mathbb{Q}(\sqrt{\gamma})$ equals $\gamma$.

i) The case $b$ is odd and $(\gamma/a) = 1$. Using Lemma 4, Lemma 2 and the observation above, we find that $c_a(n) = 1$ in case $a \equiv 1(\mod (f, n))$ and $c_a(n) = 0$ otherwise. This together with Theorem 1 and Lemma 3 implies that

\[
\phi(f)\delta(a, f, g) = \sum_{n=1}^{\infty} \frac{\mu(n)}{w(n)} + \sum_{n=1, 2|n, \Delta | \text{lcm}(n, f)} \frac{\mu(n)}{w(n)} + 2 \sum_{n=1, 2|n, \Delta | \text{lcm}(n, f)} \frac{\mu(n)}{w(n)}.
\]

where furthermore in each sum we restrict to those integers $n$ such that $a \equiv 1(\mod (f, n))$. Now, on using Proposition 4,

\[
I_1 = \sum_{d|(a-1, f)} \frac{\mu(n)}{w(n)} = \sum_{d|(a-1, f)} \frac{\mu(d)}{w(d)} \sum_{d|(a-1, f)} \frac{\mu(n)}{w(n)}
\]

\[
= \prod_{p|(a-1, f)} \left(1 - \frac{1}{w(p)}\right) \prod_{p \mid f} \left(1 - \frac{1}{w(p)}\right) = A(a, f, h).
\]

The result follows from (2) on invoking Lemma 5 and Proposition 1.
ii) The case \( b \) is odd and \( (\gamma/a) = -1 \). Using Lemma 4, Lemma 2, the observation in the beginning of this proof and (2), we find

\[
\varphi(f)\delta(a, f, g) = I_1 + S_2(b) - 2 \sum_{2|n, \Delta|\text{lcm}(f, n), a \equiv 1(\mod (f, n))} \frac{\mu(n)}{w(n)}.
\]

Now

\[
\sum_{2|n, \Delta|\text{lcm}(f, n), a \equiv 1(\mod (f, n))} \frac{\mu(n)}{w(n)} = \sum_{2|n, \Delta|\text{lcm}(f, n), a \equiv 1(\mod (f, n))} \frac{\mu(n)}{w(n)} + \sum_{2|n, \Delta|\text{lcm}(f, n), a \equiv 1(\mod (f, n))} \frac{\mu(n)}{w(n)}.
\]

In case \( \Delta \equiv 0(\mod 4) \), the latter sum is obviously zero. In case \( \Delta \equiv 1(\mod 4) \), a necessary condition for the latter sum to be non-zero is that \( a \equiv 1(\mod (f, \Delta)) \). This together with the property of Kronecker sums that \( (b/a_1) = (b/a_2) \) if \( a_1 \equiv a_2(\mod |b|) \), shows that \( (\gamma/a) = (\gamma/1) = 1 \) and hence the latter sum is also zero if \( \Delta \equiv 1(\mod 4) \). It thus follows that \( \varphi(f)\delta(a, f, g) = I_1 + S_2(b) - 2S_2(b) = I_1 - S_2(b) \).

iii) The case \( b \) is even. As in i) it follows that \( \varphi(f)\delta(a, f, g) = I_1 + S_2(b) \). Now \( I_1 = A(a, f, h) \) as we have seen and \( S_2(b) = 0 \) by Lemma 3.

\[\square\]

5 Applications

Setting \( a = f = 1 \) in Theorem 2 and taking \( g \in G \cap \mathbb{Z} \) we obtain Hooley’s theorem [2]. Setting \( a = 1 \) we obtain Theorem 4 of [5]. Notice that of all the progressions \( \mod f \), the progression 1(\mod f) is the easiest to deal with, since trivially \( c_1(n) = 1 \) for every \( n \).

Lenstra [3, Theorem 8.3] gave a sketch of a proof of the following result.

**Theorem 4** [3]. Let \( g \in G, \Delta \) denote the discriminant of \( \mathbb{Q}(\sqrt{g}) \) and let \( h \) be the largest integer such that \( g \) is an \( h \)-th power. Then \( \delta(a, f, g) = 0 \) if and only if one of the following holds

\[
i) \ (a - 1, f, h) > 1;
\]

\[
ii) \ \Delta|f \text{ and } (\Delta/a) = 1 \text{ (Kronecker symbol)};
\]

\[
iii) \ \Delta|3f, 3|\Delta, 3|h \text{ and } (\Delta/a^3) = -1.
\]

This result very easily follows from Theorem 2. We leave it to the reader to show that if \( \delta(a, f, g) = 0 \), then actually \( \mathcal{P}_{a,f,g} \) is finite.

If \( S \) is any infinite set of prime numbers, denote by \( S(x) \) the number of primes in \( S \) not exceeding \( x \). For given integers \( a \) and \( f \), denote by \( S(x; f, a) \) the number of primes in \( S \) not exceeding \( x \) that are congruent to \( a \) modulo \( f \). We say that \( S \) is weakly uniformly distributed \( \mod f \) (or WUD \mod f for short) if for every \( a \) coprime to \( f \),

\[
S(x; f, a) \sim \frac{S(x)}{\varphi(f)}.
\]
where \( \varphi(f) \) denotes Euler’s totient. The progressions \( a(\mod f) \) such that the latter asymptotic equivalence holds are said to get their fair share of primes from \( S \). Thus \( S \) is weakly uniformly distributed \( \mod f \) if and only if all the progressions mod \( f \) get their fair share of primes from \( S \). Narkiewicz [7] has written a nice survey on the state of knowledge regarding the (weak) uniform distribution of many important arithmetical sequences. Let \( D_g \) denote the set of natural numbers \( f \) such that \( P_g \) is weakly uniformly distributed modulo \( f \).

**Theorem 5** [8]. (GRH). Let \( g \in G \) and let \( h \) be the largest integer such that \( g \) is an \( h \)-th power. Write \( g = g_1g_2^2 \) with \( g_1 \in \mathbb{Z} \) squarefree and \( g_2 \in \mathbb{Q} \). Assume that not both \( g_1 = 21 \) and \( (h, 21) = 7 \), then, assuming GRH, the set \( D_g \) of natural numbers \( f \) such that the set of primes \( p \) such that \( g \) is a primitive root \( \mod p \) is weakly uniformly distributed \( \mod d \), equals

\[
\begin{align*}
\text{(i)} & \quad \{ 2^n : n \geq 0 \} \text{ if } g_1 \equiv 1(\mod 4); \\
\text{(ii)} & \quad \{ 1, 2, 4 \} \text{ if } g_1 \equiv 2(\mod 4); \\
\text{(iii)} & \quad \{ 1, 2 \} \text{ if } g_1 \equiv 3(\mod 4).
\end{align*}
\]

In the remaining case \( g_1 = 21 \) and \( (h, 21) = 7 \), we have \( D_g = \{ 2^n3^m : n, m \geq 0 \} \).

Using only a formula for \( \delta(1, f, g) \) and Theorem 1 in some special cases, this result was first deduced in [8]. Using the full force of Theorem 2, however, a shorter proof of Theorem 5 can be given.

**Proof of Theorem 5.** Put \( S_f := \{ A(a, f, h) \mid 1 \leq a \leq f, \ (a, f) = 1 \} \). Let \( b \) and \( \gamma \) be as in Theorem 2. If \( b \) is odd, put \( \chi(a) = (\gamma/a) \). Notice that \( \chi \) is a character of \((\mathbb{Z}/f\mathbb{Z})^*\). Notice that the dependence of \( \delta(a, f, g) \) on \( a \) comes in only through \( \chi(a) \) and \( A(a, f, h) \), or rather the factor \( \prod_{p \mid (a-1)f} (1 - 1/p) \) of \( A(a, f, h) \).

Let us first consider the case where \( f = 2^m \) for some \( m \geq 0 \). Notice that \( |S_{2^m}| = 1 \). If \( b \) is odd, then \( \delta(a, f, g) = A(a, f, h) \) and since \( |S_{2^m}| = 1 \) it follows that \( f = 2^m \in D_g \). If \( b \) is odd, then it is seen that \( \chi \) is the identity if and only if \( \Delta \) is odd. From these two assertions the truth of Theorem 5 in case \( f = 2^m \) easily follows. Notice that if \( f = 2^m \not\in D_g \), then the image of \( \delta(\cdot, f, g) \) has cardinality two.

It remains to deal with the case where \( f \) has an odd prime factor. First let us consider the case where \( f \) is an odd prime. Then, if \( b \) is even,

\[
\varphi(f)\delta(1, f, g) = A(1, f, h) \neq A(2, f, h) = \delta(2, f, g)\varphi(f)
\]

and we do not have equidistribution. Next assume that \( b \) is odd. If \( f = 3 \), then a short calculation shows that a necessary and sufficient condition for equidistribution to occur, is that \( 3 \mid g_1, \ (3, h) = 1, \ \mu(|b|) = -1 \), \( g_1 \equiv 1(\mod 4) \) and that the equation \( \prod_{p \mid b, p \mid h} (p - 2) \prod_{p \mid b, p \not\mid h} (p^2 - p - 1) = 5 \) has a solution with \( b \) odd. Now notice that \( g \) is a solution to this if and only if \( g_1 = 21 \) and \( (h, 21) = 7 \). Call such a \( g \) exceptional. Next let \( f \geq 5 \) (with \( f \) a prime). Then \( A(1, f, h) < A(2, f, h) = \cdots = A(f - 1, f, h) \). If \( \chi \) is the identity, then it follows from this that \( \delta(1, f, g) \neq \delta(a, f, g) \) for every \( 1 < a < f \) with \( (a, f) = 1 \). If \( \chi \) is not the identity, then it is easily seen that there exist \( 1 < a_1 < a_2 < f \), etc.
Rodier [9], in connection with a coding theoretical problem, conjectured that the density of the primes in \( P_2 \) such that \( p \equiv -1, 3 \text{ or } 19 \text{(mod 28)} \) is \( A/4 \). This would follow if 28 were to be in \( D_2 \), but 28 \( \not\in D_2 \) by Theorem 2. From Theorem 2 it follows that, under GRH, the density of the set of primes considered by Rodier is \( 21A/82 \), more precisely each of the three progressions has density \( 7A/82 \).

Let \( L \) be the set of odd primes \( \ell \) such that there are infinitely many primes with \( \ell \) a primitive root mod \( p \) and \( p \) satisfying \( p \equiv \pm 1 \text{(mod } \ell) \). If \( \ell \equiv 1 \text{(mod } 4) \) then \( \ell \not\in L \) by quadratic reciprocity. Modification of some of the arguments in [1] yields that \( L = \{ \ell : \ell \equiv 3 \text{(mod } 4) \} \) with at most two primes excluded. That \( L \) is non-empty is used in [8] to prove a weaker version of a conjecture of Lubotzky and Shalev on three-manifolds. By Theorem 2 it follows that, on GRH, the density of primes \( p \) such that \( \ell \) is a primitive root mod \( p \) and \( p \equiv \pm 1 \text{(mod } \ell) \), is \( (2\ell - 1)(\ell - 1)A/(\ell^2 - \ell - 1) \) if \( \ell \equiv 3 \text{(mod } 4) \) and is zero if \( \ell \equiv 1 \text{(mod } 4) \).

The density of primitive roots in \( \mathbb{F}_p^* \) is \( \varphi(p - 1)/(p - 1) \). Assuming that the primitive roots are equidistributed over \( [1, 2, \ldots, p - 1] \), one would perhaps expect that the number \( \pi_g(x) \) of primes \( p \leq x \) such that \( g \) is a primitive root mod \( p \), behaves asymptotically as \( \sum_{p \leq x} \varphi(p - 1)/(p - 1) \). This is a well-known and old heuristical idea. Using the Siegel-Walfisz theorem for primes in arithmetic progression, it can be easily shown unconditionally (see e.g. [10, Lemma 1]), that the latter sum is asymptotically equal to \( Ax/\log x \). Comparison with Hooley’s theorem [4], then shows that this heuristic is false in general (under GRH). Let \( h \geq 1 \) be as usual the largest integer such that \( g \) is an \( h \)-th power. A less naive heuristic arises on noting that \( g \) is not a primitive root mod \( p \) if \( g \) is a square mod \( p \) or \( (p - 1, h) = 1 \) and that there are \( \varphi(p - 1) \) primitive roots amongst the \( (p - 1)/2 \) non-squares mod \( p \). It turns out that this heuristic is asymptotically exact (under GRH), even on restricting to primes \( p \) in a prescribed arithmetic progression:

**Theorem 6** [8]. (GRH). Let \( g \in \mathbb{Z}\backslash\{-1, 0, 1\} \) and let \( h \) be the largest integer such that \( g \) is an \( h \)-th power. Then

\[
\pi_g(x; f, a) = 2 \sum_{\substack{p \leq x, \ (\frac{p}{f}) = -1 \\ p \equiv a \text{(mod } f) \\ (p-1, h) = 1}} \frac{\varphi(p - 1)}{p - 1} + O(\frac{x \log \log x}{\log^2 x}). \tag{3}
\]

The right hand side in (3) can be evaluated unconditionally by an elementary but somewhat lengthy calculation requiring little beyond the Siegel-Walfisz theorem. The left hand side of (3) is of course evaluated in this paper (on GRH). Comparison of the main terms in both expressions shows that they are equal and
Theorem 6 follows.

Note added July 2003: This paper, written around 1998, will not be published as its main result will appear as an application of the main (Galois-theoretical) result of \[4\]. For a preview of the results of \[4\] the reader is referred to the article by Stevenhagen \[11\].

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