CAPACITIES IN WIENER SPACE, QUASI-SURE LOWER FUNCTIONS, AND KOLMOGOROV’S $\varepsilon$-ENTROPY

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ABSTRACT. We propose a set-indexed family of capacities $\{\text{cap}_G\}_{G \subseteq \mathbb{R}^+}$ on the classical Wiener space $C(\mathbb{R}^+)$. This family interpolates between the Wiener measure ($\text{cap}_{\{0\}}$) on $C(\mathbb{R}^+)$ and the standard capacity ($\text{cap}_{\mathbb{R}^+}$) on Wiener space. We then apply our capacities to characterize all quasi-sure lower functions in $C(\mathbb{R}^+)$. In order to do this we derive the following capacity estimate (Theorem 2.3) which may be of independent interest: There exists a constant $a > 1$ such that for all $r > 0$,

$$\frac{1}{a} K_G(r^6) e^{-\pi^2/(8r^2)} \leq \text{cap}_G \{ f^* \leq r \} \leq a K_G(r^6) e^{-\pi^2/(8r^2)}.$$

Here, $K_G$ denotes the Kolmogorov $\varepsilon$-entropy of $G$, and $f^* := \sup_{[0,1]} |f|$.

1. INTRODUCTION

Let $C(\mathbb{R}^+)$ denote the collection of all continuous functions $f : \mathbb{R}^+ \to \mathbb{R}$. We endow $C(\mathbb{R}^+)$ with its usual topology of uniform convergence on compacts as well as the corresponding Borel $\sigma$-algebra $\mathcal{B}$. In keeping with the literature, elements of $\mathcal{B}$ are called events.

Denote by $\mu$ the Wiener measure on $(C(\mathbb{R}^+), \mathcal{B})$. Recall that an event $\Lambda$ is said to hold almost surely [a.s.] if $\mu(\Lambda) = 1$.

Next we define $U := \{U_s\}_{s \geq 0}$ to be the Ornstein–Uhlenbeck process on $C(\mathbb{R}^+)$. The process $U$ is characterized by the following requirements:

1. It is a stationary infinite-dimensional diffusion with value in $C(\mathbb{R}^+)$. 
2. Its invariant measure is $\mu$. This implies that for any fixed $s \geq 0$, $\{U_s(t)\}_{t \geq 0}$ is a standard linear Brownian motion.
3. For any given $t \geq 0$, $\{U_s(t)\}_{s \geq 0}$ is a standard Ornstein–Uhlenbeck process on $\mathbb{R}$; i.e., it satisfies the stochastic differential equation,

$$dU_s(t) = -U_s(t) \, ds + \sqrt{2} \, dX_s \quad \forall s \geq 0,$$

where $X$ is a Brownian motion.

Following P. Malliavin (1979), we say that an event $\Lambda$ holds quasi-surely [q.s.] if

$$\mathbb{P} \{ U_s \in \Lambda \text{ for all } s \geq 0 \} = 1.$$

Because $t \mapsto U_s(t)$ is a Brownian motion, any event $\Lambda$ that holds q.s. also holds a.s. The converse is not always true. For example, define $\Lambda_0$ to be the collection of all
functions $f \in C(R_+)$ that satisfy $f(1) \neq 0$ (Fukushima, 1984). Evidently, $\Lambda_0$ holds a.s. because with probability one Brownian motion at time one is not at the origin. On the other hand, $\Lambda_0$ does not hold q.s. because $\{U_s(1)\}_{s \geq 0}$ is point-recurrent. So the chances are 100% that $U_s(1) = 0$ for some $s \geq 0$.

Despite the preceding disclaimer, a number of interesting classical events of full Wiener measure do hold q.s. A notable example is a theorem of M. Fukushima (1984). We can state it, somewhat informally, as follows:

(1.3) The Law of the Iterated Logarithm (LIL) of Khintchine (1933) holds q.s. It might help to recall Khintchine’s theorem: For $\mu$-every $f \in C(R_+)$,

$$
\limsup_{t \to \infty} \frac{f(t)}{\sqrt{2t \ln \ln t}} = 1.
$$

Thus we are led to the precise formulation of (1.3): With probability one, the continuous function $f := U_s$ satisfies (1.4), simultaneously for all $s \geq 0$.

For another example consider “the other LIL” which was discovered by K. L. Chung (1948). Chung’s LIL states that for $\mu$-almost every $f \in C(R_+)$,

$$
\liminf_{t \to \infty} \sup_{u \in [0, t]} |f(u)| \sqrt{\frac{\ln \ln t}{\ln t}} = \frac{\pi}{\sqrt{8}}.
$$

Fukushima’s method can be adapted to prove that

(1.5) Chung’s LIL holds q.s.

To be more precise: With probability one, the continuous function $f := U_s$ satisfies (1.5) simultaneously for all $s \geq 0$.

T. S. Mountford (1992) has derived the quasi-sure integral test corresponding to (1.3). One of the initial aims of this article was to complement Mountford’s theorem by finding a precise quasi-sure integral test for (1.6). Before presenting this work, let us introduce the notion of “relative capacity.”

For all Borel sets $G \subseteq R_+$ and $\Lambda \in C(R_+)$ define

$$
cap_G(\Lambda) := \int_0^\infty P \{U_s \in \Lambda \text{ for some } s \in G \cap [0, \sigma] \} e^{-\sigma} d\sigma.
$$

We think of $\cap_G(\Lambda)$ as the capacity of $\Lambda$ relative to the coordinates in $G$. The special case $\cap_{R_+}(\Lambda)$ is well known and well studied (Fukushima, 1984); $\cap_{R_+}(\Lambda)$ is called the capacity on Wiener space. According to (1.2), an event $\Lambda$ holds q.s. if its complement has zero $\cap_{R_+}-capacity$.

The case where $G := \{s\}$ is a singleton is even better studied because of the simple fact that $\cap_{\{s\}}(\Lambda)$ is a multiple of the Wiener measure. Thus, $G \mapsto \cap_G(\Lambda)$ interpolates from the Wiener measure ($G = \{0\}$) to the standard capacity on Wiener space ($G = R_+$). This “interpolation” property was announced in the Abstract.

Now let $H : R_+ \to R_+$ be decreasing and measurable, and define

$$
\mathcal{L}(H) := \left\{ f \in C(R_+) : \liminf_{t \to \infty} \left[ \sup_{u \in [0, t]} |f(u)| - H(t) \sqrt{t} \right] > 0 \right\}.
$$

A decreasing measurable function $H : R_+ \to R_+$ is called an a.s.-lower function if $\mathcal{L}(H)$ holds a.s.; i.e., $\mu$-almost every $f \in C(R_+)$ is in $\mathcal{L}(H)$. Likewise, $H$ is
called a q.s.-lower function if \( \mathcal{L}(H) \) holds q.s. [The literature actually calls the function \( t \to H(t)\sqrt{t} \) an a.s.,q.s]-lower function if \( \mathcal{L}(H) \) holds a.s.q.s., but we find our parameterization here convenient.]

To understand the utility of these definitions better, consider the special case that \( H(t) = \sqrt{c} \ln \ln t \) for a fixed \( c > 0 \) \( (t \geq 0) \). In this case, Chung’s LIL \( (1.5) \) states that \( \mathcal{L}(H) \) holds a.s. if \( c < \pi/\sqrt{8} \); its complement holds a.s. if \( c > \pi/\sqrt{8} \). In fact, a precise P-a.s. integral test is known (Chung, 1948); see Corollary 1.3 below.

We aim to characterize exactly when \( (\mathcal{L}(H))^C \) has positive \( \text{cap}_G \)-capacity. Define \( K_G \) to be the Kolmogorov \( \varepsilon \)-entropy of \( G \) (Dudley, 1973; Tihomirov, 1963); i.e., for any \( \varepsilon > 0 \), \( k = K_G(\varepsilon) \) is the maximal number of points \( x_1, \ldots, x_k \in E \) such that whenever \( i \neq j \), \( |x_i - x_j| \geq \varepsilon \).

**Theorem 1.1.** Choose and fix a decreasing measurable function \( H : \mathbb{R}_+ \to \mathbb{R}_+ \), and a bounded Borel set \( G \subset \mathbb{R}_+ \). Then, \( \text{cap}_G((\mathcal{L}(H))^C) = 0 \) if and only if there exists a decomposition \( G = \bigcup_{n=1}^{\infty} G_n \) in terms of closed sets \( \{G_n\}_{n=1}^{\infty} \), such that

\[
\int_{1}^{\infty} K_{G_n}(H^6(s)) \exp \left( -\frac{\pi^2}{8H^2(s)} \right) ds < \infty \quad \forall n \geq 1. \tag{1.9}
\]

Theorem 1.1 yields the following definite refinement of (1.5).

**Corollary 1.2.** Choose and fix a decreasing measurable function \( H : \mathbb{R}_+ \to \mathbb{R}_+ \). Then, \( \mathcal{L}(H) \) holds q.s. if and only if

\[
\int_{1}^{\infty} \exp \left( -\frac{\pi^2}{8H^2(s)} \right) \frac{ds}{sH^8(s)} < \infty. \tag{1.10}
\]

Theorem 1.1 also contains the original almost-sure integral test of Chung (1948). To prove this, simply plug \( G = \{u\} \) in Theorem 1.1. Then, \( K_{(u) \cap J}(\varepsilon) \) is one if \( u \in J \) and zero otherwise. Thus we obtain the following.

**Corollary 1.3** (Chung (1948)). Choose and fix a decreasing measurable function \( H : \mathbb{R}_+ \to \mathbb{R}_+ \). Then \( \mathcal{L}(H) \) holds a.s. if and only if

\[
\int_{1}^{\infty} \exp \left( -\frac{\pi^2}{8H^2(s)} \right) \frac{ds}{sH^8(s)} < \infty. \tag{1.11}
\]

To put the preceding in perspective define

\[
H_\nu(t) := \sqrt{8 \ln 2 \ln t + \nu \ln \ln t \ln \ln t} \quad \forall t, \nu > 0. \tag{1.12}
\]

[1/0 := \infty] Then, we can deduce from Corollaries 1.2 and 1.3 that \( \mathcal{L}(H_\nu) \) occurs q.s. iff \( \nu > 5 \), whereas \( \mathcal{L}(H_\nu) \) occurs a.s. iff \( \nu > 2 \). In particular, \( \mathcal{L}(H_\nu) \) occurs a.s. but not q.s. if \( \nu \in [2, 5) \). The following is another interesting consequence of Theorem 1.1.

**Corollary 1.4.** Let \( G \subseteq [0, 1] \) be a non-random Borel set. Then,

\[
\dim_G G > \frac{\nu - 2}{3} \implies \text{cap}_G \left( (\mathcal{L}(H_\nu))^C \right) > 0, \text{ whereas } \dim_G G < \frac{\nu - 2}{3} \implies \text{cap}_G \left( (\mathcal{L}(H_\nu))^C \right) = 0. \tag{1.13}
\]

Here, \( \dim_G G \) denotes the packing dimension (Mattila, 1995) of the set \( G \).
Throughout this paper, uninteresting constants are denoted by \( a, b, \alpha, A, \) etc. Their values may change from line to line.

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## 2. Brownian Sheet, and Capacity in Wiener Space

We will be working with a special construction of the process \( U \). This construction is due to D. Williams (Meyer, 1982, Appendix).

Let \( B := \{ B(s, t) \}_{s, t \geq 0} \) denote a two-parameter Brownian sheet. This means that \( B \) is a centered, continuous, Gaussian process with

\[
\text{Cov}(B(s, t), B(s', t')) = \min((s, s') \times \min(t, t')) \quad \forall s, s', t, t' \geq 0.
\]

The Ornstein–Uhlenbeck process \( U = \{ U_s \}_{s \geq 0} \) on \( C(\mathbb{R}_+) \) is precisely the infinite-dimensional process that is defined by

\[
U_s(t) = \frac{B(e^s, t)}{e^{s/2}} \quad \forall s, t \geq 0.
\]

Indeed, one can check directly that \( U \) is a \( C(\mathbb{R}_+) \)-valued, stationary, symmetric diffusion. And for every \( t \geq 0 \), \( \{ U_s(t) \}_{s \geq 0} \) solves the stochastic differential equation (1.1) of the Ornstein–Uhlenbeck type. Furthermore, the invariant measure of \( U \) is the Wiener measure.

The following well-known result is a useful localization tool.

**Lemma 2.1.** For all bounded Borel sets \( G \subseteq \mathbb{R}_+ \) and \( \Lambda \in \mathcal{B} \), \( \text{cap}_G(\Lambda) > 0 \) iff with positive probability there exists \( s \in G \) such that \( U_s \in \Lambda \).

**Remark 2.2.** The previous lemma continues to hold even when \( G \) is unbounded.

**Proof.** Without loss of much generality, we may—and will—assume that \( G \subseteq [0, q] \) for some \( q > 0 \). Let \( p_G(\Lambda) \) denote the probability that there exists \( s \in G \) such that \( U_s \in \Lambda \). Evidently, \( \text{cap}_G(\Lambda) \leq p_G(\Lambda) \). Furthermore, \( \text{cap}_G(\Lambda) = \int_0^q P\{3s \in G \cap [0, \tau]: U_s \in \Lambda \} e^{-\tau} d\tau + e^{-q} p_G(\Lambda) \), whence the bounds,

\[
(2.3) \quad e^{-q} p_G(\Lambda) \leq \text{cap}_G(\Lambda) \leq p_G(\Lambda).
\]

The lemma follows.

Define

\[
(2.4) \quad f^* := \sup_{u \in [0, 1]} |f(u)| \quad \forall f \in C(\mathbb{R}_+).
\]

The following is the main step in the proof of Theorem 2.1. It was announced earlier in the Abstract.

**Theorem 2.3.** There exists \( \alpha > 1 \) such that for all \( r \in (0, 1) \) and all Borel sets \( G \subseteq [0, 1] \),

\[
(2.5) \quad \frac{1}{a} K_G(r^6) e^{-\pi^2/(8r^2)} \leq \text{cap}_G \{ f^* \leq r \} \leq a K_G(r^6) e^{-\pi^2/(8r^2)}.
\]

**Remark 2.4.** The constant \( a \) depends on \( G \) only through the fact that \( G \) is a subset of \([0, 1]\). Therefore, there exists \( a > 1 \) such that simultaneously for all Borel sets \( F, G \subseteq [0, 1] \),

\[
(2.6) \quad \frac{1}{a} K_F(r^6) \leq \frac{\text{cap}_F \{ f^* \leq r \}}{\text{cap}_G \{ f^* \leq r \}} \leq a K_G(r^6) \quad \forall r \in (0, 1).
\]
Remark 2.5. It turns out that for any fixed \( \varepsilon > 0 \), \( \cap_{\mathbb{R}^+} \) and \( \cap_{[0, \varepsilon]} \) are equivalent. To prove this, we can assume without loss of generality that \( \varepsilon \in (0, 1) \). [This is because \( \varepsilon \mapsto \cap_{[0, \varepsilon]}(\Lambda) \) is increasing.] Now, on one hand, \( \cap_{[0, \varepsilon]}(\Lambda) \leq \cap_{\mathbb{R}^+}(\Lambda) \).

On the other hand,

\[
\cap_{\mathbb{R}^+}(\Lambda) \leq \int_0^\infty \sum_{0 \leq j \leq \sigma/\varepsilon} \mathbb{P}\{3s \in [j\varepsilon, (j+1)\varepsilon) : U_s \in \Lambda\} e^{-\sigma} d\sigma
\]
\[
\leq \mathbb{P}\{3s \in [0, \varepsilon) : U_s \in \Lambda\} \int_0^\infty \frac{\sigma + 1}{\varepsilon} e^{-\sigma} d\sigma,
\]

by stationarity. In the notation of Lemma 2.7, the last term is \( (2/\varepsilon)\mathbb{P}_{[0, \varepsilon]}(\Lambda) \leq (2e/\varepsilon)\cap_{[0, \varepsilon]}(\Lambda) \); cf. (2.3). Thus,

\[
\frac{\varepsilon}{2e} \cap_{\mathbb{R}^+}(\Lambda) \leq \cap_{[0, \varepsilon]}(\Lambda) \leq \cap_{\mathbb{R}^+}(\Lambda) \quad \forall \Lambda \in \mathcal{B}.
\]

This proves the claimed equivalence of \( \cap_{[0, \varepsilon]} \) and \( \cap_{\mathbb{R}^+} \).

According to the eigenfunction expansion of Chung (1948),

\[
\mu\{f^* \leq r\} \sim \frac{4}{\pi} e^{-\pi^2/(8r^2)} \quad (r \to 0).
\]

Therefore, thanks to (2.3), Theorem 2.3 is equivalent to our next result.

Theorem 2.6. Recall that \( U^*_s = \sup_{t \in [0,1]} |U_s(t)| \) [eq. (2.4)]. Then, there exists a constant \( a > 1 \) such that for all \( r \in (0, 1) \) and all Borel sets \( G \subseteq [0, 1] \),

\[
\frac{1}{a} K_G(r^6) \mu\{f^* \leq r\} \leq \mathbb{P}\left\{ \inf_{s \in G} U^*_s \leq r \right\} \leq a K_G(r^6) \mu\{f^* \leq r\}.
\]

We will derive this particular reformulation of Theorem 2.3. The following result plays a key role in our analysis.

Proposition 2.7 (Lifshits and Shi (2003, Proposition 2.1)). Let \( \{X_t\}_{t \geq 0} \) denote planar Brownian motion. For every \( r > 0 \) and \( \lambda \in (0, 1] \) define

\[
\mathscr{D}_\lambda^r = \left\{(x, y) \in \mathbb{R}^2 : |x| \leq r, \left| x\sqrt{1-\lambda} + y\sqrt{\lambda}\right| \leq r\right\}.
\]

Then there exists an \( a \in (0, 1/2) \) such that for all \( r > 0 \) and \( \lambda \in (0, 1] \),

\[
\mathbb{P}\{X_t \in \mathscr{D}_\lambda^r \quad \forall t \in [0, 1]\} \leq \frac{1}{a} \mu\{f^* \leq r\} e^{-a\lambda^{1/3}/r^2}.
\]

Lemma 2.8. There exists a constant \( a \in (0, 1) \) such that for all \( 1 \geq S > s > 0 \),

\[
\mathbb{P}\{U^*_s \leq r, \ U^*_S \leq r\} \leq \frac{1}{a} \mu\{f^* \leq r\} e^{-a(S-s)^{1/3}/r^2} \quad \forall r \in (0, 1).
\]

Proof. Define \( \lambda = 1 - e^{-(S-s)} \). Then owing to (2.2), we can write

\[
U_S(t) = U_s(t)\sqrt{1-\lambda} + \frac{B(e^S, t) - B(e^s, t)}{\sqrt{e^S-e^s}}\sqrt{\lambda} := U_s(t)\sqrt{1-\lambda} + V(t)\sqrt{\lambda}.
\]

By the Markov properties of the Brownian sheet, \( X_t := (U_s(t), V(t)) \) defines a planar Brownian motion. Moreover, \( \mathbb{P}\{U^*_s \leq r, \ U^*_S \leq r\} = \mathbb{P}\{X_t \in \mathscr{D}_\lambda^r \quad \forall t \in [0, 1]\} \).

By Taylor’s expansion, \( 1 - e^{-x} \geq (x/2) (x \in [0, 1]) \). Therefore, Proposition 2.7 completes the proof. \( \square \)
Proof of Theorem 2.6: Lower Bound. Let \( k = K_G(r^6) \), and choose maximal Kolmogorov points \( s(1) < \cdots < s(k) \) such that \( s(i + 1) - s(i) \geq r^6 \). Evidently, whenever \( j > i \) we have \( s(j) - s(i) \geq (j - i)r^6 \). Now define

\[
N_r = \sum_{i=1}^{k} 1\{U^*_s(\omega) \leq r\}.
\]

According to Lemma 2.8,

\[
E[N^2_r] = k\mu \{f^* \leq r\} + 2 \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} P\{U^*_s(i) \leq r, U^*_s(j) \leq r\}
\leq k\mu \{f^* \leq r\} + \frac{2}{a} \mu \{f^* \leq r\} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \exp \left( -\frac{a(s(j) - s(i))^{1/3}}{r^2} \right)
\leq k\mu \{f^* \leq r\} + \frac{2}{a} \mu \{f^* \leq r\} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \exp \left( -a(j - i)^{1/3} \right)
\leq Ak\mu \{f^* \leq r\}.
\]

Note that \( A \) is a positive and finite constant that does not depend on \( r \). Also note that \( E[N_r] = k\mu \{f^* \leq r\} \). This and the Paley–Zygmund inequality (Khoshnevisan, 2002, Lemma 1.4.1, p. 72) together reveal that

\[
P\left( \inf_{s \in G} U^*_s \leq r \right) \geq P\{N_r > 0\} \geq \frac{(E[N_r])^2}{E[N^2_r]} \geq \frac{k}{A} \mu \{f^* \leq r\}.
\]

The definition of \( k \) implies the lower bound in Theorem 2.6. \( \square \)

Before proving the upper bound of Theorem 2.6 in complete generality, we first derive the following weak form:

**Proposition 2.9.** There exists a finite constant \( a > 1 \) such that for all \( r \in (0,1) \),

\[ P\{\inf_{s \in [0,r]} U^*_s \leq r\} \leq a\mu \{f^* \leq r\}. \]

**Proof.** Recall (2.15), and define

\[
L(s; r) = \int_0^s 1\{U^*_s(\omega) \leq r\} \, d\nu \quad \forall s, r > 0.
\]

Let \( \mathcal{F} := \{\mathcal{F}_s\}_{s \geq 0} \) denote the augmented filtration generated by the infinite-dimensional process \( \{U_s\}_{s \geq 0} \). The latter process is Markov with respect to \( \mathcal{F} \). Moreover,

\[
E\left[ L(2r^6; r + r^3) \mid \mathcal{F}_s \right] \geq \int_s^{2r^6} P\{U^*_s \leq r + r^3 \mid \mathcal{F}_s \} \, d\nu \cdot 1\{U^*_s \leq r\}.
\]

As in (2.14), if \( \nu > s \) are fixed, then we can write

\[
U^*_\nu(t) = U_s(t)e^{-(\nu-s)/2} + \frac{B(e^\nu, t) - B(e^s, t)}{\sqrt{e^\nu - e^s}} \sqrt{1 - e^{-(\nu-s)}},
\]

\[
:= U_s(t)e^{-(\nu-s)/2} + V(t)\sqrt{1 - e^{-(\nu-s)}}.
\]

\[
\text{(2.20)}
\]
We emphasize, once again, that \((U_s, V)\) is a planar Brownian motion. In addition, \(V\) is independent of \(\mathcal{F}_s\), and \(U^*_\nu \leq U^*_s + V^* \sqrt{1 - \exp\{- (\nu - s)\}}\). Consequently, as long as \(0 \leq s \leq r^6\) and \(s < \nu < 2r^6\),

\[
U^*_\nu \leq U^*_s + \frac{s^3}{\sqrt{2}} V^*.
\]

[We have used the inequality \(1 - e^{-z} \leq z/2\) valid for all \(z \in (0, 1)\).] Therefore, for all \(0 \leq s \leq r^6\),

\[
M(s) = E \left[ L(2r^6; r + r^3) \mid \mathcal{F}_s \right]
\geq \int_{s}^{2r^6} P \left\{ V^* \leq \sqrt{2} \right\} d\nu \cdot 1_{\{U^*_\nu \leq r\}}
= \mu \left\{ f^* \leq \sqrt{2} \right\} (2r^6 - s) \cdot 1_{\{U^*_\nu \leq r\}}
\geq \mu \left\{ f^* \leq \sqrt{2} \right\} r^6 \cdot 1_{\{U^*_\nu \leq r\}}.
\]

Because \(\{M(s)\}_{s \geq 0}\) is a martingale, we can apply Doob’s maximal inequality to obtain the following:

\[
P \left\{ \inf_{s \in \left[0, r^6\right]} U^*_s \leq r \right\} \leq P \left\{ \sup_{s \in \left[0, r^6\right]} M(s) \geq \mu \left\{ f^* \leq \sqrt{2} \right\} r^6 \right\}
\leq \frac{E \left[ L(2r^6; r + r^3) \right]}{\mu \left\{ f^* \leq \sqrt{2} \right\} r^6} = \frac{2\mu \left\{ f^* \leq r + r^3 \right\}}{\mu \left\{ f^* \leq \sqrt{2} \right\}}.
\]

Thanks to (2.9),

\[
\frac{\mu \left\{ f^* \leq r + r^3 \right\}}{\mu \left\{ f^* \leq r \right\}} \sim \exp \left( \frac{\pi^2}{8} \left[ \frac{1}{(r + r^3)^2} - \frac{1}{r^2} \right] \right) \to e^{\pi^2/4}. \quad (r \to 0).
\]

Thus, the left-hand side is bounded \((r \in (0, 1))\), and the proposition follows. \(\square\)

**Proof of Theorem 2.6: Upper Bound.** Define \(n = n(r)\) to be \(\lfloor r^{-6} \rfloor\), and define \(I(j; n)\) to be the interval \([j/n, (j + 1)/n)\) \((j = 0, \ldots, n)\). Then, by stationarity and Proposition 2.9,

\[
P \left\{ \inf_{s \in G} U^*_s \leq r \right\} \leq \sum_{0 \leq j \leq n; \ I(j; n) \cap G \neq \emptyset} P \left\{ \inf_{s \in I(j; n)} U^*_s \leq r \right\} \leq a\mu \left\{ f^* \leq r \right\} M_n(G),
\]

where \(M_n(G) = \# \{0 \leq j \leq n : I(j; n) \cap G \neq \emptyset\}\) defines the Minkowski content of \(G\). In the companion to this paper (2004, Proposition 2.7) we proved that \(M_n(G) \leq 3K_G(1/n)\). By monotonicity, the latter is at most \(3K_G(r^6)\), whence the theorem. \(\square\)

### 3. Proof of Theorem 1.2 and Corollaries 1.2 and 1.4

We begin with some preliminary discussions. Define

\[
\psi_H(G) := \int_{1}^{\infty} \frac{K_G(H^6(s))}{sH^2(s)} \exp \left( -\frac{\pi^2}{8H^2(s)} \right) ds, \quad \sigma(r) := \mu \left\{ f^* \leq r \right\}.
\]

Following Erdős (1942), define

\[
e_n = e^{n/\ln n}, \quad H_n = H(e_n) \quad \forall n \geq 1.
\]
Then, according to Proposition 3.1, we can write
\[ G \]
\[ \text{Proof of Theorem 1.1 in the form of Proposition 3.3.} \]
First suppose (3.3)
\[ \varepsilon_n \rightarrow 0 \] together allow us to assume without loss of generality that
\[ (3.3) \quad \frac{1}{\ln n} \leq H_n \leq \frac{2}{\ln n} \quad \forall n \geq 1. \]

From this we can conclude the existence of a constant \( a > 1 \) such that
\[ (3.4) \quad \frac{1}{a} H_n^2 e_{n+1} \leq e_n - e_{n+1} \leq a H_n^2 e_n \quad \forall n \geq 1. \]

According to our companion work (2004, eq. 2.8), for all \( r > 0 \) sufficiently small,
\[ (3.5) \quad K_G(\varepsilon) \leq 6K_G(2\varepsilon). \]

Because \( e_{n+1} \sim e_n \) as \( n \rightarrow \infty \), (2.9), (3.4), and (3.5) together imply that
\[ (3.6) \quad \sum_{n=1}^{\infty} K_G(H_n^6) \sigma(H_n) < \infty \iff \psi_H(G) < \infty. \]

The following is the key step toward proving Theorem 1.1.

**Proposition 3.1.** Let \( H : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be decreasing and measurable. Then for all non-random Borel sets \( G \subseteq [0, 1] \),
\[ (3.7) \quad \liminf_{t \rightarrow \infty} \left( \inf_{s \in G} \sup_{u \in [0, t]} |U_s(u)| - H(t)\sqrt{t} \right) = \left\{ \begin{array}{cl} +\infty, & \text{if } \psi_H(G) < \infty, \\ -\infty, & \text{if } \psi_H(G) = \infty. \end{array} \right. \]

First we assume this proposition and derive Theorem 1.1. Then, we will tidy things up by proving the technical Proposition 3.1.

Let us recall (3.1).

**Definition 3.2.** We say that \( \Psi_H(G) < \infty \) if we can decompose \( G \) as \( G = \bigcup_{n=1}^{\infty} G_n \) where \( G_1, G_2, \ldots \) are closed—such that for all \( n \geq 1 \), \( \psi_H(G_n) < \infty \). Else, we say that \( \Psi_H(G) = \infty \).

Let us first rephrase Theorem 1.1 in the following convenient, and equivalent, form.

**Proposition 3.3.** Let \( H : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be decreasing and measurable and \( G \subseteq [0, 1] \) be non-random and Borel. If \( \Psi_H(G) < \infty \), then
\[ (3.8) \quad \inf_{s \in G} \liminf_{t \rightarrow \infty} \left( \sup_{u \in [0, t]} |U_s(u)| - H(t)\sqrt{t} \right) = \infty \quad \text{P-a.s.} \]

Else, the left-hand side is P-a.s. equal to \(-\infty\).

**Proof of Theorem 1.1 in the form of Proposition 3.3.** First suppose \( \Psi_H(G) \) is finite. We can write \( G = \bigcup_{n=1}^{\infty} G_n \) where the \( G_n \)'s are closed and \( \psi_H(G_n) < \infty \) for all \( n \geq 1 \). Then, according to Proposition 3.1,
\[ (3.9) \quad \inf_{s \in G_n} \liminf_{t \rightarrow \infty} \left[ \sup_{u \in [0, t]} |U_s(u)| - H(t)\sqrt{t} \right] \geq \liminf_{t \rightarrow \infty} \inf_{s \in G} \left[ \sup_{u \in [0, t]} |U_s(u)| - H(t)\sqrt{t} \right] = \infty. \]

This proves that \( \inf_{s \in G} \liminf_{t \rightarrow \infty} (\sup_{u \in [0, t]} |U_s(u)| - H(t)\sqrt{t}) = \infty \) a.s. [P].
For the converse portion suppose \( \Psi_H(G) = \infty \), and choose arbitrary non-random closed sets \( \{G_n\}_{n=1}^{\infty} \) such that \( \cup_{n=1}^{\infty} G_n = G \). By definition, \( \psi_H(G_n) = \infty \) for some \( n \geq 1 \). Define for all \( T \geq 1 \),

\[
\mathcal{T}_T := \left\{ s \in [0, 1] : \inf_{t \geq T} \sup_{u \in [0, t]} \frac{|U_s(u)|}{H(t)\sqrt{t}} \leq 1 \right\}.
\]

Evidently, \( \mathcal{T}_T \) is a random set for each \( T \geq 0 \). Moreover, the continuity of the Brownian sheet implies that with probability one, \( \mathcal{T}_T \) is closed for all \( T \); hence, so is \( \mathcal{T}_T \cap G_n \). Because \( \psi_H(G_n) = \infty \), Proposition 3.1 implies that almost surely, \( \mathcal{T}_T \cap G_n \neq \emptyset \). Since \( \{\mathcal{T}_T \cap G_n\}_{T=1}^{\infty} \) is a decreasing sequence of non-void compact sets, they have non-void intersection. That is, \( (\cap_{T=1}^{\infty} \mathcal{T}_T) \cap G_n \neq \emptyset \) a.s. [P]. Replace \( H \) by \( H - H^3 \) to complete the proof of Proposition 3.3.

Now we derive Proposition 3.1. This completes our proof of Theorem 1.1. Our proof is divided naturally into two halves.

**Proof of Proposition 3.1** First Half. Throughout this portion of the proof, we assume that \( \psi_H(G) < \infty \).

Because \( e_{n+1} \sim e_n \) as \( n \to \infty \), Theorem 2.6 and Brownian scaling together imply that

\[
P \left\{ \inf_{s \in G} \sup_{u \in [0, e_{n-1}]} |U_s(u)| \leq H_n \sqrt{e_n} \right\} = P \left\{ \inf_{s \in G} U_s^* \leq H_n \sqrt{e_n / e_{n-1}} \right\}
\]

\[
\leq aK_G \left( H_n^6 \left[ \frac{e_n}{e_{n-1}} \right]^3 \right) \sigma \left( H_n \sqrt{\frac{e_n}{e_{n-1}}} \right).
\]

According to (3.5), \( K_G(\cdots) \leq 6K_G(H_n^6) \) for all \( n \) large. This and (3.4) together imply that for all \( n \) large,

\[
P \left\{ \inf_{s \in G} \sup_{u \in [0, e_{n-1}]} |U_s(u)| \leq H_n \sqrt{e_n} \right\}
\]

\[
\leq aK_G (H_n^6) \sigma \left( H_n \sqrt{1 + AH_n^2} \right)
\]

\[
\leq aK_G (H_n^6) \sigma \left( H_n \left[ 1 + AH_n^2 \right] \right).
\]

In accord with (2.9), for any fixed \( c \in \mathbb{R} \),

\[
\sigma (r + cr^3) = O(\sigma(r)) \quad (r \to 0).
\]

Thus, for all \( n \geq 1 \),

\[
P \left\{ \inf_{s \in G} \sup_{u \in [0, e_{n-1}]} |U_s(u)| \leq H_n \sqrt{e_n} \right\} \leq aK_G (H_n^6) \sigma (H_n).
\]

Because we are assuming that \( \psi_H(G) \) is finite, (3.6) and the Borel–Cantelli lemma together imply that almost surely, \( \inf_{s \in G} \sup_{u \in [0, e_{n-1}]} |U_s(u)| > H_n \sqrt{e_n} \) for all but a finite number of \( n \)'s. It follows from this and a standard monotonicity argument that

\[
\psi_H(G) < \infty \implies \liminf_{t \to \infty} \left[ \inf_{s \in G} \sup_{u \in [0, t]} |U_s(u)| - H(t)\sqrt{t} \right] > 0 \text{ a.s. [P].}
\]
But if \( \psi_H(G) \) were finite then \( \psi_{H^2}(G) \) is also finite; compare (3.5) and (3.13).

Thanks to (3.3), \( \lim_{t \to \infty} H^3(t) = \infty \). Therefore, the \( \liminf \) of the preceding display is infinity. This concludes the first half of our proof of Proposition 3.4. □

In order to prove the second half of Proposition 3.1 we assume that \( \psi_H(G) = \infty \), recall (3.1), and define

\[
L_n := \left\{ \inf_{s \in G} \sup_{u \in [0,n]} |U_s(u)| \leq H_n \sqrt{e_n} \right\},
\]

\[
f(z) := K_G(z^5) \sigma(z).
\]

**Lemma 3.4.** Define for all \( j \geq i \), \( \lambda_{i,j} := e_j/(e_j - e_i) \) and \( \delta_{i,j} := H_j \sqrt{\lambda_{i,j} + H_i \sqrt{\lambda_{i,j} - 1}} \). Then, there exists \( a > 1 \) such that for all \( j \geq i \), \( \Pr(L_j \mid L_i) \leq a K_G(\delta_{i,j}) \sigma(\delta_{i,j}) \).

**Proof.** Evidently, \( \Pr(L_j \mid L_i) \) is at most

\[
\Pr \left\{ \inf_{s \in G} \sup_{u \in [e_i,e_j]} |U_s(u)| \leq \sqrt{H_j e_j} \mid L_i \right\}
\]

\[
= \Pr \left\{ \inf_{s \in G} \sup_{u \in [e_i,e_j]} |U_s(u) - U_s(e_i)| \leq H_j \sqrt{e_j} \mid L_i \right\}
\]

\[
\leq \Pr \left\{ \inf_{s \in G} \sup_{u \in [e_i,e_j]} |U_s(u) - U_s(e_i)| \leq H_j \sqrt{e_j} + H_i \sqrt{e_i} \right\}.
\]

We have appealed to the Markov properties of the Brownian sheet in the last line. Because \( u \mapsto U_s(u) \) is a \( C(\mathbb{R}_+) \)-valued Brownian motion,

\[
\Pr(L_j \mid L_i) \leq \Pr \left\{ \inf_{s \in G} \sup_{u \in [e_i,e_j]} |U_s(u)| \leq H_j \sqrt{e_j} + H_i \sqrt{e_i} \right\}
\]

\[
= \Pr \left\{ U^*_s \leq \delta_{i,j} \right\}.
\]

Theorem 2.8 completes the proof. □

Our forthcoming estimates of \( \Pr(L_j \mid L_i) \) rely on the following elementary bound; see, for example, our earlier work (2003, eq. 8.30): Uniformly for all integers \( j > i \),

\[
e_j - e_i \geq e_i \left( \frac{j - i}{\ln r} \right) (1 + o(1)) \quad (i \to \infty).
\]

**Lemma 3.5.** There exist \( i_0 \geq 1 \) and a finite \( a > 1 \) such that for all \( i \geq i_0 \) and \( j \geq i + \ln^{19}(j) \),

\[
\Pr(L_j \mid L_i) \leq a \Pr(L_j).
\]

**Proof.** Thanks to (3.3) and (3.19), the following holds uniformly over all \( j > i + \ln^{19}(j) \): \( (e_j/e_i) \geq 1 + o(1) \). Thus, uniformly over all \( j > i + \ln^{19}(j) \),

\[
\sqrt{\lambda_{i,j}} = \frac{1}{\sqrt{1 - (e_i/e_j)}} \leq \frac{1}{\sqrt{1 - (1 + o(1))H_j^{30}}} = 1 + O(H_j^3),
\]

\[
H_i \sqrt{\lambda_{i,j}} - 1 = O(H_j^3) \quad (i \to \infty).
\]
Lemma 3.4 guarantees then that uniformly over all \( j > i + \ln^{19}(j) \), \( \delta_{i,j} \leq H_j + O(H_j^3) \), and the big-\( O \) and little-\( o \) terms do not depend on the \( j \)'s in question. The lemma follows from this, equations (3.5) and (3.13), and Theorem 2.6.

**Lemma 3.6.** There exist \( i_1 \geq 1 \) and \( a \in (0, 1) \) such that for all \( i \geq i_1 \) and \( j \in [i + \ln(i), i + \ln^{19}(j)] \), \( P(L_j \mid L_i) \leq (aj^n)^{-1} \).

**Proof.** Equations (3.19) and (3.3) together imply that uniformly for all \( j \geq i + \ln(i) \), (e_i/e_j) \( \leq \frac{1}{2} + o(1) \) (\( i \to \infty \)). This is equivalent to the existence of a constant \( A_i, \) such that for all \( (i, j) \) in the range of the lemma,

\[
(3.22) \quad \sqrt{\lambda_{i,j}} \vee \sqrt{\lambda_{i,j} - 1} \leq 2a.
\]

Thanks to (3.3), we can enlarge the last constant \( a \), if necessary, to ensure that for all \( (i, j) \) in the range of this lemma, \( H_i \leq aH_j \). Therefore, Lemma 3.4 then implies that \( \delta_{i,j} = O(H_j) \), and the big-\( O \) term does not depend on the range of \( j \)'s in question. Because \( G \subseteq [0, 1] \),

\[
(3.23) \quad K_G(\varepsilon) \leq K_{[0,1]}(\varepsilon) \sim 1/\varepsilon \quad (\varepsilon \to 0).
\]

Thus, Lemma 3.4 ensures that \( P(L_j \mid L_i) \leq a\delta_{i,j}^{-6}\sigma(\delta_{i,j}) \). Near the origin, the function \( \delta \to \delta^{-6}\sigma(\delta) \) is increasing. Because we have proved that over the range of \( (i, j) \) of this lemma \( \delta_{i,j} = O(H_j) \), equation (2.9) asserts the existence of a universal \( \alpha > 1 \) such that \( P(L_j \mid L_i) \) is at most \( aH_j^{-\alpha} \exp(-\alpha^{-1}H_j^{-2}) \). Equation (3.3) then completes our proof.

**Lemma 3.7.** There exist \( i_2 \geq 1 \) and \( a > 1 \) such that for all \( i \geq i_2 \) and \( j \in (i, i + \ln i) \), \( P(L_j \mid L_i) \leq ae^{-(j-i)/a} \).

**Proof.** By (3.13), \( (e_i/e_j) \leq 1 - (1 + o(1))(j - i)\ln^{-1}(i) \) (\( i \to \infty \)), where the little-\( o \) term does not depend on \( j \in (i, i + \ln i) \). Similarly, \( (e_j/e_i) \geq 1 + (1 + o(1))(j - i)\ln^{-1}(i) \). Thus, as \( i \to \infty \),

\[
(3.24) \quad \sqrt{\lambda_{i,j}} = \frac{1}{\sqrt{1 - \left(\frac{e_i}{e_j}\right)}} \leq (1 + o(1))\sqrt{\frac{\ln i}{j - i}} \leq \frac{2 + o(1)}{H_j\sqrt{j - i}},
\]

\[
(3.25) \quad \sqrt{\lambda_{i,j} - 1} = \frac{1}{\sqrt{\left(\frac{e_j}{e_i}\right) - 1}} \leq (1 + o(1))\sqrt{\frac{\ln i}{j - i}} \leq \frac{2 + o(1)}{H_j\sqrt{j - i}},
\]

by (3.3). Once again, the little-\( o \) terms are all independent of \( j \in (i, i + \ln i) \). Because \( H_i = O(H_j) \) uniformly for all \( (i, j) \) in the range considered here, Lemma 3.4 implies that uniformly for all \( j \in (i, i + \ln i) \), \( \delta_{i,j} = O(1/\sqrt{j - i}) \). Equation (3.23) bounds the first term on the right-hand side; (2.9) bounds the second. This and (3.3) together prove the existence of a constant \( \alpha > 1 \) such that for all \( i \geq i_2 \) and all \( j \in (i, i + \ln i) \), \( P(L_j \mid L_i) \leq \alpha(j - i)^{\alpha} \exp\{-\alpha^{-1}H_j^{-2}\} \). The lemma follows.

**Proof of Proposition 3.1, Second Half.** According to Theorem 2.6, for all \( n \) large enough, \( P(L_n) \geq aJ(H_n) \). Because \( \psi_H(G) = \infty \), the latter estimate and (3.6) together imply that

\[
(3.25) \quad \sum_{i=1}^{\infty} P(L_i) = \infty.
\]
Thus, our derivation is complete once we demonstrate the following:

\[ \liminf_{n \to \infty} \frac{\sum_{i=1}^{n-1} \sum_{j=i}^{n} P(L_i \cap L_j)}{(\sum_{i=1}^{n} P(L_i))^2} < \infty. \]

(3.26)

See Chung and Erdős (1952). In fact, the preceding display holds with a \( \limsup \) in place of the \( \liminf \). This fact follows from combining, using standard arguments, Lemmas 3.2 through 3.4.

Indeed, let \( I := \max(3, i_1, i_2, i_3) \) and \( s_n := \sum_{i=1}^{n} P(L_i) \). Lemma 3.5 ensures that

\[ \sum_{i=1}^{n-1} \sum_{j=i}^{n} P(L_i \cap L_j) = O \left( s_n^2 \right). \]

(3.27)

By Lemma 3.4,

\[ \sum_{i=1}^{n-1} \sum_{j=i}^{n} P(L_i \cap L_j) \leq \frac{1}{a} \sum_{i=1}^{n-1} \sum_{j=1}^{n} j^{-a} P(L_i) \]

\[ \leq \sum_{i=1}^{n} O \left( \frac{\ln^{19}(i)}{i^a} \right) P(L_i) = O \left( s_n \right). \]

(3.28)

The big-\( O \) terms do not depend on the variables \( (j, n) \).

Finally, Lemma 3.2 implies that

\[ \sum_{i=1}^{n-1} \sum_{j=i}^{n} P(L_i \cap L_j) \leq a \sum_{i=1}^{n} \sum_{j=i}^{\infty} P(L_i) e^{(j-i)/a} = O \left( s_n \right). \]

(3.29)

We have already seen that \( s_n \to \infty \). Thus, (3.27)–(3.29) imply (3.24), and hence the theorem. More precisely, we have proved so far that

\[ \psi_H(G) = \infty \implies \liminf_{t \to \infty} \left[ \inf_{s \in G} \sup_{u \in [0,t]} |U_s(u)| - H(t) \sqrt{t} \right] < 0 \text{ a.s. [P].} \]

(3.30)

Replace \( H \) by \( H + H^3 \) to deduce that the preceding \( \liminf \) is in fact \( -\infty \). This completes our proof of Proposition 3.1. \( \square \)

We conclude this section by proving the remaining Corollaries 1.2 and 1.4.

**Proof of Corollary 1.2.** By definition, \( \mathcal{L}(H) \) holds q.s. iff \( \text{cap}_{R_+} ((\mathcal{L}(H))^G) = 0 \). Thanks to Theorem 1.1, this condition is equivalent to the existence of a non-random “closed-denumerable” decomposition \( R_+ = \bigcup_{n=1}^{\infty} G_n \) such that for all \( n \geq 1 \), \( \psi_H(G_n) < \infty \). But one of the \( G_n \)’s must contain a closed interval that has positive length. Therefore, by the translation-invariance of \( G \to K_G(r) \), there exists \( \varepsilon \in (0,1) \) such that \( \psi_H([0,\varepsilon]) < \infty \).

Conversely, if \( \psi_H([0,\varepsilon]) \) is finite, then we can define \( G_n \) to be \((n-1)\varepsilon, n\varepsilon] \) \((n \geq 1)\) to find that \( \psi_H(G_n) = \psi_H([0,\varepsilon]) < \infty \). Theorem 1.1 then proves that \( \text{cap}_{R_+} ((\mathcal{L}(H))^G) = 0 \) iff there exists \( \varepsilon > 0 \) such that \( \psi_H([0,\varepsilon]) < \infty \). Because \( K_{[0,\varepsilon]}(r) \sim \varepsilon / r \) \((r \to 0)\), the corollary follows. \( \square \)
Proof of Corollary 1.4. We can change variables to deduce that $\psi_{H_\nu}(G)$ is finite iff 
\[ \int_1^\infty K_2(1/s)s^{-1-(\nu/3)}\,ds \]
converges. This and Proposition 2.8 of our companion work (2004) together imply that
\[ \inf\{\nu > 0 : \psi_{H_\nu}(G) < \infty\} = 2 + 3\overline{\dim}_{\mathcal{H}}G, \]
where $\overline{\dim}_{\mathcal{H}}$ denotes the (upper) Minkowski dimension (Mattila, 1995). By regularization (Mattila, 1995, p. 81),
\[ \inf\{\nu > 0 : \Psi_{H_\nu}(G) < \infty\} = 2 + 3\dim_{\mathcal{P}}G. \]
Theorem 1.1 now implies Corollary 1.4. □

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