Circular Peaks and Hilbert Series

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Abstract

The circular peak set of a permutation $\sigma$ is the set $\{\sigma(i) \mid \sigma(i-1) < \sigma(i) > \sigma(i+1)\}$. Let $\mathcal{P}_n$ be the set of all the subset $S \subseteq [n]$ such that there exists a permutation $\sigma$ which has the circular set $S$. We can make the set $\mathcal{P}_n$ into a poset $\mathcal{P}_n$ by defining $S \leq T$ if $S \subseteq T$ as sets. In this paper, we prove that the poset $\mathcal{P}_n$ is a simplicial complex on the vertex set $[3, n]$. We study the $f$-vector, the $f$-polynomial, the reduced Euler characteristic, the M"{o}bius function, the $h$-vector and the $h$-polynomial of $\mathcal{P}_n$. We also derive the zeta polynomial of $\mathcal{P}_n$ and give the formula for the number of the chains in $\mathcal{P}_n$. By the poset $\mathcal{P}_n$, we define two algebras $A_{\mathcal{P}_n}$ and $B_{\mathcal{P}_n}$. We consider the Hilbert polynomials and the Hilbert series of the algebra $A_{\mathcal{P}_n}$ and $B_{\mathcal{P}_n}$.

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1 Introduction

Throughout this paper, let \([m, n] := \{m, m + 1, \cdots, n\}\), \([n] := [1, n]\) and \(\mathfrak{S}_n\) be the set of all the permutations in the set \([n]\). We will write permutations of \(\mathfrak{S}_n\) in the form \(\sigma = (\sigma(1)\sigma(2)\cdots\sigma(n))\).

We say that a permutation \(\sigma\) has a circular descent of value \(\sigma(i)\) if \(\sigma(i) > \sigma(i + 1)\) for any \(i \in [n - 1]\). The circular descent set of a permutation \(\sigma\), denoted \(CDES(\sigma)\), is the set \(\{\sigma(i) \mid \sigma(i) > \sigma(i + 1)\}\). For any \(S \subseteq [n]\), define a set \(CDES_n(S)\) as \(CDES_n(S) = \{\sigma \in \mathfrak{S}_n \mid CDES(\sigma) = S\}\) and use \(cdes_n(S)\) to denote the number of the permutations in the set \(CDES_n(S)\), i.e., \(cdes_n(S) = |CDES_n(S)|\). In a join work [1], Hungyung Zhang et al. derive the explicit formula for \(cdes_n(S)\). As a application of the main results in [1], they also give the enumeration of permutation tableaux according to their shape and generalizes the results in [3]. Moreover, Robert J.Clarke et al. [2] gave the conceptions of linear peak and cyclic peak and studied some new Mahonian permutation statistics. In this paper, we are interested in the circular peaks of permutations. A permutation \(\sigma\) has a circular peak of value \(\sigma(i)\) if \(\sigma(i - 1) < \sigma(i) > \sigma(i + 1)\) for any \(i \in [2, n - 1]\). The circular peak set of a permutation \(\sigma\), denoted \(CP(\sigma)\), is the set \(\{\sigma(i) \mid \sigma(i - 1) < \sigma(i) > \sigma(i + 1)\}\). For example, the circular peak set of \(\sigma = (48362517)\) is \(\{5, 6, 8\}\). Since \(\sigma\) has no circular peaks when \(n \leq 2\), we may always suppose that \(n \geq 3\). For any \(S \subseteq [n]\), define a set \(CP_n(S)\) as \(CP_n(S) = \{\sigma \in \mathfrak{S}_n \mid CP(\sigma) = S\}\). Obviously, if \(\{1, 2\} \subseteq S\), then \(CP_n(S) = \emptyset\).

Example 1.1

\[
CP_5(\{4, 5\}) = \{14253, 14352, 24153, 34152, 24351, 34251, 15243, 15342, 25143, 35142, 25341, 35241\}
\]

Suppose that \(S = \{i_1, i_2, \cdots, i_k\}\), where \(i_1 < i_2 < \cdots < i_k\). We prove that the necessary and sufficient conditions for \(CP_n(S) \neq \emptyset\) are \(i_j \geq 2j + 1\) for all \(j \in [k]\).
Let $\mathcal{P}_n = \{S \mid CP_n(S) \neq \emptyset\}$. we can make the set $\mathcal{P}_n$ into a poset $\mathcal{P}_n$ by defining $S \leq T$ if $S \subseteq T$ as sets. We draw the Hasse diagrams of $\mathcal{P}_3$, $\mathcal{P}_4$ and $\mathcal{P}_5$ as follows.

![Hasse diagrams](image)

Fig.1. the Hasse diagrams of $\mathcal{P}_3$, $\mathcal{P}_4$ and $\mathcal{P}_5$

There is a very interesting results: $\mathcal{P}_n$ is a simplicial complex on the vertex set $[3, n]$. It is easy to obtain that the dimension of the simplicial complex $\mathcal{P}_n$ is $\left\lfloor \frac{1}{2}(n-1) \right\rfloor - 1$. As we will see, the number of the elements in $\mathcal{P}_n$ involves in the left factors of Dyck paths of length $n-1$, counted by the $(n-1)$-th central binomial coefficients $b_{n-1}$ (see Cori and Viennot [4]), where $b_{n-1} = \binom{n-1}{\frac{n-1}{2}}$; the number $p_{n,i}$ of the faces of dimension $i$ in $\mathcal{P}_n$ equals the number of the left factors of Dyck paths from $(0,0)$ to $(n-1,n-2i-3)$, counted by $b_{n-1,i+1} = \binom{n-2i-2}{i}$ [4].

We derive the recurrence relations for the poset $\mathcal{P}_n$: $\mathcal{P}_{n+1} \cong 2 \times \mathcal{P}_n$ if $n$ is even; $\mathcal{P}_{n+1} \cong (2 \times \mathcal{P}_n) \setminus \{\{1\} \times \mathcal{P}_{n,\left\lfloor \frac{n-1}{2} \right\rfloor - 1}\}$ if $n$ is odd, where the notation $\mathcal{n}$ denotes a poset formed by the set $[n]$ with its usual order.

It is very important to obtain the $f$-vector, the $f$-polynomial and the reduced Euler characteristic of a simplicial complex. The integral sequence $(p_{n,-1}, p_{n,0}, \cdots, p_{n,\left\lfloor \frac{n-1}{2} \right\rfloor - 1})$ is called the $f$-vector of $\mathcal{P}_n$. The $f$-polynomial of $\mathcal{P}_n$ is defined to be the polynomial $P_n(x) = \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} p_{n,i} x^{\left\lfloor \frac{n-1}{2} \right\rfloor - i}$. We give the recurrence relations for the $f$-vector and the $f$-polynomial of $\mathcal{P}_n$. Let $P(x,y) = \sum_{n=3}^{\infty} P_n(x) y^n$, then $P(x,y) = \frac{xy^2(x+2)-xy^2C(y^2) 1+y+xy}{x-(x+1)y^2} \frac{1}{x+1} - y^2$, where $C(y) = \frac{1-\sqrt{1-4y}}{2y}$. The reduced Euler characteristic $\tilde{\chi}(\mathcal{P}_n)$ of $\mathcal{P}_n$ satisfies that $\tilde{\chi}(\mathcal{P}_n) = 0$ if $n$ is odd; $\tilde{\chi}(\mathcal{P}_n) = \frac{2(-1)^n}{n} \left( \frac{n-2}{(n-2)} \right)$ if $n$ is even. As a corollary, the zeta poly-
nomial of $\mathcal{P}_n$ is $(m - 1)^{\lfloor \frac{n-1}{2} \rfloor} \mathcal{P}_n(\frac{1}{m-1})$, which is the number of multichains $S_{n,1} \preceq S_{n,2} \preceq \cdots \preceq S_{n,m-1}$ in $\mathcal{P}_n$ for any $m \geq 2$. The number of the chains $S_{n,1} \prec S_{n,2} \prec \cdots \prec S_{n,i}$ in $\mathcal{P}_n$ equals

$$\sum (d_1, d_2, \ldots, d_{i+1}) \frac{2d_{i+1} - n}{n},$$

where the sum is over all $(d_1, \ldots, d_{i+1})$ such that $\sum_{k=1}^{i+1} d_k = n$, $d_1 \geq 0$, $d_k \geq 1$ for all $2 \leq k \leq i$ and $d_{i+1} \geq n - \lfloor \frac{n-1}{2} \rfloor$.

We are interested in the $h$-vector and the $h$-polynomial of the simplicial complex $\mathcal{P}_n$. We obtain the explicit formula for $h$-vector and give the recurrence relations for the $h$-vector and the $h$-polynomial of $\mathcal{P}_n$.

We fix a field $\mathbb{K}$ and let $m = \left\lfloor \frac{n-1}{\frac{n}{2} - (n-1)} \right\rfloor$ and all the elements of $\mathcal{P}_n$ be $S_{n,1}, S_{n,2}, \ldots, S_{n,m}$ for any $n \geq 3$. Let $I_{\mathcal{P}_n}$ be the ideal of the polynomial ring $\mathbb{K}[x_1, \ldots, x_m]$ which is generated by all polynomials $x_i x_j$ such that $S_{n,i}$ and $S_{n,j}$ are incomparable in the poset $\mathcal{P}_n$. Let $J_{\mathcal{P}_n}$ be the ideal of the polynomial ring $\mathbb{K}[x_1, \ldots, x_m]$ which is generated by all polynomials $x_i x_j$ such that $S_{n,i}$ and $S_{n,j}$ are incomparable in $\mathcal{P}_n$ and the polynomials $x_i^2$ for all $i \in [m]$. Define the two algebras $A_{\mathcal{P}_n}$ and $B_{\mathcal{P}_n}$ as the quotients $A_{\mathcal{P}_n} = \mathbb{K}[x_1, \ldots, x_m]/I_{\mathcal{P}_n}$ and $B_{\mathcal{P}_n} = \mathbb{K}[x_1, \ldots, x_m]/J_{\mathcal{P}_n}$, respectively. We study the Hilbert polynomials and the Hilbert series of the algebras $A_{\mathcal{P}_n}$ and $B_{\mathcal{P}_n}$.

This paper is organized as follows. In Section 2, we give the necessary and sufficient conditions for $CP_n(S) \neq \emptyset$. In Section 3, we prove that the poset $\mathcal{P}_n$ is a simplicial complex and study its properties. In Section 4, we define the two algebras $A_{\mathcal{P}_n}$ and $B_{\mathcal{P}_n}$ and consider their Hilbert polynomial and Hilbert series.
The Necessary and Sufficient Conditions for $CP_n(S) \neq \emptyset$

In this section, we will give the necessary and sufficient conditions for $CP_n(S) \neq \emptyset$ for any $n \geq 3$ and $S \subseteq [n]$.

**Theorem 2.1** Suppose that $n$ is an integer with $n \geq 3$. Let $S = \{i_1, i_2, \ldots, i_k\}$ be a subset of $[n]$, where $i_1 < i_2 < \cdots < i_k$. Then the necessary and sufficient conditions for $CP_n(S) \neq \emptyset$ are $i_j \geq 2j + 1$ for all $j \in [k]$.

**Proof.** First, we suppose that $CP_n(S) \neq \emptyset$ and let $\sigma \in CP_n(S)$. For any $j \in [k]$, since all the numbers $i_1, i_2, \ldots, i_j$ are peaks of $\sigma$, we have $i_j - j \geq j + 1$, hence, $i_j \geq 2j + 1$.

Conversely, when $i_j \geq 2j + 1$ for all $j \in [k]$, let $T = [i_k] \setminus S$, then the set $T$ has at least $k + 1$ elements. So, suppose that $T = \{a_1, a_2, \ldots, a_m\}$ with $a_1 < a_2 < \cdots < a_m$. We consider $\sigma = a_1 i_1 a_2 i_2 \cdots a_k i_k a_{k+1} \cdots a_m (i_k + 1) \cdots n$. Obviously, $CP(\sigma) \subseteq S$. If $CP(\sigma) \neq S$, then there exists a minimal $j \in [k]$ such that $i_j$ is not a circular peak of $\sigma$. So, $a_{j+1} > i_j$. This implies that $i_j = 2j$, a contradiction. Hence, $CP(\sigma) = S$ and $CP_n(S) \neq \emptyset$.

**Corollary 2.1** Suppose that $n$ is an integer with $n \geq 3$ and $S \subseteq [n]$. If $CP_n(S) \neq \emptyset$, then $|S| \leq \lfloor \frac{n-1}{2} \rfloor$.

**Proof.** Suppose that $S = \{i_1, i_2, \ldots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$. Since $CP_n(S) \neq \emptyset$, Theorem 2.1 tells us that $i_k \geq 2k + 1$. $i_k \leq n$ implies that $k \leq \lfloor \frac{n-1}{2} \rfloor$.

**Corollary 2.2** Let $n \geq 3$ and $S \subseteq [n]$. Suppose that $CP_n(S) \neq \emptyset$. Then when $|S| < \lfloor \frac{n-1}{2} \rfloor$, we have $CP_{n+1}(S \cup \{n + 1\}) \neq \emptyset$; when $|S| = \lfloor \frac{n-1}{2} \rfloor$, we have $CP_{n+1}(S \cup \{n + 1\}) \neq \emptyset$ if $n$ is even; otherwise, $CP_{n+1}(S \cup \{n + 1\}) = \emptyset$. 


Proof. Suppose that \( k = |S| \). \( k < \left\lfloor \frac{n-1}{2} \right\rfloor \) implies that \( 2(k + 1) + 1 \leq 2\left\lfloor \frac{n-1}{2} \right\rfloor + 1 < n + 1 \). So, \( CP_{n+1}(S \cup \{n + 1\}) \neq \emptyset \) when \( |S| < \left\lfloor \frac{n-1}{2} \right\rfloor \). For the case \( k = \left\lfloor \frac{n-1}{2} \right\rfloor \), we have

\[
2(k + 1) + 1 = \begin{cases} n + 1 & \text{if } n \text{ is even} \\ n + 2 & \text{if } n \text{ is odd} \end{cases}
\]

By Theorem 2.1, \( CP_{n+1}(S \cup \{n + 1\}) \neq \emptyset \) if \( n \) is even; otherwise, \( CP_{n+1}(S \cup \{n + 1\}) = \emptyset \).

3 The Simplicial Complex \( \mathcal{P}_n \)

In this section, we will prove that the poset \( \mathcal{P}_n \) is a simplicial complex on the vertex set \([3, n]\) for any \( n \geq 3 \), and then study the properties of the Simplicial Complex \( \mathcal{P}_n \). Following [5], we define a simplicial complex \( \Delta \) on a vertex set \( V \) as a collection of subsets of \( V \) satisfying:

(1) If \( x \in V \), then \( \{x\} \in \Delta \), and

(2) if \( S \in \Delta \) and \( T \subseteq S \), then \( T \in \Delta \).

Theorem 3.1 Let \( n \geq 3 \). Then \( \mathcal{P}_n \) is a simplicial complex of the set \([3, n]\) and has the dimension \( \left\lfloor \frac{n-1}{2} \right\rfloor - 1 \).

Proof. Obviously, \( \emptyset \in \mathcal{P}_n \). For any \( 3 \leq x \leq n \), Theorem 2.1 implies that \( \{x\} \in \mathcal{P}_n \).

Noting that if \( CP_n(T) = \emptyset \) then \( CP_n(S) = \emptyset \) for any \( S \supseteq T \), we have if \( S \in \mathcal{P}_n \) and \( T \subseteq S \) then \( T \in \mathcal{P}_n \). Hence, \( \mathcal{P}_n \) is a simplicial complex of the set \([3, n]\).

An element \( S \in \mathcal{P}_n \) is called a face of \( \mathcal{P}_n \), and the dimension of \( S \) is defined to be \( |S| - 1 \). In particular, the void set \( \emptyset \) is always a face of \( \mathcal{P}_n \) of dimension \(-1\). Also define the dimension of \( \mathcal{P}_n \) by \( \dim \mathcal{P}_n = \max_{S \in \mathcal{P}_n} (\dim S) \).

Theorem 3.2 The simplicial complex \( \mathcal{P}_n \) has the dimension \( \left\lfloor \frac{n-1}{2} \right\rfloor - 1 \).

Proof. Taking \( S = \{3, 5, \cdots, 2\left\lfloor \frac{n-1}{2} \right\rfloor + 1\} \), by Theorem 2.1, we have \( S \in \mathcal{P}_n \). From Corollary 2.1 it follows that the dimension of \( \mathcal{P}_n \) is \( \left\lfloor \frac{n-1}{2} \right\rfloor - 1 \).
There are very close relations between the number of the elements of $\mathcal{P}_n$ and the number of left factor of Dyck path of length $n$. An $n$-Dyck path is a lattice path in the first quadrant starting at $(0, 0)$ and ending at $(2n, 0)$ with only two kinds of steps—rise step: $U = (1, 1)$ and fall step: $D = (1, -1)$. We can also consider an $n$-Dyck path $P$ as a word of $2n$ letters using only $U$ and $D$. Let $L = w_1w_2 \cdots w_{n-1}$ be a word, where $w_j \in \{U, D\}$ and $n \geq 0$. If there is another word $R$ which consists of $U$ and $D$ such that $LR$ forms a Dyck path, then $L$ is called an $n$-left factor of Dyck path. Let $\mathcal{L}_n$ denote the set of all $n$-left factor of Dyck paths.

It is well known that $|\mathcal{L}_n|$, the cardinality of $\mathcal{L}_n$, equals the $n$th Central binomial number $b_n = \binom{n}{\lfloor n/2 \rfloor}$. In the following lemma, we give a bijection $\phi$ from the sets $\mathcal{P}_n$ to $\mathcal{L}_{n-1}$.

**Lemma 3.1** There is a bijection $\phi$ between the sets $\mathcal{P}_n$ and $\mathcal{L}_{n-1}$ for any $n \geq 3$. Furthermore, the number of the elements in $\mathcal{P}_n$ is $\binom{n-1}{\lfloor n/2 \rfloor}$.

**Proof.** For any $S \in \mathcal{P}_n$, we construct a word $\phi(S) = w_1w_2 \cdots w_{n-1}$ as follows:

$$w_i = \begin{cases} D & \text{if } i + 1 \in S \\ U & \text{if } i + 1 \notin S \end{cases}$$

for any $i \in [n - 1]$. Theorem 2.1 implies that $\phi(S)$ is a $(n - 1)$-left factor of Dyck path. Conversely, for any a $n$-left factor of Dyck path $w_1w_2 \cdots w_{n-1}$, let $S = \{i + 1 \mid w_i = D\}$, then $CP_n(S) \neq \emptyset$. Hence, the mapping $\phi$ is a bijection. Note that the number of $(n - 1)$-left factor of Dyck path is $\binom{n-1}{\lfloor n/2 \rfloor}$. Hence, $|\mathcal{P}_n| = \binom{n-1}{\lfloor n/2 \rfloor}$. 

Now, we are in a position to obtain the number $p_{n,i}$ of the faces of dimension $i$ in $\mathcal{P}_n$. For any $i \geq 0$, let $\mathcal{L}_{n,i}$ denote the set of all $n$-left factor of Dyck paths from $(0, 0)$ to $(n, n - 2i)$. Define a set $\mathcal{P}_{n,i}$ as the set of all the faces of dimension $i$ in $\mathcal{P}_n$, i.e., $\mathcal{P}_{n,i} = \{S \in \mathcal{P}_n \mid |S| = i + 1\}$ for any $-1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor - 1$. Clearly, $p_{n,i} = |\mathcal{P}_{n,i}|$.

**Corollary 3.1** Let $n \geq 3$. There is a bijection between the sets $\mathcal{P}_{n,i}$ and $\mathcal{L}_{n-1,i+1}$ for any
\[-1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor - 1. \] Furthermore, we have
\[
p_{n,i} = \begin{cases} 
1 & \text{if } i = -1 \\
\frac{n-2i-2}{i+1} \binom{n-1}{i} & \text{if } 0 \leq i \leq \lfloor \frac{n-1}{2} \rfloor - 1
\end{cases}
\]

Proof. We just consider the case with \( i \geq 0 \). For any \( S \in P_{n,i} \), \(|S| = i + 1 \) implies that the number of letter \( D \) in the word \( \phi(S) \) is \( i + 1 \). Hence, \( \phi(S) \) is a left factor of Dyck path from \((0, 0)\) to \((n-1, n-2i-3)\). So, \( \phi(S) \in L_{n-1,i+1} \).

If \( P \) and \( Q \) are posets, then the direct product of \( P \) and \( Q \) is the poset \( P \times Q \) on the set \( \{(x, y) \mid x \in P \text{ and } y \in Q\} \) such that \( (x, y) \leq (x', y') \) in \( P \times Q \) if \( x \leq x' \) in \( P \) and \( y \leq y' \) in \( Q \). Recall that the poset \( n \) is formed by the set \( [n] \) with its usual order. By Corollary 2.2, we obtain a method for constructing the poset \( P_{n+1} \) from \( P_n \).

**Theorem 3.3** \( P_{n+1} \cong 2 \times P_n \) if \( n \) is even; \( P_{n+1} \cong (2 \times P_n) \setminus (\{1\} \times P_{n,\lfloor \frac{n-1}{2} \rfloor}) \) if \( n \) is odd.

By Theorem 3.3, it is easy for us to obtain the Möbius function, the recurrence relations for the \( f \)-vector and the \( f \)-polynomial of the poset \( P_n \).

**Corollary 3.2** Let \( \mu_{P_n} \) be the Möbius function of the poset \( P_n \). Then
\[
\mu_{P_n}(S, T) = (-1)^{|T|-|S|}
\]
for any \( S \preceq T \) in \( P_n \).

Proof. Obviously, \( \mu_{P_3}(\emptyset, \{3\}) = -1 \). By induction for \( n \), we assume that \( \mu_{P_n}(S, T) = (-1)^{|T|-|S|} \) for any \( S \preceq T \) in \( P_n \). Lemma 3.3 tells us that \( P_{n+1} \cong 2 \times P_n \) if \( n \) is even; \( P_{n+1} \cong (2 \times P_n) \setminus (\{1\} \times P_{n,\lfloor \frac{n-1}{2} \rfloor}) \) if \( n \) is odd. We conclude from the product theorem that
\[
\mu_{P_{n+1}}(S, T) = \begin{cases} 
\mu_{P_n}(S \setminus \{n+1\}, T \setminus \{n+1\}) & \text{if } n+1 \in S \cap T \\
\mu_{P_n}(S, T) & \text{if } n+1 \notin S \cup T \\
-\mu_{P_n}(S, T \setminus \{n+1\}) & \text{if } n+1 \notin S \text{ and } n+1 \in T
\end{cases}
\]
for any $S \prec T$. Note that $\mu_{\mathcal{P}_n}(S \setminus \{n+1\}, T \setminus \{n+1\}) = (-1)^{|T| - 1 - |S| - 1} = (-1)^{|T| - |S|}$ and $-\mu_{\mathcal{P}_n}(S, T \setminus \{n+1\}) = -(-1)^{|T| - |S|} = (-1)^{|T| - |S|}$. Hence, $\mu_{\mathcal{P}_{n+1}}(S, T) = (-1)^{|T| - |S|}$.

Corollary 3.3 Let $n \geq 3$. The sequence $p_{n,i}$ satisfies the following recurrence relation: when $n$ is even,

$$p_{n+1,i} = \begin{cases} 
p_{n,i} & \text{if } i = -1 \\
p_{n,i-1} + p_{n,i} & \text{if } i = 0, 1, \ldots, \frac{n}{2} - 2 \\
p_{n,i-1} & \text{if } i = \frac{n}{2} - 1 \end{cases}$$

when $n$ is odd,

$$p_{n+1,i} = \begin{cases} 
p_{n,i} & \text{if } i = -1 \\
p_{n,i-1} + p_{n,i} & \text{if } i = 0, 1, \ldots, \frac{n-3}{2} \end{cases}$$

with initial conditions $(p_{3,-1}, p_{3,0}) = (1, 1)$.

Proof. First, we consider the case with $n$ even. It is easy to check that $p_{n+1,-1} = p_{n-1} = 1$. For any $S \in \mathcal{P}_{n+1,\frac{n}{2}-1}$, Corollary 2.2 tells us that $n+1 \in S$. Note that $S \in \mathcal{P}_{n+1,\frac{n}{2}-1}$ if and only if $S \setminus \{n+1\} \in \mathcal{P}_{n,\frac{n}{2}-2}$. Hence, $p_{n+1,\frac{n}{2}-1} = p_{n,\frac{n}{2}-2}$. Let $0 \leq i \leq \frac{1}{2}n - 2$, obviously, $\mathcal{P}_{n,i} \subseteq \mathcal{P}_{n+1,i}$. For any $S \in \mathcal{P}_{n+1,i}$ with $n+1 \in S$, $S \setminus \{n+1\}$ can be viewed as a element of $\mathcal{P}_{n,i-1}$. Conversely, for any $S \in \mathcal{P}_{n,i-1}$, Corollary 2.2 implies that $S \cup \{n+1\} \in \mathcal{P}_{n+1,i}$. Hence, $p_{n+1,i} = p_{n,i-1} + p_{n,i}$. Similarly, we can consider the case with $n$ odd.

Theorem 3.4 The $f$-polynomial $\mathcal{P}_n(x)$ of the simplicial complex $\mathcal{P}_n$ satisfy the following recurrence relation: when $n$ is even, then

$$\mathcal{P}_{n+1}(x) = (1 + x)\mathcal{P}_n(x),$$

and when $n$ is odd, then

$$x\mathcal{P}_{n+1}(x) = (1 + x)\mathcal{P}_n(x) - \frac{2}{n+1} \binom{n-1}{\frac{n}{2}}.$$ 

for any $n \geq 3$, with initial condition $\mathcal{P}_3(x) = x + 1$. 

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Let $P(x, y) = \sum_{n\geq 3} P_n(x)y^n$. Then
\[
P(x, y) = \frac{xy^2(x+2) - xy^2C(y^2)}{x - (x+1)y^2} \frac{1 + y + xy}{x + 1} - y^2,
\]
where $C(y) = \frac{1-\sqrt{1-4y}}{2y}$.

**Proof.** Obviously, $P_3(x) = x + 1$. When $n$ is odd, we suppose that $n = 2i + 1$ with $i \geq 1$. Corollary 3.3 implies that
\[
xP_{2i+2}(x) = (1 + x)P_{2i+1}(x) - \frac{1}{(i+1)} \binom{2i}{i}.
\]
Similarly, when $n$ is even, we suppose that $n = 2i$ with $i \geq 2$. By corollary 3.3, we have
\[
P_{2i+1}(x) = (1 + x)P_{2i}(x).
\]
Let $P_{\text{odd}}(x, y) = \sum_{i \geq 1} P_{2i+1}(x)y^{2i+1}$ and $P_{\text{even}}(x, y) = \sum_{m \geq 2} P_{2i}(x)y^{2i}$, then $P_{\text{odd}}(x, y) = (x + 1)y^3 + (x + 1)yP_{\text{even}}(x, y)$ and $P(x, y) = P_{\text{odd}}(x, y) + P_{\text{even}}(x, y)$. It is easy to check that $xP_{2i+3}(x) = (1 + x)^2P_{2i+1}(x) - \frac{1}{(i+1)} \binom{2i}{i}x$. So, $P_{\text{odd}}$ satisfies the following equation
\[
xP_{\text{odd}}(x, y) = (x + 1)^2y^2P_{\text{odd}}(x, y) + x(x + 2)y^3 - xy^3C(y^2),
\]
where $C(y) = \frac{1-\sqrt{1-4y}}{2y}$. Equivalently,
\[
P_{\text{odd}}(x, y) = \frac{xy^3(x + 2) - xy^3C(y^2)}{x - (x+1)y^2}.
\]
Hence,
\[
P(x, y) = \frac{xy^2(x + 2) - xy^2C(y^2)}{x - (x+1)y^2} \frac{1 + y + xy}{x + 1} - y^2.
\]

Define the reduced Euler characteristic of $P_n$ by $\tilde{\chi}(P_n) = \sum_{i=0}^{\lfloor \frac{1}{2}(n-1) \rfloor} (-1)^{i-1}p_{n,i-1}$.  

**Corollary 3.4** For any $n \geq 3$, $\tilde{\chi}(P_n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{2(-1)^{\frac{n}{2}}n^{-1}(n-2)}{\left(\frac{n}{2}\right)^{n-1}} & \text{if } n \text{ is even} \end{cases}$
Proof. Clearly, \( \mathcal{P}_3(-1) = 0 \). Theorem 3.4 tells us that

\[
\mathcal{P}_{n+1}(-1) = \begin{cases} 
0 & \text{if } n \text{ is even} \\
\frac{2}{n+1} \binom{n-1}{\frac{1}{2}(n-1)} & \text{if } n \text{ is odd}
\end{cases}
\]

for any \( n \geq 4 \). Since \( \tilde{\chi}(\mathcal{P}_n) = (-1)^{\frac{n-1}{2}} - 1 \mathcal{P}_n(-1) \), we have

\[
\tilde{\chi}(\mathcal{P}_n) = \begin{cases} 
0 & \text{if } n \text{ is odd} \\
\frac{2(-1)^{\frac{n-2}{2}}}{n} \binom{n-2}{\frac{1}{2}(n-2)} & \text{if } n \text{ is even}
\end{cases}
\]

Let \( P \) be a finite poset. Define \( Z(P, i) \) to be the number of multichains \( x_1 \leq x_2 \leq \cdots \leq x_{i-1} \) in \( P \) for any \( i \geq 2 \). \( Z(P, i) \) is called the zeta polynomial of \( P \). We state Proposition 3.11.1a and Proposition 3.14.2 in [5] as the following lemma.

**Lemma 3.2** Suppose that \( P \) is a poset. (1) Let \( d_i \) be the number of chains \( x_1 < x_2 < \cdots < x_{i-1} \) in \( P \). Then \( Z(P, i) = \sum_{j \geq 2} d_j \binom{i-2}{j-2} \).

(2) If \( P \) is simplicial and graded, then \( Z(P, x+1) \) is the rank-generating function of \( P \).

**Corollary 3.5** Let \( n \geq 3 \) and \( m \geq 2 \). Then \( Z(\mathcal{P}_n, i) = (i - 1)^{\frac{n-1}{2}} \mathcal{P}_n\left(\frac{1}{i-1}\right) \) for any \( i \geq 2 \). Furthermore, \( Z(\mathcal{P}_n, i) \) satisfies the recurrence relations: \( Z(\mathcal{P}_{n+1}, i) = iZ(\mathcal{P}_n, i) - \varepsilon(n) \frac{2(i-1)^{\frac{n-1}{2}}(n+1)}{n+1} \binom{n-1}{\frac{1}{2}(n-1)} \), where \( \varepsilon(n) = 0 \) if \( n \) is even; \( \varepsilon(n) = 1 \) otherwise, with initial condition \( Z(\mathcal{P}_3, i) = i \).

**Proof.** Let \( \mathcal{P}_n(x) \) be the f-polynomial of \( \mathcal{P}_n \), then the rank-generating function of \( \mathcal{P}_n \) is \( x^{\frac{1}{2}(n-1)}] \mathcal{P}_n\left(\frac{1}{x}\right) \). Lemma 3.2(2) implies that \( Z(\mathcal{P}_n, i) = (i - 1)^{\frac{n-1}{2}} \mathcal{P}_n\left(\frac{1}{i-1}\right) \). The recurrence relations for \( Z(\mathcal{P}_n, i) \) follows Theorem 3.4.

Let \( d_{\mathcal{P}_n,i} \) be the number of the chains \( S_{n,1} \prec S_{n,2} \prec \cdots \prec S_{n,i} \) of \( \mathcal{P}_n \).

**Theorem 3.5** For any \( i \geq 1 \),

\[
d_{\mathcal{P}_n,i} = \sum_{d_1, d_2, \ldots, d_{i+1}} \binom{n}{d_1, d_2, \ldots, d_{i+1}} \frac{2d_{i+1} - n}{n},
\]
where the sum is over all \((d_1, \ldots, d_{i+1})\) such that \(\sum_{k=1}^{i+1} d_k = n\), \(d_1 \geq 0\), \(d_k \geq 1\) for all \(2 \leq k \leq i\) and \(d_{i+1} \geq n - \left\lfloor \frac{n-1}{2} \right\rfloor\).

**Proof.** Let \(i \geq 1\) and \(S_{n,1} \prec S_{n,2} \prec \cdots \prec S_{n,i}\) be a chain of \(\mathcal{P}_n\). Suppose that \(|S_{n,k}| = \lfloor \frac{n-1}{2} \rfloor\) for any \(k \in [i]\), then \(0 \leq j_1 < j_2 < \cdots < j_i \leq \lfloor \frac{n-1}{2} \rfloor\). There are \(p_{n,j_{i-1}}\) ways to obtain the set \(S_{n,i}\). Given \(S_{n,k}\) with \(k \geq 2\), there are \(\binom{j_k - 1}{k - 1}\) ways to form the subset \(S_{n,k-1} \subseteq S_{n,k}\). Hence,

\[
d_{\mathcal{P}_{n,i}} = \sum_{0 = j_0 \leq j_1 < j_2 < \cdots < j_i \leq \lfloor \frac{n-1}{2} \rfloor} \prod_{k=0}^{i-1} \binom{j_{k+1} - 1}{j_k} p_{n,j_{i-1}} = \sum \binom{n}{d_1, d_2, \ldots, d_{i+1}} \frac{2d_{i+1} - n}{n},
\]

where the sum is over all \((d_1, \ldots, d_{i+1})\) such that \(\sum_{k=1}^{i+1} d_k = n\), \(d_1 \geq 0\), \(d_k \geq 1\) for all \(2 \leq k \leq i\) and \(d_{i+1} \geq n - \left\lfloor \frac{n-1}{2} \right\rfloor\).

**Corollary 3.6** For any \(n \geq 3\),

\[
\mathcal{P}_n(x) = \sum_{i=2}^{\left\lfloor \frac{n-1}{2} \right\rfloor + 2} x^{i} \frac{i-2}{(i-2)!} \left(1 - \frac{2}{x} \right) \sum_{j=1}^{n} \binom{n}{d_1, d_2, \ldots, d_i} \frac{2d_i - n}{n}
\]

where the second sum is over all \((d_1, \ldots, d_i)\) such that \(\sum_{k=1}^{i} d_k = n\), \(d_1 \geq 0\), \(d_k \geq 1\) for all \(2 \leq k \leq i - 1\) and \(d_i \geq n - \left\lfloor \frac{n-1}{2} \right\rfloor\).

**Proof.** Lemma 3.2(1) implies that \(Z(\mathcal{P}_n, i) = \sum_{j=2}^{\left\lfloor \frac{n-1}{2} \right\rfloor + 2} d_{\mathcal{P}_{n,j-1}} \binom{i-2}{j-2}\). By Corollary 3.5 we have

\[
\mathcal{P}_n \left( \frac{1}{i-1} \right) = \left( \frac{1}{i-1} \right)^{\left\lfloor \frac{n-1}{2} \right\rfloor + 2} \sum_{j=2}^{\left\lfloor \frac{n-1}{2} \right\rfloor + 2} d_{\mathcal{P}_{n,j-1}} \binom{i-2}{j-2}
\]

for any \(i \geq 2\). Note that

\[
x^{i} \frac{i-2}{(i-2)!} \sum_{j=2}^{\left\lfloor \frac{n-1}{2} \right\rfloor + 2} d_{\mathcal{P}_{n,j-1}} \binom{i-2}{j-2} = \sum_{j=2}^{\left\lfloor \frac{n-1}{2} \right\rfloor + 2} x \frac{j-2}{(j-2)!} \prod_{k=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor + 2-j} (1 - kx) d_{\mathcal{P}_{n,j-1}}
\]

is a polynomial. Hence, \(\mathcal{P}_n(x) = \sum_{j=2}^{\left\lfloor \frac{n-1}{2} \right\rfloor + 2} x \frac{j-2}{(j-2)!} \prod_{k=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor + 2-j} (1 - kx) d_{\mathcal{P}_{n,j-1}}\).

\[\text{12}\]
Let $H_n(x) = P_n(x-1) = \sum_{i=0}^{\lfloor \frac{1}{2}(n-1) \rfloor} h_{n,i} x^{\lfloor \frac{1}{2}(n-1) \rfloor - i}$, then $H_n(x)$ is called $h$-polynomial of $P_n$ and the sequence $(h_{n,0}, h_{n,1}, \ldots, h_{n,\lfloor \frac{n}{2} \rfloor})$ $h$-vector of $P_n$.

**Corollary 3.7** The $h$-polynomial $H_n(x)$ of the simplicial complex $P_n$ satisfies the recurrence relation: when $n$ is even,

$$H_{n+1}(x) = xH_n(x),$$

and when $n$ is odd,

$$(x-1)H_{n+1}(x) = xH_n(x) - \frac{2}{n+1} \left( \frac{n-1}{n-1} \right),$$

for any $n \geq 3$, with initial condition $H_3(x) = x$.

Let $H(x, y) = \sum_{n \geq 3} H_n(x) y^n$, then

$$H(x, y) = \frac{[(x^2 - 1)y^2 - (x - 1)y^2C(y^2)](1 + xy)}{x(x - 1 - xy^2)} - y^2.$$  

Furthermore, let $(h_{n,0}, h_{n,1}, \ldots, h_{n,\lfloor \frac{n}{2} \rfloor})$ be the $h$-vector of $P_n$, then

$$h_{n,i} = \frac{\lfloor \frac{n}{2} \rfloor - i}{\lfloor \frac{n}{2} \rfloor + i} \left( \frac{\lfloor \frac{n}{2} \rfloor + i}{2} \right).$$

**Proof.** Since $H_n(x) = P_n(x-1)$, by Theorem 3.4 we easily obtain that if $n$ is even, then

$$H_{n+1}(x) = xH_n(x),$$

and if $n$ is odd, then

$$(x-1)H_{n+1}(x) = xH_n(x) - \frac{2}{n+1} \left( \frac{n-1}{n-1} \right),$$

for any $n \geq 3$, with initial condition $H_3(x) = x$.

Since $H(x, y) = P(x - 1, y)$, we have

$$H(x, y) = \frac{[(x^2 - 1)y^2 - (x - 1)y^2C(y^2)](1 + xy)}{x(x - 1 - xy^2)} - y^2.$$
Corollary 3.8 Let the sequence \((h_{n,0}, h_{n,1}, \cdots, h_{n,\lfloor \frac{n-1}{2} \rfloor})\) be h-vector of \(\mathcal{P}_n\). Then the sequence \(h_{n,i}\) satisfies the following recurrence relation:

\[
\begin{align*}
    h_{n+1,0} &= h_{n,0} \\
    h_{n+1,i} &= h_{n,i} + \varepsilon(n)h_{n+1,i-1} \quad \text{if} \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \\
    h_{n+1,\lfloor \frac{n}{2} \rfloor} &= \varepsilon(n)c_{\lfloor \frac{n}{2} \rfloor}
\end{align*}
\]

where \(c_m = \frac{1}{m+1} \binom{2m}{m}\) and \(\varepsilon(n) = 0\) if \(n\) is even; otherwise, \(\varepsilon(n) = 1\), with initial conditions \((h_{3,0}, h_{3,1}) = (1, 0)\). Equivalently,

\[h_{n,i} = \left\lfloor \frac{n}{2} \right\rfloor - i \left( \frac{n}{2} + i \right).
\]

**Proof.** By comparing with the coefficient in Corollary 3.7, we can obtain the desired recurrence relations. Consider \(t_{n,i} = \left\lfloor \frac{n}{2} \right\rfloor - i \left( \frac{n}{2} + i \right)\). Note that \(t_{n,i}\) satisfies the above recurrence relations as well. Hence,

\[h_{n,i} = t_{n,i} = \left\lfloor \frac{n}{2} \right\rfloor - i \left( \frac{n}{2} + i \right).
\]

**Remark 3.1** Let \(n \geq 3\). The number of left factor of Dyck path from \((0, 0)\) to \((\left\lfloor \frac{n}{2} \right\rfloor + i - 1, \left\lfloor \frac{n}{2} \right\rfloor - i - 1)\) equals \(\left\lfloor \frac{n}{2} \right\rfloor \left( \frac{n}{2} + i \right)\) for any \(0 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor\).

### 4 The Algebras \(A_{\mathcal{P}_n}\) and \(B_{\mathcal{P}_n}\) from the Poset \(\mathcal{P}_n\)

In this section, we will consider the properties of the algebras \(A_{\mathcal{P}_n}\) and \(B_{\mathcal{P}_n}\).

Let \(m = \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}\). We list all the elements of \(\mathcal{P}_n\) as \(S_{n,1}, S_{n,2}, \cdots, S_{n,m}\). For any a sequence \(S_{n,j_1}, S_{n,j_2}, \cdots, S_{n,j_t}\) in the poset \(\mathcal{P}_n\), let \(r(S_{n,j_1}, S_{n,j_2}, \cdots, S_{n,j_t}) = (r_1, \cdots, r_m)\) be a vector such that \(r_j = |\{k \mid S_{n,j_k} = S_{n,j}\}|\). Furthermore, we can obtain a monomial

\[m(S_{n,j_1}, S_{n,j_2}, \cdots, S_{n,j_i}) = x_1^{r_1}x_2^{r_2} \cdots x_m^{r_m}.
\]
A sequence $S_{n,j_1}, S_{n,j_2}, \ldots, S_{n,j_i}$ of elements in the poset $\mathcal{P}_n$ forms a multichain if and only if the monomial $m(S_{n,j_1}, S_{n,j_2}, \ldots, S_{n,j_i})$ is nonvanishing in the algebra $A_{\mathcal{P}_n}$.

For a monomial ideal $I$, the set of all monomials that do not belong to $I$ is a basis of the quotient of the polynomial ring modulo $I$, called the standard monomial basis. Thus the monomials $m(S_{n,j_1}, S_{n,j_2}, \ldots, S_{n,j_i})$, where $(S_{n,j_1}, S_{n,j_2}, \ldots, S_{n,j_i})$ ranges over the multichains of the poset $\mathcal{P}_n$, form the standard monomial basis of the algebra $A_{\mathcal{P}_n}$. The algebra $A_{\mathcal{P}_n}$ is graded. For a graded algebra $A = A^0 \oplus A^1 \oplus A^2 \cdots$, the Hilbert series of the algebra $A$, is the formal power series in $x$ given by

$$Hilb A = \sum_{i \geq 0} x^i \dim A^i;$$

there exists a polynomial $P_A(x)$ with rational coefficients (called the Hilbert polynomial of $A$) such that $P_A(k) = \dim A^i$ for all sufficiently large $i$.

**Theorem 4.1** The Hilbert polynomial $P_{A_{\mathcal{P}_n}}(x)$ of the algebra $A_{\mathcal{P}_n}$ is $x^\left\lfloor \frac{1}{2}(n-1) \right\rfloor \mathcal{P}_n(\frac{1}{x})$.

**Proof.** Note that the number of the multichains $S_{n,j_1} \preceq S_{n,j_2} \preceq \cdots \preceq S_{n,j_i}$ in $\mathcal{P}_n$ is equal to the dimension of $A^i_n$. Corollary 3.5 implies that $\dim A^i_n = i^\left\lfloor \frac{n-1}{2} \right\rfloor \mathcal{P}_n(\frac{1}{i})$ for any $i \geq 1$. Since $\deg(\mathcal{P}_n(x)) = \left\lfloor \frac{n-1}{2} \right\rfloor$, we have $P_{A_{\mathcal{P}_n}}(x) = x^\left\lfloor \frac{1}{2}(n-1) \right\rfloor \mathcal{P}_n(\frac{1}{x})$.

**Theorem 4.2** For any $n \geq 3$, the Hilbert series $Hilb A_{\mathcal{P}_n}(x)$ of the algebra $A_{\mathcal{P}_n}$ satisfies the following recurrence: when $n$ is even,

$$Hilb A_{\mathcal{P}_{n+1}}(x) = xHilb A'_{\mathcal{P}_n}(x) + Hilb A_{\mathcal{P}_n}(x);$$

when $n$ is odd,

$$Hilb A_{\mathcal{P}_{n+2}}(x) = xHilb A'_{\mathcal{P}_{n+1}}(x) + Hilb A_{\mathcal{P}_{n+1}}(x) + \frac{4n}{n+3}xHilb A'_{\mathcal{P}_{n+1}}(x)$$

$$- \frac{8n}{n+3}xHilb A'_{\mathcal{P}_n}(x) - \frac{4n}{n+3}x^2Hilb A''_{\mathcal{P}_n}(x),$$

where the notation "'" denotes differentiation of functions, with the initial condition $Hilb A_{\mathcal{P}_3}(x) = \frac{1}{(1-x)^2}$ and $Hilb A_{\mathcal{P}_4}(x) = \frac{1+x}{(1-x)^3}$.
Proof. By Theorem 4.1, we have \( \text{Hilb} A_{\mathcal{P}_n}(x) = 1 + \sum_{i \geq 1} i^{\frac{n-1}{2}} \mathcal{P}_n \left( \frac{1}{i} \right) x^i \). The results follow Theorem 3.4.

In general, we may suppose that \( \text{Hilb} A_{\mathcal{P}_n}(x) = \frac{A_{\mathcal{P}_n}(x)}{(1-x)^{\frac{n-1}{2}}} \), where \( A_{\mathcal{P}_n}(x) \) is a polynomial.

**Corollary 4.1** For any \( n \geq 3 \), \( A_{\mathcal{P}_n}(x) \) satisfies the following recurrence: when \( n \) is even,

\[
A_{\mathcal{P}_{n+1}}(x) = x(1-x)A'_{\mathcal{P}_n}(x) + \left[ \frac{1}{2}n - 1 \right] A_{\mathcal{P}_n}(x);
\]

when \( n \) is odd,

\[
(1-x)A_{\mathcal{P}_{n+3}}(x) = x(1-x)A'_{\mathcal{P}_{n+2}}(x) + \frac{(n+1)x + 2}{2} A_{\mathcal{P}_{n+2}}(x) + \frac{4n}{n+3} x(1-x)A'_{\mathcal{P}_{n+1}}(x) \\
+ \frac{2n(n+1)}{n+3} x(1-x)A_{\mathcal{P}_{n+1}}(x) - \frac{4n}{n+3} x(1-x)[(n-1)x + 2] A'_{\mathcal{P}_n}(x) \\
- \frac{n(n+1)}{n+3} x[(n-1)x + 4] A_{\mathcal{P}_n}(x) - \frac{4n}{n+3} x^2 (1-x)^2 A''_{\mathcal{P}_n}(x)
\]

with the initial conditions \( A_{\mathcal{P}_3}(x) = 1 \) and \( A_{\mathcal{P}_4}(x) = 1 + x \).

Proof. By Theorem 4.2, we immediately obtain the desired results after simple computations.

Note that a sequence \( S_{n,i_1}, S_{n,i_2}, \ldots, S_{n,i_j} \) of elements in the poset \( \mathcal{P}_n \) forms a chain if and only if the monomial \( x_{i_1} x_{i_2} \cdots x_{i_j} \) is nonvanishing in the algebra \( B_{\mathcal{P}_n} \).

**Theorem 4.3** For any \( n \geq 3 \), the Hilbert series \( \text{Hilb} B_{\mathcal{P}_n}(x) \) of the algebra \( B_{\mathcal{P}_n} \) is

\[
\text{Hilb} B_{\mathcal{P}_n}(x) = 1 + \sum_{i=1}^{\lfloor \frac{1}{2}(n-1) \rfloor + 1} \sum_{j=1}^{n} \left( d_1, d_2, \ldots, d_{i+1} \right) \frac{2d_{i+1} - n}{n} x^i,
\]

where the second sum is over all \( (d_1, \ldots, d_{i+1}) \) such that \( \sum_{k=1}^{i+1} d_k = n \), \( d_1 \geq 0 \), \( d_k \geq 1 \) for all \( 2 \leq k \leq i \) and \( d_{i+1} \geq n - \lfloor \frac{n-1}{2} \rfloor \).

Proof. Note that the number of the chains \( S_{n,j_1} \prec S_{n,j_2} \prec \cdots \prec S_{n,j_i} \) in \( \mathcal{P}_n \) is equal to the dimension of \( B_{\mathcal{P}_n}^j \). Theorem 3.5 implies that \( \dim B_{\mathcal{P}_n}^j = \sum_{d_1, d_2, \ldots, d_{i+1}} \frac{2d_{i+1} - n}{n} \) for any
\[ i \geq 1, \text{ where the sum is over all } (d_1, \cdots, d_{k+1}) \text{ such that } \sum_{k=1}^{i+1} d_k = n, d_1 \geq 0, d_k \geq 1 \text{ for all } 2 \leq k \leq i \text{ and } d_{i+1} \geq n - \lfloor \frac{n-1}{2} \rfloor. \] This completes the proof. \qed

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