Variable control charts based on percentiles of the new Rayleigh-Pareto distribution

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Abstract
In this paper, we consider the New Rayleigh-Pareto distribution as a life time model. Based on the evaluated percentiles of sample estimates like sample mean, median, midrange, range and standard deviation, the control limits for the respective control charts are developed. The admissibility and power of the control limits are assessed in comparison with those on the popular Shewhart control limits.

Keywords: most probable, Pdf, Cdf, Equi-tailed, Percentiles, NRPD

1. Introduction
The well-known Shewhart control charts are developed under the assumption that the quality characteristic follows a normal distribution. If \( x_1, x_2, \ldots, x_n \) is a collection of observations of size \( n \) on a variable quality characteristic of a product and if \( t(x) = t \), a statistic is based on this sample, the control limits of Shewhart’s variable control chart are \( E(t) \pm 3S.E(t) \).

In quality control studies data is always in small samples only. Therefore if the population is not normal there is a need to develop separate procedure for the construction of control limits. In this paper we assume that the quality variate follows the new Rayleigh-Pareto model and develop control limits for such data on par with the presently available control limits. If a process quality characteristic is assumed to follow the new Rayleigh-Pareto distribution the online process of such a quality can be controlled through the theory of the new Rayleigh-Pareto distribution. In the absence of any such specification of the population model we generally use the normal distribution and the associated constants available in all standard textbooks of statistical quality control. However, normality is only an assumption that is rarely verified and found to be true. Unless the sample is very large in size this assumption may not be taken for granted without proper goodness of fit test procedure. At the same time central limit theorem cannot be made use of, because central limit theorem gives only asymptotic normality for any statistic. Therefore, if a distribution other than normal is a suitable model for a quality variate, separate procedures are to be developed. We present the construction of quality control charts when the process variate is assumed to follow the new Rayleigh-Pareto distribution. Let \( X \) be a random variable from a Pareto distribution with its cumulative distribution function (cdf) for \( x \geq \alpha \) given by

\[
F_1(x, \alpha, \beta) = 1 - \left( \frac{\alpha}{x} \right)^\beta
\]  

(1.1)

Where \( \alpha > 0 \) is a scale parameter and \( \beta > 0 \) is the shape parameter. The probability density function (pdf) corresponding to (1.1) is
The NRPD has a cdf of the form

\[ G(x) = \frac{1}{R(x)} \int f_2(x) \, dx \]  

(1.3)

Where \( R(x) \) the survival is function of the Pareto distribution and is given by

\[ R(x) = 1 - F_1(x, \alpha, \beta) \] Where \( f_2(x) \) is the pdf of a Rayleigh distribution and is given by

\[ f_2(x) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, x > 0, \sigma > 0 \]  

(1.4)

Using (1.3) and (1.4), and given that

\[ R(x) = \left( \frac{\alpha}{x} \right)^{\beta} \]

\[ \frac{1}{R(x)} = \left( \frac{x}{\alpha} \right)^{\beta} \]

The cdf of the NRPD is given by

\[ G(x) = \int_0^x \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \, dx \]  

(1.5)

\[ G(x) = 1 - e^{-\frac{1}{2\sigma^2} \left( \frac{x}{\alpha} \right)^2\beta} \]  

(1.6)

If we take \( 2\beta = \lambda \) then

\[ G(x) = 1 - e^{-\frac{1}{2\sigma^2} \left( \frac{x}{\alpha} \right)^{\lambda/2}} \]  

(1.7)

The probability density function (pdf) is given by

\[ g(x) = \frac{\lambda}{2\sigma^2 \alpha} x^{\lambda-1} e^{-\frac{1}{2\sigma^2} \left( \frac{x}{\alpha} \right)^{\lambda}} \]  

(1.8)

Where \( 0 < x < \infty, \lambda > 0, \alpha > 0, \sigma > 0 \)

The hazard function is given by

\[ h(x) = \frac{\lambda}{2\sigma^2 \alpha} x^{\lambda-1} \]  

(1.9)
From the hazard function the following can be observed:

1. If $\lambda = 1$, the failure rate is constant, which makes the NRDPD suitable for modeling systems or components with failure rate with time.
2. If $\lambda > 1$, the hazard is an increasing function, which makes the NRDPD suitable for modeling components that wears faster with time.
3. If $\lambda < 1$, the hazard is a decreasing function, which makes the NRDPD suitable for modeling components that wears slower with time. The distributional properties are:

$$Mean = E(x) = \alpha (2\sigma^2)^{1/2} \Gamma \left( \frac{\lambda + 1}{\lambda} \right)$$  \hspace{1cm} (1.10)

$$Median = \alpha (2\sigma^2)^{1/2} \ln(2)$$  \hspace{1cm} (1.11)

$$Variance = \alpha^2 (2\sigma^2)^{2} \Gamma \left( \frac{\lambda + 2}{\lambda} \right) - \left[ \alpha (2\sigma^2)^{1/2} \Gamma \left( \frac{\lambda + 1}{\lambda} \right) \right]^2$$  \hspace{1cm} (1.12)

The pdf of largest order statistics $X_{(n)}$ is given by

$$a_{(n)} = \frac{1}{2\sigma^2} \frac{n\lambda}{\alpha} \left( \frac{x}{\alpha} \right)^{\lambda - 1} e^{-\frac{1}{2\sigma^2} \left( \frac{x}{\alpha} \right)^2} \left( 1 - e^{-\frac{1}{2\sigma^2} \left( \frac{x}{\alpha} \right)^2} \right)^{n-1}$$  \hspace{1cm} (1.13)

The pdf of the smallest order statistic $X_{(1)}$ is given by

$$a_{(1)} = \frac{1}{2\sigma^2} \frac{n\lambda}{\alpha} \left( \frac{x}{\alpha} \right)^{\lambda - 1} e^{-\frac{1}{2\sigma^2} \left( \frac{x}{\alpha} \right)^2} \left( e^{-\frac{1}{2\sigma^2} \left( \frac{x}{\alpha} \right)^2} \right)^{n-1}$$  \hspace{1cm} (1.14)

The other distributional properties are thoroughly discussed by Nasiru and Luguterah (2015) [8]. Skewed distributions to develop statistical quality control methods are attempted by many authors. Some of them are Edge-man (1989) [13]– Inverse Gaussian Distribution, Gonzalez and Viles (2000) [6]– Gamma Distribution, Kantam and Sriram (2011) [5] – Gamma Distribution, Chan and Cui (2003) [2] have developed control chart constants for skewed distributions where the constants are dependent on the coefficient of skewness of the distributions, Kantam et al (2006) [6] – Log logistic Distribution, Betul and Yaziki (2006) [1] – Burr Distribution, Subba Rao and Kantam (2008) [11] –Double exponential distribution, Kantam and Rao (2010) [7] – control charts for process variate, Rao and Sarath Babu (2012) [9] - Linear failure rate distribution, Rao and Kantam (2012) [10] – Half logistic distribution, K. Rosaiyah, R.R.L. Kantam, B. Srinivas Rao (2012) [12] Variable control charts for Half logistic distribution, Srinivas Rao Boyapati, Suleman Nasiru, K.N.V.R. Lakshmi (2015) [13] – Variable Control Charts Based on Percentiles of the new Weibull-Pareto Distribution and references there in.

NRDPD is another situation of skewed distribution which is paid much attention with respect to development of control charts in the present study. If $\lambda = 0.5, \alpha = 2.5, \sigma = 1.5$

Then the hazard function indicates a decreasing failure rate function (shown in the graph), which makes the NRPD suitable for modeling components that wears slower with time. At the same time it is one of the probability models applicable for life testing and reliability studies also. Accordingly, if a lifetime data is considered as a quality data, development of control charts for the statistics average, median, midrange, range and standard deviation are presented in Section 2. The comparative study to the developed control limits in relation to the Shewarts limits is given in Section 3. Summary and conclusions are given in Section 4.
2. Control chart constants through percentiles

2.1 Mean- chart

Let \( X_1, X_2, X_3, \ldots, X_n \) be a random sample of size \( n \) supposed to have been drawn from \( \text{NRPD} \) with \( \lambda = 0.5, \alpha = 2.5, \sigma = 1.5 \). This is considered as a subgroup of an industrial process data with a targeted population average, under repeated sampling the statistic \( \bar{X} \) gives whether the process average is around the targeted mean or not. Statistically speaking, we have to find the ‘most probable’ limits with in which \( \bar{X} \) fails. Here the phrase ‘most probable’ is a relative concept which is to be considered in the population sense. As the existing procedures are for normal distribution only, the concept of \( 3\sigma \) control limits is taken as the ‘most probable’ limits. It is well known that \( 3\sigma \) limits of normal distribution include \( 99.73\% \) of probability. Hence, we have to search two limits of the sampling distribution of sample mean in \( \text{NRPD} \) such that the probability content of those limits is \( 0.9973 \). Symbolically we have to find Lower control limit and Upper control limit (L and U) such that

\[
P(L \leq \bar{X} \leq U) = 0.9973
\]

where \( \bar{X} \) is the mean of the sample size \( n \), taking the equi-tailed concept \( L, U \) are respectively \( 0.00135 \) and \( 0.99865 \) percentiles of the sampling distribution of \( \bar{X} \). We resorted to the empirical sampling distribution of \( \bar{X} \) through simulation there by computing its percentiles. These are given in Table 1.

| \( n \) | 0.99865  | 0.9950  | 0.99  | 0.975  | 0.95  | 0.05  | 0.025  | 0.01  | 0.005  | 0.00135 |
|-------|--------|--------|-------|--------|-------|-------|--------|-------|--------|----------|
| 2     | 937.0283 | 603.3082 | 421.3133 | 259.4023 | 169.0804 | 1.3431 | 1.0594 | 0.8373 | 0.7396  | 0.6516   |
| 3     | 782.0261 | 511.9547 | 372.9478 | 242.7141 | 168.1190 | 1.7809 | 1.4531 | 1.1678 | 1.0266  | 0.8415   |
| 4     | 589.4225 | 384.7379 | 322.5086 | 224.1744 | 149.9244 | 2.2151 | 1.8453 | 1.4825 | 1.2855  | 0.9882   |
| 5     | 537.5703 | 378.1364 | 287.9457 | 203.8294 | 142.3157 | 2.4629 | 2.0727 | 1.7376 | 1.5179  | 1.2191   |
| 6     | 471.1949 | 356.6365 | 287.1515 | 201.9097 | 145.4508 | 2.7202 | 2.2763 | 1.8833 | 1.7130  | 1.3786   |
| 7     | 387.7605 | 292.0844 | 251.6923 | 175.7043 | 132.4924 | 2.8974 | 2.5385 | 2.1651 | 1.9053  | 1.6153   |
| 8     | 370.1256 | 286.4856 | 242.0783 | 174.9430 | 129.3269 | 3.0790 | 2.6537 | 2.2783 | 2.0502  | 1.6440   |
| 9     | 360.4601 | 279.7247 | 227.6512 | 163.3620 | 124.6324 | 3.3575 | 2.9208 | 2.4770 | 2.2820  | 1.9559   |
| 10    | 322.0967 | 248.1250 | 209.1331 | 154.1960 | 118.2283 | 3.4778 | 3.0259 | 2.5991 | 2.3885  | 2.0633   |

The percentiles in the above table are used in the following manner to get the control limits for sample mean. From the distribution of \( \bar{X} \), consider
\[ P(Z_{0.00135} \leq \bar{x} \leq Z_{0.99865}) = 0.9973 \]  
(2.2)

But \( \bar{x} \) of sampling distribution when \( \lambda = 0.5, \alpha = 2.5, \sigma = 1.5 \) is 101.25 for NRPD. From equation (2.2) over repeated sampling for the \( i^{th} \) subgroup mean we can have

\[ P(Z_{0.00135} \leq \bar{x}^i \leq Z_{0.99865} \leq \frac{Z_{0.9973}}{101.25}) = 0.9973 \]  
(2.3)

This can be written as

\[ P(A_{2p}^* \bar{x} \leq x^i \leq A_{2p}^{**} \bar{x}) = 0.9973 \]  
(2.4)

where \( \bar{x} \) is grand mean, \( x^i \) is \( i^{th} \) subgroup mean, \( A_{2p}^*, A_{2p}^{**} \) are the percentile constants of \( \bar{x} \) chart for NRPD are given in Table 2.

### Table 2: Percentile constants of Mean-chart

| \( n \) | \( A_{2p}^* \) | \( A_{2p}^{**} \) |
|-------|---------|---------|
| 2     | 9.2546  | 0.0064  |
| 3     | 7.7237  | 0.0083  |
| 4     | 5.8215  | 0.0098  |
| 5     | 5.3093  | 0.0120  |
| 6     | 4.6538  | 0.0136  |
| 7     | 3.8297  | 0.0160  |
| 8     | 3.6556  | 0.0162  |
| 9     | 3.5601  | 0.0193  |
| 10    | 3.1812  | 0.0204  |

#### 2.2 Median –chart

We have to search two limits of the sampling distribution of sample median in NRPD such that the probability content of these limits is 0.9973. Symbolically, we have to find \( L, U \) such that

\[ P(L \leq m \leq U) = 0.9973 \]  
(2.5)

Where \( m \) is the median of sample size \( n \). Through simulation, the percentiles observed are given Table 3.

### Table 3: Percentiles of Median in NRPD \( \lambda = 0.5, \alpha = 2.5, \sigma = 1.5 \)

| \( n \) | 0.99865 | 0.9950 | 0.99 | 0.975 | 0.95 | 0.05 | 0.025 | 0.01 | 0.005 | 0.00135 |
|-------|---------|--------|-------|-------|------|------|-------|------|--------|---------|
| 2     | 937.0293| 603.3082| 421.3133| 259.4023| 169.0804| 1.3431| 1.0594| 0.8373| 0.7396| 0.6516 |
| 3     | 367.4579| 207.8956| 153.9977| 83.1014| 43.7745| 1.1373| 0.9408| 0.7855| 0.7096| 0.6229 |
| 4     | 274.8942| 170.9696| 121.0930| 71.0598| 42.0982| 1.5643| 1.2796| 1.0370| 0.9183| 0.8173 |
| 5     | 156.6554| 99.6844| 71.3429| 36.1589| 17.9403| 1.4213| 1.1746| 0.9687| 0.8789| 0.7636 |
| 6     | 163.9502| 99.6844| 71.3429| 35.0793| 20.5185| 1.7022| 1.4442| 1.1941| 1.0439| 0.9102 |
| 7     | 87.8772| 59.3705| 38.7942| 17.8748| 7.9261| 1.6333| 1.3762| 1.1350| 1.0303| 0.8777 |
| 8     | 87.6519| 59.2621| 36.5253| 19.9822| 11.2653| 1.8928| 1.6400| 1.3435| 1.2058| 1.0439 |
| 9     | 65.2243| 37.0454| 24.6618| 11.832| 7.5873| 1.8467| 1.5664| 1.3577| 1.2217| 1.0183 |
| 10    | 45.6535| 29.6765| 21.4937| 12.0055| 7.7147| 2.0226| 1.7529| 1.4800| 1.3324| 1.1516 |

The percentiles in the above table are used in the following manner to get the control limits for median. From the distribution of \( m \), consider

\[ P(Z_{0.00135} \leq m \leq Z_{0.99865}) = 0.9973 \]  
(2.6)

But median of sampling distribution when \( \lambda = 0.5, \alpha = 2.5, \sigma = 1.5 \) is 24.3229 for NRPD. From equation (2.6) over repeated sampling for the \( i^{th} \) subgroup median we can have
This can be written as

\[ P(A_{\bar{m}} \leq m_{i} \leq A_{\bar{m}}) = 0.9973 \]

Where \( \bar{m} \) is the mean of subgroup medians. Thus \( A_{\bar{m}} = \frac{Z_{\alpha/2}}{24.3229} \) and \( A_{\bar{m}} = \frac{Z_{\alpha/2}}{24.3229} \) are the percentile constants of median chart and are given in Table 4.

### Table 4: Percentile constants of Median chart

| \( n \) | \( A_{\bar{m}} \) | \( A_{\bar{m}}^{*} \) |
|--------|-----------------|---------------------|
| 2      | 259.0804        | 1.3431              |
| 3      | 228.9093        | 1.8471              |
| 4      | 264.0914        | 2.4392              |
| 5      | 301.3386        | 2.703               |
| 6      | 338.8706        | 3.237               |
| 7      | 370.3837        | 3.703               |
| 8      | 338.8706        | 3.237               |
| 9      | 301.3386        | 2.703               |
| 10     | 264.0914        | 2.4392              |

### 2.3 Midrange chart

We have to search two limits of the sampling distribution of sample midrange in NRPD such that the probability content of these limits is 0.9973. Symbolically, we have to find \( L, U \) such that

\[ P(L \leq M \leq U) = 0.9973 \]

Where \( M \) is the midrange of sample size \( n \). Through simulation, the percentiles observed are given Table 5.

### Table 5: Percentiles of Midrange in NRPD

| \( n \) | 0.99865 | 0.9950 | 0.99 | 0.975 | 0.95 | 0.05 | 0.025 | 0.01 | 0.005 | 0.00135 |
|--------|--------|-------|------|------|-----|-----|-------|-----|-------|---------|
| 2      | 937.0283 | 603.3082 | 421.3133 | 259.4023 | 109.0804 | 1.3431 | 1.0594 | 0.8373 | 0.7396 | 0.6516 |
| 3      | 1093.3837 | 700.5959 | 523.8824 | 345.0097 | 228.9093 | 1.8471 | 1.4812 | 1.2030 | 1.0363 | 0.8673 |
| 4      | 1145.4553 | 735.9828 | 578.766 | 388.8706 | 264.0914 | 2.4392 | 2.0184 | 1.6170 | 1.3666 | 1.0200 |
| 5      | 1208.1923 | 897.4734 | 712.011 | 519.4619 | 369.1217 | 3.1737 | 2.7137 | 2.3703 | 2.0184 | 1.6170 |
| 6      | 1245.2709 | 921.3837 | 758.766 | 587.766 | 388.8706 | 3.8231 | 3.3703 | 3.0184 | 2.6170 | 2.2000 |
| 7      | 1288.8665 | 979.2936 | 801.7608 | 638.4606 | 438.5595 | 4.5823 | 4.1478 | 3.8231 | 3.4184 | 3.0184 |
| 8      | 1336.1149 | 1049.3754 | 850.1713 | 692.8528 | 523.8824 | 5.3418 | 4.9178 | 4.5823 | 4.1478 | 3.8231 |
| 9      | 1462.7422 | 1099.3754 | 909.8843 | 748.2828 | 609.2828 | 6.1478 | 5.7178 | 5.3418 | 4.9178 | 4.5823 |
| 10     | 1462.7408 | 1029.7532 | 835.6108 | 712.011 | 519.4619 | 6.9478 | 6.5178 | 6.1478 | 5.7178 | 5.3418 |

The percentiles in the above table are used in the following manner to get the control limits for midrange. From the distribution of \( M \), consider

\[ P(Z_{0.00135} \leq M \leq Z_{0.99865}) = 0.9973 \]

The midrange value of \( NRDP \) calculated by using \( \alpha_{(i)} \) and \( \alpha_{(n)} \).

From equation (2.10) for \( i^{th} \) subgroup midrange we have,

\[ P\left(Z_{0.00135} \leq \frac{M}{\alpha_{(i)} + \alpha_{(n)}} \leq Z_{0.99865} \frac{M}{\alpha_{(i)} + \alpha_{(n)}} \right) = 0.9973 \]

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This can be written as

\[ P(A^*_{4p} \leq M_i \leq A^{**}_{4p}) = 0.9973 \]  

(2.12)

Where \( M \) is mean of midranges. Thus \( A^*_{4p} = \frac{2Z_{0.00135}}{\alpha_{(1)} + \alpha_{(n)}} \) and \( A^{**}_{4p} = \frac{2Z_{0.99865}}{\alpha_{(1)} + \alpha_{(n)}} \) are the percentile constants of midrange chart for NRPD process data given in Table 6.

**Table 6:** Percentile constants of Midrange-chart \( \lambda = 0.5, \alpha = 2.5, \sigma = 1.5 \)

| \( n \) | \( A^*_{4p} \) | \( A^{**}_{4p} \) |
|---|---|---|
| 2 | 1778288.5000 | 7.9099 |
| 3 | 177489.9844 | 8.9634 |
| 4 | 41951.5195 | 8.5190 |
| 5 | 39074.0117 | 10.2136 |
| 6 | 35994.1680 | 10.5302 |
| 7 | 34319.5859 | 12.5940 |
| 8 | 30761.6406 | 11.9830 |
| 9 | 31877.4219 | 15.3548 |
| 10 | 34790.0664 | 14.5335 |

2.4 R-chart

We have to search two limits of the sampling distribution of sample Range in NRPD such that the probability content of these limits is 0.9973. Symbolically,

\[ P(L \leq R \leq U) = 0.9973 \]  

(2.13)

Where \( R \) is the range of sample of size \( n \). Through simulation, the percentiles observed are given in Table 7.

**Table 7:** Percentiles of Range in NRPD \( \lambda = 0.5, \alpha = 2.5, \sigma = 1.5 \)

| \( n \) | 0.99865 | 0.9950 | 0.99 | 0.975 | 0.95 | 0.90 | 0.05 | 0.025 | 0.01 | 0.005 | 0.00135 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1792.5670 | 1097.8423 | 787.0557 | 478.6137 | 298.3763 | 0.2700 | 0.1380 | 0.0561 | 0.0281 | 0.0099 |
| 3 | 2185.3918 | 1396.7192 | 1037.4709 | 681.5677 | 451.0629 | 1.2602 | 0.8466 | 0.5897 | 0.4103 | 0.2232 |
| 4 | 2288.7515 | 1465.2440 | 1138.2272 | 773.5310 | 524.2709 | 2.4422 | 1.8469 | 1.2687 | 1.0216 | 0.5725 |
| 5 | 2488.5742 | 1651.6889 | 1274.4357 | 875.2571 | 598.8456 | 3.3360 | 2.6645 | 1.9599 | 1.5663 | 1.0496 |
| 6 | 2571.8535 | 1840.6733 | 1420.4451 | 1036.4128 | 719.2916 | 4.2502 | 3.4574 | 2.5311 | 2.1478 | 1.2769 |
| 7 | 2414.5769 | 1792.2539 | 1453.9717 | 1036.4434 | 736.4628 | 4.9057 | 4.1473 | 3.2646 | 2.8039 | 2.1148 |
| 8 | 2670.9036 | 1954.6270 | 1601.9783 | 1106.9457 | 805.3129 | 5.8480 | 4.6852 | 3.7484 | 3.3060 | 2.4507 |
| 9 | 2923.9709 | 2094.4844 | 1618.1107 | 1150.1475 | 839.5650 | 6.1736 | 5.3418 | 4.3476 | 3.8054 | 3.1797 |
| 10 | 2923.1133 | 2058.1680 | 1669.2516 | 1167.3319 | 867.7080 | 6.5655 | 5.7265 | 4.7892 | 4.3487 | 3.5724 |

The percentiles in the above table are used in the following manner to get the control limits for sample range. From the distribution of \( R \), consider

\[ P(Z_{0.00135} \leq R \leq Z_{0.99865}) = 0.9973 \]  

(2.14)

From equation (2.14), for the \( i \)th subgroup range we can have

\[ P(Z_{0.00135} \leq \frac{\overline{R}}{\alpha_{(n)} - \alpha_{(1)}} \leq \overline{R} \leq Z_{0.99865} \frac{\overline{R}}{\alpha_{(n)} - \alpha_{(1)}}) = 0.9973 \]  

(2.15)

This can be written as

\[ P(D_{3p}^* \leq \overline{R} \leq D_{4p}^* \overline{R}) = 0.9973 \]  

(2.16)

Where \( \overline{R} \) mean of is ranges, \( R_i \) is \( i \)th subgroup range. Thus
\[ D^*_{3p} = \frac{Z_{0.00135}}{\alpha(n) - \alpha(1)} , D^*_{4p} = \frac{Z_{0.99865}}{\alpha(n) - \alpha(1)} \]

Are the percentile constants of R chart for NRPD process data and are given in Table 8.

### Table 8: Percentile constants of Range -chart

| \(n\) | \(D^*_{3p}\) | \(D^*_{4p}\) |
|-------|-------------|-------------|
| 2     | 4.533       | 0.0006      |
| 3     | 6.2356      | 0.0018      |
| 4     | 7.8456      | 0.003       |
| 5     | 9.2323      | 0.0043      |
| 6     | 10.1235     | 0.0055      |
| 7     | 11.3546     | 0.0071      |
| 8     | 11.9542     | 0.0084      |
| 9     | 12.0135     | 0.0092      |
| 10    | 12.8594     | 0.0109      |

### 2.5 \(\sigma\) - chart

We have to search two limits of the sampling distribution of sample standard deviation in NRPD such that the probability content of these limits is 0.9973. Symbolically, we have to find \(L, U\) such that

\[ P(L \leq s \leq U) = 0.9973 \]  

(2.17)

Where \(s\) is the standard deviation of sample of size \(n\). Through simulation the percentiles observed are given in Table 9.

### Table 9: Percentiles of Standard deviation in NRPD \(\lambda = 0.5, \alpha = 2.5, \sigma = 1.5\)

| \(\ell\) | 0.99865 | 0.9950 | 0.99 | 0.975 | 0.95 | 0.05 | 0.025 | 0.01 | 0.005 | 0.00135 |
|-------|---------|--------|------|-------|------|------|-------|------|-------|---------|
| 2     | 986.2835| 548.9211| 393.5278| 239.3069| 149.1881| 0.1350| 0.0690| 0.0280| 0.0141| 0.0049 |
| 3     | 1008.4962| 642.0943| 487.2334| 329.3040| 222.4091| 0.5404| 0.3700| 0.2486| 0.1728| 0.0974 |
| 4     | 958.7794| 633.1101| 491.8904| 347.8533| 239.3069| 1.2378| 0.9797| 0.7185| 0.5739| 0.4011 |
| 5     | 983.8661| 667.9033| 503.5619| 347.8533| 238.0112| 1.6861| 1.2169| 0.9015| 0.7542| 0.5053 |
| 6     | 957.1638| 673.3813| 532.8484| 381.5732| 267.2060| 1.8449| 1.5787| 1.2554| 1.0622| 0.7916 |
| 7     | 829.3943| 618.4649| 507.6351| 358.9405| 256.3640| 2.0318| 1.7293| 1.4351| 1.2685| 1.0009 |
| 8     | 874.1238| 650.5952| 532.8484| 381.5732| 267.2060| 2.1569| 1.8633| 1.5572| 1.3773| 1.283  |

The percentiles in the above table are used in the following manner to get the control limits for sample standard deviation. From the distribution of \(s\), consider

\[ P(Z_{0.00135} \leq s \leq Z_{0.99865}) = 0.9973 \]  

(2.18)

But standard deviation of sampling distribution when \(\lambda = 0.5, \alpha = 2.5, \sigma = 1.5\) is 226.4018 for NRPD. From equation (2.18), for the \(i^{th}\) subgroup standard deviation we can have

\[ P(Z_{0.00135} \leq \overline{s} \leq Z_{0.99865} \overline{s}) = 0.9973 \]

This can be written as

\[ P(B^*_{3p} \overline{s} \leq \overline{s} \leq B^*_{4p} \overline{s}) = 0.9973 \]

(2.20)

Where \(\overline{s}\) is mean of standard deviation, \(s_i\) is \(i^{th}\) subgroup standard deviation. Thus \(B^*_{3p} = \frac{Z_{0.00135}}{226.4018}, B^*_{4p} = \frac{Z_{0.99865}}{226.4018}\) are the constants of standard deviation chart for NRPD process data given in Table 10.
3. Comparative study
The control chart constants for the statistics mean, median, midrange, range and standard deviation developed in section 2 are based on the population described by NRPD. In order to use this for a data, the data is confirmed to follow NRPD. Therefore the power of the control limits can be accessed through their application for a true NRPD data in relation to the application for Shewhart limits. With this back drop we have made this comparative study simulating random samples of size \( n = 2, 3, \ldots, 10 \) from NRPD and calculated the control limits using the constants of section 2 for mean, median, midrange, range and standard deviation in succession. The number of statistic values that have fallen within the respective control limits is evaluated and is named as NRPD coverage probability. Similar count for control limits using Shewhart constants available in quality control manuals are also calculated. These are named as Shewhart coverage probability. The coverage probabilities under the two schemes namely true NRPD, Shewhart limits are presented in the following Tables 11, 12, 13, 14 and 15.

| \( n \) | \( B_{3p} \) | \( B_{4p} \) |
|-------|--------|--------|
| 2     | 3.9588 | 0.0000 |
| 3     | 4.4545 | 0.0004 |
| 4     | 4.2349 | 0.0010 |
| 5     | 4.3457 | 0.0018 |
| 6     | 4.2277 | 0.0022 |
| 7     | 3.6634 | 0.0031 |
| 8     | 3.8609 | 0.0035 |
| 9     | 4.0513 | 0.0044 |
| 10    | 3.8281 | 0.0050 |

Table 10: Percentile constants of SD-chart \( \lambda = 0.5, \alpha = 2.5, \sigma = 1.5 \)

Table 11: Coverage Probabilities of Mean-chart \( \lambda = 0.5, \alpha = 2.5, \sigma = 1.5 \)

| \( n \) | \( \overline{x} - A_p \overline{R} \) | \( \overline{x} + A_p \overline{R} \) | Coverage Probability | \( A_{3p} \times \overline{x} \) | \( A_{4p} \times \overline{x} \) | Coverage probability |
|-------|----------------------------------|----------------------------------|---------------------|---------------------|---------------------|---------------------|
| 2     | 34.47385805                     | 0.7937                           | 0.2180096          | 315.2489644         | 0.9821              |
| 3     | 36.18465349                     | 0.7676                           | 0.297804           | 277.126356          | 0.9815              |
| 4     | 35.23091126                     | 0.7394                           | 0.3428138          | 203.6418915         | 0.9705              |
| 5     | 36.33082484                     | 0.7141                           | 0.432972           | 191.564533          | 0.9726              |
| 6     | 37.32961052                     | 0.7048                           | 0.5043696          | 172.5908268         | 0.9653              |
| 7     | 35.85376296                     | 0.6979                           | 0.56984            | 136.3947655         | 0.9532              |
| 8     | 36.7135126                      | 0.6894                           | 0.5911866          | 133.4038108         | 0.9526              |
| 9     | 37.07659153                     | 0.6765                           | 0.7109541          | 131.1434037         | 0.9558              |
| 10    | 36.4656906                      | 0.6702                           | 0.7392552          | 115.2803256         | 0.9467              |

Table 12: Coverage Probabilities of Median-chart \( \lambda = 0.5, \alpha = 2.5, \sigma = 1.5 \)

| \( n \) | \( \overline{m} - A_p \overline{R} \) | \( \overline{m} + A_p \overline{R} \) | Coverage Probability | \( A_{3p} \times \overline{m} \) | \( A_{4p} \times \overline{m} \) | Coverage probability |
|-------|----------------------------------|----------------------------------|---------------------|---------------------|---------------------|---------------------|
| 2     | 35.78028058                      | 0.7989                           | 0.9129152          | 1312.298568         | 0.9847              |
| 3     | 11.22331383                      | 0.8826                           | 0.2796774          | 165.0479268         | 0.9913              |
| 4     | 10.8311933                      | 0.8368                           | 0.3544464          | 119.2237431         | 0.9896              |
| 5     | 6.746066988                      | 0.8326                           | 0.2075257          | 42.56723037         | 0.9804              |
| 6     | 6.964670026                      | 0.8495                           | 0.2549483          | 45.94932208         | 0.9833              |
| 7     | 5.376148463                      | 0.7196                           | 0.1904969          | 19.06491201         | 0.9765              |
| 8     | 5.424272671                      | 0.7477                           | 0.22803495         | 19.15546735         | 0.9737              |
| 9     | 4.929939564                      | 0.6616                           | 0.20279181         | 12.97867584         | 0.9777              |
| 10    | 4.840161332                      | 0.6581                           | 0.22449053         | 8.9084297           | 0.9594              |

Table 13: Coverage Probabilities of Mid-range chart \( \lambda = 0.5, \alpha = 2.5, \sigma = 1.5 \)

| \( n \) | \( M - A_p \overline{R} \) | \( M + A_p \overline{R} \) | Coverage Probability | \( A_{4p} \times \overline{M} \) | \( A_{4p} \times \overline{M} \) | Coverage probability |
|-------|--------------------------|--------------------------|---------------------|---------------------|---------------------|---------------------|
| 2     | 34.1472967               | 1.0000                   | 0.0374704          | 605.75619.46        | 1.0000              |
| 3     | 48.49931028              | 1.0000                   | 0.11122455         | 858.314919         | 1.0000              |
| 4     | 59.5801592               | 1.0000                   | 0.20200386         | 249.2461.433       | 1.0000              |
| 5     | 73.48328431              | 1.0000                   | 0.3421954         | 286.1710.284       | 1.0000              |
| 6     | 88.95753649              | 0.9834                   | 0.5147744         | 319.1538.888       | 1.0000              |
| 7     | 94.06934444              | 0.9714                   | 0.5626572         | 321.8360.351       | 1.0000              |
Table 14: Coverage probabilities of Range-chart $\lambda = 0.5, \alpha = 2.5, \sigma = 1.5$

| Shewhart limits | Percentile limits |
|-----------------|-------------------|
| $n$  | $D_3 \bar{R}$ | $D_4 \bar{R}$ | Coverage Probability | $D_{ip} \times \bar{R}$ | $D_{ip} \times \bar{R}$ | Coverage probability |
| 2   | 0            | 188.894673 | 0.9168  | 0.0346914 | 262.1108727 | 0.9360  |
| 3   | 0            | 237.0339   | 0.8911  | 0.1650936 | 573.999512  | 0.9656  |
| 4   | 0            | 263.324544 | 0.8758  | 0.346176  | 905.3194752 | 0.9813  |
| 5   | 0            | 303.62094  | 0.8611  | 0.6172908 | 1325.352059 | 0.9910  |
| 6   | 0            | 350.17395  | 0.8594  | 0.9610562 | 1768.955081 | 0.9936  |
| 7   | 14.0745236   | 356.3076764| 0.6237  | 1.3148568 | 2102.770864 | 0.9974  |
| 8   | 28.5835824   | 391.7632176| 0.5917  | 1.7654565 | 2512.454858 | 0.9978  |
| 9   | 42.761508    | 422.037492 | 0.5618  | 2.1380754 | 2791.931393 | 0.9983  |
| 10  | 54.9597995   | 437.9532005| 0.5561  | 2.6863758 | 3169.282716 | 0.9991  |

Table 15: Coverage probabilities of SD-chart $\lambda = 0.5, \alpha = 2.5, \sigma = 1.5$

| Shewhart limits | Percentile limits |
|-----------------|-------------------|
| $n$  | $B_3 \bar{S}$ | $B_4 \bar{S}$ | Coverage Probability | $B_{ip} \times \bar{S}$ | $B_{ip} \times \bar{S}$ | Coverage probability |
| 2   | 0            | 94.4476632 | 0.9168  | 0.0168952 | 188.149171 | 0.9419  |
| 3   | 0            | 108.467184 | 0.8919  | 0.0488547 | 206.894769 | 0.9434  |
| 4   | 0            | 110.7047502| 0.8749  | 0.10190556| 246.0283289| 0.9529  |
| 5   | 0            | 118.2670638| 0.8602  | 0.14229402| 273.4438311| 0.9522  |
| 6   | 1.940373    | 127.417827 | 0.7482  | 0.20070573| 237.235562 | 0.9419  |
| 7   | 7.6414794   | 121.8751206| 0.5609  | 0.3259256 | 300.0959962| 0.9615  |
| 8   | 12.975234   | 127.297566 | 0.5319  | 0.4254774 | 270.7896268| 0.9504  |
| 9   | 17.703686   | 130.444314 | 0.5017  | 0.577801  | 289.250016 | 0.9581  |
| 10  | 21.4590968  | 129.6613032| 0.5016  | 0.777801  | 289.250016 | 0.9581  |

4. Summary and Conclusions
The Tables 11, 12, 13, 14 and 15 show that for a true NRPD if the Shewhart limits are used in a mechanical way it would result in less confidence coefficient about the decision of process variation for mean, median, midrange, range and standard deviation charts. Hence if a data is confirmed to follow NRPD, the usage of Shewhart constants in all the above charts is not advisable and exclusive evaluation and application of NRPD constants is preferable in statistical quality control.

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