Range of the first three eigenvalues of the planar Dirichlet Laplacian

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Abstract

We conduct extensive numerical experiments aimed at finding the admissible range of the ratios of the first three eigenvalues of a planar Dirichlet Laplacian. The results improve the previously known theoretical estimates of M Ashbaugh and R Benguria. We also prove some properties of a maximizer of the ratio $\lambda_3/\lambda_1$.

1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. We consider the eigenvalue problem for the Dirichlet Laplacian,

$$-\Delta u = \lambda u \quad \text{in} \quad \Omega,$$

$$u|_{\partial \Omega} = 0.$$  

(1.1)

(1.2)

Let us denote the eigenvalues by $\lambda_1(\Omega)$, $\lambda_2(\Omega)$, $\ldots$, (we will sometimes omit explicit dependence on $\Omega$ when speaking about a generic domain), where $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$. The corresponding orthonormal basis of real eigenfunctions will be denoted $\{u_j\}_{j=1}^\infty$.

For the last fifty years, the problem of obtaining a priori estimates of the eigenvalues and their ratios has attracted a substantial attention. The existing results can be roughly divided into two groups — universal estimates,
valid, as the name suggests, for all eigenvalues and all the domains in $\mathbb{R}^n$, which do not take into account any geometric information, and isoperimetric estimates for low eigenvalues. We briefly survey some known results below; the reader is referred to the very detailed survey paper [Ash] and references therein for a full discussion.

**Universal estimates**

Probably the first, and best known, estimate of this type is the Payne-Pólya-Weinberger inequality [PPW],

$$\lambda_{m+1} \leq \lambda_m + \frac{4}{mn} \sum_{j=1}^{m} \lambda_j. \quad (PPW)$$

This was subsequently improved by Hile and Protter [HiPr], and, in 1990’s, by Hong Cang Yang [Yan], whose implicit estimate

$$\sum_{j=1}^{m} (\lambda_{m+1} - \lambda_j) \left( \lambda_{m+1} - \left(1 + \frac{4}{n}\right) \lambda_j \right) \leq 0, \quad (HCY)$$

remains the best universal estimate so far for the eigenvalues of the Dirichlet Laplacian.

The general method of obtaining (PPW) and (HCY), as well as of similar estimates for a variety of other operators, has been the use of variational principles with some ingenious choices of trial functions, see [Ash]. Recently, an alternative abstract scheme, based on the so-called commutator trace identities, which easily implies, in particular, (PPW) and (HCY), has been developed in [LevPar], see also [HarStu].

By their very nature, the universal estimates are generically non-sharp.

**Isoperimetric estimates**

Both (PPW) and (HCY) give, for $m = 1$, the estimate

$$\frac{\lambda_2}{\lambda_1} \leq 1 + \frac{4}{n}.$$

This upper bound cannot, in fact, be attained. Already, Payne, Pólya, and Weinberger conjectured that the actual optimal upper bound on the ratio of the first two eigenvalues of the Dirichlet Laplacian is

$$\frac{\lambda_2}{\lambda_1}(\Omega) \leq \frac{\lambda_2}{\lambda_1} \bigg|_{\text{n-dimensional ball}} = \frac{j_{n/2,1}^2}{j_{n/2-1,1}^2} =: K_n \quad \text{for } \Omega \subset \mathbb{R}^n \quad (AB_0)$$

(here $j_{p,q}$ stands for the $q$-th zero of the Bessel function $J_p(\rho)$, so, in a planar case $n = 2$, $K_2 \approx 2.5387$ compared with [PPW] bound $\frac{\lambda_2}{\lambda_1}(\Omega) \bigg|_{\Omega \subset \mathbb{R}^2} \leq 3$).
Conjecture (AB$_0$) was eventually proved, only in the early 1990s, by Ashbaugh and Benguria [AshBen1, AshBen3], using, in particular, symmetrization techniques going back to the Faber-Krahn inequality,

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*) ,$$

where $\Omega^*$ is an $n$-dimensional ball of the same volume as $\Omega$.

We would like to mention, in this context, extensive computational experiments designed to verify (AB$_0$) by Haeberly [Hae, HaeOve].

**Statement of the problem**

As mentioned, (AB$_0$) gives the full description of the range of the possible values of ratio of the first two eigenvalues of the Dirichlet Laplacian, $\frac{\lambda_2}{\lambda_1}$ for domains in the Euclidean space (the obvious lower bound is $\frac{\lambda_2}{\lambda_1} \geq 1$). In fact, similar results were also obtained for domains in $S^n$ and $\mathbb{H}^n$. A natural extension would be to find optimal upper bounds on the range of the ratios of the first three eigenvalues of the Dirichlet Laplacian, $\left( \frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1} \right)$, in particular for planar domains. In other words, we would like to find, for $x := \frac{\lambda_2}{\lambda_1}$ and $y := \frac{\lambda_3}{\lambda_1}$, the function

$$y^*(x) := \max_{\Omega \subset \mathbb{R}^2, \frac{\lambda_2}{\lambda_1}(\Omega) = x} \frac{\lambda_3}{\lambda_1}(\Omega)$$  \hspace{1cm} (1.3)

and the number

$$Y^* := \max_{x \in [1, K^2]} y^*(x) = \max_{\Omega \subset \mathbb{R}^2} \frac{\lambda_3}{\lambda_1}(\Omega)$$  \hspace{1cm} (1.4)

or their best possible estimates. We will use notation (1.3) and (1.4) when looking for maxima in particular classes of domains as well.

Despite an apparent simplicity of this problem, and a wide attention it has attracted, it turned out to be rather difficult. In [AshBen5, AshBen6], Ashbaugh and Benguria proved a complicated upper bound for $y^*(x)$ and also demonstrated that

$$3.1818 \approx \frac{35}{11} \leq Y^* \leq 3.83103 .$$  \hspace{1cm} (1.5)

Their estimates improve upon previous results due to Payne, Pólya, and Weinberger, Brands, de Vries, Hile and Protter, Marcellini, Chiti, Hong Cang Yang, and themselves; see [AshBen5, AshBen6] and their earlier papers [AshBen2, AshBen4] for extensive bibliography and details of proofs. We present their estimates and other known facts in the next Section; just note...
at the moment that the lower bound in (1.5) is attained when Ω is the rectangle \( R_a := [0, 1] \times [0, a] \) with \( a = \sqrt{\frac{8}{3}} \).

In the current paper, we describe extensive numerical experiments aimed at improving (1.5). We also show, using perturbation techniques, that the rectangle \( R_{\sqrt{\frac{8}{3}}} \) does not maximize the ratio \( \frac{\lambda_3}{\lambda_1} \) and indicate a class of domains among which a possible maximizer could be found.

Acknowledgements

The original suggestion to conduct numerical experiments on the range of \( \left( \frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1} \right) \) for planar domains came from Brian Davies; we would like to thank him, as well as Mark Ashbaugh and Leonid Parnovski, for valuable discussions and advice. It is the authors, nevertheless, who take full responsibility for the realization of this idea, and any criticism for possible shortcomings of this realization should be addressed to them.

2 Known results for the range of \( \left( \frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1} \right) \) for planar domains

Explicit solutions

The spectral problem (1.1), (1.2) admits a full solution by separation of variables when Ω is, for example, a disjoint union of a number of rectangles or circles. For a reference, we collect below the results on the range of \( \left( \frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1} \right) \) in these cases.

Rectangles Let \( R_a := [0, 1] \times [0, a] \) be a rectangle with the side ratio \( a \); without loss of generality \( a \geq 1 \). Then

\[
\frac{\lambda_2}{\lambda_1}(R_a) = \frac{a^2 + 4}{a^2 + 1}
\]

and

\[
\frac{\lambda_3}{\lambda_1}(R_a) = \begin{cases} 
\frac{a^2 + 9}{a^2 + 1} & \text{for } a \geq \sqrt{\frac{8}{3}}, \\
\frac{4a^2 + 1}{a^2 + 1} & \text{for } 1 \leq a \leq \sqrt{\frac{8}{3}}.
\end{cases}
\]

Thus, for rectangles, in notation (1.3) and (1.4),

\[
y(x) = y^{*}(x) \left| \text{rectangles} \right. = \begin{cases} 
\frac{8}{3}x - \frac{5}{3} & \text{for } 1 \leq x \leq \frac{20}{11}, \\
5 - x & \text{for } \frac{20}{11} \leq x \leq \frac{5}{2},
\end{cases}
\]  

(2.1)
and the maximum value of \( \frac{\lambda_3}{\lambda_1} \) is

\[
Y^*|_{\text{rectangles}} = \frac{35}{11},
\]

attained when \( a = \sqrt{\frac{8}{3}} \). Note that for this particular rectangle \( \lambda_3 \) is a degenerate eigenvalue: \( \lambda_3(R_{\sqrt{\frac{8}{3}}}) = \lambda_4(R_{\sqrt{\frac{8}{3}}}) \), and it is the only \( a \) for which \( \lambda_3(R_a) \) is not simple.

In the \((x, y)\)-plane, (2.1) corresponds to the two straight lines intersecting at the point \((\frac{20}{11}, \frac{35}{11})\).

**Circles** For a single circle, \( x = y = K_2 \). As easily checked, for a union of more than one disjoint circles of arbitrary radii,

\[
y^*(x)|_{\text{circles}} \equiv K_2, \quad \text{for} \quad 1 \leq x \leq K_2.
\]

Its graph in the \((x, y)\)-plane is a straight line parallel to the \(x\)-axis.

**Disjoint unions** The following easily checked fact shows that one cannot obtain higher values of \( y^*(x) \) by considering disjoint unions of sets from two different classes. Namely, let \( C_j \) be two arbitrary classes of domains, with corresponding functions \( y^*(x)|_{C_j} \) (not necessarily defined for all \( x \in [1, K_2] \)). Then, for any domain \( \Omega = \Omega_1 \sqcup \Omega_2 \) with \( \Omega_j \in C_j \), we have, for \( x = \frac{\lambda_2}{\lambda_1}(\Omega) \) and \( y = \frac{\lambda_3}{\lambda_1}(\Omega) \), the inequality \( y \leq \max\left(y^*(x)|_{C_1}, y^*(x)|_{C_2}, K_2\right) \).

**Other domains** There are other domains, like sectors of the annuli, ellipses, etc., for which the problem of funding the eigenvalues is reduced by separation of variables to the problem of solving some transcendental equations. However, the latter one is often not easier than the numerical solution of the original problem, so we do not treat these cases here.

The graphs of \( y(x) = y^*(x)|_{\text{rectangles}} \) and \( y^*(x)|_{\text{circles}} \) are shown in Figure 1.

**Ashbaugh-Benguria estimates**

In [AshBen1], Ashbaugh and Benguria proved, using a wide variety of methods, the following upper bounds for \( y^*(x) \):

\[
y^*(x) < K_2x \quad \text{for} \quad 1 < x \leq 1.396^-,
\]

(AB1)
\begin{align*}
y^*(x) &\leq 1 + x + \sqrt{2x - (1 + x^2)/2} \quad \text{for} \quad 1.396^- \leq x \leq 1.634^-, \quad (\text{AB}_2) \\
y^*(x) &\leq F(x) \quad \text{for} \quad 1.634^- \leq x \leq 1.676^-, \quad (\text{AB}_3) \\
y^*(x) &\leq H(x) - x \quad \text{for} \quad 1.676^- \leq x \leq 2.198^+, \quad (\text{AB}_4) \\
\text{and} \quad y^*(x) &\leq G(x) \quad \text{for} \quad 2.198^+ \leq x \leq 2.539^-, \quad (\text{AB}_5)
\end{align*}

where the functions $H(x), F(x)$ and $G(x)$ are defined by

\[
H(x) = \begin{cases} 
6 & \text{for} \quad x = 1 \\
\min_{1 \leq \eta, \xi < x} \left( \frac{4\beta(\beta + \gamma)^2(x - 1)(x - \beta\gamma/(\beta + \gamma - 1))^2}{(2\beta - 1)(2\gamma - 1)(x - \eta)(x - \xi)(4x - 2 - \eta - \xi)} \right) & \text{for} \quad x > 1
\end{cases}
\]

with $\beta = \eta + \sqrt{\eta^2 - \eta}$ and $\gamma = \xi + \sqrt{\xi^2 - \xi}$, $F(x)$ is the middle root of the cubic

\[2xy^3 - 2(5x^2 + 3x + 1)y^2 + (6x^3 + 39x^2 + 2x - 1)y - (24x^3 + 11x^2 - 4x - 1) = 0\]

and

\[G(x) = \inf_{\beta > 1/2} \left( \frac{\beta^2}{2\beta - 1} + \frac{x - \beta^2/(2\beta - 1)}{C_2(\beta)(x - \beta^2/(2\beta - 1) - 1)} \right)\]

with the infimum taken over values of $\beta$ satisfying $x > \beta^2/(2\beta - 1) + 1/C_2(\beta)$ and with

\[C_2(\beta) = \frac{2\beta - 1}{\beta} \frac{\int_0^{J_0^2}(t)dt}{\int_0^{J_0^2}(t)dt} \quad \text{for} \quad J_0^2(t) = J_{1/2}(t)
\]

We remind that $J_0(t)$ denotes the standard Bessel function of order zero and $J_{0.1}$ is its first positive zero. For the derivation of the bound (AB4) and more discussion of it, see [AshBen5]. The other bounds given above are due to Hong-Cang Yang (see [Yan] for (AB2)) and Ashbaugh and Benguria (see [AshBen2], [AshBen4] for (AB1), [AshBen5] for (AB3), and [AshBen6] for (AB5)).

The admissible region for $\left( \frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1} \right)$ defined by (AB5), obvious bounds $\frac{\lambda_2}{\lambda_1} \geq 1$ and $\frac{\lambda_3}{\lambda_1} \geq \frac{\lambda_2}{\lambda_1}$, and inequalities (AB3)–(AB5), is shown in Figure 1.

We make two remarks following [AshBen6]:
Figure 1: Admissible range (shaded) of \((\frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1})\) according to [AshBen6]. Shown for comparison are the maximum values of \(\frac{\lambda_3}{\lambda_1}\) as functions of \(\frac{\lambda_2}{\lambda_1}\) for rectangles and disjoint unions of circles.

**Remark 2.1.** The inequalities (AB1)–(AB5) all apply on broader intervals of \(x\)-values than the intervals specified explicitly with them; the given intervals indicate the range for which the corresponding inequality gives the best bound yet found.

**Remark 2.2.** The absolute maximum of the right-hand sides of (AB1)–(AB5) occurs at the point where \(F(x)\) has a maximum within the interval where it is the best bound. That happens at the point \((x, y) \approx (1.65728, 3.83103)\), and imply the best upper bound \((L.5)\) yet proven for \(\frac{\lambda_3}{\lambda_1}\).

### 3 Numerical analysis of random domains

To the best of our knowledge, there have been no large scale numerical experiments on low eigenvalues of the Dirichlet Laplacian for planar domain. In an attempt to improve the existing estimates on the range of \((\frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1})\), we conducted such experiments for a variety of domain classes.
General method

For each particular domain, the calculation of the first three eigenvalues has been conducted using a standard finite element method implementation via PDEToolbox [PDETool] and FEMLAB [FEMLAB] in Matlab, with two or three mesh refinements. For simple domains with relatively “high” values of the ratio $\frac{\lambda_3}{\lambda_1}$, optimization with respect to the parameters describing domains of the particular class was performed in order to maximize this ratio. The results of calculations for some classes of domains are described below and summarized at the end of this Section.

For each class of the domains we represent the results in the following graphical form. The range $[1, K_2]$ of possible values of $x = \frac{\lambda_2}{\lambda_1}$ is split into subintervals of length $\delta x$ (normally $\approx 0.05$). In each subinterval we choose, if it exists, a domain with maximal $y = \frac{\lambda_3}{\lambda_1}$ and plot the corresponding point $(x, y)$. For comparison, the graphs of $y(x) = y^*(x)|_{rectangles}$ and/or $y^*(x)|_{circles}$ are shown.

Triangles, quadrilaterals and ellipses

We start with cyclic calculations for all triangles, with angle step 2.5°. For the triangles with relatively high ratio of $\lambda_3/\lambda_1$ we repeat the procedure in the local neighbourhood with angle step 0.5°. The results are shown in Fig. 3.

The computational procedure for quadrilaterals is essentially the same as the one for triangles, with parameters $\alpha, \beta, \gamma$ and $\delta$ in the region $(0, \pi)$ (see Fig. 3). We choose an angle step of 2.5°. Note that quadrilaterals with negative $\alpha$ and $\beta$, or $\gamma$ and $\delta$ do not have to be considered separately — they fit into the scheme above by choosing another diagonal as a starting point and re-scaling. In case of relatively high ratio $\lambda_3/\lambda_1$ ($\geq 3$) we repeated the calculation with an angle step 0.5° in the local neighbourhood of that quadrilateral. The results are shown in Fig. 3.

The results of cyclic calculations for the ellipses, with axis ratio varying between 1 and 5 with step 0.1 (0.001 in the vicinity of the ellipse with highest $\lambda_3/\lambda_1$), are also shown in Fig. 3.

Remark 3.1. Rather surprisingly, Fig. 3 suggests that

$$y^*(x)|_{quadrilaterals} \approx y^*(x)|_{rectangles} \quad (3.1)$$

We give a partial explanation of this fact in the next Section, see Remark 4.6.
Figure 2: Parametrization of a quadrilateral.

Annuli and random sectors of annuli
The calculations for annuli with inner radius 1 and outer radius \( r \) demonstrated that the corresponding value \( \frac{\lambda_3}{\lambda_1}(r) \) is monotonically increasing from 1 to \( K_2 \) as \( r \) changes from 1 to \( \infty \) (although convergence, for large \( r \), is very slow — just logarithmic). These results are not very informative and we do not include them in the graphs or the summary table below.

In calculations for sectors of the annuli of angle \( \theta \), we choose \( r \) randomly in the interval (1,20) and \( \theta \) randomly in the interval (0.01\( \pi \), 1.99\( \pi \)). The results of calculations are shown in Fig. [\ref{fig:annuli}].

Pseudo-random polygons
For polygons with more than four vertices, cyclic calculations through all possible values of the geometric parameters with some reasonable step become impractical due to the time constraints. Instead, we choose to perform calculations for randomly generated polygons. We employ the following simple procedure for generating a pseudo-random polygon with \( N \) vertices \( v_1, \ldots, v_N \) lying inside a square \([0,1]^2\).

Vertices \( v_1, v_2, v_3 \) are chosen randomly using any pseudo-random genera-
Figure 3: $y^*(x)$ for triangles, quadrilaterals and ellipses

**Vertices $v_j$, $j = 4, \ldots, N - 1$.** We choose a possible vertex at random. If the interval $[v_j, v_j]$ intersects any of previously constructed sides $[v_{k-1}, v_k]$, $k = 1, \ldots, j - 2$, then we make another random choice.

**Vertex $v_N$** is constructed in the same manner, but we additionally check that the interval $[v_N, v_1]$ does not intersect any of the existing sides.

To avoid infinite loops, we abort the construction if the number of attempts at some stage exceeds some sufficiently big number (say, 200). We also put in place a restriction forbidding very small angles (which require special efforts in mesh generation).

The collated results of calculations for pseudo-random pentagons, hexagons and decagons are shown in Fig. 3. These results also include experiments on random perturbations of the rectangles constructed in the following way: $N$ points were randomly chosen on the sides of the rectangle $R_a$, with $a \in (1, 5)$, and $1 \leq N \leq 8$, and these points and the four vertices of the original rectangle were randomly moved by a distance not exceeding $0.1a$ to form an $(N + 4)$-gon.
Star-shaped domains (simply and non-simply connected)

The procedure described above for the polygons does not work very effectively for polygons with large number of vertices — it often takes a long time to generate a suitable vertex \( v_j \) with \( j \gtrsim 10 \). Thus, in these cases we restrict ourselves to star-shaped polygonal domains which are much easier to construct. Namely, for the vertices \( v = re^{i\theta} \), we choose the angles \( \theta \) randomly between 0 and \( 2\pi \), and the radii \( r \) randomly between given numbers \( r_1 \) and \( r_2 \). We conducted a series of experiments with a fixed number of vertices (13, 17 and 23), as well as a series of runs where the number of vertices was chosen randomly between four and thirty.

Additionally, we conducted a series of experiments of non-simply connected domains of the types \( R_{\sqrt{\pi}} \setminus S_1, S_2 \setminus R_{\sqrt{\pi}}, \) and \( S_1 \setminus S_2, \) where \( S_j \) are random star-shaped polygons such that \( S_1 \subset R_{\sqrt{\pi}} \subset S_2, \) and \( R_{\sqrt{\pi}} \) is the rectangle with the maximum \( \frac{\lambda_3}{\lambda_1} \).

The results for pseudo-random star-shaped domains are collated in Fig. 5.
Dumbbells and Jigsaw pieces

By a dumbbell we understand a domain of the type

\[ ([0, l] \times [-h, h]) \cup C((0, 0), r_1) \cup C((l, 0), r_2), \]

where \( l, h, r_1, r_2 \) are positive parameters, and \( C(v, r) \) denotes a circle with radius \( r \) centred at \( v \). By a jigsaw piece we understand a domain of the type \( R \setminus C \), where \( R \) is a rectangle, and \( C \) is a circle with a centre “near” the boundary of the rectangle. Typical dumbbell and jigsaw piece domains are shown in Fig. 6.

The results of numerical experiments on dumbbells and jigsaw pieces, with cyclical/random choice of the parameters, is shown in Fig. 7. For dumbbells, we also optimized over the parameters for domains with \( \lambda_3/\lambda_1 \approx 3.2 \), allowing additionally the centres of the circles to move in the vertical direction along the sides of rectangles. However, this did not lead to the improvement of the results.

Summary of the numerical experiments

We summarize the results of our numerical experiments in the following table.

![Figure 5: \( y^*(x) \) for pseudo-random star-shaped domains](image-url)
As seen in the last column of Table 1, in each class of domains the maximum of the ratio \( \frac{\lambda_3}{\lambda_1} \) is attained, within the accuracy of computations, on a domain with degenerate eigenvalue \( \lambda_3 \approx \lambda_4 \). The same, of course, holds for rectangles, see (2.1). This allows us to conjecture that the absolute maximum and any local maxima of \( \lambda_3 \approx \lambda_4 \) are also attained on domains with degenerate \( \lambda_3 \). We give a partial proof of this conjecture in the next
| Type of domains               | No. of experiments | $Y^*$     | $\delta_4$ |
|------------------------------|--------------------|----------|------------|
| Triangles                    | 2145               | 2.827    | 0.016      |
| Quadrilaterals               | 13222              | 3.183    | 1.6 x 10^{-5} |
| Sectors                      | 360                | 3.149    | 4.6 x 10^{-5} |
| Ellipses                     | 142                | 3.167    | 9.1 x 10^{-5} |
| Random polygons              | 26867              | 3.189    | 6.1 x 10^{-4} |
| Random star-shaped domains   | 18320              | 3.159    | 0.022      |
| Dumbbells                    | 2871               | 3.202    | 1.1 x 10^{-4} |
| Jigsaw pieces                | 1420               | 3.178    | 0.003      |
| **Total**                    | **65337**          | **3.202**| **1.1 x 10^{-4}** |

Table 1: Summary statistics for numerical experiments. The value in the fourth column is $\delta_4 := \frac{\lambda_4 - \lambda_3}{\lambda_3} (\Omega^*)$, where $\Omega^*$ is the domain which maximizes the ratio $\frac{\lambda_3}{\lambda_1} = Y^*$ in the corresponding class of domains.

Section.

The computed absolute maximum ratio $Y^* \approx 3.202$ is attained on the dumbbell-shaped domain (3.2) with $l = 1$, $h = 1.4510$, $r_1 = 0.7814$, and $r_2 = 0.7818$, see Fig. 8. Note that the maximum value $Y^*$ is only slightly higher than the corresponding value $Y^{*\text{rectangles}} \approx 3.1818$.

Remark 3.2. Additional experiments were conducted in order to check whether a maximizer is likely to be a simply connected domain. Namely, for the dumbbell-shaped domain $\Omega$ described above, we computed the eigenvalues for a number of domains obtained by removing a small hole from $\Omega$. In all the cases, the ratio of the third and the first eigenvalue for a perturbed problem was quite significantly less than that for $\Omega$.

The graph of the function $y^*(x)$ built on the basis of all numerical experiments is shown in Fig. 8.
Figure 8: $y^*(x)$ for all computed domains
4 Asymptotic results

In this Section, using standard perturbation techniques, we establish several results which, although don’t give the full answer to the question of maximizing the ratio $\frac{\lambda_3}{\lambda_1}(\Omega)$ among all planar domains $\Omega$, give some indication which of the domains may or may not be a maximizer. We first proof the following

Theorem 4.1. The rectangle $R \sqrt{\frac{\pi}{4}}$ does not maximize the $\frac{\lambda_3}{\lambda_1}$ among all planar domains.

This should be compared, however, with Remark 4.6 below.

We also give a proof of the following more general result, which justifies the remark made at the end of last Section.

Theorem 4.2. Suppose that $\Omega \subseteq \mathbb{R}^2$ is a local maximizer of $\frac{\lambda_3}{\lambda_1}(\Omega)$ among planar domains with sufficiently smooth boundary. Then $\lambda_3(\Omega) = \lambda_4(\Omega)$.

We should emphasize here that neither the statement nor the proof (found below) of Theorem 4.2 is fully rigorous. In the former, we do not discuss the requirements on the smoothness of the boundary and the concept of a local maximizer; we also do not prove that maximizers actually exist. In the latter, we rely on the following unproven, although very plausible, conjecture.

Conjecture 4.3. Let $\lambda_3$ be a simple eigenvalue of the Dirichlet Laplacian on a planar connected domain. Then not all nodal lines of the corresponding eigenfunction are closed.

Such a conjecture is not unreasonable since, in general, it is quite difficult to construct domains for which even one nodal line of a low eigenfunction is closed, see [HoHoNa].

Before giving the proofs of Theorems 4.1 and 4.2, we recall, without proof, some classical results from the domain perturbation theory. The details can be found, e.g., in [Rel, SHSP].

Domain perturbations

For simplicity, we restrict ourselves to domains in $\mathbb{R}^2$; all the results stated here hold in any dimension.

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*We were informed by Niculae Mandrache that he had independently obtained a similar result.*
Consider, for small values of real parameter $|\varepsilon|$, a family of bounded domains $\Omega^\varepsilon$ in $\mathbb{R}^2$ of variable $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$, which are transformed by the change of coordinates

\[ x = \tilde{x} + \varepsilon \mathbf{S}(x) \quad (4.1) \]

into the domain $\Omega = \Omega^0$ in $\mathbb{R}^2$ of variable $x$. We assume that the boundary $\partial \Omega$ and the vector-function $\mathbf{S}$ are sufficiently smooth.

Let $n$ be the outer unit normal to $\partial \Omega$, and denote $f = \mathbf{S} \cdot n$ (in fact, $\varepsilon f$ is a smooth function on $\partial \Omega$ which gives, up to the leading order for small $\varepsilon$, the normal distance between $\partial \Omega$ and $\partial \Omega^\varepsilon$).

Denote by $\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_j \leq \ldots$ the eigenvalues of the Dirichlet Laplacian on $\Omega$ and by $\{u_j\}$ the corresponding basis of normalized orthogonal eigenfunctions (which are chosen real). Also, denote by $\lambda_j^\varepsilon$ the eigenvalues of the Dirichlet Laplacian on $\Omega^\varepsilon$. For sufficiently small $|\varepsilon|$, the $\lambda_j^\varepsilon$ are continuous functions of $\varepsilon$ and tend to $\lambda_j$ as $\varepsilon \to 0$.

The following two results go back to Rellich.

**Proposition 4.4.** Let $\lambda_j$, $j \geq 1$, be a simple eigenvalue of the Dirichlet Laplacian on $\Omega$. Then $\lambda_j^\varepsilon$ has the asymptotic expansion

\[ \lambda_j^\varepsilon = \lambda_j + \varepsilon \tilde{\lambda}_j,1 + \varepsilon^2 \tilde{\lambda}_j,2 + \ldots, \quad (4.2) \]

as $\varepsilon \to 0$, where

\[ \tilde{\lambda}_j,1 = - \int_{\partial \Omega} f \left| \frac{\partial u_j}{\partial n} \right|^2 d\sigma. \quad (4.3) \]

The situation is slightly more complicated when $\lambda_j = \ldots = \lambda_{j+m}$ is an eigenvalue of multiplicity $m+1$. For simplicity, we consider just the case $m = 1$.

**Proposition 4.5.** Let $\lambda_k = \lambda_{k+1}$ be a double eigenvalue of the Dirichlet Laplacian on $\Omega$. Then, as $\varepsilon \to 0$, $\lambda_j^\varepsilon$ and $\lambda_{k+1}^\varepsilon$ still have the asymptotic expansions (4.2) ($j = k, k+1$) with

\[ \tilde{\lambda}_{k,1} = \frac{1}{\varepsilon} \min(\varepsilon \mu_1, \varepsilon \mu_2), \quad \tilde{\lambda}_{k+1,1} = \frac{1}{\varepsilon} \max(\varepsilon \mu_1, \varepsilon \mu_2), \quad (4.4) \]

where $\mu_1$, $\mu_2$ are two real roots of the quadratic equation

\[ (F_{k,k} + \mu)(F_{k+1,k+1} + \mu) - F_{k,k+1}^2 = 0 \quad (4.5) \]

and

\[ F_{p,q} = \int_{\partial \Omega} f \frac{\partial u_p}{\partial n} \frac{\partial u_q}{\partial n} d\sigma. \quad (4.6) \]
We will be in fact interested in the asymptotic expansion of \( \frac{\lambda_3^\varepsilon}{\lambda_1^\varepsilon} \), which follows from (4.2):

\[
\frac{\lambda_3^\varepsilon}{\lambda_1^\varepsilon} = \frac{\lambda_j}{\lambda_1} + \frac{\varepsilon}{(\lambda_1)^2}(\tilde{\lambda}_{j,1}\lambda_1 - \tilde{\lambda}_{1,1}\lambda_j) + O(\varepsilon^2),
\]

(4.7)

**Proof of Theorem 4.1**

Let \( \Omega = R \sqrt{\frac{8}{3}} \) be the rectangle \( \{(x_1, x_2) : 0 < x_1 < 1, \ 0 < x_2 < \sqrt{\frac{8}{3}}\} \). We shall construct an explicit perturbation \( \Omega^\varepsilon \) using (4.1) such that the first correction term in the asymptotic formula (4.7) is positive for \( \varepsilon > 0 \), and therefore \( \frac{\lambda_3^\varepsilon}{\lambda_1^\varepsilon} > \frac{\lambda_3}{\lambda_1} \) for sufficiently small positive \( \varepsilon \).

Let

\[
\Omega^\varepsilon = \left\{(x_1, x_2) : 0 < x_1 < 1, \ 0 < x_2 < \sqrt{\frac{8}{3}} + \varepsilon g(x_1)\right\},
\]

where

\[
g(x_1) = c_0 + \sum_{l=0}^{\infty} \sqrt{2} c_l \cos(\pi l x_1).
\]

We will choose the coefficients \( c_l \) later.

The corresponding function \( f \) appearing in the asymptotic formulae above is

\[
f(x_1, x_2) = \begin{cases} 
  g(x_1), & \text{if } x_2 = \sqrt{\frac{8}{3}}, \ 0 \leq x_1 \leq 1, \\
  0, & \text{if } (x_1, x_2) \in \partial \Omega, \ x_2 \neq \sqrt{\frac{8}{3}}.
\end{cases}
\]

Note that we shall use (4.3) for computing \( \tilde{\lambda}_{1,1} \) and (4.4) for computing \( \tilde{\lambda}_{3,1} \) and \( \tilde{\lambda}_{4,1} \), since \( \lambda_3 = \lambda_4 \) is a double eigenvalue of the unperturbed problem. Elementary but tedious calculations show that the correction terms \( \tilde{\lambda}_{k,1}, \ k = 1, 2, 3, 4 \), depend only upon the parameters \( c_j \) with \( j = 0, \ldots, 4 \). For brevity, we omit the explicit expressions.

Let us choose the parameters \( c_0, \ldots, c_4 \) in such a way that \( \tilde{\lambda}_{3,1} = \tilde{\lambda}_{4,1} \) (i.e., \( \lambda_3^\varepsilon \) remains a double eigenvalue up to the linear terms in \( \varepsilon \)). This, by Proposition 4.5, happens when \( F_{3,3} = F_{4,4} \) and \( F_{3,4} = 0 \), which in turn leads to the following conditions on coefficients \( c_j \):

\[
c_3 = c_1, \quad c_4 = 9c_2 - 8\sqrt{2}c_0.
\]

(4.8)

Under conditions (4.8), asymptotic formula (4.7) simplifies dramatically, and becomes

\[
\frac{\lambda_3^\varepsilon}{\lambda_1^\varepsilon} - \frac{\lambda_3}{\lambda_1} = \frac{96\sqrt{3}}{121} (c_2 - \sqrt{2}c_0)\varepsilon + O(\varepsilon^2),
\]

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and we can choose $c_0$ and $c_2$ in such a way that its right-hand side is positive for sufficiently small positive $\varepsilon$. This proves Theorem 4.1.

**Remark 4.6.** Let $\Omega = R_a$ be any rectangle and consider the perturbations $\Omega^\varepsilon$ as above but with function $f$ linear in $x_1, x_2$ (and, naturally, $a$ replacing $\sqrt{\frac{2}{3}}$ throughout). Thus, we are considering quadrilaterals $\Omega^\varepsilon$ which are small perturbations of the rectangle $R_a$. The same elementary calculations then imply that, for $x^\varepsilon := \frac{\lambda_2}{\lambda_1}(\Omega^\varepsilon)$ and $y^\varepsilon := \frac{\lambda_3}{\lambda_1}(\Omega^\varepsilon)$, we obtain, up to and inclusive of the terms of order $\varepsilon$, that

$$y^\varepsilon = y^*(x^\varepsilon)|_{\text{rectangles}},$$

where $y^*(x)|_{\text{rectangles}}$ is given by the right-hand side of (2.1). In other words, up to the terms of order $\varepsilon$ the rectangles are local maximizers among all quadrilaterals which are sufficiently “close” to them, cf. Remark 3.1.

**Proof of Theorem 4.2**

Suppose that $\Omega$ is a planar domain with sufficiently smooth boundary which locally maximizes the ratio $\frac{\lambda_3}{\lambda_1}$ in the following sense: for any sufficiently smooth perturbation $\Omega^\varepsilon$ determined by (4.1) we have

$$\frac{\lambda_3^\varepsilon}{\lambda_1^\varepsilon} \leq \frac{\lambda_3}{\lambda_1}. \quad (4.9)$$

Assume additionally that $\lambda_3$ is a simple eigenvalue of the Dirichlet Laplacian in the unperturbed domain $\Omega$. We shall show that this assumption leads to the contradiction with Conjecture 4.3.

Since both $\lambda_1$ and $\lambda_3$ are simple eigenvalues, the asymptotic formula (4.7) becomes, in accordance with Proposition 4.4,

$$\frac{\lambda_3^\varepsilon}{\lambda_1^\varepsilon} - \frac{\lambda_3}{\lambda_1} = \frac{\varepsilon}{(\lambda_1)^2} \int_{\partial\Omega} f \left( \lambda_3 \left| \frac{\partial u_1}{\partial n} \right|^2 - \lambda_1 \left| \frac{\partial u_3}{\partial n} \right|^2 \right) d\sigma + O(\varepsilon^2).$$

Now, as $\varepsilon$ can be chosen both positive and negative, (4.9) can hold only if

$$\int_{\partial\Omega} f \left( \lambda_3 \left| \frac{\partial u_1}{\partial n} \right|^2 - \lambda_1 \left| \frac{\partial u_3}{\partial n} \right|^2 \right) d\sigma = 0,$$

and since $f$ is an arbitrary smooth function, this requires

$$\lambda_3 \left| \frac{\partial u_1}{\partial n} \right|^2 = \lambda_1 \left| \frac{\partial u_3}{\partial n} \right|^2$$

everywhere on $\partial\Omega$. But the normal derivative of the first eigenfunction of the Dirichlet Laplacian is non-zero everywhere on the boundary, so the last formula implies that the third eigenfunction has the same property, and therefore all its nodal lines are closed, in contradiction with Conjecture 4.3.
5 Final Remarks

On the basis of the numerical computations and the results proven above we make the following conjecture, most of which is still to be established (or disproved) rigorously:

The domain maximizing the ratio $\frac{\lambda_3}{\lambda_1}$ for planar domains is close in shape to the optimal computed dumbbell-shaped domain shown in Fig. 9, is simply-connected, and has a smooth boundary. The maximal admissible value $Y^*$ is approximately equal to or is slightly greater than 3.202.

![Domain maximizing $\frac{\lambda_3}{\lambda_1}$](image)

Figure 9: Domain maximizing $\frac{\lambda_3}{\lambda_1}$ on the basis of computations

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