RATIONAL POINTS IN REGULAR ORBITS ATTACHED TO INFINITESIMAL SYMMETRIC SPACES

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Abstract. Motivated by problems arising in the relative trace formula and arithmetic invariant theory we prove the existence of rational points on orbits arising from certain infinitesimal symmetric spaces. As an application, we prove analogous results for orbits in certain global reductive symmetric spaces.

1. Introduction

Let $F$ be a characteristic zero field with fixed algebraic closure $\overline{F}$ and let $G$ be a connected reductive group over $F$ equipped with an involution (that is, an automorphism of order 2)

$$\theta : G \rightarrow G.$$ 

The Lie algebra $g$ of $G$ is then equipped with an automorphism, which we will denote as (by abuse of notation)

$$\theta : g \rightarrow g,$$

again of order 2. We let $G(1) \leq G$ be the subgroup invariant under $\theta$, and let $g(\pm 1)$ be the $\pm 1$ eigenspaces of $\theta$ acting on $g$, respectively. Thus the Lie algebra of $G(1)$ is $g(1)$, and the adjoint action of $G(1)$ on $g$ preserves $g(-1)$. One can view $g(-1)$ as an infinitesimal symmetric space, since it can be identified with the tangent space of the reductive symmetric space $G/G(1)^\circ$ at the basepoint $G(1)^\circ$ (here $\circ$ denotes the neutral component). In particular the action of $G(1)$ on $g(-1)$ can be thought of as an infinitesimal analogue of the action of $G(1)$ on $G/G(1)^\circ$ by left multiplication.

The representations

$$G(1) \rightarrow \text{Aut}(g(-1))$$

as $G$ and $\theta$ vary, show up in many contexts. The structure of the orbits is particularly interesting and is the main focus of this paper.

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Recall that an element $X \in \mathfrak{g}(-1)(\overline{F})$ is said to be relatively regular if the $G(1)_{\overline{F}}$-orbit of $X$ is of maximal dimension among all $G(1)_{\overline{F}}$-orbits. It is regular if the $G_{\overline{F}}$-orbit is of maximal dimension among all $G_{\overline{F}}$-orbits. We let

$$\mathfrak{g}(-1)^{rr} \subset \mathfrak{g}(-1)$$

be the subscheme of relatively regular elements (see §3). It is dense and $G(1)$-invariant.

For $X \in \mathfrak{g}(-1)^{rr}(\overline{F})$, let

$$O(X)(\overline{F}) := G(1)(\overline{F}) \cdot X.$$  

This is the set-theoretic orbit of $X$. If $O(X)(\overline{F})$ is invariant under $\text{Gal}(\overline{F}/F)$ then it defines a subscheme

$$O \subset \mathfrak{g}(-1)^{rr}.$$  

We refer to these subschemes as relatively regular orbits in $\mathfrak{g}(-1)$.

The $F$-points $O(F)$ of $O$ can then be concretely described as the set

$$O(F) = O(X)(\overline{F}) \cap \mathfrak{g}(-1)^{rr}(F).$$

Even though $O(X)(\overline{F})$ contains $X$ and is therefore nonempty, it is false in general that $O(F)$ is nonempty. This motivates the following question:

**Question 1.1.** Is $O(F)$ nonempty?

This question is of intrinsic number theoretic interest. Moreover, it is a problem that shows up often in representation theory and arithmetic invariant theory.

For example, if $G$ is a connected reductive group that is quasi-split with simply connected derived group over $F$ then any conjugacy class in $G$ intersects $G(F)$ by an important result of Kottwitz [Kot82] that completes work of Steinberg [Ste65]. This result is crucial for the stabilization of the trace formula [KS99, Lab99]. We expect that the work we have begun in this paper will play a similar role in the relative trace formula. We refer to [GW14] for an example of the type of comparison of relative trace formulae that would use results of this type.

Moreover the entirety of the subject of arithmetic invariant theory in the sense of Bhargava is based on a study of $F$-rational points of certain orbits like those discussed above; the thesis of Thorne is an excellent resource to consult for this point of view [Tho13].

In this paper we answer Question 1.1 in the following situation. Let $p \geq q$ be positive integers. Define the matrices

$$J_{p,q} = \begin{pmatrix} J_{p} & J_q \\ -J_q & J_p \end{pmatrix}, \quad J'_{p,q} = \begin{pmatrix} J_{p} & J_q \\ J_q & J_p \end{pmatrix}.$$
where

\[ J_r := \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}, \quad J'_r := \begin{pmatrix} (-1)^{r-1} & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & (-1)^{r-1} \end{pmatrix}. \]

For any symmetric (resp. skew symmetric) matrix \( J \in \text{GL}_n(F) \) (resp. \( J' \in \text{GL}_{2n}(F) \)) we let

\[ O(J)(R) := \{ g \in \text{GL}_n(R) : Jg^{-t}J^{-1} = g \} \]
\[ \text{Sp}(J')(R) = \{ g \in \text{GL}_{2n}(R) : J'g^{-t}J'^{-1} = g \} \]

be the associated orthogonal group and symplectic group.

We let \( G \) be one of the groups \( \text{GL}_{p+q}, O(J_{p,q}) \), or \( \text{Sp}(J'_{p,q}) \). In the orthogonal case we assume in addition that \( |p - q| \leq 1 \), and in the symplectic case we assume that \( p \) and \( q \) are even.

Note that for any of these groups, the adjoint action by \( I_{p,q} := \begin{pmatrix} I_p & -I_q \\ -I_q & I_p \end{pmatrix} \in G \) induces an automorphism \( \theta \) of \( G \), and \( G(1) \) will be a direct product of two classical groups of same type as \( G \).

The main result of this paper follows.

**Theorem 1.1.** For \( G \) and \( \theta \) defined above, any \( \text{Gal}(\overline{F}/F) \)-invariant regular \( G(1)(\overline{F}) \)-orbit in \( \mathfrak{g}(-1)(\overline{F}) \) has an \( F \)-point.

In the case of \( \text{GL}_{p+q} \), this result was obtained by Jacquet and Rallis in [JR96]; and in \( O_{p+q} \) it was obtained by J. Thorne when \( |p - q| \leq 1 \) [Tho13].

Our proof is essentially uniform in each of the above cases, and one hopes that it can be generalized to a broader setting. The ultimate goal would be to prove analogous theorems to Kottwitz’s Theorem for \( G(1) \backslash \mathfrak{g}(-1) \) under the setting of a general class in reductive groups.

As a corollary of our Main Theorem, we prove consequential results for the orbits of certain classical groups on \( M_{p,q} \), the space of \( p \times q \) matrices, which is stated as Corollary 1.2 and will be further explained in §5.

For each \( G \) above there is a natural action of \( G(1) \) on \( M_{p,q} \), written explicitly by

\[ G(1)(R) \times M_{p,q}(R) \rightarrow M_{p,q}(R) \]
\[ (g_1, g_2), X \mapsto g_1Xg_2^{-1} \]

We say an orbit in \( M_{p,q} \) is \( G(1) \)-regular if it is of maximal dimension among all \( G(1) \)-orbits.

**Corollary 1.2.** For \( G \) and \( \theta \) defined above, any \( \text{Gal}(\overline{F}/F) \)-invariant regular \( G(1) \)-orbit \( O \) in \( M_{p,q} \) has an \( F \)-point.
Let us outline the contents of this paper. In Section 2, we recall the notions of an \( \mathfrak{sl}_2 \)-triple and a Kostant-Weierstrass section. These are used to reduce the proof of Theorem 1.1 to exhibiting the existence of relatively regular nilpotent elements in \( \mathfrak{g}(-1)(F) \). This part of the argument does not rely on the fact that \( G \) is one of the three families of groups isolated above, but is also applicable in other settings. In §3 we record the dimension of the regular orbits and in §4 we use this to exhibit relatively regular nilpotent elements in \( \mathfrak{g}(-1)(F) \). Finally in §5 we prove Corollary 1.2.

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2. \( \mathfrak{sl}_2 \)-triples

In this section we apply the work of Kostant and Rallis [KR69] to reduce the proof of Theorem 1.1 to exhibiting the existence of a relatively regular nilpotent element in \( \mathfrak{g}(-1)(F) \). To make this precise let us recall the notion of an \( \mathfrak{sl}_2 \)-triple.

Definition 2.1. An \( \mathfrak{sl}_2 \)-triple in a Lie algebra \( \mathfrak{g} \) is a triple \( (e, f, h) \) of non-zero elements in \( \mathfrak{g} \) satisfying
\[
[h, e] = 2e \quad [h, f] = -2f \quad [e, f] = h.
\]

In particular, if \( (e, f, h) \) is an \( \mathfrak{sl}_2 \)-triple then its \( F \)-span is naturally isomorphic to \( \mathfrak{sl}_2 \) as a Lie algebra; this explains the terminology.

For the moment, we will assume \( F = \mathbb{Q} \). Let \( \mathfrak{g}(-1)^e \) denote the centralizer of \( e \) in \( \mathfrak{g}(-1) \).

Theorem 2.1. Suppose that \( G, \mathfrak{g} \) and \( \theta \) are defined as in the introduction. Assume that \( (e, f, h) \) is an \( \mathfrak{sl}_2 \)-triple in \( \mathfrak{g} \) such that \( e, f \in \mathfrak{g}(-1)^{rr}(\mathbb{Q}) \), \( h \in \mathfrak{g}(1)(\mathbb{Q}) \). Then the map
\[
f + \mathfrak{g}(-1)^e \to G(1) \backslash \mathfrak{g}(-1)
\]
is an isomorphism.

This theorem is mostly due to Kostant and Rallis, but requires some translation to bring to our setting. This is why we have restricted our attention to the \( F = \mathbb{Q} \) case. However, this is still strong enough to deduce our result for arbitrary \( F \) (see Corollary 2.2 below).

Proof. By faithfully flat descent to check that the given morphism is an isomorphism it suffices to check that
\[
f + \mathfrak{g}^e_c \to G(1)_c \backslash \mathfrak{g}(-1)_c
\]
is an isomorphism.

For this we note that in each case under consideration $g_C$ is a complex Lie algebra admitting a real form $g_1$ such that $\theta$, defined as in the introduction, is the associated Cartan involution. More specifically, they correspond to types $\text{AIII}$, $\text{BDI}$ and $\text{CII}$ in [Hel01, Table V, p. 518]. Thus we can apply [KR69, Theorems 8, 11, 12, and 13] to deduce the theorem. □

Remark 2.1. Technically, the Theorem in [KR69] is stated for the adjoint group $G^{\text{ad}}$ instead of $G$. However the action map

$$G \rightarrow \text{Aut}(g)$$

factors through the adjoint group of $g$, which is $G^{\text{ad}} = G/Z(G)$. Thus we have the isomorphism

$$G(1)^{\text{ad}} \backslash g(-1)_C \simeq G(1)_C \backslash g(-1)_C.$$

Corollary 2.2. Let $F = \mathbb{Q}$ and suppose that there is an $\mathfrak{sl}_2$-triple in $g$ with

$$e, f \in g(-1)^{\text{rr}}(\mathbb{Q}) \quad \text{and} \quad h \in g(1)(\mathbb{Q}).$$

Then for any characteristic zero field $k$, every $G(1)(\overline{k})$-orbit in $g(-1)^{\text{rr}}(\overline{k})$ that is fixed under $\text{Gal}(\overline{k}/k)$ intersects $g(-1)(k)$.

Proof. Let $O(\overline{k})$ be the $G(1)(\overline{k})$-orbit fixed by $\text{Gal}(\overline{k}/k)$. Its image $Y$ in $G(1) \backslash g(-1)(\overline{k})$ is in $G(1) \backslash g(-1)(k)$. Denote by

$$q : g(-1)(k) \rightarrow G(1) \backslash g(-1)(k)$$

the quotient map. The inverse image $q^{-1}(Y)$ is the $k$-points of the Zariski closure of $O(\overline{k})$. Regard $e$ and $f$ as elements in $g(k)$. We further know that $q^{-1}(Y) \cap (f + g(-1)^{\text{rr}}(k))$ is a single point which is contained in $q^{-1}(Y) \cap g(-1)^{\text{rr}}(k)$. On the other hand, by [KR69, Theorem 9],

$$q^{-1}(Y) \cap g(-1)^{\text{rr}}(k) = O(\overline{k}) \cap g(-1)^{\text{rr}}(k).$$

□

In light of the corollary, to prove our main result it suffices to exhibit an $\mathfrak{sl}_2$-triple $(e, f, h)$ as in the assumptions of Corollary 1.2. We can reduce the amount of work we have to do by proving the following lemma:

Lemma 2.1. Let $e \in g(-1)(F)$ be a relatively regular nilpotent element. Then there exists an $\mathfrak{sl}_2$-triple $(e, f, h)$ with $f \in g(-1)^{\text{rr}}(F)$ and $h \in g(1)(F)$.

Proof. The proof of [Tho13, Lemma 2.15] goes through without change in our context, but we fill in some details for the convenience of the reader. Let $e \in g(-1)(F)$ be a relatively regular element. By the Jacobson-Morosov theorem [Jac51, Theorem 3] it is an element of
an $\mathfrak{sl}_2$-triple $(e, f', h') \in \mathfrak{g}(F)$. Decompose $h' = h_1 + h_{-1}$ and $f' = f_1 + f_{-1}$ into eigenvectors under $\theta$ (with $h_i, f_i \in \mathfrak{g}(i)(F)$). Then

$$2e = [h', e] = [h_1, e] + [h_{-1}, e]$$

Since $[h_{-1}, e] \in \mathfrak{g}(1)(F)$ and $[h_1, e] \in \mathfrak{g}(-1)(F)$ we deduce that $[h_1, e] = 0$ and $[h_{-1}, e] = 2e$. Because $[e, f'] = h'$ we have $[e, f_1] = h_{-1}$ and $[e, f_{-1}] = h_1$. Thus $h := h_1$ is in the image of $\text{ad}_e$ and $[h, e] = 2e$. This implies that the pair $(e, h)$ can be completed to an $\mathfrak{sl}_2$-triple $(e, f, h)$ with $f \in \mathfrak{g}(F)$ by [Kos59, Corollary 3.5]; moreover, $f$ is uniquely determined. Since $h \in \mathfrak{g}(1)(F)$ and $e \in \mathfrak{g}(-1)(F)$ if we let $f'$ be the component of $f$ in $\mathfrak{g}(-1)(F)$ then $(e, f', h)$ is an $\mathfrak{sl}_2$-triple, so by the uniqueness result mentioned earlier we have $f' = f$. Since $e$ is relatively regular, it follows that $f$ is as well, and this completes the proof. \[\square\]

**Remark.** Strictly speaking, Kostant assumes that $\mathfrak{g}$ is semisimple and $F = \mathbb{C}$, but this is not necessary for the argument to be valid.

In view of the lemma and Corollary 2.2 to prove Theorem 1.1 it suffices to show that for every $G$ and $\theta$ as in the statement of that theorem there exists a relative regular $e \in \mathfrak{g}(-1)(\mathbb{Q})$. The latter sections of this paper construct this element $e$.

### 3. Dimensions of regular orbits

In this section we compute the dimension of the regular orbit in the cases under consideration.

A Cartan subspace $\mathfrak{t}(-1) \leq \mathfrak{g}(-1)$ is a maximal commutative subalgebra consisting of semisimple elements. Its dimension is called the **rank** of $\theta$.

**Lemma 3.1.** If $\mathfrak{t}(-1) \leq \mathfrak{g}(-1)$ is a Cartan subspace then so is $\mathfrak{t}(-1)_k \leq \mathfrak{g}(-1)_k$ for every field extension $k/F$.

**Proof.** Let $T(-1) \leq G$ be the connected subgroup whose Lie algebra is $\mathfrak{t}(-1)$. Then $T$ is a maximal $\theta$-split torus in $G$. Its base change to $k$ is therefore a maximal $\theta$-split torus in $G_k$ by [HW93, Lemma 11.1], and it follows that $\mathfrak{t}(-1)_k$ is a Cartan subspace of $\mathfrak{g}(-1)_k$. \[\square\]

Assume that $G$ is one of the groups $\text{GL}_{p+q}$, $\text{O}(J_{p,q})$ or $\text{Sp}(J'_{p,q})$ of the introduction and $\theta$ is convolution by $I_{p,q}$. In view of Lemma 3.1 the rank of $\theta$ in each of these cases is equal to the rank of the corresponding locally symmetric space, which can be found in [Hel01, Table V, §X.6]:

- **AIII** For $G = \text{GL}_{p+q}$, $\text{rank } \theta = \min(p, q)$.
- **BDI** For $G = \text{O}(J_{p,q})$, $\text{rank } \theta = \min(p, q)$.
- **CI** For $G = \text{Sp}(J'_{p,q})$, $\text{rank } \theta = \frac{1}{2} \min(p, q)$. 


In particular, note that for $G = O(J_{p,q})$ with $|p - q| \leq 1$ one has
\[ \text{rank } \theta = \text{rank } G, \]
which is equivalent to saying that $\theta$ is a stable involution in the sense of [Tho13].

The relationship between the rank and regularity is given in the following lemma. It is a combination of [KR69, Lemma 2 and Proposition 8].

**Lemma 3.2.** An element $X \in g(-1)(F)$ is relatively regular if and only if
\[ \text{rank } \theta = \dim_F g(-1)^X. \]

\[ \square \]

4. **Computations in Symmetric Spaces**

In view of Lemma 2.1 to prove Theorem 1.1 it is enough to construct a relatively regular nilpotent element in $g(-1)(F)$ for each of the cases in the introduction. This is the goal of the ongoing section. Without loss of generality, assume $p \geq q$ from now on.

Let $\epsilon_n \in gl_n$ be a regular nilpotent element of the form
\[ \epsilon_n = \begin{pmatrix} I_{n-1} \\ 0 \end{pmatrix}, \]
and let $K_n \in gl_n$ be the diagonal matrix
\[ K_n = \begin{pmatrix} 1 \\ -1 \\ \vdots \\ (-1)^{n-1} \end{pmatrix}. \]

**General Linear Groups.** For $p \geq q$ and $n = p + q$, we let $G = GL_n$, $g = gl_n$, and $\theta$ be as in the introduction. We write $0_{m,n}$ for an $m \times n$ zero matrix and omit one index when $m = n$.

We will prove that
\[ e = \begin{pmatrix} I_q \\ 0_{q,1} I_q 0_{q,p-q-1} I_q 0_{p-q,q} \end{pmatrix} \]
is a relatively regular nilpotent element in $g(-1)$.

**Remark.** In case of $p = q$, the nilpotent element is defined as
\[ e = \begin{pmatrix} I_q \\ 0_{q,1} I_q \end{pmatrix}. \]
The nilpotency is obvious, and as mentioned in §3,

\[ \text{rank } \theta = q. \]

Thus, it suffices to prove the following lemma.

**Lemma 4.1.** The dimension of the centralizer of \( e \) in \( \mathfrak{g}(-1) \) is \( q \).

**Proof.** Note that

\[ \mathfrak{g}(-1)(F) = \left\{ \begin{pmatrix} B & A \end{pmatrix} \mid A \in M_{p,q}(F) \text{ and } B \in M_{q,p}(F) \right\}. \]

The centralizer of \( e \) is given by

\[ \mathfrak{g}(-1)^e(F) = \left\{ \begin{pmatrix} B & A \end{pmatrix} \in \mathfrak{g}(-1)(F) \mid \text{ad}_e \begin{pmatrix} B & A \end{pmatrix} = 0 \right\}. \]

Hence, for any element in the centralizer, the respective \( A \) and \( B \) satisfy

\[ \begin{pmatrix} a_{q,1} & I_q & a_{q,p-q-1} \end{pmatrix} A = B \begin{pmatrix} I_q \\ 0_{p-1,q} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{q,1} & I_q & a_{q,p-q-1} \end{pmatrix} = \begin{pmatrix} I_q \\ 0_{q-p,q} \end{pmatrix} B. \]

We have

\[ \begin{pmatrix} a_{q,1} & I_q & a_{q,p-q-1} \end{pmatrix} A = \begin{pmatrix} A_2 \\ \vdots \\ A_{q+1} \end{pmatrix} \]

where \( A_r \) is the \( r \)-th row of \( A \) and \( B( \begin{pmatrix} I_q \\ 0_{p-q,q} \end{pmatrix} ) = ( B_1 \cdots B_q ) \) for \( B_r \) being the \( r \)-th column of \( B \). Thus \( b_{i,j} = a_{i+1,j} \) for any \( 1 \leq i, j \leq q \).

Observe also \( A( \begin{pmatrix} a_{q,1} & I_q & a_{q,p-q-1} \end{pmatrix} ) = ( \begin{pmatrix} a_{q,1} & A \\ a_{q,1} & \cdots \\ a_{p,q-1} & \cdots \\ 0_{p,q} \end{pmatrix} ) \) and \( \begin{pmatrix} I_q \\ 0_{q-p,q} \end{pmatrix} B = ( \begin{pmatrix} B \\ 0_{p-q,q} \end{pmatrix} ) \). This implies that \( b_{i,j} = 0 \) for \( j = 1 \) or \( q + 1 < j \leq p \), and \( a_{q,i} = 0 \) for \( q < i \leq p \). Otherwise, \( b_{i,j} = a_{i,j-1} \).

Using the two results we also conclude that \( a_{i,j} = a_{i+1,j+1} \), the matrices \( A \) and \( B \) must be of the form:

\[ A = \begin{pmatrix} a_1 & a_2 & \cdots & a_q \\ a_1 & \cdots & \vdots & \vdots \\ \vdots & \vdots & a_2 & \vdots \\ 0_{p-q,q} & a_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & a_1 & \cdots & a_q \\ 0 & a_1 & \cdots & 0_{q-p-q-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & a_1 & \cdots & 0_{q-p-q-1} \end{pmatrix}. \]

Thus, the centralizer \( \mathfrak{g}(-1)^e \) has dimension \( q \). Notice that in the case of \( p = q \), the matrix \( B \) does not have the \( (q + 1) \)-th column.

**Corollary 4.1.** The element \( e \in \mathfrak{gl}_{p+q}(-1)(F) \) is nilpotent and relatively regular.
Symplectic Groups. Let $J'_{p,q}$ as defined in the introduction. Consider the symplectic group $G = \text{Sp}(J'_{p,q})$ acting on $\mathfrak{g} = \mathfrak{sp}(J'_{p,q})$ by conjugation. The involution $\theta$ is defined by the conjugation by $I_{p,q}$, and we will focus on the action of $G(1)$ on $\mathfrak{g}(-1)$ as before.

Throughout, let $r = \frac{p-q}{2}$. If $p \neq q$, let
$$e = \begin{pmatrix} 0 & 0_{r-1,q} \\ I_q & 0_{r+1,q} \\ 0_{p,r+1} & (-1)^r I_q \\ 0_{p,r-1} \end{pmatrix},$$
and if $p = q$, let
$$e = \begin{pmatrix} \epsilon \nu & \epsilon_q \\ \epsilon_q & \epsilon \nu \end{pmatrix}.$$

As mentioned in Section 3, $\text{rank} \theta = \frac{q}{2}$.

Furthermore, we have the following lemma.

**Lemma 4.2.** The dimension of the centralizer of $e$ in $\mathfrak{g}(-1)$ equals $\frac{q}{2}$.

**Proof.** We have, by definition,
$$\mathfrak{g}(-1)^e = \{X \in \mathfrak{g}(-1) \mid [X, e] = 0, \text{ and } \theta(X) = -X\}.$$
Let $X = \begin{pmatrix} A \\ B \end{pmatrix}$ with $A \in M_{p,q}$ and $B \in M_{q,p}$. Observe that by $[X, e] = 0$, we have
$$A\begin{pmatrix} 0_{r,r+1} & (-1)^r I_q \\ 0_{p,r-1} \end{pmatrix} = \begin{pmatrix} 0_{r-1,p} \\ B \end{pmatrix} \begin{pmatrix} 0_{r+1,p} \\ 0_{r+1,q} \end{pmatrix} = \begin{pmatrix} 0_{r-1,q} \\ I_q \end{pmatrix} B \quad (4.1)$$
Thus, by comparison we have the relations
$$a_{i,j} = 0 \quad \text{if } 1 \leq i \leq r-1 \text{ or } q + r \leq i \leq p \quad \text{, and}$$
$$b_{i,j} = 0 \quad \text{if } 1 \leq j \leq r+1 \text{ or } q + r + 2 \leq j \leq p.$$
For any unspecified $1 \leq i, j \leq p$, we have $b_{i,j} = (-1)^r a_{i,j}$. This determines the centralizer of $e$ to be of the form
$$\begin{pmatrix} 0_{r-1,q} \\ A_q \\ 0_{p,r+1} (-1)^r A_q \end{pmatrix},$$
where $A_q$ is a $q \times q$ matrix and we denote its entries by $a_{i,j}$ for $1 \leq i, j \leq q$. 

Also, observe that (4.1) also provides the relation $(-1)^{r}A_{q}N_{q}' = (-1)^{r}N_{q}'A_{q}$ for the $q \times q$ matrix $A_{q}$, where

\[
N_{q}' = \begin{pmatrix} I_{q-2} \\ 0_{2} \end{pmatrix}.
\]

Written explicitly, it reads $a_{i,j} = a_{i+2,j+2}$. Also, $a_{i,j} = 0$ if $i > q - 2$ and $j \leq q - 2$ or if $j > q - 2$ and $i \leq q - 2$.

Summarizing, we have

\[
A_{q} = \begin{pmatrix} a_{1} & a_{2} & \cdots & a_{q} \\ & a_{0} & b_{1} & b_{2} & \cdots & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & \ddots \\ & & & & a_{1} & a_{2} \\ & & & & & 0 & \cdots & 0 & a_{0} \\ & & & & & & & & b_{1} \\ & & & & & & & & & & & \end{pmatrix}.
\]

Finally, we apply the constraint that the centralizer lies in $g = \mathfrak{sp}(J_{p,q}')$, that is,

\[
J_{p,q}'XJ_{p,q}' = X^t.
\]

From this equation we obtain

\[
\begin{pmatrix} J_{p}' & J_{q}' \\ 0 & 0_{q+1} \end{pmatrix} \begin{pmatrix} 0_{r-1,q} & A_{q} & 0_{q,r-1} \\ 0_{q+1,r} & (-1)^{r}A_{q} & 0_{r+1} \end{pmatrix} = \begin{pmatrix} 0_{r+1,q} & A_{q}' \\ 0_{q,r+1} & (-1)^{r}A_{q}' \end{pmatrix}.
\]

Hence, when $i + j$ is even, we have $a_{q+1-i,q+1-j} = a_{j,i}$; and when $i + j$ is odd, we have $a_{q+1-i,q+1-j} = -a_{j,i}$. This determines the form

\[
A_{q} = \begin{pmatrix} a_{1} & 0 & a_{3} & \cdots & 0 \\ a_{1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & & \ddots & \ddots \\ a_{3} & \cdots & 0 & 0 \\ a_{1} & \cdots & \ddots & \ddots \\ \end{pmatrix}.
\]

It follows that the centralizer $g(-1)e$ has dimension $\frac{q}{2}$. \qed

**Corollary 4.2.** The elements $e \in \mathfrak{sp}(J_{p,q}')(-1)(F)$, constructed respectively, are nilpotent and relatively regular.

**Orthogonal Group.** For $|p - q| \leq 1$, we let $G = O(J_{p,q})$, and $\theta$ be as in the introduction. Throughout, without the loss of generality, assume $p \geq q$. As mentioned above, in this case $\text{rank } \theta = \text{rank } G = q$, so the involution $\theta$ is stable.

Again, we construct explicit formulae for $e$, and demonstrate by computation that the centralizer of $e$ in $g(-1)$ has dimension $q$.

**Case 1**
When $p = q + 1$, our element is given by
\[ e = \begin{pmatrix} I_q \\ 0_{q,1} & I_q \end{pmatrix}. \]

**Lemma 4.3.** The dimension of the centralizer of $e$ in $\mathfrak{g}(-1)$ equals $q$.

**Proof.** The centralizer of $e$ is given by
\[ g(-1)^e(F) = \left\{ \begin{pmatrix} B & A \end{pmatrix} \in g(-1)(F) \mid \begin{pmatrix} B & A \end{pmatrix} e = e \begin{pmatrix} B & A \end{pmatrix}, B = J_q A^T J_{q+1} \right\}. \]

The condition $\begin{pmatrix} B & A \end{pmatrix} e = e \begin{pmatrix} B & A \end{pmatrix}$ is the same as in the general linear case, and the matrices $A$ and $B$ must be of the form:
\[ A = \begin{pmatrix} a_1 & a_2 & \cdots & a_q \\ 0 & a_1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & a_1 & \cdots & a_q \\ 0 & 0 & a_1 & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & a_1 \end{pmatrix}. \]

Indeed, $B = J_q A^T J_{q+1}$ and thus the centralizer of $e$ in $g(-1)$ has dimension $q$. \qed

**Case 2**

When $p = q$, our element is given by
\[ e = \begin{pmatrix} J_q (e^*)^t J_q \\ e^* \end{pmatrix}, \]
where $e^*$ is obtained by shifting the first $\lceil q/2 \rceil$ entries of the identity $I_q$ to the right by one column. For convenience, $J$ will refer to $J_q$ in the following computation.

If $p = q = 2k$, then
\[ e^* = \begin{pmatrix} 0 & 1 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 1 & \cdots & \cdots \\ \vdots & \cdots & \ddots & \ddots \\ 0 & \cdots & \cdots & 1 \end{pmatrix} = \begin{pmatrix} \epsilon_k & \lambda_{k,k} \\ 0_k & I_k \end{pmatrix}, \]

where $\epsilon_k = \begin{pmatrix} 0 & I_{k-1} \end{pmatrix}$ and $\lambda_{k,k} = \begin{pmatrix} 0 & 0_{k-1} \end{pmatrix}$.

**Lemma 4.4.** The dimension of the centralizer of $e$ in $g(-1)$ equals $2k$. 

Proof. The centralizer consists of elements of the form \( (JZJ^t)^Z \). For detailed computation, write \( Z \) as the block matrix \( Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) with \( A, B, C, D \in M_k(F) \).

From the relation
\[
\left( JZJ^t \right) \left( J(e^*)^t J e^* \right) = \left( J(e^*)^t J e^* \right) \left( JXJ^t \right),
\]
we obtain
\[
ZJ(e^*)^t = e^*JZ \quad \text{and} \quad Z^tJe^* = (e^*)^tJZ.
\]

Expanding blockwise yields
\[
\begin{pmatrix} \epsilon_k & 0_k & \lambda_k & I_k \\ 0_k & I_k \end{pmatrix} J \begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} J \begin{pmatrix} \epsilon_k & 0_k \\ \lambda_k & I_k \end{pmatrix}
\]
and
\[
\begin{pmatrix} \epsilon_k & 0_k \\ \lambda_k & I_k \end{pmatrix} J \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix} J \begin{pmatrix} \epsilon_k & \lambda_k \\ 0_k & I_k \end{pmatrix}.
\]

Solving the equalities entrywise, we obtain
\[
A = \begin{pmatrix} 0 & a_1 & \cdots & a_{k-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & a_1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad B = B_1 + B_2 + B_3,
\]
\[
C = 0_k, \quad D = \begin{pmatrix} a_1 & a_{k-1} & a_k - a_{2k} \\ a_1 & \cdots & a_{k-1} \\ \vdots & \ddots & \vdots \\ a_1 & \cdots & \cdots & a_1 \end{pmatrix}.
\]

The matrices \( B_i \), with \( i = 1, 2, 3 \), are defined as

\[
B_1 = \begin{pmatrix} a_k & a_{k+1} & \cdots & a_{2k-2} & a_{2k-1} \\ a_k & \cdots & a_{2k-2} \\ \vdots & \ddots & \ddots & \vdots \\ a_k & a_{k+1} & a_{2k} \\ a_{2k} & \cdots & \cdots & \cdots & 0 \end{pmatrix},
\]

\[
B_2 = \begin{pmatrix} 0 & a_{k-1} & \cdots & \cdots \vdots & \ddots & \ddots & \vdots \vdots & \ddots & \ddots & \cdots & a_1 \\ a_2 & a_3 & \cdots & \cdots \\ a_1 & a_2 & \cdots & a_{k-1} & 0 \end{pmatrix}, \quad \text{and}
\]
\[
B_3 = \begin{pmatrix}
0 & a_{k-1} & \cdots \\
& a_{k-1} & \cdots \\
& & a_2 & \cdots \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]

Therefore, judging from the number of independent variables, the centralizer \(g(-1)^e\) has dimension \(2k\).

\[\square\]

If \(p = q = 2k + 1\), we define

\[
e^* = \begin{pmatrix}
0 & 1 \\
& & \ddots \\
& 0 & 1 \\
& & 0 & 1 \\
& & & & \ddots \\
& & & & & 0 & 1
\end{pmatrix} = \begin{pmatrix}
\epsilon_{k+1} & \lambda_{k+1,k} \\
0_{k,k+1} & I_k
\end{pmatrix}.
\]

**Lemma 4.5.** The dimension of the centralizer of \(e\) in \(g(-1)\) equals \(2k + 1\).

**Proof.** Similar to the previous proof, the centralizer is of the form \(\begin{pmatrix} JZ^t & Z \end{pmatrix}\) where \(Z\) can be written as \(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\) with \(A \in M_{k+1}(F), B \in M_{k+1,k}(F), C \in M_{k,k+1}(F),\) and \(D \in M_k(F)\).

The equality \(\begin{pmatrix} JZ^t & Z \end{pmatrix} \begin{pmatrix} J(e^*)^t & e^* \end{pmatrix} = \begin{pmatrix} J(e^*)^t & e^* \end{pmatrix} \begin{pmatrix} JX^t & Z \end{pmatrix}\) reduces to

\begin{align*}
ZJ(e^*)^t &= e^*JZ^t \\
ZJ^t &=(e^*)^tJZ.
\end{align*}

Writing these equalities blockwise yields

\[
\begin{pmatrix}
\epsilon_{k+1} & \lambda_{k+1,k} \\
0_{k,k+1} & I_k
\end{pmatrix} J \begin{pmatrix}
A^t & C^t \\
B^t & D^t
\end{pmatrix} = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} J \begin{pmatrix}
\epsilon_{k+1} & 0_{k+1,k} \\
0_{k+1} & I_k
\end{pmatrix},
\]

and

\[
\begin{pmatrix}
\epsilon_{k+1} & \lambda_{k+1,k} \\
\lambda_{k+1,k} & I_k
\end{pmatrix} J \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
A^t & C^t \\
B^t & D^t
\end{pmatrix} J \begin{pmatrix}
\epsilon_{k+1} & \lambda_{k+1,k} \\
0_{k+1} & I_k
\end{pmatrix}.
\]
Solving the equalities entrywise, we obtain

\[
A = \begin{pmatrix}
0 & a_1 & a_2 & \cdots & a_k & 0 \\
0 & a_1 & \cdots & a_{k-1} & \vdots & \\
\vdots & \ddots & \ddots & \ddots & \vdots & \\
0 & a_1 & \cdots & a_k & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\end{pmatrix}, \quad
\]

\[
B = B_1 + B_2 + B_3,
\]

\[
C = 0_{k,k+1},
\]

\[
D = \begin{pmatrix}
a_1 & \cdots & a_{k-1} & a_k \\
a_1 & \cdots & a_{k-1} & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & a_1
\end{pmatrix},
\]

whereas

\[
B_1 = \begin{pmatrix}
a_{k+1} & a_{k+2} & \cdots & a_{2k-1} & a_{2k} \\
0 & a_{k+1} & \cdots & a_{2k-1} & \cdots & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \\
a_{k+1} & \cdots & a_{k+2} & a_{2k+1} & 0 \\
0 & a_{2k+1} & 0 & -a_{2k+1}
\end{pmatrix},
\]

\[
B_2 = \begin{pmatrix}
0 \\
a_k & \cdots \\
\vdots & \ddots & \ddots \\
a_3 & a_4 & \cdots & \cdots \\
a_2 & a_3 & \cdots & a_k & 0 \\
a_1 & a_2 & \cdots & a_{k-1} & a_{k+1}
\end{pmatrix}, \quad \text{and}
\]

\[
B_3 = \begin{pmatrix}
0 \\
a_{k-1} & \cdots \\
\vdots & \ddots & \ddots \\
a_3 & \cdots & a_{k-1} & 0 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots \\
0_{2,k}
\end{pmatrix}.
\]

We observe that the dimension of \( g(-1)^e \) is exactly \( 2k + 1 \), as claimed. \( \square \)

**Corollary 4.3.** The elements \( e \in \mathfrak{o}(J_{p,q})(-1)(F) \), constructed respectively, are nilpotent and relatively regular.

5. **Proof of Corollary 1.2**

We place ourselves in the situation of Corollary 1.2. Assume first that \( G = GL_{p+q} \). In this case the rank of a matrix in \( M_{p,q}(\bar{F}) \) is invariant under the \( G(1) = GL_p(\bar{F}) \times GL_q(\bar{F}) \)-action,
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and all matrices of a given rank are in the same orbit. Thus the corollary is trivial in this case. We can therefore assume $G = O(J_{p,q})$ or $G = Sp(J'_{p,q})$, where in the orthogonal case we assume in addition that $|p - q| \leq 1$.

One has a linear isomorphism $\mathfrak{g}(-1) \simeq M_{p,q}$ given on points in an $F$-algebra $R$ by

$$\mathfrak{g}(-1)(R) \rightarrow M_{p,q}(R)$$

$$(Y \ X) \mapsto X$$

Note that for any element in $\mathfrak{g}(-1)(R)$, $Y$ is determined by its counterpart $X$ via the relation

$$Y = J_q X^t J_p$$

when $G = O(J_{p,q})$; and

$$Y = -J'_q X^t J'_p$$

when $G = Sp(J'_{p,q})$. This proves the map is an isomorphism. It is $G(1)$-equivariant with respect to the conjugation action on the left hand side and the the action explained before the statement of the corollary on the right hand side. Thus an orbit in $M_{p,q}$ is regular if and only if its preimage is regular. The corollary follows from Theorem 1.1. □

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