ON THE ABSTRACT BOGOMOLOV-TIAN-TODOROV THEOREM

DONATELLA IACONO

Abstract. We describe an abstract version of the Theorem of Bogomolov-Tian-Todorov, whose underlying idea is already contained in various papers by Bandiera, Fiorenza, Iacono, Manetti. More explicitly, we prove an algebraic criterion for a differential graded Lie algebras to be homotopy abelian. Then, we collect together many examples and applications in deformation theory and other settings.

1. Introduction

Let \( X \) be a compact Kähler manifold. If \( X \) has trivial canonical bundle, then, the so called Bogomolov-Tian-Todorov (BTT) Theorem states that the deformations of \( X \) are unobstructed. In [Bo78], Bogolomov proved it in the particular case of complex hamiltonian manifolds; later, Tian [Ti87] and Todorov [To89] proved independently the theorem for compact Kähler manifolds with trivial canonical bundle. More algebraic proofs of BTT theorem, based on \( T^1 \)-lifting Theorem and degeneration of the Hodge spectral sequence, were given in [Ra92] for \( \mathbb{K} = \mathbb{C} \) and in [Kaw92, FM99] for any \( \mathbb{K} \) as above.

For compact Kähler manifolds with torsion canonical bundle, the theorem follows from the more general fact that the derived infinitesimal deformations are unobstructed. The guiding principle is that in characteristic zero any deformation problem is controlled by a differential graded Lie algebra (DG-Lie algebra), with quasi-isomorphic DG-Lie algebras control the same deformation problem [GM88, Ko03]. More precisely, the deformation functor associated with the geometric problem is isomorphic to the deformation functor \( \text{Def}_L \) associated with a DG-Lie algebra \( L \) via Maurer-Cartan equation up to gauge equivalence.

Therefore, it is worth to have an explicit description of a DG-Lie algebra associated with the problem. It turns out that the obstructions to the smoothness of the functor \( \text{Def}_L \) are contained in the cohomology vector space \( H^2(L) \). However, if \( L \) is an abelian DG-Lie algebra then, even if \( H^2(L) \) is not zero, the functor \( \text{Def}_L \) is smooth, i.e., it has no obstructions. In particular, it is actually enough to prove that the DG-Lie algebra is homotopy abelian, i.e., quasi-isomorphic to an abelian DG-Lie algebra, to assure that the associated deformation functor is smooth.

For a compact complex manifold \( X \), the infinitesimal deformation are controlled by the Kodaira-Spencer DG-Lie algebras \( KS_X \) (Example 2.4). If \( X \) is a compact Kähler manifold with trivial or torsion canonical bundle, then the Lie version of BTT theorem asserts that \( KS_X \) is homotopy abelian (Section 4.1). For \( \mathbb{K} = \mathbb{C} \), this was first proved in the seminal paper by W.M. Goldman and J.J. Millson [GM90], see also [Ma04]. For any algebraically
closed field $\mathbb{K}$ of characteristic 0, this was proved in a completely algebraic way in [IM10], using the degeneration of the Hodge-to-de Rham spectral sequence and the notion of Cartan homotopy.

In [IM10], the proof involves $L_\infty$-algebras and $L_\infty$-morphisms and it is based on a series of algebraic results that were also applied in other context [IM13, Ia15, BM15, FM16]. In this paper, we review some of these ideas of independent interest to establish the following criterion, that we call Abstract Bomolov-Tian-Todorov Theorem, for homotopy abelianity of a DG-Lie algebra (Theorem 3.3).

**Theorem 1.1.** (Abstract BTT Theorem) Let $L$ and $M$ be DG-Lie algebras over a field $\mathbb{K}$ of characteristic 0 and $H \subseteq M$ a DG-Lie subalgebra. Assume that there exists a linear map $i \in \text{Hom}_{\mathbb{K}}^{-1}(L, M)$, $a \mapsto i_a$, of degree $-1$ such that:

1. $[i_a, i_b] = 0$ and $i_{[a, b]} = [i_a, di_b]$ for every $a, b \in L$;
2. $di_a + i_{da} \in H$ for every $a \in L$;
3. the inclusion $H \hookrightarrow M$ is injective in cohomology;
4. the induced morphism of complexes $\bar{i} : L \rightarrow (M/H)[−1]$ is injective in cohomology.

Then, the DG-Lie algebra $L$ is homotopy abelian.

According to [FiMa09, IM10], every linear map $i : L \rightarrow M$ of degree $-1$ that satisfies the previous condition (1) is called Cartan homotopy (Section 3). According to [FM16], any pair $(i, H)$ satisfying the above conditions (1) and (2) is called Cartan calculus.

The problem is already implicitly proved in [FiMa07, IM10], using $L_\infty$-algebras and $L_\infty$-morphisms. Here, we show an alternative self contained proof based on the same ideas but involving only DG-Lie algebras. The main motivation of this paper is to gather together various ideas in one place and to provide an easier and more accessible proof (only DG-Lie algebras). Moreover, we collect many examples and applications of the Abstract BTT Theorem in deformation theory and in other derived settings (Section 4 and Section 5). Among the others, we include the classical BTT Theorem for Calabi-Yau manifolds of [IM10], the logarithmic version for log Calaby-Yau pairs $(X, D)$ of [Ia15], the DG-Lie algebra associated with a Batalin-Vilkovisky algebra with the degeneration property [FiMa12, Ia15], and the DG-Lie algebra whose associated coderivation DG-Lie algebra has the splitting principle [BM15, Ba16].

The notion of homotopy abelianity is closely related to the notion of formality. A DG-Lie algebra is formal if it is quasi-isomorphic to its cohomology and so a DG-Lie algebra is homotopy abelian if and only if it is formal and $H^*(L)$ is abelian. Then, homotopy abelianity condition is stronger than formality and this could explain why it is easier to provide a criterion for it. Indeed, there are not so many analog of the previous theorem that guarantees the formality of a DG-Lie algebra. We refer to [Ka07, Lu10, FiMa12, Ma15, Ba16] and reference therein for formality criterion.

From the geometric point of view, homotopy abelianity assures the smoothness of the associated moduli problem while formality only implies that the singularity are not too bad (see [GM88, Mar09, Ma15, Ma] for more details). If we are only interested in the
smoothness of the problem, then instead of Hypothesis (4) it is enough to require that only $H^2(i)$ is injective (Remark 3.4).

The paper goes as follows: Section 2 is included for the non expert readers and it contains the relevant definitions and properties about DG-Lie algebras. Section 3 is devoted to the proof of the Abstract BTT Theorem. Some applications and examples are collected in Section 4, while Section 5 contains some generalizations and further applications.

Throughout the paper, we work over an algebraically closed field $K$ of characteristic 0, if it is not differently specified.

2. Background on DG-Lie algebras

A differential graded Lie algebra (DG-Lie algebra) is the data of a triple $(L, d, [ , ])$, where $(L, d)$ is a differential graded vector space (DG-vector space) and $[ , ] : L \times L \to L$ is a bilinear map of degree 0 (called bracket), such that the following conditions are satisfied:

1. (graded skewsymmetry) $[x, y] = - (-1)^{ij} [y, x] \in L^{i+j}$, for every $x \in L^i$ and $y \in L^j$;
2. (graded Jacobi identity) $[x, [y, z]] = [[x, y], z] + (-1)^{ij}[y, [x, z]]$, for every $x \in L^i$, $y \in L^j$ and $z \in L$;
3. (graded Leibniz rule) $d[x, y] = [dx, y] + (-1)^i[x, dy]$, for every $x \in L^i$ and $y \in L$.

In particular, the Leibniz rule implies that the bracket of a DG-Lie algebra $L$ induces a structure of graded Lie algebra on its cohomology $H^*(L) = \bigoplus H^i(L)$. A DG-Lie algebra is called contractible if $H^*(L) = 0$. A DG-Lie algebra is called abelian if its bracket is trivial.

Example 2.1. If $L = \bigoplus L^i$ is a DG-Lie algebra, then $L^0$ is a Lie algebra in the usual sense; vice-versa, every Lie algebra is a differential graded Lie algebra concentrated in degree 0.

Example 2.2. Let $(V, d_V)$ be a differential graded vector space over $K$ and $\text{Hom}^i_K(V, V)$ the space of the linear map $V \to V$ of degree $i$. Then, $\text{Hom}^*_K(V, V) = \bigoplus \text{Hom}^i_K(V, V)$ is a DG-Lie algebra with bracket

$[f, g] = fg - (-1)^{\deg(f) \deg(g)} gf,$

and differential $d$ given by

$d(f) = [d_V, f] = d_V f - (-1)^{\deg(f)} f d_V.$

For later use, we point out that by Künneth formula, there exists a natural isomorphism

$H^*(\text{Hom}^*_K(V, V)) \cong \text{Hom}^*_K(H^*(V), H^*(V)).$

Example 2.3. Let $M$ be a DG-Lie algebra and $K[t, dt]$ the differential graded algebra of polynomial differential forms over the affine line. More precisely, $K[t, dt] = \mathbb{K}[t] \otimes \mathbb{K}[t] dt$, where $t$ has degree 0 and $dt$ has degree 1. Then, $M[t, dt] = M \otimes K[t, dt]$ is a DG-Lie algebra. As vector space $M[t, dt]$ is generated by elements of the form $m p(t) + n q(t) dt$, with $m, n \in M$ and $p(t), q(t) \in \mathbb{K}[t]$. The differential and the bracket on $M[t, dt]$ are defined as follows:

$d(mp(t) + nq(t) dt) = (dm)p(t) + (-1)^{\deg(m)}mp'(t) dt + (dn)q(t) dt,$

$[mp(t), nq(t)] = [m, n] p(t) q(t), \quad [mp(t), nq(t) dt] = [m, n] p(t) q(t) dt.$

Note that $[mdt, ndt] = 0$, for every $m, n \in M$. 

ON THE ABSTRACT BTT THEOREM 3
Example 2.4. Let $\Theta_X$ be the holomorphic tangent bundle of a complex manifold $X$. The Kodaira-Spencer DG-Lie algebra of $X$ is

$$KS_X = (\bigoplus_i \Gamma(X, A^{0,i}_X(\Theta_X))) = \bigoplus_i A^{0,i}_X(\Theta_X), d, [\ , \ ],$$

where $KS_X = A^{0,i}_X(\Theta_X)$ is the vector space of the global sections of the sheaf of germs of the differential $(0, i)$-forms with coefficients in $\Theta_X$. $d$ is the opposite of Dolbeault’s differential and the bracket is the extension of the usual bracket of vector fields. Explicitly, if $z_1, \ldots, z_n$ are local holomorphic coordinates on $X$, we have

$$d \left( f dz_1 \frac{\partial}{\partial z_i} \right) = -\overline{\partial}(f) \wedge d\overline{z}_i \frac{\partial}{\partial z_i};$$

$$\left[f \frac{\partial}{\partial z_i} dz_i, g \frac{\partial}{\partial z_j} dz_j \right] = \left(f \frac{\partial g}{\partial z_i} \frac{\partial}{\partial z_j} - g \frac{\partial f}{\partial z_j} \frac{\partial}{\partial z_i}\right) dz_i \wedge dz_j, \quad \forall \ f, g \in A^{0,0}_X.$$

Example 2.5. Let $D$ be a submanifold of a complex manifold $X$ of codimension 1. We denote by $\Theta_X(-\log D)$ the sheaf of germs of the tangent vectors to $X$ which are tangent to $D$ [Se06, Section 3.4.4]. Denoting by $J \subset \mathcal{O}_X$ the ideal sheaf of $D$ in $X$, then $\Theta_X(-\log D)$ is the subsheaf of the derivations of the sheaf $\mathcal{O}_X$ preserving the ideal sheaf $J$ of $D$. Moreover, we have the following short exact sequence

$$0 \rightarrow \Theta_X(-\log D) \rightarrow \Theta_X \rightarrow N_{D/X} \rightarrow 0.$$

Since we are in codimension 1, the sheaf $\Theta_X(-\log D)$ is dual to the sheaf $\Omega^1_X(\log D)$ of logarithmic differentials, so it is in particular locally free, see for instance [De70, p. 72], [EV92, Chapter 2] or [Vo02, Chapter 8].

Then, we can define the DG-Lie algebra of the pair $(X, D)$

$$KS_{(X,D)} = (\bigoplus_i \Gamma(X, A^{0,i}_X(\Theta_X(-\log D))), d, [\ , \ ]).$$

Note that $KS_{(X,D)}$ is a DG-Lie subalgebra of $KS_X$.

2.1. Morphisms of DG-Lie algebras. A morphism of DG-Lie algebras is a linear map $\varphi: L \rightarrow M$ that preserves degrees and commutes with brackets and differentials. A quasi-isomorphism of DG-Lie algebras is a morphism that induces an isomorphism in cohomology.

Two DG-Lie algebras $L$ and $M$ are said to be quasi-isomorphic if they are equivalent under the equivalence relation generated by quasi-isomorphisms.

Example 2.6. Let $M$ be a DG-Lie algebra and $M[t, dt]$ the DG-Lie algebra introduced in Example 2.3. Then, for every $a \in \mathbb{K}$, we can define the evaluation morphism

$$e_a : M[t, dt] \rightarrow M,$$

$$e_a \left( \sum m_i t^i + n_i dt \right) = \sum m_i a^i.$$

Note that, every $e_a$ is a morphism of DG-Lie algebras which is a left inverse of the inclusion $i : M \rightarrow M[t, dt]$, i.e., $e_a \circ i = \text{Id}_M$. In particular, $e_a$ is also a surjective quasi-isomorphism. We often use the short notation $m(a) = e_a(m(t, dt))$, for every $m(t, dt) \in M[t, dt]$.

For every two morphisms of DG-Lie algebras $f : L \rightarrow N$ and $g : M \rightarrow N$, we can consider the pull-back:
ON THE ABSTRACT BTT THEOREM

\[
\begin{array}{ccc}
L \times_N M & f' \rightarrow & M \\
\downarrow g' & & \downarrow g \\
L & f \rightarrow & N.
\end{array}
\]

Note that if \( g \) (or \( f \)) is a surjective quasi-isomorphism then \( g' \) (or \( f' \)) is also a surjective quasi-isomorphism.

**Lemma 2.7.** *(Factorisation Lemma)* Let \( f : L \rightarrow M \) be a morphism of DG-Lie algebras, then there exists a DG-Lie algebra \( P \) and a factorisation

\[
\begin{array}{ccc}
L & f \rightarrow & M \\
\downarrow i & & \downarrow g \\
P & \rightarrow & M
\end{array}
\]

such that \( g : P \rightarrow M \) is a surjective morphism and \( i : L \rightarrow P \) is an injective quasi-isomorphism (which is a right inverse of a surjective quasi-isomorphism).

**Proof.** An explicit factorisation can be defined as follow. Let \( P_f \) be the DG-Lie algebra

\[
P_f = \{(x, m(t, dt)) \in L \times M[t, dt] \mid m(1) = f(x)\};
\]

note that \( P_f \) is given by the pull back diagram

\[
\begin{array}{ccc}
P_f & \rightarrow & M[t, dt] \\
p \downarrow & & \downarrow e_1 \\
L & f \rightarrow & M,
\end{array}
\]

where \( p \) is the projection on the first factor. In particular, since \( e_1 \) is a surjective quasi-isomorphism, \( p \) is also a surjective quasi-isomorphism. Next, define

\[
i : L \rightarrow P_f \quad i(x) = (x, f(x)), \quad \forall x \in L;
\]

and

\[
g : P_f \rightarrow M \quad g(x, m(t, dt)) = m(0), \quad \forall (x, m(t, dt)) \in P_f.
\]

The morphism \( g \) is surjective: for any \( m \in M \), there exists \( (0, (1-t)m) \in P_f \) such that \( g(0, (1-t)m) = e_0((1-t)m) = m \). As regard the morphism \( i \), it is a right inverse of \( p \), i.e., \( pi = Id_L \) and so it is an injective quasi-isomorphism. Finally, we have

\[
(g \circ i)(x) = g(x, f(x)) = f(x), \quad \forall x \in L.
\]

\[\Box\]

**Corollary 2.8.** Let \( L \) and \( M \) be DG-Lie algebras. Then, \( L \) and \( M \) are quasi-isomorphic if and only if there exist a DG-Lie algebra \( P \) and two surjective quasi-isomorphisms \( p : P \rightarrow L \) and \( q : P \rightarrow M \).
Proof. The DG-Lie algebras $L$ and $M$ are quasi-isomorphic if and only if there exists a sequence of quasi-isomorphisms

By Factorisation Lemma 2.7, we can assume that all the quasi-isomorphisms in the sequence are surjective. Indeed, consider the following diagram

and apply Factorisation Lemma 2.7 to the morphism $(f, t): K \to L \times N$ in order to obtain a diagram of quasi-isomorphisms

where the two horizontal arrows are surjective. Finally, any sequence of surjective quasi-isomorphisms can be replaced by two surjective quasi-isomorphisms using fibre product and the fact that surjective quasi-isomorphisms are stable under pull backs

\[ K_1 \times_{H_1} K_2 \]

\[ L \quad K_1 \rightarrow \quad K_2 \rightarrow \quad M. \]

\[ \square \]

Definition 2.9. A DG-Lie algebra $L$ is called formal if it quasi-isomorphic to $H^*(L)$ (intended as a DG-Lie algebra with trivial differential).

A DG-Lie algebra $L$ is called homotopy abelian if it is quasi-isomorphic to an abelian DG-Lie algebra.

Example 2.10. Any DG-vector space is formal (and abelian as DG-Lie algebra). Let $(V, d_V)$ be a DG-vector space, then the DG-Lie algebra $\text{Hom}^*_K(V, V)$ is formal.

Lemma 2.11. (Transfer Lemma) Let $f : L \to M$ be a morphism of DG-Lie algebras and denote by $H^*(f) : H^*(L) \to H^*(M)$ the induced morphism in cohomology.

1. If $M$ is homotopy abelian and $H^*(f)$ is injective, then $L$ is also homotopy abelian.
(2) If \( L \) is homotopy abelian and \( H^*(f) \) is surjective, then \( M \) is also homotopy abelian.

Proof. There are various proofs of this fact, see for instance [KKP08, Proposition 4.11], [IM10, Lemma 1.10]. We follow the proof given in [FM16, Lemma 2.7]. As regard (1), since \( M \) is homotopy abelian, Corollary 2.8 implies the existence of an abelian DG-Lie algebra \( A \) and two surjective quasi-isomorphisms

\[
\begin{array}{ccc}
K & \xrightarrow{a} & A \\
\downarrow{m} & & \\
M & \xrightarrow{f} & A.
\end{array}
\]

Applying the pull back we obtain the diagram

\[
\begin{array}{ccc}
L & \xrightarrow{f} & K \\
\downarrow{m'} & & \\
L & \xrightarrow{m} & M.
\end{array}
\]

where \( m, m' \) and \( a \) are surjective quasi-isomorphisms and the morphisms \( f \) and \( f' \) induce injective morphisms in cohomology.

To conclude the proof it is enough to consider a graded vector space \( E \) with a projection \( e : A \rightarrow E \), such that \( eaf' \) is a quasi-isomorphism:

\[
\begin{array}{ccc}
L \times_M K & \xrightarrow{f'} & K \\
\downarrow{m'} & & \\
L & \xrightarrow{m} & M.
\end{array}
\]

As regard (2), since \( L \) is homotopy abelian, Corollary 2.8 implies the existence of an abelian DG-Lie algebra \( A \) with trivial differential and two surjective quasi-isomorphisms

\[
\begin{array}{ccc}
K & \xrightarrow{a} & L \\
\downarrow{i} & & \\
A & \xrightarrow{h} & A.
\end{array}
\]

Then, we can choose a graded Lie algebra \( H \) together with a morphism \( h : H \rightarrow A \) such that the composition

\[
H \xrightarrow{h} A \xrightarrow{H^*(a)^{-1}} H^*(K) \xrightarrow{H^*(f')} H^*(L) \xrightarrow{H^*(f)} H^*(M),
\]

is an isomorphism. Finally, taking the fibre product of \( h \) and \( a \), we obtain a commutative diagram

\[
\begin{array}{ccc}
H \times_A K & \xrightarrow{h'} & K \\
\downarrow{a'} & & \\
H & \xrightarrow{h} & A, \\
\downarrow{a} & & \\
L & \xrightarrow{f} & M.
\end{array}
\]
where $a, a'$ and $fh'$ are quasi-isomorphisms.

**Remark 2.12.** From the point of view of deformation theory, we could be only interested in analysing the obstruction problem. We already noticed that the obstructions of the deformation functor $\text{Def}_L$ associated with a DG-Lie algebra $L$ are contained in the vector space $H^2(L)$. Moreover, any morphism of DG-Lie algebras $f : L \to M$ induces a morphism $f : \text{Def}_L \to \text{Def}_M$ of the associated deformation functors, that behaves well with respect to the obstructions. If $M$ is homotopy abelian, then $\text{Def}_M$ is smooth. Therefore, it is enough that the morphism $H^2(f) : H^2(L) \to H^2(M)$ is injective for the smoothness of the functor $\text{Def}_L$.

**Definition 2.13.** Let $f : L \to M$ be a morphism of DG-Lie algebras, the homotopy fibre of $f$ is defined as the DG-Lie algebra $TW(f) = \{(x, m(t, dt)) \in L \times M[t, dt] \mid m(0) = 0, m(1) = f(x)\}$.

Note that the projection $TW(f) \to L$ is a morphism of DG-Lie algebras.

**Remark 2.14.** Let $f : L \to M$ be a morphism of DG-Lie algebras and $\xymatrix{L \ar[r]^i \ar[d] & P \ar[r]^g \ar[d] & M}$ the explicit factorisation, given in the proof of Factorisation Lemma 2.7. Then, $TW(f) = \ker g$. It can be proved that for any other factorisation $\xymatrix{L \ar[r]^{i'} \ar[d] & P' \ar[r]^{g'} \ar[d] & M}$, the kernel $\ker g'$ is quasi-isomorphic to $TW(f)$ [Ma, Section 6.1]. Moreover, every commutative diagram of morphisms of DG-Lie algebras:

$$
\begin{array}{ccc}
L & \xrightarrow{f} & M \\
\downarrow & & \downarrow \\
L' & \xrightarrow{f'} & M',
\end{array}
$$

induces a morphism of the homotopy fibres $TW(f) \to TW(f')$.

**Remark 2.15.** If $f : L \to M$ is an injective morphism of DG-Lie algebras, then its cokernel $M/f(L)$ is a DG vector space and the map $TW(f) \to M/f(L)[-1]$ 

$$(x, p(t)m_0 + q(t)dtm_1) \mapsto \left(\int_0^1 q(t)dt\right)m_1 \pmod{f(L)},$$

is a surjective quasi-isomorphism.

**Lemma 2.16.** (Homotopy fibre) Let $f : L \to M$ be a morphism of DG-Lie algebras.

1. If the induced morphism $H^*(f) : H^*(L) \to H^*(M)$ is injective, then $TW(f)$ is homotopy abelian.
2. If the induced morphism $H^1(f) : H^1(L) \to H^1(M)$ is injective, then $\text{Def}_{TW(f)}$ is unobstructed.

**Proof.** [IM10, Proposition 3.4] , [IM13, Lemma 2.1] or [FM16, Corollary 2.8]. As regard (1), consider the DG vector space $M[-1]$ as an abelian DG-Lie algebra and the morphism of DG-Lie algebras $\rho : M[-1] \to TW(f) \forall m \in M \quad \rho(m) = (0, dtm)$. 
According to (2) of Lemma 2.11, it is enough to show that \( \rho \) induces a surjective map in cohomology but this follows from the exact sequence
\[
\cdots \to H^{i-1}(M) \to H^i(TW(f)) \to H^i(L) \xrightarrow{H^i(f)} H^i(M) \to \cdots,
\]
since the morphisms \( H^i(f) \) are all injective by hypothesis.

As regard (2), the morphism of DG-Lie algebras \( \rho : M[-1] \to TW(f) \) induces a morphism of deformation functors \( \rho : \text{Def} M[-1] \to \text{Def}_{TW(f)} \), with \( \text{Def} M[-1] \) a smooth functor. Then, by the Standard Smoothness Criterion [FM99], [Ma09, Theorem 4.11], since \( \rho \) is injective on obstructions, if \( H^1(\rho) \) is surjective then \( \text{Def}_{TW(f)} \) is smooth. By the above exact sequence, if \( H^1(f) : H^1(L) \to H^1(M) \) injective then \( H^1(\rho) \) is surjective.

\( \square \)

**Example 2.17.** Let \( W \) be a differential graded vector space \( U \subset W \) be a DG subspace. If the induced morphism in cohomology \( H^*(U) \to H^*(W) \) is injective, then the inclusion of DG-Lie algebras

\[
f : \{ f \in \text{Hom}_K(W,W) \ | \ f(U) \subset U \} \to \text{Hom}_K(W,W)
\]

satisfies the hypothesis of Lemma 2.16 and so the DG-Lie algebra \( TW(f) \) is homotopy abelian [IM10, Example 3.5] and [FM16, Proposition 5.10].

The deformation functor associated with the DG-Lie algebra \( TW(f) \) has a natural interpretation as the local structure of the derived Grassmannian of \( W \) at the point \( U \). Therefore, the derived Grassmannian of \( W \) is smooth at the points corresponding to subspaces \( U \) such that \( H^*(U) \to H^*(W) \) is injective [FM16].

### 3. Cartan homotopies and Main Theorem

Let \( L \) and \( M \) be two DG-Lie algebras. A **Cartan homotopy** is a linear map of degree \(-1\)

\[
i : L \to M
\]
such that, for every \( a, b \in L \), we have:

\[
i_{[a,b]} = [i_a, d_M i_b] \quad \text{and} \quad [i_a, i_b] = 0.
\]

For every Cartan homotopy \( i \), it is defined the Lie derivative map

\[
l : L \to M, \quad l_a = d_M i_a + i_{d_M a}.
\]

It follows from the definition of \( i \) that \( l \) is a morphism of DG-Lie algebras and we can write the conditions of being a Cartan homotopy as

\[
i_{[a,b]} = [i_a, l_b] \quad \text{and} \quad [i_a, i_b] = 0.
\]

Note that, as a morphism of complexes, \( l \) is homotopic to 0 (with homotopy \( i \)).

**Example 3.1.** Let \( X \) be a smooth variety. Denote by \( \Theta_X \) the tangent sheaf and by \((\Omega^*_X, d)\) the algebraic de Rham complex. Then, for every open subset \( U \subset X \), the contraction of a vector with a differential form

\[
\Theta_X(U) \otimes \Omega^k_X(U) \xrightarrow{i} \Omega^{k-1}_X(U)
\]

induces a linear map of degree \(-1\)

\[
i : \Theta_X(U) \to \text{Hom}^*(\Omega^*_X(U), \Omega^*_X(U)), \quad i_\xi(\omega) = \xi \lrcorner \omega
\]
that is a Cartan homotopy. Indeed, the above conditions coincide with the classical Cartan’s homotopy formulas.

**Example 3.2.** Let $D$ be a smooth subvariety of codimension 1 of a smooth variety $X$. Let $(\Omega_X^*(\log D), d)$ be the logarithmic differential complex and $\Theta_X(-\log D)$ the logarithmic tangent sheaf. It is easy to prove explicitly that for every open subset $U \subset X$, we have

$$(\Theta_X(-\log D)(U) \to \Omega_X^*(\log D)(U)) \subset \Omega_X^{* -1}(\log D)(U).$$

Then, as above, the induced linear map of degree $-1$

$$i: \Theta_X(-\log D)(U) \to \text{Hom}^*(\Omega_X^*(\log D)(U), \Omega_X^*(\log D)(U)), \quad i_\xi(\omega) = \xi \omega$$

is a Cartan homotopy.

We are now ready to prove the main theorem.

**Theorem 3.3.** (Abstract BTT Theorem) Let $L$ and $M$ be DG-Lie algebras over a field $K$ of characteristic 0 and $H \subseteq M$ a DG-Lie subalgebra. Assume that there exists a linear map $i \in \text{Hom}_K^{-1}(L, M)$, $a \mapsto i_a$, of degree $-1$ such that:

1. $[i_a, i_b] = 0$ and $i_{[a,b]} = [i_a, di_b]$ for every $a, b \in L$;
2. $di_a + i_{da} \in H$ for every $a \in L$;
3. the inclusion $H \hookrightarrow M$ is injective in cohomology;
4. the induced morphism of complexes $i: L \to (M/H)[-1]$ is injective in cohomology.

Then, the DG-Lie algebra $L$ is homotopy abelian.

**Proof.** Let $s$ be a formal variable of degree $-1$ and consider the commutative DG-algebra $K[s]$; note that $s^2 = 0$ and the differential $d$ on $K[s]$ is defined as

$$d: Ks \to K, \quad d(s) = 1.$$

In particular, $K[s]$ is a contractible DG algebra and the inclusion $K \hookrightarrow K[s]$ is a morphism of DG algebras.

Next, we consider the DG-Lie algebra $K[s] \otimes L$. For all $s \otimes a \in K[s] \otimes L$, we have $\text{deg}(s \otimes a) = \text{deg}(a) - 1$ and

$$d(s \otimes a) = 1 \otimes a - s \otimes da.$$

Then, we can define a morphism of DG-Lie algebras by

$$\varphi: K[s] \otimes L \to M, \quad \varphi(s \otimes a) = i_a;$$

in particular, $\varphi(1 \otimes a) = \varphi(d(s \otimes a) + s \otimes da) = d(\varphi(s \otimes a)) + i_{da} = di_a + i_{da} = l_a$, for any $a \in L$, and it is contained in $H$ by Hypothesis (2).

Thus, we can construct a commutative diagram of morphisms of DG-Lie algebras

$$
\begin{array}{ccc}
L & \xrightarrow{\psi} & H \\
\downarrow{\alpha} & & \downarrow{\chi} \\
K[s] \otimes L & \xrightarrow{\varphi} & M,
\end{array}
$$
where \( \alpha(a) = 1 \otimes a \) and \( \psi(a) = l_a \), for any \( a \in L \).

According to Remark 2.14, this diagram induces a morphisms of DG-Lie algebras

\[ \phi : TW(\alpha) \to TW(\chi). \]

Hypothesis (3) and Lemma 2.16 applied to the inclusion \( \chi : H \to M \) imply that the DG-Lie algebra \( TW(\chi) \) is homotopy abelian. Moreover, according to Remark 2.15, \( TW(\chi) \) is quasi-isomorphic as a DG vector spaces to \( (M/H)[-1] \). Since \( K[s] \otimes L \) is contractible, then \( TW(\alpha) \to L \) is a quasi-isomorphism. Finally, Hypothesis (4) implies that \( \phi \) is injective in cohomology and so Lemma 2.16 implies that \( TW(\alpha) \) is homotopy abelian. It follows that \( L \) is also homotopy abelian.

**Remark 3.4.** In the above notation and using the first three hypothesis, we have constructed a diagram:

\[
\begin{array}{ccc}
TW(\alpha) & \xrightarrow{\phi} & TW(\chi) \\
\downarrow & & \downarrow \\
L & & 
\end{array}
\]

where the vertical map is a quasi-isomorphism. Then, Hypothesis (4) implies that the horizontal map is injective in cohomology. If we are only interested in the analysis of the obstruction of \( \text{Def}_L \), then Hypothesis (3) can be relaxed to the condition that only \( H^1(\chi) \) is injective and Hypothesis (4) can be relaxed to the condition that only \( H^2(\phi) \) is injective. Indeed, \( \text{Def}_{TW(\chi)} \) is unobstructed, and so for the vanishing of the obstructions of \( \text{Def}_L \) it is enough that \( H^2(\phi) \) is injective (see Section 5 for further generalizations).

**4. Examples and Applications**

In this section, we collect some applications of the main Theorem 3.3.

**4.1. Deformations of compact manifolds.** Let \( X \) be a holomorphic compact manifold and denote by \( \Theta_X \) its holomorphic tangent bundle. We introduced the Kodaira Spencer DG-Lie algebra \( KS_X \) in Example 2.4. Let \( (A_X, d) = (\bigoplus_{p,q} \Gamma(X, A^{p,q}_X), d = \partial + \overline{\partial}) \) be the De Rham complex of \( X \), where \( A^{p,q}_X \) denotes the sheaf of \((p, q)\)-differential forms on \( X \) and consider the DG-Lie Algebra \( M = \text{Hom}^\bullet_C(A^*_X, A^*_X) \).

**Corollary 4.1.** Let \( X \) be a compact manifold with torsion canonical bundle such that the Hodge-de Rham spectral sequence degenerates at \( E_1 \)-level. Then, \( KS_X \) is homotopy abelian.

**Proof.** [Ra92, Corollary 2], [Ma04a, Corollary B], [IM10, Corally 6.5]. Let us first consider the case in which the canonical bundle is trivial. The contraction of vector fields and differential forms together with the cup product defines a Cartan homotopy [Ma09, Section 6]

\[ i : KS_X \to M = \text{Hom}^*(A^*_X, A^*_X), \quad i_{\eta}(\omega) = \eta \wedge \omega, \quad \forall \ \eta \in KS_X, \ \forall \ \omega \in A_X. \]

Moreover, for any \( \eta \in KS_X, l_\eta \in H = \{ \varphi \in M \mid \varphi(A^n_X) \subset A^{n+1}_X \} \), where \( n \) is the dimension of \( X \). By the degeneration property the inclusion \( H \hookrightarrow M \) is injective in cohomology. The canonical bundle is trivial and so the cup product with a non trivial
section of it, gives the isomorphisms $H^i(\Theta_X) \cong H^i(\Omega_X^{n-1})$, where $\Omega_X^{n-1}$ denotes the sheaf of holomorphic differential $n - 1$ forms. Then, $H^*(K_S X) \to \text{Hom}^*(H^0(\Omega_X^1), H^*(\Omega_X^{n-1}))$ is injective and this implies that $K S_X \to M/H[-1]$ is also injective in cohomology. Therefore, the hypothesis of Theorem 3.3 are satisfied.

In the case of torsion canonical bundle, there exists $m > 0$ such that $K^n_X = \mathcal{O}_X$, where $K_X$ denote the canonical bundle. Next, consider the unramified $m$-cyclic cover, defined by $K_X$, i.e., $\pi : Y = \text{Spec}(\mathbb{C}[m^{-1} L^{-1}]) \to X$ (see [Pa91] for full details on abelian covers). Then, $\pi : Y \to X$ is a finite flat map of degree $m$ and $Y$ is a compact manifold with trivial canonical bundle, since $K_Y \cong \pi^* K_X \cong \mathcal{O}_Y$. Therefore, $K S_Y$ is homotopy abelian. Finally, it is enough to consider the morphism of DG-Lie algebras $K S_X \to K S_Y$ induced by pull back. This morphism is injective in cohomology and so (1) of Lemma 2.11 implies that $K S_X$ is also homotopy abelian.

\[ \square \]

**Remark 4.2.** The degeneration hypothesis is satisfied if the $\bar{\partial}Q$-Lemma holds, for instance for Kähler manifolds.

**Remark 4.3.** It is well known that the Kodaira Spencer DG-Lie algebra $K_S X$ controls the infinitesimal deformations of $X$. Then, the infinitesimal deformations of a compact Kähler manifold with torsion canonical bundle $X$ are unobstructed. This is the classical Bogomolov-Tian-Todorov Theorem.

**Remark 4.4.** Let $X$ be a smooth projective variety over an algebraically closed field $\mathbb{K}$ of characteristic 0. Then, the analogous of Corollary 4.1 holds.

In this case, we can replace the Kodaira-Spencer DG-Lie algebra with the DG-Lie algebra $\text{Tot}(\Theta_X(U))$, obtained applying the Thom-Whitney totalization to the semicosimplicial DG-Lie algebras $\Theta_X(U)$, for any affine open cover $U$ of $X$ [IM10, Theorem 5.3]. Also in this case, if the canonical bundle of $X$ is torsion, then $\text{Tot}(\Theta_X(U))$ is homotopy-abelian and so $X$ has unobstructed deformations [IM10, Theorem 6.2 and Corollary 6.5].

### 4.2. Deformations of pairs (divisor, manifold)

Let $D$ be a smooth divisor in a compact manifold $X$. We introduced the Kodaira Spencer DG-Lie algebra of the pair $K S_{(X,D)}$ in Example 2.5. Then, considering $\Theta_X(- log D)$ instead of $\Theta_X$, we can proceed as in the case of $K S_X$ of Corollary 4.1.

**Corollary 4.5.** Let $D$ be a smooth divisor in a compact manifold $X$, such that the logarithmic canonical bundle $\Omega^X_X(- log D)$ is trivial and the logarithmic Hodge-de Rham spectral sequence degenerates at $E_1$-level. Then, $K S_{(X,D)}$ is homotopy abelian.

**Proof.** [Ia15, Theorem 5.1 and Corollary 5.4]. The proof is analogous to the one of Corollary 4.1. Here, the Cartan homotopy is given by the contraction of logarithmic tangent vector and logarithmic differentials:

$$i : K S_{(X,D)} \to M = \text{Hom}^n_{\mathcal{O}_X}(A_X^{n-1}(- log D), A_X^{n-1}(log D)).$$

Then, for any $\eta \in K S_{(X,D)}, l_\eta \in H = \{ \varphi \in M \mid \varphi(A_X^{n-1}(log D)) \subset A_X^{n-1}(log D) \}$, where $n$ is the dimension of $X$.

Next, the degeneration property implies that the inclusion $H \hookrightarrow M$ is injective in cohomology. Since the logarithmic canonical bundle $\Omega^X_X(- log D)$ is trivial, the cup product with a non trivial section of it, gives the isomorphisms $H^*(\Theta_X(log D)) \cong H^*(\Omega^X_X(- log D)).$
This implies that $H^*(KS_{(X,D)}) \to \text{Hom}^*(H^0(\Omega^n_X(\log D)), H^*(\Omega^{n-1}_X(\log D)))$ is injective and so $KS_{(X,D)} \to M/H[-1]$ is also injective in cohomology. Therefore, the hypothesis of Theorem 3.3 are satisfied.

\[ \square \]

**Remark 4.6.** The degeneration hypothesis is satisfied for any globally normal crossing divisor $D$ in a compact Kähler manifold [Vo02, Theorem 8.35].

**Remark 4.7.** It can be also proved that $KS_{(X,D)}$ is homotopy abelian for a smooth divisor $D$ in a Calabi-Yau manifold $X$ [Ia15, Theorem 4.7]. In this case, the relevant spectral sequence is the one associated with the Hodge filtration

$$E_1^{p,q} = H^q(X, \Omega^p_X(\log D) \otimes \mathcal{O}_X(-D)) \Rightarrow H^{p+q}(X, \Omega^*_X(\log D) \otimes \mathcal{O}_X(-D))$$

and it degenerates at the $E_1$-level [Fuj, Section 4.3].

**Remark 4.8.** It is well known that the Kodaira Spencer DG-Lie algebra $KS_{(X,D)}$ controls the infinitesimal deformations of the $(X, D)$. Then, the infinitesimal deformations of the pair $(X, D)$ are unobstructed when $D$ is a smooth divisor in a compact Kähler manifold $X$ such that $\Omega^*_X(\log D)$ is trivial [Ia15, Corollary 4.5] and when $D$ is a smooth divisor in a compact Calabi-Yau manifold $X$ [Ia15, Corollary 4.8].

**Remark 4.9.** In general, if the ground field is an algebraically closed field of characteristic 0, we can replace the DG-Lie algebra $KS_{(X,D)}$ with the DG-Lie algebra $TW(\Theta_X(-\log D)(\mathcal{U}))$ obtained applying the Thom-Whitney realisation to the semicosimplicial DG-Lie algebras $\Theta_X(-\log D)(\mathcal{U})$, for any affine open cover $\mathcal{U}$ of $X$ [Ia15, Theorem 4.3]. Also in this case, $TW(\Theta_X(-\log D)(\mathcal{U}))$ is homotopy abelian, for a smooth divisor $D$ in a smooth projective variety $X$ such that $\Omega^*_X(\log D)$ is trivial and when $D$ is a smooth divisor in a smooth projective Calabi-Yau variety.

If $D$ is not smooth but only a simple normal crossing divisor, then $KS_{(X,D)}$ (or $TW(\Theta_X(-\log D)(\mathcal{U}))$) controls the locally trivial infinitesimal deformations of the pair $(X, D)$. Then, the computations above shows that the locally trivial infinitesimal deformations are unobstructed.

### 4.3. Differential Batalin-Vilkovisky algebras.

The main references for this example are [Te08, KKP08] and [Ia15, Section 7].

**Definition 4.10.** Let $k$ be a fixed odd integer. A **differential Batalin-Vilkovisky algebra** (dBV for short) of degree $k$ over $\mathbb{K}$ is the data $(A, d, \Delta)$, where $(A, d)$ is a differential $\mathbb{Z}$-graded commutative algebra with unit $1 \in A$, and $\Delta$ is an operator of degree $-k$, such that $\Delta^2 = 0$, $\Delta(1) = 0$ and

$$\Delta(abc) + \Delta(ab)c + (-1)^{p+q} \Delta(b)ac + (-1)^{p+q+r} \Delta(c)ab =$$

$$= \Delta(ab)c + (-1)^{p+q+r} \Delta(bc)a + (-1)^{p+r} \Delta(ac)b.$$

For any graded dBV algebra $(A, d, \Delta)$ of degree $k$, it is canonically defined a DG-Lie algebra $(L, d, [-, -])$, where: $L = A[k]$, $d_L = -d_A$ and,

$$[a, b] = (-1)^{p}(\Delta(ab) - \Delta(a)b) - a\Delta(b), \quad \forall \ a \in A^p.$$
Definition 4.11. A dBV algebra \((A,d,\Delta)\) of degree \(k\) has the degeneration property if for every \(a_0 \in A\), such that \(da_0 = 0\), there exists a sequence \(a_i, i \geq 0\), with \(\deg(a_i) = \deg(a_{i-1}) - k - 1\) and such that
\[
\Delta a_i = da_{i+1}, \quad i \geq 0.
\]

Example 4.12. Let \((V,d,\Delta)\) be a \((1,k)\)-bicomplex, where \(k\) is an odd integer, i.e., \((V,d)\) is a DG vector space and \(\Delta \in \text{Hom}_K^k(V,V)\) such that
\[
\Delta^2 = 0 \quad [d,\Delta] = d\Delta + \Delta d = 0.
\]
If the \(d\Delta\)-lemma holds, i.e.,
\[
\ker d \cap \Delta(V) = \ker \Delta \cap d(V) = d\Delta(V),
\]
then \((V,d,\Delta)\) has the degeneration property. Indeed, if \(da_0 = 0\) we have \(\Delta a_0 \in d\Delta(V)\) and then there exists \(b \in V\) such that \(d\Delta(b) = \Delta a_0\). It is sufficient to take \(a_1 = \Delta(b)\) and \(a_i = 0\) for every \(i \geq 2\). Note that the converse is not true in general. For instance, if \(d = \Delta\), then \((V,d,\Delta)\) has the degeneration property, while \(\ker \Delta \cap d(V) = d\Delta(V)\) if and only if \(d = \Delta = 0\).

Example 4.13. Let \((V,d,\Delta)\) be a \((1,k)\)-bicomplex as in Example 4.12 and suppose the existence of an operator \(f \in \text{Hom}_K^{k-1}(V,V)\) such that \(\Delta = [d,f]\), \([f,\Delta] = 0\). Then \((V,d,\Delta)\) has the degeneration property. Indeed, consider a formal parameter \(t\) and the associative graded algebra \(\text{Hom}_K^*(V,V)[[t]]\). Then, we have
\[
e^{tf}de^{-tf} = e^{[tf,-]}d = d + t[df] = d - t\Delta
\]
and therefore
\[
t\Delta e^{tf} = -e^{tf}d + de^{tf} = [d,e^{tf}].
\]
Let \(a \in V\) be such that \(da = 0\) and define the sequence \(a_i\) by the rule
\[
\sum_{i \geq 0} a_it^i = e^{tf}(a).
\]
It implies \(a_0 = a\) and
\[
\sum_{i \geq 0} t^{i+1}\Delta a_i = t\Delta e^{tf}a = de^{tf}a = \sum_{i \geq 0} t^i da_i
\]
and then \(da_{i+1} = \Delta a_i\) for every \(i\).

Theorem 4.14. Let \((A,d,\Delta)\) be a dBV algebra with the degeneration property. Then, the associated DG-Lie algebra \(L = A[k]\) is homotopy abelian.

Proof. Here we only give a sketch of the proof. The original proof can be found in [Te08, Theorem 1] or [KKP08, Theorem 4.14] for \(k = 1\) and in [Ia15, Theorem 7.6] for any odd \(k\).

Given a dBV algebra \((A,d,\Delta)\) and \(t\) a formal central variable of (even) degree 1 + \(k\), we can define the graded vector space \(A[[t]]\) of formal power series with coefficients in \(A\) and the graded vector space \(A((t)) = \bigcup_{p \in \mathbb{Z}} t^p A[[t]]\) of formal Laurent power series. Note that \(d(t) = \Delta(t) = 0\) and \(d - t\Delta\) is a well-defined differential on \(A((t))\).
Let $F^\bullet$ be the filtration on the complex $(A((t)),d-t\Delta)$ defined by $F^p = t^p A[[t]]$, for every $p \in \mathbb{Z}$. Note that $A((t)) = \bigcup_{p \in \mathbb{Z}} F^p$ and $F^0 = A[[t]]$ and the map $a \mapsto \frac{a}{t}$ induces an isomorphism of DG-vector spaces $A \rightarrow F^{-1}/F^0$.

The degeneration property implies that the inclusion $F^p \rightarrow A((t))$ is injective in cohomology, for every $p$, and, in particular, $F^0 = A[[t]] \rightarrow A((t))$ is injective in cohomology.

Consider $M = \text{Hom}_K^\ast (A((t)),A((t)))$ and $H = \{ \varphi \in M \mid \varphi(A[[t]]) \subset A[[t]] \}$; then, the degeneration property implies that $H \hookrightarrow M$ is injective in cohomology (Hypothesis (3) of Theorem 3.3).

A tedious but straightforward computation shows that the map $i : L \rightarrow M = \text{Hom}_K^\ast (A((t)),A((t)))$, $a \mapsto i_a(b) = \frac{1}{t} ab$ is a Cartan homotopy [Ia15, Lemma 7.2] and, for any $a \in L$, $l_a \in H$, i.e., Hypothesis (1) and (2) of Theorem 3.3 are satisfied. Therefore, we only need to show that the morphism of DG-vector spaces $i : L \rightarrow M/H[-1] = \text{Hom}_K^\ast \left( A[[t]], \frac{A((t))}{A[[t]]} \right)[-1]$ and so we need to prove that:

$$i : A \rightarrow \text{Hom}_K^\ast \left( A[[t]], \frac{A((t))}{A[[t]]} \right)[-k - 1]$$

is injective in cohomology. It suffices to show the injectivity for the composition with the evaluation at $1 \in A[[t]]$, i.e., the injectivity in cohomology of the morphism

$$A \rightarrow \frac{A((t))}{A[[t]]}, \quad a \mapsto \frac{a}{t},$$

or equivalently of the morphism

$$\beta : \frac{F^{-1}}{F^0} \rightarrow \frac{A((t))}{F^0}.$$ Consider the diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & F^0 & \rightarrow & F^{-1} & \rightarrow & F^{-1} \rightarrow & 0 \\
& | & \alpha & | & \beta & | & \\
0 & \rightarrow & F^0 & \rightarrow & A((t)) & \rightarrow & \frac{A((t))}{F^0} & \rightarrow & 0;
\end{array}
$$

the degeneration property implies that $\alpha$ is injective in cohomology and so $\beta$ is also injective in cohomology.

\[\square\]

Remark 4.15. In [Ba15], the author uses similar techniques to extend this result to the context of commutative $BV_{\infty}$-algebras. In this case, the degeneration property implies that the associated $L_{\infty}[1]$-algebras is homotopy abelian [Ba15, Theorem 6.16], see also [BL13].
Example 4.16. [FiMa12, Section 5] and [BM15, Section 6]. Let $X$ be a complex manifold and $\Theta_X$ it tangent bundle. An holomorphic Poisson bi-vector on $X$ is an element $\pi \in H^0(X, \wedge^2 \Theta_X)$, such that $[\pi, \pi]_{SN} = 0$, where $[,]_{SN}$ denotes the Schouten-Nijenhuis bracket of polyvectorfields [Ma]. In this case, we say that the pair $(X, \pi)$ is a holomorphic Poisson manifold. Then, let $(\mathcal{A}_X^*, d)$ be the de Rham complex, where $\mathcal{A}_X^i = \bigoplus_{p+q=i} \mathcal{A}_{X,p,q}^*$ and consider the map

$$i_\pi \in \text{Hom}^{-2}(\mathcal{A}_X^*, \mathcal{A}_X^*) \quad i_\pi(\alpha) = \pi \cdot \alpha.$$ 

Note that $[i_\pi, i_\pi] = 0$ and, setting $\Delta = [d, i_\pi]$, we have

$$[i_\pi, \Delta] = [i_\pi, [d, i_\pi]] = i_\pi([\pi, \pi]_{SN}) = 0.$$ 

According to Example 4.13, $(\mathcal{A}_X^*, d, \Delta)$ has the degeneration property and so Theorem 4.14 implies that the associated DG-Lie algebra $(\mathcal{A}_X^*[1], -d, [, ]_\pi)$ is homotopy abelian (see [FiMa12, Theorem 5.3] and [BM15, Section 6]).

4.4. The splitting property. Let $(L, d, [, ])$ be a DG-Lie algebra. Denote by $L[1]$ the shifted graded vector space: $L[1]^i = L^{i+1}$ and by $L[1]^{\otimes n}$ its $n$-th symmetric power. Note that $1 \in L[1]^{\otimes 0} = \mathbb{K}$. Then, we can consider the associated differential graded cocommutative coalgebra $(SL[1], \Delta, Q)$, where $SL[1] = \bigoplus_{n \geq 0} L[1]^{\otimes n}$, $\Delta$ is the usual coproduct and $Q$ the coderivation associated with $d$ and $[,]$.

More explicitly, let $q_1(x) = -d(x)$ and $q_2(x \odot y) = (-1)^i[x, y]$ for any $x, y \in L[1]^i$ and $y \in L[1]$, then

$$Q(v_1 \odot \ldots \odot v_n) = \sum_{k=1}^2 \sum_{\sigma \in S(k,n-k)} \epsilon(\sigma) q_k(v_{\sigma(1)} \odot \ldots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \ldots \odot v_{\sigma(n)},$$

where $S(p,q)$ denotes the set of unshuffles of type $(p,q)$ and $\epsilon(\sigma)$ is the Koszul sign. It turns out that $Q^2 = 0$.

Note that, any morphism $F : SL[1] \to SL[1]$ is uniquely determined by the corestriction $pF = (f_0, f_1, f_2, \ldots) : SL[1] \to SL[1] \to L[1]$, where $p : SL[1] \to L[1]^{\otimes 1} = L[1]$ is the natural projection and $f_i : L[1]^{\otimes i} \to L[1]$ are the components of $pF$.

Next, denote by $(\text{Coder}^*(SL[1]), [\cdot, \cdot], [\cdot, \cdot])$ the DG-Lie algebra of coderivation of $SL[1]$ and consider the surjective morphism of DG-vector spaces

$$\text{Coder}^*(SL[1]) \xrightarrow{b} L[1], \quad b(\alpha) = p\alpha(1), \quad \forall \alpha.$$

Corollary 4.17 ([Ba16a]). In the above assumption, if $b : \text{Coder}^*(SL[1]) \to L[1]$ induces a surjective morphism in cohomology, then $L$ is homotopy abelian.

Proof. Here we only give a sketch of the proof, as a consequence of the abstract BTT theorem, while the original proof contained in [Ba16a] uses the theory of derived brackets. Let $M = \text{Coder}^*(SL[1])$ and $H = \ker b \subset M$. The hypothesis implies that the inclusion $H \hookrightarrow M$ is injective in cohomology, i.e., Hypothesis (3) of Theorem 3.3 is satisfied. The map

$$i : L \to \text{Coder}^*(SL[1]), \quad a \mapsto i_a(w) = x \odot w$$

is a properly defined Cartan homotopy. Moreover, for any $a \in L, i_a \in H$, and so Hypothesis (1) and (2) are fulfilled. This also implies that it is well defined the morphism of DG vector
Remark 4.18. According to [Ba16a], we say that $L$ has the splitting property if it satisfies the hypothesis of the previous theorem. We refer to [Ba16a, Proposition 2.2] for equivalent conditions. In this paper, the author proves this result for $L_\infty[-1]$-algebras, analysing the spectral sequence computing the Chevalley-Eilenberg cohomology and showing the equivalence between the degeneration of the spectral sequence at the first page and the homotopy abelianity property. In [Ma15], the author also analyses the spectral sequence computing the Chevalley-Eilenberg cohomology of a DG-Lie algebra. In particular, he proved the equivalence between degeneration of the spectral sequence at the second page and formality property.

Remark 4.19. It can be also proved that the splitting property implies that a DG-Lie algebra $L$ is homotopy abelian if the adjoint morphism

$$ad: L \to \text{Der}_K^*(L, L), \quad ad_x(y) = [x, y],$$

is trivial in cohomology: a detailed proof will appear in the forthcoming paper [IM].

4.5. Derived brackets of Lie type. Here we follow [Ma, Section 5.6]. Let $(M, [\cdot, \cdot], d)$ be a DG-Lie algebra, such that there exist a DG-Lie subalgebra $L$ and a graded vector space $A$ satisfying the following conditions:

(i) $M = L \oplus A$ as graded vector space;

(ii) $[a, b] = 0$ for every $a, b \in A$;

(iii) $[da, b] \in A$, for every $a, b \in A$.

Consider the projection $p: M \to A$ and the operators:

$$\delta: A^i \to A^{i+1}, \quad \delta a = -pda,$$

$$\{\cdot, \cdot\}: A^{i-1} \times A^{j-1} \to A^{i+j-1}, \quad \{a, b\} := (-1)^i[da, b].$$

A straightforward computation shows that $(A[-1], \delta, \{\cdot, \cdot\})$ is a DG-Lie algebra [Ma, Proposition 5.6.6].

**Corollary 4.20.** If $L \to M$ is injective in cohomology, then the DG-Lie algebra $A[-1] = (A[-1], \delta, \{\cdot, \cdot\})$ is homotopy abelian.

**Proof.** The linear map:

$$i : A[-1] \to M \quad i_a = i(a) = a$$

is a Cartan homotopy. Indeed, for every $a, b \in A[-1]$, we have $[i_a, i_b] = [a, b] = 0$, by Condition (ii) above. Moreover, for every $a \in A[-1]^i$ and $b \in A[-1]$, we have $0 = d([a, b]) = [da, b] + (-1)^i[da, db]$ and so

$$i_{(a, b)} = \{a, b\} = -(-1)^i[da, b] = [a, db] = [i_a, di_b].$$

Note that, for every $a \in A[-1]$, we have

$$p(l_a) = p(di_a + i_\delta a) = p(da - pda) = 0$$

and so for every $a \in A[-1]$, we have $l_a \in L$, i.e., Hypothesis (2) of Theorem 3.3 is satisfied. The last two hypotheses of Theorem 3.3 are satisfied by assumption, and so we conclude that $A[-1] = (A[-1], \delta, \{\cdot, \cdot\})$ is homotopy abelian. □
Example 4.21. [Ma, Section 5.6]. Let $V$ and $W$ be two graded vector spaces over $K$ and $\pi \in \text{Hom}_K^1(W,V)$, such that $0 = d_{\text{Hom}}\pi = d_W\pi + \pi d_V$. In [Ma, Section 5.6], the author introduced the notion of the derived bracket of $\pi$ on $\text{Hom}_K^* (V,W)[-1]$ defined as:

$$[-,-]_\pi : (\text{Hom}_K^* (V,W)[-1])^i \times (\text{Hom}_K^* (V,W)[-1])^j \rightarrow (\text{Hom}_K^* (V,W)[-1])^{i+j}$$

$$[f,g]_\pi = f\pi g - (-1)^{ij} g\pi f.$$ 

On the graded vector space $\text{Hom}_K^* (V,W)[-1]$, it is also defined a differential $\delta$ as:

$$\delta(f) = -d_W f - (-1)^i f d_V, \quad \forall f \in (\text{Hom}_K^* (V,W)[-1])^i.$$ 

Then, $A[-1] = (\text{Hom}_K^* (V,W)[-1], [-,-], \delta)$ is a DG-Lie algebra. To view this example in the above setting, it is enough to consider the DG-Lie algebra $M = \text{Hom}_K^* (V \oplus W, V \oplus W)$ with differential given by $[D, -]$ where

$$D = \begin{pmatrix} d_V & -\pi \\ 0 & d_W \end{pmatrix} : V \oplus W \rightarrow V \oplus W.$$ 

It turns out that $[-,-] = [-,-]_\pi$; indeed, for every $a \in A[-1]^i$ and $b \in A[-1]^j$, we have

$$\{a,b\} = -(-1)^i [Da,b] = -(-1)^i[[D,a],b].$$

By definition,

$$[[D,a],b] = [Da - (-1)^{i-1} aD, b]$$

$$= Dab - (-1)^i aDb - (-1)^{i(j-1)} (bDa - (-1)^{i-1} baD)$$

$$= Dab - (-1)^{i-1} aDb - (-1)^{i(j-1)} bDa - (-1)^{ij} baD.$$ 

Viewing $a$ and $b$ as

$$a = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} : V \oplus W \rightarrow V \oplus W,$$

we have

$$Da = \begin{pmatrix} -\pi a & 0 \\ d_W a & 0 \end{pmatrix} ;$$

thus $Dab = abD = 0$, and

$$bDa = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \cdot \begin{pmatrix} -\pi a & 0 \\ d_W a & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -b\pi a & 0 \end{pmatrix}, \quad aDb = \begin{pmatrix} 0 & 0 \\ -a\pi b & 0 \end{pmatrix} .$$

Therefore,

$$\{a,b\} = -(-1)^i [Da,b] = -aDb + (-1)^i bDa = a\pi b - (-1)^{ij} b\pi a = [a,b]_\pi.$$ 

Finally, if the inclusion $L \rightarrow M$ is injective in cohomology, then the DG-Lie algebra $A[-1] = (\text{Hom}_K^* (V,W)[-1], [-,-], \delta)$ is homotopy abelian.
5. Further applications

According to Remark 3.4, if \( L, M \) and \( H \subseteq M \) are DG-Lie algebras, such that \( \chi : H \rightarrow M \) is injective in cohomology and \( l_a \in H \) for every \( a \in L \), we have a diagram:

\[
TW(\alpha) \xrightarrow{\phi} TW(\chi) \\
\downarrow \rho \\
L
\]

where \( TW(\chi) \) is an homotopy abelian DG-Lie algebras and the vertical arrow is a quasi isomorphism. In particular, we have a morphism:

\[
s : H^2(L) \xrightarrow{\delta} H^2(TW(\alpha)) \xrightarrow{\delta} H^2(TW(\chi))
\]

Since \( \text{Def}_{TW(\chi)} \) is smooth, \( s \) annihilates all the obstructions of \( \text{Def}_L \). This idea has been applied in various deformation cases. For instance, in [FiMa09, Proposition 4.6], the authors consider the deformations of compact K"ahler manifolds and they prove the Kodaira’s principle, ambient cohomology annihilates obstructions. A semiregularity map annihilating all the obstructions to the infinitesimal deformations of holomorphic maps of compact K"ahler manifolds with fixed codomain were analysed in [In11, Corollary 4.14]. In [IM13, Theorem 11.1], it is proved that the Bloch’s semiregularity map annihilates all the obstructions to embedded deformations of a local complete intersection, extending the proof given in [Ma07, Theorem 9.1] for the embedded deformations of a smooth submanifold.

Acknowledgement. The author wishes to thank Prof. Marco Manetti for useful discussion during the preparation of the paper and for the draft uncompleted version of [Ma]. The paper is largely inspired by the lectures held by Prof. Marco Manetti at GAP-XV at State College. The author was partially supported by Fondi di Ateneo dell’Università degli Studi di Bari: “Spazi di Moduli e Algebre di Lie Differenziali Graduate”. The author thanks the referee for improvements in the presentation of the paper and Prof. O. Fujino for pointing my attention to [Fuj].

References

[Ba15] R. Bandiera: Non abelian higher derived bracket, J. of Pure and Applied Algebra, 219, (2015), no. 8, 3292-3313; arXiv:1304.4097.
[Ba16] R. Bandiera: Formality of Kapranov’s bracket in K"ahler geometry via pre-Lie deformation theory, Int. Math. Res. Not., 21, (2016), no. 8, 6626-6655; arXiv:1307.8066.
[Ba16a] R. Bandiera: Homotopy abelian \( L_\infty \) algebras and splitting property, Rend. Mat. Appl., Serie VII, 37, (2016), 105-122.
[BM15] R. Bandiera and M. Manetti: On coisotropic deformations of holomorphic submanifolds, J. Math. Sci. Univ. Tokyo, 22 (2015), no. 1, 1-37; arXiv:1304.6000.
[Bo78] F. Bogomolov: Hamiltonian K"ahlerian manifolds, Dokl. Akad. Nauk SSSR 243 (1978) 1101-1104;
[Soviet Math. Dokl., 19, (1979), 1462-1465.
[BL13] C. Braun and A. Lazarev: Homotopy BV algebras in Poisson geometry, Trans. Moscow Math. Soc., 74, (2013), 217-227; arXiv:1304.6473.
[De70] P. Deligne: Thé Équations différentielles à points singuliers réguliers, Lecture Notes in Mathematics, 163, Springer-Verlag, New York-Berlin, (1970).
[EV92] H. Esnault and E. Viehweg: Lectures on vanishing theorems, DMV Seminar, 20, Birkhäuser, Verlag, Basel, (1992).
[FM99] B. Fantechi, M. Manetti: On the \( T^1 \)-lifting theorem, J. Algebraic Geom., 8, (1999), 31-39.
[FiMa07] D. Fiorenza and M. Manetti: $L_{\infty}$ structures on mapping cones, Algebra Number Theory, 1, (2007), 301-330; arXiv:0601312.

[FiMa09] D. Fiorenza and M. Manetti: A period map for generalized deformations, J. Noncommut. Geom., 3, (2009), 579-597; arXiv:0808.0140v1.

[FiMa12] D. Fiorenza and M. Manetti: Formality of Koszul brackets and deformations of holomorphic Poisson manifolds, Homology, Homotopy and Applications, 14, No. 2, (2012), 63-75; arXiv:1109.4309.

[FM16] D. Fiorenza and M. Manetti: Formal Abel-Jacobi maps; preprint arXiv:1610.08684.

[Fuj] O. Fujino: Notes on the weak positivity theorems, to appear in ASPM; arXiv:1406.1834.

[GM88] W.M. Goldman and J.J. Millson: The deformation theory of representations of fundamental groups of compact kähler manifolds, Publ. Math. I.H.E.S., 67, (1988), 43-96.

[GM90] W.M. Goldman and J.J. Millson: The homotopy invariance of the Kuranishi space, Illinois J. Math., 34, (1990), 337-367.

[Ka07] D. Kaledin: Some remarks on formality in families, Mosc. Math. J., 7, (2007), 643-652; arXiv:0509699.

[KKP08] L. Katzarkov, M. Kontsevich and T. Pantev: Hodge theoretic aspects of mirror symmetry, From Hodge theory to integrability and TQFT tt*-geometry, Proc. Sympos. Pure Math., 78, Amer. Math. Soc., Providence, (2008), 87-174; arXiv:0806.0107.

[Ko03] Y. Kawamata: Unobstructed deformations - a remark on a paper of Z. Ran, Math. Ann., 325, (2003), 437-445; arXiv:q-alg/9709040.

[Ia11] D. Iacono: A semiregularity map annihilating obstructions to deforming holomorphic maps, Canad. Math. Bull., 54, (2011), no. 3, 472-486; arXiv:0707.2454.

[Ia15] D. Iacono: Deformations and obstructions of pairs $(X,D)$, Internat. Math. Res. Notices (IMRN), Volume 2015, Issue 19, 9660-9695; arXiv:1302.1149.

[IM10] D. Iacono and M. Manetti, An algebraic proof of Bogomolov-Tian-Todorov theorem, Deformation Spaces, 39, (2010), 113-133; arXiv:0902.0732.

[IM13] D. Iacono and M. Manetti: Semiregularity and obstructions of complete intersections, Advances in Mathematics, 235, (2013), 92-125; arXiv:1112.0425.

[IM] D. Iacono and M. Manetti: On the adjoint map of homotopy abelian DG-Lie algebras, in preparation. 17

[Kaw92] Y. Kawamata: Unobstructed deformations - a remark on a paper of Z. Ran, J. Algebraic Geom., 1, (1992), 183-190.

[Lu10] V. Lunts: Formality of DG algebras (after Kaledin), J. of Algebra, 323, Issue 4, (2010), 878-898; arXiv:0712.0996.

[Ma04] M. Manetti: Lectures on deformations of complex manifolds, Rend. Mat. Appl., Serie VII, 24, (2004), 1-183; arXiv:0507286.

[Ma04a] M. Manetti: Cohomological constraint to deformations of compact Kähler manifolds, Adv. Math., 186, (2004), 125-142; arXiv:math.AG/0405175.

[Ma07] M. Manetti: Lie description of higher obstructions to deforming submanifolds, Ann. Sc. Norm. Super. Pisa Cl. Sci., 6, (2007), 631-659; arXiv:0507287.

[Ma09] M. Manetti: Differential graded Lie algebras and formal deformation theory, in Algebraic Geometry: Seattle 2005, Proc. Sympos. Pure Math., 80, (2009), 785-810.

[Ma15] M. Manetti: On some formality criteria for DG-Lie algebras, J. Algebra, 438, (2015), 90-118.

[Ma] M. Manetti: Lie methods in deformation theory, in preparation. 2, 8, 16, 17, 18, 19

[Mar09] E. Martinengo: Local structure of Brill-Noether strata in the moduli space of flat stable bundles, Rend. Semin. Mat. Univ. Padova, 121, (2009), 259-280; arXiv:0806.2056.

[Pa91] R. Pardini: Abelian covers of algebraic varieties, J. Reine Angew. Math., 417, (1991), 191-213.

[Ra92] Z. Ran: Deformations of manifolds with torsion or negative canonical bundle, J. Algebraic Geom., 1, (1992), 279-291.

[Se06] E. Sernesi: Deformations of Algebraic Schemes, Grundlehren der mathematischen Wissenschaften, 334, Springer-Verlag, New York Berlin, (2006), 4.

[Td08] J. Terilla: Smoothness theorem for differential BV algebras, J. Topol., 1, (2008), 693-702; arXiv:0707.1290.
[Ti87] G. Tian: *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric*, Mathematical Aspects of String Theory (San Diego, 1986), Adv. Ser. Math. Phys. 1, World Sci. Publishing, Singapore, (1987), 629-646.

[To89] A.N. Todorov: *The Weil-Petersson geometry of the moduli space of SU(n \geq 3) (Calabi-Yau) Manifolds I*, Commun. Math. Phys., 126, (1989), 325-346.

[Vo02] C. Voisin: *Hodge theory and complex algebraic geometry I*, Cambridge Studies in Advanced Mathematics, 76, Cambridge University Press, Cambridge, (2002).

Università degli Studi di Bari,
Dipartimento di Matematica,
Via E. Orabona 4, I-70125 Bari, Italy.

E-mail address: donatella.iacono@uniba.it
URL: www.dm.uniba.it/~iacono/