Abstract

Abelian quiver gauge theories provide nonsupersymmetric candidates for the conformality approach to physics beyond the standard model. Written as $\mathcal{N} = 0$, $U(N)^n$ gauge theories, however, they have mixed $U(1)_p U(1)_q^2$ and $U(1)_p SU(N)_q^2$ triangle anomalies. It is shown how to construct explicitly a compensatory term $\Delta \mathcal{L}_{\text{comp}}$ which restores gauge invariance of $\mathcal{L}_{\text{eff}} = \mathcal{L} + \Delta \mathcal{L}_{\text{comp}}$ under $U(N)^n$. It can lead to a negative contribution to the $U(1)$ $\beta$-function and hence to one-loop conformality at high energy for all dimensionless couplings.

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Introduction

One alternative to supersymmetry and grand unification is to postulate conformality, four-dimensional conformal invariance at high energy, for the non gravitational extension of the standard model. Although much less vigorously studied than supersymmetry, the conformality approach suggested [1] in 1998 has made considerable progress. Models which contain the standard model fields have been constructed [4] and a model which grand unifies at about 4 TeV [5] has been examined.

Such models are inspired by the AdS/CFT correspondence [7] specifically based on compactification of the IIB superstring on the abelian orbifold $AdS_5 \times S^5/Z_n$ with $N$ coalescing parallel D3 branes. A model is specified by $N$ and by the embedding $Z_n \subset SU(4)$ which is characterized by integers $A_m$ ($m = 1, 2, 3, 4$) which specify how the 4 of $SU(4)$ transforms under $Z_n$. Only three of the $A_m$ are independent because of the $SU(4)$ requirement that $\Sigma_m A_m = 0$ (mod $n$). The number of vanishing $A_m$ is the number $N$ of surviving supersymmetries. Here we focus on the non supersymmetric $N = 0$ case.

In [9], the original speculation [1] that such models may be conformal has been refined to exclude models which contain scalar fields transforming as adjoint representations because only if all scalars are in bifundamentals are there chiral fermions and, also only if all scalars are in bifundamentals, the one-loop quadratic divergences cancel in the scalar propagator. We regard it as encouraging that these two desirable properties select the same subset of models.

Another phenomenological encouragement stems from the observation [3] that the standard model representations for the chiral fermions can all be accommodated in bifundamentals of $SU(3)^3$ and can appear naturally in the conformality approach.

In the present article we address the issue of triangle anomalies. Although the purely non abelian anomalies involving $SU(N)^3$ subgroups of the $U(N)^n$ gauge group are cancelled, there do survive triangle anomalies of the types $U(1)_pU(1)_q^2$ and $U(1)_pSU(N)_q^2$. Since the original superstring is anomaly free, one expects such anomalies to be cancelled. This cancellation is well understood [10] in terms of the closed string axions coupling to $F \tilde{F}$. Here we shall construct a compensatory term $\Delta L_{comp}$ which is non polynomial in the bifundamental scalars and which when added to the gauge lagrangian $L$ gives rise to an effective lagrangian $L_{eff} = L + \Delta L_{comp}$ which is $U(N)^n$ gauge invariant.

In the next subsection, we shall discuss the anomaly cancellation by a compensatory term. The following section explains the explicit construction. There is then a treatment of the evolution of the $U(1)$ couplings and finally there is some discussion.
Anomaly cancellation by a compensatory term

The lagrangian for the nonsupersymmetric $Z_n$ theory can be written in a convenient notation which accommodates simultaneously both adjoint and bifundamental scalars as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu,r,r}^{ab} F_{\mu\nu,r,r}^{ba} + i \bar{\Phi}_{r+r A_4}^{ab} \gamma^\mu D^\mu \Phi_{r+r A_4}^{ba} + 2 D_\mu \Phi_{r+r A_4}^{ab} + i \bar{\Psi}_{r+r A_m}^{ab} \gamma^\mu D^\mu \Psi_{r+r A_m}^{ba}$$

$$-2ig \left[ \bar{\Psi}_{r+r A_1}^{ab} P_1 \phi_{r+r A_4}^{bc} \Phi_{r-r A_4}^{ca} - \bar{\Phi}_{r+r A_4}^{bc} P_1 \phi_{r-r A_4}^{ca} \right]$$

$$-\sqrt{2}ig \epsilon_{ijk} \left[ \bar{\Psi}_{r+r A_2}^{ab} P_2 \phi_{r+r A_4}^{bc} \Phi_{r-r A_4}^{ca} - \bar{\Phi}_{r+r A_4}^{bc} P_2 \phi_{r-r A_4}^{ca} \right]$$

$$-4g^2 \left( \Phi_{r+r A_1}^{ab} \Phi_{r-r A_4}^{bc} - \Phi_{r-r A_4}^{bc} \Phi_{r+r A_4}^{ab} \right) \left( \Phi_{r+r A_1}^{cd} \Phi_{r-r A_4}^{da} - \Phi_{r-r A_4}^{da} \Phi_{r+r A_4}^{cd} \right)$$

$$+4g^2 \left( \Phi_{r+r A_1}^{bc} \Phi_{r-r A_4}^{ca} - \Phi_{r-r A_4}^{ca} \Phi_{r+r A_4}^{bc} \right) \left( \Phi_{r+r A_1}^{cd} \Phi_{r-r A_4}^{da} - \Phi_{r-r A_4}^{da} \Phi_{r+r A_4}^{cd} \right)$$

(1)

where $\mu, \nu = 0, 1, 2, 3$ are Lorentz indices; $a, b, c, d = 1$ to $N$ are $U(N)^n$ group labels; $r = 1$ to $n$ labels the node of the quiver diagram (when the two node subscripts are equal it is an adjoint plus singlet and the two superscripts are in the same $U(N)$ group labels; when the two node subscripts are unequal it is a bifundamental and the two superscript labels transform under different $U(N)$ groups); $a_i$ ($i = \{1, 2, 3\}$) label the first three of the 6 of $SU(4)$; $A_m$ ($m = \{1, 2, 3, 4\}$) label the 4 of $SU(4)$. By definition $A_4$ denotes an arbitrarily-chosen fermion ($\lambda$) associated with the gauge boson, similarly to the notation in the $\mathcal{N} = 1$ supersymmetric case. Recall that $\sum_{m=1}^{n} A_m = 0$ (mod n).

As mentioned above we shall restrict attention to models where all scalars are in bifundamentals which requires all $a_i$ to be non zero. Recall that $a_1 = A_2 + A_3$, $a_2 = A_3 + A_1$; $a_3 = A_1 + A_2$.

The lagrangian in Eq(1) is classically $U(N)^p$ gauge invariant. There are, however, triangle anomalies of the $U(1)_p U(1)_q^2$ and $U(1)_p SU(N)_q^2$ types. Making gauge transformations under the $U(1)_r$ ($r = 1, 2, \ldots, n$) with gauge parameters $\Lambda_r$, leads to a variation

$$\delta \mathcal{L} = -\frac{g^2}{4\pi^2} \Sigma_{p=1}^{n} A_{pq} F_{\mu\nu}^{(p)} F^{(p)\mu\nu} \Lambda_q$$

(2)

which defines an $n \times n$ matrix $A_{pq}$ which is given by

$$A_{pq} = \text{Tr}(Q_p Q_q^2)$$

(3)

where the trace is over all chiral fermion links and $Q_r$ is the charge of the bifundamental under $U(1)_r$. We shall adopt the sign convention that $\mathbf{N}$ has $Q = +1$ and $\mathbf{N}^*$ has $Q = -1$.

It is straightforward to write $A_{pq}$ in terms of Kronecker deltas because the content of chiral fermions is

$$\sum_{m=1}^{n} \sum_{r=1}^{n} (N_r, N^*_{r+A_m})$$

(4)
This implies that the antisymmetric matrix \(A_{pq}\) is explicitly

\[A_{pq} = -A_{qp} = \sum_{m=1}^{m=4} (\delta_{p,q} - A_m - \delta_{p,q} + A_m)\] (5)

Now we are ready to construct \(\mathcal{L}^{(1)}_{\text{comp}}\), the compensatory term. Under the \(U(1)\) gauge transformations with gauge parameters \(\Lambda_r\) we require that

\[\delta \mathcal{L}^{(1)}_{\text{comp}} = -\delta \mathcal{L} = \frac{g^2}{4\pi} \sum_{p=1}^{p=n} A_{pq} F_{\mu\nu}^{(p)} \tilde{F}^{(p)\mu\nu} \Lambda_q\] (6)

To accomplish this property, we construct a compensatory term in the form \#2

\[\mathcal{L}^{(1)}_{\text{comp}} = \frac{g^2}{4\pi} \sum_{p=1}^{p=n} \sum_{k} B_{pk} \text{ImTrln} \left(\frac{\Phi_k}{v}\right) F_{\mu\nu}^{(p)} \tilde{F}^{(p)\mu\nu}\] (7)

where \(\Sigma_k\) runs over scalar links. We believe this form for the compensatory term to be unique\#3 because \(\mathcal{L}^{(1)}_{\text{comp}}\) must be invariant under \(SU(N)^n\). To see that \(\mathcal{L}^{(1)}_{\text{comp}}\) of Eq.(7) has such invariance rewrite \(\text{Tr ln} \equiv \exp \det\) and note that the \(SU(N)\) matrices have unit determinant. It is inconceivable that any other non-trivial function of the bifundamental, other than a closed loop of links in the quiver diagram, has the full \(SU(N)^n\) invariance but a closed loop, unlike Eq.(7), is \(U(N)^n\) invariant.

We note \textit{en passant} that one cannot take the \(v \to 0\) limit in Eq.(7); the chiral anomaly enforces a breaking of conformal invariance.

Explicit construction of the matrix \(B_{pk}\) will be the subject of the subsequent section. But first we investigate the transformation properties of the “\(\text{ImTrln}(\Phi/v)\)” term in Eq.(7). Define a matrix \(C_{kq}\) by

\[\delta \left(\sum_{p=1}^{p=n} \sum_{k} \text{ImTrln} \left(\frac{\Phi_k}{v}\right)\right) = \sum_{q=1}^{q=n} C_{kq} \Lambda_q\] (8)

whereupon Eq.(6) will be satisfied if the matrix \(B_{pk}\) satisfies \(A = BC\). The inversion \(B = AC^{-1}\) is non trivial because \(C\) is singular but \(C_{kq}\) can be written in terms of Kronecker deltas by noting that the content of complex scalar fields in the model is

\[\sum_{i=1}^{i=3} \sum_{r=1}^{r=n} (N_r, N^{\ast}_{r \pm a_i})\] (9)

which implies that the matrix \(C_{kq}\) must be of the form

\[C_{kq} = 3\delta_{kq} - \delta_{k+a_i,q}\] (10)

\#2For a related construction in a different context, see [12]
\#3Although the general form is unique, there can be a technical ambiguity in the matrix \(B\) to be discussed below.
The $U(1)_pSU(N)_q^2$ triangle anomalies necessitate the addition of a second compensatory term $\mathcal{L}_{\text{comp}}^{(2)}$. The derivation of $\mathcal{L}_{\text{comp}}^{(2)}$ is similar to, but algebraically simpler than, that for $\mathcal{L}_{\text{comp}}^{(1)}$. Under $U(1)_r$ with gauge parameter $\Lambda_r$ and $SU(N)_s$ gauge transformations the variation in $\mathcal{L}$ of Eq.(11) is

$$\delta \mathcal{L} = -\frac{g^2}{4\pi^2} \sum_{p=1}^{n} A'_{pq} i F^{(p)}_{\mu
u\alpha\beta} \tilde{F}^{(p)}_{\mu
u\alpha\beta} \Lambda_q$$

which defines an $n \times n$ matrix $A'_{pq}$ which is given by

$$A'_{pq} = \text{Tr}(Q_p n_q)$$

where the trace is over all chiral fermion links, $Q_r$ is the charge of the bifundamental under $U(1)_r$ and $n_q$ is the number of fundamentals and anti fundamentals of $SU(N)_q$ corresponding to all fermionic links between nodes $p$ and $q$. As before, we adopt the sign convention that $N$ has $Q = +1$ and $N^*$ has $Q = -1$.

It is straightforward to write $A'_{pq}$ in terms of Kronecker deltas as

$$A'_{pq} = -A'_{qp} = \sum_{m=1}^{n} (-\delta_{p,q-M_m} + \delta_{p,q+M_m})$$

$\mathcal{L}_{\text{comp}}^{(2)}$, the compensatory term for the $U(1)_pSU(N)_q^2$ triangle anomalies is

$$\mathcal{L}_{\text{comp}}^{(2)} = \frac{g^2}{4\pi^2} \sum_{p=1}^{n} \sum_{k} B'_{pk} \text{ImTrln} \left( \frac{\Phi_k}{v} \right) F_{\mu\nu\alpha}^{(p)} \tilde{F}^{(p)}_{\mu\nu\alpha} \Lambda_q$$

where $\sum_k$ runs over scalar links.

Explicit construction of the matrix $B'_{pk}$ in $\mathcal{L}_{\text{comp}}^{(2)}$ is more straightforward than $B_{pk}$ in $\mathcal{L}^{(1)}$ because when we define a matrix $C'_{kq}$ by the variation under mixed abelian-nonabelian gauge transformations

$$\delta \left( \sum_{p=1}^{n} \sum_{k} \text{ImTrln} \left( \frac{\Phi_k}{v} \right) \right) = \sum_{q=1}^{n} C'_{kq} \Lambda_q$$

we find $C'_{kq} = 3\delta_{pk}$ so $B'_{pk}$ in Eq.(14) is $B'_{pq} = \frac{1}{3} A'_{pq}$ with $A'_{pq}$ defined by Eq. (13)

Explicit construction of the matrix $B$ in $\mathcal{L}_{\text{comp}}^{(1)}$

Construction of the anomaly compensatory term $\mathcal{L}_{\text{comp}}^{(1)}$ of Eq (7) has been reduced to the explicit construction of the matrix $B_{pk}$. Although $B = AC^{-1}$ is inadequate because $\text{Rank}(C) < n$, a necessary and sufficient condition for the existence of $B$ is $\text{Rank} (A) \leq \text{Rank} (C)$. Proving this in general would be one approach but the large number of special cases will make the proof lengthy. Of course, we strongly suspect that the matrix $B$ must exist from indirect string theory arguments [10] but we shall convince the reader directly
by explicit construction of B in two extremes which we call the totally degenerate and the totally nondegenerate cases respectively.

Given the form of $A_{pq}$ in Eq.(5) and of $C_{kq}$ in Eq.(10), it is irresistible to make a corresponding ansatz for $B_{pk}$

$$B_{pk} = \sum_\eta C_\eta \delta_{p,k+\eta}$$

(16)

and this ansatz works by setting up recursion relations for the $C_\eta$ and allows explicit solution for the matrix B in any special case. Writing a general formula for B will now be demonstrated in two extreme cases.

**Totally degenerate case**

We assume $A_m = (A, A, A, -3A)$ (modulo n). In this case, from Eq.(5),

$$A_{pq} = 3\delta_{p,q-A} - 3\delta_{p,q+A} + \delta_{p,q+3A} - \delta_{p,q-3A}$$

(17)

and from Eq.(10),

$$C_{kq} = 3(\delta_{kq} - \delta_{k+2A,q})$$

(18)

Using Eq.(16) and comparing coefficients gives the series of recursion relations

$$3C_{-A} - 3C_A = 3$$

(19)

$$3C_A - 3C_{3A} = -3$$

(20)

$$3C_{-3A} - 3C_{-A} = -1$$

(21)

$$3C_{3A} - 3C_{5A} = +1$$

(22)

$$3C_{-5A} - 3C_{-3A} = 0$$

(23)

$$3C_{5A} - 3C_{7A} = 0$$

(24)

and so on, with solution $C_A = -2/3, C_{-A} = C_{3A} = 1/3$ and all other $C_A = 0$. The explicit B matrix is thus

$$B_{pk} = \frac{1}{3}(-2\delta_{p,k+A} + \delta_{p,k+3A} + \delta_{p,k-A})$$

(25)

From Eqs.(17, 25, 18) one confirms $A = BC$. 


Totally nondegenerate case

At an opposite extreme we may assume that

$$\pm A_m \text{ and } (a_i \pm A_m) \text{ are all nondegenerate integers (modulo } n)$$

Assumption (26) requires $n \gg 1$ and so is not a physical case. In this limit the recursion relations become

$$3C_{-A_m} - \Sigma_i C_{a_i-A_m} = +1$$
$$3C_{A_m} - \Sigma_i C_{a_i+A_m} = -1$$

Because the $A_m$ enter symmetrically in the model, one can put $C_{A_m} = x$ and $C_{-A_m} = y$ both independent of $m$. This yields $C_{a_i+A_m} = (x + \frac{1}{3})$ and $C_{a_i-A_m} = (y - \frac{1}{3})$. Comparing coefficients of Kronecker deltas gives $x = -2/9$ and $y = +1/9$ and hence an explicit form for $B_{pk}$ by substitution in Eq. (16).

Intermediate cases

When there are some degeneracies which violate assumption (26), there are too many special cases to permit any succinct general formula. Nevertheless, one can find fairly easily the explicit B matrix for any specific model, as we have done for dozens of cases, using Mathematica software.

Technical non uniqueness

As mentioned in an earlier footnote, although the form of $\mathcal{L}_{comp}^{(1)}$ in Eq. (7) is unique the matrix B can have technical non uniqueness which is best illustrated by a specific example.

Take $n = 6$ and the totally degenerate example $A_m = (1, 1, 1, 3)$. Following the analysis given earlier one finds for $C_{kq}$

$$C = \begin{pmatrix}
3 & 0 & -3 & 0 & 0 & 0 \\
0 & 3 & 0 & -3 & 0 & 0 \\
0 & 0 & 3 & 0 & -3 & 0 \\
0 & 0 & 0 & 3 & 0 & -3 \\
-3 & 0 & 0 & 0 & 3 & 0 \\
0 & -3 & 0 & 0 & 0 & 3
\end{pmatrix}$$

while for the matrix $B_{pk}$

$$3B = \begin{pmatrix}
0 & 1 & 0 & 1 & 0 & -2 \\
-2 & 0 & 1 & 0 & 1 & 0 \\
0 & -2 & 0 & 1 & 0 & 1 \\
1 & 0 & -2 & 0 & 1 & 0 \\
0 & 1 & 0 & -2 & 0 & 1 \\
1 & 0 & 1 & 0 & -2 & 0
\end{pmatrix}$$
Multiplication of BC gives the required

\[
A = \begin{pmatrix}
0 & 3 & 0 & 0 & 0 & -3 \\
-3 & 0 & 3 & 0 & 0 & 0 \\
0 & -3 & 0 & 3 & 0 & 0 \\
0 & 0 & -3 & 0 & 3 & 0 \\
0 & 0 & 0 & -3 & 0 & 3 \\
3 & 0 & 0 & 0 & -3 & 0 \\
\end{pmatrix}
\]  

(31)

There is, however, the following matrix \( \hat{B} \)

\[
\hat{B} = \begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
\end{pmatrix}
\]  

(32)

which has the property that \( \hat{B}C = 0 \). This means there is a one parameter family \( B' = B + \alpha \hat{B} \) with \( \alpha \) a continuous parameter which can be used in \( L_{\text{comp}} \). However, this also means that the term

\[
\hat{\mathcal{L}} = \frac{g^2}{4\pi} \sum_{p=1}^{\Sigma} \sum_k \hat{B}_{pk} \text{ImTr} \ln \left( \frac{\Phi_k}{v} \right) F_{\mu\nu}^{(p)} \tilde{F}_{\mu\nu}^{(p)}
\]

(33)

is \( U(N)^n \)-invariant and therefore need not be added for purposes of anomaly cancellation.

The non-uniqueness of the matrix \( B \) under \( B \to B + \alpha \hat{B} \) has no immediate physical interpretation but it may suggest an undiscovered residual symmetry.
String theory

In [11] the anomaly \( A_{pq} \) is written in a factorized form \( A = TU \) following from the closed string axion exchange using the Green-Schwarz mechanism so here we compare the two factorized expressions \( A = BC \) and \( A = TU \) to become convinced that there is no connection.

The expression from [11] is

\[
A_{pq} = \sum_{l=1}^{l=n} T_{pl} U_{lq} V_{l} \tag{34}
\]

where

\[
T_{pl} = \exp \left( \frac{2\pi ilp}{n} \right) \quad U_{lq} = \exp \left( -\frac{2\pi ilq}{n} \right) \tag{35}
\]

and

\[
V_{l} = \Pi_{i=1}^{l=3} \sin \left( \frac{\pi la_{i}}{n} \right) \tag{36}
\]

Let us take the example of \( n = 6 \) and \( A_{m} = (1, 1, 1, 3) \) for which the matrices \( A, B \) and \( C \) are given above. In Eq. (34) there is an ambiguity in whether \( V \) is accommodated in \( T \) or \( U \) so let us look at three possibilities: (i) \( T' = TV \) (ii) \( U' = VU \) and (iii) \( T'' = T\sqrt{V}, U'' = \sqrt{V}U \). The factorizing matrices are then, up to overall normalization which has no effect on the matrix textures, as follows. We define \( \alpha = \exp(i\pi/3) \).

\[
A = T' U = \begin{pmatrix}
\alpha & \alpha^2 & 0 & \alpha & \alpha^2 & 0 \\
\alpha^2 & -\alpha & 1 & \alpha^2 & -\alpha & 1 \\
-1 & 1 & 0 & -1 & 1 & 0 \\
-\alpha & \alpha^2 & 0 & \alpha & -\alpha^2 & 0 \\
-\alpha^2 & -\alpha & 0 & -\alpha^2 & -\alpha & 0 \\
1 & 1 & 0 & -1 & -1 & 0
\end{pmatrix} \begin{pmatrix}
-\alpha^2 & -\alpha & 1 & \alpha^2 & \alpha & 1 \\
-\alpha & \alpha^2 & 1 & -\alpha & \alpha^2 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 \\
\alpha^2 & -\alpha & 1 & \alpha^2 & -\alpha & 1 \\
\alpha & \alpha^2 & -1 & -\alpha & \alpha^2 & -1 \\
1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix} \tag{37}
\]

\[
A = T U' = \begin{pmatrix}
\alpha & \alpha^2 & -1 & -\alpha & -\alpha^2 & 1 \\
\alpha^2 & -\alpha & 1 & \alpha^2 & -\alpha & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 \\
-\alpha & \alpha^2 & 1 & -\alpha & \alpha^2 & 1 \\
-\alpha^2 & -\alpha & 1 & \alpha^2 & \alpha & 1 \\
1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix} \begin{pmatrix}
-\alpha^2 & -\alpha & 1 & \alpha^2 & \alpha & 1 \\
-\alpha & \alpha^2 & 1 & -\alpha & \alpha^2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\alpha & \alpha^2 & 1 & -\alpha & \alpha^2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{38}
\]

\[
A = T'' U'' = \begin{pmatrix}
\alpha & \alpha^2 & 0 & -i\alpha & -i\alpha^2 & 0 \\
\alpha^2 & -\alpha & 0 & i\alpha^2 & -i\alpha & 0 \\
-1 & 1 & 0 & i & -i & 0 \\
-\alpha & \alpha^2 & 0 & -i\alpha & i\alpha^2 & 0 \\
-\alpha^2 & -\alpha & 0 & i\alpha^2 & i\alpha & 0 \\
1 & 1 & 0 & i & i & 0
\end{pmatrix} \begin{pmatrix}
-\alpha^2 & -\alpha & 1 & -\alpha^2 & \alpha & 1 \\
-\alpha & \alpha^2 & 1 & -\alpha & \alpha^2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
i\alpha^2 & -i\alpha & i & i\alpha^2 & -i\alpha & i \\
i\alpha & i\alpha^2 & -i & -i\alpha & -i\alpha^2 & i
\end{pmatrix} \tag{39}
\]
By comparing the matrix textures in Eqs. (37, 38, 39) with those for matrices $C$ and $B$ in Eqs. (29, 30) we see that the factorization of the anomaly matrix is not simply related. The factorization in Eqs. (37, 38, 39) follows from the physical requirement of factorization at the closed string axion pole in a string tree diagram. There is no similar requirement that mandates our factorization $A = BC$ but these matrices have simple textures so the factorization is not surprising.

In particular, the field theoretical mechanism of anomaly cancellation discussed here has no connection to the string theoretical Green-Schwarz mechanism.
Evolution of U(1) gauge couplings.

In the absence of the compensatory term, the two independent $U(N)^n$ gauge couplings $g_N$ for SU(N) and $g_1$ for U(1) are taken to be equal $g_N(\mu_0) = g_1(\mu_0)$ at a chosen scale, e.g. $\mu_0=4$ TeV [5, 6], to enable cancellation of quadratic divergences [9]. Note that the $n$ SU(N) couplings $g_N^{(p)}$ are equal by the overall $Z_n$ symmetry, as are the $n$ U(1) couplings $g_1^{(p)}$, $1 \leq p \leq n$.

As one evolves to higher scales $\mu > \mu_0$, the renormalization group beta function $\beta_N$ for SU(N) vanishes $\beta_N = 0$ at least at one-loop level so the $g_N(\mu)$ can behave independent of the scale as expected by conformality. On the other hand, the beta function $\beta_1$ for U(1) is positive definite in the unadorned theory, given at one loop by, in the notation of [13]

$$b_1 = \frac{11N}{48\pi^2}$$

where $N$ is the number of colors. The corresponding coupling satisfies

$$\frac{1}{\alpha_1(\mu)} = \frac{1}{\alpha_1(M)} + 8\pi b_1 \ln \left( \frac{M}{\mu} \right)$$

so the Landau pole, putting $\alpha(\mu) = 0.1$ and $N = 3$, occurs at

$$\frac{M}{\mu} = \exp \left[ \frac{20\pi}{11} \right] \simeq 302$$

so for $\mu = 4$ TeV, $M \sim 1200$ TeV. The coupling becomes “strong” $\alpha(\mu) = 1$ at

$$\frac{M}{\mu} = \exp \left[ \frac{18\pi}{11} \right] \simeq 171$$

or $M \sim 680$ TeV.

We may therefore ask whether the new term $L_{comp}$ in the lagrangian, necessary for anomaly cancellation, can solve this problem for conformality?

Indeed there is the real counterpart of Eq.(7) which has the form

$$L_{comp}^{(1) real} = \frac{g_2^2}{4\pi} \sum_{p=1}^{n} \sum_{k} B_{pk} \text{ReTrln} \left( \frac{\Phi_k}{v} \right) F_{\mu\nu}^{(p)} F^{(p)\mu\nu}$$

and this contributes to the U(1) gauge propagator and to the U(1) $\beta-$function. Using Eq.[25] for $B_{pk}$, the one-loop quadratic divergence for a bifundamental scalar loop cancels because

$$\sum_{k} B_{pk} = 0$$

which confirms the cancellation found in [9].
Since the scale $v$ breaks conformal invariance, the matter fields acquire mass, so the one-loop diagram $^4$ has a logarithmic divergence proportional to

$$\int \frac{d^4p}{v^2} \left[ \frac{1}{p^2 - m_k^2} - \frac{1}{p^2 - m_{k'}^2} \right] \sim -\frac{\Delta m_{kk'}^2}{v^2} \ln \left( \frac{\Lambda}{v} \right)$$

(46)

the sign of which depends on $\delta m_{kk'}^2 = (m_k^2 - m_{k'}^2)$.

To achieve conformality of U(1), a constraint must be imposed on the mass spectrum of matter bifundamentals, viz

$$\Delta m_{kk'}^2 \propto v^2 \left( \frac{11N}{48\pi^2} \right)$$

(47)

with a proportionality constant of order one which depends on the choice of model, the $n$ of $Z_n$ and the values chosen for $A_m, m = 1, 2, 3$. This signals how conformal invariance must be broken at the TeV scale in order that it can be restored at high energy; it is interesting that such a constraint arises in connection with an anomaly cancellation mechanism which necessarily breaks conformal symmetry.

To give an explicit model, consider the case of $Z_4$ and $A_m = (1, 1, 1, 1)$ treated earlier for which one finds:

$$\Delta m_{kk'}^2 = \frac{3}{2} v^2 \left( \frac{11N}{48\pi^2} \right)$$

(48)

In a more general model, the analog of Eq.(48) involves replacement of $\frac{3}{2}$ by a generally different coefficient derivable for each case from the coefficient $B_{pk}$ in Eq.(7).

With such a constraint, the one-loop $\beta_1$ vanishes in addition to $\beta_N$ so that the couplings $\alpha_1(\mu)$ and $\alpha_N(\mu)$ can be scale invariant for $\mu \geq \mu_0$.

For such conformal invariance at high energy to be maintained to higher orders of perturbation theory probably requires a global symmetry, for example the explicit form of misaligned supersymmetry recently suggested in [14].

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$^4$The usual one-loop $\beta-$function is of order $h^2$ regarded as an expansion in Planck’s constant: four propagators each $\sim h$ and two vertices each $\sim h^{-1}$ (c.f. Y. Nambu, Phys. Lett. B26, 626 (1968)). The diagram considered is also $\sim h^2$ since it has three propagators, one quantum vertex $\sim h$ and an additional $h^{-2}$ associated with $\Delta m_{kk'}^2$. 

11
Discussion

It has been shown how a compensatory term $\mathcal{L}_{\text{comp}} = \mathcal{L}_{\text{comp}}^{(1)} + \mathcal{L}_{\text{comp}}^{(2)}$ can be constructed respectively to cancel the $U(1)_p U(1)^2_q$ and $U(1)_p SU(N)^2_q$ triangle anomalies in the quiver gauge theories with chiral fermions. We have emphasized the uniqueness of the form of the compensatory term from the requirements of invariance under $SU(N)^n \subset U(N)^n$.

Such a term can have phenomenological consequences. We expect $v$ to be at the TeV scale as in [5] and $\mathcal{L}_{\text{comp}}$ reveals new non linear coupling between the bifundamental scalars and the gauge fields expected to be significant in the TeV energy regime. Such empirical consequences merit further study.

It has further been shown that the compensatory term $\mathcal{L}_{\text{comp}}$ can lead, with a suitable mass spectrum of bifundamental matter, to vanishing one-loop $\beta$--function for the $U(1)$ gauge group, this raising the possibility of one-loop scale invariance for all dimensionless couplings which may persist at all higher loops in the presence of a global symmetry.
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