Conservation law for massive scale-invariant photons in Weyl-invariant gravity

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Abstract
It is demonstrated that a Stückelberg-type gauge theory, coupled to the scalar–tensor theory of gravity, is invariant under both gauge and Weyl transformations. Unlike the pure Stückelberg theory, this coupled Lagrangian has a genuine Weyl symmetry, with a non-vanishing current. The above is true in the Jordan frame, whereas in the Einstein frame, the same theory manifests as Proca theory in presence of pure gravity. It is found that broken scale invariance leads to simultaneous spontaneous breaking of the gauge symmetry.

Keywords: Weyl invariance, massive gauge theory, current conservation, spontaneous symmetry breaking

1. Introduction

As is well known, gauge theories display both dynamics and constraints \([1–3]\) with physical quantities being manifestly gauge-invariant. In comparison, the Weyl symmetry, although initially introduced as a gauge symmetry \([4–8]\), does not have the same pivotal role. However, it still retains intriguing features, important in the gravity sector. Recently, in the context of the scalar–tensor gravity (STG), Weyl symmetry is shown to be a ‘fake’ one, as it arises in the Jordan frame, with vanishing current \([9,10]\). This symmetry gets completely eliminated by appropriately dressing the field variables.

In the case of gauge theory, massless-ness ensures Weyl invariance. The corresponding current is conserved, as the trace of the energy–momentum tensor (EMT) vanishes \([11]\). However, in the gauge sector, gauge theories expectedly violate Weyl invariance, due to the
presence of the mass term. The generation of mass has itself been of great interest, with the Higgs mechanism \[12–16\] being the dominant theory for generation of mass for gauge excitations. Other approaches for mass generation for gauge particle have been explored, involving the topological terms \[17–20\]. However, a seemingly massive Proca theory can be imbied with gauge invariance through suitable redefinition of the field variable, by introducing a scalar mode \[21–23\].

It is important to mention here that the redefinition of metric tensor via a frame transformation is quite similar to Stueckelberg’s approach for the massive Proca theory, where the gauge invariance is not manifestly present. Due to this similarity, they are combined into a single theory, with combined Weyl and gauge symmetry. Gauge fields enter conformal supergravity models through extension of the geometric connection, leading to combination of Weyl and gauge symmetries. On the other hand, Kaluza–Klein supergravity and string theories, reducible to conformal gravity, reduce to STG as an effective theory. However, the combination of STG, containing non-minimal scalar-gravity coupling term $R \phi^2$, and Abelian gauge field, is a non-supersymmetric model \[24\], whose symmetry properties have not been analyzed in detail before. We demonstrate that it is possible to attain generalized Weyl-invariance for this massive theory, through suitable coupling with STG, the latter having ‘fake’ Weyl invariance \[10\]. Interestingly, in this process, not only the photon is massive, but the STG also acquires non-vanishing conserved generalized Weyl current, owing to its coupling with the gauge sector. Thus, the fake Weyl invariance of the STG is changed to a genuine one through interaction with massive gauge field. Earlier, it has been shown that the Weyl invariant STG plays a key role in understanding inflation in the early Universe and it yields dimensional gravitational and cosmological constants \[25–28\]. Moreover, some inflationary models based on broken scale invariance, i.e. the global limit of Weyl symmetry, have also been developed, where the usual scale symmetry of theory, with suitable choices of variables, manifests as a shift symmetry \[29\]. In pursuance of this, the present theory may have relevance also in scale-invariant cosmologies.

The paper is organized as follows. In section 2, relevant properties of the Stückelberg and Proca theory are elucidated. Section 3 deals with the symmetry properties of STG, wherein the similarity with the Stückelberg theory is depicted. The combined Stückelberg-STG theory is constructed in section 4, as a consequence of Weyl-transformed gravitational Proca theory, simultaneously obtaining a massive gauge and genuine Weyl invariance. Further, it is found that when the scale invariance is broken spontaneously, the gauge symmetry also manifests in the broken symmetry phase.

2. Massive vector field: Stueckelberg theory

The Stückelberg Lagrangian density has a gauge invariant kinetic term for the massive vector field, and can be made gauge invariant by coupling the gauge boson to a scalar field, which transforms linearly. Stueckelberg mechanism is important because mass generation and gauge symmetry coexist, without taking recourse to Higgs mechanism. Lagrangian density for the Proca theory is given by,

$$\mathcal{L}_P = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{m^2}{2} a_\mu a^\mu,$$  \hspace{1cm} (1)

where $F_{\mu \nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ is the field strength for the vector field $a_\mu$, and $m$ represents its mass. Expectedly, the above Lagrangian density does not respect the usual gauge symmetry under $\delta a_\mu = \partial_\mu \theta$, as,
The gauge invariance can be restored through Stückelberg’s approach, with the new definition of gauge field as,

$$a_\mu \rightarrow a_\mu = a_\mu + \frac{1}{m} \partial_\mu \chi,$$

where $\chi$ is the Stückelberg scalar field. Substituting the above in equation (1), we get the Stückelberg Lagrangian density:

$$\mathcal{L}_{\text{St}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} a_\mu a^\mu + ma_\mu \partial^\mu \chi + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi$$

$$= \mathcal{L}_p + ma_\mu \partial^\mu \chi + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi,$$

which is invariant under the following symmetry transformations:

$$\delta a_\mu = \partial_\mu \theta, \quad \delta \chi = -m \theta, \quad \delta \mathcal{L}_{\text{St}} = 0.$$

The Noether functions for the two fields are,

$$E_\alpha = \frac{\partial \mathcal{L}_{\text{St}}}{\partial \partial \alpha} - \partial_\alpha \left( \frac{\partial \mathcal{L}_{\text{St}}}{\partial (\partial_\alpha a_\mu)} \right) \equiv \partial_\alpha F^{\mu\nu} + ma_\nu \partial^\nu \chi,$$

$$E_\chi = \frac{\partial \mathcal{L}_{\text{St}}}{\partial \partial \chi} - \partial_\chi \left( \frac{\partial \mathcal{L}_{\text{St}}}{\partial (\partial_\chi a_\mu)} \right) \equiv -a_\mu \partial^\mu \chi - m \partial_\mu a^\mu,$$

where the second function can be obtained from the first by taking a four-divergence.

The off-shell current $X^\mu$ of this theory is obtained, by considering variation of the Lagrangian under the gauge transformation:

$$\delta \mathcal{L}_{\text{St}} = \delta a_\mu \partial_\mu a_\nu + \frac{\partial \mathcal{L}_{\text{St}}}{\partial (\partial_\mu a_\nu)} \delta (\partial_\nu a_\mu) + \frac{\partial \mathcal{L}_{\text{St}}}{\partial \partial \chi} \delta \chi + \frac{\partial \mathcal{L}_{\text{St}}}{\partial (\partial_\chi a_\mu)} \delta (\partial_\mu \chi)$$

$$= (m^2 a_\nu + m \partial^\nu \chi) \partial_\mu \theta + (\partial^\nu F_{\mu\nu}) \partial_\mu \theta + (\partial^\nu \chi + m a_\nu)(-m \partial_\mu \theta)$$

$$= 0,$$

where, in the second line, we have used expressions given in equations (4) and (5).

Thus, $X^\mu = 0$.

The on-shell current $K^\mu$ is obtained by using the equations of motion, demanding both $E_\alpha = 0$. This changes $\mathcal{L}_{\text{St}}$ as,

$$\delta \mathcal{L}_{\text{St}} = \partial_\nu \left[ \frac{\partial \mathcal{L}_{\text{St}}}{\partial (\partial_\nu a_\mu)} \delta a_\mu + \frac{\partial \mathcal{L}_{\text{St}}}{\partial (\partial_\nu \chi)} \delta \chi \right] \equiv \partial_\nu K^\mu,$$

leading to,

$$K^\mu = -(F^{\mu\nu} \partial_\nu \theta + m \partial^\nu \chi \theta + m^2 a^\nu \theta).$$

From equations (7) and (9), Noether’s current for the Stückelberg system is found to be,

$$J^\mu := X^\mu - K^\mu = F^{\mu\nu} \partial_\nu \theta + m \partial^\nu \chi \theta + m^2 a^\nu \theta,$$

satisfying the conservation law,

$$\partial_\mu J^\mu = (\partial_\mu F^{\mu\nu} + m^2 a^\nu + m \partial^\nu \chi) \partial_\nu \theta + (\partial^\nu \chi + m a_\nu)(-m \partial_\mu \theta)$$

$$= E_\alpha \delta a_\mu + (-E_\chi)(-\delta \chi) \equiv 0,$$

where, equation (6) has been applied at the end.
2.1. Finite gauge transformation in Stückelberg’s approach

Invariance of the Lagrangian density for the Stückelberg theory is independent of the gauge parameter being finite or infinitesimally small. As Stückelberg theory is obtained from the Proca theory through the finite gauge transformation,

\[ a_\mu \rightarrow a_\mu \rightarrow a_\mu - \frac{i}{g} V^{-1} \partial_\mu V, \quad V = \exp \left( \frac{i g}{m} \chi \right), \]  

(12)

which takes \( \mathcal{L}_P \) to \( \mathcal{L}_{St} \). The next set of gauge transformations,

\[ a_\mu \rightarrow a_\mu + \frac{i}{\hbar} U^{-1} \partial_\mu U, \]

\[ V \rightarrow UV, \quad U = \exp \left( - \frac{i \hbar}{m} \theta \right), \]  

(13)

yields,

\[ \mathcal{L}_{St} \rightarrow \mathcal{L}_{St} + \left(1 - \frac{h}{g}\right) (ma_\mu + \partial_\mu \chi) \partial_\mu \theta + 2 \left( \frac{1}{g} \right) \partial_\mu \theta \partial_\mu \theta. \]  

(14)

which defines a symmetry under the parametric condition \( h = g \).

The same property, namely,

\[ \mathcal{L}_P \rightarrow \mathcal{L}_{St} \rightarrow \mathcal{L}_{St}, \]  

(15)

for successive gauge transformations, with respect to both infinitesimal parameters \( \chi \) and \( \theta \) respectively, where \( \mathcal{L}_{St} \) is defined without the term quadratic in \( \chi \), prevails. Therefore, equation (15) holds for both finite and infinitesimal gauge transformations.

2.2. EMT for the Stückelberg theory

The symmetric EMT, including the Belinfanté term, is,

\[ T^{\mu\nu} = 2 \frac{\delta}{\delta g_{\mu\nu}} - g^{\mu\nu} \mathcal{L}. \]  

(16)

For the Stückelberg theory, the same is found to be,

\[ T_{St}^{\mu\nu} = - F^{\mu\nu} F_\alpha^\alpha + m^2 a^\alpha a^\alpha + \partial_\mu \chi \partial_\nu \chi + m(a^\nu \partial_\mu \chi + a^\mu \partial_\nu \chi) - g^{\mu\nu} \mathcal{L}_{St}, \]  

(17)

with the corresponding trace,

\[ g_{\mu\nu} T_{St}^{\mu\nu} = -(m^2 a^\mu a^\nu + \partial_\mu \chi \partial_\nu \chi + 2ma^\nu \partial_\mu \chi). \]  

(18)

Clearly, the theory is not Weyl-invariant, expectedly for being massive, but also for containing the scalar field \( \chi \). On adding a superpotential term \([9, 11]\), the improved EMT becomes,

\[ \Theta_{St}^{\mu\nu} = T_{St}^{\mu\nu} + \frac{1}{6} (g^{\mu\nu} \partial_\alpha \partial_\rho - \partial_\mu \partial_\rho) \chi^2, \]  

(19)

leading to the trace,

\[ g_{\mu\nu} \Theta_{St}^{\mu\nu} = -(m^2 a^\mu a^\nu + 2ma^\nu \partial_\mu \chi), \]  

(20)

after using the equation of motion for the scalar field, in the Lorentz gauge. Thus, the on-shell trace of the Stückelberg EM tensor vanishes for \( m = 0 \), as required.
3. Emergence of scalar–tensor theory from pure gravity theory

Let us now consider the $(3 + 1)$-dimensional Einstein–Hilbert action,

$$S = - \int d^4x \mathcal{L}_0 = - \int d^4x \frac{1}{12k} \sqrt{-g} R,$$

(21)

where $k = (16\pi G)$ is an overall constant and $R$ is the Ricci scalar defined as $R = g^{\mu\nu} R_{\mu\nu}$, with $g^{\mu\nu}$ and $R_{\mu\nu}$ being the metric and Riemann curvature tensor, respectively. It is important to point out that the curvature present in the free gravity theory is due to the metric tensor and mass of the background, because of which the above Lagrangian density changes under the Weyl scale transformation: $g^{\mu\nu} \rightarrow e^{2\theta} g^{\mu\nu}$, where $\theta$ is a local parameter. For $\theta$ being infinitesimally small, the Weyl symmetry becomes $\delta g^{\mu\nu} = 2\theta g^{\mu\nu}$, under which the Lagrangian density transforms as,

$$\delta \mathcal{L}_0 = \frac{2}{k} \theta \sqrt{-g} R.$$

(22)

The symmetry can be restored by redefining the metric tensor as $g_{\mu\nu} = \varphi^2 g_{\mu\nu}$, where $\varphi$ is a scalar field. This redefinition is similar to the Stückelberg’s approach, discussed in the previous section. It is important to point out that to maintain the Weyl invariance, the metric tensor has been scaled by a local scalar field, whereas in Proca theory, a derivative of scalar field has been added to redefine the vector field, for the desired gauge invariance.

Under the locally scaled metric tensor, the field variables present in the theory change accordingly as,

$$g_{\mu\nu} \rightarrow \varphi^2 g_{\mu\nu}, \quad g^{\mu\nu} \rightarrow \varphi^{-2} g^{\mu\nu},$$

with the modified Ricci scalar $[9],

$$\sqrt{-g} R \rightarrow \sqrt{-\varphi^2 g} R \varphi^2 + 6 \sqrt{-\varphi^2 g} g^{\mu\nu} \partial_{\nu} \varphi \partial_{\mu} \varphi.$$

(23)

Finally, substituting the changes in field variables, we get a modified Lagrangian, known as the scalar–tensor Lagrangian density:

$$\mathcal{L}_{ST} = \frac{1}{\kappa} \left[ \frac{1}{12} \sqrt{-g} R \varphi^2 + \frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_{\nu} \varphi \partial_{\mu} \varphi \right].$$

(24)

The infinitesimal Weyl symmetry,

$$\delta g^{\mu\nu} = 2\theta g^{\mu\nu}, \quad \delta \varphi = \theta \varphi,$$

(25)

changes the Lagrangian by a total derivative, leading to off-shell contribution to the conserved current as,

$$X^\mu = \frac{1}{2\kappa} \sqrt{-g} \varphi^2 g^{\mu\nu} \partial_{\nu} \theta.$$

(26)

The use of Euler–Lagrange equations for $\varphi$ and $g_{\mu\nu}$, re-casts the variation of the Lagrangian as a total derivative, leading to the on-shell contribution to the Weyl current,

$$K^\mu = \frac{1}{2\kappa} \sqrt{-g} \varphi^2 g^{\mu\nu} \partial_{\nu} \theta.$$

(27)

As both these contributions are equal, the conventional conserved Nöther Weyl current vanishes [10]:

$$\delta \mathcal{L}_0 = \frac{2}{k} \theta \sqrt{-g} R.$$
deeming the Weyl symmetry as a fake one. The situation is unchanged [10] upon application of Nöther’s second theorem [1, 4–7], appropriate for local symmetries, such as the Weyl symmetry here. The present aim is to obtain an extended theory having genuine generalized symmetry, instead of the ‘fake’ one, thereby obtaining a non-vanishing current. Although the matter of extended Weyl symmetries have been discussed earlier, with additional fields, having general coupling to STG [30] and massive excitations [31], the issue of the conserved current was not addressed. In section 4, we construct the simplest example of a theory, non-trivially coupled to the STG, yielding an extended, but genuine, Weyl symmetry.

3.1. Finite Weyl transformations in scalar–tensor theory

The scalar–tensor Lagrangian density \( \mathcal{L}_{ST} \) is obtained from the free gravity Lagrangian density \( \mathcal{L}_0 \), given in equation (21) through a finite Weyl transformation, given in equation (20). Let us consider the generic finite local scaling of the form,

\[
\begin{align*}
g^{\mu\nu} &\rightarrow \psi^{-n}g^{\mu\nu}, & g_{\mu\nu} &\rightarrow \psi^n g_{\mu\nu}, & \varphi &\rightarrow \psi \varphi,
\end{align*}
\]

with \( \psi \) being local and finite and \( n \) being the numerical power. Under such scaling, the terms of \( \mathcal{L}_{ST} \) transform as,

\[
\sqrt{-g} \frac{1}{12\kappa} R(g) \varphi^2 \rightarrow \sqrt{-g} \frac{1}{12\kappa} \left[ \psi^{n+2} \varphi^2 R(g) + \frac{3}{2} n (n + 4) \psi^n \varphi^2 g^{\mu\nu} \partial_{\mu} \varphi \partial_{\nu} \varphi \right. \\
&\left. + 6ng^{\mu\nu} \psi^n \partial_{\mu} \varphi \partial_{\nu} \varphi \right] - \frac{n}{4\kappa} \partial_{\nu} [\sqrt{-g} \varphi^{n+1} \partial^\mu \psi],
\]

\[
\sqrt{-g} \frac{1}{2\kappa} g^{\mu\nu} \partial_{\mu} \varphi \partial_{\nu} \varphi \rightarrow \sqrt{-g} \frac{1}{2\kappa} \psi^n g^{\mu\nu} \left[ \partial^2 \partial_{\mu} \psi \partial_{\nu} \varphi + 2 \psi \partial_{\mu} \varphi \partial_{\nu} \varphi \\
&+ \psi^2 \partial_{\mu} \varphi \partial_{\nu} \varphi \right].
\]

For the present case,

\[
\begin{align*}
g^{\mu\nu} &\rightarrow \psi^2 g^{\mu\nu}, & g_{\mu\nu} &\rightarrow \psi^{-2} g_{\mu\nu}, & \varphi &\rightarrow \psi \varphi,
\end{align*}
\]

corresponding to \( n = -2 \), the general expressions reduce to,

\[
\sqrt{-g} \frac{1}{12\kappa} R \varphi^2 \rightarrow \sqrt{-g} \left[ \frac{1}{12\kappa} R \psi^2 \varphi^2 - \frac{1}{2\kappa} \psi^{-2} \varphi^2 g^{\mu\nu} \partial_{\mu} \psi \partial_{\nu} \varphi - \frac{1}{\kappa} g^{\mu\nu} \psi^{-1} \varphi \partial_{\mu} \psi \partial_{\nu} \varphi \right] \\
&+ \partial_{\nu} \left[ \sqrt{-g} \frac{1}{2\kappa} \psi^2 \varphi g^{\mu\nu} \partial_{\mu} \psi \partial_{\nu} \varphi \right],
\]

\[
\sqrt{-g} \frac{1}{2\kappa} g^{\mu\nu} \partial_{\mu} \varphi \partial_{\nu} \varphi \rightarrow \sqrt{-g} \left[ \frac{1}{2\kappa} g^{\mu\nu} \psi^{-2} \varphi^2 \partial_{\mu} \psi \partial_{\nu} \varphi + \frac{1}{\kappa} g^{\mu\nu} \psi^{-1} \varphi \partial_{\mu} \psi \partial_{\nu} \varphi \\
&+ \frac{1}{2\kappa} g^{\mu\nu} \varphi \partial_{\mu} \varphi \partial_{\nu} \varphi \right].
\]

On combining both parts, we have,

\[
\mathcal{L}_{ST} \rightarrow \mathcal{L}_{ST} + \partial_{\nu} \left[ \sqrt{-g} \frac{1}{2\kappa} \frac{\varphi^2}{\psi} g^{\mu\nu} \partial_{\mu} \psi \right].
\]

Therefore, \( \mathcal{L}_{ST} \) changes only by a total derivative under Weyl scaling by a finite local function \( \psi(x) \), leaving the corresponding action unchanged as expected. Therefore, the STG
is invariant under both finite and infinitesimal Weyl transformations, just like the gauge transformation of Stückelberg theory. This intuitively enables us to combine both the theories through identification of the corresponding local transformation parameters. As a check, for infinitesimal Weyl transformation, \( \psi = \exp(\theta) \), \( \theta \ll 1 \), one has,

\[
\mathcal{L}_{\text{ST}} \rightarrow \mathcal{L}_{\text{ST}} + \frac{1}{2\kappa} \left[ \sqrt{-g} \frac{1}{2} \varphi^2 g^{\mu\nu} \partial_{\mu} \theta \right],
\]

yielding the expression of the off-shell contribution for the current given in equation (27) for infinitesimal Weyl scaling.

3.2. EMT for scalar–tensor theory

In this subsection, we derive the EMT for the STG and show that it is automatically symmetric, as well as traceless in nature. The EMT for a massless scalar field is given by,

\[
T^\phi_{\mu\nu} \equiv \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi,
\]

which is symmetric but not traceless. It was postulated that introduction of the additional transverse part, \( \frac{1}{4} \left( g_{\mu\nu} \nabla^\rho \nabla_\rho - \nabla^\mu \nabla^\nu \right) \varphi^2 \) makes the EMT traceless, while preserving the original Poincaré generators of the theory, thereby keeping the dynamical observables unchanged [11]. The variation of the gravitational part of the theory, with respect to the metric, leads to the corresponding contribution to the EMT as,

\[
T^g_{\mu\nu} = G^{\mu\nu} \varphi^2 + \left( g^{\mu\nu} \nabla^\rho \nabla_\rho - \nabla^\mu \nabla^\nu \right) \varphi^2;
\]

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R.
\]

Here, \( \nabla^\mu \) is the covariant derivative with respect to the gravitational metric. This yields the complete EMT of the theory,

\[
\theta^{\mu\nu} = T^\phi_{\mu\nu} + T^g_{\mu\nu}
\]

\[
\equiv \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \frac{1}{6} G^{\mu\nu} \varphi^2
\]

\[
+ \frac{1}{6} \left( g^{\mu\nu} \nabla^\rho \nabla_\rho - \nabla^\mu \nabla^\nu \right) \varphi^2.
\]

Therefore, the ‘transverse’ extension to the scalar part, required for Weyl symmetry, is automatically provided by the gravitational part. Hence, the complete scalar–tensor theory is Weyl-invariant.

4. A combined theory

Motivated by the similarity between the roles of Weyl transformation in case of gravity and that of gauge transformation in case of massive photon, it is tempting to hope for a larger picture, which can accommodate these two sectors. This has indeed been found to be possible. Starting in the Einstein frame, let us define the action,

\[
S_{\text{EHP}} = -\int d^4x \sqrt{-g} \left[ \frac{1}{12\kappa} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} g^{\alpha\beta} + \frac{m^2}{2} \partial_\mu \phi \partial^\mu \phi \right]
\]

\[
\}
\]

(38)
This keeps the STG action dimensionless, as required, provided we consider the overall constant, \( \kappa^{-1} = (16\pi G)^{-1} \) of dimension \([m]^{-2}\), with \( G \) being the gravitational constant, of dimension \([m]^2\). The corresponding action, though dimensionally admissible, is neither Weyl nor gauge invariant. More specifically, the first and third terms change under Weyl scaling, whereas the third changes under gauge transformation too. The second term is gauge invariant, and also Weyl invariant, as we are in \((3+1)\)-dimensions. The overall Weyl non-invariance is further reflected by the non-zero trace of the complete EM tensor,

\[
T^{\mu\nu} = \frac{1}{6\kappa}G^{\mu\nu} - F^{\mu\alpha}F_{\alpha}^\mu + m^2a^\mu a^\nu - g^{\mu\nu}\mathcal{L}_{\text{St}}, \tag{39}
\]

leading to,

\[
T^\mu_\mu = -\frac{1}{6\kappa}R - m^2a^\mu a_\mu. \tag{40}
\]

A simultaneous Weyl and gauge transformation, \( g_{\mu\nu} \rightarrow \varphi^2g_{\mu\nu}, \quad a_\mu \rightarrow a_\mu - \partial_\mu \text{log}\varphi \),

leads to the following Lagrangian,

\[
\mathcal{L}_{\text{STSt}} = \sqrt{-g} \left[ \frac{1}{12\kappa}R\varphi^2 + \frac{1}{2\kappa}\partial_\mu\varphi\partial^\mu\varphi g^{\mu\nu} - \frac{1}{4}F_{\mu\alpha}F_{\beta\gamma}g^{\mu\nu}g^{\alpha\beta} + m^2\varphi^2 \left( \frac{1}{2}a_\mu a^\mu + \frac{1}{2}\partial_\mu\text{log}\varphi\partial_\nu\text{log}\varphi - a_\mu\partial_\nu\text{log}\varphi \right) g^{\mu\nu} \right] \tag{42}
\]

in the Jordan frame. The above action can effectively be viewed as the sum of STG and Stückelberg actions, with a suitable coupling modifying the gauge mass term, including the field \( \varphi \). In that sense, it is different from Stückelberg theory. This action is invariant under the set of ‘combined’ transformations,

\[
g^{\mu\nu} \rightarrow \psi^2g^{\mu\nu}, \quad \varphi \rightarrow \psi\varphi, \quad a_\mu \rightarrow a_\mu + \partial_\mu \text{log}\psi, \tag{43}
\]

with \( \text{log}\psi \) being small, which is clear from the treatment of the previous sections. The combined transformations are defined in terms of a single local parameter (\( \varphi \) or \( \psi \)), with Weyl and gauge subsets being independent, as the gauge field is a Weyl scalar. Further, \( \mathcal{L}_{\text{STSt}} \) goes back to \( \mathcal{L}_{\text{EHP}} \) for \( \psi = \varphi \).

The Lagrangian in equation (42) can be re-expressed as,

\[
\mathcal{L}_{\text{STSt}} = \sqrt{-g} \left[ \frac{1}{12\kappa}R\varphi^2 + \frac{1}{2\kappa}\partial_\mu\varphi\partial^\mu\varphi g^{\mu\nu} - \frac{1}{4}F_{\mu\alpha}F_{\beta\gamma}g^{\mu\nu}g^{\alpha\beta} + \frac{m^2}{2}g^{\mu\nu}D_\mu\varphi D_\nu\varphi \right], \tag{44}
\]

with \( D_\mu = \partial_\mu - a_\mu \) being the covariant derivative corresponding to the well-known R-symmetry (or \( U(1)_R \) symmetry), analogous to that of the standard \( U(1) \) gauge theory. The last term above is invariant under the combined transformation, with the change in the term \( \sqrt{-g}g^{\mu\nu} \) compensating for the same in \( D_\mu\varphi D_\nu\varphi \). This is unlike the scalar quantum electrodynamics (QED), where complex-conjugation in \( (D_\mu\phi)^\dagger D^\mu\phi \) maintains its invariance, with \( D_\mu = \partial_\mu - ia_\mu \) and complex scalar field \( \phi \). The \( U(1) \) coupling of gauge field with complex scalar field \( \phi \) physically represents the interaction of particle–antiparticle through exchange of photon. In the present case, as the gauge transformation is aided by the Weyl
transformation to restore the overall symmetry, the real scalar field interacts, \textit{without} any charge, via $U(1)_R$ photon exchange, through its coupling with the metric (gravity). Such gauge interactions are common in supergravity models \cite{32}. Therefore, from the interaction point-of-view too, the Weyl+gauge transformation corresponds to the complete symmetry of the theory. It is worthwhile to mention that the relation between the Weyl symmetry and the gauge symmetry is similar to the color-flavor locking in quantum chromodynamics \cite{33}.

\subsection*{4.1. Equations of motion}

The combined action in equation (44) yields the respective equations of motion for the gauge ($a_\mu$), scalar ($\varphi$) and gravitational ($g_{\mu\nu}$) fields as,

\begin{align}
\nabla_\mu F^{\mu\nu} &= -m^2\varphi^2a^\nu + m^2\varphi\partial^\nu\varphi, \\
\Delta \varphi &= \frac{1}{1 + \kappa m^2} \left[ \frac{1}{6}R + \kappa m^2(\nabla_\mu a_\nu + a_\mu a_\nu)g^{\mu\nu} \right] \varphi, \\
T^{\mu\nu} &= \frac{1}{\kappa} \left[ \frac{1}{12}G_{\mu\nu}\varphi^2 + \frac{1}{2}g_{\mu\nu}\varphi\partial_\nu\varphi - \frac{1}{4}g_{\mu\nu}g^{\alpha\beta}\partial_\nu\varphi\partial_\beta\varphi + \frac{1}{12}(g_{\mu\nu}\Delta - \nabla_\mu \nabla_\nu)\varphi^2 \right] \\
&\quad - \frac{1}{2}F^\alpha_\mu F^\mu_\alpha + \frac{m^2}{2}\varphi^2\tilde{a}_\mu \tilde{a}_\nu - \frac{1}{4}g^{\mu\nu}\left( -\frac{1}{2}F^\alpha_\mu F^\alpha_\nu + m^2\varphi^2\tilde{a}_\alpha \tilde{a}^\alpha \right) \\
&\equiv 0; \\
\end{align}

with,

\begin{equation}
\Delta \varphi = \frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} g^{\mu\nu} \partial_\nu \varphi], \quad \tilde{a}_\mu = a_\mu - \partial_\mu \log \varphi. \tag{46}
\end{equation}

The trace of the \textit{off-shell} total EMT $T^\mu_\mu$, by utilizing equation of motion for the scalar field, turns out as,

\begin{equation}
T^\mu_\mu = -\frac{m^2}{2(1 + \kappa m^2)} \varphi^2 \left[ g^{\mu\nu} \nabla_\nu \varphi + g^{\mu\nu}a_\nu \partial_\nu \varphi - \frac{1}{6}R \right] - \frac{m^2}{2} g^{\mu\nu} D_\mu \varphi D_\nu \varphi. \tag{47}
\end{equation}

Thus, the presence of photon mass $m$ (equation (38)) breaks the naive Weyl invariance, as physically expected. The Lagrangian in equation (44) is further non-invariant under pure gauge transformation, in absence of scaling of the metric. The individual violations of both Weyl and gauge symmetry owe to the last term in equation (44), which corresponds to the Proca mass term, in addition to being coupled with $\varphi^2$. This term is invariant only under combined Weyl-gauge transformation of equation (43), and so is the full theory. Therefore, massless electrodynamics and gravity can co-exist with independent Weyl and gauge symmetries (in the Jordan frame). Introduction of photon mass, though breaks \textit{both} of these symmetries individually, it re-adjusts the system to be invariant under an extended symmetry, which is the combined Weyl-gauge symmetry.

The crucial role of the mass term to extend the symmetry, unique from the naive sum of Weyl and gauge symmetries, can be clearly understood from the equation of motion for the scalar field. The RHS represents a mass term contributed by gravity, in violation to the equivalence principle. Further, it includes contributions due to gauge coupling. Therefore, the theory is symmetric \textit{only} under the complete Weyl-gauge transformation, with tensor, vector and scalar parts compensating for each-other. In case of the scalar field itself being massive, it
is known to induce a contribution equivalent to the cosmological constant \[ \Box \] . For the present mass-less scalar field, no such shift occurs. Instead, it is the coupling to the gauge field that provides a mass-like contribution. The same is further reflected in the non-vanishing RHS of equation (47), making the complete EMT of the theory trace-full. However, this contribution entirely comes from the gauge-coupling, as represented by the overall multiplication by the coupling strength \( m^2 \). Indeed, for ‘pure’ Weyl invariance, the EMT must be trace-less by construction, i.e. off-shell in the gravitational sector. Therefore, although the gravitational EOM in equations (45) yields \( T_{\mu \nu} = 0 \), and thereby \( T_\mu \mu = 0 \), the theory is not Weyl-invariant. This is true for any theory coupled to gravity, as the gravitational EOM always results in vanishing of total EMT. Applying all three EOMs, an additional condition,
\[
\mathcal{D}_\mu \varphi^2 \equiv \varphi^2 g^{\mu \nu} (\nabla_\mu a_\nu + a_\mu a_\nu), \quad \mathcal{D}_\mu = \nabla_\mu - a_\mu,
\]
is obtained, merely stating that all fields are dynamically not independent [34].

Intuitively, coupling with gauge field can make the scalar field ‘massive’, but that mass will be gauge-dependent in general, as is well-known from the self-energy corrections in QED [35]. The non-minimal gravitational coupling is known to restore the Weyl invariance of the mass-less scalar field in STG. In the present case, due to gauge coupling, the scalar field acquires a mass-like term that breaks the naive Weyl invariance. But the additional Stückelberg-like gauge structure extends the same to the unique Weyl-gauge symmetry. The key to this ‘combined’ symmetry is clearly the identification of \( \varphi \) in both the Weyl and gauge sectors. The analysis of the present theory, with an appropriate gauge-fixing, is under investigation and will be reported elsewhere.

For the pure STG (equation (25)), the transverse contribution of the \( \sqrt{-g} R \varphi^2 \) to the EMT restores Weyl invariance of the dynamic scalar part. Therefore, the question arises regarding what restores the Weyl invariance of the additional dynamic scalar contribution,
\[
\sqrt{-g} \varphi^2 g^{\mu \nu} \partial_\mu \log \varphi \partial_\nu \log \varphi = \sqrt{-g} g^{\mu \nu} \partial_\mu \varphi \partial_\nu \varphi,
\]
coming from the Stückelberg part. In reality, this contribution is made Weyl invariant by the same transverse contribution, as the Weyl invariance restoration of the dynamic scalar part is on-shell, i.e. the equation of motion for the scalar field has to be used. In the combined theory the scalar field equation of motion gets contribution from both the dynamic parts, as seen in equation (45). On utilization of the same, the transverse part of the EMT exactly compensates for the Weyl non-invariance of both dynamic scalar parts.

4.2. Spontaneously broken scale symmetry

For the possibility of spontaneous symmetry breaking of the scale invariance, we now consider the global limit of Weyl symmetry (scale symmetry), after adding a potential term \( V(\varphi) = \beta \varphi^4 \), having a non-vanishing vacuum expectation value: \( \langle \varphi \rangle^2 = \Omega^2 / 2 \gamma \), for a generic Lagrangian density,
\[
\mathcal{L}_{\text{STG1}} = \sqrt{-g} \left[ \frac{\gamma}{2} R \varphi^2 + \frac{1}{2 \kappa} \partial_\mu \varphi \partial_\mu \varphi g^{\mu \nu} + V(\varphi) - \frac{1}{4} F_{\mu \nu} F_{\mu \nu} g^{\alpha \beta} g^{\mu \nu} + m^2 \varphi^2 \left( \frac{1}{2} a_\mu a_\mu + \frac{1}{2} \partial_\mu \log \varphi \partial_\mu \log \varphi - a_\mu \partial_\mu \log \varphi \right) g^{\mu \nu} \right],
\]
where \( \gamma \) is a dimensionless parameter. Although we are presently interested with global scale symmetry, the potential term \( \sqrt{-g} V(\varphi) \) is invariant also under Weyl scaling which is local,
and known to effectively represent the cosmological constant in STG models [11]. By suitably re-scaling the metric and gauge fields:

\[ g_{\mu\nu} \rightarrow \frac{\Omega^2}{2\gamma} \varphi^{-2} g_{\mu\nu}, \quad a_{\mu} \rightarrow a_{\mu} + \partial_{\mu} \log \varphi. \]  

one gets back the Lagrangian density in Einstein frame with some modifications as,

\[
\mathcal{L}_{\text{EHP1}} = \sqrt{-g} \left[ \frac{\Omega^2}{2\gamma} R + \frac{1}{2} g^{\mu\nu} \partial_{\mu} \xi \partial_{\nu} \xi + V(\xi) - \frac{1}{4} F_{\mu\alpha} F_{\nu\beta} g^{\mu\nu} g^{\alpha\beta} \right] \\
+ \frac{\Omega^2}{2\gamma} \left( \frac{m^2}{2} a_{\mu} a_{\mu} \right) g^{\mu\nu}.
\]  

where \( V(\xi) = \frac{1}{4} \gamma^{-2} \Omega^4 \varphi(\xi)^{-4} V[\varphi(\xi)] \). The relation between the original scalar (\( \varphi \)) and Einstein frame scalar (\( \xi \)) is,

\[
\varphi(\xi) = \langle \varphi \rangle \exp \left( \frac{\sqrt{\gamma} \xi}{\Omega} \right), \quad \frac{1}{\gamma} = \frac{1}{2\gamma} - 6,
\]  

with boundary condition \( \varphi(\xi = 0) = \langle \varphi \rangle \). In the Einstein frame, the original scale symmetry becomes the shift symmetry,

\[
\xi \rightarrow \bar{\xi} = \xi + \frac{\Omega}{\sqrt{\gamma}} \theta, \quad a_{\mu} = a_{\mu}.
\]  

Here, \( \theta \) is presently a global parameter, corresponding to the scale symmetry, instead of the Weyl symmetry, which is local. Under the shift symmetry, Lagrangian density in equation (50) remains invariant. Hence, \( \mathcal{L}_{\text{EHP1}} \) is the Lagrangian for the spontaneously broken scale symmetry, where the above shift symmetry is the part of the original scale invariance and \( \varphi \) is the Goldstone boson for the broken scale invariance. The above analysis is valid for in generic \( n \)-dimensions, unlike the combined Weyl-gauge symmetry before, which require \( n = (3 + 1) \)-dimensions.

### 4.3. Noether's conserved current

As the Noether current corresponds to continuous symmetries, allowing for infinitesimal transformations, one can re-parameterize the parameter for the second Weyl transformation as,

\[
\psi = \exp \left( \frac{\theta}{m} \right).
\]

A similar parameterization leads to,

\[
\varphi = \exp \left( - \frac{\chi}{m} \right).
\]

These re-definitions are adopted, as for a dimensionless Weyl scalar field, representing pure scaling of the metric, \( \theta \) and \( \chi \) must have dimension of mass (\( \text{kg}^1 \)), a fact essential for identifying \( \chi \) as the physical Nambu-Goldstone field of the Stückelberg theory [21]. Then, the infinitesimal Weyl-gauge variations become,
\[ \delta g_{\mu\nu} = 2 \frac{\theta}{m} g_{\mu\nu}, \quad \delta a_{\mu} = \frac{1}{m} \partial_{\mu} \theta, \quad \delta \chi = -\theta. \] (53)

As in section 2, the off-shell and on-shell variations, respectively, lead to,

\[ \delta_1 \mathcal{L}_{\text{STST}} = \partial_{\mu} \left( \frac{1}{\kappa} X^\mu \right), \quad X^\mu = \frac{1}{2m} \sqrt{-g} \varphi^2 g_{\mu\nu} \partial_{\nu} \theta, \]

\[ \delta_2 \mathcal{L}_{\text{STST}} = - \partial_{\mu} \left[ \sqrt{-g} \left\{ \frac{1}{m} F^{\mu\nu} \partial_{\nu} \theta + \exp \left( -2 \frac{\chi}{m} \right) m \theta (m a^\mu + \partial^\mu \chi) \right\} \right] \]

\[ + \partial_{\mu} \left( \frac{1}{\kappa} X^\mu \right). \] (54)

yielding the conserved Nöther current,

\[ \delta_1 \mathcal{L}_{\text{STST}} - \delta_2 \mathcal{L}_{\text{STST}} = \partial^\mu J_{\mu} = 0, \]

\[ J_{\mu} = \sqrt{-g} \left\{ \frac{1}{m} F_{\mu\nu} g^{\alpha\beta} \partial_\beta \theta + \exp \left( -2 \frac{\chi}{m} \right) m \theta (m a_{\mu} + \frac{1}{m} \partial_{\mu} \chi) \right\}, \] (55)

having a non-vanishing expression.

### 4.4. A parametric duality

The crucial feature of being able to construct the combined theory in equation (44) is the Weyl invariance of the mass term. It demands only that \( \sqrt{-g} \varphi^2 \) is dimensionless and \( \partial_{\mu} \varphi \) is of dimension one \( (m^1) \). This, however, leaves a freedom of choice for dimensions for both \( \sqrt{-g} \) and \( \varphi \), expected from Weyl invariance. On the other hand, dimension of \( \partial_{\mu} \varphi \) remains unaffected by any such choice, which is essential for the gauge part of the combined transformations.

From the Weyl transformation \( g_{\mu\nu} \rightarrow \varphi^2 g_{\mu\nu} \), a particular choice can be of \( \varphi \) being a \( [m^3] \) real scalar field, thereby representing physical dynamics. As a consequence, it is required that,

\[ \text{d}s^2 = g_{\mu\nu} \text{d}x^\mu \text{d}x^\nu, \] (56)

in the Einstein frame, the covariant metric tensor in the Jordan frame will now be of \( [m^{-2}] \), and \( [g^{\mu\nu}] = [m^2] \). This forces the redefinitions,

\[ \varphi = \alpha \exp(-\chi) \quad \text{and} \quad \psi = \alpha \exp(-\theta), \] (57)

with \( \alpha \) being a constant of \( [m^1] \). This further requires \( \chi \) and \( \theta \) to be dimensionless, and the prior can no more be the physical Nambu–Goldstone field of the standard Stückelberg theory. This also alters the Nöther current in equation (58) to,

\[ J_{\mu} = \sqrt{-g} \left\{ F_{\mu\nu}^{\alpha\beta} \partial_{\beta} \theta + \alpha^2 \exp(-2\chi) m^2 \theta (a_{\mu} + \partial_{\mu} \chi) \right\}, \] (58)

which is still of dimension \( [m^3] \), as required physically.

The physical difference between the previous and the present choices of field dimensions is that in the latter case, the physical scalar field is \( \varphi \), rather than \( \chi \), though the overall symmetry is intact. Thus, it can be viewed as if the roles are shifted, and the corresponding mass generation has indirectly been shifted to the metric. Although a dimensional scaling of the metric can be of deeper physical interpretation, it may find place in some special cosmological models, wherein the dual theory will be applicable.
Conclusions

In conclusion, it is shown that a massive Proca theory, in presence of gravity, can be re-cast as a modified Stückelberg theory, coupled with STG. This composite theory is both Weyl, as well as gauge-invariant, provided the transformation parameters are identified. This massive theory is invariant under generalized Weyl transformation, with necessary gravitational coupling, corresponding to a non-vanishing Weyl current. Further, the role of gravitational coupling in restoring this extended Weyl symmetry in otherwise a Weyl non-invariant theory, is explicated. The case of broken scale symmetry, as a global limit of the Weyl symmetry, is found to lead to simultaneous spontaneous breaking of gauge invariance.

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