Finite-Size Scaling for Quantum Criticality
above the Upper Critical Dimension:
Superfluid-Mott-Insulator Transition in Three Dimensions

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Abstract

Validity of modified finite-size scaling above the upper critical dimension is demonstrated for the quantum phase transition whose dynamical critical exponent is $z = 2$. We consider the $N$-component Bose-Hubbard model, which is exactly solvable and exhibits mean-field type critical phenomena in the large-$N$ limit. The modified finite-size scaling holds exactly in that limit. However, the usual procedure, taking the large system-size limit with fixed temperature, does not lead to the expected (and correct) mean-field critical behavior due to the limited range of applicability of the finite-size scaling form. By quantum Monte Carlo simulation, it is shown that the same holds in the case of $N = 1$.

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I. INTRODUCTION

Since the quantum phase transition to Mott insulator from superfluid was observed in
the optical lattice system [1], this quantum critical phenomena has been one of hot topics.
This system is effectively described by Bose-Hubbard (BH) Hamiltonian. The zero-
temperature phase diagram of BH model has been well investigated [4, 5, 6]. There are phase
transition points called multicritical points whose dynamical critical exponent is \( z = 1 \) and
line of the other type of phase transition called generic transition whose dynamical critical
exponent is \( z = 2 \) on the zero-temperature phase diagram. In this paper, we consider the
generic transition (i.e., \( z = 2 \)) in three-dimensional systems. The three dimension (\( d = 3 \))
is above the upper critical dimension \( d_u = 2 \). Therefore, this phase transition is exactly
classified and its critical exponents should be identical to those of the mean-field theory.
To estimate the locations of critical points quantitatively, we frequently apply the finite-size
scaling to the data of finite-size systems calculated using quantum Monte Carlo (QMC)
method.

Above the upper critical dimension, the finite-size scaling (FSS) should be modified due
to a dangerous irrelevant variable. In contrast to the conventional FSS below \( d_u \), the
modified finite-size scaling (MFSS) is not justified by renormalization group or scaling
theories. However, its validity has been demonstrated for the five dimensional Ising model
\( O(n) \) model [11] and \( \phi^4 \) model in large-\( N \) limit [9, 12]. For the quantum phase
transition with \( z = 1 \), below the upper critical dimension, a simple application of the FSS is
trivially possible by identifying the inverse temperature \( \beta \) as just an additional dimension.
Actually, to estimate the multicritical point quantitatively, Smakov and Sørensen [13] applied
the FSS with the additional argument \( \beta/L \) to the multicritical point in \( d = 2 \) case where
the system is below the upper critical dimension because \( d + z < 4 \). For the quantum
phase transition with \( z \neq 1 \), below the upper critical dimension, the application of FSS
is also possible with the additional argument \( \beta/L^z \) instead of \( \beta/L \) on the ground that the
ratio between the correlation time \( \xi_T \) and the correlation length \( \xi \) to the \( z \)-th power is
\( \xi_T/\xi^z = O(1) \). Zhao et al., applied the FSS to the case \( z = 2 \) and \( d = 2 \), which is
just the upper critical dimension, and succeeded in estimating the phase boundary on the
zero-temperature phase diagram of their model. The purpose of the present paper is to
demonstrate the validity of the MFSS in the case where \( d > d_u \) and \( z \neq 1 \), both by Monte
Carlo simulation and by exact solutions. We consider the case $z = 2, d = 3$, i.e., above the upper critical dimension. It seems a natural extension to add the argument $\beta/L^2$ to the scaling function of MFSS.  

\[ F_s(r, \eta, \beta, L) \sim \tilde{Y}_F(\delta L^{(d+2)/2}, \eta L^{3(d+2)/4}, \beta/L^2), \]  

(1)

with a universal scaling function $\tilde{Y}_F$, where the definition of the free energy is $F \equiv -\ln \Xi$ with the partition function $\Xi$, $r$ indicates the coefficient of the term including square of the order parameter in the Hamiltonian (e.g., the chemical potential $\mu$ or the hopping amplitude $t$ in the model (2) described below), for $\delta$ indicates the difference from the quantum critical point (e.g., $\delta = r - r_c$), and $\eta$ is the field inducing the order parameter.

The critical exponents for the finite temperature behavior at quantum critical point should be identical to those of mean-field theory, e.g., $\chi \sim T^{-3/2}$ where $\chi$ is susceptibility. However, as shown in Sec. III, the exponents derived by the limit $L \to \infty$ of scaling form (e.g., $\chi \sim T^{-5/4}$) are different from those of the mean-field theory. The reason of this apparent contradiction is that the scaling form (1) is valid only when $\beta/L^2 = O(1)$. That is, we cannot infinitize $L$ in Eq. (1) while keeping $\beta$ finite. In this paper, we show that the application of MFSS to the $z = 2$ quantum critical point is reliable, if the condition of validity is satisfied, just as well as the conventional FSS below the upper critical dimension.

In Sec. II we define $N$-component BH model. In Sec. III we focus on the $N = 1$ case and show the application of the MFSS to the numerical result of the QMC simulation. In Sec. IV we focus on the $N = \infty$ case, which is exactly solvable even for finite systems, to show that the susceptibility obeys the MFSS form under the condition $\beta/L^2 = O(1)$. In Sec. V we give a discussion and summary of this paper.

II. $N$-COMPONENT BOSE-HUBBARD MODEL

We consider the $N$-component BH model on the hypercubic lattice whose Hamiltonian is described as

\[ \mathcal{H}_N = -\frac{t}{Z} \sum_{\alpha=1}^{N} \sum_{\langle i,j \rangle} (b_{\alpha i}^\dagger b_{\alpha j} + b_{\alpha j} b_{\alpha i}^\dagger) - \mu \sum_{\alpha=1}^{N} \sum_{i} b_{\alpha i}^\dagger b_{\alpha i} + \frac{U}{2N} \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \sum_{i} b_{\alpha i}^\dagger b_{\beta i}^\dagger b_{\beta i} b_{\alpha i}, \]  

(2)
where $b_{\alpha i}^\dagger$ ($b_{\alpha i}$) creates (annihilates) a $\alpha$-type boson at site $i$, and $\langle i, j \rangle$ runs over all pairs of nearest-neighbor sites. The symbols $t$, $U$, and $\mu$, denote the hopping amplitude, the on-site interaction between bosons, and the chemical potential, respectively. The coordination number in the hypercubic lattice is $Z = 2d$. We take the lattice spacing as our unit of distance. For concreteness, we consider only three-dimensional case in this paper. (i.e., $Z = 6$.) Generalization to arbitrary dimensions should be straightforward.

Here, we define the free energy $F_\eta$ as

$$F_\eta \equiv -\frac{1}{N} \ln \text{Tr} \left[ e^{-\beta (\mathcal{H}_N - \eta \mathcal{Q})} \right],$$

with the field $\eta$ inducing the order parameter.

The $N$-component BH model (2) is solvable in the large-$N$ limit. In Sec. IV, we demonstrate that the MFSS scaling (1) exactly describes the asymptotic behavior of the model (2) in the large-$N$ limit. We note here that an exactly solvable model similar to the present one was investigated in the 1980s. [16, 17] The model was defined with Bose field operators in the continuous space. In these papers, the authors discussed the critical behavior in the thermodynamic limit near the quantum critical point. As a result, the mean-field type criticality was confirmed above the upper critical dimension. (e.g., $\chi \sim \delta^{-1}$.)

III. NUMERICAL VERIFICATION OF MODIFIED FINITE-SIZE SCALING

In this section, we apply the MFSS to the result of QMC simulation for the single component BH model [18, 19]. We focus on the superfluid to Mott insulator transition. The zero-temperature phase diagram is shown in Fig. 3 which consists of Mott lobes and a superfluid region. The phase boundary was estimated using the Mott gap. [6] At the tip of the Mott lobe, which is a the multicritical point, the dynamical critical exponent $z$ is 1 because of the asymptotic particle-hole symmetry. [4, 13] The rest of critical lines corresponds to the generic transition with the dynamical critical exponent $z = 2$. In this section, we fix the chemical potential as $\mu/U = 0.1$ and vary the hopping amplitude $t/U$. Namely, $\delta$ in the first argument of the scaling functions corresponds to $\delta = t/U - (t/U)_c$ in the present case.
We study compressibility \( \kappa \) and susceptibility \( \chi \). Their definitions are

\[
\kappa \equiv \frac{1}{\rho^2} \frac{\partial \rho}{\partial \mu},
\]

and

\[
\chi \equiv -\frac{1}{2L^d \beta} \left. \frac{\partial^2 F_\eta}{\partial \eta^2} \right|_{\eta=0},
\]

where

\[
\rho \equiv -\frac{1}{L^d \beta} \frac{\partial F_0}{\partial \mu}.
\]

The scaling forms of \( \kappa \) and \( \chi \) are derived using the scaling form of the free energy \( \eta \) as

\[
\kappa \sim \tilde{Y}_\kappa (x,y), \quad \chi \sim L^{5/2} \tilde{Y}_\chi (x,y),
\]

where

\[
x = \delta L^{5/2}, \quad y = \frac{\beta t}{L^2}.
\]

We fix the second argument as \( y = 0.375 \) and estimate the critical value of \( t/U \) as \( (t/U)_c = 0.088935(7) \) at \( \mu/U = 0.1 \) using the MFSS of \( \kappa \) and \( \chi \) as shown in Figs. 1 a) and b). In these plots, we used the mean-field values for the exponents, leaving the critical value of \( t/U \) as the only fitting parameter.

As long as \( \beta t/L^2 = O(1) \), we can use the MFSS form just as well as we do in the conventional FSS for estimating the critical value of the relevant parameter \( (t/U) \) in the present case. To compare between the estimation using Mott gap and MFSS, we estimate the Mott gap at \( t/U = 0.088935, \mu/U = 0.5 \) and plot the corresponding points on the inset of Fig. 3. As we see in the figure, the agreement is very good.

Here, a remark on the range of validity of the MFSS form is appropriate. We consider the finite temperature behavior of \( \chi \) at the quantum critical point \( \delta = 0 \). If we neglect the applicability condition of the MFSS form and take the limit \( L \to \infty \) while keeping \( \beta t \) finite, the finite temperature dependence of \( \chi \) is derived as

\[
\chi \sim L^{5/2} \left( \frac{\beta t}{L^2} \right)^{5/4} \sim T^{-5/4} \text{(error!)}.
\]
from the scaling form (8). This exponent $-5/4$ is different from that of mean-field theory $-3/2$. As shown in Sec. IV, the reason of this error is that the scaling form (8) or (11) is valid only under the condition of $\beta t/L^2 = O(1)$. To confirm the mean-field exponent, we show the finite temperature dependence of $\chi$ at the quantum critical point in Fig. 2.

The superfluid density $\rho_S$ is one of the most important quantity characterizing the superfluidity. However, it is not straightforward to derive the MFSS form of $\rho_S$ because it is not directly obtained from the free energy by simple differentiation. The superfluid density $\rho_S$ is proportional to the fluctuation of the winding number $W = (W_x, W_y, W_z)$ and defined as $\rho_S \equiv \langle W^2 \rangle / (\beta t L)$ within the framework of QMC simulation. In Appendix B we show that $\rho_S = \chi/(\beta L^d)$, for the model (2) in the large-$N$ limit under the condition, $\beta t/L^2 \geq O(1)$, $d > 2$, and $\beta t \gg 1$. From the MFSS for $\chi$, we obtain,

$$\rho_S \sim L^{-\frac{5}{2}} \tilde{Y}_{\rho_S}(x, y). \quad (11)$$

Although this form is derived only for the exactly solvable model, we believe that this holds in general for the mean-field type critical behavior. We apply this MFSS form to the result of $\rho_S$ estimated by QMC simulations. As can be seen in Fig. 1(c), the MFSS (11) describes the data well.

**IV. LARGE-$N$ LIMIT OF $N$-COMPONENT BOSE-HUBBARD MODEL**

In this section, we consider the model (2) that is known to exhibit a mean-field type critical phenomena, to see if the MFSS is applicable to such a model. We consider the model on the $d$ dimensional hypercubic lattice in the large-$N$ limit and show that the MFSS form Eq. (11) is exactly applicable to this case. To derive the self-consistent equation of $\chi$ in the large-$N$ limit, we represent the partition function as a functional integral by making use of a coherent state basis at first. Then, we use the Stratonovich-Hubbard transformation and the saddle-point method, which is also called the steepest descent method. Thus, the self consistent equation of susceptibility $\chi$ in large-$N$ limit is derived exactly as

$$\chi^{-1} = -\mu - t + \frac{U}{L^d} \sum_k \exp \left[ \beta \chi^{-1} + \frac{2\beta t}{Z} \sum_{\delta=1}^d \left(1 - \cos k_\delta\right) \right] - 1. \quad (12)$$
FIG. 1: (Color online) MFSS plots of the single component BH model where $\mu/U = 0.1$, $\beta t/L^2 = 0.375$. $\delta \equiv t/U - (t/U)_c$ with $(t/U)_c = 0.088935$: a) compressibility, b) susceptibility c) superfluid density.
FIG. 2: (Color online) Temperature dependence of $\chi$ at the quantum critical point estimated by MFSS ($\mu/U = 0.1$, $(t/U)_c = 0.088935$). The solid line is $A(T/t)^{-3/2}$ where $A = 0.89$. The data points are obtained for $L = 24, 32, 48$. There is no visible size dependence on this scale. The statistical error is smaller than the symbol size.

FIG. 3: (Color online) Zero-temperature phase diagram of single component BH model in 3D [6]. FSS indicates the result of MFSS.
See Appendix A.1 for details of the derivation. By expanding the summand with respect to 
\[ \exp \left[ - \left\{ \beta \chi^{-1} + \frac{2 \beta t}{Z} \sum_{\delta=1}^{d} (1 - \cos k_\delta) \right\} \right] \], we obtain 
\[
\chi^{-1} = -\mu - t + \frac{U}{L^d} \sum_{\nu=1}^{\infty} e^{-\nu \beta \chi^{-1}} \left[ \sum_{n=1}^{L} \exp \left[ -\frac{2 \nu \beta t}{Z} \left\{ 1 - \cos \left( \frac{2\pi n}{L} \right) \right\} \right] \right]^d.
\]
(13)

Below we show that this equation has a solution such that \( \chi \sim O((\beta t)^{(d+2)/4}) \). Therefore, we assume \( \chi \sim O((\beta t)^{(d+2)/4}) \) for \( \chi \) in the r.h.s. of (13). Then, as shown in Appendix A.2, the approximation formula
\[
\sum_{\nu=1}^{\infty} e^{-\nu \beta \chi^{-1}} \left[ \sum_{n=1}^{L} \exp \left[ -\frac{2 \nu \beta t}{Z} \left\{ 1 - \cos \left( \frac{2\pi n}{L} \right) \right\} \right] \right]^d \approx \beta^{-1} \chi,
\]
(14)
becomes exact in the limit \( \beta t \to \infty \) under the condition that \( d > 2, \beta t/L^2 \geq O(1) \). Using the self-consistent Eq. (13) and the approximation (14), we arrive at a simple equation \( \chi^{-1} = -\mu - t + U \chi \beta^{-1} L^{-d} \). Its solution can be cast into the form,
\[
\chi t \sim \left( \frac{\beta t}{Z} \right)^{\frac{d+2}{4}} \frac{P^{UZ/t}_{\chi}}{x} \left( \frac{\beta t}{L^2 Z} \right)^{-\frac{d}{4}},
\]
(15)
with a scaling function
\[
P^{u}_{\chi}(x, y) \equiv \frac{2}{x + \sqrt{x^2 + 4uy^{-1}}},
\]
(16)
At the critical point \( (\mu = -t) \), we obtain \( \chi t \sim (\beta t)^{(d+2)/4} \times (\beta t/L^2)^{-d/4} \). To make this consistent with \( \chi = O((\beta t)^{(d+2)/4}) \) assumed at first and the condition \( \beta t/L^2 \geq O(1) \), we must demand \( \beta t/L^2 = O(1) \). Thus, we have proved that Eq. (13) has a solution \( \chi = O((\beta t)^{(d+2)/4}) \) that satisfies Eq. (15), and the MFSS form (8) has been derived as a formula that is asymptotically exact under the condition of \( d > 2 \) and \( \beta t/L^2 = O(1) \).

To see the validity of the form of the scaling function (16), we demonstrate the MFSS plot of susceptibility in \( d = 3 \). We solve the self-consistent Eq. (13) without using Eq. (14) and plot on Figs. 4 a) and b). As shown in Fig 4 b), the MFSS form fits well in the region \( \beta t/L^2 = O(1) \).

V. DISCUSSION AND SUMMARY

In Sec. III and IV, we have demonstrated that the MFSS (11) is efficient in locating quantum critical points whose dynamical critical exponent is \( z = 2 \). It has been shown
FIG. 4: (Color online) MFSS plots of susceptibility at $U/(t/Z) = 1$. a) the $x$-dependence of scaling function of susceptibility $P_{x}^{UZ/t}(x,y)$ and the solution of self-consistent Eq. (13) with $y \equiv \beta t/(ZL^2) = 1$, b) $y$-dependence of $P_{x}^{UZ/t}(x,y)$ and the solution of self-consistent Eq. (13) at quantum critical point $\delta = 0$.

that the MFSS is valid only if the second argument of scaling function $\beta t/L^2$ is $O(1)$. In particular, it is not permitted to infinitize $L$ in the scaling forms (8) and (11) while keeping $\beta$ fixed. This explains the apparent contradiction between the MFSS and the mean-field critical exponents. It should be remarked here that similar situations appear in classical models. Suppose that we try to apply the MFSS to a finite-temperature phase transition of a classical system and send the system size in some (not all) of the directions to infinity.
while keeping the size in other directions fixed. Singh and Pathria considered a system of size $L^{d-d'} \times L'^d$ where $d$ is larger than the upper critical dimension and $d'$ is less than the lower critical dimension. They analyzed a spin model with $O(n)$ symmetry in the limit of $L' \to \infty$. Then they derived the scaling form of the susceptibility $\chi_0$ as

$$\chi_0 \sim L^{2(d-d')/(4-d')} Y^d \left( \tilde{t} L^{2(d-d')/(4-d')} \right),$$

where $\tilde{t} \equiv (T - T_c)/T_c$. Then, $\chi_0 \sim L^{2(d-d')/(4-d')}$ at the critical point $\tilde{t} = 0$. On the other hand, if we keep $L/L'$ is finite, the MFSS form is

$$\chi_0 \sim L^d Y^d \left( \tilde{t} L^d, L/L' \right),$$

with the additional argument $L/L'$. If we ignore the validity condition of the MFSS and take the limit $L' \to \infty$, we reach an erroneous conclusion, that is $\chi_0 \sim L^{d/2}$ at $\tilde{t} = 0$.

In summary, the MFSS is applied to the quantum critical phenomena with the dynamical critical exponent $z = 2$. Using the $N$-component BH model, the MFSS form of the susceptibility Eq. (15) is exactly derived in the large-$N$ limit with the applicability condition $d > 2$ and $\beta t/L^2 = O(1)$. We also apply the MFSS to the numerical results obtained by QMC simulations. As a result, we see that a position of quantum critical point estimated by MFSS is identical to that estimated by the Mott gap within the statistical error. Finally, note that the scaling function derived in this paper $P^\alpha(x, y)$ is in complete agreement with the scaling function of $\phi^4$ model derived in Ref. 9. While the scaling function is not justified by the renormalization group or scaling theories in contrast to the standard FSS below the upper critical dimension, the agreement strongly indicates that the mean-field scaling function above the upper critical dimension is universal.

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APPENDIX A: CALCULATION OF LARGE N LIMIT

1. Self-consistent equation of $\chi$

Here, we derive the self-consistent equation (13) of the Hamiltonian (2). The partition function is expressed as

$$Z_N = \int D\psi_i(\tau)D\psi_i^*(\tau)e^{-(S_0+S_1)},$$

$$S_0 = \int_0^\beta d\tau \left\{ \sum_i \left\{ \psi_i^*(\tau) \cdot (\partial_\tau \psi_i(\tau)) - \mu \psi_i^*(\tau) \cdot \psi_i(\tau) \right\} - \frac{t}{Z} \sum_{\langle i,j \rangle} (\psi_i^*(\tau) \cdot \psi_j(\tau) + \psi_j^*(\tau) \cdot \psi_i(\tau)) \right\},$$  \hspace{1cm} \text{(A1)}

$$S_1 = \int_0^\beta d\tau \frac{U}{2N} \sum_i (\psi_i^*(\tau) \cdot \psi_i(\tau))^2,$$

by the path-integral representation with $\psi_i(\tau)$ being an $N$-component complex field. Using Stratonovitch-Hubbard transformation, the partition function is written as

$$Z_N = \int D\psi_i(\tau)D\psi_i^*(\tau)Ds_i(\tau)e^{-S_0} \times \exp \left\{ -\int_0^\beta d\tau \left\{ \sum_i \left( \frac{N}{2} \dot{s}_i^2(\tau) - i\sqrt{U} s_i(\tau) (\psi_i^*(\tau) \cdot \psi_i(\tau)) \right) \right\} \right\},$$

$$= \int Ds_i(\tau)e^{-\frac{N}{2} \int_0^\beta d\tau \sum_i \dot{s}_i^2(\tau)} [Z_1 (\{s\})]^N, \hspace{1cm} \text{(A2)}$$

$$Z_1 (\{s\}) \equiv \int D\psi_i^\alpha(\tau)D\psi_i^{\alpha*}(\tau) \exp \left[ -\int_0^\beta d\tau \right. \left. \left\{ \sum_i \left( \psi_i^{\alpha*}(\tau) (\partial_\tau \psi_i^\alpha(\tau)) - \left( \mu + i\sqrt{U} s_i(\tau) \right) \psi_i^{\alpha*}(\tau) \psi_i^\alpha(\tau) \right) \right\} - \frac{t}{Z} \sum_{\langle i,j \rangle} (\psi_i^{\alpha*}(\tau) \psi_j^\alpha(\tau) + \psi_j^{\alpha*}(\tau) \psi_i^\alpha(\tau)) \right\}], \hspace{1cm} \text{(A3)}$$

where $s_i(\tau)$ is an auxiliary field and the integral with respect to $s_i(\tau)$ is defined as

$$\int Ds_i(\tau) = \prod_{i,\tau} \sqrt{\frac{N}{2\pi}} \int_{-\infty}^\infty ds_i(\tau). \hspace{1cm} \text{(A4)}$$

In the large-$N$ limit, using saddle-point method, the auxiliary field $s_i(\tau)$ is replaced by $\bar{s}$ (see Ref. [12] and its references), which makes the exponents of the partition function
maximum. Using Fourier transformation, we obtain

\[ Z_N = A e^{-N \beta L^d \overline{s}^2} [Z_1(\overline{s})]^N, \quad (A5) \]

\[ Z_1(\overline{s}) = \prod_k (1 - e^{-\beta \lambda_k})^{-1}, \quad (A6) \]

\[ \lambda_k = -\mu - i \sqrt{U} \overline{s} - {2t \over Z} \sum_{\delta=1}^d \cos k_{\delta} \quad (A7) \]

where \( A \) is a some real number caused by the fluctuation of \( s_i(\tau) \) from \( \overline{s} \), which does not contribute to following discussion and the product of \( k \) runs over the first Brillouin zone \( k = (2\pi/L)(m_1,\cdots,m_d) \), with \( m_i = 1,2,\cdots,L \). The stationary solution \( \overline{s} \) must satisfy

\[ \frac{\partial}{\partial \overline{s}} \left[ -\frac{N \beta L^d}{2} \overline{s}^2 - \sum_k \ln (1 - e^{-\beta \lambda_k}) \right] = 0, \quad (A8) \]

which yields,

\[ \overline{s} = i \sqrt{U} \frac{L^d}{Z} \sum_k \left[ e^{\beta \lambda_k} - 1 \right]^{-1}. \quad (A9) \]

The susceptibility \( \chi \) is related to \( \overline{s} \) by

\[ \chi \equiv \frac{1}{N} \int_0^\beta d\tau \sum_i \langle \psi_i^\ast(\tau) \cdot \psi_0(0) \rangle = (\mu - t - i \sqrt{U} \overline{s})^{-1}. \quad (A10) \]

Therefore, \( \chi \) satisfies

\[ \chi^{-1} = -\mu - t + {U \over L^d} \sum_k \exp \left[ {2\beta t \over Z} \sum_{\delta=1}^d (1 - \cos k_{\delta}) + \beta \chi \right]^{-1}. \quad (A11) \]

2. Derivation of Eq.(14)

In Sec. [IV] we derive \( \chi = O \left( (\beta t/Z)^{(2+d)/4} \right) \) by self-consistent analysis. Namely, assuming the condition \( \chi = O \left( (\beta t/Z)^{(2+d)/4} \right) \), we prove the resulting solution satisfying this condition. Here, assuming

\[ (\beta t/Z)^{(2+d)/4} \chi^{-1} = O(1), \quad (A12) \]

we provide an approximation form

\[ \sum_{\nu=1}^\infty e^{-\nu \beta \chi^{-1}} \left[ \sum_{n=1}^L \exp \left[ -\frac{2\nu \beta t}{Z} \left\{ 1 - \cos \left( \frac{2\pi n}{L} \right) \right\} \right] \right]^d \simeq \beta^{-1} \chi, \quad (A13) \]
which becomes exact under the condition that

\[ d > 2, \quad \beta t / L^2 \geq O(1), \quad (A14) \]

and

\[ \beta t \gg 1. \quad (A16) \]

To begin with, we rewrite the l.h.s. as

\[
\sum_{\nu=1}^{\infty} e^{-\nu \beta \chi^{-1}} \left[ \sum_{n=1}^{L} \exp \left[ - \frac{2 \nu \beta t}{Z} \left\{ 1 - \cos \left( \frac{2 \pi n}{L} \right) \right\} \right] \right]^d = \sum_{\nu=1}^{\infty} e^{-\nu \beta \chi^{-1}} [1 + A_{\nu}]^d,
\]

\[ = \sum_{\nu=1}^{\infty} e^{-\nu \beta \chi^{-1}} + \sum_{\nu=1}^{\infty} e^{-\nu \beta \chi^{-1}} \left[ \sum_{\alpha=1}^{d} \frac{d!}{\alpha! (d-\alpha)!} A_{\nu}^{\alpha} \right], \quad (A17) \]

where

\[ A_{\nu} \equiv e^{-\nu \beta \chi^{-1}} + 2 \sum_{n=1}^{L/2} \exp \left[ - \frac{2 \nu \beta t}{Z} \left\{ 1 - \cos \left( \frac{2 \pi n}{L} \right) \right\} \right]. \quad (A18) \]

Here we note that \( \beta \chi^{-1} \simeq 0. \) (This is because \( \beta \chi^{-1} = O((\beta t)^{-1}) \) (by the condition Eq. (A12)) and by the condition Eq. (A14) this is vanishing in the limit of Eq. (A16).) Since \( \beta \chi^{-1} \simeq 0, \) the first term of the r.h.s. of Eq. (A17) is approximated by the formula

\[
\sum_{\nu=1}^{\infty} e^{-\nu \beta \chi^{-1}} = \beta^{-1} \chi + O \left( (\beta \chi^{-1})^0 \right) = O \left( (\beta t)^{-\frac{d-2}{4}} \right). \quad (A19) \]

Below we show that the second term of Eq. (A17) is a correction term that vanishes in the limit of \( \beta t \rightarrow \infty. \) At first, \( A_{\nu} \) is bounded as

\[
0 < A_{\nu} \leq 2 \sum_{n=1}^{L/2} \exp \left[ - \frac{2 \nu \beta t}{Z} \left\{ 1 - \cos \left( \frac{2 \pi n}{L} \right) \right\} \right],
\]

\[
\leq 2 \int_{0}^{L/2} dp \exp \left[ - \frac{2 \nu \beta t}{Z} \left\{ 1 - \cos \left( \frac{2 \pi p}{L} \right) \right\} \right],
\]

\[
\leq 2 \int_{0}^{L/2} dp \exp \left[ - \frac{2 \nu \beta t}{Z} \left( \frac{8p^2}{L^2} \right) \right],
\]

\[
\leq \sqrt{\frac{\pi L^2 Z}{16 \nu \beta t}}, \quad (A20) \]
Then, the second term of Eq. \((A17)\) is evaluated as

\[
\begin{align*}
0 &< \sum_{\alpha=1}^{d} \frac{d!}{\alpha! (d-\alpha)!} \left[ \sum_{\nu=1}^{\infty} e^{-\nu \beta \chi^{-1}} A_{\nu}^\alpha \right], \\
&\leq \sum_{\alpha=1}^{d} \frac{d!}{\alpha! (d-\alpha)!} \left[ \sum_{\nu=1}^{\infty} e^{-\nu \beta \chi^{-1}} \left\{ \frac{\pi L^2 Z}{16 \nu \beta t} \right\}^{\alpha/2} \right], \\
&\leq \sum_{\alpha=1}^{d} \frac{d!}{\alpha! (d-\alpha)!} \left\{ \frac{\pi L^2 Z}{16 \beta t} \right\}^{\alpha/2} \left[ \int_{0}^{\infty} dpe^{-p \beta \chi^{-1} - p^{-\alpha/2}} \right], \\
&= \sum_{\alpha=1}^{d} \frac{d!}{\alpha! (d-\alpha)!} \left\{ \frac{\pi Z}{16 (\beta t / L^2)} \right\}^{\alpha/2} \left[ \int_{0}^{\infty} dqe^{-q \beta \chi^{-1} - q^{-\alpha/2}} \right] \left( \beta \chi^{-1} \right)^{\frac{\alpha-2}{2}}. \quad (A21)
\end{align*}
\]

Since \(\beta t / L^2 \geq O(1)\), and \(\beta \chi^{-1} \ll 1\), the \(\alpha = 1\) term is dominant. Therefore, the second term is of the same order as \((\beta t / L^2)^{-1/2} (\beta \chi^{-1})^{-1/2}\). By the condition Eq.\((A12)\), this is \(O\left((\beta t / L^2)^{-1/2} \times (\beta t)^{(d-2)/8}\right)\). Therefore, the ratio of the second and the first term of Eq.\((A17)\) becomes less than \(O\left((\beta t / L^2)^{-1/2} \times (\beta t)^{(d-2)/8}\right)\). This is vanishing because of the condition Eqs. \((A14)\), \((A15)\) and \((A16)\). Thus Eq.\((A13)\) has been derived.

APPENDIX B: SCALING FUNCTION OF SUPERFLUID DENSITY IN LARGE-N LIMIT

In this section, we provide that the MFSS form of superfluid density using the \(N\)-component BH model. The outline of this section is as follows. First, we obtain the explicit definition of superfluid density, which is estimated using the winding number in QMC simulation, with an infinitesimal twist of phase of bosonic operator. Next, we calculate the superfluid density of \(N\)-component BH model exactly. The result reveals that the superfluid density \(\rho_S\) is proportional to the susceptibility \(\chi\). Then, we derive the MFSS form of \(\rho_S\) as that of \(\chi\).

To start with, we derive an expression for the superfluid density \(\rho_S\) introducing an infinitesimal twist of phase of bosonic operators. Namely, we modify the Hamiltonian \((2)\) by

\[
\begin{align*}
b_{\alpha i}^\dagger \rightarrow b_{\alpha i}^\dagger e^{i \theta r_i^z}, \quad &b_{\alpha i} \rightarrow b_{\alpha i} e^{-i \theta r_i^z},
\end{align*}
\]

where \(r_i^z\) is the \(z\)-coordinate of the site \(i\). (Because of the periodic boundary condition, \(\theta\) should be discrete. That is, \(\theta = 2 \pi n / L\) where \(n\) is integer. However, considering a sufficiently large system, we regard \(\theta\) as a continuous real number.) Then, we define the twisted Hamiltonian \(H_{N\theta}\), the partition function \(Z_{N\theta}\) and the free energy
The superfluid density \( \rho_S \) is defined with this twisted free energy \( F_{N\theta} \) as,

\[
\mathcal{H}_{N\theta} = -\frac{t}{Z} \sum_{\alpha=1}^{N} \sum_{(i,j)} b_{\alpha i}^\dagger b_{\alpha j} e^{i\theta (r_i^z - r_j^z)} + b_{\alpha i} b_{\alpha j} e^{-i\theta (r_i^z - r_j^z)} - \mu \sum_{\alpha=1}^{N} b_{\alpha i}^\dagger b_{\alpha i} + U \frac{N}{2N} \sum_{\alpha=1}^{N} \sum_{i} b_{\alpha i}^\dagger b_{\beta i} b_{\beta i} b_{\alpha i},
\]

(B1)

\[
Z_{N\theta} = \text{Tr} \left[ e^{-\beta \mathcal{H}_{N\theta}} \right],
\]

(B2)

\[
F_{N\theta} = -\frac{1}{N} \ln Z_{N\theta}.
\]

(B3)

The derivation of the free energy is straightforward using this partition function. Then, we obtain the superfluid density is

\[
\rho_S = \frac{1}{L^d} \sum_{k} \frac{\cos k_z}{e^{\beta \lambda_k} - 1},
\]

(B8)

with \( \lambda_k \) defined in Eq. (A7). This superfluid density is smaller than the total density of particle,

\[
\rho = \frac{1}{L^d} \sum_{k} \frac{1}{e^{\beta \lambda_k} - 1},
\]

(B9)

and larger than the density of particles of \( k = 0 \),

\[
\rho_0 = \frac{1}{L^d} \frac{1}{e^{\beta \chi^{-1}} - 1}.
\]

(B10)
That is,

\[ \rho_0 \leq \rho_S \leq \rho. \]  

(B11)

As shown in Appendix A2

\[ \rho = \rho_0 = \frac{\chi}{L^d \beta}. \]  

(B12)

under the condition \( \beta t/L^2 \geq O(1), \ d > 2 \) and \( \beta t \to \infty \). Using the inequality (B11), we obtain

\[ \rho_S = \frac{\chi}{L^d \beta}. \]  

(B13)

As shown in Sec. IV we derive the MFSS form of \( \rho_S \) as

\[ \rho_S = L^{-\frac{d+2}{2}} P_{\rho_S}^{UZ/t} \left( L^{\frac{d+2}{2}} Z \left( -\frac{\mu}{t} - 1 \right), \frac{\beta t}{L^2 Z} \right), \]  

(B14)

\[ P_{\rho_S}^u(x, y) \equiv \frac{2y^{-1}}{x + \sqrt{x^2 + 4uy^{-1}}}. \]  

(B15)

The applicability condition of this MFSS form is \( d > 2 \) and \( \beta t/L^2 = O(1) \).
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