Mixing angle and phase correlations from $A_5$ with generalised CP and their prospects for discovery

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(Dated: March 27, 2015)

The observed leptonic mixing pattern could be explained by the presence of a discrete flavour symmetry broken into residual subgroups at low energies. In this scenario, a residual generalised CP symmetry allows the parameters of the PMNS matrix, including Majorana phases, to be predicted in terms of a small set of input parameters. In this article, we study the mixing parameter correlations arising from the symmetry group $A_5$ including generalised CP subsequently broken into all of its possible residual symmetries. Focusing on those patterns which satisfy present experimental bounds, we then provide a detailed analysis of the measurable signatures accessible to the planned reactor, superbeam and neutrinoless double beta decay experiments. We also discuss the role which could be played by high-precision measurements from longer term projects such as the Neutrino Factory. This work provides a concrete example of how the synergies of the upcoming experimental programme allow flavour symmetric models to be thoroughly investigated. Indeed, thanks to the rich tapestry of observable correlations, we find that each step of the experimental programme can make important contributions to the assessment of such flavour-symmetric patterns, and ultimately all patterns that we have identified can be excluded, or strong evidence found for their continued relevance.

PACS numbers: 13.30.Hv, 14.60.Pq

I. INTRODUCTION

The existence of three families of fermions in the Standard Model (SM), identical in all properties apart from their masses, is as yet unexplained by any physical principle or mechanism. Moreover, the discovery that both quarks and leptons permit complementary but distinct descriptions in terms of the flavour states which diagonalise the weak interactions and the states which diagonalise their mass terms has shown that the connection between families betrays a precise structure which is an essential component in our description of the physical world. Explaining the origins of this flavour structure has been a recurring theme in proposed extensions of the Standard Model. One such programme is the application of discrete flavour symmetries, predominately in the lepton sector, where the flavour quantum numbers are associated with a new symmetry and particles are assigned to its irreducible representations. This can provide a way to unify the three families into a single mathematical object. However, as the lepton masses are known to be distinct, any non-abelian symmetry can only be exact above the scale of mass generation. Nevertheless, its existence at high energy shapes the theory, and the residual symmetries which survive the breaking procedure at low energies can play an important role in the structure of flavour observables.

The paradigm of a non-abelian flavour symmetry breaking into residual symmetries has been used by many authors to make predictions about the six mixing angles and phases which constitute the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix and parameterise neutrino mixing: $\theta_{12}$, $\theta_{13}$, $\theta_{23}$, the Dirac phase $\delta$ and two Majorana phases $\alpha_{21}$ and $\alpha_{31}$. For a recent review of such models see e.g. Ref. [1]. Although many of the earliest models were designed to predict a very small value of $\theta_{13}$, a prediction now firmly ruled out [2], a number of models remain consistent with the current data, many of which are based on groups taken from the $\Delta(6n^2)$ family: $\Delta(96)$ [3, 4], $\Delta(150)$ [5], $\Delta(600)$ [6] and $\Delta(1536)$ [7]. This connection was strengthened in Ref. [8] which showed that, based on only a few generic model building assumptions, if the full PMNS matrix is to be specified by the symmetry alone (so-called “direct” models [1]), the only possible predictions which would agree with current data are those arising (minimally) from $\Delta(6n^2)$. In general, however, the existence of residual symmetries amongst the leptonic mass terms may not fully specify the mixing pattern. In these “semi-direct” models [9] the symmetries reduce the degrees of freedom necessary to describe the mixing parameters by defining a correlation between previously independent parameters (sometimes known as mixing sum rules [11, 14]). Often these correlations between mixing angles and phases can be derived from quite generic analyses of the residual symmetries present in a system without needing to specify a full UV-complete theory [15, 21]. As such, focusing on these relations can be an effective way to compare a wide class of models to data [17, 24].

To date, the fine structure of the PMNS matrix has been inaccessible to experiment, preventing the study of subtle parameter correlations. Current measurements of the mixing angles have $3\sigma$ uncertainties of around 6.9%
on $\theta_{12}$, 7.3% on $\theta_{13}$ and 16.7% on $\theta_{23}$ [25]. All three of the CP phases are unconstrained at this significance level; although, some low-significance hints for a maximally CP-violating value of the Dirac phase $\delta \approx 3\pi/2$ have been observed [25, 26]. Therefore the upcoming experimental work will focus on two key topics: the precision determination of the mixing angles and the first measurements of the CP phase $\delta$. This will open the door for studies of the correlations between mixing angles, and between mixing angles and phases, that are predicted by models of flavour symmetries.

In the most popular formulation of models with discrete flavour symmetries, the constraints on the mass matrices used to derive the PMNS matrix cannot remove a number of complex phase degrees of freedom. This results in an inability to predict the Majorana phases and, in general, lessens the predictivity of the model. However, by imposing a generalised CP symmetry (GCP), phase information may be accessible and dictated by the flavour structure itself. This can lead to very predictive scenarios, where all 6 mixing parameters are related to a small number of input parameters [27]. GCPs were first explored in the context of discrete and continuous groups in Ref. [28]; however, they have recently been revived due to the question of consistency between a CP symmetry and discrete flavour group [27, 29, 30]. This has lead to interesting work studying the predictions of models with imposed flavour and GCP symmetries for a number of groups such as $A_5$ [29, 31], $S_4$ [27, 32–34], $\Delta(48)$ [36, 37] and $\Delta(96)$ [38] along with more comprehensive analyses of the families $\Delta(3n^2)$ and $\Delta(6n^2)$ [39–41]. (See also Ref. [42] and Ref. [43] for further applications of GCP symmetries.)

In this article, we present a detailed analysis of a single group: the alternating group on 5 elements, $A_5$. This was first introduced to leptonic flavour physics in Ref. [44] via the study of Golden Ratio mixing [45, 46]; a possible pattern of the PMNS matrix with $\theta_{13} = 0$, $\theta_{23} = \pi/4$ and a value of $\theta_{12}$ related to the golden ratio $\varphi = 1 + \sqrt{5}/2$, tan $\theta_{12} = 1/\varphi$. This pattern has been shown to be a prediction of a number of different models based on $A_5$ [44–49]; however, it is not the only fully-specified mixing pattern associated with direct models based on this group. If a $Z_3$ subgroup is preserved among the charged leptons and a $Z_2 \times Z_2$ is preserved among the neutrinos, a pattern with $\theta_{13} = 0$, $\theta_{23} = \pi/4$ and cos $\theta_{12} = \varphi/\sqrt{3}$ can be found [21, 50, 51]. Further patterns are also possible when a $Z_2 \times Z_2$ symmetry is preserved in the charged-leptons whilst a different $Z_2 \times Z_2$ remains in the neutrino sector, which predict a large value of $\theta_{13}$, $\theta_{13} \approx 17.9^\circ$ [51].

Needless to say, the patterns above are in severe tension with the current global data by dint of their $\theta_{13}$ predictions alone; however, the possibility remains that symmetries such as these do not completely survive at low energies and that a semi-direct approach may remain viable. In this work, we consider the group $A_5$ with a GCP symmetry, deriving the most general GCP transformation which could be implemented for this group. We assume that the flavour group with GCP is broken into a set of residual symmetries at low energies insufficient to fix all of the oscillation parameters. We compute all possible predictions for the induced correlations amongst the mixing parameters. These are compared to the current data, and we identify those which are compatible with the current bounds. The viable patterns that we identify are highly predictive, expressing all six parameters of the PMNS matrix in terms of a single unphysical angle. We take particular care in assessing the phenomenology of the correlations between mixing angles and phases for these viable models: discussing their accessibility to reactor, long-baseline and neutrinoless double beta decay experiments, and highlighting particularly interesting signatures to be tested. Although we have restricted our attention to the group $A_5$, this can be seen as an illustrative choice and we would like to stress the rich but moreover readily testable phenomenology which exists in the residual symmetry framework, much of which arises from the predictions taken as a whole instead of resting on single generic types of parameter correlation.

The work presented in this paper is divided into two main parts. In Section II and Section III, we discuss the assumptions behind our framework, and explain the technical steps in our derivation of the correlations. In Section IV, we focus on the predictions themselves, presenting some simplified formulae for the correlations between observable quantities and identifying their most interesting phenomenological signatures. We also study a number of ways that the correlations can be tested by upcoming reactor, superbeam and neutrinoless double beta decay experiments, as well as possible long-term experiments such as neutrino factories. We make our concluding remarks in Section V.

II. RESIDUAL FLAVOUR AND GENERALISED CP SYMMETRIES

We assume the presence of a discrete flavour symmetry, $G$, at high energies. To unify the three flavours, we assume that the fields are assigned to a 3-dimensional irreducible representation of this group, and the general multiplets of leptons $\Psi$ transform as

$$\Psi_\alpha \to \rho(g)_{\alpha\beta} \Psi_\beta,$$

where $\rho : G \to GL(3, \mathbb{C})$ represents a unitary representation of $G$. Ensuring the existence of the 3-dimensional

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1 For alternative global analyses of oscillation data, see Ref. [26].

2 Throughout this paper we will assume that all representations are unitary and therefore all group elements are represented by unitary matrices.
irreducible representation $\rho$ restricts us to non-abelian groups.

As neutrinos are known to oscillate, they cannot have degenerate masses and therefore the non-abelian flavour group, $G$, cannot be a symmetry of our low-energy effective lagrangian. Therefore, we assume that the full flavour symmetry must be broken at low energies into two abelian residual symmetry groups, $G_e$ and $G_\nu$, which are unbroken in the charged-lepton and neutrino sectors, respectively. We denote the leptonic mass terms in the low-energy effective theory by

$$-\mathcal{L}_{\nu \nu} = (e_L)_{\alpha} (m_{\lambda})_{\alpha \beta} (e_R)_{\beta} + (e_{\ell R})_{(\lambda 1)} (m_{\lambda 1})_{\alpha \beta} (e_{\ell R})_{\beta} + \frac{1}{2} (\nu_L)_{\alpha} (m_{\nu})_{\alpha \beta} (\nu_L)_{\beta} + \frac{1}{2} (\nu_{\ell R})_{\alpha} (m_{\nu 1})_{\alpha \beta} (\nu_{\ell R})_{\beta}. \quad (1)$$

where the Greek indices are flavour indices and the mass matrices are $3 \times 3$ and complex-valued. Due to the anti-commutation of the fermionic fields, one can show that the Majorana mass matrix $m_{\nu}$ is restricted to being complex symmetric.

We assume that there exist residual symmetries acting on the left-handed charged- and neutral-leptons. If we denote a general element of these subgroups by $g_e \in G_e$ and $g_\nu \in G_\nu$, the fields transform according to the following rules:

$$(e_L)_{(\alpha)} \rightarrow \rho (g_e)_{(\alpha \beta)} (e_L)_{(\beta)} \quad \text{and} \quad (\nu_L)_{(\alpha)} \rightarrow \rho (g_\nu)_{(\alpha \beta)} (\nu_L)_{(\beta)}.$$ 

Combining these relations with the lagrangian in Eq. (1) leads us to matrix relations which the mass terms must satisfy if the residual symmetries are to be preserved at low energies:

$$m_{\lambda} m_{\lambda}^\dagger = \rho (g_e) (m_{\lambda}) (\rho (g_e))^\dagger, \quad (2)$$

$$m_{\nu} = \rho (g_\nu)^T m_{\nu} \rho (g_\nu). \quad (3)$$

These relations constrain the forms of the mass matrices and, as we will show, knowledge of their existence can be used to derive a form of the PMNS matrix, $U_{\text{PMNS}}$.

Working from a bottom-up perspective, we would like to deduce the phenomenological consequences of a given choice of residual symmetries $G_e$ and $G_\nu$. The choice of residual flavour groups is constrained in two ways. Firstly, $G_e$ and $G_\nu$ must be subgroups of the unbroken group $G$. Secondly, the possible residual flavour symmetries must be subgroups of the largest symmetry allowed by the mass terms in Eq. (1). These maximal symmetries are best identified in the basis where both mass terms are diagonal. In this basis, the most general symmetry of the charged lepton mass matrices is $U(1)^3$, which has discrete subgroups of the form $G_e = \mathbb{Z}_m$ for any $m$ or a direct product of such groups. For the neutrino residual symmetry, the argument changes due the assumed Majorana nature of the mass term. In this case the largest possible symmetry is smaller, $\mathbb{Z}_2 \times \mathbb{Z}_2$, which leaves us with only two choices, $G_\nu = \mathbb{Z}_2$ or $G_\nu = \mathbb{Z}_2 \times \mathbb{Z}_2$.

In addition to a flavour symmetry $G$, we also assume the presence of a GCP symmetry. A CP symmetry is understood as a combination of charge-conjugation and a parity transformation; however, a generalised CP symmetry is one which also acts on the flavour indices whilst making this transposition. In Ref. [29], it was shown that ensuring the consistency of a discrete flavour symmetry and a CP symmetry often requires the introduction of a non-trivial generalised CP symmetry.

We define our generalised CP symmetry to act on a set of fields $\Psi_\alpha$ as

$$\Psi_\alpha \rightarrow X_{\alpha \beta} \Psi_\beta,$$

where $X_{\alpha \beta}$ is assumed to be a unitary matrix so as to preserve the kinetic terms in the lagrangian and $\Psi^c$ denotes the conventional CP conjugate appropriate for the Lorentz representation of the field $\Psi$. If a discrete flavour symmetry is present, $G$, then a generalised CP symmetry must satisfy a consistency equation [27, 29]

$$X \rho (g)^* X^* = \rho (g'), \quad (4)$$

where $g$ and $g'$ are elements of $G$. For a faithful representation $\rho$, this relation can be seen as establishing a mapping from $g$ to $g'$ which preserves the structure of the group and therefore defines a group automorphism. In Ref. [29], it was pointed out that a physical GCP transformation must be restricted to a single irreducible representation, and is related to a class-inverting automorphism of $G$, meaning that $g'$ is mapped to an element in the conjugacy class of $g^{-1}$. We restrict our consideration to involutory GCP transformations, requiring that the application of the transformation twice is equivalent to the identity, and therefore the $X$ matrix satisfies an additional constraint

$$XX^* = 1. \quad (5)$$

As with the flavour symmetry $G$, for our GCP symmetry to leave the lagrangian invariant the mass matrices must satisfy further constraints. The GCP symmetry exchanges the hermitian conjugate terms in the lagrangian of Eq. (1), which remains invariant if the mass matrices obey the relations

$$X^T m_{\nu} X = m_{\nu}^*, \quad (6)$$

$$X^T (m_{\lambda} m_{\lambda}^\dagger) X = (m_{\lambda} m_{\lambda}^\dagger)^*. \quad (7)$$

If Eqs. (6) and (7) are unbroken relations at low energies, it can be shown that all CP violating effects of the PMNS matrix vanish [27, 29]. However, if only one of these relations is preserved, the consistency of the flavour and CP symmetries leads to novel constraints on the PMNS matrix. In this paper, we assume that the GCP symmetry is broken in the charged-lepton sector but is preserved in the neutrino sector. For this to be consistent, the $X$ matrix must map the elements of the neutrino residual symmetry to themselves,

$$X \rho (g_\nu)^* X^* = \rho (g_\nu). \quad (8)$$
In summary, we assume a discrete flavour symmetry $G$ and a GCP symmetry implemented by $X$ at high energy scales which is assumed to break into a subgroup $G_e$ acting on the charged-lepton mass terms and another subgroup $G_\nu$ which along with the GCP symmetry acts on the neutrino mass terms. This leads to a system of constraints which the mass matrices must satisfy: Eq. (2), Eq. (3), Eq. (4) and Eq. (5). In the next section we will show how knowledge of these constraints alone can be used to predict the PMNS matrix, including its Majorana phases.

### A. Constructing the PMNS matrix using symmetry constraints

Constraints on the mass matrices, as derived above, lead to restrictions on their allowed form, and subsequently to the matrices required to diagonalise them. We focus first on the charged leptons. A constraint on the charged-lepton mass matrix of the form in Eq. (3) can be rephrased as a statement of commutation

$$[ho(g_e), (m_\lambda m_\lambda^\dagger)] = 0.$$

As the matrix $\rho(g_e)$ is unitary and the matrix $H = m_\lambda m_\lambda^\dagger$ is hermitian, there exists a basis such that they are simultaneously diagonalised,

$$\exists U_e \text{ s.t. } U_e^\dagger U_e = 1, \quad \rho(g_e)_{\mathrm{d}} = U_e^\dagger \rho(g_e) U_e, \quad H_{\mathrm{d}} = U_e^\dagger H U_e,$$

where $\rho(g_e)_{\mathrm{d}}$ and $H_{\mathrm{d}}$ denote diagonal forms of the matrices $\rho(g_e)$ and $H$. As the charged leptons have distinct masses, $H$ is full rank. This implies that $U_e$ is unique (up to re-phasing and re-ordering of columns). If $\rho(g_e)$ is also known to be full rank, $U_e$ can also be found by diagonalising this operator. In this way, by insisting on the relation in Eq. (2), we can compute $U_e$ solely from the group element $\rho(g_e)$, and the symmetry alone specifies the mixing matrix. However, a complication arises if $\rho(g_e)$ is not full rank. In this case, $\rho(g_e)$ does not have a unique diagonalising matrix, as in any basis in which it takes diagonal form, further SU(2) transformations can be performed freely in its degenerate eigenspace. Without knowledge of the mass matrix, our knowledge of $\rho(g_e)$ will only allow the identification of the family of diagonalising matrices of $\rho(g_e)$, and $U_e$ must take a more general form

$$U_e = U_0 R_e(\phi, \gamma) \Phi,$$

where $U_0$ is any matrix which diagonalises $\rho(g_e)$. $R_e(\phi, \gamma)$ is a complex rotation in the degenerate subspace of $\rho(g_e)$ by an angle $\phi$ with a phase $\gamma$ and $\Phi$ is a diagonal matrix of phases.

In the neutrino sector, we have three constraints to consider on the mass terms: one from the flavour symmetry, one from the GCP symmetry and one ensuring their consistency. Under a change of flavour basis, the matrix $X$ is mapped to

$$X \rightarrow U^\dagger X U^*.$$

As $X$ is unitary and symmetric, Takagi factorization allows us to express it as $X = \Omega \Omega^T$ for some unitary matrix $\Omega$ which implies that we can choose a basis where $X$ becomes trivial [27]. In fact, this basis is not unique and the remaining freedom can be used to further diagonalise $\rho(g_\nu)$. In this basis the constraint in Eq. (6) implies that the mass matrix is real-valued

$$(\Omega^T m_\nu \Omega)_{\alpha\beta} \in \mathbb{R}.$$

As $\rho(g_\nu)$ is diagonal and commutes with this matrix, we know that the mass matrix in this basis must be diagonal up to a basis change in the degenerate subspace of $\rho(g_\nu)$. As it is purely real, the most general additional basis transformation required to bring it into diagonal form is a rotation in 2-dimensions

$$U_\nu = \Omega R_\nu(\theta),$$

where $\theta$ is the angle describing the real rotation. There remains the possibility that the diagonal mass matrix is not positive definite, in which case a diagonal re-phasing must occur. Without further knowledge of the mass matrix this cannot be predicted, and in consequence, the Majorana phases can be predicted only up $\pm \pi$ or multiples thereof.

We see that the GCP symmetry in the neutrino sector has specified a special basis in which the residual flavour symmetry elements are diagonal and the mass matrix is real. In this way, GCP symmetries help to fix some of the re-phasing degrees of freedom associated with diagonalising matrices and allow for the prediction of Majorana phases. Combining the results for the charged-lepton sector and the neutrino sector, we find the full PMNS matrix is given by

$$U_{\text{PMNS}} = \Phi R_e(\phi, \gamma) U_0^\dagger \Omega R_\nu(\theta),$$

where $R_e$ and $R_\nu$ denote two unspecified rotations ($R_e = 1$ if ord ($g_e$) $> 2$). We make two further simplifications: $\Phi$ is removed by re-phasing the charged leptons, and we note that the angles $\theta$ and $\phi$ need only be defined over the interval $\theta, \phi \in [0, \pi)$, as shifts by $\pi$ can be absorbed by unphysical redefinitions of the complex phases.

### III. MIXING PATTERNS FROM $A_5$

The preceding section showed how the assumed residual flavour and GCP symmetries can lead to expressions
for the PMNS matrix. In this section, we will derive the possible mixing matrices which arise by this method for the group $A_5$. First we will discuss the structure of $A_5$ and the possible subgroups eligible to be taken as residual symmetries. Then we will derive the form of the most general GCP transformation. In the subsequent subsections, we consider all viable combinations of CP and residual subgroups and present those patterns which are consistent with the current global data \cite{25}.

### A. Subgroups and GCP symmetries

The group $A_5$ can be defined as the group of even permutations on 5 elements. It has the abstract presentation

$$\langle S, T | S^2 = T^5 = (ST)^3 \rangle,$$

where $S$ and $T$ are the *generators* of the group and all group elements can be expressed by a word made from these distinguished elements. The structure of this group and its representation theory have been discussed in the physics literature before (see e.g. Refs. \cite{11} and \cite{18}) and we abstain from deriving the explicit representations, deferring the reader to these references instead. However, we will briefly recap those features of the group and its representations most pertinent to our subsequent analysis.

We assume that the lepton doublets are assigned to a 3-dimensional representation. $A_5$ has two distinct 3-dimensional irreducible representations, and in the following we will always work with the representation 3 of Ref. \cite{18}. We have checked that the final results do not change if we choose the alternative 3-dimensional representation instead. For our chosen representation, the generators $S$ and $T$ can be expressed by

$$S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \frac{1}{2} \begin{pmatrix} 1 & \phi & -\phi^g \\ \phi & -\phi^g & -1 \\ -\phi^g & 1 & \phi \end{pmatrix},$$

where $\phi^g = \frac{1 - \sqrt{5}}{2}$ represents the Galois conjugate\footnote{In this case Galois conjugation exchanges the two solutions of the minimal polynomial over the rationals $x^2 - x - 1 = 0$.} of $\phi$. We note that this is a real representation and in our chosen basis all group elements are real; it also forms a subgroup of SU(3).

The non-identity elements of $A_5$ are either of order 2, 3 or 5 and can be partitioned into 4 conjugacy classes: one of order 2 elements (15 members), one of order 3 elements (20 elements) and two of order 5 elements (12 members each). The centre of $A_5$ is trivial, and the identity alone forms one additional conjugacy class.

As described in Section \ref{sec:generalisedCPsymmetry} to compute the PMNS matrix from residual symmetries, we must first find $U_e$, the matrix which diagonalises the generator of the residual symmetry of the charged-lepton mass term. To do so, we must identify the eligible residual symmetry groups for $G_e$. This symmetry must be an abelian subgroup of $A_5$, of which there are 4 kinds (up to isomorphism). Three of these subgroups are cyclic groups: $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5$. These are the groups generated by a single element $g$, and its members are those powers of $g$ less than or equal to its order

$$\langle g \rangle = \{ g^n | \text{s.t. } n \leq \text{ord}(g) \}.$$

As any element can be taken as the generator of a cyclic group, there are 15 distinct subgroups of $A_5$ in $\mathbb{Z}_2, 10$ of $\mathbb{Z}_3$ and 6 of $\mathbb{Z}_5$. The diagonalizing basis for these groups is simply the basis which diagonalises the generator $\rho(g)$.

In addition to the cyclic subgroups, there are also non-cyclic abelian subgroups in $A_5$. These are isomorphic to the Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$ which is generated by distinguished pairs of order-2 elements,

$$\langle g_1, g_2 | g_1^2 = g_2^2 = (g_1g_2)^2 \rangle.$$

In fact, the 15 order-2 elements in $A_5$ can be divided into 5 triplets which (with the identity) define distinct four-element groups. For these non-cyclic groups, the diagonalizing basis is defined as that which diagonalizes the two generators simultaneously.

Therefore, the different choices for the residual symmetry of the charged-lepton mass term can be divided into four categories depending on the preserved subgroup: $G_e \in \{ \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_2 \times \mathbb{Z}_2 \}$. For the residual symmetry of the neutrino Majorana mass term, we are restricted to taking subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$, leaving us with two options: a single $\mathbb{Z}_2$ or the full Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

### B. Deriving $X$

The matrix $X$ which implements the generalised CP symmetry, as discussed in Section \ref{sec:generalisedCPsymmetry} must satisfy $XX^* = 1$ and be related to a class-inverting automorphism of the group,

$$\forall g \in A_5, \ \exists h_g \in A_5 \ \text{s.t.} \ (X^* \rho(g)X)^* = \rho(h_g^{-1})\rho(g^{-1})\rho(h_g),$$

where $\rho$ is our chosen irreducible representation, generated by the matrices $S$ and $T$. We shall derive the most general form of $X$ for the group $A_5$ by exploiting our knowledge of the automorphism structure of the group. The automorphism group of $A_5$ is $S_5$ (see e.g. \cite{22}), and we identify two important subgroups: *inner* and *outer* automorphisms. The inner automorphism group, $\text{Inn}(A_5)$, comprises those automorphisms which can be represented by conjugation by a group element,

$$\phi_h \in \text{Inn}(A_5) \iff \forall g \in A_5, \ \phi_h(g) = h^{-1}gh.$$  

This group can be found by considering the map from element ($h \in A_5$) to inner automorphism ($\phi_h(g) = h^{-1}gh$), and applying the first isomorphism theorem,

$$\text{Inn}(A_5) \cong A_5 / Z(A_5) \cong A_5,$$
where the final step uses the fact that \( A_5 \) has trivial centre, \( Z(A_5) = 1 \). Therefore, the inner automorphisms of \( A_5 \) are given by \( A_5 \) itself. The outer automorphism group is defined as the quotient of the full automorphism group by the inner automorphism group. For \( A_5 \) it follows from our discussion above that this is the unique group of two elements \( Z_2 \). Our derivation of \( X \) is greatly simplified by \( A_5 \) being an ambivalent group, where each element is conjugate to its inverse. For such groups, the class-inverting automorphisms are also class-preserving. All inner automorphisms of a group are class-preserving, but the two properties are not equivalent as there do exist class-preserving outer automorphisms for some groups \cite{31,32}. However, for the case of \( A_5 \) we have a single non-trivial outer automorphism to check, and this automorphism maps elements of order 5 from one conjugacy class to the other. Therefore, in the present case, we conclude that the class-preserving automorphisms are precisely the inner automorphisms.

We can therefore simplify our defining constraint on \( X \),

\[
\exists h \in A_5, \forall g \in A_5 \quad (X^* \rho(g)X)^* = \rho(h^{-1})\rho(g)\rho(h),
\]

where the element \( h \) is the same for all elements \( g \). As we are working with a real representation, we can always change basis so that all group elements are given by real matrices, and we use this fact with Eq. \( \theta \) to make further simplifications

\[
\forall g \in A_5 \quad X\rho(g)X^* = \rho(h^{-1})\rho(g)\rho(h),
\]

which is equivalent to a commutation relation,

\[
[\rho(h)X, \rho(g)] = 0.
\]

We can then invoke Schur’s lemma to infer that as \( \rho(h)X \) commutes with all the elements of an irreducible representation, it must be a scalar matrix: \( \rho(h)X = \lambda 1 \), for some complex constant \( \lambda \). Requiring that \( XX^* = 1 \) constrains \( \rho(h^2) = 1/|\lambda|^2 \). However, by closure the element on the left must be a member of \( A_5 \) and, as our representation is unitary, we conclude that \( \lambda \) is just a complex phase, \( \lambda = e^{i\theta} \) for \( \theta \in \mathbb{R} \). Therefore, \( h \) must be an order 2 element, and the most general form of \( X \) which implements an involutory class-inverting automorphism for \( A_5 \) is given by

\[
X = e^{i\theta} \rho(h) \quad \text{s.t.} \quad \text{ord}(h) = 2.
\]

In our basis, the consistency relation in Eq. \( \theta \) implies that the \( X \) matrix must commute with the generator \( S \) of the residual \( Z_2 \) symmetry in the neutrino sector. Therefore not all choices of \( h \) can be consistently implemented, and there will be only 3 non-trivial \( X \) matrices (up to global phases) for any given \( S \). These are the three elements of the Klein four group associated with \( S \). If we work in the basis where this group is diagonal, we find that

\[
X_1 = e^{i\theta} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]

and \( X_2 \) and \( X_3 \) can be defined as permutations of this matrix, where the row of the positive entry is denoted by the subscript. It is necessary for us to find the basis in which \( X \) is trivial. The necessary change of basis is given for \( X_1 \) by \( \Omega_i \), where

\[
\Omega_1 = e^{i\theta/2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix},
\]

and similar definitions hold for \( i = 2 \) and \( i = 3 \). In this basis, the generator of the residual symmetry in the neutrino sector is diagonal and the GCP action is trivial. However, due to the degenerate subspace in the \( Z_2 \) generator, it remains possible that the neutrino mass matrix is only block diagonal and requires an orthogonal transformation to fully diagonalise it. This rotation must be in the plane of the degenerate subspace for the matrix \( S \). Given these two elements, the most general form of the matrix which maps between neutrino flavour and mass bases is given by

\[
U_{\nu} = \Omega R(\theta),
\]

where \( R(\theta) \) is an orthogonal matrix effecting a rotation in either the 12-, 13- or 23-plane. We also note at this point that the overall phase included in our definition of \( \Omega \) can be seen to have no physical effect, and will be set to zero in what follows.

In the following subsections, we consider all possible residual symmetry groups in the charged-lepton sector. In the first three sections we consider the charged-lepton residual symmetry to be given by each of the 10 \( Z_3 \) subgroups, 6 \( Z_4 \) subgroups and 5 \( Z_2 \times Z_2 \) subgroups. For each of these subgroups, the basis is found which diagonalises its elements, \( U_e \). Due to the order of these subgroups, this diagonalising matrix is predicted exactly with no remaining degrees of freedom (e.g. \( R_e = 1 \) in Eq. \( \theta \)). The PMNS matrix is then constructed combining \( U_e \) with one of the forms of \( U_{\nu} \) found above by consideration of the residual CP symmetry,

\[
U_{\text{PMNS}} = U_e^\dagger \Omega R(\theta).
\]

Finally in Section \[13\] we consider less restrictive symmetries when \( U_e \) is not fully specified by symmetry alone (\( R_e \neq 1 \)).

For each configuration considered in this section, the arbitrariness in eigenvector ordering and phasing is accounted for by considering all permutations of rows and columns of the PMNS matrix. From these permuted matrices, we compute the mixing angles and phases, and these are compared to global data. We report all patterns of mixing angles found by this process which are consistent with the current 3\( \sigma \) regions as reported in Ref. \[25\].

C. Predictions from \( G_e = Z_3 \) and \( G_e = Z_2 \)

When the residual symmetry in the charged leptons is taken as \( Z_3 \), the diagonalising matrix of the residual sym-
Mixing angle (deg.)
Internal θ  (deg.)
Ge = Z₃
θ ₁₂
θ ₁₃
θ ₂₃
0 20 40 60 80 100 120 140 160 180

FIG. 1. Mixing angles for Z₃ as a function of the internal parameter θ. This pattern predicts |sin δ| = 1 and sin α₂₁ = sin α₃₁ = 0. The shaded regions show the 3σ allowed region for the corresponding mixing angle according to current global data [25].

The Dirac phase for this pattern depends discretely on α₂₁ and α₃₁ = 0, and 

\[
\sin^2 \theta_{12} = \frac{1}{3 - 2 \sin^2 \theta}, \quad \sin^2 \theta_{13} = \frac{2}{3} \sin^2 \theta, \quad \sin^2 \theta_{23} = \frac{1}{2}.
\]

This pattern is continuously connected to the tribimaximal mixing pattern [55] which is recovered at θ = 0, and is an explicit example of a trimaximal pattern [56] where |U_{α2}| = 1/\sqrt{3} \ \forall \ \alpha \in \{e, µ, τ\}. We have plotted these mixing angle predictions for the full range of the unobservable parameter θ in Fig. 1 where the coloured regions show the current 3σ global intervals from Ref. [25].

The Dirac phase for this pattern depends discretely on the value of θ. It can be shown that

\[
\delta = \begin{cases} 
\frac{3\pi}{2} & \theta \in (0, \frac{\pi}{2}), \\
\frac{\pi}{2} & \theta \in (\frac{\pi}{2}, \pi), 
\end{cases}
\]

whilst the Majorana phases can be shown to take CP conserving values \{α₂₁, α₃₁\} ⊆ {0, π} for all values of θ. From Fig. 1 we see that there are two intervals of the unphysical parameter which lead to mixing angles which satisfy the current global 3σ bounds. Due to the symmetry of the expressions, these two solutions offer identical predictions for values of θ₁₂, θ₁₃ and θ₂₃; however, one of these values lies in a region with δ = \frac{3\pi}{2} whilst the other predicts δ = \frac{\pi}{2}. Therefore, there are two sets of predictions from the order-3 elements, differing only in their prediction for the Dirac CP phase.

D. Predictions from Gₑ = Z₅ and Gᵥ = Z₂

If the residual charged-lepton symmetry is assumed to be Gₑ = Z₅, there are 6 subgroups which we need to consider which could lead to a viable set of mixing parameters. We find two distinct sets of correlations which are viable for some range of the unphysical parameter, leading to three distinct sets of mixing angle predictions.

The first set of predictions can be derived from the following matrix,

\[
U_{PMNS} = \begin{pmatrix}
\frac{\sqrt{2}}{2} & -\frac{i}{\sqrt{2}} & \frac{1}{2} \\
\frac{i}{\sqrt{2}} & \frac{\sqrt{2}}{2} & -\frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]

which leads to mixing angles expressed by

\[
\sin^2 \theta_{12} = \frac{1}{1 + \varphi^2 \cos^2 \theta}, \quad \sin^2 \theta_{13} = \frac{\sin^2 \theta}{1 + \varphi^2}, \quad \sin^2 \theta_{23} = \frac{1}{2}.
\]

The mixing angle predictions from this pattern are shown on the left of Fig. 2 as a function of the unphysical parameter θ. In this case, the Dirac phase is maximally CP violating with cos δ = 0. However, as with the order-3 elements, the sign of sin δ depends on the parameter θ,

\[
\delta = \begin{cases} 
\frac{3\pi}{2} & \theta \in (0, \frac{\pi}{2}), \\
\frac{\pi}{2} & \theta \in (\frac{\pi}{2}, \pi),
\end{cases}
\]

and the Majorana phases are again given by CP conserving values, \{α₂₁, α₃₁\} ⊆ {0, π}, although the precise values cannot be determined in this framework.

The second viable pattern arising from Z₅ is shown on the right panel of Fig. 2. This can be derived from a matrix similar to Eq. (11) but distinct in the relative phasing between the columns,

\[
U_{PMNS} = \begin{pmatrix}
\frac{\sqrt{2}}{2} & -\frac{i}{\sqrt{2}} & 0 \\
\frac{i}{\sqrt{2}} & \frac{\sqrt{2}}{2} & 0 \\
0 & 0 & \frac{1}{2}
\end{pmatrix}
\]

This relative phase difference, which arises from the choice of alignment between the matrix implementing the
GCP symmetry $X$ and the generator $S$ of the residual $Z_2$ symmetry of the neutrino mass term, crucially affects the mixing angle $\theta_{23}$ and leads to mixing angles which can be expressed by

$$
\sin^2 \theta_{12} = \frac{1}{1 + \varphi^2 \cos^2 \theta}, \quad \sin^2 \theta_{13} = \frac{\sin^2 \theta}{1 + \varphi^2}, \\
\sin^2 \theta_{23} = \frac{1}{2} \left( \frac{\sin \theta + \sqrt{1 + \varphi^2 \cos \theta}}{1 + \varphi^2 \cos^2 \theta} \right)^2.
$$

All CP phases take CP conserving values for this pattern of mixing parameters. The precise value of the Dirac phase again depends on $\theta$,

$$
\delta = \begin{cases} 
0 & \theta \in (0, \frac{\pi}{2}), \\
\pi & \theta \in (\frac{\pi}{2}, \pi).
\end{cases}
$$

The appearance of CP conservation can be explained as, although it was not imposed explicitly, the generalised CP symmetry remains accidentally unbroken in the charged-lepton sector. This second pattern leads to two distinct allowed intervals in $\theta$ once we restrict the mixing angles to lie in the current $3\sigma$ intervals. Due to the symmetry of the curves, these two viable sets of mixing angles are distinguished only by their predictions for $\theta_{23}$.

Both of the above mixing patterns are continuous extensions of the well known GR mixing pattern (a.k.a. GR1 or GRA) \cite{ref15,ref16}, which is found at $\theta = 0$. If we expand in the small parameter $r \equiv \sqrt{2} \sin \theta_{13}$ \cite{ref11}, we find that both patterns arising from $Z_5$ lead to the prediction for $s \equiv \sqrt{3} \sin \theta_{12} - 1$ \cite{ref11} given by,

$$
s = \sqrt{\frac{3}{2 + \varphi} - 1} + \sqrt{\frac{3}{2 + \varphi} r^2} + O(r^4) . \quad (13)
$$

The two patterns are distinguished by their predictions for $\theta_{23}$; however, both can be seen as subcases of a more general model predicting the $\theta_{12}$ correlation in Eq. (13) and further obeying a linearised atmospheric sum rule first derived in Ref. \cite{ref17},

$$
a = \frac{1 - \varphi r \cos \delta + O(r^2, a^2)}{\sqrt{2}}.
$$

From this relation, it is clear how the maximal angle of the first pattern $a = 0$ is associated with the vanishing of $\cos \delta$, while the contrasting non-trivial predictions of $\theta_{23}$ in the second pattern is due to the CP-conserving value of $\delta$, $\sin \delta = 0$. However, the correlation of maximal atmospheric mixing and CP conserving values of $\delta$ is not an artefact of linearisation and holds exactly, as is shown in the right panel of Fig. 2. We shall consider these correlated maximal predictions as a measurable signature in Section IV B.

E. Predictions from $G_c = Z_2 \times Z_2$ and $G_{\nu} = Z_2$

The only non-cyclic Abelian subgroup in $A_5$ is the Klein four group $Z_2 \times Z_2$. If we take this as the residual symmetry in the charged lepton sector, we find two patterns of mixing angles which differ only by their predictions for $\theta_{23}$. Consistent predictions exist for all of the $3\sigma$ range of $\theta_{13}$, and the predictions for the other mixing angles can be seen in Fig. 3.

The first pattern can be derived from the following mixing matrix,

$$
U_{\text{PMNS}} = \frac{1}{2} \begin{pmatrix} 
\varphi & \varphi & -1 \\
\varphi & 1 & -\varphi \\
-1 & -\varphi & \varphi
\end{pmatrix} \begin{pmatrix} 
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{pmatrix} . \quad (14)
$$

From this matrix, we find the mixing angles can be ex-
pressed by,

\[
\sin^2 \theta_{12} = \frac{1 + \varphi_e \left[ \cos^2 \theta + \sin (2\theta) \right]}{3 - \varphi_e \left[ \sin^2 \theta - \sin (2\theta) \right]},
\]

\[
\sin^2 \theta_{13} = \frac{1 + \varphi_e \left[ \sin^2 \theta - \sin (2\theta) \right]}{4},
\]

\[
\sin^2 \theta_{23} = \frac{1 + \varphi \left[ \cos^2 \theta - \sin (2\theta) \right]}{3 - \varphi \left[ \sin^2 \theta - \sin (2\theta) \right]}.
\]

For this pattern the complex phases are given by CP conserving values: \( \sin \delta = 0 \) and \( \{\alpha_{21}, \alpha_{31}\} \subseteq \{0, \pi\} \).

The true value of \( \delta \) can be shown to depend on \( \theta \)

\[
\delta = \begin{cases} 
0 & \text{if } 31.7^\circ < \theta < 58.3^\circ \text{ or } 121.7^\circ < \theta < 159.1^\circ, \\
\pi & \text{else.}
\end{cases}
\]

This dependence on \( \theta \) looks complex, but the boundaries of the \( \delta = 0 \) regions can be seen in Fig. 3 to be those values of \( \theta \) for which one mixing angle is either 0° or 90°, and closed form expressions can be derived for these values from the mixing angle formulae above. For the matrix shown here, this means that only the \( \delta = \pi \) solution is agrees with the global data. However, when considering all permutations the alternative CP conserving solution can also be found.

The prediction for \( \theta_{23} \) can be expressed as an atmospheric sum rule to first order in \( r \),

\[
a = \sqrt{\frac{1}{2} + \frac{\varphi}{1 + \varphi^2}} - 1 + \varphi \sin r + \mathcal{O}(r^2).
\]

This relation, being derived from a non-cyclic symmetry in the charged-lepton sector, has to the best of our knowledge not been presented before in general analyses of atmospheric sum rules [17].

A permutation in the \( \theta_{23} \) plane acting from the left of the PMNS matrix effects a mapping of \( \theta_{23} \to \frac{\pi}{2} - \theta_{23} \). For this reason, both of the expressions defined above have a complementary pattern with an inverted \( \theta_{23} \). These alternative patterns are shown by dashed lines in Fig. 3.

### F. Predictions with two and three degrees of freedom

So far we have analysed the cases when \( G_e \in \{Z_3, Z_5, Z_2 \times Z_2\} \) and \( G_\nu = Z_2 \). The patterns resulting from these groups have a single degree of freedom, the unphysical angle \( \theta \), controlling their mixing parameter predictions. There are however, more general scenarios where the 3 angles and 3 phases of the PMNS matrix are specified by 2 or 3 input parameters.

The cases with 2 degrees of freedom arise from the choice \( G_e = Z_2 \) while the neutrino symmetry is enlarged to the full Klein group, \( G_\nu = Z_2 \times Z_2 \). The symmetry of the charged-lepton mass terms is insufficient to uniquely specify the diagonalising matrix of the mass matrix, and requires in general a further 2-dimensional complex rotation,

\[
R_c(\theta, \gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta e^{i\gamma} \\ 0 & -\sin \theta e^{-i\gamma} & \cos \theta \end{pmatrix}.
\]

However, the neutrino symmetry has been enlarged, and so \( U_c \) is uniquely specified by the symmetry generators. We have scanned over all such combinations and found that no viable patterns arise from this scenario.

There is one more combination of residual symmetries possible in our construction: \( G_e = Z_2 \) and \( G_\nu = Z_2 \). This is an extension of the previous case, where the charged-lepton symmetry introduces two parameters but now the neutrino residual symmetry also requires a single real parameter to diagonalise the most general mass matrix. In principle, there is no reason to discount these patterns. They are consistent with the idea that the full flavour group has broken into residual subgroups implying correlations on the flavour observables. However, the increased number of parameters reduces the predictivity of the theory (3 inputs, 6 outputs). We have scanned over such groups and verified that there are eligible patterns which match all the global data. Some of these patterns feature non-constant Majorana and Dirac phase predictions, and many are not simply related to the patterns that we have found in more restrictive schemes. However, due to the larger parameter space and reduced predictivity, we will not attempt to present any results of this type.
IV. PHENOMENOLOGICAL PROSPECTS

In the preceding sections we have derived all patterns of mixing angles and phases which are possible with an $A_5$ symmetry with generalised CP broken into residual symmetries. They depend upon a single real angle, $\theta$, and can all be brought into agreement with current global data \cite{22} for a suitable restriction of its range. Eliminating the unphysical parameter $\theta$ leads to a set of correlated predictions between observables which are testable by oscillation experiments and searches for neutrinoless double-beta decay. In this section, we shall discuss the prospects for present and future experiments to constrain these patterns and derive simple versions of the predicted parameter correlations which may be useful experimentally. We would like to stress that the correlations identified in this paper will be tested at almost every stage in the experimental programme of the next few decades. Near term results from T2K \cite{57} and NOvA \cite{58} can be expected on the maximality of $\theta_{23}$ and $\delta$, and in the medium term, new reactor and long-baseline experiments such as JUNO \cite{59}, RENO-50 \cite{60}, DUNE \cite{61} and possibly ESSvB \cite{62} should bring us increased precision on $\theta_{12}$, $\theta_{23}$ and $\delta$. Finally, for the most stringent tests of the models in question, the option remains to construct a more ambitious facility such as the Neutrino Factory \cite{63}. In a complementary direction, neutrinoless double beta decay experiments will further sensitivity to this decay, providing evidence on the Majorana nature of neutrinos and, at least in principle, the first measurements of the values of Majorana phases. Many of these observations are largely independent and we can expect significant evidence either in favour of, or ruling out, the patterns identified in this paper.

A. Precision measurements of $\theta_{12}$

The viable sets of mixing parameters which we have found above predict correlations in $\theta_{12}$ and $\theta_{13}$, and therefore very precise measurements of these angles have the potential to discriminate between flavour symmetric patterns, or to rule them out entirely \cite{23}.

The upcoming medium-baseline reactor (MR) neutrino oscillation experiments, such as JUNO \cite{59} and RENO-50 \cite{60}, expect to make very precise, sub-percent measurements of the oscillation parameter $\theta_{12}$. The precision on $\theta_{13}$, currently dominated by measurements from Daya Bay \cite{2} and RENO \cite{3}, is not expected to be significantly improved by the next generation of reactor facilities. Therefore, the first significant test of the predictions of this paper will come from increased precision on $\theta_{12}$ independently of $\theta_{13}$. We have identified 3 distinct predictions for $\theta_{12}$, if we fix $\theta_{13}$ to its current best-fit \cite{24}.

\[ \theta_{12} = 35.71^\circ, \quad \theta_{12} = 32.11^\circ, \quad \theta_{12} = 35.14^\circ, \]

for preserved charged-lepton subgroups $\mathbb{Z}_3$, $\mathbb{Z}_5$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$, respectively. Given that the expected precision of the MR experiments is at the level of 0.1° or around 0.3% for $\theta_{12}$, a strong discriminatory power exists between the values of the mixing angles predicted by these correlations. The difference between the predicted values of all models under consideration is always greater than 0.26° over the current 3σ interval for $\theta_{13}$, and in many cases significantly greater. Therefore we can expect these experiments to identify with considerable confidence if any of the charged-lepton residual symmetries are consistent with observation.

In the framework discussed in this article, each model predicts a continuous correlation between the values of $\theta_{13}$ and $\theta_{12}$. If one of the predictions above appears to agree with data, it would be desirable to test the correlation between parameters itself. These correlations can be conveniently expressed as expansions in the dimensionless parameter $r = \sqrt{2} \sin \theta_{13} \cite{11}$. The current global best-fits give $\theta_{13} = 8.50^\circ \cite{24}$ which translates to $r \approx 0.2$; the second-order corrections are therefore suppressed by a factor of 1/25.Expressed in this way, the predictions for $\sin \theta_{12}$ associated with the charged-lepton subgroups $\mathbb{Z}_3$, $\mathbb{Z}_5$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ (respectively) can be expanded in the

\[
\begin{align*}
\sin \theta_{12} &\approx Z_3, \\
\sin \theta_{12} &\approx Z_5, \\
\sin \theta_{12} &\approx Z_2 \times Z_2.
\end{align*}
\]
following relations, 
\[
\sin \theta_{12} = \frac{1}{\sqrt{3}} \left( 1 + \frac{r^2}{4} \right) + O \left( r^4 \right), \\
\sin \theta_{12} = \frac{1}{\sqrt{1 + \varphi^2}} \left( 1 + \frac{r^2}{4} \right) + O \left( r^4 \right), \\
\sin \theta_{12} = \frac{\sqrt{2 + \varphi} - 2 - \sqrt{2 + \varphi}}{\sqrt{2 + \varphi}} \frac{r^2}{8} + O \left( r^4 \right).
\]

Expressing these in terms of the angles themselves, we find
\[
\theta_{12} = 35.27^\circ + 10.13^\circ r^2 + O \left( r^4 \right), \\
\theta_{12} = 31.72^\circ + 8.85^\circ r^2 + O \left( r^4 \right), \\
\theta_{12} = 36.00^\circ - 19.72^\circ r^2 + O \left( r^4 \right).
\]

The approximations in Eqs. (15), (16) and (17) have been plotted against the unapproximated expressions for \( \theta_{12} \) in Fig. 4. We see that these relations depend only slightly on \( \theta_{13} \), which first appears at the order \( O(r^2) \), leading to sub-degree level corrections.

The formulae above show that the predictions for \( \theta_{12} \) only vary by 0.07°, 0.06° and 0.13° (for \( Z_3, Z_5 \) and \( Z_2 \times Z_2 \), respectively) over the current 3σ region for \( \theta_{13} \). This is of the order of the target precision of the MR experiments, and it is therefore unlikely that the \( \theta_{12} - \theta_{13} \) correlations themselves will be tested at a significant level even if precision on \( \theta_{13} \) were to be greatly improved. There are no currently planned facilities which could further improve the precision on \( \theta_{12} \).

B. Maximal-maximal predictions for \( \theta_{23} \) and \( \delta \)

The current and upcoming generation of long-baseline experiments will be able to place important constraints on the parameters \( \theta_{23} \) and \( \delta \). Measuring \( \theta_{23} \) and \( \delta \) independently will provide valuable information on the viability of flavour symmetric models; however, in the patterns that we have identified the maximality of \( \theta_{23} \) is significantly correlated with the value of \( \delta \). In four of these patterns (excluding different Majorana phase assignments), two from \( G_e = Z_3 \) and two from \( G_e = Z_5 \), predict a maximal value of \( \theta_{23} \) and a maximal amount of CP violation,
\[
\theta_{23} = \frac{\pi}{4} \quad \text{and} \quad |\sin \delta| = 1.
\]

Therefore, the joint determination of these parameters around these maximal values would be a particularly interesting measurement from the point of view of GCP model building.

Testing the maximality of these parameters is an attainable goal for current and future oscillation experiments. After its full period of data taking, T2K expects to be able to exclude maximal \( \theta_{23} \) at the 90% C.L. for \( |\sin^2(2\theta_{23})| - 0.5 | > 0.05 - 0.07 \) largely independently of the value of \( \delta \) [57]. Measuring \( \delta \) itself is significantly harder; however, the maximal CP violating values considered here are the most accessible. T2K can expect to be able to exclude \( 0 \leq \delta \leq \pi \) \( (\pi \leq \delta \leq 2\pi) \) at the 90% C.L. for a true value of \( \delta = 3\pi/2 \) \( (\delta = \pi/2) \). This would allow T2K to distinguish between \( \delta = \pi/2 \) and \( \delta = 3\pi/2 \) if one of them is true at least the 90% C.L. NOνA can also be expected to contribute to this measurement [58] with a similar power for excluding \( \delta = \pi/2 \) and \( \delta = 3\pi/2 \). Although these exclusions are expected to be individually statistically weak, they would constitute valuable information on the validity of the models studied in this paper and would provide strong encouragement for future work by the next-generation of oscillation experiments.

In the medium term, new long-baseline experiments are expected with significantly improved sensitivities, in particular to the phase \( \delta \), allowing the maximal-maximal predictions to be further tested. To estimate the potential for excluding these models with future facilities, we have run a simulation of LBNE/DUNE using the GLoBES package [61]. Our simulation is based on the detector responses files and fluxes made available by the LBNE collaboration in Ref. [62]. We point out that thanks to its more ambitious design, it seems likely that the DUNE project can significantly improve the sensitivity computed here. However, without access to updated experimental information, making a quantitative assessment of the extent of this improvement is challenging. We assume a 700 kW beam operating at 120 GeV, a detector based on liquid Argon-TPC technology with a mass of 34 kton, and overall systematic errors of 5% for both the signal and background normalizations. The results of these simulations are shown in Fig. 5 where we present the regions of true parameter space for which the combinations of \( (\theta_{23}, \delta) = (\pi/4, \pi/2) \) and \( (\pi/4, 3\pi/2) \) can be excluded after 5 years neutrino and 5 years antineutrino running. We find that these patterns can be excluded at 3σ if the true value of \( \theta_{23} \) satisfies \( \theta_{23} \lesssim 43.0^\circ \) or \( \theta_{23} \gtrsim 48.3^\circ \), or if the true value of \( \delta \) is outside the intervals \( 90^\circ - 63^\circ \) or \( 270^\circ - 337^\circ \). Here we see the importance of observing both \( \theta_{23} \) and \( \delta \) for excluding our models. A measurement of \( \theta_{23} \) alone will not be able to distinguish between the models which predict \( \theta_{23} \)-maximality and the model from \( Z_2 \times Z_2 \) which predicts values of \( \theta_{23} \) which differ from maximality by only around 2°. However, these models have maximally distinct predictions for \( \delta \), and as we have shown, the measurement of \( \delta \) alone would be able to separate these cases at 3σ.

C. Dirac CP conserving patterns and precision measurements of \( \theta_{23} \)

Those patterns which instead make non-maximal predictions of \( \theta_{23} \) also predict CP conserving values of \( \delta \), such that \( |\cos \delta| = 1 \), and we can expect constraints to be placed on these models by the attempts to dis-
cover leptonic CP violation — a standard search for the next-generation of CP-sensitive oscillation experiments [61] [62] [66]. It has been shown that LBNO running with a beam derived from the SPS accelerator at CERN could rule out leptonic CP conservation at 3σ for around 45% (65%) of the parameter space for a detector mass of 20 kton (70 kton). This could be increased to 70% (80%) with an upgraded beam power [67]. LBNE has predicted a similar sensitivity to CP violation [66], with the ultimate reach also depending crucially on the planned series of upgrades to detector mass and beam power. With a 10 kton detector and 6 years of data using a 1.2 MW beam, the measurement could be made for 33% of the parameter space at 3σ. This rises to 40% of the parameter space at 5σ once the detector mass has been increased to 34 kton and 6 years more data has been collected. Finally, a beam power upgrade to 2.3 MW could increase this to 60% of the parameter space at 5σ [66]. The T2HK and ESSvB proposals also show strong sensitivity to CP violation, both using a megaton-scale water Čerenkov detector and MW power beams. T2HK has shown that it can expect a discovery of CP violation over 76% (58%) of the parameter space at 3σ (5σ) [70]. A similar reach is possible with ESSvB, which expects a 3σ (5σ) discovery of CP violation after 10 years of data-taking over 74% (50%) of the parameter space [62].

If the current experimental programme fails to discover CP violation in the leptonic sector and sin δ is discovered to be small, there are four distinct patterns from our model which would remain in agreement with the data. These models can be tested by the increased precision on measurements of θ_{23} expected from next generation long-baseline facilities. In all cases of this kind, we find predictions coming in pairs. The model associated with

$\theta_{23} = 45^\circ \pm 25.04^\circ r + \mathcal{O}(r^2), \quad (18)$

while the predictions for a residual $Z_2 \times Z_2$ symmetry are given by

$\theta_{23} = 31.72^\circ + 55.76^\circ r + \mathcal{O}(r^2),\quad \theta_{23} = 58.28^\circ - 55.76^\circ r + \mathcal{O}(r^2). \quad (19)$

The pairs of sum rules given by Eq. (18) or Eq. (19) are related by the octant degeneracy, $\theta_{23} \rightarrow \frac{\pi}{2} - \theta_{23}$, as can be seen clearly in Fig. 6 where the approximations above are shown against the full predictions. We note that in contrast to those for $\theta_{12}$, these relations depend on $r$ at linear order. Therefore, they are far more sensitive to the precise correlation between parameters, and the measurement of the correlation itself becomes more accessible.

The first discriminating factor between these solutions will come from improved precision on $\theta_{23}$ and the resolution of the octant degeneracy. In these models we predict non-maximal mixing, and a successful determination of the octant would provide early evidence in their favour. This would be most challenging for the model based on $Z_5$ instead predicts greater deviations from maximal mixing, and a successful determination of the octant degeneracy. In these models we predict non-maximal mixing, and a successful determination of the octant would provide early evidence in their favour.

![FIG. 5. Red (green) lines show the exclusion regions at 1, 2 and 3σ for $\theta_{23} = \pi/4$ and $\delta = 3\pi/2$ ($\delta = \pi/2$) expected at LBNE with a 34 kton LAr detector after 5 + 5 years running. In these regions outside the curves, the two sets of predictions can be excluded at the given confidence. The side panels show the appropriate marginalised $\Delta \chi^2$ and the 1, 2 and 3σ confidence levels (1 d.o.f).](image)

![FIG. 6. $\theta_{23}$ as a function of $\theta_{13}$ for the patterns which predict CP conservation (solid lines). The dashed line close to each solid line shows the linearised expression in Eqs. (18) and (19). The grey regions show the 1 and 3σ allowed regions from current global data [25].](image)
next generation of oscillation experiments. In Ref. [69] it is shown that with an exposure of 60 kton-years, LBNF could determine the octant at 3σ if the true value of $\theta_{23}$ deviates from maximality by more than 4°–5°. However, the best bounds could come from T2HK by studying atmospheric neutrino data. A 3σ determination of the octant is expected to be possible after 10 years of data-taking for true values $|\sin^2 \theta_{23} - 0.5| > 0.04–0.06$ corresponding to deviations between $2\sigma–3\sigma$ [61]. Although exclusion of the $Z_2 \times Z_2$ pattern would be unlikely, the two predictions from $Z_5$ would be distinguishable.

To go beyond the octant measurement, higher precision will be necessary to separate between the $Z_5$ and $Z_2 \times Z_2$ predictions, or indeed to test their specific correlations with $\theta_{13}$. The difference between these two predictions varies from between 2.4° to 4.8° over the current allowed regions. Therefore degree-level precision will be required to distinguish between them, even in the presence of greatly improved knowledge of $\theta_{13}$. In Ref. [69] it is shown that the minimal 10 kton LBNF configuration running for 6 years would have a precision of around 1° at 1σ for true values of $\theta_{23}$ around 51°, which increases as we approach $\theta_{23}$ maximality to a 1σ width of around 2.5°. Similarly, T2HK shows that around the point expected to give the worst sensitivity to $\theta_{23}$, the 90% C.L. width is around $2\sigma–3\sigma$ [61]. These results suggest that a significant discrimination between these models would be challenging with these set-ups; however, evidence in favour of these models would be possible at low significance and if observed in conjunction with an absence of observable CP violation, this would present a concrete hypothesis for future work.

**D. Long-term prospects**

We have seen in the previous sections that although the next generation of superbeam and reactor experiments will be able to test the consistency of the patterns that we have identified, much of their discriminatory power relies on excluding maximal angles and phases. Testing the continuous correlations predicted in our models, for example between $\theta_{12}$ and $\theta_{13}$ or between $\theta_{23}$ and $\theta_{13}$, would require higher precision.

The only proposed experiment capable of pushing the precision frontier beyond the results of the next-generation superbeams is the Neutrino Factory (NF) [73], which produces a beam with low systematic uncertainties from the decay of stored muons [62]. A NF would be able to improve our knowledge of the mixing parameters in a number of ways, but for the present purposes it serves two main roles. Firstly, such a facility would greatly increase the precision on $\delta$, with an ultimate 1σ precision estimated at around 5° [74] [75]. This could allow many of our models to be excluded independently of their other parameter correlations. Secondly, a NF would provide a high-precision determination of $\theta_{23}$ [74], allowing it to perform a very stringent test of the $\theta_{23}–\theta_{13}$ correlations discussed in the previous subsection. To quantify this possibility, we have performed a simulation of a representative Low-Energy Neutrino Factory [76] [77] to compute the regions of true parameter space which would allow two candidate models to be (individually and collectively) excluded. Our simulations assume a 2000 km baseline and a stored-muon energy of 10 GeV which is close to the optimal configuration for CP violation discovery [78–81]. For our detector, we take a Magnetized Iron Neutrino Detector (MIND) [82] [83], assumed to be in a toroidal magnetic field allowing for muon charge identification, with a fiducial mass of 100 kton. The detector response is described by a set of migration matrices provided by Ref. [84]. Backgrounds to the appearance channel signal come from both charge- and flavour-misidentified events, as well as the secondary decay products arising from $\tau^{\pm}$ decays in the detector. An overall 1% (10%) uncertainty is taken on the signal (background). Our results are shown in Fig. 7 where the blue (red) shaded regions show the area of true parameter space for which the $Z_2 \times Z_2$ ($Z_5$) correlation would not be able to be excluded. We see that the two coloured regions do not overlap, and therefore all points in this parameter space allow for the exclusion of at least one of the correlations at $3\sigma$ or higher significance.

**E. Neutrinoless double beta decay**

Over the next decade, the new generation of neutrinoless double beta ($0\nu\beta\beta$) decay experiments will significantly increase the sensitivity to this rare process. For the first time these experiments will probe the region of parameter space associated with the inverse hierarchical spectrum. These experiments aim to establish that neutrinos are Majorana in nature, but can also provide valuable information on the neutrino mass spectrum and
in principle, measure the Majorana phases themselves. The $0\nu\beta\beta$ decay rate is proportional to the effective Majorana mass $|m_{ee}|$ (see e.g. Ref. [21, 25]), which is given by

$$|m_{ee}| = \left| \sum_{k=1}^{3} U_{e\alpha}^2 m_k \right|,$$

$$= |m_1 \cos^2 \theta_{12} \cos^2 \theta_{13} + m_2 \sin^2 \theta_{12} \cos^2 \theta_{13} e^{i\alpha_{21}} + m_3 \sin^2 \theta_{13} e^{i(\alpha_{31} - 2\delta)}|,$$

where $\alpha_{21}$ and $\alpha_{31}$ are Majorana phases and $\delta$ is the Dirac phase. The predicted values of $|m_{ee}|$ depend crucially on the neutrino masses. The latter can be ordered in two ways: normal ordering (NO; $m_1 < m_2 < m_3$) or inverted ordering (IO; $m_3 < m_1 < m_2$). As the parameters $\Delta m_{21}^2$ and $|\Delta m_{31}^2|$ are known from oscillation physics, there is a single degree of freedom remaining amongst the masses. This is typically taken to be the lightest neutrino mass, $m_1$ (m3) for NO (IO), which we will denote in both cases by $m_l$ [85]. The parameter space available to $|m_{ee}|$ can be further divided into three particularly interesting regions based on the true value of $m_l$. The first is for quasi-degenerate masses (QD) where $m_l \geq 0.1$ eV, in which the splitting between masses is a small correction to approximately degenerate values. For smaller values of $m_l$ there are two parameter regions: one for normal hierarchical masses (NH; $m_1 < m_2 \ll m_3$) and the other for inverted hierarchical masses (IH; $m_3 \ll m_1 < m_2$). In order to better understand the predictions of our models, we have first computed the predicted values of $|m_{ee}|$ in the generic case, assuming only that the mixing parameters lie in their current 3σ allowed ranges [25]. These predictions for NO (IO) are shown as the blue (red) region in Fig. 8. For quasi-degenerate and IH spectra, there exist lower bounds on $|m_{ee}|$ [86]: for IH $|m_{ee}| > 0.015$ eV, and for QD $|m_{ee}| > 0.03-0.04$ eV. For NO there is no non-zero lower bound as $|m_{ee}|$ can vanish due to a cancellation between terms in Eq. (20).

There are many experiments that are searching for $0\nu\beta\beta$ decay or are in various stages of planning and construction. The most recent experiments that have set upper limits on the effective Majorana mass are CUORICINO, GERDA, EXO–200 and KamLAND-Zen. CUORICINO, based in Gran Sasso National Laboratories in Italy, was an experiment used to test the feasibility of its successor, CUORE. Using data taken from 2003–2004 they achieved a bound on $|m_{ee}|$ of 200–1100 meV [87], where as with all $0\nu\beta\beta$ experiments a key uncertainty on their limits comes from the nuclear matrix element. GERDA (The Germanium Detection Array) is also located in the Gran Sasso Laboratory LNGS in Italy [85]. During phase I of their data taking period they acquired sufficient data to attain an upper limit of $|m_{ee}| \lesssim 200–400$ meV. They intend to increase their sensitivity by a factor of approximately ten during GERDA Phase II. EXO (Enriched Xenon Observatory)–200, located in Carlsbad, New Mexico, has placed upper bounds for $|m_{ee}|$ of 69–163 meV [89]. KamLAND-Zen (Kamioka Liquid Scintillator Anti-Neutrino Detector-Zero neutrino double beta decay search) has had two phases of data acquisition and using the combined data they found an upper limit of $|m_{ee}| \lesssim 140–280$ meV [90]. Both EXO–200 and KamLAND-Zen have ambitious long term plans to upgrade their experiments in order to explore the inverted hierarchical region of parameter space. EXO–200 intends to upgrade to nEXO (next Enriched Xenon Observatory) which, with ten years of data taking, is expected to cover the full IO region [91]. In its next phase, KamLAND-Zen aims to increase its sensitivity to around 50 meV after approximately two years running time [92]. As this limit will only begin to probe the IH region, KamLAND-Zen has proposed KamLAND2-Zen. This upgraded detector (with a running time of 5 years) has a target sensitivity of $|m_{ee}| \approx 20$ meV and will allow the exploration of the majority of the IO region and all of the QD parameter space. In addition to EXO–200 and KamLAND-Zen, other future $0\nu\beta\beta$ decay experiments are CUORE, SNO+ and NEXT. CUORE plans to start taking data in 2015. Over the course of five years, they hope to reach a sensitivity of 50–120 meV [92]. SNO+, a multi-purpose experiment located in Sudbury, Canada, aim to achieve a similar upper bound on $|m_{ee}|$. After two years of data taking they expect to be able to set the upper bound $|m_{ee}| < 100$ meV [93]. The NEXT (Neutrino Experiment with Xenon TPC) experiment, based at CanFranck Underground laboratory (LSC), will commence data taking in 2018 using the NEXT-100 detector. Despite their late start compared with that of EXO–200 and KamLAND-Zen, they intend to achieve a $|m_{ee}|$ sensitivity of approximately 100 meV by 2020 [94]. The next stage will be the development of BEXT which proposes to fully cover the predicted values of $|m_{ee}|$ for IO [95]. Although they are not discussed here, there are other experiments which aim to improve the current bounds on $|m_{ee}|$: for example, COBRA [96], the Majorana Demonstrator [97], SuperNEMO [98] and the DCBA experiment [99], amongst others.

The predicted values of $|m_{ee}|$ for the case of $A_3$ with GCP can be calculated from the lepton mixing matrices of Eq. (10), Eq. (11) and Eq. (14) for both IO and NO. The complex phases only influence the predictions for NO [95]. As $|m_{ee}|$ does not depend on $\theta_{23}$, patterns which only differ by this angle will be degenerate and each preserved charged-lepton subgroup leads to a single prediction for each mass ordering and phase assignment. Fig. 8 shows the predicted values from the mixing patterns in this paper for each charged-lepton residual symmetry $G_\ell$. In these plots, we have neglected a small width to each line which comes from varying $\theta_{13}$ and the neutrino mass-squared splittings over their allowed ranges, instead fixing these at their best-fit values from Ref. [25].
ments (++) and (+−), shown on the top row of Fig. 8, predict large values of \(|m_{ee}|\), close to the upper boundary of the IO region obtained using the 3σ global data. These predictions are very similar for all models. This can be understood as the term in \(|m_{ee}|\) proportional to \(m_3\) only has a subdominant effect: it is not only multiplied by the small number \(\sin^2\theta_{13}\), but is further suppressed for IH by the small value of \(m_3\) itself. If we neglect this term, the resulting approximation at leading-order is independent of \(\theta_{12}\) up to corrections of the order \(\mathcal{O}(\Delta m^2_{23}/\Delta m^2_{31})\). It is feasible that experiments such as CUORE and KamLAND-Zen, and to a much greater extent BEXT and nEXO will be able to explore this topmost region of the IH parameter space and test these predictions. Further distinguishing between them will be beyond their scope due to the small predicted differences and substantial experimental and theoretical uncertainties on \(|m_{ee}|\). For the phase assignments (−+) and (−−), shown on the bottom row of Fig. 8, we see values of \(|m_{ee}|\) that are further suppressed and which exhibit more model dependence. Once again, the suppression of the \(m_3\) term explains the similarity between the two phase assignments. The lower values compared to the (++) case arise from the relative phase difference between the \(m_1\) and \(m_2\) terms: at leading order \(|m_{ee}| = \sqrt{|\Delta m^2_{31}|} \cos^2 \theta_{13} \cos^2(2\theta_{12})\). This effect is evident if we compare \(Z_3\) and \(Z_3\) cases: the larger \(\theta_{12}\) value of \(Z_3\) accounts for the more pronounced cancelation and therefore lower \(|m_{ee}|\) than that of \(Z_5\). These predictions are beyond the reach of many of the facilities discussed so far; although KamLAND2-Zen, aims to set limits near the predictions for \(Z_5\) and, if capable of testing the full IO region, should be accessible to nEXO and BEXT. Although lying in a region of parameter space that is harder to explore, the greater model dependence for these phase assignments would make it easier to distinguish between models than with the (++) and (+−) cases. There exists a separation of around 5 meV between the predictions for \(Z_5\) and the other subgroups; however, it is unlikely such a resolution on \(|m_{ee}|\) would be attainable in the foreseeable future.

For NO, we see quite different behaviour. In the quasi-degenerate region, the mass-squared splittings are negligible and the predictions for the IO and NO cases effectively coincide. However, in the limit of vanishing \(m_1\) the situation is very different. In this limit it is the relative phase between the \(m_2\) and \(m_3\) terms which dominates the magnitude of \(|m_{ee}|\), which leads to larger predictions for the phase assignments (++) and (−−), while suppressing the predictions of (−+) and (+−). Although

FIG. 8. \(|m_{ee}|\) versus the lightest neutrino mass for the IO (NO) for the solid (dashed) lines. The predictions in a given panel all have the same phase assignment, shown in the top left of the plot. The red (blue) shaded region shows the most general predictions for \(|m_{ee}|\) with IO (NO) obtained by varying the oscillation parameters over their current 3σ global ranges [25].
In this paper. The dimensionless parameter $r \equiv \sqrt{2} \sin \theta_{13}$ is constrained by global data to lie in the interval $0.19 \lesssim r \lesssim 0.22$ at $3\sigma$. The predictions for $\theta_{12}$ and $\theta_{23}$ shown here neglect terms of order $O(r^4)$ and $O(r^5)$, respectively. Following the method of this paper, the Majorana phases can only be predicted modulo $\pi$ and the values in the fourth column hold for all phases.

exploring the NH region experimentally is beyond the scope of any planned experiment, if $0\nu\beta\beta$ decays are not observed and oscillation physics establishes that the neutrino masses are NO, it would be of paramount importance to try and test $|m_{ee}|$ values in the NH region. Due to the rich interplay between relative phases, these models make quite different predictions across this parameter space. In fact, all mixing angle patterns discussed in this paper could accommodate a value of $|m_{ee}|$ near the top of the current NH region allowed by global data. Although such an observation would add further support to any prediction of this paper which was still consistent with experimental data, to further discriminate between these models it would be necessary to provide complementary information on the absolute mass scale.

### V. CONCLUSIONS

Assessing the viability of flavour symmetric models of the leptonic sector is an accessible target for precision measurements from present and future neutrino oscillation experiments. In this article, we have presented a detailed analysis of a particular theoretical scenario: the flavour symmetry $A_5$ with a generalised CP symmetry breaking into residual subgroups at low energies. We have identified the most general form of the generalised CP transformation, and studied the full group for consistent residual symmetries. Our analysis results in 6 distinct sets of mixing angle predictions each with an additional 8 possible combinations of phases which are shown in Table I. These depend at most on a single real parameter, and predict testable correlations between certain parameters. In addition, the Majorana phases for all of our predictions are CP conserving. These patterns can be classified by the residual symmetry in the charged-lepton mass terms: $Z_3$, $Z_5$ and $Z_2 \times Z_2$. A symmetry of $Z_3$ predicts maximal $\theta_{23}$, maximal CP violation from $\delta$ and a value of $\theta_{12}$ that lies close to the upper boundary of the $3\sigma$ global fit data. There are two distinct patterns which arise from a preserved $Z_5$ residual symmetry. These share a common $\theta_{12}$ prediction which lies close to the lower boundary of the $3\sigma$ global fit data; however, one prediction has maximal $\theta_{23}$ and a maximally CP violating value of $\delta$ whilst the other has non-maximal $\theta_{23}$ and CP conserving values of $\delta$. The patterns arising from a preserved subgroup $Z_2 \times Z_2$ also share a common $\theta_{12}$ which lies above the current $1\sigma$ region. In this case both $\theta_{23}$ predictions are non-maximal and the value of $\delta$ is CP-conserving.

We have then discussed the phenomenology of our predictions, focusing on the role which current and future reactor, superbeam and neutrinoless double beta decay experiments can play. The predictions for $\theta_{12}$ should be testable at high significance by the next generation of reactor neutrino experiments, such as JUNO and RENO-50. These experiments can be expected to distinguish between the different models; however, testing the precise correlations between $\theta_{12}$ and $\theta_{13}$ will most probably remain beyond the reach of any foreseen experiment. A particularly interesting feature of the patterns found in this paper is the correlated maximality of $\theta_{23}$ and $\delta$, and also non-maximal $\theta_{23}$ and CP conserving values of $\delta$. Testing these correlations is a feasible goal for current and future superbeam experiments. T2K and NOvA can be expected to collect early evidence if such a pattern obtains, and we have shown that DUNE will be able to identify such a pattern over a significant part of the parameter space. For the CP conserving patterns, the deviations from $\theta_{23} = \pi/4$ are expected to be measurable at $3\sigma$ by the next generation of superbeams for the preserved subgroup $Z_5$, but not for $Z_2 \times Z_2$. Ultimately separating between these models at $3\sigma$ significance across the whole parameter space could be done using a Neutrino Factory after 10 years of data taking. An attractive feature of the theoretical scenario in this work is its ability to predict Majorana phases, and therefore, observables for neutrinoless double beta decay experiments. We have seen that in the case of inverted mass ordering, two of the possible Majorana phase combinations predict the discovery of neutrinoless double beta decay at upcoming experiments. In the longer term, the exploration of the full parameter space for inverted hierarchical mass spectra could allow all of our patterns with this mass spectrum to be confirmed independently of oscillation physics.

In conclusion, we find that the combination of the flavour symmetry $A_5$ with a generalised CP symmetry allows for a number of viable predictions to be made for the mixing angles and phases. These predictions specify parameter correlations which present good targets for each stage of the next decade of the experimental programme.

**Note added:** During the final preparations of this article, preprints of two similar works were made available.

| $G_c$ | $\theta_{12}$ | $\theta_{23}$ | $\sin \alpha_{ij}$ | $\delta$ |
|------|---------------|---------------|-------------------|---------|
| $Z_3$ | $35.27^\circ + 10.13^\circ r^2$ | $45^\circ$ | $0^\circ$ | $90^\circ$ (270°) |
| $Z_5$ | $31.72^\circ + 8.85^\circ r^2$ | $45^\circ \pm 25.04^\circ r$ | $0^\circ$ | $0^\circ$ (180°) |
| $Z_2 \times Z_2$ | $36.00^\circ - 34.78^\circ r^2$ | $31.72^\circ + 55.76^\circ r$ | $0^\circ$ | $0^\circ$ (180°) |

TABLE I. Numerical predictions for the correlations found in this paper. The dimensionless parameter $r \equiv \sqrt{2} \sin \theta_{13}$ is constrained by global data to lie in the interval $0.19 \lesssim r \lesssim 0.22$ at $3\sigma$. The predictions for $\theta_{12}$ and $\theta_{23}$ shown here neglect terms of order $O(r^4)$ and $O(r^5)$, respectively. Following the method of this paper, the Majorana phases can only be predicted modulo $\pi$ and the values in the fourth column hold for all phases.
These works also study the mixing patterns arising from the residual symmetries of $A_5$ with GCP. The patterns derived in the first part of our paper confirm those found in Refs. [101] and [102]. Although, case II in Ref. [101] is omitted in our analysis as its predictions fall outside the $3\sigma$ global intervals used in this work. Our phenomenological work, however, significantly extends the analysis in these papers.

ACKNOWLEDGMENTS

We would like to thank Dr Pierre-Philippe Dechant for enlightening discussions about group theory and for his comments on various stages of this work.

This work has been supported by the European Research Council under ERC Grant “NuMass” (FP7-IDEAS-ERC ERC-CG 617143), and by the European Union FP7 ITN INVISIBLES (Marie Curie Actions, PITN-GA-2011-289442).

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