Alternative Constructs of the Lemniscate of Bernoulli

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Abstract. Until now there are at least three alternative methods of constructing lemniscate of Bernoulli. In this paper six alternative methods will be given to construct Lemniscate of Bernoulli in a simple way. Furthermore, at the end of the session will be given how to construct the incircle of Lemniscate of Bernoulli.

1. Introduction
Lemniscate of Bernoulli is a curve that is very often found because it resembles the number eight and the shape of the curve is used as an infinite symbol (1). Lemniscate of Bernoulli introduced by Jakob Bernoulli as a modification of an ellipse in the year 1694 in Acta Eruditorum (see [1], [2], [4] and [8]). The construction of Lemniscate of Bernoulli has been discussed in several books and journals (see [1], [9] and [10]). In 1784 James Watt [1] made a simple prop to construct the of Bernoulli using three bar linkage (Figure 1). In 2014 A.V Akopyan [1] also discussed the construction of Lemniscate of Bernoulli whose illustration of construction was like Figure 2.

![Figure 1](image-url)  
Figure 1. Construction of Lemniscate of Bernoulli by Watt.
In this article, we will discuss six other alternative methods to construct the Lemniscate of Bernoulli and its proof. Furthermore will be given a way of constructing the Lemniscate of Bernoulli incircle by first determining the horizontal tangent of the Lemniscate of Bernoulli.

2. Definition and Equation of Lemniscate of Bernoulli

Lemniscate has a definition like an ellipse. Based on several sources, Lemniscate of Bernoulli’s definition can be obtained, namely:

Definition 1 Lemniscate of Bernoulli is a curve such that for each point of the curve where the product of distances to foci are constant equals quarter of square of the distance between the foci (see [1], [4], and [8]).

Let $P(x_1, y_1)$ at Lemniscate of Bernoulli and there are two foci points $F_1 (-c, 0)$ and $F_2 (c, 0)$. Based on the Definition 1 we get the equation

\[
\frac{PF_1 \times PF_2}{OF_1 \times OF_2} = \frac{c^2}{OF_1 \times OF_2} = c^2
\]

\[
PF_1 \times PF_2 = \sqrt{(x - (-c))^2 + (y - 0)^2} \times \sqrt{(x - c)^2 + (y - 0)^2} = c^2
\]

\[
(x^2 + y^2)^2 = 2c^2(x^2 - y^2)
\]

Equations (1) obtained are the general equations of Lemniscate of Bernoulli with foci points $F_1 (-c, 0)$ and $F_2 (c, 0)$. The equation Lemniscate Bernoulli in polar coordinates can be determined by substituting $x = r \cos \theta$ and $y = r \sin \theta$ in Equation 1 is obtained,

\[
(x^2 + y^2)^2 = 2c^2(x^2 - y^2)
\]

\[
(r^2 \cos^2 \theta + r^2 \sin^2 \theta) = 2c^2(x^2 - y^2)
\]

\[
r^2 = 2c^2 \cos(2\theta)
\]

Equation (2) is the Lemniscate of Bernoulli equation at polar coordinates with foci points $F_1 (-c, 0)$ and $F_2 (c, 0)$. 

![Figure 2. Construction of Lemniscate of Bernoulli by Akopyan.](image-url)
3. The Lemniscate of Bernoulli Construction

The steps that will be carried out on the alternative methods of constructing the Lemniscate of Bernoulli and its proof are as follows.

3.1. Method 1.

Constructing The Lemniscate of Bernoulli in this method uses a circle with radius $\sqrt{2}c$, two parallel lines (line $h$ which passes point $F$ and point $E$ and line $i$ that goes through $O$), and line $j$ which is perpendicular to the line $h$ through the point $E$ and intersects the line $i$ at the point $H$. It will be shown that the movement of the point $H$ will form the curve of the Lemniscate of Bernoulli. The illustration of constructing the Lemniscate of Bernoulli by method 1 can be seen in Figure 3.

![Figure 3. Constructing the Lemniscate of Bernoulli by method 1](image)

Proof. Based on the construction with method 1 it is known: $FO = 2c$, $OE = \sqrt{2}c$ (radius of circle $d$), $\angle F_2OH = \angle OFE$ (are corresponding angles), $\angle EHO = 90^0$ (Line $j$ is perpendicular to line $h$ and $i$), and $EH = OB$. It will be proven that the curve formed is Lemniscate of Bernoulli using sinus rule.

![Figure 4. Proof of the Lemniscate of Bernoulli by method 1](image)
Let $\angle F_1 OH = \theta$, At $\triangle BF O$ we get,
\[
\frac{BO}{\sin \angle OFB} = \frac{FO}{\sin \angle FBO}
\]
\[
BO = \frac{FO}{\sin \angle FBO}
\]
Therefore $EH = OB$, then on $\triangle OHE$ we have,
\[
OH^2 = OE^2 - EH^2
\]
\[
OH^2 = (\sqrt{2}c)^2 - (2c \sin \theta)^2
\]
\[
OH^2 = 2c^2 \cos 2\theta
\]
obtained by Equation (2), it can be proved that the curve formed by method 1 is the Lemniscate of Bernoulli.

3.2. Method 2.
In method 2, the initial step of constructing is to make a circle $d$ with a radius of $c$ then create a tangent circle and vertical tangent to the circle. Create two circles with a radius between the intersections of vertical tangents and tangent circles to the circle. The movement of the intersection of the two circles will form the curve of the Lemniscate of Bernoulli. Illustration of constructing the Lemniscate of Bernoulli with method 3 can be seen in the Figure 5.

![Figure 5. Constructing the Lemniscate of Bernoulli by method 2](image)

Proof. based on construction using method 2, diketahui $F_1 F_2 = 2c$, $BF_1 = r_2$, $BF_2 = r_1$ dan $\angle EGC = 90^\circ$
In the $\triangle EDB$ (in Figure 6) we have
\[ BD^2 = BG^2 + GD^2 \]
\[ (r_1 + r_2)^2 = (r_1 - r_2)^2 + (2c)^2 \]
\[ r_1 r_2 = c^2 \]
\[ FF_2 \times FF_1 = c^2 \]

It can be proved that the curve formed fulfills the definition of 1, it is evident that the curve formed is the Lemniscate of Bernoulli.

3.3. Method 3.
Constructing the Lemniscate of Bernoulli curve in this method by making two lines with lengths $p$ of 2c (lines $F_1F_2$ and $AB$) and two lines with lengths of $\sqrt{2}c$ (line $AF_2$ and $BF_1$) that is connected as in Figure 7. Make a point $E$ which is the midpoint of the line $AB$ and the point $E'$ which is a reflection of the point $E$ on the diagonal of $AF_1$. Then it will show the movement point $E'$ will form the curve of the Lemniscate of Bernoulli.
**Proof.** Suppose $\angle F_1OF = \theta$ maka $\angle E'FE = 180^0 - \theta$.

![Figure 8. Proof of the Lemniscate of Bernoulli by method 3](image)

For $\triangle EF'E$ we have,

$$\frac{EE'}{\sin \angle E'FE} = \frac{FE}{\sin \angle F'FE}$$

Thus for $\triangle OHE$ we have,

$$OE'^2 = OE^2 - EE'^2$$
$$OE'^2 = (\sqrt{2}c)^2 - (2c \sin \theta)^2$$
$$OE'^2 = 2c^2 \cos 2\theta$$

obtained by Equation (2), it can be proved that the curve formed by method 3 is the Lemniscate of Bernoulli.

3.4. **Method 4.**

The first step is to circle $d$ fingers $c$ centered at $O$ then cut a circle through the point $A$ which is $\sqrt{2}c$ from the point $O$ and cut the circle at the point of $B$ and $C$. Create a *hecircle* fingers $AC$ and center at $F_2$ and circle $k$ fingers $AB$ and center at $F_1$. The movement of the intersection points of $k$ and the circle $h$ will form the curve of the Lemniscate of Bernoulli. The illustration of constructing the Lemniscate of Bernoulli with method 4 can be seen in Figure 9.
Figure 9. Construction of the Lemniscate of Bernoulli by method 4

**Proof.** Based on the construction with method 4, it is known: \( F_1 G = AB, F_2 G = AC \). To prove this method can use the secant circle theorem.

**Theorem 3** If two cut lines intersect outside the circle, then the results of the secant with the outline are the same as the results of the other secant lines with the outer line [5].

Based on the secant line theorem, obtained:
\[
AC \times AB = AF_2 \times A_1 F_1
\]
\[
F_2 G \times F_4 G = (\sqrt{2}c - c) \times (\sqrt{2}c + c)
\]
\[
F_2 G \times F_4 G = c^2
\]

Based on the Definition 1 it can be proved that the curve formed by method 1 is Lemniscate Bernoulli.

3.5. Method 5.

Construction in this method is to make a circle \( d \) with radius \( \sqrt{2}c \) which is centered at \( F_1 \) and circle through point \( F_1, F_2 \) and point \( D \) on circle \( d \). Then create a line \( i \) that is parallel to the line \( f \) so that it intersects the circle of \( e \) at the point \( F \) (Figure 10). Will be shown the movement point \( G \) which will form the curve of the Lemniscate of Bernoulli.
Figure 10. Constructing the Lemniscate of Bernoulli by method 5

Proof. Based on the construction carried out on method 6 known $F_1 D // FF_2$, $DF_1 = \sqrt{2} c$ and because $F_1 FF_2 D$ is cyclic quadrilateral then $\triangle DFF_1 \cong \triangle FF_1 F_2$ we obtained $FF_2 = DF_1 = \sqrt{2} c$ and then $FG = DG = F_1 O = F_2 O = c$. To prove the curve formed is the Lemniscate of Bernoulli in this method, the author will use Apollonius’ Theorem and Ptolemy’s Theorem.

Figure 11. Proof of the Lemniscate of Bernoulli Bernoulli by method 6

Theorem 4 If in triangle $DFG$ (in Figure 13) $FG$ is a sector of $FD$ then applies,

$$FF_2^2 + DF_2^2 = 2 \times FG + \frac{DF^2}{2}$$
Based on the theorem 4, for $\Delta DF_2$ we have

$$FF_2^2 + DF_2^2 = 2 \times F_2G + \frac{DF^2}{2}$$

$$(\sqrt{2}c)^2 + DF_2^2 = 2 \times F_2G + \frac{(2c)^2}{2}$$

$$DF_2 = \sqrt{2}c \ F_2G \quad (3)$$

For $\Delta DF_1$ (on Figure 13) hence,

$$FF_1^2 + DF_1^2 = 2 \times F_1G + \frac{DF^2}{2}$$

$$FF_1^2 + (\sqrt{2}c)^2 = 2 \times F_1G + \frac{(2c)^2}{2}$$

$$FF_1 = \sqrt{2}c \ F_1G \quad (4)$$

Then using Ptolemy’s theorem, that is

**Theorem 5** If $DF F_1 F_2$ (in Figure 13) a quadrilateral that is in a circle, then the number of two pairs of adjacent sides is the same as the result of the diagonals [5].

$$FF_1 \times DF_2 + DF_1 \times FF_2 = F_1 F_2 \times DF$$

Then based on Theorem 4 is obtained,

$$FF_1 \times DF_2 + DF_1 \times FF_2 = F_1 F_2 \times DF$$

$$\sqrt{2}c \ F_1G \times \sqrt{2}c \ F_2G + \sqrt{2}c \times \sqrt{2}c = 2c \times 2c$$

$$F_1G \times F_2G = c^2$$

Because $F_1G \times F_1G$ is equal to a quarter of the square of the distance between both points $F_1$ and $F_2$, based on the Definition 1 it is proven that the curve is the lemniscate of Bernoulli.

### 3.6. Method 6.

In constructing Lemniscate of Bernoulli in this method two points are given $F_1$ and $F_2$ which are $2c$ on the x axis with the midpoint O then make circle d with $c/2$ fingers centered on point $F_1$. Create a horizontal line f which intersects the circle at point F and intersects the y axis at point G. Create point E $(0, c/2)$ and circle e with radius GF and center at point E. Make vertical lines h and i through the intersection of points e against the x axis. Will be shown the movement of the intersection of the line h and i towards the line f will form the Lemniscateof Bernoulli curve. Illustration of the construction of Lemniscate of Bernoulli with method 7 can be seen in Figure 12.
Figure 12. Constructing the Lemniscate of Bernoulli by method 7

Proof. Based on construction using method 7 we have $F_1O = OF_2 = c$, $OE = FF_1 = \frac{c}{2}$ and $EI = EH = FG$

Figure 13. Proof of the Lemniscate of Bernoulli Bernoulli by method 6

At $\triangle EIO$ with a right angle at $O$, if $EI = p$ and we get,

$$EI^2 - IO^2 = EO^2$$

$$p^2 - m^2 = \frac{c^2}{4}$$ (5)
At $\Delta AF_1F$ with a right angle at $A$ we have the result,
\[ AF^2 = FF_1^2 - AF_1^2 \]
\[ x^2 = \frac{c^2}{4} - (c - p)^2 \]
\[ x^2 = 2cp - p^2 - \frac{3c^2}{4} \quad (6) \]

At $\Delta KIF_1$ we can substituting Equation 6 and Equation 5 is obtained
\[ KF_1^2 = KI^2 + (c - m)^2 \]
\[ KF_1^2 = 2cp - \frac{3c^2}{4} + c^2 - 2cm + m^2 - p^2 \]
\[ KF_1^2 = 2cp - 2cm \quad (7) \]

Therefore at $\Delta KIF_2$ by substituting Equation 6 and Equation 5 is obtained
\[ KF_2^2 = KI^2 + (c + m)^2 \]
\[ KF_2^2 = 2cp - \frac{3c^2}{4} + c^2 + 2cm + m^2 - p^2 \]
\[ KF_2^2 = 2cp + 2cm \quad (8) \]

By multiplying Equation 1 and Equation 2 then substituting Title 5 is obtained,
\[ KF_1^2 \times KF_2^2 = (2cp - 2cm)(2cp + 2cm) \]
\[ KF_1^2 \times KF_2^2 = 4c^2(p^2 - m^2) \]
\[ KF_1^2 \times KF_2^2 = 4c^2 \times c^2 \]
\[ KF_1 \times KF_2 = c^2 \]

Because $KF_1 \times KF_2 = c^2$ based on the Definition 1 it is proven that the curve is the lemniscate of Bernoulli.

4. Construction of Incircle the Lemniscate of Bernoulli
The idea of constructing the incircle the Lemniscate of Bernoulli is derived from the discussion carried out by Mashadi et al [7] concerning the development of the Cosmic Theorem by using incentre. In Lemniscate of Bernoulli there is an interesting discussion to be discussed, namely about the circle in Lemniscate of Bernoulli whose definition is based on the definition of the inner circle in the triangle.

Incircle of Lemniscate of Bernoulli can be defined as a circle that alludes to the Lemniscate of Bernoulli side on its pinpoint obtained from the horizontal tangent intersection with the minor axis of the Lemniscate of Bernoulli.

The construction of the Lemniscate of Bernoulli inner circle is obtained by determining the equation of the horizontal tangent of the Lemniscate of Bernoulli. For each horizontal tangent, a gradient equal to zero is obtained. The $m$ gradient can be determined by the interpretation of the geometry derivative
\[ m = \left( \frac{dy}{dx} \right)_{(x_1, y_1)} \]

If the Lemniscate of Bernoulli centered at \( O(0; 0) \) has the distance of the center point to the foci point \( c \) and by using the Equation 1 the Lemniscate of Bernoulli gradient general equation is obtained, namely,

\[
\frac{d}{dx} [(x^2 + y^2)^2] = \frac{d}{dx} [2c^2(x^2 - y^2)] \\
2 (x^2 + y^2)(2x + 2yy') = 2c^2(2x - 2yy') \\
m = y' = \frac{[(x^2+y^2)-c^2]x}{-[(x^2+y^2)+c^2]y} \tag{9}
\]

In the horizontal tangent applies \( m = 0 \), then \( y' = 0 \). To be fulfilled \( y = 0 \), it must be \( [(x^2+y^2)-c^2]x = 0 \). At \( [(x^2+y^2)-c^2] = 0 \) a solution is obtained which is \( x = 0 \) or \( (x^2+y^2) - c^2 = 0 \). If \( x = 0 \) at Equation 1 we get \( y^2 = -2c^2 \). The value of \( c \) and \( y \) that is \( c = 0 \) and \( y = 0 \), This cannot be fulfilled because if \( y = 0 \), then \( y' \) is undefined. Thus, it must be \( (x^2+y^2) - c^2 = 0 \) so that it gets \( y^2 = c^2 x^2 \).

The equation \( y^2 = c^2 x^2 \) is substituted to the Equation 1 we get [11].

\[
(x^2 + y^2)^2 = 2c^2(x^2 - y^2) \\
(x^2 + c^2 - x^2)^2 = 2c^2(x^2 - (c^2 - x^2)) \\
x = \pm \frac{c\sqrt{3}}{2}
\]

Substitute the value of \( x \) in the Equation 1 so that it can be obtained,

\[
(x^2 + y^2)^2 = 2c^2(x^2 - y^2) \\
\left(\frac{3}{4} + y^2\right)^2 = 2c^2\left(\frac{3}{4} - y^2\right) \\
16y^4 + 56c^2y^2 - 15c^4 = 0 \tag{10}
\]

The equation 8 is the equation of the horizontal tangent of the Lemniscate of Bernoulli. At Equation (10) can be obtained by the \( y \) value that satisfies that is \( y = \pm \frac{c}{2} \).

The next step to construct a circle in the lemniscate of Bernoulli is to determine the points \( A; B; C \) and \( D \) which are intersections of horizontal tangent lines with the Lemniscate of Bernoulli (Figure 14). Then circle \( m \) and \( n \) through the two intersection points in each loop on the Lemniscate of Bernoulli. The circle \( m \) and \( n \) are circles in the Lemniscate of Bernoulli.
Based on the construction of the incircle of the Lemniscate of Bernoulli, for each of the Lemniscate of Bernoulli with foci points $F_1(c; 0)$ and $F_2(c; 0)$ (Image 14) is obtained:

4.1. The maximum point of the Lemniscate of Bernoulli is,

4.1.a. $A_1(\sqrt{2}c, 0)$ and $A_2(-\sqrt{2}c, 0)$ which is the vertical tangent intersection with the Lemniscate of Bernoulli curve.

4.1.b. $A\left(\frac{\sqrt{3}c}{2}, \frac{c}{2}\right)$, $A\left(\frac{\sqrt{3}c}{2}, -\frac{c}{2}\right)$, $A\left(-\frac{\sqrt{3}c}{2}, \frac{c}{2}\right)$ and $A\left(-\frac{\sqrt{3}c}{2}, -\frac{c}{2}\right)$ which is the intersection of the horizontal tangent line with the Lemniscate of Bernoulli curve.

4.2. The Lemniscate of Bernoulli major axis is $PQ = \sqrt{3}c$

4.3. Minor axis of the Lemniscate of Bernoulli is $CD = AB = c$.

4.4. Foci Point is $F_2^2 = \left(\frac{1}{2} PQ\right)^2 + \left(\frac{1}{2} AB\right)^2$.

4.5. The centers of the incircle of the Lemniscate of Bernoulli are $\left(\frac{\sqrt{3}c}{2}, 0\right)$ and $\left(-\frac{\sqrt{3}c}{2}, 0\right)$.

4.6. Equations of the incircle the Lemniscate of Bernoulli are $\left(x + \frac{c\sqrt{3}}{2}\right)^2 + y^2 = \left(\frac{c}{2}\right)^2$ for circle $m$ and $\left(x - \frac{c\sqrt{3}}{2}\right)^2 + y^2 = \left(\frac{c}{2}\right)^2$ for circle $n$. If the equation of the incircle of the Lemniscate of Bernoulli is generalized it becomes $x^2 + y^2 \pm \sqrt{3}cx + \frac{c^2}{2} = 0$.

4.7. The area of the circle in the Lemniscate of Bernoulli is $\frac{c^2\pi}{4}$.
5. Conclusion

Based on some of the construction of Lemniscate of Bernoulli that has been done, it can be concluded that the construction of Lemniscate of Bernoulli can be done simply by using circles and lines. then, by determining the horizontal tangents of Lemniscate of Bernoulli we can obtain an incircle Lemniscate of Bernoulli.

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