PERVERSE MICROSHEAVES

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ABSTRACT. On a complex contact manifold, or complex symplectic manifold with weight-1 circle action, we construct a sheaf of stable categories carrying a $t$-structure which is locally equivalent to a microlocalization of the perverse $t$-structure.

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I. Introduction

For a complex manifold $M$, let us write $\text{sh}_{\mathbb{C}^c}(M)$ for a derived category of sheaves on $M$, whose objects are each locally constant on the strata of a locally finite stratification by complex subvarieties. Perverse sheaves are those $F$ with the following property:

\begin{equation}
\dim_{\mathbb{C}} \{ p \in M | H^i(p^*F) \neq 0 \} \leq i \quad \dim_{\mathbb{C}} \{ p \in M | H^i(p!F) \neq 0 \} \leq i
\end{equation}

Here $p^*$ and $p!$ are the restriction functors to the point $p$.

Perverse sheaves have played a pivotal role in many results in algebraic geometry and geometric representation theory. They turn to be natural both in terms of considerations of Frobenius eigenvalues in positive characteristic [3], and in terms of analytic considerations in characteristic zero, where they are the sheaves of solutions to regular holonomic differential equations, or more generally, D-modules [10, 21]. This latter equivalence highlights a key feature, not immediately apparent from the definition: despite being a seemingly arbitrarily demarcated subcategory of a category of complexes, perverse sheaves form an abelian category.

There are microlocal versions (living on $T^*M$ or $\mathbb{P}T^*M$) of the category of $D$-modules [25, 12], perverse sheaves [2, 27], and the equivalence between them [1, 23, 28]. More generally still, Kashiwara has constructed a sheaf of categories on any complex contact manifold, locally equivalent to the microlocalization of $D$-modules [11], see also [24]. The variant appropriate to conic complex symplectic geometry has allowed the methods of geometric representation theory to be extended beyond cotangent bundles to more general symplectic resolutions and similar spaces [14, 4].

The purpose of the present article is to construct categories of complex-constructible microsheaves on complex contact manifolds which naturally carry a perverse $t$-structure, globalizing the construction of Waschkies [27]. In the sequel we establish a Riemann-Hilbert equivalence with the canonical stack of $E$-modules defined by Kashiwara [11].

Our starting point is the globalization [26, 22] of the microlocal sheaf theory of Kashiwara and Schapira [15]. We recall the relevant notions in Section 4. In brief, the theory takes as input a real contact or exact symplectic manifold $V$, a choice of symmetric monoidal stable presentable coefficient category $\mathcal{C}$, and certain topological ‘Maslov data’. Maslov data is by definition a choice of trivialization for a certain canonical obstruction $V \to B^2\text{Pic}(\mathcal{C})$, and in fact agrees with the choices made in defining Floer-theoretic invariants in the same target spaces. The output of the theory is a sheaf of stable categories $\mu_{sh}V$ on $V$. For a locally closed subset $X \subset V$, we write $\mu_{sh}X \subset \mu_{sh}V|X$ for the subsheaf of full subcategories on objects locally supported in $X$. A Legendrian or conic Lagrangian $L \subset V$ determines an obstruction $L \to B\text{Pic}(\mathcal{C})$, a choice of trivialization for which (‘secondary Maslov data’) yields an equivalence $\mu_{sh}L \cong \text{Loc}_L$ with the sheaf of categories of local systems along $L$. Here we show that in the complex case:

**Theorem 1.1** (see Section 3). Let $W$ be an exact complex symplectic manifold, and $\mathcal{C}$ the category of modules over a (discrete) commutative ring $R$. Then there is a canonical Maslov datum for $W$, with respect to which secondary Maslov data for a Lagrangian $L \subset W$ are identified with spin structures on $L$.

From this and the existing theory [22], we deduce:

**Theorem 1.2** (see Section 6). Let $W$ be an exact complex symplectic manifold. For a discrete commutative ring $R$, there is a canonical sheaf of stable $R$-linear categories $\mu_{sh}W$ on $W$. For a complex Lagrangian $L \subset X$, a spin structure on $L$ determines an equivalence $\mu_{sh}L \cong \text{Loc}_L$. 
Taking \( D \) to be a disk around a smooth Lagrangian point \( x \in X \), we obtain a canonical (up to noncanonical automorphism) ‘microstalk’ functor \( \mu_x : \mu_{sh}\pi_{\ast} (W) \to \mu_{sh}\pi_{\ast} (D) \to \mathcal{R} - \text{mod} \).

We write \( \mu_{sh}\pi_{\ast} \) for the sheaf of full subcategories on objects whose (micro)support is a complex analytic Lagrangian subset of \( W \). We define \( (\mu_{sh}\pi_{\ast})^{\leq 0} \subset \mu_{sh}\pi_{\ast} \) as the sheaves of full subcategories on those objects all of whose microstalks, as elements of \( \mu_{sh}\pi_{\ast} \) manifold, we have the presheaf of categories

We write \( \pi_{\ast} \) for its sheafification.

These maps are all fully faithful on sections [27, Sec. 2.4], but we do not know when or whether (or after imposing what microsupport conditions) they are essentially surjective.

Consider the projectivized cotangent bundle, \( \pi : T^*M \setminus 0 \to \mathbb{P}^*M \). Then we have an inclusion of sheaves of full subcategories:

\[
(\pi_{\ast})_\ast \mathbb{P}\mu_{sh} := \pi_{\ast} (\gamma^\ast_{\mathbb{C}} \mu_{\mathbb{C}}_{sh}) \hookrightarrow \pi_{\ast} \mu_{sh}
\]

More generally, for \( V \) a complex contact manifold and \( \pi : \tilde{V} \to V \) its symplectization, we will identify a sheaf of full subcategories \( \mathbb{P}\mu_{sh}\pi_{\ast} V \subset \pi_{\ast} \mu_{sh}\tilde{V} \), which agrees with (2) in local charts (see Definition 5.14). We write \( \mathbb{P}\mu_{sh}\pi_{\ast} \subset \mathbb{P}\mu_{sh}\tilde{V} \) for the subsheaf of full subcategories on objects with complex Legendrian microsupport; note \( \mathbb{P}\mu_{sh}\tilde{V}_{\pi_{\ast} - c} \subset \mathbb{P}\mu_{sh}\tilde{V} \cap \pi_{\ast} \mu_{sh}\tilde{V}_{\pi_{\ast} - c} \).

For complex exact symplectic manifolds \( (W, \lambda) \), we may use the contactization \( (W \times \mathbb{C}, \lambda + dz) \) to define a sheaf of categories \( \mathbb{P}\mu_{sh}W \times \{0\} \). We do not understand the relationship of this with \( \mu_{sh}W \) in general. However, if the Liouville flow on \((W, \lambda)\) integrates to a weight-1 \( \mathbb{C}^* \)-action, then \( \mathbb{C}^* \) naturally acts on \( W \times \mathbb{C} \) by contactomorphism. Let \( \gamma_C \) be the set-theoretic identity on \( W \times \mathbb{C}^* \) (resp. on \( W \)) where the source carries the Euclidean topology but the target is endowed with the \( \mathbb{C}^* \) invariant topology. In Theorem 6.3, we show that there is a natural \( \mathbb{Z}[t, t^{-1}] \)-linear structure on \( \lambda_{\ast} \mathbb{P}\mu_{sh}W \times \{0\} \), and there are fully faithful embeddings

\[
(\gamma_{\mathbb{C}})_\ast \mathbb{P}\mu_{sh}W \times \{0\} \hookrightarrow (\gamma_{\mathbb{C}})_\ast \mu_{sh}W \otimes \mathbb{Z}[t, t^{-1}]
\]

\[
\mu_{\mathbb{C}}_{sh}W := (\gamma_{\mathbb{C}})_\ast \mathbb{P}\mu_{sh}W \times \{0\} \otimes_{\mathbb{Z}[t, t^{-1}]} \mathbb{Z} \hookrightarrow (\gamma_{\mathbb{C}})_\ast \mu_{sh}W
\]

We can now state our results on \( t \)-structures.
Theorem 1.3 (7.16). Let $V$ be a complex contact manifold. Then the pair

$$\mathbb{P} \mu_{sh}^{\geq 0}_{V,\mathbb{C}-c} := \mathbb{P} \mu_{sh} \cap \pi_* \mu_{sh}^{\geq 0}_{V,\mathbb{C}-c} \quad \quad \mathbb{P} \mu_{sh}^{< 0}_{V,\mathbb{C}-c} := \mathbb{P} \mu_{sh} \cap \pi_* \mu_{sh}^{< 0}_{V,\mathbb{C}-c}$$

determines a $t$-structure on $\mathbb{P} \mu_{sh}_{V,\mathbb{C}-c}$.

Theorem 1.4. Let $W$ be a complex exact symplectic manifold whose Liouville vector field integrates to a weight-1 $\mathbb{C}^*$ action. Then the pair

$$\mu_{C^*sh}^{\geq 0}_{W,\mathbb{C}-c} := \mu_{C^*sh} \cap \mu_{sh}^{\geq 0}_{W,\mathbb{C}-c} \quad \quad \mu_{C^*sh}^{< 0}_{W,\mathbb{C}-c} := \mu_{C^*sh} \cap \mu_{sh}^{< 0}_{W,\mathbb{C}-c}$$

determines a $t$-structure on $\mu_{C^*sh}_{W,\mathbb{C}-c}$. Moreover, the Hom sheaf of two objects in the heart is a $(\frac{1}{2} \dim W\text{-shifted})$ perverse sheaf.

Example 1.5. Take $(W, \lambda)$ as in Theorem 1.4, and suppose in addition $(W, \text{re} \lambda)$ is a Weinstein manifold admitting a complete collection of generalized cocores, so that the main theorem of [7] can be applied. Many examples coming from geometric representation theory – quiver varieties, hypertoric varieties, moduli of Higgs bundles, etc. satisfy this assumption; cf. [29]. Let $L$ be the core of $W$. Then $\mu_{sh_L}(W)$ is equivalent to (the module category of) $\text{Fuk}(W)$ [7], and so Theorem 1.4 asserts we have constructed a $t$-structure on the Fukaya category of $W$.

Let us conclude by explicitly advertising the question:

Definition 1.6. Let $W$ be a complex exact symplectic manifold whose Liouville vector field integrates to a weight-1 $\mathbb{C}^*$ action. We say a complex conic Lagrangian $L \subset W$ has only complex global microsheaves if $\mu_{C^*sh_L}(W) \rightarrow \mu_{sh_L}(W)$ is essentially surjective.

Question 1.7. Which conic complex Lagrangians have only complex global microsheaves?

We do not know that the answer is not “all of them”. (In fact, in a previous draft of this article, we blithely assumed this was the case.) Example 1.5 provides one motivation for the question, since in the notation there, whenever $L$ has only complex global microsheaves, we have constructed a $t$-structure on the Fukaya category of $W$.

Here is what little we know about Question 1.7. It is not difficult to see that $\mathbb{C}^*$-conic subsets of cotangent bundles are completely complex, and slightly more generally, that the same holds for Lagrangians which admit a cover by $\mathbb{C}^*$-conic subsets of cotangent bundles. (This implies the existence of $t$-structures on Fukaya categories of hypertoric varieties, ALE spaces, and a few other quiver varieties.) We do not presently know much more in general.

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2. Complex contact and symplectic manifolds

We review some standard properties of complex contact and symplectic manifolds. A classical reference in the contact setting is [16].

If \( X \) is a complex manifold, its holomorphic tangent bundle is denoted by \( \mathcal{T}_X \). The holomorphic cotangent bundle is denoted by \( \Omega_X \), and its exterior powers are denoted by \( \Omega_X^k := \wedge^k \Omega_X \).

**Definition 2.1.** A complex (or holomorphic) symplectic manifold is a complex manifold \( X \) along with a closed holomorphic 2-form \( \omega \in H^0(X, \Omega_X^2) \).

A complex symplectic manifold \( (X, \omega) \) determines a family, parameterized by \( h \in \mathbb{C}^* \), of real symplectic manifolds \( (X, \text{re}(h\omega)) \). By an exact complex symplectic manifold, we mean a pair \( (X, \lambda) \) where \( X \) is a complex manifold and \( \lambda \) is a holomorphic 1-form such that \( d\lambda = \partial\lambda \) is symplectic.

**Example 2.2.** Let \( X \) be a complex manifold. Then the holomorphic cotangent bundle \( \Omega_X \) carries a canonical holomorphic 1-form \( \lambda_{\text{can}, \mathbb{C}} = y dx \), where \( (x_1, \ldots, x_n, y_1, \ldots, y_n) \) are canonical holomorphic coordinates. Then \( (\Omega_X, d\lambda_{\text{can}, \mathbb{C}}) \) is a complex symplectic manifold.

Meanwhile, the real cotangent bundle \( T^*M \) carries the canonical 1-form \( \lambda_{\text{can}} \). There is a natural identification \( \Omega_X \to T^*M \) defined in local holomorphic coordinates by

\[
(x, y) \mapsto (\text{re}(x), \text{im}(x), \text{re}(y), -\text{im}(y)).
\]

One computes that this identification pulls back \( \lambda_{\text{can}} \) to \( \text{re}(\lambda_{\text{can}, \mathbb{C}}) \).

If \( (X, \omega) \) is a complex symplectic manifold, a half-dimensional complex submanifold \( L \subset X \) is said to be complex Lagrangian if \( \omega|_L = 0 \). Clearly complex Lagrangian submanifolds are automatically (real) Lagrangian with respect to \( \text{re}(h\omega) \), for all \( h \in \mathbb{C}^* \).

**Definition 2.3.** A complex (or holomorphic) contact manifold is a complex manifold \( V \) of complex dimension \( 2n + 1 \) along with a holomorphic hyperplane field \( \mathcal{H} \hookrightarrow \mathcal{T}_V \) which is maximally non-integrable. Concretely, this means that if \( \alpha \in \Omega_V(U) \) is a holomorphic 1-form for which \( \mathcal{H} = \ker \alpha \) in some local chart \( U \subset V \), then \( \alpha \wedge (d\alpha)^n \neq 0 \).

Given a complex contact manifold \( (V, \xi) \), there is a holomorphic line bundle \( \mathcal{T}_V / \mathcal{H} \to V \). Contact forms are naturally sections of \( (\mathcal{T}_V / \mathcal{H})^\vee \). Note the existence of holomorphic global sections is a rather more stringent condition than the corresponding condition in real smooth geometry; correspondingly complex contact manifolds typically do not have global contact forms.

The bundle \( (\mathcal{T}_V / \mathcal{H})^\vee \) is naturally a holomorphic sub-bundle of \( \Omega_V \) (indeed, for \( v \in V \), we have \( (\mathcal{T}_V / \mathcal{H})^\vee_v = \{ \alpha \in \Omega_{V,v} \mid \alpha(\xi) = 0 \} \subset \Omega_{V,v} \)). We consider the \( \mathbb{C}^* \)-bundle

\[
\pi : \tilde{V} := (\mathcal{T}_V / \mathcal{H})^\vee \setminus 0_v \to V
\]

which we call the complex symplectization of \( V \) and let \( \lambda_{\tilde{V}} \) denote the pullback of \( \lambda_{\text{can}, \mathbb{C}} \) under the inclusion \( \tilde{V} \hookrightarrow \Omega_V \).

We also consider the projectivized \( S^1 \)-bundle \( p : \tilde{V} / \mathbb{R}^+ \to V \) and set \( \xi_h := \ker(\text{re}(h\lambda_{\tilde{V}})) \) for \( h \in \mathbb{C}^* \). The relation between these spaces is summarized by the following diagram:

\[
\begin{array}{ccc}
\tilde{V} & \xrightarrow{q} & V / \mathbb{R}^+ \\
\pi \downarrow & & \downarrow p \\
V & \to & V
\end{array}
\]

**Lemma 2.4.** With the above notation:
(i) \((\tilde{V}, h\lambda_{\tilde{V}})\) is an exact complex symplectic manifold
(ii) \((\tilde{V}/\mathbb{R}_+, \xi_h)\) is a real contact manifold
(iii) \((\tilde{V}, \text{re}(h\lambda_{\tilde{V}}))\) is canonically isomorphic to the real symplectization of \((\tilde{V}/\mathbb{R}_+, \xi_h)\). (This isomorphism intertwines the 1-forms and the \(\mathbb{R}_+\)-bundle structure over \(\tilde{V}/\mathbb{R}_+\)).

Proof. We compute \(h\lambda_{\tilde{V}}\) locally by choosing a holomorphic contact 1-form \(\alpha\) defined on some open set \(U \subset V\). Such a choice induces a holomorphic embedding
\[
\iota_\alpha : \mathbb{C}^* \times U \hookrightarrow \tilde{V} \subset \Omega_V
\]
\[
(z, x) \mapsto z\alpha_x
\]
To compute the pullback of \(h\lambda_{\tilde{V}}\) under \(\iota_\alpha\), choose \((z, x) \in \mathbb{C}^* \times U\) and \(dt_\alpha(v) \in T\tilde{V}_{z\alpha_x} \subset T\Omega_V\). Then we have
\[
\iota_\alpha^*(h\lambda_{\tilde{V}}(dt_\alpha(v))) = h\lambda_{\text{can}, \mathbb{C}}(dt_\alpha(v)) = z\alpha_x(d\pi \circ dt_\alpha(v)).
\]
Hence
\[
\iota_\alpha^*h\lambda_{\tilde{V}} = z\alpha_x.
\]
Both (i) and (ii) can be checked immediately using (6). The proof of (iii) is an exercise in chasing definitions so is omitted.

\(\square\)

Example 2.5 (Example 2.2 continued). The complex projectivization \(V := (\Omega_X - 0_X)/\mathbb{C}^*\) is a complex contact manifold with respect to \(\lambda_{\text{can}, \mathbb{C}}\) and we have \(\tilde{V} = (\Omega_X - 0_X)\). The associated real projectivization \(\tilde{V}/\mathbb{R}_+\) carries a circle of real contact forms \(\text{re}(h\lambda_{\text{can}, \mathbb{C}})\).

Example 2.6. Given an exact complex symplectic manifold \((X, \lambda)\), then \((X \times \mathbb{C}, \lambda + dz)\) is a complex contact manifold. The contact form \(\lambda + dz\) defines a section of the \(\mathbb{C}^*\)-bundle \(\tilde{X} \times \mathbb{C} = X \times \mathbb{C} \times \mathbb{C}^* \rightarrow X \times \mathbb{C}\). Similarly, \(\text{re}(h(\lambda + dz))\) defines a section of \(\tilde{X} \times \mathbb{C}/\mathbb{R}^+ = X \times \mathbb{C} \times S^1\).

Observe that there is a fiber-preserving \(\mathbb{C}^*\)-action on \(\tilde{V}\): over some fiber \(\tilde{V}_v \subset \Omega_{\tilde{V}_v}\), it sends \(\alpha \mapsto z\alpha\) for \(z \in \mathbb{C}^*\) and \(v \in V\). For concreteness, we write \(z = e^{t+i\theta}\) for \((t, \theta) \in \mathbb{R}_+ \times S^1\) and let \(\partial_t, \partial_\theta\) denote the vector fields on \(\tilde{V}\) generated by the \(\mathbb{R}_+\) and \(S^1\) actions.

We have the following morphisms of bundles over \(\tilde{V}\):
\[
\pi^*\xi \equiv \ker(\lambda_{\tilde{V}})/\langle \partial_t, \partial_\theta \rangle \hookrightarrow \ker(\lambda_{\tilde{V}}) \hookrightarrow \ker(\text{re}(h\lambda_{\tilde{V}})) \hookrightarrow T\tilde{V}
\]

Lemma 2.7. There is a splitting of vector bundles over \(\tilde{V}\) (well-defined up to contractible choice)
\[
\ker(\text{re}(h\lambda_{\tilde{V}})) = \partial_t \oplus \langle \partial_\theta, X \rangle \oplus \pi^*\xi,
\]
where \(X\) is a non-vanishing section of \(\ker(\text{re}(h\lambda_{\tilde{V}}))/\ker(h\lambda_{\tilde{V}})\) and

(i) \(\langle \partial_\theta, X \rangle\) is a trivial 2-dimensional real symplectic vector bundle with respect to \(d(\text{re}(h\lambda_{\tilde{V}}))\)
(ii) \(\pi^*\xi\) is a complex symplectic vector bundle with respect to \(d(\lambda_{\tilde{V}})\)

Proof. The existence of this splitting is essentially a restatement of (7). The other statements can be checked locally using (6).

\(\square\)

Given a complex contact manifold \((V, \mathcal{H})\) of complex dimension \(2n + 1\), a complex submanifold \(L \subset V\) of complex dimension \(n\) which is everywhere tangent to \(\mathcal{H}\) is said to be a complex Legendrian.

Lemma 2.8. If \(L \subset (V, \xi)\) is complex Legendrian, then its preimage under \(\tilde{V} \rightarrow V\) is denoted by \(\tilde{L}\) and is complex exact Lagrangian. The quotient \(\tilde{L}/\mathbb{R}_+ \subset \tilde{V}/\mathbb{R}_+\) is a real Legendrian with respect to any of the \(\mathbb{C}^*\) of contact forms \(\text{re}(h\lambda_{\text{can}, \mathbb{C}})\).

\(\square\)
3. Grading and orientation data

The material in this section is mostly algebraic topology. We will need it in Section 5 when we discuss how the condition of being a complex contact manifold/complex Legendrian interacts with orientations/gradings in microlocal sheaf theory.

Consider the following inclusions of groups ($n/2$ ones defined only when $n$ even):

\[
\begin{array}{ccc}
U(n/2, \mathbb{H}) & \rightarrow & U(n/2) \\
\downarrow & & \downarrow \\
\sqrt{SU}(n) & \rightarrow & U(n) \\
\downarrow & & \\
O(n, \mathbb{R}) & \rightarrow & \\
\end{array}
\]

Here $\sqrt{SU}(n)$ is defined as the kernel of $U(n) \xrightarrow{det^2} U(1)$. Note that $U(n)$ is the maximal compact subgroup of both $Sp(2n, \mathbb{R})$ and $GL(n, \mathbb{C})$. Meanwhile $U(n/2, \mathbb{H})$ is also known as the ‘compact symplectic group’ $Sp(n/2) = Sp(n, \mathbb{C}) \cap U(n)$, the maximal compact subgroup of $Sp(n, \mathbb{C})$.

Definition 3.1. Consider a topological space $X$ carrying a hermitian bundle classified by a map $X \rightarrow BU(n)$. We give names to the following sorts of structures:

- A grading (or grading datum) is a lift to $B(\sqrt{SU}(n))$
- A polarization is a lift to $BO(n, \mathbb{R})$
- A quaternionic structure is a lift to $BU(n/2, \mathbb{H})$
- A complex polarization is a lift to $BU(n/2)$.

When $X$ is a symplectic manifold or contact manifold, by a polarization (etc.) on $X$, we always mean a polarization (etc.) on the symplectic tangent bundle of $X$ or the contact distribution.

Lemma 3.2. A polarization induces a grading. A quaternionic structure induces a grading. A complex polarization induces both a quaternionic structure and a polarization, each of which induces the same grading.

Remark 3.3. We emphasize that the identifications in Definition 3.1 should be understood as holding up to coherent homotopy. So, for example, a polarization for $X \rightarrow BU(n)$ is a map $X \rightarrow BO(n)$ and a homotopy between the composition $X \rightarrow BO(n) \rightarrow BU(n)$ and the original structure map $X \rightarrow BU(n)$.

Remark 3.4. Strictly speaking, a symplectic vector bundle $E$ on $X$ is classified by a map $X \rightarrow BSp(2n, \mathbb{R})$. However, $BU(n) \rightarrow BSp(2n, \mathbb{R})$ is a homotopy equivalence and we may always lift to some $X \rightarrow BU(n)$ uniquely up to contractible choice. (I.e. the space of Hermitian structures compatible with a given symplectic bundle is contractible.) Any Lagrangian distribution in $E$ will be totally real for any compatible Hermitian structure, hence define a reduction of structure to $O(n)$, i.e. what we have here called a polarization.

Likewise, a complex symplectic bundle carries a quaternionic structure, uniquely up to contractible choices. We typically omit discussion of such issues in the sequel. In particular, for $(X, \omega)$ is complex symplectic, we simply say $TX$ carries a quaternionic structure.

Remark 3.5. Let us recall the relationship between null-homotopies and lifts. Recall that a fiber sequence $P \rightarrow Q \rightarrow R$ is a Cartesian diagram $P \xrightarrow{\sim} Q \times_R \bullet$, where $\bullet$ is a point. Mapping
spaces preserve limits, so \( \text{Map}(X, P) \xrightarrow{\sim} \text{Map}(X, Q) \times_{\text{Map}(X, R)} \text{Map}(X, \bullet) \). That is, given such a fiber sequence of pointed spaces and a map \( X \to Q \to R \), a null-homotopy of the composite map \( X \to Q \to R \) is equivalent to a lift \( X \to P \).

**Example 3.6.** The sequence \( O(n) \to U(n) \xrightarrow{\text{det}^2} S^1 \) is a fiber sequence, which we may deloop to a fiber sequence \( BO(n) \to BU(n) \xrightarrow{\text{det}^2} BS^1 \). So, a symplectic vector bundle \( E \) admits a grading iff \( 2c_1(E) = 0 \). In particular, a symplectic manifold \( X \) admits a grading iff \( 2c_1(X) = 0 \).

We consider the classical stabilized compact groups

\[
Sp = \lim_{n \to \infty} U(n, \mathbb{H}) \quad U = \lim_{n \to \infty} U(n, \mathbb{C}) \quad O = \lim_{n \to \infty} O(n, \mathbb{R})
\]

and the Lagrangian Grassmannians

\[
LGr_\mathbb{C} = Sp/U \quad LGr = U/O
\]

The natural inclusion \( LGr_\mathbb{C} \to LGr \) is the limit of \( U(n, \mathbb{H})/U(n, \mathbb{C}) \to U(2n, \mathbb{C})/O(2n, \mathbb{R}) \).

There are evident stable analogues of the notions of Def. 3.1, and the stable analogue of Lemma 3.2 also holds.

**Lemma 3.7.** Let \((V, \xi)\) be a complex contact manifold and consider the real contact manifold \((\tilde{V}/\mathbb{R}_+, \xi_h)\). Then the contact distribution \( \xi_h \to \tilde{V}/\mathbb{R}_+ \) carries a stable quaternionic structure.

**Proof.** It is equivalent to prove that \( q^*\xi_h \) carries a stable quaternionic structure, where \( q : \tilde{V} \to \tilde{V}/\mathbb{R}_+ \) is the quotient map. This follows from Lemma 2.7. \( \square \)

We write \( w_2 : BO \to B^2(\mathbb{Z}/2\mathbb{Z}) \) for the map classified by the second Stiefel-Whitney class.

**Definition 3.8.** Let \( X \) be a topological space and consider a map \( \nu : X \to BU \), i.e. a stable complex or symplectic vector bundle on \( X \). Orientation data is a null-homotopy of the composite map:

\[
X \xrightarrow{\nu} BU \to B(U/O) \to B^2O \xrightarrow{Bw_2} B^3(\mathbb{Z}/2\mathbb{Z}).
\]

The canonical orientation data is the null-homotopy arising from the canonical null-homotopy of \( BU \to B(U/O) \to B^2O \).

Evidently, the space of orientation data is identified (using the canonical orientation datum) with \( \text{Maps}(X, B^2(\mathbb{Z}/2\mathbb{Z})) \); in particular, the choices up to homotopy are \( H^2(X, \mathbb{Z}/2\mathbb{Z}) \).

Fix now a stable polarization, i.e. a lift \( \bar{\nu} : BO \) of the given \( X \to BU \). In the composite map

\[
X \xrightarrow{\bar{\nu}} BO \to BU \to B(U/O) \to B^2O \xrightarrow{Bw_2} B^3(\mathbb{Z}/2\mathbb{Z})
\]

there are two evident null-homotopies, coming from the compositions \( BO \to BU \to B(U/O) \) and \( BU \to B(U/O) \to B^3O \). We term the first the polarization orientation data; the second is what we called above the canonical orientation data.

**Lemma 3.9.** Fix a stable polarization \( X \to BO \) of a given \( X \to BU \). The space of homotopies between the canonical and polarization orientation data is equivalent to the space of lifts to \( X \to BP_{in} \).

**Proof.** First we recall some general facts about null-homotopies and exact triangles. Given two null-homotopies \( n_1, n_2 \) of a given map \( f : Q \to R \), we produce a pointed map \([n_1, n_2] \in Q \times S^1 \to R\) by taking \( S^1 = [-\pi, \pi] \), taking the map \( f \) on \( Q \times 0 \) and applying the null-homotopy \( n_1 \) along
[−π, 0] and the null-homotopy $n_2$ along $[0, π]$. Note that $\text{Hom}(Q × S^1, R) = \text{Hom}(Q, ΩR)$. A homotopy between $n_1$ and $n_2$ is a null-homotopy of $[n_1, n_2] ∈ \text{Hom}(Q, ΩR)$.

Suppose now given any exact triangle $P → Q → R → BP$ in a stable category (for the application here, the stable category of spectra). The compositions $P → Q → R$ and $Q → R → BP$ give two null-homotopies of the composite map $P → BP$. The definition of exact triangles [19, Def. 1.1.2.11] promises that the comparison of these two null-homotopies is identified with the identity of $\text{Hom}(P, ΩBP) = \text{Hom}(P, P)$.

Given maps $X → P$, $P → S$, we may compose to learn that the corresponding comparison between null-homotopies of $X → P → Q → R → BP → BS$ given by $[s ◦ n_1 ◦ p, s ◦ n_2 ◦ p] ∈ \text{Hom}(X, ΩBS)$ is identified with $s ◦ p ∈ \text{Hom}(X, S)$. Here, $X$ need only be a space, not a spectrum.

Now specializing to (8), we find that the space of homotopies between polarization and canonical orientation data is identified with the space of null-homotopies of $X → BO → B^2(\mathbb{Z}/2\mathbb{Z})$, i.e., the space of lifts to $X → BPin$. □

**Definition 3.10.** Let $V$ be a symplectic or contact manifold. Fix a polarization / grading / orientation data $m$ on $V$.

Consider $L ⊂ V$ a Lagrangian or Legendrian. There is a canonical isomorphism of $T^*L$ with the restriction of the symplectic tangent bundle or contact distribution. We write $φ_L$ for the fiber polarization.

A relative polarization for $L$ is a homotopy between $m|_L$ and $φ_L$. Relative grading / orientation data are similarly homotopies between the grading / orientation data induced by $m|_L$ and $φ_L$.

Note that the obstruction to existence of a relative grading is a class in $[L, B\mathbb{Z}] = H^1(L, \mathbb{Z})$, and the space of relative gradings is a torsor for $\text{Map}(L, \mathbb{Z})$, hence in particular, is discrete. Thus we simply ask whether relative gradings are equal, rather than discuss homotopies between them.

**Lemma 3.11.** Let $V$ is a complex contact manifold, $\tilde{V} → V$ the corresponding real contact manifold equipped with the canonical grading constructed in Lemma 3.2. Let $L ⊂ \tilde{V}$ be a smooth complex Legendrian. Then there is a canonical choice for the relative grading of $\tilde{L} ⊂ \tilde{V}$.

**Proof.** On $L$, the fiber polarization provides a complex polarization of the restriction of the contact distribution, which, by Lemma 3.2, agrees with the canonical grading. □

**Lemma 3.12.** Fix the canonical orientation data on $V$. Then relative orientation data for $L ⊂ V$ is equivalent to a pin structure on $L$. (If $L$ is oriented, then a pin structure on $L$ is equivalent to a spin structure on $L$ compatible with the given orientation.)

**Proof.** This is a special case of Proposition 3.9. □

## 4. Microsheaves on real contact manifolds

Here we review ideas from the microlocal theory of sheaves as formulated for cotangent bundles of manifolds in [15] and globalized to arbitrary contact manifolds in [26, 22].

### 4.1. Sheaves on manifolds

Let $M$ be a real manifold. Fix a symmetric monoidal stable presentable category $C$. The reader will not lose much of the point of the paper taking throughout $C$ to be the derived category of dg modules over some commutative ring $R$. We write $sh(M)$ for the (stable) category of sheaves on $M$ with values in $C$. 
In this subsection we review ideas from Kashiwara and Schapira [15]. Often these were originally formulated for bounded derived categories, viewed as triangulated categories. Modern foundations [18, 19] allow one to work directly in the stable setting, and in addition for the boundedness hypothesis to be removed for many purposes; we do so when appropriate without further comment.

4.1. Microsupport. Given $F \in \text{sh}(M)$, we say that a smooth function $f : M \to \mathbb{R}$ has a cohomological $F$-critical point at $x \in M$ if $(j^1 F)_x \neq 0$ for $j : \{ f \geq 0 \} \hookrightarrow M$ the inclusion. The microsupport of $F$ (also called the singular support) is the closure of the locus of differentials of functions at their cohomological $F$-critical points. We denote it by $\text{ss}(F)$.

The microsupport is easily seen to be conical and satisfies $\text{ss}(\text{Cone}(F \to G)) \subset \text{ss}(F) \cup \text{ss}(G)$. A deep result of [15, Thm. 6.5.4] is that the microsupport is coisotropic (also called involutive; see [15, Def. 6.5.1]). For a conic subset $K \subset T^* M$, we write $\text{sh}_{K}(M)$ for the full subcategory on objects microsupported in $K$. For any subset $\Lambda \subset S^* M$, we write $\text{sh}_{\Lambda}(M) := \text{sh}_{\mathbb{R}^+ \Lambda \cap 0 M}(M)$, with $0_M$ the zero section of the cotangent bundle.

The assignment $U \mapsto \text{sh}(U)$ defines a sheaf of categories on $M$; we denote it $\text{sh}$. Similarly, $U \mapsto \text{sh}_{K \cap T^* U}(U)$ defines a subsheaf of full subcategories, we denote it $\text{sh}_K$. Similarly, $\text{sh}_{\Lambda}$.

4.1.1. Microsheaves. Let $f : M \to \mathbb{R}$ be a real analytic manifold, and $F$ a sheaf on $M$. Then the following are equivalent:

- There is a subanalytic stratification $M = \bigsqcup M_i$ such that $F|_{M_i}$ is locally constant
- $\text{ss}(F)$ is subanalytic (singular) Lagrangian

Sheaves satisfying these equivalent conditions are said to be $\mathbb{R}$-constructible. We write $\text{Sh}_{\mathbb{R},c}(M)$ for the category of $\mathbb{R}$-constructible sheaves.

4.1.2. Constructibility. Let $f : M \to \mathbb{R}$ be a complex analytic manifold, and $F$ a sheaf on $M$. Then the following are equivalent:

- There is a complex analytic stratification $M = \bigsqcup M_i$ such that $F|_{M_i}$ is locally constant
- $\text{ss}(F)$ is contained in a closed $\mathbb{C}^*$-conic subanalytic isotropic subset
- $\text{ss}(F)$ is a complex analytic (singular) Lagrangian

Sheaves satisfying these equivalent conditions are said to be $\mathbb{C}$-constructible. We write $\text{Sh}_{\mathbb{C},c}(M)$ for the category of $\mathbb{C}$-constructible sheaves.

4.2. Microsheaves on cotangent bundles. We consider the presheaf of stable categories on $T^* M$:

$$\mu \text{sh}_{T^*}^\text{pre}(U) := \text{sh}(M)/\text{sh}_{T^* M \setminus U}(M)$$

Definition 4.4. Let $\mu \text{sh}_{T^*}^\text{pre}$ be the sheaf of categories on $T^* M$ defined by sheafifying the presheaf $\mu \text{sh}_{T^*}^\text{pre}(U)$ in (9). Similarly, let $\mu \text{sh}_{S^*}^\text{pre}$ be the presheaf of categories on $S^* M$ obtained by sheafifying $\mu \text{sh}_{S^*}^\text{pre}$.

For $F \in \text{sh}(M)$, one finds that the support of the image of $F$ in $\mu \text{sh}(\Omega)$ is $\text{ss}(F) \cap \Omega$. For this reason, for any object $G \in \mu \text{sh}(\Omega)$, we sometimes write $\text{ss}(G)$ for the support of $G$.

The sheaf $\mu \text{sh}_{T^*}^\text{pre}$ is coisotropic, i.e. equivariant for the $\mathbb{R}^+$ scaling action. In particular, $\mu \text{sh}_{T^*}^\text{pre}|_{T^0 M}$ is locally constant in the radial direction. This being a contractible $\mathbb{R}^+$, we may define a sheaf of
categories \( \mu_{sh_{S^*M}} \) on \( S^*M \) equivalently by pushforward or pullback along an arbitrary section of \( T^*M \to S^*M \).

**Definition 4.5.** For conic \( K \subset T^*M \), let \( \mu_{sh_K} \subset \mu_{sh_{T^*M}} \) be the subsheaf of full subcategories on objects supported in \( K \). Similarly, for \( \Lambda \subset S^*M \), let \( \mu_{sh_{\Lambda}} \subset \mu_{sh_{S^*M}} \) be the subsheaf of full subcategories on objects supported in \( \Lambda \).

The following is shown in [15, Prop. 6.6.1]:

**Proposition 4.6.** Let \( M \) be a manifold and let \( N \subset M \) be a submanifold. Let \( L = T_N^*M \) and fix a point \( p \in L \). Then there is an equivalence of categories:

\[
\omega_p : C \xrightarrow{\sim} (\mu_{sh_{L,\phi}})_p
\]

where \( A_N \) is the image in \( \mu_{sh} \) of the constant sheaf on \( N \) with value \( A \). The corresponding result of course holds, and we use the same notations, in \( S^*M \).

**Remark 4.7.** Obviously \( \omega_p \) depends only on the germ of \( M \) near the image of \( p \); for this remark we regard \( M, N \) as germs. Consider another triple \( M', N', L' = T_{N'}^*M', p' \in L' \). If \( \dim M = \dim M' \) and \( \dim N = \dim N' \), then there is a diffeomorphism \( (M, N) \to (M', N') \) whose canonical lift to the cotangent bundle sends \( p \to p' \). (Indeed, note that \( p \) may be moved arbitrarily in \( L \setminus 0 \) by diffeomorphisms of \( M \) fixing \( N \).) Obviously the construction of \( \omega_p \) is functorial in such diffeomorphisms.

In any sheaf of categories \( \mathcal{X} \) on a topological space \( T \), given \( F, G \in \mathcal{X}(T) \), the assignment \( U \mapsto \text{Hom}_\mathcal{X}(F|_U, G|_U) \) is a sheaf on \( T \); let us denote it as \( \text{Hom}_\mathcal{X}(F, G) \). Per [15, Thm. 6.1.2]:

\[
\text{Hom}_{\mu_{sh}}(F, G) = \mu_{hom}(F, G)
\]

where \( \mu_{hom} \) is a functor explicitly defined in terms of standard sheaf theoretic operations, namely the Fourier-Sato transformation along the diagonal of the external Hom:

\[
\mu_{hom}(F, G) := \mu_{\Delta} \text{Hom}_{M \times M}(\pi_1^*F, \pi_2^*G)
\]

**Definition 4.8.** For \( M \) real analytic, we define \( \mu_{sh_{R,c}} \subset \mu_{sh} \) as the subsheaf of full subcategories on objects whose support is subanalytic and the closure of its smooth Lagrangian locus.

For \( M \) complex analytic, we define \( \mu_{sh_{C,c}} \subset \mu_{sh_{R,c}} \) as the subsheaf of full subcategories on objects whose support is complex analytic and the closure of its smooth Lagrangian locus.

### 4.3. Microsheaves on polarized real contact manifolds

A key tool of [15] is the functoriality of \( \mu_{sh} \) under (quantized) contact transformation [15, Sec. 7]. One readily obtains:

**Lemma 4.9.** Suppose given a contact manifold \( V \) and a contact embedding \( \iota : V \times T^*D^n \), for some disk \( D \). Then \( \mu_{sh_{V \times D^n}} \) is locally constant along \( D^n \), hence the pullback of a sheaf of categories along \( V \times D^n \to V \).

The basic idea of [26] is to use Lemma 4.9 as a definition of \( \mu_{sh}^{pre} \). Let \( (V, \xi) \) be a contact manifold. Suppose given a (possibly positive codimensional) contact embedding \( \iota : V \to S^*M \), and any Lagrangian distribution \( \eta \) of the symplectic normal bundle to \( \iota \). Let us write \( V(\eta) \) for a

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\(^1\)More precisely, [15] show there is a morphism of this kind for \( \mu_{sh}^{pre} \) which is an isomorphism at stalks; the stated result follows upon sheafification. See [22] for some detailed discussions about the sheafification of \( \mu_{sh}^{pre} \).
thickening of $\iota(V)$ along $\eta$ in a tubular neighborhood. Then $\iota^* \mu_{sh_{V,\eta}}$ provides a sheaf of categories on $V$. By contact transformations, it is easy to see that this sheaf of categories is locally equivalent to microsheaves on a contact cospheare bundle of the same dimension.

To eliminate the dependence on $\iota$ and $M$, one notes that Gromov's $h$-principle for contact embeddings implies the existence of high codimension embeddings of $V$ into the standard contact $\mathbb{R}^{2n+1}$ for large enough $n$, the space of which moreover becomes arbitrarily connected as $n \to \infty$. There remains the (continuous) dependence on a choice of polarization of the stable symplectic normal bundle ("stable normal polarization").

**Theorem 4.10.** [26] Given a contact manifold $(V, \xi)$ and a polarization $\eta$ of the stable symplectic normal bundle to $V$, there is a canonical sheaf of categories $\mu_{sh_{V,-\eta}}$ on $V$, locally isomorphic to $\mu_{sh_{S^*M}}$ for any $M$ of the appropriate dimension.

Above, $-\eta$ can be regarded as just a notational choice. To give it an actual meaning, note that $BO \to BU$ is a morphism of spectra, so given a stable symplectic vector bundle $E$ on some topological space $X$, say classified by some map $E : X \to BU$, and a polarization of $E$, i.e. lift to some $F : X \to BO$, then $-F$ gives a polarization of $-E$, where $-$ is the pullback by the canonical ‘inverse’ involution on $BO$ or $BU$.

In fact [22, Sec. 10.2], it is always possible to build a contact manifold, $V^{T^*LGr(\xi)}$, which is a bundle over $V$ with fibers the cotangent bundles to the Lagrangian Grassmannians of the contact distribution $\xi$. The fiberwise zero section is the Lagrangian Grassmannian bundle $V^{LGr(\xi)}$. The virtue of $V^{T^*LGr(\xi)}$ is that it has a canonical polarization of the contact distribution, hence a canonical stable normal polarization. Thus we define $\mu_{sh_{V^{T^*LGr(\xi)}}}$, and, restricting supports, $\mu_{sh_{V^{LGr(\xi)}}}$.

Let us explain how this recovers the previous notion. A polarization $\rho$ of the contact distribution is (by definition) a section $\rho : V \to V^{LGr(\xi)}$. Now, the normal bundle to $\rho(V) \subset V^{T^*LGr(\xi)}$ is $T^*LGr(\xi)$, this has the canonical polarization by the cotangent fiber. We may combine this with canonical polarization of the symplectic stable normal bundle of $V^{T^*LGr(\xi)}$ to obtain a polarization of the symplectic stable normal bundle of $V$. It’s an exercise to see that this polarization is canonically identified with $-\rho$. We conclude:

**Lemma 4.11.** For $\rho$ a polarization of the contact distribution, there is a canonical isomorphism $\rho^* \mu_{sh_{V^{LGr(\xi)}}} = \mu_{sh_{V,\rho}}$.

**Proof.** We may choose the embedding of $V$ by first embedding $V^{LGr(\xi)}$. To obtain $\mu_{sh_{V,\rho}}$, we must then thicken $V$ along a polarization of its normal bundle which stabilizes to $-\rho$. We obtain such by thickening along $V^{LGr(\xi)} \subset V^{T^*LGr(\xi)}$ and then along the canonical polarization of the normal bundle to $V^{T^*LGr(\xi)}$. But the same procedure defines $\mu_{sh_{V^{LGr(\xi)}}}$. \hfill $\square$

In case $V = S^*M$, there is a polarization $\phi$ given by the cotangent fiber. Now we have two notions of $\mu_{sh_{S^*M}}$: the original given by (9), and then the construction of Theorem 4.10.

**Lemma 4.12.** Let $M$ be a manifold, and $\nu$ the stable normal bundle of $M$. Then the stable symplectic normal bundle to $S^*M$ is $\nu \oplus \nu^*$, and the stable normal polarization by $\nu$ is canonically identified with $-\phi$.

**Corollary 4.13.** There is a canonical equivalence of categories $\mu_{sh_{S^*M}} \cong \mu_{sh_{S^*M,\phi}}$, where the LHS is defined by Equation 9 and the right hand side by Theorem 4.10.

**Proof.** Said differently, we officially define $\mu_{sh}$ by embedding into $J^1\mathbb{R}^n \subset S^*\mathbb{R}^{n+1}$, but the construction makes sense for any embedding into any cospheare bundle, e.g. the embedding of $S^*M$ into itself. We should check these give the same result.
So embed $i : M \hookrightarrow \mathbb{R}^n$; we denote the normal bundle $\nu$; note this is a representative of the stable normal bundle. Let $\lambda_{T^*\mathbb{R}^n}$ and $\lambda_{T^*M}$ be the canonical one forms on cotangent bundles. Any splitting $\sigma$ of the vector bundle map $T^*\mathbb{R}^n|_M \to T^*M$ will satisfy $\sigma^*\lambda_{T^*\mathbb{R}^n} = \lambda_{T^*M}$, hence define a contact embedding $\sigma : S^*M \hookrightarrow S^*\mathbb{R}^n$.

The symplectic normal bundle to $\sigma$ is the restriction of $T^*_M T^*\mathbb{R}^n = \nu \oplus \nu^*$. Let $S^*M(\nu)$ be a thickening of $\sigma(S^*M)$ in the direction $\nu$. By definition, $\mu sh_{S^*M(\nu)} = \sigma^*\mu sh_{S^*M(\nu)}$. Lemma 4.12 gives $\phi = -\nu$.

Meanwhile, $i^* : sh(\mathbb{R}^n) \to sh(M)$ microlocalizes to define a map $i^* : \sigma^*\mu sh_{S^*M(\nu)} \to \mu sh_{S^*M}$; using e.g. Lemma 4.9, one readily sees this is an isomorphism. □

4.4. Maslov data. It is evident from the construction that $\mu sh_{VLG_r}(\xi)$ is locally constant in the Lagrangian Grassmannian direction. We stabilize $\xi \to \xi \oplus T^*\mathbb{R}^n$; taking $n \to \infty$, we have a sheaf of categories $\mu sh_{VLG_r}$ on the (stable) Lagrangian Grassmannian bundle $V^{LG_r}$, locally constant along the Lagrangian Grassmannian direction. In fact:

**Theorem 4.14.** [22, Sec. 11] There is a map of infinite loop spaces $\mu : LG_r \to B Pic(C)$ such that the sheaf of categories $\mu sh_{VLG_r}$ descends to the $BPic(C)$ bundle over $V$ classified by the map $V \xrightarrow{\xi} BU \to BLGr \xrightarrow{B\mu_C} B^2 Pic(C)$

**Definition 4.15.** By $C$-Maslov data for $V$, we mean a null-homotopy of the map $V \to B^2 Pic(C)$.

A choice $\eta$ of $C$-Maslov data gives a trivialization of the $BPic(C)$ bundle to which Theorem 4.14 asserts that $\mu sh_{VLG_r}$ descends. We write $\mu sh_{V(\xi)}$ for the sheaf of categories on $V$ obtained by the pullback along the zero section of this trivial bundle (of groups).

A polarization $\rho$ provides a null-homotopy of the map $V \xrightarrow{\xi} BU \to BLGr$, so $B\mu_C \circ \rho$ is $C$-Maslov data. Lemma 4.11 implies that $\mu sh_{V,\rho} = \mu sh_{V,B\mu_C \circ \rho}$, where the left hand side is defined as in Theorem 4.10 and the right hand side is understood in the sense above. For this reason, given a polarization $\rho$ we will also just write $\rho$ for the Maslov data $B\mu_C \circ \rho$.

For a subset $X \subset V$ we write $\mu sh_{X,\eta} \subset \mu sh_{V,\eta}$ for the sheaf of full subcategories on objects supported in $X$. For any smooth Legendrian $L \subset V$, Theorem 4.14 plus the Weinstein neighborhood theorem imply $\mu sh_{L,\eta} \subset \mu sh_{V,\eta}$ depends only on $\eta|_L$ (and not on $V$).

**Remark 4.16.** There is a canonical choice of Maslov data on (or in a neighborhood of) a real Legendrian $L$. Indeed, canonically $\xi|_L \cong T^*L$, and taking the fiber polarization $\phi_L$ gives a canonical null-homotopy of $L \to V \to BLGr$. By composition we obtain a canonical null-homotopy $\phi_L$ of $L \to B^2 Pic(C)$.

Given Maslov data $\eta$ for $V$, we obtain another such null-homotopy $\eta|_L$, we say that a homotopy $\phi_L \sim \eta|_L$ is secondary Maslov data for $L$. Evidently, secondary Maslov data induces an isomorphism $\mu sh_{L,\eta|_L} \cong \mu sh_{L,\phi_L}$.

It is possible to further fix an isomorphism $\mu sh_{L,\phi_L} \cong Loc_L$, the right hand side being the sheaf of categories of local systems on $L$. To do so, consider the embedding $L = L \times 0 \subset L \times \mathbb{R}$. Let $L^+ \subset S^*(L \times \mathbb{R})$ be the positive conormal to $L$.

Observe that the fiber polarization of $S^*(L \times \mathbb{R})$ restricts to $\phi_{L^+}$. Taking the pushforward along the inclusion and then microlocalizing along the positive conormal gives a map $Loc_L \to \mu sh_L \cong \mu sh_{L,\phi_L}$, which we can check locally is an equivalence (e.g. using [15, Proposition 6.6.1]).

**Remark 4.17.** The category of co-orientable contact manifolds is equivalent to the category of symplectic manifolds with a free transitive $\mathbb{R}_{>0}$ action scaling the symplectic form: given a contact
manifold $V$, we may form the corresponding symplectic manifold $\mathbb{R}_{>0}V$ by taking the locus in $T^*V$ of all graphs of contact 1-forms respecting the co-orientation. An embedding $V \hookrightarrow S^*M$ always lifts to a $\mathbb{R}_{>0}$-equivariant embedding $\mathbb{R}_{>0}V \to T^*M = \mathbb{R}_{>0}S^*M$, uniquely up to shift by a function $f : V \to \mathbb{R}_{>0}$, and a thickening of $V$ canonically lifts to a thickening of $\mathbb{R}_{>0}V$, etc. Stable polarizations and Maslov data being topological, these notions are equivalent on $f$-function.

Thus by embedding, thickening, descending, etc., we find that from Maslov data $\eta$ on $V$, we obtain an $\mathbb{R}_{>0}$-equivariant sheaf of categories $\mu_{sh}R_{>0}V,\eta$ on $\mathbb{R}_{>0}V$. It is of course canonically isomorphic to the pullback of $\mu_{sh}V,\eta$ along the projection $\mathbb{R}_{>0}V \to V$.

In the discussion thus far, we have been agnostic as far as the choice of the category $C$, and we have also not needed to compute the map $\mu_C : U/O \to BPic(C)$. We now turn to this question. Note first that given a map of symmetric monoidal stable categories $C \to D$, it is clear that $\mu_D$ is the composition of $\mu_C$ with the natural map $Pic(C) \to Pic(D)$.

In particular, when $R$ is an ordinary commutative ring ($R = \pi_0(R)$), the map $\mu_{R-mod} \to Pic$ factors through $\mu_{\mathbb{Z}-mod}$. We have $Pic(\mathbb{Z} - mod) = \mathbb{Z} \oplus B(\mathbb{Z}/2\mathbb{Z})$, where $\mathbb{Z}/2\mathbb{Z}$ is really the group of invertible integers, $\{\pm 1\}$. The map $\mu_{\mathbb{Z}-mod}$ has been calculated:

**Theorem 4.18.** [8] The map $\mu_{\mathbb{Z}-mod}$ of Theorem 4.14 is $LGr \xrightarrow{\det^2 \oplus w_2} S^1 \oplus B^2(\mathbb{Z}/2\mathbb{Z})$. That is, $\mathbb{Z} - mod$ Maslov data is precisely grading and orientation data.

It follows that grading and orientation data provide $R - mod$ Maslov data for any commutative ring $R$, although of course there may be $R - mod$ Maslov data which does not arise in this way.

**Remark 4.19.** Let $C = \mathbb{Z} - mod$. Let $M$ be a manifold. Let $m_{can}$ be the Maslov datum whose grading comes from the fiber polarization, and whose orientation data is the canonical orientation data of Def. 3.8. By Lemma 3.9, Corollary 4.13, and Theorem 4.18, a spin structure on $M$ gives an equivalence of sheaves of categories on $T^c M$: $\mu_{sh}S^*M \cong \mu_{sh}S^*M,m_{can}$ where the right hand side is defined by Equation (9) and the LHS by Theorem 4.14.

If $M = N \times \mathbb{R}$ is not spin, then in fact $\mu_{sh}S^*M \not\cong \mu_{sh}S^*M,m_{can}$. Indeed, in this case the conormal Legendrian to $N \times 0$ can be seen to support a microsheaf in $\mu_{sh}S^*M$ but not in $\mu_{sh}S^*M,m_{can}$.

This corresponds to the observation of Kragh that $Fuk(T^*M) \neq Loc(M)$ when $M$ is not spin [17, Remark 1.3].

**Remark 4.20.** We return to the case of general $C$. Consider first the universal case $C = \mathbb{S} - mod$ of spectra. The map $\mu_{\mathbb{S}-mod}$ was calculated in [9]; the result is that it is a delooping of the $J$-homomorphism [9]. The map $\mathbb{S} \to \pi_0(\mathbb{S}) = \mathbb{Z}$ induces $Pic(\mathbb{S} - mod) \to \pi_{\leq 1}Pic(\mathbb{S} - mod) = Pic(\mathbb{Z} - mod)$. In particular, the inclusion of the shift $\mathbb{Z} \to Pic(\mathbb{S} - mod)$ splits and we can write $Pic(\mathbb{S} - mod) = \mathbb{Z} \oplus Pic(\mathbb{S})_0$, where $Pic(\mathbb{S})_0$ is the identity connected component. (It is known that $Pic(\mathbb{S})_0 = B\mathbb{S}^\times$ for some $\mathbb{S}^\times$ with a natural interpretation as “the units of $\mathbb{S}$”.) We write $w_{\mathbb{S}-mod}$ for the composition $LGr \to BPic(\mathbb{S} - mod) \to B^2\mathbb{S}^\times$, and define $\mathbb{S} - mod$ orientation data to be a null-homotopy of the appropriate composition with $Bw_{\mathbb{S}-mod}$. Tautologically, then, $\mathbb{S}$-mod Maslov data is the same as a grading and $\mathbb{S} - mod$ orientation data.

For more general $C$, let us write $Pic(C)_0$ for the identity connected component. Then the canonical map $\mathbb{S} - mod \to C$ induces a map $Pic(\mathbb{S})_0 \to Pic(C)_0$, and we write $w_C$ for the composition of $w_{\mathbb{S}-mod}$ with the delooping of this map. We define $C$-orientation data to be a null-homotopy of the appropriate composition with $Bw_C$. Evidently a grading and $C$-orientation datum yield a $C$-Maslov datum, though not all $C$-Maslov data need arise in this way.

Still more generally, suppose given any partition $\Pi$ of the set of isomorphism classes of objects of $C$. We write $Pic(C)_\Pi \subset Pic(C)$ for the full subcategory on invertible elements which preserve
the partition $\Pi$. By $(C, \Pi)$-orientation data, we mean a null-homotopy of the composite map to $BPic(C)_{\Pi}$. $(C$-orientation data being the case of the total partition). In particular, for $D \subset C$ a subcategory, we consider the partition into objects in versus not in $D$, and term the corresponding orientation data as $(C, D)$-orientation data.

5. Microsheaves on Complex Contact Manifolds

Let $V$ be a complex contact manifold. Recall from Section 2 that we have maps

\[ \tilde{V} \xrightarrow{q} \tilde{V}/\mathbb{R}_+ \xrightarrow{p} V, \]

where $\tilde{V}$ is an exact complex symplectic manifold with 1-form $\lambda_{\tilde{V}}$, and $\tilde{V}/\mathbb{R}_+$ carries a $\mathbb{C}^*$ of real contact forms $\xi_h := \text{re}(h\lambda_{\tilde{V}})$. Evidently $(\tilde{V}, \text{re}(h\lambda_{\tilde{V}}))$ is the real symplectization of $(\tilde{V}/\mathbb{R}_+, \xi_h)$. For specificity we fix $\hbar = 0$ (but see Remark 5.7).

For a complex Legendrian $L \subset V$, we similarly write $\tilde{L} := \pi^{-1}(L)$ and $\tilde{L}/\mathbb{R}_+ := p^{-1}(L)$.

**Definition 5.1.** Let $V$ be a complex contact manifold and $\tilde{V}$ its symplectization. Per Lemma 3.2 and Remark 3.4, there is a canonical grading for $\tilde{V}$. Choose $C$-orientation data $o$ on $V$ and pull it back to $\tilde{V}$; combining this with with the canonical grading, we obtain Maslov data. We also write $o$ for this Maslov data, and correspondingly $\mu_{sh_{\tilde{V}, o}}$ for the corresponding sheaf of categories on $\tilde{V}$.

In case $C = R - \text{mod}$ for a commutative ring $R$, then we have available the canonical orientation data; when we make this choice, we omit $o$ from the notation entirely, writing simply $\mu_{sh_{\tilde{V}}}$.

For $\tilde{\Lambda} \subset \tilde{V}$ a $\mathbb{R}_{>0}$-invariant complex-analytic subset which is the closure of its smooth Lagrangian locus, we write $\mu_{sh_{\tilde{\Lambda}}} \subset \mu_{sh_{\tilde{V}}}$ for the subsheaf of full subcategories on objects supported in $\tilde{\Lambda}$. We write $\mu_{sh_{\tilde{V},\mathbb{C}=c,o}}$ for the union of all such $\mu_{sh_{\tilde{\Lambda},o}}$.

**Remark 5.2.** For any fixed subanalytic Lagrangian $\Lambda$, the category $sh_{\Lambda}(M)$ is presentable, but $sh_{\mathbb{C}=c}(M)$ is not (arbitrary sums of constructible sheaves certainly need not be constructible). The situation for the microsheaf categories is entirely analogous. The distinction is rarely relevant: for example, in the present article, while we state theorems for $\mu_{sh_{\tilde{V},\mathbb{C}=c,o}}$, their proofs quickly reduce to statements about $\mu_{sh_{\tilde{\Lambda},o}}$.

**Remark 5.3.** A closed and complex analytic subset $\tilde{\Lambda} \subset \tilde{V}$ is $\mathbb{R}_{>0}$-invariant iff it is $\mathbb{C}^*$-invariant. We use the formulation above because we may want to consider open sets which are $\mathbb{R}_{>0}$-invariant but not $\mathbb{C}^*$-invariant, and so need a corresponding local notion.

**Remark 5.4.** Per Remark 4.17, we could equivalently have defined a category of microsheaves on the real contact manifold $\tilde{V}/\mathbb{R}_{>0}$ and pulled it back along $q : \tilde{V} \to \tilde{V}/\mathbb{R}_+$; our choice of formulation above is solely to avoid the minor cognitive dissonance of passing constantly to a non-complex manifold.

**Remark 5.5.** One could also consider the subsheaf of full subcategories of $\mu_{sh_{\tilde{V}}}$ on objects whose microsupport is complex-analytic, but this would e.g. not be closed under taking summands.

**Lemma 5.6.** Let $f : U \to V$ be a complex contact embedding, and $\tilde{f} : \tilde{U} \to \tilde{V}$ its lift to the complex symplectization. Then there is a canonical isomorphism $\mu_{sh_{U,f^*o}} = f^* \mu_{sh_{V,o}}$.

In particular, when $o$ is the canonical orientation datum, there is a canonical isomorphism $\mu_{sh_{U}} = f^* \mu_{sh_{V}}$. 

Proof. It suffices to argue that the Maslov data of \( V \) pulls back to the Maslov data of \( U \), up to canonical homotopy. The orientation data is respected by definition, while the grading is canonically induced by the complex structure, hence behaves naturally under the above pullback. \( \square \)

Remark 5.7. \( \tilde{V} \) is a \( \C^* \)-principal bundle over \( V \), and the natural \( \C^* \) action evidently satisfies \( h^*\lambda_V = h^{-1}\lambda_V \) for \( h \in \C^* \). In particular, multiplication by \( h \) descends to a real contactomorphism \( (V/\R_{>0}, \ker \text{re} \lambda) \cong (\tilde{V}/\R_{>0}, \ker \text{re} \lambda) \). The action of \( h \) is also canonically trivial on Maslov data, as this is pulled back from \( V \). Thus by similar reasoning as in Lemma 5.6, there is a canonical isomorphism \( h^*\mu_{sh}(\tilde{V},\lambda) \cong \mu_{sh}(V,\lambda) \).

Lemma 5.8. Let \( M \) be a complex manifold. Consider the complex contact manifold \( V = \mathbb{P}T^*M \), and the corresponding \( \tilde{V} = T^0M \), with canonical orientation data. A spin structure on \( M \) determines an equivalence of sheaves of categories on \( T^0M \): \( \mu_{sh} \cong \mu_{sh,T^0M} \) where the left hand side is as in (9) and the right hand side is defined as in Definition 5.1.

Proof. Follows from Corollary 4.13 by Lemma 3.11 and Lemma 3.12. \( \square \)

Even if we had used some arbitrary choice of Maslov data \( \mu \) on \( V \), not necessarily coming from orientation data and the canonical complex grading, the (complex) Darboux theorem and general functoriality of microsheaves would nevertheless imply that \( \mathbb{P}\mu_{sh_{\mathbb{P}T^*M}} \) is locally isomorphic to any \( \mathbb{P}\mu_{sh_{T^0M}} \) of the correct dimension, since the space of Maslov data, being a torsor for \( \text{Maps}(V, BPic(C)) \), is certainly connected when \( V \) is contractible. However, the resulting isomorphism is not canonical, even locally. What we gain from using the canonical grading is:

Lemma 5.9. Let \( U \) be a contractible complex contact manifold, and \( u: U \to V \) be any complex contact embedding. Then for any choices of \( C \)-orientation data \( o, o' \), there is an isomorphism \( 
\mu_{sh_{U,o}} \cong u^*\mu_{sh_{V,o'}} 
\) which is canonical up to a non-canonical natural transformation.

Proof. Using Lemma 5.6, a homotopy of orientation data \( u^*o' \sim o \) will determine such an isomorphism. This is defined up to the action of loops in the space of orientation data; as \( U \) is contractible, this is just (by definition of \( C \)-orientation data, see Remark 4.20) the connected component of the identity \( Pic(C)_0 \subset Pic(C) \), which acts by natural transformations. \( \square \)

If \( \Lambda \subset V \) is a Legendrian in a real contact manifold equipped with appropriate orientation and grading data, it follows, ultimately from Proposition 4.6, that there is a non-canonical isomorphism \( \mu_{sh_{\Lambda}} \)/\( p \cong C \) at any smooth point \( p \in \Lambda \). Any choice of such an isomorphism is termed a microstalk functor; there is a \( Pic(C) \) worth of choices.

In (10), we saw how to define a distinguished microstalk functor for Legendrians which occur as conormals to embedded submanifolds, where the orientation data is induced by the fiber polarization. However, leaving this class of submanifolds, the torsor of microstalk functors has some monodromy (classified by the map \( \mu_{C} \) from Theorem 4.14), typically involving nontrivial shifts.

A central observation of this paper is that the situation is better in the complex world:

Definition-Proposition 5.10. Let \( V \) be a complex contact manifold equipped with some orientation data \( o \). Let \( L \subset V \) be a complex Legendrian, and \( p \in L \) a smooth point with lift \( \tilde{p} \in \tilde{L} \).

Choose a codimension zero embedding \( \iota: \text{Nbd}(p) \hookrightarrow \mathbb{P}T^*M \) for a complex manifold \( M \), such that the projection \( L \cap \text{Nbd}(p) \to M \) is an embedding onto its image. Consider the composition of isomorphisms:

\[
C \xrightarrow{\omega_{\iota}(\tilde{p})} (\mu_{sh_{\iota(\tilde{L})}})_{\tilde{p}} = (\mu_{sh_{\iota(\tilde{L}),\phi}})_{\tilde{p}} \cong (\mu_{sh_{\tilde{L},o}})_{\tilde{p}}
\]
here, the first arrow is (10), the second is (the $T^\circ M$ version of) Corollary 4.13, and the last is Lemma 5.9. We have seen the first two maps are independent of choices, and the final map depends on choices only through (noncanonical) natural transformation. We denote the composition by $\omega_{\tilde{p}}$, and, henceforth, when we refer in this context to microstalk functors, we mean only those of the form $\omega_{\tilde{p}}^{-1}$.

In particular, given $F \in (\mu sh_{L, o})_{\tilde{p}}$, the object $\omega_{\tilde{p}}^{-1}(F) \in C$ is well defined (i.e. independent of the embedding $i$). We term this object the microstalk at $\tilde{p}$.

**Definition 5.11.** Fix a subcategory $D$ of our coefficient category $C$, and fix $C$-orientation data $o$ (or more generally, $(C, D)$-orientation data). Let $V$ be a complex contact manifold. We write $(\mu sh_{V, c, o})^D \subset \mu sh_{V, c, o}$ for the full subcategory on objects $F$ such that at every smooth point $p \in \text{supp}(F)$, one has $\omega_c^{-1}(F_p) \in D$.

**Remark 5.12.** By the same argument as in Lemma 5.9, the isomorphism class of $\omega_c^{-1}(F_p)$ is constant in the smooth locus of the microsupport. So we need only check $\omega_c^{-1}(F_p) \in D$ at one point in each connected component of the smooth locus.

Legendrians give objects:

**Proposition 5.13.** Let $V$ be a complex contact manifold and $L \subset V$ a complex Legendrian. Then relative orientation data on $L$ (e.g. a spin structure if we use the canonical orientation datum on $V$) determines an equivalence of categories $\mu sh_{V, c, o} \cong Loc_L$ where $Loc_L$ is the sheaf of categories of local systems.

**Proof.** As described in Remark 4.16, it suffices to provide secondary Maslov data. The grading part comes from Lemma 3.11. Lemma 3.12 explains why, for the canonical orientation datum, the corresponding notion of relative orientation data reduces is identified with spin structure.

We now introduce sheaves of categories which will carry the microlocal perverse t-structure.

**Definition 5.14.** Let $V$ be a complex-contact manifold. Given an object $K \in \pi_*\mu sh_{V}(U), U \subset V$, and a point $p \in \text{ss}(K) \cap U$, we say that $K$ is representable at $p$ by a sheaf if there exists a contact embedding $t : Nbd(p) \to \mathbb{P}^* M$ such that (the image of) $K$ in $(\pi_*\mu sh_{T^* M - 0_M})_p$ is represented by a sheaf $F_K \in sh(M)$.

**Remark 5.15.** By the discussion around [15, (11.4.8)] this definition is independent of the choice of the Darboux chart.

**Definition 5.16.** Let $V$ be a contact manifold, $o$ orientation data, and $\Lambda$ a complex (singular) Legendrian. We denote by:

$$\mathbb{P}\mu sh_{V, o} \subset \pi_*\mu sh_{V, o}, \quad \mathbb{P}\mu sh_{\Lambda, o} \subset \pi_*\mu sh_{\pi^{-1}(\Lambda), o}, \quad \mathbb{P}\mu sh_{V, c, o} \subset \pi_*\mu sh_{V, c, o}$$

the sheaves of full subcategories of $\pi_*\mu sh_{V, o}$ (resp. $\pi_*\mu sh_{\pi^{-1}(\Lambda), o}, \pi_*\mu sh_{V, c, o}$) on objects which are pointwise representable by sheaves.

**Remark 5.17.** It is a tautology that $\mathbb{P}\mu sh_{V, c}$ is a union over all complex (singular) legendrians $\Lambda$ of $\mathbb{P}\mu sh_{\Lambda}$.

\[\text{i.e. the image of } K \text{ in } (\pi_*\mu sh_{T^* M - 0_M})_p \text{ coincides with the image of } F_K \text{ under the quotient } sh(M) \to (\pi_*\mu sh_{T^* M - 0_M})_p. \text{ Here } M \text{ is any complex manifold and we recall that } \pi \text{ is the projection } T^* M - 0_M = \mathbb{P}^* M \to \mathbb{P}^* M \]
Remark 5.18. We could equivalently have pushed forward the corresponding sheaves of categories defined on the circle bundle $\tilde{V}/\mathbb{R}_{>0}$.

6. Microsheaves on complex exact symplectic manifolds

Let $X = (X, \lambda)$ be an exact complex symplectic manifold. Consider the contact thickening $(X \times \mathbb{C}, \lambda + dz)$ and the associated symplectization
\begin{equation}
(X \times \mathbb{C} \times \mathbb{C}^*, w(\lambda + dz)).
\end{equation}

Let $V_R$ denote the Liouville vector field for $(X, \lambda)$ and let $I$ denote the almost-complex structure induced by complex multiplication. Integrating the flow of $Z$ in the first component and the flow of $\dot{V}_{s1} := IV_R$ in second, we obtain an action $A : \mathbb{R} \times \mathbb{R} \times X \to X$ which satisfies $A_{t,\theta}^*(\lambda) = e^{t+2\pi i \theta} \lambda$.

If $U \subset \mathbb{C}^*$ is contractible, then we can define a action $U \times X \to X$ by lifting $U$ to $\mathbb{R} \times \mathbb{R}$ by the covering map $(t, \theta) \mapsto e^{t+2\pi i \theta}$. In many situations, the flow in the $IZ$ direction factors through $kIZ \subset \mathbb{R}, k \geq 1$; in this case, we can lift the action via the covering map $(t, \theta) \mapsto e^{k(t+2\pi i \theta)}$ to define a weight $k$ action of $\mathbb{C}^*$.

Let us now suppose that $U \subset \mathbb{C}^*$ is a ball containing 1, and consider the induced weight-1 action $U \times X \to X, (w, x) \mapsto A_w(x), A_w^*(\lambda) = w\lambda$. We record the following convenient change of variables:
\begin{equation}
(X \times \mathbb{C} \times U, \lambda + wdz) \sim (X \times \mathbb{C} \times \mathbb{C}^*, w(\lambda + dz)) \mapsto (x, z, w) \mapsto (A_{w^{-1}}(x), z, w)
\end{equation}

We let $\pi : X \times \mathbb{C} \times \mathbb{C}^* \to X \times \mathbb{C}$ be the projection which forgets the last component.

Recall that in Definition-Proposition 5.10, we show that, for a complex contact manifold $V$ and a complex Legendrian $L \subseteq V$, we can fix a canonical identification $(\mu sh_{\tilde{L}})_{\tilde{p}} \simeq C$ where $\tilde{L}$ is its Lagrangian lift and $\tilde{p}$ a smooth point. The above change of coordinates implies that a similar choice can be made for the symplectic case.

Lemma 6.1. Suppose now that $L \subset X$ is a (possibly singular) conic complex-Lagrangian. Given $p \in L$, we have
\[(\mu sh_L)_p = (\mu sh_{\pi^{-1}(L \times \{0\})})_{(p,0,1)}
\]where the left hand side is defined as in [22, Section 9.1], for any exact symplectic manifolds, and the right hand side is defined in Definition 5.1, only for symplectizations.

Proof. We have $\pi^{-1}(L \times \{0\}) = L \times T_0^* \mathbb{C} = L \times \{0\} \times \mathbb{C}^*$. As a consequence of the local splitting (14), we have
\begin{equation}
(\mu sh_{\pi^{-1}(L \times \{0\})})_{(p,0,1)} \simeq (\mu sh_L)_p \otimes (\mu sh_{T_0^* \mathbb{C}}(0,1,0) \simeq (\mu sh_L)_p
\end{equation}
\[\square\]

We now define the microstalk functor for exact complex-symplectic manifolds.

Definition 6.2. Let $(X, \lambda)$ be an exact complex-symplectic manifold equipped with some orientation data $o$. Let $L \subset X$ a (germ of a) conical complex Lagrangian submanifold. For $p \in L$ a smooth point, we have a microstalk functor defined as the composition
\[C \to (\mu sh_{\pi^{-1}(L \times \{0\}),o})_{(p,0,1)} = (\mu sh_{L,o})_p\]where the first arrow is the microstalk functor introduced in Definition-Proposition 5.10 and the equality is given by (15) above.
**Proof of Theorem 1.2.** Per Lemma 3.2 and Remark 3.4, there is a canonical grading for \( \tilde{\mathcal{V}} \). Choose the canonical orientation data. Combining these, we obtain Maslov canonical data and hence a canonical sheaf of categories \( \mu_{sh} \). Note that here we are viewing \( X = (X, re \lambda) \) as a real exact symplectic manifold, so that one can define \( \mu_{sh} \) as in [22, Section 9.1]. More precisely, we identify \( X \) with the subset \( X \times \{0\} \) of its contactization \( X \times \mathbb{R} \), and we then define \( \mu_{sh} \) as the restriction \( \mu_{sh,\{0\}} \).

Finally, if \( L \) is a smooth smooth conic complex spin Lagrangian, the canonical equivalence \( \mu_{sh,\{L\}} (-) \cong \text{loc}(L) \) is established exactly like Proposition 5.13.

Given an exact complex-symplectic manifold \( (X, \lambda) \), consider the contactization \( (X \times \mathbb{C}, \lambda + dz) \). This is a complex-contact manifold, and so we can consider \( \mathbb{P}_{\mu_{sh,\{0\}}} \subset \mathbb{P}_{\mu_{sh}} \), which is the full subcategory on objects microsupported on \( X \times \{0\} \).

We now have two sheaves of categories on \( X \), namely \( \mu_{sh} \) and \( \mathbb{P}_{\mu_{sh}} \). They are not the same: for example, if \( X \) is a point, then \( \mu_{sh} = C \) while \( \mathbb{P}_{\mu_{sh}} \) is the category of local systems on \( \mathbb{C}^* \). In general, we do not know how to relate \( \mu_{sh} \) and \( \mathbb{P}_{\mu_{sh}} \). However:

**Theorem 6.3.** When the Liouville vector field of \( (X, \lambda) \) integrates to a weight-1 \( \mathbb{C}^* \)-action, there is a natural \( \mathbb{Z} = \Omega S^1 \)-linear structure on \( \mathbb{P}_{\mu_{sh,\{0\}}} \), and there are fully faithful embeddings

\[
\mathbb{P}_{\mu_{sh,\{0\}}} \hookrightarrow \mu_{sh} \otimes \Omega S^1
\]

\[
\mu_{\mathbb{C}^*} : \mathbb{P}_{\mu_{sh,\{0\}}} \otimes \Omega S^1 \cdot \hookrightarrow \mu_{sh}
\]

Of course, when working over a discrete ring, we may replace \( \Omega S^1 \) and \( \cdot \) by their linearizations \( \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}] \) and \( \mathbb{Z} \).

**Proof.** The change of variables (14) is global:

\[
(X \times \mathbb{C} = X \times \mathbb{C} \times \mathbb{C}^*, w(\lambda + dz)) \cong (X \times \mathbb{C} \times \mathbb{C}^*, \lambda + wdz)
\]

The Liouville structure on the right hand side is a product; we have the Künneth isomorphism:

\[
\mu_{sh,\{0\} \times \{0\}^*} = \mu_{sh} \boxtimes \mu_{\mathbb{C}^* \times \{0\}^*}.
\]

Pushing forward and composing with the embedding \( \mathbb{P}_{\mu_{sh,\{0\}}} \hookrightarrow \pi_* \mu_{sh,\{0\}^*} \) gives the first assertion, from which the second follows formally. \( \square \)

**Remark 6.4.** (17) amounts to taking invariants of a topological \( \mathbb{C}^* \)-action; see [20, Sec. 6]. With respect to the split symplectic form (i.e. the right hand side of (18)), this action is \( \theta \cdot (x, z, w) \mapsto (x, \theta z, \theta^{-1} w) \). With respect to the coupled symplectic form (i.e. the left hand side of (18)), this corresponds to the diagonal \( \mathbb{C}^* \) action \( \theta(x, z, w) = (\theta \cdot x, \theta z, \theta^{-1} w) \).\(^3\) Observe that this is just the lift to the symplectization of the \( \mathbb{C}^* \)-action \( \theta \cdot (x, z) = (\theta \cdot x, \theta z) \) on \( X \times \mathbb{C} \) by contactomorphism.

**Remark 6.5.** If \( (X, \omega) \) admits a weight \( \mathbb{C}^* \)-action of weight \( k > 1 \), we expect a related statement, except that \( \mathbb{P}_{\mu_{sh,\{0\}}} \) should be a full subcategory of \( (\mu_{sh})^{\mathbb{Z}/k} \).

\(^3\) Note the slight abuse of notation: \( \theta \cdot x \) refers to the image of \( x \in X \) under the time \( \theta \) flow of the Liouville vector field, which integrates by assumption to a \( \mathbb{C}^* \) action.
7. The Perverse $t$-Structure

7.1. $t$-Structures. The notion of a $t$-structure on a triangulated category was introduced in [3]. We recall the definition and some basic properties.

Definition 7.1. Let $\mathcal{T}$ be a triangulated category. A pair of subcategories $\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}$ determine a $t$-structure if the following conditions are satisfied:

(i) For any $K' \in \mathcal{T}^{\leq 0}$ and $K'' \in \mathcal{T}^{\geq 0}$, we have $\text{Hom}(K', K''[-1]) = 0$.

(ii) If $K' \in \mathcal{T}^{\leq 0}$ then $K'[1] \in \mathcal{T}^{\leq 0}$; similarly if $K'' \in \mathcal{T}^{\geq 0}$ then $K''[-1] \in \mathcal{T}^{\geq 0}$.

(iii) Given $K \in \mathcal{T}$, there exist $K' \in \mathcal{T}^{\leq 0}$ and $K'' \in \mathcal{T}^{\geq 0}$, and a distinguished triangle

$$K' \to K \to K''[-1] \xrightarrow{[1]}$$

We write $\mathcal{T}^{\geq n} := \mathcal{T}^{\geq 0}[-n]$ and $\mathcal{T}^{\leq n} := \mathcal{T}^{\leq 0}[-n]$. The heart of the $t$-structure is $\mathcal{T}^{\heartsuit} := \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$.

It is shown that there are truncation functors $\tau^{\leq n} : \mathcal{T} \to \mathcal{T}^{\leq n}$ and $\tau^{\geq n} : \mathcal{T} \to \mathcal{T}^{\geq n}$ which are right and left adjoint to the inclusions of the corresponding subcategories. The truncation functors commute in an appropriate sense (e.g. when composed with the inclusions so as to define endomorphisms of $\mathcal{T}$). Then $H^0 := \tau^{\leq 0} \circ \tau^{\geq 0} = \tau^{\geq 0} \circ \tau^{\leq 0}$ defines a map $\mathcal{T} \to \mathcal{T}^{\heartsuit}$, and one writes $H^n : \mathcal{T} \to \mathcal{T}^{\heartsuit}$ for the appropriate composition with the shift functor. Finally, $\mathcal{T}^{\heartsuit}$ is an abelian category, closed under extensions [3, Thm. 1.3.6].

The prototypical example is when $\mathcal{T}$ is a derived category of chain complexes, and $\mathcal{T}^{\leq 0}$ consists of the complexes whose cohomology is concentrated in degrees $\leq 0$, etc.

Let us recall a result about when $t$-structures pass to quotient categories.

Lemma 7.2 (Lem. 3.3 in [5]). Let $\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}$ determine a $t$-structure on $\mathcal{T}$. Let $\mathcal{I} \subset \mathcal{T}$ be a triangulated subcategory, closed under taking direct summands (“thick subcategory”) and let $Q : \mathcal{T} \to \mathcal{T}/\mathcal{I}$ be the Verdier quotient. Then:

1. $\mathcal{T}^{\leq 0} \cap \mathcal{I}, \mathcal{T}^{\geq 0} \cap \mathcal{I}$ determine a $t$-structure if and only if $\tau^{\leq 0} \mathcal{I} \subset \mathcal{I}$

2. if the equivalent assertions of (1) hold, $Q(\mathcal{T}^{\leq 0}), Q(\mathcal{T}^{\geq 0})$ determine a $t$-structure if and only if $\mathcal{I} \cap \mathcal{T}^{\heartsuit} \subset \mathcal{T}^{\heartsuit}$ is a “Serre subcategory” (meaning it is closed under extensions, quotients and sub-objects).

The notion of $t$-structure is imported to the setting of stable categories in [19, Sec. 1.2]: By definition, a $t$-structure on a stable category is a $t$-structure on its homotopy category, which canonically carries the structure of a triangulated category. It is shown that the various properties of $t$-structures lift to the stable setting, in particular, the existence of truncation functors, and the fact that the full subcategory on objects in the heart is abelian.\footnote{Let us avoid a possible source of confusion. One might think that, insofar as stable categories generalize dg categories, the heart could be expected to have, in its hom spaces, whatever corresponds to the positive ext groups. This depends on whether or not the stable category is viewed as a usual $\infty$-category, or as an $\infty$-category enriched in spectra. Indeed, the positive ext groups (in cohomological grading conventions) correspond to negative homotopy groups, so are only manifest after the (canonical) enrichment in spectra. Here however the statement about the heart should be understood in terms of the not enriched $\infty$-category.}

Remark 7.3. When $C$ is a presentable stable category, then if either $C^{\geq 0}$ or $C^{\leq 0}$ is presentable, then so is the other, and all truncation functors are colimit preserving [19, 1.4.4.13]. In this case, the subcategory of compact objects $C^c$ is stable under the truncation functors and inherits a $t$-structure.
Indeed, \( \tau \geq 0 \) is left adjoint to the corresponding inclusion, assumed colimit preserving, hence \( \tau \geq 0 \) preserves compact objects. Taking cones, so does \( \tau \leq 0 \).

We will study sheaves of \( t \)-structures on sheaves of categories.

**Definition 7.4.** Let \( M \) be a topological space and \( \mathcal{F} \) a sheaf of stable categories on \( M \). We say a pair of sheaves of full subcategories \( \mathcal{F} \leq 0 \) and \( \mathcal{F} \geq 0 \) define a \( t \)-structure on \( \mathcal{F} \) if \( \mathcal{F} \leq 0 (U) \) and \( \mathcal{F} \geq 0 (U) \) define a \( t \)-structure on \( \mathcal{F}(U) \) for all \( U \).

**Remark 7.5.** It is easy to see that the property that \( \mathcal{F} \leq 0 \) and \( \mathcal{F} \geq 0 \) define a \( t \)-structure may be checked on sections on any base of open sets. Indeed, regarding condition (i) and (ii) of Definition 7.1, it is immediate from the sheaf condition that vanishing of Homs and containment of subcategories can be checked locally.

Regarding (iii), the key point is that for any candidate \( t \)-structure satisfying (i) and (ii), the space of fiber sequences \( K' \to K \to K'' \) as requested in (iii) is either empty or contractible. Indeed, first recall that using property (ii) to apply (i) to shifts, we find that the following strengthening of (i): for any \( K' \in T \leq 0 \) and \( K'' \in T \geq 0 \), the negative exts, aka positive homotopy groups of the hom space \( \text{Hom}(K', K''[-1]) \), must vanish. Now given any \( K' \to K \to K''[-1] \) and \( L' \to K \to L''[-1] \) both satisfying (iii), we obtain a canonical null-homotopy of the composition \( K' \to K \to L''[-1] \) hence lift of \( K' \to K \) to \( K' \to L' \), etc.

Having learned this contractibility, if (iii) holds locally, then we can canonically glue the local exact triangles to obtain (iii) globally.

Since pullbacks commute with limits, \( \mathcal{F} \cap \mathcal{F} := \mathcal{F} \leq 0 \cap \mathcal{F} \geq 0 \) defines a sheaf of \((\infty, 1)\)-categories. As hearts of \( t \)-structures, these categories are abelian, in particular, 1-categories.

We recall a comparison between various notions of sheaves of categories. For covers \( U = \bigcup U_i \), an \( \infty \)-categorical sheaf of categories \( \mathcal{C} \) satisfies the condition that

\[
\mathcal{C}(U) = \lim \left( \prod_{i \in I} \mathcal{C}(U_i) \Rightarrow \prod_{i,j \in I} \mathcal{C}(U_i \cap U_j) \cong \prod_{i,j,k \in I} \mathcal{C}(U_i \cap U_j \cap U_k) \cdots \right)
\]

The corresponding definition of stacks in the classical literature instead involves a limit of 1-categories taken in the \((2,1)\)-category of ordinary categories. (Note that for limits in \((2,1)\) categories, the compatibility condition on triple overlaps is already strict, hence the terms after triple overlaps in the above diagram are irrelevant to the limit. In the literature they are correspondingly not included in the descent condition.

The situation is analogous to the fact that the descent condition for sheaves of sets is usually given truncated after the second term.)

These notions agree: 1-categories are 1-truncated objects of \((\infty, 1)\)-categories, and, for any category \( \mathcal{C} \), the inclusion of \( k \)-truncated objects \( \tau_{\leq k} \mathcal{C} \hookrightarrow \mathcal{C} \) is limit-preserving [18, Proposition 5.5.6.5]. We summarize:

**Lemma 7.6.** The heart of a \( t \)-structure on a sheaf of stable categories is a stack of abelian categories.

7.2. The perverse \( t \)-structure on constructible sheaves. We now review from [15, Sec. 10.3] the microlocal description of the perverse \( t \)-structure on constructible sheaves.

Let \( M \) be a complex manifold. For a Lagrangian subset \( \Lambda \subset T^* M \), we write \( \Lambda^\text{cl} \) for the locus of smooth points of \( \Lambda \) where the map \( \Lambda \to M \) has locally constant rank. For a point \( p \in \Lambda^\text{cl} \), we write \( r(p) \) for this rank. Fix a \( t \)-structure \( \mathcal{C} \leq 0, \mathcal{C} \geq 0 \) on our coefficient category \( \mathcal{C} \).

We write [15, Def. 10.3.7] using the notation of (10):
Definition 7.7. [15, Def. 10.3.7] Consider the following full subcategories of \( sh(M)_{\mathbb{C}^{-c}} \):

\[
\mu sh(M)^{\leq 0}_{\mathbb{C}^{-c}} := \{ F \in sh(M)_{\mathbb{C}^{-c}} | p \in ss(F)^{d} \implies \omega_{p}^{-1}F[-2r(p)] \in C^{\leq 0} \} \\
\mu sh(M)^{\geq 0}_{\mathbb{C}^{-c}} := \{ F \in sh(M)_{\mathbb{C}^{-c}} | p \in ss(F)^{d} \implies \omega_{p}^{-1}F[-2r(p)] \in C^{\geq 0} \}
\]

It is proved in [15, Thm. 10.3.12] that the above pair agrees with the usual definition of the perverse \( t \)-structure on \( M \), and so in particular is a \( t \)-structure.\(^5\)

Lemma 7.8. If \( \Lambda \subset T^{\ast}M \) is (possibly singular) subanalytic complex Lagrangian then Definition 7.7 induces a \( t \)-structure on \( sh_{\Lambda}(M) \). Moreover, \( sh_{\Lambda}(M)^{\triangledown} \) is closed under extensions inside \( sh(M) \).

\textbf{Proof.} We wish to apply Lemma 7.2. We should therefore show that the images of the truncation functors, when applied to objects \( F \in sh_{\Lambda}(M) \), remain in \( sh_{\Lambda}(M) \). Suppose not; consider any smooth Lagrangian point in \( ss(\tau_{\leq 0}F) \setminus ss(F) \). (There must be such a point by the assumption that the microsupport is complex analytic Lagrangian.) But by construction, the truncation functors commute with \( \omega \), whence it follows that \( \omega_{p}^{-1}\tau_{\leq 0}F[-2r(p)] = \tau_{\leq 0}\omega_{p}^{-1}F[-2r(p)] = 0 \). A contradiction. The conclusion now follows from Lemma 7.2.

Now suppose given \( F', F'' \in sh_{\Lambda}(M)^{\triangledown} \) and some extension \( 0 \to F' \to F \to F'' \to 0 \). The microstalk functors are \( t \)-exact by construction so we get a corresponding extension of microstalks inside \( C^{\triangledown} \), which is closed under extensions. All microstalks of \( F' \) lie in \( C^{\triangledown} \), so \( F' \in sh_{\Lambda}(M)^{\triangledown} \). The same argument also shows that \( sh_{\Lambda}(M)^{\triangledown} \) is closed under quotients and subobjects. \( \square \)

We now review the results of Waschkies [27]. Consider \( T^{\circ}M = T^{\ast}M \setminus M \), and the map \( \pi : T^{\circ}M \to \mathbb{P}^{\ast}M \).

Lemma 7.9. The natural map

\[
\mu sh^{\preceq}(\pi^{-1}(m)) \to (\mathbb{P}\mu sh)_{m}
\]

is an equivalence.

\textbf{Proof.} We first check fully-faithfulness. It is equivalent to check that the natural map

\[
\mu sh^{\preceq}(\pi^{-1}(m)) \to (\pi_{\ast}\mu sh)_{m}
\]

is fully-faithful. Given objects \( K, L \) in the source, Waschkies [27, Sec. 2.4] shows that \( \text{hom}(K, L) \) in the source is computed by \( (\pi_{\ast}\mu \text{Hom}(K, L))_{m} \), which also computes \( \text{hom}(K, L) \) in the target.

We now check essential surjectivity. If \( \bar{K} \in (\mathbb{P}\mu sh)_{m} \), then it is represented by an object \( \bar{K} \in \mathbb{P}\mu sh(U), p \in U \), which is representable by a sheaf at \( p \). This means that (after possibly shrinking \( U \)) there is a complex-contact embedding \( \chi : U \to V \subset \mathbb{P}^{\ast}N \) so that \( \chi_{\ast}(\bar{K}) = \chi_{\ast}(\bar{K}) \in (\pi_{\ast}\mu sh)_{\chi(m)} \) is in the image of \( (\pi_{\ast}\mu sh^{\preceq})_{\chi(m)} \to (\pi_{\ast}\mu sh)_{\chi(m)} \). By standard quantization of sheaf kernels (see e.g. [27, Thm. 4.2.1]) we have

\[
(\pi_{\ast}\mu sh^{\preceq})_{m} \xrightarrow{\simeq} (\pi_{\ast}\mu sh^{\preceq})_{\chi(m)} \xrightarrow{\simeq} (\pi_{\ast}\mu sh)_{\chi(m)} = (\mathbb{P}\mu sh)_{\chi(m)}.
\]

\(^5\)The result is stated in [15] for \( C \) the bounded derived category of modules over a ring and \( t \) the standard \( t \)-structure. However, the arguments given there (or in [3]) for the existence of the perverse \( t \)-structure depend only on the general properties of the six functor formalism, and the comparison between Definition 7.7 and the usual stalk/costalk wise definition of the perverse \( t \)-structure depends only on standard properties of microsupports.
where the vertical arrows are (19). The claim follows.

\[ \]
Theorem 7.16. \((\mathbb{P}\mu sh_{V,C-c,o})^\leq, (\mathbb{P}\mu sh_{V,C-c,o})^\geq, \) determine a t-structure on \(\mathbb{P}\mu sh_{V,C-c,o}.\)

Proof. Being a t-structure is an objectwise condition. So it suffices to show, for every fixed singular Lagrangian \(\Lambda,\) that objects supported in \(\Lambda\) satisfy the corresponding conditions; it suffices therefore to show \((\mathbb{P}\mu sh_{\Lambda,o})^\leq, (\mathbb{P}\mu sh_{\Lambda,o})^\geq\) define a t-structure on \(\mathbb{P}\mu sh_{\Lambda,o}.\) We may check locally. Thus we may assume \(\Lambda\) is an arbitrarily small and contractible neighborhood of some point \(p \in \Lambda.\)

By the Kashiwara–Kawai’s general position theorem (see [13, Sec. 1.6]), we may choose a complex manifold \(M\) and an embedding \(i : \Lambda \rightarrow PT^*M\) such that \(\Lambda \rightarrow M\) is finite, hence an embedding at generic points of \(\Lambda.\) Let \(\eta\) be the Maslov datum used on \(V;\) recall that the grading part is canonical. Writing \(\phi\) for the fiber Maslov datum on \(T^*M,\) we use the canonical identification of the ‘grading’ parts of \(\eta\) and \(i^*\phi,\) and any arbitrary homotopy to identify the orientation data parts (which certainly exists since \(\Lambda\) is contractible). Thus we have an identification \(i^*\mu sh_{(\Lambda),o} \cong \mu sh_{\Lambda,o}.\) Per Definition–Proposition 5.10 and Theorem 7.10, this identification intertwines the microstalk functors used to characterize the perverse t-structure on the left hand side with the functors we used in Definition 5.11, which therefore defines a t-structure as well. \(\square\)

We immediately deduce Theorem 1.4 from the introduction.

Proof of Theorem 1.4. As noted in Remark 6.4, \(\mathbb{C}^*\) acts on \(W \times \mathbb{C}\) by contactomorphism fixing \(W \times \{0\}\) set-wise. Hence the inclusion \((\gamma_V)_*\mu sh_{\mathbb{C}^*=\{0\}} \hookrightarrow (\gamma_V)_*\pi_*\mu sh_{\mathbb{C}^*}\) is stable under this action. Taking \(\mathbb{C}^*\)-invariants, we get a fully faithful inclusion \(\mu sh_{\mathbb{C}^*} = ((\gamma_V)_*\mu sh_{\mathbb{C}^*=\{0\}})^{\mathbb{C}^*} \hookrightarrow \mu sh_{\mathbb{C}^*}.\) The \(\mathbb{C}^*\)-action manifestly preserves the subspaces \(\mu sh_{\mathbb{C}^*} \cap \mu sh_{\mathbb{C}^*=0}\) and \(\mu sh_{\mathbb{C}^*} \cap \mu sh_{\mathbb{C}^*=0}\) so the t-structure passes to the invariants (taking hearts is a pullback \((F^\circ := F^\leq \cap F^\geq)\) hence a limit, and hence commutes with taking \(G\)-invariants which is also a limit). \(\square\)

As a consequence of Theorem 7.16 and Lemma 7.6, we conclude:

Corollary 7.17. \((\mathbb{P}\mu sh_{V,C-c,o})^\mathbb{C}^*\) is a stack of abelian categories; in particular, the global sections \((\mathbb{P}\mu sh_{V,C-c,o})^\mathbb{C}^* (V)\) is an abelian category.

Remark 7.18. A t-structure is said to be nondegenerate if \(\bigcap C^{\leq 0} = 0 = \bigcap C^{\geq 0}.\) By co-isotropicty of microsupport, the vanishing of all microstalks implies the vanishing of an object. We conclude that if the t-structure on \(C\) is non-degenerate, then the t-structure on \(\mathbb{P}\mu sh_{V,C-c,o}\) is also non-degenerate.

Remark 7.19. Choose a stable (not necessarily presentable) subcategory \(D \subset C\) to which the t-structure restricts. Require \((C, C^{\leq 0}, D)\)-orientation data. Then it is evident from the definitions that the truncation functors preserve, hence define a t-structure on, the subcategory of objects with all microstalks in \(D,\) characterized by the same formulas save only with e.g. \(C^{\leq 0}\) replaced by \(C^{\leq 0} \cap D.\) For instance, we can take various bounded categories e.g. \(D = C^+ = \bigcup C^{\geq n},\) or \(D = C^- = \bigcap C^{\leq n},\) or \(D = C^0 = C^+ \cap C^-\) and ask the microstalks to be compact objects \(D = C^c.\)

Remark 7.20. Fix \(D \subset C\) as above, and assume \(D^\mathbb{C}^*\) is Artinian. (E.g., \(C = k – \text{mod}\) for a field \(k\) and \(D = C^c.\) ) Then the full subcategory of \((\mathbb{P}\mu sh_{V,o})^{D^\mathbb{C}^*}\) on objects with finitely stratified (rather than just locally finite) support is also Artinian. Indeed, any descending chain must have eventually stabilizing microsupports; we may restrict attention to one each in the finitely many connected components of the smooth locus of the support, hence by some point, all have stabilized.
Remark 7.21. Let $V$ be a complex contact manifold and let $L \subset V$ be a complex Legendrian. Then $\mathbb{P}sh_{h_L}(-)$ is a sheaf of stable categories while $\mathbb{P}erv_{V,L}(-)$ is a sheaf of abelian categories. Be warned however that $D(\mathbb{P}erv_{V,L}(-))$ is only a presheaf of stable categories. In particular, the natural map $D(\mathbb{P}erv_{V,L}(-)) \to \mathbb{P}sh_{h_L}(-)$ may restrict to an equivalence on stalks without being an equivalence on global sections. A very special case: let $V = T^*S^2 \times \mathbb{C}$ and $L = 0_{S^2} \times \{0\}$. Then $\mathbb{P}sh_{h_L}(L) = \text{loc}(S^2) \otimes \text{loc}(\mathbb{C}^*)$ while $\mathbb{P}erv_{V,L}(L) = \text{vect}_\mathbb{C} \otimes \text{loc}(\mathbb{C}^*)$, due to $S^2$ being simply connected. Similarly, $\mu_*\text{sh}_{T^*S^2,0_{S^2}}(0_{S^2}) = \text{loc}(S^2)$ while $\mu_*\text{sh}_{T^*S^2,0_{S^2}}(0_{S^2})^\vee = \text{vect}_\mathbb{C}$.

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