Low-x contribution to the Bjorken sum rule within double logarithmic $ln^2x$ approximation

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Abstract

The small-$x$ contributions to the Bjorken sum rule within double logarithmic $ln^2x$ approximation for different input parametrisations $g_1^{NS}(x, Q_0^2)$ are presented. Analytical solutions of the evolution equations for full and truncated moments of the unintegrated structure function $f^{NS}(x, Q^2)$ are used. Theoretical predictions for $\int_0^{0.003} g_1^{NS}(x, Q^2 = 10)dx$ are compared with the SMC small-$x$ data. Rough estimation of the slope $\lambda$, controlling the small-$x$ behaviour of $g_1^{NS} \sim x^{-\lambda}$ from the SMC data is performed. Double logarithmic terms $\sim (\alpha_s ln^2x)^n$ become leading when $x \to 0$ and imply the singular behaviour of $g_1^{NS} \sim x^{-0.4}$. This seems to be confirmed by recent experimental SMC and HERMES data. Advantages of the unified $ln^2x$+LO DGLAP approach and the crucial role of the running coupling $\alpha_s = \alpha_s(Q^2/z)$ at low-$x$ are also discussed.

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1 Introduction

The results of SIDIS (semi inclusive deep inelastic scattering) experiments with polarised beams and targets enable the extraction of the spin dependent quark and gluon densities. This powerful tool of studying the internal spin structure of the nucleon allows verification of sum rules. One of them is the Bjorken sum rule (BSR) in, which refers to the first moment of the nonsinglet spin dependent structure function $g_1^{NS}(x, Q^2)$. Because of $SU_f(2)$
flavour symmetry, BSR is regarded as exact. Thus all of estimations of polarised parton distributions should be performed under the assumption that the BSR is valid. Determination of the sum rules requires knowledge of spin dependent structure functions over the entire region of \( x \in (0; 1) \). The experimentally accessible \( x \) range for the spin dependent DIS is however limited \((0.7 > x > 0.003\) for SMC data \([2]\) and therefore one should extrapolate results to \( x = 0 \) and \( x = 1 \). The extrapolation to \( x \to 0 \), where structure functions grow strongly, is much more important than the extrapolation to \( x \to 1 \), where structure functions vanish. Assuming that the BSR is valid, one can determinate from existing experimental data the very small-\( x \) contribution \((0.003 > x > 0)\) to the sum rule. Theoretical analysis of the small-\( x \) behaviour of \( g_1^{NS}(x, Q^2) = g_1^p(x, Q^2) - g_1^n(x, Q^2) \) together with the broad \( x \) range measurement data allow verification of the shape of the input parton distributions. In this way one can determinate the free parameters in these input distributions. Experimental data confirm the theoretical predictions of the singular small-\( x \) behaviour of the polarised structure functions. It is well known, that the low-\( x \) behaviour of both unpolarised and polarised structure functions is controlled by the double logarithmic terms \((\alpha_s \ln^2 x)^n \) \([3],[4]\). For the unpolarised case, this singular PQCD behaviour is however overridden by the leading Regge contribution \([5]\). Therefore, the double logarithmic approximation is very important particularly for the spin dependent structure function \( g_1 \). The resummation of the \( \ln^2 x \) terms at low \( x \) goes beyond the standard LO and NLO PQCD evolution of the parton densities. The nonsinglet polarised structure function \( g_1^{NS} \), governed by leading \( \alpha_s^2 \ln^{2n} x \) terms, is a convenient function both for theoretical analysis (because of its simplicity) and for the experimental BSR tests. The small-\( x \) behaviour of \( g_1^{NS} \) implied by double logarithmic approximation has a form \( x^{-\lambda} \) with \( \lambda \approx 0.4 \). This or similar small-\( x \) extrapolation of the spin dependent quark distributions have been assumed in recent input parametrisations e.g. in \([6],[7]\). In our theoretical analysis within \( \ln^2 x \) approach we estimate the very small-\( x \) contributions \( \int_0^{x_0} g_1^{NS}(x, Q^2) dx \) and \( \int_{x_1}^{x_2} g_1^{NS}(x, Q^2) dx \) \((x_0, x_1, x_2 \ll 1)\) to the BSR. Using analytical solutions for the full and the truncated moments of the unintegrated structure function \( f^{NS}(x, Q^2) \) \([3],[8]\) we find the contributions \( \int_{x_1}^{x_2} g_1^{NS}(x, Q^2) dx \) for different input quark parametrisations: the Regge nonsingular one and the singular ones. We compare our results with the suitable experimental SMC data for BSR. In the next section we recall some of the recent theoretical developments concerning the small-\( x \) behaviour of the nonsinglet polarised structure function \( g_1^{NS} \). Section 3 is devoted to the presentation of the double logarithmic \( \ln^2 x \) approximation, in which we calculate analytically the full and the truncated moments of the nonsinglet function \( f^{NS}(x, Q^2) \). Section 4 contains our results for the very small-\( x \) contributions
2 Small-\( x \) behaviour of the nonsinglet spin dependent structure function \( g_1^{NS}(x, Q^2) \)

The small value of the Bjorken parameter \( x \), specifying the longitudinal momentum fraction of a hadron carried by a parton, corresponds by definition to the Regge limit \( (x \to 0) \). Therefore the small-\( x \) behaviour of structure functions can be described using the Regge pole exchange model \[5\]. In this model the spin dependent nonsinglet structure function \( g_1^{NS} = g_1^p - g_1^n \) in the low-\( x \) region behave as:

\[
g_1^{NS}(x, Q^2) = \gamma(Q^2) x^{-\alpha_{A_1}(0)}
\]  

where \( \alpha_{A_1}(0) \) is the intercept of the \( A_1 \) Regge pole trajectory, corresponding to the axial vector meson and lies in the limits

\[
-0.5 \leq \alpha_{A_1}(0) \leq 0
\]  

This low value of the intercept \[12\] implies the nonsingular, flat behaviour of the \( g_1^{NS} \) function at small-\( x \). The nonperturbative contribution of the \( A_1 \) Regge pole is however overridden by the perturbative QCD contributions, particularly by resummation of double logarithmic terms \( \ln^2 x \). In this way the Regge behaviour of the spin dependent structure functions is unstable against the perturbative QCD expectations, which at low-\( x \) generate more singular \( x \) dependence than that implied by \[11\] - \[12\]. Nowadays it is well known that the small-\( x \) behaviour of the nonsinglet polarised structure function \( g_1^{NS} \) is governed by the double logarithmic terms i.e. \( (\alpha_s \ln^2 x)^n \) \[3\], \[4\]. Effects of these \( \ln^2 x \) approach go beyond the standard LO and even NLO \( Q^2 \) evolution of the spin dependent parton distributions and significantly modify the Regge pole model expectations for the structure functions. From the
recent theoretical analyses of the low-\( x \) behaviour of the \( g_1^{NS} \) function \([9]\) one can find that resummation of the double logarithmic terms \((\alpha_s \ln^2 x)^n\) leads to the singular form:

\[
g_1^{NS}(x, Q^2) \sim x^{-\lambda}
\]

with \( \lambda \approx 0.4 \). This behaviour of \( g_1^{NS} \) is well confirmed by experimental data, after a low-\( x \) extrapolation beyond the measured region \([2],[10],[11]\).

3 Full and truncated moments of the unintegrated structure function \( f^{NS}(x, Q^2) \) within double logarithmic approximation

Perturbative QCD predicts a strong increase of the structure function \( g_1^{NS}(x, Q^2) \) with the decreasing parameter \( x \) \([3],[4]\) what is confirmed by experimental data \([2],[10],[11]\). This growth is implied by resummation of \( \ln^2 x \) terms in the perturbative expansion. The double logarithmic effects come from the ladder diagram with quark and gluon exchanges along the chain. In this approximation the unintegrated nonsinglet structure function \( f^{NS}(x, Q^2) \) satisfies the following integral evolution equation \([3]\):

\[
f^{NS}(x, Q^2) = f_0^{NS}(x) + \alpha_s \int_x^1 \frac{dz}{z} \int_{Q_0^2}^{Q^2} \frac{d\kappa^2}{\kappa^2} f^{NS}(\frac{x}{z}, \kappa^2)
\]

where

\[
\bar{\alpha}_s = \frac{2\alpha_s}{3\pi}
\]

and \( f_0^{NS}(x) \) is a nonperturbative contribution which has a form:

\[
f_0^{NS}(x) = \alpha_s \int_x^1 \frac{dz}{z} g_1^{0NS}(z)
\]

The input parametrisation

\[
g_1^{0NS}(x) = g_1^{NS}(x, Q^2 = Q^2_0)
\]

in the Regge model of the low-lying trajectory \( A_1 \) exchange has a small-\( x \) behaviour:

\[
g_1^{0NS}(x) \sim x^0 \div x^{0.5}
\]
More singular shape of the input function $g_{1}^{0NS}$ results from the recent experimental data and PQCD analyses:

$$g_{1}^{0NS}(x) \sim x^{-\lambda}$$  \hspace{1cm} (9)$$

where $\lambda$ has changed during last years obtaining the values from $0.2 \div 0.3, 0.5$ \[3\] to recently $0.4$ \[9\]. In our calculations in the next section we use different inputs $g_{1}^{0NS}$: the flat one and the singular ones at low-$x$ as well.

The unintegrated distribution $f^{NS}(x, Q^2)$ is related to the $g^{NS}(x, Q^2)$ via

$$f^{NS}(x, Q^2) = \frac{\partial g^{NS}(x, Q^2)}{\partial \ln Q^2}$$  \hspace{1cm} (10)$$

Relation (11) implies the following equation for the truncated Mellin moment function $\tilde{f}^{NS}(x_0, n \neq 0, Q^2)$ \[8\]:

$$\tilde{f}^{NS}(x_0, n \neq 0, Q^2) = \tilde{f}_0^{NS}(x_0, n) + \frac{\bar{\alpha}_s}{n} \int \frac{k'^2}{k^2} \tilde{f}^{NS}(x_0, n, k'^2)$$

$$+ \int \frac{Q^2}{Q^2} \left( \frac{Q^2}{k'^2} \right)^n \tilde{f}^{NS}(x_0, n, k'^2) - x_0^n \int \frac{Q^2}{Q_0^2} \tilde{f}^{NS}(x_0, 0, k'^2)]$$  \hspace{1cm} (11)$$

where

$$\tilde{f}^{NS}(x_0, n, Q^2) \equiv \int \frac{1}{x_0^0} dx x^{n-1} f^{NS}(x, Q^2)$$  \hspace{1cm} (12)$$

For $x_0 = 0$ equation (11) reduces to that for the full Mellin moment

$$\tilde{f}^{NS}(0, n, Q^2) \equiv \int \frac{1}{0} dx x^{n-1} f^{NS}(x, Q^2)$$  \hspace{1cm} (13)$$

and in this case the analytical solution for fixed $\bar{\alpha}_s$ obtained in \[3\] has a form

$$\tilde{f}^{NS}(0, n, Q^2) = \tilde{f}_0^{NS}(0, n) \frac{n\gamma}{\bar{\alpha}_s} \left( \frac{Q^2}{Q_0^2} \right)^\gamma$$  \hspace{1cm} (14)$$

where

$$\gamma = \frac{n}{2} \left[ 1 - \sqrt{1 - \left( \frac{n_0}{n} \right)^2} \right]$$  \hspace{1cm} (15)$$

$$n_0 = 2\sqrt{\bar{\alpha}_s}$$  \hspace{1cm} (16)$$
and $\bar{f}_0^{NS}(0, n)$ is obviously equal to

$$\bar{f}_0^{NS}(x_0, n) = \int_{x_0}^{1} dx x^{n-1} f_0^{NS}(x)$$

(17)

at $x_0 = 0$. Using truncated moments approach one can avoid uncertainty from the unmeasurable $x \to 0$ region and also obtain important theoretical results incorporating perturbative QCD effects at small $x$, which could be verified experimentally. Truncated moments of parton distributions have been recently used in the LO and NLO DGLAP analysis [12]. In the double logarithmic approach the analytical solution of the evolution equation (11) for fixed coupling $\bar{\alpha}_s$ has a form [8]:

$$\bar{f}^{NS}(x_0, n \neq 0, Q^2) = \bar{f}_0^{NS}(x_0, n) \left( \frac{Q^2}{Q_0^2} \right) \gamma \frac{R}{1 + (R - 1)x_0^n}$$

(18)

where

$$R \equiv R(n, \bar{\alpha}_s) = \frac{n\gamma}{\bar{\alpha}_s}$$

(19)

$\gamma$ is given in (15) and $\bar{f}_0^{NS}(x_0, n)$ is the inhomogeneous term, independent on $Q^2$:

$$\bar{f}_0^{NS}(x_0, n) = \int_{x_0}^{1} dx x^{n-1} f_0^{NS}(x) = \frac{\bar{\alpha}_s}{n} \int_{x_0}^{1} \frac{dx}{x} (x^n - x_0^n) g_1^{0NS}(x)$$

(20)

Our purpose is to calculate the truncated moments of the nonsinglet polarised structure function $g_1^{NS}$

$$I(x_1, x_2, n, Q^2) \equiv \int_{x_1}^{x_2} dx x^{n-1} g_1^{NS}(x, k^2)$$

(21)

using different input parametrisations $g_0^{0NS}$. This will allow estimation of the small-$x$ contribution to the Bjorken sum rule and comparison of the results with suitable experimental data. Our predictions for $I(x_1, x_2, n, Q^2)$ will be presented in the forthcoming section.

4 Small-$x$ contribution to the BSR within double logarithmic $ln^2x$ approximation

Dealing with truncated moments $\int_{x_1}^{x_2} dx x^{n-1} g_1^{NS}(x, Q^2)$ one can avoid uncertainties from the experimentally unavailable regions $x \to 0$ and $x \to 1$. 
Particularly important is knowledge of the small-$x$ behaviour of the structure functions. In this limit $x \to 0$ $g_1^{NS}$ increases as $x^{-0.4}$ [9], what is confirmed by a low-$x$ extrapolation of experimental data [2],[10],[11]. This extrapolation of $g_1^{NS}$ ($g_1^p$ and $g_1^n$) beyond the measured region of $x$ is necessary to compute its first moment $\Gamma_1$ and test the BSR. The BSR is a fundamental rule and must be hold as a rigorous prediction of QCD in the limit of the infinite momentum transfer $Q^2$:

$$I_{BSR} \equiv \Gamma_1^p - \Gamma_1^n = \int_0^1 dx g_1^{NS}(x, Q^2) = \frac{1}{6} \left| \frac{g_A}{g_V} \right|$$  \quad (22)

where

$$\Gamma_1^p \equiv \int_0^1 dx g_1^p(x, Q^2)$$  \quad (23)

$$\Gamma_1^n \equiv \int_0^1 dx g_1^n(x, Q^2)$$  \quad (24)

and $\left| \frac{g_A}{g_V} \right|$ is the neutron $\beta$-decay constant

$$\left| \frac{g_A}{g_V} \right| = F + D = 1.2670$$  \quad (25)

Hence the BSR for the flavour symmetric sea quarks scenario ($\Delta \bar{u} = \Delta \bar{d}$) reads:

$$I_{BSR}(Q^2) \equiv \int_0^1 dx g_1^{NS}(x, Q^2) \approx 0.211$$  \quad (26)

The small-$x$ contribution to the BSR has a form:

$$\Delta I_{BSR}(x_1, x_2, Q^2) \equiv \int_{x_1}^{x_2} dx g_1^{NS}(x, Q^2)$$  \quad (27)

Taking into account (18)-(20) and also the relation

$$g_1^{NS}(x_0, n, Q^2) \equiv \int_{x_0}^{x} dx x^{-n-1} g_1^{NS}(x, k^2) = \int_{x_0}^{x} dx x^{-n-1} g_1^{0NS}(x)$$

$$+ \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2(1 + \frac{k^2}{Q^2})} f^{NS}(x_0, n, k^2)$$

$$+ \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2(1 + \frac{k^2}{Q^2})} f^{NS}(x_0, n, k^2)$$  \quad (28)
one obtains immediately:

$$\Delta I_{BSR}(x_1, x_2, Q^2) = I(x_1, 1, 1, Q^2) - I(x_2, 1, 1, Q^2)$$

(29)

where $I(x_i, x_j, n, k^2)$ defined in [21] in a case of $x_j = 1$ has a form:

$$I_1(x_i, 1, n, Q^2) \equiv \int_{x_i}^{1} dx x^{n-1} g_{1}^{NS}(x, k^2) = g_{1}^{0NS}(x_i, n) + B(x_i, n, Q^2)[g_{1}^{0NS}(x_i, n) - x_i^n g_{1}^{0NS}(x_i, 0)]$$

(30)

$B(x_i, n, Q^2)$ in the right-hand side of (30) is defined as

$$B(x_i, n, Q^2) = \frac{\gamma(Q^2)}{1 + (R - 1)x_i^n} \int_{\ln \frac{Q^2_0}{Q^2}}^{\ln \frac{Q^2}{Q^2_0}} dt \frac{e^{\gamma t}}{1 + e^{t}}$$

(31)

and

$$g_{1}^{0NS}(x_i, n) \equiv \int_{x_i}^{1} dx x^{n-1} g_{1}^{0NS}(x)$$

(32)

In Table I we present our results for the low-$x$ contributions to the BSR [27] together with $\varepsilon(x_1, x_2)$, which is defined by the following expression:

$$\int_{x_1}^{x_2} dx g_{1}^{NS}(x, Q^2) = [1 + \varepsilon(x_1, x_2)] \int_{x_1}^{x_2} dx g_{1}^{0NS}(x)$$

(33)

In the last column we give the percentage value $p[%]$:

$$p = \frac{\Delta I_{BSR}(x_1, x_2, Q^2)}{I_{BSR}(Q^2)} \cdot 100\%$$

(34)

The predictions have been found for four different input parametrisations $g_{1}^{0NS}(x)$, chosen at $Q^2_0 = 1$GeV$^2$ (1, 2, 3 inputs) or at $Q^2_0 = 4$GeV$^2$ (4 input):

1. $g_{1}^{0NS}(x) = 0.8447(1 - x)^3$
2. $g_{1}^{0NS}(x) = 0.290x^{-0.4}(1 - x)^{2.5}$
3. $g_{1}^{0NS}(x) = \frac{1}{6}x^{-0.544}[0.4949(1 - x)^{2.84}(1 + 9.6x^{1.23}) + 0.204(1 - x)^{3.77}(1 + 14.6x^{1.36})$
Table 1: The small-$x$ contribution to the BSR (27) for different input parametrisations (35)-(38) within $\ln^2 x$ approximation.

| Input | $x_1$ | $x_2$ | $\Delta I_{BSR}(x_1, x_2, 10)$ | $\varepsilon(x_1, x_2)$ | p% |
|-------------------------|-------|-------|-------------------------------|-------------------|---|
| 0                       | 0     | $3 \cdot 10^{-3}$ | (1) 0.003837 0.5211 1.82 | (2) 0.018078 0.2242 8.57 | (3) 0.021403 0.1860 10.14 | (4) 0.045266 0.0694 21.45 |
| 0                       | 0     | $10^{-2}$ | (1) 0.011682 0.4039 5.54 | (2) 0.036448 0.2064 17.27 | (3) 0.036998 0.1844 17.53 | (4) 0.058881 0.0737 27.91 |
| $10^{-5}$               | $10^{-3}$ | 0     | (1) 0.001344 0.6102 0.64 | (2) 0.008846 0.2337 4.19 | (3) 0.011391 0.1865 5.40 | (4) 0.021620 0.0686 10.25 |
| $10^{-4}$               | $10^{-2}$ | 0     | (1) 0.011539 0.4010 5.47 | (2) 0.034050 0.2037 16.14 | (3) 0.032451 0.1840 15.38 | (4) 0.035964 0.0790 17.04 |

4. $g_1^{0\text{NS}}(x) = \frac{1}{6}[0.1138x^{-0.803}(1-x)^{2.403}(1+21.34x) + 0.0392x^{-0.81}(1-x)^{3.24}(1+30.8x)]$ (38)

Input 1 is the simple Regge form, constance as $x \to 0$; input 2 is a "toy" model, in which we have used the latest theoretical results concerning the small-$x$ behaviour $x^{-0.4}$ of the nonsinglet function $g_1^{0\text{NS}}$. Finally, input 3 [13] and input 4 [14] are parametrisations obtained lately within experimental data analysis. They are quite different because there are nowadays still no direct experimental data from the low-$x$ region. The extrapolation of $g_1^{0\text{NS}}$ to very small-$x$ region depends strongly on the assumption (input parametrisation) made for this extrapolation. In Fig.1 we plot all used inputs $g_1^{0\text{NS}}(x)$ in the small-$x$ region $[10^{-4} \div 10^{-2}]$. In Fig.2 we present the small-$x$ contribution to the BSR $\Delta I_{BSR}(0, x, 10)$ as a function of $x$ and in Fig.3 the value of $\varepsilon(0, x)$ also as a function of $x$ is shown. Numbers at each plot correspond to the suitable inputs 1 $\div$ 4. For all calculated moments $Q^2 = 10\text{GeV}^2$ and $\alpha_s = 0.18$. From these results one can read that the low-$x$ contribution to the BSR strongly depends on the input parametrisation $g_1^{0\text{NS}}$. For the
Figure 1: Input parametrisations $g_{1}^{NS}$ (35)-(38) in the small-$x$ region.

flat Regge form $\Delta I_{BSR}(0, 10^{-2}, 10)$ is equal to around 5.5% of the total $I_{BSR} = 0.211$, while for the singular inputs (36)-(38) 18.8%, 17.5% and 27.9% respectively. The most singular parametrisation 4 (38), predicting $x^{-0.8}$ behaviour of $g_{1}^{NS}$ as $x \to 0$ gives of course the largest small-$x$ contribution to the BSR in comparison to the rest of the inputs. The value of $\varepsilon(x_1, x_2)$, defined in (33) varies from 0.4 $\div$ 0.7 for the Regge input 1 to 0.07 $\div$ 0.09 for the ”subsingular” input 4. Two similar as $x \to 0$ parametrisations 2 and 3 give $\varepsilon(x_1, x_2)$ about 0.2. It means that the double logarithmic $\ln^2 x$ effects are better visible in a case of nonsingular inputs. In a case of singular input parametrisations $g_{1}^{NS} \sim x^{-\lambda} \ (\lambda \sim 0.4, 0.8$ or so) the growth of $g_{1}^{NS}$ at small-$x$, implied by the $\ln^2 x$ terms resummation, is hidden behind the singular behaviour of $g_{1}^{NS}$, which survives the QCD evolution. From the experimental SMC data [10] the low-$x$ contribution to the BSR at $Q^2 = 10\text{GeV}^2$ is equal to

$$6 \int_{0}^{0.003} g_{1}^{NS}(x, Q^2 = 10)dx = 0.09 \pm 0.09 \quad (39)$$

The above result has been obtained via an extrapolation of $g_{1}^{NS}$ to the unmeasured region of $x$: $x \to 0$. Forms of the polarised quark distributions have been fitted to SMC semi-inclusive and inclusive asymmetries. In the fitting different parametrisations of the polarised quark distributions [15] [16] have been used. In this way present experimental data give only indirectly the estimation of the small-$x$ contribution to the moments of parton distributions. The result (39) with a large statistical error and strongly fit-dependent
cannot be a final, crucial value. Nevertheless we would like to estimate the exponent $\lambda$ in the low-$x$ behaviour of $g_{1}^{NS} \sim x^{-\lambda}$ using the above SMC result for the small-$x$ contribution to the BSR. Assuming the validity of the BSR (26) at large $Q^{2} = 10 GeV^{2}$, one can find:

$$\int_{0}^{x_{0}} dx g_{1}^{NS}(x, Q^{2}) = I_{BSR}(Q^{2}) - \int_{x_{0}}^{1} dx g_{1}^{NS}(x, Q^{2})$$  \hspace{1cm} (40)

where $x_{0}$ is a very small value of the Bjorken variable. Taking into account the small-$x$ dependence of $g_{1}^{NS} \sim x^{-\lambda}$ and the experimental data for $\Delta I_{BSR}(0, 0.003, 10)$ one can obtain:

$$C \int_{0}^{0.003} x^{-\lambda} dx = 0.015 \pm 0.015$$  \hspace{1cm} (41)

The constant $C$ can be eliminated from a low-$x$ SMC data [10]:

$$C x^{-\lambda} = g_{1}^{n-p}(x, 10)$$  \hspace{1cm} (42)

Taking different small-$x$ SMC data, we have found $\lambda = 0.37 \ (x = 0.014)$; $\lambda = 0.20 \ (x = 0.008)$; $\lambda = 0.38 \ (x = 0.005)$. Comparing our predictions for $\Delta I_{BSR}(0, 0.003, 10)$ in Table I with suitable SMC data, one can read that the most probably small-$x$ behaviour of $g_{1}^{NS}$ is

$$g_{1}^{NS}(x, Q^{2}) \sim x^{-0.4}$$  \hspace{1cm} (43)
This is consistent with latest theoretical analyses [9]. The same value of $\lambda = 0.4$ was obtained in the semi-phenomenological estimation from BSR for lower $Q^2$ [17]. Small-$x$ contribution to the Bjorken sum rule resulting from the indirect SMC data analysis is equal to around 7% of the total value of the sum. Similar value 8.6% (see Table I) gives our QCD approach, which incorporates the singular input parametrisation $g_1^{0NS} \sim x^{-0.4}$ according to the double logarithmic resummation effects. In the next point we will discuss further possible improvement of our approach.

5 More realistic treatment of the nonsinglet structure function $g_1^{NS}$: $ln^2 x +$LO DGLAP evolution and running $\alpha_s$ effects

In the previous section we have presented small-$x$ contribution to the Bjorken sum rule, calculated within double logarithmic $ln^2 x$ approximation. In our approach we have used a constance (nonrunning) $\alpha_s = 0.18$. This simplification allows the analytical analysis of the suitable evolution equations for truncated and full moments of the unintegrated structure function $f^{NS}(x, Q^2)$. It has been however lately proved [9], that dealing with a very small-$x$ region one should use a prescription for the running coupling in a form $\alpha_s = \alpha_s(Q^2/z)$. This parametrisation is theoretically more justified than
\( \alpha_s = \alpha_s(Q^2) \). Namely, the substitution \( \alpha_s = \alpha_s(Q^2) \) is valid only for hard QCD processes, when \( x \sim 1 \). However the evolution of DIS structure functions at small-\( x \) needs "more running" \( \alpha_s = \alpha_s(Q^2/z) \). Taking into account this running coupling constant effects we obtain the following modified equation for the \( f^{NS}(x,Q^2) \):

\[
f^{NS}(x,Q^2) = f_0^{NS}(x) + \int_x^1 \frac{dz}{z} \alpha_s \left( \frac{Q^2}{z} \right) \frac{Q^2/z}{Q_0^2} f^{NS} \left( \frac{x}{z}, k'^2 \right)
\]

This modification should lead to smaller values of \( \bar{f}^{NS}(x_0,n,Q^2) \) and hence \( g_1^{NS}(x_0,n,Q^2) \) than for the constance \( \alpha_s = 0.18 \). We will discuss this problem in the next paper. Besides, our present analysis is adequate only for a small-\( x \) region, where the main role play the double logarithmic \( \ln^2 x \) effects. In the situation, when the present experimental data do not cover the whole region of \( x \in (0;1) \), theoretical predictions for e.g. structure functions in the unmeasured low-\( x \) region cannot be directly verified. Thus theoretical analysis should concern both the small-\( x \) physics and the available experimentally larger-\( x \) area as well. Latest experimental SMC [2],[10] and HERMES [11] data provide results for the BSR from the region \( 0.003 \leq x \leq 0.7 \) and \( 0.023 \leq x \leq 0.6 \) respectively. In the very small-\( x \) region exist only indirect, extrapolated results with large uncertainties. Thus if one wants to compare theoretical predictions with suitable real measurements, one has to use an approach, which is proper for broader range of \( x \) and \( Q^2 \). The small-\( x \) behaviour of the nonsinglet spin dependent structure function \( g_1^{NS}(x,Q^2) \) is governed by the double logarithmic \( \ln^2 x \) terms. This approximation is however inaccurate for QCD analysis at larger values of \( x \). Therefore the double logarithmic approach should be completed by LO DGLAP \( Q^2 \) evolution. Unified description of the polarised structure function \( f^{NS}(x,Q^2) \) incorporating DGLAP evolution and the double logarithmic \( \ln^2 x \) effects at low-\( x \) leads to the following equation for the truncated moments \( \bar{f}^{NS}(x_0,n,Q^2) \):

\[
\bar{f}^{NS}(x_0,n \neq 0,Q^2) = \bar{f}_0^{NS}(x_0,n) + \frac{\alpha_s}{n} \left[ \int_{Q_0^2}^{Q^2} \frac{dk'^2}{k'^2} \bar{f}^{NS}(x_0,n,k'^2) \right] + \int_{Q^2}^{Q_0^2} \frac{dk'^2}{k'^2} \left( \frac{Q^2}{k'^2} \right)^n \bar{f}^{NS}(x_0,n,k'^2)
\]

\[
- x_0^n \int_{Q_0^2}^{Q^2} \frac{dk'^2}{k'^2} \bar{f}^{NS}(x_0,0,k'^2)]
\]

\[
+ \bar{\alpha}_s \int_{Q_0^2}^{Q^2} \frac{dk'^2}{k'^2} \sum_{p=0}^{M} C_{pm} \bar{f}^{NS}(x_0,n+p,k'^2)
\]
with fixed coupling constant $\alpha_s$. Matrix elements $C_{pn}$ are given by

$$C_{pn} = \delta_{p0} \left[ \frac{3}{2} + \frac{1}{n(n+1)} - 2S_1(n) \right]$$

$$+ \sum_{k=p}^{M} \frac{(-1)^p}{p!(k-p)!} \left[ \sum_{i=1}^{n+1} \frac{(i + k - 1)!}{i!} x_0^i \right] - \frac{(n + k - 1)!}{n!} \left( x_0^n + \frac{n + k}{n + 1} x_0^{n+1} \right)$$

(46)

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i}$$

(47)

The unified description $\ln^2 x + \text{LO DGLAP}$ of the nonsinglet spin dependent structure function $g_{1NS}$ enables to determine the small-$x$ contribution to the Bjorken sum rule via medium- and large-$x$ contribution of the unintegrated structure function $f^{NS}$. Because of the relation

$$\int_0^1 f^{NS}(x, Q^2) dx = 0$$

(48)

which is discussed e.g. in [18] and [19], one can find immediately

$$\int_{x_0}^{x_1} dx g_{1NS}^0(x, Q^2) = \int_{x_0}^{x_1} dx g_{1NS}^{0NS}(x) - \int_{Q_0^2}^{x_0} \frac{d\kappa^2}{\kappa^2(1 + \kappa^2/Q^2)} \int_{x_0}^{1} dx f^{NS}(x, \kappa^2)$$

(49)

In this way, the small-$x$ part $\int_{x_0}^{x_1} dx g_{1NS}^0(x, Q^2)$ can be expressed by the well known from experimental analysis the larger-$x$ contribution $\int_{x_0}^{x_1} dx f^{NS}(x, Q^2)$.

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Improved picture of PQCD behaviour of the truncated moments of $g_{1NS}$ incorporating the running coupling effects at small-$x$ and the unified $\ln^2 x + \text{LO DGLAP}$ approach will be a topic of our next paper.

\section{Summary and conclusions}

In this paper we have estimated the contribution from the small-$x$ region to the Bjorken sum rule. We have used the analytical solutions for the full and truncated moments of the nonsinglet polarised structure function $g_{1NS}^0(x, Q^2)$ within double logarithmic $\ln^2 x$ approximation. Our predictions $\Delta I_{BSR}(x_1, x_2, Q^2)$ have been found for different input parametrisations $g_{1NS}^{0NS}(x, Q_0^2)$. 
with fixed coupling constant $\alpha_s = 0.18$. These four parametrisations describe different small-$x$ behaviour of $g_1^{0NS} = g_1^{0(p-n)}$ at $Q_0^2$: $g_1^{NS} \sim x^{-\lambda}$. We found that the low-$x$ contribution to the BSR strongly depends on the input parametrisation $g_1^{0NS}$. The percentage value $\Delta I_{BSR}(0, 10^{-2}, Q^2 = 10)$ of the total BSR $\approx 0.211$ varies from 5.5 for the flat Regge input 1 ($\lambda = 0$) to almost 28 for the most singular input 4 ($\lambda = 0.8$). Input parametrisation 2 ($\lambda = 0.4$), which incorporates latest theoretical knowledge about small-$x$ behaviour of $g_1^{NS}$ driven by $\ln^2 x$ terms, gives this $\Delta I_{BSR}$ about 17% of the total BSR. Comparing our results with the experimental SMC data one can see good agreement, particularly for the input 2, of the small-$x$ contribution $0 \leq x \leq 0.003$ to the BSR. However it must be emphasized, that SMC data for the low-$x$ region suffer from large uncertainties. Using SMC data for $g_1^{NS}$ at small-$x$ $(0.14, 5\cdot 10^{-3}, 8\cdot 10^{-3})$ we have also estimated the exponent $\lambda$ which governs the low-$x$ behaviour of $g_1^{NS}$. Thus we have obtained $\lambda = 0.20 \div 0.38$ with large uncertainties. Latest theoretical investigations suggest singular small-$x$ shape of polarised structure functions: $\sim x^{-0.4}$ for the nonsinglet case and even $\sim x^{-0.8}$ for the singlet one. Both these values are indirectly confirmed by fitted experimental HERMES data. Basing on these results, similar extrapolations of the spin dependent quark distributions towards the very low-$x$ region have been assumed in several recent input parametrisations $\Delta q(x, Q^2)$. Resummation of the $\ln^2 x$ terms generates correctly the leading small-$x$ behaviour of the polarised structure function but is inaccurate for larger values of $x$. In order to have reliable theoretical predictions for the polarised structure functions e.g. $g_1^{NS}(x, Q^2)$ and to compare them with the real and the extrapolated recent experimental data, one should use unified theoretical description of the evolution of the structure functions. Such unified approach is the formalism, which contains the resummation of the $\ln^2 x$ and the LO DGLAP $Q^2$ evolution as well. Besides, in each realistic analysis of the $Q^2$ evolution of the structure functions one should take into account the running coupling, where $\alpha_s \rightarrow \alpha_s(Q^2)$. Latest theoretical investigations imply such introduction of the running coupling, where $\alpha_s \rightarrow \alpha_s(Q^2/z)$. This is more justified in the small-$x$ region. Combined $\ln^2 x +$LO DGLAP analysis together with taken into account the running coupling effects should give a correct description of the polarised structure function e.g. $g_1^{NS}$ in the whole region of $x$. It is very important because of lack of the experimental data from the very small-$x$ region ($x < 0.003$). Agreement of the theoretical predictions e.g. for the BSR with real experimental data at medium and large $x$ may give hope, that for the very interesting small-$x$ region the suitable theoretical results are also reliable.
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