On minimal extensions of rings and applications

Francisco Franco Munoz

Abstract

We study minimal extensions of local rings and their restriction to subrings. Some applications to subrings of \( \mathbb{K}[x]/x^n \) and \( \mathbb{Z}[x]/(p^N, x^n) \) are discussed.

1 Introduction

The study of subrings of a given ring is a natural question, but one that has not been given enough attention in the literature. In [1] (and the references therein) the authors study the subrings of naturally occurring rings, such as \( \mathbb{Z}^n \) and \( \mathbb{Z}[x]/x^n \).

The purpose of this paper is to shed some light on subrings of a local rings and they way those can be built from extensions (precisely, minimal extensions). After some formal preliminaries we obtain our main result (Theorem 6) that says we can relate subrings of \( R \) with those of \( S \) through a minimal extension \( \varphi : R \rightarrow S \) in a very precise way, leading to an exact count.

After establishing some basic result on (partial)-valuations, we are able to count the subrings of \( \mathbb{K}[x]/x^n \), and obtain (section 4) results giving precise estimates in the case of finite fields. At the end we briefly extend this discussion to subrings of \( \mathbb{Z}[x]/(p^N, x^n) \) (section 5) where similar counts are obtained.

1.1 Notation

All the rings considered are commutative and unital, and all the homomorphisms are unital. By a local ring \((R, m_R)\) we mean a ring with a unique maximal ideal \(m_R\), not necessarily noetherian (sometimes called quasi-local rings).

For a ring \( R \) denote by \( R^\times \) its group of units.

1.2 On previous literature

We have relied almost entirely on [3] (which is self-contained). Some basic results of commutative algebra are assumed and we’ll refer to [7] for more extended discussions.

2 Minimal extensions

Let \((R, m_R), (S, m_S)\) be local rings. We record here few well known elementary results about surjective homomorphisms of local rings.

Proposition 1. Let \( \varphi : R \rightarrow S \) be a surjective homomorphism of local rings. Then

1. \( m_R = \varphi^{-1}(m_S) \), in other words, \( \varphi \) is a local homomorphism.
2. \( \varphi(m_R) = m_S \).
3. \( \varphi : R \to S \) induces an isomorphism on residue fields \( R/m_R \cong S/m_S \).

Proof. 1. Since \( \varphi^{-1}(m_S) \) is a proper ideal it’s contained in the maximal ideal so \( \varphi^{-1}(m_S) \subseteq m_R \). So we need \( m_R \subseteq \varphi^{-1}(m_S) \) equivalently \( \varphi(m_R) \subseteq m_S \). Now, for a surjection of rings, the image of an ideal is an ideal. In the case of a local ring, \( \varphi(m_R) \) is in fact a proper ideal: For if \( \varphi(x) = 1 \) for some \( x \in m_R \), then \( 1-x \in \ker(\varphi) \subseteq m_R \) so \( 1 \in m_R \), a contradiction. Hence \( \varphi(m_R) \subseteq m_S \).

2. From \( \varphi^{-1}(m_S) \subseteq m_R \) get that \( m_S = \varphi(\varphi^{-1}(m_S)) \subseteq \varphi(m_R) \) since \( \varphi \) is surjective, and by above \( \varphi(m_R) \subseteq m_S \) hence equality: \( \varphi(m_R) = m_S \).

3. We have \( S/m_S \cong \varphi^{-1}(S)/\varphi^{-1}(m_S) = R/m_R \) by the above. \( \square \)

Definition 1. A homomorphism \( \varphi : R \to S \) is a minimal extension if it’s surjective (in particular it’s a local homomorphism) and \( I = \ker(\varphi) \) is a minimal nonzero ideal of \( R \).

Lemma 2. A minimal ideal \( I \) satisfies \( \operatorname{Im}_R = 0 \) and so \( I \) is a vector space over the residue field \( R/m_R \). Moreover its dimension is one. Conversely, suppose that \( J \) is an ideal of \( R \) such that \( Jm_R = 0 \) and the dimension of \( J \) over \( R/m_R \) is one. Then \( J \) is a minimal ideal.

Proof. \( \operatorname{Im}_R \subseteq m_R \) and by minimality, \( \operatorname{Im}_R = 0 \). The rest is clear. \( \square \)

We need restriction of homomorphisms:

Proposition 3. Let \( T \subseteq S \) a subring, and \( \varphi : R \to S \) be a surjection of local rings, let \( \overline{T} = \varphi^{-1}(T) \). Then \( \overline{T} \) local implies \( T \) local. Consider the following statements:

1. \( m_T = m_R \cap \overline{T} \)
2. \( m_T = m_S \cap T \)
3. The composition \( T \subseteq S \to S/m_S \) is surjective
4. The composition \( \overline{T} \subseteq R \to R/m_R \) is surjective

Then \( 1 \iff 2 \implies 3 \iff 4 \implies 2 \).

If \( \varphi : R \to S \) is a minimal extension then \( \overline{T} \) local iff \( T \) local. In that case, we have that \( 4 \iff (\varphi : \overline{T} \to T \) is a minimal extension)

Proof. It’s clear that \( \overline{T} \) local implies \( T \) local, since \( \varphi \) is surjective. Assuming that, Proposition III is valid for \( \varphi : R \to S \) and \( \varphi : \overline{T} \to T \).

- \( 1 \implies 2 \): \( m_T = \varphi(m_T) = \varphi(m_R \cap \overline{T}) \subseteq \varphi(m_R) \cap \varphi(\overline{T}) = m_S \cap T \) and the other inclusion \( m_S \cap T \subseteq m_T \) is immediate.
- \( 2 \implies 1 \): \( m_T = \varphi^{-1}(m_T) = \varphi^{-1}(m_S \cap T) = \varphi^{-1}(m_S) \cap \varphi^{-1}(T) = m_R \cap \overline{T} \).
- \( 3 \iff 4 \): \( T \to S \to S/m_S \text{ surjective} \iff T/(m_S \cap T) \cong S/m_S \iff \varphi^{-1}(T)/(\varphi^{-1}(m_S) \cap \varphi^{-1}(T)) \cong \varphi^{-1}(S)/\varphi^{-1}(m_S) \iff \overline{T}/(m_R \cap \overline{T}) \cong R/m_R \iff \overline{T} \to R \to R/m_R \text{ surjective} \)
- \( 3 \implies 2 \): \( T \to S \to S/m_S \text{ surjective} \iff T/(m_S \cap T) \cong S/m_S \implies m_S \cap T \text{ is a maximal ideal} \iff m_S \cap T = m_T \).
Assume that $\varphi$ is a minimal extension. To show $T$ local implies $\bar{T}$ local. In fact, suppose that $\varphi^{-1}(m_T) \subseteq n$ where $n$ is a proper ideal of $\bar{T}$. Then $m_T = \varphi(\varphi^{-1}(m_T)) \subseteq \varphi(n) \subseteq T$ since $\varphi$ is surjective. But also being surjective implies that $\varphi(n)$ is an ideal of $T$, and the latter is a local ring. If $\varphi(n)$ is not proper, then there's $x \in n$ such that $\varphi(x) = 1$, so $x - 1 \in \text{Ker}(\varphi)$. But $\text{Ker}(\varphi)$ is square zero since it's a minimal ideal and so $(x - 1)^2 = x^2 - 2x + 1 = 0$, so $1 = x(2 - x)$ and so $x \in n \subseteq \bar{T}$ is invertible, and so $n = \bar{T}$, contradiction. Hence $\varphi(n) \subseteq m_T$, i.e. $n \subseteq \varphi^{-1}(m_T)$, and $\varphi^{-1}(m_T)$ is the maximal ideal of $T$.

Assume now $\bar{T}$ is local. To show that $4 \iff (\varphi : T \to T$ is a minimal extension).

Assume that $\bar{T} \to T$ is minimal. That means $I = \text{Ker}(\varphi)$ is a minimal nonzero ideal of $\bar{T}$, and so $m_T I = 0$. Let $x \in m_T$. Then since $x I = 0$, $[x] I = 0$, where $[x]$ is the image of $x$ in $R/m_R$. But $I$ is a nonzero vector space (in fact one dimensional) over $R/m_R$, so $[x] = [0]$ i.e. $x \in m_R$, and so $m_T \subseteq m_R \cap T$, and the other inclusion holds since $m_R \cap T$ is an ideal of $\bar{T}$, hence $m_T = m_R \cap T$. Moreover, we have $T/ (m_R \cap T) \subseteq R/m_R$ is an extension of fields, and the vector space $I$ has dimension one over both (since $\varphi : R \to S$ and $\varphi : T \to T$ are minimal), and so the fields are equal $T/(m_R \cap T) \cong R/m_R$. Conversely, if we have $T \to R \to R/m_R$ surjective then $T/(m_R \cap T) \cong R/m_R$ and so $m_T = m_R \cap T$ and the residue fields are the same. Hence the ideal $I$ is one dimensional over $T/m_T$ and the extension $\varphi : T \to T$ is minimal by Lemma 2.

The use of this proposition will be in restricting local homomorphism to subrings and applying the next two theorems.

**Theorem 4.** Let $\varphi : R \to S$ be a minimal extension of local rings.

1. Assume $R$ and $S$ contain a coefficient field $K$ (the inclusions $K \subseteq S$, $K \subseteq R$ induce an isomorphism $R/m_R \cong S/m_S \cong K$). If $A \subseteq R$ is a $K$-subalgebra mapping onto $S$, then $A$ is local and $m_A^2 \subseteq A$.

2. Assume that characteristic of $R$ is $p^N$ for some $N \geq 1$, the residue field of $S$ and $R$ is $\mathbb{F}_q$ and that $m_R$ is nilpotent. If $A \subseteq R$ is a subring that maps onto $S$, then $A$ is local and $m_A^2 + pR \subseteq A$.

**Proof.**

1. To show that $A$ is local, notice that the composition $K \to A \to A/(m_R \cap A) \cong S/m_S = K$ is a bijection, so $A = K \oplus (m_R \cap A)$, which says that $A$ is local with maximal ideal $m_R \cap A = m_A$. And $K$ is also a coefficient field for $A$. Now, the ideal $I = \text{Ker}(\varphi)$ is one dimensional over $R/m_R = A/m_A$, and so $I \cap A$ is an ideal of $A$ that’s at most one dimensional over $A/m_A$. Hence either $I \subseteq A$ or $I \cap A = 0$. In the first case, since $R/I \cong S$ and $A$ maps onto $S$, we have $A = R$ which doesn’t map isomorphically onto $S$ since $I$ is nonzero. So, $I \cap A = 0$ and the map $\varphi : A \to S$ is an isomorphism. Now as vector spaces $R = m_R \oplus K$, and the same for $A = m_A \oplus K$, and since the dimension of $I$ is one, the codimension of $m_A \subseteq m_R$ is one, and one can write $m_R = I \oplus m_A$ as vector spaces. Since $Im_R = 0$, we get $m_A^2 = m_A^2 \subseteq A$, as claimed.

2. To show that $A$ is local, notice that $A/(m_R \cap A) \cong S/m_S = \mathbb{F}_q$ so $m_R \cap A$ is a maximal ideal. Now, let $a \in A \setminus m_R \cap A$, then $a$ maps to a nonzero element of $\mathbb{F}_q$, hence there’s $l$ such that $a^l - 1 \in K(\varphi) \subseteq m_R$, but $m_R$ is nilpotent so $(a^l - 1)^m = 0$ for some $m$ which after expanding the equation gives that $a$ is invertible and it’s inverse is in $A$. So $m_R \cap A = m_A$ is the unique maximal ideal of $A$, since its complement consists of invertible elements. Moreover $A$ also has residue field $\mathbb{F}_q$. Now, the ideal $I = \text{Ker}(\varphi)$ is one dimensional over $R/m_R = A/m_A$, and so $I \cap A$ is an ideal of $A$ that’s at most one dimensional over $A/m_A$. Hence either $I \subseteq A$ or $I \cap A = 0$. In the first case, since $R/I \cong S$ and $A$ maps onto $S$, we have $A = R$ which doesn’t map isomorphically onto $S$.
since I is nonzero. So, \( I \cap A = 0 \) and the map \( \varphi : A \to S \) is an isomorphism. In particular, \( A \) is a maximal subring with the same residue field as \( R \). Indeed, a subring \( A_1 \) containing \( A \) maps onto \( S \) and the same arguments above give either that if \( A_1 \neq R \), then \( A_1 \) maps isomorphically onto \( S \) and so \( A_1 = A \). Now, using \[3\] Lemma 22, we obtain that if \( A \) is a maximal subring with the same residue field as \( R \), \( A \) contains \( m_R^2 + pR \). This finishes the proof.

Let’s record a consequence in the proof of the above theorem:

**Proposition 5.** With the same conditions as above:

1. If \( A \subseteq R \) is a \( K \)-subalgebra mapping onto \( S \) not containing \( \ker(\varphi) \), then \( A \to S \) is an isomorphism and \( A \) is a local, maximal \( K \)-subalgebra of \( R \).

2. If \( A \subseteq R \) is a subring that maps onto \( S \) not containing \( \ker(\varphi) \) then \( A \to S \) is an isomorphism and \( A \) is a local, maximal subring of \( R \) with the same residue field as \( R \).

**Observation.** Throughout this paper we haven’t made use of structural results such as Cohen’s structure theorems (see [2] tag/0323) to simplify the hypothesis of the theorems. It’s worthwhile to notice that for example, complete local rings of equal characteristic possess coefficient fields, hence the hypothesis in our theorems hold for a wide class of local rings.

Here’s the main result that completes the analysis:

**Theorem 6.** Let \( \varphi : R \to S \) be a minimal extension of local rings. Then one can describe the subrings \( A \) mapping isomorphically to \( S \) in the same cases as above:

1. Assume \( R \) and \( S \) contain a coefficient field \( K \). If \( \ker(\varphi) \subseteq m_R^2 \), then there are no \( K \)-subalgebras of \( R \) mapping isomorphically onto \( S \) under \( \varphi \). Otherwise, the set of such \( K \)-subalgebras is naturally an affine space over \( K \) of dimension \( \dim_K(m_S/m_R^2) \).

2. Assume that the characteristic of \( S \) is \( p^N \) where \( N \geq 1 \) and that the residue field of \( S \) and \( R \) is \( \mathbb{F}_q \) and that \( m_R \) is nilpotent. If \( \ker(\varphi) \subseteq m_R^2 + pR \), then there are no subrings of \( R \) that map isomorphically onto \( S \). Otherwise, the set of such subrings is naturally an affine space over \( \mathbb{F}_q \) of dimension \( \dim_{\mathbb{F}_q}(m_S/(pS + m_R^2)) \).

**Proof.**

1. By Theorem [3] any \( A \) mapping onto \( S \) satisfies \( m_R^2 \subseteq A \).

   If \( \ker(\varphi) \subseteq m_R^2 \) then \( A \) contains \( \ker(\varphi) \) so \( A = R \), which is not the case. Assume now \( \ker(\varphi) \) not contained in \( m_R^2 \), and fix a nonzero \( z \in \ker(\varphi) \). Notice that \( z \) is a generator as vector space over \( R/m_R \). Define \( V(R) = m_R/m_R^2 \). To \( A \), subalgebra mapping isomorphically onto \( S \), let’s assign the subspace \( V(A) = m_A/m_A^2 \). It’s a \( K \)-codimension one subspace in \( V(R) \) that doesn’t contain \( z \) (the image of \( z \)). Conversely, for \( V \subseteq V(R) \) a \( K \)-codimension one subspace, assign the subspace \( A(V) = K + V + m^2 \), where \( V \) is a lift of \( V \) to \( m \). Notice that \( A \) is closed under multiplication since \( V \) contains \( m^2 \) and \( KV \subseteq V + m^2 \). This provides with a 1-to-1 correspondence between the codimension one subspaces of \( V(R) \) not containing \( z \) and the \( K \)-subalgebras \( A \) mapping isomorphically onto \( S \). Now, since \( m_R = m_A + \mathbb{K}_2 \), projection in the first component gives a bijection between \( \{ V \subseteq m_R/m_R^2 \text{ codimension } 1 \mid z \notin V \} \) with \( m_A/m_A^2 \cong m_B/m_B^2 \) which is the affine space sought after.

2. By Theorem [4] any \( A \) mapping onto \( S \) satisfies \( m_R^2 + pR \subseteq A \).

   If \( \ker(\varphi) \subseteq m_R^2 + pR \) then \( A \) contains \( \ker(\varphi) \) so \( A = R \), which is not the case. By the proposition above, \( A \) is a maximal subring with the residue field as \( R \) and \( A \) doesn’t contain \( \ker(\varphi) \). Now, by [3] Theorem 28, the maximal subrings \( A \) with the same residue field as \( R \) are in one to correspondence with the codimension one subspaces of
Proposition 7. Under the following conditions, given a minimal ring extension of local rings \( \phi : R \to S \) and a local subring, \( T \subseteq S \), Theorem 6 applies to the restriction \( \varphi^{-1}(T) \to T \):

1. All \( R, S, T \) have the same coefficient field \( K \).
2. The characteristic of \( R \) is \( p^N \) for some prime \( p \) and \( N \geq 1 \), \( m_R \) is nilpotent and all \( R, S, T \) share the same residue field, a finite field \( \mathbb{F}_q \).

**Proof.** By Proposition 3 the restriction \( \varphi^{-1}(T) \to T \) to a local subring \( T \) of a minimal extension is a minimal extension provided we have the composite map \( T \to S \to S/m_S \) surjective. But this is the condition we are assuming in either case. So we only need to check the conditions of Theorem 6.

1. Since \( K \subseteq T \), \( K \subseteq \varphi^{-1}(T) \), all the algebras involved have the same coefficient field \( K \), which are the conditions of Theorem 6 part 1.
2. Since \( T \) has residue field \( \mathbb{F}_q \), so does \( \varphi^{-1}(T) \). The other conditions of Theorem 6 part 2 are satisfied.

As a result, we can compute dimensions and relate them with the existence of subalgebras.

Proposition 8. Assume that we’re in the coefficient field case of \( \varphi : R \to S \) minimal extension and that \( d(R) := \dim_K(m_R/m_R^2) \) is finite. The following are equivalent:

1. \( R \) possesses a subalgebra \( A \) mapping isomorphically onto \( B \)
2. \( \mathrm{Ker}(\varphi) \) is not contained in \( m_R^2 \)
3. \( d(R) = d(S) + 1 \)

**Proof.**

1. \( \iff \) 2. Theorem 6 part 1.

2. \( \iff \) 3. The map \( m_R \to m_S \) gives an isomorphism \( m_R/(m_R^2 + \mathrm{Ker}(\varphi)) \cong m_S/m_S^2 \) and \( m_R/(m_R^2 + \mathrm{Ker}(\varphi)) \cong \frac{m_R/m_R^2}{(m_R^2 + \mathrm{Ker}(\varphi))/m_R^2} \). From here, since \( \dim_K(\mathrm{Ker}(\varphi)) = 1 \), it’s clear \( \mathrm{Ker}(\varphi) \not\subseteq m_R^2 \iff \dim_K((m_R^2 + \mathrm{Ker}(\varphi))/m_R^2) = 1 \iff d(R) = d(S) + 1 \).

### 3 Valuations

**Definition 2.** A (commutative) partial-monoid is a set \( (M, +) \) endowed with a partial binary (commutative) operation that has a neutral element 0 (i.e. \( a + 0 = a = 0 + a \)), and it’s associative when defined (i.e. \( a + b \) and \( (a + b) + c \) are defined iff \( b + c \) and \( a + (b + c) \) are defined, and if so, \( (a + b) + c = a + (b + c) \)). A sub-partial-monoid \( N \subseteq M \) is a subset such that if \( a, b \in N \) and \( a + b \) is defined, then \( a + b \in N \). From here on all partial-monoids are commutative.
Definition 3. An ordered partial-monoid is triple \((M, +, \leq)\) where \((M, +)\) is a partial-monoid and \(\leq\) is a partial order that’s compatible with the sum, i.e. given \(a \leq c\) and \(b \leq d\), if \(c + d\) is defined then \(a + b\) is defined and \(a + b \leq c + d\).

For us, examples of interest are \([m] = \{0, \ldots, m\}\) (finite interval of non-negative numbers) with (partial) addition, and products \(\mathcal{M}_{n,N} = [n - 1] \times [N - 1]\). Both are ordered partial-monoids, the first with natural order, the second with lexicographic order: \((a, b) \leq (c, d)\) iff \(a \leq c\) or \(a = c\) and \(b \leq d\). These are total orders.

Relevant to our study we need valuation-like functions, defined on commutative rings with values in partial-monoids.

Definition 4. Let \((M, +), (N, \ast)\) be partial-monoids. A partial-homomorphism is a (total) function \(\phi : M \to N\) (i.e. everywhere defined) with the properties:

1. \(\phi(0_M) = 0_N\)
2. For all \(x, y \in M\), if both \(x \ast y\) and \(\phi(x) \ast \phi(y)\) are defined, then \(\phi(x \ast y) = \phi(x) \ast \phi(y)\).
3. For all \(x, y \in M\), if \(\phi(x) \ast \phi(y)\) is defined, then \(x \ast y\) is defined and \(\phi(x \ast y) = \phi(x) \ast \phi(y)\).

Definition 5. Let \(R\) a commutative ring. A partial-valuation is partial-homomorphism \(\nu : R \setminus \{0\} \to M\) that is surjective, whose target is an ordered partial-monoid \((M, +)\) (where \((R \setminus \{0\}, \cdot)\) is the multiplicative partial-monoid of \(R\)), that satisfies:

**Non-Archimedean condition:** If \(x + y\) is defined (i.e. not zero), then \(\nu(x + y) \geq \min\{\nu(x), \nu(y)\}\) in the following sense: for any \(\mu \in M\) such that \(\mu \leq \nu(x)\) and \(\mu \leq \nu(y)\), we have \(\nu(x + y) \geq \mu\).

\(\nu\) is called semi-strict if the partial-homomorphism \(\nu : R \setminus \{0\} \to M\) is semi-strict.

\(\nu\) is called strict if it’s semi-strict, \(M\) is totally ordered and the equality \(\nu(x + y) = \min\{\nu(x), \nu(y)\}\) holds when \(\nu(x) \neq \nu(y)\).

Observation. 1. The Non-Archimedean condition is most easily stated when \(M\) possesses infima over any finite subset. Then \(\min\{\nu(x), \nu(y)\}\) is actually an element of \(M\) and the condition simply reads that \(\nu(x + y) \geq \min\{\nu(x), \nu(y)\}\)

2. When \(M\) is totally ordered and \(\nu\) is strict, then the Non-Archimedean condition is the familiar one from valuations on fields.

Proposition 9. Let \(\nu : R \setminus \{0\} \to M\) be a semi-strict partial-valuation. Then for a subring \(S \subseteq R\), the image \(\nu(S)\) is a sub-partial-monoid. In fact, for any sub-partial-monoid of the multiplicative partial-monoid \(T \subseteq R \setminus \{0\}\), \(\nu(T)\) is a sub-partial-monoid.

Proof. Immediate because of the extra condition.

Definition 6. A semi strict partial-valuation \(\nu : R \setminus \{0\} \to M\) is monomial-like over \(R_1\), a subring of \(R\), if it satisfies the following property: for \(x, y \in R \setminus \{0\}\) such that \(\nu(x) = \nu(y)\), there is \(u \in R_1^*\) such that either \(x - uy = 0\) or \(\nu(x - uy) > \nu(x)\).

Here’s an important structural result:

Theorem 10. Let \(\nu : R \setminus \{0\} \to M\) be a partial-valuation on \(R\), monomial-like over some subring \(R_1\), where \(M\) is a finite partial-monoid. Suppose that \(a_1, \ldots, a_d\) generate \(M\) as a partial-monoid (for every element \(a \in M\) there are constants \(a_1, \ldots, a_n \in \mathbb{N}\) such that the sum \(a_1a_1 + \cdots + a_da_d\) is defined and equal to \(a\)). Let \(r_1 \in R\) be elements whose valuations are \(\nu(r_1) = a_1\). Then \(r_1\) generate \(R\) as an algebra over the subring \(R_1\).
Proof. Let $a \in M$ be a maximal element ($M$ is finite), and let $r \in R$ such that $\nu(r) = a$. Then $a = a_1 a_1 + \ldots a_d a_d$, and so the “monomial” $\tilde{r} = r_1^a \ldots r_d^a$ is nonzero since $\nu$ is semi-strict and $\nu(\tilde{r}) = a$. There’s $u \in \mathcal{R}_l$ such that $r - u\tilde{r} \neq 0$ then $\nu(r - u\tilde{r}) > \nu(r)$ which is not possible. Hence $r = u\tilde{r}$. By a standard reverse induction argument the result follows, since we’re assuming $M$ is finite.

A natural example: take any field $\mathbb{K}$ and consider the $\mathbb{K}$-algebra $\mathbb{K}[x]/x^n$, where the partial-valuation is $\nu(a_i x^i + \text{higher order terms}) = i$, taking values in $[n - 1]$. It’s easily checked that this is a strict partial valuation. This valuation is monomial-like over the coefficient field $\mathbb{K}$.

Lemma 11. Let $a, b \in \mathbb{K}[x]/x^n$ with $\nu(a) = \nu(b)$, then there is a nonzero $u \in \mathbb{K}$ such that either $a = ub$ or $\nu(a - ub)$ has valuation strictly larger than $\nu(a)$.

Proof. Let $a = a_m x^m + \ldots, b = b_m x^m + \ldots$, with $a_m, b_m$ nonzero, then take $u = a_m b_m^{-1}$ and the result follows.

Lemma 12. Define the following function on $R = \mathbb{Z}[x]/(p^N, x^n)$: Write a nonzero element $x$ as a sum of powers in increasing order $x = a_k x^k + \text{higher order terms}$, where $a_k \in \mathbb{Z}/p^N$ is nonzero, then set $\nu(x) = (k, \nu_1(a_k))$ (where $\nu_1$ is the natural partial-valuation on $\mathbb{Z}/p^N$ given by $\nu_1(\mathcal{F}^m) = m$ where $\mathcal{F}$ invertible). Then $\nu : R \setminus \{0\} \to \mathcal{M}_{n, N}$ is a strict partial-valuation, which is monomial-like over the coefficient ring $\mathbb{Z}/p^N$.

Proof. Let $z = a_j x^j + \text{higher order terms}$, $w = b_k x^k + \text{higher order terms}$. Notice that $(j, \nu_1(a_j)) + (k, \nu_1(b_k))$ is defined if and only if $j + k < n$, and $\nu_1(a_j) + \nu_1(b_k) < N$. So if this is the case, and since $\nu_1$ is a strict partial valuation (with values in $[N - 1]$), we have $a_j b_k \neq 0$, and $\nu_1(a_j b_k) = \nu_1(a_j) + \nu_1(b_k)$, so $zw = a_j b_k x^{j+k} + \text{higher order terms}$, and $\nu(zw) = (j + k, \nu_1(a_j b_k)) = (j, \nu_1(a_j)) + (k, \nu_1(b_k)) = \nu(z) + \nu(w)$. This shows it’s semi-strict. To show it’s strict, notice that $\mathcal{M}_{n, N}$ is indeed totally ordered (with lexicographic order as described before), and moreover, when $\nu(z) \neq \nu(w)$, either $j \neq k$ or $\nu_1(a_j) \neq \nu_1(b_k)$.

1. Say that $j \neq k$ and without loss of generality, $j < k$, then $z + w = a_j x^k + \text{higher order terms}$, and so $\nu(z + w) = \nu(z) = \min\{\nu(z), \nu(w)\}$ since by the definition of lexicographic order here $(j, \*) < (k, \*)$ for any $\ast, \ast$ when $j < k$.

2. Say that $j = k$, and without loss of generality, $\nu_1(a_j) < \nu_1(b_j)$. Then $z + w = (a_j + b_j) x^k + \text{higher order terms}$, and $\nu(z + w) = (j, \nu_1(a_j + b_j)) = (j, \nu_1(a_j)) = \min\{\nu(z), \nu(w)\}$ since $\nu_1$ is a strict partial valuation and using again the definition of lexicographic order.

Finally notice by definition $\nu_1$ satisfies that $\nu_1(\alpha) = \nu_1(\beta)$, for $\alpha, \beta \in \mathbb{Z}/p^N$ implies there exist $\alpha \in (\mathbb{Z}/p^N)^* \times$ such that $\alpha = u\beta$. Hence if $z = ax^m + \text{higher order terms}$, $w = bx^m + \text{higher order terms}$, and $\nu(z) = \nu(w)$, one has using the $u$ before that $z - uw = (\alpha - u\beta)x^m + \text{higher order terms}$, has only powers higher than $m$, hence if nonzero, $\nu(z - uw) > \nu(z) = \nu(w)$.

4 Subalgebras of $\mathbb{K}[x]/x^n$

4.1 Setting

Let $\mathbb{K}$ be a field and consider the $\mathbb{K}$-algebra $\mathbb{K}[x]/x^n$. Of course this is the same as $\mathbb{K}[[x]]/x^n$ and this viewpoint will be more appropriate later.

Let $R \subseteq \mathbb{K}[x]/x^n$ be a $\mathbb{K}$-subalgebra. Notice that for the prime fields $\mathbb{Q}$, $\mathbb{F}_p$, $\mathbb{K}$-subalgebra is the same as a subring. All the linear maps, bases, and subalgebras are assumed to be $\mathbb{K}$-linear, unless otherwise specified.
Definition 7. For a nonzero polynomial \( r \in \mathbb{K}[x]/x^n \) one has a unique minimal \( i \) such that \( r = a_i x^i + \ldots \) higher order terms, with \( a_i \neq 0 \in \mathbb{K} \). Define \( \nu(r) = i \). This is the strict partial-valuation defined above \( r \).

Here \( i \) is called an exponent of \( R \) and we define \( E(R) \) as the set of exponents.

Observation. Define \( \nu(0) = \infty \) as a formal symbol and with the rule \( i + \infty = \infty \) for any \( i \in [0, n - 1] \) and natural order \( i \leq \infty \) for all \( i \).

Lemma 13. 1. The set \([0, n - 1] \cup \{\infty\}\) is an ordered monoid.

2. \( \nu : \mathbb{K}[x]/x^n \to [0, n - 1] \cup \{\infty\} \) is a homomorphism: \( \nu(1) = 0, \nu(r_1 r_2) = \nu(r_1) + \nu(r_2) \).

3. The Non-Archimedean property holds: for any two elements \( r_1, r_2, \nu(r_1 + r_2) \geq \min\{\nu(r_1), \nu(r_2)\} \)
and equality holds if \( \nu(r_1) \neq \nu(r_2) \).

Proof. Immediate.

Proposition 14. \( E(R) \) is partial-monoid.

Proof. By Proposition 9, it's the image of the multiplicative monoid \( \mathbb{K}[x]/x^n \) under a monoid homomorphism.

4.2 The set of exponents \( E(R) \) and generators

A subalgebra \( R \subseteq \mathbb{K}[x]/x^n \) lifts to a subalgebra \( \bar{R} \) of \( \mathbb{K}[[x]] \) of finite codimension.

Lemma 15. The finite codimension subalgebras of \( \mathbb{K}[[x]] \) are exactly those coming from \( \mathbb{K}[x]/x^n \).

Proof. By [3] Theorem 1, those finite codimension subalgebras correspond to subalgebras of finite dimensional quotients \( \mathbb{K}[[x]]/I \) where \( I \subseteq \mathbb{K}[[x]] \) is an finite codimension ideal. Now, \( \mathbb{K}[[x]] \) is a DVR and its nonzero ideals are \( x^n \mathbb{K}[[x]] \), hence the claim.

The discrete valuation on \( \mathbb{K}[[x]] \) is: \( \nu(a_i x^i + \ldots \) higher order terms) = \( i \), the same as before. An important property of this valuation is (with proof as in Lemma 11) that this is a monomial-like valuation over \( \mathbb{K} \).

Lemma 16. We have \( E(\bar{R}) = E(R) \cup \{n, n + 1, n + 2, \ldots\} \) hence \( E(\bar{R}) \) is a numerical monoid, i.e. a submonoid of \( (\mathbb{N}, +) \) with finite complement (It’s also true that such a monoid has a unique, finite, set of minimal generators [2]).

Proof. Immediate.

From now we’ll work with finite index \( \bar{R} \subseteq \mathbb{K}[[x]] \) in this way.

Proposition 17. \( \bar{R} \) contains \( x^n \mathbb{K}[[x]] \) iff \( E(\bar{R}) \) contains \( \{n, n + 1, \ldots\} \).

Proof. The converse needs to be checked only. Assume \( \{n, n + 1, \ldots\} \subseteq E(\bar{R}) \). Then there are \( r_k \in \bar{R} \) monic such that \( \nu(r_k) = k \), for \( k \geq m \). Write \( r_m = x^n + \alpha_{n+1} x^{n+1} + \alpha_{n+2} x^{n+2} + \ldots \). There are constants \( \beta_k \) such that the sequence \( s_t = r_n - \beta_{n+1} r_{n+1} - \ldots - \beta_{n+t} r_{n+t} \) has coefficients 0 for \( x^m \) for \( n < m < t \), hence \( s_t \) converges to \( x^m \). Since \( \bar{R} \) is complete (finite index in \( \mathbb{K}[[x]] \), \( x^m \in \bar{R} \).

Consider now the map \( \pi : \mathbb{K}[[x]] \to \mathbb{K}[[x]]/x^n \). Then for \( \bar{R} \subseteq \mathbb{K}[[x]] \) containing \( x^n \mathbb{K}[[x]] \), let \( R \subseteq \mathbb{K}[[x]]/x^n \) be its image.

Lemma 18. \( m_R/m_{\bar{R}} \cong m_R/(m^2_R + x^n \mathbb{K}[[x]]) \cong \frac{m_R^2/m_{\bar{R}}^2}{(m_R^2 + x^n \mathbb{K}[[x]])/m_{\bar{R}}^2} \)
Proof. Clear since $x^n \mathbb{K}[[x]] \subseteq \hat{R}$. ■

The following result connects generators of the monoid $E(\hat{R})$ and algebra generators of $\hat{R}$:

**Proposition 19.** Let $\{a_1, \ldots, a_d\}$ be the minimal generating set for $E(\hat{R})$. The following are equivalent:

1. Then the vector space $m_\hat{R}/m_\hat{R}^2$ has a basis $\{r_1, \ldots, r_d\}$ where $\nu(r_i) = a_i$.
2. For all $n$ such that $x^n \mathbb{K}[[x]] \subseteq \hat{R}$, let $E(n) = \{a_i \mid a_i < n\}$ be the a minimal generating set for $E(R = \hat{R}/x^n)$ as partial-monoid. Then $\{r_i \mid a_i \in E(n)\}$ is a basis for $m_\hat{R}/m_\hat{R}^2$.

Proof. 1. Let the vector space $m_\hat{R}/m_\hat{R}^2$ have basis $\{r_1, \ldots, r_d\}$ where $\nu(r_i) = a_i$. Then by Lemma 16, a basis for $m_\hat{R}/m_\hat{R}^2$ is obtained consisting of those $r_i$ such that $\nu(r_i) < n$, i.e. by $\{r_i \mid a_i \in E(n)\}$.

2. Conversely, taking $n$ equal to one plus the maximum of the $a_i$, we have $x^n \mathbb{K}[[x]] \subseteq m_\hat{R}^2$ and hence the isomorphism $m_\hat{R}/m_\hat{R}^2 \cong m_\hat{R}/m_\hat{R}^2$ gives the result. ■

Theorem [10] and its proof applied to $R = \hat{R}/x^n$ says:

**Lemma 20.** If $\{a_1, \ldots, a_d\}$ generate $E(R)$ as partial-monoid. Let $r_1, \ldots, r_d$ be monic such that $\nu(r_i) = a_i$. Then $r_i$ generate $R$ as algebra. Furthermore, if $n - 1 \in E(R)$ is not a generator, then $x^{n-1} \in m_\hat{R}^2$, more precisely, $x^{n-1}$ is a nontrivial (not just one factor) monomial $r_1^{m_1} \cdots r_d^{m_d} = x^{n-1}$ where $n - 1 = a_1a_1 + \ldots + a_2a_2$.

**Proposition 21.** The valuation maps are compatible under injection and projection: Let $R \subseteq \mathbb{K}[x]/x^n$ be a $\mathbb{K}$-subalgebra.

- $x^{n-1} \in R \iff n - 1 \in E(R)$
- Let $n \geq m$ and consider the projection map $\varphi : \mathbb{K}[x]/x^n \to \mathbb{K}[x]/x^m$ given by annihilating $x^m$. Then the valuation maps are identical where defined: for any nonzero $\bar{r} \in \varphi(R)$, $\nu(\bar{r}) = \nu(r)$ for any preimage $r$ of $\bar{r}$.
- In particular, if the projection gives an isomorphism $R \cong \varphi(R)$, the sets of exponents are identical.
- In general $E(\varphi(R)) = E(R) \cap [0, m - 1]$
- The cardinality of $E(R)$ is $\#E(R) = \dim_\mathbb{K}(R)$.

Proof. Immediate by definition. ■

**Proposition 22.** For $E \subseteq [N - 1]$ a sub-partial-monoid, let $d(E)$ be the cardinality of its (unique) minimal generating set. If $R \subseteq \mathbb{K}[x]/x^n$ has $E(R) = E$, then $\dim_\mathbb{K}(m_\hat{R}/m_\hat{R}^2) \leq d(E)$.

Proof. Combine Proposition [10] and Lemma [20]. ■

### 4.3 Failure of equality in $\dim_\mathbb{K}(m_\hat{R}/m_\hat{R}^2) \leq d(E)$

Proposition [22] says that $\dim_\mathbb{K}(m_\hat{R}/m_\hat{R}^2) \leq d(E)$ and that’s the most that one can assert. Here’s a family of examples for which the equalities don’t hold.

The algebra generated by $\{1, a = x^6 + x^9, b = x^7, c = x^8\}$ inside $\mathbb{K}[x]/x^{18}$ has elements the powers

- $a = x^6 + x^9$
- $b = x^7$
Further, dim Proposition 23. Let $a > 2$ assumption $a > 2$ than 2 assumption $a > 2$ so indeed 2 hence a linear basis is \{1, x^a + x^{a+3}, b = x^{a+1}, c = x^{a+2}\}. We have:

- $a = x^a + x^{a+3}$
- $b = x^{a+1}$
- $c = x^{a+2}$
- $a^2 = x^{2a} + 2x^{2a+3}$
- $ab = x^{2a+1} + x^{2a+4}$
- $b^2 = x^{2a+2}$
- $ac = x^{2a+2} + x^{2a+5}$
- $bc = x^{2a+3}$
- $c^2 = x^{2a+4}$

Hence a linear basis is \{1, x^6 + x^9, x^7, x^8, x^{12}, x^{13}, x^{14}, x^{15}, x^{16}, x^{17}\}, and $E = \{0, 6, 7, 8, 12, 13, 14, 15, 16, 17\}$ and the generators of $E$ are \{6, 7, 8, 17\} so 17 is a generator of $E$ while $x^{17} = ac - b^2$ which belongs to $m^2$. We have also $\dim_K(m_R/m_n^2) = 3 < 4 = d(E)$. This example \(a = 6\) can be generalized to an infinite family as follows:

**Proposition 23.** Let $a > 6$, and consider the following subalgebra of $K[x]/x^n$ where $n = 2a + 6$, generated by \{1, a = x^a + x^{a+3}, b = x^{a+1}, c = x^{a+2}\}. We have:

- $a = x^a + x^{a+3}$
- $b = x^{a+1}$
- $c = x^{a+2}$
- $a^2 = x^{2a} + 2x^{2a+3}$
- $ab = x^{2a+1} + x^{2a+4}$
- $b^2 = x^{2a+2}$
- $ac = x^{2a+2} + x^{2a+5}$
- $bc = x^{2a+3}$
- $c^2 = x^{2a+4}$

A linear basis is \{1, x^a + x^{a+3}, x^a + 1, x^a + 2, x^a + 2, x^a + 3, 2a + 4, 2a + 5\} and the generators of $E$ are \{a, a + 1, a + 2, 2a + 1, 2a + 2, 2a + 3, 2a + 4, 2a + 5\} so $2a + 5$ is a generator of $E$ while $x^{2a+5} = ac - b^2$ which belongs to $m^2$. Further, $\dim_K(m_R/m_n^2) = 3 < 4 = d(E)$.

**Proof.** The only need to check is the assertion regarding $E(R)$. But this follows from the assumption \(a > 6\) which guarantees that the sum of any three nonzero elements of $E$ is larger than $2a + 5$ (in fact, the minimum of the sum of any three nonzero elements is $3a > 2a + 5$) and the sum of two nonzero elements is an element of the set \{2a, 2a + 1, 2a + 2, 2a + 3, 2a + 4\}, so indeed $2a + 5$ is not the sum of two nonzero elements, so it’s a generator.

### 4.4 Subalgebras of given shape and counting

**Lemma 24.** The extension $\varphi : K[x]/x^{n+1} \to K[x]/x^n$ is minimal. Furthermore, for any $R$ a subalgebra of $K[x]/x^{n+1}$ containing $x^n$, the extension $R \to R/x^n$ is minimal.

**Proof.** In fact $x^n K[x]/x^{n+1}$ is the unique minimal ideal of $K[x]/x^{n+1}$. The second follows as well (by applying restrictions of minimal extensions, Proposition 22).

Here’s the main use of our results on minimal extensions:

**Theorem 25.** Let $R \subseteq K[x]/x^{n+1}$ subalgebra.

1. If $x^n \in m_R^2$, there are no subalgebras mapping isomorphically onto $\varphi(R) = S$. 

10
2. Otherwise, the set of such subalgebras is parametrized by an affine space of dimension $\dim_{K}(m_S/m_N^2)$.

Proof. This is Theorem 5 part 1, since all the subalgebras involved have coefficient field $K$. ■

Corollary 26. Let $K = \mathbb{F}_q$ a finite field of $q$ elements. Let $R \subseteq K[x]/x^{n+1}$ subalgebra, and $n \in E(R)$.

1. If $x^n \in m_R^2$, there are no subalgebras mapping isomorphically onto $\varphi(R) = S$.
2. Otherwise, the number of such subalgebras is $q^d(S)$, where $d(S) = \dim_{K}(m_S/m_N^2)$.

Proof. Immediate from Theorem 25 and Proposition 21 ($n \in E(R)$ iff $x^n \in R$). ■

Corollary 27. Suppose that $A \subseteq \mathbb{F}_q[x]/x^{n+1}$ and that $n \notin E(A)$. Then the number of subalgebras of $\mathbb{F}_q[x]/x^{n+1}$ mapping isomorphically onto $\varphi(A)$ is $q^d(A)$ where $d(A) \leq d(E(A))$.

Proof. Let $S = \varphi(A), R = \varphi^{-1}(A)$. Since $n \notin E(A)$, the map $A \to S$ is an isomorphism and we’re in situation 2 of Corollary 26 and so the number of such subalgebras is $d(S) = \dim_{\mathbb{F}_q}(m_S/m_N^2) = \dim_{\mathbb{F}_q}(m_A/m_N^2) = d(A)$ and $d(A) \leq d(E(A))$. ■

### 4.4.1 Shape

**Definition 8.** Let $E \subseteq [n-1]$ a sub-partial-monoid. The collection of subalgebras of $K[x]/x^n$ of shape $E$ is the set $S_n(E) = \{R \subseteq K[x]/x^n \mid E(R) = E\}$. This is a partition of the set of subalgebras of $K[x]/x^n$.

**Proposition 28.** Let $E \subseteq [n]$ a sub-partial-monoid, $R \subseteq K[x]/x^{n+1}$ subalgebra, $E(R) = E$.

1. If $n \in E$, the mapping $R \mapsto R/x^n$ induces a bijection of sets $S_{n+1}(E) \rightarrow S_n(E \setminus \{n\})$.
2. If $n \notin E$, the mapping $R \mapsto R/((x^n) \cap R) \cong R$, induces a mapping $S_{n+1}(E) \rightarrow S_n(E)$. Moreover, for $E_1 \neq E_2$ sub-partial-monoids of $[n]$ such that $n \notin E_1, n \notin E_2$, the images of $S_{n+1}(E_1)$ and $S_{n+1}(E_2)$ are disjoint, under the mapping just described.
3. If $n \notin E$, the mapping above $S_{n+1}(E) \rightarrow S_n(E)$ is surjective $\iff$ for all $B \in S_n(E)$ the ring $R = \varphi^{-1}(B)$ possesses a subring $A$ mapping isomorphically to $B \iff$ the kernel $\ker(R \mapsto B)$ doesn’t lie in $m_R^2 \iff d(R) = d(B) + 1$.

Proof. 1. By Proposition 21 the mapping is well defined, and so is the inverse mapping $B \mapsto \varphi^{-1}(B), B \subseteq K[x]/x^n$. It’s immediate to see that the composition in both ways yields the identity on the sets $S$.
2. By Proposition 21 for $n \notin E$, the the mapping $R \mapsto R/((x^n) \cap R) \cong R$ induces an equality of sets $E(R) = E(R/((x^n) \cap R))$, hence both claims follow.
3. The last two equivalences are the content of Proposition 8. For the first one, to have a surjective map $S_{n+1}(E) \rightarrow S_n(E)$ amounts to have that $R$ is not the only ring mapping to $B$, which by theory of minimal extensions (Theorem 6), is the same as saying $R$ has a subring mapping isomorphically onto $B$.

In the setting of a finite field $K = \mathbb{F}_q$, we can make an estimate of the number of subalgebras. For $E \subseteq [n-1]$ a sub-partial-monoid, let $e(E)$ be the following quantity recursively defined, with $d(E)$ being the minimal number of generators (so $d(\{0\}) = 0$)

$$e_n(E) = \begin{cases} 
0 & \text{if } n = 1 \\
e_{n-1}(E \setminus \{n-1\}) & \text{if } n - 1 \in E, n > 1 \\
d(E) + e_{n-1}(E) & \text{if } n - 1 \notin E, n > 1
\end{cases}$$
To show \( d \) then

We have two cases: \( d \)

Assume for \( n \). And consider \( R \subseteq \mathbb{F}_q[x]/x^{n+1} \). Then \( x^n \in R \) if \( n \in E(R) \).

- \( n \in E(R) \). By Proposition \( \ref{prop:1} \) part 1, \( \#(S_{n+1}(E)) = \#(S_n(E \setminus \{n\})) \) which is at most \( q^{n(E)(n)}\).
- \( n \notin E(R) \). By Proposition \( \ref{prop:1} \) part 2, there are only two sets \( F = F_1, F_2 \) such that \( S_{n+1}(F) \) maps to \( S_n(E) \), namely, \( F_1 = E, F_2 = E \cup \{n\} \). For each \( B \in S_n(E) \), the rings \( A \in S_{n+1}(E) \) mapping to \( B \) give isomorphisms \( A \cong B \), and every such \( A \) is contained in \( \varphi^{-1}(B) \), and by Corollary \( \ref{cor:2} \) there are \( q^{d(B)} \) elements, hence \( \#(S_{n+1}(E)) \leq q^{d(E)} \#(S_n(E)) \) which by induction is \( \leq q^{d(E)} q^{n(E)} = q^{d(E)+n(E)} \) by definition.

There’s a way to make these estimates more precise with a finer partition of the set of algebras, which won’t be pursued here. However, as a corollary of the proof, we have equalities in the following cases:

**Proposition 30.** For \( E \subseteq [n] \) sub-partial-monoid:

- If \( n \in E \), \( \#(S_{n+1}(E)) = \#(S_n(E \setminus \{n\})) \)
- If \( n \notin E \), \( \#(S_{n+1}(E)) \leq q^{d(E)} \#(S_n(E)) \) and furthermore, if for all \( B \in S_n(E) \), \( d(\varphi^{-1}(B)) = d(B) + 1 \), then \( \#(S_{n+1}(E)) = q^{d(E)} \#(S_n(E)) \)

**Proof.**
- Immediate.
- If \( n \notin E \), the above proof produces the inequality and so the only thing to check is the equality: \( \#(S_{n+1}(E)) = q^{d(E)} \#(S_n(E)) \) provided for all \( B \in S_n(E) \), \( d(B) = d(E) \).

Indeed, if that’s the case, as in the proof above, the fiber of the map for all \( S_{n+1}(E) \mapsto S_n(E) \) has exactly \( q^{d(B)} = q^{d(E)} \) elements. Furthermore, it’s surjective by Proposition \( \ref{prop:1} \) part 3. Hence the equality \( \#(S_{n+1}(E)) = q^{d(E)} \#(S_n(E)) \).

**Proposition 31.** Let \( B \subseteq K[x]/x^n \) subalgebra and \( R = \varphi^{-1}(B) \). If \( d(R) = d(E(R)) \), then \( d(B) = d(E(B)) \).

**Proof.** We have two cases: \( d(R) = d(B) \) or \( d(R) = d(B) + 1 \), and in both cases, from Proposition \( \ref{prop:1} \) \( E(R) = E(B) \cup \{n\} \).
- \( d(R) = d(B) \). We need to show that \( d(E(R)) = d(E(B)) \). It’s clear that \( d(E(B)) \leq d(E(R)) \) and \( d(E(R)) = d(R) = d(B) \leq d(E(B)) \) by Proposition \( \ref{prop:2} \) hence equality \( d(E(B)) = d(B) \).
- \( d(R) = d(B) + 1 \). We need to show that \( d(E(R)) = d(E(B)) + 1 \), which is equivalent to show that \( n \) is a generator of \( E(R) \). But if not, by Lemma \( \ref{lem:3} \) then \( x^n \) would belong to \( m_R^n \) which is not the case since the condition \( d(R) = d(B) + 1 \) as we have above, says that \( \text{Ker}(\varphi) = (x^n) \) is not in \( m_R^n \). This concludes the result.

**Proposition 32.** With the same setup as above, assume \( d(R) = d(B) + 1 \). If \( d(B) = d(E(B)) \) then \( d(E(R)) = d(R) \).

**Proof.** To show \( d(E(R)) = d(E(B)) + 1 \), and the proof is the same as above.
5 Subrings of $\mathbb{Z}[x]/(p^N, x^n)$

5.1 Setting

To study the subrings of $R = \mathbb{Z}[x]/(p^N, x^n)$ in the framework of minimal extensions we need a
consider a slightly larger family of rings. Most proofs in this section are analogous to those of
$K[x]/x^n$ and will be omitted for the most part.

Definition 9. Let $n \geq 2$, $1 \leq k \leq N$. $R_{n,N,k}$ is defined as the ring $R_{n,N,k} = \mathbb{Z}[x]/(p^N, x^n, p^k x^{n-1})$.

This family “interpolates” between family $\mathbb{Z}[x]/(p^N, x^n)$ in the sense that $R_{n,N,N} = \mathbb{Z}[x]/(p^N, x^n)$
and $R_{n,N,0} = \mathbb{Z}[x]/(p^N, x^{n-1}) = R_{n-1,N,N}$. Notice that the rings $R_{n,N,k}$ decrease as $k$ decreases
from $N$ to 0, more precisely:

Lemma 33. $R_{n,N,j}$ is a quotient of $R_{n,N,k}$ for $N \geq k \geq j$.

We can extend the definition of the partial-valuation of Lemma 12 to this entire family:

Definition 10. Let $R = R_{n,N,k}$. Write a nonzero element $x$ as a sum of powers in increasing
order $x = a_1 x^k +$ higher order terms, where $a_1 \in \mathbb{Z}/p^N$ is nonzero, let $\nu$ be the function
$\nu : R_{n,N,k} \to \mathbb{M}_{n,N}$ defined by $\nu(x) = (k, \nu_1(a_k))$ (where $\nu_1$ is the natural partial-valuation on
$\mathbb{Z}/p^N$ given by $\nu_1(up^n) = m$ where $u$ invertible). Here $(k, \nu_1(a_k))$ is called an exponent of $R$
and we define $D(R)$ as the set of exponents.

Lemma 34. 1. The set $\mathbb{M}_{n,N} = [n-1] \times [N-1]$ is a totally ordered monoid with the
lexicographic order.

2. $\nu$ is a strict partial valuation on $R_{n,N,k}$ with values in the monoid $\mathbb{M}_{n,N}$ (whose image is
the set where $(a,b) \in D(R)$ and $a = n$ then $b \leq k$).

3. The Non-Archimedean property holds: for any two elements $r_1, r_2$, $\nu(r_1+r_2) \geq \min\{\nu(r_1), \nu(r_2)\}$
and equality holds if $\nu(r_1) \neq \nu(r_2)$.

4. $D(R)$ is a partial-monoid.

5.2 The set of exponents $D(R)$ and generators

Let $R \subseteq R_{n,N,k}$ a subring. Theorem 10 and its proof give:

Lemma 35. If $\{a_1, ..., a_d\}$ generate $D(R)$ as partial-monoid. Let $r_1, ..., r_d$ be monic such that
$\nu(r_j) = a_j$. Then $r_j$ generate $R$ as algebra. Furthermore, if $(n-1, k-1) \in D(R)$ is not a
generator, then $p^{k-1} x^{n-1} \in m_R^2$, more precisely, $p^{k-1} x^{n-1}$ is a monomial (not just one factor)
and $p^{k-1} x^{n-1} = p^{k-1} x^{n-1}$ where $n-1 = a_1 + ... + a_d a_d$.

Proposition 36. The valuation maps are compatible under injection and projection: Let $R \subseteq
R_{n,N,k}$ be a subring:

- $p^{k-1} x^{n-1} \in R \iff (n-1, k-1) \in D(R)$

- Let $k = j + 1$ and consider the projection map $\varphi : R_{n,N,k} \to R_{n,N,j}$. Then the valuation
map is identical where defined: for any nonzero $\tilde{r} \in \varphi(R_{n,N,k})$, $\nu(\tilde{r}) = \nu(r)$ for any
preimage $r \varphi$.

- In particular, if the projection gives an isomorphism $R \cong \varphi(R)$, the sets of exponents are
identical.

- In general $D(\varphi(R)) = D(R) \cap \nu(R_{n,N,k})$

- The cardinality of $R$ is $\#(R) = p^{\#(D(R))}$.

Proposition 37. For $D \subseteq \mathbb{M}_{n,N}$ a sub-partial-monoid, let $d(D)$ be the cardinality of its
(unique) minimal generating set. If $R \subseteq R_{n,N,k}$ has $D(R) = D$, then $\dim_{\varphi}(m_R/(m_R^2+pR)) \leq
d(D) - 1$. 13
5.3 Subrings of given shape and counting

Lemma 38. The extension $\varphi : R_{n,N,k+1} \rightarrow R_{n,N,k}$ is minimal. Furthermore, for any subring $R \subseteq R_{n,N,k+1}$ containing $p^k x^{n-1}$, the extension $R \rightarrow R/(p^k x^{n-1})$ is minimal.

Proof. In fact $(p^k x^{n-1})$ is the unique minimal ideal of $R_{n,N,k+1}$. The second follows as well (applying restrictions of minimal extensions, Proposition 39).

Here’s the main use of our results on minimal extensions:

Theorem 39. Let $R \subseteq R_{n,N,k+1}$ subring.

1. If $p^k x^{n-1} \in m_R^2 + pR$, there are no subrings of $R_{n,N,k+1}$ mapping isomorphically onto $\varphi(R) = S$.

2. Otherwise, the set of such subrings is parametrized by an affine space over $\mathbb{F}_p$ of dimension $d(S) = \dim_{\mathbb{F}_p}(m_S/(m_S^2 + pS))$, hence there are $p^{d(S)}$ of them.

Proof. This is Theorem 38, part 2, since the subrings involved have characteristic $p^N$ and residue field $\mathbb{F}_p$.

Corollary 40. Suppose that $A \subseteq R_{n,N,k+1}$ and that $(n-1,k-1) \notin D(A)$. Then the number of subrings of $R_{n,N,k}$ mapping isomorphically onto $\varphi(A)$ is $p^{d(A)}$ where $d(A) \leq d(D(A))$.

Proof. Let $S = \varphi(A), R = \varphi^{-1}(A)$. Since $(n-1,k-1) \notin D(A)$, the map $A \rightarrow S$ is an isomorphism and applying Theorem 38 the number of such subrings is $p^{d(S)}$ where $d(S) = \dim_{\mathbb{F}_p}(m_S/(m_S^2 + pS)) = \dim_{\mathbb{F}_p}(m_A/(m_A^2 + pA)) = d(A)$ and $d(A) \leq d(D(A))$.

5.3.1 Shape

Definition 11. Let $D \subseteq \mathcal{M}_{n,N}$ a sub-partial-monoid. The collection of subrings of $R_{n,N,k}$ of shape $D$ is $\delta_{n,N,k}(D) = \{ R \subseteq R_{n,N,k} \mid D(R) = D \}$, a partition of the set of subrings of $R_{n,N,k}$.

We need a simple characterization of those $D$ that are the set of exponents of a subring.

Proposition 41. $D \subseteq \mathcal{M}_{n,N}$ is the set of exponents of a subring $R \subseteq R_{n,N,k}$ $\iff$ $D$ is a partial-sub-monoid of $\mathcal{M}_{n,N}$ containing $(p^i,0)$ for all $0 \leq i \leq N-1$.

Proposition 42. Let $D \subseteq \mathcal{M}_{n,N}$ a sub-partial-monoid, $R \subseteq R_{n,N,k+1}$ subring, $D(R) = D$.

1. If $(n-1,k) \in D$, the mapping $R \rightarrow R/(p^k x^{n-1})$ induces a bijection of sets $\delta_{n,N,k+1}(D) \rightarrow \delta_{n,N,k}(D \setminus \{(n-1,k)\})$.

2. If $(n-1,k) \notin D$, the mapping $R \rightarrow R/(p^k x^{n-1})/R \cong R$, induces a mapping $\delta_{n,N,k+1}(D) \rightarrow \delta_{n,N,k}(D)$. Moreover, for $D_1 \neq D_2$ sub-partial-monoids of $\mathcal{M}_{n,N}$ such that $(n-1,k) \notin D_1, (n-1,k) \notin D_2$, the images of $\delta_{n+1}(D_1)$ and $\delta_{n+1}(D_2)$ are disjoint, under the mapping just described.

3. If $(n-1,k) \notin D$, the mapping above $\delta_{n,N,k+1}(D) \rightarrow \delta_{n,N,k}(D)$ is surjective $\iff$ for all $B \in \delta_{n,N,k}(D)$ the ring $R = \varphi^{-1}(B)$ possesses a subring $A$ mapping isomorphically to $B$ $\iff$ the kernel $\ker(R \rightarrow B)$ doesn’t lie in $m_R^2 + pR$ $\iff$ $d(R) = d(B) + 1$.

Proof. 1. By Proposition 39 the mapping is well defined, and so is the inverse mapping $B \rightarrow \varphi^{-1}(B), B \subseteq R_{n,N,k}$. It’s immediate to see that the composition in both ways yields the identity on the sets $S$.

2. By Proposition 39 for $(n-1,k) \notin D$, the the mapping $R \rightarrow R/(p^k x^{n-1}) \cong R$ induces an equality of sets $D(R) = D(R/(p^k x^{n-1}) \cap R))$, hence both claims follow.
3. The last two equivalences follow in the same way as Proposition 28 follows from Proposition 8. For the first one, to have a surjective map \( \delta_{n,N,k}(D) \rightarrow \delta_{n,N,k}(D) \) amounts to have that \( R \) is not the only ring mapping to \( B \), which by theory of minimal extensions (Theorem 6), is the same as saying \( R \) has a subring mapping isomorphically onto \( B \). 

We can proceed to estimate the number of subrings. For \( D \subseteq M_{n,N} \) a sub-partial-monoid containing \((p^i,0)\) for all \( 0 \leq i \leq N - 1 \), let \( \epsilon(D) \) be the following quantity recursively defined, with \( d(D) \) being the minimal number of generators (so \( d(\{0\}) = 0 \))

\[
\epsilon_{n,N,k}(D) = \begin{cases} 
0 & \text{if } n = 1 \\
\epsilon_{n,N,k-1}(D \setminus \{(n-1,k-1)\}) & \text{if } (n-1,k-1) \in D, n > 1 \\
d(D) - 1 + \epsilon_{n,N,k-1}(D) & \text{if } (n-1,k-1) \notin D, n > 1
\end{cases}
\]

Here’s the main counting result:

**Theorem 43.** Let \( D \subseteq \mathcal{M}_{n,N} \). Then \( \#(\delta_{n,N,k}(D)) \) is at most \( p^{\epsilon_{n,N,k}(D)} \).

**Proposition 44.** For \( D \subseteq \mathcal{M}_{n,N} \) sub-partial-monoid:
- If \((n-1,k) \in D \), \( \#(\delta_{n,N,k+1}(D)) = \#(\delta_{n,N,k}(D \setminus \{(n-1,k)\})) \)
- If \((n-1,k) \notin D \), \( \#(\delta_{n,N,k+1}(D)) \leq p^{d(D)-1} \#(\delta_{n,N,k}(D)) \) and furthermore, if for all \( B \in \delta_{n,N,k}(D) \), \( d(\varphi^{-1}(B)) = d(B) + 1 \), then \( \#(\delta_{n,N,k+1}(D)) = p^{d(D)-1} \#(\delta_{n,N,k}(D)) \)

**Proposition 45.** Let \( B \subseteq R_{n,N,k} \) subalgebra and \( R = \varphi^{-1}(B) \). If \( d(R) = d(D(R)) - 1 \), then \( d(B) = d(D(B)) - 1 \).

**Proposition 46.** With the same setup as above, assume \( d(R) = d(B)+1 \). If \( d(B) = d(D(B)) - 1 \) then \( d(R) = d(D(R)) - 1 \).

**References**

[1] S. Atanasov, N. Kaplan, B. Krakoff, and J. Menzel, Counting finite index subrings of \( \mathbb{Z}^n \) and \( \mathbb{Z}[x]/(x^n) \), arxiv.org/abs/1609.06433

[2] M.F. Atiyah, I.G. Macdonald, *Introduction to Commutative Algebra*. Addison–Wesley, Reading, MA, 1969.

[3] Franco Munoz, Subrings of finite commutative rings.

[4] Ganske, G.; McDonald, B.R., Finite local rings. Rocky Mountain J. Math. 3 (1973), no. 4, 521-540.

[5] MacDonald B. R., Finite rings with identity, M. Dekker (1974).

[6] J. C. Rosales, J. C and P. A. García-Sánchez, P. A. Numerical semigroups. Developments in Mathematics, 20. Springer, New York, 2009

[7] The Stacks Project https://stacks.math.columbia.edu/

Francisco Franco Munoz, Department of Mathematics, University of Washington, Seattle, WA 98195

E-mail address: ffm1@uw.edu