Algebraic independence and normality of the values of Mahler’s functions

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Abstract

We prove new results on algebraic independence within Mahler’s method, including algebraic independence of values at transcendental points. We also give some new measures of algebraic independence. In particular, our results furnish for \( n \geq 1 \) arbitrarily large new examples of sets \((\theta_1, \ldots, \theta_n) \in \mathbb{R}^n\) normal in the sense of G. Chudnovsky (1980).

1 Introduction

Let \( p(z) = p_1(z)/p_2(z) \) be a rational function with coefficients in \( \overline{\mathbb{Q}} \). In this paper we consider systems of functions

\[
\begin{align*}
&f_1(z), \ldots, f_n(z) \in \overline{\mathbb{Q}}[[z]]
&
\end{align*}
\]

analytic in some neighborhood \( U \) of 0, algebraically independent over \( \mathbb{C}(z) \) and satisfying the following system of functional equations:

\[
\begin{align*}
a(z)f(z) &= A(z)f(p(z)) + B(z),
\end{align*}
\]

where \( f(z) = (f_1(z), \ldots, f_n(z)) \), \( a(z) \in \overline{\mathbb{Q}}[z] \), \( A \) (resp. \( B \)) is an \( n \times n \) (resp. \( n \times 1 \)) matrix with coefficients in \( \overline{\mathbb{Q}}[z] \) and we assume that the order of vanishing of \( p(z) \) at 0 is at least 2, that is \( \text{ord}_{z=0} p_1 - \text{ord}_{z=0} p_2 \geq 2 \). More precisely, for \( \gamma \in \mathbb{C} \) we study the quantity

\[
\text{tr.deg.} \mathbb{Q}(\gamma, f_1(\gamma), \ldots, f_n(\gamma)),
\]

that is how many algebraically independent numbers are there among \( \gamma, f_1(\gamma), \ldots, f_n(\gamma) \).

Moreover, apart from results on the transcendence degree (2), we provide their quantitative refinement. The fact that

\[
\text{tr.deg.} \mathbb{Q}(\gamma, f_1(\gamma), \ldots, f_n(\gamma)) \geq k
\]
means that the corresponding point in projective space

$$(1 : \gamma : f_1(\gamma) : \cdots : f_n(\gamma)) \in \mathbb{P}^{n+1}$$

(3)
does not belong to any subvariety of $\mathbb{P}^{n+1}$ defined over $\overline{\mathbb{Q}}$ and of dimension $\leq k - 1$. We provide a lower bound for the distance from the point (3) to any subvariety $W$ of $\mathbb{P}^{n+1}$ defined over $\overline{\mathbb{Q}}$ and of dimension $\leq k - 1$. Naturally, such a lower bound depends on the degree of $W$ and on its height (we refer the reader to Chapters 5 and 7 of [5] for the definition of the height and of degree of a projective variety; here we remark only that if a projective variety $W$ defined over $\mathbb{Q}$ has a codimension 1 and has no embedded components, that is if $W$ is a zero locus of a homogeneous polynomial $P$ in $n+2$ variables with integer coefficients, then the degree of $W$ is the degree of the polynomial $P$ and the height of $W$ is the Weyl’s height of $P$). In cases when (2) is equal to $n$, $\gamma \in \overline{\mathbb{Q}}$, and we consider only projective varieties of codimension 1 without embedded components, our lower bound specializes to the classical measure of algebraic independence of the numbers

$$\gamma, f_1(\gamma), \ldots, f_n(\gamma),$$

(4)
i.e. it can be interpreted as a lower bound for non-zero polynomials $Q$ in $n+1$ variables with integer coefficients:

$$|Q(\gamma, f_1(\gamma), \ldots, f_n(\gamma))| \geq \phi(\deg(q), h(Q)),$$

where $| \cdot |$ denotes the archimedean absolute value and $\phi : \mathbb{N} \times \mathbb{R}^+ \to \mathbb{R}^+$ is a function called itself the measure of algebraic independence of numbers (4).

It is common to call Mahler’s functions the functions satisfying (1), in the name of Kurt Mahler who initiated their study introducing Mahler’s method. Algebraic independence of values of such functions also was studied by Becker, Kubota, Nishioka, Töpf er and many others [1, 3, 6, 7, 8, 14]. To advance the results in this direction we use a general method developed in [11] (see also [10]). This method requires the so-called multiplicity estimate. Recently a new multiplicity lemma for solutions of (1) was established in [15, Theorem 3.11] and [17], see Theorem 25 below. We use the new multiplicity estimate with the general method from [11] to improve previously known results and establish new facts on algebraic independence and measures of algebraic independence. The statements of the main results from this article were previously announced in [16].

This paper is organized as follows. We start by presenting our results, as well as a number of their corollaries, in Section 2. Section 2 also contains, for illustrative purposes, a few of concrete examples on algebraic independence that can be inferred from our general statements. In Section 3 we provide a criterion for algebraic independence that is a central tool in the proofs of our main results, Theorems 1, 6 and 9. This criterion, similarly
to many other theorems of this kind, relies on the existence of a sequence of polynomials with nice approximation properties at the given point. Such polynomial sequences are constructed in Section 5. To this end we use a general extrapolative construction from \[11\] (see Theorem 26 below) presented in Section 4 together with a brief revision of \(K\)-functions, which are needed to state Theorem 26. Finally, the proofs of our principal results, Theorems 1, 6 and 9, are given in Section 6.

2 Results

Throughout the text we use the following notations. For the rational fraction \(p(z) = p_1(z)/p_2(z)\), where \(p_1(z), p_2(z) \in \mathbb{Q}[z]\), we denote \(d := \deg p = \max(\deg p_1, \deg p_2)\), \(\delta := \ord_{z=0} p = \ord_{z=0} p_1 - \ord_{z=0} p_2\). For \(h \in \mathbb{N}\), we will denote by \(p^{[h]}(y)\) the \(h\)-th iterate of \(p\) at a point \(y \in \mathbb{C}\), i.e. we define recursively \(p^{[0]}(y) := y\) and \(p^{[h+1]} := p(p^{[h]}(y))\).

Note that Theorems 1 and 6 below deals with the case of a polynomial \(p(z)\), whilst Theorem 9 treats the case when \(p(z)\) is a rational fraction.

Theorem 1 Let \(f_1(z), \ldots, f_n(z)\) be analytic functions as described above. Assume that \(p(z) \in \mathbb{Q}[z]\). Let \(y \in \mathbb{Q}^*\) be such that \(p^{[h]}(y) \to 0\) as \(h \to \infty\) and no iterate \(p^{[h]}(y)\) is a zero of \(z \det A(z)\).

Then for all \(\epsilon > 0\) there is a constant \(C > 0\) such that for any variety \(W \subset \mathbb{P}_\mathbb{Q}^n\) of dimension \(k < n + 1 - \frac{\log d}{\log \delta}\), one has

\[
\log \text{Dist}(x, W) \geq -C \left( h(W) + d(W)^{\frac{n+1-k+\epsilon}{n+1-k-\log d}} \right)^{\frac{n+1-k+\epsilon}{n+1-k-\log d}} \times d(W)^{\frac{\log d}{\log d} \frac{k+1}{n-k}},
\]

where \(x = (1 : f_1(y) : \cdots : f_n(y)) \in \mathbb{P}_\mathbb{C}^n\).

Remark 2 The definition of \(\text{Dist}(x, W)\) (for a point \(x \in \mathbb{P}^n\) and a subvariety \(W\) of the same space) can be found in \[7\], Chapter 6, § 5 or \[12\], § 1.3 (see \[12\] Definition 1.17 and discussion after it)). There are two simple cases which are important for understanding of this notion. First of all, if \(W\) is a hypersurface defined by a homogeneous polynomial \(P\), then \(\text{Dist}(x, W)\) is essentially \(\|P(x)\|\) (more precisely, in this case \(\log \text{Dist}(x, W) = \log \|P(x)\| - \deg(P) \cdot \log \|x\| - \log \|P\|\)). So essentially if \(k = n - 1\) is admitted in the statement of Theorem 1, one can substitute \(\log \|P(x)\|\) in place of \(\log \text{Dist}(x, W)\) in the left hand side of \((5)\). In this
particular case one obtains the estimate which is usually considered itself as a measure of algebraic independence.

On the other hand, for all points \( x \in \mathbb{P}^n \) and all subvarieties \( W \in \mathbb{P}^n \) one has \( \text{Dist}(x, W) = 0 \) iff \( x \in W \). So if some value of \( k \) is admitted in Theorem 4 (i.e. if \( k < n + 1 - \log \frac{d}{\log \delta} \)), then at least \( k + 1 \) of values \( f_1(y), \ldots, f_n(y) \) are algebraically independent over \( \mathbb{Q} \) (as the r.h.s. of (5) is \( > -\infty \)). Using this fact we readily deduce the following two corollaries.

**Corollary 3** Assuming the conditions of Theorem 4 one has

\[
\text{tr.deg.}_{\mathbb{Q}} \mathbb{Q} (f_1(y), \ldots, f_n(y)) \geq n + 1 - \left\lceil \frac{\log d}{\log \delta} \right\rceil.
\]

**Corollary 4** Assuming the conditions of Theorem 4 and \( \frac{\log d}{\log \delta} < 2 \) one has

\[
\text{tr.deg.}_{\mathbb{Q}} \mathbb{Q} (f_1(y), \ldots, f_n(y)) = n.
\]

**Remark 5** Corollary 3 improves the lower bound

\[
\text{tr.deg.}_{\mathbb{Q}} \mathbb{Q} (f_1(y), \ldots, f_n(y)) \geq \lceil (n + 1) \frac{\log \delta}{\log d} \rceil - 1,
\]

established by Theorem 3 in [14], where \( \lceil \ast \rceil \) denotes the smallest integer bigger than \( \ast \).

**Corollary 4** improves Corollary 2 of [14], which gives only the case \( n = 1 \) of (6).

We can also give a measure of algebraic independence of values \( y, f_1(y), \ldots, f_n(y) \), for arbitrary \( y \in \mathbb{C}^* \). This type of results for transcendental \( y \) has not been considered before, though our estimates in this situation are weaker than in the case of algebraic \( y \).

**Theorem 6** Let \( f_1(z), \ldots, f_n(z) \) be analytic functions as described in the beginning of this paper. Assume that \( p(z) \in \mathbb{Q}[z] \) with \( \delta = \text{ord}_{z=0} p(z) \geq 2 \) and \( d = \text{deg} p(z) \). Let \( y \in \mathbb{C}^* \) be such that

\[
p^{[h]}(y) \to 0 \text{ as } h \to \infty
\]

and no iterate \( p^{[h]}(y) \neq 0 \) is a zero of \( z \det A(z) \).

Then for all \( \varepsilon > 0 \) there is a constant \( C \) such that for any variety \( W \subset \mathbb{P}^{n+1}_\mathbb{Q} \) of dimension \( k < n + 1 - 2\frac{\log d}{\log \delta} \), one has

\[
\log \text{Dist}(x, W) \geq -C \left( h(W) + d(W) \frac{n+1-k-\frac{\log d}{\log \delta}}{n+1-k-2\frac{\log d}{\log \delta}} \right) \times d(W) \frac{\log d}{\log \delta} \frac{n+1}{n-k},
\]

(7)
where $x = (1 : y : f_1(y) : \cdots : f_n(y)) \in \mathbb{P}_C^{n+1}$.

As before, one readily deduces two corollaries:

**Corollary 7** Assuming the conditions of Theorem 6 one has

$$\text{tr.deg}_{\mathbb{Q}} \mathbb{Q}(y, f_1(y), \ldots, f_n(y)) \geq n + 1 - \left[ \frac{\log d}{\log \delta} \right].$$

**Corollary 8** Assuming the conditions of Theorem 6 and $\frac{\log d}{\log \delta} < 3/2$ one has

$$\text{tr.deg}_{\mathbb{Q}} \mathbb{Q}(y, f_1(y), \ldots, f_n(y)) \geq n - 1.$$  \hspace{1cm} (8)

The next theorem improves Theorems 1 and 2 of [14], qualitatively and quantitatively.

**Theorem 9** Let $f_1(z), \ldots, f_n(z)$ be a collection of functions such as described in the beginning of this paper. In this statement we admit any $p(z) \in \mathbb{Q}(z)$ in the system (11) and not only $p(z) \in \mathbb{Q}[z]$ as in preceding Theorems 1 and 2. We keep the notation $d = \deg p$, $\delta = \text{ord}_{z=0} p \geq 2$. Assume that $f_i(0) = 0$, $i = 1, \ldots, n$, a number $y \in U \cap \mathbb{Q}$ satisfies $\lim_{m \to \infty} p[m](y) = 0$ and for all $m \in \mathbb{N}$ a number $p[m](y) \neq 0$ is not a zero neither $\det A(z)$ nor $a(z)$. Then there is a constant $C > 0$ such that for any variety $W \subset \mathbb{P}_Q^n$ of dimension $k < 2n + 1 - \frac{\log d}{\log \delta}(n+1)$, one has

$$\log \text{Dist}(x, W) \geq -C \left( h(W) + d(W) \frac{1}{\frac{\log d}{\log \delta} \cdot n - k + 1} \int \frac{1}{\frac{\log d}{\log \delta} \cdot n - k + 1} \right) \times d(W)^{\frac{k+1}{n-k}},$$  \hspace{1cm} (9)

where $x = (1 : f_1(y) : \cdots : f_n(y)) \in \mathbb{P}_C^n$. In particular,

$$\text{tr.deg}_{\mathbb{Q}} \mathbb{Q}(f_1(y), \ldots, f_n(y)) \geq 2n + 1 - \frac{\log d}{\log \delta}(n+1).$$  \hspace{1cm} (10)

Now we give a family of concrete examples, with sets of functions arbitrarily long and satisfying all the hypothesis of our theorems. We start with a particular case of (11) when this system has a simple diagonal form:

$$\chi_i(z) = \chi_i(p(z)) + q_i(z), \hspace{0.5cm} i = 1, \ldots, n,$$  \hspace{1cm} (11)

where $p \in \mathbb{Q}(z)$ and $q_i \in \mathbb{Q}[z]$, $i = 1, \ldots, n$. Assuming $\deg q_i \geq 1$ and $q_i(0) = 0$, $i = 1, \ldots, n$, $\text{ord}_{z=0} p \geq 2$ we obtain solutions of (11) analytic in some neighborhood of 0:

$$\chi_i(z) = \chi_i(p(z)) + q_i(z), \hspace{0.5cm} i = 1, \ldots, n.$$  \hspace{1cm} (12)
Lemma [10] below allows to verify easily the algebraic independence of \( \chi_1, \ldots, \chi_n \) over \( \mathbb{C}(z) \). It is an easy corollary of Lemma 6, [14] (as well as of Theorem 2, [4]).

**Lemma 10** Let \( n \in \mathbb{N}^* \), \( q_i \in \mathbb{C}[z] \), \( i = 1, \ldots, n \) and \( p \in \mathbb{C}[z] \) satisfying \( q_i(0) = 0, i = 1, \ldots, n \), \( p(0) = 0 \) and \( p(z) \neq z \). Let \( \chi_1, \ldots, \chi_n \in \mathbb{C}(z) \) be functions defined by (12). Suppose that \( 1, q_1, \ldots, q_n \) are \( \mathbb{C} \)-linearly independent and at least one of the following conditions is satisfied:

1. \( \deg p \nmid \deg (\sum_{i=1}^n s_i q_i(z)) \) for all \( (s_1, \ldots, s_n) \in \mathbb{C}^n \setminus \{0\} \).
2. \( \sum_{i=1}^n s_i \chi_i(z) \notin \mathbb{C}[z] \) for all \( (s_1, \ldots, s_n) \in \mathbb{C}^n \setminus \{0\} \).

Then the functions \( \chi_1, \ldots, \chi_n \) are algebraically independent over \( \mathbb{C}(z) \).

Using this lemma (especially point (11) which is due to Th.Töpfer) we can produce a large family of algebraically independent sets of functions (12). All these sets satisfy the hypothesis imposed on functions \( f_1, \ldots, f_n \) considered in this article, so we can apply Theorems 11, 10, [9] and their corollaries to them.

**Theorem 11** Let \( p \in \overline{\mathbb{Q}}(z) \), \( d = \deg p \) and \( \delta = \text{ord}_{z=0} p \geq 2 \). Let \( n \in \mathbb{N}^* \), \( q_i \in \overline{\mathbb{Q}}[z] \) \( (i = 1, \ldots, n) \) and let \( \chi_i(z) \) be functions defined by (11). Assume that \( 1, q_1, \ldots, q_n \) are \( \mathbb{C} \)-linearly independent and that at least one of the following conditions is satisfied:

1. \( \deg p \nmid \deg (\sum_{i=1}^n s_i q_i(z)) \) for all \( (s_1, \ldots, s_n) \in \mathbb{C}^n \setminus \{0\} \).
2. \( \sum_{i=1}^n s_i \chi_i(z) \notin \mathbb{C}[z] \) for all \( (s_1, \ldots, s_n) \in \mathbb{C}^n \setminus \{0\} \).

Fix a \( y \in \overline{\mathbb{Q}} \) such that \( p^{[h]}(y) \to 0 \) and \( p^{[h]}(y) \neq 0 \) for all \( h \in \mathbb{N} \). Then there exists a constant \( C > 0 \) such that for every variety \( W \subset \mathbb{P}^n_{\mathbb{Q}} \) of dimension \( k < 2n + 1 - \frac{\log d}{\log \delta}(n + 1) \), one has the measure of algebraic independence [19] at \( x = (1 : \chi_1(y) : \cdots : \chi_n(y)) \in \mathbb{P}^n_{\mathbb{C}} \).

In particular,

\[
\text{tr.deg.}_{\mathbb{Q}}(\chi_1(y), \ldots, \chi_n(y)) \geq 2n + 1 - \frac{\log d}{\log \delta}(n + 1).
\]

**Theorem 12** Let \( p \in \overline{\mathbb{Q}}[z] \), \( d = \deg p \) and \( \delta = \text{ord}_{z=0} p \geq 2 \). Let \( n \in \mathbb{N}^* \), \( q_i \in \overline{\mathbb{Q}}[z] \) \( (i = 1, \ldots, n) \) and let \( \chi_i(z) \) be functions defined by (11). Assume that \( 1, q_1, \ldots, q_n \) are \( \mathbb{C} \)-linearly independent and that at least one of the following conditions is satisfied:

1. \( \deg p \nmid \deg (\sum_{i=1}^n s_i q_i(z)) \) for all \( (s_1, \ldots, s_n) \in \mathbb{C}^n \setminus \{0\} \).
2. \( \sum_{i=1}^{n} s_i \chi_i(z) \notin \mathbb{C}[z] \) for all \((s_1, \ldots, s_n) \in \mathbb{C}^n \setminus \{0\}\).

Fix a \( y \in \mathbb{C}^* \) such that
\[
\lim_{h \to 0} p^{[h]}(y) = 0 \text{ and } p^{[h]}(y) \neq 0 \text{ for all } h \in \mathbb{N}.
\]

Then the following holds true.

1. For every \( \varepsilon > 0 \) there exists a constant \( C \) such that for every projective variety \( W \subset \mathbb{P}^{n+1}_\mathbb{Q} \) of dimension \( k < n + 1 - \frac{2 \log d}{\log \delta} \), one has the measure of algebraic independence \((7)\) at \( x = (1 : \chi_1(y) : \cdots : \chi_n(y)) \in \mathbb{P}^{n+1}_\mathbb{C} \). In particular,
\[
\text{tr.deg.}_\mathbb{Q} \mathbb{Q} (\chi_1(y), \ldots, \chi_n(y)) \geq n + 1 - \frac{[\log d]}{\log \delta}.
\]

2. Moreover, if \( y \in \mathbb{Q}^* \), then for all \( \varepsilon > 0 \) there exists a constant \( C > 0 \) such that for all projective variety \( W \subset \mathbb{P}^n_\mathbb{Q} \) of dimension \( k < n + 1 - \frac{\log d}{\log \delta} \), one has the measure of algebraic independence \((7)\) at \( x = (1 : \chi_1(y) : \cdots : \chi_n(y)) \in \mathbb{P}^n_\mathbb{Q} \). In particular,
\[
\text{tr.deg.}_\mathbb{Q} \mathbb{Q} (\chi_1(y), \ldots, \chi_n(y)) \geq n + 1 - \frac{[\log d]}{\log \delta}.
\]

Example 13
Consider the function that lies, in a sense, at the origin of Mahler’s method:
\[
M(z) = \sum_{k=0}^{\infty} z^{2^k}.
\]

Then, for all \( y \in \mathbb{C}, 0 < |y| < 1 \), in the infinite family of numbers
\[
y, M(y), M(y^3), M(y^5), \ldots, M(y^{2k+1}), \ldots \quad (13)
\]
at most three numbers are algebraically dependent over \( \mathbb{Q} \). To see this, apply Theorem \( \underline{12} \) with \( p(z) = z^2 \), \( q_i(z) = z^{2^{i+1}}, i = 1, 2, 3, \ldots, n \), where \( n \in \mathbb{N} \) is arbitrarily large. In this case, condition \( 4 \) of Theorem \( \underline{12} \) is verified, hence functions \( \chi_i(z) \) defined by \((11)\) are algebraically independent. Note that in view of our choice of \( p \) and \( q_i \), we have \( \chi_i(z) = M(z^i) \) for any \( i \in \mathbb{N} \), hence we infer the claim from Theorem \( \underline{12} \) part \( 7 \). Moreover, we find with part \( 2 \) of the same theorem that in the case if, in addition, \( y \in \mathbb{Q} \), then all the numbers
\[
M(y), M(y^3), M(y^5), \ldots, M(y^{2k+1}), \ldots \quad (14)
\]
are algebraically independent. Note that the latter claim, on algebraic independence of numbers \((14)\) can be readily deduced from the classical result by Nishioka \( \underline{J} \), whilst the example concerning numbers \((13)\) was previously unknown, as far as the author knows.
Example 14 We can produce a more exotic counterpart of Example 13 using Chebyshev’s polynomials

\[ T_n = n \sum_{k=0}^{n} (-2)^k \frac{(n + k - 1)!}{(n - k)!(2k)!} (1 - x)^k, n \in \mathbb{N}. \]

It is known that these polynomials commute under composition \[12\],

\[ T_m(T_n) = T_n(T_m) \text{ for all } m, n \in \mathbb{N}. \]

Using the same reasoning as above, we readily find with Theorem 12 the following result. Define function \( \tau \) by

\[ \tau(z) = \sum_{k=1}^{\infty} T_2^{[k]}(z), \]

where, as usual, we denote by \( T_2^{[k]} \) the \( k \)-th iteration of the second Chebyshev’s polynomial \( T_2 \). For any \( y \in \mathbb{C} \setminus \{0\} \) such that \( T_2^{[k]}(y) \neq 0 \) and

\[ \lim_{k \to \infty} T_2^{[k]}(y) = 0 \]

among the infinite set of numbers

\[ y, \tau(T_3(y)), \tau(T_5(y)), \tau(T_7(y)), \ldots, \tau(T_{2k+1}(y)), \ldots \]

at most three numbers are algebraically dependent over \( \mathbb{Q} \). If, in addition, \( y \in \overline{\mathbb{Q}} \), then all the numbers

\[ \tau(T_3(y)), \tau(T_5(y)), \tau(T_7(y)), \ldots, \tau(T_{2k+1}(y)), \ldots \]

are algebraically independent.

Actually, we can produce no more examples with the same method because, by a classical result of Ritt \[12\], all the pairs of commuting polynomials,

\[ P(Q(z) = Q(P(z)), \]

are, up to a linear homeomorphism, either both powers of \( z \), or both Chebyshev’s polynomials, or iterates of the same polynomial (the latter case applied to the construction of Examples 13 and 14 produces, evidently, algebraically dependent functions).

One more our example deals with the so called Cantor series. These functions were introduced in \[13\] and studied further in \[14\]. They are defined by

\[ \theta_i(z) = \sum_{h=0}^{\infty} \frac{1}{q_i(z)q_i(p(z)) \cdots q_i(p^{[h]}(z))}, \quad (i = 1, \ldots, n), \quad (15) \]
where \( p(z) = p_1(z)/p_2(z) \in \mathbb{Q}(z) \), \( \deg p_j = d_j \) \( (j = 1, 2) \), \( \delta = \text{ord}_{z=0} p \geq 2 \), \( q_i \in \mathbb{Q}[z] \) with \( \deg q_i \geq 1 \) and \( |q_i(0)| > 1 \), \( i = 1, \ldots, n \).

The functions \( \theta_i (i = 1, \ldots, n) \) are analytic in a neighbourhood of 0 and satisfy the functional equation

\[
\theta_i(p(z)) = q_i(z)\theta_i(z) - 1, \quad (i = 1, \ldots, n).
\]

If \( p(z) \) is a polynomial, we are in measure to apply Theorems 1 and 6 to find the following results (compare with Corollary 6 of [14])

**Theorem 15** Let polynomials \( q_1, \ldots, q_n \in \mathbb{Q}[z] \) be mutually different and let \( p \in \mathbb{Q}[z] \). Assume \( \deg q_i \geq 1 \) and \( |q_i(0)| > 1 \), \( i = 1, \ldots, n \), and define functions \( \theta_i \), \( i = 1, \ldots, n \), by (15). Assume moreover \( 2 < \deg p = d \), \( 1 \leq \deg q_i < d - 1 \), \( i = 1, \ldots, m \). Let \( y \in \mathbb{C}^n \) satisfy \( \lim_{h \to \infty} p^{[h]}(y) = 0 \) and \( q_i(p^{[h]}(y)) \neq 0 \) and \( p^{[h]}(y) \neq 0 \) for all \( h \in \mathbb{N}, i = 1, \ldots, n \). Then the following holds true.

1. For every \( \varepsilon > 0 \) there exists a constant \( C \) such that for every projective variety \( W \subset \mathbb{P}_\mathbb{Q}^{n+1} \) of dimension \( k < n + 1 - \frac{2\log d}{\log \delta} \), one has the measure of algebraic independence \( \mathbb{Q}(y, \theta_1(y), \ldots, \theta_n(y)) \) at \( x = (1 : y : \theta_1(y) : \cdots : \theta_n(y)) \in \mathbb{P}_\mathbb{C}^{n+1} \).

   In particular,

   \[
   \text{tr.deg}_{\mathbb{Q}}(y, \theta_1(y), \ldots, \theta_n(y)) \geq n + 1 - \left\lfloor \frac{2\log d}{\log \delta} \right\rfloor.
   \]

2. Moreover, if \( y \in \mathbb{Q}^n \), then for all \( \varepsilon > 0 \) there exists a constant \( C > 0 \) such that for all projective variety \( W \subset \mathbb{P}_\mathbb{Q}^n \) of dimension \( k < n + 1 - \frac{\log d}{\log \delta} \), one has the measure of algebraic independence \( \mathbb{Q}(\theta_1(y), \ldots, \theta_n(y)) \) at \( x = (1 : \theta_1(y) : \cdots : \theta_n(y)) \in \mathbb{P}_\mathbb{C}^n \).

   In particular,

   \[
   \text{tr.deg}_{\mathbb{Q}}(\theta_1(y), \ldots, \theta_n(y)) \geq n + 1 - \left\lfloor \frac{\log d}{\log \delta} \right\rfloor.
   \]

If moreover \( p(z) \in \mathbb{Q}(z) \) then we can infer from our Theorem 9 the following result, improving on Corollary 6 from [14].

**Theorem 16** Let \( q_1, \ldots, q_n \in \mathbb{Q}[z] \) be mutually different and let \( p \in \mathbb{Q}[z] \). Assume \( \deg q_i \geq 1 \) and \( |q_i(0)| > 1 \), \( i = 1, \ldots, n \), and define \( \theta_i \) \( (i = 1, \ldots, n) \) by (15). Assume moreover \( \max(2, d_2) < d_1 = d \), \( 1 \leq \deg q_i < d - 1 \) for \( i = 1, \ldots, m \). Let \( y \in \mathbb{Q}^n \) satisfies \( \lim_{h \to \infty} p^{[h]}(y) = 0 \), \( q_i(p^{[h]}(y)) \neq 0 \) and \( p^{[h]}(y) \neq 0 \) for all \( h \in \mathbb{N}, i = 1, \ldots, n \). Then, there exists a constant \( C > 0 \)
such that for every variety $W \subset \mathbb{P}^n_{\mathbb{Q}}$ of dimension $k < 2n + 1 - \log \frac{d}{\log \delta} (n + 1)$ the following holds true.

$$\log \text{Dist}(x, W) \geq -C_3 \left( h(W) + d(W)^\frac{1}{n-k+1} \right)^\frac{n-k+1}{n-k} \times d(W)^\frac{k+1}{n-k},$$

where $x = (1 : \theta_1(y) : \cdots : \theta_n(y)) \in \mathbb{P}^n_{\mathbb{C}}$. In particular,

$$\mathrm{tr.deg.}_{\mathbb{Q}} \mathbb{Q}(\theta_1(y), \ldots, \theta_n(y)) \geq 2n + 1 - \frac{\log d}{\log \delta} (n + 1).$$

Remark 17 In [2] G.V.Chudnovsky introduced the notion of "normality" of $n$-uplets $(x_1, \ldots, x_n) \in \mathbb{C}^n$. According to his definition, the $n$-tuple $(x_1, \ldots, x_n)$ is normal if it has a measure of algebraic independence of the form $\exp(-Ch(P)\psi(d(P)))$, i.e. if for all polynomial $P \in \mathbb{Q}[X_1, \ldots, X_n] \setminus \{0\}$ one has the estimate

$$|P(x_1, \ldots, x_n)| \geq \exp(-Ch(P)\psi(d(P))), \quad (16)$$

where $C > 0$ is a real constant and $\psi : \mathbb{N} \to \mathbb{R}^+$ is an arbitrary function. If one has the estimate \[(16)\] with $\psi(d) = d^\tau$ for some constant $\tau$ one says that this $n$-tuple has a measure of algebraic independence of Dirichlet's type. In this situation one also defines Dirichlet's exponent to be the infimum of $\tau$ admitted for $\psi(d) = d^\tau$ in \[(16)\]. In [2] G.V.Chudnovsky mentioned that for $n \geq 2$ the explicit examples of normal $n$-tuples are quite rare, despite the fact that almost all (in the sense of Lebesgue measure) $n$-tuples of complex numbers are normal.

Th. Töpfer gave a construction for a family of examples of normal $n$-tuples with Dirichlet's exponent $2n + 2$ (see Theorem 1 and Corollary 4 of [14]).

Our theorems improves the Dirichlet's exponent to $n + 2$ for a large sub-family of these examples and allow also to produce new examples of normal $n$-tuples (due to the condition (2) of Lemma 10).

3 Criterion for algebraic independence.

In this section we elaborate a criterion for algebraic independence, Theorem 20 adapted to our situation. We deduce it from general Theorem 5.1 in [4], see Theorem 18 below.

In what follows, we use the notions of multidegree, height and size in the multiprojectif space. These notions are a straightforward generalization of the same notions in the bi-projectif case, we refer the reader to Chapter 5 of [4] for the details.
Theorem 18 (Particular case of Theorem 5.1 in [4]) Let $K$ be a number field, $k$ an integer from $[0,n]$ and $\theta = (\theta_0, \ldots, \theta_n) \in \mathbb{C}^{n+1}$.

Let $\mu \geq 0$, $\sigma, \delta \geq 1$, $\tau \geq 0$ and $U$ be real numbers.

We suppose that there exist the following objects:

- A strictly increasing sequence of reals $(S_i)_{0 \leq i \leq l}$ satisfying:
  \[ 0 < S_0 \leq \tau + \log 2 \quad \text{and} \quad S_{l-1} < U \leq S_l, \]  
  \[ (17) \]

- for $1 \leq i \leq l$, a homogeneous polynomial $Q_i \in K[X_0, \ldots, X_n]$ such that
  1. $\deg Q_i = \delta$,
  2. $h(\omega^\delta(Q_i)) \leq \tau$, where $\omega^\delta(Q_i)$ denotes the polynomial
     \[ \sum_{|\alpha| = d} \left( \frac{d}{\alpha} \right)^{\frac{1}{2}} a_{\alpha} X^{|\alpha|} \]
     for $Q_i = \sum_{|\alpha| = d} a_{\alpha} X^{|\alpha|}$,
  3. \[ \frac{|Q_i(\theta)|}{e^{S_i - 1 + \mu}} \leq e^{-S_i}, \]
  4. the polynomial $Q_i$ has no zeros in the ball $B(\theta, e^{-(S_{i-1} + \mu)\sigma})$ of $\mathbb{P}_n(\mathbb{C})$ with centre at $\theta$ and of radius $e^{-(S_{i-1} + \mu)\sigma}$.

Let $I \subset K[X_0, \ldots, X_n]$ be a homogeneous ideal of dimension $k$.

Define $t_{\tau, \delta} \overset{\text{def}}{=} h_d(I) + (\tau + (k+1)\delta \log(n+1)) \deg_d(I)$ or $d = (\delta, \ldots, \delta) \in \mathbb{N}^{k+1}$ and assume that the following condition is realized

\[ \sigma^{k+1} \left( \frac{[K : Q]}{n_\infty} t_{\tau, \delta}(I) + \left( \mu + \log 2 + \frac{\log \delta}{2\delta^k} \right) \deg_d(I) \right) \leq U, \]  
  \[ (18) \]

where $n_\infty$ denotes the index of ramification of the valuation $| \cdot |_\infty$. Then,

\[ \log \text{Dist}(\theta, V(I)) \geq -U. \]

Proof. This is Theorem 5.1 of [4] with $\kappa = \tau$, the condition B applied for the absolute archimedean value.

Corollary 19 below is a straightforward generalization of Corollary 5.9 from [4] to the case of an arbitrary number field $K$.

Corollary 19 (see Corollary 5.9 in [4]) Let $K$ be a number field, $k$ an integer from the range to $[0,n]$ and $\theta = (1, \theta_1, \ldots, \theta_n) \in \mathbb{C}^{n+1}$.
Let $\delta$, $\tau$, $\sigma$ and $U$ be real numbers satisfying $\sigma, \delta \geq 1$ and $U > \tau \geq 3(k+1)\delta \log(n+1)$.

Assume that for every real $s$ verifying $\tau < s \leq U$, there exists a polynomial $R_s$ in $K[X_1, \ldots, X_n]$ such that

- $\deg R_s \leq \delta$,
- $h\left(\omega_{\deg R_s}(R_s)\right) \leq \tau$,
- $e^{-\sigma s+2\tau} \leq \frac{|R_s(\theta)|}{(1+|\theta|^2)^{\delta}} \leq e^{-s}$

Then for every homogeneous ideal $I \subset K[X_0, \ldots, X_n]$ of dimension $k$, of degree $D$ and of height $H$ satisfying

$$2\delta^k[K : Q](\delta H + \tau D) \leq \frac{U}{\sigma^{k+1}}$$

we have

$$\log \text{Dist}(\theta, \mathcal{V}(I)) \geq -U.$$  

**Proof.** Our proofs follows the line of proof of Corollary 5.9 in [4].

Set $\theta_0 = 1$, and let $t$ be such an index that $\theta_t = \max(\theta_0, \ldots, \theta_n)$ and $\theta' = \left(\frac{\theta_0}{\theta_t}, \ldots, \frac{\theta_n}{\theta_t}\right)$.

We are going to use the following preliminary result:

Let $Q \in K[X_0, \ldots, X_n]$ tel que $Q(X) = \sum_{|\alpha| = d} a_\alpha X^\alpha$ be a homogeneous polynomial. If this polynomial has at least one zero in the ball centred at $\theta' = \left(\frac{\theta_0}{\theta_t}, \ldots, \frac{\theta_n}{\theta_t}\right)$ and of radius $R < \frac{1}{\sqrt{n+1}}$ for the euclidean distance of the affine chart $X_t \neq 0$ then

$$\frac{|Q(\theta')|}{|\theta'|^d} \leq R^{2\delta^2-1}\sqrt{n+1} \delta \sum_{|\alpha| = \delta} \frac{|a_\alpha|^2}{\delta^2}.$$

The proof of this result can be found in [4], page 127.

We verify, using this result, that the polynomial $hR_s$ has no zeros in the ball centred at $(1, \theta_1, \ldots, \theta_n)$ and of the radius $e^{-\sigma s}$. Note that the condition $\tau \geq 3(k+1)\delta \log(n+1)$ implies $\tau > (\delta-1)\log 2 + \frac{1}{2}\log(n+1) + \log \delta$. Also we have

$$\sqrt{\sum_{|\alpha| = \delta} \frac{|a_\alpha|^2}{\delta^2}} \leq e^\tau,$$

we infer from the condition $e^{-\sigma s+2\tau} \leq \frac{|R_s(\theta')|}{(1+|\theta|^2)^{\delta}}$, the following inequality:

$$e^{-\sigma s}\sqrt{n+1} \delta \sum_{|\alpha| = \delta} \frac{|a_\alpha|^2}{\delta^2} \leq \frac{|R_s(\theta')|}{|\theta'|^\delta}.$$

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This inequality and our preliminary statement allow us to conclude that the polynomial $h R_s$ has no zeros in a ball of centre $(1, \theta_0, \ldots, \theta_n)$ and of radius $e^{-\sigma_s}$.

Let $l = \left\lceil \frac{U - S_0}{\mu} \right\rceil$, where $\mu = 2(k + 1)\delta \log(n + 1) - \log 2 - \frac{\log \delta}{2\delta^k} > 0$. Consider the following sequence:

$$
\begin{align*}
0 &< S_0 = \tau + \log 2, \\
S_i &= S_{i-1} + \mu, \quad i = 1, \ldots, l - 1, \\
S_l &= U.
\end{align*}
$$

We readily verify that for every $i$, $1 \leq i \leq l$, the polynomials $Q_i = h R_{S_i}$ verify:

- $\text{deg}(Q_i) = \delta$,
- $h(\omega_\delta^k(Q_i)) \leq \tau$,
- $\left\| \frac{Q_i(\theta')}{|\theta'|} \right\| \leq e^{-S_i}$,
- le polynôme $Q_i$ n’a pas de zéro dans la boule de centre $(1, \theta_0, \ldots, \theta_n)$ et de rayon $e^{-\sigma(S_i-1+\mu)}$.

Moreover, we have for $d = (\delta, \ldots, \delta) \in \mathbb{N}^{k+1}$

$$
\frac{[K : \mathbb{Q}]}{n_\infty} t_{\tau, d}(\mathcal{I}) + \left( \mu + \log 2 + \frac{\log \delta}{2\delta^k} \right) \deg_d(I) \\
\quad \leq \delta^k [K : \mathbb{Q}] (\delta H + (\tau + 3(k + 1)\delta \log(n + 1)) D) \\
\quad \leq 2\delta^k [K : \mathbb{Q}] (\delta H + \tau D).
$$

the hypothesis (19) implies

$$
\frac{[K : \mathbb{Q}]}{n_\infty} t_{\tau, d}(\mathcal{I}) + \left( \mu + \log 2 + \frac{\log \delta}{2\delta^k} \right) \deg_d(I) \leq \frac{U}{\sigma^{k+1}}
$$

and hence the condition (18) of Theorem 18 holds true.

Thus all the hypothesis of Theoreme 18 are verified, so we can apply this theorem with the parameters $\delta$, $\tau$, $\sigma$, $\mu$, $U$ as above. We find with this theorem

$$
\log \text{Dist} (\mathcal{I}, \mathcal{V}(\mathcal{I})) \geq -U.
$$

and this gives us the conclusion of Corollary 19.

Theorem 20 below is our principal tool in proofs of algebraic independence and establishing the measures of algebraic independence.
Theorem 20 (Criterion for the measures, [13], page 5) Let m, k, λ, δ₁, ..., δₘ ∈ N, σ, τ, U ∈ ℝ⁺₀ and x = (x₁, ..., xₘ) ∈ ℂᵐ be such that 0 ≤ k < m, λ ≥ 1, δ₁ ≥ ⋯ ≥ δₘ ≥ 1, U > τ ≥ 4(k + 1) log (δ₁ ⋯ δₘ(1 + m²)). Let K be a number field. Assume that for every real τλ ≤ s ≤ U there exists a polynomial Rs ∈ K[X₁, ..., Xₘ] of degree < e(s)δᵢ in Xᵢ, of length ≤ e⁺(s)τ and satisfying

\[ \exp(-sσ + 2e(s)τ) \leq \frac{|Rs(x)|}{\prod_{i=1}^{m}(1 + |x_i|^2)^{δ_i/2}} \leq \exp(-s - e(s)(δ₁ + ⋯ + δₘ)) \]  

(20)

where e(s) := 1 + max₁≤i≤m [degX V \frac{R_i}{δ_i}] ≤ λ. Then, for every algebraic variety V ⊂ ℂᵐ(ℂ) defined over Q, of dimension k, and satisfying

\[ [K : Q] \cdot 3λk⁺¹δ₁ ⋯ δₖ (δ_{k+1}t(V) + τ \deg(V)) \leq U/(σm)^k+1, \]  

(21)

one has

\[ \log \text{Dist}(x', V') \geq -U, \]

where x' = (1 : x₁ : ⋯ : xₘ) ∈ ℙᵐ Q and V' denotes the completion of V in ℙᵐ Q.

Proof. Consider the completion W of V in (ℙ¹)m and the point ̃x = (1 : x₁, ..., 1 : xₘ) ∈ (ℙ¹)m. Let φ : (ℙ¹)m → ℙᴺ, where N = (δ₁ + 1) ⋯ (δₘ + 1) − 1 is an embedding defined by

\[ φ(x_{0,1} : x_{1,1}, ..., x_{0,m} : x_{1,m}) = \left( \cdots : \prod_{i=1}^{m} x_{0,i}^{α_i} x_{1,i}^{δ_i - α_i} : \cdots \right)_{α ∈ ℤ^m, 0 ≤ α_i ≤ δ_i}. \]

We readily verify, as it is done in [9], III, Proposition 1 and [13], §2.3 (b), Lemma 2.13,

\[ d(φ(W)) = k! \sum_{i∈\{0,1\}^m \atop |i|=k} d_i(W)δ_i^1 ⋯ δ_i^m, \]

\[ h(φ(W)) = (k + 1)! \sum_{i∈\{0,1\}^m \atop |i|=k+1} \left( h_i(W) + \sum_{α=1 \atop \hat{i}_α \neq 0} \delta_i^1 ⋯ δ_i^m, \right) \]

where d_i(W) and h_i(W) denote the multihomogeneous degrees and heights of W, and where ̂iα is obtained from i by setting the α-th component equal to 0 (i.e. |iα| = k). In particular, k! \sum d_i(W) and (k + 1)! \sum h_i(W) are the degree and the height of V embedded in ℙ²ᵐ⁻¹ with Segre’s embedding and are equal respectively to mkd(W) and mk⁺¹h(W). Every polynome Rs,
suitably homogenised, can be represented as an inverse image by \( \phi \) of a form \( P_s \in K[X_0, \ldots, X_N] \) of degree \( e(s) \leq \delta \) and of the length \( \leq e^{e(s)\tau} \leq e^{\tau\delta} \). Moreover, if \( y = \phi(x) \) we have

\[
1 \leq \prod_{i=1}^{n}(1 + |x_i|^2)^{\delta_i/2} \leq e^{\delta_1 + \cdots + \delta_m},
\]

hence

\[
\exp(-s\sigma + 2e(s)\tau) \leq \frac{|P_s(y)|}{|y|^{\deg P_s}} \leq \exp(-s).
\]

At the same time, the condition \( \text{(21)} \) implies

\[
2\delta^{k+1}(h(\phi(W)) + (k+1)\tau d(\phi(W))) \leq \frac{U'}{\sigma^{k+1}}
\]

where \( U' = U - m^{k+1}\delta_1 \cdots \delta_k \log(\delta_1(1+m^2))d(V) \). Indeed, as \( \delta_1 \geq \cdots \geq \delta_m \), we have

\[
d(\phi(W)) = m^{k}d(V)\delta_1 \cdots \delta_k,
\]

\[
h(\phi(W)) = m^{k+1}t(V)\delta_1 \cdots \delta_{k+1}.
\]

So we can apply Corollary \( \text{[19]} \) to \( \phi(W) \) and this completes the proof, because by Proposition 3.9 of \( \text{[4]} \) we can verify

\[
\log \text{Dist}(x', V') \geq \log \text{Dist}(x, W) - \frac{k+1}{k+1} \cdot \log(\delta_1 \cdots \delta_m(1+m^2)) \cdot d(\phi(W))
\]

\[
\geq \log \text{Dist}(y, \phi(W)) - m^{k+1}\delta_1 \cdots \delta_k \log(\delta_1(1+m^2)) d(\phi(W))
\]

\[
\geq \log -U' - m^{k+1}\delta_1 \cdots \delta_k \log(\delta_1(1+m^2))d(\phi(W)) = -U,
\]

and the claim of the theorem readily follows.

\( \Box \)

**Remark 21** Theorem \( \text{[20]} \) is proved in \( \text{[11]} \) with \( Z \) in place of \( K \). The proof given there still works if we replace \( Z \) with an arbitrary number field, the only change needed is that one has to use Corollary \( \text{[14]} \) instead of Corollary 5.9 in \( \text{[7]} \) (cited in \( \text{[11]} \) as "Theorem from page 5."). Here above, we used the line of proof from \( \text{[11]} \).

### 4 \( \mathbb{K} \)-functions

In this section we present a restricted version of \( \mathbb{K} \)-functions, introduced for the first time in \( \text{[11]} \).

We start with some notations. For every integer \( t \in \mathbb{N} \) and every formal power series (for instance, for every analytic function) \( g \in \mathbb{C}((z)) \) we denote
by $D_t(g)$ the $t$-th coefficient of $g$ (we follow the notation introduced in \[11\]). Further, for every $D \subset \mathbb{N}^n$ and every $t \in \mathbb{N}$ we define

$$D_t f^D(0) = \left( D_{t'}(f_1^{\alpha_1} \cdots f_m^{\alpha_m})(0) \mid \alpha \in D, t' = 0, \ldots, t \right),$$

(22)

this is a vector from $\mathbb{Q}(t + 1)\text{card}D$. For every increasing function $\psi : \mathbb{N} \to \mathbb{N}^*$ and $\text{card}D \geq 2\psi(0)$, we also denote $t_D = t_D(\psi)$ the biggest integer such that $2t_D \psi(t_D - 1) \leq \text{card}D$.

We also use the following notation:

$$|D| := \max_{\alpha \in D} n+1 \sum_{i=1}^{n+1} |\alpha_i|.$$

**Definition 22** Let $n \in \mathbb{N}^*$, let $A$ be an infinite set of subsets of $\mathbb{N}^n$, and let functions

$$\psi : \mathbb{N} \to \mathbb{N}^*, \quad \phi : A \times \mathbb{N} \to \mathbb{R}_{\geq 0}$$

be such that for any $D \in A$ the functions $\psi(t)$ and $\phi_D(t)$ are increasing in $t$, the function $\frac{\log \phi_D(t)}{t}$ is decreasing in $t$ for $t \geq t_D$ and $\liminf_{|D| \to \infty} \frac{\log \phi_D(t_D)}{t_D} = 0$. Moreover, assume $\phi_D(t) \geq 4\sqrt{2}(t + 1)\psi(t)$ for all $t \geq t_D$.

We say that a family $(f_1, \ldots, f_m)$ of functions analytic in $B(0,1)$ form a system of $K$-functions of type $(\psi, \phi)$ if for all $t \in \mathbb{N}$ and for all $D \in A$ we have

$$D_t f(0) \subset \mathbb{Q}, \quad \left[ \mathbb{Q}(D_t f(0)) : \mathbb{Q} \right] \leq \psi(t), \quad h(D_t f^D(0)) \leq \phi_D(t).$$

**Definition 23** Let $c \geq 1$ be a real number. We say that $D \subset \mathbb{N}^{n+1}$ is $c$-admissible for a system of $K$-functions $f_1, \ldots, f_{n+1}$ if for all $Q \in \mathbb{Z}[X_1, \ldots, X_{n+1}] \setminus \{0\}$, supported by $D$ (that is of a form $\sum_{\alpha \in D} q_{\alpha} X_1^{\alpha_1} \cdots X_{n+1}^{\alpha_{n+1}}$), of length $\leq \sqrt{2} \cdot \text{card}D \cdot \phi_D(t_D)$, we have

$$\text{ord}_{z=0} Q(f_1, \ldots, f_{n+1}) \leq c \cdot \text{card}D.$$

**Remark 24** For any $D \geq 1$ a set

$$D = \{ h \in \mathbb{N}^{n+1} \mid h_i < D, \ i = 1, \ldots, n+1 \}$$

(23)

is $c$-admissible for a system of $K$-functions $f_1, \ldots, f_{n+1}$ if and only if $f = (f_1, \ldots, f_{n+1})$ satisfies multiplicity lemma with the optimal exponent, i.e. $n + 1$, and the multiplicative constant $c$).

**Theorem 25** shows that this remark is applicable in the case of Mahler’s functions (1).
Theorem 25 Let \((1, f_1(z), \ldots, f_n(z))\) be an \(n\)-tuple of functions \(C \rightarrow C\) analytic at \(z = 0\) that form a solution to the system of functional equations. Moreover, assume that \(f_1(z), \ldots, f_n(z)\) are algebraically independent over \(C(z)\). Assume also that \(\text{ord}_{z=0} p(z) \geq 2\). Then there exists a constant \(K_1\) such that for any non-zero polynomial \(P \in \mathcal{A}\) one has

\[
\text{ord}_{z=0}(P(f)) \leq K_1(\deg_X P + \deg_Y P + 1)(\deg_X P + 1)^n. \tag{24}
\]

Proof. See [17], Theorem 1.3.

Proposition 26 (A particular case of Proposition 7 in [17]). Let \(c' \geq 1\) and let \(f_1, \ldots, f_{n+1}\) be a system of \(K\)-functions of type \((\psi, \phi)\). Then,

1. for all \(D \in \mathcal{A}\) there exists a polynomial \(P \in \mathbb{Z}[X_1, \ldots, X_{n+1}] \setminus \{0\}\), supported by \(D\), of length \(\leq \sqrt{2} \cdot \text{card} D \cdot \phi_D(t_D)\) and such that \(T_0 := \text{ord}_{z=0} F(z) \geq t_D\) where

\[
F = P(f_1, \ldots, f_{n+1}); \tag{25}
\]

2. under assumption \(\frac{\log \phi_D(t_D)}{t_D} \leq \frac{1}{2T}, \) for all real numbers \(r', r''\) such that \(0 < r'' \leq r' < r^4\), where

\[
r := 1 - \frac{12 \log \phi_D(T_0)}{T_0},
\]

for every positive integer \(0 < N \leq T_0/8\) and for every point

\[
z \in \overline{B(0, r')} \setminus B(0, r'')
\]

satisfying

\[
-N \cdot \log \left(\frac{r'}{r}\right) - N \cdot \log \left(\frac{1+|z|/r'}{1+|z|/r''}\right) \geq 19 \tag{26}
\]

there exists a positive integer \(0 < t \leq N\) such that

\[
\left(\frac{r''}{4}\right)^{(c'+4)T_0} \leq |D_t F(z)| \leq \left(\frac{r'}{(1 - \sqrt{r^2})^2}\right)^{T_0/16}.
\]

Moreover, if \(D\) is \(c\)-admissible for \(f_1, \ldots, f_m\), then \(T_0 \leq c \cdot \text{card} D\).
5 Polynomial sequences

In this section we construct sequences of polynomials with nice approximation properties. Our main result in this section is Proposition 29 which is used in proofs of Theorems 1 and 6. We derive this result from general extrapolative construction, Proposition 26. The proof of Theorem 9 makes appeal to a slightly different Proposition 30 that we essentially borrow from [14].

We start with an auxiliary lemma, which is Lemma 2 from [14]. We provide a proof (following the line of the proof in [14]) for the commodity of the reader and, simultaneously, to fix minor issues with the proof given in [14].

Lemma 27 Let $p(z) = p_1(z)/p_2(z)$ be a rational function with $\delta = \text{ord}_{z=0}p(z) \geq 2$ and let $y \in \mathbb{C}$ satisfies $p^{[T]}(y) \neq 0$ for all $T \in \mathbb{N}$. Assume

$$\lim_{T \to \infty} p^{[T]}(y) = 0. \quad (27)$$

Then there exist constants $0 < c_3', c_3'' < 1$ and $T_s > 0$, depending on $p$ and $y$ only, such that

$$|c_3'|^\delta |y| \leq |p^{[T]}(y)| \leq |c_3''|^\delta |y|, \quad (28)$$

for all $T \geq T_s$.

Proof. As $p$ is a rational fraction and $z = 0$ is a zero of the order $\delta \geq 2$, we have that

$$p(z) = z^\delta g(z), \quad (29)$$

where $g$ is a rational fraction satisfying $g(0) \neq 0$. In particular, $g$ is a continuous function defined in a neighbourhood $U$ of 0. So, for a sufficiently small $U$, there exist constants $\tilde{c}_3', \tilde{c}_3'' > 0$ such that

$$\tilde{c}_3' \leq |g(z)| \leq \tilde{c}_3'' \quad (30)$$

for every $z \in U$.

Combining (29) and (30), we find

$$c_3'|z|^\delta \leq |p(z)| \leq c_3''|z|^\delta \quad (31)$$

By assumption (27), there is an index $\hat{T} \in \mathbb{N}$ such that for every $T \geq \hat{T}$ one has $p^{[T]}(y) \in U$. So iterating (31) we find (28) with $c_3', c_3'' > 0$. The upper bound $c_3', c_3'' < 1$ follows from (28) and assumption (27).

The next lemma embodies an important step in the proof of the forthcoming Proposition 29, which, in turn, plays an important role in proofs of our main results, Theorems 1 and 6. We isolate this step in order to make the reading easier.
Lemma 28 Assume the situation of Proposition 26, that is let $f_1, ..., f_{n+1}$ be a system of $K$-functions of type $(\psi, \phi)$, and assume moreover that $\phi$ is given by

$$\phi_D(t) = (t + 2)^{|D|}.$$ 

Let the function $F$ be defined by (25), for $D \geq 1$ and $D$ given by (23). Assume moreover that functions $f_1, ..., f_{n+1}$ satisfy (1) and define

$$q := \det A.$$ (32)

Then for every integer $T \geq 0$ there exists a polynomial $P_{D,T}$ with coefficients from $\mathbb{Q}$ of degree in $z$ upper bounded by

$$\deg_z P_{D,T} \leq c_4 D (d^T + \cdots + 1) \leq c_5 D d^T,$$ (33)

of degree in $X$ bounded by

$$\deg_X P_{D,T} < nD,$$ (34)

of the length not exceeding

$$L(P_{D,T}) \leq \exp(c_6 D d^T),$$ (35)

and such that for any $y \in \mathbb{C}$ one has

$$\left( \prod_{i=0}^{T-1} q(p[i](y)) \right)^{nD} \cdot F \left( p[T](y) \right) = P_{D,T}(y, f_1(y), \ldots, f_n(y)).$$ (36)

Proof. For $T = 0$, the existence of $P_{D,0}$ readily follows from (25). For $T \geq 1$ we proceed with recurrence. Assume we established the existence of $P_{D,T}$ verifying (34). Substitute $p(y)$ in place of $y$ and apply to the right hand side the equality

$$f(p(y)) = A(y)^{-1} \left( a(y) f(y) - B(y) \right),$$ (37)

which readily follows from (1). As the result, we infer from (36) the equality

$$\left( \prod_{i=1}^{T} q(p[i](y)) \right)^{nD} \cdot F \left( p[T+1](y) \right) = Q(y, f_1(y), \ldots, f_n(y),$$ (38)

where $Q = Q(y, X_1, \ldots, X_n)$ is a polynomial in $X$ of the same degree as $P_{D,T}$ (because (37) is linear in $f(z)$). So we have

$$\deg_X Q = \deg_X P_{D,T} < nD.$$ 

At the same time, $Q$ is a rational fraction in $y$, and the common denominator of its terms is a polynomial divisor of $(\det A)^{\deg_X P_{D,T}}$ (because the denominator in the right hand side of (37) is a polynomial in $z$ dividing
det \(A\)). Hence the denominator of the rational fraction \(Q\) is a polynomial from \(\mathbb{Q}[z]\) dividing \((\det A)^{nD} = q^{nD}\) (we recall the notation \((32)\)). So if we multiply both sides of \((38)\) by \(q^{nD}\), we find the equality \((36)\) for \(T + 1\), though still subject to the verification of the upper bounds \((33)\) and \((35)\) for the degree in \(z\) and for the length of \(P_{D,T+1} := q^{nD}Q\).

To verify the upper bound of degree in \(z\), we write down the definition
\[
P_{D,T+1}(y, f(y)) = q(y)^{nD}P_{D,T}(p(y), A^{-1}(y)f(y) - A^{-1}B(y)).
\] (39)

By the recurrence hypothesis we find, with the notation \(A_0(y) = q(y)A^{-1}(y)\) and \(B_0(y) = q(y)A^{-1}B(y)\) and assuming the lower bound \(c_4 \geq \max (\deg_z A_0, \deg_z B_0) + n \deg(q)\),
\[
\deg_z P_{D,T+1} \leq nD \deg q + c_4 dD (d^T + \cdots + 1) + D \max (\deg_z A_0, \deg_z B_0)
\]
\[
\leq c_4 D (d^{T+1} + \cdots + 1).
\]
So for \(c_4 = n \cdot \deg q + \max (\deg_z A_0, \deg_z B_0)\) the upper bound \((33)\) holds true by recurrence. We readily find
\[
\deg_z P_{D,T+1} \leq c_4 D d^{T+2} - 1,
\] (40)

hence
\[
\deg_z P_{D,T+1} \leq 2c_4 D d^{T+2}.
\] (41)

We proceed to the proof of the upper bound for the length \(L(P_{D,T+1})\). We deduce from \((39)\) the upper bound
\[
L(P_{D,T+1}) \leq L(P_{D,T}) \cdot L(q)^{nD} \cdot L(p)^{\deg_z P_{D,T}} \cdot (L(A_0(y)) + L(B_0(y)))^D
\]
\[
\leq L(P_{D,T}) \exp (c_0' (D + \deg_z P_{D,T})),
\]
and taking into account \((41)\) we find
\[
L(P_{D,T+1}) \leq L(P_{D,T}) \exp (c_0'' D d^T).
\]
Applying the hypothesis of the recurrence we infer
\[
L(P_{D,T+1}) \leq \exp (c_0'' D (d^T + \cdots + 1)) \leq \exp (c_0'' D d^{T+1}).
\] (42)

We conclude with recurrence that \((35)\) holds true and it completes the proof of the lemma.

Proposition 29 Let functions \(f_1, \ldots, f_{n+1}\) satisfy \((1)\) and let \(y \in \mathbb{C}\) satisfies \(p^{[T]}(y) \neq 0\) for all \(T \in \mathbb{N}\). Assume
\[
\lim_{h \to 0} p^{[T]}(y) = 0.
\] (43)
Then for any $D \geq 1$ and any $T$ big enough there exists a polynomial $P_{D,T}(z, X_1, \ldots, X_n)$ satisfying (33), (34), (35). If moreover

$$T \geq \frac{\log D + \log \log D + \log (19(n + 1)c_2) + \log \log C - \log |c_3'|}{\log \delta},$$

(44)

this polynomial satisfies as well

$$\exp(-c_7 D^{n+1}\delta^T) \leq |P_{D,T}(z)| \leq \exp(-c_8 D^{n+1}\delta^T),$$

(45)

where

$$z = (y, f_1(y), \ldots, f_n(y)) \in \mathbb{C}^{n+1}$$

and $c_7, c_8 \in \mathbb{R}^+$ are positive constants.

**Proof.** Let $D \subset \mathbb{N}^{n+1}$ be defined by (23). Lemma 12 of [14] shows that the numbers $D_{j}f(0)$ belong to a fixed number field and provides the following upper bound for heights:

$$h(D_{j}f^{D}(0)) \leq (t + 2)c_2|D|.$$  

(46)

Note that one has the following relations between the notations of [14] and our notations introduced in (22) and in text preceding this equality:

$$D_{j}f(0) = (f_{1,j}, \ldots, f_{n,j})$$

and

$$D_{j}f^{D}(0) = \left( f_{j}^{(i)} \mid j \in D \right),$$

and the height in (46) is the product of the denominator and the house of these vectors. In particular, the upper bound (46) implies that the series $f_i(z)$ converge for $|z| < 1$.

Thus for $\psi(t) = c_1, \phi_D(t) = (t + 2)c_2|D|$ the system of functions $z, f_1(z), \ldots, f_n(z)$ is a system of K-functions of type $(\psi, \phi)$.

In view of assumption (43), we can apply Lemma 27. So, for $T$ big enough we have the following double bound for $T$-th iterate of $p$:

$$|c_3'|\delta^T \leq |p^{[T]}(y)| \leq |c_3''|\delta^T,$$

(47)

where $0 < c_3', c_3'' < 1$ depend on $p$ and $y$ only.

We apply Proposition 26 with the set $D$ defined by (23), $r' = r'' = |p^{[T]}(y)|, N = 1$ and $z = p^{[T]}(y)$.

In this situation we can apply not only Proposition 26 but also Lemma 28 to get a sequence of polynomials $P_{D,T}$ with good upper bounds for their degrees and the length, (33), (34) and (35), and verifying as well (36).
Let us show that the polynomial \( P_{D,T} \) verify (45) if (44) holds true. To this end, we apply part 2 of Theorem 26. Condition (26) in our case is equivalent to

\[
\log \left( \frac{r^2 + |p[y](y)|^2}{2 \cdot r \cdot |p[y](y)|} \right) \geq 19.
\]

(48)

To prove that (48) holds true, we verify first

\[
T_0 \leq C D^{n+1}.
\]

(49)

Indeed, Theorem 25 provides us the multiplicity lemma for the functions \( z, f_1(z), \ldots, f_n(z) \), so there exists a constant \( C > 0 \) such that the set \( \mathcal{D} \) given by (23) is \( C \)-admissible for the system of \( K \)-functions \( z, f_1(z), \ldots, f_n(z) \) (see Remark 24) (recall our notation \( T_0 = \text{ord}_{z=0} F(z) \)). Now, the inequality (48) follows from (44), (49) and the upper bound in (47).

Thus we can apply part 2 of Theorem 26 (and taking into account the double bound (47)) to find that

\[
\exp(-\hat{c}_7 D^{n+1} \delta^T) \leq |F(p[y](y))| \leq \exp(-\hat{c}_8 D^{n+1} \delta^T),
\]

(50)

where \( \hat{c}_7, \hat{c}_8 \in \mathbb{R}^+ \) are two positive constants.

In view of (56), the polynomial \( P_{D,T} \) satisfies

\[
P_{D,T}(x) = \left( \prod_{i=0}^{T-1} q(p[i](y)) \right)^{nD} \cdot F(p[y](y)).
\]

(51)

At the same time, the double bound (47) implies

\[
\exp(-\hat{c}_9 D^{n+1} \delta^T) \leq \prod_{i=0}^{T-1} q(p[i](y)) \leq \hat{c}_10^n D.
\]

(52)

Combining (50), (51) and (52) we find (45) and it completes the proof of the theorem.

\[\Box\]

**Proposition 30** (See Lemma 11 in [14]) Let \( T, D \in \mathbb{N} \) and \( y \in \mathbb{C} \). Let \( K \) be a number field. Then there exists a polynome \( R_{T,D} \in K[z, X] \) satisfying

\[
\deg_z R_{T,D} \leq C_1 d^T D
\]

(53)

\[
\deg_X R_{T,D} \leq D
\]

(54)

\[
h(R_{T,D}) \leq \exp \left( C_2 D(d^T + D^n) \right).
\]

(55)

Moreover, if the following inequality holds true:

\[
\delta^T \geq C_3 D^{n+1},
\]

(56)

then

\[
\exp(-C_5 D^{n+1} \delta^T) \leq |R_{T,D}(y, f(y))| \leq \exp(-C_6 D^{n+1} \delta^T).
\]

(57)
Proof. This result follows from Lemmata 8 and 9 in [14] and from Theorem 25. We define polynomials $R_{T,D}$ as it is explained in [14] in the paragraph following Lemma 9 (see the bottom of the page 174).

Remark 31 Propositions 29 and 30 are different in two aspects. Proposition 30 is applicable in a more general situation, namely when $p(z)$ is a rational fraction and not only a polynomial as it is assumed in Proposition 29. At the same time, Proposition 29 provides a much better control over the length of the polynomials. Indeed, Proposition 29 provides essentially the same upper bound for the length of polynomials as Proposition 30 furnishes for the height of polynomial only. It is exactly the technical origin of the natural fact that in our final results, estimates in Theorem 9, dealing with the case when $p(z)$ is a rational fraction, are worse than the estimates in Theorem 6, where we consider the case of a polynomial $p(z)$. It is also the reason why our methods does not allow us to provide a counterpart of Theorem 6 with $p(z)$ been a rational fraction.

6 Proofs

Proof of Theorem 6. Let $W \subset \mathbb{P}^n$ be a variety of dimension $k < n + 1 - \frac{\log d}{\log \delta}$.

We define

$$
\theta(W) \overset{\text{def}}{=} \max \left( h(W), d(W)^{\frac{n-k+1-\frac{\log d}{\log \delta}}{n-k+1}} \right),
$$

$$
D' := c \left( \theta(W)^{1-\frac{\log d}{\log \delta}} d(W) \right)^{1/(n-k)},
$$

$$
T := \frac{1}{\log d} \log \left( \frac{\theta(W)}{d(W)} \right) + 2 \cdot \frac{\log c + \log \log c}{\log \delta},
$$

where $c$ denotes a sufficiently big constant. Let us verify (44) for all the possible couples of parameters $D \leq D'$ and $T$. As the first step, note that it is sufficient to verify this inequality for $D = D'$ and $T$. Further, with the definition of $\theta(W)$ we find

$$
(n-k+1+\varepsilon) \log d(W) \leq (n-k+1 - \frac{\log d}{\log \delta}) \log \theta(W).
$$

For $\varepsilon_1 = \frac{n-k+1-\frac{\log d}{\log \delta}}{n-k+1+\varepsilon}$ we can transform this inequality to the following form:

$$
(n-k+1) \log d(W) \leq (n-k+1 - \frac{\log d}{\log \delta} - \varepsilon_1) \log \theta(W),
$$

23
and in view of the assumption \( k < n + 1 - \frac{\log d}{\log \delta} \) we find \( \epsilon_1 > 0 \), as far as \( \epsilon > 0 \). Rearranging the terms in (60), and dividing further by \((n - k)\log d\), we rewrite (60) as

\[
\frac{1}{\log d} \left( \frac{1}{n - k} \left( \frac{\log d}{\log \delta} - 1 \right) \log \theta(W) + \frac{1}{n - k} \log d(W) \right) + \frac{\epsilon_1}{(n - k)\log d} \log \theta(W) \leq \frac{1}{\log d} \left( \log \theta(W) - \log d(W) \right). \tag{61}
\]

In view of the definitions (58) and (59) we transform (61) into

\[
\frac{1}{\log \delta} \log D' - \frac{\log c}{\log \delta} + \frac{\epsilon_1}{(n - k)\log d} \log \theta(W) \leq T - 2 \cdot \frac{\log c + \log \log c}{\log \delta}. \tag{62}
\]

To complete the verification of (44) we remark that for any choice of constants \( \epsilon_1 > 0, C_1 > 0 \) we can always find a value of \( c \) big enough to assure that the following inequality holds true for every value of \( \theta(W) \):

\[
C_1 + \frac{\log \log \theta(W)}{\delta} < \frac{\log c}{\log \delta} + \frac{\epsilon_1}{(n - k)\log d} \log \theta(W). \tag{63}
\]

Using (62) (for the values of \( c \) sufficiently big), (63) and (61) we find

\[
T \geq \frac{1}{\log \delta} \log D' - \frac{\log c}{\log \delta} + \frac{\epsilon_1}{(n - k)\log d} \log \theta(W) + 2 \cdot \frac{\log c + \log \log c}{\log \delta}
\]
\[
> \frac{1}{\log \delta} \log D' + \frac{\epsilon_1}{(n - k)\log d} \log \theta(W) + \frac{\log c}{\log \delta}
\]
\[
> \frac{1}{\log \delta} \log D' + \frac{\log (19(n + 1)c_2) - \log |\log |c''_3y||}{\log \delta},
\]

so (44) is verified.

So we can apply Proposition 29 and we deduce with this theorem the existence of polynomials \( P_{D,T} \) satisfying (33), (34), (35) and (45).

Note that the upper bounds (33) and (35) imply that for any \( y \in \mathbb{Q} \) fixed in advance the length of \( P_{D,T}(y) \in \mathbb{Q}[X_1, \ldots, X_n] \) is upper bounded by \( c''_6'Dd'^T \) (where the constant \( c''_6 \) depends on \( y \) but is independent of \( D \) and \( T \)).

Let us denote \( y_1 = y \), and let us denote by \( y_2, \ldots, y_t \) the set of conjugates of \( y \) (so we assume that \( y \) is of degree \( t \) over \( \mathbb{Q} \)). Define

\[
\hat{P}_{D,T}(X_1, \ldots, X_n) \overset{\text{def}}{=} \text{den}(y)^{t\deg_{,}(P_{D,T})} \times \prod_{i=1}^{t} P_{D,T}(y_i, X_1, \ldots, X_n),
\]
this is a polynomial with integer coefficients, of degree \( \leq t \cdot D \) in \( X_1, \ldots, X_n \) and of length
\[
L(\tilde{P}_{D,T}) \leq \exp (\tilde{c}_8 D d^T) .
\]

Bigger the constant \( c_7 \) is in the lower bound in (15), weaker is this lower bound. So we can assume that the constants \( c_7 \) is sufficiently big. For instance, \( e \) can assume \( c_7 > \frac{2c'_c c_8}{4c_6} \), where \( c' \) is the constant from (67) below.

We apply Corollary 19 to the point \( \mathbf{y}' = (f_1(y), \ldots, f_n(y)) \in \mathbb{C}^n \) with
\[
m = n, \quad 0 \leq k = \dim W < n + 1 - \frac{\log d}{\log \delta}, \quad \delta = D',
\]
\[
\tau = \tilde{c}_6 D' d^T, \quad \sigma = 4c_7 c_8, \quad U = \frac{1}{2} \tilde{c}_8 D^{n+1} \delta^T.
\]

For every real number \( s \),
\[
\tau < s \leq U,
\]
we define \( Q_s(X_1, \ldots, X_n) = \tilde{P}_{D_s,T}(X_1, \ldots, X_n) \) with \( D_s = \left[ \left( \frac{2s}{c_8 \delta^T} \right)^{1/(n+1)} \right] \) and \( T \) defined in (59). The constraints (65) imply
\[
\left[ \left( \frac{4c_6}{c_8} D' \left( \frac{d}{\delta} \right)^T \right)^{\frac{n+1}{n+1}} \right] \leq D_s = \left[ \left( \frac{2s}{c_8 \delta^T} \right)^{1/(n+1)} \right] \leq D',
\]
so the quantity \( e(s) \) of Theorem 20 satisfies
\[
e(s) = \left[ \frac{D_s}{D'} \right] + 1 \leq 2 = \lambda.
\]

We deduce (20) from (55). Note that
\[
\prod_{i=1}^{n+1} \left( 1 + |x_i|^2 \right)^{\frac{1}{2} \deg X_i} P_{D_s,T} \leq \exp \left( c'd_s d^T \right),
\]
where \( c' \) is a constant. Also, note that inequalities (65) imply (for a constant \( c \) from (58) big enough)
\[
\prod_{i=1}^{n+1} \left( 1 + |x_i|^2 \right)^{\frac{1}{2} \deg X_i} P_{D_s,T} < e^{\frac{1}{2} c_8 D_s^{n+1} \delta^T}.
\]

We verify with the definition of \( D' \) that for \( c \) big enough we have
\[
[K : \mathbb{Q}] \cdot 3 \lambda^{k+1} \delta^k (\delta_{k+1}(W) + (k+1) \tau d(W)) \leq U / ((n + 1) \sigma)^{k+1},
\]
so we can apply Theorem 19 and with this theorem we find:

\[
\log \text{Dist}(x, W) \geq -U = -C\theta(W)\frac{n+1}{n-k} \frac{\log d}{\log \delta} \frac{k+1}{n-k} d(W) \frac{\log d}{\log \delta} n+1.
\]

It completes the proof of Theorem 1. \(\blacksquare\)

**Proof of Theorem 6.** We use the same method as in the proof of Theorem 1.

Let \(W \subset \mathbb{P}_{\mathbb{Q}}^{n+1}\) be a variety of dimension \(k < n + 1 - \frac{2\log d}{\log \delta}\). Define

\[
\theta(W) \overset{\text{def}}{=} \max \left( h(W), d(W) \frac{n+1-k - \log d}{\log \delta} n+1-k - \frac{2 \log d}{\log \delta} \right),
\]

\[
D' := c \left( \theta(W) \frac{2}{\log d} d(W) \frac{\log d}{\log \delta} - 1 \right)^{1/(n-k)},
\]

\[
T := \frac{1}{\log d} \log \left( \frac{\theta(W)}{d(W)} \right) + \frac{\log c + \log \log c}{\log \delta},
\]

where \(c\) denotes a sufficiently big constant. We apply Proposition 29 to deduce the existence of polynomials \(P_{D,T}\) satisfying (33), (34), (35). The same considerations as we applied in the proof of Theorem 1 show that parameters \(D'\) and \(T\) satisfy the inequality (44). So for every \(D \leq D'\) the polynomial \(P_{D,T}\) satisfies as well (45).

We apply Theorem 20 to the point \(x = (y, f_1(y), \ldots, f_n(y)) \in \mathbb{C}^{n+1}\) with

\[
m = n + 1, \quad 0 \leq k = \dim W < n + 1 - \frac{2\log d}{\log \delta} \frac{1}{\lambda} = 2,
\]

\[
\delta_1 = \tau = c_6 D' \frac{\delta}{T}, \delta_2 = \cdots = \delta_{n+1} = D', \quad \sigma = 4c_7/c_8, \quad U = D^{n+1} \delta^T
\]

For all real \(s\),

\[
\tau \lambda < s \leq U,
\]

we define \(R_s = P_{D_s,T}\) with \(D_s = \left( \frac{2s}{c_8 \delta^T} \right)^{1/(n+1)}\) and \(T\) defined by (69). The double bound (65) implies

\[
\left( \frac{4c_6}{c_8} D' \frac{d(T)}{\delta^{1/(n+1)}} \right)^{1/(n+1)} \leq D_s = \left( \frac{2s}{c_8 \delta^T} \right)^{1/(n+1)} \leq D'.
\]

Hence the quantity \(e(s)\) in the statement of Theorem 20 satisfies

\[
e(s) \leq 2 = \lambda.
\]

So we have (20).
We verify with the definition of $D'$
\[
3\lambda^{k+1}\delta_1 \cdots \delta_k (\delta_{k+1}t(W) + (k+1)\tau d(W)) \leq U/((n+1)\sigma)^{k+1}
\]
And we infer with Theorem 20:
\[
\log \text{Dist} (x, W) \geq -U = -C\theta(W)^{1-\log \frac{1}{\log \delta} \frac{1}{2-n-k}} d(W) \log \frac{1}{\log \delta} \frac{1}{2-n-k} \cdot \frac{n+1}{n-k},
\]
so establishing Theorem 6.

Proof of Theorem 9. We use the same method of proof as for Theorems 1 and 6. The only modifications we need is to replace Proposition 29 by Proposition 30, adjust the values of parameters according to bounds given by this new proposition and verify, once again, the hypothesis of Theorem 20.

So, let $W \subset \mathbb{P}^n$ be a variety of dimension $k < 2n + 1 - \log \frac{1}{\log \delta} (n+1)$, define
\[
\theta(W) \overset{\text{def}}{=} \max \left( h(W), d(W) \log \frac{1}{\log \delta} \frac{1}{2-n-k+1} \right),
\]
\[
D := c \left( \theta(W)^{1-\log \frac{1}{\log \delta} d(W) \log \frac{1}{\log \delta}} \log \frac{1}{\log \delta} \frac{1}{2-n-k} \right)^{1/(n-k)},
\]
\[
T := \frac{1}{\log \delta} \log \left( \frac{\theta(W)}{d(W)} \right) + \frac{(n+1) \log c + \log \log c}{\log \delta},
\]
where $c$ denotes a sufficiently big constant.

We apply Theorem 20 at the point $x = (f_1(y), \ldots, f_n(y)) \in \mathbb{C}^n$ with
\[
m = n, \quad 0 \leq k = \dim W < n, \quad \lambda = 2,
\]
\[
\tau = C_2' D d^{\tau T}, \quad \delta_1 = \delta_2 = \cdots = \delta_n = D, \quad \sigma = C_5/C_6, \quad U = \frac{1}{2} C_6 D^{n+1} \delta^T,
\]
where $C_2' = (C_1 + C_2) \left( 1 + \frac{1}{\log \delta} \right)$.

By Proposition 30, there exist polynomials $R_{T,D} \in \mathbb{Q}[z, X]$ satisfying (53) and (55).

Substituting the values of $D$ of $T$ defined by (73) and (74) to (56), we readily verify that this hypothesis is satisfied for any $k < 2n+1 - \log \frac{1}{\log \delta} (n+1)$.

Indeed, in this case we have $\delta^T = \left( \theta(W)^{\log \frac{1}{\log \delta} \frac{n+1}{n-k}} \right) \times c^{n+1} \log c$ and $D^{n+1} = \theta(W)^{1-\log \frac{1}{\log \delta} \frac{n+1}{n-k}} d(W)^{\log \frac{1}{\log \delta} \frac{n+1}{n-k}} \times c^{n+1}$ hence (56) is satisfied with $c > e^{C_3}$ and $\theta(W)$ defined by (72).

We verify also $\tau \lambda < U$ and for every real
\[
\tau \lambda < s \leq U
\]
(75)
we define \( R_s(X) = R_{T,D_s}(y,X) \in \mathbb{Q}(y)[X] \), where \( D_s = \left[ \left( \frac{2s}{C_6 \delta^T} \right)^{1/(n+1)} \right] \) and \( T \) is defined by (74). The constraints (75) imply

\[
\left[ \left( \frac{4C_4^2D}{C_6} \cdot \left( \frac{d}{\delta} \right)^T \right)^{n+1} \right] \leq D_s = \left[ \left( \frac{2s}{C_6 \delta^T} \right)^{1/(n+1)} \right] \leq D,
\]

so the quantity \( e(s) \) in Theorem 20 satisfies

\[
e(s) \leq \lambda.
\]

Moreover, (76) implies that for \( k < 2n+1 - \frac{\log d}{\log \delta} (n+1) \) the condition (56) is satisfied, hence the polynomials \( R_{T,D_s} \) satisfy (57). Thus the hypothesis (20) of Theorem 20 is verified.

We verify with the definition of \( D \) and \( T \), adjusting if necessary the constant \( c \),

\[
[\mathbb{Q}(y) : \mathbb{Q}] \cdot 3\lambda^{k+1} \delta_1 \cdots \delta_k (\delta_{k+1} t(W) + (k + 1) \tau d(W)) \leq U / ((n + 1)\sigma)^{k+1}
\]

and we infer from Theorem 20

\[
\log \text{Dist}(\mathbf{x}, W) \geq -U = -C\theta(W) \frac{\log d}{\log \delta} \frac{\delta^{k+1}}{n-k} d(W) \frac{\log \delta}{\log d} \frac{\delta^{k+1}}{n-k},
\]

finding the conclusion of Theorem 9. 

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