Non-abelian resonance: product and coproduct formulas

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Abstract We investigate the resonance varieties attached to a commutative differential graded algebra and to a representation of a Lie algebra, with emphasis on how these varieties behave under finite products and coproducts.

1 Introduction

Resonance varieties emerged as a distinctive object of study in the late 1990s, from the theory of hyperplane arrangements. Their usefulness became apparent in the past decade, when a slew of applications in geometry, topology, group theory, and combinatorics appeared.

The idea consists of turning the cohomology ring of a space $X$ into a family of cochain complexes, parametrized by the first cohomology group $H^1(X, \mathbb{C})$, and extracting certain varieties $R^i_{m}(X, \mathbb{C})$ from these data, as the loci where the cohomology of those cochain complexes jumps. Part of the importance of these resonance varieties is their close connection with a different kind of jumping loci: the characteristic varieties of $X$, which record the jumps in homology with coefficients in rank 1 local systems.

In recent years, various generalizations of these notions have been introduced in the literature, for instance in [3, 2, 7, 5]. The basic idea now is to...
replace the cohomology ring of a space by an algebraic analogue, to wit, a commutative, differential graded algebra \((A, d)\), and to replace the coefficient group \(\mathbb{C}\) by a finite-dimensional vector space \(V\), endowed with a representation \(\theta: \mathfrak{g} \to \mathfrak{gl}(V)\), for some finite-dimensional Lie algebra \(\mathfrak{g}\). In this setting, the parameter space for the higher-rank resonance varieties, \(R^i_m(A, \theta)\), is no longer \(H^1(A)\), but rather, the space of flat, \(\mathfrak{g}\)-valued connections on \(A\), which, according to the results of Goldman and Millson from [4], is the natural replacement for the variety of rank 1 local systems on \(X\).

In a previous paper, [6, §13], we established some basic product and coproduct formulas for the classical resonance varieties. In this note, we extend those results to the non-abelian case, using some of the machinery developed in [5]. In Theorem 1, we give a general upper bound on the varieties \(R^i_1(A \otimes A, \theta)\) in terms of the resonance varieties of the factors and the space of \(\mathfrak{g}\)-flat connections on the tensor product of the two \(\text{cdga}\)'s. In Theorem 2, we improve this bound to an equality of a similar flavor, in the case when the respective \(\text{cdga}\)'s have zero differentials, and \(\mathfrak{g}\) is either \(\mathfrak{sl}_2\) or \(\mathfrak{sol}_2\). Finally, in Corollary 5 and Theorem 3 we give precise formulas for the varieties \(R^i_1(A \vee A, \theta)\) associated to the wedge sum of two \(\text{cdga}\)'s.

2 Flat connections and holonomy Lie algebras

We start by introducing some basic notions (\(\text{cdga}\)'s, flat connections, holonomy Lie algebras), following in rough outline the exposition from [5].

2.1 Differential graded algebras and Lie algebras

Let \(A = (A^*, d)\) be a commutative, differential graded algebra (\(\text{cdga}\)) over the field of complex numbers, that is, a positively-graded \(\mathbb{C}\)-vector space \(A = \bigoplus_{i \geq 0} A^i\), endowed with a graded-commutative multiplication map \(\cdot: A^i \otimes A^j \to A^{i+j}\) and a differential \(d: A^i \to A^{i+1}\) satisfying \(d(a \cdot b) = da \cdot b + (-1)^i a \cdot db\), for every \(a \in A^i\) and \(b \in A^j\).

We will assume throughout that \(A\) is connected, i.e., \(A^0 = \mathbb{C}\), and of finite \(q\)-type, for some \(q \geq 1\), i.e., \(A^i\) is finite-dimensional, for all \(i \leq q\). Let \(Z^i(A) = \ker(d: A^i \to A^{i+1})\), \(B^i(A) = \text{im}(d: A^{i-1} \to A^i)\), and \(H^i(A) = Z^i(A)/B^i(A)\). For each \(i \leq q\), the dimension of this vector space, \(b_i(A) = \dim_{\mathbb{C}} H^i(A)\), is finite.

Now let \(\mathfrak{g}\) be a Lie algebra over \(\mathbb{C}\). On the vector space \(A \otimes \mathfrak{g}\), we may define a bracket by \([a \otimes x, b \otimes y]\) = \(ab \otimes [x, y]\) and a differential given by \(\partial(a \otimes x) = da \otimes x\), for \(a, b \in A\) and \(x, y \in \mathfrak{g}\). This construction produces a differential graded Lie algebra \((\text{dglg}, A \otimes \mathfrak{g} = (A^* \otimes \mathfrak{g}, \partial))\). It is readily verified that the assignment \((A, \mathfrak{g}) \mapsto A \otimes \mathfrak{g}\) is functorial in both arguments.
2.2 Flat, g-valued connections

Definition 1. An element $\omega \in A^1 \otimes g$ is called an infinitesimal, g-valued flat connection on $(A, d)$ if $\omega$ satisfies the Maurer–Cartan equation,

$$\partial \omega + [\omega, \omega]/2 = 0. \quad (1)$$

We will denote by $F(A, g)$ the subset of $A^1 \otimes g$ consisting of all flat connections. A typical element in $A^1 \otimes g$ is of the form $\omega = \sum_j \eta_j \otimes x_j$, with $\eta_j \in A^1$ and $x_j \in g$; the flatness condition amounts to

$$\sum_j d\eta_j \otimes x_j + \sum_{j<k} \eta_j \eta_k \otimes [x_j, x_k] = 0. \quad (2)$$

In the rank one case, i.e., the case when $g = \mathbb{C}$, the space $F(A, \mathbb{C})$ may be identified with the vector space $H^1(A) = \{ \omega \in A^1 \mid d\omega = 0 \}$. In particular, if $d = 0$, then $F(A, \mathbb{C}) = A^1$.

The bilinear map $P: A^1 \times g \to A^1 \otimes g$, $(\eta, g) \mapsto \eta \otimes g$ induces a map $P: H^1(A) \times g \to F(A, g)$. The essentially rank one part of the set of flat g-connections on $A$ is the image of this map:

$$F^1(A, g) = P(H^1(A) \times g). \quad (3)$$

2.3 Holonomy Lie algebra

An alternate view of the parameter space of flat connections is as follows. Let $A_1 = \text{Hom}(A^1, \mathbb{C})$ be the dual vector space. Let $\nabla: A_2 \to A_1 \wedge A_1$ be the dual to the multiplication map $A_1 \wedge A_1 \to A_2$, and let $d_1: A_2 \to A_1$ be the dual of the differential $d_1: A^1 \to A^2$.

Definition 2 ([5]). The holonomy Lie algebra of a cdga $A = (A^*, d)$ is the quotient of the free Lie algebra on the $\mathbb{C}$-vector space $A_1$ by the ideal generated by $\partial_A = d_1 + \nabla$:

$$\mathfrak{h}(A) = \text{Lie}(A_1)/(\text{im}(\partial_A)). \quad (4)$$

Remark 1. In the case when $d = 0$, the above definition coincides with the classical holonomy Lie algebra $\mathfrak{h}(A) = \text{Lie}(A_1)/(\text{im}(\nabla))$ of K.T. Chen [1]. In this situation, $\mathfrak{h}(A)$ inherits a natural grading from the free Lie algebra, compatible with the Lie bracket. Consequently, $\mathfrak{h}(A)$ is a finitely-presented, graded Lie algebra, with generators in degree 1 and relations in degree 2.

In general, though, the ideal generated by $\text{im}(\partial_A)$ is not homogeneous, and the Lie algebra $\mathfrak{h}(A)$ is not graded. Here is a concrete example, extracted from [5].
Example 1. Let $A$ be the exterior algebra on generators $x, y$ in degree 1, endowed with the differential given by $dx = 0$ and $dy = y \wedge x$, and let $\mathfrak{so}_2$ be the Borel subalgebra of $\mathfrak{so}_2$. Then $h(A) \cong \mathfrak{so}_2$, as (ungraded) Lie algebras.

The next lemma (see [5, §4] for details) identifies the set of flat, $\mathfrak{g}$-valued connections on a cdga $(A, d)$ with the set of Lie algebra morphisms from the holonomy Lie algebra of $(A, d)$ to $\mathfrak{g}$.

Lemma 1. The canonical isomorphism $A^1 \otimes \mathfrak{g} \cong \text{Hom}(A_1, \mathfrak{g})$ restricts to isomorphisms $\mathcal{F}(A, \mathfrak{g}) \cong \text{Hom}_{\text{Lie}}((h(A), \mathfrak{g})$ and $\mathcal{F}^1(A, \mathfrak{g}) \cong \text{Hom}_{\text{Lie}}^1((h(A), \mathfrak{g})$.

Here, $\text{Hom}_{\text{Lie}}^1((h(A), \mathfrak{g})$ denotes the subset of Lie algebra morphisms with at most 1-dimensional image.

3 Resonance varieties

In this section, we recall the definition of the Aomoto complexes associated to a cdga $(A, d)$ and a representation of a Lie algebra $\mathfrak{g}$, as well as the resonance varieties associated to these data, following the approach from [3, 2, 5].

3.1 Twisted differentials

Let $\theta: \mathfrak{g} \to \text{gl}(V)$ be a representation of our Lie algebra $\mathfrak{g}$ in a finite-dimensional, non-zero $\mathbb{C}$-vector space $V$. For each flat connection $\omega \in \mathcal{F}(A, \mathfrak{g})$, we make $A \otimes V$ into a cochain complex,

$$(A \otimes V, d_\omega): A^0 \otimes V \xrightarrow{d_\omega} A^1 \otimes V \xrightarrow{d_\omega} A^2 \otimes V \xrightarrow{d_\omega} \cdots,$$

using as differential the covariant derivative

$$d_\omega = d \otimes \text{id}_V + \text{ad}_\omega,$$

where $\text{ad}_\omega$ is defined via the Lie semi-direct product $V \rtimes_\theta \mathfrak{g}$. The flatness condition insures that $d_\omega^2 = 0$. In coordinates, if $\omega = \sum_j \eta_j \otimes x_j$, then

$$d_\omega(\alpha \otimes v) = d\alpha \otimes v + \sum_j \eta_j \alpha \otimes \theta(x_j)(v),$$

for all $\alpha \in A$ and $v \in V$.

It is readily seen that the multiplication map

$$\mu: (A, d) \otimes (A \otimes V, d_\omega) \to (A \otimes V, d_\omega), \quad a \otimes (b \otimes v) \mapsto ab \otimes v$$
defines the structure of a differential $A$-module on the Aomoto complex $(A^* \otimes V, d)$. In particular, the graded vector space $H^*(A \otimes V, d)$ is, in fact, a graded module over the ring $H^*(A)$.

### 3.2 Resonance varieties of a cdga

Associated to the above data are the resonance varieties

$$R^i_m(A, \theta) = \{ \omega \in \mathcal{F}(A, \mathfrak{g}) \mid \dim_{\mathbb{C}} H^i(A \otimes V, d) \geq m \}. \tag{9}$$

If $\mathfrak{g}$ is finite-dimensional, the sets $R^i_m(A, \theta)$ are Zariski closed subsets of $\mathcal{F}(A, \mathfrak{g})$, for all $i \leq q$ and $m \geq 0$. In the case when $\mathfrak{g} = \mathbb{C}$ and $\theta = \text{id}_{\mathbb{C}}$, we will simply write $R^i_m(A)$ for these varieties, viewed as algebraic subsets of $H^1(A)$. Clearly,

$$0 \in R^i_1(A, \theta) \iff 0 \in R^i_1(A) \iff H^i(A) \neq 0. \tag{10}$$

When $d = 0$, the varieties $R^i_m(A)$ are homogeneous subsets of $A^1$. This happens in the classical case, when $X$ is a path-connected space, and $A = H^*(X, \mathbb{C})$ is its cohomology algebra, endowed with the zero differential.

In general, though, the resonance varieties of a cdga are not homogeneous sets, even in the rank 1 case.

**Example 2.** Let $(A, d)$ be the cdga from Example 1. Then $H^1(A) = \mathbb{C}$, while $R^1_1(A) = \{0, 1\}$.

**Lemma 2.** Let $\omega = \eta \otimes g \in \mathcal{F}^1(A, \mathfrak{g})$.

1. If $\omega \in R^i_1(A, \theta)$, then $A^i \neq 0$.
2. Suppose $A^i \neq 0$ and $d = 0$. Then $\omega \in R^i_1(A, \theta)$ if and only if either $\eta \in R^i_1(A)$ or $\det(\theta(g)) = 0$.

**Proof.** The first claim is clear. When $d = 0$, recall that the rank one resonance variety $R^1_1(A)$ is homogeneous. The second claim then follows from [5, Corollary 3.6].

### 3.3 Resonance varieties of a Lie algebra

Let $\mathfrak{h}$ be a finitely generated Lie algebra, and let $\theta: \mathfrak{g} \to \text{gl}(V)$ be a representation of another Lie algebra. Associated to these data are the resonance varieties

$$R^i_m(\mathfrak{h}, \theta) = \{ \varphi \in \text{Hom}_{\text{Lie}}(\mathfrak{h}, \mathfrak{g}) \mid \dim_{\mathbb{C}} H^i(\mathfrak{h}, V_{\theta \varphi}) \geq m \}. \tag{11}$$
where \( V_{\theta \varphi} \) denotes the \( \mathbb{C} \)-vector space \( V \), viewed as a module over the enveloping algebra \( U(\mathfrak{h}) \) via the representation \( \theta \circ \varphi : \mathfrak{h} \rightarrow \mathfrak{gl}(V) \).

Now suppose \( \mathfrak{g} \) is finite-dimensional. Then the resonance varieties \( R^i_m(\mathfrak{h}, \theta) \) are Zariski-closed subsets of \( \text{Hom}_{\text{Lie}}(\mathfrak{h}, \mathfrak{g}) \), for all \( i \leq 1 \) and \( m \geq 0 \).

**Lemma 3 ([5]).** For each \( i \leq 1 \) and \( m \geq 0 \), the canonical isomorphism \( \mathcal{F}(\mathfrak{A}, \mathfrak{g}) \cong \text{Hom}_{\text{Lie}}(\mathfrak{h}(\mathfrak{A}), \mathfrak{g}) \) restricts to an isomorphism

\[
R^i_m(\mathfrak{A}, \theta) \cong R^i_m(\mathfrak{h}(\mathfrak{A}), \theta).
\]

**Example 3.** Let \( x_1, \ldots, x_n \) be a basis for \( \mathfrak{A}_1 \). Using Lemma 3 and [5, Lemma 2.3], we find that

\[
R^0_0(\mathfrak{h}(\mathfrak{A}), \theta) = \left\{ \varphi \in \text{Hom}_{\text{Lie}}(\mathfrak{h}(\mathfrak{A}), \mathfrak{g}) \mid \bigcap_{i=1}^n \ker(\theta \circ \varphi(x_i)) \neq 0 \right\}.
\]

### 4 Products

In this section, we study the way the various constructions outlined so far behave under (finite) product operations.

#### 4.1 Holonomy Lie algebra and products

Let \((\mathfrak{A}, d)\) and \((\mathfrak{A}, \bar{d})\) be two cdga's. The tensor product of these two \( \mathbb{C} \)-vector spaces, \( \mathfrak{A} \otimes \mathfrak{A} \), is again a cdga, with grading \( (\mathfrak{A} \otimes \mathfrak{A})^q = \bigoplus_{i+j=q} \mathfrak{A}_i \otimes \mathfrak{A}_j \), multiplication \( (a \otimes \bar{a}) \cdot (b \otimes \bar{b}) = (-1)^{|a||b|}(ab \otimes \bar{a}\bar{b}) \), and differential \( D \) given on homogeneous elements by \( D(a \otimes \bar{a}) = da \otimes \bar{a} + (-1)^{|a|}a \otimes \bar{d}a \).

The definition is motivated by the cartesian product of spaces, in which case the Künneth formula gives an isomorphism

\[
(H^*(X \times \overline{Y}, \mathbb{C}), D = 0) \cong (H^*(X, \mathbb{C}), d = 0) \otimes (H^*(\overline{Y}, \mathbb{C}), \overline{D} = 0).
\]

In [3, §9], we gave a product formula for holonomy Lie algebras in the 1-formal case. We now extend this formula to cdga's with non-zero differential.

**Proposition 1.** Let \( \mathfrak{A} \) and \( \mathfrak{A} \) be two connected cdga's. Then the Lie algebra \( \mathfrak{h}(\mathfrak{A} \otimes \mathfrak{A}) \) is generated by \( \mathfrak{A}_1 \oplus \mathfrak{A}_1 \), subject to the relations \( \partial_\mathfrak{A}(\mathfrak{A}_2) = 0 \), \( \partial_{\overline{\mathfrak{A}}}(\mathfrak{A}_2) = 0 \), and \([\mathfrak{A}_1, \mathfrak{A}_1] = 0\).

**Proof.** By construction, \( (\mathfrak{A} \otimes \mathfrak{A})^1 = \mathfrak{A}^1 \oplus \mathfrak{A}^1 \) and \( (\mathfrak{A} \otimes \mathfrak{A})^2 = \mathfrak{A}^2 \oplus \mathfrak{A}^2 \oplus (\mathfrak{A}^1 \otimes \mathfrak{A}^1) \). Plainly, \( D^1 \) restricts to \( d^1 \) on \( \mathfrak{A}^1 \) and to \( \overline{d}^1 \) on \( \mathfrak{A}^1 \). It is readily seen that the multiplication map on \( \mathfrak{A} \otimes \mathfrak{A} \) restricts to the multiplication maps on
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A\^1 \land A\^1 on \bar{A}\^1 \land \bar{A}\^1, respectively, and to the identity map on A\^1 \otimes \bar{A}\^1. By taking duals, we conclude that h(A \otimes \bar{A}) has the asserted presentation.

**Corollary 1.** The holonomy Lie algebra of a tensor product of cdga’s is isomorphic to the (categorical) product of the respective holonomy Lie algebras,

\[ h(A \otimes \bar{A}) \cong h(A) \times h(\bar{A}). \]

### 4.2 Flat connections and products

Proposition 1 also yields a formula for the representation variety of a tensor product of cdga’s.

**Corollary 2.** For any Lie algebra g,

\[
\text{Hom}_{\text{Lie}}(h(A \otimes \bar{A}), g) = \{ (\varphi, \bar{\varphi}) \in \text{Hom}_{\text{Lie}}(h(A), g) \times \\
\text{Hom}_{\text{Lie}}(h(\bar{A}), g) \mid [[\varphi(x), \bar{\varphi}(\bar{x})] = 0, \forall(x, \bar{x}) \in A_1 \oplus \bar{A}_1 \}\}.
\]

Furthermore, if g is abelian, then

\[
\text{Hom}_{\text{Lie}}(h(A \otimes \bar{A}), g) = \text{Hom}_{\text{Lie}}(h(A), g) \times \text{Hom}_{\text{Lie}}(h(\bar{A}), g).
\]

For the simple Lie algebra \( \mathfrak{sl}_2 \) and its Borel subalgebra \( \mathfrak{so}_2 \), the above corollary can be made more explicit.

**Corollary 3.** If \( g = \mathfrak{sl}_2 \) or \( \mathfrak{so}_2 \), then

\[
\text{Hom}_{\text{Lie}}(h(A \otimes \bar{A}), g) = \{ 0 \} \times \text{Hom}_{\text{Lie}}(h(A), g) \cup \\
\text{Hom}_{\text{Lie}}(h(A), g) \times \{ 0 \} \cup \text{Hom}_{\text{Lie}}^1(h(A \otimes \bar{A}), g).
\]

**Proof.** The inclusion \( \supseteq \) is clear. To prove the reverse inclusion, fix bases \( \{ x_1, \ldots, x_n \} \) and \( \{ \bar{x}_1, \ldots, \bar{x}_m \} \) for \( A_1 \) and \( \bar{A}_1 \). Let \( \varphi : h(A \otimes \bar{A}) \to g \) be a morphism of Lie algebras, and suppose there are indices \( i \) and \( j \) such that \( \varphi(x_i) \neq 0 \) and \( \varphi(\bar{x}_j) \neq 0 \). We need to prove that the family \( \{ \varphi(x_1), \ldots, \varphi(x_n), \varphi(\bar{x}_1), \ldots, \varphi(\bar{x}_m) \} \) has rank 1.

We know from Corollary 2 that \( [[\varphi(x_k), \varphi(\bar{x}_l)]] = 0 \), for all \( k \in [n] \) and \( l \in [m] \). Now note that, for any \( 0 \neq g, h \in g \), the following holds: \( [g, h] = 0 \) if and only if \( g = \lambda h \), for some \( \lambda \in \mathbb{C}^\times \). The desired conclusion is now immediate.
4.3 Resonance and products

We now turn to the jump loci of a tensor product of cdga’s. We start with a general upper bound for the depth 1 resonance varieties.

**Theorem 1.** For any representation \( \theta : \mathfrak{g} \to \mathfrak{gl}(V) \),

\[
\mathcal{R}_1^q(A \otimes \tilde{A}, \theta) \subseteq \left( \bigcup_{i \leq q} \mathcal{R}_1^i(A, \theta) \right) \times \left( \bigcup_{j \leq q} \mathcal{R}_1^j(\tilde{A}, \theta) \right) \cap \mathcal{F}(A \otimes \tilde{A}, \mathfrak{g}).
\]

**Proof.** By Lemma 1 and Corollary 2, every element \( \Omega \in \mathcal{F}(A \otimes \tilde{A}, \mathfrak{g}) \) can be written as \( \Omega = \omega + \tilde{\omega} \), for some \( \omega \in \mathcal{F}(A, \mathfrak{g}) \) and \( \tilde{\omega} \in \mathcal{F}(\tilde{A}, \mathfrak{g}) \). Setting up a first-quadrant double complex with \( E^{i,j}_0 = A^i \otimes \tilde{A}^j \otimes V \), horizontal differential \( d_\omega : E^{i,j}_0 \to E^{i+1,j}_0 \), and vertical differential \( d_{\tilde{\omega}} : E^{i,j}_0 \to E^{i,j+1}_0 \), we obtain spectral sequences starting at

\[
\text{hor } E^{i,j}_1 = H^i(A \otimes V, d_\omega) \otimes \tilde{A}^j \quad \text{and} \quad \text{vert } E^{i,j}_1 = A^i \otimes H^j(\tilde{A} \otimes V, d_{\tilde{\omega}}),
\]

respectively, and converging to \( H^{i+j}(A \otimes \tilde{A} \otimes V, d_{\Omega}) \). See (7).

Consequently, if either \( H^{\leq q}(A \otimes V, d_\omega) \) or \( H^{\leq q}(\tilde{A} \otimes V, d_{\tilde{\omega}}) \) vanishes, then \( H^q(A \otimes \tilde{A} \otimes V, d_{\Omega}) = 0 \). In view of definition (9), this completes the proof.

In general, the inclusion from Theorem 1 is strict. We illustrate this phenomenon with a simple example.

**Example 4.** Let \( A \) be the exterior algebra on a single generator in degree 1, let \( \mathfrak{g} = \mathfrak{gl}_2 \), and let \( \theta = \text{id}_\mathfrak{g} \). Using Example 3 and Corollary 2, we see that \( \mathcal{R}_1^0(A, \theta) = \{ (g, h) \in \mathfrak{gl}_2 \times \mathfrak{gl}_2 \mid \det(g) = 0 \} \), yet

\[
\mathcal{R}_1^0(A \otimes \tilde{A}, \theta) = \{ (g, h) \in \mathfrak{gl}_2 \times \mathfrak{gl}_2 \mid [g, h] = 0, \rank(g \mid h) < 2 \},
\]

which is a proper subset of \( (\mathcal{R}_1^0(A, \theta) \times \mathcal{R}_1^0(\tilde{A}, \theta)) \cap \mathcal{F}(A \otimes A, \mathfrak{g}) \).

4.4 Product formulas for resonance

Under certain additional hypotheses, the upper bound from Theorem 1 may be improved to an equality. First, as shown in [6, Proposition 13.1], such an equality holds in the formal, rank 1 case.

**Proposition 2 ([6]).** Assume both \( A \) and \( \tilde{A} \) have zero differential. Then

\[
\mathcal{R}_1^q(A \otimes \tilde{A}) = \bigcup_{i+j=q} \mathcal{R}_1^i(A) \times \mathcal{R}_1^j(\tilde{A}).
\]

Using this result, we now show that an analogous resonance formula holds for the non-abelian Lie algebras \( \mathfrak{sl}_2 \) and \( \mathfrak{so}_2 \).
Therefore, \( \eta \) and \( \bar{\eta} \) done. Otherwise, Proposition 2 implies that \( \eta \) then we must have claimed.

2. Consequently, \( H \) that \( \eta \)

5 Coproducts

In this final section, we study the way our various constructions behave under (finite) coproducts.

5.1 Holonomy Lie algebras and coproducts

Let \( A = (A^*, d) \) and \( \bar{A} = (\bar{A}^*, \bar{d}) \) be two connected cdga’s. Their wedge sum, \( A \vee \bar{A} \), is a new connected cdga, whose underlying graded vector space in
positive degrees is $A^+ \oplus \bar{A}^+$, with multiplication $(a, \bar{a}) \cdot (b, \bar{b}) = (ab, \bar{a}b)$, and differential $D = d + \bar{d}$.

The definition is motivated by the wedge operation on pointed spaces, in which case we have a well-known isomorphism

$$(H^*(X \vee \bar{X}), D = 0) \cong (H^*(X), d = 0) \vee (H^*(\bar{X}), \bar{d} = 0).$$  

We now extend the coproduct formula for 1-formal spaces from [3, §9], as follows.

**Proposition 3.** The holonomy Lie algebra $h(A \vee \bar{A})$ is generated by $A_1 \oplus \bar{A}_1$, with relations $\partial A_2 = 0$ and $\partial \bar{A}_2 = 0$.

**Proof.** By construction, $(A \vee \bar{A})^1 = A^1 \oplus \bar{A}^1$, $(A \vee \bar{A})^2 = A^2 \oplus \bar{A}^2$, and $D^1 = d^1 \oplus \bar{d}^1$. Moreover, the multiplication map on $A \vee \bar{A}$ restricts to the multiplication maps on $A^1 \wedge A^1$ and $\bar{A}^1 \wedge \bar{A}^1$, respectively, and is zero when restricted to $A^1 \otimes \bar{A}^1$. The conclusion follows at once.

**Corollary 4.** The holonomy Lie algebra of a wedge sum of cdga’s is isomorphic to the (categorical) coproduct of the respective holonomy Lie algebras,

$$h(A \vee \bar{A}) \cong h(A) \coprod h(\bar{A}).$$

### 5.2 Resonance and coproducts

As shown in [6, Proposition 13.3], the classical resonance varieties behave nicely with respect to wedges of spaces. Let us recall this result, in a form adapted to our purposes.

**Proposition 4 ([6]).** Assume both $A$ and $\bar{A}$ have zero differential. Then, for all $i > 1$,

$$R^i_1(A \vee \bar{A}) = R^i_1(A) \times H^1(\bar{A}) \cup H^1(A) \times R^i_1(\bar{A}).$$

If, moreover, $b_1(A) > 0$ and $b_1(\bar{A}) > 0$, then

$$R^i_1(A \vee \bar{A}) = H^1(A) \times H^1(\bar{A}).$$

Our goal for the rest of this section will be to extend the above proposition to the non-abelian setting, for cdga’s with non-zero differential. To that end, let $\mathfrak{g}$ be a Lie algebra, and let $\omega \in A^1 \otimes \mathfrak{g}$ and $\bar{\omega} \in \bar{A}^1 \otimes \mathfrak{g}$. Set $\Omega = \omega + \bar{\omega} \in (A \vee \bar{A})^1 \otimes \mathfrak{g}$.

**Lemma 4.** $\Omega$ is a flat connection if and only if both $\omega$ and $\bar{\omega}$ are flat.

**Proof.** By definition of multiplication in $A \vee \bar{A}$, we have that $a \cdot \bar{a} = 0$ for every $a \in A^+$ and $\bar{a} \in \bar{A}^+$. Hence, $[\omega, \bar{\omega}] = 0$, and the conclusion follows.
Now let \( \theta : g \to \mathfrak{gl}(V) \) be a representation. Given an element \( \omega \in F(A, g) \), we write \( Z^i_\omega = \ker(d_\omega: A^i \otimes V \to A^{i+1} \otimes V) \) and \( B^i_\omega = \text{im}(d_\omega: A^{i-1} \otimes V \to A^i \otimes V) \), and set \( H^i_\omega = Z^i_\omega/B^i_\omega \).

**Lemma 5.** For \( i > 0 \),
\[
d^i_\Omega = d^i_\omega \oplus d^i_\bar{\omega} : (A^i \otimes V) \oplus (\bar{A}^i \otimes V) \longrightarrow (A^{i+1} \otimes V) \oplus (\bar{A}^{i+1} \otimes V),
\]
while for \( i = 0 \)
\[
d^0_\Omega = (d^0_\omega, d^0_\bar{\omega}) : (A \vee \bar{A})^0 \otimes V \cong V \longrightarrow (A^1 \otimes V) \oplus (\bar{A}^1 \otimes V).
\]

**Proof.** Both claims follow from (7) and the construction of \( A \vee \bar{A} \), by straightforward direct computation.

**Corollary 5.** For each \( i > 1 \) and for any representation \( \theta : g \to \mathfrak{gl}(V) \),
\[
\mathcal{R}^i_\Omega(A \vee \bar{A}, \theta) = \mathcal{R}^i_\Omega(A, \theta) \times F(A, g) \cup F(A, g) \times \mathcal{R}^i_\Omega(\bar{A}, \theta).
\]

**Proof.** By Lemma 5, \( H^i_\Omega \cong H^i_\omega \oplus H^i_\bar{\omega} \). Using this isomorphism, the desired conclusion follows from Lemma 4.

### 5.3 A coproduct formula for degree 1 resonance

To conclude, we compute the degree 1 resonance variety of a wedge sum, \( \mathcal{R}^1_\Omega(A \vee \bar{A}, \theta) \). We start with two lemmas.

**Lemma 6.** There is a surjective homomorphism
\[
H^1((A \vee \bar{A}) \otimes V, d_\Omega) \overset{\Phi}{\longrightarrow} H^1(A \otimes V, d_\omega) \oplus H^1(\bar{A} \otimes V, d_{\bar{\omega}}),
\]
whose kernel is isomorphic to \( (B^1_\omega + B^1_{\bar{\omega}})/\text{im}((d^0_\omega + d^0_{\bar{\omega}}) \circ \Delta) \), where \( \Delta : V \to V \oplus V \) is the diagonal map.

**Proof.** Follows from Lemma 5.

**Lemma 7.** The homomorphism \( \Phi \) is injective if and only if \( V = Z^0_\omega + Z^0_{\bar{\omega}} \).

**Proof.** Start by noting that \( V \oplus V = \text{im}(\Delta) \oplus (V \times 0) \). A standard linear algebra argument, then, finishes the proof.

**Theorem 3.** Suppose both \( b_1(A) \) and \( b_1(\bar{A}) \) are positive, and at least one of them is greater than 1. Then, for any representation \( \theta : g \to \mathfrak{gl}(V) \),
\[
\mathcal{R}^1_\Omega(A \vee \bar{A}, \theta) = F(A \vee \bar{A}, g).
\]
Proof. Set $r = \dim V$. Using our hypothesis, we may assume that $b_1(A) > 1$ and $b_1(\bar{A}) \geq 1$. Supposing $H^1_{\Omega} = 0$ for some $\Omega = \omega + \bar{\omega} \in F(A \lor \bar{A}, g)$, we derive a contradiction, as follows.

Lemma 6 implies that $Z^1_\omega = B^1_\omega$ and $Z^1_{\bar{\omega}} = B^1_{\bar{\omega}}$. Furthermore, the discussion from §3.1 shows that $Z^1(A) \otimes Z^0_\omega \subseteq Z^1_\omega$ and $Z^1(\bar{A}) \otimes Z^0_{\bar{\omega}} \subseteq Z^1_{\bar{\omega}}$. Hence,

$$r - \dim Z^0_\omega = \dim B^1_\omega = \dim Z^1_\omega \geq b_1(A) \cdot \dim Z^0_\omega,$$

and so $\dim Z^0_\omega \leq r/(b_1(A) + 1) < r/2$. Similarly, $\dim Z^0_{\bar{\omega}} \leq r/2$.

Using again Lemma 6, we deduce that $\Phi$ must be injective. By Lemma 7,

$$r = \dim(Z^0_\omega + Z^0_{\bar{\omega}}) \leq \dim Z^0_\omega + \dim Z^0_{\bar{\omega}} < r,$$

a contradiction.

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References

1. K.-T. Chen, *Extension of $C^\infty$ function algebra by integrals and Malcev completion of $\pi_1$*, Adv. in Math. 23 (1977), 181–210.
2. A. Dimca, S. Papadima, *Nonabelian cohomology jump loci from an analytic viewpoint*, to appear in Communications in Contemporary Mathematics, available at arXiv:1206.3773v3.
3. A. Dimca, S. Papadima, A. Suciu, *Topology and geometry of cohomology jump loci*, Duke Math. Journal 148 (2009), no. 3, 405–457.
4. W. Goldman, J. Millson, *The deformation theory of representations of fundamental groups of compact Kähler manifolds*, Inst. Hautes Études Sci. Publ. Math. 67 (1988), 43–96.
5. A. Măcinic, S. Papadima, R. Popescu, A. Suciu, *Flat connections and resonance varieties: from rank one to higher ranks*, preprint arXiv:1312.1439v1.
6. S. Papadima, A. Suciu, *Bieri–Neumann–Strebel–Renz invariants and homology jumping loci*, Proc. London Math. Soc. 100 (2010), no. 3, 795–834.
7. S. Papadima, A. Suciu, *Jump loci in the equivariant spectral sequence*, preprint arXiv:1302.4075v2.