Mean-Field Transmission Power Control in Dense Networks

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Abstract—We consider uplink power control in wireless communication when a large number of users compete over the channel resources. The code-division multiple-access (CDMA) protocol, as a supporting technology of 3G networks accommodating signals from different sources over the code domain, represents the orthogonal multiple-access techniques. With the development of fifth-generation wireless networks, nonorthogonal multiple access (NOMA) is introduced to improve the efficiency of channel allocation. Our goal is to investigate whether the power-domain NOMA protocol can introduce performance improvement when the users interact with each other in a noncooperative manner. It is compared with the CDMA protocol, where the fierce competition among users jeopardizes the efficiency of channel usage. In this work, we conduct analysis with an aggregative game model, and show the existence and uniqueness of an equilibrium strategy. Next, we adopt the social welfare of the population as the performance metric, which is the average utility achieved by the user population. It is shown that under the corresponding equilibrium strategies, NOMA outperforms CDMA by the higher efficiency of channel access for uplink communications.

Index Terms—Aggregative game, code-division multiple access (CDMA), fifth-generation (5G), nonorthogonal multiple access (NOMA), successive interference cancellation (SIC).

I. INTRODUCTION

POWER allocation has been widely employed in wireless communications and wireless sensor networks. As a variant of resource allocation, it mainly deals with the tradeoff between the performance achieved and the power consumption. Moreover, recent advances in the fifth-generation (5G) communication networks [1] have led to a resurgence of interest in transmission power control.

Earlier works in wireless communications optimize the performance through appropriate energy allocation. Decentralized approaches are frequently investigated, among which game-theoretic methods are powerful tools for modeling noncooperative channel access behaviors, especially for uplink users. Alpcan et al. [2] and Huang et al. [3] considered a channel access game where each user strives for a better quality of service. Centralized power allocation for maximizing the sum data rate is considered by Fischione et al. [4].

Recent studies address transmission power allocation for a large number of users at the mean-field limits. Huang et al. [5], [6] investigated uplink power control for a large number of players within a mean-field model. Moreover, Semasinghe and Hossain [7] dealt with downlink power allocation and compared the mean-field equilibrium performance under different utility functions. Typically, mean-field games consider dynamic models with an infinite number of players, where the impact from the opponents is modeled collectively as a mean-field term, as illustrated by Caines et al. [8]. As an alternative of game-theoretic model with mean-field type coupling, an aggregative game model [9] featuring static decision with either finite or infinite players is more appropriate for resource allocation in a wireless communication system.

Aggregative game has been widely adopted in wireless communications and networked games. For example, decentralized channel access among secondary users in a cognitive radio system is modeled by Pang et al. [10] as an aggregative game. A similar model is also employed for a large-scale channel access with incentive design at the base station, as investigated by Zhou et al. [11]. These works adopt conventional mean-field modeling where each agent faces the same aggregate of the opponents’ strategy. However, in a more recent work on networked games by Parise and Ozdaglar [12], where the mean-field aggregate faced by different agents is dependent on its neighboring network topology, variational inequalities are employed for the equilibrium analysis.

In 5G communication networks, the users are able to share the same portion of physical resource with nonorthogonal multiple access (NOMA) protocol. The receivers under the NOMA protocol adopt successive interference cancellation (SIC) [13], which improves the chance of successful decoding at the receiver.

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Our results are partially related to the existence and uniqueness of equilibrium strategies under code division multiple access (CDMA), as investigated by Alpcan et al. [2] as well as Aziz and Caines [14], and we have extended the equilibrium analysis to NOMA. The main difference between the modeling and literature, in this work, is twofold. First, we employed an aggressive game model with mean-field limits for NOMA, which to the best of our knowledge is new. Second, we have formulated the power control game under CDMA and NOMA in a unified framework, which enables a qualitative performance comparison. Following the direction of Xu and Cumanan [15] as well as Wei et al. [16], this work further discusses the welfare comparison of uplink channel access between the conventional orthogonal multiple access (OMA) schemes and NOMA. Past works [17]–[19] only investigated the performance comparison with centralized resource allocation, where all the users have a common objective function and their channel access behaviors are dictated by the base station. In this article, we consider a population of selfish users accessing the shared wireless channel with a noncooperative game model. This is suitable when the channel users are interested in pursuing their individual objectives and are accommodated by a micro base station in the 5G network.

The contributions of this article are summarized as follows.

1) We model the interactions among a large number of players under CDMA and NOMA as aggregative games, where the opponents’ action is modeled as a mean-field effect (i.e., the interference). The equilibria under CDMA and NOMA are characterized in Theorems 1 and 2. To the best of our knowledge, such an aggregative game framework with mean-field limit modeling for NOMA (with nonidentical mean-field interferences faced by different users) is new, and the existence and uniqueness of the equilibrium power control strategy under NOMA has not been addressed in the literature. The results we have obtained do not depend on a specific channel fading model, i.e., the distribution of the channel gain, and can be applied to a wide range of physical environments.

2) Through establishing a contraction property of the strategy update at each user in the population, a distributed algorithm (Algorithm 1) is proposed for CDMA (Theor. 1) and NOMA (Theor. 2) such that the strategy profile of the user population is guaranteed to converge to the unique equilibrium strategy.

3) We compare the social welfare at the equilibria between CDMA and NOMA. Due to the difficulties in calculating the equilibrium strategies and directly evaluating the corresponding values of the social welfare, we propose a perturbation-based approach to find the trend of change in the equilibrium performance from CDMA to NOMA. The social welfare achieved under the equilibrium of power control game with NOMA strictly dominates that with CDMA (Theor. 3).

The remainder of this article is organized as follows. The preliminaries on wireless communications are introduced in Section II. In Section III, we formulate the interaction between channel users as an aggregative game. Sections IV and V characterize the equilibrium strategies under CDMA and NOMA protocols, respectively. In Section VI, we compare the social welfare at the equilibria. Section VII studies the individual behaviors at the equilibrium. To illustrate the social welfare comparison results, we present the numerical simulations in Section VIII. Finally, Section IX concludes this article.

To highlight the structure and contribution of this article, a flowchart is displayed in Fig. 1, where the comparison between CDMA and NOMA constitutes a key contribution.

Notations: We denote the set of non-negative real numbers by $\mathbb{R}_+$, the set of positive real numbers by $\mathbb{R}_{++}$, and the set of non-negative integers by $\mathbb{N}_+$. The abbreviation “a.e.” is adopted for “almost everywhere.”

II. PRELIMINARIES ON UPLINK WIRELESS COMMUNICATION

In this article, we consider a transmission power allocation problem that arises in wireless networks where multiple signal sources are users of a single wireless communication channel. The block diagram of the problem is presented in Fig. 2 and the details of each component are elaborated as follows.

A. Data Sources

There are $N$ independent data sources as selfish users of the wireless channel. The signal provided by each data source is modeled as a continuous-time waveform $x_i(t)$ ($t \in \mathbb{R}_+$), where $i \in \mathcal{N} := \{1, 2, \ldots, N\}$. Assume that each of the signal source $x_i(t)$ has a unit average power level throughout the time horizon, i.e., $\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}[|x_i(t)|^2] dt = 1$ holds for any $i \in \mathcal{N}$. This is for notational convenience in describing the power level of the transmitted signal at the antennas.

For transmission through digital devices, a sampled version $x_i[k]$ ($k \in \mathbb{N}_+$) of the original signal $x_i(t)$ ($t \in \mathbb{R}_+$) is generated. Hence, we obtain the discrete-time signal sources
\[ x_i[k] := x_i(k \cdot \Delta t) \text{ for each } i \in \mathcal{N}. \] Since each signal source contains different amount of information, they feature different data rates. We denote the data rate of source \( i \) as \( R_i \) bits/s (\( R_i > 0 \)).

### B. Gaussian Channel and Transmission Antennas

The communication channel between the data sources and the base station can be modeled as an additive white Gaussian noise (AWGN) channel as follows:

\[ y[k] = \sum_{i=1}^{N} \sqrt{a_i} h_i x_i[k] + w[k] \]  

where \( x_i[k] \) is the transmitted signal with transmission power level \( a_i \), \( w[k] \sim \mathcal{C}\mathcal{N}(0, N_0) \) is a complex Gaussian white noise process, and \( y[k] \) is the aggregate received signal at the base station. In addition, we assume time-invariant flat-fading channel at each data source and denote the uplink channel gain for data source \( i \) as a \( C \)-valued random variable \( h_i \). The probability density function (PDF) of the squared magnitude \( ||h_i||^2 \geq 0 \) of the channel gain is \( f_i(x) \). Obviously, \( f_i(x) \geq 0 \) for any \( x \in \mathbb{R}_+ \), and \( \int_{\mathbb{R}_+} f_i(x) \, dx = 1 \).

In this article, we assume that the channel gains \( ||h_1||^2, ||h_2||^2, \ldots, ||h_N||^2 \) are independent identically distributed random variables, for which the PDF is simply denoted as \( f_i \). Intuitively, all the local users are located within the coverage area of a base station, so that they share similar environments. For more generality, we do not take a specific form of distribution for the channel gain \( h_i \).

**Assumption 1:** The squared norm of the channel gain has a finite first-order moment: \( \mathbb{E}[||h_i||^2] := \int_{\mathbb{R}_+} x f(x) \, dx < \infty \).

### C. Base Station

A base station is located at the end of the uplink Gaussian channel, which is capable of sensing the channel gains of all communication links. The signal received from each data source \( i \) will be processed independently at the base station. The specific physical model adopted for describing the fundamental limits in signal decoding as well as two different decoding strategies is introduced in the following.

The data rate of source \( i \) is \( R_i > 0 \). Based on Shannon’s theorem [20], in order to reliably decode the signal from data source \( i \) with decoding error going to zero asymptotically as the block length increases, it is necessary for the instantaneous channel capacity of this link to exceed the corresponding data rate \( R_i \) [21] as follows:

\[ \log_2 (1 + ||h_i||^2 S I N R_i) > R_i. \]  

We define the indicator for the successful decoding of the signal from data source \( i \) as \( \gamma^{(i)} \in \{ 0, 1 \} \), where \( \gamma^{(i)} = 1 \) indicates successful decoding.

The distribution for the decoding outcomes \( \gamma^{(i)} \) relies on the distribution of \( ||h_i|| \), and is determined by the outage probability

\[ p_{\text{out}}^{(i)}(R_i) := \mathbb{P} \left\{ \log_2 (1 + ||h_i||^2 S I N R_i) < R_i \right\}. \]  

Accordingly, the probability of a packet arrival can be expressed as \( \mathbb{P} \{ \gamma^{(i)} = 1 \} = 1 - p_{\text{out}}^{(i)}(R_i) \).

For a Gaussian channel, there are mainly two decoding strategies [22] when multiple signal sources are transmitting through their uplink channels: single-user decoding (SUD) in CDMA and multipacket reception (MPR) in NOMA. The power allocation for the data sources will differ when different decoding strategies are employed. Details of these protocols are elaborated in Sections IV and V.

### III. Problem Formulation: An Aggregative Game Model for Uplink Power Control

Each data source \( i \) aims at uploading its local information to the base station, which can be considered as different users sharing a common computer network attempting to transmit its data to a distant server, such as cloud storage or cloud computing services. Due to the selfish nature, each data source aims only at improving the chance of successfully decoding its own signal source at the base station while consuming less power.

The channel sharing behavior among these data sources can be modeled in a noncooperative game-theoretic framework. To reduce the computational complexity for equilibrium analysis when a large number of players are involved, we adopt the idea of aggregative games [9], where the impact of opponents’ actions on each player is modeled as a collective effect. Then, each player chooses its action as a response to this collective effect. Now, we present the aggregative game model in detail.

#### A. Aggregative Game Model

In dense networks, a large number of users will attempt to access the Internet through a single access point. We employ a nonatomic game [23] model in this work, where the effect of a single player on the overall population is negligible as the number of players \( N \to \infty \).

In this article, the interactive behavior among different users when competing over the uplink wireless channel is modeled as an aggregative game \( \mathcal{G} = (M, A, u) \), of which each component is elaborated as follows.

1) **Set of Players \( M \):** We adopt symbolic representations for each type of users. With a slight abuse of notations, the realization of the random variable \( h_i \) is denoted as \( h_i \in \mathbb{C} \) as well, which is the channel gain of user \( i \). Then, the identifier of this user is defined as \( \theta_i = ||h_i||^2 \in \mathbb{R}_+ \). We define the set of all possible identifiers as \( M \subset \mathbb{R}_+ \) such that any possible identifiers \( \theta_i \) belong to \( M \). To avoid triviality, we assume \( M \neq \emptyset \) and \( 0 \notin M \). Thus, the set of all possible identifiers is a subset of positive real numbers, i.e., \( M \subset \mathbb{R}_+ \). Also, a Borel \( \sigma \)-algebra is generated for the set \( M \) and is denoted as \( \mathcal{B}(M) \).

In practice, for players with different channel gains, each gain takes up a certain ratio of presence in the population. We define a probability measure \( P \) over the measurable space \( (M, \mathcal{B}(M)) \) to model the population of players, where the probability measure \( P \) is induced by the PDF \( f(x) \) given in Theorem 1. To be specific, we have \( P : \mathcal{B}(M) 

\rightarrow [0, 1] \) such that for any \( A \in \mathcal{B}(M) \), we have \( P(A) := \int_{x \in A} f(x) \, dx \) and \( P(M) = 1 \). Hence, we choose \( M \subset \mathbb{R}_+ \) as the set of players.
**Assumption 2:** $f(x) > 0$ for any $x \in M$, i.e., the PDF $f$ has positive value on the set of all possible identifiers $M \subset \mathbb{R}_{++}$.

2) **Strategy Space A:** Now we define the strategy space of all players in the game.

Since the transmission antennas are controlled by analog circuits, the maximum transmission power is bounded. We define the feasible set of transmission power levels as a compact and convex set $\mathcal{E} := [E_{\min}, E_{\max}]$ ($0 \leq E_{\min} < E_{\max} < \infty$) without loss of generality.

We denote the strategy of all the players in $M$ as a mapping $p : M \rightarrow \mathcal{E}$ such that the action chosen by the player $\theta_i \in M$ is $a_i := p(\theta_i) \in \mathcal{E}$.

Before giving the formal definition of the feasible strategy space of the players in $M$, we introduce a new measure with which the aggregate effect of the players' strategies can be evaluated. For $M \subset \mathbb{R}_{++}$, a measure space is defined as $(M, \mathcal{B}(M), \nu)$ based on the Lebesgue measure $\lambda$. We define a new measure $\nu$ as follows:

$$\nu(A) := \int_A w(x) d\lambda(x) \quad \forall A \in \mathcal{B}(M)$$  \hspace{1cm} (4)

where $w(x) := xf(x) \geq 0$, $x \in M$ is a weight function.

Consequently, a new measure space $(M, \mathcal{B}(M), \nu)$ is generated, based on which we define the feasible strategy space as the set of functions

$$\mathcal{A} := \{p : (p : M \rightarrow \mathbb{R}) \& (p = \nu - a.e. \mathcal{E}-valued)\}$$  \hspace{1cm} (5)

where $p$ is a Lebesgue measurable function.

**Definition 1:** For any Lebesgue measurable function $g : M \rightarrow \mathbb{R}$, we introduce the norm

$$\|g\|_{\nu}^2 := \int_{x \in M} |g(x)| d\nu(x) = \int_{x \in M} |g(x)| w(x) d\lambda(x).$$  \hspace{1cm} (6)

**Definition 2:** For a bounded Lebesgue measurable function $f : M \rightarrow \mathbb{R}$, we define its essential supremum norm based on the measure $\nu$ as $\|f\|_{\nu} := \inf\{C > 0 : |f(x)| \leq C \nu-a.e.\}$.

3) **Utility Function $u$:** The utility function of a player with identifier $\theta_i \in M$ is denoted as $u(a_i, p, \theta_i)$, where it chooses an action $a_i \in \mathcal{E}$ in response to $p \in \mathcal{A}$, which is adopted by the opponent. Its utility function takes a tradeoff between the achieved data rate and energy consumptions, which has the form

$$u(a_i, p, \theta_i) = \log_2 (1 + \theta_i \cdot \text{SINR}(p, \theta_i)) - \beta_i a_i \quad \forall \theta_i \in M$$  \hspace{1cm} (7)

where $\beta_i > 0$ is the power penalty parameter at player $i$. As CDMA and NOMA adopt different decoding algorithms, for a fixed strategy profile $p \in \mathcal{A}$, the receiving signal-to-interference-plus-noise ratio (SINR) at the receiver’s side will have different values under these two protocols.

**Assumption 3:** The power penalty parameter is identical for each uplink user $u \in \mathcal{N}$, i.e., $\beta_1 = \beta_2 = \cdots = \beta_N = \beta > 0$.

We will adopt this common power penalty parameter $\beta$ for all the users in the following analysis.

4) **Information Set $I$:** Though the realization of the channel gain $h_i$ is only disclosed to the user itself, the PDF $f(x)$ of the user identity $\theta_i = \|h_i\|^2$ in the population is common knowledge. The realized value of the channel gain for each user can be interpreted as sampled independently from this distribution $f(x)$.

For user $\theta_i \in M$, the available information set is

$$I(\theta_i) = \{\theta_i, N_0\}, \quad \theta_i \in M$$  \hspace{1cm} (8)

where $N_0$ is the power spectrum density of the Gaussian noise in the channel, and $f(\theta)$ is the PDF of the user identifier.

B. **Solution Concept for a Game**

Assume all players are rational in the sense that each of them will seek to optimize its own utility function (i.e., self-interested). In addition, the rationality of the players’ behaviors is common knowledge among all players participating in the game $\mathcal{G}$, as illustrated by Gibbons [24]. Then, each player $i$ will choose a power level from $\mathcal{E}$ to optimize its utility based on its own information set. We call it a decision-making process if a player $\theta_i \in M$ predicts the strategies of other players based on the information available and chooses an action in response to the predicted actions.

Different from an optimization problem, in a game-theoretic setup, the decision-making process of each player $i$ is in the form of a “best response” to the action profile of its opponents. Under this decision pattern, it is of interest to find an action profile of all players on which they agree. Thus, we proceed to introduce the concept of mean-field equilibrium.

**Definition 3 (Mean-Field Equilibrium):** A strategy profile $p^* \in \mathcal{A}$ is a mean-field equilibrium of a game with an infinite number of players if for any $\theta_i \in M$, we have

$$u(p^*(\theta_i), p^*, \theta_i) \geq u(a_i^*, p^*, \theta_i) \quad \forall a_i \in \mathcal{E}.\hspace{1cm}(9)$$

For the aggregate game $\mathcal{G}$, we define the best response of user $\theta_i \in M$ to the opponents’ strategy $p \in \mathcal{A}$ as a set-valued mapping $BR : M \times A \rightarrow 2^E$. For each $\theta_i \in M$, we have

$$BR(\theta_i, p) := \left\{ a_i^* \in \mathcal{E} : u(a_i^*, p, \theta_i) \geq u(a_i, p, \theta_i) \forall a_i \in \mathcal{E} \right\}.\hspace{1cm}(10)$$

Therefore, a strategy $p^* \in \mathcal{A}$ is a mean-field equilibrium if and only if for any $\theta_i \in M$, $p^*(\theta_i) \in BR(\theta_i, p^*)$.

IV. **CDMA Transmission Power Game**

A. **SUD in CDMA**

1) **Descriptions of CDMA Protocol and SUD Decoding Algorithm:** Under the CDMA communication protocol, each user is allocated a unique signature sequence so that its transmitted signal can be spread over different subcarriers (i.e., code chips) in order to mitigate the interference between different users.

As mentioned by Tse and Hanly [25] and Ferrante and Di Benedetto [26], through appropriate selection of signature sequences of length $n_s$, the squared cross-correlation between the signature sequences $s_k$ and $s_j$ of users $k$ and $j$ ($k \neq j$) can be expressed as $\rho_{k,j} = \langle s_k, s_j \rangle^2 \approx \frac{1}{n_s} = \frac{\nu}{E}$, which is the gain of the interference induced by user $k$ to the received signal from user $j$. Since each user in CDMA is assigned a distinct signature sequence, the length $n_s$ satisfies $n_s \geq N$. bliss.
i.e., $0 < \alpha \leq 1$. In practice, as $\rho_{k,j} = \frac{1}{L}$, it is preferred to have a larger length $n_k$ in order to reduce the cross-correlation between signature sequences assigned to different users, and then limit the interference introduced. Therefore, the parameter $\alpha$ is often chosen to satisfy $0 < \alpha \ll 1$.

The SINR of the signal from data source $i$ is expressed as
\[
SINR_i = \frac{a_i}{\sum_{j \neq i} \rho_j a_j \| h_j \|^2 + N_0} = \frac{a_i}{\sum_{j \neq i} \alpha_j a_j \| h_j \|^2 + N_0}
\]  
(11)
for any $i \in \{1, 2, \ldots, N\}$. The receiver will attempt to decode the signal from each communication link independently.

2) Utility Functions of the Aggregative Game Using SUD: When the population size $N \to \infty$, if the user $\theta_i \in M$ chooses $a_i \in \mathcal{E}$ as its transmission power, its received SINR is given as
\[
SINR(p, \theta_i) = \lim_{N \to \infty} \frac{a_i}{\sum_{j \neq i} \rho_j a_j \theta_j + N_0}
\]
where almost sure convergence holds by Theorem 1 and Kolmogorov’s Strong Law of large numbers.

Then, the utility function of any user $\theta_i \in M$ in the case of CDMA can be expressed as
\[
u(a_i, p, \theta_i) = \log_2 \left( 1 + \frac{\theta_i a_i}{\alpha \mathbb{E}[\rho(\theta_j) \theta_j] + N_0} \right) - \beta a_i.
\]  
(12)

B. Analysis of the CDMA Transmission Power Equilibrium

Before we attempt to find the best response of player $\theta_i$, we first define a projection operator on $\mathbb{R}$ according to Bertsekas and Tsitsiklis [27].

Definition 4: For any given closed interval $X \subset \mathbb{R}$, define an orthogonal projection operator $P_X : \mathbb{R} \to X$ such that
\[
P_X(x) := \arg \min_{z \in X} |z - x| \forall x \in \mathbb{R}.
\]  
(13)

Given the utility function (12) and a fixed strategy $p \in \mathcal{E}$ of the opponents $M \setminus \{\theta_i\}$, we can obtain the optimal action $a_i^* \in \mathcal{E}$ of the player $\theta_i$ based on the best response operator in (10)
\[
a_i^* \in \mathcal{E}(\theta_i, p)
\]
\[
= \left\{ P_\mathcal{E} \left( \frac{1}{\beta \ln 2} - \frac{\alpha \mathbb{E}[\rho(\theta_j) \theta_j] + N_0}{\theta_i} \right) \right\}, \quad \theta_i \in M.
\]  
(14)

As the utility function $\nu(a_i, p, \theta_i)$ of the player is strictly concave with respect to the power control action $a_i \in \mathcal{E}$, there is a unique maximizer $a_i^*$ of the utility function, as indicated in [28, Th. 9.17]. Accordingly, the set of the best response of any player $\alpha_i \in M$ is a singleton.

Then, the space of all strategy profiles that induced a bounded interference term $\alpha \mathbb{E}[\rho(\theta_j) \theta_j]$ under the CDMA protocol is a vector space defined as
\[
L^1(M, \mathbb{R}, \nu) := \{ p : (p : M \to \mathbb{R}) \& (\|p\|_1^\nu < \infty) \}. 
\]  
(15)

The operator to be defined in the following will perform the truncation of any strategies in the space $(L^1(M, \mathbb{R}, \nu), \| \cdot \|_1^\nu)$ to the set of feasible strategies $A \subset L^1(M, \mathbb{R}, \nu)$, due to the power limitations of the circuits of the transmitter.

Definition 5: Given the set of feasible strategies $A \subset L^1(M, \mathbb{R}, \nu)$, the truncation operator is the mapping $T : L^1(M, \mathbb{R}, \nu) \to A$ such that for an arbitrary $p \in L^1(M, \mathbb{R}, \nu)$, we have that the resulting strategy profile
\[
\hat{p} := T(p)
\]  
(16)
which satisfies $\hat{p}(x) = P_\mathcal{E}(p(x)), \forall x \in M$.

A property of the operator $T$ is given in the following lemma.

Lemma 1: The operator $T : L^1(M, \mathbb{R}, \nu) \to L^1(M, \mathbb{R}, \nu)$ is nonexpansive, i.e., for two arbitrarily picked elements $p^{(1)}, p^{(2)} \in L^1(M, \mathbb{R}, \nu)$, we have
\[
\| T\left( p^{(1)} \right) - T\left( p^{(2)} \right) \|_1^\nu \leq \|p^{(1)} - p^{(2)}\|_1^\nu.
\]  
(17)

Proof: This lemma is a direct extension of the projection theorem in Euclidean space [27]. See Lemma 1 in the full version [29] of this article for details.

Now we establish the existence and uniqueness of mean-field equilibrium in the case of CDMA communication protocol.

Theorem 1: Assume $\alpha < 1$. Then for CDMA, there exists a unique mean-field equilibrium $p^* \in \mathcal{A} \subset L^1(M, \mathbb{R}, \nu)$ with single user detection and utility function (12). Moreover, starting from any initial strategy $p_0 \in \mathcal{A}$, the unique mean-field equilibrium $p^*$ can be obtained through a recursive update based on the best response operator, i.e., $\lim_{k \to \infty} p_k = p^*$. 

Proof: The theorem can be proved by establishing that the best response operator under CDMA is a contraction mapping in the space $L^1(M, \mathbb{R}, \nu)$. See Appendix I of the full version [29] of this article.

Moreover, we have the following result on the continuity and monotonicity of the equilibrium strategy profile under the CDMA protocol.

Corollary 1: The mean-field equilibrium strategy $p^* : M \to \mathcal{E}$ for CDMA is continuous and monotonically increasing with respect to the identifier $\theta_i \in M$.

Proof: See Appendix II of the full version [29] of this article.

Remark 1: Since our model uses an infinite number of users to approximate the behavior of a large but finite user population, it is natural to raise the question on how accurate this approximation is. We take CDMA with a finite population of $N$ users as an instance. Then, we define the mean-field interference as the average receiving power level of the user population over the shared Gaussian channel, i.e., $\nu := (1/N) \sum_{j=1}^{N} \rho(\theta_j) \theta_j$, which is identical for all channel users. On the other hand, each user $\theta_i$ chooses its optimal action in response to the interference $\nu$, i.e., $p(\theta_i) = a_i^* = \arg \max_{a_i \in \mathcal{E}} [\nu(a_i, p, \theta_i)] = \frac{1}{\beta \ln 2} - \frac{\alpha \mathbb{E}[\rho(\theta_j) \theta_j] + N_0}{\theta_i}$, where we assume the set of feasible power levels $\mathcal{E}$ is sufficiently large such that no truncation is performed. Hence, by calculating the fixed point of the mean-field term $\nu$, a closed-form expression of the equilibrium strategy can be obtained as $p^*(\theta_i) := \frac{1}{\beta \ln 2} - \frac{\alpha \mathbb{E}[\rho(\theta_j) \theta_j] + N_0}{\theta_i}$.
\[ \frac{1}{N} \left( \frac{N_i}{1 + \alpha} + \frac{1}{(1 + \alpha)^{N_i}} \right) \sum_{i=1}^{N_i} \left( \frac{1}{N} \sum_{j=1}^{N} \| h_{ij} \|^2 \right) \] for \( i = 1, 2, \ldots, N \). As \( N \to \infty \), based on the Strong Law of large number, we have \( \frac{1}{N} \sum_{i=1}^{N} \| h_{ij} \|^2 = \mathbb{E}[\| h_{ij} \|^2] \). According to [30, Theor. 2.5.7], the speed of almost sure convergence is faster than \( N^{-0.5} (\log N)^{0.5+\varepsilon} \) for any \( \varepsilon > 0 \).

V. NOMA TRANSmission POWER GAME

A. MPR IN NOMA

1) Power-Domain NOMA and SIC Decoding: For power-domain NOMA, we still adopt the spread-spectrum technique. For convenience and fairness of social welfare comparison between CDMA and NOMA, the same set of signature sequences \( \{ s_j \}_{j=1}^{N} \) is allocated to the users, hence \( 0 < \alpha \leq 1 \). The main difference between CDMA and NOMA is reflected in the specific decoding algorithm adopted by the receiver.

Now we explicitly give the procedures of the SIC decoding algorithm for power-domain NOMA. This requires a proper determination of the decoding order of signals from different sources. Consider the decoding order of the signals as a vector \( v := (v_1, v_2, \ldots, v_N) \), where each index \( v_j \in \{1, 2, \ldots, N\} \) is distinct. Then, the SINR of each signal source upon its turn of decoding can be expressed as

\[ \text{SINR}_{v_i} = \frac{p(v_i) \sum_{j \in \{1, 2, \ldots, N\}/v_i} \mathbb{E}[h_{ij}^2]}{N_0} \]  \hspace{1cm} (18)

where \( I_d(v_i) \) is the set of successfully decoded signal sources before \( v_i \).

By Xia et al. [31], the signal that was not successfully decoded will be treated as interference in the subsequent decoding procedures. In this case, the successful decoding set from the perspective of user \( v_i \) is expressed as

\[ I_d(v_i) = \{ v_j : 1 \leq j \leq i - 1 \text{ and } \gamma^{(j)} = 1 \} \]  \hspace{1cm} (19)

and the decoding procedure terminates after all data sources \( i \in \{1, 2, \ldots, N\} \) have been attempted for decoding.

Though the complexity introduced by SIC cannot be ignored [32], the online implementation of SIC under the power-domain NOMA considered in this article is feasible due to the following reasons. First, recent improvements in the computational capability have enabled the implementations of SIC at user equipment [32], such as the network-assisted interference cancellation and suppression terminals investigated by Zhou et al. [33]. Second, the noncooperative channel access model corresponds to the scenario of a micro base station serving a moderate user population size, typically around a hundred users. For such a user population, it is feasible to adopt SIC decoding during online implementation.

Remark 2: For convenience of analysis, we have assumed that a fixed decoding order (i.e., the descending order of the channel gain) is employed at the base station. However, from the perspective of a practical implementation, as the channel users are noncooperative decision makers, it is in general difficult to regulate different users to choose their actions based on the predetermined SIC decoding order broadcasted by the base station. The requirement that each channel user will automatically adopt the decoding order broadcasted by the base station can be interpreted as a communication protocol programmed into the user equipment.

On the other hand, NOMA with a fixed SIC decoding order can be considered as a benchmark for possible performance achievable by NOMA. Specifically, if an optimal decoding order for NOMA exists, it will perform no worse than the benchmark. Hence, we can without loss of generality analyze NOMA with a fixed decoding order.

2) Utility Functions for the Aggregative Game Using SIC: In this article, we restrict our consideration to the case where the SIC algorithm at the base station follows the descending order of the squared norm of the channel gain.

For simplicity of analysis, we assume that at each step of SIC, the interference caused by users decoded prior to this step is perfectly canceled regardless of their decoding outcomes, which is similar to the model in [18]. Thus, the SINR is approximated by

\[ \text{SINR} = \frac{p}{\sum_{j \neq i} \mathbb{E}[h_{ij}^2]} \]  \hspace{1cm} (20)

Similar to the analysis of CDMA, when the number of players \( N \to \infty \) under the NOMA protocol, the SINR of the received signal from player \( \theta_i \in M \) is expressed as

\[ \text{SINR}(p, \theta_i) = \lim_{N \to \infty} \frac{\mathbb{E}[h_{ij}^2]}{\sum_{j \neq i} \mathbb{E}[h_{ij}^2]} \]  \hspace{1cm} (21)

The utility function of any user \( \theta_i \in M \) under the NOMA protocol is

\[ u(\theta_i, p, \theta_i) = \log_2 \left( 1 + \frac{\mathbb{E}[h_{ij}^2]}{\sum_{j \neq i} \mathbb{E}[h_{ij}^2]} \right) - \beta \gamma_i \]  \hspace{1cm} (22)

B. Analysis of the NOMA Transmission Power Equilibrium

Player \( \theta_i \)’s optimal action against opponents’ aggregate actions is given by the best response operator (10), i.e., \( \forall \theta_i \in M \), there is

\[ a^*_i \in \bar{\mathcal{B}}^R_{\text{ordered}}(\theta_i, p) \]  \hspace{1cm} (23)

Again, we are going to adopt the Banach fixed point theorem to establish the existence and uniqueness of the mean-field equilibrium strategy \( p^* \).

According to Theorem 2, a vector space consisting of all strategies with bounded power allocation is defined as

\[ L^\infty(M, \mathbb{R}, \nu) := \{ p : (p : M \to \mathbb{R}) \land (\| p \|_\infty < \infty) \} \]  \hspace{1cm} (24)

Algorithm 1: Distributed Equilibrium-Seeking Algorithm

Input: Number of users $N$; 
Number of iterations $NUM_{IT}$.

Result: The strategy profile $p_i^t(\theta_i)$ (CDMA) 
or $p_{\text{ordered}}^t(\theta_i)$ (NOMA), where $i = 1, 2, \ldots, N$.

Initialization: Fix an arbitrarily chosen initial power 
allocation strategy $p_{0}(\theta_i) \in A$ for users $i = 1, 2, \ldots, N$;
for $k = 0$ to $NUM_{IT} - 1$ do
   for $i = 1$ to $N$ do
      $p_{k+1}(\theta_i) = \arg\max_{p \in A} J(p, z^k) (\text{CDMA})$ 
      or $p_{k+1}(\theta_i) = \arg\max_{p \in \text{ordered}} J(p, z^k) (\text{NOMA})$;
      (Parallel updates for users $i = 1, 2, \ldots, N$.)
   end
end

The strategy profile obtained is $p_i^t(\theta_i) \in A$ (CDMA) 
or $p_{\text{ordered}} \in A$ (NOMA), for $i = 1, 2, \ldots, N$.

The space $(L^\infty(M, \mathbb{R}, \nu), \| \cdot \|_\infty)$ is a Banach space. The set of 
feasible strategies $A$ is a subset of $L^\infty(M, \mathbb{R}, \nu)$.

Now, we give the main result concerning the equilibrium strategy 
for NOMA, which establishes the existence and uniqueness of 
equilibrium strategy.

**Theorem 2:** Assume $\alpha < 1$. Then, for the game $G$ that adopts 
utility function (21) and NOMA with SIC decoding by descending 
order of the channel gains $\| h_i \|^2$, there exists a unique mean-field 
equilibrium profile $p_{\text{ordered}} \in A \subset L^\infty(M, \mathbb{R}, \nu)$. Moreover, starting 
from any initial strategy $p_0 \in A$, the unique mean-field 
equilibrium profile $p_{\text{ordered}}$ can be obtained through a recursive update 
based on the best response operator, i.e., $\lim_{k \to \infty} p_k = p_{\text{ordered}} \subset L^\infty(M, \mathbb{R}, \nu)$, where $p_{k+1}(\theta_i) \in \text{BR}_{\text{ordered}}(\theta_i, p_k)$ for 
any $\theta_i \in M$ and $k \geq 0$.

**Proof:** Similar to Theorem 1, this theorem can be proved by 
establishing that the best response operator under NOMA is 
a contraction mapping in the space $L^\infty(M, \mathbb{R}, \nu)$. See 
Appendix III of the full version [29] of this article.

**Corollary 2:** For the game $G$ adopting NOMA, the unique 
equilibrium strategy profile $p_{\text{ordered}} : M \to \mathcal{E}$ characterized in Theorem 
2 is continuous with respect to $\theta_i \in M$.

**Proof:** See Appendix IV of the full version [29] of this article.

Based on Theorems 1 and 2, it is seen that the strategy update 
with a best response operator is a contraction mapping under both 
CDMA and NOMA protocols. According to Banach fixed-point theorem [27], 
this has directly led to a distributed algorithm for strategy updates at each player such that the convergence to the 
unique equilibrium strategy is guaranteed. The distributed 
algorithm is given in Algorithm 1.

Next, we will compare between OMA and NOMA for a 
given noncooperative user population. Since a subcarrier (e.g., 
frequency band, time slots, signature sequences, etc.) needs to be 
allocated to each user before we model the channel interference, 
we employ CDMA with SUD as a benchmark of OMA 
protocols. Due to the averaging effect introduced by the spread 
spectrum techniques, the channel interference faced by users can 
be described by a mean-field term. The same spread spectrum 
techniques is adopted in the subcarrier allocation of the NOMA 
scheme for the fairness of comparison. In this article, we take 
power-domain NOMA with SIC as the representative of general 
NOMA schemes. The results are expected to be valid for other 
types of subcarriers as well.

VI. SOCIAL WELFARE COMPARISON BETWEEN 
CDMA AND NOMA

We aim at comparing the effectiveness of NOMA for 
performance improvement in power control game in 5G networks 
against CDMA. In this article, we focus more on qualitative 
analysis than quantitative analysis. In an aggregative game with 
a large number of players, we define the performance criterion 
in terms of social welfare, i.e., the expected utility achieved by 
all players

$$J(p, z) := \mathbb{E}[\tilde{u}(p(\theta_i), z, \theta_i)] = \int_{\theta_i \in M} \tilde{u}(p(\theta_i), z, \theta_i) f(\theta_i) d\theta_i,$$  (24)

where $p \in A$ and $z : M \to \mathbb{R}$ is $\nu$-measurable, with the individual 
utility $\tilde{u}$ corresponding to each player $\theta_i$, defined as a function of the action $p(\theta_i)$ taken by player $\theta_i$ and the interference effects $z$. The expressions of individual utilities $\tilde{u}$, based on the formulation of the aggregative game, can be expressed as

$$\tilde{u}(a_i, z, \theta_i) := \log_2 \left( 1 + \frac{a_i}{\alpha z(\theta_i) + N_0} \right) - \beta a_i.$$  (25)

**Remark 3:** For finite users, the social welfare can be defined as 
the average utility achieved in the population [34]. When 
the number of users $N$ approaches infinity, the social welfare 
converges to the expectation of individual utilities with respect 
to the distribution of the user identity $\theta_i \in M$.

The only difference between CDMA and NOMA is on the 
interference effects $z : M \to \mathbb{R}$.

First, we consider the CDMA protocol. As the channel is 
shared in an orthogonal manner with spread spectrum tech-
niques, the interference level is identical for different channel 
users, i.e.,

$$z(\theta_i) = \mathbb{E}[p(\theta_j) \theta_j] \forall \theta_i \in M.$$  (26)

Next, we consider the NOMA protocol. Since the SIC 
decoding algorithm is used by the base station, as introduced in 
Section V, the interferences faced by different channel users $\theta_i$ 
in $M$ are nonidentical. With slight abuse of notations, we arbitrarily 
fix a user identity $\theta_i$ to be a constant rather than a random variable. Due to the assumption on perfect interference cancellation and fixed decoding order (i.e., descending order of the channel gain), the interference faced by each user under NOMA is

$$z(\theta_i) = \mathbb{E}[p(\theta_j) \mathbf{1}_{\{\theta_j < \theta_i\}}] \forall \theta_i \in M$$  (27)

where the indicator function $\mathbf{1}_{\{\theta_j < \theta_i\}}$ results from recursive 
cancellation of previously decoded signal.

To compare the equilibrium social welfare under CDMA and 
NOMA, it is difficult to calculate their equilibrium strategies in 
closed form and evaluate the corresponding equilibrium social 
wellfare (24). Hence, it is necessary for us to propose approaches 
for analyzing the trend of changes in the equilibrium performance 
as we switch the protocol from CDMA to NOMA.

Given an interference profile $z(\theta_i)$ ($\theta_i \in M$), if we plan to 
predict the outcome of power control game among channel
users and evaluate the corresponding performance, we need to restrict the strategies to satisfy the definition of mean-field equilibrium (Theorem 3), i.e.,

$$u(p(\theta_i), z, \theta_i) \geq u(a_i, z, \theta_i) \quad \forall a_i \in \mathcal{E}, \, \theta_i \in M. \quad (28)$$

With the social welfare $J(p, z)$ as the performance criterion of this game, our objective in this section is to theoretically compare $J(p, z)$ achieved under the equilibrium strategies of CDMA and NOMA, respectively. We employ a perturbation-based approach on the social welfare $J(p, z)$ as a functional with respect to $(p, z)$ for characterizing the trend at which the equilibrium social welfare changes when we switch from CDMA to NOMA. Details are given in Theorem 3 and illustrated in Fig. 3.

Given the unique equilibrium power allocation strategies $p^*$ and $p_{\text{ordered}}^{\ast}$ in power control game $G$ under CDMA and NOMA, the corresponding interference functions are given by $z^*(\theta_i) = \mathbb{E}[p^*(\theta_i)\theta_j]$ and $z_{\text{ordered}}^{\ast}(\theta_i) = \mathbb{E}[p_{\text{ordered}}^{\ast}(\theta_j)\theta_j, 1(\theta_i < \theta_j)]$ for any $\theta_i \in M$, respectively.

**Theorem 3 (NOMA Outperforms CDMA Under Equilibria):** NOMA can achieve a strictly better social welfare than CDMA at the corresponding equilibrium strategies, i.e., $J(p_{\text{ordered}}^{\ast}, z_{\text{ordered}}^{\ast}) > J(p^*, z^*)$.

**Proof:** See the Appendix.

**VII. INDIVIDUAL BEHAVIORS AT THE EQUILIBRIUM**

According to Theorems 1 and 2, denote the unique equilibrium strategy for CDMA as $p^* \in \mathcal{A}$ and the equilibrium strategy for NOMA as $p_{\text{ordered}}^{\ast} \in \mathcal{A}$. For an unbounded set $M$ of user identities, we obtain an additional property such that for users with sufficiently large uplink channel gains, their equilibrium transmission power under CDMA and NOMA can be arbitrarily close.

**Proposition 1 (Convergence Behavior for High-Gain Users):** Assume the player set $M$ is unbounded from above, i.e., $\forall L > 0, \, \exists \theta_i \in M \, \forall \theta_i > L$. Then, we have

$$\lim_{\theta_i \to \infty} |p^*(\theta_i) - p_{\text{ordered}}^{\ast}(\theta_i)| = 0. \quad (29)$$

**Proof:** See Proposition 1 in the full version [29] of this article for details.

Next, we show that pointwise improvement in the equilibrium data rate for different types of users is not achievable by NOMA, which means there is “no free lunch” in employing NOMA to improve for all individual users.

**Proposition 2 (Infeasibility of Pointwise Improvement):** Assume $E_{\text{min}} = 0$. Then, the curve of equilibrium power strategy $p^*$ for CDMA crosses $p_{\text{ordered}}^{\ast}$ for NOMA, i.e., $\exists \theta_i \in M$ s.t. $p^*(\theta_i) > p_{\text{ordered}}^{\ast}(\theta_i)$ and $\exists \theta_i < \theta_i$ with $p^*(\theta_i) - p_{\text{ordered}}^{\ast}(\theta_i) \neq 0$ in the equilibrium data rate achieved by NOMA in comparison with CDMA.

**Proof:** We assume there is no crossing at the curve of equilibrium power control strategy and prove by contradiction. Then, this crossing is also shown to exist in the curve of equilibrium data rate. See Proposition 2 in the full version [29] of this article for details.

**VIII. SIMULATIONS**

In this section, we numerically illustrate the results concerning the properties of the equilibrium strategy profile under both CDMA (with fierce competition) and NOMA (with regulating effects among the user population).

First, we introduce some parameters and setups adopted in the simulation. We suppose that the channel gain $h_i$ for each user follows Rayleigh fading. Specifically, for an arbitrary user, the PDF for the squared magnitude of its channel gain $\theta_i = ||h_i||^2$ is

$$f(\theta_i) = \frac{1}{\sigma} \exp\left(-\frac{\theta_i}{\sigma^2}\right), \quad \theta_i \geq 0, \quad \text{Otherwise.} \quad (30)$$

For the simulation, we take the parameter $\sigma = 5$. It is intractable to evaluate the behaviors of an infinite number of players for a numerical simulation. Hence, the results we present in the following are generated with $N = 1000$ players. The white noise process $w[k]$ in the AWGN channel features a power spectrum density $N_0 = 5$, and the spread spectrum parameter $\alpha = \frac{\Delta}{\sigma} = 0.25$ applies to both the case of CDMA and NOMA.

In the aggregative game, we assume the set of feasible power levels is $\mathcal{E} = [0, 150]$.

Now, we calculate the equilibrium power allocation strategy of the game $G$ as well as the corresponding data rates when CDMA and NOMA are employed, respectively, as shown in Fig. 4.

The improvement in user fairness achieved by NOMA can be observed from the curves of achieved data rate in Fig. 4. With the same power penalty parameter $\beta > 0$ for unit power consumption, NOMA features a better fairness in achieved data rates than CDMA, which is discussed as follows.

For a rigorous comparison, we employ Jain’s fairness index [35] to evaluate the fairness in equilibrium data rate of the user population. In our simulation with $N = 1000$ users, we denote the data rate achieved by user $i$ ($1 \leq i \leq N$) as $d_i \geq 0$. Jain’s fairness index is defined as

$$J(d_1, d_2, \ldots, d_N) := \left(\sum_{i=1}^{N} d_i^2\right)^2 / \left(N \cdot \sum_{i=1}^{N} d_i^2\right) = \frac{\sigma^2}{d^2} \in (0, 1] \quad (31)$$
where a larger value of the index implies better fairness. For the values of $\beta$ considered in Fig. 4, the values of Jain’s fairness for CDMA and NOMA in the simulation are listed in Table I.

As indicated by Fig. 4, though NOMA outperforms CDMA in terms of fairness in equilibrium data rate, the increase in $\beta$ is undesirable for NOMA due to decreasing fairness. Aside from that, the level of achieved data rates in general only slightly decreases with a larger $\beta$.

**Remark 4:** Based on these comparisons, we summarize some empirical findings concerning the applicability of CDMA and NOMA.

1) For the cases with a small cost for power consumption (i.e., $\beta > 0$ takes a small value), NOMA is preferable for its advantages in the fairness achieved.

2) For the case of costly power resources (i.e., $\beta > 0$ takes a large value), the performance gap between CDMA and NOMA is negligible, whereas CDMA is more convenient for implementation.

The qualitative comparison between equilibrium social welfare under CDMA and NOMA has been done in Theorem 3. Thus, we numerically evaluate the expected utility of all participants in Fig. 5, i.e., the objective function $J(p, z) = \mathbb{E}[u(p, \theta_i), p, \theta_i]$ defined in (24).

It is observed from Fig. 5 that NOMA can indeed achieve a higher social welfare than CDMA, confirming the theoretical results in Theorem 3. In addition, we observe that when the value of $\beta > 0$ is small, more effective performance improvement is achieved by NOMA, which is consistent with our intuitive analysis in Theorem 4. When $\beta > 0$ takes a large value, the regulating effects of the energy cost dominates, and the benefits of implementing the NOMA protocol gradually shrink.
IX. CONCLUSION

We considered an uplink power control problem for wireless communication when a large number of users compete for the channel resources. Both power-domain CDMA and NOMA were investigated. We performed equilibrium analysis of the noncooperative channel access with an aggregative game model so that the opponents’ actions are captured collectively. The existence and uniqueness of an equilibrium strategy are established for both CDMA and NOMA. Moreover, performance evaluation has been conducted under the equilibrium strategies. It turns out that NOMA achieves a better social welfare in the noncooperative user population at its equilibrium. In addition, simulation results show an improved fairness in the equilibrium data rate under NOMA.

APPENDIX

PROOF OF THEOREM 3

Proof: To build up a bridge between the equilibrium performance of the power control game under CDMA and NOMA, we relax the conditions on interference profile $z(\theta_i)$ such that only lower bounds are imposed, i.e., $z(\theta_i) \geq \mathbb{E}[p(\theta_j)\theta_j] \forall \theta_i \in M$ for CDMA and $z(\theta_i) \geq \mathbb{E}[p(\theta_j)\theta_j \mathbb{1}_{\theta_i,\theta_j}] \forall \theta_i \in M$ for NOMA.

Under the relaxed interference profile $z$, given the common best response condition (28) for both protocols, the space of feasible variable pairs $(p, z)$ in CDMA is a subset of that in NOMA due to the fact that $\mathbb{E}[p(\theta_j)\theta_j \mathbb{1}_{\theta_i,\theta_j}] \leq \mathbb{E}[p(\theta_j)\theta_j] \forall \theta_i \in M$, as indicated in Fig. 3.

Then, we show that the maximum social welfare in the relaxed space of feasible variable pairs $(p, z)$ under either CDMA or NOMA can only be obtained at the boundary of the relaxed conditions on interference $z$. For the convenience of presentation, given the relaxed variable space and the social welfare criterion, we construct two auxiliary optimization problems (Theor. 1 and 2) to search for the best possible performance. Then, it is equivalent to showing that for both protocols, the maximum social welfare we seek can only be achieved on the boundary of the feasible sets.

**Problem 1 (CDMA - Relaxed):**

$$\max_{p \in \mathcal{A}, z \in \mathcal{M}_{\text{R}}(M)} J(p, z)$$

subject to

$$z(\theta_i) \geq \mathbb{E}[p(\theta_j)\theta_j] \quad \forall \theta_i \in M \quad (32)$$

$$u(p(\theta_j), z, \theta_i) \geq u(a_i, z, \theta_i) \quad \forall a_i \in \mathcal{A}, \ \theta_i \in M.$$  

**Problem 2 (NOMA - Relaxed):**

$$\max_{p \in \mathcal{A}, z \in \mathcal{M}_{\text{R}}(M)} J(p, z)$$

subject to

$$z(\theta_i) \geq \mathbb{E}[p(\theta_j)\theta_j \mathbb{1}_{\theta_i,\theta_j}] \quad \forall \theta_i \in M$$

$$u(p(\theta_j), z, \theta_i) \geq u(a_i, z, \theta_i) \quad \forall a_i \in \mathcal{A}, \ \theta_i \in M. \quad (33)$$

In the following, we establish the necessary optimality condition for NOMA, whereas a similar approach applies to CDMA.

To begin with, we focus on the constraint $z(\theta_i) \geq \mathbb{E}[p(\theta_j)\theta_j \mathbb{1}_{\theta_i,\theta_j}]$. We aim at showing that any pair of optimal solution $(p^*, z^*)$ to Theorem 2 satisfies $z^*(\theta_i) = \mathbb{E}[p^*(\theta_j)\theta_j \mathbb{1}_{\theta_i,\theta_j}]$ a.e. in $M$.

From the feasible set of Theorem 2, we pick up a pair of decision variables $(p, z)$ such that $u(p(\theta_i), z, \theta_i) \geq u(a_i, z, \theta_i) \forall \theta_i \in M$ and there exists a bounded set $\bar{M} \subset M$ satisfying $P(\bar{M}) > 0$ and for any $\theta_i \in \bar{M}$, there is $z(\theta_i) > \mathbb{E}[p(\theta_j)\theta_j \mathbb{1}_{\theta_i,\theta_j}]$. Assume $(p, z)$ is an optimal solution to Theorem 2.

Define a measurable function $\epsilon : \bar{M} \to \mathbb{R}$ such that $\epsilon(\theta_i) := z(\theta_i) - \mathbb{E}[p(\theta_j)\theta_j \mathbb{1}_{\theta_i,\theta_j}] > 0$. Since the space $M \subset \mathbb{R}$ is a metric space, according to Lemma 4.1 (Lusin’s theorem) in Chapter II of [36], for $\epsilon_2 := \frac{1}{2} P(\bar{M}) > 0$, there exists a closed set $M_2 \subset \bar{M}$ such that $\nu(M_2 \setminus M_2) \leq \epsilon_2$ and the restriction of the measurable function $\epsilon$ on the set $M_2$, which is denoted as $\epsilon_{M_2} : M_2 \to \mathbb{R}$, is continuous. Since $\bar{M}$ is bounded, the closed set $M_2 \subset \bar{M}$ is compact. Hence, based on Weierstrass extreme value theorem [37], there exists a $\theta' \in M_2$ such that $\inf_{\theta \in M_2} \epsilon_{M_2}(\theta) = \epsilon_{M_2}(\theta') = \epsilon(\theta') > 0$.

Then, we construct a new perturbed variable $\tilde{z}$ such that

$$\tilde{z}(\theta_i) := \begin{cases} z(\theta_i) - \frac{1}{K} \epsilon(\theta_i), & \theta_i \in M_2 \\ z(\theta_i), & \text{Otherwise} \end{cases}$$

where $K > 1$ is a scaling factor.

In order for the constructed variable pair $\tilde{z}$ to satisfy the constraint $u(p(\theta_i), z, \theta_i) \geq u(a_i, z, \theta_i) \forall \theta_i \in M$, we obtain an updated version of the optimal power control variable $\tilde{p}$ in response to the change in the interference term from $z$ to $\tilde{z}$. Since the individual utility function $u(a_i, z, \theta_i)$ in the optimization problem is strictly concave with respect to the variable $\alpha_i$, it has a unique maximizer in terms of $\alpha_i$ when other variables are fixed. Then, the perturbed version of the optimal power control strategy $\tilde{p}(\theta_i)$ is expressed as

$$\tilde{p}(\theta_i) := \arg \max_{\alpha_i \in \mathcal{A}} u(a_i, \tilde{z}, \theta_i)$$

$$= \begin{cases} \tilde{p}(\theta_i), & \theta_i \in M_2 \\ \tilde{p}(\theta_i), & \text{Otherwise} \end{cases}$$

(34)

It remains to verify the existence of a scaling factor $K > 1$ such that the pair $(\tilde{p}, \tilde{z})$ satisfies the constraint $z(\theta_i) \geq \mathbb{E}[p(\theta_j)\theta_j \mathbb{1}_{\theta_i,\theta_j}]$ for any $\theta_i \in M$. By definition of $(\tilde{p}, \tilde{z})$, it suffices to show that $\tilde{z}(\theta_i) \geq \mathbb{E}[\tilde{p}(\theta_j)\theta_j \mathbb{1}_{\theta_i,\theta_j}]$ for any $\theta_i \in \bar{M}$.

By the derivations in (34), for any $\theta_i \in M$, $|p(\theta_j) - \tilde{p}(\theta_j)| \leq \frac{\alpha_i z(\theta_i) - \tilde{z}(\theta_i)}{K \epsilon(\theta_i)}$. Hence, for any $\theta_i \in M$, $|\mathbb{E}[p(\theta_j) - \tilde{p}(\theta_j)\theta_j \mathbb{1}_{\theta_i,\theta_j}]| \leq |\mathbb{E}[p(\theta_j) - \tilde{p}(\theta_j)\theta_j]| = \frac{\mathbb{E}[\epsilon(\theta_i)]}{K}$.

As $\epsilon(\theta_i) > 0$ for any $\theta_i \in M_2$ and $0 \leq \mathbb{E}[\epsilon(\theta_i)] < \infty$ is a constant, there exists a sufficiently large $K$ such that

$$\tilde{z}(\theta_i) - \mathbb{E}[\tilde{p}(\theta_j)\theta_j \mathbb{1}_{\theta_i,\theta_j}]$$

$$= z(\theta_i) - \frac{1}{K} \epsilon(\theta_i) - \mathbb{E}[p(\theta_j)\theta_j \mathbb{1}_{\theta_i,\theta_j}]$$

$$+ \mathbb{E}[p(\theta_j) - \tilde{p}(\theta_j)]\theta_j \mathbb{1}_{\theta_i,\theta_j}.$$
\[
\begin{align*}
&\geq \frac{K - 1}{K} \epsilon(\theta_i) - |\mathbb{E}[(p(\theta_j) - \tilde{p}(\theta_j))\theta_j 1_{\{\theta_i < \theta_j\}}]| \\
&\geq \frac{K - 1}{K} M^2(\theta) - \frac{\alpha}{K} \mathbb{E}[\epsilon(\theta_j)] > 0 \forall \theta_i \in M_2
\end{align*}
\]
where the last inequality holds due to \(\epsilon(\theta_i) = \epsilon_{M_2}(\theta_i) \geq \inf_{\theta_i \in M_2} \epsilon_{M_2}(\theta_i) = M(\theta)\) for any \(\theta_i \in M_2\) and by a fixed choice of \(K > \frac{\alpha}{\epsilon(\theta_j)} + 1 \geq 1\). Thus, the feasibility of the constructed variable pair \((\tilde{p}, \tilde{z})\) is successfully shown.

Now, since \(\tilde{z}(\theta_i) < z(\theta_i)\) by definition, we can obtain the following inequalities:
\[
\begin{align*}
&\tilde{u}(\tilde{p}(\theta_i), \tilde{z}, \theta_i) = \log_2 \left( 1 + \frac{p(\theta_i)}{\alpha z(\theta_i) + N_0} \right) - \beta \tilde{p}(\theta_i) \\
&< \log_2 \left( 1 + \frac{\tilde{p}(\theta_i)}{\alpha z(\theta_i) + N_0} \right) - \beta \tilde{p}(\theta_i) \\
&\leq \log_2 \left( 1 + \frac{\tilde{p}(\theta_i)}{\alpha z(\theta_i) + N_0} \right) - \beta \tilde{p}(\theta_i)
\end{align*}
\]
where the last inequality is due to the fact that \(a_i = \tilde{p}(\theta_i)\) is a maximizer of \(\tilde{u}(a_i, \tilde{z}, \theta_i)\).

Therefore, the social welfare under the perturbed variable pair \((\tilde{p}, \tilde{z})\) satisfies
\[
J(\tilde{p}, \tilde{z}) = \int_{\theta_i \in M} \tilde{u}(\tilde{p}(\theta_i), \tilde{z}, \theta_i) f(\theta_i) d\theta_i
\]
and hence it is impossible for the original pair of decision variables \((p, z)\) to be optimal.

Therefore, that \(z(\theta_i) = \mathbb{E}(p(\theta_j)\theta_j 1_{\{\theta_i < \theta_j\}})\) holds almost everywhere in \(M\) is a necessary optimality condition of Theorem 2. In light of the proof above, a necessary condition for optimality can be obtained for Theorem 1 such that \(z(\theta_i) = \mathbb{E}[p(\theta_j)\theta_j]\) holds almost everywhere in \(M\).

Denote the optimal solution to Theorem 1 as \((p_1^*, z_1^*)\) and the optimal solution to Theorem 2 as \((p_2^*, z_2^*)\). Take into account the inclusive relationship between the relaxed spaces of feasible variables \((p, z)\) for CDMA and NOMA, we can obtain \(J(p_2^*, z_2^*) \geq J(p_1^*, z_1^*)\). Next, we show that the above inequality is strict.

Assume \(J(p_2^*, z_2^*) = J(p_1^*, z_1^*)\), since the optimal variable pair \((p_1^*, z_1^*)\) in Theorem 1 is also feasible for Theorem 2. Then, the optimal social welfare for NOMA can also be achieved at \((p_1^*, z_1^*)\). However, since there exists a nonzero measure set \(M_+ \subset M\) such that \(z_1^*(\theta_i) = \mathbb{E}[p_1^*(\theta_j)\theta_j] \neq \mathbb{E}[p_1^*(\theta_j)\theta_j 1_{\{\theta_j < \theta_i\}}], \forall \theta_i \in M_+\), this contradicts with the necessary optimality condition for Theorem 2. Thus, we have \(J(p_2^*, z_2^*) > J(p_1^*, z_1^*)\).

Finally, it can be shown that the social welfare achieved by the variable pairs \((p_1^*, z_1^*)\), \((p_2^*, z_2^*)\) equals that achieved under the equilibrium of CDMA, i.e., \((p^*, z^*)\), and NOMA, i.e., \((\overline{p}^{\text{ordered}}, \overline{z}^{\text{ordered}})\), respectively, as detailed in Theorem 3 in the full version [29] of this article. Therefore, the equilibrium social welfare of NOMA is strictly better than CDMA, i.e., \(J(\overline{p}^{\text{ordered}}, \overline{z}^{\text{ordered}}) > J(p^*, z^*)\). □

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