A STURM-LIOUVILLE APPROACH FOR CONTINUOUS AND DISCRETE MITTAG-LEFFLER KERNEL FRACTIONAL OPERATORS

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Abstract. In this work, we use integration by parts formulas derived for fractional operators with Mittag-Leffler kernels to formulate and investigate fractional Sturm-Liouville Problems (FSLPs). We analyze the self-adjointness, eigenvalue and eigenfunction properties of the associated Fractional Sturm-Liouville Operators (FSLOs). The discrete analogue of the obtained results is formulated and analyzed by following nabla analysis.

1. Introduction. Fractional calculus has been studied in the last two decades or so. It has been employed successfully in the modelling of many problems in lots of fields of science and engineering. It has described successfully the properties of global complex systems [41, 45, 47, 49]. On the other hand, the discrete fractional calculus was of interest among several mathematicians [1, 2, 3, 4, 5, 6, 30, 31, 32, 37, 38, 46] and has been developing rapidly. In order to have more fractional operators with better-behaved kernels, recently, some researchers in fractional calculus have introduced and analyzed new non-local fractional operators with non-singular kernels and have used them in modeling some real world problems [7, 8, 23, 24, 25, 35, 36, 44]. Those with Mittag-Leffler kernels are called Atangana-Baleanu (AB) fractional operators. The extension to higher order fractional operators and their Lyapunov type inequalities have been investigated in [9, 10]. The proposed kernels are non-singular such as those with Mittag-Leffler

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kernels. The approach in defining such operators is different from the one of classical fractional operators which is through an iterative process of either the usual integration or differentiation. The idea behind this is to define the fractional derivatives first by imposing a non-singular kernel depending on the degree $\nu$ so that as $\nu \to 1$ the usual derivative is obtained and then by applying a Laplace transform method to find their corresponding fractional integrals. A point of interest of $AB$-fractional derivatives is that the $AB$-fractional integrals possess Riemann-Liouville fractional integrals as a part. Such operators have advantages over other types in developing more significant algorithms in solving fractional dynamical systems [20, 26, 27, 28, 29, 39, 40]. Later, the discrete counterparts of these fractional operators were introduced, studied, and their monotonicity properties were analyzed [11, 12, 13, 14, 15, 16, 17, 50].

The $SLEs$, which were investigated a long time ago, have many applications in various areas of science, engineering, and mathematics [34, 53]. However, its fractional formulation and numerical methods for the solutions have started very recently [18, 19, 21, 22, 33, 43, 48, 51, 52]. The classical Sturm-Liouville problem (SLP) for a linear second order differential equation is a boundary value problem (BVP) of the form:

$$\frac{d}{ds} \left( p_1(s) \frac{d\xi}{ds} \right) + q_1(s)\xi(s) = \mu l(s)\xi(s), \ s \in [a, b],$$

$$c_1\xi(a) + c_2\xi'(a) = 0,$$

$$d_1\xi(b) + d_2\xi'(b) = 0,$$

where all the functions $p_1, p'_1, q_1, l$ are in the space $C[a, b] = \{ \theta : [a, b] \to \mathbb{R} : \theta \text{ is continuous} \}$, such as $p_1(s) > 0$, $l(s) > 0$ on $[a, b]$. The above equation can be expressed shortly as:

$$L(\xi) = \mu l(s)\xi,$$

where $L(\xi) = -[p_1(s)\xi']' + q_1(s)\xi$. A $\mu$ for which the above $BVP$ has a nontrivial solution is called an eigenvalue, and the corresponding solution, an eigenfunction.

Motivated by what we have mentioned above, we formulate and investigate fractional $SLEs$ in the frame of fractional operators with Mittag-Leffler kernels together with their discrete versions. The corresponding fractional operator $L$ is introduced so that it contains left and right sided fractional operators with Mittag-Leffler kernels which makes it possible to use the appropriate fractional integration by parts discussed earlier in [7, 12].

This article is organized as follows: In the rest of this section, we recall some basic concepts concerning the classical fractional calculus, classical nabla discrete fractional calculus, fractional operators with Mittag-Leffler kernels, and discrete fractional operators with discrete Mittag-Leffler kernels. In section 2, we state the main results which is divided into two parts. In the first part, we discuss the $SLEs$ with fractional operators with Mittag-Leffler kernels and in the second part, we deal with the $SLEs$ with nabla discrete fractional operators with discrete Mittag-Leffler kernels. Finally, in section 3, we present an open problem for the higher order discrete fractional $SLE$ of order $\nu \in (1, \frac{3}{2})$.

Below, we first recall some basic concepts from classical fractional calculus.

**Definition 1.1.** ([47]) The Mittag-Leffler function of one parameter is defined by

$$E_{\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\nu k + 1)}, \ z, \nu \in \mathbb{C}, \text{Re}(\nu) > 0,$$
and the one with two parameters \( \nu \) and \( \beta \) by
\[
E_{\nu,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\nu k + \beta)}, \quad z, \nu, \beta \in \mathbb{C}, \quad \text{Re}(\nu) > 0, \text{Re}(\beta) > 0,
\]
where \( E_{\nu,1}(z) = E_{\nu}(z) \).

**Definition 1.2.** ([41]) The generalized Mittag-Leffler function of three parameters is defined by
\[
E_{\nu,\beta}^\rho(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k z^k}{k! \Gamma(\nu k + \beta)}, \quad z, \nu, \beta, \rho \in \mathbb{C}, \quad \text{Re}(\nu) > 0, \text{Re}(\beta) > 0, \text{Re}(\rho) > 0,
\]
where \((\rho)_k = \frac{\Gamma(\rho + k)}{\Gamma(\rho)}\).

Notice that \((1)_k = k!\) so \(E_{\nu,\beta}^1(z) = E_{\nu,\beta}(z)\).

- The left fractional integral of order \( \nu > 0 \) starting at \( a \) has the following form
  \[
  \left( _aI^\nu f \right)(t) = \frac{1}{\Gamma(\nu)} \int_{a}^{t} (t-s)^{\nu-1} f(s)ds.
  \]
- The right fractional integral of order \( \nu > 0 \) ending at \( b \) is defined by
  \[
  \left( I_b^\nu f \right)(t) = \frac{1}{\Gamma(\nu)} \int_{t}^{b} (s-t)^{\nu-1} f(s)ds.
  \]
- The left Riemann-Liouville fractional derivative of order \( 0 < \nu < 1 \) starting at \( a \) is given by
  \[
  \left( _aD^\nu f \right)(t) = \frac{d}{dt} \left( _aI^{1-\nu} f \right)(t).
  \]
- The right Riemann-Liouville fractional derivative of order \( 0 < \nu < 1 \) ending at \( b \) has the form
  \[
  \left( D^\nu f \right)(t) = \frac{-d}{dt} \left( I_b^{1-\nu} f \right)(t).
  \]

**Definition 1.3.** ([23]) Let \( \phi \in H^1(a, b) \), \( a < b \), \( \nu \in [0, 1] \). Then the left Caputo fractional derivative with Mittag-Leffler kernel (Atangana-Baleanu in sense of Caputo) is given by
\[
_a^{ABC} D^\nu \phi(\zeta) = \frac{B(\nu)}{1-\nu} \int_{a}^{\zeta} \phi'(x) E_{\nu} \left( \frac{-\nu}{1-\nu} (\zeta - x)^\nu \right) dx,
\]
and the left Riemann-Liouville one (Atangana-Baleanu in sense of Riemann-Liouville) by
\[
_a^{ABR} D^\nu \phi(\zeta) = \frac{B(\nu)}{1-\nu} \frac{d}{d\zeta} \int_{a}^{\zeta} \phi(x) E_{\nu} \left( \frac{-\nu}{1-\nu} (\zeta - x)^\nu \right) dx,
\]
where \( B(\nu) > 0 \) is a normalization function vanishing at 0 and 1. Further, the corresponding fractional integral is given by
\[
_a^{AB} I^\nu \phi(\zeta) = \frac{1-\nu}{B(\nu)} \phi(\zeta) + \frac{\nu}{B(\nu)} \int_{a}^{\zeta} \phi(x) dx.
\]

For a function \( \phi \) defined on \([a, b]\), the action of the \( Q^{-} \) operator is expressed by \((Q\phi)(\tau) = \phi(a + b - \tau)\). From classical fractional calculus, we have \( (_aI^\nu Q\phi)(\tau) = Q(\_aI^\nu \phi)(\tau) \) and \( (_aD^\nu Q\phi)(\tau) = Q(\_aD^\nu \phi)(\tau) \). In [7], by using the \( Q^{-} \) operator, the authors defined the right versions of the \( ABR \) and \( ABC \) fractional derivatives and their corresponding integrals as follows:
Definition 1.4. ([7]) Let $\phi \in H^1(a,b), \ a < b, \ \nu \in [0,1]$. The right Caputo fractional derivative with Mittag-Leffler kernel is defined by

$$\text{ABC} D_{b}^{\nu} \phi(\zeta) = -\frac{B(\nu)}{1-\nu} \int_{\zeta}^{b} \phi'(x)E_{\nu} \left( -\frac{\nu}{1-\nu} (x-\zeta)^{\nu} \right) dx,$$

and the right Riemann-Liouville one by

$$\text{ABR} D_{b}^{\nu} \phi(\zeta) = -\frac{B(\nu)}{1-\nu} \frac{d}{d\zeta} \int_{\zeta}^{b} \phi(x)E_{\nu} \left( -\frac{\nu}{1-\nu} (x-\zeta)^{\nu} \right) dx.$$

Further, the associated fractional integral is formulated by

$$\text{AB} I_{b}^{\nu} \phi(\tau) = \frac{1-\nu}{B(\nu)} \phi(\tau) + \frac{\nu}{B(\nu)} I_{b}^{\nu} \phi(\tau).$$

To prove an integration by parts formula for $\text{ABR}$ fractional derivatives, in [7], the authors introduced the function spaces

$$\text{AB} I_{\nu}(L_p) = \left\{ f : f = \text{AB} I_{\nu}^{\nu} \varphi, \ \varphi \in L_p(a,b) \right\},$$

and

$$\text{AB} I_{\nu}^{\nu}(L_p) = \left\{ f : f = \text{AB} I_{\nu}^{\nu} \phi, \ \phi \in L_p(a,b) \right\},$$

where $p \geq 1$ and $\nu > 0$ are constants. In [7, 23], the authors showed that

$$\text{AB} D_{a}^{\nu} \text{AB} I_{a}^{\nu} f(\zeta) = f(\zeta), \ \text{AB} D_{b}^{\nu} \text{AB} I_{b}^{\nu} f(\zeta) = f(\zeta),$$

and also

$$\text{AB} I_{a}^{\nu} \text{AB} D_{a}^{\nu} f(\zeta) = f(\zeta), \ \text{AB} I_{b}^{\nu} \text{AB} D_{b}^{\nu} f(\zeta) = f(\zeta).$$

Hence, from (1), it follows that $\text{AB} I_{\nu}(L_p)$ and $\text{AB} I_{\nu}^{\nu}(L_p)$ are nonempty.

Theorem 1.5. ([7]) Let $\nu > 0, \ p \geq 1, \ q \geq 1$, and $\frac{1}{p} + \frac{1}{q} = 1 + \nu$ when $p \neq 1$ and $q \neq 1$

- If $\varphi(x) \in L_p(a,b)$ and $\psi(x) \in L_q(a,b)$, then
  $$\int_{a}^{b} \varphi(x) \text{AB} I_{a}^{\nu} \psi(x) dx = \int_{a}^{b} \psi(x) \text{AB} I_{a}^{\nu} \varphi(x) dx.$$

- If $f(x) \in \text{AB} I_{\nu}^{\nu}(L_p)$ and $g(x) \in \text{AB} I_{\nu}^{\nu}(L_q)$, then
  $$\int_{a}^{b} f(x) \text{AB} D_{a}^{\nu} g(x) dx = \int_{a}^{b} g(x) \text{AB} D_{a}^{\nu} f(x) dx.$$

From [23], we recall the following relation between the left $\text{ABR}$ and $\text{ABC}$ fractional derivatives as

$$\text{ABC} D_{a}^{\nu} f(\zeta) = \text{ABR} D_{a}^{\nu} f(\zeta) - \frac{B(\nu)}{1-\nu} f(0)E_{\nu} \left( -\frac{\nu}{1-\nu} \zeta^{\nu} \right).$$

Right version of (2) was proved in [7] by using the $Q$ operator as follows:

$$\text{ABC} D_{b}^{\nu} f(\zeta) = -\text{ABR} D_{b}^{\nu} f(\zeta) - \frac{B(\nu)}{1-\nu} f(0)E_{\nu} \left( -\frac{\nu}{1-\nu} (b-\zeta)^{\nu} \right).$$

In [42], the left generalized fractional integral operator was defined by

$$\text{E}^\nu_{\nu,\beta,\omega,a+} \varphi(x) = \int_{a}^{x} (x-\zeta)^{\beta-1} E_{\nu,\beta}^{\nu}(\omega(x-\zeta)^{\nu}) \varphi(\zeta) d\zeta, \ x > a.$$
Similarly, its right version can be defined by
\[ E^{\rho}_{\nu, \beta, \omega, b}(x) = \int_{x}^{b} (\zeta - x)^{\beta - 1} E^{\rho}_{\nu, \beta}(\omega(\zeta - x)^{\nu}) \varphi(\zeta) d\zeta, \ x < b \quad (5) \]
(see also [7]).

**Remark 1.** Using (4) and (5), we can write the ABR and ABC fractional derivatives as follows:
\[
\begin{align*}
ABR D^\nu_a f(t) &= \frac{B(\nu)}{1 - \nu} \int_{a}^{1} E^{1}_{\nu, 1, \frac{\nu}{1 - \nu}, a^+} f(t), \\
ABR D^\nu_b f(t) &= \frac{-B(\nu)}{1 - \nu} \int_{a}^{1} E^{1}_{\nu, 1, \frac{\nu}{1 - \nu}, b^-} f(t), \\
ABC D^\nu_a f(t) &= \frac{B(\nu)}{1 - \nu} \int_{a}^{1} E^{1}_{\nu, 1, \frac{\nu}{1 - \nu}, a^+} f'(t), \\
ABC D^\nu_b f(t) &= \frac{-B(\nu)}{1 - \nu} \int_{a}^{1} E^{1}_{\nu, 1, \frac{\nu}{1 - \nu}, b^-} f'(t).
\end{align*}
\]

**Proposition 1.** ([7]) Let \( \phi, \psi \in H^1(a, b) \) and \( 0 < \nu < 1 \). We have
- \( \int_{a}^{b} \psi(\zeta) A^{\nu}_{a} D^{\nu}_{a} \phi(\zeta) d\zeta = \int_{a}^{b} \phi(\zeta) A^{\nu}_{a} D^{\nu}_{a} \psi(\zeta) d\zeta + \frac{B(\nu)}{1 - \nu} \phi(\zeta) E^{1}_{\nu, 1, \frac{\nu}{1 - \nu}, b^-} \psi(\zeta)|_{a}^{b} \).
- \( \int_{a}^{b} \psi(\zeta) A^{\nu}_{a} D^{\nu}_{b} \phi(\zeta) d\zeta = \int_{a}^{b} \phi(\zeta) A^{\nu}_{a} D^{\nu}_{b} \psi(\zeta) d\zeta - \frac{B(\nu)}{1 - \nu} \phi(\zeta) E^{1}_{\nu, 1, \frac{\nu}{1 - \nu}, b^-} \psi(\zeta)|_{a}^{b} \).

The proof of Proposition 1 was presented in [7] by making use of the relations (2) and (3) and the ABR integration by parts formula in Theorem 1.5. Below, we present an alternative proof for Proposition 1 by using an integration by parts formula for the generalized fractional integral operators defined in (4) and (5) and the ordinary integration by parts.

**Lemma 1.6.** Let \( \nu > 0, p \geq 1, q \geq 1, \) and \( \frac{1}{p} + \frac{1}{q} = 1 + \nu \) (\( p \neq 1 \) and \( q \neq 1 \) when \( \frac{1}{p} + \frac{1}{q} = 1 + \nu \)). If \( \varphi(x) \in L_{p}(a, b) \) and \( \psi(x) \in L_{q}(a, b) \), then
\[
\int_{a}^{b} \varphi(t) E^{1}_{\nu, 1, \frac{\nu}{1 - \nu}, a^+} \psi(t) dt = \int_{a}^{b} \psi(t) E^{1}_{\nu, 1, \frac{\nu}{1 - \nu}, b^-} \varphi(t) dt.
\]
**Proof.** The proof is clearly done by using the definition of the generalized fractional integral operators and interchanging the order of integration. \( \square \)

We now give the alternative proof of Proposition 1:

**Proof.** By using Remark 1, Lemma 1.6, and the usual integration by parts, it follows that
\[
\begin{align*}
\int_{a}^{b} \psi(\zeta) A^{\nu}_{a} D^{\nu}_{a} \phi(\zeta) d\zeta &= \frac{B(\nu)}{1 - \nu} \int_{a}^{b} \psi(\zeta) E^{1}_{\nu, 1, \frac{\nu}{1 - \nu}, a^+} \phi'(\zeta) d\zeta \\
&= \frac{B(\nu)}{1 - \nu} \int_{a}^{b} \phi'(\zeta) E^{1}_{\nu, 1, \frac{\nu}{1 - \nu}, b^-} \psi(\zeta) d\zeta \\
&= \frac{B(\nu)}{1 - \nu} \phi(\zeta) E^{1}_{\nu, 1, \frac{\nu}{1 - \nu}, b^-} \psi(\zeta)|_{a}^{b} \\
&- \frac{B(\nu)}{1 - \nu} \phi(\zeta) \frac{d}{d\zeta} E^{1}_{\nu, 1, \frac{\nu}{1 - \nu}, b^-} \psi(\zeta) d\zeta \\
&= \frac{B(\nu)}{1 - \nu} \phi(\zeta) E^{1}_{\nu, 1, \frac{\nu}{1 - \nu}, b^-} \psi(\zeta)|_{a}^{b}.
\end{align*}
\]
Similarly, the second part can be proved as follows:
\[
\int_a^b \psi(\zeta) ABR D^\nu \phi(\zeta) d\zeta = -B(\nu) \int_a^b \psi(\zeta) \frac{\Gamma(\nu)}{\Gamma(\nu - 1)} \phi(\zeta) d\zeta + \int_a^b \phi(\zeta) ABR D^\nu \psi(\zeta) d\zeta.
\]

Now, we recall some notations and basic definitions concerning the classical nabla discrete fractional operators. To see more details, we may refer the reader to [1, 2, 4, 32, 37] and the references therein.

We assume that our functions are defined on the discrete sets
\[ N_a = \{ a, a + 1, a + 2, \ldots \}, \quad \mathbb{b}N = \{ \ldots, b - 2, b - 1, b \}, \]
where \( a, b \in \mathbb{R} \), or a set of the form
\[ N_{a,b} = \{ a, a + 1, a + 2, \ldots, b \}, \]
where \( b - a \) is a positive integer.

**Definition 1.7** ([32, 37]). (i) For a natural number \( n \) and \( t \in \mathbb{R} \), the \( n \) rising (ascending) factorial of \( s \) is defined by
\[
s^n = \prod_{i=0}^{n-1} (t + i), \quad s^0 = 1.
\]
(ii) For any real number \( \nu \), the (generalized) rising function is defined by
\[
s^\nu = \frac{\Gamma(s + \nu)}{\Gamma(s)}, \quad s \in \mathbb{R} \setminus \{ \ldots, -2, -1, 0 \}, \quad 0^\nu = 0.
\]

**Definition 1.8** ([1, 2]). For a function \( \phi : N_a \to \mathbb{R} \), the nabla left fractional sum of order \( \nu > 0 \) (starting from \( a \)) is given by
\[
\nabla_a^{-\nu} \phi(\zeta) = \frac{1}{\Gamma(\nu)} \sum_{j=a+1}^\zeta (\zeta - \rho(j))^{\nu - 1} \phi(j), \quad \zeta \in N_{a+1}.
\]
The nabla right fractional sum of order \( \nu > 0 \) (ending at \( b \)) for \( \phi : \mathbb{b}N \to \mathbb{R} \) is defined by
\[
\nabla_b^{-\nu} \phi(\zeta) = \frac{1}{\Gamma(\nu)} \sum_{j=\zeta}^{b-1} (j - \rho(\zeta))^{\nu - 1} \phi(j) = \frac{1}{\Gamma(\nu)} \sum_{j=\zeta}^{b-1} (\sigma(j) - \zeta)^{\nu - 1} \phi(j), \quad \zeta \in b_{-1}N.
Definition 1.9 ([1, 2]). For a function \( \phi : \mathbb{N}_a \to \mathbb{R} \), the nabla left Riemann-Liouville fractional difference of order \( 0 < \nu < 1 \) (starting from \( a \)) is defined by
\[
_a \nabla^\nu \phi(\zeta) = \nabla_a \nabla^{-(1-\nu)} \phi(\zeta) = \nabla \left[ \frac{1}{\Gamma(1-\nu)} \sum_{j=a+1}^{\zeta} (\zeta - \rho(j))^\nu \phi(j) \right], \quad \zeta \in \mathbb{N}_{a+1},
\]

and for \( f : \mathbb{N} \to \mathbb{R} \), the nabla right Riemann-Liouville fractional difference of order \( 0 < \nu < 1 \) (ending at \( b \)) by
\[
_b \nabla^\nu \phi(\zeta) = \nabla_b \nabla^{-(1-\nu)} \phi(\zeta) = -\nabla \left[ \frac{1}{\Gamma(1-\nu)} \sum_{j=\zeta}^{b-1} (j - \rho(j))^\nu \phi(j) \right], \quad \zeta \in b-1 \mathbb{N}.
\]

In the above, \( \rho \) and \( \sigma \) are the backward and forward jump operators, respectively.

Definition 1.10 ([1, 2]). For a function \( \phi : \mathbb{N}_a \to \mathbb{R} \), the nabla left Caputo fractional difference of order \( 0 < \nu < 1 \) (starting from \( a \)) is defined by
\[
\left( \nabla_a^\nu \phi \right)(\zeta) = \nabla(1-\nu) \nabla^\nu \phi(\zeta), \quad \zeta \in \mathbb{N}_{a+1},
\]

and for \( \phi : \mathbb{N} \to \mathbb{R} \), the nabla right Caputo fractional difference of order \( 0 < \nu < 1 \) (ending at \( b \)) by
\[
\left( \nabla_b^\nu \phi \right)(\zeta) = \nabla(1-\nu) \nabla^\nu \phi(\zeta), \quad \zeta \in b-1 \mathbb{N}.
\]

Definition 1.11. ([1, 2, 3]) For \( \lambda \in \mathbb{R} \), \( |\lambda| < 1 \), and \( \nu, \sigma \in \mathbb{C} \) with \( \text{Re}(\nu) > 0 \), the nabla discrete Mittag-Leffler function is defined by
\[
E_\nu^\sigma(\lambda, z) = \sum_{j=0}^{\infty} \lambda^j \frac{z^{\nu j+\sigma}}{\Gamma(\nu j+\sigma)}, \quad z \in \mathbb{C}.
\]

For \( \sigma = 1 \), we write
\[
E_\nu^1(\lambda, z) = E_\nu^{\nu 1}(\lambda, z) = \sum_{j=0}^{\infty} \lambda^j \frac{z^{\nu j}}{\Gamma(\nu j+1)}, \quad z \in \mathbb{C}.
\]

Definition 1.12. ([12]) The nabla discrete generalized Mittag-Leffler function of three parameters \( \nu, \sigma, \rho \) is defined by
\[
E_\nu^\rho^\sigma(\lambda, z) = \sum_{j=0}^{\infty} \lambda^j (\rho)_j \frac{z^{\nu j+\sigma}}{\Gamma(\nu j+\sigma)},
\]

Notice that \( E_\nu^1(\lambda, z) = E_\nu^{\nu 1}(\lambda, z) \). Now, we review some main concepts concerning the nabla discrete fractional differences with discrete Mittag-Leffler kernels following the notations in [12].

Definition 1.13. ([12]) Assume \( \phi : \mathbb{N}_a \to \mathbb{R} \) and \( \nu \in (0, 1/2) \). Then the nabla discrete left Caputo fractional difference in the sense of Atangana and Baleanu is defined by
\[
_a A^BC \nabla^\nu \phi(\zeta) = \frac{B(\nu)}{1-\nu} \sum_{j=a+1}^{\zeta} \nabla \phi(j) E_\nu \left( \frac{-\nu}{1-\nu}, \zeta - \rho(j) \right), \quad \zeta \in \mathbb{N}_{a+1},
\]

and in the left Riemann-Liouville sense by
\[
_a A^BR \nabla^\nu \phi(\zeta) = \frac{B(\nu)}{1-\nu} \nabla \phi \left( \zeta \right) E_\nu \left( \frac{-\nu}{1-\nu}, \zeta - \rho(j) \right), \quad \zeta \in \mathbb{N}_{a+1}.
\]
In addition, the corresponding fractional sum is given by
\[
A^\nu_a \nabla^{-\nu} \phi(\zeta) = \frac{1-\nu}{B(\nu)} \phi(\zeta) + \frac{\nu}{B(\nu)} a \nabla^{-\nu} \phi(\zeta), \quad \zeta \in \mathbb{N}_{a+1}.
\]

Similar to the continuous case, for a function \( \phi \) defined on \( \mathbb{N}_{a,b} \), the Q operator is defined by \( (Q \phi)(\zeta) = \phi(a + b - \zeta). \) From classical discrete fractional calculus, we have \( (a \nabla^{-\nu} Q \phi)(\zeta) = Q(a \nabla^{-\nu} \phi)(\zeta) \) and \( (a \nabla^{\nu} Q \phi)(\zeta) = Q(a \nabla^{\nu} \phi)(\zeta). \) In [12], by using the Q operator, the authors defined the right versions of the ABR and ABC nabla fractional differences and their corresponding sum as follows:

**Definition 1.14.** ([12]) Assume \( \phi : \mathbb{N} \to \mathbb{R} \) and \( \nu \in (0, 1/2) \). Then the nabla discrete right Riemann-Liouville fractional difference with discrete Mittag-Leffler kernel is given by
\[
A^\nu_b \nabla^{-\nu} \phi(\zeta) = \frac{B(\nu)}{1-\nu} \zeta \sum_{j=\zeta}^{b-1} \phi(j) E_\nu \left( \frac{-\nu}{1-\nu}, j - \rho(\zeta) \right), \quad \zeta \in \mathbb{N}_{b-1},
\]
and the right Caputo one by
\[
A^\nu_b \nabla^{-\nu} \phi(\zeta) = \frac{B(\nu)}{1-\nu} \zeta \sum_{j=\zeta}^{b-1} \Delta \phi(j) E_\nu \left( \frac{-\nu}{1-\nu}, j - \rho(\zeta) \right), \quad \zeta \in \mathbb{N}_{b-1}.
\]
In addition, the associated fractional sum is defined by
\[
A^\nu_b \nabla^{-\nu} \phi(\zeta) = \frac{1-\nu}{B(\nu)} \phi(\zeta) + \frac{\nu}{B(\nu)} \nabla^{-\nu} \phi(\zeta), \quad \zeta \in \mathbb{N}_{b-1}.
\]

From [12], it is known that ABR–fractional differences and their AB–fractional sums satisfy
\[
A^\nu_a \nabla^{\nu} A^\nu_a \nabla^{-\nu} \phi(\zeta) = \phi(\zeta), \quad A^\nu_a \nabla^{\nu} A^\nu_b \nabla^{-\nu} \phi(\zeta) = \phi(\zeta),
\]
and also
\[
A^\nu_a \nabla^{-\nu} A^\nu_a \nabla^{\nu} \phi(\zeta) = \phi(\zeta), \quad A^\nu_b \nabla^{-\nu} A^\nu_b \nabla^{\nu} \phi(\zeta) = \phi(\zeta).
\]

From [12], it is also known that the left ABR and ABC nabla fractional differences has the following relation
\[
A^\nu_a \nabla^{\nu} \phi(\zeta) = a A^\nu_a \nabla^{\nu} \phi(\zeta) - \phi(a) \frac{B(\nu)}{1-\nu} E_\nu \left( \frac{-\nu}{1-\nu}, \zeta - a \right).
\]
Right version of (6) was proved in [12] by making use of the Q-operator as follows:
\[
A^\nu_a \nabla^{\nu} \phi(\zeta) = A^\nu_b \nabla^{\nu} \phi(\zeta) - \phi(b) \frac{B(\nu)}{1-\nu} E_\nu \left( \frac{-\nu}{1-\nu}, b - \zeta \right).
\]

The following two theorems state the integration by parts for fractional sums and differences in the nabla sense.

**Theorem 1.15.** ([12]) Assume \( f, g : \mathbb{N}_{a,b} \to \mathbb{R} \) and \( \nu \in (0, 1/2) \). Then we have
\[
\sum_{j=a+1}^{b-1} g(j) A^\nu_a \nabla^{-\nu} f(j) = \sum_{j=a+1}^{b-1} f(j) A^\nu_b \nabla^{-\nu} g(j).
\]

**Theorem 1.16.** ([12]) Assume \( f, g : \mathbb{N}_{a,b} \to \mathbb{R} \) and \( \nu \in (0, 1/2) \). Then we have
\[
\sum_{j=a+1}^{b-1} f(j) A^\nu_a \nabla^{\nu} g(j) = \sum_{j=a+1}^{b-1} g(j) A^\nu_b \nabla^{\nu} f(j).
\]
Prior to the presentation of an integration by parts for the left \( ABC \) fractional differences, we recall the discrete forms of the left and right generalized fractional integrals stated in (4) and (5).

**Definition 1.17.** ([12])

- The discrete (left) generalized fractional integral operator is defined by

\[
E_{\nu,\beta,\omega,a}^1 \varphi(\zeta) = \sum_{j=a+1}^{\zeta} (\zeta - \rho(j))^{\nu-1} E_{\nu,\beta}(\omega, \zeta - \rho(j)) \varphi(j), \quad \zeta \in \mathbb{N}_a.
\]

- The discrete (right) generalized fractional integral operator is defined by

\[
E_{\nu,\beta,\omega,b}^1 \varphi(\zeta) = \sum_{j=\zeta}^{b-1} (j - \rho(\zeta))^{\nu-1} E_{\nu,\beta}(\omega, j - \rho(\zeta)) \varphi(j), \quad \zeta \in \mathbb{N}_b.
\]

**Remark 2.** By means of Definition 1.17, the \( ABR \) and \( ABC \) fractional differences can be expressed as:

\[
\begin{align*}
ABR_a \nabla^\nu \phi(\zeta) &= \frac{B(\nu)}{1-\nu} \nabla E_{\nu,1,\frac{-\nu}{1-\nu},a}^1 \phi(\zeta), \\
ABR_b \nabla^\nu \phi(\zeta) &= \frac{-B(\nu)}{1-\nu} \Delta E_{\nu,1,\frac{-\nu}{1-\nu},b}^1 \phi(\zeta), \\
ABC_a \nabla^\nu \phi(\zeta) &= \frac{B(\nu)}{1-\nu} E_{\nu,1,\frac{-\nu}{1-\nu},a}^1 \nabla \phi(\zeta), \\
ABC_b \nabla^\nu \phi(\zeta) &= \frac{-B(\nu)}{1-\nu} E_{\nu,1,\frac{-\nu}{1-\nu},b}^1 \Delta \phi(\zeta).
\end{align*}
\]

**Lemma 1.18.** Assume \( \phi, \psi : \mathbb{N}_{a,b} \to \mathbb{R} \) and \( \nu \in (0, 1/2) \). Then we have

\[
\sum_{j=a+1}^{b-1} \phi(j) E_{\nu,1,\frac{-\nu}{1-\nu},a}^1 \psi(j) = \sum_{j=a+1}^{b-1} \psi(j) E_{\nu,1,\frac{-\nu}{1-\nu},b}^1 \phi(j).
\]

**Proof.** The proof is straightforward and it follows by interchanging the order of generalized summations.

In [12], the authors presented integration by parts formulas for the \( ABC \) nabla fractional differences by using Theorem 1.16, Eq.(6) and Eq.(7). By making use of Remark 2, Lemma 1.18, and the integration by parts in the usual difference calculus, we shall present an integration by parts formula for the left \( ABC \) fractional differences .

**Theorem 1.19.** Assume \( \phi, \psi : \mathbb{N}_{a,b} \to \mathbb{R} \) and \( \nu \in (0, 1/2) \). Then we have

\[
\sum_{j=a+1}^{b-1} \phi(j) ABC \nabla^\nu \psi(j)
\]

\[
= \sum_{j=a+1}^{b-1} \psi(j-1) ABR \nabla^\nu \phi(j-1) + \psi(\zeta) \frac{B(\nu)}{1-\nu} E_{\nu,1,\frac{-\nu}{1-\nu},b}^1 \phi(\zeta)|_{a}^{b-1}.
\]

\]
Remark 3. Another form of the integration by parts formula in Theorem 1.19 is:
\[
\sum_{j=a+1}^{b} \phi(j) \, _{a}^{B}D_{\nu}^{\nu} \psi(j) = \sum_{j=a+1}^{b} \psi(j-1) \, _{a}^{BR}D_{\nu}^{\nu} \phi(j-1) + \psi(\zeta) \, \frac{B(\nu)}{1-\nu} \, E_{1,1}^{\nu,1,\nu-1,1-b+1-\phi(\zeta)}|_{a}^{b}.
\]

2. Main results. In this section, we consider discrete fractional Sturm-Liouville equations containing the left- and right-sided ABC and ABR fractional difference operators. Our approach depends on the discrete integration by part formulas. Let
\[
L_{1}\xi(\zeta) = _{a}^{ABR}D_{\nu}^{\nu}(p(\zeta) \, _{a}^{ABR}D_{\nu}^{\nu}\xi(\zeta)) + q(\zeta)\xi(\zeta).
\]
Consider the fractional SLE
\[
_{a}^{ABR}D_{\nu}^{\nu}(p(\zeta) \, _{a}^{ABR}D_{\nu}^{\nu}\xi(\zeta)) + q(\zeta)\xi(\zeta) = \lambda r(\zeta)\xi(\zeta), \quad \zeta \in (a, b), \quad (8)
\]
where \(\nu \in (0, 1), p(\zeta) > 0, r(\zeta) > 0 \forall \zeta \in [a, b], p, q, r \) are in the space \(C[a, b]\).

**Theorem 2.1.** The fractional SL operator \(L_{1}\) is self-adjoint with respect to the inner product
\[
< \eta, \theta > = \int_{a}^{b} \overline{\eta(\zeta)}\theta(\zeta) \, d\zeta.
\]

**Proof.** From
\[
\theta(\zeta)L_{1}\overline{\theta}(\zeta) = \theta(\zeta) \, _{a}^{ABR}D_{\nu}^{\nu}(p(\zeta) \, _{a}^{ABR}D_{\nu}^{\nu}\overline{\theta}(\zeta)) + q(\zeta)\overline{\theta}(\zeta)
\]
and
\[
\overline{\theta}(\zeta)L_{1}\theta(\zeta) = \overline{\theta(\zeta)} \, _{a}^{ABR}D_{\nu}^{\nu}(p(\zeta) \, _{a}^{ABR}D_{\nu}^{\nu}\theta(\zeta)) + q(\zeta)\theta(\zeta),
\]
we obtain
\[
\theta(\zeta)L_{1}\overline{\theta}(\zeta) - \overline{\theta(\zeta)}L_{1}\theta(\zeta) = \theta(\zeta) \, _{a}^{ABR}D_{\nu}^{\nu}(p(\zeta) \, _{a}^{ABR}D_{\nu}^{\nu}\overline{\theta}(\zeta)) - \overline{\theta(\zeta)} \, _{a}^{ABR}D_{\nu}^{\nu}(p(\zeta) \, _{a}^{ABR}D_{\nu}^{\nu}\theta(\zeta)). \quad (9)
\]
Integrating (9) over the interval \([a, b]\), we see that
\[
\int_{a}^{b} (\theta(\zeta)L_{1}\overline{\theta}(\zeta) - \overline{\theta(\zeta)}L_{1}\theta(\zeta)) \, d\zeta =
\]
\[
\int_{a}^{b} (\theta(\zeta) \, _{a}^{ABR}D_{\nu}^{\nu}(p(\zeta) \, _{a}^{ABR}D_{\nu}^{\nu}\overline{\theta}(\zeta)) - \overline{\theta(\zeta)} \, _{a}^{ABR}D_{\nu}^{\nu}(p(\zeta) \, _{a}^{ABR}D_{\nu}^{\nu}\theta(\zeta))) \, d\zeta.
\]
From Theorem 1.5, it now follows that
\[
\int_{a}^{b} (\theta(\zeta)L_{1}\overline{\theta}(\zeta) - \overline{\theta(\zeta)}L_{1}\theta(\zeta)) \, d\zeta = \int_{a}^{b} p(\zeta) \, _{a}^{ABR}D_{\nu}^{\nu}\overline{\theta}(\zeta) \, _{a}^{ABR}D_{\nu}^{\nu}\theta(\zeta) \, d\zeta
\]
\[
- \int_{a}^{b} p(\zeta) \, _{a}^{ABR}D_{\nu}^{\nu}\theta(\zeta) \, _{a}^{ABR}D_{\nu}^{\nu}\overline{\theta}(\zeta) \, d\zeta = 0.
\]
Hence, \(< L_{1}\eta, \theta > = < \eta, L_{1}\theta >\). That is, \(L_{1}\) is self-adjoint.

**Theorem 2.2.** Equation (8) has real eigenvalues.
Proof. Suppose that $\lambda$ is the eigenvalue of (8) associated with the eigenfunction $\xi$. Then $\xi$ and its complex conjugate $\bar{\xi}$ satisfy

$$L_1\xi(\zeta) = \lambda r(\zeta)\xi(\zeta)$$

(10)

and

$$L_1\bar{\xi}(\zeta) = \bar{\lambda}r(\zeta)\bar{\xi}(\zeta),$$

(11)

respectively. Multiplying (10) by $\bar{\xi}(\zeta)$ and (11) by $\xi(\zeta)$, respectively, and then subtracting, we get

$$(\bar{\lambda} - \lambda)r(\zeta)\xi(\zeta)\bar{\xi}(\zeta) = \xi(\zeta)L_1\bar{\xi}(\zeta) - \bar{\xi}(\zeta)L_1\xi(\zeta).$$

(12)

$$\int_a^b \lambda - \lambda \int_a^b r(\zeta)|\xi(\zeta)|^2 d\zeta = 0.$$

Since $\xi$ is nontrivial and $r(\zeta) > 0$, it follows that $\lambda = \bar{\lambda}$. \hfill \qed

**Theorem 2.3.** The eigenfunctions, corresponding to distinct eigenvalues of (8) are orthogonal with respect to the weight function $r$ on $[a, b]$ that is

$$<\xi_{\lambda_1}, \xi_{\lambda_2}>= \int_a^b r(\zeta)\xi_{\lambda_1}(\zeta)\xi_{\lambda_2}(\zeta) d\zeta = 0, \quad \lambda_1 \neq \lambda_2,$$

when the functions $\xi_{\lambda_i}$ correspond to eigenvalues $\lambda_i$, $i = 1, 2$.

Proof. Let $\lambda_1$ and $\lambda_2$ be two distinct eigenvalues of (8) and let $\xi_{\lambda_1}$ and $\xi_{\lambda_2}$ be the corresponding eigenfunctions. Then we have

$$a \text{ABR}D(\nu)(p(\zeta) \text{ABR}D(\nu)\xi_{\lambda_1}(\zeta)) + q(\zeta)\xi_{\lambda_1}(\zeta) = \lambda_1 r(\zeta)\xi_{\lambda_1}(\zeta)$$

(13)

and

$$a \text{ABR}D(\nu)(p(\zeta) \text{ABR}D(\nu)\xi_{\lambda_2}(\zeta)) + q(\zeta)\xi_{\lambda_2}(\zeta) = \lambda_2 r(\zeta)\xi_{\lambda_2}(\zeta).$$

(14)

Multiplying (13) and (14) by $\xi_{\lambda_2}(\zeta)$ and $\xi_{\lambda_1}(\zeta)$, respectively, and then subtracting, we get

$$(\lambda_1 - \lambda_2)r(\zeta)\xi_{\lambda_1}(\zeta)\xi_{\lambda_2}(\zeta) = \xi_{\lambda_2}(\zeta) a \text{ABR}D(\nu)(p(\zeta) a \text{ABR}D(\nu)\xi_{\lambda_1}(\zeta))$$

(15)

$$- \xi_{\lambda_1}(\zeta) a \text{ABR}D(\nu)(p(\zeta) a \text{ABR}D(\nu)\xi_{\lambda_2}(\zeta)).$$

Now, integrating (16) over the interval $[a, b]$ and then using Theorem 1.5, we obtain

$$(\lambda_1 - \lambda_2) \int_a^b r(\zeta)\xi_{\lambda_1}(\zeta)\xi_{\lambda_2}(\zeta) d\zeta = 0.$$

Since $\lambda_1 \neq \lambda_2$, we have

$$\int_a^b r(\zeta)\xi_{\lambda_1}(\zeta)\xi_{\lambda_2}(\zeta) d\zeta = 0,$$

which completes the proof. \hfill \qed

Now, consider the nabla discrete fractional SL operator

$$L_2\xi(\zeta) = a \text{ABR}\nabla(\nu)(p(\zeta) a \text{ABR}\nabla(\nu)\xi(\zeta)) + q(\zeta)\xi(\zeta),$$

and the corresponding SLE

$$a \text{ABR}\nabla(\nu)(p(\zeta) a \text{ABR}\nabla(\nu)\xi(\zeta)) + q(\zeta)\xi(\zeta) = \lambda r(\zeta)\xi(\zeta), \quad \zeta \in \mathbb{N}_{a+1,b-1},$$

(16)

where $\nu \in (0, 1/2)$, $p(\zeta) > 0$, $r(\zeta) > 0 \forall \zeta \in [a, b]$, $p, q, r$ are real valued functions on $\mathbb{N}_{a,b}$. 

A STURM-LIOUVILLE APPROACH FOR MITTAG-LEFFLER KERNEL OPERATORS 11

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The text appears to be a mathematical proof involving eigenvalue problems and the use of Sturm-Liouville operators. The proof involves showing orthogonality of eigenfunctions corresponding to distinct eigenvalues. The notation and concepts are typical in the field of functional analysis and differential equations.
Proof. By Proposition 1, we have

\[ \langle \eta, \theta \rangle = \sum_{\zeta = a + 1}^{b-1} \overline{\eta(\zeta)} \theta(\zeta). \]

Proof. The proof is as in Theorem 2.1 and the result follows directly by using Theorem 1.16.

\[ \text{Theorem 2.5.} \quad \text{The eigenvalues of (16) are real.} \]

Proof. The proof is as in Theorem 2.2.

\[ \text{Theorem 2.6.} \quad \text{The eigenfunctions, corresponding to distinct eigenvalues of (16) are orthogonal with respect to the weight function } r \text{ on } \mathbb{N}_{a,b}, \text{ that is} \]

\[ \langle \xi_{\mu_1}, \xi_{\mu_2} \rangle = \sum_{\zeta = a + 1}^{b-1} r(\zeta) \xi_{\mu_1}(\zeta) \xi_{\mu_2}(\zeta) = 0, \quad \mu_1 \neq \mu_2, \]

when the functions \( \xi_\lambda \) correspond to eigenvalues \( \mu_\zeta, \zeta = 1, 2. \)

Proof. The proof is as in Theorem 2.3, whereas it follows from Theorem 1.16.

Next, for the purpose of the investigation of discrete mixed ABC-ABR SLEs with appropriate boundary conditions, we consider

\[ C L_1 \xi(\zeta) = A_{ABC} D^\nu(r(\zeta)) A_{ABR} D^\nu_\zeta(\xi(\zeta)) + q(\zeta) \xi(\zeta) \]

and consider the ABC type fractional SLE

\[ A_{ABC} D^\nu(p(\zeta)) A_{ABR} D^\nu_\zeta(\xi(\zeta)) + q(\zeta) \xi(\zeta) = \lambda r(\zeta) \xi(\zeta), \quad \zeta \in (a, b), \]

where \( \nu \in (0, 1), p(\zeta) \neq 0, r(\zeta) > 0 \forall \zeta \in [a, b], p, q, r \) are in \( C[a, b] \) together with the boundary conditions:

\[ c_1 E_{\nu, 1, \frac{b-a}{\nu}} \xi(a) + c_2 A_{ABR} D^\nu_\zeta(\xi(a)) = 0, \]

\[ d_1 E_{\nu, 1, \frac{b-a}{\nu}} \xi(b) + d_2 A_{ABR} D^\nu_\zeta(\xi(b)) = 0, \]

where \( c_1^2 + c_2^2 \neq 0 \) and \( d_1^2 + d_2^2 \neq 0. \)

\[ \text{Theorem 2.7.} \quad \text{The eigenvalues of (17)-(19) are real.} \]

Proof. By Proposition 1, we have

\[ \int_a^b \eta(\zeta) C L_1 \theta(\zeta) d\zeta = \int_a^b q(\zeta) \eta(\zeta) \theta(\zeta) d\zeta + \]

\[ \int_a^b p(\zeta) A_{ABR} D^\nu_\zeta(\theta(\zeta)) A_{ABR} D^\nu_\zeta(\eta(\zeta)) d\zeta + \frac{B(\nu)}{1 - \nu} \int_a^b \eta(\zeta) A_{ABR} D^\nu_\zeta(\theta(\zeta)) d\zeta. \]

Let \( \lambda \) be an eigenvalue of (17)-(19) and let \( \xi \) be the corresponding eigenfunction. Then \( \xi \) and its complex conjugate \( \xi \) satisfy

\[ C L_1 \xi(\zeta) = \lambda r(\zeta) \xi(\zeta), \]

\[ c_1 E_{\nu, 1, \frac{b-a}{\nu}} \xi(a) + c_2 A_{ABR} D^\nu_\zeta(\xi(a)) = 0, \]

\[ d_1 E_{\nu, 1, \frac{b-a}{\nu}} \xi(b) + d_2 A_{ABR} D^\nu_\zeta(\xi(b)) = 0, \]

and

\[ C L_1 \overline{\xi}(\zeta) = \overline{\lambda r(\zeta) \xi(\zeta)}, \]
Multiplying (28) by $c_2 ABR D^\nu_{\nu} \xi(a)$ and (29) by $d_1 ABR D^\nu_{\nu} \xi(a) = 0$, respectively, we see that

$$
(\lambda - \nu) r(\zeta) \xi(\zeta) \bar{\xi}(\zeta) = \xi(\zeta) C L_1 \xi(\zeta) - \bar{\xi}(\zeta) C L_1 \bar{\xi}(\zeta).
$$

Integrating (27) over the interval $[a, b]$ and applying (20) with $\eta(\zeta) = \xi(\zeta)$ and $\theta(\zeta) = \bar{\xi}(\zeta)$ and vice versa, we get

$$
(\lambda - \nu) \int_a^b r(\zeta) |\xi(\zeta)|^2 d\zeta = 0.
$$

Since $\xi$ is nontrivial and $r(\zeta) > 0$, we have $\lambda = \bar{\lambda}$.

Theorem 2.8. The eigenfunctions, belonging to distinct eigenvalues of (17)-(19) are orthogonal with respect to the weight function $r$ on $[a, b]$, namely

$$
< \xi_{\lambda_1}, \xi_{\lambda_2} > = \int_a^b r(\zeta) \xi_{\lambda_1}(\zeta) \xi_{\lambda_2}(\zeta) d\zeta = 0, \quad \lambda_1 \neq \lambda_2,
$$

when the functions $\xi_{\lambda_i}$ correspond to eigenvalues $\lambda_i$, $i = 1, 2$.

Proof. Assume $\lambda_1$ and $\lambda_2$ are distinct eigenvalues of (17)-(19) and let $\xi_{\lambda_1}$ and $\xi_{\lambda_2}$ be the corresponding eigenfunctions. Then we have

$$
C L_1 \xi_{\lambda_1}(\zeta) = \lambda_1 r(\zeta) \xi_{\lambda_1}(\zeta),
$$

(28) and

$$
C L_1 \xi_{\lambda_2}(\zeta) = \lambda_2 r(\zeta) \xi_{\lambda_2}(\zeta),
$$

(31) and

$$
C L_1 \xi_{\lambda_1}(\zeta) = \lambda_1 r(\zeta) \xi_{\lambda_1}(\zeta),
$$

(29) and

$$
C L_1 \xi_{\lambda_2}(\zeta) = \lambda_2 r(\zeta) \xi_{\lambda_2}(\zeta),
$$

(32) and

$$
C L_1 \xi_{\lambda_1}(\zeta) = \lambda_1 r(\zeta) \xi_{\lambda_1}(\zeta),
$$

(33)

Multiplying (28) by $\xi_{\lambda_2}(\zeta)$ and (31) by $\xi_{\lambda_1}(\zeta)$, respectively, and the subtracting, we get

$$
\xi_{\lambda_2}(\zeta) C L_1 \xi_{\lambda_1}(\zeta) - \xi_{\lambda_1}(\zeta) C L_1 \xi_{\lambda_2}(\zeta) = (\lambda_1 - \lambda_2) r(\zeta) \xi_{\lambda_1}(\zeta) \xi_{\lambda_2}(\zeta).
$$

Integrating (34) over the interval $[a, b]$ and applying (20) with $\eta(\zeta) = \xi_{\lambda_2}(\zeta)$ and $\theta(\zeta) = \xi_{\lambda_1}(\zeta)$ and vice versa, we have

$$
(\lambda_1 - \lambda_2) \int_a^b r(\zeta) \xi_{\lambda_1}(\zeta) \xi_{\lambda_2}(\zeta) d\zeta =
$$

$$
\frac{B(\nu)}{1 - \nu} p(b) \left[ E^{1}_{\nu,1, \frac{\nu}{\nu}} - \xi_{\lambda_2}(b) ABR D^\nu_{\nu} \xi_{\lambda_1}(b) - E^{1}_{\nu,1, \frac{\nu}{\nu}} - \xi_{\lambda_1}(b) ABR D^\nu_{\nu} \xi_{\lambda_2}(b) \right] +
$$

$$
\frac{B(\nu)}{1 - \nu} p(a) \left[ E^{1}_{\nu,1, \frac{\nu}{\nu}} - \xi_{\lambda_2}(a) ABR D^\nu_{\nu} \xi_{\lambda_1}(a) - E^{1}_{\nu,1, \frac{\nu}{\nu}} - \xi_{\lambda_1}(a) ABR D^\nu_{\nu} \xi_{\lambda_2}(a) \right] = 0.
$$

Proof. Assume $\lambda_1$ and $\lambda_2$ are distinct eigenvalues of (17)-(19) and let $\xi_{\lambda_1}$ and $\xi_{\lambda_2}$ be the corresponding eigenfunctions. Then we have

$$
C L_1 \xi_{\lambda_1}(\zeta) = \lambda_1 r(\zeta) \xi_{\lambda_1}(\zeta),
$$

(28) and

$$
C L_1 \xi_{\lambda_2}(\zeta) = \lambda_2 r(\zeta) \xi_{\lambda_2}(\zeta),
$$

(31) and

$$
C L_1 \xi_{\lambda_1}(\zeta) = \lambda_1 r(\zeta) \xi_{\lambda_1}(\zeta),
$$

(29) and

$$
C L_1 \xi_{\lambda_2}(\zeta) = \lambda_2 r(\zeta) \xi_{\lambda_2}(\zeta),
$$

(32) and

$$
C L_1 \xi_{\lambda_1}(\zeta) = \lambda_1 r(\zeta) \xi_{\lambda_1}(\zeta),
$$

(33)

Multiplying (28) by $\xi_{\lambda_2}(\zeta)$ and (31) by $\xi_{\lambda_1}(\zeta)$, respectively, and the subtracting, we get

$$
\xi_{\lambda_2}(\zeta) C L_1 \xi_{\lambda_1}(\zeta) - \xi_{\lambda_1}(\zeta) C L_1 \xi_{\lambda_2}(\zeta) = (\lambda_1 - \lambda_2) r(\zeta) \xi_{\lambda_1}(\zeta) \xi_{\lambda_2}(\zeta).
$$

Integrating (34) over the interval $[a, b]$ and applying (20) with $\eta(\zeta) = \xi_{\lambda_2}(\zeta)$ and $\theta(\zeta) = \xi_{\lambda_1}(\zeta)$ and vice versa, we have

$$
(\lambda_1 - \lambda_2) \int_a^b r(\zeta) \xi_{\lambda_1}(\zeta) \xi_{\lambda_2}(\zeta) d\zeta =
$$

$$
\frac{B(\nu)}{1 - \nu} p(b) \left[ E^{1}_{\nu,1, \frac{\nu}{\nu}} - \xi_{\lambda_2}(b) ABR D^\nu_{\nu} \xi_{\lambda_1}(b) - E^{1}_{\nu,1, \frac{\nu}{\nu}} - \xi_{\lambda_1}(b) ABR D^\nu_{\nu} \xi_{\lambda_2}(b) \right] +
$$

$$
\frac{B(\nu)}{1 - \nu} p(a) \left[ E^{1}_{\nu,1, \frac{\nu}{\nu}} - \xi_{\lambda_2}(a) ABR D^\nu_{\nu} \xi_{\lambda_1}(a) - E^{1}_{\nu,1, \frac{\nu}{\nu}} - \xi_{\lambda_1}(a) ABR D^\nu_{\nu} \xi_{\lambda_2}(a) \right] = 0.
$$
For this purpose, we define the inner product
\[
\frac{B(\nu)}{1 - \nu} p(a) \left[ E^{1}_{\nu, 1} b^c \cdot \zeta_{\lambda_1}(a) A^{BR} D^\nu b \cdot \zeta_{\lambda_2}(a) - E^{1}_{\nu, 1} b^c \cdot \zeta_{\lambda_2}(a) A^{BR} D^\nu b \cdot \zeta_{\lambda_1}(a) \right].
\]

From the boundary conditions (29), (30), (32), and (33), it now follows that
\[
(\lambda_1 - \lambda_2) \int_a^b r(\zeta) \zeta_{\lambda_1}(\zeta) \zeta_{\lambda_2}(\zeta) d\zeta = 0,
\]
and hence, as \( \lambda_1 \neq \lambda_2, \zeta_{\lambda_1}, \zeta_{\lambda_2} \geq 0 \). \( \square \)

**Remark 4.** If we employ the integration by parts formula in part two of Proposition 1 then we may have the following SLE:
\[
A^{BC} D^\nu p(\zeta) A^{BR} D^\nu \zeta(\zeta) + q(\zeta) \zeta(\zeta) = \lambda r(\zeta) \zeta(\zeta), \ \zeta \in (a, b),
\]
with the boundary conditions:
\[
c_1 E^{1}_{\nu, 1} b^c \cdot a^c \cdot \zeta(a) + c_2 A^{BR} D^\nu \zeta(a) = 0,
\]
\[
d_1 E^{1}_{\nu, 1} b^c \cdot a^c \cdot \zeta(b) + d_2 A^{BR} D^\nu \zeta(b) = 0,
\]
where \( c_1^2 + c_2^2 \neq 0 \) and \( d_1^2 + d_2^2 \neq 0 \). Similar results can also be proved for (35)-(37) as proved for (17)-(19).

For further detailed investigation of mixed ABC-ABR SLE, we consider
\[
d^{BC} L_1 \zeta(\zeta) = A^{BC} D^\nu p(\zeta) A^{BR} D^\nu \zeta(\zeta) + q(\zeta) \zeta(\zeta)
\]
and consider the nabla discrete ABC type SLE
\[
A^{BC} D^\nu p(\zeta) A^{BR} D^\nu \zeta(\zeta) + q(\zeta) \zeta(\zeta) = \lambda r(\zeta) \zeta(\zeta), \ \zeta \in N_{a+1, b-1},
\]
where \( \nu \in (0, 0.5), p(\zeta) \neq 0, r(\zeta) > 0 \forall \zeta \in N_{a,b-1}, p, q : N_{a,b-1} \to \mathbb{R} \), together with the boundary conditions:
\[
c_1 E^{1}_{\nu, 1} b^c \cdot a^c \cdot \zeta(a) + c_2 A^{BR} D^\nu \zeta(a) = 0,
\]
\[
d_1 E^{1}_{\nu, 1} b^c \cdot a^c \cdot \zeta(b) + d_2 A^{BR} D^\nu \zeta(b) = 0,
\]
where \( c_1^2 + c_2^2 \neq 0 \) and \( d_1^2 + d_2^2 \neq 0 \). By using Theorem 1.19, we now prove the corresponding results for (38)-(40). For this purpose, we define the inner product
\[
< \phi, \psi > = \sum_{\zeta = a+1}^{b-1} r(\zeta) \phi(\zeta) \psi(\zeta).
\]

**Remark 5.** In order to use Remark 3, we consider the SLE
\[
d^{BC} L_2 \zeta(\zeta) = A^{BC} D^\nu p(\zeta) A^{BR} D^\nu \zeta(\zeta) + q(\zeta) \zeta(\zeta) = \lambda r(\zeta) \zeta(\zeta), \ \zeta \in N_{a+1, b},
\]
where \( \nu \in (0, 0.5), p(\zeta) \neq 0, r(\zeta) > 0 \forall t \in N_{a,b}, p, q : N_{a,b} \to \mathbb{R} \), together with the conditions:
\[
c_1 E^{1}_{\nu, 1} b^c \cdot a^c \cdot \zeta(a) + c_2 A^{BR} D^\nu \zeta(a) = 0,
\]
\[
d_1 E^{1}_{\nu, 1} b^c \cdot a^c \cdot \zeta(b) + d_2 A^{BR} D^\nu \zeta(b) = 0,
\]
where \( c_1^2 + c_2^2 \neq 0 \) and \( d_1^2 + d_2^2 \neq 0 \) and the inner product
\[
< \phi, \psi > = \sum_{\zeta = a+1}^{b} r(\zeta) \phi(\zeta) \psi(\zeta).
\]
Theorem 2.9. The eigenvalues of (38)-(40) are real.
Proof. The proof is as in Theorem 2.7 and the result follows from Theorem 1.19. □

Theorem 2.10. The eigenfunctions, belonging to distinct eigenvalues of (38)-(40) are orthogonal via the weight function \( r \) on \([a, b]\) that is
\[
\langle \xi_{\lambda_1}, \xi_{\lambda_2} \rangle = \sum_{\zeta=a+1}^{b-1} r(\zeta) \xi_{\lambda_1}(\zeta) \xi_{\lambda_2}(\zeta) = 0, \quad \lambda_1 \neq \lambda_2,
\]
where the functions \( \xi_{\lambda_i} \) correspond to eigenvalues \( \lambda_\zeta, \zeta = 1, 2. \)

Proof. The proof is as in Theorem 2.8 and it follows directly from Theorem 1.19 by making use of the inner product (41). □

Remark 6. The proofs of Theorem 2.7, Theorem 2.9 and Remark 5 indicate that \( ^aC^1L_1, \ ^dC^1L_1 \) and \( \ ^dC^1L_2 \) are self-adjoint, respectively.

3. The higher order discrete fractional SLE and an open problem. In the previous section, the values of \( \nu \) in the discrete SLE are taken in the interval \((0, \frac{1}{2})\) in order to handle the convergence for the discrete Mittag-Leffler kernel used in defining the \( \text{ABR} \) and \( \text{ABC} \) fractional differences. Therefore, the ordinary difference SLE can not be obtained as \( \nu \rightarrow 1^- \). For this purpose, we shall recommend for a discrete fractional SLE with the values of \( \nu \in (1, \frac{3}{2}) \) so that the ordinary difference case can be obtained as \( \nu \rightarrow 1^+ \). For the main concepts regarding to higher order fractional calculus with discrete Mittag-Leffler kernels, we refer to [16]. For higher order fractional operators with non-singular Mittag-Leffler kernels, we refer the reader to [9], where a Lyapunov type inequality was formulated for a \( BV \) of order \( 2 < \nu < 3 \) and the ordinary Lyapunov inequality was obtained as \( \nu \rightarrow 2^+ \). Using Definition 2.1 in [16] and Definition 1.17, Lemma 1.18, and Remark 2 in our paper that if \( \nu \in (1, 2), \beta = \nu - 1 \in (0, 1), \lambda_\beta = \frac{1}{1-\beta} = -\frac{\nu-1}{2-\nu} \) and \( |\lambda_\beta| < 1 \) if \( \nu \in (1, \frac{3}{2}) \), then the following four results hold:
\[
\begin{align*}
\ ^a\text{ABC}\nabla^\nu \phi(\zeta) &= \ ^a\text{ABC}\nabla^\beta \nabla^2 \phi(\zeta) = \frac{B(\nu-1)}{2-\nu} \ E_{\beta,1,\lambda_\beta, a+}^1 \nabla^2 \phi(\zeta), \\
\ ^a\text{ABR}\nabla^\nu \phi(\zeta) &= \ ^a\text{ABR}\nabla^\beta \nabla \phi(\zeta) = \frac{B(\nu-1)}{2-\nu} \ \nabla \ E_{\beta,1,\lambda_\beta, a+}^1 \nabla \phi(\zeta), \\
\ ^a\text{ABC}\nabla^\nu_\phi(\zeta) &= \ ^a\text{ABC}\nabla^\nu_\beta (-\Delta \phi)(\zeta) = \frac{B(\nu-1)}{2-\nu} \ \Delta \ E_{\beta,1,\lambda_\beta, b-}^1 \Delta \phi(\zeta), \\
\ ^a\text{ABR}\nabla^\nu_\phi(\zeta) &= \ ^a\text{ABR}\nabla^\nu_\beta (-\Delta \phi)(\zeta) = \frac{B(\nu-1)}{2-\nu} \ \Delta \ E_{\beta,1,\lambda_\beta, b-}^1 \Delta \phi(\zeta).
\end{align*}
\]

Remark 7. Notice that the ordinary SLE can be obtained in the non-discrete case as \( \nu \rightarrow 1^- \). In fact, since non-discrete Mittag-Leffler kernels do not have a convergence problem, the \( \text{ABR} \) and \( \text{ABC} \) fractional operators become well-defined for any \( \nu \in (0, 1] \). For that reason, we present the following open problem in the discrete case.

Open problem: Depending on the above discussion, can we formulate integration by parts formulas for the \( \text{ABR} \) and \( \text{ABC} \) fractional differences of order \( \nu \in (1, \frac{3}{2}) \) which will be used to prove the essential properties of a proper fractional difference SLEs with suitable assigned boundary conditions?
4. Conclusion. Fractional operators with nonsingular Mittag-Leffler kernels, which have been introduced recently by Atangana and Baleanu depending on a limit process via dirac delta functions, have been started to be applied extensively by many researchers to model real world problems. On the other hand, the Sturm-Liouville problems are always of priority to be investigated by mathematicians working in dynamical systems and boundary value problems. They have many applications in several areas of science, engineering, and mathematics. However, few works are observed regarding their fractional formulation in the frame of the Riemann-Liouville and Caputo type derivatives and no works have been noticed, up to our knowledge, for the case of Atangana-Baleanu fractional operators. Motivated by the above observations, in this work we have establish the self-adjointness, the eigenvalues and the eigenfunction of the Sturm-Liouville problems in the frame of Atangana-Baleanu fractional operators in both the continuous and discrete senses.

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