Radiation from a Charged Particle Bunch Exiting an Open-Ended Dielectric-Loaded Circular Waveguide: Rigorous Approach

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An elegant and convenient rigorous approach for solving circular open-ended dielectric-loaded waveguide diffraction problems is presented. It uses the solution of corresponding Wiener-Hopf-Fock equation to obtain an infinite linear system for reflection coefficients (S-parameters) of the waveguide. This system can be efficiently solved numerically using the reduction technique. As a specific example, diffraction of a TM symmetrical mode at the open end of a circular waveguide with uniform dielectric filling is considered. A series of such modes represent the wakefield (Cherenkov radiation field) generated by a charged particle bunch during its passage through the waveguide. Calculated S-parameters were compared with those obtained from COMSOL simulation and an excellent agreement is shown. Advantages of using this method for investigation of various waveguide structures prospective in context of modern beam-driven Terahertz radiation sources development is discussed.

INTRODUCTION

Recently, a high interest has been shown toward radiation in the Terahertz (THz) frequency range, which is a very promising tool for studying various structures and materials [1, 2]. To generate this radiation, both classical methods of microwave electronics and methods known in quantum radiation generators are developed. Several innovative ideas are also being considered. One of these ideas is the application of Cherenkov radiation (CR) generated by a specially prepared relativistic bunch (or bunch train) in a special dielectric waveguide structure [3]. The expected intensity of radiation exceeds by several orders of magnitude the intensity of traditional THz oscillators and backward-wave tubes [4]. To improve this scheme, it is necessary to investigate not only regular waveguides but also waveguides with an open end, since one of the main tasks here is efficient extraction of the generated CR into the open space.

At present, there are technologies for creating metalized waveguides with diameters of the order of a millimeter and less (“capillaries”) having layers of different dielectrics [5]. Technologies were also developed for generating electron bunches with size of hundreds of microns or less [6]. The bunch passing along the channel in such capillaries generates CR whose high-order fundamental modes can have frequencies in the THz range. After that, CR should be extracted into the free space through the open end of such a structure. There are recent experimental results, which indicate that this approach is promising [7, 8].

It should be noted that the rigorous theory of waveguides with an open end was actively developed in many works (see, for example, Refs. [9, 10]. In particular, the case of a planar waveguide with dielectric filling was considered [11]. Waveguides having a non-orthogonal open cut (the so-called Vlasov antennas) were considered less strictly [12]. To our knowledge, cylindrical waveguides (which are of most practical importance) with an open end and a dielectric loading have been studied much less frequently and in approximate formulation only [13, 14]. Reliable rigorous results were recently obtained only for an “embedded” (closed) structure where an open-ended waveguide was placed inside an infinite collinear waveguide of larger radius [15–18] Therefore, questions remain about the limits of the applicability of the aforementioned approximate methods, the error in the results and the possible ways to improve their accuracy. The answers can be given by direct comparison between their results with those obtained using a rigorous approach. Such an approach – which is elegant and efficient – is presented in this paper.

PROBLEM FORMULATION AND GENERAL SOLUTION

Let us consider a semi-infinite cylindrical waveguide with radius a filled with a dielectric (ε > 1) (Fig. 1). We suppose that single TM00 waveguide mode incidents the orthogonal open end (cylindrical frame ρ, φ, z is used):

\[ H^{(i)}_{ωφ} = M^{(i)} J_1(\rho_0 a) e^{i k_0 z}, \]
\[ E_{ωρ} = \frac{1}{ik_0 ε} \frac{∂H_{ωφ}}{∂z}, \quad E_{ωz} = \frac{i}{k_0 ε} \left( \frac{H_{ωφ}}{ρ} + \frac{∂H_{ωφ}}{∂ρ} \right), \]
where \( J_0(j_{0L}) = 0 \), \( k_{z\perp} = \sqrt{k_0^2 - j_{0L}^2}, \) \( \text{Im} \, k_{z\perp} > 0 \), \( k_0 = \omega/c + i\delta \) (\( \delta \to 0 \), which is equivalent to infinitely small dissipation in all areas). Connection between (1) and CR wakefield generated by a charged particle bunch small dissipation in all areas). Connection between (1) and CR wakefield generated by a charged particle bunch should be determined. The vacuum area is divided into two subareas “1” and “2” (see Fig. 1), where the field is determined using continuity of \( \frac{\partial E}{\partial z} \), for example, \( E(z) \). One can see that the function to the left of the “=” sign is regular in the area \( \text{Im} \, \alpha > -\delta \) while the function to the right possesses pole singularity for \( \alpha = \alpha_p, \, p = 1, 2, \ldots \) in this area. Since the singularity at the right-hand side should be canceled, we obtain the following requirement:

\[
\Phi_+^{(1)}(a, \alpha) = \frac{i\pi}{4\pi} J_1(j_{0p}) \times \left[ \delta_{lp} M(i) \frac{k_{zp}}{\varepsilon} - \alpha_p \right] - M_p \left( \frac{k_{zp}}{\varepsilon} + \alpha_p \right),
\]

where \( \delta_{lp} \) is the Kronecker symbol.

Using general solution for the area “2” (10), coefficient \( C_2 \) (13) and continuity conditions (14) we arrive at the Wiener-Hopf-Fock equation:

\[
0 = \frac{2i\Phi_+^{(2)}(a, \alpha)}{\kappa G(\alpha)} + \Phi_+^{(2)}(a, \alpha) + \frac{i}{2\pi} \times \left[ M(i) \frac{k_{zp}}{\varepsilon} - \alpha_p \right] - M_p \left( \frac{k_{zp}}{\varepsilon} + \alpha_p \right),
\]

where \( G(\alpha) = \pi \kappa J_0(ak) H_0^{(1)}(ak) \). Performing factorization, \( \kappa = \kappa^+ \kappa^- \), \( \kappa^\pm = \sqrt{k_0 \pm \alpha}, \) \( G(\alpha) = G_+(\alpha) G_-(\alpha) \), we obtain from (17) after multiplication by \( \kappa_+ G_+ \) and consequent decomposition of corresponding functions into a sum of “+” and “−” summands (standard formulas from [10] can be used):

\[
0 = \frac{2\Phi_+^{(2)}(a, \alpha)}{\kappa G(\alpha)} + \kappa_- G_-(\alpha) \Psi_+^{(2)}(a, \alpha) + \times \left[ M(i) \left[ \eta_+ (\alpha) + \eta_-(\alpha) \right] J_1(j_{0L}) - \right.
\]

\[
\left. - \sum_{m=1}^\infty M_m \left( \zeta_+(\alpha) + \zeta_-(\alpha) \right) J_1(j_{0m}) \right],
\]

where \( \alpha_m = \sqrt{k_0^2 - j_{0m}^2}, \) \( \text{Im} \alpha_m > 0 \) are longitudinal wavenumbers of vacuum waveguide, \( C_{1,2} \) are unknown coefficients. In particular, one obtains \( \Phi_+^{(1)}(a, \alpha) = C_1 \kappa J_0(ak) \), \( \Phi_+^{(2)}(a, \alpha) = C_2 \kappa H_0^{(1)}(ak) \). Performing factorization, \( \kappa = \kappa^+ \kappa^- \), \( \kappa^\pm = \sqrt{k_0 \pm \alpha}, \) \( G(\alpha) = G_+(\alpha) G_-(\alpha) \), we obtain from (17) after multiplication by \( \kappa_+ G_+ \) and consequent decomposition of corresponding functions into a sum of “+” and “−” summands (standard formulas from [10] can be used):

\[
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\]

\[
\left. - \sum_{m=1}^\infty M_m \left( \zeta_+(\alpha) + \zeta_-(\alpha) \right) J_1(j_{0m}) \right],
\]
where
\[ \eta_l(\alpha) = \kappa_-(\alpha)G_-(\alpha) \frac{k_{zl}}{\varepsilon - \alpha} \frac{1}{\alpha l^2 - \alpha^2}, \]
\[ \zeta_m(\alpha) = \kappa_-(\alpha)G_-(\alpha) \frac{k_{zm}}{\varepsilon - \alpha} \frac{1}{\alpha m^2 - \alpha^2}, \]
\[ \eta_+(\alpha) = \kappa_+(\alpha)G_+(\alpha) \frac{k_{zl}}{2\alpha_l(\alpha_l + 1)} + \alpha l, \]
\[ \zeta_+(\alpha) = \kappa_+(\alpha m)G_+(\alpha) \frac{k_{zm}}{2\alpha m(\alpha m + \alpha)} - \alpha m. \]

Equation (18) is solved in a common way: one should separate “+” and “−” terms into different parts of the equation:
\[ P(\alpha) = \frac{2M(19)}{\kappa_+G_+(\alpha)} + \frac{M(19)\eta_+(\alpha)J_1(j_{p1})}{2\pi} + \frac{M(19)\eta_-(\alpha)J_1(j_{p2})}{2\pi} + \frac{M(19)\zeta_m(\alpha)\Psi(2)(a,\alpha)}{2\pi} + \frac{M(19)\zeta_m(\alpha)\Psi(2)(a,\alpha)}{2\pi}, \]

where a polynomial function \( P(\alpha) \) has been written based on analytic continuation theorem [10]. To determine \( P(\alpha) \) one should estimate asymptotic behaviour of all terms in (21) for \(|\alpha| \to \infty, -\delta < \text{Im} \alpha < \delta \). Based on Meixner edge condition [10] we have:
\[ \Phi_+(2)(a,\alpha) |_{|\alpha| \to \infty} \sim \alpha^{-1/2-\tau}, \quad \tau = \frac{1}{\pi} \arcsin \left( \varepsilon - \frac{1}{\varepsilon + 1} \right), \]
\[ M_m \sim m^{-1-\tau}, \quad \Psi(2)(a,\alpha) |_{|\alpha| \to \infty} \sim \alpha^{-3/2}, \]

therefore all terms in (21) decrease in accordance with power law and therefore \( P(\alpha) = 0 \). Formal solution of the Wiener-Hopf-Fock equation then reads
\[ \Phi_+(2)(a,\alpha) = \frac{-\kappa_+(\alpha)G_+(\alpha)}{4\pi} \times \left[ M(19)\eta_+(\alpha)J_1(j_{p1}) + \sum_{m=1}^{\infty} M_m\zeta_m(\alpha)J_1(j_{p2}) \right]. \]

It should be noted that (23) contains unknown coefficients \( M_m \). To resolve this, one should substitute (23) into (16). After that we obtain the following infinite linear system for \( M_m \):
\[ \sum_{m=1}^{\infty} W_{pm}M_m = w_p, \quad p = 1, 2, \ldots, \]

where
\[ W_{pm} = J_1(j_{p1}) \left[ \zeta_m(\alpha_p) + \delta_m ig \frac{k_{zm}}{\kappa_+(\alpha_p)G_+(\alpha_m)} \right], \]
\[ w_p = M(19)J_1(j_{p1}) \left[ \zeta_m(\alpha_p) + \delta_0 ig \frac{k_{zm}}{\kappa_+(\alpha_p)G_+(\alpha_p)} \right]. \]

It can be easily shown that for \( \varepsilon = 1 \) this system is analytically solved and the solution coincides with well-known result for open-ended vacuum waveguide [9]. For \( \varepsilon \neq 1 \) system (24) can be solved numerically using the reduction technique. Corresponding examples are given below.

**NUMERICAL RESULTS**

For the case of \( \varepsilon \neq 1 \), we solve (24) by reducing it to the finite system of \( M_{\text{max}} \) equations, where \( M_{\text{max}} \) was chosen around 2-3 times as much as the total number of propagating modes in the waveguide at given frequency. After that \( M_m, m = 1, 2, \ldots, M_{\text{max}} \) are immediately calculated, for example, in Matlab. For convenient comparison between analytical results and results of numerical simulation, we have calculated powers carrying by incident mode and each reflected mode through the waveguide cross-section,
\[ \Sigma_m^{(i)} = c^2 a^2/(8k_0\varepsilon)J_f^2(j_{p1}) \left| M^{(i)} \right|^2 \text{Re}(k_{zm}), \]
\[ \Sigma_m^{(r)} = c^2 a^2/(8k_0\varepsilon)J_f^2(j_{p2}) \left| M_m \right|^2 \text{Re}(k_{zm}), \]

and constructed corresponding \( S \)-parameters:
\[ S_{ml} = \sqrt{\Sigma_m^{(r)}/\Sigma_m^{(i)}}, \]

which also can be expressed in dB, \( S_{ml}^{\text{dB}} = 20\log S_{ml} \).

Numerical simulations were performed in RF module of Comsol Multiphysics package. A two dimensional frequency domain solver was utilized. An input end of the waveguide was supported by a series of numerical ports, one separate port for each propagating mode. The port which corresponds to the incident mode was set to be active and option “active port feedback” has been disabled. Corresponding eigenmodes were determined numerically, with analytically calculated longitudinal wavenumbers \( k_{zm} \) being used as guess values. An open end of the waveguide was surrounded by a hemisphere with scattering boundary condition applied. The length of the waveguide and the radius of damping hemisphere radius were of the same order, at least several tens of maximum wavelength inside the waveguide.

For calculations presented below, the mode frequency was chosen to be equal to the frequency of CR mode \( f_{l1}^{\text{CR}} \) with numbers \( l = 5, l = 10 \) and \( l = 20 \) produced by a moving charge having its Lorentz factor \( \gamma = 7 \) [13]:
\[ \omega_{l1}^{\text{CR}} = 2\pi f_{l1}^{\text{CR}} = c\beta j_0/\sqrt{a \sqrt{\varepsilon \beta^2 - 1}}, \]

where \( \beta = \sqrt{1 - \gamma^{-2}} \). To clarify this choice of the frequency let us discuss the relation of the obtained results to the problem of diffraction of a charged particle bunch wakefield at the open-end of the discussed dielectric-loaded waveguide. Wakefield is a CR generated inside a waveguide as an infinite set of discrete frequencies (28),
while each frequency contribution to the total field is usually referred to as a CR mode. A CR mode can be presented (after simplification) in the following form [19]:

\[ H_{\varphi l}^{\text{CR}} = \text{Im} \left[ i H_{\varphi 0 l} J_1 \left( \frac{\rho_{0 l}}{a} \right) e^{i \omega_{\text{CR}} t} e^{-i \omega_{\text{CR}} t} \right], \tag{29} \]

where \( c/\beta \) is bunch velocity, \( H_{\varphi 0 l} \) is some constant. Since for \( \omega = \omega_{\text{CR}} \) we have \( k_{\text{CR}} = \omega_{\text{CR}} / (c/\beta) \), an incident mode (1) corresponds to the \( l \)-th CR mode if \( M^{(i)} \) is chosen appropriately.

Figure 2 shows comparison between S-parameters calculated via presented rigorous analytical approach and obtained from Comsol simulations. As one can see, the agreement between results is excellent. This fact proves the presented theory and also shows correctness of Comsol simulation procedure. One can see that typically the reflected mode with the number of incident mode dominates (it has the largest S-parameter), therefore the overall diffraction process is similar to a single mode reflection. However, other modes (especially those with close numbers) can be significant and therefore can alter mentioned “close to single mode” regime.

In conclusion, we have presented an elegant and convenient rigorous analytical approach for calculation of various diffraction processes at the open end (with orthogonal cut) of a circular waveguide with dielectric loading. The obtained results have been compared to the results of simulations with commercial code Comsol and an excellent agreement has been observed.

For simplicity and clearness of the presentation, in this short paper we have considered the problem with uniform dielectric filling. However, a series of other more complicated problems which are closer to possible applications can be also considered using this powerful approach. For example, excitation by a charged particle bunch (in full formulation including both wakefield and Coulomb field) can be incorporated into the solution and layered dielectric filling or corrugation of the waveguide wall can be investigated. It is worth noting that computation resources used by the Matlab code based on analytical formulas is much smaller then those occupied by Comsol. For example, even for \( n = 20 \) and around 1THz frequency our code took about 30 seconds to calculate S-parameters shown in Fig. 2 (typical PC based on Intel Core i7 processor and Matlab Parallel Computing were used). Comsol model took several hours to do the same task, depending on used machine. Therefore, the discussed rigorous approach can be extremely useful for further development of beam-driven radiation sources based on dielectrically loaded or corrugated cylindrical waveguides. Besides mentioned open-ended waveguides with straight cut, this method can be also useful for investigation of structures with non-orthogonal cut. Since in this case the rigorous theory can be marginally applied, development of reliable approximate methods is the most substantial idea (see, for example, [13]). Such reliable methods can be benchmarked and adjusted at simpler structures with orthogonal end cut.

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\[ S_{m5} \]

\[ S_{m10} \]

\[ S_{m20} \]

Figure 2. Comparison between S-parameters (in dB) obtained via the presented analytical approach and via Comsol simulations: \( S_{ml} \) corresponds to frequency \( f_{ml}^{\text{CR}} \) (28) and incident mode with number \( l \). We have 7 propagating modes for \( l = 5 \) (\( f_{5}^{\text{CR}} = 300\text{GHz} \)), 14 for \( l = 10 \) (\( f_{10}^{\text{CR}} = 615\text{GHz} \)) and 28 for \( l = 20 \) (\( f_{20}^{\text{CR}} = 1.247\text{THz} \)). Other parameters: \( a = 0.24\text{cm}, \varepsilon = 2 \).

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