On an intrinsic formulation of time-variant Port Hamiltonian systems

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Abstract

In this contribution we present an intrinsic description of time-variant Port Hamiltonian systems as they appear in modeling and control theory. This formulation is based on the splitting of the state bundle and the use of appropriate covariant derivatives, which guarantees that the structure of the equations is invariant with respect to time-variant coordinate transformations. In particular, we will interpret our covariant system representation in the context of control theoretic problems. Typical examples are time-variant error systems related to trajectory tracking problems which allow for a Hamiltonian formulation. Furthermore we will analyze the concept of collocation and the balancing/interaction of power flows in an intrinsic fashion.

Key words: Nonlinear control systems, Differential geometric methods, Mathematical systems theory, Tracking applications, Mechanical systems

1 Introduction

Hamiltonian systems are the object of analysis for a long period and they have been investigated from many different points of view and in many different scientific areas. In the last two decades, in mathematical physics especially field theoretic aspects of Hamiltonian systems without control input are of importance, see [5,6,7]. In field theory the use of bundles to distinguish dependent and independent coordinates is commonly used and since time-variant lumped parameter systems can be seen as a special case of field theory with only one independent coordinate the use of bundles also applies to time-variant systems where the fibration is accomplished with respect to the time-coordinate.

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Beside field theory, also in classical mechanics, especially in the time-invariant setting, the geometric interpretation of the Hamiltonian picture is well established, see for example [1] for many details concerning this subject. From the control theoretic point of view, the class of Port Hamiltonian systems are a well-analyzed class, see for example [11,16] and references therein, where both the theoretical point of view and, of course, the physical applications play a prominent role. Roughly speaking, the main idea of many passivity based control approaches is to maintain the Hamiltonian structure of the system by feedback since this structure has some pleasing properties concerning the stability proof also in the nonlinear scenario.

In the literature most of these approaches for the lumped parameter scenario concerning control theoretic aspects present system analysis, modeling and control for time-invariant systems, whereas the time-variant case is analyzed very rarely. We believe that the main difficulty in the time-variant scenario is the fact that the geometric picture of the equations changes considerably. In the time-invariant setting the role of the time is solely to be the curve parameter, which is not true in the time-variant scenario. Contributions which treat the time-variant case especially with regard to control theoretic problems are for example [3,4] where the authors consider, what they call generalized Hamiltonian systems and canonical transformations which might be time-dependent.

Two important applications where time-variant systems arise quite naturally based on a time-variant change of coordinates should be mentioned at this stage: Firstly, the introduction of displacement coordinates with respect to a system trajectory as it arises for instance when the analysis of the tracking error is the objective, see for example [4]. And secondly, in mechanics/robotics floating/accelerated frames of reference are commonly used with respect to an inertial one.

The main contribution of this paper are that: (i) an intrinsic definition of time-variant (Port) Hamiltonian systems is given based on a covariant derivative induced by a connection; (ii) this intrinsic description is analyzed in a differential geometric way; (iii) for the system class of time-variant (controlled) Hamiltonian mechanics, a covariant version of the power balance relation including collocation is developed; (iv) for the special case where (beside a possible feed-forward) the connection can be expressed as an additive Hamiltonian the results of [3] are recovered.

It is worth mentioning that in our opinion a time-variant (Port) Hamiltonian system has to be introduced using covariant derivatives, which differs significantly from the definition in [3,4]. We identify 'covariant' with the fact that system properties do not depend on the chosen coordinate chart, i.e., we formulate systems in an intrinsic way. The key idea is the use of a connection
which induces a covariant derivative, see [5]. Partially, results in this paper have been presented preliminarily in [13,15].

2 The Time-Invariant Case

This introductory section is a reminder of time-invariant Port Hamiltonian systems [8,11,16] including also the arising matching conditions when state transformations and affine input transformations are considered. It serves as a basis for the generalization to the time-variant case and is also used to introduce the differential geometric language which is then extensively exploited in the time-variant scenario. The notation is similar to the one in [5], where the interested reader can find much more details about this geometric machinery.

To keep the formulas short and readable we will use tensor notation and especially Einsteins convention on sums where we will not indicate the range of the used indices when they are clear from the context. We use the standard symbol $\otimes$ for the tensor product, $d$ is the exterior derivative, $\cdot$ the natural contraction between tensor fields and $\circ$ denotes the composition of maps. By $\partial^B_A$ are meant the partial derivatives with respect to coordinates with the indices $A_B$.

To study the time-invariant case of Port Hamiltonian systems in a geometric fashion we introduce the state manifold $\mathcal{X}$ equipped with coordinates $(x^\alpha)$, where $\alpha = 1, \ldots, \dim(\mathcal{X})$ and we consider diffeomorphisms (in the sequel also called transition functions) of the type $\bar{x} = \varphi(x)$ where $\bar{x}$ denotes the states in the transformed coordinate system. Standard differential geometric constructions, see [1,5,9,12] lead to the tangent bundle $T(\mathcal{X})$ and the cotangent bundle $T^*(\mathcal{X})$, which possess the induced coordinates $(x^\alpha, \dot{x}^\alpha)$ and $(x^\alpha, \dot{x}_\alpha)$ with respect to the holonomic bases $\partial_\alpha$ and $dx^\alpha$. Typical elements of $T(\mathcal{X})$ (vector fields) and $T^*(\mathcal{X})$ (1-forms) read in local coordinates as $w = \dot{x}^\alpha(x)\partial_\alpha$ and $\omega = \dot{x}_\alpha(x)dx^\alpha$, respectively. To introduce in- and outputs we consider the vector bundle $U \to \mathcal{X}$ with the coordinates $(x^\alpha, u^i)$ for $U$ and the base $e_i$ for the fibres where $i = 1, \ldots, \dim(U_{\mathcal{X}})$, where $U_{\mathcal{X}}$ denotes the fibres of the input bundle (vector spaces) as well as the dual output vector bundle $Y \to \mathcal{X}$ possessing the coordinates $(x^\alpha, y_i)$ and the fibre base $e^i$. Greek indices will correspond to the components of the coordinates of the state manifold and induced structures. Latin indices correspond to the components of the input and the output variables (fibres of the dual bundles $U \to \mathcal{X}$ and $Y \to \mathcal{X}$). Let us consider the maps $J, R : T^*(\mathcal{X}) \to T(\mathcal{X})$ which are contravariant tensors that are given by the local coordinate expressions

$$J = J^{\alpha\beta}\partial_\alpha \otimes \partial_\beta, \quad R = R^{\alpha\beta}\partial_\alpha \otimes \partial_\beta$$

(1)

with $J^{\alpha\beta}, R^{\alpha\beta} \in \mathcal{C}^\infty(\mathcal{X})$ where $J$ is skew-symmetric, i.e. $J^{\alpha\beta} = -J^{\beta\alpha}$ and $R$
is symmetric $R^{\alpha \beta} = R^{\beta \alpha}$ and positive-semidefinite. Furthermore we introduce the bundle map $G : U \to \mathcal{T}(\mathcal{X})$ which is a tensor that has the local coordinate expression $G = G^i_\alpha e^i \otimes \partial _\alpha$ with $G^\alpha_i \in C^\infty(\mathcal{X})$. Having the maps $J, R$ and $G$ at our disposal a time-invariant Port Hamiltonian system (with dissipation), see [8,11,16] can be constructed as

$$\dot{x} = (J - R)\, dH + Gu$$
$$y = G^*\, dH$$

where the function $H \in C^\infty(\mathcal{X})$ denotes the Hamiltonian and $G^* : \mathcal{T}^*(\mathcal{X}) \to \mathcal{Y}$ the adjoint (dual) map of $G$. The local coordinate expression of (2) reads as

$$\dot{x}^\alpha = \left( J^{\alpha \beta} - R^{\alpha \beta} \right) \partial _\beta H + G^\alpha_i u^i$$
$$y_i = G^\alpha_i \partial _\alpha H.$$  

(3)

We want to analyze structure preserving transformations for the system (2). To allow for affine input transformations we can replace the input bundle by an affine one $\mathcal{Z} \to \mathcal{X}$ (with underlying vector bundle $U \to \mathcal{X}$), for the geometric properties of affine bundles see for example [5] and references therein. The transition functions for the vector bundle and the affine bundle read as

$$\bar{u} = Mu, \quad \bar{u}^3 = M^3_i u^i$$
$$\bar{u} = Mu + g, \quad \bar{u}^3 = M^3_i u^i + g^3$$

(4)

(5)

with $M^3_i, g^3 \in C^\infty(\mathcal{X})$ where $\bar{u}$ denotes the transformed input coordinates and we restrict ourselves to regular transformations (i.e. $M$ is invertible). The geometric representation of the system leads to the observation that the structure of (2) is preserved by a diffeomorphism of the type $\bar{x} = \varphi(x)$ together with (4). The case of an affine input bundle is more challenging since the preservation of the structure demands to solve a partial differential equation. See also [2] in this context, where the problem of general feedback equivalence of nonlinear systems to Port Hamiltonian systems is discussed and so called matching conditions appear.

**Lemma 1** Consider the system (2) together with the diffeomorphism $\bar{x} = \varphi(x)$ and (5). The structure of (2) is preserved if and only if we can find a solution $\bar{H} \in C^\infty(\mathcal{X})$ of the partial differential equations

$$\left( \bar{J}^{\alpha \beta} - \bar{R}^{\alpha \beta} \right) \partial _\beta \bar{H} - \left( \partial _\alpha \varphi^\delta G^\alpha_i \dot{M}^i_j g^3 \right) \circ \dot{\varphi} = 0.$$  

(6)

Here $\bar{J}^{\alpha \beta}$ and $\bar{R}^{\alpha \beta}$ are the components of the transformed tensors (1) with respect to $\bar{x} = \varphi(x)$. The inverse maps are denoted by $x = \varphi(\bar{x})$ and $\dot{M}^i_j \dot{M}^j_i = \delta^i_j$ where $\delta$ is the Kronecker delta.
Remark 2 The partial differential equations (6) are written in the coordinates \( \bar{x} \) but it is readily observed that it can be formulated in the original coordinates \( x \), as well.

The proof of this Lemma is a straightforward calculation in local coordinates. If in Lemma 1 a solution for \( \bar{H} \) can be obtained, then the following Corollary is an immediate consequence.

**Corollary 3** Suppose (6) is met, then the system (3) in the new coordinates reads as

\[
\begin{align*}
\dot{x}^\alpha & = \left( J^{\alpha \beta} - R^{\alpha \beta} \right) \partial_{\beta}(\bar{H} - \bar{\dot{H}}) + \bar{G}^\alpha_i \bar{u}^i \\
\bar{y}_i & = \bar{G}^\alpha_i \partial_{\alpha}(\bar{H} - \bar{\dot{H}})
\end{align*}
\]

with \( \bar{G}^\alpha_j = \left( \partial_{\alpha} \varphi^\alpha \bar{G}^\alpha_i \dot{M}^i_j \right) \circ \varphi \) and \( \bar{H} = H \circ \varphi \). The output

\[
\bar{y}_j = \dot{M}^i_j (y_i - (\partial_{\alpha} \bar{H}) G^\alpha_i).
\]

and the Hamiltonian are transformed affine and in general \( u^i y_i \neq \bar{u}^j \bar{y}_j \) is met.

**Example 4** Let us consider the shifting of a nonzero (but constant) equilibrium point \((x_e, u_e)\) of the system (2). This leads to affine relations of the form \( \bar{x} = x - x_e \) and \( \bar{u} = u - u_e \). For a different interpretation concerning Port Hamiltonian systems with nonzero equilibrium consider [10] whereas in [8] the construction of Lyapunov functions is treated in this case.

3 The Time-Variant Case

Time-variant systems arise when the time coordinate is treated as an additional coordinate in contrast to the case where it is solely a curve parameter. This has the consequence for (Port) Hamiltonian systems that the maps \( J, R, G \) and the Hamiltonian \( H \) may depend on the time coordinate and that also time-dependent coordinate changes have to be taken into account. Concerning time-dependent changes of coordinates in mathematical physics, especially in mechanics a covariant treatment of the equations requires to use frames of references formulated on bundles, see [5].

**Remark 5** A special case of a time-variant coordinate transformation is the introduction of displacement coordinates with respect to a system trajectory of the (time-invariant) system (3).

We will apply intrinsic concepts to model Hamiltonian systems. To motivate how these covariant concepts arise let us consider a time-variant system modeled on an extended state manifold given by the direct product \( \mathcal{B} \times \mathcal{X} \) that
includes the time coordinate \( t^0 \) for \( B \) (the index 0 will always correspond to the time coordinate in the sequel) as it is analyzed for instance in [3] and by a slight abuse of notation we have

\[
\partial_0 x^\alpha(t) = J^{\alpha\beta}(t, x(t))\partial_\beta H(t, x(t)) + G^\alpha_i(t, x(t))u^i(t)
\]

\[
y_i(t) = G^\alpha_i(t, x(t))\partial_\alpha H(t, x(t)).
\]

The crucial point is now how to interpret the time derivative \( \partial_0 x^\alpha(t) \) either using the structure of jet spaces (jet bundles) which are affine or using tangent structures. Let us consider a transformation of the form \( \bar{x}^\alpha = \varphi^\alpha(x^\beta, t^0) \) with \( \bar{t}^0 = t^0 \). We have

\[
\dot{x}^\alpha = \partial_0 \varphi^\alpha \bar{t}^0 + \partial_\beta \varphi^\alpha \dot{x}^\beta
\]

and consequently the vector field \( \partial_0 \) is mapped to the vector field \( \partial_0 + \partial_0 \varphi^\alpha \partial_\alpha \). This follows from \( \bar{t}^0 = 1 \) together with (8) and the main reason for that fact is the observation that a trivial product bundle structure of the form \( B \times X \to B \) is not preserved by time-variant transformations and therefore not the adequate choice for time-variant systems. To overcome this problem the system has to be formulated on a bundle \( E \to B \). The choice of the trivial one \( B \times X \to B \) corresponds to a specific trivialization \( E \cong B \times X \), i.e. to the choice of a certain reference frame. An intrinsic approach demands a formulation on the bundle \( E \to B \) and the choice of a connection, which can be seen as a reference frame.

3.1 Geometric Setting

To obtain an appropriate geometric picture for the time-variant case we consider the bundle \( E \to B \), where we use coordinates \((t^0, x^\alpha)\) for \( E \) and obtain the following geometric structures. The tangent bundle \( T(E) \) with coordinates \((t^0, x^\alpha, t^0, \dot{x}^\alpha)\), the cotangent bundle \( T^*(E) \) equipped with coordinates \((t^0, x^\alpha, t^0_0, \dot{x}_0^\alpha)\), as well as the vertical bundle \( V(E) \) with coordinates \((t^0, x^\alpha, \dot{x}^\alpha)\) and \( V^*(E) \) with \((t^0, x^\alpha, \dot{x}_0^\alpha)\), where \( V(E) \) possess the induced bases \( \partial_\alpha \) and since \( V^*(E) \) does not possess a canonical base without the choice of a connection it is denoted as \( d\dot{x}^\alpha \) at this stage, see [5].

To be able to deal with time-derivatives of sections \( s : B \to E \), i.e. \( x = s(t) \) we introduce the first jet manifold \( J^1(E) \) with the adapted coordinates \((t^0, x^\alpha, x^\alpha_0)\), see [5,12]. The coordinates \( x^\alpha_0 \) are often called derivative coordinates and the transition functions with respect to the bundle morphism \( \bar{x}^\alpha = \varphi^\alpha(t, x), \bar{t}^0 = t^0 \) (we do not consider time-reparametrization) read as \( \bar{x}^\alpha = \partial_0 \varphi^\alpha + \partial_\beta \varphi^\alpha \dot{x}_0^\beta \).

The key object for a covariant system representation will be a connection together with the associated covariant differential. The main philosophy behind a connection is the fact that \( T(E) \) possesses a canonical subbundle,
namely the vertical tangent bundle $\mathcal{V}(\mathcal{E})$ but there is no canonical horizontal complement such that $\mathcal{T}(\mathcal{E}) = \mathcal{V}(\mathcal{E}) \oplus \mathcal{H}(\mathcal{E})$ holds, unless one specifies a horizontal subbundle and this is exactly what a connection does. By duality one derives analogously a decomposition of the cotangent bundle as $\mathcal{T}^*(\mathcal{E}) = \mathcal{V}^*(\mathcal{E}) \oplus \mathcal{H}^*(\mathcal{E})$, where in this case $\mathcal{H}^*(\mathcal{E})$ is canonically given and the connection defines $\mathcal{V}^*(\mathcal{E})$.

3.1.1 Connection

Given a bundle $\mathcal{E} \to \mathcal{B}$ whose fibration induces the vertical tangent bundle $\mathcal{V}(\mathcal{E})$ it is the desire to obtain a splitting of the form $\mathcal{T}(\mathcal{E}) = \mathcal{V}(\mathcal{E}) \oplus \mathcal{H}(\mathcal{E})$. This can be achieved by a connection $\Gamma$ which in local coordinates can be represented as a tensor of the form
\[
\Gamma = dt^0 \otimes (\partial_0 + \Gamma_0^a \partial_a)
\] (9)
with the connection coefficients $\Gamma_0^a \in \mathcal{C}^\infty(\mathcal{E})$, see [5,12]. The transition functions for $\Gamma$ when no time-reparametrization (i.e. $\bar{t} = t$) takes place read as
\[
\bar{\Gamma}_0^\bar{a} = \partial_0 \varphi^\bar{a} + \Gamma_0^a \partial_a \varphi^\bar{a}
\] (10)
where $\bar{\Gamma}_0^\bar{a}$ are the connection coefficients in the transformed coordinate system.

This connection (9) can be used to define a covariant derivative and to split the tangent bundle $\mathcal{T}(\mathcal{E}) \to \mathcal{E}$, i.e. $\mathcal{T}(\mathcal{E}) = \mathcal{V}(\mathcal{E}) \oplus \mathcal{H}(\mathcal{E})$ as stated above, where $\mathcal{H}(\mathcal{E})$ denotes the horizontal subbundle. We have the following coordinate representations for the splitting for a typical element of $\mathcal{T}(\mathcal{E})$ and $\mathcal{T}^*(\mathcal{E})$, respectively
\[
\dot{x}^0 \partial_0 + \dot{x}^a \partial_a = \dot{x}^0 w^H_0 + \left(\dot{x}^a - \dot{t}^0 \Gamma_0^a \right) \partial_a
\]
\[
i_0 dt^0 + \dot{x}_a dx^a = \left(i_0 + \dot{x}_a \Gamma_0^a \right) dt^0 + \dot{x}_a \omega_V^a
\] (11)
with the vector field $w^H_0$ and the 1-form $\omega_V^a$
\[
w^H_0 = \partial_0 + \Gamma_0^a \partial_a, \quad \omega_V^a = dx^a - \Gamma_0^a dt^0
\] (12)
such that $w^H_0$ and $\omega_V^a$ qualify as bases for $\mathcal{H}(\mathcal{E})$ and $\mathcal{V}^*(\mathcal{E})$, respectively.

Remark 6 It is readily observed that a time-variant transformation $\dot{x}^a = \varphi^a(t, x)$ converts a trivial connection $\Gamma_0^a = 0$ in one coordinate chart to a non-trivial one in the transformed coordinates $\bar{\Gamma}_0^\bar{a} = \partial_0 \varphi^\bar{a}$, see (10).

It should be noted that the concept of vertical and horizontal parts of several objects will become important in the sequel and we will indicate this using $\mathcal{V}$ for vertical and $\mathcal{H}$ for horizontal. However, $\mathcal{H}$ should not be confused with the Hamiltonian $H$. 

7
3.1.2 Covariant Derivative

Given the connection (9) a covariant differential relative to the connection \( \Gamma \) can be introduced as a map \( \nabla^\Gamma : J^1(E) \to T^*(B) \otimes \mathcal{V}(E) \) which reads in coordinates as

\[
\nabla^\Gamma = (x^\alpha_0 - \Gamma^\alpha_0) \, dt^0 \otimes \partial_\alpha. \tag{13}
\]

If the connection is trivial, i.e. \( \Gamma^\alpha_0 = 0 \) then the covariant differential on the bundle \( E \to B \) corresponds to the classical time derivative. Based on the covariant differential one can define the covariant derivative of a section \( s : B \to E \) which follows as

\[
\nabla^\Gamma(s) = (\partial_0 s^\alpha - \Gamma^\alpha_0 \circ s) \, dt^0 \otimes \partial_\alpha.
\]

Given the vector field \( \partial_0 : B \to T(B) \), the contraction

\[
\partial_0 | \nabla^\Gamma(s) = (\partial_0 s^\alpha - \Gamma^\alpha_0 \circ s) \, \partial_\alpha
\]

is said to be the covariant derivative of \( s : B \to E \) along \( \partial_0 \), see [5].

3.2 System Representation

We define a time-variant (Port) Hamiltonian system on the bundle \( Z \to E \to B \) together with a connection (9) such that the structure of the system is invariant with respect to bundle morphisms of \( E \to B \) that explicitly depend on the time coordinate but we do not consider time-reparametrization (i.e., \( \bar{x} = \varphi(t, x) \) and \( \bar{t} = t \)). The input bundle \( \mathcal{U} \to E, (t^0, x^\alpha, u^i) \to (t^0, x^\alpha) \) is a vector bundle, and \( Z \to E \) is used in the affine case, analogous to the time-invariant case.

Applying a time-variant transformation does not destroy the structure of a (Port) Hamiltonian system if it is defined in an intrinsic manner. But in contrast to [3], where only the relative motion with respect to a frame should be expressed in a Port Hamiltonian framework, it is necessary in general to adopt the involved differential operators to preserve the structure of the system. This will be done by applying covariant derivatives. The choice of \( \Gamma \) corresponds to the selection of a frame of reference, i.e. starting in an inertial frame (with trivial connection) a time-variant transformation induces a non-trivial connection. An example how a non-trivial connection can arise by changing the frame of reference for a mechanical system is given in Section 4.2.

**Definition 7** Given a connection (9) then a Hamiltonian system on a bundle \( Z \to E \to B \) is given by

\[
\partial_0 | \nabla^\Gamma = (J - R) | dH + G | u \tag{14}
\]
with \( J, R : \mathcal{V}^*(\mathcal{E}) \to \mathcal{V}^*(\mathcal{E}) \) where \( J \) is skew symmetric and \( R \) is symmetric and positive semidefinite. Additionally we have the bundle map \( G : \mathcal{U} \to \mathcal{V}(\mathcal{E}) \) and the Hamiltonian \( H \in \mathcal{C}^\infty(\mathcal{E}) \). In local coordinates we obtain the expression

\[
x_0^\alpha - \Gamma_0^\alpha = (J^{\alpha\beta} - R^{\alpha\beta}) \partial_\beta H + G_\alpha^i u^i.
\]

where

\[
v_{H,V}^\alpha = (J^{\alpha\beta} - R^{\alpha\beta}) \partial_\beta H + G_\alpha^i u^i
\]

is referred as the vertical part of the Hamiltonian vector field \( v_{H,V} = v_{H,V}^\alpha \partial_\alpha \) with \( H, G_\alpha^\alpha, J^{\alpha\beta}, R^{\alpha\beta} \in \mathcal{C}^\infty(\mathcal{E}) \).

To show that in the time-variant scenario \( J \) and \( R \) are maps from the vertical cotangent bundle to the vertical tangent bundle \( J, R : \mathcal{V}^*(\mathcal{E}) \to \mathcal{V}(\mathcal{E}) \) we have to observe that the total differential of the Hamiltonian \( dH = \partial_\alpha H dt^0 + \partial_\beta H dx^\beta \) can be decomposed according to (11) as

\[
dH = \partial_\beta H \omega_V^\beta + w_{0}^H(H) dt^0
\]

with the 1-form \( \omega_V^2 \) and the vector field \( w_{0}^H \) from (12) from which we deduce the result, together with (13).

**Remark 8** Since the exterior derivative \( d \) can also be decomposed into a vertical and a horizontal one \( d = d_V + d_H \), see [5,12] one can equivalently use \( d_V H \) instead of \( dH \) in the relation (14).

Let us consider the total time change \( d_0(H) \) of the Hamiltonian \( H \) along solutions. For a time-invariant system we obtain a relation of the form

\[
d_0(H) = v_{H}(H) = -(\partial_\alpha H)R^{\alpha\beta}(\partial_\beta H) + u^i y_i
\]

where \( v_{H}(H) \) denotes the Lie-derivative of the Hamiltonian \( H \) along the Hamiltonian vector field \( v_{H} = (J^{\alpha\beta} - R^{\alpha\beta}) \partial_\beta H + G_\alpha^i u^i \), see (3) and the total time derivative (time-invariant case) reads as \( d_0 = \dot{x}^\alpha \partial_\alpha \).

The intrinsic version in the time-variant case is given in the following corollary.

**Corollary 9** The total time change of the Hamiltonian leads to a decomposition of the form

\[
d_0(H) = w_{0}^H(H) + v_{H,V}(H)
\]

or in local coordinates

\[
\overbrace{\partial_0(H) + \Gamma_0^\alpha \partial_\alpha (H)}_{w_{0}^H(H)} - \underbrace{\langle \partial_\alpha H \rangle R^{\alpha\beta}(\partial_\beta H) + \partial_\alpha (H)G_\alpha^i u^i}_{v_{H,V}(H)}
\]

\(^2\) It should be noted that \( \omega_V^2 = dx^\alpha - \Gamma_0^\alpha dt^0 \) serves as an adapted basis for \( \mathcal{V}^*(\mathcal{E}) \) induced by the connection \( \Gamma \). This justifies \( J, R : \mathcal{V}^*(\mathcal{E}) \to \mathcal{V}(\mathcal{E}) \) as we have claimed.
where the vector field $w_0^H$ is induced by the connection $\Gamma$, see (12), the vector field $v_{H,V}$ from (15) and $d_0 = \partial_0 + x_0^\alpha \partial_\alpha$. It is worth noting that we have a decomposition of the derivative operator into a horizontal $w_0^H$ and a vertical $v_{H,V}$ component where the horizontal part degenerates to $\partial_0$ only if the connection is trivial, i.e. $\Gamma_0^\alpha = 0$.

This follows by a direct calculation based on the intrinsic system representation. The discussion of collocation will be performed using a special class of systems in the next section.

### 3.3 A special class of Port Hamiltonian Systems

Mechanical systems in a Port Hamiltonian representation are distinguished since the bundle $E \to B$ has an even richer geometric structure. The main difference is the separation of the $x^\alpha$ coordinates in positions $q^\alpha$ and momenta $\dot{q}^\alpha$. To obtain the correct geometric picture we introduce the bundle $Q \to B$ with coordinates $(t^0, q^\alpha)$ for $Q$. This bundle structure induces again some tangent structures where we make use of the dual vertical cotangent bundle $V^*(Q) \to Q$ with coordinates $(t^0, q^\alpha, \dot{q}^\alpha)$. Furthermore we will utilize $T(V^*(Q))$ which possesses the adapted bases $(\partial_0, \partial_\alpha, \dot{\gamma}^\alpha)$ with $\dot{\gamma}^\alpha = \frac{\partial}{\partial \dot{q}^\alpha}$ and to be conform with most of the literature we make the identification $\dot{q}_\alpha = p_\alpha$.

#### 3.3.1 The Composite Bundle Structure

From the bundles introduced so far $V^*(Q) \to Q$ and $Q \to B$ we can construct the composite bundle structure $V^*(Q) \to Q \to B$.

The additional fibration $V^*(Q) \to B$, $(t^0, q^\alpha, p_\alpha) \to (t^0)$ plays the role of the state bundle $E \to B$. We choose a connection $\gamma$ that corresponds to the selection of a frame of reference that splits $T(Q)$

$$\gamma = dt^0 \otimes (\partial_0 + \gamma^\alpha_0 \partial_\alpha), \gamma^\alpha_0 \in C^\infty(Q).$$

(18)

Based on the connection $\gamma$ the so-called covertical connection, see [5]

$$\Gamma_H = dt^0 \otimes \left( \partial_0 + \gamma^\alpha_0 \partial_\alpha - (\partial_\rho \gamma^\beta_0) p_\beta \dot{\partial}^\rho \right)$$

(19)

can be constructed that splits the tangent bundle $T(V^*(Q))$ with respect to the fibration $V^*(Q) \to B$ and this is the connection that corresponds to $\Gamma$, see (9) that splits $T(E)$.

In classical mechanics it is well known that the Hamiltonian vector field can be defined using Symplectic and Poisson structures, see [1,5]. These concepts can also be generalized to the time-variant scenario. We base the construction of
the Hamiltonian vector field using the Hamilton form $\omega_H$ which follows from
the canonical Liouville 1-form (by a pull-back)

$$\omega_H = p_\alpha dq^\alpha - (H + p_\alpha \gamma_0^\alpha) dt^0$$

with $H \in C^\infty(V^*(Q))$.

From the relations $v_H|d\omega_H = 0$ and $v_H|dt^0 = 1$ the autonomous Hamiltonian vector field $v_H : V^*(Q) \to T(V^*(Q))$ follows with $H_\gamma = H + p_\beta \gamma_0^\beta$ as

$$v_H = \partial_0 + \dot{\partial}^\alpha H_\gamma \partial_\alpha - \partial_\alpha H_\gamma \dot{\partial}^\alpha.$$  \hspace{1cm} (20)

**Remark 10** If the connection $\gamma$ is trivial, i.e. $\gamma_0^\alpha = 0$ holds then the Hamiltonian vector field reads as $v_H = \partial_0 + \dot{\partial}^\alpha H_\gamma \partial_\alpha - \partial_\alpha H \dot{\partial}^\alpha$ since then $H_\gamma = H$ which is the standard result in mechanics, see [1].

### 3.3.2 Power Balance Equation for Controlled Mechanical Systems

The connection (19) enables us to split the Hamiltonian vector field (20) into
a vertical and horizontal part, respectively, see also [13]. It follows that (analogously to (11)) we have a decomposition of the form

$$v_{H,V} = \dot{\partial}^\alpha H \partial_\alpha - \partial_\alpha H \dot{\partial}^\alpha$$

$$v_{H,H} = \partial_0 + \gamma_0^\alpha \partial_\alpha - (\partial_\rho \gamma_0^\beta) p_\beta \dot{\partial}^\rho$$  \hspace{1cm} (21)

according to the Hamiltonian connection (19). Let us inspect the relation (17)
in the case of a mechanical system where it is obvious that now the vector field $w_0^\rho$ from (12) is replaced by $v_{H,H}$ due to our richer bundle structure.

We consider the autonomous case first and since no dissipation is present we obtain

$$v_H(H) = v_{H,H}(H)$$  \hspace{1cm} (22)

where $v_H(H)$ denotes the Lie-derivative of the Hamiltonian $H$ with respect
to the vector field $v_H$. To include control inputs we consider an extended
Hamiltonian corresponding to a controlled Hamiltonian, see [9], of the form

$$H = H_0 - H_{c,\rho} u^\rho$$  \hspace{1cm} (23)

with $H_0$, $H_{c,\rho} \in C^\infty(V^*(Q))$ and the input functions $u^\rho \in C^\infty(B)$.

Based on (23) and the relation (22) we can state the following Theorem concerning a covariant formulation of the power flows as well as the concept of collocation.
\textbf{Theorem 11} The change of the free Hamiltonian $H_0$ along solutions of the Hamiltonian system decomposes as

$$v_H(H_0) = v_{H,H}(H_0) + v_{H,V}(H_{c,\rho})u^\rho$$

where we used the controlled Hamiltonian $H = H_0 - H_{c,\rho}u^\rho$ and the decomposition (21).

Obviously the choice of the output $y_\rho = v_{H,V}(H_{c,\rho})$ allows a physical interpretation of the power flows of the system, since $v_{H,H}(H_0)$ corresponds to the power caused by the free Hamiltonian $H_0$ and the product $y_\rho u^\rho$ describes the power flow into the system caused by the input. Theorem 11 is an intrinsic version of the balancing/interaction of power flows and it reduces to the well known formula when the connection is trivial, i.e. $\gamma_0^\alpha = 0$. It is worth mentioning that the splitting of the Hamiltonian field as in (21) is essential to obtain a coordinate free representation.

\section{Applications}

This section is devoted to a short discussion of two possible applications of the presented theory. Firstly, we focus on the description of the error system arising when the stabilization of the trajectory tracking error is the objective in a Port Hamiltonian framework and discuss the role of the connection in this context. As a second application we present the equations of motion of a mass particle observed from a rotating frame of reference, in a time-variant Hamiltonian formulation since this example demonstrates how to calculate the non-trivial connections coefficients and additionally one can show that this connection is used to formulate conservations laws and/or power balance relations.

\subsection{Trajectory tracking}

This section deals with time-variant Port Hamiltonian systems as they arise quite naturally when a feed-forward based approach is applied to a time-independent Port Hamiltonian system and the control objective is the stabilization of the error system using techniques of passivity. It is evident that the error system is constructed using a time-variant transformation which enables us to discuss the developed machinery on this concrete example. It should be pointed out that compared to the approach in [4] we use a different definition of a time-variant Port Hamiltonian system (see Definition 7) which is coordinate independent and more general due to the connection term. If the
connection is interpreted as an additive term in a modified Hamiltonian, then our results coincide with the results of [4].

Let us consider a Port Hamiltonian system of the form

\[ x_0^\alpha = \left( J^{\alpha \beta} - R^{\alpha \beta} \right) \partial_\beta H + G_i^{\alpha} u^i. \]  

(24)

The control system (24) is modeled on a bundle \( E \rightarrow B \) where the coordinates are obviously adapted to the connection, such that the connection coefficients then read as \( \Gamma_0^\alpha = 0 \). If a desired trajectory \( c_d(t) \) and a corresponding input \( \eta_d(t) \), which produces this trajectory, is given

\[ \partial_0 c_d^\alpha = \left( (J^{\alpha \beta} - R^{\alpha \beta}) \partial_\beta H + G_i^{\alpha} \eta_d^i \right) \circ c_d \]  

(25)

then a transformation with respect to a reference trajectory can be stated as

\[ \bar{x}^\alpha = \varphi^\alpha (t^0, \bar{x}^\beta) = \delta^\alpha_\alpha (x^\alpha - c_d^\alpha) \]  

(26)

\[ \bar{u}^j = M_i^j (u^i - \eta_d^i), \quad M_i^j \in C^\infty (X). \]  

(27)

**Remark 12** The relations (26) and (27) are of course only a trivial choice leading to an error system. But for our purposes, the geometric interpretation of time-variant Port Hamiltonian systems, the exact structure of (26) and (27) is of minor importance, since the intrinsic system formulation is independent of the special choice of the transition functions. For a more general discussion of error systems in the Port Hamiltonian context, see [4].

It is readily observed that the connection coefficients read as

\[ \Gamma_0^\alpha = \partial_0 \varphi^\alpha = - (\partial_0 c_d^\alpha) \delta^\alpha_\alpha. \]  

(28)

with \( \varphi^\alpha \) from (26).

The combination of the results of Lemma 1 together with the fact that in the time-variant case a connection appears that leads to an additional affine term, we obtain the following Corollary.

**Corollary 13** Given \( c_d(t) \) and \( \eta_d(t) \) fulfilling (25) then the transformations (26) and (27) applied to the system (24) lead to a representation as

\[ \bar{x}_0^\alpha = (\bar{J}^{\alpha \beta} - \bar{R}^{\alpha \beta}) \partial_\beta (\bar{H} + \tilde{H}) + G_i^{\alpha} \bar{u}^i, \]  

(29)

if and only if the partial differential equations

\[ (\bar{J}^{\alpha \beta} - \bar{R}^{\alpha \beta}) \partial_\beta \tilde{H} = \tilde{\Gamma}_0^\alpha + \bar{\delta}^{\alpha} G_i^{\alpha} \eta_d^i \]

allow a solution for \( \tilde{H} \).
From Corollary 13 we deduce that if the connection * and the feed-forward part ** are expressed as an additive Hamiltonian, one obtains a Port-Hamiltonian representation which is beneficial when the control objective is to stabilize error systems that arise typically when trajectory tracking is the demand.

4.2 Rotating Frame of Reference

To show that a time-variant transformation preserves the Hamiltonian structure we will consider the equations of motion of a mass particle using two different frames, i.e. an inertial one and a rotating one with respect to the inertial frame. Let us consider an inertial system with Euclidean coordinates \((q^\alpha, t^0)\) together with the bundle structure \(Q \rightarrow B\). The canonical equations of motion for a mass particle with mass \(m \in \mathbb{R}^+\) read as

\[
\partial_0 s^\alpha = \partial^\alpha H, \quad \partial_0 (p_\alpha \circ s) = -\partial_\alpha H
\]

with \(s: B \rightarrow Q\). For this example the Hamiltonian is given as

\[
H = \frac{1}{2m} p_\alpha \delta^{\alpha\beta} p_\beta
\]

where \(\delta\) denotes the Kronecker delta.

A rotating coordinate chart with respect to the inertial one can be constructed, using \(\bar{q}^\alpha = R^\alpha_\beta(t^0)q^\beta = \varphi^\alpha(q^\beta, t^0)\) with \(R^\alpha_\beta \delta^{\alpha\beta} R^\beta_\alpha = \delta^{\alpha\beta}\). The non-trivial connection in the floating reference system can be computed as in (10) and reads as \(\bar{\gamma}_0^\alpha = \partial_0 (R^\alpha_\beta) R^\beta_\rho \bar{q}^\rho = \Omega^\alpha_\rho \bar{q}^\rho\).

The equations of motion in the rotating coordinate system follow as, see (20) or [14] for a detailed exposition concerning covariant derivatives

\[
\partial_0 \bar{s}^\alpha - \bar{\gamma}_0^\alpha = \partial^\alpha \bar{H}
\]

\[
\partial_0 (\bar{p}_\alpha \circ \bar{s}) + \bar{p}_\beta \partial_\alpha \bar{\gamma}_0^\beta = -\partial_\alpha \bar{H}.
\]

Comparing (30) with (31) it is evident that they both are Hamiltonian representations of the same physical problem, but in contrast to the inertial frame where the left hand side consists of partial time derivatives only in the rotating frame a differential operator induced by the connection (19) has to be applied. In the concrete example where the connection reads as \(\bar{\gamma}_0^\alpha = \Omega^\alpha_\rho \bar{q}^\rho\) together with \(\bar{s}: \bar{B} \rightarrow \bar{Q}\) the equations follow as

\[
\partial_0 \bar{s}^\alpha - \Omega^\alpha_\rho \bar{q}^\rho = \frac{1}{m} \bar{p}_\beta \delta^{\alpha\beta}
\]

\[
\partial_0 (\bar{p}_\alpha \circ \bar{s}) + \bar{p}_\beta \Omega^\beta_\rho \bar{s}^\rho = 0,
\]
when no other forces are applied.

**Remark 14** To apply Theorem 11 it should be noted that $\bar{v}_{\bar{H}, \bar{H}}$ has to be constructed with the connection $\bar{z}_0^\alpha = \Omega_\rho^\gamma \bar{q}^\rho$. Then the Hamiltonian $\bar{H}$ is a conserved quantity as the conservation law $\bar{v}_{\bar{H}, \bar{H}}(\bar{H}) = 0$ (see Theorem 11 with $\bar{H}_0 = \bar{H}$) is met. If we apply forces to control the system then the full relation of Theorem 11 has to be applied including the additional expression $\bar{v}_{\bar{H}, \bar{V}}$, see (21).

Combining the equations (32) and (33) we obtain

$$\partial_0 (\partial_0 s^\gamma - \Omega_\rho^\gamma \bar{s}^\rho) \delta_{\gamma\bar{\alpha}} + (\partial_0 s^\gamma - \Omega_\rho^\gamma \bar{s}^\rho) \delta_{\gamma\bar{\beta}} \Omega_\alpha^{\bar{\beta}} = 0$$

which can be rewritten as

$$\partial_0 \delta s^\gamma - \partial_0 \Omega_\rho^{\bar{\gamma}} \bar{s}^\rho - 2 \Omega_\rho^{\bar{\gamma}} \partial_0 \bar{s}^\rho + \Omega_\gamma^{\bar{\gamma}} \Omega_\rho^{\bar{\gamma}} \bar{s}^\rho = 0$$

where we have exploited the skew symmetry of the so-called angular velocity tensor $\Omega$. This is a classical result in mechanics, i.e. this is an expression for the acceleration including the Coriolis and the Centrifugal acceleration. The relations (32) and (33) are the covariant version in a Hamiltonian point of view.

Summarizing we can state, that time-variant transformations preserve the Hamiltonian structure, if a (Port) Hamiltonian system is introduced covariantly as in Definition 7. The system properties can be expressed in a time-variant frame which may lead to non trivial connections, which then requires the use of covariant derivatives. Also the conservation laws and the power balance laws have to be formulated in an intrinsic fashion, see Theorem 11.

Finally it should be stressed again, that the interpretation of the connection in the subsections (4.1) and (4.2) is completely different. In subsection (4.1) the connection is absorbed in a modified Hamiltonian, whereas in subsection (4.2) the connection is explicitly part of the covariant differential.

## 5 Conclusion

In this paper we investigated the geometry of time-variant (Port) Hamiltonian systems and used an intrinsic description based on connections and covariant derivatives. We were interested in two concrete applications where time-variant systems arise quite naturally, namely error systems with regard to trajectory tracking in control theory as well as time-variant mechanics. Concerning control theory we described that the formulation of the error system in a Hamiltonian fashion requires to absorb the nontrivial connection and the
feed-forward part in a modified Hamiltonian. Here the key problem was the
definition of a time-variant Hamiltonian system itself, i.e. if the tensors are
time-dependent and/or the coordinate chart is moving. The interpretation of
these two scenarios is significant for a correct understanding of time-variant
systems. Furthermore time-variant mechanics has been analyzed in a covari-
ant way, where we showed how collocation and the balancing/interaction of
power flows can be formulated in an intrinsic way.

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