Error Analysis of Douglas-Rachford Algorithm for Linear Inverse Problems: Asymptotics of Proximity Operator for Squared Loss

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Abstract—Proximal splitting-based convex optimization is a promising approach to linear inverse problems because we can use some prior knowledge of the unknown variables explicitly. In this paper, we firstly analyze the asymptotic property of the proximity operator for the squared loss function, which appears in the update equations of some proximal splitting methods for linear inverse problems. The analysis shows that the output of the proximity operator can be characterized with a scalar random variable in the large system limit. Moreover, we investigate the asymptotic behavior of the Douglas-Rachford algorithm, which is one of the famous proximal splitting methods. From the asymptotic result, we can predict the evolution of the mean-square-error (MSE) in the algorithm for large-scale linear inverse problems. Simulation results demonstrate that the MSE performance of the Douglas-Rachford algorithm can be well predicted by the analytical result in compressed sensing with the \( \ell_1 \) optimization.

Index Terms—Linear inverse problems, convex optimization, proximity operator, Douglas-Rachford algorithm, asymptotic analysis

I. INTRODUCTION

LINEAR inverse problems, i.e., the reconstruction of an unknown vector from its linear measurements, often appear in the field of signal processing. The linear inverse problem is called underdetermined when the number of measurements is less than that of elements of the unknown vector. In such underdetermined problems, we often use some prior knowledge of the unknown vector to obtain a good reconstruction result. In compressed sensing [1]–[4], for example, we reconstruct an unknown sparse vector from its underdetermined linear measurements by using the sparsity effectively. Another example of the underdetermined problems is the overloaded signal detection in wireless communications [5], [6], where we can use the discreteness of the unknown vector.

A common approach to underdetermined linear inverse problems is to solve an optimization problem using some regularizer based on the prior knowledge of the unknown vector. In compressed sensing, the most popular optimization problem is the \( \ell_1 \) optimization, which utilizes the \( \ell_1 \) norm as the regularizer to promote the sparsity. Although the objective function is not differentiable due to the \( \ell_1 \) norm, some proximal splitting methods, e.g., the iterative soft thresholding algorithm (ISTA) [7]–[9] and the fast iterative soft thresholding algorithm (FISTA) [10], can solve the problem with feasible computational complexity. Alternating direction method of multipliers (ADMM) [11], [12] and the Douglas-Rachford algorithm [13]–[15] can also be applied to the \( \ell_1 \) optimization problem. Although these algorithms require a matrix inversion for linear inverse problems, they can converge faster than some gradient-based methods such as ISTA in some cases [16], [17].

The asymptotic performance of several optimization-based approaches for linear inverse problems has been analyzed by using some techniques [18]–[20]. In particular, the recently developed convex Gaussian min-max theorem (CGMT) [21], [22] enables us to obtain the asymptotic error in a precise manner for various optimization problems. For example, the asymptotic mean-square-error (MSE) of various regularized estimators has been analyzed in [23], [24]. The asymptotic symbol error rate has also been analyzed for the discrete-valued vector reconstruction in [24], [25]. These analyses focus on the performance of the optimizer, whereas the application of CGMT to the analysis of optimization algorithms has been considered in [26]. In [26], the asymptotic behavior of ADMM for compressed sensing has been analyzed. The analytical result can be used to obtain appropriate parameters in ADMM.

In this paper, we analyze the asymptotic behavior of the Douglas-Rachford algorithm for linear inverse problems. In the analysis, we consider the objective function with the squared loss function and a separable regularizer. Since the proximity operator of the squared loss function is not element-wise, we investigate its asymptotic property via the CGMT framework. Specifically, we show that the output of the proximity operator for the squared loss function can be regarded as a scalar random variable in the large system limit. We then utilize the result to obtain the evolution of the asymptotic MSE in the Douglas-Rachford algorithm. Simulation results show that the empirical performance of the Douglas-Rachford algorithm is close to its theoretical prediction in compressed sensing with the \( \ell_1 \) optimization.

In this paper, we use the following notations. We denote the transpose by \((\cdot)^\top\) and the identity matrix by \(I\). For a vector \(\bm{u} = [u_1 \cdots u_N]^\top \in \mathbb{R}^N\), the \(\ell_1\) norm and the \(\ell_2\) norm are given by \(\|\bm{u}\|_1 = \sum_{n=1}^N |u_n|\) and \(\|\bm{u}\|_2 = \sqrt{\sum_{n=1}^N u_n^2}\), respectively. \(\text{sign}(\cdot)\) denotes the sign function. For a lower semicontinuous convex function \(\phi : \mathbb{R}^N \rightarrow \mathbb{R}\)
where \( \gamma \) is the operator of the function \( f \) on the \( \ell_2 \)-norm for the reconstruction of a sparse vector. The regularization parameter \( \lambda \) is a popular convex regularizer for the reconstruction of a linear function in this paper. The function \( f \) is known vector \( x \) and \( \lambda \) is a convex regularizer to utilize the prior knowledge of the unknown vector \( x \). In some applications, the unknown vector has other structures such as boundedness and discreteness [27], [28].

Convex optimization is a promising approach to underdetermined linear inverse problems because we can flexibly design the objective function to utilize the structure of the unknown vector \( x \). In this paper, we consider the convex optimization problem given by

\[
\min_{s \in \mathbb{R}^N} \{ L(s) + \lambda f(s) \},
\]

where we call \( L(s) = \frac{1}{2} \| y - Ax \|_2^2 \) as the squared loss function in this paper. The function \( f(\cdot) : \mathbb{R}^N \to \mathbb{R} \cup \{ +\infty \} \) is a convex regularizer to utilize the prior knowledge of the unknown vector \( x \). For example, \( \ell_1 \) regularization \( f(s) = \| s \|_1 \) is a popular convex regularizer for the reconstruction of a sparse vector. The regularization parameter \( \lambda (>0) \) controls the balance between the squared loss function \( L(s) \) and the regularization term \( \lambda f(s) \).

The Douglas-Rachford algorithm [13], [15] solves the optimization problem in (2) by using the proximity operators of \( L(\cdot) \) and \( f(\cdot) \). The update equations of the algorithm with the iteration index \( k (= 0, 1, 2, \ldots) \) can be written as

\[
s^{(k+1)} = \text{prox}_{\gamma L}(z^{(k)}),
\]

\[
z^{(k+1)} = z^{(k)} + \rho_k \left( \text{prox}_{\gamma \lambda f} \left( 2s^{(k+1)} - z^{(k)} \right) - s^{(k+1)} \right),
\]

where \( \gamma (>0) \) and \( \rho_k \in [\varepsilon, 2-\varepsilon] (\varepsilon \in (0, 1)) \) are the parameters in the algorithm. By definition, the proximity operator of the function \( L(\cdot) \) is given by

\[
\text{prox}_{\gamma L}(z) = \arg \min_{s \in \mathbb{R}^N} \left\{ \frac{1}{2} \| y - As \|^2 + \frac{1}{2\gamma} \| s - z \|^2 \right\}
\]

\[
= \left( A^T A + \frac{1}{\gamma} I \right)^{-1} \left( A^T y + \frac{1}{\gamma} z \right),
\]

where \( z \in \mathbb{R}^N \) and \( \gamma > 0 \). The proximity operator of \( f(\cdot) \) can also be computed efficiently for various regularizers. When \( f(s) = \| s \|_1 \), for example, the proximity operator of the function \( f(\cdot) \) can be written as

\[
[\text{prox}_{\gamma \lambda f}(r)]_n = \text{sign}(r_n) \max(|r_n| - \gamma \lambda, 0),
\]

where \( r = [r_1 \cdots r_N]^T \in \mathbb{R}^N \), and \([\cdot]_n \) denotes the \( n \)-th element of the vector. By computing (3) and (4) iteratively, we can obtain a sequence \( \{ s^{(k)} \}_{k=1,2,\ldots} \) converging to the solution of the optimization problem in (2).

III. MAIN RESULTS

A. Asymptotics of Proximity Operator for Squared Loss Function

We firstly analyze the output of the proximity operator \( \text{prox}_{\gamma L}(\cdot) \) in (6). In the analysis, we assume the large system limit \( M, N \to \infty (M/N = \Delta) \), where we consider the sequence of problems with \( \{ x, A, v \} \) indexed by \( N \) as in several high-dimensional analyses [22]. We also use the following assumptions.

Assumption III.1. The unknown vector \( x \in \mathbb{R}^N \) consists of independent and identically distributed (i.i.d.) random variables with a known distribution \( p_X \) with some finite mean and variance. The measurement matrix \( A \in \mathbb{R}^{M \times N} \) consists of i.i.d. Gaussian random variables with zero mean and variance \( 1/N \). Moreover, the additive noise vector \( v \in \mathbb{R}^M \) is also Gaussian with zero mean and the covariance matrix \( \sigma_v^2 I \).

Remark III.1. In Assumption III.1, we assume that the elements of the measurement matrix \( A \) are Gaussian variables. This is because we require the Gaussian assumption in the CGMT-based analysis [22]. However, the universality of random matrices [29]-[31] suggests that the analytical result is valid when the measurement matrix is drawn from some other distributions.

Under Assumption III.1 we can obtain the following result on the asymptotic behavior of the proximity operator \( \text{prox}_{\gamma L}(\cdot) \).

Proposition III.1. We assume that \( x, A, \) and \( v \) satisfy Assumption III.1. We consider the output of the proximity operator given by

\[
\hat{s} = \text{prox}_{\gamma L}(z),
\]

where \( z \) has i.i.d. elements with a distribution \( p_Z(z) \). We here assume that the optimization problem

\[
\min_{\alpha > 0, \beta > 0} \left\{ \alpha \beta \sqrt{\Delta} \frac{\Delta}{2} + \beta \sigma_v^2 \sqrt{\Delta} \frac{2}{2\alpha} - \frac{1}{2} \beta^2 + E[J(\alpha, \beta; Z)] \right\}
\]

\[
(9)
\]
has a unique optimizer \((\alpha^*, \beta^*)\), where we define
\[
J(\alpha, \beta; Z) = \frac{\beta \sqrt{\Delta}}{2\alpha} \left( \hat{S}(\alpha, \beta; Z) - X \right)^2 \\
- \beta H \left( \hat{S}(\alpha, \beta; Z) - X \right) \\
+ \frac{1}{2\gamma} \left( \frac{\beta \sqrt{\Delta}}{\alpha} \left( X + \frac{\alpha}{\sqrt{\Delta}} H \right) + \frac{1}{\gamma} Z \right)^2,
\]
(10)
\[
\hat{S}(\alpha, \beta; Z) = \frac{\beta \sqrt{\Delta}}{\alpha} + \frac{1}{\gamma} \left( \frac{\beta \sqrt{\Delta}}{\alpha} \left( X + \frac{\alpha}{\sqrt{\Delta}} H \right) + \frac{1}{\gamma} Z \right)^2,
\]
(11)
with the random variables \(X \sim p_X, H \sim N(0, 1), \) and \(Z \sim p_Z\). In \([9]\), the expectation is taken over all random variables \(X, H, \) and \(Z\). Then, the following statements hold:

1) The asymptotic MSE of \(\hat{s}\) in \((8)\) is given by
\[
\lim_{N \to \infty} \frac{1}{N} \left\| \hat{s} - x \right\|_2^2 = (\alpha^*)^2 - \sigma_y^2.
\]
(12)

2) Let \(\mu_{\hat{s}}\) be the empirical distribution of \(\hat{s} = [\hat{s}_1 \cdots \hat{s}_N]^T \in \mathbb{R}^N\) corresponding to the cumulative distribution function (CDF) given by \(P_{\hat{s}} = \frac{1}{N} \sum_{n=1}^{N} I(\hat{s}_n < s)\), where \(I(\hat{s}_n < s) = 1\) if \(\hat{s}_n < s\) and otherwise \(I(\hat{s}_n < s) = 0\). Then, the distribution \(\mu_{\hat{s}}\) converges weakly in probability to the distribution \(\mu_S\) of \(S = \hat{S}(\alpha^*, \beta^*; Z)\), i.e., \(\int g d\mu_{\hat{s}} \to \int g d\mu_S\) holds for any continuous compactly supported function \(\mu : \mathbb{R} \to \mathbb{R}\).

**Sketch of proof.** By definition, the vector \(s\) is characterized as the solution of the optimization problem \((5)\). In a similar way to \([26]\), we can prove the statement of the proposition by using the standard approach with CGMT \([22]\). Specifically, we can obtain Proposition III.1 by replacing \(z(k) - w(k), Z_k - W_k\) and \(\rho\) in \([26]\) Theorem III.1 with \(z, Z\) and \(1/\gamma\), respectively.

The first statement in Proposition III.1 means that the asymptotic MSE for the output of the proximity operator can be predicted by solving the scalar optimization problem in \((9)\). The second one implies that the distribution of the elements of the vector \(s\) can be characterized by the random variable \(S = \hat{S}(\alpha^*, \beta^*; Z)\). We can thus consider \(\hat{S}(\alpha^*, \beta^*; Z)\) as a decoupled version of the proximity operator in \((8)\) intuitively.

The optimization of \(\alpha\) and \(\beta\) in \((9)\) can usually be performed by using searching techniques such as the ternary search and the golden section search \([52]\). Since the exact computation of the expectation in \((9)\) is difficult in general, we need to approximate it with the average of many realizations of \(X, H,\) and \(Z\).

**B. Application to Analysis of Douglas-Rachford Algorithm**

By using Proposition III.1, we can analyze the asymptotic behavior of the tentative estimate of \(x\) in the Douglas-Rachford algorithm. In the analysis, we assume the separability of the regularizer \(f(\cdot)\) as follows.

**Assumption III.2.** Let the regularizer \(f(\cdot) : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}\) be a lower semicontinuous convex function. Moreover, we assume that \(f(\cdot)\) is separable and can be expressed as \(f(s) = \sum_{n=1}^{N} f(s_n)\) with a function \(f(\cdot) : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}\). With the slight abuse of notation, we sometimes use the same \(f(\cdot)\) for the corresponding function \(f(\cdot)\).

**Remark III.2.** For the separable regularizer \(f(\cdot)\) in Assumption III.2, the proximity operator \(\text{prox}_{\gamma \lambda f}(\cdot) : \mathbb{R}^N \to \mathbb{R}^N\) becomes an element-wise function, i.e., the \(n\)-th element of the output depends only on the \(n\)-th element of the input.

Under Assumptions III.1 and III.2, we have the following result.

**Proposition III.2.** Suppose that Assumptions III.1 and III.2 hold. We consider the stochastic process given by
\[
S_{k+1} = \hat{S}(\alpha_k^*, \beta_k^*; Z_k)
\]
(13)
\[
Z_{k+1} = Z_k + \rho_k \left( \text{prox}_{\gamma \lambda f}(2S_{k+1} - Z_k) - S_{k+1} \right)
\]
(14)
with the index \(k\). We here assume that the optimization problem
\[
\min_{\alpha > 0, \beta > 0} \left\{ \frac{\alpha \beta \sqrt{\Delta}}{2} + \frac{\beta \sigma_y^2 \sqrt{\Delta}}{2\alpha} - \frac{1}{2} \beta^2 + E[J(\alpha, \beta; Z_k)] \right\}
\]
(15)
has a unique optimizer \((\alpha_k^*, \beta_k^*)\). Then, the following statements hold:

1) The asymptotic MSE of the tentative estimate \(s(k+1)\) in \((3)\) is given by
\[
\lim_{N \to \infty} \frac{1}{N} \left\| s(k+1) - x \right\|_2^2 = (\alpha_k^*)^2 - \sigma_y^2.
\]
(16)

2) The empirical distribution of \(s(k+1)\) converges weakly in probability to the distribution of \(S_{k+1}\).

**Sketch of proof.** We can prove the result with the same procedure in the proof of \([26]\) Theorem III.1 and Corollary III.1. The distribution of the element of \(s(k+1)\) in \((3)\) can be characterized by that of the random variable \(S_{k+1}\) in \((13)\) from Proposition III.1 when the element of \(z(k)\) can be regarded as the random variable \(Z_k\). Also, the update of \(z(k)\) in \((14)\) can be decoupled into \((15)\) because the update is element-wise under Assumption III.2. We can thus prove the statement of Proposition III.2 by induction.

**Proposition III.2** means that the evolution of the asymptotic MSE in the Douglas-Rachford algorithm can be obtained by solving the optimization problem in \((15)\) for each \(k\). The updates of \(S_k\) and \(Z_k\) in \((13)\) and \((14)\) can be considered as a decoupled version of the update of \(s(k)\) and \(z(k)\) in \((3)\) and \((4)\), respectively.

From the theoretical result in Proposition III.2, we can tune the parameters in the Douglas-Rachford algorithm to achieve fast convergence. Since the relation between the parameters and the MSE is complicated, it is difficult to obtain the explicit expression of the optimal parameters. However, we can numerically predict the performance of the algorithm and select the parameter achieving the fast convergence in the asymptotic regime.
In this section, we demonstrate the validity of our analysis via computer simulations. We consider an unknown vector $x$ whose distribution is given by the Bernoulli-Gaussian distribution as

$$ p_X(x) = p_0 \delta_0(x) + (1 - p_0) p_H(x), \quad (17) $$

where $p_0 \in (0, 1)$ is the probability for 0, $\delta_0(\cdot)$ denotes the Dirac delta function, and $p_H(\cdot)$ is the probability density function of the standard Gaussian distribution. The unknown vector $x$ is sparse when $p_0$ is large. For the reconstruction of such sparse vector, we consider the $\ell_1$ optimization with $f(s) = \|s\|_1$, i.e.,

$$ \minimize_{s \in \mathbb{R}^N} \left\{ \frac{1}{2} \|y - As\|^2 + \lambda \|s\|_1 \right\}, \quad (18) $$

which is the most popular convex optimization problem for compressed sensing and satisfies Assumption [III.2]. In the simulations, the measurement matrix $A$ and the noise vector $v$ satisfy Assumption [III.1].

We compare the empirical MSE performance of the Douglas-Rachford algorithm for (18) and its prediction obtained from Proposition [III.2]. Figure 1 shows that the MSE performance versus the number of iterations in the algorithm, where $\Delta = 0.7$, $p_0 = 0.9$, and $\sigma_x^2 = 0.001$. The parameters of the Douglas-Rachford algorithm are set as $\gamma = 10$ and $\rho_k = 1$. The initial value of $z^{(k)}$ is given by $z^{(0)} = 0$. In the figure, ‘empirical’ denotes the empirical MSE performance of the Douglas-Rachford algorithm in (3) and (4) when $N = 50, 100$, and 500. The empirical performance is obtained by averaging the results for 500 independent realizations of $x$, $A$, and $v$. Also, ‘prediction’ represents the theoretical prediction obtained by Proposition [III.2] in the large system limit. To obtain the theoretical prediction, we compute 300, 000 realizations of the random variables $(S_k, Z_k)$ and obtain the approximation of $(\alpha_k, \beta_k)$ because the exact computation of the distribution of $(S_k, Z_k)$ is difficult. As a reference, we also show the asymptotic MSE of the optimizer of the optimization problem in (18), which can be obtained from the standard CGMT approach as in [24]. The parameter $\lambda$ in (18) is determined by minimizing the asymptotic MSE of the optimizer. Figure 1 shows that the empirical performance is close to the theoretical prediction when $N$ is sufficiently large. We can also see that they converge to the asymptotic MSE of the optimizer of the $\ell_1$ optimization problem. To be precise, there is a slight difference between the empirical performance and its prediction. This is partly because we evaluate the empirical performance for finite $N$, whereas we assume the large system limit $N \to \infty$ in the asymptotic analysis. Another reason might be that we use many realizations of $(S_k, Z_k)$ for the theoretical prediction instead of computing their exact distributions.

Next, we evaluate the MSE performance for different parameters $\gamma$ in the Douglas-Rachford algorithm. Figure 2 shows the MSE performance versus $\gamma$, where $N = 500$, $M = 350$, $p_0 = 0.9$, $\sigma_x^2 = 0.001$, and $\rho_k = 1$. From the figure, we can observe that the performance is improved as the iteration index $k$ increases. The figure also implies that the value of $\gamma$ significantly affects the performance of the algorithm. Since the empirical performance is well predicted for all $\gamma$, we can tune the parameter by using the theoretical prediction. The figure shows that we should choose $\gamma$ between 10 and 15 in this case. Although the parameter $\rho_k$ is fixed in this paper for simplicity, we can also tune $\rho_k$ by using the theoretical result.

V. CONCLUSION

In this paper, we have analyzed the asymptotic behavior of the proximity operator for the squared loss function in linear inverse problems. We have also shown that the theoretical result in Prop. [III.1] can be applied to the asymptotic analysis of the Douglas-Rachford algorithm. The analytical result enables us to predict the performance of the algorithm for large-scale linear inverse problems. Simulation results show that the empirical performance of the Douglas-Rachford algorithm can be well predicted in large-scale compressed sensing with hundreds of unknown variables.

Fig. 1. MSE performance for different $N$ in sparse vector reconstruction ($\Delta = 0.7$, $p_0 = 0.9$, $\sigma_x^2 = 0.001$, $\gamma = 10$, $\rho_k = 1$).

Fig. 2. MSE performance versus $\gamma$ in sparse vector reconstruction ($N = 500$, $M = 350$, $p_0 = 0.9$, $\sigma_x^2 = 0.001$, $\rho_k = 1$).
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