GENERAL CHEEGER INEQUALITIES FOR
\textit{p}-LAPLACIANS ON GRAPHS

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ABSTRACT. We prove Cheeger inequalities for \textit{p}-Laplacians on finite and infinite weighted graphs. Unlike in previous works, we do not impose boundedness of the vertex degree, nor do we restrict ourselves to the normalized Laplacian and, more generally, we do not impose any boundedness assumption on the geometry. This is achieved by a novel definition of the measure of the boundary which is using the idea of intrinsic metrics. For the non-normalized case, our bounds on the spectral gap of \textit{p}-Laplacians are already significantly better for finite graphs and for infinite graphs they yield non-trivial bounds even in the case of unbounded vertex degree. We, furthermore, give upper bounds by the Cheeger constant and by the exponential volume growth of distance balls.

1. Introduction

Cheeger inequalities have a long history and are relevant for both pure mathematics and applied mathematics. The pure mathematical interest stems from the fact that they connect geometry and spectral theory. In applications they are used to partition the underlying space in an efficient way.

From the perspective of pure mathematics the history of our topic starts with the work of Cheeger [Che70]. On compact manifolds Cheeger used an isoperimetric constant to estimate the first non-trivial eigenvalue of the Laplacian from below. This isoperimetric constant – thus called Cheeger constant ever since – serves as measure for separating the manifold into two approximately equally sized parts.

Similar ideas for finite graphs were independently found shortly afterwards in the pioneering work of Fiedler [Fie73], where the first non-trivial eigenvalue of the graph Laplacian is shown to be a quantitative measure of the graph’s connectedness. The first “genuine” Cheeger estimates on graphs are due to Dodziuk [Dod84] and Alon/Milman [AM85].

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Since then these estimates have been improved and various variants have been shown. However, it was only until recently that non-trivial estimates for unbounded graph Laplacians were available. Specifically, in previous investigations either it was the normalized graph Laplacian (which is always a bounded operator) that was considered, or else an upper bound on the vertex degree appeared in the denominator of the lower bound, thus making the inequality trivial whenever the degree is unbounded. In [BKW15] a novel measure of the boundary of a set has been introduced using the concepts of intrinsic metrics for non-local Dirichlet forms. These metrics have first been systematically studied by Frank/Lenz/Wingert [FLW14] for general regular Dirichlet forms. Since then they have proven a very efficient tool, see e.g. the recent survey [Kel15] on graphs.

The history sketched above for the classical case of the linear Laplacian has inspired analogs in non-linear theory. After Cheeger [Che70] treated the linear case \( p = 2 \) and Yau [Yau75] proved an equality for \( p = 1 \), Kawohl/Fridman [KF03] generalized Cheeger’s inequality to the \( p \)-Laplace-Beltrami operators, for \( p > 1 \). Cheeger inequalities for the \( p \)-Laplacian (or the normalized \( p \)-Laplacian) on finite graphs can be found in [Amg03, Tak03, BH09].

The applications perspective is converse. Here, one is interested in finding graph partitions. While computing the Cheeger constant of a graph is an NP-hard problem, see e.g. [Kai04], the computation of the first non-trivial eigenvalue and the corresponding eigenfunction is rather efficient, by simple variational methods. Thus, Fiedler’s intuition [Fie73, Fie75] had far-reaching repercussions in theoretical computer science. In particular, the supports of the positive and negative part of the first non-trivial eigenfunction of the graph Laplacian (or \( p \)-Laplacian) suggest a reasonable splitting of the graph. In view of the Cheeger inequalities this splitting is close to the optimal Cheeger cut. Indeed, several machine learning tasks – like clustering of data sets or segmenting of pictures – can actually be reduced to the study of eigenvalues of the Laplacian on the associated graphs. Since the pioneering investigations in [DH73, Fie73], spectral methods on graphs or manifolds associated with data sets have become rather popular in computer science, cf. e.g. [NJW01, Lux07, GP10, Bol13].

The \( p \)-Laplacians have recently aroused interest in applications to computer science mostly because their lowest nontrivial eigenvalue converges to the Cheeger constant as \( p \to 1 \). This was shown in the continuous case in the remarkable work of Kawohl/Fridman [KF03], which was later proven by Bühler/Hein [BH09] in the graph setting. Hence,
the Cheeger constant can be approximated by means of a sequence of convex optimization problems.

As in the linear case, the known estimates for the $p$-Laplacian on graphs are proven either for the normalized Laplacian, or else they involve an upper bound on the vertex degree in the estimate. In the second case, this leads to non-optimal estimates for finite graphs that have only few vertices of very large degree, like real-life scale-free networks. In the case of infinite graphs with unbounded degree the estimates known so far even become trivial.

In this paper, we adapt the ideas of intrinsic metrics from [BKW15] to the non-linear case of $p$-Laplacians to improve the estimates known so far. The techniques use a novel definition of the boundary measure of a set. In particular, not only the weight of an edge is taken into account but also its length. This length stems from a metric whose $p^*$-norm, with $1/p + 1/p^* = 1$, of the "discrete gradient" is less than one. In the linear case it can be motivated by the distances attributed to a diffusion on the graph, see [Kel15]. One instance of such a metric can be obtained involving the inverses of the vertex degrees, see Example 2.1.

From this perspective the vertex degrees are part of the minimization itself and do not enter as a uniform upper bound.

Our main perspective is rather the one of pure mathematics, that is we look for estimates of spectral quantities in terms on geometric ones. Nevertheless, the Cheeger constant defined by these novel metrics might be of applicative interest on its own right, as it encodes relevant geometric data of the underlying graph.

We also prove upper bounds for the first non-trivial eigenvalue. Such estimates are known as Buser inequalities in the case of manifolds. Unlike in the manifold case there is classically no curvature notion whatsoever entering the upper estimate in the graph case [AM85, Moh88]. However, we get an upper bound involving a constant related to uniform discreteness of the space. Indeed, it turns out that this estimate becomes often trivial when the vertex degree is unbounded. This once again suggests that a lower bound on the curvature in the case of manifolds corresponds to an upper bound on the vertex degree for graphs.

We, furthermore, give an alternative proof of Bühler/Hein’s approximation result for finite graphs. Finally, we show an upper bound for bottom of the spectrum in terms of exponential volume growth of balls. Classically this is known as Brook’s theorem [Bro81] and was shown for regular Dirichlet forms in [HKW13] in the linear case (i.e., $p = 2$). Our proof uses again an adaption of the concept of intrinsic metrics to the case of general $p$. 
The paper is structured as follows. In the next section we introduce the set-up with all relevant quantities. This is followed by Section 3 where we formulate and prove the Cheeger inequalities. In Subsections 3.4 and 3.5 of Section 3 we prove upper bounds in the sense of a Buser inequality and the convergence result in the case of finite graphs. In Section 4 we prove the upper bounds by exponential volume growth in the spirit of a Brooks-type theorem. Finally, in the appendix we discuss the interpretation of our variational results as estimates on the spectral gap of discrete $p$-Laplacians.

2. Set up

2.1. Graphs and the energy functional. Let $X$ be a discrete countably finite or countably infinite set. We denote the set of real-valued functions on $X$ by $C(X)$ and its subset of finitely supported functions by $C_c(X)$.

Let a symmetric function $b : X \times X \rightarrow [0, \infty)$ with zero diagonal be given such that

$$\sum_{y \in X} b(x, y) < \infty, \quad x \in X.$$ 

Such a function has an interpretation in classical graph theory: the elements of $X$ are vertices and two vertices $x, y \in X$ are connected by an edge with weight $b(x, y)$ if and only if $b(x, y) > 0$; we write $x \sim y$ in this case.

Let $m : X \rightarrow (0, \infty)$ be a function which extends to a measure via additivity. Moreover, we denote the spaces of $p$-summable real valued functions on $X$ with respect to the measure $m$ by $\ell^p(X, m)$ and the corresponding norm by $\| \cdot \|_{m,p}, p \in [1, \infty)$. The dual pairing for functions $f \in \ell^p(X, m)$ and $g \in \ell^q(X, m)$ with $p, q \in (1, \infty)$ and $1/p + 1/q = 1$ is given by

$$\langle f, g \rangle_m := \sum_X fg m = \sum_{x \in X} f(x)g(x)m(x).$$

Following Nakamura/Yamasaki [NY76], for $p \in [1, \infty)$, we introduce a convex energy functional $\mathcal{E}_p : C(X) \rightarrow [0, \infty]$ by

$$\mathcal{E}_p(f) := \frac{1}{2} \sum_{x,y \in X} b(x, y)|f(x) - f(y)|^p, \quad f \in C(X).$$

In this paper we are interested in giving estimates on the quantity

$$\lambda_p^{(0)} := \inf_{0 \neq \varphi \in C_c(X)} \frac{\mathcal{E}_p(\varphi)}{\|\varphi\|^p_{m,p}}.$$
and in the case \( m(X) < \infty \)

\[
\lambda_p^{(1)} := \inf_{\text{const} \neq f \in \ell^p(X, \mu)} \inf_{\gamma \in \mathbb{R}} \| f - \gamma \|_{m,p}^{p}.
\]

The latter is of course in particular relevant in the case of finite graphs.

2.2. A non-linear generalization of intrinsic metrics. In recent years the notion of intrinsic metrics has been developed for graphs; it has various strong applications for the case \( p = 2 \). Here we are going to extend it to general \( p \). For our purposes we do not need to enforce the triangle inequality: This suggests to introduce the set

\[
R_p(b, \mu) := \{ d : X \times X \to [0, \infty) \mid \text{symmetric and} \sum_{y \in X} b(x, y)d(x, y)^{p/(p-1)} \leq \mu(x), \text{for all } x \in X \}
\]

for \( p \in (1, \infty) \) and

\[
R_1(b, \mu) := \{ d : X \times X \to [0, \infty) \mid \text{symmetric and } d(x, y) \leq 1 \text{ if } x \sim y \}
\]

for \( p = 1 \).

Let us give two examples: on one hand we show that \( R_p(b, \mu) \) does indeed contain non-vanishing functions for any graph structure \( b \) and any measure \( \mu \); on the other hand we show how to embed a classical object of the literature – the combinatorial graph distance – in our theoretical setting.

**Example 2.1.** (a) For a given graph \( b \) over \((X, \mu)\) and \( p \in (1, \infty) \) we define a function \( d_p : X \times X \to [0, \infty) \) by

\[
d_p(x, y) := (\text{Deg}(x) \lor \text{Deg}(y))^{-(p-1)/p}, \quad x, y \in X,
\]

where \( \text{Deg}(x) \) is the weighted degree of the vertex \( x \in X \) with respect to \( b \) and \( \mu \) given by

\[
\text{Deg}(x) = \frac{1}{\mu(x)} \sum_{y \in X} b(x, y), \quad x \in X.
\]

The function \( d_p \) can be seen to be in \( R_p(b, \mu) \) for any \( b \) and \( \mu \) by direct calculations. Furthermore, we can easily construct a pseudo metric from \( d_p \) by considering the associated path metric.

(b) Let a graph \( b \) over \( X \) be given. In the case where \( \mu \) is chosen to be the normalizing measure, i.e.,

\[
\mu(x) = \sum_{y \in X} b(x, y),
\]
and, hence, Deg $\equiv 1$, then the combinatorial graph distance $d_{\text{comb}}$ defines a function in $R_p(b, m)$ for all $p \geq 1$. This is case most usually considered in the literature, see e.g. [Amg03, Tak03].

**Remark 2.2.** For $p = 2$, a function in $R_p(b, m)$ that is additionally a pseudo metric, i.e., satisfies the triangle inequality, is called an *intrinsic metric*. This concept was first studied systematically for general regular Dirichlet forms by Frank/Lenz/Wingert in [FLW14] and used since then in various contexts, see e.g. [BKWi15, Folli11, GHM12, HKMW13]. Here, we replace 2 by $q = p/(p - 1)$ which is the conjugate of $p$, i.e., $1/p + 1/q = 1$.

### 3. Cheeger Inequalities

We start by defining the isoperimetric constants with the novel definition of the boundary measure of a set. This is inspired by [BKWi15] where this was used in the case $p = 2$.

Below, we introduce the isoperimetric constants $h^{(0)}_p$ and $h^{(1)}_p$ which are each tailored for the quantities $\lambda^{(0)}_p$ and $\lambda^{(1)}_p$. We then state our main results: Theorem 3.1 for $\lambda^{(0)}_p$ and Theorem 3.2 for $\lambda^{(1)}_p$. The latter result is in particular relevant for finite graphs.

For the proof we show an abstract Cheeger estimate, Theorem 3.5 from which we derive both Theorem 3.1 and Theorem 3.2.

Afterwards, we proceed by showing upper bounds for the quantities $\lambda^{(0)}_p$ and $\lambda^{(1)}_p$ in terms of $h^{(0)}_p$ and $h^{(1)}_p$ and a uniform discreteness constant. Finally, we turn to finite graphs and give an alternative proof of the convergence of $\lambda^{(1)}_p$ to $\lambda_1^{(1)}$ as $p \to 1$.

#### 3.1. Isoperimetric constant and results.

In this subsection we define the isoperimetric constants and state the results.

Let $W \subseteq X$. We define the boundary $\partial W$ of the set $W$ by

$$\partial W := W \times (X \setminus W).$$

The *measure of the boundary* with respect to a function $w : X \times X \to [0, \infty)$ is defined as

$$|\partial W|_w := w(\partial W) = \sum_{(x,y) \in \partial W} w(x,y).$$

Whenever a graph structure on $X$ given by some $b$ along with a measure $m$ is considered, this definition will be used with $w = bd$, where $d$ is a function in $R_p(b, m)$. In this case the sum over $\partial W$ above is effectively only over the edges leaving $W$. This definition of the measure of the boundary generalizes the classical theory which considers $w = b$, 


i.e., $d = 1$, only. This generalization is the key idea so that we do not have to impose any boundedness assumptions, neither by assuming bounded weighted vertex degree nor by restricting ourselves to the case of the normalizing measure (cf. Example 2.1).

We define for $p \in [1, \infty)$ the $p$-isoperimetric numbers $h_p^{(0)}$ and $h_p^{(1)}$ by

$$ h_p^{(0)} := \sup_{d \in R_p(b,m)} h^{(0)}(d) \quad \text{with} \quad h^{(0)}(d) := \inf_{W \subseteq X \text{ finite}} \frac{\left| \partial W \right|_{bd}}{m(W)} $$

and, only in the case $m(X) < \infty$,

$$ h_p^{(1)} := \sup_{d \in R_p(b,m)} h^{(1)}(d) \quad \text{with} \quad h^{(1)}(d) := \inf_{W \subseteq X \atop m(W) \leq m(X)/2} \frac{\left| \partial W \right|_{bd}}{m(W)} $$

In the case of finite graphs one always has $h_p^{(0)} = 0$ since one can choose $W = X$ in the definition of $h^{(0)}(d)$. Furthermore, choosing $m$ as the normalizing measure, see Example 2.1(b), the constants $h_p^{(0)}$ and respectively $h_p^{(1)}$ agree with the classical Cheeger constants in the case of infinite and respectively finite graphs.

Having introduced the relevant quantities we are in the position to state our main results. These are two Cheeger-type inequalities relating the isoperimetric numbers and the spectral gaps.

**Theorem 3.1.** For all $p \in (1, \infty)$,

$$ \frac{2p-1}{p^p} h_p^{(0)p} \leq \lambda_p^{(0)}. $$

For $p \in [1, \infty)$, let

$$ D_p = \{ f \in C(X) \mid E_p(f) < \infty \} \cap \ell^p(X,m). $$

We say a function $f \in D_p$ is a weak solution for $\lambda$ if for all $g \in D_p$

$$ E'_p(f)g := \frac{1}{2} \sum_{x,y \in X} b(x,y)|f(x) - f(y)|^{p-2}(f(x) - f(y))(g(x) - g(y)) $$

$$ = \lambda \sum_X |f|^{p-2}fgm. $$

**Theorem 3.2.** Let $m(X) < \infty$. Let there be a non-constant weak solution $f \in D_p$ for $\lambda_p^{(1)}$ for some $p \in (1, \infty)$. Then,

$$ \frac{2p-1}{p^p} h_p^{(1)p} \leq \lambda_p^{(1)}. $$
Remark 3.3. (a) Using the Fréchet differentiability of $E_p$ on suitable Banach spaces, see e.g. [Mug13], one can prove the following: In the case weak solutions for $\lambda_p^{(1)}$ exist, they are exactly the minimizers of the functional $E_p$. This was already shown in [BH09, Thm. 3.1] for finite graphs. Indeed, $\lambda_p^{(0)}$ and respectively $\lambda_p^{(1)}$ can be understood as the bottom of the spectrum and respectively first non-trivial eigenvalue of $p$-Laplacian under Dirichlet and respectively Neumann conditions. These $p$-Laplacians are restrictions of the general $p$-Laplacian that is introduced in the appendix.

(b) Theorem 3.2 can be seen as a generalization of the corresponding estimates in [Amg03, Tak03] and [BH09] in the case of finite graphs. For $b$ over $(X, m)$ let the classical Cheeger constant be given by

$$h := \inf_{m(W) \leq m(X/2)} \frac{|\partial W|_b}{m(W)}.$$ 

In [Amg03, Tak03] the case
- $b : X \times X \to \{0, 1\}$,
- $m(x) = \sum_{y \in X} b(x, y) = \# \{ y \in X \mid x \sim y \}$, $x \in X$,

was considered and the bound

$$2^{p-1} \left( \frac{h}{p} \right)^p \leq \lambda_p^{(1)}$$

was obtained. This is a special case of Theorem 3.2 since $h = h^{(1)}(d_{comb})$ with $d_{comb}$ being the combinatorial graph metric which is in $R_p(b, m)$ for $b$ and $m$ as chosen above.

Furthermore, for
- $b : X \times X \to \{0, 1\}$,
- $m \equiv 1$

we improve the bound in [BH09, Theorem 4.3], where the inequality

$$\left( \frac{2}{M} \right)^{p-1} \left( \frac{h}{p} \right)^p \leq \lambda_p^{(1)}$$

was proven with $M := \sup_{x \in X} \# \{ y \in X \mid x \sim y \}$. Observe that the combinatorial graph metric is not in $R_p(b, 1)$ apart from the trivial case of a graph consisting of isolated vertices and edges. To see that our estimate is sharper, one can choose the weight $d_p(x, y) := (\text{Deg}(x) \lor \text{Deg}(y))^{-(p-1)/p}$ from Example 2.1. In Example 3.4 below we give explicit constructions of graphs where our estimate is seen to be significantly sharper than the one of [BH09].
Example 3.4. We consider a finite $k$-regular graph, i.e., $b_0 : X \times X \to \{0, 1\}$ such that $\sum_{y \in X} b(x, y) = \#\{y \sim x\} = k$ for all $x \in X$. Furthermore, let $m \equiv 1$. Let $W_0$ be a set that minimizes

$$h_0 := \inf_{\#W \leq \#X/2} \frac{\# \partial W_{b_0}}{\#W}$$

and set $N_0 := \#W_0 - 1$. We may assume that the graph is such that $N_0 \geq k$.

Now, we let $b$ be the graph over $X$ which has the edges of $b_0$ and we choose an arbitrary vertex $w \in W_0$ and connect it to every other vertex in $W_0$ by an edge. Obviously, the Cheeger constant $h := \inf_{\#W \leq \#X/2} \frac{\# \partial W_b}{\#W}$ of the new graph $b$ equals $h_0$ and $W_0$ is still a minimizing set since $\# \partial W_b \geq \# \partial W_{b_0}$ for any $W \subseteq X$ and $\# \partial W_{b_0} = \# \partial W_0_{b_0}$. In other words the conductivity $\frac{\# \partial W_{b_0}}{m(W)}$ of a set $W$ depends on the connectivity to its complement $X \setminus W$ as well as to its measure, but not on the internal structure of $W$. So, [BH09] yields the estimate

$$\left(\frac{2}{N_0}\right)^{p-1} \left(\frac{h}{p}\right)^p \leq \lambda_p^{(1)}.$$

To compare this to our estimate, we choose the function $d_0 := bk^{-1/q}$ with $q = p/(p - 1)$. It is easy to see that $d_0 \in R_p(b, m)$ and $\# \partial W_{bd_0} = k^{-1/p'} \# \partial W_{b_0}$. Hence, $h^{(1)}(d_0) = h_0 = h$ and the estimate

$$\frac{2^{p-1}}{k^{1/q}} \left(\frac{h}{p}\right)^p \leq \lambda_p^{(1)}$$

improves the bound of [BH09] above by a factor $k^{1/q}/N_0^{p-1}$.

Of course, the choice of $d_0$ above required a rather detailed knowledge of the structure of the graph. Below we show that even for the generic function $d$ from Example 2.1(a) the bound is improved.

So, consider the function from Example 2.1(a)

$$d(x, y) = (\text{Deg}(x) \lor \text{Deg}(y))^{-1/q}$$

with $q = p/(p - 1)$ and $\text{Deg}$ given by $\text{Deg}(x) = \sum_{y \in X} b(x, y)$ in the case $m \equiv 1$. Then, for any $W \subseteq X$ with $w \in W$ we have

$$\# \partial W_{bd} = \sum_{(x, y) \in W \times X \setminus W, x \neq w} bd(x, y) + \sum_{y \in X \setminus W} bd(w, y)$$

$$= (k + 1)^{-1/q} \sum_{(x, y) \in W \times X \setminus W, x \neq w} b(x, y) + (k + N_0)^{-1/q} \sum_{y \in X \setminus W} b(w, y)$$

$$\geq (k + 1)^{-1/q} (\# \partial W_{b_0} - k + 1)$$

and
for any $W \subseteq X$ with $w \in X \setminus W$, we have analogously
\[ |\partial W|_{\text{bd}} \geq (k + 1)^{-1/q}(|\partial W|_{b0} - k + 1). \]
Hence, with $c = (k + 1)^{-1/q}(k - 1)$ we obtain
\[ h_p^{(1)}(d) = \inf_{\#W \leq \#X/2} \frac{|\partial W|_{\text{bd}}}{\#W} \geq (k + 1)^{-1/q} \inf_{\#W \leq \#X/2} \frac{|\partial W|_{b} - c}{\#W}. \]
So, whenever we have a graph where $N_0 = \#W_0 - 1$ is significantly larger than $(k + 1)^{1/q}h_0^{-1}$, then $h_p^{(1)}(d)$ is close to $(k + 1)^{-1/q}h_0^{-1}$. In such cases our estimate is still significantly better than the one of [BH09] above, namely by the factor $k^{1/q}/N_0^{p-1}$.

3.2. A general isoperimetric inequality. In this subsection we prove a general isoperimetric inequality from which we will deduce Theorem 3.1 and Theorem 3.2.

Let $p \in [1, \infty)$ be given. We extend as usual a symmetric function $w : X \times X \to [0, \infty)$ to a measure by
\[ w(U) := \sum_{(x,y) \in U \times U} w(x, y), \quad U \subseteq X \times X. \]
Furthermore, let $m : X \to (0, \infty)$ be given. Moreover, for a linear subspace $G \subseteq C(X)$, we let
\[ Y_G = \{ \Omega_t(|f|^p) \mid f \in G, t > 0 \}, \]
where, for a function $g \in C(X)$, the level sets $\Omega_t(g)$ are given by
\[ \Omega_t(g) := \{ x \in X \mid g(x) > t \}. \]

Given the ingredients $p, w, m$ and $G$, we define a general isoperimetric constant
\[ h_{p,w,m,G} := \inf_{W \in Y_G} \frac{w(\partial W)}{m(W)}. \]

For example if $G = C_c(X)$, then $Y_{C_c(X)}$ consists of all finite sets. Furthermore, for $G = \ell^p(X, m), p \in [1, \infty)$, the set $Y_{\ell^p(X, m)}$ is the set of all finite measure subsets of $X$.

**Theorem 3.5.** Let $p \in (1, \infty), b$ be a graph over $(X, m), w, \sigma : X \times X \to [0, \infty)$ such that $w \leq b\sigma, G \subseteq C(X)$ and
\[ k(x) := \sum_{y \in X} (b\sigma^{p/(p-1)})(x, y), \quad x \in X. \]
Then, for all \( g \in G \cap \ell^p(X, m) \)
\[
\frac{2^{p-1}}{p^p} h^p_{p,w,m,G} \|g\|_{p,m}^2 \leq \|g\|_{k,p}^{p(p-1)} \mathcal{E}_p(g),
\]
where both sides may take the value \(+\infty\).

The inequality in Theorem 3.5 bears some resemblance to the interpolation inequality of Gagliardo/Nirenberg on domains. Its proof is based on a co-area formula and the area formula (or Cavalieri’s principle). For a proof of the following two lemmata see [KL10, Theorem 12 and 13].

**Lemma 3.6** (Co-area formula). Let \( w : X \times X \to [0, \infty) \) and \( f : X \to [0, \infty) \). Then,
\[
\frac{1}{2} \sum_{x,y \in X} w(x, y) |f(x) - f(y)| = \int_0^\infty w(\partial \Omega_t(f)) dt,
\]
where both sides may take the value \( \infty \).

**Lemma 3.7** (Area formula). Let \( m : X \to [0, \infty) \) and \( f : X \to [0, \infty) \). Then,
\[
\sum_{x \in X} m(x) f(x) = \int_0^\infty m(\Omega_t(f)) dt,
\]
where both sides may take the value \( \infty \).

In contrast to the continuous setting there is no chain rule in the discrete. The lemma below serves as a proxy of the chain rule. It is due to S. Amghibech, [Amg03, Lemma 3]. For the sake of being self-contained we give a proof which is slightly different from Amghibech’s proof and owes to [HS97].

**Lemma 3.8.** Let \( f : X \to [0, \infty) \) and \( x, y \in X \). Then, for all \( p \in [1, \infty) \),
\[
|f^p(x) - f^p(y)| \leq p \left( \frac{f^p(x) + f^p(y)}{2} \right)^{(p-1)/p} |f(x) - f(y)|.
\]

**Proof.** The statement is trivial for \( p = 1 \), so assume \( p \in (1, \infty) \). We assume without loss of generality \( f(y) \leq f(x) \) and denote \( a = f(y), b = f(x) \). Furthermore, the only non-trivial case is \( 0 < a < b \) which we assume so forth. As the function \( x \mapsto x^{p/(p-1)} \) is convex on \([0, \infty)\), we obtain by Jensen’s inequality
\[
\left( \frac{1}{p} b^p - a^p \right)^{\frac{1}{p-1}} = \left( \int_a^b t^{p-1} dt \right)^{\frac{p}{p-1}} \leq \frac{\int_a^b t^p dt}{b - a} = \frac{1}{p+1} \frac{b^{p+1} - a^{p+1}}{b - a}.
\]
We proceed by identity
\[ b^{p+1} - a^{p+1} = (b - a) (b^p + a^p) + ab (b^{p-1} - a^{p-1}) \]
which leaves us to estimate the term \( ab(b^{p-1} - a^{p-1}) \). Note that this term is non-negative for all \( p \geq 1 \). Moreover, the function \( t \mapsto t^{-p} \) is convex on \((0, \infty)\) and, thus, its image lies below the line segment connecting the points \((b^{-1}, b^{p-1})\) and \((a^{-1}, a^{p-1})\). Therefore, for \( p > 1 \), we estimate
\[
\frac{1}{(p-1)} (b^{p-1} - a^{p-1}) = \int_{b^{-1}}^{a^{-1}} t^{-p} dt \\
\leq (a^{-1} - b^{-1}) \left( \frac{(b^p - a^p)}{2} + a^p \right) \\
= \frac{1}{2ab} (b - a) (b^p + a^p).
\]
The inequality combined with the inequality and the equality above yields the statement.

\[ \square \]

**Proof of Theorem 3.5.** We calculate using the co-area formula, Lemma 3.7, and the area formula, Lemma 3.6, with \( f = |g|^p \)
\[
h_{p,w,m,G} \|g\|_p^p = h_{p,w,m,G} \sum_{x \in X} m(x) |g(x)|^p \\
= h_{p,w,m,G} \int_0^\infty m(\Omega_t(|g|^p)) dt \\
\leq \int_0^\infty w(\partial \Omega_t(|g|^p)) dt \\
= \frac{1}{2} \sum_{x,y \in X} w(x,y) ||g(x)||^p - ||g(y)||^p.
\]
Applying Lemma 3.8 we conclude
\[
\ldots \leq \frac{p}{2} \sum_{x,y \in X} w(x,y) \left( \frac{|g(x)|^p + |g(y)|^p}{2} \right)^{(p-1)/p} ||g(x)| - |g(y)|| \\
\leq \frac{p}{2} \sum_{x,y \in X} b(x,y) \sigma(x,y) \left( \frac{|g(x)|^p + |g(y)|^p}{2} \right)^{(p-1)/p} |g(x) - g(y)|,
\]
where the last inequality follows from the assumption \( w \leq b \sigma \) and \( ||b|| - ||a|| \leq |b-a| \). Applying Hölder’s inequality and using the definition
k(x) = \sum_{y \in X} (b \sigma^{p/(p-1)}(x, y)) \text{ yields}

\ldots \leq \frac{p}{2} \left( \sum_{x \in X} k(x)|g(x)|^p \right)^{(p-1)/p} \left( \sum_{x,y \in X} b(x,y)|g(x) - g(y)|^p \right)^{1/p}

= \frac{p}{2} \|g\|_{k,p}^{(p-1)} (2 \mathcal{E}_p(g))^{1/p}.

The statement follows now by taking the \( p \)-th power and dividing by \( p^p / 2^{p-1} \).

3.3. Proof of the main results.

Proof of Theorem 3.1. Let \( p \in (1, \infty) \), \( q = p/(p-1) \) and \( d \in R_p(b, m) \). With the notation of Theorem 3.3 let \( \sigma = d \) and \( w = br \). Then, \( k(x) = \sum_{y \in X} b(x,y)d^p(x,y), \) \( x \in X \). As \( d \in R_p(b, m) \), we estimate for \( \varphi \in \mathcal{C}_c(X) \)

\[ \|\varphi\|_{k,p}^p = \sum_{x \in X} |\varphi(x)|^p \sum_{y \in X} b(x,y)d^p(x,y) \leq \sum_{x \in X} |\varphi(x)|^p m(x) = \|\varphi\|_{m,p}^p \]

By Theorem 3.3 applied with \( G = \mathcal{C}_c(X) \), we obtain for all \( \varphi \in \mathcal{C}_c(X) \)

\[ \frac{2^{p-1}}{p^p} h_{p,bd,m,\mathcal{C}_c(X)}^p \|\varphi\|_{m,p}^2 \leq \|\varphi\|_{k,p}^{p(p-1)} \mathcal{E}_p(\varphi) \leq \|\varphi\|_{m,p}^{p(p-1)} \mathcal{E}_p(\varphi) \]

and by definition \( h_{p,bd,m,\mathcal{C}_c(X)} = h^{(0)}(d) \). Hence,

\[ \frac{2^{p-1}}{p^p} h^{(0)}(d)^p \leq \mathcal{E}_p(\varphi). \]

By taking the supremum over all \( d \in R_p(b, m) \) and the infimum over all \( \varphi \in \mathcal{C}_c(X) \), we arrive at the statement \( \frac{2^{p-1}}{p^p} h^{(0)}_p \leq \lambda_p^{(0)} \). \( \square \)

Proof of Theorem 3.2. Let \( p \in (1, \infty) \). Let \( f \in D_p \) be a non-constant weak solution for \( \lambda_p^{(1)} \). As \( \mathcal{E}_p(f_\pm) \leq \mathcal{E}_p(f) \) and \( \|f_\pm\|_{m,p} \leq \|f\|_{m,p} \) for all \( p \in [1, \infty) \), we find that the positive and negative parts \( f_\pm \) of \( f \) are in \( D_p \) as well. With the elementary estimate

\[ |r_+ - s_+|^p \leq |r - s|^{p-2} (r - s)(r_+ - s_+), \quad r, s \in \mathbb{R}, \]

we get for the positive part \( f_+ \)

\[ \mathcal{E}_p(f_+) \leq \mathcal{E}_p'(f) f_+ = \lambda_p^{(1)}(f_+, |f|^{p-2} f)_m = \lambda_p^{(1)} \|f_+\|_{m,p}^p \]

Since we assumed that the solution is non-constant, we deduce \( \lambda_p^{(1)} \neq 0 \) and because \( 1 \in D_p \) we infer by the definition of weak solutions

\[ \sum_X |f|^{p-2} f m = \sum_X |f|^{p-2} f 1 m = \frac{1}{\lambda_p^{(1)}} \mathcal{E}_p'(f) 1 = 0. \]
Hence, \( f \) has non-definite sign and we can assume without loss of generality that the positive part \( f_+ = f \vee 0 \) of \( f \) satisfies \( m(\text{supp}f_+) \leq m(X)/2 \). For \( G = \{ g \in D_p \mid m(\text{supp}g) \leq m(X)/2 \} \) we have that \( Y_G \subseteq \{ A \subseteq X \mid m(A) \leq m(X)/2 \} \). Hence, \( f_+ \in G \) and \( h_{p,bd,m,G} \geq h^{(1)}(d) \) for any \( d \in R_p(b,m) \). By Theorem 3.10 applied \( \sigma = d \) and \( w = b\sigma \), and the considerations above we get

\[
\frac{2^{p-1}}{p^p} h^{(1)}(d)^p \| f_+ \|_{m,p}^2 \leq \| f_+ \|_{k,p}^{p(p-1)} \mathcal{E}_p(f_+) \leq \lambda^{(1)}_p \| f_+ \|_{m,p}^2,
\]

where we used \( \| f_+ \|_{k,p} \leq \| f_+ \|_{m,p} \) in the last estimate which is implied by \( d \in R_p(b,m) \). Since \( f \) has non-definite sign as discussed above, we have \( f_+ \neq 0 \). Thus, dividing by \( \| f_+ \|_{m,p}^2 \) and taking the supremum over all \( d \in R_p(b,m) \) yields the result. \( \square \)

3.4. Upper bounds. In this subsection we show Buser-type upper bounds for \( \lambda^{(0)}_p \) and \( \lambda^{(1)}_p \) in terms of isoperimetric constants \( h^{(0)}(d) \) and \( h^{(1)}(d) \) for arbitrary functions \( d \). To this end, for a given graph \( b \), we define for \( d : X \times X \to [0, \infty) \)

\[
\delta(d) := \inf \{ d(x,y) \mid x, y \in X \text{ with } b(x,y) > 0 \}.
\]

**Theorem 3.9.** For any \( p \in [1, \infty) \) and any function \( d : X \times X \to [0, \infty) \) with \( \delta(d) > 0 \), we have

\[
\lambda^{(0)}_p \leq h^{(0)}(d) / \delta(d).
\]

**Proof.** The inequality directly follows from \( b \leq bd / \delta(d) \) and the equality

\[
\frac{\mathcal{E}_p(1_W)}{\| 1_W \|_{m,p}^p} = \frac{|\partial W|^1_b}{m(W)}
\]

with \( 1_W \) being the characteristic function of a set \( W \subseteq X \). \( \square \)

**Theorem 3.10.** Let \( m(X) < \infty \). For any \( p \in [1, \infty) \) and any function \( d : X \times X \to [0, \infty) \) with \( \delta(d) > 0 \) we have

\[
\lambda^{(1)}_p \leq 2^{p-1} h^{(1)}(d) / \delta(d).
\]

**Proof.** For any set \( W \subseteq X \) with \( m(W) \leq m(X)/2 \) we let

\[
f_W = m(X \setminus W) 1_W - m(W) 1_{X \setminus W}.
\]

Then, \( f_W \in \ell^\infty(X) \) and in fact \( f \in \ell^p(X, m) \) for all \( p \in [1, \infty] \) since \( m(X) < \infty \). Therefore,

\[
\langle f_W, 1 \rangle_m = m(W)m(X \setminus W) - m(W)m(X \setminus W) = 0.
\]
which implies \( \min_{\gamma \in \mathbb{R}} \| f - \gamma \|_{m,p} = \| f \|_{m,p} \). Moreover,
\[
\mathcal{E}_p(f_W) \frac{\| \partial W \|_b (m(X \setminus W) + m(W))^p}{\| f_W \|_{m,p}^p} = \frac{|\partial W|_b (m(X \setminus W) + m(W))^p}{m(W)m(X \setminus W)^p + m(X \setminus W)m(W)^p}.
\]
By \( m(X) < \infty \) we have \( \mathcal{E}_p(f_W) < \infty \) and \( f \in \ell^p(X, m) \) and, therefore, \( f \in D_p \). Together with the observations above this yields that \( \mathcal{E}_p(f_W) \frac{\| \partial W \|_b (m(W))^p}{\| f_W \|_{m,p}^p} \) is larger than \( \lambda_p^{(1)} \). Moreover, we apply the inequality
\[
(\alpha + \beta)^p \leq 2^{p-1}(\alpha^p + \beta^p), \quad \alpha, \beta \geq 0 \text{ and } m(W) \leq m(X)/2 \leq m(X \setminus W)
\]
to conclude
\[
\lambda_p^{(1)} \leq 2^{p-1} \frac{|\partial W|_b}{m(W)} \leq 2^{p-1} \frac{|\partial W|_b}{\delta(d)m(W)}
\]
which yields the statement. \( \square \)

**Remark 3.11.** Let us comment on the constant \( \delta(d) \) in the denominator. The upper bound becomes trivial whenever \( \delta(d) = 0 \). Suppose that \( d \in R_p(b, m) \), in which case \( h^{(0)}(d) \) and \( h^{(1)}(d) \) also appears in the lower bounds. In this case \( \delta(d) = 0 \) whenever \( \text{Deg} \) is unbounded, i.e., \( M = \sup_{x \in X} \text{Deg}(x) = \infty \): Indeed, using the definition \( \text{Deg}(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y) \), we see
\[
\text{Deg}(x) \leq \delta(d)^{-\frac{p}{p-1}} \frac{1}{m(x)} \sum_{y \in X} b(x, y)d(x, y)^{\frac{1}{p-1}} \leq \delta(d)^{-\frac{p}{p-1}}.
\]
Thus, \( M \leq \delta(d)^{-\frac{p}{p-1}} \).

If one considers the isoperimetric constant \( h = \inf_{m(W) \leq m(X)/2} \frac{|\partial W|_b}{m(W)} = h^{(1)}(1) \) from Remark 3.3(a) instead, then \( \delta(1) = 1 \) and one gets the upper bound,
\[
\lambda_p^{(1)} \leq 2^{p-1}h^{(1)}(1)
\]
cf. [BH09], which does not depend on \( M \). However, as already discussed in Remark 3.3(a) this comes at the expense of the worse lower bound \((2/M)^{p-1}(h^{(1)}(1)/p)^p\).

### 3.5. Convergence results for finite graphs.

In this section we give an alternative proof of the convergence result \( \lambda_p^{(1)} \to \lambda_1^{(1)} \), \( p \to 1 \) for finite graphs which is originally due to Bühler/Hein, [HB10].

**Theorem 3.12.** If \( X \) is finite, then \( \lim_{p \downarrow 1} \lambda_p^{(1)} = \lambda_1^{(1)} \).

**Proof.** We pick the function \( d \) given by
\[
d_p(x, y) = (\text{Deg}(x) \vee \text{Deg}(y))^{-(p-1)/p}, \quad x, y \in X,
\]
with $\text{Deg}(x) = \sum_{y \in X} b(x, y) / m(x)$. As discussed in Example 2.1(a), $d \in R_p(b, m)$. By the lower bound in Theorem 3.2 and the upper bound in Theorem 3.10, we get
\[
2^{p-1} \frac{1}{p^p} h_p^{(1)}(d_p)^p \leq \lambda_p^{(1)} \leq h^{(1)}(1) = \lambda_1^{(1)},
\]
where a proof of the equality on the right hand side can be carried over verbatim from [Chu97, Theorem 2.6] replacing the normalizing measure by general $m$. Since $X$ is finite, there are only finitely many subsets $W$ with $m(W) \leq m(X)/2$. So, since $d_p \to 1$ for $p \to 1$, we deduce $h_p^{(1)}(d_p) \to h^{(1)}(1)$ for $p \to 1$. Thus, it follows $\lambda_p^{(1)} \to \lambda_1^{(1)}$ for $p \to 1$. □

4. **Brook’s theorem**

In this section we show an estimate on $\lambda_p^{(0)}$ from above in terms of the volume growth of the graph. A result of this type was first proven by Brooks [Bro81] on manifolds and it was later improved and generalized in [LW01, Stu94]. Similar results were proven for the normalized Laplacian on graphs in [DK88, Fuj96, OU94] in the case $p = 2$ and in [Tak03] for general $p$. In [HKW13] a corresponding result for regular Dirichlet forms is proven which unifies all the above results for $p = 2$. Here, we show a analogous result for general $p$ and general $p$-Laplacians.

We define the exponential volume growth of $X$ by
\[
\mu = \liminf_{r \to \infty} \inf_{o \in X} \frac{1}{r} \log \frac{m(B_r^{(d)}(o))}{m(B_1^{(d)}(o))},
\]
where $B_r^{(d)}(o)$ is the *distance ball* with center $o$ and radius $r$ with respect to a given pseudo metric $d$.

**Theorem 4.1.** Let $p \in [1, \infty)$ and let $d$ be a pseudo metric such that all distance balls are finite and such that
\[
\sum_{y \in X} b(x, y)d(x, y)^p \leq m(x), \; x \in X.
\]
Then,
\[
\lambda_p^{(0)} \leq \frac{\mu^p}{2^{p^2}}.
\]

First, let $d$ be an arbitrary pseudo metric and $\mu$ be the exponential volume growth defined above. To ease notation we denote the $r$-balls with center $x_0 \in X$ by $B_r := B_r^{(d)}(x_0)$ whenever the pseudo metric $d$ and the center does not vary.
Next, we construct the test functions following the ideas of [HKW13]. For \( r \in \mathbb{N} \), \( x_0 \in X \), \( \alpha > 0 \), define

\[
f_{r,x_0,\alpha} : X \to [0, \infty), \quad x \mapsto (e^{\alpha r} - e^{(2r-d(x_0,x))}) - 1 \vee 0.
\]

Obviously, we have \( f|_{B_r} \equiv e^{\alpha r} - 1 \), \( f|_{B_2r \setminus B_r} = e^{\alpha(2r-r(x_0,\cdot))} - 1 \) and \( f|_{X \setminus B_{2r}} \equiv 0 \). Clearly, \( f \) is spherically homogeneous, i.e., there exists \( h : [0, \infty) \to [0, \infty) \) such that \( f(x) = h(d(x_0, x)) \).

Moreover, for \( r \in \mathbb{N} \), \( x_0 \in X \), \( \alpha > 0 \), let \( g_{r,x_0,\alpha} : X \to [0, \infty) \), be given by

\[
g_{r,x_0,\alpha} = (f_{r,x_0,\alpha} + 2)1_{B_{2r}}.
\]

In [HKW13] the following lemma is proven for \( p = 2 \). However, with the obvious modifications the proof carries over verbatim to the case of general \( p \).

**Lemma 4.2** (Lemma 2.2 in [HKW13].) Let \( p \in [1, \infty) \) If \( \alpha > \mu/p \), then there are sequences \( (x_k) \) in \( X \) and \( (r_k) \) in \( \mathbb{N} \) such that \( f_k = f_{r_k,x_k,\alpha}, g_k = g_{r_k,x_k,\alpha} \in L^p(X, m) \) and we have that \( \|g_k\|_{m,p}/\|f_k\|_{m,p} \to 1 \) as \( k \to \infty \).

The following lemma is also found in [HKW13] for \( p = 2 \). We sketch a short proof for general \( p \).

**Lemma 4.3** (Lemma 2.5 in [HKW13].) Let \( p \in [1, \infty) \), \( r \in \mathbb{N} \), \( x_0 \in X \), \( \alpha > 0 \) and set \( f := f_{r,x_0,\alpha}, g := g_{r,x_0,\alpha} \). Then, for \( x, y \in X \)

\[
(f(x) - f(y))^p \leq \frac{\alpha^p}{2}(g(x)^p + g(y)^p)d(x,y)^p.
\]

**Proof.** From the second part of the proof of Lemma 3.8 we see that

\[
|s^k - t^k| \leq k|s^{k-1} - t^{k-1}| |s - t|/2
\]

and, therefore,

\[
|e^s - e^t| \leq \sum_{k=1}^{\infty} \frac{1}{k!}|s^k - t^k| \leq \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{(k-1)!} (s^{k-1} + t^{k-1}) |s - t|
\]

\[
\leq \frac{(e^s + e^t)}{2} |s - t|
\]

for \( s, t \geq 0 \). By Jensen’s inequality we conclude

\[
|e^s - e^t|^p \leq \frac{1}{2}(e^{sp} + e^{tp}) |s - t|^p
\]

Without loss of generality we can assume \( f(x) \geq f(y) \). Now, we distinguish the six cases \( x, y \in B_r \) and \( x \in B_r, y \in B_{2r} \setminus B_r \) and \( x \in B_r, y \in X \setminus B_{2r} \) and \( x, y \in B_{2r} \) and \( x \in B_{2r} \setminus B_r, y \in X \setminus B_{2r} \) and \( x, y \in X \setminus B_{2r} \) to finish the proof. \( \square \)
Proof of Theorem 4.1. Let $p \in [1, \infty)$ and $d$ be a pseudo metric such that all distance balls have finite cardinality and such that

$$\sum_{y \in X} b(x, y)d(x, y)^p \leq m(x) \quad \text{for all } x \in X.$$ 

Let $\alpha > \mu/p$ and let $(f_k), (g_k)$ be the sequences of functions given by Lemma 4.2. We see that $f_k, g_k \in C_c(X)$, by the assumption of finiteness of the balls. Using the definition of $\lambda_p^{(0)}$, Lemma 4.3 and the assumption on $d$, we obtain

$$\lambda_p^{(0)} \frac{\|f_k\|_{m,p}^p}{\|g_k\|_{m,p}^p} \leq \frac{1}{2} \sum_{x,y \in X} b(x, y)|f_k(x) - f_k(y)|^p$$

$$\leq \frac{\alpha^p}{2} \sum_{x \in X} g_k(x)^p \sum_{y \in X} b(x, y)d(x, y)^p$$

$$\leq \frac{\alpha^p}{2} \frac{\|g_k\|_{m,p}^p}{\|g_k\|_{m,p}^p}.$$ 

Since by Lemma 4.2 $\|f_k\|_{m,p}/\|g_k\|_{m,p} \to 1$ as $k \to \infty$ we now deduce $\lambda_p^{(0)} \leq \alpha^p/2$ for all $\alpha > \mu/p$. Thus, the statement of the theorem follows. \[\square\]

5. Appendix

The discrete $p$-Laplacian $L_p$ is a quasi-linear (linear for $p = 2$) operator defined by

$$L_p f(x) = \frac{1}{m(x)} \sum_{y \in X} b(x, y)|f(x) - f(y)|^{p-2}(f(x) - f(y)).$$

on

$$F_p := \{ f \in C(X) \mid \sum_{y \in X} b(x, y)|f(y)|^{p-1} < \infty \text{ for all } x \in X \}.$$ 

The parabolic equation

$$\frac{d}{dt} u(t, x) + L_p u(t, x) = 0, \quad t > 0, \ x \in X,$$

associated with this operator has received some attention lately, cf. [Mug13, HM15] and the references therein. One can naturally regard this as an evolution equation in the Hilbert space $\ell^2(X; m)$: mimicking the techniques of [DUV04, § 4.4] one sees that its long-time behavior is determined by the Rayleigh-type quotient

$$\frac{\mathcal{E}_p(\varphi)}{\|\varphi\|_{m,2}^2}.$$
evaluated at $\varphi = u_0$, the initial data of the above problem. Observe that this functional is not homogeneous, and in fact this is not the quotient we have considered throughout this paper. Indeed, homogeneity is an important property of energy functionals and its lack significantly complicates the parabolic theory of (5.1), which suggests to introduce the relevant functional of this paper,

$$(5.2) \quad \frac{\mathcal{E}_p(\varphi)}{\|\varphi\|_{m,p}^p}.$$ 

In search of homogenization procedures that would allow for a nonlinear extension of linear Harnack inequalities, it was observed in [Tru68] that (5.1) should be replaced by what is now occasionally called the Trudinger equation: in our setting it reads

$$(5.3) \quad \frac{d}{dt}(u^{p-1})(t, x) + \mathcal{L}_p u(t, x) = 0, \quad t > 0, \ x \in X ,$$

whose corresponding eigenvalue equation (3.1) is associated with the functional in (5.2). In analogy with the classical theory of Laplacians on domains of $\mathbb{R}^d$, we actually can think of the quantities $\lambda_p^{(0)}$ and $\lambda_p^{(1)}$ introduced in this paper as the spectral gap of the Dirichlet and Neumann realizations of the $p$-Laplacian $\mathcal{L}_p$.

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