Parametrizing the lepton mixing matrix in terms of deviations from tri-bimaximal mixing

S. F. King

School of Physics and Astronomy, University of Southampton, Southampton, SO17 1BJ, U.K.

Abstract

We propose a parametrization of the lepton mixing matrix in terms of an expansion in powers of the deviations of the reactor, solar and atmospheric mixing angles from their tri-bimaximal values. We show that unitarity triangles and neutrino oscillation formulae have a very compact form when expressed in this parametrization, resulting in considerable simplifications when dealing with neutrino phenomenology. The parametrization, which is completely general, should help to establish possible relations between the deviations of the reactor, solar and atmospheric mixing angles from their tri-bimaximal values, and hence enable models which predict such relations to be more directly compared to experiment.

1E-mail: sfk@hep.phys.soton.ac.uk
Over the last decade neutrino physics has undergone a revolution with the measurement of neutrino mass and lepton mixing from a variety of solar, atmospheric and terrestrial neutrino oscillation experiments [1]. Lepton mixing is described by the $3 \times 3$ matrix [2]

$$U = \begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} \\
U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\
U_{\tau 1} & U_{\tau 2} & U_{\tau 3}
\end{pmatrix}. \quad (1)$$

The Particle Data Group (PDG) parameterization of the lepton mixing matrix (see e.g. [3]) is:

$$U = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-c_{23}s_{12} - s_{13}s_{23}c_{12}e^{i\delta} & c_{23}c_{12} - s_{13}s_{23}s_{12}e^{i\delta} & s_{23}c_{13} \\
s_{23}s_{12} - s_{13}c_{23}c_{12}e^{i\delta} & -s_{23}c_{12} - s_{13}c_{23}s_{12}e^{i\delta} & c_{23}c_{13}
\end{pmatrix}P, \quad (2)$$

where $s_{13} = \sin\theta_{13}$, $c_{13} = \cos\theta_{13}$ with $\theta_{13}$ being the reactor angle, $s_{12} = \sin\theta_{12}$, $c_{12} = \cos\theta_{12}$ with $\theta_{12}$ being the solar angle, $s_{23} = \sin\theta_{23}$, $c_{23} = \cos\theta_{23}$ with $\theta_{23}$ being the atmospheric angle, $\delta$ is the (Dirac) CP violating phase which is in principle measurable in neutrino oscillation experiments, and $P = \text{diag}(e^{i\alpha_1/2}, e^{i\alpha_2/2}, 0)$ contains additional (Majorana) CP violating phases $\alpha_1, \alpha_2$. Current data is consistent with the tri-bimaximal mixing (TBM) form [4]

$$U \approx \begin{pmatrix}
\sqrt{2}/3 & \frac{1}{\sqrt{3}} & 0 \\
-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}
\end{pmatrix}P. \quad (3)$$

Many models can account for TBM lepton mixing [5, 6, 7, 8, 9, 10, 11]. However there is no convincing reason for TBM to be exact, and in the future deviations from it are expected to be observed. With this in mind it is clearly useful to develop a parametrization of the lepton mixing matrix in which such deviations are manifest, and in which the predictions of models for deviations from tri-bimaximal mixing can naturally be expressed. Such a parametrization must be model independent, and completely general so that it can be used by experimentalists and phenomenologists in performing analyses of neutrino experiments. It must also be sufficiently simple to be useful and yet accurate enough to be reliable.

In this paper we discuss a parametrization of the lepton mixing matrix which possesses all of the above desirable features. The parametrization exploits the empirical observed closeness of lepton mixing to the TBM form, and is analogous to the Wolfenstein parametrization of quark mixing [12]. Just as the Wolfenstein parametrization is an

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1 Sometimes an alternative phase convention is chosen in which the third row of $U_{\text{MNS}}$ has its signs reversed.
expansion about the unit matrix, so the present parametrization is an expansion about the tri-bimaximal matrix. Unlike the Wolfenstein parametrization, we introduce three small parameters parametrizing the deviations of the reactor, solar and atmospheric angles from their tri-bimaximal values. The expansion works since all three parameters are empirically small, having magnitude of order the Wolfenstein parameter $\lambda \approx 0.227$ or less. A related proposal to expand the lepton mixing matrix elements about the tri-bimaximal matrix elements, using a different parametrization from that introduced here, was discussed in [13]. Other related proposals to parametrize the lepton mixing matrix have been considered in [14, 15, 16, 17, 18].

Without loss of generality we define

$$s_{13} = \frac{r}{\sqrt{2}}, \quad s_{12} = \frac{1}{\sqrt{3}}(1 + s), \quad s_{23} = \frac{1}{\sqrt{2}}(1 + a),$$

(4)

where we have introduced the three real parameters $r, s, a$ to describe the deviations of the reactor, solar and atmospheric angles from their tri-bimaximal values. Global fits of the conventional mixing angles [19] can be translated into the $2\sigma$ ranges

$$0 < r < 0.22, \quad -0.11 < s < 0.04, \quad -0.12 < a < 0.13.$$  

(5)

The empirical smallness of these parameters suggests that we consider an expansion of the lepton mixing matrix in powers of $r, s, a$ about the tri-bimaximal form. To first order in $r, s, a$ the lepton mixing matrix can be written

$$U \approx \begin{pmatrix}
\frac{\sqrt{2}}{3}(1 - \frac{1}{2}s) & \frac{1}{\sqrt{3}}(1 + s) & \frac{1}{\sqrt{2}}r e^{-i\delta} \\
-\frac{1}{\sqrt{6}}(1 + s - a + r e^{i\delta}) & \frac{1}{\sqrt{3}}(1 - \frac{1}{2}s - a - \frac{1}{2}r e^{i\delta}) & \frac{1}{\sqrt{2}}(1 + a) \\
-\frac{1}{\sqrt{6}}(1 + s + a - r e^{i\delta}) & -\frac{1}{\sqrt{3}}(1 - \frac{1}{2}s + a + \frac{1}{2}r e^{i\delta}) & \frac{1}{\sqrt{2}}(1 - a)
\end{pmatrix} P. \quad (6)$$

As in the Wolfenstein parametrization, the above parametrization of the lepton mixing matrix avoids the introduction of mixing angles, instead dealing directly with elements of the mixing matrix. Accordingly the parametrization results in considerable simplifications when dealing with neutrino phenomenology. For example, the complex elements of the quark mixing matrix can be visualized using unitarity triangles [20], which, when normalized, only depend on two parameters. The same proves to be true when using the above parametrization of the lepton mixing matrix. The sides of the unitarity triangles enter into the neutrino oscillation formulae, and consequently these are also considerably

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2I am grateful to S. Parke, Z.Z. Zing and P. Harrison for informing me about their work [14, 15, 17].

3Note that $r$ must be positive definite, while $s, a$ can take either sign. Indeed there is a preference for $s$ to be negative.

4The second order corrections are expected to be very small, of order one per cent or less, depending on the (presently constrained but undetermined) values of $r, s, a$. Throughout the main text where results are presented to first order in $r, s, a$, the second order corrections are given in Appendix [A].
Figure 1: The $\nu_2,\nu_3$ unitarity triangle. The angle $\gamma$ is equal to the CP phase $\delta$ to first order. The unknown Majorana phases just rotate the triangle in the complex plane. The rescaled triangle is oriented as shown with the opening angles unchanged, the horizontal side having unit length, and the shortest side having length $r$ to first order. Currently $0 < r < 0.22$ at $2\sigma$, and the opening angles $\alpha$, $\beta$ and $\gamma$ are all undetermined.

simplified by the new parametrization. In the remainder of the paper we shall discuss unitarity triangles and neutrino oscillation formulae using the above parametrization.

CP violation is described by the Jarlskog [21] invariant which to leading order is

$$J \approx \frac{r}{6} \sin \delta.$$ (7)

Leptonic unitarity triangles [22] may be constructed using the orthogonality of different pairs of columns or rows of the mixing matrix. Only the opening angles, side lengths and areas of the triangles have physical significance. For example the area of each unitarity triangle is $A = \frac{1}{2} |J|$ and CP violation implies that the longest side of each unitarity triangle is smaller than the sum of the other two. Current solar, reactor and atmospheric experiments directly constrain the elements $U_{e2}$, $U_{e3}$ and $U_{\mu3}$, which have a particularly simple parametrization in Eq.6. The most important unitarity triangles should therefore include all of the elements $U_{e2}$, $U_{e3}$ and $U_{\mu3}$. There are two such unitarity triangles, the $\nu_2,\nu_3$ one [16] corresponding to the orthogonality of the second and third column, and the $\nu_e,\nu_\mu$ one [23] corresponding to the orthogonality of the first and second row. Each of them has a simple expression in terms of the new parametrization, as we now discuss.

The $\nu_2,\nu_3$ triangle in Fig[1] corresponds to the unitarity relation

$$U_{e2}U_{e3}^* + U_{\mu2}U_{\mu3}^* + U_{\tau2}U_{\tau3}^* = 0.$$ (8)

To first order the sides of this unitarity triangle are given by

$$S_1 = U_{e2}U_{e3}^* \approx \frac{1}{\sqrt{6}} r e^{i \delta},$$

$$S_2 = U_{\mu2}U_{\mu3}^* \approx \frac{1}{\sqrt{6}} (1 - \frac{s}{2} - \frac{r}{2} e^{i \delta}),$$

$$S_3 = U_{\tau2}U_{\tau3}^* \approx -\frac{1}{\sqrt{6}} (1 - \frac{s}{2} + \frac{r}{2} e^{i \delta}).$$ (9)
Figure 2: The $\nu_e.\nu_\mu$ unitarity triangle. The angle $\gamma'$ is equal to the CP phase $\delta$ to first order. The unknown Majorana phases cancel. The rescaled triangle is oriented as shown with the opening angles unchanged, the horizontal side having unit length, and the shortest side having length $\frac{3}{2}r$ to first order. Currently $0 < r < 0.22$ at $2\sigma$ and the opening angles $\alpha'$, $\beta'$ and $\gamma'$ are all undetermined.

Clearly $S_1 + S_2 + S_3 = 0$ to first order. The invariant $J$ is

$$J = Im(S_1 S_2^*) = Im(S_3 S_1^*) = Im(S_2 S_3^*)$$

which yields Eq[4]. To first order the sides of this triangle are only sensitive to the solar and reactor parameters $s$ and $r$ and the phase $\delta$, with the atmospheric parameter $a$ only appearing at second order. One may rescale the sides by $S_3$

$$S_1' = \frac{U_{e2} U_{e3}^*}{U_{\tau2} U_{\tau3}^*} \approx -re^{i\delta}$$
$$S_2' = \frac{U_{\mu2} U_{e3}^*}{U_{\tau2} U_{\tau3}^*} \approx -1 + re^{i\delta}$$
$$S_3' = 1. \quad (11)$$

To first order the rescaled triangle is only sensitive to the reactor parameter $r$ and the phase $\delta$, which is the anticipated result. To second order the solar parameter $s$ (but not the atmospheric parameter $a$) appears.

The other unitarity triangle of interest is $\nu_e.\nu_\mu$ in Fig[2] corresponding to the unitarity relation

$$U_{\mu1} U_{e1}^* + U_{\mu2} U_{e2}^* + U_{\mu3} U_{e3}^* = 0. \quad (12)$$

To first order the sides of this unitarity triangle are given by

$$T_1 = U_{\mu1} U_{e1}^* \approx \frac{1}{3}(1 + \frac{s}{2} - a + re^{i\delta})$$
$$T_2 = U_{\mu2} U_{e2}^* \approx \frac{1}{3}(1 + \frac{s}{2} - a - \frac{r}{2} e^{i\delta})$$
$$T_3 = U_{\mu3} U_{e3}^* \approx \frac{1}{2} re^{i\delta}. \quad (13)$$
Clearly $T_1 + T_2 + T_3 = 0$ to first order. The invariant $J$ is

$$J = Im(T_3T_2^*) = Im(T_1T_3^*) = Im(T_2T_1^*)$$

(14)

which again yields Eq.7. Unlike the previous case, the sides of this triangle are sensitive to the atmospheric parameter $a$ at first order. One may rescale the sides by

$$T_1' = 1$$

$$T_2' = \frac{U_{\mu 2} U_{e 2}^*}{U_{\mu 1} U_{e 1}^*} \approx -1 + \frac{3}{2} r e^{i \delta}$$

$$T_3' = \frac{U_{\mu 3} U_{e 3}^*}{U_{\mu 1} U_{e 1}^*} \approx -\frac{3}{2} r e^{i \delta}.$$  

(15)

As in the previous case, to first order the rescaled triangle is only sensitive to the reactor parameter $r$ and the phase $\delta$, which is the anticipated result. To second order the solar parameter $s$ and the atmospheric parameter $a$ appear.

We now turn to the application of the parametrization in Eq.1 to neutrino oscillations. Let us denote by $P_{\alpha \beta} = P(\nu_\alpha \rightarrow \nu_\beta)$ the probability of transition from a neutrino flavour $\alpha$ to a neutrino flavour $\beta$. Then expanding to second order in the parameters $r, s, a$ and $\Delta_{21}$, where it is assumed that $\Delta_{21} \ll 1$ as in [24], we find considerably simplified vacuum oscillation probabilities.

The electron anti-neutrino disappearance probability relevant for a reactor experiment [25] is given to second order in $r, s, a$ and $\Delta_{21}$ as

$$P_{ee} = 1 - 2 r^2 \sin^2 \Delta_{31} - \frac{8}{9} \Delta_{21}^2$$

(16)

where $\Delta_{ij} = 1.27 \Delta m^2_{ij} L/E$ with $L$ the oscillation length in km, $E$ the beam energy in GeV, and $\Delta m^2_{ij} = m_i^2 - m_j^2$ in eV$^2$. Note that this disappearance probability is independent of the solar and atmospheric parameters $s, a$, as well as the phase $\delta$, to this order.

The electron neutrino appearance probability relevant for a forthcoming long baseline muon neutrino beam experiment [26] is given to second order in $r, s, a$ and $\Delta_{21}$ as

$$P_{\mu e} = r^2 \sin^2 \Delta_{31} + \frac{4}{9} \Delta_{21}^2 + \frac{4}{3} r \Delta_{21} \sin \Delta_{31} \cos (\Delta_{31} + \delta).$$

(17)

It is also independent of the solar and atmospheric parameters $s, a$ and only depends on the reactor parameter $r$ and the phase $\delta$ to this order. The reason is that each of the terms is second order in the parameters $r, \Delta_{21}$, so any deviations from tri-bimaximal solar

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5 Similar considerations apply to oscillations in the presence of matter as discussed in Appendix B.
or atmospheric mixing only appear at third order. The muon neutrino disappearance probability is given to second order in $r, s, a$ and $\Delta_{21}$ as

$$P_{\mu\mu} = 1 - (1 - 4a^2) \sin^2 \Delta_{31} - \frac{2}{9} (1 + 3 \cos 2\Delta_{31}) \Delta_{21}^2$$

$$+ \frac{2}{3} (1 - s - r \cos \delta) \Delta_{21} \sin 2\Delta_{31}. \quad (18)$$

Muon neutrino disappearance is clearly sensitive to deviations from tri-bimaximal mixing, since all three parameters $r, s, a$ and the phase $\delta$ appear. For example the prospects for measuring deviations from maximal atmospheric mixing in the next generation of long baseline muon neutrino beam experiments has recently been discussed [27]. Similarly the tau neutrino appearance probability is given to second order in $r, s, a$ and $\Delta_{21}$ as

$$P_{\mu\tau} = (1 - 4a^2 - r^2) \sin^2 \Delta_{31} - \frac{2}{9} (1 - 3 \cos 2\Delta_{31}) \Delta_{21}^2$$

$$- \frac{2}{3} (1 - s) \Delta_{21} \sin 2\Delta_{31} + \frac{4}{3} r \Delta_{21} \sin^2 \Delta_{31} \sin \delta. \quad (19)$$

We emphasize that the parametrization discussed here is completely general and is not based on the ansatz of tri-bimaximal mixing, any more than the Wolfenstein parametrization [12] is based on the ansatz that the quark mixing matrix is equal to the unit matrix. Just as the Wolfenstein parametrization is an expansion about the unit matrix, so this parametrization is an expansion about the tri-bimaximal matrix. Unlike the Wolfenstein parametrization, there are three small parameters $r, s, a$ parametrizing the reactor, solar and atmospheric deviations from tri-bimaximal mixing. The expansion works since the deviations from tri-bimaximal mixing are empirically small parameters with $r, s, a$ all having magnitude of order the Wolfenstein parameter $\lambda \approx 0.227$ or less. Indeed these parameters are sufficiently small that the first order approximation is accurate enough for many purposes, resulting in quite a simple looking lepton mixing matrix in Eq. 6, for example. Unitarity triangles and neutrino oscillation formulae also have a very simple form when expressed in this parametrization.

The three parameters $r, s, a$ are not determined at the present time, and it is even possible that one or more of them (possibly all of them) are zero, although this seems a priori unlikely. However, as mentioned, many speculations appear in the literature as to the origin and nature of tri-bimaximal mixing and the deviations from it, and these speculations naturally find expression in this parametrization. For example certain classes of unified flavour models [5] predict a sum rule which relates $s$ to $r$ and $\delta$, namely $s \approx r \cos \delta$, where $r \approx \lambda/3$ and $a = O(\lambda^2)$. Alternatively it has been suggested [16] that trimaximal solar mixing is exact, $s = 0$, with $a \approx -\frac{1}{2} r \cos \delta$ and $r$ unspecified. Clearly an important goal of the next generation of neutrino experiments must be to show that the parameters $r, s, a$ differ from zero. Subsequent high precision neutrino experiments will then be required to accurately measure the values of the parameters $r, s, a$, as well
as δ, to investigate their possible relationships to each other and to the Wolfenstein parameter \( \lambda \).

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**Appendix**

**A Second order corrections**

In this appendix we list the second order corrections to all the results given in the main text. The second order corrections to the first order MNS matrix elements in Eq\[5\] are

\[
\begin{align*}
\Delta U_{e1} & \approx \sqrt{\frac{2}{3}} (-\frac{1}{4} r^2 - \frac{3}{8} s^2) \\
\Delta U_{e2} & \approx \frac{1}{\sqrt{3}} (-\frac{1}{4} r^2) \\
\Delta U_{e3} & \approx 0 \\
\Delta U_{\mu_1} & \approx -\frac{1}{\sqrt{6}} \left( \frac{1}{2} r se^{i\delta} - ra e^{i\delta} + sa + a^2 \right) \\
\Delta U_{\mu_2} & \approx \frac{1}{\sqrt{3}} \left( -\frac{1}{2} r se^{i\delta} - \frac{1}{2} rae^{i\delta} + \frac{1}{2} sa - \frac{3}{8} s^2 - a^2 \right) \\
\Delta U_{\mu_3} & \approx \frac{1}{\sqrt{2}} (-\frac{1}{4} r^2) \\
\Delta U_{\tau_1} & \approx \frac{1}{\sqrt{6}} \left( \frac{1}{2} r se^{i\delta} + rae^{i\delta} + sa \right) \\
\Delta U_{\tau_2} & \approx -\frac{1}{\sqrt{3}} \left( \frac{1}{2} r se^{i\delta} - \frac{1}{2} rae^{i\delta} - \frac{1}{2} sa - \frac{3}{8} s^2 \right) \\
\Delta U_{\tau_3} & \approx \frac{1}{\sqrt{2}} (-\frac{1}{4} r^2 - a^2). 
\end{align*}
\]

The second order correction to the Jarlskog CP invariant in Eq\[7\] is

\[
\Delta J \approx \frac{rs}{12} \sin \delta. 
\]
The second order corrections to the unscaled sides of the $\nu_2.\nu_3$ unitarity triangle in Eq.9 are

$$\Delta S_1 \approx \frac{1}{\sqrt{6}} s r e^{i\delta}$$

$$\Delta S_2 \approx -\frac{1}{\sqrt{6}} \left( \frac{r^2}{4} + 2a^2 + \frac{3}{8} s^2 + a r e^{i\delta} + \frac{1}{2} s r e^{i\delta} \right)$$

$$\Delta S_3 \approx \frac{1}{\sqrt{6}} \left( \frac{r^2}{4} + 2a^2 + \frac{3}{8} s^2 + a r e^{i\delta} - \frac{1}{2} s r e^{i\delta} \right). \quad (22)$$

The second order corrections to the normalized sides of the $\nu_2.\nu_3$ unitarity triangle in Eq.11 are

$$\Delta S'_1 \approx \frac{r^2}{2} e^{2i\delta} - \frac{3}{2} s r e^{i\delta}$$

$$\Delta S'_2 \approx -\frac{r^2}{2} e^{2i\delta} + \frac{3}{2} s r e^{i\delta}$$

$$\Delta S'_3 = 0. \quad (23)$$

The second order corrections to the unscaled sides of the $\nu_e.\nu_\mu$ unitarity triangle in Eq.13 are

$$\Delta T_1 \approx \frac{r^2}{12} + \frac{7}{24} s^2 + \frac{a^2}{3} + \frac{sa}{6} + \frac{sr}{6} e^{i\delta} - \frac{ar}{3} e^{i\delta}$$

$$\Delta T_2 \approx -\frac{r^2}{12} - \frac{7}{24} s^2 - \frac{a^2}{3} - \frac{sa}{6} - \frac{sr}{6} e^{i\delta} - \frac{ar}{3} e^{i\delta}$$

$$\Delta T_3 \approx \frac{ar}{2} e^{i\delta}. \quad (24)$$

The second order corrections to the normalized sides of the $\nu_e.\nu_\mu$ unitarity triangle in Eq.15 are

$$\Delta T'_1 = 0$$

$$\Delta T'_2 \approx -\frac{3}{4} r e^{i\delta} (2r e^{i\delta} + s - 4a)$$

$$\Delta T'_3 \approx \frac{3}{4} r e^{i\delta} (2r e^{i\delta} + s - 4a). \quad (25)$$

**B Neutrino oscillations in matter**

In this appendix we present the complete formulae for neutrino oscillations in the presence of matter of constant density to second order in the quantities $r, s, a$ and $\Delta_{21}$, where
it is assumed that $\Delta_{21} \ll 1$ as in [24]. Following [24] we write $\Delta = \Delta_{31}$, $\alpha = \frac{\Delta m_{31}^2}{\Delta m_{21}^2}$ and $A = \frac{V L}{2 \Delta}$ where $V$ is the potential expressed in units of eV as

$$V \approx 7.56 \times 10^{-14} \rho Y_e$$

where $\rho$ is the matter density of the Earth in units of g/cm$^3$ and $Y_e \approx 0.5$ is the number of electrons per nucleon in the Earth. The constant density approximation is good when the neutrino beam only passes through the Earth’s crust where $\rho \approx 3$ g/cm$^3$ or the Earth’s mantle where $\rho \approx 45$ g/cm$^3$.

The complete set of neutrino oscillation probabilities for electron neutrino or muon neutrino beams in the presence of matter of constant density to second order in the parameters $r, s, a$ and $\alpha$ are

$$P_{e\nu} = 1 - \frac{8}{9} \alpha^2 \sin^2 \frac{A \Delta}{A^2} - 2 r^2 \sin^2 (A - 1) \Delta \frac{A}{(A - 1)^2}.$$  \hfill (27)

$$P_{e\mu} = \frac{4}{9} \alpha^2 \sin^2 \frac{A \Delta}{A^2} + r^2 \sin^2 (A - 1) \Delta \frac{A}{(A - 1)^2} + \frac{4}{3} r \alpha \cos (\Delta - \delta) \sin \frac{A \Delta}{A} \sin \frac{(A - 1) \Delta}{(A - 1)}.$$  \hfill (28)

$$P_{e\tau} = \frac{4}{9} \alpha^2 \sin^2 \frac{A \Delta}{A^2} + r^2 \sin^2 (A - 1) \Delta \frac{A}{(A - 1)^2} - \frac{4}{3} r \alpha \cos (\Delta - \delta) \sin \frac{A \Delta}{A} \sin \frac{(A - 1) \Delta}{(A - 1)}.$$  \hfill (29)

$$P_{\mu e} = \frac{4}{9} \alpha^2 \sin^2 \frac{A \Delta}{A^2} + r^2 \sin^2 (A - 1) \Delta \frac{A}{(A - 1)^2} + \frac{4}{3} r \alpha \cos (\Delta + \delta) \sin \frac{A \Delta}{A} \sin \frac{(A - 1) \Delta}{(A - 1)}.$$  \hfill (30)
\[ P_{\mu\nu} = 1 - (1 - 4a^2) \sin^2 \Delta + \frac{2}{3} (1 - s) \alpha \Delta \sin 2\Delta \]

\[
- \frac{4}{9} \alpha^2 \frac{\sin^2 A\Delta}{A^2} - \frac{4}{9} \alpha^2 \Delta^2 \cos 2\Delta \\
+ \frac{4}{9} \alpha^2 \frac{1}{A} \left( \sin \Delta \frac{\sin A\Delta}{A} \cos(A - 1)\Delta - \frac{\Delta}{2} \sin 2\Delta \right) \\
- r^2 \sin^2 (A - 1)\Delta \]

\[
\frac{1}{A - 1} r^2 \left( \sin \Delta \cos A\Delta \frac{\sin(A - 1)\Delta}{(A - 1)} - \frac{A}{2} \Delta \sin 2\Delta \right) \\
- \frac{4}{3} r\alpha \cos \delta \cos A\Delta \sin(A - 1)\Delta \]

(31)

\[ P_{\mu\tau} = (1 - 4a^2) \sin^2 \Delta - \frac{2}{3} (1 - s) \alpha \Delta \sin 2\Delta + \frac{4}{9} \alpha^2 \Delta^2 \cos 2\Delta \]

\[
- \frac{4}{9} \alpha^2 \frac{1}{A} \left( \sin \Delta \frac{\sin A\Delta}{A} \cos(A - 1)\Delta - \frac{\Delta}{2} \sin 2\Delta \right) \\
+ \frac{1}{A - 1} r^2 \left( \sin \Delta \cos A\Delta \frac{\sin(A - 1)\Delta}{(A - 1)} - \frac{A}{2} \Delta \sin 2\Delta \right) \\
+ \frac{4}{3} r\alpha \sin \delta \sin A\Delta \sin(A - 1)\Delta \]

(32)

References

[1] For recent reviews see e.g. J. N. Bahcall. [arXiv:physics/0406040], R. N. Mohapatra et al., [arXiv:hep-ph/0510213], R. N. Mohapatra and A. Y. Smirnov, Ann. Rev. Nucl. Part. Sci. 56 (2006) 569 [arXiv:hep-ph/0603118]; S. F. King, Rept. Prog. Phys. 67 (2004) 107 [arXiv:hep-ph/0310204]; G. Altarelli and F. Feruglio, New J. Phys. 6 (2004) 106 [arXiv:hep-ph/0405048].

[2] Z. Maki, M. Nakagawa and S. Sakata, Prog. Theo. Phys. 28 (1962) 247; B. W. Lee, S. Pakvasa, R. E. Shrock and H. Sugawara, Phys. Rev. Lett. 38 (1977) 937 [Erratum-ibid. 38 (1977) 1230].

[3] W.-M. Yao et al. [Particle Data Group Collaboration], J. Phys. G 33 (2006) 1.

[4] P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B 530 (2002) 167 [arXiv:hep-ph/0202074]; P. F. Harrison and W. G. Scott, Phys. Lett. B 535 (2002) 163 [arXiv:hep-ph/0203209]; P. F. Harrison and W. G. Scott, Phys. Lett. B 557 (2003) 76 [arXiv:hep-ph/0302025]; an earlier related ansatz was proposed by: L. Wolfenstein, Phys. Rev. D 18 (1978) 958.
[5] S. F. King, JHEP 0508 (2005) 105 [arXiv:hep-ph/0506297]; I. Masina, Phys. Lett. B 633 (2006) 134 [arXiv:hep-ph/0508031]; S. Antusch and S. F. King, Phys. Lett. B 631 (2005) 42 [arXiv:hep-ph/0508044]; S. Antusch, P. Huber, S. F. King and T. Schwetz, JHEP 0704 (2007) 060 [arXiv:hep-ph/0702286].

[6] P. H. Frampton, S. T. Petcov and W. Rodejohann, Nucl. Phys. B 687 (2004) 31 [arXiv:hep-ph/0401206]; A. Dighe, S. Goswami and W. Rodejohann, Phys. Rev. D 75 (2007) 073023 [arXiv:hep-ph/0612328]; F. Plentinger and W. Rodejohann, Phys. Lett. B 625 (2005) 264 [arXiv:hep-ph/0507143]; R. N. Mohapatra and W. Rodejohann, Phys. Rev. D 72 (2005) 053001 [arXiv:hep-ph/0507312]; K. A. Hochmuth, S. T. Petcov and W. Rodejohann, arXiv:0706.2975 [hep-ph].

[7] G. Altarelli, F. Feruglio and Y. Lin, Nucl. Phys. B 775 (2007) 31 [arXiv:hep-ph/0610165]; G. Altarelli and F. Feruglio, Nucl. Phys. B 741 (2006) 215 [arXiv:hep-ph/0512103]; G. Altarelli and F. Feruglio, Nucl. Phys. B 720 (2005) 64 [arXiv:hep-ph/0504165]; F. Feruglio, C. Hagedorn, Y. Lin and L. Merlo, Nucl. Phys. B 775 (2007) 120 [arXiv:hep-ph/0702194].

[8] E. Ma, arXiv:0709.0507 [hep-ph]; E. Ma, arXiv:hep-ph/0701016; E. Ma, Mod. Phys. Lett. A 22 (2007) 101 [arXiv:hep-ph/0610342]; E. Ma, Mod. Phys. Lett. A 21 (2006) 2931 [arXiv:hep-ph/0607190]; E. Ma, Mod. Phys. Lett. A 21 (2006) 1917 [arXiv:hep-ph/0607056]; E. Ma, H. Sawanaka and M. Tanimoto, Phys. Lett. B 641 (2006) 301 [arXiv:hep-ph/0606103]; E. Ma, Phys. Rev. D 73 (2006) 057304; B. Adhikary, B. Brahmachari, A. Ghosal, E. Ma and M. K. Parida, Phys. Lett. B 638 (2006) 345 [arXiv:hep-ph/0603059]; E. Ma, Mod. Phys. Lett. A 20 (2005) 2601 [arXiv:hep-ph/0508099]; E. Ma, Phys. Rev. D 72 (2005) 037301 [arXiv:hep-ph/0505209]; S. L. Chen, M. Frigerio and E. Ma, Nucl. Phys. B 724 (2005) 423 [arXiv:hep-ph/0504181]; E. Ma, Phys. Rev. D 70 (2004) 031901 [arXiv:hep-ph/0404199].

[9] I. de Medeiros Varzielas and G. G. Ross, Nucl. Phys. B 733 (2006) 31 [arXiv:hep-ph/0507176]; I. de Medeiros Varzielas, S. F. King and G. G. Ross, Phys. Lett. B 644 (2007) 153 [arXiv:hep-ph/0512313]; I. de Medeiros Varzielas, S. F. King and G. G. Ross, Phys. Lett. B 648 (2007) 201 [arXiv:hep-ph/0607045]; S. F. King and M. Malinsky, Phys. Lett. B 645 (2007) 351 [arXiv:hep-ph/0610250]; S. F. King and M. Malinsky, JHEP 0611 (2006) 071 [arXiv:hep-ph/0608021]; C. Luhn, S. Nasri and P. Ramond, Phys. Lett. B 652 (2007) 27 [arXiv:0706.2341 [hep-ph]].

[10] P. F. Harrison and W. G. Scott, Phys. Lett. B 557 (2003) 76 [arXiv:hep-ph/0302025]; P. F. Harrison and W. G. Scott, Phys. Lett. B 535 (2002) 163 [arXiv:hep-ph/0203209]; R. N. Mohapatra, S. Nasri and H. B. Yu, Phys. Lett. B 639 (2006) 318 [arXiv:hep-ph/0605020]; R. N. Mohapatra and H. B. Yu,
Phys. Lett. B 644 (2007) 346 [arXiv:hep-ph/0610023]; M. C. Chen and K. T. Mahanthappa, Phys. Lett. B 652 (2007) 34 [arXiv:0705.0714 [hep-ph]]; C. I. Low and R. R. Volkas, Phys. Rev. D 68 (2003) 033007 [arXiv:hep-ph/0305243]; X. G. He, Nucl. Phys. Proc. Suppl. 168 (2007) 350 [arXiv:hep-ph/0612080]; A. Aranda, arXiv:0707.3661 [hep-ph].

[11] A. H. Chan, H. Fritzsch and Z. z. Xing, arXiv:0704.3153 [hep-ph]; Z. z. Xing, Phys. Lett. B 618 (2005) 141 [arXiv:hep-ph/0503200]; Z. z. Xing, H. Zhang and S. Zhou, Phys. Lett. B 641 (2006) 189 [arXiv:hep-ph/0607091]; S. K. Kang, Z. z. Xing and S. Zhou, Phys. Rev. D 73 (2006) 013001 [arXiv:hep-ph/0511157]; S. Luo and Z. z. Xing, Phys. Lett. B 632 (2006) 341 [arXiv:hep-ph/0509065]; M. Hirsch, E. Ma, J. C. Romao, J. W. F. Valle and A. Villanova del Moral, Phys. Rev. D 75 (2007) 053006 [arXiv:hep-ph/0606082]; N. N. Singh, M. Rajkhowa and A. Borah, arXiv:hep-ph/0603189; X. G. He and A. Zee, Phys. Lett. B 645 (2007) 427 [arXiv:hep-ph/0607163]; N. Haba, A. Watanabe and K. Yoshioka, Phys. Rev. Lett. 97 (2006) 041601 [arXiv:hep-ph/0603116].

[12] L. Wolfenstein, Phys. Rev. Lett. 51 (1983) 1945.

[13] N. Li and B. Q. Ma, Phys. Rev. D 71 (2005) 017302 [arXiv:hep-ph/0412126].

[14] Talk by S. Parke at WIN’05, 20th International Workshop on Weak Interactions and Neutrinos, European Cultural Center, Delphi, Greece, June 6–11, 2005, http://conferences.phys.uoa.gr/win05/

[15] Z. z. Xing, Phys. Lett. B 533 (2002) 85 [arXiv:hep-ph/0204049].

[16] J. D. Bjorken, P. F. Harrison and W. G. Scott, Phys. Rev. D 74 (2006) 073012 [arXiv:hep-ph/0511201];

[17] Talk by P. F. Harrison, “Deviations from Tri-bimaximal Mixing”, Rutherford Appleton Laboratory, U.K., April 24 - 28, 2006, http://www.hep.ph.ic.ac.uk/uknfic/iss0406/physics.html

[18] A. Datta, L. Everett and P. Ramond, Phys. Lett. B 620 (2005) 42 [arXiv:hep-ph/0503222]; L. L. Everett, Phys. Rev. D 73 (2006) 013011 [arXiv:hep-ph/0510256].

[19] M. Maltoni, T. Schwetz, M. A. Tortola and J. W. F. Valle, New J. Phys. 6 (2004) 122 [arXiv:hep-ph/0405172].

[20] L. L. Chau and W. Y. Keung, Phys. Rev. Lett. 53 (1984) 1802; J. D. Bjorken, Phys. Rev. D 39 (1989) 1396; C. Jarlskog and R. Stora, Phys. Lett. B 208 (1988) 268; G. C. Branco and L. Lavoura, Phys. Lett. B 208 (1988) 123.
[21] C. Jarlskog, Phys. Rev. Lett. 55 (1985) 1039; C. Jarlskog, Z. Phys. C 29 (1985) 491; For a recent review see: C. Jarlskog, Phys. Scripta T127 (2006) 64 [arXiv:hep-ph/0606050].

[22] H. Fritzsch and Z. z. Xing, Prog. Part. Nucl. Phys. 45 (2000) 1 [arXiv:hep-ph/9912358]; H. Zhang and Z. z. Xing, Eur. Phys. J. C 41 (2005) 143 [arXiv:hep-ph/0411183]; Z. z. Xing and H. Zhang, Phys. Lett. B 618 (2005) 131 [arXiv:hep-ph/0503118].

[23] Y. Farzan and A. Y. Smirnov, Phys. Rev. D 65 (2002) 113001 [arXiv:hep-ph/0201105]; J. A. Aguilar-Saavedra and G. C. Branco, Phys. Rev. D 62 (2000) 096009 [arXiv:hep-ph/0007025].

[24] E. K. Akhmedov, R. Johansson, M. Lindner, T. Ohlsson and T. Schwetz, JHEP 0404 (2004) 078 [arXiv:hep-ph/0402175].

[25] M. Apollonio et al. [CHOOZ Collaboration], Phys. Lett. B 466 (1999) 415 [arXiv:hep-ex/9907037]; M. Goodman, Nucl. Phys. Proc. Suppl. 145 (2005) 186 [arXiv:hep-ph/0501206].

[26] Y. Hayato [T2K Collaboration], Nucl. Phys. Proc. Suppl. 147 (2005) 9. D. S. Ayres et al. [NOvA Collaboration], [arXiv:hep-ex/0503053]

[27] S. Antusch, P. Huber, J. Kersten, T. Schwetz and W. Winter, Phys. Rev. D 70 (2004) 097302 [arXiv:hep-ph/0404268].