LETTER TO THE EDITOR

Solvable nonlinear discrete-time evolutions and Diophantine findings

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Certain nonlinearly-coupled systems of $N$ discrete-time evolution equations are identified, which can be solved by algebraic operations; and some remarkable Diophantine findings are thereby obtained. These results might be useful to test the accuracy of numerical routines yielding the $N$ roots of polynomials of arbitrary degree $N$.

1. Introduction and notation

As the reader will easily see, the results of this paper amount to a transfer—from continuous to discrete time, via the approach introduced in [1]—of the findings reported in [2] and [3] (see also Chapters 3 and 7 of [4]).

Throughout this paper the following notation is used: $N$ and $L$ are two arbitrary positive integers ($N \geq 2, L \geq 2$), the indices $n$ and $m$ run from 1 to $N$, the discrete-time variable $\ell = 0, 1, 2, \ldots$ takes all nonnegative integer values, the $N$ dependent variables $z_n(\ell)$ are generally complex numbers and, being generally defined (see below) as the $N$ zeros of a polynomial of degree $N$ in its (complex) argument $z$, they are the elements of an unordered set of $N$ elements identified hereafter with the notation $\tilde{z}(\ell)$; likewise in the following the notation $\tilde{f}$ denotes the unordered set of $N$ elements $f_n$.

2. Results

Proposition 2.1. Consider the system of $N$ second-order discrete-time evolution equations

$$2 \prod_{m=1}^{N} [z_n(\ell + 2) - z_m(\ell + 1)] - \prod_{m=1}^{N} [z_n(\ell + 2) - z_m(\ell)] = 0 ; \quad (2.1a)$$

note that this formula provides the unordered set $\tilde{z}(\ell + 2)$, the elements of which are the $N$ values $z_n(\ell + 2)$, as the $N$ zeros of the polynomial of degree $N$ in $z$ defined in terms of the two unordered sets $\tilde{z}(\ell)$ and $\tilde{z}(\ell + 1)$ as follows:

$$P_N (z ; \tilde{z}(\ell) , \tilde{z}(\ell + 1) ) = 2 \prod_{m=1}^{N} [z - z_m(\ell + 1)] - \prod_{m=1}^{N} [z - z_m(\ell)] . \quad (2.1b)$$

Let this system of second-order discrete-time evolution equations, (2.1a), be complemented by the following assignments of the two unordered sets $\tilde{z}(0)$ respectively $\tilde{z}(1)$ of $2N$ initial data $z_n(0)$
respectively \( z_n(1) \): (i) the \( N \) data \( z_n(0) \) are assigned arbitrarily; (ii) the \( N \) data \( z_n(1) \) are defined—in terms of the parameter \( L, (L \neq 1) \), the unordered set \( \tilde{z}(0) \), and the unordered set \( \tilde{f} \) the elements of which are \( N \) arbitrarily assigned (generally complex) numbers \( f_m \)—by the \( N \) algebraic equations

\[
\prod_{m=1}^{N} [z_n(1) - z_m(0)] + \frac{(-1)^N}{L-1} \prod_{m=1}^{N} [z_n(1) - f_m] = 0 ;
\]

(2.2a)

hence these \( N \) data \( z_n(1) \) are the \( N \) roots of the polynomial \( p_{N}^{(1)}(z;\tilde{z}(0),\tilde{f};L) \), of degree \( N \) in \( z \), defined as follows in terms of the two unordered sets \( \tilde{z}(0) \) and \( \tilde{f} \):

\[
p_{N}^{(1)}(z;\tilde{z}(0),\tilde{f};L) = \prod_{m=1}^{N} [z - z_m(0)] + \frac{(-1)^N}{L-1} \prod_{m=1}^{N} [z - f_m] .
\]

(2.2b)

The solution \( z(\ell) \) of the system of second-order discrete-time evolution equations (2.1a) is then given by the \( N \) roots of the following polynomial of degree \( N \) in \( z \):

\[
\psi_{N}(z;\tilde{z}(0),\tilde{f};L; \ell) = \left( \frac{L-\ell}{L} \right) \prod_{m=1}^{N} [z - z_m(0)] + \left( \frac{\ell}{L} \right) (-1)^N \prod_{m=1}^{N} (z - f_m) .
\]

(2.3)

Proposition 2.1 is proven in the following Section. In the meantime the reader may immediately verify the validity of the formula (2.3) at \( \ell = 0 \) and—via (2.2b)—at \( \ell = 1 \).

**Remark 2.1.** Note that—while in the formulation of this Proposition 2.1 we considered the system of \( N \) equations (2.2) as determining the \( N \) elements of the unordered set \( \tilde{z}(1) \) in terms of the \( 2N \) elements of the two, arbitrarily assigned, unordered sets \( \tilde{z}(0) \) and \( \tilde{f} \), this system (2.2) of \( N \) algebraic equations might as well be considered to define the \( N \) elements \( f_m \) of the unordered set \( \tilde{f} \) in terms of the \( 2N \) elements of the two—both then arbitrarily assigned—unordered sets \( \tilde{z}(0) \) and \( \tilde{z}(1) \). □

**Corollary 2.1.** At \( \ell = L \) the unordered set \( \tilde{z}(L) \) coincides with the unordered set \( \tilde{f} \):

\[
\tilde{z}(L) \equiv \tilde{f} .
\]

(2.4)

The validity of this Corollary 2.1 is an immediate consequence of the Proposition 2.1, being obtained by setting \( \ell = L \) in (2.3). And it has an obvious Diophantine implication if the \( N \), a priori arbitrary, numbers \( f_m \) are chosen to be integers or rationals.

### 3. Proof

The starting point of the proof of Proposition 2.1 is the definition (2.3) of the polynomial \( \psi_{N}(z;\ell) \). The consistency of this definition with the assignment of the initial data \( \tilde{z}(0) \) and \( \tilde{z}(1) \) has already been noted above. What remains to be proven is that the formula

\[
\psi_{N}(z;\ell) = \prod_{n=1}^{N} [z - z_n(\ell)]
\]

(3.1)

—which, with \( \psi_{N}(z;\ell) \) defined by (2.3), clearly coincides with the statement of Proposition 2.1—implies that the \( N \) zeros \( z_n(\ell) \) satisfy the evolution equation (2.1a). To this end we note that since by
definition (see (2.3)) the dependence of $\psi_N(z;\ell)$ on the discrete-time variable $\ell$ is linear, $\psi_N(z;\ell)$ satisfies identically the linear second-order difference equation

$$
\psi_N(z;\ell + 2) - 2\psi_N(z;\ell + 1) + \psi_N(z;\ell) = 0 .
$$

(3.2)

For $z = z_n(\ell + 2)$, via (3.1), this formula implies (2.1a).

Q. E. D.

4. Envoy

The result reported in the above Proposition 2.1 is likely to look, at least at first sight, somewhat remarkable, especially in view of the arbitrariness of the assignment of the $2N$ numbers $z_n(0)$ and $f_n$ (or, equivalently, $z_n(0)$ and $z_n(1)$; see the above Remark 2.1). But of course, after its validity has been proven, it shall be considered obvious—as all valid mathematical results in some sense are. A potential application of this finding is as a tool to test the accuracy of numerical routines to compute the zeros of polynomials of arbitrary degree $N$: by comparing, with the simple explicit outcome detailed in the above Corollary 2.1, the results yielded by the application of such routines in order to solve numerically—from the initial data detailed in Proposition 2.1, up to $\ell = L$—the discrete-time evolution (2.1); which indeed requires finding the zeros of appropriate polynomials of degree $N$ at every step of this discrete-time evolution. In this context the flexibility implied by the possibility to assign arbitrarily the two integers $N$ and $L$ and the $2N$, generally complex, numbers $z_n(0)$ and $f_n$ might be quite useful. Specialists in numerical analysis might be interested to explore in detail the vistas implied by such possibilities: note for instance that, for $N = 20$ and $f_m = m$, Corollary 2.1—for any arbitrary assignment of the parameters $L$ and $x_n(0)$—yields the 20 zeros of the perfidious Wilkinson polynomial [5].

An extension of the findings reported in this paper to the case in which the finite positive integer $N$ is replaced by $\infty$ is of course possible, see [2].

A (perhaps less elegant) variant of the approach described in this paper—characterized by the replacement of the system of second-order discrete-time evolution equations (2.1a) by systems of first-order discrete-time evolution equations—is of course possible, in analogy to the treatments of the continuous-time cases, see [1], [2] and Chapter 3 of [4].

References

[1] O. Bihun and F. Calogero, “Generations of solvable discrete-time dynamical systems”, *J. Math. Phys.* **58**, 052701 (21 pages) (2017); DOI: 10.1063/1.4982959.

[2] F. Calogero, “Finite and infinite systems of nonlinearly coupled ordinary differential equations the solutions of which feature remarkable Diophantine findings”, *J. Nonlinear Math. Phys.* **25** (3) (2018) 433–441.

[3] F. Calogero, “Novel differential algorithm to evaluate all the zeros of any generic polynomial”, *J. Nonlinear Math. Phys.* **24**, 469–472 (2017). DOI: 10.1080/14029251.2017.1375685.

[4] F. Calogero, *Zeros of Polynomials and Solvable Nonlinear Evolution Equations*, Cambridge University Press, Cambridge, U.K., 2018 (in press).

[5] J.H. Wilkinson, “The perfidious polynomial”, in *Studies in Numerical Analysis*, vol. **24**, pp. 1–28, 1984; G.H. Golub (editor), Mathematical Association of America, Washington DC, USA.