ROBUST THRESHOLDS: COUNTING TRIANGLE FACTORS AND
A SHORTER PROOF OF THE ROBUST CORRÁDI–HAJNAL THEOREM

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ABSTRACT. We show that the distribution of perfect matchings in an \( r \)-uniform hypergraph on \( n \) vertices induced by the edges of an \( r \)-partite super-regular graph is \( O(1/n^{r-1}) \)-spread. This yields a short proof of the robust Corrádi–Hajnal Theorem recently obtained in [1], that there is \( C > 0 \) such that, for a graph \( G \) with minimum degree \( 2n/3 \), a random subgraph of \( G \) where each edge is retained with probability \( C(\log n)^{1/3}n^{-2/3} \) contains a triangle factor with high probability. We also show that the number of triangle factors in a graph with minimum degree \( 2n/3 \) is at least \( (cn)^{2n/3} \) for a constant \( c > 0 \), addressing a question in [1].

1. INTRODUCTION

Conditions that guarantee the existence of graph factors have been extensively studied in extremal and probabilistic combinatorics. In extremal combinatorics, the Corrádi–Hajnal theorem [2] says that a graph \( G \) on \( n \) vertices with \( 3|n \) and minimum degree at least \( 2n/3 \) contains a triangle factor. This minimum degree condition is tight, as shown by the graph which consists of an independent set \( A \) of size \( n/3 + 1 \) and a clique \( B \) of size \( 2n/3 - 1 \) and all edges between \( A \) and \( B \). On the other hand, in probabilistic combinatorics, a seminal result of Johansson, Kahn and Vu [5] says that a random graph \( G(n, p) \) with \( p \geq C(\log n)^{1/3}n^{-2/3} \) contains a triangle factor with high probability. The bound on \( p \) is sharp up to the constant \( C \). Recently, building on the argument of [5], Kahn derived sharp asymptotics for the threshold of containing a triangle factor [6] and later an even stronger hitting time result [7].

In a very recent nice work, Allen et al. [1] developed an argument that shares some similarities with the argument of Johansson, Kahn and Vu but with several key differences to show a common strengthening of the Corrádi–Hajnal theorem and the result of Johansson, Kahn and Vu.

**Theorem 1.1.** Let \( G \) be a graph with minimum degree at least \( 2n/3 \) and \( 3|n \). Let \( H \) be a random subgraph of \( G \) obtained by choosing each edge independently with probability \( p \geq C(\log n)^{1/3}n^{-2/3} \). Then \( H \) contains a triangle factor with high probability.

In [1], by combining the regularity lemma with a careful stability analysis of the extremal cases, Theorem 1.1 is reduced to the following result. We say that a bipartite graph \( G \) is \((d, \delta)\)-super-regular if the minimum degree of \( G \) is at least \( dn \) and, furthermore, for every subset \( S \subseteq A \) and \( T \subseteq B \) of size at least \( \delta n \), \(|E(S, T) - d|S||T|| \leq \delta|S||T|\). We say that a tripartite graph is \((d, \delta)\)-super-regular if the bipartite graphs corresponding to each pairs of parts are \((d, \delta)\)-super-regular.

**Theorem 1.2.** Let \( G = (V_1, V_2, V_3, E) \) be a \((d, \delta)\)-super-regular tripartite graph with \(|V_i| = n \) for \( i \in [3] \). Assume that \( \delta \) is sufficiently small in \( d \). Let \( H \) be a random subgraph of \( G \) obtained by choosing each edge independently with probability \( p \geq C(\log n)^{1/3}n^{-2/3} \). Then \( H \) contains a triangle factor with high probability.

This is the key technical result of [1], which was proved through an involved analysis using entropy of a scheme to build the triangle factor one triangle at a time, similar in spirit to the argument of [5],
but with key features that allow the argument to handle general non-complete host graphs $G$ instead of the complete graph. As is the case with the original proof of Johansson, Kahn and Vu, the proof is quite involved. In the case where $G$ is the complete graph, where the resolution of the Kahn–Kalai conjecture [8] (or its weaker fractional version [3]) allow one to give a much simpler proof of the main result of [5]. We say that $H \subseteq 2^X$ is $p$-small if there exists $H' \subseteq 2^X$ such that any set in $H$ contains a set from $H'$ and $\sum_{H' \subseteq H} p^{|H'|} < 1/2$. We say that $H \subseteq 2^X$ is fractionally $p$-small if for any $w : 2^X \to [0,1]$ such that $\sum_{S \subseteq H} w(S) \geq 1$ and $\sum_{S \subseteq X} w(S)p^{|S|} < 1/2$. It is immediate that $H$ is fractionally $p$-small whenever it is fractionally $p$-small. Using linear programming duality, Talagrand [11] observed that $H$ is not fractionally $p$-small if and only if there exists a distribution $\mu$ supported on $H$ for which $\mu(\{W \subseteq X : S \subseteq W\}) \leq 2p^{|S|}$ for all $S \subseteq X$. We say that such $\mu$ is $p$-spread. The following theorem is the main result of [8], whose immediate corollary in the fractional setting was also earlier obtained in [3].

**Theorem 1.3.** There exists $C > 0$ such that the following holds. Let $H \subseteq 2^X$ be so that $H$ is not $p$-small. Then a random subset of $X$ where each element is sampled independently with probability $Cp\log |X|$ contains a set in $H$ with high probability.

In particular, the same conclusion holds if there exists a distribution $\mu$ supported on $H$ which is $p$-spread.

Theorem 1.3 allows to determine the order of the answer to Shamir’s problem, which asks for the density threshold at which a random 3-uniform hypergraph likely contains a perfect matching. The following result of Riordan [9] and Heckel [4] allow us to connect Shamir’s problem to the problem of triangle factors. As it will be useful for us later, we state the result of Riordan and Heckel in the more general robust setting. One notes that the argument of Riordan [9] and Heckel [4] goes through identically in this setting.

**Theorem 1.4.** Let $r \geq 3$. There exists $\epsilon = \epsilon(r) > 0$ such that for any $p \leq n^{-2/r+\epsilon}$ the following holds. Let $G$ be a graph on $n$ vertices. Let $G_r$ be the $r$-uniform hypergraph whose edges are the $r$-tuple of vertices $(v_1, \ldots, v_r)$ where $\{v_i, v_j\} \in E(G)$ for all $i \neq j$. For some $\pi \sim p^{C(r)}$, there is a joint distribution $\lambda$ of a graph $H$ and an $r$-uniform hypergraph $H_r$ defined on the same set of vertices $[n]$ such that the following holds. The marginal of $H$ is the same as a random subgraph of $G$ where each edge is sampled independently with probability $p$, and the marginal of $H_r$ is the same as a random $r$-uniform subgraph of $G_r$ where each hyperedge is sampled independently with probability $\pi$. Furthermore, with high probability, for every hyperedge of $H_r$, there is a copy of $K_r$ in $H$ with the same vertex set.

Theorem 1.4 reduces the computation of the threshold for containment of a triangle factor in a random graph to the threshold for containment of a perfect matching in a random hypergraph. In the case the random hypergraph is obtained from selecting edges of the complete hypergraph, verifying the existence of an $O(1/n^2)$-spread measure supported on the collection of perfect matchings of the complete 3-uniform hypergraph is easy, as exact counting can be done. This yields an upper bound on the threshold of containing a 3-uniform hypergraph perfect matching of order $O((\log n)/n^2)$ via Theorem 1.3, which implies via Theorem 1.4 the tight upper bound on the threshold of the random graph $G(n,p)$ containing a triangle factor of $O((\log n)^{1/3}/n^{2/3})$.

In the robust setting of Theorem 1.2, the corresponding version of Shamir’s problem is captured in the following theorem.

**Theorem 1.5.** Let $G = (V_1, V_2, V_3, E)$ be a $(d, \delta)$-super-regular tripartite graph with $|V_i| = n$ for $i \in [3]$. Assume that $\delta$ is sufficiently small in $d$. Let $G_3$ be the 3-uniform hypergraph whose edges are $(v_1, v_2, v_3)$ where $v_i \in V_i$ and $\{v_i, v_j\} \in E(G)$ for all $i, j \in [3]$. Let $H_3$ be a random subgraph of $G_3$ obtained by choosing each hyperedge independently with probability $p \geq C(\log n)n^{-2}$. Then $H$ contains a perfect matching with high probability.
It now becomes much more nontrivial to show the existence of a measure supported on matchings of $G_3$ that is spread, and this is the key difficulty of the robust setting. Our main result is to verify the existence of such measure.

**Theorem 1.6.** Let $G = (V_1, V_2, V_3, E)$ be a $(d, \delta)$-super-regular tripartite graph for $\delta$ sufficiently small in $d$ and $|V_i| = n$ for $i \in [3]$. Let $G_3$ be the 3-uniform hypergraph whose edges are $(v_i, v_2, v_3)$ where $v_i \in V_i$ and $\{v_i, v_j\} \in E(G)$ for all $i \neq j \in [3]$. There exists a measure $\mu$ supported on perfect matchings of $G_3$ which is $O_d(1/n^2)$-spread.

In fact, our proof easily generalizes to higher uniformity of the hypergraph.

**Theorem 1.7.** Let $r \geq 2$. Let $G = (V_1, V_2, \ldots, V_r, E)$ be a $(d, \delta)$-super-regular $r$-partite graph for $\delta$ sufficiently small in $d$ and $|V_i| = n$ for $i \in [r]$. Let $G_r$ be the $r$-uniform hypergraph whose edges are $(v_1, v_2, \ldots, v_r)$ where $v_i \in V_i$ and $\{v_i, v_j\} \in E(G)$ for all $i \neq j \in [r]$. There exists a measure $\mu$ supported on perfect matchings of $G_r$ which is $O_d(1/n^{r-1})$-spread.

The main tool in the proof of Theorem 1.6 and Theorem 1.7 is the corresponding result for perfect matchings in bipartite graphs.

**Theorem 1.8.** Let $d > 0$ and $\delta$ sufficiently small in $d$. Let $G$ be a $(d, \delta)$-regular bipartite graph with parts of size $n$. There exists a distribution $\mu$ on perfect matchings in $G$ which is $O_d(1/n)$-spread.

Theorem 1.6 together with Theorem 1.3 imply Theorem 1.5. Together with Theorem 1.4, we immediately obtain a new short proof of Theorem 1.2. We also obtain the higher uniformity analog of Theorem 1.2.

**Theorem 1.9.** Let $G = (V_1, V_2, \ldots, V_r, E)$ be a $(d, \delta)$-super-regular $r$-partite graph with $|V_i| = n$ for $i \in [r]$. Assume that $\delta$ is sufficiently small in $d$. Let $H$ be a random subgraph of $G$ obtained by choosing each edge independently with probability $p \geq C((\log n)^{1/2}/n^{2/r})$. Then $H$ contains a $K_r$-factor with high probability.

As such, the remaining part of the paper is devoted to the proof of Theorem 1.6 and Theorem 1.8. As a byproduct of Theorem 1.6, we also obtain a tight counting result for the number of triangle factors in a super-regular graph, showing that there are at least $(cn)^{2n/3}$ such factors.

**Theorem 1.10.** Let $G$ be a graph on $n$ vertices with minimum degree at least $2n/3$. Then the number of triangle factors of $G$ is at least $(cn)^{2n/3}$ for a constant $c > 0$.

This improves a result in [1] that gives a lower bound of $(cn/\sqrt{\log n})^{2n/3}$ and addresses an open question in [1] in the case of triangle factors.

In concurrent work, Sah, Sawhney and Simkin [10] also show the same results on spreadness of perfect matchings in the robust setting via vertex absorption.

2. **Optimal spread of perfect matchings**

Given a bipartite graph $G = (A, B, E)$ with $|A| = |B| = n$, we consider the following subgraph of $G$. For each vertex $v$ of $G$, choose a uniform and independent random set of $C$ neighbors of $v$ (with repetitions). Let $H$ be the graph containing all of the chosen edges.

**Lemma 2.1.** Let $G$ be $(d, \delta)$-super-regular for $\delta$ sufficiently small in $d$. Then, with high probability, for $C$ sufficiently large depending only on $d$, the subgraph $H$ contains a perfect matching.
Proof. We will prove that $H$ satisfies Hall’s condition with high probability. In particular, we want to show that there is no subset $T$ of $B$ of size $k$ for which there is a subset $S$ of $A$ of size $k + 1$ and the neighborhood of any vertex in $S$ is contained in $T$; and similarly for $T$ a subset of $A$ of size $k$ and $S$ a subset of $B$ of size $k + 1$. Note that if all vertices in $S$ have their neighborhood contained in $T$, then all vertices in $B \setminus T$ have their neighborhood contained in $A \setminus S$. Thus, by symmetry, we only need to consider the case $k \leq n/2$, since for $k > n/2$ we have $|A \setminus S| < n/2$. In the following, let $T$ be a subset of $B$ of size $k$, and $S$ a subset of $A$ of size $k + 1$. We bound the probability that $N_H(v) \subseteq T$ for all $v \in S$. Let $\eta = 4\delta$.

Case 1: $k \in (\eta n, n/2)$. In this case, by the assumption the $G$ is $(d, \delta)$-regular, for $\epsilon = \delta/d$, the number of vertices $v$ with $|N_G(v) \cap T| > (k/n + \epsilon)n$ is at most $\delta n$. Thus there are at least $k - \delta n$ vertices $v$ in $S$ with $|N_G(v) \cap T| \leq (k/n + \epsilon)n$. For each such $v$, the chance that $N_H(v) \subseteq T$ is at most $(k/n + \epsilon)^C$. Hence, the probability that $N_H(S) \subseteq T$ is at most $(k/n + \epsilon)^{C(k-\delta n)}$. By the union bound, the probability that there exists $S$ and $T$ with $N_H(S) \subseteq T$ is at most

$$\left(\frac{n}{k}\right) \left(\frac{n}{k+1}\right) (k/n + \epsilon)^{C(k-\delta n)} \leq \left(\frac{e^2n^2}{k^2}\right)^k \left(\frac{k}{n} + \epsilon\right)^{C(k-\delta n)} \leq \left(\frac{e^2n^2}{k^2}\right)^k \left(\frac{k}{n} + \epsilon\right)^{Ck/2} \leq \left(\frac{e^2}{k^2} \cdot \left(\frac{k}{n} + \epsilon\right)^{C/2}\right)^k.$$ 

Note that $\frac{k}{n} + \epsilon \leq \min(2/3, 2(k/n)/d)$, and hence

$$\left(\frac{e^2}{k^2} \cdot \left(\frac{k}{n} + \epsilon\right)^{C/2}\right)^k \leq \left(\frac{e^2n^2}{k^2} \cdot \left(\frac{k}{n} + \epsilon\right)^2 \cdot (2/3)^{C/2-2}\right)^k \leq \left(\frac{e^2}{d^2} \cdot (2/3)^{C/2-2}\right)^k < 2^{-k},$$

assuming that $C$ is sufficiently large in $d$.

Case 2: $k \leq \eta n$. In this case, for each $v \in S$, $|N_G(v) \cap T| \leq k \leq \eta n$. Hence, the chance that $N_H(v) \subseteq T$ is at most $(k/(dn))^C$. Hence, the probability that $N_H(S) \subseteq T$ is at most $(k/(dn))^C$. By the union bound, the probability that there exists $S$ and $T$ with $N_H(S) \subseteq T$ is at most

$$\left(\frac{n}{k}\right) \left(\frac{n}{k+1}\right) (k/(dn))^C k \leq \left(\frac{e^2}{d^2} \cdot \left(\frac{k}{n} + \epsilon\right)^C\right)^k.$$ 

Combining the cases, by the union bound, the probability that $H$ does not satisfy Hall’s condition is at most

$$2 \sum_{k \leq n} \left(\frac{e\eta^{C-2}}{d^2}\right)^k + 2^{-\eta n + 1} = o(1).$$

\[\square\]

Using Lemma 2.1, we can prove Theorem 1.8.

Proof of Theorem 1.8. From $G$ pick a subgraph $H$ as in Lemma 2.1, which has a perfect matching with high probability. Pick and output an arbitrary perfect matching $W$ of $H$. This induces a distribution $\mu$ on perfect matchings of $G$. We show that $\mu$ is $O_d(1/n)$-spread. Indeed, given any subset $S$ of edges of $G$, if $S$ is not a matching, then $\mu(W \supseteq S) = 0$. If $S$ is a matching, $W$ can only contain $S$ if for each edge $e = \{x, y\} \in S$, either $x$ or $y$ picks the other vertex as one of the $C$ neighbors, which happens with probability at most $2C/n$. Furthermore, the above events are independent across different edges of the matching $S$. Hence, $\mu(W \supseteq S) \leq (2C/n)^{|S|}$. Thus, the perfect matchings of $G$ are $(2C/n)$-spread. \[\square\]
3. Robust Corrádi–Hajnal Theorem

Given a tripartite graph $G = (V_1, V_2, V_3, E)$ with $|V_1| = |V_2| = |V_3| = n$, we denote by $G_1$ the graph between $V_2$ and $V_3$, and similarly define $G_2$ and $G_3$.

**Proof of Theorem 1.6.** We can assume that all connected $v_2 \in V_2, v_3 \in V_3$ have at least $(d^2 - \delta^{1/2})n$ common neighbors in $V_1$. Indeed, by the regularity condition, for each $v_3 \in V_3, v_3$ has at least $dn$ neighbors in $V_1$ and the number of bad vertices $v_2 \in V_2$ with less than $(d^2 - \delta^{1/2})n$ common neighbors with $v_3$ is at most $\delta n$. We can remove all edges between $v_3$ and the bad vertices in $V_2$. After removal, the minimum degree of vertices in $V_2$ and $V_3$ is at least $d - \delta$ and the graph remains $2\delta$-regular. Hence, we can replace $d$ by $d - \delta$ and replace $\delta$ by $2\delta$, and assume that all connected $v_2 \in V_2, v_3 \in V_3$ have at least $(d^2 - \delta^{1/2})n$ common neighbors in $V_1$.

Given the bipartite graph $G_1$ between $V_2$ and $V_3$, applying Lemma 2.1, we have a distribution $\mu_1$ on perfect matchings between $V_2$ and $V_3$ such that $\mu_1$ is $O_d(1/n)$-spread. For any given perfect matching $M_1$ sampled from $\mu_1$, we construct the following graph $\Gamma_{M_1}$, with vertex sets $E(M_1)$ and $V_1$, and edges $(e, v)$ if $v$ is adjacent to both endpoints of $e$. Then the minimum degree of $\Gamma_{M_1}$ is at least $d^2 - \delta^{1/2}$.

Let $\eta = (\log \delta^{-1})^{-1/2}$. We next prove that $\Gamma_{M_1}$ is $4\eta^{-1/2}$-regular with high probability. Indeed, for each subset $S_1$ of $V_1$ of size $\rho n$ and $\rho \geq 4\eta^{1/2}$, we say that $e \in E(G_1)$ is bad if the number of common neighbors of the endpoints of $e$ in $S_1$ is not in $(d^2 \pm \eta)\rho n$, and say that $v_i \in V_i$ is bad for $i \in \{2, 3\}$ if the number of neighbors of $v_i$ in $S_1$ is not in $(d \pm \eta)\rho n$. The number of bad vertices is at most $\delta n$. For each $v_i$ which is not bad, the number of bad $e$ adjacent to $v_i$ is at most $\delta n$. Consider the subgraph $H$ of $G_1$ chosen by selecting $C$ independent edges of $G_1$ adjacent to each vertex in $V_2 \cup V_3$ (as in Lemma 2.1), noting that $C$ only depends on $d$. The number of bad edges selected by a good vertex $v_i$ to place in $H$ is stochastically dominated by a sum of $C$ Bernoulli random variables with parameter $\delta$. Hence, the number of bad edges adjacent to good vertices in $H$ is dominated by a sum of $2Cn$ Bernoulli random variables with parameter $\delta$. By the Chernoff bound, the probability that there are at least $\eta n$ bad edges adjacent to good vertices in $H$ is at most

$$\left(\frac{e^{\eta/(2C\delta)}-1}{(\eta/(2C\delta))^{\eta/(2C\delta)}}\right)^{2Cn\delta} \leq \exp(-\log(\delta^{-1})\eta n/4) = \exp(-(\log \delta^{-1})^{1/2} n/4).$$

Note that if there are at most $\eta n$ bad edges in $E(M_1)$, then the number of vertices in $E(M_1)$ whose number of edges to $S_1$ is not $(d \pm \eta)\rho n$ is at most $(\eta + 2\delta)n$. In that case, for any subset $T$ of $E(M_1)$ of size at least $\rho n$, the number of edges between $T$ and $S_1$ is $(d \pm 2(\eta + 2\delta)\rho^{-1})\rho n |T|$. Hence, by the union bound over $S_1$, we obtain that $\Gamma_{M_1}$ is $4\eta^{-1/2}$-regular with probability at least

$$2^n \exp(\log \delta^{-1})^{1/2}/n/4 < 2^{-n}.$$  

Now, if $\Gamma_{M_1}$ is $4\eta^{-1/2}$-regular, assuming that $\delta$ is sufficiently small in $d$, we can apply Lemma 2.1 and output a matching $\tilde{M}$ between $V_1$ and $E(M_1)$, which corresponds to a 3-uniform perfect matching $M$ of $G_3$. (If $\Gamma_{M_1}$ is not $4\eta^{-1/2}$-regular, we do not output and resample $M_1$.) We now verify that $M$ is $O_d(1/n^2)$-spread. We can assume that $S$ is a matching. Let $S_1$ be the matching of $G_1$ induced by $S$. If $M \supseteq S$, then $M \supseteq S_1$, which holds with probability at most $(C/n)^{|S|}$ for some $C$ depending only on $d$. Furthermore, conditioned on a consistent realization of $M_1$, we need the matching $M$ to contain a corresponding set of edges of $E(\Gamma_{M_1})$ of size $|S|$, which holds with probability at most $(C'/n^2)^{|S|}$. Hence, the probability that $M$ contains $S$ is at most $(C'/n^2)^{|S|}$. Thus, the perfect matchings of $G_3$ is $O_d(1/n^2)$-spread.

The argument easily generalizes to the higher uniformity case.
Proof of Theorem 1.7. We prove the result by induction on \( r \). The case \( r = 2 \) and \( r = 3 \) are shown in Theorem 1.8 and Theorem 1.6.

Let \( H_1 \) be an \((r-1)\)-uniform hypergraph on \( V_2 \cup \ldots \cup V_r \) whose edges are the tuples \((v_2, \ldots, v_r)\) with \( \{v_i, v_j\} \in E(G) \) for all \( i \neq j \). As in the proof of Theorem 1.6, we can assume that for all \( v_2 \in V_2, \ldots, v_r \in V_r \) so that \((v_2, \ldots, v_r) \in H_1\), they have at least \((d^r - \delta^1/2)n\) common neighbors in \( V_1 \). By the inductive hypothesis, we have a distribution \( \mu_1 \) on perfect matchings of \( H_1 \) which is \( O_d(1/n^r-2) \)-spread. For any given perfect matching \( M_1 \) sampled from \( \mu_1 \), we construct the following graph \( \Gamma_{M_1} \), with vertex sets \( E(M_1) \) and \( V_1 \), and edges \((e, v)\) if \( v \) is adjacent to all endpoints of \( e \). Then the minimum degree of \( \Gamma_{M_1} \) is at least \( d^r-1 - \delta^{1/2} \). We will then select a matching \( M \) between \( V_1 \) and \( \Gamma_{M_1} \) if \( \Gamma_{M_1} \) is \( 4r\eta^{1/2}\)-regular for \( \eta = (\log \delta^{-1})^{-1/2} \), and output the corresponding \( r \)-uniform perfect matching \( M \).

We say that \((v_2, \ldots, v_r)\) is good if the number of common neighbors of \( v_2, \ldots, v_r \) in \( V_1 \) is \((d^r-1 \pm \delta)n\). For each subset \( S_1 \) of \( V_1 \) of size \( \rho n \) and \( \rho \geq 4r\eta^{1/2} \), we say that \( e \in E(H_1) \) is bad if the number of common neighbors of the endpoints of \( e \) in \( S_1 \) is not \((d^r-1 \pm \eta)n\). Then the number of bad edges in \( E(H_1) \) is at most \( r\delta \rho n d^{r-1} \). Say that a vertex \( v_i \in V_i \) is bad if it is adjacent to more than \( \delta^{1/2} n^{-2} \) bad edges, and \( v_i \in V_i \) is good otherwise. Then the number of bad vertices is at most \( r^2 \delta^{1/2} n^{-2} \). We say that a perfect matching \( M_1 \) of \( H_1 \) is bad if it contains at least \( \eta n \) bad edges which are adjacent only to good vertices. The number of bad perfect matchings is at most

\[
\binom{n}{\eta n} (\delta^{1/2} n^{r-2})^{\eta n} \cdot (n^{r-2})^{n-\eta n}.
\]

For each such perfect matching, the probability (under \( \mu_1 \)) that it is realized is at most \((C/n^{r-2})^n\) where \( C \) can only depend on \( d \). Hence, the probability that there are at least \( \eta n \) bad edges adjacent to good vertices selected in \( M_1 \) is at most

\[
\binom{n}{\eta n} (\delta^{1/2} n^{r-2})^{\eta n} \cdot (n^{r-2})^{n-\eta n} \cdot (C/n^{r-2})^n \leq \exp((\log \delta^{-1})^{1/2} \eta n) C^n \exp(\eta n \log(\delta^{-1})/2)
\leq \exp(- (\log \delta^{-1})^{1/2} n /4).
\]

Note that if there are at most \( \eta n \) bad edges adjacent to good vertices in \( M_1 \), then the number of vertices in \( E(M_1) \) whose number of edges to \( S_1 \) is not \((d^r-1 \pm \eta)\rho n \) is at most \((r^2 \delta^{1/2} + \eta) n \). In that case, for any subset \( T \) of \( E(M_1) \) of size at least \( \rho n \), the number of edges between \( T \) and \( S_1 \) is \((d^{r-1} \pm 2(\eta + r^2 \delta^{1/2}) \rho^{-1}) \rho n |T| \). Hence, by the union bound over \( S_1 \), we obtain that \( \Gamma_{M_1} \) is \( 4r\eta^{1/2} \)-regular with probability at least

\[
2^n \exp(- (\log \delta^{-1})^{1/2} n /4) < 2^{-n}.
\]

When \( \Gamma_{M_1} \) is \( 4r\eta^{1/2} \)-regular, assuming that \( \delta \) is sufficiently small in \( d \), we can apply Lemma 2.1 and output a matching \( \tilde{M} \) between \( V_1 \) and \( E(M_1) \), which corresponds to a \( r \)-uniform perfect matching \( M \) of \( G_r \). (If \( \Gamma_{M_1} \) is not \( 4r\eta^{1/2} \)-regular, we do not output and resample \( M_1 \).) We now verify that \( M \) is \( O_d(1/n^{r-1}) \)-spread. Fix a subset \( S \) of hyperedges. As before, we can assume that \( S \) is a matching. Let \( S_1 \) be the matching of \( H_1 \) induced by \( S \). If \( M \supseteq S \), then \( M_1 \supseteq S_1 \), which holds with probability at most \((C/n^{r-2})^{|S|}\) for some \( C \) depending only on \( d \). Furthermore, conditioned on a consistent realization of \( M_1 \), we need the matching \( M \) to contain a corresponding set of edges of \( E(\Gamma_{M_1}) \) of size \(|S|\), which holds with probability at most \((C/n)^{|S|}\). Hence, the probability that \( M \) contains \( S \) is at most \((C'/n^{r-1})^{|S|}\). Thus, the perfect matchings of \( G_r \) is \( O_d(1/n^{r-1}) \)-spread.

\[\square\]

4. Counting triangle factors

To prove Theorem 1.10, we follow the strategy in Section 9 of [1] to reduce the general problem to the super-regular setting, and apply the following simple corollary of Theorem 1.6.
Proposition 4.1. Let $G = (V_1, V_2, V_3, E)$ be a $(d, \delta)$-super-regular tripartite graph for $\delta$ sufficiently small in $d$ and $|V_1| = |V_2| = |V_3| = n$. Then the number of triangle factors of $G$ is at least $(cn)^{2n}$.

Proof. Let $G_3$ be the 3-uniform hypergraph whose edges are $(v_1, v_2, v_3)$ where $v_i \in V_i$ and $\{v_i, v_j\} \in E(G)$ for all $i, j \in [3]$. By Theorem 1.6, there exists a measure $\mu$ supported on perfect matchings of $G_3$ which is $C_d/n^2$-spread. In particular, by considering $S$ to be a perfect matching of $G_3$ with the largest probability under $\mu$, since $\mu$ is $C_d/n^2$-spread, we have $\mu(W = S) \leq (C_d/n^2)^3$. Hence the number of perfect matchings of $G_3$ is at least $(n^2/C_d)^n$. Thus, the number of triangle factors of $G$ is at least $(cn)^{2n}$. \hfill $\Box$

Theorem 1.10 follows from the following three lemmas, which are analogs of Lemma 9.1, Lemma 9.2 and Lemma 9.3 in [1].

Lemma 4.2. For every sufficiently small $\mu > 0$, there exists $c > 0$ and $0 < d \leq \mu$ such that the following holds. Let $n \in 3\mathbb{N}$ and suppose $G$ is an $n$-vertex graph with $\delta(G) \geq (2/3 - d/2)n$ such that there is no $S \subseteq V(G)$ of size at least $(1/3 - 2\mu)n$ with $\Delta(G[S]) \leq 2dn$. Then $G$ contains at least $(cn)^{2n/3}$ triangle factors.

Lemma 4.3. For every sufficiently small $\mu > 0$, there exists $c > 0$ and $0 < \tau, d \leq \mu$ such that the following holds for all $n \in 3\mathbb{N}$. Suppose $G$ is an $n$-vertex graph with $\delta(G) \geq 2n/3$, and suppose $S$ is a subset of $V(G)$ with $|S| \geq (1/3 - \tau)n$ and $\Delta(G[S]) \leq \tau n$. Suppose further that there is no $S' \subseteq V(G) \setminus S$ of size at least $(1/3 - 2\mu)n$ with $\Delta(G[S']) \leq 2dn$. Then $G$ contains at least $(cn)^{2n/3}$ triangle factors.

Lemma 4.4. There exists $c, \tau > 0$ such that the following holds for all $n \in 3\mathbb{N}$. Suppose $G$ is an $n$-vertex graph with $\delta(G) \geq 2n/3$, and suppose $S_1, S_2$ are disjoint subsets of $V(G)$ with $|S_i| \geq (1/3 - \tau)n$ and $\Delta(G[S_i]) \leq \tau n$ for $i = 1, 2$. Then $G$ contains at least $(cn)^{2n/3}$ triangle factors.

The proof of all of the above lemmas are essentially identical to Section 9 of [1] upon immediate replacement of appropriate lemmas showing existence of triangle matchings in a random sparse subgraph with results for counting triangle matchings. Instead of repeating the reduction of [1], in Appendix 4, we highlight the main differences that have to be changed to prove Lemmas 4.2, 4.3 and 4.4.

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References

[1] P. Allen, J. Böttcher, J. Corsten, E. Davies, M. Jenssen, P. Morris, B. Roberts and J. Skokan, A robust Corrádi-Hajnal Theorem, arXiv preprint (2022), arXiv:2209.01116. 1, 3, 6, 7, 8
[2] K. Corrádi and A. Hajnal, On the maximal number of independent circuits in a graph, Acta Math. Acad. Sci. Hungar. 14 (1963), 423–439. 1
[3] K. Frankston, J. Kahn, B. Narayana, and J. Park, Thresholds versus fractional expectation-thresholds, Ann. of Math. 194(2) (2021), 475–495. 2
[4] A. Heckel, Random triangles in random graphs, Random Struct. Algorithms 59(4) (2021), 616–621. 2
[5] A. Johansson, J. Kahn, and V. H. Vu, Factors in random graphs, Random Struct. Algorithms 33(1) (2008), 1–28. 1, 2
[6] J. Kahn, Asymptotics for Shamir’s problem, arXiv preprint (2019), arXiv:1909.06834. 1
[7] J. Kahn, Hitting times for Shamir’s problem, Trans. Amer. Math. Soc. 375(1) (2022), 627–668. 1
[8] J. Park and H. T. Pham, A proof of the Kahn-Kalai conjecture, arXiv preprint (2022), arXiv:2203.17207. 2
[9] O. Riordan, Random cliques in random graphs and sharp thresholds for $F$-factors, Random Struct. Algorithms, to appear. 2
[10] A. Sah, M. Sawhney and M. Simkin, Robustness of the Dirac condition for hypergraph perfect matchings, upcoming. 3
[11] M. Talagrand, Are many small sets explicitly small?, Proceedings of the 2010 ACM International Symposium on Theory of Computing (2010), 13–35. 2
Lemma 4.7. For all $\mu > 0$, there exists $C > 0$ such that the following holds. Let $k, n \in \mathbb{N}$ and let $G$ be an $n$-vertex graph.

(1) Assume that for every $X \subseteq V(G)$ with $|X| \geq 3k$, $G[X]$ contains at least $\mu n^3$ triangles, then $G$ contains at least $(\mu n^3)_{n^3}^{3-k} / (n - 3k)^{n^3-k}$ triangle matchings of size $n/3 - k$.

(2) Assume that $n_0 \geq k$ and $V(G) = V_1 \cup V_2 \cup V_3$ is a partition into sets of size at least $n_0$ so that for every $X_i \subseteq V_i$ with $|X_i| \geq k$ for all $i \in [3]$, $G[X_1, X_2, X_3]$ contains at least $\mu n^3$ triangles. Then $G$ contains at least $(\mu n^3)_{n^3-k}^{3-k} / (3n_0 - 3k)^{n^3-k}$ triangle matchings of size $n_0 - k$. 

Proof. Note that one can iteratively select triangles from $G$ and remove the vertices of selected triangles until there are at most $3k$ remaining vertices. This yields a triangle matching of size $n/3 - k$. In each step, one has $\mu n^3$ choices, and each triangle matching is overcounted at most $(n/3 - k)!3^{n^3-3-k}$ times. This immediately yields the bound in part (1). The bound in part (2) follows similarly. \qed

We will replace Lemma 8.1 of [1] by the following lemma.

Lemma 4.6. For any $0 < \mu < 1/100$, there exists $C > 0$ such that the following holds for every $n \in \mathbb{N}$. Let $G$ be an $n$-vertex graph, and let $v_1, \ldots, v_\ell \in V(G)$ be distinct vertices with $\ell \leq \mu^2 n$. For each $i \in [\ell]$, let $E_i \subseteq \text{Tr}_{v_i}(G)$ be a set of edges that form a triangle with $v_i$ such that $|E_i| \geq \mu n^2$. Moreover, suppose $A_1, \ldots, A_\ell \subset V(G) \setminus \{v_1, \ldots, v_\ell\}$ are disjoint sets for some $\ell \in \mathbb{N}$. Then, there are at least $(\mu n^2 / 2)^{\ell}$ triangle matchings $T = \{T_1, \ldots, T_\ell\}$ in $G$ such that for each $i \in [\ell]$, the triangle $T_i$ consists of $v_i$ joined to an edge of $E_i$, and $|A_k \cap V(T)| \leq 12\mu|A_k| + 1$ for all $k \in [\ell]$. 

Proof. We iteratively go through vertices $v_1, \ldots, v_\ell$ and, in step $i$, pick a triangle $T_i$ containing $v_i$ using edges in $E_i$. We guarantee that $T_i$ does not contain any vertex $v_j$ with $j \neq i$ and any vertex in $T_j$ with $j < i$. Furthermore, we say that $A_k$ is full at time $i$ if $|A_k \cap V(\{T_1, \ldots, T_{i-1}\})| \geq 12\mu|A_k|$, and let $X_i$ be the union of the sets $A_k$ that are full at time $i$. We will guarantee that $T_i$ does not contain any vertex in $X_i$. Note that $|X_i| \leq \frac{3\ell}{12\mu} \leq \frac{\ell}{4} n$, and there are at most $3\ell n \leq 3\mu^2 n^2$ edges of $E_i$ which are adjacent to $v_j$ for $j \neq i$ or $T_j$ for $j < i$. Hence, for all $i$, at step $i$, we have at least $\mu n^2 / 2$ choices for $T_i$. \qed

We will replace Lemma 8.4 Lemma 8.4 with the following lemma.

Lemma 4.7. For any $0 < \mu < 1/1000$, there exists $C > 0$ such that the following holds for all $\ell, \delta, \delta_1, \delta_2 \in \mathbb{N}$ and $n$-vertex graphs $G$.

(1) Let $X_1, X_2, X_3 \subset V(G)$ be disjoint sets of size at least $n/10$, and let $E \subseteq E(G[X_1])$ be a set of edges such that $\deg_{E}(v) \geq \delta$ for all $v \in X_1$ and $\deg_{E}(e; X_i) \geq \mu n$ for all $e \in E$ and $i = 2, 3$. Let $n_2, n_3 \in \mathbb{N}$ with
Lemma 4.7 follows easily from the following lemma.

**Lemma 4.8.** Let $0 < \mu < 1/1000$. Let $G$ be a graph on $n$ vertices with minimum degree at least $\delta \leq \mu^5n$. Then there are at least $(\mu^3n)^\delta$ matchings in $G$ of size $\delta$.

**Proof.** Let $B$ be the set of vertices in $G$ with degree at least $\mu^2n$. If $|B| \geq \delta$, then we arbitrarily order vertices in $B$ as $b_1, \ldots, b_{|B|}$, and for each of the first $\delta$ vertices, we match $b_i$ with a vertex $\pi(b_i) \notin \{b_1, \ldots, b_{i-1}, \pi(b_1), \ldots, \pi(b_{i-1})\}$. Note that there are always at least $\mu^2n - 2\delta$ available choices for $\pi(b_i)$ for each $i \leq \delta$. Hence, there are at least $(\mu^2n - 2\delta)^\delta > (\mu^3n)^\delta$ matchings in $G$ of size $\delta$ in this case.

Otherwise, $|B| < \delta$. Note that for each vertex $v \in V(G)$, the degree of $v$ in $V(G) \setminus B$ is at least $\delta - |B|$. We consider the following procedure. For each $i \leq \delta - |B|$, we select a vertex $v_i \in V(G) \setminus B$ and match $v_i$ with a neighbor $\pi(v_i)$ in $V(G) \setminus (B \cup \{v_1, \ldots, v_{i-1}, \pi(v_1), \ldots, \pi(v_{i-1})\})$. By double counting, the number of vertices $v \in V(G) \setminus B$ with more than $\mu\delta$ neighbors in $\{v_1, \ldots, v_{i-1}, \pi(v_1), \ldots, \pi(v_{i-1})\}$ is at most $\frac{2\mu^2n(i-1)}{\mu\delta} < 2\mu n$. Hence, at least $(1 - 3\mu)n$ vertices in $V(G) \setminus B$ have degree less than $\mu\delta$ to $\{v_1, \ldots, v_{i-1}, \pi(v_1), \ldots, \pi(v_{i-1})\}$. We then have at least $(1 - 3\mu)n$ choices for $v_i$ and $(1 - \mu)\delta$ choices for $\pi(v_i)$. In particular, we have at least

$$\frac{(1 - 3\mu)n \cdot (1 - \mu)\delta^{\delta - |B|}}{(\delta - |B|)!2^{\delta - |B|}}$$

choices for a matching in $V(G) \setminus B$ of size $\delta - |B|$. After having selected this matching, we then iteratively go over vertices of $B$ and select for each vertex a neighbor that has not been chosen in the matching, having at least $\mu^2n - 2\delta$ choices for each vertex of $B$. Thus, in total, the number of matchings of size $\delta$ in $G$ is at least

$$\frac{(1 - 3\mu)n \cdot (1 - \mu)\delta^{\delta - |B|}}{(\delta - |B|)!2^{\delta - |B|}} \cdot (\mu^2n - 2\delta)^{|B|} > (\mu^3n)^\delta.$$

\[ \square \]

**Proof of Lemma 4.7.** Both parts of the lemma follow easily from Lemma 4.8: In part (1), one can extend each edge in a matching on $X_1$ to a corresponding triangle using a vertex in either $X_2$ and $X_3$ with at least $(\mu - \mu^5)n$ choices per edge. Similarly, in part (2), one can extend each edge of in a matching on $X_1$ to a corresponding triangle using a vertex in $X_{3-i}$ with at least $(\mu - \mu^5)n$ choices per edge.

\[ \square \]