On the Origin of Polytropic Behavior in Space and Astrophysical Plasmas

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Abstract

It is shown that the polytropic behavior—a specific power-law relationship among the thermal plasma moments—restricts the functional form of the distribution of particle velocities and energies. Surprisingly, the polytropic behavior requires the statistical mechanics of the plasma particles to obey the framework of kappa distributions. An already known interesting property of these distributions is that they can lead to the polytropic relationship. New results show that the reverse derivation is also true, thus, the polytropic behavior has the role of a mechanism generating kappa distributions. Ultimately, an observation of a polytropic behavior in plasma particle populations constitutes a possible indirect observation of kappa velocity or energy distributions. Finally, it is discussed how the derived equivalence between the polytropic behavior and the kappa distribution function can be used in further modeling and data analyses in space and astrophysical plasmas.

Key words: methods: analytical – plasmas – Sun: heliosphere

1. Introduction

The meaning of polytrope ("\( p \propto \rho^{\gamma} \rho \propto \rho^{\gamma} \)") has its origin in the many ("\( p \propto \rho^{\gamma} \rho \propto \rho^{\gamma} \)") of the thermodynamic state of a system. It stands for a family of thermodynamic processes that follow a specific relationship among thermal observables, e.g., density \( n \), temperature \( T \), and thermal pressure \( p \), that is,

\[
p(r) \propto n(r)^\gamma, \quad \text{or} \quad n(r) \propto T(r)^\nu,
\]

where the polytropic indices \( \gamma \) and \( \nu \) are independent of the position vector and related with

\[
\gamma = 1 + 1/\nu \quad \text{or} \quad \nu = 1/(\gamma - 1).
\]

Frequently, space and astrophysical plasmas present positive correlations between density and temperature, caused by such a polytropic relationship, \( T \propto n^{\gamma-1} \), of some positive exponent \( \nu = 1/(\gamma - 1) > 0 \), or, with a polytropic index ranging between the values of the isothermal (\( \gamma = 1 \)) and isochoric (\( \gamma \to +\infty \)) processes. Several space plasmas have their most frequent polytropic index close to the value of the adiabatic process (\( \gamma = 5/3 \)). Some examples are the solar wind plasma (e.g., Totten et al. 1995; Newbury et al. 1997; Nicolau et al. 2014a; Livadiotis & Desai 2016; Livadiotis 2018a), solar flares (e.g., Garcia 2001; Wang et al. 2015, 2016), and planetary bow shocks (Tatrallyay et al. 1984; Winterhalter et al. 1984). Often, polytropic indices have sub-adiabatic (1 < \( \gamma < 5/3 \)) or super-adiabatic values (5/3 < \( \gamma < +\infty \)); for example in coronal mass ejections (e.g., Liu et al. 2006; Mishra & Wang 2018), coronal plasma (e.g., Prasad et al. 2018), Earth’s plasma sheet (e.g., Zhu 1990; Goertz & Baumjohann 1991; Borovsky et al. 1998), planetary magnetospheres (e.g., Dialynas et al. 2018), and even in the galaxy clusters and superclusters (e.g., Markovitch et al. 1998; Ettori et al. 2000; Grandi & Molendi 2002; Bautz et al. 2009).

Interestingly, there are cases where these plasmas present negative correlations between density and temperature (Livadiotis 2017, Chapter 5), characterized by a polytropic relationship of a negative exponent \( \gamma-1 < 0 \), with a polytropic index closer to the value of the isobaric process (\( \gamma = 0 \)); some examples are the magnetic clouds (e.g., Osherovich et al. 1993, 1998, 1999; Farrugia et al. 1995; Fainberg et al. 1996; Osherovich & Burlaga 1997; Sittler & Burlaga 1998), the planetary magnetosheaths (e.g., Sckopke et al. 1981; Nicolau et al. 2014b, 2015; Pang et al. 2016), and the inner heliosheath (Livadiotis et al. 2011, 2013; Livadiotis & McComas 2012, 2013a; Livadiotis 2016). Also, there are several other cases with negative indices closer to the isothermal value (e.g., Pang et al. 2015; Dialynas et al. 2018). Nevertheless, there are disagreements over how to interpret the observed negative correlations between temperature and density measurements, because in some cases it could be generated by non-thermodynamic mechanisms (e.g., Hammond et al. 1996; Gosling 1999).

It has been shown that there is a strong connection between the polytropic index and the governing parameter of the kappa distributions, the kappa index (e.g., Scudder 1992; Meyer-Vernet et al. 1995; Moncuquet et al. 2002; Tsallis et al. 2004; Livadiotis 2017, Chapters 5 and 11; Livadiotis 2018b; Livadiotis et al. 2018). According to this, higher kappa indices, which characterize stationary states near the classical Maxwell–Boltzmann distribution, correspond to polytropic indices closer to the isothermal value of \( \gamma = 1 \); on the other hand, lower kappa indices, which characterize states farther from the Maxwell–Boltzmann distribution, correspond to polytropic indices closer to the isobaric \( \gamma = 0 \) or isochoric values \( \gamma \to +\infty \); (the concept of thermodynamic distance of a stationary state, as well as the characterization near/far the classical thermal equilibrium, was presented in Livadiotis & McComas 2010). In general, the relationship among the polytropic and kappa indices has been shown theoretically and verified in several cases such as, for instance, in the case of solar wind near Earth (e.g., Livadiotis et al. 2018; Livadiotis 2018b).

The particle potential energy varies along the streamlines and plays the role of the connecting link between the profiles of thermal observables, which leads to the polytropic behavior.
Density, temperature, and thermal pressure depend on the potential energy and vary along the streamlines in a way that must be determined by the energy distribution function. In the classical case of Boltzmann–Gibbs (BG) statistical mechanics, the distribution at thermal equilibrium is given by the Boltzmannian exponential function, which can be separated as a product of two independent distributions, that is, the marginal distributions of kinetic and potential energy. As a result, the extracted polytropic behavior that originates from the exponential distribution is the trivial isothermal process. Moreover, in the case of generalized thermal equilibrium (Abe 2001; Livadiotis 2018c), where correlations among particles may be significant (Livadiotis & McComas 2011), the distribution is given by the kappa function (Livadiotis 2015a, 2017). The formalism of kappa distributions does not allow the separation of potential and kinetic energy, while it addresses the density and temperature profiles in such a way that predicts exactly the whole spectrum of polytropic behavior.

The formalism of kappa distributions can lead to the polytropic relationship, but this does not imply the reverse reasoning, i.e., that the “true” velocity distribution is necessarily given by a kappa function. This argument was first stated by Moncuquet et al. (2002), and it leads us to the following question. What is the most general distribution function of particle velocities and energies consistent with plasmas exhibiting polytropic behavior? Or, in short, what is the origin of the polytropic behavior in space and astrophysical plasmas?

The paper shows that the kappa distribution function constitutes the most general formalism for describing polytropes. The origin of any polytropic behavior is interwoven with the generalized thermal equilibrium, in which the particle velocities and energies are described by kappa distributions. Section 2 presents the formalism of kappa distributions in the presence of a potential energy, which leads to the polytropic relationship expressed in terms of the kappa index. Section 3 presents the Euler’s momentum equation (starting from the corresponding Navier–Stokes momentum equation in the presence of a conservative external field), and shows how the Maxwell–Boltzmann and kappa distributions obey this equation. Section 4 shows the derivation of the most general density profile function that is consistent with polytropes, using Euler’s equation. Section 5 shows the same thing, but using the Bernoulli integral. Section 6 shows the derivation in the presence of a magnetic field. Section 7 puts the last piece from the puzzle by showing the most general phase-space distribution function that is consistent with polytropes. Section 8 discusses the derived equivalence between the polytropic behavior and the kappa distribution functions and several interesting applications. Our conclusions in Section 9 summarize the results.

2. Formalism of Kappa Distributions in the Presence of a Potential Energy

First, we briefly present the formalism of kappa distributions in the presence of a potential energy, which leads to the derivation of the polytropic index expressed in terms of the kappa index and the degrees of freedom (see also Livadiotis 2015b, 2017, Chapters 3 and 4).

The Hamiltonian function is \( H(r, u) = \varepsilon_K(u) + \Phi(r) \), where \( \varepsilon_K(u) = \frac{1}{2}m \cdot u^2 \) is the kinetic energy and \( \Phi(r) \) the potential energy (that depends only on the position vector). The kappa phase-space distribution of a Hamiltonian gives the probability distribution of a particle having its position and velocity in the infinitesimal intervals \([r, r + dr]\) and \([u, u + du]\), respectively, that is,

\[
P(r, u; \kappa, T) \propto \left[ 1 + \frac{1}{\kappa} \cdot \frac{H(r, u) - \langle H \rangle}{k_B T} \right]^{\kappa - 1} = \left[ 1 + \frac{1}{\kappa} \cdot \frac{\varepsilon_K(u) + \Phi(r) - (\langle \varepsilon_K \rangle + \langle \Phi \rangle)}{k_B T} \right]^{\kappa - 1}.
\]

The mean Hamiltonian defines the total degrees of freedom or dimensionality, \( d \), summing the kinetic and potential degrees of freedom,

\[
\frac{1}{2}d = \frac{\langle H \rangle}{k_B T} = \frac{1}{2}d_K + \frac{1}{2}d_\phi,
\]

where the potential degrees of freedom are defined similar to the kinetic ones,

\[
\frac{1}{2}d_K = \frac{\langle \varepsilon_K \rangle}{k_B T}, \quad \frac{1}{2}d_\phi = \frac{\langle \Phi \rangle}{k_B T}.
\]

Note that by definition \( d_\phi \) can be either positive or negative; alternatively, it could be defined to be non-negative by \( \frac{1}{2}d_\phi \cdot \text{sign}(\Phi) = \Phi/(k_B T) \), and \( \frac{1}{2}d = \frac{1}{2}d_K + \frac{1}{2}d_\phi \cdot \text{sign}(\Phi) \).

The kappa index depends on the dimensionality as \( \kappa = \kappa(d) = \text{constant} + \frac{1}{2}d \), so that the difference \( \kappa(d) - \frac{1}{2}d \) remains invariant under changes of the dimensionality \( d \). Hence, the invariant kappa index \( \kappa_0 \) is defined by \( \kappa_0 \equiv \kappa - \frac{1}{2}d \); hence, \( \kappa(d) = \kappa_0 + \frac{1}{2}d \). The physical meaning of the thermodynamic parameter kappa is better carried by its invariant value \( \kappa_0 \), because this is independent of the degrees of freedom (Livadiotis & McComas 2011, 2013b; Livadiotis 2015a, 2015c, 2017, Chapter 1). Throughout this analysis, we use the notion of the invariant kappa index \( \kappa_0 \), but the typical three-dimensional index can be easily retrieved, \( \kappa_3 = \kappa_0 + \frac{3}{2} \).

Then, the phase-space distribution (2) is rewritten as

\[
P(r, u; \kappa_0, T) \propto \left[ 1 + \frac{1}{\kappa_0} \cdot \frac{\varepsilon_K(u) + \Phi(r)}{k_B T} \right]^{-\kappa_0 - 1 - \frac{1}{2}d}.
\]

The marginal distributions come from the integration over the velocity or the positional spaces,

\[
P(r) = \int_{-\infty}^{+\infty} P(r, u) du \quad \text{and} \quad P(u) = \int_{-\infty}^{+\infty} P(r, u) dr,
\]

which constitute the positional and velocity distributions, respectively.

If we integrate over the velocity space, instead of the whole phase-space, then we derive the positional kappa distribution

\[
P(r; \kappa_0, T) \propto \int_{-\infty}^{+\infty} \left[ 1 + \frac{1}{\kappa_0} \cdot \frac{\varepsilon_K(u) + \Phi(r)}{k_B T} \right]^{-\kappa_0 - 1 - \frac{1}{2}d} du \propto \left[ 1 + \frac{1}{\kappa_0} \cdot \frac{\Phi(r)}{k_B T} \right]^{-\kappa_0 - 1 - \frac{1}{2}d_\phi}.
\]
position \( \mathbf{n}_0 \) where the potential turns zero:

\[
n(r) = n_0 \left[ 1 + \frac{1}{\kappa_0} \cdot \frac{\Phi(r)}{k_B T} \right]^{-\kappa_0 - \frac{1}{2}d_\Phi}.
\]

(7)

The temperature profile is determined by the local mean kinetic energy (that is, without the integration over the positional space):

\[
\frac{1}{2}d_k k_B T(r) = \frac{\int_{-\infty}^{+\infty} P(r, \mathbf{u}; \kappa_0, T)\partial \Phi(\mathbf{u})d\mathbf{u}}{\int_{-\infty}^{+\infty} P(r, \mathbf{u}; \kappa_0, T)d\mathbf{u}},
\]

(8)

where we find,

\[
T(r) = T_0 \cdot \left[ 1 + \frac{1}{\kappa_0} \cdot \frac{\Phi(r)}{k_B T} \right], \quad \text{with} \quad T_0 = T \cdot \frac{\kappa_0}{\kappa_0 + \frac{1}{2}d_\Phi}.
\]

(9)

Then, the thermal pressure, \( p(r) = n(r)k_B T(r) \), becomes

\[
p(r) = p_0 \cdot \left[ 1 + \frac{1}{\kappa_0} \cdot \frac{\Phi(r)}{k_B T} \right]^{-\kappa_0 - \frac{1}{2}d_\Phi}, \quad \text{with} \quad p_0 = n_0k_B T_0.
\]

(10)

In the classical case of BG statistical mechanics, it is rather trivial to obtain the positional distribution and the positional dependence of thermal observables. The integration of the exponential of Hamiltonian (Boltzmann distribution) over the velocity space gives the exponential distribution of the potential energy (Boltzmannian density profile)

\[
P(r; \kappa_0 \to \infty, T) = \int_{-\infty}^{+\infty} P(r, \mathbf{u}; \kappa_0 \to \infty, T) \times d\mathbf{u} \propto \int_{-\infty}^{+\infty} e^{-\frac{\Phi(r)}{k_B T}}d\mathbf{u} \propto e^{-\frac{\Phi(r)}{k_B T}},
\]

(11a)

and

\[
\frac{1}{2}d_k k_B T(r) = \frac{\int_{-\infty}^{+\infty} \frac{1}{2}e^{-\frac{\Phi(r)}{k_B T}}\partial \Phi(\mathbf{u})d\mathbf{u}}{\int_{-\infty}^{+\infty} \frac{1}{2}e^{-\frac{\Phi(r)}{k_B T}}d\mathbf{u}} = \frac{1}{2}d_k k_B T,
\]

(11b)

thus, we obtain

\[
n(r) = n_0 \cdot e^{-\frac{\Phi(r)}{k_B T}}, \quad T(r) = T,
\]

\[
p(r) = n_0k_B T \cdot e^{-\frac{\Phi(r)}{k_B T}}.
\]

(11c)

Therefore, in the Boltzmannian case we obtain \( p(r) \propto n(r) \), hence \( \gamma = 1 \) (or \( \nu \to \pm \infty \)), corresponding to a single thermodynamic process, the isothermal one.

In the case of kappa distributions, we obtain

\[
p(r) \propto n(r)^{\frac{\nu + \frac{1}{2}d_\Phi}{\kappa_0 + \frac{1}{2}d_\Phi + 1}} \quad \text{or} \quad n(r) \propto \frac{T(r)^{\nu - \frac{1}{2}d_\Phi}}{T_{\text{eq}}^{\nu - \frac{1}{2}d_\Phi + 1}},
\]

(12a)

leading to the relationship between kappa and polytropic indices:

\[
\gamma = \frac{\kappa_0 + \frac{1}{2}d_\Phi}{\kappa_0 + \frac{1}{2}d_\Phi + 1} \quad \text{or} \quad \nu = \kappa_0 + \frac{1}{2}d_\Phi + 1.
\]

(12b)

Note that we may express this relationship in terms of the \( d \)-dimensional kappa index, i.e., \( \nu = -\kappa + \frac{1}{2}d_k - 1 \); for \( d_k = 3 \) it becomes \( \nu = -\kappa + \frac{1}{2} \), that is, the formulation derived in the earlier studies (e.g., Meyer-Vernet et al. 1995; Moncuquet et al. 2002); care must be shown though, as the involved kappa index \( \kappa = \kappa_0 + \frac{1}{2}d_\Phi \) differs from the standard three-dimensional one, \( \kappa_3 = \kappa_0 + \frac{3}{2} \), because of the nonzero potential degrees of freedom \( \frac{1}{2}d_\Phi \).

### 3. Hydrodynamics: Euler’s Momentum Equation

The Navier–Stokes momentum equation in the presence of a conservative external field of nonzero potential energy \( \Phi \) is given by

\[
\rho \left[ \frac{d}{dt} \left( \mathbf{u} \cdot \nabla \right) \mathbf{u} \right] = -\nabla p - n \nabla \Phi + \nabla \cdot \mathbf{R}.
\]

(13)

This becomes Euler’s equation when the viscosity tensor \( \mathbf{R} \) is neglected. If the velocity vector field is also independent of the position vector, then the convective acceleration term vanishes, \( (\mathbf{u} \cdot \nabla) \mathbf{u} = 0 \). Finally, if the velocity vector field is stationary, then \( d\mathbf{u}/dt = 0 \). Therefore, Equation (13) becomes

\[
\nabla p(r) = -n(r) \cdot \nabla \Phi(r).
\]

(14)

We observe that both Boltzmannian and kappa distributions obey the hydrostatic relationship of Equation (14) and the polytropic relationship in Equation (1). Indeed, in the classical case of Boltzmann distribution, we have the thermal observables shown in Equation 11(c). Then, we find

\[
\nabla p(r) = -n_0 \cdot e^{-\frac{\Phi(r)}{k_B T}} \cdot \nabla \Phi(r) = -n(r) \cdot \nabla \Phi(r).
\]

(15a)

In the case of kappa distributions, the pressure is given by Equation (10); then, using also Equations (8), (9), we find

\[
\nabla p(r) = -n_0 \cdot \left[ 1 + \frac{1}{\kappa_0} \cdot \frac{\Phi(r)}{k_B T} \right]^{-\kappa_0 - \frac{1}{2}d_\Phi - 1} \cdot \left( \frac{T_0}{T} \cdot \frac{\kappa_0 + \frac{1}{2}d_\Phi}{\kappa_0} \right) \cdot \nabla \Phi(r).
\]

(15b)

Therefore, both Maxwell–Boltzmann and kappa distributions obey to Euler’s Equation (14) and the polytropic relationship of Equation (1). It is an important and interesting question whether these are the only two distribution functions that obey Euler’s equation. What is the most general stationary distribution of particle velocities and energies, consistent with an inviscid and polytropic particle flow under a conservative field? First, we will find the most general density profile, and then we derive the corresponding phase-space distribution.

### 4. Most General Density Profile Consistent with Polytopes

The density and temperature are respectively written in terms of an arbitrary function \( f \),

\[
n(r) = n_0 \cdot \left\{ f \left( \frac{\Phi(r)}{k_B T} \right) \right\}^\nu,
\]

\[
T(r) = T_0 \cdot f \left( \frac{\Phi(r)}{k_B T} \right).
\]

(16a)

with \( f(0) = 1 \) (again, \( n_0 \) and \( T_0 \) are derived at the position \( \mathbf{n}_0 \), for which the potential turns zero).
The polytropic relationship still holds:
\[
\left[ \frac{n(r)}{n_0} \right] = \left[ \frac{T(r)}{T_0} \right]^\gamma. \tag{16b}
\]

Then, the thermal pressure becomes
\[
p(r) = n_k B T_0 \cdot \left\{ \frac{\Phi(r)}{k B T} \right\}^{r+1}, \tag{16c}
\]
and applying Equations 16(a), (c) in Euler’s Equation (14), we find
\[
\nabla p(r) = (\nu + 1) \frac{T_0}{T} \cdot f' \left[ \frac{\Phi(r)}{k B T} \right] \cdot n(r) \cdot \nabla \Phi(r)
\]
\[
= -n(r) \cdot \nabla \Phi(r), \text{ or}
\]
\[
\left\{ 1 + (\nu + 1) \frac{T_0}{T} \right\} \cdot f' \left[ \frac{\Phi(r)}{k B T} \right] \cdot n(r) \cdot \nabla \Phi(r) = 0, \tag{17}
\]
leading to
\[
f' \left[ \frac{\Phi(r)}{k B T} \right] = -\frac{1}{(\nu + 1) \frac{T_0}{T}} = \text{constant.} \tag{18}
\]

The right-hand side is independent of the position, and, given that \( f(0) = 1 \), we find
\[
f' \left[ \frac{\Phi(r)}{k B T} \right] = 1 + \frac{\Phi(r)}{\nu + 1} \frac{T_0}{T} \cdot \Phi(r). \tag{19}
\]

Then, we have
\[
n(r) = n_0 \cdot \left[ 1 + \frac{1}{(\nu + 1) \frac{T_0}{T}} \cdot \Phi(r) \right]^\gamma, \tag{20a}
\]
\[
T(r) = T_0 \cdot \left[ 1 + \frac{\Phi(r)}{\nu + 1} \frac{T_0}{T} \right]. \tag{20b}
\]

Averaging the temperature profile over the whole positional space gives the global temperature \( T(\langle r \rangle) = T \). Then, from the relationship between \( T(r) \) and \( \Phi(r) \), shown in Equation (20), we obtain
\[
T = \langle T(r) \rangle = T_0 \cdot \left[ 1 + \frac{1}{(\nu + 1) \frac{T_0}{T}} \cdot \langle \Phi(r) \rangle \right]
\]
\[
= T_0 - T \cdot \frac{1}{\nu + 1}. \tag{21}
\]

Using the definition of the potential degrees of freedom, \( \frac{1}{\nu + 1} \frac{T_0}{T} = \frac{\langle \Phi(r) \rangle}{(k_B T)} \). Then, the constant in Equation (18) with \( \kappa_0 \equiv -(\nu + 1) \frac{T_0}{T} \). Then, substituting in Equations (20), (22), we find:
\[
T_0 = T \cdot \frac{\kappa_0}{\frac{\kappa_0}{\frac{\gamma}{\nu + 1}}} \quad \text{and} \quad \kappa_0 = -(\nu + 1) \frac{T_0}{T}, \tag{23}
\]
and the kappa distribution density and temperature profiles
\[
n(r) = n_0 \cdot \left[ 1 + \frac{\Phi(r)}{k_B T} \right]^{\kappa_0 - 1 - \frac{1}{\nu + 1} \frac{T_0}{T}}, \tag{24a}
\]
\[
T(r) = T_0 \cdot \left[ 1 + \frac{\Phi(r)}{k_B T} \right]. \tag{24b}
\]

### 5. Using the Bernoulli Integral

The Bernoulli’s integral of energy \( E \) for an incompressible hydrodynamic fluid (i.e., \( n(r) = n_0 \) and \( B(r) = 0 \)), expressed in the comoving reference frame along with the fluid, for a point in the plasma flow streamline with position vector \( r \), is given by
\[
\Phi(r) + \frac{p(r)}{n(r)} = \text{constant} \equiv E. \tag{25a}
\]

For a compressible hydrodynamic fluid, the ratio of the thermal term, \( p/n \), is substituted by the integral \( \int dp/n \), i.e., applying the polytropic relationship between the two different streamline points with position vectors \( r \) and \( n_0 \) (for which \( \Phi = 0 \)), that is, \( p = p_0 \cdot (n/n_0)^\gamma \), we obtain
\[
\int_{p_0}^{p} dp/n = [\gamma/(\gamma - 1)] p_0 n_0^{-\gamma} (n^{-\gamma} - n_0^{-\gamma})
\]
\[
= [\gamma/(\gamma - 1)] (p/n - p_0/n_0). \tag{25b}
\]

Hence, the Bernoulli integral is
\[
\Phi(r) + \frac{\gamma}{\gamma - 1} \int \frac{p(r)/n(r) - p_0/n_0}{\Phi(r)} = \text{constant} \equiv E. \tag{25c}
\]

We determine the value of the constant \( E \) at the position \( n_0 \), where \( \Phi(n_0) = 0 \), \( n(r) = n_0 \), \( T(r) = T_0 \), i.e., \( E = [\gamma/(\gamma - 1)] \cdot p_0/n_0 \). Then,
\[
\Phi(r) + \frac{\gamma}{\gamma - 1} \int \frac{p(r)/n(r) - p_0/n_0}{\Phi(r)} = \Phi(r) + (\nu + 1) \frac{T_0}{T} \cdot \Phi(r) = 0. \tag{25d}
\]

Substituting \( T(r) \) from Equation (16), we find,
\[
\frac{\Phi(r)}{k_B T} + (\nu + 1) \frac{T_0}{T} \cdot f' \left[ \frac{\Phi(r)}{k_B T} \right] = (\nu + 1) \frac{T_0}{T}, \tag{26}
\]
which leads again to Equation (19) and the profiles in Equations 24(a), (b).

Another way to show Equations 24(a), (b) is as follows. Let the identity
\[
\frac{\gamma}{\gamma - 1} \left[ \frac{p}{n} - \frac{p_0}{n_0} \right] = \frac{\gamma}{\gamma - 1} \cdot \frac{p}{n} \cdot \left[ 1 - \left( \frac{n}{n_0} \right)^{1-\gamma} \right]
\]
\[
= \gamma \cdot \frac{p}{n} \cdot \ln \left( \frac{n}{n_0} \right) \tag{27}
\]

The functions \( \exp_q(x) \) and \( \ln_q(x) \) are called “\( q \)-deformed” exponential and logarithm, respectively (e.g., Silva et al. 1998; Yamano 2002; Livadiotis & McComas 2009; Livadiotis 2017, Chapter 1),
\[
\ln_q(x) = \frac{1 - x^{1-q}}{q - 1}, \exp_q(x) = [1 + (1 - q) \cdot x]^{1/q}. \tag{28}
\]
and they are inverse to each other, i.e., \( \ln_n \exp_n(x) = x \).

Using Equation (27), we rewrite Equation 25(c) as

\[
\Phi(r) + \gamma \cdot k_B T(r) \cdot \ln_n \left[ \frac{n(r)}{n_0} \right] = 0. \tag{29}
\]

Its inverse gives the density profile, namely:

\[
\frac{n(r)}{n_0} = \exp_n \left[ - \frac{\Phi(r)}{\gamma k_B T(r)} \right] = \left[ 1 + \frac{\Phi(r)}{(\nu + 1)k_B T(r)} \right]^{-\nu}. \tag{30}
\]

Then, substituting \((\nu + 1)k_B T(r)\) from Equation 25(c), Equation (30) becomes

\[
\frac{n(r)}{n_0} = \left[ 1 + \frac{\Phi(r)}{(\nu + 1)k_B T_0 - \Phi(r)} \right]^{-\nu} = \left[ 1 + \frac{1}{(\nu + 1)\frac{k_B T}{T}} \cdot \frac{\Phi(r)}{\Phi(r)} \right]^{-\nu}, \tag{31}
\]

leading again to the kappa distribution density profile shown in Equation (20), and thus to Equation (24).

6. In the Presence of a Magnetic Field

For a compressible magneto-hydrodynamic fluid, the thermal pressure \( p \) also includes the magnetic pressure and magnetic energy density, both equal to \( B^2(r)/(2\mu); \) namely, the Bernoulli integral becomes

\[
\Phi(r) + \frac{p(r)}{\rho(r)} + \gamma \cdot \frac{\rho(r)}{\rho(r)} = E, \quad \text{or} \quad (\nu + 1)k_B T(r) + \Phi_{\text{eff}}(r) = (\nu + 1)k_B T_0, \tag{32a}
\]

where the effective potential energy is defined by the summation of the external potential energy and the magnetic energy,

\[
\Phi_{\text{eff}}(r) \equiv \Phi(r) + \frac{B^2(r)}{\mu n(r)}. \tag{33}
\]

Hereafter, the steps are those followed in the previous section, but now they are expressed in terms of the effective potential energy; then, we find

\[
n(r) = n_0 \cdot \left[ 1 + \frac{1}{\kappa_0} \cdot \frac{\Phi_{\text{eff}}(r)}{k_B T} \right]^{-\nu},
\]

\[
T(r) = T_0 \cdot \left[ 1 + \frac{1}{\kappa_0} \cdot \frac{\Phi_{\text{eff}}(r)}{k_B T} \right], \tag{34}
\]

where the potential degrees of freedom are defined by \( \frac{1}{2}d_{\Phi_{\text{eff}}} = (\Phi_{\text{eff}}(r))/(k_B T) \), with \( \kappa_0 + \nu + 1 + \frac{1}{2}d_{\Phi_{\text{eff}}} = 0 \).

7. Most General Phase-space Distribution Consistent with Polytropes

Having shown that the polytropic behavior requires a density profile described by kappa distributions, it is straightforward now to search for the last piece of the puzzle, that is, to show that the kappa distribution density profile, shown in Equation (24) or (34), is coming from the standard phase-space kappa distribution. In other words, we need to show that the phase-space kappa distribution is the unique distribution function that has a positional marginal distribution described by the mentioned kappa distribution profile.

The positional kappa distribution is given after the integration in the velocity space,

\[
P(r; \kappa_0, T) = \int_{-\infty}^{+\infty} \left[ 1 + \frac{1}{\kappa_0} \cdot \frac{\Phi(r)}{k_B T} \right]^{-\kappa_0 - 1 - \frac{1}{2}d_k - \frac{1}{2}d_0} du \propto \left[ 1 + \frac{1}{\kappa_0} \cdot \frac{\Phi(r)}{k_B T} \right]^{-\kappa_0 - 1 - \frac{1}{2}d_k - \frac{1}{2}d_0}. \tag{35a}
\]

This integration can be expressed in terms of the kinetic and potential energies treated as variables:

\[
\int_{0}^{\infty} \left[ 1 + \frac{1}{\kappa_0} \cdot \frac{\Phi(r)}{k_B T} \right]^{-\kappa_0 - 1 - \frac{1}{2}d_k - \frac{1}{2}d_0} d\Phi \propto \left( 1 + \frac{1}{\kappa_0} \cdot \frac{\Phi(r)}{k_B T} \right)^{-\kappa_0 - 1 - \frac{1}{2}d_k - \frac{1}{2}d_0}. \tag{35b}
\]

The latter can be written as

\[
\int_{0}^{\infty} \left[ 1 + \frac{1}{\kappa_0} \cdot \frac{\Phi(r)}{k_B T} \right]^{-\kappa_0 - 1 - \frac{1}{2}d_k - \frac{1}{2}d_0} d\Phi \propto \left( 1 + \frac{1}{\kappa_0} \cdot \frac{\Phi(r)}{k_B T} \right)^{-\kappa_0 - 1 - \frac{1}{2}d_k - \frac{1}{2}d_0}. \tag{35c}
\]

Therefore, the question we wish to answer is, what is the most general functional form \( f \), that is involved in the following integral equation:

\[
\int_{0}^{\infty} f(x+y) \frac{1}{2} dk^{-1} dx \propto \left( 1 + \frac{1}{\kappa_0} \cdot \frac{\Phi(r)}{k_B T} \right)^{-\kappa_0 - 1 - \frac{1}{2}d_k - \frac{1}{2}d_0}. \tag{36}
\]

We can rewrite the above as follows:

\[
\int_{-\infty}^{\infty} f(y-x) \frac{1}{2} dk^{-1} \Theta(-x) dx \propto \left( 1 + \frac{1}{\kappa_0} \cdot \frac{\Phi(r)}{k_B T} \right)^{-\kappa_0 - 1 - \frac{1}{2}d_k - \frac{1}{2}d_0}. \tag{37}
\]

This is a special case of the Fredholm integral equation of the first kind (Arfken 1985), written as

\[
\{ f \circ h \}(y) = \int_{-\infty}^{\infty} f(y-x) h(x) dx \propto g(y), \tag{38}
\]

where

\[
h(x) = x \frac{1}{2} dk^{-1} \Theta(-x), \quad g(x) = g_0 \cdot \left( 1 + \frac{1}{\kappa_0} \cdot \frac{\Phi(r)}{k_B T} \right)^{-\kappa_0 - 1 - \frac{1}{2}d_k - \frac{1}{2}d_0}, \tag{39}
\]

\( (g_0 \text{ includes the normalization constants. This type of integral equation is a convolution of } f \text{ and } h \text{ and it can be solved in order to find } f \text{ as follows. We apply the Fourier transformation } F \text{ in both sides of Equation (38)}, \)

\[
\hat{F} \{ f(x) \} \cdot \hat{F} \{ h(x) \} = \hat{F} \{ g(x) \}, \tag{40}
\]

then the function \( f \) is expressed as

\[
f(x) = \hat{F}^{-1} \left\{ \frac{\hat{F} \{ g(x) \} \{ \xi \}}{\hat{F} \{ h(x) \} \{ \xi \}} \right\}(x). \tag{41}
\]

We have the identities

\[
\hat{F} \{ x^{\frac{1}{2} d_k^{-1} \Theta(-x)} \} \{ \xi \} \propto \xi^{-\frac{1}{2} d_k}, \quad \hat{F}^{-1} \{ x^{\frac{1}{2} d_k^{-1} \Theta(-x)} \} \propto x^{-\frac{1}{2} d_k - 1}, \tag{42}
\]
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thus,

\[ f(x) \propto \hat{b}^{-1} \left\{ \hat{F} \left( g(x) \right) \left( \xi, \xi^2 \right) \right\}(x) \propto g(x) \propto \chi^{-1} \xi^{4k-1}, \]

where we defined the convolution function

\[ g(x) \propto \chi^{-1} \xi^{4k-1} = \int_{0}^{\infty} g(x-y) y^{-1} \xi^{4k-1} dy. \]

Thus, we have

\[ g(x) \propto \chi^{-1} \xi^{4k-1} \]

\[ \int_{0}^{\infty} \left[ 1 + \frac{1}{\kappa_0} \cdot (x-y) \right]^{-\kappa_0-1} y^{-1} \xi^{4k-1} dy \]

\[ \propto \left( 1 + \frac{1}{\kappa_0} \chi \right)^{-\kappa_0-1} \xi^{-1} \xi^{4k-1} \]

\[ \times \int_{0}^{\infty} (1 - t)^{-\kappa_0-1} \xi^{4k-1} dt, \]

where \( t = y \left/ \left( 1 + \frac{1}{\kappa_0} \chi \right) \right. \), i.e.,

\[ g(x) \propto \chi^{-1} \xi^{4k-1} \left( 1 + \frac{1}{\kappa_0} \chi \right)^{-\kappa_0-1} \xi^{-1} \xi^{4k-1}. \]

Hence, substituting Equation (46) into (43), we find that

\[ f(x) = \left( 1 + \frac{1}{\kappa_0} \chi \right)^{-\kappa_0-1} \xi^{-1} \xi^{4k-1} \xi^{-1}. \]

Namely, it is finally shown that the function \( f \) of Equation (36) is exactly the one shown in Equation 35(c), that is, the standard kappa distribution function. Therefore, the only function \( f \) involved in the density profile,

\[ P(r, \kappa_0, T) = \int_{-\infty}^{\infty} f \left\{ \Phi_{k}(a) + \Phi_{k}(b) \right\} \left( -\kappa_0^{-1} \xi^{4k-1} \right) \]

\[ \times du \left[ 1 + \frac{1}{\kappa_0} \cdot \Phi_{k}(r) \right] ^{-\kappa_0-1} \xi^{4k-1}, \]

is the phase-space kappa distribution function (4), as shown in Equation 35(a). Hence, the equivalence:

\[ P(r, u) \propto \left( 1 + \frac{1}{\kappa_0} \right) \cdot \left( -\kappa_0^{-1} \xi^{4k-1} \right) \]

\[ \Leftrightarrow n(r) \propto \left( 1 + \frac{1}{\kappa_0} \right) \cdot \Phi_{k}(r) ^{-\kappa_0-1} \xi^{4k-1}. \]

Notes. Similar derivation steps can lead to a phase-space distribution with anisotropy in the velocities. Also, other non-dynamical effects may also cause the polytropic behavior, e.g., the spherical expansion of the solar wind plasma, where the total density profile would be given by the convolution of the two terms.

8. Discussion

The derived equivalence between the polytropic behavior and the kappa distribution functions can be valuable for further modeling and data analyses. The following are some example applications:

(1) Polytropic plasmas are described by kappa distributions (or combinations thereof). An observation of the polytropic behavior of plasma populations in the presence of an external field, constitutes an indirect observation of their distribution function.

(2) The existing Rankine–Hugoniot jump conditions at shock discontinuities do not include the kappa index, because the conservation of mass, momentum, and energy equations are independent of the kappa index (Livadiotis 2015d). The strong connection between polytropic and kappa indices introduces the new type of jump condition. Assuming the same potential upstream (1) and downstream (2) the shock, we find the new jump condition:

\[ \kappa_1 + \nu_1 = \kappa_2 + \nu_2. \]

This relationship can be more complicated when the magnetic field is included (as in Equation (33)). The polytropic index may vary spatially, even across a shock (e.g., Nicolaou & Livadiotis 2017). In such a case, the difference of the polytropic indices \( \nu \) can determine the difference on the kappa indices.

(3) The density and temperature profiles, expressed in terms of the potential and magnetic energies (as shown in Equation (34)), can be used to model possible correlations among the plasma moments and the magnetic field. For example, if the magnetic energy prevails over the potential energy (e.g., for low plasma beta parameter—the ratio of plasma thermal pressure to the magnetic pressure), then if we solve in terms of the magnetic field in Equation (34), we obtain

\[ \frac{B(r)}{\mu n_n(k_B T)} = \left[ \frac{T(r)}{T_0} \right]^{\gamma} - \frac{n(r)}{n_0}. \]

The magnetic field is plotted in Figure 1 in terms of the density profile, or the temperature profile derived from the polytropic relation 16(b),

\[ \frac{B(r)}{\mu n_n(k_B T)} = \left[ \frac{T(r)}{T_0} \right]^{\gamma} - \frac{n(r)}{n_0}. \]

In the approximation of low density, \( n(r) \ll n_0 \), the density profile gives \( B(r) \propto n(r)^\gamma \), leading to \( B(r) \propto \mu n_n(k_B T) \); hence, we also have that the plasma beta \( \beta(r) = \rho(r) \left[ \frac{1}{2} B(r)^2 \right] \) is constant along the plasma streamlines.

(4) The estimation of the polytropic index may lead to the indirect derivation of the kappa index through \( \kappa_0 = -\nu - \frac{1}{2} d_b - 1 \) or \( \kappa_3 = -\nu + \frac{1}{2} d_b + \frac{1}{2} \). The application requires the knowledge of the potential energy (Equation (33)); examples are the power-law attractive potentials, \( \Psi_{\text{eff}}(r) = \pm \frac{1}{b} \cdot r \cdot r^\pm \), where \( \frac{1}{2} d_b = \pm \frac{1}{2} d \), indicating the dimensionality of the position vector (e.g., \( d_e = \left[ \frac{1}{2} \right] \) for a 3D system); hence, \( \kappa_3 = \frac{1}{2} \pm \frac{1}{2} \left[ \frac{1}{2} \right] \) (Livadiotis 2017, Chapters 3–5). Finding the polytropic indices requires only the moments of density and temperature. On the other hand, finding the kappa indices is a harder problem that involves analysis of the velocity/energy distributions. In addition, the data sets of distributions are usually not publicly accessible, making this methodology of finding kappa quite valuable.

(5) The estimation of the polytropic and kappa indices may lead to the determination of the potential degrees of freedom, which can help with figuring the local potential energy that applies in the examined plasma particles (e.g., Livadiotis 2018b). In this case, the analysis should be restricted to data sets with slope \( -1 \) between the estimated kappa and polytropic indices, according to \( \kappa_0 = -\frac{1}{2} d_b - 1 - \nu \), or, for power-law attractive
potentials becomes, \( \kappa_0 = \frac{d}{b} - 1 - \nu \). Then, the intercept \( \frac{d}{b} - 1 \) gives the value of \( b \), from which we can determine the type of the potential.

### 9. Conclusions

Kappa distributions have been ambitiously used in studies of space and astrophysical plasmas. While it was known that these distributions can lead to polytropes—particle systems with a polytropic behavior—it was unexpected that polytropes could restrict the functional form of the distribution of particle velocities and energies. However, this paper has shown that the reverse is also true: polytropes are consistent only with kappa distributions! In other words, not only the kappa distributions lead to polytropic behavior, but also polytropic behavior leads to kappa distributions (Livadiotis et al. 2018).

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