\[ L_f^2 \]-harmonic 1-forms on smooth metric measure spaces with positive \( \lambda_1(\Delta_f) \)

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**Abstract.** In this paper, we study vanishing and splitting results on a complete smooth metric measure space \((M^n, g, e^{-f} dv)\) with various negative \(m\)-Bakry-Émery Ricci curvature lower bounds in terms of the first eigenvalue \(\lambda_1(\Delta_f)\) of the weighted Laplacian \(\Delta_f\), i.e., \(\text{Ric}_{m,n} \geq -a\lambda_1(\Delta_f) - b\) for \(0 < a \leq \frac{m}{m-1}, b \geq 0\). In particular, we consider three main cases for different \(a\) and \(b\) with or without conditions on \(\lambda_1(\Delta_f)\). These results are extensions of Dung and Vieira, and weighted generalizations of Li-Wang, Dung-Sung, and Vieira.

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**1. Introduction.** One of the central problems in differential geometry is the relation between the geometry and topology of a manifold. The study of \(L^2\)-harmonic forms is such kind of problem, see, for example, [1, 2, 11] etc. and references therein. Let \(M\) be a complete Riemannian manifold and \(\lambda_1(M)\) be the first eigenvalue of the Laplacian on \(M\), which can be characterized by

\[ \lambda_1(M) = \inf_{\psi \in C_0^\infty(M)} \frac{\int_M |\nabla \psi|^2 \, dv}{\int_M \psi^2 \, dv}. \]

In the work of Li and Wang [8], they proved the following vanishing theorem for \(L^2\)-harmonic 1-forms on manifolds whose Ricci curvature is bounded below by a negative multiple of the first eigenvalue.

**Theorem 1.1** ([8, Theorem 4.2]). Let \(M\) be an \(n\)-dimensional complete Riemannian manifold with \(\lambda_1(M) > 0\) and

\[ \text{Ric}_M \geq -\frac{n}{n-1} \lambda_1(M) + \delta \]
for some \( \delta > 0 \). Then \( \mathcal{H}^1(L^2(M)) = 0 \), where \( \mathcal{H}^1(L^2(M)) \) denotes the space of \( L^2 \) integrable harmonic 1-forms on \( M \).

After that, Lam [7] generalized Li-Wang’s theorem to manifolds with a weighted Poincaré inequality.

Later, Dung [3] asked: what is the geometric structure of \( M \) if \( \delta = 0 \)? Dung and Sung [5] get the following theorem.

**Theorem 1.2** ([5, Theorem 2.2]). Let \( M \) be a complete Riemannian manifold of dimension \( n \geq 3 \). Suppose that \( \lambda_1(M) > 0 \) and 

\[
\text{Ric}_M \geq -\frac{n}{n-1} \lambda_1(M).
\]

Then either

1. \( \mathcal{H}^1(L^2(M)) = 0 \); or
2. \( \tilde{M} = \mathbb{R} \times N \), where \( \tilde{M} \) is the universal cover of \( M \) and \( N \) is a manifold of dimension \( n - 1 \).

More generally, Dung [3] considered the smooth metric measure space \( (M, g, e^{-f} dv) \), which is a smooth Riemannian manifold \( (M, g) \) together with a smooth function \( f \) and a measure \( e^{-f} dv \). For any constant \( m \geq n = \dim M \), we have the \( m \)-dimensional Bakry-Émery Ricci curvature

\[
\text{Ric}_{m,n} = \text{Ric} + \nabla^2 f - \frac{\nabla f \otimes \nabla f}{m - n},
\]

which is just the Ricci tensor if \( m = n \), and usually we denote by \( \text{Ric}_f = \text{Ric} + \nabla^2 f \) the \( \infty \)-dimensional Bakry-Émery Ricci curvature. Hence, in the following, we are actually dealing with \( m > n \).

Denote by \( \lambda_1(\Delta_f) \) the first eigenvalue of the \( f \)-Laplacian on \( M \), which can be similarly characterized by

\[
\lambda_1(\Delta_f) = \inf_{\psi \in C_0^\infty(M)} \frac{\int_M |\nabla \psi|^2 \cdot e^{-f} dv}{\int_M \psi^2 \cdot e^{-f} dv}.
\]

Then Dung [3] proved the following result which is concerned with vanishing for the space of \( L^2_f \)-harmonic functions and the splitting of \( M \).

**Theorem 1.3** ([3, Theorem 1.3]) Let \( (M, g, e^{-f} dv) \) be a complete non-compact smooth metric measure space of dimension \( n \geq 3 \) with positive eigenvalue \( \lambda_1(\Delta_f) > 0 \). Assume that

\[
\text{Ric}_{m,n} \geq -\frac{m}{m-1} \lambda_1(\Delta_f).
\]

Then either

1. \( \mathcal{H}(L^2_f(M)) = \mathbb{R} \), where \( \mathcal{H}(L^2_f(M)) \) is the space of \( f \)-harmonic functions with finite \( f \)-energy; or
2. \( M = \mathbb{R} \times N \) with the warped product metric

\[
ds_M^2 = dt^2 + \eta^2(t) ds_N^2,
\]

where \( \eta(t) \) is a positive function and \( N \) is an \((n - 1)\)-dimensional manifold.
For the space of $L^2_f$-harmonic 1-forms, Vieira proved in the following [12].

**Theorem 1.4** ([12, Theorem 1.1]). Let $(M, g, e^{-f} dv)$ be a complete non-compact smooth metric measure space with non-negative $\infty$-Bakry-Émery Ricci curvature. If the space of $L^2_f$-harmonic 1-forms is non-trivial, then the weighted volume of $M^n$ is finite, that is,

$$\text{vol}_f(M^n) = \int_{M^n} e^{-f} dv < \infty,$$

and the universal covering splits isometrically as $\tilde{M}^n = \mathbb{R} \times N^{n-1}$.

As a corollary, Vieira obtained ([12, Corollary 1.2]) that with the same curvature assumption, if the first eigenvalue of the $f$-Laplacian is positive, then the space of $L^2_f$-harmonic 1-forms is trivial.

Inspired by Li-Wang, Vieira, Dung, and Dung-Sung’s work, in this paper, we extend Vieira’s Theorem 1.4 by relaxing the curvature condition to be $\text{Ric}_{m,n} \geq -a\lambda_1(\Delta_f)$, and generalize Dung-Sung’s Theorem 1.2 to complete non-compact smooth metric measure spaces, which can also be considered as an extension of Dung’s Theorem 1.3. More precisely, we get the following results.

**Theorem 1.5** (Theorem 3.1 in this paper). Let $(M, g, e^{-f} dv)$ be a complete non-compact smooth metric measure space of dimension $n \geq 3$ with $m$-Bakry-Émery Ricci curvature satisfying

$$\text{Ric}_{m,n} \geq -a\lambda_1(\Delta_f),$$

where $0 < a < \frac{m}{m-1}$. If the space of $L^2_f$-harmonic 1-forms is non-trivial, then the weighted volume of $M$ is finite, $\lambda_1(\Delta_f) = 0$, and the universal covering splits isometrically as $\tilde{M}^n = \mathbb{R} \times N^{n-1}$.

A corollary is that, under the same curvature assumption, if $\lambda_1(\Delta_f) > 0$, then $H^1(L^2_f(M)) = \{0\}$.

**Theorem 1.6** (Theorem 3.3 in this paper). Let $(M, g, e^{-f} dv)$ be a complete non-compact smooth metric measure space of dimension $n \geq 3$ with positive first eigenvalue $\lambda_1(\Delta_f)$. Assume that the $m$-Bakry-Émery Ricci curvature satisfies

$$\text{Ric}_{m,n} \geq -\frac{m}{m-1}\lambda_1(\Delta_f).$$

Then either

(1) $H^1(L^2_f(M)) = 0$; or

(2) $\tilde{M} = \mathbb{R} \times N$, where $\tilde{M}$ is the universal cover of $M$ and $N$ is a manifold of dimension $n-1$.

Finally, if $\lambda_1(\Delta_f)$ has some positive lower bound, then the conditions on $\text{Ric}_{m,n}$ can be further relaxed (Theorem 3.5), which is in the spirit of [13, Theorem 6 and 7].
2. Preliminaries. For a smooth metric measure space \((M, g, e^{-f}dv)\), analogously to \(L^2\)-differential forms, a differential form \(\omega\) is called an \(L^2_f\)-differential form if

\[
\int_M |\omega|^2 e^{-f}dv < \infty.
\]

By [1], the formal adjoint of the exterior derivative \(d\) with respect to the \(L^2_f\)-inner product is

\[
\delta_f = \delta + i \nabla f.
\]

Then the \(f\)-Hodge Laplacian operator is defined as

\[
\Delta f = -(d\delta_f + \delta_f d).
\]

Since the first eigenvalue of the weighted Laplacian \(\Delta_f\) is given by

\[
\lambda_1(\Delta_f) = \inf_{\psi \in C_0^\infty(M)} \frac{\int_M |\nabla \psi|^2 \cdot e^{-f}dv}{\int_M \psi^2 \cdot e^{-f}dv},
\]

by the variational principle, we have the following Poincaré type inequality

\[
\lambda_1(\Delta_f) \int_M \psi^2 \cdot e^{-f}dv \leq \int_M |\nabla \psi|^2 \cdot e^{-f}dv \quad \text{for} \quad \psi \in C_0^\infty(M).
\]

Next, we establish and recall some lemmas to be used later. By a smart application of the elementary inequality

\[
(a + b)^2 \geq \frac{a^2}{1 + \alpha} - \frac{b^2}{\alpha}, \quad \forall \alpha > 0,
\]

(2.1)

Li obtained a Bochner type inequality for \(f\)-harmonic functions, which is [10, Lemma 2.1]. Here we adopt Li’s idea [10] to get the following Bochner type inequality for \(L^2_f\)-harmonic 1-forms, which will play a key role in this paper.

**Lemma 2.1.** Let \(\omega\) be an \(L^2_f\)-harmonic 1-form on an \(n\)-dimensional complete smooth metric measure space \((M, g, e^{-f}dv)\) and \(m \geq n\) be any constant. Then

\[
|\omega| \Delta_f |\omega| \geq \frac{|\nabla |\omega||^2}{m - 1} + \text{Ric}_{m,n}(\omega, \omega).
\]

(2.2)

Equality holds iff

\[
(\omega_{i,j}) = \begin{pmatrix}
-(m - 1)\mu & 0 & 0 & \ldots & 0 \\
0 & \mu & 0 & \ldots & 0 \\
0 & 0 & \mu & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \mu
\end{pmatrix},
\]

where \(\mu = \frac{\langle \nabla f, \omega \rangle}{n - m}\).
Proof. By the weighted Bochner formula [11, Equation 2.10] and [12, Lemma 3.1])
\[
\frac{1}{2} \Delta f |\omega|^2 = |\nabla \omega|^2 + \Delta_f \omega \cdot \omega + \text{Ric}_f(\omega, \omega),
\]
we have
\[
\frac{1}{2} \Delta f |\omega|^2
\leq |\nabla \omega|^2 + \langle \Delta_f \omega, \omega \rangle + \text{Ric}_{m,n}(\omega, \omega) + \frac{\nabla f \otimes \nabla f}{m - n}(\omega, \omega). \tag{2.3}
\]
It also holds that
\[
\frac{1}{2} \Delta f |\omega|^2 = |\omega| \Delta_f |\omega| + |\nabla| |\omega|^2.
\]
Hence, if \( \Delta_f \omega = 0 \), we have
\[
|\omega| \Delta_f |\omega| = |\nabla \omega|^2 - |\nabla| |\omega|^2 + \text{Ric}_{m,n}(\omega, \omega) + \frac{\nabla f \otimes \nabla f}{m - n}(\omega, \omega).
\]
By [12, Lemma 2.2], \( \Delta_f \omega = 0 \) is equivalent to
\[
\begin{cases}
\omega_{i,j} = \omega_{j,i}, & i, j = 1, \ldots, n, \\
\sum_{i=1}^{n} \omega_{i,i} = \langle \nabla f, \omega \rangle.
\end{cases} \tag{2.4}
\]
Choose an appropriate local frame such that \( \omega_1 = \frac{\omega}{|\omega|} \), then
\[
|\nabla| |\omega|^2 = \sum_{j=1}^{n} \omega_{1,j}^2.
\]
Since
\[
|\nabla \omega|^2 = \omega_{1,1}^2 + \sum_{j=2}^{n} \omega_{1,j}^2 + \sum_{j=2}^{n} \omega_{j,1}^2 + \sum_{i=2}^{n} \omega_{i,i}^2 + \sum_{i,j=2, i \neq j}^{n} \omega_{i,j}^2
\geq \omega_{1,1}^2 + 2 \sum_{j=2}^{n} \omega_{1,j}^2 + \frac{1}{n-1} \left( \sum_{i=2}^{n} \omega_{i,i} \right)^2
= \omega_{1,1}^2 + 2 \sum_{j=2}^{n} \omega_{1,j}^2 + \frac{1}{n-1} (-\omega_{1,1} + \langle \nabla f, \omega \rangle)^2, \tag{2.5}
\]
where the Cauchy-Schwarz inequality and the fact that $\omega$ is $\Delta_f$-harmonic are being used in (2.5), by (2.1), we have for any positive number $\alpha$,

\[
|\nabla \omega|^2 - |\nabla |\omega||^2 \geq \sum_{j=2}^{n} \omega_{1,j}^2 + \frac{1}{n-1} (-\omega_{1,1} + \langle \nabla f, \omega \rangle)^2
\]

\[
\geq \sum_{j=2}^{n} \omega_{1,j}^2 + \frac{1}{n-1} \left( \frac{\omega_{1,1}^2}{1+\alpha} - \frac{\langle \nabla f, \omega \rangle^2}{\alpha} \right)
\]

\[
\geq \frac{1}{(1+\alpha)(n-1)} \sum_{j=2}^{n} \omega_{1,j}^2 - \frac{1}{\alpha(n-1)} \langle \nabla f, \omega \rangle^2.
\]

Hence, if we choose $\alpha = \frac{m-n}{m-1}$,

\[
|\omega|\Delta_f |\omega|
\]

\[
\geq \frac{1}{(1+\alpha)(n-1)} \sum_{j=2}^{n} \omega_{1,j}^2 + \text{Ric}_{m,n}(\omega,\omega) + \frac{\langle \nabla f, \omega \rangle^2}{m-n} - \frac{\langle \nabla f, \omega \rangle^2}{\alpha(n-1)}
\]

\[
= \frac{|\nabla |\omega||^2}{m-1} + \text{Ric}_{m,n}(\omega,\omega).
\]

In addition, equality in (2.2) holds if and only if equality in (2.5),(2.6),(2.7) holds simultaneously. Then “=” in (2.5) implies $\omega_{i,j} = 0$ for $2 \leq i \neq j \leq n$ and $\omega_{2,2} = \omega_{3,3} = \cdots = \omega_{n,n}$; “=” in (2.6) implies $\omega_{1,1} = \frac{m-1}{m-n} \langle \nabla f, \omega \rangle$; “=” in (2.7) implies $\omega_{1,j} = 0$ for $j = 2,\ldots,n$. Hence, by (2.4) and letting $\mu = \frac{\omega_{2,2}}{n-m}$, we finish the proof. \qed

In the proofs of the sequel, we will always use the following cut-off function

\[
\phi_R = \begin{cases} 
1 & \text{on } B(R), \\
0 & \text{on } M \setminus B(2R), 
\end{cases}
\]

such that $|\nabla \phi_R|^2 \leq \frac{C}{R^2}$ on $B(2R) \setminus B(R)$.

The following lemma is a weighted version of the corresponding [7, Lemma 3.1], which can be found as [4, Lemma 4.1].

**Lemma 2.2.** Let $h$ be a non-negative function satisfying the differential inequality

\[
h \Delta_f h \geq -ah^2 + b|\nabla h|^2,
\]

in the weak sense, where $a, b$ are constants and $b > -1$. For any $\varepsilon > 0$, we have the estimate

\[
[b(1-\varepsilon) + 1] \int_M |\nabla (\phi h)|^2 \cdot e^{-f} dv
\]

\[
\leq \left( b \left( \frac{1}{\varepsilon} - 1 \right) + 1 \right) \int_M h^2 |\nabla \phi|^2 \cdot e^{-f} dv + a \int_M \phi^2 h^2 \cdot e^{-f} dv
\]
for any compactly supported smooth function $\phi \in C^\infty_0(M)$. In addition, if
\[
\int_{B_\rho(R)} h^2 \cdot e^{-f} dv = o(R^2),
\]
then
\[
\int_M |\nabla h|^2 \cdot e^{-f} dv \leq \frac{a}{b+1} \int_M h^2 \cdot e^{-f} dv. \tag{2.9}
\]
In particular, $h$ has a finite $f$-Dirichlet integral if $h \in L^2_f(M)$.

We will also need the following result,

**Lemma 2.3.** For an $L^2_f$-integrable function $h$ on $(M, g, e^{-f} dv)$ satisfying the differential inequality
\[
h \Delta_f h \geq -ah^2 + b|\nabla h|^2,
\]
we have
\[
\lim_{R \to \infty} \int_M \phi_R h \langle \nabla \phi_R, \nabla h \rangle \cdot e^{-f} dv = 0, \tag{2.10}
\]
\[
\lim_{R \to \infty} \int_M |\nabla (\phi_R h)|^2 \cdot e^{-f} dv = \int_M |\nabla h|^2 \cdot e^{-f} dv. \tag{2.11}
\]
Moreover,
\[
\lambda_1(\Delta_f) \int_M h^2 \cdot e^{-f} dv \leq \int_M |\nabla h|^2 \cdot e^{-f} dv. \tag{2.12}
\]

**Proof.** Since $h$ is $L^2_f$-integrable, by (2.9) in Lemma 2.2, we have
\[
\int_M |\nabla h|^2 \cdot e^{-f} dv < \infty.
\]
In addition,
\[
\int_M |\nabla (\phi_R h)|^2 \cdot e^{-f} dv
\]
\[
= \int_M h^2 |\nabla \phi_R|^2 \cdot e^{-f} dv + \int_M \phi_R^2 |\nabla h|^2 \cdot e^{-f} dv + 2 \int_M \langle h \nabla \phi_R, \phi_R \nabla h \rangle \cdot e^{-f} dv,
\]
where
\[ \left| \int_M h^2 |\nabla \phi_R|^2 \cdot e^{-f} dv + 2 \int_M \langle h \nabla \phi_R, \phi_R \nabla h \rangle \cdot e^{-f} dv \right| \]
\[ \leq \frac{C}{R^2} \int_M h^2 \cdot e^{-f} dv + \frac{2}{R} \left( \int_M h^2 |\nabla \phi_R|^2 \cdot e^{-f} dv \right)^{\frac{1}{2}} \left( \int_M \phi_R^2 |\nabla h|^2 \cdot e^{-f} dv \right)^{\frac{1}{2}} \]
\[ \leq \frac{C}{R^2} \int_M h^2 \cdot e^{-f} dv + \frac{2\sqrt{C}}{R} \left( \int_M h^2 \cdot e^{-f} dv \right)^{\frac{1}{2}} \left( \int_M \phi_R^2 |\nabla h|^2 \cdot e^{-f} dv \right)^{\frac{1}{2}}. \]

Hence, letting \( R \to \infty \), one gets
\[ \lim_{R \to \infty} \int_M |\nabla (\phi_R h)|^2 \cdot e^{-f} dv = \int_M |\nabla h|^2 \cdot e^{-f} dv. \]

From the proof, we see that (2.10) holds. By the variational principle,
\[ \lambda_1(\Delta f) \int_M (\phi_R h)^2 \cdot e^{-f} dv \leq \int_M |\nabla (\phi_R h)|^2 \cdot e^{-f} dv, \]
and letting \( R \to \infty \), we obtain (2.12).

\[ \square \]

**3. Metric measure spaces with positive first eigenvalue.** In this section, we present several vanishing and splitting results. The following is an extension of [12, Theorem 1.1].

**Theorem 3.1.** Let \((M, g, e^{-f} dv)\) be a complete non-compact smooth metric measure space of dimension \( n \geq 3 \) with \( m \)-Bakry-Émery Ricci curvature satisfying
\[ \text{Ric}_{m,n}(x) \geq -a \lambda_1(\Delta_f), \]
where \( 0 < a < \frac{m}{m-1} \). If the space of \( L^2_f \)-harmonic 1-forms is non-trivial, then the weighted volume of \( M \) is finite, \( \lambda_1(\Delta_f) = 0 \), and the universal covering splits isometrically as \( \tilde{M}^n = \mathbb{R} \times N^{n-1} \).

**Proof.** Choose a non-trivial \( L^2_f \)-harmonic 1-form \( \omega \), and let \( h = |\omega| \). Then by (2.2) and \( \text{Ric}_{m,n} \geq -a \lambda_1(\Delta_f) \), we have
\[ h \Delta_f h \geq \frac{|\nabla h|^2}{m-1} - a \lambda_1(\Delta_f) h^2. \quad (3.1) \]

We multiply by the cut-off function \( \phi_R^2 \) on both sides of (3.1) and by integration by parts, we get
\[ \frac{1}{m-1} \int_M \phi_R^2 |\nabla h|^2 \cdot e^{-f} dv - a \lambda_1(\Delta_f) \int_M \phi_R^2 h^2 \cdot e^{-f} dv \]
\[ \leq - \int_M \phi_R^2 |\nabla h|^2 \cdot e^{-f} dv - 2 \int_M \phi_R h \langle \nabla \phi_R, \nabla h \rangle \cdot e^{-f} dv, \]
for which we have used the Poincaré’s inequality, so

$$\frac{m}{m-1} \int_M \phi_R^2 |\nabla h|^2 \cdot e^{-f} \, dv$$

$$\leq a \lambda_1(\Delta f) \int_M \phi_R^2 \cdot e^{-f} \, dv - 2 \int_M \phi_R h \langle \nabla \phi_R, \nabla h \rangle \cdot e^{-f} \, dv$$

$$\leq a \int_M |\nabla (\phi_R h)|^2 \cdot e^{-f} \, dv - 2 \int_M \phi_R h \langle \nabla \phi_R, \nabla h \rangle \cdot e^{-f} \, dv.$$  

Hence, by Lemma 2.3, letting $R \to \infty$, we obtain

$$\left( \frac{m}{m-1} - a \right) \int_M |\nabla h|^2 \cdot e^{-f} \, dv \leq 0.$$  

Since $a < \frac{m}{m-1}$, $h$ must be a constant. Then

$$\text{vol}_f(M) = \int_M h^2 \cdot e^{-f} \, dv < \infty.$$  

If $\lambda_1(\Delta f) > 0$,

$$\lambda_1(\Delta f) \int_M \phi_R^2 \cdot e^{-f} \, dv \leq \int_M |\nabla \phi_R|^2 \cdot e^{-f} \, dv \leq \frac{C}{R^2} \text{vol}_f(M) \to 0, \text{ as } R \to \infty.$$  

This forces $\phi_R \equiv 0$, which contradicts with the choice of $\phi_R$. Hence, $\lambda_1(\Delta f) = 0$. Then the curvature condition becomes $\text{Ric}_{m,n} \geq 0$, and combining this with the weighted Bochner formula (2.3), we get

$$0 = |\nabla \omega|^2 + \text{Ric}_{m,n}(\omega, \omega) + \frac{\langle \nabla f, \omega \rangle^2}{m-n} \geq |\nabla \omega|^2 + \frac{\langle \nabla f, \omega \rangle^2}{m-n},$$

which implies that $\nabla \omega = 0$ and $\langle \nabla f, \omega \rangle = 0$, i.e., $\omega$ is a parallel 1-form. By lifting $\omega$ to the universal cover $\tilde{M}$ of $M$, we get a non-trivial parallel 1-form $\tilde{\omega}$, which concludes the splitting of $\tilde{M}$ by the de Rham decomposition theorem (see [6, Theorem 6.2]). \qed

A direct corollary is the following vanishing result

**Corollary 3.2.** Let $(M,g,e^{-f} \, dv)$ be a complete non-compact smooth metric measure space of dimension $n \geq 3$ with positive first eigenvalue of the $f$-Laplacian. Assume that the $m$-Bakry-Émery Ricci curvature satisfies

$$\text{Ric}_{m,n}(x) \geq -a \lambda_1(\Delta f),$$

where $0 < a < \frac{m}{m-1}$, then the space of $L^2_f$-harmonic 1-forms is trivial.

Under the same assumption of Corollary 3.2, if $a = \frac{m}{m-1}$, we obtain a generalization of [5, Theorem 2.2].
Theorem 3.3. Let $M$ be a complete metric measure space of dimension $n \geq 3$. Suppose that $\lambda_1(\Delta_f) > 0$ and

$$\text{Ric}_{m,n} \geq -\frac{m}{m-1}\lambda_1(\Delta_f).$$

Then either

(1) $H^1(L^2_f(M)) = 0$; or

(2) $\tilde{M} = \mathbb{R} \times N$, where $\tilde{M}$ is the universal cover of $M$ and $N$ is a manifold of dimension $n - 1$.

Proof. If $H^1(L^2_f(M)) = 0$, there is nothing to prove. Otherwise, let $\omega$ be a non-trivial $L_f$ harmonic 1-form, and let $h = |\omega|$. Then $h$ is $L^2_f$-integrable. Hence,

$$\lim_{R \to \infty} \int_M (\phi_R h)^2 \cdot e^{-f} \, dv = \int_M h^2 \cdot e^{-f} \, dv.$$  

By inequality (2.2) and the assumption on $\text{Ric}_{m,n}$, we have

$$h\Delta_f h \geq -\frac{m\lambda_1(\Delta_f)}{m-1} h^2 + \frac{1}{m-1} |\nabla h|^2.$$  (3.2)

By Lemma 2.3,

$$\lim_{R \to \infty} \int_M |\nabla (\phi_R h)|^2 \cdot e^{-f} \, dv = \int_M |\nabla h|^2 \cdot e^{-f} \, dv,$$

$$\lambda_1(\Delta_f) \int_M h^2 \cdot e^{-f} \, dv \leq \int_M |\nabla h|^2 \cdot e^{-f} \, dv.$$  (3.3)

Similar to the proof of [5, Theorem 2.2], if ">" holds in (3.3), then for sufficiently large $R$, there exists a positive number $\eta$ such that

$$(\lambda_1(\Delta_f) + \eta) \int_M (\phi_R h)^2 \cdot e^{-f} \, dv \leq \int_M |\nabla (\phi_R h)|^2 \cdot e^{-f} \, dv.$$  

By Lemma 2.2, for $a = \frac{m\lambda_1(\Delta_f)}{m-1}, b = \frac{1}{m-1}$, we have for any $\varepsilon > 0$ and sufficiently large $R$,

$$\left[\frac{1}{m-1}(1 - \varepsilon) + 1\right] (\lambda_1(\Delta_f) + \eta) \int_M \phi_R^2 h^2 \cdot e^{-f} \, dv$$

$$\leq \left[\frac{1}{m-1}(1 - \varepsilon) + 1\right] \int_M |\nabla (\phi_R h)|^2 \cdot e^{-f} \, dv$$

$$\leq \frac{m\lambda_1(\Delta_f)}{m-1} \int_M \phi_R^2 h^2 \cdot e^{-f} \, dv + \left[\frac{1}{m-1}\left(\frac{1}{\varepsilon} - 1\right) + 1\right] \int_M h^2 |
abla \phi_R|^2 \cdot e^{-f} \, dv.$$
Hence,
\[
\frac{\eta(1-\varepsilon)}{m-1} \int_{B(R)} h^2 \cdot e^{-f} \, dv \\
\leq \frac{\varepsilon \lambda_1(\Delta f)}{m-1} \int_{B(R)} \phi_R^2 h^2 \cdot e^{-f} \, dv + \left[ \frac{1}{m-1} \left( \frac{1}{\varepsilon} - 1 \right) + 1 \right] \frac{C}{R^2} \int_{B(2R) \setminus B(R)} h^2 \cdot e^{-f} \, dv.
\]
Since \( h \) is \( L^2_f \)-integrable, by letting \( R \to \infty \), and then \( \varepsilon \to 0 \), we have that
\[
\int_M h^2 \cdot e^{-f} \, dv \leq 0.
\]
Hence, \( h \equiv 0 \), which contradicts with the assumption that “\(<" holds in (3.3). If “=” holds in (3.3), equality in (3.2) holds, so by Lemma 2.2,
\[
(\omega_{i,j}) = \begin{pmatrix}
-(m-1)\mu & 0 & 0 & \ldots & 0 \\
0 & \mu & 0 & \ldots & 0 \\
0 & 0 & \mu & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \mu
\end{pmatrix}.
\]
The splitting argument is the same as that of Li and Wang [9, page 946], or Dung and Sung [5, page 1788], so we omit it here. \( \square \)

We need the following lemma for the proof of Theorem 3.5.

**Lemma 3.4** ([9, Lemma 4.1]). Let \( M \) be a complete Riemannian manifold of dimension \( n \geq 2 \). Assume that the Ricci curvature of \( M \) satisfies the lower bound
\[
\text{Ric}_M(x) \geq -(n-1)\tau(x)
\]
for all \( x \in M \). Suppose \( f \) is a non-constant harmonic function defined on \( M \). Then the function \( \nabla f \) must satisfy the differential inequality
\[
\Delta |\nabla f| \geq -(n-1)\tau |\nabla f| + \frac{||\nabla \nabla f||^2}{(n-1)|\nabla f|}
\]
in the weak sense. Moreover, if equality holds, then \( M \) is given by \( M = \mathbb{R} \times N \) with the warped product metric
\[
ds_M^2 = dt^2 + \eta^2(t) ds_N^2
\]
for some positive function \( \eta(t) \), and some \( (n-1) \)-dimensional manifold \( N \). In this case, \( \tau(t) \) is a function of \( t \) alone satisfying
\[
\eta''(t)\eta^{-1}(t) = \tau(t).
\]
If the \( m \)-Bakry-Émery Ricci curvature condition is further relaxed, we will need an extra condition on \( \lambda_1(\Delta f) \).
Theorem 3.5 Let \((M, g, e^{-f} dv)\) be a complete metric measure space of dimension \(n \geq 3\). Suppose that \(\lambda_1(\Delta_f) \geq \frac{b}{m-1} - a\) and
\[
\text{Ric}_{m,n} \geq -a\lambda_1(\Delta_f) - b,
\]
where \(0 < a < \frac{m}{m-1}\) and \(b > 0\). Then either

1. \(H^1(L^2_{f}(M)) = 0\); or
2. \(\tilde{M} = \mathbb{R} \times N\), where \(\tilde{M}\) is the universal cover of \(M\) and \(N\) is a manifold of dimension \(n - 1\).

Proof For any \(L^2_f\)-harmonic 1-form \(\omega\), let \(h = |\omega|\), so we have
\[
h\Delta_f h \geq \frac{1}{m-1} |\nabla h|^2 - a\lambda_1(\Delta_f) h^2 - bh^2. \tag{3.4}
\]
Multiplying by the cut-off function \(\phi_R^2\) on both sides of (3.4) and by integration by parts, one gets
\[
\frac{1}{m-1} \int_M \phi_R^2 |\nabla h|^2 \cdot e^{-f} dv - a\lambda_1(\Delta_f) \int_M \phi_R^2 h^2 \cdot e^{-f} dv - \int_M \phi_R^2 h^2 \cdot e^{-f} dv \leq -\int_M \phi_R^2 |\nabla h|^2 \cdot e^{-f} dv - 2 \int_M \langle \phi_R \nabla h, h \nabla \phi_R \rangle \cdot e^{-f} dv.
\]
Combining with the variational principle, one obtains
\[
\frac{m}{m-1} \int_M \phi_R^2 |\nabla h|^2 \cdot e^{-f} dv \leq a \int_M |\nabla (\phi_R h)|^2 \cdot e^{-f} dv + b \int_M \phi_R^2 h^2 \cdot e^{-f} dv - 2 \int_M \langle \phi_R \nabla h, h \nabla \phi_R \rangle \cdot e^{-f} dv.
\]
By Lemma 2.3, when \(R \to \infty\), we have
\[
\int_M |\nabla h|^2 \cdot e^{-f} dv \leq \frac{b}{m-1} - a \int_M h^2 \cdot e^{-f} dv. \tag{3.5}
\]
Suppose \(\lambda_1(\Delta_f) > \frac{b}{m-1} - a\), if \(\omega\) is non-trivial, i.e., \(h \neq 0\), then (3.5) implies
\[
\lambda_1(\Delta_f) \leq \frac{b}{m-1} - a,
\]
which is a contradiction. Hence, if \(\lambda_1(\Delta_f) > \frac{b}{m-1} - a\), then
\[
H^1(L^2_{f}(M)) = \{0\}.
\]
Suppose \(\lambda_1(\Delta_f) = \frac{b}{m-1} - a\) and \(H^1(L^2_{f}(M))\) is non-trivial, then equality holds in (3.5). Hence, equality holds in (3.4), i.e.,
\[
h\Delta_f h = \frac{1}{m-1} |\nabla h|^2 - a\lambda_1(\Delta_f) h^2 - bh^2. \tag{3.6}
\]
Lift the metric of $M$ to the universal cover $\tilde{M}$ and the harmonic 1-form is lifted to a harmonic 1-form $\tilde{\omega}$ on $\tilde{M}$. Since $\tilde{M}$ is simply connected, $\tilde{\omega}$ is exact, i.e., there exists a smooth function $\zeta$ such that $\tilde{\omega} = d\zeta$. Hence, $\zeta$ is a non-constant harmonic function on $\tilde{M}$ such that
\[
|\nabla \zeta| \Delta |\nabla \zeta| = -(a\lambda_1(\Delta_f) + b)|\nabla \zeta|^2 + \frac{1}{n-1}|\nabla|\nabla \zeta||^2.
\]
Applying [9, Lemma 4.1] (see Lemma 3.4 in this paper) with
\[
\tau = \frac{1}{n-1}(a\lambda_1(\Delta_f) + b),
\]
we get the splitting of $\tilde{M}$. \hfill \Box

From the proof, we see that, if $\lambda_1(\Delta_f) > \frac{b}{\frac{m}{m-1} - a}$, $\mathcal{H}^1(L^2_f(M))$ vanishes, and the splitting case only happens when $\lambda_1(\Delta_f) = \frac{b}{\frac{m}{m-1} - a}$.

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