On Reachability in Parameterized Phaser Programs

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Abstract. We address the problem of statically checking safety properties (such as assertions or deadlocks) for parameterized phaser programs. Phasers embody a non-trivial and modern synchronization construct used to orchestrate executions of parallel tasks. This generic construct supports dynamic parallelism with runtime registrations and deregistrations of spawned tasks. It generalizes many synchronization patterns such as collective and point-to-point schemes. For instance, phasers can enforce barriers or producer-consumer synchronization patterns among all or subsets of the running tasks. We consider in this work programs that may generate arbitrarily many tasks and phasers. We study different formulations of the verification problem and propose an exact procedure that is guaranteed to terminate for some reachability problems even in the presence of unbounded phases and arbitrarily many spawned tasks. In addition, we prove undecidability results for several problems on which our procedure cannot be guaranteed to terminate.

1 Introduction

We focus on the parameterized verification problem of parallel programs that adopt the phasers construct for synchronization \cite{16}. This coordination construct unifies collective and point-to-point synchronization. Parameterized verification is particularly relevant for mainstream parallel programs as the number of interdependent tasks in many applications, from scientific computing to web services or e-banking, may not be known apriori. Parameterized verification of phaser programs is a challenging problem due to the arbitrary numbers of involved tasks and phasers. In this work, we address this problem and provide an exact symbolic verification procedure. We identify parameterized problems for which our procedure is guaranteed to terminate and prove the undecidability of several variants on which our procedure cannot be guaranteed to terminate in general.

Phasers build on the clock construct from the X10 programming language \cite{5} and are implemented in Habanero Java \cite{4}. They can be added to any parallel programming language with a shared address space. Conceptually, phasers are synchronization entities to which tasks can be registered or unregistered. Registered tasks may act as producers, consumers, or both. Tasks can individually
issue \textit{signal}, \textit{wait}, and \textit{next} commands to a phaser they are registered to. Intuitively, a \textit{signal} command is used to inform other tasks registered to the same phaser that the issuing task is done with its current phase. It increments the \textit{signal} value associated to the issuing task on the given phaser. The \textit{wait} command on the other hand checks whether all \textit{signal} values in the phaser are greater than the number of \textit{wait}s issued by this task, i.e. all registered tasks have passed the issuing task’s \textit{wait} phase. It then increments the \textit{wait} value associated to the task on the phaser. As a result, the \textit{wait} command might block the issuing task until other tasks issue enough \textit{signals}. The \textit{next} command consists in a \textit{signal} followed by a \textit{wait}. The \textit{next} command may be associated to a sequence of statements that are to be executed in isolation by one of the registered tasks participating in the command. A program that does not use this feature of the \textit{next} statement is said to be \textit{non-atomic}. A task deregisters from a phaser by issuing a \textit{drop} command on it.

The dynamic synchronization allowed by the construct suits applications that need dynamic load balancing (e.g., for solving non-uniform problems with unpredictable load estimates [18]). Dynamic behavior is enabled by the possible runtime creation of tasks and phasers and their registration/de-registration. Moreover, the spawned tasks can work in different phases, adding flexibility to the synchronization pattern. The generality of the construct makes it also interesting from a theoretical perspective, as many language constructs can be expressed using phasers. For example, synchronization barriers of Single Program Multiple Data programs, the Bulk Synchronous Parallel computation model [17], or promises and futures constructs [3] can be expressed using phasers.

This paper provides general (un)decidability results that we believe will guide verification of other synchronization constructs. We identify combinations of features (e.g., unbounded differences between \textit{signal} and \textit{wait} phases, atomic statements) and properties to be checked (e.g., assertions, deadlocks) for which the parameterized verification problem becomes undecidable. These help identify synchronization constructs with enough expressivity to result in undecidable parameterized verification problems. We also provide a symbolic verification procedure that terminates even on fragments with arbitrary phases and numbers of spawned tasks. We get back to possible implications in the conclusion. We summarize our contributions:

– We show an operational model for phaser programs based on [16,14,10].
– We propose an exact symbolic verification procedure for checking reachability of sets of configurations for non-atomic phaser programs even when arbitrarily many tasks and phasers may be generated.
– We prove undecidability results for several reachability problems.
– We show termination of our procedure when checking assertions for non-atomic programs even when arbitrary many tasks may be spawned.
– We show termination of our procedure when checking deadlock-freedom and assertions for non-atomic programs in which the difference between \textit{signal} and \textit{wait} phases is bounded, even when arbitrary many tasks may be spawned.
Related work. The closest work to ours is [9], which is the only work on automatic and static formal verification of phaser programs. The work in [9] does not consider the parameterized case. For instance, this work can decide whether some program assertion is violated even in the presence of arbitrary many tasks with arbitrary large phaser gaps. This is well beyond [9] and requires a more complex symbolic representation with a deeper termination argument. The work of [6] considers the dynamic verification of phaser programs and can therefore only be used to detect deadlocks at runtime. The work in [2] uses Java Path Finder [11] to explore all execution paths of the application. It is however also restricted to work on one concrete input at a time. A more general description of the phasers mechanism of Habanero Java has also been formalized in Coq [7].

Outline. We describe the phasers construct in Sec. 2. We then formally introduce the construct and show the associated general reachability problem to be undecidable in Sec. 3. We describe in Sec. 4 our symbolic representation and state some of its non-trivial properties. We use the representation in Sec. 5 to define a verification procedure and establish decidability results. We refine our undecidability results in Sec. 6 and summarize our findings in Sec. 7.

Fig. 1. An unbounded number of producers and consumers are synchronized using two phasers. In this construction, each consumer requires all producers to be ahead of it (wrt. the \( p \) phaser) in order for it to consume their respective products. At the same time, each consumer needs to be ahead of all producers (wrt. the \( c \) phaser) in order for the producers to be able to move to the next phase and produce new items.

2 Motivating example

The program listed in Fig. 1 uses Boolean shared variables \( B = \{a, done\} \). The main task creates two phasers (line 4-5). When creating a phaser, the task gets automatically registered to it. The main task also creates an unbounded number of other task instances (lines 7-8). When a task \( t \) is registered to a phaser \( p \), a pair \( (w_{p}^{t}, s_{p}^{t}) \) in \( \mathbb{N}^{2} \) can be associated to the couple \( (t, p) \). The pair represents the individual wait and signal phases of task \( t \) on phaser \( p \).
Registration of a task to a phaser can occur in one of three modes: \texttt{Sig\_Wait}, \texttt{Wait} and \texttt{Sig}. In \texttt{Sig\_Wait} mode, a task may issue both \texttt{signal} and \texttt{wait} commands. In \texttt{Wait} mode, a task may only issue \texttt{wait} commands on the phaser. Finally, when registered in \texttt{Sig} mode, a task may only issue \texttt{signal} commands. Issuing a \texttt{signal} command by a task on a phaser results in the task incrementing its signal phase associated to the phaser. This command is non-blocking. On the other-hand, issuing a \texttt{wait} command by a task on a phaser \texttt{p} will block until all tasks registered to \texttt{p} get signal values on \texttt{p} that are strictly larger than the wait value of the issuing task on the same phaser. In this case, the wait phase of the issuing task is incremented. Intuitively, a signal command allows the issuing task to state that other tasks need not wait for it to complete its signal phase. In retrospect, a \texttt{wait} command allows a task to make sure all registered tasks have moved past its wait phase.

Upon creation of a phaser, wait and signal phases are initialized to 0 (except in \texttt{Wait} mode where no signal phase is associated to the task in order to not block other waiters). The only other way a task may get registered to a phaser is if an already registered task spawns and registers it in the same mode (or in \texttt{Wait} or \texttt{Sig} if the registrar is registered in \texttt{Sig\_Wait}). In this case, wait and signal phases of the newly registered task are initialized to those of the registrar. Tasks are therefore dynamically registered (e.g., lines 7-8). They can also dynamically deregister themselves (e.g., line 10-11).

In this example, an unbounded number of producers and consumers are synchronized using two phasers. Consumers require producers to be ahead of them (wrt. the phaser they point to with \texttt{p}) in order for them to consume their products. At the same time, consumers need to be ahead of all producers (wrt. the phaser pointed to with \texttt{c}) in order for these to produce their items. It should be clear that phasers can be used as barriers for synchronizing dynamic subsets of concurrent tasks. Observe that tasks need not, in general, proceed in a lock step fashion. The difference between the largest signal value and the smallest wait value can be arbitrarily large (several signals before waits catch up). Tasks have then more flexibility to proceed at their own speeds.

We are interested in checking: (a) control reachability as in assertions (e.g., line 20), race conditions (e.g., mutual exclusion of lines 20 and 33) or registration errors (e.g., signaling a dropped phaser), and (b) plain reachability as in deadlocks (e.g., a producer at line 19 and a consumer at line 30 with equal phases waiting for each other). Intuitively, both problems concern themselves with the reachability of target sets of program configurations. The difference is that control state reachability defines the targets with the states of the tasks (their control locations and whether they are registered to some phasers). Plain reachability can, in addition, constrain values of the phases in the target configurations (e.g., requiring equality between wait and signal values for deadlocks). Observe that control state reachability depends on the values of the actual phases, but these values are not used to define the target sets. For example, assertions are expressed as predicates over Boolean variables (e.g., line 20). Validity of such assertions may depend on respecting phasers synchronizations.
3 Phaser programs and reachability

We define the syntax and semantics of a core phaser programs language. We make sure the simplified language presented here is representative of the general purpose languages using phasers so that our results have a practical impact. A phaser program $\text{prg} = (B, V, T)$ involves a set $T$ of tasks including a unique “main” task $\text{main()}\{\text{stmt}\}$. Arbitrary many instances of each task might be spawned during a program execution. All task instances share a set $B$ of Boolean variables and make use of a set $V$ of phaser variables that are local to individual task instances. Arbitrary many phasers might also be generated during program execution. Syntax of programs is as follows.

$$
\text{prg} ::= \text{bool } b_1, \ldots, b_p; \\
\text{   task}_1(v_{1_1}, \ldots, v_{k_1})\{\text{stmt}_1\}; \\
\text{   \ldots} \\
\text{   task}_n(v_{1_n}, \ldots, v_{k_n})\{\text{stmt}_n\}; \\
\text{stmt} ::= v = \text{newPhaser}(); | \text{asynch}(\text{task}, v_{1}, \ldots, v_k); | v.\text{drop}(); | v.\text{signal}(); \\
\text{   | v.wait(); | v.next(); | v.next()\{\text{stmt}\}; | b := \text{cond}; | \text{assert(}\text{cond}; \\
\text{   | while(}\text{cond}\{\text{stmt}\}; | \text{stmt stmt | exit; \\
\text{cond} ::= \text{ndet()} | \text{true} | \text{false} | b | \text{cond } \lor \text{cond} | \text{cond } \land \text{cond} | \sim \text{cond}
$$

Initially, a unique task instance starts executing the $\text{main()}\{\text{stmt}\}$ task. A phaser can recall a pair of values (i.e., wait and signal) for each task instance registered to it. A task instance can create a new phaser with $v = \text{newPhaser}()$, get registered to it (i.e., gets zero as wait and signal values associated to the new phaser) and refer to the phaser with its local variable $v$. We simplify the presentation by assuming all registrations to be in $\text{Sig}_W\text{Wait}$ mode. Including the other modes is a matter of depriving $\text{Wait}$-registered tasks of a signal value (to ensure they do not block other registered tasks) and of ensuring issued commands respect registration modes. We use $V$ for the union of all local phaser variables. A task $\text{task}(v_1, \ldots, v_k)\{\text{stmt}\}$ in $T$ takes the phaser variables $v_1, \ldots, v_k$ as parameters (write $\text{paramOf(task)}$ to mean these parameters). A task instance can spawn another task instance with $\text{asynch(task, v_1, \ldots, v_n)}$. The issuing task instance registers the spawned task to the phasers pointed to by $v_1, \ldots, v_n$, with its own wait and signal values. Spawner and Spawnee execute concurrently. A task instance can deregister itself from a phaser with $v.\text{drop}()$.

A task instance can issue signal or wait commands on a phaser referenced by $v$ and on which it is registered. A wait command on a phaser blocks until the wait value of the task instance executing the wait on the phaser is strictly smaller than the signal value of all task instances registered to the phaser. In other words, $v.\text{wait}()$ blocks if $v$ points to a phaser such that at least one of the signal values stored by the phaser is equal to the wait value of the task that tries to perform the wait. A signal command does not block. It only increments the signal value.
of the task instance executing the signal command on the phaser. \texttt{v.next()} is syntactic sugar for a signal followed by a wait. Moreover, \texttt{v.next()}\{\texttt{stmt}\} is similar to \texttt{v.next()} but the block of code \texttt{stmt} is executed atomically by exactly one of the tasks participating in the synchronization before all tasks continue the execution that follows the barrier. \texttt{v.next()}\{\texttt{stmt}\} thus requires all tasks to be synchronized on exactly the same statement and is less flexible. Absence of a \texttt{v.next()}\{\texttt{stmt}\} makes a program non-atomic.

Note that assignment of phaser variables is excluded from the syntax; additionally, we restrict task creation \texttt{async(task,v_1,...,v_n)} and require that parameter variables \(v_i\) are all different. This prevents two variables from pointing to the same phaser and avoids the need to deal with aliasing; we can reason on the single variable in a process that points to a phaser. Extending our work to deal with aliasing is easy but would require heavier notations.

We will need the notions of configurations, partial configurations and inclusion in order to define the reachability problems we consider in this work. We introduce them in the following and assume a phaser program \(\texttt{prg} = (B,V,T)\).

**Configurations.** Configurations of a phaser program describe valuations of its variables, control sequences of its tasks and registration details to the phasers. 

**Control sequences.** We define the set \(\texttt{Suff}\) of control sequences of \(\texttt{prg}\) to be the set of suffixes of all sequences \texttt{stmt} appearing in some statement \texttt{task}(...){\texttt{stmt}}. In addition, we define \(\texttt{UnrSuff}\) to be the smallest set containing \(\texttt{Suff}\) in addition to the suffixes of all (i) \(s_1;\texttt{while(cond)}\{s_1\};s_2\) if \texttt{while(cond)}\{\texttt{s_1}\};\texttt{s_2} is in \(\texttt{UnrSuff}\), and of all (ii) \(s_1;s_2\) if \texttt{if(cond)}\{\texttt{s_1}\};\texttt{s_2} is in \(\texttt{UnrSuff}\), and of all (iii) \(s_1;\texttt{v.next()}\{\}\};s_2\) if \texttt{v.next()}\{\texttt{s_1}\};\texttt{s_2} in \(\texttt{UnrSuff}\), and finally of all (iv) \texttt{v.signal()};\texttt{v.wait()};\texttt{s_2} if \texttt{v.next()}\{\}\};\texttt{s_2} is in \(\texttt{UnrSuff}\). We write \(\texttt{hd(s)}\) and \(\texttt{tl(s)}\) to respectively mean the head and the tail of a sequence \(s\).

**Partial configurations.** Partial configurations allow the characterization of sets of configurations by partially stating some of their common characteristics. A partial configuration \(c\) of \(\texttt{prg} = (B,V,T)\) is a tuple \((T,\mathcal{P},bv,seq,phase)\) where:

- \(T\) is a finite set of task identifiers. We let \(t,u\) range over the values in \(T\).
- \(\mathcal{P}\) is a finite set of phaser identifiers. We let \(p,q\) range over the values in \(\mathcal{P}\).
- \(bv : B \to B^\ast\) fixes the values of some of the shared variables\(^3\)
- \(seq : T \to \texttt{partialFunctions}(\mathcal{P},V^{\ast}) \times (N^2 \cup \{\ast,\ast\},nreg))\) is a mapping that associates to each task \(t\) in \(T\) a partial mapping stating which phasers are known by the task and with which registration values.

Intuitively, partial configurations are used to state some facts about the valuations of variables and the control sequences of tasks and their registrations. Partial configurations leave some details unconstrained using partial mappings or the symbol *. For instance, if \(bv(b) = *\) in a partial configuration \(\langle T,\mathcal{P},bv,seq,phase\rangle\), then the partial configuration does not constrain the value of the shared variable \(b\). Moreover, a partial configuration does not constrain

\(^3\) For any set \(S\), \(S^{\{a,b,\ldots\}}\) denotes \(S \cup \{a,b,\ldots\}\).
the relation between a task \( t \) and a phaser \( p \) when \( \text{phase}(t)(p) \) is undefined. Instead, when the partial mapping \( \text{phase}(t) \) is defined on phaser \( p \), it associates a pair \( \text{phase}(t)(p) = (\text{var}, \text{val}) \) to \( p \). If \( \text{var} \in \mathcal{V}(\neg \rightarrow) \) is a variable \( v \in \mathcal{V} \) then the task \( t \) in \( \mathcal{T} \) uses its variable \( v \) to refer to the phaser \( p \) in \( \mathcal{P} \). If \( \text{var} \) is the symbol \( − \) then the task \( t \) does not refer to \( v \) with any of its variables in \( \mathcal{V} \). If \( \text{var} \) is the symbol \(*\), then the task might or might not refer to \( p \). The value \( \text{val} \) in \( \text{phase}(t)(p) = (\text{var}, \text{val}) \) is either the value \( \text{nreg} \) or a pair \((w, s)\). The value \( \text{nreg} \) means the task \( t \) is not registered to phaser \( p \). The pair \((w, s)\) belongs to \((\mathbb{N} \times \mathbb{N}) \cup \{(*, *)\} \). In this case, task \( t \) is registered to phaser \( p \) with a symbolic wait phase \( w \) and a symbolic signal phase \( s \). The value \(*\) means that the wait phase \( w \) (resp. signal phase \( s \)) can be any value in \( \mathbb{N} \). For instance, \( \text{phase}(t)(p) = (v, \text{nreg}) \) means variable \( v \) of the task \( t \) refers to phaser \( p \) but the task is not registered to \( p \). On the other hand, \( \text{phase}(t)(p) = (−, (*, *)) \) means the task \( t \) does not refer to \( p \) but is registered to it with a arbitrary wait and signal phases.

**Concrete configurations.** A concrete configuration (or configuration for short) is a partial configuration \((\mathcal{T}, \mathcal{P}, \mathcal{bv}, \mathcal{seq}, \mathcal{phase})\) where \( \text{phase}(t) \) is total for each \( t \in \mathcal{T} \) and where the symbol \(*\) does not appear in any range. It is a tuple \((\mathcal{T}, \mathcal{P}, \mathcal{bv}, \mathcal{seq}, \mathcal{phase})\) where \( \mathcal{bv} : \mathcal{B} \rightarrow \mathcal{B} \), \( \mathcal{seq} : \mathcal{T} \rightarrow \text{UnrSuff} \), and \( \mathcal{phase} : \mathcal{T} \rightarrow \text{totalFunctions}(\mathcal{P}, \mathcal{V}(\neg \rightarrow) \times ((\mathbb{N} \times \mathbb{N}) \cup \{\text{nreg}\})) \). For a concrete configuration \((\mathcal{T}, \mathcal{P}, \mathcal{bv}, \mathcal{seq}, \mathcal{phase})\), we write \( \text{isReg}(\mathcal{phase}, t, p) \) to mean the predicate \( \text{phase}(t)(p) \notin (\mathcal{V}(\neg \rightarrow) \times \{\text{nreg}\}) \). The predicate \( \text{isReg}(\mathcal{phase}, t, p) \) captures whether the task \( t \) is registered to phaser \( p \) according to the mapping \( \mathcal{phase} \).

**Inclusion of configurations.** A configuration \( c' = (\mathcal{T}', \mathcal{P}', \mathcal{bv}', \mathcal{seq}', \mathcal{phase}') \) includes a partial configuration \( c = (\mathcal{T}, \mathcal{P}, \mathcal{bv}, \mathcal{seq}, \mathcal{phase}) \) if renaming and deleting tasks and phasers from \( c' \) can give a configuration that “matches” \( c \). More formally, \( c' \) includes \( c \) if \(((\mathcal{bv}(b) \neq \mathcal{bv}'(b)) \implies (\mathcal{bv}(b) = *))\) for each \( b \in \mathcal{B} \) and there are injections \( \tau : \mathcal{T} \rightarrow \mathcal{T}' \) and \( \pi : \mathcal{P} \rightarrow \mathcal{P}' \) s.t. for each \( t \in \mathcal{T} \) and \( p \in \mathcal{P} : (1) (\tau(t) \neq \mathcal{seq}'(\tau(t))) \implies (\mathcal{seq}(t) = *)\), and either (2.a) \( \text{phase}(t)(p) \) is undefined, or (2.b) \( \text{phase}(t)(p) = (\text{var}, \text{val}) \) and \( \mathcal{phase}'(\tau(t))(\pi(p)) = (\text{var}', \text{val}') \) with \((\text{var} \neq \text{var}') \implies (\text{var} = *)\) and either \((\text{val} = \text{val}' = \text{nreg})\) or \(\text{val} = (w, s)\) and \(\text{val}' = (w', s')\) with \((\text{val} \neq \text{val}') \implies (w = *)\) and \((s \neq s') \implies (s = *)\).

**Semantics and reachability.** Given a program \( \text{prg} = (\mathcal{B}, \mathcal{V}, \mathcal{T}) \), the main task \( \text{main}()\{\text{stmt}\} \) starts executing \( \text{stmt} \) from an initial configuration \( c_{init} = (\mathcal{T}_{init}, \mathcal{P}_{init}, \mathcal{bv}_{init}, \mathcal{seq}_{init}, \mathcal{phase}_{init}) \) where \( \mathcal{T}_{init} \) is a singleton, \( \mathcal{P}_{init} \) is empty, \( \mathcal{bv}_{init} \) sends all shared variables to \( \text{false} \) and \( \mathcal{seq}_{init} \) associates \( \text{stmt} \) to the unique task in \( \mathcal{T}_{init} \). We write \( c \xrightarrow{t} c' \) to mean a task \( t \) in \( c \) can fire statement \( \text{stmt} \) and result in configuration \( c' \). See Fig. 2 for a description of the operational semantics. We write \( c \xrightarrow{\text{stmt}} c' \) if \( c \xrightarrow{t} c' \) for some task \( t \), and \( c \xrightarrow{c'} c' \) if \( c \xrightarrow{\text{stmt}} c' \) for some statement \( \text{stmt} \). We also write \( \xrightarrow{\rightarrow}^+ \) for the transitive closure of \( \xrightarrow{\rightarrow} \) and let \( \rightarrow^* \) be the reflexive transitive closure of \( \xrightarrow{} \). Fig. 3 identifies erroneous configurations.

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4 The uniqueness of this variable is due to the absence of aliasing discussed above.
Operational semantics of phaser statements without errors. Each transition corresponds to a task \( t \in T \) executing a statement from a configuration \( (T, \mathcal{P}, \mathcal{B}, \mathcal{S}, \mathcal{P}) \). For instance, the \texttt{drop} transition corresponds to a task \( t \) executing \texttt{v.drop()} when registered to phaser \( p \in \mathcal{P} \) (with phases \( (w, s) \)) and referring to it with variable \( v \). The result is the same configuration where task \( t \) moves to its next statement without being registered to \( p \) (albeit still referring to \( p \) with \( v \)).
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\[
\begin{align*}
\text{hd}(\text{seg}(t)) &= \text{assert}(\text{cond}) \quad \text{[cond]}_0 = \text{false} \quad \text{assertion : errors} \\
\langle T', P', bv, \text{seq}, \text{phase} \rangle &\in \text{AssertErrors} \\
\text{hd}(\text{seg}(u)) &\in \{b := \text{cond}.\text{if}(\text{cond})\{\text{stmt}\}, \text{while(}\text{cond}\{\text{stmt}\}, \text{assert(}\text{cond})\} \quad \text{(race : errors)} \\
\langle T', P', bv, \text{seq}, \text{phase} \rangle &\in \text{RaceErrors} \\
\text{hd}(\text{seg}(t)) &\in \{\text{async(task}, \ldots \text{[stmt]}, \text{v.signal()}, \text{v.wait()}, \text{v.drop()}\} \quad \text{(registration : errors)} \\
\langle T', P', bv, \text{seq}, \text{phase} \rangle &\in \text{RegisterErrors} \\
\text{hd}(\text{seg}(t)) &\in \{\text{assert(task}, \ldots \text{v.signal()}, \text{v.wait()}, \text{v.drop()}\} \quad \text{(deadlock : errors)} \\
\langle T', P', bv, \text{seq}, \text{phase} \rangle &\in \text{DeadlockErrors}
\end{align*}
\]

Fig. 3. Definition of error configurations. Starting from \(\langle T, P, bv, \text{seq}, \text{phase} \rangle\), error configurations are obtained when tasks execute the above statements under certain conditions. For instance, a deadlock is obtained if tasks in a subset \(\{t_0, \ldots, t_n\} \subseteq T\) form a cycle where each \(t_i\) blocks (with its signal phase \(s'_i\)) the wait being executed by \(t_{i+1}\%\(n+1\)\) on phaser \(P_{i+1}\%\(n+1\)\) (with wait phase \(w_{i+1}\%\(n+1\)\).

We are interested in the reachability of sets of configurations (i.e., checking safety properties). We differentiate between two reachability problems depending on whether the target sets of configurations constrain the registration phases or not. The plain reachability problem constrains the registration phases of the target configurations. The control reachability problem only constrains registration status, control sequences, or variable values. We will see that decidability of the two problems can be different. The two problems are defined in the following.

Plain reachability. First, we define equivalent configurations. A configuration \(c = \langle T, P, bv, \text{seq}, \text{phase} \rangle\) is equivalent to configuration \(c' = \langle T', P', bv', \text{seq}', \text{phase}' \rangle\) if \(bv = bv'\) and there are bijections \(\tau : T \rightarrow T'\) and \(\pi : P \rightarrow P'\) such that, for all \(t \in T, p \in P\) and \(\text{var} \in V^{-1}\), \(\text{seq}(t) = \text{seq}'(\tau(t))\) and there are some integers \((k_p)_{p \in P}\) such that \(\text{phase}(t)(p) = (\text{var}, (w, s))\) iff \(\text{phase}'(\tau(t))(\pi(p)) = (\text{var}, (w + k_p, s + k_p))\). We write \(c \sim c'\) to mean that \(c\) and \(c'\) are equivalent. Intuitively, equivalent configurations simulate each other. We can establish the following:

Lemma 1 (Equivalence). Assume two configurations \(c_1\) and \(c_2\). If \(c_1 \rightarrow c_2\) and \(c'_1 \sim c_1\) then there is a configuration \(c'_2\) s.t. \(c'_2 \sim c_2\) and \(c'_1 \rightarrow c'_2\).

Observe that if the wait value of a task \(t\) on a phaser \(p\) is equal to the signal of a task \(t'\) on the same phaser \(p\) in some configuration \(c\), then this is also the case, up to a renaming of the phasers and tasks, in all equivalent configurations. This is particularly relevant for defining deadlock configurations where a number of tasks are waiting for each other. The plain reachability problem is given a program and a target partial configuration and asks whether a configuration (equivalent
to a configuration) that includes the target partial configuration is reachable. More formally, given a program \( prg \) and a partial configuration \( c \), let \( c_{\text{init}} \) be the initial configuration of \( prg \), then \( \text{reach}(prg, c) \) if and only if \( c_{\text{init}} \rightarrow^* c_1 \) for \( c_1 \sim c_2 \) and \( c_2 \) includes \( c \).

**Definition 1 (Plain reachability).** For a program \( prg \) and a partial configuration \( c \), decide whether \( \text{reach}(prg, c) \) holds.

**Control reachability.** A partial configuration \( c = (T, \mathcal{P}, \text{bv}, \text{seq}, \text{phase}) \) is said to be a control partial configuration if for all \( t \in T \) and \( p \in \mathcal{P} \), either \( \text{phase}(t)(p) \) is undefined or \( \text{phase}(t)(p) \in (V^{-*}) \times \{(\ast, \ast), \text{nreg}\} \). Intuitively, control partial configurations do not constrain phase values. They are enough to characterize, for example, configurations where an assertion is violated (see Fig. 3).

**Definition 2 (Control reachability).** For a program \( prg \) and a control partial configuration \( c \), decide whether \( \text{reach}(prg, c) \) holds.

Observe that plain reachability is at least as hard to answer as control reachability since any control partial configuration is also a partial configuration. It turns out the control reachability problem is undecidable for programs resulting in arbitrarily many tasks and phasers as stated by the theorem below. This is proven by reduction of the state reachability problem for 2-counter Minsky machines. A 2-counter Minsky machine \( (S, \{x_1, x_2\}, \Delta, s_0, s_F) \) has a finite set \( S \) of states, two counters \( \{x_1, x_2\} \) with values in \( \mathbb{N} \), an initial state \( s_0 \) and a final state \( s_F \). Transitions may increment, decrement or test a counter. For example \( (s_0, \text{test}(x_1), s_F) \) takes the machine from \( s_0 \) to \( s_F \) if the counter \( x_1 \) is zero.

**Theorem 1 (Minsky machines [15]).** Checking whether \( s_F \) is reachable from configuration \( (s_0, 0, 0) \) for 2-counter machines is undecidable in general.

**Theorem 2.** Control reachability is undecidable in general.

*Proof sketch.* State reachability of an arbitrary 2-counters Minsky machine is encoded as the control reachability problem of a phaser program (captured in Fig. 4). The phaser program has three tasks \( \text{main}, \text{xUnit} \) and \( \text{yUnit} \). It uses Boolean shared variables to encode the state \( s \in S \) and to pass information between different task instances. The phaser program builds two chains, one with \( \text{xUnit} \) instances for the \( x \)-counter, and one with \( \text{yUnit} \) instances for the \( y \)-counter. Each chain alternates a phaser and a task and encodes the values of its counter with its length. The idea is to have the phaser program simulate all transitions of the counter machine, i.e., increments, decrements and tests for zero. Answering state reachability of the counter machine amounts to checking whether there are reachable configurations where the boolean variables encoding the counter machine can evaluate to the target machine state \( s_F \). This can be captured with a control partial configuration.
```c
bool s1, s2,..., sF;
bool xInc, xDec, yInc, yDec,
temp;

main(){
xPh = newPhaser();
yPh = newPhaser();
while(true){
temp = false;
//(q_i:inc(x):q_j)
if (ndet() ∨ si){
xInc =true;
xPh.signal();
xPh.wait();
if(!temp){
asynch(xTask, xPh);
}
else{
temp = false;
}
xInc = false;
si=false;
sj=true;
}
//(q_i:dec(x):q_j)
if (ndet() & si){
Dec = true;
xPh.signal();
xPh.wait();
while(!temp){
temp =false;
xDec = false;
si=false;
sj=true;
}
//(q_i:test(x):q_j)
if (ndet() & sj){
}
}
}
}
```

**Fig. 4.** For the proof of Theorem 2. Encoding a 2-counter Minsky machine with the counters \{x, y\} using two task-phaser-chains with the lengths of the chains capturing the values of each counter. The messages among the tasks are used to orchestrate the simulation and are transmitted in the Boolean variables temp, xInc, xDec, yInc, and yDec. For instance, xInc stands for incrementing the counter x.
4 A gap-based symbolic representation

The symbolic representation we propose builds on the following intuitions. First, observe the language semantics impose, for each phaser, the invariant that signal values are always larger or equal to wait values. We can therefore assume this fact in our symbolic representation. In addition, our reachability problems from Sec. 3 are defined in terms of reachability of equivalence classes, not of individual configurations. This is because configurations violating considered properties (see Fig. 3) are not defined in terms of concrete phase values but rather in terms of relations among them (in addition to the registration status, control sequences and variable values). Finally, we observe that if a wait is enabled with smaller gaps on a given phaser, then it will be enabled with larger ones. We therefore propose to track the gaps of the differences between signal and wait values wrt. to an existentially quantified level (per phaser) that lies between wait and signal values of all registered tasks (to the considered phaser).

We formally define our symbolic representation and describe a corresponding entailment relation. We also establish a desirable property (namely that of being a well-quasi-ordering) on some classes of representations. This property is crucial for the decidability of certain reachability problems (see Sec. 5).

Named gaps. A named gap is associated to a task-phaser pair. It consists in a tuple \((\text{var}, \text{val})\) in \(G = (V^{(-, +)} \times \left(\mathbb{N}^4 \cup \left(\mathbb{N}^2 \times \{\infty\}\right)\right) \cup \{\text{nreg}\})\). Like for partial configurations in Sec. 3, \(\text{var} \in V^{(-, +)}\) constrains variable values. The \(\text{val}\) value describes task registration to the phaser. If registered, then \(\text{val}\) is a 4-tuple \((1w, 1s, uw, us)\). This intuitively captures, together with some level \(l\) common to all tasks registered to the considered phaser, all concrete wait and signal values \((w, s)\) satisfying \(1w \leq (l - w) \leq uw\) and \(1s \leq (s - l) \leq us\). A named gap \((\text{var}, (1w, 1s, uw, us))\) is said to be free if \(uw = us = \infty\). It is said to be \(B\)-gap-bounded, for \(B \in \mathbb{N}\), if both \(uw \leq B\) and \(us \leq B\) hold. A set \(G \subseteq G\) is said to be free (resp. \(B\)-gap-bounded) if all its named gaps are free (resp. \(B\)-gap-bounded). The set \(G\) is said to be \(B\)-good if each one of its named gaps is either free or \(B\)-gap-bounded. Finally, \(G\) is said to be good if it is \(B\)-good for some \(B \in \mathbb{N}\). Given a set \(\tilde{G}\) of named gaps, we define the partial order \(\leq\) on \(\tilde{G}\), and write \((\text{var}, \text{val}) \leq (\text{var}', \text{val}')\), to mean (i) \(\text{var} \neq \text{var}' \Rightarrow \text{var} = +\), and (ii) \((\text{val} = \text{nreg}) \iff (\text{val}' = \text{nreg})\), and (iii) if \(\text{val} = (1w, 1s, uw, us)\) and \(\text{val}' = (1w', 1s', uw', us')\) then \(1w \leq 1w'\), \(1s \leq 1s'\), \(uw \leq uw'\) and \(us \leq us'\).

Intuitively, named gaps are used in the definition of constraints in order to capture relations (i.e., reference, registration and possible phases) of tasks and phasers. The partial order \((\text{var}, \text{val}) \leq (\text{var}', \text{val}')\) ensures the relations allowed by \((\text{var}', \text{val}')\) are also allowed by \((\text{var}, \text{val})\).

Constraints. A constraint \(\phi\) of \(\text{prg} = (B, V, T)\) is a tuple \((T, P, bv, seq, gap, egap)\) that denotes a possibly infinite set of configurations. Intuitively, \(T\) and \(P\) respectively represent a minimal set of tasks and phasers that are required in any configuration denoted by the constraint. In addition:
- \( bv : B \rightarrow \mathbb{B}^{(\ast)} \) and \( seq : T \rightarrow \text{UnrSuff}^{(\ast)} \) respectively represent, like for partial configurations, a valuation of the Boolean variables and a mapping of tasks to their control sequences.

- \( gap : T \rightarrow \text{totalFunctions}(\mathcal{P}, \mathcal{G}) \) constrains relations between \( T \)-tasks and \( \mathcal{P} \)-phasers by associating to each task \( t \) a mapping \( gap(t) \) that defines for each phaser \( p \) a named gap \( \{ \text{var}, \text{val} \} \in \mathcal{G} \) capturing the relation of \( t \) and \( p \).

- \( egap : \mathcal{P} \rightarrow \mathbb{N}^{2} \) associates lower bounds \( \{ \text{ew}, \text{es} \} \) on gaps of tasks that are registered to \( \mathcal{P} \)-phasers but which are not explicitly captured by \( T \). This is described further in the deonotations of a constraint below.

We write \( \text{isReg}(\text{gap}, t, p) \) to mean the task \( t \) is registered to the phaser \( p \), i.e., \( \text{gap}(t)(p) \notin (\mathbb{V}^{1-\ast} \times \{ \text{nreg} \}) \). A constraint \( \phi \) is said to be free (resp. \( B \)-gap-bounded or \( B \)-good) if the set \( \mathcal{G} = \{ \text{gap}(t)(p) \mid t \in T, p \in \mathcal{P} \} \) is free (resp. \( B \)-gap-bounded or \( B \)-good). The dimension of a constraint is the number of its phasers (i.e., \( |\mathcal{P}| \)). A set of constraints \( \Phi \) is said to be free, \( B \)-gap-bounded, \( B \)-good or \( K \)-dimension-bounded if each of its constraints are.

**Denotations.** We write \( c \models \phi \) to mean constraint \( \phi = \{ \mathcal{T}_{\phi}, \mathcal{P}_{\phi}, bv_{\phi}, seq_{\phi}, gap_{\phi}, egap_{\phi} \} \) denotes configuration \( c = (\mathcal{T}_{c}, \mathcal{P}_{c}, bv_{c}, seq_{c}, phase_{c}) \). Intuitively, the configuration \( c \) should have at least as many tasks (captured by a surjection \( \tau \) from a subset \( \mathcal{T}^{1}_{c} \) of \( \mathcal{T}_{c} \) to \( \mathcal{T}_{\phi} \) and phasers (captured by a bijection \( \pi \) from a subset \( \mathcal{P}^{1}_{c} \) of \( \mathcal{P}_{c} \) to \( \mathcal{P}_{\phi} \)). Constraints on the tasks and phasers in \( \mathcal{T}^{1}_{c} \) and \( \mathcal{P}^{1}_{c} \) ensure target configurations are reachable. Additional constraints on the tasks in \( \mathcal{T}^{2}_{c} = \mathcal{T}_{c} \setminus \mathcal{T}^{1}_{c} \) ensure this reachability is not blocked by tasks not captured by \( \mathcal{T}_{\phi} \). More formally:

1. For each \( b \in B \), \( (bv_{\phi}(b) \neq bv_{c}(b)) \implies (bv_{\phi}(b) = \ast) \), and
2. \( \mathcal{T}_{c} \) and \( \mathcal{P}_{c} \) can be written as \( \mathcal{T}_{c} = \mathcal{T}^{1}_{c} \uplus \mathcal{T}^{2}_{c} \) and \( \mathcal{P}_{c} = \mathcal{P}^{1}_{c} \uplus \mathcal{P}^{2}_{c} \), with
3. \( \tau : \mathcal{T}^{1}_{c} \rightarrow \mathcal{T}_{\phi} \) is a surjection and \( \pi : \mathcal{P}^{1}_{c} \rightarrow \mathcal{P}_{\phi} \) is a bijection, and
4. For \( t_{c} \in \mathcal{T}^{1}_{c} \) with \( t_{\phi} = \tau(t_{c}) \), \( (seq_{\phi}(t_{\phi}) \neq seq_{c}(t_{c})) \implies (seq_{\phi}(t_{\phi}) = \ast) \), and
5. For each \( p_{\phi} = \pi(p_{c}) \), there is a natural level \( l : 0 \leq l \) such that:
   a) if \( t_{c} \in \mathcal{T}^{1}_{c} \) with \( t_{\phi} = \tau(t_{c}) \), \( \text{phase}_{c}(t_{c})(p_{c}) = (\text{var}_{c}, \text{val}_{c}) \) and \( \text{gap}_{\phi}(t_{\phi})(p_{\phi}) = (\text{var}_{\phi}, \text{val}_{\phi}) \), then it is the case that:
      i. \( (\text{var}_{c} \neq \text{var}_{\phi}) \implies (\text{var}_{\phi} = \ast) \), and
      ii. \( (\text{val}_{c} = \text{nreg}) \iff (\text{val}_{\phi} = \text{nreg}) \), and
     iii. if \( (\text{var}_{c} = (w, s)) \) and \( (\text{val}_{c} = 1w, 1s, uw, us) \) then \( 1w \leq l - w \leq uw \) and \( 1s \leq s - l \leq us \).
   b) if \( t_{c} \in \mathcal{T}^{2}_{c} \), then for each \( p_{\phi} = \pi(p_{c}) \) with \( \text{phase}_{c}(t_{c})(p_{c}) = (\text{var}_{c}, (w, s)) \) and \( \text{egap}(p_{\phi}) = \{ \text{ew}, \text{es} \} \), we have: \( (\text{es} \leq s - l) \) and \( (\text{ew} \leq l - w) \).

We say in this case that \( \tau \) and \( \pi \) witness the denotation of \( c \) by \( \phi \). Intuitively, for each phaser, the bounds given by \( \text{gap} \) constrain the values of the phases belonging to tasks captured by \( \mathcal{T}_{\phi} \) (i.e., those in \( \mathcal{T}^{1}_{c} \)) and registered to the given phaser. This is done with respect to some non-negative level, one per phaser. The same level is used to constrain phases of tasks registered to the phaser but not captured by \( \mathcal{T}_{\phi} \) (i.e., those in \( \mathcal{T}^{2}_{c} \)). For these tasks, lower bounds are enough as we only want to ensure they do not block executions to target sets of configurations.

We write \([\phi] \) for \( \{ c \mid c \models \phi \} \).
Entailment. We write $\phi_a \subseteq \phi_b$ to mean $\phi_a = (T_a, P_a, b_{\nu_a}, seq_a, gap_a, egap_a)$ is entailed by $\phi_b = (T_b, P_b, b_{\nu_b}, seq_b, gap_b, egap_b)$. This will ensure that configurations denoted by $\phi_b$ are also denoted by $\phi_a$. Intuitively, $\phi_b$ should have at least as many tasks (captured by a surjection $\tau$ from a subset $T_b^1$ of $T_b$ to $T_a$) and phasers (captured by a bijection $\pi$ from a subset $P_b^1$ of $P_b$ to $P_a$). Conditions on tasks and phasers in $T_b^1$ and $P_b^1$ ensure the conditions in $\phi_b$ are met. Additional conditions on the tasks in $T_b^1 = T_b \setminus T_b^1$ ensure at least the $egap_a$ conditions in $\phi_a$ are met. More formally:

1. We have that, for each $b \in B$ and $\phi_a \subseteq \phi_b$,
2. $T_b$ and $P_b$ can be written as $T_b = T_b^1 \uplus T_b^2$ and $P_b = P_b^1 \uplus P_b^2$ with $\tau : T_b^1 \to T_a$ a surjection and $\pi : P_b^1 \to P_a$ a bijection, and
3. $\pi \circ \tau$ is a bijection and $\pi : P_b^1 \to P_a$ is a bijection, and
4. $\mathbf{(seq}_b(t_b) \neq \mathbf{seq}_a(t_a)) \implies (\mathbf{seq}_b(t_b) = \ast)$ for each $t_b \in T_b^1$ with $t_a = \tau(t_b)$, and
5. for each phaser $p = \pi(p_b)$ in $P_a$:
   a. if $egap_a(p_a) = (e_{w_a}, e_{s_a})$ and $egap_b(p_b) = (e_{w_b}, e_{s_b})$ then $e_{w_a} \leq e_{w_b}$ and $e_{s_a} \leq e_{s_b}$
   b. for each $t_b \in T_b^1$ with $t_a = \tau(t_b)$ and $gap_a(t_a)(p_a) = (var_a, val_a)$, and $gap_b(t_b)(p_b) = (var_b, val_b)$, it is the case that:
      i. $\mathbf{(var}_a \neq \mathbf{var}_b) \implies (\mathbf{var}_a = \ast)$, and
      ii. $\mathbf{(val}_a = \mathbf{nreg}) \implies (\mathbf{val}_a = \mathbf{nreg})$, and
      iii. if $\mathbf{val}_a = (1w_a, ls_a, uw_a, us_a)$ and $\mathbf{val}_b = (1w_b, ls_b, uw_b, us_b)$, then $1w_a \leq 1w_b$, $(ls_a \leq ls_b)$, $(uw_a \leq uw_b)$, and $(us_a \leq us_b)$.
   c. if $\mathbf{egap}_a(p_a) = (e_{w_a}, e_{s_a})$, both $e_{w_a} \leq 1w_b$ and $e_{s_a} \leq 1s_b$ hold.

We say in this case that $\tau$ and $\pi$ witness the entailment of $\phi_a$ by $\phi_b$.

**Lemma 2 (Constraint entailment).** $\phi_a \subseteq \phi_b$ implies $[[\phi_b]] \subseteq [[\phi_a]]$

**Proof.** Assume a configuration $c = (T_c, P_c, b_{\nu_c}, seq_c, phase_c)$ is denoted by $\phi_b = (T_b, P_b, b_{\nu_b}, seq_b, gap_b, egap_b)$ with $\phi_a = (T_a, P_a, b_{\nu_a}, seq_a, gap_a, egap_a)$. We show $c$ is also denoted by $\phi_a$.

By assumption, we can write $T_c$ as a partition $T_{c,m} \uplus T_{c,c}$ and $P_c$ as a partition $P_{c,m} \uplus P_{c,c}$ such that a surjection $\tau_c : T_{c,m} \to T_b$ and a bijection $\pi_c : T_{c,c} \to P_b$ witness the denotation of $c$ by $\phi_b$. Also, we can write $T_b$ as a partition $T_{b,m} \uplus T_{b,c}$ and $P_b$ as a partition $P_{b,m} \uplus P_{b,c}$ such that a surjection $\tau_b : T_{b,m} \to T_a$ and a bijection $\pi_b : P_{b,m} \to P_a$ witness the entailment of $\phi_b$ by $\phi_a$. Let us write $T_c$ as the partition $T_{c,m} \uplus T_{c,c}$ where $T_{c,m} = \tau_c^{-1}(T_{b,m})$ and $T_{c,c}$ as the partition $P_{c,m} \uplus P_{c,c}$ where $P_{c,m} = \pi_c^{-1}(P_{b,m})$. We define $\tau$ to be the restriction of $\tau_b \circ \tau_c$ to $T_{c,m}$, i.e., $\tau : T_{c,m} \to T_a$ with $\tau(t) = \tau_b(\tau_c(t))$ for each $t$ in $T_{c,m}$. Observe that $\tau$ is a well defined surjection. In addition, we write $\pi$ to mean the restriction of $\pi_b \circ \pi_c$ to $P_{c,m}$. Observe that $\pi$ is a well defined bijection. We show that $\tau$ and $\pi$ witness the denotation of $c$ by $\phi_a$.

1. we have that, for each $b \in B$, both $(b_{\nu_b}(b) \neq b_{\nu_c}(b)) \implies (b_{\nu_b}(b) = \ast)$ and $(b_{\nu_a}(b) \neq b_{\nu_c}(b)) \implies (b_{\nu_a}(b) = \ast)$ hold. Hence, $(b_{\nu_c}(b) \neq b_{\nu_b}(b)) \implies (b_{\nu_a}(b) = \ast)$ also holds.
2. \( T = T^{m,m} \cup (T^{m,e} \cup T^e) \) and \( P_c = P_c^{m,m} \cup (P_c^{m,e} \cup P_c^e) \), with

3. \( \tau : T^{m,m} \rightarrow T = T^{m,m} \) and \( \pi : P_c^{m,m} \rightarrow \Phi \), is a bijection, such that

4. we have that \( (seq(t_b) \neq \Phi) \implies (seq(t_a) = \Phi) \) for each \( t_a, t_b \) in \( T \) with \( t_b = \tau(t_a) \) in \( T \), and \( (seq(t_a) \neq \Phi) \implies (seq(t_a) = \Phi) \) for each \( t_b \) in \( T \) with \( t_a = \Phi(t_b) \) in \( T \). Since \( T^{m,m} \subseteq T^{m,m} \) and the surjection \( \tau : T^{m,m} \rightarrow T \) is the restriction of \( \Phi \) to \( T^{m,m} \), we deduce: \( (seq(t_a) = \Phi) \implies (seq(t_a) = \Phi) \) for each \( t_a \) in \( T^{m,m} \) with \( t_a = \Phi(t_b) \) in \( T \).

5. for each \( p_a = \Phi(p_b) = \Phi(\Phi(p_c)) = \pi(p_c) \), there is a \( l : 0 \leq l \) s.t.:

(a) if \( t_a \in T^{m,m} \) with \( t_a = \Phi(t_c) \) with \( phase(t_a)(p_c) = (var,c, val,c), gap_b(t_b)(p_b) = (var_b, val_b) \) and \( gap_a(t_a)(p_a) = (var_a, val_a) \), then:

i. we have \( (var_a = var_b) \implies (var_a = var_b) \implies (var_a, var_b) = (\Phi_a, \Phi_b) \implies (var_a = \Phi_b) \).

ii. \( (val_a = \Phi_a) \implies (val_b = \Phi_b) \implies (val_a = \Phi_a) \).

iii. \( (\Phi_a, \Phi_b) \implies (\Phi_a, \Phi_b) \implies (\Phi_a, \Phi_b) \).

(b) if \( t_a \in T^{m,m} \) with \( t_b = \Phi(t_c) \) with \( phase(t_a)(p_c) = (var,c, val,c), gap_b(t_b)(p_b) = (var_b, val_b) \) and \( gap_a(t_a)(p_a) = (\Phi_a, \Phi_b) \), then:

i. if \( (val_a = (w, s)) \) and \( (val_b = (l_a, l_b, u_a, u_b)) \) and \( (\Phi_a, \Phi_b) \) then \( \Phi_a \leq l_a \leq l - w \) and \( \Phi_a \leq l - s \).

(c) if \( t_a \in T^{m,m} \) with \( phase(t_a)(p_c) = (var,c, val,c) \) and \( gap_a(t_a)(p_a) = (\Phi_a, \Phi_b) \) and \( \Phi_a \leq \Phi_b \), then \( \Phi_a \leq \Phi_b \leq l - w \).

The remaining part of this section aims to establish the following theorem:

**Theorem 3.** \((\Phi, \sqsubseteq)\) is WQO if \( \Phi \) is \( K \)-dimension-bounded and \( B \)-good for some pre-defined \( K, B \in \mathbb{N} \).

The idea is to propose an encoding for each constraint \( \phi = (T, P, bv, seq, gap, egap) \) wrt. some arbitrary total orders \( <_v \) and \( <_p \). We write \( enc(\phi, <_v, <_p) \) for the encoding of \( \phi \). We also define an entailment relation \( \sqsubseteq_e \) on encodings. Then, we show in Lem. 7 that \( enc(\phi, <_v, <_p) \sqsubseteq_e enc(\phi', <_v', <_p') \) implies \( \phi \sqsubseteq \phi' \).

Finally, we show in Lem. 3 that \( \sqsubseteq_e \) is WQO if the encoded \( K \)-dimension-bounded constraints are \( B \)-good for some pre-defined \( K, B \in \mathbb{N} \). We start with the named gaps. It is not difficult to show the following lemma:

**Lemma 3.** If \( \mathcal{G} \) is a good set of named gaps then \((\mathcal{G}, \sqsubseteq)\) is WQO.

A task state of dimension \( K \) is any tuple in \((\mathcal{UnrSuff} \times \mathcal{G}^K)\) where \( K \) is a natural in \( \mathbb{N} \) (corresponding to the number of phasers in the constraint to be encoded). We write \((s, g_1, \ldots, g_K) \sqsubseteq (s', g_1, \ldots, g_K)\) for two task states to mean that they have the same dimension (i.e., \( K = K' \)), that \( s \neq s' \Rightarrow s = \ast \), and that \( g_k \sqsubseteq g_k' \) for each \( k : 1 \leq k \leq K \). Using Higman’s lemma [12] and Lem. 3 we can show the following:

**Lemma 4.** \((\mathcal{UnrSuff} \times \mathcal{G}^K), \sqsubseteq)\ is WQO in case \( \mathcal{G} \) is good.
Let $\mathcal{M}(\text{UnrSuff} \times g^K)$ be the set of finite multisets over $(\text{UnrSuff} \times g^K)$. We write $A \preceq^\leq B$, for $A$ and $B$ two multisets in $\mathcal{M}(\text{UnrSuff} \times g^K)$, if each element $a \in A$ can be mapped to an element $b \in B$ for which $a \preceq b$. By adapting Higman’s lemma \[12\] and using Lem. \[4\] we can show the following lemma:

**Lemma 5.** $(\mathcal{M}(\text{UnrSuff} \times g^K), \preceq^\leq)$ is WQO if $g$ is good.

We write $A \preceq \supseteq B$, for $A$ and $B$ finite multisets in $\mathcal{M}(\text{UnrSuff} \times g^K)$, to mean that each $b \in B$ can be mapped to some $a \in A$ for which $a \preceq b$. Rado’s structure \[13\] shows that $(\mathcal{M}(S), \preceq)$ need not be WQO just because $\preceq$ is WQO over $S$. Still, we establish the following result:

**Lemma 6.** $(\mathcal{M}(\text{UnrSuff} \times g^K), \preceq^\leq)$ is WQO if $g$ is good.

**Proof.** We proceed by contradiction. Assume, without loss of generality, an infinite sequence $(A_1, A_2, \ldots)$ of $\leq$-minimal multisets in $\mathcal{M}(\text{UnrSuff} \times g^K)$ such that $A_j \preceq \supseteq A_i$ for all $1 \leq j < i$. Notice that:

1. for each $i : 1 \leq j < i$, we can identify an element $a_{i,j}^\ast \in A_i$ such that $a_j \not\preceq a_{i,j}^\ast$
   for any $a_j \in A_j$. We sometimes write $a_{i,j}^\ast$ as $(a_{i,j}^\ast[1], a_{i,j}^\ast[2], \ldots, a_{i,j}^\ast[K])$
   where $a_{i,j}^\ast[k]$ is the $k$th component of $a_{i,j}^\ast$.

2. by the definition of $(\text{UnrSuff} \times g^K)$ and the fact that it is a WQO, we can extract a subsequence $(A_{1'}, A_{2'}, \ldots)$ of $(A_1, A_2, \ldots)$ associated to a sequence of partial mappings $(b_j : \{1, \ldots, K\} \to \mathbb{N})$ s.t. for each $j : 1 \leq j$:
   (a) the $j$-sequence $a_{1,j}^\ast \preceq a_{2,j}^\ast \preceq \ldots$ is \(\leq\)-monotone,
   (b) each $j$-sequence has a constant control, i.e., $s^\ast_j = s^\ast_{1,j} = s^\ast_{2,j} = \ldots$ and
   (c) for each dimension $k : 1 \leq k \leq K$, the projection of a $j$-sequence $a_{1,j}^\ast \preceq a_{2,j}^\ast \preceq \ldots$ on $k$ is either constant or strictly increasing, i.e.,:
   i. $k \in \text{dom}(b_j)$ implies $a_{1,j}^\ast[k] = b_j(k)$ for all $i : j < i$
   ii. $k \notin \text{dom}(b_j)$ implies $a_{1,j}^\ast[k] \preceq a_{2,j}^\ast[k]$ but $a_{1,j}^\ast[k] \not\preceq a_{2,j}^\ast[k]$ for any $i, i' : 1 \leq j < i'$

We rename the sequence $(A_{1'}, A_{2'}, \ldots)$ into $(A_1, A_2, \ldots)$. We obtain a $\preceq \supseteq$-bad sequence $(A_1, A_2, \ldots)$ of $\leq$-minimal sets satisfying the constraints depicted in the following figure:

\[
\begin{array}{cccccccccccc}
A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & \cdots & A_i & \cdots \\
\neg A_1 & a_2^1 & \preceq & a_3^1 & \preceq & a_4^1 & \preceq & a_5^1 & \preceq & a_6^1 & \preceq & a_7^1 & \preceq & \cdots & A_i & \cdots \\
\neg A_2 & a_3^2 & \preceq & a_4^2 & \preceq & a_5^2 & \preceq & a_6^2 & \preceq & a_7^2 & \preceq & \cdots & A_i & \cdots \\
\neg A_3 & a_4^3 & \preceq & a_5^3 & \preceq & a_6^3 & \preceq & a_7^3 & \preceq & \cdots & A_i & \cdots \\
\neg A_4 & a_5^4 & \preceq & a_6^4 & \preceq & a_7^4 & \preceq & \cdots & A_i & \cdots \\
\neg A_5 & a_6^5 & \preceq & a_7^5 & \preceq & \cdots & A_i & \cdots \\
\neg A_6 & a_7^6 & \preceq & \cdots & A_i & \cdots \\
\neg A_i & \cdots & a_i^\ast & \cdots \\
\end{array}
\]
Some observations about the $a_i^{\sim j}$ elements:

1. For any $i, j: 1 \leq j < i$, $a_i^{\sim j} \in A_i$ and $a_j \not\leq a_i^{\sim j}$ for any $a_j \in A_j$.
2. For any $i, i', j: 1 \leq j < i < i'$, $a_i^{\sim j} \leq a_{i'}^{\sim j}$ with $a_i^{\sim j}[k] = a_{i'}^{\sim j}[k]$ iff $k \in \text{dom}(b_j)$.
3. For any $i, j, j': 1 \leq j' < j < i$ and $i' : j \leq i'$, $a_{i'}^{\sim j'} \not\leq a_i^{\sim j}$.

Observations (1) and (2) are obtained by construction. Suppose observation (3) does not hold, i.e., suppose $a_{i'}^{\sim j'} \leq a_i^{\sim j}$ for some $i, j, j': 1 \leq j' < j < i$ and $i' : j \leq i'$. Observation (2) and reflexivity of $\leq$ ensure $a_{i'}^{\sim j'} \leq a_i^{\sim j}$. By transitivity of $\leq$, we get $a_{i'}^{\sim j'} \leq a_i^{\sim j}$. This contradicts observation (1). Hence observation (3) also holds.

Now, observe that the domain $D_j$ of $b_j$, for any $j : 1 \leq j$, is a subset of $\{1, \ldots, K\}$. Since this domain is finite, there is an infinite number of indices $j : 1 \leq j$ with the same domain $D \subseteq \{1, \ldots, K\}$. In other words, an infinite number of j-sequences are constant (with possibly different values) on the same dimensions $D$. We can therefore extract a sequence $\langle j_1, j_2, \ldots \rangle$ of j-sequences that is increasing on each dimension in $D$ (i.e., $j_a \leq j_b \implies$ for each $k \in D, b_{j_a}(k) \leq b_{j_b}(k)$). In order for observation (3) to hold (i.e., $a_{i'}^{\sim j'} \not\leq a_i^{\sim j}$ for any $i, j, j': 1 \leq j' < j < i$ and $i' : j \leq i'$), we need to have some dimensions on which the j-sequences $(a_i^{\sim j})_{j_1, j_2, \ldots}$ do not increase, these would be the dimensions in $\{1, \ldots, K\} \setminus D$. But for these dimensions, $(a_i^{\sim j})_{i: 1, 2, \ldots}$ is strictly increasing with $i$. This again contradicts observation (3).

Lem. 6 will be used in Lem. 8 to show an entailment relation on encodings of $K$-dimension-bounded and $B$-good constraints corresponds to a stronger relation than $\sqsubseteq$ and is WQO, hence establishing Thm. 8. First, we introduce constraints encodings and an entailment relation on them.

**Encodings of constraints.** Given a finite set $Q$ and an associated total order $<_Q$, we write $Q[i] \in Q$ to mean the element of $Q$ with $<_Q$-index $i \in \{1, \ldots, |Q|\}$.

For instance, given a finite set of phasers $P$ and an associated total order $<_P$, we write $P[k]$ to mean the phaser with $<_P$-index $k$ in $P$. The encoding of a constraint $\phi = (T, P, bv, seq, gap, egap)$ with respect to total orders $<_P$ and $<_Q$, written $\text{enc}(\phi, <_P, <_Q)$, is a tuple $(bv, acc, env)$ where:

1. $bv : B \to B^{(*)}$ is the same as in $\phi$.
2. $acc : \{1, \ldots, |T|\} \to (\text{UnrSuff} \times G[|P|])$ associates each task $T[i]$ to a tuple $\text{acc}(i) = (s, g_1, \ldots, g_{|P|})$ where $\text{seq}(T[i]) = s$ and $\text{gap}(T[i])(P[j]) = g_j$ for each phaser $P[j]$ with index $j$ in $P$.
3. $env : \{1, \ldots, |P|\} \to \mathbb{N}^2$ associates to each phaser $P[j]$ in $P$ the pair $env(j) = egap(P[j])$.

\footnote{the $<_Q$-index of an element $q$ in $Q$ is $1 + |\{q' \mid q' \in Q \text{ and } q' <_Q q\}|$}
Observe that if two constraints result in the same encoding, then they can be obtained from each other by renaming the tasks and the phasers. As a consequence, if a constraint is free (resp. B-gap-bound or B-good), then all constraints resulting in the same encoding will also be free (resp., B-gap-bound or B-good). We define the dimension of an encoding \((bv, acc, env)\) to be the size of the domain of \(env\) (i.e., the dimension of an encoded constraint). A (possibly infinite) set of encodings \(E\) is said to be free (resp. B-gap-bound or B-good) if all constraints encoded by any of its elements are free (resp. B-gap-bound or B-good). The set is said to be \(K\)-dimension-bound if there is natural \(K\) in \(\mathbb{N}\) that is larger than the dimension of any of its elements.

**Entailment of encodings.** Assume two encodings \((bv, acc, env)\) and \((bv', acc', env')\) with \(acc : \{1, \ldots, L\} \to \text{UnrSuff} \times G^M\), \(env : \{1, \ldots, M\} \to \mathbb{N}^2\), \(acc' : \{1, \ldots, L'\} \to \text{UnrSuff} \times G^{M'}\), \(env : \{1, \ldots, M'\} \to \mathbb{N}^2\). Write \((bv, acc, env) \sqsubseteq_e (bv', acc', env')\) iff:

1. for each \(b \in B\), \((bv(b) \neq bv'(b)) \implies (bv(b) = *)\), and
2. \(M' = M\) and there is a surjection \(h : \{1, \ldots, L'\} \to \{1, \ldots, L\}\) such that:
   (a) \(acc(h(i)) \leq acc'(i)\) for each index \(i \in \{1, \ldots, L'\}\),
   (b) \(env(j) \leq env'(j)\) for each index \(j \in \{1, \ldots, M'\}\),

**Lemma 7.** Let \(\phi = (T, P, bv, seq, gap, egap)\) and \(\phi' = (T', P', bv', seq', level', gap', egap')\).

If \(\text{enc}(\phi, <_{T}, <_{P}) = (bv, acc, env)\) and \(\text{enc}(\phi', <_{T'}, <_{P'}) = (bv', acc', env')\), then \((bv, acc, env) \sqsubseteq_e (bv', acc', env')\) implies \(\phi \sqsubseteq \phi'\).

**Proof.** From \((bv, acc, env) \sqsubseteq_e (bv', acc', env')\) we deduce \(|P| = |P'|\) and the existence of a surjection \(h : \{1, \ldots, |T'|\} \to \{1, \ldots, |T|\}\), such that \(acc(h(i)) \leq acc'(i)\) for each \(i \in \{1, \ldots, |T'|\}\), and \(env(j) \leq env'(j)\) for each \(j \in \{1, \ldots, |P'|\}\).

1. \(bv = bv'\), hence, for each \(b \in B\), \((bv(b) \neq bv'(b)) \implies (bv(b) = *)\).
2. let \(\tau : T' \to T\) with \(\tau(T'[i]) = T[h(i)]\) for each \(T'[i]\) in \(T'\). Let \(\pi : P' \to P\) with \(\pi(P'[i]) = P[i]\) for each \(P'[i]\) in \(P'\). Observe \(\tau\) is surjective and \(\pi\) is bijective.
3. for each \(i \in \{1, \ldots, |T'|\}\), we have that \(acc(h(i)) \leq acc'(i)\). By definition, \(acc(h(i))\) is the tuple \((seq(T[h(i)]), gap(T[h(i)])(P_1), \ldots, gap(T[h(i)])(P_{|P|}))\) and \(acc'(i)\) is the tuple \((seq(T'[i]), gap(T'[i])(P'[1]), \ldots, gap(T'[i])(P'[|P'|]))\) where \(P = \{P_1, \ldots, P_{|P|}\}\) and \(P' = \{P'[1, \ldots, P'[|P'|]\}\). By definition of \(\leq\), we get:
   (a) \(seq(T[h(i)]) \neq seq'(T'[i]) \implies seq(T[h(i)]) \neq seq'(T'[i])\),
   (b) \(gap(T[h(i)])(P[i]) \leq gap'(T'[i])(P'[i])\) for each \(i \in \{1, \ldots, |P'|\}\). Since \(P[i] = P'[i]\) and if \(gap(T[h(i)])(P[i]) = (var, val)\) and \(gap'(T'[i])(P'[i]) = (var', val')\), we deduce that:
   i. \((var \neq var') \implies (var = *)\)
   ii. \((val = \text{nreg}) \iff (val' = \text{nreg})\)
   iii. if \(val = (lw, ls, uw, us)\) and \(val' = (lw', ls', uw', us')\), then \(lw \leq lw'\) and \(ls \leq ls'\) and \(uw \leq uw'\) and \(us \leq us'\).

4. for each \(j \in \{1, \ldots, |P'|\}\), we have \(env(j) \leq env'(j)\). By definition, \(env(j) = egap(P[j])\) and \(env'(j) = egap'(P'[j])\). Since \(\pi(P'[j]) = P[j]\), we deduce that \(egap(P[j]) \leq egap'(P'[j])\) for each \(j \in \{1, \ldots, |P'|\}\).
Lemma 8. \((E, \sqsubseteq_c)\) is WQO if the set \(E\) of encodings is \(K\)-dimension-bounded and \(B\)-good for some pre-defined \(K, B \in \mathbb{N}\).

Proof. Assume a \(K\)-dimension-bounded set \(E\) of \(B\)-good encodings and an infinite sequence \(S_1 = \langle (bv_1, acc_1, env_1), (bv_2, acc_2, env_2), \ldots \rangle\). We show the existence of \(i, j : 1 \leq i < j\) for which \((bv_i, acc_i, env_i) \sqsubseteq c (bv_j, acc_j, env_j)\). Dimension-boundedness of \(E\) ensures there are infinitely many encodings in \(S_1\) with the same dimension, say \(K\). We extract the subsequence \(S_2\) consisting in all encodings with dimension \(K\) in \(S_1\). In addition, observe that the set of possible valuations of the Boolean variables is finite. We can therefore extract from \(S_2\) an infinite subsequence \(S_3\) where all elements share the same valuation of the Boolean variables. Let us rewrite \(S_3\), for simplicity, as the sequence \(((bv_1, acc_1, env_1), (bv_2, acc_2, env_2), \ldots \rangle\). For each \(i : 1 \leq i\), we can represent the mapping \(env_i\) as the tuple \(\langle env_i(1), env_i(2), \ldots , env_i(K) \rangle\) in \((\mathbb{N}^2)^K\). Using Higman’s lemma, we can extract from \(S_3\) a subsequence \(S_4\), also renamed to \(((bv_1, acc_1, env_1), (bv_2, acc_2, env_2), \ldots \rangle\) for simplicity, where \(env_i(k) \leq 2 env_j(k)\) for any \(i, j : 1 \leq i < j\) and \(k : 1 \leq k \leq K\).

For each mapping \(acc_i : \{1, \ldots , L_i\} \rightarrow \text{UnrSuff} \times g^K\) in \(S_4\), we write \(m_{acc_i}\) to mean the multiset over \((\text{UnrSuff} \times g^K)\) where the number of occurrences of an element \((s, g_1, \ldots , g_K)\) coincides with the number of indices \(j\) in \(\{1, \ldots , L_i\}\) for which \(acc_i(j) = (s, g_1, \ldots , g_K)\). Consider the sequence \(\langle m_{acc_1}, m_{acc_2}, m_{acc_3}, \ldots \rangle\) of elements in \(\mathcal{M} \rightarrow \text{UnrSuff} \times g^K\). Using the fact that \(E\) is \(B\)-good together with Lem. 5 and 6, we deduce the existence of \(i, j : i < j\) for which \(m_{acc_i} \preceq \exists \ m_{acc_j}\) and \(m_{acc_i} \preceq_\forall m_{acc_j}\). We can therefore build a surjection \(h : \{1, \ldots , L_j\} \rightarrow \{1, \ldots , L_i\}\) such that \(acc_i(h(l)) \preceq acc_j(l)\) for each \(l\) in \(\{1, \ldots , L_j\}\).

5 A symbolic verification procedure

We use the constraints from Sec. 4 as a symbolic representation in the adaptation of the classical working-list based backward procedure described below. This procedure corresponds to an instantiation of the framework of Well-Structured-Transition-Systems [118]. The procedure takes as arguments a program \(\text{prg}\) and a \(\subseteq\)-minimal set \(\Phi\) of constraints denoting the targeted set of configurations. Such sets can be easily built from the partial configurations described in Fig. 8.

The predecessor computation rules in Figures 7 and 8 that are used by the procedure need to first concretize the given constraint to explicitly contain the required task(s), phaser(s), phaser variable(s), and control sequence(s). For this purpose, they make use of the rules \texttt{concretizeTask} (a wrapper for \texttt{concretizeTask} and \texttt{concretizePhaser}), \texttt{concretizeSeq}, and \texttt{concretizeVar}. Each concretization rule returns a (possibly empty) set of concrete constraints. Intuitively, \texttt{concretizeTask}(\(\phi, t, \text{Un}, \mathcal{F}\)) makes sure either \(t\) that is going to be used in the computation is already in \(\phi\), or adds it as a fresh task. \texttt{concretizePhaser}(\(\phi, p, \text{Un}, \mathcal{F}\)) concretizes the phaser \(p\). \texttt{concretizeSeq}(\(\phi, t, s\)) makes sure the sequence of the task \(t\) is \(s\), which is a requirement for the predecessor computation rule. Otherwise, it will return an empty set. Finally, \texttt{concretizeVar}(\(\phi, t, p, v\)) ensures \(t\)
uses the variable \( v \) to reference \( p \), which is again required by the predecessor computation rule. In the following sections we introduce and then prove some of the characteristics of the concretization functions, predecessor computation functions, and the verification procedure.

### 5.1 Concretization

In this section, we introduce the concretization functions and prove some properties about them. Assume a constraint \( \phi = (T, P, bv, seq, gap, egap) \). We say a set \( \mathcal{U}n \) is in \( \phi \) if \( \mathcal{U}n \subseteq T \) and a set \( \mathcal{F} \) is fresh for \( \phi \) if \( \mathcal{F} \cap T = \emptyset \). We say \texttt{concretizePhaser}(\( \phi, p, \mathcal{U}n, \mathcal{F} \)) (Figure 5) concretizes the phaser \( p \) in \( \phi \). Rule \texttt{concretize phaser} 1 is used when phaser \( p \) is already concrete and the input constraint will be returned without any modification. Rule \texttt{concretize phaser} 2 adds a new phaser \( p \) to the constraint \( \phi \) and non-deterministically registers a set \( T_2 \) of tasks in the concrete constraint that map to \( T_1 \subseteq T \) with \( p \). Moreover, it non-deterministically chooses a set of tasks \( T_2' \) that map to \( T_2 \subseteq T \) to not be registered with \( p \). The task mappings \( \tau_1 \) and \( \tau_2 \) are used to map the tasks from the concrete constraint to constraint \( \phi \).

\texttt{concretizeTask}(\( \phi, t_\phi, \mathcal{U}n, \mathcal{F} \)) (Figure 5) concretizes the task \( t_\phi \) in \( \phi \). Rule \texttt{concretize task} 1 is used when task \( t_\phi \) is already concrete and the input constraint will be returned without any modification. Rules \texttt{concretize task} 2 and 3 consider the case when a new task \( t_\phi \) will be added that copies a task \( u \in T \). The difference between \texttt{concretize task} 2 and 3 is in the fact that \texttt{concretize task} 3, unlike \texttt{concretize task} 2, concretizes \( u \) (after being renamed to \( t' \)) as well as \( t \). Rule \texttt{concretize task} 4 concretizes a task that is non-deterministically registered with a subset of the phasers in \( P_2 \subseteq P \) and not registered to others. Such task needs to have phase bounds that respect the environment of the phasers in \( P_2 \).

\texttt{concTasksPhasers}(\( \phi, A_\phi, B_\phi, \mathcal{U}n, \mathcal{F} \)) (Figure 5) concretizes the phasers \( A_\phi \) and the tasks \( B_\phi \) in \( \phi \) using \texttt{concretizePhaser}(\( \phi, p, \mathcal{U}n, \mathcal{F} \)) and \texttt{concretizeTask}(\( \phi, t_\phi, \mathcal{U}n, \mathcal{F} \)).

Rule \texttt{concretize seq} concretizes the control sequence of a task and rule \texttt{concretize var} concretizes the phaser name in the fact that is required for referencing a phaser. Some facts about the the concretizations will follow.

#### Lemma 9

For a given constraint \( \phi = (T, P, bv, seq, gap, egap) \), a set \( \mathcal{U}n \) that is in \( \phi \), a set \( \mathcal{F} \) that is fresh for \( \phi \), and a phaser \( p \) that is possibly in \( P \), \texttt{concretizePhaser}(\( \phi, p, \mathcal{U}n, \mathcal{F} \)) always terminates. Each tuple \((\phi', \mathcal{U}n', \mathcal{F}')\) in \texttt{concretizePhaser}(\( \phi, p, \mathcal{U}n, \mathcal{F} \)) satisfies the following:

- \( \phi \leq \phi' \),
- \( \mathcal{F}' \subseteq \mathcal{F} \) and \( \mathcal{F}' \cap T_{\phi'} = \emptyset \),
- \( \mathcal{U}n' = \mathcal{U}n \cup (\mathcal{F} \setminus \mathcal{F}') \) and \( \mathcal{U}n' \subseteq T_{\phi'} \),

**Proof.** By definition of \texttt{concretize phaser} 1 and 2 in Figure 5.
\[
\begin{align*}
t \in T & \quad seq(t) = s \quad (\text{concretize seq 1}) \\
(t, \phi, bv, seq, gap, gap) \in & \text{concretizeSeq}(T, \phi, bv, seq, gap, gap), \quad t, s) \\
(t \in T & \quad seq(t) = s \quad seq' = seq(t \leftarrow s) & \quad (\text{concretize seq 2}) \\
(t, \phi, bv, seq', gap', gap) \in & \text{concretizeSeq}(T, \phi, bv, seq, gap, gap), \quad t, s) \\
t \in T & \quad \text{in} \quad \text{concretizeTask}(T, \phi, bv, seq, gap, gap), \quad \text{tn, } \text{tn')} \quad (\text{concretize task 1}) \\
(t' \in T' & \quad \text{tn} \in T \setminus \text{tn} \quad \text{tn'} = T \cup \{t\} \\
gap' = gap\{t \leftarrow gap(u) \mid p \in \phi\} & \quad (\text{concretize task 2}) \\
(t', \phi, bv, seq', gap', gap', \text{tn'}, \text{tn}') \in & \text{concretizeTask}(T, \phi, bv, seq, gap, gap), \quad t, \text{tn}, \text{tn'}) \quad (\text{concretize task 3}) \\
 t \not\in t' & \quad \{t, t'\} \subseteq T \quad u \in T' \setminus \text{tn} \quad \text{tn'} = (T' \setminus \{u\}) \cup \{t, t'\} \\
gap' = gap\{t \leftarrow map(t) \mid p \in \phi\} & \quad (\text{concretize task 4}) \\
p \not\in \phi & \quad (\text{concretize phaser 1}) \\
(t', \phi, bv, seq', gap', gap', \text{tn'}, \text{tn}') \in & \text{concretizePhaser}(T, \phi, bv, seq, gap, gap), \quad p, \text{tn}, \text{tn'}) \quad (\text{concretize phaser 2}) \\
t \in T & \quad p \not\in \phi \quad gap(t)(p) = (v, val) \quad (\text{concretize phaser var 1}) \\
(t', \phi, bv, seq', gap', gap', t, p, v) \in & \text{concretizeVar}(T, \phi, bv, seq, gap, gap), \quad t, p, v) \\
t \in T & \quad p \not\in \phi \quad gap(t)(p) = (v, val) \quad (\text{concretize phaser var 2}) \\
\forall v \in \phi \quad (\text{gap}(t)(q) = (var, val) \Rightarrow \text{gap}(t)(p) = (v, val)) \\
\forall v \in \phi \quad (\text{gap}(t)(q) = (var, val) \Rightarrow \text{gap}(t)(p) = (v, val)) \\
\end{align*}
\]

**Fig. 5.** Auxiliary functions used in the predeccessor computation of the parameterized Phaser instructions (Part I). The function `concreteTasksPhasers(\phi, A, B, tn, f)` concretizes the set of phasers A and the set of tasks B in the constraint \(\phi\). Those phasers or tasks in A or B that are already in \(\phi\) will be preserved. \(\mathcal{U}\text{tn} is a subset of B, which is already concretized to unique tasks. \(f\) is a set of fresh task identifiers that does not intersect B.
5 do not change the tasks, phasers, and phases. One can show that after calling concTasksPhasers is satisfies the following:

By definition of always terminates. Each tuple for a given constraint \( \phi \) that is in \( \mathcal{F} \), a set \( T \) that is fresh for \( \phi \), and a task \( t \in \mathcal{U} \cap \mathcal{F} \), concretizeTask(\( \phi, t, \mathcal{U} \), \( \mathcal{F} \)) always terminates. Each tuple \( (\phi', \mathcal{U}' \), \( \mathcal{F}' \)) in concretizeTask(\( \phi, p, \mathcal{U} \), \( \mathcal{F} \)) satisfies the following:

- \( \phi \preceq \phi' \)
- \( \mathcal{F}' \subseteq \mathcal{F} \) and \( \mathcal{F}' \cap \mathcal{T}_{\phi'} = \emptyset \)
- \( \mathcal{U}' = \mathcal{U} \cup (\mathcal{F} \setminus \mathcal{F}') \) and \( \mathcal{U}' \subseteq \mathcal{T}_{\phi'} \)

**Proof.** By definition of concretize task 1, 2, 3, and 4 in Figure 5.

Observe that concretizeVar(\( \phi, t, p, v \)) and concretizeSeq(\( \phi, t, s \)) in Figure 5 do not change the tasks, phasers, and phases. One can show that after calling them, the identity relations on \( T \) and \( \mathcal{F} \) witness \( \phi \preceq \phi' \).

**Lemma 11.** For a given constraint \( \phi = \left( \mathcal{T}_\phi, \mathcal{P}_\phi, \mathcal{B}_\phi, \mathcal{V}_\phi, \mathcal{G}_\phi \right) \), a set \( \mathcal{U} \) that is in \( \phi \), a set \( \mathcal{F} \) that is fresh for \( \phi \), a task \( t \in \mathcal{T}_\phi \), a phaser variable \( v \), a phaser \( p \in \mathcal{P}_\phi \), and a \( \mathcal{S} \), concretizeVar(\( \phi, t, p, v \)) and concretizeSeq(\( \phi, t, s \)) always terminate and will respectively generate a singleton \( \{\phi'\} \) such that the constraint \( \phi' \) satisfies \( \phi \preceq \phi' \).

**Proof.** By definition of concretizeVar(\( \phi, t, p, v \)) and concretizeSeq(\( \phi, t, s \)) in Figure 5.

**Lemma 12.** Assume \( \phi = \left( \mathcal{T}_\phi, \mathcal{P}_\phi, \mathcal{B}_\phi, \mathcal{V}_\phi, \mathcal{G}_\phi \right) \), \( A_\phi \) that is a set of phaser identifiers possibly intersecting \( \mathcal{P}_\phi \), a set \( \mathcal{U} \) that is in \( \phi \), a set \( \mathcal{F} \) that is fresh for \( \phi \), and \( \mathcal{B}_\phi \subseteq \mathcal{U} \cup \mathcal{F} \). conTasksPhasers(\( \phi, A_\phi, \mathcal{B}_\phi, \mathcal{U} \), \( \mathcal{F} \)) always terminates and returns a finite set of tuples. Each tuple \( (\phi', \mathcal{U}' \), \( \mathcal{F}' \)) in concTasksPhasers(\( \phi, A_\phi, \mathcal{B}_\phi, \mathcal{U} \), \( \mathcal{F} \)) satisfies

\[
\begin{align*}
(\phi', \mathcal{U}' \), \( \mathcal{F}' \) & \in \text{concTasksPhasers}(\phi, A, \mathcal{U}, \mathcal{F}) \\
(\phi'', \mathcal{U}'' \), \( \mathcal{F}'' \) & \in \text{concretizePhaser}(\phi', \mathcal{U}', \mathcal{F}') \\
(\phi''', \mathcal{U}''' \), \( \mathcal{F}''' \) & \in \text{concTasksPhasers}(\phi, A \cup \{p\}, \mathcal{U}, \mathcal{F}) \\
(\phi''', \mathcal{U}'''' \), \( \mathcal{F}'''' \) & \in \text{concretizeTask}(\phi', \mathcal{T}, \mathcal{U}, \mathcal{F}) \\
(\phi''', \mathcal{U}'''' \), \( \mathcal{F}'''' \) & \in \text{concTasksPhasers}(\phi, A \cup \{t\}, \mathcal{U}, \mathcal{F}) \\
(\phi, \mathcal{U}, \mathcal{F}) & \in \text{concTasksPhasers}(\phi, \emptyset, \emptyset, \mathcal{U}, \mathcal{F}) \\
\phi = (T, \mathcal{F}, \mathcal{B}, \mathcal{S}, \mathcal{G}, \mathcal{V}) & \in \text{conTasksPhasers}(\phi, A, B) \quad \mathcal{B} = \mathcal{U} \cup \mathcal{F} \quad \mathcal{U} \subseteq \mathcal{F} \quad \mathcal{F} \cap \mathcal{F} = \emptyset
\end{align*}
\]
among which there is a tuple \((F, \phi)\) that is in \(\phi\).

By Lemmas 9 and 10 and induction on \(A\).

**Proof.** By Lemmas 9 and 10 and induction on \(A\).

**Lemma 13.** Assume a configuration \(c = (T, \rho, \text{bv}_c, \text{seq}_c, \text{phase}_c)\) and constraint \(\phi = (T, \rho, \text{bv}_c, \text{seq}_c, \text{gap}_c, \text{egap}_c)\) such that \(\tau\) and \(\pi\) witness \(c \in [\phi]\), a task \(t \in T\) is said to be \(\tau\)-uniquely-mapped iff \(\tau(u) = \tau(t) \Rightarrow u = t\).

**Definition 3 (\(\tau\)-uniquely-mapped).** For a given configuration \(c = (T, \rho, \text{bv}_c, \text{seq}_c, \text{phase}_c)\) and constraint \(\phi = (T, \rho, \text{bv}_c, \text{seq}_c, \text{gap}_c, \text{egap}_c)\) such that \(\tau\) and \(\pi\) witness \(c \in [\phi]\), a set \(\text{Un}\) that is in \(\phi\) such that \(\tau\) uniquely maps \(\tau^{-1}(\text{Un})\), and a set \(\text{Un}\) that is fresh for \(\phi\). Let \(p_c\) be an arbitrary task in \(T\) and \(p_\phi\) be \(\pi(p_c)\) if \(p_c\) is mapped by \(\pi\) or a fresh phaser, otherwise. A tuple \((\phi', \text{Un}', F')\) exists among \(\text{concretizePhaser}(\phi, p_\phi, \text{Un}, F)\) such that:

- \(c \in [\phi']\) and \(\tau'\) and \(\pi'\) witness the denotation,
- \(\pi' = \pi[p_c \leftarrow p_\phi]\),
- \(\tau'\) uniquely maps the tasks in \(\tau'^{-1}(\text{Un}')\).

**Proof.** The concretization of \(p_\phi\) will be performed in one of the following ways:

- If \(p_c\) is mapped to \(p_\phi\) by \(\pi\), the rule (concretize phaser 1) will return \((\phi, \text{Un}, F)\), which is the desired tuple. Observe that the tasks are not altered, hence, the returned tuple satisfies the conditions in the lemma.
- If \(p_c\) is not mapped by \(\pi\), the rule (concretize phaser 2) generates a set of tuples \((\phi', \text{Un}', F')\) in which the concrete constraints account for all possible registrations of the tasks in \(T_0\) to the new phaser \(p_\phi\). A constraint \(\phi' = (T_0', \rho, \text{bv}_c, \text{seq}_c, \text{gap}_c, \text{egap}_c)\) generated by the rule will capture
  
  - the tasks in \(T_1 \subseteq T\) that were mapped by \(\tau\) but were not registered with \(p_c\) can be mapped to \(T_1' \subseteq T_0\) by the task mapping \(\tau_1(\tau)\).
  - the tasks in \(T_2 \subseteq T\) that were mapped by \(\tau\) and are registered with \(p_c\) can be mapped to \(T_2' \subseteq T_0\) by the task mapping \(\tau_2(\tau)\).

The task mapping \(\tau'\) that maps the tasks in \(T_1\) using \(\tau_1(\tau)\) and those in \(T_2\) using \(\tau_2(\tau)\) and the phaser mapping \(\pi' = \pi[p_c \leftarrow p_\phi]\) witness \(c \models \phi'. \tau'\) by construction uniquely maps the tasks in \(\tau'^{-1}(\text{Un}')\).

**Lemma 14.** Assume a configuration \(c = (T, \rho, \text{bv}_c, \text{seq}_c, \text{phase}_c)\) and constraint \(\phi = (T, \rho, \text{bv}_c, \text{seq}_c, \text{gap}_c, \text{egap}_c)\) such that \(\tau\) and \(\pi\) witness \(c \in [\phi]\), a set \(\text{Un}\) that is in \(\phi\) such that \(\tau\) uniquely maps \(\tau^{-1}(\text{Un})\), a set \(\text{Un}\) that is fresh for \(\phi\), and a set \(B_c = (B_{c_1} \cup B_{c_2}) \subseteq T\) that is uniquely maps \(B_{c_1}\) and does not uniquely map \(B_{c_2}\). Let \(t_c\) be an arbitrary task in \(B_c\) and \(t_\phi\) be \(\tau(t_c)\) if \(t_c\) is in \(\text{Un}\) or a task not in \(\text{Un}\), otherwise. \(\text{concretizeTask}(\phi, t_\phi, \text{Un}, F)\) will generate a set of tuples among which there is a tuple \((\phi', \text{Un}', F')\) such that
Lemma 15. \(c \in [\phi']\) and \(\tau'\) and \(\pi\) witness the denotation,
\(- \tau'\) uniquely maps the tasks in \(\tau'^{-1}(\mathcal{tIn}')\).

Proof. The concretization of \(t_\phi\) will be performed in one of the following ways:

- If \(t_c\) is in \(B_{c_1}\), it is already uniquely mapped to \(t_\phi\), the rule (concretize task 1) will return \((\phi, \mathcal{tIn}, \mathcal{f})\), which is the desired tuple.
- If \(t_c\) is mapped by \(\tau\) but is in \(B_{c_2}\), it is mapped to \(\tau(t_c)\) but not uniquely.
  The rule (concretize task 2) (respectively, concretize task 3) return in this case a tuple \((\phi', \mathcal{tIn}', \mathcal{f}')\) for which the mappings \(\tau' = \tau|_{t_c} \leftarrow t_\phi\) (respectively, \(\tau' = \tau|_{t_c} \leftarrow t_\phi|_{u_c} \leftarrow u_\phi\) given that \(t_c\) and \(u_c\) are both mapped to \(\tau(t_c)\)) and \(\pi\) witnesses \(c \in [\phi']\). The other claims hold by construction.
- If \(t_c\) is not mapped by \(\tau\), it is in \(B_{c_3}\). Let \(\mathcal{p}_m^n \subseteq \mathcal{p}_c\) be the set of mapped phasers by \(\pi\) and \(\mathcal{p}_m^{n,r} \subseteq \mathcal{p}_c\) be the set of mapped phasers that \(t_c\) is registered with. In this case, the rule (concretize task 4) returns a set of tuples among which there is a tuple \((\phi', \mathcal{tIn}', \mathcal{f}')\) in which \(t_\phi\) is a fresh task that is registered with \(\pi(\mathcal{p}_m^{n,r})\) and not with \(\pi(\mathcal{p}_m^n \setminus \mathcal{p}_m^{n,r})\). Hence, the task and phaser mappings \(\tau' = \tau|_{t_c} \leftarrow t_\phi\) and \(\pi\) witnesses \(c \models \phi'\). The other claims hold by construction.

Definition 4 (Match). Assume a tuple \((c, A_c, B_c)\) with \(c = (\mathcal{T}_c, \mathcal{p}_c, \mathcal{bv}_c, \mathcal{seq}_c, \mathcal{phase}_c)\), \(A_c \subseteq \mathcal{p}_c\), and \(B_c \subseteq \mathcal{T}_c\) and \((\phi, A_\phi, B_\phi, \mathcal{tIn}, \mathcal{f})\) with \(\phi = (\mathcal{T}_\phi, \mathcal{p}_\phi, \mathcal{bv}_\phi, \mathcal{seq}_\phi, \mathcal{gap}_\phi, \mathcal{egap}_\phi)\), \(A_\phi\) that is a set of phaser identifiers possibly intersecting \(\mathcal{p}_\phi\), \(B_\phi = \mathcal{tIn} \cup \mathcal{f}\), a set \(\mathcal{tIn}\) that is in \(\phi\), and a set \(\mathcal{f}\) that is fresh for \(\phi\). We say the tuple \((c, A_c, B_c)\) matches \((\phi, A_\phi, B_\phi, \mathcal{tIn}, \mathcal{f})\) with respect to \(\tau\) and \(\pi\) if

- \(\tau\) and \(\pi\) witness \(c \in [\phi]\),
- \(\tau^{-1}(\mathcal{tIn})\) is \(\tau\)-uniquely-mapped,
- \(|A_c| = |A_\phi|\) and \(|B_c| = |B_\phi|\),
- \(B_c = B_{c_1} \cup B_{c_2}\),
- \(\tau|_{B_{c_1}} \subseteq \mathcal{tIn}\), i.e., \(B_{c_1}\) is \(\tau\)-uniquely-mapped,
- \(\tau\) does not uniquely map \(B_{c_2}\),
- \(|B_{c_2}| = |\mathcal{f}|\).

Lemma 15. Given \(c = (\mathcal{T}_c, \mathcal{p}_c, \mathcal{bv}_c, \mathcal{seq}_c, \mathcal{phase}_c)\), \(\phi = (\mathcal{T}_\phi, \mathcal{p}_\phi, \mathcal{bv}_\phi, \mathcal{seq}_\phi, \mathcal{gap}_\phi, \mathcal{egap}_\phi)\), and \(\tau\) and \(\pi\) that witness \(c \in [\phi]\), for any \(A_c \subseteq \mathcal{p}_c\) and \(B_c \subseteq \mathcal{T}_c\), \((c, A_c, B_c)\) matches \((\phi, A_\phi, B_\phi, \mathcal{tIn}, \mathcal{f})\) with respect to \(\tau\) and \(\pi\) if

- \(|A_c| = |A_\phi|\) and \(|B_c| = |B_\phi|\),
- \(A_\phi = \pi(A_{c|\phi}) \cup \phi(\mathcal{dom}(\pi))\),
- \(B_\phi = \mathcal{tIn} \cup \mathcal{f}\), \(\mathcal{tIn} \subseteq \mathcal{T}_\phi\), \(\mathcal{f} \cap \mathcal{T}_\phi = \emptyset\),
- \(\tau^{-1}(\mathcal{tIn})\) is \(\tau\)-uniquely-mapped.

Proof. By Definition 4 and construction of \(c \in [\phi]\).

Example 1. Let \(c \in [\phi]\) such that \(c = (\mathcal{T}_c, \mathcal{p}_c, \mathcal{bv}_c, \mathcal{seq}_c, \mathcal{phase}_c)\), \(p_c \in \mathcal{p}_c\), \(t_c \in \mathcal{T}_c\), \(\phi = (\mathcal{T}_\phi, \mathcal{p}_\phi, \mathcal{bv}_\phi, \mathcal{seq}_\phi, \mathcal{gap}_\phi, \mathcal{egap}_\phi)\), and \(\tau\) and \(\pi\) witness the denotation. \((c, \{p_c\}, \{t_c\})\) will match some \((\phi, \{p_\phi\}, \{t_\phi\}, \mathcal{tIn}, \mathcal{f})\) for \(\mathcal{tIn} \cup \mathcal{f} = \{t_\phi\}\).
Lemma 16. Assume a configuration \( c = (T_c, \mathcal{P}_c, \mathcal{bv}_c, \mathcal{seq}_c, \mathcal{phase}_c) \), a constraint \( \phi = (T_\phi, \mathcal{P}_\phi, \mathcal{bv}_\phi, \mathcal{seq}_\phi, \mathcal{gap}_\phi, \mathcal{egap}_\phi) \), the mappings \( \tau \) and \( \pi \), and the sets \( A_c, B_c, A_\phi, \) and \( B_\phi \) such that \((c, A_c, B_c)\) matches \((\phi, A_\phi, B_\phi, \mathcal{un}, \mathcal{f})\) with respect to \( \tau \) and \( \pi \). For all the tuples \((\phi', \mathcal{un}', \mathcal{f}')\) in \text{concTasksPhasers}(\phi, A_\phi, B_\phi, \mathcal{un}, \mathcal{f})\) we have that \( \mathcal{un}' = B_\phi \) and \( \mathcal{f}' = \emptyset \) and there exists one such tuple that:

- \( \pi' \colon \text{dom}(\pi) \cup A_c \to T_{\phi'} \),
- \( \tau' \colon \text{dom}(\tau) \cup B_c \to T_{\phi'} \),
- \( \pi' \) uniquely maps \( B_c \),
- \( \pi' \) and \( \tau' \) witness \( c \in [\phi'] \).

Proof. By Lemmas 13 and 14 and induction on \( |A_\phi| + |B_\phi| \).

Lemma 17. Assume a configuration \( c = (T_c, \mathcal{P}_c, \mathcal{bv}_c, \mathcal{seq}_c, \mathcal{phase}_c) \) and a constraint \( \phi = (T_\phi, \mathcal{P}_\phi, \mathcal{bv}_\phi, \mathcal{seq}_\phi, \mathcal{gap}_\phi, \mathcal{egap}_\phi) \) such that \( \tau \) and \( \pi \) witness \( c \in [\phi] \). Assume also that for some task \( t_c \) and phaser \( p_c \), \( \tau(t_c) = t_\phi, \pi(p_c) = p_\phi, \mathcal{seq}_c(t_c) = s, \) and phase\(_c\)(\( p_c \)) = \( (v, \text{val}) \). \text{concretizeVar}(\phi, t_\phi, p_\phi, v) \) and \text{concretizeSeq}(\phi, t_\phi, s) \) will respectively generate a singleton \( \{\phi'\} \) such that the constraint \( \phi' \) satisfies \( c \in [\phi'] \).

Proof. By definition of \text{concretizeVar}(\phi, t, p, v) \) and \text{concretizeSeq}(\phi, t, s) \) in Figure 6.

5.2 Predecessor Computation

In this section, we formally define the predecessor computation functions in Figures 7 and 8 and prove their soundness and relative completeness.

Theorem 4. Each predecessor computation rule \( \xrightarrow{\text{stat}} \) in Figures 5 and 7 is sound with respect to the semantic rules of Figure 2.

Proof. Assume a configuration \( c \) and a constraint \( \phi \) such that \( c \in [\phi] \).

\text{newPhaser.} Assume \( c' \xrightarrow{v:=\text{newPhaser}()} t_c \) for some \( c' \) where \( p_c \) is the new phaser
id in \( c \). We exhibit \( \phi' \) as well as the task and phaser mappings that witness \( c' \in [\phi'] \) such that \( \phi \xrightarrow{v:=\text{newPhaser}()} \phi' \) for some \( t_\phi \). The rule starts by concretizing \( \phi \) according to \( c \) and \( A_c = \{t_c\} \) and \( B_c = \{p_c\} \). Lemmas 15 and 17 ensure that after concretizations, a concrete constraint \( \phi_{\text{conc}} \) is generated such that \( c \in [\phi_{\text{conc}}] \) with respect to some task and phaser mappings \( \tau \) and \( \pi \) so that \( t_c \) is uniquely mapped to \( t_\phi \) by \( \tau \) and \( p_c \) is mapped to some \( p_\phi \) by \( \pi \). The task \( t_c \) in \( c' \) creates \( p_c \) hence, it must be the only task registered with \( p_\phi \) or referencing it. This, combined with the definition of \( \tau \) and \( \pi \), guarantees that \( t_\phi \) is the only task in \( \phi \) which is registered with \( p_\phi \) or referencing it. The new mappings \( \tau \) and \( \pi' = \pi \setminus \{p_c\} \) witness \( c' \in [\phi'] \).
signal. Assume \( t_{c'} \xrightarrow{\text{v.signal}()} t_{c} \) for some \( c' \). We exhibit \( \phi' = (\tau', \pi', b\nu_{\phi'}, \text{seq}_{\phi'}, \text{gap}_{\phi'}, \text{egap}_{\phi'}) \) as well as the task and phaser mappings that witness \( c' \in [\phi'] \) such that \( \phi \xrightarrow{\text{v.signal}()} \phi' \) for some \( t_{\phi} \). The rule starts by concretizing \( t_{\phi} \) and \( p_{\phi} \) in \( \phi \). Lemmas 16 and 17 ensure that a concrete constraint \( \phi_{\text{conc}} = (\tau_{\phi}, \pi_{\phi}, b\nu_{\phi}, \text{seq}_{\phi}, \text{gap}_{\phi}, \text{egap}_{\phi}) \) is generated such that \( c \in [\phi_{\text{conc}}] \) with respect to some task and phaser mappings \( \tau \) and \( \pi \) so that \( t_{c} \) is uniquely mapped to some \( t_{\phi} \) by \( \tau \), and \( p_{c} \) is mapped to some \( p_{\phi} \) by \( \pi \). The task \( t_{c} \) in \( c' \) has just incremented the signal phase of the task \( t_{c} \) on the phaser \( p_{c} \). There is a level \( l > 0 \) that separates the signal phases of the tasks registered with \( \{ \text{concTasks} \} \) from their wait phases. Incrementing the signal \( s_{t_{c}} \) of task \( t_{c} \) on \( p_{c} \) satisfies the phase bounds in \( \phi_{\text{conc}} \). The case in which \( l < s_{t_{c}} \) can be captured by \( \text{signal II} \) (here, \( u_{\text{signal}} > 0 \) and \( l' = l \)). The case where \( l = s_{t_{c}} \) can be captured with \( \text{signal I} \) (here, \( u_{\text{signal}} > 0 \) and \( l' = l - 1 \).

wait. Assume \( t_{c'} \xrightarrow{\text{v.wait}()} t_{c} \) for some \( c' \). We exhibit \( \phi' \) as well as the task and phaser mappings that witness \( c' \in [\phi'] \) such that \( \phi \xrightarrow{\text{v.wait}()} \phi' \) for some \( t_{\phi} \). Similar to the signal rule, this rule starts by concretizing \( t_{c} \) and \( p_{c} \) in \( \phi \) and uniquely maps them to \( t_{\phi} \) and \( p_{\phi} \) in \( \phi_{\text{conc}} \). The task \( t_{c} \) in \( c' \) has just incremented the wait phase of the task \( t_{c} \) on the phaser \( p_{c} \). Let \( \text{gap}_{\phi}(t_{\phi})(p_{\phi}) = (v, (l_{\text{signal}}, l_{w}, u_{\text{wait}}, u_{\text{signal}})) \). A level \( l > 0 \) exists that shows the signal and wait phases of the tasks registered with \( p_{c} \) respect the phase bounds in \( \phi_{\text{conc}} \). The same level can be used to show that phases of \( c' \) respect the phase bounds in \( \phi' \) that is generated by the rule.

drop. Assume \( t_{c'} \xrightarrow{\text{v.drop}()} t_{c} \) for some \( c' \). We exhibit \( \phi' = (\tau', \pi', b\nu_{\phi'}, \text{seq}_{\phi'}, \text{gap}_{\phi'}, \text{egap}_{\phi'}) \) such that \( c' \in [\phi'] \) and \( \phi \xrightarrow{\text{v.drop}()} \phi' \) for some task \( t_{\phi} \). Similar to the signal rule, this rule starts by concretizing \( t_{c} \) and \( p_{c} \) in \( \phi \) and uniquely maps them to \( t_{\phi} \) and \( p_{\phi} \) in \( \phi_{\text{conc}} \). Since \( c \in [\phi_{\text{conc}}] \), there exists a level \( l \) that separates signal and wait phases of the tasks registered with \( \phi_{\phi} \). The task \( t_{c} \) in \( c' \) has just dropped the phaser \( p_{c} \). By existence of \( c' \), a \( l' \) exists that takes into account the phases of \( t_{\phi} \) as well. If \( \delta < -l_{\text{max}} \) (respectively, \( \delta > l_{\text{max}} \)), we can show \( l - l_{\text{max}} \) (respectively, \( l + l_{\text{max}} \)) also separates the phases in \( c' \). Otherwise, \( l' + \delta \) with \( -l_{\text{max}} \leq \delta \leq l_{\text{max}} \) separates the phases in \( c' \). We adapt the bounds for each such \( \delta \) in this case. In a similar manner, if \( u_{\text{min}} \leq \infty \) and \( u_{\text{min}} \leq \infty \) for some task, then a level \( l + \delta \) with \( -u_{\text{max}} \leq \delta \leq u_{\text{min}} \) must separate the phases in \( c' \). We adapt the phase bounds to this case.

exit. Assume \( t_{c} \xrightarrow{\text{exit}} t_{\phi} \) for some \( c' \). Proof of soundness of the rule exit has a similar approach to that of the drop rule. The difference is that exit intuitively has to iterate through all the phasers in \( \phi \), with which \( t_{c} \) is registered and drop them. The mappings \( \tau' = \tau \cup \{ t_{c} \rightarrow t_{\phi} \} \) and \( \pi \) witness the denotation.
**asynch.** Assume $c \xrightarrow{\text{asynch}(\text{task}, v_1, \ldots, v_n)(s)} \phi$ for some $c'$ where $\text{paramOf}(\text{task}) = (u_1, \ldots, u_k)$. The task $t_c$ has just spawned $u_c$ in $c$. We exhibit $\phi' = (\tau_{\phi'}, P_{\phi'}, bv_{\phi'}, seq_{\phi'}, gap_{\phi'}, egap_{\phi'})$ such that $c' \in [\phi']$ and $\phi \xrightarrow{\text{asynch}(\text{task}, v_1, \ldots, v_n)(s)} \phi'$. The rule starts by concretizing $t_{\phi}$, $u_{\phi}$, and $p_{i_{\phi}}$ for every $i : 1 \leq i \leq k$ in $\phi$. Lemmas 10 and 11 ensure that a concrete constraint $\phi_{\text{conc}} = (T_{\phi}, P_{\phi}, bv_{\phi}, seq_{\phi}, gap_{\phi}, egap_{\phi})$ is generated such that $c \in [\phi_{\text{conc}}]$ with respect to some task and phaser mappings $\tau$ and $\pi$ so that $t_c$ and $u_c$ are uniquely mapped to some $t_{\phi}$ and $u_{\phi}$ by $\tau$, and $p_{i_c}$ is mapped to some $p_{i_{\phi}}$ by $\pi$ for each $i : 1 \leq i \leq k$. The task $t_{c'}$ in $c'$ has just spawned the task $u_c$, $t_c$ and $u_c$ are registered with each $p_{i_c}$ for $i : 1 \leq i \leq k$ and have the same phases in $\text{gap}_{\phi'}$. $c \in [\phi_{\text{conc}}]$ ensures that for each $i : 1 \leq i \leq k$, $\text{gap}_{\phi'}$ can be constrained so that $t_{\phi}$ and $u_{\phi}$ are registered with each $p_{i_{\phi}}$ in $\text{gap}_{\phi'}$ and have the same phases. Hence, the meet of the signal and wait gaps of $t_{\phi}$ and $u_{\phi}$ in $\text{gap}_{\phi'}$ is not empty. This meet will actually be the phase of $t_{\phi}$ on phasers $p_{i_{\phi}}$ for $i : 1 \leq i \leq k$. $\text{phase}'$ is then obtained from $\text{phase}$ by removing the phases of $u_c$. $\text{gap}'_{\phi'}$ is also obtained from $\text{gap}_{\phi'}$ by removing $u_{\phi}$. The task and phaser mappings $\tau' = \tau \setminus \{u\}$ and $\pi$ witness that $\phi'$ denotes $c'$.

We define $c_0 \xrightarrow{\text{stat}} c_n$ to mean a sequence $c_0 \xrightarrow{\text{stat}} c_1 \ldots c_{n-1} \xrightarrow{\text{stat}} c_n$ where $\mathcal{T} = \{t_1, \ldots, t_n\}$ are the tasks in $c_0$. Observe that this is in the transitive closure of $\text{stat}$.

**Theorem 5.** Each predecessor computation rule $\xrightarrow{\text{stat}}$ in Figures 8 and 7 except for the rule $\text{newPhaser}$ is complete with respect to the semantic rules of those in Figure 2 and $\xrightarrow{\text{stat}}$ defined above. The rule $\text{newPhaser}$ is complete only when the task $\text{main}$ executes it.

**Proof.** Assume $\phi \xrightarrow{\text{stat}} \phi'$ according to Figures 8 and 7. For any configuration $c' = (T_{c'}, P_{c'}, bv_{c'}, seq_{c'}, \text{phase}_{c'})$ where $c' \in [\phi']$, we exhibit a configuration $c = (T_c, P_c, bv_c, seq_c, \text{phase}_c)$ such that $c \xrightarrow{\text{stat}} c$ and $c \in [\phi]$. Note that any predecessor computation rule starts by concretizing $\phi$ to some concrete constraint $\phi_{\text{conc}} = (T_{\phi}, P_{\phi}, bv_{\phi}, seq_{\phi}, gap_{\phi}, egap_{\phi})$. Then, the predecessor $\phi'$ is obtained from the $\phi_{\text{conc}}$. Lemmas 11 and 12 ensure $\phi \preceq \phi_{\text{conc}}$ for any concrete constraint $\phi_{\text{conc}}$. Hence, for every rule we show $c \in [\phi_{\text{conc}}]$. This implies $c \in [\phi]$.

Assume $\tau$ and $\pi$ witness $c' \in [\phi']$. We show that using $\tau^{-1}(t_{\phi})$ for $c' \xrightarrow{\text{stat}} c$ generates the desired configuration $c$. The intuition is that if all tasks in $T_{c'}$ that are associated to $t_{\phi}$ by $\tau$ execute $\text{stat}$, we obtain a configuration $c$ that is denoted by some concrete constraint $\phi_{\text{conc}}$, hence, denoted by $\phi$.

**newPhaser.** This rule is only complete when $\tau^{-1}(t_{\phi})$ is a singleton. To simplify the presentation, we show completeness when $\text{main}$ executes $\text{newPhaser}$, because
then we are sure \(\tau^{-1}(t_\phi)\) is a singleton. Assume \(t_\phi\) is the task main in some \(\phi\) and \(\phi \xrightarrow{v:=\text{newPhaser}()} t_\phi\). Assume also \(p_\phi\) is the new phaser id. Let \(p_c \not\in \mathcal{P}_c\) and \(\mathcal{P}_c = \mathcal{P}_c \cup \{p_c\}\). \(\text{phase}_c\) is obtained from \(\text{phase}_c\) by assigning \(\text{phase}_c(t_c)(p_c) = (v, (0, 0))\) where \(t_c\) is the task main in \(c\). Since \((0, 0) \models \text{gap}_\phi(t_\phi)(p_\phi)\), \(t_c\) can again be associated with \(t_\phi\) in \(\phi_{\text{conc}}\) and \(\text{phase}_c(u_c)(p_c) = (-, \text{nreg})\) for all other tasks \(u_c \in \mathcal{T}_c\setminus\{t_c\}\). Moreover, \(\text{phase}_c(u_c)(p_\phi) = (-, \text{nreg})\) for all tasks \(u_c \in \mathcal{T}_c\setminus\{t_c\}\).

Hence, for \(c = (\mathcal{T}_c, \mathcal{P}_c, \mathbf{bv}_c, \mathbf{seq}_c, \text{phase}_c)\), we have \(c \xrightarrow{v:=\text{newPhaser}()} t_c\) and the task and phaser mappings \(\tau\) and \(\pi = \{p_c \rightarrow p_\phi\}\) witness \(c \in [\phi_{\text{conc}}]\).

**signal.** Assume \(\phi \xrightarrow{v:\text{signal}()} t_\phi\) for some \(\phi\). By definition of \(c' \in [\phi']\), there exists \(p_c\) such that \(\pi(p_c) = p_\phi\) and all tasks in \(\tau^{-1}(t_\phi)\) are associated with \(t_\phi\) by \(\tau\). \(c'\) is denoted by \(\phi'\) and is obtained via **signal I** or **signal II**. There is therefore a level \(l \geq 0\) which shows that the signal and wait phases of the tasks registered in \(c'\) respect the phase bounds in \(\phi'\). We can show that \(l + 1\) captures that the signal and wait phases of the tasks registered in \(c\) respect the phase bounds in \(\phi\) if \(\phi'\) is obtained via **signal I**, and \(l\) shows that the signal and wait phases of the tasks registered in \(c\) respect the phase bounds in \(\phi\) if \(\phi'\) is obtained via **signal II**.

In both cases, the task and phaser mappings \(\tau\) and \(\pi\) should be used for the denotation.

**wait.** Assume \(\phi \xrightarrow{v:\text{wait}()} t_\phi\) for some \(\phi\). By definition of \(c' \in [\phi']\), there exists \(p_c\) such that \(\pi(p_c) = p_\phi\) and each task \(t \in \tau^{-1}(t_\phi)\) is associated with \(t_\phi\) by \(\tau\). If \(l\) witnesses \(c' \in [\phi']\), then the same level witnesses \(c \in [\phi]\) with the task and phaser mappings \(\tau\) and \(\pi\).

**drop.** Assume \(\phi \xrightarrow{v:\text{drop}()} t_\phi\) for some \(\phi\). By definition of \(c' \in [\phi']\), there exists \(p_c\) such that \(\pi(p_c) = p_\phi\) and each task \(t \in \tau^{-1}(t_\phi)\) is associated with \(t_\phi\) by \(\tau\). \(\text{phase}_c\) is obtained from \(\text{phase}_c\) by 1) not modifying the phases of the tasks that are not associated with \(t_\phi\), 2) assigning \(\text{phase}_c(t)(p_c) = (-, \text{nreg})\) for each \(t \in \tau^{-1}(t_\phi)\). A level \(l \geq 0\) witnesses these tasks respect the phase bounds in \(\phi'\) (which is obtained for a certain \(\delta\)). We can show \(l - \delta\) witnesses denotation of \(c\) by \(\phi_{\text{conc}}\). The third group of tasks to consider are those in \(\mathcal{T}_c \setminus \tau^{-1}(t_\phi)\) which are registered with \(p_c\). The phases for these tasks are not modified. Hence, the task and phaser mappings \(\tau\) and \(\pi\) witness \(c \in [\phi_{\text{conc}}]\).

**exit.** This proof is very similar to the proof of completeness of **drop**. The only difference is that the proof of **drop** needs to be extended by iterating through all phasers dropped in \(c'\). The mappings \(\tau' = \tau \setminus \tau^{-1}(t_\phi)\) and \(\pi\) witness \(c \in [\phi_{\text{conc}}]\).

**asynch.** Assume \(\phi \xrightarrow{v:\text{asynch}(\{t, v_1, \ldots, v_s\})} t_\phi\) for some \(\phi\). Let \(u_\phi\) be the newly spawned task, and \(p_{1_\phi}, \ldots, p_{k_\phi}\) be the phasers passed to \(u_\phi\). By definition of
Proof. By soundness of \( \phi \rightarrow \phi' \) (Theorem 4), the procedure \( \text{check}(\text{prg}, \phi) \) is sound.

Lemma 19 (Relative completeness). If the procedure \( \text{check}(\text{prg}, \phi) \) returns a trace \( \phi_0; \text{stmt}_0; \ldots; \phi_{n-1}; \text{stmt}_{n-1}; \phi_n \) in which only the task main has executed newPhaser, there are \( c_0, \ldots, c_n \) with \( c_0 = c_{\text{strt}}, c_n \in [\phi_{\text{bad}}] \) and \( c_{i-1} \xrightarrow{\text{stmt}_{i-1}} c_i \) for \( 1 \leq i \leq n \).

Proof. By relative completeness of \( \phi \rightarrow \phi' \) (Theorem 5), the procedure \( \text{check}(\text{prg}, \phi) \) is complete relative to the runs which is which only the task main is allowed to execute newPhaser.
**Input:** $\text{prg} = (\mathcal{B}, \mathcal{V}, \mathcal{T})$ and a $\square$-minimal target set $\Phi$

**Output:** A symbolic run to $\Phi$ or the value unreachable

1. Initialize both $\text{Working}$ and $\text{Visited}$ to $\{(\phi, \phi) \mid \phi \in \Phi\}$;
2. while there exists $(\phi, \text{trace}) \in \text{Working}$ do
   3. remove $(\phi, \text{trace})$ from $\text{Working}$;
   4. let $(T, P, \text{bv}, \text{seq}, \text{gap}, \text{egap}) = \phi$;
   5. if $c_{\text{init}} \models \phi$ then return $\text{trace}$;
   6. foreach $t \in T \cup \{u\}$ where $u \notin T$ do
      7. foreach $\phi' \text{ s.t. } \phi \xrightarrow{\text{stmt}}^{t} \phi'$ do
         8. if $\psi \nsubseteq \phi'$ for all $(\psi, \omega) \in \text{Visited}$ then
            9. Remove from $\text{Working}$ and $\text{Visited}$ each $(\psi, \omega) \text{ s.t. } \phi' \subseteq \psi$;
         10. Add $(\phi', \phi' \cdot \text{stmt} \cdot \text{trace})$ to both $\text{Working}$ and $\text{Visited}$;
   11. return unreachable;

**Procedure** $\text{check(prg,}\Phi)$, a simple working list procedure for checking constraints reachability.

We can also show the procedure to terminate if we only manipulate $K$-bounded-dimension and $B$-good constraints.

**Lemma 20 (Termination).** $\text{check(prg,}\Phi)$ terminates if there are $K, B \in \mathbb{N}$ s.t. all constraints in $\text{Visited}$ are $K$-dimension-bounded and $B$-good.

**Proof sketch.** Suppose the procedure does not terminate. The infinite sequence of constraints passing the test at line 8 violates Thm. 3.

**Theorem 6.** Control reachability for non-atomic phaser programs generating a finite number of phasers is decidable.

**Proof sketch.** Systematically drop, in the backward procedure, constraints violating $K$-dimension-boundedness (as none of the denoted configurations is reachable) ensures $K$-boundedness. Also, the set of target constraints is free (since we are checking control reachability) and this is preserved by the pre computation in Fig. 3, 7 and 8. Finally, Lemmas 15, 19 and 20 ensure soundness, relative completeness, and termination.

**Theorem 7.** Plain reachability for non-atomic phaser programs generating at most $K$ phasers with, for each phaser, $B$-bounded gaps is decidable.

**Proof sketch.** Systematically drop, in the backward procedure, constraints requiring more than $K$ phasers or larger than $B$ gaps-values for some phaser gaps (as none of the denoted configurations is reachable) ensures $K$-dimension-boundedness and $B$-goodness. Finally, Lemmas 15, 19 and 20 ensure soundness, relative completeness, and termination.
6 Limitations of deciding reachability

Assume a program \( \text{prg} = (B, V, T) \) and its initial configuration \( c_{\text{init}} \). We show a number of parameterized reachability problems to be undecidable. First, we address checking control reachability when restricting to configurations with at most \( K \) task-referenced phasers. We call this \( K \)-control-reachability.

**Definition 5 (K-control-reachability).** Given a partial control configuration \( c \), we write \( \text{reach}_K(\text{prg}, c) \), and say \( c \) is \( K \)-control-reachable, to mean there are \( n + 1 \) configurations \( (c_i)_{0 \leq i \leq n} \), each with at most \( K \) reachable phasers (i.e., phasers referenced by at least a task variable) s.t. \( c_{\text{init}} = c_0 \) and \( c_i \rightarrow c_{i+1} \) for \( i : 0 \leq i < n - 1 \) with \( c_n \) equivalent to a configuration that includes \( c \).

**Theorem 8.** \( K \)-control-reachability is undecidable in general.

*Proof sketch.* Encode state reachability of an arbitrary Minsky machine with counters \( x \) and \( y \) using \( K \)-control-reachability of a suitable phaser program. The program (see [10]) has five tasks: \( \text{main}, \text{xTask}, \text{yTask}, \text{child1} \) and \( \text{child2} \). Machine states are captured with shared variables and counter values with phasers \( x\text{Ph} \) for counter \( x \) (resp. \( y\text{Ph} \) for counter \( y \)). Then, (1) spawn an instance of \( \text{xTask} \) (resp. \( \text{yTask} \)) and register it to \( x\text{Ph} \) (resp. \( y\text{Ph} \)) for increments, and (2) perform a wait on \( x\text{Ph} \) (resp. \( y\text{Ph} \)) to test for zero. Decrementing a counter, say \( x \), involves asking an \( \text{xTask} \), via shared variables, to exit (hence, to deregister from \( x\text{Ph} \)). However, more than one task might participate in the decrement operation. For this reason, each participating task builds a path from \( x\text{Ph} \) to \( \text{child2} \) with two phasers. If more than one \( \text{xTask} \) participates in the decrement, then the number of reachable phasers of an intermediary configuration will be at least five. As a result, the phaser program will reach a configuration corresponding to \( s_F \) via configurations having at most 4 reachable phasers iff the counter machine reaches a configuration with state \( s_F \).

**Theorem 9.** Control reachability of phaser programs generating a finite number of phasers is undecidable if atomic statements are allowed.

*Proof sketch.* We encode state reachability problem of an arbitrary Minsky machine with counters \( x \) and \( y \) using a phaser program with atomic statements. The phaser program (captured in Fig. [10]) has three tasks: \( \text{main}, \text{xTask} \) and \( \text{yTask} \). The idea is to associate a phaser \( x\text{Ph} \) to counter \( x \) (resp. \( y\text{Ph} \) to counter \( y \)) and to perform a signal followed by a wait on \( x\text{Ph} \) (resp. \( y\text{Ph} \)) to test for zero on counter \( x \) (resp. counter \( y \)). Incrementing and decrementing is performed by asking spawned tasks to spawn a new instance (incrementing) or to deregister (decrementing). Atomic-next statements are used to ensure exactly one task is spawned or deregistered. As a result, the phaser program will reach a configuration sending the variable \( s \) to \( s_F \) iff the counter machine reaches a configuration with state \( s_F \).

Finally, even with finite numbers of tasks and phasers, but with arbitrary gap-bounds, we can show [9] the following.
Theorem 10. Plain reachability of non-atomic phaser programs generating a finite number of phasers is undecidable if the generated gaps are not bounded.

7 Conclusion

We have studied parameterized plain (e.g., deadlocks) and control (e.g., assertions) reachability problems for phaser programs. We have proposed an exact verification procedure for non-atomic programs. The procedure can be used for answering both control and plain reachability problems. We summarize our findings in Table 7. The procedure is guaranteed to terminate, even for programs that may generate arbitrary many tasks but finitely many phasers, when checking control reachability or when checking plain reachability with bounded gaps. These results were obtained using a non-trivial symbolic representation for which termination had required showing an $\exists \preceq \forall$ preorder on multisets on gaps on natural numbers to be a WQO. We are working on a tool that implements the procedure in order to verify programs that dynamically spawn tasks and synchronize them with phasers. We believe our general decidability results are useful to reason about synchronization constructs other than phasers. For instance, a traditional static barrier can be captured with one phaser and with bounded gaps (in fact one). Similarly, one phaser with one producer and arbitrary many consumers can be used to capture futures where a “get” instruction can be modeled with a wait. We believe our negative results can also be used. For instance, atomic instructions can be modeled using test-and-set operation and may result in the undecidability of the reachability problem. This suggests more general applications of the work are to be investigated.

| Arbiary numbers of tasks | Finite dimension | $K$-reachability | Arbitrary dimension |
|--------------------------|------------------|------------------|-------------------|
| Bounded gaps             | ctrl atomic $\times$ | plain non-atomic $\checkmark$ | ctrl non-atomic $\times$ |
|                          | (Thm.9)           | (Thm.7)          | (Thm.6)           |
| Arbitrary gaps            | ctrl non-atomic $\checkmark$ | plain non-atomic $\times$ | ctrl non-atomic $\times$ |
|                          | (Thm.5)           | (From [9])       | (Thm.2)           |

Table 1. Findings summary: ctrl stands for control reachability and plain for plain reachability; atomic stands for allowing the \texttt{v.next()}\{\texttt{stmt}\} atomic instruction and non-atomic for forbidding it (resulting in non-atomic programs). Recall the dimension of a phaser program is the number of dynamically generated phasers.

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On Reachability in Parameterized Phaser Programs

Fig. 7. Derivation rules for computing predecessors wrt. parameterized Phaser instructions newPhaser(), v.drop() and exit. For each rule, a task \( t \) executes the statement. If \( t \) does not belong to \( \mathcal{F} \), it will be added by concretizing \( \phi \). Otherwise, the concretization will preserve it. In newPhaser() and v.drop(), a phaser \( p \) is required, which is either added by concretization or is preserved by it.
Derivation rules for predecessor computation wrt. parameterized Phaser in-

...
bool s1, s2, ..., sF;
bool xDec, yDec;

main()
{
exPh = newPhaser();
yPh = newPhaser();

while (true)
{
    // (q_i: inc(x):q_j)
    if (ndet() && si)
    {
        asynch(xTask, xPh);
        si = false;
        sj = true;
    }

    // (q_i: inc(y):q_j)
    if (ndet() && si)
    {
        asynch(yTask, yPh);
        si = false;
        sj = true;
    }

    // (q_i: dec(x):q_j)
    if (ndet() && si)
    {
        xDec = true;
        xPh. next();
        xPh. next();
        xPh. next();
        assert(!xDec);
        si = false;
        sj = true;
    }

    // (q_i: dec(y):q_j)
    if (ndet() && si)
    {
        yDec = true;
        yPh. next();
        yPh. next();
        yPh. next();
        assert(!yDec);
        si = false;
        sj = true;
    }

    // (q_i: test(x):q_j)
    if (ndet() && si)
    {
        xPh. signal();
        xPh. wait();
        xPh. wait();
        xPh. wait();
        si = false;
        sj = true;
    }

    // (q_i: test(y):q_j)
    if (ndet() && si)
    {
        yPh. signal();
        yPh. wait();
        yPh. wait();
        yPh. wait();
        si = false;
        sj = true;
    }
}

// (end of main)

// ****** xTask *******
xTask(xPh)
{
    while (true)
    {
        if (xDec)
        {
            if (ndet())
            {
                xDec = false;
                p1 = newPhaser();
                xPh. next();
                exit;
            }
            xPh. next();
            xPh. next();
        }
    }
}

// ****** yTask *******
yTask(yPh)
{
    while (true)
    {
        if (yDec)
        {
            if (ndet())
            {
                yDec = false;
                p1 = newPhaser();
                yPh. next();
                exit;
            }
            yPh. next();
            yPh. next();
        }
    }
}

Fig. 9. For the proof of Thm. Encoding a Minsky machine with the two counters \(\{x,y\}\). The value of counter \(x\) is represented by the number of instances of xTask tasks registered to phaser xPh. The construction ensures that runs trying to decrement by more than 1 will result in configurations with larger number of phasers.
bool xDec, yDec, s1, s2, ..., sF;

main () {
  xPh = new Phaser();
  yPh = new Phaser();
  while (true) {
    // (q:i:inc(x):qj)
    if (ndet() && si) {
      async (xTask, xPh);
      si = false;
      sj = true;
    }
  }
    // (q:i:dec(x):qj)
    if (ndet() && si) {
      xDec = true;
      xPh.next();
      xDec = false;
      exit;
    }
    // (q:i:test(x):qj)
    if (ndet() && si) {
      xTask(xPh);  
      while (true) {
        if (xDec) {
          xPh.next();
          xDec = false;
          exit;
        }
      }
    }  
    // **** xTask *****
    xTask(xPh);  
    while (true) {
      if (xDec) {
        xPh.next();
        xDec = false;
        exit;
      }
    }
  }

Fig. 10. In proof of Thm.9 for control reachability of atomic phaser programs. Encoding a Minsky machine with the two \( \{x, y\} \). The value of counter \( x \) is represented by the number of instances of \texttt{xTask} tasks registered to phaser \texttt{xPh}. The construction ensures increments or decrements involve exactly one task.