Quantum Pattern Recognition

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Received November 4, 2002; accepted January 25, 2003

I review and expand the model of quantum associative memory that I have recently proposed. In this model binary patterns of n bits are stored in the quantum superposition of the appropriate subset of the computational basis of n qubits. Information can be retrieved by performing an input-dependent rotation of the memory quantum state within this subset and measuring the resulting state. The amplitudes of this rotated memory state are peaked on those stored patterns which are closest in Hamming distance to the input, resulting in a high probability of measuring a memory pattern very similar to it. The accuracy of pattern recall can be tuned by adjusting a parameter playing the role of an effective temperature. This model solves the well-known capacity shortage problem of classical associative memories, providing a large improvement in capacity.

KEY WORDS: Quantum search algorithms; associative memories.

PACS: 03.67.-a.

1. INTRODUCTION

The power of quantum computation(1) is mostly associated with the speed-up in computing time it can provide with respect to its classical counterpart, the paramount examples being Shor’s factoring algorithm(2) and Grover’s search algorithm.(3) There is, however, another aspect of quantum computation which represents a big improvement upon its classical counterpart.(4) This leads to a large increase in a particular memory capacity rather than speed. In this paper I will review and expand the main aspects of this new application of quantum information theory. Further aspects of it can be found in Ref. 5.

In traditional computers the storage of information requires setting up a lookup table (RAM). The main disadvantage of this address-oriented

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memory system lies in its rigidity. Retrieval of information requires a precise knowledge of the memory address and, therefore, incomplete or corrupted inputs are not permitted.

This is definitely not how our own brain works. When trying to recognize a person from a blurred photo it is totally useless to know that it is the 17384th person you met in your life. Rather, the recognition process is based on our strong power of association with stored memories that resemble the given picture. Association is what we use every time we solve a crossword puzzle and is distinctive of the human brain.

Given the superior power of associative pattern recognition for complex tasks, the shortcomings of RAM memories were addressed by introducing models of associative (or content-addressable) memories. Here, recall of information is possible on the basis of partial knowledge of their content, without knowing the storage location. Apart from a recently proposed, non-conventional realization based on DNA computing, the traditional models of associative memories are mostly examples of collective computation on neural networks, the best known example being the Hopfield model and its generalization to a bidirectional associative memory.

While these models solve the problem of recalling incomplete or noisy inputs, they suffer from a severe capacity shortage. Due to the phenomenon of crosstalk, which is essentially a manifestation of the spin glass transition in the corresponding spin systems, the maximum number of binary patterns that can be stored in a Hopfield network of $n$ neurons is $p_{\text{max}} \simeq 0.14 \cdot n$. While various possible improvements can be introduced, the maximum number of patterns remains linear in the number of neurons, $p_{\text{max}} = O(n)$.

Another model of associative memory based on neural networks is due to Kanerva. This model can be formulated as a feed-forward neural network with a hidden layer of $l$ units in addition to the input and output layers of $n$ units each. While the capacity of the Kanerva associative memory is exponential in $n$, it is also bounded by the number $l$ of hidden layer neurons. This amounts again to a linear limit analogous to the one characterizing the Hopfield model.

Quantum mechanics offers a way out from the impossibility of reconciling the association power of content-addressable memories with the requirement of large storage capacity without increasing exponentially the hardware complexity. Indeed, quantum mechanical entanglement provides a natural mechanism for both improving dramatically the storage capacity of associative memories and retrieving corrupted or incomplete information.

The basic idea is to store the given $p$ binary patterns of $n$ bits in a quantum superposition of the corresponding subset of the computational
basis of \( n \) qbits. The number of binary patterns that can be stored in such a quantum associative memory is exponential in the number \( n \) of qbits, \( p_{\text{max}} = 2^n \), i.e. it is optimal in the sense that all binary patterns that can be formed with \( n \) bits can be stored.

The basic idea of the information retrieval mechanism is very simple. Given an input pattern, the memory quantum state is rotated within the subspace defined by the stored patterns so that the resulting amplitudes are peaked on the stored patterns which are closest in Hamming distance to the input. A measurement of the rotated memory quantum state provides the output pattern.

An efficient way to perform this rotation is to embed the memory quantum state in a larger Hilbert space by adding \( b \) control qbits. The full state is then rotated in the enlarged Hilbert space. After this rotation one is interested only in the projection of the rotated state onto a specific subspace of the enlarged Hilbert space. This projection can be obtained either by repeated measurement or by rotating the state (approximately) to the desired subspace using the amplitude amplification technique. Either way one has to repeat a certain algorithm a number of times and measure the control register to check if the desired projection has been obtained. The information retrieval mechanism is thus probabilistic, with postselection of the measurement result. This means that one has to repeat an algorithm until a threshold \( T \) is reached or the measurement of a control register yields a given result. In the former case the input is not recognized. In the latter case, instead, the output is determined itself by a probability distribution on the memory which is peaked around the stored patterns closest in Hamming distance to the input.

The accuracy of this information retrieval mechanism depends on the distribution of the stored patterns. Recognition efficiency is best when the number of stored patterns is very large while identification efficiency is best for isolated patterns which are very different from all other ones, both very intuitive features. Both efficiencies can be tuned to prescribed accuracy levels. The recognition efficiency can be varied by changing the threshold \( T \): the higher \( T \), the larger the number of qbits that can be corrupted without affecting recognition. The identification efficiency, instead, can be tuned by varying the number \( b \) of control qbits in the memory. As we shall see, \( b = 1/t \) plays the role of an inverse effective temperature \( t \). The lower \( t \), the more concentrated is the corresponding effective Boltzmann distribution on the states closest (in Hamming distance) to the input and the better becomes the identification.

By averaging over the distribution of stored patterns one can eliminate the dependence on the stored pattern distribution and derive the effective statistical mechanics of quantum associative memories by introducing the
usual thermodynamic potentials. In particular, the free energy $F(t)$ describes the average behavior of the recall mechanism at temperature $t$ and provides concrete criteria to tune the accuracy of the associative memory. By increasing $b$ (lowering $t$), the associative memory undergoes a phase transition from a disordered phase with no correlation between input and output to an ordered phase with minimal Hamming distance between the input and the output. This extends to quantum information theory the relation with Ising spin systems known in error-correcting codes\(^{(14)}\) and in public key cryptography.\(^{(15)}\)

This paper is mainly a review of the original material of Ref. 4. However, I present here new versions of both the storage and the information retrieval algorithms which do not require an auxiliary register from which to copy input patterns but rather store these in unitary operators. Also, I include a new, detailed section on how to store the quantum memory for repeated use. Finally, in the conclusion, I establish an analogy with the classical Hopfield model.

### 2. STORING INFORMATION

Let me start by describing the elementary quantum gates\(^{(1)}\) that I will use in the rest of the paper. First of all there are the single-qbit gates represented by the Pauli matrices $\sigma_i$, $i = 1 \ldots 3$. The first Pauli matrix $\sigma_1$, in particular, implements the NOT gate. Another single-qbit gate is the Hadamard gate $H$, with the matrix representation

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Then, I will use extensively the two-qbit XOR (exclusive OR) gate, which performs a NOT on the second qbit if and only if the first one is in state $|1\rangle$. In matrix notation this gate is represented as $\text{XOR} = \text{diag}(1, \sigma_1)$, where 1 denotes a two-dimensional identity matrix and $\sigma_1$ acts on the components $|01\rangle$ and $|11\rangle$ of the Hilbert space. The 2XOR, or Toffoli gate is the three qbit generalization of the XOR gate: it performs a NOT on the third qbit if and only if the first two are both in state $|1\rangle$. In matrix notation it is given by $2\text{XOR} = \text{diag}(1, 1, 1, \sigma_1)$. In the storage algorithm I shall make use also of the nXOR generalization of these gates, in which there are $n$ control qbits. This gate is also used in the subroutines implementing the oracles underlying Grover’s algorithm\(^{(1)}\) and can be realized using unitary maps affecting only
a few qbits at a time,\(^{(16)}\) which makes it feasible. All these are standard gates. In addition to them I introduce the two-qbit controlled gates

\[ CS_i = |0\rangle\langle 0| \otimes 1 + |1\rangle\langle 1| \otimes S_i, \]

\[ S_i = \begin{pmatrix} \frac{\sqrt{i-1}}{i} & \frac{1}{\sqrt{i}} \\ -\frac{1}{\sqrt{i}} & \frac{\sqrt{i-1}}{i} \end{pmatrix} \]  

for \( i = 1, \ldots, p \). These have the matrix notation \( CS_i = \text{diag}(1, S_i) \). For all these gates I shall indicate by subscripts the qbits on which they are applied, the control qbits always come first.

Given \( p \) binary patterns \( p_i \) of length \( n \), it is not difficult to imagine how a quantum memory can store them. Indeed, such a memory is naturally provided by the following superposition of \( n \) entangled qbits:

\[ |m\rangle = \frac{1}{\sqrt{p}} \sum_{i=1}^{p} |p_i\rangle \]  

(3)

The only real question is how to generate this state unitarily from a simple initial state of \( n \) qbits. In Ref. 4 I presented an algorithm which achieves this by loading sequentially the classical patterns into an auxiliary quantum register from which they are then copied into the actual memory register. Here I will generalize this algorithm by constructing also the unitary operator which generates the memory state (3) directly from the state \( |0, \ldots, 0\rangle \).

Let me begin by reviewing the sequential algorithm of Ref. 4. I shall use three registers: a first register \( p \) of \( n \) qbits in which I will subsequently feed the patterns \( p_i \) to be stored, a utility register \( u \) of two qbits prepared in state \( |01\rangle \), and another register \( m \) of \( n \) qbits to hold the memory. This latter will be initially prepared in state \( |0_1, \ldots, 0_n\rangle \). The full initial quantum state is thus

\[ |\psi_0\rangle = |p_1^1, \ldots, p_n^1; 01; 0_1, \ldots, 0_n\rangle \]  

(4)

The idea of the storage algorithm is to separate this state into two terms, one corresponding to the already stored patterns, and another ready to process a new pattern. These two parts will be distinguished by the state of the second utility qbit \( u_2; |0\rangle \) for the stored patterns and \( |1\rangle \) for the processing term.
For each pattern $p^i$ to be stored one has to perform the operations described below:

$$|\psi_1^i\rangle = \prod_{j=1}^{n} 2XOR_{p^j|m_j} |\psi_0^i\rangle$$

(5)

This simply copies pattern $p^i$ into the memory register of the processing term, identified by $|u_2\rangle = |1\rangle$.

$$|\psi_2^i\rangle = \prod_{j=1}^{n} NOT_{m_j} XOR_{p^j|m_j} |\psi_1^i\rangle,$$

$$|\psi_3^i\rangle = nXOR_{m_1...m_{u_1}} |\psi_2^i\rangle$$

(6)

The first of these operations makes all qbits of the memory register $|1\rangle$'s when the contents of the pattern and memory registers are identical, which is exactly the case for the processing term only. Together, these two operations change the first utility qbit $u_1$ of the processing term to a $|1\rangle$, leaving it unchanged for the stored patterns term.

$$|\psi_4^i\rangle = CS_{u_1u_2} |\psi_3^i\rangle$$

(7)

This is the central operation of the storing algorithm. It separates out the new pattern to be stored, already with the correct normalization factor.

$$|\psi_5^i\rangle = nXOR_{m_1...m_{u_1}} |\psi_4^i\rangle,$$

$$|\psi_6^i\rangle = \prod_{j=n}^{1} XOR_{p^j|m_j} NOT_{m_j} |\psi_5^i\rangle$$

(8)

These two operations are the inverse of Eqs. (6) and restore the utility qbit $u_1$ and the memory register $m$ to their original values. After these operations one has

$$|\psi_6^i\rangle = \frac{1}{\sqrt{p}} \sum_{k=1}^{i} |p^j; 00; p^k\rangle + \frac{\sqrt{p-i}}{p} |p^j; 01; p^i\rangle$$

(9)

With the last operation,

$$|\psi_7^i\rangle = \prod_{j=n}^{1} 2XOR_{p^j|u_2m_j} |\psi_6^i\rangle$$

(10)

one restores the third register $m$ of the processing term, the second term in Eq. (9) above, to its initial value $|0_1, \ldots, 0_n\rangle$. At this point one can load a
new pattern into register $p$ and go through the same routine as just described. At the end of the whole process, the $m$-register is exactly in state $|m\rangle$, Eq. (3).

Any quantum state can be generically obtained by a unitary transformation of the initial state $|0, \ldots, 0\rangle$. This is true also for the memory state $|m\rangle$. In the following I will explicitly construct the unitary operator $M$ which implements the transformation $|m\rangle = M|0, \ldots, 0\rangle$.

To this end I introduce first the single-qbit unitary gates

$$ U_j^i = \cos \left( \frac{\pi}{2} p_j^i \right) 1 + i \sin \left( \frac{\pi}{2} p_j^i \right) \sigma_2 $$

where $\sigma_2$ is the second Pauli matrix. These operators are such that their product over the $n$ qbits generates pattern $p^i$ out of $|0, \ldots, 0\rangle$:

$$ |p^i\rangle = P^i |0, \ldots, 0\rangle, $$

$$ P^i \equiv \prod_{j=1}^{n} U_j^i \quad (11) $$

I now introduce, in addition to the memory register proper, the same two utility qbits as before, also initially in the state $|0\rangle$. The idea is, exactly as in the sequential algorithm, to split the state into two parts, a storage term with $|u_2\rangle = |0\rangle$ and a processing term with $|u_2\rangle = |1\rangle$. Therefore I generalize the operators $P^i$ defined above to

$$ CP^i_{u_2} \equiv \prod_{j=1}^{n} CU_{u_2 j}^i \quad (13) $$

which loads pattern $p^i$ into the memory register only for the processing term.

It is then easy to check that

$$ |m; 00\rangle = M|0, \ldots, 0; 00\rangle, $$

$$ M = \prod_{i=1}^{p} \left[ \left( CP^i_{u_2} \right)^{-1} NOT_{u_1} CS_{u_1 u_2}^{p+1-i} XOR_{u_2 u_1} CP^i_{u_2} \right] \times NOT_{u_2} \quad (14) $$

The storage algorithm is thus efficient in the sense that the number $p(2n + 3) + 1$ of elementary one- and two-qbit gates needed to implement it, is linear in $n$ for fixed $p$. Note that, by construction, there are no restrictions on the loading factor $p/n$. However, the storage algorithm is efficient in an absolute sense only for $p$ polynomial in $n$. 
3. REMEMBERING

A memory is of real value only if it can be used repeatedly. This poses a problem since, as we shall see in the next section, an output of the memory is obtained by measuring the memory register and the rules of quantum mechanics imply that, when the memory register is measured, all the information about the entangled superposition of stored patterns is lost. If one does not want to forget everything after the first information retrieval one must therefore find a way to store the information for repeated use.

In quantum mechanics there are many choices to do this, since information can be stored, with various degrees of compression, both in quantum states and in unitary operators. The most compressed storage would be a quantum state: in this case up to $2^n$ patterns can be stored using only $n$ (quantum) degrees of freedom. To this end, however, one would have to keep a master copy of the memory and produce copies out of it when needed. Unfortunately, this is impossible since the linearity of quantum mechanics forbids exact universal cloning of quantum states. Universal cloning has two disadvantages: first of all the copies to be used for information retrieval are imperfect, though optimal; secondly, the quality of the master copy decreases with each recall, i.e., the memory is quickly washed out.

This leaves state-dependent cloning as the only viable option if one wants to store at least part of the information in a quantum state. State-dependent cloners are designed to reproduce only a finite number of states and this is definitely enough for our purposes. The simplest option in this setting is to use a probabilistic cloning machine. To this end it is sufficient to consider any dummy state $|d\rangle$ different from $|m\rangle$ (for more than two states the condition would be linear independence) and to construct a probabilistic cloning machine for these two states. This machine reproduces $|m\rangle$ with probability $p_m$ and $|d\rangle$ with probability $p_d$; a flag tells us exactly when the desired state $|m\rangle$ has been obtained. In order to obtain an exact copy of $|m\rangle$ one needs then $1/p_m$ trials on average. The master copy is exactly preserved.

The cloning efficiencies of the probabilistic cloner of two states are bounded as follows:

$$p_m + p_d \leq \frac{2}{1 + \langle d|m \rangle} \quad (15)$$

This bound can be made large by choosing $|d\rangle$ as nearly orthogonal to $|m\rangle$ as possible. A simple way to achieve this for a large number of patterns is to
encode also the state

$$|d\rangle = \frac{1}{\sqrt{p}} \sum_{i=1}^{p} (-1)^{i+1} |p^i\rangle$$

(16)

together with $|m\rangle$ when storing information. This can be done simply by using alternately the operators $S^i$ and $(S^i)^{-1}$ in the storing algorithm of Section 2. For binary patterns which are all different from one another one has then

$$\langle d|m\rangle = 0, \quad p \text{ even,}$$

$$\langle d|m\rangle = \frac{1}{p}, \quad p \text{ odd}$$

(17)

and the bound for the cloning efficiencies is very close to its maximal value 2 in both cases.

The quantum network for the probabilistic cloner of two states has been developed in Ref. 21. It can be constructed exclusively out of the two simple distinguishability transfer (D) and state separation (S) gates. Note that these gates embody information about the two states to be cloned. Part of the memory, therefore, actually resides in the cloning network, which is unavoidable if one wants to use a quantum memory repeatedly. This is then a mixed solution in which part of the information resides in a quantum state and another part in a unitary operator implemented as a probabilistic cloning network.

At the other end of the spectrum I can store the information about the $p$ patterns entirely in the unitary operator $M$ in Eq. (14). Each time I need to retrieve information I prepare then an initial quantum state $|0, \ldots, 0\rangle$ and I transform it into $|m\rangle$ by applying $M$ to it. In this case I need $pn$ bits of information to store the $p$ patterns, exactly as in the classical Hopfield model. Indeed, this way of storing the patterns is very close in spirit to the Hopfield model since a unitary operator can always be represented as the exponential of an Hamiltonian operator, which is the quantum generalization of an energy functional. However, as I now show in the quantum case there are no restrictions on the number of patterns that can be stored and retrieved.

4. RETRIEVING INFORMATION

Assume now one is given a binary input $i$, which might be, e.g., a corrupted version of one of the patterns stored in the memory. The retrieval
algorithm requires also three registers. The first register \(i\) of \(n\) qbits contains the input pattern; the second register \(m\), also of \(n\) qbits, contains the memory \(|m\rangle\); finally there is a control register \(c\) with \(b\) qbits all initialized in the state \(|0\rangle\).

The full initial quantum state is thus:

\[
|\psi_0\rangle = \frac{1}{\sqrt{p}} \sum_{k=1}^{p} |i; p^k; 0, \ldots, 0\rangle
\]  

where \(|i\rangle = |i_1, \ldots, i_n\rangle\) denotes the input qbits, the second register, \(m\), contains the memory\(^{(3)}\) and all \(b\) control qbits are in state \(|0\rangle\). Applying the Hadamard gate to the first control qbit one obtains

\[
|\psi_1\rangle = \frac{1}{\sqrt{2p}} \sum_{k=1}^{p} |i; p^k; 0, \ldots, 0\rangle + \frac{1}{\sqrt{2p}} \sum_{k=1}^{p} |i; p^k; 1, \ldots, 0\rangle
\]  

I now apply to this state the following combination of quantum gates:

\[
|\psi_2\rangle = \prod_{j=1}^{n} \text{NOT}_{mj} \text{XOR}_{ijmj} |\psi_1\rangle
\]  

As a result of the above operation the memory register qbits are in state \(|1\rangle\) if \(i_j\) and \(p_j^k\) are identical and \(|0\rangle\) otherwise:

\[
|\psi_2\rangle = \frac{1}{\sqrt{2p}} \sum_{k=1}^{p} |i; d^k; 0, \ldots, 0\rangle + \frac{1}{\sqrt{2p}} \sum_{k=1}^{p} |i; d^k; 1, \ldots, 0\rangle
\]  

where \(d^k_j = 1\) if and only if \(i_j = p^k_j\) and \(d^k_j = 0\) otherwise.

Consider now the following Hamiltonian:

\[
\mathcal{H} = (d_H)_m \otimes (\sigma_3)_c,
\]

\[
(d_H)_m = \sum_{j=1}^{n} (\sigma_3 + \frac{1}{2})_{mj}
\]  

where \(\sigma_3\) is the third Pauli matrix. \(\mathcal{H}\) measures the number of 0’s in register \(m\), with a plus sign if \(c_1\) is in state \(|0\rangle\) and a minus sign if \(c_1\) is in state \(|1\rangle\). Given how I have prepared the state \(|\psi_2\rangle\), this is nothing else than the number of qbits which are different in the input and memory registers \(i\) and \(m\). This quantity is called the Hamming distance and represents the (squared) Euclidean distance between two binary patterns.
Every term in the superposition (21) is an eigenstate of $\mathcal{H}$ with a different eigenvalue. Applying thus the unitary operator $\exp(i\pi \mathcal{H}/2n)$ to $|\psi_2\rangle$ one obtains

$$|\psi_3\rangle = e^{i\pi/2n}\mathcal{H}|\psi_2\rangle,$$

where $d_H(i, p^k)$ denotes the Hamming distance between the input $i$ and the stored pattern $p^k$.

In the final step I restore the memory register to the state $|m\rangle$ by applying the inverse transformation to Eq. (20) and then I apply the Hadamard gate to the control qbit $c_1$, thereby obtaining

$$|\psi_4\rangle = H_{c_1}\prod_{j=n}^{1}XOR_{i,j}NOT_{m_j}|\psi_3\rangle,$$

$$|\psi_4\rangle = \frac{1}{\sqrt{p}}\sum_{k=1}^{p} \cos\left(\frac{\pi}{2n}d_H(i, p^k)\right)|i; p^k; 0_1, \ldots, 0_b\rangle$$

$$+ \frac{1}{\sqrt{p}}\sum_{k=1}^{p} \sin\left(\frac{\pi}{2n}d_H(i, p^k)\right)|i; p^k; 1_1, \ldots, 0_b\rangle \tag{24}$$

The idea is now to repeat the above operations sequentially for all $b$ control qbits $c_1$ to $c_b$. This gives

$$|\psi_{\text{fin}}\rangle = \frac{1}{\sqrt{p}}\sum_{k=1}^{p} \sum_{l=0}^{b} \cos^{b-l}\left(\frac{\pi}{2n}d_H(i, p^k)\right) \sin^l\left(\frac{\pi}{2n}d_H(i, p^k)\right) \sum_{\{j\}} |i; p^k; J^l\rangle \tag{25}$$

where $\{J^l\}$ denotes the set of all binary numbers of $b$ bits with exactly $l$ bits 1 and $(b - l)$ bits 0.

As in the case of the storing algorithm, there is a version of the information retrieval algorithm in which the input is not loaded into an auxiliary quantum register but rather into a unitary operator. Indeed, the auxiliary quantum register is needed only by the operator (20) leading from
The same result (apart from an irrelevant overall sign) can be obtained by applying

\[ I = \prod_{j=1}^{n} U_j, \]

\[ U_j = \sin\left(\frac{\pi}{2} i_j\right) 1 + i \cos\left(\frac{\pi}{2} i_j\right) \sigma_2 \]

(26)
directly on the memory state \(|m\rangle\). The rest of the algorithm is the same, apart the reversing of the operator (20) which needs now the operator \(I^{-1}\).

The end effect of the information retrieval algorithm represents a rotation of the memory quantum state in the enlarged Hilbert space obtained by adding \(b\) control qbits. Note that the overall effect of this rotation is an amplitude concentration on memory states similar to the input if there is a large number of \(|0\rangle\) control qbits and an amplitude concentration on states different to the input if there is a large number of \(|1\rangle\) control qbits. As a consequence, the most interesting state for information retrieval purposes is the projection of \(|\psi_{\text{fin}}\rangle\) onto the subspace with all control qbits in state \(|0\rangle\).

There are two ways of obtaining this projection. The first is to repeat the above algorithm and measure the control register several times, until exactly the desired state for the control register is obtained. If the number of such repetitions exceeds a preset threshold \(T\) the input is classified as “non-recognized” and the algorithm is stopped. Otherwise, once \(|c_1, \ldots, c_b\rangle = |0_1, \ldots, 0_b\rangle\) is obtained, one proceeds to a measurement of the memory register \(m\), which yields the output pattern of the memory.

The second method is to apply \(T\) steps of the amplitude amplification algorithm(13) rotating \(|\psi_{\text{fin}}\rangle\) towards its projection onto the “good” subspace formed by the states with all control qbits in state \(|0\rangle\). To this end it is best to use the versions of the storing and retrieving algorithms which do not need any auxiliary quantum register for inputs. Let me define as \(R(i)\) the input-dependent operator which rotates the memory state in the Hilbert space enlarged by the \(b\) control qbits towards the final state \(|\psi_{\text{fin}}\rangle\) in Eq. (25) (where I now omit the auxiliary register for the input):

\[ |\psi_{\text{fin}}\rangle = R(i) |m; 0_1, \ldots, 0_b\rangle \]

(27)

By adding also the two utility qbits needed for the storing algorithm one can then obtain \(|\psi_{\text{fin}}\rangle\) as a unitary transformation of the initial state with all qbits in state \(|0\rangle\):

\[ |\psi_{\text{fin}}; 00\rangle = R(i) \ M |0, \ldots, 0; 0_1, \ldots, 0_b; 00\rangle \]

(28)
The amplitude amplification rotation of $|\psi_{\text{fin}}; 00\rangle$ towards its “good” subspace in which all $b$ control qubits are in state $|0\rangle$ is then obtained\(^{13}\) by repeated application of the operator

$$Q = -R(i)MS_0M^{-1}R^{-1}(i)S$$

(29)

on the state $|\psi_{\text{fin}}; 00\rangle$. Here $S$ conditionally changes the sign of the amplitude of the “good” states with the $b$ control qubits in state $|0\rangle$, while $S_0$ changes the sign of the amplitude if and only if the state is the zero state $|0, \ldots, 0; 0_1, \ldots, 0_b; 00\rangle$. As before, if a measurement of the control register after the $T$ iterations of the amplitude amplification rotation yields $|0_1, \ldots, 0_b\rangle$ one proceeds to a measurement of the memory register; otherwise the input is classified as “non-recognized”.

Since the expected number of repetitions needed to measure the desired control register state is $1/P_{\text{rec}}^b$, with

$$P_{\text{rec}}^b = \frac{1}{p} \sum_{k=1}^{p} \cos^{2b} \left( \frac{\pi}{2n} d_H(i; p^k) \right)$$

(30)

the probability of measuring $|c_1, \ldots, c_n\rangle = |0_1, \ldots, 0_n\rangle$, the threshold $T$ governs the recognition efficiency of the input patterns. Note, however, that amplitude amplification provides a quadratic boost\(^{13}\) to the recognition efficiency since only $1/\sqrt{P_{\text{rec}}^b}$ steps are required to rotate $|\psi_{\text{fin}}\rangle$ onto the desired subspace. Accordingly, the threshold $T$ can be lowered to $\sqrt{T}$ with respect to the method of projection by measurement.

Once the input pattern $i$ is recognized, the measurement of the memory register yields the stored pattern $p^k$ with probability

$$P_b(p^k) = \frac{1}{Z} \cos^{2b} \left( \frac{\pi}{2n} d_H(i; p^k) \right).$$

(31)

$$Z = pP_{\text{rec}}^b = \sum_{k=1}^{p} \cos^{2b} \left( \frac{\pi}{2n} d_H(i; p^k) \right)$$

(32)

Clearly, this probability is peaked around those patterns which have the smallest Hamming distance to the input. The highest probability of retrieval is thus realized for that pattern which is most similar to the input. This is always true, independently of the number of stored patterns. In particular there are never spurious memories: the probability of obtaining as output a non-stored pattern is always zero. As a consequence there are no restrictions on the loading factor $p/n$ coming from the information retrieval algorithm.

In addition to the threshold $T$, there is a second tunable parameter, namely the number $b$ of control qubits. This new parameter $b$ controls the
identification efficiency of the quantum memory since, increasing $b$, the probability distribution $P_b(p^x)$ becomes more and more peaked on the low $d_H(i, p^x)$ states, until

$$\lim_{b \to \infty} P_b(p^x) = \delta_{kk_{\text{min}}}$$  \hspace{1cm} (33)

where $k_{\text{min}}$ is the index of the pattern (assumed unique for convenience) with the smallest Hamming distance to the input.

The probability of recognition is determined by comparing (even) powers of cosines and sines of the distances to the stored patterns. It is thus clear that the worst case for recognition is the situation in which there is an isolated pattern, with the remaining patterns forming a tight cluster spanning all the largest distances to the first one. As a consequence, the threshold needed to recognize all patterns diminishes when the number of stored patterns becomes very large, since, in this case, the distribution of patterns becomes necessarily more homogeneous. Indeed, for the maximal number of stored patterns $p = 2^n$ the recognition efficiency becomes maximal, as it should be.

While the recognition efficiency depends on comparing powers of cosines and sines of the same distances in the distribution, the identification efficiency depends on comparing the (even) powers of cosines of the different distances in the distribution. Specifically, it is best when one of the distances is zero, while all others are as large as possible, such that the probability of retrieval is completely peaked on one pattern. As a consequence, the identification efficiency is best when the recognition efficiency is worst and vice versa.

The role of the parameter $b$ becomes familiar upon a closer examination of Eq. (31). Indeed, the quantum distribution described by this equation is equivalent to a canonical Boltzmann distribution with (dimensionless) temperature $t = 1/b$ and (dimensionless) energy levels

$$E^k = -2 \log \cos \left( \frac{\pi}{2n} d_H(i, p^x) \right)$$  \hspace{1cm} (34)

with $Z$ playing the role of the partition function.

The appearance of an effective thermal distribution suggests studying the average behavior of quantum associative memories via the corresponding thermodynamic potentials. Before this can be done, however, one must deal with the different distributions of stored patterns characterizing each individual memory. To this end I propose to average also over this distribution, by keeping as a tunable parameter only the minimal Hamming distance $d$ between the input and the stored patterns. In doing so, one
obtains an average description of the average memory. This is essentially the replica trick used to derive the behavior of spin glasses\(^{(10)}\) and classical Hopfield models.\(^{(6)}\)

As a first step it is useful to normalize the pattern representation by adding (modulo 2) to all patterns, input included, the input pattern \(i\). This clearly preserves all Hamming distances and has the effect of normalizing the input to be the state with all qbits in state \(|0\rangle\). The Hamming distance \(d_{H}(i, p^{k})\) becomes thus simply the number of qbits in pattern \(p^{k}\) with value \(|1\rangle\). For loading factors \(p/n \to 0\) in the limit \(n \to \infty\) the partition function for the average memory takes then a particularly simple form:

\[
Z_{\text{av}} = \frac{p}{N_{\lambda}} \sum_{\{\lambda\}} \sum_{j=0}^{n} \lambda_{j} \cos^{2b} \left( \frac{\pi j}{2n} \right)
\]

where \(\lambda_{j}\) describes an unconstrained probability distribution such that \(\sum_{j=0}^{n} \lambda_{j} = 1\), \(\{\lambda\}\) is the set of such distributions and \(N_{\lambda}\) the corresponding normalization factor. For finite loading factors, instead, the probabilities \(\lambda_{j}\) become subject to constraints which make things more complicated.

I now introduce the free energy \(F(b, d)\) by the usual definition

\[
Z_{\text{av}} = p \, e^{-bF(b, d)} = Z_{\text{av}}(b = 0) \, e^{-bF(b, d)}
\]

where I have chosen a normalization such that \(\exp(-bF)\) describes the deviation of the partition function from its value for \(b = 0\) (high effective temperature). Since \(Z/p\), and consequently also \(Z_{\text{av}}/p\) posses a finite, non-vanishing large-\(n\) limit, this normalization ensures that \(F(b, d)\) is intensive, exactly like the energy levels (34), and scales as a constant for large \(n\). This is the only difference with respect to the familiar situation in statistical mechanics.

The free energy describes the equilibrium of the system at effective temperature \(t = 1/b\) and has the usual expression in terms of the internal energy \(U\) and the entropy \(S\):

\[
F(t, d) = U(t, d) - tS(t, d),
\]

\[
U(t, d) = \langle E \rangle_{t}, \quad S(t, d) = \frac{-\partial F(t, d)}{\partial t}
\]

Note that, with the normalization I have chosen in (36), the entropy \(S\) is always a negative quantity describing the deviation from its maximal value \(S_{\text{max}} = 0\) at \(t = \infty\).
By inverting Eq. (34) with $F$ substituting $E$ one can also define an effective (relative) input/output Hamming distance $D$ at temperature $t$:

$$D(t, d) = \frac{2}{\pi} \arccos e^{-F(t, d)/2}$$

This corresponds exactly to representing the recognition probability of the average memory as

$$\left( P_{\text{rec}}^{\text{av}} \right) = \cos^2 \left( \frac{\pi}{2} D(b, d) \right)$$

which can also be taken as the primary definition of the effective Hamming distance.

The function $D(b, d)$ provides a complete description of the behavior of quantum associative memories in the limit $p/n \ll 1$. This can be used to tune their performance. Indeed, suppose that one wants the memory to recognize and identify inputs with up to $n$ corrupted inputs with an efficiency of $\varepsilon(0 \leq \varepsilon \leq 1)$. Then one must choose a number $b$ of control qubits sufficiently large that $(D(b, \varepsilon n) - \varepsilon) \leq (1 - \varepsilon)$ and a threshold $T$ of repetitions satisfying $T \geq 1/\cos^2 b (\pi/2D(b, \varepsilon n))$, as illustrated in Fig. 1 below.

A first hint about the general behavior of the effective distance function $D(b, d)$ can be obtained by examining closer the energy eigenvalues (34). For small Hamming distance to the input these reduce to

$$E^k \simeq \frac{\pi^2}{4} \left( \frac{d_H(i, p^k)}{n} \right)^2, \quad \frac{d_H(i, p^k)}{n} \ll 1$$

Choosing again the normalization in which $|iangle = |0\ldots0\rangle$ and introducing a “spin” $s^k_i$ with value $s^k_i = -1/2$ if qbit $i$ in pattern $p^k$ has value $|0\rangle$ and $s^k_i = +1/2$ if qbit $i$ in pattern $p^k$ has value $|1\rangle$, one can express the energy levels for $d_H/n \ll 1$ as

$$E^k = \frac{\pi^2}{16} + \frac{\pi^2}{4n^2} \sum_{i,j} s^k_i s^k_j + \frac{\pi^2}{4n} \sum_i s^k_i$$

Apart from a constant, this is the Hamiltonian of an infinite-range antiferromagnetic Ising model in presence of a magnetic field. The antiferromagnetic term favors configurations $k$ with half the spins up and half down, so that $s^k_{\text{tot}} = \sum_i s^k_i = 0$, giving $E^k = \pi^2/16$. The magnetic field, however, tends to align the spins so that $s^k_{\text{tot}} = -n/2$, giving $E^k = 0$. Since this is lower than $\pi^2/16$, the ground state configuration is ferromagnetic, with all qbits having value $|0\rangle$. At very low temperature (high $b$), where the
energy term dominates the free energy, one expects thus an ordered phase of
the quantum associative memory with $D(t, d) = d/n$. This corresponds to a
perfect identification of the presented input. As the temperature is raised ($b$
reduced) however, the thermal energy embodied by the entropy term in the
free energy begins to counteract the magnetic field. At very high
temperatures (low $b$) the entropy approaches its maximal value
$S(t = \infty) = 0$ (with the normalization chosen here). If this value is
approached faster than $1/t$, the free energy will again be dominated by
the internal energy. In this case, however, this is not any more determined by
the ground state but rather equally distributed on all possible states, giving

\[
F(t = \infty) = U(t = \infty) = \frac{-1}{1 - (d/n)} \int_{d/n}^1 dx \log \cos \left( \frac{\pi x}{2} \right)
\]

\[
= \left(1 + \frac{d}{n}\right) \log 2 + O\left(\frac{d^2}{n^2}\right) \quad (42)
\]
and leading to an effective distance

\[ D(t = \infty, d) = \frac{2}{3} - \frac{2 \log 2}{\pi \sqrt{3}} \frac{d}{n} + O\left(\frac{d}{n}^2\right) \]  

(43)

This value corresponds to a disordered phase with no correlation between input and output of the memory.

A numerical study of the thermodynamic potentials in (37) and (38) indeed confirms a phase transition from the ordered to the disordered phase as the effective temperature is raised. In Fig. 1, I show the effective distance \( D \) and the entropy \( S \) for 1 Mb \((n = 8 \times 10^6)\) patterns and \( d/n = 1\% \) as a function of the inverse temperature \( b \) (the entropy is rescaled to the interval \([0, 1]\) for ease of presentation). At high temperature there is indeed a disordered phase with \( S = S_{\text{max}} = 0 \) and \( D = 2/3 \). At low temperatures, instead, one is in the ordered phase with \( S = S_{\text{min}} \) and \( D = d/n = 0.01 \). The effective Hamming distance plays the role of the order parameter for the phase transition.

The phase transition occurs at \( b_{cr} \approx 10^{-1} \). The physical regime of the quantum associative memory \((b = \text{positive integer})\) begins just above this transition. For a good accuracy of pattern recognition one should choose a temperature low enough to be well into the ordered phase.

Having described at length the information retrieval mechanism for complete, but possibly corrupted patterns, it is easy to incorporate also incomplete ones. To this end assume that only \( q < n \) qbits of the input are known and let me denote these by the indices \( \{k_1, \ldots, k_q\} \). After assigning the remaining qbits randomly, there are two possibilities. One can just treat the resulting complete input as a noisy one and proceed as above or, better, one can limit the operator \( (d_H)_m \) in the Hamiltonian (22) to

\[ (d_H)_m = \sum_{i=1}^{q} \left( \frac{\sigma_3 + 1}{2} \right)_{m_{k_i}} \]  

(44)

so that the Hamming distances to the stored patterns are computed on the basis of the known qbits only. After this the pattern recall process continues exactly as described above. This second possibility has the advantage that it does not introduce random noise in the similarity measure but it has the disadvantage that the operations of the memory have to be adjusted to the inputs.

5. EFFICIENCY, COMPLEXITY, AND MEMORY TUNING

As anticipated in Sec. 4, the effective input/output Hamming distance can be used to tune the quantum associative memory to prescribed accuracy
levels. Typically, it is to be expected that increasing this accuracy will lead to an enhanced complexity level. Before I even begin addressing this issue, however, I will show that the information retrieval algorithm is efficient.

First of all I would like to point out that, in addition to the standard NOT, H (Hadamard), XOR, 2XOR (Toffoli) and nXOR gates\(^{(1)}\) I have introduced only the two-qbit gates \(CS_i\) in Eq. (2) and the unitary operator \(\exp(i\pi H/2n)\). This latter can, however also be realized by simple gates involving only one or two qbits. To this end I introduce the single-qbit gate

\[
U = \begin{pmatrix}
e^{i\pi/2n} & 0 \\
0 & 1
\end{pmatrix}
\]

and the two-qbit controlled gate

\[
CU^{-2} = |0\rangle\langle 0| \otimes 1 + |1\rangle\langle 1| \otimes U^{-2}
\]

It is then easy to check that \(\exp(i\pi H/2n)\) in Eq. (21) can be realized as follows:

\[
e^{i\pi/2n}H\psi_2 = \prod_{i=1}^n (CU^{-2})_{cm} \prod_{j=1}^n U_{m_j} |\psi_2\rangle
\]

where \(c\) is the control qbit for which one is currently repeating the algorithm. Essentially, this means that I implement first \(\exp(i\pi d_H/2n)\) and then I correct by implementing \(\exp(-i\pi d_H/2n)\) on that part of the quantum state for which the control qbit \(|c\rangle\) is in state \(|1\rangle\).

Using this representation for the Hamming distance operator I can count the total number of simple gates that I must apply in order to implement one step of the information retrieval algorithm. This is given by \((6n + 2)\) using the auxiliary register for the input and by \((4n + 2)\) otherwise. This retrieval step has then to be repeated for each of the \(b\) control qbits. Therefore, implementing the projection by repeated measurements, the overall complexity \(C\) of information retrieval is bounded by

\[
C \leq Tb(6n + 2)C_{\text{clon}}
\]

where \(C_{\text{clon}}\) is the complexity of the (probabilistic) cloning machine that prepares a copy of the memory state.

The computation of the complexity is easier for the information retrieval algorithm which uses the amplitude amplification technique. In this case the initial memory is prepared only once by a product of the operators \(M\), with complexity \(p(2n + 3) + 1\) and \(R(i)\), with complexity \(b(4n + 2)\). Then one applies \(T\) times the operator \(Q\), with complexity \(p(4n + 6)\)
+b(8n + 4) + 2 + C_S + C_{S_0}, \text{ where } C_S \text{ and } C_{S_0} \text{ are the polynomial complexities of the oracles implementing } S \text{ and } S_0. \text{ This gives}

\begin{equation}
C = T[p(4n + 6) + b(8n + 4) + 2 + C_S + C_{S_0}]
+ p(2n + 3) + b(4n + 2) + 1
\end{equation}

As expected, this result depends on both $T$ and $b$, the parameters governing the recognition and identification efficiencies. This represents exactly the unavoidable tradeoff between accuracy and complexity.

Suppose now one would like to recognize on average inputs with up to $1\%$ of corrupted or missing bits and identify them with high accuracy. The effective input/output Hamming distance $D$ shown in Fig. 1 can then be used to determine the values of the required parameters $T$ and $b$ needed to reach this accuracy for the average memory with $p/n \ll 1$. For $b = 10^4$, e.g., one has $D = 0.018$, which gives the average input/output distance (in percent of total qbits) if the minimum possible input/output distance is 0.01. For this value of $b$ the recognition probability is $1.5 \cdot 10^{-4}$. With the measurement repetition technique one should thus set the threshold $T = 0.6 \cdot 10^4$. Using amplitude amplification, however, one needs only around $T = 80$ repetitions.

I would like to conclude by stressing that the values of $b$ and $T$ obtained by tuning the memory with the effective input/output Hamming distance become $n$-independent for large values of $n$. This is because they are intensive variables unaffected by this “thermodynamic limit”. For any $p$ polynomial in $n$ the information retrieval can then be implemented efficiently and the overall complexity is determined by the accuracy requirements via the $n$-independent parameters $T$ and $b$.

6. CONCLUSION

I would like to conclude this review by stressing the reason why a quantum associative memory works better than its classical counterpart.

In classical associative memories, the information about the patterns to recall is typically stored in an energy functional. When retrieving information, the input configuration evolves to the corresponding output, driven by the memory functional. The capacity shortage is due to a phase transition in the statistical ensemble governed by the memory energy functional. Spurious memories, i.e., spurious metastable minima not associated with any of the original patterns become important for loading factors $p/n > 0.14$ and wash out completely the memory. So, in the low $p/n$ phase the memory works perfectly in the sense that it outputs always the
stored pattern which is most similar to the input. For \( p/n > 0.14 \), instead, there is an abrupt transition to total amnesia caused by spurious memories.

Quantum associative memories work better than classical ones since they are free from spurious memories. The easiest way to see this is in the formulation

\[
|m\rangle = M |0\rangle
\]

(50)

All the information about the stored patterns is encoded in the unitary operator \( M \). This is such that all quantum states that do not correspond to stored patterns have exactly vanishing amplitudes.

An analogy with the classical Hopfield model \((6)\) can be established as follows. Instead of generating the memory state from the initial zero state one can start from a uniform superposition of the computational basis. This is achieved by the operator \( MW \) defined by

\[
|m\rangle = MW \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} |j\rangle
\]

\[
W = \prod_{j=1}^{n} H_j
\]

(51)

Now, the same result can also be obtained by Grover’s algorithm, or better by its generalization with zero failure rate \((22)\). Here the state \( |m\rangle \) is obtained by applying to the uniform superposition of the computational basis \( q \) times the search operator \( X \) defined by

\[
|m\rangle = X^q \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} |j\rangle
\]

\[
X = -WJ_0WJ
\]

(52)

where \( J \) rotates the amplitudes of the states corresponding to the patterns to be stored by a phase \( \phi \) which is very close to \( \pi \) (the original Grover value) for large \( n \) and \( J_0 \) does the same on the zero state. Via the two Eqs. (51) and (52), the memory operator \( M \) provides an implicit realization of the phase shift operator \( J \). Being a unitary operator, this can always be written as an exponential of an hermitian Hamiltonian \( \mathcal{H} \), which is the quantum generalization of a classical energy functional. By defining \( J \equiv \exp(-i \mathcal{H}) \) one obtains an energy operator which is diagonal in the computational basis. The energy eigenvalues of this operator are such that the patterns to be stored have energy \( E = -\phi \simeq -\pi \) while all others have energy \( E = 0 \).
This formulation is the exact quantum generalization of the Hopfield model; the important point is that the operator $M$ realizes efficiently a dynamics in which the patterns to be stored are always, for all numbers $p$ of patterns, the exact global minima of a quantum energy landscape, without the appearance of any spurious memories. The price to pay is the additional probabilistic nature of the information retrieval mechanism caused by quantum mechanics. As always in quantum mechanics, the dynamics determines only the evolution of probability distributions and the probabilistic aspect is brought in by the collapse of this probability distributions upon measurement. Therefore, contrary to the classical Hopfield model in the low $p/n$ phase, one does not always have the absolute guarantee that an input is recognized, and identified correctly as the stored pattern most similar to the input, even if this state has the highest probability of being measured.

REFERENCES

1. M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000); A. O. Pittenger, An Introduction to Quantum Computing Algorithms (Birkhäuser, Boston, 2000).
2. P. W. Shor, SIAM J. Computing 26, 1484 (1997).
3. L. Grover, Phys. Rev. Lett. 79, 325 (1997).
4. C. A. Trugenberger, Phys. Rev. Lett. 87, 067901 (2001); C. A. Trugenberger, Phys. Rev. Lett. 89, 277903 (2002).
5. M. Sasaki, A. Carlini, and R. Jozsa, Phys. Rev. A 64, 022317 (2001); M. Sasaki and A. Carlini, quant-ph/0202173; R. Schützhold, quant-ph/0208063.
6. B. Müller and J. Reinhardt, Neural Networks (Springer-Verlag, Berlin, 1990); T. Kohonen, Self-Organization and Associative Memory (Springer-Verlag, Berlin, 1984).
7. E. Baum, Science 268, 583 (1995).
8. J. J. Hopfield, Proc. Natl. Acad. Sci. USA 79, 2554 (1982).
9. B. Kosko, IEEE Trans. on Systems, Man and Cybernetics 18, 49 (1988).
10. M. Mezard, G. Parisi, and M. A. Virasoro, Spin Glass Theory and Beyond (World Scientific, Singapore, 1987).
11. P. Kanerva, in Associative Neural Memories: Theory and Implementation, M. H. Hassoun (ed.) (Oxford University Press, New York, 1993).
12. P. A. Chou, in Neural Information Processing Systems, D. Z. Anderson (ed.) (American Institute of Physics, New York, 1988); P. A. Chou, IEEE Trans. Info. Theory 35, 281 (1989).
13. G. Brassard, P. Hoyer, M. Mosca and A. Tapp, quant-ph/0005055.
14. N. Sourlas, Nature 339, 693 (1989); I. Kanter and D. Saad, Phys. Rev. Lett. 83, 2660 (1999); Y. Kabashima, T. Murayama, and D. Saad, Phys. Rev. Lett. 84, 1355 (2000).
15. Y. Kabashima, T. Murayama, and D. Saad, Phys. Rev. Lett. 84, 2030 (2000).
16. A. Barenco, C. Bennett, R. Cleve, D. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. Smolin, and H. Weinfurter, Phys. Rev. A 52, 3457 (1995).
17. W. Wootters and W. Zurek, Nature 299, 802 (1982).
18. V. Buzek and M. Hillery, Phys. Rev. A 54, 1844 (1996).
19. N. Gisin and S. Massar, *Phys. Rev. Lett.* 79, 2153 (1997); D. Bruss, A. K. Ekert, and C. Macchiavello, *Phys. Rev. Lett.* 81, 2598 (1998).
20. L.-M. Duan and G.-C. Guo, *Phys. Rev. Lett.* 80, 4999 (1998).
21. A. Chefles and S. M. Barnett, *Phys. Rev. A* 60, 136 (1999).
22. G. L. Long, quant-ph/0106071.