Footprints and footprint analysis for atmospheric dispersion problems

Niklas Brännström and Leif Å Persson
Swedish Defence Research Agency, FOI
SE-901 82 Umeå
Sweden
September 19, 2014

Contents

1 Introduction 2
2 An atmospheric dispersion problem, and its adjoint formulation 3
3 Detectability and Non-detectability 5
4 Posterior and prior footprints, posterior and prior zero footprints 9
5 Footprint analysis, composite footprints 14
6 Conclusion 16

Abstract

Footprint analysis, also known as the study of Influence areas, is a first order method for solving inverse atmospheric dispersion problems. We revisit the concept of footprints giving a rigorous definition of the concept (denoted posterior footprints and posterior zero footprints) in terms of spatio-temporal domains. The notion of footprints is then augmented the to the forward dispersion problem by defining prior footprints and prior zero footprints. We then study how posterior footprints and posterior zero footprints can be combined to
reveal more information about the source, and how prior footprints
and prior footprints can be combined to yield more information about
the measurements.

1 Introduction

In inverse atmospheric dispersion problems the task is to use sensor data
(e.g. concentration readings of a pollutant) to characterise the source of
the pollutant. As inverse problems are usually hard due the problem being
over-determined and having non-unique solutions many different methods to
solve the problem have been suggested. A common feature of these different
methods, however, is the tendency to specialise on certain parameterisations
of the problem. That is, the methods are usually tailored to the problem at
hand. This is of course a natural path to follow when solving a particular
problem, but in [1] an alternative view was presented: therein a measure the-
etric framework for studying inverse atmospheric dispersion problems was
developed. While not solving any particular inverse dispersion problem, the
framework allows for general conclusions to be drawn about general inverse
dispersion problems. (If a particular problem is to be studied, the framework
can be suitably parameterised to cope with the situation). The framework it-
self relies on a measure theoretic description of the dispersion problem (and
its adjoint) which is reviewed in Section 2. In [1] this approach was em-
ployed to derive necessary and sufficient conditions for when the inverse can
be solved using the method of least-squares. In the current paper we will use
the same approach to study another method for solving inverse atmospheric
dispersion problems: footprint analysis (which is also known as ’area of in-
fluence’). In essence, footprint analysis is a rough method for solving the
inverse problem where the information contents of each sensor measurement
is superpositioned to gain a rough idea of the characteristics of the unknown
source function.

With the risk of oversimplifying it seems footprint analysis can be divided
into two parts, where the first is to establish a relationship between the source
and the sensor and the second to use it to solve the inverse problem, footprint
analysis. In atmospheric dispersion models the analysis of how the source in-
fluences the sensor is called footprints [4]. The ideas to study footprints (but
without the nomenclature of today’s literature) were introduced in [9]. Since
then a whole body of literature has emerged, with the survey article [15] be-
ing a good starting point. Initially the focus lay on finding 2D footprints, i.e.
areas where the part of the source that influences the sensor most is located.
From the start the problem was set in terrains that were flat and smooth but
later the emphasis has been placed on studying the phenomena over rougher terrain, in particular forest canopies, [13], 4, 7, 11. In 15 the footprint analysis is generalised to 3D footprints (three spatial dimensions). To our knowledge there are no spatio-temporal studies of the footprint despite many dispersion problems being set both in space and time. In the present paper we augment the notion of footprints to the spatio-temporal setting. The term footprint analysis (or 'area of influence') is also used to refer to a first order method for solving the inverse problem of estimating source parameters, see e.g. 10 and 12. Here the idea is that using the adjoint formulation of the dispersion model one can compute ”back trajectories” starting from each sensor and evolving backwards in time. The source, one then concludes, is located where all these back trajectories intersect. The two uses of the word footprint stems from exactly the same idea, namely to describe where the source is located, but in the case of trying to solve the inverse problem it is always the adjoint formulation of the dispersion model that is being used, and secondly, if several measurements are available then this information is put to use to solve the inverse problem (by intersecting the back trajectories). In this paper we build on these ideas and use the framework of 11 to put these concepts on a rigorous footing. Indeed, the footprints referred to in the footprint literature are closely related to what we will term posterior footprints, and footprint analysis, used as a first order method for solving the inverse problem, makes use of both posterior footprints and posterior zero footprints. The nomenclature regarding footprints will be explained in Section 4 but first we have to revise the setting of the underlying problem.

2 An atmospheric dispersion problem, and its adjoint formulation

The atmospheric dispersion problem that we are interested in can be formulated in terms of a transition probability \( p(t, x; s, y) \), where \((s, y), (t, x) \in T \times V \) where \( T \subset \mathbb{R} \) is a time interval and \( V \subset \mathbb{R}^3 \) is a spatial domain. The transition probability expresses the probability for a particle released at the time-space point \((s, y)\) to reside in the time-space point \((t, x)\) for \( t \geq s \). We note that \( p = 0 \) when \( t < s \). The particles whose dispersion is governed by this transition probability is assumed to originate from a source \( S \). The source \( S \) is assumed to be a positive measure on \( T \times V \). In this way the total mass \( M \) released from the source is given by integrating the source measure \( S \) over its support

\[
M = \int_T \int_V dS(s, y).
\]
The quantity that is usually desired as output from a dispersion model is the concentration of the pollutant in a given space-time point. Since $S$ has its support on $T \times V$ and the transition probability describes the dynamics of the released substance the concentration $c(t, x)$ is obtained by weighing all released particles (released at some $(s, y)$ with $s < t$) with the probability that they have been transported from $(s, y)$ to $(t, x)$

$$c(t, x) = \int_T \int_V p(t, x; s, y) dS(s, y).$$

(2)

While $c(t, x)$ is the predicted concentration at the space time point $(t, x)$ the sensor may not have the resolution to make an ideal measurement from the concentration field $c(t, x)$, indeed the sensor may perform some form of averaging in both space and time to yield the sensor response $\tau(t, x)$. We assume that the averaging process in the sensor can be described by a probability measure $S^*$ (usually referred to as the sensor-filter function) on $T \times V$, and hence we express the sensor response as

$$\tau = \int_T \int_V c(t, x) dS^*(t, x).$$

(3)

Let us now use the definition of $c(t, x)$ to rewrite this expression in the following way

$$\tau = \int_T \int_V c(t, x) dS^*(t, x)$$

$$= \int_T \int_V \int_T \int_V p(t, x; s, y) dS(s, y) dS^*(t, x)$$

(4)

$$= \int_T \int_V \left( \int_T \int_V p(t, x; s, y) dS^*(t, x) \right) dS(s, y).$$

By defining the adjoint concentration field $c^*(s, y)$ as

$$c^*(s, y) = \int_T \int_V p(t, x; s, y) dS^*(t, x)$$

(5)

we get

$$\tau = \int_T \int_V c^*(s, y) dS(s, y).$$

(6)

Hence we have two equivalent ways of calculating the sensor response

$$\tau = \int_T \int_V c(t, x) dS^*(t, x) = \int_T \int_V \int_V c^*(s, y) dS(s, y)$$

(7)
which is the dual relationship between the forward and the adjoint description of the dispersion problem. We note that equation (5) describing the adjoint concentration field is evolving backwards in time: we may view the transition probability as moving adjoint particles released by $S^*$ backwards in time and space. The main advantage of using the adjoint representation in inverse dispersion modelling is computational efficiency. This is a well-documented fact, see for example [8]. We also remark that the adjoint concentration field $c^*$ is independent of the source function $S$, and the concentration field $c$ is independent of the sensor-filter function $S^*$.

To better model real world sensors we assume that all sensors have a threshold value $\tilde{c}_\text{lim}$ (the threshold value depends on the specific sensor) below which any sensor response $\tau$ will be put to zero

$$\tau = \begin{cases} \tau, & \text{if } \tau \geq \tilde{c}_\text{lim} \\ 0, & \text{otherwise} \end{cases}.$$ 

### 3 Detectability and Non-detectability

The dispersion problem predicts how a pollutant from a source spreads in the atmosphere. From an abstract point of view this problem can be seen as a problem of mapping of measures: the source $S$ can be viewed as a measure in the spatio-temporal domain $T \times V$ that is being mapped via the dispersion equations into a scalar function $c$ (the concentration), from which we make measurements represented by a probability measure $S^*$, defining the averaging of the concentration function $c$. From this level of abstraction the adjoint version of the problem is very similar. In this case the adjoint equations maps a probability measure $S^*$ on $T \times V$ representing a measurement in a sensor to a scalar function $c^*$ (adjoint 'concentration') from which we can make ”adjoint measurements” using a source measure $S$ acting on the adjoint 'concentration' $c^*$. (Depending on the scaling of the problem the adjoint 'concentration' $c^*$ may not be a proper concentration dimensionally.) In view of this light asking questions about the sensor response in the forward problem or asking questions about the source in the inverse problem are very similar.

Before heading into footprints we begin by studying the notion of detectability, that is, when a source can be detected by sensor measurements.

**Definition 1** A measurement $S^*$ is said to have sensitivity $k$ at $(s, y) \in T \times V$ if $c^*(s, y) \geq k$.

**Definition 2** A measurement $S$ is said to detect the source $S$ at detection level $c_{\text{lim}}$ if $\langle S, c^* \rangle \geq c_{\text{lim}}$. 


To connect detection level to sensitivity we must assume a minimum mass of the source. It follows from these definitions that

**Proposition 3** If an instantaneous point source of mass $M$ at $(s, y)$ is detected by measurement $S^*$, then $S^*$ has sensitivity $k = c_{\lim}/M$. To detect an instantaneous point source with mass at least $M_{\min}$, a sensitivity of $c_{\lim}/M_{\min}$ is required at the source location.

A sensor detects on sensitivity level $k$ by weighting the concentration field $c$ in a spatio-temporal neighbourhood of the sensor using the measurement $S^*$. We state the some properties of this measurement in general terms in the following theorem.

**Theorem 4** Assume that $S$ is a positive measure on a domain $D \subseteq \mathbb{R}^n$ and $c^*$ a nonnegative measurable function on $D$, integrable with respect to $S$. Then for all $k \geq 0$

$$kS \{c^* \geq k\} + \int_{\{c^* < k\}} c^* dS \leq \int c^* dS$$

and

$$\int c^* dS \leq kS \{c^* \leq k\} + \int_{\{c^* > k\}} c^* dS$$

Moreover, if there is equality in (3) then $S \{c^* > k\} = 0$ (i.e., $c^* = k$, $S$-almost everywhere on $\{c^* \geq k\}$). Finally, if there is equality in (4), then $S \{c^* < k\} = 0$ (i.e., $c^* = k$, $S$-almost everywhere on $\{c^* \leq k\}$).

**Proof.** We have

$$\int c^* dS = \int_{\{c^* \geq k\}} c^* dS + \int_{\{c^* < k\}} c^* dS \geq k \int_{\{c^* \geq k\}} dS + \int_{\{c^* < k\}} c^* dS$$

which proves equation (3). If there is equality in equation (3) we have

$$\int_{\{c^* \geq k\}} c^* dS = k \int_{\{c^* \geq k\}} dS$$

which implies that $S \{c^* > k\} = 0$ (cf. [13], Theorem 1.39 therein). The proof of equation (4) is similar, with all inequalities reversed.

**Definition 5** A source $S$ is said to be $S^*$-detectable (with detection level $c_{\lim}$) if

$$\int c^* dS \geq c_{\lim}$$

and $S^*$-nondetectable if

$$\int c^* dS < c_{\lim}$$
Theorem 4 gives necessary and sufficient conditions for $S$ to be $S^*$ detectable and $S^*$ nondetectable in terms of masses on level sets $\{c^* \geq k\}$ etc., which is exploited in the following propositions.

**Proposition 6** (Necessary conditions for detection) Assume that $S$ is $S^*$ detectable and $k > 0$. Then

1. $$kS \{c^* \leq k\} + \int_{\{c^* > k\}} c^* dS \geq c_{\text{lim}}$$ (13)

2. $$kS \{c^* < k\} + \sup c^* S \{c^* \geq k\} \geq c_{\text{lim}}$$ (14)

3. If there is a constant $\alpha \geq 0$ such that $S \{c^* < k\} \leq \alpha S \{c^* \geq k\}$ then
   $$S \{c^* \geq k\} \geq \frac{c_{\text{lim}}}{k\alpha + \sup c^*}$$ (15)

4. If $\sup c^* = \infty$, there is no positive lower bound on $S \{c^* \geq k\}$, i.e., there are $S^*$ detectable sources with arbitrarily small mass $S \{c^* \geq k\}$.

**Proof.**

1. Equation (13) follows immediately from equations (9) and (11).

2. Since $c^* \leq \sup c^*$ we get from (13) that
   $$c_{\text{lim}} \leq kS \{c^* \leq k\} + \int_{\{c^* > k\}} c^* dS \leq kS \{c^* \leq k\} + \sup c^* S \{c^* > k\}$$ (16)
   Moreover we have
   $$kS \{c^* \leq k\} + \sup c^* S \{c^* > k\} \leq kS \{c^* < k\} + \sup c^* S \{c^* \geq k\}$$ (17)
   (with equality if $S \{c^* = k\} = 0$, in particular, if $k > \sup c^*$), which proves (14).

3. We get from (14) and the additional condition $S \{c^* < k\} \leq \alpha S \{c^* \geq k\}$ that
   $$c_{\text{lim}} \leq kS \{c^* < k\} + \sup c^* S \{c^* \geq k\} \leq (k\alpha + \sup c^*) S \{c^* \geq k\}$$ (18)
   which proves (15).

7
4. Take a sequence \( t_j, x_j \) of release times and locations such that \( c^* (t_j, x_j) \to \infty \), and for each \( j \) let \( S_j \) be an instantaneous point source at \( (t_j, x_j) \) with mass \( M_j = c_{\lim} / c^* (t_j, x_j) \). Then \( \int c^* dS_j = c_{\lim} \), so each \( S_j \) is \( S^* \)-detectable.

Proposition 7 (Sufficient conditions for detection) Assume \( k > 0 \) and at least one of the following conditions 1-3 is satisfied:

1. \[ kS \{c^* \geq k\} + \int_{\{c^* < k\}} c^* dS \geq c_{\lim} \] (19)

2. \[ kS \{c^* \geq k\} + \inf c^* S \{c^* < k\} \geq c_{\lim} \] (20)

3. There is a constant \( \beta \geq 0 \) such that
\[
S \{c^* < k\} \geq \beta S \{c^* \geq k\} \text{ and } S \{c^* \geq k\} \geq \frac{c_{\lim}}{k + \beta \inf c^*} \] (21)

Then \( S \) is \( S^* \)-detectable.

Proof.

1. Equation (11) follows immediately from equations (19) and (8).

2. Equation (19) follows from (20) since \( \inf c^* S \{c^* < k\} \leq \int_{\{c^* < k\}} c^* dS \).

3. Equation (20) follows from (21) since
\[
kS \{c^* \geq k\} + \inf c^* S \{c^* < k\} \geq (k + \beta \inf c^*) S \{c^* \geq k\} \geq c_{\lim}
\]

Proposition 8 (Necessary conditions for nondetection) Assume that \( S \) is \( S^* \)-nondetectable and that \( k > 0 \). Then

1. \[ kS \{c^* \geq k\} + \int_{\{c^* < k\}} c^* dS < c_{\lim} \] (22)
2. 
\[ k S \{ c^* \geq k \} + \inf c^* S \{ c^* < k \} < c_{\text{lim}} \]  
(23)

3. If there is a constant \( \gamma \geq 0 \) such that \( S \{ c^* < k \} \geq \gamma S \{ c^* \geq k \} \) then 
\[ S \{ c^* \geq k \} < \frac{c_{\text{lim}}}{k + \gamma \inf c^*} \]  
(24)

**Proof.** Contrapositive of Proposition 7. ■

**Proposition 9** (Sufficient conditions for non–detection) Assume \( k > 0 \) and at least one of the following conditions 1–3 are satisfied:

1. 
\[ k S \{ c^* \leq k \} + \int_{\{c^*>k\}} c^* dS < c_{\text{lim}} \]  
(25)

2. 
\[ k S \{ c^* < k \} + \sup c^* S \{ c^* \geq k \} < c_{\text{lim}} \]  
(26)

3. There is a constant \( \varepsilon \geq 0 \) such that 
\[ S \{ c^* < k \} \leq \varepsilon S \{ c^* \geq k \} \text{ and } \] 
\[ S \{ c^* \geq k \} < \frac{c_{\text{lim}}}{k \varepsilon + \sup c^*} \]  
(27)

Then \( S \) is \( S^* \)–nondetectable.

**Proof.** Contrapositive of Proposition 6. ■

4 Posterior and prior footprints, posterior and prior zero footprints

We want to define the notions of footprint and zero footprint. A footprint is, loosely speaking, a subset \( F \) of spacetime where the total source mass is larger than a specified limit \( M_{\text{lim}} \), i.e.,
\[ \int_F dS(t, x) \geq M_{\text{lim}} \]  
(28)
for all source measures \( S \) in a given admissible class \( S \). Likewise, a zero footprint is a subset \( Z \) of spacetime where the total source mass is smaller than a specified limit
\[ \int_Z dS(t, x) < M_{\text{lim}} \]  
(29)
for all $S \in \mathcal{S}$. To be of interest, the footprints and zero footprints should be associated not only to a fixed set $\mathcal{S}$ of admissible sources, but moreover restricted to subsets of $\mathcal{S}$ determined by conditions on measured values. Hence, given an $m$-tuple of measurements $(S_1^*, \ldots, S_m^*)$ and corresponding adjoint fields $c_j^*(s, y) = \int_{T \times V} p(s, y; t, x) dS_j^*(t, x)$ we consider conditions on the form

$$\langle S, c_j^* \rangle \geq \tilde{c} \lim_{j} \quad \text{or} \quad \langle S, c_j^* \rangle < \tilde{c} \lim_{j} \quad \text{for } j = 1, \ldots, m$$

(30)

where $\tilde{c} \lim_{j} > 0$ are given limits (sensor thresholds). We could work with these conditions in the form stated, but for the application we have in mind (and for the sake of brevity) it is convenient to rewrite these conditions in a form where the inequalities in both conditions (30) go in the same direction. We achieve this by letting $\hat{c} \lim_{j} := \tilde{c} \lim_{j} \text{ if } \langle S, c_j^* \rangle \geq \tilde{c} \lim_{j}$, and $\check{c} \lim_{j} := -\tilde{c} \lim_{j} \text{ if } \langle S, c_j^* \rangle < \tilde{c} \lim_{j}$, thus we have

$$\langle S, c_j^* \rangle - \hat{c} \lim_{j} > 0$$

(31a)

for $\hat{c} \lim_{j} > 0$, and

$$-\langle S, c_j^* \rangle - \check{c} \lim_{j} > 0$$

(32)

for $\check{c} \lim_{j} < 0$. By letting

$$c \lim_{j} = \begin{cases} \hat{c} \lim_{j} & \text{if } \langle S, c_j^* \rangle \geq \tilde{c} \lim_{j} \\ \check{c} \lim_{j} & \text{if } \langle S, c_j^* \rangle < \tilde{c} \lim_{j} \end{cases}$$

we combine (31a) and (32) into

$$c \lim_{j} \geq 0 \text{ or } c \lim_{j} < 0 \text{ and } \text{sign}(c \lim_{j}) \left( \langle S, c_j^* \rangle - |c \lim_{j}| \right) \geq 0 \text{ for } j = 1, \ldots, m$$

(33)

where we define

$$\text{sign}(c) = 1_{\{c \geq 0\}} - 1_{\{c < 0\}}$$

(34)

We note that the limit $c \lim_{j}$ has the same physical interpretation as $\tilde{c} \lim_{j}$, the value of $c \lim_{j}$ is the limit (threshold) while the sign of $\tilde{c} \lim_{j}$ tells whether the limit is exceeded (+) or not (-). Hence we represent lower limits by positive values of $c \lim_{j}$ and upper limits by negative values of $c \lim_{j}$. We define a footprint set $F$ by requiring a logical implication between the footprint mass condition, equation (28), and the measurement condition, equation (33). Likewise, we define a zero footprint set $Z$ by requiring a logical implication between the zero footprint mass condition, equation (29), and equation (33). If the mass condition is necessary for the measurement condition, we say that we have a posterior footprint or posterior zero footprint; if the mass condition is sufficient, we say that we have a prior footprint or prior zero
footprint. Hence, posterior footprints and posterior zero footprints are used to deduce facts about the released masses, given the measurements, whilst prior footprints are used to deduce facts about the measurements, given facts about the released masses.

More precisely, we have

**Definition 10** A subset $F \subset T \times V$ is said to be a posterior $(S^*, c_{\text{lim}}, S, M_{\text{lim}})$–footprint (or posterior footprint when the parameters are understood) if $S \in S$ and \( \text{sign} (c_{\text{lim},j}) \left( \langle S, c^*_j \rangle - |c_{\text{lim},j}| \right) \geq 0 \) for $j = 1, ..., m$ implies that $S \{ c^*_j \} \geq M_{\text{lim}}$. $F \subset T \times V$ is said to be a prior $(S^*, c_{\text{lim}}, S, M_{\text{lim}})$–footprint (or prior footprint when the parameters are understood) if $S \in S$ and $S \{ c^*_j \} \geq M_{\text{lim}}$ implies that $\text{sign} (c_{\text{lim},j}) \left( \langle S, c^*_j \rangle - |c_{\text{lim},j}| \right) \geq 0$ for $j = 1, ..., m$.

**Definition 11** A subset $Z \subset T \times V$ is said to be a posterior $(S^*, c_{\text{lim}}, S, M_{\text{lim}})$–zero footprint (or posterior zero footprint when the parameters are understood) if $S \in S$ and $S \{ c^*_j \} < M_{\text{lim}}$ implies that $\text{sign} (c_{\text{lim},j}) \left( \langle S, c^*_j \rangle - |c_{\text{lim},j}| \right) \geq 0$ for $j = 1, ..., m$. $Z \subset T \times V$ is said to be a prior $(S^*, c_{\text{lim}}, S, M_{\text{lim}})$–zero footprint (or prior zero footprint when the parameters are understood) if $S \in S$ and $S \{ c^*_j \} < M_{\text{lim}}$ implies that $\text{sign} (c_{\text{lim},j}) \left( \langle S, c^*_j \rangle - |c_{\text{lim},j}| \right) \geq 0$ for $j = 1, ..., m$.

**Remark 12** Note that the vector $c_{\text{lim}}$ in the previous definition can hold both positive and negative elements, thus we are handling measurements (positive elements) and non-measurements (negative elements) simultaneously.

To see some examples, consider the case of one measurement.

**Proposition 13** If $m = 1$, $c_{\text{lim}} > 0$, $k > 0$, $\alpha \geq 0$, $\sup c^* < \infty$ and

$$ S = \{ S \in M^+ : S \{ c^* < k \} \leq \alpha S \{ c^* \geq k \} \} $$

and

$$ M_{\text{lim}} = \frac{c_{\text{lim}}}{k\alpha + \sup c^*} $$

then $\{ c^* \geq k \}$ is a posterior $(S^*, c_{\text{lim}}, S, M_{\text{lim}})$–footprint and a prior $(S^*, -c_{\text{lim}}, S, M_{\text{lim}})$–zero footprint.

**Proof.** Proposition 6 and Proposition 9

**Proposition 14** If $m = 1$, $c_{\text{lim}} < 0$, $k > 0$, $\beta \geq 0$ and

$$ S = \{ S \in M^+ : S \{ c^* < k \} \geq \beta S \{ c^* \geq k \} \} $$

and

$$ M_{\text{lim}} = \frac{c_{\text{lim}}}{k\beta + \sup c^*} $$

then $\{ c^* \geq k \}$ is a posterior $(S^*, c_{\text{lim}}, S, M_{\text{lim}})$–footprint and a prior $(S^*, -c_{\text{lim}}, S, M_{\text{lim}})$–zero footprint.

**Proof.** Proposition 6 and Proposition 9
\[ M_{\text{lim}} = \frac{-c_{\text{lim}}}{k + \beta \inf c^*} \]

then \( \{c^* \geq k\} \) is a prior \((S^*, -c_{\text{lim}}, S, M_{\text{lim}})\)-footprint and a posterior \((S^*, c_{\text{lim}}, S, M_{\text{lim}})\)-zero footprint. Note that if \( \beta = 0 \) then \( S = M^+ \).

**Proof.** Proposition 7 and Proposition 8.

The fact that we obtained pairs of prior/posterior footprints/zero footprints in the preceding propositions is not a coincidence. Indeed, we have

**Proposition 15** Assume that \( c_{\text{lim}}' \neq c_{\text{lim}} \) and that \( c_{\text{lim},j}' = \pm c_{\text{lim},j} \). Then

1. If \( A \) is a posterior \((S^*, c_{\text{lim}}, S, M_{\text{lim}})\)-footprint then \( A \) is a prior \((S^*, c_{\text{lim}}', S, M_{\text{lim}})\)-zero footprint.
2. If \( A \) is a prior \((S^*, c_{\text{lim}}, S, M_{\text{lim}})\)-footprint then \( A \) is a posterior \((S^*, c_{\text{lim}}', S, M_{\text{lim}})\)-zero footprint.
3. If \( A \) is a posterior \((S^*, c_{\text{lim}}, S, M_{\text{lim}})\)-zero footprint then \( A \) is a prior \((S^*, c_{\text{lim}}', S, M_{\text{lim}})\)-footprint.
4. If \( A \) is a prior \((S^*, c_{\text{lim}}, S, M_{\text{lim}})\)-zero footprint then \( A \) is a posterior \((S^*, c_{\text{lim}}', S, M_{\text{lim}})\)-footprint.

**Proof.** The condition \( \text{sign} \left( c_{\text{lim},j} \right) \left( \langle S, c^*_j \rangle - |c_{\text{lim},j}| \right) \geq 0 \) is not fulfilled for all \( j \) if and only if it is violated for at least one component, i.e., \( \text{sign} \left( c_{\text{lim},j}' \right) \left( \langle S, c^*_j \rangle - |c_{\text{lim},j}'| \right) \geq 0 \) for some \( c_{\text{lim}}' \neq c_{\text{lim}} \) with \( c_{\text{lim},j}' = \pm c_{\text{lim},j} \), so the results follows by contraposition.

Let us now investigate how we can construct new footprints from old ones by set theory operations. Some are obvious, collected in the following

**Proposition 16**

1. If \( F \) is a posterior \((S^*, c_{\text{lim}}, S, M_{\text{lim}})\)-footprint, \( S' \subseteq S, M'_{\text{lim}} \leq M_{\text{lim}} \) and \( F' \supseteq F \), then \( F' \) is a \((S^*, c_{\text{lim}}, S', M'_{\text{lim}})\)-footprint.
2. If \( F \) is a prior \((S^*, c_{\text{lim}}, S, M_{\text{lim}})\)-footprint, \( S' \subseteq S, M'_{\text{lim}} \geq M_{\text{lim}} \) and \( F' \subseteq F \), then \( F' \) is a prior \((S^*, c_{\text{lim}}, S', M'_{\text{lim}})\)-footprint.
3. If \( Z \) is a posterior \((S^*, c_{\text{lim}}, S, M_{\text{lim}})\)-zero footprint, \( S' \subseteq S, M'_{\text{lim}} \geq M_{\text{lim}} \) and \( F' \subseteq F \), then \( F' \) is a posterior \((S^*, c_{\text{lim}}, S', M'_{\text{lim}})\)-zero footprint.
4. If \( Z \) is a prior \((S^*, c_{\text{lim}}, S, M_{\text{lim}})\)-zero footprint, \( S' \subseteq S \), \( M'_{\text{lim}} \leq M_{\text{lim}} \) and \( F' \supseteq F \), then \( F' \) is a prior \((S^*, c_{\text{lim}}, S', M'_{\text{lim}})\)-zero footprint.

**Definition 17**

1. A set \( F_{\text{min}} \in T \times V \) is said to be a minimal posterior footprint if there is no other posterior footprint \( F \) (with the same parameters) with \( F \subset F_{\text{min}} \).

2. A set \( F_{\text{max}} \in T \times V \) is said to be a maximal prior footprint if there is no other prior footprint \( F \) (with the same parameters) with \( F \supseteq F_{\text{max}} \).

3. A set \( Z_{\text{max}} \in T \times V \) is said to be a maximal posterior zero footprint if there is no other posterior zero footprint \( Z \) (with the same parameters) with \( Z \supseteq Z_{\text{max}} \).

4. A set \( Z_{\text{min}} \subseteq T \times V \) is said to be a minimal prior zero footprint if there is no other prior zero footprint \( Z \) (with the same parameters) with \( Z \supset Z_{\text{min}} \).

In the following proposition it is understood that all footprints are taken with respect to the same parameters \((S^*, c_{\text{lim}}, S, M_{\text{lim}})\).

**Proposition 18**

1. For every nonempty posterior footprint \( F \) there is a minimal posterior footprint \( F_{\text{min}} \) with \( F_{\text{min}} \subseteq F \).

2. For every nonempty prior footprint \( F \) there is a maximal prior footprint \( F_{\text{max}} \) with \( F \subseteq F_{\text{max}} \).

3. For every nonempty posterior zero footprint \( Z \) there is a maximal posterior footprint \( Z_{\text{max}} \) with \( Z \subseteq Z_{\text{max}} \).

4. For every nonempty prior zero footprint \( Z \) there is a minimal prior zero footprint \( Z_{\text{min}} \) with \( Z_{\text{min}} \subseteq Z \).

**Proof.** Let \( \mathcal{F} \) denote the class of all posterior \((S^*, c_{\text{lim}}, S, M_{\text{lim}})\)-footprints. By Proposition 16, \( \mathcal{F} \) is a partially ordered set with respect to set inclusion. Consider a nonempty \( F \in \mathcal{F} \) and a nest \( \mathcal{N} \) containing \( F \), i.e., a subset \( \mathcal{N}' \subset \mathcal{F} \) such that if \( F_1, F_2 \in \mathcal{N} \), then either \( F_1 \subset F_2 \) or \( F_2 \subset F_1 \). By the Hausdorff Maximal Principle (see [2], p. 32) \( \mathcal{N} \) can be extended to a maximal nest in \( \mathcal{F} \) (i.e., no other nest in \( \mathcal{F} \) contains \( \mathcal{N} \)). Hence \( F_{\text{min}} = \cap_{F \in \mathcal{N}} F \in \mathcal{F} \) is a minimal element in \( \mathcal{F} \) contained in \( F \), i.e. there is no other \( F \in \mathcal{F} \) contained in \( F_{\text{min}} \) as a proper subset. Likewise, \( F_{\text{max}} = \cup_{F \in \mathcal{N}} F \in \mathcal{F} \) is a maximal element in \( \mathcal{F} \) containing \( F \). The proof for zero footprints is similar.

These concepts are perhaps best illustrated for the case where the source measures are point masses.
Proposition 19 Let $\mathcal{S}$ be the class of point masses on $T \times V$, $k > 0$, $m = 1$ and $c_{\lim} > 0$. Then $Z = \{c^* \geq k\}$ is a maximal posterior $(S^*, c_{\lim}, S, k/c_{\lim})$–zero footprint, and any singleton set $F = \{(s, y)\} \subset \{c^* \geq k\}$ is a minimal posterior $(S^*, c_{\lim}, S, k/c_{\lim})$–footprint.

**Proof.** $Z$ is a posterior $(S^*, c_{\lim}, S, c_{\lim}/k)$–zero footprint by Proposition 14 and $Z$ cannot be extended by some point $(s, y)$ with $c^*(s, y) < k$, since then we would have $(k/c_{\lim} \delta_{s,y}), c^*) = c_{\lim} c^*(s, y)/k < c_{\lim}$, and hence there would be a point mass $S$ with slightly larger mass than $c_{\lim}/k$ which would still fulfill the bound $(S, c^*) < c_{\lim}$. $F$ is a posterior $(S^*, c_{\lim}, S, c_{\lim}/k)$–footprint by Proposition 13 and is clearly minimal since it is a singleton set.

\section{Footprint analysis, composite footprints}

Posterior and prior footprints as well as posterior and prior zero footprints carries (spatio-temporal) information about the source measure and the measurements. These footprints may have been calculated for subsets of measurements separately which then gives rise to the question of whether these pieces of information can be combined to give a more complete picture? Let us begin by studying finite unions and finite intersections of footprints:

Proposition 20 If $F_j$ are posterior $(S^*_j, c_{\lim,j}, S, M_{\lim,j})$–footprints for $j = 1, \ldots, m$ and $F = \bigcup_j F_j$ then $F$ is a posterior $(S^*, c_{\lim}, S, \max_j M_{\lim,j})$–footprint.

**Proof.** If $\text{sign}(c_{\lim,j}) \left(\langle S, c^*_j \rangle - |c_{\lim,j}|\right) \geq 0$ then $S(F_j) \geq M_{\lim,j}$, and $S(F) \geq S(F_j)$ for all $j$, so $S(F) \geq \max_j M_{\lim,j}$. ■

Proposition 21 If $Z_j$ are posterior $(S^*_j, c_{\lim,j}, S, M_{\lim,j})$–zero footprints for $j = 1, \ldots, m$ and $Z = \bigcup_j Z_j$ then $Z$ is a posterior $(S^*, c_{\lim}, S, \sum_j M_{\lim,j})$–zero footprint.

**Proof.** If $\text{sign}(c_{\lim,j}) \left(\langle S, c^*_j \rangle - |c_{\lim,j}|\right) \geq 0$ then $S(Z_j) < M_{\lim,j}$, so $S(Z) \leq \sum_j S(F_j) < \sum_j M_{\lim,j}$. ■

Proposition 22 If $F_j$ are prior $(S^*_j, c_{\lim,j}, S, M_{\lim,j})$–footprints for $j = 1, \ldots, m$ and $F = \bigcap_j F_j$ then $F$ is a prior $(S^*, c_{\lim}, S, \max_j M_{\lim,j})$–footprint.

**Proof.** If $S(F) \geq \max_j M_{\lim,j}$ then $S(F_j) \geq M_{\lim,j}$ for all $j$, and hence $\text{sign}(c_{\lim,j}) \left(\langle S, c^*_j \rangle - |c_{\lim,j}|\right) \geq 0$ for all $j$. ■
**Proposition 23** If $Z_j$ are prior $(S^*, c_{lim,j}, S, M_{lim,j})$-zero footprints for $j = 1, ..., m$ and $Z = \bigcup_j Z_j$ then $Z$ is a prior $(S^*, c_{lim}, S, \min_j M_{lim,j})$-zero footprint.

**Proof.** If $S(Z) < \min_j M_{lim,j}$ then $S(Z_j) < M_{lim,j}$ for all $j$, and hence
\[
\text{sign} \left( c_{lim,j} \left( \langle S, c^*_j \rangle - |c_{lim,j}| \right) \right) \geq 0 \text{ for all } j. \]

Concerning the set difference between a footprint and a zero footprint, we have

**Proposition 24** If $F$ is a $(S^*_F, c_{F,lim}, S, M_{F,lim})$-posterior footprint and $Z$ is a $(S^*_Z, c_{Z,lim}, S, M_{Z,lim})$-posterior footprint, then $F \setminus Z$ is a $(S^*_F, \langle S, c^*_Z \rangle - |c_{Z,lim}|, S, M_{F,lim} - M_{Z,lim})$-posterior footprint.

**Proof.** If $\text{sign} \left( c_{F,lim,j} \left( \langle S, c^*_{F,j} \rangle - |c_{F,lim,j}| \right) \right) \geq 0$ and $\text{sign} \left( c_{Z,lim,j} \left( \langle S, c^*_{Z,j} \rangle - |c_{Z,lim,j}| \right) \right) \geq 0$ for all applicable $j$, then $S(F) \geq M_{F,lim}$ and $S(Z) < M_{Z,lim}$, so $S(F) \setminus S(Z) = S(F) - S(F \cap Z) \geq (S(F) - S(Z)) > M_{F,lim} - M_{Z,lim}$. ■

The following theorem shows the information that can be obtained from level sets.

**Theorem 25** Assume that $k_j > 0$ and $0 \leq \beta_j \leq \alpha_j$ for $j = 1, ..., m$,
\[
S = \bigcap_{j=1}^m \{ S \in M^+ : \beta_j S \{ c^*_j \geq k_j \} \leq S \{ c^*_j < k_j \} \leq \alpha_j S \{ c^*_j \geq k_j \} \}
\]
and
\[
M_{lim,j} = \max \left( \frac{c_{lim,j}}{k_j \alpha_j + \sup c^*_j}, \frac{-c_{lim,j}}{k_j + \beta_j \inf c^*_j} \right)
\]
Then
\[
Z = \bigcup_{c_{lim,j} < 0} \{ c^*_j \geq k_j \}
\]
is a posterior $(S^*, c_{lim, S}, M_{Z,lim})$-zero footprint with
\[
M_{Z,lim} = \sum_{c_{lim,j} < 0} M_{lim,j} = \sum_{c_{lim,j} < 0} \frac{-c_{lim,j}}{k_j + \beta_j \inf c^*_j}
\]
and
\[
F = \bigcup_{c_{lim,j} > 0} \{ c^*_j \geq k_j \}
\]
is a posterior $(S^*, c_{lim, S}, M_{F,lim})$-footprint with
\[
M_{F,lim} = \max_{c_{lim,j} > 0} M_{lim,j} = \max_{c_{lim,j} > 0} \frac{c_{lim,j}}{k_j \alpha_j + \sup c^*_j}
\]
Finally, \( F \setminus Z \) is a posterior \((S^*, c_{\lim}, S, M_{\lim})\)-footprint with

\[
M_{\lim} = M_{F,\lim} - M_{Z,\lim} = \max_{c_{\lim,j} > 0} \frac{c_{\lim,j}}{k_j \alpha_j} + \sup_{c_{\lim,j} < 0} c_{\lim,j} + \sum_{c_{\lim,j} < 0} \frac{c_{\lim,j}}{k_j + \beta_j \inf c_{\lim,j}}
\]

**Proof.** If \( c_{\lim,j} < 0 \), the set \( \{ c_j^* \geq k_j \} \) is a posterior \((S_j^*, c_{\lim,j}, S, M_{\lim,j})\)-zero footprint by Proposition 14, and hence \( Z \) is a posterior \((S^*, c_{\lim}, S, M_{Z,\lim})\)-zero footprint by Proposition 21. Moreover, if \( c_{\lim,j} > 0 \) then the set \( \{ c_j^* \geq k_j \} \) is a posterior \((S_j^*, c_{\lim,j}, S, M_{\lim,j})\)-footprint by Proposition 13, so \( F \) is a \((S^*, c_{\lim}, S, M_{F,\lim})\)-footprint by Proposition 20. Finally, \( F \setminus Z \) is a \((S^*, c_{\lim}, S, M_{\lim})\)-footprint by Proposition 24. ■

## 6 Conclusion

Using the measure theoretic framework introduced in [1] we have provided rigorously defined the concept of footprints. Indeed, we have defined posterior footprints, posterior zero footprints, prior footprints and prior zero footprints. These footprints are all defined as spatio-temporal domains. Based on the definitions we presented some basic properties of the footprints, like the pairwise occurrence of prior/posterior footprints/zero footprints, and maximal and minimal footprints. We then studied how the information contents in single footprints can be synthesised by taking finite unions and intersections of footprints. The main result, Theorem 25, shows how the posterior zero footprint and posterior footprint are related to level lines of the adjoint concentration fields \( c_j^* \). Having adjoint concentration fields \( c_j^* \) is a common starting point of many methods of finding solutions to inverse problems. Using Theorem 25 allows us to immediately conclude in which part of the spatio-temporal domain we can expect the source measure to have most of its (effective) weight (the posterior footprint), to have least of its (effective) weight (the posterior zero footprint), and how to combine these footprints in an attempt to further limit the spatio-temporal domain where most of the (effective) weight of the source is located (the set difference of the posterior footprint and the posterior zero footprint). We believe that this fast, albeit rough, estimate of the source measure’s spatio-temporal support will be very useful in decision support systems that aid blue light forces when handling CBRN events. Theorem 25 gives a first idea of what the hazard area looks like, information that may be very desirable while the more sophisticated inverse methods are busy calculating more refined hazard areas and source estimates.
References

[1] Brännström, Niklas and Persson, Leif Å.: A measure theoretic approach to linear inverse atmospheric dispersion problems, Preprint, http://arxiv.org/abs/1305.6906??, June 2013, revised Sept 2014, submitted.

[2] Kelly, John L.: General Topology. Van Nostrand, 1955.

[3] Cai, X., Leclerc, M. L. (2007) Forward-in-time and Backward-in-time Dispersion in the Convective Boundary Layer: the Concentration Footprint. Boundary-Layer Meteorology, 123, 201-218.

[4] Flesch, T. K., Wilson, J. E., Yee, E. (1995) Backward-Time Lagrangian Stochastic Dispersion Models and Their Application to Estimate Gaseous Emissions. Journal of Applied Meteorology, 34, 1320-1332.

[5] Franklin, J. N. (1970) Well-Posed Stochastic Extensions of Ill-Posed Linear Problems. Journal of mathematical analysis and applications, 31, 682-716.

[6] Keats, A., Yee, E., Lien, F.-S. (2007) Bayesian inference for source determination with applications to a complex urban environment. Atmospheric Environment, 41, 465-479.

[7] Kljun, N, Rotach, M. W., Schmid H. P. (2002) A three-dimensional backward lagrangian footprint model for a wide range of boundary-layer stratifications. Boundary-Layer Meteorology, 103, 205-226.

[8] Marchuk, G.I. (1986) Mathematical models in environmental problems. Studies in mathematics and its applications, vol 16.

[9] Pasquill, F. (1972) Some aspects of boundary layer description. Quart. J. Roy. Meteorol. Soc. 98, 469-494.

[10] Pudykiewicz, J. A. (1998) Application of adjoint tracer transport equations for evaluating source parameters. Atmospheric Environment, 32, 3039-3050.

[11] Rannik, Ü, Aubinet, M., Kurbamuradov, O., Sabelfeld, K. K., Markkanen, T., Vesala T. (2000) Footprint analysis for measurements over a heterogeneous forest, Boundary-Layer Meteorology, 97, 137-166.
[12] Robertson, L. (2004) Extended back-trajectories by means of adjoint equations. RMK No. 105, Swedish Meteorological and Hydrological Institute.

[13] Rudin W (1966) Real and Complex Analysis, McGraw-Hill.

[14] Schmid, H.P. (1994) Source areas for scalars and scalar fluxes. Boundary-Layer Meteorology, 67, 293-318.

[15] Schmid, H.P. (2002) Footprint modeling for vegetation atmosphere exchange studies: a review and perspective. Agricultural and Forest Meteorology, 113, 159-183.

[16] Stuart, A. M. (2010) Inverse problems: a Bayesian perspective. Acta Numerica, 19.

[17] Yee, E. (2007) Bayesian Inversion of Concentration Data for an Unknown Number of Contaminant Sources. Technical Report DRDC Suffield TR 2007-085.

[18] Yee, E., Flesch, T. K. (2010) Inference of emission rates from multiple sources using Bayesian probability theory. Journal of Environmental Monitoring, 12, 622-634.

[19] Yee, E. (2012) Probability Theory as Logic: Data Assimilation for Multiple Source Reconstruction. Pure and Applied Geophysics, 169, 499-517.