Integrable systems on semidirect product Lie groups

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Abstract
We study integrable systems on the semidirect product of a Lie group and its Lie algebra as the representation space of the adjoint action. Regarding the tangent bundle of a Lie group as phase space endowed with this semidirect product Lie group structure, we construct a class of symplectic submanifolds equipped with a Dirac bracket on which integrable systems (in the Adler–Kostant–Symes sense) are naturally built through collective dynamics. In doing so, we address other issues such as factorization, Poisson–Lie structures and dressing actions. We show that the procedure becomes recursive for some particular Hamilton functions, giving rise to a tower of nested integrable systems.

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1. Introduction

Integrable Hamiltonian systems find a natural setting in the realm of Lie algebras and Lie groups, where their equations of motion are realized as Lax pairs. This has become a standard framework for many integrable systems following the seminal work by Arnold [3], with the equations of motion associated with the rigid body and incompressible fluid encoded in this framework. This setting fits perfectly for systems which are strongly symmetric, in such a way that the configuration space can be identified with the symmetry group.

Systems with a Lie group as the configuration space have the cotangent bundle of this Lie group as the phase space, which in turn can be identified with the Cartesian product of the
group itself and the dual of its Lie algebra. They are symplectic manifolds, meaning that the Poisson brackets are nondegenerate, enjoying many nice properties related to the symmetry issues [1, 14]. The restriction of this bracket to functions on the dual of the Lie algebra produces the Lie–Poisson bracket, whose symplectic leaves coincide with coadjoint orbits. When the Lie algebra is supplied with a nondegenerate Ad-invariant bilinear form, one can translate the Poisson structure to the Lie algebra and the equations of motion turn into equations in Lax pair form.

It is surprising that when symmetries are broken, one may still find a Lie groups–Lie algebras setting encoding these kinds of systems by considering semidirect products of Lie groups [24]. This is the case when the potential energy is included as, for instance, in the rigid body system and its N-dimensional analogues [17–19]. The general setting is such that reduction of the cotangent bundle of a Lie group by the action of some Lie subgroup makes semidirect products arise. A deep understanding of this connection began to appear in [8], and it was fully clarified in [15], where the heavy top and compressible flow are presented as motivating examples. There are many other dynamical systems falling in this scheme such as the Kirchhoff equation for the motion of a rigid body in an ideal incompressible fluid moving under a potential and at rest at infinity, and the Leggett equation for the magnetic moment in the low temperature phases of $^3$He [7, 16], where other issues for these kinds of systems are analyzed. Since then, further developments and applications have been carried out, widening the involvement of semidirect product Lie groups in dynamical systems [5, 9, 10, 13].

In this work we investigate integrable systems on semidirect product Lie groups from the point of view of the Adler–Kostant–Symes (AKS) [2, 11, 23] and Reyman and Semenov-Tian-Shansky [20, 21] theory. In this context, nontrivial integrable systems are generated by a restriction procedure: given a Lie algebra $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, with $\mathfrak{g}_+$, $\mathfrak{g}_-$ Lie subalgebras of $\mathfrak{g}$, the restriction of the set of $Ad$-invariant functions on a coadjoint orbit of $\mathfrak{g}$ to the coadjoint orbit of one of its components, for instance $\mathfrak{g}_+$, gives rise to a nontrivial set of Poisson commuting functions, so the Arnold–Liouville theorem holds. These ideas are extended to cotangent bundles of the Lie groups $G$, $G_+$, $G_-$ associated with $\mathfrak{g}$, $\mathfrak{g}_+$, $\mathfrak{g}_-$, and the solution of the equations of motion on the cotangent bundle of the factor $G_+$, for instance, arises as the $G_+$-factor of an exponential curve in $G$. The main difference of the construction developed in this work with the above mentioned integrable systems on semidirect product Lie groups is that integrability is not necessarily tied to geodesic flows, since it is obtained by a restriction procedure which breaks the Ad-invariance.

We consider, as the general framework, the semidirect product of a Lie group with the underlying vector space of its Lie algebra regarded as the representation space of the adjoint action and, for dynamical issues, the Dirac scheme as presented in [4], focusing our attention on a family of symplectic submanifolds in $G \times \mathfrak{g}$ defined as the level set of the projection map $\Psi : G \times \mathfrak{g} \longrightarrow G_+ \times \mathfrak{g}_-$.

We study Hamiltonian systems on the tangent bundle of a semidirect product Lie group and work out the structure of the Hamilton equations of motion by writing them out in terms of its original components. In particular, the collective dynamics and factorization will connect our construction with integrable systems. Because the factorization arises from a Poisson–Lie group, dressing vector fields are strongly involved in the dynamics, making it relevant to analyze the induced Poisson–Lie structure on the semidirect products and the associated dressing actions.

It must be stressed that, although the AKS theorem does not require $\mathfrak{g}$ to be semisimple, it is frequently assumed to be so, in order to simplify the discussion related to the issue; additionally, this requirement plays a fundamental role in linking these kinds of systems with Lax pair formulation. Thus, when working with semidirect product groups, it is necessary to
provide a pairing between the Lie algebra and its dual, namely, a replacement for the Killing form. In the present work a proposal for this pairing is given, showing that the system is integrable via a factorization procedure. In view of the fact that taking the semidirect product with the Lie algebra is an operation that can be applied to any Lie group, and that the same can be said for the proposed pairing, this yields a scheme that can be carried out recursively. The AKS procedure applied to semidirect product Lie groups and the recursive construction are the main results of this work.

Nevertheless, the construction produces a set of equations of motion resembling those of the heavy top on just substituting the adjoint action by the dressing one, and they can be solved by factorization following the AKS ideas. Also, as we said above, for some particular collective Hamiltonians this construction can be promoted to a recursive procedure on iterated semidirect products, leading to a tower of integrable systems. Some issues of this developments are shown in an explicit example built on $SL(2, \mathbb{C})$ and its Iwasawa decomposition.

The work is ordered as follows. In section 2 we fix the algebraic tools of the problem, and in section 3 we describe the phase spaces involved, with the corresponding Dirac brackets, working out symmetries, the additional Poisson–Lie structure and the associated dressing actions. In section 4 we focus on the dynamical systems, collective dynamics and integrability by factorization, describing also the nested equation of motion inherited from the nested actions. In section 5, we build up a tower of integrable systems for a particular collective Hamiltonian, and finally, in section 6, we present an example on $SL(2, \mathbb{C})$ showing the explicit solution obtained by factorization.

2. Semidirect products with the adjoint representation

Let $H_1$ be a Lie group and $\mathfrak{h}_1$ its Lie algebra, which we assume to be equipped with a nondegenerate symmetric bilinear form

$$\kappa_1 : \mathfrak{h}_1 \otimes \mathfrak{h}_1 \longrightarrow \mathbb{R}.$$  

We consider, associated with them, the semidirect product $H_2 = H_1 \ltimes \mathfrak{h}_1$, where the vector space $\mathfrak{h}_1$, with the trivial Lie algebra structure, is regarded as the representation space for the adjoint action of $H_1$ in the right action structure of the semidirect product, so the Lie group structure for $(a, X), (b, Y) \in H_2$ is

$$(a, X) \bullet (b, Y) = (ab, \text{Ad}^1_{b^{-1}} X + Y),$$  

where $\text{Ad}^1$ stands for the adjoint action of $H_1$ on its Lie algebra $\mathfrak{h}_1$.

By using the right and left translations lifted to $TH_2$, we have that the right and left invariant vector fields on $H_2 = H_1 \ltimes \mathfrak{h}_1$ at a point $(a, V)$ can be defined as

$$\{(X, U)^R\big|_{(a, V)} := (R^*_{(a, V)})(X, U) = (X a, \text{Ad}^1_{a^{-1}} U)\big|_{(a, V)},$$

$$\{(X, U)^L\big|_{(a, V)} := (L^*_{(a, V)})(X, U) = (a X, [V, X] + U)\big|_{(a, V)},$$

where $(X, U) \in h_2 = \mathfrak{h}_1 \oplus \mathfrak{h}_1$. The exponential map $\text{Exp}^* : h_2 \longrightarrow H_2$ is

$$\text{Exp}^*(t(X, Y)) = \left(e^{tX} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} t^n (\text{ad}^1_X)^{n-1} Y\right)$$  

and the adjoint action of $H_2$ on $h_2$ is given by

$$\text{Ad}^2_{(a, Z)}(X, Y) = (\text{Ad}^1_a X, \text{Ad}^1_a ([Z, X]_1 + Y)).$$  

The Lie algebra structure on $h_2 = \mathfrak{h}_1 \ltimes \mathfrak{h}_1$ is

$$[\{Y, V\}, \{X, Z\}]_2 = \text{ad}^2_{\{X, V\}}(X, Z) = ([Y, X], [Y, Z]_1 + [V, X],_1).$$
which underlies the Lie group semidirect product structure of \( H_2 = H_1 \ltimes h_1 \). The coadjoint action becomes

\[
\text{ad}^2_{(x,y)}(\eta, \xi) = (\text{ad}^1_{x} \eta + \text{ad}^1_{y} \xi, \text{ad}^2_{x} \xi) .
\]

### 2.1. Bilinear forms and factorization

We now consider the Lie algebra \( h_1 \) equipped with the symmetric nondegenerate Ad\(^1\)-invariant bilinear form \( k_1 : h_1 \otimes h_1 \rightarrow \mathbb{R} \), and we define on \( h_2 \) the nondegenerate symmetric bilinear form \( k_2 : h_2 \otimes h_2 \rightarrow \mathbb{R} \) as

\[
k_2((X, U), (Y, V)) := k_1(X, V) + k_1(Y, U)
\]

which is also Ad\(^2\)-invariant.

Let us assume that there exists a factorization \( H_1 = H^+_1 \ltimes H^-_1 \) where \( H^+_1 \) and \( H^-_1 \) are Lie subgroups of \( H_1 \), implying that every \( h_1 \in H_1 \) can be written as

\[
h_1 = h^+_1 h^-_1
\]

for some \( h^+_1 \in H^+_1 \) and \( h^-_1 \in H^-_1 \). Consequently, \( h_1 = h^+_1 \oplus h^-_1 \) with \( h^+_1 \) and \( h^-_1 \) being Lie subalgebras of \( h_1 \). The factorization at this level can be seen in the crossed Lie bracket

\[
[X^-_1, X^+_1] = (X^+_1)^{X^-_1} + (X^-_1)^{X^+_1},
\]

where \( X^+_1, (X^+_1)^{X^-_1} \in h^+_1 \) and \( X^-_1, (X^-_1)^{X^+_1} \in h^-_1 \). Also we assume that \( h^+_1, h^-_1 \) are isotropic subspaces in relation with \( k_1 : h_1 \otimes h_1 \rightarrow \mathbb{R} \), such that \( (h_1, h^+_1, h^-_1) \) compose a Manin triple. In this way, each Lie algebra is also a Lie bialgebra and the associated groups are Poisson–Lie groups.

The factorization of \( H_1 \) induces the factorization \( H_2 = H^+_2 \ltimes H^-_2 \), where \( H^+_2 = H^+_1 \ltimes h^+_1 \), such that any element \((h_1, X_1) \in H_2 \) can be written as

\[
(h_1, Y_1) = (h^+_1, X^+_1) \ltimes (h^-_1, X^-_1) = (h^+_1 h^-_1, \text{Ad}^1_{(h^-_1)} X^+_1 + X^-_1)
\]

with

\[
\begin{align*}
\text{h}_1 &= h^+_1 h^-_1 \\
X^+_1 &= \Pi_+ (\text{Ad}^1_{h^-_1} Y_1) \\
X^-_1 &= \text{Ad}^1_{(h^-_1)^{-1}} \Pi_- \text{Ad}^1_{h^-_1} Y_1.
\end{align*}
\]

**Remark.** Observe that the above factorization is fully determined by the factorization in the previous step: let \( h_2 = (h_1, Y_1) \in H_2 \), \( h_1 \in H_1 \) and \( Y_1 \in h_1 \); then

\[
h_2 = (h^+_1, \Pi_+ \text{Ad}^1_{h^-_1} Y_1) \cdot (h^-_1, \text{Ad}^1_{(h^-_1)^{-1}} \Pi_- \text{Ad}^1_{h^-_1} Y_1)
\]

where \( h^+_1 \) and \( h^-_1 \) are such that \( h_1 = h^+_1 h^-_1 \).

The Lie algebra \( h_2 \) decomposes into a direct sum as \( h_2 = h^+_2 \oplus h^-_2 \), with \( h^+_2 = h^+_1 \ltimes h^+_1 \) being Lie subalgebras of \( h_2 \) such that for \((X, V) \in h_2 \),

\[
(X, V) = (X^+_2, V^+_2) + (X^-_2, V^-_2).
\]

Moreover, \( h^+_2 \) and \( h^-_2 \) are isotropic subspaces in relation to \( k_2 \), implying the bijections \( \gamma (h^+_2) = (h^+_2)^\text{\#} \) and the identifications \( (h^+_2)^\text{\#} = h^+_2 \). For the sake of simplicity, we shall use \( \gamma \) to generically denote the bijections induced by the bilinear forms \( k_i \), for any \( i \); no confusion will arise from this ambiguity since it becomes clear which map must be used from the labels of its arguments.
Let us quote some useful relations holding for every Lie bialgebra and Poisson–Lie group. They can be obtained from the interplay of the relation (2.6) and the bilinear forms $k_i$:

$$X_i^+ = -\gamma^{-1}(a d_{h_i}^+ \gamma(X_+)),$$

$$X_i^- = \gamma^{-1}(A d_{h_i}^- \gamma(X_-))$$

(2.8)

and

$$A d_{h_i}^- Y_+ = h_i^{-1} h_i^+ \gamma + \gamma^{-1}(A d_{h_i}^+ \gamma(X_-)).$$

(2.9)

We remark that $A d$ denotes the adjoint action of the groups $H = H^+ H^-$, while $A d$ denotes the corresponding adjoint actions of the factors $H^+$ or $H^-$. 

2.2. Poisson–Lie structure on $H_2^+$

The Lie algebras $h_i^+$ and $h_i^-$, besides the bilinear form $k_i : h_i \otimes h_i \rightarrow \mathbb{R}$, for $i = 1, 2$, constitute Manin triples $(h_i, h_i^+, h_i^-)$, so each Lie algebra is indeed a Lie bialgebra, and the associated Lie groups are Poisson–Lie groups. For instance, on $H_2^+$ the Poisson–Lie structure $\pi^+_2$ is defined as (see [12])

$$\langle\gamma(X^-_2), V^-_2 \rangle = k_2 (\Pi_+ a d_{h_i^+}^2 \pi_1^+(h_i^+), \Pi_+ a d_{h_i^-}^2 \pi_1^-(h_i^-)).$$

Having in mind the expression (2.3), and the Poisson–Lie structure in $h_i^+$,

$$\langle\gamma(X^-_i), V^-_i \rangle (h_i^+)^{-1} \otimes \gamma(Y^-_i), (h_i^-)^{-1}, \pi^+_i (h_i^-) \rangle \equiv k_i (\Pi_+ a d_{h_i^+}^2 \pi_1^+(h_i^+), \Pi_+ a d_{h_i^-}^2 \pi_1^-(h_i^-)),$$

we may write

$$k_2 ((h_i^+)^{-1} \otimes (h_i^-)^{-1}, (R_{h_i^+} h_i^-)^{-1} \pi^+ \pi^- (h_i^+), (h_i^-)^{-1}, (R_{h_i^-} h_i^+)^{-1} \pi^+ \pi^- (h_i^-))$$

$$= k_1 (h_i^+)^{-1} \otimes (h_i^-)^{-1}, (R_{h_i^+} h_i^-)^{-1} \pi^+ \pi^- (h_i^+), (h_i^-)^{-1}, (R_{h_i^-} h_i^+)^{-1} \pi^+ \pi^- (h_i^-))$$

(2.10)

where $k_i (A \otimes B, C \otimes D) = k_i (A, C) k_i (B, D)$. In terms of the components in $h_i^+ \oplus h_i^-$, this turns into

$$k_2 ((h_i^+)^{-1} \otimes (h_i^-)^{-1}, (R_{h_i^+} h_i^-)^{-1} \pi^+ \pi^- (h_i^+), (h_i^-)^{-1}, (R_{h_i^-} h_i^+)^{-1} \pi^+ \pi^- (h_i^-))$$

$$= k_1 (h_i^-)^{-1} \otimes (h_i^+)^{-1}, (R_{h_i^-} h_i^+)^{-1} \pi^+ \pi^- (h_i^-), (h_i^+)^{-1}, (R_{h_i^+} h_i^-)^{-1} \pi^+ \pi^- (h_i^+))$$

$$+ k_1 (h_i^-)^{-1} \otimes (h_i^+)^{-1}, (R_{h_i^-} h_i^+)^{-1} \pi^+ \pi^- (h_i^-), (h_i^+)^{-1}, (R_{h_i^+} h_i^-)^{-1} \pi^+ \pi^- (h_i^+))$$

$$- k_1 (h_i^-)^{-1} \otimes (h_i^+)^{-1}, (R_{h_i^-} h_i^+)^{-1} \pi^+ \pi^- (h_i^-), (h_i^+)^{-1}, (R_{h_i^+} h_i^-)^{-1} \pi^+ \pi^- (h_i^+))$$

The Poisson–Lie bracket for a couple of functions $F, G$ on $H_2^+$ is

$$\{ F, G \}^+_2 (h_i^+) = \langle d F \otimes d G, \pi^+_2 (h_i^+) \rangle.$$  (2.11)

In order to take account of the structure of semidirect product of $H_2^+$, we write its differentials as $d F = (d F, \delta F) \in T^* H_2^+ \oplus h_i^+$. Due to the relation between $k_i$ and $k_c$, (2.4), we have that the bijections $\gamma_i : h_i \rightarrow (h_i)^*$, $i = 1, 2$, are related as

$$\gamma_2 (X_1, X_2) = (\gamma_1 (X_2), \gamma_1 (X_1)).$$

Also, we need the right translations on the cotangent bundle:

$$R_{h_i}^+ (\alpha, \beta) = (R_{h_i} \alpha, A d_{h_i}^+ \beta)$$
for \((\alpha, \beta) \in T^*_{\text{orb}(\mathbf{H})} H_2\). So, comparing (2.11) with the expression (2.10), we get
\[
\begin{aligned}
[dF, dG]_2^\gamma (h_1^+, Z_1^+) &= \langle \delta \gamma, ((L_{(h_1^+)_{-1}})_* \otimes \text{id}) \pi_1^+(h_1^+) \rangle \\
&+ \langle \delta \gamma, (\text{id} \otimes (L_{(h_1^+)_{-1}})_*) \pi_1^+(h_1^+) \rangle \\
&+ \langle \delta \gamma, (\text{ad}_{Z_1^+})^{\otimes 2} (L_{(h_1^+)_{-1}})_* \otimes \pi_1^+(h_1^+) \rangle.
\end{aligned}
\]
This expression suggests writing the Poisson–Lie bivector in a block matrix form on \(h_1^+ \oplus h_1^+\), as
\[
\pi_1^+(h_1^+, Z_1^+) = \begin{pmatrix}
0 & (\text{id} \otimes (L_{(h_1^+)_{-1}})_*) \pi_1^+(h_1^+) \\
(id \otimes (L_{(h_1^+)_{-1}})_*) \pi_1^+(h_1^+) & (\text{ad}_{Z_1^+})^{\otimes 2} (L_{(h_1^+)_{-1}})_* \otimes \pi_1^+(h_1^+)
\end{pmatrix}.
\tag{2.12}
\]
Let us introduce the map \(\pi_2^{2R} : H_2^+ \longrightarrow h_2^+ \oplus h_2^+\) by composing the PL bivector, regarded as a section of \(T \otimes 2 H_2^+\), with the right translation to the neutral element
\[
\pi_2^{2R}(h_1^+, Z_1^+) = (R_{(h_1^+)_{-1}} Z_1^+ ) \otimes \pi_2^+(h_1^+).
\]
The differential of this map at the neutral element, namely \(\delta := (\pi_2^{2R})_{\text{er}} : h_2^+ \longrightarrow h_2^+ \oplus h_2^+\), is a linear map that supplies \((h_2^+)^*\) with the Lie algebra structure
\[
([\eta_2^+ \otimes \xi_2^+, X_1^+] = \langle \eta_2^+ \otimes \xi_2^+, \delta(X_1^+) \rangle
\]
for \(\eta_2^+, \xi_2^+ \in (h_2^+)^*\) and \(X_2^+ \in h_2^+\). The Jacobi property of the Lie bracket in \((h_2^+)^*\) is guaranteed by requiring that \(\delta\) be a cocyle. Then using the expression (2.12) we get
\[
\pi_2^{2R}(h_1^+, Z_1^+) = \begin{pmatrix}
0 & I \\
I & (\text{ad}_{Z_1^+})^{\otimes 2}
\end{pmatrix} \pi_2^+(h_1^+).
\]
An important case happens whenever \(\delta\) is a coboundary, giving rise to the r-matrix approach to integrable system [20]. In this framework, we may see that in general a coboundary at the level 1 fails to produce a coboundary at level 2.

### 2.3. Dressing vectors

We can think of the Poisson–Lie bivector on \(H_2^+\) as providing a linear map from \(T^* H_2^+ \) to \(TH_2^+\). It is well known that this map defines a Lie algebra antihomomorphism from \((h_2^+)^*\) to \(X(H_2^+)^*\), the Lie algebra of vector fields on \(H_2^+\). It can be translated to a Lie algebra antihomomorphism from \(h_2^+\) to \(X(H_2^+)^*\) by regarding the bijection \(\gamma : h_2^+ \longrightarrow (h_2^+)^*\) in such a way that, for \(h_1^+ \in H_2^+\) and \(X_1^- \in h_2^+\), the linear map
\[
(X_1^-)^\gamma := ((R_{(h_1^+)_{-1}})_* X_1^- = (\text{id} \otimes (R_{(h_1^+)_{-1}})_* \gamma(X_1^-)) \pi_2^+(h_1^+)
\]
is a Lie algebra antihomomorphism. These vector fields are the dressing infinitesimal generators associated with the factorization \(H_2 = H_2^+ H_2^-\), and the integral submanifold of this distribution coincides with the symplectic leaves of the Poisson–Lie structure (2.12) [22]. The dressing vector right translated to the tangent space at the neutral element becomes
\[
(R_{(h_1^+)_{-1}})_* (Y_1^-, W_1^-) h_2^+ (h_1^+, Z_1^+) = \langle (\text{id} \otimes \gamma) (Y_1^-, W_1^-) \rangle (R_{(h_1^+)_{-1}} Z_1^+) \otimes \pi_2^+(h_1^+, Z_1^+).
\tag{2.13}
\]
Let us come back to expression (2.10) and decompose it in the summands of \(h_1^+ \oplus h_1^+\). Since \(H_2^+\) is also a Poisson–Lie group, we may write the above expression in terms of the corresponding dressing vectors:
\[
(\text{id} \otimes (R_{(h_1^+)_{-1}})_* \gamma(Y_1^-)) \pi_1^+(h_1^+) = (h_1^+)^\gamma.
\]
therefore, the right-hand side in (2.13) reduces to

\[
(id \otimes \gamma(Y^+_1, W^-_1))(R(h^+_1, Z^+_1)) = \left((h^+_1)^Y, (h^+_1)^W, (h^+_1)^\gamma, (h^+_1)^{\gamma^{-1}}\right).
\]

To obtain the dressing infinitesimal generators, this element of \(\mathfrak{h}_1^+ \) has to be right translated to \((h^+_1, Z^+_1) \in H^+_1\); thus we have the assignment \((Y^+_1, W^-_1) \mapsto (Y^+_1, W^-_1)_{H^+_1} \in X(H^+_2)\)

\[
(Y^+_1, W^-_1)_{H^+_1}(h^+_1, Z^+_1) = \left((h^+_1)^Y, \Pi_+ (\text{Ad}_{(h^+_1)^Y})_W, [Y^+_1, Z^+_1]\right).
\]

(2.14)

Analogously, for the reciprocal action \((h^+_1, Z^+_1)^{H^+_1, X^-_1}\),

\[
(X^+_1, Y^-_1)_{H^+_1}(h^+_1, Z^+_1) = \left((h^+_1)^X, \text{Ad}_{(h^+_1)^X}, X^+_1 \in \text{Ad}_{(h^+_1)^X}, Z^+_1 \in \text{Ad}_{(h^+_1)^X}\right).
\]

Moreover, they are Lie algebra (anti)morphisms:

\[
[(X^-_1, Y^-_1), (X'_1, Y'_1)]_{H^+_1} = 0, [(X^-_1, Y^-_1), (X'_1, Y'_1)]_{H^+_2} = (X^+_1, Y^+_1), (X'_1, Y'_1)]_{H^+_1}.
\]

2.4. The Hamiltonian dressing action

Besides the Poisson–Lie structure on \(H^+_1 = H^+_1 \times \mathfrak{h}_1^+ \) described in (2.2), one may profit from the identification \(H^+_1 \times \mathfrak{h}_1^+ = T^* H^+_1\) by supplying \(H^+_1\) with a symplectic structure borrowed from the canonical one in \(T^* H^+_1\). In fact, let us assume that \(H_1\) is equipped with nondegenerate symmetric bilinear form \((\cdot, \cdot)\) such that its restriction to \(\mathfrak{h}_1^+\) is also nondegenerate, and let \(\sigma : \mathfrak{h}_1 \rightarrow \mathfrak{h}_1^+\) be the induced the linear bijection. Then, \(H^+_1\) is a symplectic manifold with the symplectic form \(\omega^+_1\) defined as

\[
\omega^+_1 = -\omega(X^+_1, (h^+_1)^{-1}w^+_1) = \omega(X^+_1, \omega^+_1, X^-_1) + \omega(X^+_1, (h^+_1)^{-1}w^+_1) = \omega(X^-_1, (h^+_1)^{-1}v^+_1, (h^+_1)^{-1}w^+_1)
\]

for \((h^+_1, Z^+_1) \in H^+_1 \times \mathfrak{h}_1^+\) and \((v^+_1, X^+_1), (w^+_1, Y^+_1) \in T(h^+_1, Z^+_1) \mathfrak{h}_1^+\).

We now seek for a Hamiltonian function associated with the infinitesimal generators of the dressing actions of \(H^+_2\) on \(H^+_1\), given in equation (2.14), relative to the above symplectic structure. In doing so, for a vector \((X^-_1, Y^-_1) \in \mathfrak{h}_1^+\), we split the action as the composition of actions of \((X^-_1, 0)\) followed by the action of \((0, Y^-_1)\).

From the expression of the infinitesimal generator (2.14) and using the relations (2.8), (2.9), we write

\[
(X^-_1, 0)_{H^+_1}(h^+_1, Z^+_1) = (h^+_1, \Pi_+ \text{Ad}_{(h^+_1)^Y} X^-_1, \Pi_+ \text{Ad}_{(h^+_1)^Y} X^-_1, Z^+_1).
\]

(2.15)

This vector field coincides with the lift of the infinitesimal generators of the dressing action of \(H^+_1\) on \(H^+_1\) to the tangent bundle \(T H^+_1\). These vector fields are Hamiltonian provided the bilinear form \((\cdot, \cdot)\) is also \(\text{Ad}^1\)-invariant and, in such a case, the Hamilton function \(\theta_{X^-}\) associated with \(X^-_1 \in \mathfrak{h}_1^+\) is

\[
\theta_{X^-} = (h^+_1, Z^+_1) = (Z^+_1, \Pi_+ \text{Ad}_{(h^+_1)^Y} X^-_1, \Pi_+ \text{Ad}_{(h^+_1)^Y} X^-_1, Z^+_1).
\]

where \(\Theta : H^+_1 \rightarrow (\mathfrak{h}_1^+)\) is the \(\text{Ad}^1\)-equivariant momentum map

\[
\Theta(h^+_1, Z^+_1) = \gamma(\Pi_+ \text{Ad}_{(h^+_1)^Y} X^-_1, (\sigma(Z^+_1))^{-1}).
\]
The remaining term in the infinitesimal generator (2.14) is
\[ (0, W_{1}^{-})_{H_1} (h_{+}, Z_{+}) = (0, \Pi_{+} \text{Ad}_{[h_{+}^{-1}]^{-1}} W_{1}^{-}); \]
observe that
\[ t_{(0, W_{1}^{-})_{H_1}} \omega_{1}^{+} = -(h_{1}^{+})^{-1} \sigma ((h_{1}^{+})^{-1} (h_{1}^{+}) W_{1}^{-}) \]
and the condition \( d t_{(0, W_{1}^{-})_{H_1}} \omega_{1}^{+} = 0 \) imposes strong restrictions on \( \sigma \) so, in general, the Lie derivative \( L_{(0, W_{1}^{-})_{H_1}} \omega_{1}^{+} \neq 0 \), meaning that the infinitesimal transformation induced by the vector field \((0, W_{1}^{-})_{H_1}\) is not a symplectomorphism. One may see that, related to this fact, the left action of \( H_1 \) on itself is in general not Hamiltonian relative to the symplectic form of \( H_1 \times h_1 \).

In the following sections we study some integrable systems on these phase spaces with flows described by a dressing vector field like (2.15).

3. Phase spaces in \( H_2 \)

We now consider \( H_2 \) as phase space equipped with the nondegenerate Poisson bracket inherited from the canonical symplectic form of \( T^* H_1 \) and, following the Dirac procedure developed in [4], we construct a class of phase subspaces of \( H_2 \) on which nontrivial integrable systems naturally arise.

The nondegenerate Poisson bracket that we consider here is obtained using an approach analogous to that in the previous subsection, by using the inner product \( \langle \cdot, \cdot \rangle_1 \) and the induced bijection \( \sigma : h_1 \longrightarrow h_1^* \), with its restrictions \( \sigma | h_1^\pm : h_1^\pm \longrightarrow h_1^{\pm*} \) being also linear bijections. Thus the symplectic structure in \( H_2 \) is
\[ \langle \omega_2, (v, X) \otimes (w, Y) \rangle_{(h_1, Z_1)} = -(X, h_1^{-1} w)_{1} + (Y, h_1^{-1} v)_{1} + (Z_1, [h_1^{-1} v, h_1^{-1} w])_{1} \]
(3.1)
for \((h_1, Z_1) \in H_1 \times h_1, (v, X), (w, Y) \in T_{(h_1, Z_1)} (H_1 \times h_1) \). Let \( F, \mathcal{H} \) be functions on \( H_1 \times h_1 \) and let us write their differential as \( dF = (dF, \delta F) \in T^* H_1 \otimes h_1^* \); then the associated Poisson bracket is
\[ [F, \mathcal{H}]_{2} (h_1, Z_1) = (dF, h \sigma^{-1} (\delta \mathcal{H})) - (d\mathcal{H}, h \sigma^{-1} (\delta F)) - (\sigma (Z_1), [\sigma^{-1} (\delta F), \sigma^{-1} (\delta \mathcal{H})]); \]
From this expression we get the Hamiltonian vector field of \( \mathcal{H} \):
\[ V_{\mathcal{H}} (h_1, Z_1) = (h_1 \sigma^{-1} (\delta \mathcal{H}), \sigma^{-1} (\text{ad} h_1^{*, -1} (\delta \mathcal{H}) \sigma (Z_1) - h d \mathcal{H})); \]
and the Poisson bracket \([F, \mathcal{H}]_{2}\) just means the Lie derivative of the function \( F \) along the vector field \( V_{\mathcal{H}} \).

The main idea of the approach to integrability that we use here is to obtain nontrivial integrable systems as the reduction of an almost trivial system defined in a phase space to some submanifold. It fits perfectly in the realm of the Dirac method for constrained systems since it produces Lie derivatives of functions on the whole phase space along the projection of the Hamiltonian vector fields on the tangent space of the constrained submanifold, giving rise to a representation of these vectors fields in terms of the geometrical data in the total phase space [6]. Integral curves of this projected vector field are the trajectories of the constrained Hamiltonian system. In the rest of the current section we adapt the approach developed in [4] to the framework of the semidirect product and factorization as introduced above.
3.1. Fibration of symplectic submanifolds in $H_2$

The starting point is the phase space $(H_2, [, ]_2)$, and we introduce the fibration

$$\Psi_2 : H_1 \times h_1 \longrightarrow H^+_1 \times h^+_1$$

$$(h_1, X_1) \longmapsto (h^+_1, X^+_1)$$

such that the fiber on $(h^+_1, X^+_1) \in H^+_1 \times h^+_1$ is described as

$$\mathcal{N}_2(h^+_1, X^+_1) = \Psi_2^{-1}(h^+_1, X^+_1) = \{(h^+_1 h_1, X^+_1 + X_1)/h^+_1 \in H^+_1, X^+_1 \in h^+_1\}.$$  

In particular, $\mathcal{N}_2(e, 0) = H^+_1 \times h^+_1$. Each fiber $\mathcal{N}_2(h^+_1, X^+_1)$ can be supplied with a nondegenerate Dirac bracket constructed following [4]. We shall use the linear bijection $\sigma : h_1 \longrightarrow h^+_1$ in order to translate those results to the current framework.

With the purpose of simplifying the notation we introduce the projectors $\mathcal{A}^k(h), k = 1, 2,$ defined as

$$\mathcal{A}^k(h) := \text{Ad}^k \cdot \Pi \cdot \text{Ad}^k$$

such that

$$\left\{ \begin{aligned}
\mathcal{A}^k(h)\mathcal{A}^k(h) &= \mathcal{A}^k(h) \\
\mathcal{A}^k(h)\mathcal{A}^k(h) &= 0 \\
\mathcal{A}^k(h) + \mathcal{A}^k(h) &= \text{Id}
\end{aligned} \right.$$  

which will be used in the following.

Therefore the Dirac bracket on $\mathcal{N}_2(h^+_1, X^+_1)$ for $\mathcal{F}, \mathcal{H} \in C^\infty(H_1 \times h_1)$ is

$$[\mathcal{F}, \mathcal{H}]_2^0(h_1, Z_1) = [h_1 \text{d}\mathcal{F}, \mathcal{A}_+^1(h^+_1)\sigma^{-1}(\delta\mathcal{H})] - [h_1 \text{d}\mathcal{H}, \mathcal{A}_+^1(h^+_1)\sigma^{-1}(\delta\mathcal{F})]$$

$$-\{\sigma(Z_1), [\mathcal{A}_+^1(h^+_1)\sigma^{-1}(\delta\mathcal{F}), \mathcal{A}_+^1(h^+_1)\sigma^{-1}(\delta\mathcal{H})]\},$$  

(3.2)

where we have denoted the differential of a function $\mathcal{F}$ as $\text{d}\mathcal{F} = (\text{d}\mathcal{F}, \delta\mathcal{F})$ for its components in $T^*_0(H_1 \times h_1) = T^*_0 H_1 \times h^+_1$. In the particular case $h^+_1 = e, Z^+_1 = 0$, we have $\mathcal{N}_2(e, 0) = H^+_2$ and the Dirac bracket reduces to

$$[\mathcal{F}, \mathcal{H}]_2^0(h^+_1, Z^+_1) = (h^+_1 \text{d}\mathcal{F}, \Pi_+\sigma^{-1}(\delta\mathcal{H})) - (h^+_1 \text{d}\mathcal{H}, \Pi_+\sigma^{-1}(\delta\mathcal{F}))$$

$$-\{Z^+_1, [\Pi_+\sigma^{-1}(\delta\mathcal{F}), \Pi_+\sigma^{-1}(\delta\mathcal{H})]\},$$

as expected. Let us consider a generic Hamiltonian function $\mathcal{H}$ on $H_1 \times h_1$; from the Poisson–Dirac bracket (3.2), one may obtain the corresponding Hamiltonian vector field which turns out to be

$$V^N_{\mathcal{H}}(h, Z) = (h(\mathcal{A}_+^1(h^+_1)\sigma^{-1}(\delta\mathcal{H})),$$

$$\sigma^{-1}(\gamma(\mathcal{A}_+^1(h^+_1)((\gamma^{-1}(\sigma(Z)), \mathcal{A}_+^1(h^+_1)\sigma^{-1}(\delta\mathcal{H}) - \gamma^{-1}(h\mathcal{D}\mathcal{H})))))$$

The Hamilton equations are then

$$\left\{ \begin{aligned}
h^{-1}_1 &= \mathcal{A}_+^1(h^+_1)\sigma^{-1}(\delta\mathcal{H}) \\
Z_t &= \sigma^{-1}(\gamma(\mathcal{A}_+^1(h^+_1)((\gamma^{-1}(\sigma(Z)), \mathcal{A}_+^1(h^+_1)\sigma^{-1}(\delta\mathcal{H}) - \gamma^{-1}(h\mathcal{D}\mathcal{H}))))
\end{aligned} \right.$$  

(3.3)

The Hamiltonian vector field for left invariant functions associated with this structure is

$$V_{\mathcal{H}_L}(h, Z) = \gamma(\mathcal{A}_+^1(h^+_1)((\gamma^{-1}(\sigma(Z)), \mathcal{A}_+^1(h^+_1)\sigma^{-1}(\delta\mathcal{H}), \gamma^{-1}(\sigma(Z))))).$$
3.2. The left action of $H_1$ on $H_2$ and its restriction to $\mathcal{N}(h_1^-, \eta_1^-)$

In body coordinates for $TH_1$, the left translation is $(L_{h_1})_* (h_1, Z_1) = (g_1 h_1, Z_1)$, with the infinitesimal generator

$$X_{H_1}(h_1, Z_1) := \frac{d}{dt} (e^{tX_1} h_1, Z_1) \Big|_{t=0} = (X_1 h_1, 0).$$

This action is the Hamiltonian referred to the symplectic form (3.1), with associated $A_d$-equivariant momentum map $\Phi_B : H_2 \longrightarrow h_1^*$ given by

$$\Phi_B(h_1, Z_1) = \gamma (A_d h_1^* \gamma^{-1}(\sigma(Z_1))).$$

Hence, the momentum functions

$$\phi_{X_1}(h_1, Z_1) := \{ \gamma (A_d h_1^* \gamma^{-1}(\sigma(Z_1))), X_1 \} = (Z_1, A_d h_1^*, X_1)$$

produce the Lie derivative of a function $F$ on $H_2$ along the projection of the vector field $X_{H_1}$ on $N_2(h_1^-, \eta_1^-)$ through the Dirac bracket (3.2), namely

$$\{F, \phi_{X_1}\}_{\gamma} (h_1, Z_1) = (dF, h_1 A_d h_1^* (h_1^*) X_1)$$

$$- (\delta F, \sigma^{-1}(\gamma(A_d h_1^* | (\gamma^{-1}(\sigma(Z_1))), A_d h_1^*, \Pi \gamma^{-1}(\sigma(Z_1))) - X_1))).$$

In particular, the Dirac bracket between momentum functions $\phi_X, \phi_Y$ is

$$\{\phi_X, \phi_Y\}_{\gamma} (h_1, Z_1) = \phi_{X Y} (h_1, Z_1) = (Z_1, [A_d h_1^*, \Pi \gamma^{-1}(\sigma(Z_1)) - X, A_d h_1^*, \Pi \gamma^{-1}(\sigma(Z_1)) - Y])$$

for $X, Y \in h_1$. Observe that the second term vanishes whenever $\Pi \gamma^{-1}(\sigma(Z_1))$ is a character of $h_1^*$, meaning that the set of Hamiltonian vector fields $\{V_{\phi_X} \} \in \mathfrak{h}$ forms a Lie subalgebra of $\mathfrak{X}(N_2(h_1^-, \eta_1^-))$:

$$[V_{\phi_X}, V_{\phi_Y}] = - V_{\phi_{X Y}}$$

and defines a foliation in $N_2(h_1^-, Z_1^-)$. It is just in this case that the map $X \mapsto V_{\phi_X}$ defines an infinitesimal left action of $h_1^+$ on $N_2(h_1^-, Z_1^-)$; this produces the infinitesimal generators of the action $H_1 \times N_2(h_1^-, \eta_1^-) \longrightarrow N_2(h^-_1, \eta^-_1)$

$$d(g, (h_1^+, \eta_1^+, \eta^-_1)) = (g A_d (h_1^+), \Pi \gamma^{-1}(\sigma(Z_1)) - g h_1^+, \sigma^{-1}(\gamma(A_d h_1^* | (\gamma^{-1}(\sigma(Z_1)) - g h_1^+, \Pi \gamma^{-1}(\sigma(Z_1)) - X_1))))$$

for $g \in H_1$.

4. The dynamics and integrable systems on $H_2$

4.1. Hamilton equations and the collective dynamics on $\mathcal{N}(g_1^+, \eta_1^-)$

The Hamilton equations of motion on $\mathcal{N}(g_1^+, \eta_1^-)$ were derived from the Dirac bracket (3.2) in (3.3), for any Hamiltonian function $H$ on $H_1 \times h_1 = H_2$. In order to study integrable systems, we shall consider a general collective Hamiltonian

$$\mathcal{H}(h_1, Z_1) := h(\Phi_B(h_1, Z_1))$$

where $h : h_1^+ \longrightarrow \mathbb{R}$ is $A_d$-invariant; observe that collective Hamiltonians with $A_d$-invariant

$h : h_1^+ \longrightarrow \mathbb{R}$ are naturally left invariant.

Now let us introduce the Legendre transform of $h : h_1^+ \longrightarrow \mathbb{R}$, defined as the map

$$L_h : h_1^+ \longrightarrow \mathfrak{h}$$

with

$$k_1(L_h(\eta_1), X_1) = \frac{d}{dt} h(\eta_1 + t \gamma(X_1)) \Big|_{t=0}. \quad \text{10}$$
Because of the Ad-invariance of $\mathfrak{h}$ we have that $dH_{|(h_1,z_1)} = (0, \sigma(\mathcal{L}_h(\sigma(z_1))))$ and the Hamilton equations (3.3) turn into
\[
\begin{aligned}
&\hat{h}_i^- = L_{\gamma(\mathfrak{g}(h_i^-))}(\sigma(Z_1)) \\
&\hat{Z}_i = -\gamma^{-1}(\mathfrak{g}(h_i^-)(\mathfrak{g}(\sigma(z_1)))) \\
&\mathcal{L}_{\gamma(\mathfrak{g}(h_i^-))}(\sigma(z_1)).
\end{aligned}
\]
It is interesting to see that they can be decomposed into the dynamical contents on each factor of $H_1 = H_1^+ H_1^-$ and $h_1 = h_1^+ \oplus h_1^-$, yielding
\[
\begin{aligned}
&\hat{h}_1^+ = \Pi_+ \mathcal{L}_{\gamma(\mathfrak{g}(h_1^+))}(\sigma(Z_1)) \\
&\hat{Z}_1 = -\gamma^{-1}(\mathfrak{g}(h_1^+)(\mathfrak{g}(\sigma(z_1)))) \\
&\mathcal{L}_{\gamma(\mathfrak{g}(h_1^+))}(\sigma(z_1))
\end{aligned}
\]

### 4.2. $(\Omega, \Gamma)$ coordinates

The equations (4.1) are easily handled after introducing the variables
\[
\begin{aligned}
&\Omega_1 = \mathcal{L}_{\gamma(\mathfrak{g}(\Gamma_1))} \\
&\Gamma_1 = \mathcal{L}_{\gamma^{-1}(\mathfrak{g}(\sigma(Z_1)))}
\end{aligned}
\]
which are related as
\[
\Omega_1 = \mathcal{L}_{\gamma(\mathfrak{g}(\Gamma_1))}
\]
Also, observe that, because $\Pi_+ \sigma(\mathfrak{g}(\sigma(Z_1)))$ is a character of $\mathfrak{h}_1^+$, $\gamma(\Pi_+ \mathcal{L}_{\Gamma_1}) = \Pi_+ \gamma(\mathcal{L}_{\Gamma_1})$ is a character of $\mathfrak{h}_1^-$. Equations (4.1) state that $\hat{Z}_1^- = 0$, so $\Gamma_1^- = 0$. Therefore, the collective Hamilton equations (4.1) can be written as
\[
\begin{aligned}
&\hat{h}_1^+ = \Omega_1^+ \\
&\hat{\Gamma}_1^- = -\Pi_- [\Gamma_1, \Omega_1^-]
\end{aligned}
\]
which, in some sense, resembles the Euler equation of motion for a rigid body moving under dressing action in place of adjoint action.

The Ad-invariance of $\mathfrak{h}$ translates, in terms of the new variables $\Omega_1, \Gamma_1$, as
\[
[\Gamma_1, \Omega_1^+] = -[\Gamma_1, \Omega_1^-]
\]
or, equivalently, $[\Gamma_1, \Omega_1^-] = 0$. Using this result, the Hamilton equations (4.3) now read
\[
\begin{aligned}
&\hat{h}_1^+ = \Omega_1^+ \\
&\hat{\Gamma}_1^- = -\Pi_- [\Gamma_1, \Omega_1^-]
\end{aligned}
\]

### Remark

Since $\Pi_+ \sigma(\mathfrak{g}(\sigma(Z_1)))$ is a character of $\mathfrak{h}_1^+$, so is $\Pi_+ \gamma(\mathfrak{g}(\sigma(Z_1))) = \gamma(\mathfrak{g}(\sigma(Z_1)))$.

From the fact that $\gamma(\Gamma_1^+)$ is a character of $\mathfrak{h}_1^+$, we have
\[
\Pi_+ [\Gamma_1, \Omega_1^-] = 0
\]
which allows us to write the above Hamilton equations as
\[
\begin{aligned}
&\hat{h}_1^+ = \Omega_1^+ \\
&\hat{\Gamma}_1^- = -[\Gamma_1, \Omega_1^-]
\end{aligned}
\]
4.3. Solving by factorization

We now see how to solve the Hamilton equations (4.5) by factorization, showing that the solution curves are orbits of the action of the factor curves in $H^+_1$ and $H^-_1$ of an exponential curve in $H_1$.

Let us start with the last version of the Hamilton equation in terms of $(\Omega, \Gamma)$, namely equation (4.5). As is well known, the second of these equations has the solution

$$\Gamma_1(t) = \text{Ad}^t_{\gamma_1(t)} \Gamma_1_0$$

for some initial value $\Gamma_1(t_0) = \Gamma_1_0$, provided that the curve $g^-_1(t) \in H^-_1$ solves the differential equation

$$\dot{g}_1^- (g_1^-)^{-1} = \Omega^-_1.$$  \hspace{1cm} (4.6)

In addition, this yields another important consequence: the problem can be solved by factorization. Let us show how this idea works: adding the first equations (4.5) and (4.6) we get

$$(h^+_1)^{-1} h^+_1 + \dot{g}_1^- (g_1^-)^{-1} = \Omega^-_1,$$

which is equivalent to

$$(h^+_1 g_1^-)^{-1} \frac{d}{dt}(h^+_1 g_1^-) = \text{Ad}_{\gamma_1^-}\Omega^-_1. \hspace{1cm} (4.7)$$

On introducing the curve $k_1(t) := h^+_1(t) g_1^- (t)$ and the map $\Theta_1 : H_1 \times \mathfrak{h}_1 \rightarrow \mathfrak{h}_1$ as

$$(h^+_1 g_1^-, \Gamma_1) \mapsto \Theta_1(k_1, \Gamma_1) = \mathcal{L}_h(\gamma(\text{Ad}_{\gamma_1^-}\Gamma_1_0)) = \text{Ad}_{\gamma_1^-} \Omega^-_1,$$

equation (4.7) becomes

$$k_1^{-1} k_1 = \Theta_1(k_1, \Gamma_1). \hspace{1cm} (4.8)$$

Therefore, solving the differential equation (4.8) and decomposing the solution in its factors in $H^+_1$ and $H^-_1$, one solves the original problem. However, for general $\Theta_1$, this equation can hardly be integrated. The next result is crucial for the success of the AKS procedure.

**Proposition.** The vector field on $H_1 \times \mathfrak{h}_1$ defined by the assignment

$$(k_1, \Gamma_1) \mapsto \mathcal{X}(k_1, \Gamma_1) := (\Theta_1(k_1, \Gamma_1), \Pi^-_1 \mathcal{L}_h(\gamma(\Gamma_1)))$$

is in the null distribution of the differential $\Theta_{1*}$ of $\Theta_1$.

**Proof.** Let us calculate it as follows:

$$\Theta_{1*}(\mathcal{X}(k_1, \Gamma_1)) = \frac{d}{dt} \mathcal{L}_h(\gamma(\text{Ad}_{\gamma_1^-}\Gamma_1(t)))|_{t=0}$$

$$= \mathcal{L}_h(\frac{d}{dt}(\text{Ad}_{\gamma_1^-}\Gamma_1(t)))|_{t=0}.$$  \hspace{1cm} (4.9)

Since

$$\frac{d}{dt}(\text{Ad}_{\gamma_1^-}\Gamma_1(t))|_{t=0} = -\text{Ad}_{\gamma_1^-}\text{Ad}_{\gamma_1^-}\Gamma_1 + \text{Ad}_{\gamma_1^-} \Gamma_1$$

$$= -\text{Ad}_{\gamma_1^-}(\mathcal{X}_1 + [\Gamma_1, \Omega^-_1])$$

$$= 0,$$

we conclude that

$$\Theta_{1*}(\mathcal{X}(k_1, \Gamma_1)) = 0,$$

as stated. \hfill \Box
**Corollary.** The map \( \Theta_1 : H_1 \times b_1 \rightarrow b_1 \) is constant along the solution curves of the system of Hamilton equations
\[
\begin{align*}
\dot{k}_1^{-1}k_1 &= \Theta_1(k_1, \Gamma_1) \\
\dot{\Gamma}_1 &= -[\Gamma_1, \Omega_1^-].
\end{align*}
\]

This makes the differential equation on \( H_1 \) (4.8) easily integrable: because \( \Theta_1(k_1, \Gamma_1) \) is a constant of motion \( \Theta_{1c} \in b_1 \), it has the exponential solution
\[
k(t) = e^{\Theta_{1c}};
\]
the decomposition of this solution \( k(t) \) into its factors on \( H_1^+ \) and \( H_1^- \), as
\[
e^{\Theta_{1c}} = h_1^+(t) g_1^-(t)
\]
gives the full solution of the original problem. In fact,
\[
\Omega_1^+ = (h_1^+)^{-1} \dot{h}_1^+
\]
\[
\Omega_1^- = \dot{g}_1^- (g_1^-)^{-1}
\]
and
\[
\Gamma_1(t) = \text{Ad}_{h_1^+(t)} \Gamma_{10}
\]  
(4.9)
for some initial value \( \Gamma_1(t_0) = \Gamma_{10} \).

**4.4. The nested equation of motion**

One may think of phase spaces on semidirect product Lie groups as a way of packing many variables in a well suited fashion in order to formulate a complicated problem in a simpler way. This becomes evident if we consider the Hamilton equation on \( H_1 = H_0 \times b_0 \) and we express it in terms of the \( H_0 \) and \( b_0 \) variables. In doing so, we consider now the equations (4.1) on \( \mathcal{N}(h_1^-, Z_1^-) \), in terms of the variables \( (\Omega_1, \Gamma_1) \) as in equations (4.3), (4.4) and (4.5), and write them in terms of the variables \( H_0, b_0 \). Let us first establish some notational convention regarding the elements of \( H_1 \) as objects in \( H_0 \times b_0 \):
\[
(h_1, Z_1) := ((h_0, Z_0), (X_0, Y_0))
\]
\[
\gamma^{-1}(\sigma(Z_1)) := (\gamma^{-1}(\sigma(Y_0)), \gamma^{-1}(\sigma(X_0)))
\]
\[
\mathcal{L}_h(\sigma(Z_1)) := (\mathcal{L}_h(\sigma(Z_1)), \mathcal{L}_h(\sigma(Z_1))).
\]
Thus, on introducing the variables
\[
\tilde{\Omega}_0 = \text{Ad}_{h_0}^\sigma \mathcal{L}_h(\sigma(X_0, Y_0))
\]
\[
N_0 = \text{Ad}_{h_0}^\sigma \mathcal{L}_h(\sigma(X_0, Y_0))
\]
\[
\tilde{\Gamma}_0 = \text{Ad}_{h_0}^\sigma \gamma^{-1}(\sigma(Y_0))
\]
\[
M_0 = \text{Ad}_{h_0}^\sigma \gamma^{-1}(\sigma(X_0))
\]
\[
R_0 = \text{Ad}_{h_0}^\sigma Z_0,
\]  
(4.10)
the expressions for \( (\Omega_1, \Gamma_1) \) become
\[
\Omega_1 = (\tilde{\Omega}_0, [R_0^{-1}, \tilde{\Omega}_0] + N_0)
\]
\[
\Gamma_1 = (\tilde{\Gamma}_0, [R_0^{-1}, \tilde{\Gamma}_0] + M_0).
\]
Also, the (translated) evolution vector field reads
\[
(h_1^+)^{-1}(t) \dot{h}_1^+(t) = ((h_1^+)^{-1}(t) \dot{h}_1^+(t), [h_1^+(t)]^{-1} \dot{h}_1^+(t), Z_0^+(t)] + \dot{Z}_0^+(t)
\]
and 
\[ [\Gamma_1, \Omega^+] = ([\tilde{\Gamma}_0, \tilde{\Omega}^+_0], [\tilde{\Gamma}_0, \Pi_+ [R^+_0, \tilde{\Omega}^+_0]] + [\tilde{\Gamma}_0, \Pi_+ [R^-_0, \tilde{\Omega}^-_0]]) + [M_0, \tilde{\Omega}^+_0]). \]

Therefore, the equations of motion (4.3) for the \( H^+_0, h^+_0 \) coordinates are
\[
\begin{align*}
(h^+_0)^{-1}(t)h^+_0(t) &= \tilde{\Omega}^+_0 \\
\dot{Z}^+_0(t) &= -[\tilde{\Omega}^+_0, Z^+_0] + [R^+_0, \tilde{\Omega}^+_0] + N_0 \\
\frac{d}{dt} \tilde{\Gamma}_0 &= \Pi_+ [\tilde{\Gamma}_0, \tilde{\Omega}^+_0] \\
M_0 &= \Pi_+ [R^+_0, \tilde{\Omega}^+_0] + [\Pi_+ [\tilde{\Gamma}_0, \tilde{\Omega}^+_0], R^+_0] + [\Pi_+ [\tilde{\Gamma}_0, \tilde{\Omega}^+_0], R^-_0] + [M_0, \tilde{\Omega}^+_0].
\end{align*}
\]

**Remark.** Observe that the last two equations become simpler and more familiar upon setting \( R^-_0 = 0 \) and making the substitutions
\[
\begin{align*}
\Pi_+ [\tilde{\Gamma}_0, \tilde{\Omega}^+_0] &= \Pi_+ [\tilde{\Gamma}_0^-, \tilde{\Omega}^+_0] = (\tilde{\Gamma}_0^-)^\Omega^+_0, \\
\Pi_+ [M_0, \tilde{\Omega}^+_0] &= \Pi_+ [M_0^-, \tilde{\Omega}^+_0] = (M_0^-)^\tilde{\Omega}^+_0,
\end{align*}
\]
where \((Y^-_0)^\tilde{\Omega}^+_0\) means the dressing action of \( h^+_0 \) on \( h^-_0 \) (see (2.6) and [12]). Namely, these equations turn into
\[
\begin{align*}
\frac{d}{dt} \tilde{\Gamma}_0^- &= (\tilde{\Gamma}_0^-)^\tilde{\Omega}^+_0 \\
\frac{d}{dt} M^-_0 &= (\tilde{\Gamma}_0^-)^N^+_0 + (M_0^-)^\tilde{\Omega}^+_0,
\end{align*}
\]
resembling the Euler–Poisson equations of the heavy rigid body. In fact, they are
\[
\begin{align*}
\dot{\Gamma} &= [\Gamma, \Omega] \\
M &= [M, \Omega] + [\Gamma, N]
\end{align*}
\]
where \( \Omega \) is the angular velocity, \( M \) is the angular momenta, \( \Gamma \) is the gravitational force as seen from the moving frame, and \( N \) is the vector joining the center of mass with a fixed point. In these equations the evolution is generated by the adjoint action of the Lie algebra \( \mathfrak{so}_3 \) on itself, while in our case it is generated by the dressing action of \( h^+_0 \) on \( h^-_0 \).

In order to solve the equations (4.11) we just need to factorize the exponential curve \( k_1(t) \) on \( H^1 \):
\[
k_1(t) = e^{t\Theta_1} = h^+_1(t)g^-_1(t)
\]
with
\[
\Omega_1(t) = Ad^{1}_k(t)\Theta_1.
\]
These factors decompose on \( H^+_0 \) and \( h^+_0 \) as
\[
\begin{align*}
h^+_1(t) &= (h^+_0(t), Z^+_0(t)) \\
g^-_1(t) &= (g^-_0(t), Z^-_0(t))
\end{align*}
\]
so, from the solution (4.9),
\[
\Gamma_1(t) = Ad^{1}_k(t)\Gamma_1 = (Ad^{0}_{k(t)} \tilde{\Gamma}_0^-, Ad^{0}_{k(t)} ([Z^+_0(t), \tilde{\Gamma}_0^-] + \tilde{\Gamma}^-_0))
\]
we get
\[
\Gamma_1(t) = (\tilde{\Gamma}_0(t), [R^+_0, \tilde{\Gamma}_0^+(t)] + M_0(t)).
\]
which implies

$$\mathbf{\Gamma}_0(t) = \text{Ad}^0_{g(t)} \mathbf{\Gamma}_0^0$$

and

$$\mathbf{M}_0(t) = [\text{Ad}^0_{g(t)} \mathbf{Z}_0(t) - \mathbf{R}_0^0, \text{Ad}^0_{g(t)} \mathbf{\Gamma}_0^0] + \text{Ad}^0_{g(t)} \mathbf{\Gamma}_0^0$$

for some initial value $\mathbf{\Gamma}_0 = (\mathbf{\Gamma}_0^0, \mathbf{\Gamma}_0^m)$.

5. The tower of integrable systems

The above construction can be made recursive for appropriate Hamiltonians, giving rise thus to a tower of integrable systems on semidirect products, all of them solvable by factorization.

First, let us suppose that $G$ is a semisimple Lie group, $\mathfrak{g}$ is its Lie algebra, and $(\cdot, \cdot)_G$ its Killing form. For every $m = 1, 2, \ldots$, we define $J_m := J_{m-1} \ltimes J_{m-1}$, and we make the identifications $J_0 = G, J_0 = \mathfrak{g}$ and $K_0 = (\cdot, \cdot)_G$. Then we have:

**Lemma.** Let $\mathfrak{h}_m$ be a Lie algebra equipped with a nondegenerate $\text{Ad}^{m-1}$-invariant bilinear form $k_{m-1} : \mathfrak{h}_{m-1} \otimes \mathfrak{h}_{m-1} \rightarrow \mathbb{K}$. On $\mathfrak{h}_m := \mathfrak{h}_{m-1} \ltimes \mathfrak{h}_{m-1}$ the bilinear form $k_m : \mathfrak{h}_m \times \mathfrak{h}_m \rightarrow \mathbb{K}$ given by

$$k_m((X, Y), (X', Y')) := \frac{1}{2}(k_{m-1}(X, Y') + k_{m-1}(Y, X'))$$

for all $(X, Y), (X', Y') \in \mathfrak{h}_m$, is nondegenerate and $\text{Ad}^m$-invariant.

**Proof.** It follows by direct substitution of

$$\text{Ad}^m_{(b, Z)}(X, Y) = (\text{Ad}^{m-1}_b X, \text{Ad}^{m-1}_b ([Z, X]_{m-1} + Y))$$

for $(b, Z) \in H_m$ and $(X, Y), (X', Y') \in \mathfrak{h}_m$, in

$$k_m(\text{Ad}^m_{(b, Z)}(X, Y), \text{Ad}^m_{(b, Z)}(X', Y')),$$

and using the $\text{Ad}^{m-1}$-invariance of the bilinear form $k_{m-1}$, to reach

$$k_m(\text{Ad}^m_{(b, Z)}(X, Y), \text{Ad}^m_{(b, Z)}(X', Y')) = k_m((X, Y), (X', Y'))$$

as expected.

As we saw above, in equations (2.5) and (2.7), the factorization

$$\begin{cases}
H_m = H^+_m H^-_m \\
\mathfrak{h}_m = \mathfrak{h}^+_m \oplus \mathfrak{h}^-_m
\end{cases}$$

implies that $H_{m+1}$ factorizes according to

$$H_{m+1} = H^+_{m+1} H^-_{m+1} = (H^+_m \ltimes \mathfrak{h}^+_m) \cdot (H^-_m \ltimes \mathfrak{h}^-_m).$$

Explicitly, this means that $(h, Y) \in H_{m+1} = H_m \ltimes \mathfrak{h}_m$ admits the factorization

$$(h, Y) = (h_+, X_+) \cdot (h_-, X_-)$$

with

$$\begin{cases}
h = h_+ h_- \\
X_+ = \Pi_+ (\text{Ad}^m_{h_+} Y) \\
X_- = \text{Ad}^m_{h_-} \Pi_- \text{Ad}^m_{h_-} Y.
\end{cases}$$
The chain of semidirect products of Lie groups produces a chain of semidirect sums of Lie algebras. Each term factorizes as
\[ h_{m+1} = h_{m+1}^+ \ltimes h_{m+1}^- \]
and so for every \((X, Y) \in h_{m+1}^- = h_m \ltimes h_m\), there exists a pair of elements \((X_+, Y_+) \in h_{m+1}^+ = h_m^+ \ltimes h_m^+\) such that
\[(X, Y) = (X_+, Y_+) \oplus (X_-, Y_-).\]
Let us choose the Hamiltonian on \(N_{m+1}^- (h_m^- , Z_m^-) \subset H_{m+1}\) as
\[ \mathcal{H}_{m+1} (h_m^- , Z_m^-) := h_m^{(m)} (\Phi^{(m+1)} (h_m^- , Z_m^-)) , \]
for the momentum map
\[ \Phi^{(m+1)} (h_m^- , Z_m^-) = \gamma (A_{h_m^-} \gamma^{-1} (\sigma (Z_m^-))) = \text{Ad}^{(m)}_{h_m^-} \sigma (Z_m^-) \]
\[ \text{and } h^{(m)} : h_m^+ \rightarrow \mathbb{R} \text{ given by} \]
\[ h^{(m)} (\eta) = \frac{1}{2} k_m (\gamma^{-1} (\eta), \gamma^{-1} (\eta)). \]
Then its Legendre transform \(L_{h^{(m)}} : h_m^+ \rightarrow h_m \) results as
\[ L_{h^{(m)}} (\eta_m) = \gamma^{-1} (\eta_m) = (\gamma^{-1} (\eta_{m-1}), \gamma^{-1} (\eta_{m-1})). \]
In terms of the variables \((\Omega, \Gamma)\) (see (4.2)), we have
\[ \Omega_m = \text{Ad}^{(m)}_{h_m^-} \gamma^{-1} (\sigma (Z_m^-)) = \Gamma_m \]
and the collective Hamilton equations (4.1) are
\[ \begin{cases} (h_m^+)^{-1} h_m^+ = \Omega_m^+ \\ \Omega_m = \Pi_{\Omega} [\Omega_m, \Omega_m^-] \end{cases} \]
To write down the dynamics in terms of the coordinates of \(H_{m-1}^+ \) and \(h_{m-1}^+ \), regarding
\[ (h_{m-1}^+, Z_{m-1}) = ((h_{m-1}^- , Z_{m-1}), (X_{m-1} , Y_{m-1})), \]
we can introduce the variables
\[ \hat{\Omega}_{m-1} = \hat{\Gamma}_{m-1} = \text{Ad}^{m-1}_{h_{m-1}^-} \gamma^{-1} (\sigma (X_{m-1})) \]
\[ N_{m-1} = \text{Ad}^{m-1}_{h_{m-1}^-} \gamma^{-1} (\sigma (Y_{m-1})) \]
\[ M_{m-1} = \text{Ad}^{m-1}_{h_{m-1}^-} \gamma^{-1} (\sigma (Y_{m-1})) \]
\[ R_{m-1} = \text{Ad}^{m-1}_{h_{m-1}^-} Z_{m-1} \]
such that \(\Omega_m\) and \(\Gamma_m\) are now
\[ \Omega_m = (\hat{\Omega}_{m-1} , [R_{m-1}^- , \hat{\Omega}_{m-1}^-] + N_{m-1}), \]
\[ \Gamma_m = (\hat{\Gamma}_{m-1} , [R_{m-1}^- , \hat{\Gamma}_{m-1}^-] + M_{m-1}), \]
and the corresponding equations of motion (4.11) are
\[ \begin{cases} (h_{m-1}^+)^{-1} h_{m-1}^+ (t) = \hat{\Omega}_{m-1} \\ \dot{Z}_{m-1}^+ (t) = -[\hat{\Omega}_{m-1}^- , Z_{m-1}^+] + [R_{m-1}^- , \hat{\Omega}_{m-1}^-] + N_{m-1} \\ \frac{d}{dt} \hat{\Gamma}_{m-1} = \Pi_{\hat{\Gamma}} [\hat{\Gamma}_{m-1} , \hat{\Omega}_{m-1}^-] \\ M_{m-1} = \Pi_- [\hat{\Gamma}_{m-1} , N_{m-1}^- + [M_{m-1} , \hat{\Omega}_{m-1}^-] + [M_{m-1} , \hat{\Omega}_{m-1}^-], R_{m-1}^-] \\ + \Pi_- [\hat{\Gamma}_{m-1} , \Pi_+ [R_{m-1}^- , \hat{\Omega}_{m-1}^-] + \Pi_- [R_{m-1}^- , \hat{\Omega}_{m-1}^-] \hat{\Omega}_{m-1}^-] \end{cases} \]
A simpler set of equations is obtained by setting the constant point \( Z_{-1}^- = 0 \), which implies that \( R_{m-1}^- = \text{Ad}_{h_{m-1}^0}^{-1}Z_{m-1}^- = 0 \),

\[
\begin{align*}
(h_{m-1}^+)^{-1}(t)h_{m-1}^+(t) &= \bar{\Omega}_{m-1}^+ \\
\dot{Z}_{m-1}^+(t) &= -[\bar{\Omega}_{m-1}^+, Z_{m-1}^+] + N_{m-1} \\
\frac{d}{dt}\tilde{\Gamma}_{m-1} &= \Pi_+ [\tilde{\Gamma}_{m-1}, \bar{\Omega}_{m-1}^+] \\
M_{m-1} &= \Pi_+ [\tilde{\Gamma}_{m-1}, N_{m-1}^+] + \Pi_- [M_{m-1}, \bar{\Omega}_{m-1}^+] 
\end{align*}
\]

besides the equations \( d\tilde{\Gamma}_{m-1}/dt = dM_{m-1}/dt = 0 \).

Thus, the dynamical system (5.1) replicates one level down in the tower by projecting on the second component of the semidirect product, and all of them are solvable by factorization.

6. Example: \( SL(2, \mathbb{C}) \)

6.1. Iwasawa decomposition of \( SL(2, \mathbb{C}) \)

Let us consider the Lie algebra \( sl_2(\mathbb{C}) \) with the Iwasawa decomposition of \( \mathfrak{h}_0 = sl_2(\mathbb{C})^\mathbb{R} \) by taking \( \mathfrak{h}_0^+ = \mathfrak{su}_2 \) and \( \mathfrak{h}_0^- = \mathfrak{b} \), where \( \mathfrak{su}_2 \) is the real subalgebra of \( sl_2(\mathbb{C}) \) of anti-Hermitian matrices, and \( \mathfrak{b} \) is the subalgebra of upper triangular matrices with real diagonal and null trace. For \( \mathfrak{su}_2 \) we take the basis

\[
X_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},
\]

and in \( \mathfrak{b} \) this one:

\[
E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad iE = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

We also introduce the dual basis \( \{x_1, x_2, x_3\} \subset \mathfrak{su}_2^* \) and \( \{e, \bar{e}, h\} \subset \mathfrak{b}^* \). This decomposition translates to the group giving \( SL(2, \mathbb{C}) = SU(2) \times B \), where now \( B \) is the group of \( 2 \times 2 \) upper triangular matrices with real diagonal and determinant 1. So, in order to fit the previous notation, we make the identifications \( \mathfrak{h}_0 = sl_2(\mathbb{C}) \), \( \mathfrak{h}_0^+ = \mathfrak{su}_2 \), \( \mathfrak{h}_0^- = \mathfrak{b} \), and \( H_0^+ = SU(2) \) and \( H_0^- = B \).

The Killing form for \( sl_2(\mathbb{C}) \) is

\[
\kappa(X, Y) := \text{tr} (ad(X)ad(Y)) = 4 \text{tr} (XY),
\]

and the restriction to \( \mathfrak{su}_2 \) is negative definite. Hence we take

\[
k_0(X, Y) = -\frac{1}{4} \text{Im} \kappa(X, Y),
\]

which is a symmetric nondegenerate Ad-invariant bilinear form turning \( \mathfrak{b} \) and \( \mathfrak{su}_2 \) into isotropic subspaces. It induces the linear bijection \( \gamma : \mathfrak{su}_2 \to \mathfrak{b}^* \) defined on the basis as

\[
\gamma(X_1) = -e, \quad \gamma(X_2) = \bar{e}, \quad \gamma(X_3) = -2h
\]
or its dual \( \gamma^* : \mathfrak{b} \to \mathfrak{su}_2^* \):

\[
\gamma^*(E) = -x_1, \quad \gamma^*(iE) = x_2, \quad \gamma^*(H) = -2x_3.
\]

On \( \mathfrak{su}_2 \) there is the standard Killing form \( \kappa \) defined as

\[
\kappa_{\mathfrak{su}_2}(X, Y) := 4 \text{tr}(XY),
\]

such that on the basis \( \{X_1, X_2, X_3\} \) it is \( (\kappa_{\mathfrak{su}_2})_{ij} = -8 \delta_{ij} \). Then, we may use it to set a linear bijection \( \xi : \mathfrak{su}_2 \to \mathfrak{su}_2^* \) such that

\[
(\xi(X), Y) = \kappa_{\mathfrak{su}_2}(X, Y)
\]
This morphism is symmetric, \( \langle \sigma(X), Y \rangle = \langle \sigma(Y), X \rangle \), because it is defined from a symmetric bilinear form.

### 6.2. The semidirect product of \( SL(2, \mathbb{C}) \) with \( sl_2(\mathbb{C}) \)

Then, we shall work on \( H_0 = SL(2, \mathbb{C}) \) and its Lie algebra \( h_0 = sl_2(\mathbb{C}) \) by considering the semidirect product Lie group \( H_1 = SL(2, \mathbb{C}) \ltimes sl_2(\mathbb{C}) \). We assume that \( sl_2(\mathbb{C}) \) is equipped with a nondegenerate symmetric bilinear form (6.3)

\[
\kappa_0(X, Y) = -\frac{1}{4} \text{Im} \kappa(X, Y) = -\text{Im} \text{tr} (XY).
\]

Let \( h_0^+, h_0^- \) be generic elements of \( SU(2) \) and \( B \), respectively; then

\[
h_0^+ = \begin{pmatrix} a & \beta \\ -\beta & \alpha \end{pmatrix}, \quad h_0^- = \begin{pmatrix} a & b + ic \\ 0 & a^{-1} \end{pmatrix},
\]

with \( \alpha, \beta \in \mathbb{C} \), satisfying \(|\alpha|^2 + |\beta|^2 = 1\), \( a \in \mathbb{R}_{>0} \), and \( b, c \in \mathbb{R} \). We have also an expression analogous to (2.9) for the adjoint action of \( b \) on \( sl_2(\mathbb{C}) \):

\[
\text{Ad}^b_{h_0^+} X_h^0 = (h_0^- X_h^0 (h_0^+)^{-1} + \gamma^{-1} (\text{Ad}^b_{(h_0^-)^{-1}} Y (X_h^0))),
\]

and each term in the right-hand side can be calculated explicitly, namely, if for \( X \in sl_2(\mathbb{C}) \) we write

\[
X = x_1 X_1 + x_2 X_2 + x_3 X_3 + x_H H + x_E E + x_{i(E)} (iE),
\]

then

\[
\text{Ad}^b_{h_0^+} X = x_1 \frac{1}{a^2} X_1 + x_2 \frac{1}{a^2} X_2 + \left( x_1 - x_2 \frac{c}{a}, x_3 \right) X_3 + \left( 2bcx_1 + \frac{b^2 - c^2 + a^2}{a^2} - \frac{1}{a^2} \right) x_2 + 2acx_3 + x_E a^2 - 2x_H ba \right) E + x_1 \left( c^2 - b^2 + a^2 - \frac{1}{a^2} \right) x_2 + 2bc - x_3 2ab + x_{i(E)} a^2 - 2x_H ca \right) (iE) + x_1 \left( c \frac{a}{a} + x_2 \frac{b}{a} - x_H \right) H.
\]

The operators \( \mathcal{A}_+^0 (h_--) \) are

\[
\mathcal{A}_+^0 (h^-) X = \Pi_+ X + \frac{1}{a^2} \left( \left( b^2 + c^2 + \frac{1}{a^2} - a^2 \right) x_2 - 2acx_3 \right) E + \frac{1}{a^2} \left( \left( b^2 + c^2 + \frac{1}{a^2} - a^2 \right) x_1 + 2abx_3 \right) (iE) + \frac{1}{a} (x_1 c + x_2 b) H.
\]
The last result of the previous subsections allows us to write the Dirac bracket on \( A \) and \( J. \) Phys. A: Math. Theor. Integrable systems are built in fibers on points \( H \) means that \( H \) upper triangular matrices with positive real diagonal elements and determinant equal 1. This describing in terms of the variables \( b \)

\[
\text{Integrable systems are built in fibers on points } \langle h^0, Z^0 \rangle \quad \text{with } \Pi^+ \sigma (Z^0) \text{ being a character of } b. \quad \text{Since } B \text{ has no nontrivial character, it results that } \Pi^+ \sigma (Z^0) = 0.
\]

6.3. Dirac brackets on \( N(Z_1, 0) \subset SL(2, C) \times s l_2(C) \)

The last result of the previous subsections allows us to write the Dirac bracket on \( N(Z_1, 0) \), given in (3.2), for \( Z_1 \) a character of \( a \), as

\[
\{ F, H \}^D (h, Z_+) = \langle h F, H^0_+ (h) \Pi_+ \sigma^{-1} (\delta \mathcal{F}) \rangle - \langle h H, A^0_+ (h) \Pi_+ \sigma^{-1} (\delta \mathcal{F}) \rangle - \langle \sigma (Z_+), \Pi_+ [H^0_+ (h) \Pi_+ \sigma^{-1} (\delta \mathcal{F}), A^0_+ (h) \Pi_+ \sigma^{-1} (\delta \mathcal{F})] \rangle.
\]

The last term in the right-hand side can be written as

\[
\langle \sigma (Z_+), \Pi_+ [H^0_+ (h) \Pi_+ \sigma^{-1} (\delta \mathcal{F}), A^0_+ (h) \Pi_+ \sigma^{-1} (\delta \mathcal{F})] \rangle = 16 \langle \delta \mathcal{F}, (\delta \mathcal{F}) \cdot e_{ijk} (z_k - B_k, h, Z_+) \rangle,
\]

where

\[
B = \begin{pmatrix} b a z_3 X_1 - \frac{c}{a} b z_3 X_2 + \frac{b}{a} b z_1 - \frac{c}{a} c z_2 - \left( \frac{b^2}{a^2} + \frac{c^2}{a^2} + \frac{1}{a^2} - 1 \right) z_3 
\end{pmatrix} X_3.
\]

Observe that

\[
\nabla \times B = 0,
\]

\[
\nabla \cdot B = -\left( \frac{b^2}{a^2} + \frac{c^2}{a^2} + \frac{1}{a^2} - 1 \right),
\]

so it can be regarded as a static magnetic field associated with a magnetic monopole charge density

\[
\rho_m = -\frac{1}{a^2} \left( b^2 + c^2 \right) = \frac{1}{a^2} \text{tr} (h H h^0_-).
\]

For left invariant functions the Poisson–Dirac bracket reduces to

\[
\{ F^L, H^L \}^D (h, Z) = -16 \langle \delta \mathcal{F}, (\delta \mathcal{F}) \cdot e_{ijk} (z_k - B_k, h, Z_+) \rangle.
\]

Now we shall consider the semidirect product Lie group \( H_1 = H_1 \times b_1 \), where \( H_1 = SL(2, C) \times s l_2(C) \). As in previous sections, in both cases we use the right action structure of the semidirect product (2.1). We define on \( b_1 \) the nondegenerate symmetric bilinear form

\[
k_1 (X, U, (Y, V)) = \frac{1}{2} (k_0 (X, V) + k_0 (U, Y)).
\]

We regard \( H_2 = H_1 \times b_1 \) as a phase space, where \( H_1 \) inherits the factorization of \( SL(2, C) = H^0_+ \times H^0_- \) where \( H^0_+ = SU(2) \) and \( H^0_- = B \), the group of \( 2 \times 2 \) complex upper triangular matrices with positive real diagonal elements and determinant equal 1. This means that \( H_2 = H^+_1 \times H^-_1 \) where \( H^+_1 = H^+_1 \times b^+_1 \).

In this framework, we shall study a dynamical system on the fibration \( H_2 \rightarrow H^-_2 \) by applying the Dirac procedure explained above. We use the results obtained in section (4.4), describing in terms of the variables \( (\Omega_1, \Gamma_1) \) (see (4.2)) a collective system on \( N(h_1, Z^-_1) \) with equations of motion (4.11) and (4.1). Regarding \( h_1 \) as the semidirect product \( H_1 = H_0 \times b_0 \), we introduce in this context the variables \( (\Omega_0, N_0, \Gamma_0, M_0, R^0_0) \) as defined in (4.10), with \( Z_0^- = 0 \) and \( R^0_0 = 0 \).
6.4. Solving by factorization

In particular, we consider a collective Hamiltonian on $H_2$:

$$\mathcal{H}^{(2)}(h_1, Z_1) := h^{(2)}(\Phi_B^{(2)}(h_1, Z_1)),$$

with

$$\Phi_B^{(2)}(h_1, Z_1) = \gamma(\text{Ad}_h^B \gamma^{-1}(\sigma(Z_1))) = \text{Ad}_{h_1}^B \sigma(Z_1)$$

and the Hamilton function on $h_1^*$ given as $h^{(2)} : h_1^* \rightarrow \mathbb{R}$:

$$h^{(2)}(X_0, Y_0) = -\frac{i}{\hbar} \text{Re} \langle X_0, Y_0 \rangle,$$

which is $\text{Ad}^B$ since it is naturally $\text{Ad}^0$-invariant. The Killing form on $\kappa : sl_2 \mathbb{C} \times sl_2 \mathbb{C} \rightarrow \mathbb{C}$ is defined in (6.2). Its Legendre transform, $\mathcal{L}_{h^{(2)}} : sl_2 \mathbb{C} \oplus sl_2 \mathbb{C} \rightarrow sl_2 \mathbb{C} \oplus sl_2 \mathbb{C}$, is

$$\mathcal{L}_{h^{(2)}}(X_0, Y_0) = \frac{1}{i}(X_0, Y_0).$$

Remembering that $\sigma(Z_1)$ is a character of $h_1^*$, we conclude that $(\sigma(X_0^+), \sigma(Y_0^+)) \in \text{ch}h_1^* \oplus \text{ch}h_1^*$, and we saw that in $h_1^*$ there are no nontrivial characters, so $X_0^+ = Y_0^+ = 0$; therefore

$$\mathcal{L}_{h^{(2)}}(X_0, Y_0) = \frac{1}{i}(X_0^+, Y_0^+).$$

The AKS scheme states that the $H_1^\pm$ factors of the exponential curve

$$k(t) = e^{\mathcal{L}_{h^{(2)}}(X_0, Y_0)} = e^{i(2 X_0^+, Y_0^+)}$$

solve the original system of differential equations, for some time independent $(X_0^+, Y_0^+) \in h_1^*$. In order to analyze what is happening at the level of $H_0$ and $h_0$, we write these exponentials by the definition of the exponential, (2.2):

$$e^{i(2 X_0^+, Y_0^+)} = \left( e^{iX_0^+} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left( \frac{i}{2} \right)^n (\text{ad}_{X_0^+}^n)^{n-1} Y_0^+ \right).$$

Using the result (2.7), we have

$$\begin{cases}
\Pi_+ e^{i(2 X_0^+, Y_0^+)} = \left( h_0^+(t), -\text{Ad}_{X_0^+}^0 (k_0^-) \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left( \frac{i}{2} \right)^n (\text{ad}_{X_0^+}^n)^{n-1} Y_0^+ \right)
\Pi_- e^{i(2 X_0^+)} = \left( k_0^-(t), -\Pi_- A_0^-(k_0^-) \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left( \frac{i}{2} \right)^n (\text{ad}_{X_0^+}^n)^{n-1} Y_0^+ \right)
\end{cases}$$

where we wrote $e^{iX_0^+} = h_0^+(t) k_0^-(t)$.

The exponential for $X_0^+ \in su_2$ can be explicitly computed via

$$\exp \left( i \frac{X_0^+}{2} \right) = \cosh \left( \frac{t}{2} \right) + i x_0^+ \sinh \left( \frac{t}{2} \right)$$

where it was assumed that $||X|| = \sqrt{\text{det}X} = 1$. In order to obtain the factors $h_0^+(t)$ and $k_0^-(t)$, we write the time independent vector $X_0^+ = a_1 X_1 + a_2 X_2 + a_3 X_3$, so $||X|| = 1$ is equivalent to $a_1^2 + a_2^2 + a_3^2 = 1$. Thus, the curve in $SL(2, \mathbb{C})$

$$\exp \left( i \frac{X_0^+}{2} \right) = \begin{pmatrix}
\cosh(t/2) - a_3 \sinh(t/2) & -(a_1 - ia_2) \sinh(t/2) \\
-(a_1 + ia_2) \sinh(t/2) & \cosh(t/2) + a_3 \sinh(t/2)
\end{pmatrix}$$

can be factorized as $\exp \left( i \frac{X_0^+}{2} \right) = h_0^+(t) k_0^-(t)$, for $h_0^+(t) \subset SU(2)$ and $k_0^-(t) \subset B$, with

$$h_0^+(t) = \begin{pmatrix}
\cosh(t/2) - a_3 \sinh(t/2) & (a_1 - ia_2) \sinh(t/2) \\
-(a_1 + ia_2) \sinh(t/2) & \cosh(t/2) + a_3 \sinh(t/2)
\end{pmatrix}.$$
\[ k_0(t) = \begin{pmatrix} \sqrt{\cosh t - a_3 \sinh t} & - (a_1 - ia_2) \sinh t \\ \sqrt{\cosh t - a_3 \sinh t} & (\sqrt{\cosh t - a_3 \sinh t})^{-1} \end{pmatrix}. \] (6.8)

Let us now address the second component in the exponential solution (6.6). In order to handle it more easily, we shall introduce some notation. For the basis \{X_1, X_2, X_3\} on \( su(2) \), \( (6.1) \), we introduce the following notation: we write each element \( X \) in \( su(2) \) as

\[ X = \mathbf{x} \cdot \mathbf{X} := x_1 X_1 + x_2 X_2 + x_3 X_3, \]

where \( \mathbf{x} := (x_1, x_2, x_3) \) is the real 3-vector formed by the components of \( X \). Let us assume additionally that the 3-vector \( \mathbf{x} := (x^1, x^2, x^3) \) has unit norm \( \mathbf{x} \cdot \mathbf{x} = 1 \); then we can establish the following.

**Lemma.** The map \((\text{ad}_{k_0})^n : su(2) \rightarrow su(2)\), for arbitrary \( n \in \mathbb{N} \), has the formula

\[ (\text{ad}_{k_0})^n(Y_0^+) = \begin{cases} (-1)^{\frac{n}{2} + 1} 2^n [(\mathbf{x} \cdot \mathbf{y}) X_0^+ - Y_0^+] & \text{for } n \text{ even} \\ (-1)^{\frac{n}{2}} 2^{-n} [X_0^+, Y_0^+] & \text{for } n \text{ odd} \end{cases} \]

where \( Y_0^+ := \mathbf{y} \cdot \mathbf{X} \).

Thus, the second component in the right-hand side of (6.6) is

\[
- \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left( \frac{it}{2} \right)^n (\text{ad}_{k_0})^{n-1} Y_0^+
= \frac{i}{2} (t - \sinh t) (\mathbf{x} \cdot \mathbf{y}) X_0^+ + \frac{i}{2} (\sinh t) Y_0^+ + \frac{1}{4} (\cosh t - 1) [X_0^+, Y_0^+]
\]

and, finally, the factorization of this term is obtained by following the relations (2.7), to get

\[ X_i^+(t) = \Pi_+ \text{Ad}_{k_0}(t) \left( \frac{i}{2} (t - \sinh t) (\mathbf{x} \cdot \mathbf{y}) X_0^+ + \frac{i}{2} (\sinh t) Y_0^+ + \frac{1}{4} (\cosh t - 1) [X_0^+, Y_0^+] \right) \] (6.9)

and

\[ X_i^-(t) = \Pi_- (k_0(t)) \left( \frac{i}{2} (t - \sinh t) (\mathbf{x} \cdot \mathbf{y}) X_0^+ + \frac{i}{2} (\sinh t) Y_0^+ + \frac{1}{4} (\cosh t - 1) [X_0^+, Y_0^+] \right). \] (6.10)

Using an expression analogous to (2.9) for the adjoint action of \( b \) on \( su_2 \),

\[ \text{Ad}_{k_0}(t) Z_0^+ = t(k_0) Z_0^+ (k_0)^{-1} + \gamma^{-1} (\text{Ad}_{(k_0)^{-1} \gamma} (Z_0^+)), \]

we have that, for \( Z_0^+ \in su_2 \),

\[ \begin{cases} \Pi_+ \text{Ad}_{k_0}(t) \Pi_+ (k_0(t)) Z_0^+ = \gamma^{-1} (\text{Ad}_{(k_0)^{-1} \gamma} (Z_0^+)) \\ \Pi_- (k_0(t)) Z_0^+ = (k_0)^{-1} (k_0)^{-1} Z_0^+ \end{cases}. \]

In summary, joining the explicit forms of curves (6.7) and (6.8) on the group factors with the curves (6.9) and (6.10) we get the solution for the Hamiltonian system (4.11) with the Hamilton function \( H^{(2)}(h_1, Z_1) \) defined at the beginning of this subsection and assuming \( R_0 = Z_0 = 0 \).
7. Conclusions

Starting from a Poisson–Lie group, we have constructed integrable systems on the semidirect product with its Lie algebra showing that most of the standard issues of integrability and factorization are well suited in this framework, allowing for wider classes of systems. The construction of integrable systems on the fibers of $\Psi : T^*H \rightarrow T^*H_-$ by using the Dirac brackets introduces, through the character of $h_-$, some extra parameters in the systems which can be regarded as a nontrivial background endowing the dynamical system [4]. We have obtained the Poisson–Lie structure on the semidirect product having a very simple relation with the original one, although it does not map coboundaries into coboundaries. This construction allows us to supply each tangent bundle with two Poisson structures: a nondegenerate one which is derived from the canonical symplectic structure on the associated cotangent bundle through some linear bijection, and a Poisson–Lie one inherited from the semidirect product Lie group structure on the trivialization of the tangent bundle. Moreover, the construction can be iterated on iterated semidirect products, giving rise to a chain of phase spaces sharing both the Poisson structures.

On this chain, we get collective systems from the left translation momentum map on the whole Lie group, and by means of the Dirac brackets we project the collective dynamics onto a class of phase spaces which are isomorphic to the tangent bundle of one of the factors. That means that the Dirac bracket produces nontrivial integrable systems on these phase subspaces from the collective Hamiltonians on the whole phase space. The associated Hamiltonian vector fields turn out to be dressing vectors associated with the Poisson–Lie structures of the previous step in the chain, and they fail to be Hamiltonian in relation to the Poisson–Lie structure of the corresponding level.

The systems thus obtained are integrable by factorization of the Lie group in the previous step, and this can be traced back to the factorization of the initial Lie group of the chain. Moreover, for some special Hamiltonians we get a tower of integrable systems where the dynamical system at each level replicates one level down by projecting on the second component of the semidirect product.

Thus, the construction presented provides a setting for new class of systems that are integrable by factorization which are obtained from semidirect products of Lie groups.

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