A Differential Game Control Problem in Finite Horizon with an Application to Portfolio Optimization

Brahim El Asri* and Hafid Lalioui†

Abstract

This paper considers a new class of deterministic finite-time horizon, two-player, zero-sum differential games (DGs) in which the maximizing player is allowed to take continuous and impulse controls whereas the minimizing player is allowed to take impulse control only. We seek to approximate the value function, and to provide a verification theorem for this class of DGs. We first, by means of dynamic programming principle (DPP) in viscosity solution (VS) framework, characterize the value function as the unique VS to the related Hamilton-Jacobi-Bellman-Isaacs (HJBI) double-obstacle equation. Next, we prove that an approximate value function exists, that it is the unique solution to an approximate HJBI double-obstacle equation, and converges locally uniformly towards the value function of each player when the time discretization step goes to zero. Moreover, we provide a verification theorem which characterizes a Nash-equilibrium for the DG control problem considered. Finally, by applying our results, we derive a new continuous-time portfolio optimization model, and we provide related computational algorithms.

Keywords Zero-sum differential game, Impulse control, Viscosity solution, Discrete approximation, Verification theorem, Nash-equilibrium, Continuous-time portfolio optimization.

MS Classifications (2020) 49K35, 49L20, 49L25, 49N70, 49N90, 91G10.

JEL Classifications (2020) C61, C62, C63, C72, C73, G11.

1 Introduction

Optimal control (OC) theory is an important field of research due to its connections with partial differential equations (PDEs) and many fields of engineering such as mathematical finance. As a consequence, OC problems can be used for designing numerical algorithms to nonlinear PDEs arising from many optimization problems, we refer for the instant to Bensoussan and Lions [10], Fleming and Rishel [38], Fleming and Soner [39] and Pham [52] (see also [3, 5, 46]). Impulse control and differential game (DG) problems appear in many practical situations, for example in mathematical finance one can consider the option pricing and the control of exchange rate problems by Bernard [11], Bernard et al. [12] and Bertola et al. [13] (see also Barles [4], Dharmatti [25, 26], Yong [59], Zhang [60] and [10] for more information). The rigorous mathematical study of OC problems and DGs gives rise to some non-linear partial differential equation (PDE), usually called Hamilton-Jacobi-Bellman (HJB) equation for classic OC problems and Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation for DGs, satisfied

*Ibn Zohr University, Lab. LISAD, Equipe Aide à la decision, ENSA, B.P. 1136, Agadir, Morocco. E-mail: b.elasri@uiz.ac.ma.
†Ibn Zohr University, Lab. LISAD, Equipe Aide à la decision, ENSA, B.P. 1136, Agadir, Morocco. E-mail: hafid.lalioui@edu.uiz.ac.ma.
by the value function related to the control problem or DG. In most cases, even in very simple, these value functions are not sufficiently smooth, then the related PDE needs to be studied in viscosity solution (VS) framework. Introduced in 1980s by Crandall and Lions [22] (see also Crandall et al. [23, 24]) to circumvent the fact that the value function of control problems or DGs is not smooth enough, the notion of VS provides very powerful means to study in great generality and gives a rigorous formulation of the related PDEs to these control problems or DGs. The notion of value function has then a key role in the theory of OC problems and DGs, and the related PDEs should be considered in the viscosity sense.

Differential Game Control Problem. In a previous work El Asri and Lalioui [28], a deterministic two-player, zero-sum, DG where each player uses both continuous and impulse controls in infinite-time horizon was studied (see also El Asri et al. [29]). In [28], we proved under Isaac’s condition that the lower and upper value functions coincide. In [29], the zero-sum deterministic impulse controls game problem we have considered involves only impulse controls in infinite-time horizon, where a new Hamilton-Jacobi-Bellman-Isaacs (HJBI) quasi-variational inequality (QVI) was defined to prove, under a proportional property assumption on the maximizing player cost, that the value functions coincide and turn out to be the unique VS to the defined new HJBI QVI. The problem considered in this paper and the obtained results extend those in [28, 29], and provides an application to continuous-time portfolio optimization problem.

This paper studies a new class of deterministic finite-time horizon, two-player, zero-sum, continuous and impulse controls DG, defined by the $\mathbb{R}^n$-valued state vector $y_{t,x}(s)$ solution of the dynamical equation ($E$) below:

$$
y_{t,x}(s) = x + \int_t^s b(r, y_{t,x}(r); \theta(r)) dr + \sum_{m \geq 1} g(\tau_m, y_{t,x}(\tau_m^-); \xi_m) \mathbb{I}_{[\tau_m, T]}(s) \prod_{k \geq 1} \mathbb{I}_{[\tau_m, T]}(s) + \sum_{k \geq 1} g(\rho_k, y_{t,x}(\rho_k^-); \eta_k) \mathbb{I}_{[\rho_k, T]}(s),$$

for time variables $T \in (0, +\infty)$, $t \in [0, T]$ and $s \in [t, T]$, with initial state $y_{t,x}(t^-) = x \in \mathbb{R}^n$, where $y_{t,x}(t^-) := \lim_{t' \to t} y_{t,x}(t')$. In the differential form, for $s \neq \tau_m, s \neq \rho_k$ and the initial state $x$, the dynamical equation ($E$) is governed by the following controlled ordinary differential equation (ODE):

$$\dot{y}_{t,x}(s) = b(s, y_{t,x}(s); \theta(s)), \text{ and } y_{t,x}(t^-) = x,$$

where

$$\dot{y}_{t,x}(s) := \frac{dy_{t,x}(s)}{ds},$$

$\theta(.) \in \Theta(t, T)$ being the continuous control in $\Theta(t, T)$, the space of measurable functions from $[t, T] \subset \mathbb{R}_+$ into $\mathbb{R}_t$, and $b$ is a function that satisfies the following assumption:

[Hb] Dynamic The function $b : (s, y, \theta) \in [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^l \to b(s, y; \theta) \in \mathbb{R}^n$ is continuous w.r.t. $s$ uniformly in $y$ and $\theta$, Lipschitz-continuous w.r.t. $y$ uniformly in $s$ and $\theta$ with constant $C_b > 0$, and continuous w.r.t. $\theta$. Moreover, $b$ satisfies $\|b(s, y; \theta)\|_{\infty} \leq M$ for any $(s, y, \theta) \in [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^l$ and some positive constant $M$.

The state vector $y_{t,x}(s)$, in addition to the continuous evolution due to the ODE above, undergoes impulses (jumps) $\xi_m$ and $\eta_k$ at certain impulse stopping times $\tau_m$ and $\rho_k$, respectively, that is:

$$\left\{
\begin{aligned}
y_{t,x}(\tau_m^+) &= y_{t,x}(\tau_m^-) + g(\tau_m, y_{t,x}(\tau_m^-); \xi_m) \prod_{k \geq 1} \mathbb{I}_{[\tau_m, \rho_k]}(s), t \leq \tau_m \leq T, \xi_m \neq 0; \\
y_{t,x}(\rho_k^+) &= y_{t,x}(\rho_k^-) + g(\rho_k, y_{t,x}(\rho_k^-); \eta_k), t \leq \rho_k \leq T, \eta_k \neq 0,
\end{aligned}\right.$$

where $y_{t,x}(s^-) := \lim_{s' \uparrow s} y_{t,x}(s')$ and $y_{t,x}(s^+) := \lim_{s' \downarrow s} y_{t,x}(s')$, under the following assumption on the two functions $g$ and $h$:
\[ (\text{Impulses Form}) \] The function \( g_\xi : (s, y, \xi) \in [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow g_\xi(s, y; \xi) \in \mathbb{R}^n \) (resp. \( g_\eta : (s, y, \eta) \in [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^q \rightarrow g_\eta(s, y; \eta) \in \mathbb{R}^q \)) is Lipschitz-continuous w.r.t. \( s \), uniformly in \( y \) and \( \xi \) (resp. \( \eta \)), with constant \( \hat{C}_g > 0 \) (resp. \( \hat{C}_g > 0 \)), and Lipschitz-continuous w.r.t. \( y \), uniformly in \( s \) and \( \xi \) (resp. \( \eta \)), with constant \( C_{g_\xi} > 0 \) (resp. \( C_{g_\eta} > 0 \)).

**HJBI Equation.** Initiated in the 1950s by Bellman [9], the dynamic programming principle (DPP) leads, for our deterministic finite-time horizon, two-player, zero-sum, DG control problem, to a non-linear PDE satisfied by the Elliott-Kalton [34,35] value function of the game, and given by the following system:

\[
\begin{align*}
\max \left\{ \min \left[ -\frac{\partial}{\partial s} v(s, y) + \lambda v(s, y) + H(s, y, D_y v(s, y)) , v(s, y) - \mathcal{H}^c_{\sup} v(s, y) \right] ; \right. \\
\left. v(s, y) - \mathcal{H}^c_{\inf} v(s, y) \right] = 0, \quad \text{on } [t, T) \times \mathbb{R}^n; \\
\end{align*}
\]

(HJBI)

where the Hamiltonian \( H \) and the two non-local cost operators \( \mathcal{H}^c_{\sup}, \mathcal{H}^c_{\inf} \) are classic expressions given, in Sect. 2.3 below, by (\( H \)) and (\( \mathcal{H}^c_{\sup}, \mathcal{H}^c_{\inf} \)), respectively. The above system (HJBI) called Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation, or dynamic programming equation (DPE), and we will refer to it as HJBI equation. By combining the notion of VS for the HJBI equation (HJBI), with comparison principle for these solutions, we characterize the value function of the zero-sum DG control problem studied as the unique VS of the HJBI equation (HJBI), and this can then be used to obtain further results. Indeed, our paper will provide a discrete-time approximation for the HJBI equation (HJBI). Let \( h > 0 \) be the time discretization step, \( h_0 \) be a positive number, and \( \Phi(h) \) be a continuous function such that \( \Phi(0) = 1 \) and \( 0 < \Phi(h) < 1 \) for \( 0 < h < h_0 \), the approximate equation (HJBI\(_h\)) of the HJBI equation (HJBI) will be given by the following system:

\[
\begin{align*}
\max \left\{ \min \left[ H_h(s, y, v_h(s, y)), v_h(s, y) - \Phi(h) \mathcal{H}^c_{\sup} v_h(s, y) \right] ; v_h(s, y) - \Phi(h) \mathcal{H}^c_{\inf} v_h(s, y) \right] = 0, \quad \text{on } [t, T) \times \mathbb{R}^n; \\
v_h(T, y) = G(y) \text{ for all } y \in \mathbb{R}^n,
\end{align*}
\]

(HJBI\(_h\))

where the approximate Hamiltonian \( H_h \) is defined, in Sect. 2.3 below, by (\( H_h \)). We may use this approximate equation (HJBI\(_h\)) to give some computational aspects for our zero-sum DG control problem. Indeed, the convergence of the approximate value function, unique solution of the approximate equation (HJBI\(_h\)), to the unique bounded uniformly continuous VS of the HJBI equation (HJBI), leads to a numerical approach for the considered DG control problem illustrated by the Nash-equilibrium (NE) of Sect. 5 and the algorithms of Sect. 6.3.

**Previous Literature.** Regarding the literature on optimal impulse control problems and DGs with impulse controls, one my find a number of variants of OC problems with impulse, for example Liu et al. [47] have considered the case where the number of jump instants is fixed, and Reddy et al. [53] have studied a problem when the impulse instants are known a priori, the literature on DGs with impulse controls is sparse, zero-sum games with one player using piece-wise continuous controls and the other using impulses were studied in a deterministic setting in [4,59], and in a stochastic setting in [60] and Azimzadeh [2] (see also Issacs [44]). Barles et al. [6] and [11,12] introduced impulse control in zero-sum DGs to study an option pricing problem. Impulse control problems are typically solved using two main approaches, one based on Bellman’s DPP [9], and another using Pontryagin’s maximum principle [15] to compute the value function (see e.g. [13,14]). Recent papers by Cosso [21] and El Asri and Mazid [30] consider dynamic programming approach for zero-sum stochastic DGs where both players use only impulse control. Works by Aid et al. [1], Basei et al. [8], Campi and De Santis [18] and Sadana et al. [54,55] study some nonzero-sum DGs with impulse controls. We mention that in [1] authors studied a DG between two notions that have different targets for the currency exchange rate, and provided a system of QVIs that needs to be solved in order to compute the NE. In [54] the necessary and sufficient conditions for the existence of an open-loop NE for a class of DGs with impulse control were formulated. Impulse control
problems are typically solved using two main approaches, one based on Bellman’s [9] DPP, and another using Pontryagin’s maximum principle [15] to compute the value function (see e.g. [13, 14]). Regarding discrete-time approximation of HJB equation of deterministic control theory, we cite the works by Falcone [37], Gonzalez and Rofman [41,42], Capuzzo-Dolcetta [19], Capuzzo-Dolcetta and Ishii [20], and recent works by El Farouq [32,33] related to deterministic impulse control problems (see also [7, 16, 17, 56, 57]).

Application in Mathematical Finance. The use of OC methods to analyze financial market models has expanded at a remarkable rate after the revolutionary works by Markowitz [48] and Merton [49–51]. Many researches have dealt with the role of OC in portfolio optimization, including Eastham and Hastings [27], Hastings [43] and Korn [45]. In Sect. 6, a deterministic finite-time horizon, two-player, zero-sum, impulse controls DG approach for continuous-time portfolio optimization will be given. We first adjust the functions $b$, $g_e$ and $g_i$ of the dynamical equation $(E)$ to our portfolio optimization problem, then, for time variable $s \in [t, T]$ and a fixed positive real discount factor $\lambda$, we consider the following discounted terms:

1. A running gain/cost of integral type $\int_t^T f^\pi(s, y^\pi_{t,x}(s); \theta(s)) \exp(-\lambda(s-t))dr$, giving by the running gain/cost function $f^\pi := L^\pi - U^\pi$, where $L^\pi$ and $U^\pi$ denote, respectively, the investor’s stocks holding cost and his instantaneous utility function;

2. The total jump costs $\sum_{m \geq 1} c^\pi(\tau_m, y^\psi_{t,x}(\tau_m^-); \xi_m) \exp(-\lambda(\tau_m - t)) \mathbb{1}_{\{\tau_m \leq T\}} \prod_{k \geq 1} \mathbb{1}_{\{\tau_m \neq \rho_k\}}$,

and $\sum_{k \geq 1} \chi^\pi(\rho_k, y^\psi_{t,x}(\rho_k^-); \eta_k) \exp(-\lambda(\rho_k - t)) \mathbb{1}_{\{\rho_k \leq T\}}$ for the maximizing player—$\xi$ (market) and the minimizing player—$\eta$ (investor), respectively, with impulse stopping times $\tau_m$, $\rho_k$ and impulse values $\xi_m$, $\eta_k$;

3. A terminal gain/cost $G^\pi(y^\psi_{t,x}(T)) \exp(-\lambda(T-t))$ giving by the function $G^\pi$,

with the assumption that the flow of funds is between the market and the investor who reacts immediately to the market whereas the market is not so quick in reacting to the investor’s moves. We note that $\psi := (\theta(.), u := (\tau_m, \xi_m)_{m \in \mathbb{N}^+}$ represents the admissible continuous-impulse control for maximizing player—$\xi$ (market) and $v := (\rho_k, \eta_k)_{k \in \mathbb{N}^+}$ is the admissible impulse control for minimizing player—$\eta$ (investor). Thus we make our deterministic finite-time horizon, DG framework for the continuous-time portfolio optimization problem. Using the three discounted terms in the above, we can define an Elliott-Kalton [34,35] value function $v(t, w)$ for our portfolio optimization problem which represents the investor’s lost in the worst-case scenario, we then apply our results to derive a new continuous-time portfolio optimization model. Following [20,37,41,42], we derive some computational aspects for $v(t, w)$ from the approximate equation (HJB)$_h$).

Contribution. By establishing existence and uniqueness results for the considered class of DGs in viscosity sense, providing discrete-time approximation method of their HJB equation (HJBI) which leads to a NE, and applying to mathematical finance, our paper contributes to both the theory and applications of DGs with impulse controls, and leads to a new continuous-time portfolio optimization model where the investor tries to counteract to dangerous scenarios that can happen because of market price fluctuations. To the best of our knowledge the literature on deterministic DGs does not provide any theoretical or computational means to study the class of DGs we have considered in this paper.

The outline of the paper is the following: in Sect. 2, we formulate the zero-sum DG control problem studied and we define its value function, then we give the DPP and regularity results. In Sect. 3, by means of the VS framework, we investigate the HJB equation (HJBI) that characterizes the value function of the game studied. Sect. 4 deals with the approximate equation (HJBI)$_h$) and discusses the convergence of the approximation scheme. More precisely, we prove that the approximate value function converges, as the discretization step goes to zero, locally uniformly towards the value function of the considered game. In Sect. 5, we expose a verification theorem for identifying a NE strategies derived from the convergence result of Sect. 4. Finally, in Sect. 6, we apply the theory we have developed to derive a new continuous-time portfolio optimization model where the market is playing against the investor and wishes to maximize his discounted terminal cost, we give a portfolio strategy.
2 Formulation of the Game Problem and Preliminary Results

2.1 Deterministic Zero-Sum Differential Game Control Problem

We will be given the precise statement of our zero-sum differential game (DG) control problem, the definition of its related value functions and some preliminary results. The state vector \( y_{t,x}(s) \) of the deterministic two-player, zero-sum continuous and impulse controls DG considered is given, for finite-time horizon with time variables \( T \in (0, +\infty), t \in [0, T] \), by the solution of the following dynamical system:

\[
\begin{align*}
\dot{y}_{t,x}(s) &= b(s, y_{t,x}(s); \theta(s)), \quad s \neq \tau_m, s \neq \rho_k, s \in [t, T]; \\
y_{t,x}(\tau_m^+) &= y_{t,x}(\tau_m^-) + g_{\xi}(\tau_m, y_{t,x}(\tau_m^-); \xi_m) \prod_{k \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_k\}}, \quad \tau_m \in [t, T], \quad \xi_m \neq 0; \\
y_{t,x}(\rho_k^+) &= y_{t,x}(\rho_k^-) + g_{\eta}(\rho_k, y_{t,x}(\rho_k^-); \eta_k), \quad \rho_k \in [t, T], \quad \eta_k \neq 0; \\
y_{t,x}(T^-) &= x \in \mathbb{R}^n \text{ (initial state)},
\end{align*}
\]

the evolution of the state system (S), described by the mapping \( y_{t,x} : [t, T] \to \mathbb{R}^n \), is controlled by two players:

i. A maximizing player—\( \xi \) who uses both continuous control \( \theta(.) \) and impulse control \( u := (\tau_m, \xi_m)_{m \in \mathbb{N}^*}; \)

ii. A minimizing player—\( \eta \) who adopts only impulse control \( v := (\rho_k, \eta_k)_{k \in \mathbb{N}^*}, \)

where \( \mathbb{N}^* := \mathbb{N} \setminus \{0\} \). The mapping \( y_{t,x} : [t, T] \to \mathbb{R}^n \) is called the response or the state corresponding to controls \( \theta(.,), u \) and \( v \). These controls are defined, for our zero-sum DG control problem, as follows:

**Definition 2.1 (Continuous and Impulse Controls)** We let the continuous control \( \theta(.) \) and the impulses controls \( u \) and \( v \), related to the zero-sum DG control problem studied, be defined by:

i. A continuous control \( \theta(.) \in \Theta(t, T) \) is giving by a map \( \theta : [t, T] \to \mathbb{R}^l \), where \( \Theta(t, T) \) denotes the set of all measurable functions of \([t, T] \subset \mathbb{R}_+ \) to \( \mathbb{R}^l \);

ii. An impulse control \( u := (\tau_m, \xi_m)_{m \in \mathbb{N}^*} \in U(t, T) \) for player—\( \xi \) (resp. \( v := (\rho_k, \eta_k)_{k \in \mathbb{N}^*} \in V(t, T) \) for player—\( \eta \)) is defined by the non-decreasing impulse time sequence \( \{\tau_m\}_{m \in \mathbb{N}^*} \) (resp. \( \{\rho_k\}_{k \in \mathbb{N}^*} \)) of \([t, T] \), and by the impulse value (or size) sequence \( \{\xi_m\}_{m \in \mathbb{N}^*} \) (resp. \( \{\eta_k\}_{k \in \mathbb{N}^*} \)) of elements of \( U \subset \mathbb{R}^p \) (resp. \( V \subset \mathbb{R}^q \)), where \( U(t, T) \) (resp. \( V(t, T) \)) is the space of all impulse controls \( u \) (resp. \( v \)).

We denote, for notational brevity, \( \Theta = \Theta(t, T), U = U(t, T), V = V(t, T) \) and \( \Psi = \Theta \times U \). In the system (S), the product \( \prod_{k \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_k\}} \) signifies that when the two players act together on the system at the same time, only the action of minimizing player—\( \eta \) is tacking into account. Assumptions on the data, related to system (S), were given in Sect. 1.

**Remark 2.1** By assumptions \( H_0 \) and \( H_1 \), the existence of a unique global solution of the above dynamical system (S) is guaranteed and will be denoted by \( y_{t,x}^{\psi,v}(s) \) at time \( s \), for \( \psi := (\theta(.,), u) \in \Psi \) and \( v \in \mathbb{V} \).

The gain (resp. cost) functional \( J \) for maximizing player—\( \xi \) (resp. minimizing player—\( \eta \)) is defined, for \( \psi := (\theta(.,), u) \in \Psi \) and \( v \in \mathbb{V} \) being the admissible controls for the two players, as follows:

\[
J(t, x; \psi, v) := \int_t^T f(s, y_{t,x}^{\psi,v}(s); \theta(s)) \exp(-\lambda(s-t))ds \\
- \sum_{m \geq 1} c(\tau_m, y_{t,x}^{\psi,v}(\tau_m^-); \xi_m) \exp(-\lambda(\tau_m - t)) \mathbb{I}_{\{\tau_m \leq T\}} \prod_{k \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_k\}} \\
+ \sum_{k \geq 1} \chi(\rho_k, y_{t,x}^{\psi,v}(\rho_k^-); \eta_k) \exp(-\lambda(\rho_k - t)) \mathbb{I}_{\{\rho_k \leq T\}} \\
+ G(y_{t,x}^{\psi,v}(T)) \exp(-\lambda(T-t)),
\]
where \( y_{t,x}^{\psi,v}(s) \) is the response to controls \( \psi \) and \( v \) at time \( s \). The functional \( J \) will be considered under the following classical assumptions on the given running gain/cost function \( f \), impulse cost functions \( c, \chi \), and terminal gain \( G \), where \( \lambda \) is a fixed positive real that represents the discount factor:

**[H] (Running Gain)** We assume that the function \( f : [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R} \) is continuous w.r.t. \( s \) uniformly in \( y \) and \( \theta \), Lipschitz-continuous w.r.t. \( y \) uniformly in \( s \) and \( \theta \) with constant \( C_f > 0 \), and continuous w.r.t. \( \theta \). Moreover, \( f \) satisfies \( \|f(s, y; \theta)\|_\infty \leq M \) for any \( (s, y, \theta) \in [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^l \) and some positive constant \( M \);

**[Hc,\chi] (Impulses Cost)** The impulse cost functions \( c : [0, +\infty) \times \mathbb{R}^n \times U \subset \mathbb{R}^p \to \mathbb{R}_+ \) and \( \chi : [0, +\infty) \times \mathbb{R}^n \times V \subset \mathbb{R}^q \to \mathbb{R}_+ \) are from \( [0, +\infty) \times \mathbb{R}^n \) and two convex cones \( U \) and \( V \), respectively, into \( \mathbb{R}_+ \), nonnegative, and satisfy \( c(., .; 0) = \chi(., .; 0) = 0 \) and the zero lower bound property given by:

\[
\inf_{(s, y, \xi) \in [0, +\infty) \times \mathbb{R}^n \times U \setminus \{0\}} c(s, y; \xi) > 0, \quad \text{and} \quad \inf_{(s, y, \eta) \in [0, +\infty) \times \mathbb{R}^n \times V \setminus \{0\}} \chi(s, y; \eta) > 0.
\]

The function \( c \) (resp. \( \chi \)) is Lipschitz-continuous w.r.t. \( y \), uniformly in \( s \) and \( \xi \) (resp. \( \eta \)), with constant \( C_c > 0 \) (resp. \( C_\chi > 0 \)) and continuous w.r.t. \( s \) and \( \xi \) (resp. \( \eta \)). Moreover, for all \( (s, y) \in [0, +\infty) \times \mathbb{R}^n \), \( \xi_1, \xi_2 \in U \) and \( \eta_1, \eta_2 \in V \), we let the impulse cost functions satisfy the following:

\[
\begin{align*}
\{ & c(s, y; \xi_1 + \xi_2) \leq c(s, y; \xi_1) + c(s, y; \xi_2); \\
& \chi(s, y; \eta_1 + \eta_2) \leq \chi(s, y; \eta_1) + \chi(s, y; \eta_2); \\
& \chi(s, y; \eta_1) \leq \chi(s, y; \eta_2); \\
& \chi(s, y; \eta_2) \leq \chi(s, y; \eta_1);
\end{align*}
\]

**[HG] (Terminal Gain)** We let the function \( G : \mathbb{R}^n \to \mathbb{R} \) be bounded, Lipschitz-continuous with constant \( C_G > 0 \) and satisfies, for all \( y \in \mathbb{R}^n \) at time \( T \), the following no terminal impulse condition:

\[
\sup_{\xi \in U} \left\{ G(y + g_\xi(T, y; \xi)) - c(T, y; \xi) \right\} \leq G(y) \leq \inf_{\eta \in V} \left\{ G(y + g_\eta(T, y; \eta)) + \chi(T, y; \eta) \right\}.
\]

Note that the functional \( J \) represents a gain for the maximizing player and a cost for the minimizing, it is the criterion which player—\( \xi \) wants to maximize and player—\( \eta \) wants to minimize. In the other words, \( -J \) is the cost player—\( \eta \) has to pay, so the sum of the costs of the two players is null, which explains the name zero-sum.

**Remark 2.2** Assumptions \( H_f, H_c, \chi \) and \( H_G \) provide the classical framework for the study, in the viscosity solution (VS) framework, of the zero-sum DG control problem considered in this paper. In the rest of the paper:

1. We let \( n, p, q \) and \( l \) be some fixed positive integers, \( k, m \in \mathbb{N}^* \), \( T \in (0, +\infty) \), \( t \in [0, T] \) and \( s \in [t, T] \);

2. We denote by \( |.| \) and \( \|.| \) the Euclidean vector norm in \( \mathbb{R} \) and \( \mathbb{R}^n \), respectively, and by \( \|.|_\infty \) the infinite norm in the space of bounded continuous functions. \( \square \)

Before moving to the notions of non-anticipative strategy and value function, we first give the following proposition:

**Proposition 2.1** (Estimates on the Trajectories) Assume \( H_f \) and \( H_g \). Then we have, for all \( x, x' \in \mathbb{R}^n \), \( t \in [0, T] \) and \( t' \in [t, T] \), the following estimates on the trajectories:

i. \( \|y_{t,x}^{\psi,v}(s) - x\| \leq M(s - t) \) for any \( s \in [t, T] \);

ii. \( \|y_{t',x'}^{\psi,v}(s) - y_{t,x}^{\psi,v}(s)\| \leq \exp(C(T - t'))(\|x' - x\| + M(t' - t)) \) for any \( s \in [t', T] \),

for all \( \psi := (\theta(., u) \in \Psi \) and \( v \in \mathcal{V} \), where \( C \) and \( M \) are two real positive constants. \( \square \)

**Proof.** The proof of this result is classic.
2 FORMULATION OF THE GAME PROBLEM AND PRELIMINARY RESULTS

We now assume that one player knows just the current and past choices of the control made by his opponent. Thus, following Elliott and Kalton [34,35], we are given an information pattern for the two players by introducing the notion of non-anticipative strategy for our zero-sum DG control problem (see also Evans and Souganidis [36]) as follows:

**Definition 2.2 (Non-Anticipative Strategy)** A strategy for player $-\xi$ is a map $\alpha : \mathcal{V} \to \Psi$; it is non-anticipative, if, for any $v_1, v_2 \in \mathcal{V}$, $T > 0$ and $t \in [0,T]$, $v_1 \equiv v_2$ on $[t,T]$ implies $\alpha(v_1) \equiv \alpha(v_2)$ on $[t,T]$, i.e., if $\alpha(v_1) := (\theta_1(\cdot), u_1) \in \Psi$ and $\alpha(v_2) := (\theta_2(\cdot), u_2) \in \Psi$ with $v_1 \equiv v_2$ then $\theta_1(s) = \theta_2(s)$ and $u_1 \equiv u_2$ for any $t \leq s \leq T$. We denote by $\mathcal{A}$ the set of all non-anticipative strategies $\alpha$ for player $-\xi$.

Similarly, the set of all non-anticipative strategies $\beta$ for player $-\eta$ is denoted by $\mathcal{B}$ as

$$
\mathcal{B} := \left\{ \beta : \Psi \to \mathcal{V} \mid \theta_1(s) = \theta_2(s) \text{ and } u_1 \equiv u_2 \text{ on } [t,T] \text{ for all } \theta_1(\cdot), \theta_2(\cdot) \in \Theta, \ u_1, u_2 \in \mathcal{U}, \right.

T > 0, \ t \in [0,T] \text{ implies } v_1 := \beta(\theta_1(s), u_1) \equiv v_2 := \beta(\theta_2(s), u_2) \text{ on } [t,T] \text{ for all } t \leq s \leq T \right\}.
$$

We then give the definitions of the lower and the upper value functions related to our problem.

**Definition 2.3 (Value Functions)** The definitions of the lower value function ($V^-$) and the upper value function ($V^+$) of the zero-sum DG control problem with the gain/cost functional $J : [0,T] \times \mathbb{R}^n \times \Psi \times \mathcal{V} \to \mathbb{R}$, related to system (S), are given by the following expressions:

$$
V^-(t,x) := \inf_{\beta \in \mathcal{B}} \sup_{\psi \in \Psi} J(t,x;\psi;\beta(\psi)); \quad (V^-)
$$

$$
V^+(t,x) := \sup_{\alpha \in \mathcal{A}} \inf_{v \in \mathcal{V}} J(t,x;\alpha(v),v). \quad (V^+)
$$

If $V^-(t,x) = V^+(t,x)$ we say that the game, with initial point $x \in \mathbb{R}^n$ at initial time $t \in [0,T]$, has a value. We denote the value function of the zero-sum DG control problem by:

$$
V(t,x) := V^-(t,x) = V^+(t,x). \quad (V)
$$

Next, we give some properties concerning the value functions.

**Proposition 2.2 (Boundedness)** Assume $H_b$, $H_g$, $H_f$, $H_{c,\chi}$ and $H_G$. Then the lower value ($V^-$) and the upper value ($V^+$) are bounded in $[0,T] \times \mathbb{R}^n$.

**Proof.** The proof of this result is classic, see e.g. [6].

**Proposition 2.3 (Time-Continuity)** Assume $H_b$, $H_g$, $H_f$, $H_{c,\chi}$ and $H_G$. Then the lower value ($V^-$) and the upper value ($V^+$) are continuous with respect to time variable.

**Proof.** The proof of this result is classic, see e.g. [6].

The next section is devoted to announcing some regularity results for the value functions with respect to the state variable.
2.2 Dynamic Programming Principle and Regularity Results

We first give the Bellman’s [9] dynamic programming principle (DPP) for the zero-sum DG control problem considered:

**Theorem 2.1** (Dynamic Programming Principle) Assume $H_b, H_g, H_f, H_{c,\lambda}$ and $H_G$. For all $x \in \mathbb{R}^n$ and $t' \in [t, T]$, the lower value ($V^-$) and the upper value ($V^+$) satisfy, respectively,

\[
V^-(t, x) = \inf_{\beta \in \mathcal{B}} \sup_{\psi \in \Psi} \left\{ \int_t^{t'} f(s, y_{t,x}^\psi(s); \theta(s)) \exp(-\lambda(s - t)) ds + \sum_{m \geq 1} c(\tau_m, y_{t,x}^\psi(\tau_m); \xi_m) \exp(-\lambda(\tau_m - t)) \mathbb{I}_{\{\tau_m \leq t'\}} \prod_{k \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_k\}} + \sum_{k \geq 1} \chi(\rho_k, y_{t,x}^\psi(\rho_k^-); \eta_k) \exp(-\lambda(\rho_k - t)) \mathbb{I}_{\{\rho_k \leq t\}} + V^-(t', y_{t,x}^\psi(t')) \exp(-\lambda(t' - t)) \right\}
\]

and

\[
V^+(t, x) = \sup_{\alpha \in \mathcal{A}} \inf_{v \in \mathcal{V}} \left\{ \int_t^{t'} f(s, y_{t,x}^{\alpha,v}(s); \theta(s)) \exp(-\lambda(s - t)) ds + \sum_{m \geq 1} c(\tau_m, y_{t,x}^{\alpha,v}(\tau_m); \xi_m) \exp(-\lambda(\tau_m - t)) \mathbb{I}_{\{\tau_m \leq t'\}} \prod_{k \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_k\}} + \sum_{k \geq 1} \chi(\rho_k, y_{t,x}^{\alpha,v}(\rho_k^-); \eta_k) \exp(-\lambda(\rho_k - t)) \mathbb{I}_{\{\rho_k \leq t\}} + V^+(t', y_{t,x}^{\alpha,v}(t')) \exp(-\lambda(t' - t)) \right\}
\]

**Proof.** The proof of this result is classic, see e.g. [28]. \hfill \Box

**Proposition 2.4** (State-Continuity) Assume $H_b, H_g, H_f, H_{c,\lambda}$ and $H_G$. Then there exists a real positive constant $M$ such that for all $x, x' \in \mathbb{R}^n$, and $t \in [0, T]$, the lower value ($V^-$) and the upper value ($V^+$) satisfy

\[
|v(t, x') - v(t, x)| \leq M||x' - x||.
\]

**Proof.** We give only the proof for the lower value ($V^-$), similarly for the upper value ($V^+$). Let $t \in [0, T]$, fix $x, x' \in \mathbb{R}^n$ and an arbitrary $\varepsilon > 0$, and first pick a non-anticipative strategy $\beta^\varepsilon \in \mathcal{B}$ for minimizing player $-\eta$ such that the following inequality holds true:

\[
V^-(t, x') \geq \sup_{\psi \in \Psi} J(t, x'; \psi, \beta^\varepsilon(\psi)) - \frac{\varepsilon}{2},
\]

then we choose $\psi^\varepsilon := (\theta^\varepsilon(.), u^\varepsilon := (\tau^\varepsilon_m, \xi^\varepsilon_m)_{m \in \mathbb{N}^+}) \in \Psi$, an admissible continuous-impulse control for maximizing player $-\xi$, such that

\[
V^-(t, x) \leq \sup_{\psi \in \Psi} J(t, x; \psi, \beta^\varepsilon(\psi)) \leq J(t, x; \psi^\varepsilon, \beta^\varepsilon(\psi^\varepsilon)) + \frac{\varepsilon}{2}.
\]

Thus we get

\[
V^-(t, x) - V^-(t, x') \leq J(t, x; \psi^\varepsilon, \beta^\varepsilon(\psi^\varepsilon)) - J(t, x'; \psi^\varepsilon, \beta^\varepsilon(\psi^\varepsilon)) + \varepsilon.
\]
It follows, for $\beta^\varepsilon := (\rho_k^\varepsilon, \eta_k^\varepsilon)_{k \in \mathbb{N}^*} \in \mathcal{V}$, that

$$V^-(t, x) - V^-(t, x') \leq \int_t^{t'} \left[ f(s, y_{t,x}^\varepsilon, \beta^\varepsilon(s); \theta^\varepsilon(s)) - f(s, y_{t,x'}^\varepsilon, \beta^\varepsilon(s); \theta^\varepsilon(s)) \right] \exp(-\lambda(s-t)) ds$$

$$- \sum_{m \geq 1} c(\tau_m^\varepsilon, y_{t,x}^\varepsilon, \beta^\varepsilon(\tau_m^\varepsilon); \xi_m^\varepsilon) \exp(-\lambda(\tau_m^\varepsilon - t)) \mathbb{I}_{\{\tau_m^\varepsilon \leq t\}} \prod_{k \geq 1} \mathbb{I}_{\{\tau_m^\varepsilon \neq \rho_k^\varepsilon\}}$$

$$+ \sum_{k \geq 1} \chi(\rho_k^\varepsilon, y_{t,x}^\varepsilon, \beta^\varepsilon(\rho_k^\varepsilon)); \eta_k^\varepsilon) \exp(-\lambda(\rho_k^\varepsilon - t)) \mathbb{I}_{\{\rho_k^\varepsilon \leq t\}}$$

$$+ \sum_{m \geq 1} c(\tau_m^\varepsilon, y_{t,x'}^\varepsilon, \beta^\varepsilon(\tau_m^\varepsilon); \xi_m^\varepsilon) \exp(-\lambda(\tau_m^\varepsilon - t)) \mathbb{I}_{\{\tau_m^\varepsilon \leq t\}} \prod_{k \geq 1} \mathbb{I}_{\{\tau_m^\varepsilon \neq \rho_k^\varepsilon\}}$$

$$- \sum_{k \geq 1} \chi(\rho_k^\varepsilon, y_{t,x'}^\varepsilon, \beta^\varepsilon(\rho_k^\varepsilon)); \eta_k^\varepsilon) \exp(-\lambda(\rho_k^\varepsilon - t)) \mathbb{I}_{\{\rho_k^\varepsilon \leq t\}}$$

$$+ \left[ G(y_{t,x}^\varepsilon, \beta^\varepsilon(T)) - G(y_{t,x'}^\varepsilon, \beta^\varepsilon(T)) \right] \exp(-\lambda(T - t)) + \varepsilon.$$

Then, from the DPP property (2.2) for $t' > t$, we get

$$V^-(t, x) - V^-(t, x') \leq \int_t^{t'} \left[ f(s, y_{t,x}^\varepsilon, \beta^\varepsilon(s); \theta^\varepsilon(s)) - f(s, y_{t,x'}^\varepsilon, \beta^\varepsilon(s); \theta^\varepsilon(s)) \right] \exp(-\lambda(s-t)) ds$$

$$- \sum_{m \geq 1} c(\tau_m^\varepsilon, y_{t,x}^\varepsilon, \beta^\varepsilon(\tau_m^\varepsilon); \xi_m^\varepsilon) \exp(-\lambda(\tau_m^\varepsilon - t)) \mathbb{I}_{\{\tau_m^\varepsilon \leq t\}} \prod_{k \geq 1} \mathbb{I}_{\{\tau_m^\varepsilon \neq \rho_k^\varepsilon\}}$$

$$+ \sum_{k \geq 1} \chi(\rho_k^\varepsilon, y_{t,x}^\varepsilon, \beta^\varepsilon(\rho_k^\varepsilon)); \eta_k^\varepsilon) \exp(-\lambda(\rho_k^\varepsilon - t)) \mathbb{I}_{\{\rho_k^\varepsilon \leq t\}}$$

$$+ \sum_{m \geq 1} c(\tau_m^\varepsilon, y_{t,x'}^\varepsilon, \beta^\varepsilon(\tau_m^\varepsilon); \xi_m^\varepsilon) \exp(-\lambda(\tau_m^\varepsilon - t)) \mathbb{I}_{\{\tau_m^\varepsilon \leq t\}} \prod_{k \geq 1} \mathbb{I}_{\{\tau_m^\varepsilon \neq \rho_k^\varepsilon\}}$$

$$- \sum_{k \geq 1} \chi(\rho_k^\varepsilon, y_{t,x'}^\varepsilon, \beta^\varepsilon(\rho_k^\varepsilon)); \eta_k^\varepsilon) \exp(-\lambda(\rho_k^\varepsilon - t)) \mathbb{I}_{\{\rho_k^\varepsilon \leq t\}}$$

$$+ \left[ V^-(t', y_{t,x}^\varepsilon, \beta^\varepsilon(t')) - V^-(t', y_{t,x'}^\varepsilon, \beta^\varepsilon(t')) \right] \exp(-\lambda(t' - t)) + \varepsilon.$$

Thus, by assumptions on functions $f$, $c$ and $\chi$, we get

$$V^-(t, x) - V^-(t, x') \leq \int_t^{t'} C_f \|y_{t,x}^\varepsilon, \beta^\varepsilon(s) - y_{t,x'}^\varepsilon, \beta^\varepsilon(s)\| \exp(-\lambda(s-t)) ds$$

$$- \sum_{m \geq 1} C_c \|y_{t,x}^\varepsilon, \beta^\varepsilon(\tau_m^\varepsilon) - y_{t,x'}^\varepsilon, \beta^\varepsilon(\tau_m^\varepsilon)\| \exp(-\lambda(\tau_m^\varepsilon - t)) \mathbb{I}_{\{\tau_m^\varepsilon \leq t\}} \prod_{k \geq 1} \mathbb{I}_{\{\tau_m^\varepsilon \neq \rho_k^\varepsilon\}}$$

$$+ \sum_{k \geq 1} C_\chi \|y_{t,x}^\varepsilon, \beta^\varepsilon(\rho_k^\varepsilon) - y_{t,x'}^\varepsilon, \beta^\varepsilon(\rho_k^\varepsilon)\| \exp(-\lambda(\rho_k^\varepsilon - t)) \mathbb{I}_{\{\rho_k^\varepsilon \leq t\}}$$

$$+ \left[ V^-(t', y_{t,x}^\varepsilon, \beta^\varepsilon(t')) - V^-(t', y_{t,x'}^\varepsilon, \beta^\varepsilon(t')) \right] \exp(-\lambda(t' - t)) + \varepsilon.$$

By Propositions 2.1 and 2.2, we deduce that there exist some constants $C > 0$ and $C_v > 0$ such that

$$V^-(t, x) - V^-(t, x') \leq C_f \|x - x'\| \int_t^{t'} \exp ((C - \lambda)(s-t)) ds$$

$$- C_c \|x - x'\| \sum_{m \geq 1} \exp ((C - \lambda)(\tau_m^\varepsilon - t)) \mathbb{I}_{\{\tau_m^\varepsilon \leq t\}} \prod_{k \geq 1} \mathbb{I}_{\{\tau_m^\varepsilon \neq \rho_k^\varepsilon\}}$$

$$+ C_\chi \|x - x'\| \sum_{k \geq 1} \exp ((C - \lambda)(\rho_k^\varepsilon - t)) \mathbb{I}_{\{\rho_k^\varepsilon \leq t\}}$$

$$+ 2C_v \exp (-\lambda(t' - t)) + \varepsilon.$$
Now, if $C < \lambda$ the sums in the right-hand side of (2.4) are finite, then there exists a positive constant $K$ such that we have
\[ V^-(t, x) - V^-(t, x') \leq \frac{C_f}{C - \lambda} \|x - x'\| \left[ \exp((C - \lambda)(t' - t)) - 1 \right] \]
\[ + K \|x - x'\| + 2C_v \exp(-\lambda(t' - t)) + \varepsilon, \]
(2.5)
taking into account the boundedness of $\exp((C - \lambda)(t' - t))$ and $\exp(-\lambda(t' - t))$ for any $t \in [0, T]$, the arbitrariness of $\varepsilon$ and the fact that $x$ and $x'$ play symmetrical roles in the left hand side of the above inequality one might deduce the existence of a positive constant $M$ which satisfies
\[ |V^-(t, x) - V^-(t, x')| \leq M \|x - x'\| \text{ for all } t \in [0, T]. \]
In the case where $\lambda < C$, we choose $t'$ such that $\exp(-C(t' - t)) = \|x - x'\|^{1/2}$ with $\|x - x'\| < 1$. Hence, in the right-hand side of (2.5), the first term equals to
\[ \frac{C_f}{C - \lambda} \|x - x'\|^{1/2} \left[ \exp(-\lambda(t' - t)) - \|x - x'\|^{1/2} \right], \]
while the term $2C_v \exp(-\lambda(t' - t))$ is bounded for any $t \in [0, T]$. We then deduce from the arbitrariness of $\varepsilon$ that there exist a positive constant $M_1$ which satisfies
\[ V^-(t, x) - V^-(t, x') \leq M_1 \|x - x'\| \text{ for all } t \in [0, T], \]
again the roles of $x$ and $x'$ being symmetrical, we then conclude. Finally, in the case where $C = \lambda$, it suffice to let some constant $\hat{\lambda} < \lambda = C$, so we go back to the above inequality (2.4) and we proceed, since
\[ \exp((C - \lambda)(t' - t)) < \exp((C - \hat{\lambda})(t' - t)) \text{ and } \exp(-\lambda(t' - t)) < \exp(-\hat{\lambda}(t' - t)), \]
as above with the case $\hat{\lambda} \neq C$. Thus the lower value function is Lipschitz-continuous w.r.t. state variable, which completes the proof. \[ \square \]

Next, we prove the uniform continuity of the functions $x \to \mathcal{H}^X_{\text{inf}} v(t, x)$ and $x \to \mathcal{H}^c_{\text{sup}} v(t, x)$ for $t \in [0, T]$.

**Proposition 2.5** Let $t \in [0, T]$ and $x \to v(t, x)$ be a uniformly continuous function in $\mathbb{R}^n$. Then the two functions $x \to \mathcal{H}^X_{\text{inf}} v(t, x)$ and $x \to \mathcal{H}^c_{\text{sup}} v(t, x)$ are uniformly continuous in $\mathbb{R}^n$.

**Proof.** Let $t \in [0, T]$ and $x \to v(t, x)$ be a uniformly continuous function. We give the proof for the function $x \to \mathcal{H}^X_{\text{inf}} v(t, x)$, similarly for $x \to \mathcal{H}^c_{\text{sup}} v(t, x)$. Let $x, x' \in \mathbb{R}^n$, and choose an arbitrary $\varepsilon > 0$ and $(\rho_\varepsilon, \eta_\varepsilon) \in \{t, T\} \times \mathcal{V}$ such that
\[ \mathcal{H}^X_{\text{inf}} v(t, x') + \varepsilon \geq v(t, x + g_\eta(\rho_\varepsilon, x'; \eta_\varepsilon)) + \chi(\rho_\varepsilon, x'; \eta_\varepsilon), \]
thus
\[ \mathcal{H}^X_{\text{inf}} v(t, x) - \mathcal{H}^X_{\text{inf}} v(t, x') \leq v(t, x + g_\eta(\rho_\varepsilon, x; \eta_\varepsilon)) + \chi(\rho_\varepsilon, x; \eta_\varepsilon) \]
\[ - v(t, x' + g_\eta(\rho_\varepsilon, x'; \eta_\varepsilon)) - \chi(\rho_\varepsilon, x'; \eta_\varepsilon) + \varepsilon. \]
It follows from assumptions on functions $\varepsilon$ and $\chi$ the existence of a positive constant $C$ such that
\[ \mathcal{H}^X_{\text{inf}} v(t, x) - \mathcal{H}^X_{\text{inf}} v(t, x') \leq C \|x - x'\| + \varepsilon, \]
since $x$ and $x'$ play symmetrical roles and $\varepsilon$ is arbitrary, then the function $x \to \mathcal{H}^X_{\text{inf}} v(t, x)$ is uniformly continuous in $\mathbb{R}^n$. \[ \square \]
2.3 Hamilton-Jacobi-Bellman-Isaacs Equation and Approximate Equation

Since in the definition of the lower value \((V^-)\) the inf is taken over non-anticipative strategies whereas in the definition of the upper value \((V^+)\) it is taken over admissible controls, and similarly the sup is taken over different sets in the definitions of the lower and the upper value functions, then the inequality \(V^+(t, x) \leq V^-(t, x)\) for all \((t, x) \in [0, T] \times \mathbb{R}^n\) is false in general. In addition the inequality \(V^-(t, x) \leq V^+(t, x)\) for all \((t, x) \in [0, T] \times \mathbb{R}^n\) is not obvious at first glance. We then prove, in a rather indirect way by using the associated HJBI equation (HJBI), that the zero-sum DG control problem studied has a value. The dynamic programming equation (DPE) associated to our deterministic finite-time horizon, two-player, zero-sum DG control problem, which turns out to be the same for the two value functions because the two players cannot act simultaneously on the system, is derived from DPP and is given by the following expression:

\[
\max \left\{ \min \left[ -\frac{\partial}{\partial s} v(s, y) + \lambda v(s, y) + H(s, y, D_y v(s, y)), v(s, y) - \mathcal{H}^c_{\sup} v(s, y) \right] \right\} = 0, \quad \text{on } [t, T] \times \mathbb{R}^n; \\

v(T, y) = G(y) \quad \text{for all } y \in \mathbb{R}^n,
\]

where \(\frac{\partial}{\partial s} v(s, y)\) denotes the time derivative, \(D_y v(s, y)\) the spatial gradient of the function \(v(s, y) : [t, T] \times \mathbb{R}^n \to \mathbb{R}\), with \(D_y := (\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n})^T\). The associated first-order Hamiltonian \((H)\) and the two obstacles, defined through the use of the maximum and minimum non-local cost operators \(\mathcal{H}^c_{\sup}\) and \(\mathcal{H}^c_{\inf}\), respectively, are given by the following:

**Definition 2.4** (Hamiltonian and Cost Operators) For any function \(v : [t, T] \times \mathbb{R}^n \to \mathbb{R}\), we define the first-order Hamiltonian \(H\) by:

\[
H(s, y, D_y v(s, y)) := \inf_{\theta \in \mathbb{R}^l} \left\{ -D_y v(s, y), b(s, y; \theta) - f(s, y; \theta) \right\}, \quad (H)
\]

where \(\cdot, \cdot\) denotes the inner product in \(\mathbb{R}^n\), and the two non-local cost operators \(\mathcal{H}^c_{\sup}\) and \(\mathcal{H}^c_{\inf}\) by:

\[
\mathcal{H}^c_{\sup} v(s, y) := \sup_{\xi \in \mathcal{V}} \left\{ v(s, y + g_\xi(s, y; \xi)) - c(s, y; \xi) \right\}; \quad (\mathcal{H}^c_{\sup})
\]

\[
\mathcal{H}^c_{\inf} v(s, y) := \inf_{\eta \in \mathcal{V}} \left\{ v(s, y + g_\eta(s, y; \eta)) + \chi(s, y; \eta) \right\}; \quad (\mathcal{H}^c_{\inf})
\]

We propose in this paper, for all \((s, y) \in [t, T] \times \mathbb{R}^n\), the approximate equation (HJBI\(_h\)) an approximation of the classic HJBI equation (HJBI). Let \(h\) be the time discretization step for the approximation we will be given, \(h_0\) be a positive number, and \(\Phi(h)\) be a continuous function such that \(\Phi(0) = 1\) and \(0 < \Phi(h) < 1\) for \(0 < h < h_0\):

\[
\max \left\{ \min \left[H_h(s, y, v_h(s, y)), v_h(s, y) - \Phi(h) \mathcal{H}^c_{\sup} v_h(s, y) \right] ; v_h(s, y) - \Phi(h) \mathcal{H}^c_{\inf} v_h(s, y) \right\} = 0, \quad \text{on } [t, T] \times \mathbb{R}^n; \\
v_h(T, y) = G(y) \quad \text{for all } y \in \mathbb{R}^n,
\]

where the approximate Hamiltonian \(H_h\) is defined as follows:

**Definition 2.5** (Approximate Hamiltonian) For any function \(v_h : [t, T] \times \mathbb{R}^n \to \mathbb{R}\), we define the approximate Hamiltonian \(H_h\) by:

\[
H_h(s, y, v_h(s, y)) := \inf_{\theta \in \mathbb{R}^l} \left\{ v_h(s, y) - (1 - \lambda h_v)(s + h, y + h b(s, y; \theta)) - h f(s, y; \theta) \right\}. \quad (H_h)
\]
Remark 2.3 The contribution of the paper is four-fold:

1. First, we prove that the lower value ($V^-$) and the upper value ($V^+$) are viscosity solutions to the HJBI equation (HJBI). Then we show, by proving a comparison principle, that the HJBI equation (HJBI) has a unique solution in viscosity sense, i.e., the zero-sum DG control problem studied admits a value ($V$);

2. Second, we prove that an approximate value function $v_h$ exists, that it is the unique solution of the approximate equation (HJBI$_h$). Then we show that $v_h$ converges, as the time discretization step $h$ goes to zero, locally uniformly towards the value function ($V$) of the zero-sum DG control problem;

3. Third, we prove a verification theorem for the zero-sum DG control problem considered, that is, the game has a Nash-equilibrium (NE) strategies;

4. Fourth, we apply our theory to continuous-time portfolio optimization problem to derive a new optimization model which leads to a new portfolio strategy.

3 Viscosity Characterization of the Value Functions

The value function of an OC problem is a solution to the corresponding HJB (or, HJBI) equation whenever it has sufficient regularity (see e.g. [39]). In other word, it requires that the HJB (or, HJBI) equation admits classical solutions, meaning that the solutions be smooth enough. Unfortunately, this is not necessarily the case even in very simple cases. To overcome this difficulty, the so-called viscosity solution (VS) was introduced in the early 80’s [22–24]. This new notion is a kind of non-smooth solutions, where if the value function is continuous, then, it is a solution to the HJB (or, HJBI) equation in the VS sense, whose key feature is to replace the conventional derivatives while maintaining the uniqueness of solutions under very mild conditions. These make the theory a powerful tool in tackling OC problems and DGs [3, 36, 39, 52, 56, 57]. We recall here the definition of a VS of the HJBI equation (HJBI) following [22–24]:

Definition 3.1 (Viscosity Solution) Let $v : [t, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function such that $v(T, y) = G(y)$ for any $y \in \mathbb{R}^n$. $v$ is called:

i. A viscosity sub-solution of the Hamilton-Jacobi-Bellman-Isaacs equation (HJBI) if for any $(\bar{s}, \bar{y}) \in [t, T] \times \mathbb{R}^n$ and any function $\phi \in C^{1,1}([t, T] \times \mathbb{R}^n)$ such that $v(\bar{s}, \bar{y}) = \phi(\bar{s}, \bar{y})$ and $(\bar{s}, \bar{y})$ is a local maximum point of $v - \phi$, we have

$$\max \left\{ \min \left[ - \frac{\partial \phi}{\partial s}(\bar{s}, \bar{y}) + \lambda v(\bar{s}, \bar{y}) + H(\bar{s}, \bar{y}, D_y \phi(\bar{s}, \bar{y})), v(\bar{s}, \bar{y}) - \mathcal{H}^c_{\text{sup}} v(\bar{s}, \bar{y}) \right] ; 
\v(\bar{s}, \bar{y}) - \mathcal{H}^c_{\text{inf}} v(\bar{s}, \bar{y}) \right\} \leq 0;$$

ii. A viscosity super-solution of the Hamilton-Jacobi-Bellman-Isaacs equation (HJBI) if for any $(s, y) \in [t, T] \times \mathbb{R}^n$ and any function $\phi \in C^{1,1}([t, T] \times \mathbb{R}^n)$ such that $v(s, y) = \phi(s, y)$ and $(s, y)$ is a local minimum point of $v - \phi$, we have

$$\max \left\{ \min \left[ - \frac{\partial \phi}{\partial s}(s, y) + \lambda v(s, y) + H(s, y, D_y \phi(s, y)), v(s, y) - \mathcal{H}^c_{\text{sup}} v(s, y) \right] ; 
\v(s, y) - \mathcal{H}^c_{\text{inf}} v(s, y) \right\} \geq 0;$$

iii. A viscosity solution of the Hamilton-Jacobi-Bellman-Isaacs equation (HJBI) if it is both a viscosity sub-solution and super-solution of the Hamilton-Jacobi-Bellman-Isaacs equation (HJBI).

The remainder of this section deals with the first contribution of the paper, as it is mentioned in Remark 2.3.
3.1 Existence of Viscosity Solutions for the HJBI Equation

The following theorem shows that the value functions of the zero-sum DG control problem studied satisfy the HJBI equation (HJBI) in VS sense.

**Theorem 3.1** Assume $H_b, H_g, H_f, H_{c,x}$ and $H_G$. The lower value ($V^-$) and the upper value ($V^+$) are viscosity solutions to the Hamilton-Jacobi-Bellman-Isaacs equation (HJBI).

**Proof.** We give only the proof for the lower value ($V^-$), similarly for the upper value ($V^+$). The proof of the theorem requires the following technical Lemmas 3.1 and 3.2:

**Lemma 3.1** Let assumptions $H_b, H_g, H_f, H_{c,x}$ and $H_G$ hold. Given any $(s,y) \in [t,T) \times \mathbb{R}^n$, the lower value ($V^-$) and the upper value ($V^+$) satisfy the following equation:

$$\max \left\{ \min \left[ 0, v(s,y) - \mathcal{H}_{\text{sup}} v(s,y) \right], v(s,y) - \mathcal{H}_{\text{inf}} v(s,y) \right\} = 0.$$

**Proof.** The proof can be found in the Appendix A.1.

**Remark 3.1** From the above Lemma 3.1 one may deduce, for any $(s,y) \in [t,T) \times \mathbb{R}^n$, that the lower value ($V^-$) and the upper value ($V^+$) satisfy the following:

i. $v(s,y) \leq H_{\text{inf}} v(s,y)$.

ii. If $v(s,y) < H_{\text{inf}} v(s,y)$ then $v(s,y) \geq H_{\text{sup}} v(s,y)$.

So we may regard $H_{\text{sup}} v(s,y)$ as a lower obstacle and $H_{\text{inf}} v(s,y)$ as an upper obstacle.

**Lemma 3.2** Let assumptions $H_b, H_g, H_f, H_{c,x}$ and $H_G$ hold. Given any function $\phi \in C^{1,1}([t,T) \times \mathbb{R}^n)$ such that for all $(s,y) \in [t,T) \times \mathbb{R}^n$ we have

$$-\frac{\partial}{\partial s} \phi(s,y) + \lambda \phi(s,y) + H(s,y,D_y \phi(s,y)) = \gamma > 0,$$

then there exists a non-anticipative strategy $\beta^\gamma \in \mathcal{B}$ for minimizing player $-\eta$ such that, for any $\psi := (\theta(\cdot),u) \in \Psi$ and $s$ tends to $t$, we have that

$$\int_t^s \left\{ \frac{\partial}{\partial r} \phi(r,y_{t,x}(r)) - \lambda \phi(r,y_{t,x}(r)) + D_y \phi(r,y_{t,x}(r)),b(r,y_{t,x}(r)) \theta(r) + f(r,y_{t,x}(r)) \theta(r) \right\} \exp(-\lambda \gamma(r-t)) dr \leq \frac{\gamma}{4}(s-t),$$

where $\beta^\gamma \psi \in \mathcal{V}$.

**Proof.** The proof can be found in the Appendix A.2.

**Proof of Theorem 3.1.** We give the proof for the lower value ($V^-$), similarly for the upper value ($V^+$). The proof is inspired from [3,4] and based on DPP. We start by proving the sub-solution property. Let $\phi$ be a function in $C^{1,1}([t,T) \times \mathbb{R}^n)$ and $(\bar{t},\bar{x}) \in [t,T) \times \mathbb{R}^n$ be such that $V^- - \phi$ achieves a local maximum at $(\bar{t},\bar{x})$ and $V^-(\bar{t},\bar{x}) = \phi(\bar{t},\bar{x})$. If $V^-(\bar{t},\bar{x}) - \mathcal{H}_{\text{sup}} V^- (\bar{t},\bar{x}) \leq 0$ the proof is finished, since from Remark 3.1 $V^-(\bar{t},\bar{x}) \leq H_{\text{inf}} V^- (\bar{t},\bar{x})$. Otherwise, for $\varepsilon > 0$ and without loss of generality, we assume that $V^-(\bar{t},\bar{x}) - \mathcal{H}_{\text{sup}} V^- (\bar{t},\bar{x}) \geq \varepsilon > 0$, then we proceed by contradiction. Since, from Remark 3.1, we have $V^-(\bar{t},\bar{x}) \leq H_{\text{inf}} V^- (\bar{t},\bar{x})$, we now explore the result of Lemma 3.2 by assuming first that

$$-\frac{\partial \phi}{\partial s}(\bar{t},\bar{x}) + \lambda \phi(\bar{t},\bar{x}) + H(\bar{t},\bar{x},D_y \phi(\bar{t},\bar{x})) = \gamma > 0,$$
then one can find a non-anticipative strategy $\beta^\gamma \in \mathcal{B}$ for minimizing player $-\eta$ such that, for any $\psi := (\theta(\cdot), u) \in \Psi$ and $s$ tends to $t$, we have that

\[
\int_t^s \left\{ \frac{\partial}{\partial r} \phi(r, y^\psi_{\bar{t}, \bar{x}}(r)) - \lambda \phi(r, y^\psi_{\bar{t}, \bar{x}}(r)) + D_y \phi(r, y^\psi_{\bar{t}, \bar{x}}(r)) b(r, y^\psi_{\bar{t}, \bar{x}}(r); \theta(r)) \right. \\
+ f(r, y^\psi_{\bar{t}, \bar{x}}(r); \theta(r)) \right\} \exp(-\lambda(r-t)) dr \leq -\frac{\gamma}{4} (s-t),
\]

where $\beta^\gamma(\psi) \in \mathcal{V}$. Thus,

\[
\int_t^s f(r, y^\psi_{\bar{t}, \bar{x}}(r); \theta(r)) \exp(-\lambda(r-t)) dr + \exp(-\lambda(s-t)) \phi(s, y^\psi_{\bar{t}, \bar{x}}(s)) - \phi(\bar{t}, \bar{x}) \leq -\frac{\gamma}{4} (s-t). \quad (3.2)
\]

Since $V^- - \phi$ has a local maximum at $(\bar{t}, \bar{x})$ and $V^- (\bar{t}, \bar{x}) = \phi(\bar{t}, \bar{x})$ we have, for $t = \bar{t}$ and $s - t$ small enough, that

\[
\|y_{\bar{t}, \bar{x}}^\psi(\psi)(s) - \bar{x}\| \to 0,
\]

which yields

\[
\exp(-\lambda(s-t)) \phi(s, y_{\bar{t}, \bar{x}}^\psi(\psi)(s)) - \phi(\bar{t}, \bar{x}) \geq \exp(-\lambda(s-t)) V^-(s, y_{\bar{t}, \bar{x}}^\psi(\psi)) - V^- (\bar{t}, \bar{x}).
\]

By plugging this into the inequality (3.2) we obtain, for $s = t'$ and $t' - t$ small enough,

\[
\inf_{\beta \in \mathcal{B}} \sup_{\psi \in \Psi} \left\{ \int_t^{t'} f(r, y^\psi_{\bar{t}, \bar{x}}(r); \theta(r)) \exp(-\lambda(r-t)) dr \\
+ V^-(t', y^\psi_{\bar{t}, \bar{x}}(t')) \exp(-\lambda(t'-t)) \right\} - V^- (\bar{t}, \bar{x}) \leq -\frac{\gamma}{4} (t' - t) < 0,
\]

which, without loss of generality when $t = \bar{t}$ and $t' - t < \tau_0 \wedge \rho_0$ and taking into account the DPP (2.2), yields a contradiction. Hence the lower value $V^-$ is a viscosity sub-solution to the HJBI equation (HJBI).

Next, we show the super-solution property. Let $\phi$ be a function in $C^{1,1}([t, T] \times \mathbb{R}^n)$ and $(\bar{t}, \bar{x}) \in [t, T] \times \mathbb{R}^n$ be such that $V^- - \phi$ achieves a local minimum at $(\bar{t}, \bar{x})$ in $I \times B_\delta(\bar{x})$, where $B_\delta(\bar{x})$ is the open ball of radius $\delta > 0$ centered at $\bar{x}$ and $I := [\bar{t} - \delta / 4, \bar{t} + \delta / 4]$, and $V^-(\bar{t}, \bar{x}) = \phi(\bar{t}, \bar{x})$. Now, for $\varepsilon > 0$ and without loss of generality, we assume that $V^- (\bar{t}, \bar{x}) - \mathcal{H}^V_\inf V^- (\bar{t}, \bar{x}) < \varepsilon < 0$ on $I \times B_\delta(\bar{x})$, otherwise, i.e., $V^- (\bar{t}, \bar{x}) = \mathcal{H}^V_\inf V^- (\bar{t}, \bar{x})$, the proof is finished. Therefore Remark 3.1 leads us to $V^- (\bar{t}, \bar{x}) \geq \mathcal{H}^V_\sup V^- (\bar{t}, \bar{x})$. We define

\[
s' = \inf \left\{ s \geq 0 : s \notin I \text{ and } y_{\bar{t}, \bar{x}}(s) \notin B_\delta(\bar{x}) \right\},
\]

then we let $t < s \leq s'$, and we proceed by contradiction. Assume that

\[
-\frac{\partial}{\partial s} (\bar{t}, \bar{x}) + \lambda \phi(\bar{t}, \bar{x}) + H (\bar{t}, \bar{x}, D_y \phi(\bar{t}, \bar{x})) = -\gamma < 0.
\]

By the definition of the first-order Hamiltonian $H$, one can find an element $\theta$ of $\mathbb{R}^l$ such that

\[
-\frac{\partial}{\partial s} (\bar{t}, \bar{x}) + \lambda \phi(\bar{t}, \bar{x}) - D_y \phi(\bar{t}, \bar{x}) b(\bar{t}, \bar{x}; \theta) - f(\bar{t}, \bar{x}; \theta) \leq -\gamma.
\]

Thus, there exists a non-anticipative strategy $\alpha^\gamma \in \mathcal{A}$ for maximizing player $-\xi$ such that for any $v \in \mathcal{V}$, $\alpha^\gamma (v) = \psi^\gamma := (\theta(\cdot), u)$, and for $s - t$ small enough and any $\beta \in \mathcal{B}$, we have that

\[
-\frac{\partial}{\partial s} \phi(s, y_{\bar{t}, \bar{x}}(\psi^\gamma)) + \lambda \phi(s, y_{\bar{t}, \bar{x}}(\psi^\gamma)) - D_y \phi(s, y_{\bar{t}, \bar{x}}(\psi^\gamma)) b(s, y_{\bar{t}, \bar{x}}(\psi^\gamma)) \leq -\frac{\gamma}{2}.
\]
Now we multiply both sides of the last inequality by \( \exp(-\lambda(s-t)) \) and integrate from \( t \) to \( t' \) to obtain

\[
\phi(t, x) - \exp(-\lambda(t'-t))\phi(t', y_{t,x}^{\psi, \beta(\psi)}) - \int_t^{t'} f(s, y_{s,x}^{\psi, \beta(\psi)}; \theta(s)) \exp(-\lambda(s-t))ds \leq -\frac{\gamma}{4}(t'-t). \tag{3.3}
\]

Since \( V^- - \phi \) has a local minimum at \((t, x)\) and \( V^-(t, x) = \phi(t, x) \) we have, for \( t = \xi \) and \( s - t \) small enough, that

\[
\|y_{t,x}^{\psi, \beta(\psi)}(s) - x\| \to 0,
\]

which gives

\[
\exp(-\lambda(s-t))\phi(s, y_{t,x}^{\psi, \beta(\psi)}(s)) - \phi(t, x) \leq \exp(-\lambda(s-t))V^-(y_{t,x}^{\psi, \beta(\psi)}(s)) - V^-(t, x),
\]

thus

\[
\exp(-\lambda(t'-t))V^-(y_{t,x}^{\psi, \beta(\psi)}(t')) + \int_t^{t'} f(s, y_{s,x}^{\psi, \beta(\psi)}; \theta(s)) \exp(-\lambda(s-t))ds \geq \frac{\gamma}{2}(t'-t) + V^-(t, x).
\]

By plugging this into (3.3), for \( t' - t \) small enough, we obtain

\[
\inf_{\beta \in B} \sup_{\psi \in \Psi} \left\{ \int_t^{t'} f\left(y_{s,x}^{\psi, \beta(\psi)}(s); \theta(s)\right) \exp(-\lambda(s-t))ds + V^-(t', y_{t,x}^{\psi, \beta(\psi)}(t')) \exp(-\lambda(t'-t)) \right\} - V^-(t, x) > 0,
\]

which, without loss of generality when \( t = \xi \) and \( t' - t < \tau_0 \land \rho_0 \) and taking into account the DPP (2.2), yields a contradiction, then the lower value \( V^- \) is a viscosity super-solution to the HJBI equation (HJBI). Hence deducing the thesis.

\[
\Box
\]

### 3.2 Uniqueness of Viscosity Solutions for the HJBI Equation

This section proves the comparison principle of viscosity solutions to the HJBI equation (HJBI), and shows that this equation has a unique bounded uniformly continuous VS. As a consequence, the value functions coincide, since they are viscosity solutions to the HJBI equation (HJBI). Thus the zero-sum DG considered has a value. We state the comparison principle as follows:

**Theorem 3.2** (Comparison Theorem) Assume \( H_b, H_g, H_f, H_{c,\chi} \) and \( H_G \). If \( u \) and \( v \) are, respectively, a bounded uniformly continuous sub-solution and super-solution to the Hamilton-Jacobi-Bellman-Isaacs equation (HJBI), satisfying \( u(T, \cdot) \leq v(T, \cdot) \), then we have

\[
\forall (t, x) \in [0, T] \times \mathbb{R}^n : u(t, x) \leq v(t, x).
\]

**Proof.** We start by giving the useful Proposition 3.1, for which the proof can be found in the Appendix A.3.

**Proposition 3.1** The Hamilton-Jacobi-Bellman-Isaacs equation (HJBI) is equivalent to the following equation:

\[
\begin{cases}
\lambda v(s, y) = \min_{i \in \{0,1\}} \left\{ \left(1 - i\right) \max_{j \in \{0,1\}} \left[ (1 - j) \left( \frac{\partial}{\partial s} v(s, y) + \sup_{\theta \in \mathbb{R}^l} \left\{ D_y v(s, y), b(s, y; \theta) + f(s, y; \theta) \right\} \right) 
+ j \lambda H^c_{\text{sup}} v(s, y) \right] + i \lambda H^\chi_{\text{inf}} v(s, y) \right\}, \quad \text{on } [t, T] \times \mathbb{R}^n. \\
v(T, y) = G(y) \text{ for all } y \in \mathbb{R}^n.
\end{cases}
\]

\[
\Box
\]
Remark 3.2 By using Proposition 3.1, the Definition 3.1 of the viscosity solution of the Hamilton-Jacobi-Bellman-Isaacs equation (HJBI) could be rewritten as the following:

A continuous function \( v \) in \( [t, T] \times \mathbb{R}^n \) which satisfies \( v(T, y) = G(y) \) for any \( y \in \mathbb{R}^n \), is a viscosity sub-solution (resp. super-solution) of the equation if and only if for any function \( \phi \in C^{1,1}([t, T] \times \mathbb{R}^n) \) and \(( \overline{s}, \overline{y}) \) (resp. \(( \underline{s}, \underline{y}) \)) \( \in [t, T] \times \mathbb{R}^n \) a local maximum (resp. minimum) point of \( v - \phi \) such that \( v(\overline{s}, \overline{y}) = \phi(\overline{s}, \overline{y}) \) (resp. \( v(\underline{s}, \underline{y}) = \phi(\underline{s}, \underline{y}) \)), we have

\[
\lambda v(\overline{s}, \overline{y}) \leq \min_{i \in \{0, 1\}} \left\{ (1 - i) \max_{j \in \{0, 1\}} \left[ (1 - j) \left( \frac{\partial \phi}{\partial s}(\overline{s}, \overline{y}) + \sup_{\theta \in \mathbb{R}} \left\{ D_y \phi(\overline{s}, \overline{y}), b(\overline{s}, \overline{y}; \theta) + f(\overline{s}, \overline{y}; \theta) \right\} \right) + j \lambda \mathcal{H}^c_{\sup} v(\overline{s}, \overline{y}) \right\}
\]

and

\[
\lambda v(\underline{s}, \underline{y}) \geq \min_{i \in \{0, 1\}} \left\{ (1 - i) \max_{j \in \{0, 1\}} \left[ (1 - j) \left( \frac{\partial \phi}{\partial s}(\underline{s}, \underline{y}) + \sup_{\theta \in \mathbb{R}} \left\{ D_y \phi(\underline{s}, \underline{y}), b(\underline{s}, \underline{y}; \theta) + f(\underline{s}, \underline{y}; \theta) \right\} \right) + j \lambda \mathcal{H}^c_{\sup} v(\underline{s}, \underline{y}) \right\}
\]

\( \square \)

The Lemma 3.3 below, for which the proof is obvious, will be useful later for deducing the thesis.

Lemma 3.3 If a continuous function \( v \) is a viscosity solution to the HJBI equation (HJBI) such that \( v(T, y) = G(y) \) for any \( y \in \mathbb{R}^n \), then for any \( 0 < \mu < 1 \) the function \( \mu v \) is a viscosity solution to the equation (HJBI\( \mu \)), given by:

\[
\begin{cases}
\max \left\{ \min \left\{ -\frac{\partial}{\partial s} v(s, y) + \lambda v(s, y) + H_\mu(s, y, D_y v(s, y)), v(s, y) - \mathcal{H}^c_{\sup} v(s, y) \right\}; \\
v(s, y) - \mathcal{H}^c_{\inf} v(s, y) = 0, \text{ on } [t, T) \times \mathbb{R}^n; \\
v(T, y) = \mu G(y) \text{ for all } y \in \mathbb{R}^n,
\end{cases}
\]

(HJBI\( \mu \))

where

\[
\mathcal{H}^c_{\sup} v(s, y) := \sup_{\xi \in U} \left\{ v(s, y + g_\xi(s, y; \xi)) - \mu c(s, y; \xi) \right\},
\]

\[
\mathcal{H}^c_{\inf} v(s, y) := \inf_{\eta \in V} \left\{ v(s, y + g_\eta(s, y; \eta)) + \mu c(s, y; \eta) \right\},
\]

and

\[
H_\mu(s, y, D_y v(s, y)) := \inf_{\theta \in \mathbb{R}^l} \left\{ -D_y v(s, y), b(s, y; \theta) - \mu f(s, y; \theta) \right\}.
\]

\( \square \)

Proof of Theorem 3.2. Now we are in a position to give the proof of the comparison result which is inspired from [6, 33]. Let \( u \) and \( v \) be, respectively, a bounded uniformly continuous viscosity sub-solution and supersolution to the HJBI equation (HJBI). Recalling, for all \( 0 < \mu < 1 \), Proposition 3.1 and Lemma 3.3, to get that \( \mu v \) is a viscosity sub-solution to the following equation:

\[
\begin{cases}
\lambda u(s, y) = \min_{i \in \{0, 1\}} \left\{ (1 - i) \max_{j \in \{0, 1\}} \left[ (1 - j) \left( \frac{\partial}{\partial s} u(s, y) + \sup_{\theta \in \mathbb{R}^l} \left\{ D_y u(s, y), b(s, y; \theta) + f(s, y; \theta) \right\} \right) + j \lambda \mathcal{H}^c_{\sup} u(s, y) \right\} \\
u(T, y) = \mu G(y) \text{ for all } y \in \mathbb{R}^n.
\end{cases}
\]
where the operators $\mathcal{H}^\chi_{inf}$ and $\mathcal{H}^\chi_{sup}$ are defined as in Lemma 3.3 and the function $u$ is from $[t, T] \times \mathbb{R}^n$ into $\mathbb{R}$. First, we assume that $M = \sup_{(t,x) \in [0,T] \times \mathbb{R}^n} (u(t,x) - v(t,x)) > 0$, if it is not the case, i.e., $M \leq 0$, the proof is then finished. If $\|u\|_\infty = 0$ we have

$$M_\mu = \sup_{(t,x) \in [0,T] \times \mathbb{R}^n} (\mu u(t,x) - v(t,x)) > 0,$$

otherwise, by letting $1 - M/(2\|u\|_\infty) \leq \mu < 1$ we also get that $M_\mu > 0$. Next, we divide the proof into the following three steps:

**Step 1.** Let $\varepsilon > 0$, $\beta > 0$ and consider for all $t \in (0, T)$ and $x, y \in \mathbb{R}^n$ the following test function:

$$\Gamma_{\mu,\varepsilon,\beta}(t, x, y) = \mu u(t, x) - v(t, y) - \frac{\|x-y\|^2}{\varepsilon^2} - \beta(\|x\|^2 + \|y\|^2).$$

Since $\Gamma_{\mu,\varepsilon,\beta}$ is a continuous function going to infinity when $x$ or $y$ does, then it admits a maximum point $(t_m, x_m, y_m)$ satisfying $M_{\mu,\varepsilon,\beta} = \Gamma_{\mu,\varepsilon,\beta}(t_m, x_m, y_m)$. We have for all $t \in (0, T)$ and $x, y \in \mathbb{R}^n$,

$$\mu u(t_m, x_m) - v(t_m, y_m) - \frac{\|x_m-y_m\|^2}{\varepsilon^2} - \beta(\|x_m\|^2 + \|y_m\|^2) \geq \mu u(t, x) - v(t, y) - \frac{\|x-y\|^2}{\varepsilon^2} - \beta(\|x\|^2 + \|y\|^2).$$

(3.5)

- Firstly, using inequality (3.5) with $(t, y) = (t_m, y_m)$, we get that $(t_m, x_m)$ is a maximal point of $\mu u(t, x) - \phi_u(t, x)$, where

$$\phi_u(t, x) = \frac{\|x-y_m\|^2}{\varepsilon^2} + \beta\|x\|^2,$$

then, since $\mu u$ is viscosity sub-solution of (3.4), we get

$$\lambda \mu u(t_m, x_m) \leq \min_{i \in \{0, 1\}} \left\{ (1-i) \max_{j \in \{0, 1\}} \left( 1-j \right) \sup_{\theta \in \mathbb{R}^l} \left\{ \lambda \left( \frac{2\|x_m-y_m\|}{\varepsilon^2} + 2\beta x_m, b(t_m, x_m; \theta) \right) + \mu f(t_m, x_m; \theta) \right\} + j \lambda \mathcal{H}^c_{sup} u(t_m, x_m) \right\} + i \lambda \mathcal{H}^\chi_{inf} u(t_m, x_m).$$

(3.6)

- Secondly, using inequality (3.5) with $(t, x) = (t_m, x_m)$, we get that $(t_m, y_m)$ is a minimal point of $v(t, y) - \phi_v(t, y)$, where

$$\phi_v(t, y) = -\frac{\|x_m-y\|^2}{\varepsilon^2} - \beta\|y\|^2,$$

then, since $v$ is viscosity super-solution of the HJBI equation (HJBI), by applying Proposition 3.1 we get

$$\lambda v(t_m, y_m) \geq \min_{i \in \{0, 1\}} \left\{ (1-i) \max_{j \in \{0, 1\}} \left( 1-j \right) \sup_{\theta \in \mathbb{R}^l} \left\{ \lambda \left( \frac{2\|x_m-y_m\|}{\varepsilon^2} - 2\beta y_m, b(t_m, y_m; \theta) \right) + f(t_m, y_m; \theta) \right\} + j \lambda \mathcal{H}^c_{sup} v(t_m, y_m) \right\} + i \lambda \mathcal{H}^\chi_{inf} v(t_m, y_m) \right\}.$$

(3.7)

Hence, using above inequalities (3.6) and (3.7), we get

$$\lambda (\mu u(t_m, x_m) - v(t_m, y_m)) \leq \min_{i \in \{0, 1\}} \left\{ (1-i) \max_{j \in \{0, 1\}} \left( 1-j \right) \sup_{\theta \in \mathbb{R}^l} \left\{ \lambda \left( \frac{2\|x_m-y_m\|}{\varepsilon^2} + 2\beta x_m, b(t_m, x_m; \theta) \right) + \mu f(t_m, x_m; \theta) \right\} + j \lambda \mathcal{H}^c_{sup} u(t_m, x_m) \right\} + i \lambda \mathcal{H}^\chi_{inf} u(t_m, x_m) \right\}$$

$$+ \max_{i \in \{0, 1\}} \left\{ (1-i) \min_{j \in \{0, 1\}} \left( 1-j \right) \inf_{\theta \in \mathbb{R}^l} \left\{ -\lambda \left( \frac{2\|x_m-y_m\|}{\varepsilon^2} - 2\beta y_m, b(t_m, y_m; \theta) \right) + f(t_m, y_m; \theta) \right\} - j \lambda \mathcal{H}^c_{sup} v(t_m, y_m) \right\} - i \lambda \mathcal{H}^\chi_{inf} v(t_m, y_m) \right\},$$

17
then we get
\[
\lambda(\mu u(t_m, x_m) - v(t_m, y_m)) \leq \max_{i \in \{0, 1\}} \left\{ \left(1 - i\right) \min_{j \in \{0, 1\}} \left(1 - j\right) \inf_{\theta \in \mathbb{R}} \left\{ \frac{2\|x_m - y_m\|}{\varepsilon^2}, \right. \right.
\]
\[
\left. b(t_m, x_m; \theta) - b(t_m, y_m; \theta) + 2\beta \left(\mu u(t_m, x_m) - \mu u(t_m, y_m)\right) + \mu f(t_m, x_m; \theta) - f(t_m, y_m; \theta) \right) + j\lambda \left(\mathcal{H}_{\text{sup}}^{\mu} \mu u(t_m, x_m) - \mathcal{H}_{\text{sup}}^{\mu} v(t_m, y_m)\right)
\]
\[
+ i\lambda \left(\mathcal{H}_{\text{inf}}^{\mu} \mu u(t_m, x_m) - \mathcal{H}_{\text{inf}}^{\mu} v(t_m, y_m)\right) \right\}.
\]

Thus from standing assumptions
\[
\lambda(\mu u(t_m, x_m) - v(t_m, y_m)) \leq \max \left\{ \min \left[ 2C_b \frac{\|x_m - y_m\|^2}{\varepsilon^2} + 2\beta\|b\|_{\infty} \left(\|x_m\| + \|y_m\|\right) + (1 - \mu)\|f\|_{\infty}, \right. \right.
\]
\[
\lambda \left(\mathcal{H}_{\text{sup}}^{\mu} \mu u(t_m, x_m) - \mathcal{H}_{\text{sup}}^{\mu} v(t_m, y_m) + \|\left(\mathcal{H}_{\text{sup}}^{\mu} \mu u - \mathcal{H}_{\text{sup}}^{\mu} v\right)\|_{\infty}\right); \right.
\]
\[
\lambda \left(\mathcal{H}_{\text{inf}}^{\mu} \mu u(t_m, x_m) - \mathcal{H}_{\text{inf}}^{\mu} v(t_m, y_m) + \|\left(\mathcal{H}_{\text{inf}}^{\mu} \mu u - \mathcal{H}_{\text{inf}}^{\mu} v\right)\|_{\infty}\right) \right\}.
\]

The following two steps investigate the right-hand side of inequality (3.8):

**Step 2.** We prove hereafter that
\[
\forall \eta > 0, \exists \varepsilon > 0, \beta_0 > 0, \forall \varepsilon \leq \varepsilon_0, \beta \leq \beta_0 : \frac{\|x_m - y_m\|^2}{\varepsilon^2} + \beta \left(\|x_m\|^2 + \|y_m\|^2\right) \leq \eta. \tag{3.9}
\]

We use inequality (3.5) for \(x = y\) in the right-hand side, then we get \(M_{\Gamma, u, \varepsilon, \beta} \geq \mu u(t, x) - v(t, x) - 2\beta\|x\|^2\). Further, we let \(\sup_{(t, x) \in [0, T] \times \mathbb{R}^n} (\mu u(t, x) - v(t, x))\) be reached in a point \((t^*, x^*)\), within \(\delta > 0\) arbitrary small, thus \(\mu u(t^*, x^*) - v(t^*, x^*) \geq M_{\mu} - \delta\). Now we choose \(\delta\) and \(\beta\) such that \(M_{\mu} - \delta - 2\beta\|x^*\|^2 > 0\), which is possible since \((t^*, x^*)\) depends only on \(\delta\). Thus we deduce
\[
M_{\Gamma, u, \varepsilon, \beta} \geq \mu u(t^*, x^*) - v(t^*, x^*) - 2\beta\|x^*\|^2
\]
\[
\geq M_{\mu} - \delta - 2\beta\|x^*\|^2
\]
\[
> 0. \tag{3.10}
\]

By letting \(r^2 = \mu\|u\|_{\infty} + \|v\|_{\infty}\), we get
\[
\|u\|_{\infty} \leq M_{\Gamma, u, \varepsilon, \beta} \leq r^2 - \frac{\|x_m - y_m\|^2}{\varepsilon^2} - \beta \left(\|x_m\|^2 + \|y_m\|^2\right),
\]
and
\[
\|x_m - y_m\| \leq r\varepsilon. \tag{3.11}
\]

Therefore, we introduce the following increasing function:
\[
m(w) = \max_{t \in [0, T], \|x - y\| \leq w} |v(t, x) - v(t, y)|,
\]
then, combining with (3.11), we obtain
\[
\mu u(t_m, x_m) - v(t_m, y_m) = \mu u(t_m, x_m) - v(t_m, x_m) + v(t_m, x_m) - v(t_m, y_m) \leq M_{\mu} + m(r\varepsilon).
\]
Thus, from (3.10) using the definition of $M_{\Gamma,\mu,\varepsilon,\beta}$, we get

$$M_{\mu} - \delta - 2\beta \|x^*\|^2 \leq M_{\Gamma,\mu,\varepsilon,\beta} \leq M_{\mu} + m(\varepsilon) - \frac{\|x_m - y_m\|^2}{\varepsilon^2} - \beta(\|x_m\|^2 + \|y_m\|^2),$$

then

$$\frac{\|x_m - y_m\|^2}{\varepsilon^2} + \beta(\|x_m\|^2 + \|y_m\|^2) \leq \delta + 2\beta \|x^*\|^2 + m(\varepsilon).$$

Now, we choose $\eta < 4M_{\mu}/3$ and we take $\delta = \eta/4$ and $\beta_0 = 1$ if $\|x^*\| = 0$, $\beta_0 = \varepsilon/(4\|x^*\|^2)$ if $\|x^*\| \neq 0$, to get the desired inequality (3.9). The proof is then complete. We also get for any $\beta \leq \beta_0$,

$$0 < M_{\mu} - \frac{3\eta}{4} \leq M_{\mu} - 2\beta \|x^*\|^2 \leq M_{\Gamma,\mu,\varepsilon,\beta} \leq \mu u(t_m, x_m) - v(t_m, y_m). \tag{3.12}$$

**Step 3.** To complete the proof it remains to show contradiction. By (3.9), for $\varepsilon = \varepsilon_0$ and $\beta = \beta_0$ we have

$$2\varepsilon_0 \|x_m - y_m\|^2/\varepsilon^2 \leq 2\varepsilon_0 \|x_m\| \leq \sqrt{\beta_0}, \quad \text{and} \quad \beta \|y_m\| \leq \sqrt{\beta_0}.$$ 

Then, for all $\beta \leq \beta_1 = \min\{\beta_0, \eta/\|b\|^2\}$, we get $2\beta \|b\| (\|x_m\| + \|y_m\|) \leq 4\eta$. Moreover, for all $\varepsilon \leq \varepsilon_1 = \min\{\varepsilon_0, \sqrt{\eta}/C_f\}$, we have $C_f (\|x_m - y_m\|) \leq \eta$. By Proposition 2.5, the two functions $x \to \mathcal{H}_{\mu,\beta}^\chi(t, x)$ and $x \to \mathcal{H}_{\sup}^\chi(t, x)$ are uniformly continuous for any $t \in [0, T)$, then, taking into account (3.11), we can find $\varepsilon_2 \leq \varepsilon_1$ such that for $\varepsilon \leq \varepsilon_2$,

$$\mathcal{H}_{\inf,\beta}^\chi(t_m, x_m) - \mathcal{H}_{\inf}^\chi(t_m, y_m) \leq \eta, \quad \text{and} \quad \mathcal{H}_{\sup}^{\chi/\beta}(t_m, x_m) - \mathcal{H}_{\sup}^\chi(t_m, y_m) \leq \eta.$$ 

Thus, from (3.8) for all $\varepsilon \leq \varepsilon_2$ and $\beta \leq \beta_1$, we get

$$\lambda(\mu u(t_m, x_m) - v(t_m, y_m)) \leq \max \left\{ \min \left[ (1 - \mu) \|f\|_\infty, \lambda \|H_{\sup}^{\chi,\mu} - H_{\sup}^\chi \|_\infty \right]; \right.$$

$$\lambda \|H_{\inf,\beta}^{\chi,\mu} - H_{\inf}^\chi \|_\infty \right\} + (5 + 2\varepsilon_0 + \lambda) \eta,$$

from (3.12) and the fact that $\eta$ is arbitrary we deduce

$$\lambda(\mu u - v)^+ \leq \max \left\{ \min \left[ (1 - \mu) \|f\|_\infty, \lambda \|H_{\sup}^{\chi,\mu} - H_{\sup}^\chi \|_\infty \right]; \right.$$

$$\lambda \|H_{\inf,\beta}^{\chi,\mu} - H_{\inf}^\chi \|_\infty \right\},$$

thus

$$\lambda(\mu u - v)^+ \leq \max \left\{ (1 - \mu) \|f\|_\infty, \lambda \|H_{\inf,\beta}^{\chi,\mu} - H_{\inf}^\chi \|_\infty \right\}. \tag{3.13}$$

Since for all $(s, y) \in [t, T) \times \mathbb{R}^n$,

$$\mathcal{H}_{\inf,\beta}^{\chi,\mu}(s, y) - \mathcal{H}_{\inf}^\chi(s, y) \leq \sup_{\eta \in V} \left( \mu u(s, y + g_\eta(s, y; \eta)) - v(s, y + g_\eta(s, y; \eta)) \right) + \sup_{\eta \in V} \left( (\mu - 1) \chi(s, y; \eta) \right), \tag{3.14}$$

and, from standing assumptions for all $(s, y) \in [t, T) \times \mathbb{R}^n$ and $\eta \in V \setminus \{0\}$, we have $\chi(s, y; \eta) > 0$. We then deduce, from (3.14) for $0 < \mu < 1$, that

$$\left\| (\mathcal{H}_{\inf,\beta}^{\chi,\mu} - \mathcal{H}_{\inf}^\chi) \right\|_\infty < \|\mu u - v\|^+.$$ 

Therefore, combining the two inequalities (3.13) and (3.15) yield that

$$\lambda(\mu u - v)^+ \leq (1 - \mu) \|f\|_\infty.$$ 

Hence, by letting $\mu \to 1$ and using the fact that $f$ is bounded, we get $\|\mu u - v\|^+ \leq 0$, which leads us to a contradiction and gives the desired comparison result, for any $(t, x) \in [0, T] \times \mathbb{R}^n$, $u(t, x) \leq v(t, x).$
4 DISCRETE APPROXIMATION OF THE HJBI EQUATION

Theorem 3.3 Assume $H_b, H_g, H_f, H_{c,\chi}$ and $H_G$. The Hamilton-Jacobi-Bellman-Isaacs equation (HJBI) has a unique bounded uniformly continuous viscosity solution.

Proof. Assume that $u$ and $v$ are two viscosity solutions to the HJBI equation (HJBI). We first use $u$ as a bounded uniformly continuous viscosity sub-solution and $v$ as a bounded uniformly continuous viscosity super-solution and we recall the comparison principle. Then we change the role of $u$ and $v$ to get $u(t, x) = v(t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$.

Let us now give a corollary which summarizes the principal results of this section, thus it gives the first contribution of the paper.

Corollary 3.1 Assuming $H_b, H_g, H_f, H_{c,\chi}$ and $H_G$, the lower value ($V^-$) and the upper value ($V^+$) coincide and the value function ($V$) of the deterministic finite-time horizon, two-player, zero-sum DG control problem is the unique bounded uniformly continuous viscosity solution to the Hamilton-Jacobi-Bellman-Isaacs equation (HJBI).

Next, we focus on the second contribution of the paper as mentioned in Remark 2.3. Using the fact that the value function ($V$) is the unique VS to the HJBI equation (HJBI) and studying the approximate equation (HJBI$_h$), a family of value functions converging to the value function ($V$) of each player is introduced, for which the limit when the time discretization step $h$ goes to zero is characterized either as the unique VS to the HJBI equation (HJBI), or as the limit when $h$ goes to zero of the unique solution of the approximate equation (HJBI$_h$).

4 Discrete Approximation of the HJBI Equation

This section discusses an approximation scheme to the solution of the HJBI equation (HJBI). In the other words, it gives an approximation scheme to the value function ($V$) of the zero-sum DG control problem studied. We mainly prove that the approximate equation (HJBI$_h$) has, for any time discretization step $0 < h < 1/\lambda$, a unique bounded continuous solution $v_h$ which converges locally uniformly towards the value function ($V$) when $h$ goes to zero. Such a result will be useful to characterize, by means of a verification theorem, some NE strategies for both players. This will be the subject of Sect. 5 which leads to some numerical aspects for computing the value function ($V$) and the related optimal controls of NE.

4.1 Uniqueness of the Approximate Value Function

We begin by giving the proposition below, then we prove that the approximate equation (HJBI$_h$) has a unique bounded continuous solution for any time discretization step $0 < h < 1/\lambda$.

Proposition 4.1 Solving the approximate Hamilton-Jacobi-Bellman-Isaacs equation (HJBI$_h$) is equivalent to solve the following equation:

$$v_h(s, y) = \min_{i \in \{0, 1\}} \left\{ (1 - i) \max_{j \in \{0, 1\}} \left[ (1 - j) \sup_{\theta \in \mathbb{R}^l} \left\{ (1 - \lambda h) v_h(s + h, y + hb(s, y; \theta)) + h f(s, y; \theta) \right\} ight] ight\}$$

$$+ j \Phi(h) H_{sup} \psi(s, y) + i \Phi(h) H_{inf} \psi(s, y), \text{ on } [t, T] \times \mathbb{R}^n;$$

$$v_h(T, y) = G(y) \text{ for all } y \in \mathbb{R}^n.$$  

Proof. Similar to the proof of Proposition 3.1.

Now we give the proof of the following theorem:
**Theorem 4.1** For any time discretization step $0 < h < 1/\lambda$, there exists a unique bounded continuous function $v_h$ solution to the approximate Hamilton-Jacobi-Bellman-Isaacs equation ($HJBI_h$).

**Proof.** We first rewrite the approximate equation ($HJBI_h$) as a fixed-point problem. Let $(s, y) \in [t, T] \times \mathbb{R}^n$ and $v_h(T, y) = G(y)$, from Proposition 4.1 we get that the equation ($HJBI_h$) is equivalent to $Fv_h(s, y) = v_h(s, y)$, where $F$ is a function from the space of bounded continuous (BC) functions on $[t, T] \times \mathbb{R}^n$ into the same space defined as follows:

$$
\begin{align*}
Fv(s, y) = \min_{i \in \{0, 1\}} \left\{ (1 - i) \max_{j \in \{0, 1\}} & \left( (1 - j) \sup_{\theta \in \mathbb{R}} \left( (1 - \lambda h)v(s + h, y + hb(s, y; \theta)) + hf(s, y; \theta) \right) \\
& + j \Phi(h) H_{\max} v(s, y) \right\}, \text{ on } [t, T) \times \mathbb{R}^n; \\
Fv(T, y) = G(y) \text{ for all } y \in \mathbb{R}^n.
\end{align*}
$$

Next, we let $v_1$ and $v_2$ be two functions in $BC([t, T] \times \mathbb{R}^n)$, then for any $(s, y) \in [t, T) \times \mathbb{R}^n$ we have that

$$
Fv_1(s, y) - Fv_2(s, y) \leq \max_{i \in \{0, 1\}} \left\{ (1 - i) \min_{j \in \{0, 1\}} \left( (1 - j) \inf_{\theta \in \mathbb{R}} \left( (1 - \lambda h) \left(v_1(s + h, y + hb(s, y; \theta)) \right. \right. \right. \right. \\
- \left. \left. \left. \left. \left. v_2(s + h, y + hb(s, y; \theta)) \right) \right) \right) \right) \right) \right) \right) \right) + j \Phi(h) \inf_{\xi \in \mathbb{R}} \left( v_1(s, y + \xi) - v_2(s, y + \xi) \right)
$$

thus

$$
Fv_1(s, y) - Fv_2(s, y) \leq \max \left\{ 1 - \lambda h, \Phi(h) \right\} \| v_1 - v_2 \|_{\infty}.
$$

We proceed similarly to get

$$
\| Fv_2(s, y) - Fv_1(s, y) \|_{\infty} \leq \max \left\{ 1 - \lambda h, \Phi(h) \right\} \| v_1 - v_2 \|_{\infty}.
$$

Finally, by the contraction mapping principle for any $0 < h < 1/\lambda$, there exists a unique BC function $v_h$ solution of the approximate equation ($HJBI_h$). \qed

**Remark 4.1** The function $v_h$, unique bounded continuous solution of the approximate Hamilton-Jacobi-Bellman-Isaacs equation ($HJBI_h$), will be called the approximate value function. \qed

### 4.2 Convergence of the Approximate Value Function

In this section, we prove the convergence result for the approximate value function $v_h$. We mainly prove that the limit, when $h$ tends to zero, of $v_h$ is a viscosity solution to HJBI equation ($HJBI$). We give first the following lemma:

**Lemma 4.1** Let $v_h$ be the approximate value function, the family $\{v_h\}$ is uniformly equicontinuous with respect to state variable and uniformly bounded in $[t, T] \times \mathbb{R}^n$ by $\|f\|_{\infty}/\lambda$.

**Proof.** The proof can be found in the Appendix B. \qed

**Theorem 4.2** The approximate value function $v_h$, as the time discretization step $h$ goes to zero, converges locally uniformly towards the value function ($V$) of the zero-sum DG control problem.
Proof. Let $v_h$ be the approximate value function. From Lemma 4.1, the family $\{v_h\}$ is uniformly equicontinuous and uniformly bounded, we then get, using the Ascoli-Arzelà theorem (see e.g. [3]), that from any sequence $h_r$ converging towards 0, there exists a sub-sequence $h_s$ of $h_r$ and a bounded uniformly continuous function $v$ such that $v_{h_s}$ converges locally uniformly in $[t, T] \times \mathbb{R}^n$ towards $v$. Now, we only need to prove that $v$ is a VS to the HJBI equation (HJBI). Let $\phi \in C^1([t, T) \times \mathbb{R}^n)$ and $(\overline{t}, \overline{x})$ be a strict local maximum point of $v - \phi$. Then there exists $B_\delta(\overline{x})$ a closed ball in $\mathbb{R}^n$ of radius $\delta > 0$ centered at $\overline{x}$ such that

$$(v - \phi)(t, \overline{x}) > (v - \phi)(t, x),$$

for all $(t, x) \in I \setminus \{\overline{t}\} \times B_\delta(\overline{x}) \setminus \{\overline{x}\}$, where $I := [\overline{t} - \delta, \overline{t} + \delta] \subset [0, T]$. Let $(\overline{t}_{h_s}, \overline{x}_{h_s})$ be a maximum point of $v_{h_s} - \phi$ over $I \times B_\delta(\overline{x})$, let $\overline{t}_0$ and $\overline{x}_0$ be clusters point of the sequences $\{\overline{t}_{h_s}\}$ and $\{\overline{x}_{h_s}\}$, respectively, and denote $\{\overline{t}_{h_s}\}$ and $\{\overline{x}_{h_s}\}$ two sub-sequences converging to $\overline{t}_0$ and $\overline{x}_0$, respectively. By definition we have

$$(v_{h_{sp}} - \phi)(\overline{t}_{h_{sp}}, \overline{x}_{h_{sp}}) \geq (v_{h_{sp}} - \phi)(t, x),$$

for all $(t, x) \in I \times B_\delta(\overline{x})$. Using the continuity of $v_{h_{sp}}$ and $\phi$, and the fact that $v_{h_s}$ converges locally uniformly towards $v$, we get

$$(v - \phi)(\overline{t}_0, \overline{x}_0) \geq (v - \phi)(t, x),$$

for all $(t, x) \in I \times B_\delta(\overline{x})$. Thus, by the uniqueness of the maximum, $(\overline{t}_0, \overline{x}_0) = (\overline{t}, \overline{x})$ which means that the clusters point $\overline{t}_0$ and $\overline{x}_0$ are unique, we then get that the whole sequences $\overline{t}_{h_s}$ and $\overline{x}_{h_s}$ converge toward $\overline{t}$ and $\overline{x}$, respectively. Since $h_s$ is a small number and $b$ is assumed to be bounded, we get that the points $\overline{t}_{h_s} + h_s$ and $\overline{x}_{h_s} + h_s b(\overline{t}_{h_s}, \overline{x}_{h_s}; \theta)$ remain in $I$ and $B_\delta(\overline{x})$, respectively, for all $\theta \in \mathbb{R}^l$. Then it follows

$$v_{h_s}(\overline{t}_{h_s}, \overline{x}_{h_s}) - \phi(\overline{t}_{h_s}, \overline{x}_{h_s}) \geq v_{h_s}(\overline{t}_{h_s} + h_s, \overline{x}_{h_s} + h_s b(\overline{t}_{h_s}, \overline{x}_{h_s}; \theta)) - \phi(\overline{t}_{h_s} + h_s, \overline{x}_{h_s} + h_s b(\overline{t}_{h_s}, \overline{x}_{h_s}; \theta)).$$

Since $v_h$ is a solution to the approximate equation (HJBI), the last inequality combined to the expression of the approximate equation (HJBI) in the proof of Proposition 4.1 gives

$$\max_{i \in \{0, 1\}} \left\{ (1 - i) \min_{j \in \{0, 1\}} \left[ (1 - j) \inf_{\theta \in \mathbb{R}^l} \left\{ \phi(\overline{t}_{h_s}, \overline{x}_{h_s}) - \phi(\overline{t}_{h_s} + h_s, \overline{x}_{h_s} + h_s b(\overline{t}_{h_s}, \overline{x}_{h_s}; \theta)) \right. \ight. \\
+ \lambda h_s v_{h_s}(\overline{t}_{h_s} + h_s, \overline{x}_{h_s} + h_s b(\overline{t}_{h_s}, \overline{x}_{h_s}; \theta)) - h_s f(\overline{t}_{h_s}, \overline{x}_{h_s}; \theta) \right] \\
+ j \left( v_{h_s}(\overline{t}_{h_s}, \overline{x}_{h_s}) - \phi(\overline{t}_{h_s}, \overline{x}_{h_s}) \right) - \phi(\overline{t}_{h_s} + h_s, \overline{x}_{h_s} + h_s b(\overline{t}_{h_s}, \overline{x}_{h_s}; \theta)) + i \right\} \leq 0.$$
5 VERIFICATION THEOREM

Then the result in Proposition 4.1 yields to the following inequality

$$
\max_{i \in \{0,1\}} \left\{ (1-i) \min_{j \in \{0,1\}} \left[ (1-j) \inf_{\theta \in \mathbb{R}} \left\{ \frac{\partial \phi}{\partial s}(s, y) + \lambda v_{h_s}(\overline{t}_{h_s} + h_s, \overline{x}_{h_s} + h_s b(\overline{t}_{h_s}, \overline{x}_{h_s}; \theta)) \\
- D_y \phi(s, y). b(s, y; \theta) - f(\overline{t}_{h_s}, \overline{x}_{h_s}; \theta) \right\} + j(v_{h_s}(\overline{t}_{h_s}, \overline{x}_{h_s}) - \Phi(h_s) H^c_{\sup} v_{h_s}(\overline{t}_{h_s}, \overline{x}_{h_s})) \right\}
+ i(v_{h_s}(\overline{t}_{h_s}, \overline{x}_{h_s}) - \Phi(h_s) H^c_{\inf} v_{h_s}(\overline{t}_{h_s}, \overline{x}_{h_s})) \right\} \leq 0,
$$

we then let $h_s$ goes to zero and use Proposition 2.5 to get the convergence of the terms $H^c_{\sup} v_{h_s}(\overline{t}_{h_s}, \overline{x}_{h_s})$ and $H^c_{\inf} v_{h_s}(\overline{t}_{h_s}, \overline{x}_{h_s})$ toward $H^c_{\sup} v(\overline{t}, \overline{x})$ and $H^c_{\inf} v(\overline{t}, \overline{x})$, respectively, to finally deduce that

$$
\max_{i \in \{0,1\}} \left\{ (1-i) \min_{j \in \{0,1\}} \left[ (1-j) \inf_{\theta \in \mathbb{R}} \left\{ -\frac{\partial \phi}{\partial s}(\overline{t}, \overline{x}) + \lambda v(\overline{t}, \overline{x}) - D_y \phi(\overline{t}, \overline{x}). b(\overline{t}, \overline{x}; \theta) - f(\overline{t}, \overline{x}; \theta) \right\} + j(v(\overline{t}, \overline{x}) - H^c_{\sup} v(\overline{t}, \overline{x})) \right\}
+ i(v(\overline{t}, \overline{x}) - H^c_{\inf} v(\overline{t}, \overline{x})) \right\} \leq 0.
$$

The last inequality shows, using the expression of HJBI equation (HJBI) given in the proof of Proposition 3.1, that the function $v$ is a viscosity sub-solution to the HJBI equation (HJBI). Similarly we prove the viscosity super-solution property. The proof of the theorem is then finished.

\hspace{1cm} \Box

**Corollary 4.1** Assuming $H_b, H_g, H_f, H_{c,\chi}$ and $H_G$, the value function $(V)$ of the zero-sum DG control problem is the limit, when $h$ goes to zero, of the approximate value function $v_h$, i.e., the limit of the unique bounded continuous solution of the approximate Hamilton-Jacobi-Bellman-Isaacs equation (HJBI$_h$).

\hspace{1cm} \Box

5 Verification Theorem

This section uses the fact that the approximate value function converges to the value function $(V)$ of the considered DG control problem to provide a NE strategy for this game, whose definition is given by the following:

**Definition 5.1** (Nash-Equilibrium) Given $(t, x) \in [0, T] \times \mathbb{R}^n$, we say that the zero-sum DG control problem studied admits $(\psi^*, v^*) \in \Psi \times \mathcal{V}$ as a NE if the two strategies $\psi^*$ and $v^*$ satisfies:

$$
\begin{align*}
J(t, x; \psi^*, v^*) &\geq J(t, x; \psi, v^*) \text{ for all } \psi \in \Psi; \\
J(t, x; \psi^*, v^*) &\leq J(t, x; \psi^*, v) \text{ for all } v \in \mathcal{V}.
\end{align*}
$$

\hspace{1cm} \Box

In view of the above definition, the value function of the NE $(\psi^*, v^*) \in \Psi \times \mathcal{V}$ is defined for all $(t, x) \in [0, T] \times \mathbb{R}^n$ by

$$
V(t, x) := J(t, x; \psi^*, v^*).
$$

We will be concerned here with the optimal strategies for our deterministic two-player, zero-sum DG continuous and impulse controls problem. We first suppose, for $(t, x) \in [0, T] \times \mathbb{R}^n$, that a classical solution $v(t, x)$ of the HJBI equation (HJBI) and an approximate value function $v_h(t, x)$ exist and satisfy, for all $y \in \mathbb{R}^n$, $v(T, y) = G(y)$ and $v_h(T, y) = G(y)$, respectively. Next, let $h$ be a constant which tends to zero and $\Phi$ be defined as in Sect. 2.3. Then we construct the optimal strategies of each player $\psi^* := (\theta^*(\cdot), u^* := (\tau^*_m, \xi^*_m)_{m \in \mathbb{N}^+})$ and $v^* := (\rho^*_k, \eta^*_k)_{k \in \mathbb{N}^+}$ in an inductive way as follows:

$$
\begin{align*}
\theta^*(t) &= \theta^*_0 \in \mathbb{R}^d \text{ initial value of the optimal continuous control; } \\
\theta^*(s) := \begin{cases} \\
\theta^*(s) \in \mathbb{R}^d : v_h(s, y^s_{t,x} \psi^* v^*(s) - (1 - \lambda h)v_h(s + h, y^s_{t,x} \psi^* v^*(s) + h b(s, y^s_{t,x} \psi^* v^*(s); \theta^*)) \\
- h f(s, y^s_{t,x} \psi^* v^*(s); \theta^*) = 0 \end{cases}, \text{ where } s \in (t, T) \text{ and } s \neq \tau^*_m, \rho^*_k \text{ for all } m, k \geq 1; \\
(\theta^*)
\end{align*}
$$

\hspace{1cm} (\theta^*)
Next, we show that the above strategies are optimal and form a NE for the value function \((V)\) when the time discretization step \(h\) goes to zero.

The following theorem announces a NE for the game problem we have considered in this paper, it gives a verification result and confirms that \((\psi^*, v^*)\) defined in the above are optimal strategies for both players:

**Theorem 5.1** (Verification Theorem) Assuming that \((\psi^*, v^*) \in \Psi \times V\) and letting \(h\) goes to zero, if the value function \((V)\) of the zero-sum DG control problem is in \(C^{1,1}([0, T] \times \mathbb{R}^n)\), then it satisfies

\[
V(t, x) = J(t, x; \psi^*, v^*) \text{ for any } (t, x) \in [0, T] \times \mathbb{R}^n.
\]

**Proof.** We begin the proof by assuming that both HJBI equation \((\text{HJBI})\) and approximate equation \((\text{HJBI}_h)\) have solutions denoted, respectively, by \(V(t, x)\) and \(v_h(t, x)\) for \((t, x) \in [0, T] \times \mathbb{R}^n\). Next, we consider the following related discrete-time DG problems involving continuous and impulse controls:

\[
V_h^-(t, x) := \inf_{\beta \in \mathcal{B}_h} \sup_{\psi \in \Psi_h} J_h(t, x; \psi, \beta(\psi)); \quad (V_h^-)
\]

\[
V_h^+(t, x) := \sup_{\alpha \in \mathcal{A}_h} \inf_{v \in \mathcal{V}_h} J_h(t, x; \alpha(v), v), \quad (V_h^+)
\]

where, for the time discretization step \(h\) and \(d \in \mathbb{D} := \{0, 1, 2, \ldots, \frac{T}{h} - 1\}\), the discrete-time mapping \(y_{t,x}^h : \mathbb{D} \to \mathbb{R}^n\) depends on controls \(\psi\) and \(v\), and determines the discrete-time state of the DG control problems.
(V_h^-) and (V_h^+) by the following recursion:

\[ y_{t,x}^h(0) = x; \]
\[ y_{t,x}^h(d + 1) = y_{t,x}^h(d) + hb(t + dh, y_{t,x}^h(d); \theta(t + dh)) \prod_{m \geq 1} \mathbb{I}_{\{\tau_m \notin [t + dh, t + (d + 1)h]\}} \prod_{k \geq 1} \mathbb{I}_{\{\rho_k \notin [t + dh, t + (d + 1)h]\}} \]
\[ + \sum_{m \geq 1} q_m(\tau_m, y_{t,x}^h(d); \xi_m) \mathbb{I}_{[t + dh, t + (d + 1)h]}(\tau_m) \prod_{k \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_k\}}; \]
\[ + \sum_{k \geq 1} \lambda_k(\rho_k, y_{t,x}^h(d); \eta_k) \mathbb{I}_{[t + dh, t + (d + 1)h]}(\rho_k), \]  

the discrete-time gain/cost functional \( J_h \) is given by

\[ J_h(t, x; \psi, v) := h \sum_{d \in \mathbb{D}} f(t + dh, y_{t,x}^h(d); \theta(t + dh))(1 - \lambda h)^d \]
\[ - \sum_{m \geq 1} \sum_{d \in \mathbb{D}} c(\tau_m, y_{t,x}^h(d); \xi_m)(1 - \lambda h)^d \mathbb{I}_{[t + dh, t + (d + 1)h]}(\tau_m) \prod_{k \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_k\}} \]
\[ + \sum_{k \geq 1} \chi(\rho_k, y_{t,x}^h(d); \eta_k)(1 - \lambda h)^d \mathbb{I}_{[t + dh, t + (d + 1)h]}(\rho_k) \]
\[ + G(y_{t,x}^h(T - \frac{t}{h}))(1 - \lambda h)^\frac{T - t}{h}, \]

\( B_h := \left\{ \text{Subset of } \Psi \text{ consisting of all controls with constant values on each interval } [t + dh, t + (d + 1)h] \right\}; \)

\( \Psi_h := \left\{ \text{Subset of } \Psi \text{ consisting of all controls with constant values on each interval } [t + dh, t + (d + 1)h] \right\}; \)

Similarly we define the set \( A_h \). Following [19] one might deduce, for any \( t \in [0, T] \) and \( x \in \mathbb{R}^n \), the representation formula \( v_h(t, x) = V_h^-(t, x) = V_h^+(t, x) \). Hence, by focusing only on the discrete-time control problem \((V_h^-)\), we can deduce the optimal strategies. In other words, we use the fact that \( v_h \) is the unique bounded continuous solution to the approximate equation \((HJB)\), the formula \( v_h = V_h^- \) and the convergence \( v_h \rightarrow V \) to define some discrete-time optimal controls

\[ \psi_h := (\theta_h^*, u_h^*) = (\tau_m^*, \xi_m^*)_{m \in \mathbb{N}^*} \in \Psi_h \times \mathcal{U}_h, \text{ and } v_h^* := (\rho_k^*, \eta_k^*)_{k \in \mathbb{N}^*} \in \mathcal{V}_h, \]

for both discrete-time DG control problems \((V_h^-)\) and \((V_h^+)\). Since the function \( v_h \) separate the domain \([t, T] \times \mathbb{R}^n\) into many regions including the following continuation region:

\( \mathcal{C} := \left\{ (s, y) \in [t, T] \times \mathbb{R}^n : H_h(s, y) = 0, v_h(s, y) - \Phi(h)H_{\text{sup}}v_h(s, y) > 0 \right\}, \)

\( \text{and } v_h(s, y) - \Phi(h)H_{\text{inf}}v_h(s, y) < 0 \right\}, \)

then the expressions of the optimal impulse stopping times \( \tau_m^*, \rho_k^* \) and values \( \xi_m^*, \eta_k^* \) follow immediately and were given, respectively, by the aforementioned expressions \((u^*)\) and \((v^*)\) for \( h \) tends to zero. We now focus on the optimal continuous control \( \theta_h^*(\cdot) \) by assuming, without loss of generality, that there are no impulse controls for both players, i.e. \( \tau_1 = \rho_1 = T \), and proceeding as in [20]. It will be useful in what follows to consider the piece-wise constant extension \( g_{t,x}^h(\cdot) \) to \([t, T]\) of the mapping \( s \rightarrow y_{t,x}^h(s/h) \) defined, for \( k \in \{0, 1, 2, \ldots, \frac{T}{h} - 1\} \), on \([t + kh] \) by \( g_{t,x}^h(s) = y_{t,x}^h([s/h]) \), where \([s/h]\) denotes the largest integer which is less than or equal to \( s/h \).
From the definition of the continuous region \((C)\) we deduce that there exists a function \(\theta^*_h : \mathbb{R}^n \to \mathbb{R}^l\), such that for all \((s, y) \in [t, T] \times \mathbb{R}^n\) we have
\[
v_h(s, y) - (1 - \lambda h)v_h(s + h, y + hb(s, y; \theta^*_h(y))) - hf(s, y; \theta^*_h(y)) = 0, \tag{5.4}
\]
define then a discrete-time state mapping \(y_{t,x}^h : \mathbb{D} \to \mathbb{R}^n\) by
\[
y_{t,x}^h(0) = x, \quad \text{and} \quad y_{t,x}^h(d + 1) = y_{t,x}^h(d) + h b(t + dh, y_{t,x}^h(d); \theta^*_h(y_{t,x}^h(d))),
\]
and a function \(\tilde{\theta}^*_h : [t, T] \to \mathbb{R}^l\) by
\[
\tilde{\theta}^*_h(s) = \theta^*_h\left(y_{t,x}^h([s/h])\right), \quad \text{for all} \; s \in [t, T].
\]
Equation (5.4) leads, for \(d \in \mathbb{D}\), to
\[
v_h(s, y) = (1 - \lambda h)^d v_h(s + dh, y_{t,x}^h(d)) + h \sum_{i=1}^{d-1} (1 - \lambda h)^i f\left(s, y_{t,x}^h(i); \theta^*_h(y_{t,x}^h(i))\right).
\]
The fact that the control \(\tilde{\theta}^*_h(.)\) has constant values on each interval \([t + dh, t + (d + 1)h]\) and the boundedness of \(v_h\) confirm that \(\tilde{\theta}^*_h(.)\) is the optimal continuous control for the problem with no impulses, for which the expression was giving by \((\theta^*)\). Therefore, using the representation formula \(v_h = V^-\) we get \(v_h(t, x) = J_h(t, x; \psi^*_h, v^*_h)\).

Now, following [20], we write
\[
\lim_{h \to 0} J_h(t, x; \psi^*_h, v^*_h) = \lim_{h \to 0} J(t, x; \psi^*_h, v^*_h),
\]
we then get, from the convergence \(v_h \to V\), that \(\lim_{h \to 0} \tilde{\theta}^*_h(.)\) represents the optimal continuous control for \(V\). Thus \(V(t, x) = J(t, x; \psi^*, v^*)\) for optimal controls given by \((\theta^*), (u^*)\) and \((v^*)\). Hence we obtain the thesis. \(\square\)

Hence, the third contribution of the paper as mentioned in Remark 2.3. The obtained results make us ready to introduce a new continuous-time portfolio optimization model as an application, this will be the subject of Sect. 6 below.

### 6 Application to Continuous-Time Portfolio Optimization

An interesting framework of the theory of deterministic finite-time horizon, two-player, zero-sum, DGs involving continuous and impulse controls, developed in the present paper, is provided by the continuous-time portfolio optimization problem. In this section we address an application of our results to the analysis of a new continuous-time portfolio optimization model, in which the investor plays against the market and wishes to maximize his discounted terminal payoff, or to minimize a given cost. In Sect. 6.1 below the dynamical system \((S^\pi)\) describes the investor’s wealth at time \(s \in [t, T]\), while the functional \((J^\pi)\) represents his discounted terminal cost. On one hand, the market (maximizing player—\(\xi\)) wishes to minimize the investor’s (minimizing player—\(\eta\)) discounted terminal payoff (i.e., maximize the cost functional \((J^\pi)\)), where, on the other hand, the investor uses an impulse control to re-balance his portfolio in order to minimize the given cost functional \((J^\pi)\). Thus, the value function represents the investor’s lost in the worst-case scenario. Hence, our results can be used to derive a new continuous-time portfolio optimization model.
6.1 Formulation of a New Continuous-Time Portfolio Optimization Model

We describe hereafter our deterministic two-player, zero-sum, DG approach for continuous-time portfolio optimization problem in finite-time horizon. We first adjust the expressions of functions $b$, $g_\xi$ and $g_\eta$ in the standing dynamical system (S) to get a new one (S*), which characterizes the investor’s wealth at each instant $s$ between the initial time $t$ and the horizon $T$. Next, we approach the resulted continuous-time portfolio optimization problem by the non-linear HJBI equation (HJBI) and its approximate equation (HJBf).

**Dynamic of the Portfolio’s State.** Our finite-time horizon deterministic DG approach leads to a new continuous-time portfolio optimization model in which the investor’s wealth is described by the following dynamical system:

$$
\begin{align*}
& W_{t,w}(s) = W_{t,w}(s) \sum_{i=1}^{N} \omega_i^\pi(s) \hat{R}_i(s), \ s \neq \tau_m, \ s \neq \rho_k, \ s \in [t,T], \text{ where } t \geq 0, \ T \in (0, +\infty); \\
& W_{t,w}(\tau_m^+) = W_{t,w}(\tau_m^-) \left( 1 + \sum_{i=1}^{N} \omega_i^\xi dR_i(\tau_m) \prod_{k \geq 1} \| \tau_m = \rho_k \right), \ \tau_m \in [t,T], \ [\omega_1^\xi, \ldots, \omega_N^\xi]^\top \neq 0; \\
& W_{t,w}(\rho_k^+) = W_{t,w}(\rho_k^-) \left( 1 + \sum_{i=1}^{N} \omega_i^\eta dR_i(\rho_k) \right), \ \rho_k \in [t,T], \ [\omega_1^\eta, \ldots, \omega_N^\eta]^\top \neq 0; \\
& W_{t,w}(t^-) = w \text{ (investor’s initial wealth)}. 
\end{align*}
$$

Here $N = l = p = q$ is the number of stocks in the market, $\top$ denotes transpose, and $R_i(s)$ is the function that describes the cumulative return of $i$–th stock up to time $s$ starting from $t$, where $dR_i(s) = \frac{dP_i(s)}{P_i(s)}$ for $P_i(s)$ being the price of $i$–th stock at time $s$. The mapping $W_{t,w} : [t, T] \to \mathbb{R}_+$ represents the investor’s wealth at time $s \in [t, T]$ with initial value $w > 0$. The wealth $W_{t,w}(s)$ gives the state of the investor’s portfolio $\pi$ at time $s$ which is controlled by:

i. A continuous control $\omega^\pi(.) := [\omega_1^\pi(.), \ldots, \omega_N^\pi(.)]^\top$ which represents the investor’s instantaneous portfolio composition, i.e., the portfolio’s weights vector due to the market fluctuations. Thus, $\omega^\pi(s)$ combined with the cumulative returns vector $[R_1(s), \ldots, R_N(s)]^\top$ characterizes the investor’s wealth at time $s$;

ii. Two Impulse controls

$$
u := (\tau_m, \omega_m^\xi := [\omega_1^\xi, \ldots, \omega_N^\xi]^\top)_{m \in \mathbb{N}^+}, \text{ and } v := (\rho_k, \omega_k^\eta := [\omega_1^\eta, \ldots, \omega_N^\eta]^\top)_{k \in \mathbb{N}^+},$$

which describe new investor’s portfolio compositions at some jump instants $\tau_m$ and $\rho_k$, respectively. That is whenever the continuous control $\omega^\pi(.)$ doesn’t perform, the market (player–$\xi$) uses a new optimal portfolio composition determined at each impulse instant $\tau_m$ by the impulse value $\omega^\xi_m$, while the investor (player–$\eta$) adjusts his portfolio at each impulse instant $\rho_k$ using the impulse value $\omega^\eta_k$ to outperform the market. Where $\omega^\xi_m$ and $\omega^\eta_k$ are two weights vectors.

**Remark 6.1** (Another Formulation) If $r_i(s) := \hat{R}_i(s) = \frac{dR_i(s)}{ds}$ denotes the instantaneous return of $i$–th stock, i.e., $R_i(s) := \int_s^t r_i(\tau)d\tau$ is the cumulative return of the $i$–th stock on $[t, s]$ satisfying $R_i(t) = 0$, then our
dynamical system \( S^\pi \) can be rewritten as follows:

\[
\begin{align*}
\dot{W}_{t,w}(s) &= W_{t,w}(s) \sum_{i=1}^{N} \omega_i^\pi(s) r_i(s), \ s \neq \tau_m, \ s \neq \rho_k, \ s \in [t,T], \text{ where } t \geq 0, \ T \in (0, +\infty); \\
\dot{W}_{t,w}(\tau_m) &= W_{t,w}(\tau_m) \sum_{i=1}^{N} \omega_{i,m}^\xi r_i(\tau_m) \prod_{k \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_k\}}, \ \tau_m \in [t,T], \ [\omega_{1,m}^\xi, \ldots, \omega_{N,m}^\xi]^\top \neq 0; \\
\dot{W}_{t,w}(\rho_k) &= W_{t,w}(\rho_k) \sum_{i=1}^{N} \omega_{i,k}^\eta r_i(\rho_k), \ \rho_k \in [t,T], \ [\omega_{1,k}^\eta, \ldots, \omega_{N,k}^\eta]^\top \neq 0; \\
W_{t,w}(t^-) &= w \text{ (investor's initial wealth),}
\end{align*}
\]

where

\[
\sum_{i=1}^{N} \omega_i^\pi(\cdot) = \sum_{i=1}^{N} \omega_{i,m}^\xi = \sum_{i=1}^{N} \omega_{i,k}^\eta = 1. \tag{C}
\]

\[\square\]

**Continuous-Time Portfolio Optimization Problem.** Our deterministic zero-sum DG approach consists then in defining the investor’s wealth \( W_{t,w}(s) \) at time \( s \in [t,T] \) by the following dynamical equation:

\[
W_{t,w}(s) = w + \int_t^s W_{t,w}(\tau) \sum_{i=1}^{N} \omega_i^\pi(\tau) \frac{dP_i(\tau)}{P_i(\tau)} + \sum_{m \geq 1} W_{t,w}(\tau_m) \sum_{i=1}^{N} \omega_{i,m}^\xi \frac{dP_i(\tau_m)}{P_i(\tau_m)} \mathbb{I}_{[\tau_m,T]}(s) \prod_{k \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_k\}} + \sum_{k \geq 1} W_{t,w}(\rho_k) \sum_{i=1}^{N} \omega_{i,k}^\eta \frac{dP_i(\rho_k)}{P_i(\rho_k)} \mathbb{I}_{[\rho_k,T]}(s). \tag{W}
\]

We denote by \( \psi := (\omega^\pi(\cdot), u := (\tau_m, \omega_{m}^\xi)_{m \in \mathbb{N}^+} \in \Psi \) and \( v := (\rho_k, \omega_{k}^\eta)_{k \in \mathbb{N}^+} \in \mathcal{V} \) the continuous-impulse control for player – \( \xi \) (market) and the impulse control for player – \( \eta \) (investor), respectively, and we assume that the investor reacts immediately to the market whereas the market is not so quick in reacting to investor’s moves, i.e., the investor’s impulse action comes first whenever the impulse times for the two players coincide. Moreover, we assume that the investor does not consume wealth in the process of investing but is only interested to maximize his discounted terminal payoff, that is minimizing the following cost functional:

\[
J^\pi(t, w; \psi, v) := \int_t^T f^\pi(s, W_{t,w}^\psi,v(s); \omega^\pi(s)) \exp(-\lambda(s-t)) ds - \sum_{m \geq 1} c^\pi(\tau_m, W_{t,w}^\psi,v(\tau_m), \omega_{m}^\xi) \exp(-\lambda(\tau_m - t)) \prod_{k \geq 1} \mathbb{I}_{\{\tau_m \neq \rho_k\}} \tag{J^\pi}
\]

\[
+ \sum_{k \geq 1} \chi^\pi(\rho_k, W_{t,w}^\psi,v(\rho_k), \omega_{k}^\eta) \exp(-\lambda(\rho_k - t)) \mathbb{I}_{\{\rho_k \leq T\}} + G^\pi(W_{t,w}^\psi,v(T)) \exp(-\lambda(T-t)),
\]

where the functional \( J^\pi \) represents the investor’s cost, with the following components:

i. The running gain/cost of integral type giving by the investor’s stokes holding cost \( I^\pi \) minus his instantaneous utility function \( w^\pi \), that is \( f^\pi(\cdot, \cdot, \cdot) := (I^\pi - w^\pi)(\cdot, \cdot, \cdot) \);

ii. The maximizing player’s (market) (resp. minimizing player’s (investor)) cost function \( c^\pi \) (resp. \( \chi^\pi \)) which corresponds to the cost of selling or buying stokes at impulse instants \( \tau_m \) (resp. \( \rho_k \)).
iii. The terminal gain/cost giving by the function \( G^\pi \).

Our model is then related to either one of the following optimization problems:

\[
\begin{cases}
\inf_{\beta \in B} \sup_{\psi \in \Psi} J^\pi(t, w; \beta(\psi)), \\
\sup_{\alpha \in A} \inf_{\pi \in \pi} J^\pi(t, w; \alpha(v), v);
\end{cases}
\]

Subject to Equation (W) and Constraint (C).

### 6.2 Main Results and Portfolio Strategy

We assume that the market moves according to the continuous control \( \omega^\pi(\cdot) \), creates jumps at impulse instants \( \tau_m \) and tries to maximize the cost functional (\( J^\pi \)), and that the investor creates jumps at impulse instants \( \rho_k \), obviously, tries to minimize the cost functional (\( J^\pi \)). We also make the assumption that the flow of funds is between the investor and the market which makes our zero-sum DG framework.

**Main Results.** Tacking into account the fact that the dynamical function \( (s, w, \omega) \in [t, T] \times \mathbb{R}_+ \times \mathbb{R}^N \to \omega \cdot \omega \cdot P(s) \in \mathbb{R}_+ \) satisfies, for a bounded function \( P(s) \), the assumptions \( H_b \) and \( H_g \), and assuming that \( H_f, H_{c,\lambda} \) and \( H_r \) hold for the functions \( f^\pi, c^\pi, \lambda^\pi \) and \( G^\pi \), respectively, we might use previous results to conclude that the investor’s maximum discounted terminal cost (i.e., value function) can be characterized:

i. As the unique VS to the DPE (HJBI);

ii. Or, as the limit of the approximate value function, i.e., the limit of the unique solution of the approximate equation (HJBI);

iii. Or, by the optimal strategies of the NE of the deterministic zero-sum DG control problem.

**Portfolio Strategy.** The following corollary summarizes the discussion in the above by giving the portfolio strategy for the investor and the related maximal lost provided by the model we have developed:

**Corollary 6.1** A portfolio strategy \( \Pi(s) \) for the investor is given, at time \( s \in [t, T] \) in finite-time horizon \( T \) and for some initial wealth \( w \) at initial time \( t \) in a market with \( N \) stocks, by:

\[
\Pi(s) := \left\{ \left( \omega^\pi(\cdot) \right)_{t \leq s \leq s'} : \sum_{m \geq 1} \omega^\pi_m \Pi_{[\tau_m, T]}(s), \sum_{k \geq 1} \omega^\pi_k \Pi_{[\rho_k, T]}(s) \right\};
\]

where

\[
\omega^\pi(s) := \left[ \omega^\pi_1(s), \omega^\pi_2(s), \ldots, \omega^\pi_N(s) \right]^T;
\]

\[
\omega^\pi_m := \left[ \omega^\pi_{1,m}, \omega^\pi_{2,m}, \ldots, \omega^\pi_{N,m} \right]^T;
\]

\[
\omega^\pi_k := \left[ \omega^\pi_{1,k}, \omega^\pi_{2,k}, \ldots, \omega^\pi_{N,k} \right]^T.
\]

The optimal portfolio strategy \( \Pi^*(s) \) is then described by the following sequences of elements of \( \mathbb{R}^N \):

\[
(\omega^\pi^*(s))_{t \leq s \leq T}, \quad (\omega^\pi_k^*)_{m \geq 1}, \quad \text{and} \quad (\omega^\pi_k^*)_{k \geq 1},
\]

for which the expressions are given by the verification theorem of Sect. 5.

The investor’s maximal lost is the value function \( v(t, w) \) of the game generated by the optimal portfolio strategy \( \Pi^*(s) \) for \( s \in [t, T] \).

We now provide a computational algorithm for our portfolio optimization model.

### 6.3 Computational Algorithm

We give numerical aspects describing the value functions for both market and investor, and their NE optimal strategies. More precisely, we propose a computational algorithm to find the approximate value function \( v_h \), and then deduce, when the time discretization step \( h \) goes to zero, the value function (V) for both players: \( v_h(t, w) \) tends to \( v(t, w) \) for any \( (s, y) \in [t, T] \times \mathbb{R}^n \). The algorithm is as follows:
Algorithm 1 COMPUTE THE APPROXIMATE VALUE FUNCTION \( v_h \) OF THE GAME AND THE NE

\[ \text{Input} \text{ Time interval } [t, T], \text{ initial wealth } w_0 = W_{t,w_0}(t) \in \mathbb{R} \text{ and value } V(T, w) = G(w) \text{ at } T \text{ for any } w \in \mathbb{R}^+. \text{ Initial composition of the market portfolio } \theta^*_t \in \mathbb{R}^N \text{ at } t \text{ and the investor's portfolio } \omega^*_t \in \mathbb{R}^N \text{ at } \rho_1 = t^{-}. \text{ The numbers } N \text{ and } I, J \text{ are, respectively, the number of stocks in the market and the number of time-space discretization steps. Time and space grid } (s_i, w_j) \text{ where } s_i \in \{ s_1 = t, s_2, \ldots, s_I = T \} \text{ and } w_j \in \{ w_1 = w_0, w_2, \ldots, w_J = W \in \mathbb{R}^+ \}. \text{ Functions } f^*, c^*, \chi^*, G^\ast \text{ and } \Phi, \text{ and discount factor } \lambda.\]

\[ \text{Output} \text{ Portfolio strategy } \Pi^* (s_i) \text{ given by } \omega_{\pi}^* (s_i), \{ \tau_{m}^*, \omega_{m}^* \}_{m \in \mathbb{N}^+} \text{ and } \{ \rho_{k}^*, \omega_{k}^* \}_{k \in \mathbb{N}^+}. \text{ Approximate value at initial time } t \text{ and initial state } w_0 \text{ given by } v_h(s_1, w_1), \text{ and value } V(s_1, w_1) = \lim_{h \to 0} v_h(s_1, w_1) \text{ (investor's maximal cost at } t). \]

\[ h \leftarrow (T - t)/n \quad \triangleright \text{ n large enough (i.e., } h \text{ goes to zero) } \]

\[ s_i \leftarrow T \]

**for** \( j \) equal 1 to \( J \) **do**

\[ v_h(s_i, w_j) \leftarrow G(w_j) \quad \triangleright \text{ Terminal value given by } G \]

**end for**

**while** \( s_i \) not equal to \( t \) **do**

\[ s_i \leftarrow s_i - h \]

**for** \( l \) equal 1 to \( I \) **do**

**for** \( j \) equal 1 to \( J \) **do**

\[ \triangleright \text{ } W^h_{t,w_0} (j) \text{ depends on } \psi^*, v^* \text{ and defined as in (5.2) } \]

Compute \( \omega_{\pi}^* \) solution of:

\[ v_h(s_i, W^h_{t,w_0} (j)) - (1 - \lambda h)v_h(s_{i+1}, W^h_{t,w_0} (j)) + h f^* (s_i, W^h_{t,w_0} (j); \omega_{\pi}^* ) - \chi^* (s_i, W^h_{t,w_0} (j); \omega_{\pi}^* ) = 0; \]

Compute, if exists for \( (\tau^*_m, \xi^*_m) = (s_i, 0) \) and \( (\rho^*_k, \eta^*_k) = (s_i, \eta_1) \), the sequences \( \{ \tau^*_m, \omega^*_{m} \}_{m=2,3,\ldots,1-i+2} \) and \( \{ \rho^*_k, \omega^*_k \}_{k=3,4,\ldots,1-i+2} \) through verification result:

**for** \( d \) equal \( i \) to \( l + 1 \) **do**

\[ \text{if} \quad v_h(s_d, W^h_{t,w_0} (j)) > \Phi(h) \inf_{\omega_{k-1}^* \in V} \{ v_h(s_d, W^h_{t,w_0} (j)) + g_{\eta}^* (s_d, W^h_{t,w_0} (j); \omega_{\pi}^* ) \} \]

\[ \text{else if} \quad v_h(s_d, W^h_{t,w_0} (j)) < \Phi(h) \sup_{\omega_{k-1}^* \in U} \{ v_h(s_d, W^h_{t,w_0} (j)) + g_{\eta}^* (s_d, W^h_{t,w_0} (j); \omega_{\pi}^* ) \} \]

\[ \text{then} \quad \rho^*_k = s_d \text{ and } \omega^*_k = \]

**end if**

**end for**

Compute \( v_h(s_i, w_i) = J_h(s_i, w_i; \psi^*, v^*) \)

**end for**

Compute \( w_i = W^h_{s_i,w_i} (i) \) then deduce \( v_h(s_i, w_i) = J_h(s_i, w_i; \psi^*, v^*) \)

**end while**
7 Conclusion

In this paper, we have considered a new class of deterministic finite-time horizon, two-player, zero-sum DGs, where the maximizing player takes continuous and impulse controls, while the minimizing player uses impulse control only, aims were to optimize a discounted terminal gain/cost functional, approximate the value function, and describe an optimal strategy for the two players. After studying the related Hamilton-Jacobi-Bellman-Isaacs double-obstacle equation (HJBI) in the VS framework, we have proposed a discrete-time approximation scheme for this class of DGs given by the equation (HJBI). We have further given a verification result which analytically characterizes the equilibrium timing and level of impulses, and describes the optimal continuous actions. Our major contributions are the comparison principle, the convergence result for the approximate value function, and the verification theorem. Moreover, we have given some dynamics \(b, g_\xi\) and \(g_\eta\) to apply our results to continuous-time portfolio optimization problem, where the investor takes priority actions (impulses) only occasionally, while the market makes decisions both continuously and in specific impulse times, in such situation our results have been successfully implemented to derive a new optimization model. Moreover, we have provided a computational procedure to numerically determine the value function and the corresponding NE strategies.

We intend to develop this work in two main directions in the future:

1. It would be interesting to consider a problem with feedback continuous control (i.e., \(\theta\) depends on \(y_{t,x}(s)\)), thus \(\dot{y}_{t,x}(s)\) the instantaneous evolution of \(y\) at time \(s\), and the running gain/cost function \(f\) at time \(s\) become, respectively,

\[
 b(s, y_{t,x}(s); \theta(s, y_{t,x}(s))) \quad \text{and} \quad f(s, y_{t,x}(s); \theta(s, y_{t,x}(s))) ;
\]

2. Another extension of our work would be to adopt a machine learning approach based on generative adversarial networks (GANs) to deep generate the corresponding value function and the corresponding NE in the mini-max game framework of GANs (see e.g. [40, 58]).

Declarations

The second author’s research is financially supported by national center for scientific and technical research CNRST, Rabat, Morocco (Grant 17 UIZ 19). The authors declare no conflict of interest.

Appendix A

A.1 Proof of Lemma 3.1

We give only the proof for the lower value \((V^-)\), similarly for the upper value \((V^+)\). Let \((s, y) \in [t, T] \times \mathbb{R}^n\), \(\psi := (\theta(\cdot), u := (\tau_m, \xi_m)_{m \in \mathbb{N}^*}) \in \Psi\), then consider the non-anticipative strategy \(\beta \in \mathcal{B}\) for player – \(\eta\) where \(\beta(\psi) = v := (\rho_k, \eta_k)_{k \in \mathbb{N}^*} \in \mathcal{V}\). Now choose \(\beta'(\psi, u) := (s, \eta; \rho_2, \eta_2; \rho_3, \eta_3; \ldots)\), we then obtain

\[
 V^-(s, y) \leq \sup_{\psi \in \Psi} J(s, y; \psi, \beta'(\psi)),
\]

thus

\[
 V^-(s, y) = \sup_{\psi \in \Psi} J(s, y + g_\eta(s, y; \eta; \psi, \beta'(\psi)) + \chi(s, y; \eta),
\]

from which we get

\[
 V^-(s, y) \leq V^-(s, y + g_\eta(s, y; \eta)) + \chi(s, y; \eta).
\]

Then from the arbitrariness of \(\eta\) we get

\[
 V^-(s, y) \leq \mathcal{H}_{inf}^\chi V^-(s, y).
\]
Next, we assume that $V^-(s, y) < \mathcal{H}_{\inf}^X V^-(s, y)$ for some $(s, y) \in [t, T) \times \mathbb{R}^n$. The dynamic programming property (2.2) for the lower value, when $T = 0$, yields

$$V^-(s, y) = \inf_{\rho_0 \in \{s, T\}, \eta \in V} \sup_{\theta(.) \in \Theta \atop \tau_0 \in \{s, T\}, \xi \in U} \left[ -c(s, y; \xi) \mathbb{I}_{\{\tau_0 = s\}} \mathbb{I}_{\{\rho_0 = T\}} + \chi(s, y; \eta) \mathbb{I}_{\{\rho_0 = s\}} \right] + \sup_{\theta(.) \in \Theta \atop \tau_0 \in \{s, T\}, \xi \in U} \left[ -c(s, y; \xi) \mathbb{I}_{\{\tau_0 = s\}} + V^-(s, y + g_\xi(s, y; \xi) \mathbb{I}_{\{\tau_0 = s\}}) \mathbb{I}_{\{\rho_0 = s\}} \right].$$

therefore

$$V^-(s, y) = \inf_{\rho_0 \in \{s, T\}} \left[ \inf_{\eta \in V} \left[ \chi(s, y; \eta) + V^-(s, y + g_\eta(s, y; \eta)) \right] \mathbb{I}_{\{\rho_0 = s\}} \right] + \sup_{\theta(.) \in \Theta \atop \tau_0 \in \{s, T\}, \xi \in U} \left[ -c(s, y; \xi) \mathbb{I}_{\{\tau_0 = s\}} + V^-(s, y + g_\xi(s, y; \xi) \mathbb{I}_{\{\tau_0 = s\}}) \right] \mathbb{I}_{\{\rho_0 = T\}} \right].$$

From the fact that $V^-(s, y) < \mathcal{H}_{\inf}^X V^-(s, y)$, we get

$$V^-(s, y) = \sup_{\tau_0 \in \{s, T\}, \xi \in U} \left[ -c(s, y; \xi) \mathbb{I}_{\{\tau_0 = s\}} + V^-(s, y + g_\xi(s, y; \xi) \mathbb{I}_{\{\tau_0 = s\}}) \right].$$

Hence

$$V^-(s, y) \geq \sup_{\xi \in U} \left[ V^-(s, y + g_\xi(s, y; \xi)) - c(s, y; \xi) \right],$$

which completes the proof.

### A.2 Proof of Lemma 3.2

Let $(s, y) \in [t, T) \times \mathbb{R}^n$ and $\phi \in C^{1,1}([t, T) \times \mathbb{R}^n)$ be such that

$$-\frac{\partial}{\partial s} \phi(s, y) + \lambda \phi(s, y) + H(s, y, D_y \phi(s, y)) = \gamma > 0. \quad (A.1)$$

Following [3], we define for $s' > 0, z \in \mathbb{R}^n, \theta(.) \in \Theta$,

$$\Gamma(s', z; \theta(s')) = -\frac{\partial}{\partial s} \phi(s', z) + \lambda \phi(s', z) - D_y \phi(s', z) D_y \phi(s', z) - f(s', z; \theta(s')).$$

By (A.1) and the definition of the Hamiltonian $H$ we get

$$\inf_{\theta \in \mathbb{R}^l} \Gamma(s, y; \theta) = \gamma,$$

then for any $\theta(.) \in \Theta$ we have $\Gamma(s, y; \theta(t)) \geq \gamma$. Since $\theta \to \Gamma(s, y; \theta)$ is uniformly continuous in $\mathbb{R}^l$, we have in fact

$$\Gamma(s, y; \zeta(.) \geq \frac{3\gamma}{4} \text{ for all } \zeta(.) \in B_{r_1}(\theta(.) \cap \Theta),$$

where $B_{r_1}(\theta(.))$ denotes the open ball of radius $r(.) := r(\theta(.)) > 0$ centered at $\theta(.)$. Without loss of generality, for $\kappa$ a compact subset of $\mathbb{R}^l$ and $\Theta$ being $\kappa$-valued, there exist finitely many points $(\theta_1(\cdot), \theta_2(\cdot), \ldots, \theta_n(\cdot))$ and $(r_1(\cdot), r_2(\cdot), \ldots, r_n(\cdot))$ such that $\theta_i(.) \subset \kappa, r_i(.) > 0$ for $i = 1, 2, \ldots, n$ and

$$\Theta \subseteq \bigcup_{i=1}^n B_{r_i}(\theta_i(\cdot)).$$
where $r_i(.) := r_i(\theta_i(.)) > 0$, and
\[ \Gamma(s, y; \zeta(.)) \geq \frac{3\gamma}{4} \text{ for all } \zeta(.) \in B_{r_i(.)}(\theta_i(.)) \cap \Theta. \]

By the continuity of $\Gamma$ and Proposition 2.1 there exists $t' > 0$ such that
\[ \Gamma(s, y_{t,r}(s); \theta(s)) \geq \frac{\gamma}{2} \text{ for all } t \leq s \leq t' \text{ and all } \theta(.) \in \Theta. \]

Finally we multiply both sides of the last inequality by $\exp(-\lambda s)$ and integrate from $t$ to $t'$ to obtain the result for $t' - t$ small enough.

**A.3 Proof of Proposition 3.1**

For any positive numbers $a$, $b$, $a'$ and $b'$, solving an equation of the form $\max\{\min[A, B]; C\} = 0$ is equivalent to solve the equation
\[ \max_{i \in \{0, 1\}} \left\{ (1 - i) \min_{j \in \{0, 1\}} \left[ (1 - j) a \min \left[ (1 - j) a' A + j b' B \right] + i b C \right] \right\} = 0, \]
(A.2)

the same for the inequalities
\[ \max\{\min[A, B]; C\} \leq 0, \text{ and } \max\{\min[A, B]; C\} \geq 0. \]

We use (A.2), for $a = a' = 1$ and $b = b' = \lambda$, to rewrite the equation (HJBI) as follows
\[ \max_{i \in \{0, 1\}} \left\{ (1 - i) \min_{j \in \{0, 1\}} \left[ (1 - j) \inf_{\theta \in \mathbb{R}^l} \left\{ -\frac{\partial}{\partial s} v(s, y) + \lambda v(s, y) - D_y v(s, y) \cdot b(s, y; \theta) - f(s, y; \theta) \right\} 
+ j \lambda \left( v(s, y) - \mathcal{H}_{sup}^{c} v(s, y) \right) \right\} + i \lambda \left( v(s, y) - \mathcal{H}_{inf}^{c} v(s, y) \right) \right\} = 0, \]

where $v$ being a continuous function in $[t, T] \times \mathbb{R}^n$. We then get
\[ \max_{i \in \{0, 1\}} \left\{ (1 - i) \min_{j \in \{0, 1\}} \left[ \lambda v(s, y) - j \lambda v(s, y) + (1 - j) \inf_{\theta \in \mathbb{R}^l} \left\{ -\frac{\partial}{\partial s} v(s, y) - D_y v(s, y) \cdot b(s, y; \theta) - f(s, y; \theta) \right\} 
+ j \lambda \left( v(s, y) - \mathcal{H}_{sup}^{c} v(s, y) \right) \right\} + i \lambda \left( v(s, y) - \mathcal{H}_{inf}^{c} v(s, y) \right) \right\} = 0, \]
thus
\[ \max_{i \in \{0, 1\}} \left\{ (1 - i) \min_{j \in \{0, 1\}} \left[ \lambda v(s, y) - (1 - j) \sup_{\theta \in \mathbb{R}^l} \left\{ \frac{\partial}{\partial s} v(s, y) + D_y v(s, y) \cdot b(s, y; \theta) + f(s, y; \theta) \right\} 
- j \lambda \mathcal{H}_{sup}^{c} v(s, y) \right\} + i \lambda \left( v(s, y) - \mathcal{H}_{inf}^{c} v(s, y) \right) \right\} = 0. \]

Then it follows
\[ \min_{i \in \{0, 1\}} \left\{ (1 - i) \max_{j \in \{0, 1\}} \left[ -\lambda v(s, y) + (1 - j) \sup_{\theta \in \mathbb{R}^l} \left\{ \frac{\partial}{\partial s} v(s, y) + D_y v(s, y) \cdot b(s, y; \theta_2) + f(s, y; \theta_2) \right\} 
+ j \lambda \mathcal{H}_{sup}^{c} v(s, y) \right\} - i \lambda \left( v(s, y) - \mathcal{H}_{inf}^{c} v(s, y) \right) \right\} = 0, \]
from which we deduce
\[
\min_{i \in \{0, 1\}} \left\{ -\lambda v(s, y) + (1 - i) \max_{j \in \{0, 1\}} \left[ (1 - j) \sup_{\theta \in \mathbb{R}^l} \left\{ \frac{\partial}{\partial s} v(s, y) + D_y v(s, y) b(s, y; \theta) + f(s, y; \theta) \right\} + j \lambda \mathcal{H}^{c}_{sup} v(s, y) \right]\right. \\
+ \left. i \lambda \mathcal{H}^{c}_{inf} v(s, y) \right\} = 0.
\]

Finally we deduce the desired expression for the HJBI equation (HJBI)
\[
\lambda v(s, y) = \min_{i \in \{0, 1\}} \left\{ (1 - i) \max_{j \in \{0, 1\}} \left[ (1 - j) \left( \frac{\partial}{\partial s} v(s, y) + \sup_{\theta \in \mathbb{R}^l} \left\{ D_y v(s, y) b(s, y; \theta) + f(s, y; \theta) \right\} \right) \right. \\
+ \left. j \lambda \mathcal{H}^{c}_{sup} v(s, y) \right. \right. \\
+ \left. i \lambda \mathcal{H}^{c}_{inf} v(s, y) \right\},
\]
(A.3)

which completes the proof.

**Appendix B**

**Proof of Lemma 4.1.** First, let a function \( v^0 \) be non-negative and bounded uniformly continuous with respect to \( y \in \mathbb{R}^n \). We have, for any \( s \in [t, T] \), that
\[
\forall \varepsilon > 0, \exists \delta_0 > 0 \text{ such that } \forall y_1, y_2 \in \mathbb{R}^n, \|y_1 - y_2\| < \delta_0, \text{ implies } |v^0(s, y_1) - v^0(s, y_2)| < \varepsilon/2.
\]
Define now, for \( 0 < h < 1/\lambda, (s, y) \in [t, T] \times \mathbb{R}^n \) and \( F \) defined as in (4.1), a family of functions \( v^h_n(s, y) = F^n v^0(s, y) \) which converges uniformly towards the unique solution \( v_h \) to approximate equation (HJBI_h). We have, for any \( s \in [t, T] \), that
\[
|v^h_n(s, y_1) - v^h_n(s, y_2)| \leq \max_{j \in \{0, 1\}} \left[ (1 - j) \inf_{\theta \in \mathbb{R}^l} \left\{ (1 - \lambda h) v^0(s + h, y_1 + hb(s, y_1; \theta)) \right. \\
- v^0(s + h, y_2 + hb(s, y_2; \theta)) \right. \\
+ \left. h \left( f(s, y_1; \theta) - f(s, y_2; \theta) \right) \right\} \\
+ j \Phi(h) \inf_{\xi \in \mathbb{R}^l} \left\{ v^0(s, y_1 + g_1(s, y_1; \xi)) - v^0(s, y_2 + g_1(s, y_2; \xi)) \right\} + \left. |c(s, y_1; \xi) - c(s, y_2; \xi)| \right\} \\
+ i \Phi(h) \sup_{\eta \in \mathbb{R}^l} \left\{ v^0(s, y_1 + g_1(s, y_1; \eta)) - v^0(s, y_2 + g_1(s, y_2; \eta)) \right\} + \left. \chi(s, y_1; \eta) - \chi(s, y_2; \eta) \right\}.
\]

By letting \( \delta_1 = \min \{\delta_0/(1 + C_\theta/\lambda), \lambda \varepsilon/2C_f, \delta_0/C_{g_1}, \delta_0/C_{g_2}, \varepsilon/2C_c, \varepsilon/2C_X\} \), we get for all \( 0 < h < 1/\lambda \), for all \( s \in [t, T] \) and for all \( \|y_1 - y_2\| \leq \delta_1 \), that \( |v^h_n(s, y_1) - v^h_n(s, y_2)| \leq \varepsilon/3 \). The family \( \{v^h_n\} \) is then uniformly equicontinuous with respect to state variable and by induction for all \( n \geq 1 \), the family \( \{v^h_n\} \) is uniformly equicontinuous. Now we prove that the family \( \{v^h_n\} \) is also uniformly equicontinuous. Let \( s \in [t, T] \), since we have that
\[
\forall \varepsilon > 0, \exists \delta > 0, \forall y_1, y_2 \in \mathbb{R}^n, \|y_1 - y_2\| \leq \delta, \text{ implies } \|v^h_n(s, y_1) - v^h_n(s, y_2)\| < \varepsilon/3,
\]
and \( \forall \varepsilon > 0, \exists N > 0, \forall n > N, \|y_1 - y_2\| \leq \delta, \text{ implies } \|v^h_n(s, y_1) - v_h(s, y_1)\| < \varepsilon/3, \text{ and } \|v^h_n(s, y_2) - v_h(s, y_2)\| < \varepsilon/3. \) It follows that the family \( \{v^h\} \) is uniformly equicontinuous.

Since \( v^h_n = F^n v^0 \) tends to the approximate value function \( v_h \) and \( f \) is non-negative, we get that \( v_h \) is non-negative, we then use the fact that \( F v_h = v_h \) to deduce that
\[
\|v_h\|_\infty \leq (1 - \lambda h)\|v_h\|_\infty + h\|f\|_\infty,
\]
thus \( v_h \) results to be uniformly bounded by \( \|f\|_\infty/\lambda \).
References

[1] Aid, R., Basei, M., Callegaro, G., Campi, L., & Vargiolu, T.: Nonzero-sum stochastic differential games with impulse controls: A verification theorem with applications. Mathematics of Operations Research, 45(1), pp. 205-232 (2020)

[2] Azimzadeh, P.: Zero-sum stochastic differential game with impulses, precommitment and unrestricted cost functions. Appl. Math. Optim., pp. 1-32 (2017)

[3] Bardi, M., & Capuzzo-Dolcetta, I.: Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations. Birkhäuser, Boston (1997)

[4] Barles, G.: Deterministic impulse control problems. SIAM J. Control Optim., 23, pp. 419-432 (1985).

[5] Barles, G.: Solutions de viscosité des équations de Hamilton-Jacobi. Collection SMAI, Springer-Verlag, Berlin (1994)

[6] Barles, G., Bernhard, P., & El Farouq, N.: Deterministic minimax impulse control. Appl. Math. Optim., 61, pp. 353-378 (2010)

[7] Barles, G., & Souganidis, P. E.: Convergence of approximation schemes for fully nonlinear second order equations. In: Asymptotic Anal., 4(3), pp. 271-283 (1991)

[8] Basei, M., Cao, H., & Guo, X.: Nonzero-sum stochastic games and mean-field games with impulse controls. Mathematics of Operations Research (2021)

[9] Bellman, R.: Dynamic programming, Princeton Univ. Press, Princeton (1957)

[10] Bensoussan, A., & Lions, J. L.: Impulse control and quasi-variational inequalities. Bordes, Paris (1984)

[11] Bernard, P.: A robust control approach to option pricing including transaction costs. Annals of the ISDG., 7, pp. 391-416, Birkhäuser, Basel (2005)

[12] Bernard, P., El Farouq, N., & Thiery, S.: An impulsive differential game arising in finance with interesting singularities. Annals of the ISDG., 8, pp. 335-363, Birkhäuser, Basel (2006)

[13] Bertola, G., Runggaldier, W., & Yasuda, K.: On classical and restricted impulse stochastic control for the exchange rate. Appl. Math. Optim., 74, pp. 423-454 (2016)

[14] Blaquière, A.: Impulsive optimal control with finite or infinite time horizon. Journal of Optimization Theory and Applications, 46(4), pp. 431-439 (1985)

[15] Boltyanskii, V. G., Gamkrelidze, R. V., Mishchenko, E. F., & Pontryagin, L. S.: The mathematical theory of optimal processes. Interscience, New York (1962).

[16] Camilli, F., & Falcone, M.: Analysis and approximation of the infinite-horizon problem with impulsive controls. Autom. Remote Control, 58, pp. 1203-1215 (1997)

[17] Camilli, F., & Falcone, M.: Approximation of control problems involving ordinary and impulsive controls. ESAIM: Control Optim. Calc. Var., 4, pp. 159-176 (1999)

[18] Campi, L., & De Santis, D.: Nonzero-sum stochastic differential games between an impulse controller and a stopper. Journal of Optimization Theory and Applications, 186(2), pp. 688-724 (2020)

[19] Capuzzo-Dolcetta, I.: On a discrete approximation of the Hamilton-Jacobi equation of dynamic programming. Appl. Math. Optim., 10, pp. 367-377 (1983)
[20] Capuzzo-Dolcetta, I., & Ishii, H.: Approximate solutions of the Bellman equation of deterministic control theory. Appl. Math. Optim., 11, pp. 161-181 (1984)

[21] Cosso, A.: Stochastic differential games involving impulse controls and double-obstacle quasi-variational inequalities. SIAM J. Control Optim., 51(3), pp. 2102-2131 (2013)

[22] Crandall, M. G., & Lions, P. L.: Viscosity solutions of Hamilton-Jacobi equations. Trans. Amer. Math. Soc., 277, pp. 1-42 (1983)

[23] Crandall, M. G., Evans, L. C., & Lions, P. L.: Some properties of viscosity solutions of Hamilton-Jacobi equations. Trans. Amer. Math. Soc., 282, pp. 487-502 (1984)

[24] Crandall, M., Ishii, H., & Lions, P. L.: Users guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc., 27, pp. 1-67 (1992)

[25] Dharmatti, S., & Shaiju, A.J.: Differential games with continuous, switching and impulse controls. Nonlinear Anal., 63, pp. 23-41 (2005).

[26] Dharmatti, S. & Ramaswamy, M.: Zero-sum differential games involving hybrid controls. J. Optim. Theory Appl., 128, pp. 75-102 (2006).

[27] Eastham, J. F., & Hastings, K. J.: Optimal impulse control of portfolios. Mathematics of Operations Research, 13(4), pp. 588-605 (1988)

[28] El Asri, B., & Lalioui, H.: Deterministic differential games in infinite horizon involving continuous and impulse controls. Preprint, arXiv:2107.03524 [math.OC] (2021)

[29] El Asri, B., Lalioui, H., & Mazid, S.: A zero-sum deterministic impulse controls game in infinite horizon with a new HJBI QVI. Preprint, arXiv:2101.11669 [math.OC] (2021)

[30] El Asri, B., & Mazid, S.: Zero-sum stochastic differential game in finite horizon involving impulse controls. Appl. Math. Optim., pp. 1-33 (2018)

[31] El Asri, B. & Mazid, S.: Stochastic impulse control problem with state and time dependent cost functions. Mathematical Control and Related Fields, 10(4), pp. 855-875 (2020).

[32] El Farouq, N.: Degenerate first-order quasi-variational inequalities: an approach to approximate the value function. SIAM J. Control Optim., 55, pp. 2714-2733 (2017)

[33] El Farouq, N.: Deterministic impulse control problems: Two discrete approximations of the quasi-variational inequality. Journal of Computational and Applied Mathematics, 309, pp. 200-218 (2017)

[34] Elliott, R. J., & Kalton, N. J.: The existence of value in differential games. Mem. Amer. Math. Soc., 126 (1972)

[35] Elliott, R. J., & Kalton, N. J.: Cauchy problems for certain Issacs-Bellman equations and games of survival. Trans. Amer. Math. Soc., 198, pp. 45-72 (1974)

[36] Evans, L. C., & Souganidis, P. E.: Differential games and representation formulas for solutions of Hamilton-Jacobi-Issaacs equations. Indiana Univ. Math. J., 33(5), pp. 773-797 (1984)

[37] Falcone, M.: A numerical approach to the infinite horizon problem of deterministic control theory. App. Math. Optim., 15, pp. 1-13 (1987)

[38] Fleming, W. H., & Rishel, R. W.: Deterministic and stochastic optimal control. Springer-Verlag, Berlin, Heidelberg, New York (1975)
REFERENCES

[39] Fleming, W. H., & Soner, H. M.: Controlled Markov processes and viscosity solutions, 2nd ed. Springer, New York (2006)

[40] Goodfellow, I., Pouget-Abadie, J., Mirza, M., Xu, B., Warde-Farley, D., Ozair, S., Courville, A., & Bengio, Y.: Generative adversarial nets. In Z. Ghahramani, M. Welling, C. Cortes, N. D. Lawrence, and K. Q. Weinberger, editors, Advances in Neural Information Processing Systems 27, pp. 2672-2680. Curran Associates, Inc. (2014)

[41] Gonzalez, R., & Rofman, E.: On deterministic control problem: An approximation procedure for the optimal cost I the stationary problem. SIAM J. Control Optimization, 23(2), pp. 242-266 (1985)

[42] Gonzalez, R., & Rofman, E.: On deterministic control problem: An approximation procedure for the optimal cost II the non-stationary case. SIAM J. Control Optimization, 23(2), pp. 267-285 (1985)

[43] Hastings, K. J.: Impulse control of portfolios with jumps and transaction costs. Communications in Statistics. Stochastic Models, 8(1), pp. 59-72 (1992)

[44] Isaacs, R.: Differential games. A mathematical theory with applications to warfare and pursuit, control and optimization. John Wiley & Sons, Inc., New York-London-Sydney (1965)

[45] Korn, R.: Portfolio optimization with strictly positive transaction costs and impulse control. Finance Stochast., 2, pp. 85-114 (1998)

[46] Lions, P. L.: Generalized solution of Hamilton-Jacobi equations. Pitman, Boston (1982)

[47] Liu, Y., Teo, K.L., Jennings, L.S., & Wang, S.: On a class of optimal control problems with state jumps. Journal of Optimization Theory and Applications, 98(1), pp. 65-82 (1998)

[48] Markowitz, H.: Portfolio selection. Journal of Finance, 7, pp. 77-91 (1952)

[49] Merton, R. C.: Lifetime portfolio selection under uncertainty: the continuous-time model. Rev. Econ. Statist., 51, pp. 247-257 (1969)

[50] Merton, R. C.: Optimum consumption and portfolio rules in a continuous-time model. J. Econ. Theory, 3, pp. 373-413 (1971)

[51] Merton, R. C.: Continuous-Time Finance. Cambridge, MA: Blackwell (1990)

[52] Pham, H.: Continuous time stochastic control and optimization with financial applications. Springer-Verlag, Berlin, Heidelberg (2009)

[53] Reddy, P. V., Wraczek, S., & Zaccour, G.: Quality effects in different advertising models - An impulse control approach. European Journal of Operational Research, 255(3), pp. 984-995 (2016)

[54] Sadana, U., Reddy, P. V., & Zaccour, G.: Nash equilibria in nonzero-sum differential games with impulse control. Eur. J. Oper. Res., 295(2), pp. 792-805 (2021)

[55] Sadana, U., Reddy, P. V., Başar, T., & Zaccour, G.: Sampled-data Nash equilibria in differential games with impulse controls. Journal of Optimization Theory and Applications, 190(3), pp. 999-1022 (2021)

[56] Souganidis, P. E.: Approximation schemes for viscosity solutions of Hamilton-Jacobi equations. J. Diff. Eqns., 57, pp. 1-43 (1985)

[57] Souganidis, P. E.: Approximation schemes for viscosity solutions of Hamilton-Jacobi equations with applications to differential games. J. Non. Anal., TMA 9, pp. 217-257 (1985)
[58] Wiese, M., Knobloch, R., Korn, R., & Kretschmer, P.: Quant GANs: deep generation of financial time series. Quant. Finance 20, pp. 1419-1440 (2020)

[59] Yong, J. M.: Zero-sum differential games involving impulse controls. Appl. Math. Optim., 29, pp. 243-261 (1994)

[60] Zhang, F.: Stochastic differential games involving impulse controls. ESAIM: Control Optim. Calc. Var., 17(3), pp. 749-760 (2011)