Duality rotations in supersymmetric nonlinear electrodynamics revisited

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Abstract

We revisit the U(1) duality-invariant nonlinear models for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ vector multiplets coupled to off-shell supergravities. For such theories we develop new formulations which make use of auxiliary chiral superfields (spinor in the $\mathcal{N} = 1$ case and scalar for $\mathcal{N} = 2$) and are characterized by the remarkable property that U(1) duality invariance is equivalent to the manifest U(1) invariance of the self-interaction. Our construction is inspired by the non-supersymmetric approach that was proposed by Ivanov and Zupnik a decade ago and recently re-discovered in the form of twisted self-duality.
1 Introduction

Motivated by patterns of duality in extended supergravity theories \[1, 2\] (for a recent comprehensive review, see \[3\]), and also extending the famous 1981 work by Gaillard and Zumino \[4\], the general theory of duality-invariant models for nonlinear electrodynamics in four dimensions was developed in the mid-1990s \[5, 6, 7, 8\]. Given such a model described by a Lorentz invariant Lagrangian \( L(F_{ab}) \), with \( F_{ab} \) the electromagnetic field strength, the condition for invariance under U(1) duality rotations\(^1\)

\[
\delta F_{ab} = \lambda G_{ab} , \quad \delta G_{ab} = -\lambda F_{ab} \tag{1.1}
\]

proves to be equivalent to the requirement that the Lagrangian should obey the equation

\[
G^{ab} \tilde{G}_{ab} + F^{ab} \tilde{F}_{ab} = 0 , \tag{1.2}
\]

where

\[
\tilde{G}_{ab}(F) := \frac{1}{2} \varepsilon_{abcd} G^{cd}(F) = 2 \frac{\partial L(F)}{\partial F_{ab}} , \quad G(F) = \tilde{F} + \mathcal{O}(F^3) . \tag{1.3}
\]

The self-duality equation (1.2) was originally derived by Gibbons and Rasheed in 1995 \[5\]. Two years later, it was re-derived by Gaillard and Zumino \[7\] with the aid of their formalism developed back in 1981 \[4\] but originally applied only in the linear case. The self-duality equation (1.2) can be reformulated in a form suitable for theories with higher derivatives \[14\] (see also \[15\] for a recent discussion with examples).

As field theories, the models for nonlinear electrodynamics with U(1) duality invariance possess very interesting properties \[7, 8\] reviewed in \[14\] and later in \[3\]. First of all, the energy-momentum tensor is duality-invariant. Secondly, the action is automatically invariant under a Legendre transformation, and this is one of the reasons why the duality-invariant theories may be called self-dual. Thirdly, although the Lagrangian is not invariant under the duality rotations (1.1),

\[
\delta L = \frac{1}{4} \lambda (G^{ab} \tilde{G}_{ab} - F^{ab} \tilde{F}_{ab}) , \tag{1.4}
\]

the partial derivative \( \partial L / \partial g \) with respect to any duality-inert parameter \( g \) is invariant under (1.1). In fact, the duality invariance of the energy-momentum tensor is a corollary of this general statement. It is worth pointing out that for any solution \( L(F_{ab}) \) of the self-duality equation and a real parameter \( g \), the Lagrangian

\[
\hat{L}(F_{ab}) := \frac{1}{g^2} L(g F_{ab}) \tag{1.5}
\]

\(^1\)For early approaches to electromagnetic duality rotations see \[9, 10\]. For alternative formulations of duality symmetric actions see \[11, 12, 13\] and references therein.
is also a solution of (1.1).

The concept of self-dual nonlinear electrodynamics was generalized to the cases of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ rigid supersymmetric theories in [16]. This generalization has turned out to be very useful, since the families of actions obtained include all the known models for partial breaking of supersymmetry based on the use of a vector Goldstone multiplet. In particular, the $\mathcal{N} = 1$ supersymmetric Born-Infeld action [17], which is a Goldstone multiplet action for partial supersymmetry breakdown $\mathcal{N} = 2 \to \mathcal{N} = 1$ [18, 19] is, at the same time, a solution to the $\mathcal{N} = 1$ self-duality equation [14, 16]. Furthermore, the model for partial breaking of supersymmetry $\mathcal{N} = 4 \to \mathcal{N} = 2$ [20], which nowadays is identified with the $\mathcal{N} = 2$ supersymmetric Born-Infeld action, was first constructed in [14] as a unique solution to the $\mathcal{N} = 2$ self-duality equation possessing a nonlinearly realized central charge symmetry. The models for self-dual nonlinear supersymmetric electrodynamics [14, 16] were generalized to $\mathcal{N} = 1$ supergravity in [21] and recently to $\mathcal{N} = 2$ supergravity [22].

The self-duality equation (1.2) is a nonlinear differential equation on the Lagrangian, and thus its general solutions are difficult to construct explicitly. The most famous exact solution of (1.2) is the Born-Infeld Lagrangian [23]

$$L_{\text{BI}}(F_{ab}) = \frac{1}{g^2} \left\{ 1 - \sqrt{-\text{det}(\eta_{ab} + gF_{ab})} \right\} ,$$

(1.6)

with $g$ the coupling constant.

A decade ago, Ivanov and Zupnik [24, 25] proposed a reformulation of nonlinear electrodynamics, $L(F_{ab}) \to \tilde{L}(F_{ab}, V_{ab})$, which makes use of an auxiliary bivector $V_{ab} = -V_{ba}$, the latter being equivalent to a pair of symmetric spinors, $V_{\alpha\beta} = V_{\beta\alpha}$ and its conjugate $\tilde{V}_{\dot{\alpha}\dot{\beta}}$. The new Lagrangian $\tilde{L}$ is at most quadratic with respect to the electromagnetic field strength $F_{ab}$, while the self-interaction is described by a nonlinear function of the auxiliary variables, $L_{\text{int}}(V_{ab})$,

$$\tilde{L}(F_{ab}, V_{ab}) = \frac{1}{4} F^{ab} F_{ab} + \frac{1}{2} V^{ab} V_{ab} - V^{ab} F_{ab} + L_{\text{int}}(V_{ab}) .$$

(1.7)

The original theory $L(F_{ab})$ is obtained from $\tilde{L}(F_{ab}, V_{ab})$ by integrating out the auxiliary variables. In terms of $\tilde{L}(F_{ab}, V_{ab})$, the condition of U(1) duality invariance was shown [24, 25] to be equivalent to the requirement that the self-interaction

$$L_{\text{int}}(V_{ab}) = L_{\text{int}}(\nu, \bar{\nu}) , \quad \nu := V^{\alpha\beta} V_{\alpha\beta}$$

(1.8)

is invariant under linear U(1) transformations $\nu \to e^{i\varphi} \nu$, with $\varphi \in \mathbb{R}$, and thus

$$L_{\text{int}}(\nu, \bar{\nu}) = f(\nu\bar{\nu}) ,$$

(1.9)
where $f$ is a real function of one real variable. As a result, the Ivanov-Zupnik formulation allows one to generate, in principle, all solutions of the self-duality equation. At first sight, this approach appears somewhat mysterious. However its origin becomes more transparent if we recall some general features of all solutions of (1.2) discussed in [5, 14].

First of all, it is worth recalling another useful representation for the Lagrangian $L(F_{ab})$ and for the self-duality equation, following [14]. Since in four dimensions the electromagnetic field has only two independent invariants,

$$\alpha = \frac{1}{4} F_{ab} F_{ab} , \quad \beta = \frac{1}{4} F_{ab} \tilde{F}_{ab} , \quad (1.10)$$

the Lagrangian $L(F_{ab})$ can be considered as a real function of one complex variable

$$L(F_{ab}) = L(\omega, \bar{\omega}) , \quad \omega = \alpha + i \beta . \quad (1.11)$$

The theory is parity invariant iff $L(\omega, \bar{\omega}) = L(\bar{\omega}, \omega)$. If the theory is duality invariant, then $L(\omega, \bar{\omega})$ can be shown [14] to have the form

$$L(\omega, \bar{\omega}) = -\frac{1}{2} (\omega + \bar{\omega}) + \omega \bar{\omega} \Lambda(\omega, \bar{\omega}) , \quad \Lambda(\omega, \bar{\omega}) = \text{const} + \mathcal{O}(|\omega|) , \quad (1.12)$$

where the interaction $\Lambda(\omega, \bar{\omega})$ is a real analytic function. The self-duality equation (1.2) turns into

$$\text{Im} \left\{ \partial_\omega (\omega \Lambda) - \bar{\omega} \left( \partial_{\bar{\omega}} (\omega \Lambda) \right)^2 \right\} = 0 , \quad (1.13)$$

with $\partial_\omega = \partial / \partial \omega$. For the Born-Infeld Lagrangian (1.6), we have

$$L_{\text{BI}}(\omega, \bar{\omega}) = \frac{1}{g^2} \left( 1 - \sqrt{1 + g^2(\omega + \bar{\omega}) + \frac{1}{4} g^4(\omega - \bar{\omega})^2} \right) , \quad \Lambda_{\text{BI}}(\omega, \bar{\omega}) = \frac{g^2}{1 + \frac{1}{2} g^2(\omega + \bar{\omega}) + \sqrt{1 + g^2(\omega + \bar{\omega}) + \frac{1}{4} g^4(\omega - \bar{\omega})^2}} . \quad (1.14)$$

It is a simple exercise to check that $\Lambda_{\text{BI}}$ is a solution of (1.13).

Now, we reproduce verbatim a paragraph from section 2 in [14] (a similar discussion appeared earlier in [5]).

In perturbation theory one looks for a parity invariant solution of the self-duality equation by considering the Ansatz

$$\Lambda(\omega, \bar{\omega}) = \sum_{n=0}^{\infty} \sum_{p+q=n} C_{p,q} \omega^p \bar{\omega}^q , \quad C_{p,q} \in \mathbb{R} , \quad (1.15)$$

and

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where $n = p + q$ is the level of the coefficient $C_{p,q}$. It turns out that for odd level the self-duality equation uniquely expresses all coefficients recursively. If, however, the level is even, the self-duality equation uniquely fixes the level-$n$ coefficients $C_{p,q}$ with $p \neq q$ through those at lower levels, while $C_{r,r}$ remain undetermined. This means that a general solution of the self-duality equation involves an arbitrary real analytic function of one real argument, $f(\omega \bar{\omega})$.

Given a real analytic function

$$f(\omega \bar{\omega}) = \sum_{r=0}^{\infty} C_{r,r} (\omega \bar{\omega})^r,$$

the self-duality equation (1.13) uniquely determines the entire self-interaction (1.15). This means that there exists a one-to-one map $\pi : f(\omega \bar{\omega}) \to \Lambda(\omega, \bar{\omega})$, where $\Lambda(\omega, \bar{\omega})$ corresponds to a duality-invariant theory. In other words, the duality-invariant theories can be formulated in terms of the function $f(\omega \bar{\omega})$. This is actually the same function which appears within the Ivanov-Zupnik approach, eq. (1.9). Their approach is essentially a scheme to formulate self-dual theories in terms of such a real function, $f_{IZ}(x)$, uniquely related to $f(x)$ in (1.16).

Recently, there has been a revival of interest in the duality-invariant dynamical systems [26, 27, 15, 28] inspired by the desire to achieve a better understanding of the UV properties of extended supergravity theories. The authors of [26, 27, 15] have put forward the so-called “twisted self-duality constraint” as a systematic procedure to generate duality-invariant theories. However, it has been demonstrated [29] that the non-supersymmetric construction of [26, 27, 15] naturally originates within the more general approach previously developed in [24, 25]. Specifically, the twisted self-duality constraint correspond to an equation of motion in the approach of [24, 25].

The authors of [28] studied perturbative solutions of the $\mathcal{N} = 2$ supersymmetric self-duality equation [16] by combining the perturbative analysis of [14] with the idea of twisted self-duality. In the present paper, we give $\mathcal{N} = 1$ and $\mathcal{N} = 2$ locally supersymmetric extensions of the Ivanov-Zupnik approach. In the rigid supersymmetric limit, our results provide an off-shell extension of the approach pursued in [28].

This paper is organized as follows. In section 2 we give a brief summary of self-dual models for $\mathcal{N} = 1$ supersymmetric electrodynamics coupled to supergravity. Here two off-shell realizations for $\mathcal{N} = 1$ supergravity are used: the old minimal formulation [30, 31] and the new minimal formulation [32]. In section 3 we develop a novel description of
the $\mathcal{N} = 1$ supersymmetric duality-invariant theories \cite{16,14,21} that makes use of an auxiliary covariantly chiral spinor superfield and its conjugate. Section 4 is devoted to a novel description of the $\mathcal{N} = 2$ supersymmetric duality-invariant theories \cite{16,14,22} that employs an auxiliary covariantly chiral scalar superfield and its conjugate. A few concluding comments are given in section 5. The main body of the paper is accompanied by two technical appendices devoted to aspects of the superspace differential geometry of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supergravities.

2 Duality rotations in $\mathcal{N} = 1$ supersymmetric nonlinear electrodynamics

Unless otherwise specified, in this and the next sections we use the old minimal formulation for $\mathcal{N} = 1$ supergravity. Our notation and conventions mostly follow \cite{33} (which are similar to those adopted in \cite{34}) with the only exception that we use different symbols for the full superspace and for the chiral integration measures. A summary concerning the Wess-Zumino superspace geometry is given in Appendix A.

Consider a theory of an Abelian $\mathcal{N} = 1$ vector multiplet in curved superspace generated by an action $S[W, \bar{W}]$. The covariantly chiral spinor field strength $W_\alpha$ and its conjugate $\bar{W}_{\dot{\alpha}}$ are defined as

$$W_\alpha = -\frac{1}{4}(D^2 - 4R)D_\alpha V, \quad \bar{W}_{\dot{\alpha}} = -\frac{1}{4}(D^2 - 4\bar{R})\bar{D}_{\dot{\alpha}} V, \quad (2.1)$$

in terms of a real unconstrained prepotential $V$. The field strengths $W_\alpha$ and $\bar{W}_{\dot{\alpha}}$ obey the Bianchi identity

$$D^\alpha W_\alpha = \bar{D}_{\dot{\alpha}} \bar{W}_{\dot{\alpha}}. \quad (2.2)$$

In many cases $S[W, \bar{W}]$ can unambiguously be defined as a functional of an unrestricted covariantly chiral superfield $W_\alpha$ and its conjugate $\bar{W}_{\dot{\alpha}}$. Then, defining

$$i M_\alpha := 2 \frac{\delta}{\delta W_\alpha} S[W, \bar{W}], \quad (2.3)$$

the equation of motion for $V$ is

$$D^\alpha M_\alpha = \bar{D}_{\dot{\alpha}} \bar{M}_{\dot{\alpha}}. \quad (2.4)$$

\footnote{As pointed out in \cite{16,14}, this is always possible if $S[W, \bar{W}]$ does not involve the combination $D^\alpha W_\alpha$ as an independent variable.}
Here the variational derivative $\frac{\delta S}{\delta W^\alpha}$ is defined by
\[
\delta S = \int d^4x d^2\theta \mathcal{E} \frac{\delta S}{\delta W^\alpha} \frac{\delta S}{\delta W^\alpha} + \text{c.c.} ,
\]
where $\mathcal{E}$ denotes the chiral integration measure, and $W_\alpha$ is assumed to be an unrestricted covariantly chiral spinor.

Since the Bianchi identity (2.2) and the equation of motion (2.4) have the same functional form, one may consider U(1) duality rotations
\[
\delta W_\alpha = \lambda M_\alpha , \quad \delta M_\alpha = -\lambda W_\alpha ,
\]
with $\lambda$ a constant parameter. The condition for duality invariance is the self-duality equation
\[
\text{Im} \int d^4x d^2\theta \mathcal{E} \left\{ W^2 + M^2 \right\} = 0 ,
\]
in which $W_\alpha$ is chosen to be an unrestricted covariantly chiral spinor.

For any vector multiplet model $S[W, \bar{W}]$, one can develop a dual formulation. This is achieved by introducing the auxiliary action
\[
S[W, W, W_D, \bar{W}_D] = -\frac{i}{2} \int d^4x d^2\theta \mathcal{E} W^\alpha W_D^\alpha + \text{c.c.} + S[W, \bar{W}] ,
\]
where $W_\alpha$ is now an unrestricted covariantly chiral spinor superfield, and $W_D^\alpha$ the dual field strength
\[
W_D^\alpha = -\frac{1}{4} \left( \mathcal{D}^2 - 4R \right) D_\alpha V_D , \quad V_D = V_D ,
\]
with $V_D$ a dual gauge prepotential. This model is equivalent to the original model, since the equation of motion for $V_D$ implies that $W$ satisfies the Bianchi identity (2.2) and the action (2.8) reduces to $S[W, \bar{W}]$. On the other hand, under quite general conditions on the structure of $S[W, \bar{W}]$, one can integrate out from $S[W, \bar{W}, W_D, \bar{W}_D]$ the auxiliary variables $W_\alpha$ and $\bar{W}_\dot{\alpha}$ and end up with a dual action $S_D[W_D, \bar{W}_D]$. By construction the models $S[W, \bar{W}]$ and $S_D[W_D, \bar{W}_D]$ are related to each other by a Legendre transformation.

If $S[W, \bar{W}]$ is a solution of the self-duality equation (2.7), then the dual action has the same functional form as the original action,
\[
S_D[W, \bar{W}] = S[W, \bar{W}] .
\]
Therefore the theory is self-dual under the superfield Legendre transformation.
The most general self-dual model with no more than two derivatives at the component level was constructed in the rigid supersymmetric case in [16] and extended to supergravity a few years later in [21]. The corresponding action has the form

\[
S_{\text{SED}} = \frac{1}{4} \int d^4x d^2\theta E W^2 + \text{c.c.} + \frac{1}{4} \int d^4x d^2\theta d^2\bar{\theta} E W^2\bar{W}^2\Lambda(u, \bar{u}) ,
\]

where

\[
u := \frac{1}{8} (D^2 - 4\bar{R}) W^2 .
\]

For this model the self-duality equation (2.7) amounts to

\[
\text{Im} \int d^4x d^2\theta d^2\bar{\theta} E W^2\bar{W}^2 \left\{ \Gamma - \bar{u} \Gamma^2 \right\} = 0 , \quad \Gamma := \partial_u (u \Lambda) .
\]

In this equation the covariantly chiral spinor \(W_\alpha\) has to be completely arbitrary, and therefore we conclude that

\[
\text{Im} \left\{ \Gamma - \bar{u} \Gamma^2 \right\} = 0 .
\]

The component structure of the theory (2.11) was studied in [35]. In the rigid supersymmetric case, the bosonic sector of (2.11) was originally analyzed in [14]. Upon switching off the auxiliary field of the vector multiplet, \(D = 0\), which always is a solution of the corresponding equation of motion, the bosonic action reduced to that describing self-dual nonlinear electrodynamics. The self-interaction \(\Lambda\), which determines the bosonic Lagrangian (1.12), coincides with that appearing in the supersymmetric action (2.11), see [14, 35] for more details.

The supercurrent multiplet corresponding to the self-dual theory (2.11) was computed in [21] and shown to be duality-invariant.

The action (2.11) describes supersymmetric nonlinear electrodynamics in old minimal supergravity. The model can also be coupled to new minimal supergravity [32] or to non-minimal supergravity [36, 37]. Here we recall, following [21], how this is achieved in the case of new minimal supergravity.

As is known, each off-shell formulation for \(\mathcal{N} = 1\) supergravity can be realized as a super-Weyl invariant coupling of old minimal supergravity to certain compensator(s) [33, 38, 39]. Super-Weyl transformations [40] are local scale and U(1) transformations of the covariant derivatives of the form

\[
\delta_\sigma D_\alpha = (\bar{\sigma} - \frac{1}{2} \sigma) D_\alpha + (D\beta \sigma) M_{\alpha\beta} , \quad \delta_\sigma \bar{D}_{\dot{\alpha}} = (\sigma - \frac{1}{2} \bar{\sigma}) \bar{D}_{\dot{\alpha}} + (D\dot{\beta} \bar{\sigma}) \bar{M}_{\dot{\beta}\dot{\alpha}} ,
\]

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with \( \sigma \) an arbitrary covariantly chiral scalar parameter, \( \mathcal{D}_\sigma \sigma = 0 \). The compensator for new minimal supergravity is a real covariantly linear scalar \( \mathbb{L} \),

\[
(\mathcal{D}^2 - 4R)\mathbb{L} = 0, \quad \bar{\mathbb{L}} = \mathbb{L},
\]

which is required to be nowhere vanishing. Its super-Weyl transformation law is

\[
\delta_\sigma \mathbb{L} = (\sigma + \bar{\sigma}) \mathbb{L}.
\]

Recalling the super-Weyl transformation of \( W_\alpha \),

\[
\delta_\sigma W_\alpha = \frac{3}{2} \sigma W_\alpha,
\]

one may see that the following combination

\[
(\mathcal{D}^2 - 4\bar{R})\left(\frac{W^2}{\mathbb{L}^2}\right)
\]

is super-Weyl invariant. This implies that the vector-multiplet action \([21]\)

\[
S[W, \bar{W}, \mathbb{L}] = \frac{1}{4} \int d^4x d^2\theta E W^2 + \text{c.c.}
\]

\[
+ \frac{1}{4} \int d^4x d^2\theta d^2\bar{\theta} E \frac{W^2 \bar{W}^2}{\mathbb{L}^2} \Lambda\left(\frac{u}{\mathbb{L}^2}, \frac{\bar{u}}{\mathbb{L}^2}\right),
\]

is super-Weyl invariant. Moreover, it is not difficult to check that \( S[W, \bar{W}, \mathbb{L}] \) solves the self-duality equation \((2.7)\). The action \((2.20)\) described self-dual supersymmetric electrodynamics coupled to new minimal supergravity.

3 New realization

We now turn to presenting a new formulation for the self-dual models of the \( \mathcal{N} = 1 \) vector multiplet described in the previous section. This representation is inspired by the non-supersymmetric construction of \([24, 25]\).

3.1 General setup

Consider an auxiliary action of the form

\[
S[W, \bar{W}, \eta, \bar{\eta}] = \int d^4x d^2\theta E \left\{ \eta W - \frac{1}{2} \eta^2 - \frac{1}{4} W^2 \right\} + \text{c.c.} + S_{\text{int}}[\eta, \bar{\eta}].
\]
Here the spinor superfield $\eta_\alpha$ is constrained to be covariantly chiral, $\mathcal{D}_\beta \eta_\alpha = 0$, but otherwise it is completely arbitrary. By definition, the second term on the right, $\mathcal{G}_{\text{int}}[\eta, \bar{\eta}]$, contains cubic, quartic and higher powers of $\eta_\alpha$ and its conjugate.

The above model is equivalent to a theory with action

$$S[W, \bar{W}] = \frac{1}{4} \int d^4x d^2\theta \mathcal{E} W^2 + \text{c.c.} + S_{\text{int}}[W, \bar{W}] ,$$

(3.2)

describing the dynamics of the vector multiplet. Indeed, the equation of motion for $\eta_\alpha$ is

$$W_\alpha = \eta_\alpha - \frac{\delta}{\delta \eta_\alpha} \mathcal{G}_{\text{int}}[\eta, \bar{\eta}] .$$

(3.3)

In perturbation theory, this equation can be used to express $\eta_\alpha$ as a functional of $W_\alpha$ and its conjugate, $\eta_\alpha = \Psi_\alpha[W, \bar{W}]$. Plugging this functional and its conjugate into the action (3.1), we end up with some vector-multiplet model of the form (3.2). As a result, we have two equivalent realizations of the same theory, in terms of the action (3.1) or in terms of $S[W, \bar{W}]$.

Suppose $S[W, \bar{W}]$ is a solution of the self-duality equation (2.7). We need to understand what the implications of self-duality are on the structure of (3.1). Let us compute $M_\alpha$ by using the two actions $S[W, \bar{W}, \eta, \bar{\eta}]$ and $S[W, \bar{W}]$:

$$i M_\alpha := 2 \frac{\delta}{\delta W_\alpha} S = 2\eta_\alpha - W_\alpha$$

(3.4a)

$$= W_\alpha + 2 \frac{\delta}{\delta W_\alpha} S_{\text{int}}[W, \bar{W}] .$$

(3.4b)

Now, if we make use of the equation of motion for $\eta$, eq. (3.3), the self-duality equation (2.7) turns into

$$\text{Im} \int d^4x d^2\theta \mathcal{E} \eta_\alpha \frac{\delta}{\delta \eta_\alpha} \mathcal{G}_{\text{int}}[\eta, \bar{\eta}] = 0 .$$

(3.5)

This condition means that the self-interaction $\mathcal{G}_{\text{int}}[\eta, \bar{\eta}]$ is invariant under rigid U(1) phase transformations of $\eta_\alpha$ and its conjugate,

$$\mathcal{G}_{\text{int}}[e^{i\varphi} \eta, e^{-i\varphi} \bar{\eta}] = \mathcal{G}_{\text{int}}[\eta, \bar{\eta}] , \quad \varphi \in \mathbb{R} .$$

(3.6)

The duality rotation (2.6) acts on the chiral spinor $\eta_\alpha = \frac{1}{2}(W_\alpha + iM_\alpha)$, eq. (3.4a), as

$$\delta \eta_\alpha = -i \lambda \eta_\alpha .$$

(3.7)

From (3.4a) and (3.4b) we have

$$\eta_\alpha = W_\alpha + \frac{\delta}{\delta W_\alpha} S_{\text{int}}[W, \bar{W}] .$$

(3.8)
This relation allows us to express $W_\alpha$ as a functional of $\eta_\alpha$ and its conjugate, $W_\alpha = W_\alpha[\eta, \bar{\eta}]$, and then reconstruct the self-interaction $\mathcal{G}_{\text{int}}[\eta, \bar{\eta}]$ starting from the action (3.2).

Given a manifestly U(1) invariant self-interaction $\mathcal{G}_{\text{int}}[\eta, \bar{\eta}]$, we can construct a duality-invariant theory $S[W, \bar{W}]$ by starting for the action (3.1) and then integrating out the auxiliary variables $\eta_\alpha$ and $\bar{\eta}_\dot{\alpha}$.

Suppose that $S[W, \bar{W}]$ is self-dual under Legendre transformation, eq. (2.10). In terms of the auxiliary action (3.1) this condition proves to be equivalent to

$$\mathcal{G}_{\text{int}}[i \eta, -i \bar{\eta}] = \mathcal{G}_{\text{int}}[\eta, \bar{\eta}] \quad (3.9)$$

We see that the self-duality under the superfield Legendre transformation is equivalent to the fact that the self-interaction $\mathcal{G}_{\text{int}}[\eta, \bar{\eta}]$ is $\mathbb{Z}_4$ invariant.

In general, self-duality under a Legendre transformation is known to be a pretty mild condition [41].

### 3.2 Two-derivative models

We now turn to duality-invariant supersymmetric theories with at most two derivatives at the component level. Consider an auxiliary action of the form

$$S[W, \bar{W}, \eta, \bar{\eta}] = \int d^4x d^2\theta E \left\{ \eta W - \frac{1}{2} \eta^2 - \frac{1}{4} W^2 \right\} + \text{c.c.}$$

$$+ \frac{1}{4} \int d^4x d^2\theta d^2\bar{\theta} E \eta^2 \bar{\eta}^2 \mathfrak{F}(v, \bar{v}) \quad (3.10)$$

where

$$v := \frac{1}{8}(\mathcal{D}^2 - 4\mathcal{R})\eta^2 \quad (3.11)$$

and $\mathfrak{F}(v, \bar{v})$ is a real analytic function. We would like to integrate out from (3.10) the auxiliary spinor variables $\eta_\alpha$ and $\bar{\eta}_\dot{\alpha}$ in order to bring the action to the form (2.11). The equation of motion for $\eta^\alpha$ is

$$W_\alpha = \eta_\alpha \left\{ 1 + \frac{1}{8}(\mathcal{D}^2 - 4\mathcal{R}) \left[ \bar{\eta}^2 \left( \mathfrak{F} + \frac{1}{8}(\mathcal{D}^2 - 4\mathcal{R})(\eta^2 \partial_\alpha \mathfrak{F}) \right) \right] \right\} \quad (3.12)$$

Its immediate implications are

$$\eta W = \eta^2 \left[ 1 + \frac{1}{8}(\mathcal{D}^2 - 4\mathcal{R}) \left\{ \bar{\eta}^2 \partial_\alpha (v \mathfrak{F}) \right\} \right] \quad (3.13a)$$

$$W^2 = \eta^2 \left[ 1 + \frac{1}{8}(\mathcal{D}^2 - 4\mathcal{R}) \left\{ \bar{\eta}^2 \partial_\alpha (v \mathfrak{F}) \right\} \right]^2 \quad (3.13b)$$

$$W^2\bar{W}^2 = \eta^2 \bar{\eta}^2 \left[ 1 + \partial_\alpha (v \bar{v} \mathfrak{F}) \right]^2 \left[ 1 + \partial_\alpha (v \bar{v} \mathfrak{F}) \right]^2 \quad (3.13c)$$
Eq. (3.13b) implies that

\[ u \approx v[1 + \partial_v(v\bar{\mathcal{F}})]^2. \]  

(3.14)

Here and below, the symbol \( \approx \) is used to indicate that the result holds modulo terms proportional to \( \eta_\alpha \) and \( \bar{\eta}_\dot{\alpha} \) (or, equivalently, to \( W_\alpha \) and \( \bar{W}_{\dot{\alpha}} \)).

The identities (3.13) may be used to derive several integral relations

\[
\int d^4x d^2\theta \mathcal{E} W^2 = \int d^4x d^2\theta \mathcal{E} \eta^2 \\
- \int d^4x d^2\bar{\theta} \mathcal{E} \eta^2 \mathcal{E} \eta^2 \left\{ \partial_v(v\bar{\mathcal{F}}) + \frac{1}{2} \bar{v}[\partial_v(v\bar{\mathcal{F}})]^2 \right\}, 
\]

(3.15a)

\[
\int d^4x d^2\theta \mathcal{E} \eta W = \int d^4x d^2\bar{\theta} \mathcal{E} \eta^2 - \frac{1}{2} \int d^4x d^2\bar{\theta} \mathcal{E} \eta^2 \eta^2 \partial_v(v\bar{\mathcal{F}}). 
\]

(3.15b)

With the aid of these relations, the action (3.10) takes the form

\[
S[W, \bar{W}] = \frac{1}{4} \int d^4x d^2\theta \mathcal{E} W^2 + \text{c.c.} + \frac{1}{4} \int d^4x d^2\bar{\theta} \mathcal{E} W^2 \bar{W}^2 \Lambda(u, \bar{u}), 
\]

(3.16)

where we have introduced

\[
\Lambda(u, \bar{u}) := \frac{\mathcal{F} + \bar{v}[\partial_v(v\bar{\mathcal{F}})]^2 + v[\partial_v(\bar{v}\mathcal{F})]^2}{[1 + \partial_v(v\bar{\mathcal{F}})]^2[1 + \partial_v(\bar{v}\mathcal{F})]^2}. 
\]

(3.17)

The right-hand side of (3.17) is uniquely determined in terms of \( \mathcal{F} \) and its partial derivatives. In order to read off the function in the left-hand side of (3.17), we have to know the expression for \( u \) as a functional of \( v \) and \( \bar{v} \) which, in accordance with (3.13b), is quite complicated:

\[
u = \frac{1}{8}(D^2 - 4\bar{R}) \eta^2 \left[ 1 + \frac{1}{8}(\bar{D}^2 - 4R) \left\{ \bar{\eta}^2 \partial_v(v\bar{\mathcal{F}}) \right\}^2 \right]. 
\]

(3.18)

However, since \( \Lambda(u, \bar{u}) \) appears in the action (3.17) multiplied by \( W^2\bar{W}^2 \), when evaluating \( \Lambda(u, \bar{u}) \) we can replace (3.14) with the “effective” relation \( u = v[1 + \partial_v(v\bar{\mathcal{F}})]^2 \). The latter allows us to express \( v \) in terms of \( u \) and \( \bar{u} \), and therefore to read off the function \( \Lambda(u, \bar{u}) \) in (3.18).

Now we have to learn how to carry out an inverse transformation, that is how to reconstruct the self-coupling \( \mathcal{F}(v, \bar{v}) \) starting from the action (3.16). Making use of the action (3.10) leads to (3.4a), and hence

\[
\eta_\alpha = \frac{1}{2}(W_\alpha + iM_\alpha). 
\]

(3.19)
On the other hand, we may compute $M_\alpha$, eq. (2.3), from the action (3.16).

$$iM_\alpha = W_\alpha \left\{ 1 - \frac{1}{4}(\mathcal{D}^2 - 4R) \left[ \bar{W}^2 \left( \Lambda + \frac{1}{8}(\mathcal{D}^2 - 4R)(W^2 \partial_u \Lambda) \right) \right] \right\} .$$

(3.20)

Combining the two results, we obtain

$$\eta_\alpha = W_\alpha \left\{ 1 - \frac{1}{8}(\mathcal{D}^2 - 4R) \left[ \bar{W}^2 \left( \Lambda + \frac{1}{8}(\mathcal{D}^2 - 4R)(W^2 \partial_u \Lambda) \right) \right] \right\} .$$

(3.21)

An important result may be seen by comparing the relations (3.12) and (3.21). We obtain

$$[1 - \partial_u (u\bar{u}\Lambda)] \approx \frac{1}{1 - \partial_{\bar{u}} (\bar{u}u\Lambda)} \approx 1 .$$

(3.22)

There are three simple implications of (3.21):

$$\eta W = W^2 \left[ 1 + \frac{1}{8}(\mathcal{D}^2 - 4R) \left\{ \bar{W}^2 \partial_u (u\Lambda) \right\} \right] ,$$

(3.23a)

$$\eta^2 = W^2 \left[ 1 - \frac{1}{8}(\mathcal{D}^2 - 4R) \left\{ \bar{W}^2 \partial_u (u\Lambda) \right\} \right]^2 .$$

(3.23b)

$$\eta^2 \bar{\eta}^2 = W^2 \bar{W}^2 [1 - \partial_u (u\bar{u}\Lambda)]^2 \approx \frac{1}{1 - \partial_{\bar{u}} (\bar{u}u\Lambda)} \approx 1 .$$

(3.23c)

These results lead to

$$v \approx u[1 - \partial_u (u\bar{u}\Lambda)]^2 ,$$

(3.24a)

$$v[1 + \partial_{\bar{u}} (v\bar{v}\bar{\mathcal{F}})] \approx u[1 - \partial_u (u\bar{u}\Lambda)] .$$

(3.24b)

Due to (3.22), the relation (3.13b) is equivalent to (3.23c). The relations obtained allow us to restore the self-interaction

$$\mathcal{F}(v, \bar{v}) = \frac{\Lambda - \bar{u}[\partial_u (u\Lambda)]^2 - u[\partial_u (\bar{u}\Lambda)]^2}{[1 - \partial_u (u\bar{u}\Lambda)]^2 [1 - \partial_{\bar{u}} (\bar{u}u\Lambda)]^2} .$$

(3.25)

Starting from the action (3.16), it is now trivial to restore the self-interaction $\mathcal{F}(v, \bar{v})$ by making use of eq. (3.25) in conjunction with the effective relation $v = u[1 - \partial_u (u\bar{u}\Lambda)]^2$. It is instructive to compare the functional forms of the transformation $\Lambda(u, \bar{u}) \to \mathcal{F}(v, \bar{v})$, eq. (3.25), and its inverse (3.17).

The last point to analyze is $U(1)$ duality invariance. Suppose that the action (3.16) is a solution of the self-duality equation (2.7). Using the above relations, one may show that

$$W^2 \bar{W}^2 (\Gamma - \bar{\Gamma}^2) = \eta^2 \bar{\eta}^2 (\mathcal{F} + v\partial_v \mathcal{F}) .$$

(3.26)
Therefore, the self-duality condition (2.13) is equivalent to

$$ (v \partial_v - \bar{v} \partial_{\bar{v}}) \mathcal{F} = 0 \iff \mathcal{F}(v, \bar{v}) = f(v\bar{v}) \quad (3.27) $$

We conclude that the self-interaction $\mathcal{F}(v, \bar{v})$ must be invariant under the U(1) transformations (3.7).

The model (3.10) can naturally be coupled to new minimal supergravity. For this we postulate the super-Weyl transformation of $\eta_\alpha$ to be (compare with (2.18))

$$ \delta_{\sigma} \eta_\alpha = \frac{3}{2} \sigma \eta_\alpha \quad (3.28) $$

and replace the action (3.10) with

$$ S[W, \bar{W}, \eta, \bar{\eta}, L] = \int d^4x d^2\theta d^{2}\bar{\theta} \mathcal{E} \left\{ \eta W - \frac{1}{2} \eta^2 - \frac{1}{4} W^2 \right\} + \text{c.c.} $$

$$ + \frac{1}{4} \int d^4x d^2\theta d^2\bar{\theta} E \frac{\eta^2 \bar{\eta}^2}{L^2} \mathcal{F} \left( \frac{v}{L}, \frac{\bar{v}}{L} \right) \quad (3.29) $$

This action is obviously super-Weyl invariant.

4 Duality rotations in $\mathcal{N} = 2$ supersymmetric nonlinear electrodynamics

Finally, we give a new realization for the duality-invariant $\mathcal{N} = 2$ supersymmetric theories presented in [16, 22]. The superspace formulation for $\mathcal{N} = 2$ conformal supergravity developed in [42] is used throughout this section.

We denote by $S[W, \bar{W}]$ an action functional which generates the dynamics of an $\mathcal{N} = 2$ vector multiplet. The Abelian vector multiplet coupled to $\mathcal{N} = 2$ conformal supergravity can be described by its covariantly chiral field strength $W$,

$$ \mathcal{D}_{\tilde{a}} W = 0 \quad (4.1) $$

subject to the Bianchi identity

$$ \left( \mathcal{D}^{ij} + 4 S^{ij} \right) W = \left( \mathcal{D}^{ij} + 4 S^{ij} \right) W \quad (4.2) $$

where $\mathcal{D}^{ij} := \mathcal{D}^{\alpha(i} \mathcal{D}^{j)}_{\alpha}$ and $\mathcal{D}^{ij} := \mathcal{D}_{\tilde{a}}^{(i} \mathcal{D}^{j)\tilde{a}}$; $S^{ij}$ and its conjugate $S_{ij} = \varepsilon_{ij} \varepsilon_{kl} S^{kl}$ are special dimension-1 components of the torsion, see Appendix B. In the flat superspace limit, the Bianchi identity reduces to that given in [43].
To realize \( W \) as a gauge-invariant field strength, we make use of a curved-superspace extension of Mezincescu’s prepotential [44] (see also [45]), \( V_{ij} = V_{ji} \), which is an unconstrained real SU(2) triplet, \( (V_{ij})^* = V^{ij} = \epsilon^{ik}\epsilon^{lj}V_{kl} \). The expression for \( W \) in terms of \( V_{ij} \) was found in [46] to be

\[
W = \bar{\Delta} \left( D^{ij} + 4S^{ij} \right) V_{ij} .
\] (4.3)

Here \( \bar{\Delta} \) is the covariantly chiral projection operator \( (\text{B.4}) \).

Starting from the action \( S[W, \bar{W}] \), we introduce a covariantly chiral scalar superfield \( M \) defined as

\[
i M := 4 \frac{\delta}{\delta W} S[W, \bar{W}] , \quad \bar{D}_{\bar{i}} M = 0 .
\] (4.4)

In terms of \( M \) and its conjugate \( \bar{M} \), the equation of motion for \( V_{ij} \) is

\[
\left( D^{\alpha(i} D_{\alpha)} + 4S^{ij} \right) M = \left( \bar{D}_{\dot{\alpha}(i} \bar{D}^{\dot{\alpha})j} + 4\bar{S}^{ij} \right) \bar{M} .
\] (4.5)

Here we have used the chiral integration rule \( (\text{B.5}) \).

Since the Bianchi identity \( (\text{4.2}) \) and the equation of motion \( (\text{4.5}) \) have the same functional form, one can consider infinitesimal \( U(1) \) duality rotations

\[
\delta W = \lambda M , \quad \delta M = -\lambda W ,
\] (4.6)

with \( \lambda \) a constant parameter. The theory under consideration is duality invariant under the condition \( [22] \)

\[
\text{Im} \int d^4x d^4\theta \mathcal{E} \left( W^2 + M^2 \right) = 0 .
\] (4.7)

In the rigid superspace limit, this reduces to the \( \mathcal{N} = 2 \) self-duality equation \([16]\).

All \( \mathcal{N} = 2 \) locally supersymmetric theories, which solve the equation \( (\text{4.7}) \), are self-dual under a Legendre transformation \([16, 22]\).

At this point, an important comment is in order. We realize \( \mathcal{N} = 2 \) Poincaré supergravity as a super-Weyl invariant coupling of conformal supergravity to certain compensators. In such an approach, any matter action, in particular \( S[W, \bar{W}] \), must be super-Weyl invariant, \( \delta_{\sigma} S[W, \bar{W}] = 0 \). In the case of duality-invariant theories, the self-duality equation \( (\text{4.7}) \) has to be super-Weyl invariant. Let us check that this is indeed true. Under the super-Weyl transformation \( (\text{B.3}) \), \( W \) varies as \([32]\)

\[
\delta_{\sigma} W = \sigma W .
\] (4.8)
This is induced by the following variation of Mezincescu’s prepotential \[22]\:

\[
\delta_{\sigma}V_{ij} = -(\sigma + \bar{\sigma})V_{ij} .
\] (4.9)

Making use of (4.8) and the super-Weyl transformation of the chiral density \[42]\,

\[
\delta_{\sigma}\mathcal{E} = -2\sigma\mathcal{E} ,
\] (4.10)
we obtain the super-Weyl transformation of \( M \):

\[
\delta_{\sigma}M = \sigma M .
\] (4.11)

Since the chiral scalars \( W \) and \( M \) have the same super-Weyl transformation law, the
duality rotation (4.6) is well defined.

We are interested in developing an alternative formulation for the theory described by
\( S[W, \bar{W}] \). For this we consider an auxiliary action of the form

\[
S[W, \bar{W}, \eta, \bar{\eta}] = \frac{1}{2} \int d^4x d^4\theta \mathcal{E}\left\{ \eta W - \frac{1}{2} \eta^2 - \frac{1}{4} W^2 \right\} + \text{c.c.} + \mathcal{G}_{\text{int}}[\eta, \bar{\eta}] ,
\] (4.12)
in which the scalar superfield \( \eta \) is only constrained to be covariantly chiral, \( \bar{D}_{\beta} \eta = 0 \).
Here \( \mathcal{G}_{\text{int}}[\eta, \bar{\eta}] \) contains terms of third and higher orders in powers of \( \eta \) and its conju-
gate. We require \( S[W, \bar{W}, \eta, \bar{\eta}] \) to be super-Weyl invariant, and therefore the super-Weyl
transformation of \( \eta \) is

\[
\delta_{\sigma}\eta = \sigma \eta .
\] (4.13)

For \( \mathcal{G}_{\text{int}}[\eta, \bar{\eta}] \) to be super-Weyl invariant,

\[
\delta_{\sigma}\mathcal{G}_{\text{int}}[\eta, \bar{\eta}] = 0 ,
\] (4.14)
it may explicitly depend on the supergravity compensators. Consider the equation of
motion for \( \eta \):

\[
W = \eta - 2 \frac{\partial}{\partial \eta} \mathcal{G}_{\text{int}}[\eta, \bar{\eta}] .
\] (4.15)

In perturbation theory, this equation may be used to express \( \eta \) as a functional of the field
strength \( W \) and its conjugate, \( \eta = \eta[W, \bar{W}] \). As a result, we end up with the action

\[
S[W, \bar{W}] = S[W, \bar{W}, \eta, \bar{\eta}] \bigg|_{\eta = \eta[W, \bar{W}]} = \frac{1}{8} \int d^4x d^4\theta \mathcal{E} W^2 + \text{c.c.} + S_{\text{int}}[W, \bar{W}] ,
\] (4.16)
which describes the dynamics of the vector multiplet.
Now, we have two expressions for $M$ derived from the actions (4.12) and (4.16):

$$iM = -W + 2\eta \quad (4.17a)$$

$$= W + 4 \frac{\delta}{\delta W} S_{\text{int}}[W, \bar{W}] . \quad (4.17b)$$

An important corollary of eqs. (4.17a) and (4.17b) is

$$\eta = W + 2 \frac{\delta}{\delta W} S_{\text{int}}[W, \bar{W}] . \quad (4.18)$$

This relation may be used to determine $W$ as a functional of $\eta$ and its conjugate, $W = W[\eta, \bar{\eta}]$, and then reconstruct the self-interaction $S_{\text{int}}[\eta, \bar{\eta}]$ starting from the action (4.16).

Eq. (4.17a) tells us that $\eta = \frac{1}{2}(W + iM)$, and hence the duality rotation (4.6) acts on $\eta$ by the rule

$$\delta \eta = -i\lambda \eta . \quad (4.19)$$

Making use of the relation (4.17a) and the equation of motion for $\eta$, (4.15), one may see that the self-duality equation (4.7) is equivalent to

$$\text{Im} \int d^4x d^4\theta E \eta \frac{\delta}{\delta \eta} S_{\text{int}}[\eta, \bar{\eta}] = 0 . \quad (4.20)$$

This means that the self-interaction $S_{\text{int}}[\eta, \bar{\eta}]$ is invariant under the rigid U(1) transformations (4.19).

Given an arbitrary real functional $S_{\text{int}}[\eta, \bar{\eta}] = \mathcal{O}(|\eta|^3)$ such that

$$S_{\text{int}}[e^{i\varphi} \eta, e^{-i\varphi} \bar{\eta}] = S_{\text{int}}[\eta, \bar{\eta}] , \quad \varphi \in \mathbb{R} , \quad (4.21)$$

eq (4.12) defines a U(1) duality-invariant theory. This is the fundamental significance of the representation (4.12).

5 Concluding comments

From the point of view of perturbative quantum theory, the important features of the new representation $S[W, \bar{W}] \to S[W, \bar{W}, \eta, \bar{\eta}]$ are that (i) it can be carried out under a path integral; (ii) it makes the action at most quadratic in the physical vector multiplet and shifts all self-interaction to the sector of auxiliary chiral variables $\eta$ and $\bar{\eta}$. Within the background-field method applied to $S[W, \bar{W}, \eta, \bar{\eta}]$, we can integrate out the physical vector
multiplet without losing duality invariance (ignoring the issue that models for nonlinear electrodynamics are non-renormalizable). For a recent discussion of duality symmetry in perturbative quantum theory see [47].

The novel representation for duality-invariant $\mathcal{N} = 2$ supersymmetric theories presented in section 4 may be useful for the construction of an $\mathcal{N} = 2$ supersymmetric Born-Infeld action. The perturbative scheme to derive such an action was formulated in [14]. The $\mathcal{N} = 2$ Born-Infeld action should be (i) self-dual; (ii) reduce to the $\mathcal{N} = 1$ supersymmetric Born-Infeld action [17] under $\mathcal{N} = 2 \to \mathcal{N} = 1$ reduction; and (iii) possess a rigid shift symmetry of the form

$$W \to c + \mathcal{O}(|W|) , \quad c \in \mathbb{C} ,$$

where the field-dependent part should be consistent with (the rigid version of) the Bianchi identity (4.2). If we work within the representation (4.12), the condition (i) is easy to implement. However, the condition (iii) remains highly non-trivial. In this formulation, the shift symmetry (5.1) turns into

$$W \to c + \mathcal{O}(|\eta|) , \quad \eta \to c + \mathcal{O}(|\eta|) .$$

It would be really interesting to revisit the problem of constructing the $\mathcal{N} = 2$ Born-Infeld action by using the representation (4.12).

In conclusion, we wish to emphasize that the approaches developed [14] and [21] apply to arbitrary rigid and locally supersymmetric duality invariant theories. This is in stark contradistinction with the bosonic approaches of [5] which literally apply only to the theories without higher derivatives, i.e. when the Lagrangian does not involve derivatives of the field strengths, $L = L(F_{ab})$. An extension of the formalism of [5] to the case of $\mathcal{N} = 0$ duality invariant theories with higher derivatives trivially follows from the $\mathcal{N} = 1$ supersymmetric approach of [16]. To derive such an extension, one may start from the rigid version of the $\mathcal{N} = 1$ supersymmetric self-duality equation (2.7) and switch off all the fermionic and auxiliary fields. This will lead to the following self-duality equation for an electromagnetic field theory with action $S[F]$

$$\int d^4x \left\{ G^{ab} \tilde{G}_{ab} + F^{ab} \tilde{F}_{ab} \right\} = 0 ,$$

where we have defined [14]

$$\tilde{G}_{ab}[F] = 2 \frac{\delta S[F]}{\delta F_{ab}} .$$
In these relations $F_{ab}$ has to chosen to be an arbitrary bivector, $F_{ab} = - F_{ba}$, which is not subject to the Bianchi identity. In the case of theories without higher derivatives, $S[F] = \int d^4 x \ L(F)$, the self-duality equation (5.3) is equivalent to (1.2). Ref. [14] clearly sketched the steps leading to the equation (5.3) (at the end of section 2), although this equation was not given explicitly. It appears that it had escaped the attention of the authors of the recent publication [15] that the procedure which leads to eq. (5.3) was already outlined in [14].

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A The Wess-Zumino superspace geometry

The superspace geometry is described by covariant derivatives of the form

$$D_A = (D_a, D_\alpha, \bar{D}^{\dot{\alpha}}) = E_A + \Omega_A .$$  

(A.1)

Here $E_A$ denotes the inverse vielbein, $E_A = E_A^M \partial_M$, and $\Omega_A$ the Lorentz connection, $\Omega_A = \Omega_A^{\beta\gamma} M_{\beta\gamma} + \Omega_A^{\dot{\beta}\dot{\gamma}} \bar{M}_{\dot{\beta}\dot{\gamma}}$, with $M_{\beta\gamma}$ and $\bar{M}_{\dot{\beta}\dot{\gamma}}$ the Lorentz generators. The covariant derivatives obey the following anti-commutation relations:

$$\{D_a, \bar{D}_a\} = -2i D_{a\dot{a}} ,$$  

(A.2a)

$$\{D_\alpha, D_\beta\} = -4 R M_{\alpha\beta} , \quad \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 4 R \bar{M}_{\dot{\alpha}\dot{\beta}} ,$$  

(A.2b)

$$[\bar{D}_{\dot{a}}, D_{\dot{b}}] = -i \varepsilon_{\dot{a}\dot{b}} \left( R D_{\dot{b}} + G_{\dot{b}}^\gamma \bar{D}_\gamma - (\bar{D}^\gamma G_{\dot{b}}^\gamma) \bar{M}_\gamma \delta + 2 W_{\dot{b}}^\gamma \gamma \delta \right) + i (D_{\beta} R) \bar{M}_{\dot{\alpha}\dot{\beta}} ,$$  

(A.2c)

$$[D_a, D_{\beta\dot{\beta}}] = i \varepsilon_{\alpha\beta} \left( R \bar{D}_{\dot{\beta}} + G_{\dot{\beta}}^\gamma D_\gamma - (D^\gamma G_{\dot{\beta}}^\gamma) M_{\gamma\delta} + 2 W_{\dot{\beta}}^\gamma \gamma \delta \bar{M}_\delta \right) + i (\bar{D}_{\dot{\beta}} R) M_{\alpha\beta} .$$  

(A.2d)

Here the torsion tensors $R, G_a = \bar{G}_a$ and $W_{\alpha\beta\gamma} = W_{(\alpha\beta\gamma)}$ satisfy certain Bianchi identities [33, 34]. In particular, $R$ and $W_{\alpha\beta\gamma}$ are covariantly chiral.

The chiral integration rule in $\mathcal{N} = 1$ supergravity is

$$\int d^4 x \ d^2 \theta d^2 \bar{\theta} \ E \ U = - \frac{1}{4} \int d^4 x \ d^2 \theta \ \mathcal{E} \ (D^2 - 4 R) U , \quad E^{-1} = \text{Ber}(E_A^M) ,$$  

(A.3)

with $\mathcal{E}$ the chiral density.
B \ N = 2 \ conformal\ supergravity

In this appendix we give a summary of the superspace formulation for \( \mathcal{N} = 2 \) conformal supergravity developed in \cite{42}.

The structure group is chosen to be \( \text{SL}(2, \mathbb{C}) \times \text{SU}(2) \), and the covariant derivatives \( \mathcal{D}_A = (\mathcal{D}_a, \bar{\mathcal{D}}^i, \bar{\mathcal{D}}^i_\alpha) \) read

\[
\mathcal{D}_A = E_A + \Phi_A^{kl} J_{kl} + \Omega_A^{\beta\gamma} M_{\beta\gamma} + \bar{\Omega}_A^{\dot{\beta}\dot{\gamma}} \bar{M}_{\dot{\beta}\dot{\gamma}}.
\]

(B.1)

Here \( J_{kl} \) are the generators of the group SU(2), and \( \Phi_A^{kl} \) the corresponding connection.

The spinor covariant derivatives obey the anti-commutation relations \cite{42}

\[
\{ \mathcal{D}^i_\alpha, \mathcal{D}^j_\beta \} = 4\delta^{ij} M_{\alpha\beta} + 2\varepsilon^{ij\gamma\delta} Y_{\gamma\delta} M_{\alpha\beta} + 2\varepsilon^{ij\gamma\delta} \bar{W}^{i\bar{\gamma}\bar{\delta}} \bar{M}_{\bar{i}\bar{\gamma}\bar{\delta}}
+ 2\varepsilon_{\alpha\beta}^{\gamma\delta} \bar{g}^{kl} J_{kl} + 4Y_{\gamma\delta} J^{ij},
\]

(B.2a)

\[
\{ \bar{\mathcal{D}}^i_\dot{\alpha}, \bar{\mathcal{D}}^j_\dot{\beta} \} = -4\bar{S}_{ij} \bar{M}_{\dot{\alpha}\dot{\beta}} - 2\varepsilon_{ij} \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{Y}_{\dot{\alpha}\dot{\beta}} \bar{M}_{\dot{i}\dot{\gamma}\dot{\delta}} - 2\varepsilon_{ij} \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{W}_{\dot{i}\dot{\gamma}\dot{\delta}} M_{\dot{i}\dot{\gamma}\dot{\delta}}
- 2\varepsilon_{ij} \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{S}^{kl} J_{kl} - 4\bar{Y}_{\dot{\alpha}\dot{\beta}} J^{ij},
\]

(B.2b)

\[
\{ \mathcal{D}^i_\alpha, \bar{\mathcal{D}}^j_\dot{\beta} \} = -2\delta^{ij}(\sigma^c)_\alpha^\dot{\beta} D_c + 4\delta^{ij} G^{\dot{\alpha}\dot{\beta}} M_{\alpha\dot{\beta}} + 4\delta^{ij} G_{\alpha\dot{\gamma}} \bar{M}_{\dot{i}\dot{\gamma}\dot{\beta}} + 8G_{\alpha\dot{\beta}} J^{ij}.
\]

(B.2c)

Here the real four-vector \( G_{\alpha\dot{\beta}} \), the complex symmetric tensors \( S^{ij} = S^{ji}, \ W_{\alpha\beta} = W_{\beta\alpha}, \ Y_{\alpha\beta} = Y_{\beta\alpha} \) and their complex conjugates \( \bar{S}_{ij} = \bar{S}_{ji}, \ \bar{W}_{\dot{\alpha}\dot{\beta}} := \bar{W}_{\dot{\beta}\dot{\alpha}}, \ \bar{Y}_{\dot{\alpha}\dot{\beta}} := \bar{Y}_{\dot{\beta}\dot{\alpha}} \) obey additional differential constraints implied by the Bianchi identities \cite{18, 42}.

An infinitesimal super-Weyl transformation of the covariant derivatives \cite{42} is

\[
\delta_\sigma \mathcal{D}^i_\alpha = \frac{1}{2} \bar{\sigma} \mathcal{D}^i_\alpha + (\mathcal{D}^i_\alpha) \mathcal{D}^j_\alpha - (\mathcal{D}^i_\alpha) \mathcal{D}^j_\alpha J^{ki},
\]

(B.3a)

\[
\delta_\sigma \mathcal{D}^i_\dot{\alpha} = \frac{1}{2} \sigma \mathcal{D}^i_\dot{\alpha} + (\mathcal{D}^i_\dot{\alpha}) \mathcal{D}^j_\dot{\alpha} + (\mathcal{D}^i_\dot{\alpha}) \mathcal{D}^j_\dot{\alpha} J^{ki},
\]

(B.3b)

where the parameter \( \sigma \) is an arbitrary covariantly chiral superfield, \( \mathcal{D}^i_\dot{\alpha} \sigma = 0 \).

The covariantly chiral projection operator \cite{49} is

\[
\Delta = \frac{1}{96} \left( (\mathcal{D}^i + 16S^{ij}) \mathcal{D}^j - (\mathcal{D}^i + 16S^{ij}) \mathcal{D}^j \right)
= \frac{1}{96} \left( \mathcal{D}^{ij} (\mathcal{D}^i + 16S^{ij}) - \mathcal{D}^{ij} (\mathcal{D}^i + 16S^{ij}) \right),
\]

(B.4)

where \( \mathcal{D}^{ij} := \mathcal{D}^k (\mathcal{D}^{ij})^k \). The fundamental property of \( \Delta \) is that \( \Delta U \) is covariantly chiral, for any scalar and isoscalar superfield \( U \), that is \( \mathcal{D}^{i_\dot{\alpha}} \Delta U = 0 \). This operator relates an integral over the full superspace to that over its chiral subspace:

\[
\int d^4x d^4\theta d^4\bar{\theta} \ E U = \int d^4x d^4\theta \ E \bar{\Delta} U, \quad E^{-1} = \text{Ber}(E_A^M),
\]

(B.5)

with \( \mathcal{E} \) the chiral density. A derivation of (B.5) is given \cite{50}.
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