ON THE $q$-ALGEBRA $su_q(2)$ AND ITS CONNECTION WITH THE QUANTUM THEORY OF ANGULAR MOMENTUM: A SURVEY

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Abstract. In the present paper we review the $q$-analog of the Quantum Theory of Angular Momenta based on the $q$-algebra $su_q(2)$, with an special emphasis on the representation of the Clebsch-Gordan coefficients in terms of $q$-hypergeometric series. This representation allows us to obtain several known properties of the Clebsch-Gordan coefficients in an unified and simple way.

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1. Introduction

It is well known the important role that the representation theory of groups and algebras plays in Physics (see e.g. [25, 26]), and in particular in the Quantum Theory of Angular Momenta (QTAM) [29]. In fact a deep knowledge of the group theory (and, in particular, of the representation theory) allows us to understand a lot of phenomena of physical systems as it is shown, for example, in the already classical monographs [7, 8].

So it is understandable that after the appearance of the so-called quantum groups and $q$-algebras at the end of the XX century, there were an increasing interest in understanding if they can have any role in solving some physical problems. In fact, after the publication of the first $q$-analogues of physical systems (like the harmonic oscillator [6, 13]), there were a lot of research papers exploring the connection between those new models with the $q$-algebras. One of the most interesting study was the one devoted to the construction of a suitable $q$-analogue of the Quantum Theory of Angular Momenta (for a review on the QTAM we refer to the reader to the classical books [7, 29] and references therein).

Among all possible constructions of QTAM there is one based on the projection operators (PO), those defined by von Neumann in his seminal book [31, page 50]. This approach has been used by Lowdin for the first time in 1964, further developed by Shapiro in 1965 for $su(2)$, and by Smirnov et al. starting from 1968 for $su(3)$, etc. (for a full history of this method see [28]). In fact, the construction of the $q$-analog of the Quantum Theory of Angular Momenta by using the projection operators method was developed by Smirnov, Tolstoy, and Kharitonov in a series of papers started with [22, 23]. Since these papers are hard to find, we will describe here the method used in [22, 23] for the $su_q(2)$ algebra and, in a future contribution, also the case of the $su_q(1, 1)$. For a more recent review on the PO method see [27] and for the representation theory of $su_q(2)$ and $su_q(1, 1)$ $q$-algebras see [10].

We will also go further by exploring the connection of the Clebsch-Gordan coefficients (CGC) with certain special functions. The connection between $q$-algebras and $q$-special functions is well documented in the literature (see e.g. [11, 12, 30]). Here we will focus our attention in the connection with the $q$-hypergeometric function $3F_2$ introduced by Nikiforov and Uvarov (see e.g. [14, page 138]), which is a symmetric version of

Date: June 1, 2022.
the well known basic hypergeometric series \(3\varphi_2\) (see e.g. [9]). In fact, using some properties of the basic series (that can be easily translated to the \(q\)-hypergeometric function \(3F_2\)) it is possible to obtain in very easy way the properties of the corresponding CGC. As examples of this we will derive the symmetry properties of the CGCs, properties that require an elaborate proof by other methods (see [23]). Finally, we will establish the connection of the Clebsch-Gordan coefficients with a certain \(q\)-analog of the Hahn polynomials by using the obtained relation with the \(q\)-hypergeometric function \(3F_2\).

The structure of the paper is as follows. In Section 2 we will include the notation as well as the needed preliminary results that are relevant for our purposes. In Section 3 we will introduce the \(q\)-algebra \(SU_q(2)\) and we discuss some of its properties, including the construction of the projection operators that will allow us to obtain an explicit formula for computing the Clebsch-Gordan coefficients of this algebra. Finally, in Section 4, we will obtain the explicit expressions for the CGC of the \(SU_q(2)\) \(q\)-algebra as well as several of their properties, including the symmetry properties and some recurrence relations.

2. Some preliminary results

Let be \(q \in \mathbb{R} \setminus \{\pm 1\}\) and \(x \in \mathbb{R}\). The symmetric quantum number \([x]_q\) is given by

\[
[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.
\]

It is clear from the above equation that \([x]_q \to x\) when \(q \to 1\).

Following [14] we will define for all \(a \in \mathbb{R}\), the symmetric \(q\)-Pochhammer by

\[
(a|q)_0 := 1, \quad (a|q)_n = \prod_{m=0}^{n-1} [a + m]_q, \quad n = 1, 2, 3, \ldots
\]

The special case when \(a = 1\) leads to the symmetric \(q\)-factorial

\[
(1|q)_n = [1|q][2|q] \cdots [n - 1|q][n]_q =: [n]_q!.
\]

The symmetric \(q\)-factorial symbol \([n]_q!\) can be expressed through the symmetric \(q\)-Gamma function \(\tilde{\Gamma}(s)\) defined in [14, Eq. (3.2.24) page 67] by the formula \([n]_q! = \tilde{\Gamma}(n + 1)\), where

\[
\tilde{\Gamma}(s) = q^{-\frac{(s - 1)(s - 2)}{4}} \Gamma_q(s),
\]

and the classical \(q\)-Gamma function \(\Gamma_q(s)\) is given by [14, Eq. (3.6.3) page 79]

\[
\Gamma_q(s) = \begin{cases} 
(1 - q)^{1-s} \prod_{k \geq 0} (1-q^k)^{\frac{(s-1)(s-2)}{2}} \Gamma_q(1/q), & 0 < |q| < 1, \\
q^{\frac{(s-1)(s-2)}{2}} \Gamma_q(1/q), & |q| > 1.
\end{cases}
\]

Notice that

\[
\lim_{q \to 1}[n]_q! = n!, \quad \lim_{q \to 1} (a|q)_k = (a)_k,
\]

where \((a)_n\) is the Pochhammer symbol defined by

\[
(a)_0 := 1, \quad (a)_n = a(a + 1) \cdots (a + n - 1), \quad n \in \mathbb{N}.
\]

Moreover, if \(a \in \mathbb{N}\) then

\[
(a|q)_n = \frac{[a + n - 1]!}{[a - 1]!}, \quad (-a|q)_n = \begin{cases} 
(-1)^n [a]! & \text{if } a \geq n, \\
[a - n]! & \text{if } a < n.
\end{cases}
\]

We will also use the \(q\)-symmetric analog of the binomial coefficients

\[
\begin{bmatrix} n \\ k \end{bmatrix}_q := [n]_q! [k]_q! [n - k]_q!.
\]

We will use the symmetric \(q\)-hypergeometric function introduced in [14, page 138]

\[
p + 1 F_p \left( \begin{array}{c} a_1, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{array} \mid q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1|q)_k (a_2|q)_k \cdots (a_{p+1}|q)_k}{(b_1|q)_k (b_2|q)_k \cdots (b_p|q)_k} \frac{z^k}{(1|q)_k},
\]
which is one of the $q$-analogues of the the generalized hypergeometric function (see e.g. [4])

$$
 p+1 F_p \left( \begin{array}{l}
 a_1, a_2, \ldots, a_{p+1} \\
 b_1, b_2, \ldots, b_q
 \end{array} \right| z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_{p+1})_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{z^k}{k!}.
$$

For convenience in writing, we will also use the notation

$$
 p+1 \varphi_p \left( \begin{array}{l}
 a_1, a_2, \ldots, a_{p+1} \\
 b_1, b_2, \ldots, b_q
 \end{array} \right| q, z \right) := p+1 F_p \left( \begin{array}{l}
 a_1, a_2, \ldots, a_{p+1} \\
 b_1, b_2, \ldots, b_q
 \end{array} \right| q; z
$$

In our work we will restrict ourselves to the case when one of the $a_i$, $i = 1, 2, \ldots, p+1$, is a negative integer, so the series (8) is always terminating. Also notice that when one of the $a_i = 0$, the series (8) is equal to 1.

Before continue, it is convenient to point out that there is another $q$-analogue of the hypergeometric function (9), the so-called basic (hypergeometric) series defined by

$$
 p+1 \varphi_p \left( \begin{array}{l}
 a_1, a_2, \ldots, a_{p+1} \\
 b_1, b_2, \ldots, b_q
 \end{array} \right| q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_{p+1})_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{z^k}{(q)_k},
$$

where

$$
 (a; q)_k = \prod_{n=0}^{k-1} (1 - a q^n).
$$

There are two mainly reasons for using the $q$-hypergeometric series (8) instead the basic series (10). The first reason is that the former is invariant with respect to the change $q \to 1/q$ (that is why it is called symmetric), and the second one is that we have the following straightforward limit

$$
 \lim_{q \to 1} p+1 F_p \left( \begin{array}{l}
 a_1, \ldots, a_{p+1} \\
 b_1, \ldots, b_q
 \end{array} \right| q, z \right) = p+1 F_p \left( \begin{array}{l}
 a_1, \ldots, a_{p+1} \\
 b_1, \ldots, b_q
 \end{array} \right| z
$$

Since the $q$-hypergeometric series (8) is related with the basic series (10) by the expression [14, page 139]

$$
 p+1 \varphi_p \left( \begin{array}{l}
 a^{a_1}, a^{a_2}, \ldots, a^{a_{p+1}} \\
 q^{b_1}, q^{b_2}, \ldots, q^{b_q}
 \end{array} \right| q, z \right) = p+1 F_p \left( \begin{array}{l}
 a_1, \ldots, a_{p+1} \\
 b_1, \ldots, b_q
 \end{array} \right| q^{1/2}, z q^{(a_1 + \cdots + a_{p+1} - b_1 - \cdots - b_q)/2},
$$

one can obtain several transformation and summation formulas for the $q$-hypergeometric series from the already-known ones for the basic series. Notice that the symmetric $q$-hypergeometric function $p+1 F_p$ in the right hand side of the previous identity is defined for $q^{1/2}$, so this detail should be taking into account when $p+1 \varphi_p$ is evaluated on $z$ depending on $q$, such as in the case of certain summation and transformation formulas that will be used in the present work.

We will start writing the following transformation formulas for the $q$-hypergeometric function $3 F_2$, which follows from Eqs. (III.11) and (III.12) of [9, page 330], respectively

$$
 3 F_2 \left( \begin{array}{l}
 -n, a, b \\
 d, e
 \end{array} \right| q, q^{-(a+b-n-d-e-1)} \right) = \frac{q^{\pm an} (e-a|q)_n}{(c|q)_n} 3 F_2 \left( \begin{array}{l}
 -n, a, d - b \\
 d, a - e - n + 1
 \end{array} \right| q, q^{\pm (b-e)}
$$

$$
 3 F_2 \left( \begin{array}{l}
 -n, a, b \\
 d, e
 \end{array} \right| q, q^{-(a+b-n-d-e-1)} \right) = \frac{(d-a|q)_n (e-a|q)_n}{(d|q)_n (e|q)_n} 3 F_2 \left( \begin{array}{l}
 -n, a, a + b - d - e - n + 1 \\
 a - d - n + 1, a - e - n + 1
 \end{array} \right| q, q^{\pm b}
$$

We will also need some summation formulas. Using the $q$-analogue of the Vandermonde formula [9, Eq. (1.5.3), page 14] we have, for $n \in \mathbb{N}$,

$$
 2 F_1 \left( \begin{array}{l}
 -n, b \\
 c
 \end{array} \right| q, q^{\pm (b-c-n+1)} \right) = \frac{(c-b|q)_n}{(c|q)_n} q^{\pm nb},
$$

which has some restrictions that should be taken into account, specially in our work when both $b$ and $c$ are integer numbers. First of all, in the $2 F_1$ function (15) it is assumed that, if $b$ is a negative integer, then $n < |b|$. Moreover, if $c$ is also a negative integer, then $n < \min(|b|, |c|)$, otherwise there will be a zero factor in the denominator of some terms of the series.
In the following we will consider $n, b, c \in \mathbb{Z}^+$, such that $n < \min(b, c)$. From (15) and (6), the following useful summation formula follows

$$2F_1 \left( \begin{array}{c} -n, b \\ c \end{array} \middle| q, q^{+(b-c-n+1)} \right) = \frac{[c-b+1+n]_q! [c-1]_q!}{[c-b]_q! [c-1+n]_q!} q^{b n}, \quad c > b,$$

or, equivalently,

$$\sum_{r=0}^{n} \frac{(-1)^r [b-1+r]_q!}{[c-1+r]_q! [n-r]_q!} q^{r(b-c-n+1)} = \frac{[c-b+1+n]_q! [b-1]_q!}{[n]_q! [c-b-1]_q! [c-1+n]_q!} q^{b n}, \quad c > b.$$

If we make the change $c \rightarrow -c$ and $b \rightarrow -b$ in (15) and use (6), we find the very useful formula

$$2F_1 \left( \begin{array}{c} -n, -b \\ -c \end{array} \middle| q, q^{+(b-c-n-1)} \right) = (-1)^n \frac{[c-n]_q! [b+n-c-1]_q!}{[c]_q! [b-c-1]_q!} q^{b n}, \quad b > c,$$

or, equivalently,

$$\sum_{r=0}^{n} \frac{(-1)^r [c-r]_q!}{[b-r]_q! [n-r]_q!} q^{r(b-c+n-1)} = (-1)^n \frac{[c-n]_q! [b+n-c-1]_q!}{[b]_q! [n]_q! [b-c-1]_q!} q^{b n}, \quad b > c,$$

where we recall that $b$ and $c$ are positive integers.

Another important summation formula is the Jackson's terminating $q$-analogue of Dixon's sum [9, Eq. II.15, page 355], that in terms of the symmetric $q$-hypergeometric function reads

$$3F_2 \left( \begin{array}{c} -2n, b, c \\ 1-2n-b, 1-2n-c \end{array} \middle| q, q \right) = \frac{q^n [2n]_q! [b+c+n]_q!}{[n]_q! [b+n]_q! [c+n]_q!}, \quad n = 0, 1, 2, \ldots$$

The following lemma, which can be proved by induction, will be useful in the next section.

**Lemma 1.** Let us consider three operators $A_+, A_-$, and $B$ such that $[B, A_\pm] = \pm A_\pm$, then

$$B^\ell A_\pm = A_\pm (B \pm I)^\ell,$$

and

$$[\nu B + \eta]_q A_\pm^\ell = A_\pm^\ell [\nu B + \eta \pm \nu r]_q, \quad \nu, \eta \in \mathbb{R},$$

for any $\ell, r \in \mathbb{Z}^+$. Moreover,

$$[B, A_\pm^\ell] = \pm r A_\pm^\ell,$$

and

$$[A_\pm, A_\mp^\ell] = \mp A_\pm^\ell [r]_q [2B \mp (r - 1)]_q$$

if $[A_\pm, A_\mp] = \pm (2B)_q$, \hspace{1cm} (24)

$$[A_\pm, A_\mp^\ell] = \pm A_\mp^\ell [r]_q [2B \mp (r - 1)]_q$$

if $[A_\pm, A_\mp] = \mp (2B)_q$. \hspace{1cm} (25)

3. The $su_q(2)$ Algebra

In this section we will introduce some basic facts about the $su_q(2)$ algebra. The main related references are [11, 22, 23].

### 3.1. Basic facts on the unitary representations of the $su_q(2)$ Algebra

The $su_q(2)$ algebra is generated by the operators $J_+, J_-$, and $J_0$, which fulfill the relations

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = [2J_0]_q,$$

and the adjointness properties

$$J_\pm^\dagger = J_\mp, \quad J_0^\dagger = J_0.$$

A basis of any irreducible and unitary representation of finite dimension $D^j$ is given by

$$|jm\rangle, \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, m = -j, -j+1, \ldots, j-1, j.$$

The action of the generators on the basis is given by

$$J_0 |jm\rangle = m |jm\rangle,$$

$$J_- |jm\rangle = \sqrt{|j+m\rangle [j-m+1]_q} |jm+1\rangle,$$

$$J_+ |jm\rangle = \sqrt{|j-m\rangle [j+m+1]_q} |jm-1\rangle,$$
\[ J_+ |jm\rangle = \sqrt{|j-m|_q |j+m+1|_q} |jm+1\rangle. \]

Hence the explicit form of matrix elements of the irreducible representation \( D^j \) is determined by the equations
\[ \langle jm'|J_0 |jm\rangle = m\delta_{m'm}, \]
\[ \langle jm'|J_- |jm\rangle = \sqrt{|j+m|_q |j-m+1|_q} \delta_{m'm-1}, \]
\[ \langle jm'|J_+ |jm\rangle = \sqrt{|j-m|_q |j+m+1|_q} \delta_{m'm+1}. \]

By induction it is easy to see that
\[ (27) \quad J_-^r |jm\rangle = \sqrt{\frac{|j+m|_q!}{|j-m|_q!} \frac{|j-m+r|_q!}{|j+m-r|_q!}} |j m-r\rangle \quad \text{and} \quad J_+^r |jm\rangle = \sqrt{\frac{|j-m|_q!}{|j+m|_q!} \frac{|j+m+r|_q!}{|j-m-r|_q!}} |j m+r\rangle. \]

In particular, the matrix elements for powers of the generators are
\[ \langle jm'|J_-^r |jm\rangle = \sqrt{\frac{|j+m|_q!}{|j-m|_q!} \frac{|j-m+r|_q!}{|j+m-r|_q!}} \delta_{m'm-r} \quad \text{and} \quad \langle jm'|J_+^r |jm\rangle = \sqrt{\frac{|j-m|_q!}{|j+m|_q!} \frac{|j+m+r|_q!}{|j-m-r|_q!}} \delta_{m'm+r}. \]

Using the previous identities it is clear that
\[ J_-^{-m} |jj\rangle = \sqrt{\frac{|j-m|_q!}{|j+m|_q!}} |jm\rangle \quad \text{and} \quad J_+^{+m} |j-j\rangle = \sqrt{\frac{|j+m|_q!}{|j-m|_q!}} |jm\rangle, \]

so
\[ |jm\rangle = \sqrt{\frac{|j-m|_q!}{|j+m|_q!}} J_+^{+m} |j-j\rangle \quad \text{or} \quad |jm\rangle = \sqrt{\frac{|j+m|_q!}{|j-m|_q!}} J_-^{-m} |jj\rangle. \]

Next we define the Casimir operator of second order for \( su_q(2) \) algebra by
\[ C = J_- J_+ + |J_0 + 1/2|_q^2 = J_+ J_- + |J_0 - 1/2|_q^2, \]
which is self-adjoint, \( C^\dagger = C \), due to \[ \langle j'm'|C |jm\rangle = \langle jm|C |j'm\rangle \].

A fundamental property of the Casimir operator is its commutativity with the operators \( J_0 \) and \( J_\pm \). A straightforward calculations show that
\[ [C, J_0] = 0 \quad \text{and} \quad [C, J_\pm] = 0. \]

Now, taking into account that
\[ C |jm\rangle = |j + 1/2|_q^2 |jm\rangle, \]
i.e., all vectors of the family \( |jm\rangle \) are eigenvectors of the Casimir operator with the common eigenvalue \( |j + 1/2|_q^2 \) (which is the eigenvalue of the maximum weight vector \( |jj\rangle \)), Eq. (26) as well as that \( [C, J_0] = 0 \), then operators \( C \) and \( J_0 \) share a common orthonormal system of eigenvectors which is the family \( |jm\rangle \) itself
\[ \langle j'm'|jm\rangle = \delta_{j'}\delta_{m'm}. \]

3.2. **Projection operator for \( su_q(2) \) algebra.** Let any vector \( |j'm\rangle \) of the basis of an unitary irreducible representation \( D^{j'} \) such that \( m' = j \geq 0 \). We define an application
\[ \begin{align*}
P_{jj}^{j'} : \quad D^{j'} & \quad \rightarrow \quad \text{span}\{|jj\rangle\} \\
|j'm\rangle = j \quad \mapsto \quad \delta_{j'}|jj\rangle.
\end{align*} \]

Since we have chosen \( m' = j \geq 0 \), there will exist an unitary irreducible representation \( D^j \subseteq D^{j'} \). The action is the following one: from all possible vectors \( |j'm' = j\rangle \) of \( D^{j'} \), the application \( P_{jj}^{j'} \) extracts the maximum weight vector \( |jj\rangle \) of the representation \( D^j \).

First, we note that the projection operator commutes with the generator \( J_0 \) when it is applied to a vector \( |j'j\rangle \), i.e., \[ P_{jj}^{j'} J_0 |j'j\rangle = 0, \] so if we want to write the projection operator as an expansion in terms of the generators \( J_- \) and \( J_+ \), they have to be raised to the same power, i.e.,
\[ P_{jj}^{j'} = \sum_{r=0}^{\infty} c_r J_-^r J_+^r. \]
Actually previous expansion is not a series but a finite sum because if we apply the projection operator to a vector $|j’j’⟩$, then there will be an index $r_0$ such that $J^{r_0}_+ |j’j’⟩ = 0$.

Let us now obtain an explicit expression for the coefficients $c_r$. It is clear by (28) that

$$P^j_+ |jj⟩ = \begin{cases} |jj⟩, \\ c_0 |jj⟩ + c_1 J_- J_+ |jj⟩ + \cdots = c_0 |jj⟩, \end{cases} \implies c_0 = 1.$$  

For the remainder coefficients we note that $J_+ P^j_+ |j’j⟩ = 0$ and, on the other hand, by the equations (24) and (22) of Lemma 1

$$0 = J_+ P^j_+ |j’j⟩ = \sum_{r=0}^{∞} c_r J_+ J^r_+ |j’j⟩ = \sum_{r=0}^{∞} c_r (J^r_+ J^r_+^{−1} + J^r_+^{−1} [2J_0 − r + 1] q J^r_+) |j’j⟩ = \sum_{r=0}^{∞} (c_r + c_{r+1} q [2J_0 + r] q J^r_+ J^r_+^{−1} |j’j⟩.$$  

Therefore, by (27) we get

$$0 = \sum_{r=0}^{∞} \left( c_r + c_{r+1} [r + 1] q J^r_+ J^r_+^{−1} |j’j⟩ \right) = \frac{[j’ − j] q!}{[j’ + j] q!} \sum_{r=0}^{∞} \left( c_r + c_{r+1} [r + 1] q J^r_+ J^r_+^{−1} |j’j⟩ \right) [j’ − j] q.$$

If we take $j’ = j + a$, with $a \in N$, we obtain on the right hand side

$$\frac{[a − 1] q!}{[2j + a] q!} \sum_{r=0}^{∞} \frac{[2j + r + 1] q!}{[a − 1] q!} (c_r + c_{r+1} [r + 1] q J^r_+ J^r_+^{−1} |j’j⟩) [j + a, j + 1].$$

We note that the factor

$$\frac{[a − 1] q!}{[a − 1] q!} = [1] q [2] q \cdots [a − r − 1] q [a − r] q$$

vanishes if $r ≥ a$, so actually the series is a sum from $r = 0$ to $r = a − 1$, i.e., previous identity is

$$\frac{[a − 1] q!}{[2j + a] q!} \sum_{r=0}^{a−1} \frac{[2j + r + 1] q!}{[a − r − 1] q!} (c_r + c_{r+1} [r + 1] q J^r_+ J^r_+^{−1} |j’j⟩) [j + a, j + 1] = 0,$$

which implies

$$\sum_{r=0}^{a−1} \frac{[2j + r + 1] q!}{[a − r − 1] q!} (c_r + c_{r+1} [r + 1] q J^r_+ J^r_+^{−1} |j’j⟩) = 0.$$

For $a = 1$ we have

$$[2j + 2] q! (c_0 + c_1 [1] q [2j + 2] q) = 0 \iff c_1 = \frac{−c_0}{[2j + 2] q} = \frac{1}{[2j + 2] q},$$

and, in general,

$$[2(j + n)] q! (c_{n−1} + c_n [n] q [2j + n + 1] q) = 0 \iff c_n = (−1)^n \frac{[2j + 1] q!}{[n] q! [2j + n + 1] q},$$

that can be checked by induction. Therefore, $P^j_+ = \sum_{r=0}^{∞} (−1)^r \frac{[2j + 1] q!}{[r] q! [2j + r + 1] q} J^r_+ J^r_+^{−1}.$

It is not complicate to proof that $⟨j’m’|P^j_+ |jm⟩ = ⟨jm|P^j_+ |j’m’⟩$, so $(P^j_+)^\dagger = P^j_-$, that is, the projection operator $P^j_-$ is self-adjoint.
Let us consider now any vector \(|j'm'\rangle\) of the basis of the unitary irreducible representation \(D^{j'}\) such that \(m' \leq j\), for \(j \geq 0\). We define the application

\[
P^{j}_{mm'} : D^{j'} \rightarrow \text{span}\{|jm\rangle\}
\]
such that

\[
P^{j}_{mm'} |j'm'\rangle = \sqrt{\frac{[j + m]_q!}{[2j]_q!|j - m\rangle_q!}} P_{jj'}^{-m} \sqrt{\frac{[j + m']_q!}{[2j]_q!|j - m'\rangle_q!}} |j'm'\rangle.
\]

We note that in the case \(m = m' = j\) we recover the aforementioned projection operator, so we can understand this operator as a generalized projection operator. Here we must indicate that it is possible to define the case related to \(m' > j\) by means of the already known identity

\[
|j'm\rangle = \sqrt{\frac{[j' - m]_q!}{[2j']_q!|j' + m\rangle_q!}} |j' - j\rangle,
\]

which allows us to define

\[
P^{j}_{mm'} |j'm'\rangle = \sqrt{\frac{[j + m]_q!}{[2j]_q!|j - m\rangle_q!}} P_{jj'}^{-m} \sqrt{\frac{[j + m']_q!}{[2j]_q!|j - m'\rangle_q!}} |j'm'\rangle
\]

\[
= \sqrt{\frac{[j + m]_q!}{[2j]_q!|j - m\rangle_q!}} P_{jj'}^{-m} \sqrt{\frac{[j + m']_q!}{[2j]_q!|j - m'\rangle_q!}} |j' - j\rangle
\]

\[
+ \sqrt{\frac{[j + m]_q!}{[2j]_q!|j - m\rangle_q!}} P_{jj'}^{-m} \sqrt{\frac{[j + m']_q!}{[2j]_q!|j - m'\rangle_q!}} |j' + j\rangle.
\]

The generalized projection operator \(P^{j}_{mm'}\) fulfills the property \(\langle P^{j}_{mm'} |j'm'\rangle = P^{j}_{mm'}\). Indeed, a direct calculation shows

\[
\langle jm|P^{j}_{mm'}|j'm\rangle = \langle j'm'|P^{j}_{mm'}|jm\rangle.
\]

Moreover, applying this generalized projection operator to the elements of the basis we have, after some cumbersome but straightforward calculations, that

\[
P^{j}_{mm'} |j'm\rangle = \delta_{jj'} |jm\rangle, \quad m' = -j', -j' + 1, \ldots, j' - 1, j',
\]

that is, \(P^{j}_{mm'} |j'm\rangle = \delta_{jj'} |jm\rangle\).

Thanks to the above identity we are able to prove a very interesting property of the generalized projection operator \(P^{j}_{mm'}\). Given any linear combination \(|m\rangle = \sum_{j} a_{j'm'} |j'm\rangle\) of vectors \(|j'm\rangle\), for a given \(m'\),

\[
m' = -j', -j' + 1, \ldots, j' - 1, j',
\]

then

\[
P^{j}_{mm'} |m\rangle = P^{j}_{mm'} \sum_{j'} a_{j'm'} |j'm\rangle = \sum_{j'} a_{j'm'} P^{j}_{mm'} |j'm\rangle = \sum_{j'} a_{j'm'} \delta_{jj'} |jm\rangle = a_{jm'} |jm\rangle.
\]

So, the generalized projection operator \(P^{j}_{mm'}\) applied on an arbitrary linear combination \(\sum_{j'} a_{j'm'} |j'm\rangle\) is proportional to \(|jm\rangle\). Finally, let us point out that

\[
P^{j}_{mm'} P^{j'}_{m'm''} = \delta_{jj'} P^{j}_{m'm''}.
\]

4. The Clebsch-Gordan coefficients

Let us consider the direct product \(D^{j_1} \otimes D^{j_2}\) of two representations \(D^{j_1}\) and \(D^{j_2}\). In general one have that the direct product of two representations can be expressed as a direct sum of the irreducible representations, i.e.,

\[
D^{j_1} \otimes D^{j_2} = \sum_{j} \oplus D^{j},
\]

where the sums runs in a certain set of discrete values related with \(j_1\) and \(j_2\) as we will see later on (we will not consider the case of the continuous spectra)
Let \(|j_1m_1\rangle\) and \(|j_2m_2\rangle\) the orthogonal basis vectors of \(D^{j_1}\) and \(D^{j_2}\), respectively, and let \(|j_1j_2; jm\rangle\) be basis vectors of \(D^{j}\). Then, a typical situation is when one expand the vectors \(|j_1j_2; jm\rangle\) in the basis \(|j_1m_1\rangle |j_2m_2\rangle\)

\[
|j_1, j_2 : jm\rangle = \sum_{m_1, m_2} C^{j_1m_1, j_2m_2}_{j_1m_1, j_2m_2} |j_1m_1\rangle |j_2m_2\rangle = \sum_{m_1, m_2} \langle j_1m_1, j_2m_2; jm | j_1m_1\rangle |j_2m_2\rangle.
\]

The coefficients \(C^{j_1m_1, j_2m_2}_{j_1m_1, j_2m_2}\) of the above expansion is called the Clebsch-Gordan coefficients and are usually denoted by \((j_1m_1, j_2m_2; jm)\). Our aim here is to compute them.

Notice that one also can write the vectors \(|j_1m_1\rangle |j_2m_2\rangle\) in the basis \(|j_1, j_2 : jm\rangle\)

\[
|j_1m_1\rangle |j_2m_2\rangle = \sum_{j', m'} \tilde{C}^{j_1m_1', j_2m_2'}_{j', m'} |j_1, j_2 : j'm'\rangle.
\]

Since the basis vectors \(|j_1m_1\rangle\) and \(|j_2m_2\rangle\) can be always assume to be orthonormal then, from (31), one find that

\[
\langle j_1m_1, j_2m_2; jm | j_1m_1\rangle |j_2m_2\rangle = \langle j_1m_1 | j_2m_2 | j_1, j_2 : jm\rangle.
\]

Notice that one also can write the vectors \(|j_1m'_1\rangle |j_2m'_2\rangle\) in the basis \(|j_1, j_2 : j'm'\rangle\)

\[
|j_1m'_1\rangle |j_2m'_2\rangle = \sum_{j', m'} \tilde{C}^{j_1m'_1, j_2m'_2}_{j', m'} |j_1, j_2 : j'm'\rangle.
\]

But we know that applying the generalized projection operator \(P^{j}_{m'm'}\) (for the \(q\)-algebras we are interested in) to a linear combination (in \(j'\)) of vectors \(|j_1, j_2 : j'm'\rangle\) a vector proportional to \(|j_1, j_2 : jm\rangle\) is obtained, thus

\[
P^{j}_{m'm'} |j_1m'_1\rangle |j_2m'_2\rangle = a_{j'm'} \langle j_1, j_2 : jm | j_1m'_1\rangle |j_2m'_2\rangle \iff \langle j_1, j_2 : jm | j_1m'_1\rangle |j_2m'_2\rangle = \frac{P^{j}_{m'm'} |j_1m'_1\rangle |j_2m'_2\rangle}{\|P^{j}_{m'm'} |j_1m'_1\rangle |j_2m'_2\rangle\|},
\]

where, since, in our case \((P^{j}_{m'm'})^\dagger = P^{j}_{m'm}\) and using (30) we find

\[
\|P^{j}_{m'm'} |j_1m'_1\rangle |j_2m'_2\rangle\|^2 = \left\langle \langle j_1m'_1\rangle |j_2m'_2\rangle \left( (P^{j}_{m'm'})^\dagger P^{j}_{m'm} |j_1m'_1\rangle |j_2m'_2\rangle \right) \right\rangle = \left\langle \langle j_1m'_1\rangle |j_2m'_2\rangle \|P^{j}_{m'm'} |j_1m'_1\rangle |j_2m'_2\rangle\|^2 \right\rangle,
\]

if we want \(|j_1, j_2 : jm\rangle\) to be normalized to 1.

From the above it follows that the Clebsch-Gordan coefficients can be find in terms of the projection operator by the expression

\[
\langle j_1m_1, j_2m_2; jm | j_1m_1\rangle |j_2m_2\rangle = \frac{\left\langle \langle j_1m_1 | j_2m_2 \| P^{j}_{m'm'} |j_1m'_1\rangle |j_2m'_2\rangle\right\rangle}{\|P^{j}_{m'm'} |j_1m'_1\rangle |j_2m'_2\rangle\|}.
\]

This is a matter of fact that both \(m'_1\) and \(m'_2\) are free parameters, so we can choose the most appropriate values in each case to make easier the computation of the Clebsch-Gordan coefficients.

Let us now apply all of this to the irreducible representations of the \(q\)-algebra \(su_q(2)\).

4.1. Clebsch-Gordan coefficients for the \(su_q(2)\). Before calculate the Clebsch-Gordan coefficients with the formula (32), we will discuss some preliminary results (for more details see e.g. [22]).

Let us consider two different irreducible and unitary representations of order \(j_1\) and \(j_2\), i.e,

\[
D^{j_i}, \quad |j_i m_i\rangle, \quad m_i = -j_i, -j_i + 1, \ldots, j_i - 1, j_i, \quad i = 1, 2,
\]

which has dimension \(2j_i + 1\) each one and whose generators are \(J_{\pm}(i)\) and \(J_0(i)\). The direct product representation \(D^{j_1} \otimes D^{j_2}\) has a basis \(|j_1m_1\rangle |j_2m_2\rangle\) of dimension \((2j_1 + 1)(2j_2 + 1)\) and it is generated by the operators

\[
J_0(12) = J_0 \otimes 1 + 1 \otimes J_0 \quad \text{and} \quad J_{\pm}(12) = J_{\pm} \otimes q^{J_0} + q^{-J_0} \otimes J_{\pm},
\]

which, for convenience, we will rewrite as

\[
J_0(12) = J_0(1) + J_0(2) \quad \text{and} \quad J_{\pm}(12) = J_{\pm}(1)q^{J_0(2)} + q^{-J_0(1)}J_{\pm}(2).
\]
Here the number $i = 1, 2$ indicate to which representation $D_j^i$ the operators $J_0(i), J_{\pm}(i)$, are acting. Notice that these operators satisfy $(J_0(12))^\dagger = J_0$, $(J_-(12))^\dagger = J_+(12)$,
\[ q a_{J_0(i)} J_0(i) = J_0(i) q a_{J_0(i)} \quad \text{and} \quad q a_{J_0(i)} J_\pm(i) = J_\pm(i) q a_{J_0(i) \pm 1} \]
and
\[ [J_0(12), J_\pm(12)] = \pm J_\pm(12) \quad \text{and} \quad [J_+(12), J_-(12)] = [2J_0(12)]_q, \]
so we can apply Lemma 1 with the operators $J_-(12)$, $J_+(12)$ and $J_0(12)$. This will be an important fact in order to construct the corresponding projection operator $P_{mn}^j(12)$.

We will also need to know the expression of the powers of $J_\pm(12)$ and $J_0(12)$. In fact, by induction it can be shown that for all $r \in \mathbb{N}$,
\[ J_0(12) \langle j_1 m_1 \rangle_0 \langle j_2 m_2 \rangle = (m_1 + m_2) \dagger \langle j_1 m_1 \rangle_0 \langle j_2 m_2 \rangle, \]
as well as
\[ J_\pm(12) \langle j_1 m_1 \rangle_0 \langle j_2 m_2 \rangle = \sum_{\ell=0}^r \frac{[\ell]_q^{\alpha}}{[\ell]_q^{\alpha} [\ell - \ell]_q^{\beta}} J_{\pm}^\ell(12) J_{\pm}^{r-\ell}(2) q^{\ell J_0(2) - (r-\ell) J_0(1)}. \]

Following the form of the Casimir operator for a single representation, we define the Casimir operator by
\[ C(12) = J_-(12) J_+(12) + [J_0(12) + 1/2]_q^2 = J_+(12) J_-(12) - [2J_0(12)]_q + [J_0(12) - 1/2]_q^2. \]

Now we are ready for computing the CGC. For doing that we will use (32) and we will choose $m'_1 = j_1$ and $m'_2 = j - j_1$. With this choice $m' = m'_1 + m'_2 = j$ and then
\[ \langle j_1 m_1, j_2 m_2 | j m \rangle = \left( \langle j_1 m_1, j_2 m_2 | P_{m_j}^j | j_1 j_1 \rangle \langle j_2 - j - j_1 \rangle \right), \]
where the generalized projection operator (29) is given by
\[ P_{m_j}^j = \sqrt{\frac{[j + m'_1]_q!}{[2j]_q! [j - m]_q!}} J_{\pm}^{-m}(12) P_{m_j}^j = \sqrt{\frac{[j + m'_1]_q!}{[2j]_q! [j - m]_q!}} \sum_{r=0}^{\infty} \frac{(-1)^r [2j + 1]_q!}{[r]_q! [2j + r + 1]_q!} J_{\pm}^{r + j-m}(12) J_{\pm}^r(12). \]

From (34) it is clear that $j_1 - j_2 \leq j \leq j_1 + j_2$. But if we choose $m'_2 = j_2$ and $m'_1 = j - j_2$ in (32), then we get $j_2 - j_1 \leq j \leq j_1 + j_2$ from where it follows that
\[ D_j^i \otimes D_j^j = \sum_{j=|j_1-j_2|}^{j_1+j_2} \otimes D_j^j. \]

We will start computing the numerator of (34). By the binomial expansion (33)
\[ J_{\pm}^r(12) \langle j_1 j_1 \rangle \langle j_2 - j - j_1 \rangle = \sum_{\ell=0}^r \frac{[\ell]_q^{\alpha}}{[\ell]_q^{\alpha} [r - \ell]_q^{\beta}} J_{\pm}^\ell(12) J_{\pm}^{r-\ell}(2) q^\ell J_0(2) - (r-\ell) J_0(1) \langle j_1 j_1 \rangle \langle j_2 - j - j_1 \rangle \]
\[ = \sum_{\ell=0}^r \frac{[\ell]_q^{\alpha} [j_2 - j - j_1 + r]_q^{\gamma}}{[\ell]_q^{\alpha} [r - \ell]_q^{\beta}} \left( J_{\pm}^\ell(12) \langle j_1 j_1 \rangle \langle j_2 - j - j_1 \rangle \right) \left( J_{\pm}^{r-\ell}(2) \langle j_2 - j - j_1 \rangle \right) \]
\[ = \sum_{\ell=0}^r \frac{[j_2 - j + j_1 + r]_q^{\gamma}}{[j_2 + j - j_1]_q^{\gamma} [j_2 - j + j_1 + r]_q^{\gamma}} q^{r-j_1} \langle j_1 j_1 \rangle \langle j_2 - j - j_1 + r \rangle. \]

The last equality follows from (27) and the fact that $\langle j_1 j_1 \rangle$ is the maximal weight vector so $J_{\pm}^0(1) \langle j_1 j_1 \rangle = 0$ for all $\ell > 0$. Notice also that if $r > j_1 + j_2 - j$, the above expression vanishes.

In a similar fashion we compute
\[ J_{\pm}^{r+j-m}(12) \langle j_1 j_1 \rangle \langle j_2 - j - j_1 + r \rangle = \sum_{\ell=0}^{r+j-m} \frac{[r + j - m]_q^{\alpha}}{[\ell]_q^{\alpha} [r + j - m - \ell]_q^{\beta}} \left( J_{\pm}^\ell(12) J_{\pm}^{r+j-m-\ell}(2) q^\ell J_0(2) - (r+j-m-\ell) J_0(1) \right) \times \langle j_1 j_1 \rangle \langle j_2 - j - j_1 + r \rangle \]
\[
\frac{r+j-m}{[\ell]_q ! [r+j-m-\ell]_q !} \sqrt{[2j_1]_q ! [j_2 + j - j_1 + r]_q ! [j_2 + j_1 - m - \ell]_q ! \times q^{(j+r)-(r+j-m)j_1} |j_1 j_1 - \ell| |j_2 m - j_1 + \ell| .
\]

In fact, in the above expression, all terms for \( \ell > 2j_1 \) and \( \ell < j_1 - j_2 - m \) vanish. Then, for the numerator on \((34)\) we get

\[
\sqrt{[2j]_q ! [j - m]_q ! \sum_{r=0}^{j_1+j_2-j} (-1)^j [2j_1 + 1]_q ! \min(r+j-m,2j_1) [r+j-m]_q ! \sum_{\ell=\max(0,j_2-j-m)}^{j_1} [\ell]_q ! [r+j-m-\ell]_q ! \sqrt{[2j_1]_q ! [j_1-j_2+r+1]_q ! [j_2+j_1-j_2-r]_q ! \times q^{-j_1(j_1+1)(j_2-j+j_1+1)/2} .}
\]

Let us now compute the denominator of \((34)\). It is clear that it is the same as the numerator but with parameters \(m_1 = j_1\), \(m_2 = j - j_1\), so \(m = j\), so it is equal to

\[
\sqrt{[2j+1]_q ! [j_2-j-j_1+1]_q ! q^{(j_2+j-j_1+1)(j_2-j+j_1+1)/2} .}
\]

Putting Eqs. (36) and (38) together we obtain the following expression for the Cebesch-Gordan coefficients for the \( su_q (2) \)

\[
\langle j_1 j_1, j_2 m_2 | j m \rangle = \sqrt{[2j+1]_q ! [j_1+j_2+1]_q ! \frac{[j+j_1-j_2]_q ! [j_1+j_2-j]_q ! [j+m]_q ! [j_2-m_2]_q !}{[j_2-j-j_1+1]_q ! [j_1+m] _q ! [j_1-m_1]_q ! [j_2+m_2]_q !} \times q^{m_1-m_2-j-(j_2+j-j_1+1)(j_2-j+j_1+1)/2} .
\]

If we make the change \( r \to j_1 + j_2 - j - r \), we recover the expression in \((22)\)

\[
\langle j_1 j_1, j_2 m_2 | j m \rangle = \sqrt{[2j+1]_q ! [j_1+j_2+1]_q ! \frac{[j+j_1-j_2]_q ! [j_1+j_2-j]_q ! [j+m]_q ! [j_2-m_2]_q !}{[j_2-j-j_1+1]_q ! [j_1+m] _q ! [j_1-m_1]_q ! [j_2+m_2]_q !} \times q^{j_1 m_2 - j_2 m_1 - \frac{1}{2}(j_1+j_2-j)(j_1+j_2+j+1)} \sum_{r=0}^{\min(j_2-m_2,j_1+j_2-j)} (-1)^{j_1+j_2-j+r} [j_1+j_2-m-r]_q ! [j_2-r]_q ! q^{(j_1+m_1) r} .
\]
4.2. The representation as a terminating $3F_2$ $q$-hypergeometric function and its consequences.

We are interested in writing the last sum in terms of the symmetric terminating $q$-hypergeometric function $3F_2(-n, a, b; d, e; q, z)$ given in (8), so we should fix the value of the nonnegative integer $n$. A direct calculation shows that we can fix $n = j_{1} + j_{2} - j$ or $n = j_{2} - m_{2}$ independently of which one is bigger. The main reason is that, independently of the choice we make, all the extra terms in the sum will vanish. Taking this into account, we can write the above expression (40) as a symmetric terminating $q$-hypergeometric function (notice that $j_{1} + j_{2} + j + 1$ is always bigger than max$(j_{2} - m_{2}, j_{1} + j_{2} - j)$)

$$
\langle j_{1}m_{1}, j_{2}m_{2}|jm\rangle_{q} = \frac{(-1)^{j_{1}+j_{2}+j}q^{1+m_{1}m_{2}j_{1}m_{1}j_{2}m_{2}-\frac{1}{2}(j_{1}+j_{2})+1+j_{1}+j_{2}+j+1}}{\sqrt{[j_{1}+m_{1}]}_{q}[j_{2}+m_{2}]}_{q}[j_{2}+j_{1}]_{q}[j_{2}+m_{2}]_{q}[j_{2}+m_{2}^{-}\frac{1}{2}j_{1}]_{q}^{j_{2}+j_{1}+j_{2}+1}3F_{2}\left(\begin{array}{c}
 j_{1}-j_{2}, j_{2}-j_{1}, j_{2}+j_{1}, j_{1}+j_{2} - j_{2} - j_{1} - j_{2} - 1
 m_{1} - j_{1} - j_{2}, -j_{2}
\end{array} | q, q^{j_{1}+m_{1}}\right).
$$

For convenience we will write the last formula as follows

$$
\langle j_{1}m_{1}, j_{2}m_{2}|jm\rangle_{q} = (-1)^{j_{1}+j_{2}+j} \Gamma_{m_{1}, m_{2}}^{j_{1}, j_{2} = j} F_{2}(j_{1} - j_{2}, m_{2} - j_{2}, -j - j_{1} - j_{2} - 1 | q, q^{j_{1}+m_{1}}),
$$

where

$$
\Gamma_{m_{1}, m_{2}}^{j_{1}, j_{2} = j} = \frac{q^{-m_{2}j_{2}m_{1}j_{1}m_{2}^{-j}}[2j_{1}+1][j_{2}+m_{1}]}{[j_{1}+j_{2}+j+1]}[j_{1}+j_{2}-j]_{q}^{-j}_{q}^{j_{1}+j_{2}+j+1}3F_{2}(j_{1}-j_{2}, m_{2} - j_{2}, -j - j_{1} - j_{2} - 1 | q, q^{j_{1}+m_{1}}).
$$

In the last formula (41), as well as in (40) we explicitly write down the base $q$ we are using. This will be important when we discuss the symmetry properties of the CGC coefficients.

**Symmetry properties.** Next we will obtain the symmetry properties for the CGC of the $su_{q}(2)$ algebra. In order to that, we will use the representation formula (41).

If we use the transformation (13) and the formulas (6) with the choice $n = j_{1} + j_{2} - j$, $a = -j - j_{1} - j_{2} - 1$, $b = m_{2} - j_{2}$, $d = m - j_{1} - j_{2}$ and $e = -2j_{2}$, then we find, after a straightforward calculation the following symmetry formula

$$
\langle j_{1}m_{1}, j_{2}m_{2}|jm\rangle_{q} = (-1)^{j_{1}+j_{2}-j} \langle j_{2}m_{2}, j_{1}m_{1}|jm\rangle_{1/q},
$$

Similarly, if instead of (13) we use (14) we find

$$
\langle j_{1}m_{1}, j_{2}m_{2}|jm\rangle_{q} = \langle j_{2} - m_{2}, j_{1} - m_{1}|j_{2} - j_{1} - m_{1}| 1/q,\n$$

where the last equality follows by using the symmetry (42).

In a similar way we now set $n = m_{2} - j_{2}$, $a = -j - j_{1} - j_{2} - 1$, $b = j_{1} + j_{2} - j$, $d = -2j_{2}$, and $e = m - j_{1} - j_{2}$ and use the transformation formula (13) with the minus sign we obtain

$$
\langle j_{1}m_{1}, j_{2}m_{2}|jm\rangle_{q} = (-1)^{j_{2}+m_{2}q^{-m_{2}}} \sqrt{\frac{[2j_{1}+1]}{[2j_{2}+1]}} \langle j_{1} - m_{2}, j_{2}m_{2}|j_{1} - m_{1}\rangle_{1/q}.
$$

If in the right hand side of (44) we apply consecutively the symmetries (42) and (43) we obtain the following relation

$$
\langle j_{1}m_{1}, j_{2}m_{2}|jm\rangle_{q} = (-1)^{j_{2}+m_{2}q^{-m_{2}}} \sqrt{\frac{[2j_{1}+1]}{[2j_{2}+1]}} \langle j_{1} - m_{1}, j_{2}m_{2}|j_{1}m_{1}\rangle_{1/q}.
$$

Combining properly all the above formulas we can find a lot of new symmetries. For example, we can rewrite the last formula (45) by interchanging the indexes $1 \leftrightarrow 2$ as well as $q \rightarrow 1/q$ to get

$$
\langle j_{1}m_{1}, j_{2}m_{2}|jm\rangle_{q} = (-1)^{j_{1}+m_{1}q^{m_{1}}} \sqrt{\frac{[2j_{1}+1]}{[2j_{2}+1]}} \langle j_{1} - m_{1}, j_{1}m_{1}|j_{2}m_{2}\rangle_{1/q},
$$

and then use (42) in the right and left sides of the obtained equation to get another symmetry property

$$
\langle j_{1}m_{1}, j_{2}m_{2}|jm\rangle_{q} = (-1)^{j_{1}+m_{1}q^{m_{1}}} \sqrt{\frac{[2j_{1}+1]}{[2j_{2}+1]}} \langle jm, j_{2} - m_{2}|j_{1}m_{1}\rangle_{1/q}.
$$
Formulas (45) and (47) have been obtained in [23] by a much more complicated method that the one used here.

If we now in (46) make the change $m_1 \to -m_1, m_2 \to -m_2, m \to -m, q \to 1/q$ and apply to the obtained CGC on the left hand side the first equality in (43) we find another relevant symmetry

$$\langle j_1 m_1, j_2 m_2 | jm \rangle_q = (-1)^{j_1 - m_1} q^{m_1} \sqrt{[2j+1]_q} \langle j_1 m_1, j - m | j_2 - m_2 \rangle_{1/q}. \tag{48}$$

It can be also established, by a direct calculation, that the right hand side of (40) or (41) is invariant under the change $j_1 \to j_1 + j_2 + m_1 + m_2, m_1 \to j_1 - j_2 - m_1 - m_2, j_2 \to j_1 + j_2 - m_1 - m_2, m_2 \to j_1 - j_2 - m_1 + m_2, j \to j, m \to j_1 - j_2$ so

$$\langle j_1 m_1, j_2 m_2 | jm \rangle_q = \left\langle j_1 + j_2 + m_1 + m_2, j_1 - j_2 + m_1 - m_2, j_1 + j_2 - m_1 - m_2, j_1 - j_2 - m_1 + m_2 | j_1 - j_2 \right\rangle_q, \tag{49}$$

which is one of the 72 symmetry properties for the CGC discovered by Regge for the case $q \to 1$ [17] (for the $q$-case see [23, §4.3]). The other 71 can be obtained by combining the above symmetry (49) with (42), (43) and (44).

Finally, let us point out that all the obtained symmetries becomes into the classical ones [29, §8.4.3] when $q \to 1$.

**Another explicit formula for the CGCs.** Let us apply (13) to the $3F_2$ in (41) setting $n = j_1 + j_2 - j, a = m_2 - j_2, b = -j - j_1 - j_2 - j - 1, d = m_1 - j_1 - j_2, \text{ and } e = -2j_2$. This yields to the equivalent representation formula

$$\langle j_1 m_1, j_2 m_2 | jm \rangle_q = \sqrt{[2j+1]_q} \langle j_2 - m_1 | j_1 \rangle_q \langle j_2 + m_2 | j \rangle_q \langle j_1 | j_2 \rangle_q \langle j \rangle_q$$

$$\times (-1)^{j_1 + j_2 - j} \frac{q^{j_1} m_1 - j_2 - m_1 - (j_1 + j_2 - j) (j_1 + j_2 + j + 1)}{[j - j_2 + m_1]_q [j_1 - j_2 + m_2]_q} 3F_2 \left( \begin{array}{c} j - j_1 - j_2, m_2 - j_2, -m_1 - j_1 \\ j - j_2 + m_1 + 1, j - j_1 + m_2 + 1 \end{array} | q, q^{j_1 + j_2 + j + 1} \right). \tag{50}$$

If in the above relation we interchange the indexes 1 and 2, change $q \to 1/q$ and use the symmetry relation (42) we get the representation

$$\langle j_1 m_1, j_2 m_2 | jm \rangle_q = \sqrt{[2j+1]_q} \langle j_2 - m_1 | j_1 \rangle_q \langle j_2 + m_2 | j \rangle_q \langle j_1 | j_2 \rangle_q$$

$$\times \frac{q^{j_2} m_2 - j_2 - m_1 + (j_1 + j_2 - j) (j_1 + j_2 + j + 1)}{[j - j_2 + m_1]_q [j_1 - j_2 + m_2]_q} 3F_2 \left( \begin{array}{c} j - j_1 - j_2, m_1 - j_1, -m_2 - j_2 \\ j - j_2 + m_1 + 1, j - j_1 - m_2 + 1 \end{array} | q, q^{j_1 - j_2 - j - 1} \right), \tag{51}$$

from where, by using (6), we obtain

$$\langle j_1 m_1, j_2 m_2 | jm \rangle_q = q^{j_1 m_2 - j_2 m_1 - (j_1 + j_2 - j) (j_1 + j_2 + j + 1)} \sqrt{[2j+1]_q}$$

$$\times \frac{[j_1 + m_1]_q [j_2 - j_1 + m_2]_q [j_2 - j_1]_q [j_2 + m_2]_q [j_2 - j_2]_q [j_2 + j_1 + 1]_q}{[j_1 + j_2 + j + 1]_q}$$

$$\times \sum_{r=0}^{\infty} r!_q [j_2 - j - r]_q [j_2 + m_2 - r]_q [j_1 - m_1 - r]_q [j_1 - j_2 + j_2 - j_2]_q [j + j_2 - j_2]_q [j]_q [j - j_1 - m_2 + r]_q,$$

which is the $q$-analog of the the Racah formula for the CGCs of the $su_q(2)$ algebra [29, Eq. (3), page 238]. Here, the summation index are the integers for which all the factorial arguments and nonnegative.

The last formula can be written in terms of the $q$-analoge of the binomial coefficients (7), so

$$\langle j_1 m_1, j_2 m_2 | jm \rangle_q = q^{j_1 m_2 - j_2 m_1 - (j_1 + j_2 - j) (j_1 + j_2 + j + 1)} \sqrt{[2j+1]_q}$$

$$\times \sum_{r=0}^{\infty} (-1)^r \frac{[j_1 + j_2 - j]_q [j + j_2 - j]_q [j_1 - j_2]_q [j_2 + m_2 - j_2]_q \cdots [j]_q \cdots [j - j_1 - m_2 + r]_q [j_1 - j_2]_q [j_2 + m_2 - r]_q [j_2 + j_1 + r]_q [j_1 + j_2 + j + 1]_q}{[j_1 + j_2 + j + 1]_q \cdots [j_1 + j_2 + j + 1]_q \cdots [j_1 + j_2 + j + 1]_q}, \tag{53}$$

which is seem to be, at least for the limit case $q \to 1$, very convenient for computing the explicit values of the CGCs [19]. Let us also point out that the symmetry formulas (43), (42), and (48) can be directly
obtained from the $q$-analog of the Racah formula (53) in a similar way as it is done for the limit case $q \to 1$ in [18, pages 40–41].

Some special values. From the representation of the $q$-CGC in terms of the $3F_2$ we can obtain some relevant values of the CGC.

The first one is when $j$ reaches its maximal value, i.e., $j = j_1 + j_2$. In this case the $3F_2$ function in (41) is equal to one, so

$$\langle j_1 m_1, j_2 m_2 | j + j_2 m \rangle = q^{j_1 m_2 - j_2 m_1} \sqrt{\frac{[2j_1 q] ! [2j_2 q] ! [j_1 + j_2 + m] ! [j_1 + j_2 - m] q !}{[2j_1 + 2j_2 q] ! [j_1 + m_1] ! [j_1 - m_1] q ! [j_2 + m_2] ! [j_2 - m_2] q !}}. $$

If now set $j = j_1 - j_2$ (we assume without loss of generality that $j_1 \geq j_2$), i.e., the case when $j$ reaches its minimal value, then using (18) with $n = j_2 - m_2$, $b = 2j_1 + 1$, $c = j_1 + j_2 - m$, we find

$$\langle j_1 m_1, j_2 m_2 | j_1 - j_2 m \rangle = (-1)^{j_2 + m_2} q^{-j_1 m_2 - j_2 m_1 - m_2} \times \sqrt{\frac{[2j_1 + 2j_2 + 1] q ! [j_1 + m_1] ! [j_1 - m_1] q !}{[j_1 + j_2 + m_2] q ! [j_1 + j_2 - m] q ! [j_2 - m_2] q !}}. $$

If we put $m = j$ in (41) and use the formula (18) with $n = j_2 - m_2$, $b = j_1 + j_2 + j + 1$, $c = 2j_2$, we get

$$\langle j_1 m_1, j_2 m_2 | j j \rangle = (-1)^{j_1 - m_1} q^{j(j_1 + j_2 - j) (j_1 + j_2 - j + 1)/2} \times \sqrt{\frac{[j_1 + j_2 + j + 1] q ! [j_1 - j_2 - j] q ! [j_1 + j_2 + j + 1] q ! [j_1 - m_1] q ![j_2 - m_2] q !}{j_1 + j_2 + j + 1] q ![j_1 - j_2 - j] q ![j_1 - m_1] q ![j_2 - m_2] q !}}. $$

Putting $j = 0$ in the above formula and taking into account that, in this case, $j_1 = j_2$, $m_1 + m_2 = 0$, we obtain the value

$$\langle j_1 m_1, j_1 - m_1 | 00 \rangle = \delta_{j_1, j_2} \delta_{m_1 - m_2} \frac{(-1)^{j_1 - m_1} q^{m_1}}{\sqrt{2j_1 + 1] q !}}. $$

Notice also that, when $j = j_1 + j_2$ and $m = j_1 + j_2 (m_1 = j_1, m_2 = j_2)$, then $\langle j_1 m_1, j_2 m_2 | j_1 + j_2 + j_2 \rangle = 1$, which is the standard normalization for the CGCs.

The $3F_2$ function in (41) is equal to one also when $j_2 = m_2$, thus we find

$$\langle j_1 m_1, j_2 m_2 | j m \rangle = (-1)^{j_1 + j_2 - j} q^{j_2 (j_1 - m_1) - j_1 (j_1 + j_2 - j + 1)/2} \times \sqrt{\frac{[j_1 + j_2 + j + 1] q ![j_1 - j_2 - j] q ![j_1 + j_2 + j + 1] q ![j_1 - m_1] q ![j_2 - m_2] q !}{j_1 + j_2 + j + 1] q ![j_1 - j_2 - j] q ![j_1 - m_1] q ![j_2 - m_2] q !}}. $$

If we now put $m_1 = j_1$ in (41) and use the formula (18) with $n = j_1 + j_2 - j$, $b = j_1 + j_2 + j + 1$ and $c = 2j_2$, we get

$$\langle j_1 j_2, j_2 m_2 | j m \rangle = q^{-j_1 (j_2 - m_2) + j_2 (j_1 + j_2 - j)} q^{j_2 (j_1 - m_1) - j_1 (j_1 + j_2 - j + 1)/2} \times \sqrt{\frac{[j_1 + j_2 + j + 1] q ![j_1 - j_2 - j] q ![j_1 + j_2 + j + 1] q ![j_1 - m_1] q ![j_2 - m_2] q !}{j_1 + j_2 + j + 1] q ![j_1 - j_2 - j] q ![j_1 - m_1] q ![j_2 - m_2] q !}}. $$

Using the formulas (50) and (51) we find the following expressions for the special values of the CGCs when $m_1 = -j_1$ and $m_2 = -j_2$, respectively

$$\langle j_1 - j_1, j_2 m_2 | j m \rangle = (-1)^{j_1 + j_2 - j} q^{j_1 (j_2 + m_2) + j_2 (j_1 + j_2 - j)} q^{j_2 (j_1 + j_2 - j + 1)/2} \times \sqrt{\frac{[j_1 + j_2 + j + 1] q ![j_1 - j_2 - j] q ![j_1 + j_2 + j + 1] q ![j_1 - m_1] q ![j_2 - m_2] q !}{j_1 + j_2 + j + 1] q ![j_1 - j_2 - j] q ![j_1 - m_1] q ![j_2 - m_2] q !}}. $$

and

$$\langle j_1 m_1, j_2 - j_2 m | j m \rangle = q^{-j_2 (j_1 + m_1) + j_1 (j_2 + j_2 - j)} q^{j_2 (j_1 + j_2 - j + 1)/2} \times \sqrt{\frac{[j_1 + j_2 + j + 1] q ![j_1 - j_2 - j] q ![j_1 + j_2 - j] q ![j_1 + j_2 - j] q ![j_1 + m_1] q ![j_1 - m_1] q ![j_2 - m_2] q !}{j_1 + j_2 + j + 1] q ![j_1 - j_2 - j] q ![j_1 - m_1] q ![j_2 - m_2] q !}}. $$

1Here we correct a misprint in [22].
Before going ahead, let us point out that all the formulas from (54)–(60) becomes into the classical ones [29] by taking the limit \( q \to 1 \).

**Connection with the \( q \)-Hahn polynomials.** In this section we will obtain the connection of the \( q \)-CGC with the \( q \)-Hahn polynomials introduced in [14, Eq. (3.11.53), page 150] and detailed studies in [5]. For more details on the theory of orthogonal polynomials on nonuniform lattices developed by Nikiforov and Uvarov see e.g. [2, 14].

Our starting point will be different from the one used in [5] or the one used for the classical case in [14, §5.2.2.3 pages 245–246]. Our idea is to exploit the representation of the \( q \)-hypergeometric function \( _3F_2 \) given in (41).

So, we start from the hypergeometric representation (41)

\[
(61) \quad (j_1 m_1, j_2 m_2 | j m)_q = (-1)^{j_1+j_2-j} \Gamma_{j_1,j_2,j}^{m_1,m_2,j} F_2 \left( \begin{array}{c} m_2 - j_2, -j - j_1 - j_2 - 1, j - j_1 - j_2 \\ m - j_1 - j_2, -2j_2 \end{array} \mid q, q^{j_1+m_1} \right),
\]

and use the transformation (13) with \( n = j_2 - m_2, a = -j - j_1 - j_2 - 1, b = j - j_1 - j_2, d = m - j_1 - j_2 \) and \( e = -2j_2 \) to get

\[
(62) \quad (j_1 m_1, j_2 m_2 | j m)_q = (-1)^{j_1+j_2-j} \tilde{\Gamma}_{j_1,j_2,j}^{m_1,m_2,j} F_2 \left( \begin{array}{c} m_2 - j_2, -j - j_1 - j_2 - 1, m - j \\ m - j_1 - j_2, m_2 - j - j_1 \end{array} \mid q, q^{-(j_1+j_2)} \right),
\]

where

\[
\tilde{\Gamma}_{j_1,j_2,j}^{m_1,m_2,j} = q^{(j_2 - m_2)(j_1 + j_2 + 1)} \frac{(j + j_1 - j_2 + 1) \Gamma_{j_2-m_2}^{m_2} \Gamma_{j_1-m_1}^{j_2-m_2} \Gamma_{j_1,j_2,j}^{m_1,m_2,j}}{(-2j_2 \mid q, q^{j_2-m_2} \Gamma_{j_1,j_2,j}^{m_1,m_2,j}}.
\]

Next, we apply to the last expression for the \( q \)-CGC the transformation (13) again, but this time with parameters \( n = j_2 - m_2, a = m - j, b = -j - j_1 - j_2 - 1, d = m - j_1 - j_2 \) and \( e = m_2 - j - j_1 \). This leads to

\[
(63) \quad (j_1 m_1, j_2 m_2 | j m)_q = (-1)^{j_1+j_2-j} \tilde{\Gamma}_{j_1,j_2,j}^{m_1,m_2,j} F_2 \left( \begin{array}{c} m_2 - j_2, m + j + 1, m - j \\ m - j_1 - j_2, m_1 + j_2 + 1 \end{array} \mid q, q^{-(j_2+1)} \right),
\]

where, now,

\[
\tilde{\Gamma}_{j_1,j_2,j}^{m_1,m_2,j} = q^{(j_2 - m_2)(m + j_1 + j_2 + 1)} \frac{(j + j_1 - j_2 + 1) \Gamma_{j_2-m_2}^{m_2} \Gamma_{j_1-m_1}^{j_2-m_2} \Gamma_{j_1,j_2,j}^{m_1,m_2,j}}{(-2j_2 \mid q, q^{j_2-m_2} \Gamma_{j_1,j_2,j}^{m_1,m_2,j}}.
\]

Before finding the connection with the \( q \)-Hahn polynomials it is convenient to introduce some properties of such family of \( q \)-polynomials. As it is shown in [14] (see also [2]), there are several analogs of the classical Hahn polynomials. Among them, we will use the one introduced in [14, Eq. (3.11.53), page 150] and detailed studied in [5]. The main reason is related to the fact that the limit \( q \to 1 \) leads directly (without introducing any rescaling factor) to obtain any classical property, so in the following we will use the formulas obtained in [5]. Moreover, we will write the Hahn polynomials in terms of the \( q \)-hypergeometric function \( _3F_2 \) instead of the basic series for same aforementioned reason.

The \( q \)-Hahn polynomials on the \( q \)-linear lattice \( x(s) = (q^{2s - 1})/(q^2 - 1) \) can be written in terms of the \( q \)-symmetric hypergeometric function (8) as follows [14, Eq. (3.11.53), page 150]

\[
(64) \quad h_n^{\alpha,\beta}(x(s), N)_q = (-1)^n q^{\frac{n(n+1)}{2}} F_2 \left( \begin{array}{c} N-1 \\ n \end{array} \mid q, q^{x-N} \right)
\]

where it is assumed that \( N \) is a nonnegative integer. To obtain the expression (64) we have corrected a typo in [5, Eq. (4.38), page 32]. From the above representation it can be shown (see [16, page 233]) that the Hahn polynomials are polynomials on \( x(s) = (q^{2s - 1})/(q^2 - 1) \) of degree \( n \). Notice also that when \( q \to 1 \), \( x(s) \to s \), and the \( q \)-Hahn polynomials defined in (63)-(64) become into the classical ones [14, Eq. (2.7.19) page 52].
The $q$-Hahn polynomials are defined on the interval $s = 0, 1, \ldots, N - 1$, and, if $\alpha, \beta > -1$ they are orthogonal with respect to a weight function $\rho$, i.e.,
\begin{equation}
\sum_{s=0}^{N-1} h_n^{\alpha,\beta}(x(s), N) q h_m^{\alpha,\beta}(x(s), N) \rho(s) \triangle x(s - 1/2) = \delta_{n,m} d_n^2.
\end{equation}
They satisfy a linear difference equation on $s$
\begin{equation}
\xi(s)y(s + 1) + [\lambda_n - \zeta(s) - \xi(s)]y(s) + \zeta(s)y(s - 1) = 0,
\end{equation}
where
\begin{equation}
\xi(s) = \frac{\sigma(s) + \tau(s) \triangle x(s - \frac{1}{2})}{\triangle x(s - \frac{1}{2}) \nabla x(s)}, \quad \zeta(s) = \frac{\sigma(s)}{\triangle x(s - \frac{1}{2}) \nabla x(s)},
\end{equation}
as well as a three-term recurrence relation (which is also a second order difference equation but in $n$)
\begin{equation}
x(s)h_n^{\alpha,\beta}(x(s), N) q = \alpha_n h_{n+1}^{\alpha,\beta}(x(s), N) q + \beta_n P_{n+1}(x(s)) + \gamma_n h_{n-1}^{\alpha,\beta}(x(s), N) q, \quad n \geq 0,
\end{equation}
with $h_{-1}^{\alpha,\beta}(x(s), N) q := 0$ and $h_0^{\alpha,\beta}(x(s), N) q = 1$.

All the characteristics of the $q$-Hahn polynomials are given in Table 1.

**Table 1.** Main data for the $q$-Hahn polynomials in the lattice $x(s) = \frac{q^s - 1}{q - 1}$.

| $h_n^{\alpha,\beta}(x(s), N) q$ | \(\sigma(s)\) | $-q^{-\frac{\alpha}{\beta + 1}} x(s)^2 + q^{-\frac{\alpha}{\beta}} [N + \alpha] q x(s)$ |
| --- | --- | --- |
| \(\sigma(s) + \tau(s) \triangle x(s - \frac{1}{2})\) | $-q^{\frac{\alpha + 1}{\beta + 1}} \left[ s + \beta + 1 \right] q \left[ s - N + 1 \right] q$ |
| \(\lambda_n\) | $q^{\frac{\alpha + \beta}{\beta + 1}} \left[ n + \alpha + \beta + 1 \right] q$ |
| \(\rho(s)\) | $q^{\frac{\alpha}{\beta + 1}} \frac{\Gamma_q(s + \beta + 1) \Gamma_q(N + \alpha - s)}{\Gamma_q(s + 1) \Gamma_q(N - s)}$ |
| \(d_n^2\) | $q^{\frac{\beta}{\alpha + 1}} N^{\frac{\alpha + 1}{2}} \left[ n + \alpha + \beta + 1 \right] q \left[ n + \alpha + \beta + 1 \right] q \prod_{j=0}^{n-1} \left[ 2j + \alpha + \beta + 1 \right] q$ |
| \(\alpha_n\) | $q^{\frac{\alpha + \beta + 1}{2}} \frac{\left[ n + 1 \right] q \left[ n + \alpha + \beta + 1 \right] q \left[ 2n + \alpha + \beta + 1 \right] q}{\left[ 2n + \alpha + \beta + 1 \right] q}$ |
| \(\beta_n\) | $q^{\frac{\alpha + \beta + 1}{2}} \frac{\left[ n + \alpha + \beta + 1 \right] q \left[ n + \alpha + \beta + 1 \right] q \left[ N - n - 2 \right] q}{\left[ 2n + \alpha + \beta + 1 \right] q \left[ 2n + \alpha + \beta + 1 \right] q}$ |
| \(\gamma_n\) | $q^{\frac{\alpha + \beta + 1}{2}} \frac{\left[ n + \alpha \right] q \left[ n + \alpha + \beta + 1 \right] q \left[ N - n - 1 \right] q}{\left[ 2n + \alpha + \beta + 1 \right] q \left[ 2n + \alpha + \beta + 1 \right] q \left[ 2n + \alpha + \beta + 1 \right] q}$ |

If we compare the $3F_2$ function in (62) with the one in (63) we see that they coincide if we make the choice
\begin{equation}
s = j_2 - m_2, \quad n = j - m, \quad N = j_1 + j_2 - m + 1, \quad \alpha = m - j_1 + j_2, \quad \beta = m + j_1 - j_2.
\end{equation}
Therefore, it is straightforward to see that, by substituting the $3F_2$ function in (62) by the one in (63) that the following relation holds

$$(-1)^{j_1-\alpha} \langle j_1 m_1, j_2 m_2 | jm \rangle_q = \sqrt{\rho(s) \triangle x(s - \frac{1}{2})} h_{n^\alpha}^{\alpha} (x(s), N)_q.$$  

Notice that this is not the connection formula given in [5]. To obtain the one given in [5] we can do the following:

First we use the symmetry relation (42) and then use twice the transformation formula (13) choosing the appropriate parameters. This leads to a similar expression to (62) but now the hypergeometric function $3F_2$ is given by

$$3F_2 \left( \begin{array}{c} m_1 - j_1, m + j + 1, m - j \\ m - j_1 - j_2, m + j_2 - j_1 + 1 \end{array} \mid q, q^{j_1+m_1+1} \right).$$

Comparing the above function with the $3F_2$ given in (63) we obtain the connection formula

$$(-1)^{j_1-\alpha+j-m} \langle j_1 m_1, j_2 m_2 | jm \rangle_q = \sqrt{\rho(s) \triangle x(s - \frac{1}{2})} h_{n^\alpha}^{\alpha} (x(s), N)_{1/q},$$

where now

$$s = j_1 - m_1, \quad N = j_1 + j_2 - m + 1, \quad \alpha = m + j_1 - j_2, \quad \beta = m - j_1 + j_2, \quad n = j - m.$$ 

Notice that in (71) the $q$-Hahn polynomials are defined for $q^{-1}$.

There are several relevant consequences of the above connection formulas. For example, the TTRR (68) leads to the recurrence relation [24]

$$q^{-\frac{1}{2}} \left( q^{\frac{1}{2}}(j+1)|jm|_q - q^{\frac{3}{2}}(j+1)|j-1m|_q \right) \langle j_1 m_1, j_2 m_2 | jm \rangle_q = 0.$$ 

and the difference equation (67) gives [22]

$$q^{-1} \sqrt{[m_2 - j_2 - 1]|jm|_q \langle jm | j_1 m_1 + 1 \rangle_q + \sqrt{[m_2 + j_2 + 1]|jm|_q \langle jm | j_1 m_1 - 1 \rangle_q + [m_2 + j_2 + 1]|jm|_q \langle jm | j_1 m_1 \rangle_q} = 0.$$ 

which can be used for the numerical computation of the $q$-CGC.

Before finishing this section let us point out that the representation (62) can be also used for connecting the $q$-CGC with the so-called $q$-dual Hahn in the lattice $x(s) = [s]_q [s + 1]_q$. This have been done in [3] and will be not considered here. In fact, there are several other relations involving the $q$-Hahn polynomials different from the (67) and (68) that lead to recurrence relations for the $q$-CGC. The interested reader is referred to the aforementioned papers [3, 5].

**Concluding remarks**

As we see, the use of the special function theory, in this case the $q$-hypergeometric functions (or, equivalently, the basic series) can be very useful for the study of the Clebsch-Gordon coefficients of the $su_q(2)$ algebra. As it has been shown, there several properties that are quite complicated to obtain by using the representation theory tools that can be easily obtained exploiting the representation of the $q$-CGC in terms of the symmetric $q$-hypergeometric function $3F_2$ (see formula (4.2)). In particular, all the obtained results transforms into the classical ones for the algebra (group) $su(2)$ just by taking the limit $q \to 1$. It is worth
to point out that a similar analysis can be done also for the non compact algebra $su_q(1,1)$. The results in this case will be published elsewhere.

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