An effective method of calculating the non-Markovianity $\mathcal{N}$ for single channel open systems

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We propose an effective method which can simplify the optimization of the increase of the trace distance over all pairs of initial states in calculating the non-Markovianity $\mathcal{N}$ for single channel open systems. For the amplitude damping channel, we can unify the results of Breuer et al. [Phys. Rev. Lett. 103, 210401 (2009)] in the large-detuning case and the results of Xu et al. [Phys. Rev. A 81, 044105 (2010)] in the resonant case; furthermore, for the general off-resonant cases we can obtain a very tight lower bound of $\mathcal{N}$. As another application of our method, we also discuss $\mathcal{N}$ for the non-Markovian depolarizing channel.

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I. INTRODUCTION

Inevitable interaction with external environment may lead to the phenomenon of decoherence for open quantum systems. In general, the assumption of Markovian approximation usually was applied to dynamical evolution of system. However, recently people found that the Markovian processes without memory and non-Markovian processes with memory can lead to distinctly different effects on decoherence and disentanglement of open systems. Thus the non-Markovian dynamics have become increasingly important and are extensive studied in both discrete variable [1–9] and continuous-variable [10–13] systems. It has been found that the non-Markovian effect of environment can extend significantly the entanglement time of the qubits [1] and has been experimentally observed [2]. Because of the importance of the non-Markovian effect of environment, some authors [16–19] have developed some measures to detect the non-Markovianity of open systems from different points of view. Breuer et al. [17] proposed a computable measure $\mathcal{N}$ to detect the non-Markovianity of open systems. The idea is based on the distinguishability of quantum states which results from information flow between the open system and its environment. Rivas et al. [18] also proposed a measure of non-Markovianity which is based on the fact that negative rates are linked to whether entanglement between the system and an ancilla can increase. Lu et al. [19] defined a measure of non-Markovianity using quantum fisher information flow. As we know that evaluation of $\mathcal{N}$ requires optimization of the total increase of the trace distance over all pairs of initial states, which is very difficult to accomplish. In Ref. [17] Breuer et al. considered a two-level system interacting with a reservoir which possesses the Lorentzian spectral property, and in the large detuning case they found by numerical simulation the pair of initial states $\rho_1(0) = |e\rangle\langle e|$ and $\rho_2(0) = |g\rangle\langle g|$ which make the optimization of the total increase of the trace distance (It should be noted that in Ref. [17] they use $|+\rangle$ and $|–\rangle$ instead of $|e\rangle$ and $|g\rangle$ to represent the excited and ground states respectively). Later, for the same two initial states Li et al. [20] obtained the analytical expression of the trace distance $D(\rho_1(t), \rho_2(t)) = |h(t)|^2$. Very recently for the same model in the resonant case, Xu et al. [21] found two different initial states which make the optimization of the total increase of the trace distance by using an analytical method and the corresponding trace distance $D(\rho_1(t), \rho_2(t)) = |h(t)|^2$. It should be noted that in Ref. [21] they use $b(t)$ instead of $h(t)$ to represent the amplitude damping of the excited state $|e\rangle$. It is intriguing that these two results are quite different, that is, in these two cases the two initial states which make the optimization are different, and the corresponding trace distance are also different. Because the optimization is very difficult to accomplish, until now we have not seen any reports about the non-Markovianity for the general off-resonant case. In Refs. [17, 21] the authors separately dealt with the optimization by different methods, one is numerical and another is analytical. For the same model only for different parameter regimes the results are quite different, therefore an unified understanding of these results is in demands.

In this paper, we propose an effective method which can easily optimize the increase of the trace distance over all pairs of initial states in calculating the non-Markovianity $\mathcal{N}$ for single channel open systems. For the amplitude damping channel, we analytically re-derive the results of Ref. [17] in the large-detuning case and the results of Ref. [21] in the resonant case; furthermore, in the general off-resonant cases we can obtain a very tight lower bound for $\mathcal{N}$. Thus an unified understanding of the results of Ref. [17] and Ref. [21] is given. As another application of our method, we also discuss $\mathcal{N}$ for the non-Markovian depolarizing channel.

The paper is organized as follows. In Sec. II, we introduce our method. The non-Markovian amplitude damping channel and the non-Markovian depolarizing channel

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are examined by our method in Sec. III. Finally, we give the conclusion of our results in Sec. IV.

II. METHOD OF OPTIMIZING THE TOTAL INCREASE OF TRACE DISTANCE

Recently, Breuer et al. [17] proposed a measure to detect the non-Markovian behavior of quantum processes in open systems based on the distinguishability of quantum states. The trace distance $D$ describing the distinguishability between the two states is defined as [22]

$$D(p_1, p_2) = \frac{1}{2} \text{tr}|p_1 - p_2|,$$

where $|M| = \sqrt{M^\dagger M}$, and $0 \leq D \leq 1$. If $D = 0$, the two states are the same, and if $D = 1$, the two states are totally distinguishable.

Considering a quantum process $\Phi(t)$, $\rho(t) = \Phi(t)\rho(0)$, where $\rho(0)$ and $\rho(t)$ denote the density operators at time $t = 0$ and at any time $t > 0$, respectively, then the non-Markovianity $N$ is defined as

$$N(\Phi) = \max_{p_1, p_2(0)} \int_{\sigma > 0} dt \sigma(t, p_1, p_2(0)),$$

where $\sigma(t, p_1, p_2(0))$ is the rate of change of the trace distance defined as

$$\sigma(t, p_1, p_2(0)) = \frac{d}{dt}D(p_1(t), p_2(t)).$$

As we known that $\sigma(t, p_1, p_2(0)) \leq 0$ corresponding to all dynamical semigroups and all time-dependent Markovian processes, and a process is non-Markovian if there exists a pair of initial states and at certain time $t$ such that $\sigma(t, p_1, p_2(0)) > 0$. Physically, this means that for non-Markovian dynamics the distinguishability of the pair of states increases at certain times.

In view of Eq. (3), the non-Markovianity $N$ also can be written as the following form

$$N(\Phi) = \max_{p_1, p_2(0)} \sum_n \left[D(p_1(\tau_n^{\text{max}}), p_2(\tau_n^{\text{max}})) - D(p_1(\tau_n^{\text{min}}), p_2(\tau_n^{\text{min}}))\right],$$

where $\tau_n^{\text{max}}$ and $\tau_n^{\text{min}}$ correspond to the time points of the local maximum and minimum of $D(p_1(t), p_2(t))$, respectively. $N(\Phi)$ can be calculated as follows: One first derives the increment of the trace distance over each time interval $[\tau_n^{\text{min}}, \tau_n^{\text{max}}]$ for any pairs of initial states, then sums up the total contributions of all intervals, finally performs the maximization for all pairs of initial states.

Generally speaking it is very difficult to make the maximization in Eq. (4). In this paper we want to find easy ways to maximize the increase of trace distance. Our idea is like this: First we find the two specific initial states which make the maximization of the quantity $N_n(\Phi)$ at each time interval. The $N_n(\Phi)$ is defined as the difference between the local maximum and local minimum of the trace distance for arbitrary time interval $[\tau_n^{\text{min}}, \tau_n^{\text{max}}]$. That is

$$N_n(\Phi) = \max_{p_1, p_2(0)} \left[D(p_1(\tau_n^{\text{max}}), p_2(\tau_n^{\text{max}})) - D(p_1(\tau_n^{\text{min}}), p_2(\tau_n^{\text{min}}))\right].$$

Apparently it is much easier to find the two initial states which makes the maximization in Eq. (5) than to find the two initial states which makes the summation in Eq. (4) maximal. Then for specific non-Markovian channel we try to prove that the two initial states we found can also make the summation in Eq. (4) maximal. Generally speaking it is not easy to prove this. If it can not be proved, we still believe that the pair of initial states which make the increase of the trace distance in single time interval maximal, will also make the optimization of the summation in $N$. Of course this is not rigorous. If we are strict enough, at least in this case we can find a lower bound of $N(\Phi)$, and we argue that this lower bound is tight. We will show in the following that this method is very effective.

III. APPLICATIONS

Based on the above idea, we can calculate the non-Markovianity $N$ for the non-Markovian amplitude damping channel and the non-Markovian depolarizing channel.

A. Non-Markovian amplitude damping channel

We consider a two-level system (qubit) interacting with a zero temperature reservoir. The Hamiltonian of the total system under the rotating wave approximation is given by ($\hbar = 1$)

$$\hat{H} = \omega_0 \hat{\sigma}_+ + \hat{a}_k^{\dagger} \hat{a}_k + \sum_{k=1}^{N} (g_k \hat{\sigma}_- \hat{a}_k^{\dagger} + g_k^* \hat{\sigma}_+ \hat{a}_k),$$

where $\hat{\sigma}_+ = |e\rangle\langle g|$ and $\hat{\sigma}_- = |g\rangle\langle e|$, are the Pauli raising and lowering operators for the two-level system, respectively. $\omega_0$ is the Bohr frequency of the two-level system, $\hat{a}_k$ and $\hat{a}_k^{\dagger}$ are the annihilation and creation operators for reservoir mode $k$, $\omega_k$ is the frequency of the mode $k$ of the reservoir, and $g_k$ is the coupling constant. The Hamiltonian of Eq.(6) can describe various systems. For concrete discussion we take a two-level atom interacting with the reservoir formed by the quantized modes of a high-$Q$ cavity. The dynamics of the reduced density matrix for the two-level atom can be written as [23]

$$\rho^S(t) = \begin{pmatrix} \rho_{ee}^S(0) & |h(t)|^2 \\ |h(t)|^2 & \rho_{gg}^S(0) \end{pmatrix},$$

in the basis $\{|e\rangle, |g\rangle\}$, where the superscript $S$ represents the two-level atom. Corresponding $h(t)$ denotes the amplitude of the upper level $|e\rangle$ of the atom initially prepared in $|e\rangle$ and satisfies the following integrodifferential
\[
\frac{d}{dt} h(t) = -\int_0^t dt_1 f(t-t_1)h(t_1),
\]
(8)
where the kernel \( f(t-t_1) = \int d\omega J(\omega) \exp[i(\omega_0-\omega)(t-t_1)] \)
is related to the spectral density \( J(\omega) \) of the reservoir. The model describes the damping of a two-level atom in a cavity. In this paper we restrict ourselves to the case that the atom-cavity system has only one excitation, and suppose that \( J(\omega) \) takes the Lorentzian spectral density \( \frac{1}{\pi(\omega_0-\delta-\omega)^2+\lambda^2} \).

Here \( \delta = \omega_0 - \omega_c \) is the detuning of the center frequency of the cavity \( \omega_c \) and the Bohr frequency of the two-level atom \( \omega_0 \), the parameter \( \lambda \) defines the spectral width of the coupling, which is associated with the reservoir correlation time by the relation \( \tau_B = \lambda^{-1} \) and the parameter \( \gamma_0 \) is related to the relaxation time scale \( \tau_R \) by the relation \( \tau_R = \gamma_0^{-1} \). Therefore the analytic expression of \( h(t) \) can be obtained as
\[
h(t) = e^{-(\lambda-i\delta)t/2}[\cosh(dt/2) + (\lambda-i\delta) \sinh(dt/2)/d]
\]
with \( d = \sqrt{(\lambda-i\delta)^2 - 2\gamma_0\lambda} \).

Based on the Hermiticity, and unit trace of a physical density matrix, any pair of initial states can be defined as \( \rho(0) \)
\[
\rho_1(0) = \begin{pmatrix}
\alpha & \beta \\
\beta^* & 1-\alpha
\end{pmatrix}
\]
\[
\rho_2(0) = \begin{pmatrix}
\mu & \nu \\
\nu^* & 1-\mu
\end{pmatrix}
\]
with \( |\beta|^2 \leq \alpha(1-\alpha), |\nu|^2 \leq \mu(1-\mu) \) corresponding to the semipositivity of a density matrix, \( \beta, \nu \in \mathbb{C}, 0 \leq \alpha, \mu \leq 1 \), and \( \alpha, \mu \in \mathbb{R} \). Thus, the evolution of the corresponding density matrix can be obtained
\[
\rho_1(t) = \begin{pmatrix}
\alpha |h(t)|^2 & \beta h(t) \\
\beta^* h^*(t) & 1-\alpha |h(t)|^2
\end{pmatrix}
\]
\[
\rho_2(t) = \begin{pmatrix}
\mu |h(t)|^2 & \nu h(t) \\
\nu^* h^*(t) & 1-\mu |h(t)|^2
\end{pmatrix}
\]
(12)
The combination of Eqs. (1) and (12) immediately provides the expression of the trace distance at any time \( t \geq 0 \)
\[
D(\rho_1(t), \rho_2(t)) = \sqrt{|h(t)|^4(\alpha-\mu)^2 + |h(t)|^2|\beta-\nu|^2}
\]
which has been obtained in Ref. [21]. It is noted that the maximization of trace distance in the resonant case has been given in Ref. [21], however, their method cannot be extended to the general off-detuning case because the condition \( |h(\tau_{min})| = 0 \) cannot always be guaranteed at each local minima. Now using Eq. (5), we can easily achieve the maximization of \( \mathcal{N} \) analytically, namely the optimization of the trace distance difference between the local maximum and local minimum in the time interval \([\tau_{min}^1, \tau_{max}^1] \) by choosing two specific initial states. According to Eqs. (5) and (13), \( \mathcal{N} \) can be written as
\[
\mathcal{N}_n = \max_{\rho_{1,2}(0)} |h(\tau_{max}^n)| \sqrt{|h(\tau_{max}^n)|^2(\alpha-\mu)^2 + |\beta-\nu|^2}
\]
\[
- |h(\tau_{min}^n)| \sqrt{|h(\tau_{min}^n)|^2(\alpha-\mu)^2 + |\beta-\nu|^2}
\]
(14)
When \( t = 0, |h(0)| = 1 \), and from Eq. (13) \( D \equiv \sqrt{(\alpha-\mu)^2 + |\beta-\nu|^2} \leq 1 \). Furthermore, the condition is equivalent to these parameterized conditions \( (\alpha-\mu) = \cos\theta, \beta-\nu = \cos\theta \sin\theta \) \( (r \leq 1, \theta \in [0, 2\pi]) \) and \( \phi \in [0, \pi] \). Substituting these parameterized conditions into Eq. (14) and considering that the maximization is over all pairs of initial states, we can obtain that the maximization condition requires \( \sqrt{(\alpha-\mu)^2 + |\beta-\nu|^2} = 1 \) corresponding to \( r = 1 \). Then the problem becomes the maximization of the following \( \mathcal{N}^0_n(\theta) \)
\[
\mathcal{N}^0_n(\theta) = |h(\tau_{max}^n)| \sqrt{|h(\tau_{max}^n)|^2 \cos^2\theta + \sin^2\theta}
\]
\[
- |h(\tau_{min}^n)| \sqrt{|h(\tau_{min}^n)|^2 \cos^2\theta + \sin^2\theta}
\]
(15)
From the following equation
\[
\frac{\partial \mathcal{N}^0_n(\theta)}{\partial \theta} = 0
\]
(16)
we can obtain the extrema, which are \( \mathcal{N}^{\alpha}_{n1} = A^2 - B^2 \)
when \( \theta = 0; \mathcal{N}^{\alpha}_{n2} = A - B \) when \( \theta = \pi/2 \) or \( \theta = 3\pi/2; \mathcal{N}^{\alpha}_{n3} = A^2 \sqrt{1-(A^2+B^2)^{-1}} - B^2 \sqrt{1-(A^2+B^2)^{-1}} \)
when \( \theta = \arccos \sqrt{\frac{(A^2+B^2)^{-1}(A^2+B^2)}{A^2+B^2}} \), where
\[
A = |h(\tau_{max}^n)|, B = |h(\tau_{min}^n)| \) and \( \) A > B. So \( \mathcal{N} \) can be represented as
\[
\mathcal{N}_n = \max\{\mathcal{N}^\alpha_{n1}, \mathcal{N}^\alpha_{n2}, \mathcal{N}^\alpha_{n3}\}
\]
(17)
From numerical calculation, we find that for any A and B satisfying 0 < B < A < 1, \( \mathcal{N}^\alpha_{n1} \) is always less than \( \mathcal{N}^\alpha_{n1} \) and \( \mathcal{N}^\alpha_{n2} \), so \( \mathcal{N} = \max\{\mathcal{N}^\alpha_{n1}, \mathcal{N}^\alpha_{n2}\} \).

(i) In the resonant case, that is \( \delta = 0 \), it is obvious that \( A + B = |h(\tau_{max}^n)| + |h(\tau_{min}^n)| < 1 \) because \( B = |h(\tau_{min}^n)| = 0 \) at \( \tau_{min}^n = 2[\pi - \arctan(d'(t)/2)]/d' \)
with \( n=1,2,3,..., \) and \( d' = \sqrt{|X_1^2 - 2\tau_{min}^n|} \). So in any time interval \([\tau_{min}^n, \tau_{max}^n] \), \( \mathcal{N}_n = \max\{\mathcal{N}^\alpha_{n1}, \mathcal{N}^\alpha_{n2}\} = \mathcal{N}^\alpha_{n2} = A - B = |h(\tau_{max}^n)| \).
For \( \mathcal{N}^\alpha_{n2}, \theta = \pi/2 \) or \( 3\pi/2\), and the two initial states correspond to \( \alpha = \mu \) and \( \beta - \nu = 1 \). Because in this case \( \mathcal{N} \) reaches its maximum for any n, the same two initial states, that is \( \alpha = \mu \) and \( \beta - \nu = 1 \), are also the two initial states which make the summation in Eq. (4) maximal. It is worth noting that these conditions \( \alpha = \mu, |\beta - \nu| = 1 \) together with \( |\beta|^2 \leq \alpha(1-\alpha) \) and \( |\nu|^2 \leq \mu(1-\mu) \) are equivalent to the conditions \( \alpha = \mu = 1/2, |\beta| = |\nu| = 1/2 \) and \( |\beta - \nu| = 1 \) obtained in Ref. [21], which can be explained as follows.
Our conditions can be changed into $\alpha = \mu$, $|\beta - \nu| = 1$, $(\alpha - 1/2)^2 + |\beta|^2 \leq (1/2)^2$ and $(\mu - 1/2)^2 + |\nu|^2 \leq (1/2)^2$. From the geometric point of view, the new conditions indicate that the two points $(\alpha, |\beta|)$ and $(\mu, |\nu|)$ is in (or on the circumference of) the same circle

$$(x - 1/2)^2 + |y|^2 = (1/2)^2$$

(18)

with $x \in \mathbb{R}$ and $y \in \mathbb{C}$. It is easy to check that the conditions $\alpha = \mu$, and $|\beta - \nu| = 1$ are just $\alpha = \mu = 1/2$, $|\beta| = |\nu| = 1/2$, $|\beta - \nu| = 1$. In summary, in the resonant case our results are consistent with the results of Ref. [21].

(ii) In the off-resonant case, that is $\delta \neq 0$, $A + B = |h(r_{n_{\text{max}}})| + |h(r_{n_{\text{min}}})|$ may be less than 1, equal to 1, or more than 1 depending on the values of $\gamma_0$, $\lambda$ and $\delta$. We have proved that depending on the value of $A + B$ there are only two pairs of initial states which make the maximization of $N_n$ for each $n$. Next we give our effective and practical method to calculate $N$ for any fixed parameters $\gamma_0$, $\lambda$ and $\delta$. From Eqs. (2) and (3) we can use the two pairs of initial states to obtain $N_1$ and $N_2$ respectively, $N_1 = \int_{t_\sigma > 0} dt \sigma_1(t, \theta = 0)$, $N_2 = \int_{t_\sigma > 0} dt \sigma_2(t, \theta = \pi/2$ or $3\pi/2)$. Correspondingly, the expressions of $\sigma_1(t, \theta = 0)$ and $\sigma_2(t, \theta = \pi/2$ or $3\pi/2)$ are given by

$$\sigma_1(t) = e^{-\lambda t}\{\mu|\cosh(at) - \cos(bt)| + \nu \sinh(at) - \xi \sin(bt)\},$$

(19)

$$\sigma_2(t) = \frac{e^{-\lambda t}}{2\sqrt{2\lambda}}\{\mu|\cosh(at) - \cos(bt)| + \nu \sinh(at) - \xi \sin(bt)\},$$

(20)

where $a$ and $b$ denote the real part and imaginary part of $d$ respectively; $\mu = \frac{1}{2|d|^2}\{(\lambda \alpha^2 - \lambda \beta^2 - \lambda \delta^2 - 2ab\delta), \nu = \frac{1}{2|d|^2}\{(\nu \alpha^2 - \nu \beta^2 - \nu \delta^2 - 2ab\delta), \xi = \frac{1}{2|d|^2}\{(\nu \alpha^2 - \nu \beta^2 - \nu \delta^2 - 2ab\delta), \eta = \frac{1}{2|d|^2}\{(2\lambda - 2b\delta), \chi = \frac{1}{2|d|^2}\{(2\lambda + 2a\delta), \kappa = \frac{1}{2|d|^2}\{(\lambda + \delta)^2 + \alpha^2 + \beta^2), \zeta = \frac{1}{2|d|^2}\{(\lambda + \delta)^2 + \alpha^2 + \beta^2), |d|$ denotes the absolute value of $d$. It is worth noting that Eq. (19) has been obtained in Ref. [21]. We can not prove but we believe that one of the two pairs of initial states we found can also make the optimization in the summation of Eq. (4), thus $N' = \max\{N_1, N_2\}$. If we are strict enough, at least it is a very tight lower bound (TLD) for the genuine $N'$,

$$N_{\text{TLD}} = \max\{N_1, N_2\}.$$

(21)

We plot $N_{\text{TLD}}$, $N_1$, and $N_2$ as functions of detuning $\delta$ for $\lambda = 0.1 \gamma_0$ in Fig. 1, and $N_{\text{TLD}}$, $N_1$, and $N_2$ as functions of $\lambda$ for $\delta = 0.1 \gamma_0$ in Fig. 2. From Fig. 1 we can see that there exists a critical point for $\delta$ at which the pair of initial states change from $\theta = \pi/2$ or $\theta = 3\pi/2$ to $\theta = 0$. More specifically when $\delta < \delta_c$, $N_{\text{TLD}} = N_2$ corresponding to the two initial states $\theta = \pi/2$ or $\theta = 3\pi/2$; when $\delta = \delta_c$, $N_{\text{TLD}} = N_1 = N_2$ corresponding to the two initial states $\theta = 0$, and $\theta = \pi/2$ or $3\pi/2$; when $\delta > \delta_c$, $N_{\text{TLD}} = N_1$ corresponding to the two initial states $\theta = 0$. Similarly, it can be seen from Fig. 2 that there also exists a critical point for $\lambda$ at which the pair of initial states change from $\theta = 0$ to $\theta = \pi/2$ or $\theta = 3\pi/2$.

It is worth noting that in the large detuning case by numerical simulation Breuer et al. [17] have performed the optimization of the total increase of the trace distance. In the large detuning case from Eq. (10) we know $A + B = |h(r_{n_{\text{max}}})| + |h(r_{n_{\text{min}}})| > 1$ for any $n = 1, 2, 3, ...$. Therefore, we can obtain $N_n = \max\{N_{n_{\text{max}}}, N_{n_{\text{min}}}\} = \int_{t_\sigma > 0} dt \sigma_n(t, \theta = 0)$, and the two initial states satisfy $\alpha - \mu = 1$ and $|\beta - \nu| = 0$ corresponding to $\theta = 0$. Because the two initial states ($\theta = 0$) make the optimization of $N_n$ for any $n$, they also make the optimization of the summation in Eq. (4). Similar to the resonant case, the optimization conditions including $\alpha - \mu = 1$, $|\beta - \nu| = 0$, $|\beta|^2 \leq \alpha(1 - \alpha)$, and $|\nu|^2 \leq \mu(1 - \mu)$ can be simplified into $\alpha = 1$, $\mu = 0$ and $\beta = \nu = 0$. 

![Fig. 1](image1.png)

FIG. 1: (Color online) $N_{\text{TLD}}$, $N_1$, and $N_2$ as a function of $\delta$, $\lambda = 0.1 \gamma_0$.

![Fig. 2](image2.png)

FIG. 2: (Color online) $N_{\text{TLD}}$, $N_1$, and $N_2$ as a function of $\lambda$, $\delta = 0.1 \gamma_0$. 


Evidently, the two initial states are \( \rho_1(0) = |e\rangle\langle e| \) and \( \rho_2(0) = |g\rangle\langle g| \) which has been obtained in Ref. [17]. It is noted that the condition is obtained by numerical simulation in [17], but here we obtain the same condition using an analytic method. Then we can also obtain the trace distance \( D = |\Delta(t)|^2 \) for these two initial states which is given by Ref. [20]. In summary, in the large-detuning case our results are consistent with the results reported in Refs. [17, 20].

### B. Non-Markovian depolarizing channel

As another application of our method, we consider the non-Markovianity \( \mathcal{N} \) for a non-Markovian depolarizing channel. The dynamical property of this system and in particular the conditions of complete positivity of the map corresponding to a master equation have been studied in detail by Daffer et al. in Ref. [24]. For this model the time-dependent Hamiltonian, that corresponds to a two-level system subjected to random telegraph noise, is

\[
H(t) = \hbar \sum_{i=1}^{3} \Gamma_i(t) \sigma_i, \tag{22}
\]

where \( \Gamma_i(t) = a_i n_i(t) \) are independent random variables, and \( \sigma_i \) are the usual Pauli operators. \( n_i(t) \) has a Poisson distribution with a mean equal to \( t/2\tau_i \), while \( a_i \) is an independent coin-flip random variable taking the values \( \pm a_i \). For the time-dependent Hamiltonian of Eq. (22), the corresponding equation of motion for the density operator is governed by the von Neumann equation

\[
\dot{\rho}(t) = -i[H(t), \rho] = -i\sum_k \Gamma_k(t)[\sigma_k, \rho] \tag{23}
\]

Substituting the formal solution Eq. (23) into the von Neumann equation and performing a stochastic average, one can obtain the following memory kernel master equation

\[
\dot{\rho}(t) = -\int_0^t \sum_k e^{-(t-t')/\tau_k} \sigma_k^2[\sigma_k, \rho(t')] dt', \tag{24}
\]

where the correlation function of the random telegraph signal \( \Gamma_j(t') \Gamma_k(t')' \) contributes to the memory kernel. It has been pointed out [24] that the system density operator with an exponential memory kernel obeys a homogeneous Volterra equation after averaging over the reservoir variables, and also proved that when two of \( a_i \) are zero, only one direction having the noise, the map \( \Phi(\rho) \) can be written as Kraus operator form [24], namely

\[
\rho'(t) = \Phi(t, \rho) = \sum_{k=1}^4 A_k^\dagger \rho A_k, \tag{25}
\]

For simplicity, in this paper we only consider the case the \( z \) direction with noise, \( x \) and \( y \) directions without noise. Therefore, \( A_1 = 0, A_2 = 0, A_3 = \sqrt{[1 - \Lambda(\nu)]/2\tau_3}, \) and \( A_4 = \sqrt{[1 + \Lambda(\nu)]/2I} \), where \( \Lambda(\nu) = \exp(-\nu)[\cos(\mu\nu) + \sin(\mu\nu)/\mu] \) with \( \mu = \sqrt{(4\pi^2 - 1)\tau}, \) and \( \nu = t/2\tau \) is a dimensionless time. \( a \) is the coupling strength of the system with the external environment while \( \tau \) determines which frequencies the system prefers most. For convenience we let \( \lambda = 1/\tau \), then \( \Lambda(\nu) \) can be rewritten as

\[
\Lambda(t) = \begin{cases} 
  e^{-\lambda t/2}[\cosh(\frac{t}{2\lambda}) + \frac{\mu}{2} \sinh(\frac{t}{2\lambda})] & (16a^2 < \lambda^2) \\
  e^{-\lambda t/2}[1 + \frac{\mu^2}{2}] & (16a^2 = \lambda^2) \\
  e^{-\lambda t/2}[\cosh(\frac{t}{2\lambda}) + \frac{\mu}{2} \sinh(\frac{t}{2\lambda})] & (16a^2 > \lambda^2), 
\end{cases} \tag{26}
\]

where \( \epsilon = \sqrt{16a^2 - \lambda^2} \).

Using the same two initial states of Eq. (11), from Eq. (25) we can obtain the evolutions of the two density matrices, respectively

\[
\rho^S_1(t) = \left( \begin{array}{cc} \alpha \Lambda(t) & \beta \Lambda(t) \\
\beta^* \Lambda^*(t) & 1 - \alpha 
\end{array} \right),
\rho^S_2(t) = \left( \begin{array}{cc} \mu & \nu \Lambda(t) \\
\nu^* \Lambda^*(t) & 1 - \mu 
\end{array} \right). \tag{27}
\]

Therefore the trace distance can be obtained

\[
D(\rho^S_1(t), \rho^S_2(t)) = \sqrt{(a - \mu)^2 + |\Lambda(t)|^2|\beta - \nu|^2}. \tag{28}
\]

From Eqs. (5) and (28), \( N_n \) can be expressed as

\[
N_n = \max_{\rho_{1,2}(0)} \sqrt{(a - \mu)^2 + |\Lambda(\tau_{\text{max}})|^2|\beta - \nu|^2 - (a - \mu)^2 + |\Lambda(\tau_{\text{min}})|^2|\beta - \nu|^2}. \tag{29}
\]

As in the subsection A, after parameterizing \( (a - \mu) \) and \( |\beta - \nu| \), the problem becomes the maximization of the following \( N_n^\alpha(\theta) \)

\[
N_n^\alpha(\theta) = \sqrt{\cos^2\theta + |\Lambda(\tau_{\text{max}})|^2\sin^2\theta} - \sqrt{\cos^2\theta + |\Lambda(\tau_{\text{min}})|^2\sin^2\theta}. \tag{30}
\]

Then we can obtain the extrema of \( N_n^\alpha \): \( N_n^\alpha(\theta = 0) = 0 \) when \( \theta = 0; N_{n1}^\alpha = A - B \) when \( \theta = \pi/2 \) or \( 3\pi/2, \) where \( A = |\Lambda(\tau_{\text{max}})|, B = |\Lambda(\tau_{\text{min}})| \) and \( A > B. \) Obviously, \( N_n = \max\{N_{n1}^\alpha, N_{n2}^\alpha\} = N_{n2}^\alpha = A - B. \) Because the pair of initial states corresponding to \( \theta = \pi/2 \) or \( 3\pi/2 \) makes the increase of trace distance \( N_n^\alpha \) maximal for any \( n, \) it also makes the summation in Eq. (4) maximal. The condition \( \theta = \pi/2 \) or \( 3\pi/2 \) means \( a - \mu = 0 \) and \( |\beta - \nu| = 1, \) and similar to the discussion in subsection A the condition together with \( |\beta|^2 \leq \alpha(1 - \alpha) \) and \( |\nu|^2 \leq \mu(1 - \mu) \) is also equivalent to \( a = \mu = 1/2, |\beta| = |\nu| = 1/2, |\beta - \nu| = 1. \) The trace distance after choosing the two initial states can be given as

\[
D = |\Lambda(t)|. \tag{31}
\]
It is noted that there are some similarity between the non-Markovian depolarizing channel and the amplitude damping channel. In the non-Markovian depolarizing channel the maximal trace distance is a function of the decoherence factor $|\Lambda(t)|$, while in the amplitude damping channel it is a function of the amplitude damping factor $|h(t)|$. It is very interesting that the trace distance of the non-Markovian depolarizing channel for the specific pair of initial states which make the optimization, is very similar to that of the amplitude damping channel in the resonant case, that is for the former the trace distance for the specific pair of initial states is equal to the decoherence factor $|\Lambda(t)|$, while for the latter it is equal to the amplitude damping factor $|h(t)|$; furthermore, in both cases the two initial states which makes the optimization are the same.

We can obtain the rate of the change of the trace distance using Eq. (3)

$$\sigma(t, \rho_{1,2}(0)) = \frac{\Lambda(t) \frac{d}{dt} \Lambda(t)}{|\Lambda(t)|}. \quad (32)$$

Having these preparations, we can easily calculate the non-Markovianity $\mathcal{N}$ from Eq. (2). We plot $\mathcal{N}$ as a function of $a$ in Fig. 3. From Fig. 3 we can see that there is a threshold $a_c = \lambda/4$. When $a < a_c$, $\mathcal{N} = 0$, which means that the process is Markovian. While when $a > a_c$, $\mathcal{N}$ increases with $a$, which means that the process is non-Markovian. This can be easily understood: because $a$ represents the coupling strength of the system and the reservoir, it is obvious that the non-Markovianity $\mathcal{N}$ will become more and more strong with the increasing of $a$ in the non-Markovian regime. It is worth noting that recently Mazzola et al. [26] elucidated that the memory kernel master equation does not ensure the presence of non-Markovian behavior in the evolution of dynamics. For example, in Ref. [26] they have shown that the non-Markovian behavior does not appear in the memory kernel master equations Eqs.(4) and (10) of the same reference. $\xi_M(R, t)$ and $\xi_P(R, t)$ are two functions corresponding to two models Eq. (4) and Eq. (10) of Ref. [26] respectively, which play the central role in the dynamics of the system. It has been proved that $\xi_M(R, t)$ and $\xi_P(R, t)$ are positive, monotonically decreasing functions under the condition of positivity of the dynamical maps [27]. Thus the quantum processes corresponding to Eqs.(4) and (10) of Ref. [26] are Markovian. However, the model we used can also be described by the memory kernel master equation Eq. (24), but we clearly see that the function $\Lambda(t)$ which plays a central role in our model, isn’t a monotonous function with time but is a damped oscillating function in some parameter regimes, which can be seen from Eq. (26). And in this case the quantum process we consider is non-Markovian.

As we know that the definition of the non-Markovianity $\mathcal{N}$ needs an optimization over all pairs of initial states, and generally it is very hard to do this. In this paper we proposed a method which can simplify this optimization. The main idea is like this: First we find the pair of initial states which make the maximization of the difference between the local maximum and local minimum of the trace distance in arbitrary nth time interval, then we try to prove that this pair of initial states can also make the optimization of the summation over all pairs of initial states in calculating the non-Markovianity $\mathcal{N}$. Using this method we have analytically obtained the pair of initial states which make the optimization in Eq. (4) and the corresponding trace distance for the amplitude damping channel in both the resonant case and the large-detuning case, and unified the results of Breuer et al. [17] and Xu et al. [21]; and we also have analytically obtained the pair of initial states which make the optimization in Eq. (4) and the corresponding trace distance for the non-Markovian depolarizing channel. Generally speaking it can not always be proved that the pair of initial states which make the maximization of the nth difference between the local maximum and local minimum of the trace distance, also make the optimization of the summation in $\mathcal{N}$. But here we argue that the pair of initial states which make the increase of the trace distance maximal in one time interval, will also makes the optimization of the summation in $\mathcal{N}$. We can not prove this in this paper, and this needs further investigations. If we are strict enough, at least we can obtain a very tight lower bound for $\mathcal{N}$. For example, for the amplitude damping channel, we have found only two pairs of initial states depending on the system parameters which make the increase of the trace distance maximal in arbitrary single time interval. Then it is easy to calculate $\mathcal{N}$ for these two pairs of initial states, and the larger one is the tight lower bound of $\mathcal{N}$. In this paper we mainly focus on the single channel case and the generalization to the general multi-channel case may be a more challenging task. In one word we have simplified the problem of finding a pair of initial states which optimizing the summation of the increase of trace distance in many time interval into a problem of optimizing the increase of trace distance in

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3.png}
\caption{The non-Markovianity $\mathcal{N}$ as a function of $a$.}
\end{figure}
just one single time interval.

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