Quantum Current Algebra for the $AdS_5 \times S^5$ Superstring

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Abstract

The sigma model describing the dynamics of the superstring in the $AdS_5 \times S^5$ background can be constructed using the coset $PSU(2,2|4)/SO(4,1) \times SO(5)$. A basic set of operators in this two dimensional conformal field theory is composed by the left invariant currents. Since these currents are not (anti) holomorphic, their OPE's is not determined by symmetry principles and its computation should be performed perturbatively. Using the pure spinor sigma model for this background, we compute the one-loop correction to these OPE's. We also compute the OPE's of the left invariant currents with the energy momentum tensor at tree level and one loop.
1 Introduction

During the last few years a number of results were obtained in \( N = 4 \) super Yang-Mills using integrability techniques\(^5\) culminating in a general system of equations that predicts the anomalous dimension for all operators at any value of the coupling constant \([1]\). Despite all these results, the quantum properties of the dual string theory, namely strings in \( AdS_5 \times S^5 \), still remain elusive.

The classical integrability of the \( AdS_5 \times S^5 \) sigma model was established in the paper \([2]\) and later is was shown to hold also in the pure spinor description \([3]\). Using cohomological and algebraic renormalization techniques, Berkovits argued that the sigma model still has an infinite number of conserved charges when quantum effects are taken into account \([4]\).

Besides these general results, not much is known about the sigma model. The one loop conformal invariance was proved in \([5, 6]\) and the argument for all loop conformal invariance was presented in \([4]\). The one loop effective action was computed recently in \([7]\) where it was shown that the “level” of the CFT is not renormalized at one loop, which in turn means that the relation between the ’t Hooft coupling \( \lambda \) and the \( AdS \) radius does not change at one loop. Besides, it was also shown that, using the prescription given in \([4]\), the effective action does not get any correction at all (neither local or non-local). Regarding the integrability of the model, a detailed study of the transfer matrix of the worldsheet was done in \([8]\), where it was shown to be a well defined operator in quantum theory.

In this work we consider the one-loop correction of the OPE’s of the left invariant currents. This is one particularly interesting set of operators in the worldsheet. Since these currents are not gauge invariant they are not expected to be primary fields of the CFT, nevertheless they are invariant under global \( PSU(2,2|4) \) transformations and are used to construct integrated massless vertex operators \([9]\) and also appear in massive unintegrated vertex operators. Another complication is that these currents are not holomorphic even in the classical limit, so their OPE’s cannot be deduced from general arguments. Therefore, if one wants to compute spacetime observables using worldsheet techniques, a perturbative knowledge of these OPE’s is mandatory. Besides this practical application, the knowledge of this current algebra in the worldsheet may shed light into more general aspects of the theory, such as the apparent quantum integrability. The tree level OPE’s of these currents were computed in \([10]\) (see also \([8]\) and \([11]\)) while the algebra of the left and right currents for a principal chiral model have been computed in \([12]\) and \([13]\).

Surprisingly, most of the possible one-loop corrections vanish due to spacetime supersymmetry and the result obtained here corroborates with the effective action result obtained in \([7]\). Thus this serves as further evidence that the relation between the ’t Hooft coupling and the \( AdS \) radius is not renormalized.

\(^5\)The literature on this subject is very large, and we did not attempt to give a list of references.
We also compute the OPE’s of the left-invariant currents with the worldsheet energy momentum tensor. Although the currents are not primary fields, their tree level OPE with the energy momentum gives the expected result coming from gauge covariance. The results we found are compatible with general assumptions of CFT but they are not as simple as in the case of a chiral current algebra. Furthermore, at 1-loop we show that there is no correction to the tree level OPE for the bosonic currents. This is a surprising result since the left-invariant currents are not protected by any symmetry argument. On the other hand, the fermionic currents get anomalous dimension contributions. However, this is not inconsistent, the two types of fermionic currents get contributions that cancel when combined into a single operator, so the stress energy tensor still has zero anomalous dimension.

**Organization** The structure of this paper is as follows. In section two we review the pure spinor superstring formalism. The case of $AdS_5 \times S^5$ background is discussed in section 3. In section four the methods to compute the OPE’s is described. In section five we compute the one-loop contributions to the OPE’S. Section 6 contains the computation of the OPE’s between the left-invariant currents with the energy momentum tensor. In the section 7 we summarize and comment our results. The appendices contain some technical details which were omitted in the main text.

## 2 Pure Spinor Type II Superstring in Curved Backgrounds

In a curved background, the pure spinor sigma model action for the type II superstring is obtained by adding to the flat action the integrated vertex operator for supergravity massless states and then covariantizing with respect to ten dimensional $N = 2$ superreparameterization invariance. The result of this procedure is

$$S = \frac{1}{2\pi\alpha'} \int d^2z \left( \frac{1}{2} \partial Z^M \overline{\partial} Z^N (G_{NM} + B_{NM}) + d_\alpha \partial Z^M E^\alpha_M + \overline{d}_{\dot\alpha} \partial Z^M E^{\dot\alpha}_M + \lambda^\alpha \omega\partial Z^M \Omega_{M\alpha}^\beta + \overline{\lambda}^{\dot\alpha} \overline{\omega}_{\dot\beta} \partial Z^M \Omega_{M\dot\alpha}^{\dot\beta} \right) + S_{pure} + S_{FT},$$

(1)

where $E_M^A$ is the supervielbein and $Z^M$ are the curved superspace coordinates, $B_{NM}$ is the super two-form potential. $S_{pure}$ is the action for the pure spinor ghosts and is the same as in the flat space case. The pure spinor condition means that they satisfy $\lambda^\alpha \gamma^\beta \partial_\alpha \gamma^\beta = 0$ and $\overline{\lambda}^{\dot\alpha} \gamma^\beta \partial_{\dot\alpha} \gamma^\beta = 0$, where $c = 0, \ldots, 9$ is a tangent space bosonic index.

As was shown in [14], the gravitini and the dilatini fields are described by the lowest $\theta$-components of the superfields $C^\beta_{\alpha}$ and $\overline{C}^{\dot\beta}_{\dot\alpha}$, while the Ramond-Ramond field strengths
are in the superfield $P^{a\dot{b}}$. The dilaton is the theta independent part of the superfield $\Phi$ which defines the Fradkin-Tseytlin term

$$S_{FT} = \frac{1}{2\pi} \int d^2 z \: r \: \Phi,$$

(2)

where $r$ is the world-sheet curvature. Because of the pure spinor constraints, the superfields in (1) cannot be arbitrary. In fact, it is necessary that

$$\Omega_{\alpha}^\beta = \Omega_{\alpha}^{(s)} \delta_\alpha^\beta + \frac{1}{4} \Omega_{\alpha\beta}^{(\gamma^{ab})} \gamma^{\alpha\beta}, \quad \tilde{\Omega}_{\dot{\alpha}}^{\dot{\beta}} = \tilde{\Omega}_{\dot{\alpha}}^{(s)} \delta_{\dot{\alpha}}^{\dot{\beta}} + \frac{1}{4} \tilde{\Omega}_{\dot{\alpha}\dot{\beta}}^{(\gamma^{ab})} \gamma^{\dot{\alpha}\dot{\beta}},$$

(3)

$$C_\alpha^{\beta\gamma} = C_\alpha^{\hat{\gamma}} \delta_\alpha^\beta + \frac{1}{4} C_{\alpha\beta}^{\hat{\gamma}} \gamma^{\alpha\beta}, \quad \tilde{C}_\alpha^{\dot{\gamma}\dot{\delta}} = \tilde{C}_\alpha^{\hat{\gamma}} \delta_\alpha^\dot{\delta} + \frac{1}{4} \tilde{C}_{\alpha\dot{\delta}}^{\hat{\gamma}} \gamma^{\alpha\dot{\delta}},$$

$$S_{a\dot{a}}^{\beta\dot{\alpha}} = S_{a\dot{a}}^{\beta\dot{\alpha}} \delta_\alpha^\dot{\beta} \delta_{\dot{\alpha}}^\beta + \frac{1}{4} S_{a\dot{a}}^{\beta\dot{\alpha}} \delta_\alpha^\dot{\beta} \delta_{\dot{\alpha}}^\beta + \frac{1}{4} S_{a\dot{a}}^{\beta\dot{\alpha}} \delta_\alpha^\dot{\beta} \delta_{\dot{\alpha}}^\beta + \frac{1}{16} S_{a\dot{a}}^{\beta\dot{\alpha}} \delta_\alpha^\dot{\beta} \delta_{\dot{\alpha}}^\beta.$$

The engineering dimensions, i.e dimensions in units of space-time length, for the world-sheet fields in (1) are:

$$[X^m] = 1, \quad [\theta^\mu] = \frac{1}{2}, \quad [d_\alpha] = [\tilde{d}_{\dot{\alpha}}] = \frac{3}{2}, \quad [\lambda^\alpha \omega_\beta] = [\tilde{\lambda}^\alpha \tilde{\omega}_\beta] = 2.$$

(4)

3 Review of the Pure Spinor Superstring in $AdS_5 \times S^5$

As was shown for the first time in [15], the superstring in $AdS_5 \times S^5$ background can be described using some currents defined in the superalgebra $psu(2,2|4)$. Those currents, which are defined in a left-invariant way, are given by $J^A = (g^{-1}\partial g)^A = \partial Z^M E^A_M$, $\bar{J}^A = (g^{-1}\partial g)^A = \overline{\partial} Z^M E^A_M$ for $g$ an element in the coset supergroup $PSU(2,2|4)/SO(4,1) \times SO(5)$. The index $A$ denotes $(a, \alpha, \dot{\alpha},) \quad$ and $a = 0, \ldots 4$ for $AdS_5$, $a' = 5, \ldots 9$ for $S^5$, $\alpha = 1, \ldots 16$, $\dot{\alpha} = 1, \ldots 16$ and $a$ denotes both $a$ and $a'$.

Another way of obtaining the action for the superstring in the $AdS_5 \times S^5$ background is by replacing the values that the superfields of the action (1) take on that background, as shown in [16] and [17]. In the following we will review that procedure.

Using the supervielbein and the definition of the currents given above, one can check that the term which contains $G_{MN}$ can be written as

$$\frac{1}{2} \partial Z^M \overline{\partial} Z^N G_{MN} = \frac{1}{2} J^a \overline{\partial} J_{ab},$$

(5)

In $AdS_5 \times S^5$ the only non-zero component of $B_{MN}$ is $B_{a\dot{b}} = \frac{1}{2} (Ng_s)^{a\dot{b}} \sqrt{\alpha'} \delta_{a\dot{b}}$, where $\delta_{a\dot{b}} = (\gamma^{01234})^{a\dot{b}}$. Then the term containing $B_{MN}$ in the action will lead to

$$\frac{1}{2} \partial Z^M \overline{\partial} Z^N B_{MN} = \frac{1}{2} (\partial_{\dot{\beta}} \bar{J}^a B_{a\dot{b}} + \partial_\alpha J_{\dot{\beta}} B_{a\dot{b}}) = \frac{1}{4} \sqrt{\alpha'} (Ng_s)^a \dot{b} (\partial_{\dot{\beta}} \bar{J}^a + \partial_\alpha J_{\dot{\beta}}) \delta_{a\dot{b}}.$$

(6)
From the definitions of the currents $J^A$, the terms containing explicitly $E_M^\alpha$ and $E_M^{\hat{\alpha}}$ in (1) will give

$$d_\alpha \tilde{\Omega}^M E_M^\alpha = d_\alpha \tilde{J}^\alpha, \quad \tilde{d}_\alpha \partial^M E_M^{\hat{\alpha}} = \tilde{d}_\alpha J^{\hat{\alpha}}. \quad (7)$$

By computing the flux of the five-form Ramond-Ramond field-strength one finds that

$$P^{\alpha\hat{\beta}} = \frac{\delta^{\alpha\hat{\beta}}}{\sqrt{\alpha' (N_{gs})}}, \quad (8)$$

where $\delta^{\alpha\hat{\beta}} = (\gamma^{01234})^{\alpha\hat{\beta}}$ and actually (8) sets the value for $B_{\alpha\hat{\beta}}$ written above, as can be proven by using the field-strength $H = dB$ and the constraints of [14] . The values of the Superfields $C^\beta_\alpha$ and $\tilde{C}^\beta_\alpha\gamma$ are zero in the $AdS_5 \times S^5$, as well as $\Omega^{(a)}_M$ and $\tilde{\Omega}^{(a)}_M$ because they are related to derivatives of the superfield containing the dilaton, which is constant for this background. Now, the terms containing the spin connections will lead to

$$\lambda^\alpha \omega_{\beta} \tilde{\Omega}^M \Omega_M^{\alpha\beta} = N_{ab} \tilde{J}^{ab}, \quad \tilde{\lambda}^\alpha \tilde{\omega}_\beta \partial^M \tilde{\Omega}_M^{\alpha\beta} = \tilde{N}_{ab} J^{ab}, \quad (9)$$

where $J^{ab} = \frac{1}{2} \Omega^M \tilde{\Omega}_{Mab}$, $\tilde{J}^{ab} = \frac{1}{2} \tilde{\Omega}^M \Omega_{Mab}$ and $N^{ab} = \frac{1}{2} (\lambda^\alpha \omega_{\beta}, \tilde{N}^{ab} = \frac{1}{2} (\tilde{\lambda}^\alpha \tilde{\omega}_\beta)$ are the pure spinors Lorentz currents. Finally, the term containing $C^\beta_\alpha$ is related to the space-time curvature as shown in [14], which is constant for the $AdS_5 \times S^5$ space. More specifically,

$$R_{abcd} = - \frac{1}{R^2} \eta_{[a} \eta_{d]} \quad R_{a'b'c'd'} = \frac{1}{R^2} \eta_{a'[c} \eta_{d']b]}, \quad (10)$$

where $R$ is the radius of $AdS_5$ and $S^5$ and the minus sign in the first equation is because $AdS_5$ has negative curvature. Thus, replacing the values of the background fields given above, the following action is found

$$S = \frac{1}{2 \pi \alpha'} \int d^5z \frac{1}{2} J^{\alpha} \tilde{J}^{\hat{\alpha}} \eta_{ab} + \frac{1}{4} (N_{gs}) \sqrt{\alpha'} \delta^{\alpha\hat{\beta}} (J^\alpha \tilde{J}^{\hat{\beta}} + J^\hat{\beta} \tilde{J}^\alpha) + d_\alpha \tilde{J}^\alpha + \tilde{d}_\alpha J^{\hat{\alpha}}$$

$$+ \frac{1}{(N_{gs}) \sqrt{\alpha'}} d_\alpha \tilde{d}_\hat{\alpha} \delta^{\alpha\hat{\beta}} + N_{ab} \tilde{J}^{ab} + \tilde{N}_{ab} J^{ab} - \frac{1}{R^2} N_{ab} \tilde{N}^{ab} + \frac{1}{R^2} N_{a'b'} \tilde{N}^{a'b'} + S_\lambda + S_{\tilde{\lambda}}. \quad (11)$$

Note by now that the engineering dimensions are

$$[J^a] = [\tilde{J}^\alpha] = 1, \quad [J^\alpha] = [J^{\hat{\alpha}}] = [\tilde{J}^\alpha] = [\tilde{J}^{\hat{\alpha}}] = \frac{1}{2}, \quad (12)$$

$$[N_{ab}] = [\tilde{N}_{ab}] = 2, \quad [d_\alpha] = [\tilde{d}_\alpha] = \frac{3}{2}, \quad [J^{ab}] = [\tilde{J}^{ab}] = 0.$$ 

By defining $\alpha^{-1} = (N_{gs}) \frac{1}{\sqrt{\alpha'}}$, using the equations of motion for $d_\alpha$ and $\tilde{d}_\alpha$ and performing the scalings

$$(J_a, \tilde{J}_a^\alpha \rightarrow \alpha^{-1} (J_a, \tilde{J}_a^\alpha), \quad (J^\alpha, J^{\hat{\alpha}}, \tilde{J}^\alpha, \tilde{J}^{\hat{\alpha}}) \rightarrow 2 (\alpha') \frac{1}{\sqrt{\alpha'}} (J^\alpha, J^{\hat{\alpha}}, \tilde{J}^\alpha, \tilde{J}^{\hat{\alpha}}) \quad (13).$$
$$(N_{ab}, \tilde{N}_{ab}) \to \frac{1}{\alpha R}(N_{ab}, \tilde{N}_{ab}), \quad (J^{ab}, \tilde{J}^{ab}) \to \sqrt{\alpha'}(J^{ab}, \tilde{J}^{ab}),$$

we find the action

$$S = \frac{1}{2 \pi \alpha' \alpha^2} \int d^2 z \left( \frac{1}{2} J^{\alpha \beta} \tilde{\eta}_{\alpha \beta} + \delta_{\alpha \beta}(J^\alpha \tilde{J}^\beta - 3J^\alpha \tilde{J}^\beta) \right)$$

\[+ N_{ab} \tilde{J}^{ab} + \tilde{N}_{ab} J^{ab} - N_{ab} \tilde{N}^{ab} + N_{a'b'} \tilde{N}^{a'b'} + S_\Lambda + S_{\tilde{\Lambda}}, \tag{14}\]

which coincides with "usual" action for the superstring written in terms of the $psu(2,2|4)$ currents [16] [17]. Note also that in (14) all $J$'s, $\tilde{J}$'s, and pure spinor Lorentz currents has engineering dimension one. So, by choosing units in which $2\pi \alpha' = 1$ the action is given in terms of dimensionless worldsheet fields.

Because of their definition, $(J^A, \tilde{J}^A)$ satisfy the Maurer-Cartan identities $\partial \tilde{J}^A - \partial J^A + [J, \tilde{J}]^A = 0$, so by making a variation of the action and using those identities, we can find the equations of motion

$$\nabla J_2 = -[J_1, \tilde{J}_1] + \frac{1}{2}[N, \tilde{J}_2] - \frac{1}{2}[J_2, \tilde{N}], \tag{15}$$

$$\nabla J_2 = [J_3, \tilde{J}_3] - \frac{1}{2}[J_2, \tilde{N}] + \frac{1}{2}[N, \tilde{J}_2] \tag{16}$$

$$\nabla \tilde{J}_1 = \frac{1}{2}[N, \tilde{J}_1] - \frac{1}{2}[J_1, \tilde{N}], \tag{17}$$

$$\nabla J_1 = [J_2, \tilde{J}_3] + [J_3, \tilde{J}_2] + \frac{1}{2}[N, \tilde{J}_1] - \frac{1}{2}[J_1, \tilde{N}] \tag{18}$$

$$\nabla J_3 = \frac{1}{2}[N, \tilde{J}_3] - \frac{1}{2}[J_3, \tilde{N}], \tag{19}$$

$$\nabla \tilde{J}_3 = -[J_2, \tilde{J}_1] - [J_1, \tilde{J}_2] + \frac{1}{2}[N, \tilde{J}_3] - \frac{1}{2}[J_3, \tilde{N}], \tag{20}$$

where $\nabla = \partial + [J_0, \cdot]$ and $\nabla = \partial + [\tilde{J}_0, \cdot]$. We have supressed the index $A$ and introduced a sub-index 0, 1, 2, 3 for the currents. This notation stands for $J_0 = J^{ab} M_{ab}$, $J_1 = J^a Q_a$, $J_2 = J^a P_a$, $J_3 = J^a \tilde{Q}_a$ and similarly for the $\tilde{J}$ currents. This $Z_4$ grading for the superalgebra was noted in [5]. Note that we have written the currents in terms of the generators of $psu(2,2|4)$, whose structure constants different from zero are

$$f^e_{\alpha \beta} = 2 \delta^e_{\alpha \beta}, \quad f^e_{\alpha \beta} = 2 \delta^e_{\alpha \beta},$$

\[f^{[ef]}_{\alpha \beta} = (\gamma^{ef})_{\alpha} \gamma^e_\beta = -\gamma^{ef})_{\beta} \gamma^e_\alpha = f^{[ef]}_{\beta \alpha}, \quad f^{[e\beta]}_{\alpha} = -\gamma^{e}\gamma^f_{\alpha}, \quad f_{\beta \alpha}^{[e \beta]} = f_{\alpha \beta}^{[e \beta]} =$$

$$f^{[e \beta]}_{\alpha} = \frac{1}{2}(\gamma^{ef})_{\alpha \beta} \delta^{\beta \gamma}, \quad f^{[\gamma \delta \epsilon \sigma]}_{\alpha \beta} = \frac{1}{2}(\gamma^{ef})_{\alpha \beta} \delta^{\beta \gamma} \delta^{\epsilon \sigma},$$

$$f^{[\beta \alpha]}_{\epsilon \sigma} = \frac{1}{2}(\gamma^{ef})_{\alpha \beta} \delta^{\beta \gamma} \delta^{\epsilon \sigma}, \quad f^{[e \beta]}_{\gamma \delta \epsilon \sigma} = \frac{1}{2}(\gamma^{ef})_{\alpha \beta} \delta^{\beta \gamma} \delta^{\epsilon \sigma},$$

$$f^{[\beta \alpha]}_{\epsilon \sigma} = \frac{1}{2}(\gamma^{ef})_{\alpha \beta} \delta^{\beta \gamma} \delta^{\epsilon \sigma}, \quad f^{[\beta \alpha]}_{\epsilon \sigma} = \frac{1}{2}(\gamma^{ef})_{\alpha \beta} \delta^{\beta \gamma} \delta^{\epsilon \sigma}.$$

6
If we define the quantum field by $g$ that a coset parametrization without fixing $X$ component is just a local gauge transformation of currents $J$ formula to write significant change using another coset parametrization.

As it should. Although our results are not gauge invariant, they are gauge covariant, so we do not expect any significant change using another coset parametrization.

The pure spinors have also equations of motion, given by $\nabla N = \frac{1}{2}[N, \dot{N}]$ and $\nabla \dot{N} = \frac{1}{2}[N, \dot{N}]$.

4 OPE’S in momentum space and dimensional regularization

In this section it is described the kind of calculations we intend to do. We are going to calculate contributions to the expectation values $\langle J^\alpha(y) J^\beta(z) \rangle$, $\langle J^\alpha(y) J^\alpha(z) \rangle$, etc... perturbatively, including double contractions (one loop) with no contributions of classical fields. The traditional way to calculate this kind of expectation values is to perform a background field expansion as in [5], [6] and [18]. That is, we choose a classical background given by an element $g_0$ in the supergroup and parametrize the quantum fluctuations by $X$ as $g = g_0 e^{aX}$, where $a$ is the coupling constant defined in the last section. Then, the currents can be written as

$$J = g^{-1} \partial g = e^{-aX} J_0 e^{aX} + e^{-aX} \partial e^{aX},$$

$$\overline{J} = g^{-1} \partial g = e^{-aX} \overline{J}_0 e^{aX} + e^{-aX} \overline{\partial} e^{aX}.$$ 

The exponentials in (22) can be expanded, giving rise to expressions involving commutators, which can be evaluated by using the structure constants of the $psu(2,2|4)$ Lie superalgebra (21), that is,

$$J = J_0 + \alpha (\partial X + [J_0, X]) + \frac{\alpha^2}{2} (\partial X + [J_0, X]) + [J_0, X] + \frac{\alpha^3}{3!} (\partial X + [J_0, X]) + \ldots,$$

and similarly for $\overline{J}$. In the last expression $J_0$ denotes the classical part of $J$ and not the index of the $Z_4$ grading. That sub-index will be dropped out, so it will be understood that the currents which appears in this type of expansion are classical. In the appendix, the expansion of the terms in the action (14) is written up to cubic terms in the quantum

\[ J = J_0 + \alpha (\partial X + [J_0, X]) + \frac{\alpha^2}{2} (\partial X + [J_0, X]) + [J_0, X] + \frac{\alpha^3}{3!} (\partial X + [J_0, X]) + \ldots, \]

Note that we have made the choice $X = X_2 + X_1 + X_3$ for the parametrization of the coset. Here we have used the $SO(1,4) \times SO(5)$ gauge invariance to fix $X_0 = 0$. Suppose we do not use the gauge invariance to fix this component and use another parametrization $X' = X'_2 + X'_1 + X'_3 + X'_0$. We can use the Baker-Campbell-Hausdorff formula to write $e^{X'} = e^{X_2 + X_1 + X_3} e^{X_0}$ and find the field redefinitions from from $X'$ to $X_2 + X_1 + X_3$ and $X_0$. If we define the quantum field by $g = g_0 e^{X_2 + X_1 + X_3} e^{X_0}$ the expanded action will be independent of $X_0$, so this component is just a local gauge transformation of currents $J_i \rightarrow e^{-X_0} J_i e^{X_0}$ for $i = 1, 2$ and $3$. This implies that a coset parametrization without fixing $X_0$ to vanish is related to our choice by a gauge transformation, as it should. Although our results are not gauge invariant, they are gauge covariant, so we do not expect any significant change using another coset parametrization.
fields, since this is the relevant order for the one-loop computation of the current’s OPE’s. We will focus on the matter part of the OPE’s. In Section 5 they do not enter at all, since there is no diagram that mixes matter with ghosts. However, in Section 6, they do enter at tree and one loop level.

Replacing those expansions of the appendix in (14), one can identify the kinetic piece $S_p$ of the action

\[ S_p = \int d^2z \left( \frac{1}{2} \partial X^a \partial X^a_{\mu b} + 4 \delta_{ab} \partial X^a \partial X^b \right), \tag{24} \]

from which we obtain the propagators in coordinate space

\[ X^a(y)X^b(z) \rightarrow -\eta^{ab} \ln |y - z|^2, \quad X^{a'}(y)X^{b'}(z) \rightarrow -\delta^{a'b'} \ln |y - z|^2 \tag{25} \]

\[ X^\alpha(y)X^{\bar{\beta}}(z) \rightarrow -\frac{1}{4} \delta^{\alpha\bar{\beta}} \ln |y - z|^2. \]

The reminder terms of the background expansion will provide the vertices of the theory. It is then straightforward to write down coordinate-space expressions for the Feynman rules of the diagrams which will appear in the remaining of the paper, and calculate the contribution of each OPE, like the tree level calculations of [10]. However, things are different at one loop. There are divergences which produce ambiguities in the coordinate-space integrals. The basic techniques for dealing with such a problem, involving this kind of calculation, were developed a long time ago in [19], [20], [21], when it were used momentum space Feynman rules with a prescription for worldsheet dimensional regularization. Then the results could be written in coordinate space by using an inverse Fourier transformation.

The two dimensional prescription for dimensional regularization consists in keeping all the interactions in exactly two dimensions, but the kinetic terms, and hence the denominators of the propagators will be in $d=2-2\epsilon$ dimensions.

We are going to use the definition $d^2k = \frac{dk_xdk_y}{\pi}$. With this choice there is no $\pi$ dependence in the results and the Green function $G(y, z)$ is represented as

\[ G(y, z) = \int d^2k e^{ik(y-z)+ik(\bar{y}-\bar{z})} \frac{\delta^2(k+l)}{k^2}. \tag{26} \]

The momentum space propagators look like

\[ X^a(k)X^b(l) \rightarrow \eta^{ab} \frac{\delta^2(k+l)}{|k|^2}, \quad X^{\alpha}(k)X^{\bar{\beta}}(l) \rightarrow \frac{1}{4} \delta^{\alpha\bar{\beta}} \frac{\delta^2(k+l)}{|k|^2}. \tag{27} \]

To work out the corresponding expression for the OPE’s in momentum space we use the dimensional regularization prescription and include a factor $\Gamma(1-\epsilon)(4\pi)^{-\epsilon}(2\pi)^{2\epsilon}$ for each loop. This will remove the Euler constant (the G-scheme [22]). All the integrals we
need to compute in the momentum space come from the formula

\[
\int d^d p \frac{p^a p^b}{(|p|^2)^\alpha (|p-k|^2)^\beta} = \\
k^{a+1-\alpha-\beta}k^{b+1-\alpha-\beta} \left[ \frac{k^2}{\mu^2} \right]^{-\epsilon} \times \sum_{i=0}^{\alpha} \binom{a}{i} \frac{\Gamma(2-\alpha-\beta+b+i-\epsilon)}{\Gamma(2-2\epsilon-\alpha-\beta+i+b)} \times \frac{\Gamma(\alpha+\beta-1-i+\epsilon)}{\Gamma(1+\epsilon) \mu^{-2\epsilon}} \Gamma(1-\epsilon-\beta+i),
\]

(28)

where \(\mu\) is the usual mass parameter of the dimensional regularization and the measure \(d^d p\) is the standard d-dimensional measure divided by \(\pi\). Using this regularization, integrals like \(\int \frac{d^d k}{|k|^2}\) vanish due to the cancelation between ultraviolet and infrared divergences in two dimensions. In order to check whether infrared and ultraviolet divergences cancel separately, we should replace the propagator in each infrared diagram by [23, 24, 19]

\[
\frac{1}{|k|^2} \rightarrow \frac{1}{|k|^2} + \frac{1}{\eta} \delta^2(k)
\]

and by taking \(\epsilon = \eta\) we could subtract out all infrared divergences. Since we are not evaluating expectation values of conserved currents, the result may depend on \(\epsilon\) and this procedure is important. However, for the sake of simplicity we are not going to do this in this paper and we postpone this discussion to a future work.\(^7\)

Next, we need to calculate all diagrams in momentum space using (28) with the dimensional regularization prescription, and afterwards reexpress the results in coordinate space using the following:

\[
\frac{k}{\bar{k}} \leftrightarrow -\frac{1}{(y-z)^2}, \quad \frac{\bar{k}}{k} \leftrightarrow -\frac{1}{(\bar{y}-\bar{z})^2},
\]

\[
\frac{k}{\bar{k}} \epsilon k + \frac{\bar{k}}{k} \left( 1 - \log \frac{|k|^2}{\mu^2} \right) \leftrightarrow -\ln \frac{|y-z|^2}{(y-z)^2},
\]

\[
\frac{\bar{k}}{k} \epsilon k + \frac{k}{\bar{k}} \left( 1 - \log \frac{|k|^2}{\mu^2} \right) \leftrightarrow -\ln \frac{|y-z|^2}{(\bar{y}-\bar{z})^2}.
\]

5 OPE’S without classical part

First note that from the expansions in the appendix, collecting the terms with three quantum fields and no classical field we obtain

\[
S(X^3) = \frac{\alpha}{4} \int d^2 z [\partial X^\alpha \bar{\partial} X^\alpha X^\beta (\gamma_\alpha)_{\alpha\beta} - \partial X^\alpha \bar{\partial} X^\alpha X^\beta (\gamma_\alpha)_{\dot{\alpha}\dot{\beta}} - \bar{\partial} X^\dot{\alpha} \partial X^\dot{\alpha} X^\beta (\gamma_{\dot{\alpha}})_{\alpha\beta} + \bar{\partial} X^\dot{\alpha} \partial X^\dot{\alpha} X^\beta (\gamma_{\dot{\alpha}})_{\dot{\alpha}\dot{\beta}}]
\]

(31)

\(^7\)If we keep the parameters \(\epsilon\) and \(\eta\) as independent parameters the infrared divergences can be read from the \(\frac{1}{\epsilon} - \frac{1}{\eta}\) coefficients and the ultraviolet from \(\frac{1}{\epsilon}\) coefficients.
\begin{equation}
+2X^a \partial X^\alpha \overline{\partial} X^\beta (\gamma_a)_{\alpha\beta} - 2X^a \partial X^\alpha \overline{\partial} \bar{X}^\beta (\gamma_a)_{\alpha\beta}.
\end{equation}

Integrating by parts the first line we obtain
\begin{equation}
S(X^3) = \alpha \int d^2 z [X^a \partial X^\alpha \overline{\partial} X^\beta (\gamma_a)_{\alpha\beta} - X^a \partial X^\alpha \overline{\partial} \bar{X}^\beta (\gamma_a)_{\alpha\beta}].
\end{equation}

The last expression gives the vertices used in the computation detailed in the next subsection.

### 5.1 One-loop computations

We can use the expansions of the appendix to compute perturbatively the OPE’s of the various currents $J^A$ and $\bar{T}^A$. We will give in detail the computation of $\langle J^a(y)J^b(z) \rangle$ leaving the method clear and explaining how to get the rest of the results.

Restricting the expansion (23) to the case without classical currents, we can write

\begin{equation}
\langle J^a(y)J^b(z) \rangle = \alpha^2 \langle \partial X^a(y)\partial X^b(z) \rangle - \alpha^3 \langle \partial X^a(y)\partial X^\alpha X^\beta(z) \rangle \gamma^b_{\alpha\beta} - \alpha^3 \langle \partial X^a(y)\partial X^\alpha X^\beta(z) \rangle \gamma^b_{\alpha\beta} - \alpha^3 \langle \partial X^a(y)\partial X^\alpha X^\beta(z) \rangle \gamma^b_{\alpha\beta} \gamma^b_{\alpha\beta},
\end{equation}

With the first term in (33) we can form a one-loop diagram by using the two terms in the right hand side of (32), which will come from the expansion of the exponential of minus the action at second order. This one-loop diagram is shown below.

\begin{align*}
\text{Graph 1.}
\end{align*}

So, in momentum space, using the contractions (27) the first diagram gives

\begin{equation}
\alpha^2 \langle \partial X^a(y)\partial X^b(z) \rangle = -\frac{\alpha^4}{8} \int d^d p \frac{p \bar{p} (k - p)(\bar{k} - \bar{p})}{|p|^2 |k - p|^2}.
\end{equation}

\footnote{In all the diagrams crosses indicate vertices coming from the currents, double lines indicate background fields and single lines indicate quantum fields.}
The coefficient deserves an explanation. There is a $\frac{1}{2}$ coming from the expansion of $\exp -S$ at the second order in $S$, also there is symmetry factor of 2 from the different possibilities of contracting the bosonic indices. Another factor of two comes from the double product in (32) when computing $S(X^3)^2$. Finally, there is a $\frac{1}{12}$ from the fermionic propagator. It can be easily checked that $(\gamma_c)_{\alpha\beta}(\gamma_d)_{\hat{\alpha}\hat{\beta}}\delta^{\alpha\hat{\alpha}}\delta^{\beta\hat{\beta}} = 16 \eta_{cd}$. Therefore, using the results of the integrals summarized in the appendix, we obtain

$$\alpha^2 \langle \partial X^2(y) \partial X^2(z) \rangle = -2 \alpha^4 \eta^{ab} \frac{k}{|k|} + \frac{k}{|k|} \left( \frac{1}{e} + 1 - \ln \frac{|k|^2}{\mu^2} \right).$$

(36)

Now let’s consider the remaining terms in (33). Both the second and third terms in (33) can be represented by the diagram

\begin{center}
\includegraphics[width=1in]{graph2.png}
\end{center}

**Graph 2.**

(37)

It can be checked that this diagram cancels because the second term in (33) cancels with the third. The reason for this cancelation is as follows: to form the diagram the second term in (33) contracts with the second term in (32), while the third term in (33) contracts with the first term in (32). Since those terms in (32) have opposite signs then $-\alpha^3 \langle \partial X^2(y) \partial X^3(z) \rangle \gamma^{b}_{\alpha\beta}$ cancels with $-\alpha^3 \langle \partial X^2(y) \partial X^3(z) \rangle \gamma^{b}_{\hat{\alpha}\hat{\beta}}$. Using the same reasoning one can check that $-\alpha^3 \langle \partial X^3(y) \partial X^2(z) \rangle \gamma^{a}_{\alpha\beta}$ cancels with $-\alpha^3 \langle \partial X^3(y) \partial X^2(z) \rangle \gamma^{a}_{\hat{\alpha}\hat{\beta}}$. That means the following diagram also cancels

\begin{center}
\includegraphics[width=1in]{graph3.png}
\end{center}

**Graph 3.**

(38)
Finally the term \( \alpha^4 \langle \partial X^\alpha X^\beta(y) \partial X^\delta(z) \rangle \gamma_{\alpha \beta} \gamma_{\gamma \delta} \) in (33), which is represented by the diagram

![Diagram](Image)

Graph 4. \( \alpha^4 \langle \partial X^\alpha X^\beta(y) \partial X^\delta(z) \rangle \gamma_{\alpha \beta} \gamma_{\gamma \delta} \) gives

\[
\alpha^4 \langle \partial X^\alpha X^\beta(y) \partial X^\delta(z) \rangle \gamma_{\alpha \beta} \gamma_{\gamma \delta} = \frac{\alpha^4}{4\pi^2} \delta^\alpha_\beta \delta^\gamma_\delta \int d^d p \frac{p^2}{|p|^2 |k-p|^2} \frac{1}{2} - \frac{\alpha^4}{4\pi^2} \delta^\alpha_\beta \delta^\gamma_\delta \int d^d p \frac{p^2 (k-p)}{|k-p|^2} \frac{1}{2} \]

\[
= \alpha^4 \eta^\rho_\delta \left( \frac{k}{k} + \frac{k}{k} \left( \frac{1}{\epsilon} + 1 - \ln \frac{|k|^2}{\mu^2} \right) \right)
\]

and the last term in (33) gives the same result. Because of this fact, (36) cancels with two times the result in (40), or in other words, the first diagram cancels with the forth. Then, the one-loop correction to \( \langle J^\mu(y) J^\rho(z) \rangle \) without classical field vanishes.

Let’s consider now \( \langle J^\alpha(y) J^\beta(z) \rangle \) at one loop and also without classical currents contributions. Then

\[
\langle J^\alpha(y) J^\beta(z) \rangle = \alpha^2 \langle \partial X^\alpha(y) \partial X^\beta(z) \rangle + 1 \frac{\alpha^3}{4} \langle \partial X^\alpha(y) \partial X^\beta z^\gamma(z) \rangle \gamma_\gamma_\delta \delta_\delta \]

\[
+ \frac{1}{4} \alpha^3 \langle \partial X^\alpha(y) \partial X^\beta z^\gamma(z) \rangle \gamma_\gamma_\delta \delta_\delta - \frac{1}{4} \alpha^3 \langle \partial X^\beta(y) \partial X^\beta z^\gamma(z) \rangle \gamma_\gamma_\delta \delta_\delta - \frac{1}{4} \alpha^3 \langle \partial X^\gamma(y) \partial X^\beta z^\gamma(z) \rangle \gamma_\gamma_\delta \delta_\delta
\]

\[
= \frac{1}{16} \alpha^4 \langle \partial X^\alpha(y) \partial X^\beta z^\gamma(z) \rangle \gamma_\gamma_\delta \delta_\delta + \frac{1}{16} \alpha^4 \langle \partial X^\beta(y) \partial X^\beta z^\gamma(z) \rangle \gamma_\gamma_\delta \delta_\delta + \frac{1}{16} \alpha^4 \langle \partial X^\gamma(y) \partial X^\beta z^\gamma(z) \rangle \gamma_\gamma_\delta \delta_\delta
\]

The result will be analog in this case. The first term, represented by diagram 1 gives

\[
\alpha^2 \langle \partial X^\alpha(y) \partial X^\beta(z) \rangle = \frac{5}{16} \alpha^4 \delta^\alpha_\beta \frac{1}{k} \left( \frac{k}{k} + 1 - \ln \frac{|k|^2}{\mu^2} \right)
\]

(42)
while the second term cancels the third in (41), as well as the fourth cancels the fifth. Again, the second and third diagrams cancel independently. Also, in this case those cancellations are due to the sign difference in the two terms of (32). The last four terms in (41) are represented by the fourth diagram. The sixth term in (41) gives

\[ -\frac{1}{16} \alpha^4 \langle \partial X^\gamma \partial X^\delta \rangle \langle \gamma_{\alpha \beta} \rangle \delta_{\alpha \delta} = -\frac{5}{32} \alpha^4 \delta_{\alpha \beta} \frac{k}{\kappa (1 + \ln \frac{|k|^2}{\mu^2})}, \]  

which is also the result of the eighth term. Finally, the seventh term in (41) gives

\[ \frac{1}{16} \alpha^4 \langle \partial X^\gamma \partial X^\delta \rangle \langle \gamma_{\alpha \beta} \rangle \delta_{\alpha \delta} = -\frac{5}{32} \alpha^4 \delta_{\alpha \beta} \frac{k}{\kappa}, \]

which is the same result for the ninth term. Then, twice (43) plus twice (44) cancels with (42), or again, the first diagram cancels with the fourth.

Let’s now consider \( \langle J^{ab}(y)J^{cd}(z) \rangle \). For this case only diagram 4 contributes.

\[ \langle J^{ab}(y)J^{cd}(z) \rangle = \frac{\alpha^4}{4} \langle \partial X^\alpha X^\beta(y) \partial X^\alpha X^\beta(z) \rangle \langle \gamma_{\alpha \beta} \rangle \delta_{\alpha \beta} \delta_{\delta \alpha} \]

\[ -\frac{\alpha^4}{4} \langle \partial X^\alpha X^\beta(y) \partial X^\alpha X^\beta(z) \rangle \langle \gamma_{\alpha \beta} \rangle \delta_{\alpha \beta} \delta_{\delta \alpha} \]

\[ + \frac{\alpha^4}{4} \langle \partial X^\alpha X^\beta(y) \partial X^\alpha X^\beta(z) \rangle \langle \gamma_{\alpha \beta} \rangle \delta_{\alpha \beta} \delta_{\delta \alpha} \]

Each term can be computed either in momentum or coordinate space without ambiguities. The first term gives

\[ \langle J^{ab}(y)J^{cd}(z) \rangle = \frac{\alpha^4}{4} \langle \partial X^\alpha X^\beta(y) \partial X^\alpha X^\beta(z) \rangle \langle \gamma_{\alpha \beta} \rangle \delta_{\alpha \beta} \delta_{\delta \alpha} \]

The second gives

\[ -\frac{\alpha^4}{4} \langle \partial X^\alpha X^\beta(y) \partial X^\alpha X^\beta(z) \rangle \langle \gamma_{\alpha \beta} \rangle \delta_{\alpha \beta} \delta_{\delta \alpha} \]

The third gives the same result as the second and the fourth gives the same result as the first. Finally, the fifth term gives

\[ \frac{\alpha^4}{4} \langle \partial X^\alpha X^\beta(y) \partial X^\alpha X^\beta(z) \rangle \delta_{\alpha \beta} \delta_{\delta \alpha} \]

Thus

\[ \langle J^{ab}(y)J^{cd}(z) \rangle = -\frac{3}{8} \alpha^4 \frac{\delta_{\alpha \beta} \delta_{\delta \alpha}}{(y-z)^2} (1 + \ln |y - z|^2). \]

One can easily check, given the vertices of (32) that there is no way to form one loop diagrams without classical current contributions for \( \langle J^{ab}(y)J^{\beta}(z) \rangle, \langle J^{ab}(y)J^{\beta}(z) \rangle, \langle J^{ab}(y)J^{\beta}(z) \rangle, \langle J^{ab}(y)J^{\beta}(z) \rangle, \langle J^{ab}(y)J^{\beta}(z) \rangle, \langle J^{ab}(y)J^{\beta}(z) \rangle, \langle J^{ab}(y)J^{\beta}(z) \rangle, \langle J^{ab}(y)J^{\beta}(z) \rangle, \langle J^{ab}(y)J^{\beta}(z) \rangle, \langle J^{ab}(y)J^{\beta}(z) \rangle \).
Let’s compute now \( \langle J^a(y) \bar{J}^h(z) \rangle \)

\[
\langle J^a(y) \bar{J}^h(z) \rangle = \alpha^2 \langle \partial X^a(y) \bar{J}X^h(z) \rangle - \alpha^3 \langle \partial X^a(y) \bar{J} X^\alpha X^\beta(z) \rangle \gamma^{\beta}_{\alpha \beta} - \alpha^3 \langle \partial X^a(y) \bar{J} X^\delta X^\beta(z) \rangle \gamma^{\beta}_{\alpha \beta} \\
- \alpha^3 \langle \partial X^a X^\delta(y) \bar{J} X^h(z) \rangle \gamma^{\delta}_{\alpha \beta} - \alpha^3 \langle \partial X^a X^\beta(y) \bar{J} \bar{X}^h(z) \rangle \gamma^{\beta}_{\alpha \beta} + \alpha^4 \langle \partial X^a X^\beta(y) \bar{J} \bar{X}^\gamma X^\delta(z) \rangle \gamma^{\beta}_{\alpha \beta} \gamma^{\gamma}_{\alpha \beta},
\]

(50)

In this case the first term will give

\[
\alpha^2 \langle \partial X^a(y) \bar{J} X^h(z) \rangle = - \alpha^4 \eta^a_{\beta} \eta^b_{\alpha \beta} \gamma^{\beta}_{\alpha \beta} \gamma^{\delta}_{\alpha \beta} \gamma^{\beta}_{\alpha \beta} \gamma^{\gamma}_{\alpha \beta} \int d^4p \frac{p^2}{|p|^2} \frac{(k-p)(k-\bar{p})}{|k-p|^2}
\]

(51)

\[
- \delta^{\alpha \beta} \delta^{\alpha \beta} \int d^4p \frac{p^2}{|p|^2} \frac{(k-p)^2}{|k-p|^2}
\]

\[
- 2 \alpha^4 \eta^a_{\beta} \left[ 1 + \left( \frac{1}{\epsilon} + 1 - \ln \frac{|k|^2}{\mu^2} \right) \right].
\]

As in the case of \( \langle J^a(y) \bar{J}^h(z) \rangle \) the second term cancels with the third and the fourth with the fifth, i.e. the second and third diagrams cancel independently. Nevertheless, the sixth term gives

\[
\alpha^4 \langle \partial X^a X^\beta(y) \bar{J} \bar{X}^\gamma X^\delta(z) \rangle \gamma^{\beta}_{\alpha \beta} \gamma^{\gamma}_{\alpha \beta} \gamma^{\gamma}_{\alpha \beta} = - \alpha^4 \eta^a_{\beta} \left[ 1 + \left( \frac{1}{\epsilon} + 1 - \ln \frac{|k|^2}{\mu^2} \right) \right],
\]

(52)

and the seventh term in (50) gives the same result. So, differently from \( J^a(y) \bar{J}^h(z) \) where the first and fourth diagrams canceled, they add up for \( \langle J^a(y) \bar{J}^h(z) \rangle \), giving

\[
\langle J^a(y) \bar{J}^h(z) \rangle = - 4 \alpha^4 \eta^a_{\beta} \left[ 1 + \left( \frac{1}{\epsilon} + 1 - \ln \frac{|k|^2}{\mu^2} \right) \right],
\]

(53)

which in coordinate space is

\[
\langle J^a(y) \bar{J}^h(z) \rangle = 4 \alpha^4 \eta^a_{\beta} \left[ \delta^{(2)}(y, z) \ln |y - z|^2 - \frac{1}{|y - z|^2} \right],
\]

(54)

In a completely analog way \( \langle J^a(y) \bar{J}^h(z) \rangle \) gives

\[
\langle J^a(y) \bar{J}^h(z) \rangle = \frac{5}{4} \alpha^4 \delta^{(2)}(y, z) \ln |y - z|^2 - \frac{1}{|y - z|^2},
\]

(55)

and

\[
\langle J^{ah}(y) \bar{J}^{cd}(z) \rangle = \frac{3}{8} \alpha^4 \eta^a_{\beta} \eta^c_{\alpha \beta} \left[ \delta^{(2)}(y, z) \ln |y - z|^2 - \frac{1}{|y - z|^2} \right].
\]

(56)

Summarizing, the only non-vanishing one-loop results are (49), (54), (55) and (56), which are consistent with the results found in [7].
6 OPEs of the Energy momentum tensor with the currents

The energy momentum tensor is

\[ T = -\frac{1}{\alpha^2} \left( \frac{1}{2} J^a J^b \eta_{ab} - 4 \delta_{\alpha\beta} J^\alpha J^\beta J^\alpha + 2 N_{ab} J^a J^b + 2 \omega_4 \partial \lambda^\alpha, \right) \] (57)

6.1 Tree level

In this subsection we will compute \( T(y)J^A(z) \) at tree level. Let’s start with \( J^a \). The result is

\[
\langle T(y)J^a(z) \rangle = \frac{J^a(z)}{(y-z)^2} + \frac{1}{y-z} \left( \partial J^a(z) + [J_0, J_2]^a_2(z) - \frac{1}{2}[N, J_2]^a_2(z) \right) (58)
\]

\[
+ \frac{(y-z)}{(y-z)^2} \left( \partial J^a(z) + [J_0, J_2]^a_2(z) - [J_3, J^a_2](z) - \frac{1}{2}[N, J_2]^a_2(z) - \frac{1}{2}[N, J_2]^a_2(z) \right)
\]

Note that the second line of the equations above vanishes by the use of the classical equations of motion, so there is no inconsistency from the fact that \( \partial T = 0 \). We will now explain how to arrive to this result. From the first term in the energy momentum tensor we obtain

\[-\frac{1}{2\alpha^2} \langle J^b \mathcal{J} \rangle \eta_{ab}(y) J^a(z) \rangle = -\frac{1}{2} \langle \partial X^2 \eta_{ab}(y) \partial X^a(z) \rangle - \frac{1}{2} \langle [J, X] \eta_{ab}(y) \partial X^a(z) \rangle - \frac{1}{2} \langle \partial X \eta_{ab}(y) [J, X]^a_2(z) \rangle \]

(59)

Contracting using the propagator in the first term of the right hand side we obtain the double pole, as well as the terms with \( \partial J^a \) and \( \mathcal{J} J^a \) in (58). Now, the expansion of the action contains terms of the form \( \partial X^2 \mathcal{J} \partial X^a \eta_{ab} \) and \( \mathcal{J} \partial X^2 \mathcal{J} \partial X^a \eta_{ab} \). Specifically, those terms come from the expansion of \( \eta_{ab} J^a J^b \). Those terms can contribute at tree level when contracting with the first term in (59). The first gives the \( [J_0, J_2]^a_2 \) in (58), while the second gives a \(-[J_0, J_2]^a_2 \) which exactly cancels with the second term in (59). The third term in (59) gives the \([J_0, J_2]^a_2 \) which appears in (58).

From the second term in the energy momentum tensor we obtain

\[
\frac{4}{\alpha^2} \delta_{\alpha\beta} \langle J^\alpha \mathcal{J} \rangle \langle J^\beta \rangle = 4 \delta_{\alpha\beta} \langle \partial X^\beta (y) \partial X^a(z) \rangle - 4 \delta_{\alpha\beta} \langle \partial X^\alpha (y) \partial X^a(z) \rangle + (60)
\]

\[
4 \delta_{\alpha\beta} \langle [J, X] \rangle \langle \partial X^\alpha (y) \partial X^a(z) \rangle - 4 \delta_{\alpha\beta} \langle [J, X] \rangle \langle \partial X^\alpha (y) \partial X^a(z) \rangle + 4 \delta_{\alpha\beta} \langle \partial X^\beta (y) [J, X]^a_2(z) \rangle
\]

Expanding \( \delta_{\alpha\beta} J^\alpha \mathcal{J} \) and \(-3 \delta_{\alpha\beta} J^\beta \mathcal{J} \) in the action (14) we can form tree level diagrams with the first term in (60) whose result vanishes. Nevertheless, the tree level diagrams formed with those expansions and the second term in (60) gives the \([J_3, \mathcal{J}_3] \) in (58). The
remaining terms in (60) vanish because they give contributions of the form $J^\alpha J^\beta \gamma^\mu_{\alpha\beta}$ or $J^\alpha J^\beta \gamma^\mu_{\alpha\beta}$.

From the third term in the energy momentum tensor we can easily obtain $-[N, J]^{2}(y - z)^{-1}$, while using the first term in (59) and the expansion of $\mathcal{T}^{ab} N_{ab}$ in the action (14) we obtain $\frac{1}{2}[N, J]^{2}(y - z)^{-1}$, giving at the end the term $[N, J]^{2}$ in (58). Similarly, the first term in (59) contracted with the expansion of $\hat{N}_{ab} J^{ab}$ gives the $[\hat{N}, J]^{2}$ in (58). Finally, the last term in the energy momentum tensor contracted with $J^{ab}$ will give a tree level contribution by forming tree-level diagram contracting with $N^{(1)}_{ab} \mathcal{T}^{X} X^{a}$, which comes from the expansion of $N_{ab} \mathcal{T}^{ab}$. This contribution will be the $[N, J]^{2}$ in (58). Note that using the classical equations of motion, the second line in (58) vanishes. Then, classically $J^{ab}$ is not a primary field.

Similarly, we obtain

$$\langle T(y)J^{a}(z) \rangle = \frac{J^{a}(z)}{(y - z)^{2}} + \frac{1}{y - z} \left( \partial J^{a}(z) + [J_{0}, J_{1}]^{a}(z) - \frac{1}{2} [N, J_{1}]^{a}(z) \right)$$

and an analog expression for $\langle T(y)\bar{J}^{a}(z) \rangle$. Nevertheless, it can be easily checked that at tree level, $T(y)\bar{J}^{(ab)}$ is regular.

It is also interesting to know $\langle T(y)\mathcal{T}^{a}(z) \rangle$. Following the same computation described in detail for $\langle T(y)J^{a} \rangle$, we found

$$\langle T(y)\mathcal{T}^{a}(z) \rangle = -J^{a}(y)\delta^{(2)}(y - z) - \frac{1}{2} \frac{[N, J_{1}]^{a}(z)}{y - z} + \frac{1}{2} \frac{[N, J_{2}]^{a}(z)}{y - z} + \frac{1}{2} \frac{[\hat{N}, J_{2}]^{a}(z)}{y - z} ,$$

$$\langle T(y)\bar{J}^{a}(z) \rangle = -J^{a}(y)\delta^{(2)}(y - z) + \frac{1}{2} \frac{[N, J_{1}]^{a}(z)}{y - z} + \frac{1}{2} \frac{[N, J_{2}]^{a}(z)}{y - z} - \frac{1}{2} \frac{[\hat{N}, J_{2}]^{a}(z)}{y - z} ,$$

$$\langle T(y)\mathcal{T}^{ab}(z) \rangle = \frac{[J_{2}, J_{1}]^{ab}(z)}{y - z} - \frac{[J_{1}, J_{3}]^{ab}(z)}{y - z} - \frac{[J_{2}, J_{1}]^{ab}(z)}{y - z} ,$$

Note that these results are not inconsistent with $\partial T = 0$, since, as usual, this derivative only gives contact terms in the right-hand side.

### 6.2 One-loop

Next, we calculate the OPE’s between the energy momentum tensor and the current $J^{a}$ at one loop. We are going to show that there is no contribution to this OPE. To this aim, we need to go up to one classical field in the action and current expansions. In particular, we need to evaluate terms with one classical field and three quantum fields in the action.

We are not going to show the details like in the last subsection and we just list the contribution of each diagram directly in coordinate space.
The unique contribution to \( \langle T(y) J(z) \rangle \) OPE come from \(-\frac{1}{2\alpha^2} \langle \eta_{ab} J^a(y) J^b(z) \rangle\) and \(\frac{4}{\alpha^2} \langle \delta_{\alpha\beta} J^\alpha J^\beta(z) \rangle\). It will be shown now that these OPE's cancel separately. Let us start with the first one. Expanding \( J^a(y) J^a(y) \) and \( J^c(z) \) up to one classical current, the expectation values we need to calculate come out as follow:

\[
-\frac{1}{2\alpha^2} \langle J^a(y) J^a(y) J^c(z) \rangle = -\eta_{ab} \langle J^b(y) \langle \partial X^b(y) \partial X^c(z) \rangle \rangle - \alpha \eta_{ab} \gamma_{\hat{a}\hat{b}} \langle J^b(y) \langle \partial X^\hat{a} \partial X^\hat{b}(y) X^c X^\hat{c}(z) \rangle \rangle \\
- \frac{\alpha^2}{4} \eta_{ab} f_{c(gh)} J^a(y) \langle \partial X^a \partial X^c \langle \partial X^{g}(y) X^h X^\hat{a}(z) \rangle \rangle \\
- \frac{\alpha}{2} \eta_{ab} \langle \partial X^a \partial X^b(y) \partial X^c(z) \rangle + \frac{\alpha^2}{4} \eta_{ab} \gamma_{\hat{a}\hat{b}} \langle \partial X^a \partial X^\hat{a}(y) \partial X^\hat{a} X^\hat{b}(z) \rangle \\
+ \alpha \eta_{ab} \gamma_{\hat{a}\hat{b}} \langle J^b \langle \partial X^a(y) \partial X^\hat{a} X^\hat{a}(z) \rangle \rangle, 
\]

where we are using the notation: \( \alpha \rightarrow (\alpha, \hat{\alpha})\). For the sake of simplicity we don’t write explicity the structure constants \( f_{dabc} \) and \( f_{d(ab)} \). The first term is given by diagram five

\[\text{Graph 5.}\] (66)

The result is

\[-\alpha^2 \eta_{ab} J^a(y) \langle \partial X^b(y) \partial X^c \rangle = -\frac{2\alpha^2 J^b(y)}{(y - z)^2} \left( 1 + \ln \frac{1}{|y - z|^2} \right)\] (67)

The next term is computed by evaluating diagram six

\[\text{Graph 6.}\] (68)
and the result is zero. The contribution for the third term comes from diagram seven.

Graph 7.

The result is

\[ -\frac{\alpha^4}{4} \eta_{ab} f^c_{(gh)} J^d(z) \left\langle \partial X^a(y) \partial X^b(y) X^e X^f(z) \right\rangle = -\alpha^2 \frac{f^c_{(gh)} f^{[ab]} J^e(z)}{2(y - z)^2} \]  

The next two terms could be calculated by evaluating diagrams eight and nine, but there are no possible contractions and they do not contribute.

Graph 8.

Graph 9.
The contribution for the sixth term comes from diagram ten and gives

\[ + \alpha^2 \eta_{ab} J^a \left\langle \gamma_{\alpha\beta} \partial X_\alpha^\beta(y) \gamma_{\alpha\beta} \partial X_\alpha^\beta(z) \right\rangle = \frac{2\alpha^2 J^a(y)}{(y-z)^2} (1 + \ln |y-z|). \]  (74)

The seventh term is calculated from diagram eleven and the result is zero.

Finally, the fourth term also contributes to diagram twelve

\[ - \frac{\alpha}{2} \eta_{ab} \left\langle \partial X^\alpha(y) \partial X^\beta(y) \partial X^\gamma(z) \right\rangle = \alpha^2 \frac{f_{[ab]}^\alpha f_{[ab]}^\beta J^\omega(z)}{2(y-z)^2}. \]  (77)
So, we conclude that $-\frac{1}{2\alpha^2} \langle \eta_{ab} J^a J^b(y) J^c(z) \rangle = 0$.

Now we will show that $\frac{4}{\alpha^2} \langle \delta^{\alpha \beta} J^\beta J^\alpha(y) J^c(z) \rangle$ is also zero. Again we need to expand the currents up to one classical field and calculate each expectation value. As the relevant diagrams are the same, we are not going to put the results for each expectation value and we will use the notation $I_n$ for the $n$-th diagram, and just list the result of the diagrams that contribute, as follows

\begin{align*}
I_6 &= 0 \\
I_7 &= -\frac{4\alpha^2 J^c(z)}{(y - z)^2} \\
I_8 &= \frac{2\alpha^2 J^c(z) \ln |y - z|^2}{(y - z)^2} + \frac{3\alpha^2 J^c(z)}{(y - z)^2} \\
I_9 &= \frac{2\alpha^2 J^c(z)}{(y - z)^2} (2 + \ln |y - z|^2) \\
I_{10} &= -\frac{4\alpha^2 J^c(y)}{(y - z)^2} (1 + \ln |y - z|^2) \\
I_{12} &= \frac{\alpha^2 J^c(z)}{(y - z)^2}
\end{align*}

After evaluating the background fields at the point $(z, \bar{z})$, the sum of the diagrams is null. The derivative terms of the $J^c$ don't appear in the results because they can be written as bilinear terms in the classical fields due to the equations of motion, and they will not enter in this one classical field calculation. Therefore, one can see that the result of the one loop calculation is

$$<T(y) J^c(z)> = 0$$

(79)

Now, for the currents $J_1$ and $J_3$ the results are different. The one-loop results for $<-\frac{1}{2\alpha^2} \eta_{ab} J^a J^b(y) J^c(z) >$ are

\begin{align*}
I_6 &= 5 \frac{\alpha^2 J^c(z)}{4(y - z)^2} [1 + \ln|y - z|^2], \\
I_7 &= 5 \frac{\alpha^2 J^c(z)}{4(y - z)^2}, \\
I_8 &= -5 \frac{\alpha^2 J^c(z)}{4(y - z)^2} [\frac{3}{2} + \ln |y - z|^2], \\
I_9 &= -5 \frac{\alpha^2 J^c(z)}{4(y - z)^2} [2 + \ln |y - z|^2], \\
I_{10} &= 5 \frac{\alpha^2 J^c(z)}{4(y - z)^2} [1 + \ln |y - z|^2], \\
I_{12} &= \frac{5\alpha^2 J^c(z)}{8(y - z)^2},
\end{align*}

(80)
then in one-loop order \((-\frac{1}{2}\eta_{ab}J^a J^b(y) J^\gamma(z))\) vanishes. Nevertheless, computing \((-4\delta_{\alpha\beta}J^\alpha J^\hat{\beta}(y) J^\gamma(z))\) we found the following results for each diagram

\[
\begin{align*}
I_5 &= \frac{5}{4} \alpha^2 \frac{J^\gamma(z)}{(y-z)^2} [1 + \ln |y-z|^2], \\
I_6 &= 0, \\
I_7 &= \frac{5}{4} \alpha^2 \frac{J^\gamma(z)}{(y-z)^2} , \\
I_8 &= -\frac{5}{4} \alpha^2 \frac{J^\gamma(z)}{(y-z)^2} \left[\frac{3}{2} + \ln |y-z|^2\right], \\
I_9 &= -\frac{5}{4} \alpha^2 \frac{J^\gamma(z)}{(y-z)^2} \left[2 + \ln |y-z|^2\right], \\
I_{10} &= 0, \\
I_{11} &= \frac{5}{4} \alpha^2 \frac{J^\gamma(z)}{(y-z)^2} [1 + \ln |y-z|^2], \\
I_{12} &= \frac{5}{16} \alpha^2 \frac{J^\gamma(z)}{(y-z)^2},
\end{align*}
\]

(81)

then \(T(y)J_1(z)\) does not cancel and indeed gives

\[
\langle T(y)J^\gamma(z) \rangle = \frac{5}{16} \alpha^2 \frac{J^\gamma(z)}{(y-z)^2}.
\]

(82)

Something similar happens for \(T(y)J_3(z)\). Computing \(-\frac{1}{2}\eta_{ab}(J^a J^b)(y) J^\hat{\gamma}(z))\) we found

\[
\begin{align*}
I_6 &= -\frac{5}{4} \alpha^2 \frac{J^\hat{\gamma}(z)}{(y-z)^2} [1 + \ln |y-z|^2], \\
I_7 &= \frac{5}{4} \alpha^2 \frac{J^\hat{\gamma}(z)}{(y-z)^2} , \\
I_8 &= 0, \\
I_9 &= 0, \\
I_{10} &= \frac{5}{4} \alpha^2 \frac{J^\hat{\gamma}(z)}{(y-z)^2} [1 + \ln |y-z|^2], \\
I_{12} &= -\frac{5}{4} \alpha^2 \frac{J^\gamma(z)}{(y-z)^2},
\end{align*}
\]

(83)

so, \((-\frac{1}{2}\eta_{ab}J^a J^b(y) J^\hat{\gamma}(z))\) cancels. Nevertheless the diagram results for \((-4\delta_{\alpha\beta}J^\alpha J^\hat{\beta}(y) J^\gamma(z))\)
\[ I_5 = \frac{5}{4} \alpha^2 \frac{J^5(z)}{(y-z)^2} \left[ 1 + \ln|y-z|^2 \right], \]
\[ I_6 = 0, \]
\[ I_7 = \frac{5}{4} \alpha^2 \frac{J^5(z)}{(y-z)^2}, \]
\[ I_8 = 0, \]
\[ I_9 = 0, \]
\[ I_{10} = 0, \]
\[ I_{11} = -\frac{5}{4} \alpha^2 \frac{J^5(z)}{(y-z)^2} \left[ 1 + \ln|y-z|^2 \right], \]
\[ I_{12} = -\frac{5}{16} \alpha^2 \frac{J^5(z)}{(y-z)^2}, \]
\[ (84) \]

So,
\[ \langle T(y) J^5(z) \rangle = \frac{5}{16} \alpha^2 \frac{J^5(z)}{(y-z)^2}. \]
\[ (85) \]

Finally, let’s consider \( \langle T(y) J^{cd}(z) \rangle \). Computing \( \langle -\frac{1}{2\alpha^2} \eta_{\alpha\beta} J^\alpha J^\beta (y) J^{[cd]} (z) \rangle \) we found
\[ I_7 = \alpha^2 \frac{J^{[cd]}(z)}{(y-z)^2}, \]
\[ I_9 = -\alpha^2 \frac{J^{[cd]}(z)}{(y-z)^2} \left[ 2 + \ln|y-z|^2 \right], \]
\[ I_{10} = \alpha^2 \frac{J^{[cd]}(z)}{(y-z)^2} \left[ 1 + \ln|y-z|^2 \right], \]
\[ (86) \]

and the same result with opposite sign for \( \langle -\frac{1}{2\alpha^2} \eta_{\alpha\beta} J^\alpha J^\beta (y) J^{[cd]} (z) \rangle \) so in one loop order \( \langle -\frac{1}{2\alpha^2} \eta_{\alpha\beta} J^\alpha J^\beta (y) J^{[cd]} (z) \rangle \) cancels.

Similarly, computing \( \langle -\frac{4}{\alpha^2} \delta_{\alpha\beta} J^\alpha J^\beta (y) J^{[cd]} (z) \rangle \) we found
\[ I_7 = 4\alpha^2 \frac{J^{[cd]}(z)}{(y-z)^2}, \]
\[ I_9 = -4\alpha^2 \frac{J^{[cd]}(z)}{(y-z)^2} \left[ 2 + \ln|y-z|^2 \right], \]
\[ I_{10} = 4\alpha^2 \frac{J^{[cd]}(z)}{(y-z)^2} \left[ 1 + \ln|y-z|^2 \right], \]
\[ (87) \]

and the same results with opposite sign for \( \langle -\frac{4}{\alpha^2} \delta_{\alpha\beta} J^\alpha J^\beta (y) J^{[cd]} (z) \rangle \). Then \( \langle -\frac{4}{\alpha^2} \delta_{\alpha\beta} J^\alpha J^\beta (y) J^{[cd]} (z) \rangle \) cancels at one loop order. Considering the term \( N_{\alpha\beta} J^{\alpha\beta} \) in the energy momentum tensor,
we find that the diagram 10 contributes

\[ I_{10} = \frac{3}{4} \alpha^2 \frac{N_{cd}}{(y - z)^2} [1 + \ln|y - z|^2], \]

and this result is directly related to (49). This last result is canceled by computing the one loop contribution coming from the contraction of the last term in the energy momentum tensor (57) with the term \( N_{ab}^{(1)} \) coming from the expansion of the action. In conclusion, the one loop contribution for \( \langle T(y) J_{cd}(z) \rangle \) cancels.

7 Summary of Results

In this work we showed that at one loop, there are non trivial cancelations in the possible corrections to the double pole of the product of the currents \( J^a(y) J^b(z) \) and \( J^\alpha(y) J^\beta(z) \). These results are in agreement with [7]. On the other hand, we found the following one loop corrections to the double pole corrections

\[ \langle J^{ab}(y) J_{cd}(z) \rangle = -\frac{3}{8} \alpha^4 \eta^{ab} \frac{N_{cd}}{(y - z)^2} (1 + \ln|y - z|^2), \]

(89)

\[ \langle J^a(y) J^b(z) \rangle = -4 \alpha^4 \eta^{ab} \left[ \frac{1}{|y - z|^2} - \delta^{(2)}(y, z) \ln|y - z|^2 \right], \]

(90)

\[ \langle J^\alpha(y) J^\beta(z) \rangle = -\frac{5}{4} \alpha^4 \delta^{\alpha\beta} \left[ \frac{1}{|y - z|^2} - \delta^{(2)}(y, z) \ln|y - z|^2 \right], \]

(91)

and

\[ \langle J^{ab}(y) J^{cd}(z) \rangle = -\frac{3}{8} \alpha^4 \eta^{ab} \eta^{cd} \left[ \frac{1}{|y - z|^2} - \delta^{(2)}(y, z) \ln|y - z|^2 \right]. \]

(92)

We also found that there is no way to form one loop diagrams without classical current contributions, i.e double pole corrections for \( \langle J^a(y) J^b(z) \rangle, \langle J^a(y) J^b(z) \rangle, \langle J^\alpha(y) J^\beta(z) \rangle, \langle J^\alpha(y) J^\beta(z) \rangle, \langle J^\alpha(y) J^{\beta c}(z) \rangle, \langle J^\alpha(y) J^{\beta c}(z) \rangle, \langle J^\alpha(y) J^{\beta c}(z) \rangle, \langle J^\alpha(y) J^{\beta c}(z) \rangle \).

About the product of the energy momentum tensor with the currents we found the following results on-shell

\[ \langle T(y) J^a(z) \rangle = \frac{J^a(z)}{(y - z)^2} + \frac{1}{y - z} \left( \partial J^a(z) + [J_0, J^a(z)] - \frac{1}{2} [N, J^a(z)] \right), \]

(93)

where we found a non trivial cancellation in the possible one-loop contribution to the double pole. On the other hand, for the fermionic currents we found

\[ \langle T(y) J^\alpha(z) \rangle = (1 - \frac{5}{16} \alpha^2) \frac{J^\alpha(z)}{(y - z)^2} + \frac{1}{y - z} \left( \partial J^\alpha(z) + [J_0, J^\alpha(z)] - \frac{1}{2} [N, J^\alpha(z)] \right), \]

(94)

\[ \langle T(y) J^{\bar{\alpha}}(z) \rangle = (1 + \frac{5}{16} \alpha^2) \frac{J^{\bar{\alpha}}(z)}{(y - z)^2} + \frac{1}{y - z} \left( \partial J^{\bar{\alpha}}(z) + [J_0, J^{\bar{\alpha}}(z)] - \frac{1}{2} [N, J^{\bar{\alpha}}(z)] \right). \]

(95)
Thus, there are one loop corrections in the double poles. However, forming a single operator $J^a J^\dot{a}$ those corrections cancels, which means that the energy momentum tensor still has zero anomalous dimension. It is worth to note that for $\langle T(y) J^\mu(z) \rangle$ we found regular terms at tree level, while at one loop the possible corrections to the double pole term cancel. In this cancelation plays a key role the result (89) and the pure spinors. We also computed $\langle T(y) T^4(z) \rangle$ at tree level, whose results were written at the end of subsection 6.1.

In the one loop level of this work, we focused on the corrections to the double poles. We leave the study of the possible corrections to single poles for future work.

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A Background Field Expansion

Here we use the background field expansion described in section 4 and write the expansion of the terms in the action (14) up to cubic terms in the quantum fields, since this is the order relevant for the one-loop computation of the current’s OPE’s. For the pure spinors Lorentz currents one expands

$$N_{ab} = N_{ab}^{(0)} + \alpha N_{ab}^{(1)} + \alpha^2 N_{ab}^{(2)},$$

and similarly for $\tilde{N}_{ab}$. Now, the pure spinor Lorentz currents have the following behaviour

$$N_{ab}^{(1)}(y) N_{cd}^{(1)}(z) \rightarrow \frac{\eta_{cb} N_{ad}^{(0)}(z) - \eta_{da} N_{cb}^{(0)}(z)}{y - z},$$

$$N_{ab}^{(2)}(y) N_{cd}^{(2)}(z) \rightarrow -3 \frac{\eta_{da} \eta_{cb}}{(y - z)^2}.$$

B Explicit expansion of the action

In this subsection we will write down the expansion of the matter part of action containing three quantum fields and one classical current.

The contributions proportional to $J^\mu$ and $J^2$ are
\[
\frac{\alpha^3}{2} \int d^2z \left[ \frac{2}{3} \partial X^a X^b X^c \partial \eta_{a'b'de} - \frac{2}{3} \partial X^a X^b X^c \partial' \eta_{a'b'de'} - \frac{2}{3} \partial X^a X^b X^c J^d \eta_{a'b'de} \right] + \frac{2}{3} \sqrt{g} X^a X^b X^c J^d \eta_{a'b'de} \tag{99}
\]

\[
- \frac{2}{3} \sqrt{g} X^a X^b X^c J^d \eta_{a'b'de} \partial \frac{1}{3} \partial X^a X^b X^c \partial \eta_{a'b'de} + \frac{5}{4} (\gamma_{a'd')} \partial_\gamma \partial \eta_{a'b'de} + \frac{5}{4} (\gamma_{a'd'}) \partial_\gamma \partial \eta_{a'b'de} - \frac{1}{4} (\gamma_{a'd'}) \partial_\gamma \partial \eta_{a'b'de} \tag{100}
\]

\[
- \frac{1}{3} \sqrt{g} X^a X^b X^c J^d \eta_{a'b'de} \partial X^a X^b X^c \partial \eta_{a'b'de} - \frac{1}{3} \partial X^a X^b X^c \partial \eta_{a'b'de} = \frac{5}{4} (\gamma_{a'd'}) \partial_\gamma \partial \eta_{a'b'de} - \frac{1}{4} (\gamma_{a'd'}) \partial_\gamma \partial \eta_{a'b'de} \tag{101}
\]

\[
- \frac{1}{3} \sqrt{g} X^a X^b X^c J^d \eta_{a'b'de} \partial X^a X^b X^c \partial \eta_{a'b'de} = \frac{5}{4} (\gamma_{a'd'}) \partial_\gamma \partial \eta_{a'b'de} - \frac{1}{4} (\gamma_{a'd'}) \partial_\gamma \partial \eta_{a'b'de} \tag{102}
\]

\[
C \text{ List of integrals}
\]

\[
\int d^4m \left| \frac{1}{m^2 - m - k^2} \right| = \frac{2}{m} \left( \frac{1}{e} - \ln \left| \frac{k^2}{\mu^2} \right| \right).
\]

\[
\int d^4m \left| \frac{m}{m^2 - m - k^2} \right| = \frac{m}{m - k} = 1.
\]

\[
\int d^4m \left| \frac{m}{m^2 - m - k^2} \right| = - \frac{1}{\pi} \frac{1}{k} \frac{\Gamma(1 - \epsilon) \Gamma(1 - 2\epsilon)}{\Gamma(1 - 2\epsilon)} = - \frac{1}{k} \left( \frac{1}{e} - \ln \left| \frac{k^2}{\mu^2} \right| \right).
\]

\[
\frac{1}{\mu^2} \text{ for } k - m - k^2 + \frac{1}{\pi} \frac{1}{k} \frac{\Gamma(1 - \epsilon) \Gamma(1 - 2\epsilon)}{\Gamma(1 - 2\epsilon)} = \frac{1}{k} \left( \frac{1}{e} - \ln \left| \frac{k^2}{\mu^2} \right| \right).
\]
\[ \int d^d m \frac{\bar{m}}{|m|^2|m-k|^2} = -\frac{1}{\pi^d} k^{-d} \frac{\Gamma(1-\epsilon)^2 \Gamma(\epsilon)}{\Gamma(2-2\epsilon)} = -\frac{1}{k} \left( \frac{1}{\epsilon} - \ln \frac{|k|^2}{\mu^2} \right). \tag{105} \]

\[ \int d^d m \frac{m^2}{|m|^2|m-k|^2} = -\frac{1}{\pi^d} k \frac{\Gamma(1-\epsilon)^2 \Gamma(1-\epsilon) \Gamma(\epsilon)}{\Gamma(2-2\epsilon)} = -\frac{k}{\epsilon} \left( \frac{1}{\epsilon} + 1 - \ln \frac{|k|^2}{\mu^2} \right). \tag{106} \]

\[ \int d^d m \frac{\bar{m}^2}{|m|^2|m-k|^2} = -\frac{1}{\pi^d} k \frac{\Gamma(1-\epsilon)^2 \Gamma(1-\epsilon) \Gamma(\epsilon)}{\Gamma(2-2\epsilon)} = -\frac{k}{\epsilon} \left( \frac{1}{\epsilon} + 1 - \ln \frac{|k|^2}{\mu^2} \right). \tag{107} \]

\[ \int d^d m \frac{m^2 \bar{m}}{|m|^2|m-k|^2} = \frac{1}{2\pi^d} k \frac{\Gamma(1-\epsilon)^2 \Gamma(1+\epsilon)}{\Gamma(2-2\epsilon)} = \frac{k}{2}. \tag{108} \]

\[ \int d^d m \frac{m \bar{m}^2}{|m|^2|m-k|^2} = \frac{1}{2\pi^d} k \frac{\Gamma(1-\epsilon)^2 \Gamma(1+\epsilon)}{\Gamma(2-2\epsilon)} = \frac{\bar{k}}{2}. \tag{109} \]

\[ \int d^d m \frac{m^2 \bar{m}^2}{|m|^2|m-k|^2} = \frac{1}{\pi^d} k \frac{\Gamma(1-\epsilon)^2 \Gamma(1+\epsilon)}{\Gamma(4-2\epsilon)} = \frac{k \bar{k}}{6}. \tag{110} \]

\[ \int d^d m \frac{\bar{m} m}{|m|^2|m-k|^2} = -\frac{1}{\pi^d} \frac{1}{kk} \left( 1 + \frac{2}{\epsilon} - 2\ln \frac{|k|^2}{\mu^2} \right). \tag{111} \]

\[ \int d^d m \frac{m \bar{m}}{|m|^2|m-k|^2} = -\frac{1}{\pi^d} \frac{1}{k} \left( \frac{1}{\epsilon} - \ln \frac{|k|^2}{\mu^2} \right). \tag{112} \]

\[ \int d^d m \frac{m \bar{m}}{|m|^2|m-k|^2} = -\frac{1}{\pi^d} \frac{1}{k} \left( \frac{1}{\epsilon} - 1 - \ln \frac{|k|^2}{\mu^2} \right). \tag{113} \]

\[ \int d^d m \frac{m^2 \bar{m}^2}{|m|^2|m-k|^2} = \frac{3}{2}. \tag{114} \]

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