2-Clean Rings *

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Abstract. A ring $R$ is said to be $n$-clean if every element can be written as a sum of an idempotent and $n$ units. The class of these rings contains clean ring and $n$-good rings in which each element is a sum of $n$ units. In this paper, we show that for any ring $R$, the endomorphism ring of a free $R$-module of rank at least 2 is 2-clean and that the ring $B(R)$ of all $\omega \times \omega$ row and column-finite matrices over any ring $R$ is 2-clean. Finally, the group ring $RC_n$ is considered where $R$ is a local ring.

Key words: 2-clean rings, 2-good rings, free modules, row and column-finite matrix rings, group rings.

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1. Introduction

The question of when the automorphism group of a module additively generates its endomorphism ring has been of interest for many years. A ring is called $n$-good [12] if every element is a sum of $n$ units. In 1953 Wolfson [14] and in 1954 Zelinsky [17] showed, independently, that every element of the ring of all linear transformations of a vector space over a division ring of characteristic not 2 is 2-good. In 1985 Goldsmith [4] proved that the endomorphism ring of a complete module over a complete discrete valuation ring is 2-good. In [13] Wans considered free $R$-modules where $R$ is a PID, and showed that if the rank of $M$ is finite and greater than 1, then $\text{End}_R(M)$ is 2-good. Meehan [8] further showed that the endomorphism ring of a free $R$-module of rank at least 2 is 2-good where $R$ is a PID. Moreover, the above question is considered by many authors on abelian groups (see [2],[7],[8]) and on general ring with an identity (see [3],[6],[11]).

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In 1977 Nicholson [10] introduced the concept of a clean ring (1-clean) which contains unit-regular rings and semiperfect rings, and showed that every clean ring must be exchange. Camillo and Yu [1] further proved that a clean ring with 2 invertible is 2-good. Recently, Xiao and Tong [16] called a ring $R$ $n$-clean if every element of $R$ is the sum of an idempotent and $n$ units. The class of these rings contains clean rings and $n$-good rings. In 1974 Henriksen [6] found that for any ring $R$ and $n > 1$, the matrix ring $M_n(R)$ is 3-good. Moreover, Vámos [12] proved that for any ring $R$, the endomorphism ring of a free $R$-module of rank at least 2 is 3-good. Motivated by the result of Henriksen and Vámos, we conjecture that for any ring $R$, the endomorphism ring of a free $R$-module of rank at least 2 is 2-clean.

In this paper, we answer the question in the positive. In fact, we proved that for any ring $R$, the endomorphism ring of a free $R$-module of rank at least 2 is 2-clean. It is also proved that the ring $B(R)$ of all $\omega \times \omega$ row and column-finite matrices over any ring $R$ is 2-clean. Finally, the group ring $RC_n$ is considered where $R$ is a local ring.

Throughout this paper, rings are associative with identity and modules are unitary. $J(R)$ and $U(R)$ denote the Jacobson radical and the group of units of $R$, respectively.

### 2. BASIC PROPERTIES OF $n$-CLEAN RINGS

An element of a ring is called $n$-clean if it can be written as the sum of an idempotent and $n$ units. A ring is called $n$-clean if each of its elements is $n$-clean. In this section, some properties of $n$-clean rings are given.

**Proposition 1.** Let $R$ be a ring and let $a \in R$. Then the following statements hold:

1. if $a$ is $n$-clean then it is also $l$-clean for all $n \leq l$.
2. every $n$-good ring is $n$-clean; if $R$ is $n$-clean with $2 \in U(R)$ then it is $(n+1)$-good.

**Proof.** (1) We only need to prove that $a$ is $n+1$-clean. Let $a \in R$ be $n$-clean: $a = e + u_1 + u_2 + \cdots + u_n$ where $e^2 = e \in R$ and $u_1, u_2, \cdots, u_n \in U(R)$. Note that $e = (1 - e) + (2e - 1)$, thus we have $a = (1 - e) + (2e - 1) + u_1 + \cdots + u_n$ where $2e - 1 \in U(R)$.

(2) It is clear that every $n$-good ring is $n$-clean. The second statement is due to Xiao and Tong (see [16]).
Let $S(R)$ be the nonempty set of all proper ideal of $R$ generated by central idempotents. An ideal $P \in S(R)$ is called a Pierce ideal of $R$ if $P$ is a maximal (with respect to inclusion) element of the set $S(R)$. If $P$ is a Pierce ideal of $R$, then the factor ring $R/P$ is called a Pierce stalk of $R$. The next result shows that the $n$-clean property needs to be checked only by for indecomposable rings or Pierce stalks.

**Proposition 2.** Let $R$ be a ring. Then the following are equivalent:

1. $R$ is $n$-clean.
2. every factor ring of $R$ is $n$-clean.
3. every indecomposable factor ring of $R$ is $n$-clean.
4. every Pierce stalk of $R$ is $n$-clean.

**Proof.** (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are directly verified.

(3) $\Rightarrow$ (1). Suppose that (3) holds and $R$ is not $n$-clean, then there is an element $a \in R$ which is not $n$-clean. Now let $\mathcal{S}$ be the set of all proper ideals $I$ of $R$ such that $\mathfrak{p}$ is not $n$-clean in $R/I$. Clearly, $0 \in \mathcal{S}$ and the set $\mathcal{S}$ is not empty. Define a partial ordering on $\mathcal{S}$ by "$\subseteq". If $\{I_{\alpha} : \alpha \in \Lambda\}$ is a chain in $\mathcal{S}$, let $I = \cup_{\alpha \in \Lambda} I_{\alpha}$. We will show that $\mathfrak{p}$ is not $n$-clean in $R/I$. Suppose that $\mathfrak{p}$ is $n$-clean in $R/I$. Then there exist $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_n \in U(R/I)$ (with inverses $\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n$, respectively) and $\overline{e}^2 = \overline{e} \in R/I$ such that $\overline{e} = \overline{u}_1 + \overline{u}_2 + \cdots + \overline{u}_n$. Note that $e^2 - e \in \cup_{\alpha \in \Lambda} I_{\alpha}$ and $u_i v_i - 1, v_i u_i - 1 \in \cup_{\alpha \in \Lambda} I_{\alpha}$, so $e^2 - e \in I_{\alpha}$, $u_i v_i - 1 \in I_{\alpha}$ and $v_i u_i - 1 \in I_{\alpha}'$ for $\alpha, \alpha' \in \Lambda$. Because $\{I_{\alpha} : \alpha \in \Lambda\}$ is a chain in $\mathcal{S}$, there is a maximal $I_s$ in the set $\{I_{\alpha_0}, I_{\alpha_1}, \ldots, I_{\alpha_n}, I_{\alpha_1}', I_{\alpha_1}', \ldots, I_{\alpha_n}'\}$ such that $I_{\alpha_0}, I_{\alpha_1}, I_{\alpha_1}' \subseteq I_s$. That is, $\mathfrak{p}$ is $n$-clean in $R/I_s$, a contradiction. This implies that $I \in \mathcal{S}$ is a upper bound of the chain. Because $\mathcal{S}$ is an inductive set and, by Zorn’s Lemma, $\mathcal{S}$ has a maximal element $I_0$. By (3) $R/I_0$ is decomposable as a ring. Write $R/I_0 \cong R/I_1 \oplus R/I_2$ where both the ideals $I_1, I_2$ strictly contain $I_0$ and so by the choice of $I_0$, $\mathfrak{p}$ is $n$-clean in $R/I_1$ and $R/I_2$. But then $\mathfrak{p}$ is $n$-clean in $R/I_0$, a contradiction.

(4) $\Rightarrow$ (1). Let $\mathcal{S}$ be the set of all proper ideals $I$ of $R$ such that $I$ is generated by central idempotents and the ring $R/I$ is not $n$-clean. Assume that $R$ is not $n$-clean. Then $0 \in \mathcal{S}$ and the set $\mathcal{S}$ is not empty. It is directly verified as above that the union of every ascending chain of ideals from $\mathcal{S}$ belongs to $\mathcal{S}$. By Zorn’s Lemma, the set $\mathcal{S}$ contains a maximal element $P$. By condition (4), it is sufficient to prove that $P$ is a Pierce ideal. Assume that
contrary. By the definition of the Pierce ideal, there is a central idempotent \( e \) of \( R \) such that \( P + eR \) and \( P + (1 - e)R \) are proper ideals of \( R \) which properly contain the ideal \( P \). Since ideals \( P + eR \) and \( P + (1 - e)R \) do not belong to \( S \) and are generated by central idempotents, \( R/(P + eR) \) and \( R/(P + (1 - e)R) \) are \( n \)-clean. Note that \( R/P \approx (R/(P + eR)) \times (R/(P + (1 - e)R)) \), it can be verified that \( R \) is \( n \)-clean. \( \square \)

3. MATRIX RINGS AND ENDMORPHISM RINGS OF FREE MODULES

In this section, we will consider the \( 2 \)-cleaness of the endomorphism ring of a free \( R \)-module of rank at least 2. First we give the following simple and interesting decomposition.

**Lemma 3.** Over any ring, the \( 2 \times 2 \) and \( 3 \times 3 \) matrices are \( 2 \)-clean.

**Proof.** Let \( R \) be a ring and let \( A = \left( \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) \in M_2(R) \). Put \( E = \left( \begin{array}{cc} a_{11} - 1 & 2 - a_{11} \\ a_{11} - 1 & 2 - a_{11} \end{array} \right) \). It is checked easily that then \( E^2 = E \). Thus we have

\[
A - E = \left( \begin{array}{cc} 1 & a_{12} + a_{11} - 2 \\ a_{21} - a_{11} + 1 & a_{22} + a_{11} - 2 \end{array} \right).
\]

Observing the above matrix, and then there exist invertible matrices

\[
P = \left( \begin{array}{ccc} 1 & 0 & 0 \\ a_{11} - a_{21} - 1 & 1 & 1 \\ 0 & 1 & 1 \end{array} \right) \quad \text{and} \quad Q = \left( \begin{array}{ccc} 1 & 2 - a_{11} - a_{12} & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{array} \right)
\]

such that

\[
P(A - E)Q = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{array} \right) = \left( \begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{array} \right) + \left( \begin{array}{ccc} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & c \end{array} \right),
\]

where \( c = a_{11}^2 + a_{11}a_{12} - a_{21}a_{12} - a_{21}a_{11} - 2a_{11} + 2a_{21} - a_{12} + a_{22} \). This shows that \( A = P^{-1} \left( \begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right) Q^{-1} + P^{-1} \left( \begin{array}{cc} 0 & -1 \\ -1 & 0 \end{array} \right) Q^{-1} + E \) is \( 2 \)-clean.

Now let \( B = \left( \begin{array}{ccc} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{array} \right) \) be a \( 3 \times 3 \) matrix over \( R \). We first construct an idempotent in order to show \( 2 \)-cleaness of \( B \). Set

\[
F = \left( \begin{array}{ccc} b_{11} - 1 & b_{22} - 1 & 3 - b_{11} - b_{22} \\ b_{11} - 1 & b_{22} - 1 & 3 - b_{11} - b_{22} \\ b_{11} - 1 & b_{22} - 1 & 3 - b_{11} - b_{22} \end{array} \right).
\]
It is directly verified that $F^2 = F$. Thus

$$B - F = \begin{pmatrix} 1 & b_{12} - b_{22} + 1 & b_{13} + b_{11} + b_{22} - 3 \\ b_{21} - b_{11} + 1 & 1 & b_{23} + b_{11} + b_{22} - 3 \\ b_{31} - b_{11} + 1 & b_{32} - b_{22} + 1 & b_{33} + b_{11} + b_{22} - 3 \end{pmatrix}.$$  

We only need to show that $B - F$ is 2-good. Observing the above matrix, and then there exist invertible matrices

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b_{11} - b_{31} - 1 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & b_{22} - b_{12} - 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 - b_{23} - b_{11} - b_{22} \\ 0 & 0 & 1 \end{pmatrix}$ such that

$$VT(B - F)W = \begin{pmatrix} * & 0 & * \\ * & 1 & 0 \\ 0 & * & * \end{pmatrix} = \begin{pmatrix} 0 & 1 & * \\ 0 & 0 & 1 \\ 1 & * & * \end{pmatrix} + \begin{pmatrix} * & -1 & 0 \\ * & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}.$$  

Consider the two matrices $U_1$, $U_2$ occurring in the decomposition above of $VT(B - F)W$. It is straightforward to verify that the two matrices are invertible in $M_3(R)$. Thus we obtain immediately a 2-clean expression of $B$, i.e.,

$$B = T^{-1}V^{-1}U_1W^{-1} + T^{-1}V^{-1}U_2W^{-1} + F.$$  

This completes the proof. □

**Remark 4.** (1). For the matrix ring $M_n(R)$, it is customary to write $GL_n(R)$ for $U(M_n(R))$. An elementary matrix is the result of an elementary row operation performed on the identity matrix. We denote by $E_n(R)$ the subgroup of $GL_n(R)$ generated by the elementary matrices, permutation matrices and -1. Observing the decompositions of the $2 \times 2$ and $3 \times 3$ matrices above, we see that, these matrices can be written as the sum of an idempotent matrix and two elements of $E_n(R)$.

(2). For any ring $R$, $R$ can be embedded in the $2 \times 2$ matrix ring $M_2(R)$. That is, all rings can be embedded in a 2-clean ring by Lemma 3.

(3). We know that 2-clean rings contain clean rings and 2-good rings. However, the converse is not true. For example, the matrix ring $M_2(\mathbb{Z})$ is not clean since $\mathbb{Z}$ is not a exchange ring, and the matrix ring $M_2(\mathbb{Z}[x])$ is not 2-good (see [12, Proposition 8]).

(4). It is well known that for a clean ring $R$, idempotents can be lifted modulo $J(R)$. However, a 2-clean ring has not this property in general. Let
\[ R = \mathbb{Z}_{(2)} \cap \mathbb{Z}_{(3)} = \{ \frac{m}{n} \in \mathbb{Q} : m, n \in \mathbb{Z}, 2 \nmid n \text{ and } 3 \nmid n \} \] and set \( S = M_2(R) \).

Then \( J(S) = J(M_2(R)) = M_2(J(R)) = M_2(6R) \). Let \( F = \begin{pmatrix} 3 & 0 \\ 6 & 3 \end{pmatrix} \). Then \( F^2 - F \in J(S) \), but there is no idempotent \( E \) of \( S \) such that \( F - E \in J(S) \) since non-trivial idempotents of \( S \) are only of form \( \begin{pmatrix} a & b \\ c & 1 - a \end{pmatrix} \) where \( bc = a - a^2 \) for \( a, b, c \in R \). Thus \( S \) is 2-clean by Lemma 3 but there exists an idempotent which cannot be lifted modulo \( J(S) \).

**Lemma 5.** Let \( R \) be a ring, \( m, n \geq 1 \) and \( k \geq 2 \). If the matrix rings \( M_n(R) \) and \( M_m(R) \) are both \( k \)-clean, then so is the matrix ring \( M_{n+m}(R) \).

**Proof.** Let \( A \in M_{n+m}(R) \) be a typical \( (n+m) \times (n+m) \) matrix which we will write in the block decomposition form

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},
\]

where \( A_{11} \in M_n(R), A_{22} \in M_m(R) \) and \( A_{12}, A_{21} \) are appropriately sized rectangular matrices. By hypothesis, there exist invertible \( n \times n, m \times m \) matrices \( U_1, U_2, \ldots, U_k \) and \( V_1, V_2, \ldots, V_k \), and idempotent matrices \( E_1, E_2 \) such that \( A_{11} = U_1 + U_2 + \cdots + U_k + E_1 \) and \( A_{22} = V_1 + V_2 + \cdots + V_k + E_2 \). Thus the decomposition

\[
\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} U_1 & A_{12} \\ & \end{pmatrix} + \begin{pmatrix} U_2 & O \\ O & V_1 \end{pmatrix} + \cdots + \begin{pmatrix} U_k & O \\ O & V_k \end{pmatrix} + \begin{pmatrix} E_1 & O \\ O & E_2 \end{pmatrix}
\]

shows that \( A \) is \( k \)-clean. \( \square \)

**Corollary 6.** Let \( k \geq 1 \). If \( R \) is a \( k \)-clean ring, then so the matrix ring \( M_n(R) \) for any positive integer \( n \).

**Proof.** For \( k = 1 \), it follows from [5, Corollary 1]. Assume that \( k \geq 2 \), it is clear by induction and by Lemma 5. \( \square \)

**Theorem 7.** Let \( R \) be a ring and let the free \( R \)-module \( F \) be (isomorphic to) the direct sum of \( \alpha \geq 2 \) copies of \( R \) where \( \alpha \) is a cardinal number. Then the ring of endomorphisms \( E \) of \( F \) is 2-clean.

**Proof.** Assume first that \( \alpha \geq 2 \) is finite so \( E \cong M_\alpha(R) \). Then \( E \) is 2-clean for \( \alpha = 2, 3 \) by Lemma 3 and the values of \( \alpha < \omega \) for which \( E \) is 2-clean are closed under addition by Lemma 5. So \( E \) is 2-clean for all finite \( \alpha \).

Assume now that \( \alpha \) is infinite. Then \( E \cong M_2(E) \) follows from \( F \cong F \oplus F \), and so \( E \) is 2-clean by Lemma 3. \( \square \)
4. ROW AND COLUMN-FINITE MATRIX RINGS

Let $B(R)$ be the ring of all $\omega \times \omega$ row and column-finite matrices over a ring $R$. Fix a free $R$-module $F = \bigoplus_{i=1}^{\infty} f_i R$ on countably many generators, and for each $k \in \mathbb{N}$ let $F_k = \bigoplus_{i=k}^{\infty} f_i R$. A moment’s reflection, using the standard correspondence between $R$-endomorphisms of $F_R$ and $\omega \times \omega$ column-finite matrices over $R$ relative to the basis $\{f_i\}_{i=1}^{\infty}$, confirms that

$$B(R) \cong \{ \phi \in \text{End}_R(F) : \text{for each } k \in \mathbb{N}, \exists m \in \mathbb{N} \text{ with } \phi(F_m) \subseteq F_k \}.$$

Hence we identify $B(R)$ with this ring of transformations. Next we will consider the 2-cleanness of $B(R)$. The proof of the following result is a modification of that in [8, Theorem 3.5].

**Theorem 8.** Let $R$ be a ring. Then the row and column-finite matrix ring $B(R)$ is 2-clean.

**Proof.** Note that $B(R) \cong B(M_2(R))$, so we may assume that $R$ is 2-clean by Lemma 3. Let $\phi \in B(R)$. Recall that $\varphi$ is defined by

(a) $\alpha$-endomorphism if $\varphi(f_i R) \subseteq \bigoplus_{k<i} f_i R$ for all $i < \omega$;
(b) $\beta$-endomorphism if $\varphi(f_i R) \subseteq \bigoplus_{i=1}^{t-1} f_i R$ for all $i < \omega$;
(c) $d$-endomorphism if $\varphi(f_i R) \subseteq f_i R$ for all $i < \omega$.

Then $\phi$ can obviously be expressed as

$$\phi = \eta + \rho + \delta,$$

where $\eta$ is an $\alpha$-endomorphism, $\rho$ is a $\beta$-endomorphism and $\delta$ is a $d$-endomorphism. Since $\phi \in B(R)$, for each $k \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $\phi(F_m) \subseteq F_k$. By the definitions of $\eta$, $\rho$ and $\delta$, we check easily that $\eta(F_m) \subseteq F_k$, $\rho(F_m) \subseteq F_k$ and $\delta(F_m) \subseteq F_k$. For the $\alpha$-endomorphism $\eta$, by [8, Proposition 3.2], there exists a strictly ascending sequence of integers $0 < r_0 < r_1 < r_2 < \cdots$ such that $\eta(f_i R) \subseteq \bigoplus_{k=i+1}^{r_i} f_k R$ for all $r_s \leq i < r_{s+1}$. Using this sequence we define endomorphisms $\eta_1$, $\eta_2$ of $F$ as follows

$$\eta_1 f_i = \begin{cases} \eta f_i & \text{for } r_{2t} \leq i < r_{2t+1}; \\ 0 & \text{for } r_{2t+1} \leq i < r_{2t+2}, \end{cases}$$

and

$$\eta_2 f_i = \begin{cases} 0 & \text{for } r_{2t} \leq i < r_{2t+1}; \\ \eta f_i & \text{for } r_{2t+1} \leq i < r_{2t+2}. \end{cases}$$

Clearly, $\eta_1$ and $\eta_2$ are $\alpha$-endomorphisms of $F$ with $\eta = \eta_1 + \eta_2$, and for each $k \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $\eta_1(F_m) \subseteq F_k$ and $\eta_2(F_m) \subseteq F_k$. By [8,
Lemma 3.4], we have that $\eta_1, \eta_2$ are both locally nilpotent. Next we decompose the $\beta$-endomorphism $\rho$. For each $i < \omega$, we have

$$\rho f_i = \sum_{k < i} f_k r_{ik} = \sum_{k < i, k \in I_1} f_k r_{ik} + \sum_{k < i, k \in I_2} f_k r_{ik},$$

where $I_1 = \bigcup_{t < \omega}\{k \mid r_{2t} \leq k < r_{2t+1}\}$ and $I_2 = \bigcup_{t < \omega}\{k \mid r_{2t+1} \leq k < r_{2t+2}\}$.

We define $\rho_1, \rho_2$ correspondingly, i.e.,

$$\rho_1 f_i = \sum_{k < i, k \in I_1} f_k r_{ik} \quad \text{and} \quad \rho_2 f_i = \sum_{k < i, k \in I_2} f_k r_{ik}.$$

Clearly, $\rho = \rho_1 + \rho_2$ and $\rho_1, \rho_2$ are both locally nilpotent. We check easily that for each $k \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $\rho_1(F_m) \subseteq F_k$ and $\rho_2(F_m) \subseteq F_k$.

Note that $\rho_1 \eta_2 = 0 = \rho_2 \eta_1$ by definitions of $\eta_1, \eta_2, \rho_1, \rho_2$, so $\eta_1 + \rho_2$ and $\eta_2 + \rho_1$ are also locally nilpotent. Now we consider the $d$-endomorphism $\delta$. For each $i < \omega$, there exists an element $r_i$ of $R$ such that $\delta f_i = f_i r_i$. Since $R$ is 2-clean, there are $e_i^2 = e_i \in R$ and units $u_{i1}, u_{i2}$ of $R$ such that

$$\delta f_i = f_i u_{i1} + f_i u_{i2} + f_i e_i.$$

defining $\delta_e f_i = f_i e_i$ and $\delta_j f_i = f_i u_{ij}$ ($i < \omega, j = 1, 2$). So $\delta = \delta_1 + \delta_2 + \delta_e$ and $\delta_1, \delta_2, \delta_e$ are $d$-endomorphisms of $F$. Note that for each $k \in \mathbb{N}$, set $m = k$, we get $\delta_1(F_m) \subseteq F_k, \delta_2(F_m) \subseteq F_k$ and $\delta_e(F_m) \subseteq F_k$. Thus we consider the decomposition of $\phi$

$$\phi = \eta + \rho + \delta$$
$$= \eta_1 + \eta_2 + \rho_1 + \rho_2 + \delta_1 + \delta_2 + \delta_e$$
$$= (\eta_1 + \rho_2 + \delta_1) + (\eta_2 + \rho_1 + \delta_2) + \delta_e$$
$$= \delta_1(\delta_1^{-1}(\eta_1 + \rho_2) + 1) + \delta_2(\delta_2^{-1}(\eta_2 + \rho_1) + 1) + \delta_e.$$

Note that $\delta_1^{-1}(\eta_1 + \rho_2)$ is locally nilpotent since $\delta_1^{-1}$ is $d$-endomorphism and $\eta_1 + \rho_2$ is locally nilpotent, and so $\delta_1^{-1}(\eta_1 + \rho_2) + 1$ is an automorphism of $F$. Hence $\delta_1(\delta_1^{-1}(\eta_1 + \rho_2) + 1)$ is also an automorphism of $F$. Similarly, $\delta_2(\delta_2^{-1}(\eta_2 + \rho_1) + 1)$ is an automorphism of $F$. Clearly, by the definitions of $\delta_e, \delta_e$ is idempotent endomorphism of $F$. It is checked easily that $\eta_1 + \rho_2 + \delta_1, \eta_2 + \rho_1 + \delta_2, \delta_e \in B(R)$ since $B(R)$ is a ring. Thus we complete the proof. \hfill \Box

**Remark 9.** From the proof of Theorem 8, we may consider row and column-finite matrix rings over a 2-good ring similarly. In fact, we obtain that if $R$
is 2-good then so is the row and column-finite matrix ring $B(R)$, and that for any ring $R$ the row and column-finite matrix ring $B(R)$ is 3-good.

5. 2-CLEAN GROUP RINGS

Given a group $G$ and a ring $R$, denote the group ring by $RG$. In this section, we consider the group ring $RC_n$ where $R$ is a local ring and $C_n$ is a cyclic group of order $n$. Some results of Xiao and Tong [16] are extended.

Theorem 10. Let $R$ be a local ring with $\overline{R} = R/J(R)$ and let $C_n$ be a cyclic group of order $n$. If $\text{char}\overline{R} \neq 2$, then $RC_n$ is 2-good.

Proof. If $\text{char}\overline{R} = 0$ or $(\text{char}\overline{R}, n) = 1$, then $n$ and $2$ are invertible in $R$. Note that $R$ is a division ring, then $RC_n$ is semisimple from $n \cdot 1 = n \in U(R)$, and so $RC_n$ is clean. This implies that $RC_n$ is 2-good by [1, Proposition 10]. We know that if $G$ is locally finite then $J(G) \subseteq J(RG)$ by [15]. Clearly, $J(R)C_n \subseteq J(RC_n)$, and then $RC_n \cong RC_n/J(R)C_n \twoheadrightarrow RC_n/J(RC_n)$. So the factor ring $RC_n/J(RC_n)$ is 2-good since 2-good rings are closed under factor rings. By [12, Proposition 3], $RC_n$ is also 2-good. If $n = mp^k$ where $\text{char}\overline{R} = p \neq 2$, $k \geq 1$, and $(m, p) = 1$. Then $C_n \cong C_{p^k} \times C_m$, and so $RC_n \cong (RC_{p^k})C_m$. By [9, Theorem], $RC_{p^k}$ is also a local ring and $\text{char}RC_{p^k} = p$. The rest is proved similarly as above since $(p, m) = 1$. Thus we complete the proof. \square

By Theorem 10, we obtain the following corollary immediately

Corollary 11. Let $R$ be a local ring with $\overline{R} = R/J(R)$ and let $C_n$ be a cyclic group of order $n$. If $\text{char}\overline{R} \neq 2$, then $RC_n$ is 2-clean.

Corollary 12. ([16, Theorem 2.3]) If $C_3$ is a cyclic group of order 3, then the group ring $\mathbb{Z}(p)C_3$ is 2-clean for any prime number $p \neq 2$.

Remark 13. The group ring $RC_n$ which satisfies the conditions of Theorem 10 need not be clean. In [5], Han and Nicholson showed that the group ring $\mathbb{Z}_{(7)}C_3$ is not clean where $\mathbb{Z}_{(7)} = \{m/n \in \mathbb{Q} : 7 \nmid n\}$.

Let $C_m = \{1, g, g^2, \cdots, g^{m-1}\}$ with $g^m = 1$ where $m$ is odd. Set $S = \{1, 2, \cdots, m-1\}$. Define $\sigma : S \rightarrow S$ by $i \mapsto 2i \mod m$. It is checked easily that $\sigma$ is a permutation of $\{1, 2, \cdots, m-1\}$. Let $F$ be a field with $\text{char}F = 2$ and let $e = e_0 + e_1g + \cdots + e_{m-1}g^{m-1} \in FC_m$ be an idempotent. Note that $2 = 0$ and $g^n = 1$, so $e^2 = e_0^2 + e_1g^{\sigma(1)} + \cdots + e_{\sigma(m-1)}g^{\sigma(m-1)}$. Suppose that $\sigma$
Theorem 14. Let $R$ be a local ring with $\text{char } R = 2$ and let $C_n$ be a cyclic group of order $n$. Write $n = m \cdot 2^k$ ($k \geq 0$) where $(m, 2) = 1$. If $\overline{R}$ is a field and $\sigma$ is a cyclic permutation of $\{1, 2, \ldots, m-1\}$, then the group ring $RC_n$ is semiperfect.

Proof. Suppose $k \geq 1$. Then $C_n \cong C_{2^k} \times C_m$ from $(m, 2) = 1$, and so $RC_n \cong (RC_{2^k})C_m$. By [9, Theorem], $RC_{2^k}$ is local. Since $\overline{R}$ is a field and $(RC_{2^k}) \to \overline{RC_{2^k}}$ is a ring epimorphism, $\overline{RC_{2^k}}$ is a field and $\text{char } \overline{RC_{2^k}} = \text{char } R = 2$. Hence we may assume $n = m$. Note that $\overline{RC_m}$ is semisimple by $(m, 2) = 1$ and $J(R)C_m \subseteq J(\overline{RC_m})$, so $J(R)C_m = J(\overline{RC_m})$. This shows that $\overline{RC_m} \cong \overline{RC_m}$ with $\text{char } \overline{R} = 2$. Since $\overline{R}$ is a field and $\sigma$ is a cyclic permutation of $\{1, 2, \ldots, m-1\}$, $\overline{RC_m}$ has only four idempotents, and so all idempotents in $\overline{RC_m}$ are $0$, $1$, $\overline{1 + g + \cdots + g^{m-1}}$, $\overline{g + g^2 + \cdots + g^{m-1}}$. We find easily idempotents in $RC_m$, $f_1 = 0$, $f_2 = 1$, $f_3 = m^{-1}(1 + g + \cdots + g^{m-1})$, $f_4 = m^{-1}((m-1) - g - g^2 - \cdots - g^{m-1})$ such that $\overline{f_1} = 0$, $\overline{f_2} = 1$, $\overline{f_3} = \overline{1 + g + \cdots + g^{m-1}}$, $\overline{f_4} = \overline{g + g^2 + \cdots + g^{m-1}}$. This shows that $RC_m$ is semiperfect.

The following result is immediate by Theorem 14 and by [1, Theorem 9].

Corollary 15. Let $R$ be a local ring with $\text{char } R = 2$ and let $C_n$ be a cyclic group of order $n$. Write $n = m \cdot 2^k$ ($k \geq 0$) where $(m, 2) = 1$. If $\overline{R}$ is a field and $\sigma$ is a cyclic permutation of $\{1, 2, \ldots, m-1\}$, then the group ring $RC_n$ is clean.

Corollary 16. ([16, Theorem 3.2]) If $C_3$ is a cyclic group of order 3, then the group ring $\mathbb{Z}(2)C_3$ is clean.

Remark 17. The condition which $\sigma$ is cyclic in Theorem 14 can not be removed. In fact, it is determined only by $m$ whether the permutation $\sigma$ of $\{1, 2, \ldots, m-1\}$ is cyclic. We calculate that $\sigma$ is cyclic in the case $m = 3, 5, 11, 13, \ldots$. However, set $m = 7$ or 9, $\sigma$ is not cyclic. Here, $\mathbb{Z}(2)C_7$ is not semiperfect. In fact, in $\mathbb{Z}(2)[X]$, $X^7 - X = (X + T)(X^3 + X - T)(X^3 + X^2 + T)$. But in $\mathbb{Z}(2)[X]$, $X^7 - 1 = (X - 1)(X^6 + X^5 + X^4 + X^3 + X^2 + X + 1)$ and $X^6 + X^5 + X^4 + X^3 + X^2 + X + 1$ is irreducible. So $\mathbb{Z}(2)C_7$ is not semiperfect by [15, Theorem 5.8]. Note that $\overline{\mathbb{Z}(2)C_7}$ is semisimple, then idempotents cannot be lifted modulo $J(\mathbb{Z}(2)C_7)$, and so $\mathbb{Z}(2)C_7$ is not clean.
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