Approximate analytical descriptions of the stationary single-vortex Marangoni convection inside an evaporating sessile droplet of capillary size

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Three versions of an approximate analytical description of the stationary single vortex Marangoni convection in an axially symmetrical sessile drop of capillary size are studied for arbitrary contact angle and compared with the results of numerical simulations. The first approach is heuristic extension of the well-known lubrication approximation. Two other descriptions are developed here and named $\eta$- and $r_z$-description. They are free from most of restrictive assumptions of the lubrication approach. For droplets with large contact angles they result in better accuracy compared to the heuristic extension of the lubrication approach, which still gives reasonable results within the accuracy 10–30 per cent. For droplets with small contact angles all three analytical descriptions well agree with the numerical data.

I. INTRODUCTION

The drying of a liquid droplet was studied since Maxwell time [1–3], and has attracted much attention over the last decade and a half in view of its role in various engineering applications, the advent of nanotechnology and progress in understanding of the evaporation process. In particular, the structure of the fluid flow generated by Marangoni forces inside an evaporating droplet has been intensively studied (see, for example, [4] and references therein).

In order to calculate the fluid dynamics in an evaporating sessile droplet, one has to solve numerically the coupled system of equations which contains the nonstationary vapor diffusion equation, the thermal conduction equation, the Navier-Stokes equations and to recalculate the droplet shape at each step due to the evaporative mass loss for the respective time interval [5]. The self-consistent solution should take into account an inhomogeneous evaporating flux density in the boundary conditions for the thermal conduction equation, since it is related to the heat transfer and hence to the temperature gradient at the droplet surface. The variation of temperature over the droplet surface affects the boundary conditions for the fluid dynamics, since the surface tension depends on temperature. In addition, the velocity field can influence the thermal conduction as a result of the effects of heat convection.

The calculation of the fluid flows in the droplet can be considerably simplified under the following conditions.

a) The capillary number $Ca = \eta \overline{u}/\sigma$ and the Bond number $Bo = g \rho h_0 R/(2\sigma \sin \theta)$ are much smaller than unity (see the notations in Table I). In this case, the sessile drop shape $h(r, t)$ can be approximated with high accuracy by the spherical cap approximation (see right panel in Fig. 1)

$$h(r, t) = \frac{R(\cos \phi(r, t) - \cos \theta(t))}{\sin \theta(t)}; \quad \phi(r, t) = \arcsin \left( \frac{r \sin \theta(t)}{R} \right). \quad (1)$$

Here $\theta$ is the droplet contact angle, $\overline{u}$ is the characteristic velocity and $h_0 = h(0, t)$ is the droplet height. The condition $Ca \ll 1$ signifies that the viscous forces, which generally enter the boundary condition for the pressure, are much smaller than capillary forces, and the hydrostatic Young-Laplace equation can be used in order to determine the droplet shape. Under the condition $Bo \ll 1$, influence of gravitational forces on the droplet shape is also small (see, for example, [5]).

b) The Inverse Stanton number $St^{-1} = \overline{u} R/\kappa$ is much smaller than unity. In this case the rate of the convective heat transfer is much smaller than conductive heat transfer, and, hence, the velocity field does not influence the thermal conduction (see, for example, [6]).

c) The transient time for heat transfer $t_{heat} = R h_0/\kappa$, transient time for momentum transfer $t_{mom} = \rho R h_0/\eta$ and transient time for vapor phase mass transfer $t_{mass} = \rho_{vap}/\rho f \cdot t_f$ should be much smaller than the total drying time $t_f \approx 0.2 \rho R h_0/(D u_s)$. This allows to describe the quasistationary stage of the evaporation process disregarding the time derivatives in the heat conduction equation, Navier-Stokes equation and the diffusion equation.

d) The dimensionless number $\rho g h^2 \beta/(\overline{u} \sigma')$, where $\beta$ is thermal expansion coefficient, is much smaller than unity. In this case buoyancy-induced convection is much weaker than Marangoni flow [12].
The laminar character of the flow should be also assumed. A turbulent regime arises in the droplet only for very large values of Marangoni and Reynolds numbers \[13\]. Therefore, we will not assume that the Reynolds number \( Re = \rho \pi R / \eta \) is much smaller than unity, i.e. the consideration in this paper will not be limited to the consideration of Stokes flow.

It is naturally to start from the vapor diffusion equation, which can be solved independently from the other equations. The analytical solution to the stationary vapor diffusion equation with appropriate boundary conditions gives the inhomogeneous evaporation flux from the surface of the evaporating droplet of a spherical shape \[7\]

\[
J_s(r) = \frac{Du_s}{R} \left( \frac{\sin \theta}{2} + \sqrt{2} x(r) + \cos \theta \right)^{3/2} \int_0^\infty \frac{\cosh(\theta \tau)}{\cosh(\pi \tau)} \tau \tanh\left((\pi - \theta)\tau\right) P_{-1/2+i\tau}(x(r)) d\tau,
\]

where \( x(r) = \left( r^2 \cos \theta / R^2 + \sqrt{1 - r^2 \sin^2 \theta / R^2} \right) / (1 - r^2 / R^2) \) and \( P_{-1/2+i\tau}(x) \) is the Legendre polynomial. It was shown in \[7\] that \( J_s \) can be approximated with high accuracy as

\[
J_s(r) = J_0(\theta) (1 - r^2 / R^2)^{-\lambda(\theta)},
\]

where \( \lambda(\theta) = 1/2 - \theta / \pi \) (see also \[8\]).

The nonstationary character of the inhomogeneous diffusion from a fixed surface of a sessile drop results in the following corrections to \[2\] \[3\]

\[
J_s(r, t) = J_s(r, \infty) \left( 1 + \frac{AR}{2\sqrt{Dt}} \right),
\]

where \( A \) is a constant which is close to 1. Another and yet quantitatively more important contribution to the nonstationarity in \( J_s(r, t) \) is induced by the dependence of the contact angle \( \theta \) on time due to the reduction of the mass of the evaporating droplet.

Under the condition b) specified above, the heat conduction equation inside the droplet can be solved independently, provided that the evaporation flux, which is connected to the temperature gradient at the droplet surface through the boundary condition \( \partial T / \partial n = -L J_s(r) / k \), is determined by \[2\]. We will carry out such numerical calculations and, following \[9\], fit the obtained quasistationary temperature at the droplet surface with the relation

\[
\frac{T}{\Delta T} = a \left( \frac{r}{R} \right)^b + (1-a) \left( \frac{r}{R} \right)^2 + c,
\]

where \( \Delta T \) is the temperature difference between the apex of the droplet and the substrate, and the fitting parameters \( a, b \) and \( c \) will be specified in Section \[V\]

The next step is the calculation of the fluid velocities in the droplet, where the obtained profile of the surface temperature enters the boundary condition at the droplet surface through the corresponding surface tension. This is the central problem of the present paper. It was found in \[6, 7, 9\] that this problem allows an approximate analytical description at least for the case of relatively flat droplets (see Sec. \[II\]). The derivation of the analytical description of the fluid flow employs only the boundary conditions and the mass balance equations and does not use the Navier-Stokes equations. In this paper we will generalize the existing approximate analytical descriptions and will compare in detail the analytical and the numerical results for various droplets.

Sec. \[II\] contains a brief review of the lubrication approximation derived in \[6, 9\], also it contains the heuristic extension of the lubrication approximation. The new analytical descriptions which result in better accuracy compared to other analytical approaches are developed in Sections \[III\] and \[IV\]. Sec. \[V\] contains discussion of the obtained analytical and numerical results.

II. THE LUBRICATION APPROXIMATION

The lubrication approximation for an evaporating droplet of capillary size was derived in \[6, 9\]. The derivation includes three basic assumptions which are justified for \( \theta \ll 1 \):

a) The radial velocity \( u_r(z) \) at each value of \( r \) has a quadratic dependence on \( z \).

b) The droplet free surface is approximated as a parabola \( h(\tilde{r}, t) = h(0, t)(1 - \tilde{r}^2) \), where \( \tilde{r} = r / R \).

c) The total shear stress at the droplet free surface is approximated by the \( rz \)-component of the stress tensor.
Consider an axially symmetrical column in the droplet (see left panel in Fig. 1). The mass balance equation states that the rate of mass change in the given volume element is equal to the net flux of mass into the element:

$$\frac{d}{dt}(\delta m) = - \oint \rho \mathbf{u} \cdot d\mathbf{f},$$

where the vector \(d\mathbf{f}\) is perpendicular to the surface of the element, its absolute value is equal to the area of a small part of the boundary of the element, and the amount of mass which evaporates each second from the surface of the element is assumed to be also properly included in the surface integral in the right-hand side of (6). An easy consequence of the mass balance equation for the column was obtained in [7]:

$$\pi_r(r, t) = -\frac{1}{\rho r h} \int_0^r dr r \left( J_s(r, t) \sqrt{1 + \left( \frac{\partial h}{\partial r} \right)^2 + \rho \frac{\partial h}{\partial t}} \right),$$

(7)

where \(\pi_r(r, t)\) is a height-averaged velocity in the column. Let \(r = r/R\), \(\tilde{z} = z/h_0\), \(\tilde{u}_r = u_r t_f/R\), \(\tilde{u}_z = u_z t_f/h_0\), \(\tilde{t} = t/t_f\), \(\tilde{h} = h(r)/h_0\), \(M_a = -\sigma_f \Delta T t_f/\eta R\), \(\tilde{J} = -J_0/\rho h_0\). Substituting in (7) the free surface of the droplet as a parabola \(h(r, t) = h_0(t)(1 - \tilde{r}^2)\) and the corresponding relations \(dh_0/dt = 2(dm/dt)/(\rho \pi R^2) \approx h_0/(t - t_f)\), \(dm/dt = \pi J_0 R^2/(1 - \lambda)\), Hu and Larson obtained in [8] the relation

$$\frac{\pi_r t_f}{R} = \frac{1}{4t} \frac{1}{1 - t} \left( (1 - \tilde{r}^2) - (1 - \tilde{r}^2)^{-\lambda} \right).$$

(8)

This relation for the height-averaged velocity for relatively flat evaporating droplets is the basis for the lubrication approximation for the droplets. Using the above assumptions a), b), c) and Eqn. (9), Hu and Larson derived the main lubrication equation for \(u_r\), which takes the form

$$\tilde{u}_r = \frac{3}{8} \frac{1}{1 - t} \left[ (1 - \tilde{r}^2) - (1 - \tilde{r}^2)^{-\lambda} \right] \left( \frac{\tilde{z}^2}{h^2} - 2 \frac{\tilde{h}}{h} \right) + \frac{\tilde{r}^2 h_0^2}{R^2} \left( \tilde{J} \lambda (1 - \tilde{r}^2)^{-\lambda - 1} + 1 \right) \left( \tilde{z}^2 - \frac{\tilde{z}^2}{2h^2} \right) + \frac{M_a h_0 \tilde{h} d\tilde{T}}{2R} \left( \tilde{z}^2 - 2 \frac{\tilde{z}^2}{h^2} \right),$$

(9)

and implies \(\tilde{h}(\tilde{r}, \tilde{t}) = \tilde{h}_0(\tilde{t})(1 - \tilde{r}^2)\). As seen, \(\tilde{u}_r(\tilde{r}, \tilde{z})\) is the quadratic function of \(\tilde{z}\) that can be considered as a result of expansion in powers of small parameter \(\tilde{z}\) for droplets with small contact angles. Eqn. (9) is applicable for arbitrary function \(T(r)\) and at the same time it coincides with Eqn. (28) in [8] provided that the surface temperature distribution is described by Eqn. (3).

The final step of the derivation in [8] is to find \(\tilde{u}_z\) using the obtained \(\tilde{u}_r\) and the continuity equation for the incompressible fluid. Here we represent the result in the form

$$\tilde{u}_z = \frac{3}{4} \frac{1}{1 - t} \left( 1 + \lambda (1 - \tilde{r}^2)^{-\lambda - 1} \right) \left( \frac{\tilde{z}^3}{3h^2} - \frac{\tilde{z}^2}{h} \right) - \frac{3}{4} \frac{1}{1 - t} \left[ (1 - \tilde{r}^2) - (1 - \tilde{r}^2)^{-\lambda} \right] \left( \frac{\tilde{z}^2}{2h^2} - \frac{\tilde{z}^3}{3h^2} \right) \frac{1}{\tilde{r}} \frac{\partial h}{\partial \tilde{r}} - \frac{h_0^2}{R^2} \left( \tilde{J} \lambda (1 - \tilde{r}^2)^{-\lambda - 1} + 1 \right) \left( \tilde{z}^2 - \frac{\tilde{z}^3}{h} \right) - \frac{\tilde{r}^2 h_0^2}{R^2} \left( \tilde{J} \lambda (1 - \tilde{r}^2)^{-\lambda - 1} + 1 \right) \frac{\tilde{z}^3}{2h^2} \frac{\partial h}{\partial \tilde{r}} - \frac{M_a h_0 \tilde{h} d\tilde{T}}{4R} \left( \frac{d^2 \tilde{T}}{d\tilde{r}^2} + \frac{1}{\tilde{r}} \frac{d\tilde{T}}{d\tilde{r}} \right) \left( \tilde{z}^2 - \frac{\tilde{z}^3}{h} \right) - \frac{M_a h_0 \tilde{h} d\tilde{T}}{4R} \frac{\tilde{z}^3}{h} \frac{\partial h}{\partial \tilde{r}}.$$

(10)

Substituting \(\tilde{h}(\tilde{r}, \tilde{t}) = \tilde{h}_0(\tilde{t})(1 - \tilde{r}^2)\) and (3) in Eqn. (10), one obtains Eqn. (29) in [8]. The assumption \(\tilde{h}(\tilde{r}, t) = \tilde{h}_0(t)(1 - \tilde{r}^2)\) is inherent to the lubrication approximation and, in particular, to Eqn. (8) above and to Eqns. (28) and (29) in [8].

The main purpose of this work is to develop analytical descriptions of fluid flows in an evaporating droplet for arbitrary contact angles. As a first step, we will test the heuristic extension of the lubrication approximation, substituting in Eqns. (9) and (10) the functions \(\tilde{h}(\tilde{r})\) and \(\partial \tilde{h}/\partial \tilde{r}\), which are obtained from (11), i.e., which correspond to the spherical cap profile of the sessile drop. We note that the continuity equation is precisely satisfied for arbitrary \(h(r)\) and \(T(r)\), if \(\tilde{u}_r\) and \(\tilde{u}_z\) are represented in the form (9), (10), as opposed to original form of lubrication equations, where the continuity equation is approximately satisfied for small contact angles. As we will see in Sec. [17] the heuristic extension of the lubrication approximation formally allows to obtain the velocity field for \(\theta \lesssim 1\), and often describes fluid flows with reasonable accuracy beyond the region of its strict validity. We will compare the results of Eqns. (9) and (10) with the numerical simulations.
III. DERIVATION OF THE DESCRIPTION IN THE $n\tau$-COORDINATE SYSTEM

In this section the analytical approach for calculating the fluid velocities in an evaporating droplet will be consistently and explicitly developed without using the assumptions $a), b), c)$ of Section II. The boundary conditions at the droplet surface will be considered assuming its spherical profile.

Consider the orthogonal $n\tau$-coordinate system, where $n$ is the distance between a point and the center of the sphere which contains the droplet surface (see right panel in Fig. 1). We also have $d\tau = nd\phi$; at the substrate we have $n = n_1 = R \cot \theta / \cos \phi$, and at the droplet surface we have $n = n_2 = R / \sin \theta$. The $n\tau$-coordinates are connected with the cylindrical $rz$-coordinates via the following relations:

$$r = n \sin \phi; \quad z = n \cos \phi - R \cot \theta; \quad n = \sqrt{r^2 + (z + R \cot \theta)^2}; \quad \tau = n \phi; \quad \phi = \arcsin(r / n).$$

The total mass of the shaded element in right panel of Fig. 1 is

$$\delta m = \rho \int_{n_1}^{n_2} 2\pi r(n)d\tau(n)dn = 2\pi \rho \int_{n_1}^{n_2} (n \sin \phi)(n \cos \phi)dn = \frac{2\pi R^3 \sin \phi}{\sin^3 \theta} \left(1 + \frac{\cos^2 \phi}{\cos^3 \phi}\right). \tag{11}$$

It follows from the mass balance equation (6) for the element that

$$\frac{d}{dt}(\delta m) = -J_s(\tau, t) \cdot 2\pi(n_2 \sin \phi)d\tau - 2\pi \rho \cdot d\phi \frac{d}{d\phi} \left(\sin \phi \int_{n_1}^{n_2} n_\tau dn\right), \tag{12}$$

where $J_s$ can be obtained with (3). Therefore,

$$2\pi \rho R^3 \sin \phi \cdot d\phi \frac{\cos \theta \cdot \theta'(t)}{\sin^4 \theta} \left(\frac{\cos \theta}{\cos^3 \phi} - 1\right) = -J_0(\theta) \left(1 + \frac{\sin^2 \phi}{\sin^2 \theta}\right)^{-\lambda(\theta)} - 2\pi R^2 \sin \phi \cdot d\phi \frac{d}{d\phi} \left(\sin \phi \int_{n_1}^{n_2} n_\tau dn\right). \tag{13}$$

Hence,

$$\int_{n_1}^{n_2} n_\tau dn = I(\phi) = \frac{-R^2}{\sin^2 \theta \sin \phi} \int_0^\phi \left(\frac{J_0}{\rho} \left(1 + \frac{\sin^2 \phi}{\sin^2 \theta}\right)^{-\lambda} + \frac{R \cos \theta \cdot \theta'(t)}{\sin^2 \theta} \left(\frac{\cos \theta}{\cos^3 \phi} - 1\right)\right) \sin \phi \cdot d\phi. \tag{14}$$

Here one can use the approximation $\theta'(t) \approx \theta / t_f$, where $t_f$ is remaining time of evaporation. Eqn. (14) results in the singularity in $u_\tau$ at the contact line, where $n_1(\phi) \to n_2$, which is a consequence of the singularity in evaporation rate.
at the contact line. Similar singularity is inherent in all known analytical models which are based on no-slip boundary condition at the substrate, and enters, in particular, Eqs. (7)–(10). The singularity influences the velocity field only at the contact line. Similar singularity is inherent in all known analytical models which are based on no-slip boundary condition at the substrate.

At the droplet surface we have the following boundary condition (see Appendix B in \[ \text{Eqn. (15)} \]):

\[
\frac{d\sigma}{d\tau} = -\frac{M_n \cos \phi \, d\tilde{T}}{t_f} = \frac{\partial u_r}{\partial n} + \frac{\partial u_n}{\partial \tau} - u_r \frac{d\phi}{d\tau}.
\]

Additional condition at the droplet surface can be derived from the continuity equation:

\[
u_n \cos \phi = u_r \tan \phi + u_z = -u_r h'(r) - \frac{1}{r} \int_0^{h(r)} \frac{\partial(r u_r)}{\partial r} \, dz = -\frac{1}{r} \frac{\partial(r h\pi_r)}{\partial r},
\]

hence, using the relation \[ \text{Eqn. (16)} \], one obtains at the droplet surface

\[
u_n = \frac{\partial h(r, t)}{\partial \tau} \cos \phi + \frac{J_0(\theta)}{\rho} \left(1 - \frac{\sin^2 \phi}{\sin^2 \theta}\right)^{-\lambda(\theta)}.
\]

Substituting \[ \text{Eqn. (17)} \] in \[ \text{Eqn. (15)} \], one gets

\[
u_n = \frac{R(\cos \phi - \cos \theta)\theta'(t)}{\sin^2 \theta} + \frac{J_0(\theta)}{\rho} \left(1 - \frac{\sin^2 \phi}{\sin^2 \theta}\right)^{-\lambda}, \quad \frac{\partial u_n}{\partial \tau} = \frac{\sin \theta}{R} \frac{\partial u_n}{\partial \phi} = \frac{\sin \phi}{R \sin \theta} \left(\frac{2J_0 \lambda \cos \phi}{\rho} \left(1 - \frac{\sin^2 \phi}{\sin^2 \theta}\right)^{-\lambda - 1} - R\theta'(t)\right).
\]

We will use the following approximation for \( u_r \):

\[
u_r(n, \phi) = (n - n_1)p(\phi)A(\phi) + (n - n_1)B(\phi).
\]

Here \( p(\phi) \) is a trial function, which will be chosen later. The coefficient functions \( A(\phi) \) and \( B(\phi) \) will be specified based on the boundary condition \[ \text{Eqn. (15)} \] and the mass balance relation \[ \text{Eqn. (14)} \]. Eqn. \[ \text{Eqn. (20)} \] automatically satisfies the no-slip boundary condition at the substrate.

Using \[ \text{Eqn. (20)} \], Eqn. \[ \text{Eqn. (15)} \] can be rewritten as

\[
A(\phi)(n_2 - n_1)p(\phi)\left(p(\phi) - 1 + \frac{n_1}{n_2}\right) + B(\phi)\frac{n_1}{n_2} + \frac{\partial u_n}{\partial \tau} + \frac{M_n \cos \phi \, d\tilde{T}}{t_f} = 0.
\]

where the right side of \[ \text{Eqn. (19)} \] can be used instead of \( \partial u_n / \partial \tau \).

Eqn. \[ \text{Eqn. (14)} \] gives the second linear relationship between \( A(\phi) \) and \( B(\phi) \):

\[
\frac{A(\phi)}{(p + 1)(p + 2)}(n_2 - n_1)^{p+1}(n_2(p + 1) + n_1) + \frac{1}{6}B(\phi)(n_2 - n_1)^2(n_1 + 2n_2) = I(\phi).
\]

We note that the integral in the right side of \[ \text{Eqn. (14)} \] can be obtained exactly, because

\[
\int_0^\phi \sin \phi \left(\frac{\cos \theta}{\cos^3 \phi} - 1\right) \, d\phi = \cos \phi - 1 + \frac{1}{2} \cos \theta \tan^2 \phi,
\]

\[
\int \sin \phi \left(1 - \frac{\sin^2 \phi}{\sin^2 \theta}\right)^{-\lambda} \, d\phi = (\cos \theta + \cos \phi) \left(1 - \frac{\sin^2 \phi}{\sin^2 \theta}\right)^{-\lambda} \, _2F_1 \left(1, \lambda; 2\lambda; \frac{2 \cos \theta}{\cos \theta - \cos \phi}\right) \frac{\Gamma(2\lambda - 1)}{\Gamma(2\lambda)} + C,
\]

where \( _2F_1(a, b; c; z) \) is the hypergeometric function and \( \Gamma(z) \) is the gamma function. Therefore,

\[
I(\phi) = \frac{-R^2J_0}{\rho \sin^2 \theta} (\cos \theta + \cos \phi) \left(1 - \frac{\sin^2 \phi}{\sin^2 \theta}\right)^{-\lambda} \, _2F_1 \left(1, \lambda; 2\lambda; \frac{2 \cos \theta}{\cos \theta - \cos \phi}\right) \frac{\Gamma(2\lambda - 1)}{\Gamma(2\lambda)} +
\]

\[
+ \frac{R^2J_0(\cos \theta + 1)}{\rho \sin^2 \theta} \, _2F_1 \left(1, \lambda; 2\lambda; \frac{2 \cos \theta}{\cos \theta - 1}\right) \frac{\Gamma(2\lambda - 1)}{\Gamma(2\lambda)} +
\]

\[
+ \frac{R^3 \cos \theta \cdot \theta'(t)}{\sin^4 \theta \sin \phi} \left(1 - \cos \phi - \frac{1}{2} \cos \theta \tan^2 \phi\right).
\]
Eqns. (21), (22) are the system of two linear relationships between $A(\phi)$ and $B(\phi)$. The solution is:

$$A = -\frac{(-n_1 + n_2)^{-1-p}(1 + p)(2 + p)}{(p - 1)(2n_2^2(p + 1)(p + 2) + n_1^2(p + 4) + n_1n_2(p + 1)(p + 4))} \left(6n_1 + \frac{u_n}{\partial n}(n_1 - n_2)^2n_2(n_1 + 2n_2) \right) + \frac{\partial \tau}{\partial n} M_a(n_1 - n_2)^2n_2(n_1 + 2n_2) \cos \phi \frac{t_f}{(p - 1)(2n_2^2(p + 1)(p + 2) + n_1^2(p + 4) + n_1n_2(p + 1)(p + 4))}$$

$$B = \frac{6}{(n_1 - n_2)^2(p + 1)(p + 2) + n_1^2(p + 4) + n_1n_2(p + 1)(p + 4)} \left( (\partial u_n/\partial n) + (\partial \tau/\partial n) M_a \cos \phi \right)$$

Thus, we have obtained the analytical description of $u_r(n, \phi)$ based on (20). In order to complete the approximate analytical description of the velocity field, we need to obtain $u_n(n, \phi)$ inside the droplet. The continuity equation for the incompressible fluid $\text{div} \, u = 0$ takes the following form in the $n\tau$-coordinate system:

$$\frac{\partial u_n}{\partial n} + \frac{2u_n}{n} + \frac{\partial u_r}{\partial \tau} + \frac{\cot \phi}{n} u_r = 0.$$  

Therefore,

$$\frac{1}{n} \frac{\partial (n^2 u_n(n, \phi))}{\partial n} = \frac{-\partial u_r}{\partial n} - \cot \phi \cdot u_r = -A'(\phi)(n - n_1)^{p(\phi)} - B'(\phi)(n - n_1) + B(\phi)(n - n_1) \cot \phi + A(\phi)n_1' \phi(n - n_1)^{p(\phi) - 1} - A(\phi)(n - n_1)^{p(\phi)}(p'(\phi) \log(n - n_1) + \cot \phi),$$

$$u_n(n, \phi) = -A'(\phi)(n - n_1)^{p(\phi)} n_1 + np + n \frac{n^2}{(p + 1)(p + 2)} - B'(\phi) \frac{n^2}{(p + 1)(n - n_1)^2(n_1 + 2n)} + \frac{B(\phi)}{2n^2} \frac{n^2}{(n - n_1)^2} n_1' \phi - B(\phi)(2n + n_1)(n - n_1)^2 \cot \phi + A(\phi)(pn + n_1)(n - n_1)^p n_1' \phi - A(\phi) \cot \phi \cos p n_1' \phi(p + 1)^{p - 1} + \frac{A(\phi)p'(\phi)(n - n_1)^{p - 1}}{n^2} \left( \frac{n}{(p + 2)} + n_1(2p + 3) \frac{n}{(p + 1)(p + 2)} + \frac{p + 1}{(p + 1)(p + 2)} \right).$$

The velocities in the cylindrical $rz$-coordinate system can be obtained as

$$u_r(r, z) = u_r(n, \phi) \cos \phi + u_n(n, \phi) \sin \phi,$$

$$u_z(r, z) = -u_r(n, \phi) \sin \phi + u_n(n, \phi) \cos \phi,$$

where $n = \sqrt{r^2 + (z + R \cot \theta)^2}$ and $\phi = \arcsin(r/n)$.

IV. DERIVATION OF THE DESCRIPTION IN THE CYLINDRICAL $r\theta$-COORDINATE SYSTEM

Here, as in previous section, the analytical approach for calculating the fluid velocities in an evaporating droplet will be consistently developed without using the assumptions $a), b), c)$ of Section III and with careful consideration of the boundary conditions. The cylindrical $r\theta$-coordinate system will be used in this section. Namely, we will now consider the approximation for $u_r$ based on

$$u_r(r, z) = A(r)z^{p(r)} + B(r)z$$

instead of Eq. (20), where $p(r)$ is a trial function. We note that the $rz$- and $n\tau$- power law approximations (20) and (23) generally are not equivalent to each other. It follows from (23)

$$\frac{\partial u_r}{\partial z} = p(r)A(r)z^{p(r) - 1} + B(r).$$

The boundary condition (15) can be rewritten as

$$-M_a \cos \phi \frac{d \tilde{T}}{d r} = \left( \frac{\partial u_r}{\partial \tau} + \frac{\partial u_r}{\partial z} \right) \cos 2\phi + \left( \frac{\partial u_r}{\partial r} - \frac{\partial u_r}{\partial z} \right) \sin 2\phi.$$
The linear relation between \( A(r) \) and \( B(r) \) follows from (7), where one can use (3) and (11). Integrating Eq. (7), we obtain
\[
\pi_r(r,t) = \frac{J_0(\theta)R(\sin \theta)^{2\lambda-1}}{h\rho(2\lambda - 1) \sin \phi} 2F_1 \left( \lambda - \frac{1}{2}; \lambda; \frac{1}{2}; \cos^2 \theta \right) - \\
\frac{J_0(\theta)R}{h\rho(2\lambda - 1) \sin \phi} \left( \frac{\sin \phi}{\cos \phi} \right)^{2\lambda-1} 2F_1 \left( \lambda - \frac{1}{2}; \lambda; \frac{1}{2}; \frac{\cos^2 \phi}{\cos^2 \phi} \right) - \frac{R^3}{2hr \sin^4 \theta} \left( \sin^2 \phi - 2 \cos \theta(1 - \cos \phi) \right) \theta'(t),
\]
where \( 2F_1 \) is the hypergeometric function. This gives us the linear relation between \( A(r) \) and \( B(r) \), because \( \pi_r(r,t) = A(r)h(r)p/(p + 1) + B(r)h(r)/2 \), hence
\[
B(r) = \frac{2\pi_r(r)}{h(r)} - \frac{2A(r)h(r)p^{-1}}{p + 1}. \tag{37}
\]

Also, at the droplet surface we have the relations (18), (19), and thus we know the values of
\[
u_n = u_r \sin \phi + u_z \cos \phi, \tag{38}
\]
\[
\frac{\partial u_n}{\partial \tau} = \left( \frac{\partial u_r}{\partial r} - \frac{\partial u_z}{\partial z} \right)_{r=h(r,t)} \cos \phi \cdot \sin \phi - \frac{\partial u_r}{\partial z} \sin^2 \phi + \frac{\partial u_z}{\partial r} \cos^2 \phi. \tag{39}
\]

In addition, it follows from the continuity equation for the incompressible fluid that
\[
\frac{\partial u_r}{\partial r} - \frac{\partial u_z}{\partial z} = 2\frac{\partial u_r}{\partial \tau} + \frac{u_r}{r} = \left( 2A'(r) + A(r) \left( \frac{1}{r} + 2p'(r) \log z \right) \right) z^{p(r)} + \left( 2B'(r) + \frac{B(r)}{r} \right) z. \tag{40}
\]

Substituting (40), (34) and (19) into (33), one finds the value of \( \partial u_z/\partial r \) at the droplet surface. Substituting (40), (34) and the obtained value of \( \partial u_z/\partial r \) into (35) and using (37), one finds the following equation:

\[
A'(r) + A(r)f(r) + g(r) = 0, \tag{41}
\]

where
\[
f(r) = \frac{1}{2r} + \frac{(2 + p) \cot 2\phi - 2h'(r)}{h(r)} + \frac{p'(r)(2 + (p^2 - 1) \log h(r))}{p^2 - 1}, \tag{42}
\]
\[
g(r) = \frac{p + 1}{p - 1} \frac{4rt_f \pi_r(\cot 2\phi - h'(r)) + h \left( M_a r \cos \phi \cdot \cot \phi \cdot dt/\delta + 2t_f (r \cot 2\phi \cdot \partial u_n/\partial r + \pi_r + 2r\pi_r'(r)) \right)}{2rt_fh^{p+1}}. \tag{43}
\]

The solution of Eq. (41) satisfying the boundary condition \( \lim_{r \to 0} A(r) = 0 \) is
\[
A(r) = - \exp \left( \int_r^{R/2} f(x)dx \right) \int_0^r g(y) \exp \left( - \int_y^{R/2} f(x)dx \right) dy. \tag{44}
\]

Thus, we have obtained the velocity field in the droplet: \( u_r(r, z) \) is defined by (33), where \( A(r) \) and \( B(r) \) are determined by (41) and (37); also it follows from (33) and from the continuity equation for the incompressible fluid that
\[
u_z(r, z) = - \left( A'(r) + \frac{A(r)}{r} - \frac{A(r)p'(r)}{p + 1} + A(r)p'(r) \log z \right) \frac{z^{p+1}}{p + 1} - \left( B'(r) + \frac{B(r)}{r} \right) \frac{z^2}{2}. \tag{45}
\]

V. NUMERICAL RESULTS AND DISCUSSION

Table I shows the parameter values that were used in the calculations. The fluid and vapor properties were taken from (10). Table II shows the contact angle, contact line radius and fitting parameters corresponding to Eqn. (6) for the 25 droplets. For each droplet, we have \( c = T_0/\Delta T - 1 \), and the value of \( a \) is obtained with the least squares fit for
TABLE I: The parameter values used in the calculations.

| Droplet | Initial temperature $T_0$ (K) | toluene | ethanol | 2-propanol | octane | 1-butanol | 1-hexanol |
|---------|-------------------------------|---------|---------|------------|--------|-----------|-----------|
| Contact line radius $R$ (cm) | 0.2 | 0.154 | 0.168 | 0.175 | 0.157 | 0.155 |
| Fluid | Density $\rho$ (g/cm$^3$) | 0.87 | 0.789 | 0.786 | 0.703 | 0.8098 | 0.8136 |
| Molar mass $\mu$ (g/mole) | 92.14 | 46.07 | 60.1 | 114.23 | 74.122 | 102.17 |
| Thermal conductivity $k$ (W/(cm·K)) | 1.311·10$^{-3}$ | 1.69·10$^{-3}$ | 1.35·10$^{-3}$ | 1.28·10$^{-3}$ | 1.54·10$^{-3}$ | 1.50·10$^{-3}$ |
| Heat capacity $c_p$ (J/(mole·K)) | 156.7 | 112.3 | 156.5 | 254.6 | 177.2 | 240.4 |
| Isochoric heat capacity $c_v$ (J/g·K) | 1.286 | 1.787 | 1.931 | 1.791 | 1.83 | 1.889 |
| Thermal diffusivity $\kappa$ (cm$^2$/s) | 8.86·10$^{-4}$ | 8.79·10$^{-4}$ | 6.60·10$^{-4}$ | 8.17·10$^{-4}$ | 7.95·10$^{-4}$ | 7.84·10$^{-4}$ |
| Dynamic viscosity $\eta$ (g/(cm·s)) | 5.6·10$^{-3}$ | 1.074·10$^{-2}$ | 2.038·10$^{-2}$ | 5.08·10$^{-3}$ | 2.544·10$^{-2}$ | 4.578·10$^{-2}$ |
| Surface tension $\sigma$ (g/s$^2$) | 28.3049 | 21.97 | 20.93 | 21.14 | 24.93 | 25.81 |
| $-\partial$(surface tension)/$\partial T$ (g/(s$^2$·K)) | 0.1189 | 0.0832 | 0.0788 | 0.095 | 0.0898 | 0.08 |
| Latent heat of evap. $L_e$ (J/g) | 300.0 | 918.6 | 755.2 | 363.2 | 706.27 | 603.0 |

| Vapor | Diffusion constant $D$ (cm$^2$/s) | 0.1449 | 0.1181 | 0.1013 | 0.0616 | 0.0861 | 0.0621 |
| Saturated vapor density $u_s$ (g/cm$^3$) | 1.27·10$^{-4}$ | 1.46·10$^{-4}$ | 1.38·10$^{-4}$ | 8.57·10$^{-5}$ | 2.76·10$^{-5}$ | 6.55·10$^{-6}$ |
| Local evap. rate at apex $j_0$ (g/(cm$^2$·s)) | 9.2·10$^{-5}$ | 1.12·10$^{-4}$ | 8.2·10$^{-5}$ | 2.9·10$^{-5}$ | 1.55·10$^{-5}$ | 2.7·10$^{-6}$ |

For $n\pi$- and $rz$-approaches, we have chosen a comparatively simple trial functions $p(\phi)$ and $p(r)$ respectively, which do not depend on the liquid properties and at the same time allow to achieve a good accuracy in describing the velocity field in various droplets. We found the trial function $p(\phi)$ which does not depend on the contact angle, but we did not succeed in obtaining a suitable function $p(r)$ with such property. The function $p(\phi)$ takes the form

$$p(\phi) = \begin{cases} 
2, & \text{for } \phi/\theta \leq 1/2, \\
2 + 4\delta/\pi \left(1 - \cos \left(\pi (\phi/\theta - 1/2)/(2\delta)\right)\right), & \text{for } 1/2 < \phi/\theta \leq 1/2 + \delta, \\
1 + 2\phi/\theta - 2(1 - 2/\pi)\delta, & \text{for } \phi/\theta > 1/2 + \delta,
\end{cases}$$

(47)

where $\delta = 10^{-2}$. This function is equal to 2 for $\phi \leq \theta/2$, it smoothly changes from 2 to 3 when $\phi$ changes from $\theta/2$ to $\theta$ and its derivative is continuous. For the $rz$-approach, we have used the trial function $p(r) = 2 + \theta/3 - 1.7 \cdot \theta^2$. Particularly, for $\theta = 20^\circ$ we took $p(r) = 2$, and for $\theta = 50^\circ$ we took $p(r) = 1.0001$.

The numerical simulation for the fluid flows was carried out with the method described in [3], where the droplet surface was considered to be fixed, the surface temperature was taken in accordance with Table II and Eqn. (5), and the heat convection was switched off. Although the inverse Stanton number for many of the considered droplets is quite large, we switch off the heat convection, since all the three approaches obviously do not take into account such effects. Among the droplets considered, the inverse Stanton number is sufficiently small for a hexanol droplet with a contact angle $20^\circ$. It will also be small for droplets of a smaller size. During the numerical calculation, the array of velocity values is obtained from the stream function $\psi$ using the relations

$$v_r(i,j) = \frac{\psi_{i+1,j} - \psi_{i,j}}{\eta \cdot h_x \cdot h_y}, \quad v_z(i,j) = \frac{\psi_{i,j} - \psi_{i+1,j}}{(i + 1/2) \cdot h_x \cdot h_y}.$$  

(48)

Therefore, we will estimate the deviation of the numerical velocity field from the analytically obtained velocity field
FIG. 2: The surface velocity for the hexanol droplet. Left panel: $\theta = 50^\circ$, right panel: $\theta = 20^\circ$. Blue curve is the numerically obtained surface velocity. Green curve: the surface velocity in the heuristic extension of the lubrication approximation. Purple curve: the surface velocity in the $n\tau$-description derived in Sec. III. Red curve: the surface velocity in the $rz$-description derived in Sec. IV.

FIG. 3: The velocity field for the hexanol droplet with $\theta = 50^\circ$: numerically obtained velocity field, the velocity field in the heuristic extension of the lubrication approximation, in the $n\tau$-description and in the $rz$-description.

using the following mean-square deviations:

$$
\sigma_r = \frac{1}{N} \sqrt{\frac{1}{N-1} \sum_{i,j=0}^{N-1} (v_r(i,j) - u_r(i \cdot h_x, (j + 1/2) \cdot h_y))^2},
$$

$$
\sigma_z = \frac{1}{N} \sqrt{\frac{1}{N-1} \sum_{i,j=0}^{N-1} (v_z(i,j) - u_z((i + 1/2) \cdot h_x, j \cdot h_y))^2}.
$$

Here $N = 200$ is the mesh size, $h_x = R/N$, $h_y = h/N$. One more characteristic value of the velocity field is $u_{max}$, the absolute value of maximal velocity at the surface of the droplet. Tables III and IV show $u_{max}$ obtained with numerical simulation, with heuristic extension of the lubrication approximation, and with $n\tau$- and $rz$- approximations correspondingly. They also show the values of $\sigma_r$ and $\sigma_z$ for the analytical descriptions. Table IV contains the results for the droplet of 2-propanol, where the size of the droplet is varied. Also, table IV contains the comparison of analytical and numerical results for the droplet of virtual liquid with variable viscosity, where all other characteristics coincide with those of 2-propanol.

Table III and right panels in Figs. 2 and 4 show for the case $\theta < 30^\circ$, when the droplets are relatively flat, that all the three approximate analytical descriptions, including the lubrication approximation, agree well with the numerically obtained velocity field, though the $nt$-approach is the most precise.

For large contact angles, the accuracy of the $rz$- and $nt$-descriptions exceeds that of the heuristic extension of lubrication approximation in about 1.5 times, and also they result in a much more precise value of the maximal surface velocity $u_{max}$ (see Table III). Interestingly, the heuristic extension of the lubrication approximation still works within 10–30 per cent for droplets with large contact angles, where the assumptions a), b) and c) of Sec II are not justified. Since the $n\tau$-approach does not require numerical integration and its trial function (47) does not depend on the value of contact angle, and in agreement with Figs. 2 3 4 5 and Tables III IV the $n\tau$ analytical description for the fluid flow seems to be favourable.
FIG. 4: The surface velocity for the toluene droplet. Left panel: $\theta = 50^\circ$, right panel: $\theta = 20^\circ$. Blue curve is the numerically obtained surface velocity, which shows the "bottleneck effect" arising in the numerical results only when the Marangoni number exceeds 3000. Green curve: the surface velocity in the heuristic extension of the lubrication approximation. Purple curve: the surface velocity in the $n\tau$-description derived in Sec. III. Red curve: the surface velocity in the $rz$-description derived in Sec. IV.

FIG. 5: The velocity field for the toluene droplet with $\theta = 50^\circ$: numerically obtained velocity field, which shows the "bottleneck effect" arising in the numerical results only when the Marangoni number exceeds 3000, the velocity field in the heuristic extension of the lubrication approximation, in the $n\tau$-description and in the $rz$-description.

We find that the effect of nonzero $J_0$ and $\theta'(t)$ in (25) and (36) on the fluid flows is quite small compared to the effect of Marangoni forces, except that (25) and (36) result in the singularity in $u_\tau$ in a small vicinity of the contact line. We will not concern this singularity, which is known to arise at the contact line due to no-slip boundary condition at the substrate [11].

The droplets under consideration have a wide spread of values of the Marangoni number $\tilde{M}_a = -\sigma' \Delta T \rho_h/\eta k$, starting from 10 for the droplet of 1-hexanol, up to 3450 for the toluene droplet. For droplets of toluene, propanol, octane and ethanol, the droplet size is much smaller than the Marangoni cell size [5, 12] on the flat fluid film containing the same liquid of the same height, while for droplets of butanol and hexanol, the droplet size and the Marangoni cell size are of the same order.

The comparison shows that the discrepancy between the numerical results and the analytical descriptions is considerably large only for droplets with huge Marangoni numbers and, therefore, large contact angles. Numerical results show that when the Marangoni number exceeds 3000, the "bottleneck effect" will take place. This means that the vortex center becomes sufficiently close to the symmetry axis due to large Marangoni forces. This results in substantial increase of downward velocities along the symmetry axis. This is shown in the left panels of Figs. 4 and 5. Evidently, such an effect cannot be quantitatively described without employing the Navier-Stokes equations. Moreover, it seems very probable that for such huge Marangoni numbers, Marangoni forces would rather destroy the axial symmetry of the droplet, which would result in a more complicated three-dimensional velocity field. We observed the "bottleneck effect" only for the large droplet of toluene and for the large droplet of virtual liquid with viscosity in 4 or 8 times smaller than that of 2-propanol (see the fourth and fifth rows in Table IV). For other droplets there is no "bottleneck effect". For example, Figs. 2 and 3 show the velocity field for the droplet of 1-hexanol. The velocity fields for all other droplets with contact angles $50^\circ$ and $20^\circ$ are very similar to those shown in Figs. 2 and 3. Thus, the numerical results for axially symmetrical velocity field demonstrate the "bottleneck effect" only if the Marangoni number exceeds 3000.
VI. CONCLUSION

In summary, three versions of an approximate analytical description of the stationary single-vortex Marangoni convection inside an axially symmetrical evaporating sessile drop have been derived for arbitrary contact angle, compared with the results of numerical calculations and systematically analyzed. The first approach is heuristic extension of the well-known lubrication approximation, which is found to still give reasonable results within the accuracy 10–30 per cent even for droplets with large contact angles. Two other new descriptions named $n\tau$- and $rz$-description are consistently and explicitly derived, they result in better accuracy, and, in particular, describe better the surface velocity distribution. The $n\tau$-approach is most convenient, since it does not require numerical integration and its trial function does not depend on the contact angle. For droplets with small contact angles all three analytical descriptions well agree with the numerical data. The descriptions have been tested and compared with the numerical results for droplets containing various liquids and having various values of the contact angle.

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TABLE II: Fitting parameters for calculation of surface temperature.

|         | θ     | R, cm | ΔT, K | a     | b     | c     |
|---------|-------|-------|-------|-------|-------|-------|
| 1-butanol | 50°   | 0.157 | 0.529506 | 0.3793 | 10   | 552.629 |
| 1-butanol | 20°   | 0.157 | 0.154023 | 0.324516 | 15   | 1902.29 |
| ethanol  | 50°   | 0.154 | 4.55357  | 0.379299 | 10   | 63.3781 |
| ethanol  | 20°   | 0.154 | 1.32454  | 0.324514 | 15   | 220.321 |
| octane   | 50°   | 0.175 | 0.727793 | 0.379299 | 10   | 401.793 |
| octane   | 20°   | 0.175 | 0.2117   | 0.324512 | 15   | 1383.74 |
| 2-propanol | 50°  | 0.168 | 3.7995   | 0.379299 | 10   | 76.1549 |
| 2-propanol | 20°  | 0.168 | 1.1052   | 0.324514 | 15   | 264.246 |
| toluene  | 50°   | 0.2   | 2.04597  | 0.379299 | 10   | 142.281 |
| toluene  | 20°   | 0.2   | 0.595135 | 0.324515 | 15   | 491.577 |
| 1-hexanol | 10°   | 0.155 | 0.010597 | 0.29475 | 24   | 27662.5 |
| 1-hexanol | 15°   | 0.155 | 0.016353 | 0.337253 | 16  | 17925.4 |
| 1-hexanol | 20°   | 0.155 | 0.023109 | 0.324521 | 15  | 12684.5 |
| 1-hexanol | 25°   | 0.155 | 0.030568 | 0.317682 | 14  | 9589.09 |
| 1-hexanol | 30°   | 0.155 | 0.038757 | 0.317114 | 13  | 7562.79 |
| 1-hexanol | 35°   | 0.155 | 0.047704 | 0.323616 | 12  | 6144.19 |
| 1-hexanol | 40°   | 0.155 | 0.057442 | 0.337952 | 11  | 5102.41 |
| 1-hexanol | 45°   | 0.155 | 0.068007 | 0.3489  | 11  | 4309.59 |
| 1-hexanol | 50°   | 0.155 | 0.079445 | 0.379306 | 10  | 3688.97 |
| 1-hexanol | 55°   | 0.155 | 0.091807 | 0.403695 | 10  | 3192.11 |
| 1-hexanol | 60°   | 0.155 | 0.105159 | 0.434731 | 10  | 2786.68 |
| 2-propanol | 50°  | 0.2   | 3.7995   | 0.379299 | 10   | 76.1549 |
| 2-propanol | 50°  | 0.05  | 3.7995   | 0.379299 | 10   | 76.1549 |
| 2-propanol | 50°  | 0.02  | 3.7995   | 0.379299 | 10   | 76.1549 |
| 2-propanol | 50°  | 0.01  | 3.7995   | 0.379299 | 10   | 76.1549 |
TABLE III: Accuracy of the analytical descriptions. The values of $u_{\text{max}}$, $\sigma_r$ and $\sigma_z$ are in cm/s.

| $\theta$ | Numerical | Lubrication | Approx. $n, \tau$ | Approx. $r, z$ |
|----------|-----------|-------------|------------------|----------------|
|          | $u_{\text{max}}$ | $\sigma_r$ | $\sigma_z$ | $u_{\text{max}}$ | $\sigma_r$ | $\sigma_z$ | $u_{\text{max}}$ | $\sigma_r$ | $\sigma_z$ |
| 1-butanol | $5^\circ$ | 0.12 | 0.18 | 0.0098 | 0.0064 | 0.12 | 0.0077 | 0.0047 | 0.12 | 0.0082 | 0.0049 |
| 1-butanol | $20^\circ$ | 0.0123 | 0.0129 | $8.7 \cdot 10^{-4}$ | $1.9 \cdot 10^{-4}$ | 0.0124 | $8.3 \cdot 10^{-4}$ | $1.8 \cdot 10^{-4}$ | 0.0120 | $8.2 \cdot 10^{-4}$ | $1.8 \cdot 10^{-4}$ |
| ethanol | $50^\circ$ | 2.29 | 3.43 | 0.221 | 0.189 | 2.31 | 0.160 | 0.122 | 2.22 | 0.169 | 0.117 |
| ethanol | $20^\circ$ | 0.232 | 0.244 | 0.017 | 0.0037 | 0.234 | 0.016 | 0.0034 | 0.227 | 0.016 | 0.0035 |
| octane | $50^\circ$ | 0.876 | 1.324 | 0.0812 | 0.065 | 0.889 | 0.059 | 0.040 | 0.856 | 0.062 | 0.039 |
| octane | $20^\circ$ | 0.089 | 0.094 | 0.0064 | 0.0014 | 0.090 | 0.0061 | 0.0013 | 0.088 | 0.0060 | 0.0013 |
| 2-propanol | $50^\circ$ | 0.929 | 1.429 | 0.078 | 0.053 | 0.960 | 0.060 | 0.035 | 0.924 | 0.064 | 0.036 |
| 2-propanol | $20^\circ$ | 0.096 | 0.101 | 0.0069 | 0.0015 | 0.098 | 0.0066 | 0.0014 | 0.094 | 0.0065 | 0.0014 |
| toluene | $50^\circ$ | 3.14 | 4.23 | 0.49 | 0.62 | 2.84 | 0.41 | 0.54 | 2.73 | 0.41 | 0.53 |
| toluene | $20^\circ$ | 0.285 | 0.30 | 0.020 | 0.0046 | 0.288 | 0.019 | 0.0041 | 0.279 | 0.019 | 0.0043 |

TABLE IV: Accuracy of the analytical descriptions. Dependence on the viscosity and on the contact line radius. The values of $u_{\text{max}}$, $\sigma_r$ and $\sigma_z$ are in cm/s.

| $\theta$ | $R$, cm | $\eta/\eta_0$ | Numerical | Lubrication | Approx. $n, \tau$ | Approx. $r, z$ |
|----------|---------|----------------|-----------|-------------|------------------|----------------|
|          | $u_{\text{max}}$ | $\sigma_r$ | $\sigma_z$ | $u_{\text{max}}$ | $\sigma_r$ | $\sigma_z$ | $u_{\text{max}}$ | $\sigma_r$ | $\sigma_z$ |
| 2-propanol | $50^\circ$ | 0.168 | 2 | 0.46 | 0.71 | 0.039 | 0.025 | 0.48 | 0.030 | 0.018 | 0.46 | 0.032 | 0.019 |
| 2-propanol | $50^\circ$ | 0.168 | 1 | 0.929 | 1.429 | 0.078 | 0.053 | 0.960 | 0.060 | 0.035 | 0.924 | 0.064 | 0.036 |
| 2-propanol | $50^\circ$ | 0.168 | 0.5 | 1.90 | 2.86 | 0.182 | 0.153 | 1.92 | 0.131 | 0.097 | 1.85 | 0.139 | 0.093 |
| 2-propanol | $50^\circ$ | 0.168 | 0.25 | 4.31 | 5.72 | 0.70 | 0.90 | 3.84 | 0.59 | 0.79 | 3.69 | 0.59 | 0.78 |
| 2-propanol | $50^\circ$ | 0.168 | 0.125 | 8.23 | 11.43 | 1.70 | 2.35 | 7.68 | 1.49 | 2.16 | 7.39 | 1.44 | 2.13 |
| 2-propanol | $50^\circ$ | 0.05 | 1 | 0.92 | 1.43 | 0.077 | 0.051 | 0.960 | 0.060 | 0.036 | 0.924 | 0.064 | 0.038 |
| 2-propanol | $50^\circ$ | 0.02 | 1 | 0.92 | 1.43 | 0.077 | 0.050 | 0.960 | 0.061 | 0.037 | 0.924 | 0.065 | 0.039 |
| 2-propanol | $50^\circ$ | 0.01 | 1 | 0.92 | 1.42 | 0.077 | 0.050 | 0.960 | 0.061 | 0.037 | 0.924 | 0.065 | 0.039 |