Collapsing and Expanding Cylindrically Symmetric Fields with Light-like Wave-Fronts in General Relativity

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The dynamics of collapsing and expanding cylindrically symmetric gravitational and matter fields with lightlike wave-fronts is studied in General Relativity, using the Barrabés-Israel method. As an application of the general formulae developed, the collapse of a matter field that satisfies the condition \( R_{AB} g^{AB} = 0 \), \((A, B = z, \varphi)\), in an otherwise flat spacetime background is studied. In particular, it is found that the gravitational collapse of a purely gravitational wave or a null dust fluid cannot be realized in a flat spacetime background. The studies are further specified to the collapse of purely gravitational waves and the general conditions for such collapse are found. It is shown that after the waves arrive at the axis, in general, part of them is reflected to spacelike infinity along the future light cone, and part of it is focused to form spacetime singularities on the symmetry axis. The cases where the collapse does not result in the formation of spacetime singularities are also identified.

I. INTRODUCTION

Gravitational collapse of a realistic body has been one of the most thorny and important problems in Einstein’s theory of General Relativity. Particularly, in 1991 Shapiro and Teukolsky [1] studied numerically the problem of a dust spheroid, and found that only the spheroid is compact enough, a black hole can be formed. Otherwise, the collapse most likely ends with a naked singularity in violation of Penrose's cosmic censorship conjecture [2]. Later, by studying the collapse of a cylindrical shell that is free of both gravitational and matter radiation and is made of counter-rotating particles, Apostolatos and Thorne (AT) [3] showed analytically that the centrifugal forces associated with an arbitrarily small amount of rotation, by themselves, without the aid of any pressure, can halt the collapse at some non-zero, minimum radius, and the shell will then oscillate until it settles down at some final, finite radius, whereby a spacetime singularity is prevented from forming on the symmetry axis. Soon after AT’s work, Shapiro and Teukolsky [4] studied numerically the gravitational collapse of rotating spheroids, and found that the rotation indeed significantly modifies the evolution when it is sufficiently large. But, for small enough angular momentum, their simulations showed that spindle singularities appeared to arise without apparent horizons, too. Hence, it is still possible that even spheroids with some angular momentum naked singularities may be formed. However, it should be noted that the possibility of the formation of a horizon at late times cannot be excluded in the ST simulations, since their numerical calculations always terminate right at the formation of the spacetime singularities.

Quite recently, we have studied analytically the problem of a collapsing cylindrical shell that radiates gravitational waves and massless particles when it is collapsing, and found two physically distinguishable final states [5]. In one case, after the shell collapses to a minimal non-zero radius, it starts to expand, that is, the angular momentum of the dust particles is strong enough to halt the collapse, so that a spacetime singularity is prevented from forming on the symmetry axis. However, in the other case the rotation is not strong enough to halt the collapse at a finite non-zero radius, and, as a result, a spacetime singularity is finally formed on the symmetry axis. As shown numerically in [1,4], during the late stages of the collapse, the spheroid tends to fall at nearly the speed of light. Therefore, it is reasonable to consider the collapsing body as consisting of null fluid at its late stages of collapse. In this paper, we shall generalize our previous studies [5] to the gravitational collapse of a cylinder with a light-like wave-front. Specifically, the paper is organized as follows: In Sec. II, we consider the general matching of two cylindrically symmetric regions across a null hypersurface, using the Barrabés-Israel (BI) method...
II. DYNAMICS OF CYLINDRICAL NULL SHELLS WITHOUT ROTATION

Both static and dynamic cylindrical symmetric and timelike thin shells with zero total angular momentum have been studied previously. In this section, we shall give a general treatment for cylindrically symmetric null shells, which connects two cylindrical, otherwise arbitrary regions. In some cases the null shell can be made disappear, and the corresponding null hypersurface becomes a boundary surface. In the following we shall consider the latter as a particular case of the former and study them all together.

Assume that the shell, located on the hypersurface \( \Sigma \), divides the whole spacetime into two regions, \( V^\pm \). Let \( V^+ \) denote the region outside the shell, and \( V^- \) denote the region inside the shell. In \( V^+ \), the metric takes the form

\[
ds^2_+ = f^+(T, R)dt^2 - g^+(T, R)dr^2 - h^+(T, R)dz^2 - l^+(T, R)d\varphi^2, \quad (R \geq R_0(T)),
\]

where \( \{x^{+\mu}\} = \{T, R, z, \varphi\} \) with \(-\infty < T, z < +\infty, 0 \leq \varphi \leq 2\pi\), and \( R = R_0(T) \) is the location of the shell in the coordinates \( x^{+\mu} \). If the sources are confined within a finite region in the radial direction, then the spacetime should be asymptotically flat in the radial direction as \( R \to \infty \), where \( R \) denotes the proper radial distance.

In \( V^- \), the metric takes the form

\[
ds^2_- = f^-(t, r)dt^2 - g^-(t, r)dr^2 - h^-(t, r)dz^2 - l^-(t, r)d\varphi^2, \quad (r \leq r_0(t)),
\]

where \( \{x^{-\mu}\} = \{t, r, z, \varphi\} \) are the usual cylindrical coordinates chosen in the region \( V^- \). In this region we shall choose the radial coordinate \( r \) such that \( r = 0 \) represents the symmetry axis. The hypersurface \( \Sigma \) in this region is described by \( r = r_0(t) \). To have the cylindrical symmetry be well defined, the spacetime in this region has to satisfy several conditions:

(i) The existence of an axially symmetric axis: The spacetime that has an axially symmetric axis is assured by the condition

\[
||\partial \varphi|| = |g_{\varphi\varphi}| \to O(r^2),
\]

as \( r \to 0^+ \).

(ii) The elementary flatness on the axis: This condition requires that the spacetime be locally flat on the axis, which in the present case can be expressed as

\[
\frac{X_{\alpha\beta}g^{\alpha\beta}}{4X} \to 1,
\]

as \( r \to 0^+ \), where \( X \) is given by \( X = ||\partial \varphi|| = |g_{\varphi\varphi}| \).

Since the trajectory of the shell is null, we have

\[
R'_0(T) = \epsilon \left( \frac{f^+}{g^+} \right)^{1/2} \left|_{R=R_0(T)} \right.,
\]

\[
r'_0(t) = \epsilon \left( \frac{f^-}{g^-} \right)^{1/2} \left|_{r=r_0(t)} \right.,
\]

where a prime denotes the ordinary differentiation with respect to the indicated argument, and \( \epsilon = \pm 1 \), with the sign “+” corresponding to expanding shells, and the sign “−” to collapsing shells. On the shell, the intrinsic coordinates will be chosen as \( \{\xi^a\} = \{w, z, \varphi\} \), \( (a = 1, 2, 3) \), where \( w \) denotes a null coordinate, which will be defined explicitly below. In terms of \( \xi^a \), the metric on the shell takes the form

\[
ds^2|_{\Sigma} = g_{\cdot\cdot}d\xi^a d\xi^b = -h(w)dw^2 - l(w)d\varphi^2,
\]

where \( g_{\cdot\cdot} \) denotes the induced metric on the hypersurface. The first junction condition that the metrics in both sides of the shell reduce to the same metric \((2.4)\) on \( \Sigma \) requires that

\[
h(w) \equiv h^+(T, R_0(T)) = h^-(t, r_0(t)),
\]

\[
l(w) \equiv l^+(T, R_0(T)) = l^-(t, r_0(t)),
\]

where on the hypersurface \( \Sigma \) we have \( T = T(w) \) and \( t = t(w) \). The normal vector to the hypersurface \( \Sigma \) is given in \( V^+ \) and \( V^- \), respectively, by

\[
n^+_\mu = \{-R'_0(T), 1, 0, 0\},
\]

\[
n^-_\mu = \{-r'_0(t), 1, 0, 0\}.
\]

Following Barrabés and Israel (BI), let us introduce the vectors \( N^\pm_\mu \) via the relations

\[
N^{\pm\lambda}N^{\pm}_\lambda = 0, \quad N^{\pm\lambda}n^{\pm}_\lambda = 1,
\]

we find that in the present case they take the form,
\[ N^+ = \frac{1}{2} \left\{ -\epsilon \left( \frac{g^+}{f^+} \right)^{1/2} \delta^\mu_T + \delta^\mu_R \right\}, \]
\[ N^- = \frac{1}{2} \left\{ -\epsilon \left( \frac{g^-}{f^-} \right)^{1/2} \delta^\mu_T + \delta^\mu_R \right\}. \] (2.10)

Defining the quantities, \( e^{\pm^\mu}_{(a)} \) by \( e^{\pm^\mu}_{(a)} \equiv \partial x^{\pm^\mu}/\partial \xi^a \), we obtain
\[ e^{\pm^\mu}_{(1)} = \frac{dT}{dw} \{ \delta^\mu_T + R^\nu_0(T) \delta^\mu_R \}, \]
\[ e^{\pm^\mu}_{(3)} = \delta^\mu_\varphi, \]
\[ e^{\mu}_{(2)} = \frac{dt}{dw} \{ \delta^\mu_t + \nu_0(t) \delta^\mu_\varphi \}, \]
\[ e^{\mu}_{(3)} = \delta^\mu_\varphi. \] (2.11)

To make sure that \( N^\mu_\lambda \), defined on the two faces of the hypersurface \( \Sigma \), do represent the same vector \( N^\mu \), we impose the conditions
\[ N^\mu_\lambda e^{\pm^\lambda}_{(a)} \big|_{\Sigma^\pm} = N^\mu_\lambda \left. e^{\pm^\lambda}_{(a)} \right|_{\Sigma^-}, \quad (a = 1, 2, 3). \] (2.12)

In the present case, it can be shown that the above conditions reduce to
\[ \left( f^+ g^+ \right)^{1/2} \left. \frac{dT}{dw} \right|_{\Sigma^+} = \left( f^- g^- \right)^{1/2} \left. \frac{dt}{dw} \right|_{\Sigma^-}. \] (2.13)

The above equation can be also considered as defining the relation between \( T \) and \( t \) on the hypersurface. Following BI [3], we define the extrinsic curvature of the hypersurface \( \Sigma \) as,
\[ R_{ab} = -N_\lambda e^{\nu}_{(b)} \left( \nabla_\epsilon e^{\lambda}_{(a)} \right) \]
\[ = -N_\lambda e^{\nu}_{(b)} \left\{ e^{\lambda}_{(a,\nu)} + \Gamma^\lambda_{\nu\rho} e^{\rho}_{(a)} \right\}, \] (2.14)

which in the present case has the following non-vanishing components,
\[ R_{++} = \epsilon \left( f^+ g^+ \right)^{1/2} \left\{ \frac{d^2 T}{dw^2} + \frac{1}{2} \left( \frac{dT}{dw} \right)^2 \left( \frac{f_T^+ + \frac{g_T^+}{g^+}}{f^+} \right) \right\} \]
\[ + \left( \frac{dT}{dw} \right)^2 \frac{f_T^+}{f^+}; \]
\[ R_{zz} = \frac{1}{4} \left\{ \epsilon \left( g^+ \right)^{1/2} \frac{h_{+T}^T - h_{-R}^+}{h_{+T}^T} \right\}, \]
\[ R_{\varphi\varphi} = \frac{1}{4} \left\{ \epsilon \left( g^+ \right)^{1/2} \frac{l_{-R}^T - l_{+T}^+}{l_{-R}^T} \right\}, \] (2.15)

where \( f_T^+ \equiv \partial f^+ / \partial T \) etc., and \( R_{ab} \) can be obtained from the above expressions by the replacement
\[ f^+, g^+, h^+, l^+, R_0(T), T, R \]
\[ \rightarrow f^-, g^-, h^-, l^-, r_0(t), t, r. \] (2.16)

Introducing the null vector \( l^a \) along the direction \( dw \) by \( l^a = \delta^a_w \), it can be shown that the surface energy-momentum tensor \( \tau_{ab} \) is given by
\[ \tau_{ab} = -\frac{1}{2\kappa} \left\{ g_{ab} l^pl^d + g^a_{*} l^d l^c \gamma_{cd} \right\}, \] (2.17)

where \( \kappa = 8\pi G/c^4 \),
\[ \gamma_{ab} \equiv \left( R_{ab}^+ - R_{ab}^- \right), \] (2.18)

and
\[ \left[ g_{ab} \right] = \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -l(w) \\ 0 & 0 & -l^{-1}(w) \end{array} \right], \] (2.19)

Inserting Eqs. (2.13), (2.15) and the corresponding expressions for \( R_{ab} \) into the above equations, we find that \( \tau_{ab} \) can be written in the form
\[ \tau_{ab} = \rho l^al^b + p \left( \varphi^a \varphi^b + \varphi^a \varphi^b \right), \quad (a, b = w, z, \varphi), \] (2.20)

where
\[ \rho = -\frac{1}{2\kappa} \left[ \frac{\gamma_{zz}}{h(w)} + \frac{\gamma_{\varphi\varphi}}{l(w)} \right] \]
\[ = -\frac{1}{4\kappa} \left\{ \epsilon \left( \frac{g^+}{f^+} \right)^{1/2} \left( \frac{h_{+T}^{-1}}{h_{+T}^T} + \frac{l_{+T}^T}{l_{+T}^T} \right) \right\} \]
\[ - \left( \frac{g^-}{f^-} \right)^{1/2} \left( \frac{h_{-T}^{-1}}{h_{-T}^T} + \frac{l_{-T}^T}{l_{-T}^T} \right) \]
\[ = \left[ \frac{h_{-T}^{-1}}{h_{+T}^T} + \frac{l_{+T}^T}{l_{+T}^T} \right] \left[ \frac{h_{+T}^{-1}}{h_{-T}^T} + \frac{l_{-T}^T}{l_{-T}^T} \right] \right\}, \] (2.21)

and
\[ z_a = h^{1/2}(w) \delta^a_w, \quad \varphi_a = h^{1/2}(w) \delta^a_w. \] (2.22)

Clearly, the surface energy-momentum tensor given by Eq. (2.20) represents a Type II null fluid in the classifications of Hawking and Ellis [2], where the null fluid is moving along the lines defined by the tangential vector \( l^a \) with its isotropic pressure \( p \) in the \( dz \) and \( d\varphi \) directions. It is interesting to note that when \( f^+ = g^+ \) and \( f^- = g^- \), we have \( p = 0 \).
III. GRAVITATIONAL COLLAPSE OF MATTER FIELDS SATISFYING $R_{AB}G^{AB} = 0$

As an application of the formulae developed above, let us consider some specific models that represent gravitational collapse. Therefore, from now on we shall consider only the case where $\varepsilon = -1$. The wave front of a collapsing cylinder is given by $R = R_0(t)$ in the $x^{\mu}$ coordinates and $r = r_0(t)$ in the $x^{-\mu}$ coordinates. Before the wave arrives, we assume that the spacetime is flat, that is,

$$ds_\Sigma^2 = dt^2 - dr^2 - dz^2 - r^2 d\varphi^2, \quad (r \leq r_0(t)). \quad (3.1)$$

Clearly, in the present case the cylindrical symmetry of the spacetime is well defined, and the axis $r = 0$ is free of any kind of spacetime singularities [cf. Fig. 1(a)].

![Diagram](image)

**FIG. 1.** The matching of a Minkowskian region $M_4$ with an exterior $V^+: (a)$ In the $z = \text{Const.}$ plane. The hypersurface $\Sigma$ is a null surface represented by the circle $r = r_0(t)$. (b) In the $(t, r)$-plane. The hypersurface $\Sigma$ now is represented by the straight line $t + r = 0$. In the region $t + r < 0$, the spacetime is Minkowskian, while in the region $t + r > 0$, the spacetime is curved.

Then, the condition (2.3) yields,

$$r_0(t) = -(t - t_0), \quad (3.2)$$

where $t_0$ is a constant. By a translation transformation in $t$, we can always set $t_0 = 0$. In the following, we shall always assume that this is the case. Since $r_0(t) \geq 0$, we must have $t \leq 0$ on the hypersurface $\Sigma$. Then, we can see that the history of this hypersurface in the $(t, r)$-plane is represented by the part of the light cone $t + r = 0$, $t \leq 0$, as shown by Fig. 1(b).

In the region $R \geq R_0(t)$, where collapsing cylinder is present, the metric, without loss of generality, can be cast in the form

$$ds_+^2 = e^{-M} (dt^2 - dR^2) - e^{-U} (e^V dz^2 + e^{-V} d\varphi^2), \quad (3.3)$$

where $M$, $U$ and $V$ are functions of $T$ and $R$ only. Then, Eqs. (2.3), (2.7) and (2.13) yield

$$\begin{align*}
R_0(T) &= -T \geq 0, \quad V(T, R_0(T)) = U(T, R_0(T)), \\
r_0(t) &= e^{-U(T, R_0(T))}, \quad dt = e^{-M(T, R_0(T))}dT. \quad (3.4)
\end{align*}$$

Inserting the above equations into Eq. (2.21), we find that

$$\rho = -\frac{1}{2\kappa} \left\{ U_{,T} + U_{,R} + e^U \right\}_{R=-T}, \quad p = 0. \quad (3.5)$$

To further study the problems, in the following we shall consider only the case where the cylinder is made of matter that satisfies the condition

$$R_{AB}g^{AB} = 0, \quad (A, B = z, \varphi), \quad (3.6)$$

where $R_{\mu\nu}$ denotes the Ricci tensor. It can be shown that several matter fields, such as, null dust fluid, massless scalar field, and electromagnetic field, satisfy the above condition [14,15]. For the metric given by (3.3), Eq. (3.6) yields

$$U_{,TT} - U_{,T}^2 - U_{,RR} + U_{,R}^2 = 0, \quad (3.7)$$

which has the general solution

$$U = -\ln \left[ a(T + R) + b(T - R) \right], \quad (3.8)$$

where $a$ and $b$ are arbitrary functions of their indicated arguments, subject to

$$a + b \geq 0. \quad (3.9)$$

Inserting the above expressions into Eq. (2.23), we find that

$$\rho = \frac{2a' - 1}{2\kappa(a + b)} \bigg|_{R=-T}, \quad p = 0. \quad (3.10)$$

According to

$$\begin{align*}
a) \quad a'b' < 0, & \quad b) \quad a'b' > 0, \\
c) \quad a' = 0, & \quad b' \neq 0, \\
d) \quad a' \neq 0, & \quad b' = 0, \quad (3.11)
\end{align*}$$
the solutions will have physically different interpretations, thus, in the following let us consider them separately.

**Case a)** \( a'b' < 0 \): In this case the normal vector to the hypersurfaces \( \Phi \equiv e^{-U} = \text{Const.} \) is spacelike. As a matter of fact, it is easy to show that

\[
\Phi_{\alpha} \Phi_{\beta} g^{\alpha\beta} = 4a'b'e^M < 0.
\]

Without loss of generality, now we can choose

\[
a = \frac{1}{2}(T + R), \quad b = -\frac{1}{2}(T - R).
\]

In fact, introducing two new coordinates \( \bar{T} \) and \( \bar{R} \) via the relations,

\[
\bar{R} = a + b, \quad \bar{T} = a - b,
\]

we find that the corresponding metric takes the form,

\[
\begin{align*}
d\bar{s}_+^2 &= e^{-\bar{M}} (d\bar{T}^2 - d\bar{R}^2) - \bar{R} \left( e^V d\bar{s}^2 + e^{-V} d\varphi^2 \right),
\end{align*}
\]

where \( e^{-\bar{M}} \equiv e^{-M}/(-4a'b') \), and \( \bar{M} \) and \( V \) now are functions of \( \bar{T} \) and \( \bar{R} \) only. Clearly, the above form of metric is the same as that given by Eqs.(3.3) with the choice of Eq.(3.13), from which we find that

\[
e^{-U} = a + b = R.
\]

Combining Eqs.(3.2) and (3.4), we find that the first junction condition now requires

\[
\begin{align*}
r_0(t) &= R_0(T) = -t = -T \geq 0, \\
V(T, R_0(T)) &= -\ln [R_0(T)], \quad M(T, R_0(T)) = 0.
\end{align*}
\]

Substituting Eq.(3.13) into Eq.(3.10), on the other hand, we find that

\[
\rho = 0 = p.
\]

That is, in this case the matching across the hypersurface \( \Sigma \) is always smooth, and no matter shell appears on the hypersurface. Using the \( C \)-energy defined by Thorne [27],

\[
C_E \equiv \frac{1}{8} \left( 1 - \frac{e^{M-V}}{R} \right),
\]

we find from Eq.(3.17) that on the hypersurface \( \Sigma \) we have

\[
C_E(T, R_0(T)) = 0.
\]

That is, the total \( C \)-energy inside the cylinder \( R = R_0(T) \) is zero. This is what one would expect, as the hypersurface \( \Sigma \) now is free of matter shells and the spacetime, before the wave front arrives, is flat.

**Case b)** \( a'b' > 0 \): In this case it can be shown that the normal vector to the hypersurfaces \( \Phi \equiv e^{-U} = \text{Const.} \) is timelike. Following a similar argument as that given in the last case, one can show that now we can choose

\[
a = -\frac{1}{2}(T + R), \quad b = -\frac{1}{2}(T - R),
\]

so that

\[
e^{-U} = a + b = -T \geq 0.
\]

Inserting Eq.(3.21) into Eq.(3.10), we find

\[
\rho = -\frac{1}{\kappa(-T)}, \quad p = 0.
\]

Since on the hypersurface \( \Sigma \) we have \( T \leq 0 \), from the above we can see that the energy density \( \rho \) is always negative. This is not physical and we shall not consider this case further in the following discussions.

**Case c)** \( a' = 0, \ b' \neq 0 \): In this case, without loss of generality, we can set \( a = 0 \). Then, it can be shown that the hypersurfaces \( \Phi \equiv e^{-U} = b(T - R) = \text{Const.} \) now become null, and the function \( b \) is positive and otherwise arbitrary. Hence, from Eq.(3.10) we find

\[
\rho = -\frac{1}{2kb(2R)}, \quad p = 0,
\]

from which we can see that the energy density \( \rho \) is always negative, too, and this case should be also discarded in the following discussions.

**Case d)** \( a' \neq 0, \ b' = 0 \): Similar to the last case, it can be shown that now the hypersurfaces \( \Phi \equiv e^{-U} = \text{Const.} \) are null, and the function \( a \) is arbitrary, subject to \( a \geq 0 \), with the choice \( b = 0 \). Then, the corresponding surface energy density is given by

\[
\rho = \frac{2a'(0) - 1}{2ka(0)}.
\]

On the other hand, from Eq.(3.4) we find that

\[
r_0(t) = e^{-U(T, R_0(T))} = a(0),
\]

which is not consistent with Eq.(3.2). Thus, this case has to be discarded as representing gravitational collapse, too.

It is interesting to note that in the last two cases the only possible matter field is a null dust fluid with the energy-momentum tensor (EMT) being given by

\[
T_{\mu\nu} = \rho k_\mu k_\nu,
\]

where \( k_\mu \) is a null vector. In the last case it is given by \( k_\mu = \delta^T_\mu + \delta^R_\mu \), and the corresponding EMT represents an ingoing null dust fluid moving along the hypersurfaces \( T + R = \text{Const.} \), while in Case c), it is given by \( k_\mu = \delta^T_\mu - \delta^R_\mu \), and the corresponding EMT represents an outgoing null dust fluid moving along the hypersurfaces \( T - R = \text{Const.} \). In both of the two cases, an, ingoing
in the last case, and outgoing in Case c), purely gravitational cylindrical wave is present, except for the case where \( V = \text{Const} \).

In review of all the above, we find that the gravitational collapse in the last three cases in an otherwise flat and cylindrical spacetime background cannot be realized. The only case that is relevant to gravitational collapse is Case a). This result is a little bit of surprise, and mainly due to the requirement that, before the wave front arrives, the spacetime be flat and take the cylindrical form Eq.(3.1). Although Birkhoff’s theorem for the spherically symmetric case is not applicable to the cylindrically symmetric case [14], from the physical point of view this requirement seems quite reasonable.

To further study the problem, let us now turn to consider gravitational collapse of cylindrical purely gravitational waves.

IV. GRAVITATIONAL COLLAPSE OF PURELY GRAVITATIONAL WAVES

As shown in the last section, the only case that is relevant to gravitational collapse in an otherwise flat background is Case a), in which the null hypersurface \( r = r_0(t) \) [or \( R = R_0(T) \)] is free of matter, \( \tau_{ab} = 0 \), and the function \( U(T, R) \) is chosen as

\[
U = - \ln R. \tag{4.1}
\]

For this gauge, it can be shown that only three of the vacuum Einstein field equations \( R_{\mu\nu} = 0 \) are independent and can be written in the form

\[
\Omega_{,R} = \frac{R}{2} \left( V_T^2 + V_R^2 \right), \tag{4.2}
\]

\[
\Omega_{,T} = RV_{,T}V_{,R}, \tag{4.3}
\]

\[
V_{,RR} + \frac{1}{R} V_{,R} - V_{,TT} = 0, \tag{4.4}
\]

where

\[
\Omega(T, R) = \frac{1}{2} \ln R - M(T, R). \tag{4.5}
\]

Thus, once the solution of Eq.(4.4) is known, the function \( \Omega \) can be obtained by the quadrature from Eqs.(4.2) and (4.3). The general solution of Eq.(4.4) is given by [18,19]

\[
V = \int_0^\infty \left[ A(\omega) \sin(\omega T) + B(\omega) \cos(\omega T) \right] J_0(\omega R) d\omega
+ \int_0^\infty \left[ C(\omega) \sin(\omega T) + D(\omega) \cos(\omega T) \right] N_0(\omega R) d\omega
+ \alpha T + \beta \ln R + \sum_{k=1}^{N} h_k \ln \left( \frac{\mu_k}{R} \right), \tag{4.6}
\]

where

\[
\mu_k \equiv T_k - T \pm \left[ (T_k - T)^2 - R^2 \right]^{1/2}, \tag{4.7}
\]

\( \alpha, \beta, A(\omega), B(\omega), C(\omega), D(\omega), \omega, h_k \) and \( T_k \) are arbitrary constants, and \( N \) is an integer. The functions \( J_0(\omega R) \) and \( N_0(\omega R) \) denote the Bessel and Neumann functions of zero order, respectively. The integration constants \( T_i \) can be real or complex, and in general takes the form

\[
T_k = T_k^0 + i w_k, \quad (k = 1, 2, \ldots, N), \tag{4.8}
\]

where \( T_k^0 \)’s and \( w_k \)’s are real. If one starts with a complex term, say, \( \mu_i \), then its complex conjugate \( \mu_i^* \) should be also included, in order to make the metric coefficients to be real. Note that when \( T_k \) is real, the solution is defined only in the regions \( (T_k - T)^2 \geq R^2 \) [cf. Fig. 2].

![FIG. 2. The function \( \mu_k \) appearing in the expression (4.6) is defined only in the regions \( (T_k - T)^2 \geq R^2 \) when \( T_k \) is real. On the light cone, we have \( \mu_k = R \).](image)

On the light cone \( (T_k - T)^2 = R^2 \), we have \( \ln(\mu_k / R) = 0 \). Then, from Eq.(4.6) we can see that \( V(T, R) \) matches continuously to the solution that does not include \( \mu_k \) in the region \((T_k - T)^2 \leq R^2\). Belinskii and Francaviglia interpreted this as gravitational shock-wave solutions [20], but later Gleiser showed that in general the light cone also support a null dust fluid [21]. Yet, Curir et al. found that sometimes the solutions have curvature singularities
on the light cone. In any case, since here we are mainly concerned with gravitational collapse of gravitational waves only, we shall not consider this possibility any more, and simply assume that all the $\mu_k$’s are complex, as in the latter case there are no such restrictions and the solutions are valid in the whole spacetime. The term related to $\mu_k$ is usually referred to as generalized soliton solutions \[23\].

On the hypersurface $\Sigma$ where $T = - R$, the first junction condition Eq. (4.17) requires

\[ V(-R,R) = -\ln R, \quad M(-R,R) = 0. \]  

Since for the general solution of Eq. (4.3), the function $M(T,R)$ is not known, in the following let us consider the above condition only in the two limits: $R \to 0$ and $R \to \infty$. When $R \to 0$, we have

\[ J_0(\omega R) \to 1, \quad N_0(\omega R) \to -2 \pi \ln \left(\frac{\omega R}{2}\right), \]

\[ \mu_k^+ \to 2T_k, \quad \mu_k^- \to \frac{R^2}{2T_k}. \]  

(4.10)

We assume, without loss of generality, that in Eq. (4.6) the first $n$ $\mu_k$’s take the values of $\mu_k^+$ and the rest $(N-n)$ $\mu_k$’s take the values of $\mu_k^-$. Then we find that

\[ V(-R,R) \to (\beta + h^- + d) \ln R, \]  

(4.11)

as $R \to 0$, where

\[ h^- = \sum_{k=1}^{n} h_k, \quad d = \frac{2}{\pi} \int_{0}^{\infty} D(\omega) d\omega. \]  

(4.12)

The junction condition Eq. (4.11) requires

\[ \beta + h^- + d = -1. \]  

(4.13)

When $R \to \infty$, we have

\[ J_0(\omega R) \to \left(\frac{2}{\pi \omega R}\right)^{1/2} \cos \left(\omega R - \frac{\pi}{4}\right), \]

\[ N_0(\omega R) \to \left(\frac{2}{\pi \omega R}\right)^{1/2} \sin \left(\omega R - \frac{\pi}{4}\right), \]

\[ \mu_k^\pm \to R, \]  

(4.14)

from which we find that

\[ V(-R,R) \to -\alpha R + \beta \ln R. \]  

(4.15)

Thus, the first junction condition Eq. (4.13) in this limit requires that

\[ \alpha = 0, \quad \beta = -1. \]  

(4.16)

Combining Eqs. (4.13) and (4.16) we find that

\[ V(-R,R) = -\ln R, \quad M(-R,R) = 0. \]  

(4.17)

To further study the problem, now let us turn to study the spacetime outside the hypersurface $\Sigma$, i.e., region $V^+$. In particular, following \[23\], we shall concentrate ourselves in the four different regions: (i) The spacelike infinity, $|T| \ll R \to \infty$; (ii) the future null infinity, $R \sim T \to \infty$; (iii) the timelike future infinity, $R \ll T \to \infty$; and (iv) the focusing region, $R \sim T \to 0^+$ [cf. Fig. 3].

(i) The spacelike infinity, $|T| \ll R \to \infty$: In this limit, let us first consider the first two terms in Eq. (4.6), which will be denoted by $V_{FB}(T,R)$. Using Eq. (4.14), we find that

\[ V_{FB} \to \left(\frac{1}{2\pi R}\right)^{1/2} \int_{0}^{\infty} \left\{ E^+(\omega) \cos \left[\omega(R-T) + \phi^+(\omega)\right] + E^-(\omega) \cos \left[\omega(R+T) + \phi^-(\omega)\right] \right\} \frac{d\omega}{\omega^{1/2}}, \]  

(4.18)

where

\[ E^\pm(\omega) = \left\{ [A(\omega) \pm D(\omega)]^2 + [C(\omega) \mp B(\omega)]^2 \right\}^{1/2}, \]

\[ \phi^\pm(\omega) = \tan^{-1} \left[ \frac{C(\omega) \mp B(\omega)}{A(\omega) \pm D(\omega)} \right]. \]  

(4.19)

It is clear that, as $R \to \infty$, these two terms represent a superposition of two traveling gravitational waves along the radial direction, one is ingoing represented by the term $E^-\omega$ and the other is outgoing represented by the term $E^+\omega$. When the waves have only a single wavelength, say, $\lambda_0$, it can be shown that the corresponding function $M(T,R)$ takes the form \[24\],

\[ M_{FB} \to \frac{1}{2} \ln R - \frac{2\omega_0}{\pi} \left[ (A_0^2 + B_0^2 + C_0^2 + D_0^2) R + 2(B_0C_0 - A_0D_0) T \right. \]

\[ - \frac{1}{2\pi} \left\{ E_0^- \cos \left[2\omega_0(T + R) - \theta_0^-\right] \right\} + E_0^+ \cos \left[2\omega_0(R - T) - \theta_0^+\right] \}, \]  

(4.20)

where $\omega_0 = 2\pi/\lambda_0$,

\[ \theta_0^\pm \equiv \tan^{-1} \left[ \frac{2(A_0 \pm D_0)(C_0 \mp B_0)}{(A_0 \pm D_0)^2 - (C_0 \mp B_0)^2} \right]. \]  

(4.21)

and $A_0 \equiv A(\omega_0)$, etc. It can be shown that the spacetime is singular in this limit. Since we are considering gravitational collapse, the more interesting case would be that the spacetime is asymptotically flat in the radial direction, and it is possibly singular only at $R = 0$ for $T > 0$. Then, we are sure that such singularities are indeed formed by the collapse. Therefore, in the following we shall assume that these two terms vanish identically,

\[ A(\omega) = B(\omega) = C(\omega) = D(\omega) = 0. \]  

(4.22)

Hence, from Eq. (4.12) we find that $d = 0$. Substituting it into Eq. (4.17) we obtain

\[ \alpha = 0, \quad \beta = -1, \quad d + h^- = 0. \]  

(4.17)
\[
\sum_{k=1}^{n} h_{k}^{-} = 0. \tag{4.23}
\]

The third term in Eq. (4.18) vanishes too, because of the first junction condition (4.17). Thus, in the following let us concentrate ourselves only on the fourth and last (soliton) terms with \( \beta = -1 \). When the soliton term vanishes, the spacetime becomes Minkowskian.

As we mentioned previously, in this paper we consider only the case where \( T_k \) is complex. For the complex \( T_k \) it is found convenient to introduce the following quantities \[23\],

\[
\mu_{k}^{+} = R\sigma_{k}^{1/2}e^{i\gamma_{k}}, \quad \mu_{k}^{-} = \frac{R^{2}}{\mu_{k}^{+}} = R\sigma_{k}^{-1/2}e^{-i\gamma_{k}}, \tag{4.24}
\]

where

\[
\sigma_{k} = L_{k} + (L_{k}^{2} - 1)^{1/2},
\]

\[
L_{k} = \frac{L_{k}^{0}}{R^{2}} + \left[ 1 - \frac{2L_{k}^{0}}{R^{2}} + \left( \frac{L_{k}^{0}}{R^{4}} \right)^{2} \right]^{1/2},
\]

\[
L_{k}^{0} \equiv T_{k}^{0} \pm w_{k}^{2}, \quad T_{k} = T_{k}^{0} - T,
\]

\[
\gamma_{k} = \cos^{-1} \left[ \frac{2\mu_{k}^{1/2}T_{k}}{R(1 + \sigma_{k})} \right], \tag{4.25}
\]

If we further assume that

\[
h_{k+n/2}^{-} = (h_{k}^{-})^{*} = \alpha_{k}^{-} - i \beta_{k}^{-},
\]

\[
h_{k+n+(N+n)/2}^{-} = (h_{k+n}^{-})^{*} = \alpha_{k}^{+} - i \beta_{k}^{+}, \tag{4.26}
\]

where \( \alpha^{\pm} \)'s and \( \beta^{\pm} \)'s are real constants, we find that

\[
V_{\text{soliton}} = -\ln R + \sum_{k=1}^{N} h_{k} \ln \left( \frac{\mu_{k}}{R} \right)
\]

\[
= -\ln R + \sum_{k=1}^{s} \left( \frac{g_{k}}{2} \ln \sigma_{k} + f_{k}\gamma_{k} \right), \tag{4.27}
\]

where \( s \equiv N/2 \), and

\[
g_{k} = \begin{cases} 
-2\alpha_{k}^{-}, & 1 \leq k \leq n/2 \\
2\alpha_{k+n/2}^{-}, & n/2 + 1 \leq k \leq N/2
\end{cases},
\]

\[
f_{k} = \begin{cases} 
2\beta_{k}^{-}, & 1 \leq k \leq n/2 \\
2\beta_{k+n/2}^{-}, & n/2 + 1 \leq k \leq N/2
\end{cases}. \tag{4.28}
\]

In terms of \( \alpha_{k}^{-} \)'s, Eq. (4.23) takes the form

\[
\sum_{k=1}^{n/2} \alpha_{k}^{-} = 0. \tag{4.29}
\]

The general solution (4.27) was studied in details in [23], thus in the following we shall only summarize its main properties, and for the details, we refer readers to [23]. In particular, it can be shown that

\[
\sigma_{k} \rightarrow 1 + \frac{2\omega_{k}}{R} + O(R^{-2}),
\]

\[
\gamma_{k} \rightarrow \frac{\pi}{2} - \frac{T_{k}}{R} + O(R^{-3}), \tag{4.30}
\]

as \( R \rightarrow \infty \). Thus, from the above we can see that the terms that are proportional to \( \sigma_{k} \) are negligible, while the terms that proportional to \( \gamma_{k} \) have the following distributions to the metric coefficients,

\[
V_{\text{soliton}} \rightarrow -\ln R + \sum_{k=1}^{s} f_{k}\gamma_{k},
\]

\[
M_{\text{soliton}} \rightarrow \frac{f^{2}}{2} \ln R, \tag{4.31}
\]

where

\[
f = \sum_{k=1}^{s} f_{k} = 2 \left( \sum_{k=1}^{n/2} \beta_{k}^{-} + \sum_{k=(n+2)/2}^{(N+n)/2} \beta_{k}^{+} \right). \tag{4.32}
\]

Thus, in this limit the solution is singular unless \( f = 0 \).

With the same reason as that given following Eq. (4.21), in the following we shall assume that

\[
\sum_{k=1}^{n/2} \beta_{k}^{-} + \sum_{k=(n+2)/2}^{(N+n)/2} \beta_{k}^{+} = 0, \quad (f = 0). \tag{4.33}
\]

Then, we can see that the solution will represent the gravitational collapse of purely gravitational cylindrical waves with lightlike wave-front, on which no dust shell appears. The spacetime is asymptotically flat in the radial direction and locally flat near the symmetry axis.

(ii) The future null infinity, \( R \sim T \rightarrow \infty \): In this limit, it can be shown that

\[
\sigma_{k} \rightarrow 1 + O \left( R^{-1/2} \right),
\]

\[
\gamma_{k} \rightarrow \pi + O \left( R^{-1/2} \right), \tag{4.34}
\]

and that the metric coefficients behave like

\[
V(T, R) \rightarrow -\ln R + O \left( R^{-1/2} \right),
\]

\[
M(T, R) \rightarrow O \left( R^{-1/2} \right). \tag{4.35}
\]

Thus, in this region the spacetime is asymptotically Minkowskian, too. A detailed study of the Riemann tensor showed that the spacetime in this region is Petrov type N and represents an outgoing gravitational wave along the light cone \[23\].

(iii) The timelike future infinity, \( R \ll T \rightarrow \infty \): The study of the solutions in this region will give us some information about the final state of the collapse. It can be shown that

\[
\sigma_{k} \rightarrow \frac{4T^{2}}{R^{2}},
\]

\[
\gamma_{k} \rightarrow \pi + O \left( \frac{R^{2}}{4T^{2}} \right), \tag{4.36}
\]
and
\[ V_{\text{soliton}} \to -(1 + g^+) \ln R + g^+ \ln T, \]
\[ M_{\text{soliton}} \to -\frac{1}{2} g^+ (2 + g^+) \ln R \\
+ g^+ (1 + g^+) \ln T, \]  
(4.37)
where
\[ g^+ \equiv \frac{\sum_{k=1+n/2}^{(N+n)/2} \alpha^+_k}{2}. \]  
(4.38)

When \( g^+ = 0 \), we can see that the collapsing gravitational waves are reflected completely from the symmetry axis to spacelike infinity along the future light cone, and nothing is left behind, so that the final state of the spacetime is Minkowskian.

(iii) The focusing region, \( R \sim T \to 0^+ \): This is the region where the collapsing gravitational waves just arrive at the axis, and start to accumulate there. It can be shown that
\[ \sigma_k \to 4 \left( \frac{T^0_k}{R^2} \right)^2 + \frac{w_k^2}{R^2}, \]
\[ \gamma_k \to \gamma^0_k, \]  
(4.40)
and
\[ V_{\text{soliton}} \to -(1 + g^+) \ln R, \]
\[ M_{\text{soliton}} \to -\frac{1}{2} g^+ (2 + g^+) \ln R, \]  
(4.41)
where \( \gamma^0_k \) is a constant. Thus, the spacetime in this region is singular, unless the conditions given by Eq.(4.39) hold.

In review of all the above, we have the following: The only solutions that represent gravitational collapse of purely gravitational cylindrical waves in an otherwise flat spacetime background are given by Eq.(4.27) subject to the conditions of Eqs.(4.29) and (4.33). After the waves arrive at the symmetry axis, part of them is reflected to spacelike infinity along the light cone \( R \sim T \to \infty \), and part of them is focused to form spacetime singularity on the axis, except for the cases defined by Eq.(4.39). The corresponding Penrose diagram is given by Fig. 3.

\[ \text{FIG. 3. The corresponding Penrose diagram for gravitational collapse of purely gravitational cylindrical waves. The wave-front is denoted by the straight line } T + R = 0. \text{ Before the waves arrives, the spacetime is flat, denoted by } M_4. \text{ The spacetime is asymptotically flat at spacelike infinite } |T| < R \to \infty \text{ in the radial direction. When the wave-front arrives at the symmetry axis } (T, R) = (0, 0), \text{ in general part of the wave is reflected to infinity along the light cone } T - R = 0 \text{ and part of it is focused to spacetime singularity on the axis, denoted by the curved vertical line.} \]

Then, we find that the spacetime in this region is always singular, except for the cases
\[ (i) \ 0 \leq g^+ \leq 1, \quad \text{or} \quad (ii) \ g^+ = -2. \]  
(4.39)

\[ \text{In this paper, the general matching of two cylindrically symmetric regions across a null hypersurface is studied using the Bärrabé-Israël method. It is shown that in general the hypersurface supports a null dust shell. As an application of these formulae, we study the collapse of a matter field that satisfies the condition } R_{AB}g^{AB} = 0, \ (A, B = z, \varphi), \text{ in an otherwise flat spacetime background. It is found that the only case that represents gravitational collapse is that where the normal vector to the hypersurface } \Phi = \text{Const.} \text{ is space-like, where } \Phi \equiv (g_{zz} g_{\varphi \varphi})^{1/2}. \text{ As a result, the gravitational collapse of a purely gravitational cylindrical wave or a pure null dust fluid cannot be realized in a flat spacetime background.} \]

The general formulae are further applied to the collapse of purely gravitational cylindrical waves in a flat spacetime background and the general conditions for such collapse are found. It is shown that after the waves arrive at the axis, part of them is reflected to spacelike infinity along the future light cone, and part of it is focused to form spacetime singularities on the symmetry axis. The
only cases where no spacetime singularity is formed are these given by Eq. (4.39).

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