Quantum critical point and scaling in a layered array of ultrasmall Josephson junctions.

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Abstract

We have studied a quantum Hamiltonian that models an array of ultrasmall Josephson junctions with short range Josephson couplings, $E_J$, and charging energies, $E_C$, due to the small capacitance of the junctions. We derive a new effective quantum spherical model for the array Hamiltonian. As an application we start by approximating the capacitance matrix by its self-capacitive limit and in the presence of an external uniform background of charges, $q_x$. In this limit we obtain the zero–temperature superconductor–insulator phase diagram, $E_J^{\text{crit}}(E_C, q_x)$, that improves upon previous theoretical results that used a mean field theory approximation. Next we obtain a closed–form expression for the conductivity of a square array, and derive a universal scaling relation valid about the zero–temperature quantum critical point. In the latter regime the energy scale is determined by temperature and we establish universal scaling forms for the frequency dependence of the conductivity.

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Quantum phase transitions have attracted a significant amount of interest in recent years. There are different physical systems where quantum phase transitions can be studied, like in magnetic systems, the quantum Hall effect, superconducting films, and Josephson junction arrays (JJA). There have been significant advances in lithographic techniques that allow the fabrication of arrays of ultra-small superconducting islands, with charging energy, $E_C$, that can dominate the Josephson coupling energy, $E_J$, making quantum fluctuation effects of paramount importance. Because the junction parameters can be controlled accurately JJA offer a unique system where one can test the nature of quantum phase transitions and critical phenomena, in particular the superconductor–insulator (SI) phase transition induced by quantum fluctuations. Therefore JJA can be a prototype system that can display well controlled quantum SI phase transitions.

In the JJA problem, the two relevant low temperature energy scales are $E_J$, that permits tunneling of Cooper pairs between islands, and $E_C$ that tends to localize the charge carriers in the islands. When $E_J$ is much larger then $E_C$ the phases in the islands are well defined. In this regime only the phases determine the properties of the JJA. In the opposite limit, i.e. $E_C >> E_J$ the superconducting phase is perturbed by strong zero point quantum fluctuations due to the Coulomb blockade that localizes the charge carriers to the islands. Several theoretical and experimental studies have considered the competition between the $E_J$ dominated phase and the $E_C$ dominated charging energy regions in periodic JJA. It has been established that for sufficiently large charging energy the quantum phase fluctuations lead to a complete suppression of long–range phase coherence even at zero temperature. This type of quantum phase transition has attracted significant interest in recent years (for a review see Ref. [10] and also [11]). Generally, these transitions take place at zero temperature where crossing the phase boundary is accompanied by a change in the ground state of the system. This transition is induced at zero temperature by changing some external parameter in the Hamiltonian of the system, for example the charging energy in the Josephson-junction arrays (given by the quantum parameter $\alpha \equiv E_C/E_J$). Let the critical temperature be $T_c$ and denote the distance from criticality as $\delta T = 1 - T/T_c$. The critical behavior is asymptotically close to the critical point which is entirely determined by classical physics. This is because the characteristic frequency associated with the critical fluctuations, $\omega_c \sim 1/\xi \tau$, vanishes at criticality, and the characteristic correlation time for relaxation towards equilibrium $\xi \tau \sim \xi^z$ diverges at the critical point (here $\xi \sim |t|^{-\nu}$ is the diverging characteristic correlation length in the system.) A quantum system behaves classically if the temperature is larger than all frequencies of interest, i.e. as long as $\hbar \omega_c << k_B T$, which is always the case when the system is sufficiently close to the critical point with a non zero $T_c$. In the critical region various physical quantities can have scaling forms (i.e. given in terms of homogeneous functions) as a consequence of the diverging $\xi$ and $\xi \tau$. Since the experimentally accessible systems are necessarily at non-zero temperature, one needs to understand how the quantum behavior is modified at finite temperature. The important observation is that at finite temperature the partition function has a finite temporal dimension, since the effective classical system extends to an extra dimension of size $\beta \hbar$.

At $T = 0$, the 2D JJA problem goes into a 3–dimensional classical XY model with the critical behavior of the 3D XY model. Moreover, there should be a clear signature in the
nature of the correlation functions (like for example in linear response) when crossing over from the 2D XY model at high temperatures to the 3D XY model as $T \to 0$. Specifically, the response of the quantum critical point to a small temperature perturbation will be determined by the location of the quantum–to–classical crossover. An especially interesting quantum critical regime appears when the relaxational frequency satisfies $\hbar \omega \approx k_B T$. In this regime the only relevant scale is given by the temperature and the dynamical scaling functions will only depend on the ratio $\hbar \omega / k_B T$. As the ratio $\hbar \omega / k_B T$ is varied in an experiment we expect to see a crossover between temperature– and frequency–dominated scaling regimes. For the finite temperature experiments one can scale the data by using the knowledge of the frequency dependent scaling functions to test the existence or probing of a quantum critical point. The experimental identification of a quantum phase transition in JJA will rely upon finding the scaling behavior with the relevant parameters as temperature, frequency and the wavelength dependence of various observables. The specific signatures are given by universal values of certain dimensionless amplitudes as, for example, the resistivities at the critical point in 2D JJA systems. Motivated by these issues, it is the goal of the present paper to study the superconductor–insulator transition in 2D JJA in the quantum critical regime, by focusing on the scaling properties of the directly measurable electromagnetic conductivities in JJA systems.

The outline of the rest of the paper is the following: In Section II we define the model Hamiltonian, followed by a novel path integral formulation of the problem. In section III we present as a test of our quantum spherical model approach for the self-capacitance model. There we show that the spherical technique gives a better answer than the one obtained from mean field theory. In Section IV we present our calculation of the scaling properties of the conductivity in the quantum critical regime. Finally, in Section V we briefly discuss our results.

II. THE MODEL

A. Quantum phase Hamiltonian

A Josephson junction array (JJA) can be modeled by a periodic lattice of superconducting islands separated by insulating barriers. Each island becomes independently superconducting about the bulk transition temperature $T_{c0}$ and it is characterized by an order parameter $\psi(r_i) = |\psi_0(r_i)| e^{i\phi(r_i)}$, where $r_i$ is a two–dimensional vector denoting the position of each island. The magnitude of the order parameter, $|\psi_0(r_i)|$, is non–fluctuating when the temperature is lowered further and the onset of long range phase coherence due to the tunneling of Cooper pairs between the islands is responsible for the zero resistance drop in the arrays. The phase coherence onset temperature in JJAs can be significantly reduced by making the junctions ultrasmall. When the junction’s capacitance is small the charging energy (i.e. the energy necessary to transfer Cooper pair charges between the islands) increases to the point where no pairs can tunnel anymore, completely quenching the Josephson current. The competition between the Josephson tunneling and the charging (Coulomb) energy, without dissipation, can be modeled by the Hamiltonian

$$H = H_C + H_J,$$
\[ H_C = -\frac{1}{2} \sum_r \langle C^{-1} \rangle_{rr'} \hat{Q}_r \hat{Q}_{r'}, \]
\[ H_J = \sum_{\langle r_1, r_2 \rangle} J(|r_1 - r_2|) [1 - \cos(\phi_{r_1} - \phi_{r_2})]. \]  

Here, \( \hat{Q}_r = (2e/i)\partial/\partial \phi_r \) is the charge operator while \( \hat{\phi}_r \) represents the superconducting phase operator of the grain at the site \( r \). \( J(|r_1 - r_2|) \) is the site–dependent Josephson coupling and \( \langle C^{-1} \rangle_{rr'} \) is the inverse capacitance matrix.

### B. Euclidean action and path integral representation

It is useful to derive a field-theoretic representation of the partition function for Eq (1) to study the quantum nature of the superconductor-insulating (SI) phase transition. A convenient procedure is to introduce a path–integral representation in a basis diagonal in \( \phi_j \). In this representation the partition function is expressed as

\[ Z = \int \prod_r \mathcal{D}\phi_r e^{-S[\phi]} . \]  

Here the functional integral is evaluated over the phases restricted to the compact interval \([0, 2\pi]\) and with an effective action \( (\hbar = 1) \)
\[ S[\phi] = S_C[\phi] + S_J[\phi], \]
\[ S_C[\phi] = \frac{1}{8e^2} \int_0^\beta d\tau \sum_{r,r'} \langle C^{-1} \rangle_{rr'} \left( \frac{\partial \phi_r}{\partial \tau} \right) \left( \frac{\partial \phi_{r'}}{\partial \tau} \right), \]
\[ S_J[\phi] = \int_0^\beta d\tau \sum_{\langle r_1, r_2 \rangle} J(|r_1 - r_2|) \{1 - \cos[\phi_{r_1}(\tau) - \phi_{r_2}(\tau)]\}. \]  

Because the values of the phases which differ by \( 2\pi \) are equivalent, the path integral can be written in terms of non–compact phase variables \( \theta_r(\tau) \), defined on the unrestricted interval \([-\infty, +\infty]\), and by a set of winding numbers \{\( n_r \)\}, which are integers running from \(-\infty\) to \(+\infty\). Consequently, the path–integral representation in Eq.(2) also includes a summation over winding numbers\[ \phi_r(\tau) = \theta_r(0) + 2\pi n_r \tau / \beta + \theta_r(\tau) \] which reflects the discreteness of the charge, so that the integration measure reads

\[ \int \mathcal{D}\phi_r \equiv \sum_{\{n_r\}} \prod_r \int_0^{2\pi} d\theta(0) \int_{\theta(0)}^{\theta(0)+2\pi n_r} \mathcal{D}\theta_r(\tau). \]  

This concludes the formulation of the model. In the following section we will transform this representation to one where the order parameter field is expressed in a novel quantum spherical model self–consistent scheme.

### III. Quantum Spherical Model Approach

To study the JJA model it appears at first natural to use a description in terms of an effective Ginsburg–Landau functional derived from the microscopic model of Eq. (3).
Several studies of JJA have followed this route, also known as the coarse grained approach first developed by Doniach. The key point of this method is to introduce a complex field order parameter \( \psi_r \) (or equivalently a two component real field) whose expectation value is proportional to the \( S_r(\phi) \) vector defined by

\[
S_r(\phi) = [S^x_r(\phi), S^y_r(\phi)] \equiv [\cos(\phi_r), \sin(\phi_r)].
\] (5)

The non-zero thermal average \( \langle S^x(\phi) \rangle = \langle \cos \phi_i \rangle \) describes the “phase-locking” or long-range phase ordering in the model. The system is governed by the Ginzburg-Landau functional as long as the order parameter is small and the decoupling of the Josephson energies is valid. This is a serious shortcoming of the coarse grained approach since the method is restricted to the region of parameters in the vicinity of the critical point and does not offer a self-consistent description of the full problem.

There are no nontrivial exact solutions of the model, except for some molecular field-type approaches. It is therefore reasonable to develop new approximate mappings of the model which may admit exact solutions, both in the ordered in disordered phases. From the definition of the pseudo-spin variables \( S_r \), the following rigid constraint holds for each one,

\[
|S_r(\phi)|^2 = [S^x_r(\phi)]^2 + [S^y_r(\phi)]^2 = \cos^2(\phi_r) + \sin^2(\phi_r) \equiv 1.
\] (6)

The relation in Eq. (6) also implies that a weaker condition also applies, namely:

\[
\sum_r |S_r(\phi)|^2 = N.
\] (7)

The main idea of our approach is to attempt to generate an effective partition function from the original one with cosine interaction, which incorporates the constrained nature of the original variables. This leads us to the formulation of the problem in terms of the spherical model by introducing the appropriate constrained order parameter field. The name of the model comes from the observation that in Eq.(7) the allowed states of the spherical model are all points on the surface of a hypersphere of radius \( N \). Although the spherical model was originally introduced as an approximation to the classical Ising model, the “sphericalization” technique may be readily applied to a variety of other problems (for a quantum generalization of the spherical model, see Ref. 20); the one key necessary ingredient is the existence of an inherent constraint on the Hamiltonian variables as in Eq.(5). The model defined by Eq.(7) is in fact a hybrid of the spherical and the two component (\( M = 2 \)) vector models. The global constraint of Eq. (7) may be introduced on the continuous order parameter field by using the functional analogue of the Dirac–delta function

\[
1 \equiv \int \left[ \prod_r D\psi_r D\psi^*_r \right] \delta \left( \sum_r |\psi_r(\tau)|^2 - N \right) \prod_r \delta \left[ \text{Re} \psi_r(\tau) - S^x_r(\phi(\tau)) \right] \delta \left[ \text{Im} \psi_r(\tau) - S^y_r(\phi(\tau)) \right],
\] (8)

where the \( \psi_r(\tau) \) are the complex \( c \)-number fields, which satisfy the quantum periodic boundary condition \( \psi_r(\beta) = \psi_r(0) \), and taken as continuous variables, i.e., \( -\infty < \psi_r(\tau) < +\infty \), but constrained (on the average) to have unit length. The partition function of Eq. (2) then reads
One may wonder whether the introduction of the spherical condition on the order parameter fields in Eq. (7) is essential since the global constraint is automatically fulfilled by virtue of the exact relation (6). Indeed, in a rigorous treatment of the partition function of Eq. (2), the constraint will be spurious; in any approximate treatment of the functional integral for $Z$, however, (as for example in a coarse-graining approach) the introduction of the unrestricted order parameter field generally will lead to a violation of Eq. (6). In such a case the introduction of the spherical constraint on the order parameter field in the partition function simply reflects the constrained nature of the original pseudo-spin variables.

Integrating over the phase variables leaves the statistical sum entirely written in terms of the constrained continuous order parameter field $\psi_r(\tau)$, so that the partition function becomes

$$Z = \int \left[ \prod_r \mathcal{D}\psi_r \mathcal{D}\psi^*_r \right] \delta \left( \sum_r |\psi_r(\tau)|^2 - N \right) e^{-S_{J}[\psi]} \times$$

$$\times \int \left[ \prod_r \mathcal{D}\phi_r \right] e^{-S_{C}[\phi]} \prod_r \delta \left[ \text{Re} \psi_r(\tau) - S^x_r(\phi(\tau)) \right] \delta \left[ \text{Im} \psi_r(\tau) - S^y_r(\phi(\tau)) \right],$$

(9)

where the effective order parameter action reads

$$S_{\text{eff}}[\psi] = S_{J}[\psi] + \Phi_{C}[\psi],$$

(11)

with

$$S_{J}[\psi] = \int_0^\beta d\tau \sum_{\langle r_1, r_2 \rangle} J(|r_1 - r_2|) \psi^*_r(\tau) \psi_{r_2}(\tau),$$

(12)

and

$$e^{-\Phi_{C}[\psi]} = \int \left[ \prod_r \frac{\mathcal{D}\mu_r}{2\pi i} \right] \exp \left[ - \left( \sum_r \int_0^\beta d\tau \mu_r(\tau) \cdot \psi_r(\tau) - W[\mu] \right) \right],$$

(13)

where the generating functional of the cumulant multipoint phase correlators of the “non-interacting” system (i.e., involving only charging energy terms) is

$$W[\mu] = \ln \int \left[ \prod_r \mathcal{D}\phi_r \right] \exp \left( \sum_r \int_0^\beta d\tau \mu_r(\tau) \cdot S_r(\phi(\tau)) + S_{C}[\phi] \right)$$

(14)

with the variables $\mu$ acting as the source fields. It can be seen that from Eq. (14) that $e^{-\Phi_{C}[\psi]}$ is just the functional Fourier transform of $e^{-W[\mu]}$. The standard way to proceed is to calculate $\Phi_{C}[\psi]$ using the saddle point method and a subsequent loop expansion in terms of the powers of the order parameter using Eq. (14). The structure of this expansion is briefly described in the Appendix.

It is convenient to employ the functional Fourier representation of the $\delta$-functional to resolve the spherical constraint in Eq. (10):
\[
\delta [g(\tau)] = \int_{-i\infty}^{+i\infty} \left[ \frac{D\xi(\tau)}{2\pi i} \right] \times \exp \left[ \int_0^\beta d\tau \xi(\tau) g(\tau) \right].
\]

(15)

Accordingly, we can write
\[
\delta \left( \sum_r |\psi_r(\tau)|^2 - N \right) \equiv \int_{-i\infty}^{+i\infty} \left[ \frac{D\lambda(\tau)}{2\pi i} \right] \times \exp \left[ \int_0^\beta d\tau \lambda(\tau) \left( \sum_r |\psi_r(\tau)|^2 - N \right) \right].
\]

(16)

We may factor the trace over each \( \psi_r(\tau) \) at the expense of introducing a new integral over one component field \( \lambda(\tau) \). Expanding part of the effective action \( \Phi_C[\psi] \) to second order in \( \psi_r(\tau) \) we get from Eq.(10)
\[
Z_{QSA} = \int \left[ \prod_r D\psi_r D\psi_r^* \right] \int \left[ \frac{D\lambda}{2\pi i} \right] e^{-S_{QSA}[\psi,\lambda]},
\]

(17)

where
\[
S_{QSA}[\psi, \lambda] = \int_0^\beta d\tau d\tau' \left\{ \sum_{(r_1,r_2)} \left[ (|J| r_1 - r_2| + \lambda(\tau) \delta_{r_1,r_2}) \delta(\tau - \tau') \right.ight.
\]
\[
+ \left. \Gamma_{02}^{-+}(r_1 \tau; r_2 \tau') \right\} \psi^*_{r_1}(\tau) \psi_{r_2}(\tau') - N\lambda(\tau) \delta(\tau - \tau') \right\}.
\]

(18)

is the effective action of the quantum spherical approximation (QSA). Subsequent improvements over the spherical model approach may be obtained by considering the saddle point corrections to the Lagrangian parameter \( \lambda \). This can be conveniently done in terms of a loop expansion as described in the Appendix. Furthermore, \( \Gamma_{02}^{-+}(r_1 \tau; r_2 \tau') = [W_{02}^{-1}]^{-+}(r_1 \tau; r_2 \tau') \) is the two–point phase vertex correlator and
\[
W_{02}^{-+}(r_1 \tau; r_2 \tau') = \frac{1}{Z_0} \sum_{\{n_r\}} \prod_r \int_0^{2\pi} d\theta(0) \int_{\theta(0)}^{\theta(0)+2\pi n_r} D\theta_r(\tau) e^{i[\theta_{r_1}(\tau) - \theta_{r_2}(\tau')]},
\]

(19)

with
\[
Z_0 = \sum_{\{n_r\}} \prod_r \int_0^{2\pi} d\theta(0) \int_{\theta(0)}^{\theta(0)+2\pi n_r} D\theta_r(\tau) e^{-S_C[\theta]},
\]

(20)

is the partition function for the “non–interacting” system. Eq. (18) incorporates the quadratic term proportional to \( \lambda(\tau) \) and the integration in Eq.(17) takes place over all configurations of the order parameter field. The model is now unconstrained and quadratic, so all quantities can be readily computed. In the thermodynamic limit, \( N \to \infty \), we can calculate the functional integral in Eq.(17) by the steepest–descent method. To proceed, we introduce propagators associated with the order parameter field defined by
\[
G(r_1 \tau; r_2 \tau') = \langle \psi^*_{r_1}(\tau) \psi_{r_2}(\tau') \rangle_{QSA}.
\]

(21)

The condition that the integrand in Eq.(17) has a saddle point \( \lambda(\tau) = \lambda_0 \) is that
\[ 1 = \frac{1}{N} \sum_r G(r\tau; r\tau + 0^+), \quad (22) \]

which becomes an implicit equation for \( \lambda_0 \). The QSA ensemble average is now defined by

\[
\langle \ldots \rangle_{\text{QSA}} = \frac{\int \prod_r D\psi_r D\psi^*_r \ldots e^{-S_{\text{QSA}}[\psi, \lambda_0]}}{\int \prod_r D\psi_r D\psi^*_r e^{-S_{\text{QSA}}[\psi, \lambda_0]}}.
\quad (23) \]

A Fourier transform of Eq. (18) in momentum and frequency space enables one to write the spherical constraint (22) explicitly as

\[
1 = \frac{1}{N\beta} \sum_k \sum_{\ell} \frac{1}{\lambda - J(k) + [W_{02}(\omega_\ell)]^{-1}}, \quad (24) \]

with \( J(k) \) the Fourier transform of the Josephson interactions \( J(|r_1 - r_2|), W_{02}(\omega_\ell) \) the frequency transformed phase–phase correlator of Eq. (19) and \( \omega_\ell = 2\pi\ell/\beta \) (\( \ell = 0, \pm 1, \pm 2, \ldots \)) the (Bose) Matsubara frequencies. As usual in a spherical model analysis the critical behavior is determined by the denominator of the summand in the spherical constraint equation of Eq. (22). Specifically, when \([1/G(k = 0, \omega_\ell = 0)] = 0\), where \( G(k, \omega_\ell) \) is the Fourier transformed order parameter correlation of Eq. (21), the system displays a critical point at

\[
\lambda_0 - J(k = 0) + [W_{02}(\omega_\ell = 0)]^{-1} = 0, \quad (25) \]

provided the momentum and frequency summations in the constraint in Eq. (22) converge. The system does not show a phase transition if the frequency–momentum sum in Eq.(22) diverges for \( N \to \infty \) at the point defined by the criticality condition (25). The universal critical properties only depend on the low–frequency behavior of \([W_{02}(\omega_\ell)]^{-1}\) and the long–wavelength properties of the interaction \( J(k) \). Formally, the \( T = 0 \) critical behavior will be identical to that of interacting quantum rotors with a kinetic energy proportional to \( k^2 \) in \( d = 3 \) dimensions, i.e. the transition will be in the universality class of the tree dimensional (3D) XY model.

**A. Zero temperature ground capacitance model.**

Offset charges are an important ingredient in the experimental array samples made of ultrasmall junctions. In this subsection we reconsider this problem as a test for the usefulness of the quantum spherical model approach. The offset charges, or an external gate voltage applied between the array and the substrate, behave like a chemical potential for injection of Cooper pairs into the array. Several authors have shown that static background charges can have a pronounced effect on the SI transition at zero temperature\(^{12,13}\). Including the offset charges, \( q_x \), in the charging energy of Eq. (3) gives

\[
S_C[\phi] = \frac{1}{8e^2} \int_0^\beta d\tau \sum_{r,r'} [C^{-1}]_{rr'} \left( \frac{\partial \phi_r}{\partial \tau} - q_x \right) \left( \frac{\partial \phi_{r'}}{\partial \tau} - q_x \right). \quad (26) \]

Furthermore, we simplify the model to include only the background capacitance (or self–charging) model. In this case \([C^{-1}]_{rr'} = \delta_{r,r'}/C_0\) and \( E_C' = \frac{1}{2} e^2 [C^{-1}]_{rr} \equiv e^2/2C_0\). Of course
the approximation that \( C_0 \gg C_1 \), where \( C_1 \) is the mutual capacitance between the islands, is a first step in the analysis that leads to physical insights into the general problem. Finally, we assume a square array characterized by the nearest–neighbor Josephson coupling \( E_J \) with 
\[
J(k) = E_J \{ \cos(k_x a) + \cos(k_y a) \},
\]
with \( a \) the lattice constant. We obtain the corresponding density of states
\[
\rho(E) = \frac{1}{N} \sum_k \delta(E - J(k)) = \frac{1}{\pi^2 E_J} K \left( \sqrt{1 - \frac{E^2}{4E_J^2}} \right) \Theta(2E_J - |E|),
\]
(27)
where \( K(x) \) is the complete elliptic integral of the first kind. The phase–phase correlation function becomes
\[
W_{02}(\omega) = \frac{8E_C}{Z_0} \sum_q \frac{\exp \left[ -4\beta E_C(q - q_x)^2 \right]}{(4E_C)^2 - [8E_C(q - q_x) - i\omega\ell]^2},
\]
(28)
where \( Z_0 = \sum_q \exp \left[ -4\beta E_C(q - q_x)^2 \right] \), and the summation is performed over all integer–valued charge states \( q = 0, \pm 1, \pm 2, \ldots \), which makes the function \( W_{02}(\omega) \) periodic. At low temperatures the sum over \( q \) in Eq.(28) is dominated by the charge \( q \) which makes the exponent in the numerator of Eq.(28) smallest. For \( T = 0 \) this value is \( q = 0 \) and the equation for the critical line \( (E/E_C) \) vs. \( q_x \) is obtained from the implicit equation
\[
1 = \frac{1}{\beta} \sum_\ell \int_{-\infty}^{+\infty} dE \frac{\rho(E)}{\lambda_0 + 2E_C - E - 8E_C \left( q_x + \frac{i\omega\ell}{8E_C} \right)^2},
\]
(29)
valid for \(-\frac{1}{2} \leq q_x \leq \frac{1}{2}\) (other values of \( q_x \) are included by a periodic extension with the period equal to one). The criticality condition (cf. Eq.(25)) reads
\[
\lambda_0 = 2E_J - 2E_C + 8E_C q_x^2.
\]
(30)
By formally setting \( \lambda_0 = 0 \) in Eq.(30) we obtain the coarse-grained mean–field parabolic solution:
\[
\frac{E_J}{E_C} = 1 - 4q_x^2.
\]
(31)
By substituting the value of \( \lambda_0 \) from Eq.(30) into (28) within the QSA method, after performing the summation over Matsubara frequencies, one obtains the \( T \to 0 \) limit result
\[
1 = \mathcal{P} \int_{-\infty}^{+\infty} dE \rho(E) \sqrt{\frac{E_C}{2(\lambda_0 + 2E_C - E)}} \left[ \text{sign} \left( 4q_x \frac{E_C}{E_J} + \frac{\sqrt{2(\lambda_0 + 2E_C - E)E_C}}{E_J} \right) \right.
\]
\[
- \text{sign} \left( 4q_x \frac{E_C}{E_J} - \frac{\sqrt{2(\lambda_0 + 2E_C - E)E_C}}{E_J} \right) \left],
\]
(32)
where \( \mathcal{P} \) denotes the principal value of the integral. After Eq.(27) we finally obtain the phase boundary of the insulating Mott lobe at zero temperature.
\[ 1 = \frac{\sqrt{2}}{\pi^2} \int_{-2}^{2} dy \frac{K\left(\sqrt{1-y^2/4}\right)}{\sqrt{(2-y)\frac{E_J}{E_C} + 8q_x^2}}. \] (33)

In Fig.(1) we plot the resulting phase boundary in the \( (E_J/E_C) \)-vs-\( q_x \) plane at \( T = 0 \). We recognize the periodic lobes in \( q_x \) of the insulating phase separated by regions of phase coherent superconducting state. For \( q_x = 1/2 \) the superconducting state extends down to arbitrarily small values of \( E_J/E_C \). We can compare the phase diagram resulting from the QSA to the one obtained from mean-field theory via the coarse grained approach (cf. Eq.(31)). Whereas QSA gives the value \( E_J(q_x = 0)/E_C \approx 1.653 \), mean-field theory gives \( E_J(q_x = 0)/E_C = 1 \), which underestimates the critical value of \( E_J \). It is also instructive to compare the QSA result for \( E_J(q_x = 0)/E_C \) to the recent third-order perturbation expansion in \( E_J/E_C \):

\[ q_x = \frac{1}{2} - \frac{E_J}{4E_C} - \frac{3}{128} \left( \frac{E_J}{E_C} \right)^2 - \frac{11}{1025} \left( \frac{E_J}{E_C} \right)^3 + O\left( \left( \frac{E_J}{E_C} \right)^4 \right), \] (34)

which gives \( E_J(q_x = 0)/E_C \approx 1.590 \), giving a 3.8\% of the difference from the critical value of \( E_J/E_C \) at \( q_x = 0 \) obtained in the QSA.

IV. CONDUCTIVITY SCALING

The conductivity is an experimentally measurable quantity in JJA. There are other studies of the superconductor–Mott–insulator and its universal conductance in JJA. Cha et al. (see, Ref. 21) carried out a \( 1/N \) expansion and Monte Carlo analysis. Van Otterlo et. al. used the coarse-graining approach and Fazio and Zappela did an \( \epsilon \)-expansion. Here, we are interested in another aspect of the SI transition, namely, the scaling of the conductivity in the vicinity of the quantum critical point.

To evaluate the conductivity, we need to add an external field in terms of a minimally coupled order parameter to the vector potential \( A(r, \tau) \). The Josephson coupling term in Eq. (3) then becomes

\[ J(|r_i - r_j|) \rightarrow J(|r_i - r_j|) \exp \left( i\frac{2e}{\hbar} \int_{r_i}^{r_j} A \cdot dl \right). \] (35)

The phase shift on each junction is determined by the vector potential \( A \) of the magnetic applied field and in a typical experimental situation it can be entirely ascribed to the external field.

The imaginary–time frequency dependent linear-response conductivity is given by

\[ \sigma_{\mu\nu}(\omega, \mathbf{q}) = \frac{\hbar}{\omega} \int d^2\mathbf{r} \int_0^\beta d\tau \frac{\delta^2 \ln Z}{\delta A_\mu(\tau, \mathbf{r}) \delta A_\nu(0, 0)} e^{i\omega_\nu \tau + iq\mathbf{r}}. \] (36)

For vanishing magnetic field and offset charge the longitudinal component of \( \sigma(\omega) \equiv \sigma_{xx}(\omega, \mathbf{q} = 0) \) in the QSA is
\[
\sigma(\omega_{\nu}) = \frac{4e^2E_j^2}{\hbar\omega_{\nu}} \int_{-\pi/a}^{\pi/a} \frac{dk_xdk_y}{(2\pi/a)^2} \frac{1}{\beta} \sum_{\omega_{\ell}} \sin^2(ak_x) \left[ G^2(k,\omega_{\ell}) - G(k,\omega_{\ell})G(k,\omega_{\nu} + \omega_{\ell}) \right], \tag{37}\]

where \(G^{-1}(k,\omega_{\ell}) = [\lambda - J(k) + 2E_C + \omega_{\ell}^2/8E_C]^{-1}\), and \(\lambda\) is determined self-consistently from the constraint equation (\([7]\)). To proceed it is convenient to obtain the generalized density of states for the 2-D square lattice as

\[
\rho_2(E) = \frac{2}{\pi^2E_j} \left[ E \left( \sqrt{1 - \frac{E^2}{4E_j^2}} \right) - \left( \frac{E}{2E_j} \right)^2 K \left( \sqrt{1 - \frac{E^2}{4E_j^2}} \right) \right] \Theta(4E_j^2 - E^2) \tag{38}\]

where \(E(x)\) is the elliptic integral of the second kind. In terms of Eq. (\([8]\)) the conductivity of Eq. (\([37]\)) becomes

\[
\sigma(\omega_{\nu}) = \frac{4e^2E_j^2}{\hbar\omega_{\nu}} \int_{-\infty}^{+\infty} \frac{dE}{\beta} \frac{1}{\lambda} \sum_{\omega_{\ell}} \rho_2(E)G(E,\omega_{\ell}) \left[ G(E,\omega_{\ell}) - G(E,\omega_{\nu} + \omega_{\ell}) \right]. \tag{39}\]

Here, \(G^{-1}(E,\omega_{\ell}) = [\delta_\lambda - E + 2E_J + \omega_{\ell}^2/8E_C]^{-1}\), where \(\delta_\lambda = \lambda - \lambda_{\text{crit}}\). The zero-temperature critical boundary of the phase coherent state is signaled by \(\lambda(E_j/E_C,T)\) reaching the value \(\lambda_{\text{crit}}(E_{J_{\text{crit}}}^2/E_C,T = 0)\). The parameter \(\delta_\lambda\) plays the important role of energy scale (which vanishes in the superconducting phase). We now consider the \(T = 0\) phase transition between the long-range ordered coherent phase state and the insulating phase by varying the coupling constant \(E_J\) through the critical value \(E_{J_{\text{crit}}}^2\), where there is a diverging correlation length \(\xi \sim |E_J - E_{J_{\text{crit}}}^2|\). At finite temperatures, the deviations from \(T = 0\) behavior are distinguished by the scale set by the thermal coherence length \(\xi_T \sim T^{-1/z}\) (where \(z\) is the dynamic critical exponent). The quantum-critical region is defined by the inequality \(\xi_T < \xi\). In this case the system feels the finite value of the temperature before becoming sensitive to the deviations of \(E_J\) from \(E_{J_{\text{crit}}}^2\). In this regime the dynamic conductivity turns out to be remarkably universal. To proceed, we need to determine the \(\delta_\lambda\) dependence of temperature and \(E_J/E_C\). The spherical constraint of Eq. (\([7]\)) takes the form

\[
1 = \frac{1}{\beta} \sum_{\ell} \int_{-\infty}^{+\infty} \frac{dE}{\delta_\lambda + 2E_J - E - 8E_C \left( \frac{\omega_{\nu}}{8E_C} \right)^2}. \tag{40}\]

The near-critical properties of the spherical model are essentially determined by the structure of the spectrum of the interaction matrix \(J(|\mathbf{r}_1 - \mathbf{r}_2|)\) in the neighborhood of its upper boundary, specifically by the behavior of the density of states associated with \(J(|\mathbf{r}_1 - \mathbf{r}_2|)\). We therefore expand the square lattice density of states (\([7]\)) about the upper limit of the spectrum of \(J(k)\):

\[
\frac{1}{E_J\pi^2} K \left[ \sqrt{1 - \frac{(E - 2E_J)^2}{4E_J^2}} \right] = \frac{1}{2E_J\pi} + \frac{E}{8E_J^2\pi} + O(E^2). \tag{41}\]

The subsequent summation frequency yields
This constraint equation can be explicitly solved for the dependence of $\delta_\lambda$ on $T, E_J$, and $E_C$ giving

$$\delta_\lambda = \frac{(k_B T)^2}{2 E_C} \left\{ \text{arsinh} \left[ e^{-2 \pi \beta E_J} \sinh \left( \beta \sqrt{8 E_C E_J} \right) \right] \right\}^2,$$

and near the $T = 0$ quantum critical point

$$\delta_\lambda = \frac{(k_B T)^2}{2 E_C} \left\{ \frac{1}{2} \exp \left( -\pi \beta (E_J - E_J^{\text{crit}}) \right) \right\}^2,$$

where $E_J^{\text{crit}} = 2 E_C / \pi^2$. We extract the conductivity, as a function of the real frequencies $\omega$ in the form of real and imaginary parts of $\sigma(\omega) = \sigma_{\text{sing}}(\omega) + \sigma_{\text{reg}}(\omega)$, after performing the sum over Matsubara frequencies in Eq. (43) followed by analytic continuation to real frequencies $i \omega_t \to \omega + i 0^+$ getting

$$\text{Im} \sigma_{\text{sing}}(\omega) = \frac{4 e^2 E_J^2}{h} \int_{-\infty}^{+\infty} dE \rho_2(E) \frac{1}{\omega} \frac{8 \beta E_C \cosh \left[ \frac{\beta}{2} \sqrt{8 E_C (\delta_\lambda - E + 2 E_J)} \right]}{\delta_\lambda - E + 2 E_J},$$

$$\text{Im} \sigma_{\text{reg}}(\omega) = \frac{4 e^2 E_J^2}{h} \int_{-\infty}^{+\infty} dE \rho_2(E) \frac{1}{4} \sqrt{\frac{1}{8 E_C (\delta_\lambda - E + 2 E_J)^3}} \frac{\omega \coth \left( \frac{\beta}{2} \sqrt{8 E_C (\delta_\lambda - E + 2 E_J)} \right)}{\omega E - 4 (\delta_\lambda - E + 2 E_J)}$$

and

$$\text{Re} \sigma_{\text{sing}}(\omega) = \frac{4 e^2 E_J^2 \pi}{h} \int_{-\infty}^{+\infty} dE \rho_2(E) \delta(\omega) \frac{8 \beta E_C \cosh \left[ \frac{\beta}{2} \sqrt{8 E_C (\delta_\lambda - E + 2 E_J)} \right]}{\delta_\lambda - E + 2 E_J}$$

$$\times \delta \left[ \frac{\omega^2}{8 E_C} - 4 (\delta_\lambda - E + 2 E_J) \right].$$

The real part of the conductivity contains two contributions: the first, $\text{Re} \sigma_{\text{sing}}(\omega)$, is singular since it is proportional to $\delta(\omega)$ and the second finite–frequency is regular, $\text{Re} \sigma_{\text{reg}}(\omega)$, which arises from the electromagnetic field induced transitions to excited states. The singular part in turn is due to the free acceleration of quasiparticles. This is so since the JJA model considered here contains no dissipation mechanism which would e.g. arise from a coupling of the phase degrees of freedom to the normal electrons (Ohmic damping). At $T = 0$ the singular part vanishes while the regular one can be evaluated explicitly by performing an energy integration in Eq. (44) with the result

$$\text{Re} \sigma_{\text{reg}}(\omega, \delta) = \frac{(2 e^2)^2}{\hbar} \left[ 1 + \frac{\delta}{2} \left( 1 - \frac{\omega^2}{\omega_c^2} \right) \right] \Theta \left[ \left( \frac{\omega}{\omega_c} - 1 \right) \left( \sqrt{1 + 4 / \delta} - \frac{\omega}{\omega_c} \right) \right].$$

where $F(x) = E \left( \sqrt{1 - x^2} \right) - x^2 K \left( \sqrt{1 - x^2} \right)$, $\bar{\omega} = \omega / \omega_c$, with $\omega_c = \sqrt{32 E_C \delta_\lambda}$ and $\delta = \delta_\lambda / E_J$. Here $\omega_c$ denotes the threshold frequency above which the particle–hole excitations
can be created and the real part of the conductivity is finite, while \( \text{Re} \sigma_{\text{reg}}(\omega) \) vanishes for \( \omega < \omega_c \) indicating a Mott–insulating phase. Letting the gap parameter \( \delta \) go to zero, while keeping the ratio \( \omega/\omega_c \) finite, and using the result

\[
\lim_{x \to 0} \frac{2}{x \pi^2} F(1 - x) = \frac{1}{\pi},
\]

we obtain from the general QSA result (47) (valid for arbitrary distance \( \delta \lambda \) away from the critical point) the long–wave length limit near–critical form of the conductivity:

\[
\lim_{\delta \to 0} \text{Re} \sigma_{\text{reg}}(\omega, \delta) = \left( \frac{2e^2}{h} \right) \left( 1 - \frac{\omega^2}{\omega_c^2} \right) \Theta \left( \frac{\omega}{\omega_c} - 1 \right).
\]

A plot of this result for different values of \( \delta \) is shown in Fig. (2), where we clearly see an asymptotic gap as we approach the critical point at zero temperature. This analytical result was previously derived in studies that relied on the coarse–grained and loop–expansion approaches.

From Eq.(49) at the transition, where the thrashed frequent \( \omega_c \) vanishes, a finite dc conductance \( \sigma^* = (\pi/8) \cdot 4e^2/h \) emerges which is the universal zero temperature conductivity earlier obtained by Cha et al. In practice, however, all experiments are performed at low but nonzero temperature.

We will now present the scaling analysis satisfied by \( \sigma(\omega) \) in the vicinity of the quantum phase transition \( E_J = E_J^{\text{crit}} \). The temperature is taken to be nonzero but must obey \( k_B T << E_J \). A nonzero \( T \) implies the absence of long–range phase coherence and the scaling properties will depend upon a variable which measures the distance of the superconducting ground state from criticality. The behavior of the conductivity in this regime can be understood in terms of the scaling forms

\[
\text{Re} \sigma(\omega) = \frac{(2e)^2}{h} g' \left( \frac{\omega}{k_B T}, \frac{(k_B T)^{1/\nu z}}{\delta E_J} \right),
\]

\[
\text{Im} \sigma(\omega) = \frac{(2e)^2}{h} g'' \left( \frac{\omega}{k_B T}, \frac{(k_B T)^{1/\nu z}}{\delta E_J} \right),
\]

where \( g'(X,Y) = g'_{\text{sing}}(X,Y) + g'_{\text{reg}}(X,Y) \) and \( g''(X,Y) = g''_{\text{sing}}(X,Y) + g''_{\text{reg}}(X,Y) \) are highly non–trivial but universal two–parameter functions. From equations (45) and (46) and the explicit solution (44) of the spherical constraint equation we derive

\[
g'_{\text{sing}}(X,Y) = \frac{\pi}{2} \delta(X) \Xi(Y) \int_1^\infty du \left( u - \frac{1}{u} \right) \Sigma(Y, u),
\]

\[
g'_{\text{reg}}(X,Y) = \frac{\pi}{8} \left[ 1 - \frac{\Xi^2(Y)}{X^2} \right] \coth \left( \frac{X}{4} \right) \Theta \left[ \frac{X^2}{\Xi^2(Y)} - 1 \right],
\]

where

\[
\Xi(Y) = 4 \ln \left( \frac{1}{2} \sqrt{\frac{e^{-2\pi/Y}}{4} + \frac{1}{2} e^{-\pi/Y}} \right)
\]

\[
\Sigma(Y, u) = \frac{4}{\left[ \left( \frac{1}{2} \sqrt{\frac{e^{-2\pi/Y}}{4} + \frac{1}{2} e^{-\pi/Y}} \right) u - \left( \frac{1}{2} \sqrt{\frac{e^{-2\pi/Y}}{4} + \frac{1}{2} e^{-\pi/Y}} \right)^u \right]^2},
\]
and

\[ g''_{\text{sing}}(X, Y) = \frac{\pi \Xi^2(Y)}{2 X} \int_1^\infty du \left( u - \frac{1}{u} \right) \Sigma(Y, u) \]
\[ g''_{\text{reg}}(X, Y) = \frac{1}{4} \frac{X}{\Xi(Y)} \int_1^\infty du \left( 1 - \frac{1}{u^2} \right) \frac{\Omega(Y, u)}{X^2 - u^2} \]  

with

\[ \Omega(Y, u) = \frac{\left( \frac{1}{2} \sqrt{e^{-2\pi/Y} + 4 + \frac{1}{2} e^{-\pi/Y}} \right)^u + \left( \frac{1}{2} \sqrt{e^{-2\pi/Y} + 4 + \frac{1}{2} e^{-\pi/Y}} \right)^{-u}}{\left( \frac{1}{2} \sqrt{e^{-2\pi/Y} + 4 + \frac{1}{2} e^{-\pi/Y}} \right)^u - \left( \frac{1}{2} \sqrt{e^{-2\pi/Y} + 4 + \frac{1}{2} e^{-\pi/Y}} \right)^{-u}}. \]  

(53)

We are in the quantum–critical regime by setting \( \delta E_j \) to zero \( (Y = \infty) \), while keeping the temperature small but finite \( (X \neq 0) \), with \( z = 2 \), and \( z \nu = 1 \), with the critical fluctuations quenched in a universal way by temperature. Here, \( k_B T \) appears to be the dominant energy which determines the physics of the problem. At large frequencies, \( \hbar \omega \gg k_B T \), the finite temperature effects do not become manifest and the system displays the behavior of the zero–temperature critical point \( (\delta E_j = 0, T = 0) \). However, at small enough frequencies, \( \hbar \omega \sim k_B T \), the finite temperature effects of \( \delta \) become apparent which introduces a new energy scale into the problem. While fluctuations with frequencies larger then \( k_B T / \hbar \) are unaffected those with \( \hbar \omega < k_B T \) will behave classically. We show the regular parts of the nontrivial scaling functions \( g'(X, Y) \) and \( g''(X, Y) \) in Figs. (3) and (4).

**V. SUMMARY**

In this paper we have developed a microscopic analysis of a coupled array of ultrasmall Josephson junctions in the presence of charging energy effects. Our analysis is based on the quantum spherical model approach and the path–integral formulation of quantum mechanics explicitly tailored for the microscopic JJA Hamiltonian. The effective action formalism allows for an explicit implementation of the Coulomb and offset voltage effects into our consideration. Using this formalism we have considered the zero–temperature phase transition to try to understand the finite temperature behavior of the system in terms of scaling functions for the electromagnetic response of the array. This is important since the experimental analysis of quantum phase transitions relies on the scaling behavior of observables (eg. the conductivity) for relevant physical parameters (like temperature and frequency). A relevant question which naturally arises is: To what extent does this phenomena depend on the model used to describe the JJA? Future theoretical studies on quantum critical behavior should consider disorder effects (random offset charges), coupling to quasiparticles (dissipation), magnetic field frustration and finite size effects, just to call a few relevant issues in this problem. The formalism presented in this paper can be used to look at these specific situations.

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APPENDIX A: LOOP EXPANSION FOR THE EFFECTIVE ACTION

The generating functional of Eq. (14) for the connected cumulant functions $W_{0m}(x_1, \ldots, x_m)$ is given by the functional expansion

$$W[\mu] = \sum_{m=1}^{\infty} \frac{1}{m!} \int dx_1 \ldots dx_m W_{0m}(x_1, \ldots, x_m) \mu(x_1) \ldots \mu(x_m),$$  \hspace{1cm} (A1)

where we have for convenience introduced the short-hand notation

$$x_m \equiv (r_m, \tau_m, a_m) \quad \int dx \ldots \equiv \int_0^\beta d\tau_m \sum r_m \sum a_m \ldots,$$  \hspace{1cm} (A2)

to obtain

$$W_{0m}(x_1, \ldots, x_m) = \langle S_{r_1}^\alpha[\phi(\tau_1)] \ldots S_{r_m}^\alpha[\phi(\tau_m)] \rangle_0^{\text{cum}}.$$  \hspace{1cm} (A3)

For a given set of operators $A, B$ and $C$ the cumulant averages are defined as: $\langle AB \rangle_0^{\text{cum}} = \langle AB \rangle_0 - \langle A \rangle_0 \langle B \rangle_0$; $\langle ABC \rangle_0^{\text{cum}} = \langle ABC \rangle_0 - \langle A \rangle_0 \langle BC \rangle_0 - \langle B \rangle_0 \langle AC \rangle_0 - \langle C \rangle_0 \langle AB \rangle_0 + 2 \langle A \rangle_0 \langle B \rangle_0 \langle C \rangle_0$, \ldots etc... where

$$\langle \ldots \rangle_0 = \frac{\int [\Pi_r D\phi_r] \ldots e^{-S_C[\phi]} \langle \ldots \rangle}{\int [\Pi_r D\phi_r] e^{-S_C[\phi]}},$$  \hspace{1cm} (A4)

which is closely related to the multi-point vertex function $\Gamma_{0m}(x_1, \ldots, x_m)$. The generating functional reads:

$$\Gamma[\psi] = \sum_{m=1}^{\infty} \frac{1}{m!} \int dx_1 \ldots dx_m \Gamma_{0m}(x_1, \ldots, x_m) \psi(x_1) \ldots \psi(x_m).$$  \hspace{1cm} (A5)

The relation between $W[\mu]$ and $\Gamma[\psi]$ is then given by the Legendre transform

$$\Gamma[\psi] + W[\mu] = \int dx \mu(x) \psi(x).$$  \hspace{1cm} (A6)

We may now evaluate the action $\Phi_C[\psi]$ defined in Eq. (13), by introducing in the exponential of Eq. (13) a formal parameter $\zeta$ to order the perturbation expansion ($\zeta$ will be set to unity at the end of calculation). We obtain

$$e^{-\Phi_C[\psi]} = \int [\Pi_r \frac{D\mu_r}{2\pi i}] \exp \left[-\zeta \left( \sum_r \int_0^\beta d\tau \mu_r(\tau) \cdot \psi_r(\tau) - W[\mu] \right) \right].$$  \hspace{1cm} (A7)

The steepest descent procedure is the standard method to obtain the expansion for $\Phi_C[\psi]$. The first step is to determine the saddle point $\mu_0$ of the integrand in Eq. (A7) followed by a systematic expansion in terms of fluctuations about the saddle point. The procedure is by now standard (see, e.g. Ref. 24) and we quote the final result:
\( e^{-\Phi_C[\psi]} = \exp \left[ -\zeta \left( \int dx \mu_0(x) \psi(x) - W[\mu_0] \right) - \frac{1}{2} \text{tr} \ln W^{(2)}(\psi) \right] \times \)

\[
\times \left\{ 1 + \frac{1}{\zeta} \left[ -\frac{1}{8} \int dx_1 \ldots dx_4 W^{(4)}(x_1, \ldots, x_4; \psi) D(x_1, x_2; \psi) D(x_3, x_4; \psi) + \int dx_1 \ldots dy_1 W^{(3)}(x_1, x_2, x_3; \psi) W^{(3)}(y_1, y_2, y_3; \psi) \left( \frac{1}{8} D(x_1, x_2; \psi) \times D(y_1, y_2; \psi) D(x_3, y_3; \psi) + \frac{1}{12} D(x_1, y_1; \psi) D(x_2, y_2; \psi) D(x_3, y_3; \psi) \right) \right] + O(\zeta^{-2}) \right\}, \tag{A8}
\]

where

\[
\psi(x) = \frac{\delta W[\mu]}{\delta \mu(x)} \bigg|_{\mu=\mu_0(x; \psi)} + \frac{1}{2\zeta} \int dy dy_1 W^{(3)}(x, y, y_1; \psi) D(y, y_1; \psi) + O(\zeta^{-2}),
\]

\[
\mu_0(x; \psi) = \psi(x) + \frac{1}{2\zeta} \int dy dy_1 W^{(3)}(x, y, y_1; \psi) D(y, y_1; \psi) D(y_1, y_2; \psi) + O(\zeta^{-2}). \tag{A9}
\]

Here the propagator \( D(x_1, x; \psi) \) is defined by

\[
\int dx \zeta D(x_1, x; \psi) W^{(2)}(x, x_2) = \delta(x_1 - x_2), \tag{A10}
\]

while

\[
W^{(m)}(x_1, \ldots, x_m) = \left. \frac{\delta^m W}{\delta \mu(x_1) \ldots \delta \mu(x_m)} \right|_{\mu=\mu_0(x; \psi)}.
\tag{A11}
\]

(cf. definition (A1)). Restricting ourselves to the lowest order in the expansion (A8) (only including the saddle point terms proportional to \( \zeta \)) we obtain the form of the effective action used in the QSA approach (see, Eq.(18)).
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the small parameter $\zeta$ (see, Appendix Eq. (A7)). The $1/M$ corrections in the $M \to \infty$ limit will be analogously given by loop contributions to the saddle point.

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FIGURES

FIG. 1. Zero-temperature phase phase boundary in the $E_J/E_C$-vs-$q_x$ plane separating the Mott-insulating and superconducting states obtained from the quantum spherical approach (solid line), and the coarse-graining method (broken line). Note that there is no cusp at $q_x = 0$ since $dE_J(q_x)/dq_x|_{q_x=0} = 0$ in the QSA.

FIG. 2. Regular contribution of the frequency dependent conductivity at $T = 0$ (real part) for several values of the dimensionless gap parameter $\delta$ (that measures the distance from the critical point): a) $\delta = 0.5$, b) $\delta = 0.1$, c) $\delta = 0.05$ and d) $\delta = 0.0$.

FIG. 3. Plot of the two parameter universal scaling function (real part) of the regular contribution to the frequency dependent conductivity in the quantum critical regime.

FIG. 4. Same as in Fig. 3 for the imaginary part.
Fig. 1 Quantum critical point ..., T.K. Kopec & J.V. Jose
Fig. 2  Quantum critical point ...,  
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$g_{\text{reg}}'(X,Y)$

Fig. 3  Quantum critical point ...,  
T.K. Kopec & J.V. Jose
Fig. 4 Quantum critical point ..., T.K. Kopec & J.V. Jose