KAZHDAN-LUSZTIG TENSORING AND HARISH-CHANDRA CATEGORIES

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Abstract. We use the Kazhdan-Lusztig tensoring to define affine translation functors, describe annihilating ideals of highest weight modules over an affine Lie algebra in terms of the corresponding VOA, and to sketch a functorial approach to “affine Harish-Chandra bimodules”.

1. Introduction

Our original motivation was to answer the question: “What is a Harish-Chandra bimodule over an affine Lie algebra?” Although we have not yet been able to give a complete answer, we can state a conjecture and we can produce objects which are remarkably (and non-trivially) reminiscent of the principal series representations of a complex group. Along the way we get a couple of results (on annihilating ideals of highest weight modules, and on equivalence of categories) which are apparently interesting by themselves.

1.1. Representations of complex groups. To make things clearer, we first review a categorical approach to Harish-Chandra bimodules over a simple Lie algebra following the beautiful paper by Bernstein and S.Gelfand [2]. Let $\mathcal{A}$ be a category. Then one can consider the category $\text{Funct}(\mathcal{A})$ of functors on $\mathcal{A}$, objects being functors, morphisms being natural transformations of functors. In general, there is no reason to think that $\text{Funct}(\mathcal{A})$ is abelian even if $\mathcal{A}$ is so. Here is, however, an important example when $\text{Funct}(\mathcal{A})$ contains an abelian complete subcategory.

Let $\text{Mod}(\mathfrak{g})$ be the category of modules over a simple complex Lie algebra $\mathfrak{g}$ and $\text{Mod}(\mathfrak{g} - \mathfrak{g})$ the category of $\mathfrak{g}$-bimodules. “Module” will always mean a space carrying a left action of $\mathfrak{g}$; “bimodule” will always mean a space carrying a left and a right action commuting with each other. Any $H \in \text{Mod}(\mathfrak{g} - \mathfrak{g})$ gives rise to the functor

$$\Phi_H : \text{Mod}(\mathfrak{g}) \to \text{Mod}(\mathfrak{g}); \quad \Phi_H(M) = H \otimes_{\mathfrak{g}} M.$$
It is well-known that
\[
\text{Hom}_{\text{Mod}(g-g)}(H_1, H_2) = \text{Hom}_{\text{Funct}(\text{Mod}(g))}(\Phi_{H_1}, \Phi_{H_2}).
\]
Therefore \(\text{Mod}(g-g)\) is a complete abelian subcategory of \(\text{Funct}(\text{Mod}(g))\).

Any \(g\)-bimodule is a \(g\)-module with respect to the diagonal action (that is, the left action minus the right action). A Harish-Chandra module is a bimodule such that under the diagonal action it decomposes in a direct sum of finite dimensional \(g\)-modules occurring with finite multiplicities. Consider the category of Harish-Chandra bimodules \(HCh\), and the \(O\) category of \(g\)-modules. The condition imposed on the diagonal action ensures that if \(H\) is a Harish-Chandra bimodule, then \(\Phi_H\) preserves \(O\). Therefore the construction we just discussed gives an embedding \(HCh \hookrightarrow \text{Funct}(O)\) as a complete subcategory.

Further, indecomposable projective Harish-Chandra bimodules are exactly those corresponding to direct indecomposable summands of the functor of tensoring by a finite dimensional \(g\)-module \(V\):
\[
V \otimes \cdot : O \to O, \ M \mapsto V \otimes M.
\]
Such functors are called projective.

Having classified projective functors, it is relatively easy to establish an equivalence (see (2) below) of (sub)categories of \(HCh\) and \(O\).

To be more precise, observe that \(HCh\) and \(O\) admit direct product decompositions with respect to the action of the center of the universal enveloping \(U(g)\). Namely,
\[
HCh = \bigoplus_{\theta_l, \theta_r} HCh(\theta_l, \theta_r), \ O = \bigoplus_{\theta} O_{\theta},
\]
where \(\theta_l, \theta_r, \theta\) are central characters, \(HCh(\theta_l, \theta_r) \subset HCh\) is a complete subcategory of Harish-Chandra bimodules admitting left central character \(\theta_l\) and right central character \(\theta_r\); \(O_{\theta} \subset O\) is a complete subcategory defined in a similar way. It is easy to see that \(HCh(\theta_l, \theta_r)\) is empty unless \(\lambda_l - \lambda_r\) is integral, where \(\lambda_l\) (resp. \(\lambda_r\)) is a dominant weight related to \(\theta_l\) (resp. \(\theta_r\)); this condition will be tacitly assumed from now no.

Another convention to be adopted for simplicity is that all central characters in question are assumed to be regular, i.e. all corresponding weights are off the walls of the Weyl chambers.

One of the main results of [2] is that the functor
\[
(2) \quad HCh(\theta_l, \theta_r) \to O_{\theta_l}, \ H \mapsto \Phi_H(M_{\lambda_r}),
\]
is an equivalence of categories. Here \(M_{\lambda_r}\) is the Verma module with the highest dominant weight \(\lambda_r\).
From this one gets the principal series representations $H_w \in HCh(\theta_l, \theta_r), w \in W$, as preimages of the Verma modules $M_{w, \lambda}$ under (2):

$$H_w = \text{Hom}_C(M_{\lambda_r}, M_{w, \lambda})^{\text{fin}},$$

where $\text{Hom}_C(M_{\lambda_r}, M_{w, \lambda})^{\text{fin}}$ is understood as a $\mathfrak{g}$-bimodule with respect to the obvious bimodule structure and $\text{Hom}_C(M_{\lambda_r}, M_{w, \lambda})^{\text{fin}} \subset \text{Hom}_C(M_{\lambda_r}, M_{w, \lambda})$ is the maximal submodule locally finite with respect to the diagonal action. Thus we get, in particular, that simple Harish-Chandra bimodules are labelled by the elements of the Weyl group $W$.

Another important corollary of (2) is the following description of the 2-sided ideal lattice of $U(\mathfrak{g})_\theta := U(\mathfrak{g})/U(\mathfrak{g})\text{Ker}(\theta)$. Denote by $\Omega(U(\mathfrak{g})_\theta)$ the 2-sided ideal lattice of $U(\mathfrak{g})_\theta$ and by $\Omega(M_\lambda)$ the submodule lattice of $M_\lambda$, where $\lambda$ is the dominant weight related to $\theta$. Then the map

$$\Omega(U(\mathfrak{g})_\theta) \to \Omega(M_\lambda), I \mapsto IM_\lambda$$

is a lattice equivalence. Indeed, $U(\mathfrak{g})_\theta$ is an algebra containing $\mathfrak{g}$, and hence a $\mathfrak{g}$-bimodule; its 2-sided ideals as algebra are its submodules as bimodule. Under the equivalence (2) $U(\mathfrak{g})_\theta$ goes to $M(\lambda)$, because $U(\mathfrak{g})_\theta \otimes_{\mathfrak{g}} M(\lambda) = M(\lambda)$. Thus submodule lattices of $U(\mathfrak{g})_\theta$ and $M(\lambda)$ are equivalent. A little extra work is needed to find the explicit form (3) of this equivalence.

The last result of [2] which we want to review here is another equivalence of categories based on the notion of a translation functor. Let the central characters $\theta_1, \theta_2$ be such that the difference of the corresponding dominant highest weights $\lambda_1 - \lambda_2$ is integral. Denote by $\lambda$ the dominant weight lying in the $W$-orbit of $\lambda_1 - \lambda_2$, and by $V_\lambda$ the simple $\mathfrak{g}$-module with highest weight $\lambda$. For any $\theta$ denote by $p_\theta : \mathcal{O} \to \mathcal{O}_\theta$ the natural projection. Then the functor

$$T_{\theta_1}^{\theta_2} : \mathcal{O}_{\theta_2} \to \mathcal{O}_{\theta_1}, T_{\theta_2}^{\theta_1}(M) = p_{\theta_1}(V_\lambda \otimes M)$$

is an equivalence of categories. The functor $T_{\theta_2}^{\theta_1}$ is called translation functor.

We finish our review of the semi-simple case by remarking that many results of [2] are based on, refine and generalize the earlier work, see e.g. [7, 8, 21, 22].
1.2. An affine analogue. There are many reasons why it is difficult to give an intelligent definition of a Harish-Chandra bimodule over an affine Lie algebra $\hat{\mathfrak{g}}$. (For one thing, it follows from our results that one should rather define a Harish-Chandra bimodule over the corresponding vertex operator algebra.) We find it easiest to adopt a functorial point of view.

Thus we are looking for an interesting subcategory in $\text{Funct}(\tilde{O}_k)$, $\tilde{O}_k$ being the Bernstein-Gelfand-Gelfand category of $\hat{\mathfrak{g}}$-modules at level $k$ satisfying the additional condition that the modules are semi-simple over $\mathfrak{g} \subset \hat{\mathfrak{g}}$. As an analogue of the functor $V \otimes \cdot$, we choose

$$V_k^\lambda \otimes \cdot : \tilde{O}_k \rightarrow \tilde{O}_k, \ A \mapsto V_k^\lambda \otimes A,$$

where $\otimes : \tilde{O}_k \times \tilde{O}_k \rightarrow \tilde{O}_k$ is the Kazhdan-Lusztig tensoring [17, 18, 19], and $V_k^\lambda$ is the Weyl module (generalized Verma module in another terminology) induced in a standard way from the finite dimensional $\mathfrak{g}$-module $V_\lambda$. (There seems to be no other reasonable choice.)

The Kazhdan-Lusztig tensoring is a subtle thing and many obvious properties of $V \otimes \cdot$ are hard to carry over to the case of $V_k^\lambda \otimes \cdot$. For example, the functor $V_k^\lambda \otimes \cdot$ does not seem to be exact in general. There is, however, a case when the analogy is precise – the affine version of a translation functor. By [3, 20], there is a direct sum decomposition

$$\tilde{O}_k = \bigoplus_{(\lambda,k) \in P_k^+} \tilde{O}_k^\lambda,$$

and thus a projection

$$p_\lambda : \tilde{O}_k \rightarrow \tilde{O}_k^\lambda,$$

where $P_k^+$ is the set of dominant weights at level $k + h^\vee \in \mathbb{Q}_>$. (This is an analogue of the central character decomposition for $\mathfrak{g}$.) We can therefore define an affine translation functor

$$T_\mu^\lambda : \tilde{O}_k^\mu \rightarrow \tilde{O}_k^\lambda,$$

by adjusting definition (4) to the affine case (most notably by replacing $\otimes$ with $\hat{\otimes}$ and the finite dimensional $\mathfrak{g}$-module with an appropriate Weyl module, for details see [4.1]). This construction was first proposed in [10] in the case of negative level $(k + h^\vee < 0)$ representations.

The basic properties of affine translation functors are collected in Proposition 4.3.1. They are summarized by saying that a Weyl module with a dominant highest weight is rigid and the functor of Kazhdan-Lusztig tensoring with such a module is exact. These properties easily imply that $T_\mu^\lambda : \tilde{O}_k^\mu \rightarrow \tilde{O}_k^\lambda$ is an equivalence of categories (c.f. (4)). This theorem refines results of [3], where a different version of translation functors was defined (in the framework of a general symmetrizable
Kac-Moody algebra) by using the standard tensoring with an integrable module.

The study of Kazhdan-Lusztig tensoring is not easy but rewarding. A simple translation of Proposition 4.3.1 in the language of vertex operator algebras (see 5.1.1, 5.2) gives the following affine analogue of the equivalence (3). Recall that by [13], there is a vertex operator algebra (VOA) $(\mathfrak{g}, Y(v, t))$ attached to $\hat{\mathfrak{g}}$. The Fourier components of the fields $Y(v, t), v \in \mathfrak{g}^0$ span a Lie algebra, $U(\hat{\mathfrak{g}})_{\text{loc}}$. We prove (Theorem 5.3.1) that the ideal lattice of $U(\hat{\mathfrak{g}})_{\text{loc}}$ as VOA is equivalent to the submodule lattice of the Weyl module $V^k_\lambda$ with a dominant highest weight $(\lambda, k), k + h^\vee \in \mathbb{Q}_>$. Observe that the crucial difference between this statement and (3) is that the associative algebra $U(\mathfrak{g})_\theta$ is replaced by a huge Lie algebra $U(\hat{\mathfrak{g}})_{\text{loc}}$. Theorem 5.3.1 generalizes and refines the well-known result that Fourier components of the field $e^\theta(t)^{k+1}$ annihilate all integrable modules at a positive level $k$; here $e^\theta \in \mathfrak{g}$ is a highest root vector.

Having found two affine analogues of two corollaries of the equivalence (3), we return to the problem of affinizing the notion of a Harish-Chandra module. We conjecture (for details see sect.5) that the functor

$$\tilde{O}_k^0 \rightarrow \text{Funct}(\tilde{O}_k^\lambda), \; A \mapsto p_\lambda \circ (A \otimes ?)$$

realizes $\tilde{O}_k^0$ as a complete subcategory of $\text{Funct}(\tilde{O}_k^\lambda)$. Realized in this way $\tilde{O}_k^0$ becomes a precise analogue of $HCh(\lambda, \lambda)$. (We are forced to change notation from $HCh(\theta, \theta)$ to $HCh(\lambda, \lambda)$ as introducing the notion of a central character is troublesome in the affine case.) The theorem on affine translation functors then shows that (affine) $HCh(\lambda, \lambda)$ is equivalent to $\tilde{O}_k^\lambda$ as it should in light of (3). As a supporting evidence we show that the natural map

$$Hom_\tilde{\mathfrak{g}}(A, V^k_{w,0}) \rightarrow Hom_\tilde{\mathfrak{g}}(p_\lambda(A \otimes V^k_\lambda), p_\lambda(V^k_{w,0} \otimes V^k_\lambda))$$

is an isomorphism. Therefore there is an injection

$$Hom_\tilde{\mathfrak{g}}(A, V^k_{w,0}) \hookrightarrow Hom_{\text{Funct}(\tilde{O}_k^\lambda)}(p_\lambda(A \otimes ?), p_\lambda(V^k_{w,0} \otimes ?)).$$

(The conjecture would imply that this map is an isomorphism.) Thus the functor

$$p_\lambda \circ (V^k_{w,0} \otimes ?) : \tilde{O}_k^\lambda \rightarrow \tilde{O}_k^\lambda,$$

is indeed very reminiscent of the principal series representation $H_{w}$, insofar as the Weyl module $V^k_{w,0}$ is analogous to the Verma module $M_{w,0}$.
We finally observe that all these have analogues for the category $O_k \supset \tilde{O}_k$ obtained by dropping the condition of $\mathfrak{g}$-semi-simplicity. In this way we get objects better modelling principal series representations in the affine case.

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2. preliminaries

2.1. The following is a list of essentials which will be used but will not be explained.

- $\mathfrak{g}$ is a simple finite dimensional Lie algebra with a fixed triangular decomposition; in particular with a fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$;
- the action ($\lambda \mapsto w\lambda$) and the shifted (by $\rho$) action ($\lambda \mapsto w\cdot\lambda$) action of the Weyl group $W$ on $\mathfrak{h}^*$ preserving the weight lattice $P \in \mathfrak{h}^*$; denote by $C$ the Weyl chamber – a fundamental domain for the shifted action attached to the fixed triangular decomposition; $P^+ = P \cap C$;
- the $O$ category of $\mathfrak{g}$-modules attached to the triangular decomposition;
- a Verma module $M_\lambda \in O$, $\lambda \in \mathfrak{h}$ and a simple finite dimensional module $V_\lambda$, $\lambda \in P^+ \subset P$;
- the affine Lie algebra $\hat{\mathfrak{g}} = \mathbb{C}((z)) \oplus \mathbb{C}K$ and the “generalized” Borel subalgebra $\hat{\mathfrak{g}}_\geq = \mathfrak{g} \otimes \mathbb{C}[z] \oplus \mathbb{C}K$;
- $O_k$ – the category of modules at level $k$ (i.e. $K \mapsto k$), and the full subcategory $O_k \subset \tilde{O}_k$ consisting of $\hat{\mathfrak{g}}$-modules semisimple over $\mathfrak{g} \subset \hat{\mathfrak{g}}$;
- $M_\lambda^k \in O_k$, $\lambda \in \mathfrak{h}^*$ is a Verma module; $V_\lambda^k = Ind_{\hat{\mathfrak{g}}_\leq}^{\hat{\mathfrak{g}}_\geq} \in \tilde{O}_k$, $\lambda \in P^+$ is a Weyl module; more generally, if $V$ is a $\mathfrak{g}$-module, then $V^k \in \tilde{O}_k$ is a $\hat{\mathfrak{g}}$-module obtained by inducing from $V$; obviously, $V_\lambda^k$ is a quotient of $M_\lambda^k$; each simple module is a quotient of $M_\lambda^k$; denote it by $L_\lambda^k$;
- if $k \in \mathbb{Q}$, then $\tilde{O}_k$ is semi-simple, each object being a direct sum of Weyl modules; there is an obvious analogue of this statement for $O_k$;
- for $k + h^\vee = p/q \in \mathbb{Q}_\geq$ consider an affine Weyl group $W_k = pQ \ltimes W$, where $Q$ is a root lattice of $\mathfrak{g}$; there is the usual and the dotted (shifted) action of $W_k$ on $\mathfrak{h}^*$; the fundamental domain for the latter is $C_{aff} = C \cap \{ \lambda : 0 < (\lambda + \rho, \theta) < p \}$, where $\theta$ is the highest root of $\mathfrak{g}$; set $P^+_k = P^+ \cap C$; call $\lambda \in P^+_k$ (sometimes $(\lambda, k)$ if $\lambda$ satisfies this condition) dominant;
by \([3, 15, 20]\), \(\tilde{O}_k = \oplus_{\lambda \in P^+} \tilde{O}_k^\lambda\), where \(\tilde{O}_k^\lambda\) is a full subcategory consisting of modules whose composition series contain only irreducible modules \(L_{w, \lambda}^k, w \in W_k\); similar decomposition is true for \(\tilde{O}_k\).

**Duality Functors.** Given a vector space \(W\), denote by \(W^d\) its total dual. If \(W\) is a Lie algebra module, then so is \(W^d\).

Given a vector space \(W\) carrying a gradation by finite dimensional subspaces, denote by \(D(W)\) its restricted dual.

Objects of \(\tilde{O}_k\) are canonically graded. Denote by \(D: \tilde{O}_k \rightarrow \tilde{O}_k\), \(M \mapsto D(M)\) the functor such that the \(\hat{g}\)-module structure is defined by precomposing the canonical action on the dual space with an automorphism \(\hat{g} \rightarrow \hat{g}, g \otimes z^n \mapsto g \otimes (z)^{-n}\).

The functors \(d, D(\cdot)\) are exact.

There is an involution \(\bar{\cdot}: P^+ \rightarrow P^+\) so that \(V_{\bar{\lambda}} = V_{\lambda}\).

2.2. **Geometry of weights.** The following is proved in \([14]\) Lemma 7.7.

**Lemma 2.2.1.** Suppose:

(i) \((\lambda, k), (\mu, k) \in P^+_k\) are regular;

(ii) \(\bar{w} \in W\) satisfies \(\bar{w}(\lambda - \mu) \in P^+\);

(iii) \(\nu\) is a weight of \(V_{\bar{w}(\lambda - \mu)}\) such that \(w_1 \cdot \lambda = w \cdot \mu + \nu\) for some \(w, w_1 \in W_k\).

Then: \(w_1 = w\) and \(\nu \in W(\lambda - \mu)\).

3. THE KAZHDAN-LUSZTIG TENSORING

Kazhdan and Lusztig \([17, 18, 19]\) (inspired by Drinfeld \([4]\)) defined a covariant bifunctor

\[\tilde{O}_k \times \tilde{O}_k \rightarrow \tilde{O}_k, A, B \mapsto A \hat{\otimes} B.\]

We shall review its definition and main properties.

3.1. **Definition.**

3.1.1. **The set-up.** The notation to be used is as follows:

- \(z\) is a once and for all fixed coordinate on \(\mathbb{CP}^1\);
- \(L_0^P, P \in \mathbb{CP}^1\) is the loop algebra attached to \(P\); in other words, \(L_0^P = \mathfrak{g} \otimes C((z - P)), P \in \mathbb{C}\), and \(L_0^\infty = \mathfrak{g} \otimes C((z^{-1}))\);

more generally, if \(P = \{P_1, \ldots, P_m\} \subset \mathbb{CP}^1\), then

\[L_0^P = \oplus_{i=1}^m L_0^{P_i};\]

\[\hat{g}^P = L_0^P \oplus CK, P \in \mathbb{CP}^1\] is the affine algebra attached to the point \(P\) – the canonical central extension of \(L_0^P\); of course, \(\hat{g}^0 = \hat{g}\);
more generally, if \( P = \{P_1, \ldots, P_m\} \subset \mathbb{CP}^1 \), then \( \hat{g}^P \) is the direct sum of \( \hat{g}^{P_i} \), \( i = 1, \ldots, m \) modulo the relation: all canonical central elements \( K \) (one in each copy) are equal each other;

\[ \Gamma = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}, (z - 1)^{-1}] ; \Gamma \text{ is obviously a Lie algebra.} \]

The Laurent series expansions at points \( \infty, 1, 0 \) produce the Lie algebra homomorphism

\[ \epsilon : \Gamma \to L\mathfrak{g}_{\{\infty, 1, 0\}}. \]

**Lemma 3.1.1.** The map \( \epsilon \) lifts to a Lie algebra homomorphism \( \Gamma \to \hat{\mathfrak{g}}_{\{\infty, 1, 0\}}. \)

Proof consists of using the residue theorem, see [18].

By pull-back, any \( \hat{\mathfrak{g}}_{\{\infty, 1, 0\}} \)-module is canonically a \( \Gamma \)-module. Further, any \( A \in \hat{\mathcal{O}}_k \) is canonically a \( \hat{\mathfrak{g}}^P \)-module for any \( P \) – by the obvious change of coordinates; refer to this as attaching \( A \) to \( P \in \mathbb{CP}^1 \). Given \( A, B, C \in \hat{\mathcal{O}}_k \), we shall regard \( A \otimes B \otimes C \) as a \( \hat{\mathfrak{g}}_{\{\infty, 1, 0\}} \)-module meaning that \( \hat{g}^\infty \) acts on \( A \), \( \hat{g}^1 \) on \( B \), \( \hat{g}^0 \) on \( C \). (There is an obvious ambiguity in this notation.) There arises the space of coinvariants

\[ (A \otimes B \otimes C)_{\Gamma} = (A \otimes B \otimes C) / \Gamma(A \otimes B \otimes C). \]

This construction easily generalizes to the case when instead of three points – \( \infty, 1, 0 \) – there are \( m \) points, \( m \) modules and instead of \( \Gamma \) one considers the Lie algebra of rational functions on \( \mathbb{CP}^1 \) with \( m \) punctures with values in \( \mathfrak{g} \). We shall be mostly interested in the case \( m = 3 \) and sometimes in the case \( m = 2 \). If \( m = 2 \), then \( \Gamma \) becomes \( \hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}] \).

**Lemma 3.1.2.** Suppose \( D(B) \) is attached to \( \infty \), \( A \) to 0. Then

\[ \text{Hom}_\hat{\mathfrak{g}}(A, B) = ((D(B) \otimes A)\hat{\mathfrak{g}})^d. \]

Proof can be found in [18]; the reader may also observe that the arguments from [3.2.2] are easily adjusted to this case.

3.1.2. **Definition.** Let \( \hat{\Gamma} \) be the central extension of \( \Gamma \), the cocycle being defined as usual except that one takes the sum of residues at \( \infty \) and 1. Let \( \Gamma(0) \subset \hat{\Gamma} \) be the subalgebra consisting of functions vanishing at 0. Obviously, \( \Gamma(0) \) can also be regarded as a subalgebra of \( \Gamma \).

Consider the (total) dual space \( (A \otimes B)^d \); it is naturally a \( \hat{\Gamma} \)-module. \( (A \otimes B)^d \) carries the increasing filtration \( \{(A \otimes B)^d(N)\} \), where

\[ (A \otimes B)^d(N) = \{x \in (A \otimes B)^d : \gamma_1 \cdots \gamma_N x = 0 \text{ if all } \gamma_i \in \Gamma(0), x \in (A \otimes B)\}. \]
The space $\bigcup_{N \geq 1} (A \otimes B)^d(N)$ is naturally a $\hat{\mathfrak{g}}$-module. The passage from $(A \otimes B)^d$ to $\bigcup_{N \geq 1} (A \otimes B)^d(N)$ (or its obvious versions) is often called a functor of smooth vectors.

Define

$$A \otimes B = D\left(\bigcup_{N \geq 1} (A \otimes B)^d(N)\right).$$

Lemma 3.1.3. The functor $\otimes : \hat{\mathcal{O}}_k \times \hat{\mathcal{O}}_k \to \hat{\mathcal{O}}_k$ is right exact in each variable.

Proof (see loc. cit.) The functor $\otimes$ is a composition of two dualizations, $d$ and $D(,)$, and the functor of smooth vectors. It is enough to remark that the first two are exact while the last is only left exact. \qed

3.2. Some properties of $\otimes$.

3.2.1. For the future reference we collect some of the properties of $\otimes$ in the following

Theorem 3.2.1. (i)

$$\text{Hom}_{\hat{\mathfrak{g}}}(A \hat{\otimes} B, D(C)) = \text{Hom}_{\hat{\mathfrak{g}}}(D(\hat{\otimes} B), C) = ((A \otimes B \otimes C)^d)_\Gamma.$$

(ii) If $A, B \in \hat{\mathcal{O}}_k$ have a Weyl filtration, then $A \otimes B$ has also. (Here by Weyl filtration we mean a filtration such that its quotients are Weyl modules.)

(iii) If $k \not\in \mathbb{Q}$, then $V^k \hat{\otimes} V_k^k = (V^k \otimes V^k)_\Gamma$.

(iv) For any $k \in \mathbb{C}$, $V^k \hat{\otimes} V^k$ has a Weyl filtration (see (ii)), the multiplicity of $V^k$ being equal $(V^k \otimes V^k : V^k)$ (c.f. (iii)).

(v) There is an isomorphism $A \hat{\otimes} V^k_0 \to A$ for any $A \in \hat{\mathcal{O}}_k$.

(vi) There are commutativity and associativity morphisms $A \hat{\otimes} B \approx B \hat{\otimes} A$ and $(A \hat{\otimes} B) \hat{\otimes} C \approx A \hat{\otimes} (B \hat{\otimes} C)$ which endow $\hat{\mathcal{O}}_k$ with the structure of a braided monoidal category.

3.2.2. Morphisms and coinvariants. The description of morphisms in terms of coinvariants (see Theorem 3.2.1(i)) is the hallmark of this theory. Let us briefly explain why (i) holds. There is the obvious isomorphism of vector spaces

$$(A \otimes B \otimes C)^d \to \text{Hom}_C(C, (A \otimes B)^d).$$

It induces the map

$$((A \otimes B \otimes C)^d_\Gamma \to \text{Hom}_C(C, (A \otimes B)^d).$$

By $\hat{\Gamma}$-linearity, it actually gives the map
\[(A \otimes B \otimes C)_\Gamma^d \to \text{Hom}_\Gamma(C, \bigcup_{N \geq 1} (A \otimes B)^d(N)).\]

It remains to look at (7) and note that $\hat{\Gamma}$ is dense in $\hat{g}$.

3.2.3. Using the spaces of coinvariants. A lot about the functor $\hat{\otimes}$ easily follows from Theorem 3.2.1(i). As an example, let us derive (v). By (i),

\[\text{Hom}_{\hat{g}}(A \hat{\otimes} V_k, B) = ((A \otimes V_0^k \otimes D(B))_\Gamma)^d\]

for any $B \in \tilde{O}_k$.

As $V_0^k = \text{Ind}_{g_{\geq 1}}^g C$, the Frobenius reciprocity gives

\[(A \otimes V_0^k \otimes B)_\Gamma = (A \otimes D(B))_{g_{\geq 1}},\]

the latter space being $\text{Hom}_{\hat{g}}(A, B)$ by Lemma 3.1.2. We see that the spaces of morphisms of the modules $A$ and $A \hat{\otimes} V_0^k$ are equal, hence so are the modules.

Replacing in this argument $C$ with a suitable finite dimensional $g$-module and repeating it three times one gets

\[(8) \quad \text{Hom}_{\hat{g}}(V^k_\lambda \hat{\otimes} V^k_\mu, D(V^k_\nu)) = \text{Hom}_g(V_\lambda \otimes V_\mu, V_\nu).\]

As for generic $k$ $D(V^k_\nu) \approx V^k_\nu$ (see 2.1), (8) along with Theorem 3.2.1(i) implies Theorem 3.2.1(iii).

4. Affine translation functors

4.1. Definition. For any $(\lambda, k) \in P^+_k$ denote by $\tilde{O}_k^\lambda$ the full subcategory of $\tilde{O}_k$ consisting of modules whose composition factors all have highest weights lying in the orbit $W_k \cdot (\lambda, k)$. There arises the projection

\[p_\lambda : \tilde{O}_k \to \tilde{O}_k^\lambda.\]

This all has been reviewed in 2.1.

Given $(\lambda, k), (\mu, k) \in P^+_k$, pick $\bar{w} \in W$ so that $\bar{w}(\lambda - \mu) \in P^+$. It is easy to see that then $(\bar{w}(\lambda - \mu), k) \in P^+_k$. Define the translation functor

\[(9) \quad T^\lambda_\mu : \tilde{O}_k^\mu \to \tilde{O}_k^\lambda A \mapsto p_\lambda(V^k_{\bar{w}(\lambda - \mu)} \hat{\otimes} A).\]

This functor was first introduced by Finkelberg [10] who, however, considered it only for $k < 0$. 
As an immediate corollary of the definition, one has

\[ T_\mu^\lambda = p_\mu \circ ((V^d_{\bar{w}(\lambda - \mu)})^k \hat{\otimes} e) \]

4.2. Rigidity of Weyl modules with dominant highest weight.

Lemma 4.2.1. If \((\lambda, k), (\mu, k)\) are regular (i.e. off the affine walls) and \(w \in W_k\) satisfies \(w \cdot \mu \in P^+\), then

\[ T_\mu^\lambda (V^k_{w, \mu}) = V^k_{w, \lambda}. \]

Proof. By Theorem 3.2.1 (iv), \(T_\mu^\lambda (V^k_{w, \mu})\) has a filtration with quotients isomorphic to \(V^k_{w_1, \lambda}\), \(w_1 \in W_k\) such that \(w_1 \cdot \lambda = w \cdot \mu + \nu, \nu\) being a weight of \(V_{\bar{w}(\lambda - \mu)}\). By Lemma 2.2.1, \(w_1 = w\). This implies that this filtration has only one term, \(V^k_{w, \lambda}\). \(\square\)

Corollary 4.2.2. If \((\lambda, k) \in P^+_k\) is regular, then \(V^k_0\) is a direct summand of \(V^k_\Lambda \hat{\otimes} V^k_\Lambda\).

Proof. Of course \((0, k)\) is dominant regular and \(p_0 A\) is a direct summand of \(A\). It remains to observe that \(T^0_\lambda V^k_\Lambda = p_0 (V^k_\Lambda \hat{\otimes} V^k_\Lambda)\) and use Lemma 4.2.1 to get \(T^0_\lambda V^k_\Lambda = V^k_0\). \(\square\)

We get the maps

\[ i_\lambda : V^k_0 \to V^k_\Lambda \hat{\otimes} V^k_\Lambda, e_\lambda : V^k_\Lambda \hat{\otimes} V^k_\Lambda \to V^k_0. \]

Observing that the maps between \(\hat{\otimes}\)-products of Weyl modules are uniquely determined by the induced maps of the corresponding finite dimensional \(\mathfrak{g}\)-modules (Theorem 3.2.1 and (8)), we see that we can normalize \(i_\lambda, e_\lambda\) so that the compositions

\[ V^k_\Lambda = V^k_0 \hat{\otimes} V^k_\Lambda \xrightarrow{i_\lambda \hat{\otimes} id} V^k_\Lambda \hat{\otimes} V^k_\Lambda \xrightarrow{id \hat{\otimes} e_\lambda} V^k_\Lambda \]

\[ V^k_\Lambda = V^k_0 \hat{\otimes} V^k_\Lambda \xrightarrow{id \hat{\otimes} i_\lambda} V^k_\Lambda \hat{\otimes} V^k_\Lambda \xrightarrow{id \hat{\otimes} e_\lambda} V^k_\Lambda \]

are equal to the identity. By definition (see e.g. [19] III, Appendix) we have

Corollary 4.2.3. If \((\lambda, k) \in P^+_k\), then \(V^k_\Lambda\) and \(V^k_\Lambda\) are rigid.

Consider the functor \(V^k_\Lambda \hat{\otimes} ? : \mathcal{O}_k \to \mathcal{O}_k, M \mapsto V^k_\Lambda \hat{\otimes} M\).

Corollary 4.2.4. (i) If \((\lambda, k) \in P^+_k\), then the functors \(V^k_\Lambda \hat{\otimes} ?\) and \(V^k_\Lambda \hat{\otimes} ?\) are adjoint, i.e. there is a functor isomorphism

\[ \text{Hom}_\mathfrak{g}(V^k_\Lambda \hat{\otimes} A, B) = \text{Hom}_\mathfrak{g}(A, V^k_\Lambda \hat{\otimes} B). \]

(ii) If \((\lambda, k) \in P^+_k\), then the functors \(V^k_\Lambda \hat{\otimes} ?\) and \(V^k_\Lambda \hat{\otimes} ?\) are exact, i.e. send exact short sequences to exact ones.
Proof is standard; for the reader’s convenience we reproduce the one from [13] III, Appendix. To prove (i), consider two composition maps

\[ \phi : \text{Hom}_\theta(V^k_\lambda \hat{\otimes} A, B) \rightarrow \text{Hom}_\theta(V^k_\lambda \hat{\otimes} V^k_\lambda \hat{\otimes} A, V^k_\lambda \hat{\otimes} B) \xrightarrow{\psi \circ} \text{Hom}_\theta(A, V^k_\lambda \hat{\otimes} B), \]

\[ \psi : \text{Hom}_\theta(A, V^k_\lambda \hat{\otimes} B) \rightarrow \text{Hom}_\theta(V^k_\lambda \hat{\otimes} A, V^k_\lambda \hat{\otimes} V^k_\lambda \hat{\otimes} B) \xrightarrow{\phi} \text{Hom}_\theta(V^k_\lambda \hat{\otimes} A, B). \]

By (1), the compositions \( \phi \circ \psi \) and \( \psi \circ \phi \) are equal to the identity.

(ii) is an easy consequence of (i): we have to prove that \( B \rightarrow B_2 \) is a monomorphism implies that \( V^k_\lambda \hat{\otimes} B_1 \rightarrow V^k_\lambda \hat{\otimes} B_2 \) is also, or, equivalently, that for any \( A \in \tilde{O}_k \) the induced map

\[ \text{Hom}_\theta(A, V^k_\lambda \hat{\otimes} B_1) \rightarrow \text{Hom}_\theta(A, V^k_\lambda \hat{\otimes} B_2) \]

is also a monomorphism. By (i), it is equivalent to proving that

\[ \text{Hom}_\theta(V^k_\lambda \hat{\otimes} A, B_1) \rightarrow \text{Hom}_\theta(V^k_\lambda \hat{\otimes} A, B_2) \]

is a monomorphism, but this is an obvious corollary of injectivity of the map \( B_1 \rightarrow B_2 \).

\[
4.3. \text{Properties of affine translation functors.} \text{ Recall that there is the notion of a formal character } chA \text{ for any } A \in \tilde{O}_k^\lambda, \text{ see e.g. [3]. There arises an abelian group of characters, each of the following sets being a topological basis in it:}
\{chV^k_{w,\lambda}, w \in W_k\}, \{chL^k_{w,\lambda}, w \in W_k\}. \text{ Of course the symbols } chV^k_{w,\lambda}, chL^k_{w,\lambda} \text{ should be ignored unless } w \cdot \lambda \in P^+. \text{ Observe that}
\]

\[ chA = \sum_{w \geq w_o} \bar{n}_w chL^k_{w,\mu} \Leftrightarrow chA = \sum_{w \geq w_o} n_w chV^k_{w,\mu} \]

Proposition 4.3.1. Let \((\lambda, k), (\mu, k)\) be regular dominant.

(i) \( T^\lambda_\mu \) is exact;

(ii) \( T^\lambda_\mu, T^\mu_\lambda \) are adjoint to each other;

(iii) If \( chA = \sum_{w \in W_k} n_w chV^k_{w,\mu} \), then \( chT^\lambda_\mu A = \sum_{w \in W_k} n_w chV^k_{w,\lambda} \).

(iv) \( T^\lambda_\mu(L^k_{w,\mu}) = L^k_{w,\lambda} \).

(v) More generally, \( T^\lambda_\mu(\cdot) \) establishes an equivalence of the submodule lattices of \( V^k_{w,\mu} \) and \( V^k_{w,\lambda} \).

Proof. (i) \( T^\lambda_\mu \) is exact as a composition of the exact functors \( p_\lambda \) and \( V^d_{w(\lambda-\mu)} \hat{\otimes} ?, \text{ see Corollary 4.2.4 (ii).}

(ii) By Corollary 4.2.4 (i), one has for any \( A \in \tilde{O}_k^\mu, B \in \tilde{O}_k^\lambda \)
\[ \text{Hom}^\hat{g}(T^\mu A, B) = \text{Hom}^\hat{g}(p_\lambda(V^k_{\bar{w}(\lambda-\mu)} \otimes A), B) = \text{Hom}^\hat{g}(V^k_{\bar{w}(\lambda-\mu)} \otimes A, B) = \text{Hom}^\hat{g}(A, V^d_{\bar{w}(\lambda-\mu)} \otimes B) = \text{Hom}^\hat{g}(A, T^\mu_{\lambda}B). \]

(iii) follows at once from (i) (if one uses the local composition series, see e.g. [3]).

(iv) Let \( T^\lambda_{\mu}(L^k_{\bar{w}0 \cdot \mu}) \) be reducible. There arises an exact sequence with non-zero \( N \):

\[ 0 \to N \to T^\lambda_{\mu}(L^k_{\bar{w}0 \cdot \mu}) \to L^k_{\bar{w}0 \cdot \lambda} \to 0. \]

Applying \( T^\mu_{\lambda} \) to it one gets

\[ 0 \to T^\mu_{\lambda}(N) \to T^\mu_{\lambda}(T^\lambda_{\mu}(L^k_{\bar{w}0 \cdot \mu})) \to T^\mu_{\lambda}(L^k_{\bar{w}0 \cdot \lambda}) \to 0. \]

By (iii) and (12), \( \text{ch}(T^\mu_{\lambda}(T^\lambda_{\mu}(L^k_{\bar{w}0 \cdot \mu}))) = \text{ch}L^k_{\bar{w}0 \cdot \mu} \) and \( \text{ch}T^\mu_{\lambda}(N) \neq 0; \) therefore \( \text{ch}T^\mu_{\lambda}(L^k_{\bar{w}0 \cdot \lambda}) < \text{ch}L^k_{\bar{w}0 \cdot \mu}. \) Contradiction.

(v) Here proof is an obvious version of that of (iv). By (ii) it is enough to show that if \( A \subset B \subset V^k_{\bar{w} \cdot \mu} \), then \( T^\lambda_{\mu}(A) \subset T^\lambda_{\mu}(B) \subset V^k_{\bar{w} \cdot \lambda}. \) Using (12) and passing to quotients, if necessary, the problem is reduced to the case when \( B \) is a highest weight module. In this case the arguments of (ii) go through practically unchanged. \( \square \)

4.4. **Theorem 4.4.1.** The functor \( T^\lambda_{\mu} : \tilde{\mathcal{O}}^\mu_k \to \tilde{\mathcal{O}}^\lambda_k \) is an equivalence of categories.

**Proof.** It is enough show that \( T^\lambda_{\mu} \circ T^\mu_{\lambda} : \tilde{\mathcal{O}}^\lambda_k \to \tilde{\mathcal{O}}^\lambda_k \) is equivalent to the identity. In other words, we want to show that \( \text{id} : A \to A, A \in \tilde{\mathcal{O}}^\lambda_k \) is transformed into an isomorphism in \( \text{Hom}^\hat{g}(T^\lambda_{\mu} \circ T^\mu_{\lambda}(A), A) \). We already know this when \( A \) is simple, see Corollary 4.3.1 (ii). Using (12) and passing to quotients, if necessary, the problem is reduced to the case where \( B \) is a highest weight module. In this case the arguments of (ii) go through practically unchanged. \( \square \)

4.5. **Generalizing from \( \tilde{\mathcal{O}}_k \) to \( \mathcal{O}_k \).** Our two key results – Proposition 1.3.1 and Theorem 1.4.1 – can be carried over to the category \( \mathcal{O}_k \). Let us briefly explain it. We will be using subcategories \( \mathcal{O}^\lambda_k \subset \mathcal{O}_k \) (see 2.1) only when \( k + h^\vee \in \mathbb{Q}_> \) and \( \lambda \) is integral, although the last condition can be easily relaxed.
It is non-trivial (if at all meaningful) to carry the Kazhdan-Lusztig tensoring over to the entire $\mathcal{O}_k$. (An intelligent way to do something like it requires introducing additional structures, see [9].) It is however straightforward to extend it to the functor

$$\hat{\otimes} : \hat{\mathcal{O}}_k \times \mathcal{O}_k \to \mathcal{O}_k,$$

as proposed by Finkelberg [10]. One basic property of this operation absolutely analogous (along with the proof) to Theorem 3.2.1 (iv) is as follows.

As $V_\lambda \otimes M_\mu$ has a filtration by Verma modules in the category of $\mathfrak{g}$-modules, $V^k_\lambda \hat{\otimes} M^k_\mu$ has a filtration by Verma modules in $\mathcal{O}_k$; further the multiplicities are the same as in the finite dimensional case:

$$(V^k_\lambda \hat{\otimes} M^k_\mu : M^k_\nu) = (V_\lambda \otimes M_\mu : M_\nu).$$

Given this one can easily inspect our exposition of affine translation functors and observe that quite a lot carries over to the setting of $\mathcal{O}_k$ word for word except that at the appropriate places Weyl modules are to be changed for the corresponding Verma modules. Here are some examples:

(i) definition of $T^\lambda_\mu : \mathcal{O}^\mu_k \to \mathcal{O}^\lambda_k$ if $\lambda, \mu$ belong to the same Weyl chamber;

(ii) the Verma filtration of $V^k_\lambda \hat{\otimes} M^k_{w \cdot \mu}, w \in W_k$ and Lemma 2.2.1 imply that $T^\lambda_\mu(M^k_{w \cdot \mu}) = M^k_{w \cdot \lambda}$ if $(\mu, k), (\lambda, k)$ are regular (c.f. Lemma 1.2.1); observe that we can now drop the condition that $w \cdot \mu \in P^+$;

(iii) therefore Proposition 4.3.1 holds with the indicated changes.

We get

**Theorem 4.5.1.** The functor $T^\lambda_\mu : \mathcal{O}^\mu_k \to \mathcal{O}^\lambda_k$ is an equivalence of categories if $\lambda, \mu$ are integral and both belong to the same Weyl chamber.

5. **Annihilating Ideals of Highest Weight Modules**

5.1. **Vertex operators and ...** The usual tensor functor $\otimes : M, N \mapsto M \otimes N$ has the following fundamental (and trivial) property: there is a natural map

$$(14) \quad N \to \text{Hom}_\mathfrak{g}(M, M \otimes N)$$

$$n \mapsto n(.) \text{ such that } n(m) = m \otimes n.$$ 

Here we shall explain the $\hat{\otimes}$-analogue of this map
5.1.1. By Theorem 3.2.1 (v), \( A \otimes V_0^k \approx A \) for any \( A \in \tilde{O}_k \). Therefore by Theorem 3.2.1 (i), there is a natural isomorphism

\[(A \otimes V_0^k \otimes D(A))_\Gamma^d \approx \text{Hom}_{\tilde{g}}(A, B),\]

for any \( B \in \tilde{O}_k \).

Recall that the space \((A \otimes V_0^k \otimes D(A))_\Gamma^d\) was defined by means of \( \Gamma \), the latter being defined by choosing three points, \( \infty, 1, 0 \), see the end of 3.2.3. The choice of points was, of course, rather arbitrary. Keeping \( \infty, 0 \) fixed and \( A, D(B) \) attached to \( \infty, 0 \) resp., we shall allow the third point to vary. We get then the family of Lie algebras \( \Gamma_t, t \in \mathbb{C}^* \) and the family of the one-dimensional spaces (c.f. 3.1.1)

\[< A, V_0^k, D(B) >_t := ((A \times V_0^k \times D(B))_\Gamma^d), \quad t \in \mathbb{C}^*.\]

These naturally arrange in a trivial line bundle over \( \mathbb{C}^* \), the fiber being isomorphic to

\[< A, V_0^k, D(B) >_t := (A \otimes D(B))_\tilde{g} = \text{Hom}_{\tilde{g}}(A, B),\]

by the arguments using Frobenius reciprocity as in 3.2.3. Pick a section of this bundle by choosing \( \phi \in \text{Hom}_{\tilde{g}}(A, B) \).

Hence we get a trilinear functional (depending on \( t \in \mathbb{C}^* \))

\[\Phi^\phi_t \in < A, V_0^k, D(B) >_t \subset (A \otimes V_0^k \otimes D(B))^d.\]

Reinterpret it as the linear map:

(15) \[\Phi^\phi_t(\cdot) : V_0^k \rightarrow (A \otimes D(B))^d,\]

or, equivalently,

(16) \[\tilde{\Phi}^\phi_t(\cdot) : V_0^k \rightarrow \text{Hom}_\mathbb{C}(A, D(B))^d, \quad t \in \mathbb{C}^*.\]

The latter map is an analogue of \( N \rightarrow \text{Hom}_\mathbb{C}(M, M \otimes N) \) mentioned above. To analyze its properties observe that there is an obvious embedding \( B \rightarrow (D(B))^d \). It does not, of course, allow us to interprete \( \tilde{\Phi}^\phi_t(v), \ v \in V_0^k \) as an element of \( \text{Hom}_\mathbb{C}(A, B) \) depending on \( t \). But, as the following lemma shows, Fourier coefficients of \( \tilde{\Phi}^\phi_t(v), \ v \in V_0^k \) are actually elements of \( \text{Hom}_\mathbb{C}(A, B) \). To formulate this lemma observe that there is a natural gradation on \( A \) and \( B \) consistent with that of \( \tilde{g} \); e.g. \( A = \oplus_{n \geq 0} A[n], \ \text{dim}A[n] < \infty. \)

**Lemma 5.1.1.** Let \( B \) be either \( A \) or a quotient of \( A \), \( \text{id} : A \rightarrow B \) be the natural projection. Then:

(i) \( \Phi^\text{id}_t(\text{vac})(x, y) = y(x) \), where \( \text{vac} \) is understood as the generator of \( V_0^k \);
(ii) more generally, if \( v \in V^k_0[n] \), \( x \in A[m] \), \( y \in D(B)[l] \), then
\[
\Phi^id_t(v)(x, y) \in C \cdot t^{-l+m-n}.
\]

**Proof.** Given \( g \in \mathfrak{g} \), denote by \( g_n \in \hat{\mathfrak{g}}^P \) the element \( g \otimes (z - P)^n \) or \( g \otimes z^{-n} \) if \( P = \infty \). (It should be clear from the context which \( P \) is meant.) Thus \( g_n x = (g \otimes z^{-n}) x \) if \( x \in A \), the \( A \) being attached to \( \infty \); similarly, \( g_n x = (g \otimes z^n) x \) if \( x \in D(B) \), the \( D(B) \) being attached to 0.

(i) can be proved by an obvious induction on the degree of \( x \) and \( y \) using the following formula (which follows from the definition of \((A \times V^k_0 \times D(B))_\Gamma d\) and the Laurent expansions of \( z^{-n} \) at \( \infty \) and 0):
\[
\Phi^id_t(v)(g_n x, y) = -\Phi^id_t(v)(x, g_n y).
\]

To prove (ii) observe, first, that (i) is a particular case of (ii) when \( v = \text{vac} \). One then proceeds by induction on \( n \) using the formula (which again follows from the definition of \((A \times V^k_0 \times D(B))_\Gamma d\) and the Laurent expansions of \((z - t)^{-n}\) at \( \infty \) and 0):
\[
(-1)^{n-1}(n-1)! \Phi^id_t(g_n v)(x, y) = \left( \frac{d}{dt} \right)^{n-1} \{ \sum_{i=1}^{\infty} t^{i-1} \Phi^id_t(v)(g_i x, y) - \sum_{i=0}^{\infty} t^{-i-1} \Phi^id_t(v)(x, g_i y) \}.
\]

\[\square\]

Observe that the spaces \( A, B \) being graded, the space \( Hom_C(A, D(B)) \) is also. Lemma \[5.1.1\] means that although the map \( \Phi^id_t(.) \) from \[10\] cannot be interpreted as an element of \( Hom_C(A, B) \), its Fourier components can because they are homogeneous. To compare with \[12\] introduce the following notation: for any \( v \in V^k_0[n] \) set
\[
Y(v, t) = \sum_{i \in \mathbb{Z}} v_i t^{-i-n},
\]
where
\[
v_i := \oint \Phi^id_t(v) t^{i+n-1} \ dt : A[l] \to B[l + i],
\]
for all \( l \geq 0 \), and call the generating functions \( Y(v, t) \) fields. For example, it easily follows from the formulae above that
\[
x(t) := Y(x_{-1} \text{vac}, t) = \sum_{i \in \mathbb{Z}} x_i t^{-i-1},
\]
producing the famous current \( x(t) \). Another fact easily reconstructed from the formulae above (especially from the proof of Lemma 5.1.1) is that

\[
(-1)^{n-1}(n-1)! Y(x_{-n}v, t) =: x(t)^{(n-1)} Y(v, t) ;
\]

where we set

\[
: x(t)^{(n-1)} Y(v, z) := (x(z)^{(n-1)})_{-} Y(v, t) + Y(v, t)(x(z)^{(n-1)})_{+},
\]

\((x(z)^{(n-1)})_{\pm}\) being defined as usual (see e.g. [13]). It follows that all fields are infinite combinations of elements of \( \hat{\mathfrak{g}} \).

The expressions \( Y(v, t) \) are not only formal generating functions. In this notation Lemma 5.1.1 can be rewritten as follows.

**Corollary 5.1.2.** Under the assumptions of Lemma 5.1.1,

\[
\Phi^v_t(x, y) = y(Y(v, t)x).
\]

5.1.2. The considerations of 5.1.1 are easily generalized as follows. (We shall skip the proofs as they essentially repeat those in 5.1.1.) Replace \( V^k_0 \) with \( V^k_\lambda \) and pick \( A, B \in \mathcal{O}_k \) so that the space \( < A, V^k_\lambda, D(B) >_t \neq 0 \). For any \( \phi \in < A, V^k_\lambda, D(B) >_t \) we get a map

\[
Y(\cdot, t) : V^k_\lambda \to Hom_{\mathbb{C}}(A, B((t, t^{-1})))
\]

\[
V^k_\lambda \ni v \mapsto Y(v, t) = \sum_{i \in \mathbb{Z}} v_i t^{-i-\tilde{v}}, \quad v_i \in Hom_{\mathbb{C}}(A, B).
\]

\( Y(v, t), \quad v \in V^k_\lambda \) is a generating function having all properties its counterpart from 5.1.1 with one notable exception. Consider the “upper floor” of \( V^k_\lambda \): \( V_\lambda \subset V^k_\lambda \). The Fourier components of the fields \( Y(v, t), \quad v \in V_\lambda \), \( \lambda \neq 0 \) generate a \( \hat{\mathfrak{g}} \)-submodule of \( Hom_{\mathbb{C}}(A, B) \) isomorphic to the loop module \( L(V_\lambda) = V_\lambda \otimes \mathbb{C}[z, z^{-1}] \). Strange as it may seem to be, if \( \lambda = 0 \), then instead of \( \mathbb{C}[z, z^{-1}] \) this construction gives simply \( \mathbb{C} \) – this was explained above.

The embedding \( L(V_\lambda) \subset Hom_{\mathbb{C}}(A, B) \) is called a **vertex operator**. It is easy to see that all vertex operators are obtained via the described construction.

5.2. **...and vertex operator algebras.** We now recall that a vertex operator algebra (VOA) is defined to be a graded vector space \( \bigcup_{i \in \mathbb{Z}} V[i] \), \( dim V_i < \infty \) along with a map

\[
Y(\cdot, t) : V \to End(V)((t, t^{-1})),
\]
satisfying certain axioms among which we mention *associativity* and *commutativity* axioms, see e.g. [12, 13]. Similarly one defines the notion of a module (submodule) over a VOA. A VOA is a module over itself; call an ideal of a VOA a submodule of a VOA as a module over itself. Observe that it follows from the associativity axiom that the Fourier components of fields \( Y(v, t), v \in V \) close in a Lie algebra, \( \text{Lie}(V) \). In this way, an ideal of a VOA \( V \) produces an ideal of \( \text{Lie}(V) \) in the Lie algebra sense. Not any ideal of \( \text{Lie}(V) \) can be obtained in this way. Refer to such an ideal an ideal of \( \text{Lie}(V) \) as VOA.

It follows from [13] that the constructions of 5.1.1 give: \((V^k_0, Y(., t))\) is a vertex operator algebra and each \( A \in \mathcal{O}_k \) is a module over it. \( \text{Lie}(V^k_0) \) is habitually denoted \( \hat{U}(\hat{\mathfrak{g}})_{\text{loc}} \) and called a local completion of \( \hat{U}(\hat{\mathfrak{g}}) \), even though it is not an associative algebra! A moment’s thought shows that the ideal lattice of \( \hat{U}(\hat{\mathfrak{g}})_{\text{loc}} \) as VOA is isomorphic with the submodule lattice of \( V^k_0 \) as a \( \hat{\mathfrak{g}} \)-module.

5.3. Here we prove the following theorem – one of the main results of this paper.

**Theorem 5.3.1.** Let \( k \in \mathbb{Q}>\), \((\lambda, k), (0, k) \in P^+_k\) be regular. Denote by \( \Omega(V^k_{\lambda}) \) the submodule lattice of \( V^k_{\lambda} \), and by \( \Omega(\hat{U}(\hat{\mathfrak{g}})_{\text{loc}}) \) the ideal lattice of \( \hat{U}(\hat{\mathfrak{g}})_{\text{loc}} \) as VOA at the level \( k \). There is a lattice equivalence

\[
\omega : \Omega(\hat{U}(\hat{\mathfrak{g}})_{\text{loc}}) \to \Omega(V^k_{\lambda}),
\]

\[
\Omega(\hat{U}(\hat{\mathfrak{g}})_{\text{loc}}) \ni I \mapsto IV^k_{\lambda}.
\]

**Proof.**

First of all, by definition \ref{5.2} \( \omega \) is equivalently reinterpreted as a map of the submodule lattices of the \( \hat{\mathfrak{g}} \)-modules: \( \omega : \Omega(V^k_0) \to \Omega(V^k_{\lambda}) \). In what follows we shall make use of this reinterpretation.

Consider the translation functor: \( T^\lambda_0 \). If \( N \subset V^k_0 \) is a submodule, then on the one hand we have

\[
T^\lambda_0(V^k_0) = V^k_{\lambda} \hat{\otimes} V^k_0 (= V^k_{\lambda}),
\]

and therefore

\[
T^\lambda_0(V^k_0/N) = V^k_{\lambda} \hat{\otimes} (V^k_0/N).
\]

By Theorem \ref{3.2.1} and Corollary \ref{5.1.2},

\[
\text{Hom}_\hat{\mathfrak{g}}(T^\lambda_0(V^k_0/N), ?) = <V^k_0/N, V^k_{\lambda}, D(?)>_{\text{t}} = \text{Hom}_\hat{\mathfrak{g}}(V^k_{\lambda}/\omega(N), ?).
\]

On the other hand, by Proposition \ref{1.3.1} (i)

\[
T^\lambda_0(V^k_0/N) = V^k_{\lambda}/T^\lambda_0(N).
\]
We conclude immediately that $\omega(N) = T^\lambda_0(N)$. It remains to recollect that $T^\lambda_0$ is an isomorphism of the submodule lattices by Proposition 4.3.1 (v).

An application of this result to annihilating ideals of admissible representations is as follows. Recall that if $k + h^\vee \in \mathbb{Q}_{>0}$, $(\lambda, k) \in P_k^+$ is regular, then $L^k_\lambda$ is called admissible [16]. $L^k_\lambda$ is an irreducible quotient of $V^k_\lambda$ be a submodule $N^k_\lambda$ generated by one singular vector, see also [16]. By Theorem 5.3.1, $\omega(N^k_\lambda) = N^k_\lambda$. We get

**Corollary 5.3.2.** The annihilating ideal of an admissible representation equals $\text{Lie}(N^k_\lambda)$; in particular, it is generated (as VOA) by one singular vector of $V^k_\lambda$.

**Remarks.**

(i) In the case $g = \mathfrak{sl}_2$, Corollary 5.3.2 follows from the more general results of [3], see also [3].

(ii) If the Feigin-Frenkel conjecture on the singular support of $L^k_0$ (theorem in the $\mathfrak{sl}_2$-case, see [4]) were correct, then Corollary 5.3.2 would imply its validity for any admissible representation from $\mathcal{O}_k$ and thus would give a new example of rational conformal field theory.

(iii) Another way to think of Corolary 5.3.2 is that $L^k_0$ is a VOA and $L^k_\lambda$ is a module over it; in the $\mathfrak{sl}_2$-case, this point of view is adopted in [1, 5].

6. **What is a Harish-Chandra bimodule over an affine Lie algebra?**

6.1. **Restricted Harish-Chandra category.** Our approach to defining affine Harish-Chandra bimodules will heavily rely on the properties of affine translation functors. We begin in the framework of the category $\mathcal{O}_k$, see Proposition 4.3.1 and Theorem 4.4.1. Call a triple of weights $\lambda_l, \lambda_r, \lambda \in P_k^+$ a translation datum if $\lambda_l - \lambda \in W \cdot \lambda_r$. There arises the translation functor $T^\lambda_{\lambda_l} = p_{\lambda_l} \circ (V^k_{\lambda_r} \hat{\otimes}?)$.

Let $\text{Funct}(\mathcal{O}_k^{\lambda_r}, \mathcal{O}_k^{\lambda_l})$ be the category of functors from $\mathcal{O}_k^{\lambda_l}$ to $\mathcal{O}_k^{\lambda_r}$. There is a functor

$$\Phi : \mathcal{O}_k \rightarrow \text{Funct}(\mathcal{O}_k^{\lambda_r}, \mathcal{O}_k^{\lambda_l}),$$

$$\Phi(A) : B \mapsto p_{\lambda_l}(A \hat{\otimes} B).$$

Setting for the sake of brevity $\mathcal{F}^\lambda_{\lambda_l} = \text{Funct}(\mathcal{O}_k^{\lambda_r}, \mathcal{O}_k^{\lambda_l})$, we get the natural map

$$i : \text{Hom}_{\mathcal{F}^\lambda_{\lambda_l}}(F, G) \rightarrow \text{Hom}_{\mathfrak{g}}(F(V^k_{\lambda_r}), G(V^k_{\lambda_l})).$$
where $i(\psi)$ is simply the value of the functor morphism $\psi$ on $V^k_{\lambda_r}$.

**Conjecture 6.1.1.** If $(\lambda_r, \mu, \lambda_l)$ and $(\lambda_r, \nu, \lambda_l)$ are translation data and $A \in \tilde{O}_k^{\mu}$, $B \in \tilde{O}_k^{\nu}$, then the map

$$\text{Hom}_{\tilde{\mathcal{F}}_{\lambda_l}^{\lambda_r}}(\Phi(A), \Phi(B)) \to \text{Hom}_{\tilde{\mathfrak{g}}}(\Phi(A)(V^k_{\lambda_r}), \Phi(B)(V^k_{\lambda_r})).$$

is an isomorphism (c.f. Theorem 3.5 in [2]).

To provide a supporting evidence, we prove surjectivity in the case $\mu = \nu$. As $(\lambda_r, \mu, \lambda_l)$ and $(\lambda_r, \nu, \lambda_l)$ are translation data, $\Phi(\lambda_r, \mu, \lambda_l)$ and $\Phi(\lambda_r, \nu, \lambda_l)$ are translation data, $\Phi(A)(V^k_{\lambda_r}) = T^\lambda_{\mu}(A)$ and $\Phi(B)(V^k_{\lambda_r}) = T^\lambda_{\nu}(B)$. By Theorem 4.4.1 we get an isomorphism

$$T^\lambda_{\mu} : \text{Hom}_{\tilde{\mathfrak{g}}}(\Phi(A)(V^k_{\lambda_r}), \Phi(B)(V^k_{\lambda_r})) \approx \text{Hom}_{\tilde{\mathfrak{g}}}(A, B).$$

It follows that any $\phi \in \text{Hom}_{\tilde{\mathfrak{g}}}(A, B)$ gives rise to $\Phi(\phi) \in \text{Hom}_{\tilde{\mathcal{F}}_{\lambda_l}^{\lambda_r}}(\Phi(A), \Phi(B))$ and, of course, the value of the functor morphism $\Phi(\phi)$ on $V^k_{\lambda_r}$ corresponds to $T^\lambda_{\mu}(\phi) \in \text{Hom}_{\tilde{\mathfrak{g}}}(\Phi(A)(V^k_{\lambda_r}), \Phi(B)(V^k_{\lambda_r}))$:

$$i(\Phi(\phi)) = T^\lambda_{\mu}(\phi). \quad \Box$$

**Definition.** Let $(\lambda_r, \lambda, \lambda_l)$ be a translation datum. Define the restricted affine Harish-Chandra category $\tilde{\mathcal{H}}_{\lambda_l}^{\lambda_r}$ to be the complete subcategory $\Phi(\tilde{O}_k^{\lambda_l}) \subset \tilde{\mathcal{F}}_{\lambda_l}^{\lambda_r}$.

Conjecture 6.1.1 implies that $\tilde{\mathcal{H}}_{\lambda_l}^{\lambda_r}$ is equivalent to $\tilde{O}_k^{\lambda_l}$ and, in particular, independent of $\lambda$. This all is in precise analogy with the Bernstein-Gelfand theorem, see equivalence (2) in Introduction. As a corollary, we get that the simple objects of $\tilde{\mathcal{H}}_{\lambda_l}^{\lambda_r}$ are in one-to-one correspondence with the subset of the affine Weyl group $W_k$:

$$\{ w \in W_k : w \cdot \lambda_l \in \mathbb{Z}^+ \}.$$

Similarly, the functors $p_{\lambda_l} \circ (V^k_{w \cdot \lambda_l})$ are obvious analogues of the principal series representations.

A drawback of our definition is that $\tilde{\mathcal{H}}_{\lambda_l}^{\lambda_r}$ is defined only if there is $\lambda$ such that $(\lambda_r, \lambda, \lambda_l)$ is a translation datum. The simplest example when $\tilde{\mathcal{H}}_{\lambda_l}^{\lambda_r}$ is not defined is when $\mathfrak{g} = \mathfrak{sl}_2, k = 2, \lambda_l = 1, \lambda_r = 2$. This drawback, however, is not as serious as it may seem to be. In the case of special interest $\lambda_l = \lambda_r$, the triple $(\lambda_l, 0, \lambda_r)$ is a translation datum.
6.2. Non-restricted case. One would prefer to have as many simple objects as there are elements in the entire affine Weyl group. To achieve that we use the results of sect. 4.5.

Just as it was above, for a translation datum \((\lambda_l, \lambda, \lambda_r)\) we have the category of functors \(\mathcal{F}_{\lambda_l}^{\lambda} \) from \(\mathcal{O}_k^{\lambda_r} \) to \(\mathcal{O}_k^{\lambda_l} \) and the functor
\[
\Phi : \mathcal{O}_k^{\lambda} \to \mathcal{F}_{\lambda_l}^{\lambda},
\]
\[
\Phi(A) : B \mapsto p_{\lambda}(A \otimes B).
\]

There again arises the natural map
\[
i : \text{Hom}_{\mathcal{F}_{\lambda_l}^{\lambda}}(F, G) \to \text{Hom}_{\mathcal{O}_k^{\lambda}}(F(V_{\lambda}^k), G(V_{\lambda}^k)),
\]
where \(i(\psi)\) is the value of the functor morphism \(\psi\) on \(V_{\lambda}^k\).

Conjecture 6.2.1. If \((\lambda_r, \mu, \lambda_l)\) and \((\lambda_r, \nu, \lambda_l)\) are translation data and \(A \in \mathcal{O}_k^{\mu}, B \in \mathcal{O}_k^{\nu},\) then the map
\[
\text{Hom}_{\mathcal{F}_{\lambda_l}^{\lambda}}(\Phi(A), \Phi(B)) \to \text{Hom}_{\mathcal{O}_k^{\lambda}}(\Phi(A)(V_{\lambda}^k), \Phi(B)(V_{\lambda}^k)).
\]
is an isomorphism.

Surjectivity of the map in Conjecture 6.2.1 in the case \(\mu = \nu\) is proved just like surjectivity of the map in Conjecture 6.1.1 except that instead of Theorem 4.4.1, one uses Theorem 4.5.1.

We then define the Harish-Chandra category \(HCh(\lambda_l, \lambda_r)\) as a complete subcategory of \(\mathcal{F}_{\lambda_l}^{\lambda} \) generated by \(\Phi(\mathcal{O}_k^{\lambda})\) if \((\lambda_l, \lambda, \lambda_r)\) is a translation datum. Provided Conjecture 6.2.1 is valid, this category is isomorphic to \(\mathcal{O}_k^{\lambda} \). Analogues of the principal series representations are, therefore, \(\Phi(M^k_{\lambda l, \lambda}) , w \in W_k \).

References

[1] Adamovic D., Milas A. MRL 2 (1995) 563-575
[2] Bernstein J.N., Gelfand S.I. Compositio Mathematica 41, 2, (1980) 245-285
[3] Deodhar V.V., Gabber O., Kac V.G., Adv.in Math. 45 (1982) 92-116
[4] Drinfeld V.G. Algebra Anal. 2 (1990) 149-181
[5] Dong C., Li H., Mason G., Vertex operator algebras associated to admissible representations of \(\hat{sl}_2\) [1-algebra/9509026]
[6] Duflo M. Lect. Notes in Math. 497 (1975) 26-88
[7] Enright T. Ann. of Math. 110 (1979) 1-82
[8] Feigin B., Malikov F. Lett.in Math.Phys. 31 (1994) 315-325
[9] Feigin B., Malikov F. Cont.Math. 202 “Operads: Proceedings of Renaissance Conferences” (ed. by Loday, Stasheff and Voronov)
[10] Finkelberg M. Fusion categories, Ph.D. thesis, Harvard university, 1993
[11] Frenkel E., Kac V., Wakimoto M. Comm.Math.Phys. 147(1992) 295-328
[12] Frenkel I.B., Lepowsky J., Meurman A. *Vertex Operator Algebras and the Monster*, Academic Press, Inc 1988
[13] Frenkel I.B., Zhu Y., Duke Math. Journal 66 (1992) 123-168
[14] Jantzen J.C. Representations of Algebraic Groups, Pure and Applied Mathematics 131, Academic Press (1987)
[15] Kac V.G., Kazhdan D.A., Adv.in Math. 34 (1979) 97-108
[16] Kac V.G., Wakimoto M. Proc. Nat’l Acad. Sci. USA 1988 4956
[17] Kazhdan D., Lusztig G. Duke Math.J. 62 21-29
[18] Kazhdan D., Lusztig G. JAMS 6 no. 4 (1993)
[19] Kazhdan D., Lusztig G. JAMS 6 no. 5 (1993)
[20] Rocha A., Wallach N. Math.Z. 180 (1982) 151-177
[21] Vogan D. Duke Math.J. 46 (1979)
[22] Zuckerman G. Ann. of Math. 106 (1977) 295-308

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