ENERGY ESTIMATES FOR MINIMIZERS TO A CLASS OF ELLIPTIC SYSTEMS OF ALLEN-CAHN TYPE AND THE LIOUVILLE PROPERTY

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Abstract. We prove a theorem for the growth of the energy of bounded, globally minimizing solutions to a class of semilinear elliptic systems of the form $\Delta u = \nabla W(u)$, $x \in \mathbb{R}^n$, $n \geq 2$, with $W : \mathbb{R}^m \to \mathbb{R}$, $m \geq 1$, nonnegative and vanishing at exactly one point (at least in the closure of the image of the considered solution $u$). As an application, we can prove a Liouville type theorem under various assumptions.

1. Introduction and statement of the main results

Consider the semilinear elliptic system

$$\Delta u = \nabla W(u) \text{ in } \mathbb{R}^n, \quad n \geq 2,$$

(1.1)

where $W : \mathbb{R}^m \to \mathbb{R}$, $m \geq 1$, is sufficiently smooth and nonnegative. This system has variational structure, as solutions (in a smooth, bounded domain $\Omega \subset \mathbb{R}^n$) are critical points of the energy

$$E(v; \Omega) = \int_{\Omega} \left\{ \frac{1}{2} |\nabla v|^2 + W(v) \right\} dx$$

(1.2)

(subject to their own boundary conditions), where $|\nabla v|^2 = \sum_{i=1}^{n} |v_{x_i}|^2$. A solution $u \in C^2(\mathbb{R}^n; \mathbb{R}^m)$ is called globally minimizing if

$$E(u; \Omega) \leq E(u + \varphi; \Omega)$$

(1.3)

for every smooth, bounded domain $\Omega \subset \mathbb{R}^n$ and for every $\varphi \in W^{1,2}_0(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m)$ (see also [24] and the references therein).

If $m \geq 2$, two are the main categories of such potentials $W$:

- Those that vanish only on a discrete set of points (usually finite); in this case (1.1) is known as the vectorial Allen-Cahn equation and models multi-phase transitions (see [7], [11], [21] and the references that follow).
- Those that vanish on a continuum of points, as in the Ginzburg-Landau system (see [10]) or the elliptic system modeling phase-separation in [9] or the one in [15].

This article is motivated from the first class. In this setting, an effective way to construct entire, nontrivial solutions to (1.1) is to assume that $W$ is symmetric with respect to a finite reflection group and to look for equivariant solutions. Under proper assumptions, this roughly amounts to studying bounded, globally minimizing solutions to (1.1) such that the closure of their image contains at most one global minimum of $W$. In the scalar case, that is $m = 1$, this approach has been utilized, among others, in [12]. On the other side, recent progress has been made in the vector case in [4], [8], and [23]. In our opinion, the main obstruction in the vector case is the lack of the maximum principle. This short discussion motivates our main result:
Theorem 1.1. Assume that \( W \in C^1(\mathbb{R}^m; \mathbb{R}) \), \( m \geq 1 \), and that there exists \( a \in \mathbb{R}^m \) such that
\[
W > 0 \quad \text{in} \quad \mathbb{R}^m \setminus \{a\} \quad \text{and} \quad W(a) = 0. \tag{1.4}
\]
If \( u \in C^2(\mathbb{R}^n; \mathbb{R}^m) \), \( n \geq 2 \), is a bounded, globally minimizing solution to the elliptic system (1.1), we have that
\[
\lim_{R \to \infty} \left( \frac{1}{R^{n-1}} \int_{B_R} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} \, dx \right) = 0, \tag{1.5}
\]
where \( B_R \) stands for the \( n \)-dimensional ball of radius \( R \) and center at the origin.

We emphasize that there is no assumption for the behavior of \( W \) near \( a \). To the best of our knowledge, besides the ordinary differential equation case \( n = 1 \) (see [32], [38]), this is the first nontrivial result for the vector case in this generality.

Our proof of Theorem 1.1 is based on an adaptation to this setting of the famous “bad discs” construction of [10] from the study of vortices in the Ginzburg-Landau model. We stress that, to the best of our knowledge, this is the first application of this powerful technique to the study of the vector Allen-Cahn equation.

Moreover, we can provide a quantitative version of Theorem 1.1.

Theorem 1.2. Assume that \( W \in C^2(\mathbb{R}^m; \mathbb{R}) \), \( m \geq 1 \), satisfies (1.4) and that there exist \( C_0, r_1 > 0 \) and \( q \geq 2 \) such that
\[
W(a + r\nu) \geq C_0 r^q, \quad \text{where} \quad \nu \in \mathbb{S}^{m-1}, \quad \text{for} \quad r \in (0, r_1). \tag{1.6}
\]
If \( u \in C^2(\mathbb{R}^n; \mathbb{R}^m) \), \( n \geq 2 \), is a bounded, globally minimizing solution to the elliptic system (1.1), given positive \( \tau < \frac{2}{qn} \), there exists \( C(\tau) > 0 \) such that
\[
\int_{B_R} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} \, dx \leq C(\tau) R^{n-1 - \tau}, \quad R > 0.
\]

As an application of Theorem 1.1, we can prove the following Liouville type theorem.

Theorem 1.3. Assume that \( W \) and \( u \) are as in Theorem 1.1. Then, it holds that
\[ u \equiv a, \]
provided that one of the following additional conditions holds:

(a): \( m = 1 \) and \( W \in C^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}) \); or \( m \geq 1 \) and \( u \) is radially symmetric; or \( m \geq 1 \) and Modica’s gradient bound holds, that is
\[
\frac{1}{2} |\nabla u|^2 \leq W(u) \quad \text{in} \quad \mathbb{R}^n. \tag{1.7}
\]

(b): \( n = 2 \) and there exists small \( r_0 > 0 \) such that the functions
\[
r \mapsto W(a + r\nu), \quad \text{where} \quad \nu \in \mathbb{S}^{m-1}, \quad \text{are strictly increasing for} \quad r \in (0, r_0]; \tag{1.8}
\]
or \( n = 2 \), \( W \in C^{1,1}_{\text{loc}}(\mathbb{R}^m; \mathbb{R}) \), and the above functions are nondecreasing for \( r \in (0, r_0] \); or \( n = 2 \) and \( m = 1 \).

(c): \( W \in C^2(\mathbb{R}^m; \mathbb{R}) \) and
\[
W_{uu}(a)\nu \cdot \nu > 0 \quad \text{for all} \quad \nu \in \mathbb{S}^{m-1}, \tag{1.9}
\]
where \( \cdot \) stands for the Euclidean inner product in \( \mathbb{R}^m \).
The above theorem was originally proven by different techniques in [24] (see also the earlier paper [23]), under the conditions that \( W \in C^2(\mathbb{R}^m; \mathbb{R}) \) and \( u \) satisfy the assumptions of Theorem 1.1, and that the functions in (1.8) have a strictly positive second derivative in \((0, r_0)\). In particular, the approach of the latter references is based on a quantitative refinement of the replacement lemmas in [4] and [22]. If \( W \) additionally satisfies the stronger assumption (1.9), this theorem was recently re-proven in [6] by extending to this setting the density estimates of [14]. In the aforementioned references, the Liouville type theorem was proven by an application of a basic pointwise estimate. However, it is not difficult to convince oneself that the opposite direction is also possible, that is the pointwise estimate follows from the Liouville property (see also [37] for this viewpoint). We note that the pointwise estimate is the one that is directly applicable in relation to the discussion preceding Theorem 1.1. This pointwise estimate roughly says that if \( W \) (as in Theorem 1.1) is such that the Liouville type theorem holds, then a globally minimizing solution, defined in a sufficiently large ball (with the appropriate modifications in the definition), and bounded independently of the size of the ball, has to be close to \( a \) in the ball of half the radius.

In the scalar case, under the assumptions of the first part of Case (a) above, this theorem can also be proven by using radial barriers as in [37]. On the other side, this powerful, but intrinsically scalar technique, does not seem to work under the minimal assumptions of the last part of Case (b).

In our opinion, three are the main advantages of our approach: Firstly, we can treat in a unified and coordinate way various situations. Secondly, in our opinion, our approach is considerably simpler than those in the aforementioned references. Lastly, to the best of our knowledge, it provides the strongest available result when \( n = 2 \) for any \( m \geq 1 \), even for the extensively studied scalar case. This may seem too restrictive at first, but keep in mind that the dimensions \( n = 2, 3 \) are the ones with physical interest. In fact, the majority of papers on the subject deals exclusively with these dimensions (see [1], [2], [11], [33] for \( n = 2 \) and [26] for \( n = 3 \)).

The proof of Theorem 1.3 is based on combining Theorem 1.1 with a variety of diverse results that are available in the literature.

In the sequel, we will provide the proofs of our main results. We will close the paper with two appendixes that are used in the proofs, but are also of independent interest and contain new results.

2. Proof of the main results

2.1. Proof of Theorem 1.1.

Proof. Throughout this proof, we will denote the energy density of \( u \) by
\[
e(x) = \frac{1}{2} |\nabla u(x)|^2 + W(u(x)), \quad x \in \mathbb{R}^n.
\]
(2.1)

Firstly, note that standard elliptic regularity theory and Sobolev imbeddings [18, 25], in combination with the fact that \( u \) is bounded, yield that
\[
\|u\|_{C^{1,\alpha}(\mathbb{R}^n; \mathbb{R}^m)} \leq C_1,
\]
(2.2)
for some \( \alpha \in (0, 1) \) and \( C_1 > 0 \) (in fact, it holds for any \( \alpha \in (0, 1) \) provided that \( C_1 = C_1(\alpha) > 0 \)). We point out that this is the only place where we use that \( W \in C^2 \).
Since $u$ is a globally minimizing solution, by comparing its energy to that of a suitable test function which agrees with $u$ on $\partial B_R$ and is identically $a$ in $B_{R-1}$, we find that
\[
\int_{B_R} e(x) dx \leq C_2 R^{n-1}, \quad R \geq 1,
\]
for some $C_2 > 0$ (see also [14]).

Therefore, by (2.3), the co-area formula (see for instance [18, Ap. C]), the nonnegativity of $W$, and the mean value theorem, there exist
\[
S_R \in (R, 2R)
\]
such that
\[
\int_{\partial B_{S_R}} e(x) dS(x) \leq C_3 R^{n-2}, \quad R \geq 1,
\]
for some $C_3 > 0$ (actually, we can take $C_3 = \frac{C_2}{2}$).

Let $\epsilon > 0$ be any small number. By virtue of (2.2), we can infer that the subset of $\partial B_{S_R}$ where $e(x)$ is above $\epsilon$ is contained in at most $O(R^{n-2})$ number of geodesic balls of radius 1 as $R \to \infty$ (the so-called “bad discs”, see [10]). More precisely, there exist $N_{\epsilon, R} \geq 0$ points \( \{x_{R,1}, \cdots, x_{R,N_{\epsilon, R}}\} \) on $\partial B_{S_R}$ such that
\[
e(x) \geq \epsilon \quad \text{if} \quad x \in U_R(x_{R,i}, 1), \quad i = 1, \cdots, N_{\epsilon, R},
\]
and
\[
e(x) \leq \epsilon \quad \text{if} \quad x \in \partial B_{S_R} \setminus \bigcup_{i=1}^{N_{\epsilon, R}} U_R(x_{R,i}, 1),
\]
for $R \gg 1$, where $U_R(p,r) \subset \partial B_{S_R}$ stands for the geodesic ball with center at $p$ and radius $r$. Moreover, we have that
\[
N_{\epsilon, R} \leq M_\epsilon R^{n-2}, \quad R \gg 1 \quad \text{(with $M_\epsilon > 0$ independent of $R$)}.
\]

In the sequel, we will prove the above properties by adapting some arguments from [10]. Firstly, we prove a clearing-out property. Note that (2.2) implies that there exists $\mu_\epsilon < \epsilon$ such that
\[
\int_{U_R(y,2)} e(x) dS(x) < \mu_\epsilon \quad \text{for some} \quad y \in \partial B_{S_R}
\]
implies that
\[
e(x) \leq \epsilon, \quad x \in U_R(y, 1),
\]
for $R \geq 1$. Indeed, suppose that
\[
e(z) \geq \epsilon \quad \text{for some} \quad y \in \partial B_{S_R} \quad \text{and} \quad z \in U_R(y, 1).
\]
From (2.2), there exists $C_4 > 0$ such that
\[
\|e\|_{C^{0,\alpha}(\mathbb{R}^n; \mathbb{R})} \leq C_4.
\]
It then follows that
\[
e(x) \geq \epsilon - C_4 d^\alpha, \quad x \in B(z, d) = z + B_d,
\]
for all $d < \min\left\{1, \left(\frac{\epsilon}{2C_4}\right)^\frac{1}{\alpha}\right\}$ (see also [41, Lem. 2.3]). Since $\epsilon \geq 0$, we find that
\[
\int_{U_R(y,2)} e(x) dS(x) \geq \int_{U_R(z,d)} e(x) dS(x) \geq (\epsilon - C_4 d^\alpha) |U_R(z,d)| \geq \frac{\epsilon}{2} |U_R(z,d)| = \frac{\epsilon}{2} |S^{n-1}| d^{n-1}.
\]
Hence, we can exclude the scenario (2.8) by choosing
\[ \mu_\epsilon = \frac{\epsilon}{2^n |\mathbb{S}^{n-1}|} \left( \min \left\{ 1, \left( \frac{\epsilon}{2C_4} \right)^{\frac{1}{\alpha}} \right\} \right)^{n-1}. \]

Next, consider a finite family of geodesic balls \( U_R(x_i, 1)_{i \in I_R} \) such that
\[ U_R \left( x_i, \frac{1}{4} \right) \cap U_R \left( x_k, \frac{1}{4} \right) = \emptyset \text{ if } i \neq k, \]
\[ \bigcup_{i \in I_R} U_R(x_i, 1) \supset \partial B_{S_R}, \]
for all \( R \geq 1 \) (having suppressed the obvious dependence of \( x_i \) on \( R \)). This is indeed possible by the Vitali’s covering theorem (see [17, Sec. 1.5] and keep in mind that \( \partial B_{S_R} \) becomes a metric space when equipped with the geodesic distance). We say that the ball \( U_R(x_i, 1) \) is a good ball if
\[ \int_{U_R(x_i, 2)} e(x) \, dS(x) < \mu_\epsilon, \]
and that \( U_R(x_i, 1) \) is a bad ball if
\[ \int_{U_R(x_i, 2)} e(x) \, dS(x) \geq \mu_\epsilon. \]

The collection of bad balls is labeled by
\[ J_R = \{ i : U_R(x_i, 1) \text{ is a bad ball} \}. \]

The main observation is that, by virtue of (2.10), there is a universal constant \( C_5 > 0 \) (independent of both \( \epsilon \) and \( R \)) such that
\[ \sum_{i \in I_R} \int_{U_R(x_i, 2)} e(x) \, dS(x) \leq C_5 \int_{\partial B_{S_R}} e(x) \, dS(x), \]
since each point on \( \partial B_{S_R} \) is covered by at most \( C_5 \) geodesic balls \( U_R(x_i, 2) \). The latter property plainly follows by observing that all such balls that contain the same point are certainly contained in a geodisc ball of radius 10, and from the basic fact that any \((n - 1)\)-dimensional ball of radius 10 can contain only a finite number of disjoint balls of radius \( \frac{1}{4} \).

Making use of (2.5), we infer that
\[ \text{card} J_R \leq \frac{C_5 C_3^2}{\mu_\epsilon} R^{n-2}, \quad R \geq 1. \]

Now, let \( x \in \partial B_{S_R} \setminus \bigcup_{i \in J_R} U_R(x_i, 1) \). By (2.11), there exists some \( k \in I_R \setminus J_R \) such that \( x \in U_R(x_k, 1) \) which is a good ball. It follows from the definition of \( \mu_\epsilon \) that
\[ e(x) \leq \epsilon, \]
as desired.

In view of (1.4) and (2.6), we have that
\[ |\nabla u(x)|^2 \leq 2\epsilon \quad \text{and} \quad |u(x) - a| \leq m_\epsilon \quad \text{if} \quad x \in \partial B_{S_R} \setminus \bigcup_{i=1}^{N_{\epsilon,R}} U_R(x_{R,i}, 1), \quad R \gg 1, \]
(2.13)
where

\[ m_\varepsilon \to 0 \text{ as } \varepsilon \to 0, \tag{2.14} \]

(we point out that \( m_\varepsilon \) depends only on \( \varepsilon \)).

We consider the function \( v_R \in W^{1,2}(B_{S_R}; \mathbb{R}^m) \cap L^\infty(B_{S_R}; \mathbb{R}^m) \) which is defined, in terms of polar coordinates, as follows:

\[
v_R(r, \theta) = \begin{cases} 
  u(S_R, \theta) + (a - u(S_R, \theta)) (S_R - r), & r \in [S_R - 1, S_R], \ \theta \in S^{n-1}, \\
  a, & r \in [0, S_R - 1], \ \theta \in S^{n-1},
\end{cases}
\]

(having slightly abused notation, keep in mind that \( x = r\theta \)). We note that \( v_R \) belongs in \( W^{1,2} \) because it is the composition of a smooth function with a Lipschitz continuous one (see [28, pg. 54] and keep in mind that we only use the polar coordinates away from the origin).

Clearly, we have that

\[ v_R = u \text{ on } \partial B_{S_R}. \tag{2.15} \]

Let

\[ \mathcal{A}_R = B_{S_R} \setminus B_{(S_R-1)} \text{ and } \mathcal{C}_R = \bigcup_{i=1}^{N_{\varepsilon, R}} (\bar{B}_{10}(x_{R,i}) \cap \mathcal{A}_R). \]

If \( x = r\theta \in \mathcal{A}_R \setminus \mathcal{C}_R \), via (2.13), it holds that

\[ |v_R(x) - a| \leq 2 |u(S_R, \theta) - a| \leq 2m_\varepsilon. \tag{2.16} \]

Moreover, for such \( x \), we find that

\[
|\nabla_{\mathbb{R}^n} v_R|^2 = |u(S_R, \theta) - a|^2 + \frac{1}{R^2} |(1 + r - S_R) \nabla_{S^{n-1}} u(S_R, \theta)|^2
\]

using (2.4), (2.13) : \[ \leq m_\varepsilon^2 + \frac{9}{S_R} |\nabla_{S^{n-1}} u(S_R, \theta)|^2 \tag{2.17} \]

using again (2.13) : \[ \leq m_\varepsilon^2 + 9 |\nabla_{\mathbb{R}^n} u(S_R \theta)|^2 \]

where we made repeated use of the identity

\[ |\nabla_{\mathbb{R}^n} v|^2 = |v_r|^2 + \frac{1}{R^2} |\nabla_{S^{n-1}} v|^2 \text{ on } \partial B_R, \ R > 0, \]

(see [42, Ch. 8]). It follows that

\[
\int_{B_{S_R}} \left\{ \frac{1}{2} |\nabla v_R|^2 + W(v_R) \right\} dx = \int_{\mathcal{A}_R} \left\{ \frac{1}{2} |\nabla v_R|^2 + W(v_R) \right\} dx
\]

using (2.2) : \[ \leq C_6 N_{\varepsilon, R} + \int_{\mathcal{A}_R \setminus \mathcal{C}_R} \left\{ \frac{1}{2} |\nabla v_R|^2 + W(v_R) \right\} dx \]

using (2.16), (2.17) : \[ \leq C_6 N_{\varepsilon, R} + \left( \frac{m_\varepsilon^2}{2} + 9\varepsilon + C_7 m_\varepsilon \right) |\mathcal{A}_R \setminus \mathcal{C}_R| \]

\[ \leq C_6 N_{\varepsilon, R} + C_8 (m_\varepsilon + \varepsilon) S^{n-1}_{R}, \]
where $C_6, C_7, C_8 > 0$ are independent of both $\epsilon$ and $R$. Since $u$ is a globally minimizing solution, thanks to (2.15), we obtain that
\[
\int_{B_{SR}} e(x)dx \leq C_6 N_{\epsilon,R} + C_8 (m_\epsilon + \epsilon) S_R^{n-1}
\]
(2.18)
using (2.4), (2.7) : \(\leq 2^{n-2} C_6 M_\epsilon R^{n-2} + 2^{n-1} C_8 (m_\epsilon + \epsilon) R^{n-1}, \ R \gg 1.\)
Since $\epsilon > 0$ is arbitrary, in light of (2.14), we infer that (1.5) holds, as desired. □

**Remark 2.1.** The assertion of Theorems 1.3 holds for any bounded solution of (1.1) provided that $W$ is assumed to be globally convex (see for example [40]).

### 2.2. Proof of Theorem 1.2

**Proof.** The proof is based on a bootstrap argument. Assume that
\[
\int_{B_R} e(x)dx \leq C(k) R^k, \ R > 0, \text{ for some } k \in (0, n - 1],
\]
(recall the notation (2.1)). In the proof of Theorem 1.1, choose
\[
\epsilon_R = R^{-\beta},
\]
for $R \gg 1$, with $\beta > 0$ to be chosen. In analogy to (2.4)-(2.5), we now take $S_R \in (R, 2R)$ such that
\[
\int_{\partial B_{SR}} e(x)dS(x) \leq \frac{1}{2} C(k) R^{k-1}, \ R > 0.
\]
Let $N_R$ be the minimal number of geodesic balls of radius 1 on $\partial B_{SR}$ which contain the set where $e(x)$ is above $\epsilon_R = R^{-\beta}$. In view of (2.9), the corresponding quantity can be chosen so that it satisfies
\[
\mu_\epsilon R \geq C_{11} \epsilon R^n = C_{11} R^{-\beta n}
\]
for $R \gg 1$, where $C_{11} > 0$. In turn, as in (2.12), we obtain that
\[
N_R \leq C_{12} R^{k-1+\beta n}, \ R \gg 1,
\]
where $C_{12} > 0$. By virtue of (1.6), the corresponding quantity in (2.13) satisfies
\[
m_\epsilon R \leq C_{13} \epsilon R^{\frac{1}{q}} = C_{13} R^{-\frac{\beta}{q}}, \ R \gg 1,
\]
for some $C_{13} > 0$. Substituting the above in the analog of (2.18), yields that
\[
\int_{B_{SR}} e(x)dx \leq C_{14} \left( R^{n-1-\frac{2q}{q+2}} + R^{k-1+\beta n} \right), \ R \gg 1,
\]
for some $C_{14} > 0$. We will choose $\beta$ so that the two exponents in the above relation are equal, this gives
\[
\beta = \frac{q(n-k)}{qn + 2}.
\]
We have arrived at
\[
\int_{B_R} e(x)dx \leq C_{15} R^{\gamma(k)}, \ R > 0,
\]
where
\[
\gamma(k) = n - 1 - \frac{2(n-k)}{qn + 2}.
\]
To conclude, observe that the mapping $k \rightarrow \gamma(k)$ is a contraction with $n - 1 - \frac{2}{qn}$ as a fixed point.

2.3. Proof of Theorem 1.3.

Proof. Case (a) If $u$ satisfies (1.7), since $W \geq 0$, it is known that the following strong monotonicity formula holds:

$$
\frac{d}{dR} \left( \frac{1}{R^{n-1}} \int_{B_R} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dx \right) \geq 0, \quad R > 0,
$$

(2.19)

(see [13], [31] for $m = 1$ and [3], [15] for arbitrary $m \geq 1$). We point out that the fact that $u$ is minimal is not used for this. Hence, for any positive $r < R$, we have that

$$
\frac{1}{r^{n-1}} \int_{B_r} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dx \leq \frac{1}{R^{n-1}} \int_{B_R} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dx.
$$

By virtue of Theorem 1.1, letting $R \to \infty$ in the above relation yields that $u \equiv a$. For the reader’s convenience, we will present the proof of a seemingly new monotonicity formula in Appendix A which can also be used to reach the same result.

To complete the proof in this case, we note that the gradient estimate (1.7) was shown in [20] to hold for any bounded, entire solution when $m = 1$ and $W \in C^{1,1}_{loc}(\mathbb{R}; \mathbb{R})$ is nonnegative (see [13], [30] for earlier proofs which required higher regularity on $W$). Lastly, it is easy to show that any radially symmetric solution satisfies this gradient bound for any $m \geq 1$ and $W \in C^{1}$ nonnegative (see [39]).

Case (b) Here we partly follow [39]. Since $n = 2$, by working as in (2.5), and using the assertion of Theorem 1.1, we arrive at

$$
\int_{\partial B_{SR}} W(u(x)) dS(x) \to 0, \quad \text{for some } S_R \in (R, 2R), \quad \text{as } R \to \infty.
$$

By making use of just the $C^1$-bound in (2.2), and working as we did in order to exclude (2.8), we deduce that

$$
\max_{|x|=S_R} |u(x) - a| \to 0 \quad \text{as } R \to \infty. \quad (2.20)
$$

Under the assumptions of the first part of Case (b), a recent variational maximum principle from [5] implies that

$$
\max_{|x| \leq S_R} |u(x) - a| \leq \max_{|x|=S_R} |u(x) - a|,
$$

(see also Appendix B herein). Moreover, as we will prove in Appendix B, this variational maximum principle also holds under the assumptions of the second part of Case (b). In light of (2.20), by letting $R \to \infty$ in the above relation, we can conclude that the first two assertions in Case (b) hold.

We will establish the validity of the last assertion of Case (b) by borrowing some ideas from [43], while adopting a more explanatory viewpoint. To this end, we will argue by contradiction. Without loss of generality, we may assume that there exists a sequence $R_j \to \infty$ and a $\delta > 0$ such that

$$
u(x_j) = \max_{|x| \leq S_{R_j}} u(x) \geq a + \delta, \quad j \geq 1,$$
for some \(x_j \in B_{S_{R_j}}\). In particular, there exists \(d \in (0, \delta)\) such that
\[
W(a + d) < W(u(x_j)), \quad j \geq 1.
\]
By virtue of (2.20), we may further assume that
\[
\max_{|x| = S_{R_j}} u(x) \leq a + \frac{d}{2} \quad j \geq 1. \tag{2.21}
\]
Let \(u_j \in [a + d, u(x_j))\) be such that
\[
W(u_j) = \min_{u \in [a + d, u(x_j)]} W(u). \tag{2.22}
\]
Consider the competitor function
\[
V_j(x) = \min\{u(x), u_j\}, \quad x \in B_{S_{R_j}},
\]
which belongs in \(W^{1,2}(B_{S_{R_j}}; \mathbb{R}^m) \cap L^\infty(B_{S_{R_j}}; \mathbb{R}^m)\) (see for instance [16]) and, thanks to (2.21), agrees with \(u\) on \(\partial B_{S_{R_j}}\). To conclude, we will show that
\[
E(V_j; B_{S_{R_j}}) < E(u; B_{S_{R_j}}),
\]
which contradicts the minimality character of \(u\). To this aim, let
\[
D_j = \left\{ x \in B_{S_{R_j}} : u(x) > u_j \right\}.
\]
Observe that \(D_j\) is nonempty (since it contains \(x_j\)) and strictly contained in \(B_{S_{R_j}}\) (from (2.21)). Then, note that
\[
E(V_j; B_{S_{R_j}} \setminus D_j) = E(u; B_{S_{R_j}} \setminus D_j) \quad \text{and} \quad E(V_j; D_j) = E(u_j; D_j) < E(u; D_j),
\]
since (2.22) holds and there exists a connected component \(E_j\) of \(D_j\), say the one containing \(x_j\), where \(u\) is nonconstant (note that \(u = u_j\) on \(\partial D_j\)).

Case (c) Let
\[
\varepsilon = \frac{1}{R} \quad \text{and} \quad u_\varepsilon(y) = u\left(\frac{y}{\varepsilon}\right), \quad y \in \mathbb{R}^n.
\]
We have that
\[
\varepsilon^2 \Delta u_\varepsilon = \nabla W(u_\varepsilon) \quad \text{in} \quad \mathbb{R}^n.
\]
Moreover, thanks to Theorem 1.1 and (1.9), we have that
\[
\frac{1}{\varepsilon} \int_{B_2} |u_\varepsilon(y) - a|^2 dy \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]
In particular, this implies that
\[
\|u_\varepsilon - a\|_{L^1(B_2)} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]
If \(m = 1\), we can obtain the uniform convergence
\[
\|u_\varepsilon - a\|_{L^\infty(B_1)} \to 0 \quad \text{as} \quad \varepsilon \to 0,
\]
by appealing to the results of [14]. The latter results were recently extended to cover the case \(m \geq 2\) in [6]. Therefore, we deduce that the above uniform estimate holds under the assumptions of Case (c). We can then conclude by switching back to \(u\) and \(R\) and letting \(R \to \infty\), similarly to Case (b). \(\square\)
APPENDIX A. A NEW MONOTONICITY FORMULA FOR SOLUTIONS TO THE ELLIPTIC SYSTEM $\Delta u = \nabla W(u)$

In this appendix, we will prove a seemingly new monotonicity formula which can be used in the proof of the first case of Theorem 1.3.

**Theorem A.1.** If $u \in C^2(\mathbb{R}^n; \mathbb{R}^m)$, $n \geq 2, m \geq 1$, solves (1.1) with $W \in C^1(\mathbb{R}^m; \mathbb{R})$ nonnegative, we have the weak monotonicity formula:

$$
\frac{d}{dR} \left( \frac{1}{R^{n-2}} \int_{B_R} \left\{ \frac{n-2}{2} |\nabla u|^2 + nW(u) \right\} dx \right) \geq 0, \quad R > 0.
$$

(A.1)

In addition, if $u$ satisfies Modica’s gradient bound (1.7), we have the strong monotonicity formula:

$$
\frac{d}{dR} \left( \frac{1}{R^{n-1}} \int_{B_R} \left\{ \frac{n-2}{2} |\nabla u|^2 + nW(u) \right\} dx \right) \geq 0, \quad R > 0.
$$

(A.2)

**Proof.** By means of a direct calculation, it was shown in [3] that, for solutions $u$ to (1.1), the stress energy tensor $T(u)$, which is defined as the $n \times n$ matrix with entries

$$
T_{ij} = u_{,i} \cdot u_{,j} - \delta_{ij} \left( \frac{1}{2} |\nabla u|^2 + W(u) \right), \quad i, j = 1, \ldots, n, \quad \text{(where } u_{,i} = u_{x_i}, \text{)}
$$

satisfies

$$
\text{div} T(u) = 0 \quad \text{(A.3)}
$$

using the notation $T = (T_1, T_2, \ldots, T_n)^\top$ and $\text{div} T = (\text{div} T_1, \text{div} T_2, \ldots, \text{div} T_n)^\top$, (see also [35]). Observe that

$$
\text{tr} T = - \left( \frac{n-2}{2} |\nabla u|^2 + nW(u) \right), \quad \text{(A.4)}
$$

and that

$$
T + \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) I_n = (\nabla u)^\top (\nabla u) \geq 0 \quad \text{(in the matrix sense)}, \quad \text{(A.5)}
$$

where $I_n$ stands for the $n \times n$ identity matrix.

As in [34], writing $x = (x_1, \ldots, x_n)$, and making use of (A.3), we calculate that

$$
\sum_{i,j=1}^n \int_{B_R} (x_i T_{ij})_j \, dx = \sum_{i,j=1}^n \int_{B_R} \{ \delta_{ij} T_{ij} + x_i (T_{ij})_j \} \, dx = \sum_{i=1}^n \int_{B_R} T_{ii} \, dx. \quad \text{(A.6)}
$$

On the other side, from the divergence theorem, denoting $\nu = x/R$, and making use of (A.5), we find that

$$
\sum_{i,j=1}^n \int_{B_R} (x_i T_{ij})_j \, dx = R \sum_{i,j=1}^n \int_{B_R} \nu_i T_{ij} \nu_j \, dS \geq -R \int_{\partial B_R} \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) \, dS. \quad \text{(A.7)}
$$

Since $W$ is nonnegative, if $n \geq 3$, we have that

$$
\frac{1}{2} |\nabla u|^2 + W(u) \leq \frac{1}{n-2} \left( \frac{n-2}{2} |\nabla u|^2 + nW(u) \right). \quad \text{(A.8)}
$$

Let

$$
f(R) = \int_{B_R} \left( \frac{n-2}{2} |\nabla u|^2 + nW(u) \right) \, dx, \quad R > 0.
$$
By combining (A.4), (A.6), (A.7) and (A.8), for \( n \geq 3 \), we arrive at
\[
-f(R) \geq -\frac{R}{n-2} \frac{d}{dR} f(R), \quad R > 0,
\]
which implies that
\[
\frac{d}{dR} (R^{2-n} f(R)) \geq 0, \quad R > 0,
\]
(clearly this also holds for \( n = 2 \)). We have thus shown the first assertion of the theorem.

Suppose that \( u \) additionally satisfies Modica’s gradient bound (1.7). Then, we can strengthen (A.8), for \( n \geq 2 \), by noting that
\[
\frac{1}{2} |\nabla u|^2 + W(u) = \frac{1}{n-1} \left( \frac{n-2}{2} |\nabla u|^2 + \frac{1}{2} |\nabla u|^2 + (n-1)W(u) \right) \leq \frac{1}{n-1} \left( \frac{n-2}{2} |\nabla u|^2 + nW(u) \right).
\]
Now, by combining (A.4), (A.6), (A.7) and the above relation, we arrive at
\[
-f(R) \geq -\frac{R}{n-1} \frac{d}{dR} f(R), \quad R > 0,
\]
which implies that
\[
\frac{d}{dR} (R^{1-n} f(R)) \geq 0, \quad R > 0,
\]
as desired. \(\square\)

Remark A.2. The weak monotonicity formula (A.1) for the special case of the Ginzburg-Landau system was stated (without proof) in [19]. If \( m = 1 \) and \( n = 2 \), the strong monotonicity formula (A.2) was proven recently, by different and intrinsically two dimensional techniques, in [36].

Remark A.3. Mingfeng Zhao [44] kindly informed me that all of the monotonicity formulas in this paper can also be derived from Pohozaev’s identities for systems (see [35] for the case of the Ginzburg-Landau system).

Appendix B. On a maximum principle for vector minimizers to the Allen-Cahn energy

In the recent paper [5], the authors proved the following variational maximum principle:

Theorem B.1. Let \( W : \mathbb{R}^m \to \mathbb{R} \) be \( C^1 \) and nonnegative. Assume that \( W(a) = 0 \) for some \( a \in \mathbb{R}^m \) and that there is \( r_0 > 0 \) such that (1.8) holds. Let \( \Omega \subset \mathbb{R}^n \) be an open, connected, bounded set, with \( \partial \Omega \) minimally smooth (Lipschitz continuous is enough), and suppose that \( u \in W^{1,2}(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m) \) is minimal, in the sense that (1.3) is satisfied.

If there holds
\[
|u(x) - a| \leq r \quad \text{for} \quad x \in \partial \Omega,
\]
for some \( r \in (0, \frac{r_0}{2}) \), then it also holds that
\[
|u(x) - a| \leq r \quad \text{for} \quad x \in \Omega.
\]

The main idea of the proof is that if the assertion is violated at some point, then one can construct a suitable competitor function which agrees with \( u \) on \( \partial \Omega \) and has strictly less energy, which is impossible.

In this appendix, under the slight additional regularity assumption that \( W \in C^{1,1}_{loc} \) (which is consistent with most applications), we will show that one can conclude just by showing
that the aforementioned competitor function has less or equal energy. Our main observation is to apply the unique continuation principle for linear elliptic systems (see [29] for other applications). As a result, under the slight additional assumption that $W \in C_{\text{loc}}^{1,1}$, we can simplify the corresponding proof in [5]. Moreover, we can allow for the functions in (1.8) to be merely nondecreasing which is crucial for establishing the second assertion of Case (b) in Theorem 1.3. More precisely, we have the following theorem.

**Theorem B.2.** Assume that $W : \mathbb{R}^m \to \mathbb{R}$ is $C_{\text{loc}}^{1,1}$, nonnegative, such that $W(a) = 0$ for some $a \in \mathbb{R}^m$ and that the functions in (1.8) are nondecreasing. Moreover, assume that

\begin{align*}
W(u) > 0 \text{ if } |u - a| < 2r_0 \text{ and } u \neq a.
\end{align*}

Then, the assertion of Theorem B.1 remains true.

**Proof.** Firstly, by standard elliptic regularity theory, we have that $u$ is a smooth solution to the elliptic system in (1.1) in $\Omega$ and continuous up to the boundary (under reasonable assumptions on $\partial \Omega$). Without loss of generality, we take $a = 0$. As in [5], we set

\[ \rho(x) = |u(x)| \text{ in } \Omega \text{ and } \nu(x) = \frac{u(x)}{\rho(x)} \text{ in } \Omega_{+} = \{ x \in \Omega : \rho > 0 \}. \]

We also set $\Omega_0 = \{ x \in \Omega : \rho = 0 \}$ (actually, it can be shown that $\Omega_0 = \emptyset$ but it is not important for the proof). It has been shown in [5] that the energy of $u$ equals

\[ E(u; \Omega) = \frac{1}{2} \int_{\Omega} |\nabla \rho|^2 dx + \frac{1}{2} \int_{\Omega_{+}} \rho^2 |\nabla \nu|^2 dx + \int_{\Omega} W(\rho \nu) dx. \]

Let

\[ \tilde{u}(x) = \begin{cases} 
\min \{ \rho(x), r \} \alpha(\rho(x)) \nu(x), & x \in \Omega_{+} \cap \{ \rho < 2r \}, \\
0, & x \in \Omega_0 \cup \{ \rho \geq 2r \}, 
\end{cases} \]

where $\alpha(\cdot)$ is the auxiliary function

\[ \alpha(\tau) = \begin{cases} 
1, & \tau \leq r, \\
\frac{2r-\tau}{r}, & r \leq \tau \leq 2r, \\
0, & \tau \geq 2r.
\end{cases} \]

It was shown in [5] that $\tilde{u} \in W^{1,2}(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m)$ and that its energy equals

\[ E(\tilde{u}; \Omega) = \frac{1}{2} \int_{\Omega} |\nabla \tilde{\rho}|^2 dx + \frac{1}{2} \int_{\tilde{\Omega}_{+}} \tilde{\rho}^2 |\nabla \tilde{\nu}|^2 dx + \int_{\Omega} W(\tilde{\rho} \tilde{\nu}) dx, \]

where $\tilde{\rho}(x) = |\tilde{u}(x)|$ and $\tilde{\Omega}_{+} = \{ x \in \Omega : \tilde{\rho} > 0 \}$. Note that, thanks to (B.1), we have

\[ u = \tilde{u} \text{ on } \partial \Omega \text{ and } |\tilde{u}| \leq r \text{ a.e. in } \Omega. \quad (B.3) \]

It follows readily that

\[ E(\tilde{u}; \Omega) \leq E(u; \Omega), \]

see also the proof in [5]. Consequently, $\tilde{u}$ is also a minimizer in $\Omega$ subject to the same boundary conditions as $u$. Hence, the function $\tilde{u}$ is smooth and satisfies

\[ \Delta \tilde{u} = \nabla W(\tilde{u}) \text{ in } \Omega. \]
We are now set to show that assertion (B.2) holds. Suppose, to the contrary, that
\[ |u(x_0)| > r \quad \text{for some } x_0 \in \Omega. \] (B.4)

We will first exclude the case
\[ r \leq \rho(x) \leq 2r \quad \text{for all } x \in \Omega. \]

If not, the function
\[ \hat{u} = r \nu(x) \in W^{1,2}(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m) \]
would have strictly less energy than \( u \) (because \( \int_\Omega |\nabla \rho|^2 dx > 0 \)) while \( \hat{u} = u \) on \( \partial \Omega \), which is impossible. Next, we exclude entirely the case
\[ r \leq \rho(x), \quad x \in \Omega. \]

If not, there would exist \( x_1 \in \Omega \) such that \( \rho(x_1) > 2r \). This implies that \( \tilde{u} = 0 \) on a set of positive measure containing \( x_1 \). Since \( \nabla W(u) \) is locally Lipschitz continuous, we see that \( \tilde{u} \) satisfies the linear system
\[ \Delta \tilde{u} = Q(x)\tilde{u} \quad \text{in } \Omega, \quad \text{where } Q(x) = \int_0^1 W_{uu}(t\tilde{u}) dt \text{ is bounded in norm.} \]

On the other hand, because \( \tilde{u} = 0 \) on a set of positive measure, by the unique continuation principle for linear elliptic systems (see [27]), we infer that \( \tilde{u} \equiv 0 \) which is clearly impossible (otherwise \( |u| \geq 2r \) in \( \Omega \)). Therefore, we may assume that there exists a set \( D \subset \Omega \) with positive measure such that
\[ u = \tilde{u} \quad \text{in } D. \]

As before, by considering the linear system for the difference \( u - \tilde{u} \), we conclude that \( \tilde{u} \equiv u \).

We have thus arrived at a contradiction, because of (B.3) and (B.4). \( \square \)

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