Optical diagnostics fractal structures: methodical aspects

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Abstract. The properties of fractal structures defined by complex functions are tested. The amplitude-phase characteristics and spatial spectra of probing light beams are considered. An important methodic aspect of spectral analysis of structures with fractal geometry is the ability to expand the range of information obtained by parallel amplitude and phase processing. In deciphering the Fourier images of fractals, one can use the discovered fact that the intensity maxima and phase singularities coincide. As a positive point, it should be noted a high degree of the spatial spectrum stability to the influence of optical noise. The registered degree of asymmetry of the Fourier image structure can be used to determine the level of phase disturbances in the initial light field. The performed work expands the knowledge of factors requiring consideration in the optical diagnostics of fractal objects.

1. Introduction

The study of the fractal properties is important for the development of a new optical element base. The creation of fractal optical elements often relies on the principles of building the Cantor set \cite{1}, the numerical sequence of Fibonacci \cite{2}, the Koch and the Weierstrass curves \cite{3}-\cite{4}, as well as some other fractal structures. Such elements cause amplitude modulation of the transverse structure of the plane wave falling on them and form in it self-similar fragments. Fractal perturbation of the flat wave front can be observed when the radiation is reflected from the rough surface \cite{5}. The results of these works, in addition to scientific, are of practical importance. For example, an annular Cantor plate allows multiple focusing of radiation along the axis, as well as the construction of optical images \cite{1}. By determining the fractal dimension of radiation diffusely reflected from the surface, one can estimate its roughness \cite{5}. Waves after fractal plates are characterized by a high degree of reliability of signal transmission, since their individual fragments contain sufficient information to recreate the entire original signal \cite{6}. The structural features of light waves reflected from fractal images play a large role in the processing of optical information in the cerebral cortex \cite{7}. Significantly less literature information on the transformation of the structure of light beams when they pass through fractal objects, which simultaneously cause modulation of the amplitude and phase of the wave. Thin sections of biological tissues can play the role of such objects \cite{8}. A fractal analysis of the speckle fields of the transmitted laser radiation makes it possible to identify tissues with malignant formations. For an analytical description of such objects, complex functions should be used. This distinguishes them from fractal elements that cause only amplitude modulation and are described by real functions.

The purpose of this work is to determine the basic laws that determine the transformation of the amplitude-phase profile of the probe light beams by fractal structures. The analysis is carried out in the
context of the most important methodical problem for fractal optics about the interconnection of the object characteristics and its Fourier image. Particular attention is paid to assessing the scaling characteristics of fractal and fractal-like objects and their spatial spectra. The solution of these methodical issues is of great importance for improving the methods of optical diagnostics of fractal formations [9] and improving the quality control methods of optical elements [10]. In assessing the relationship of fractal of structures described by complex functions with spectral characteristics, the Mandelbrot-Weierstrass function (M-W) was used as a test. In assessing the relationship between fractal features of structures described by complex functions and spectral characteristics, the M-W function was used as a test function. The literature describes both its mathematical properties and practical applications [11]-[15].

2. Characteristics of the one-dimensional M-W function

M-W function is defined as [11]

\[ w_k = \sigma \sum_{n=-N}^{N} \frac{(1-e^{i\theta_k})e^{i\psi(n)}}{b^{(2-D)n}}, \]

where \( D \) – the fractal dimension; \( b,s \) are scaling parameters; \( \sigma \) is the normalization factor; \( 2N+1 \) – the number of terms in the formula (1); \( \psi(n) \) – phases (in the general case, random); \( k \) is the number of the significant point in the digital representation of the function; \( i = \sqrt{-1} \). Dependence \( W_k = |w_k| \) and the Fourier spectrum of the function \( w_k \) are graphically presented in figure 1. The continuous curve in figure 1, b is the module \( |F_q| \), the dotted line is the phase \( P_q = \arg(F_q) \), \( q \) – spatial frequencies. Values \( W_k \) and \( |F_q| \) are presented in relative units.

![Figure 1. Graphs of the function \( W_k \) (a) and the Fourier coefficients \( F_q \) (b).](image-url)
The following parameter values were used: $D = 1.65$, $b = 2$, $s = 3$, $\sigma = 1$, $N = 5$; $\psi(n)$ varies randomly from 0 to $\pi/15$. For the used integer value of the parameter $b = 2$, the dependence $W_k$ is quasiperiodic. The value of parameter $b$ also determines the scaling coefficient $\zeta$ of the analyzed dependences. This can be seen from the ratio of the segment lengths limiting the sizes of similar fragments on the graphs shown in the figure 1. The ratio of the segments $(0b/0a, 0c/0b, 0d/0c, 0e/0d)$ and $(e_1c_1/e_1d_1, e_1b_1/e_1c_1, e_1a_1/e_1b_1)$, equal to the scaling coefficient $\zeta$, coincides with good accuracy with the value of the parameter $b$ ($\zeta = b = 2$). Note that the spectrum of the complex function $w_k$ is asymmetric. This is due to the different parity of its real and imaginary parts [16]. Real functions do not possess this property – their spectra are symmetric with respect to zero. Figure 1, b indicates the correspondence of the amplitude and phase parts of the Fourier spectrum of the function M-W. It can be seen from the graphs that the amplitude maxima correspond to edge phase dislocations with sharp changes of the phase values by $\pi$. Thus, scaling is directly manifested in the amplitude and phase components of the spectrum. Calculations show that the indicated scaling property is observed both for deterministic and random phase values $\psi(n)$ (in the latter case, the average level of fluctuations in the $|F_q|$ distribution should not exceed the value of spectral peaks). The effect of the imaginary part of the M-W function on the structure of the amplitude Fourier spectrum was also analyzed. The imaginary part of the M-W complex function was varied by multiplying it by the coefficient $K$, which changed from 0 to 1. Then, the asymmetry coefficient $A$ was calculated. It was equal to the ratio of the part of the Fourier image located in the negative frequency region to the part of the Fourier image in the region of positive frequencies. Research showed that the imaginary part introduces a significant asymmetry in the Fourier transform, namely: the greater the relative magnitude of the imaginary part, the higher the asymmetry of the Fourier spectrum. Thus, a change in the coefficient $K$ from 0 to 1 leads to an approximately linear increase in the asymmetry coefficient $A$ from 0 to 1.8.

The stability of spatial spectra to the influence of optical noise arising during the Fourier transform of the original structure was investigated. The noise level depended on the range of variation of the random phase $\psi(n)$. When $\psi(n)$ varies in the range from 0 to $\pi/50$, the noise is negligible compared to the maximum values of the function $W_k$; if the phase values vary in the range from 0 to $\pi$, then the noise level is comparable with the signal level. But even in the latter case, the correlation coefficient $C_r$ of the shape of the Fourier spectrum with the one corresponding to small noise has the value $C_r = 0.97$. Such a significant value of the correlation coefficient proves a high degree of the spectral stability of the considered fractal objects.

3. Features of two-dimensional spatial spectra

If we use function (1) as a generator, we can construct a two-dimensional fractal structure. Its transverse field will be characterized by the following complex function:

$$W_{k,m} = \sigma \sum_{i=0}^{V} \sum_{n=-N}^{N} \left(1 - e^{i\beta s\left(k\cos(\alpha z_i) + m\sin(\alpha z_i)\right)}\right) e^{i\psi(n)} \frac{b^{(2-D)m}}{b^{(2-D)n}}. \tag{2}$$

Here $k, m$ are the numbers of significant points along the transverse coordinates; $V$ is the total number of azimuthal rotations of the coordinate system, $\nu$ is the number of an individual rotation, $\alpha$ – its value. Amplitude dependence $W_{k,m} = |w_{k,m}|$ and the structure of the spatial spectrum $P_{p,q}$ of the considered fractal ($p, q$ are spatial frequencies) are shown in figure 2. The calculation was carried out for the following parameter values: $N = 5$; $s = 3$; $V = 8$; $K = 127$; $b = 2$; $\alpha = \pi/8$; $k, m = -K \ldots K$; $D = 1.65$. The figure 2 shows that the most significant maxima of the presented distributions are
located around the circles with radii $R_1, R_2, \ldots, R_5$. The ratio of the circle radii, which determines the value of the scaling coefficient, is equal to the parameter $b = 2$. It can be seen that the spectra, unlike the images, do not have central symmetry. In the part of the spectrum where the maxima appear most clearly, they are also located in circles. The ratio of the circle sizes is equal to the ratio of the radii of the maxima location in the initial amplitude distribution (figure 2, a). An analysis of the phase distribution of the wave in the Fourier plane showed that the points of the intensity maxima correspond to the points of phase singularities where the phase changes stepwise by $\pi$. This is largely analogous to the behavior of the phase in the one-dimensional case (figure 1, b).

Figure 2. The distribution of the amplitude $W_{k,m}$ (a) and the structure of the spatial spectrum $P_{p,q}$ (b).

The scaling characteristics of the amplitude distribution in the radiation field and in its spectrum do not always correspond to each other. This correspondence can be violated by complicating the initial structure by changing the phase relations between partial waves. For example, in expression (2), replace the random phase $\psi(n)$, which depends only on the index $n$, by the random phase $\psi(n,\nu)$, which depends on the indices $n$ and $\nu$. The distribution of the light field $W_{k,m}$ modified in this way in the initial plane is shown in figure 3, a. Despite the disordered speckle-like light field, its spatial spectrum does not undergo any significant changes compared to the spectrum of the initially specified field (see figure 2, b).

Figure 3. Images of beams with spatial spectrum scaling.
The shape of the spatial frequency distribution the image of the beam is preserved if it is converted to another form using its phase profile \( \Phi_{k,m} = \arg(W_{k,m}) \). In this case, a smooth non-fractal change in amplitude along the image field can be ensured. An example of such an image is shown in figure 3, b. It was built using the formula

\[
w_{k,m} = cke^{i\Phi_{k,m}},
\]

where \( c \) is the normalization factor. Function \( W_{k,m} = |w_{k,m}| \) for the beam amplitude is shown in figure 3, b.

The last two examples indicate that images that do not have self-similar fragments may possess fractal spectra with a certain scaling coefficient.

4. Conclusions
The testing of fractal structures in a complex representation significantly enhances the understanding of the interconnection of amplitude-phase characteristics and spatial radiation spectra. An important methodical aspect of the spectral analysis of such structures is the ability to expand the range of obtained information by parallel processing of data on the distribution of amplitude and phase. Processing efficiency will be higher if we use the detected coincidence of intensity maxima and phase singularities. The development of diagnostic methods requires taking into account the fact that, in some cases, a direct relationship between the spatial and spectral characteristics of radiation may be disturbed. A positive fact is the established high degree of spectra stability to the influence of optical noise. The recorded degree of asymmetry Fourier image due to the presence of the imaginary component in the initial light field can be used to estimate the level of initial phase disturbance.

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