Semi-Hopf Algebra and Supersymmetry

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Abstract

We define a semi-Hopf algebra which is more general than a Hopf algebra. Then we construct the supersymmetry algebra via the adjoint action on this semi-Hopf algebra. As a result we have a supersymmetry theory with quantum gauge group, i.e., quantised enveloping algebra of a simple Lie algebra. For the example, we construct the Lagrangian $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetry.

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I. Introduction

Supersymmetry was introduced in the seventy decades by Golfand-Likhtman, Volkov-Akulov, Wess-Zumino, and Salam-Strathdee[1]. This supersymmetry has been the subject of intense research in particle physics which promises the grand unification theory, also in field and string theory. The recent development in supersymmetric field theory is to find a generalized supersymmetry theory via the notion of duality. It was begin by Seiberg and Witten[3] who considered the $\mathcal{N}=2$ supersymmetric $SU(2)$ Yang-Mills theory.

On the other hand, Drinfel’d and Jimbo[4] generalized the Lie group (called quantum group) via noncommutative and non-co-commutative Hopf algebra. This quantum group has been applied in gauge theory[8,9,10]. This implies that the field theory become no longer commutative. So it is uneasy to define the variation of the action and its quantum theory. A different approach introduced in [14] has successfully handled this noncommutative problem in defining the variation of the action.

In this paper, we start with the definition of a semi-Hopf algebra which is more general than a Hopf algebra, then we construct the supersymmetry algebra via this semi-Hopf algebra. As a result we have a supersymmetry theory with quantum gauge group $U_q(g)$, i.e., quantised enveloping algebra of a simple Lie algebra $g$ which is defined by Lyubashenko and Sudbery[7]. The field (or superfield) become noncommutative that we handle by defining a noncommutative invariant scalar product between two fields (or superfield). So we can construct the Lagrangian $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetry by the scalar product between two superfield and do not worry about the noncommutative factor. It is also straightforward to define the variation of the action and then get the usual equation of motion. We use the notation for the supersymmetry as was given by Wess-Bagger[2].

This paper is organized as follows. In section II we briefly review the definition of a Hopf algebra given by Tjin[6] and then we define a semi-Hopf algebra which is more general structure than a Hopf algebra. Then we construct the supersymmetry algebra via this semi-Hopf algebra and define the general Jacobi identity in section III. The definition a noncommutative invariant scalar product and the construction of $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetry Lagrangian via superspace formalism, and the $\mathcal{N}=2$ prepotential are
presented in section IV. Section V is devoted for conclusion and outlook.

II. Semi-Hopf Algebra

Before defining the notion of semi-Hopf algebra, we start with the definition and elementary properties of Hopf algebra[5]. This Hopf algebra contains all the information on the algebraic structure of the Lie group while a semi-Hopf algebra contains all the information on the algebraic structure of the super-Lie group (as we will see later).

In this paper we use the notation and the definition of a Hopf algebra given by Tjin[6] and briefly introduce Hopf algebraic structure. Then we define a semi-Hopf algebra which is more general than a Hopf algebra.

Definition 1 Let \( A \) is a linear space over the ground field \( \mathbb{C} \). A bi-algebra is the set \((A, m, \Delta, \eta, \varepsilon)\) with the following properties:

1) \((A, m, \eta)\) is unital associative algebra, where

\[ m : A \otimes A \rightarrow A \]  \hspace{1cm} (2.1)

\[ \eta : \mathbb{C} \rightarrow A \] \hspace{1cm} (2.2)

2) A map \( \Delta : A \rightarrow A \otimes A \) satisfies

i) \( \Delta \) is linear

ii) \( \Delta \) is an algebra homomorphism

iii) \( (\Delta \otimes \text{id}) \Delta = (\text{id} \otimes \Delta) \Delta \) \hspace{1cm} (co-associativity)

3) A map \( \varepsilon : A \rightarrow \mathbb{C} \) such that

\[ (\text{id} \otimes \varepsilon) \Delta = (\varepsilon \otimes \text{id}) \Delta = \text{id} \] \hspace{1cm} (2.3)

Definition 2 A Hopf algebra is a bi-algebra \((A, m, \Delta, \eta, \varepsilon)\) together with a map

\[ S : A \rightarrow A \] \hspace{1cm} (2.4)

with the following properties.
\[ m(S \otimes id) \Delta = m(id \otimes S) \Delta = \eta \circ \varepsilon \]  \hspace{1cm} (2.5)

where \( S \) is an antipode.

Some example of a Hopf algebra can be seen in [4-9]. We interested in constructing an algebra which contain the algebraic structure of the super-Lie group. So we define an algebra (which we called semi-Hopf algebra) as follows :

**Definition 3** A semi-Hopf algebra is a bi-algebra \((B, m, \Delta, \eta, \varepsilon)\) with the following properties :

1) There exist \( q \in B \) such that

\[ m(id \otimes id) \Delta(q) = \eta \circ \varepsilon(q) \]  \hspace{1cm} (2.6)

2) There exist \( b \in B \) and \( b \neq q \) such that

\[ m(S \otimes id) \Delta(b) = m(id \otimes S) \Delta(b) = \eta \circ \varepsilon(b) \]  \hspace{1cm} (2.7)

where \( S \) is an antipode.

We can choose \( \eta \circ \varepsilon(a) = \varepsilon(a) I_B \) for \( a \in B \) with \( I_B \) is an identity element of \( B \) [4]. Both of the above condition does not compatible to each other. It means that if \( q \in B \) satisfies the condition (2.6), then \( q \) does not satisfy the condition (2.7) and vice versa.

The definition 3 implies the following proposition :

**Proposition 4** A Hopf algebra is subalgebra of a semi-Hopf algebra.

In the next section, we will construct a supersymmetry algebra via a semi-Hopf algebra where the odd elements, as we will define later, satisfies the condition (2.6)
III. Semi-Hopf Algebra and Supersymmetry Algebra

In this section we start to construct supersymmetry algebra via the semi-Hopf algebra as we introduced above.

Let $B$ be a bi-algebra generated by $P_\mu, J_{\mu\nu}, Q^I_\alpha, \tilde{Q}_{\dot{\alpha}I}, E_r, F_r, q^{\pm H_r}, I, \mathfrak{J}$ with coproduct $\Delta$, antipode $S$, and counit $\varepsilon$ defined by

$$
\Delta (P_\mu) = P_\mu \otimes I + I \otimes P_\mu, \quad \Delta (J_{\mu\nu}) = J_{\mu\nu} \otimes I + I \otimes J_{\mu\nu}, \quad (3.1)
$$

$$
\Delta (Q^I_\alpha) = Q^I_\alpha \otimes I + 3 \otimes Q^I_\alpha, \quad \Delta (\tilde{Q}_{\dot{\alpha}I}) = \tilde{Q}_{\dot{\alpha}I} \otimes I + \mathfrak{J} \otimes \tilde{Q}_{\dot{\alpha}I},
$$

$$
\Delta (E_r) = E_r \otimes q^{-H_r} + q^{H_r} \otimes E_r, \quad \Delta (F_r) = F_r \otimes q^{-H_r} + q^{H_r} \otimes F_r,
$$

$$
\Delta (q^{\pm H_r}) = q^{\pm H_r} \otimes q^{\pm H_r}, \quad \Delta (\mathfrak{J}) = \mathfrak{J} \otimes \mathfrak{J}, \quad \Delta (I) = I \otimes I,
$$

$$
S (P_\mu) = -P_\mu, \quad S (J_{\mu\nu}) = -J_{\mu\nu}, \quad S (Q^I_\alpha) = -Q^I_\alpha, \quad (3.2)
$$

$$
S (\tilde{Q}_{\dot{\alpha}I}) = -\tilde{Q}_{\dot{\alpha}I}, \quad S (E_r) = -q^{-1}_r E_r, \quad S (F_r) = -q_r F_r,
$$

$$
S (q^{\pm H_r}) = q^{\mp H_r}, \quad S (\mathfrak{J}) = \mathfrak{J}, \quad S (I) = I,
$$

$$
\varepsilon (P_\mu) = \varepsilon (J_{\mu\nu}) = \varepsilon (Q^I_\alpha) = \varepsilon (\tilde{Q}_{\dot{\alpha}I}) = 0, \quad (3.3)
$$

$$
\varepsilon (E_r) = \varepsilon (F_r) = \varepsilon (q^{\pm H_r}) = \varepsilon (\mathfrak{J}) = \varepsilon (I) = 1,
$$

where $q$ is a fixed elements of ground field $C$, $q_r = q^{(H_r,H_r)}$ and $\langle , \rangle$ denotes the Killing form in Cartan subalgebra. The operator $\mathfrak{J}$ acts on the generator of $B$ as

$$
\mathfrak{J} Q^I_\alpha = -Q^I_\alpha, \quad \mathfrak{J} \tilde{Q}_{\dot{\alpha}I} = -\tilde{Q}_{\dot{\alpha}I}, \quad (3.4)
$$

$$
\mathfrak{J} P_\mu = P_\mu, \quad \mathfrak{J} J_{\mu\nu} = J_{\mu\nu}, \quad \mathfrak{J} E_r = E_r,
$$

$$
\mathfrak{J} F_r = F_r, \quad \mathfrak{J} q^{\pm H_r} = q^{\mp H_r}.
$$
It can be seen from the above equation that the basis element of $B$ consist of the odd elements, i.e., $Q^I_\alpha$ and $\tilde{Q}_{\dot{\alpha}I}$ because they have $-1$ parity and the rest are even, i.e., $P_\mu$, $J_{\mu\nu}$, $E_r$, $F_r$, $q^{\pm H_r}$ because they have $+1$ parity.

The odd elements, i.e., $Q^I_\alpha$ and $\tilde{Q}_{\dot{\alpha}I}$, turn out to satisfy the condition (2.6) while the even elements, i.e., $P_\mu$, $J_{\mu\nu}$, $E_r$, $F_r$, $q^{\pm H_r}$, satisfy the condition (2.7). So it is sufficient for $B$ to be a semi-Hopf algebra.

We define the adjoint action of $B$ on itself, given by $x \rightarrow \text{ad}_x \in \text{End}_\mathfrak{e} B$ where

$$\text{ad} \ x \ (y) \equiv \sum x_{(1)} \ y \ S \ (x_{(2)}) \ = \ [x, y] \ , \quad (3.5)$$

if $\Delta \ (x) = \sum x_{(1)} \otimes x_{(2)}$. For odd-odd element the adjoint action become

$$\begin{align*}
\left[ Q^I_\alpha, \tilde{Q}_{\dot{\beta}J} \right] & = Q^I_\alpha \tilde{Q}_{\dot{\beta}J} + \tilde{Q}_{\dot{\beta}J} Q^I_\alpha \ , \\
\left[ Q^I_\alpha, Q^J_\beta \right] & = Q^I_\alpha Q^J_\beta + Q^J_\beta Q^I_\alpha \ , \\
\left[ \tilde{Q}_{\dot{\alpha}I}, \tilde{Q}_{\dot{\beta}J} \right] & = \tilde{Q}_{\dot{\alpha}I} \tilde{Q}_{\dot{\beta}J} + \tilde{Q}_{\dot{\beta}J} \tilde{Q}_{\dot{\alpha}I}
\end{align*}$$

and for odd-even element (the even element only $P_\mu$, $J_{\mu\nu}$) the adjoint action become

$$\begin{align*}
\left[ Q^I_\alpha, P_\mu \right] & = Q^I_\alpha P_\mu - P_\mu Q^I_\alpha \ , \\
\left[ Q^I_\alpha, J_{\mu\nu} \right] & = Q^I_\alpha J_{\mu\nu} - J_{\mu\nu} Q^I_\alpha \ , \quad (3.7)
\end{align*}$$

and for even-even element (only $P_\mu$, $J_{\mu\nu}$) similar to equation (3.7). Thus, we can identify $Q^I_\alpha$ and $\tilde{Q}_{\dot{\alpha}I}$ as supersymmetry generator, $P_\mu$ and $J_{\mu\nu}$ as the energy-momentum operators, and the Lorentz rotation generators (antisymmetric tensor) respectively with $\mu, \nu = 0, 1, 2, 3$ (spacetime index with the metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$), $I$ is the number of supersymmetry generators, and the fermionic index $\alpha(\dot{\alpha}) = 1(\dot{1}), 2(\dot{2})$. The rest even elements, i.e., $E_r$, $F_r$, $q^{\pm H_r}$ are generator of the simply-connected quantised enveloping algebra $U_q(\mathfrak{g})$ and $\mathfrak{g}$ is a simple Lie algebra[7], where $r$ is the number of fundamental roots of $\mathfrak{g}$. A quantum Lie algebra $\mathfrak{g}_q$ is a subspace of $U_q(\mathfrak{g})$ satisfying certain properties[7,8]. So we can construct the elements of $\mathfrak{g}_q$ (denoted $T^a$
with \( a \) is the numbers of basis element) from \( E_r, F_r, q^\pm H_r \). In this case the coproduct of the elements of \( g_q \) are the form [7,8]

\[
\Delta (T^a) = T^a \otimes C + u^a_b \otimes T^b ,
\]

and

\[
\text{ad} u^a_b (T^c) = \sigma^{ac}_{db} T^d ,
\]

where \( C \) is a central element of \( U_q (g) \) and \( \sigma \) is the quantum flip operator (a deformation of the classical flip operator: \( \sigma^{ac}_{db} = \delta^c_d \delta^a_b \) when \( q =1 \)).

As we see from equation (3.6) and (3.7), the adjoint action of \( B \) have a \( Z_2 \) graded structure to preserve the closure property. Thus we can define the adjoint action on the basis elements of \( B \) as

\[
[P_\mu, P_\nu] = [P_\mu, T^a] = [J_\mu_\nu, T^a] = 0 ,
\]

\[
\begin{align*}
[Q^I_\alpha, \bar{Q}^I_\beta] & = 2 \sigma^I_{\alpha\beta} P_\mu \delta^I_{\nu}, \\
[Q^I_\alpha, Q^J_\beta] & = \varepsilon_\alpha^I_\beta Z^{IJ} , \\
[\bar{Q}^I_\alpha, \bar{Q}^J_\beta] & = \varepsilon^I_\alpha^J_\beta Z^{IJ} ,
\end{align*}
\]

\[
\begin{align*}
[Q^I_\alpha, Z^{IJ}] & = [\bar{Q}^I_\alpha, Z^{IJ}] = [P_\mu, Z^{IJ}] = 0 , \\
[J_\mu\nu, Z^{IJ}] & = [T^a, Z^{IJ}] = 0 , \\
[Q^I_\alpha, P_\mu] & = c (\gamma_\mu)_\alpha^\beta Q^I_\beta , \\
[Q^I_\alpha, J_\mu\nu] & = (b_{\mu\nu})_\alpha^\beta Q^I_\beta ,
\end{align*}
\]

\[
\begin{align*}
[Q^I_\alpha, T^a] & = l (s^a)_f^I J Q^J_\alpha , \\
[T^a, Q^I_\alpha] & = r (s^a)_f^I J Q^J_\alpha , \\
[Z^{IJ}, Z^{JK}] & = 0 ,
\end{align*}
\]
\[ [T^a, T^b] = f_{ab}^c T^c, \]
\[ [P_\mu, J_{\rho\nu}] = \eta_{\mu\rho} P_{\nu} - \eta_{\mu\nu} P_\rho, \]
\[ [J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\mu\sigma} J_{\nu\rho} + \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\sigma} J_{\mu\rho}, \]

where \( c \) is a constant, \( f_{ab}^c \) is a structure constant, \( Z_{IJ} \) are the central charge with its conjugate \( Z_{IJ}^\ast \), \( \sigma^\mu \) are the Pauli matrices, \( \varepsilon_{\alpha\beta} \) and \( \varepsilon_{\alpha\beta}^\ast \) are the Levi-Civita symbol, \( l \) and \( r \) denote left and right, and \( \gamma_\mu \) and \( b_{\mu\nu} \) are matrix vector and matrix antisymmetric tensor, respectively. The central charges \( Z_{IJ} \) are even elements of \( B \). The set of the adjoint action in equation (3.10) through (3.13) is called general supersymmetry algebra. The second and third equation in equation (3.10) are set to be zero in order to preserve the Coleman-Mandula theorem\[12\].

Let \( x_1 = P_\mu, x_2 = J_{\mu\nu}, x_3 = Q^I_\alpha, x_4 = \tilde{Q}_{\dot{\beta}J}, x_a = T^a \) with \( a = 1, \ldots, N \) and \( N \) is the numbers of generator of the gauge group. The general supersymmetry algebra satisfies:

1. The left Jacobi identity

\[
\left[x_i, \left[x_j, x_k\right]\right] = \Gamma_{i,j,k}^{\ell,m} \left[x_\ell, \left[x_m, x_k\right]\right].
\]

2. The right Jacobi identity

\[
\left[x_i, \left[x_j, x_k\right]\right] = \Gamma_{j,k}^{\ell,m} \left[x_\ell, \left[x_i, x_m\right]\right].
\]

\( \Gamma_{i,j,k}^{\ell,m} \) is the antisymmetriser with \( i, j, k, l, m = 1, 2, 3, 4, a \), which will reduce to the antisymmetriser defined by Lyubashenko-Sudbery\[7\] if \( \ell = a, m = b, j = c, k = d \) (where \( a, b, c, d = 1, \ldots, N \)), i.e., \( \Gamma_{cd}^{ab} = \gamma_{cd}^{ab} \).

The antisymmetriser \( \Gamma \) is restricted by the general supersymmetry algebra (equation (3.10) through (3.13)) for which \( c = 0, b_{\mu\nu} \) form a representation of Lorentz algebra, and \( l(s^a), r(s^a) \) represent of internal symmetry algebra.
IV. $U_q(\mathfrak{g})$ Supersymmetry

In the previous section, we have the general supersymmetry algebra (equation (3.10) through (3.13)), i.e., the supersymmetry algebra with quantum gauge group $U_q(\mathfrak{g})$. This algebra implies that the massless and massive multiplet are precisely the same as the supersymmetry with classical gauge group. The difference is that in this theory we have noncommutative fields (or superfields) in each multiplet. So we need to define the noncommutative factor between two different fields (or superfields). Some example for the gauge theory are in references [8,9,10]. It is a complicated problem to define the variation of action of the theory. Fortunately, the noncommutative invariant property of the scalar product between two fields (or superfields) bring us out of this problem.

A. Noncommutative factor and Metric

**Definition 5** Let $\Phi^\alpha_1$ and $\Psi^\beta_2$ are two different kind of field (or superfield) in $\rho_1$ and $\rho_2$ representation, respectively. Then there exist a factor $B^\alpha_1^\beta_2$ such that

$$\Phi^\alpha_1 \Psi^\beta_2 = B^\alpha_1^\beta_2 \Psi^\beta_2 \Phi^\alpha_1,$$  \hspace{1cm} (4.1)

$B^\alpha_1^\beta_2$ is called the noncommutative factor.

The noncommutative factor $B^\alpha_1^\beta_2$ is an $Nn \times Nn$ matrix with $N$ is the dimension of $\rho_1$ representation and $n$ is the dimension of $\rho_2$ representation.

If $\Phi$ and $\Psi$ have the same representation, i.e., $\rho_1 = \rho_2 = \rho$ then the noncommutative factor become

$$\Phi^\alpha \Psi^\beta = \lambda^{\alpha \beta}_{\rho, \epsilon d} \Psi^\epsilon \Phi^\delta,$$  \hspace{1cm} (4.2)

where $\lambda^{\alpha \beta}_{\rho, \epsilon d}$ is an $N^2 \times N^2$ matrix with $N$ is the dimension of the $\rho$ representation. For special case, if $\rho$ is an adjoint representation then $\lambda^{ab}_{\rho, \epsilon d} = \sigma^{ab}_{\epsilon d}$, where $\sigma$ is the quantum flip operator defined in [7,8].
**Definition 6** Let $\Phi$ and $\Psi$ are two different kind of fields (or superfields) in the $\rho$ representation. The product of two fields (or superfields)

$$g_{\bar{a}\bar{b}}\Phi^\bar{a}\Psi^\bar{b} \equiv \Phi^\bar{a}\Psi_\bar{a}$$

is called a noncommutative invariant scalar product if it satisfies

$$\Phi^\bar{a}\Psi_\bar{a} = \Psi^\bar{a}\Phi_\bar{a} . \quad (4.3)$$

In the rest of this paper, the noncommutative invariant scalar product will be simply called the scalar product. The function $g_{\bar{a}\bar{b}}$ has form

$$g_{\bar{a}\bar{b}} = \text{Tr} \left( u e_{\bar{a}} e_{\bar{b}} \right) , \quad (4.4)$$

where $e_{\bar{a}}$ and $e_{\bar{b}}$ are basis of the $\rho$ representation, $u = \sum S (R_2) R_1$ being the quantum trace element of $U_q (g)$ and $R = \sum R_1 \otimes R_2$ its universal $R$-matrix. We have that the function $g_{\bar{a}\bar{b}}$ contracting the upper index to the lower index, that is

$$g_{\bar{a}\bar{b}} \Phi^\bar{b} = \Phi^\bar{a} , \quad (4.5)$$

and vice versa. The function $g_{\bar{a}\bar{b}}$ also satisfies

$$g^{\bar{a}\bar{b}} g_{\bar{b}\bar{c}} = \delta^{\bar{a}}_{\bar{c}} . \quad (4.6)$$

**Proposition 7** The function $g_{\bar{a}\bar{b}}$ satisfies $g_{\bar{a}\bar{b}} \lambda^{\bar{a}\bar{b}}_{\rho, \bar{c}\bar{d}} = g_{\bar{c}\bar{d}}$.

For the adjoint representation, we have

$$g_{ab} = \text{Tr} \left( u \text{ad}T^a \text{ad}T^a \right) ,$$

and for the $SU_q (2)$ see Sudbery\[7,8\].

Note that it is straightforward to define the variation of the Lagrangian: Let $\mathcal{L} (\Psi, \Psi_{,\mu})$ be a Lagrangian, then we define the variation of $\mathcal{L}$ as

$$\delta\mathcal{L} = \delta\Psi^a \frac{\partial \mathcal{L}}{\partial \Psi^a} + \delta\Psi_{,\mu}^a \frac{\partial \mathcal{L}}{\partial \Psi_{,\mu}^a} , \quad (4.7)$$

where $\Psi$ is a field and $\Psi_{,\mu} = \partial \Psi / \partial x^\mu$. We cancel the surface term and then get the usual equation of motion.
B. $\mathcal{N}=1$ Scalar and Vector Multiplet

1. $\mathcal{N}=1$ Scalar Multiplet

The $\mathcal{N}=1$ scalar multiplet is represented by a chiral superfield $\Phi = \Phi^a e_a$ and can be expanded as

$$\Phi (y, \theta) = \phi (y) + \sqrt{2} \theta \psi (y) + \theta^2 F (y) \ , \tag{4.8}$$

where $y^\mu = x^\mu + i \theta \sigma^\mu \bar{\theta}$, $x$ are spacetime coordinates, $\theta$, $\bar{\theta}$ are anticommuting coordinates, and $\sigma^\mu$ are the Pauli matrices. Here, $\phi$ and $\psi$ are the scalar and fermionic components respectively and $F$ is an auxiliary field required for the off-shell representation.

Similarly, an anti-chiral superfield is represented by $\bar{\Phi} = \Phi^{\dagger a} e_a$ and can be expanded as

$$\bar{\Phi} (y^\dagger, \bar{\theta}) = \bar{\phi} (y^\dagger) + \sqrt{2} \bar{\theta} \bar{\psi} (y^\dagger) + \bar{\theta}^2 \bar{F} (y^\dagger) \ , \tag{4.9}$$

where $y^{\dagger \mu} = x^{\dagger \mu} - i \theta \sigma^\mu \bar{\theta}$.

The Lagrangian for the scalar multiplet is the scalar product of a chiral and an anti-chiral superfield

$$\mathcal{L} = \int \Phi^{\dagger a} \Phi_a \, d^4 \theta = (\partial^\mu \phi^{\dagger a}) (\partial^\mu \phi_a) - i \bar{\psi}^{\dagger a} \sigma^\mu \partial^\mu \psi + F^{\dagger a} F_a \ , \tag{4.10}$$

where $d^4 \theta = d^2 \theta \, d^2 \bar{\theta}$.

2. $\mathcal{N}=1$ Vector Multiplet

This multiplet is represented by a real superfield satisfying $V^{\dagger} = V$. Using the abelian gauge transformation

$$V \rightarrow V + a \Lambda + b S \, (\Lambda^{\dagger}) \ , \tag{4.11}$$

where $\Lambda (\Lambda^{\dagger})$ are chiral (anti-chiral) superfield, $a, b$ are coefficient, and $S$ is an antipode, we can write $V$ as

$$V = -\theta \sigma \bar{\theta} A_\mu + i \theta^2 \bar{\theta} \lambda - i \bar{\theta}^2 \theta \lambda + \frac{1}{2} \theta^2 \bar{\theta}^2 D \ , \tag{4.12}$$
the so called Wess-Zumino gauge. The abelian field strength is defined by

\[ W_\alpha = -\frac{1}{4} \bar{D}^2 D_\alpha V, \quad \bar{W}_\dot{\alpha} = -\frac{1}{4} D^2 \bar{D}_{\dot{\alpha}} V, \quad (4.13) \]

where \( D^2 = D^a D_a \) and \( \bar{D}^2 = \bar{D}_\dot{\alpha} \bar{D}^{\dot{\alpha}} \), and \( D^a, \bar{D}_{\dot{\alpha}} \) are the super-covariant derivatives[2]. \( W_\alpha \) is a chiral superfield.

In the non-abelian case, \( V \) belongs to the adjoin representation of the gauge group \( U_q(\mathfrak{g}) : V = V_a \text{ ad } T^a \). The gauge transformations are now implemented by

\[ e^{-2V} \rightarrow \sum h_{(1)} (\bar{\Lambda}) \, e^{-2V} \, S(h_{(2)} (\Lambda)) \, , \quad (4.14) \]

\[ e^{2V} \rightarrow \sum h_{(1)} (\Lambda) \, e^{2V} \, S(h_{(2)} (\bar{\Lambda})) \, , \]

if \( \Delta (h) = \sum h_{(1)} \otimes h_{(2)} \), where \( \Lambda = \Lambda_a \text{ ad } T^a \) and \( \bar{\Lambda} = \Lambda^\dagger_a \text{ ad } T^a \). The non-abelian gauge field strength is defined by

\[ W_\alpha = \frac{1}{8} \bar{D}^2 e^{2V} D_\alpha e^{-2V}, \quad \bar{W}_{\dot{\alpha}} = \frac{1}{8} D^2 e^{2V} \bar{D}_{\dot{\alpha}} e^{-2V}, \quad (4.15) \]

and as we expect they transform as

\[ W'_\alpha \rightarrow \sum h_{(1)} (\Lambda) \, W_\alpha \, S(h_{(2)} (\Lambda)) \, . \quad (4.16) \]

In components, it takes the form

\[ W_\alpha = \left( -i \lambda_{a,\alpha} + \theta_a D_a - \frac{i}{2} (\sigma^\mu \bar{\sigma}^\nu \theta)_{a, \alpha} F_{a, \mu \nu} + \theta^2 (\sigma^\mu \nabla_{\mu} \bar{\lambda}_{a, \alpha}) \right) \text{ ad } T^a, \quad (4.17) \]

where

\[ F^{a}_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + C f^a_{bc} A^b_\mu A^c_\nu, \]

\[ \nabla_{\mu} \bar{\lambda}^a = \partial_\mu \bar{\lambda}^a + C f^a_{bc} A^b_\mu \bar{\lambda}^c, \]

and \( C \) is the central element of \( U_q(\mathfrak{g}) \).
The non-abelian supersymmetric Lagrangian (with the $F\tilde{F}$-term) is given by

$$L = \frac{1}{8\pi} \text{Im} \left( \tau \int d^2\theta W^{a,\alpha} W_{a,\alpha} \right), \quad (4.18)$$

where $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$ (see [13]), $\theta$ is a real parameter, and $g$ is a coupling constant.

3. $\mathcal{N} = 1$ Vector Multiplet couple with Matter

Let $\Phi$ be a chiral superfield belong to a $\rho$ representation of the gauge group $U_q(g)$. The kinetic term $\Phi^{\dagger, a} \Phi_{\bar{a}}$ is invariant under the global gauge transformation $\Phi' = h(\Lambda) \Phi$ and $\bar{\Phi}' = \bar{\Phi} S(h(\Lambda))$ with $\Lambda$ a real parameter. In the local case, to insure that $\Phi'$ and $\bar{\Phi}'$ remain a chiral and anti-chiral superfield, $\Lambda$ and $\bar{\Lambda}$ have to be a chiral and anti-chiral superfield, respectively. So the local gauge transformation of a chiral and an anti-chiral superfield become

$$\Phi' = h(\Lambda) \Phi, \quad \bar{\Phi}' = \bar{\Phi} S(h(\Lambda)),$$

respectively. The supersymmetric gauge invariant kinetic term is given by

$$\bar{\Phi} e^{-2gV} \Phi.$$  \hspace{1cm} (4.20)

Then the Lagrangian $\mathcal{N} = 1$ vector multiplet couple with the kinetic matter term is

$$L = \frac{1}{8\pi} \text{Im} \left( \tau \int d^2\theta W^{a,\alpha} W_{a,\alpha} \right) + \int d^4\theta \bar{\Phi} e^{-2gV} \Phi.$$  \hspace{1cm} (4.21)

In terms of the superfield components, the above Lagrangian takes the form

$$L = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a}_{\mu\nu} + \frac{\theta}{32\pi^2} F^a_{\mu\nu} F_{\mu\nu} - \frac{i}{2g^2} \lambda^a \sigma^\mu \nabla_\mu \bar{\lambda}_a.$$  \hspace{1cm} (4.22)
\[ + \frac{i}{2g^2} \bar{\lambda}^a \sigma^\mu \nabla_\mu \lambda_a + \frac{1}{2g^2} D^a D_a + (\nabla_{\mu}^{(\rho)} \bar{\phi})(\nabla^{(\rho), \mu} \phi) \]

\[ - i \bar{\psi} \sigma^\mu \nabla_{\mu}^{(\rho)} \psi - \bar{\phi} D \phi - i \sqrt{2} \bar{\phi} \lambda \psi + i \sqrt{2} \bar{\psi} \bar{\lambda} \phi + \bar{F} F , \]

where

\[ \nabla_{\mu}^{(\rho)} \bar{\phi} = \partial_{\mu} \bar{\phi} + ig \bar{\phi} A_{\mu} \, , \quad \nabla_{\mu}^{(\rho)} \phi = \partial_{\mu} \phi + ig A_{\mu} \phi . \]

The above Lagrangian is invariant under the supersymmetry variations, \( \delta_c = \epsilon^a Q_a + \bar{\epsilon}_\dot{a} \bar{Q}^{\dot{a}} \), i.e.,

\[ \begin{align*}
\delta_c A_{\mu} &= i \lambda^a \sigma^\mu \bar{\epsilon} - i \epsilon \sigma^\mu \bar{\lambda}^a , \\
\delta_c D^a &= - \nabla_\mu \lambda^a \sigma^\mu \bar{\epsilon} - \epsilon \sigma^\mu \nabla_\mu \bar{\lambda}^a , \\
\delta_c \lambda^a &= \sigma^{\mu\nu} \epsilon F_{\mu\nu}^a + i \epsilon D^a , \\
\delta_c \bar{\lambda}^a &= \bar{\epsilon} \bar{\sigma}^{\nu\mu} F_{\mu\nu}^a - i \bar{\epsilon} D^a , \\
\delta_c \phi &= \sqrt{2} \epsilon \psi , \\
\delta_c \bar{\psi} &= \sqrt{2} \bar{\epsilon} \bar{F} + i \sqrt{2} \sigma^\mu \bar{\epsilon} \nabla_{\mu}^{(\rho)} \phi , \\
\delta_c \bar{\phi} &= \sqrt{2} \bar{\epsilon} \bar{\phi} - i \sqrt{2} \bar{\epsilon} \sigma^\mu \nabla_{\mu}^{(\rho)} \bar{\phi} , \\
\delta_c F &= -i \sqrt{2} \nabla_{\mu}^{(\rho)} \psi \sigma^\mu \bar{\epsilon} - 2i \phi \bar{\epsilon} \bar{\lambda} .
\end{align*} \]

We see from the above equation that the supersymmetry variations of field in this theory are precisely the same as the classical internal symmetry group of supersymmetry[2].

C. \( \mathcal{N} = 2 \) Vector Multiplet

\( \mathcal{N} = 2 \) vector multiplet consist of fields \( \phi, \psi \) and \( A_{\mu}, \lambda \) in a single multiplet. This means that all fields must be in the same representation of the gauge group \( U_q (g) \) as \( A_{\mu} \), i.e., in the adjoint representation. So, the Lagrangian \( \mathcal{N} = 2 \) vector multiplet is the Lagrangian (4.21)( or (4.22)) with the matter multiplet in the adjoint representation, i.e.,

\[ \mathcal{L} = \text{Im} \left( \frac{\tau}{8\pi} \left[ \int d^2 \theta W^{a, \alpha} W_{a, \alpha} + 2 \int d^4 \theta \bar{\Phi} e^{-2gV} \Phi \right] \right) , \quad (4.24) \]
with scaling $\Phi \to \Phi/g$.

The Lagrangian $\mathcal{N}=2$ vector multiplet can be constructed by using $\mathcal{N}=2$ superspace formalism [11]. The $\mathcal{N}=2$ chiral superfield is introduced as follows

$$\Psi = \Phi (\tilde{y}, \theta) + \sqrt{2} \tilde{\theta} \alpha W^\alpha (\tilde{y}, \theta) + \tilde{\theta}^2 G (\tilde{y}, \theta),$$

where $\tilde{y}^\mu = x^\mu + i \theta \sigma^\mu \bar{\theta} + i \bar{\theta} \sigma^\mu \tilde{\theta}$ and $\bar{\theta}, \tilde{\theta}$ are addition anti-commuting coordinates. Then the Lagrangian (4.24) can be written as

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \left( \int d^2 \theta d^2 \bar{\theta} \frac{1}{2} \tau \Psi^a \Psi_a \right),$$

with

$$G (\tilde{y}, \theta) = \int d^2 \bar{\theta} \Phi (\tilde{y}^{\mu+} + i \theta \sigma^\mu \bar{\theta}, \theta, \bar{\theta}) \exp (\frac{-2g \mathcal{V} (\tilde{y}^\mu - i \theta \sigma^\mu \bar{\theta}, \theta, \bar{\theta})) .$$

(4.27)

The most general Lagrangian for $\mathcal{N}=2$ vector multiplet is

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \left( \int d^2 \theta d^2 \bar{\theta} \mathcal{F} (q, \Phi) \right),$$

(4.28)

$$= \frac{1}{8\pi} \text{Im} \left( \left[ \int d^2 \theta W^{a, a} W^b_a \mathcal{F}_{ab} (q, \Phi) + 2 \int d^4 \theta \left( \Phi e^{-2g \mathcal{V}} \right)^a \mathcal{F}_a (q, \Phi) \right] \right).$$

Here, $\mathcal{F}_a (q, \Phi) = \partial \mathcal{F} / \partial \Phi^a$, $\mathcal{F}_{ab} (q, \Phi) = \partial^2 \mathcal{F} / \partial \Phi^a \partial \Phi^b$, and $\mathcal{F}$ is referred to as the $\mathcal{N}=2$ prepotential [3]. The second term in the above equation has the Kähler Potential $\text{Im} (\tilde{\Phi}^a \mathcal{F}_a (q, \Phi))$ which gives a metric on the space of fields. Seiberg and Witten [3] have determined exactly this function $\mathcal{F}$ for the $SU(2)$ gauge group.

V. Conclusion and Outlook

We have defined a semi-Hopf algebra which is more general than a Hopf algebra and we construct the supersymmetry algebra via the adjoint action on this semi-Hopf algebra. As a result we have a supersymmetry theory with quantum gauge group $U_q (\mathfrak{g})$, i.e., quantised enveloping algebra of a simple
Lie algebra $\mathfrak{g}$. The field (or superfield) become noncommutative that we handled by defining the noncommutative invariant scalar product between two fields (or superfield). We then construct the Lagrangian of the $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetry by the scalar product between two superfield and do not worry about the noncommutative factor. We find that the $\mathcal{N}=2$ prepotential depends on a fixed element $q$ of the ground field $\mathbb{C}$ and the chiral superfield $\Phi$. We left the problems of finding R-matrix for the quasi-semi-Hopf algebra and the extended Seiberg-Witten theory of quantum gauge group for further research.

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