Research Article

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Cahn–Hilliard equation on the boundary with bulk condition of Allen–Cahn type

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Abstract: The well-posedness of a system of partial differential equations with dynamic boundary conditions is discussed. This system is a sort of transmission problem between the dynamics in the bulk $\Omega$ and on the boundary $\Gamma$. The Poisson equation for the chemical potential and the Allen–Cahn equation for the order parameter in the bulk $\Omega$ are considered as auxiliary conditions for solving the Cahn–Hilliard equation on the boundary $\Gamma$. Recently, the well-posedness of this equation with a dynamic boundary condition, both of Cahn–Hilliard type, was discussed. Based on this result, the existence of the solution and its continuous dependence on the data are proved.

Keywords: Cahn–Hilliard equation, bulk condition, dynamic boundary condition, well-posedness

MSC 2010: 35K61, 35K25, 35D30, 58J35, 80A22

1 Introduction

In this paper, we treat the Cahn–Hilliard equation [1] on the boundary of some bounded smooth domain. Let $0 < T < +\infty$ be some fixed time and let $\Omega \subset \mathbb{R}^d$, $d = 2$ or $3$, be a bounded domain occupied by a material, where the boundary $\Gamma$ of $\Omega$ is supposed to be smooth enough. We start from the following equations of Cahn–Hilliard type on the boundary $\Gamma$:

\begin{align}
\partial_t u_{\Gamma} - \Delta_{\Gamma} m_{\Gamma} &= -\partial_{\nu} m \\
mu_{\Gamma} &= -\Delta_{\Gamma} u_{\Gamma} + W_\Gamma'(u_{\Gamma}) - f_{\Gamma} + \partial_{\nu} u
\end{align}

where $\partial_t$ denotes the partial derivative with respect to time, and $\Delta_{\Gamma}$ denotes the Laplace–Beltrami operator on $\Gamma$ (see, e.g., [21, Chapter 3]). Here, the unknowns $u_{\Gamma}$ and $m_{\Gamma} : \Sigma \to \mathbb{R}$ stand for the order parameter and the chemical potential, respectively. In the right-hand sides of (1.1) and (1.2), the outward normal derivative $\partial_{\nu}$ on $\Gamma$ acts on functions $\mu, u : Q := (0, T) \times \Omega \to \mathbb{R}$ that satisfy the following trace conditions:

\begin{align}
\mu_{|\Sigma} &= \mu_{\Gamma}, \quad u_{|\Sigma} = u_{\Gamma}
\end{align}

where $\mu_{|\Sigma}$ and $u_{|\Sigma}$ are the traces of $\mu$ and $u$ on $\Sigma$. Moreover, these functions $\mu$ and $u$ solve the following equations in the bulk $\Omega$:

\begin{align}
-\Delta \mu &= 0 \quad \text{in } Q, \\
\tau \partial_t u - \Delta u + W'(u) &= f \quad \text{in } Q
\end{align}

where $\tau > 0$ is a positive constant and $\Delta$ denotes the Laplacian.
Note that the nonlinear terms \( W'_\mu \) and \( W' \) are the derivatives of the functions \( W_\mu \) and \( W \), usually referred as double-well potentials, with two minima and a local unstable maximum in between. The prototype model is provided by \( W_\mu (r) = \mu(r) = (1/4)(r^2 - 1)^2 \), so that \( W'_\mu (r) = W''(r) = r^3 - r, r \in \mathbb{R} \), is the sum of an increasing function with a power growth and another smooth function which breaks the monotonicity properties of the former and is related to the non-convex part of the potential \( W_\mu \) or \( W \).

Therefore, we can say that system (1.1)–(1.2) yields the Cahn–Hilliard equation on a smooth manifold \( \Gamma \), and equations (1.4) and (1.5) reduce to the Poisson equation for \( \mu \) and the Allen–Cahn equation for \( u \) in the bulk \( \Omega \), as auxiliary conditions for solving (1.1)–(1.2). In other word, (1.1)–(1.5) is a sort of transmission problem between the dynamics in the bulk \( \Omega \) and the one on the boundary \( \Gamma \). With the initial conditions

\[
u_\Gamma(0) = u_{0\Gamma} \quad \text{on} \quad \Gamma, \quad u(0) = u_0 \quad \text{in} \quad \Omega,
\]

problem (1.1)–(1.6) becomes an initial value problem of a Poisson–Allen–Cahn system with a dynamic boundary condition of Cahn–Hilliard type, named (P). Indeed, the interaction between \( \mu \) and \( u \) appears only on (1.1)–(1.2), whereas (1.4) and (1.5) are independent equations. As a remark, if \( \tau = 0 \), then problem (P) turns out to be a quasi-static system. From (1.1), (1.4) and (1.6), we easily see that the mass conservation on the boundary holds as follows:

\[
\int_{\Gamma} u_t(t) \, d\Gamma = \int_{\Gamma} u_{0\Gamma} \, d\Gamma \quad \text{for all} \quad t \in [0, T].
\]

Let us mention some related results: the papers [11–14, 16] consider some quasi-static systems with dynamic boundary conditions (see also [4, 6]), the contributions [3, 5, 8, 9, 15, 17, 20, 28] set a Cahn–Hilliard equation on the boundary as the dynamic boundary condition, and a more complicated system of Cahn–Hilliard type on the boundary with a mass conservation condition is investigated in [18]. Especially, in this paper we will exploit the previous result in [5], on which equations (1.1)–(1.5) were replaced by

\[
\begin{align*}
\varepsilon \partial_t \mu - \Delta \mu &= 0 \quad \text{in} \quad Q, \\
\varepsilon \mu &= \tau \partial_t u - \Delta u + W'(u) - f \quad \text{in} \quad Q, \\
u_{t\Sigma} &= u_t, \quad \mu_{t\Sigma} = \mu_t \quad \text{on} \quad \Sigma, \\
\partial_t u_{\Sigma} + \partial_x \mu - \Delta \mu_{\Sigma} &= 0 \quad \text{on} \quad \Sigma, \\
\mu_{t\Sigma} &= \partial_x u - \Delta u_{\Sigma} + W'_\mu(u_{\Sigma}) - f_{\Sigma} \quad \text{on} \quad \Sigma,
\end{align*}
\]

where \( \varepsilon > 0 \). Then from system (1.8)–(1.12) with (1.6), we obtain the following total mass conservation:

\[
\varepsilon \int_{\Omega} u(t) \, dx + \int_{\Gamma} u_{\Gamma}(t) \, d\Gamma = \varepsilon \int_{\Omega} u_0 \, dx + \int_{\Gamma} u_{0\Gamma} \, d\Gamma \quad \text{for all} \quad t \in [0, T].
\]

Our essential idea of the existence proof is to be able to pass to the limit as \( \varepsilon \searrow 0 \) in system (1.8)–(1.12), with (1.6). To be more precise about our arguments, let us introduce a brief outline of the paper along with a short description of the various items.

In Section 2, we present the main results of the well-posedness of system (1.1)–(1.6). A solution to problem (P) is suitably defined. The main theorems are concerned with the existence of the solution (Theorem 2.3) and the continuous dependence on the given data (Theorem 2.4), the second theorem entailing the uniqueness property.

In Section 3, we consider the approximate problem for (P), with two approximation parameters \( \varepsilon \) and \( \lambda \), by substituting the maximal monotone graphs with their Yosida regularizations in terms of the parameter \( \lambda \). Moreover, we obtain uniform estimates with suitable growth order. Here, we can apply the results that have been shown in [5].

In Section 4, we prove the existence result. The proof is split in several steps. In the first step, we obtain uniform estimates with respect to \( \varepsilon \). Then, combining them with the previous estimates of Section 3, we can pass to the limit as \( \varepsilon \searrow 0 \). In the second step, we improve suitable estimates in order to make them independent of \( \lambda \). Then we can pass to the limit as \( \lambda \searrow 0 \) and conclude the existence proof. The last part of this section is devoted to the proof of the continuous dependence.

Finally, in Appendix A, the approximate problem for (P) and some auxiliary results are discussed.
2 Main results

In this section, our main result is stated. At first, we give our target system (P) some equations and conditions as follows: for any fixed constant $\tau > 0$, we have

\begin{align}
-\Delta \mu &= 0 \quad \text{a.e. in } Q, \quad (2.1) \\
\tau \partial_t u - \Delta u + \xi + \pi(u) &= f, \quad \xi \in \beta(u) \quad \text{a.e. in } Q, \quad (2.2) \\
u_t = u_{|\Gamma}, \quad \mu_t = \mu_{|\Gamma}, \quad \partial_t u_T + \partial_T \mu - \Delta_T \mu_T &= 0 \quad \text{a.e. on } \Sigma, \quad (2.3) \\
\mu_t = \partial_T u - \Delta_T u_T + \xi_T + \pi_T(u_T) - f_T, \quad \xi_T \in \beta_T(u_T) \quad \text{a.e. on } \Sigma, \quad (2.4) \\
u(0) &= u_0 \quad \text{a.e. in } \Omega, \quad u_T(0) = u_{0|\Gamma} \quad \text{a.e. on } \Gamma, \quad (2.5)
\end{align}

where $f : Q \to \mathbb{R}, f_T : \Sigma \to \mathbb{R}$ are given sources, $u_0 : \Omega \to \mathbb{R}, u_{0|\Gamma} : \Gamma \to \mathbb{R}$ are known initial data, $\beta$ stands for the subdifferential of the convex part $\beta$ and $\pi$ stands for the derivative of the concave perturbation $\pi$ of a double well potential $\mathcal{W}(r) = \beta(r) + \pi(r)$, defined for all $r$ in the domain of $\beta$. The same setting holds for $\beta_T$ and $\pi_T$.

Typical examples of the nonlinearities $\beta, \pi$ are given by

- $\beta(r) = r^3, \pi(r) = -r$ for all $r \in \mathbb{R}$, with $D(\beta) = \mathbb{R}$ for the prototype double well potential
  \[ \mathcal{W}(r) = \frac{1}{4}(r^2 - 1)^2. \]

- $\beta(r) = \ln((1 + r)/(1 - r)), \pi(r) = -2cr$ for all $r \in D(\beta)$, with $D(\beta) = (-1, 1)$ for the logarithmic double well potential
  \[ \mathcal{W}(r) = ((1 + r) \ln(1 + r) + (1 - r) \ln(1 - r)) - cr^2, \]
  where $c > 0$ is a large constant which breaks convexity.

- $\beta(r) = \partial I_{[-1,1]}(r), \pi(r) = -r$ for all $r \in D(\beta)$, with $D(\beta) = [-1, 1]$ for the singular potential
  \[ \mathcal{W}(r) = I_{[-1,1]}(r) - \frac{1}{2} r^2, \]
  where $I_{[-1,1]}$ is the indicator function on $[-1, 1]$.

Similar choices can be considered for $\beta_T, \pi_T$ and the related potential $\mathcal{W}_T$. What is important in our approach is that the potential on the boundary should dominate the potential in the bulk, that is, we prescribe a compatibility condition between $\beta$ and $\beta_T$ (see the later assumption (A5)) that forces the growth of $\beta$ to be controlled by the growth of $\beta_T$. A similar approach was taken in previous analyses, see [2, 4–6, 9, 28].

As a remark, $\tau > 0$ plays the role of a viscous parameter. Indeed, if $\tau = 0$, then equations (2.1) and (2.2) become the stationary problem in $Q$, namely, the quasi-static system. A natural question arises whether one can investigate also the case $\tau = 0$ or, in our framework, also study the singular limit as $\tau \searrow 0$. In our opinion, this is not a trivial question and deserves some attention and efforts. For the moment, we can just highlight it as open problem.

2.1 Definition of the solution

We treat problem (P) by a system of weak formulations. To do so, we introduce the spaces $H := L^2(\Omega), H_T := L^2(\Gamma), V := H^1(\Omega), V_T := H^1(\Gamma), W := H^2(\Omega), W_T := H^2(\Gamma)$, with usual norms and inner products; we denote them by $| | \cdot | |_H, | \cdot |_B, (\cdot, \cdot)_H, (\cdot, \cdot)_B$, and so on. Concerning these inner products and norms, we use the same notation for scalar and vectorial functions (e.g., typically gradients).

Moreover, we put $H := H \times H_T$ and $V := \{(z, z_T) \in V \times V_T : z_T = z_{|\Gamma}, \text{a.e. on } \Gamma\}$. Then $H$ and $V$ are Hilbert spaces with the inner products

\begin{align}
(u, z)_H &:= (u, z)_H + (u_T, z_T)_{H_T} \quad \text{for all } u := (u, u_T), z := (z, z_T) \in H, \\
(u, z)_V &:= (u, z)_V + (u_T, z_T)_{V_T} \quad \text{for all } u := (u, u_T), z := (z, z_T) \in V
\end{align}

and related norms. As a remark, if $z := (z, z_T) \in V$, then $z_T$ is the trace $z_{|\Gamma}$ of $z$ on $\Gamma$, while if $z := (z, z_T) \in H$,
then \( z \in H \) and \( z_t \in H_t \) are independent. Hereafter, we use the notation of a bold letter like \( z \) to denote the pair which corresponds to the letter, that is, \( (z, \ z_t) \) for \( z \).

It is easy to see that problem (P) has a structure of volume conservation on the boundary \( \Gamma \). Indeed, integrating the last equation in (2.3) on \( \Sigma \), and using (2.1) and (2.5), we obtain

\[
\int_\Gamma u_t(t) \, d\Gamma = \int_\Gamma u_0 \, d\Gamma \quad \text{for all } t \in [0, T];
\]  

(2.6)

hereafter, we put

\[
m_\Gamma := \frac{1}{|\Gamma|} \int_\Gamma u_0 \, d\Gamma,
\]

(2.7)

where \(|\Gamma| := \int_\Gamma 1 \, d\Gamma\). The space \( V^* \) denotes the dual of \( V \), and \( \langle \cdot, \cdot \rangle_{V^*} \) denotes the duality pairing between \( V^* \) and \( V \). Moreover, it is understood that \( H \) is embedded in \( V^* \) in the usual way, i.e., \( \langle u, \ z \rangle_{V^*} = (u, \ z)_H \) for all \( u \in H, \ z \in V \). Then we obtain \( V \hookrightarrow H \hookrightarrow V^* \), where \( \hookrightarrow \) stands for the dense and compact embedding, namely, \( (V, H, V^*) \) is a standard Hilbert triplet.

Under this setting, we define the solution of (P) as follows.

**Definition 2.1.** The triplet \((u, \mu, \xi)\) is called the solution of (P) if \( u = (u, \ u_t) \), \( \mu = (\mu, \mu_t) \), \( \xi = (\xi, \xi_t) \) satisfy

\[
\begin{align*}
&u \in H^1(0, T; H) \cap C([0, T]; V) \cap L^2(0, T; W), \\
u_t \in H^1_0(0, T; V_t^*) \cap L^\infty(0, T; V_t) \cap L^2(0, T; W_t), \\
\mu \in L^2(0, T; V), \quad \mu_t \in L^2(0, T; V_t), \\
\xi \in L^2(0, T; H), \quad \xi_t \in L^2(0, T; H_t), \\
u_{|z} = u_t, \quad \mu_{|z} = \mu_t \quad \text{a.e. on } \Sigma, \\
\xi \in B(u) \quad \text{a.e. in } Q, \quad \xi_t \in B(\mu_t) \quad \text{a.e. on } \Sigma,
\end{align*}
\]

solve

\[
\langle u_t^*, z_t \rangle_{V_t^*, V_t} + \int_\Omega \nabla u(t) \cdot \nabla z \, dx + \int_\Gamma \nabla_\Gamma u(t) \cdot \nabla_\Gamma z_t \, d\Gamma = 0
\]

(2.8)

for all \( z := (z, \ z_t) \in V \) and a.a. \( t \in (0, T) \), and

\[
\begin{align*}
\tau \partial_t u - \Delta u + \xi + \pi(u) &= f \quad \text{a.e. in } Q, \\
\mu_t &= \partial_t u - \Delta_t u_t + \xi_t + \pi_t(u_t) - f_t \quad \text{a.e. on } \Sigma, \\
u(0) &= u_0 \quad \text{a.e. in } \Omega, \quad u_t(0) = u_{0t} \quad \text{a.e. on } \Gamma.
\end{align*}
\]

(2.9) \hfill (2.10) \hfill (2.11)

**Remark 2.2.** Taking \( z := (1, 1) \) in the weak formulation (2.8), we see that (2.8) and (2.11) imply the mass conservation (2.6) on the boundary. Moreover, for any \( z \in \mathcal{D}(\Omega) \), taking \( z := (0, 0) \) in (2.8) and using the trace condition on \( \mu \), we deduce that

\[
-\Delta \mu(t) = 0 \quad \text{a.e. in } \Omega, \quad \mu_{|z}(t) = \mu_t(t) \quad \text{a.e. on } \Gamma,
\]

for a.a. \( t \in (0, T) \), whence the regularities \( \mu \in L^2(0, T; V) \), \( \mu_t \in L^2(0, T; V_t) \) allow us to infer the higher regularity \( \mu \in L^{3/2}(0, T; H^{3/2}(\Omega)) \). Moreover, the boundedness \( \Delta \mu (= 0) \) in \( L^2(0, T; H) \) gives us the property \( \partial_v \mu \in L^2(0, T; H_t) \), as well as

\[
\langle u_t^*, \partial_v \mu(t), z_t \rangle_{V_t^*, V_t} + \int_\Gamma \nabla_\Gamma \mu(t) \cdot \nabla_\Gamma z_t \, d\Gamma = 0 \quad \text{for all } z_t \in V_t,
\]

this is the weak formulation of (2.3).

### 2.2 Main theorems

The first result states the existence of the solution. To this end, we assume the following:

(A1) \( f \in L^2(0, T; H) \) and \( f_t \in W^{1,1}(0, T; H_t) \).

(A2) \( u_0 := (u_0, u_{0t}) \in V \).

The following theorems provide the existence and uniqueness of the solution of (P) for any \( T > 0 \).
The maximal monotone graphs \( \beta \) and \( \beta_T \) in \( \mathbb{R} \times \mathbb{R} \) are the subdifferentials \( \beta = \partial \hat{\beta} \) and \( \beta_T = \partial \hat{\beta}_T \) of some proper lower semicontinuous and convex functions \( \hat{\beta} \) and \( \hat{\beta}_T : \mathbb{R} \to [0, +\infty] \) satisfying \( \hat{\beta}(0) = \hat{\beta}_T(0) = 0 \), with some effective domains \( D(\hat{\beta}) \supset D(\hat{\beta}_T) \supset D(\hat{\beta_T}), \) respectively. This implies that \( 0 \in \beta(0) \) and \( 0 \in \beta_T(0) \).

Under assumptions (A2)–(A5), the maximal monotone graphs \( \beta \) and \( \beta_T \) are obtained from this theorem. Here, we just use the following regularity properties on the data:

\[ |\beta^r(r)| \leq g(|\beta^r_T(r)|) + c_0 \quad \text{for all } r \in D(\beta_T), \quad (2.12) \]

where \( \beta^r \) and \( \beta^r_T \) denote the minimal sections of \( \beta \) and \( \beta_T \).

The minimal section \( \beta^r \) of \( \beta \) is specified by \( \beta^r(r) = \{ r^* \in \beta(r) : |r^*| = \min_{s \in \beta(r)} |s| \} \) and the same definition applies to \( \beta^r_T \). These assumptions are the same as in [2, 5]. Concerning assumption (A1), let us note that the regularity conditions for \( f \) and \( f_T \) are not symmetric. In fact, we need more regularity for the source term on the boundary, since the equation on the boundary is of Cahn–Hilliard type, while the equation on the bulk turns out to be of the simpler Allen–Cahn type. Of course, here the condition \( \tau > 0 \) plays a role, and if the term \( \tau \partial_t u \) is not present in (2.2), then we would certainly need higher regularity for \( f \).

**Theorem 2.3.** Under assumptions (A1)–(A6), there exists a solution of problem \( (P) \).

The second result states the continuous dependence on the data. The uniqueness of the component \( u \) of the solution is obtained from this theorem. Here, \( u \) uses the following regularity properties on the data:

(A1)$'$ \( f \in L^2(0, T; V^*) \) and \( f_T \in L^2(0, T; V^*_T) \),

(A2)$'$ \( u_0 \in H \) and \( u_{0T} \in V^*_T \).

Then we obtain the continuous dependence on the data as follows:

**Theorem 2.4.** Under assumptions (A3)–(A4), let, for \( i = 1, 2, \beta^{\delta(i)}, \) \( u^{\delta(i)}_0 \) satisfy (A1)$'$, (A2)$'$ and assume that the corresponding solutions \( (u^{\delta(i)}, \mu^{\delta(i)}, \xi^{\delta(i)}) \) exist. Then there exists a positive constant \( C > 0 \), depending on \( L, L_T \) and \( T \), such that

\[
|u^{(1)}(t) - u^{(2)}(t)|_{H^1}^2 + |u^{(1)}_T(t) - u^{(2)}_T(t)|_{V^*_T}^2 + \int_0^t |u^{(1)}(s) - u^{(2)}(s)|_V^2 \, ds + \int_0^t |u^{(1)}_T(s) - u^{(2)}_T(s)|_{V^*_T}^2 \, ds \\
\leq C \left\{ |u^{(1)}_0 - u^{(2)}_0|_{H^1}^2 + |u^{(1)}_{0T} - u^{(2)}_{0T}|_{V^*_T}^2 + \int_0^t |u^{(1)}(s) - f^{(2)}(s)|_V^2 \, ds + \int_0^t |f^{(1)}(s) - f^{(2)}(s)|_{V^*_T}^2 \, ds \right\}.
\]

(2.13)

for all \( t \in [0, T] \).

## 3 Approximate problem and uniform estimates

In this section, we first consider an approximate problem for (P), and then we obtain uniform estimates. For each \( \delta, \lambda \in (0, 1) \), we introduce an approximate problem \( (P; \lambda, \delta) \) where the proof of the well-posedness of \( (P; \lambda, \delta) \) is given in Appendix A.

### 3.1 Moreau–Yosida regularization

For each \( \lambda \in (0, 1) \), we define \( \beta_{\lambda}, \beta_{T, \lambda} : \mathbb{R} \to \mathbb{R} \), along with the associated resolvent operators \( J_{\lambda}, J_{T, \lambda} : \mathbb{R} \to \mathbb{R} \), by

\[
\beta_{\lambda}(r) := \frac{1}{\lambda}(r - J_{\lambda}(r)) := \frac{1}{\lambda}(r - (I + \lambda \beta)^{-1}(r)), \\
\beta_{T, \lambda}(r) := \frac{1}{\lambda\rho}(r - J_{T, \lambda}(r)) := \frac{1}{\lambda\rho}(r - (I + \lambda \beta_T)^{-1}(r)).
\]
for all $r \in \mathbb{R}$, where $\rho > 0$ is the same constant as in assumption (2.12). Note that the two definitions are not symmetric, since in the second one, it is $\lambda \rho$ and not directly $\lambda$ to be used as approximation parameter; let us note that this adaptation comes from the previous work [2]. Now, we easily have $\beta_A(0) = \beta_{T, A}(0) = 0$. Moreover, the related Moreau–Yosida regularizations $\tilde{\beta}_A, \tilde{\beta}_{T, A}$ of $\beta, \tilde{\beta}_T : \mathbb{R} \to \mathbb{R}$ fulfill
\[
\tilde{\beta}_A(r) := \inf_{s \in \mathbb{R}} \left\{ \frac{1}{2\lambda} |r - s|^2 + \beta(s) \right\} = \frac{1}{2\lambda} |r - f_A(r)|^2 + \tilde{\beta}(f_A(r)) = \int_0^r \beta_A(s) \, ds,
\]
\[
\tilde{\beta}_{T, A}(r) := \inf_{s \in \mathbb{R}} \left\{ \frac{1}{2\lambda} |r - s|^2 + \tilde{\beta}(s) \right\} = \int_0^r \beta_{T, A}(s) \, ds \quad \text{for all } r \in \mathbb{R}.
\]
It is well known that $\beta_A$ is Lipschitz continuous with Lipschitz constant $1/\lambda$ and $\beta_{T, A}$ is also Lipschitz continuous with constant $1/(\lambda \rho)$. In addition, for each $\lambda \in (0, 1)$, we have the standard properties
\[
|\beta_A(r)| \leq |\beta(r)|, \quad |\beta_{T, A}(r)| \leq |\beta_T(r)|, \quad 0 \leq \beta_A(r) \leq \beta(r), \quad 0 \leq \beta_{T, A}(r) \leq \beta_T(r) \quad \text{for all } r \in \mathbb{R}. \tag{3.1}
\]
Let us point out that, using [2, Lemma 4.4], we have
\[
|\beta_A(r)| \leq \rho |\beta_{T, A}(r)| + c_0 \quad \text{for all } r \in \mathbb{R}, \tag{3.2}
\]
for all $\lambda \in (0, 1)$, with the same constants $\rho$ and $c_0$ as in (2.12).

Now for each $\varepsilon \in (0, 1)$, let $f_{\varepsilon} := (f_\varepsilon, f_{T, \varepsilon})$ and $u_{0, \varepsilon} := (u_{0, \varepsilon}, u_{0T, \varepsilon})$ be smooth approximations for $f$ and $u_0$, so that $f_{\varepsilon} \in H^1(0, T; \mathcal{H})$ with $f_{\varepsilon}(0) \in \mathcal{V}$ and $u_{0, \varepsilon} \in W \cap \mathcal{V}$ with $(-\Delta u_{0, \varepsilon}, \partial u_{0, \varepsilon} - \Delta_T u_{0T, \varepsilon}) \in \mathcal{V}$ satisfying
\[
f_{\varepsilon} \to f \quad \text{strongly in } L^2(0, T; \mathcal{H}) \quad \text{as } \varepsilon \to 0, \tag{3.3}
\]
\[
|f_{T, \varepsilon} - f_T|_{L^2(0, T; \mathcal{H})} \leq \varepsilon^{1/2} C_0 \quad \text{for all } \varepsilon \in (0, 1), \tag{3.4}
\]
\[
u_{0, \varepsilon} \to u_0 \quad \text{strongly in } \mathcal{V} \quad \text{as } \varepsilon \to 0, \tag{3.5}
\]
\[
\int_{\Omega} \tilde{\beta}_A(u_{0, \varepsilon}) \, dx \leq c_0, \quad \int_{\Gamma} \tilde{\beta}_{T, A}(u_{0T, \varepsilon}) \, d\Gamma \leq \left( 1 + \frac{\varepsilon^{1/2}}{\lambda} \right) c_0 \quad \text{for all } \varepsilon \in (0, 1), \tag{3.6}
\]
where $C_0$ is a positive constant independent of $\varepsilon, \lambda \in (0, 1)$. Indeed, $f_{\varepsilon}$ and $u_{0, \varepsilon}$ satisfying (3.3)–(3.6) are given in Appendix A. Hereafter, we use $C^* := (1 + \varepsilon^{1/2}/\lambda)^{1/2}$, which satisfies $C^* \searrow 1$ as $\varepsilon \searrow 0$ for $\lambda \in (0, 1)$. Then we can solve the following auxiliary problem.

**Proposition 3.1.** Under assumptions (A1)–(A6), for each $\varepsilon, \lambda \in (0, 1)$, there exists a unique pair
\[
u_{\lambda, \varepsilon} := (u_{\lambda, \varepsilon}, u_{T, \lambda, \varepsilon}) \in W^{1, \infty}(0, T; \mathcal{H}) \cap H^1(0, T; \mathcal{V}) \cap L^\infty(0, T; \mathcal{W}), \tag{3.7}
\]
\[
u_{\lambda, \varepsilon} := (\mu_{\lambda, \varepsilon}, \mu_{T, \lambda, \varepsilon}) \in L^\infty(0, T; \mathcal{V}) \cap L^2(0, T; \mathcal{W}), \tag{3.8}
\]

satisfying
\[
\varepsilon \partial_T u_{\lambda, \varepsilon} - \Delta u_{\lambda, \varepsilon} = 0 \quad \text{a.e. in } Q, \tag{3.9}
\]
\[
\varepsilon \mu_{\lambda, \varepsilon} = \tau \partial_T u_{\lambda, \varepsilon} - \Delta u_{\lambda, \varepsilon} + \beta_A(u_{\lambda, \varepsilon}) + \pi(u_{\lambda, \varepsilon}) - f_{\varepsilon} \quad \text{a.e. in } Q, \tag{3.10}
\]
\[
\partial_T u_{T, \lambda, \varepsilon} + \partial_T \mu_{\lambda, \varepsilon} - \Delta u_{T, \lambda, \varepsilon} + \beta_{T, A}(u_{\lambda, \varepsilon}) + \pi_T(u_{\lambda, \varepsilon}) - f_{T, \varepsilon} \quad \text{a.e. on } \Sigma, \tag{3.11}
\]
\[
u_{T, \lambda, \varepsilon} = \varepsilon \partial_T u_{\lambda, \varepsilon} + \partial_T \mu_{\lambda, \varepsilon} + \beta_{T, A}(u_{T, \lambda, \varepsilon}) + \pi_T(u_{T, \lambda, \varepsilon}) - f_{T, \varepsilon} \quad \text{a.e. on } \Sigma, \tag{3.12}
\]
\[
u_{\lambda, \varepsilon}(0) = u_{0, \varepsilon} \quad \text{a.e. in } \Omega, \quad u_{T, \lambda, \varepsilon}(0) = u_{0T, \varepsilon} \quad \text{a.e. on } \Gamma. \tag{3.13}
\]
As a remark, the trace conditions is included in the regularities (3.7)–(3.8). The strategy of the proof of this proposition is based on the previous work [5, Theorems 2.2, 4.2]. It is given in Appendix A.
3.2 A priori estimates

In order to obtain the uniform estimates independent of the approximate parameter $\varepsilon$ and $\lambda$, we use the following type of Poincaré–Wirtinger inequalities (see, e.g., [22, 29]): there exists a positive constant $C_F$ such that

$$|z|_{H^2}^2 \leq C_F \left\{ \int_{\Omega} |\nabla z|^2 \, dx + \int_{\Gamma} |z_t|^2 \, d\Gamma \right\} \quad \text{for all } z \in V, \quad (3.14)$$

$$|z_{t1}|_{H^2}^2 \leq C_F \left\{ \int_{\Gamma} |\nabla z_{t1}|^2 \, d\Gamma \right\} \quad \text{for all } z_{t1} \in V_{t1}, \quad (3.15)$$

$$|z|_{V}^2 \leq C_F \left\{ \int_{\Omega} |\nabla z|^2 \, dx + \int_{\Gamma} |\nabla z_t|^2 \, d\Gamma \right\} \quad \text{for all } z \in V, \quad (3.16)$$

Moreover, from the compactness inequality, recalled in [26, Chapter 1, Lemma 5.1] or [30, Section 8, Lemma 8], for each $\delta > 0$, there exists a positive constant $C_\delta$ depending on $\delta$ such that

$$|z|_{H_{1-\delta}(\Omega)}^2 \leq \delta |z|_V^2 + C_\delta |z|_{H}^2 \quad \text{for all } z \in V \text{ and all } \alpha \in (0, 1), \quad (3.17)$$

$$|z_{t1}|_{H_{1-\delta}(\Gamma)}^2 \leq \delta |z_{t1}|_{V_{t1}}^2 + C_\delta |z_{t1}|_{H_{1-\delta}(\Gamma)}^2 \quad \text{for all } z_{t1} \in V_{t1}, \quad (3.18)$$

because we have the compact embeddings $V \hookrightarrow \hookrightarrow H^{1-\alpha}(\Omega) \hookrightarrow \hookrightarrow H$ for $\alpha \in (0, 1)$, (see, e.g., [27, Chapter 1, Theorem 16.1]) and $V_t \hookrightarrow \hookrightarrow H_t \hookrightarrow \hookrightarrow V_t^*$, respectively. Next, from the standard theorem for the trace operators $\gamma_0(z) := z_t$ of $\gamma_0 : V \to H^{1/2}(\Gamma)$ and $\gamma_1 : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$, $\gamma_1 : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \subset H$ for $\alpha \in (0, 1/2)$ (see, e.g., [23, Chapter 2, Theorem 2.24], [29, Chapter 2, Theorem 5.5]), we see that there exists a positive constant $c_1$ such that

$$|z|_{H^{1/2}(\Gamma)}^2 \leq |z|_{V}^2 \quad \text{for all } z \in V, \quad (3.19)$$

for $\alpha \in (0, 1/2)$. Moreover, the boundedness of some recovering operator $\mathcal{R} : H^{1/2}(\Gamma) \to V$ of the trace $\gamma_0$ gives us

$$|\mathcal{R} z_{t1}|_{V}^2 \leq c_2 |z_{t1}|_{H^{1/2}(\Gamma)}^2 \leq c_2 |z_{t1}|_{V_{t1}}^2 \quad \text{for all } z_{t1} \in V_{t1}, \quad (3.20)$$

where $c_2$ is a positive constant (see, e.g., [23, Chapter 2, Theorem 2.24], [29, Chapter 2, Theorem 5.7]).

**Lemma 3.2.** There exist two positive constants $M_1$ and $M_2$, depending on $\tau$ but independent of $\varepsilon$ and $\lambda \in (0, 1)$, such that

$$|u_{\lambda,e}|_{L^\infty(0,\tau;V)} \leq C^* M_1, \quad |u_{\Gamma,e}|_{L^\infty(0,\tau;H_t)} \leq C^* M_1, \quad (3.21)$$

$$|\partial_s u_{\lambda,e}|_{L^1(0,\tau;H)} + \varepsilon^{1/2} |\partial_t u_{\Gamma,e}|_{L^2(0,\tau;H_t)} + |\nabla_t u_{\Gamma,e}|_{L^2(0,\tau;H_t)} + |\tilde{\beta}_\lambda(u_{\lambda,e})|_{L^2(0,\tau;L^2(\Gamma))}$$

$$+ |\tilde{\beta}_\lambda(u_{\Gamma,e})|_{L^2(0,\tau;L^2(\Gamma))} + |\nabla_t u_{\Gamma,e}|_{L^2(0,\tau;H_t)} + |\nabla_t u_{\Gamma,e}|_{L^2(0,\tau;H_t)} \leq C^* M_2. \quad (3.22)$$

**Proof.** We test (3.10) at time $s$ by $\partial_s u_{\lambda,e}$, the time derivative of $u_{\lambda,e}$. Integrating the resultant over $\Omega \times (0, t)$ leads to

$$\int_0^t \left[ \partial_s u_{\lambda,e}(s)|_H^2 \right] \, ds + \frac{1}{2} \int_{\Omega} |\nabla u_{\lambda,e}(t)|_{H_t}^2 \, dx + \int_{\Omega} \tilde{n}(u_{\lambda,e}(t)) \, dx$$

$$- \int_0^t \left( \partial_s u_{\lambda,e}(s), \partial_s u_{\lambda,e}(s) \right)_H \, ds - \int_0^t (f_\varepsilon(s), \partial_s u_{\lambda,e}(s))_H \, ds$$

$$= \varepsilon \left( \mu_{\lambda,e}(s), \partial_s u_{\lambda,e}(s) \right)_H \, ds + \frac{1}{2} \int_{\Omega} |\nabla u_{\lambda,e}|_{H_t}^2 \, dx + \int_{\Omega} \tilde{n}(u_{\lambda,e}) \, dx \quad (3.23)$$
for all \( t \in [0, T] \). Next, testing (3.12) by \( \partial_u u_{\lambda,e} \) and integrating the resultant over \( \Gamma \times (0, t) \), we obtain

\[
- \int_0^t \left( \partial_u u_{\lambda,e}(s), \partial_u u_{\lambda,e}(s) \right)_H ds = \int_0^t \left( \partial_u u_{\lambda,e}(s), \partial_u u_{\lambda,e}(s) \right)_H ds - \int_0^t \left( \mu_{\lambda,e}(s), \partial_u u_{\lambda,e}(s) \right)_H ds
+ \frac{1}{2} | \nabla u_{\lambda,e}(t) |^2_H - \frac{1}{2} | \nabla u_{0,e} |^2_H + \int \bar{\Omega}_{\lambda}(u_{\lambda,e}(t)) d\Gamma
+ \int \tilde{\eta}_\tau(u_{\lambda,e}(t)) d\Gamma - \int \tilde{\beta}_{\lambda}(u_{0,e}) d\Gamma
- \int \tilde{\eta}_\tau(u_{0,e}) d\Gamma - \int (f_{\tau,e}(s), \partial_u u_{\lambda,e}(s))_H ds \tag{3.24}
\]

for all \( t \in [0, T] \). On the other hand, testing (3.9) by \( \mu_{\lambda,e} \), testing (3.11) by \( \mu_{\tau,e} \), and adding them, we infer that

\[
\int_0^t \left( \mu_{\lambda,e}(s), \partial_u u_{\lambda,e}(s) \right)_H ds + \int_0^t \left( \mu_{\tau,e}(s), \partial_u u_{\lambda,e}(s) \right)_H ds = - \int_0^t | \nabla \mu_{\lambda,e}(s) |^2_H ds - \int_0^t | \nabla \mu_{\tau,e}(s) |^2_H ds \tag{3.25}
\]

for all \( t \in [0, T] \). Combining (3.23)–(3.25) and using (3.1), we have

\[
\int_0^t \left( \partial_u u_{\lambda,e}(s) \right)_H ds + \int_0^t \left( \partial_u u_{\lambda,e}(s) \right)_H ds + \frac{1}{2} | \nabla u_{\lambda,e}(t) |^2_H + \int \bar{\Omega}(u_{\lambda,e}(t)) dx
+ \frac{1}{2} | \nabla u_{0,e} |^2_H + \int \tilde{\eta}(u_{\lambda,e}(t)) dx + \int \tilde{\eta}(u_{0,e}) dx
+ \frac{1}{2} | \nabla u_{0,e} |^2_H + \int \bar{\Omega}_{\lambda}(u_{\lambda,e}(t)) d\Gamma + \int \tilde{\eta}(u_{\lambda,e}(t)) d\Gamma + \int \tilde{\eta}(u_{0,e}) d\Gamma
+ \int (f_{\tau,e}(s), \partial_u u_{\lambda,e}(s))_H ds + \int (f_{\tau,e}(s), \partial_u u_{\lambda,e}(s))_H ds \tag{3.26}
\]

for all \( t \in [0, T] \). Therefore, in order to estimate the right-hand side of (3.26), we prepare the estimate of \(|u_{\lambda,e}(s)|_H \). Indeed, from the Young inequality, we see that

\[
\frac{1}{2} |u_{\lambda,e}(s)|^2_H = \int_0^t ( \partial_u u_{\lambda,e}(s), u_{\lambda,e}(s) )_H ds + \frac{1}{2} |u_{0,e} |^2_H
\leq \frac{\delta}{2} \int_0^t | \partial_u u_{\lambda,e}(s) |^2_H ds + \frac{1}{2} |u_{\lambda,e} |^2_H ds + \frac{1}{2} |u_{0,e} |^2_H \tag{3.27}
\]

for all \( t \in [0, T] \) and some \( \delta > 0 \). Now, from (A4), we can use the fact that \(|\pi(r)| = |\pi(r) - \pi(0)| \leq L|r| \), and then we deduce

\[
|\tilde{\eta}(r)| \leq \int_0^r |\eta(l) | dl \leq \frac{L}{2} r^2 \quad \text{for all } r \in \mathbb{R}.
\]

Therefore, by taking \( \delta := \tau/(5L) \) in (3.27), we have

\[
\int_\Omega |\tilde{\eta}(u_{\lambda,e}(t)) | dx \leq \int_\Omega |u_{\lambda,e}(t) |^2 dx \leq \frac{\tau}{10} \int_0^t | \partial_u u_{\lambda,e}(s) |^2_H ds + \frac{5L^2}{2\tau} \int_0^t |u_{\lambda,e}(s) |^2_H ds + \frac{L}{2} |u_{0,e} |^2_H \tag{3.28}
\]
and, analogously,
\[
\left| \tilde{\eta}(u_{0,\varepsilon}) \right| dx \leq \frac{L}{2} \int_\Omega |u_{0,\varepsilon}|^2 dx, \quad \int_\Gamma |\tilde{\eta}_\Gamma(u_{0,\varepsilon})| d\Gamma \leq \frac{L}{2} \int_\Gamma |u_{0,\varepsilon}|^2 d\Gamma. \tag{3.29}
\]

Additionally, by using (3.17), (3.19) and (3.27) with \( \delta := \tau/(10C_\delta) \), we get
\[
\int_\Gamma |\tilde{\eta}_\Gamma(u_{\Gamma,\lambda,\varepsilon}(t))| d\Gamma \leq \frac{L}{2} \int_\Gamma |u_{\Gamma,\lambda,\varepsilon}(t)|^2 d\Gamma \leq \frac{c_1L\tau}{2} |u_{\lambda,\varepsilon}(t)|^2_{H^{1-\alpha}(\Omega)} \leq C_\delta |u_{\lambda,\varepsilon}(t)|^2_H \leq \delta |u_{\lambda,\varepsilon}(t)|^2_V + \frac{10C_\delta^2}{\tau} \int_0^t |\partial_s u_{\lambda,\varepsilon}(s)|^2_H ds + C_\delta |u_{0,\varepsilon}|^2_H \tag{3.30}
\]
for some \( \alpha \in (0, 1/2) \) and \( \delta > 0 \). Moreover, from the Young inequality, it turns out that
\[
\int_0^t (f_\varepsilon(s), \partial_s u_{\lambda,\varepsilon}(s)) ds \leq \frac{\tau}{10} \int_0^t |\partial_s u_{\lambda,\varepsilon}(s)|^2_H ds + \frac{5}{2\tau} \int_0^t |f_\varepsilon(s)|^2_H ds, \tag{3.31}
\]
\[
\int_0^t (f_{\Gamma,\lambda}(s), \partial_s u_{\Gamma,\lambda,\varepsilon}(s))_H ds \leq \frac{\varepsilon}{2} \int_0^t |\partial_s u_{\Gamma,\lambda,\varepsilon}(s)|^2_H ds + \frac{1}{2\varepsilon} \int_0^t |f_{\Gamma,\varepsilon}(s)|^2_H ds \tag{3.32}
\]
for all \( t \in [0, T] \). Therefore, collecting (3.28)–(3.32), adding \( (1/2)|u_{\lambda,\varepsilon}(s)|^2_H \) to both sides of (3.26) and using (3.27) with \( \delta := \tau/5 \), we obtain
\[
\frac{\tau}{2} \int_0^t |\partial_s u_{\lambda,\varepsilon}(s)|^2_H ds + \frac{\varepsilon}{2} \int_0^t |\partial_s u_{\Gamma,\lambda,\varepsilon}(s)|^2_H ds + \frac{1}{2} |u_{\lambda,\varepsilon}(t)|^2_V + \int_\Omega \tilde{\beta}_\lambda(u_{\lambda,\varepsilon}(t)) dx
\]
\[
+ \frac{1}{2} |\nabla u_{\Gamma,\lambda,\varepsilon}(t)|^2_H + \int_\Gamma \tilde{\beta}_{\Gamma,\lambda}(u_{\Gamma,\lambda,\varepsilon}(t)) d\Gamma + \int_0^t |\nabla u_{\lambda,\varepsilon}(s)|^2_H ds + \int_0^t |\nabla u_{\Gamma,\lambda,\varepsilon}(s)|^2_H ds \leq \frac{5L^2}{2\tau} \int_0^t |u_{\lambda,\varepsilon}(s)|^2_H ds + L|u_{0,\varepsilon}|^2_H
\]
\[
+ \frac{5}{2\tau} \int_0^t |f_\varepsilon(s)|^2_H ds + \frac{1}{2} |u_{0,\varepsilon}|^2_V + \int_\Gamma \tilde{\beta}_{\Gamma,\lambda}(u_{0,\varepsilon}) d\Gamma + \delta |u_{\lambda,\varepsilon}(t)|^2_V
\]
\[
+ \frac{10C_\delta^2}{\tau} \int_0^t |u_{\lambda,\varepsilon}(s)|^2_H ds + C_\delta |u_{0,\varepsilon}|^2_H + \frac{L}{2} |u_{0,\varepsilon}|^2_H + \frac{1}{2\varepsilon} \int_0^t |f_{\Gamma,\varepsilon}(s)|^2_H ds \tag{3.33}
\]
for all \( t \in [0, T] \). Thus, taking \( \delta := 1/4 \), and using (3.3)–(3.6) and the Gronwall inequality, we see that there exists a positive constant \( M_1 \), depending on \( |u_0|_V, C_0, L, |f|_{L^2(0, T; H)}, |u_{0,r}|_V \), \( L, |f|_{L^2(0, T; H)} \) and \( \tau \), in which \( M_1 \to +\infty \) as \( \tau \to 0 \), independent of \( \varepsilon \) and \( \lambda \), such that
\[
|u_{\lambda,\varepsilon}|_{L^\infty(0, T; V)} \leq C^* M_1.
\]
Moreover, using this, (3.19) implies
\[
|u_{\Gamma,\lambda,\varepsilon}|_{L^\infty(0, T; H)} \leq C_1^{1/2} C^* M_1,
\]
and from (3.33) we obtain estimate (3.22). \( \square \)
By comparison, the following estimates for $\Delta\mu_{\lambda,\varepsilon}$ and $u'_{\lambda,\varepsilon} := (\partial_t u_{\lambda,\varepsilon}, \partial_t u_{\Gamma,\lambda,\varepsilon})$ are obtained.

**Lemma 3.3.** Let $M_2$ be the same constant as in Lemma 3.2. Then, for each $\varepsilon$ and $\lambda \in (0, 1]$, the following estimates hold:

\[
|\Delta\mu_{\lambda,\varepsilon}|_{L^2(0,T;H)} \leq \varepsilon C^* M_2, \\
|u'_{\lambda,\varepsilon}|_{L^2(0,T;V')} \leq 2\varepsilon C^* M_2.
\]  

**Proof.** From (3.9), we easily see that

\[
|\Delta\mu_{\lambda,\varepsilon}|_{L^2(0,T;H)} = |\varepsilon \partial_t u_{\lambda,\varepsilon}|_{L^2(0,T;H)} \leq \varepsilon C^* M_2.
\]

Next, we separate $u'_{\lambda,\varepsilon}$ as follows:

\[
(u'_{\lambda,\varepsilon}) = ((1 - \varepsilon) \partial_t u_{\lambda,\varepsilon}, 0) + (\varepsilon \partial_t u_{\lambda,\varepsilon}, \partial_t u_{\Gamma,\lambda,\varepsilon}).
\]

Then the first term of the right-hand side is estimated as follows:

\[
|((1 - \varepsilon) \partial_t u_{\lambda,\varepsilon}, 0)|_{L^2(0,T;V')} = |(1 - \varepsilon) \partial_t u_{\lambda,\varepsilon}|_{L^2(0,T;H)} \leq (1 - \varepsilon) C^* M_2 \leq C^* M_2.
\]  

Moreover, for the second term, we see that

\[
|(\varepsilon \partial_t u_{\lambda,\varepsilon}, \partial_t u_{\Gamma,\lambda,\varepsilon})|_{L^2(0,T;V')} \leq C^* M_2.
\]  

Indeed, from (3.9) and (3.11), we have

\[
\int_0^T (\varepsilon \partial_t u_{\lambda,\varepsilon}(t), \eta(t))_H \, dt + \int_0^T (\partial_t u_{\Gamma,\lambda,\varepsilon}(t), \eta_\Gamma(t))_{H_\Gamma} \, dt = \int_0^T (\nabla u_{\lambda,\varepsilon}(t), \nabla \eta(t))_H \, dt + \int_0^T (\nabla u_{\Gamma,\lambda,\varepsilon}(t), \nabla_\Gamma \eta(t))_{H_\Gamma} \, dt
\]

for all $\eta \in L^2(0,T;V)$. Therefore, estimate (3.22) for $\mu_{\lambda,\varepsilon}$ and $\mu_{\Gamma,\lambda,\varepsilon}$ imply (3.37). Thus, using (3.36)–(3.37), we show (3.35). \qed

We have obtained the uniform estimates (3.21), (3.22), (3.34), (3.35) provided in Lemmas 3.2 and 3.3 independent of $\varepsilon$ and $\lambda \in (0, 1]$, actually $C^* \searrow 1$ as $\varepsilon \searrow 0$. They can be used throughout this paper by considering the limiting procedure $\varepsilon \searrow 0$ for each fixed $\lambda \in (0, 1]$, and next the limiting procedure $\lambda \searrow 0$.

## 4 Proof of the main theorem

In this section, we prove Theorem 2.3.

### 4.1 Additional uniform estimate independent of $\varepsilon \in (0, 1]$

In this subsection, we obtain additional uniform estimates independent of $\varepsilon \in (0, 1]$, which may depend on $\lambda \in (0, 1]$. Therefore, we use them only in the next subsection to consider the limiting procedure as $\varepsilon \searrow 0$.

**Lemma 4.1.** There exist two positive constants $M_3(\lambda)$ and $M_6(\lambda)$, depending on $\lambda \in (0, 1]$ but independent of $\varepsilon \in (0, 1]$, such that

\[
|\beta_{\lambda}(u_{\lambda,\varepsilon})|_{L^2(0,T;H)} \leq M_3(\lambda), \quad |\beta_{\Gamma,\lambda}(u_{\Gamma,\lambda,\varepsilon})|_{L^2(0,T;H)} \leq M_3(\lambda), \quad (4.1)
\]

\[
|\mu_{\Gamma,\lambda,\varepsilon}|_{L^2(0,T;V)} \leq M_6(\lambda), \quad |\mu_{\lambda,\varepsilon}|_{L^2(0,T;V)} \leq M_6(\lambda). \quad (4.2)
\]

**Proof.** From the Lipschitz continuity of $\beta_{\lambda}$ and $\beta_{\Gamma,\lambda}$, we see from (3.21) that there exists a positive constant $M_3(\lambda)$, which is proportional to the Lipschitz constants $1/\lambda$ of $\beta_{\lambda}$ and $1/(\lambda \theta)$ of $\beta_{\Gamma,\lambda}$, such that (4.1) holds.
Next, let us point out the variational equality, deduced from (3.10) and (3.12):
\[
\varepsilon(\mu_{\lambda,e}(s), z)_H + (\mu_{\Gamma,\lambda,e}(s), z)_H = \tau(\partial_s u_{\lambda,e}(s), z)_H + (\nabla u_{\lambda,e}(s), \nabla z)_H \\
+ (\beta_{\lambda}(u_{\lambda,e}(s)) + \tau u_{\lambda,e}(s)) - f(e)(s), z)_H \\
+ \varepsilon(\partial_\lambda u_{\Gamma,\lambda,e}(s), z)_H + (\nabla u_{\Gamma,\lambda,e}(s), \nabla z)_H \\
+ (\beta_{\Gamma,\lambda}(u_{\Gamma,\lambda,e}(s)) + \tau u_{\Gamma,\lambda,e}(s)) - f_{\Gamma,\lambda}(s), z)_H
\]
(4.3)
for all \( z \in V \) and a.a. \( s \in (0, T) \). Taking now \( z = \mu_{\lambda,e}(s) \) in (4.3), integrating with respect to time, we infer, with the help of Poincaré–Wirtinger inequality (3.14), that
\[
\varepsilon \int_0^t |\mu_{\lambda,e}(s)|^2_H ds + \int_0^t |\mu_{\Gamma,\lambda,e}(s)|^2_H ds
\]
\[
\leq C_T \| \varepsilon \| \| \beta_{\lambda}(u_{\lambda,e}) + \tau u_{\lambda,e} - f_{\varepsilon}(s) \|_{L^2(0,T;H)}^2 + |\nabla u_{\lambda,e}(s), n|_{L^2_H(0,T;H)} \\
+ C_T |\beta_{\lambda}(u_{\lambda,e}) + \tau u_{\lambda,e} - f_{\varepsilon}(s)\|_{L^2(0,T;H)}^2 + |\nabla u_{\lambda,e}(s), n|_{L^2_H(0,T;H)} \\
+ \varepsilon |\partial_\lambda u_{\Gamma,\lambda,e}(s), n|_{L^2(0,T;H)}^2 + |\nabla u_{\Gamma,\lambda,e}(s), n|_{L^2_H(0,T;H)}^2 \\
+ |\beta_{\Gamma,\lambda}(u_{\Gamma,\lambda,e}(s)) + \tau u_{\Gamma,\lambda,e}(s) - f_{\Gamma,\lambda}(s), n|_{L^2_H(0,T;H)}^2
\]
for all \( t \in [0, T] \). Then, using the Young inequality along with (3.3), (3.21), (3.22) and (4.1), we deduce the uniform estimate of \( |\mu_{\lambda,e}(s)|_{e(0,1)} \) in \( L^2(0, T; V_1) \). Next, with the help of the Poincaré–Wirtinger inequality (3.14) again, we deduce the uniform estimate of \( |\mu_{\lambda,e}(s)|_{e(0,1)} \) in \( L^2(0, T; H) \). Thus, combining the resultant with (3.22), we see that there exists a positive constant \( M_5(\lambda) \), depending on \( M_3(\lambda) \) but independent of \( \varepsilon \in (0, 1) \), such that (4.2) holds.

**Lemma 4.2.** There exists a positive constant \( M_5(\lambda) \), depending on \( \lambda \in (0, 1) \) but independent of \( \varepsilon \in (0, 1) \), such that
\[
|\mu_{\lambda,e}(s)|_{L^2(0, T; W^1_2)} + |\mu_{\Gamma,\lambda,e}(s)|_{L^2(0, T; W^1_2)} \leq M_5(\lambda).
\]

**Proof.** We can compare the terms in (3.10) and conclude that \( |\Delta u_{\lambda,e}|_{L^2(0, T; H)} \) is uniformly bounded. Hence, applying the theory of elliptic regularity (see, e.g., [23, Theorem 3.2, p. 1.79]), we have that
\[
|\mu_{\lambda,e}(s)|_{L^2(0, T; H^{1/2}(\Omega))} \leq \tilde{M}_5(\lambda)
\]
for some positive constant \( \tilde{M}_5(\lambda) \), and owing to both the uniform bounds, we see that
\[
|\partial_\lambda u_{\lambda,e}|_{L^2(0, T; H)} \leq \tilde{M}_5(\lambda)
\]
Next, by comparison in (3.12), \( |\Delta u_{\Gamma,\lambda,e}|_{L^2(0, T; H)} \) is uniformly bounded and, consequently (see, e.g., [21, Section 4.2]),
\[
|\mu_{\Gamma,\lambda,e}(s)|_{L^2(0, T; W^1_2)} \leq (|\mu_{\Gamma,\lambda,e}(s)|_{L^2(0, T; W^1_2)}^2 + |\Delta u_{\Gamma,\lambda,e}|_{L^2(0, T; H)}^2)^{1/2} \leq M_5(\lambda)
\]
for some constant \( M_5(\lambda) \). Then, using the theory of elliptic regularity (see, e.g., [23, Theorem 3.2, p. 1.79]), we get (4.4).

**4.2 Passage to the limit as \( \varepsilon \searrow 0 \)**

In this subsection, we pass to the limit in the approximating problem as \( \varepsilon \searrow 0 \). Indeed, owing to the estimates stated in Lemmas 3.2, 3.3, 4.1 and 4.2, there exist a subsequence of \( \varepsilon \) (not relabeled) and some limit functions...
as \( \varepsilon \searrow 0 \). From (4.5) and (4.7), using well-known compactness results (see, e.g., [30, Section 8, Corollary 4]), we obtain

\[
\begin{align*}
\mu_{\Gamma, \lambda, e} &\to \mu_{\Gamma, \lambda} \quad \text{weakly in } L^2(0, T; \mathcal{V}), \\
\mu_{\Gamma, \lambda, e} &\to \mu_{\Gamma, \lambda} \quad \text{weakly in } L^2(0, T; \mathcal{V}^*), \\
\mu_{\Gamma, \lambda, e} &\to \mu_{\Gamma, \lambda} \quad \text{weakly in } L^2(0, T; \mathcal{V}^*_T), \\
\mu_{\Gamma, \lambda, e} &\to \mu_{\Gamma, \lambda} \quad \text{weakly in } L^2(0, T; V^*_T). 
\end{align*}
\]

Moreover, from the a priori estimates shown in Lemmas 3.2 and 3.3, we have

\[
\begin{align*}
|\mu_{\Gamma, \lambda}|_{L^\infty(0, T; \mathcal{V})} &\leq M_1, \\
|\mu_{\Gamma, \lambda}|_{L^\infty(0, T; \mathcal{V}^*)} &\leq c_1^{1/2} M_1, \\
|\mu_{\Gamma, \lambda}|_{L^\infty(0, T; \mathcal{V}^*_T)} &\leq M_2, \\
|\mu_{\Gamma, \lambda}|_{L^2(0, T; \mathcal{V}^*_T)} &\leq 2M_2, \\
|\mu_{\Gamma, \lambda}|_{L^2(0, T; \mathcal{V}^*_T)} &\leq 2M_2. 
\end{align*}
\]

for all \( \varepsilon \searrow 0 \). As a remark, taking \( z = 1, z_T = 1 \) in (4.18) and using (4.11), (4.17), we obtain that

\[
\int_\Gamma u_{\lambda}(t) \, d\Gamma = \int_\Gamma u_{0\Gamma} \, d\Gamma \quad \text{for all } t \in [0, T].
\]
4.3 Proof of Theorem 2.3

In this subsection, we prove the main theorem. To do so, we are going to produce estimates independent of \( \lambda \), and then pass to the limit as \( \lambda \searrow 0 \). The point of emphasis is the effective usage of the mean value zero function.

**Lemma 4.3.** There exists a positive constant \( M_6 \), independent of \( \lambda \in (0, 1] \), such that

\[
|\beta_\lambda(u_\lambda)|_{L^2(0,T;L^1(\Omega))} + |\beta_{\Gamma,\lambda}(u_{\Gamma,\lambda})|_{L^2(0,T;L^1(\Gamma))} \leq M_6.
\]

**(Proof.** Recall (2.7) and take different condition. Moreover, due to (3.14) and (3.15), there exists a positive constant \( (\cdot, \cdot) \), \( r_1 \), and \( \mu_{\Gamma,\lambda}(s), u_{\Gamma,\lambda}(s) - m_{\Gamma} \) in (4.19). Then we have

\[
\begin{align*}
\tau(\partial_t u_\lambda(s), u_\lambda(s) - m_{\Gamma})_H + |\nabla u_\lambda(s)|_H^2 + (\beta_\lambda(u_\lambda(s)), u_\lambda(s) - m_{\Gamma})_H \\
+ (\pi(u_\lambda(s) - f(s), u_\lambda(s) - m_{\Gamma})_H + |\nabla u_{\Gamma,\lambda}(s)|_H^2 + (\beta_{\Gamma,\lambda}(u_{\Gamma,\lambda}(s)), u_{\Gamma,\lambda}(s) - m_{\Gamma})_{H_\Gamma} \\
+ (\pi_{\Gamma}(u_{\Gamma,\lambda}(s)) - f_{\Gamma}(s), u_{\Gamma,\lambda}(s) - m_{\Gamma})_{H_\Gamma} = (\mu_{\Gamma,\lambda}(s), u_{\Gamma,\lambda}(s) - m_{\Gamma})_{H_\Gamma}.
\end{align*}
\]

(4.25)

Let now \((y, y_{\Gamma})\) be the solution of the following problem:

\[
\begin{align*}
\int_\Omega \nabla y \cdot \nabla z \, dx + \int_\Gamma \nabla y_{\Gamma} \cdot \nabla z_{\Gamma} \, d\Gamma &= (u_{\Gamma,\lambda} - m_{\Gamma})z_{\Gamma} \, d\Gamma \quad \text{for all } z \in V, \\
y_{\Gamma} &= y_{\Gamma} \quad \text{a.e. on } \Gamma, \quad \int_\Gamma y_{\Gamma} \, d\Gamma = 0 \quad \text{a.e. in } (0, T).
\end{align*}
\]

(4.26)

This problem has one and only one solution, see [5, Appendix, Lemma A] and repeat the proof with the different condition. Moreover, due to (3.14) and (3.15), there exists a positive constant \( \hat{M}_6 \), independent of \( \lambda \in (0, 1] \), such that

\[
|y(s)|_V^2 + |y_{\Gamma}(s)|_{V_{\Gamma}}^2 \leq \hat{M}_6 |u_{\Gamma,\lambda}(s) - m_{\Gamma}|_{H_\Gamma}^2
\]

(4.27)

for a.e. \( s \in (0, T) \). Taking \( z := \mu_{\lambda}(s), z_{\Gamma} := \mu_{\Gamma,\lambda} \) in (4.26), we see that the right-hand side of (4.26) is equal to

\[
(\nabla y(s), \nabla \mu_{\lambda}(s))_H + (\nabla y_{\Gamma}(s), \nabla \mu_{\Gamma,\lambda}(s))_{H_\Gamma}
\]

and, consequently, we use (4.18) with \( z := y(s), z_{\Gamma} := y_{\Gamma}(s) \) to conclude that

\[
(\mu_{\Gamma,\lambda}(s), u_{\Gamma,\lambda}(s) - m_{\Gamma})_{H_\Gamma} = -(\partial_s u_{\Gamma,\lambda}(s), y_{\Gamma}(s))_{H_\Gamma}.
\]

Then, from (4.25) and the properties

\[
\beta_\lambda(r) - (r - m_{\Gamma}) \geq \delta_0|\beta_\lambda(r)| - c_3, \quad \beta_{\Gamma,\lambda}(r) - (r - m_{\Gamma}) \geq \delta_0|\beta_{\Gamma,\lambda}(r)| - c_3
\]

for all \( r \in \mathbb{R}, \lambda \in (0, 1] \) and some positive constants \( \delta_0 \) and \( c_3 \) which are provided from [19, Section 5] with the assumptions \( D(\beta_{\Gamma}) \subseteq D(\beta) \) of (A5) and \( m_{\Gamma} \in \text{int } D(\beta_{\Gamma}) \) of (A6), we deduce

\[
|\nabla u_\lambda(s)|_H^2 + \delta_0|\beta_\lambda(u_\lambda(s))|_{L^1(\Omega)} + |\nabla u_{\Gamma,\lambda}(s)|_{H_\Gamma}^2 + \delta_0|\beta_{\Gamma,\lambda}(u_{\Gamma,\lambda}(s))|_{L^1(\Gamma)} \\
\leq c_3 T(|\Omega + |\Gamma|) + |\tau(\partial_t u_\lambda(s), u_\lambda(s) - m_{\Gamma})_H + |\nabla u_\lambda(s)|_H^2 + f(s)|_{H_\Gamma}u_\lambda(s) - m_{\Gamma}|_H \\
+ |\pi_{\Gamma}(u_{\Gamma,\lambda}(s)) - f_{\Gamma}(s)|_{H_\Gamma}u_{\Gamma,\lambda}(s) - m_{\Gamma}|_{H_\Gamma} + |\partial_s u_{\Gamma,\lambda}(s)|_{V_{\Gamma}}|y_{\Gamma}(s)|_{V_{\Gamma}}.
\]

Now, squaring both sides, we obtain

\[
\begin{align*}
& (\delta_0|\beta_\lambda(u_\lambda(s))|_{L^1(\Omega)} + \delta_0|\beta_{\Gamma,\lambda}(u_{\Gamma,\lambda}(s))|_{L^1(\Gamma)})^2 \\
& \leq 4c_3^2 T^2(|\Omega + |\Gamma|) + 12(\tau^2|\partial_t u_\lambda(s)|_H^2 + |\pi(u_\lambda(s))|_H^2 + |f(s)|_H^2|u_\lambda(s) - m_{\Gamma}|_H^2 \\
& + 8(|\pi_{\Gamma}(u_{\Gamma,\lambda}(s))|_{H_\Gamma}^2 + |f_{\Gamma}(s)|_{H_\Gamma}^2)|u_{\Gamma,\lambda}(s) - m_{\Gamma}|_{H_\Gamma}^2 + |\partial_s u_{\Gamma,\lambda}(s)|_{V_{\Gamma}}^2|y_{\Gamma}(s)|_{V_{\Gamma}}^2.
\end{align*}
\]

(4.28)
for a.a. \( s \in (0, T) \) Here, by virtue of (4.20), (3.15), (4.21) and (4.27), we have that
\[
|u_{t} - m_{t}|_{L_{\infty}(0, T; H)}^{2} \leq 2M_{1}^{2} + 2m_{t}|Q|,
\]
\[
|u_{t, \lambda} - m_{t}|_{L_{\infty}(0, T; H^{1})}^{2} \leq C_{P}M_{2}^{2},
\]
\[
|y_{t}|_{L_{\infty}(0, T; V_{t})}^{2} \leq C_{P}M_{2}^{2}M_{6},
\]
because \( \int_{\Gamma}(u_{t, \lambda}(s) - m_{t})\,d\Gamma = 0 \) for a.a. \( s \in (0, T) \). Therefore, we integrate (4.28) over \( [0, T] \) with respect to time. Then the right-hand side can be bounded due to (4.20), (4.21) and (4.22). Thus, there exists a positive constant \( M_{6} \), independent of \( \lambda \in (0, 1) \), such that (4.24) holds.

Put
\[
\omega_{t}(t) := \frac{1}{|\Gamma|} \int_{\Gamma} \mu_{t, \lambda}(t)\,dt
\]
for a.a. \( t \in (0, T) \). Then we obtain the following estimate.

**Lemma 4.4.** There exist two positive constants \( M_{7} \) and \( M_{8} \), independent of \( \lambda \in (0, 1) \), such that
\[
|\omega_{t}|_{L^{2}(0, T)} \leq M_{7},
\]
\[
|\mu_{t, \lambda}|_{L^{2}(0, T; V_{t})} \leq M_{7},
\]
\[
|\mu_{t, \lambda}|_{L^{2}(0, T; V)} \leq M_{8}.
\]

**Proof.** Taking \( z := 1/|\Gamma| \) and \( z_{T} := 1/|\Gamma| \) in (4.19), we obtain
\[
|\omega_{t}(t)|^{2} = \frac{7}{|\Gamma|^{2}} \left\{ \tau^{2}|\partial_{t}u_{t}(t)|^{2}_{L^{2}(\Omega)} + |\beta_{t}(u_{t}(t))|^{2}_{L^{2}(\Omega)} + |\pi_{t}(u_{t}(t))|^{2}_{L^{2}(\Omega)} + |f_{t}(t)|^{2}_{L^{2}(\Omega)} + |\mathcal{H}_{t, \lambda}(u_{t}(t))|^{2}_{L^{2}(T)} + |\mathcal{H}_{t, \lambda}(u_{t}(t))|^{2}_{L^{2}(\Gamma)} \right\}
\]
for a.a. \( t \in (0, T) \), that is, there exists a positive constant \( M_{7} \), independent of \( \lambda \in (0, 1) \), such that (4.29) holds. Next, using (3.15) we obtain
\[
|\mu_{t, \lambda}(t)|^{2}_{H_{t}} \leq 2|\mu_{t, \lambda}(t) - \omega_{t}(t)|^{2}_{H_{t}} + 2|\omega_{t}(t)|^{2}_{H_{t}} \leq 2C_{P}|\nabla_{t}\mu_{t, \lambda}(t)|^{2}_{H_{t}} + 2|\Gamma||\omega_{t}(t)|^{2}
\]
for a.a. \( t \in (0, T) \). Thus, (4.21) and (4.29) imply the first estimate of (4.30) for some positive constant \( M_{8} \) independent of \( \lambda \in (0, 1) \). Moreover, by using (3.14) with the above, (4.21) ensures the validity of the second estimate in (4.30).

**Lemma 4.5.** There exist a positive constant \( M_{9} \), independent of \( \lambda \in (0, 1) \), such that
\[
|\beta_{t}(u_{t})(t)|_{L^{2}(0, T; H)} + |\beta_{t}(u_{t, \lambda})(t)|_{L^{2}(0, T; H^{1})} \leq M_{9},
\]
\[
|\Delta u_{t}|_{L^{2}(0, T; H)} + |u_{t}|_{L^{2}(0, T; H^{1}/2(\Omega))} + |\partial_{\nu}u_{t}|_{L^{2}(0, T; H^{1})} \leq M_{9},
\]
\[
|\beta_{t, \lambda}(u_{t, \lambda})(t)|_{L^{2}(0, T; H)} \leq M_{9},
\]
\[
|\nabla_{t}u_{t, \lambda} + |u_{t, \lambda}|_{L^{2}(0, T; W)} + |u_{t, \lambda}|_{L^{2}(0, T; W_{t})} \leq M_{9}.
\]

Estimates (4.31)–(4.32) take advantage from writing (4.19) as the combination of the equations
\[
\tau\partial_{t}u_{t} - \Delta u_{t} + \beta_{t}(u_{t}) + \pi_{t}(u_{t}) = f \quad \text{a.e. in } Q,
\]
\[
\mu_{t, \lambda} = \partial_{\nu}u_{t} - \Delta_{t}u_{t, \lambda} + \beta_{t, \lambda}(u_{t, \lambda}) + \pi_{t}(u_{t, \lambda}) - f_{t} \quad \text{a.e. on } \Gamma,
\]
which are rigorous due to the regularity of \( u_{t} \) and \( u_{t, \lambda} \) stated in (4.4). Recalling [2, Lemmas 4.4, 4.5], we observe that the proof is essentially the same as in these lemmas. Therefore, we omit the details for (4.31)–(4.34).

We have collected all information which enables us to pass to the limit as \( \lambda \downarrow 0 \).

**Proof of Theorem 2.3.** Thanks to (4.20)–(4.22), (4.30), (4.31), (4.33) and (4.34), we see that there exist a subsequence of \( \lambda \) (not relabeled) and some limit functions \( u, u_{t}, \mu, \mu_{t}, \xi \) and \( \xi_{t} \) such that
\[
u_{t} \to u \quad \text{weakly star in } H^{1}(0, T; H) \cap L^{\infty}(0, T; V) \cap L^{2}(0, T; W),
\]
\[
u_{t} \to \mu \quad \text{weakly in } L^{2}(0, T; V),
\]
\[
u_{t, \lambda} \to u_{t} \quad \text{weakly star in } H^{1}(0, T; V^{*}) \cap L^{\infty}(0, T; V_{t}) \cap L^{2}(0, T; W_{t}).
\]
\[
\mu_{\Gamma, \lambda} \to \mu_\Gamma \quad \text{weakly in } L^2(0, T; V_\Gamma), \tag{4.40}
\]
\[
\beta_\lambda(u_\lambda) \to \xi \quad \text{weakly in } L^2(0, T; H), \tag{4.41}
\]
\[
\beta_{\Gamma, \lambda}(u_{\Gamma, \lambda}) \to \xi_\Gamma \quad \text{weakly in } L^2(0, T; H_\Gamma) \tag{4.42}
\]
as \(\lambda \searrow 0\). From (4.37) and (4.39), using well-known compactness results (see, e.g., [30, Section 8, Corollary 4]), we obtain
\[
u_\lambda \to u \quad \text{strongly in } C([0, T]; H) \cap L^2(0, T; V), \tag{4.43}
\]
\[
u_{\Gamma, \lambda} \to u_\Gamma \quad \text{strongly in } C([0, T]; H_\Gamma) \cap L^2(0, T; V_\Gamma), \tag{4.44}
\]
which imply that
\[
\pi(u_\lambda) \to \pi(u) \quad \text{strongly in } C([0, T]; H), \tag{4.45}
\]
\[
\pi_{\Gamma}(u_{\Gamma, \lambda}) \to \pi_{\Gamma}(u_\Gamma) \quad \text{strongly in } C([0, T]; H_\Gamma) \tag{4.46}
\]
as \(\lambda \searrow 0\) and
\[
\xi \in \beta(u) \quad \text{a.e. in } Q, \quad \xi_\Gamma \in \beta_{\Gamma}(u_\Gamma) \quad \text{a.e. on } \Sigma. \tag{4.47}
\]
due to the maximal monotonicity of \(\beta\) and \(\beta_{\Gamma}\). Finally, we can pass to the limit as \(\lambda \searrow 0\) in the variational equality (4.18) as well as in (4.35)–(4.36) obtaining (2.8)–(2.10), and in the initial conditions (4.17) obtaining (2.11). Thus, we arrive at the conclusion. \(\square\)

**Remark 4.6.** Thanks to the strong convergence (4.44) and equality (4.23), we deduce that
\[
\int_\Gamma u(t) \, d\Gamma = \int_\Gamma u_{0\Gamma} \, d\Gamma \quad \text{for all } t \in [0, T]. \tag{4.48}
\]

### 4.4 Proof of Theorem 2.4

**Proof of Theorem 2.4.** Let us note that the solutions \(u^{(i)} := (u^{(i)}_1, u^{(i)}_2), \mu^{(i)} := (\mu^{(i)}_1, \mu^{(i)}_2), \xi^{(i)} := (\xi^{(i)}_1, \xi^{(i)}_2)\) of (P) satisfy
\[
\begin{align*}
(\partial_t u^{(i)} (s, z_\Gamma), v_\Gamma)_H + (\nabla u^{(i)} (s, z_\Gamma), \nabla v_\Gamma)_H &+ (\mu^{(i)}_1 (s, z_\Gamma), v_\Gamma)_H = 0, \\
(\mu^{(i)}_1 (s, z_\Gamma), v_\Gamma)_H &+ (\partial_t z^{(i)}_\Gamma(s, z_\Gamma), v_\Gamma)_H \\
&+ (\nabla u^{(i)} (s, z_\Gamma), \nabla v_\Gamma)_H + (\xi^{(i)}_1(s) + \pi(u^{(i)}_1(s)) - f(t(s), z_\Gamma)_H
\end{align*}
\]
for all \(z := (z, z_\Gamma) \in V\) and \(i = 1, 2\). Let us use the notation
\[
\begin{align*}
\tilde{u} &:= u^{(1)} - u^{(2)}, \quad \tilde{u}_\Gamma := u^{(1)}_1 - u^{(2)}_1, \quad \tilde{\mu} := \mu^{(1)}_1 - \mu^{(2)}_1, \\
\tilde{\xi} &:= \xi^{(1)} - \xi^{(2)}, \quad \tilde{\xi}_\Gamma := \xi^{(1)}_1 - \xi^{(2)}_1, \quad \tilde{f} := f^{(1)} - f^{(2)}, \quad \tilde{f}_\Gamma := f^{(1)}_1 - f^{(2)}_1, \\
\tilde{u}_0 &:= u^{(1)}_0 - u^{(2)}_0, \quad \tilde{u}_{0\Gamma} := u^{(1)}_{0\Gamma} - u^{(2)}_{0\Gamma}.
\end{align*}
\]

We take the difference of equations (4.50), test it by \((\tilde{u}, \tilde{u}_\Gamma)\) and integrate over \([0, t]\) with respect to \(s\), obtaining
\[
\begin{align*}
\frac{t}{2} |\tilde{u}(t)|_H^2 + \int_0^t |\tilde{\nabla} \tilde{u}(s)|_H^2 \, ds + \int_0^t |\tilde{V}_\Gamma \tilde{u}(s)_H^2|_H \, ds + \int_0^t (\tilde{\xi}(s), \tilde{u}(s)_H)_H \, ds + \int_0^t (\tilde{\xi}_\Gamma(s), \tilde{u}_\Gamma(s)_H)_H \, ds &= \int_0^t (\tilde{\mu}(s), \tilde{u}_\Gamma(s)_H)_H \, ds \\
&- \int_0^t (\tilde{\pi}_1(u_1^{(1)}(s)) - \pi(u_1^{(2)}(s)), \tilde{u}(s)_H)_H \, ds - \int_0^t (\tilde{\pi}_1(u_1^{(1)}(s)) - \pi(u_1^{(2)}(s)), \tilde{u}_\Gamma(s)_H)_H \, ds \\
&+ \int_0^t (\tilde{f}(s), \tilde{u}(s)_H)_H \, ds + \int_0^t (\tilde{f}_\Gamma(s), \tilde{u}_\Gamma(s)_H)_H \, ds \tag{4.51}
\end{align*}
\]
for all \( t \in [0, T] \). We start by discussing the last term in the left-hand side of (4.51). Let \((\bar{y}, \bar{y}_r) \in H^1(0, T; V)\) be a solution of the problem

\[
\int_\Omega \nabla \bar{y} : \nabla z \, dx + \int_\Gamma \nabla \bar{y}_r : \nabla z_r \, d\Gamma = \langle \bar{u}_r, z_r \rangle_{V_r^*, V_r}
\]  

(4.52)

for all \( z := (z, z_r) \in V \), with

\[
\int_\Gamma \bar{y}_r \, d\Gamma = 0
\]  

(4.53)

a.e. in \((0, T)\). Now, testing (4.52) by \( z = (\bar{\mu}(s), \bar{\mu}_r(s)) \), we have

\[
- \int_0^t \langle \bar{\mu}_r(s), \bar{u}_r(s) \rangle_{H_r} \, ds = - \int_0^t \langle \nabla \bar{y}(s), \nabla \bar{\mu}(s) \rangle_{H} \, ds - \int_0^t \langle \nabla \bar{y}_r(s), \nabla \bar{\mu}_r(s) \rangle_{H_r} \, ds
\]

for all \( t \in [0, T] \). On the other hand, considering the difference of (4.49) and testing then by \( z = (\bar{y}(s), \bar{y}_r(s)) \), we infer that

\[
- \int_0^t \langle \nabla \bar{\mu}(s), \nabla \bar{y}(s) \rangle_{H} \, ds - \int_0^t \langle \nabla \bar{\mu}_r(s), \nabla \bar{y}_r(s) \rangle_{H_r} \, ds = \int_0^t \langle \partial_s \bar{u}_r(s), \bar{y}_r(s) \rangle_{V_r^*, V_r} \, ds
\]

for all \( t \in [0, T] \). Therefore, we can differentiate (4.52) with respect to time, then test by \( z = (\bar{y}(s), \bar{y}_r(s)) \) and integrate the resultant over \([0, t]\) with respect to \( s \), obtaining

\[
\frac{1}{2} \| \nabla \bar{y}(t) \|_{H}^2 + \frac{1}{2} \| \nabla \bar{y}_r(t) \|_{H_r}^2 - \frac{1}{2} \| \nabla \bar{y}(0) \|_{H}^2 - \frac{1}{2} \| \nabla \bar{y}_r(0) \|_{H_r}^2 = \int_0^t \langle \partial_s \bar{u}_r(s), \bar{y}_r(s) \rangle_{V_r^*, V_r} \, ds
\]

for all \( t \in [0, T] \). Thus, we have the contribution

\[
\frac{1}{2} \| \nabla \bar{y}(t) \|_{H}^2 + \frac{1}{2} \| \nabla \bar{y}_r(t) \|_{H_r}^2 \geq c_4 \| \bar{u}_r(t) \|_{V_r}^2
\]  

(4.54)

on the left-hand side and

\[
\frac{1}{2} \| \nabla \bar{y}(0) \|_{H}^2 + \frac{1}{2} \| \nabla \bar{y}_r(0) \|_{H_r}^2 \leq c_5 \| \bar{u}_r(t) \|_{V_r}^2
\]  

(4.55)

on the right-hand side for all \( t \in [0, T] \), for some positive constants \( c_4 \) and \( c_5 \). Notice that (4.54) follows from (4.52), since we have

\[
\| \bar{u}_r(t) \|_{V_r} = \sup_{z_r \in V_r} \| \langle \bar{u}_r(t), z_r \rangle_{V_r^*, V_r} \|
\]

\[
\leq \sup_{z_r \in V_r} \left( \int_\Omega \nabla \bar{y}(t) : \nabla z_r \, dx + \int_\Gamma \nabla \bar{y}_r(t) : \nabla z_r \, d\Gamma \right)
\]

\[
\leq \sup_{z_r \in V_r} \left\{ c_2 \| \nabla \bar{y}(t) \|_{H} \| z_r \|_{V_r} + \| \nabla \bar{y}_r(t) \|_{H_r} \| z_r \|_{V_r} \right\}
\]

for all \( t \in [0, T] \), where we have to use (3.20) and the fact that \((\nabla z_r, z_r) \in V\) for all \( z_r \in V_r \). Moreover, (4.55) follows from (3.16) and (4.52) at time 0, taking \( z = (\bar{y}(0), \bar{y}_r(0)) \); of course, \( c_5 \) depends on \( C_P \). Next, for all \( t \in [0, T] \), we note that

\[
\int_0^t \langle \bar{\xi}(s), \bar{u}(s) \rangle_H \, ds \geq 0, \quad \int_0^t \langle \bar{\xi}_r(s), \bar{u}_r(s) \rangle_{H_r} \, ds \geq 0,
\]

by the monotonicity,

\[
\int_0^t \langle \pi(u^{(1)}(s)) - \pi(u^{(2)}(s)), \bar{u}(s) \rangle_H \, ds \leq L \int_0^t \| \bar{u}(s) \|_{H}^2 \, ds
\]
and this will be treated by the Gronwall inequality,

\[
\int_0^t (\pi_t(u_t^{(1)}(s)) - \pi_t(u_t^{(2)}(s)), \tilde{u}_t(s))_{H_t} \, ds \leq L_t \int_0^t [\tilde{u}_t(s)]^2_{V_t} \, ds \leq \delta \int_0^t [\tilde{u}_t(s)]^2_{V_t} \, ds + C_\delta \int_0^t [\tilde{u}_t(s)]^2_{V_t} \, ds,
\]

due to the compactness inequality (3.18), also for this term we will use the Gronwall inequality (the last two terms can be simply treated by the Young inequality), and finally

\[
\int_0^t (\tilde{f}(s), \tilde{u}(s))_{H_t} \, ds \leq \delta \int_0^t [\tilde{u}(s)]^2_{V_t} \, ds + \frac{1}{4\delta} \int_0^t [f(s)]^2_{V_t} \, ds,
\]

for all \( \delta, \tilde{\delta} > 0 \). Now we take advantage of (3.16) Indeed, \( \tilde{u}_t \) satisfies the zero mean value condition from (4.48). Moreover, by (4.55), we can recover from (4.51) the following inequality:

\[
\frac{t}{2} [\tilde{u}(t)]^2_{H_t} + c_4 |\tilde{u}_t(t)|^2_{V_t} + \frac{1}{2C_P} \int_0^t [\tilde{u}(s)]^2_{V_t} \, ds + \frac{1}{2C_P} \int_0^t [\tilde{u}_t(s)]^2_{V_t} \, ds \leq \frac{t}{2} |\tilde{u}_0|^2_{H_t} + c_5 |\tilde{u}_0|^2_{V_t} + L \int_0^t [\tilde{u}(s)]^2_{V_t} \, ds + C_\delta \int_0^t [\tilde{u}_t(s)]^2_{V_t} \, ds + \frac{C_P}{2} \int_0^t [f(s)]^2_{V_t} \, ds + C_P \int_0^t [\tilde{f}_t(s)]^2_{V_t} \, ds
\]

for all \( t \in [0, T] \), that is, \( \delta := 1/(4C_P) \) and \( \tilde{\delta} := 1/(2C_P) \). Then the continuous dependence (2.13) follows from the application of the Gronwall inequality.

\[\Box\]

A Appendix

We use the same notation as in the previous sections. We also use the following notation of function spaces. For each fixed \( \epsilon \in (0, 1) \), define a subspace \( H_0^\epsilon \) of \( H \) by \( H_0^\epsilon := \{ z \in H : m_\epsilon(z) = 0 \} \), where \( m_\epsilon : H \rightarrow \mathbb{R} \),

\[
m_\epsilon(z) := \frac{1}{\epsilon |\Omega| + |\Gamma|} \left( \epsilon \int_\Omega z \, dx + \int_\Gamma z \, d\Gamma \right) \quad \text{for all} \ z \in H.
\]

Moreover, define an inner product of \( H \) by

\[
\langle z, \tilde{z} \rangle_H := \epsilon (z, \tilde{z})_H + (z, \tilde{z})_{H_\epsilon} \quad \text{for all} \ z \in H.
\]

Then we see that the induced norm \( \| \cdot \|_H \) and the standard norm \( | \cdot |_H \) are equivalent, because

\[
\|z\|_H^2 \leq |z|_H^2 \leq \frac{1}{\epsilon} \|z\|_H^2 \quad \text{for all} \ z \in H.
\]

Next, we define \( V_0^\epsilon := V \cap H_0^\epsilon \), with \( |z|_{V_0^\epsilon} := \sqrt{a(z, z)} \) for all \( z \in V_0^\epsilon \), where we use \( a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R} \), a bilinear form defined by

\[
a(z, \tilde{z}) := \int_\Omega \nabla z \cdot \nabla \tilde{z} \, dx + \int_\Gamma [\nabla_\Gamma z \cdot \nabla_\Gamma \tilde{z}] \, d\Gamma \quad \text{for all} \ z, \tilde{z} \in V.
\]

Let us define a linear bounded operator \( F : V_0^\epsilon \rightarrow (V_0^\epsilon)^* \) by

\[
\langle Fz, \tilde{z} \rangle_{(V_0^\epsilon)^*} := a(z, \tilde{z}) \quad \text{for all} \ z, \tilde{z} \in V_0^\epsilon.
\]

Then there exists a positive constant \( c_p^{(\epsilon)} \) such that

\[
\|z\|_H^2 \leq |z|_H^2 \leq c_p^{(\epsilon)} |z|_{V_0^\epsilon}^2 \quad \text{for all} \ z \in V_0^\epsilon,
\]
Firstly, we see that
\[
\begin{align*}
\mathcal{D}(\mathcal{A}) & = \{ z \in H \mid (z, \mathcal{A}^* z)_{L^2(\Omega)} = 0 \}
\end{align*}
\]

Next, we infer that
\[
\begin{align*}
\mathcal{D}(\mathcal{A}) & = \{ z \in H \mid (z, \mathcal{A}^* z)_{L^2(\Omega)} = 0 \}
\end{align*}
\]

Therefore, we deduce that
\[
\begin{align*}
\| z - P_z z \|_H & = \| (z - P_z z, z - y - P_z z) \|_H \\
& = m_z(z) \left\{ \varepsilon \int_{\Omega} (y-z) \, dx + \int_{\Gamma} (y_z - z_z) \, d\gamma \right\} + m_z(z)^2 \{ \varepsilon |\Omega| + |\Gamma| \} \\
& = 0 \quad \text{for all } z \in H.
\end{align*}
\]

Next, we infer that
\[
\begin{align*}
\langle z - P_z z, y - P_z z \rangle_H & = \langle m_z(z) \mathbf{1}, y - z + m_z(z) \mathbf{1} \rangle_H \\
& = m_z(z) \left\{ \varepsilon \int_{\Omega} (y-z) \, dx + \int_{\Gamma} (y_z - z_z) \, d\gamma \right\} + m_z(z)^2 \{ \varepsilon |\Omega| + |\Gamma| \} \\
& = 0 \quad \text{for all } y \in H_0^\varepsilon.
\end{align*}
\]

Therefore, we deduce that
\[
\begin{align*}
\| z - P_z z \|_H^2 & = \langle (z - P_z z, z - y + y - P_z z) \rangle_H \\
& = \langle (z - P_z z, z - y) \rangle_H \\
& \leq \frac{1}{2} \| z - P_z z \|_H^2 + \frac{1}{2} \| z - y \|_H^2,
\end{align*}
\]
i.e., (A.1) holds. \(\square\)

We also easily obtain the following conditions:
\[
\begin{align*}
\langle z^*, P_z z \rangle_H & = \langle z^*, z \rangle_H \quad \text{for all } z^* \in H_0^\varepsilon \text{ and } z \in H, \\
|P_z z|_{H_0^\varepsilon} & \leq |z|_V \quad \text{for all } z \in V.
\end{align*}
\]

Incidentally, we note that another possibility of projection is given by
\[
P_{\mathcal{P}} z := \left( z - \frac{1}{|\Omega|} \int_{\Omega} z \, dx, z_\Gamma - \frac{1}{|\Gamma|} \int_{\Gamma} z_\Gamma \, d\gamma \right)
\]
for all \( z \in H, \)
and \( P_{\mathcal{P}} \) is actually the projection from \( H \) to \( H_0^\varepsilon \) with respect to the standard norm (cf. (A.1)). However, this choice is not suitable from the viewpoint of the trace condition. Indeed, \( P_{\mathcal{P}} z \notin V \) even if \( z \in V. \)

Next, we prepare suitable approximations for \( f \) and \( u_0, \) which satisfy assumptions (A1) and (A2), respectively.
Lemma A.2. For each $\varepsilon \in (0, 1]$, the solution of the following Cauchy problem
\begin{equation}
\begin{cases}
\varepsilon f_\varepsilon'(s) + f_\varepsilon(s) = f(s) & \text{in } H, \text{ for a.a. } s \in (0, T), \\
f_\varepsilon(0) = 0 & \text{in } H
\end{cases}
\tag{A.2}
\end{equation}
satisfies (3.3) and (3.4).

Proof. Note that $f_\varepsilon(0) = 0 \in V$. Multiplying the above equation by the solution $f_\varepsilon := (f_\varepsilon, f_{1\varepsilon}) \in H^1(0, T; H)$, integrating it over $[0, t]$ and using the Young inequality, we have
\begin{equation}
\frac{\varepsilon}{2} |f_\varepsilon(t)|_H^2 + \frac{1}{2} \int_0^t |f_\varepsilon(s)|_H^2 \, ds \leq \frac{1}{2} \int_0^t |f(s)|_H^2 \, ds
\end{equation}
for all $t \in [0, T]$. Thus, there exist a subsequence of $\varepsilon$ (not relabeled) and some limit function $\tilde{f} \in L^2(0, T; H)$ such that
\begin{align*}
f_\varepsilon & \rightarrow \tilde{f} \quad \text{weakly in } L^2(0, T; H) \\
\varepsilon f_\varepsilon & \rightarrow \tilde{f} \quad \text{weakly in } L^2(0, T; H), \\
f_\varepsilon & \rightarrow 0 \quad \text{strongly in } C([0, T]; H)
\end{align*}
as $\varepsilon \downarrow 0$, that is, $\tilde{f} = 0$. From (A.2), this implies that $\tilde{f} = f$. Therefore, by using (A.3) again, we have
\begin{equation}
\limsup_{\varepsilon \to 0} |f_\varepsilon|_{L^2(0, T; H)} \leq |f|_{L^2(0, T; H)}
\end{equation}
which gives us the convergence of norms and consequently the strong convergence (3.3). Next we show the required order of the convergence (3.4). Recall the equation for $f_{t\varepsilon} \in H^1(0, T; H_t)$ as follows:
\begin{equation}
\begin{cases}
\varepsilon f_{t\varepsilon}'(s) + f_{t\varepsilon}(s) = f_t(s) & \text{in } H_t, \text{ for a.a. } s \in (0, T), \\
f_{t\varepsilon}(0) = 0 & \text{in } H_t
\end{cases}
\tag{A.4}
\end{equation}
From (A.4) we see that
\begin{equation}
|f_t - f_{t\varepsilon}|_{L^2(0, T; H_t)} = \varepsilon |f_{t\varepsilon}'|_{L^2(0, T; H_t)}
\end{equation}
therefore it is enough to prove that there exists a positive constant $C_0$ such that
\begin{equation}
\varepsilon^{1/2} |f_{t\varepsilon}'|_{L^2(0, T; H_t)} \leq C_0.
\end{equation}
Testing (A.4) by $f_{t\varepsilon}'(s)$ and integrating the resultant with respect to time, we obtain
\begin{equation}
\varepsilon \int_0^t \left| f_{t\varepsilon}'(s) \right|^2_{H_t} \, ds + \frac{1}{2} \int_0^t \left| f_{t\varepsilon}(s) \right|^2_{H_t} \, ds = (f_t(t), f_{t\varepsilon}(t))_{H_t} - \int_0^t (f_{t\varepsilon}'(s), f_{t\varepsilon}(s))_{H_t} \, ds
\end{equation}
for all $t \in [0, T]$. Therefore, using the Gronwall inequality, we see that there exists a positive constant $C_0$, depending on $|f_t|_{C([0, T]; H_t)}$ and $|f_{t\varepsilon}'|_{L^2(0, T; H_t)}$ but independent of $\varepsilon \in (0, 1]$, such that
\begin{equation}
\varepsilon^{1/2} |f_{t\varepsilon}'|_{L^2(0, T; H_t)} + |f_{t\varepsilon}|_{L^2(0, T; H_t)} \leq C_0
\end{equation}
for all $\varepsilon \in (0, 1]$. 
\qed
Lemma A.3. For each \( \varepsilon \in (0, 1) \), the solution \( u_{0,\varepsilon} := (u_{0,\varepsilon}, u_{0\Gamma,\varepsilon}) \in W \cap V \) of the following elliptic system

\[
\begin{align*}
 u_{0,\varepsilon} - \varepsilon \Delta u_{0,\varepsilon} &= u_0 & \text{a.e. in } \Omega, \\
 (u_{0,\varepsilon})_t &= u_{0\Gamma,\varepsilon} & \text{a.e. on } \Gamma,
\end{align*}
\]

satisfies (3.5) and (3.6) with the required regularity \((-\Delta u_{0,\varepsilon}, \partial_u u_{0,\varepsilon} - \Delta \Gamma u_{0\Gamma,\varepsilon}) \in V \).}

Proof. The strategy of the proof is similar to the one of [7, Lemma A.1]. By virtue of [5, Lemma C], there exists \( u_{0,\varepsilon} := (u_{0,\varepsilon}, u_{0\Gamma,\varepsilon}) \in W \cap V \) such that \( u_{0,\varepsilon} \) satisfies system (A.5)–(A.6). Moreover, assumption (A2) gives us the required regularity \((-\Delta u_{0,\varepsilon}, \partial_u u_{0,\varepsilon} - \Delta \Gamma u_{0\Gamma,\varepsilon}) \in V \). Next we show (3.5). Indeed, testing equation (A.5) by \( u_{0,\varepsilon} \) and the second equation in (A.6) by \( u_{0\Gamma,\varepsilon} \), adding them, and using the Young inequality, we obtain

\[
\frac{1}{2} \int_\Omega |u_{0,\varepsilon}|^2 dx + \frac{1}{2} \int_\Gamma |u_{0\Gamma,\varepsilon}|^2 d\Gamma + \varepsilon \int_\Omega |\nabla u_{0,\varepsilon}|^2 dx + \varepsilon \int_\Gamma |\nabla u_{0\Gamma,\varepsilon}|^2 d\Gamma \leq \frac{1}{2} \int_\Omega |u_0|^2 dx + \frac{1}{2} \int_\Gamma |u_0\Gamma|^2 d\Gamma.
\]

Therefore, \( \{u_{0,\varepsilon}\}_{\varepsilon \in (0, 1)} \) is bounded in \( H \), \( \{u_{0\Gamma,\varepsilon}\}_{\varepsilon \in (0, 1)} \) is bounded in \( H_\Gamma \), \( \{\varepsilon^{1/2} \nabla u_{0,\varepsilon}\}_{\varepsilon \in (0, 1)} \) is bounded in \( H \), and \( \{\varepsilon^{1/2} \nabla u_{0\Gamma,\varepsilon}\}_{\varepsilon \in (0, 1)} \) is bounded in \( H_\Gamma \), respectively. These give us (see (A.5)–(A.6))

\[
\begin{align*}
 u_{0,\varepsilon} &\to u_0 \text{ weakly in } H, & \varepsilon u_{0,\varepsilon} &\to 0 \text{ strongly in } V, \\
u_{0\Gamma,\varepsilon} &\to u_{0\Gamma} \text{ weakly in } H_\Gamma, & \varepsilon u_{0\Gamma,\varepsilon} &\to 0 \text{ strongly in } V_\Gamma
\end{align*}
\]
as \( \varepsilon \searrow 0 \). Moreover, we have

\[
\limsup_{\varepsilon \to 0} \left( \int_\Omega |u_{0,\varepsilon}|^2 dx + \int_\Gamma |u_{0\Gamma,\varepsilon}|^2 d\Gamma \right) \leq \int_\Omega |u_0|^2 dx + \int_\Gamma |u_0\Gamma|^2 d\Gamma,
\]

which entails that

\[
u_{0,\varepsilon} \to u_0 \text{ strongly in } H
\]
as \( \varepsilon \searrow 0 \). Next, from the definition of the subdifferential together with (A.5) and (A.6), we see that

\[
\int_\Omega \tilde{\beta}_A(u_{0,\varepsilon}) dx - \int_\Omega \tilde{\beta}_A(u_0) dx \leq \int_\Omega (u_{0,\varepsilon} - u_0) \beta_A(u_{0,\varepsilon}) dx
\]

\[
= - \int_\Omega \varepsilon \beta'_A(u_{0,\varepsilon}) |\nabla u_{0,\varepsilon}|^2 dx + \varepsilon \int_\Gamma \nabla u_{0,\varepsilon} \beta_A(u_{0,\varepsilon}) d\Gamma
\]

\[
\leq \int_\Gamma (u_{0\Gamma} - u_{0\Gamma,\varepsilon}) \beta_A(u_{0\Gamma,\varepsilon}) d\Gamma - \varepsilon \beta'_A(u_{0\Gamma}) |\nabla u_{0\Gamma,\varepsilon}|^2 d\Gamma
\]

\[
\leq \int_\Gamma \tilde{\beta}_A(u_{0\Gamma}) d\Gamma - \int_\Gamma \tilde{\beta}_A(u_{0\Gamma,\varepsilon}) d\Gamma,
\]

that is,

\[
\int_\Omega \tilde{\beta}_A(u_{0,\varepsilon}) dx + \int_\Gamma \tilde{\beta}_A(u_{0\Gamma,\varepsilon}) d\Gamma \leq \int_\Omega \tilde{\beta}_A(u_0) dx + \int_\Gamma \tilde{\beta}_A(u_{0\Gamma}) d\Gamma
\]

for all \( \varepsilon \in (0, 1) \), where (3.1) has been used. Now, due to (3.2), we obtain that

\[
\tilde{\beta}_A(r) = \int_0^r \beta_A(s) ds \leq \varrho \int_0^r \beta_{r,\Lambda}(s) ds + c_0 |r| = \varrho \tilde{\beta}_{r,\Lambda}(r) + c_0 |r|
\]

for all \( r \in \mathbb{R} \). Indeed, in the case \( r \geq 0 \), because of the fact that \( \beta_A(0) = \beta_{r,\Lambda}(0) = 0 \), we see from (3.2) that

\[
\int_0^r \beta_A(s) ds = \int_0^r |\beta_A(s)| ds \leq \varrho \int_0^r |\beta_{r,\Lambda}(s)| ds + c_0 r = \varrho \int_0^r \beta_{r,\Lambda}(s) ds + c_0 |r|.
\]
In the case $r < 0$, we have
\[
\int_r^0 |\beta_\lambda(s)| \, ds \leq \int_0^0 |\beta_{\Gamma,\lambda}(s)| \, ds - c_0 r = q \int_0^r |\beta_{\Gamma,\lambda}(s)| \, ds + c_0 |r|.
\]
Then it turns out that
\[
\int_{\Gamma} \tilde{\beta}_\lambda(u_{0\epsilon}) \, d\Gamma \leq q \int_{\Gamma} \tilde{\beta}_{\Gamma,\lambda}(u_{0\epsilon}) \, d\Gamma + c_0 |u_{0\epsilon}|_{L^1(\Gamma)} \leq q \int_{\Gamma} \tilde{\beta}_\lambda(u_{0\epsilon}) \, d\Gamma + c_0 |u_{0\epsilon}|_{L^1(\Gamma)}.
\]
Therefore, $\tilde{\beta}(u_{0\epsilon}) \in L^1(\Gamma)$, due to the Fatou lemma and the almost everywhere convergence of $\tilde{\beta}_\lambda(u_{0\epsilon})$ to $\tilde{\beta}(u_{0\epsilon})$. Concerning our approximation, testing (A.5) by $-\Delta u_{0,\epsilon}$ and using (A.6), we get
\[
\int_\Omega \nabla(u_{0,\epsilon} - u_0) \cdot \nabla u_{0\epsilon} \, dx + \frac{1}{\epsilon} \int_\Gamma |u_{0\epsilon} - u_{0\epsilon}|^2 \, d\Gamma + \epsilon \int_\Omega |\Delta u_{0\epsilon}|^2 \, dx + \int_\Omega \nabla u_{0\epsilon} \cdot \nabla u_{0\epsilon} \, dx = 0.
\]
Therefore, using the Young inequality, we deduce
\[
\frac{1}{2} \int_\Omega |\nabla u_{0\epsilon}|^2 \, dx + \frac{1}{2} \int_\Gamma |\nabla_{\Gamma} u_{0\epsilon}|^2 \, d\Gamma \leq \frac{1}{2} \int_\Omega |\nabla u_0|^2 \, dx + \frac{1}{2} \int_\Gamma |\nabla_{\Gamma} u_0|^2 \, d\Gamma.
\]
Thus, we obtain that there exists a positive constant $\hat{C}_0$ such that
\[
|u_{0\epsilon} - u_0|_{H^1} \leq \epsilon^{1/2} \hat{C}_0, \quad \epsilon^{1/2}|\Delta u_{0\epsilon}|_{H^1} \leq \hat{C}_0 \quad \text{for all } \epsilon \in (0, 1), \quad u_{0\epsilon} \rightharpoonup u_0 \quad \text{strongly in } V
\]
as $\epsilon \searrow 0$, because, $u_{0\epsilon} \rightharpoonup u_0$ weakly in $V$ and $|u_{0\epsilon}|_V \rightarrow |u_0|_V$ as $\epsilon \searrow 0$. Then, from (A.6), we can also infer that
\[
\epsilon^{1/2} |\nabla u_{0\epsilon} - \Delta \Gamma u_{0\epsilon}|_{H^1} \leq \hat{C}_0.
\]
Now, if we test (A.6) by $\beta_{\Gamma,\lambda}(u_{0\epsilon})$, then we obtain
\[
\int_{\Gamma} \tilde{\beta}_{\Gamma,\lambda}(u_{0\epsilon}) \, d\Gamma - \int_{\Gamma} \tilde{\beta}_{\Gamma,\lambda}(u_{0\epsilon}) \, d\Gamma \leq \int_{\Gamma} (-\epsilon \nabla u_{0\epsilon} + \epsilon \Delta u_{0\epsilon}) \beta_{\Gamma,\lambda}(u_{0\epsilon}) \, d\Gamma
\]
\[
\leq \hat{C}_0 \epsilon^{1/2} |\beta_{\Gamma,\lambda}(u_{0\epsilon})|_{H^1} \leq \frac{\hat{C}_0 \epsilon^{1/2}}{\lambda} |u_{0\epsilon}|_{H^1},
\]
hence, from (3.1), there exists a positive constant $\hat{C}_0$, independent of $\epsilon, \lambda \in (0, 1)$, such that
\[
\int_{\Gamma} \tilde{\beta}_{\Gamma,\lambda}(u_{0\epsilon}) \, d\Gamma \leq \int_{\Gamma} \tilde{\beta}_{\Gamma,\lambda}(u_{0\epsilon}) \, d\Gamma + \frac{\hat{C}_0 \epsilon^{1/2}}{\lambda} \leq \int_{\Gamma} \tilde{\beta}_\lambda(u_{0\epsilon}) \, d\Gamma + \frac{\hat{C}_0 \epsilon^{1/2}}{\lambda}.
\]
Thus, we get the conclusion. \[\square\]

**Proof of Proposition 3.1.** For each $\epsilon, \lambda \in (0, 1)$, let us consider the Cauchy problem for the following equivalent evolution equation:
\[
\dot{v}(t) + M_v(t) = 0 \quad \text{in } (V_0^\alpha)^*, \quad \text{for a.a. } t \in (0, T), \quad (A.7)
\]
\[
M_{\mu(t)} = M \dot{v}(t) + \lambda M \Delta v(t) + \beta_{\lambda}(u(t)) + \mu(t) - f_\lambda(t) \quad \text{in } H_\alpha, \quad \text{for a.a. } t \in (0, T), \quad (A.8)
\]
\[
u(t) = v(t) + m_{e_\lambda}(u_0(e)), \quad \nu(0) = v_0 := u_0(e) - m_{e_\lambda}(u_0(e)) \quad \text{in } H_0^{\epsilon}, \quad (A.9)
\]
(cf. [5, Section 2.3], see also [24, 25]), where
\[
M := \left( \begin{array}{cc} \epsilon & 0 \\ 0 & 1 \end{array} \right), \quad T := \left( \begin{array}{cc} \tau & 0 \\ 0 & \epsilon \end{array} \right),
\]
and we define $\varphi: H_0^\epsilon \to [0, +\infty]$ by
\[
\varphi(z) := \begin{cases}
\frac{1}{2} \int_\Omega |\nabla z|^2 \, dx + \frac{1}{2} \int_\Gamma |\nabla \Gamma z|^2 \, d\Gamma & \text{if } z \in V_0^\epsilon, \\
+\infty & \text{if } z \in H_0^\epsilon \setminus V_0^\epsilon.
\end{cases}
\]
Then we see that $\varphi$ is proper, lower semicontinuous and convex on $H_0^\epsilon$, and the subdifferential $\partial \varphi$ on $H_0^\epsilon$ is characterized by $\partial \varphi(z) = (-1/\epsilon) \nabla \varphi z + \nabla \varphi z_\Gamma$ with $z \in D(\partial \varphi) = W \cap V_0^\epsilon$. The Cauchy problem (A.7)–(A.9) can be solved by applying [5, Sections 4.2–4.4], based on the abstract theory of doubly nonlinear evolution equation [10], because all assumptions for $f$, and $u_{0,\epsilon}$ in order to obtain the strong solution are satisfied.

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