Dueling RL: Reinforcement Learning with Trajectory Preferences

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Abstract

We consider the problem of preference-based reinforcement learning (PbRL), where, unlike traditional reinforcement learning (RL), an agent receives feedback only in terms of 1 bit (0/1) preferences over a trajectory pair instead of absolute rewards for it. The success of the traditional reward-based RL framework crucially depends on how accurately a system designer can express an appropriate reward function, which is often a non-trivial task. The main novelty of our framework is the ability to learn from preference-based trajectory feedback that eliminates the need to hand-craft numeric reward models. This paper sets up a formal framework for the PbRL problem with non-Markovian rewards, where the trajectory preferences are encoded by a generalized linear model of dimension $d$. Assuming the transition model is known, we propose an algorithm with a regret guarantee of $\tilde{O}\left(SHd\log(T/\delta)\sqrt{T}\right)$. We further extend the above algorithm to the case of unknown transition dynamics and provide an algorithm with regret $O((\sqrt{d}+H^2+|S|)\sqrt{dT+\sqrt{|S||A|TH}})$. To the best of our knowledge, our work is one of the first to give tight regret guarantees for preference-based RL problem with trajectory preferences.

1 INTRODUCTION

Classical reinforcement learning (RL) with absolute reward feedback is a well-studied framework which is a sequential experience-driven learning process to optimize an accumulated long-term reward (Sutton and Barto 2018; Auer et al. 2009; Singh et al. 2002). Over the years, several works have addressed RL in terms of both the optimal sample complexity for finding the best policy (Azar et al. 2013; Dann and Brunskill 2015; Dann et al. 2017; Domingues et al. 2020a; Lattimore and Hutter 2012) and minimizing regret via balancing exploration and exploitation (Zhang and Ji 2019; Azar et al. 2017; Ortner 2020; Lilebied and Maillard 2018; Efroni et al. 2020; Domingues et al. 2020b).

However, a major limitation of the standard RL setting is that its success crucially depends on the prior knowledge encoded into the definition of the reward function. The learned policy can often be sensitive to small changes of the reward, possibly yielding very different behaviors depending on the relative values of the rewards. The choice of reward function in applications such as robotics consequently entails a high amount of non-trivial effort in reward engineering, leading to challenges such as reward shaping, reward hacking, infinite rewards, and multi-objective outcomes (Wirth and Fürnkranz 2013; Wirth et al. 2017).

The framework of Preference-based Reinforcement Learning (PbRL) (Busa-Fekete et al. 2014; Wirth et al. 2016; Wirth et al. 2017) has been proposed as a fix to this problem, to enforce learning from non-numerical, relative feedback which need not suffer from issues due to the inaccuracy of reward modeling or engineering. This framework widely applies to multiple areas including robot training, stock-prediction, recommender systems, clinical trials, etc. (Novoseller et al. 2019; Sadigh et al. 2017; Christiano et al. 2017; Kupcsik et al. 2018; Jain et al. 2013; Wirth et al. 2017).

While the problem of PbRL was introduced almost a decade ago, most work in it has been primarily ap-
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To the best of our knowledge, we are the first to formulate and analyze the finite time regret guarantee for preference-based linear bandits problem with non-Markovian reward models (Sec. 2).

2. We propose an algorithm for known transition dynamics which is shown to yield a regret guarantee of $\tilde{O}\left(\sqrt{d} \log(T/\delta) \sqrt{T}\right)$ (Sec. 3).

3. We further generalize our algorithm to the case of unknown models and propose an algorithm with regret guarantee $\tilde{O}(\sqrt{d} + H^2 + |S|)\sqrt{dT} + \sqrt{|S||A||H|}$ (Sec. 4).

Related Work. Over the last two decades the problem of learning from preference feedback in bandits, known as dueling bandits, has gained much attention (Yue et al., 2012; Zoghi et al., 2014b, 2015). Dueling bandits generalizes the standard multi-armed bandit (MAB) (Auer et al., 2002). The goal is to identify a set of ‘good’ arms from a larger fixed set of arms by querying preference feedback for pairs of actively chosen arms. Yue and Joachims (2009, 2011); Saha and Krishnamurthy (2022); Ghoshal and Saha (2022); Saha and Gopalan (2018a) The setting is relevant in various real-world systems which aim to collect information from user preferences, including recommender systems, retail management, search engine optimization, job scheduling, etc. Towards these goals, several algorithms have been proposed (Ailon et al., 2014; Zoghi et al., 2014a; Komiyama et al., 2015; Gajane et al., 2015; Saha and Gopalan (2018b, 2019).

Though there has been a fair amount of research for preference-based bandits (no state information), few works consider incorporating preference feedback in the reinforcement learning (RL) framework, which considers the problem of long-term objectives over a Markov decision process (Singh et al., 2002; Ng et al., 2006; Talebi and Maillard, 2018; Ortner, 2020; Zhang and Ji, 2019; Zanette and Brunskill, 2019). However the classical RL setup assumes access to reward feedback for each state-action pair which might be impractical in many real world scenarios. Few very recent works considered training RL agents based on general trajectory-based reward which are available only at the end of each trajectory (Efroni et al., 2021; Chatterji et al., 2021), but their setting still assumes access to absolute reward feedback, unlike the case in PbRL. Some initial works consider the applied PbRL problem inspired by the problems of reward hacking, reward shaping, difficulty to model infinite rewards or multi-objective trade-offs (Busa-Fekete et al., 2014; Wirth et al., 2016, 2017; Christiano et al., 2017) etc.

Novoseller et al. (2019) made the first attempt to analyze the finite $T$-round regret guarantee for the PbRL

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1. The notation $\tilde{O}(\cdot)$ hides logarithmic factors in $T, H, |A|, |S|$. 

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Problem Setup (informal). Consider a $T$-round, $H$-horizon MDP $(P, S, A, H)$, with $S$ and $A$ being the finite sets of states and actions, respectively, and $P$ representing the transition dynamics of the MDP. We consider a real-valued score function $s(\tau)$, which is neither known nor queryable, that scores a given trajectory $\tau$. We assume preference of any two trajectories $\tau_1$ and $\tau_2$ is determined by their underlying score difference, i.e. $P(\tau_1 \succ \tau_2) = \sigma(s(\tau_1) - s(\tau_2))$, $\sigma : \mathbb{R} \mapsto [0, 1]$ being a suitable link-function. In particular, we assume $s(\cdot)$ is an (unknown) linear function of the trajectory-feature $\phi(\tau) \in \mathbb{R}^d$, and the link-function $\sigma$ is the sigmoid [Li et al., 2017]. The goal is to minimize the regret with respect to the optimal policy.

An important thing to note is that in our setting the trajectory features $\phi(\tau) \in \mathbb{R}^d$ are not necessarily sum-decomposable (over individual state-action features of the trajectory) and the underlying reward function is non-Markovian. In this case, the optimal policy may be history dependent. This is more general than assuming the trajectory reward is a linear function of the sum of per-state features, e.g. in [Novoseller et al., 2019]. Under the latter more limiting assumption, the traditional bandit techniques can be easily used to derive regret guarantees. Since the number of history dependent policies is super exponential, to deal with this more general setting we first show the log-covering number of the history dependent policies that are an optimal policy of an MDP with a trajectory score specified by our form of trajectory feedback is upper bounded by a polynomial quantity. Our specific contributions are as follows:

1. To the best of our knowledge, we are the first to formulate and analyze the finite time regret guarantee for preference-based linear bandits problem on $H$-length trajectories.
problem with trajectory preference feedback, where the learner is allowed to run two independent trajectories in parallel and receive 0/1 preference feedback after every such \( h \)-length roll out. Assuming an underlying MDP model, the preference between two \( h \)-length trajectories is modeled as being proportional to the accumulated reward of the corresponding trajectories. The authors propose a Double Posterior Sampling (DPS) technique with asymptotically sublinear regret.

Xu et al. models reward-free trajectory preferences and analyses the sample complexity of finding the \( \epsilon \)-best-policy. Their proposed algorithm crucially depends on an underlying dueling bandit black box whose performance guarantee is restricted to preference structures like Strong Stochastic Transitivity and Stochastic Triangle Inequality. Furthermore, the algorithms proposed in this work are not shown to enjoy provably optimal sample complexity, and, moreover, the fundamental performance limit of sample complexity is also not explicitly analyzed.

The literature of multi-agent reinforcement learning in Markov games closely relates to the setup of PbRL which attempts the problem of reaching Nash equilibrium of a simultaneous move markov game based on per-state win-loss feedback of the two (or multiple) players. Bai and Jin (2020); Bai et al. (2020); Liu et al. (2021) address the problems from finite action two player markov games, while Xie et al. (2020) extended this setting to zero sum games with linear function approximation. However, all these works analyzed the episodic sample complexity of the learning algorithm towards finding an \( \epsilon \)-approximate Nash equilibrium which is fairly unrelated to the regret objective of PbRL problem we considered in this paper.

Another closely related sub-field of RL, imitation learning, addresses the objective of learning optimal behavior from trajectories suggested by an expert. In Ng et al. (2000); Boularias et al. (2011); Neu and Szepesvári (2012); Wulfmeier et al. (2015), inverse reinforcement learning problems have been considered, where the objective is to extract (unknown) reward function from the trajectories given by an oracle or expert. Once the reward functions are computed, any RL algorithm could, in principle, be applied to compute the optimal policy. Ho and Ermon (2016) propose a generative adversarial network based imitation learning algorithm that computes the optimal policy directly from the trajectories of expert. Our work is fundamentally different in the sense that we do not receive trajectories or optimal actions from an expert. Instead, we get preferences over sample trajectories that are posed as queries to a system expert for preference feedback.

2 PROBLEM SETUP

Notation. Let \( [n] \) denote the set \( \{1, 2, \ldots, n\} \). Given a set \( A \), for any two items \( x, y \in A \), we denote that \( i \) is preferred over \( j \) by \( x \succ y \). By \( B_r(d) \) we denote the \( \ell_2 \)-norm ball of radius \( r \) in dimension \( d \). Lower case bold letters denote vectors, upper case bold letters denote matrices.

RL Model. Consider a \( T \)-episode, \( H \)-horizon RL setup \( (\mathcal{P}, \mathcal{S}, \mathcal{A}, H, \rho) \), \( \mathcal{S} \) is a finite set of states, \( \mathcal{A} \) is a set of actions, \( \mathcal{P}(\cdot | s, a) \) is the MDP transition dynamics given a state and action pair \( (s, a) \), \( H \in \mathbb{N} \) is the length of an episode, \( \rho \) denotes the initial distribution over states.

We denote a trajectory by concatenation of all states and actions visited during \( H \) steps \( \tau \) := \((s_1, a_1, \cdots, s_H, a_H)\). In general let \( \tau_h: H \) := \((s_h, a_h, \cdots, s_H, a_H)\) denote the states and actions from step \( h \) until the end of the episode. We denote by \( \tau_h \) to be all the states and actions taken up to step \( h \) and define \( \tau_0 = \emptyset \). Let \( \Gamma \) be the set of all possible trajectories of length \( H \), similarly \( \Gamma_h \) denotes the set of all sub-trajectories up to step \( h \). We use the superscript \( t \) as in \( \tau^t \) to denote a trajectory sampled during the \( t \)-th episode. At the start of each episode, we assume the initial state \( s_1 \) is drawn from a fixed distribution \( \rho \) known to the learner apriori (for example concentrated on an initial state \( s_0 \)).

Trajectory embedding. For any trajectory \( \tau \) we assume the existence of a trajectory embedding function \( \phi: \Gamma \rightarrow \mathbb{R}^d \). We denote by \( \phi(\tau) \) to the \( d \)-dimensional embedding of trajectory \( \tau \). The map \( \phi \) is known to the learner. One special case of such a trajectory-dependent feature map is a decomposed embedding, where \( \phi(\tau) = \sum_{h=1}^{H} \phi(s_h, a_h) \) and \( \phi: \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}^d \) is a mapping from state-actions pairs to \( \mathbb{R}^d \). Examples of these trajectory embeddings can be borrowed from the Behavior Guided class of algorithms for policy optimization found in Pacchiano et al. (2020). Many practically relevant trajectory or state-action embeddings can be found defined in Pacchiano et al. (2020); Parker-Holder et al. (2020). It is conceivable the preference model may be based on one of these embedding maps.

Policy embedding. The above feature embedding also leads to a natural mean embedding of any policy \( \pi: \mathcal{S} \rightarrow \mathcal{A} \) given by \( \phi(\pi) := \mathbb{E}_{x \sim \pi}[\phi(x)] \).

Preference modeling. Assuming \( \mathbf{w}^* \in \mathbb{R}^d \) to be an unknown vector, we define the pairwise-preference of trajectory \( \tau_1 \) over \( \tau_2 \) as:

\[
\mathbb{P}(\tau_1 \succ \tau_2) = \sigma((\phi(\tau_1) - \phi(\tau_2), \mathbf{w}^*)) = \frac{\exp(\phi(\tau_1)^\top \mathbf{w}^*)}{\exp(\phi(\tau_1)^\top \mathbf{w}^*) + \exp(\phi(\tau_2)^\top \mathbf{w}^*)}
\]
where $\sigma : \mathbb{R} \rightarrow [0, 1]$ is the logistic link function, i.e. $\sigma(z) = \frac{1}{1 + e^{-z}}$. We can ‘lift’ the definition of a comparison from trajectories to policies by setting,
\[
P(\pi_1 > \pi_2) = \sigma(\phi(\pi_1) - \phi(\pi_2), w^*) \tag{2}
\]
Equation (2) says the probability of any trajectory $\tau_1$ being preferred over $\tau_2$ is essentially proportional to the score difference of the individual trajectories, assuming the score for any trajectory $s$ is defined as the function
\[
s(\tau) := \langle \phi(\tau), w^* \rangle.
\]
The linear score of any policy $\pi$ (expectation over trajectories) can be similarly defined as $s(\pi) := \mathbb{E}_{\tau \sim \pi}[\langle \phi(\tau), w^* \rangle]$ and therefore $P(\pi_1 > \pi_2) = \sigma(s(\pi_1) - s(\pi_2))$.

**Non markovian policy class.** The performance of all our algorithms will be measured against the policy that maximizes $s(\tau)$. Since $s(\tau)$ may be a non-markovian function of the trajectory, the policy optimizing this objective need not be markovian. We therefore set $\Pi$ as the set of all history dependent policies. In contrast with standard markovian RL works, this is one of the main sources of technical complexity of our setting.

**Assumption 1.** [Bounded parameter] We assume that $\|w^*\| \leq W$ for some known $W > 0$.

**Assumption 2.** [Bounded feature maps] For all trajectories $\tau$ we assume that $\|\phi(\tau)\| \leq B$ for some known $B > 0$.

**Definition 1.** The degree of non-linearity of the sigmoid $\sigma$ over the parameter space (denoting the first derivative of $\sigma$ by $\sigma'$) is given by
\[
\kappa := \sup_{x \in B_{\theta}(d), w \in B_{s}(d)} \frac{1}{\sigma'(w^T x)}.
\]

**Objective:** Alternative: The objective of the learner is to minimize regret by finding policies to maximize the sum of their expected scores over $T$ rounds. At each round $t$, the learner proposes two policies, $\pi_1^t$ and $\pi_2^t$, which are executed in the MDP generating trajectories $\tau_1^t$ and $\tau_2^t$. The learner then receives feedback in the form of the Bernoulli variable $o_t \in \{0, 1\}$ which specifies whether $\tau_1^t$ is preferred ($o_t = 1$) or $\tau_2^t$ is preferred ($o_t = 0$). The preference feedback $o_t$ is distributed according to $P(\tau_1^t > \tau_2^t)$. We measure the learner’s performance via its pseudo-regret w.r.t. policy class $\Pi$, which we define as:
\[
R_T^{\text{pref}} := \max_{\pi \in \Pi} \sum_{t=1}^{T} \left[ (2\phi(\pi) - \phi(\pi_1^t) - \phi(\pi_2^t))^T w^* \right]
\]
where $w^* := \max_{\pi \in \Pi} s(\pi)$. This essentially measures the performance of the learner at round $t$ in terms of average score of the played policies $\pi_1^t, \pi_2^t$ w.r.t. the score maximizing policy $\pi^*$.

**Remark 1.** An important thing to note is that representing any trajectory pair $(\tau_1, \tau_2)$ by the feature $\langle \phi(\tau_1) - \phi(\tau_2) \rangle \in \mathbb{R}^d$, our underlying preference model is similar to reward model of Chatterji et al. (2021) (see Assumption 2.1). The fundamental difference between our setting and that of Chatterji et al. (2021) is the nature of the dueling feedback. In our work, the only way to gather any information when interacting with the world is by comparing the trajectories of two different policies. This adds a layer of complexity not present in the per trajectory feedback model from Chatterji et al. (2021), that makes their algorithms not immediately applicable to our setting.

One may think of using our preference model (Equation 2) to define an alternative notion of regret:
\[
R_T^{\text{pref}} := \max_{\pi \in \Pi} \sum_{t=1}^{T} P(\pi > \pi_1^t) + P(\pi > \pi_2^t) - 1. \tag{4}
\]

Fortunately, these two notions of regret can be shown to be ‘equivalent’ in the following sense,

**Claim 1.** Let $\pi^* \in \arg\max_{\pi \in \Pi} s(\pi)$. Then $\pi^*$ also achieves the max in Eqn. (4).

The logistic link function is increasing w.r.t. its argument, thus for any $\pi, \pi'$ we have $P(\pi^* > \pi') = \sigma(s(\pi^*) - s(\pi')) \geq \sigma(s(\pi) - s(\pi')) = P(\pi > \pi')$. It follows that for all $t$ and all $\pi$:
\[
P(\pi^* > \pi_1^t) + P(\pi^* > \pi_2^t) \geq P(\pi > \pi_1^t) + P(\pi > \pi_2^t)
\]
thus establishing the claim.

This argument can also be used to show $R_T^{\text{pref}}$ and $R_T^{\text{pref}}$ are equivalent up to constant factors when $B, W \leq 1$.

The proof is given in Appendix A.

**Claim 2.** $\frac{R_T^{\text{pref}}}{2\kappa(c+\gamma)} \leq R_T^{\text{pref}} \leq \frac{R_T^{\text{pref}}}{2}$.

We conclude that a strategy that attains sublinear $R_T^{\text{pref}}$ regret also has sublinear $R_T^{\text{pref}}$ regret.
3 KNOWN MODEL: ALGORITHM AND ANALYSIS

In this section, we introduce and analyze an algorithm for solving the preference-based RL problem when the transition model, $P$, that governs the probability of transitioning to a next state is known to the learner. In this case, it becomes possible to directly compute expected features induced by policies; however, the difficulty of learning based only on preference feedback as opposed to rewards remains. This is because we have access to feedback only through relative preferences on the trajectories rather than an assumed known reward function. Before stating the algorithm, we first detail a method of estimating the underlying parameter $w_*$ in the logistic model. This procedure serves as a basis for the algorithm.

3.1 Maximum Likelihood Estimation

In the logistic model, a natural way of computing an estimator $w_t$ of $w^*$ given trajectory pairs $\{(\tau^1_t, \tau^2_t)\}_{t=1}^T$ and preference feedback values $\{o_t\}_{t=1}^T$ is via maximum likelihood estimation. At time $t$ the regularized log-likelihood (or negative cross-entropy loss) of a parameter $w$ can be written as:

$$ L_t^\lambda(w) = \sum_{t=1}^{T} \left( o_t \log((\phi(\tau^1_t) - \phi(\tau^2_t), w)) - \frac{\lambda}{2} \|w\|_2^2 \right) + (1 - o_t) \log(\mu((\phi(\tau^1_t) - \phi(\tau^2_t), w)), \lambda > 0 is a regularization parameter. The function $L_t^\lambda$ is strictly concave for $\lambda > 0$.

The maximum likelihood estimator $\hat{w}_t^{\text{MLE}}$ can be written as $\hat{w}_t^{\text{MLE}} = \arg\max_{w \in \mathbb{R}^d} L_t^\lambda(w)$. Unfortunately, $\hat{w}_t^{\text{MLE}}$ may not satisfy the boundedness Assumption 1, so we instead make use of a projected version of $\hat{w}_t^{\text{MLE}}$. Following Faury et al. (2020), and recalling Assumption 1, we define a data matrix and a transformation of $\hat{w}_t^{\text{MLE}}$ given by:

$$ V_t = \kappa \Lambda d + \sum_{t=1}^{T} \left( \phi(\tau^1_t) - \phi(\tau^2_t) \right) \left( \phi(\tau^1_t) - \phi(\tau^2_t) \right)^T $$

$$ g_t(w) = \sum_{t=1}^{T} \sigma((\phi(\tau^1_t) - \phi(\tau^2_t), w)) \left( \phi(\tau^1_t) - \phi(\tau^2_t) \right) + \lambda w $$

Then, the projected parameter, along with its confidence set, is given by:

$$ w_t^* = \arg\min_{w \text{ s.t. } \|w\| \leq W} \|g_t(w) - g_t(\hat{w}_t^{\text{MLE}})\|_{V_t^{-1}} $$

$$ C_t(\delta) = \{w \text{ s.t. } \|w - w_t^*\|_{V_t} \leq 2\kappa \beta_t(\delta) \} $$

where $\beta_t(\delta) = \sqrt{\kappa W} + \sqrt{\log(1/\delta) + 2d \log \left( 1 + \frac{TB}{\kappa d} \right)}$. We restate a bound by Faury et al. (2020) that shows the probability of $w_*$ being in $C_t(\delta)$ for all $t \geq 1$ can be lower bounded.

**Lemma 1.** [Lemma 1 from Faury et al. (2020)] Let $\delta \in (0, 1]$ and define the event that $w_*$ is in the confidence interval $C_t(\delta)$ for all $t \in \mathbb{N}$:

$$ E_\delta = \{\forall t \geq 1, w_* \in C_t(\delta)\} $$

Then $\mathbb{P}(E_\delta) \geq 1 - \delta$.

3.2 Algorithm and Analysis

We are now ready to state the Logistic Preference based Reinforcement Learning (LPbRL) algorithm with known model, shown in Algorithm 1.

Before any interaction or feedback, we initialize identical data matrices $V_1 = V_1 = \kappa \Lambda d$, $\lambda > 0$ being a regularization parameter. $V_t$, as defined before, is designed to track the exact covariates used in the maximum likelihood estimation. $V_t$ (Line 10) on the other hand tracks a similar quantity, but instead uses the expected features under a given policy.

At each round $t$, we then compute an estimate $w_t^*$ and determine a set of candidate policies $\Pi_t$ for which no other policy $\pi$ significantly outperforms a member of $\Pi_t$. The threshold for what constitutes “significant” is determined by the uncertainty in the estimate of $w_t^*$. We then search over this set to identify two policies, $\pi^1_t$ and $\pi^2_t$, with expected features that maximize the uncertainty determined by $V_t$, precisely by choosing $(\pi^1_t, \pi^2_t) = \arg\max_{\pi^1, \pi^2 \in \Pi_t} \|\phi(\pi^1) - \phi(\pi^2)\|_{V_t^{-1}}$. Both policies are deployed, inducing trajectories $\tau^1_t$ and $\tau^2_t$ and feedback $o_t$ is received. We then update the data matrices $V_t$ and $V^t$ with the trajectory features $\phi(\tau^1_t) - \phi(\tau^2_t)$ and expected features $\phi(\pi^1_t) - \phi(\pi^2_t)$, respectively.

The procedure is repeated for each round $t \in [T]$.

**Theorem 1.** Let $\delta \leq 1/e$ and $\lambda \geq \frac{B}{\kappa}$. Then, with probability at least $1 - \delta$, the expected regret of Algorithm 1 can be bounded by

$$ R_t \leq (4\kappa \beta_t(\delta) + 2\alpha_{d,T}(\delta)) \sqrt{2Td \log \left( 1 + \frac{TB}{\kappa d} \right)} $$

Note there is no dependence on the size of the state or action spaces on account of the model being known in this setting. Furthermore, we note that any dependence on the horizon $H$ is effectively accounted for in the size
of the constant $B$ that bounds the norm of the trajectory features $\phi(\tau)$. For example, if $\phi(\tau)$ decomposes in a per-timestep fashion as $\phi(\tau) = \sum_{h \in [H]} \phi(s_h, a_h)$ where each $h$ satisfies $\|\phi(s_h, a_h)\| \leq B'$, then a trivial bound would give $B \leq B'H$. However, Assumption 2 allows for greater generality.

Algorithm 1 LPbRL: Regret minimization (Known Model)

1: **input:** Regularization parameter $\lambda$, Learning rate $\eta > 0$, exploration length $t_0 > 0$
2: Define $a_{d,T}(\delta) = 20BW\sqrt{d\log(T(1+2T)/\delta)}$ and $\gamma(\delta) = 2\kappa^2\beta_t(\delta) + a_{d,T}(\delta)$.
3: Initialize $\overline{V}_t = \kappa \lambda \eta$
4: for $t = 1, 2, \ldots, T$ do
5: Compute $w_t^2$ (see Eqn. Eq. (5))
6: Set $\Pi_t = \{\pi^1(\phi(\pi^1) - \phi(\pi^2))w_t^2 + \gamma(\delta)||\phi(\pi^1) - \phi(\pi^2)||\overline{V}^{-1} \geq 0 \forall \pi\}$
7: Compute $(\pi^1_t, \pi^2_t) = \arg\max_{\pi^1, \pi^2 \in \Pi_t} ||\phi(\pi^1_t) - \phi(\pi^2)||\overline{V}^{-1}$.
8: Sample $\tau^1_t \sim \pi^1_t$ and $\tau^2_t \sim \pi^2_t$.
9: Play the duel $(\tau^1_t, \tau^2_t)$ and receive $a_t = \mathbb{I}(\tau^1_t$ beats $\tau^2_t)$
10: Update $\overline{V}_{t+1} = \overline{V}_t + (\phi(\pi^1_t) - \phi(\pi^2_t))(\phi(\pi^1_t) - \phi(\pi^2_t))\top$
11: end for

Remark 2. Theorem 1 shows that for a sufficiently large choice of the regularization parameter $\lambda$, the pseudo-regret of Algorithm 1 is at most $R_t = O\left(W\sqrt{\kappa B} + WB\right)d\log(TB/\delta)\sqrt{T}$. Importantly, the regret scales nearly optimally with $d\sqrt{T}$ dependence given existing $\Omega(d\sqrt{T})$ lower bounds for linear bandits [Lattimore and Szepesvári 2020] and known reductions between the standard and preference regret [Saha 2021]. Assuming $\kappa$ to be constant, we pay the additional factors in $B$ and $W$ due to non-Markovian rewards which are only indirectly revealed to the learner in terms of preferences.

3.3 Regret Analysis: Proof Sketch of Thm. 1

We now sketch the proof of Theorem 1. Details and proofs of supporting results can be found in Appendix B.1. The main idea of the proof is to ensure that $\Pi_t$ contains only candidate policies that are predicted to be “sufficiently good” under the learned model $w_t^2$ using the size of the confidence set $C_t(\delta)$. We must also verify that $\Pi_t$ always contains the optimal policy $\pi^*$. Thus, as long as the set $C_t(\delta)$ shrinks at a sufficiently fast rate, our algorithm will have sublinear regret.

However, in order to judge the uncertainty in predictions of the expected feature $\phi(\pi)\top w_t^2$ of a policy $\pi$, we must relate the data matrix $V_t$ that controls the accuracy of the learned parameter $w_t^2$ (see Lemma 1), and its expected counterpart $\overline{V}_t$ (used to define $\Pi_t$). The set $\Pi_t$ is characterized via $\overline{V}_t$ because this way it allows us to relate it to the algorithm’s regret, a quantity that depends on the expected features of the played policies. Corollary 1 establishes that distances $||w_t^2 - w^*||\overline{V}_t$ weighted by $\overline{V}_t$ are not too far from the same distances $||w_t^2 - w^*||V_t$ weighted by $V_t$. Let

$$E_{\text{prec}} = \{\overline{V}_T \leq 2V_T + 84B^2d\log((1+2T)/\delta)\delta\}.$$ 

Corollary 1. Under Assumption 7 conditioned on event $\mathcal{E}_\delta \cap E_{\text{prec}}$, for any $t \in [T]$

$$||w^* - w_t^2||\overline{V}_t \leq 4\kappa \beta_t(\delta) + a_{d,T}(\delta),$$

where $a_{d,T}(\delta) = 20BW\sqrt{d\log(T(1+2T)/\delta)}$. Furthermore, if $\delta \leq 1/\epsilon$, then $P(\mathcal{E}_\delta \cap E_{\text{prec}}) \geq 1 - \delta - \delta \log_2 T$.

The proof of above is given in Appendix B.2. Leveraging this relationship, we can establish that the confidence set of policies $\Pi_t$ defined in line 5 of Algorithm 1 will contain the optimal policy.

Lemma 2. Conditioned on event $\mathcal{E}_\delta \cap E_{\text{prec}}$, $\pi^* \in \Pi_t$.

The remainder of the proof now consists of showing the instantaneous regret can be bounded in terms of the size of the confidence sets and the uncertainty values $||\phi(\pi^1_t) - \phi(\pi^2_t)||\overline{V}_t$. We defer the final details to Appendix B.3.

4 UNKNOW MODEL: ALGORITHM AND ANALYSIS

Algorithm description. The LPbRL algorithm for unknown dynamics models works in a similar way to Algorithm 1. The main differences lay in the definition of the set $\Pi_t$. Whereas in Algorithm 1 this set of policies can be defined without taking into account the model uncertainty, in this case the set of policies to optimize over needs to be carefully constructed in such a way that it can be shown to contain $\pi_\star$ (see Lemma 3). With this in mind we start by introducing the necessary technical tools that will be used throughout this section to deal with model uncertainty.

4.1 Analysis of instantaneous regret:

For any policy $\pi$ and any MDP model $P$, we denote by $\phi^{\pi}(\pi)$ to the mean feature of policy $\pi$ in model $P$. 

We use the notation $N_t(s, a)$ to denote the number of samples of action $a$ at state $s$ the learner has collected up to time $t$. We use the notation $\tilde{\mathbb{P}}_t$ to denote the empirical model at time $t$. We use an ‘empirical’ version of $\tilde{\mathbb{V}}_t$ defined using the average features computed using the model available at time $t$:

$$
\tilde{\mathbb{V}}_t = \kappa \Lambda_d + \sum_{t=1}^{t-1} \left( \phi_\pi^t(\pi_t^i) - \phi_\pi^t(\pi_t^j) \right) \left( \phi_\pi^t(\pi_t^i) - \phi_\pi^t(\pi_t^j) \right) ^ {\top}
$$

(7)

Our confidence intervals will use a Mahalanobis norm defined by this covariance matrix. Throughout this section we will make heavy use of some of the results from [Chatterji et al. (2021)]. With that in mind we will define a variety of bonus terms. Given any $\eta > 0$ define,

$$
\xi^{(t)}_{s, a}(\eta, \delta) = \min \left( 2\eta, 4\eta \sqrt{ \frac{U}{N_t(s, a)} } \right)
$$

s.t. $U = H \log(|S|) \log \left( \frac{6 \log(N_t(s, a))}{\delta} \right)$. We define the following ‘bonus’ function corresponding to the expectation of these bonus terms summed over a trajectory sampled from a policy $\pi$ in the model $\tilde{\mathbb{P}}_t$,

$$
\tilde{B}_t(\pi, \eta, \delta) = \mathbb{E}_{s_1, \ldots, s_T \sim \tilde{\mathbb{P}}_t(\cdot|s_1)} \left[ \sum_{h=1}^{H-1} \xi^{(t)}_{s_h, a_h}(\eta, \delta) \right].
$$

Similar to the previous theorem, we must relate $\|w^* - w_t^\pi\|_{\tilde{\mathbb{V}}_t}$ and $\|w^* - w_t^\pi\|_{\mathbb{V}_t}$. We do this via a series of Lemmas.

**Lemma 3.** Let $\tilde{E}_0$ be the event that for all $t \in \mathbb{N}$,

$$
\|w_t^\pi - w^*\|_{\mathbb{V}_t} \leq \sqrt{2} \|w_t^\pi - w^*\|_{\tilde{\mathbb{V}}_t} + \sqrt{ \sum_{\ell=1}^{t-1} \left( \tilde{B}_t \left( \pi, 2WB, \frac{\delta'}{8\delta^3 |A|^2} \right) \right)^2 \frac{1}{t} }.
$$

where $\delta' = \frac{\delta}{(t+1)^2}$ and $\epsilon = \frac{1}{c_\pi \lambda + 4B^2 + \sigma^2}$. Then $P(\tilde{E}_0) \geq 1 - \delta$.

The proof of Lemma 3 is in Appendix C.2. We now proceed to define the set of plausible policies $\Pi_t$. To do so, it will be useful to introduce the following confidence radius multiplier

$$
\gamma_t = \sqrt{2} (4\kappa \beta_t(\delta) + \alpha_d T(\delta)) + \frac{1}{t} + 
$$

$$
2 \sum_{\ell=1}^{t-1} \tilde{B}_t^2 \left( \pi_t^\ell, 2WB, \frac{\delta'}{8\delta^3 |A|^2} \right) + B_t^2 \left( \pi_t^\ell, 2WB, \frac{\delta'}{8\delta^3 |A|^2} \right).
$$

Finally,

$$
\Pi_t = \left\{ \pi \mid (\phi_\pi^t(\pi^i_t) - \phi_\pi^t(\pi^j_t))^\top w_t^\pi + \gamma_t \|\phi_\pi^t(\pi^i_t) - \phi_\pi^t(\pi^j_t)\|_{\mathbb{V}_t} \right\} 
$$

$$
+ \left\{ \pi \mid \gamma_t \|\phi_\pi^t(\pi^i_t) - \phi_\pi^t(\pi^j_t)\|_{\mathbb{V}_t} + \tilde{B}_t \left( \pi, 2SB, \frac{\delta}{2|A|} \right) \right\} \geq 0, \forall \pi \}
$$

**Algorithm 2** LPbRL: Regret minimization (Unknown Model)

1: **input:** Learning rate $\eta_t > 0$, exploration length $t_0 > 0$
2: **initialize** empirical model $\tilde{\mathbb{P}}_1$.
3: **for** $t = 1, 2, \ldots, T$ **do**
4: **compute** $w^\pi_t$ and $\Pi_t$.
5: **compute**

$$
\left( \pi^1_t, \pi^2_t \right) = \text{arg max}_{\pi^1, \pi^2 \in \Pi_t} \gamma_t \|\phi_\pi^t(\pi^i_t) - \phi_\pi^t(\pi^j_t)\|_{\mathbb{V}_t} + \tilde{B}_t \left( \pi^1_t, 2SB, \delta \right) + \tilde{B}_t \left( \pi^2_t, 2SB, \delta \right)
$$

6: **Sample** $\tau^1_t \sim \pi^1_t$ and $\tau^2_t \sim \pi^2_t$.
7: **Play** the duel $(\tau^1_t, \tau^2_t)$ and receive $o_t = 1(\tau^1_t \text{ beats } \tau^2_t)$.
8: **update**

$$
\mathbb{V}_{t+1} = \mathbb{V}_t + \left( \phi_\pi^t(\tau^1_t) - \phi_\pi^t(\tau^2_t) \right) \left( \phi_\pi^t(\tau^1_t) - \phi_\pi^t(\tau^2_t) \right)^\top.
$$

9: **update** empirical model and build $\tilde{\mathbb{P}}_{t+1}$.
10: **end for**

Algorithm 2 shares the structure of Algorithm 1. The main difference lies in the definition of $\Pi_t$ and in the optimization problem to find $(\pi^1_t, \pi^2_t)$. We can prove a result similar to Lemma 2 and show that $\pi^* \in \Pi_t$.

**Lemma 4.** Let $\tilde{E}_{t-1}$ be the event that $\pi^* \in \Pi_t$ for all $t \in \mathbb{N}$. Then $P(\tilde{E}_{t-1}) \geq 1 - 5\delta$.

The proof of Lemma 4 can be found in Appendix C.3. The next step in the proof is to exhibit a bound on the instantaneous regret.

**Lemma 5.** Let $\tilde{E}_2$ be the event that for all $t \in \mathbb{N}$,

$$
2r_t \leq (\phi_\pi^t(\pi^*) - \phi_\pi^t(\pi_t^i))^\top w^* + (\phi_\pi^t(\pi^*) - \phi_\pi^t(\pi_t^j))^\top w^* + \tilde{B}_t(\pi^*, 4WB, \delta) + \tilde{B}_t(\pi_t^i, 2WB, \delta) + \tilde{B}_t(\pi_t^j, 2WB, \delta)
$$

Then $P(\tilde{E}_2) \geq 1 - 2\delta$. 

Proof. Note that we can write:

\[
2r_t = (\phi(\pi^*) - \phi(\pi^+_t))^T w^* + (\phi(\pi^*) - \phi(\pi^+_t))^T w^* + 2(\phi(\pi^*) - \phi(\pi^+_t))^T w^* + \\
(\phi(\pi^-_t) - \phi(\pi^+_t))^T w^* + (\phi(\pi^-_t) - \phi(\pi^+_t))^T w^*
\]

By Lemma 4 (Lemma B.1 in Chatterji et al., 2021), we conclude that with probability at least \(1 - \delta\), for all \(t \in \mathbb{N}\), setting \(\eta = 4WB\),

\[
2(\phi(\pi^*) - \phi(\pi^+_t))^T w^* \leq \hat{B}_{tl}(\pi^*, 4WB, \delta)
\]

Similarly, as a consequence of Lemma 12 and a union bound, setting \(\eta = 2WB\), with probability at least \(1 - 2\delta\),

\[
(\phi(\pi^-_t) - \phi(\pi^+_t))^T w^* \leq \hat{B}_{tl}(\pi^-_t, 2WB, \delta)
\]

The result follows.

Armed with the results of Lemma 4 we can show the following bound for the regret.

Lemma 6. With probability at least \(1 - 15\delta\) the regret is bounded by,

\[
R_T \leq 2\gamma_T \sqrt{2Td \log \left(1 + \frac{TB}{d}\right)} + \\
\sum_{t \in [T]} 4\hat{B}_{tl}(\pi^+_t, 4WB, \delta) + 4\hat{B}_{tl}(\pi^-_t, 2WB, \delta)
\]

The proof of Lemma 6 can be found in Appendix C.4. The derivation follows from a repeated use of the instantaneous regret upper bound derived from Lemma 4.

The rest of the proof is dedicated to bound the \(\hat{B}_{tl}(\cdot)\) terms. The general idea is to relate these bonus expectations under the empirical model with an expected sum of bonus terms under the true model and sampled according to policies \(\pi^+_t\) and \(\pi^-_t\). Once this is achieved we have reduced the problem to bound a sum of vanishing markovian errors under the sampling distribution defined by the policies that were selected during optimization. This can be done via a similar argument as many existing RL works. Finally, we also show the \(\gamma_T\) term can be bounded by a term of the form \(\hat{O}(k\beta_1(\delta) + \alpha_dT(\delta) + \text{poly}(H, |S|, |A|))\), hides logarithmic factors in \(\delta, |S|\) and \(|A|\). A detailed discussion of these arguments can be found in Appendix C. Our final main result (simplified) is thus,

Theorem 2. The regret of LPbRL satisfies,

\[
R_T \leq \hat{O}(kd\sqrt{T} + H^{3/2} \sqrt{|S||A|dTH} + H|S|\sqrt{|A|dTH}).
\]

For all \(T \in \mathbb{N}\) simultaneously with probability at least \(1 - 15\delta\). Where \(\hat{O}\) hides logarithmic factors in \(\delta, |S|\) and \(|A|\).

The complete version of Theorem 2 can be found in Appendix C. Similar to Theorem 1, the leading term in the regret scales as \(\hat{O}(d\sqrt{T})\) due to estimation based on the preferences. In addition to this, we now have dependence on \(|S|\) and \(|A|\) unlike before. These arise due to the tabular nature of the problem since the transition dynamics are unknown in this case.

5 DISCUSSIONS AND FUTURE SCOPES

In this work we addressed the problem of reinforcement learning from relative preference feedback where the agent does not get to see the absolute reward of actions taken at each state but instead observes the relative preferences between trajectories. We modeled the preference feedback in terms of the underlying non-Markovian linear reward model and proposed algorithms for both known as well as unknown MDP transition models. Precisely the regret guarantees of our proposed algorithms are analyzed to be respectively \(\hat{O}(d\log(T/\delta)\sqrt{T})\) and \(\hat{O}((\sqrt{d} + H^2 + |S|)\sqrt{dT} + \sqrt{|S||A|dTH})\) for the case of known and unknown transition models.

As discussed in the introduction, preference-based reinforcement learning has applications in several fields including training robots, stock market, recommender systems, two player games, chatbot interactions, etc. Thus there are plenty of scopes to extend the above setup to incorporate the corresponding system requirements, e.g. generalizing dueling trajectory preferences to subsets, considering alternative preference feedback without assuming an underlying reward model, extending to infinite horizon settings with more complex state-actions spaces, etc. Analyzing the fundamental performance limits of the PbRL regret minimization problem and designing algorithms with tighter performance guarantees would also be another interesting direction to investigate.

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Supplementary for Dueling RL: Reinforcement Learning with Trajectory Preferences

A Appendix for Section 2

Claim 2. \( \frac{R_T^\text{pref}}{\sqrt{c+1}} \leq R_T^\text{pref} \leq \frac{R_T^\text{pref}}{2} \).
A.1 Proof of Claim 2

**Proof.** Recall by Eq. (3), Eq. (4) and Claim 1.

\[
R^{\text{rst}}_T = \sum_{t=1}^T \frac{2s(\pi^*) - \left(s(\pi^1_t) + s(\pi^2_t)\right)}{2},
\]

\[
R^{\text{pref}}_T := \sum_{t=1}^T \frac{P(\pi^* \succ \pi^1_t) + P(\pi^* \succ \pi^2_t) - 1}{2}.
\]

Now assume \( S, B < 1 \). Then for any two policies \( \pi_1 \) and \( \pi_2 \in \Pi \), such that \( s(\pi_1) \geq s(\pi_2) \), we have:

\[
P(\pi_1, \pi_2) - 1/2 = \frac{e^{s(\pi_1)} - e^{s(\pi_2)}}{2(e^{s(\pi_1)}) + e^{s(\pi_2)}}
\]

\[
\geq \frac{e^{s(\pi_1)} - s(\pi_2) - 1}{2(e^{s(\pi_1)} - s(\pi_2) + 1)}
\]

\[
> \frac{s(\pi_1) - s(\pi_2)}{2(e + 1)}
\]

On the other hand denoting \( x = s(\pi_1) - s(\pi_2) \in (0, 1) \) we get:

\[
P(\pi_1, \pi_2) - 1/2 = \frac{e^x - 1}{2(e^x + 1)} < \frac{(x + x^2/2! + x^3/3! + \ldots)}{4}
\]

\[
< \frac{x(1 + x/2 + x^2/2^2 + \ldots)}{4} < x/2
\]

The claim now follows combining the above two inequalities and noting that by definition \( \pi^* := \arg\max_{\pi \in \Pi} s(\pi) \).

B Appendix for Section 3

By first-order optimality conditions, \( \bar{w}^{\text{MLE}}_t \) is the point in \( \mathbb{R}^d \) satisfying:

\[
\nabla_w L^\lambda_t(\bar{w}^{\text{MLE}}_t) = \sum_{\ell=1}^{t-1} o_\ell \left( \phi(\tau^1_\ell) - \phi(\tau^2_\ell) \right)
\]

\[
- \sum_{\ell=1}^{t-1} \sigma(\phi(\tau^1_\ell) - \phi(\tau^2_\ell), w) \left( \phi(\tau^1_\ell) - \phi(\tau^2_\ell) \right) + \lambda w.
\]

B.1 Proof of Corollary 1

The primary mechanism behind Corollary 1 is the following lemma for matrix concentration.

**Lemma 7.** Let \( \delta \leq e^{-1} \). Then, with probability \( 1 - \delta \log_2 T \), for all \( t \in [T] \), it holds that

\[
|w^* - w^t_\ell|_V^2 \leq 2|w^* - w^t_\ell|_{\nabla_t}^2 + 84B^2d\log(T(1 + 2T)/\delta)\|w^* - w^t_\ell\|_V^2
\]

(8)
The proof of the corollary now follows immediately as a consequence.

Proof. Fix $v \in \mathbb{R}^d$ such that $\|v\|_2 = 1$. For $\ell \in [T]$, let $X_{\ell} = v^T (\phi(\tau_1^\ell) - \phi(\tau_2^\ell)) (\phi(\tau_1^\ell) - \phi(\tau_2^\ell))^T v$. Furthermore define $X_0 = \kappa \lambda$. Observe that $X_{\ell} - E_{\ell-1}X_{\ell}$ for $\ell = 0, \ldots, T$ is an $\{F_\ell\}$-adapted martingale difference sequence where $E[\cdot | F_{\ell}]$ denotes the conditional expectation $E[\cdot]$. Let $\mathcal{C}$ be the closest vector in the cover so that $\|v - \mathcal{C}\|_2 \leq B$. Let $\tilde{V}_t = \kappa \lambda I + \sum_{\ell \in [t]} E_{\ell-1} (\phi(\tau_1^\ell) - \phi(\tau_2^\ell)) (\phi(\tau_1^\ell) - \phi(\tau_2^\ell))^T$.

By (Bartlett et al., 2008 Lemma 2), we have, with probability at least $1 - \delta_2 T$,

$$v^T \tilde{V}_T v = \sum_{\ell=0}^T E_{\ell-1} X_{\ell} \leq \sum_{\ell=0}^T X_{\ell} + \sqrt{16 \log(1/\delta) \sum_{\ell=0}^T \text{var}_{\ell-1}(X_{\ell}) + 2B^2 \log(1/\delta)}$$

where the third line applied the AM-GM inequality. Rearranging shows that

$$\frac{1}{2} v^T \tilde{V}_T v \leq v^T V_T v + 34B^2 \log(1/\delta)$$

This holds for a fixed $v$. We now show that it approximately holds for all $v$ such that $\|v\|_2 = 1$ via a covering argument.

Let $\mathcal{C}$ be a minimal $\epsilon$-cover of $\Pi^{d-1} = \{v \in \mathbb{R}^d : \|v\|_2 = 1\}$. A standard result states that $|\mathcal{C}_\epsilon| \leq (1 + 2/\epsilon)^d$. Then, by the union bound, with probability $1 - \delta \log_2 T$, for all $v \in \mathcal{C}_\epsilon$,

$$\frac{1}{2} v^T \tilde{V}_T v \leq v^T V_T v + 34B^2 d \log((1 + 2/\epsilon)/\delta)$$

Let $A = \frac{1}{2} V_T - \tilde{V}_T$. Note that $\|A\| \leq 4B^2 T$ by definition. Let $v \in \Pi^{d-1}$ be arbitrary and let $u_v \in \mathcal{C}_\epsilon$ be the closest vector in the cover so that $\|v - u_v\|_2 \leq \epsilon$. Then,

$$v^T Av = v^T A v + u_v^T A u - u_v^T A u$$

under the good event and choosing $\epsilon = 1/T$. Since this holds for all $v \in \Pi^{d-1}$, we conclude that

$$\tilde{V}_T \leq 2V_T + 84B^2 d \log((1 + 2T)/\delta)I_d$$

with probability at least $1 - \delta \log_2 T$. Finally, by Jensen’s inequality we have $V_T \leq \tilde{V}_T$. Then, we apply the union bound over $t \in [T]$, which gives the result.

The proof of the corollary now follows immediately as a consequence.
Corollary 1. Under Assumption $[\text{I}]$, conditioned on event $\mathcal{E}_\delta \cap \mathcal{E}_{\text{prec}}$, for any $t \in [T]$,

$$\|w^* - w^*_t\|_{\nabla_t} \leq 4\kappa\beta(\delta) + \alpha_{d,T}(\delta),$$

where $\alpha_{d,T}(\delta) = 20BW\sqrt{d\log(T(1 + 2T)/\delta)}$. Furthermore, if $\delta \leq 1/e$, then $\mathbb{P}(\mathcal{E}_\delta \cap \mathcal{E}_{\text{prec}}) \geq 1 - \delta - \delta \log_2 T$.

Proof of Corollary $[\text{I}]$. Assuming that $\mathcal{E}_\delta$ holds, we have that $\|w^* - w^*_t\|_{\nabla_t} \leq 2\kappa\beta(\delta)$. Furthermore, Lemma $[\text{II}]$ gives

$$\|w^* - w^*_t\|_{\nabla_t} \leq \sqrt{2}\|w^* - w^*_t\|_{\nabla_t} + 10B\sqrt{d\log(T(1 + 2T)/\delta)}\|w^* - w^*_t\|_2 \leq 4\kappa\beta(\delta) + 20BS\sqrt{d\log(T(1 + 2T)/\delta)}$$

Also, $\mathbb{P}(\mathcal{E}_\delta \cap \mathcal{E}_2) \geq 1 - \delta - \delta \log_2 T$ follows from above combined with the claim of Lemma $[\text{I}]$.

B.2 Proof of Lemma $[\text{II}]$

Lemma 2. Conditioned on event $\mathcal{E}_\delta \cap \mathcal{E}_{\text{prec}}$, $\pi^* \in \Pi_t$.

Proof. Condition on $\mathcal{E}_\delta \cap \mathcal{E}_{\text{prec}}$. By definition of $\pi^*$, we have $(\phi(\pi^*) - \phi(\pi))^\top w^* \geq 0$ for any arbitrary $\pi$. This implies

$$0 \leq (\phi(\pi^*) - \phi(\pi))^\top w^*_t + \|\phi(\pi^*) - \phi(\pi)\|_{\nabla_t^{-1}} \cdot \|w^* - w^*_t\|_{\nabla_t}$$

$$\leq (\phi(\pi^*) - \phi(\pi))^\top w^*_t + (4\kappa\beta(\delta) + \alpha_{d,T}(\delta)) \cdot \|\phi(\pi^*) - \phi(\pi)\|_{\nabla_t^{-1}}$$

where the second line follows from Corollary $[\text{I}]$.

B.3 Proof of Theorem $[\text{II}]$

We require a standard determinant bound to complete the proof.

Lemma 8. Let $\lambda \geq B$. Consider the sequence $v_1, \ldots, v_T \in \mathbb{R}^d$ such that $\|v_i\| \leq B$ and define $V_t = \lambda I + \sum_{s \in [t]} v_s v_s^\top$. Then,

$$\sum_{t \in [T]} \|v_t\|^2_{V_{t-1}^{-1}} \leq 2d\log \left(1 + \frac{TB}{d}\right)$$

Proof. Since $\lambda \geq B$, we have that $\|v_t\|_{V_{t-1}^{-1}} \leq 1$. Therefore, from Lemma 19.4 of Lattimore and Szepesvári (2020)

$$\sum_{t \in [T]} \|v_t\|^2_{V_{t-1}^{-1}} \leq \sum_{t \in [T]} \log \left(1 + \|v_t\|^2_{V_{t-1}^{-1}}\right) \leq 2d \log \left(\frac{d\lambda + TB^2}{d\lambda}\right) \leq 2d \log \left(1 + \frac{TB}{d}\right)$$

Theorem 1. Let $\delta \leq 1/e$ and $\lambda \geq \frac{B}{\kappa^2}$. Then, with probability at least $1 - \delta$, the expected regret of Algorithm $[\text{II}]$ can be bounded by

$$R_t \leq (4\kappa\beta(\delta) + 2\alpha_{d,T}(\delta)) \sqrt{2Td\log \left(1 + \frac{TB}{kd}\right)}.$$
Proof of Theorem 1. Armed with the supporting results, we now focus on completing the proof of Theorem 1. The result may be shown by bounding the instantaneous regret. Condition on the event $E_{\delta}$. Then,

$$2r_t := (\phi(\pi^*) - \phi(\pi^*_t))^T w^* + (\phi(\pi^*) - \phi(\pi^*_t))^T w^*$$

$$= (\phi(\pi^*) - \phi(\pi^*_t))^T w^*_t + (\phi(\pi^*) - \phi(\pi^*_t))^T (w^* - w^*_t) + (\phi(\pi^*) - \phi(\pi^*_t))^T w^*_t$$

$$+ (\phi(\pi^*) - \phi(\pi^*_t))^T (w^* - w^*_t)$$

$$\leq (\phi(\pi^*) - \phi(\pi^*_t))^T w^*_t + (\phi(\pi^*) - \phi(\pi^*_t))^T w^*_t$$

$$+ \|w^* - w^*_t\|v_t \cdot \|\phi(\pi^*) - \phi(\pi^*_t)\|v_t^{-1} + \|w^* - w^*_t\|v_t \cdot \|\phi(\pi^*) - \phi(\pi^*_t)\|v_t^{-1}$$

The last two terms in the above sum can be bounded using Corollary 1 as follows:

$$\|w^* - w^*_t\|v_t \cdot \|\phi(\pi^*) - \phi(\pi^*_t)\|v_t^{-1} + \|w^* - w^*_t\|v_t \cdot \|\phi(\pi^*) - \phi(\pi^*_t)\|v_t^{-1}$$

The first two terms leverage the optimistic bonus, using the fact that $\pi^*_t, \pi^*_t \in S_t$:

$$\leq (2\kappa T_d(\delta) + \alpha T_d(\delta)) \cdot \left( \|\phi(\pi^*) - \phi(\pi^*_t)\|v_t^{-1} + \|\phi(\pi^*) - \phi(\pi^*_t)\|v_t^{-1} \right)$$

In summary, we have that the instantaneous regret is upper bounded as

$$2r_t \leq 2(2\kappa T_d(\delta) + \alpha T_d(\delta)) \cdot \left( \|\phi(\pi^*) - \phi(\pi^*_t)\|v_t^{-1} + \|\phi(\pi^*) - \phi(\pi^*_t)\|v_t^{-1} \right)$$

$$\leq 4(2\kappa T_d(\delta) + \alpha T_d(\delta)) \cdot \|\phi(\pi^*_t) - \phi(\pi^*_t)\|v_t^{-1}$$

where the last inequality follows from the fact that $\pi^* \in S_t$ by Lemma 2 and since $\pi^*_t$ and $\pi^*_t$ were chosen the maximizer of the weighted difference $\|\phi(\pi^*_t) - \phi(\pi^*_t)\|v_t$. The regret is therefore

$$R_T = \sum_{t \in [T]} r_t$$

$$\leq 2(2\kappa T_d(\delta) + \alpha T_d(\delta)) \cdot \sum_{t \in [T]} \|\phi(\pi^*_t) - \phi(\pi^*_t)\|v_t^{-1}$$

$$\leq 2(2\kappa T_d(\delta) + \alpha T_d(\delta)) \cdot \sqrt{T} \sum_{t \in [T]} \|\phi(\pi^*_t) - \phi(\pi^*_t)\|^2v_t^{-1}$$

$$\leq 2(2\kappa T_d(\delta) + \alpha T_d(\delta)) \cdot \sqrt{2T \log \left(1+ \frac{TB}{d} \right)}$$

where the second inequality follows from Cauchy-Schwarz and the last inequality applies Lemma 8.

C Appendix for Section 4

In this section we will use the notation $N_t(s,a)$ to denote the number of times action $a$ was executed at state $s$ up to time $t - 1$. Recall the bonus terms,

Given any $\eta > 0$ define,

$$\xi_{s,a}(\eta, \delta) = \min \left(2\eta, 4\eta \sqrt{\frac{U}{N_t(s,a)}} \right)$$

s.t. $U = H \log(||S||A) + \log \left(\frac{6 \log(N_t(s,a))}{\delta} \right)$.
and the empirical average of $\xi^{(t)}_{s,a}(\eta, \delta)$ bonuses,

$$\hat{B}_t(\pi, \eta, \delta) = \mathbb{E}_{s_1 \sim \rho, \tau \sim \hat{\phi}^*_T(\cdot | s_1)} \left[ \sum_{h=1}^{H-1} \xi^{(t)}_{s_h, a_h}(\eta, \delta) \right].$$

Additionally we also define the error terms

$$\xi^{(t)}_{s,a}(\epsilon, \eta, \delta) = \min \left( 2\eta, 4\eta \sqrt{\frac{U}{N_t(s,a)}} \right)$$

s.t. $U = H \log(|S||A|H) + |S| \log \left( \left\lfloor \frac{4\eta H}{\epsilon} \right\rfloor \right) + \log \left( \frac{6 \log(N_t(s,a))}{\delta} \right)$.

In contrast with the definition of bonus $\xi^{(t)}_{s,a}(\eta, \delta)$ this quantity depends on an extra parameter $\epsilon$. These error terms induce the following ‘bonus’ function,

$$B_t(\pi, \eta, \delta, \epsilon) = \mathbb{E}_{s_1 \sim \rho, \tau \sim \hat{\phi}^*(\cdot | s_1)} \left[ \sum_{h=1}^{H-1} \xi^{(t)}_{s_h, a_h}(\epsilon, \eta, \delta) \right].$$

Here the expectation is under the true MDP dynamics.

Once we have established the validity of Lemma 9 and therefore that with probability at least $1 - 15\delta$,

$$R_T \leq 2\gamma_T \sqrt{2Td \log \left( 1 + \frac{TB}{d} \right) + \sum_{t \in [T]} 4\hat{B}_t(\pi^1, 4WB, \delta) + 4\hat{B}_t(\pi^2, 4WB, \delta)}$$

it remains to show the $\hat{B}_t(\cdot)$ terms are small. We’ll do so by showing that for any $\eta > 0$ and $\delta \in (0, 1)$ and for all policies $\pi$ simultaneously we can bound the empirical expected bonuses $\hat{B}_t(\pi, \eta, \delta)$ in terms of the population quantities $B_t(\pi, \eta H, \delta)$.

**Lemma 9.** Let $\eta, \epsilon > 0$. For all $\pi$ simultaneously and for all $t \in \mathbb{N}$, with probability $1 - \delta$,

$$\hat{B}_t(\pi, \eta, \delta) \leq 2B_t(\pi, 2H\eta, \delta, \epsilon) + \epsilon$$

**Proof.** Recall that,

$$\hat{B}_t(\pi, \eta, \delta) = \mathbb{E}_{s_1 \sim \rho, \tau \sim \hat{\phi}^*_T(\cdot | s_1)} \left[ \sum_{h=1}^{H-1} \xi^{(t)}_{s_h, a_h}(\eta, \delta) \right].$$

Let $f : \Gamma \to \mathbb{R}$ be defined as,

$$f(\tau) = \sum_{h=1}^{H-1} \xi^{(t)}_{s_h, a_h}(\eta).$$

It is easy to see that $f(\tau) \in (0, 2\eta H]$ for all $\tau \in \Gamma$. Therefore, a direct application of Lemma 13 implies that with probability at least $1 - \delta$ and simultaneously for all $\pi$, and $t \in \mathbb{N}$,

$$\hat{B}_t(\pi, \eta, \delta) \leq \mathbb{E}_{s_1 \sim \rho, \tau \sim \hat{\phi}^*(\cdot | s_1)} \left[ \sum_{h=1}^{H-1} \xi^{(t)}_{s_h, a_h}(\eta, \delta) \right] + B_t(\pi, 2H\eta, \delta, \epsilon) + \epsilon$$

Since $\xi^{(t)}_{s,a}(\epsilon, \eta, \delta) \geq \xi^{(t)}_{s,a}(\eta, \delta)$ for all $\epsilon > 0, s, a \in S \times A$ and $\xi^{(t)}_{s,a}(\epsilon, \eta, \delta)$ is monotonic in $\eta$ we conclude that,

$$\mathbb{E}_{s_1 \sim \rho, \tau \sim \hat{\phi}^*(\cdot | s_1)} \left[ \sum_{h=1}^{H-1} \xi^{(t)}_{s_h, a_h}(\eta, \delta) \right] \leq \mathbb{E}_{s_1 \sim \rho, \tau \sim \hat{\phi}^*(\cdot | s_1)} \left[ \sum_{h=1}^{H-1} \xi^{(t)}_{s_h, a_h}(\epsilon, \eta, \delta) \right] \leq \mathbb{E}_{s_1 \sim \rho, \tau \sim \hat{\phi}^*(\cdot | s_1)} \left[ \sum_{h=1}^{H-1} \xi^{(t)}_{s_h, a_h}(\epsilon, 2H\eta, \delta) \right] = B_t(\pi, 2H\eta, \delta, \epsilon)$$
Combining these inequalities the result follows.

Let \( \bar{\mathcal{E}}_3 \) such that for all \( T \in \mathbb{N} \) be the event that,
\[
\sum_{t \in [T]} 4\bar{B}_t(\pi_1^1, 4SB, \delta) + 4\bar{B}_t(\pi_1^2, 4SB, \delta) \leq \epsilon T + \sum_{t \in [T]} 8B_t(\pi_1^1, 8HSB, \delta, \epsilon) + 8B_t(\pi_1^2, 8HSB, \delta, \epsilon)
\]
Invoking Lemmas 6 and 9 we can show \( \bar{\mathcal{E}}_3 \) occurs with probability at least \( 1 - 2\delta \). Let’s bound the sum \( \sum_{t \in [T]} 8B_t(\pi_1^1, 8HSB, \delta, \epsilon) + 8B_t(\pi_1^2, 8HSB, \delta, \epsilon) \). Consider the martingale difference sequences \( \{B_t(\pi_1^1, 8HSB, \delta, \epsilon) - \sum_{h=1}^{H-1} \xi_{s_{t,h}^1, a_{t,h}^1}^t (\epsilon, 8HSB, \delta)\}_{t=1}^\infty \) and \( \{B_t(\pi_1^2, 8HSB, \delta, \epsilon) - \sum_{h=1}^{H-1} \xi_{s_{t,h}^2, a_{t,h}^2}^t (\epsilon, 8HSB, \delta)\}_{t=1}^\infty \) each with norm upper bound \( 32H^2SB \). By an anytime Hoeffding inequality (see Lemma 16) (since \( \xi_{s,a}(\epsilon, \eta, \delta) \leq 2\eta \) and therefore \( \sum_h \xi_{s_h, a_h}(\epsilon, \eta, \delta) \leq 2H\eta \)) applied to with probability at least \( 1 - 2\delta \) for all \( T \in \mathbb{N} \) simultaneously
\[
\sum_{t \in [T]} 8B_t(\pi_1^1, 4SB, \delta, \epsilon) + 8B_t(\pi_1^2, 4SB, \delta, \epsilon) \leq 8 \sum_{t \in [T]} \left( \sum_{h=1}^{H-1} \xi_{s_{t,h}^1, a_{t,h}^1}^t (\epsilon, 8HSB, \delta) + \sum_{h=1}^{H-1} \xi_{s_{t,h}^2, a_{t,h}^2}^t (\epsilon, 8HSB, \delta) \right) + I.
\]
Where \( I = 128HSB \sqrt{TH \log \left( \frac{6\log(TH)}{\delta} \right)} \). In order to bound the remaining empirical error terms, we to the following standard result,

**Lemma 10.** For \( i \in \{1, 2\} \) the empirical sum of errors satisfies the following bound
\[
\sum_{t \in [T]} \sum_{h=1}^{H-1} \xi_{s_{t,h}^i, a_{t,h}^i}^t (\epsilon, 8HSB, \delta) \leq 64HSB \sqrt{\left( H \log(|S||A|T) + |S| \log \left( \frac{32H^2SB}{\epsilon} \right) + \log \left( \frac{6\log(TH_s)}{\delta} \right) \right) |S||A|TH}.
\]

**Proof.** Let’s rewrite this sum by instead summing over states and actions,
\[
\sum_{t \in [T]} \sum_{h=1}^{H-1} \xi_{s_{t,h}^i, a_{t,h}^i}^t (\epsilon, 8HSB, \delta) = \sum_{s \in S} \sum_{a \in A} \min_{t=1}^{N_{T+1}(s,a)} \left( 16HSB, 32HSB \sqrt{\frac{H \log(|S||A|H) + |S| \log \left( \frac{32H^2SB}{\epsilon} \right) + \log \left( \frac{6\log(T)}{\delta} \right) }{t}} \right)
\]
\[
\leq 32HSB \sqrt{H \log(|S||A|H) + |S| \log \left( \frac{32H^2SB}{\epsilon} \right) + \log \left( \frac{6\log(T)}{\delta} \right) \sum_{s \in S} \sum_{a \in A} \sum_{t=1}^{N_{T+1}(s,a)} \frac{1}{\sqrt{t}}}
\]
\[
\leq 32HSB \sqrt{H \log(|S||A|H) + |S| \log \left( \frac{32H^2SB}{\epsilon} \right) + \log \left( \frac{6\log(T)}{\delta} \right) \sum_{s \in S} \sum_{a \in A} 2 \sqrt{N_{T+1}(s,a)}}
\]
\[
\leq 64HSB \sqrt{H \log(|S||A|H) + |S| \log \left( \frac{32H^2SB}{\epsilon} \right) + \log \left( \frac{6\log(T)}{\delta} \right) |S||A|TH}.
\]
The result follows.

We will use Lemma 10 with \( \epsilon = 1/T \). As a consequence of Lemma 10 and Lemma 6 we see that when \( \bar{\mathcal{E}}_3 \cap \bar{\mathcal{E}}_2 \cap \bar{\mathcal{E}}_0 \cap \bar{\mathcal{E}}_{-1} \cap \bar{\mathcal{E}}_2 \cap \bar{\mathcal{E}}_3 \) holds
\[
R_T \leq 2\gamma_T \sqrt{2T \log \left(1 + \frac{TB}{d}\right)} + \tilde{O}\left(H^{3/2}\sqrt{|A||S|TH + H|S|\sqrt{|A||TH + H\sqrt{TH}}}\right).
\]

Now it remains to bound term \(\gamma_T\). From now on let’s set \(\epsilon = \min(1/T, 8HSB)\) and let \(\tilde{E}_4\) be the event that for all \(t \in \mathbb{N}\) and all \(i \in \{1, 2\},\)
\[
\tilde{B}_t \left(\pi^r_1, 2SB, \frac{\delta'}{8\epsilon^3 A^3}\right) \leq 2B_t \left(\pi^r_1, 4HSB, \frac{\delta'}{8\epsilon^3 A^3}, \epsilon\right) + \epsilon
\]
for all \(t \in \mathbb{N}\) and all \(i \in \{1, 2\}.\) As a consequence of Lemma \(9\) we can bound \(P(\tilde{E}_4) \geq 1 - 2\delta\). Squaring both sides,
\[
\left(\tilde{B}_t(\pi^r_1, 2SB, \delta'_i)\right)^2 \leq \left(2B_t(\pi^r_1, 4HSB, \delta'_i, \epsilon) + \epsilon\right)^2
\]
\[
\leq 4 \left(2B_t(\pi^r_1, 4HSB, \delta'_i, \epsilon)\right)^2 + 16\epsilon HSB + \epsilon^2
\]
\[
\leq 4 \left(B_t(\pi^r_1, 4HSB, \delta'_i, \epsilon)\right)^2 + 24\epsilon HSB
\]
Where \(\delta'_i = \frac{\delta'}{8\epsilon^3 A^3}\). Inequality (i) used that \(B(\pi, \eta, \delta, \epsilon) \leq 2H\eta\). The last inequality holds because \(\epsilon \leq 8HSB\). Therefore if \(\tilde{E}_4\) holds for all \(t \in \mathbb{N},\)
\[
\gamma_T \leq \sqrt{\frac{2}{\pi} (4\kappa_\beta T (\delta) + \alpha_{d,T}(\delta)) + \frac{1}{t} + 4 + \frac{\sum_{t=1}^{t-1} \sum_{i=1}^{t-1} \left(2B_t(\pi^r_1, 4HSB, \delta'_i, \epsilon)\right)^2 + \left(B_t(\pi^r_1, 4HSB, \delta'_i, \epsilon)\right)^2 + \left(B_t(\pi^r_1, 4HSB, \delta'_i, \epsilon)\right)^2}{4\epsilon^2 HSB}}.
\]
Where \(I = 96(t-1)\epsilon HSB\). We are just left with bounding the sum of squares \(\sum_{t=1}^{t-1} \sum_{i=1}^{t-1} \left(2B_t(\pi^r_1, 4HSB, \delta'_i, \epsilon)\right)^2 + \left(B_t(\pi^r_1, 4HSB, \delta'_i, \epsilon)\right)^2 + \left(B_t(\pi^r_1, 4HSB, \delta'_i, \epsilon)\right)^2\).

**Lemma 11.** Let \(\eta, \epsilon > 0\) and \(\delta, \delta' \in (0, 1)\) and define \(\tilde{E}_5(\delta')\) to be the event that for all \(t \in \mathbb{N}\) and \(i \in \{1, 2\}\)
\[
\sum_{t=1}^{t-1} \sum_{i=1}^{t-1} \left(B_t(\pi^r_1, \eta, \delta \delta', \epsilon)\right)^2 \leq 12\eta^2 H^2 \left(1.4 \ln \left(2 \max \left(4\eta^2 H t, 1\right)\right) + \ln \left(\frac{5.2}{\delta'} + 1\right)\right) + 64\eta^2 H \left(H \log(|S|A|H| + |S| \log \left(\left\lfloor \frac{4\eta H}{\epsilon}\right\rfloor\right) + \left(\log \left(\frac{5H}{\delta}\right)\right)\right) \log(T|H + |S||A|)
\]
Then \(P(\tilde{E}_5(\delta')) \geq 1 - 2\delta'.\)

**Proof.** Observe that,
\[
\left(B_t(\pi^r_1, \eta, \delta \delta', \epsilon)\right)^2 + \left(B_t(\pi^r_1, \eta, \delta \delta', \epsilon)\right)^2 = \left(\mathbb{E}_{s^1_1 \sim \rho, \eta, \delta \delta', \epsilon} \left[H \sum_{h=1}^{H-1} \xi^{(t)}_{s^1_h, a^r_h}(\epsilon, \eta, \delta \delta')\right]\right)^2 + \left(\mathbb{E}_{s^2_1 \sim \rho, \eta, \delta \delta', \epsilon} \left[H \sum_{h=1}^{H-1} \xi^{(t)}_{s^2_h, a^r_h}(\epsilon, \eta, \delta \delta')\right]\right)^2
\]
\[
\leq H \mathbb{E}_{s^1_1 \sim \rho, \eta, \delta \delta', \epsilon} \left[H \sum_{h=1}^{H-1} \xi^{(t)}_{s^1_h, a^r_h}(\epsilon, \eta, \delta \delta')\right]^2 + H \mathbb{E}_{s^2_1 \sim \rho, \eta, \delta \delta', \epsilon} \left[H \sum_{h=1}^{H-1} \xi^{(t)}_{s^2_h, a^r_h}(\epsilon, \eta, \delta \delta')\right]^2
\]
Where inequality (i) is a consequence of \(\left(\mathbb{E} \left[\sum_{h=1}^{H} a_h\right]\right)^2 \leq H \mathbb{E} \left[\sum_{h=1}^{H} a^2_h\right].\) Define the martingale-difference sequences for \(i \in \{1, 2\},\)
\[
D^{(i)}_t = \mathbb{E}_{s^1_1 \sim \rho, \eta, \delta \delta', \epsilon} \left[H \sum_{h=1}^{H-1} \left(\xi^{(t)}_{s^1_h, a^r_h}(\epsilon, \eta, \delta \delta')\right)^2\right] - \mathbb{E}_{s^2_1 \sim \rho, \eta, \delta \delta', \epsilon} \left[H \sum_{h=1}^{H-1} \left(\xi^{(t)}_{s^2_h, a^r_h}(\epsilon, \eta, \delta \delta')\right)^2\right]
\]
Since $\xi_{s,a}^{(l)}(\epsilon, \eta, \delta) \leq 2\eta$, we see that $|D_{t}^{(i)}| \leq 8\eta^{2}H$. Observe that for $i \in \{1, 2\}$,

$$
\operatorname{Var}_{t}^{(i)} \left( \sum_{h=1}^{H-1} \left( \xi_{s_h, a_h}^{(l)}(\epsilon, \eta, \delta) \right)^2 \right) \leq \mathbb{E}_{s_{1} \sim \rho, \tau \sim \nu_{s_{1}}^{(\epsilon, \eta, \delta)}} \left[ \sum_{h=1}^{H-1} \left( \xi_{s_h, a_h}^{(l)}(\epsilon, \eta, \delta) \right)^2 \right]^{(i)} 
\leq 4\eta^{2}H \mathbb{E}_{s_{1} \sim \rho, \tau \sim \nu_{s_{1}}^{(\epsilon, \eta, \delta)}} \left[ \sum_{h=1}^{H-1} \left( \xi_{s_h, a_h}^{(l)}(\epsilon, \eta, \delta) \right)^2 \right]^{(i)} 
\leq 16\eta^{4}H^{2}
$$

Where (i) follows follows because for $\xi_{s,a}^{(l)}(\epsilon, \eta, \delta) \leq 2\eta$.

Since the variance can be bounded by the mean, we can make use of a Uniform empirical Bernstein Bound from Lemma 17. Let $S_{t}^{(i)} = \sum_{t=1}^{t} D_{t}^{(i)}$ for $i \in \{1, 2\}$ and $W_{t}^{(i)} = \sum_{t=1}^{t} \operatorname{Var}_{t}^{(i)} \left( \sum_{h=1}^{H-1} \left( \xi_{s_h, a_h}^{(l)}(\epsilon, \eta, \delta) \right)^2 \right)$. Let $c = 8\eta^{2}H$ and $m = 4\eta^{2}H$. With probability $1 - \delta'$ for all $t \in \mathbb{N}$,

$$
\sum_{t=1}^{t-1} D_{t}^{(i)} \leq \max_{\ell = 1}^{\ell-1} \left( 4\eta^{2}H \sum_{\ell = 1}^{t-1} \mathbb{E}_{s_{1} \sim \rho, \tau \sim \nu_{s_{1}}^{(\epsilon, \eta, \delta)}} \left[ \sum_{h=1}^{H-1} \left( \xi_{s_h, a_h}^{(l)}(\epsilon, \eta, \delta) \right)^2 \right] \right) + 3.28\eta^{2}H \eta \left( 1.4 \ln (2 \max (4\eta^{2}H, 1)) + \ln \frac{5.2}{\delta'} \right)
\leq \left( 4\eta^{2}H \sum_{\ell = 1}^{t-1} \mathbb{E}_{s_{1} \sim \rho, \tau \sim \nu_{s_{1}}^{(\epsilon, \eta, \delta)}} \left[ \sum_{h=1}^{H-1} \left( \xi_{s_h, a_h}^{(l)}(\epsilon, \eta, \delta) \right)^2 \right] + 4\eta^{2}H \right) \left( 1.4 \ln (2 \max (4\eta^{2}H, 1)) + \ln \frac{5.2}{\delta'} \right)
+ 3.28\eta^{2}H \left( 1.4 \ln (2 \max (4\eta^{2}H, 1)) + \ln \frac{5.2}{\delta'} \right)
$$

Since $\sqrt{ab} \leq \frac{a+b}{2}$,

$$
\sum_{t=1}^{t-1} D_{t}^{(i)} \leq 2 \mathbb{E}_{s_{1} \sim \rho, \tau \sim \nu_{s_{1}}^{(\epsilon, \eta, \delta)}} \left[ \sum_{h=1}^{H-1} \left( \xi_{s_h, a_h}^{(l)}(\epsilon, \eta, \delta) \right)^2 \right] + 2\eta^{2}H + (3.28\eta^{2}H + 2\eta^{2}H) \left( 1.4 \ln (2 \max (4\eta^{2}H, 1)) + \ln \frac{5.2}{\delta'} \right).
$$

Therefore with high probability for $i \in \{1, 2\}$,

$$
\mathbb{E}_{s_{1} \sim \rho, \tau \sim \nu_{s_{1}}^{(\epsilon, \eta, \delta)}} \left[ \sum_{h=1}^{H-1} \left( \xi_{s_h, a_h}^{(l)}(\epsilon, \eta, \delta) \right)^2 \right] \leq 2 \sum_{l=1}^{t-1} \sum_{h=1}^{H-1} \left( \xi_{s_h, a_h}^{(l)}(\epsilon, \eta, \delta) \right)^2 + 4\eta^{2}H + 6\eta^{2}H \left( 1.4 \ln (2 \max (4\eta^{2}H, 1)) + \ln \frac{5.2}{\delta'} \right).
$$
Therefore with probability $1 - 2\delta'$,

$$\sum_{t \in \{1, 2\}} \sum_{\ell=1}^{t-1} (B_t(\pi^{(\ell)}_t, \eta, \delta/\ell^{3}))^2 \leq 2H \sum_{t \in \{1, 2\}} \sum_{\ell=1}^{t-1} \sum_{h=1}^{H-1} \left( \epsilon^{(t)}_{s_h, a_h} (\epsilon, \eta, \delta) \right)^2 + 12\eta^2 H^2 \left( 1.4 \ln \left( 2 \left( \max \left( 4\eta^2 Ht, 1 \right) \right) \right) + \ln \frac{5.2}{\delta'} + 1 \right).$$

We are left with the task of bounding the terms $\sum_{t=1}^{t-1} \sum_{h=1}^{H-1} \left( \epsilon^{(t)}_{s_h, a_h} (\epsilon, \eta, \delta) \right)^2$.

Let's rewrite this sum by instead summing over states and actions,

$$\sum_{t \in \{1\}} \sum_{h=1}^{H-1} \left( \epsilon^{(t)}_{s_h, a_h} (\epsilon, \eta, \delta) \right)^2 = \sum_{s \in S} \sum_{a \in A} \sum_{t=1}^{N_{T+1}(s, a)} \min \left( 4\eta^2, 16\eta^2 \frac{H \log(|S||A|H) + |S| \log \left( \frac{4H\eta}{\epsilon} \right) + \log \left( \frac{6 \log(t)}{\delta} \right)}{t} \right)$$

$$= \sum_{s \in S} \sum_{a \in A} \sum_{t=1}^{N_{T+1}(s, a)} \frac{16\eta^2 H \log(|S||A|H) + |S| \log \left( \frac{4H\eta}{\epsilon} \right) + \log \left( \frac{6 \log(t)}{\delta} \right)}{t}$$

$$32\eta^2 \left( H \log(|S||A|H) + |S| \log \left( \frac{4H\eta}{\epsilon} \right) + \log \left( \frac{6 \log(t)}{\delta} \right) \right) \sum_{s \in S} \sum_{a \in A} \log \left( N_{T+1}(s, a) + 1 \right)$$

$$\leq 32\eta^2 \left( H \log(|S||A|H) + |S| \log \left( \frac{4H\eta}{\epsilon} \right) + \log \left( \frac{6 \log(t)}{\delta} \right) \right) |S||A| \log(TH + |S||A|)$$

Therefore with probability $1 - 2\delta'$,

$$\sum_{t \in \{1, 2\}} \sum_{\ell=1}^{t-1} (B_t(\pi^{(\ell)}_t, \eta, \delta/\ell^{3}))^2 \leq 12\eta^2 H^2 \left( 1.4 \ln \left( 2 \left( \max \left( 4\eta^2 Ht, 1 \right) \right) \right) + \ln \frac{5.2}{\delta'} + 1 \right) + 64\eta^2 H \left( H \log(|S||A|H) + |S| \log \left( \frac{4H\eta}{\epsilon} \right) + \log \left( \frac{6 \log(t)}{\delta} \right) \right) |S||A| \log(TH + |S||A|)$$

The result follows. \qed

The main takeaway from this lemma is that the sum of the square errors grows only logarithmically in $T$. Applying this bound to $\gamma_T$ and setting $\epsilon = O(1/T)$ we obtain,

$$\gamma_T \leq \sqrt{2} \left( 4\eta \beta_T(\delta) + \alpha_{d,T}(\delta) \right) + 2 \sqrt{\omega_T(\delta)} + \frac{1}{T} = \tilde{O} \left( \kappa \sqrt{d} + H^2 \sqrt{|S||A|} + H^{3/2} |S| \sqrt{|A|} \right)$$

Where

$$\omega_T(\delta) = 192H^4 S^6 2B^2 \left( 1.4 \ln \left( 2 \left( \max \left( 64H^3 S^2 B^2 t, 1 \right) \right) \right) + \ln \frac{5.2}{\delta'} + 1 \right) + 1024H^3 S^2 B^2 \left( H \log(|S||A|H) + |S| \log \left( \frac{64H^3 S^2 B^2}{\epsilon} \right) + \log \left( \frac{6 \log(t)}{\delta} \right) \right) |S||A| \log(TH + |S||A|)$$
Applying this bound to \( \gamma_T \) and setting \( \epsilon = 1/96T \) we obtain,

\[
\gamma_T \leq \sqrt{2} (4\kappa_\beta T(\delta) + \alpha_{d,T}(\delta)) + 2\sqrt{\omega_T(\delta) + HSB + \frac{1}{T}}
\]

Combining these observations we can derive our main result,

**Theorem 3** (Formal version of Theorem 2). If \( \mathcal{E}_\delta \cap \mathcal{E}_2 \cap \tilde{\mathcal{E}}_0 \cap \tilde{\mathcal{E}} \cap \tilde{\mathcal{E}}_2 \cap \tilde{\mathcal{E}}_2 \cap \tilde{\mathcal{E}}_2 \cap \tilde{\mathcal{E}}_2 \cap \tilde{\mathcal{E}}_2 \cap \tilde{\mathcal{E}}_5(\frac{\delta}{2}) \) holds then the regret of LPbRL satisfies,

\[
R_T \leq 2 \left( 4\kappa_\beta T(\delta) + \alpha_{d,T}(\delta) + 2\sqrt{\omega_T(\delta) + 96(t-1)\epsilon HSB + \frac{1}{T}} + \right)
\]

\[
128H^{3/2}SB \sqrt{T H \log \left( \frac{6\log(T)}{\delta} \right)} +
\]

\[
1024HSB \sqrt{\left( \log(|S| A H) + |S| \log \left( \frac{32H^2SB}{\epsilon} \right) \right) + \log \left( \frac{6\log(T H)}{\delta} \right)} \text{ SATH.}
\]

For all \( T \in \mathbb{N} \) simultaneously. Where \( \mathbb{P} \left( \mathcal{E}_\delta \cap \mathcal{E}_2 \cap \tilde{\mathcal{E}}_0 \cap \tilde{\mathcal{E}} \cap \tilde{\mathcal{E}}_2 \cap \tilde{\mathcal{E}}_2 \cap \tilde{\mathcal{E}}_2 \cap \tilde{\mathcal{E}}_2 \cap \tilde{\mathcal{E}}_2 \cap \tilde{\mathcal{E}}_5(\frac{\delta}{2}) \right) \geq 1 - 15\delta.

**C.1 Supporting Related Work Lemmas**

We will make use of the following Lemma (see Lemma B.1 in Chatterji et al. (2021)),

**Lemma 12.** For any fixed policy \( \pi \), and any function \( f : \Gamma \rightarrow \mathbb{R} \) satisfying \( \max_{\tau \in \Gamma} |f(\tau)| \leq \eta \), with probability at least \( 1 - \delta \) for all \( t \in \mathbb{N} \),

\[
\mathbb{E}_{s_1 \sim \rho, \tau \sim \pi(\cdot|s_1)} [f(\tau)] - \mathbb{E}_{s_1 \sim \rho, \tau \sim \pi(\cdot|s_1)} [f'(\tau)] \leq B_t(\pi, \eta, \delta)
\]

We will also make use of the following Lemma (see Lemma B.2 from Chatterji et al. (2021) ) corresponding to the uniform version of lemma 12.

**Lemma 13** (Uniform version of Lemma 12). Let \( \epsilon > 0 \). For any function \( f : \Gamma \rightarrow \mathbb{R} \) satisfying \( \max_{\tau \in \Gamma} |f(\tau)| \leq \eta \), for all policies \( \pi \) simultaneously and all \( t \in \mathbb{N} \),

\[
\mathbb{E}_{s_1 \sim \rho, \tau \sim \pi(\cdot|s_1)} [f(\tau)] - \mathbb{E}_{s_1 \sim \rho, \tau \sim \pi(\cdot|s_1)} [f'(\tau)] \leq B_t(\pi, \eta, \delta, \epsilon) + \epsilon.
\]

We will make use of the following standard bound on the covering number of the \( l_2 \) ball.

**Lemma 14.** For any \( \epsilon \in (0,1] \) the \( \epsilon \)-covering number of the Euclidean ball in \( \mathbb{R}^d \) with radius \( r > 0 \) i.e., \( \{ x \in \mathbb{R}^d : ||x||_2 \leq r \} \) is upper bounded by \( (\frac{1+2r}{\epsilon})^d \).

**C.2 Proof of Lemma 3**

**Lemma 3.** Let \( \tilde{\mathcal{E}}_0 \) be the event that for all \( t \in \mathbb{N} \),

\[
||w_t^T - \omega_*||_{\mathcal{V}_t} \leq \sqrt{2}||w_t^T - \omega_*||_{\mathcal{V}_t} + \sum_{t=1}^{T-1} 4 \left( \frac{\delta^t}{8\delta^t |A|} \right) + \frac{1}{t}.
\]

where \( \delta^t = \frac{\delta}{(1+2r)^d} \) and \( \epsilon = \frac{1}{1+2r^2 T^2} \). Then \( \mathbb{P} (\tilde{\mathcal{E}}_0) \geq 1 - \delta \).

**Proof.** Recall that as a result of assumption 1 and the definition of \( w_t^T \) we can bound \( ||w_t^T - \omega_*|| \leq 2S \). Let \( \mathbf{v} \) be such that \( ||\mathbf{v}|| \leq 2S \).
Let’s consider $\mathbf{v}^\top \tilde{\mathbf{V}}_t \mathbf{v}$,

$$
\mathbf{v}^\top \tilde{\mathbf{V}}_t \mathbf{v} = \kappa \lambda \|\mathbf{v}\|^2 + \sum_{\ell=1}^{t-1} \left( \langle \mathbf{v}, \phi^\ell_1(\pi^1_\ell) - \phi^\ell_2(\pi^2_\ell) \rangle \right)^2
$$

Let’s focus on a single summand $\langle \mathbf{v}, \phi^\ell_1(\pi^1_\ell) - \phi^\ell_2(\pi^2_\ell) \rangle^2$ with $\ell \in [t-1]$.

By Lemma 12, with probability at least $1 - \frac{\delta'}{4t^2}$ for all $\pi \in \Pi$ simultaneously,

$$
\langle \phi^\ell_1(\pi), \mathbf{v} \rangle \leq \langle \phi(\pi), \mathbf{v} \rangle + \tilde{B}_t \left( \pi, 2SB, \frac{\delta'}{8\ell^3A^3} \right).
$$

And

$$
-\langle \phi^\ell_1(\pi), \mathbf{v} \rangle \leq -\langle \phi(\pi), \mathbf{v} \rangle + \tilde{B}_t \left( \pi, 2SB, \frac{\delta'}{8\ell^3A^3} \right).
$$

Then

$$
\left| \langle \phi^\ell_1(\pi^1_\ell) - \phi^\ell_2(\pi^2_\ell), \mathbf{v} \rangle \right| \leq \left| \langle \phi(\pi^1_\ell) - \phi(\pi^2_\ell), \mathbf{v} \rangle \right| + \tilde{B}_t \left( \pi^1_\ell, 2SB, \frac{\delta'}{8\ell^3A^3} \right) + \tilde{B}_t \left( \pi^2_\ell, 2SB, \frac{\delta'}{8\ell^3A^3} \right).
$$

And therefore,

$$
\langle \phi^\ell_1(\pi^1_\ell) - \phi^\ell_2(\pi^2_\ell), \mathbf{v} \rangle^2 \leq 2\langle \phi(\pi^1_\ell) - \phi(\pi^2_\ell), \mathbf{v} \rangle^2 + 4 \left( \tilde{B}_t \left( \pi^1_\ell, 2SB, \frac{\delta'}{8\ell^3A^3} \right) \right)^2 + 4 \left( \tilde{B}_t \left( \pi^2_\ell, 2SB, \frac{\delta'}{8\ell^3A^3} \right) \right)^2.
$$

Therefore with probability at least $1 - \frac{\delta}{2t^2}$, for all $t \in \mathbb{N}$ simultaneously,

$$
\mathbf{v}^\top \tilde{\mathbf{V}}_t \mathbf{v} = \kappa \lambda \|\mathbf{v}\|^2 + \sum_{\ell=1}^{t-1} \left( \langle \mathbf{v}, \phi^\ell_1(\pi^1_\ell) - \phi^\ell_2(\pi^2_\ell) \rangle \right)^2
\leq 2\kappa \lambda \|\mathbf{v}\|^2 + 2 \sum_{\ell=1}^{t-1} \left( \langle \mathbf{v}, \phi(\pi^1_\ell) - \phi(\pi^2_\ell) \rangle \right)^2 + 4 \left( \tilde{B}_t \left( \pi^1_\ell, 2SB, \frac{\delta'}{8\ell^3A^3} \right) \right)^2 + 4 \left( \tilde{B}_t \left( \pi^2_\ell, 2SB, \frac{\delta'}{8\ell^3A^3} \right) \right)^2
= 2\mathbf{v}^\top \tilde{\mathbf{V}}_t \mathbf{v} + \sum_{\ell=1}^{t-1} 4 \left( \tilde{B}_t \left( \pi^1_\ell, 2SB, \frac{\delta'}{8\ell^3A^3} \right) \right)^2 + 4 \left( \tilde{B}_t \left( \pi^2_\ell, 2SB, \frac{\delta'}{8\ell^3A^3} \right) \right)^2.
$$

Consider an $\epsilon$-cover of the $2S$ ball in $\mathbb{R}^d$ and let’s see that for any other $\mathbf{v}$ in the $2S$ ball, the closest vector in the covering $\check{\mathbf{v}}$ satisfies, $\|\mathbf{v} - \check{\mathbf{v}}\| \leq \epsilon$ and therefore,

$$
\left| \mathbf{v}^\top \tilde{\mathbf{V}}_t \mathbf{v} - \check{\mathbf{v}}^\top \tilde{\mathbf{V}}_t \check{\mathbf{v}} \right| = \left| \mathbf{v}^\top \tilde{\mathbf{V}}_t \mathbf{v} - \mathbf{v}^\top \tilde{\mathbf{V}}_t \check{\mathbf{v}} + \mathbf{v}^\top \tilde{\mathbf{V}}_t \check{\mathbf{v}} - \check{\mathbf{v}}^\top \tilde{\mathbf{V}}_t \check{\mathbf{v}} \right|
\leq \left| \mathbf{v}^\top \tilde{\mathbf{V}}_t \mathbf{v} - \mathbf{v}^\top \tilde{\mathbf{V}}_t \check{\mathbf{v}} \right| + \left| \mathbf{v}^\top \tilde{\mathbf{V}}_t \check{\mathbf{v}} - \check{\mathbf{v}}^\top \tilde{\mathbf{V}}_t \check{\mathbf{v}} \right|
\leq \|\tilde{\mathbf{V}}_t \mathbf{v}\| \|\mathbf{v} - \check{\mathbf{v}}\| + \|\tilde{\mathbf{V}}_t \check{\mathbf{v}}\| \|\mathbf{v} - \check{\mathbf{v}}\|
\leq \epsilon \left( \kappa \lambda + 4B^2(t-1) \right).
$$
Similarly

\[ |v^\top \tilde{V}_t v - \tilde{v}^\top \tilde{V}_t \tilde{v}| \leq \epsilon (\kappa \lambda + 4B^2(t - 1)) \]

Invoking Lemma 14 and setting \( \delta' = \frac{\delta}{(1 + 2\delta)^2} \) applying it to \( w_t^L - w_* \)

\[ \|w_t^L - w_*\|_V^2 \leq 2\|w_t^L - w_*\|_V^2 + \sum_{t'=1}^{t-1} 4 \left( \tilde{B}_t \left( \pi_{t'}, 2SB, \frac{\delta'}{8\ell^3 A^2} \right) \right)^2 + 4 \left( \tilde{B}_t \left( \pi_{t'}, 2SB, \frac{\delta'}{8\ell^3 A^2} \right) \right)^2 + 2\epsilon (\kappa \lambda + 4B^2(t - 1)). \]

Setting \( \epsilon = \frac{1}{(\pi \kappa + 4B^2)^2} \) and using the fact that all \( a, b, c \geq 0 \) we have \( \sqrt{a^2 + b^2 + c^2} \leq a + b + c, \)

\[ \|w_t^L - w_*\|_V \leq \sqrt{2}\|w_t^L - w_*\|_V + \sum_{t'=1}^{t-1} 4 \left( \tilde{B}_t \left( \pi_{t'}, 2SB, \frac{\delta'}{8\ell^3 A^2} \right) \right)^2 + 4 \left( \tilde{B}_t \left( \pi_{t'}, 2SB, \frac{\delta'}{8\ell^3 A^2} \right) \right)^2 + \frac{1}{t}. \]

Since the inequality holds for any \( t \in \mathbb{N} \) with probability at least \( 1 - \frac{\delta}{2t^2} \), by the union bound, the inequality holds for all \( t \in \mathbb{N} \) simultaneously with probability at least \( 1 - \delta. \)

\[ \square \]

C.3 Proof of Lemma 4

**Lemma 4.** Let \( \tilde{E}_{-1} \) be the event that \( \pi^* \in \Pi_t \) for all \( t \in \mathbb{N} \). Then \( \mathbb{P}(\tilde{E}_{-1}) \geq 1 - 5\delta. \)

**Proof.** Let’s start by conditioning on \( E_\delta, \tilde{E}_0 \) and \( E_2 \) (see Corollary 1 for a definition of \( E_2 \)). By Lemmas 1 and 3 \( \mathbb{P}(E_\delta \cap \tilde{E}_0 \cap E_2) \geq 1 - 4\delta \)

By definition of \( \pi_* \), \( (\phi(\pi^*) - \phi(\pi))^\top w^* \geq 0 \) for any arbitrary \( \pi \). Therefore,

\[ 0 \leq (\phi(\pi^*) - \phi(\pi))^\top w^* \]

By Lemma 12 with probability at least \( 1 - \delta \) for all \( \pi_1, \pi_2 \in \Pi \) simultaneously and all \( t \in \mathbb{N} \)

\[ (\phi(\pi_1) - \phi(\pi_2))^\top w^* \leq (\phi_t(\pi_1) - \phi_t(\pi_2))^\top w^* + \tilde{B}_t \left( \pi_1, 2SB, \frac{\delta}{2A^2} \right) + \tilde{B}_t \left( \pi_2, 2SB, \frac{\delta}{A^2} \right), \tag{23} \]

In particular this implies that with probability at least \( 1 - \delta \) for \( \pi^* \) and any \( \pi \),

\[ 0 \leq (\phi(\pi^*) - \phi(\pi))^\top w^* \leq (\phi_t(\pi^*) - \phi_t(\pi))^\top w^* + \tilde{B}_t \left( \pi^*, 2SB, \frac{\delta}{2A^2} \right) + \tilde{B}_t \left( \pi, 2SB, \frac{\delta}{A^2} \right), \tag{24} \]

Let’s bound the term \( (\phi_t(\pi^*) - \phi_t(\pi))^\top w^* \)

\[ (\phi_t(\pi^*) - \phi_t(\pi))^\top w^* = (\phi_t(\pi^*) - \phi_t(\pi))^\top w_t^L + (\phi_t(\pi^*) - \phi_t(\pi))^\top (w^* - w_t^L) \]

\[ \leq (\phi_t(\pi^*) - \phi_t(\pi))^\top w_t^L + \|\phi_t(\pi^*) - \phi_t(\pi)\|_V \|w^* - w_t^L\|_V, \tag{25} \]
Thus implying $\pi$

Proof. Since we are conditioning on $\pi$ Lemma 15.

\[
\|w_t^* - w_*\|_{V_t} \leq \sqrt{2}\|w_t^* - w_*\|_{V_t} + \sqrt{\sum_{t=1}^{L-1} 4\left(\hat{B}_t \left(\pi, 2SB, \frac{\delta'}{8\delta^3 A^3}\right)^2 + \frac{1}{t}\right)}.
\]

Since $\mathcal{E}_2$ is assumed to hold Corollary 1 implies that $\|w_t^* - w_*\|_{V_t} \leq 4\gamma_0(\delta) + \alpha_{d,T}(\delta)$ and therefore,

\[
\|w_t^* - w_*\|_{V_t} \leq \sqrt{2}(4\gamma_0(\delta) + \alpha_{d,T}(\delta)) + \sqrt{\sum_{t=1}^{L-1} 4\left(\hat{B}_t \left(\pi, 2SB, \frac{\delta'}{8\delta^3 A^3}\right)^2 + \frac{1}{t}\right)} + \frac{1}{t},
\]

Since $\gamma_t = \sqrt{2}(4\gamma_0(\delta) + \alpha_{d,T}(\delta)) + \sqrt{\sum_{t=1}^{L-1} 4\left(\hat{B}_t \left(\pi, 2SB, \frac{\delta'}{8\delta^3 A^3}\right)^2 + \frac{1}{t}\right)} + \frac{1}{t}$, combining these results with Equations 25 and 24 yields,

\[
0 \leq (\hat{\phi}_t^*(\pi) - \phi_t^*(\pi))^\top w_t^* + \gamma_t||\hat{\phi}_t^*(\pi) - \phi_t^*(\pi)||_{V_t} + \hat{B}_t \left(\pi, 2SB, \frac{\delta}{2A^3}\right) + \hat{B}_t \left(\pi, 2SB, \frac{\delta}{A^2}\right).
\]

Thus implying $\pi^* \in \Pi_t$. Taking a union bound between $\mathcal{E}_2 \cap \mathcal{E}_0 \cap \mathcal{E}_1$ and the $1 - \delta$ probability event from Equation 25 yields the result.

\[
\square
\]

### C.4 Proof of Lemma 6

Full version of Lemma 6

**Lemma 15.** If $\mathcal{E}_2 \cap \mathcal{E}_2 \cap \mathcal{E}_0 \cap \mathcal{E}_1 \cap \mathcal{E}_2$ the regret is bounded by,

\[
R_T \leq 2\gamma_T \sqrt{2Td \log \left(1 + \frac{T B}{d}\right)} + \sum_{t \in [T]} 4\hat{B}_t(\pi^t_1, 4WB, \delta) + 4\hat{B}_t(\pi^t_1, 4WB, \delta)
\]

**Proof.** We first condition on $\mathcal{E}_2 \cap \mathcal{E}_2 \cap \mathcal{E}_0 \cap \mathcal{E}_1$. Let’s start by showing the following bound on the instantaneous regret,

\[
2\gamma_t \leq 2\gamma_t \|\phi_t^*(\pi^t_1) - \hat{\phi}_t^*(\pi^t_1)\|_{V_t^t} + 4\hat{B}_t(\pi^t_1, 4SB, \delta) + 4\hat{B}_t(\pi^t_1, 4SB, \delta)
\]

Since we are conditioning on $\mathcal{E}_2$, by Lemma 5 follows that for all $t$,

\[
2\gamma_t \leq (\phi_t^*(\pi^*) - \hat{\phi}_t^*(\pi^*_1))^\top w^* + (\phi_t^*(\pi^*) - \hat{\phi}_t^*(\pi^*_1))^\top w^* + \hat{B}_t(\pi^*, 4SB, \delta) + \hat{B}_t(\pi^*_1, 2SB, \delta) + \hat{B}_t(\pi^*_1, 2SB, \delta).
\]

Let’s focus on bounding the term $(\phi_t^*(\pi^*) - \hat{\phi}_t^*(\pi^*_1))^\top w^*$.

\[
(\phi_t^*(\pi^*) - \hat{\phi}_t^*(\pi^*_1))^\top w^* \leq (\phi_t^*(\pi^*) - \phi_t^*(\pi^*_1))^\top w^* + (\phi_t^*(\pi^*) - \phi_t^*(\pi^*_1))^\top (w^* - w^t_1)
\]

\[
+ (\phi_t^*(\pi^*) - \phi_t^*(\pi^*_1))^\top w^t_1 + (\phi_t^*(\pi^*) - \phi_t^*(\pi^*_1))^\top (w^* - w^t_1)
\]

\[
\leq (\phi_t^*(\pi^*) - \phi_t^*(\pi^*_1))^\top w^t_1 + (\phi_t^*(\pi^*) - \phi_t^*(\pi^*_1))^\top (w^* - w^t_1)
\]

\[
+ \|w^* - w^t_1\|_{\bar{V}} \cdot ||\phi_t^*(\pi^*) - \phi_t^*(\pi^*_1)||_{\bar{V}} + \|w^* - w^t_1\|_{\bar{V}} \cdot ||\phi_t^*(\pi^*) - \phi_t^*(\pi^*_1)||_{\bar{V}}.
\]

Thus proving the lemma.

\[
2\gamma_t \leq 2\gamma_t \|\phi_t^*(\pi^t_1) - \hat{\phi}_t^*(\pi^t_1)\|_{V_t^t} + 4\hat{B}_t(\pi^t_1, 4SB, \delta) + 4\hat{B}_t(\pi^t_1, 4SB, \delta)
\]
Then with probability

We will make use of the following Lemmas

Where the last inequality follows from Lemma 8.

The regret is therefore upper bounded by,

\[ R_T = \sum_{t \in [T]} 2r_t \]

\[ \leq \sum_{t \in [T]} 2\gamma_t \left( \|\phi_{t}^\ast - \phi_{t}^\dagger\|_{\tilde{V}_{t}^{-1}} + \|\phi_{t}^\ast - \phi_{t}^\dagger\|_{\tilde{V}_{t}^{-1}} \right) + 4\tilde{B}_t({\pi}_t^1, 4SB, \delta) + 4\tilde{B}_t({\pi}_t^2, 4SB, \delta) \]

\[ \leq 2\gamma T \sqrt{T \sum_{t \in [T]} \|\phi_{t}^\ast - \phi_{t}^\dagger\|_{\tilde{V}_{t}^{-1}}^2} + 4\tilde{B}_t({\pi}_t^1, 4SB, \delta) + 4\tilde{B}_t({\pi}_t^2, 4SB, \delta) \]

\[ \leq 2\gamma T \sqrt{2Td \log \left( 1 + \frac{TB}{d} \right)} + \sum_{t \in [T]} 4\tilde{B}_t({\pi}_t^1, 4SB, \delta) + 4\tilde{B}_t({\pi}_t^2, 4SB, \delta) \]

Where the last inequality follows from Lemma 8.

\[ \Box \]

D Miscelaneous Technical Lemmas

We will make use of the following Lemmas

Lemma 16 (Hoeffding Inequality). Let \( \{x_t\}_{t=1}^\infty \) be a martingale difference sequence with \( |x_t| \leq \zeta \) and let \( \delta \in (0, 1] \). Then with probability \( 1 - \delta \) for all \( T \in \mathbb{N} \)

\[ \sum_{t=1}^T x_t \leq 2\zeta \sqrt{T \ln \left( \frac{6\ln T}{\delta} \right)} \].
Proof. Observe that \( \frac{|x_t|}{\zeta} \leq 1 \). By invoking a time-uniform Hoeffding-style concentration inequality (Howard et al., 2020, Equation (11)) we find that

\[
\Pr \left[ \forall t \in \mathbb{N} : \sum_{t=1}^{T} \frac{x_t}{\zeta} \leq 1.7 T \left( \log \log(T) + 0.72 \log \left( \frac{5.2}{\delta} \right) \right) \right] \geq 1 - \delta.
\]

Rounding up the constants for the sake of simplicity we get

\[
\Pr \left[ \forall t \in \mathbb{N} : \sum_{t=1}^{T} x_t \leq 2 \zeta \sqrt{T \left( \log \left( \frac{6 \log(T)}{\delta} \right) \right)} \right] \geq 1 - \delta,
\]

which establishes our claim.

Lemma 17 (Uniform empirical Bernstein bound). In the terminology of Howard et al. (2021), let \( S_t = \sum_{i=1}^{t} Y_i \) be a sub-\( \psi_p \) process with parameter \( c > 0 \) and variance process \( W_t \). Then with probability at least \( 1 - \delta \) for all \( t \in \mathbb{N} \)

\[
S_t \leq 1.44 \sqrt{\max(W_t, m) \left( 1.4 \ln \ln \left( 2 \left( \max \left( \frac{W_t}{m}, 1 \right) \right) \right) + \ln \frac{5.2}{\delta} \right)} + 0.41 c \left( 1.4 \ln \ln \left( 2 \left( \max \left( \frac{W_t}{m}, 1 \right) \right) \right) + \ln \frac{5.2}{\delta} \right)
\]

where \( m > 0 \) is arbitrary but fixed.