Calculus domains modelled using an original bool algebra based on polygons

E Oanta¹,², C Panait¹, A Raicu¹, M Barhalescu¹ and T Axinte¹
¹Constanta Maritime University, Faculty of Naval Electro-Mechanics, 104 Mircea cel Batran Street, 900663, Constanta, Romania

E-mail: eoanta@yahoo.com

Abstract. Analytical and numerical computer based models require analytical definitions of the calculus domains. The paper presents a method to model a calculus domain based on a bool algebra which uses solid and hollow polygons. The general calculus relations of the geometrical characteristics that are widely used in mechanical engineering are tested using several shapes of the calculus domain in order to draw conclusions regarding the most effective methods to discretize the domain. The paper also tests the results of several CAD commercial software applications which are able to compute the geometrical characteristics, being drawn interesting conclusions. The tests were also targeting the accuracy of the results vs. the number of nodes on the curved boundary of the cross section. The study required the development of an original software consisting of more than 1700 computer code lines. In comparison with other calculus methods, the discretization using convex polygons is a simpler approach. Moreover, this method doesn’t lead to large numbers as the spline approximation did, in that case being required special software packages in order to offer multiple, arbitrary precision. The knowledge resulted from this study may be used to develop complex computer based models in engineering.

1. Introduction
Geometric modelling is a long run concern of the authors in the past 25 years. The geometric models were used in various projects: definition of the calculus domains, development of computer aided dimensioning original software, experimental data reduction in photoelasticimetry and others. Calculus of the mechanical stresses is closely related to the geometric model of the cross section and, further on, to the calculus of the geometrical characteristics of the sections. So far, the section was modelled as a set of geometric entities, hollow or solid, which were either added one to the other, or subtracted one from the other. In this way there were modelled both homogeneous and inhomogeneous sections. In the initial studies the simple geometrical entities used to define the bool algebra, to compute the mechanical stresses and to develop the according software were rectangles and ring-type sections. Therefore, regions of the frontier of the section were boundaries of the aforementioned geometrical entities [1]. According to a more general approach, the boundary was discretized using spline functions, being developed general methods to define the geometrical characteristics and to solve the integrals [2]. However, the spline discretization of the boundary has some shortcomings, in some cases being preferred a linear discretization of the boundary. In this case the domain was divided in triangles, a general located triangle being considered as a result of some Boolean operations with rectangles and right angled triangles [3]. After a more thorough
documentation, we concluded that the cross section, i.e. the domain, may be discretized in a general way using polygons.

2. Calculus of the geometrical characteristics of a polygon
Let us consider a polygon defined by \( N \) vertices, i.e. \( (y_1, z_1), \ldots, (y_{N-1}, z_{N-1}), (y_N, z_N) \). [4]

The area of the polygon is:

\[
A = \frac{1}{2} \sum_{i=1}^{N} (y_i \cdot z_{i+1} - y_{i+1} \cdot z_i) .
\]  (1)

The coordinates of the centroid are:

\[
\begin{align*}
Y_C &= \frac{1}{6 \cdot S} \sum_{i=1}^{N} (y_i + y_{i+1}) \cdot (y_i \cdot z_{i+1} - y_{i+1} \cdot z_i), \\
Z_C &= \frac{1}{6 \cdot S} \sum_{i=1}^{N} (z_i + z_{i+1}) \cdot (y_i \cdot z_{i+1} - y_{i+1} \cdot z_i).
\end{align*}
\]  (2)

The second moments of area may be computed using the relations:

\[
\begin{align*}
I_y &= \frac{1}{12} \sum_{i=1}^{N} \left( z_i^2 + y_i \cdot z_{i+1} + z_{i+1}^2 \right) \cdot \left( y_i \cdot z_{i+1} - y_{i+1} \cdot z_i \right), \\
I_z &= \frac{1}{12} \sum_{i=1}^{N} \left( y_i^2 + y_i \cdot y_{i+1} + y_{i+1}^2 \right) \cdot \left( y_i \cdot z_{i+1} - y_{i+1} \cdot z_i \right),
\end{align*}
\]  (3)  (4)

The product moment of area is:

\[
I_{yz} = \frac{1}{24} \sum_{i=1}^{N} \left( y_i \cdot z_{i+1} + 2 \cdot y_i \cdot z_i + 2 \cdot y_{i+1} \cdot z_{i+1} + y_{i+1} \cdot z_i \right) \cdot \left( y_i \cdot z_{i+1} - y_{i+1} \cdot z_i \right). 
\]  (5)

As it can be noticed, the previous relations are simple, they use only the coordinates of the vertices and their implementation is facile.

3. Discussion
The previous relations were subjected to several tests in order to verify their correctness, to master the details and to conceive the most effective implementation in terms of simplicity and flexibility.

3.1. Tests for simple shapes
The previous relations were tested for three simple shapes: a triangle, a rectangle and a circle. For these simple shapes we have direct calculus relations of the geometrical characteristics. Vertices were generated in clockwise direction, as well as in counter clockwise direction. The software developed at this stage has more than 270 computer code lines.

Conclusions regarding the previous relations:
- in the ‘for’ loops, when \( i = N \), then \( i + 1 = N + 1 = 1 \), i.e. \( y[N+1] \leftarrow y[1] \) and \( z[N+1] \leftarrow z[1] \);
- when the vertices are generated in clockwise direction it results positive values for \( S > 0 \), \( I_y > 0 \), \( I_z > 0 \), \( I_{yz} > 0 \); when the vertices are generated in counter clockwise direction we have \( S < 0 \), \( I_y < 0 \), \( I_z < 0 \), \( I_{yz} < 0 \);
- the coordinates of the centroid, \( (Y_C, Z_C) \), have the same signs regardless the direction in which the vertices are generated, clockwise or counter clockwise;
- the relations are correct, therefore they may be used in the development of a more complex software.
For the circle, the relative errors $\varepsilon$ for different numbers of vertices, $N$, i.e. values of the angles in which the circle is divided, $\Delta \alpha = \frac{360}{N}$, are presented in the following figure.

![Figure 1](image.png)

**Figure 1.** Variation of the relative errors with respect to the total number of vertices in which the circle is divided. In the left side is the relative error of the area and in the right side are the relative errors of the second moments of area.

According to the results of the program, the relative error in the calculus of the centroid’s coordinates are around $10^{-14}$, that is practically a null value. The relative error of the area is in the left side of the previous figure and the relative error of the second moment of area is in the right side of the same figure. The relative error of the product moment of area is smaller than the error of the second moment of area and it has the same variation.

To conclude, for angles $\Delta \alpha < 10^\circ$, the relative errors are $|\varepsilon| < 0.5\%$. This conclusion is important for practical reasons, i.e. the discretization of the fillets using an appropriate $\Delta \alpha$ angle.

### 3.2. Tests for solid and hollow polygons

Let us consider that a calculus domain may be divided in $\text{No\_Polygons}$ number of polygons. We assign $\text{Sign} = +1$ to the solid polygons and $\text{Sign} = -1$ to the hollow ones. The coordinates of the centroids $Y_{C_P_j}, Z_{C_P_j}$ may be computed using relations (2) and the according first moments of area are:

$$
\begin{align*}
S_{Y_P_j} &= A_{P_j} \cdot Z_{C_P_j} \\
S_{Z_P_j} &= A_{P_j} \cdot Y_{C_P_j}
\end{align*}
$$

(6)

The centroid of the entire cross section is:

$$
\begin{align*}
Y_C &= \frac{\sum_{P_j=P_1}^{\text{No\_Polygons}} \text{Sign}_{P_j} \cdot A_{P_j} \cdot Y_{C_P_j}}{\sum_{P_j=P_1}^{\text{No\_Polygons}} \text{Sign}_{P_j} \cdot A_{P_j}} \\
Z_C &= \frac{\sum_{P_j=P_1}^{\text{No\_Polygons}} \text{Sign}_{P_j} \cdot A_{P_j} \cdot Z_{C_P_j}}{\sum_{P_j=P_1}^{\text{No\_Polygons}} \text{Sign}_{P_j} \cdot A_{P_j}}
\end{align*}
$$

(7)

The geometrical characteristics of the sections may be computed using two methods.
According to the first method, the second moments of area \( I_{Y_{P_j}} \), \( I_{Z_{P_j}} \) and the product moment of area \( I_{YZ_{P_j}} \) in the initial system of axes may be computed using the relations (3), (4) and (5). These geometrical characteristics with respect to their local centroid axes are computed using the parallel axes theorem:

\[
\begin{align*}
I_{Y_{P_j}}^C &= I_{Y_{P_j}} - Z_{C_{P_j}}^2 \cdot A_{P_j}, \\
I_{Z_{P_j}}^C &= I_{Z_{P_j}} - Y_{C_{P_j}}^2 \cdot A_{P_j}, \\
I_{YZ_{P_j}}^C &= I_{YZ_{P_j}} - Y_{C_{P_j}} \cdot Z_{C_{P_j}} \cdot A_{P_j}.
\end{align*}
\] (8)

Further on, the second moments of area and the product moment of area in the centroid system of axes of the section are calculated using again the parallel axes theorem:

\[
\begin{align*}
I_{Y_{P_j}}^C &= I_{Y_{P_j}}^C + (Z_{C_{P_j}} - Z_C)^2 \cdot A_{P_j}, \\
I_{Z_{P_j}}^C &= I_{Z_{P_j}}^C + (Y_{C_{P_j}} - Y_C)^2 \cdot A_{P_j}, \\
I_{YZ_{P_j}}^C &= I_{YZ_{P_j}}^C + (Y_{C_{P_j}} - Y_C) \cdot (Z_{C_{P_j}} - Z_C) \cdot A_{P_j}.
\end{align*}
\] (9)

The previous geometrical characteristics of the entire section are

\[
\begin{align*}
I_Y &= \sum_{P_j \in P_i} \text{Sign}_{j} \cdot I_{Y_{P_j}}^C, \\
I_Z &= \sum_{P_j \in P_i} \text{Sign}_{j} \cdot I_{Z_{P_j}}^C, \\
I_{YZ} &= \sum_{P_j \in P_i} \text{Sign}_{j} \cdot I_{YZ_{P_j}}^C.
\end{align*}
\] (10)

The extreme values of the vertices’ coordinates in the centroid system of axes of the section are:

\[
\begin{align*}
y_{\text{min}} &= \min_{i} (y_i - Y_C), & y_{\text{max}} &= \max_{i} (y_i - Y_C), \\
z_{\text{min}} &= \min_{i} (z_i - Z_C), & z_{\text{max}} &= \max_{i} (z_i - Z_C).
\end{align*}
\] (11)

The distances from the centroid axes to the most remote vertices of the section are

\[
\begin{align*}
Y_{\text{max}} &= \max_i (|y_{\text{min}}|, |y_{\text{max}}|), \\
Z_{\text{max}} &= \max_i (|z_{\text{min}}|, |z_{\text{max}}|).
\end{align*}
\] (12)

The section moduli may be calculated using the relations

\[
\begin{align*}
W_Y &= \frac{I_Y}{Z_{\text{max}}}, \\
W_Z &= \frac{I_Z}{Y_{\text{max}}}.
\end{align*}
\] (13)

According to the second method, the coordinates of the vertices are expressed in the centroid system of axes:

\[
\begin{align*}
y_i &= y_i - Y_C, \\
z_i &= z_i - Z_C.
\end{align*}
\] (14)

and the algorithm is simpler, faster and the number of round off errors are fewer and smaller. In the centroid system of axes, after all the coordinates are translated, the centroid has the coordinates
\[
\begin{align*}
Y_C = 0 \\
Z_C = 0
\end{align*}
\]  
(20)

The geometrical characteristics in the centroid system of axes of the section are calculated using relations (3), (4) and (5). The rest of the algorithm is similar to the first one.

The computing methods were tested using the discretizations presented in the above figure. The results are practically identical. There may be remarked the following aspects:

- the values generated by the both methods are not very large, therefore there is not necessary to use special libraries which offer multiple arbitrary precision, such as GMP;
- the results of the both methods confirm the correctness of the algorithms, being compared with the values resulted from the classic calculus;
- the tests were necessary in order to validate the original computer code; further on the code may be used as a solver of complex and general case studies.

3.3. Test for a complex shape

Once the original software is validated, it must be tested to verify the results for a complex shape. We selected the cross section of the UIC60 rail as a case study for our test. We were able to find the drawing and some of the geometrical characteristics. However, the drawings were not presenting all the details regarding the geometry of the section.

In order to have a visual validation of the analytically-defined section, there was developed a function which is automatically generating the drawing of the section in AutoCAD, using script files. Because the boundary is approximated by segment lines, there were followed two stages in the modelling of the section. First stage replaced the fillets of the section with segment lines and allowed us to visually validate the so-called ‘coarse’ approximation model, figure 5. The results of the calculi allowed us to evaluate the ratio ‘boundary-refinement’ vs. accuracy of the geometrical characteristics. The second model replaced the fillets with segment lines, the arcs of the circle being discretized using an angle, \( \Delta \alpha \), which may be set as a variable of the program. We considered that \( \Delta \alpha = 1^\circ \) leads to a fair accurate approximation of the boundary, figure 6.
For the unrefined model, as well as for the refined model the calculi were performed using both methods.

The results are:

For the unrefined model:
\[
A_{\text{unrefined}, M_1} = A_{\text{unrefined}, M_2} = 7828.67 \text{ mm}^2;
I_{Y_{\text{unrefined}, M_1}} = I_{Y_{\text{unrefined}, M_2}} = 3.0459 \cdot 10^7 \text{ mm}^4;
I_{Z_{\text{unrefined}, M_1}} = I_{Z_{\text{unrefined}, M_2}} = 5.1171 \cdot 10^6 \text{ mm}^4;
I_{YZ_{\text{unrefined}, M_1}} = -0.000394884 \text{ mm}^4; \quad I_{YZ_{\text{unrefined}, M_2}} = 0 \text{ mm}^4.
\]

For the refined model:
\[
A_{\text{refined}, M_1} = A_{\text{refined}, M_2} = 7790.63 \text{ mm}^2;
I_{Y_{\text{refined}, M_1}} = I_{Y_{\text{refined}, M_2}} = 3.07849 \cdot 10^7 \text{ mm}^4;
I_{Z_{\text{refined}, M_1}} = I_{Z_{\text{refined}, M_2}} = 5.19829 \cdot 10^6 \text{ mm}^4;
I_{YZ_{\text{refined}, M_1}} = 0.00176793 \text{ mm}^4; \quad I_{YZ_{\text{refined}, M_2}} = 2.47383 \cdot 10^{-10} \text{ mm}^4.
\]

By comparing the results of the product moment of area, one can notice that method 2 uses fewer calculi, therefore the round off errors are fewer and smaller. However, we are allowed to consider that \( I_{YZ} = 0 \) in comparison with the large values of the other geometrical characteristics.

The according relative errors with respect to the refined model are:
\[
\epsilon_A = -0.488\% \quad \epsilon_{I_y} = \epsilon_{W_y} = 1.059\% \quad \epsilon_{I_z} = \epsilon_{W_z} = 1.562\% \quad \epsilon_{I_{yz}} = 0\%.
\]

If we consider as references the values offered by the manufacturers of the UIC60 rail, \( A = 7670 \text{ mm}^2, \quad I_y = 3.0383 \cdot 10^7 \text{ mm}^4, \quad I_z = 5.123 \cdot 10^6 \text{ mm}^4, \quad W_y = 3.336 \cdot 10^6 \text{ mm}^4, \quad W_z = 6.83 \cdot 10^4 \text{ mm}^4 \), the according errors are: \( \epsilon_A^{\text{unrefined}} = -2.069\% \quad \epsilon_{I_y}^{\text{unrefined}} = -0.250\% \quad \epsilon_{I_z}^{\text{unrefined}} = -0.115\% \quad \epsilon_{W_y}^{\text{unrefined}} = 0.123\% \quad \epsilon_{W_z}^{\text{unrefined}} = 0.123\% \), respectively \( \epsilon_A^{\text{refined}} = -1.573\% \quad \epsilon_{I_y}^{\text{refined}} = -1.323\% \quad \epsilon_{I_z}^{\text{refined}} = -1.470\% \quad \epsilon_{W_y}^{\text{refined}} = -1.149\% \quad \epsilon_{W_z}^{\text{refined}} = -1.479\% \).

4. Conclusions
The paper investigates the possibility to use a bool algebra based on polygons for the definition of the cross sections. The values of the geometrical characteristics of the analytically-defined sections are accurate and the original software is general and it may be used to solve many other case studies. An important accomplishment regards the usefulness of the polygons, which, based on the results of the
study, may be included in the library of simple-shapes that will be used to model a complex shape calculus domain.

Computer based analytic models offer reliable and fast instruments of investigation that can be used either as standalone research resources, or may be used in the broader context of the hybrid modelling. From this standpoint, the analytic model of a cross section may be used for the fast and accurate calculus of the geometrical characteristics, as well as for subsequent calculi of the stresses, of the displacements and for the buckling phenomenon studies. Moreover, from a bi-dimensional definition of the sections, there may be developed three-dimensional definitions of the calculus domains, i.e. of the beams of a structure.

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