SOLVING AN INVERSE SOURCE PROBLEM FOR A TIME FRACTIONAL DIFFUSION EQUATION BY A MODIFIED QUASI-BOUNDARY VALUE METHOD

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Abstract. In this paper, we propose a modified quasi-boundary value method to solve an inverse source problem for a time fractional diffusion equation. Under some boundedness assumption, the corresponding convergence rate estimates are derived by using an a priori and an a posteriori regularization parameter choice rules, respectively. Based on the superposition principle, we propose a direct inversion algorithm in a parallel manner.

1. Introduction. In this paper, we consider an inverse source problem for a time fractional diffusion equation with time independent source term as follows:

\[
\begin{align*}
\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} &= (Lu)(x, t) + f(x), \ x \in \Omega, \ t > 0, \ 0 < \alpha < 1, \\
u(x, t) &= 0, \ x \in \partial\Omega, \ t \geq 0, \\
u(x, 0) &= 0, \ x \in \bar{\Omega},
\end{align*}
\]

with the final data

\[
g(x) = u(x, T), \ x \in \bar{\Omega},
\]

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where the time fractional derivative is the Caputo derivative defined by
\[
\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial u(x,\tau)}{\partial \tau} d\tau, \quad 0 < \alpha < 1.
\]

\(-L\) is a symmetric uniformly elliptic operator defined on \(D(-L) = H_0^1(\Omega) \cap H^2(\Omega)\) given by
\[
Lu(x) = \sum_{i=1}^d \frac{\partial}{\partial x_i} \left( \sum_{j=1}^d \theta_{i,j} \frac{\partial}{\partial x_j} u(x) \right) - c(x)u(x),
\]
i.e., there exists a constant \(v > 0\), such that
\[
v \sum_{i=1}^d \xi_i^2 \leq \sum_{i,j=1}^d \theta_{i,j} \xi_i \xi_j, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^d.
\]
The coefficients satisfy:
\[
\theta_{i,j} \in C^1(\overline{\Omega}), \quad \theta_{i,j} = \theta_{j,i},
\]
\[
c(x) \in C(\overline{\Omega}), \quad c(x) \geq 0, \quad \forall x \in \overline{\Omega}.
\]
In practical measurement, noise is inevitable, we can only obtain the terminal observation data \(g_{\delta}(x)\) with noise polluted, where \(\delta\) is the noise level such that
\[
\|g_{\delta}(x) - g(x)\| \leq \delta.
\]

We notice that the inverse problems for the time fractional diffusion equation have attracted more and more researchers’ attention. In [3], Cheng et al. gave a uniqueness result of determining the order of fractional derivative and diffusion coefficient in a fractional diffusion equation. In [25, 19, 15], the method of the eigenfunction expansion, the integral equation method and the separation of variables method were adopted respectively to recover the space-dependent or time-dependent source term for time fractional diffusion equation. In [9, 21, 17, 24], some backward problems were investigated. For inverse potential problems, we refer to [8, 10, 22].

The quasi-boundary value method, also called the non-local boundary value problem method in [7], has been extensively used to solve the inverse problems of different equations, such as time fractional diffusion equation [20, 18, 23], parabolic equation [7, 2, 4], elliptic and hyper-parabolic equations [6, 5, 14]. The main idea is to approximate the ill-posed problem by a well-posed problem. In this study, we propose a modified quasi-boundary value method to solve an inverse source problem for a time fractional diffusion equation. The main idea is by constructing a coupled inverse source problem as a regularised problem to approximate the originally inverse source problem. We derive two kinds of convergence rate estimates by using an a priori and an a posteriori regularization parameter choice rule, respectively.

The paper is organized as follows. In Section 2, we give some preliminary results and conditional stability conclusion of the inverse source problem. In Section 3, we propose a modified quasi-boundary value method and derive the convergence estimates under an a priori assumption for the exact solution and an a posteriori regularization parameter choice rule, respectively. In Section 4, we give a direct inversion algorithm for the modified quasi-boundary value method. In final, several numerical examples are given to test the performance of the inversion algorithm.

2. Preliminaries and conditional stability. Throughout this paper, we use the following definition.
Definition 2.1. The two-parameters Mittag-Leffler function is
\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ z \in \mathbb{C}, \]
where \(\alpha\) and \(\beta\) are arbitrary constants, \(\Gamma(\cdot)\) is the gamma function.

For convenient, we collect some properties of Mittag-Leffler function as follows [11].

Lemma 2.2. (a) For \(0 < \alpha < 1\) and \(\eta > 0\),
\[ 0 \leq E_{\alpha,1}(-\eta) < 1, \quad \frac{d^\alpha}{d\eta^\alpha} E_{\alpha,1}(-\lambda \eta^\alpha) = -\lambda E_{\alpha,1}(-\lambda \eta^\alpha) . \]
Moreover, \(E_{\alpha,1}(-\eta)\) is fully monotonic — i.e. \((-1)^n \frac{d^n}{d\eta^n} E_{\alpha,1}(-\eta) / d\eta^n > 0\).

(b) For \(\lambda > 0\), \(0 < \alpha < 1\), we have
\[ \frac{d}{dt} E_{\alpha,1}(-\lambda \eta^\alpha) = -\lambda \eta^\alpha E_{\alpha,1}(-\lambda \eta^\alpha), \ t > 0 . \]

(c) For any \(\lambda_k\) such that \(\lambda_k \geq \lambda_1 > 0\), there have two positive constants \(c_-, c_+\) only depending on \(\alpha\), \(T\), \(\lambda_1\) such that
\[ \frac{c_-}{\lambda_k} \leq E_{\alpha,1}(-\lambda_k T^\alpha) \leq \frac{c_+}{\lambda_k}. \]

We prove the following two lemmas which will be used for the proof of convergence rate estimates.

Lemma 2.3. For any constants \(\mu > 0\), \(\beta_0 > 0\), \(p > 0\), \(s \geq \lambda_1 > 0\), we have
\[ F(s) = \frac{\mu s^{1-p} \Gamma(\frac{p}{2})}{\mu s + \beta_0} \leq \begin{cases} C_1 \mu^p, & 0 < p \leq 2, \\ C_2 \mu, & p > 2, \end{cases} \]
where \(C_1, C_2\) are two positive constants which are only dependent on \(\lambda_1, \lambda_2\) and \(\beta_0\).

Proof. If \(0 < p \leq 2\), we have \(\lim_{s \to 0} F(s) = 0\) and \(\lim_{s \to \infty} F(s) = 0\). Therefore, there exists \(s_0 \in (0, +\infty)\) such that \(F(s) \leq \sup_{s \in (0, +\infty)} F(s) \leq F(s_0)\). Thus \(F'(s_0) = 0\). It is not difficult to prove that \(s_0 = (2-p) \beta_0 / pp \) \(> 0\), so we have
\[ F(s) \leq F(s_0) = \frac{\mu ((2-p) \beta_0)^{1-p} \Gamma(\frac{p}{2})}{\mu (2-p) \beta_0 + \beta_0} = C_1(p, \beta_0) \mu^p. \]

If \(p > 2\), then for \(s \geq \lambda_1 > 0\), we have
\[ F(s) = \frac{\mu}{(\mu s + \beta_0) s^{1-p}} \leq \frac{\mu}{\beta_0 \lambda_1^{1-p}} = C_2(p, \beta_0, \lambda_1) \mu. \]
The proof is finished.

Lemma 2.4. For any constants \(\mu > 0\), \(\beta_1, \beta_2 > 0\), \(p > 0\), \(s \geq \lambda_1 > 0\), we have
\[ F(s) = \frac{\mu^2 s^{1-p} \Gamma(\frac{p}{2})}{(\mu s + \beta_1)(\mu s + \beta_2)} \leq \begin{cases} C_3 \mu^{1+p} \frac{\Gamma(\frac{p}{2})}{(\mu s + \beta_1)(\mu s + \beta_2)}, & 0 < p \leq 2, \\ C_4 \mu^p, & p > 2, \end{cases} \]
where \(C_3, C_4\) are two positive constants which are only dependent on \(\lambda_1, \beta_1\) and \(\beta_2\).
Proof. The same argument as used in lemma 2.3 can be used to prove the lemma. Here we omit the details.

Denote $\lambda_k$ as the eigenvalues of the symmetric uniformly elliptic operator $-L$ and $\varphi_k(x) \in H_0^1(\Omega) \cap H^2(\Omega)$ as the corresponding eigenfunctions, i.e.,

$$-L\varphi_k(x) = \lambda_k\varphi_k(x), \ \ k = 1, \cdots, \infty.$$  

Since $-L$ is the symmetric uniformly elliptic operator, we can assume $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots, \lim_{n \to \infty} \lambda_n = +\infty,$ and $\{\varphi_n(x)\}_{n=1}^{\infty}$ becomes an orthonormal basis of space $L^2(\Omega)$.

Define

$$D((-L)^{\frac{p}{2}}) = \{\psi \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^p |\langle \psi, \varphi_k \rangle|^2 < \infty\},$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(\Omega)$. It is not difficult to derive that the space $D((-L)^{\frac{p}{2}})$ is a Hilbert space with the following norm

$$\|\psi\|_{D((-L)^{\frac{p}{2}})} = \left( \sum_{k=1}^{\infty} \lambda_k^p |\langle \psi, \varphi_k \rangle|^2 \right)^{\frac{1}{2}}.$$

For any given source function $f(x) \in L^2(\Omega)$, by solving Eq. (1) we can formally define a forward linear operator $K : f(x) \mapsto u(x,T;f)$, i.e., $Kf(x) = g(x)$. From the lemma 2.1 in [12] or theorem 2.1 in [3], we know $g(x) \in H^2(\Omega)$. Then compactness of the map $K$ follows by Sobolev compact embedding theorem 6.61 in [1]. The forward operator $K$ is a smoothing operator and hence the associated inverse problem is ill-posed.

From Theorem 3.1 in [20], we know that there exists a unique solution $u(x,t)$ and $f(x)$ for the inverse source problem (1)-(2) and it has the following conditional stability.

**Theorem 2.5.** Suppose $f(x) \in D((-L)^{\frac{p}{2}})$ satisfy

$$\|f\|_{D((-L)^{\frac{p}{2}})} \leq E, \ p > 0,$$  

then we have

$$\|f\| \leq C_5 E^\frac{p}{2+p} \|g\|^\frac{p}{2+p}, \ p > 0,$$  

where $C_5$ is a positive constant only dependent on $\alpha$, $T$, $p$, $\lambda_1$.

3. A modified quasi-boundary value method and convergence rate estimates. In this section, a modified quasi-boundary value method is proposed to solve problem (1)-(2). The main idea of the modified quasi-boundary value method is replacing inverse source problem (1)-(2) with a coupled inverse source problem (8)-(9) which is well posed, then using the solutions of the coupled problem to construct approximate solutions of (1)-(2).

Let $\{u_\mu(x,t), \ v_\mu(x,t), \ f_\mu(x)\}$ be the solution of the following coupled inverse source problem

\[
\begin{align*}
\frac{\partial^\alpha}{\partial t^\alpha} u_\mu(x,t) &= (Lu_\mu)(x,t) + f_\mu(x), \ x \in \Omega, \ t > 0, \\
u_\mu(x,t) &= 0, \ x \in \partial\Omega, \ t \geq 0, \\
u_\mu(x,0) &= 0, \ x \in \bar{\Omega}, \\
u_\mu(x,T) &= \mu(Lv_\mu(x,T_1) + g(x), \ x \in \bar{\Omega},
\end{align*}
\]  

(8a)  

(8b)  

(8c)  

(8d)
coupled with
\[ \begin{aligned}
  \partial_t^\alpha v_\mu(x,t) &= (L_{v_\mu})(x,t) + f_\mu(x), \quad x \in \Omega, \ t > 0, \\
  v_\mu(x,t) &= 0, \quad x \in \partial\Omega, \ t \geq 0, \\
  v_\mu(x,0) &= 0, \quad x \in \bar{\Omega},
\end{aligned} \tag{9} \]

where \( \alpha \in (0, 1) \) is an arbitrary constant, \( \mu \) is a regularization parameter, \( T_1 \) is an arbitrary positive constant. Generally, we take \( T_1 < T \).

**Theorem 3.1.** If \( g(x) \in H^2(\Omega) \cap H^1_0(\Omega) \), then the solutions \( u_\mu(x,t) \in C((0,T]; H^2(\Omega) \cap H^1_0(\Omega)) \), \( v_\mu(x,t) \in C((0,T]; H^2(\Omega) \cap H^1_0(\Omega)) \), \( f_\mu(x) \in L^2(\Omega) \) of problem (8)-(9) are unique.

**Proof.** By the method of separation of variables, we can derive the solution to (9) in the form of series as follows
\[ v_\mu(x,t) = \sum_{k=1}^{\infty} f_{\mu,k} \frac{1 - E_{\alpha,1}(\lambda_k t^\alpha)}{\lambda_k} \varphi_k(x), \tag{10} \]

where \( f_{\mu,k} = \langle f_\mu(x), \varphi_k(x) \rangle, \ k = 1, \cdots, \infty \). In the same way, we known that the solution \( u_\mu(x,t) \) to (8a)-(8c) can be expressed as follows
\[ u_\mu(x,t) = \sum_{k=1}^{\infty} f_{\mu,k} \frac{1 - E_{\alpha,1}(\lambda_k t^\alpha)}{\lambda_k} \varphi_k(x). \tag{11} \]

By theorem 2.1 in [3], we know that \( u_\mu(x,t) \), \( v_\mu(x,t) \in C((0,T]; H^1_0(\Omega) \cap H^2(\Omega)) \). By (8d), we have
\[ f_{\mu,k} \frac{1 - E_{\alpha,1}(\lambda_k t^\alpha)}{\lambda_k} = -\mu f_{\mu,k} \frac{1 - E_{\alpha,1}(\lambda_k T_1^\alpha)}{\lambda_k} \lambda_k + g_k, \ k = 1, \cdots, \infty, \]

where \( g_k = \langle g(x), \varphi_k(x) \rangle, \ k = 1, \cdots, \infty \). So, we get
\[ f_{\mu,k} = \frac{\lambda_k g_k}{(1 - E_{\alpha,1}(\lambda_k T_1^\alpha)) + \mu \lambda_k (1 - E_{\alpha,1}(\lambda_k T_1^\alpha))}, \ k = 1, \cdots, \infty. \tag{12} \]

From (12), we know if \( g(x) = 0 \), we have \( f_{\mu}(x) = 0 \). Further, by (10) and (11), we known \( u_\mu(x,t) = 0, \ v_\mu(x,t) = 0 \). This ends the proof.

**Remark 1.** If \( T_1 \to 0 \), the coupled regularization problem restores to the inverse source problem (1) - (2) itself. If \( T_1 \to \infty \), the regularization problem is identical to the standard quasi-boundary value method for the inverse source problem (1) - (2). Therefore, in some sense, the regularized approach proposed in this paper can be understood as a generalization of the standard quasi-boundary value method.

### 3.1. Convergence rate estimate under an a priori regularization parameter choice strategy.

We denote \((u_\mu^\delta(x,t), v_\mu^\delta(x,t), f_\mu^\delta(x))\) as the solutions to the following regularized problem
\[ \begin{aligned}
  \partial_t^\alpha u_\mu^\delta(x,t) &= (L_{u_\mu^\delta})(x,t) + f_\mu^\delta(x), \quad x \in \Omega, \ t > 0, \tag{13a} \\
  u_\mu^\delta(x,t) &= 0, \quad x \in \partial\Omega, \ t \geq 0, \tag{13b} \\
  u_\mu^\delta(x,0) &= 0, \quad x \in \bar{\Omega}, \tag{13c} \\
  u_\mu^\delta(x,T) &= \mu (L_{u_\mu^\delta})(x,T_1) + g^\delta(x), \quad x \in \bar{\Omega}, \tag{13d}
\end{aligned} \]
coupled with
\[
\begin{cases}
\frac{\partial ^\alpha}{\partial t^\alpha} \nu ^\delta _\mu (x,t) = (L \nu ^\delta _\mu (x,t)) + f ^\delta _\mu (x), \quad x \in \Omega, \quad t > 0, \\
\nu ^\delta _\mu (x,t) = 0, \quad x \in \partial \Omega, \quad t \geq 0, \\
\nu ^\delta _\mu (x,0) = 0, \quad x \in \bar{\Omega}.
\end{cases}
\]
(14)

In the same way, we can obtain
\[
\nu ^\delta _\mu (x,t) = \sum _{k=1} ^\infty f ^\delta _{\mu,k} \frac{1 - E_{\bar{\alpha},1}(\cdot)}{\lambda _k} \varphi _k(x),
\]
\[
f ^\delta _{\mu,k} (x) = \sum _{k=1} ^\infty f ^\delta _{\mu,k} \varphi _k(x),
\]
where
\[
f ^\delta _{\mu,k} = \frac{\lambda _k g ^\delta _k}{(1 - E_{\alpha,1}(\cdot)) + \mu \lambda _k (1 - E_{\bar{\alpha},1}(\cdot))}, \quad k = 1, \ldots, \infty.
\]
\(\) (16)

**Theorem 3.2.** If the function \(f(x)\) satisfies the a priori bounded condition (6), and the noise estimate (3) holds, then we have

(a) If \(0 < p \leq 2\) and choose \(\mu = (\frac{\delta}{\bar{\delta}})\frac{2}{p^2} - \), we have the following convergence rate estimate
\[
\| f ^\delta _\mu (\cdot) - f(\cdot) \| \leq \tilde{C} _1 \tilde{\delta} ^\frac{p}{2}.
\]
(b) If \(p > 2\) and choose \(\mu = (\frac{\delta}{\bar{\delta}})\frac{1}{p} \), we have the following convergence rate estimate
\[
\| f ^\delta _\mu (\cdot) - f(\cdot) \| \leq \tilde{C} _2 \tilde{\delta} ^\frac{p}{2},
\]
where \(\tilde{C} _1, \tilde{C} _2\) are two positive constants independent of \(\tilde{\delta}\), but may be dependent on \(\alpha, \bar{\alpha}, T, T_1, \lambda_1, c_+, c_-, p, E\).

**Proof.** By direct computation and lemma 2.3, we can get
\[
\begin{align*}
&\| f ^\delta _\mu (\cdot) - f(\cdot) \|^2 \\
&= \| \sum _{k=1} ^\infty (f _{\mu,k} - f_k) \varphi _k(\cdot) \|^2 \\
&\leq \sum _{k=1} ^\infty \left( \frac{\lambda _k g _k}{(1 - E_{\alpha,1}(\cdot)) + \mu \lambda _k (1 - E_{\bar{\alpha},1}(\cdot))} \right)^2 \\
&\leq \sum _{k=1} ^\infty \left( \frac{\lambda _k g _k}{(1 - E_{\alpha,1}(\cdot))} \right)^2 \\
&\leq \sum _{k=1} ^\infty (|f_k| \lambda _k ^{\frac{1}{p} - \frac{1}{2}}) ^2 (1 - E_{\alpha,1}(\cdot)) + \mu \lambda _k (1 - E_{\bar{\alpha},1}(\cdot)).
\end{align*}
\]
\[
\begin{align*}
&\leq \left\{ \begin{array}{ll}
(C _0 E \mu) ^2, & 0 < p \leq 2, \\
(C _\gamma E \mu) ^2, & p > 2,
\end{array} \right. \\
&\leq \tilde{C} _2 \tilde{\delta} ^\frac{p}{2}.
\end{align*}
\]
From expression (19), we can get the following conclusion.

As we know, the boundedness of constant $\rho_s$ of constant when considering a priori regularization parameter selection strategy. The unknown theorem 3.2 holds.

By the triangle inequality, we get

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If we choose

then theorem 3.2 holds.

3.2. Convergence rate estimate under an a posterior regularization parameter choice strategy. As we know, the boundedness $E$ is not easy to obtain when considering a priori regularization parameter selection strategy. The unknown constant $E$ hampers the selection of the regularization parameter by the a priori rule. We now derive the convergence rate estimate for an a posteriori regularisation parameter choice strategy, based on the following Morozov’s discrepancy principle:

If we choose

then theorem 3.2 holds.

\[
\|f(\cdot) - f^\delta(\cdot)\| \leq \|f(\cdot) - f_\mu(\cdot)\| + \|f_\mu(\cdot) - f^\delta(\cdot)\|
\]

\[
\leq \begin{cases}
C_6 E \mu^{\frac{2}{p}} + \frac{1}{1 - E_{\alpha,1}(\lambda_1 T_1^\alpha)}, & 0 < p \leq 2, \\
C_7 E \mu + \frac{1}{1 - E_{\alpha,1}(\lambda_1 T_1^\alpha)}, & p > 2.
\end{cases}
\]

\[
\mu = \left\{ \begin{array}{ll}
\left( \frac{\delta}{E} \right)^{\frac{2}{p}}, & 0 < p \leq 2, \\
\left( \frac{\delta}{E} \right)^{\frac{1}{2}}, & p > 2.
\end{array} \right.
\]

\[
\|\mu(\mu I + K)^{-1}(K f^\delta_{\mu} - g^\delta)\| = \rho \delta.
\]  

Set $\varrho(\mu) = \|\mu(\mu I + K)^{-1}(K f^\delta_{\mu} - g^\delta)\|$, then from expressions (15) and (16), we can get

\[
\varrho(\mu) = \left( \sum_{k=1}^{\infty} \frac{\mu^2 \lambda_k^2 (1 - E_{\alpha,1}(\lambda_k T_1^\alpha))(1 - E_{\alpha,1}(\lambda_k T_1^\alpha))}{(\mu \lambda_k + 1 - E_{\alpha,1}(\lambda_k T_1^\alpha))(1 - E_{\alpha,1}(\lambda_k T_1^\alpha)) + \mu \lambda_k (1 - E_{\alpha,1}(\lambda_k T_1^\alpha))} \right)^{\frac{1}{2}}.
\]  

From expression (19), we can get the following conclusion.

Lemma 3.3. \((a)\) $\varrho(\mu)$ is a continuous function, and is a strictly increasing function over $(0, \infty)$. \((b)\) $\lim_{\mu \rightarrow 0} \varrho(\mu) = 0$, $\lim_{\mu \rightarrow \infty} \varrho(\mu) = \|g^\delta(\cdot)\|$.

Theorem 3.4. Assume the a prior condition (6) and noise data inequality (3) satisfy, and there exists a constant $\rho > 1$ satisfying $\|g^\delta(\cdot)\| > \rho \delta$. If the regularization parameter $\mu$ is selected by the discrepancy principle (18), then

(a) If $0 < p \leq 2$, we have

\[
\|f^\delta_{\mu}(\cdot) - f(\cdot)\| \leq C_3 \delta^{\frac{1}{p + 2}}.
\]
(b) If $p > 2$, we have
\[
\|f^\delta_\mu(\cdot) - f(\cdot)\| \leq \tilde{C}_3 \delta^{\frac{2}{p}},
\]
where the two constants $\tilde{C}_3$, $\tilde{C}_4$ are positive and are only dependent on $\alpha$, $\tilde{\alpha}$, $T$, $T_1$, $\lambda_1$, $c_+$, $c_-$, $p$, $\rho$, $E$.

Proof:
\[
\|f^\delta_\mu(\cdot) - f(\cdot)\| \\
= \left\| \sum_{k=1}^{\infty} f_k \frac{\mu \lambda_k(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha))}{1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha) + \mu \lambda_k(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha))} \varphi_k(\cdot) \right\| \\
= \left\| \sum_{k=1}^{\infty} (\lambda_k^{-1}) \frac{\mu \lambda_k(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha))}{1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha) + \mu \lambda_k(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha))} \tilde{C}_3 \delta^{\frac{2}{p}} \right\| \\
\cdot \left\| \left(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha) + \mu \lambda_k(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha)) \right)^{-\frac{p}{2}} \lambda_k^p f_k \varphi_k(\cdot) \right\| \\
\leq \left\| \sum_{k=1}^{\infty} (\lambda_k^{-1}) \frac{\mu \lambda_k(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha))}{1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha) + \mu \lambda_k(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha))} \tilde{C}_3 \delta^{\frac{2}{p}} \right\| \\
\cdot \left\| \left(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha) + \mu \lambda_k(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha)) \right)^{-\frac{p}{2}} \lambda_k^p f_k \varphi_k(\cdot) \right\| \\
\leq \left\| \sum_{k=1}^{\infty} \left(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha) + \mu \lambda_k(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha)) \right)^{-\frac{p}{2}} \lambda_k^p f_k \varphi_k(\cdot) \right\| \\
\cdot \left\| \sum_{k=1}^{\infty} \frac{\mu \lambda_k(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha))}{1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha) + \mu \lambda_k(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha))} \frac{g_k}{1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha)} \varphi_k(\cdot) \right\| \\
\leq \left\| \sum_{k=1}^{\infty} \frac{\mu \lambda_k(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha))}{1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha) + \mu \lambda_k(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha))} \frac{g_k}{1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha)} \varphi_k(\cdot) \right\| \\
\cdot \left\| \sum_{k=1}^{\infty} \lambda_k^p f_k \varphi_k(\cdot) \right\| \\
\leq \left\| \sum_{k=1}^{\infty} \left(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha) + \mu \lambda_k(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha)) \right)^{-\frac{p}{2}} \lambda_k^p g_k \varphi_k(\cdot) \right\| \\
\cdot \left\| \sum_{k=1}^{\infty} \lambda_k^p f_k \varphi_k(\cdot) \right\| \\
\leq \left\| \sum_{k=1}^{\infty} \left(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha) + \mu \lambda_k(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha)) \right)^{-\frac{p}{2}} \lambda_k^p g_k \varphi_k(\cdot) \right\| \\
\cdot \left\| \sum_{k=1}^{\infty} \lambda_k^p f_k \varphi_k(\cdot) \right\| \\
\leq E^{\frac{p}{2}} \left\| \sum_{k=1}^{\infty} \left(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha) + \mu \lambda_k(1 - E_{\tilde{\alpha},1}(-\lambda_k T^\alpha)) \right)^{-\frac{p}{2}} \lambda_k^p g_k \varphi_k(\cdot) \right\| \\
\cdot \left\| \sum_{k=1}^{\infty} \lambda_k^p f_k \varphi_k(\cdot) \right\| \\
\leq C_8 (1 + \rho)^{\frac{p}{2}} E^{\frac{p}{2}} \delta^{\frac{2}{p}}.
\]

From inequality (17), we have
\[
\|f^\delta_\mu(\cdot) - f(\cdot)\| \leq \frac{1}{1 - E_{\tilde{\alpha},1}(-\lambda_1 T^\alpha)} \delta^{\frac{2}{p}}.
\]
From (19), there holds
\[
\rho \tilde{\delta} = \|\mu(K f^\delta - g^\delta(\cdot))\|
\]
\[
= \left( \sum_{k=1}^{\infty} \left[ (\mu \lambda_k + 1 - E_{\bar{\alpha},1}(\lambda_k T^2)) g_k \right]^2 \right) ^{\frac{1}{2}}
\]
\[
\leq \left( \sum_{k=1}^{\infty} \left[ \frac{\mu \lambda_k (1 - E_{\bar{\alpha},1}(\lambda_k T^2)) (g_k - \bar{g}_k)}{1 - E_{\bar{\alpha},1}(\lambda_k T^2)} \right]^2 \right) ^{\frac{1}{2}}
\]
\[
+ \left( \sum_{k=1}^{\infty} \left[ (\mu \lambda_k + 1 - E_{\bar{\alpha},1}(\lambda_k T^2)) g_k \right]^2 \right) ^{\frac{1}{2}}
\]
\[
\leq \tilde{\delta} + \left( \sum_{k=1}^{\infty} \left( \frac{\lambda_k}{1 - E_{\bar{\alpha},1}(\lambda_k T^2)} \right)^2 \right) ^{\frac{1}{2}}
\]
\[
\cdot \sup_k \frac{\lambda_k^{\frac{1}{2}} \mu^2 (1 - E_{\bar{\alpha},1}(\lambda_k T^2)) (1 - E_{\bar{\alpha},1}(\lambda_k T^2))}{(\mu \lambda_k + 1 - E_{\bar{\alpha},1}(\lambda_k T^2)) (1 - E_{\bar{\alpha},1}(\lambda_k T^2)) + \mu \lambda_k (1 - E_{\bar{\alpha},1}(\lambda_k T^2))}
\]
\[
\leq \tilde{\delta} + E \sup_k \frac{\lambda_k^{\frac{1}{2}} \mu^2}{(\mu \lambda_k + 1 - E_{\bar{\alpha},1}(\lambda_k T^2)) (1 - E_{\bar{\alpha},1}(\lambda_k T^2)) + \mu \lambda_k (1 - E_{\bar{\alpha},1}(\lambda_k T^2))}
\]
\[
(a) \text{ If } 0 < p \leq 2, \text{ from lemma 2.4, we have}
\]
\[
\tilde{\delta} \leq C_0 E \mu^{1+\frac{p}{2}}.
\]
where \(C_0\) is a positive constant only dependent on \(\alpha, \bar{\alpha}, T, T_1, \lambda_1, p\). Therefore, we have
\[
\frac{1}{\mu} \leq (C_0 E)^{\frac{2}{p+2}} (\frac{1}{\delta})^{\frac{p}{p+2}}.
\]
Substituting (23) to (21), we get
\[
\|f^\delta_\mu - f_\mu\| \leq \frac{1}{1 - E_{\bar{\alpha},1}(\lambda_1 T^2)} (C_0 E)^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}.
\]
(24)
The case (a) arrives when considering (20), (24) and combining with the following triangle inequality (25)
\[
\|f(\cdot) - f^\delta_\mu(\cdot)\| \leq \|f(\cdot) - f_\mu(\cdot)\| + \|f_\mu(\cdot) - f^\delta_\mu(\cdot)\|.
\]
(b) For \(p > 2\), because space \(D((-L)^{\frac{p}{2}})\) is embedded into space \(D((-L))\), we can get the conclusion. This ends the proof. \(\square\)

4. Inversion algorithm. We take the finite element technique to solve the inverse source problem. Triangulating the domain \(\Omega\) with a regular triangulation of simplicial elements. Let \(T_h\) be a quasi-uniform triangulation of \(\Omega\), \(\{p_i\}_{i=0}^{N}\) be the set of the nodes. By the interpolation of finite element, the source term \(f(x)\) can be approximated in the finite element form of
\[
f(x) \approx f_h(x) = \sum_{i=0}^{N} f_i \psi_i(x),
\]
where \(f_i := f(p_i)\), \(\psi_i\) is the pyramid function, i.e.,
\[
\psi_i(p_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.
\]
So, solving the inverse source problem is transformed into determining the \((N_1 + 1)\)-dimensional real vector \(\mathcal{F} = [f_0, \cdots, f_{N_1}]^T\).

Based on the superposition principle, the regularization equation (13d) can be approximately written into the following weak form:

\[
< U_\mu^\delta(x, T), \chi > + \mu a \left( V_\mu^\delta(x, T), \chi \right) = < g^\delta(x), \chi >, \quad \forall \chi \in H_0^1(\Omega),
\]

where \(a \left( V_\mu^\delta(x, T), \chi \right) = \int_{\Omega} \left( \sum_{i,j=1}^d \theta_{i,j} \frac{\partial V_\mu^\delta(x, T_1)}{\partial x_i} \cdot \chi_{x_j} + c(x)V_\mu^\delta(x, T) \cdot \chi \right) dx, U_\mu^\delta(x, T) = \sum_{i=0}^{N_1} f_i u_{\mu,i}^\delta(x, T), V_\mu^\delta(x, T_1) = \sum_{i=0}^{N_1} f_i v_{\mu,i}^\delta(x, T_1). \) \(u_{\mu,i}^\delta(x)\) and \(v_{\mu,i}^\delta(x)\) are the solutions to the problem (1) and (9) with \(f(x) = \psi_i(x)\) and \(f_\mu(x) = \psi_i(x), i = 0, 1, 2, \cdots, N_1,\) respectively. After some simplification, we get

\[\mathcal{A}^\mu \mathcal{F} = \mathcal{G},\]

where \(\mathcal{A}_{i,j} = < v_{\mu,j}^\delta(x, T), \psi_i > + \mu a \left( v_{\mu,j}^\delta(x, T_1), \psi_i \right), \mathcal{G}_i = < g^\delta, \psi_i >, i = 0, 1, \cdots, N_1.\) Hence, we can formulate the inversion algorithm as follows.

**Algorithm 1** Modified quasi-boundary value approach

1. Given \(T, T_1\) and observation data \(g^\delta(t)\).
2. Compute the right side column vector \(\mathcal{G}\) in equation (26).
3. Compute direct problem (1) and direct problem (9) with \(f(x) = \psi_i(x)\) and \(f_\mu(x) = \psi_i(x), i = 0, 1, \cdots, N_1,\) in parallel.
4. Compute elements \(\mathcal{A}_{i,j}\) of matrix \(\mathcal{A}^\mu, i, j = 0, 1, \cdots, N_1,\) in parallel.
5. Obtain the regularization solution \(\mathcal{F}^\mu = [f_{\mu,0}, \cdots, f_{\mu,N_1}]^T\) by solving system (26) with an appropriate regularization parameter \(\mu.\) Thereby, we get the regularized solution as \(f_{\mu,h}(x) = \sum_{i=0}^{N_1} f_{\mu,i} \psi_i(x).\)

5. **Numerical examples.** In this section two numerical examples in one dimensional and two dimensional domains are tested by the proposed algorithm 1. First we will give the configuration for the numerical tests, which include computational domain, data generation, and how to choose the regularization parameter. Second we take two numerical examples to show the performance of the algorithm.

5.1. **Experiments setting.** We take the computational domain \(\Omega = (0, 1)\) and \(\Omega = (0, 1) \times (0, 1)\) for one dimensional and two dimensional example, respectively. The finite element space \(V_h\) (for final time measurement) is continuous \(P_1\) over triangulations \(T_h.\) In one dimensional case, \(T_h\) is chosen as a uniform partition with \(h = \frac{1}{100}\). In two dimensional case, we let \(T_h\) with 256 triangles. The final time is chosen as \(T = 1.\) We take the discrete Galerkin finite element method to compute the final time measurement and the temporal discretization parameter \(\tau\) is \(\frac{1}{100}\) and \(\frac{1}{80}\) in one and two dimensional case, respectively. The fraction time derivative \(\frac{\partial^\alpha u(x, t)}{\partial t^\alpha}\) at \(t_k\) is approximated as

\[
\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} \bigg|_{t=t_k} = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_k} (t_k - \eta)^{-\alpha} \frac{\partial u(x, \eta)}{\partial \eta} d\eta = \frac{1}{\Gamma(1-\alpha)} \sum_{l=1}^k \int_{t_{l-1}}^{t_l} (t_k - \eta)^{-\alpha} \frac{\partial u(x, \eta)}{\partial \eta} d\eta
\]
Table 1. Inversional results for example 1 with different relative errors

| $\delta$  | 0.05% | 0.1%  | 0.2%  | 0.4%  | 0.8%  | 1.6%  |
|-----------|-------|-------|-------|-------|-------|-------|
| $T_1=0.1$ | $e(f,\delta)$ | 2.1%  | 2.9%  | 4.0%  | 5.5%  | 7.9%  | 10.1% |
|           | $C_r$  | 0.34  | 0.33  | 0.31  | 0.38  | 0.24  |
| $T_1=0.2$ | $e(f,\delta)$ | 2.1%  | 3.1%  | 4.1%  | 5.9%  | 8.3%  | 11.0% |
|           | $C_r$  | 0.38  | 0.28  | 0.35  | 0.33  | 0.28  |
| $T_1=0.4$ | $e(f,\delta)$ | 2.2%  | 3.2%  | 4.2%  | 6.2%  | 8.5%  | 11.5% |
|           | $C_r$  | 0.39  | 0.26  | 0.37  | 0.31  | 0.30  |
| $T_1=0.8$ | $e(f,\delta)$ | 2.2%  | 3.3%  | 4.3%  | 6.3%  | 8.6%  | 11.8% |
|           | $C_r$  | 0.40  | 0.25  | 0.38  | 0.30  | 0.31  |
| $T_1=1$   | $e(f,\delta)$ | 2.2%  | 3.3%  | 4.3%  | 6.4%  | 8.6%  | 11.8% |
|           | $C_r$  | 0.40  | 0.25  | 0.38  | 0.29  | 0.31  |

\[
\approx \frac{\tau^{-\alpha}}{1-(2-\alpha)} \sum_{l=1}^{k} \left( u(x,t_l) - u(x,t_{l-1}) \right) \left( (k+1-l)^{1-\alpha} - (k-l)^{1-\alpha} \right)
\]
\[
= \frac{\tau^{-\alpha}}{1-(2-\alpha)} \sum_{l=1}^{k} \omega_l(u(x,t_{k+1-l}) - u(x,t_{k-l})),
\]

where $\omega_l = l^{1-\alpha} - (l-1)^{1-\alpha}$, $l = 1, \ldots, L$. To obtain the (noisy) additional data $g^\delta$, we first give the true solution $f(x)$ and solve the direct problem (1), then add pointwise noise by $g^\delta(p_i) = u^L_h(p_i) \ast (1+\delta \xi)$, where $p_i$ is the nodal point in $T_h$, $u^L_h(p_i)$ is the finite element solution of direct problem (1) at the point $(x,t) = (p_i,T)$. $\delta$ is the relative noise level and $\xi$ is a uniform random variable in $[-1,1]$. The corresponding absolute noise level is $\tilde{\delta} = \delta \|u^L_h(\cdot)\|$.

Meanwhile, the regularization parameter $\mu$ is very crucial for the inversion algorithm. Here we apply a continuation strategy for the regularization parameter $\mu$ (cf. [16, 13]), i.e., given a decreasing sequence $\{\mu_k\}$, $\mu_k = \mu_0 r^k (0 < r < 1)$, then we solve the regularization problem (26) with $\mu = \mu_k$. Once the discrepancy principle (18) is satisfied, we stop the algorithm and let $f_{\mu,h}(x)$ be $f(x)$.

5.2. Numerical tests. To observe the performance of the inversion algorithm, we define the error function as

\[
E_\delta(x) = f_{\mu,h}(x) - f(x), \quad x \in \Omega,
\]

and the $L^2$ relative error

\[
e(f,\delta) = \| f_{\mu,h}(\cdot) - f(\cdot) \| / \| f(\cdot) \|.
\]

Define the convergence rate $C_r$ as follows:

\[
C_r = \log \frac{e(f,2\delta)}{e(f,\delta)}.
\]

Example 1. Let $\alpha = 0.7$, and the exact source function to be $\sin(\pi x)$.

Error curves of numerical inversion for example 1 are showed in figure 1 with relative noise level 0.05%, 0.1%, 0.2% (left) and 0.4%, 0.8%, 1.6% (right), respectively. Table 1 and Table 2 include the relative errors and convergence orders of the inversion solution for Example 1 and Example 2, respectively, with different relative error levels for the additional data $g^\delta(x)$. The numerical error is decreasing as the level of relative noise becomes smaller and the convergence order is about 0.5, which is consistent to our convergence estimate.
Example 2. Let $\alpha = 0.5$, and the exact source function to be $x^2 y \sin(\pi x) \sin(\pi y)$.

Figure 2 presents the exact source function and the error surface of the inversion solution solved by algorithm 1 with relative error level 0.05%, 0.1% and 0.2%, respectively. From figure 1 and 2, it can be observed that the proposed inversion algorithm gives stable numerical reconstructions for both one dimensional and two dimensional examples.

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Figure 2. Error surfaces for example 2
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