A locally trivial quantum Hopf bundle

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Abstract

We describe a locally trivial quantum principal $U(1)$-bundle over the quantum space $S^2_{pq}$ which is a noncommutative analogue of the usual Hopf bundle. We also provide results concerning the structure of its total space algebra (irreducible $\ast$-representations and topological $K$-groups) and its Galois aspects (Galois property, existence of a strong connection, non-cleftness).

1 Introduction

In this note, we describe an example of a principal bundle in the setting of noncommutative geometry, which meets two possible (still provisional) definitions: It is a locally trivial quantum principal bundle in the sense of [BK96] as well as a Hopf-Galois extension [M-S93]. Besides giving the definition of these notions and a description of the bundle [CM00], [CM02], we provide a list of results obtained in [HMS] concerning the structure of the total space algebra and the Galois aspects of the bundle.

2 Quantum principal bundles

2.1 Hopf-Galois extensions

Dualizing the corresponding classical structure “à la Gelfand-Neumark”, one arrives at the following items which show up in the definition of quantum principal bundles:

- There is some algebra $P$ replacing the total space of a principal bundle.
- There is some Hopf algebra $H$ replacing the structure group, coacting on $P$ on the right, i.e., there is an algebra homomorphism $\Delta_R : P \to P \otimes H$ with $(\Delta_R \otimes \text{id}) \circ \Delta_R = (\text{id} \otimes \Delta) \circ \Delta_R$ and $(\text{id} \otimes \varepsilon) \circ \Delta_R = \text{id}$.
- There is another algebra replacing the base space, which coincides with the subalgebra of coinvariants of the coaction of $H$ on $P$, $B = P^{coH} := \{ p \in P \mid \Delta_R(p) = p \otimes 1 \}$. The bundle projection is the embedding $B \subset P$, denoted by $\iota : B \to P$.

$B \subset P$ is called $H$-extension in the above context [M-S93]. For a classical principal bundle with base space $M$, total space $P$ and structure group $G$, the right action is assumed to be free. This assumption can be restated as bijectivity of the map $X \times G \to X \times_M X$, $(x, g) \mapsto (x, xg)$. At the level of algebras, this means bijectivity of the map

$$\text{can} : P \otimes_B P \longrightarrow P \otimes H, \; p \otimes p' \mapsto pp'_{(0)} \otimes p'_{(1)}.$$ 

Here we use Sweedler notation, $\Delta_R(p) = p_{(0)} \otimes p_{(1)}$. An $H$-extension is called Hopf-Galois if $\text{can}$ is bijective. This is essentially the notion of an algebraic quantum principal bundle (see, e.g., [BM93]).
2.2 Locally trivial quantum principal bundles

There is another approach to quantum principal bundles emphasizing the idea of gluing which is behind the definition of classical fibre bundles [BK96]. In order to state this definition, we need an algebraic notion of covering:

A covering of an algebra \( B \) is a family \( (J_i)_{i \in I} \) of ideals with zero intersection. Let \( \pi_i : B \to B_i := B/J_i \), \( \pi_i^j : B_i \to B_{ij} := B/(J_i + J_j) \) be the quotient maps. A covering \( (J_i)_{i \in I} \) is called complete if the homomorphism \( B \ni b \mapsto (\pi_i(b))_{i \in I} \in \prod_{i \in I} B_i | \pi_i^j(b_i) = \pi_i^j(b_j) \) is surjective (it is always injective). Finite coverings by closed ideals in \( \mathbb{C}^* \)-algebras and two-element coverings are always complete. A locally trivial \( H \)-extension is an \( H \)-extension \( B \subset P \) supplied with the following local data:

(i) \( B \) has a complete finite covering \( (J_i)_{i \in I} \).

(ii) There are given surjective homomorphisms \( \chi_i : P \to B_i \otimes H \) (local trivializations) such that

- \( \chi_i \circ \iota = \pi_i \otimes 1 \) (\( \iota : B \to P \)),

- \( (\chi_i \otimes \text{id}) \circ \Delta_R = (\text{id} \otimes \Delta) \circ \chi_i \) (right colinearity),

- \( \ker \chi_i \) is a complete covering of \( P \).

As in the classical situation, locally trivial bundles can be reconstructed from transition functions related to the covering of the base algebra. More precisely, every locally trivial principal fibre bundle with fixed base algebra \( B \) and Hopf algebra \( H \) is determined by the following data:

- a complete finite covering \( (J_i)_{i \in I} \)
- a family of transition functions, i.e., of homomorphisms \( \tau_{ij} : H \to Z(B_{ij}) \) (center) fulfilling \( \tau_{ii} = 1_\mathbb{C} \), \( \tau_{ji} \circ S = \tau_{ij} \) (\( S \) the antipode of \( H \)), and the cocycle condition \( \tau_{ki} \circ \tau_{ij} = m_{B_{ijk}} \circ ((\pi_{ik} \circ \tau_{ij}) \otimes (\pi_{i}^j \circ \tau_{kj})) \circ \Delta \).

The total space algebra is then given as the gluing

\[
P = \{(f_i)_{i \in I} \in \oplus_{i \in I} B_i \otimes H \mid (\pi_i^j \otimes \text{id})(f_i) = \varphi_{ij} \circ (\pi_i^i \otimes \text{id})(f_j)\},
\]

where \( \varphi_{ij}(b \otimes h) = b\tau_{ji}(h_{(1)}) \otimes h_{(2)} \). The remaining data of the corresponding locally trivial \( H \)-extension are as follows:

\[
\Delta_R((f_i)_{i \in I}) = ((\text{id} \otimes \Delta)(f_i))_{i \in I}, \quad \chi_i((f_i)_{i \in I}) = f_i, \quad \iota(b) = (\pi_i(b) \otimes 1)_{i \in I}.
\]

3 Description of the locally trivial \( U(1) \)-bundle \( S^3_{pq} \to S^2_{pq} \)

3.1 Quantum discs

We use the following subfamily of a two-parameter family of quantum discs defined in [KL93] whose \( * \)-algebra is \( \mathcal{O}(D_q) := \mathbb{C}\langle x, x^* \rangle/(x^*x - qx + 1 - (1 - q)) \), \( 0 < q < 1 \). The irreducible \( * \)-representations of \( \mathcal{O}(D_q) \) are an \( S^1 \)-family of one-dimensional representations, given by \( \pi_\theta(x) = e^{i\theta} \) (classical points), and an infinite-dimensional representation \( \pi_q \) in a separable Hilbert space representing the generator \( x \) as a one-sided weighted shift.

The classical points define an embedding of \( S^1 \) into \( D_q \), i.e., \( \mathcal{O}(D_q) \ni x \overset{\phi_\theta}{\longrightarrow} u \in \mathcal{O}(S^1) := \mathbb{C}\langle u, u^* \rangle/(u^*u - 1, uu^* - 1) \). Since \( \| \pi(x) \| = 1 \) for any \( * \)-representation of \( \pi \) in some \( B(\mathcal{H}) \), the \( \mathbb{C}^* \)-closure \( C(D_q) \) of \( \mathcal{O}(D_q) \) is well-defined (using bounded \( * \)-representations). One knows that \( C(D_q) \simeq \mathcal{T} \) (Toeplitz or shift algebra). Using the above-mentioned irreducible \( * \)-representations, one may heuristically interprete \( D_q \) as a diffuse membrane spanned by a classical \( S^1 \).
3.2 Quantum two-spheres (quantum cones)

They are defined as a gluing of two quantum discs along the classical “boundary” $S^1$: $\mathcal{O}(S^3_{pq}) := \mathcal{O}(D_p) \oplus_\phi \mathcal{O}(D_q) = \{(f, g) \in \mathcal{O}(D_p) \oplus \mathcal{O}(D_q) \mid \phi_p(f) = \phi_q(g)\}$, $0 < p, q < 1$. The $*$-algebra $\mathcal{O}(S^2_{pq})$ can be identified with the quotient of the free algebra generated by $f_1, f_0^*$, $f_0$ by the ideal $J$ defined by the relations $f_0^2 = f_0$, $f_1^* f_1 - q_0 f_1 f_1^* = (p - q)f_0 + (1 - p)1$, $(1 - f_0)(f_1 f_1^* - f_0) = 0$. There are an $S^1$-family of one-dimensional and two nonequivalent infinite dimensional $*$-representations in a separable Hilbert space. The latter represent the structural $\mathcal{O}(U(1))$-extension $\mathcal{O}(S^3_{pq}) \subset \mathcal{O}(S^3_{pq})$.

3.3 The $\mathcal{O}(U(1))$-extension $\mathcal{O}(S^2_{pq}) \subset \mathcal{O}(S^3_{pq})$

Note that $\mathcal{O}(S^2_{pq}) = \mathcal{O}(D_p) \oplus_\phi \mathcal{O}(D_q)$ has a canonical covering consisting of the kernels of the first and second projections, $J_1 = \ker pr_1$, $J_2 = \ker pr_2$. One has canonical identifications $\mathcal{O}(S^2_{pq})/J_1 = \mathcal{O}(D_p)$, $\mathcal{O}(S^2_{pq})/J_2 = \mathcal{O}(D_q)$, $\mathcal{O}(S^2_{pq})/(J_1 + J_2) = \mathcal{O}(S^1)$. These are the data of the base algebra. The desired extension results from gluing $\mathcal{O}(D_p) \otimes \mathcal{O}(U(1))$ and $\mathcal{O}(D_q) \otimes \mathcal{O}(U(1))$ by means of one transition function $\tau : \mathcal{O}(U(1)) \longrightarrow \mathcal{O}(S^1)$, $u \mapsto u$, following the general method of Subsection 2.2. The corresponding gluing $\mathcal{O}(S^3_{pq})$ of two quantum solid tori along their set $\mathbb{T}^2$ of classical points is fully analogous to the geometrical picture in the case of the usual $U(1)$-Hopf bundle (Heegard splitting of $S^3$). It turns out that $\mathcal{O}(S^3_{pq})$ is isomorphic to the quotient of the free $*$-algebra generated by $a, b$ by the ideal generated by the relations

\[ ab = ba, \quad ab^* = b^* a, \quad a^* b = b^* a^*, \quad a^* b = ba^*, \]

\[ a^* a - qaa^* = 1 - q, \quad b^* b - pbb^* = 1 - p, \]

\[ (1 - aa^*)(1 - bb^*) = 0. \]

The structural $*$-homomorphisms of the locally trivial $U(1)$-extension in terms of the generators $a, b$ are:

\[ \Delta_R(a) = a \otimes u, \quad \Delta_R(b) = b \otimes u^*, \]

\[ \chi_p(a) = 1 \otimes u, \quad \chi_p(b) = x \otimes u^*, \quad \chi_q(a) = y \otimes u, \quad \chi_q(b) = 1 \otimes u^*, \]

\[ \iota(f_1) = ba, \quad \iota(f_0) = bb^*. \]

4 Further results

4.1 Structure of $S^3_{pq}$

- The classes of irreducible $*$-representations of $\mathcal{O}(S^3_{pq})$ in bounded operators are classified: There is a $\mathbb{T}^2$-family of one dimensional representations and two $S^1$-families of infinite-dimensional representations in a separable Hilbert space. In the first of these two families, $a$ is a multiple of the unit operator, and $b$ is a one-sided weighted shift. In the second family $a$ and $b$ exchange their roles. Since again the norms of $a$ and $b$ are 1 in any bounded representation, one can define the $C^*$-algebra $\mathcal{C}(S^3_{pq})$ using such representations.

- A vector space basis of $\mathcal{O}(S^3_{pq})$ can be exhibited.
• $C(S^3_{pq})$ is a 2-graph $C^*$-algebra.

• The $K$-groups of $C(S^3_{pq})$ coincide with the $K$-groups of the classical $S^3$, i.e., $K_0(C(S^3_{pq})) = K_1(C(S^3_{pq})) = \mathbb{Z}$.

### 4.2 Hopf-Galois (bundle) aspects

• The $\mathcal{O}(U(1))$-extension $\mathcal{O}(S^2_{pq}) \subset \mathcal{O}(S^3_{pq})$ has the Galois property. (Idea of proof: Find a lift $l$ of the translation map and use a general argument of Schneider.)

• The lift $l$ of the translation map is a strong connection in the sense of [H-PM96]. Consequently, the $\mathcal{O}(U(1))$-extension $\mathcal{O}(S^2_{pq}) \subset \mathcal{O}(S^3_{pq})$ is relatively projective [BH].

• As a further consequence of the existence of a strong connection, all associated modules (vector bundles) are finitely generated projective. In particular, using the strong connection one can for any winding number give explicitly a projector matrix corresponding to the associated line bundle.

• The $\mathcal{O}(U(1))$-extension $\mathcal{O}(S^2_{pq}) \subset \mathcal{O}(S^3_{pq})$ is non-cleft (not a crossed product). This is proved using a trace on $\mathcal{O}(S^2_{pq})$, which is defined as the operator trace composed with the difference of the two irreducible infinite dimensional representations. The Chern-Connes pairing of this trace with the $K_0$-class of the projector defining the associated line bundle with winding number -1 just gives this number, which proves the above claim (cf. [HM99]).

**Acknowledgements:** This work was supported by the Deutsche Forschungsgemeinschaft and the Mathematisches Forschungsinstitut Oberwolfach, where this note was completed during a stay under the Research in Pairs programme. Also, it is a pleasure to thank D. Calow, P.M. Hajac and W. Szymanski for many hours of discussion and joint work.

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