Denominators in Lusztig’s asymptotic Hecke algebra via the Plancherel formula

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Abstract

Let $\tilde{W}$ be an extended affine Weyl group, $H$ be the corresponding affine Hecke algebra over the ring $\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$, and $J$ be Lusztig’s asymptotic Hecke algebra, viewed as a based ring with basis $\{t_w\}$. Viewing $J$ as a subalgebra of the $(q^{-\frac{1}{2}})$-adic completion of $H$ via Lusztig’s map $\phi$, we use Harish-Chandra’s Plancherel formula for $p$-adic groups to show that the coefficient of $T_x$ in $t_w$ is a rational function of $q$, with denominator depending only on the two-sided cell containing $w$, and dividing a power of the Poincaré polynomial of the finite Weyl group. As an application, we conjecture that these denominators encode more detailed information about the failure of the Kazhdan-Lusztig classification at roots of the Poincaré polynomial than is currently known.

Along the way, we show that upon specializing $q = q > 1$, the map from $J$ to the Harish-Chandra Schwartz algebra is injective. As an application of injectivity, we give a novel criterion for an Iwahori-spherical representation to have fixed vectors under a larger parahoric subgroup in terms of its Kazhdan-Lusztig parameter.

Keywords— Asymptotic Hecke algebra, Iwahori-Hecke algebra, Plancherel formula, parahoric subgroup

Contents

1 Introduction ................................................................. 2
  1.1 The asymptotic Hecke algebra and $p$-adic groups ............... 3
    1.1.1 Denominators in the affine Hecke algebra and injectivity of $\eta$ .......... 3
    1.1.2 Denominators in the affine Hecke and the Kazhdan-Lusztig classification at roots of unity .... 5
    1.1.3 Application: representations with parahoric-fixed vectors ................. 5
  1.2 Outline of the argument .............................................. 6
  1.3 The affine Hecke algebra ........................................... 7
  1.4 The asymptotic Hecke algebra ..................................... 7
    1.4.1 Deformations of the group ring .................................. 8
  1.5 Representation theory of $H$ and $J$ ................................. 9
    1.5.1 Involutions on $H$ .................................................. 9
  1.6 Acknowledgements ................................................... 10

2 Harish-Chandra’s Plancherel formula .................................. 11
  2.1 Tempered and discrete series representations ..................... 11
    2.1.1 Formal degrees of discrete series representations ...................... 12
  2.2 Harish-Chandra’s canonical measure ................................ 12
    2.2.1 Unramified characters ............................................. 12
    2.2.2 Unitary unramified characters .................................... 13
    2.2.3 Action by twisting and the canonical measure ....................... 13

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1 Introduction

Let \( \tilde{W} \) be an affine Weyl group or extended affine Weyl group, and let \( H \) be its associated Hecke algebra over \( A := \mathbb{C}[q^{1/2}, q^{-1/2}] \), where \( q \) is a formal variable. The representation theory of \( H \) is very well understood, behaving well and uniformly when \( q \) is specialized to any \( q \in \mathbb{C}^\times \) that is not a root of the Poincaré polynomial \( P_W \) of the finite Weyl group \( W \subset \tilde{W} \).

When \( q \) is specialized to a prime power \( q \), the category of finite-dimensional modules over the specialized algebra \( H \) is equivalent to the category of admissible representations with nonzero Iwahori-fixed vector of some \( p \)-adic group. A form of local Langlands correspondence, the Deligne-Langlands conjecture, has been established by Kazhdan and Lusztig in [KL87], where they classified modules over the generic algebra \( H \) using algebraic \( K \)-theory. A slightly different approach to this classification due to Ginzburg is explained in [CG97].

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last being defined over \( \mathbb{C} \). Moreover, he showed that when \( P_W(q) \neq 0 \), the map \( \phi_q \) induces an isomorphism

\[
(\phi_q)_*: K_0(J - \text{Mod}) \to K_0(H|_{q=q} - \text{Mod})
\]

of Grothendieck groups. The map \( \phi \) becomes a bijection after completing \( H \) and \( J \otimes_{\mathbb{C}} A \) by replacing \( A \) with \( \mathbb{C}((q^{-1/2})) \) and allowing infinite sums convergent in the \( (q^{-1/2}) \)-adic topology. In this way one can write a basis element \( t_w \) as an infinite sum

\[
()^\dagger \circ \phi^{-1}(t_w) = \sum_{x \in \tilde{W}} a_{x,w} T_x,
\]

where each \( a_{x,w} \) is a formal Laurent series in \( q^{-1/2} \), and \( ()^\dagger \) is the Goldman involution.

In light of the above, it is natural to ask how \( a_{x,w} \) behaves when \( q \) is specialized to a root of unity.

1.1 The asymptotic Hecke algebra and \( p \)-adic groups

This paper is prompted by the work of Braverman and Kazhdan in [BK18], who related the asymptotic Hecke algebra to harmonic analysis on \( p \)-adic groups. Specifically, in [Lus89a], Lusztig relates simple \( J \)-modules to certain \( H \otimes A \mathbb{C}(q^{-1/2}) \)-modules termed \( \text{tempered} \) because their definition is made in analogy with Casselman’s criterion for temperedness of \( p \)-adic groups. In [BK18], Braverman and Kazhdan showed essentially that the analytic meaning of the word “tempered” can be substituted into Lusztig’s results from [Lus89a].

More precisely, let \( G \) be a connected reductive group defined and split over a non-archimedean local field \( F \) whose extended affine Weyl group is \( \tilde{W} \). Then in [BK18], Braverman and Kazhdan define a map expressing \( J \) as sitting between the Iwahori-Hecke algebra of \( G = G(F) \) and the Harish-Chandra Schwartz algebra \( \mathbb{C} \), and propose a spectral characterization of \( J \) via the operator Paley-Wiener theorem, obtaining a diagram

\[
\begin{array}{ccc}
H(G, I) & \xrightarrow{\phi_q(\cdot)^\dagger} & \mathcal{C}^I \times I \\
\sim \downarrow & & \sim \downarrow \\
\mathcal{E}^I & \xrightarrow{\eta} & \mathcal{E}_J^I \xrightarrow{\bar{\eta}} \mathcal{E}^I,
\end{array}
\]

where the outer vertical maps are Fourier transform \( f \mapsto \pi(f) \) and the rings \( \mathcal{E}^I \) and \( \mathcal{E}_J^I \) of endomorphisms of forgetful functors to vector spaces are as described by the operator Paley-Wiener theorem, as we recall in Section 2.3.

The map \( \eta \) is defined in [BK18] and we will recall its definition below. In particular, it induces the map \( \tilde{\phi} \), which then associates a Harish-Chandra Schwartz function to every element of \( J \) such that \( \eta(j) = \pi(\tilde{\phi}(j)) \), giving another way of associating to \( t_w \) an expression similar to (1).

This prompts several questions: whether \( \eta \) (equivalently \( \tilde{\phi} \)) is injective, whether it is surjective, and the nature of the relationship between the Schwartz function \( \tilde{\phi}(t_w) \) and the expression (1).

1.1.1 Denominators in the affine Hecke algebra and injectivity of \( \eta \)

In the first part of this paper we prove that \( \eta \) is injective. Along the way, we prove in Proposition 3 that \( \tilde{\phi} \) is essentially the map \( \phi^{-1} \). Our strategy is to determine that the Schwartz functions \( f_w \) on the \( p \)-adic group \( G \) satisfy the statements of Theorem 2 below, and are in addition sufficiently well-behaved so as to lift to elements of a certain completion \( \mathcal{H}^- \) of \( H \), thus defining a map

\[
\phi_1: J \otimes_{\mathbb{C}} A \to \mathcal{H}^-.
\]

We therefore obtain two maps \( J \subset J \otimes_{\mathbb{C}} A \to \mathcal{H}^- \): the inverse \( ()^\dagger \circ \phi^{-1} \) of Lusztig’s map, and our map \( \phi_1 \) induced by the construction in [BK18]. We prove in Proposition 3 that these maps agree, at which point Theorems 2 and 13.
In particular, $\phi_1$ is injective. Using that the representation theory of $J$ is sufficiently independent of $q$, we then show in Corollary 4 that $\tilde{\phi}$ is injective for any $q > 1$. This is obviously equivalent to

**Theorem 1 (Corollary 4).** The map $\eta$ is injective for any $q > 1$.

A weaker form of the following Theorem, which is the main result of this paper, was conjectured by Kazhdan.

**Theorem 2.** Let $\tilde{W}$ be an affine Weyl group, $H$ its affine Hecke algebra over $A$, and $J$ its asymptotic Hecke algebra. Let $\phi: H \rightarrow J \otimes_C A$ be Lusztig’s map.

1. For all $x, w \in \tilde{W}$, $a_{x,w}$ is a rational function of $q$. The denominator of $a_{x,w}$ is independent of $x$. As a function of $w$, it is constant on two-sided cells.

2. There exists $N_{\tilde{W}} \in \mathbb{N}$ such that upon writing

$$\gamma^\dagger \circ \phi^{-1}(t_w) = \sum_{x \in \tilde{W}} a_{x,w} T_x,$$

we have

$$P_W(q)^{N_{\tilde{W}}} a_{x,w} \in A$$

for all $x, w \in \tilde{W}$.

3. If $d$ is a distinguished involution in the lowest two-sided cell, then $a_{1,d} = 1/P_W(q)$ exactly.

In [Daw21], the author proved Theorem 2 in type $\tilde{A}_1$, but with different conventions. To translate to the conventions of this paper, the reader should replace $j$ with the Goldman involution $(\gamma)^\dagger$, and the completion of $H$ with respect to the $C_w$ basis and positive powers of $q^{1/2}$ with the completion of $H$ with respect to the basis $\{(−1)^{\ell(w)}C_w\}_{w \in \tilde{W}}$ and negative powers of $q^{1/2}$. Note also that we write $a_{x,w}$ instead of $a_{w,x}$ as in [Daw21]. In [Neu06], Neumhöffer described the coefficients $a_{x,w}$ for finite Weyl groups.

In future, it would be desirable to also treat the case of unequal parameters, where a result like that of [BDD23] governing denominators is not yet available. On the other hand, when $G = \text{GL}_n$, we are able to be slightly more precise than Theorem 2, while also not appealing to [BDD23]. For this reason we treat the case $G = \text{GL}_n$ separately as Theorem 13.

Our main tool is Harish-Chandra’s Plancherel formula for the $p$-adic group $G$ associated to $H$ and the surjection of cocenters induced by Lusztig’s map $\phi$ after inverting $P_W(q)$ proved in [BDD23]. We invoke [BDD23] only at the very end of our argument, which, absent [BDD23], still proves that $a_{x,w}$ are rational functions with denominator depending only the two-sided cell containing $w$. We do so with an eye to future work dealing with Hecke algebras with unequal parameters.

In [Daw21], the author related a conjecture of Kazhdan concerning the positivity of some coefficients related to the coefficients $a_{x,w}$. Historically, proofs of such positivity phenomena have also provided interpretations of the positive quantities in question. While we cannot currently prove the conjecture in [Daw21], our results in Section 4 hint at a possible interpretation of $a_{1,d}$ for certain distinguished involutions $d$.

**Remark 1.** It is tempting to conjecture the following more precise version of Theorem 2, based on the factorization $P_{\text{Mat}}(q)P_{G/P}(q) = P_{G/B}(q)$: for every Levi subgroup $\text{Mat}$ of $G$ and all $\omega \in \mathcal{E}_2(\text{Mat})$, the formal degree $d(\omega)$ is a rational function of $q$ the denominator of which divides a power of $P_{\text{Mat}}(q)$. The integral over all induced twists $\text{Ind}_B^G(\nu \otimes \omega)$ is a rational function of $q$ with denominator dividing a power of the Poincaré polynomial of the partial flag variety $(G/P)(\mathbb{C})$. For example, if $G = \text{GL}_6(F)$ and $M = \text{GL}_3(F) \times \text{GL}_3(F)$. In this case the integral itself (omitting the factor $C_M$ in the notation of Section 2.5) is

$$\frac{1}{2\pi i} \int_T \int_T \frac{(z_1 - z_2)(z_1 - z_3)}{(z_1 - q^{-3}z_2)(z_1 - q^{-3}z_3)} \frac{dz_1}{z_1} \frac{dz_2}{z_2} = \frac{(1 - q^3)^2}{(1 - q^6)^2q^4} + 1 = \frac{(1 - q^3)q^{-3}}{1 + q^3} + 1,$$

\(^1\)In fact, as proved by Bezrukavnikov-Braverman-Kazhdan in the appendix of loc. cit., $\phi_q$ induces an isomorphism whenever $q$ is not a root of unity, but we will not need this.
and by [BT82] Proposition 23.1, (with \(q = t^2\)) we have

\[ P_{G/P}(q) = P_{G(3,6)}(q) = (1 + q^2)(1 + q + q^2 + q^3)(1 + q^3). \]

In examples such as the above, this does indeed happen, but only after cancellation with some terms in the numerator. In general, we will not track numerators precisely enough to show this version of Theorem 2. We shall however see a limited demonstration of this behaviour in Corollary 1.

1.1.2 Denominators in the affine Hecke and the Kazhdan-Lusztig classification at roots of unity

The affine Hecke has a filtration by two-sided ideals

\[ H^{Z_1} = \text{span} \{C_w \mid a(w) \geq i\}, \]

where \(a\) is Lusztig’s \(\alpha\)-function. As such, for any \(q \in \mathbb{C}^\times\) and any simple \(H = H_{[q=x]}\)-module \(M\) there is an integer \(a(M)\) such that \(H^{Z_1}M \neq 0\) but \(H^{Z_1+1}M = 0\). Define \(a(M) = i\) to be this integer. One can also define \(a(E) = a(c(E))\) where \(E\) is a simple \(J\)-module and \(J_{c(E)}\) is the unique two-sided ideal not annihilating \(E\).

The algebra \(J\) linearizes the above filtration into an honest direct sum, and implements the almost-independence on \(q \in \mathbb{C}^\times\) of the representation theory of \(H = H_{[q=x]}\) as follows.

**Theorem 3** [{XI06}]. 1. Suppose that \(q\) is not a root of the Poincaré polynomial of \(G\). Then for each simple \(J\)-module \(E\), the \(H\) module \(\alpha \ast E\) has a unique simple quotient \(L\) such that \(a(E) = a(L)\). For all other simple subquotients \(L'\) of \(E\), we have \(a(L') < a(E)\).

   Equivalently, for all admissible triples \((u, s, \rho)\), the representation \(K(u, s, \rho, q)\) of \(H\) has a unique nonzero simple quotient \(L = L(u, s, \rho, q)\) such that \(a(L) = a(c(u))\). That is, the Deligne-Langlands conjecture is true for \(H_{[q=x]}\).

2. If \(q\) is a root of the Poincaré polynomial of \(G\), then the Deligne-Langlands conjecture is false for the lowest cell. That is, if \(u = \{1\}\), then every simple subquotient \(L'\) of \(K(u, s, \rho, q)\) has \(a(L') < a(c_0)\).

By Theorem 2, the coefficients \(a_{1,d}\) have poles at every root of \(P_W\), for all distinguished involutions \(d\) in the lowest two-sided cell \(c_0\). On the other hand, as we show in Example 7, there do exist cells \(c \neq c_0\) such that the coefficients \(a_{x,w}\) are nonsingular at certain roots of \(P_W\), for all \(w \in c\) and \(x \in \hat{W}\). We encode the hope that this is no accident as

**Conjecture 1.** Let \(\hat{W}\) be an affine Weyl group, and let \(q \in \mathbb{C}^\times\) be a root of \(P_W\). Let \(c\) be a two-sided cell such that if \(w \in c\), then \(a_{x,w}\) does not have a pole at \(q = q\) for any \(x \in \hat{W}\). Let \(u = u(c)\). Then every standard module \(K(u, s, \rho, q)\) has a unique simple quotient \(L = L(u, s, \rho, q)\) such that \(a(L) = a(E)\), where \(E\) is the simple \(J\) module corresponding to \(K(u, s, \rho, q)\), in the notation of [{Lus89a}]. Two such simple modules are isomorphic if and only if their corresponding triples are conjugate.

Note that in type \(A_n\), the number of two-sided cells grows as \(e^{\sqrt{n}}\), whereas the number of subsets of roots of \(P_W\) is \(2^n(n+1)/2\). For example in type \(A_1\), there is only one root of \(P_W = q + 1\), but there are two two-sided cells (and both are singular at \(q = -1\)). However, already in type \(A_3\), one can see from Theorem 11 that the two-sided cell corresponding to the partition \(4 = 2 + 2\) is not singular at two of the roots of \(P_{A_3}(q) = (1 + q)(1 + q + q^2)(1 + q + q^2 + q^3)\); see Example 7.

1.1.3 Application: representations with parahoric-fixed vectors

In Section 4, we use the existence of the action of the asymptotic Hecke algebra on tempered \(G\)-representations to give a simple criterion for the existence of vectors fixed under a parahoric subgroup of \(G\): them existence of parahoric-fixed vectors

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Theorem 14. Let $\pi = K(u, s, \rho)$ be a simple tempered $I$-spherical representation of $G$. Let $P$ be a parahoric subgroup of $G$ and let $w_P$ be the longest element in the corresponding subgroup of $W$. Let $E^i_u$ be the Springer fibre for $u$.

1. If $\ell(w_P) > \dim E^i_u$, then $\pi^P = \{0\}$.

2. Conversely, let $w_P$ be the unipotent conjugacy class corresponding to the two-sided cell containing $w_P$. Then there exists a semisimple element $s \in \mathbb{Z}_G^+(w_P)$, a Levi subgroup $M^\vee$ of $G^\vee$ minimal such that $(w_P, s) \in M^\vee$, and a discrete series representation $\omega \in \mathcal{E}_2(M)$ such that

$$\pi^P = i_{P_M}^G(\omega \otimes \nu)^P \neq \{0\}$$

for all $\nu$ non-strictly positive and the parameter of $\pi$ is $(w_P, s)$.

Thus starting from the regular unipotent class, $\pi(t_{w_P}) = 0$ until reaching the unipotent attached to $w_P$. At this unipotent, $P$-fixed vectors are first encountered, and $t_{w_P}$ acts by a nonzero projector with image contained in $\pi^P$. For lower cells, it may still be the case that $\pi^P \neq 0$, but $t_{w_P}$ will act by zero on such representations, too. Therefore the nonzero action of $t_{w_P}$ detects the precisely the “most regular” unipotent attached to $P$-spherical representations, in the sense that if a representation $\pi$ such that $\pi^P \neq 0$ has the unipotent part of its parameter equal to $u$, then $a(u) \geq a(wp)$. In this way the distinguished involutions $t_{w_P}$ are more exact versions of the corresponding indicator functions $1_P$, at the expense of being more complicated to understand.

Remark 2. Recall from Section 1.1.2 that for every simple $H$-module $M$ there is a number $a(M)$ such that $H^{>i}M = 0$ for all $i > a(M)$. However, if $(u, s, \rho)$ is the $KL$-parameter of $M$, then without knowing that $M$ extends to a simple $J$-module, it does not follow that $a(M) = \dim E^i_u$.

Remark 3. By [Lus89a, Theorem 4.8(d)] and the proof of [Lus89a, Lemma 5.5], every two-sided cell contains a distinguished involution contained in a finite parabolic subgroup of $\tilde{W}$, but not every distinguished involution of a finite Coxeter group is the longest word of a parabolic subgroup, i.e. is of the form $w_P$; approximately half of two-sided cells of the finite Weyl group $W \subset \tilde{W}$ do not contain any distinguished involutions contained in proper parabolic subgroups, because of the cell-preserving bijection $w \mapsto u_0w$. For example, this happens for the second-lowest cell for $E_8$.

The existence of parahoric-fixed vectors is a rigid question, in the sense of the rigid cocentre of Ciubotaru-He [CH14]. We investigate this connection further in forthcoming work.

Some time after completing the present paper, we became aware of [GP19], which also studies the connection between the asymptotic Hecke algebra and the Plancherel theorem in type $G_2$, for unequal parameters. In op. cit. the authors speculate that the “asymptotic Plancherel measure” of op. cit. should be related to the perspective of [BK18] on $J$. We defer investigation of this to future work, but note that in light of both the classic work [Mor93] of Morris, and recent work [Sol22] of Solleveld, the unequal parameters case is relevant even to split $p$-adic groups. In particular, establishing results similar to those of the present paper for unequal parameters may provide an effective way to study the algebra $J$ of Braverman and Kazhdan given in Definition 1.9 of [BK18].

1.2 Outline of the argument

This paper is organized according to our strategy for proving Theorems 2, Theorem 13, and Corollary 4.

These results are each simple corollaries of computations with the Plancherel formula and some of Lusztig’s results on $J$. The remainder of this section will introduce $H$ and $J$ precisely, and recall their basic representation theory. In Section 2, we introduce Harish-Chandra’s Plancherel formula in detail, along with all the numerical constants that appear in it. In Section 2.4, we recall the results of Braverman-Kazhdan from [BK18]. There is no original material in the first two sections. In Section 2.8, we prepare to apply the Plancherel formula by proving that, if $f_w$ is the Schwartz function associated by Braverman-Kazhdan to $t_w$, and $\pi$ is a tempered representation, then trace $(\pi, f_w)$ is sufficiently regular so as not to complicate the
denominators of $a_{x,w}$. This section is also mostly a recollection of standard material, the only original result being Lemma 4.

In Section 3, we prove most of our main results. As we are able to be more precise in type $\tilde{A}_n$, we perform each step in parallel for type $\tilde{A}_n$ and for other types: in Sections 3.1 and 3.2 we prove statements like those of Theorem 2 for the Schwartz functions $f_w$. In these sections $q$ is specialized to a prime power $q$. In Section 3.3 we relate the functions $f_w$ to the basis elements $t_w$, turning statements that hold for all prime powers $q$ into statements that hold for the formal variable $q$. We are then able to prove our main results.

In Section 4, we state our application about the existence of parahoric-fixed vectors.

1.3 The affine Hecke algebra

Let $F$ be a non-archimedean local field, $O$ its ring of integers and $\varpi$ be a uniformizer. Let $q$ be the cardinality of the residue field. Then $q = p^v$ is a prime power. We write $| \cdot |_F$ for the $p$-adic absolute value on $F$; when necessary, $| \cdot |_\infty$ will denote the archimedean absolute value on $\mathbb{C}$.

Let $G$ be a connected reductive algebraic group defined and split over $F$, $A$ a maximal torus of $G$, and $X_*$ the cocharacter lattice of $A$. Let $N$ be unipotent radical of a chosen Borel subgroup $B$, so that $B = AN$. Let $W$ be the finite Weyl group of $G$, and $\tilde{W} = W \times X_*(A)$ be the extended affine Weyl group. Write $S$ for the set of simple reflections in $W$. Let $G^\vee$ be the Langlands dual group of $G$, taken over $\mathbb{C}$. We write $G = G(F)$, $A = A(F)$, etc. Where there is no danger of confusion, we also write $G^\vee$, $G^\vee(\mathbb{C})$, $M^\vee$, etc. Let $K$ be the maximal compact subgroup $G(O)$. Also let $I$ be the Iwahori subgroup of $G$ that is the preimage of $B(\mathbb{F}_q)$. We sometimes write $P_{G/B}$ for $P_W$, as this polynomial is also the Poincaré polynomial of the flag variety $(G/B)(\mathbb{C})$.

We write $H$ for the affine Hecke algebra of $\tilde{W}$. It is a unital associative algebra over the ring $A = \mathbb{C}[q^\pm 1, q^{-\frac{1}{2}}]$ (in fact, it is defined over $\mathbb{Z}[q^\pm 1, q^{-\frac{1}{2}}]$ but we will work over $\mathbb{C}$ to avoid having to introduce extra notation later), where $q^\pm 1$ is a formal variable. We will think of $\mathbb{C}^\times$ as $\text{Spec} A$. The algebra $H$ has the Coxeter presentation with standard basis $\{T_w\}_{w \in \tilde{W}}$ with $T_w T_{w'} = T_{ww'}$ if $\ell(ww') = \ell(w) + \ell(w')$ and quadratic relation $(T_s + 1)(T_s - q) = 0$ for $s \in S$. We write $\theta_s$ for the generators of the Bernstein subalgebra.

Recall from [KL79] the two Kazhdan-Lusztig bases $\{C_w\}_{w \in \tilde{W}}$ and $\{C'_w\}_{w \in \tilde{W}}$ of $H$ and write $C_x C_y = \sum_{z \in \tilde{W}} h_{x,y,z} C_z$. The inverse Kazhdan-Lusztig polynomials $Q_{y,x}$ are the unique family of polynomials satisfying

$$T_x = \sum_{y \leq x} (-1)^{\ell(x) - \ell(y)} q^{\frac{\ell(y)}{2}} Q_{y,x}(q) C'_y,$$

or equivalently,

$$\sum_{x \leq y \leq x} (-1)^{\ell(x) - \ell(y)} Q_{y,x}(q) P_{y,x}(q) = \delta_{x,x}$$

along with some restrictions on their degrees. For example, we shall use in Section 3.3.2 that $\deg Q_{y,x} \leq \frac{1}{2}(\ell(x) - \ell(y) - 1)$. See [BB05] for further exposition.

It is well-known that there is an isomorphism of associative $\mathbb{C}$-algebras

$$H|_{q=q} := H \otimes_A \mathbb{C} \cong C_c^\infty(G)_{\mathfrak{l}} =: H,$$

where $q$ acts on $\mathbb{C}$ by multiplication by $q$.

1.4 The asymptotic Hecke algebra

**Definition 1.** Lusztig’s $a$-function $a : \tilde{W} \to \mathbb{Z}_{\geq 0}$ is defined such that $a(w)$ is the minimal value such that $q^{a(w)} h_{x,y,w} \in A^+$ for all $x, y \in \tilde{W}$.

The $a$-function is constant of two-sided cells of $\tilde{W}$. Obviously, $a(1) = 1$, and the $a$-function obtains its maximum, equal to the number of positive roots, on the two-sided cell containing the longest word $w_0 \in \tilde{W}$.

In general, under the bijection between two-sided cells $c$ of $\tilde{W}$ and unipotent conjugacy classes $u = u(c)$ in $G^\vee$ of [Lus89a], we have

$$a(c) = \dim_{\mathbb{C}}(B_u^\vee),$$
where $B'_u$ is the Springer fibre of $u$. We have $a(w) \leq \ell(w)$ for all $w \in \tilde{W}$.

In [Lus87a], Lusztig defined an associative algebra $J$ over $\mathbb{C}$ equipped with an injection $\phi: H \hookrightarrow J \otimes_\mathbb{C} A$ which becomes an isomorphism after taking a certain completion, to be recalled in Section 3.3.1, of both sides. As an abelian group, $J$ is free with basis $\{t_w\}_{w \in \tilde{W}}$. The structure constants of $J$ are obtained from those of $H$ written in the $\{C_w\}_{w \in \tilde{W}}$-basis under the following procedure: first, the integer $\gamma_{x,y,z}$ is defined by the condition

$$q^\gamma h_{x,y,z}^{-1} - \gamma_{x,y,z} \in qA^+.$$  

One then defines

$$t_xt_y = \sum_{z \in \tilde{W}} \gamma_{x,y,z} t_z^{-1}.$$  

Surprisingly, this defines a unital associative algebra with unit

$$1_J = \sum_{d \in D} t_d,$$

where $D$ is the (finite) set of distinguished involutions [Lus87a]. The elements $t_d$ are orthogonal idempotents in $J$, which decomposes as a direct sum indexed by the two-sided cells of $\tilde{W}$ in the sense of [Lus85]. Each left cell, again in the sense of op. cit., contains a single distinguished involution which is the unit in the ring $t_d J t_d$. If $c$ is a two-sided cell, then $J_c$ is a unital ring with unit

$$1_{J_c} = \sum_{d \in D \cap c} t_d.$$

Lusztig further defined a map of algebras

$$\phi: H \rightarrow J \otimes_\mathbb{C} A$$

given by

$$\phi(C_w) = \sum_{z \in \tilde{W}, d \in D, a(z) = a(d)} h_{x,y,z} t_z.$$  

Write $\phi_q$ for the specialization of this map when $q = q$. It is known [Lus87b, Proposition 1.7] that $\phi_q$ is injective for all $q \in \mathbb{C}^\times$.

### 1.4.1 Deformations of the group ring

Upon setting $q = 1$, $H_{|q=1}$ is isomorphic to $\mathbb{C}[\tilde{W}]$, and so $H$ is a deformation of the group algebra of $\tilde{W}$.

Let, temporarily, $W$ be any finite Coxeter group. Then one can define its Hecke algebra $H$, an algebra over $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ which deforms the group ring $\mathbb{Z}[W]$. Let $q \in \mathbb{C}^\times$. For all but finitely-many values of $q$, all roots of unity, the algebras $H_{|q=q}$ are trivial deformations of $\mathbb{C}[W]$, and hence are all isomorphic. However, this isomorphism requires choosing a square root of $q$. The affine Hecke algebra provides a canonical isomorphism: away from finitely-many $q$, we have that $H_{|q=q}$ is isomorphic to $J$, and $J$ is defined over $\mathbb{Z}$ (although, as stated above, we will view it as a $\mathbb{C}$- algebra to unburden notation), see [Lus03, Section 20.1 (e)].

**Example 1.** Let $W = \langle 1, s | s^2 = 1 \rangle$ be the Weyl group of type $A_1$. The Kazhdan-Lusztig $C_{w}$-basis elements are $C_1 = T_1$ and $C_s = q^{-\frac{1}{2}}T_s - q^{\frac{1}{2}}T_1$. There are two two-sided cells in $W$, and one can easily check that

$$\phi: C_1 \mapsto t_1 + t_s$$

and

$$\phi: C_s \mapsto -\left(q^{\frac{1}{2}} + q^{-\frac{1}{2}}\right) t_s.$$  

Specializing $q = q$, we see that $\phi$ becomes an isomorphism whenever $\left(q^{\frac{1}{2}} + q^{-\frac{1}{2}}\right) \neq 0$, that is, whenever $q \neq -1$. 

8
1.5 Representation theory of $H$ and $J$

Recall the classification of finite-dimensional $H$-modules given in [KL87]. For an extended exposition with slightly different conventions, we refer the reader to [CG97]. The primary difference between the setup we require and that of [CG97] is that we must be able to defer specializing $q$ until the last possible moment, whereas specializing $q$ is the first step of the construction as given in [CG97]. In particular, let $u \in G^\vee(\mathbb{C})$ be a unipotent element, and $s \in G^\vee(\mathbb{C})$ be a semisimple element such that $us = su$. Let $\rho$ be an irreducible representation of the simultaneous centralizer $Z_{G^\vee}(s, u)$. The standard $H$-module $K(s, u, \rho)$ is a certain, and in general reducible, $H$-module defined using the geometry of the flag variety of the Langlands dual group. It may be the zero module; we say that $(u, s, \rho)$ is admissible when this does not happen. Having fixed $s$ and $u$, we say that $\rho$ is admissible if $(u, s, \rho)$ is.

We now recall an algebraic version of the Langlands classification. As we do not have access to the notion of absolute value of $q$, the classical definitions of tempered and discrete-series representations of the corresponding $p$-adic group $G$ are not available to us. However, Kazhdan-Lusztig provide the following algebraic generalization. Let $\kappa = \mathbb{C}(q^{-1/2})$ and $\bar{\kappa}$ be the algebraic closure of $\kappa$. We write $H_{\bar{\kappa}}$ for $H \otimes_{\mathbb{C}} \bar{\kappa}$ and recall another definition of Lusztig’s from [Lus89a].

**Definition 2.** Let $M$ be a $H_{\bar{\kappa}}$-module finite-dimensional over $\bar{\kappa}$. Say that $m \in M$ is an eigenvector if $\theta_x \cdot m = \chi_m(x)m$ for all dominant $x$ in $X_s$. As $\chi_m$ is a character of the coweight lattice, it corresponds to an element $\sigma_m \in A^\vee(\bar{\kappa})$ in the sense that, for all cocharacters $x$ of $A$, we have

$$\chi_m(x) = x(\sigma_m)$$

where $x$ is viewed as a character of $A^\vee$. Then $M$ is of constant type if there is a semisimple element $s' \in G^\vee(\bar{\kappa})$ and a morphism of algebraic groups

$$\phi' : SL_2(\mathbb{C}) \to Z^0_{G^\vee}(s')$$

such that for all eigenvectors $m$ of $M$, the element $\sigma_m$ is $G^\vee(\bar{\kappa})$-conjugate to

$$\phi'(\text{diag}(q^{1/2}, q^{-1/2}))s' ,$$

where by abuse of notation we have written $\phi'$ again for the base-change to $\bar{\kappa}$.

The idea of the name of the definition is that $s' \in G^\vee(\mathbb{C})$ is a “constant element” not depending on $q$.

Next is a generalization of Casselman’s criterion, which as such, requires a choice of dominant weights. The definitions in both [KL87] and [Lus89a] are both given under the choice of positive roots to be those associated to the opposite Borel (the so-called geometric choice of dominance). On the other hand, for purposes of harmonic analysis, Casselman’s criterion as usually stated uses the standard choice of positive roots.

Following [Lus89a], we choose a morphism of groups $V : \bar{\kappa}^\times \to \mathbb{R}$ such that $V(q^{-1/2}) = 1$ and $V(aq^{-1/2} + b) = 0$ for all $a \in \bar{\kappa}$, $b \in \mathbb{C}^\times$.

**Definition 3** ([Lus89a], c.f. [KL87]). Let $M$ be any finite-dimensional $H_{\bar{\kappa}}$-module and fix the geometric notion of dominance for $X_s(\mathbb{A})$. We say that $M$ is $V$-geometrically anti-tempered if all eigenvalues $\nu$ of $\theta_\lambda$ for all geometrically dominant $\lambda \in X_s(\mathbb{A})$ satisfy $V(\nu) \geq 0$. We say that $M$ is $V$-geometrically anti-square-integrable if $V(\nu) > 0$.

When $q$ is specialized a prime power $q \in \mathbb{C}^\times$ and $V(z) = \log |z|$, we recover that our anti-tempered for the geometric choice of dominance modules are tempered in the traditional analytic sense after twisting by the Goldman involution, and likewise for square-integrability, by the Casselman criterion.

The representation theory of $J$ is very well understood. We shall recall some notation and then state some major classification results of Lusztig, which relate the representation theory of $J$ to certain $H$-modules defined by Kazhdan-Lusztig.

**Definition 4.** Let $E$ be a $J$-module. Then $E \otimes_{\mathbb{C}} \kappa$ is a $J \otimes_{\mathbb{C}} \kappa$-module. Hence $H_{\kappa}$ acts on $E$ via $\phi$. Denote the resulting $H_{\kappa}$ module by $^\phi E$. 
1.5.1 Involution on $H$

**Definition 5.** Let $j : H \to H$ be the ring (and not $A$-algebra) involution of $H$ defined by $j(\sum_a \frac{a}{T}) = \sum_a \frac{-a}{T} - a(T)q^{-1}$. The $j$-involution exchanges the $\{C_w\}$ and $\{C'_w\}$-bases [KL79].

The Goldman involution $h \mapsto h^\dagger$ is the $A$-algebra involution of $H$ defined by setting

$$T^\dagger_w = (-1)^{\ell(w)}q^{\ell(w)T_w^{-1}}.$$ 

**Lemma 1.** We have

1. \[ \overline{h^\dagger} = \overline{h}^\dagger = j(h) \]

as $A$-antilinear involutions. In particular, $C'_w = j(C_w) = (-1)^{\ell(w)}C'_w$.

2. Let $\lambda$ be dominant for $\Delta$ and let $\theta^\Delta_\lambda$ and $\theta_{\lambda}^{-\Delta}$ denote Bernstein elements with respect to $\Delta$ and $\Delta^{-}$, respectively. Then

$$\theta^\Delta_\lambda = (-1)^{\ell(\lambda)}\theta_{\lambda}^{-\Delta}.$$ 

**Proof.** We compute

$$\sum_x b_x T^\dagger_x = \sum_x \bar{b}_x(1)^{\ell(x)}q^{\ell(x)T_x} = \sum_x \bar{b}_x(1)^{\ell(x)}q^{-\ell(x)T_x} = j\left(\sum_x b_x T_x\right)$$

whereas

$$\left(\sum_x b_x T_x\right)^\dagger = \left(\sum_x \bar{b}_x(1)^{\ell(x)}q^{-\ell(x)T_x}\right)^\dagger.$$ 

Thus we have $C'_w = (-1)^{\ell(w)}C'_w$, whence the first claim.

Next,

$$\theta^\Delta_\lambda = (-1)^{\ell(\lambda)}q^{-\ell(\lambda)}T^{-1}(-\lambda) = (-1)^{\ell(\lambda)}\theta_{\lambda}^{-\Delta},$$

noting that $-\lambda$ is dominant with respect to $\Delta^{-}$ and so can appear as a label with respect to $\Delta^{-}$. 

We now summarize the relationship between representations of $H$ and of $J$.

**Theorem 4 ([Lus89], Prop. 2.11, Thm. 4.2, Prop. 4.4).** Fix the geometric notion of dominance for $X_\ast(A)$. Then there are bijections of sets

$$\begin{align*}
(u, s, \rho) & \quad \{ (u, s, \rho) \mid \rho \text{ admissible, } us = su \}/G^\vee(C) \\
K(s, u, \rho) \otimes_A \kappa & \quad \{ M \in H_\kappa \text{ - Mod } \mid M \otimes \kappa \text{ simple, } V - \text{geometrically anti-tempered} \} \\
\phi E = E \otimes_C \kappa & \quad H_\kappa - \text{module of constant type} \\
E & \quad \{ E \in J \text{ - Mod } \mid E \text{ is simple} \}.
\end{align*}$$
Moreover, for a simple $J$-module $E$,

1. $E$ is finite-dimensional over $\mathbb{C}$;
2. There is a unique two-sided cell $c = c(E)$ of $\hat{W}$ such that $\text{trace}(E, t_w) \neq 0$ implies $w \in c$.
3. $\text{trace}(E, t_w)$ is the constant term of the polynomial
   $$ (-q^{1/2})^{-a(c(E))} \text{trace}(M, C_w) \in \mathbb{C}[q^{\pm}] $$
   where $M \simeq \phi E$.

In particular, $\text{trace}(E, t_w)$ is independent of $q$, and upon specializing $q = q$ a prime power, will be a regular function in the twisting character in the setting of the Payle-Wiener theorem for the Iwahori-Hecke algebra of the $p$-adic group $G$, as we will explain in greater detail below.

That is, in our conventions, simple tempered $\mathbf{H}_{|q=q}$-modules in the sense of harmonic analysis will extend to simple $J$-modules via $\phi \circ ()^\dagger$.

We will comment even further on the necessity of twisting by the Goldman involution in Section 2.4.

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### 2 Harish-Chandra’s Plancherel formula

We recall the notation and classical results we will need about the Plancherel formula. For Iwahori-biinvariant Schwartz functions, the Plancherel formula is known explicitly for all connected reductive groups, and is due to Opdam in [Opd04]. In the case of $G = \text{GL}_n(F)$, we shall refer instead to [AP05] (where in fact the Plancherel formula is computed explicitly in its entirety for $\text{GL}_n$). In the case $G = \text{Sp}_4$, we shall refer to the unpublished work [AK00] of Aubert and Kim. For $G = G_2$, we will refer to Parkinson [Par14].

In this section $q$ is a prime power (or at least a real number of absolute value strictly greater than 1). The formal variable $q$ will not appear in this section.

#### 2.1 Tempered and discrete series representations

**Definition 7.** A smooth representation $\pi$ of $G$ is tempered if $\pi$ is unitary, of finite length, and all matrix coefficients of $\pi$ belong to $L^{2+\epsilon}(G/Z(G))$ for all $\epsilon > 0$. We write $\mathcal{M}_t(G)$ for the category of tempered representations of $G$.

In particular, the central character of such a representation takes values in the circle group $\mathbb{T} \subset \mathbb{C}^\times$.

**Definition 8.** A smooth finite-length representation $\omega$ of $G$ belongs to the discrete series if all matrix coefficients of $\omega$ are square-integrable modulo $Z(G)$. We write $\mathcal{E}_2(G)$ for the space of unitary discrete series, and $\mathcal{E}_2(G)^\dagger$ for the space of unitary discrete series with nontrivial Iwahori-fixed vectors.

The tempered representations are built from the discrete series according to the following theorem of Harish-Chandra, as related in [Wal03, Prop. III.4.1].
Theorem 5 (Harish-Chandra). Let $P$ be a parabolic subgroup of $G$ with Levi subgroup $M$, and let $\omega \in E_2(M)$. Let $\nu$ be a unitary character of $M$. Then $\pi = \text{Ind}^G_M(\omega \otimes \nu)$ is a tempered representation. Every simple tempered representation is a direct summand of a representation of this form.

By induction we always understand normalized induction.

2.1.1 Formal degrees of discrete series representations

We will soon study the Plancherel decomposition $f = \sum_M f_M$ of the Schwartz function $f$ determined by an element of $J$ as explained in Section 2.4. As will be explained below, each function $f_M$ is given by an integral formula that involves several constants that depend on the Levi subgroup $M$, or are functions on the discrete series of $M$. These constants are rational functions of $q$, the most sensitive of which is the formal degree $d(\omega)$ of $\omega \in E_2(M)^I$. Much is known about formal degrees for $I$-spherical $\omega$; the most general current result seems to be

Theorem 6 ([Sol21], [FOS21], Theorem 5.1 (b), [GR10] Proposition 4.1). Let $G$ be connected reductive over $F$. Let $\omega$ be any unipotent—in particular, any Iwahori-spherical—discrete series representation of $G = G(F)$. Then $d(\omega)$ is a rational function of $q$, the numerator and denominator of which are products of factors of the form $q^{m/2}$ with $m \in \mathbb{Z}$ and $(q^n - 1)$ with $n \in \mathbb{N}$. Moreover, there is a polynomial $\Delta_G$ depending only on $G$ and $F$ such that $\Delta_G d(\omega)$ is a polynomial in $q$.

This result is proven by first proving that the Hiraga-Ichino-Ikeda conjecture [HII08] holds for unipotent discrete series representations. Note that [Sol21], [FOS21] and [HII08] all use the normalization of the Haar measure on $G$ defined in [HII08]. This normalization gives in our setting $\mu_{\text{HII}}(K) = q^{\dim G} \# G(F_q)$. Hence, noting that $\# G(F_q) = P_G/B(q) \cdot \# B(F_q)$ and that, as $F_q$ is perfect, $\# B(F_q)$ is a polynomial in $q$, we have

$$\mu_I = \frac{1}{q^{\dim G} \cdot \# B(F_q)} \mu_{\text{HII}},$$

and so this question of normalization cannot affect the denominators of $d(\omega)$, for any Levi subgroup.

In the Iwahori-spherical case, Opdam showed the above result in [Opd04, Proposition 3.27 (v)], although with less control over the possible factors appearing in the numerator and denominator of $d(\omega)$. We emphasize that op. cit. does not make the splitness assumption we allow ourselves.

Remark 4. Proposition 4.1 of [GR10] studies not the $\gamma$-factor we are interested in, but rather its quotient by the $\gamma$-factor for the Steinberg representation. However, accounting for the use of the Euler-Poincaré measure $\mu_{\text{HII}}$ on $G$, and known formula for the formal degree of the Steinberg representation, one may recover our desired statement about formal degrees from the main theorem and equation (61) of [GR10].

2.2 Harish-Chandra’s canonical measure

In this section, we recall the standard coordinates used in [AP05], and [Wal03]. We follow both references closely. These conventions differ slightly from the original [HC84b]. Everything in this section is standard, but we include details because we do require explicit measures with which to compute. We state the below for general $G$, for application to each standard Levi subgroup of $G$.

2.2.1 Unramified characters

Let $X(G) = \text{Hom}(G, \mathbb{G}_m)$ denote the rational characters of $G$ defined over $F$. Let $a_G := (X^*(A) \otimes \mathbb{R})^* = (X^*(G) \otimes \mathbb{R})^*$ be the real Lie algebra of the maximal split central torus $A = A_G$ of $G$ and let $a_{G\mathbb{C}}$ be its complexification. We have a map

$$X(G) \to \text{Hom}(G/G^1, \mathbb{C}^\times) =: \mathcal{X}(G)$$

given by $\chi \mapsto |\chi|_F$, where $|\chi|_F(q) = |\chi(q)|_F$ and $G^1 = \bigcap_{\chi \in X(G)} \ker |\chi|_F$. This gives the unramified characters $\mathcal{X}(G)$ a complex manifold structure under which $\mathcal{X}(G) \simeq (\mathbb{C}^\times)^{\dim a_G}$. For indeed, we have
\[ a_{GC} \overset{\cdot}{\longrightarrow} \mathcal{X}(G) \overset{\cdot}{\longrightarrow} 1 \]

given by
\[ \chi \otimes s \mapsto (g \mapsto |\chi(g)|_p^p = q^{-s \text{val}(\chi(g))}). \]

The kernel is spanned by all \( \chi \otimes s \) such that \( s \text{val}(\chi(g)) \in \frac{2\pi i}{\log q} \mathbb{Z} \) for all \( g \in G \). Hence the kernel is \( \frac{2\pi i}{\log q} R \), where \( R \subset X^*(A) \otimes \mathbb{Z} \) is a lattice. In this way the quotient \( \mathcal{X}(G) \) is a complex manifold.

### 2.2.2 Unitary unramified characters

Denote \( \text{Im} \mathcal{X}(G) \) the set of unitary unramified characters taking values in the unit circle \( \mathbb{T} \subset \mathbb{C}^\times \). This notation is justified, as if \( |\chi|_F = |\chi'|_F \), then Re(\( \chi \)) = Re(\( \chi' \)) \in \mathcal{A}_G \). Hence we can define \( \text{Im} \mathcal{X}(G) \) to be the unramified characters coming from pure imaginary elements of \( a_{GC} \).

The surjection \( \text{Im} \mathcal{X}(G) \to \text{Im} \mathcal{X}(A) \) has finite kernel, and \( \text{Im} \mathcal{X}(A) \) is compact. We chose the Haar measure on it with volume one.

### 2.2.3 Action by twisting and the canonical measure

The group \( \mathcal{X}(G) \) acts on admissible representations of \( G \) by twisting: \( \omega \mapsto \omega \otimes \nu \) for \( \nu \in \mathcal{X}(G) \). This restricts to an action on the unitary unramified characters on \( \mathcal{E}_2(G) \).

Pulling this action back to \( ia_G \), and given a representation \( \omega \), let \( L^* \) be its stabilizer in \( ia_G \), so that the orbit \( \sigma \) of \( \omega \) is identified as \( ia_G/L^* \). This gives \( \sigma = \text{Im} \mathcal{X}(G) \cdot \omega \) the structure of a real submanifold of the larger orbit \( \sigma_C = a_{GC}/L^* = \mathcal{X}(G) \cdot \omega \).

**Definition 9.** Given an orbit \( \sigma \subset \mathcal{E}_2(G) \), the Harish-Chandra canonical measure on \( \sigma \) is the Euclidean measure on \( \sigma \) whose pullback to \( \text{Im} \mathcal{X}(G) \) agrees with the pullback of the Haar measure on \( \text{Im} \mathcal{X}(A) \).

Hence even though the construction of the canonical measure is slightly involved, in practice it will be easy to recognize as being essentially the Haar measure on the compact torus \( \text{Im} \mathcal{X}(A) \).

**Example 2.** If \( G = SL_2(F) \) and \( M = A \) is the diagonal torus, we have \( a_{GC} \simeq \mathbb{R} \) and \( R = \mathbb{Z} \) so that a fundamental domain for \( a_{GC}/\frac{2\pi}{\log q} R \) is \( \left[ -\frac{\pi}{\log q}, \frac{\pi}{\log q} \right] \) and the canonical measure \( d\nu = \frac{\log q}{2\pi} dx \), where \( dx \) is the Lebesgue measure. To obtain quasicharacters of \( G \), we associate to \( \nu \in a_{GC} \) the quasicharacter \( \chi_\nu(g) = q^{i\nu \langle \nu, \mathfrak{H}_G(g) \rangle} \).

Therefore to compute the integral of a function \( f \) on \( \mathcal{E}_2(G) \) supported on the unramified unitary characters of \( A \), we compute
\[
\int_{\mathcal{E}_2(A)} f(\omega) d\omega = \int f(\chi_\nu) d\nu = \frac{\log q}{2\pi} \int_{-\frac{\pi}{\log q}}^{\frac{\pi}{\log q}} f(e^{it \log q}) dt = \frac{1}{2\pi i} \int_{\mathbb{T}} f(z) \frac{dz}{z}
\]
if \( t \mapsto e^{it \log q} = q^s =: z \) (here \( s = it \)) parameterizes the unit circle \( \mathbb{T} \).

In general, \( \mathcal{E}_2(M_P) \) is a disjoint union of compact tori, and the Plancherel density descends to the quotients of these compact tori by certain finite groups, namely, the Weyl groups of \( (P, A_P) \). The set \( \mathcal{E}_2(M_P)^I \) is finite up to twist by unramified characters, by a result of Harish-Chandra [Wal03].

### 2.3 The Harish-Chandra Schwartz algebra

Let \( \mathcal{C} = \mathcal{C}(G) \) be the Harish-Chandra Schwartz algebra of \( G \); see [HC84a], [Wal03], or [BK18] for the definition and associated notation. In particular, we will record for future comparison with the argument in the proof of Proposition 2 that
\[
q^{\frac{d(w)}{2}} q^{-\#W} \leq \Delta(IwI) \leq q^{\frac{d(w)}{2}}.
\]

Let \( \mathcal{C}^{I\times I} \) the subalgebra of Iwahori-biinvariant functions. As explained in [Wal03], the Fourier transform \( f \mapsto \pi(f) \) defines an endomorphism of \( \pi \) for every \( f \in \mathcal{C} \) and every tempered representation \( \pi \).
The Plancherel theorem is the statement (see [Wal03, Thm. VIII.1.1]) that this assignment defines an isomorphism of rings

\[ \mathcal{C} \to \mathcal{E}_I(G), \]

where \( \mathcal{E}_I(G) \) is the subring of the endomorphism ring of the forgetful functor \( \mathcal{M}_I(G) \to \text{Vect}_\mathbb{C} \) defined by the following conditions:

1. For all \( \pi = \text{Ind}^G_{M}(\nu \otimes \omega) \), the endomorphism \( \eta_{\pi} = \eta_{\nu,\omega} \) is a smooth function of the unramified unitary character \( \nu \) and \( \omega \in \mathcal{E}_2(M_P) \);

2. The endomorphism \( \eta_{\pi} \) is \( I \times I \)-biinvariant with respect to some open compact subgroup of \( G \).

We have the obvious inclusion \( \iota: H \to C^{I \times I} \).

Define the subring \( \mathcal{E}(G) \) of the endomorphism ring of the forgetful functor \( \mathcal{M}(G) \to \text{Vect}_\mathbb{C} \) from the category of all smooth representations, by replacing, in (1) above, \( \mathcal{E}_2(M) \) with \( \mathcal{M}(M) \), unitary characters with all unramified characters, and “smooth” with ”algebraic.” The matrix Paley-Wiener theorem of Bernstein [Ber92] says that \( f \mapsto \pi(f) \) is an isomorphism from the full Hecke algebra of \( G \) onto \( \mathcal{E}(G) \). Denote \( \mathcal{E}^I \) and \( \mathcal{E}_I \) the subrings of \( I \)-invariant endomorphisms.

For computational purposes such as ours, we require that these isomorphisms be explicit. In the Iwahori-spherical case, harmonic analysis on \( C^{I \times I} \) can be phrased internally to \( H \) and various completions of \( H \). In this setting Opdam gave an explicit Plancherel formula in [Opd04]. In more general settings there are explicit formulas for \( \text{GL}_n(F) \), \( \text{Sp}_4(F) \), and \( G_2(F) \), which we will also make use of.

### 2.4 The algebra \( J \) as a subalgebra of the Schwartz algebra.

In [BK18], Braverman and Kazhdan constructed a map of \( \mathbb{C} \)-algebras \( J \to C^{I \times I} \). We shall review this construction now.

**Definition 10** ([BK18], Section 1.7). Let \( P = M_P N_P \). A character \( \chi: M_P \to \mathbb{C}^\times \) of \( M_P \) is non-strictly positive if for all root subgroups \( U_\alpha \subset N_P \), we have \( |\chi(\alpha^\vee(x))|_\infty \geq 1 \) for \( |x|_F \geq 1 \).

We say a non-strictly positive character \( \chi \) is strictly positive if for all root subgroups \( U_\alpha \subset N_P \), we have \( |\chi(\alpha^\vee(x))|_\infty > 1 \) for \( |x|_F \geq 1 \).

Of course, it suffices to test this for \( x = \varpi^{-1} \). The condition for \( \chi^{-1} \) to be non-strictly positive is then that \( |\chi(\alpha^\vee(\varpi))|_\infty \geq 1 \).

**Example 3.** For \( G = \text{SL}_2 \), in the conventions fixed in Example 2, an unramified character \( \chi \) of \( A \) is non-strictly positive if \( \chi^{-1} \) corresponds to \( z \) such that \( |z| \geq 1 \).

If \( G = \text{GL}_n \) and \( \nu \) corresponds to the vector \( (z_1, \ldots, z_n) \in \mathbb{C}^n \), then the condition that \( \nu^{-1} \) is non-strictly positive translate to \( |z_n| \geq |z_{n-1}| \geq \cdots \geq |z_1| \). Such conditions divide \( \mathbb{C}^n \) into chambers, on which the Weyl group \( \mathfrak{S}_n \) clearly acts simply-transitively. Interior points correspond to \( \nu \) such that \( \nu^{-1} \) is strictly positive.

Following op. cit., let \( \mathcal{E}^I_J(G) \) denote the subring of \( \mathcal{E}_I(G) \) defined by the following conditions on the endomorphisms \( \eta_{\pi} \):

1. For all \( \pi = \text{Ind}^G_{M}(\nu \otimes \omega) \), the endomorphism \( \eta_{\pi} = \eta_{\nu,\omega} \) is a rational function of \( \nu \), regular on the set of \( \nu \) such that \( \nu^{-1} \) is non-strictly positive.

2. The endomorphism \( \eta_{\pi} \) is \( I \times I \)-biinvariant.

**Theorem 7** ([BK18], Theorem 1.8). Let \( G \) be a connected reductive group defined and split over \( F \). Then the following statements hold:

1. Let \( \pi \) be a tempered representation of \( G \). Then the action of \( H \) on \( \pi^I \) extends uniquely to \( J \).

2. Let \( P = MN \) be a parabolic subgroup of \( G \) with Levi subgroup \( M \) and let \( \omega \) an irreducible tempered representation of \( M \). Let \( \nu \) be a non-strictly positive character of \( M \) and let \( \pi = \text{Ind}^G_{M}(\omega \otimes \nu^{-1}) \). Then the action of \( H(G, I) \) on \( \pi^I \) extends uniquely to an action of \( J \).
3. The map \( t_w \mapsto (\eta_\pi(w))_{\pi \in M_i(G)} \) defines a homomorphism 
\[ \eta: J \to \mathcal{E}_I^f(G). \]

Composing this isomorphism with the inverse Fourier transform, Braverman and Kazhdan define an algebra map 
\[ \tilde{\phi}: J \to C^{I \times I} \]
sending 
\[ t_w \mapsto (\eta_\pi(w))_{\pi \in M_i(G)} \mapsto f_w = \sum_{x \in W} A_{x,w} T_x \in C^{I \times I}, \]
where \( A_{x,w} = f_w(IxI) \). By definition, \( \eta_\pi(w) = \pi(f_w) \) as endomorphisms of \( \pi \). We will show later that \( \tilde{\phi} \) is essentially the map \( \phi^{-1} \).

Remark 5. There are gaps in the proofs of injectivity and, as pointed out to us by R. Bezrukavnikov and I. Karpov after an early version of the present paper was completed, of surjectivity of the map \( \eta \) in [BK18]. In the present paper we prove injectivity by proving injectivity of \( \tilde{\phi} \) in Corollary 4. We prove surjectivity in [Daw23] for all but a small number of cells for exceptional groups; [BKK23] proves that \( \tilde{\phi} \) is essentially the map \( \phi^{-1} \). Implicit in [BK18] is

Lemma 2. We have the commutative diagram (2).

Proof. Let \( \pi \) be a tempered representation of \( G \). By [BK18], the \( H \)-action on it extends to a \( J \)-action, and by Corollary 2.6 of loc. cit., we have \( \eta_\pi(\phi_q(f^1)) = \pi(f) \) in \( \mathcal{E}_I^f \) for any \( f \in H \). Therefore \( \tilde{\phi} \circ \phi_q(f^1) = f \) by the Plancherel theorem. \( \square \)

Remark 6. The presence of the Goldman involution is necessary for tempered representations of \( J \) to restrict to tempered representations of \( H \). For example, in [BK18], Braverman and Kazhdan showed for \( G = SL_2(F) \), that \( \mathcal{E}_I^f(G) = \text{End}(St^I) \oplus \text{End}_{\Phi,X_1}(S^I) \), matching the decomposition \( J = C \mathfrak{t}_1 \oplus J_0 \), where \( \Phi \) is the Fourier transform on the Schwartz space of the basic affine space \( S \) [BK99], and \( St \) is the Steinberg representation. In this case \( J \) has three one-dimensional representations, whereas \( H \) has one one-dimensional discrete series representation \( St \), two tempered one-dimensional representations not belonging to the discrete series, and one non-tempered one-dimensional representation, the trivial representation. It is easy to compute using the formulas of [Daw21] that \( \phi_q(T_s^1)t_1 = qt_1 \), whereas \( \phi_q(T_s^{1*})t_1 = -t_1 \). Thus for the Steinberg representation to restrict to the Steinberg representation, the twist is required.

2.5 The Plancherel formula for \( GL_n \)

For \( G = GL_n(F) \), we have access to an explicit Plancherel measure and its Bernstein decomposition, thanks to [AP05].

Recall that for \( G = GL_n(F) \), we have bijections 
\[ \{\text{partitions of } n\} \leftrightarrow \{\text{Standard Levi subgroups } M \text{ of } GL_n(F)\} \]
\[ \leftrightarrow \{\text{unipotent conjugacy classes in } GL_n(\C)\} \]
\[ \leftrightarrow \mathcal{N}/GL_n(\C) \]
\[ \leftrightarrow \{2 \text{-sided cells } \mathfrak{c} \text{ in } \mathring{W}\} \]
\[ \leftrightarrow \{\text{direct summands } J_c \text{ of } J\} \]

where (3) \( \leftrightarrow \) (4) sends a unipotent conjugacy class \( u \) to the standard Levi \( M \) such that a member of \( u \) is distinguished in \( M^\vee \), and (4) \( \leftrightarrow \) (5) is Lusztig’s bijection from [Lus89a].

Definition 11. Let \( u \) be a unipotent element of a semisimple group \( S \) over the complex numbers. Then \( u \) is distinguished in \( S \) if \( Z_S(u) \) contains no nontrivial torus.
Let $P = M_p N_p$ be a parabolic subgroup of $G$ and let $\mathcal{O}$ be an orbit in $E_2(M_p)$ under the action of the unitary unramified characters of $M_p$ as explained in Section 2.2. Write $W_{M_p} \subset W$ for the finite Weyl group of $(M_p, A_P)$. Let $\text{Stab}_{W_M}(\mathcal{O})$ be the stabilizer of $\mathcal{O}$. Recall that a parabolic subgroup of $G$ is said to be semistandard if it contains $A$. Then the Plancherel decomposition reads

$$f = \sum_{(P=M_p N_p, \mathcal{O})/\text{association}} f_{M_p, \mathcal{O}},$$

where $f \in C(G)$, the sum is taken over semistandard parabolic subgroups $P$ up to association, and

$$f_{M_p, \mathcal{O}}(g) = c(G/M)^{-2\gamma(G/M)} \text{Stab}_{W_M}(\mathcal{O})^{-1} \int_{\mathcal{O}} \mu_{G/M}(\omega) d(\omega) \text{trace}(\pi, R_g(f)) d\omega,$$

where $R_g(f)(x) = f(xg)$ is the right translation of $f$ and $\pi = \text{Ind}_{P}^{G}(\nu \otimes \omega)$ is the normalized parabolic induction of the twist of $\omega$ by a unitary unramified character $\nu$ of $M$. In [AP05], each term above is explicitly calculated as a rational function of $q$.

**Lemma 3.** There is a finite set $S = \{(M, \omega) \mid \omega \in E_2(M)^{I_M}\}$ such that $\text{trace}(\text{Ind}_{P}^{G}(\omega \otimes \nu), f)$ is nonzero only for $\omega \in S$, for all $I$-biinvariant Schwartz functions $f$.

**Proof.** This is entirely standard. As $E_2(M)^{I_M}$ is finite for every $M$, and there are finitely-many standard parabolics of $G$, we need only show that $\text{Ind}_{P}^{G}(\omega \otimes \nu)^{I_M} \neq 0$ only if $\omega^{I_M} \neq 0$, where $I_M$ is the Iwahori subgroup of the reductive group $M$ relative to the Borel subgroup $M(\mathbb{F}_q) \cap B(\mathbb{F}_q)$ of $M(\mathbb{F}_q)$. Note that $I_M$ is naturally a subgroup of $I$. For any representation $\sigma$ of $M$, if $f \in \text{Ind}_{P}^{G}(\sigma)$ is $I$-fixed, then for $i_M \in I_M$, we have

$$f(i_M) = \sigma(i_M) \delta_P^{I_M}(i_M) f(1) = f(1),$$

and so if $\delta_P^{I_M}(i_M) = 1$, then it must be that $f(1) \in \sigma^{I_M}$. The Iwahori factorization

$$I_M = N_P^{\circ}(\pi \mathcal{O}) A(\mathcal{O}) N_M(\mathcal{O})$$

and the fact that $\text{Ad}_{n_P}$ is a representation implies that

$$\delta_P^{I_M}(i_M) = |\det \text{Ad}_{n_P}(N_P(i_M) A(i_M) N_M(i_M))|_F = |\det \text{Ad}_{n_P}(A(i_M))|_F = 1$$

by unipotence of the groups $N_P^{\circ}$ and $N_M$ and the fact that the image of $i_M$ in $M(\mathbb{F}_q)$ lies in the Borel. \qed

Let $\pi = \text{Ind}_{P}^{G}(\omega \otimes \nu)$ be a tempered representation and let $(u, s)$ be the KL-parameter of its discrete support. Then by [KL87, Theorem 8.3], $M_P^{\nu}$ is minimal such that $(u, s) \in M_P^{\nu}$. By op. cit., this condition is equivalent to $Z_M(s)$ being semisimple and $u$ being distinguished in $Z_M(s)$.

**Proposition 1** (c.f. [Opd04] Proposition 8.3). The Plancherel decomposition is compatible with the decomposition $J = \bigoplus_a J_a$ in the sense that if $w \in \mathfrak{c}$, $f = f_w$ and $u = u(c)$ under Lusztig’s bijection, then $f_M \neq 0$ only for those $M$ such that there exists $s \in M^\vee(\mathbb{C})$ such that $Z_{M^\vee}(s)$ is semisimple and $Z_{M^\vee}(s) \cap Z_{M^\vee}(u)$ contains no nontrivial torus.

**Proof.** Let $\pi := \text{Ind}_{P}^{G}(\nu \otimes \omega)$ be a tempered irreducible representation of $G$ induced as usual from a standard parabolic subgroup $P$ with Levi subgroup $M$. Then $\pi^I$ is a tempered irreducible $H$-module, and is of the form $K(u, s, \rho, q)$ for $u, s \in G^\vee(\mathbb{C})$ and $\rho$ a representation of $\pi_0(Z_{G^\vee}(u))$ with $s$ compact. By [BK18], Corollary 2.6, $K(u, s, \rho, q)$ extends to a $J$-module. By definition of Lusztig’s bijection, $\pi(f_w) \neq 0$ only if $w$ is in the two-sided cell $c = c(u)$ of $W$ corresponding to $u$. On the other hand, $K(u, s, \rho, q)$ is induced from a square-integrable standard module $K_M(u, s, \tilde{\rho}, q)$ of $H(M, I_M)$. But now [KL87, Theorem 8.3] says that $(u, s)$ must be precisely as in the statement of the proposition. Thus only such summands $(f_w)_M$ in the Plancherel decomposition of $f_w$ are nonzero. \qed

Of course, when $G = \text{GL}_n$, the bijections (3), (4), and (5), imply that there is a unique nonzero summand $(f_w)_M$ for each $w$. 

16
2.6 Plancherel measure for $GL_n$

We refer to [AP05, Section 5], for a summary of the Bernstein decomposition of the tempered irreducible representations of $GL_n$, in particular we use the description in loc. cit. of the Bernstein component parameterizing the $L$-spherical representations of $G$.

As we shall be applying the Plancherel formula only to Iwahori-biinvariant functions, it suffices to consider only irreducible tempered representations with Iwahori-fixed vectors. For $GL_n$, the only such representations are of the form

$$\pi = \text{Ind}_P^G(\nu_1 \text{St}_1 \times \cdots \times \nu_k \text{St}_k)$$

where $\text{St}_i$ is the Steinberg representation of $GL_i$, and $P \supset M = GL_{l_1} \times \cdots \times GL_{l_k}$.

These representations are parameterized as follows. Let $M$ be a Levi subgroup corresponding to the partition $l_1 + \cdots + l_k = n$, and recall that we write $T$ for the circle group. Define $\gamma \in \mathfrak{S}_n$ by $\gamma = (1 \ldots l_1)(1 \ldots l_2) \cdots (1 \ldots l_k)$, so that the fixed-point set $(\mathbb{T}^n)^\gamma = \{(z_1, \ldots, z_1, \ldots, z_k, \ldots, z_k)\} \cong \mathbb{T}^k$. Then the irreducible tempered representations with Iwahori-fixed vectors induced from $M$ are parameterized by the compact orbifold $(\mathbb{T}^n)^\gamma/\mathbb{Z}_{\mathfrak{S}_n}(\gamma)$.

**Theorem 8** ([AP05], Remark 5.6). Let $G = GL_n$ and $M = GL_{l_1} \times \cdots \times GL_{l_k}$ be a Levi subgroup. Then the Plancherel measure of $H$ on $(\mathbb{T}^n)^\gamma/\mathbb{Z}_{\mathfrak{S}_n}(\gamma)$ is

$$d\nu_H(\omega) = \prod_{i=1}^k q^{l_i - l_i}(q - 1)^{l_i} \cdot q^{\frac{n-2}{2}} \cdot \prod_{(i,j,g)} q^{2g+1}(z_i - q^g z_j)(z_i - q^{-g} z_j)(z_i - q^{-g+1} z_j),$$

where the tuples $(i, j, g) \in \mathbb{Z} \times \mathbb{Z} \times \frac{1}{2}\mathbb{Z}$ are tuples such that $1 \leq i < j \leq k$ and $|g_i - g_j| \leq g \leq g_i + g_j$, where $g_i = \frac{l_i - 1}{2}$.

This is the measure that we will integrate against, by successively applying the residue theorem. When carrying out explicit calculations, we will usually elide the constant

$$\prod_{i=1}^k q^{l_i - l_i}(q - 1)^{l_i} \cdot q^{\frac{n-2}{2}}$$

as it depends only on $M$. We shall abbreviate

$$\Gamma_{i,j,g} := q^{-2g-1} \left| \Gamma_F \left( q^{-g} \frac{z_i}{z_j} \right) \right|^2 = \frac{(z_i - q^g z_j)(z_i - q^{-g} z_j)}{(z_i - z_j q^{-g})(z_i - q^{g+1} z_j)},$$

and recall that, as noted in the proof of Theorem 5.1 in [AP05], the function $(z_1, \ldots, z_k) \mapsto \prod_{(i,j,g)} \Gamma_{i,j,g}$ is $\mathbb{Z}_{\mathfrak{S}_n}(\gamma)$-invariant. Hence for the purposes of integration, we may allow ourselves to integrate simply over $\mathbb{T}^k$. Moreover, there are many cancellations between the $\Gamma_{i,j,g}$ for a fixed pair $i < j$ as $g$ varies. Indeed, putting $q_{ij} = q^{|g_i - g_j|}$, $q^{ij} = q^{g_i + g_j + 1}$, and

$$\Gamma^{ij} := \frac{(z_i - q_{ij} z_j)(z_i - (q_{ij})^{-1} z_j)}{(z_i - q^{ij} z_j)(z_i - (q^{ij})^{-1} z_j)},$$

we have

$$\prod \Gamma_{i,j,g} = \Gamma^{ij},$$

where the product is taken over all integers $g$ appearing in triples $(i, j, g)$ for $i < j$ fixed. We set

$$c_M := \prod_{(i,j,g)} q^{2g+1} \cdot \prod_{i=1}^k q^{l_i - l_i}(q - 1)^{l_i} \cdot q^{\frac{n-2}{2}},$$

where the first product is taken over $(i, j, g)$ such that $1 \leq i < j \leq k$ and $|g_i - g_j| \leq g \leq g_i + g_j$. 

17
2.7 Beyond type $A$: the Plancherel formula following Opdam

Beyond type $A$, we still have available Opdam’s explicit Plancherel formula for the Iwahori-Hecke algebra [Opd04]. Let $G$ be a connected reductive algebraic group defined and split over $F$, of Dynkin type other than type $A$ (to avoid redundancy). Let $f$ be an Iwahori-biinvariant Schwartz function on $G$ and let $M$ be a Levi subgroup of $G$. Given a parabolic subgroup $P$ of $G$, let $R_{1,+}$ and $R_{P,1,+}$ be defined as in [Opd04], Section 2.3. Recall that the group of unramified characters of a Levi subgroup $M$ has the structure of a complex torus, and is in fact a maximal torus of $M^\vee(\mathbb{C})$. In particular, if $P$ is a parabolic subgroup and $\alpha$ is a root of $(M_P, A_P)$, then it makes sense to write $\alpha(\nu)$ for any unramified character $\nu$ of $M$. Then, altering Opdam’s notation to match our own from Section 2.5, the Plancherel formula reads

**Theorem 9** ([Opd04] Thm. 4.43).

$$f_{M,\mathcal{O}}(1) = \frac{q^{-\ell(w^P)}}{\#\text{Stab}_{W_M}(\mathcal{O})} \int_{\mathcal{O}} d(\omega) \prod_{\alpha^\vee \in R_{1,+} \setminus R_{P,1,+}} \frac{|1 - \alpha^\vee(\nu)|^2}{|1 + q_2^2 \alpha^\vee(\nu)^{1/2}|^2|1 - q_2^2 q_3 \alpha^\vee(\nu)^{1/2}|^2} \text{trace}(\pi, f) \, d\omega, \quad (6)$$

where $\pi = \text{Ind}_P^G(\omega \otimes \nu)$, $P \supseteq M$, and where $q_3$ and $q_2$ are powers of $q$, and $w^P$ is the longest element in the complement $W^P$ to the parabolic subgroup $W_P$ of $W$.

Note that whenever $q_2 = 1$, which holds whenever $\alpha^\vee \notin 2X_+$, the factor for $\alpha$ reduces to

$$\frac{|1 - \alpha^\vee(\nu)|^2}{|1 - q_3 \alpha^\vee(\nu)|^2}. \quad (7)$$

In types $A$ (as we have used above) and $D$, this simplification always occurs. In types $B$ and $C$, it happens for all roots except $\alpha = 2\varepsilon_i \in R_{1,+}(B_n)$ and $\alpha = 4\varepsilon_i \in R_{1,+}(C_n)$, where $\varepsilon_i$ is the character $\text{diag}(a, \ldots, a) \mapsto a_i$.

For explicit evaluation we rewrite (6) in coordinates as follows. Recalling the setup of Section 2.2.1, a chosen basis $\{\beta_i^\vee\}$ of the coweight lattice of $G$. Then we obtain coordinates $z_i$, such that if $\alpha^\vee = \sum_i \varepsilon_i \beta_i^\vee$, then the factor in (6) labelled by $\alpha^\vee$ is

$$\frac{|1 - z_1^{\epsilon_1} \cdots z_n^{\epsilon_n}|^2}{|1 + q_2^2 (z_1^{\epsilon_1} \cdots z_n^{\epsilon_n})^{1/2}|^2|1 - q_2^2 q_3 (z_1^{\epsilon_1} \cdots z_n^{\epsilon_n})^{1/2}|^2}. \quad (8)$$

When integrating, the coordinates $z_i$ are restricted to the residual coset corresponding to $\mathcal{O}$, in the sense of [Opd04]. For $G = \text{GL}_n$, we used the basis afforded by the characters $\varepsilon_i$.

2.8 Regularity of the trace

In order to extract information about the expansion of the elements $t_w$ in terms of the basis $T_x$ via the Plancherel formula, we must establish a regularity property of $\text{trace}(\pi, f_{w})$ where $\pi$ is an irreducible tempered representation of $G$. The needed property follows trivially from Theorem 7.

2.8.1 Intertwining operators

The goal is to use the property that elements of $\mathcal{E}_f^{\mathcal{I}}(G)$ commute with all intertwining operators in $\mathcal{M}_f(G)$, and regularity of the trace for unitary and non-strictly positive characters of $M$ to deduce regularity of the trace at all characters of $M$.

Let $\omega$ be a discrete series representation of a Levi subgroup $M$ of $G$, and let $\nu$ be any unramified character of $M$, not necessarily unitary. Then we may form the representation $\pi = \text{Ind}_P^G(\nu \otimes \omega)$ of $G$, where $P$ is a parabolic with Levi factor $M$. We will now recall some well-known facts about the action of the Weyl group of $M$ on such representations $\pi$. Let $\theta$ and $\theta'$ be two subsets of $\Delta$ corresponding to Levi subgroups $M$ and $M'$. Let $w \in W$ be such that $w\theta = \theta'$. Then there is an intertwining operator

$$J_{P|P'}(\omega, \nu): \text{Ind}_P^G(\nu \otimes \omega) \to \text{Ind}_{P'}^G(\nu \otimes \omega)$$
for each \( w \in W(\theta, \theta') = \{ w \in W \mid w(\theta) = \theta' \} \). For \( G = \text{GL}_n \), this set is nonempty only if

\[
M_\theta = \text{GL}_{l_1} \times \cdots \times \text{GL}_{l_N} \quad M_{\theta'} = \text{GL}_{l'_1} \times \cdots \times \text{GL}_{l'_{N'}}
\]

and \( \{ l_1, \ldots, l_N \} = \{ l'_1, \ldots, l'_{N'} \} \) are equal multisets. In this case, \( W(\theta, \theta') \simeq \mathfrak{S}_N \) can be viewed as acting by permuting the blocks of \( M \). It is well-known that \( J_{\alpha_{P,P'}}(\omega, \nu) \) is a meromorphic function of \( \nu \) with simple poles. The poles of these operators have been studied by Shahidi in [Sha81], and in the language of modules over the full Hecke algebra by Arthur in [Art89]. The results will be stated for certain renormalizations \( A(\nu, \omega, w) \) of the operators \( J_{\alpha_{P,P'}}(\omega, \nu) \) as explained in equation (2.2.1) of [Sha81].

### 2.8.2 Conventions on parabolic subgroups

We now recall the notation of Shahidi [Sha81]. Given a subset \( \theta \), we set \( \Sigma_\theta = \text{span} \theta \), and \( \Sigma_\theta^+ = \Psi^+ \cap \Sigma_\theta \) and likewise define \( \Sigma_\theta^- \). Let \( \Sigma(\theta) \) be the roots of \( (P, A_P) \). Define the positive roots \( \Sigma^+(\theta) \) to be the roots obtained by restriction of an element of \( \Psi^+ \setminus \Sigma_\theta^- \).

Given two subsets \( \theta, \theta' \subset \Delta \), following [Sha81] we set

\[
W(\theta, \theta') = \{ w \in W \mid w(\theta) = \theta' \},
\]

and then for \( w \in W(\theta, \theta') \), we define

\[
\Sigma(\theta, \theta', w) = \{ [\beta] \in \Sigma^+(\theta) \mid \beta \in \Psi^+ - \Sigma_\theta^- \text{ and } w(\beta) \in \Psi^- \},
\]

and then

\[
\Sigma^+(\theta, \theta', w) := \{ [\beta] \in \Sigma(\theta, \theta', w) \mid w(\beta) \in W(\alpha_P) \}.
\]

**Remark 7.** In [Sha81] additional care about the relative case is taken in the notation. In our simple case this is of course unnecessary, and we omit it.

In the case \( G = \text{GL}_n \), this specializes as follows. Let \( \alpha_{ij} : \text{diag}(t_i) \mapsto t_i t_j^{-1} \) be characters of \( T \). The \( \alpha_{ij} \) for \( j > i \) are the positive roots of \( (B, A) \) and \( \beta_i := \alpha_{ii+1} \) are our chosen simple roots. If \( P \) corresponds to the partition \( n_1 + \cdots + n_p \) of \( n \) and subset \( \theta \subset \Delta \), then

\[
\Sigma(\theta) = \text{span} \{ \beta_{n_1}, \beta_{n_1+n_2}, \ldots \},
\]

where we view \( \beta_i \) as restricted to \( a_P \mapsto a_{\beta_i} \). Note that all the positive roots \( \Sigma^+(\theta) \) of \( (P_\theta, N_\theta) \) are in \( N_P \).

Denoting by \( [\alpha] \) the coset representing a root \( \alpha \) of \( G \) restricted to \( P \), the positive roots in \( N_P \) are the \( \alpha_{ij} \) such that \( [\alpha_{ij}] = [\beta_{n_1+\cdots+n_k}] \).

**Example 4.** If \( G = \text{GL}_6 \) and \( P \) be the parabolic subgroup of block upper-triangular matrices corresponding to the partition \( 6 = 2 + 2 + 1 + 1 \) and \( \theta = \{ \alpha_{12}, \alpha_{34} \} \), then we have

\[
A_P = \{ \text{diag}(t_1, t_1, t_2, t_2, t_3, t_4) \}.
\]

The positive simple roots are \( \Sigma^+(\theta) = \{ [\beta_2], [\beta_4], [\beta_5] \} \). The Weyl group \( W(A_P) \simeq \mathfrak{S}_2 \times \mathfrak{S}_2 \) acts by permuting the blocks. Note that the simple reflection sending \( \beta_4 \mapsto -\beta_4 \) does not arise by permuting the blocks (i.e. \( 4 \leadsto 5 \) does not send blocks to blocks), hence \( w(\beta_4) \not\in W(A_P) \). Hence for any \( w \in W(\theta, \theta') \) we have \( \Sigma^+(\theta, \theta', w) \subseteq \{ [\beta_2], [\beta_5] \} \).

### 2.8.3 Regularity of the trace

We have the following information about the poles of intertwining operators, due, according to [Sha81], to Harish-Chandra:

**Theorem 10.** (Sha81, Theorem 2.2.1). Let \( \omega \) be an irreducible unitary representation of \( M \). Say that \( \omega \) is a subrepresentation of \( \text{Ind}^M_P(\omega_\ast) \) for a parabolic subgroup \( P = M_s N_s \) and \( \omega_\ast \) is an irreducible supercuspidal representation of \( M_s \). Let \( \theta_\ast \subset \Delta \) be such that \( P_\ast = P_0 \ast \) as parabolic subgroups of \( M_s \).
Then the operator
\[
\prod_{\alpha \in \Sigma_2(\theta, w \theta, w)} (1 - \chi_{\omega, \nu}(h_{\alpha})) A(\nu, \omega, w)
\]
is holomorphic on \(a_{\theta}C^*\). Here \(\chi_{\omega, \nu}\) is the central character of the twisted representation \(\omega \otimes q^{\nu, H_\theta(-)}\).

In particular for the purposes of the Plancherel formula, the only relevant \(\omega\) are unitary, hence have unitary central characters. Therefore \(A(\nu, \omega, w)\) is holomorphic at \(\nu\) if \(|q^{\nu, H_\theta(-)}| \neq 1\), or equivalently if
\[
\Re((\nu, H_\theta(-))) \neq 0.
\]
In particular, there is a finite union of hyperplanes away from which each operator \(A(\nu, \pi, w)\) is holomorphic, for any \(w \in W\).

**Lemma 4.** Let \(M\) be a Levi subgroup of \(G = G(F)\) and let \(\omega\) be a discrete series representation of \(M\). Let \(P = M_0 A_P N_P\) be the standard parabolic subgroup containing \(M = M_0\) and let \(k = \text{rk} A_P\). Let \(z_1, \ldots, z_k \in (C^\times)^k = X^*(A_P) \otimes \mathbb{Z} C\) define an unramified quasicharacter \(\nu = \nu(z_1, \ldots, z_k)\) of \(A_P\) as in Section 2.2. Let \(\pi = \text{Ind}_P^G(\omega \otimes \nu)\). Let \(f \in J\). Then
\[
\text{trace}(\pi, f) \in \mathbb{C}[z_1, \ldots, z_k, z_1^{-1}, \ldots, z_k^{-1}].
\]
That is, the trace is a regular function on \((C^\times)^k\).

**Proof.** We know a priori that
\[
\text{trace}(\pi, f) \in \mathbb{C}(z_1, \ldots, z_k)
\]
is a rational function of \(\nu\), as the operator \(\pi(f)\) itself depends rationally on the variables \(z_i\) by Theorem 7. Therefore
\[
\text{trace}(\pi, f) = \frac{p(\nu)}{h(\nu)} \in \mathbb{C}(z_1, \ldots, z_k).
\]
By Theorem 10 and the discussion following it, there is an open subset \(U\) of the unitary characters, such that for all \(\nu \in U\), and all \(w \in W\) we have
\[
\frac{p(\nu)}{h(\nu)} = \frac{p(w(\nu))}{h(w(\nu))}.
\]
Holding \(z_2, \ldots, z_n\) constant and in \(U\), (9) becomes an equality of meromorphic functions of \(z_1\) that holds on a set with an accumulation point, and hence (9) holds for all \(z_1 \in C^\times\). Now holding \(z_1 \in C^\times\) constant and arbitrary, and \(z_3, \ldots, z_k\) constant and in \(U\), we see that (9) holds also for all \(z_2 \in C^\times\). Therefore (9) actually holds for all \(\nu\), i.e. \(\text{trace}(\pi, f)\) is a \(W\)-invariant rational function of \(\nu\).

When \(\nu\) is non-strictly positive with respect to \(M\), by Theorem 7, \(\text{trace}(\pi, f)\) has poles only of the form \(z_1^{n_1} = 0\). The claim now follows from the \(W\)-invariance, and thus \(\text{trace}(\pi, f)\) is a regular function on \((C^\times)^k\).

We will therefore allow ourselves to write \(\text{trace}(\pi, f)\) for functions \(f \in J\) even for \(\nu\) such that the operator \(\pi(f)\) itself is not defined.

**Lemma 5.** Let \(d \in \tilde{W}\) be a distinguished involution and \(\pi\) be a tempered representation induced from one of the Levi subgroups attached to the two-sided cell containing \(d\) in the sense of Proposition 1. Then \(\text{trace}(\pi, f_d)\) is constant and a natural number. In fact, the same is true for any idempotent in \(J\).

**Proof.** Let \(j\) be an idempotent in \(J\). We have \(\text{trace}(\pi, f_j) = \text{rank}(\pi(f_j))\). The trace is continuous in \(\nu\) by Lemma 4, and the present lemma follows as \(T\) is connected.
3 Proof of Theorem 13 for $GL_n$, and of Theorem 2 for general $G$

3.1 The functions $f_\omega$ for $GL_n$

In this section, $q > 1$.

To compute with the Plancherel formula, we will need to apply the residue theorem successively in each variables $z_i$, and in doing so will we will need to sum over a certain tree that will track, for each variable, at which residues we evaluated. Upon integrating with respect to each variable $z_i$, we will have poles of the form $z_i = 0$ or $z_i = (q^{ij})^{-1}z_j$. For example, if we have 4 variables $z_1, z_2, z_3, z_4$ corresponding to a Levi subgroup $GL_1 \times GL_2 \times GL_3 \times GL_4$, then some of the summands obtained by successively applying the residue theorem are labelled by paths on the tree

$$
\begin{aligned}
z_1 &= 0 \\
z_2 &= (q^{23})^{-1}z_3 \\
z_3 &= (q^{43})^{-1}z_4 \\
z_4 &= (q^{34})^{-1}z_3
\end{aligned}
$$

Of course, to evaluate the entire integral for $M$, we would also need to consider trees whose roots are decorated with $z_1 = (q^{12})^{-1}z_2$, and so on, for a total of four trees.

Definition 12. Given a Levi subgroup $M$ with $N + 1$ blocks, a bookkeeping tree $T$ for $M$ is a rooted tree with $N$ levels such that the vertices on the $i$-th level below the root each have $N - i$ child vertices, and each vertex is decorated with an equation of the form $z_i = 0$ or $z_i = (q^{ij})^{-1}z_j$, where the index $j$ does not appear along the path from the vertex to the root, and the parent root is decorated with an equation $z_k = (q^{ki})^{-1}z_i$ for some $k$. Moreover, we require that the root be decorated with an equation of the form $z_i = 0$ or $z_i = (q^{ij})^{-1}z_j$ for $i$ minimal. A branch of $T$ is a simple path in $T$ from the root to one of the leaves.

Definition 13. Given a branch $B$ of a bookkeeping tree, a clump in $B$ is an ordered subset of indices $i$ appearing in the decorations of successive parent-child vertices, such that no decoration of the form $z_i = 0$ occurs along the path from the closest index to the root to the farthest index from the root. We write $C \prec B$ if $C$ is a clump of $B$.

Example 5. The sets of indices $\{3, 4\}$, $\{2, 3, 4\}$, $\{2, 3\}$, $\{2, 4\}$, and $\{2, 4, 3\}$ (note the ordering) are all the clumps of the above tree. The sets $\{1, 2, 3\}$ and $\{1, 2, 4\}$ are not clumps.

Theorem 11. Let $G = GL_n(F)$ and let $d$ be a distinguished involution such that the two-sided cell containing $d$ corresponds to the Levi subgroup $M$. Let $N + 1$ be the number of blocks in $M$ such that the $i$-th block has size $l_i$. Let $m_j$ be the number of $l_i$ that are equal to $j$. Define for $r \leq k$

$$Q_{rk} = q^{i_k i_{k+1}}q^{j_k j_{k+1}} \cdots q^{j_r j_{r+1}}$$

in the notation of 2.6. Then if $f_d$ is as in Sections 2.4,

$$f_d(1) = \frac{\text{rank}(\pi(f_d))}{m_1! \cdots m_n!} \sum_{\text{trees } T} \sum_{\text{branches } B} \prod_{C \prec B} \prod_{C = \{i_0, \ldots, i_t\}} \frac{(1 - q^{j_0})(1 - q^{j_1})}{1 - q^{j_0 + i_1}} \cdot \prod_{k=1}^{t-1} \frac{(1 - q^{l_k + i_{k+1}})}{(1 - q^{l_k + i_{k+1}})} \cdot \prod_{r=0}^{k-1} \frac{R_{rk} q^{j_r j_{r+1}}}{1 - Q_{rk} q^{j_r j_{r+1}}}$$

where

$$R_{rk} = \begin{cases}
1 - Q_{rk} q^{g_r - g_k} & \text{if } k < t - 1 \\
(1 - Q_{r+1,t-1} q^{g_r - g_{r+1}})(1 - Q_{r,t-1} q^{g_k - g_r}) & \text{if } k = t - 1
\end{cases}$$

(10)
Corollary 1. The denominator of $f_d(1)$ divides a power of the Poincaré polynomial $P_{G/B}(q) = P_{\mathcal{S}_n}(q)$ of $G$. Moreover, when $d$ is in lowest two-sided cell, corresponding to $M = T$, $f_d(1) = \frac{\text{rank}(\pi(f_d))}{P_{G/B}(q)}$ exactly.

Proof of Corollary 1. We will show that each of the three forms of denominator that appear in the conclusion of Theorem 11 divide $P_{G/B}(q)$, and thus that their product divides a power of $P_{G/B}(q)$. The denominators of $c_M$ are all of the form $1 + q + \cdots + q^{l_i - 1}$, and so divide $P_{G/B}(q)$ as $l_i \leq n$ for all $i$. Note that as $l_i + l_{i+1} \leq n$, the leftmost denominators in (10) satisfy the conclusion of the corollary also. Finally, $q^{rk+1} = q^{l_{i+1} + l_i + \cdots + l_r}$, and $P_{r_{12}}$ is likewise always a polynomial in $q$ (as opposed to $q^{-1}$) divisible by $1 - q$. Again using that $l_{i+1} + l_i + \cdots + l_r \leq n$, we are done with the first statement.

We now take up the second statement, the proof of which does not require any computations at all and which holds for any $w \in \mathfrak{c}_0$. Recalling that the only $K$-spherical tempered representations of $G$ are principal series representations [Mac71], let $\mu_K$ be the Haar measure on $G$ such that $\mu_K(K) = 1$, $\mu_I$ be the Haar measure such that $\mu_I(I) = 1$, and denote temporarily $d\pi_I$, the Plancherel measure normalized according to a chosen Haar measure $\mu$ on $G$. Likewise write $\pi_K(f)$ for the Fourier transform with respect to $\mu_K$ and $\pi_I(f)$ for the Fourier transform with respect to $\mu_I$.

As we have

$$\mu_I = P_{G/B}(q)\mu_K,$$

and formal degrees scale inversely to the Haar measure, we have

$$d\pi_I = \frac{1}{P_{G/B}(q)}d\pi_K$$

for tempered principal series. Now let $w \in \mathfrak{c}_0$. By Lemma 4 and the Satake isomorphism, there is a function $h_w$ in the spherical Hecke algebra such that for all principal series $\pi$,

$$\text{trace}(\pi_I(f_w)) = \text{trace}(\pi_K(h_w))$$

as regular Weyl-invariant functions of the Satake parameter. By the Plancherel formula and (11),

$$f_w(1) = \int \text{trace}(\pi_I(f_w))d\pi_I = \int \text{trace}(\pi_K(h_w))d\pi_I = \frac{1}{P_{G/B}(q)}\int \text{trace}(\pi_K(h_w))d\pi_K = \frac{h_w(1)}{P_{G/B}(q)}.$$  

(12)

In particular, if $w = d$ is a distinguished involution in the lowest two-sided cell, then

$$\text{trace}(\pi_I(f_d)) = \text{rank}(\pi_I(f_d)) = \text{rank}(\pi_I(f_d)) \cdot \text{trace}(\pi_K(1_K)),$$

and (12) becomes

$$f_d(1) = \frac{\text{rank}(\pi(f_d))}{P_{G/B}(q)} = \frac{1}{P_{G/B}(q)}.$$  

(13)

Here the rank is given by [Xi90], Proposition 5.5. (In op. cit. there is the assumption of simple-connectedness, but it is easy to see that the distinguished involutions for the extended affine Weyl group $\tilde{W}(G(F))$ of the universal cover $\tilde{G}$ are distinguished involutions for $\tilde{W}$ using the definition in [Lus87a] and uniqueness of the $\{C_w\}$-basis, and that the lowest cell is just the lowest cell of $\tilde{W}(G(F))$ intersected with $\tilde{W}$.)

Once we have established injectivity of $\phi$, we will show by a counting argument that $\text{rank}(\pi(t_d)) = 1$ for any distinguished involution $d$, in the case $G = GL_n$, see Theorem 13. Note that (13) is an example of the behaviour conjectured in Remark 1.

Remark 8. It would be interesting to find $I$-biinvariant Schwartz functions $h$ playing the role of $h_w$ for the other two-sided cells, namely such that $\text{trace}(\pi, h)$ was regular, nonzero only for a single pair $(M, \omega)$, and the value $h(1)$ was known as a function of $q^{1/2}$.

\[^2\text{We thank A. Braverman for explaining to us the observation whose obvious generalization we present below.}\]
Remark 9. If $P$ is a maximal parahoric subgroup of $G$, then the longest word $w_P$ is a distinguished involution in the lowest two-sided cell. As above, and as we will explain again in Section 4, $\pi(f_{w_P})$ is nonzero only for the principal series representations, and its image is contained in $\pi_P$. Thus after we will have shown injectivity, [Key82] gives another proof that $\text{rank}(\pi(f_{w_P})) = 1$, and more generally that $\pi(t_d)$ has rank 1 on the principal series for any distinguished involution $d \in c_0$.

Lemma 6. Let $e_0, \ldots, e_n \in \mathbb{Z}$. Then

$$\int_{\mathbb{T}} \cdots \int_{\mathbb{T}} z_0^{e_0} \cdots z_N^{e_N} \prod_{i<j}^N \Gamma^{ij} \, dz_0 \cdots dz_N = 0$$

unless $e_0 + \cdots + e_N = -N$.

Although we do not use the Lemma in the sequel, we include it because it illustrates an efficient way to compute the functions $f_w$ in practice, as we explain after its proof.

Corollary 2. Let $w \in \hat{W}$. Then $f_w(1)$ is a rational function of $q$ with denominator dividing a power of the Poincaré polynomial of $G$. The numerator is a Laurent polynomial $p_1(q^{1/2}) + p_2(q^{-1/2})$ in $q^{1/2}$, where the degree of $p_1$ is bounded uniformly in terms of $\hat{W}$. The denominator of $f_w(1)$ depends only on the two-sided cell containing $w$.

Remark 10. In light of Lemma 4, Lemma 6 and Corollary 2 have the following interpretation. Let $\Gamma$ be a left cell in a two-sided cell $c$. Then in [Xi02], Xi shows that all the rings $J_{\Gamma \cap \Gamma^{-1}}$ are isomorphic to the representation ring of the associated Levi subgroup $M_c$, and $J_c$ is a matrix algebra over $J_{\Gamma \cap \Gamma^{-1}}$. Therefore $w \in \Gamma \cap \Gamma^{-1}$ are labelled by dominant weights of $M_c$, and if $t_\lambda$ is such an element, Xi’s results show that if $q = q$ and $\pi = \text{Ind}_B^G(\nu)$ is an irreducible representation of $J_c$, then

$$\text{trace}(\pi, t_\lambda) = \text{trace}(V(\lambda), \nu),$$

where we view $\nu$ as a semisimple conjugacy class in $M_c$, and $V(\lambda)$ is the irreducible representation of $M_c$ of highest weight $\lambda$. Then we have that $f_\lambda(1) \neq 0$ only if $\lambda$ is of height 0 with respect to the basis $\varepsilon_i; \text{diag}(a_1, \ldots, a_n) \mapsto a_i$.

The proofs of Lemma 6 and Corollary 2 will use the notation of the proof of Theorem 11, and we defer them until after the proof of the theorem.

3.1.1 Example computations and a less singular cell

In this section we provide two example computations to elucidate the coming proof of Theorem 11.

Example 6. Let $G = \text{GL}_2$ and $M = A$. Then $l_1 = l_2 = 1$ and $g_1 = g_2 = 0$. Let $d = s_0$ or $s_1$. Then

$$f_d(1) = \frac{1}{2} \frac{1}{2\pi i} q^{-1} \int \int q \frac{(z_1 - z_2)(z_1 - z_2)}{(z_1 - q^{-1} z_2)(z_1 - q z_2)} \frac{dz_1}{z_1} \frac{dz_2}{z_2}$$

$$= \frac{1}{2} \frac{1}{2\pi i} \int \text{Res}_{z_1 = z_2} \frac{(z_1 - z_2)(z_1 - z_2)}{(z_1 - q^{-1} z_2)(z_1 - q z_2)} \frac{1}{z_1} + \text{Res}_{z_1 = 0} \frac{(z_1 - z_2)(z_1 - z_2)}{(z_1 - q^{-1} z_2)(z_1 - q z_2)} \frac{1}{z_2}$$

$$= \frac{1}{2} \frac{1}{2\pi i} \int \frac{(q^{-1} z_2 - z_2)(q^{-1} z_2 - z_2)}{(q^{-1} z_2 - q z_2)(q^{-1} z_2 - 2)} + \frac{z_2^2}{z_2} \frac{dz_2}{z_2}$$

$$= \frac{1}{2} \frac{1}{2\pi i} \int \frac{(q^{-1} - 1)^2}{q^{-2} - 1} + \frac{1}{z_2} \frac{dz_2}{z_2}$$

$$= \frac{1}{q + 1}.$$
We claim that the value of this integral is
\[
7 \sum_{\text{trees}} (1 + 1 - q) \frac{2}{1 + q} = \frac{1}{1 + q}.
\]

Now we give an example of a less singular cell.

Example 7. Let \( G = \text{GL}_4 \), and let \( c \) be the two-sided cell corresponding to the partition \( 4 = 2 + 2 \). Then \( P_W(q) = (1 + q)(1 + q + q^2)(1 + q + q^2 + q^3) \) has five distinct roots \( q = -1, \pm i, \zeta_1, \zeta_2 \), where \( \zeta_i \) are primitive third roots of unity. We compute \( R_c \) using the Plancherel formula. We have
\[
L_1 = L_2 = 2, \quad q = 1, \quad q^2 = q^3,
\]
and
\[
c_M = \frac{q^4(q - 1)^2}{2(q + 1)^2},
\]
and
\[
\left( \frac{1}{2\pi i} \right)^2 \int_{T^2} \Gamma_{12} \frac{dz_1}{z_1} \frac{dz_2}{z_2} = \left( \frac{1}{2\pi i} \right)^2 \int_{T^2} \frac{(z_1 - z_2)(z_1 - z_2)}{(z_1 - q^2 z_2)(z_1 - q^{-2} z_2)} \frac{dz_1}{z_1} \frac{dz_2}{z_2} = 1 + \frac{1}{2\pi i} J \left( \frac{g(z - 1)^2}{z} \right) = 1 + \frac{(1 + q)(1 - q^2)}{1 + q + q^2 + q^3} = \frac{2}{1 + q^2}.
\]

Accounting for \( c_M \), we see that \( J_c \) is regular at \( q = \zeta_1, \zeta_2 \).

3.1.2 Proofs of Theorem 11, Lemma 6, and Corollary 2

Proof of Theorem 11. Let \( n = l_0 + \cdots + l_N \). We may assume that \( l_0 \leq l_2 \leq \cdots \leq l_N \). It suffices to evaluate the integral
\[
\left( \frac{1}{2\pi i} \right)^N \int_{T^N} \prod_{i,j,g} d_z \int_{T^N} \prod_{i,j,g} d_z = \left( \frac{1}{2\pi i} \right)^N \int_{T^N} \prod_{i,j,g} d_z.
\]
We claim that the value of this integral is
\[
c_M \sum_{\text{trees}} \sum_{\text{branches}} \prod_{i,j,g} \sum_{C \subset B} \prod_{k=0}^{l-1} \frac{1 - q^{i+1} q^{j+1} q^{k+1}}{1 - (q^{i+1})^2} \prod_{r=0}^{k-1} \frac{1 - Q_{r,k} q^{i+r+1}}{1 - Q_{r,k} q^{i+1}} = 1 - Q_{r,k} q^{i+r+1}
\]
where the sum over trees is taken over all bookkeeping trees for the integral. When \( k = 0 \), we interpret the product over \( r \) as being empty.

First we explain how (15) simplifies to (10). All cancellations will take place within the same clump \( C \) of some branch \( B \), which we now fix. We have
\[
1 - Q_{r,k} q^{i+r+1} = 1 - q^{g_{ik+1} + g_{ik+1}} q^{g_{ik+1} + \cdots + g_{ik+1}} q^{g_{ir+1}} q^{g_{ir+1} - g_{ir+1}} = 1 - Q_{r,k-1} q^{g_{ik+1} - g_{ir+1}}.
\]
which is one of the factors in the product \((1 - Q_{r,k-1}q_{i,k}) (Q_{r,k-1}(q_{i,k})^{-1})\). The surviving factor in the numerator at index \((k - 1, r)\) is then equal to \((1 - Q_{r,k-1}q^{\theta ri}g_k)\). In short, the above factors in the denominator cancel with a numerator occurring with the same \(r\)-index but \(k\)-index one lower. Such a factor occurs whenever \(r < k - 1\) (note that this inequality does not hold when \(k = 1\) and \(r = 0\)). When \(r = k - 1\), we have

\[
1 - Q_{k-1,k}(q_{i,k+1})^{-1} = 1 - q_{i,k+1}^{-1} q_{i,k}^{-1} = q_{i,k} q_{i,k+1}^{-1} = 1 - q_{i,k+1}^{2},
\]

which is one of the factors in \((1 - q_{i,k+1}^{2}) (Q_{i,k+1}^{-1})\). The cancellation leaves behind the factor \(1 - q_{i,k+1}^{2}\) in the numerator, except for \(k = 0\); this term keeps both its denominators. At this point we have shown that the factor corresponding to \(C\) in (15) simplifies to

\[
\frac{(1 - q_{i,0}^{2})(1 - q_{i,1}^{2})}{1 - q_{i,0}^{2} z_{i1} \cdots z_{i1}} \prod_{k=1}^{t-1} \frac{(1 - q_{i,k+1}^{2})}{1 - q_{i,k+1}^{2} z_{i1} \cdots z_{i1}} \prod_{r=0}^{k-1} \frac{R_{r,k}}{1 - Q_{r,k} q_{r;k+1}^{2}},
\]

where

\[
R_{r,k} = \begin{cases} 1 - Q_{r,k} q_{r;k}^{-1} & \text{if } k < t - 1 \\ (1 - Q_{r,t-1} q_{r,t-1}^{-1})(1 - Q_{t,t-1} q_{t,t-1}^{-1}) & \text{if } k = t - 1. \end{cases}
\]

This means that (15) simplifies to (10).

To prove (15), we will use the residue theorem for each variable consecutively, keeping track of the constant expressions in \(q\) that we extract after integrating with respect to each variable \(z_i\). More precisely, we will track what happens in a single summand corresponding to some set of successive choices of poles to take residues at. Note that all the rational functions that will appear, namely \(\Gamma^ij\) or the rational functions that result from substitutions into the \(\Gamma^ij\), become equal to 1 once \(z_i\) or \(z_j\) is set to zero. Therefore poles at \(z_i = 0\) serve simply to remove all factors involving \(z_i\) from inside the integrand (we will see what these rational functions are below). It follows that the summand whose branch we are computing is a product over clumps in the corresponding branch, so it suffices to compute the value of a given clump for some ordered subsets \(\{i_0, i_1, \ldots, i_l\}\) of the indices \(\{0, \ldots, N\}\). As we are inside a clump, we will consider only poles occurring at nonzero complex numbers. Thus we are left only to determine what happens within a single clump.

We first integrate with respect to the variable \(z_{i0}\). The residue theorem gives

\[
\left(\frac{1}{2\pi i}\right)^{l+1} \int_{\Gamma} \cdots \int_{\Gamma} \prod_{i < j} \Gamma^ij \frac{dz_{i0}}{z_{i0}} \cdots \frac{dz_{i1}}{z_{i1}} = \left(\frac{1}{2\pi i}\right)^{l} \int_{\Gamma} \cdots \int_{\Gamma} \prod_{i \neq i0} \text{Res}_{z_{i0} = (q_{i,0})^{-1} z_{i1}} \frac{1}{z_{i0}} \prod_{i < j} \Gamma^ij \frac{dz_{i0}}{z_{i0}} \cdots \frac{dz_{i1}}{z_{i1}} + \left(\frac{1}{2\pi i}\right)^{l} \int_{\Gamma} \cdots \int_{\Gamma} \prod_{i \neq i0} \Gamma^ij \frac{dz_{i0}}{z_{i0}} \cdots \frac{dz_{i1}}{z_{i1}}.
\]

As noted above, the second integral belongs to a different branch (in fact, in the case of \(i_0\), to a different tree); our procedure will deal with it separately, and we will now consider what happens with the first integral.

For the first integral, consider one of the summands corresponding to \(z_{i0} = (q_{i,0})^{-1} z_{i1}\) for some \(i_1\). We have

\[
\left(\frac{1}{2\pi i}\right)^{l} \int_{\Gamma} \cdots \int_{\Gamma} \text{Res}_{z_{i0} = (q_{i,0})^{-1} z_{i1}} \frac{1}{z_{i0}} \prod_{i < j} \Gamma^ij \frac{dz_{i0}}{z_{i0}} \cdots \frac{dz_{i1}}{z_{i1}} = \frac{1}{1 - (q_{i,0})^{-1} (1 - q_{i,0})^{-1}} \left(\frac{1}{2\pi i}\right)^{l} \int_{\Gamma} \cdots \int_{\Gamma} \prod_{j \neq i1, i0} \frac{z_{i1} - q_{i,0} q_{i,j} z_{j}}{(z_{i1} - q_{i,0} q_{i,j} z_{j}) (z_{i1} - q_{i,0} q_{i,j} z_{j})} \frac{dz_{i1}}{z_{i1}} \cdots \frac{dz_{i1}}{z_{i1}}.
\]

\[
\prod_{i < j} \Gamma^ij \frac{dz_{i1}}{z_{i1}} \frac{dz_{i1}}{z_{i1}} \cdots \frac{dz_{i1}}{z_{i1}}.
\]
Recall that, if \( i_1 > 2 \), even though formally we have defined the symbols \( q_{ij} \) and \( q^{ij} \) only for \( i < j \), the symmetry of the factors allows us to write \( q_{ij} \) even if \( i > j \), thanks to the factor with \( (q_{ij})^{-1} \) also present in the numerator.

Now we integrate with respect to \( z_i \). Observe that the leftmost product over \( j \neq i_1, i_0 \) does not contribute poles. Indeed, the first factor each denominator does not have its zero contained in \( T \), and the second factor in each denominator has its zero at \( z_i = q^{i_0 i_1} (q^{i_0 i_2})^{-1} z_{i_2} \). The power of \( q \) appearing is

\[
g_{i_0} + g_{i_1} + 1 - (g_{i_0} + g_{i_2} + 1) = g_{i_1} - g_{i_2},
\]

and so \( q^{i_0 i_1} (q^{i_0 i_2})^{-1} \) is equal to \( q_{i_1 i_2} \) or \( (q_{i_1 i_2})^{-1} \), whichever is defined. Thus this zero cancels with a zero in the numerator of \( \Gamma^{i_1 i_2} \) or \( \Gamma^{i_2 i_1} \), whichever is defined. Therefore we need only consider the simple poles at \( z_i = (q^{i_1 i_2})^{-1} z_{i_2} \) for \( i_1 < i_2 \) and \( z_i = (q^{i_2 i_1})^{-1} z_{i_2} \) for \( i_2 < i_1 \). Observe that the residues will be the same for either inequality. In the case \( i_2 < i_1 \), for example, \( \Gamma^{i_2 i_1} \) needs to be rewritten so that its simple pole inside \( T \) is in the correct format to calculate the residue by substitution:

\[
\text{Res}_{z_i} = \frac{(z_i - (q^{i_1 i_2} z_{i_1} - (q^{i_1 i_2} z_{i_1})^{-1} z_{i_2})^2}{z_i - (q^{i_1 i_2} z_{i_1})^{-1} z_{i_2}}\frac{1}{z_i - (q^{i_1 i_2} z_{i_1})^{-1} z_{i_1}}\frac{1}{z_i - (q^{i_1 i_2} z_{i_1})^{-1} z_{i_1}} = \frac{(1 - q^{i_1 i_2} z_{i_1}^{-1}) (1 - q^{i_1 i_2} (q^{i_1 i_2}^{-1})_1}{1 - (q^{i_1 i_2})^2}.
\]

Therefore after integrating within the clump at hand with respect to \( z_{i_0} \) and then \( z_{i_1} \), we have the expression

\[
\frac{1 - q^{i_1 i_2} q_{i_0 i_2}}{1 - (q^{i_1 i_2})^2} (1 - q^{i_1 i_2} q_{i_0 i_2})^{-1} (1 - q^{i_1 i_2} q_{i_0 i_2})^{-1} (1 - q^{i_1 i_2} q_{i_0 i_2})^{-1} (1 - q^{i_1 i_2} q_{i_0 i_2})^{-1} (1 - q^{i_1 i_2} q_{i_0 i_2})^{-1} \tag{16}
\]

Now we integrate with respect to \( z_{i_2} \). Again, only poles from the product of \( \Gamma^{ij} \)'s occur with nonzero residues: in total we have simple poles contained in \( T \) possibly at \( z_{i_2} = q^{i_1 i_2} q^{i_0 i_1} (q^{i_0 i_2})^{-1} z_{i_2} \), at \( z_{i_2} = q^{i_1 i_2} (q^{i_1 i_2})^{-1} z_{i_2} \) and at \( z_{i_2} = (q^{i_2 i_1})^{-1} z_{i_2} \) for \( j \neq i_1, i_1, i_2 \). It may happen that these poles are not all distinct, but all zeros in the denominator of the former two types are in fact cancelled by zeros of denominator anyway. The necessary factors occur in the product immediately adjacent on the right. Indeed, we have

\[
g_{i_1} + g_{i_2} + 1 + g_{i_0} + g_{i_1} + 1 - g_{i_0} - g_{j} - 1 = g_{i_1} + g_{i_2} + 1 + g_{i_1} - g_{j}
\]

and so either \( q^{i_1 i_2} q_{i_0 i_2}^{-1} = q^{i_1 i_2} q_{i_1 i_2} \) or \( q^{i_1 i_2} q_{i_0 i_2}^{-1} = q^{i_1 i_2} (q_{i_1 i_2})^{-1} \) (or \( q_{i_1 i_2} \) or \( (q_{i_1 i_2})^{-1} \)).

Likewise we have

\[
g_{i_1} + g_{i_2} + 1 + g_{i_1} - g_{j} - 1 = g_{i_2} - g_{j}
\]

as happened when we integrated with respect to \( z_{i_0} \).

Now we see the following pattern: At each stage, we integrate with respect to the variable we took a residue at in the previous step. When integrating with respect to \( z_{i_0} \), there will be \( r + 1 \) products of rational functions, and each rational function in the leftmost \( r \) products will contribute a pole, in addition to the pole contributed by \( \Gamma^{i_1 r+3} \) or \( \Gamma^{r+1 i_1} \), whichever is defined. However, each is nullified by having zero residue thanks to a denominator in the product immediately to the right. Integrating with respect to \( z_{i_1} \) will result in
extracting \( r + 1 \) new rational factors, each of the form claimed in the theorem. When it comes to integrating with respect to \( z_i \), all variables \( z_i \) will cancel from the remains of the Gamma functions by homogeneity, and the factor \( \frac{1}{z_i} \) will result in the final integral contributing just the remaining \( l + 1 \) rational factors. \hfill \Box

Lemma 6 is a version of the preceding proof.

Proof of Lemma 6. We will evaluate the integral (14) by applying the residue theorem successively for each variable as in the proof of Theorem 11. Note that as functions of any variable \( z_i \), the functions \( \prod_{i<j} \Gamma^{ij} \) and all the other products, for example those appearing in (16), have numerator and denominator with equal degrees. Thus the overall sum of powers of all \( z_i \) in the integrand of (14) is \( e_0 + \cdots + e_N \), and in general, the sum of powers of all \( z_i \) in the integrand of an expression like (16) is the sum of the degrees of the monomial \( z_i \) terms.

When evaluating (14) along a single branch, we find again that the only poles that appear are of the form \( z_i = (q^{ij})^{-1} z_j \), or \( z_i^{-1} = 0 \) for \( e_i < 0 \). We will track the effect that evaluating each successive residue has on the total degree of the integrand, and then conclude using the fact that \( \int_T z^d \, dz = 2\pi i \delta_{r,-1} \). First, observe that evaluating a residue of the form \( z_i = (q^{ij})^{-1} z_j \) increases the sum of all powers by 1; a factor \( z_j \) is contributed to the resulting integrand. The sum of all powers is likewise increased by 1 when evaluating the residue at a simple pole of \( z_i^{-1} \) at 0. To compute the residue at a pole of \( z_i^{-e_i} \) at 0 for \( e_i > 1 \), consider the Taylor expansion at 0 of \( \Gamma^{ij} \). We have

\[
\frac{1}{z_i - q^{ij} z_j} = -\frac{1}{q^{ij} z_j} - \frac{z_i}{(q^{ij} z_j)^2} - \frac{z_i^2}{(q^{ij} z_j)^3} - \cdots
\]

and

\[
\frac{1}{z_i - (q^{ij})^{-1} z_j} = \frac{q^{ij}}{z_j} - \frac{(q^{ij})^2}{z_j^2} z_i - \frac{(q^{ij})^{3}}{z_j^3} z_i^2 - \cdots.
\]

Multiplying these series and further multiplying by the denominator \( z_i^n - (q^{ij})^{-1} z_i z_j + z_j \), it follows that the Taylor expansion of \( \Gamma^{ij} \) is

\[
1 + \frac{q^{ij} + (q^{ij})^{-1} - q_{ij} - q_{ij}^{-1}}{z_j} \frac{(q^{ij})^2 + 2 + (q^{ij})^{-2} - (q_{ij} + q_{ij}^{-1})(q^{ij} + q^{-ij})}{z_j} z_i^2 + \cdots + \frac{-(q_{ij} + q_{ij}^{-1})((q^{ij})^{n-1} + \cdots + (q^{ij})^{-n+1})}{z_j^n} + \frac{(q^{ij})^{-n+2} + \cdots + (q^{ij})^{n+2} + (q^{ij})^n + \cdots + (q^{ij})^{-n}}{z_i^n} + \cdots. \tag{17}
\]

It is clear that the salient point of (17), that \( z_i^n \) appears with a coefficient proportional to \( z_j^{-e_i} \), holds also for all the products like those in (16). Thus computing a residue at the pole of \( z_i^{-e_i} \) at 0 will result in a new integrand, the total degree of which has increased by \( e_i - e_i + 1 = 1 \). It now follows that after integrating with respect to \( N - 1 \) variables, the final integral to be computed will be a constant times \( (2\pi i)^{-1} \int_T z_0^{e_0} \cdots z_N^{e_N} - 1 dz_N \). This is nonzero if and only if \( e_0 + \cdots + e_N = -N \). Summing over all branches of a bookkeeping tree, we see that (14) is nonzero only if \( e_0 + \cdots + e_N = -N \). \hfill \Box

The lemma is helpful for computing explicit examples, as it points out that one can always avoid dealing with higher-order poles. Indeed, given an integral of the form (14), we may assume by Lemma 6 that \( e_0 + \cdots + e_N = -N \). If \( e_i = -1 \) for all \( i \) then the integral (14) is just the integral from Theorem 11. Otherwise there is some \( e_{i_0} \geq 0 \). We may assume that \( i_0 = 0 \). Then the only poles in \( z_0 \) are of the form \( z_0 = (q^{0i_1})^{-1} z_{i_1} \) for indices \( i_1 > 0 \). Therefore we compute that (14) is equal to a sum of terms of the form

\[
(1 - q^{0i_1} q_{i_1})(1 - q^{0i_1} q_{0i_1}^{-1})(q^{0i_1})^{-e_0 - 1} \left(1 - \frac{1}{2\pi i} \right)^{N-1} \int_\mathcal{T} \cdots \int_\mathcal{T} \prod_{i=1}^{N} z_i^{e_i + e_0 + e_i / 2, \ldots, e_N} \prod_{0 < i < j} \prod_{i,j \neq 0} \Gamma^{ij} dz_{i_1} \cdots dz_{i_N}. \tag{18}
\]
The total degree of the integrand is now \(e_0 + \cdots + e_N + 1 = -(N - 1)\). Therefore either \(e_{i_0} + e_{i_1} + 1 = e_{i_2} = \cdots = e_{-N} = -1\), or we may again assume without loss of generality that the exponent of some \(z_{i_j}\) is nonnegative, and proceed with evaluating (18) by integrating with respect to \(z_{i_j}\). We may continue in this way, never having to deal with more than a simple pole at 0. (Of course, the resulting order of integration need not be the same as the order used in the proofs of Theorem 11 and Corollary 2.)

We can now Corollary 2.

**Proof of Corollary 2.** In light of Lemma 4, it is enough to prove that the conclusions of the present corollary hold for integrals of the form (14). We again follow the algorithm from the proof of Theorem 11, so it is sufficient to consider a single branch, and for this it suffices to observe that, given, for example, a variant 

\[
\left(\frac{1}{2\pi i}\right)^{l-1} \int_T \cdots \int_T \prod_{j \neq i_0, i_1, i_2} \frac{(z_{i_2} - q^{i_1i_2}q^{i_0i_1}q^{i_0j}z_j)(z_{i_2} - q^{i_1i_2}q^{i_0i_1}(q^{i_0j} - 1)z_j)}{(z_{i_2} - q^{i_1i_2}q^{i_0i_1}z_j)(z_{i_2} - q^{i_1i_2}q^{i_0i_1}(q^{i_0j} - 1)z_j)} \prod_{i < j} \Gamma^{i_j} \frac{\alpha'_{i_2}}{z_{i_2}} \cdots \frac{\alpha'_{i_l}}{z_{i_l}} \, dz_{i_2} \cdots dz_{i_l}
\]

of (16) in which \(e_{i_2} < -1\), the quotient rule gives that the residue at \(z_{i_2} = 0\) is a equal to a linear combination over \(\mathbb{Q}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]\) of integrals of the form

\[
\left(\frac{1}{2\pi i}\right)^{l-2} \int_T \cdots \int_T \prod_{i,j \neq i_0, i_1, i_2} \Gamma^{i_j} \frac{\alpha'_{i_3}}{z_{i_3}} \cdots \frac{\alpha'_{i_l}}{z_{i_l}} \, dz_{i_3} \cdots dz_{i_l}
\]

for new exponents \(\alpha'_{i_k}\). This and the calculations in the proof of Theorem 11 make clear that the only positive powers of \(q^{1/2}\) that appear in the any denominator are those that appeared as in Theorem 11. These are controlled by the possible block sizes of \(M\), and hence are bounded in terms of \(\tilde{W}\). Throughout this procedure, the denominator has been contributed to only by the Plancherel density itself, which depends only on \(M\). The last claim of the corollary now follows from Proposition 1.

\[\square\]

### 3.2 The functions \(f_w\) for general \(G\)

For general \(G\), we will follow the same plan as for \(G = GL_n\). The only difference is that we have less control over which denominators can appear. Indeed, this is true even for formal degrees, but complications are also introduced by residual coset we integrate over, or equivalently, by lack of explicit control of the Satake parameter of \(\pi = i_{P}^{G}(\omega \otimes \nu)\) for arbitrary discrete series representations \(\omega \in \mathcal{E}_{\pi}^{d}(M)\).

**Theorem 12.** Let \(w \in \tilde{W}\). Then \(f_w(1)\) is a rational function of \(q\) with poles drawn from a finite set of roots of unity depending only on \(\tilde{W}\). The numerator is a Laurent polynomial in \(q^{1/2}\). The denominator depends only on the two-sided cell containing \(w\). If \(w\) is in the lowest two-sided cell, then the denominator divides the Poincaré polynomial of \(G\).

In the proof, we do not attempt to record in any information about degrees of the numerators. It will therefore be necessary to control the possible numerators of \(f_w(1)\) in a different manner than for \(G = GL_n\) in order to prove Proposition 2 below.

**Proof of Theorem 12.** First, we note that the reasoning for the lowest cell used in the proof of Corollary 1 holds for arbitrary \(G\). Therefore (12) proves the last claim just as for \(GL_n\).

More generally, let \(w \in \tilde{W}\) and \(M_P\) be a Levi subgroup corresponding to the two-sided cell containing \(w\). Let \(N = \text{rk} A_P\). Given a coroot \(\alpha'\) and a basis of the cocharacter lattice as explained in Section 2.7, we write \(\alpha' = \alpha_1' \cdots \alpha_n'\) for integers \(e_i = e_i(\alpha)\).
By Lemma 4 and Theorem 6, it suffices to show the conclusions of the theorem hold for integrals of the form

\[
\left(\frac{1}{2\pi i}\right)^n \int_{T} \prod_{\alpha \in R_{i,s} \setminus R_{i,t,+}} \frac{q_{\alpha}(z_{1}^{\alpha}z_{2}^{\alpha} \cdots z_{n}^{\alpha} - q_{\omega})(z_{1}^{\alpha^{'}}z_{2}^{\alpha^{'}} \cdots z_{n}^{\alpha^{'}} - q_{\omega}^{-1})}{(z_{1}^{\alpha}z_{2}^{\alpha} \cdots z_{n}^{\alpha} - q_{\omega}q_{\alpha})(z_{1}^{\alpha^{'}}z_{2}^{\alpha^{'}} \cdots z_{n}^{\alpha^{'}} - q_{\omega}^{-1}q_{\alpha^{'}})} dz_{1} \cdots dz_{n},
\]

where \(f_{i} \in \mathbb{Z}\) and \(q_{\omega} = q_{\omega, \alpha}\) is the value of \(\alpha^{'}/\alpha\) on the Satake parameter of \(\omega\)—this is a positive power of \(q^{1/2}\) (of \(q\) if \(\alpha^{'}/\alpha\) is a coroot) by Sections 7 and 8 of [Odp04]. We have \(q_{2\alpha} = 1\) and the resulting simplification \((7)\) whenever \(\alpha^{'}/\alpha\) is not a coroot.

With notation fixed, the theorem is essentially an observation. Indeed, suppose that we integrate with respect to \(z_{1}\), and wish to compute the contribution to the residue at a pole

\[
z_{1} = \xi q_{\omega}^{1/2} q_{\alpha} - \xi q_{\omega}^{1/2} q_{\alpha} z_{2}^{\alpha} \cdots z_{n}^{\alpha}
\]

arising from the factor

\[
\frac{q_{\alpha}(z_{1}^{\alpha}z_{2}^{\alpha} \cdots z_{n}^{\alpha} - q_{\omega})(z_{1}^{\alpha}z_{2}^{\alpha} \cdots z_{n}^{\alpha} - q_{\omega}^{-1})}{(z_{1}^{\alpha}z_{2}^{\alpha} \cdots z_{n}^{\alpha} - q_{\omega}q_{\alpha})(z_{1}^{\alpha^{'}}z_{2}^{\alpha^{'}} \cdots z_{n}^{\alpha^{'}} - q_{\omega}^{-1}q_{\alpha^{'}})} z_{1} = \frac{1}{\xi q_{\omega}^{1/2} q_{\alpha} - \xi q_{\omega}^{1/2} q_{\alpha} z_{2}^{\alpha} \cdots z_{n}^{\alpha}} \frac{1}{\xi q_{\omega}^{1/2} q_{\alpha} - \xi q_{\omega}^{1/2} z_{2}^{\alpha} \cdots z_{n}^{\alpha}} \prod_{\alpha \neq \xi}(z_{1} - \xi q_{\omega}^{1/2} q_{\alpha} - \xi q_{\omega}^{1/2} z_{2}^{\alpha} \cdots z_{n}^{\alpha}),
\]

where \(\xi\) and the \(\zeta\) are primitive \(e_{i}\)-st roots of unity. The contribution is then

\[
\frac{q_{\alpha}(q_{\omega}^{-1} q_{\alpha} - q_{\omega}^{-1} z_{2}^{\alpha} \cdots z_{n}^{\alpha}) (q_{\omega}^{-1} q_{\alpha} - q_{\omega} z_{2}^{\alpha} \cdots z_{n}^{\alpha})}{(q_{\omega}^{-1} q_{\alpha} - q_{\omega} z_{2}^{\alpha} \cdots z_{n}^{\alpha}) (q_{\omega}^{-1} q_{\alpha} - q_{\omega}^{-1} z_{2}^{\alpha} \cdots z_{n}^{\alpha})} \prod_{\alpha \neq \xi}(z_{1} - \xi q_{\omega}^{1/2} q_{\alpha} - \xi q_{\omega}^{1/2} z_{2}^{\alpha} \cdots z_{n}^{\alpha})
\]

which simplifies to

\[
\frac{(1 - q_{\omega})^{2}(1 - q_{\alpha})}{(1 - q_{\omega}^{2} q_{\alpha}) \prod_{\alpha \neq \xi}(1 - \xi^{-1} \xi)}.
\]

We have used no special properties of the integers \(e_{i}\) or \(f_{j}\), and thus after integrating with respect to \(z_{1}\), (19) is equal to a sum of integrals with respect to \(z_{2}, \ldots, z_{n}\) again of the form (19), except the factors in the denominator will now involve powers \(q_{\omega}(\alpha)q_{\omega}(\alpha')^{1/2}\) and \(q_{\omega}(\alpha)q_{\omega}(\alpha')^{-1/2}\), as for \(G = GL_{n}\). The coefficients of the this sum are the form

\[
\frac{Q}{(1 \pm q^{e_{i}})^{c_{i}}(1 \pm q^{e_{i}^{'}})^{c_{i}^{*}}},
\]

where \(Q\) is a Laurent polynomial in \(q^{\frac{1}{2}}\), \(r_{i} \in \mathbb{N}\) with, \textit{a priori}, complex coefficients, and \(c_{i} \in \mathbb{N}\). Indeed, as we never required any cancellations with the numerator to extract rational functions of \(q\) of the required form, it is clear that the simplification of (20) to (21) works essentially the same way for any higher order poles that appear—again thanks to the quotient rule—and that when integrating with respect to subsequent variables, the additional rational functions of \(q\) appearing have the same shape as in the proof of Theorem 11. It is also clear that the factors corresponding to non-reduced roots behave similarly.

Therefore (19) has poles in \(q\) only at a finite number of roots of unity. Again by the quotient rule, exponents \(r_{i}\) and \(c_{i}\) depend only on \(M\). Clearly there are only finitely-many exponents \(r_{i}\) that appear for any \(M\). The theorem now follows. \(\square\)
3.3 Relating $t_w$ and $f_w$

We will now relate the Schwartz functions $f_w$ on $G$ to the elements $\phi^{-1}(t_w)$ of a completion $\mathcal{H}^-$ of $H$, whose definition we will now recall. In this section $G$ is general.

3.3.1 Completions of $H$ and $J \otimes_{\mathbb{C}} A$

Let $\hat{A} = \mathbb{C}((q^{-1/2}))$ and $\hat{A}^- = \mathbb{C}[[q^{-1/2}]]$. Write $\mathcal{H}^-$ for the $\hat{A}$-algebra

$$\mathcal{H}^- := \left\{ \sum_{x \in W} b_x T_x \mid b_x \in \hat{A}, \ b_x \to 0 \text{ as } \ell(x) \to \infty \right\},$$

where we say that $b_x \to 0$ as $\ell(x) \to \infty$ if for all $N > 0$, $b_x \in (q^{-1/2})^N \hat{A}^-$ for all $x$ sufficiently long.

Consider also the completions

$$\mathcal{H}_{C'}^- := \left\{ \sum_{x \in W} b_x C'_x \mid b_x \in \hat{A}, \ b_x \to 0 \text{ as } \ell(x) \to \infty \right\}$$

and

$$\mathcal{H}_C^- := \left\{ \sum_{x \in W} b_x C_x \mid b_x \in \hat{A}, \ b_x \to 0 \text{ as } \ell(x) \to \infty \right\}$$

of $H$ (note the difference between $C'_x$ and $C_x$), as well as the completion

$$\mathcal{J} := \left\{ \sum_{x \in W} b_x t_x \mid b_x \in \hat{A}, \ b_x \to 0 \text{ as } \ell(x) \to \infty \right\}$$

of $J \otimes_{\mathbb{C}} A$.

In [Lus87a], Lusztig shows that $\phi$ extends to an isomorphism of $\hat{A}$-algebras $\mathcal{H}_{C'}^- \to \mathcal{J}$. In this way the elements $t_w \in J \subset \mathcal{J}$ may be identified with elements of $\mathcal{H}^-$.  

**Lemma 7.** We have $\mathcal{H}_{C'}^- \subset \mathcal{H}^-$. The inclusion is continuous. 

**Proof.** Given an infinite sum $\sum_{x} b_x C'_x$, upon rewriting this sum in the standard basis, the coefficient of some $T_y$ is

$$a_y := \sum_{x \geq y} b_x q^{-\ell(x)/2} P_{y,x}(q).$$

As $\deg P_{y,x} \leq \frac{1}{2}(\ell(x) - \ell(y) - 1)$, we have that $q^{-\ell(x)/2} P_{y,x}(q)$ is a polynomial in $q^{-1/2}$. Therefore the above sum defines a formal Laurent series. Moreover, as $\ell(y) \to \infty$, it is clear that $a_y \to 0$. Continuity of the inclusion is clear from the formula for $a_y$. \qed

3.3.2 The functions $f_w$ and the basis elements $t_w$

We shall now explain how the map $\tilde{\phi} : J \to \mathcal{C}(G)^I$ induces a map of $A$-algebras $\phi_1 : J \otimes_{\mathbb{C}} A \to \mathcal{H}^-$. 

**Proposition 2.** There is a map of $A$-algebras $\phi_1 : J \otimes_{\mathbb{C}} A \to \mathcal{H}^-$ such that if $\phi_1(t_w) = \sum_x a_{x,w} T_x$, then $a_{x,w}(q) = f_w(x)$. Moreover, there is a constant $N$ depending only on $W$ such that $a_{x,w} \in (q^{1/2})^N \hat{A}^-$ for all $x, w \in W$.

The most difficult part of the proof of the proposition is showing that $a_{x,w} \to 0$ as $\ell(x) \to \infty$. To this end we have
Lemma 8. Let $w \in \hat{W}$. Then the degree in $q$ of the numerator of $f_w(1)$ is bounded uniformly in $w$ by some $N$ depending only on $\hat{W}$, and hence in the notation of Proposition 2 above, we have $a_{1,w} \in (q^{1/2})^N \hat{A}^\perp$ for all $w \in \hat{W}$.

Remark 11. In type $A$, the Lemma follows immediately from the order of integration given after the proof of Lemma 6.

Proof of Lemma 8. Let $\pi_w = \text{Ind}^G_P(\nu \otimes \omega)$ be a tempered $I$-spherical representation of $G$ arising by induction from a parabolic $P$ such that $\text{rank} A_P = k$. Let $\mathbb{T}^k_\omega$ be the compact torus parametrizing twists of $\omega$, and let $t_w$ be given. By Lemma 4, $\text{trace} (\pi, f_w)$ is a regular function on $\mathbb{T}^k$, i.e. a Laurent polynomial in the coordinates $z_1, \ldots, z_k$ on $\mathbb{T}^k$. The coefficients of this Laurent polynomial are independent of $q$, as $J$ and its representation theory are independent of $q$. As we have

$$|1 - q^r| \leq |z_i - q^r z_1^{e_1} \cdots z_k^{e_k}| \leq |1 + q^r|$$

for all $z_i \in \mathbb{T}$ and $r$, we may bound, for large $q$, the absolute value $|\mu_{M_P}(z)|$ of the Plancherel density for $M_P$ by a rational function $U_{M_P,\omega}(q)$. By Theorem 6, we may do the same for formal degrees $d(\omega)$, bounding $|d(\omega)|$ by a rational function $D_{\omega}(q)$. Thus we define a rational function of $q$ by

$$\frac{h_1(q) + h_2(q^{-1})}{k_1(q) + k_2(q^{-1})} = \sum_{\omega} U_{M_P,\omega}(q) D_{\omega}(q) \max_{z \in \mathbb{T}^k_\omega} |\text{trace} (\pi_w, f_w)|,$$

where $h_1, k_1 \in \mathbb{C}[q]$ and $h_2, k_2 \in q^{-1}\mathbb{C}[q^{-1}]$, and the sum is over the finitely many, up to unitary twist, pairs $(M, \omega)$, for $\omega \in \mathcal{E}_2(M)$, such that $\text{trace} (\pi_w, f_w) \neq 0$.

Fix $w \in \hat{W}$. Then

$$|f_w(1)| = \left| \sum_{(P,\omega)} \int_{O(\omega)} \text{trace} (\pi_w, f_w) d(\omega) d\mu_{M_P}(z) \right| \leq \frac{h_1(q) + h_2(q^{-1})}{k_1(q) + k_2(q^{-1})}.$$

Crucially, this expression holds for all $q \gg 1$.

On the other hand, by Corollary 2 and Theorem 12, $f_w(1)$ is a rational function of $q$, and for $q$ sufficiently large, we may write

$$|f_w(1)| = \frac{f_1(q) + f_2(q^{-1})}{d_1(q) + d_2(q^{-1})}$$

where $f_1, d_1 \in \mathbb{C}[q]$ and $f_2, d_2 \in q^{-1}\mathbb{C}[q^{-1}]$. Therefore for all $q \gg 1$ we have

$$|(f_1(q) + f_2(q^{-1}))(k_1(q) + k_2(q^{-1}))| \leq |(d_1(q) + d_2(q^{-1}))(h_1(q) + h_2(q^{-1}))|.$$  \hspace{1cm} (22)

We claim that this implies

$$|f_1(q)k_1(q)| \leq |d_1(q)h_1(q)|$$  \hspace{1cm} (23)

for $q$ sufficiently large. Indeed, let $\epsilon > 0$ be given and choose $q \gg 1$ such that

$$||(f_1(q) + f_2(q^{-1}))(k_1(q) + k_2(q^{-1}))| - |f_1(q)k_1(q)|| \leq \epsilon$$

and

$$|(d_1(q) + d_2(q^{-1}))(h_1(q) + h_2(q^{-1}))| - |d_1(q)h_1(q)| \leq \epsilon,$$

and (22) holds. Then we have

$$|f_1(q)||k_1(q)| \leq |(f_1(q) + f_2(q^{-1}))(k_1(q) + k_2(q^{-1}))| + \epsilon \leq |(d_1(q) + d_2(q^{-1}))(h_1(q) + h_2(q^{-1}))| + \epsilon \leq |d_1(q)||h_1(q)| + 2\epsilon,$$
which proves (23).

Therefore
\[ \deg f_1 \leq \deg f_1 + \deg k_1 \leq \deg d_1 + \deg h_1. \]

Now, the denominator of \( f(w)(1) \), and hence \( \deg d_1 \), depends only on the two-sided cell containing \( w \), again by Corollary 2 and Theorem 12. We can also bound \( \deg h_1 \) uniformly in terms of \( \tilde{W} \), as it depends only on the Plancherel measure and the finitely many possible formal degrees appearing in the parametrization of the \( I \)-spherical part of the tempered dual of \( G \). This proves the lemma.

If \( t_w \in t_d \mathcal{H}_{d'} \) for \( d \neq d' \), then \( t_w \) is a commutator. We record this observation as

Lemma 9. We have that \( f(w)(1) = 0 \) unless \( f_d * f_w * f_d \neq 0 \) for some distinguished involution \( d \).

Now we can prove the proposition.

Proof of Proposition 2. Let \( w \in \tilde{W} \). Write \( \tilde{\phi}(t_w) = f_w = \sum_x A_{x,w} T_x \) as Schwartz functions on \( G \) so that \( A_{x,w} = f_w(x) \). We must show that there is a unique element \( a_{x,w} \in \hat{A} \) such that \( a_{x,w}(q) = A_{x,w} \) as complex numbers. We will then check that \( a_{x,w} \to 0 \) rapidly enough as \( \ell(x) \to \infty \) for \( \sum_x a_{x,w} T_x \) to define an element of \( H^- \).

By Corollary 2 and Theorem 12, there is a formal power series in \( \hat{A}^- \) with constant term equal to 1 that specializes to the denominator of \( A_{1,w} \) when \( q = q \). Moreover, there is a unique formal Laurent series \( a_{1,w} \in \hat{A} \) such that \( a_{1,w}(q) = A_{1,w} \) for all prime powers. Indeed, \( a_{1,w} \) is convergent for \( q = q \), and the difference of any two such series defines a meromorphic function of \( q^{-1/2} \) outside the unit disk with zeros at \( q = p^r \) for every \( r \in \mathbb{N} \). As these prime powers accumulate at \( \infty \), such a meromorphic function must be identically zero.

If \( f \in C^{I \times I} \) is a Harish-Chandra Schwartz function, then
\[
q^{-\ell(x)}(f * T_{x^{-1}})(1) = q^{-\ell(x)} \int_G f(g) T_{x^{-1}}(g^{-1}) \, d\mu_I(g) = q^{-\ell(x)} \int_{I \times I} f(g) \, d\mu_I(g) = q^{-\ell(x)} \mu_I(I \times I) f(x) = f(x).
\]
By definition, $f_w(x) = A_{x,w}$. On the other hand, according to Lemma 2, we have

$$q^{-\ell(x)}(f_w \ast T_{z^{-1}})(1) = q^{-\ell(x)} \left( \tilde{\phi}(t_w) \ast \tilde{\phi} \left( \phi_q(T_{z^{-1}}) \right) \right)(1) = q^{-\ell(x)} \tilde{\phi} \left( t_w \phi_q(T_{z^{-1}}) \right)(1) = q^{-\ell(x)} \tilde{\phi} \left( t_w \phi_q \left( \sum_{y \leq x^{-1}} q^{\ell(y)} \left( -1 \right)^{\ell(x^{-1}) - \ell(y)} Q_{y,x^{-1}}(q)(C_y') \right) \right)(1) \quad (24)$$

$$= q^{-\ell(x)} \tilde{\phi} \left( t_w \phi_q \left( \sum_{y \leq x^{-1}} q^{\ell(y)} \left( -1 \right)^{\ell(x^{-1})} Q_{y,x^{-1}}(q) \sum_{a(d)=a(r)} h_{y,d,r} t_r \right) \right)(1) \quad (26)$$

$$= q^{-\ell(x)} \tilde{\phi} \left( \sum_{y \leq x^{-1}} q^{\ell(y)} \left( -1 \right)^{\ell(x^{-1})} Q_{y,x^{-1}}(q) \sum_{\gamma \sim_L d \sim_R a(w)} h_{y,d,r} t_r \right)(1) \quad (27)$$

where $d_w$ is the unique distinguished involution in the right cell containing $w$.

In line (24), we rewrote $T_{z^{-1}}$ in terms of the $C'$-basis of $H$, using the inverse Kazhdan-Lusztig polynomials $Q_{y,x^{-1}}$. In line (25), we applied the Goldman involution (see Lemma 1). In line (26) we applied Lusztig’s map $\phi_q$, and then in line (27), we applied the map $\tilde{\phi}$. Also in line (27), we used that left (respectively right) cells give left (respectively right) ideals of $J$, and so $t_w t_r$ is an integral linear combination of $t_{z^{-1}}$, with $d \sim_L z^{-1} \sim_R d_w$. By lemma 9, $f_{z^{-1}}(1) \neq 0$ only if $d = d_w$, that is, $z^{-1} \sim_L d_w$.

We use (27) to define $a_{x,w} \in \mathcal{A}$. By the same arguments as above, $a_{x,w}$ is unique and defines a meromorphic function of $q^{-1/2}$. It remains to show that as $\ell(x) \to \infty$, $a_{x,w} \to 0$ in the $(q^{-1/2})$-adic topology. This follows in fact from (27). Indeed, the product $f_w \ast f_r$ is an $\mathbb{N}$-linear combination of functions $f_x$, and the values $f_x(1)$ are rational functions of $q$, the numerators of which have uniformly bounded degree in $q$ by Lemma 8. The polynomials $h_{y,d,w,r}$ have bounded degree in $q$ (for example in terms of the $a$-function). Finally, the degree in $q$ of

$$q^{-\ell(x)} q^{\frac{\ell(y)}{2}} Q_{y,x^{-1}}(q)$$

is at most

$$q^{-\ell(x)} q^{\frac{\ell(y)}{2}} q^{\frac{\ell(y)-\ell(q)-1}{2}} = q^{-\ell(x)} q^{\frac{\ell(y)}{2}} q^{\frac{\ell(q)-\ell(y)-1}{2}} = q^{-\ell(x)-1} \to 0 \quad (28)$$

as $\ell(x) \to \infty$. This completes the definition of $\phi_1$ as a map of $\mathcal{A}$-modules.

It is easy to see that $\phi_1$ is a morphism of rings, essentially because $\tilde{\phi}$ is. Indeed, we have

$$\phi_1(t_w t_w') = \sum_z \gamma_{w,w',z^{-1}} \sum_x a_{x,z} T_x \quad (29)$$

while on the other hand

$$\phi_1(t_w) \phi_1(t_{w'}) = \sum_x a_{x,w} T_x \cdot \sum_y a_{y,w'} T_y \quad (30)$$
and when $q = q$, we have that (30) becomes by definition
\[ \tilde{\phi}(t_w) \ast \tilde{\phi}(t_{w'}) = \tilde{\phi} \left( \sum_z \gamma_{w,w',z-1} t_z \right) = \sum_z \sum_x \gamma_{w,w',z-1} A_{x,z} T_x. \]
Hence for infinitely many prime powers we have that the specializations of (29) agrees with those of (30), and hence (29) is equal to (30) in $\mathcal{H}^-$. A similar argument shows that $\phi_1$ preserves units.

Remark 12. The proof, specifically (28), gives a necessary condition for an element of $\mathcal{H}^-$ to belong to the image of $\phi_1$: the coefficients must decay asymptotically at least as fast as $q^{-\ell(x)/2}$.

**Proposition 3.** There is a commutative diagram
\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\phi} & J \otimes_{\mathbb{C}} \mathcal{A} \\
\phi_1 & \downarrow \quad \phi^{-1} & \downarrow \quad \phi^{-1} \\
\mathcal{H}^- & \xrightarrow{\phi_1} & \mathcal{H}^- \\
\end{array}
\]
and we have $\phi_1 = (\dagger) \circ \phi^{-1}$ as morphisms of $\mathcal{A}$-algebras $J \otimes_{\mathbb{C}} \mathcal{A} \to \mathcal{H}^-$. In particular, $a_{x,w}$ has integer coefficients for all $x, w \in \hat{W}$.

**Proof.** The second claim follows from the first if we show that $\phi_1$ extends to a continuous morphism $J \to \mathcal{H}^-$, by density of $\phi(H)$ in $J \simeq \mathcal{H}_C$, and the third claim follows from the second and the fact that the completions we consider are actually defined over $\mathbb{Z}$ [Lus87a, Thm. 2.8].

Note that as $\tilde{\phi} \circ \phi_q = (\dagger)$ on $H$ for all $q$, we have that $\phi_1 = (\dagger) \circ \phi^{-1}$ on $\phi(H)$. This says that the diagram commutes.

We now show that $\phi_1$ extends to a continuous map $J \to \mathcal{H}_C^-$. Let $\sum_w b_{w,t_w}$ define an element of $J$ and define
\[ \phi_1 \left( \sum_w b_{w,t_w} \right) = \sum_y b'_y T_y, \]
where
\[ b'_y = \sum_w b_w a_{y,w}. \]

We must first show that this infinite sum of elements of $\mathcal{A}$ is well-defined. By Lemma 8 and (27), we have that there is $M \in \mathbb{N}$ such that $a_{y,w} \in (q^{1/2})^M A^-$ for all $y, w$. Therefore $b'_y$ is well-defined, and as $a_{y,w} \to 0$ as $\ell(y) \to \infty$, we have $b'_y \to 0$ as $\ell(y) \to \infty$. Therefore $\phi_1$ extends to $J$.

To show continuity, it suffices to show that if $\{\sum_w b_{w,t_w}\}_n$ is a sequence of elements of $J$ tending to $0$ as $n \to \infty$, then
\[ \sum_w b_{w,n} \phi_1(t_w) = \sum_y b'_{y,n} T_y \to 0 \]
as $n \to \infty$ in $\mathcal{H}^-$, where $b'_{y,n} = \sum_w b_{w,n} a_{y,w}$. For all $R > 0$, there is $N > 0$ such that $n > N$ implies $b_{x,n} \in (q^{-1/2})^R A^-$ for all $x$. We have seen that there is $M$ depending only on $W$ such that $a_{y,w} \in (q^{1/2})^M A^-$ for all $w, y$. Therefore $b'_{y,n} \to 0$ as $n \to \infty$, because $b_{w,n} \to 0$ as $n \to \infty$.

Note that Proposition 3 means in particular that $\phi_1(J) \subset \mathcal{H}_C^-$. 

**Corollary 3.** We have
\[ (\phi \circ (\dagger)^{-1})(t_1) = f_1(1) \sum_{w \in \hat{W}} (-1)^{\ell(w)} q^{-\ell(w)} T_w. \]

**Proof.** If $w = 1$, everything in (27) reduces to $r = d_w = y = 1$, and we need only recall that $Q_{1,x}(q) = 1$ for any $x \in \hat{W}$. This follows from unicity of the inverse Kazhdan-Lusztig polynomials and the identity $\sum_x (-1)^{\ell(x)} P_{x,w}(q) = 0$ for $w \neq 1$ [BB05, Exercise 5.17].
Corollary 4. The map \( \tilde{\phi} \) defined in [BK18] and recalled in diagram (2) is injective.

Proof. Let \( q > 1 \) and let \( j \neq 0 \) be an element of \( J \). We must show that \( \tilde{\phi}(j) \neq 0 \). By injectivity of \( \phi_1 \), we have \( \phi_1(j) \neq 0 \). By definition of the map \( \phi_1 \), this means that there exists \( q_0 > 1 \), \( u, s \in G^\ast \) such that \( us = su \) with \( s \) compact, and a representation \( \rho \) of \( \pi_0(Z_{G^\ast}(u, s)) \) such that \( jK(u, s, \rho, q_0) \neq 0 \). But then \( jK(u, s, \rho, q) \neq 0 \), \( K(u, s, \rho, q) \) being a different specialization of the restriction of the same \( J \)-module \( E(u, s, \rho) \) as for \( K(u, s, \rho, q_0) \), and \( K(u, s, \rho, q) \) is also tempered. It follows that \( \tilde{\phi}(j) \neq 0 \).

3.4 Proof of Theorem 13 for \( GL_n \)

Theorem 13. Let \( \tilde{W} \) be of type \( \tilde{A}_n \). Then statements 1 and 2 in the statement of Theorem 2 are true, together with a stronger version of statement 3: Let \( u = (r_1, \ldots, r_k) \) be a unipotent conjugacy class in \( GL_n(\mathbb{C}) \). Let \( d \) be a distinguished involution in the two-sided cell \( c(u) \subset W(GL_n) \) corresponding to \( u \). Let \( \pi \) be the unique family of parabolic inductions that \( t_d \) does not annihilate. Then \( \text{rank } \pi(t_d) = 1 \).

Proof. By Corollary 2 and Propositions 2 and 3, \( a_{1,x} \) is a rational function of \( q \) for all \( w \). Then equation (27) implies that \( a_{x,w} \), being a sum of rational functions with Laurent polynomial coefficients, is a rational function of \( q \) for all \( x \). The same equation, together with the fact that \( J_c \) is a two-sided ideal for each cell \( c \) shows that the denominator of \( a_{x,w} \) depends only on the two-sided cell containing \( w \). This proves the first claim. The second claim now follows from the first claim and the first statement of Corollary 2 and the fact that \( \tilde{W} \) has finitely-many two-sided cells.

Finally, let \( u = (r_1, \ldots, r_k) \). We have \( \pi = \text{Ind}_{P_1}^{G}(\text{St}_{M_1} \otimes \nu) \), for the unique Levi \( M_1 \) such that \( u \) is distinguished in \( M_1 \), and as \( \dim \text{St}_{M_1} = 1 \), clearly we have

\[
\dim \pi_l = \frac{n!}{r_1! \cdots r_k!}.
\]

This is also the number of distinguished involutions in \( c(u) \), by [Shi06]. The claim follows from Corollary 4.

3.5 Proof of Theorem 2

Proof of Theorem 2. Theorem 12 together with Propositions 2 and 3 show that \( a_{1,w} \) is a rational function of \( q \) with denominator depending only on the two-sided cell containing \( w \). Equation (27) again shows that \( a_{x,w} \) is a rational function of \( q \) with denominator depending only on the two-sided cell containing \( w \); up to twists the set \( E_2(M)^I \) is finite for every Levi subgroup \( M \), so we may multiply through to include the denominators of all required formal degrees, which are in fact rational of the correct form by Theorem 6. Therefore there is a polynomial \( P_G(q) \) that clears denominators of all \( a_{x,w} \). This proves the first statement of the Theorem.

Now, by Proposition 9 and Remark 2 of [BDD23], \( \phi \) induces a surjection

\[
\tilde{\phi} : H/[H,H] \left[ \frac{1}{P_W(q)} \right] \to J/[J,J] \otimes A.
\]

Therefore for every \( j \in J \), there is \( N = N(j) \in \mathbb{N} \) and \( h \in H \) such that

\[
j \equiv h \frac{1}{P_W(q)^N}
\]

in \( J/[J,J] \otimes A \). Considering traces and invoking Proposition 3, we see that the denominator of every \( a_{x,w} \) divides a power of \( P_W(q) \). Therefore there is \( N = N_W \) depending only \( \tilde{W} \) such that we can take \( P_{G}^{1}(q) = P_{W}(q)^N \). This proves the second statement of the Theorem. The third statement was proven in Corollary 1.
4 Representations with fixed vectors under parahoric subgroups

In this section we will give an application of the $J$-action on the tempered $H$-modules. The first statement is an immediate corollary of the existence of this action, but the second statement relies on Corollary 4.

**Theorem 14.** Let $\mathcal{P}$ be a parahoric subgroup of $G$. Let $\pi$ be a simple tempered representation of $G$ with $I$-fixed vectors with Kazhdan-Lusztig parameter $(u, s, \rho)$. Let $w_P$ be the longest element in the parabolic subgroup of $\mathcal{W}$ defined by $\mathcal{P}$ and $B_u'$ be the Springer fibre for $u$.

1. If 
\[ \ell(w_P) > a(u) = \dim_C B_u', \]
then 
\[ \pi^P = \{0\}. \]

2. Conversely, let $w_P$ be the unipotent conjugacy class corresponding to the two-sided cell containing $w_P$. Then there exists $s \in Z_{G^v}(w_P)$, a Levi subgroup $M^v$ of $G^v$ minimal such that $(u_P, s) \in M^v$, and a discrete series representation $\omega \in \mathcal{E}_2(M)$ such that 
\[ \pi^P = i_{P,m}^G(\omega \otimes \nu)^P \neq \{0\} \]
for all $\nu$ non-strictly positive and the parameter of $\pi$ is $(u_P, s)$.

**Proof.** Let $\mathcal{P}$ and $w_P$ be as in the statement. Then $C_{w_P}'$ is proportional by a power of $q$ to the indicator function $1_P$ in $H$. Moreover, $w_P$ is a distinguished involution, with $a(w_P) = \ell(w_P)$. By Proposition 3 and the fact that $\phi$ is “upper-triangular” with respect to the $a$-function, we have $(\phi \circ (\cdot)) (C_{w_P}') J_c = 0$ for $c$ corresponding to $u$ if $a(w_P) > a(u)$.

For the second statement, by Corollary 4, there is a tempered representation $\pi = \text{Ind}_P^G(\omega \otimes \nu)$ with $\pi(t_{w_P}) \neq 0$; the unipotent part of its parameter is $w_P$. In particular, there is a vector $v \in \pi$ such that $\pi(t_{w_P}) v = v$. We have 
\[ (\phi \circ (\cdot)) (C_{w_P}') t_{w_P} = (-1)^{\ell(w_P)} \sum_{z \sim_L d} h_{w_P, d, z} t_{w_P} = (-1)^{\ell(w_P)} \sum_z h_{w_P, w_P, z} t_z = \text{vol}(P) t_{w_P}, \]
as $t_z t_{w_P} \neq 0$ only if $z \sim_L w_P$, and $C_{w_P} C_{w_P} = (-1)^{\ell(w_P)} \text{vol}(P) C_{w_P}$. Thus $v \in \pi^P$. As trace $(\pi, t_d)$ is constant with respect to $\nu$, the last part of the claim follows.

A version of this statement for enhanced parameters in the case $\mathcal{P} = G(\mathcal{O})$ appears in [ABPS17, Prop. 10.1].

**Example 8.** The principal series representations have $u = \{1\}$, maximal $a$-value, and fixed vectors under every maximal compact subgroup of $G$. On the other extreme, the Steinberg representation has $a$ regular, and does not have fixed vectors under any proper parahoric.

In addition to the interpretation given in Section 1.1.3, in the examples below, the $J$-action also detects which direct summands of a reducible $\mathcal{P}$-spherical tempered representation are themselves $\mathcal{P}$-spherical.

**Example 9.** Let $G = \text{SL}_2(F)$. As remarked in [Daw21], the distinguished involutions in the lowest two-sided cell are the simple reflections $s_0$ and $s_1$, and each $t_{s_i}$ is invariant under one of the two conjugacy classes of maximal parahoric subgroup of $G$. For unitary principal series representations $\pi$, one has trace $(\pi, t_{s_0}) = \text{trace} (\pi, t_{s_1}) = 1$. At the quadratic character, the corresponding principal series representation is reducible, and each summand contains fixed vectors under precisely one of the maximal parahorics. Indeed, in [Daw21] this computation is carried out at the level of the Schwartz space of the basic affine space.

**Example 10.** Let $G = \text{SO}_5(F) = \text{PGSp}_4(F)$, with affine Dynkin diagram labelled as

\[
\begin{array}{ccc}
0 & 1 & 2
\end{array}
\]
There are five conjugacy classes of parahoric subgroup, each obtained by projection from $\mathrm{GSp}_4(F)$: the maximal parahoric $\mathrm{PGSp}_4(\mathcal{O})$, the image of the paramodular group $K$ corresponding to $\{0,2\}$, the image of the Siegel parahoric subgroup $\mathrm{Si}$ corresponding to $\{1\}$, the image of the Klingen parahoric $\mathrm{Kl}$ corresponding to $\{0\}$, and the Iwahori subgroup.

The columns of the below table give all the $I$-spherical tempered representations of $G$ by denoting the representation $\text{Ind}_G^P(\omega)$ by $\omega$. We recall the few cases of reducibility of these inductions immediately below. The rows list unipotent conjugacy classes in $G^\vee = \mathrm{Sp}_4(\mathbb{C})$ such that all tempered standard modules $K(u,s,\rho)$ are in row $u$. That is, the rows record which summand of $J$ acts on each representation.

| Cell \ Levi | $\nu \times \mathrm{GL}_1$ | $\nu \times \mathrm{SO}_3^+$ | $\xi \mathrm{St}_{\mathrm{GL}_2}$ | $\pm \tau_2$ | $\pm \mathrm{St}$ |
|------------|--------------------------|--------------------------------|---------------------------------|--------------|----------------|
| $(1, \ldots, 1)$ | | | | | |
| $(2, 1, 1)$ | $\nu$ | $\nu \otimes \pm \mathrm{St}_{\mathrm{SO}_3}$ | | | |
| $(2, 2)$ | | | $\xi \mathrm{St}_{\mathrm{GL}_2}$ | $\pm \tau_2$ | | $\pm \mathrm{St}$ |
| $(4)$ | | | | | |

The discrete series are as in [Ree94]. The only reducibility, by [Mat10, Prop. 3.3], is the reducibility

$$\text{Ind}_G^P(\xi \mathrm{St}_{\mathrm{GL}_2}) = \tau_{\text{triv}} \oplus \tau_{\text{sgn}}$$

for $\xi^2 = 1$.

We can now compute the traces of some elements $t_d$ using the description of the simple $J$-modules given in [Xi94]. For the cells $(1, \ldots, 1)$, $(2, 1, 1)$, and $(4)$, we have $\text{trace}(\pi, t_d) = 1$ for all $\pi$. We have

$$\text{trace}(\tau_{\text{triv}}, t_{s_0}) = \text{trace}(\tau_2, t_{s_0}) = 1 = \text{trace}(\tau_{\text{sgn}}, t_{s_1}) = \text{trace}(\tau_{\text{triv}}, t_{s_1}).$$

In the below table, trace $(\pi, t_d)$ is recorded in bold face, whereas the dimension of $\pi^P$, taken from [RS07, Table A.15], is recorded in normal face. Representations attached to the same cell but belonging to different packets are separated with a dotted line.

| $t_{WP}$ \ $\pi = \text{Ind}_P^G(-)$ | $a \nu$ | $\nu \otimes \pm \mathrm{St}_{\mathrm{SO}_3}$ | $\tau_{\text{triv}}$ | $\tau_{\text{sgn}}$ | $\pm \tau_2$ | $\pm \mathrm{St}$ | $\mathcal{P}$ |
|-----------------|--------|-------------------------------|-----------------|-----------------|-----------------|-----------------|-----|
| $t_{w_0}$ | 4 1 | | | | | |
| $t_{s_0,s_2}$ | 2 2 | 1 | | | | $\mathrm{PGSp}_4(\mathcal{O})$ |
| $t_{s_0}$ | 1 4 | 2 | 1 | | $\tau_{\text{sgn}}$ | $\pm \tau_2$ | $\mathcal{P}$ |
| $t_{s_1}$ | 1 4 | 1 | 1 | 1 | | $\pm \mathrm{St}$ |
| $t_1$ | 0 8 | 4 | 3 | 1 | 4 | 1 | $\mathcal{I}$ |

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