Density perturbations in Kantowski-Sachs models with a cosmological constant

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Abstract. The growth of density perturbations in Kantowski-Sachs cosmologies with a positive cosmological constant is studied, using the 1+3 and 1+1+2 covariant formalisms. For each wave number a closed system for eight scalars is obtained. These are formed from quantities that are zero on the background and hence are gauge invariant. As an example a numerical solution describing the evolution of density perturbations on a background that experiences a bounce is presented. Typically the density gradient in the bouncing directions experiences a local maximum at or slightly after the bounce.

1. Introduction
In this work we study the growth of density perturbations in perfect fluid Kantowski-Sachs universes with positive cosmological constant with special attention to the behaviour close to a bounce, using the 1+3 and 1+1+2 covariant splits of spacetime \cite{1, 2, 3, 4}. As inhomogeneity variables we use the spatial gradients of the density, the expansion, the shear scalar and one more auxiliary scalar to close the system. These quantities are all covariantly defined and are also zero on the background \cite{1, 5}. Hence they are gauge invariant \cite{6}. By projecting along the preferred and orthogonal directions and taking divergences along the preferred and orthogonal directions respectively and making harmonic decompositions of the spatial derivatives, the system is reduced to a first order system of ordinary differential equations in time for eight scalar quantities for each wave number.

2. The 1+3 and 1+1+2 covariant formalisms
In \cite{1, 2}, to which the reader is referred for details, a covariant formalism for the 1+3 split of spacetimes with a preferred timelike vector, $u^a$, was developed. The projection operator onto the perpendicular 3-space is given by $h^b_a = g^b_a + u^a u^b$. With the help of this vectors and tensors can be covariantly decomposed in "spatial" and "timelike" parts. The covariant time derivative and projected spatial derivative are given by

$$\dot{\psi}_{a...b} = u^c \nabla_c \psi_{a...b} \quad \text{and} \quad \tilde{\nabla}_c \psi_{a...b} = h^f_c h^d_a ... h^e_b \nabla_f \psi_{d...e}$$

respectively. The covariant derivative of the 4-velocity, $u^a$, can be decomposed as

$$\nabla_a u_b = -u_a \dot{u}_b + \tilde{\nabla}_a u_b = -u_a \dot{u}_b + \frac{1}{3} \theta h_{ab} + \omega_{ab} + \sigma_{ab}$$

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where the kinematic quantities of \( u^a \), acceleration, expansion, shear and vorticity are defined by \( \dot{u}_a \equiv u^b \nabla_b u_a \), \( \theta \equiv \nabla_a u^a \), \( \sigma_{ab} \equiv \nabla_{[a} u_{b]} \), and \( \omega_{ab} \equiv \nabla_{[a} u_{b]} \) respectively. These quantities, together with the Ricci tensor (that in the present case is given by the energy density, \( \mu \), and the pressure, \( p \)) and the electric, \( E_{ab} \), \( \text{and} \) the magnetic, \( H_{ab} \), and the electric, \( C_{abcd} \), \( \text{and} \) the magnetic, \( C_{abcd} \), parts of the Weyl tensor, are then used as dependent variables. From the Ricci and Bianchi identities one obtains propagation equations in the \( u^a \) direction and constraints.

A formalism for a further split (1+2) with respect to a spatial vector \( n^a \) (with \( u^a n_a = 0 \)) was developed in \([3, 4]\). Projections perpendicular to \( n^a \) are made with \( N_a^b = g_a^b + u_a u^b - n_a n^b \). Derivatives along and perpendicular to \( n^a \) are given by

\[
\psi_{a...b} \equiv n^c \nabla_c \psi_{a...b} = n^c h^f_{a...} h^d_{b...} \nabla_f \psi_{d...e} \quad \text{and} \quad \delta_{c} \psi_{a...b} \equiv N_c^f N_a^d ... N_b^e \nabla_f \psi_{d...e}
\]

(3) respectively. Similarly to the decomposition of \( \nabla_a u_b \), \( \nabla_a n_b \), and \( n_a \) can be decomposed into further ‘kinematical’ quantities of \( n^a \). The set of equations is now decomposed into propagation equations in the \( u^a \) and \( n^a \) directions respectively and constraints.

### 3. Perturbations on Kantowski-Sachs

As backgrounds we use the Kantowski-Sachs cosmologies, \([7]\), that are spatially homogeneous and locally rotationally symmetric (LRS). With zero vorticity the line-element can be written as

\[
ds^2 = -dt^2 + a_1^2(t) d\xi^2 + a_2^2(t) \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right)
\]

(4) with the 4-velocity of matter given by \( u = \frac{\partial}{\partial t} \) and the direction of anisotropy by \( n = \frac{1}{a_1} \frac{\partial}{\partial \varphi} \). The expansion and the only independent component of the shear are given by

\[
\theta = \frac{\dot{a}_1}{a_1} + 2 \frac{\dot{a}_2}{a_2} \quad \text{and} \quad \Sigma \equiv \sigma_{nn} = \frac{2}{3} \left( \frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2} \right),
\]

(5) respectively.

Given an equation of state and the cosmological constant, \( \Lambda \), the Kantowski-Sachs models are completely determined in terms of shear, \( \Sigma \), expansion, \( \theta \), and energy density, \( \mu \). The electric part of the Weyl tensor is given algebraically as \( E = -\frac{2}{3} \mu - \frac{1}{3} \Lambda - \Sigma^2 + \frac{2}{3} \theta^2 + \frac{1}{3} \Sigma \theta \). The following evolution equations are obtained

\[
\dot{\Sigma} = \frac{2}{3} (\mu + \Lambda) + \frac{1}{2} \Sigma^2 - \theta^2 - \frac{2}{9} \theta^2, \quad \dot{\Sigma} = \theta (\mu + p), \quad \dot{\theta} = -\frac{1}{3} \theta^2 - \frac{1}{2} \left( \mu + 3p - 2\Lambda \right) - \frac{3}{2} \Sigma^2.
\]

(6)

The inhomogeneities will be described by quantities that are zero on the background, and hence are gauge invariant \([6]\). The primary variable is the density gradient

\[
D_a \equiv \frac{a \nabla_a \mu}{\mu},
\]

(7)

where \( a \) is the average scale factor, defined from \( \theta = 3 \dot{a}/a \). The density fluctuations \( \frac{\delta \mu}{\mu} \) on a dimensionless comoving length scale \( l_0 \) are related to the quantity \( D_a \) through \( \frac{\delta \mu}{\mu} \sim (D_a D^a)^{1/2} l_0 \). To close the system three auxiliary quantities, that we choose as

\[
Z_a \equiv a \nabla_a \theta, \quad T_a \equiv a \nabla_a \sigma^2 \quad \text{and} \quad S_a \equiv a \nabla_a (\sigma^{ab} S_{ab}),
\]

(8)

will be needed. Here \( S_{ab} \) is the traceless part of the 3-Ricci tensor. In the present case it can be rewritten as \( S_{ab} = E_{ab} + \sigma_{<a} \sigma_{b>} - \frac{1}{3} \theta a_{ab} \) in terms of covariant quantities.
The gradients are then decomposed into components parallel and perpendicular to \( n^a \) like \( \mathcal{D} \equiv D_a n^a \) and \( \mathcal{D}_a \equiv D_b N^b_a \) respectively and equivalently for the other quantities in (8). To treat the spatial derivatives appearing in the equations we will do an harmonic decomposition. For this purpose it is suitable to get the spatial derivatives in the form of two Laplace-like operators, acting on our variables. To obtain this we define new variables \( \hat{\mathcal{D}} \equiv n^a \nabla_a \mathcal{D} \) and \( \mathcal{P} \equiv \delta^a \delta_a \), and similarly for the other variables. Finally, to remove some singular terms, \( \mathcal{T}_a \) is redefined according to

\[
\mathcal{T}_a = \Sigma^2 \mathcal{S}_a + \Sigma S_a, \tag{10}
\]

where \( \mathcal{S} \) to zeroth order is given by \( \mathcal{S} = -\frac{3}{2} \mu - \frac{3}{2} \Lambda - \frac{3}{2} \Sigma^2 + \frac{2}{3} \theta^2 = -\frac{2}{3} K < 0 \) (\( K \) is the curvature of the 2-spheres \( dt = dz = 0 \)). The system for the hat variables then becomes

\[
\begin{align*}
\dot{\hat{\mathcal{D}}} &= \left[ \left( \frac{2}{3} \mu + \frac{1}{2} \right) - 2 \Sigma \right] \hat{\mathcal{D}} - \left( 1 + \frac{p}{\mu} \right) \hat{\mathcal{T}} \\
\dot{\hat{\mathcal{Z}}} &= - (1 + 2 \Sigma) \hat{\mathcal{Z}} + \left[ \frac{1}{2} \mu + \frac{3}{2} \mu p \right] \left( \frac{3}{2} \Sigma^2 \right) \hat{\mathcal{Z}} - \left( 1 + \frac{p}{\mu} \right) \hat{\mathcal{T}} \\
\dot{\hat{\mathcal{T}}} &= - \left( \frac{1}{3} \mu + 2 \Sigma + \frac{1}{3} \Sigma^2 \right) \hat{\mathcal{T}} - \left( \frac{1}{3} \mu + p \right) \left( \theta - 2 \Sigma \right) \hat{\mathcal{T}} \\
\dot{\mathcal{S}} &= \left[ \mu \Sigma^2 + \frac{\mu p}{\mu + p} \hat{\mathcal{S}} \right] \left( \frac{5}{2} \theta \Sigma + \frac{3}{2} \Sigma^3 \right) \hat{\mathcal{S}} - \left( \frac{2}{3} \theta \Sigma + \frac{5}{2} \hat{\mathcal{S}} \right) \Sigma \hat{\mathcal{Z}} + \left( \Sigma^4 + 2 \hat{\mathcal{S}}^2 \right) \hat{\mathcal{T}} + \left( \frac{1}{2} \Sigma - \frac{1}{2} \theta \right) \hat{\mathcal{T}} + \left( \frac{1}{2} \Sigma - \frac{1}{2} \theta \right) \hat{\mathcal{S}} - \left[ \frac{1}{2} \Sigma + 2 \Sigma^2 \right] \hat{\mathcal{D}} \\
&= - \Sigma^2 \left( \frac{1}{2} \Sigma - \frac{1}{2} \theta \right) \hat{\mathcal{D}} + \left( \frac{1}{2} \Sigma - \frac{1}{2} \theta \right) \hat{\mathcal{S}} + \Sigma \hat{\mathcal{D}} \hat{\mathcal{T}} - \left( \frac{1}{2} \Sigma - \frac{1}{2} \theta \right) \hat{\mathcal{D}} + \left( \frac{1}{2} \Sigma - \frac{1}{2} \theta \right) \hat{\mathcal{S}}.
\end{align*}
\]

A similar system is obtained for the variables \( \mathcal{P} \), see [8].

By making a harmonic decomposition \( \Psi = \sum_{k_{\parallel}, k_{\perp}} \Psi_{k_{\parallel}k_{\perp}} P_{k_{\parallel}} Q_{k_{\perp}} \) where

\[
\Delta P_{k_{\parallel}} = - \frac{k_{\parallel}^2}{a_1^2} P_{k_{\parallel}}, \quad \delta_a P_{k_{\parallel}} = \hat{P}_{k_{\parallel}} = 0 \quad \text{and} \quad \delta^2 Q_{k_{\parallel}} = - \frac{k_{\parallel}^2}{a_2^2} Q_{k_{\perp}}, \quad \hat{Q}_{k_{\perp}} = \hat{Q}_{k_{\perp}} = 0 \tag{12}
\]

the system (11) and the corresponding system for the slashed variables are rewritten in terms of the constant comoving wave numbers \( k_{\parallel} \) and \( k_{\perp} \).

Due to the definitions of \( \hat{\mathcal{D}} \) and \( \mathcal{P} \) in terms of the spatial derivatives, the time dependences of the variables \( \mathcal{D}_{\parallel} = a_1 \mathcal{D} \) and \( \mathcal{D}_{\perp} = a_2 \mathcal{P} \) give a better representation of the development of the relative density contrast \( \delta_H / \mu \).

4. Numerical solutions

An example of a numerical solution is given here. This background, where the equation of state is given by \( p = \mu / 3 \) (radiation), starts in a state where the directions perpendicular to the direction
of anisotropy have small and decreasing expansion rates that become almost negligible for a period of time. After this the expansion starts again to finally reach a constant value $\theta_\perp = \sqrt{\Lambda/3}$. The anisotropy direction starts from a contracting state, goes through a bounce and then starts expanding. Eventually its expansion also approaches $\theta_\parallel = \sqrt{\Lambda/3}$, giving a total expansion rate of $\theta = \sqrt{3\Lambda}$. The initial state is close to the critical point $-X$ (contracting Kasner) and for an intermediate period the solution is close to another critical point $+X$ (expanding Kasner), but for large times the solution approaches the sink point de Sitter $+dS$. The evolutions of density, $\mu_0$, expansion, $\theta_0$, expansion in the anisotropy direction, $\theta_\parallel$, expansion in one of the perpendicular directions, $\theta_\perp$ and shear $\Sigma_0$ are depicted in figure 1.

In figure 2 the growth of the density gradient in the direction of anisotropy is shown for different values of the comoving wave numbers $k_\parallel$ and $k_\perp$. For $k = 0$ the density gradient in the direction of anisotropy reaches a small maximum after the bounce and after this a small minimum before it starts growing unboundedly. For higher values of the wave number $k$ the density gradient in the anisotropy direction shows an oscillatory behaviour with an initially increasing amplitude that later on decreases, but does not fall of to zero. This behaviour with a local maximum in the density gradient at or slightly after the bounce in the bouncing direction seems to be typical. In the perpendicular directions (not depicted here) the gradient for the $k = 0$ case is roughly constant before it also starts growing. For higher $k$ values the oscillations initially have an approximately constant amplitude that with time slowly starts growing.

![Figure 1](image1.png)

**Figure 1.** A radiation background that experiences a bounce. It starts close to $-X$ and ends at $+dS$. Initial values at $t_0 = 1$ are given by $\mu_0 = 0.02$, $\theta_0 = -0.6$, $\Sigma_0 = -0.6$, $\theta_\parallel = -0.8$ and $\theta_\perp = 0.1$.

![Figure 2](image2.png)

**Figure 2.** The growth of the density perturbations $\mathcal{D}_\parallel$ in the background given by figure 1 for the wave numbers $k_\parallel/a_10 = k_\perp/a_20 = 0, 1, 5$ and 20. Initially, at $t_0 = 1$, $\mathcal{D} = \mathcal{P} = 0.001$. ($a_10 = a_1(t_0)$, $a_20 = a_2(t_0)$.)

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