Kishimoto’s Conjugacy Theorems in simple $C^*$-algebras of tracial rank one

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Abstract

Let $A$ be a unital separable simple amenable $C^*$-algebra with finite tracial rank which satisfies the Universal Coefficient Theorem (UCT). Suppose $\alpha$ and $\beta$ are two automorphisms with the Rokhlin property that induce the same action on the $K$-theoretical data of $A$. We show that $\alpha$ and $\beta$ are strongly cocycle conjugate and uniformly approximately conjugate, that is, there exists a sequence of unitaries $\{u_n\} \subset A$ and a sequence of strongly asymptotically inner automorphisms $\sigma_n$ such that

$$\alpha = \text{Ad} u_n \circ \sigma_n \circ \beta \circ \sigma_n^{-1}$$

and that the converse holds. We then give a $K$-theoretic description as to exactly when $\alpha$ and $\beta$ are cocycle conjugate, at least under a mild restriction. Moreover, we show that given any $K$-theoretical data, there exists an automorphism $\alpha$ with the Rokhlin property which has the same $K$-theoretical data.

1 Introduction

Let $A$ be a unital separable simple $C^*$-algebra and let $\text{Aut}(A)$ be the automorphism group of $A$. Kishimoto ([6] and [11]) studied those automorphisms with the Rokhlin property (see definition 2.1 below). Suppose $\alpha, \beta \in \text{Aut}(A)$ have the Rokhlin property and are asymptotically unitarily equivalent. Kishimoto showed, under the assumption that $A$ is a unital simple $\text{AT}$-algebra of real rank zero, $\alpha$ and $\beta$ are cocycle conjugate, i.e., there exists a unitary $u \in A$ and $\sigma \in \text{Aut}(A)$ such that

$$\alpha = \text{Ad} u \circ \sigma \circ \beta \circ \sigma^{-1}.$$ (e 1.1)

Kishimoto also showed that automorphisms with the Rokhlin property are abundant. Kishimoto’s work actually revealed much more. Matui ([16]) found out, using a homotopy lemma, that the above-mentioned cocycle conjugate result of Kishimoto holds for all unital separable amenable $C^*$-algebras with tracial rank zero satisfies the Universal Coefficient Theorem (UCT). Furthermore, combining Kishimoto’s argument with the condition of $\mathcal{Z}$-stability, Sato recently proved that, given any $\alpha \in \text{Aut}(A)$, where $A$ is a unital separable simple $C^*$-algebra with tracial rank zero, there is $\beta \in \text{Aut}(A)$ with the Rokhlin property such that $\alpha$ and $\beta$ are asymptotically unitarily equivalent. This further shows that automorphisms with the Rokhlin property are abundant. We will also mention that Kishimoto’s result also holds for the case that $A$ is a purely infinite simple $C^*$-algebra (see [18]).

We are lead to consider the following questions that this paper attempts to answer: 1) Could Kishimoto’s result work for more general stably finite simple $C^*$-algebras? 2) Is it possible to remove the unitary $u$ in (e 1.1)? Or, is it possible to obtain an approximate conjugacy result? 3) Is there a $KK$-theoretical description of cocycle conjugacy (or approximate conjugacy)?

To answer the first question, we will consider the case that $A$ is a unital separable amenable simple $C^*$-algebra with tracial rank at most one. We further assume that $A$ satisfies the UCT.
These $C^*$-algebras include all unital simple $AT$-algebras, as well as algebras that may not have real rank zero. By a classification result, these $C^*$-algebras are precisely those unital simple $AH$-algebras with slow dimension growth. Kishimoto’s argument works very well in the case that $A$ has real rank zero. However, there is a significant technical problem in generalising Kishimoto’s method to the $C^*$-algebras with infinite exponential length, which is the case for general unital simple AH-algebra with slow dimension growth. Kishimoto’s method relies on something called the Basic Homotopy Lemma (2) which has a controlled length of the homotopy. The control of the length is essential in the argument and plays a very important role. The Basic Homotopy Lemma has been extended into a much more general situation (see [10] and [11]) which implies the case when the $C^*$-algebras are allowed to have tracial rank one instead of zero. However, for that extension, one loses control of the length of the homotopy. This cannot be improved since the exponential length is infinite in unital simple AH-algebras whenever it has real rank other than zero. Nevertheless it has been demonstrated that it is possible to control the exponential length of some special unitaries, namely those unitaries which are not only in the path-connected component of the identity but are also in the closure of the commutator subgroup. In this paper, we first present a very special homotopy lemma for some special unitaries which are in the closure of the commutator subgroup. By proving an existence type of result, we will show that the special homotopy lemma mentioned above would be sufficient if one is willing to pay the toll for a narrow passage into Kishimoto’s method. We will prove that Kishimoto’s conjugacy theorem holds for unital separable amenable simple $C^*$-algebras satisfying the UCT.

For the second question, upon examining Kishimoto’s proof, one realizes that, when $\alpha$ and $\beta$ are asymptotically unitarily equivalent and satisfy the Rokhlin property, $\alpha$ and $\beta$ are cocycle conjugate in a stronger sense: one can choose $\sigma$ to be strongly asymptotically inner. Furthermore, by examining his proof further, one notices that Kishimoto’s argument provided a way to remove $u$, or at least make $u$ as close to the identity as possible, so that a conclusion of being approximately conjugate is possible, if certain obstacles disappeared. We identify the obstacle as the quotient group $K_1(A)/H_1(K_0(A), K_1(A))$ (see 2.9 below). When $H_1(K_0(A), K_1(A)) = K_1(A)$, $\alpha$ and $\beta$ are (strongly) asymptotically unitrily equivalent and satisfy the Rokhlin property implies that $\alpha$ and $\beta$ are not only cocycle conjugate in the strong sense but are also approximately conjugate uniformly. This is our answer to the second question (see 7.4).

Since Kishimoto was primarily concerned with approximately inner automorphisms at the time, he did not provide $K$-theoretical description of cocycle conjugacy. It should be noted that his cocycle conjugacy is not equivalent to asymptotic unitary equivalence even in the class of automorphisms with the Rokhlin property. However, strong cocycle conjugacy is equivalent to asymptotic unitary equivalence which, by recent developments in Elliott’s classification program, can be characterized by $K$-theoretical data (7.3). In the case that $K_1(A) = H_1(K_0(A), K_1(A))$, this also gives a $K$-theoretical description of strong cocycle conjugacy and uniformly approximate conjugacy (see 7.9). To have a better $K$-theoretical description of cocycle conjugacy for automorphisms with the Rokhlin property, we impose some mild restriction (for example, we assume that $K_1(A)/\text{Tor}(K_1(A))$ is free). We then introduce a $K$-theory related group on invariant sets. We show, under the restriction, $\alpha$ and $\beta$ are cocycle conjugate if and only if their $K$-theoretical invariant $\tilde{\alpha}$ and $\tilde{\beta}$ are conjugate (see 7.12).

Briefly, this paper is organized as follows. The next section is a list of definitions and notations that will be used in this paper. In section 3 we present a version of the homotopy lemma which controls the length of the path. Section 4 uses results section 3 to present a proof of Kishimoto’s conjugacy theorem for unital separable amenable simple $C^*$-algebras with tracial rank at most one which satisfy the UCT. Section 5 serves as a preparation for the proof of section 6. In section 6, using results in section 5 and Sato’s refinement of Kishimoto’s argument, we show that, given any automorphism $\alpha$ on a unital separable amenable simple $C^*$-algebra $A$
satisfying the UCT, there exists an automorphism \( \hat{\alpha} \) on \( A \) which has the Rokhlin property and which is strongly asymptotically unitarily equivalent to \( \alpha \). This result is also used in the section that follows. Section 7, the last section, contains the conclusion of the paper. We give the \( K \)-theoretical description of (strong) cocycle conjugacy and uniformly approximate conjugacy for automorphisms with the Rokhlin property on \( A \) mentioned above.

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### 2 Preliminaries

**Definition 2.1.** Let \( A \) be a unital \( C^* \)-algebra. Denote by \( \text{Aut}(A) \) the group of automorphisms of \( A \). Let \( \alpha \in \text{Aut}(A) \). Following Kishimoto (Definition 4.1 of [6]), we say that \( \alpha \) has the Rokhlin property if for any integer \( k \in \mathbb{N} \), any \( \epsilon > 0 \) and any finite subset \( F \subset A \), there exists a family of projections \( \{e_1,0,e_1,1,...,e_{1,k−1},e_2,0,e_2,1,...,e_{2,k}\} \) in \( A \) such that

\[
\sum_{j=0}^{k-1} e_{1,j} + \sum_{j=0}^{k} e_{2,j} = 1, \tag{e 2.2}
\]

\[
\|\alpha(e_{i,j}) - e_{i,j+1}\| < \epsilon, j = 0,1,...,k + i - 1, i = 0,1, \tag{e 2.3}
\]

\[
\|[e_{i,j},x]\| < \epsilon, j = 0,1,...,k + i - 1, i = 0,1. \tag{e 2.4}
\]

As pointed out by Kishimoto, one has

\[
\|\alpha(e_{1,k−1} + e_{2,k}) - (e_{1,0} + e_{2,0})\| < (2k−1)\epsilon. \tag{e 2.5}
\]

The set of those elements in \( \text{Aut}(A) \) with the Rokhlin property will be denoted by \( \text{Aut}_R(A) \).

**Definition 2.2.** We say \( \alpha \) and \( \beta \) are asymptotically unitarily equivalent, if there exists a continuous path of unitaries \( \{u(t) : t \in [0, \infty)\} \subset A \) such that

\[
\alpha(x) = \lim_{t \to \infty} u(t)^* \beta(x) u(t) \quad \text{for all} \quad x \in A. \tag{e 2.6}
\]

We say that \( \alpha \) and \( \beta \) are strongly asymptotically unitarily equivalent if, in addition, \( u(1) = 1_A \) in (e 2.6).

We say that \( \alpha \) is (strongly) asymptotically inner if \( \alpha \) is (strongly) asymptotically unitarily equivalent to \( \text{id}_A \).

**Definition 2.3.** Let \( \alpha \) and \( \beta \) be two automorphisms on \( A \). Automorphisms \( \alpha \) and \( \beta \) are cocycle conjugate if there exists an automorphism \( \sigma \in \text{Aut}(A) \) and a unitary \( u \in A \) such that

\[
\alpha = \text{Ad} \ u \circ \sigma^{-1} \circ \beta \circ \sigma.
\]

Cocycle conjugacy is an equivalent relation. Denote by \( \text{Aut}_R(A)/\sim_{cc} \) the cocycle conjugate classes of automorphisms in \( \text{Aut}_R(A) \).

We say \( \alpha \) and \( \beta \) are strongly cocycle conjugate if they are cocycle conjugate with \( \sigma \) being strongly asymptotically inner. Denote by \( \text{Aut}_R(A)/\sim_{scc} \) the strongly cocycle conjugacy classes of automorphisms in \( \text{Aut}_R(A) \).

**Definition 2.4.** Two automorphisms \( \alpha, \beta \in \text{Aut}(A) \) are said to be cocycle conjugate and uniformly approximately conjugate if there exists a sequence of a unitaries \( u_n \in U(A) \) and a sequence of automorphisms \( \sigma_n \in \text{Aut}(A) \) such that

\[
\alpha = \text{Ad} \ u_n \circ \sigma_n^{-1} \circ \beta \circ \sigma_n \quad \text{and} \quad \lim_{n \to \infty} \|u_n − 1\| = 0. \tag{e 2.7}
\]
Note, in this case,
\[
\lim_{n \to \infty} \|\alpha - \sigma_n \circ \beta \circ \sigma_n^{-1}\| = 0. \tag{e 2.8}
\]

Two automorphisms \(\alpha, \beta \in \text{Aut}(A)\) are said be strongly cocycle conjugate and uniformly approximately conjugate if there exists a sequence of unitaries \(u_n \in U(A)\) and a sequence of strongly asymptotically inner automorphisms \(\sigma_n \in \text{Aut}(A)\) such that
\[
\alpha = \text{Ad} \; u_n \circ \sigma_n^{-1} \circ \beta \circ \sigma_n \quad \text{and} \quad \lim_{n \to \infty} \|u_n - 1\| = 0. \tag{e 2.9}
\]

Denote by \(\text{Aut}_R(A)/\sim_{\text{aucc}}\) the cocycle conjugate and uniformly approximately conjugate classes of automorphisms in \(\text{Aut}_R(A)\), and denote by \(\text{Aut}_R(A)/\sim_{\text{sauc}}\) the strongly cocycle conjugate and uniformly approximately conjugate classes of automorphisms.

**Definition 2.6.** Let \(A\) be a unital \(C^*\)-algebra. Denote by \(T(A)\) the tracial state space of \(A\). Denote by \(\rho_A : K_0(A) \to \text{Aff}(T(A))\) the homomorphism defined by \(\rho_A([p])(\tau) = \tau(p)\) for all projections \(p \in M_\infty(A)\) and for all \(\tau \in T(A)\).

Denote by \(U(A)\) the unitary group and \(U_0(A)\) the normal subgroup of \(U(A)\) which consists of those unitaries in the path connected component containing the identity. Denote by \(DU(A)\) the commutator subgroup of \(U(A)\) and by \(CU(A)\) the closure of \(DU(A)\). Let \(u \in U_0(A)\) and choose a piece-wise smooth and continuous path \(\{u(t) : t \in [0, 1]\}\) of unitaries in \(A\) with \(u(0) = u\) and \(u(1) = 1_A\). Denote by
\[
\Delta(u)(\tau) = \int_0^1 \tau\left(\frac{du(t)}{dt}\right)^* u(t)^* dt \quad \text{for all} \quad \tau \in T(A).
\]

the de la Harpe and Skandalis determinant. We use \(\Delta : U_0(A)/DU(A) \to \text{Aff}(T(A))/\ker \rho_A(K_0(A))\) for the induced isomorphism. We will use \(\overline{\Delta} : U_0(A)/CU(A) \to \text{Aff}(T(A))/\rho_A(K_0(A))\) for the induced isomorphism. Note that, when \(A\) has stable rank one, \(\Delta\) and \(\overline{\Delta}\) are isomorphisms. In this case, K. Thomsen provided the following splitting short exact sequence:
\[
0 \to \text{Aff}(T(A))/\rho_A(K_0(A)) \xrightarrow{\overline{\Delta}} U(A)/CU(A) \xrightarrow{\pi_1} K_1(A) \to 0, \tag{e 2.10}
\]
where \(\pi_1 : U(A)/CU(A) \to U(A)/U_0(A) \cong K_1(A)\) is the quotient map and \(s_1\) is a fixed splitting homomorphism. Please note that \(\pi_1\) and \(s_1\) will be used later.

**Definition 2.6.** Let \(A\) be a unital \(C^*\)-algebra and let \(A^1\) be the unit ball of \(A\). Denote by \(A^q\) the image of \(A_{s,a}\) of the map given by \(a \mapsto \hat{a}\), where \(\hat{a}(\tau) = \tau(a)\) for all \(\tau \in T(A)\) and for all \(a \in A_{s,a}\). Denote by \(A^{q,1}\) the image of \(A^1\) in \(A^q\) and by \(A^{q,1}_+\) the image of \(A^1_+\) in \(A^q\), respectively.

**Definition 2.7.** Let \(a \in A\) be an element such that \(\|a^* a - 1\| < 1/8\) and \(\|aa^* - 1\| < 1/8\). Then \(a(a^* a)^{-1/2}\) is a unitary in \(A\). Set \(\langle a \rangle = a(a^* a)^{-1/2}\). Note that \(\|\langle a \rangle - a\| < 1\). Moreover, \(\|\langle a \rangle - a\|\) is small if \(\|a^* a - 1\|\) is small (which does not depend on \(A\) or \(a\)). We will use this notation through the paper.

**Definition 2.8.** Throughout this paper \(\mathcal{Z}\) is the Jiang-Su algebra which is a unital projectionless simple ASH-algebra with a unique tracial state (\(\mathcal{T}\)). \(\mathcal{Z}\) is a strongly self-absorbing algebra. So \(\otimes_{n \in \mathbb{Z}} \mathcal{Z} \cong \mathcal{Z}\). Denote by \(\sigma_0 : \otimes_{n \in \mathbb{Z}} \mathcal{Z} \to \otimes_{n \in \mathbb{Z}} \mathcal{Z}\) the shift, i.e., \(\sigma_0(\cdots \otimes a_{-1} \otimes a_0 \otimes a_1 \otimes \cdots) = (\cdots \otimes a_{-2} \otimes a_{-1} \otimes a_0 \otimes \cdots)\) for all \((\cdots \otimes a_{-1} \otimes a_0 \otimes a_1 \otimes \cdots) \in \otimes_{n \in \mathbb{Z}} \mathcal{Z}\). By identifying \(\otimes_{n \in \mathbb{Z}} \mathcal{Z}\) with \(\mathcal{Z}\), we view \(\sigma_0\) as an automorphism on \(\mathcal{Z}\).

We then write \(\mathcal{Z} = \otimes_{n \in \mathbb{N}} \mathcal{Z}\) and denote by \(\sigma\) the \(\otimes_{n \in \mathbb{N}} \sigma_0\).

It is a strongly asymptotically inner automorphism.
Definition 2.9. Let $A$ be a unital $C^*$-algebra. Recall that
\[ H_1(K_0(A), K_1(A)) = \{ x \in K_1(A) : \psi([1_A]) = x \text{ for some } \psi \in \text{Hom}(K_0(A), K_1(A)) \}. \]
If $K_0(A) = \mathbb{Z} \cdot [1_A] \oplus G$ for some abelian group $G$, then $H_1(K_0(A), K_1(A)) = K_1(A)$. If $K_1(A)$ is divisible, then $H_1(K_0(A), K_1(A)) = K_1(A)$.

3 Homotopy Lemmas with controlling of length

This section contains some refinement of homotopy lemmas in [15].

Lemma 3.1. Let $C = PM_r(C(X))P$ for some compact metrizable space $X$, and let $\nabla : (0, 1) \to (0, 1)$ be a non-decreasing function and $\eta > 0$ such that
\[ \mu_{\tau \circ \varphi}(O_a) > \nabla(a) \text{ for all } \tau \in T(A) \]
and for any open ball $O_a$ of $X$ with radius $a > \eta$.

Let $\mathcal{F} \subseteq C$, $\mathcal{G}' \subseteq C \otimes C(\mathbb{T})$, $\mathcal{H} \subseteq C \otimes C(\mathbb{T})$ be finite subsets, and let $\epsilon > 0$. Then there are $\delta > 0$ and a finite subset $\mathcal{G} \subseteq C$ such that for any $C^*$-algebra $A$ which is tracially approximately divisible, any homomorphism $\varphi : C \to A$, any unitary $u \in A$ with
\[ \|[(\varphi(c), u)]\| < \delta \quad \forall c \in \mathcal{G}, \]
there exist unitaries $w_1, w_2 \in A$, a path of unitaries \{w(t); t \in [0, 1]\} $\subset A$ with $w(0) = 1$ and $w(1) = w_1 w_2 w_1^* w_2^* =: w$, and a completely positive $\mathcal{G}'$-$\epsilon$-multiplicative linear maps $L_1, L_2 : C \otimes C(\mathbb{T}) \to A$ such that
\[
\begin{align*}
\|w_i, \varphi(a)\| &< \epsilon \text{ for all } a \in \mathcal{F}, \quad i = 1, 2, \quad (e.3.11) \\
\|w(t), u\| &< \epsilon, \quad \|w(t), \varphi(a)\| < \epsilon, \quad \text{for all } a \in \mathcal{F}, \quad \text{and } t \in [0, 1], \quad (e.3.12) \\
\|L_1(a \otimes z) - (\varphi(a)uw)\| &< \epsilon, \quad \|L_1(a \otimes 1) - \varphi(a)\| < \epsilon, \quad \text{for all } a \in \mathcal{F}, \quad (e.3.13) \\
\|L_2(a \otimes z) - (\varphi(a)w)\| &< \epsilon, \quad \|L_2(a \otimes 1) - \varphi(a)\| < \epsilon, \quad \text{for all } a \in \mathcal{F}, \quad (e.3.14) \\
|\tau \circ L_1(g) - \tau \circ L_2(g)| &< \epsilon, \quad \text{for all } g \in \mathcal{H}, \quad \text{for all } \tau \in T(A), \quad (e.3.15)
\end{align*}
\]
and
\[ \mu_{\tau \circ L_i}(B_a) > \nabla_0(a), \quad i = 1, 2, \quad \text{for all } \tau \in T(A) \]
and for any open ball $B_a$ of $X \times \mathbb{T}$ with radius $a > 3\sqrt{2}\eta$.

Moreover,
\[ \text{length}\{w(t)\} \leq \pi. \quad (e.3.16) \]

Furthermore, one may require that $w(t) \in CU(A)$ for all $t \in [0, 1]$ but with
\[ \text{length}\{w(t)\} \leq 2\pi. \quad (e.3.17) \]

Proof. This lemma is the same statement as that of Lemma 3.7 of [15] except the last part beginning with “Furthermore”. So the proof is exactly the same but we need to justify \(e.3.16\) (as well as \(e.3.17\)) above.

Note that in the proof of Lemma 3.7 of [15],
\[ \varphi(\mathcal{F} \cup \mathcal{F}_1 \cup \mathcal{G} \cup \mathcal{G}) \in \epsilon'' B' \cap A \text{ and } u \in \epsilon'' B' \cap A. \]
Moreover, \( w'(t) \in B \) for all \( t \in [0,1] \) and \( B \cong M_k \) for some integer \( k \geq 1 \). Therefore the length of \( \{w'(t)\} \) could be made no more than \( \pi \). Since \( w(t) = (1 - p) + w'(t) \), as in the proof of Lemma 3.7 of \([15]\), the length of \( \{w(t)\} \) could be controlled by \( \pi \).

Note that \( w(1) = w \in CU(B) \). We may write \( w = \exp(\i h) \) with \( h \in B \), \( \|h\| \leq 2\pi \) and \( \tau(h) = 0 \) for all \( \tau \in T(B) \). One can then choose \( w'(t) = \exp(\i(1 - t)h) \) for \( t \in [0,1] \). Since \( \tau(th) = t\tau(h) = 0 \) for all \( t \in [0,1] \), \( w'(t) \in CU(B) \subseteq CU(A) \) for all \( t \in [0,1] \).

**Theorem 3.2.** Let \( C = C(X) \) with \( X \) a compact metric space and let \( \nabla : (0,1) \to (0,1) \) be a non-decreasing map. For any \( \epsilon > 0 \) and any finite subset \( F \subseteq C \), there exists \( \delta > 0 \), \( \eta > 0 \), \( \gamma > 0 \), a finite subsets \( G \subseteq C \), \( P \subseteq \overline{K}(C) \), a finite subset \( Q = \{x_1, x_2, ..., x_m\} \subseteq K_0(C) \) which generates a free subgroup and \( x_i = [p_i] - [q_i] \), where \( p_i, q_i \in M_n(C) \) (for some integer \( n \geq 1 \)) are projections, satisfying the following:

Suppose that \( A \) is a unital simple \( C^* \)-algebra with \( TR(A) \leq 1 \), \( \varphi : C \to A \) is a unital homomorphism and \( u \in A \) is a unitary, and suppose that

\[
\|\varphi(c), u\| < \delta, \quad \text{for all } c \in G \quad \text{and} \quad \Bott(\varphi, u)\|P = 0,
\]

\[
\mu_{\tau\varphi}(O_a) \geq \nabla(a) \quad \text{for all } \tau \in T(A),
\]

where \( O_a \) is any open ball in \( X \) with radius \( \eta \leq a < 1 \) and \( \mu_{\tau\varphi} \) is the Borel probability measure defined by \( \tau \circ \varphi \), and, for each \( 1 \leq i \leq m \), there is \( v_i \in CU(M_n(A)) \) such that

\[
\|((1_n - \varphi(p_i) + \varphi(q_i)1_n \otimes u)(1_n - \varphi(q_i) + \varphi(q_i)(1_n \otimes u^*))) - v_i\| < \gamma. \tag{e.3.18}
\]

Then there is a continuous path of unitaries \( \{u(t) : t \in [0,1]\} \) in \( A \) such that

\[
u(0) = u, u(1) = 1, \quad \text{and} \quad \|\varphi(c), u(t)\| < \epsilon
\]

for any \( c \in F \) and for any \( t \in [0,1] \).

Moreover,

\[
\text{length}(\{u(t)\}) \leq 2\pi + \epsilon. \tag{e.3.19}
\]

One can also require that

\[
dist(u(t), CU(A)) < \epsilon \tag{e.3.20}
\]

with \( \text{cel}(\{u(t)\}) \leq 4\pi + \epsilon \).

**Proof.** Note that, by Lemma 3.1 of this paper, in the proof of Theorem 3.9 of \([15]\), the length of \( \{w'(t)\} \) is at most \( \pi \). Thus the length of \( \{w(t) : t \in [1/2,1]\} \) in the proof of 3.9 of \([15]\) is at most \( \pi \) and the length of \( \{w(t) : t \in [0,1/4]\} \) in the proof of Theorem 3.9 of \([15]\) is at most \( \pi \). We also note that the length of \( \{w(t) : t \in [1/4,1/2]\} \) is no more than

\[
2 \arcsin(\epsilon_0/2).
\]

Note that, at the beginning of the proof of Theorem 3.9 of \([15]\), we assume \( \epsilon_0 < \epsilon/2 \). This will imply that \( 2 \arcsin(\epsilon_0/2) < \epsilon \) for \( 0 < \epsilon_0 < \pi/4 \). But we certainly can choose \( \epsilon_0 \) so that \( 2 \arcsin(\epsilon_0/2) < \epsilon \) at the proof of 3.9 of \([15]\). So the total length of \( \{w(t) : t \in [0,1]\} \) is at most \( 2\pi + \epsilon \) (for any given \( \epsilon \)).

Note that, in the proof of Theorem 3.9 of \([15]\), as in the proof of Lemma 3.1, \( \{w'(t)\} \) can be chosen to be in \( CU(A) \) with length at most \( 2\pi \). Thus \( w(t) \in CU(A) \) for all \( t \in [1/2,1] \) and \( w(t) \in CU(A) \) for \( t \in [0,1/4] \). However, the length of \( \{w(t) : t \in [1/2,1]\} \) and the length of \( \{w(t) : t \in [0,1/4]\} \) will be both bounded by \( 2\pi \). It follows from this and the above estimates that (e.3.19) and (e.3.20) hold.\[\square\]
**Definition 3.3.** Let $X$ be a compact metric space and $P \in M_r(C(X))$ be a projection, where $r \geq 1$ is an integer. Put $C = PM_r(C(X))P$ and suppose $\tau \in T(C)$. It is known that there exists a probability measure $\mu_\tau$ on $X$ such that

$$
\tau(f) = \int_X t_x(f(x))d\mu_\tau(x),
$$

where $t_x$ is the normalized trace on $P(x)M_rP(x)$ for all $x \in X$.

**Remark 3.4.** Regard $C(X)$ as the center of $C = PM_r(C(X))P$, and denote by $\iota : C(X) \to C$ the embedding. Then the measure $\mu_\tau$ is in fact induced by the trace $\tau \circ \iota$ on $C(X)$.

**Corollary 3.5.** Let $C = PM_r(C(X))P$, where $X$ is a compact subset of a finite CW complex, $r \geq 1$ is an integer and $P \in M_r(C(X))$ is a projection. Then Theorem 3.2 holds for $C$ using the measure $\mu_\tau$ as in 3.3.

**Proof.** We will use 3.2 and the proof of Theorem 3.10 in [15].

The following is an easy fact.

**Proposition 3.6.** Let $X$ be a compact metric space and $C = PM_r(C(X))P$, where $r \geq 1$ is an integer and $P \in M_r(C(X))$ is a projection. Let $\nabla_0 : C^{n+1}_r \{0\} \to (0,1)$ be a positive map. Then there is a non-decreasing function $\nabla : (0,1) \to (0,1)$ such that, for any $\gamma > 0$, there is a finite subset $H \subset C^1_+ \{0\}$ satisfying the following: If $\varphi : C \to A$ is a unital injective homomorphism, where $A$ is a unital simple $C^*$-algebra with $T(A) \neq \emptyset$, such that

$$
\tau \circ \varphi(h) \geq \nabla_0(\hat{h}) \text{ for all } \tau \in T(A)
$$

and for all $h \in H$, then

$$
\mu_{\tau \circ \varphi}(O_r) \geq \nabla(r),
$$

for all $\tau \in T(A)$ and for all $r \geq \gamma$.

The next is a restatement of 3.5 using 3.6.

**Corollary 3.7.** Let $X$ be a compact subset of a finite CW complex and let $C = PM_r(C(X))P$, where $r \geq 1$ is an integer and $P \in M_r(C(X))$ is a projection. Let $\nabla_0 : C^{n+1}_r \{0\} \to (0,1)$ be a positive map. For any $\gamma > 0$ and any finite subset $F \subset C$, there exists $\delta > 0$, $\gamma > 0$, a finite subsets $\mathcal{G} \subset C$, $\mathcal{P} \subset \mathcal{K}(C)$, a finite subset $Q = \{x_1, x_2, ..., x_m\} \subset K_0(C)$ which generates a free subgroup and $x_i = |p_i| - |q_i|$, where $p_i, q_i \in M_n(C)$ (for some integer $n \geq 1$) are projections and there exists a finite subset $H \subset C^1_+ \{0\}$ satisfying the following:

Suppose that $A$ is a unital simple $C^*$-algebra with $TR(A) \leq 1$, $\varphi : C \to A$ is a unital homomorphism and $u \in A$ is a unitary, and suppose that

$$
||\varphi(c), u|| < \delta, \text{ for all } c \in \mathcal{G} \quad \text{and} \quad \text{Bott}(\varphi, u)_p = 0,
$$

$$
\tau \circ \varphi(h) \geq \nabla_0(\hat{h}) \text{ for all } \tau \in T(A)
$$

and for all $h \in H$, and, for each $1 \leq i \leq m$, there is $v_i \in CU(M_n(A))$ such that

$$
||\left(1_n - \varphi(p_i) + \varphi(p_i)1_n \otimes u\right)\left(1_n - \varphi(q_i) + \varphi(q_i)1_n \otimes u^*\right) - v_i|| < \gamma, \quad \text{(e.3.21)}
$$

Then there is a continuous path of unitaries $\{u(t) : t \in [0,1]\}$ in $A$ such that

$$u(0) = u, u(1) = 1, \text{ and } ||\varphi(c), u(t)|| < \epsilon$$
for any $c \in \mathcal{F}$ and for any $t \in [0,1]$.

Moreover,

$$\text{length}(\{u(t)\}) \leq 2\pi + \epsilon. \quad (e\,3.22)$$

One can also require that

$$\text{dist}(u(t), \text{CU}(A)) < \epsilon \quad (e\,3.23)$$

with $\text{cel}(\{u(t)\}) \leq 4\pi + \epsilon$.

The following follows immediately from 3.7.

**Theorem 3.8.** Let $C$ be a unital AH-algebra and let $\nabla : C_{+}^{0,1} \setminus \{0\} \to (0,1)$ be a non-decreasing map. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subseteq C$, there exists $\delta > 0$, $\gamma > 0$, a finite subsets $\mathcal{G} \subseteq C$, $\mathcal{P} \subseteq K(C)$, a finite subset $\mathcal{Q} = \{x_{1}, x_{2}, \ldots, x_{m}\} \subseteq K_{0}(C)$ which generates a free subgroup and $x_{i} = [p_{i}] - [q_{i}]$, where $p_{i}, q_{i} \in M_{n}(C)$ (for some integer $n \geq 1$) are projections, and a finite subset $\mathcal{H} \subset C_{+}^{1} \subset \{0\}$ satisfying the following: Suppose that $A$ is a unital simple C*-algebra with $\text{TR}(A) \leq 1$, $\varphi : C \to A$ is a unital homomorphism and $u \in A$ is a unitary, and suppose that

$$\|\varphi(c), u\| < \delta, \ \forall c \in \mathcal{G} \text{ and } \text{Bott}(\varphi, u)|_{\mathcal{P}} = 0,$$

$$\tau \circ \varphi(h) \geq \nabla(\hat{h})\tau \in T(A) \quad (e\,3.24)$$

and all $h \in \mathcal{H}$, and, for each $1 \leq i \leq m$, there is $v_{i} \in \text{CU}(M_{n}(A))$ such that

$$\|(1_{n} - \varphi(p_{i}) + \varphi(p_{i})1_{n} \otimes u)(1_{n} - \varphi(q_{i}) + \varphi(q_{i})(1_{n} \otimes u^{*}) - v_{i}\| < \gamma. \quad (e\,3.25)$$

Then there is a continuous path of unitaries $\{u(t) : t \in [0,1]\}$ in $A$ such that

$$u(0) = u, u(1) = 1 \text{ and } \|\varphi(c), u(t)\| < \epsilon$$

for any $c \in \mathcal{F}$ and for any $t \in [0,1]$.

Moreover,

$$\text{length}(\{u(t)\}) \leq 2\pi + \epsilon. \quad (e\,3.26)$$

One can also require that

$$\text{dist}(u(t), \text{CU}(A)) < \epsilon \text{ for all } t \in [0,1] \quad (e\,3.27)$$

with $\text{cel}(\{u(t)\}) \leq 4\pi + \epsilon$.

**Lemma 3.9.** Let $A$ be a unital separable simple amenable C*-algebra with $\text{TR}(A) \leq 1$. There is a positive map $\nabla : A_{+}^{0,1} \setminus \{0\} \to (0,1)$ satisfies the following: For any unital homomorphism $\varphi : A \to B$, where $B$ is a unital separable C*-algebra with $T(B) \neq \emptyset$,

$$\tau \circ \varphi(h) \geq \nabla(\hat{h}) \text{ for all } \tau \in T(A)$$

and for all $h \in A_{+} \setminus \{0\}$. 

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Proof. Let \( h \in A_+^1 \setminus \{0\} \). Define
\[
\hat{\varphi}(h) = \inf_{\tau \in T(A)} \tau(h).
\]
Then, since \( A \) is simple, \( \hat{\varphi}(h) > 0 \). Now let \( \varphi : A \to B \) be any unital homomorphism, \( \varphi : A \to B \). If \( \tau \in T(B) \), the \( \tau \circ \varphi \in T(A) \). It follows that
\[
\tau \circ \varphi(h) \geq \hat{\varphi}(h) \text{ for all } \tau \in T(B).
\]

\[\square\]

**Theorem 3.10.** Let \( C \) be a unital separable simple amenable \( C^* \)-algebra with \( TR(C) \leq 1 \) which satisfies the UCT. For any \( \varepsilon > 0 \) and any finite subset \( \mathcal{F} \subseteq C \), there exists \( \delta > 0 \), \( \gamma > 0 \), a finite subsets \( \mathcal{G} \subseteq C \), \( \mathcal{P} \subseteq K(C) \), a finite subset of projections \( p_1, p_2, \ldots, p_m \in C \) satisfying the following:

Suppose that \( A \) is a unital simple \( C^* \)-algebra with \( TR(A) \leq 1 \), \( \varphi : C \to A \) is a unital homomorphism and \( u \in A \) is a unitary, and suppose that
\[
\|\|\varphi(c), u\|\| < \delta, \quad \forall c \in \mathcal{G} \quad \text{and} \quad \text{Bott}(\varphi, u)_{\mathcal{P}} = 0,
\]
and for each \( 1 \leq i \leq m \), there is \( v_i \in CU(A) \) such that
\[
\|\|1 - \varphi(p_i) + \varphi(p_i)u - v_i\|\| < \gamma, \quad i = 1, 2, \ldots, m.
\]
Then there is a continuous path of unitaries \( \{u(t) : t \in [0, 1]\} \) in \( A \) such that
\[
u(0) = u, u(1) = 1, \quad \text{and} \quad \|\|\varphi(c), u(t)\|\| < \varepsilon
\]
for any \( c \in \mathcal{F} \) and for any \( t \in [0, 1] \).

Moreover,
\[
\text{length}(\{u(t)\}) \leq 2\pi + \varepsilon.
\]

One can also require that
\[
\text{dist}(u(t), CU(A)) < \varepsilon \quad \text{for all } t \in [0, 1]
\]
with \( \text{cel}(\{u(t)\}) \leq 4\pi + \varepsilon \).

**Proof.** This follows from \( \text{3.8} \). In fact, it follows from the classification result in \( \text{3.8} \) that \( C \) is isomorphic to a unital simple AH-algebra with no dimension growth. Note \( C \) is simple. So \( \text{3.9} \) applies. In other words, \( \nu \) can be given and \( \text{3.21} \) applies. \( \square \)

**Remark 3.11.** Note that all paths \( \{u(t)\}, \{w(t)\} \) in this section can be made not only continuous but also piecewise smooth. Furthermore, if \( \text{length}(\{u(t) : t \in [0, 1]\}) \leq C \), then we can also assume that
\[
\|u(t) - u(t')\| \leq C |t - t'| \quad \text{for all } t, t' \in [0, 1].
\]

The \( C^* \)-algebra \( C \) below is a unital AH-algebra with no dimension growth which has stable rank one by the classification theorem. Thus the next follows immediately from Theorem 3.13 in [15].

**Theorem 3.12.** Let \( C \) be a unital separable amenable simple \( C^* \)-algebra with \( TR(C) \leq 1 \) which satisfies the UCT and let \( G \subseteq K_0(C) \) be a finitely generated subgroup generated by projections \( p_1, p_2, \ldots, p_n \in C \). Let \( A \) be a unital separable simple \( C^* \)-algebra with \( TR(A) \leq 1 \). Suppose \( \varphi : C \to A \) is a unital homomorphism. Then, for any \( \varepsilon > 0 \), any \( \gamma > 0 \), any finite subset \( \mathcal{F} \subseteq C \), any finite subset \( \mathcal{P} \subseteq K(C) \), and any homomorphism \( \Gamma : G \to U_0(A)/CU(A) \), there is a unitary \( u \in U(A) \) such that
\[
\|\|\varphi(x), u\|\| < \varepsilon \quad \text{for all } x \in \mathcal{F}, \quad \text{Bott}(\varphi, u)_{\mathcal{P}} = 0 \quad \text{and} \quad \text{dist}(\{(1 - \varphi(p_i) + \varphi(p_i)u), \Gamma(|p_i|)\}) < \gamma, \quad 1 \leq i \leq n.
\]
4 Kishimoto’s conjugacy theorem

The following serves as an alternative for Kishimoto’s stability theorem. The proof uses the special homotopy lemmas in the previous section and a construction due to Kishimoto.

Lemma 4.1. Let $A$ be a unital separable amenable $C^*$-algebra with $\text{TR}(A) \leq 1$ and let $\alpha \in \text{Aut}(A)$ be an automorphism with the Rokhlin property. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists $\delta > 0$ and $\lambda > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{P} \subset K(A)$, and a finite subset of projections $\{p_1, p_2, ..., p_m\} \subset \mathcal{A}$ satisfying the following: If $u \in U(A)$ such that

$$\|a, u\| < \delta \text{ for all } a \in \mathcal{G}, \, \text{Bott}(id_A, u)\mathcal{P} = 0 \text{ and}$$

$$\text{dist}(\{(1 - p_i) + p_iu\}, \text{CU}(A)) < \lambda, \quad i = 1, 2, ..., m,$$

then there is a continuous path of unitaries $\{V(t) : t \in [0, 1]\} \subset \mathcal{A}$ such that $V(1) = 1$,

$$\|u - V(0)\alpha(V(0)^*)\| < \epsilon \text{ and } \|b, V(t)\| < \epsilon \text{ for all } b \in \mathcal{F}$$

and for all $t \in [0, 1]$, and

$$\|V(t) - V(t')\| \leq (4\pi + 1)|t - t'| \text{ for all } t, t' \in [0, 1].$$

Moreover, a version for $\mathcal{F} = \emptyset$ holds as follows: For any $\epsilon > 0$ and $u \in \text{CU}(A)$, there exists a continuous path of unitaries $\{U(t) : t \in [0, 1]\} \subset U(A)$ such that $U(1) = 1$,

$$\|u - U(0)\alpha(U(0)^*)\| < \epsilon \text{ and } \|U(t) - U(t')\| \leq (4\pi + 1)|t - t'|$$

for all $t, t' \in [0, 1]$.

Proof. Choose $N \geq 1$ such that $\pi/(N - 1) < \epsilon/(2^{10} \cdot 5)$. Let $\mathcal{F}_1 = \bigcup_{j = -N+1}^{N+1} \alpha^j(\mathcal{F})$. Let $\epsilon > 0$. Let $\delta_1 > 0$ (in place of $\delta$), $\lambda_1 > 0$ (in place of $\lambda$), $\mathcal{G}_1 \subset A$ (in place of $\mathcal{G}$) be a finite subset and $\mathcal{P}_1 \subset K(A)$ be a finite subset and $\{p_1, p_2, ..., p_m\} \subset \mathcal{A}$ be a finite subset of projections required by for $\eta = \epsilon/2^{10}(N + 1)^2$ (in place of $\epsilon$) and $\mathcal{F}_1$ (in place of $\mathcal{F}$). Let $\mathcal{G} = \bigcup_{j = -N+1}^{N+1} \alpha^j(\mathcal{G}_1)$, $\mathcal{P} = \bigcup_{j = -N+1}^{N+1} (\alpha^j(\mathcal{P}_1))$ and $\{q_1, q_2, ..., q_K\}$, where $\{q_1, q_2, ..., q_K\}$ is a finite subset of projections in $A$ which includes $\{p_i, \alpha^j(p_i) : i = 1, 2, ..., m \text{ and } j = \pm 1, \pm 2, ..., \pm N\}$ as well as $1_A$. We may assume that $\mathcal{F}_1 \subset \mathcal{G}$. Let $\delta = \frac{\min\{\epsilon, \delta_1\}}{2^{10}(N + 1)^2}$ and $\lambda = \lambda_1/2^{10}$.

Now suppose that $u \in U(A)$ which satisfies the assumption for the above $\delta$, $\lambda$, $\mathcal{G}$, $\mathcal{P}$ and $\{p_1, p_2, ..., p_m\}$. Define

$$u_0 = 1, \quad u_1 = u, \quad u_k = u\alpha(u) \cdots \alpha^{k-1}(u), \quad k = 2, ..., N + 1.$$  

Then, we estimate, for $x \in \mathcal{G}_1$,

$$\|[u_k, x]\| < \frac{\min\{\epsilon, \delta_1\}}{2^{10}(N + 1)^2}. \quad (e 4.39)$$

Note that, by the assumption,

$$\text{Bott}(id_A, u_k)|\mathcal{P}_1 = 0. \quad (e 4.40)$$

Moreover, we note that

$$\langle (1 - p_i) + p_i\alpha^j(u) \rangle = \langle (1 - \alpha^j(\alpha^{-j}(p_i))) + \alpha^j(\alpha^{-j}(p_i)u) \rangle$$

$$= \langle \alpha^j(1 - \alpha^{-j}(p_i)) + \alpha^{-j}(p_i)v \rangle.$$  

(e 4.41)
Since $\alpha$ is an automorphism, $\alpha(CU(A)) = CU(A)$. It follows that
\[
\text{dist}(((1 - p_i) + p_i \alpha^j(u))CU(A)) < \lambda, \ i = 1, 2, ..., m \ and \ j = 0, 1, ..., N + 1 \quad (e 4.43)
\]
Therefore,
\[
\text{dist}(((1 - p_i) + p_i u_k)CU(A)) < \lambda_1, \ i = 1, 2, ..., m \ and \ k = 0, 1, 2, ..., N + 1. \quad (e 4.44)
\]
By applying (3.10), we obtain piecewise smooth and continuous paths of unitaries $\{u_j(t) : t \in [0,1]\}$ of $A$ such that $u_j(1) = u_j, u_j(0) = 1_A$.
\[
||u_j(t), x|| < \eta \ for \ all \ x \in \bigcup_{j=-N-1}^{N+1} \mathcal{F} \ and \ for \ all \ t \in [0,1],
\]
\[
||u_j(t) - u_j(t')|| \leq (4\pi + 1)|t - t'| \quad (e 4.46)
\]
for all $t, t' \in [0,1], j = 1, 2, ..., N + 1$. Hence
\[
||\alpha^k(u_j(t))x - x\alpha^k(u_j(t))|| = ||\alpha^k(u_j(t))\alpha^{-k}(x) - \alpha^{-k}(x)u_j(t)||
\]
\[
= ||u_j(t)\alpha^{-k}(x) - \alpha^{-k}(x)u_j(t)|| < \eta \quad (e 4.47)
\]
\[
||\alpha^k(v(t))x - x\alpha^k(v(t))|| < \eta \ and \ ||\alpha^k(w(t))x - x\alpha^k(w(t))|| < \eta \quad (e 4.50)
\]
for all $x \in \mathcal{F}, t \in [0,1]$ and $k = 0, \pm 1, \pm 2, ..., \pm N$, and $j = 1, 2, ..., N + 1$. Set $v(t) = u_N(t)$ and $w(t) = u_{N+1}(t)$ for $t \in [0,1]$. Thus,
\[
v_j = v(j/(N-1)), \ j = 0, 1, ..., N - 1 \ and \ w_j = w(j/N), \ j = 0, 1, ..., N. \quad (e 4.51)
\]
Define $v_j(t) = v\left(\frac{j(1-t)}{N-1}\right), \ j = 0, 1, ..., N - 1$, and $w_j(t) = w\left(\frac{j(1-t)}{N}\right), \ j = 0, 1, ..., N$ for $t \in [0,1]$.
There is an integer $N_1 \geq 1$ and $0 = t_0 < t_1 < \cdots < t_{N_1} = 1$ such that
\[
||u_k(t) - u_k(t)|| < \epsilon/2^{13}(N + 1)^2, \ k = 1, 2, ..., N, \quad (e 4.52)
\]
\[
||v_j(t) - v_j(t)|| < \epsilon/2^{13}(N + 1)^2, \ j = 0, 1, ..., N - 1 \ and \quad (e 4.53)
\]
\[
||w_j(t) - w_j(t)|| < \epsilon/2^{13}(N + 1)^2, \ j = 0, 1, ..., N, \quad (e 4.54)
\]
for all $t \in [t_{i-1}, t_i], i = 1, 2, ..., N_1$. Put
\[
\mathcal{W} = \bigcup_{i=0}^{N_1}\{u_k(t_i) : 1 \leq k \leq N\} \cup \{v_j(t_i) : 0 \leq j \leq N - 1\} \cup \{w_j(t_i) : 0 \leq j \leq N\}.
\]
Let $\{e_{1,0}, e_{1,1}, ..., e_{1,N-1}, e_{2,0}, ..., e_{2,N}\}$ be a family of mutually orthogonal projections in the definition of the Rokhlin property in (2.1) such that
\[
\sum_{j=1}^{N-1} e_{1,j} + \sum_{j=1}^{N} e_{2,j} = 1, \quad (e 4.55)
\]
\[
||e_{i,j}y - ye_{i,j}|| < \epsilon/2^{10}(N + 1)^2 \ for \ all \ y \in \mathcal{H}, \ i = 0, 1, \quad (e 4.56)
\]
\[
||\alpha(e_{1,j}) - e_{1,j+1}|| < \epsilon/2^{10}(N + 1)^2, \ j = 0, 1, ..., N - 2 \ and \quad (e 4.57)
\]
\[
||\alpha(e_{2,j}) - e_{2,j+1}|| < \epsilon/2^{10}(N + 1)^2, \ j = 0, 1, 2, ..., N - 1, \quad (e 4.58)
\]
where $\mathcal{H} \supset \mathcal{F} \cup \{u_k, 1 \leq k \leq N + 1, \alpha^{N+1}(v_j), \alpha^{-N}(w_j), 0 \leq j \leq N\} \cup \mathcal{W}$. It follows that (see 4.2 of (2))
\[
||\alpha(e_{1,N-1} + e_{2,N}) - (e_{1,0} + e_{2,0})|| < (2N - 1)(\epsilon/2^{10}(N + 1)^2) < \epsilon/2^9(N + 1). \quad (e 4.59)
\]
Therefore
\[ \|e_{1,j} \alpha(e_{1,N-1})\| < \epsilon/2^9(N + 1), \ j = 1, 2, \ldots, N - 1, \quad (e 4.60) \]
\[ \|e_{2,j} \alpha(e_{1,N-1})\| < \epsilon/2^9(N + 1), \ j = 1, 2, \ldots, N, \quad (e 4.61) \]
\[ \|e_{1,j} \alpha(e_{2,N})\| < \epsilon/2^9(N + 1), \ j = 1, 2, \ldots, N - 1 \ 	ext{and} \]
\[ \|e_{2,j} \alpha(e_{2,N})\| < \epsilon/2^9(N + 1), \ j = 1, 2, \ldots, N. \quad (e 4.63) \]

Moreover
\[ \|[e_{i,j}, w_k(t)]\| < \epsilon/2^{10}(N + 1)^2, \ k = 1, 2, \ldots, N, \quad (e 4.64) \]
\[ \|[e_{i,j}, v_k(t)]\| < \epsilon/2^{10}(N + 1)^2, \ k = 0, 1, \ldots, N - 1, \quad (e 4.65) \]
\[ \|[e_{i,j}, w_k(t)]\| < \epsilon/2^{10}(N + 1)^2, \ k = 0, 1, \ldots, N \quad (e 4.66) \]

for all \( t \in [0, 1] \) and for \( j = 1, 2, \ldots, N + i - 2, \ i = 0, 1 \).

Define
\[ U(t) = \sum_{j=0}^{N-1} u_{j+1}(t) \alpha^{j-N+1} (v_j(t))^* e_{1,j} + \sum_{j=0}^{N} u_{j+1}(t) \alpha^{j-N} (w_j(t))^* e_{2,j}. \quad (e 4.67) \]

One checks that, by (e 4.56), (e 4.65) and (e 4.66),
\[ U(t)^* U(t) = \sum_{i,j} e_{1,j} \alpha^{j-N+1} (v_j(t)) u_{j+1}(t)^* u_{i+1}(t) \alpha^{i-N+1} (v_i(t))^* e_{1,i} \]
\[ + \sum_{i,j} e_{2,j} \alpha^{j-N} (w_j(t)) u_{j+1}(t)^* u_{i+1}(t) \alpha^{i-N} (w_i(t))^* e_{2,i} \]
\[ + \sum_{i,j} e_{1,j} \alpha^{j-N+1} (v_j(t)) u_{j+1}(t) \alpha^{i-N+1} (v_i(t))^* e_{1,i} \]
\[ + \sum_{i,j} e_{2,j} \alpha^{j-N} (w_j(t)) u_{j+1}(t) \alpha^{i-N+1} (w_i(t))^* e_{2,i} \]
\[ \approx 4(N-1)N \eta + 4(N-1)N \eta + 4N^2 \eta + 4N^2 \eta \sum_{j=0}^{N-1} e_{1,j} \alpha^{j-N+1} (v_j(t)) u_{j+1}(t)^* u_{j+1}(t) \alpha^{j-N+1} (v_j(t))^* e_{1,j} \]
\[ + \sum_{j=0}^{N} e_{2,j} \alpha^{j-N} (w_j(t)) u_{j+1}(t)^* u_{j+1}(t) \alpha^{j-N} (w_j(t))^* e_{2,j} \]
\[ = \sum_{j=0}^{N-1} e_{1,j} + \sum_{j=0}^{N} e_{2,j} = 1 \quad (e 4.74) \]

for all \( t \in [0, 1] \). Thus
\[ \|U(t)^* U(t) - 1\| < 4^2(\epsilon/2^{10}) = \epsilon/2^6 \text{ for all } t \in [0, 1]. \quad (e 4.75) \]

Similarly,
\[ \|U(t) U(t)^* - 1\| < \epsilon/2^6 \text{ for all } t \in [0, 1]. \quad (e 4.76) \]
One also checks that, if $x \in \mathcal{F}$, by (e 4.56), (e 4.47), (e 4.50) and (e 4.45),

\[
U(t)x = \sum_{j=0}^{N-1} u_{j+1}(t)\alpha^j N^+1(v_j(t)^*)e_{1,j}x + \sum_{j=0}^{N} u_{j+1}(t)\alpha^j N^+N(w_j(t)^*)e_{2,j}x \quad \text{(e 4.77)}
\]

\[
\approx 2N\eta \sum_{j=0}^{N-1} u_{j+1}\alpha^j N^+1(v_j(t)^*)xe_{1,j} + \sum_{j=0}^{N} u_{j+1}(t)\alpha^j N^+N(w_j(t)^*)xe_{2,j} \quad \text{(e 4.78)}
\]

\[
\approx 2N\eta \sum_{j=0}^{N-1} u_{j+1}(t)\alpha^j N^+1(v_j(t)^*)e_{1,j} + \sum_{j=0}^{N} u_{j+1}(t)\alpha^j N^+N(w_j(t)^*)e_{2,j} \quad \text{(e 4.79)}
\]

\[
\approx 2N\eta \sum_{j=0}^{N-1} xu_{j+1}(t)\alpha^j N^+1(v_j(t)^*)e_{1,j} + \sum_{j=0}^{N} xu_{j+1}(t)\alpha^j N^+N(w_j(t)^*)e_{2,j} = xU(t). \quad \text{(e 4.80)}
\]

Therefore

\[
||[U(t), x]|| < 6\epsilon/2^{10}(N + 1) < \epsilon/2^7 \quad \text{for all} \quad x \in \mathcal{F}. \quad \text{(e 4.81)}
\]

We have, by (e 4.57), (e 4.60), (e 4.56), (e 4.46), (e 4.51) and (e 4.56),

\[
\sum_{i,j} u_{j+1}\alpha^j N^+1(v_j^*)e_{1,j}\alpha(e_{1,i})\alpha^j N^+2(v_i)\alpha(u_{i+1}^*) \quad \text{(e 4.82)}
\]

\[
\approx N(N-1)\eta + N\epsilon/2^b(N+1) \sum_{j=1}^{N-1} u_{j+1}\alpha^j N^+1(v_j^*)e_{1,j}\alpha^j N^+1(v_j-1)\alpha(u_j^*) + u_1e_{1,0}\alpha(e_{1,N-1}) \quad \text{(e 4.83)}
\]

\[
\approx (4\pi+1)/(N-1) \sum_{j=1}^{N-1} e_{1,j}u_{j+1}\alpha^j N^+1(v_j^*v_j-1)\alpha(u_j^*)e_{1,j} + u_1e_{1,0}\alpha(e_{1,N-1}) \quad \text{(e 4.85)}
\]

\[
\approx (N-1)\eta \sum_{j=1}^{N-1} u_1e_{1,j} + u_1e_{1,0}\alpha(e_{1,N-1}). \quad \text{(e 4.87)}
\]

Thus

\[
|| \sum_{i,j} u_{j+1}\alpha^j N^+1(v_j^*)e_{1,j}\alpha(e_{1,i})\alpha^j N^+2(v_i)\alpha(u_{i+1}^*) - \left( \sum_{j=1}^{N-1} u_1e_{1,j} + u_1e_{1,0}\alpha(e_{1,N-1}) \right) || < \frac{\epsilon}{2^7}. \quad \text{(e 4.88)}
\]
We estimate that, by \((e \, 4.58), (e \, 4.61), (e \, 4.56), (e \, 4.51), (e \, 4.46)\) and \((e \, 4.56)\),

\[
\sum_{i,j} u_{j+1} \alpha^{j-N} (w_j^*) e_{2,j} \alpha (e_{2,i}) \alpha^{i-N+1} (w_i) \alpha (u_{i+1}^*)
\] (e 4.89)

\[
\approx_{N(N+1)\eta + N\epsilon/2^{10} (N+1)} \sum_{j=1}^{N} u_{j+1} \alpha^{j-N} (w_j^*) e_{2,j} \alpha^{j-N} (w_{j-1}) \alpha (u_j^*) + u_1 e_{2,0} \alpha (e_{2,N}) \alpha (u_{N+1})
\] (e 4.90)

\[
\approx_{4N\eta} \sum_{j=1}^{N} e_{2,j} u_{j+1} \alpha^{j-N} (w_j^*) \alpha^{j-N} (w_{j-1}) \alpha (u_j^*) e_{2,j} + u_1 e_{2,0} \alpha (e_{2,N})
\] (e 4.92)

\[
\approx_{(4\pi+1)/(N-1)} \sum_{j=1}^{N} e_{2,j} u_{j+1} \alpha (u_j^*) e_{2,j} + u e_{2,0} \alpha (e_{2,N})
\] (e 4.93)

\[
\approx_{N\eta} \sum_{j=1}^{N} u e_{2,j} + u e_{2,0} \alpha (e_{2,N}).
\] (e 4.94)

Thus

\[
\| \sum_{i,j} u_{j+1} \alpha^{j-N} (w_j^*) e_{2,j} \alpha (e_{2,i}) \alpha^{i-N+1} (w_i) \alpha (u_{i+1}^*) \|_\infty < \frac{\epsilon}{2^7}.
\] (e 4.95)

Moreover, by \((e \, 4.58), (e \, 4.62), (e \, 4.57)\) and \((e \, 4.61)\),

\[
\sum_{i,j} u_{j+1} \alpha^{j-N+1} (v_j^*) e_{1,j} \alpha (e_{2,i}) \alpha^{i-N+1} (w_i) \alpha (u_{i+1}^*)
\] (e 4.96)

\[
\approx_{N^2\eta} \sum_{j=0}^{N-1} u_{j+1} \alpha^{j-N+1} (v_j^*) e_{1,j} \alpha (e_{2,N}) \alpha (w_N) \alpha (u_{N+1}^*)
\] (e 4.97)

\[
\approx_{N\epsilon/2^{10}(N+1)} u_1 \alpha^{-N+1} (v_0^*) e_{1,0} \alpha (e_{2,N})
\] (e 4.98)

\[= u e_{1,0} \alpha (e_{2,N}) \text{ and}
\] (e 4.99)

\[
\sum_{i,j} u_{j+1} \alpha^{j-N} (w_j^*) e_{2,j} \alpha (e_{1,i}) \alpha^{i-N+2} (v_i^*) \alpha (u_{i+1}^*)
\] (e 4.100)

\[
\approx_{(N+1)\eta N} \sum_{j=0}^{N} u_{j+1} \alpha^{j-N} (w_j^*) e_{2,j} \alpha (e_{1,N-1}) \alpha (v_{N-1}^*) \alpha (u_N)
\] (e 4.101)

\[
\approx_{N\epsilon/2^{10}(N+1)} u_1 \alpha^{-N} (w_0^*) e_{2,0} \alpha (e_{1,N-1})
\] (e 4.102)

\[= u e_{2,0} \alpha (e_{1,N-1}).
\] (e 4.103)

In other words,

\[
\| \sum_{i,j} u_{j+1} \alpha^{j-N+1} (v_j^*) e_{1,j} \alpha (e_{2,i}) \alpha^{i-N+1} (w_i) \alpha (u_{i+1}^*) - u e_{1,0} \alpha (e_{2,N}) \| < \epsilon/2^9 \text{ and (e 4.104)}
\]

\[
\| \sum_{i,j} u_{j+1} \alpha^{j-N} (w_j^*) e_{2,j} \alpha (e_{1,i}) \alpha^{i-N+2} (v_i^*) \alpha (u_{i+1}^*) - u e_{2,0} \alpha (e_{1,N-1}) \| < \epsilon/2^9.
\] (e 4.105)
By \( e^{4.88} \), \( e^{4.95} \), \( e^{4.104} \) and \( e^{4.105} \),
\[
U(0)\alpha(U(0)^*) = \sum_{i,j} u_{j+1}\alpha^{j-N+1}(v_j^*)e_{1,j}\alpha(e_{1,i})\alpha^{i-N+2}(v_i)\alpha(u_{i+1}^*) \tag{4.106}
\]
\[
+ \sum_{i,j} u_{j+1}\alpha^{j-N}(w_j^*)e_{2,j}\alpha(e_{2,i})\alpha^{i-N+1}(w_i)\alpha(u_{i+1}^*) \tag{4.107}
\]
\[
+ \sum_{i,j} u_{j+1}\alpha^{j-N+1}(v_j^*)e_{1,j}\alpha(e_{2,i})\alpha^{i-N+1}(w_i)\alpha(u_{i+1}^*) \tag{4.108}
\]
\[
+ \sum_{i,j} u_{j+1}\alpha^{j-N}(w_j^*)e_{2,j}\alpha(e_{1,i})\alpha^{i-N+2}(v_i)\alpha(u_{i+1}^*) \tag{4.109}
\]
\[
\approx_{\epsilon/64+\epsilon/4+\epsilon/64} \sum_{j=1}^{N-1} u e_{i,j} + u e_{1,0}\alpha(e_{1,N-1}) + \sum_{j=1}^{N} u e_{2,j} + u e_{2,0}\alpha(e_{2,N}) \tag{4.110}
\]
\[
+ u e_{1,0}\alpha(e_{2,N}) + u e_{2,0}\alpha(e_{1,N-1}) \tag{4.111}
\]
\[
= \sum_{j=1}^{N-1} u e_{i,j} + u e_{1,0}(\alpha(e_{1,N-1}) + \alpha(e_{2,N})) + \sum_{j=1}^{N} u e_{2,j} + u e_{2,0}(\alpha(e_{1,N-1}) + \alpha(e_{2,N})) \tag{4.112}
\]
\[
\approx_{2\epsilon/2^{10}(N+1)} \sum_{j=0}^{N-1} u e_{1,j} + \sum_{j=0}^{N} u e_{2,j} = u. \tag{4.113}
\]

Therefore
\[
\|U(0)\alpha(U(0)^*) - u\| < \epsilon/16 + \epsilon/2^9(N + 1). \tag{4.114}
\]

Let \( V(t) = U(t)(U(t)^*)U(t)^{-1/2} \) for \( t \in [0,1] \). Then \( \{V(t) : t \in [0,1]\} \) is a continuous path of unitaries in \( A \). Since \( U(1) = 1 \), \( V(1) = 1 \). Moreover,
\[
\|V(0)\alpha(V(0)^*) - u\| < \epsilon \text{ and } \|[V(t), x]\| < \epsilon \text{ for all } x \in \mathcal{F} \tag{4.115}
\]
and for all \( t \in [0,1] \).

For last part (with \( \mathcal{F} = \emptyset \)), we note that, in the above, \( u_k \in CU(A) \), \( k = 0, 1, \ldots, N + 1 \). Note that in this case, \( CU(A) \subset U_0(A) \). Moreover, by [14], \( \text{cel}(u_k) \leq 2\pi \), \( k = 0, 1, \ldots, N + 1 \). So \( V(t) \) can be constructed (without worrying about the set \( \mathcal{F} \)).

**Theorem 4.2.** Let \( A \) be a unital separable simple amenable \( C^* \)-algebra with \( TR(A) \leq 1 \) which satisfies the UCT. Suppose that \( \alpha, \beta \in \text{Aut}(A) \) have the Rokhlin property. Then the following holds:

1. If \( \alpha \circ \beta^{-1} \) is asymptotically inner, then there exists a unitary \( u \in U(A) \) and a strongly asymptotically inner automorphism \( \sigma \) such that
   \[
   \alpha = \text{Ad } u \circ \sigma \circ \beta \circ \sigma^{-1}.
   \]

2. If \( \alpha \circ \beta^{-1} \) is strongly asymptotically inner, then there exists a sequence of unitaries \( u_n \in U(A) \) and a sequence \( \{\sigma_n\} \) of strongly asymptotically inner automorphisms of \( A \) such that
   \[
   \alpha = \text{Ad } u_n \circ \sigma_n \circ \beta \circ \sigma_n^{-1} \text{ and } \lim_{n \to \infty} \|u_n - 1\| = 0.
   \]

**Proof.** By the assumption there exists a continuous path of unitaries \( \{v(t) : t \in [0,\infty)\} \subset U(A) \) such that
\[
\alpha \circ \beta^{-1}(a) = \lim_{t \to \infty} v_t^*a v_t \text{ for all } a \in A. \tag{4.116}
\]
By replacing $\beta$ by $\text{Ad} v_0 \circ \beta$, without loss of generality, we may assume that $v_0 = 1$. Note, and it is important, that if $\alpha \circ \beta^{-1}$ is strongly asymptotically inner, then, one may always assume that $v_0 = 1$ without replacing $\beta$ by $\text{Ad} v_0 \circ \beta$.

Let $\{F_n\}$ be an increasing sequence of finite subsets in the unit ball of $A$ such that $\bigcup_{n=1}^{\infty} F_n$ is dense in the unit ball of $A$. We assume that $1_A \in F_1$. Let $1 > \epsilon > 0$. We choose $t_1 > 0$ such that

$$\alpha \approx_{\epsilon/2^4} \text{Ad} v_t \circ \beta \quad \text{on} \quad F_1. \quad (\text{e} \text{4.117})$$

It follows from $3.12$ that there is a unitary $v'_t \in U(A)$ such that

$$v'_t v_t \in CU(A) \quad \text{and} \quad \|\alpha(x), v'_t\| < \epsilon/2^4 \quad \text{for all} \quad x \in \alpha(F_1) \quad (\text{e} \text{4.118})$$

(by considering $\Gamma : \{[1_A]\} \rightarrow U(A)/CU(A)$ by $\Gamma([1_A]) = \overline{v'_t}$. Let $V_t = v'_t v_t \in CU(A)$. Then

$$\alpha \approx_{\epsilon/2^3} \text{Ad} V_t \circ \beta \quad \text{on} \quad F_1. \quad (\text{e} \text{4.119})$$

It follows from the last part of $4.11$ that there is a unitary $u_1, w_1 \in U_0(A)$ such that

$$V_1 = w_1 u_1 \beta(u_1^*) \quad \text{and} \quad \|w_1 - 1\| < \epsilon/4. \quad (\text{e} \text{4.120})$$

Put

$$\beta_1 = \text{Ad} w_1 \circ \text{Ad} u_1 \circ \beta \circ \text{Ad} u_1^*$$

and $v^{(1)}_t = v_{t+t_1} V_1^*$. Then (see $\text{e} \text{4.117}$)

$$\alpha = \lim_{t \to \infty} \text{Ad} v^{(1)}_t \circ \beta_1, \quad v^{(1)}_0 = (v'_1)^* \quad \text{and} \quad (\text{e} \text{4.121})$$

$$\alpha \approx_{\epsilon/2^4} \text{Ad} v^{(1)}_t \circ \beta_1 \quad \text{on} \quad F_1 \quad \text{for all} \quad t. \quad (\text{e} \text{4.122})$$

Let $F'_1 = F_1 \cup \text{Ad} u_1(F_1)$. Let $\delta_1 > 0$ (in place of $\delta$), $\gamma_1 > 0$ (in place of $\gamma$), let $G_1 \subset A$ (in place of $G$) be a finite subset, $P_1 \subset K(A)$ (in place of $P$) be a finite subset and let $\{p_{1,1}, p_{1,2}, ..., p_{1,p(1)}\}$ (in place of $\{p_1, p_2, ..., p_n\}$) of $A$ required by $3.10$ for $\epsilon/4^3$ (in place of $\epsilon$) and $F'_1$ (in place of $F$) (with $C = A$ and $\varphi = \text{id}_A$). We may assume that $F'_1 \cup \{p_{1,j} : 1 \leq j \leq p(1)\} \subset G_1$ and $\{[p_{1,j}] : 1 \leq j \leq p(1)\} \subset P_1$. Without loss of generality, we may also assume that $p_1 = 1_A$. Let $G^{(1)}_0$ be the subgroup of $K_0(A)$ generated by $\{p_{1,j} : 1 \leq j \leq p(1)\}$. We may assume that $\delta_1$ is sufficiently small so that $\Gamma([p_{1,j}]) = ((1 - p_{1,j}) + p_{1,j} W)$ defines a homomorphism from $G^{(1)}_0$ into $U(A)/CU(A)$, and $\text{Bott}(\text{id}_A, W)|_{P_1}$ is well defined, where $W$ is any unitary in $A$ such that $\|[x, W]\| < \delta_1$ for all $x \in G_1$. Moreover, we may further assume that

$$\text{Bott}(\text{id}_A, W_1 W_2)|_{P_1} = \text{Bott}(\text{id}_A, W_1)|_{P_1} + \text{Bott}(\text{id}_A, W_2)|_{P_1} \quad \text{and} \quad (\text{e} \text{4.123})$$

$$((1 - p_{1,j}) + p_{1,j} W_1 W_2) = ((1 - p_{1,j}) + p_{1,j} W_1)((1 - p_{1,j}) + p_{1,j} W_2), \quad (\text{e} \text{4.124})$$

provided that

$$\|[y, W_k]\| < \delta_1 \quad \text{for all} \quad y \in G_1, \quad k = 1, 2, \ldots. \quad (\text{e} \text{4.125})$$

Put $\epsilon_1 = \{\epsilon, \delta_1/2, \gamma_1/2\}$. Since $\beta_1 = \lim_{t \to \infty} \text{Ad} (v^{(1)}_t)^* \circ \alpha$, we choose $t_2 > 0$ such that for any $t \geq t_2$,

$$\beta_1 \approx_{\epsilon_1/4^3} \text{Ad} (v^{(1)}_t)^* \circ \alpha \quad \text{on} \quad \beta_1^{-1}(G_1) \cup F'_1. \quad (\text{e} \text{4.126})$$
It follows from (e 4.131) again that there is $v_2' \in U(A)$ such that
\[ v_{t_2}^{(1)} v_2' \in C(U(A), \|x, v_2'\| < \epsilon_1/4^4 \text{ for all } x \in \alpha(\beta_1^{-1}(G_1)) \cup F'_1 \] (e.4.127)
and $\text{Bott}(\text{id}_A, v_2')|_{P_1} = 0$. (e.4.128)

Let $V_2 = v_{t_2}^{(1)} v_2' \in C(U(A)$. Then
\[ \beta_1 \approx_{\epsilon_1/4^3} \text{Ad} V_2^* \circ \alpha \text{ on } \beta_1^{-1}(G_1) \cup F'_1. \] (e.4.129)

By the last part of (4.1) we choose $u_2, w_2 \in U_0(A)$ such that
\[ V_2^* = w_2 u_2 \circ \alpha u_2^* \text{ and } \|w_2 - 1\| < \epsilon/4^2. \] (e.4.130)

Set $v_t^{(2)} = (v_{t_2+t_2})^* V_2$ and $\alpha_2 = \text{Ad} w_2 \circ \text{Ad} u_2 \circ \alpha \circ \text{Ad} u_2^*$. Then, by (e.4.129),
\[ \beta_1 \approx_{\epsilon_1/4^3} \alpha_2 \text{ on } F'_1 \cup \beta_1^{-1}(G_1). \] (e.4.131)

Moreover (see (e.4.129))
\[ v_0^{(2)} = v_2', \quad \beta_1 = \lim_{t \to \infty} \text{Ad} (v_t^{(2)})^* \circ \alpha_2 \text{ and } \] (e.4.132)
\[ \beta_1 \approx_{\epsilon_1/4^3} \text{Ad} (v_t^{(2)})^* \circ \alpha_2 \text{ on } \beta_1^{-1}(G_1) \cup F'_1 \text{ for all } t \geq 0. \] (e.4.133)

Let $F_2^{(0)} = F_1 \cup F_2$ and $F_2' = F_2^{(0)} \cup \text{Ad} u_1(F_2^{(0)}) \cup \text{Ad} u_2(F_2^{(0)})$.

Let $\delta_2 > 0$ (in place of $\delta$), $\gamma_2 > 0$ (in place of $\gamma$), let $G_2 \subset A$ (in place of $G$), let $P_2 \in \mathcal{K}(A)$ (in place of $P$) be a finite subset and let $\{p_2, p_2, \ldots, p_2, p(2)\}$ (in place of $\{p_1, p_2, \ldots, p_n\}$) of $A$ required by (3.10) for $\epsilon/4^3$ (in place of $\epsilon$) and $F_2'$ (in place of $F$). We may assume that $P_1 \cup \{p_2, j \mid 1 \leq j \leq p(2)\} \subset P_2$ and $F_2' \cup \{p_2, j \mid 1 \leq j \leq p(2)\} \subset G_2$. We may assume that $G_2 \supset \alpha_2 \circ \beta_1^{-1}(G_1) \cup F'_2$. Without loss of generality, we may also assume that $p_2, 1 = A$. Let $G_0^{(2)}$ be the subgroup of $K_0(A)$ generated by $\{p_2, j \mid 1 \leq j \leq p(2)\}$. We may assume that $\delta_2$ is sufficiently small so that $\Gamma(p_2, j) = (1 - p_2, j + p_2, j, W)$ defines a homomorphism from $G_0^{(2)}$ into $U(A)/CU(A)$, and Bott(id$_A$, W)$_{p_2}$ is well defined, where $W$ is any unitary in $A$ such that $\|x, W\| < \delta_2$ for all $x \in G_2$. Moreover, we may further assume that
\[ \text{Bott}(\text{id}_A, W_1 W_2)|_{P_2} = \text{Bott}(\text{id}_A, W_1)|_{P_2} + \text{Bott}(\text{id}_A, W_2)|_{P_2} \] and
\[ (1 - p_2, j + p_2, j, W_1 W_2) = (1 - p_2, j + p_2, j, W_1) \langle (1 - p_2, j + p_2, j, W_2), \] provided that
\[ \|y, W_k\| < \delta_2 \text{ for all } y \in G_1, \quad k = 1, 2, \ldots. \] (e.4.134)

Let $\epsilon_2 = \min\{\epsilon_1, \delta_2/2, \lambda_2/2\}$.

Since $\alpha_2 = \lim_{t \to \infty} \text{Ad} v_t^{(2)} \circ \beta_1$, we choose $t_3 > 0$ such that for $t \geq t_3$,
\[ \alpha_2 \approx_{\epsilon_2/4^6} \text{Ad} v_t^{(2)} \circ \beta_1 \text{ on } \alpha_2^{-1}(G_2) \cup F'_2. \] (e.4.137)

It follows from (e.4.131) and (e.4.133), for $x \in \beta_1^{-1}(G_1)$,
\[ \beta_1(x) \approx_{\epsilon_1/4^3} \alpha_2(x) \approx_{\epsilon_2/4^6} \text{Ad} v_t^{(2)}(\beta_1(x)) \text{ for all } t \geq 0. \] (e.4.138)

In other words, for $t \geq 0$,
\[ \|y, v_t^{(2)}\| < \delta_1/4^3 \text{ for all } y \in G_1. \] (e.4.139)
Note that $v_0^{(2)} = v_2'$. Therefore, by (e 4.128) and (e 4.139),
\[
\text{Bott}(\text{id}_A, v_t^3)|_{p_1} = \text{Bott}(\text{id}_A, v_2')|_{p_1} = 0. 
\] (e 4.140)

By the choices of $\delta_1$ and $G_1$, let $\Gamma_1 : G_0^{(1)} \rightarrow U(A)/CU(A)$ be the homomorphism defined by $\Gamma_1([p_{1,j}]) = \langle (1 - p_{1,j}) + p_{1,j}v_t^{(2)} \rangle$, $j = 1, 2, \ldots, p(1)$. Then, since $\{p_{1,j} : 1 \leq j \leq p(1)\} \subset \mathcal{P}_1$, by (e 4.140), $\Gamma_1$ maps $G_0^{(1)}$ into $U_0(A)/CU(A)$. It follows from (3.12) that there is a unitary $v_t' \in U(A)$ such that
\[
\|y, v_t'\| < \epsilon_2/4^6 \text{ for all } y \in \alpha_2^{-1}(G_2) \cup \mathcal{F}_2' \cup \beta_1(\alpha_2^{-1}(G_2) \cup \mathcal{F}_2'), 
\] (e 4.141)
\[
\text{Bott}(\text{id}_A, v_t^3)|_{p_2} = 0 \text{ and } 
\] (e 4.142)
\[
\text{dist}((1 - p_{1,j}) + p_{1,j}v_t^3), \Gamma_1([p_{1,j}]) < \gamma_1/2, \quad j = 1, 2, \ldots, p(1). 
\] (e 4.143)

Define $V_3 = v_t'v_t^3$. Then (since $G_1 \subset \beta_1(\alpha_2^{-1}(G_1))$),
\[
\alpha_2 \approx \epsilon_2/4^5 \text{ Ad } V_3 \circ \beta_1 \text{ on } \alpha_2^{-1}(G_2) \cup \mathcal{F}_2' \text{ and } 
\] (e 4.144)
\[
\|y, V_3\| < \delta_1/2 \text{ for all } y \in G_1. 
\] (e 4.145)

Let $z_t^{(3)} = v_t'v_t^3$. Then $z_t = V_3$. By (e 4.139) and (e 4.141),
\[
\|y, z_t^{(3)}\| < \delta_1/4 \text{ for all } y \in G_1 \text{ and for all } t \geq 0. 
\] (e 4.146)

Since $z_0^{(3)} = v_3'v_0^{(2)} = v_3'v_2'$, by (e 4.146), (e 4.123), (e 4.128) and (e 4.142),
\[
\text{Bott}(\text{id}_A, V_3)|_{p_1} = \text{Bott}(\text{id}_A, z_t^{(3)})|_{p_1} = \text{Bott}(\text{id}_A, v_t'v_2')|_{p_1} 
= \text{Bott}(\text{id}_A, v_t')|_{p_1} + \text{Bott}(\text{id}_A, v_2')|_{p_1} = 0. 
\] (e 4.147)

Furthermore, by (e 4.123),
\[
\langle (1 - p_{1,j}) + p_{1,j}V_3 \rangle = \langle (1 - p_{1,j}) + p_{1,j}v_t'v_2' \rangle 
= \langle (1 - p_{1,j}) + p_{1,j}v_t^3 \rangle \langle (1 - p_{1,j}) + p_{1,j}v_t^{(2)} \rangle 
= \langle (1 - p_{1,j}) + p_{1,j}v_t^3 \rangle \Gamma_1([p_{1,j}]). 
\] (e 4.149)

It follows from (e 4.143) that
\[
\text{dist}((1 - p_{1,j}) + p_{1,j}V_3), CU(A)) < \lambda_1, \quad j = 1, 2, \ldots, p(1). 
\] (e 4.152)

It follows from (e 4.145), (e 4.148) and (e 4.152), and by the choice of $G_1$, $\mathcal{P}_1$, $\delta_1$, $\lambda_1$ and $\{p_{1,j} : 1 \leq j \leq p(1)\}$, by applying (4.11) there is a unitary $u_3, w_3 \in U(A)$ and a continuous path of unitaries $\{u_3(t) : t \in [0, 1]\}$ such that $u_3(0) = u_3, u_3(1) = 1_A,$
\[
V_3 = w_3u_3\beta_1(u_3^*), \quad \|w_3 - 1\| < \epsilon/4^3 \text{ and } \|x - \text{Ad } u_3(t)(x)\| < \epsilon/4^3 \text{ for all } x \in \mathcal{F}_1 \quad (e 4.153)
\]
and $t \in [0, 1]$. Set $v_t^{(3)} = v_t^{(2)}V_3^*$ and $\beta_3 = \text{Ad } w_3 \circ u_3 \circ \beta_1 \circ \text{Ad } u_3^*$. Then, by (e 4.144),
\[
\alpha_2 \approx \epsilon_2/4^5 \beta_3 \text{ on } \mathcal{F}_2'. 
\] (e 4.154)

Moreover (see (4.137))
\[
v_0^{(3)} = v_2', \quad \alpha_2 = \lim_{t \to \infty} \text{Ad } (v_t^{(3)}) \circ \beta_3 \text{ and } 
\] (e 4.155)
\[
\alpha_2 \approx \epsilon_2/4^5 \text{ Ad } (v_t^{(3)}) \circ \beta_3 \text{ on } \alpha_2^{-1}(G_2) \cup \mathcal{F}_2' \text{ for all } t \geq 0. 
\] (e 4.156)
We now construct $\alpha_4$, $w_4$ and $u_4$.

Let $F_3^{(0)} = F_3 \cup F_3$, and let

$$F_3' = F_3^{(0)} \cup \text{Ad} u_1(F_3^{(0)}) \cup \text{Ad} u_2(F_3^{(0)}) \cup \text{Ad} u_3(F_3^{(0)}) \cup \text{Ad} u_4(F_3^{(0)}).$$

Let $\delta_3 > 0$ (in place of $\delta$), $\gamma_3 > 0$ (in place of $\gamma$), let $G_3 \subset A$ (in place of $G$) be a finite subset, $P_3 \in K(A)$ (in place of $P$) be a finite subset and let $\{p_{3,1}, p_{3,2}, ..., p_{3,\beta(3)}\}$ (in place of $\{p_1, p_2, ..., p_n\}$) of $A$ required by \ref{e.4.10} for $\varepsilon/4^t$ (in place of $\epsilon$) and $F_3'$ (in place of $F$).

We may assume that $P_3 \cup \{p_{3,j} : 1 \leq j \leq p(3)\} \subset P_3$ and $F_3' \cup \{p_{3,j} : 1 \leq j \leq p(3)\} \subset G_3$. We may assume that $G_3 \supset \beta_3 \circ \alpha_2^{-1}(G_3) \cup F_3'$. Without loss of generality, we may also assume that $p_{3,1} = 1_A$. Let $G_0^{(3)}$ be the subgroup of $K_0(A)$ generated by $\{p_{3,j} : 1 \leq j \leq p(3)\}$. We may assume that $\delta_3$ is sufficiently small so that $\Gamma(p_{3,j}) = \{1 - p_{3,j} + p_{3,j}W\}$ defines a homomorphism from $G_0^{(3)}$ into $U(A)/CU(A)$, and $\text{Bott}(id_A, W)|_{P_3}$ is well defined, where $W$ is any unitary in $A$ such that $\|x, W\| < \delta_3$ for all $x \in G_3$. Moreover, we may further assume that

$$\text{Bott}(id_A, W_1 W_2)|_{P_3} = \text{Bott}(id_A, W_1)|_{P_3} + \text{Bott}(id_A, W_2)|_{P_3}$$

and

$$\langle (1 - p_{3,j} + p_{3,j} W_1) \rangle = \langle (1 - p_{3,j} + p_{3,j} W_1) \rangle.$$  

provided that

$$\|x, W_k\| < \delta_1$$

for all $x \in G_3$, $k = 1, 2, \ldots$ \quad \{(e 4.157)\}

Let $\varepsilon_3 = \min \{\varepsilon_2, \delta_3/2, \lambda_3/2\}$.

Since $\beta_3 = \lim_{t \to \infty} \text{Ad}(v_t^{(3)})^* \circ \alpha_2$, we choose $t_4 > 0$ such that for $t \geq t_4$,

$$\beta_3 \approx_{\varepsilon_2/4^t} \text{Ad}(v_t^{(3)})^* \circ \alpha_2 \text{ on } \beta_3^{-1}(G_3) \cup F_3'.$$

It follows from \ref{e.4.141} and \ref{e.4.156}, for $x \in \alpha_2^{-1}(G_2)$,

$$\alpha_2(x) \approx_{\varepsilon_2/4^t} \beta_3(x) \approx_{\varepsilon_2/4^t} \text{Ad}(v_t^{(3)})^*(\alpha_2(x))$$

for all $t \geq 0$ \quad \ref{e.4.161}

In other words, for $t \geq 0$,

$$\|y, v_t^{(3)}\| < \delta_2/4^5$$

for all $y \in G_2$. \quad \ref{e.4.162}

Note that $v_0^{(3)} = (v_3')^*$. Therefore, by \ref{e.4.162} and \ref{e.4.139},

$$\text{Bott}(id_A, v_t^{(3)})|_{P_2} = \text{Bott}(id_A, (v_3')^*)|_{P_2} = 0.$$ \quad \ref{e.4.163}

By the choices of $\delta_2$ and $G_2$, let $\Gamma_2 : G_0^{(2)} \to U(A)/CU(A)$ be the homomorphism defined by

$$\Gamma_2([p_{2,j}]) = \{1 - p_{2,j} + p_{2,j}v_t^{(3)}\}, \quad j = 1, 2, \ldots, p(2).$$

Then, since $\{[p_{2,j}] : 1 \leq j \leq p(2)\} \subset P_2$, by \ref{e.4.163}, $\Gamma_2$ maps $G_0^{(2)}$ into $U_0(A)/CU(A)$. It follows from \ref{3.12} that there is a unitary $v_4' \in U(A)$ such that

$$\|y, v_4'\| < \varepsilon_2/4^8$$

for all $y \in \alpha_2(\beta_3^{-1}(G_3)) \cup F_3'$, \quad \ref{e.4.164}

$$\text{Bott}(id_A, v_4')|_{P_2} = 0$$

and

$$\text{dist}(\{(1 - p_{2,j} + p_{1,j}(v_4')^*)\}, \Gamma_2([p_{2,j}])) < \gamma_2/2, \quad j = 1, 2, \ldots, p(1).$$ \quad \ref{e.4.165}

Define $V_4 = v_4^{(3)} v_4'$. Then (since $G_2 \supset \alpha_2(\beta_3^{-1}(G_3))$),

$$\beta_3 \approx_{\varepsilon_2/4^7} \text{Ad} V_4^* \circ \alpha_2 \text{ on } \beta_3^{-1}(G_3) \cup F_3'$$

and

$$\|y, V_4\| < \delta_2/2$$

for all $y \in G_2$. \quad \ref{e.4.166}
Let $z_t^{(4)} = v_t^{(3)} v'_4$. Then $z_t^{(4)} = V_4$. By (e 4.161) and (e 4.162),
\[ ||[y, z_t^{(4)}]] < \delta_2/4^2 \text{ for all } y \in \mathcal{G}_2 \text{ and } \text{ for all } t \geq 0. \tag{e 4.169} \]
Since $z_0^{(4)} = v_0^{(3)} v'_4 = (v'_3)^* v_4'$, by (e 4.169), (e 4.134), (e 4.142) and (e 4.165),
\[ \text{Bott}(\text{id}_A, V_4)|_{\mathcal{P}_2} = \text{Bott}(\text{id}_A, v_3^{(4)})|_{\mathcal{P}_2} = \text{Bott}(\text{id}_A, v_3'v'_4)|_{\mathcal{P}_2} = \text{Bott}(\text{id}_A, v_3')|_{\mathcal{P}_2} + \text{Bott}(\text{id}_A, v_4')|_{\mathcal{P}_2} = 0. \tag{e 4.170} \]
Furthermore, by (e 4.135),
\[ \langle (1 - p_{2,j}) + p_{2,j} V_4 \rangle = \frac{\langle (1 - p_{2,j}) + p_{2,j} v^{(3)}_t v'_4 \rangle}{\langle (1 - p_{2,j}) + p_{2,j} v^{(3)}_t v'_4 \rangle} \tag{e 4.172} \]
\[ = \frac{\langle (1 - p_{2,j}) + p_{2,j} v^{(3)}_t \rangle \langle (1 - p_{1,j}) + p_{1,j} v'_4 \rangle}{\Gamma_2([p_{2,j}]) \langle (1 - p_{2,j}) + p_{2,j} v'_4 \rangle}. \tag{e 4.174} \]
It follows from (e 4.166) that
\[ \text{dist}(\langle (1 - p_{2,j}) + p_{2,j} V_4 \rangle, CU(A)) < \lambda_2, \text{ } j = 1, 2, \ldots, p(2). \tag{e 4.175} \]
It follows from (e 4.169), (e 4.171) and (e 4.175), and by the choice of $\mathcal{G}_2$, $\mathcal{P}_2$, $\delta_2$, $\lambda_2$ and \{p_{2,j} : 1 \leq j \leq p(2)\}, by applying (4.1) there is a unitary $u_4, w_4 \in U(A)$ and a continuous path of unitaries \{u_4(t) : t \in [0, 1]\} in $A$ such that $u_4(0) = u_4$ and $u_4(1) = 1_A$,
\[ V_4 = w_4 u_4 \alpha_2 (u'_4), \text{ } ||w_4 - 1|| < \epsilon/4^4 \text{ and } ||x - \text{Ad} u_4(x)|| < \epsilon/4^5 \text{ for all } x \in \mathcal{F}_2'. \tag{e 4.176} \]
Set $v_t^{(4)} = v_t^{(3)} V_4^*$ and $\alpha_4 = \text{Ad } w_4 \circ u_4 \circ \alpha_2 \circ \text{Ad } u_4^*$. Then, by (e 4.167),
\[ \beta_3 \approx_{\epsilon/4^7} \alpha_4 \text{ on } \mathcal{F}_3' \cup \beta_3^{-1}(G_3). \tag{e 4.177} \]
Then (see (e 4.160))
\[ v_0^{(4)} = v'_4, \text{ } \beta_3 = \lim_{t \to \infty} \text{Ad}(v_t^{(4)}) \circ \alpha_4 \text{ and } \tag{e 4.178} \]
\[ \beta_3 \approx_{\epsilon/4^8} \text{Ad}(v_t^{(4)})^* \alpha_4 \text{ on } \beta_3^{-1}(G_3) \cup \mathcal{F}_3'\text{ for all } t \geq 0. \tag{e 4.179} \]
We repeat this and obtain
\[ \beta_1, \alpha_2, \beta_3, \alpha_4, \ldots, \tag{e 4.180} \]
\[ u_1, u_2, u_3(t), u_4(t), u_5(t), u_6(t), \ldots, \tag{e 4.181} \]
\[ w_1, w_2, w_3, w_4, \ldots, \tag{e 4.182} \]
which satisfy the following:
(i) $||[x, u_i]|| < 1/4^{i+1}$, $i = 1, 2$, $||[x, u_n(t)]|| < 1/4^{n+1}$ for all $x \in \mathcal{F}_n'$ and $t \in [0, 1]$, $n = 3, 4, \ldots$.
(ii) $||w_n - 1|| < 1/4^n$, $n = 1, 2, \ldots$.
(iii) $\alpha_{2k} = \text{Ad } w_{2k} \circ \text{Ad } u_{2k} \circ \alpha_{2k-2} \circ \text{Ad } u_{2k}^*$, $k = 1, 2, 3, \ldots$, (with $\alpha_0 = \alpha$),
(iv) $\beta_{2k+1} = \text{Ad } w_{2k+1} \circ \text{Ad } u_{2k+1} \circ \alpha_{2k+1} \circ \text{Ad } w_{2k+1}^*$, $k = 0, 1, 2, \ldots$, with $\beta_1 = \beta$,
(v) $\alpha_{2n} \approx_{\epsilon/4^{2n+1}} \beta_{2n+1}$ on $\mathcal{F}_{2n}'$, $n = 1, 2, \ldots$,
(vi) $\beta_{2n-1} \approx_{\epsilon/4^{2n+1}} \alpha_{2n} \text{ on } \mathcal{F}_{2n-1}'$, $n = 1, 2, \ldots$.
Define $\gamma_{k,0}, \gamma_{k,0} \in \text{Aut}(A)$ by
\[ \gamma_{k,0} = \text{Ad } u_{2k} u_{2(k-1)} \cdots u_2 \text{ and } \gamma_{k,1} = \text{Ad } u_{2k+1} u_{2k} \cdots u_1. \tag{e 4.183} \]
Fix $k_0 \geq 1$,
\[ \gamma_{k,0}(\mathcal{F}_{k_0}) \subset \mathcal{F}'_{k+1}. \] (e 4.184)

Since $$\|[x, u_n]\| < \epsilon/4^n$$ for $x \in \mathcal{F}'_n$ and $0 < \epsilon < 1$, we have, if $m > k$,
\[ \|[\gamma_{m,0}(x) - \gamma_{k,0}(x)]\| \leq \sum_{j=k}^{m} \epsilon/4^j < 1/4^{k-1} \] (e 4.185)
for all $x \in \mathcal{F}_{k_0}$. Since $\|\gamma_{n,0}\|$ is bounded and $\bigcup_{k=1}^{\infty} \mathcal{F}_k$ is dense in the unit ball, it follows that \{\gamma_{k,0}(x)\} is Cauchy for each $x \in A$. This gives a linear map \( \gamma_0 \) on \( A \) such that
\[ \gamma_0(x) = \lim_{n \to \infty} \gamma_{n,0}(x) \text{ for all } x \in A. \]

Since each $\gamma_{n,0}$ is an automorphism, \( \gamma_0 \) is a unital injective homomorphism. Exactly the same reason shows that $\gamma_{k,0}^{-1}$ converges point-wisely to another unital injective homomorphism \( \gamma_0' \). It is easy to check that $\gamma_0' = \gamma_0^{-1}$. So \( \gamma_0 \) is an automorphism.

Define \{\( W(t) : t \in [2, \infty) \)\} as follows:
\[ W(2) = u_2, \ W_0(t) = u_{2k}(k-t)u_{2(k-1)}u_{2(k-2)} \cdots u_2 \text{ for all } t \in (k, k+1), \ k = 2, 3, \ldots. \]

Note \{\( W(t) : t \in [1, \infty) \)\} is a continuous path of unitaries in \( A \) with \( W(2) = u_2 \in U_0(A) \). Since \( u_2 \in U_0(A) \), we may assume that \( W(t) \) defined on \([1, \infty)\), with \( W(1) = 1_A \). Then, by (1) above,
\[ \gamma_0(x) = \lim_{t \to \infty} \text{Ad} W(t)(x) \text{ for all } x \in A. \] (e 4.186)

Therefore \( \gamma_0 \) is strongly asymptotically inner. Similarly, \( \gamma_1(x) = \lim_{n \to \infty} \gamma_{n,1}(x) \) for all \( x \in A \) defines another strongly asymptotically inner automorphism on \( A \).

Define \( W_{2,0} = w_2 \) and
\[ W_{2k,0} = w_{2k}u_{2k}W_{2k-2}u_{2k}^*, \ k = 2, 3, \ldots. \] (e 4.187)

We estimate that
\[ \|[W_{2k,0} - 1]\| \leq \|[w_{2k} - 1]\| + \|[u_{2k}W_{2k-2}u_{2k}^* - 1]\| \] (e 4.188)
\[ < \epsilon/4^{2k} + \|[W_{2k-2} - 1]\|. \] (e 4.189)

It follows from the induction that
\[ \|[W_{2k,0} - 1]\| < \sum_{j=1}^{k} \epsilon/4^{2j}, \ k = 1, 2, \ldots. \] (e 4.190)

It follows that \( \lim_{k \to \infty} W_{2k,0} = W_0 \) is a unitary in \( A \) with
\[ \|[W_0 - 1]\| < \epsilon/2. \] (e 4.191)

Define \( W_{1,1} = u_1 \) and
\[ W_{2k+1,1} = w_{2k+1}u_{2k+1}W_{2k-1,1}u_{2k+1}^*, \ k = 1, 2, \ldots. \] (e 4.192)

As above, \( W_1 = \lim_{k \to \infty} W_{2k+1,1} \) is a unitary in \( A \) with
\[ \|[W_1 - 1]\| < \epsilon/2. \] (e 4.193)
From (iii) above,
\[ \alpha_{2k} = \text{Ad} w_{2k} \circ \alpha_{2k-2} \circ \text{Ad} u_{2k}^* \]  
\[ = \text{Ad} W_{2k,0} \circ \gamma_{k,0} \circ \alpha \circ \gamma_{k,0}. \]  
(e 4.194)  
(e 4.195)

Similarly, by (iv),
\[ \beta_{2k+1} = \text{Ad} w_{2k+1} \circ \alpha_{2k+1} \circ \beta_{2k-1} \circ \text{Ad} u_{2k-1}^* \]  
\[ = \text{Ad} W_{2k+1,1} \circ \gamma_{k,1} \circ \beta \circ \gamma_{k,1}^{-1}. \]  
(e 4.196)  
(e 4.197)

It follows from (v) and (vi) that the sequence \( \beta_1, \alpha_2, \beta_3, ... \) converges point-wisely. By (e 4.194) and (e 4.197) that the sequence converges to
\[ \text{Ad} W_0 \circ \gamma_0 \circ \alpha \circ \gamma_0^{-1} = \text{Ad} W_1 \circ \gamma_1 \circ \beta \circ \gamma_1^{-1}. \]  
(e 4.198)

Therefore
\[ \alpha = \gamma_0^{-1} \circ \text{Ad} W_0^* \circ \text{Ad} W_1 \circ \gamma_1 \circ \beta \circ \gamma_1^{-1} \circ \gamma_0 \]  
\[ = \text{Ad} \gamma_0^{-1}(W_0^*W_1) \circ \gamma_0^{-1} \circ \gamma_1 \circ \beta \circ \gamma_1^{-1} \circ \gamma_0. \]  
(e 4.199)  
(e 4.200)

Note that, since \( \gamma_0 \) and \( \gamma_1 \) are strongly asymptotically inner, so is \( \gamma_1^{-1} \circ \gamma_0 \). Moreover, if both \( \gamma_0 \) and \( \gamma_1 \) are strong asymptotically inner, so is \( \gamma_1^{-1} \circ \gamma_0 \). Finally, let \( w = \gamma_0^{-1}(W_0^*W_1) \), \( \sigma = \gamma_0^{-1} \circ \gamma_1 \). Then \( \sigma^{-1} = \gamma_1^{-1} \circ \gamma_0 \) and
\[ \|w - 1\| = \|\gamma_0^{-1}(W_0^*W_1) - 1\| = \|W_0^*W_1 - 1\| < \epsilon/2 + \epsilon/2 = \epsilon. \]  
(e 4.201)

Moreover,
\[ \alpha = \text{Ad} w \circ \sigma \circ \beta \circ \sigma^{-1}. \]  
(e 4.202)

The theorem follows.

5 Projections in \( M_n \otimes \mathcal{Z} \)

Define a continuous function \( h_\lambda \) on \([0,1]\) as follows
\[ h_\lambda = \begin{cases} 
0, & 0 \leq t \leq \lambda \\
t/\lambda - 1, & \lambda \leq t < 2\lambda \\
1, & 2\lambda \leq t \leq 1.
\end{cases} \]

The following is a refinement of Theorem 5.4 of [8].

**Lemma 5.1.** Let \( A \) be a unital simple \( C^* \)-algebra with \( TR(A) \leq 1 \). Let \( \epsilon > 0 \), \( \eta > 0 \), \( F \subset A \) be a finite subset, \( N \geq 1 \) be an integer and \( \eta > 0 \). There is a projection \( p \in A \) and a finite dimensional \( C^* \)-subalgebra \( B \cong M_N \) with \( 1_B = p \) and a finite subset \( F_1 \subset A \) such that
\[ \|p, x\| < \epsilon \text{ for all } x \in F, \ yb = by \text{ for all } y \in F_1 \text{ and } b \in B, \]  
\[ pxp \in \epsilon F_1 \text{ for all } x \in F \text{ and } \tau(1 - p) < \eta \text{ for all } \tau \in T(A). \]  
(e 5.203)  
(e 5.204)
Proof. Let \( N_1 \) be an integer such that \( N/N_1 < \eta/4 \). Without loss of generality, we may assume that \( F \) is in the unit ball of \( A \). Since \( TR(A) \leq 1 \), as in the proof of Theorem 5.4 of \( [S] \), there exists a projection \( p_1 \in A \) and a finite dimensional \( C^* \)-subalgebra \( C \cong \bigoplus_{j=1}^J M_{r(j)} \) with \( r(j) \geq N \cdot N_1 \) and with \( 1_C = p_1 \) such that

\[
\| [p_1, x] \| < \epsilon/16(N + 1)^2 \quad \text{for all} \quad x \in F, \quad \tau(1 - p) < \eta/4 \quad \text{for all} \quad \tau \in T(A), \tag{5.205}
\]

\[
\| [p_1xp_1, c] \| < \epsilon/16(N + 1)^2 \quad \text{for all} \quad x \in F \quad \text{and} \quad c \in C \quad \text{with} \quad \| c \| \leq 1. \tag{5.206}
\]

Write \( r(j) = R_j N + r_j \), where \( R_j \geq N_1 \) and \( 0 \leq r_j < N \) are integers, \( j = 1, 2, \ldots, J \). Thus \( C \) has a projection \( p \) such that \( pCp \cong M_N \otimes C_1 \), where \( C_1 \) is a finite dimensional \( C^* \)-subalgebra and

\[
t(p) > 1 - \eta/4 \quad \text{for all} \quad t \in T(C). \tag{5.207}
\]

Since \( p \in C \), by \( (5.205) \) and \( (5.206) \),

\[
\| [p, x] \| < \epsilon/8(N + 1)^2 \quad \text{for all} \quad x \in F \quad \text{and} \tag{5.208}
\]

\[
\| [xp, c] \| < \epsilon/8(N + 1)^2 \quad \text{for all} \quad x \in F \quad \text{and} \quad c \in pCp. \tag{5.209}
\]

For each \( x \in F \), let \( c_x \in pCp \) with \( \| c \| \leq \| x \| \) such that

\[
\| c_x - xp \| < \eta/8(N + 1)^4. \tag{5.210}
\]

Let \( \{ e_{i,j}' : 1 \leq i, j \leq N \} \) be a matrix unit for \( M_N \) and let \( e_{i,j} = e_{i,j}' \otimes 1_{C_1} \). Define

\[
\Phi(xp) = (1/N) \sum_{1 \leq i, j \leq N} e_{i,j} c_x e_{i,j} \quad \text{for all} \quad x \in F. \tag{5.211}
\]

By \( (5.209) \), we estimate that

\[
\Phi(xp) \approx \eta/8(N + 1)^2 (1/N) \sum_{i=1}^N e_{i,i} c_x e_{i,i} \approx \eta/8(N+1)^2 c_x. \tag{5.212}
\]

We also have

\[
\Phi(xp)b = b\Phi(xp) \quad \text{for all} \quad x \in F \quad \text{and} \quad b \in M_N \otimes C \cdot 1_{C_1} \cong M_N. \tag{5.213}
\]

Now let \( F_1 = \{ \Phi(xp) : x \in F \} \). Then

\[
xp \in \epsilon F_1. \tag{5.214}
\]

Finally, we note that \( (5.207) \) and \( (5.206) \) imply that

\[
\tau(p) > 1 - \eta \quad \text{for all} \quad \tau \in T(A). \tag{5.214}
\]

\[\square\]

**Lemma 5.2.** Let \( a \in Z \) be a non-zero element with \( 0 \leq a \leq 1 \). For any \( \epsilon > 0 \) and \( 1 > \eta > 0 \), there is an integer \( N \geq 1 \) satisfying the following: If \( n \geq N \), there is a projection \( E \in M_n \) and a projection \( p \in A = EM_n E \otimes Z \) such that

\[
\frac{\text{rank} E}{n} > 1 - \epsilon, \quad \| ph\eta(E \otimes a) - h\eta(E \otimes a) \| < \epsilon, \tag{5.215}
\]

\[
p \in (1 \otimes a) A (1 \otimes a) \quad \text{and} \quad (t \otimes \tau)(p) > d_{r}(a) - \epsilon, \tag{5.216}
\]

where \( \tau \) is the unique tracial state on \( Z \) and \( t \) is the normalized trace on \( M_n \).

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Then (by Lemma 2.5.3 in \[7\]) there exists a projection $p \in (1_Q \otimes a)B(1_Q \otimes a)$ such that
\[
\|qh_\eta(1 \otimes a) - h_\eta(1 \otimes a)\| < \epsilon/2 \quad \text{and} \quad (t_0 \otimes \tau)(q) > d_{t_0 \otimes \tau}(1 \otimes a) - \epsilon/16 = d_\tau(a) - \epsilon/16, \tag{e.5.218}
\]
where $t_0$ is the unique tracial state on $Q$. Write $Q = \lim_{n \to \infty} (M_{n!}, \nu_n)$. There exists $N_0 \geq 1$ such that there is a projection $p' \in M_{N_1} \otimes Z$ such that
\[
\|p' - q\| < \eta/4 \quad \text{and} \quad 1/N_0 < \epsilon/16, \tag{e.5.219}
\]
where $N_1 = N_0!$ and we identify $M_{N_1}$ as a unital $C^*$-subalgebra of $Q$. Since $q \in (1_Q \otimes a)C(1_Q \otimes a)$, there is a continuous function $f \in C_0((0, 1])$ with $0 \leq f \leq 1$ such that
\[
\|qf(1_Q \otimes a) - q\| < \epsilon/8. \tag{e.5.220}
\]
It follows that
\[
\|p'f(1_Q \otimes a) - p'\| < \epsilon/4. \tag{e.5.221}
\]
View $p' \in M_{N_1} \otimes Z$, we also have
\[
\|p'f(1_{M_{N_1}} \otimes a) - p'\| < \epsilon/4 \quad \text{and} \quad \|p'h_\eta(1_{M_{N_1}} \otimes a) - h_\eta(1_{M_{N_1}} \otimes a)\| < \epsilon. \tag{e.5.222}
\]
Put $b = f(1_{M_{N_1}} \otimes a)$. Then
\[
\|bp'b - p'\| < \eta/2. \tag{e.5.223}
\]
Then (by Lemma 2.5.3 in \[7\]) there exists a projection $p_0 \in b(M_{N_1} \otimes Z)b$ such that
\[
\|p_0 - p'\| < \eta. \tag{e.5.224}
\]
Since $\eta < 1$, by (e.5.221) and (e.5.219),
\[
(t_1 \otimes \tau)(p_0) = (t_1 \otimes \tau)(p') = (t_0 \otimes \tau)(p') = (t_0 \otimes \tau)(q) > d_\tau(a) - \epsilon/16, \tag{e.5.225}
\]
where $t_1$ is the tracial state on $M_{N_1}$. Note that $b \in (1_{M_{N_1}} \otimes a)C(1_{M_{N_1}} \otimes a)$. Let $N = (N_0 + 1)!$. If $n \geq N$, we may write $n = dN_0 + r$, where $d \geq N_0!$ and $0 \leq r < N_0$. There is a projection $E \in M_n$ with $\text{rank}E = dN_0$. Then $EM_nE \cong M_d \otimes M_N$. Moreover
\[
\frac{\text{rank}E}{n} > 1 - \frac{r}{n} > 1 - \epsilon.
\]
Put $A = M_n \otimes Z$. The proof above shows that there exists a projection $p \in (E \otimes a)A(E \otimes a)$ such that
\[
(t_E \otimes \tau)(p) > d_\tau(a) - \epsilon/16, \tag{e.5.226}
\]
where $t_E$ is the tracial state on $EM_nE$. Clearly $p \in (1_{M_n} \otimes a)A(1_{M_n} \otimes a)$. We compute that
\[
(t \otimes \tau)(p) > (d_\tau(a) - \epsilon/16)(1 - r/n) \geq 1 - \epsilon/16, \tag{e.5.227}
\]
where $t$ is the tracial state on $M_n$. Note, by (e.5.222),
\[
\|ph_\eta(E \otimes a) - h_\eta(E \otimes a)\| < \epsilon. \tag{e.5.228}
\]
The following is not used in its full strength. When $A$ has sufficiently many projections, for example, $A$ has real rank zero, a version of the following is proved in [21](Lemma 3.5). An early version of this can be found in Lemma 2.8 of [19] and may be traced back to the proof of Theorem 4.5 of [5].

**Lemma 5.3.** Let $A$ be a unital simple $C^*$-algebra with $TR(A) \leq 1$ and let $e, f \in A$ be two projections such that $2\tau(e) < \tau(f)$ for all $\tau \in T(A)$. Then, for any $1 > \epsilon > 0$, there exists a projection $p \leq f$ such that

\[
\|pe\| < \epsilon, \quad \tau(p) > \tau(e) \quad \text{and} \quad \tau(p) > \tau(f) - \tau(e) - \epsilon \quad \text{for all} \quad \tau \in T(A).
\]

(e 5.229)

In particular, there is a partial isometry $w \in A$ such that

\[
w^*w = e, \quad ww^* \leq f \quad \text{and} \quad \|w^2\| < \epsilon.
\]

Moreover, one may requires that

\[
\|ph^e_{\epsilon^2/32}(fe)\| < \epsilon.
\]

(e 5.231)

**Proof.** Since $TR(A) \leq 1$ and $A$ has strict comparison (see 4.7 of [8]), we have two mutually orthogonal and mutually equivalent projections $f_1, f_2 \in A$ such that $f_1 + f_2 \leq f$ and $f_1$ is equivalent to $e$. Since $A$ has property (SP) (see 3.2 and of [8]), there are mutually orthogonal and mutually equivalent non-zero projections $f_3, f_4, f_5, f_6 \in (f - f_1 - f_2)A(f - f_1 - f_2)$. Put $f_0 = f - \sum_1^6 f_i$.

Let $N \geq 1$ be an integer such that $1/N < \min\{\max_{\tau \in T(A)} \{\tau(f_3)\}, \epsilon\}/8$ for all $\tau \in T(A)$. There are partial isometries $v_1, v_2, v_3, u_1, u_2, ..., u_5 \in A$ such that

\[
v_1^*v_1 = e, \quad v_2^*v_2 = e, \quad v_3^*v_3 = f_1, \quad v_3v_3^* = f_2,
\]

and $u_j u^*_j = f_{j+1}$, $j = 1, 2, ..., 5$.

(e 5.232)

(e 5.233)

Choose $\eta > 0$ such that

\[
\|h^e_{\eta^2/32}(a) - h^e_{\eta^2/32}(b)\| < \epsilon/16
\]

(e 5.234)

provided that $0 \leq a, b \leq 1$ and $\|a - b\| < \eta$, where $a, b \in A$. We may assume that $\eta < \min\{1, \epsilon, \delta/16\}$.

If $ef = 0$, the lemma follows easily. So we assume that $ef \neq 0$. Let $\delta = \min\{\epsilon^2/128, \|ef\|/4, 1/4\}$. Since $A$ is simple, there are $x_1, x_2, ..., x_n \in A$ such that

\[
\sum_{j=1}^n x_j^* h_\delta(ef)x_j = 1.
\]

(e 5.235)

Let $M = \max\{\|x_i\| : 1 \leq i \leq n\}$ and let

\[F = \{f, e, fe, ef, fe, v_1, v_2, v_3, u_1, ..., u_5, x_1, x_1^*, \}, \]

Fix $0 < \eta' \leq \eta$. Since $TR(A) \leq 1$, using $F$, one obtains a projection $P \in A$ and a $C^*$-subalgebra $C \cong \bigoplus_{j=1}^J M_{r(j)}(C(I_j))$ with $1_C = P$ and with $r(j) \geq N$, where $I_j$ is either a point or $I_j = [0, 1]$, such that

\[
\tau(1 - P) < \min\{\tau(f_3), \epsilon\}/16 \quad \text{for all} \quad \tau \in T(A),
\]

(e 5.236)

\[
\|xP - Px\| < \eta'/98(16Mn) \quad \text{for all} \quad x \in F,
\]

(e 5.237)

\[
\|f_i - (f'_i + f''_i)\| < \eta/96(16Mn), \quad \|e - (e' + e'')\| < \eta/96(16Mn),
\]

(e 5.238)

\[
e', f'_i \in C, \quad e'', f''_i \in (1 - P)A(1 - P), \quad i = 0, 1, ..., 6,
\]

(e 5.239)
where \( e', f'_i, f''_i, e'' \) are non-zero projections and \( f'_i f'_j = 0, f''_i f''_j = 0 \), if \( i \neq j \). We may further assume that

\[
(t(e')) = (t(f'_i), t(f'_j), t(f'_3), t(f'_1) = t(f'_5) = t(f'_6) \text{ for all } t \in T(C), \quad i = 1, 2. \quad (e.5.240)
\]

Put \( f' = \sum_{i=0}^{6} f'_i \).

Let \( \delta = \min\{\epsilon^2/128, 1/4\} \) and put \( a = f'e'f' \). Note that

\[
\| \sum_{i=1}^{n} y_i h_\delta(f'e'f')y_i - P \| < \delta/16 \quad (e.5.241)
\]

for some \( y_i \in C, \quad i = 1, 2, ..., n \), if we choose \( \eta' \) sufficiently small (depending only additionally on \( \delta \)).

We see that \( h_\delta(a) \) is invertible (in each summand of \( C \)). Let \( R(j) \) be the least rank of \( h_\delta(a) \) and \( R(j)' \) be the largest rank of \( h_{\delta/2}(a) \) in \( f'M_{r(j)}(I_j)f' \), \( j = 1, 2, ..., J \). Note the rank of \( e' \) is at least \( R(j)' \) in \( j \)-th summand. So \( f'_2 \) has rank at most \( R(j)' \) in \( j \)-th summand.

It follows from Lemma C of [1] that there is a projection \( q \in f'Cf' \) such that

\[
qh_\delta(a) = h_\delta(a) \quad (e.5.242)
\]

and the rank of \( q \) at each summand \( f'M_{r(j)}(I_j)f' \) is at most \( R(j)' + 1 \). Put \( p_1 = f' - q \in f'Cf' \).

Then

\[
p_1 h_\delta(a) = 0. \quad (e.5.243)
\]

Moreover

\[
\tau(p_1) > \tau(f') - \tau(e') - 1/N > \tau(f) - \tau(e') - \epsilon/16 - \tau(1 - P) \quad (e.5.244)
\]

\[
> \tau(f) - \tau(e) - \epsilon/4 \text{ for all } \tau \in T(A). \quad (e.5.245)
\]

We also have

\[
\| h_\delta(a)f'e' - f'e' \|^2 \quad (e.5.246)
\]

\[
= \| h_\delta(a)f'e'f'h_\delta(a) - h_\delta(a)f'e'f'h_\delta(a) + f'e'f'h_\delta(a) + f'e'f' \| \quad (e.5.247)
\]

\[
\leq 2\| h_\delta(f'e'f')f'e'f' - f'e'f' \| \leq 2\delta. \quad (e.5.248)
\]

It follows that

\[
\| p_1 e' \| = \| qf'e' - f'e' \| \quad (e.5.249)
\]

\[
\leq \| qf'e' - qh_\delta(a)f'e' \| + \| qh_\delta(a)f'e' - h_\delta(a)f'e' \| + \| h_\delta(a)f'e' - f'e' \| \quad (e.5.250)
\]

\[
< \sqrt{2\delta} + \sqrt{2\delta} + \epsilon/64. \quad (e.5.251)
\]

Now we count rank of \( p_1 \). Since \( f' = \sum_{i=0}^{6} f'_i \) and the rank of \( q \) in \( j \)-th summand is not more than that of \( f'_i \), \( f'_2 \) has the same rank as that of \( f'_1 \) which has the rank at that of \( e' \) at each summand of \( C \), \( p_1 = f' - q \) has rank great than that of \( e' \) at each summand of \( C \). So there is a partial isometry \( w_1 \in C \) such that

\[
w_1^* w_1 = e' \text{ and } w_1 w_1^* \leq p_1. \quad (e.5.253)
\]

It follows (from \( e.5.252 \)) that

\[
\| w_1 w_1 \| = \| w_1 e' p_1 w_1 \| < \epsilon/64 \quad (e.5.254)
\]
Note that
\[ \tau(1 - P) < \tau(f'_p) \leq \tau(p_1 - w_1 w_1^*) \] for all \( \tau \in T(A) \). (e 5.255)

There is a partial isometry \( w_2 \in A \) such that
\[ w_2^* w_2 = e'' \quad \text{and} \quad w_2 w_2^* \leq p_1 - w_1 w_1^*. \] (e 5.256)

Since \( p_1 - w_1 w_1^* \in PAP \), we have \( w_2^2 = 0 \). Moreover
\[ w_1 w_2 = w_1 e'(p_1 - w_1 w_1^*) w_2 = 0 \quad \text{and} \quad w_2 w_1 = w_2 e'' p_1 w_1 = 0. \] (e 5.257)

Put \( w' = w_1 + w_2 \). Then, by (e 5.254) and (e 5.257)
\[ (w')^* w' = e' + e'', \quad w'(w')^* \leq p_1 \quad \text{and} \quad \|w'(w')^*\| < \epsilon/64, \] (e 5.258)

Since
\[ \|fp_1 - p_1\| < \epsilon/16, \] (e 5.259)
there is a projection \( p \in A \) such that \( p \leq f \) and
\[ \|p_1 - p\| < \epsilon/8. \] (e 5.260)

It follows from (e 5.258), (e 5.258) and (e 5.260) that there exists a partial isometry \( w \in A \) such that
\[ w^* w = e, \quad ww^* \leq f \quad \text{and} \quad \|w^2\| < \epsilon. \] (e 5.261)

From (e 5.253) and (e 5.255), we have
\[ \tau(e) = \tau(e') + \tau(e'') \leq \tau(w_1 w_1^*) + \tau(p_1 - w_1 w_1^*) = \tau(p) \quad \text{for all} \quad \tau \in T(A). \] (e 5.262)

We also have, (using (e 5.245) for (e 5.264),
\[ \|pe\| < \epsilon/8 + \|p_1 (e' + e'')\| = \eta/8 + \|p_1 e'\| < \eta/8 + \epsilon/16 < \epsilon \quad \text{and} \quad \tau(p) = \tau(p_1) > \tau(f) - \tau(e) - \epsilon/4 \quad \text{for all} \quad \tau \in T(A). \] (e 5.263)

Finally, by the choice of \( \eta \), one computes that
\[ ph_{1/32}(f e f) \approx \epsilon/8 \quad \text{p(h}_{1/32}(a) + h_{1/32}(f'' e'' f'')) \approx \epsilon/8 \quad \text{p_1 h_8(a) = 0}. \]

This completes the proof.

The following probably holds in much more general setting. Recall that a sequence \( \{a_n\} \) is said to be central in \( A \), if \( \lim_{n \to \infty} \|a_n b - b a_n\| = 0 \) for all \( b \in A \).

**Proposition 5.4.** Let \( A \) be a unital separable nuclear simple \( C^* \)-algebra with \( TR(A) \leq 1 \). Let \( \{e_n\} \) and \( \{p_n\} \) be two central sequences for \( A \) satisfying
\[ \lim_{n \to \infty} \sup_{\tau \in T(A)} \tau(e_n) = 0 \quad \text{and} \quad \liminf_{n \to \infty} \min_{\tau \in T(A)} \tau(p_n) > 0. \] (e 5.265)

Then there exists a central sequence of partial isometries \( \{w_n\} \) in \( A \) such that
\[ \lim_{n \to \infty} \|w_n^* w_n - e_n\| = 0, \quad \lim_{n \to \infty} \|p_n w_n - w_n\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \|w_n^2\| = 0. \] (e 5.266)
Proof. We first note, by Corollary 8.4 of [13], since \( TR(A) \leq 1 \), \( A \) is \( Z \)-stable. Let \( \{F_n\} \) be an increasing sequence of finite subsets of the unit ball of \( A \) such that \( \bigcup_{n=1}^{\infty} F_n \) is dense in the unit ball of \( A \). Since \( \{p_n e_n p_n\} \) is a central sequence for \( A \), for each \( k \), there exists \( n(k) \) such that
\[
\|h_{1/32k^2}(p_n e_n) p_n x - x h_{1/32k^2}(p_n e_n p_n)\| < 1/k \quad \text{for all } \ x \in F_k
\] (e 5.267)
and for all \( n \geq n(k), \ k = 1, 2, \ldots \). We may assume that \( n(k+1) > n(k), \ k = 1, 2, \ldots \). Define \( a_1 = 1/32, \ a_j = 1/32k^2 \), if \( n(k) \leq j < n(k+1), \ k = 1, 2, \ldots \). Then \( \{h_{n_k}(p_n e_n p_n)\} \) is also a central sequence for \( A \).

For each \( n \), defined \( y_n = p_n - h_{a_n}(p_n e_n p_n) \). Then \( 0 \leq y_n \leq 1, \ n = 1, 2, \ldots, \) and \( \{y_n\} \) is a central sequence for \( A \). By applying 5.3 one has a projection \( q_n \in p_n \) such that
\[
\|q_n h_{a_n}(p_n e_n p_n)\| < 1/k \quad \text{and } \tau(q_n) \geq \tau(p_n) - \tau(e_n) - 1/k \quad \text{for all } \ n \geq n(k),
\] (e 5.268)
\( k = 1, 2, \ldots \). We also have
\[
\lim_{n \to \infty} \|h_{a_n}(p_n e_n p_n) p_n e_n - p_n e_n\|^2 = 0.
\] (e 5.269)

It follows that
\[
\lim_{n \to \infty} \|y_n e_n\| = \lim_{n \to \infty} \|(p_n - h_{a_n}(p_n e_n p_n)) e_n\| = \lim_{n \to \infty} \|p_n e_n - h_{a_n}(p_n e_n p_n)\| = 0.
\] (e 5.270)

For \( n \geq n(k) \),
\[
\|q_n y_n - q_n\| = \|q_n h_{a_n}(p_n e_n p_n)\| < 1/k, \ k = 1, 2, \ldots.
\] (e 5.272)

Therefore, for any integer \( m \geq 1 \), if \( n \geq n(k) \),
\[
\|y_n^m q_n y_n^m - q_n\| < 2m/k, \ k = 1, 2, \ldots.
\] (e 5.273)

Thus, if \( n \geq n(k) \),
\[
\tau(y_n^m) > \tau(y_n^m q_n y_n^m) > \tau(q_n) - 2m/k,
\] (e 5.274)
\( k = 1, 2, \ldots \). It follows that
\[
\inf_{m \in \mathbb{N}} \lim_{n \to \infty} \min_{\tau \in T(A)} \tau(y_n^m) \quad \text{and} \quad \inf_{m \in \mathbb{N}} \lim_{n \to \infty} \min_{\tau \in T(A)} \tau(q_n) = \inf_{m \in \mathbb{N}} \lim_{n \to \infty} \min_{\tau \in T(A)} \tau(p_n) > 0.
\] (e 5.275)

By Theorem 4.2 of [17], since \( A \) is \( Z \)-stable, there is a central sequence \( \{s_n\} \) for \( A \) such that
\[
\lim_{n \to \infty} \|s_n s_n - e_n\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \|y_n s_n - s_n\| = 0.
\] (e 5.277)

It is standard that there exists a central sequence of partial isometries \( \{w_n\} \) of \( A \) such that
\[
w_n^* w_n = e_n \quad \text{and} \quad w_n w_n^* \leq f_n \quad \text{and} \quad \|w_n - s_n\| = 0.
\] (e 5.278)

Therefore \( \{w_n\} \) is a central sequence of \( A \). By (e 5.271) and (e 5.271),
\[
\lim_{n \to \infty} \|w_n^2\| = \lim_{n \to \infty} \|s_n^2\| = \lim_{n \to \infty} \|s_n e_n y_n s_n\| = \lim_{n \to \infty} \|e_n y_n\| = 0.
\] (e 5.279)
6 Existence of automorphisms with the Rokhlin property

The following fact will be used without further notices.

**Lemma 6.1.** Let $A$ be a unital simple $C^*$-algebra with $T(A) \neq \emptyset$. Suppose that $\alpha \in \text{Aut}(A)$. Then, for any positive element $a \in A$,

$$\inf \{ \tau \circ \alpha(a) : \tau \in T(A) \} = \inf \{ \tau(a) : \tau \in T(A) \} \quad \text{and} \quad \sup \{ \tau \circ \alpha(a) : \tau \in T(A) \} = \sup \{ \tau(a) : \tau \in T(A) \}. \quad (e 6.282)$$

In particular, if $1 > \eta > 0$ and $e \in A$ is a projection such that $\tau(e) < \eta$ for all $\tau \in T(A)$, then

$$\tau(\alpha(e)) < \eta \quad \text{for all} \quad \tau \in T(A). \quad (e 6.284)$$

**Proof.** This follows from the fact that $\tau \circ \alpha$ is a tracial state of $A$ for all $\tau \in T(A)$.\hfill $\square$

**Lemma 6.2.** Let $A$ be a unital separable amenable simple $C^*$-algebra with $\text{TR}(A) \leq 1$. Let $\alpha \in \text{Aut}(A)$ and $\sigma \in \text{Aut}(Z)$ as defined in (2.8). Then $\alpha \otimes \sigma$ has tracial Rokhlin property (see (20)). Moreover, one has the following: Let $k \geq 2$ be an integer. Let $\epsilon > 0$ and $1/k > \eta > 0$, let $F \subset A \otimes Z$ be a finite subset. There are mutually orthogonal projections $p_0, p_1, \ldots, p_{k-1} \in A \otimes Z$ such that

$$\|p_j, x\| < \epsilon \quad \text{for all} \quad x \in F, \quad (\alpha \otimes \sigma)^j(p_0) = p_j, \quad (e 6.285)$$

$$\tau(p_j) > 1/k - \eta \quad \text{for all} \quad \tau \in T(A), \quad j = 0, 1, 2, \ldots, k-1, \quad (e 6.286)$$

$$\tau(1 - \sum_{j=0}^{k-1} (\alpha \otimes \sigma)^j(p_0)) < \eta \quad \text{and} \quad \tau((\alpha \otimes \sigma)(p_{k-1})) > 1/k - \eta \quad \text{for all} \quad \tau \in T(A). \quad (e 6.287)$$

**Proof.** Note that, by Corollary 8.4 of (13), $A \otimes Z \cong A$. We may prove the part of the statement in the theorem after “Moreover” for $F \subset A$.

Write $Z = \bigotimes_{n \in \mathbb{N}} Z$ as an inductive limit of $\bigotimes_{1 \leq j \leq n} Z$. To simplify notation, if necessary, without loss of generality, we may assume that $F$ is in the unit ball of $A \bigotimes \bigotimes_{1 \leq j \leq n} Z$. Put $\beta_0 = \alpha \otimes \bigotimes_{1 \leq j \leq n} Z_0$. Note that $\beta(x) = \beta_0(x)$ for all $x \in F$. We will continue to use $\sigma$ for $\sigma\big|_{\bigotimes_{m \geq n+1} Z}$ and identify $\bigotimes_{m \geq n+1} Z$ with $Z$ whenever it is convenient.

Let $1/4k > \eta > 0$. It follows from Proposition 4.4 of (22) that there are mutually orthogonal elements $f_0, f_1, \ldots, f_{k-1} \in Z$ with $0 \leq f_j \leq 1$ ($0 \leq j \leq k$) such that

$$\sigma(f_j) = f_{j+1}, \quad j = 0, 1, \ldots, k-1 \quad \text{and} \quad \tau_Z(1 - \sum_{j=1}^{k} f_j) < \eta/16, \quad (e 6.288)$$

where $\tau_Z$ is the unique tracial state on $Z$.

Put $f_k = \sigma(f_{k-1})$. Note that $\sigma$ is asymptotically inner. Therefore $\tau(f_k) = \tau(f_0)$ for all $\tau \in T(A)$.

Fix $\epsilon_1 > 0$ with $\epsilon_1 < \min\{\eta, \epsilon\}$. There is $N$ given by (5.2) for $f_k$ and $\epsilon_1/64k$ (instead of $\epsilon$).

Define

$$F_1 = \{ \beta_0^j(x) : x \in F \quad \text{and} \quad -k \leq j \leq k \}. \quad (e 6.289)$$

Put $A_1 = A \bigotimes \bigotimes_{m \geq n+1} Z$. Note that $A_1 \cong A \otimes Z \cong A$. It follows from (5.1) that there is a projection $P_1 \in A_1$, a finite dimensional $C^*$-subalgebra $B_1 = M_N$ with $1_{B_1} = P_1$ and a finite subset $F_2$ in the unit ball of $A_1$ such that

$$\tau(1 - P_1) < \epsilon_1/64 \quad \text{for all} \quad \tau \in T(A), \quad (e 6.289)$$

$$\|P_1, x\| < \epsilon_1/64, \quad P_1xP_1 \in \eta/64 F_2 \quad \text{for all} \quad x \in F_1 \quad \text{and} \quad yb = by \quad \text{for all} \quad x \in F_2 \quad \text{and} \quad b \in B_1. \quad (e 6.291)$$
Put \( C_1 = B_1 \otimes Z \). By applying \(5.2\) we obtain projections \( p_k \in (P_1 \otimes f_k)C_1(P_1 \otimes f_k) \) such that

\[
(t \otimes \tau_Z)(p_k) > d_{\tau_Z}(f_k) - \epsilon_1/64k = d_{\tau_Z}(f_0) - \epsilon_1/64k
\]  
(e 6.292)

for all tracial state \( t \) on \( B_1 \). Let \( D_j \) be hereditary \( C^* \)-subalgebra generated by \( 1_A \otimes f_j \).

Put \( \beta = \beta_0 \otimes \sigma \) and put \( p_{k-j} = \beta^{-j}(p_k) \). Then \( p_j \in D_j \) and \( p_0, p_1, ..., p_{k-1} \) are mutually orthogonal and

\[
\beta(p_j) = p_{j+1}, \ j = 0, 1, ..., k - 1.
\]  
(e 6.293)

For each \( \tau \in T(A) \), define \( t(b) = \frac{1}{\tau(P_1)} \tau(b) \) for \( b \in B_1 \). Then \( t \) is the unique tracial state on \( B_1 \). We also estimate that

\[
(\frac{1}{\tau(P_1)})(\tau \otimes \tau_Z)(p_0) > (t \otimes \tau_Z)(p_0) > d_{\tau_Z}(f_0) - \eta/64k.
\]  
(e 6.294)

It follows that

\[
(\tau \otimes \tau_Z)(p_0) > (1 - \eta/64)(d_{\tau_Z}(f_0) - \eta/64k)
\]  
(e 6.295)

for all \( \tau \in T(A) \). Therefore

\[
(\tau \otimes \tau_Z)(\sum_{j=0}^{k-1} \beta^j(p_0)) > (1 - \eta/64)(t \otimes \tau_Z)(\sum_{j=0}^{k-1} \beta^j(p_0))
\]  
(e 6.296)

\[
> (1 - \eta/64)(\sum_{j=0}^{k-1} \tau_Z(f_j) - \eta/64)
\]  
(e 6.297)

\[
> (1 - \eta/64)(1 - \eta/16)
\]

\[
> 1 - \eta/4 \text{ for all } \tau \in T(A).
\]  
(e 6.298)

\[
> 1 - \eta/4 \text{ for all } \tau \in T(A).
\]  
(e 6.299)

Finally, we note that since \( p_k \in C_1 \), for any \( y \in F_2 \),

\[
p_ky = yp_k \text{ for all } y \in F_2.
\]  
(e 6.300)

Let \( x \in F \). There is \( y \in F_2 \) such that

\[
\|y - P_1\beta^j_0(x)P_1\| < \eta_1/64.
\]  
(e 6.301)

It follows that, for \( x \in F \),

\[
p_{k-j}x = \beta^{-j}(p_k)\beta^{-j}(p_1)\beta^{-j}(\beta^j(x)) = \beta^{-j}(p_kP_1\beta^j(x)) \quad \text{ (e 6.302)}
\]

\[
\approx_{\eta_1/64} \beta^{-j}(p_kP_1\beta^j(x)P_1) \approx_{\eta_1/64} \alpha^{-j}(p_ky) \quad \text{ (e 6.303)}
\]

\[
\approx_{\eta_1/64} \beta^{-j}(\beta^j(x)p_k) = x\beta^{-j}(p_k) = xp_{k-j}, \quad \text{ (e 6.304)}
\]

\[
j = 1, 2, ..., k.
\]

The proof of the following is based on, again, an argument of Kishimoto which was also further clarified in Lemma 4.3 of \([21]\). We state here for the case that \( A \) is not assumed to have real rank zero. We should notice that if \( \alpha \) has the Rokhlin property then \( \text{Ad } u \circ \alpha \) may not have the Rokhlin property in general.
Lemma 6.3. Let $A$ be a unital separable amenable simple $C^*$-algebra with $TR(A) \leq 1$, let $\alpha \in \text{Aut}(A)$ be an automorphism and let $k \geq 1$ be an integer. Suppose $\alpha$ has the following property: there exists a central sequence of projections $\{p_n\}$ of $A$ such that
\[
\lim_{n \to \infty} \|\alpha^i(p_n)\alpha^j(p_n)\| = 0 \text{ if } i \neq j, \quad 0 \leq i, j \leq k - 1, \quad \text{and}
\]
\[
\lim_{n \to \infty} \sup_{\tau \in T(A)} (1 - \sum_{j=0}^{k-1} \alpha^j(p_n)) = 0.
\]

Then, for any $\epsilon > 0$ and any finite subset $F \subset A$, there exist projections
\[
e_1(0), \ne_2(0), ..., \ne_k(0), \ne_1(1), \ne_2(1), ..., \ne_k(1) \subset A
\]
with $\sum_{j=1}^{k-1} \ne_j(0) + \sum_{j=1}^k \ne_j(1) = 1_A$, a continuous path of unitaries $\{U(t) : t \in [0,1]\} \subset A$ such that
\[
U(0) = 1_A, \quad \|[x, U(t)]\| < \epsilon \text{ for all } x \in F \text{ and } t \in [0,1],
\]
\[
\|\text{Ad} U(1) \circ \alpha(\ne_j(0)) - \ne_j(1)\| < \epsilon, \quad j = 0, 1, 2, ..., k + i - 2, \quad i = 0, 1
\]
\[
\|[x, \ne_j(1)]\| < \epsilon \text{ for all } x \in F, \quad j = 0, 1, ..., k + i - 2, \quad i = 0, 1.
\]

Proof. Let $l^\infty(A) = \prod_{n=1}^{\infty} A$ be the $C^*$-algebra product of infinite copies of $A$ and $c_0(A) = \bigoplus_{n=1}^{\infty} A$ be the $C^*$-algebra direct sum of infinite copies of $A$. Denote by $A_d = \{(a, a, ..., a, ...) : a \in A\}$, a $C^*$-subalgebra of $l^\infty(A)$. Let $q(A) = l^\infty(A) / c_0(A)$ and let $\pi : l^\infty(A) \to c_0(A)$ be the quotient map. It is standard that there are, for each $n$, mutually orthogonal projections $\{p_{j,n} : 0 \leq j \leq k\}$ such that
\[
\pi(\{p_{j,n}\}) = \pi(\{\alpha^j(p_n)\}), \quad j = 0, 1, 2, ..., k.
\]

Let $q_n = 1 - \sum_{j=1}^{k-1} p_{j,n}, n = 1, 2, ...$. Then $\pi(\{p_{j,n}\})$ and $\pi(\{q_n\})$ are in $q(A) \cap \pi(A_d)'$. Note that, by 6.1
\[
\inf_{\tau \in T(A)} \tau(p_{j,n}) = \inf_{\tau \in T(A)} \tau(p_n) \geq 1/k \text{ and } \lim_{n \to \infty} \sup_{\tau \in T(A)} \tau(q_n) = 0.
\]

It follows from 5.4 that there exists a sequence of partial isometries $\{w_n\} \in l^\infty(A)$ such that
\[
\lim_{n \to \infty} \|w_n^* w_n - q_n\| = 0, \quad \lim_{n \to \infty} \|p_{k,n} w_n - w_n\| = 0 \text{ and } \lim_{n \to \infty} \|w_n^2\| = 0.
\]

Let $w = \pi(\{w_n\})$. Then $w$ is a partial isometry in $q(A)$ such that
\[
w^* w = \pi(\{q_n\}), \quad w w^* \leq \pi(\{p_{k,n}\}) \quad \text{and} \quad w^2 = 0.
\]

Let $V = w + w^* + 1 - w^* w = w + w^* + (1 - q_n - w_n w_n^*)$, $n = 1, 2, ...$. Note that $\pi(\{u'_n\}) = V$. Then $V$ is a unitary in $q(A)$. Moreover, $V \in q(A) \cap \pi(A_d)'$. Put $E = \pi(\{q_n\} + \{p_{0,n}\})$. Then $E$ is a projection in $q(A) \cap \pi(A_d)$ It is important to note, since $w^2 = 0, w w^* \pi(\{q_n\}) = \pi(\{q_n\}) w w^* = 0$. Since $p_{0,n}, p_{1,n}, ..., p_{k-1,n}$ mutually orthogonal and
\[
\pi(\{q_n\}) + \sum_{j=1}^{k-1} \pi(\{p_{j,n}\}) = 1_{q(A)},
\]
\[
w w^* \leq \sum_{j=1}^{k-1} \pi(\{p_{j,n}\}).
\]
Since
\[ \pi(\{\alpha(p_{j,n})\})\pi(\{\alpha(p_{k-1,n})\}) = 0 \text{ for all } j = 0, 1, ..., k - 2, \quad (e \, 6.317) \]
\[ w w^* \pi(\{p_{j,n}\}) = \pi(\{p_{j,n}\})w w^* = 0 \text{ for all } j = 1, 2, ..., k - 2. \quad (e \, 6.318) \]
Combining (e 6.316) and (e 6.317), we have
\[ w w^* \leq p_{0,n}. \quad (e \, 6.319) \]
It follows that \( V(1 - E) = 1 - E \). By (e 6.314), we also have
\[ \pi(\{q_n\}) \leq w^* \pi(\alpha(\{p_{k-1,n}\}))w. \quad (e \, 6.320) \]
Since \( V \) is selfadjoint, \( sp(V) \subset \{-1, 1\} \). It follows that there is a projection \( F \in q(A) \) such that \( V = F - (1 - F) = 2F - 1 \). It follows that \( F \in q(A) \cap \pi(\{A_j\})' \). Therefore there exists a central sequence of projections \( \{f_n\} \) in \( A \) such that \( \pi(\{f_n\}) = F \). For each \( n \), define \( U_n(t) = f_n + e^{\pi it}(1 - f_n) \) for \( t \in [0, 1] \). Then \( \{U_n(t) : t \in [0, 1]\} \) is a continuous path of unitaries in \( A \) such that \( U_n(0) = 1_A \) and \( U_n(1) = f_n - (1 - f_n) \). Moreover \( \pi(\{U_n(1)\}) = V \). Define a contractive completely positive linear map \( \Lambda_n : A \to A \) by \( \Lambda_n(a) = \alpha^{-1}(u_n^* a(u_n)^*) \) for all \( a \in A \). Define \( \tilde{e}_{j}^{(1)} = \pi(\Lambda_{n-j}(q_n)), j = 0, 1, ..., k \). Note that \( e_j^{(1)} \in q(A) \) is a projection and \( e_j^{(0)} = \pi(\{q_n\}) \). We compute that
\[ \tilde{e}_{j-1}^{(1)} = \pi(\{\alpha^{-1}(u_n^* q_n(u_n)^*)\}) \leq \pi(\{\alpha^{-1} \circ \alpha(\{p_{k-1,n}\})\}) = \pi(\{p_{k-1,n}\}). \quad (e \, 6.321) \]
\[ j = 2, 3, ..., k - 1. \]
In particular, \( \{\tilde{e}_{j}^{(1)} : 1 \leq j \leq k\} \) is a set of mutually orthogonal projections. Therefore, we obtain, for each \( j \), a sequence of projections \( e^{(1)}_{j,n} \in A \) such that \( e^{(1)}_{j,n} \leq p_{j,n} \) for all \( n \), \( \pi(\{e^{(1)}_{j,n}\}) = \tilde{e}_{j}^{(1)}, \) and \( j = 0, 1, 2, ..., k - 1 \) and \( e^{(0)}_{k,n} = q_n \), \( n = 1, 2, ..., k \). In particular, \( \{e^{(1)}_{j,n} : 0 \leq j \leq k\} \) is a set of mutually orthogonal projections for each \( n \). Moreover,
\[ \pi(\{\text{Ad} u_n' \circ \alpha(e_{j,n}^{(1)})\}) = \pi(\{e_{j+1,n}^{(1)}\}), \quad j = 0, 1, ..., k - 1. \quad (e \, 6.322) \]
Define
\[ e_{j,n}^{(0)} = p_{j,n} - e_{j,n}^{(1)}, \quad j = 0, 1, ..., k - 1. \]
By (e 6.322) and (e 6.311),
\[ \text{Ad} V(\pi(\{\alpha(e_{j,n}^{(0)})\})) = \pi(\{e_{j+1,n}^{(0)}\}), \quad j = 0, 1, 2, ..., k - 2. \quad (e \, 6.323) \]
Note that
\[ \sum_{j=1}^{k-1} e_{j,n}^{(0)} + \sum_{j=1}^{k} e_{j,n}^{(1)} = 1. \quad (e \, 6.324) \]
Now, fix an \( \epsilon > 0 \) and a finite subset \( F \), the lemma follows by taking \( e_{j,n}^{(0)}, e_{j,n}^{(1)} \) and \( U_n(t) \) for sufficiently large \( n \).
6.4. Let $B_k = \bigotimes_{j=1}^k Z$, the tensor product of $k$ copies of $Z$. Denote $C_k = \bigotimes_{j \geq k+1} Z$. $D_n = \bigotimes_{j=1}^n B_j$, $n = 1, 2, \ldots$

In what follows, we write $Z = \bigotimes_{n \in \mathbb{N}} Z = D_k \otimes C_k$. Define $\sigma_k^i = \sigma|_{C_k}$, $k = 2, 3, \ldots$

**Theorem 6.5.** Let $A$ be a unital separable amenable simple $C^*$-algebra with $\text{TR}(A) \leq 1$ and let $\alpha \in \text{Aut}(A)$. Then there exists an automorphism $\tilde{\alpha} \in \text{Aut}(A)$ which has the Rohklin property such that $\alpha$ and $\tilde{\alpha}$ are strongly asymptotically unitarily equivalent.

**Proof.** Since $A \cong A \otimes Z$ and $Z \otimes Z$ is asymptotically unitarily equivalent to $Z$, $\alpha$ and $\alpha \otimes \sigma$ are strongly asymptotically unitarily equivalent.

For each $k$, let $\mathcal{F}_{n,k}$ be an increasing sequence of finite subsets of the unit ball of

$$A \otimes D_n \otimes B_k \otimes C_{n+k}.$$

We may assume, without loss of generality, that $\mathcal{F}_{n,k+1} \supset \mathcal{F}_{m,k}$ for all $k$ and $n \geq m$, and $\cup_{k=1}^\infty \mathcal{F}_{n,k}$ is the unit ball of $A$.

Put $\alpha_{1,1} = \alpha \otimes \sigma$ and $\alpha_{1}^\prime = \alpha_{1,1}|_{A \otimes B_1}$. Suppose that we have constructed two sets of projections $\{e_{j,k}^{(0)}, j = 0, 1, \ldots, k-1, \} \textrm{ and } \{e_{j,k}^{(1)}, j = 0, 1, \ldots, k \}$, in $A \otimes D_n \otimes B_k \otimes 1_{C_{n+k}}$, a continuous path of unitaries $\{w_{n,k}(t) : t \in [0, 1]\}$ with $w_{n,k}(0) = 1$ in $A \otimes D_n \otimes B_k \otimes 1_{C_{n+k}}$, an automorphism $\alpha_{n,k}$ on $A \otimes D_n \otimes B_k \otimes 1_{C_{n+k}}$ and a finite subset $\mathcal{G}_{n,k} \subset A \otimes D_n \otimes B_k \otimes 1_{C_{n+k}}$ such that

$$\sum_{j=1}^{k-1} e_{j,n,k}^{(0)} + \sum_{j=1}^{k} e_{j,n,k}^{(1)} = 1; \quad (e \ 6.325)$$

$$\|x, e_{j,n,k}^{(i)}\| < 1/(n+k) \quad \text{for all } x \in \mathcal{G}_{n,k-1}, \ j = 0, 2, \ldots, k-i+1 \text{ and } i = 0, 1, \quad (e \ 6.326)$$

$$\|\text{Ad } w_{n,k} \circ \alpha_{n,k-1}(e_{j,n,k}^{(i)}) - e_{j+1,n,k}^{(i)}\| < 1/(n+k), \ j = 0, 1, \ldots, k-2+i, \ i = 0, 1, \quad (e \ 6.327)$$

$$\|w_{n,k}(t), x\| < 1/2^{n+k}, \ x \in \mathcal{G}_{n,k-1} \text{ and } t \in [0, 1], \quad (e \ 6.328)$$

$$\mathcal{G}_{n,k} \supset \cup_{j=1}^k \mathcal{F}_{n,j} \cup \alpha_{n,k}(\mathcal{G}_{n,k-1}) \cup \{\alpha_{n,k}(e_{i,n,k}^{(i)} : i, j\}, \quad (e \ 6.329)$$

where $\alpha_{k}' = (\text{Ad } w_{n,k} \circ \alpha_{n,k-1}'|_{C_{n,k+1}}) \otimes \sigma_0$ (defined on $A \otimes D_n \otimes B_k \otimes 1_{C_{n+k+1}}$), where we identify $B_k$ with $B_{k-1} \otimes Z$, $\alpha_{n,m} = \alpha_{n,m+1}', \ k = 1, 2, \ldots, n_1 \leq n$, $n_1 = 1, 2, \ldots, m$. It is important to note that $\alpha_{n,m}^\prime$ satisfies the assumption of 6.3 for $k = n_1$, by applying 6.3 on $\alpha_{n,m_1}'$, we obtain $\{e \ 6.325\}, \{e \ 6.326\}, \{e \ 6.327\}, \{e \ 6.328\}$ and $\{e \ 6.329\}$ for $k = n_1 + 1$. By induction, we obtain $\{e \ 6.325\}, \{e \ 6.326\}, \{e \ 6.327\}, \{e \ 6.328\}$ and $\{e \ 6.329\}$ for all $k = n_1 + 1$, if $n_1 < n$. If $k = n_1 = n$, we also set $\alpha_{n+1,0}' = \alpha_{n,n}$ and, as above, obtain $\{w_{n+1,1}(t) : t \in [0, 1]\}$ with $w_{n+1,1}(0) = 1$ and $\alpha_{n+1,1}' = \text{Ad } w_{n+1,1}(1) \circ \alpha_{n+1,0}' \otimes \sigma_0$.

Define $\alpha_{n,k} = \alpha_{k}' \otimes \sigma|_{C_{n+k}}$.

Consider the sequence of automorphisms $\{\beta_m\}$ as follows: $\alpha_{1,1}, \alpha_{2,1}, \alpha_{2,2}, \ldots, \alpha_{n-1,n-1} = \alpha_{n,0}, \alpha_{n,n}, \alpha_{n+1}, \ldots, \alpha_{n,n} = \alpha_{n+1,0}, \alpha_{n+1,1}, \ldots$.

We will show that $\beta_m(x)$ is Cauchy for all $x \in A \otimes Z$. In fact, if $x \in \mathcal{G}_{n,k-1}$ and $m \geq n > k$, by $\{e \ 6.328\}$,

$$\|\alpha_{m,k}(x) - \alpha_{m,k+1}(x)\| < 1/2^{m+k} \text{ and if } m = n = k, \quad (e \ 6.330)$$

$$\|\alpha_{n+1,0}(x) - \alpha_{n+1,1}(x)\| < 1/2^{m+1}. \quad (e \ 6.331)$$

It follows that, with $m \geq k', n$ and $n \geq k$,

$$\|\alpha_{m,k'}(x) - \alpha_{n,k}(x)\| < 1/2^{n-1} \text{ for all } x \in \mathcal{G}_{n,k-1}. \quad (e \ 6.332)$$

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Since \( \cup_{n=1}^{\infty} \cup_{k=1}^{n} G_{n,k} \) is dense in the unit ball of \( A \), we conclude that \( \{ \beta_m(x) \} \) is Cauchy for all \( x \in A \).

Define \( \tilde{\alpha}(x) = \lim_{m \to \infty} \beta_m(x) \). It is an endomorphism. Similar to the above estimates as in (6.330) and (6.331), if \( x \in G_{n,k} \) and \( m \geq n > k \), then

\[
\| \alpha_{m,k}^{-1}(x) - \alpha_{m,k}^{-1} \circ \text{Ad} w_{n,k+1}^*(1)(x) \| < 1/2^{m+k} \quad \text{and if } n = k, \quad (e.333)
\]

\[
\| \alpha_{n+1,0}^{-1}(x) - \alpha_{n+1,0}^{-1} \circ \text{Ad} w_{n+1,1}^*(1)(x) \| < 1/2^{n+1} \quad \text{and if } n = k, \quad (e.334)
\]

Note that \( \alpha_{n,k+1}^{-1} = \alpha_{n,k}^{-1} \circ \text{Ad} w_{n,k+1}^* \) (if \( n > k \)) and \( \alpha_{n+1,1}^{-1} = \alpha_{n+1,0}^{-1} \circ \text{Ad} w_{n+1,1}^* \). It follows that, if \( m \geq n, k \) and \( n \geq k' \),

\[
\| \alpha_{m,k}^{-1}(x) - \alpha_{n,k'}^{-1}(x) \| < 2^{-n}. \quad (e.335)
\]

Therefore \( \{ \beta_m^{-1}(x) \} \) is Cauchy for all \( x \in A \). We obtain an endomorphism \( \tilde{\alpha}^{-1} \) such that \( \tilde{\alpha}^{-1}(x) = \lim_{m \to \infty} \beta^{-1}_m(x) \) for all \( x \in \cup_{k=1}^{\infty} G_{k,k} \). It follows that \( \tilde{\alpha}^{-1} \circ \tilde{\alpha} = \text{id}_A = \tilde{\alpha} \circ \tilde{\alpha}^{-1} \). Thus \( \tilde{\alpha} \) is an automorphism.

Fix \( k \geq 1, \varepsilon > 0 \) and a finite subset \( F \), by (6.326), there exists an integer \( n_1 \geq 1 \) such that

\[
\| e^{(i)}_{j,n,k} \| < \varepsilon \quad j = 0, 1, \ldots, k + i - 1, \quad i = 0, 1, \quad (e.336)
\]

for all \( n \geq n_1 \). We may also assume that \( 1/n_1 < \varepsilon/4 \) and \( n_1 \geq k \). Then, if \( n \geq n_1 \),

\[
e^{(i)}_{j+1,n,k} \approx 1/(n+k) \text{ Ad } w_{n,k}(1) \circ \alpha_{n,k-1}(e^{(i)}_{j,n,k}) = \alpha_{n,k}(e^{(i)}_{j,n,k}) \quad (e.337)
\]

\[
\approx 2^{n+k+1} \text{ Ad } w_{n+1,1}(1) \circ \alpha_{n,k}(e^{(i)}_{j,n,k}) = \alpha_{n+1,1}(e^{(i)}_{j,n,k}), \quad (e.338)
\]

\[
\quad j = 0, 1, \ldots, k + i - 2, \quad i = 0, 1. \quad (e.339)
\]

\[
\alpha_{n+1,1}(e^{(i)}_{j,n,k}), \quad (e.340)
\]

\[
\quad j = 0, 1, \ldots, k + i - 2, \quad i = 0, 1. \quad (e.341)
\]

\[
\| e^{(i)}_{j+1,n,k} - \beta_m(e^{(i)}_{j,n,k}) \| < 1/(n + k) + 1/2^{n+k} \quad (e.342)
\]

for \( j = 0, 1, \ldots, k + i - 2, \quad i = 0, 1. \) Therefore

\[
\| \tilde{\alpha}(e^{(i)}_{j,n,k}) - e^{(i)}_{j+1,n,k} \| < 2/(n + k). \quad (e.343)
\]

Thus \( \tilde{\alpha} \) has the Rokhlin property. To show \( \tilde{\alpha} \) is asymptotically unitarily equivalent to \( \alpha \otimes \sigma \), define \( w(t) \) as follows:

\[
w(t) = w_{n,k}(nt - n(n-1) - (k-1))w_{n,k-1}(1) \cdots w_{1,1} \quad (e.344)
\]

\[
\text{for } n-1 + \frac{k-1}{n} \leq t < n-1 + \frac{k}{n}, \quad (e.345)
\]

where we identify \( w_{n+1,0}(t) = w_{n,n}(t) \) for \( t \in [0,1] \). Note that \( \{ w(t) : t \in [0, \infty) \} \) is a continuous path of unitaries in \( A \otimes \mathbb{Z} \). It follows from (6.328) that

\[
\tilde{\alpha}(x) = \lim_{t \to \infty} \alpha \otimes \sigma(x) \quad \text{for all } x \in A \otimes \mathbb{Z}. \quad (e.346)
\]

Since \( w_{n,k}(0) = 1 \), we conclude that \( w(0) \in U_0(A) \). Therefore \( \tilde{\alpha} \) and \( \alpha \otimes \sigma \) are strongly asymptotically unitarily equivalent.
7 Classification of automorphisms with Rokhlin property

**Definition 7.1.** Let $A$ be a unital separable amenable C*-algebra which satisfies the UCT. At the present, we also assume that $A$ has stable rank one and simple. Let $KK_e^{-1}(A,A)^{++}$ be the subset of those elements $\kappa$ in $KK(A,A)$ which is strictly positive, i.e., $\kappa(K_0(A)_+ \setminus \{0\}) \subset K_0(A)_+ \setminus \{0\}$, preserving the identity, $\kappa([1_A]) = [1_A]$, invertible, i.e., there is $\kappa^{-1}$ such that $\kappa \times \kappa^{-1} = \kappa^{-1} \times \kappa = [id_A]$. Let $\gamma : T(A) \to T(A)$ be an affine homeomorphism and $\lambda : U_0(A)/CU(A) \to U(A)/CU(A)$ be a continuous isomorphism. Let $\rho_A : K_0(A) \to Aff(T(A)$ be the order preserving homomorphism defined by $\rho_A([p]) = (\tau(p))$ for all projections $p \in M_\infty(A)$.

We say $\kappa$ and $\gamma$ are compatible if $\gamma(\tau(p)) = \tau(\rho_A(\kappa([p])))$ for all projections $p \in M_\infty(A)$ and $\tau \in T(A)$. Let $\gamma^* : Aff(T(A) \to Aff(T(A)$ be the continuous affine isomorphism induced by $\gamma$, i.e.,

$$\gamma^*(f)(\tau) = f(\gamma(\tau)) \text{ for all } f \in Aff(T(A)$$

and for all $\tau \in T(A)$. If $\gamma$ is compatible with $\kappa$, denote by

$$\gamma^* : Aff(T(A)/\rho_A(K_0(A)) \to Aff(T(A)/\rho_A(K_0(A))$$

the isomorphism induced by $\gamma^*$. We say $\lambda$ is compatible with $\kappa$, if $\lambda$ maps $U_0(A)/CU(A)$ into $U_0(A)/CU(A)$ and $q_1 \circ \lambda(\bar{u}) = \kappa([u])$ for all unitaries $u \in U(A)$, where $q_1 : U(A)/CU(A) \to K_1(A)$ is the quotient map. We say the triple $(\kappa, \gamma, \lambda)$ are compatible, if $\gamma$ and $\lambda$ are compatible with $\kappa$ and

$$\Delta_A \circ \lambda \circ \Delta_A^{-1} = \gamma^*,$$

where $\Delta_A : U_0(A)/CU(A) \to Aff(T(A)/\rho_A(K_0(A)))$ is the de la Harp and Skandalis determinant.

Denote by $KKUT_{e^{-1}}(A,A)^{++}$ the set of all compatible triples $(\kappa, \gamma, \lambda)$.

For each $\varphi \in Aut(A)$, there is a triple $([\varphi]), \varphi_T, \varphi^1)$, where $[\varphi] \in KK_{-1}(A,A)^{++}$ is induced $KK$-element, $\varphi_T : T(A) \to T(A)$ is the affine isomorphism induced by $\varphi$ and defined by $\varphi_T(\tau)(a) = \tau(\varphi(a))$ for all selfadjoint elements $a \in A$ and for all $\tau \in T(A)$, and where $\varphi^1 : U(A)/CU(A) \to U(A)/CU(A)$ is the isomorphism induced by $\varphi$ and defined by $\varphi^1(\bar{u}) = \varphi(\bar{u})$ for all unitaries $u \in U(A)$. Therefore there is a map $\mathfrak{R} : Aut(A) \to KKUT_{e^{-1}}(A,A)^{++}$ defined by $\mathfrak{R}(\varphi) = ([\varphi], \varphi_T, \varphi^1)$.

**Definition 7.2.** Let $\varphi, \psi : A \to A$ be two unital homomorphisms. Let

$$M_{\varphi, \psi} = \{ f \in C([0,1], A) : f(0) = \varphi(a), f(1) = \psi(a) \text{ for some } a \in A \}.$$

There is a homomorphism

$$R_{\varphi, \psi} : K_1(M_{\varphi, \psi}) \to Aff(T(A)$$

defined by

$$R_{\varphi, \psi}([u])(\tau) = \frac{1}{2\pi i} \int_0^1 \tau\left(\frac{du(t)}{dt}u(t)^*dt\right).$$

When $[\varphi] = [\psi]$ in $KK(A,A)$, there is a splitting short exact sequence

$$0 \to K_0(A) \to M_{\varphi, \psi} \xrightarrow{\eta} K_1(A) \to 0.$$

We will also use $R_{\varphi, \psi}$ for the induced map $R_{\varphi, \psi} : K_1(A) \to Aff(T(A))$. See, for example, Definition 3.2 of [12] for details.

Let $R_0 \subset Hom(K_1(A), Aff(T(A))$ be the subset of those $\eta$ for which there is a homomorphism $h : K_1(A) \to K_0(A)$ such that $\eta = \rho_A \circ h$.

We now assume that $\mathfrak{R}(\varphi) = \mathfrak{R}(\psi)$. Then, by Lemma 9.2 of [12],

$$R_{\varphi, \psi} \in Hom(K_1(A), \rho_A(K_0(A))).$$

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Denote by $\overline{R_{\varphi,\psi}}$ the element in $Hom(K_1(A), Aff(T(A))/\mathcal{R}_0$. If $\overline{R_{\varphi,\psi}} = 0$, then there exists $\Theta \in Hom_M(K(A), K(M_{\varphi,\psi}))$ such that $[\pi_0] \circ \Theta = [id_A]$ and

$$R_{\varphi,\psi} \circ \Theta = 0,$$

where $\pi_0 : M_{\varphi,\psi} \to A$ is the point-evaluation at $t = 0$ (see 3.2 of \cite{12}).

**Theorem 7.3.** Let $A$ be a unital separable simple amenable $C^*$-algebra with $TR(A) \leq 1$ which satisfies the UCT. Let $\alpha$ and $\beta$ be two automorphisms with the Rokhlin property. Then the following are equivalent:

1. $\alpha$ and $\beta$ are strongly cocycle conjugate;
2. $\alpha$ and $\beta$ are asymptotically unitarily equivalent and
3. $\mathfrak{R}(\alpha) = \mathfrak{R}(\beta)$ and $\overline{R_{\alpha,\beta}} = 0$.

**Proof.** We have proved “(2) $\Rightarrow$ (1)” in \cite{42}. That of “(2) $\iff$ (3)” follows from Theorem 7.2 of \cite{12}.

Now assume that $\alpha$ and $\beta$ are strongly cocycle conjugate. Then there is a unitary $u \in U(A)$ and a strongly asymptotically inner automorphism $\sigma$ such that $\alpha = \text{Ad}u \circ \sigma^{-1} \circ \beta \circ \sigma$. It follows that $\sigma \circ \text{Ad}u^* \circ \alpha = \beta \circ \sigma$. Since $\sigma$ is (strongly) asymptotically inner, there is a continuous path of unitaries $\{u(t) : t \in [1, \infty)\} \subset A$ such that

$$\sigma(x) = \lim_{t \to \infty} u(t)xu(t)$$

for all $x \in A$.

It follows that

$$\beta \circ \sigma(x) = \lim_{t \to \infty} \beta(u(t))^*\beta(x)\beta(u(t))$$

for all $x \in A$.

It follows that $\beta \circ \sigma$ is (strongly) asymptotically unitarily equivalent to $\beta$. Similarly $\sigma \circ \text{Ad}u^* \circ \alpha$ is asymptotically unitarily equivalent to $\alpha$ (note that $u^*$ may not be in $U_0(A)$). Therefore $\alpha$ and $\beta$ are asymptotically unitarily equivalent. This proves that “(1) $\Rightarrow$ (2)”.

\qed

The following also follows from \cite{42}.

**Theorem 7.4.** Let $A$ be a unital separable simple amenable $C^*$-algebra with $TR(A) \leq 1$ which satisfies the UCT and let $\alpha$ and $\beta$ be two automorphisms on $A$ with the Rokhlin property. Then $\alpha$ and $\beta$ are strongly cocycle conjugate and uniformly approximately conjugate if and only if $\alpha$ and $\beta$ are strongly asymptotically unitarily equivalent.

**7.5.** Let $\langle \alpha \rangle, \langle \beta \rangle \in \text{Aut}_R(A)/\sim_{soc}$ be two elements represented by automorphisms $\alpha$ and $\beta$ with the Rokhlin property. One can define a multiplication by $\langle \alpha \rangle \circ \langle \beta \rangle = \langle \alpha \circ \beta \rangle$, where $\alpha \circ \beta$ is an automorphism with the Rokhlin property which is strongly asymptotically unitarily equivalent to $\alpha \circ \beta$ given by \cite{65}. By \cite{42} it is well-defined. This makes $\text{Aut}_R(A)/\sim_{soc}$ a group with the identity represented by an asymptotically inner automorphism which has the Rokhlin property.

Similarly, $\text{Aut}_R(A)/\sim_{saucc}$ is also a group with the identity represented by a strongly asymptotically inner automorphism which has the Rokhlin property.

If $(\kappa_1, \gamma_1, \lambda_1), (\kappa_2, \gamma_2, \lambda_2) \in KKUT^{-1}_2(A, A)^{++}$, define

$$(\kappa_1, \gamma_1, \lambda_1) \times (\kappa_2, \gamma_2, \lambda_2) = (\kappa_2 \times \kappa_1, \gamma_2 \circ \gamma_1, \lambda_1 \circ \lambda_2).$$

This makes $KKUT^{-1}_e(A, A)^{++}$ a group. Let $\alpha, \beta \in \text{Aut}(A)$. Then

$$\mathfrak{R}(\alpha \circ \beta) = ([\alpha \circ \beta], (\alpha \circ \beta)_T, (\alpha \circ \beta)^{\dagger}) = ([\beta] \times [\alpha], \beta_T \circ \alpha_T, \alpha^{\dagger} \circ \beta^{\dagger}) = \mathfrak{R}(\alpha) \times \mathfrak{R}(\beta).$$

Thus, by \cite{73} $\mathfrak{R}$ gives a group homomorphism from $\text{Aut}_R(A)/\sim_{soc}$ into $KKUT^{-1}_e(A, A)^{++}$.
Theorem 7.6. Let $A$ be a unital separable simple amenable $C^*$-algebra with $TR(A) \leq 1$ which satisfies the UCT. Then one has the following short exact sequence of groups:

$$1 \to \text{Hom}(K_1(A), \rho_A(K_0(A)))/R_0 \to \text{Aut}_R(A)/\sim_{scc} \xrightarrow{\delta} KK^{-1}(A, A)^{++} \to 1. \quad (e \text{7.347})$$

Proof. Theorem [6.5] shows that every automorphism $\alpha \in \text{Aut}(A)$ is strongly asymptotically unitarily equivalent to an automorphism in $\text{Aut}_R(A)$. Thus the theorem follows from this, Theorem [7.3] and Corollary 9.10 of [12].

Definition 7.7. Let $G_0$ be the subset of all those elements which are cocycle conjugate to some asymptotically inner automorphisms which have the Rokhlin property. Let $G_0$ be the image of $G_0$ in $\text{Aut}_R(A)/\sim_{scc}$. Suppose that $\alpha, \beta$ are two automorphisms with the Rokhlin property whose image in $\text{Aut}_R(A)/\sim_{scc}$ are in $G_0$. Let $\text{Ad} v \circ \sigma_1^{-1} \circ \alpha \circ \sigma_1 = \delta_1$ and $\text{Ad} v \circ \sigma_2^{-1} \circ \alpha \circ \sigma_2 = \delta_2$, where $\delta_1, \delta_2$ are asymptotically inner automorphisms with the Rokhlin property, $u, v \in U(A)$, $\sigma_1$ and $\sigma_2$ are asymptotically inner automorphisms. Consider $\alpha \circ \beta$. Then

$$\text{Ad} v \circ \sigma_2^{-1} \circ (\alpha \circ \beta) \circ \sigma_2 = \text{Ad} v \circ \sigma_2^{-1} \circ \alpha \circ \sigma_2 \circ \text{Ad} v^* \circ \text{Ad} v \circ \sigma_2^{-1} \circ \beta \circ \sigma_2 = \text{Ad} v \circ \sigma_2^{-1} \circ \alpha \circ \sigma_2 \circ \text{Ad} v^* \circ \delta_2$$

which is strongly asymptotically unitarily equivalent to $\text{Ad} v \circ \sigma_2^{-1} \circ \alpha \circ \sigma_2 \circ \text{Ad} v^*$. Therefore

$$\langle \alpha \circ \beta \rangle = \langle \text{Ad} v \circ \sigma_2^{-1} \circ \alpha \circ \sigma_2 \circ \text{Ad} v^* \rangle. \quad (e \text{7.351})$$

However, $\text{Ad} v \circ \sigma_2^{-1} \circ \alpha \circ \sigma_2 \circ \text{Ad} v^*$ is strongly cocycle conjugate to $\delta_1$. It follows that $\langle \alpha \circ \beta \rangle \in G_0$. This shows that $G_0$ is a subgroup. Clearly it is a normal subgroup. It is also clear that $(\text{Aut}_R(A)/\sim_{scc})/G_0 = \text{Aut}_R(A)/\sim_{cc}$. This also implies that $\text{Aut}_R(A)/\sim_{cc}$ is a group with the identity is the class of asymptotically inner automorphisms with the Rokhlin property.

Theorem 7.8. Let $A$ be a unital separable simple amenable $C^*$-algebra with $TR(A) \leq 1$ which satisfies the UCT. Then there is a short exact sequence of groups:

$$1 \to K_1(A)/H_1(K_0(A), K_1(A)) \to \text{Aut}_R(A)/\sim_{scc} \to \text{Aut}_R(A)/\sim_{scc} \to 1. \quad (e \text{7.352})$$

Proof. Denote $\pi : \text{Aut}_R(A)/\sim_{scc} \to \text{Aut}_R(A)/\sim_{cc}$ the quotient map. Let $\alpha$ be a strongly asymptotically inner automorphism on $A$ with the Rokhlin property. There exists a continuous path of unitaries $\{u(t) : t \in [1, \infty)\} \subset A$ with $u(1) = 1_A$ such that

$$\alpha(a) = \lim_{t \to \infty} \text{Ad} u(t)(a) \quad \text{for all } a \in A. \quad (e \text{7.353})$$

Let $\beta$ be any asymptotically inner automorphism with the Rokhlin property. There exists a continuous path of unitaries $\{v(t) : t \in [1, \infty)\} \subset A$ such that

$$\beta(a) = \lim_{t \to \infty} \text{Ad} v(t)(a) \quad \text{for all } a \in A. \quad (e \text{7.354})$$

Let $Q : K_1(A) \to K_1(A)/H_1(K_0(A), K_1(A))$ be the quotient map. Define $\Omega(\beta) = Q([v(1)^*])$ in $K_1(A)/H_1(K_0(A), K_1(A))$. Define $w(t) = v(t)^* u(t)$ for all $t \in [1, \infty)$. Note that $w(1) = v(1)^*$. Then

$$\alpha \circ \beta^{-1}(a) = \lim_{t \to \infty} \text{Ad} w(t)(a) \quad \text{for all } a \in A. \quad (e \text{7.355})$$
By the proof of [1.2], there exists $v \in U(A)$ with $[v] = [v(1)^*]$ in $U(A)/U_0(A)$ and $\sigma \in \text{Aut}(A)$ which is strongly asymptotically inner such that $\alpha = \text{Ad} v \circ \sigma^{-1} \circ \beta \circ \sigma$. Suppose that there exists another continuous path of unitaries $\{z(t) : t \in [1, \infty)\} \subset A$ such that

$$\beta(a) = \lim_{t \to \infty} \text{Ad} z(t)(a) \text{ for all } a \in A.$$  

(e 7.356)

Then $\text{Ad} z(1)^* \circ \beta$ is strongly asymptotically unitarily equivalent to $\text{id}_A$. By (e 7.354), $\text{Ad} v(1)^* \circ \beta$ is also strongly asymptotically unitarily equivalent to $\text{id}_A$. It follows that $\text{Ad} v(1)^* \circ \beta$ is strongly asymptotically unitarily equivalent to $\beta$. It follows from Proposition 12.3 of [9] that $[v(1)z(1)^*] \in H_1(K_0(A), K_1(A))$. In other words, $Q([v(1)^*]) = Q([z(1)^*])$. Thus $\mathcal{U}(\beta)$ is well defined homomorphism from $\ker \pi$ into $K_1(A)/H_1(K_0(A), K_1(A))$.

If $\mathcal{U}(\beta) = Q([1_A])$, then, by the proof of Lemma 10.4 and 10.5 of [12], $\text{Ad} v(1)^* \circ \beta$ is strongly asymptotically unitarily equivalent to $\beta$. But, form the above, $\text{Ad} v(1)^* \circ \beta$ is strongly asymptotically inner. It follows that $\beta$ is strongly asymptotically inner. Therefore $\beta$ and $\alpha$ are strongly cocycle conjugate and uniformly approximately conjugate.

So far, we have shown that $\mathcal{U}$ is injective. Choose any unitary $w \in U(A)$, by [6.3] there exists an automorphism $\gamma$ with the Rokhlin property which is strongly asymptotically unitarily equivalent to $\text{Ad} w \circ \alpha$. Therefore $\gamma$ is asymptotically inner. Then $\mathcal{U}(\gamma) = Q([w^*])$. This implies that $U$ is an isomorphism.

□

**Corollary 7.9.** Let $A$ be a unital separable simple amenable $C^*$-algebra with $TR(A) \leq 1$ which satisfies the UCT. Suppose that $K_1(A) = H_1(K_0(A), K_1(A))$. Then two automorphisms $\alpha, \beta \in \text{Aut}_R(A)$ are strongly cocycle conjugate and uniformly approximately conjugate if and only if

$$\mathfrak{F}(\alpha) = \mathfrak{F}(\beta) \text{ and } \mathcal{R}_{\alpha, \beta} = 0.$$  

Moreover,

$$\text{Aut}_R(A)/\sim_{scc} = \text{Aut}_R(A)/\sim_{saucc}$$  

(e 7.357)

and there is a short exact sequence:

$$1 \to \text{Hom}(K_1(A), \overline{\rho_1(A)}/\rho_0(A))/\mathcal{R}_0 \to \text{Aut}_R(A)/\sim_{saucc} \to KKUT^{e-1}_e(A, A)^{++} \to 1.$$  

(e 7.358)

**Definition 7.10.** Let $A$ be a unital separable simple amenable $C^*$-algebra with $TR(A) \leq 1$. We assume that the following short exact sequence splits:

$$0 \to \text{Tor}(K_1(A)) \to K_1(A) \to K_1(A)/\text{Tor}(K_1(A)) \to 0,$$  

(e 7.359)

or $\rho_1(A)$ is divisible. Denote by $s_f : K_1(A)/\text{Tor}(K_1(A)) \to K_1(A)$ a fixed splitting map for [6.3]. Let $\pi_f : K_1(A) \to K_1(A)/\text{Tor}(K_1(A))$ be the quotient map. So $\pi_f \circ s_f = \text{id}_{K_1(A)/\text{Tor}(K_1(A))}$. Put $FK_1(A) = \text{im} s_f$. Then $s_f \circ \pi_f|_{FK_1(A)} = \text{id}_{FK_1(A)}$. We also write $K_1(A) = \text{Tor}(K_1(A)) \oplus FK_1(A)$.

Consider the short exact sequence:

$$0 \to CU(A)/DU(A) \to U(A)/DU(A) \to U(A)/CU(A) \to 0.$$  

(e 7.360)

Note that

$$CU(A)/DU(A) \cong \overline{\rho_1(A)/\rho_0(A)} \subset \text{Aff}(T(A))/\rho_1(A).$$
is divisible. Therefore the above short exact sequence splits. Denote by \( s_{uc} : U(A)/CU(A) \to U(A)/DU(A) \) a fixed splitting map for (e 7.360). Consider the short exact sequence:

\[
0 \to U_0(A)/CU(A) \to U(A)/CU(A) \xrightarrow{\pi_1} K_1(A) \to 0. \tag{e 7.361}
\]

Note also that

\[
U_0(A)/CU(A) \cong \text{Aff}(T(A))/\rho_A(K_0(A)) \tag{e 7.362}
\]

is a divisible group. Therefore the short exact sequence in (e 7.361) is also splitting. Denote by \( \phi : A \to A \) a homomorphism. Let \( \varphi : U(A)/DU(A) \to U(A)/DU(A) \) be an isomorphism. Define

\[
\hat{\varphi} : K_1(A)/\text{Tor}(K_0(A)) \to U_0(A)/DU(A) \cong \text{Aff}(T(A))/\rho_A(K_0(A))
\]

by

\[
\hat{\varphi} = \varphi \circ s_{uc} \circ \varphi \circ s_f - s_{uc} \circ \varphi \circ s_f,
\]

when \( s_f \) exists. In this case, we may view \( \hat{\varphi} \) is a homomorphism from \( FK_1(A) \) and

\[
\hat{\varphi} = \varphi \circ s_{uc} \circ (s_1)|_{FK_1(A)} - s_{uc} \circ s_1 \circ (\varphi \circ s_f)|_{FK_1(A)}.
\]

In the case that \( \rho_A(K_0(A)) \) is divisible, \( \text{Aff}(T(A))/\rho_A(K_0(A)) \) is torsion free. Thus \( \varphi \circ s_{uc} \circ s_1 - s_{uc} \circ s_1 \circ \varphi \circ s_f \) maps \( K_1(A) \) into \( U_0(A)/DU(A) \cong \text{Aff}(T(A))/\rho_A(K_0(A)) \) which vanishes on \( \text{Tor}(K_0(A)) \) and which induces a homomorphism \( \varphi : K_1(A)/\text{Tor}(K_1(A)) \to U_0(A)/DU(A) \). So, under the assumption on \( A \), the map \( \varphi \) is well-defined.

Let \( \kappa \in KK e^{-1}(A, A)^{++} \) and \( \gamma : T(A) \to T(A) \) be an affine homeomorphism which is compatible with \( \kappa \). Denote by \( KK e^{-1}(A, A)^{++} \) the set of triples \((\kappa, \gamma, \zeta)\), where \( \zeta : K_1(A)/\ker(K_0(A)) \to U_0(A)/DU(A) \cong \text{Aff}(T(A))/\rho_A(K_0(A)) \) is a homomorphism. Let \( \gamma^* : \text{Aff}(T(A)) \to \text{Aff}(T(A)) \) be induced by \( \gamma \). Then \( \gamma^*(\rho_A(K_0(A))) = \rho_A(K_0(A)) \) and \( \gamma^*(\rho_A(K_0(A))) = \rho_A(K_0(A)) \). Therefore \( \gamma \) induces an isomorphism

\[
\tilde{\gamma}^* : \text{Aff}(T(A))/\rho_A(K_0(A)) \to \text{Aff}(T(A))/\rho_A(K_0(A)).
\]

Denote by \( \kappa : K_1(A)/\text{Tor}(K_1(A)) \to K_1(A)/\text{Tor}(K_1(A)) \) the isomorphism induced by \( \kappa \). Define a product on \( KK e^{-1}(A, A)^{++} \) as follows:

\[
(\kappa_1, \gamma_1, \zeta_1) \times (\kappa_2, \gamma_2, \zeta_2) = (\kappa_2 \times \kappa_1, \gamma_1 \circ \gamma_2, \zeta_2 \circ \kappa_1 + \kappa_1 \circ \zeta_1) \tag{e 7.365}
\]

\( KK e^{-1}(A, A)^{++} \) becomes a group with the identity \((\id_A, \id_T(A), 0)\) and, if \( (\kappa, \gamma, \zeta) \in KK e^{-1}(A, A)^{++} \), then

\[
(\kappa, \gamma, \zeta)^{-1} = (\kappa^{-1}, \gamma^{-1}, -\gamma^* \circ \zeta \circ \kappa^{-1}) \tag{e 7.366}
\]

If \( \alpha \in \text{Aut}(A) \), define \( \hat{\alpha}(\alpha) = ([\alpha], \alpha_T, \alpha^{\dagger}) \). Then \( \hat{\alpha} \) is a homomorphism from \( \text{Aut}(A) \) into \( KK e^{-1}(A, A)^{++} \).

**Theorem 7.11.** Let \( A \) be a unital separable simple amenable \( C^* \)-algebra with \( TR(A) \leq 1 \). Suppose that \( A \) satisfies the UCT and either \( K_1(A)/\text{Tor}(K_1(A)) \) is a free group, or \( \rho_A(K_0(A)) \) is divisible. Suppose that \( (\kappa, \gamma, \zeta) \in KK e(A, A)^{++} \). Then there exists a unital homomorphism \( \varphi : A \to A \) such that

\[
([\varphi], \varphi_T, \varphi^\dagger) = (\kappa, \gamma, \zeta).
\]

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Proof. Note, with the assumption on \(A\), \(7.10\) applies. Let \(\tilde{\psi} : K_1(A) / \text{Tor}(K_1(A)) \to U_0(A) / \text{CU}(A)\) be the homomorphism induced by \(\zeta\). In what follows in this proof, we will identify \(U_0(A) / \text{DU}(A)\) with \(\text{Aff}(T(A)) / \rho_A(K_0(A))\), and \(\text{CU}(A) / \text{DU}(A)\) with \(\rho_A(K_0(A)) / \rho_A(K_0(A))\) whenever it is convenient. Let \(\bar{T} : \text{Aff}(T(A)) / \rho_A(K_0(A)) \to \text{Aff}(T(A)) / \rho_A(K_0(A))\) be the isomorphism induced by \(\gamma\). Define \(\lambda : U(A) / \text{CU}(A) \to U(A) / \text{CU}(A)\) by

\[
\lambda(x) = s_1 \circ \kappa \circ \pi_1(x) + (\zeta \circ \pi_f \circ \pi_1(x) + \bar{T} \circ \Delta(x - s_1(x))) \quad \text{for all } x \in U(A) / \text{CU}(A)
\]

Then \(\lambda\) defines a continuous homomorphism from \(U(A) / \text{CU}(A)\) to \(U(A) / \text{CU}(A)\). Since both \(\kappa\) and \(\bar{T}\) are isomorphism, it is easy to check that \(\lambda\) is an isomorphism. It follows that

\[
(\kappa, \gamma, \lambda) \in KKUT_e^{-1}(A, B)^{++}.
\]

It follows from Theorem 8.6 of \([12]\) that there exits a unital homomorphism \(\psi : A \to A\) such that

\[
([\psi], \psi_T, \psi^\dagger) = (\kappa, \gamma, \lambda).
\]

As in 9.10 of \([12]\), we may assume that \(\varphi\) is an automorphism. Define \(\eta_1 : K_1(A) / \text{Tor}(K_1(A)) \to \text{CU}(A) / \text{DU}(A)\) by

\[
\eta_1(x) = \zeta(x) \tilde{\psi}^{\dagger}(-x) \quad \text{for all } x \in K_1(A) / \text{Tor}(K_1(A)).
\]

This gives a homomorphism. If \(K_1(A) / \text{Tor}(K_1(A))\) is free, one obtains a homomorphism \(\eta_2 : K_1(A) / \text{Tor}(K_1(A)) \to \rho_A(K_0(A))\) such that \(\pi_a \circ \eta_2 = \eta_1\), where

\[
\pi_a : \rho_A(K_0(A)) \to \rho_A(K_0(A)) / \rho_A(K_0(A)) \cong \text{CU}(A) / \text{DU}(A)
\]

is the quotient map. If \(\rho_A(K_0(A))\) is divisible, the short exact sequence

\[
0 \to \rho_A(K_0(A)) \to \rho_A(K_0(A)) / \rho_A(K_0(A)) / \rho_A(K_0(A)) \to 0
\]

splits. Let \(s_a\) is a splitting map. Define \(\eta_2 = s_a \circ \eta_1\). In both case we obtain a homomorphism \(\eta_2 : K_1(A) / \text{Tor}(K_1(A)) \to \rho_A(K_0(A))\) such that \(\eta_1 = \pi_a \circ \eta_2\). Define \(\eta : K_1(A) \to \rho_A(K_0(A))\) by

\[
\eta(x) = \eta_2 \circ \pi_f(x) \quad \text{for all } x \in K_1(A).
\]

It follows from 9.10 of \([12]\) and 9.8 of \([12]\) that there is an automorphism \(\alpha\) such that

\[
([\alpha], \alpha_T, \alpha^\dagger) = ([\id_A], \id_T, \id_A^\dagger)
\]

and

\[
\overline{R}_{\id, \alpha} = \eta \circ (\psi_1^{-1}),
\]

where \(\eta \in \text{Hom}(K_1(A), \rho_A(K_0(A))) / \mathcal{R}_0\) is the image of \(\eta\) in the quotient. Define \(\varphi = \alpha \circ \psi\). Then

\[
([\varphi], \varphi_T) = (\kappa, \gamma).
\]

We will show that

\[
\tilde{\varphi}^{\dagger} = \zeta.
\]
We first consider the case that $K_1(A)/\text{Tor}(K_1(A))$ is free. Then

$$
\tilde{\varphi}^\dagger = (\alpha \circ \psi)^\dagger \circ s_{uc} \circ s_f - s_{uc} \circ s_1 \circ (\alpha \circ \psi)_{s_1} \circ s_f \quad (e\,7.375)
$$

$$
= \alpha^\dagger(\psi)^\dagger \circ s_{uc} \circ s_f - s_{uc} \circ s_1 \circ \psi_{s_1} \circ s_f
$$

$$
= \alpha^\dagger \circ \tilde{\psi}^\dagger + \alpha^\dagger \circ s_{uc} \circ s_1 \circ \psi_{s_1} \circ s_f
$$

$$
= \tilde{\psi}^\dagger + \alpha^\dagger \circ s_{uc} \circ s_1 \circ \psi_{s_1} \circ s_f
$$

$$
- s_{uc} \circ s_1 \circ \psi_{s_1} \circ s_f
$$

$$
\quad (e\,7.377)
$$

$$
\quad (e\,7.378)
$$

$$
\quad (e\,7.379)
$$

$$
\quad (e\,7.380)
$$

For any $u \in U(A)$,

$$
\alpha^\dagger \circ s_{uc} \circ s_1 \circ \psi_1([u]) - s_{uc} \circ s_1 \circ \psi_{s_1}([u]) = \alpha(\psi(u))\psi(u)^* \quad (e\,7.381)
$$

$$
\quad (e\,7.382)
$$

(Recall that $\alpha(\psi(u))\psi(u)^*$ is the image of $\alpha(\psi(u))\psi(u)^*$ in $U(A)/DU(A)$. However

$$
\Delta(\alpha(\psi(u))\psi(u)^*) = R_{id_A,0}(\psi_{s_1}([u])) = \eta([u]) \mod \mathcal{R}_0. \quad (e\,7.383)
$$

It follows that

$$
\tilde{\varphi}^\dagger = \tilde{\psi}^\dagger + \eta_1 = \zeta. \quad (e\,7.384)
$$

Almost the identical proof shows that $\tilde{\psi}^\dagger$ also holds in the case that $\rho_A(K_0(A))$ is divisible.

\begin{proof}
\end{proof}

**Theorem 7.12.** Let $A$ be a unital separable simple amenable C*-algebra with $TR(A) \leq 1$ which satisfies the UCT. Suppose that $K_1(A)/\text{Tor}(K_1(A))$ is free, or $\rho_A(K_0(A))$ is divisible.

Then two automorphisms $\alpha, \beta \in \text{Aut}_R(A)$ are cocycle conjugate, i.e., there exists a unitary $u \in U(A)$ and $\sigma \in \text{Aut}(A)$ such that

$$
\alpha = \text{Ad} \, u \circ \sigma \circ \beta \circ \sigma^{-1} \quad (e\,7.385)
$$

if and only if $\tilde{\alpha}$ and $\tilde{\beta}$ are conjugate in $KKFT_c^{-1}(A, A)^{++}$, i.e., there exists $\zeta \in KKFT_c^{-1}(A, A)^{++}$ such that

$$
\tilde{\alpha} = \zeta \times \tilde{\beta} \times \zeta^{-1}. \quad (e\,7.386)
$$

Moreover, when $\tilde{\alpha}$ holds, one can require that $\tilde{\alpha}(\sigma) = \zeta$.

If, in addition, $H_1(K_0(A), K_1(A)) = K_1(A)$, then there exists a sequence of unitaries and a sequence $\{\sigma_n\} \subset \text{Aut}(A)$ such that

$$
\alpha = \text{Ad} \, u_n \circ \sigma_n \circ \beta \circ \sigma_n^{-1} \quad (e\,7.387)
$$

if and only if there exists $\zeta \in KKFT_c^{-1}(A, A)^{++}$ such that

$$
\tilde{\alpha} = \zeta \times \tilde{\beta} \times \zeta^{-1}. \quad (e\,7.388)
$$

Moreover, when $\tilde{\alpha}$ holds, one can require that $\tilde{\alpha}(\sigma_n) = \zeta$ for all $n$. 

\begin{proof}
\end{proof}
Proof. Let \( \alpha \) and \( \beta \) be two automorphisms with the Rokhlin property such that
\[
\alpha = \text{Ad} \ u \circ \sigma \circ \beta \circ \sigma^{-1}
\]
for some unitary \( u \in U(A) \) and some \( \sigma \in \text{Aut}(A) \). Note that
\[
(\text{Ad} \ u, (\text{Ad} \ u) \tau, (\text{Ad} \ u) \tau^\dagger) = ([\text{id}_A], \text{id}_T, \text{id}_{U(A)/DU(A)}).
\]
It follows that
\[
\tilde{\mathcal{R}}(\alpha) = \tilde{\mathcal{R}}(\sigma) \times \tilde{\mathcal{R}}(\beta) \times \tilde{\mathcal{R}}(\sigma)^{-1}.
\] (e 7.389)

Now suppose that there exists \( \zeta \in KKFT_{+1}(A, A)^{++} \) such that
\[
\tilde{\mathcal{R}}(\alpha) = \zeta \times \tilde{\mathcal{R}}(\beta) \times \zeta^{-1}.
\] (e 7.390)
Write \( \zeta = (\zeta_K, \zeta_T, \lambda) \), where \( \zeta_K \in KK_{-1}(A, A)^{++} \), \( \zeta_T : T(A) \to T(A) \) is an affine homeomorphism and \( \lambda : K_1(A)/\text{Tor}(K_1(A)) \to \rho_A(K_1(A))/\rho_A(K_0(A)) \) is a homomorphism. It follows from (e 7.11) that there exists \( \sigma' \in \text{Aut}(A) \) such that
\[
([\sigma'], \sigma_T, (\sigma')^t) = (\zeta_K, \zeta_T, \lambda).
\] (e 7.391)
Let \( \beta_1 = \sigma' \circ \beta \circ (\sigma')^{-1} \). Then
\[
([\alpha], \alpha_T, \alpha^t) = ([\beta_1], (\beta_1)_T, \beta_1^t).
\] (e 7.392)
In particular,
\[
\alpha^t = \beta_1^t.
\] (e 7.393)
We will now show that
\[
\overline{R}_{\alpha, \beta_1} = 0.
\] (e 7.394)
First we note, since \([\alpha] = [\beta_1]\), there is homomorphism \( \theta : \overrightarrow{K}(A) \to \overrightarrow{K}(M_{\alpha, \beta_1}) \) such that
\[
\theta \circ [\pi_0] = [\text{id}_A],
\] (e 7.395)
where \( \pi_0 : M_{\alpha, \beta_1} \to A \) is the point-evaluation of the mapping torus at \( t = 0 \). Let \( u \in U(A) \) be a unitary. Let \( z \in U(M_{\alpha, \beta_1}) \) be a unitary such that \( z(0) = \alpha(u) \) and \( z(1) = \beta_1(u) \). Moreover, we may assume that \( z \) is piece-wise smooth. Define \( z_1(t) = \beta_1(u)^*z(t) \) for \( t \in [0, 1] \). Then \( z_1 \) is a continuous and piece-wise smooth with \( z_1(0) = \alpha(u)\beta_1(u)^* \) and \( z_1(1) = 1_A \). By (e 7.392),
\[
\frac{1}{2\pi i} \int_0^1 \tau \frac{dz_1(t)}{dt} z_1(t)^* dt \in \rho_A(K(A)),
\] (e 7.396)
where \( \tau \in T(A) \). It follows that
\[
R_{\alpha, \beta_1}([z]) \in \rho_A(K_0(A)).
\] (e 7.397)
On the other hand there exists two projections \( p, q \in M_l(A) \) (for some integer \( l \geq 1 \)) such that
\[
\theta([u]) = [(z + 1_{M_l(A)})v],
\] (e 7.398)
where \( v(t) = (e^{2\pi t}p + (1_{M_l(A)} - p))(e^{-2\pi t}q + (1_{M_l(A)} - q)) \) for \( t \in [0, 1] \). Then
\[
R_{\alpha, \beta_1}([(z + 1_{M_l(A)})v]) = R_{\alpha, 1}([z]) + R_{\alpha, \beta_1}([v]) \in \rho_A(K_0(A)).
\] (e 7.399)
It follows that
\[ R_{\alpha, \beta_1} \circ \theta \in \text{Hom}(K_1(A), \rho_A(K_0(A))). \] (e 7.400)

Since \( \rho_A(K_0(A)) \) is always torsion free, \( R_{\alpha, \beta_1} \circ \theta \) factors \( K_1(A)/\text{Tor}(K_1(A)) \), i.e., there is \( R : K_1(A)/\text{Tor}(K_1(A)) \to \rho_A(K_1(A)) \) such that
\[ R \circ \pi_f = R_{\alpha, \beta_1} \circ \theta. \] (e 7.401)

(Recall that \( \pi_f : K_1(A) \to K_1(A)/\text{Tor}(K_1(A)) \) is the quotient map.) If \( K_1(A)/\text{Tor}(K_1(A)) \) is free, there is \( \delta : K_1(A)/\text{Tor}(K_1(A)) \to K_0(A) \) such that
\[ \rho_A \circ \delta = R. \] (e 7.402)

Put \( \theta_1 = \theta|_{K_1(A)} - \delta \circ \pi_f \). Then
\[ \theta_1 \circ (\pi_0)_*1 = \text{id}_{K_1(A)}. \] (e 7.403)

Note that, since \([\alpha] = [\beta]\),
\[ R_{\alpha, \beta_1}|_{K_0(A)} = \rho_A. \] (e 7.404)

It follows that
\[ R_{\alpha, \beta_1} \circ \theta_1 = 0. \] (e 7.405)

If \( \rho_A(K_0(A)) \) is divisible, there is \( s_\rho : \rho_A(K_0(A)) \to K_0(A) \) such that
\[ \rho_A \circ s_\rho = \text{id}_{\rho_A(K_0(A))}. \] (e 7.406)

In this case, define
\[ \theta_1 = \theta|_{K_1(A)} - s_\rho \circ R_{\alpha, \beta_1} \circ s_1. \] (e 7.407)

We have \( \theta_1 \circ (\pi_0)_*1 = \text{id}_{K_1(A)} \). We also have that
\[ R_{\alpha, \beta_1} \circ \theta_1 = 0. \] (e 7.408)

Thus, in both cases,
\[ \overline{R_{\alpha, \beta_1}} = 0. \] (e 7.409)

It follows from \( (e \text{ 7.409}), (e \text{ 7.392}), (e \text{ 7.393}) \) and Theorem 7.2 of [12] that \( \alpha \) and \( \beta_1 \) are asymptotically unitarily equivalent.

It follows from Theorem 4.2 that there exists a unitary \( u \in U(A) \) and a strongly asymptotically inner automorphism
\[ \alpha = \text{Ad} u \circ \sigma_1 \circ \beta_1 \circ \sigma_1^{-1}, \] (e 7.410)
\[ \text{Ad} u \circ \sigma \circ \beta \circ \sigma^{-1}, \] (e 7.411)

where \( \sigma = \sigma_1 \circ \sigma' \). Note that
\[ \tilde{\mathcal{R}}(\sigma) = \tilde{\mathcal{R}}(\sigma_1) \times \tilde{\mathcal{R}}(\sigma') = \zeta. \] (e 7.412)

In the case that \( K_1(A) = H_1(K_0(A), K_1(A)) \), by [12] there exists a sequence of unitaries \( \{u_n\} \subset U_0(A) \) and a sequence of strongly asymptotically inner automorphisms \( \sigma''_n \in \text{Aut}(A) \) such that
\[ \alpha = \text{Ad} u_n \circ \sigma''_n \circ \beta_1 \circ (\sigma''_n)^{-1} \text{ and } \lim_{n \to \infty} \|u_n - 1\| = 0. \] (e 7.413)

Put \( \sigma_n = \sigma''_n \circ \sigma' \). Theorem follows.
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