SYMPLICIAL LEVEL-RANK DUALITY VIA TENSOR CATEGORIES

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Abstract. We give two proofs of a level-rank duality for braided fusion categories obtained from quantum groups of type C at roots of unity. The first proof uses conformal embeddings, while the second uses a classification of braided fusion categories associated with quantum groups of type C at roots of unity. In addition we give a similar result for non-unitary braided fusion categories quantum groups of types B and C at odd roots of unity.

1. Introduction

There are a number of results which connect classical affine Lie algebras with interchanged level and rank; such phenomena are generally called level-rank duality. One manifestation of this is that the braided fusion categories associated with such Lie algebras are closely related. In type A a result of this type was proved in [23]. It states that there is a braid-reversing tensor equivalence between $C(\mathfrak{sl}_n)_k^0$ and $C(\mathfrak{sl}_k)_n^0$ where $C(\mathfrak{sl}_n)_k^0$ is the adjoint subcategory of the modular tensor category $C(\mathfrak{sl}_n)_k$ obtained from the affine Lie algebra $\widehat{\mathfrak{sl}}_n$ at level $k$. Recall that an alternative construction of $C(\mathfrak{sl}_n)_k$ is via the semisimplification of the category of tilting modules of the quantum group $U_q\mathfrak{sl}_n$ specialized at $q = e^{\pi i/(n+k)}$, typically denoted $C(\mathfrak{sl}_n, n+k)$.

In this paper we examine the analogous situation for the type C case associated with symplectic Lie algebras $\mathfrak{sp}_{2n}$. Just as in the type A case, the unitary modular category $C(\mathfrak{sp}_{2n})_k$ may be constructed in (at least) two ways: 1) as the semisimplification $C(\mathfrak{sp}_{2n}, 2k+2n+2)$ of the subcategory of tilting modules in $Rep(U_q\mathfrak{sp}_{2n})$ for the choice $q = e^{\pi i/(2k+2n+2)}$ and 2) as the category $C(\mathfrak{sp}_{2n})_k$ of level $k$ representations of the affine Lie algebra $\widehat{\mathfrak{sp}}_{2n}$ under the level preserving tensor product (see [1]).
monoidal equivalence of the categories constructed from these two approaches can be found in [9, 12]. Despite this equivalence we will use both notations to distinguish the approaches.

From any braided fusion category $C$ we may obtain a new braided fusion category $C^{\text{rev}}$ with the same underlying fusion category by replacing the braiding isomorphisms $c_{X,Y}$ with $c_{Y,X}^{-1}$. For any $\mathbb{Z}/2$-graded braided fusion category $C = C_0 \oplus C_1$ we may also obtain a new braided structure by replacing the braiding $c_{X,Y}$ for $X, Y \in C_1$ by $-c_{X,Y}$, leaving the braiding unchanged if either of $X$ or $Y$ are in $C_0$. One way to construct this new braided category directly is to take the diagonal of the $\mathbb{Z}/2 \times \mathbb{Z}/2$-graded category $C \boxtimes s\text{Vec}$. We will denote by $C^-$ the braided fusion category obtained in this way. We prove the following:

**Theorem.** There is a braid-reversing equivalence between $C(\mathfrak{sp}_{2n})_k$ and $C(\mathfrak{sp}_{2k})^-_n$.

We provide two proofs of this theorem. The first, direct, proof follows the strategy of [23] using the conformal embedding $(\mathfrak{sp}_{2n})_k \oplus (\mathfrak{sp}_{2k})_n \subset (\mathfrak{so}_{4nk})_1$. The second employs reconstruction techniques for braided fusion categories with fusion rules of type $C$, found in [27]. While the latter proof is shorter, it invokes some heavy categorical machinery. On the other hand, this categorical approach allows us to prove a related result for $C(\mathfrak{sp}_{2n}, \ell)$ for odd $\ell$ where now the category $C(\mathfrak{so}_{2k}, \ell)$ with $\ell = 2k + 2n + 1$ plays the role of the dual. These are typically non-unitarizable categories, and hence cannot be constructed from affine Lie algebras in the standard way.

It would be interesting to extend our results to the orthogonal case where the level-rank duality connects affine Lie algebras $(\mathfrak{so}_n)_k$ and $(\mathfrak{so}_k)_n$. However this case seems to be technically more involved than the symplectic case and it is not considered in this paper. We refer the reader to [11, 15, 22] for some interesting results in this direction.

This article is organized as follows. In Section 2 we lay the basic Lie theoretic and combinatorial groundwork and in Section 3 we describe the key conformal embedding. Section 4 contains two proofs of our level-rank duality theorem. The appendix contains a detailed proof of the Kac-Peterson formula in the symplectic case.

## 2. Preliminaries

### 2.1. Combinatorics.

Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ be a partition of $|\lambda| := \lambda_1 + \cdots + \lambda_n$ with $n$ parts (see e.g. [20]). We will identify $\lambda$ with its corresponding Young diagram. Let $I_{n,k}$ be the set of all partitions $\lambda$ whose Young diagram fits into an $n \times k$ rectangle; in other words $I_{n,k}$ consists of partitions with $n$ parts and $\lambda_1 \leq k$. We will denote by $I^0_{n,k}$ (respectively $I^1_{n,k}$) the subset of $I_{n,k}$ consisting of partitions $\lambda$ such that $|\lambda|$ is even (respectively odd).

Denote by $\lambda^t$ the transposed partition of $\lambda$. Clearly, $\lambda \in I_{n,k}$ implies $\lambda^t \in I_{k,n}$. Denote by $\lambda^c$ the transpose of the complement of $\lambda$ in an $n \times k$ rectangle. Again, $\lambda \in I_{n,k}$ implies
Let $\lambda \in I_{k,n}$. We will also consider a composition $(\lambda^t)^c = (\lambda^c)^t =: \lambda'^c$ which preserves the set $I_{n,k}$.

**Example 2.1.** Let $\lambda = (2,1,1) \in I_{3,2}$. Then $\lambda^t = (3,1) \in I_{2,3}$, $\lambda' = (2,0) \in I_{2,3}$, and $\lambda'^c = (1,1,0) \in I_{3,2}$.

Let $C_{n,k} := \{(k_0,k_1,\ldots,k_n) \in \mathbb{N}^{n+1} \mid \sum_i k_i = k\}$ be the set of dominant weights for $\mathfrak{sp}_{2n}$ of level $k$. We identify the sets $C_{n,k}$ and $I_{n,k}$ via the mutually inverse bijections $c_{n,k} : I_{n,k} \rightarrow C_{n,k}$, $c_{n,k}(\lambda) := (k - \lambda_1, \lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{n-1} - \lambda_n, \lambda_n)$ and $d_{n,k} : C_{n,k} \rightarrow I_{n,k}$, $d_{n,k}(k_0,k_1,\ldots,k_n) = (k_1 + \cdots + k_n, k_2 + \cdots + k_n, \ldots, k_n)$. With these identifications the bijection $C_{n,k} \rightarrow C_{k,n}$ corresponding to $\lambda \mapsto \lambda'^c$ can be described as follows (see [16]): if we consider $n + k$ dots, first represent $(k_0,\ldots,k_n) \in C_{n,k}$ by picking $n$ dots to be black (leaving the other $k$ dots white) so that there are $k_0$ white dots before the first black dot, $k_1$ white dots between the first and second black dot, and continue this until the last $k_n$ dots are after the last black dot. In this way the black dots partition the white dots according to an element of $C_{n,k}$. Its corresponding element in $C_{k,n}$ is how the white dots partition the black dots (see example below).

**Example 2.2.** Let $n = 7$, $k = 6$ consider $n + k = 13$ dots

\[
\bullet \bullet \bullet \bullet \bullet \circ \circ \circ \circ \circ \circ \circ \circ
\]

$C_{7,6} \ni (0,0,1,0,0,0,3,2) \mapsto (2,4,0,0,1,0,0) \in C_{6,7}$.

The corresponding partitions are $(6,6,5,5,5,2) \in I_{7,6}$ and $(5,1,1,1) \in I_{6,7}$.

The bijection $\lambda \mapsto \lambda'^c$ also has a simple interpretation in this language: one has to read the diagram of black and white dots representing $\lambda$ backwards.

2.2. Symplectic Lie algebra. Let $\mathfrak{sp}_{2N}$ be the Lie algebra of $2N \times 2N$ symplectic matrices over $\mathbb{C}$, see e.g. [14].

**2.2.1. The root system of $\mathfrak{sp}_{2N}$.** Following [14], let $e_1, e_2, \ldots, e_N$ be an orthonormal basis for $\mathfrak{h}^*$ such that the simple roots are given by

\[
e_1 - e_2, e_2 - e_3, \ldots, e_{N-2} - e_{N-1}, e_{N-1} - e_N, 2e_N.
\]

The positive long roots are therefore given by

\[2e_1, 2e_2, \ldots, 2e_N\]

Let $W$ be the Weyl group of $\mathfrak{sp}_{2N}$. We recall that the group $W$ identifies with the group of signed permutations of the basis $e_1, e_2, \ldots, e_N$, see [14].
2.2.2. Affinization. Let \( \hat{\mathfrak{sp}}_{2N} \) be the affinization of \( \mathfrak{sp}_{2N} \) (see e.g. [17 Chapter 7]). Thus
\[
\hat{\mathfrak{sp}}_{2N} = (\mathfrak{sp}_{2N} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}K \text{ where the element } K \text{ is central and (for } \kappa \text{ the Killing form)}
\]
\[
[x \otimes t^a, y \otimes t^b] = [x, y] \otimes t^{a+b} + a\kappa(x, y)\delta_{a-b}K.
\]

2.2.3. The root system of \( \hat{\mathfrak{sp}}_{2N} \). Let \( \hat{W} \) be the affine Weyl group. Let \( T \subset \hat{W} \) be the subgroup of translations, so \( \hat{W} = W \rtimes T \) ([17 Proposition 6.5]). Write \( \hat{\Delta} \) and \( \hat{\Delta}^+ \) for the affine root system and its positive roots respectively. Let \( \delta \) be the indivisible positive imaginary root. Then \( w(\delta) = \delta \) for all \( w \in \hat{W} \) and the imaginary roots are nonzero multiples of \( \delta \). The real roots of \( \hat{\Delta} \) are \( \Delta + \mathbb{Z}\delta \), with \( \hat{\Delta}^+ = \Delta^+ \cup (\Delta + \mathbb{Z}_{>0}\delta) \).

2.2.4. Representations. Let \( \mathcal{C}(\mathfrak{sp}_{2n})_k \) be the category of highest weight integrable \( \hat{\mathfrak{sp}}_{2n} \) modules of level \( k \), see e.g. [17 Chapter 10]. The simple objects of \( \mathcal{C}(\mathfrak{sp}_{2n})_k \) are labeled by partitions \( \lambda \in I_{n,k} \). Write \( \lambda \) for the simple \( \mathcal{C}(\mathfrak{sp}_{2n})_k \) module of highest weight \( c_{n,k}(\lambda) \). We will also denote by \( \hat{\Lambda} \) the corresponding irreducible finite dimensional module over \( \mathfrak{sp}_{2n} \); thus the highest weight of \( \hat{\Lambda} \) is \( (k_1, \ldots, k_n) \) if \( c_{n,k}(\lambda) = (k_0, k_1, \ldots, k_n) \).

2.3. Orthogonal Lie algebra. Let \( \mathfrak{so}_N \) be the Lie algebra of form preserving endomorphisms on an \( N \)-dimensional vector space equipped with a symmetric non-degenerate bilinear form and let \( \hat{\mathfrak{so}}_N \) be its affinization. We will use the same notations as in Section 2.2 to discuss the highest weight integrable representations of \( \hat{\mathfrak{so}}_N \). Recall that in the case when \( N \) is divisible by 4 the simple objects of \( \mathcal{C}(\mathfrak{so}_N)_1 \) are \( \hat{\Lambda}_0, \hat{\Lambda}_1, \hat{\Lambda}_+, \hat{\Lambda}_- \) where \( \hat{\Lambda}_0 \) is the trivial \( \mathfrak{so}_N \)-module, \( \hat{\Lambda}_1 \) is the natural \( \mathfrak{so}_{4nk} \)-module of dimension \( N \), and \( \hat{\Lambda}_\pm \) are two half-spinor \( \mathfrak{so}_N \)-modules.

Now assume that \( N = 4nk \) and that the symmetric bilinear form on \( \mathbb{C}^N \) is the tensor product of the symplectic forms on the spaces \( \mathbb{C}^{2n} \) and \( \mathbb{C}^{2k} \). Thus we have an embedding \( \mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2k} \subset \mathfrak{so}_{4nk} \) and we can distinguish between \( \hat{\Lambda}_+ \) and \( \hat{\Lambda}_- \) in the following way: \( \hat{\Lambda}_+ \) has some nonzero \( \mathfrak{sp}_{2n} \)-invariant vector while \( \hat{\Lambda}_- \) has no nonzero \( \mathfrak{sp}_{2n} \)-invariant vectors, see Remark 3.4 below.

**Warning:** Notation convention for \( \hat{\Lambda}_\pm \) may change if we interchange the roles of \( \mathfrak{sp}_{2n} \) and \( \mathfrak{sp}_{2k} \), see Remark 3.4.

2.4. Conformal embeddings and Kac-Peterson formula. It is a well known fact and a consequence of general results in [16] that the embedding \( \mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2k} \subset \mathfrak{so}_{4nk} \) induces a conformal embedding
\[
(\hat{\mathfrak{sp}}_{2n})_k \oplus (\hat{\mathfrak{sp}}_{2k})_n \subset (\hat{\mathfrak{so}}_{4nk})_1.
\]
This means that a restriction of a highest weight integrable \( \hat{\mathfrak{so}}_{4nk} \)-module of level 1 to \( \hat{\mathfrak{sp}}_{2n} \oplus \hat{\mathfrak{sp}}_{2k} \) is again a highest weight integrable module on which the central elements of \( \mathfrak{sp}_{2n} \) and \( \mathfrak{sp}_{2k} \) act as \( k\text{Id} \) and \( n\text{Id} \) respectively.
The following result is [16, Proposition 2]. Since the proof is omitted in loc. cit. we provide its derivation from [16, Proposition 1] in the appendix (Section 5).

**Proposition 2.3.** There is an isomorphism of \((\mathfrak{sp}_{2n})_k \oplus (\mathfrak{sp}_{2k})_n\) modules:

\[
\hat{\Lambda}_+ \oplus \hat{\Lambda}_- \cong \bigoplus_{\lambda \in I_{n,k}} \hat{\lambda} \boxtimes \hat{\lambda}^c.
\]

### 2.5. Tensor categories.

Recall that a monoidal category \((\mathcal{C}, \otimes)\) is *braided* if there is a natural bifunctor isomorphism \(c_{X,Y} : X \otimes Y \rightarrow Y \otimes X\) called a braiding subject to the hexagon axioms (see e.g. [8, Definition 8.1.1]). For every braided tensor category \(\mathcal{C}\), there is a reversed braiding on \(\mathcal{C}\) given by \(c^\text{rev}_{X,Y} = c^{-1}_{Y,X}\). A braided tensor category \(\mathcal{C}\) endowed with the reversed braiding will be denoted \(\mathcal{C}^\text{rev}\). A monoidal functor (a functor \(T : \mathcal{C} \rightarrow \mathcal{D}\), together with a natural isomorphism \(J : T(- \otimes -) \rightarrow T(-) \otimes T(-)\)) is said to be a braided equivalence of categories if \(c_{T(X),T(Y)}J_{X,Y} = J_{Y,X}T(c_{X,Y})\) and \(T\) is an equivalence of the underlying categories (see e.g. [8, Definition 8.1.7]). A *braid–reversing* equivalence of \(\mathcal{C}\) and \(\mathcal{D}\) is a braided equivalence \(\mathcal{C} \simeq \mathcal{D}^\text{rev}\). Two objects \(X, Y\) in a braided tensor category \(\mathcal{C}\) are said to mutually centralize each other, in the sense of [21], if \(c_{X,Y}c_{Y,X} = \text{id}_{X \otimes Y}\).

It is a deep and important fact (see e.g. [1, Chapter 7]) that the categories \(\mathcal{C}(\mathfrak{sp}_{2n})_k\) and \(\mathcal{C}(\mathfrak{so}_{4nk})_1\) have a natural structure of *modular tensor categories*. In particular they are braided rigid monoidal categories which are *non-degenerate* in the sense of [8, 8.19]. We will often use the following special property: in the categories \(\mathcal{C}(\mathfrak{sp}_{2n})_k\) and \(\mathcal{C}(\mathfrak{so}_{4nk})_1\) every object is self-dual.

It is well known that the category \(\mathcal{C}(\mathfrak{sp}_{2n})_k\) is \(\mathbb{Z}/2\)–graded (see [8, 4.14]); the grading is given by the weights of an object modulo the root lattice. Thus the parity of the object \(\hat{\lambda}\) is \(|\lambda| \mod 2\).

Similarly the category \(\mathcal{C}(\mathfrak{so}_{4nk})_1\) is graded by the quotient of the weight lattice of \(\mathfrak{so}_{4nk}\) modulo the root lattice; this group is known to be \(\mathbb{Z}/2 \oplus \mathbb{Z}/2\). This implies that the category \(\mathcal{C}(\mathfrak{so}_{4nk})_1\) is *pointed* (i.e. all simple objects are invertible) with group of simple objects isomorphic to \(\mathbb{Z}/2 \oplus \mathbb{Z}/2\). The unit object of the category \(\mathcal{C}(\mathfrak{so}_{4nk})_1\) is \(\hat{\Lambda}_0\). It is a well known and easy to check fact that the object \(\hat{\Lambda}_1 \in \mathcal{C}(\mathfrak{so}_{4nk})_1\) is a *fermion*. This means that the braiding \(c_{\hat{\Lambda}_1,\hat{\Lambda}_1}\) equals \(-\text{id}_{\hat{\Lambda}_1,\hat{\Lambda}_1}\) or, equivalently, that the subcategory of \(\mathcal{C}(\mathfrak{so}_{4nk})_1\) generated by \(\hat{\Lambda}_1\) is equivalent to the category of super vector spaces as a braided fusion category. This gives a simple construction of the category \(\mathcal{C}^-\) which appears in the Introduction: let \(\mathcal{C} = \mathcal{C}^0 \oplus \mathcal{C}^1\) be a \(\mathbb{Z}/2\)–graded braided fusion category. Consider the subcategory \(\mathcal{C}_0 = \mathcal{C}^0 \boxtimes \hat{\Lambda}_0 \oplus \mathcal{C}^1 \boxtimes \hat{\Lambda}_1\) of \(\mathcal{C} \boxtimes \mathcal{C}(\mathfrak{so}_{4nk})_1\). It is clear that \(\mathcal{C}_0\) is closed under the tensor product; moreover we have a braided equivalence \(\mathcal{C}^- \simeq \mathcal{C}_0\) which sends \(X \in \mathcal{C}_0\) to \(X \boxtimes \hat{\Lambda}_0\) and \(X \in \mathcal{C}_1\) to \(X \boxtimes \hat{\Lambda}_1\).
2.6. Étale algebras from conformal embeddings. An étale algebra in a semisimple braided tensor category \( \mathcal{C} \) is defined to be an object \( A \in \mathcal{C} \) endowed with an associative commutative unital multiplication and that the category \( \mathcal{C}_A \) of right \( A \)-modules is semisimple, see [4, Definition 3.1]. An étale algebra \( A \) is called connected if the unit object appears in \( A \) with multiplicity 1. For a connected étale algebra \( A \in \mathcal{C} \) the category \( \mathcal{C}_A \) with operation \( \otimes_A \) of tensor product over \( A \) is naturally a fusion category, see e.g. [4, Section 3.3]. The category \( \mathcal{C}_A \) contains a full tensor Serre subcategory \( \mathcal{C}^{dys}_A \) of dyslectic modules, which is also naturally braided. See for example [4, Section 3.5].

A general result observed in [18, Theorem 5.2] states that for any conformal embedding the pullback of the vacuum module is an étale algebra (see [13, 3] for a proof); moreover taking pullbacks is a braided equivalence with the category of dyslectic modules over this algebra, see [2] for a proof. Specializing this to the conformal embedding (1) we get

**Theorem 2.4.** Let \( A \) be the restriction of \( \hat{\Lambda}_0 \) under the embedding (1). Then \( A \) is a connected étale algebra in \( \mathcal{C}(\mathfrak{sp}_{2n})_k \otimes \mathcal{C}(\mathfrak{sp}_{2k})_n \). Moreover, the restriction functor is a braided equivalence \( \mathcal{C}(\mathfrak{so}_{4nk})_1 \simeq (\mathcal{C}(\mathfrak{sp}_{2n})_k \otimes \mathcal{C}(\mathfrak{sp}_{2k})_n)^{dys}_A \). Hence the pullbacks of \( \hat{\Lambda}_1, \hat{\Lambda}_+ \) and \( \hat{\Lambda}_- \) are precisely the simple dyslectic \( A \)-modules in \( \mathcal{C}(\mathfrak{sp}_{2n})_k \otimes \mathcal{C}(\mathfrak{sp}_{2k})_n \).

2.7. Lagrangian algebras in products. We recall that a connected étale algebra \( A \) in a non-degenerate braided fusion category \( \mathcal{C} \) is Lagrangian if the category \( \mathcal{C}^{dys}_A \) is trivial or, equivalently, \( \text{FPdim}(A)^2 = \text{FPdim}(\mathcal{C}) \), see [4, Definition 4.6].

The following result is a version of [5, Theorem 3.6].

**Theorem 2.5.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be two non-degenerate braided fusion categories and let \( A \in \mathcal{C} \otimes \mathcal{D} \) be a Lagrangian algebra. Let \( A_1 = A \cap (\mathcal{C} \otimes 1) \) and let \( A_2 = A \cap (1 \otimes \mathcal{D}) \). Then \( A_1 \in \mathcal{C} \) and \( A_2 \in \mathcal{D} \) are connected étale algebras and there exists a braid reversing equivalence \( \phi : \mathcal{C}^{dys}_{A_1} \simeq \mathcal{D}^{dys}_{A_2} \) such that \( A = \bigoplus_{M \in \text{Irr}(\mathcal{C}^{dys}_{A_1})} M^* \otimes \phi(M) \) as an object of \( \mathcal{C} \otimes \mathcal{D} \) (the summation is over the isomorphism classes of simple objects of \( \mathcal{C}^{dys}_{A_1} \)).

Conversely, given two connected étale algebras \( A_1 \in \mathcal{C} \) and \( A_2 \in \mathcal{D} \) and a braid reversing equivalence \( \phi : \mathcal{C}^{dys}_{A_1} \simeq \mathcal{D}^{dys}_{A_2} \), the object \( A = \bigoplus_{M \in \text{Irr}(\mathcal{C}^{dys}_{A_1})} M^* \otimes \phi(M) \in \mathcal{C} \otimes \mathcal{D} \) has the structure of a Lagrangian algebra.

**Example 2.6.** Assume that \( A \in \mathcal{C} \otimes \mathcal{D} \) is a Lagrangian algebra such that \( A_1 = A \cap (\mathcal{C} \otimes 1) = 1 \otimes 1 \). Then Theorem 2.5 says that there exists a braid reversing equivalence \( \phi : \mathcal{C} \simeq \mathcal{D}^{dys}_{A_2} \). If we write \( A = \bigoplus_{X \in \text{Irr}(\mathcal{C})} X^* \otimes Y_i \) then the functor \( \phi \) sends \( X_i \) to \( A_2 \)-module which is \( Y_i \) as an object of \( \mathcal{D} \).
3. Branching rules

In this Section we will present the branching rules for conformal embedding \([\mathfrak{1}]\).

The unit object \(1_{n,k}\) of \(C(\mathfrak{sp}_{2n})_k\) corresponds to the empty Young diagram in \(I_{n,k}\) and \(1^c_{n,k}\) is a unique nontrivial invertible object corresponding in \(I_{k,n}\) to the unique diagram with \(kn\) boxes. For \(\lambda \in I_{n,k}\), tensoring in \(C(\mathfrak{sp}_{2k})_n\) is given by

\[
1^c_{n,k} \otimes ^\wedge \lambda \cong ^\wedge \lambda^c,
\]

see [10]. We will just write \(1\) and \(1^c\) for convenience.

**Lemma 3.1.** The \(A\)-modules \((1 \boxtimes 1^c) \otimes A\) and \((1^c \boxtimes 1) \otimes A\) are simple and dyslectic. Moreover there exists a sign \(s = \pm\) such that \((1 \boxtimes 1^c) \otimes A \cong \hat{\Lambda}_s^c\).

**Proof.** By Proposition 2.3, \(1 \boxtimes 1^c\) and \(1^c \boxtimes 1\) are direct summands of \(\hat{\Lambda}_+^c \oplus \hat{\Lambda}_-^c\). Hence the result follows from Theorem 2.4. \(\square\)

**Warning:** We do not claim that these two modules are (or are not) isomorphic. We will see that this depends on the values of \(n\) and \(k\).

Recall that the category \(C(\mathfrak{sp}_{2n})_k\) is \(\mathbb{Z}/2\)-graded. We denote by \(C(\mathfrak{sp}_{2n})_k^0\) and \(C(\mathfrak{sp}_{2n})_k^1\) its trivial and nontrivial components. Recall that \(C(\mathfrak{sp}_{2n})_k^0\) coincides with the centralizer of the object \(1^c\).

**Corollary 3.2.** (i) The object \(\hat{\Lambda}_0 = A\) is contained in \(C(\mathfrak{sp}_{2n})_k^0 \boxtimes C(\mathfrak{sp}_{2k})_n^0\).

(ii) The object \(\hat{\Lambda}_1\) is contained in \(C(\mathfrak{sp}_{2n})_k^1 \boxtimes C(\mathfrak{sp}_{2k})_n^1\).

**Proof.** By Lemma 3.1 and [6, Lemma 3.15] \(A\) centralizes \(1 \boxtimes 1^c\) and \(1^c \boxtimes 1\). This implies (i). It is clear that \(\hat{\Lambda}_1\) contains \(\hat{\lambda}_1 \boxtimes \hat{\lambda}_1\) where \(\hat{\lambda}_1\) is the one box Young diagram; hence \(\hat{\Lambda}_1\) is a direct summand of \((\hat{\lambda}_1 \boxtimes \hat{\lambda}_1) \otimes A\). Thus (i) implies (ii). \(\square\)

We arrive at the main result of this section:

**Theorem 3.3 (cf [11]).** There are isomorphisms of \((\mathfrak{sp}_{2n})_k \oplus (\mathfrak{sp}_{2k})_n\)-modules:

\[
\hat{\Lambda}_0 \cong \bigoplus_{\lambda \in I_{n,k}^0} \hat{\lambda} \boxtimes \hat{\lambda}^c,
\]

\[
\hat{\Lambda}_1 \cong \bigoplus_{\lambda \in I_{n,k}^1} \hat{\lambda} \boxtimes \hat{\lambda}^c,
\]

\[
\hat{\Lambda}_+ \cong \bigoplus_{\lambda \in I_{n,k}^0} \hat{\lambda} \boxtimes \hat{\lambda}_c,
\]

\[
\hat{\Lambda}_- \cong \bigoplus_{\lambda \in I_{n,k}^1} \hat{\lambda} \boxtimes \hat{\lambda}_c.
\]
Proof. Let $s = \pm$ be as in Lemma 3.1. The multiplication rules in the group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ imply that

$$\hat{\Lambda}_s \otimes (\hat{\Lambda}_+ \oplus \hat{\Lambda}_-) \cong \hat{\Lambda}_0 \oplus \hat{\Lambda}_1,$$

where the tensor product is taken in the category $\mathcal{C}(\mathfrak{so}_{4nk})_1$. Using Theorem 2.4, the definition of $s$, and Proposition 2.3 we have equivalently

$$\hat{\Lambda}_0 \oplus \hat{\Lambda}_1 \cong \hat{\Lambda}_s \otimes_A (\hat{\Lambda}_+ \oplus \hat{\Lambda}_-)$$

$$\cong ((1 \boxtimes 1^c) \otimes A) \otimes_A \bigoplus_{\lambda \in I_{n,k}} (\hat{\lambda} \boxtimes \hat{\lambda}^c)$$

$$\cong (1 \boxtimes 1^c) \otimes \bigoplus_{\lambda \in I_{n,k}} (\hat{\lambda} \boxtimes \hat{\lambda}^c)$$

$$\cong \bigoplus_{\lambda \in I_{n,k}} \hat{\lambda} \boxtimes (1^c \otimes \hat{\lambda}^c)$$

$$\cong \bigoplus_{\lambda \in I_{n,k}} \hat{\lambda} \boxtimes \hat{\lambda}^t,$$

where the tensor product in the third and fourth line is taken in $\mathcal{C}(\mathfrak{sp}_{2n})_k \boxtimes \mathcal{C}(\mathfrak{sp}_{2k})_n$ and in $\mathcal{C}(\mathfrak{sp}_{2k})_n$ respectively. Using Corollary 3.2 we get the required decompositions of $\hat{\Lambda}_0$ and $\hat{\Lambda}_1$. Now by definition of $s$ we get $\hat{\Lambda}_s = (1 \boxtimes 1^c) \otimes A \cong \bigoplus_{\lambda \in I_{n,k}} \hat{\lambda} \boxtimes \hat{\lambda}^c$ and therefore $\hat{\Lambda}_{-s} = \bigoplus_{\lambda \in I_{n,k}} \hat{\lambda} \boxtimes \hat{\lambda}^c$. Then by Remark 3.4 we see that $s = +$ and the result follows.

Remark 3.4. (i) We see that $1^c \boxtimes 1$ appears in the decomposition of the same $\hat{\Lambda}_+$ if and only if $nk$ is even. Thus the modules $(1 \boxtimes 1^c) \otimes A$ and $(1^c \boxtimes 1) \otimes A$ from Lemma 3.1 are isomorphic if and only if $nk$ is even.

(ii) It was observed by Hasegawa in [11] that all summands in the decompositions of $\hat{\Lambda}_\pm$ have the same conformal dimension. It follows that we have similar decompositions

$$\Delta_+ \cong \bigoplus_{\lambda \in I_{n,k}^0} \lambda \boxtimes \lambda^c,$$

$$\Delta_- \cong \bigoplus_{\lambda \in I_{n,k}^1} \lambda \boxtimes \lambda^c$$

for the branching under the finite dimensional algebras embedding $\mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2k} \subset \mathfrak{so}_{4nk}$ (see Proposition 5.5 for a closely related statement). In particular the summand of $\Delta_+$ corresponding to the empty Young diagram contains vectors invariant under the action of $\mathfrak{sp}_{2n}$, which justifies our choice for labeling of $\hat{\Lambda}_\pm$ in Section 2.3. Note that the conformal dimensions of the summands of $\hat{\Lambda}_0$ and $\hat{\Lambda}_1$ are not constant.
4. Level rank duality

In this section we prove the main result of this paper. Recall the construction of the category $C(\mathfrak{sp}_{2n})_k$ from the introduction.

**Theorem 4.1.** There is a braid-reversing monoidal equivalence

$$C(\mathfrak{sp}_{2n})_k \simeq C(\mathfrak{sp}_{2k})^{-}$$

sending $\hat{\lambda} \mapsto \hat{\lambda}^t$. In particular, $C(\mathfrak{sp}_{2n})_k^0$ and $C(\mathfrak{sp}_{2k})_k^0$ are braid reversing equivalent.

4.1. First proof.

**Proof.** By Theorems 2.4 and 3.3 we see that an object

has the structure of a connected étale algebra such that we have a braided equivalence $\mathcal{C}(\mathfrak{sp}_{2n})_k \simeq D$. Thus by Example 2.6 we have a braid reversing equivalence $\hat{\lambda} \mapsto \hat{\lambda}^t \mapsto \hat{\lambda}^t \mapsto \hat{\lambda}$. In particular, $\mathcal{C}(\mathfrak{sp}_{2n})_k^0$ and $\mathcal{C}(\mathfrak{sp}_{2k})_k^0$ are braid reversing equivalent.

Let us use Theorem 2.5 again with $A = \bigoplus_{\lambda \in I_{n,k}} \hat{\lambda} \otimes \hat{\lambda}^t \in C(\mathfrak{sp}_{2n})_k \boxtimes C(\mathfrak{sp}_{2k})_n$,

$$A = C(\mathfrak{sp}_{2n})_k \boxtimes C(\mathfrak{sp}_{2k})_n$$

has the structure of a connected étale algebra such that we have a braided equivalence $\mathcal{C}(\mathfrak{sp}_{2n})_k \boxtimes C(\mathfrak{sp}_{2k})_n)_{\text{dys}} \simeq C(\mathfrak{so}_{4nk})_1$. Thus by Theorem 2.5 we get a Lagrangian algebra

$$\tilde{A} \in C(\mathfrak{sp}_{2n})_k \boxtimes C(\mathfrak{sp}_{2k})_n \boxtimes C(\mathfrak{so}_{4nk})_1$$

given by

$$\tilde{A} = \bigoplus_{\lambda \in I_{n,k}} \hat{\lambda} \otimes \hat{\lambda}^t \otimes \hat{\lambda}_0 \oplus \bigoplus_{\lambda \in I_{n,k}} \hat{\lambda} \otimes \hat{\lambda}^t \otimes \hat{\lambda}_1 \oplus \bigoplus_{\lambda \in I_{n,k}} \hat{\lambda} \otimes \hat{\lambda}^c \otimes \hat{\lambda}_+ \oplus \bigoplus_{\lambda \in I_{n,k}} \hat{\lambda} \otimes \hat{\lambda}^c \otimes \hat{\lambda}_-.$$

Let us use Theorem 2.5 again with $C = C(\mathfrak{sp}_{2n})_k$ and $D = C(\mathfrak{sp}_{2k})_n \boxtimes C(\mathfrak{so}_{4nk})_1^{\text{rev}}$. We have $\tilde{A} \cap (C \boxtimes 1) = 1 \boxtimes 1 \hat{\lambda}_0$ and $\tilde{A} \cap (1 \boxtimes D) = 1 \boxtimes 1 \hat{\lambda}_0 \oplus 1 \boxtimes \hat{\lambda}_0 \otimes \hat{\lambda}_h =: B$ (note that the second summand is invertible of order 2). Thus by Example 2.6 we have a braid reversing equivalence $C \simeq D_B^{\text{dys}}$ sending $\hat{\lambda}$ to the $B$-module $\hat{\lambda}^t \boxtimes \hat{\lambda}_0 \otimes \hat{\lambda}_c \boxtimes \hat{\lambda}_+ = (\hat{\lambda}^t \boxtimes \hat{\lambda}_0) \otimes B$ if $|\lambda|$ is even and to $\hat{\lambda}^t \boxtimes \hat{\lambda}_1 \oplus \hat{\lambda}^c \boxtimes \hat{\lambda}_- = (\hat{\lambda}^t \boxtimes \hat{\lambda}_1) \otimes B$ if $|\lambda|$ is odd.

Now consider the additive subcategory $\mathcal{D}_0$ of $\mathcal{D} = C(\mathfrak{sp}_{2k})_n \boxtimes C(\mathfrak{so}_{4nk})_1^{\text{rev}}$ generated by the objects $\hat{\lambda} \otimes \hat{\lambda}_0$ with $\lambda \in I_{k,n}^0$ and $\hat{\lambda} \otimes \hat{\lambda}_1$ with $\lambda \in I_{k,n}^1$. It is clear that $\mathcal{D}_0$ is a subcategory of $\mathcal{D}$; moreover since the object $\hat{\lambda}_1$ is a fermion (see Section 2.6), the category $\mathcal{D}_0$ identifies with the category $C(\mathfrak{sp}_{2k})_n^{\text{rev}}$. On the other hand, the free module functor $X \mapsto X \otimes B$ gives a braided equivalence $\mathcal{D}_0 \simeq D_B^{\text{dys}}$. Thus we get a chain of equivalences

$$C(\mathfrak{sp}_{2n})_k = C \simeq D_B^{\text{dys}} \simeq \mathcal{D}_0 = C(\mathfrak{sp}_{2k})_n^{\text{rev}}.$$
sending $\hat{\lambda}$ to $\hat{\lambda}'$. This completes the proof. 

\[\square\]

\textit{Remark 4.2.} The proof above gives no information on uniqueness of the structure of braided tensor functor on the functor sending $\hat{\lambda}$ to $\hat{\lambda}'$.

4.2. \textbf{Second Proof.} In [27] braided fusion categories with the same fusion rules as $\mathcal{C}(\mathfrak{sp}_{2n}, 2n + 2k + 2)$ and $\mathcal{C}(\mathfrak{sp}_{2n}, 2n + 2k + 1)$ are reconstructed from their Grothendieck (semi-)rings and the eigenvalues of the braiding $c_{X,X}$ on the generating object $X$ analogous to the 2n-dimensional representation of $\mathfrak{sp}_{2n}$, i.e. with highest weight $(1,0,\ldots,0)$. In particular they prove the following (cf. [27, Section 9.3]):

\textbf{Theorem 4.3.} Let $\mathcal{C}$ and $\tilde{\mathcal{C}}$ be braided fusion categories with Grothendieck rings isomorphic to that of $\mathcal{C}(\mathfrak{sp}_{2n}, \ell)$. Denote by $X$, $\hat{X}$ the generating objects of $\mathcal{C}, \tilde{\mathcal{C}}$ corresponding to $X_{(1)} \in \mathcal{C}(\mathfrak{sp}_{2n}, \ell)$, and $1, \tilde{1}, Y, \tilde{Y}$ and $W, \tilde{W}$ the objects corresponding to $X_{(0)}, X_{(2)}$ and $X_{(1,1)}$ in $\mathcal{C}(\mathfrak{sp}_{2n}, \ell)$. If the eigenvalues of $c_{X,X}$ and $c_{\hat{X},\hat{X}}$ coincide on $[1, Y, W]$ and $[\tilde{1}, \tilde{Y}, \tilde{W}]$ then $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are equivalent as braided fusion categories.

We remark that if $\ell$ is even we are in the symplectic case, while for $\ell$ odd this is called the ortho-symplectic case. Notice also that in this subsection we denote the weights by the corresponding Young diagrams, i.e. $\lambda$ etc. rather than as $\hat{\lambda}$ as is customary in the quantum group approach.

We now provide an alternate proof to Theorem 4.1 using Theorem 4.3.

\textit{Proof.} Let $\ell = 2k + 2n + 2$, and denote by $X_{\lambda}$ the simple objects in $\mathcal{C}(\mathfrak{sp}_{2n}, \ell)$ and by $\hat{X}_{\mu}$ the simple objects in $\tilde{\mathcal{C}}(\mathfrak{sp}_{2k}, \ell)$, where $\lambda$ is a Young diagram with at most $n$ rows and at most $k$ columns and $\mu$ is a Young diagram with at most $k$ rows and at most $n$ columns. Notice that $\mathcal{C}(\mathfrak{sp}_{2k}, \ell)^{-\text{rev}}$ and $\mathcal{C}(\mathfrak{sp}_{2k}, \ell)$ are identical as fusion categories.

We first must verify that $\mathcal{C}(\mathfrak{sp}_{2k}, \ell)$ has the same fusion rules as $\mathcal{C}(\mathfrak{sp}_{2n}, \ell)$. Clearly there is a bijection between these label sets given by transpose. To see that transpose provides an isomorphism of Grothendieck rings we note by [27, Proposition 8.6] that the fusion rules for $\mathcal{C}(\mathfrak{sp}_{2n}, \ell)$ and $\mathcal{C}(\mathfrak{sp}_{2k}, \ell)$ are determined by the rule for tensoring with $X := X_{(1)}$ (resp. $\hat{X} := \hat{X}_{(1)}$). These are as follows: $X_{(1)} \otimes X_{\lambda}$ is the direct sum of all $X_{\mu}$ whose Young diagram has one box more or one box less than $\hat{\lambda}$ with the same rule for $\hat{X}$. Clearly this rule is preserved when taking the transpose. Notice that under this isomorphism the object $X_{(2)}$ is mapped to $\hat{X}_{(1,1)}$ and $X_{(1,1)}$ is mapped to $\hat{X}_{(2)}$.

Next we must compare the eigenvalues of the braiding isomorphisms $c_{X,X}$ and $c_{\hat{X},\hat{X}}$.

Observe that $X \otimes X \cong 1 \oplus X_{(1,1)} \oplus X_{(2)}$. The eigenvalues of $c_{X,X}$ on $X_{\lambda} \subset X \otimes X$ are computed as in [25]: $\pm q^{\lambda \cdot \rho - c_{(1)}}$ where $c_{\lambda} := \langle \lambda + 2\rho, \lambda \rangle$ and where we take $+$ if $X_\lambda$ appears in $S^2 X$ as representations of $U_q \mathfrak{sp}_{2n}$ and $-$ otherwise. In this case $X_{(2)} \subset S^2 X$
while $1 = X_{(0)}, X_{(1,1)} \subset \Lambda^2 X$. It follows that the eigenvalues for $c_{X,X}$ on $[1, X_{(2)}, X_{(1,1)}]$ are $[-q^{-2n-1}, q, -q^{-1}]$, whereas for $c_{\tilde{X}, \tilde{X}}$ on $[1, \tilde{X}_{(2)}, \tilde{X}_{(1,1)}]$ are $[-q^{-2k-1}, q, -q^{-1}]$. The effect of transposing diagrams, followed by reversing the braiding and then an overall sign change takes $[-q^{-2k-1}, q, -q^{-1}]$ to $[-q^{2k+1}, q, -q^{-1}]$, and since $q^{2k+1} = -q^{-2n-1}$ we have verified the eigenvalues match.

\[\square\]

In [26] it is shown that the non-unitary ribbon category $\mathcal{C}(\mathfrak{sp}_{2n}, 2n + 2k + 1)$ has the same fusion rules as $\mathcal{C}(\mathfrak{so}_{2k+1}, 2n + 2k + 1)$. We will prove a more precise result, but first we need some additional notation. One may construct ribbon fusion categories from $\mathcal{C}$ with simple objects labeled by integer weights forms a modular subcategory, and we need some additional notation. One may construct ribbon fusion categories from $\mathcal{C}$ with simple objects labeled by integer weights forms a modular subcategory, and we need some additional notation. One may construct ribbon fusion categories from $\mathcal{C}$.

Moreover, we compute $1$ for type $B$.

We first establish the following:

Lemma 4.4. $\mathcal{C}(\mathfrak{sp}_{2n}, Q, \ell)$ and $\mathcal{C}(\mathfrak{so}_{2k+1}, Q, \ell)$ are $\mathbb{Z}/2$-graded, with modular trivially graded component, for any choice of $Q$. In fact we have $\mathcal{C}(\mathfrak{sp}_{2n}, Q, \ell) \cong \mathcal{C}(\mathfrak{sp}_{2n}, Q, \ell) \otimes \mathcal{C}(\mathbb{Z}/2, P_1)$ and $\mathcal{C}(\mathfrak{so}_{2n}, Q, \ell) \cong \mathcal{C}(\mathfrak{so}_{2n}, Q, \ell) \otimes \mathcal{C}(\mathbb{Z}/2, P_2)$ where $\mathcal{C}(\mathbb{Z}/2, P_i)$ are pointed ribbon categories associated with the pre-metric group $(\mathbb{Z}/2, P_i)$.

Proof. $\mathcal{C}(\mathfrak{sp}_{2n}, Q, \ell) \otimes \mathcal{C}(\mathfrak{so}_{2k+1}, Q, \ell)$ each have one non-trivial invertible object (see [26]), labeled by the weight $(\ell - 2n, 0, \ldots, 0)$ for type $C$ and $\frac{1}{2}(\ell - 2k, \ldots, \ell - 2k)$ for type $B$. Let us denote these by $\eta$ for type $C$ and $\gamma$ for type $B$.

Computing the twists and dimension we find that $\eta$ is a non-unitary fermion $(\dim(\eta) = -1, \theta_\eta = 1)$ if $Q^\ell = 1$ and a non-unitary boson $(\dim(\eta) = -1, \theta_\eta = -1)$ if $Q^\ell = -1$. Moreover, we compute

$$c_{\eta,X_{(1)}} c_{X_{(1)}} \eta = \frac{\theta_\eta \theta_{X_{(1)}}}{\theta_{\eta \otimes X_{(1)}}} Id_{X_{(1)} \otimes \eta} = Id_{X_{(1)} \otimes \eta}$$

so that $\eta$ is transparent regardless of the value of $Q^\ell$, and in fact generates the M"uger center of $\mathcal{C}(\mathfrak{sp}_{2n}, Q, \ell)$. In particular the subcategory with simple objects labeled by Young diagrams with an even number of boxes $\mathcal{C}(\mathfrak{sp}_{2n}, Q, \ell)_0$ does not contain $\eta$ but has centralizer $(\eta)$, giving the desired factorization and modularity $\mathcal{C}(\mathfrak{sp}_{2n}, Q, \ell)_0$.

Similarly we compute that $\gamma$ is a transparent boson $(\dim(\gamma) = \theta_\gamma = \pm 1)$ or fermion $(\dim(\gamma) = -\theta_\gamma = \pm 1)$ if $k$ is even or if $k$ is odd and $Q^\ell = 1$ and a semion $(\dim(\gamma) = \pm 1$ and $\theta_\gamma = \pm i$) otherwise. In either case this shows that the subcategory $\mathcal{C}(\mathfrak{so}_{2k+1}, Q, \ell)_0$ with simple objects labeled by integer weights forms a modular subcategory, and we have the desired factorization.
We can now prove the following:

**Theorem 4.5.** Let $\ell = 2k + 2n + 1$ and $q = e^{\pi i/\ell}$. Then there are braid-reversing equivalences between

1. $\mathcal{C}(\mathfrak{sp}_{2n}, q, \ell)_0$ and $\mathcal{C}(\mathfrak{so}_{2k+1}, q^{\ell+1}, \ell)_0$
2. $\mathcal{C}(\mathfrak{sp}_{2n}, q^2, \ell)_0$ and $\mathcal{C}(\mathfrak{so}_{2k+1}, q, \ell)_0$.

**Proof.** We again employ Theorem 4.3. The Grothendieck rings of $\mathcal{C}(\mathfrak{so}_{2k+1}, \ell)$ and $\mathcal{C}(\mathfrak{sp}_{2n}, \ell)$ were already established to be isomorphic in [26], denote that isomorphism by $\Phi$. To avoid confusion we will denote simple objects in $\mathcal{C}(\mathfrak{so}_{2k+1}, Q, \ell)$ by $Y$ and those in $\mathcal{C}(\mathfrak{sp}_{2n}, P, \ell)$ by $X_\mu$. Under the isomorphism $\Phi$ we have that $\Phi(Y^{(1)}) = X^{(1)}_1 \otimes X^{(1)}_\eta = X^{(1)}_{(\ell - 2n - 1, 0, ..., 0)} = X^{(1)}'$, and these objects generate the respective modular subcategories $\mathcal{C}(\mathfrak{so}_{2k+1}, Q, \ell)_0$ and $\mathcal{C}(\mathfrak{sp}_{2n}, P, \ell)_0$ as in Lemma 4.4. Notice that it is enough to identify the eigenvalues of the braidings on $Y^{(1)}_0$ and $(X^{(1)}')^{(2)}$: we may lift this identification to $\mathcal{C}(\mathfrak{so}_{2k+1}, Q, \ell)$ and $\mathcal{C}(\mathfrak{sp}_{2n}, P, \ell)$ by tensoring with pointed categories $\mathcal{C}(\mathbb{Z}/2, P_i)$ and then apply Theorem 4.3.

The eigenvalues of $c_{Y^{(1)}, Y^{(1)}}$ on $[1, Y^{(2)}, Y^{(1, 1)}]$ for $\mathcal{C}(\mathfrak{so}_{2k+1}, Q, \ell)$ are found in [19, Table 6.1], the are $[Q^{-4k}, Q^2, -Q^{-2}]$. We have $\Phi(Y^{(2)}) = X^{(1, 1)}_1$ and vice versa, so transpose followed by braid-reversing gives us the eigenvalues $[Q^{4k}, -Q^2, Q^{-2}]$.

For $\mathcal{C}(\mathfrak{sp}_{2n}, P, \ell)$ the eigenvalues of $e_{X^{(1)}, X^{(1)}}$ on $[1, X^{(2)}, X^{(1, 1)}]$ are $[-P^{-2n-1}, P, -P^{-1}]$ [19, Table 6.1]. If $P^\ell = -1$ then $\eta$ is a boson so the braiding eigenvalues on $X'$ and $X^{(1)}$ are identical. Thus if $P = q = e^{\pi i/\ell}$ we must take $Q = q^{k\ell+1}$ so that $Q^{4k} = q^{2k(\ell+1)} = q^{2k} = -q^{2n-1}$, and $-Q^2 = -(q^{4k+1}) = q$ and $Q^{-2} = -q^{-1}$ so that the eigenvalues match. If $P^\ell = 1$ then $\eta$ is a fermion so that the braiding eigenvalues on $X'$ differ from those on $X^{(1)}$ by an overall sign, giving: $[P^{-2n-1}, -P, -P^{-1}]$. Thus if we take $P = q^2 = e^{2\pi i/\ell}$ we choose $Q = q$ so that $Q^{4k} = P^{2k} = P^{-2n-1}$, $-Q^2 = -P$ and $Q^{-2} = P^{-1}$.

We close this section with some remarks on the advantages of this categorical approach.

**Remark 4.6.**

1. The non-unitary categories $\mathcal{C}(\mathfrak{sp}_{2n}, 2n + 2k + 1)$ and $\mathcal{C}(\mathfrak{so}_{2k+1}, 2n + 2k + 1)$ cannot be constructed from affine Lie algebras, to our knowledge. Twisted affine Lie algebras at fractional levels (see e.g. [17]) provide similar combinatorics, but there is no level-preserving fusion product.

2. The results of [27] provide a description of the categories $\mathcal{C}(\mathfrak{sp}_{2n}, \ell)$ via generators and relations. One deduces that the functor between $\mathcal{C}(\mathfrak{sp}_{2k}, 2k + 2n + 2)$ and $\mathcal{C}(\mathfrak{sp}_{2n}, 2k + 2n + 2)$ sending $X$ to $\tilde{X}$ has a unique braided tensor structure up...
to isomorphism of tensor functors, see [7]. Similar uniqueness holds for functors from Theorem [4,5] provided that the functor sends \( X \) to \( X' \).

5. Appendix: Kac-Peterson formula in the symplectic case

5.1. The main goal of this Section is to give a proof of Proposition 2.3 based on [16, Proposition 1].

Let us recall the setup. Let \( g \) be a simple finite dimensional Lie algebra and let \( \theta \) be an automorphism of \( g \) of order 2. Let \( t = \{ x \in g | \theta(x) = x \} \) be the subspace of \( \theta \)-invariant vectors and let \( p = \{ x \in g | \theta(x) = -x \} \). The following result is standard:

**Lemma 5.1** (Cartan decomposition). We have a decomposition \( g = t \oplus p \) satisfying the following

(i) \( t \) is a reductive Lie subalgebra of \( g \) and \( [t, p] \subset p \).

(ii) The restricted Killing form \( \kappa|_{p \times p} \) is a non-degenerate symmetric bilinear form preserved by the action of \( t \).

(iii) The action of \( t \) on \( p \) gives an embedding \( t \hookrightarrow so(p, \kappa|_{p \times p}) \cong so_{\text{dim}p} \).

We will be interested in the following special case.

**Example 5.2.** Let \( \mathbb{C}^{2n+2k} = \mathbb{C}^{2n} \oplus \mathbb{C}^{2k} \) be a direct sum of two symplectic spaces. Let \( \theta \in GL(\mathbb{C}^{2n+2k}) \) be the linear operator acting by -1 on \( \mathbb{C}^{2n} \) and by 1 on \( \mathbb{C}^{2k} \). It is clear that \( \theta \) preserves the symplectic form, so it acts by conjugations on \( g = sp(\mathbb{C}^{2n+2k}) = sp_{2n+2k} \).

Then \( t \cong sp_{2n} \oplus sp_{2k} \) and \( p \) must have dimension \( 4nk \) so we get an embedding \( sp_{2n} \oplus sp_{2k} \subset so_{4nk} \) from Section 2.3.

This example enjoys the following extra property: there exists a Cartan subalgebra \( h \) of \( g \) which is contained in \( t \). We choose and fix such a subalgebra. Let \( \Delta \) and \( \Delta_t \) denote the root systems of \( g \) and \( t \) with respect to \( h \). Let \( W \) and \( W_t \) be the corresponding Weyl groups. We choose a set \( \Delta_t^+ \) of positive roots for \( \Delta_t \); then \( \Delta_t^+ = \Delta_t \cap \Delta \) is a set of positive roots for \( \Delta_t \). Let \( \rho \) and \( \rho_t \) be the sums of fundamental weights. Finally let \( \hat{W}, \hat{W}_t, \hat{\Delta}, \hat{\Delta}_t \) etc denote the affine versions of the notions above.

The set

\[
W_t^1 = \{ w \in W | \Delta_t^+ \subset w\Delta^+ \} = \{ w \in W | w^{-1}\Delta_t^+ \subset \Delta^+ \}
\]

and its affine counterpart

\[
\hat{W}_t^1 = \{ w \in \hat{W} | \hat{\Delta}_t^+ \subset w\hat{\Delta}^+ \} = \{ w \in \hat{W} | w^{-1}\hat{\Delta}_t^+ \subset \hat{\Delta}^+ \}
\]

will play a significant role in what follows in view of the following result:
Theorem 5.3. Let $g$, $t$ and $h$ be as above.

(i) (Lemma 2.2 of [24]) Let $S$ be the spinor representation of $so_{dim\mathfrak{p}}$ restricted to $t$ and let $L(\mu)$ be the irreducible $t$-module of highest weight $\mu$. Then there is an isomorphism of $t$-modules

$$S \cong \bigoplus_{w \in W^1} L(w(\rho) - \rho_\lambda).$$

(ii) (Proposition 1 of [16]) The affinization of the embedding $t \subset so_{dim\mathfrak{p}}$ is a conformal embedding $\hat{t} \subset (\hat{so}_{dim\mathfrak{p}}).$ Let $\hat{S}$ be the spinor representation of $\hat{so}_{dim\mathfrak{p}}$ and let $\hat{L}(\mu)$ be the irreducible $\hat{t}$-module of highest weight $\mu$. Then there is an isomorphism of $\hat{t}$-modules

$$\hat{S} \cong \bigoplus_{w \in \hat{W}^1} \hat{L}(w(\hat{\rho}) - \hat{\rho}_\lambda).$$

Remark 5.4. (i) The levels of affine Lie algebras appearing in Theorem 5.3 (ii) can be computed as follows: Let $t_1$ be a direct summand of the Lie algebra $t$. Then the level of $\hat{t}_1$ is the ratio of the Killing form of $g$ restricted to $t_1$ and the Killing form of $t_1$. In particular in the setup of Example 5.2 we get the conformal embedding (1).

(ii) Note that the dimension of space $p$ is even (this is twice the cardinality of $\Delta^+ \setminus \Delta^+_t$). Thus both $S$ and $\hat{S}$ are the sums of two half-spinor modules. In particular in the setup of Example 5.2 we have $\hat{S} = \hat{\Lambda}_+ \oplus \hat{\Lambda}_-.$

5.2. The symplectic case. We will make Theorem 5.3 explicit in the setup of Example 5.2. Let

$$\{e_1 - e_2, e_2 - e_3, \ldots, e_{n+k-1} - e_{n+k}, 2e_{n+k}\}$$

be the simple roots of $sp_{2n+2k}$. Then the simple roots of $sp_{2n}$ are

$$\{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, 2e_n\}$$

and the simple roots of $sp_{2k}$ are

$$\{e_{n+1} - e_{n+2}, e_{n+2} - e_{n+3}, \ldots, e_{n+k-1} - e_{n+k}, 2e_{n+k}\}.$$
\[
t^{-1}w^{-1}(w(\delta - 2e_i)) = t^{-1}(\delta) - t^{-1}(2e_i) = \delta - 2e_i + 2m_i\delta.
\]

Since \(w(\delta - 2e_i) = \delta - w(2e_i) \in \Delta^+_t\) and \(w' \in \hat{W}^t_1\), it follows \(m_i \geq 0\), see Section 2.2.3.

Similarly \(w(\delta + 2e_i) \in \Delta^+_t\) and \((w')^{-1}w(\delta + 2e_i) = \delta + 2e_i - 2m_i\delta\). Thus \(w' \in \hat{W}^t_1\) implies \(m_i \leq 0\). Hence \(m_i = 0\). \qed

**Remark 5.6.** In view of Theorem 5.3 the Proposition 5.5 says that the branching rules for the finite dimensional and affine cases are “the same”, cf Remark 3.4 (ii).

Let us describe \(W^t_1\). Recall that the group \(W\) is the group of signed permutations; let \(S_{n+k} \subset W\) be the subgroup of permutations without signs. Recall that \(\Delta^\text{long} \subset \Delta^+_t\).

Thus we have

\[
W^t_1 = \{w \in W \mid w^{-1}\Delta^\text{long} = \Delta^\text{long}, w^{-1}(e_i - e_{i+1}) \in \Delta^+_t, \text{for all } i \neq n\}.
\]

Observe that the the first condition implies that \(w^{-1} \in S_{n+k}\). Thus we have

**Lemma 5.7.** The set \(W^t_1\) is contained in \(S_{n+k} \subset W\). A permutation \(s \in S_{n+k}\) is in \(W^t_1\) if and only if

\[
s^{-1}(1) < s^{-1}(2) < \cdots < s^{-1}(n) < s^{-1}(n+1) < s^{-1}(n+2) < \cdots < s^{-1}(n+k).
\]

Now if we consider \(n + k\) dots on a straight line and paint the dots numbered \(s^{-1}(1), s^{-1}(2), \ldots, s^{-1}(n)\) in black and the dots numbered \(s^{-1}(n+1), s^{-1}(n+2), \ldots, s^{-1}(n+k)\) in white we get precisely “black and white dots diagram” as in Example 2.2. Conversely from such a diagram we get a unique permutation as in Lemma 5.7. Thus we constructed a bijection between \(W^t_1\) and the set \(C_{n,k} = I_{n,k}\) from Section 2.1.

**Example 5.8.** The permutation \(s^{-1}\) corresponding to the diagram

\[
\begin{array}{cccccccccccc}
& & & & & & \bullet & \bullet & \circ & \bullet & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13
\end{array}
\]

from example 2.2 is

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
1 & 2 & 4 & 5 & 6 & 7 & 11 & 3 & 8 & 9 & 10 & 12 & 13
\end{pmatrix}.
\]

It remains to compute the weights \(w(\rho) - \rho_t\) for \(w \in W^t_1\) (we can restrict ourselves to the finite case in view of Proposition 3.3). We have

\[
\rho = (n + k)e_1 + (n + k - 1)e_2 + \ldots + 2e_{n+k-1} + e_{n+k} = \sum_{i=1}^{n+k} (n + k + 1 - i)e_i,
\]

and

\[
\rho_t = ne_1 + (n - 1)e_2 + \ldots + e_n + ke_{n+1} + (k - 1)e_{n+2} \ldots + e_{n+k}.
\]
For a permutation \( s \in S_{n+k} \) we have
\[
s \rho = \sum_{i=1}^{n+k} (n + k + 1 - i) e_{s(i)} = \sum_{i=1}^{n+k} (n + k + 1 - s^{-1}(i)) e_i.
\]

Thus we have
\[
s \rho - \rho_t = \sum_{i=1}^{n} (k + i - s^{-1}(i)) e_i + \sum_{i=1}^{k} (n + i - s^{-1}(i)) e_{n+i}.
\]

Here the first summand represents the weight of \( \mathfrak{sp}_{2n} \subset \mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2k} = \mathfrak{t} \) and the second summand represents the weight of \( \mathfrak{sp}_{2k} \subset \mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2k} = \mathfrak{t} \). It is clear that the second weight is computed similarly to the first one with replacement of the sequence \( s^{-1}(1) < s^{-1}(2) < \cdots < s^{-1}(n) \) by the sequence \( s^{-1}(n+1) < s^{-1}(n+2) < \cdots < s^{-1}(n+k) \)
or, equivalently, by replacing all the black dots by the white ones and vice versa. This is precisely the description of the bijection \( \lambda \mapsto \lambda^c \) in the language of diagrams. Thus Proposition 2.3 is proved.

References

[1] B. Bakalov, A. Kirillov Jr, Lectures on Tensor Categories and Modular Functors, University Lecture Series. American Mathematical Society volume 21, Providence, RI, (2001).
[2] T. Creutzig, S. Kanade, R. McRae, Tensor categories for vertex operator superalgebra extensions, arXiv:1705.05017.
[3] T. Creutzig, S. Kanade, R. McRae, Glueing vertex algebras, arXiv:1906.00119.
[4] A. Davydov, M. Müger, D. Nikshych, V. Ostrik, The Witt group of non-degenerate braided fusion categories, J. Reine Angew. Math. 677 (2013), 135–177.
[5] A. Davydov, D. Nikshych and V. Ostrik, On the structure of the Witt group of braided fusion categories, Selecta Math. 19(1), 237–269 (2013)
[6] V. Drinfeld, S. Gelaki, D. Nykshych, V. Ostrik, On braided fusion categories I, Selecta Math 16 (2010), 1-119.
[7] C. Edie-Michell, Auto-equivalences of the modular tensor categories of type A, B, C and G, arXiv:2002.03220.
[8] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, Tensor categories, Mathematical surveys and monographs, AMS 205 (2015).
[9] M. Finkelberg, An equivalence of fusion categories. Geom. Funct. Anal. 6 (1996), no. 2, 249-267.
[10] J. Fuchs, The connections between Wess-Zumino-Witten models and free field theories, Nucl. Phys. B (Proc. Suppl.) 6 (1989) 157-159.
[11] J.E. Hasegawa, Spin Module Versions of Weyl’s Reciprocity Theorem for Classical Kac-Moody Lie Algebras, Publ. RIMS Kyoto Univ. 25(1989) 741-828.
[12] Y.-Z. Huang, Vertex operator algebras, the Verlinde conjecture and modular tensor categories, Proc. Natl. Acad. Sci. USA 102 (2005) no. 15, 5352–5356.
[13] Y.-Z. Huang, A. Kirillov, J. Lepowsky, Braided tensor categories and extensions of vertex operator algebras, Comm. Math. Phys. 337 (2015), no. 3, 1143–1159.
[14] J. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, Springer 1972.

[15] C. Jiang, C. H. Lam, *Level-rank duality for vertex operator algebras of types B and D*, Bull. Inst. Math. Acad. Sin. (N.S.) 14 (2019), no. 1, 31–54.

[16] V.G. Kac and D.H. Peterson, *Spin and Wedge Representations of Infinite-Dimensional Lie Algebras and Groups*, Proc. Natl. Acad. Sci. USA 78 No. 6 (1981) 3308–3312.

[17] V.G. Kac, *Infinite dimensional Lie algebras*, Cambridge University Press, 3rd-Edition, 1990.

[18] A. Kirillov and V. Ostrik, *On a q-analogue of the McKay correspondence and the ADE classification of $\hat{\mathfrak{sl}}_n$, conformal field theories*, Advances in Mathematics 171 (2002), no. 2, 183-227.

[19] M. J. Larsen, E. C. Rowell, Z. Wang, *The N-eigenvalue problem and two applications* Int. Math. Res. Not. 2005, no. 64, 3987–4018.

[20] I. Macdonald, *Symmetric Functions and Hall Polynomials*, Second Edition, Oxford Science Publications, Oxford 1995.

[21] M. M{"u}ger, *On the structure of modular categories*, Proc. London Math. Soc. 87(2003), no.2, pp 291–308.

[22] S. Mukhopadhyay, *Rank-level duality of conformal blocks for odd orthogonal Lie algebras in genus 0*, Trans. Amer. Math. Soc. 368 (2016), no. 9, 6741–6778.

[23] V. Ostrik and M. Sun, *Level rank duality via tensor categories*, Comm. Math. Phys. 326 (2014), no. 1, 49–61.

[24] R. Parthasarathy, *Dirac operator and the discrete series*, Annals of Math. 96 (1972), no. 1, 1–30.

[25] N. Y. Reshetikhin, *Quantized universal enveloping algebras, the Yang-Baxter equation, and invariants of links*, I, LOMI Preprint E-4-87, 1987.

[26] E. C. Rowell, *On a family on non-unitarizable ribbon categories*, Math. Z. 250 no. 4 (2005), 745-774.

[27] I. Tuba, H. Wenzl, *On braided tensor categories of type BCD*, J. Reine Angew. Math. 581 (2005), pp. 31–69.

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