Hypersurfaces and generalized deformations

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Abstract

The moduli space of generalized deformations of a Calabi-Yau hypersurface is computed in terms of the Jacobian ring of the defining polynomial. The fibers of the tangent bundle to this moduli space carry algebra structures, which are identified using subalgebras of a deformed Jacobian ring.

1 Introduction

1.1 Background

In his landmark paper [Wit92] “Mirror Manifolds and Topological Field Theories,” Witten discussed his $B(X)$ model topological field theory (TFT) associated to a Calabi-Yau manifold $X$ and introduced the problem of deforming $B(X)$ as a TFT. Denote by $\mathcal{M}(X)$, or just $\mathcal{M}$, the moduli space of deformations of $B(X)$. Witten concluded that the deformations of $B(X)$ are parametrized by $H^\bullet(\Theta^*_X)$. Not surprisingly, the deformations of the complex structure of $X$—these correspond to elements of $H^1(\Theta_X) \subseteq H^\bullet(\Theta^*_X)$—are among the deformations of $B(X)$. For this reason, elements of $\mathcal{M}$, are called generalized, or extended, deformations of $X$.

It is natural to look for a mathematical deformation problem (inspired by, but without using, physical field theory) that gives rise to $\mathcal{M}$ as a moduli space. One is led into formal deformation theory [Sch68] and its Lie algebraic expression [SS79, SS85, GM88]. Following Witten, Ran [Ran96] wrote about an algebraic deformation problem associated to a Calabi-Yau manifold $X$ whose first order deformations are in one to one correspondence with $H^\bullet(\Theta_X)$.

Independently, Boris Dubrovin [Dub96] studied deformations of abstract topological field theories and discovered that their moduli spaces always carry additional structures, which he axiomatically treated and named $\textit{Frobenius}$
Barannikov and Kontsevich [BK98] considered a deformation problem, different from Ran’s, with moduli space $H^\bullet(\Theta^\bullet_X)$ and proved that this space is a Frobenius manifold. The Barannikov and Kontsevich theory is connected via the formality theorem [Kon97] to the moduli of $A_\infty$ deformations of the category of coherent sheaves on a complex manifold $X$ introduced in [Kon95].

1.2 Contents

In the case that $X$ is a Calabi-Yau projective hypersurface, one expects that the extended deformation theory, the moduli space $\mathcal{M}$, and some of the additional structures on $\mathcal{M}$ will have algebraic descriptions in terms of the defining polynomial $f$. We give these algebraic descriptions in this paper.

In section 2, we review the Barannikov-Kontsevich construction of $\mathcal{M}(X)$ for an arbitrary Calabi-Yau $X$ and recall some of the structures implied by $\mathcal{M}$’s status as a Frobenius manifold. In particular, each fiber of $T\mathcal{M}$ has an associative product.

In section 3, we discuss the case in which $X$ is a hypersurface with defining polynomial $f$. The main results of the paper are stated in this section. We describe $\mathcal{M}(X)$ in terms of a subvector space of the Jacobian algebra of $f$. Then, a (primitive) element $a \in \mathcal{M}$ can be identified with a polynomial $g$ and the algebra structure on $T_a\mathcal{M}$ is identified with a subalgebra of the Jacobian ring of $f + g$. There are similarities here to the deformation theory of a function and to the complex deformations of a projective hypersurface. However, we know of no other place where the specific algebras described here have appeared.

The remaining sections are devoted to constructing a deformation theory, equivalent to the one introduced in [BK98], that yields the polynomial descriptions presented in section 3.

The work in [BK98] relies on a Dolbeault resolution of $H^\bullet(\Theta^\bullet_X)$. Section 4 lays the foundation for using Čech cochains instead. Even though section 4 is used in this paper only as a means to verify the presentation in section 3, we consider the successful replacement of the Dolbeault complex by the Čech complex to be a main result of the paper.

Section 5 describes an algebraic model for $\Theta_X$ in the case that $X$ is a hypersurface. We describe a sheaf of differential graded Lie algebras $(\mathcal{L}, df, [\cdot, \cdot])$ that is quasi-isomorphic to the sheaf of holomorphic vector fields on $X$. The achievement of this model is that it describes $\Theta_X$ as the cohomology of a simple complex where all the information about $X$ is carried in the differential. This makes the model well suited to being deformed.
In section 6, we use \( L \) to form \((C, D)\)—the complex of Čech cochains with values in powers of \( L \). This complex is our analogue of the Dolbeault resolution \( H^\ast(\Theta_X^\ast) \) used in [BK98]. We then give the complex \((C, D)\) the structure of an \( L_\infty \) algebra. The theorems stated in section 3 follow from studying the deformation theory controlled by this \( L_\infty \) algebra.

Concluding remarks are contained in section 7.

2 A picture of \( \mathcal{M} \)

2.1 \( \mathcal{M} \) as a moduli space of generalized deformations

Let \( X \) be a Calabi-Yau manifold, let \( \Theta_x \) denote the holomorphic tangent sheaf of \( X \), and let \( \Theta_X^\ast \) denote \( \oplus_{p=0}^{\dim(X)} \Lambda^p \Theta_X \). Consider the Dolbeault resolution \((g, \bar{\partial})\) of \( H^\ast(\Theta_X^\ast) \):

\[
g = \bigoplus_k g^k, \quad g^k = \bigoplus_{k=p+q-1} \Gamma(\Theta_X^p \otimes \Omega_X^q).
\]

Throughout this paper, \( g \) will denote the graded vector space defined above. Note that \( H^\ast(\Theta_X^\ast) = H(g, \bar{\partial}) \), as the cohomology of a graded vector space, inherits a grading. With this grading, a homogeneous element \( \alpha \in H^q(\Theta_X^p) \) has degree \( p + q - 1 \). We will not view \( H^\ast(\Theta_X^\ast) \) as a bigraded object.

There is a Schouten bracket \([\, , \,]\) on \( g \) extended from the ordinary bracket of vector fields which makes \((g, \bar{\partial}, [\, , \,])\) into a differential graded (dg) Lie algebra.

It is a general fact that for any dg Lie algebra \((\mathfrak{h}, d, [\, , \,])\), there is a deformation theory controlled by \((\mathfrak{h}, d, [\, , \,])\), represented by a moduli space. Informally, the differential \( d \) is deformed and the moduli space consists of (equivalence classes of) elements \( \alpha \) making

\[
d_{\alpha} \overset{\text{def}}{=} d + \text{ad}(\alpha)
\]

into a differential. The deformed differential \( d_{\alpha} \) may contain parameters (from the maximal ideal \( m \) of an Artin local ring: \( \alpha \in \mathfrak{h} \otimes m \)). The map \( d_{\alpha} \) is always a derivation and the condition that \( d_{\alpha}^2 = 0 \) translates into the condition on \( \alpha \):

\[
d\alpha + \frac{1}{2}[\alpha, \alpha] = 0,
\]

which is sometimes called the Mauer-Cartan equation, or the master equation.

More formally, to any Artin local ring \( A \) with maximal ideal \( m \), one defines a functor \( Def_{\mathfrak{h}} \) from Artin local algebras with residue field \( k \) to sets by

\[
Def_{\mathfrak{h}}(A) = \{ \alpha \in (\mathfrak{h} \otimes m)^1 : d\alpha + \frac{1}{2}[\alpha, \alpha] = 0 \} / \sim
\]
where $\sim$ is the equivalence generated infinitesimally by $(\mathfrak{h} \otimes m)^0$, which acts on $\alpha \in (\mathfrak{h} \otimes m)^1$ by $\beta \cdot \alpha = [\beta, \alpha] - d\beta$. One may extend the functor $\text{Def}_{\mathfrak{h}}$ to one acting on graded local Artin rings, which we denote by $\text{Def}_{\mathfrak{h}}^\mathbb{Z}$. In [Sch68], the problem of representing a functor of Artin rings is discussed. The functor $\text{Def}_{\mathfrak{h}}$ (or $\text{Def}_{\mathfrak{h}}^\mathbb{Z}$), is said to be represented by $\mathfrak{O}$, a projective limit of local Artin rings, if $\text{Def}_{\mathfrak{h}}(A) = \text{Hom}(\mathfrak{O}, A)$ (or $\text{Def}_{\mathfrak{h}}^\mathbb{Z}(A) = \text{Hom}(\mathfrak{O}, A)$). In the case that $\mathfrak{O} = \mathcal{O}_N$ is the ring of local functions of a pointed space $N$, then $N$ is called the moduli space for $\text{Def}_{\mathfrak{h}}$ (or $\text{Def}_{\mathfrak{h}}^\mathbb{Z}$).

We refer the reader to [SS79, SS85, GM88, BK98, Bar99, Kon97] for more details, or to [Kon97] or section 4.1.4 of this paper, where a more general deformation theory governed by an $L_\infty$ algebra is discussed.

**Definition 1.** Define $\mathcal{M}$ to be the moduli space for $\text{Def}_{\mathfrak{h}}^\mathbb{Z}$.

### 2.2 Algebraic structures on $\mathcal{M}$

We now review two theorems which appear in [BK98].

The set $\text{Def}_{\mathfrak{h}}^\mathbb{Z}(k[t]/t^2)$ is the set of infinitesimal deformations. It has a natural vector space structure which is isomorphic to $H(\mathfrak{h})$. Provided the moduli space $N$ exists, the infinitesimal deformations are naturally isomorphic to $\text{hom}(\mathcal{O}_N, k[t]/t^2)$ which can be identified with $T_0 N$. However, the moduli space may not exist as a smooth manifold and infinitesimal deformations may be obstructed. But the dg Lie algebra $\mathfrak{g}$ is special:

**Theorem 2.1.** $\mathcal{M}$ can be identified with a formal neighborhood of zero in the graded vector space $H(\mathfrak{g}, \bar{\partial})$.

Now there is some additional algebraic structure on $\mathcal{M}$ which we describe. The associative $\wedge$ product on $\Gamma(\Omega^*_X)$ brings an additional multiplication making $(\mathfrak{g}, \bar{\partial}, [\ , \ ]\ , \wedge)$ a differential Gerstenhaber algebra. This means that $(\mathfrak{g}, \bar{\partial}, [\ , \ ]\ , \wedge)$ is a differential graded Lie algebra, that $(\mathfrak{g}, \bar{\partial}, \wedge)$ is a differential graded commutative associative algebra (after a degree shift), and that these structures are compatible via the odd Poisson identity:

$$[\alpha, \beta \wedge \gamma] = [\alpha, \beta] \wedge \gamma + (-1)^{(i+1)j} \beta \wedge [\alpha, \gamma] \quad \text{for } \alpha \in \mathfrak{g}^i, \beta \in \mathfrak{g}^j, \gamma \in \mathfrak{g}.$$ 

Let $a \in H(\mathfrak{g}, \bar{\partial})$. Viewed as a deformation of $\bar{\partial}$, there corresponds a differential $\bar{\partial}_a$ with $[\alpha] = a$. The compatibilities among $[\ , \ ]\ , \bar{\partial}$, and $\wedge$ ensure that the shifted cohomology $H(\mathfrak{g}, \bar{\partial}_a)$ inherits an associative product from $(\mathfrak{g}, \wedge)$.

**Theorem 2.2.** For all $\alpha \in \mathcal{M}$, $H(\mathfrak{g}, \bar{\partial}_a) \simeq H(\mathfrak{g}, \bar{\partial})$ as vector spaces.
Note that unless the degree of $\alpha$ is $+1$, the differential $\bar{\partial}_\alpha$ will not be a homogeneous derivation and hence $H(\mathfrak{g}, \bar{\partial}_\alpha)$ will not be graded. The isomorphism in theorem 2.2 is only as ungraded vector spaces.

Since, as vector spaces, $H(\mathfrak{g}, \bar{\partial}_\alpha) \simeq H(\mathfrak{g}, \bar{\partial})$ and $H(\mathfrak{g}, \bar{\partial}_\alpha)$ is an associative algebra, theorems 2.1 and 2.2 may be assembled to view $\mathcal{M}$ as a (formal neighborhood of zero in $a$) vector space with an associative product on each fiber of the tangent bundle $T\mathcal{M}$: simply identify $T_a\mathcal{M}$ with $H(\mathfrak{g}, \bar{\partial}_\alpha)$.

### 2.3 $\mathfrak{g}$ as a dGBV algebra

An effective setting in which to explain these two theorems is to regard $\mathfrak{g}$ as a differential Gerstenhaber-Batalin-Vilkovisky (dGBV) algebra. See the remark at the end of section 7 in [BK98] and the elaboration in [Man99b].

The BV suffix is added because there is another degree one differential $\Delta : \mathfrak{g} \to \mathfrak{g}$ that commutes with $\bar{\partial}$, is a derivation of the bracket, and, together with $\wedge$, “generates” the bracket:

$$(-1)^i \Delta(\alpha \wedge \beta) - (-1)^i \Delta(\alpha) \wedge \beta - \alpha \wedge \Delta(\beta) = [\alpha, \beta] \text{ for } \alpha \in \mathfrak{g}^i, \beta \in \mathfrak{g}.$$  \hspace{2cm} (1)

This behavior of $\Delta$ together with the fact that $(\mathfrak{g}, \bar{\partial}, [, ], \wedge)$ is a Gerstenhaber algebra, makes the quintuple $(\mathfrak{g}, \bar{\partial}, \Delta, [, ], \wedge)$ what is called a dGBV algebra (there is a nice survey of dGBV algebras in [Sch98]). Furthermore, there is an additional homological property relating the two differentials that make the particular dGBV algebra $(\mathfrak{g}, \bar{\partial}, \Delta, [, ], \wedge)$ extraordinary:

**$\Delta - \bar{\partial}$ Lemma.** $\text{Im}(\bar{\partial}) \cap \text{Ker}(\Delta) = \text{Im}(\Delta) \cap \text{Ker}(\bar{\partial}) = \text{Im}(\bar{\partial} \Delta)$.

**Corollary 2.3.** The dg Lie maps,

$$(\text{Ker}(\Delta), \bar{\partial}, [, ]) \to (\mathfrak{g}, \bar{\partial}, [, ]) \text{ and } (\text{Ker}(\Delta), \bar{\partial}, [, ]) \to (H(\mathfrak{g}, \Delta), 0, 0)$$

induce isomorphisms in cohomology.

Any map of dg Lie algebras inducing an isomorphism in cohomology provides an isomorphism of deformation functors [SS79] [SS85] [GM88] [Kon97] [BK98] [Man99a]. Corollary 2.3 then, implies that the moduli space associated to $(\mathfrak{g}, [, ], \bar{\partial})$ is equal to the moduli space associated to $(H(\mathfrak{g}, \Delta), 0, 0)$. Because both the differential and the bracket in $H(\mathfrak{g}, \Delta)$ vanish, the moduli space associated to $(H(\mathfrak{g}, \Delta), 0, 0)$ can be simply identified with a formal neighborhood of zero in the graded vector space $H(\mathfrak{g}, \Delta)$. So, $\mathcal{M}$, defined as the moduli space associated to $(\mathfrak{g}, [, ], \bar{\partial})$, can be identified with $H(\mathfrak{g}, \Delta)$. The $\bar{\partial} - \Delta$ Lemma

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reveals that $H(g, \Delta) \cong H(g, \bar{\partial})$ by establishing natural isomorphisms between both cohomologies and the quotient

$$(\text{Ker}(\Delta) \cap \text{Ker}(\bar{\partial})) / (\text{Image} \, \Delta \bar{\partial}).$$

Given $a \in M$, there corresponds an $[\alpha] \in H(g, \Delta)$ with $\Delta(\alpha) = \bar{\partial}(\alpha) = 0$. The identification of $a \in M$ with $[\alpha] \in H(g, \bar{\partial})$ gives Theorem 2.1.

It can be checked that if $\Delta(\alpha) = 0$ then $\bar{\partial} - \Delta$ Lemma $\Rightarrow \bar{\partial}_\alpha - \Delta$ Lemma. So, Theorem 2.2 follows from having established that each element in $a \in M$ can be represented by an element $[\alpha] \in H(g, \bar{\partial})$ with $\alpha \in \text{Ker}(\Delta)$. We have quasi-isomorphisms

$$(H(g, \Delta), 0, 0) \xrightarrow{\bar{\partial} - \Delta \text{ lemma}} (g, \bar{\partial}_\alpha, [\ , \ ]).$$

Therefore, as (ungraded) vector spaces $H(g, \bar{\partial}_\alpha) \cong H(g, \bar{\partial})$, since they are both isomorphic, as vector spaces, to $H(g, \Delta)$. Note that this diagram does not imply a stronger isomorphism between $H(g, \bar{\partial}_\alpha)$ and $H(g, \bar{\partial})$; $\Delta$ is not a derivation of $\wedge$ and hence $H(g, \Delta)$ is not an associative algebra—the arrows are dgLie maps.

3 Hypersurfaces

Let $S = \mathbb{C}[x_0, \ldots, x^n]$. If $g \in S$ is a homogeneous polynomial, denote its polynomial degree by $\text{wt}(g) = k$ and call it the weight of $g$. So, $S = \bigoplus_k S^{[k]}$, where $S^{[k]} = \{g \in S : \text{wt}(g) = k\}$. Suppose $f \in S^{[\nu]}$ is nonsingular. Denote $\frac{\partial f}{\partial x_i}$ by $f_i$ and let $J_f$ be the ideal of $S$ generated by $\{f_i\}_{i=0}^n$. We have a subalgebra of $S/J_f$ defined by

$$R = \text{Image} \left( \bigoplus_k S^{[k\nu]} \to S/J_f \right).$$

Note that $J_f$ is a homogeneous ideal in $S$ so $R$ (like $S$) is weighted: $R = \bigoplus_k R^k$ where $R^k := \text{Image}(S^{[k\nu]} \to S/J_f)$. Let

$$X = \{x \in \mathbb{P}^n : f(x) = 0\}.$$

From now on, we assume $X$ is a Calabi-Yau manifold. This assumption means that $n + 1 = \nu$ and, as we already assume, $f$ is nonsingular. So, $X$ is a degree $n + 1$ hypersurface of dimension $n - 1$ in $\mathbb{P}^n$.

For $g \in \bigoplus_k S^{[k\nu]}$, define the deformed algebra

$$R_{f+g} = \text{Image} \left( \bigoplus_k S^{[k\nu]} \to S/J_{f+g} \right).$$
where $J_{f+g}$ is the ideal generated by $\left\{ \frac{\partial (f+g)}{\partial x_i} \right\}$. Note that $f + g$ is not, in general, homogeneous and so $R_{f+g}$ is not weighted.

**Definition 2.** Define a graded $\mathbb{C}$ algebra $\tilde{R}$ and an ungraded $\mathbb{C}$ algebra $\tilde{R}_{f+g}$ by

$$\tilde{R} = R \oplus \mathbb{C}e_0 \oplus \cdots \oplus \mathbb{C}e_{n-1}$$

and

$$\tilde{R}_{f+g} = R_{f+g} \oplus \mathbb{C}e_0 \oplus \cdots \oplus \mathbb{C}e_{n-1}$$

with products extended from the products in $R$ and $R_{f+g}$ by

$$e_i \cdot e_j = e_i \cdot [h] = 0 \text{ for all } i, j = 0, \ldots n-1 \text{ and } h \in S^{[k\nu]}, k > 0.$$ 

The grading on $\tilde{R}$ is defined by $\text{deg}([h]) = 2k$ for $h \in S^{[k\nu]}$ and $\text{deg}([e_i]) = n-1$.

We now state the main theorems (proved in sections 4 and 6) in this paper. They follow from analyzing the deformation theory governed by an $L_\infty$ algebra $(\mathcal{C}, Q^\mathcal{C})$ that we construct (cf. definition 4).

**Theorem 3.1.** $(\mathcal{C}, Q^\mathcal{C})$ is quasi-isomorphic to $(\mathfrak{g}, \tilde{\partial}, [, ])$. Hence, $\mathcal{M}(X)$ can be identified with a formal neighborhood of zero in the vector space $\tilde{R}$.

Our $L_\infty$ algebra carries an associative product compatible with $D$, the differential term of $Q^\mathcal{C}$, and the cohomology of $(\mathcal{C}, Q^\mathcal{C})$ is naturally isomorphic to $\tilde{R}$ as an algebra. An analysis of the shifted cohomology rings, yields:

**Theorem 3.2.** For each $[\alpha] \in \mathcal{M}(X)$ there corresponds a differential $D_{[\alpha]}$, a deformation of $D$, and a polynomial $g \in S$ so that as (ungraded) associative algebras $H(\mathcal{C}, D_{[\alpha]}) \simeq \tilde{R}_{f+g}$.

We make a remark about each theorem.

**Remark 1.** Since

$$\oplus_{p,q} H^q(\Theta^n_X) \simeq \oplus_{p,q} H^q(\Omega^{n-1-q}_X) \simeq \oplus_{p,q} H^{p,q}(X) \simeq \oplus_i H^i_{DR}(X),$$

all of the tangent cohomology of $\oplus_{p,q} H^q(\Theta^n_X)$ appears as DeRham cohomology of $X$. Using Hodge theory and residues [Gri69] one can identify the primitive (middle) cohomology $H^{n-1-k,k}(X)$ with $\tilde{R}^k$. This accounts for the “primitive” tangent cohomology contained in $\oplus_k H^k(\Theta^k)$. The rest of the tangent cohomology is inherited from projective space as $\oplus_k H^k(\Theta^{n-k-1}_X)$ and each inherited
piece $H^k(\Theta_X^{n-1-k})$ is one dimensional, adjoined to $R$ as $C_{ek}$. It is not surprising that $\tilde{R} \simeq H^*(\Theta_X^\bullet) \simeq H(\mathfrak{g}, \bar{\partial})$, and the reader may recognize the grading on $\tilde{R}$ as the grading acquired from an identification of $\tilde{R}$ with $H^*(\Theta_X^\bullet)$. However, in this paper, $\tilde{R}$ arises as the cohomology of a graded algebra and as a moduli space—Theorem 3.1 is revealed in a natural setting.

**Remark 2.** Theorem 3.2 is stated in terms of the fairly simple algebras $\tilde{R}_{f+g}$.

As cohomologies of shifted differentials, they can be assembled to give algebra structures on the fibers of $TM$. From the definitions of $\tilde{R}_{f+g}$, however, it is not even apparent that $\tilde{R}_{f+g}$ has the same vector space dimension as $\tilde{R}$. This should be considered a corollary of the theorem 3.2 and its proof.

### 4 Čech versus Dolbeault

As mentioned earlier, the Dolbeault resolution $(\Gamma(\Theta_X^\bullet \otimes \bar{\Omega}_X^\bullet), \bar{\partial})$ of $H^*(\Theta_X^\bullet)$ becomes a differential graded Lie algebra with the addition of the Schouten bracket. Its cousin $(C^*(\Theta_X^\bullet), \delta)$, the complex of Čech cochains with values in $\Theta_X^\bullet$, is also a resolution of $H^*(\Theta_X^\bullet)$ but does not carry a Lie bracket. It does, however, have an $L_\infty$ structure, which, for the purposes of deformation theory, is just as good.

#### 4.1 $L_\infty$ algebras

##### 4.1.1 Two reminders about vector spaces

For a graded vector space $V$, we denote by $S V$ the graded cofree cocommutative coalgebra generated by $V$. That is, $S V$ is the subcoalgebra of the graded tensor coalgebra $\bigoplus_{k \geq 0} V \otimes^k$, with the standard comultiplication, invariant under the action of the symmetric group $\Sigma_k$ on $V \otimes^k$.

For $i \in \mathbb{Z}$ there is the shift functor $[i]$ acting on a graded vector space $V = \bigoplus_k V^k$ defined by $V[i] = \bigoplus_k V[i]^k$ where $V[i]^k = V^{i+k}$. The action of the shift functor $[i]$ can be considered as tensoring with the trivial vector space concentrated in degree $i$.

The exterior and symmetric products are related in this graded environment by $S^i(V[1]) \simeq (\Lambda^i V)[i]$.

##### 4.1.2 $L_\infty$ algebras

**Definition 3.** An $L_\infty$ algebra consists of a pair $(V, Q)$ where $V$ is a graded vector space and $Q$ is a degree one codifferential (square zero coderivation) on $S(V[1])$. 

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Because $S(V[1])$ is cofree, any coderivation $Q$ on $S(V[1])$ is determined by the pieces (projections followed by restrictions) $Q_i : S^i V[1] \to V[2 - i]$. The condition that $Q^2 = 0$ imposes constraints on the linear maps $Q_i$. The constraints involving the first two components of $Q$ are as follows: $Q_1 : V \to V[1]$ must be a differential on $V$; $Q_2 : \Lambda^2 V \to V$ is a skew symmetric bilinear operator on $V$ and $Q_1$ is a derivation of $Q_2$; and $Q_2$ satisfies the Jacobi identity up to a homotopy defined by $Q_3 : \Lambda^3 V \to V[-1]$. Often, $L_\infty$ algebras are called “strong homotopy Lie algebras” or “$sh$ Lie algebras.” The introduction for physicists in [LS93] is a good introduction for mathematicians, too.

Any differential graded Lie algebra can be regarded as an $L_\infty$ algebra. In particular, $g = \Gamma(\Theta_X^* \otimes \Omega_X)$ has an $L_\infty$ structure given by $Q$ with $Q_1 = \bar{\partial}$, $Q_2 = [\cdot, \cdot]$, and $Q_3 = Q_4 = \cdots = 0$. However, even when studying dg Lie algebras, it can be important to view them as $L_\infty$ algebras when studying maps between them.

4.1.3 $L_\infty$ morphisms

Definition 4. An $L_\infty$ morphism $(V, Q) \to (V', Q')$ is a differential coalgebra map $\mu : (S(V[1]), Q) \to (S(V'[1]), Q')$. We say that $\mu$ is a quasi-isomorphism if the first component of $\mu$, which is a map of complexes $\mu_1 : (V, Q_1) \to (V', Q'_1)$, induces an isomorphism $\mu^*_1 : H(V, Q_1) \to H(V', Q'_1)$. We say that two $L_\infty$ algebras $(V, Q)$ and $(V', Q')$ are quasi-isomorphic if they are equivalent under the equivalence relation generated by quasi-isomorphisms.

It is a standard theorem, see for example [SS85, Kon97], that if

$$\mu : (S(V[1]), Q) \to (S(V'[1]), Q')$$

is a quasi-isomorphism, then there exists a quasi-isomorphism

$$\mu' : (S(V'[1]), Q') \to (S(V[1]), Q).$$

The equivalence relation generated by quasi-isomorphisms as stated in the definition of quasi-isomorphic above is quite natural.

Given an $L_\infty$ algebra $(V, Q)$, one can form the differential graded Lie algebra, $Coder(S(V[1]), Q)$, which consists of coderivations of $S(V[1])$, a differential given by $ad(Q)$, and the usual Lie bracket of coderivations. Then, an $L_\infty$ morphism $(S(V[1]), Q) \to (S(V'[1]), Q')$ is equivalent to a dg Lie morphism

$$Coder(S(V[1]), Q) \to Coder(S(V'[1]), Q').$$

This description of $L_\infty$ algebras is similar to the “geometric” picture promoted in [Kon97, Bar99, Mer00] of $Q$ as an odd, square zero vector field on a formal pointed graded manifold.
4.1.4 Deformation theory

Given an $L_\infty$ algebra $(V,Q)$, the functor $\text{Def}(V,Q)$ acting from the category of graded local Artin algebras to the category of sets is defined by

$$\text{Def}(V,Q)(A) = \{\text{coalgebra maps } f : m^* \to SV[1] \text{ with } Q(f(m^*)) = 0\} / \sim$$

where $m^*$ is the dual of the maximal ideal of $A$ and $\sim$ is a gauge equivalence, described geometrically in [Bar99, Kon97]. If $(V,Q)$ and $(V',Q')$ are quasi-isomorphic, then the functors $\text{Def}(V,Q)$ and $\text{Def}(V',Q')$ are canonically equivalent. In situations where $\text{Def}(V,Q)$ is representable; that is $\text{Def}(V,Q) = \text{Hom}(\cdot, \mathcal{O}_N)$ for some algebra $\mathcal{O}_N$, then we call $\mathcal{N}$ the moduli space. The conclusion [SS85, Kon97] needed for this paper is:

**Theorem 4.1.** If $(V,Q)$ and $(V',Q')$ are quasi-isomorphic $L_\infty$ algebras and $\mathcal{M}$ is the moduli space for $\text{Def}(V,Q)$ then $\mathcal{M}$ is the moduli space for $\text{Def}(V',Q')$.

We add an opinion here about Lie versus $L_\infty$. Some traditional deformation theories are concerned with $\text{Def}(V,Q)$ in the case that $(V,Q)$ is a differential graded Lie algebra; that is, $Q_3 = Q_4 = \cdots = 0$. In this case,

$$\text{Def}(V,Q)(A) = \{\gamma \in (V \otimes m)^1 \mid Q_1(\gamma) + \frac{1}{2}Q_2(\gamma, \gamma) = 0\} / \sim .$$

Here, $\text{Def}(V,Q)$ reduces to the functor described in section 2.1. But even when $(V,Q)$ is a dgLie algebra, one cannot effectively study $\text{Def}(V,Q)$ entirely within the more comfortable category of dgLie algebras. It is necessary to involve $L_\infty$ structures because $L_\infty$ morphisms provide transformations of (dg Lie) deformation functors—dg Lie maps are not plentiful enough. It seems inevitable, then, that the dg Lie algebra which controls the deformation theory should be considered an $L_\infty$ object to begin with.

4.2 $L_\infty$ structure on Čech cochains

In general, the vector space of Čech cochains with values in a sheaf of Lie algebras does not form a Lie algebra, but rather an $L_\infty$ algebra, canonical up to $L_\infty$ quasi-isomorphism. The lemma below explains this phenomenon in some generality. We will apply it to the space of Čech cochains with values in a holomorphic vector bundle whose space of sections carries a Lie bracket.

**Lemma 4.2.** Suppose $(V,Q)$ is an $L_\infty$ algebra and $(V',d)$ is a complex. If $(V',d)$ and $(V,Q_1)$ are quasi-isomorphic as complexes, then there exists a degree one differential $Q'$ on $SV'[1]$ with $Q'_1 = d$ making $(V,Q)$ and $(V',Q')$ quasi-isomorphic as $L_\infty$ algebras.
Proof. Here we sketch a simple existence proof. If \((V',d)\) and \((V,Q_1)\) are quasi-isomorphic as complexes, then the dg Lie algebras

\[(\text{Coder}(SV'[1]), \text{ad}(d))\] and \[(\text{Coder}(SV[1]), \text{ad}(Q_1))\]

will be quasi-isomorphic as dgLie algebras (one must use the cofreeness of the symmetric product). Since \(\text{ad}(Q_1)\) extends to an integrable differential \(\text{ad}(Q)\) on \((\text{Coder}(SV'[1]), \text{ad}(Q))\), so must \(\text{ad}(d)\) extend to a differential \(D'_Q = \text{ad}(Q')\) making the dg Lie algebras

\[(\text{Coder}(SV'[1]), \text{ad}(Q'))\] and \[(\text{Coder}(SV[1]), \text{ad}(Q))\]

quasi-isomorphic. \(\square\)

Suppose that \(A\) is a holomorphic vector bundle with \((\Gamma(A), [, ])\) a Lie algebra. The bracket on \(\Gamma(A)\) can be extended using the associative \(\wedge\) product on \(\Gamma(\Omega^\bullet)\) to a Lie bracket on \(\Gamma(A \otimes \Omega^\bullet)\). Since \((A \otimes \bar{\Omega}^\bullet, \bar{\partial})\) is a fine resolution of \(A\), the Čech complex \((C^\bullet(A), \bar{\partial})\) and the Dolbeault complex \((\Gamma(A \otimes \bar{\Omega}^\bullet), \bar{\partial})\) are quasi-isomorphic. Then, Lemma 4.2 can be used to export the Lie algebra structure from the Dolbeault complex to an \(L_\infty\) structure (which will not in general be Lie), to the Čech complex.

Applying this when \(A = \Theta^\bullet_X\), we obtain the following theorem. Recall that \(g\) denotes \(\Gamma(\Theta^\bullet_X \otimes \bar{\Omega}^\bullet_X)\).

**Theorem 4.3.** There exists an \(L_\infty\) structure \(\hat{Q}\) on \(C^\bullet(\Theta_X^\bullet)\), with \(\hat{Q}_1 = \hat{\delta}\), so that

\[(C^\bullet(\Theta_X^\bullet), \hat{Q}) \text{ and } (g, \bar{\partial}, [, ])\]

are quasi-isomorphic.

**4.2.1 The \(\hat{Q}_2\) term**

It is possible to give a constructive proof of Lemma 4.2. That is, \(d\) can be extended to \(Q' = d + Q'_2 + Q'_3 + \cdots\) perturbatively. See, for example, the articles \[\text{BFLS98, Mer00}\] for other explicit, perturbative constructions of \(L_\infty\) structures. Here is an outline for producing \(Q'_2\).

For simplicity, and because this is the case of interest, let \((V, Q = Q_1 + Q_2)\) be a differential graded Lie algebra and suppose that \(\phi : (V', d) \to (V, Q_1)\) is a quasi-isomorphism of complexes. Then, there exists a quasi-isomorphism \(\psi : (V, Q_1) \to (V', d)\) which is a homotopy inverse of \(\phi\). In particular, there exists an \(s : V \to V[-1]\) with

\[\phi \psi - 1 = sQ_1 + Q_1s.\]
Now, one can define a skew-symmetric bilinear map \( Q'_2 : V' \otimes V' \to V' \) by

\[
Q'_2(v_1, v_2) = \psi(Q_2(\phi(v_1), \phi(v_2))).
\]

Because \( \psi \) and \( \phi \) are inverses only up to homotopy, \( Q'_2 \) will not satisfy the Jacobi identity. However \( Q'_2 \) will satisfy Jacobi up to homotopy, which is all that is required for the second term of an \( L_\infty \) structure on \( V' \). The higher homotopies are constructed inductively. Also, note that \( \phi \) is not a homomorphism in the sense that \( \phi(Q'_2(v_1, v_2)) \neq Q_2(\phi(v_1), \phi(v_2)) \); but \( \phi \) does define an \( L_\infty \) morphism, which can be thought of as a homomorphism up to homotopy.

In fact, the collection of maps \( F_k : \Lambda^k V' \to V \) given by

\[
F_1 = \phi \\
F_2 = s \circ Q_2 \circ (\phi \otimes \phi) \\
\vdots
\]

assemble to define an \( L_\infty \) morphism from \( (V', d + Q'_2 + \cdots) \) to \( (V, Q_1 + Q_2) \).

This outline describes \( Q'_2 \) modulo the precise information about the quasi-isomorphisms between \( (V, Q_1) \) and \( (V', d) \). For the example of Čech cochains, \( \hat{Q}_2 \) can be described explicitly once the quasi-isomorphisms between \( (C^\bullet(A), \delta) \) and \( (\Gamma(A \otimes \bar{\Omega}^\bullet), \bar{\partial}) \) are analyzed.

**Remark.** The construction of \( Q'_2 \) above depends on the choice of \( \psi \) and \( s \). However, a change of choice induces a quasi-isomorphism between the respective \( L_\infty \) structures.

### 4.2.2 The quasi-isomorphisms between Čech and Dolbeault

As before, assume that \( A \) is a holomorphic vector bundle and that \((\Gamma(A), [\cdot, \cdot])\) is a Lie algebra. Consider the resolution of \( A \) by the complex of fine sheaves \((A^\bullet, \bar{\partial})\) where \( A^p = A \otimes \bar{\Omega}^p_X \). The inclusions

\[
(C^\bullet(A^\bullet), D = \delta + \bar{\partial}) \\
(C^\bullet(A), \delta) \\
(\Gamma(A^\bullet), \bar{\partial})
\]

are quasi-isomorphisms [GH78]. Each map is invertible, up to homotopy, so we have maps

\[
\phi : (C^\bullet(A), \delta) \to (\Gamma(A^\bullet), \bar{\partial}) \\
\psi : (H^0(A^\bullet), \bar{\partial}) \to (C^\bullet(A), \delta)
\]
inducing (inverse) isomorphisms in cohomology.

The map $\phi$ is straightforward to describe on the subspace $Z^\bullet(A)$. Let $a \in C^p(A)$ with $\tilde{\delta}(a) = 0$. One can find (using a partition of unity) an $\alpha_{p-1} \in C^{p-1}(A^0)$ with $\delta(\alpha_{p-1}) = -a$. Then,

$$a + D(\alpha_{p-1}) = \bar{\delta}(\alpha_{p-1}) \in C^{p-1}(A^1).$$

Now, $\partial(\alpha_{p-1})$ is $\tilde{\delta}$ closed since

$$\tilde{\delta}(\partial(\alpha_{p-1})) = \partial\tilde{\delta}(\alpha_{p-1}) = -\partial(a) = 0.$$

Again, (using a partition of unity), there exists an $\alpha_{p-2} \in C^{p-2}(A^1)$ with $\tilde{\delta}(\alpha_{p-2}) = -\partial(\alpha_{p-1})$. One gets

$$a + D(\alpha_{p-1} + \alpha_{p-2}) = \bar{\delta}(\alpha_{p-2}) \in C^{p-2}(A^2) \text{ with } \delta(\alpha_{p-2}) = 0.$$

Continuing, one arrives at $\alpha_0 \in C^0(A^{p-1})$, with

$$a + D(\alpha_{p-1} + \alpha_{p-2} + \cdots + \alpha_0) = \partial(\alpha_0) \text{ and } \partial \partial \alpha_0 = 0.$$

Since $\delta(\bar{\delta}(\alpha_0)) = 0$, $\bar{\delta}(\alpha_0) \in C^0(A^p)$ gives a global section $\alpha \in \Gamma(A^p)$. This describes the map $\phi$:

$$\phi(a) = \alpha.$$

The map $\psi$ is defined on the cycles in $\Gamma(A^\bullet)$ in a similar fashion. If $\beta \in \Gamma(A^p)$ with $\bar{\partial}(\beta) = 0$, then one can find (using the Poincaré lemma) elements

$$b_{p-k} \in C^{k-1}(A^{p-k}) \text{ with } \bar{\partial}(b_{p-k}) = -\delta(b_{p-k+1}).$$

Then

$$\beta + D(b_{p-1} + b_{p-2} + \cdots + b_0) = \bar{\delta}(b_0) \in C^p(A^0).$$

Since $\partial \delta b_0 = 0$, $\delta b_0$ can be identified with an element $b \in C^p(A)$. We have

$$\psi(\beta) = b.$$

### 4.2.3 Computing $\tilde{Q}_2$

Now, $\tilde{Q}_2 : Z^\bullet(A) \otimes Z^\bullet(A^\bullet) \to Z^\bullet(A^\bullet)$ is defined by

$$\tilde{Q}_2(a, b) = \psi[\phi(a), \phi(b)]$$

where the bracket on the right is the Schouten bracket in $\Gamma(A^\bullet)$.

**Lemma 4.4.** For $a = \{a^i\} \in Z^0(A^\bullet)$ and $b = \{b^{i_0 \cdots i_p}\} \in Z^p(A^\bullet)$

$$\tilde{Q}_2(a, b) = c = \{c^{i_0 \cdots i_p}\} \in Z^p(A) \text{ where } c^{i_0 \cdots i_p} = [a^{i_0}, b^{i_0 \cdots i_p}].$$
Proof. This is a computation. We introduce as a notational aid, the bilinear function $[|,|] : C^p(A^\bullet) \times C^q(A^\bullet) \to C^{p+q}(A^\bullet)$, defined for $\alpha \in C^p(A^\bullet)$ and $\beta \in C^q(A^\bullet)$ by

$$[\alpha, \beta] = \gamma = \{\gamma^i_{i_0 \cdots i_{p+q}}\} \text{  where  } \gamma^i_{i_0 \cdots i_{p+q}} = [\alpha^i_{i_0 \cdots i_p}, \beta^i_{i_{p+1} \cdots i_{p+q}}].$$

While $[|,|]$ does not satisfy the Jacobi identity, nor the Jacobi identity up to homotopy (it is not even skew-symmetric for arbitrary elements of $C^p(A^\bullet) \otimes C^q(A^\bullet)$), it does behave well with respect to the differentials—both differentials are derivations of $[|,|]$:

$$\delta [a, b] = [\delta(a), b] + (-1)^{|b|} [a, \delta(b)],$$

$$\bar{\partial} [a, b] = [\bar{\partial}(a), b] + (-1)^{|b|} [a, \bar{\partial}(b)].$$

In this notation, the lemma asserts that for the special case of $a \in Z^0(A^\bullet)$ and $b \in Z^p(A^\bullet)$, we have $\bar{Q}_2(a, b) = [a, b]$.

Let $a = \{a_i\} \in Z^0(A)$ and $b = \{b^i_{i_0 \cdots i_p}\} \in Z^p(A)$. Select elements $\beta_0, \beta_1, \ldots, \beta_{p-1}$ with

$$\beta_k \in C^k(A^{p-k-1}), \  \delta \beta_{k-1} = \bar{\partial} \beta_k, \ \delta \beta_{p-1} = b, \ \beta = \bar{\partial} \beta_0.$$  

We have $\phi(b) = \beta \in \Gamma(A^p)$ and $\phi(a) = \alpha \in \Gamma(A^0)$ identified with $a \in C^0(A)$. Then $[\alpha, \bar{\partial} \beta_0] \in \Gamma(A^p)$ can be considered as the cocycle $[|,|,\bar{\partial} \beta_0] \in C^0(A^p)$. Now, we compute $c = Q_2'(a, b) = \psi([\alpha, \beta_0])$. We set $\gamma = [\alpha, \bar{\partial} \beta_0] \in \Gamma(A^p)$ and find elements $\gamma_0, \gamma_1, \ldots, \gamma_{p-1}$ satisfying

$$-\delta \gamma_k = \bar{\partial} \gamma_{k+1} \text{  and  } \bar{\partial} \gamma_0 = \gamma.$$  

Then $-\delta \gamma_{p-1} = \psi(\gamma) = c \in C^p(A)$. One finds that

$$\gamma_0 = [\alpha, \bar{\partial} \beta_0]$$

$$= \bar{\partial} [a, \beta_0]$$

$$= \bar{\partial} \gamma_0 \text{  for  } \gamma_0 = [a, \beta_0].$$

then

$$\delta \gamma_0 = \delta [a, \beta_0]$$

$$= [a, \delta \beta_0]$$

$$= -[a, \bar{\partial} \beta_1]$$

$$= -\bar{\partial} [a, \beta_1]$$

$$= \bar{\partial} \gamma_1, \text{  for  } \gamma_1 = [a, \beta_1].$$

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and so on.

One finds that \( \gamma_k = [\alpha, \beta_k] \) and at the end

\[-\delta \gamma_{p-1} = \delta [a, \beta_{p-1}] = [a, \delta \beta_{p-1}] = [a, b] \]

as claimed.

It is also straightforward, but not necessary for our purposes, to calculate \( \tilde{Q}_2(a, b) \) for \( a \in Z^p(A), \ p > 0 \). One finds that \( \tilde{Q}_2 \) can be expressed in terms of \( [\cdot, \cdot] \), but not with just one such term: For example, if \( \alpha, \beta \in Z^1(A) \),

\[ \tilde{Q}_2(\alpha, \beta) = \frac{1}{2} ( [\alpha, \beta] + [\beta, \alpha] ) \tag{2} \]

or, equivalently,

\[ \tilde{Q}_2(\alpha, \beta) = \gamma \in Z^2(A) \quad \text{where} \quad \gamma^{ijk} = \frac{1}{2} \left( [\alpha^{ij}, \beta^{jk}] + [\beta^{ij}, \alpha^{jk}] \right). \tag{3} \]

The reader may recognize this as classical bracket in the Kodaira-Spencer theory of deformations of complex structure \[\text{Kod86}\].

4.3 Associative algebra structures

If the space of sections of \( A \) carries an associative product, then both the Dolbeaut and Čech complexes become associative algebras and the maps \( \phi \) and \( \psi \) are algebra maps.

Since \( \Gamma \left( A \otimes \bar{\Omega}^* \right) \) is simply the tensor product of two associative algebras (\( \bar{\Omega}^* \) has the wedge product), it is an associative algebra. The standard way to make the Čech cochains \( C^\bullet(A) \) into an associative algebra is as follows:

For \( a = \{a^{i_0 \cdots i_p}\} \in C^p(A) \) and \( b = \{b^{i_0 \cdots i_q}\} \in C^q(A) \)

define

\[ a \cdot b = c \in C^{p+q}(A), \quad \text{where} \quad c^{i_0 \cdots i_{p+q-1}} = a^{i_0 \cdots i_p} \cdot b^{i_p \cdots i_{p+q}}. \]

Both differentials are derivations of these algebra structures; so \( \Gamma(A \otimes \bar{\Omega}^*, \bar{\partial}, \wedge) \) and \( (C^\bullet(A), \delta, \cdot) \) are quasi-isomorphic as differential graded associative algebras.

In the case that the sections of \( A \) have both an associative product and a Lie bracket satisfying the Poisson identity equation \[\text{2.2}\] then the Dolbeaut
resolution $A \otimes \Omega^\bullet$ will also be a Gerstenhaber algebra. The $Q_2$ term of the $L_\infty$ structure on $C^\bullet(A)$, being a sum of terms of the form $[|, |]$, also satisfies a Poisson identity.

One way that $C^\bullet(A)$ becomes an algebra is when $A = \Lambda^\bullet B$ for some bundle $B$. Then the wedge product on sections of $A$ induces a wedge product on $C^\bullet(A)$, which is usually also denoted by $\wedge$. This is the case for $A = \Theta^\bullet X$. A common example occurs for $A = \Lambda^\bullet \Omega X$ and the maps $\phi$ and $\psi$ induce multiplicative isomorphisms between $H^\bullet(\Omega^\bullet_X)$ and $H^\bullet\cdot(\cdot)$.

5 A model for $\Theta_X$

In this section, we construct a sheaf of differential graded Lie algebras $L$ on $\mathbb{P}^n$ that is quasi-isomorphic to $\Theta_X$ (with the zero differential and the ordinary bracket of vector fields).

Let $T = S[y] = S[x^0, \ldots, x^n, y]$ and with weights and degrees as follows:

- $\text{wt}(x) = 1$
- $\text{deg}(x) = 0$
- $\text{wt}(y) = \nu$
- $\text{deg}(y) = -1$.

As before, we assume that $f \in S$ has been chosen, that $\text{wt}(f) = \nu = n + 1$, and that $f$ is nonsingular. Consider $\text{Der}(T)$. We use the notation $\partial_i$ for $\partial_{x^i}$, $\partial$ for $\frac{\partial}{\partial y}$, and $g_i$ for $\partial_i g$. We have weights and degrees:

- $\text{wt}(\partial_i) = -1$
- $\text{deg}(\partial_i) = 0$
- $\text{wt}(\partial) = -\nu$
- $\text{deg}(\partial) = 1$.

We define\footnote{One way to view $d_f$ is to define a differential in $T$ by $d(y) = f$ and $d(x^i) = 0$. Then $d_f = \text{ad}(d)$ in $\text{Der}(T)$.} an $S$ linear, weight zero, degree one, differential $d_f : \text{Der}(T) \to \text{Der}(T)$ by

$$
\begin{align*}
    d_f(y \partial_i) &= f \partial_i - f_i y \partial \\
    d_f(\partial_i) &= f_i \partial \\
    d_f(y \partial) &= f \partial.
\end{align*}
$$

The triple $(\text{Der}(T), d_f, [\, , \,])$ is a differential graded Lie algebra, where the grading is by degree and $[\, , \,]$ is the ordinary bracket of derivations. That $\text{Der}(T)$ is weighted is a feature that will be useful for sheaf computations later.
Now, adjoin a weight zero, degree zero element $e$ with

$$d_f(e) = \sum_{k=0}^{n} x^k \partial_k - \nu y \partial$$

$$[e, a] = [a, e] = \text{wt}(a)$$ for any $a \in \text{Der}(T) \oplus Se$.

The result $(L, d_f)$ where $L = \text{Der}(T) \oplus Se$ is a graded Lie algebra and a complex (and the weight zero subcomplex $\{a \in \text{Der}(T) \oplus Se \mid \text{wt}(a) = 0\}$ is a differential graded Lie algebra when considered with the bracket).

### 5.1 Cohomologies of $L$

Here is a chart to help keep track of degrees and weights:

|            | $e$   | $y \partial$ | $\partial$ | $y \partial$ | $\partial$ |
|------------|-------|--------------|------------|--------------|------------|
| degree     | $-1$  | $-1$         | $0$        | $0$          | $1$        |
| weight     | $0$   | $\nu - 1$   | $-1$       | $0$          | $-\nu$     |
| $d_f$      | $\sum_{k=0}^{n} x^k \partial_k - \nu y \partial$ | $f \partial - f_i y \partial$ | $f_i \partial$ | $f \partial$ | $0$        |

One reads $H^{-1}(L) = 0$ and $H^1(L) = S \partial / J_f \partial = S / J_f$ and

$$H^0(L) = \left\{ \sum_{i=0}^{n} g^i \partial_i \mid \sum_{i=0}^{n} g^i f_i = h f \right\} / \left\{ p \left( \sum_{i=0}^{n} x^i \partial_i \right) \right\}$$

where $g^i, h, p \in S$.

Let $L$ denote the sheaf of differential graded Lie algebras on $\mathbb{P}^n$ associated to $L$:

$$\Gamma(L, U_i) = \left\{ \frac{\alpha}{(x^i)^r} : \alpha \in L, \text{wt}(\alpha) = r \Leftrightarrow \text{wt} \left( \frac{\alpha}{(x^i)^r} \right) = 0 \right\}.$$

Note that $\mathcal{H}^0(L) \simeq \Theta_X$. Here calligraphic $\mathcal{H}$ stands for the kernel of $d_f$ modulo the image of $d_f$ (not the hypercohomology of $L$). Since $H^1(L)$ is finite dimensional over $\mathbb{C}$, for the complex of sheaves $L$, we have $\mathcal{H}^1(L) = 0$. So $\mathcal{H}^{-1}(L) = \mathcal{H}^1(L) = 0$ giving $\mathcal{H}(L) = \mathcal{H}^0(L)$. That is,

$$\mathcal{H}(L) \simeq \Theta_X.$$
5.2 A simple formality for $L$

For any three term sheaf of dgLa’s

\[ 0 \to \mathfrak{A}^{-1} \xrightarrow{d^0} \mathfrak{A}^0 \xrightarrow{d^1} \mathfrak{A}^1 \to 0 \]

with $H^{-1}(\mathfrak{A}) = H^1(\mathfrak{A}) = 0$, the vertical maps $\eta$ and $\zeta$

\[ \begin{array}{c}
\zeta \\
\downarrow \\
\zeta
\end{array} \quad \begin{array}{c}
\zeta \\
\downarrow \\
\zeta
\end{array} \]

are quasi-isomorphisms of sheaves of dgLa’s (meaning the induced map on $H$ is an isomorphism). This is the situation for L.

**Theorem 5.1.** $(L, d_f, [,])$ is quasi-isomorphic as a sheaf of dgLa’s to $(\Theta_X, 0, [,])$.

6 The complex $(\mathcal{C}, D)$ and the $L_\infty$ algebra $(\mathcal{C}, Q^\mathcal{C})$

Let

\[ F = \bigoplus_{k=0}^{n-1} F^k, \quad F^k = S^k(L[1]) \simeq \left( \Lambda^k L \right)[k]. \]

The differential on $L$ extends (by the Leibnitz rule) to $F$ and the bracket on $L$ extends to a Schouten–type bracket on $F$. They are determined by

- for $\alpha \in F^i$, $\beta \in F^j$, $d_f(\alpha \wedge \beta) = d_f(\alpha) \wedge \beta + (-1)^i \alpha \wedge d_f(\beta)$,
- for $g \in F^0 = T = S[x^0, \ldots, x^n, y]$ and $\alpha \in F^1 = L[1]$, $[\alpha, g] = \alpha(g)$,
- for $\alpha \in F^i$, $\beta \in F^j$, $\gamma \in F$, $[\alpha \wedge \beta, \gamma] = \alpha \wedge [\beta, \gamma] + (-1)^{i+j}[\alpha, \gamma] \wedge \beta$.

Again, here is a chart of degrees and weights. Note that $x^i, y \in F^0 \subset F$ and remember that $L$ was shifted:

|   | $y$ | $x^i$ | $e$ | $y\partial_i$ | $\partial_i$ | $y\partial$ | $\partial$ | $\partial^k$ |
|---|-----|------|----|--------------|-------------|------------|--------|----------|
| deg | $-1$ | $0$ | $0$ | $0$ | $1$ | $1$ | $2$ | $2l$ |
| w   | $\nu$ | $1$ | $0$ | $\nu - 1$ | $-1$ | $0$ | $-\nu$ | $-l \nu$ |
Let \( \delta \) denote the sheaf associated to \( F \) and denote by \( C^p(\delta) \), the Čech \( p \)-cochains with values in \( \delta \). Let \( \delta : C^p(\delta) \to C^{p+1}(\delta) \) denote the Čech differential. We have a double complex whose \((p,q)\) term is \( C^p(\delta^q) \) and a single complex \((\mathcal{C}, D)\) where \( C^k = \oplus_{p+q=k} C^p(\delta^q) \) and \( D = \delta + d_f \).

**Theorem 6.1.** The complexes \((\mathcal{C}, D)\) and \((g, \bar{\partial})\) are quasi-isomorphic.

**Proof.** From theorem 5.1, the complexes of sheaves \((\mathcal{L}, d_f)\) and \((\Theta_X, 0)\) are quasi-isomorphic. Therefore, by taking exterior powers, \((\delta, d_f)\) and \((\Theta_X^\bullet, 0)\) are quasi-isomorphic (as complexes of sheaves). Since quasi-isomorphic complexes induces isomorphisms in hypercohomology, \((C^\bullet(\Theta_X^\bullet), \delta + 0)\) are quasi-isomorphic as complexes.

By theorem 4.3, we have a quasi-isomorphism between \((C^\bullet(\Theta_X^\bullet), \delta)\) and \((\Gamma(\Theta_X^\bullet \otimes O_X^\bullet), \bar{\partial})\), establishing the theorem. \( \Box \)

There are advantages to working with \((\mathcal{C}, D)\). For one, we are able to compute the cohomology of \((\mathcal{C}, D)\). Another is that \( \mathcal{C} \) is defined independently of \( X \)—the polynomial \( f \), and hence all of the geometry of \( X \), is carried in the differential \( D \). When the differential \( D \) is deformed, it is not difficult to interpret the shifted cohomology rings.

**Theorem 6.2.** \( H(\mathcal{C}, D) \simeq \tilde{R} \).

**Proof.** Consider the double complex \((C^\bullet(\delta^\bullet), \delta, d_f)\) and the filtration on the single complex \((C^\bullet, D)\):

\[
F^p C^\bullet = C^0(\delta^\bullet) \oplus C^1(\delta^{\bullet-1}) \oplus \cdots \oplus C^{-p}(\delta^p).
\]

Let \( \{E_r, \delta_r\} \) denote the spectral sequence associated to this filtered complex. We will see that spectral sequence degenerates at the \( E_2 \) term: 
\[
E^{p,q}_2 = E^{p,q}_2 \simeq H^q_{d_f}(H^p_\delta(\delta)).
\]

We compute \( E^{1,q}_1 \simeq H^q(\delta^p, \delta) \). To describe the classes in \( E_1 = \oplus_q H^q_\delta(\delta) \) that survive in \( E_2 \) (see figure 1), first note that since the \( \delta^p \) are locally free sheaves on \( \mathbb{P}^n \),

\[
H^q_\delta(\delta) = 0, \text{ unless } q = 0 \text{ or } n.
\]

**Case** \( q = 0 \). The (homogeneous) elements generating \( H^0(\delta) \) that are \( d_f \) closed and not necessarily \( d_f \) exact are given by \( a = \{a^i\} \in C^0(U_i, \delta) \) where

\[
a^i = \frac{x^i g(x) \partial^p}{x^i} \in \Gamma(U_i, \delta), \quad g(x) \in S^\nu.
\]
$q$ (= Čech deg)

\[ e_k \]

\[ n \]

\[ n - 1 \]

\[ \cdots \]

\[ 1 \]

\[ \cdots \]

\[ 0 \]

\[ -1 \quad 0 \quad 2 \quad 4 \quad 6 \quad p \quad (= \text{deg in } \mathfrak{F}) \]

Figure 6.1: $E_{1}^{p,q}$

An $a$ as above is a $d_f$ boundary if and only if $g(x) = \sum_{k=0}^{n} g^i(x)f_i(x)$ for some $g^i \in S^{(n-1)\nu+1}$. So,

\[ H_{d_f}^2(H^n_\delta(\mathfrak{F})) \cong R^p = \text{Image } \left( S^{[\nu]} \to S/ < f_1, \ldots, f_n > \right) \]

and

\[ \oplus_p H_{d_f}^2(H^n_\delta(\mathfrak{F})) \cong \oplus_p R^p. \]

**Case** $q = n$. Each class in $H^n_\delta(\mathfrak{F})$ is of the form

\[ e_k \overset{\text{def}}{=} \frac{y e^k}{x^0 \cdots x^n}, \quad k = 0, 1, \ldots, n - 1. \]

Now,

\[ d_f \left( \frac{y e^k}{x^0 \cdots x^n} \right) = \frac{f e^k - y k e^{k-1} \left( \sum_{i=0}^{n} x^i \partial_i \right)}{x^0 \cdots x^n}. \]

Because $d_f(e_k)$ can be expressed as a sum of terms, each of which has a positive power of an $x^i$ in the numerator,

\[ d_f(e_k) \in \delta (C^{n-1}(\mathfrak{F})). \]

Therefore, each $e_k$ is a class in $E_2$ of total degree $n - 1$ (Čech degree $n$ and degree $-1$ in $\mathfrak{F}$) and

\[ \oplus H_{d_f}^{-1}(H^n_\delta(\mathfrak{F})) \cong \mathbb{C}e_k. \]

Note, too, that it is impossible for $D(e_k)$ to kill any of the Čech degree zero classes, hence $E_2 = E_3 = E_4 = \cdots$.

Together, we have

\[ H(C, D) \cong \oplus_p H_{d_f}^2(H^n_\delta(\mathfrak{F})) \oplus H_{d_f}^{-1}(H^n_\delta(\mathfrak{F})) \cong \oplus_p R^p \oplus_{k=0}^{n-1} \mathbb{C}e_k = \tilde{R}. \]

\[ \square \]
6.0.1 Moduli

Since, \((C, D)\) is quasi-isomorphic to \((C(\Theta^\bullet_X, \tilde{\delta}))\), there exists an \(L_\infty\) structure, call it \(Q^C : SC[1] \to SC[1]\), with \(Q_1^C = D\) such that \((C, Q^C)\) is quasi-isomorphic to \((C(\Theta^\bullet_X, \tilde{\delta}))\). Since \((C(\Theta^\bullet_X, \tilde{\delta}))\) is quasi-isomorphic to \((\mathfrak{g}, \tilde{\partial}, [\cdot, \cdot])\), we have

**Theorem 6.3.** As \(L_\infty\) algebras, \((C, Q^C)\) and \((\mathfrak{g}, \tilde{\partial}, [\cdot, \cdot])\) are quasi-isomorphic.

**Corollary 6.4.** The moduli space for \(\text{Def}(\mathfrak{g}, \tilde{\partial}, [\cdot, \cdot])\) equals the moduli space for \(\text{Def}(C, Q^C)\).

This completes the proof of theorem 3.1.

In order to complete the proof of theorem 3.2, we analyze the cohomology of \(H(C, D_b)\) for the shifted differentials \(D_b\).

6.0.2 Shifting \(D\)

We take \([a] \in H(C, D) \simeq \tilde{R}\) and obtain a shifted \(L_\infty\) algebra \((C, Q^{C,a})\). The linear part of \(Q^{C,a}\) is a shifted differential \([Mer00]\) \(D_a := Q_1^{C,a}\) defined by

\[
Q_1^{C,a} = D + Q_2^C(a, \cdot) + \frac{1}{2!}Q_3^C(a, a, \cdot) + \frac{1}{3!}Q_4^C(a, a, a, \cdot) + \cdots
\]

The classes in \(H(C, D) \simeq \tilde{R}\) come in two types:

1. the primitive elements \([a]\),
   \[
a^i = \frac{x^i g(x) \partial^p}{x^i} \in \Gamma(U_i, \tilde{\mathfrak{g}}), \quad g(x) \in S^{p\nu}
\]
   have Čech degree zero and even degree \(2p\) in \(\tilde{\mathfrak{g}}\).

2. the nonprimitive elements \([e_k]\) have Čech degree \(n\) and degree \(-1\) in \(\tilde{\mathfrak{g}}\).

For a primitive \(a\), a simple calculation shows that

\[
Q_2^C(Q_2^C(a, a), b) + Q_2^C(Q_2^C(b, a), a) + Q_2^C(Q_2^C(a, b), a) = 0.
\]

That is, \(Q_2^C\) satisfies the Jacobi identity (exactly, not just up to homotopy) when two of the three elements are Čech zero cocycles in \(C\). This implies that \(Q\) may be taken with

\[
Q_3^C(a, a, b) = Q_4^C(a, a, a, b) = \cdots = 0.
\]

So, \(D_a\) reduces to the familiar

\[
D_a = D + Q_2^C(a, \cdot).
\]
For the nonprimitive elements $e_k$, degree restrictions imply that

$$Q_3^C(e_k, e_k, b) = Q_4^C(e_k, e_k, e_k, b) = \cdots = 0,$$

and again we have

$$D_{e_k} = D + Q_2^C(e_k, \cdot).$$

Now, we recompute the cohomologies $H(C, D_a)$ and $H(C, D_{e_k})$:

Consider a $(0, 2p)$ cocycle $a \in C$ and let $g \in S^{[p\nu]}$ as in Equation 4. For $b \in C$, we can determine $Q_2(a, b)$ by $[a^{i_0}, b^{i_1..i_q}]$ where this bracket is just the bracket in $\mathfrak{F}$. Since $\Gamma(U, \mathfrak{F})$ is generated by symmetric products of elements of $\Gamma(U, \mathfrak{F}^i)$, $i = 0, 1$, (and $[,]$ satisfies the Liebnitz property with respect to these symmetric products) we need only compute $[a, b]$ for generators $b$. The results

$$[g \partial^p, y \partial^q] = \left(g \partial^q - \frac{\partial g}{\partial x^i} y \partial^i\right) \partial^{p-1}$$

$$[g \partial^p, \partial^q] = \left(\frac{\partial g}{\partial x^i} \partial^i\right) \partial^{p-1}$$

$$[g \partial^p, y \partial^q] = (g \partial^q) \partial^{p-1}$$

$$[g \partial^p, \partial^q] = 0$$

yield

**Lemma 6.5.** For $a$ as in Equation 4

$$D_a := \tilde{\delta} + df + Q^C_2(g, \cdot) = \tilde{\delta} + df + g$$

where $df + g(b) = df(b) + d_g(b) \partial^{p-1}$.

Lemma 6.5 above and the same spectral sequence computation (cf., the proof of theorem 6.2) that gives $H(C, D) \simeq \tilde{R}$, except with $f$ replaced by $f + g \partial^p$, proves that $H(C, D_a) \simeq \tilde{R}_{f+g}$. 

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To determine $Q_2(e_k, b)$, we need only look at $Q_2(e_k, b)$ for $b \in H^0_{\delta} (\mathfrak{F})$, since $Q_2(e_k, b) \in \text{Image}(D)$ if the Čech degree of $b > 0$:

\[
\begin{align*}
\frac{ye^k}{x^0 \ldots x^n} y \partial_i &= 0 \\
\frac{ye^k}{x^0 \ldots x^n} x^j \partial_i &= \frac{x^j ye^k}{x^0 \ldots x^n} \in \delta (C^{n-1}(\mathfrak{F})) \\
\frac{ye^k}{x^0 \ldots x^n} y \partial &= -\frac{ye^k}{x^0 \ldots x^n} \\
\frac{ye^k}{x^0 \ldots x^n} g \partial &= -\frac{ge^k}{x^0 \ldots x^n} \in \delta (C^{n-1}(\mathfrak{F})) \\
\frac{ye^k}{x^0 \ldots x^n} e &= 0.
\end{align*}
\]

Therefore, $D$ closed elements are $D_{e^k}$ closed and $D$ exact elements are $D_{e^k}$ exact elements:

\[H(\mathcal{C}, D_{e^k}) = H_{d_f + \text{ad}(e_k)} H_{\delta}^*(\mathfrak{F}) = H_{d_f} H_{\delta}^*(\mathfrak{F}) = H(\mathcal{C}, D) = \tilde{R}.\]

Together,

**Theorem 6.6.** $H(\mathcal{C}, D_{e^k}) \simeq \tilde{R}$ and $H(\mathcal{C}, D_{a}) \simeq \tilde{R}_{f+g}$.

## 7 Concluding remarks

While we have stated results for Calabi-Yau hypersurfaces in $\mathbb{P}^n$, the methods presented here can be extended to handle Calabi-Yau hypersurfaces in weighted projective spaces, and probably have applications to hypersurfaces in toric varieties.

One can generalize in another direction by replacing the field of complex numbers by an arbitrary field of characteristic zero. There is an obstacle, however, to using a field of characteristic $p$. Namely, our construction of the $L_\infty$ structure $\tilde{Q}$ on the space of Čech cochains requires division by a factorial. Equation 2, for example, has a factor of $\frac{1}{2}$. Our model is defined for large primes, $p > n!$.

As mentioned in the introduction, the moduli space $\mathcal{M}$ is a Frobenius manifold. We have described an algebra structure on the tangent spaces $T_a \mathcal{M}$. A Frobenius structure implies a great deal more. The products on $T_a \mathcal{M}$ are connected by a potential function. That is, there exist coordinates on $\mathcal{M}$ so that the structure constants of the shifted cohomology rings are the third derivatives of a single function satisfying the WDVV equations. We are interested what
the special basis of \( R \) must be and how the potential function appears in our construction and will return to this point in a later paper.

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