Strong gravity effects of charged $Q$-clouds and inflating black holes

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Abstract

In this paper, we re-examine charged $Q$-clouds around spherically symmetric, static black holes. In particular, we demonstrate that for fixed coupling constants two different branches of charged scalar clouds exist around Schwarzschild black holes. This had not been noticed previously. We find that the new solutions possess a “hard wall” at maximal possible gauge coupling. This wall separates the interior (containing the black hole horizon), in which the scalar field is trapped in the “false vacuum”, from the “true vacuum” exterior. When taking back-reaction onto the space-time into account, we find that at maximal possible back reaction, the black hole solutions corresponding to these two branches either become extremal black holes with diverging scalar field derivative on the horizon or inflating black holes with a second, “cosmological” horizon which – outside this second horizon – correspond to extremal Reissner-Nordström black holes.

1 Introduction

Asymptotically flat black hole solutions of the (electro-)vacuum Einstein(-Maxwell) equations can be characterized by a very small number of parameters: their mass, charge and angular momentum and consequently uniqueness theorems for these type of black holes have been proven [1]. Moreover, a number of No-hair theorems for scalar fields exist as well [2], but these are not general. In fact, possibilities to evade the requirements for these theorems exist and hence black holes can carry scalar hair. An interesting observation in this respect is that when the scalar field possesses a harmonic time-dependence of the form $\sim \exp(i\omega t)$, $\omega = \omega_R + i\omega_I$, a scalar wave impinging on a rotating black hole can be amplified if the real part of the frequency of the time-dependence $\omega_R$ is smaller than $m\Omega$, where $\Omega$ is the horizon velocity and $m$ an integer [3]. When, additionally, a bounding potential exists that traps the scalar modes, the black hole can become super-radiantly unstable [4]. These phenomena have led to the observation that when fulfilling the so-called “synchronization condition” $\omega_R = m\Omega$, $\omega_I = 0$, Kerr black holes with scalar hair exist [5]. Interestingly, these black holes possess a globally regular limit when letting the horizon tend to zero: so-called boson stars [6]. These are objects made of massive scalar fields (that can be self-interacting) and can be interpreted as macroscopic Bose-Einstein condensates. When considering static and charged black holes, i.e. Reissner-Nordström black holes, an equivalent condition exists for scalar fields to be amplified. In this case, $\omega_R < gV(r_h)$, where $g$ is the U(1) charge of the scalar field and $V(r_h)$ the value of the electric potential on the horizon [7]. However, a super-radiant instability of the RN solution is not possible [8]. Fulfilling the “synchronization condition” $\omega_R = gV(r_h)$ leads to the existence of charged black holes with scalar hair [11] [12] (and also [13]). In [11], the authors studied scalar clouds on Reissner-Nordström black holes, while scalar clouds on Schwarzschild black holes as well as the back-reaction of the scalar clouds on the space-time have been addressed in [12]. In particular, it was shown that the coupling to the electromagnetic field as well as self-interaction of the scalar field are crucial ingredients for the existence of these clouds. In this paper, we re-examine and extend the results of [12]. In particular, we are interested in charged clouds on Schwarzschild black holes and how these
back-react on the space-time. We will show in the following, that two different types of clouds exist in the Schwarzschild background. Moreover, the back-reaction of these clouds leads to very different phenomena in the strong gravity regime.

## 2 The model

In the following, we will discuss black hole solutions in General Relativity minimally coupled to an electromagnetic and complex scalar field, respectively, in a (3+1)-dimensional space-time. The Lagrangian density reads:

$$
\mathcal{L} = \frac{\mathcal{R}}{16\pi G} - D_{\mu} \Psi \bar{D}_{\mu} \Psi - U(|\Psi|) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu},
$$

(1)

where $\mathcal{R}$ is the Ricci scalar and $\Psi$ is a complex scalar field with potential $U(|\Psi|)$ that is minimally coupled to a $U(1)$ gauge field. $D_{\mu} \Psi = \partial_{\mu} \Psi - igA_{\mu} \Psi$ and $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ then denote the covariant derivative of the complex scalar field and field strength tensor of the gauge field, respectively. As show in [12], $Q$-clouds exist only for massive and self-interacting scalar fields. The potential we will use in the following is given by:

$$
U(\Psi) = \mu^2 \Psi^2 - \lambda \Psi^4 + \nu \Psi^6.
$$

(2)

Due to appropriate rescalings (see below), we will be able to choose $\mu = \lambda = 1$ without loosing generality. The only parameter in the potential to vary will then be related to $\nu$. For $\mu = \lambda = 1$, $\nu = 1/4$, the potential possesses three degenerate minima at $\Psi = 0$, $\Psi = 2$. On the other hand, for $\mu = \lambda = 1$, $\nu = 1/3$, the potential has a saddle point at $\Psi = 1$ and for $\nu > 1/3$ would be monotonically increasing. In our numerical construction, we will choose a value in between these two limiting values, see more details below. We would like to discuss static, spherically symmetric solutions to the equations resulting from the variation of the action associated to (1) and thence use the following Ansatz for the metric and matter fields:

$$
ds^2 = -(\sigma(r))^2 N(r) dt^2 + \frac{1}{N(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad A_{\mu} dx^\mu = V(r) dt, \quad \Psi = \psi(r) \exp(i\omega t).
$$

(3)

In the following, we will choose the gauge $\omega = 0$, which due to the synchronization condition $\omega = gV(r_h)$ amounts to choosing $V(r_h) = 0$. Using the rescalings $x = \mu r$, $v = \frac{\sqrt{\gamma}}{\mu} V$, $\psi = \frac{\sqrt{\gamma}}{\mu} \Psi$ we are left with three dimensionless couplings:

$$
\alpha^2 = \frac{4\pi G \mu^2}{\lambda}, \quad \beta^2 = \frac{\nu \mu^2}{\lambda^2}, \quad e = \frac{g}{\sqrt{\lambda}}.
$$

(4)

The equations of motion then read (the prime denotes the derivative with respect to $x$):

$$
N' = -2\alpha^2 \left[ \frac{v'^2}{2\sigma^2} + N\psi'^2 + U(\psi) + \frac{(e\psi \psi')^2}{N^2 \sigma^2} \right] - \frac{N - 1}{x}, \quad \sigma' = 2\alpha^2 x \sigma \left[ \psi'^2 + \frac{(e\psi \psi')^2}{N^2 \sigma^2} \right],
$$

(5)

$$
v'' = -\left( \frac{2}{x} - \frac{\sigma'}{\sigma} \right) v' + \frac{2e^2 v^2 \psi^2}{N},
$$

(6)

$$
\psi'' = -\left( \frac{2}{x} + \frac{N'}{N} + \frac{\sigma'}{\sigma} \right) \psi' - \frac{e^2 v^2 \psi}{N^2 \sigma^2} + \frac{1}{2N} \frac{dU}{d\psi}.
$$

(7)

The boundary conditions for a black hole solution with regular matter fields on the event horizon at $x = x_h$ as well as finite energy and asymptotic flatness are:

$$
N(x_h) = 0, \quad N' \psi'|_{x=x_h} = \frac{1}{2} \frac{dU}{d\psi} \bigg|_{\psi = \psi_h}, \quad v(x_h) = 0
$$

(8)

with $\psi_h = \psi(x_h)$ and

$$
N(x \rightarrow \infty) \rightarrow 1, \quad \psi(x \rightarrow \infty) \rightarrow 0, \quad v(x \rightarrow \infty) \rightarrow v_{\infty}.
$$

(9)
The physical parameters of the solutions are the (dimensionless) mass \( M \), the (dimensionless) electric charge \( Q \) and the (dimensionless) Noether charge \( Q_N \). These can be read off from the metric and matter field functions at infinity

\[
N(x \gg 1) = 1 - \frac{2M}{x} + \frac{\alpha^2 Q^2}{x^2} + \ldots \quad , \quad v(x) = v_\infty - \frac{Q}{x} + \ldots
\]

and the following integral of the \( t \)-component of the locally conserved Noether current:

\[
Q_N = \int dx \frac{2a^2 e v^2 \psi^2}{N\sigma}.
\]

This globally conserved quantity can be interpreted as the number of scalar bosons and is related to the electric charge \( Q \). In fact, using the equation (6) for \( v \), it is easy to show that for globally regular solutions \( eQ_N \equiv Q \), i.e. the total charge is the charge of \( Q_N \) individual scalar bosons that each carry charge \( e \).

For black holes, this is no longer true, since the horizon at \( x = x_h \) presents a surface, on which regularity (boundary) conditions have to be imposed. The total electric charge is then a sum of the electric charge of \( Q_N \) individual charges \( e \) and the horizon electric charge given by \( -E_x(x_h)x_h^2/\sigma(x_h) \), where \( E_x(x) = -v'(x) \) is the radial electric field. The scalar field has an exponential decay at infinity, which reads

\[
\psi(x \to \infty) \sim \frac{\exp(-\mu_{\text{eff},\infty} x)}{x} + \ldots \quad , \quad \mu_{\text{eff},\infty} = \sqrt{1 - e^2 v_\infty^2}.
\]

Hence, the scalar field has an (asymptotic) effective mass given by the difference between the “bare” mass \( \mu \equiv 1 \) and the electric potential energy. Obviously, the requirement of \( ev_\infty < 1 \) then is the condition that the electric potential energy should be lower than the threshold to produce scalar particles of mass \( \mu \).

We can also define the temperature of the black hole, which will be important in the strong gravity regime. For our choice of metric this is given by

\[
T_H = (4\pi)^{-1}\sigma(x_h)N'|_{x=x_h}.
\]

### 3 Q-clouds around static black holes

In the following, we assume \( \alpha^2 \equiv 0 \), i.e. neglect the back-reaction of the scalar and gauge field, respectively, on the space-time. There are two cases to discuss here, which turn out to be very different in the strong gravity limit. For Reissner-Nordström black holes, the electric field is completely fixed and does not interact dynamically with the scalar field. We will hence discuss this case first. When choosing a Schwarzschild black hole background, the gauge field interacts dynamically with the scalar field and we will show in the following that this has interesting consequences, in particular when letting these charged Q-clouds back-react on the space-time. In the following, we will also be interested in the mass of the Q-cloud, \( M_Q \) as defined in [12] :

\[
M_Q = \frac{1}{4\pi} \int d^3x \sqrt{-g} \left( T^i_i - T^i_i \right) , \quad i = 1, 2, 3.
\]

Using the explicit expressions for the energy-momentum tensor and the gauge field equation, we find that

\[
M_Q = 2v_\infty Q - \int_{x_h}^{\infty} dx \frac{v'^2 x^2}{\sigma} - 2 \int_{x_h}^{\infty} dx x^2 \sigma U(\psi).
\]

The second term on the (rhs) of this relation is the energy stored in the electric field of the cloud, since the integrand is nothing else but \( F_{\mu\nu} F^{\mu\nu} \sqrt{-g} \) \( dx \). Hence, the first two terms are related to the electromagnetic field. The part of the “bare” scalar field that gravitates is the scalar potential energy only.
3.1 Reissner-Nordström black holes

Rescaling \( v \to v/\alpha \) and \( e \to \alpha e \) and letting \( \alpha = 0 \) leads to the decoupling of the scalar field equation from the remaining equations \([12]\). The gravity and electromagnetic field equations have a well known solution, the Reissner-Nordström (RN) solution, which is a non-extremal (extremal, respectively) black hole solution for charge \( Q \) smaller (equal) to the mass \( M \) and in this case reads:

\[
v(r) = \frac{Q}{x_h} - \frac{Q}{x}, \quad \sigma \equiv 1, \quad N(x) = 1 - \frac{2M}{x} + \frac{Q^2}{x^2}.
\]

\( x_h \) denotes the radius of the outer (event) horizon of this black hole given by \( x_h = M + \sqrt{M^2 - Q^2} \), which becomes \( x_h^{(ex)} = M = Q \) for the extremal case \( M = Q \). Note that here \( v_\infty = Q/x_h \) is fixed by the space-time. We then consider the scalar field equation in the background of this black hole solution. The scalar field equation in the presence of the fixed metric and electromagnetic fields can be interpreted as that of a scalar field in an effective potential \( U_{\text{eff}} = \mu_{\text{eff}}^2 \psi^2 - \psi^4 + \beta^2 \psi^6 \), where

\[
\mu_{\text{eff}}^2 = 1 - \frac{e^2 v^2}{N} = 1 - \frac{e^2 Q^2(x_h - x)}{x_h(Q^2 - x x_h)}
\]

is the effective “mass” of the scalar field, which is \( \mu_{\text{eff}}^2 = 1 - eQ^2/(x_h^2 - Q^2) + \cal{O}(e^2) \) close to the horizon at \( x = x_h + e \) and asymptotically becomes \( \mu_{\text{eff}} \infty \) (see \([12]\)). The scalar field equation then has to be solved numerically subject to the regularity condition on the horizon (see \([8]\)):

\[
\psi'_{|x=x_h} = \frac{x_h^3}{2(x_h^2 - Q^2)} \frac{dU}{d\psi_{|\psi=\psi_h}},
\]

where \( \psi_h \) denotes the value of the scalar field on the horizon. \( \psi(x = x_h) \). The condition \([18]\) tells us that the value of the scalar field on the black hole horizon, \( \psi_h \), has to be fine-tuned (using the gauge coupling \( e \)) such that the scalar field decays exponentially according to \([12]\). This case has been studied in detail in \([11]\), so we refer the reader for more details to this paper. Here, we have cross-checked our numerics with that in the aforementioned paper and find perfect agreement.

We have chosen \( \beta^2 = 9/32 \) throughout this paper. As \([12]\) demonstrates, scalar clouds can only exist close to a RN black hole if \( e < \epsilon_{\text{max}} = x_h/Q \). Moreover, the coupling of the scalar field to the electromagnetic field needs to be sufficiently strong, i.e. \( e \geq \epsilon_{\text{min}} \neq 0 \). In fact, decreasing \( e \) from its maximal possible value \( x_h/Q \), the cloud becomes increasingly extended. We find numerically that it extends to arbitrarily large \( x \) for \( e \rightarrow \epsilon_{\text{min}} \). The reason for this is that the second local minimum of the effective potential \( U_{\text{eff}} \) located at \( \psi_{\text{min},2} = (1 + \sqrt{1 - 3\beta^2 \mu_{\text{eff}}^2})/(3\beta^2) \) becomes degenerate in its potential value with the local minimum \( \psi_{\text{min},1} = 0 \) for \( \mu_{\text{eff}} = 1/(2\beta) \). This imposes a constraint on how strong the scalar field can be coupled to the electromagnetic field of the black hole. Using \( \mu_{\text{eff}}^2 \approx 1 - e^2 Q^2/x_h^2 \) we find that

\[
\epsilon_{\text{min}} \approx \frac{x_h}{Q} \sqrt{1 - 1/(4\beta^2)}.
\]

The scalar field profiles of scalar clouds for some values of \( e \) between \( \epsilon_{\text{max}} \) and \( \epsilon_{\text{min}} \) are shown in Fig. 11 (left) for \( x_h = 0.15 \) and \( Q = 0.09 \). For this RN black holes, which has 60\% of its maximal possible charge, we get from the formulae above that \( \epsilon_{\text{max}} = 5/3 \approx 1.67 \) and \( \epsilon_{\text{min}} \approx 5/9 \approx 0.56 \). Note that our numerical results agree very well with these values, we find \( \epsilon_{\text{min}} \approx 0.6 \) numerically, see Fig. 11 (right), where we give \( \psi(x_h) = \psi_h \), \( \psi'_{|x=x_h} \) as well as \( \mu_{\text{eff}} \infty \) in function of \( e \). Obviously, both the scalar field and its derivative on the horizon tend to zero for \( e \rightarrow \epsilon_{\text{max}} \). That one follows from the other can, in fact, be understood when remembering the regularity condition on the horizon, see \([18]\). Approaching \( \epsilon_{\text{min}} \), \( \psi_h \) tends to the value of the local minimum of the “bare” potential \( U(\psi) \) (remember that \( \psi(x_h) = 0 \)), which for our choice of parameters is \( \psi \approx 1.286 \). This implies – again using \([18]\) – that \( \psi'_{|x=x_h} \) should approach zero. We find
that $\mu_{\text{eff},\infty}$ tends to zero at $e \to e_{\text{max}}$ and to $\approx 1/(2\beta) \approx 0.94$ at $e \to e_{\text{min}}$. As our reasoning above suggests, both $e_{\text{min}}$ and $e_{\text{max}}$ decrease for increasing charge $Q$ at fixed horizon radius. Close to the extremely charged RN solution with $Q \to x_h$ we find that $e_{\text{max}} \to 1$ and $e_{\text{min}} \to 1/3$, which again agrees very well with our results. In other words, two black holes with equal event horizon radius, but different charge differ in the required coupling strength $e$ to allow for the existence of $Q$-clouds. The larger the charge of the black hole, the smaller the gauge coupling can be chosen for $Q$-clouds to exist. This is related to the electromagnetic repulsion of the cloud which consists of scalar boson that are coupled to the electromagnetic field of the RN black hole. However, note that extremal RN black holes with $x_h = Q$ cannot have regular scalar clouds, as (18) clearly demonstrates. The mass $M_Q$ (see (15)) of the cloud becomes

$$M_Q = \frac{Q^2}{x_h} - 2 \int_{x_h}^{\infty} dx\ x^2\ U(\psi) ,$$

(20)

i.e. is determined by the properties of the black hole and the integral of the scalar potential energy, respectively. Also note that a constant, non-vanishing value of the scalar field leads to a decrease of $M_Q$ when the interval in $x$ on which $\psi$ is constant (and non-zero) increases.

### 3.2 Schwarzschild black holes

In this section, we consider the equations for the gauge and scalar field, respectively, in the background of a Schwarzschild black hole, i.e. we set $\alpha^2 = 0$ in (5) such that the gravitational field is not sourced by any matter field. The result is the Schwarzschild solution with $N(x) = 1 - 2M/x$, $\sigma \equiv 1$ with an event horizon at radius $x = x_h = 2M$. In contrast to the RN case, the gauge field and scalar field equations are now coupled. The appropriate boundary conditions for the equations (6) and (7) are:

$$\left| \frac{d\psi}{dx} \right|_{x = x_h} = \frac{x_h}{2} \frac{dU}{d\psi} |_{\psi = \psi_h} , \quad v(x_h) = 0 , \quad \psi(x \to \infty) = 0 , \quad v(x \to \infty) = v_\infty .$$

(21)

We have then fixed the strength of the attractive self-interaction, $\beta^2$, the value of the gauge field at infinity, $v_\infty$, and the radius of the black hole horizon, $x_h$, and studied the dependence of the solutions on the strength of the gauge coupling $e$, i.e. the coupling between the gauge field and the scalar field. Charged $Q$-clouds

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Figure 1: Left: We show the profile of the scalar field function $\psi(x)$ for scalar clouds on a RN black hole with horizon radius $x_h = 0.15$ and charge $Q = 0.09$ for different values of the gauge coupling $e$. Right: The values of the scalar field, $\psi$ and its derivative $\partial_x \psi \equiv \psi'$ on the horizon as well as $\mu_{\text{eff},\infty}$ in function of $e$ for scalar clouds on this same RN black hole.
around Schwarzschild black holes have been studied before in [12], however, we find that not one, but two solutions exist for one choice of parameters $\beta^2$, $v_\infty$, $x_h$ and $e$. These two solutions differ in the size of the scalar cloud and consequently also in their energy $M_Q$, the electric charge $Q$ and the value of the radial electric field $E_x = -v'$ on the horizon. This is shown in Fig. 2 for $x_h = 0.15$, $\beta^2 = 9/32$ and two different values of $v_\infty$. Both branches exist on a finite interval of the gauge coupling $e$ with maximal coupling strength given by $1 - e^2_{\text{max}}v^2_\infty = 0$ (see [12]). In agreement with this reasoning, we find $e_{\text{max}} = 0.1$ (resp. $e_{\text{max}} = 0.2$) for $v_\infty = 10$ ($v_\infty = 5$). At the minimal value of $e$, $e_{\text{min}}$, the two branches of solutions join. The value of $e_{\text{min}}$ can only be determined numerically and we find that $e_{\text{min}} \approx 0.061$ (resp. $e_{\text{min}} \approx 0.183$) for $v_\infty = 10$ ($v_\infty = 5$). To state it differently : the larger $v_\infty$, the smaller we have to choose the value of the gauge coupling in order to allow for charged $Q$-clouds to exist. On the other hand, the larger $v_\infty$, the larger the interval $e_{\text{max}} - e_{\text{min}}$ on which charged $Q$-clouds exist. For $v_\infty \approx 4$, this interval shrinks to zero, i.e. scalar clouds on Schwarzschild black holes do no longer exist if the potential difference between the horizon and infinity is too small. At fixed gauge coupling $e$, the clouds on the second branch have higher mass $M_Q$ and higher electric charge $Q$ than those of the first, already known branch. Moreover, the radial electric field on the horizon, $E_x(x_h)$, is smaller in absolute value. It is also interesting to consider the ratio between the total electric charge of the $Q_N$ individual bosons making up the cloud, $eQ_N$, and the total electric charge $Q$ of the solution. This ratio is shown in dependence of the mass $M_Q$ in Fig. 2 (right). We find for both values of $v_\infty$ studied that the increase of the ratio leads to an increase of the mass $M_Q$. Remembering that the total electric charge $Q$ is a sum of the horizon electric charge and the charge in the $Q_N$ bosons in the cloud, we conclude that transfer of electric charge from the horizon to the scalar cloud increases the mass of the solution. Moreover, there are limits (depending on $v_\infty$) how much transfer can be done. For $v_\infty = 5$, we need at least $\approx 75\%$ of the electric charge $Q$ to be in the cloud and are allowed maximally $\approx 96\%$. For $v_\infty = 10$, the lower bound is $\approx 32\%$, while the upper bound is $\approx 99\%$, i.e. very close to the limit, where all electric charge is in the cloud. Remember that only globally regular solutions, so-called boson stars, have $eQ_N = Q$.

![Figure 2](image1.png)

**Figure 2:** *Left:* We show the dependence of the electric charge $Q$, the radial electric field at the horizon $E_x(x_h) = -v'(x_h)$ as well as the mass $M_Q$ of charged $Q$-clouds around Schwarzschild black holes in dependence of the gauge coupling $e$ for two values of $v_\infty$. *Right:* We show the ratio of $eQ_N$, i.e. the total charge of the $Q_N$ individual bosonic particles making up the cloud, and the total electric charge $Q$ as function of the mass $M_Q$ for two values of $v_\infty$.

How the transfer of charge to the cloud is possible becomes clear when considering the two solutions available at the same values of the couplings. We have plotted the profiles of the radial electric field $E_x$ as well as the scalar field $\psi$ of the two solutions available at $v_\infty = 10$ and $e = 0.08$ in Fig[3]. The solution on the second branch has a practically constant, non-vanishing scalar field outside the horizon, i.e. is non-vanishing.
for a much larger interval of the radial coordinate than the corresponding solution on the first branch. Hence, the extend of the scalar cloud is larger for solutions on the second branch. Note also that very similar to what we have observed for clouds on RN black holes the value of the scalar field on the horizon approaches the local minimum of the “bare” potential $\psi_{\text{min},2} \approx 1.286$. This value extends to increasingly larger values of $x$ along the branches and shows a sudden drop to the global minimum $\psi_{\text{min},1} = 0$. In other words: the scalar clouds possess an interior (containing the black hole), in which the scalar field is trapped in the “false vacuum” of the potential $\psi_{\text{min},2}$. The explanation of the existence of the second branch and the behaviour of the scalar field becomes clear when remembering that $e = g/\sqrt{\lambda}$. On the first branch of solutions, varying $e$ can be interpreted as varying the gauge coupling $g$ and fixing $\lambda$. This can be done down to a minimal value of $e$, as discussed above. Now, increasing $e$ from this minimal value on the second branch can be interpreted as keeping $g$ fixed and decreasing $\lambda$.

![Diagram](image_url)

Figure 3: We compare the scalar ($\psi$) and radial electric field ($E_x$) profiles of the two $Q$-cloud solutions in a Schwarzschild space-time with $x_h = 0.15$ for $v_\infty = 10$ and $e = 0.08$.

The fact that two different solutions exist for the same choice of coupling constants can also be observed when fixing $e$ and varying $v_\infty$. This is shown in Fig. 4 for $e = 0.08$ and $x_h = 0.15$, where we give the mass $M_Q$ and the Noether charge $Q_N$ in dependence of $ev_\infty$. Again, both branches extend all the way back to $ev_\infty = 1$, where the effective mass of the scalar field $1 - e^2v_\infty^2$ becomes zero. Interestingly, we find that the solutions on the first branch have $M_Q > Q_N$, while the ones on the second branch have $M_Q < Q_N$. In fact, we can approximately calculate the value of $M_Q$ for the solutions on the second branch with pronounced “hard wall”. First, assume that $\psi \equiv \psi_0$ for $x \in [x_h : \hat{x}]$ and $\psi \equiv 0$ for $x \in [\hat{x} : \infty[$. Inserting this into the gauge field equation (\ref{eq:6}), this gives

$$v(x) \sim \begin{cases} 
  v_\infty - \frac{Q}{\hat{x}} \exp(-\sqrt{2}e\psi_0 x) & \text{for } x \in [x_h : \hat{x}] \\
  v_\infty - \frac{Q}{\hat{x}} & \text{for } x \in [\hat{x} : \infty[.
\end{cases}$$

(22)

For $x \in [\hat{x} : \infty[$, this is the standard electric potential of a point charge, for $x \in [x_h : \hat{x}]$, this is the electric...
potential of a screened point charge. Now using the form of \( \psi \) and \( v \) and inserting this into (15), we find that

\[
M_Q = 2Qv_\infty - \frac{Q}{x} - \frac{Q^2e\psi_0}{\sqrt{2}} \left[ \exp(-2\sqrt{2}e\psi_0x_h) - \exp(-2\sqrt{2}e\psi_0\tilde{x}) \right]
- Q^2 \left[ \frac{\exp(-2\sqrt{2}e\psi_0x_h)}{x_h} - \frac{\exp(-2\sqrt{2}e\psi_0\tilde{x})}{\tilde{x}} \right] - \frac{2}{3}U(\psi_0) (\tilde{x}^3 - x_h^3)
\]

As a final remark, let us state that all charged \( Q \)-clouds on Schwarzschild black holes possess charge larger than their charged black hole counterparts with the same horizon radius. We find for all solutions that \( Q > x_h \) or equivalently \( v_\infty > v_\infty,RN = Q/x_h \).

4 Back-reaction of \( Q \)-clouds on space-time

The question is then how these scalar clouds, once formed, back-react on the space-time. This is what we will discuss here. We have solved the system of coupled differential equations (5), (6) and (7) subject to the appropriate boundary conditions. Considering the scalar cloud in the background of a Schwarzschild black hole, we have then increased the back-reaction of this cloud onto the space-time by increasing the parameter \( \alpha \) from zero. As mentioned above, two solutions exist for the same values of the parameters and our results indicate that, when increasing the back-reaction, these solutions behave very differently in the strong gravity regime. Increasing the gravitational back-reaction for the scalar cloud with lower mass at fixed gauge coupling, we find that the temperature \( T_H \) of the black hole decreases and approaches zero at a critical value of \( \alpha \), \( \alpha_{1,cr} \). This suggests that the limiting solution is an extremal black hole. Indeed, we observe that the metric function \( N(x) \) develops a double zero at \( x = x_h \), while \( \sigma(x_h) \) stays perfectly finite. This can be seen in Fig. 5 (left), where we plot the metric functions \( N(x) \), \( \sigma(x) \), the radial electric field \( E_x(x) \) as well as the scalar field \( \psi(x) \) for \( x_h = 0.15 \), \( e = 0.08 \), \( v_\infty = 10 \) and the value of \( \alpha \) close to
the maximal possible value, $\alpha_{1,cr}^2 \approx 0.0095$. Note that this extremal black hole, however, does not possess regular scalar hair as the derivative of the scalar field, $\psi'$, diverges on the extremal horizon. That this should be so (and is confirmed by our numerics) is obvious when remembering the boundary conditions (8). On the other hand, when increasing $\alpha$ for a solution on the second branch, the metric function $N(x)$ develops a local minimum at $x \approx \tilde{x} > x_h$ which corresponds to the value of $x$ where $\psi$ drops from its roughly constant and non-vanishing value to zero. At the maximal possible back-reaction $\alpha = \alpha_{cr,2}$ the value of this minimum drops to zero and the solution forms a second horizon at $x_{h,2} \approx \tilde{x}$. For $x \in [x_{h,2} : \infty]$ the solution possesses no scalar field, $\psi \equiv 0$, and the space-time corresponds to an extremal Reissner-Nordström solution. This is shown in Fig. 5 (right). In the region $x \in [x_h, x_{h,2}]$, the solution possesses a constant, but non-vanishing scalar field $\psi \equiv \psi_0$. Hence, the scalar potential sources the space-time in a non-trivial way. Using the fact that our numerics indicates $\sigma \equiv \sigma_0 \neq 1$ for $x \in [x_h : \tilde{x}]$ we can combine the gravity equations (5) to find

$$N(x) = 1 - \frac{2M}{x} - \frac{\alpha^2}{x\sigma_0^2} \int dx (xv')^2 - \frac{2\alpha^2}{3} U(\psi_0)x^2 \quad \text{for} \quad x \in [x_h : \tilde{x}].$$

(23)

Using the gauge field $v$ for a screened charge $Q$ (see (22)) the integration leads to:

$$N(x) = 1 - \frac{2M}{x} + \left[ \frac{\alpha^2 Q^2}{\sigma_0^2 x^2} + \frac{\sqrt{2\alpha^2 Q^2\epsilon_0}}{2x} \right] \exp \left( -2\sqrt{2}v\psi_0 x \right) - \frac{2\alpha^2}{3} U(\psi_0)x^2 \quad \text{for} \quad x \in [x_h : \tilde{x}].$$

(24)

which becomes the RN solution for $\psi_0 = 0$, $\sigma_0 = 1$. The metric function describes a space-time with mass $M$ and a screened electric charge $Q$ as well as a cosmological constant term with $\Lambda = 2\alpha^2 U(\psi_0)$. The space-time is hence a RN-de Sitter (RNdS) solution with screened electric charge. For large $x$ this space-time has a cosmological horizon at $x_\Lambda = 1/(\alpha\sqrt{2U(\psi_0)})$. For the solution presented in Fig. 5 (right), we find that $\alpha_{cr,2} \approx 0.0124$, and $\psi_0 \approx 1.28$ such that $x_\Lambda \approx 14.5$. Of course, this does not agree with the result we find exactly, since also the mass term and the screened charge have to be taken into account when computing the horizon, but it gives a good approximation to our result. So, the extremal horizon for the exterior solution corresponds to a cosmological horizon for the interior solution. In other words, the limiting solution is an inflating black hole with carries a screened charge and looks like an extremal Reissner-Nordström black hole from the outside.

5 Conclusions

In this paper, we have discussed scalar clouds on static, spherically symmetric black holes as well as the back-reaction of these clouds onto the space-time. Studies of these solutions have been done before in [11, 12, 13], however, the strong gravity limit had not been discussed previously. Investigating the charged clouds on a fixed Schwarzschild background, we find a new branch of solutions that had not been noticed previously. In contrast to scalar clouds on RN black holes, where the cloud disperses to spatial infinity at minimal coupling to the electric field of the black hole, the extend of charged scalar clouds on Schwarzschild black holes is limited and forms a “hard wall” on which the energy density associated to the scalar field abruptly drops to zero. We also find that for these type of solutions we can transfer practically all electric charge to the scalar cloud when choosing the potential difference between the horizon and infinity large enough. When investigating the back-reaction of these clouds onto the space-time, we observe that at the “hard wall” a horizon forms. The scalar field has a value close to the second minimum of the scalar potential, i.e. is trapped inside the “false vacuum” of the model. This generates a huge amount of scalar potential energy that back-reacts onto the space-time in the form of a positive cosmological constant. For strong enough back-reaction, this leads to the formation of an extremal black hole. This phenomenon is reminiscent to that of an inflating monopole in the context of topological inflation [14, 15], where the inside of the monopole core is trapped inside the false vacuum of the potential. The energy associated to this false vacuum leads to the collapse of the monopole to a magnetically charged RN solutions at sufficiently strong back-reaction.
Figure 5: Left: We shown the profiles of the metric functions $N$, $\sigma$, the radial electric field $E_x$ and the scalar field $\psi$ for a scalar cloud on the first branch of solutions for $x_h = 0.15$, $v_\infty = 10$, $e = 0.08$ and at maximal possible back-reaction $\alpha^2_{\text{cr},1} \approx 0.0095$. Right: Same as left, but for a scalar cloud on the second branch of solutions close to the maximal possible back-reaction $\alpha^2_{\text{cr},2} \approx 0.0124$.

Although we have discussed a scalar field theory that possesses an unbroken U(1) symmetry (in contrast to the monopole case), the scalar field gets trapped inside the “false vacuum” of the potential and leads to an inflating black hole solution. It would certainly be interesting to see whether such a phenomenon could also be observed in the case of rotating black holes and whether our results have any relevance to the theory of inflation.

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