Research Article

Optimal Form for Compliance of Membrane Boundary Shift in Nonlinear Case

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Abstract

In this work, we search the existence shifting compliance optimal form of some boundary membrane, which is not elastic and not isotropic, generating nonlinear PDE. An optimal form of the elastic membrane described by the $p$-Laplacian is investigated. The boundary perturbation method due to Hadamard is applied in Sobolev spaces.

1. Introduction and Preliminaries

In this work we will study the geometric shape optimization of forms, where the main idea is to vary the edge position of a form, without changing its topology which remains the same. We use a membrane model as shown in Figure 1. At rest the membrane occupies a reference domain $\Omega$ whose edge is divided into three disjoint parts:

$$\partial \Omega = \Gamma \cup \Gamma_N \cup \Gamma_D$$

where $\Gamma$ is the free variable part, $\Gamma_D$ is the fixed part of the boundary (Dirichlet boundary conditions), and $\Gamma_N$ is also the free part of the boundary on which we apply the efforts $g \in L^p(\Gamma_N)^N$ (Neumann boundary condition). The three parts of the boundary are supposed to be nonzero surface measurements, as we suppose that the free boundary variable $\Gamma$ responds to homogenous Neumann condition. So the vertical displacement $u$ is the solution of the following membrane model:

$$-\Delta u = 0 \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \Gamma_D$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \Gamma_N$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma$$

We want to minimize the compliance defined by $J(\Omega) = \int_{\Gamma_N} g u \, dx$ whenever $x \in \Omega$.

The shape optimization problem is $\inf_{\Omega \in U} J(\Omega)$ where it remains to define the set of admissible forms.

1.1. Existence under a Condition of Regularity. The main idea of this section is to apply a regularity constraint on all the admissible forms $U_{ad}$, to demonstrate a result of existence of optimal forms. The results and demonstrations are mainly due to F. Murat and J. Simons [1, 2]. It rests on a very significant restriction of $U_{ad}$; in other words, $\Omega$ is obtained by applying a regular diffeomorphism $T$ to the reference domain $\Omega_0$. We first define a diffeomorphism set:

$$\tau = \{T \text{ such that } (T - I_d) \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N); (T^{-1} - I_d) \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)\}$$

Then we define a set of the admissible forms obtained by deformation of $\Omega_0$:

$$C(\Omega_0) = \{ \Omega \text{ such that } \exists T \in \tau; \Omega = T(\Omega_0) \}$$

Finally we introduce a pseudo-distance on $C(\Omega_0)$:

$$\delta(\Omega_1, \Omega_2) = \inf_{T \in \tau; T(\Omega_1) = \Omega_2} \left( \| (T - I_d) \|_{W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)} + \| (T^{-1} - I_d) \|_{W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)} \right)$$
Lipschitz diffeomorphism and 
Definition 3. Let \( \Omega \) be a regular bounded open set of \( \mathbb{R}^N \). For all \( \Omega_0 \), \( \Omega \) open sets close to \( \Omega_0 \) in the sense of pseudo-distance; for each \( R > 0 \) we pose \( \Psi_{ad} = \{ \Omega \in C(\Omega_0) \text{ such that } d(\Omega, \Omega_0) \leq R; \Gamma_D \cup \Gamma_N \subset \partial \Omega \text{ and } \int_\partial \omega = V_0 \} \) where \( V_0 \) is an imposed volume. The result is the following theorem.

Theorem 1. For all objective functions, the shape optimization problem \( \inf_{\Omega \in \Psi_{ad}} J(\Omega) \) admits at least a minimum point.

1.2. Derivation from the Domain. The boundary variation method that we study is a classical idea well known and used before by Hadamard [3] in 1907 and many others as [4–12]. We will adopt the same representation as F. Murat and J. Simons [1]. In fact, let \( \Omega_0 \) be an open regular bounded referential domain of \( \mathbb{R}^N \) and the admissible form class \( C(\Omega_0) \) composed of the open sets such as \( \Omega = T(\Omega_0) \) where \( T \) is a Lipschitz diffeomorphism and

\[
T = Id + \theta
\]

with \( \theta \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N) \)

where \( Id \) is the identity application, and we note \( \Omega = (Id + \theta)(\Omega_0) \) defined by

\[
\Omega = \{ x + \theta(x) \text{ such that } x \in \Omega_0 \}
\]

Lemma 2 (See [13]). For all \( \theta \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N) \) satisfying \( \|\theta\|_{W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)} < 1 \), the function \( T = Id + \theta \) is a bijection of \( \mathbb{R}^N \) and \( T \in \tau \).

Definition 3. Let \( f \) be application from \( C(\Omega_0) \) to \( \mathbb{R} \). One says that it is differentiable with respect to the domain at \( \Omega_0 \) if the function

\[
\Theta \longrightarrow J((Id + \theta)(\Omega_0))
\]

is Frechet differentiable at 0 in the Banach space \( W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N) \). I.e., \( \exists \), a linear continuous form on \( W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N) \), such that

\[
J((Id + \theta)(\Omega_0)) = J(\Omega_0) + J'(\Omega_0) + o(\theta),
\]

with \( \lim_{\theta \to 0} \frac{o(\theta)}{\|\theta\|_{W^{1,\infty}}} = 0. \)

The linear form \( J'(\Omega_0) \) depends only on the normal component of \( \theta \) on the boundary of \( \Omega_0 \).

Proposition 4. Let \( \Omega_0 \) be a regular bounded open set of \( \mathbb{R}^N \). Let \( f \) be a differentiable application on \( \Omega_0 \). If \( \theta_1, \theta_2 \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N) \) are such that \( \theta_1 \rightarrow \theta_2 \in C^1(\mathbb{R}^N, \mathbb{R}^N) \) such that \( \theta_1, n = \theta_2, n \text{ on } \partial \Omega_0 \), then the derivative \( J'(\Omega_0) \) is verifying:

\[
J'(\Omega_0)(\theta_1) = J'(\Omega_0)(\theta_2)
\]

1.3. Derivation of Integrals. Since the compliance \( J \) is defined by surface or volume integrals then its differentiation devotes the following tools.

Lemma 5 (see [1, 7]). Let \( \Omega_0 \) be an open set of \( \mathbb{R}^N \). Let \( T \in \tau \) and \( 1 \leq p \leq +\infty \). Then \( f \in L^p(T(\Omega_0)) \) iff \( f o T \in L^p(\Omega_0) \) and one has

\[
\int_{T(\Omega_0)} f dx = \int_{\Omega_0} f o T |\det V T| dx
\]

and

\[
\int_{T(\Omega_0)} f |\det(V T)|^{-1} dx = \int_{\Omega_0} f o T dx
\]

On the other hand \( f \in W^{1,p}(T(\Omega_0)) \) iff \( f o T \in W^{1,p}(T(\Omega_0)) \) and one has

\[
(\nabla f) o T = (\nabla(T^{-1})) (\nabla f o T)
\]

Proposition 6 (See [13]). Let \( \Omega_0 \) be a regular bounded open set of \( \mathbb{R}^N \). Let \( f \in W^{1,1}(\mathbb{R}^N) \) and \( \Theta \) be an application from \( C(\Omega_0) \) to \( \mathbb{R} \) defined by \( J(\Omega) = \int_{\Omega} f(x) dx \). Then \( J' \) is differentiable in \( \Omega_0 \) and

\[
J'(\Omega_0)(\theta) = \int_{\Omega_0} \text{div} (\theta(x) f(x)) dx
\]

with \( \theta \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N) \)

Now we move to a lemma on the change of variables in surfaces integrals.

Lemma 7 (See [1, 7]). Let \( \Omega_0 \) be an open set of \( \mathbb{R}^N \). Let \( T \in \tau \) be a class \( C^2 \) diffeomorphism of \( \mathbb{R}^N \), and \( 1 \leq p \leq +\infty \). Then \( f \in L^p(T(\Omega_0)) \) iff \( f o T \in L^p(\Omega_0) \) and one has

\[
\int_{\partial T \Omega_0} f o T |\det(V T)|^{-1} (\nabla(T^{-1})) \cdot n dS = \int_{\partial \Omega_0} f o T |\det(V T)|^{-1} (\nabla(T^{-1})) \cdot n dS
\]
The surface integral derivative of a function with respect to the domain is given by the following proposition.

**Proposition 8** (See [13]). Let \( \Omega_0 \) be a regular bounded open set of \( \mathbb{R}^N \). Let \( f \in W^{2,1}(\mathbb{R}^N) \) and \( J \) be an application from \( C(\Omega_0) \) to \( \mathbb{R} \) defined by \( J(\Omega) = \int_{\Omega} f(x)dx \). Then \( J \) is differentiable in \( \Omega_0 \) and

\[
J'(\Omega_0)(\theta) = \int_{\Omega_0} \left( \nabla f \cdot \nabla \theta + f(\nabla \theta) \right) d\Omega
\]

with \( \theta \in C^3(\mathbb{R}^N, \mathbb{R}^N) \)

where \( H \) is the average curvature of \( \partial \Omega_0 \) defined by \( H = \text{div}n \).

1.4. Derivation of a Domain Dependent Function. In this section we try to derive a function depending on the domain; for this we use the Eulerian \( U \) or Lagrangian \( L \) derivative. The second is a more reliable concept than the first. Let \( u(x, \Omega) \) be a function defined for all \( x \in \Omega \) and depending on \( \Omega \). It represents a solution of an PDE posed in \( \Omega \). In a point \( x \) belonging to both \( \Omega \) and \( \Omega = (I + \theta)(\Omega_0) \), we can calculate the differential \( u(\Omega, x) \):

\[
u((I + \theta)(\Omega_0), x) = u(\Omega_0, x) + U(\theta, x) + o(\theta)
\]

such that

\[
\lim_{\theta \to 0} \frac{\|o(\theta)\|}{\|\theta\|} = 0 \tag{16}
\]

\( U \) is a linear continuous form in \( \theta \); it represents a directional derivative in the direction \( \theta \). This definition makes sense in the case where \( x \in \Omega \), but it poses a problem if \( x \in \Omega_0 \). Then in this case we use the Lagrangian derivative; for this we build the transported \( \Omega(\theta) \) on \( \Omega_0 \).

By changing variables we obtain \( \Omega(\theta, x) = \Omega((1 + \theta)(\Omega_0), o(I + \theta)) = u((1 + \theta)(\Omega_0), x + \theta(x)) \).

To arrive at the derivative Lagrangian by drifting \( \Omega(\theta, x) \) with respect to \( \theta \):

\[
\overline{\Omega}(\theta, x) = u(\theta, x) + L(\theta, x) + o(\theta)
\]

such that

\[
\lim_{\theta \to 0} \frac{\|o(\theta)\|}{\|\theta\|} = 0 \tag{17}
\]

\( L \) is a linear continuous form in \( \theta \); it represents a directional derivative in the direction \( \theta \).

There is a relation between these two derivatives \( L(\theta, x) = U(\theta, x) + \theta \cdot \nabla u(\Omega_0, x) \).

**Proposition 9.** Let \( \Omega_0 \) be a regular bounded open set of \( \mathbb{R}^N \).

Let \( J(\Omega) \) be an application from \( C(\Omega_0) \) to \( L^1(\mathbb{R}) \); one defines its transpose from \( W^{1,1}(\mathbb{R}^N) \) to \( L^1(\mathbb{R}) \) \( \overline{J}(\theta, x) = J-f((1 + \theta)(\Omega_0), o(I + \theta)) \) which we suppose to be derivable in \( 0 \) and \( L \) is considered as its derivative. So the application \( J \) from \( C(\Omega_0) \) to \( L \) defined by \( J(\Omega) = \int_{\Omega} f(\Omega)dx \) is differentiable in \( \Omega_0 \) and for all \( \theta \in W^{1,1}(\mathbb{R}^N, \mathbb{R}^N) \) one has

\[
J'_1(\Omega)(\theta) = \int_{\Omega} \left( f(\Omega) \cdot \nabla \theta + L(\theta) \right) d\Omega \tag{18}
\]

In the same way, if \( \overline{J}(\theta) \) is derivable as an application from \( \mathbb{C}^1(\mathbb{R}^N, \mathbb{R}^N) \) to \( L^1(\mathbb{R}) \), so the application \( J \) from \( C(\Omega_0) \) to \( \mathbb{R} \) defined by \( J(\Omega) = \int_{\Omega} f(\Omega)dx \) is differentiable in \( \Omega_0 \) and for all \( \theta \in \mathbb{C}^1(\mathbb{R}^N, \mathbb{R}^N) \) one has

\[
J'_1(\Omega_0)(\theta) = \int_{\Omega} (f(\Omega) \cdot \nabla \theta - \nabla \cdot Y(\theta)) d\Omega \tag{19}
\]

2. Deriving an Equation with respect to the Domain

2.1. Dirichlet Conditions. We consider the following equation with Dirichlet boundary conditions:

\[
-\Delta_p u = f \quad \text{in} \ \Omega \tag{20}
\]

\[
u = 0 \quad \text{on} \ \partial \Omega
\]

With \( \Omega_0 \) a regular bounded open set in \( \mathbb{R}^N \), \( f \in W^{1,1}(\mathbb{R}^N) \) and \( \Delta_p u = div(|\nabla u|^{p-2} \nabla u) \) with \( 1 < p < \infty \).

Equation (20) admits a unique solution in \( W^{1}_{0} \).

Remark. For \( p = 2 \) we obtain the linear operator "Laplacian".

The variational formulation of problem (20) is as follows:

\[
\int_{\Omega} -\Delta_p u.v = \int_{\Omega} f.v = \int_{\Omega} \nabla F + f(\nabla F) \quad \text{it implies that}
\]

\[
f \cdot \nabla F = \text{div} (f \nabla F - \nabla F) \]

So \( \int_{\Omega} -\Delta_p u.v = \int_{\Omega} -\text{div} (|\nabla u|^{p-2} \nabla u) . v \)

\[
= \int_{\Omega} -\text{div} (v. |\nabla u|^{p-2} \nabla u)
\]

\[
+ \int_{\Omega} \nabla u. \nabla |\nabla u|^{p-2} \nabla u
\]

Using the Green formula we obtain

\[
= \int_{\Omega} v. |\nabla u|^{p-2} \nabla u. n d\Omega + \int_{\Omega} \nabla |\nabla u|^{p-2} \nabla u v d\Omega \tag{22}
\]

but \( v \in W^{1}_{0} (\Omega) \); it implies that \( v = 0 \) on \( \partial \Omega \). So the first term equals zero. Then

\[
= \int_{\Omega} |\nabla u|^{p-2} \nabla u v \forall \Omega \in W^{1,1}_{0} (\Omega) \tag{23}
\]

**Proposition 10.** Let \( \Omega = (I + \theta)(\Omega_0) ; u(\Omega) \) is the solution of the problem (20). We define its transported on \( \Omega_0 \) by

\[
\overline{u}(\theta) = u((I + \theta)(\Omega_0) \circ (I + \theta)) \in W^{1,1}_{0} (\Omega_0) \tag{24}
\]

Then the application \( \theta \rightarrow \overline{u}(\theta) \) from \( W^{1,1}_{0}(\mathbb{R}^N, \mathbb{R}^N) \) to \( W^{1,1}_{0}(\Omega_0) \) is derivable in \( 0 \) and its directional derivative
called Lagrangian derivative $L = \langle \textbf{u}(0), \theta \rangle$ is the unique solution of
\[ -\Delta_p L - \text{div} \left[ \alpha \nabla L (\theta, x) \right] \]
\[ - \text{div} \left( (\nabla L (\theta, x))^p \nabla \text{det} (\nabla \theta) \right) = \text{div} \left( f - \theta \right) \quad \text{in} \quad \Omega \]
\[ L = 0 \quad \text{on} \quad \partial \Omega \]
with $g' = -\text{div} \left( (\lambda_{x, \theta} + \alpha \phi_{\theta}) \nabla \text{det} (\nabla \theta) \right)$.

**Proof.** We consider a test function $w = \nu (I + \theta)$ in $W_0^{1,p} (\Omega) \implies $ \[ \nu = w (I + \theta)^{-1} \in W_0^{1,p} (\Omega). \]
Let $U \in W_0^{1,p} (\Omega)$ such that $u$ is a solution of problem (20) satisfying $u = U \circ (I + \theta)^{-1}$. We remark that $w$ and $U$ are independent of $\theta$. By a change of variable $x = T(y)$ and the Lemma 5, (23) becomes
\[ (23) \iff \int_{\Omega_0} |U|^{p-2} u \nabla u \nabla x \iff \int_{\Omega_0} f \nabla x \]
\[ \int_{\Omega_0} |U|^{p-2} |w| \nabla u \nabla d y \]
\[ = \int_{\Omega_0} f \nabla x \quad \text{and} \quad \text{det} \nabla \theta \]
\[ = |U|^{p-2} |w| \nabla u \nabla \theta \nabla d y \]
\[ \text{with} \quad \text{det} \nabla \theta = \sqrt{\lambda} - \theta I - \nabla \theta \cdot (\nabla \theta)^T \]
\[ \text{and} \quad \text{det} \nabla \theta = \sqrt{\lambda} - \theta I - \nabla \theta \cdot (\nabla \theta)^T \]
\[ A (\theta, \text{det} \nabla \theta) = |\text{det} (I + \nabla \theta)| \left| \left( (I + \nabla \theta)^{-1} \right)^T \nabla (\text{det} \nabla \theta) \right|^{p-2} \]
\[ \cdot (I + \nabla \theta)^{-1} \left( (I + \nabla \theta)^{-1} \right)^T \nabla \text{det} \nabla \theta \]
Then (23)$\iff \int_{\Omega_0} A (\theta, \text{det} \nabla \theta) \nabla u \text{det} \nabla y = \int_{\Omega_0} f \circ T \cdot |w| \text{det} \nabla \theta d y; \]
\[ \text{then we drift with respect to} \theta \text{in 0.} \]
\[ \text{On the other hand the application} \theta \rightarrow A (\theta, \text{det} \nabla \theta) \text{from} \]
\[ W^{1,\infty} (\mathbb{R}^N, \mathbb{R}^N) \rightarrow \text{to} \]
\[ L^{\infty} (\mathbb{R}^N, \mathbb{R}^N) \text{is differentiable in 0.} \]
\[ \text{In fact} \quad \text{det} (I + \nabla \theta) = 1 + \text{div} (\theta) + o (\theta) \quad \text{with} \] \[ \lim_{\theta \rightarrow 0} o (\theta) = 0. \quad \text{Therefore} \quad \text{det} (I + \nabla \theta) = 1 + \text{div} (\theta) + o (\theta) \quad \text{because} \theta \text{is small enough.} \]
\[ \text{We have} \quad (I + \nabla \theta)^{-1} = I - \nabla \theta + o (\theta) \]
\[ \text{and} \quad (I + \nabla \theta)^{-1} = I - (\nabla \theta)^T + o (\theta) \quad \text{(28)} \]
\[ \text{And} \quad \nabla \theta (x, \theta) = \nabla \theta (0, x) + \nabla L (\theta, x) + o (\theta) \implies \]
\[ \nabla \theta (x, \theta) = \nabla \theta (0, x) + \nabla L (\theta, x) + o (\theta) \]
\[ \text{By using [14] we find} \]
\[ \left| (1 - (\nabla \theta)^T)^T \nabla \theta^{-1} \right|^{p-2} = \left| (1 - (\nabla \theta)^T + o (\theta)) \right|^{p-2} \]
\[ - (\nabla \theta)^T \nabla \theta (0, x) + \nabla L (\theta, x) + o (\theta) \]
\[ = |\text{det} (I + \nabla \theta)| (I + \nabla \theta)^{-1} \text{div} (\theta) \]
\[ \text{On the other hand} \quad |\text{det} (I + \nabla \theta)| (I + \nabla \theta)^{-1} \text{div} (\theta) \]
\[ = (1 + \text{div} (\theta)) I - \nabla \theta - (\nabla \theta)^T + o (\theta). \]
\[ \text{Thus} \]
\[ A (\theta, \text{det} \nabla \theta) = B (\theta, \text{det} \nabla \theta) \left[ (1 + \text{div} (\theta)) I - \nabla \theta - (\nabla \theta)^T + o (\theta) \right] \]
\[ + o (\theta) \left[ |\text{det} \nabla \theta (0, x) + \nabla L (\theta, x) + o (\theta)| \right] = B (\theta, \text{det} \nabla \theta) \]
\[ \cdot |\text{det} \nabla \theta (0, x) + \nabla L (\theta, x) + o (\theta)| \]
\[ + (\text{div} \theta I - \nabla \theta - (\nabla \theta)^T) \nabla \theta (0, x) + \nabla L (\theta, x) \]
\[ + o (\theta) \right) = B (\theta, \text{det} \nabla \theta) \left[ |\text{det} \nabla \theta (0, x) + \nabla L (\theta, x) + o (\theta)| \right] \]
\[ = |\text{det} \nabla \theta (0, x) + \nabla L (\theta, x) + o (\theta)| \]
\[ + (\text{div} \theta I - \nabla \theta - (\nabla \theta)^T) \nabla \theta (0, x) + \nabla L (\theta, x) \]
\[ = \left| (\nabla \theta)^T \nabla \theta (0, x) \right|^{p-2} \left| (\nabla \theta)^T \nabla \theta (0, x) \right|^{p-2} \]
\begin{align*}
\cdot \nabla \left( u(0, x) + |\nabla L(\theta, x)|^{-2} \nabla u(0, x) \right) \\
+ |\nabla L(\theta, x)|^{-2} \nabla L(\theta, x) + |\nabla L(\theta, x)|^{-2} \left( \text{div}(\theta) I - \nabla \theta - (\nabla \theta)^I \right) \\
- (\left( A(\theta, x) \nabla L(\theta, x) \right) - |\nabla L(\theta, x)|^{-2} \nabla L(\theta, x)) \right) \nabla \omega + \circ(\theta) = |\nabla \theta(0, x)|^{-2}
\end{align*}

Therefore, we have

\begin{align*}
\int_{\Omega} A(\theta, x) \nabla \omega - \int_{\Omega} |\nabla \theta(0, x)|^{-2} \nabla \omega - \nabla \omega + \left( |\nabla \theta(0, x)|^{-2} \nabla L(\theta, x) \right) + |\nabla L(\theta, x)|^{-2} \left( I + \text{div}(\theta) I - \nabla \theta - (\nabla \theta)^I \right) \\
- |\nabla \theta(0, x)|^{-2} \left( I + \text{div}(\theta) I - \nabla \theta - (\nabla \theta)^I \right) \nabla L(\theta, x) + |\nabla \theta(0, x)|^{-2} \left( \text{div}(\theta) I - \nabla \theta - (\nabla \theta)^I \right) \nabla L(\theta, x)
\end{align*}

(30)

Thus \( \forall \omega \in W_{0\Omega}^{1,p}(\Omega) \) the Lagrangian of \( \theta \) is a solution of the following differential equation:

\begin{align*}
-\Delta_p L - \text{div} \left( |\rho_0 \nabla \theta(\theta, x)|^{-2} \left( I + \rho_0 \nabla \theta(\theta, x) \right) \right) + \text{div}(f, \theta) \omega = g \quad \text{in } \Omega \\
L = 0 \quad \text{on } \partial\Omega
\end{align*}

(31)

Remark II. When \( p > 2 \) we will have:

(i) \( \lambda_{(x, \theta)} \nabla \theta(\theta, x) = o(\theta) \)

(ii) \( |L(\theta, x)|^{-2} \rho_0 = o(\theta) \)

(iii) \( \lambda_{(x, \theta)} \rho_0 = o(\theta) \)

Then \( L \), the Lagrangian of \( \theta \), will be solution of the following reduced differential equation:

\begin{align*}
-\Delta_p L - \text{div} \left[ |\alpha_x \nabla \theta(\theta, x)|^{-2} \left( I + \rho_0 \nabla \theta(\theta, x) \right) \right] + \text{div}(f, \theta) \omega = g' + \text{div}(f, \theta) \\
L = 0 \quad \text{on } \partial\Omega
\end{align*}

(32)


Then afterwords we put

\begin{align*}
\rho_0 & = \text{div}(\theta) - \nabla \theta - (\nabla \theta)^I \\
\lambda_{(x, \theta)} & = |\nabla \theta(0, x)|^{-2} \\
\alpha_x & = |\nabla \theta(0, x)|^{-2}
\end{align*}

And \( g = -\text{div} \left[ (\lambda_{(x, \theta)}(I + \rho_0) + \alpha_x \rho_0) \nabla \theta(0, x) \right] \)

\begin{align*}
\int_{\Omega} A(\theta, x) \nabla \omega - \int_{\Omega} |\nabla \theta(0, x)|^{-2} \nabla \omega - \nabla \omega + \left( |\nabla \theta(0, x)|^{-2} \nabla \omega \right) + |\nabla \theta(0, x)|^{-2} \left( I + \text{div}(\theta) I - \nabla \theta - (\nabla \theta)^I \right) \\
- |\nabla \theta(0, x)|^{-2} \left( I + \text{div}(\theta) I - \nabla \theta - (\nabla \theta)^I \right) \nabla \omega = |\nabla \theta(0, x)|^{-2} \left( \text{div}(\theta) I - \nabla \theta - (\nabla \theta)^I \right) \nabla \omega
\end{align*}

(33)

Thus \( L \), the Lagrangian of \( \theta \), will be solution of the following reduced differential equation:

\begin{align*}
-\Delta_p L - \text{div} \left[ |\alpha_x \nabla \theta(\theta, x)|^{-2} \left( I + \rho_0 \nabla \theta(\theta, x) \right) \right] + \text{div}(f, \theta) \omega = g' + \text{div}(f, \theta) \\
L = 0 \quad \text{on } \partial\Omega
\end{align*}

(34)

with \( g' = -\text{div} \left[ (\lambda_{(x, \theta)} + \alpha_x \rho_0) \nabla \theta(0, x) \right] \)

2.2. Neumann Conditions. We consider the following equation with Neumann boundary conditions (see [15]):

\begin{align*}
-\Delta_p (u) + |u|^{p-2} u = f \quad \text{in } \Omega \\
|\nabla u|^{p-2} \frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega
\end{align*}

(35)
where $\Omega_0$ a regular bounded open set in $\mathbb{R}^N$, $f \in W^{1,p}(\Omega_0)$, $g \in W^{2,p}(\Omega_0)$, and $\Delta_p(u) = \text{div}((|\nabla u|^{p-2} \nabla u))$ with $1 < p < \infty$.

The variational formulation of problem (35) is to find $u \in W^{1,p}(\Omega)$ such that $\forall v \in W^{1,p}(\Omega)$ (v represents a test function)
\[
\int_\Omega \nabla u \cdot \nabla v + \int_\Omega |\nabla u|^{p-2} |\nabla^2 u| = \int_\Omega f \cdot v + \text{div}(f) = \int_\Omega F_0 + \text{div}(F).
\]

It implies that $f \text{div}(F) = \text{div}(f(F)) - \nabla F$:
\[
\int_\Omega -\text{div}((|\nabla u|^{p-2} \nabla u)) \cdot v + \int_\Omega |\nabla u|^{p-2} u \cdot v
\]
\[
= \int_\Omega f \cdot v \iff (36)
\]
\[
\int_\Omega |\nabla u|^{p-2} u \cdot v \cdot \nabla v - \int_\Omega \text{div}(v |\nabla u|^{p-2} \nabla u)
\]
\[
+ \int_\Omega |\nabla u|^{p-2} u \cdot v \cdot u \cdot v = \int_\Omega f \cdot v dx.
\]

By using the Green formula we find
\[
\int_\Omega |\nabla u|^{p-2} u \cdot v \cdot \nabla v - \int_\Omega \text{div}(v |\nabla u|^{p-2} \nabla u)
\]
\[
+ \int_\Omega |\nabla u|^{p-2} u \cdot v \cdot u \cdot v = \int_\Omega f \cdot v dx
\]
\[
(37)
\]
\[
\int_\Omega |\nabla u|^{p-2} u \cdot v \cdot \nabla v + \int_\Omega |\nabla u|^{p-2} u \cdot u \cdot v = \int_\Omega f \cdot v dx.
\]

Proposition 12. Let $\Omega = (1d + \theta)(\Omega_0); u(\Omega)$ is the solution of problem (35). We define its transport on $\Omega_0$ by $\overline{u}(\theta) = u(1d + \theta)(\Omega_0)$. Then the application $\theta \rightarrow \overline{u}(\theta)$ from $C^1(\mathbb{R}^N, \mathbb{R}^N)$ to $W^{1,\infty}(\Omega_0)$ is differentiable in $0$ on the direction $\theta$ and its directional derivative called Lagrangian derivative of $u$ is $L = \overline{u}(0, \theta) > L \in W^{1,p}(\Omega_0)$ is the unique solution of

\[
-\Delta_p L + \text{div}(\nabla L(\theta, x)) = F
\]
\[
+ \text{div}((\alpha - \lambda_{(x, \theta)}) \nabla L(\theta, x))
\]
\[
+ \text{div}([\nabla L(\theta, x)]^{p-2} (I + \rho_0) \nabla L(\theta, x))
\]
\[
- |L(\theta, x)|^{p-2} \nabla L(\theta, x) \cdot \nabla L(\theta, x)
\]
\[
- \left( \|\nabla L(\theta, x)\|^{p-2} + \|v L(\theta, x)\|^{p-2} \right)
\]
\[
\cdot \left( \nabla L(\theta, x) \cdot \nabla \theta \right) \text{ div } \theta \text{ in } \Omega
\]
\[
G_\theta - (\alpha - \lambda_{(x, \theta)}) \text{ div } \theta - |\nabla L(\theta, x)|^{p-2}
\]
\[
\cdot (I + \rho_0) \frac{\partial L(\theta, x)}{\partial n} - |L(\theta, x)|^{p-2} \frac{\partial \lambda_{(x, \theta)}}{\partial n} = 0
\]
on $\partial \Omega$

where $F = -\text{div}(\lambda_{(x, \theta)}(I + \rho_0) + \alpha_{(x, \theta)} \lambda_{(x, \theta)} \nabla L(\theta, x) - f \theta)$.

Proof. We make a change of variable $x = y + \theta(y)$ where $y \in \Omega_0$ and $x \in \Omega$ in the variational formulation (37). We pose $w(y) = v(x) \iff w(y) = v(Id + \theta)(y) \iff v = w \circ (Id + \theta)^{-1}$ noticing that $w$ does not depend on $\theta$. Thus by drifting the variational formulation, we obtain by using Lemmas 5 and 7

\[
(37) \iff
\]
\[
\int_\Omega A(\theta, \overline{\theta}) \cdot \nabla w \cdot dy
\]
\[
+ \int_\Omega |\nabla \theta|^{p-2} \nabla \overline{\theta}(\theta, x) \cdot w \cdot |\text{det } VT| \cdot dy
\]
\[
= \int_\Omega (f \cdot T) \cdot w \cdot |\text{det } VT| \cdot dy
\]
\[
+ \int_\Omega g \cdot T \cdot w \cdot |\text{det } VT| \cdot (|\text{det } VT|^{-1}) \cdot |\overline{\theta}(\theta, x)| \cdot dS
\]

Or
\[
\int_\Omega A(\theta, \overline{\theta}) \cdot \nabla w \cdot dy
\]
\[
= \int_\Omega |\nabla \overline{\theta}(\theta, x)|^{p-2} \nabla \overline{\theta}(\theta, x) \cdot \nabla w
\]
\[
+ \int_\Omega |\nabla (\theta, x)|^{p-2} \nabla (\theta, x) \cdot \nabla w
\]
\[
+ \int_\Omega \left( (\alpha - \lambda_{(x, \theta)}) \nabla (\theta, x) \cdot \nabla w
\]
\[
+ \int_\Omega \left( |\nabla (\theta, x)|^{p-2} (I + \rho_0) \right) \nabla \theta(\theta, x) \cdot \nabla w
\]
\[
- \left( \lambda_{(x, \theta)} (I + \rho_0) + \alpha_{(x, \theta)} \theta \right) \nabla \theta(\theta, x) \cdot \nabla w
\]

We have by using [14]$_\Omega$

\[
\int_\Omega |\nabla \theta(\theta, x)|^{p-2} \nabla \theta(\theta, x) \cdot w \cdot |\text{det } VT| \cdot dy
\]
\[
= \int_\Omega |\nabla \theta(\theta, x) + L(\theta, x)|^{p-2}
\]
\[
\cdot (\theta(\theta, x) + L(\theta, x)) \cdot w \cdot (1 + \text{div} (\theta)) \cdot dy + \cdot (\theta)
\]
\[
= \int_\Omega \left( |\nabla \theta(\theta, x)|^{p-2} + |L(\theta, x)|^{p-2} \right)
\]
\[
\cdot (\theta(\theta, x) + L(\theta, x)) \cdot w \cdot (1 + \text{div} (\theta)) \cdot w \cdot (1 + \text{div} (\theta)) \cdot dy + \cdot (\theta)
\]
\[
= \int_\Omega |\nabla \theta(\theta, x)|^{p-2} \nabla \theta(\theta, x) \cdot w \cdot dy + \int_\Omega |L(\theta, x)|^{p-2}
\]
\[
\cdot L(\theta, x) \cdot w \cdot dy + \int_\Omega |\nabla \theta(\theta, x)|^{p-2}
\]
\[
\cdot w \cdot dy + \int_\Omega |\nabla \theta(\theta, x)|^{p-2} \nabla \theta(\theta, x) \cdot w \cdot dy
\]
\[ + \int_{\Omega_0} \left( |\overline{\nabla}(0,x)|^p - 2 + |L(\theta,x)|^p - 2 \right) \cdot (\overline{\nabla}(0,x) + L(\theta,x)) \cdot w \cdot \text{div}(\theta) \, dy + o(\theta) \]

And also
\[ \int_{\Omega_0} (f \circ T) \cdot w \cdot |\text{det }\nabla T| \, dy - \int_{\Omega_0} f \, dy \]
\[ = \int_{\Omega_0} \text{div}(f \theta) \, dy + o(\theta) \]

Then
\[ \int_{\partial \Omega_0} g \cdot Tw \cdot |\text{det }\nabla T| \left|((\nu T)^{-1})^T \nu \right|_{\partial \Omega_0} \, dS \]
\[ - \int_{\partial \Omega_0} gw \, dS = \int_{\partial \Omega_0} (\nabla g \theta + g \cdot (\text{div } \theta - \nabla \theta n \cdot n)) \cdot w \, dS + o(\theta) \]

And
\[ \int_{\Omega_0} A(\theta, \overline{\nabla}) \nabla w \, dy + \int_{\Omega_0} |\overline{\nabla}(\theta,x)|^{p-2} \cdot \overline{\nabla}(\theta,x) \cdot w \cdot \text{det }\nabla T \, dy \]
\[ - \int_{\Omega_0} |\overline{\nabla}(\theta,x)|^{p-2} \cdot \overline{\nabla}(\theta,x) \cdot w \cdot \text{div} \nabla \theta \, dy \]
\[ + \int_{\Omega_0} |L(\theta,x)|^{p-2} \cdot L(\theta,x) \cdot w \, dy \]
\[ + \int_{\Omega_0} \left( |\overline{\nabla}(0,x)|^p - 2 + |L(\theta,x)|^p - 2 \right) \cdot (\overline{\nabla}(0,x) + L(\theta,x)) \cdot \text{div}(\theta) \, dy \]
\[ - \int_{\Omega_0} (\lambda_{(x,\theta)} (I + \rho_0) + \alpha_x \rho_0) \cdot \overline{\nabla}(0,x) \cdot w \, dy \]
\[ + \int_{\Omega_0} |L(\theta,x)|^{p-2} \cdot L(\theta,x) \cdot w \, dy + \int_{\Omega_0} |\overline{\nabla}(\theta,x)|^{p-2} \cdot \overline{\nabla}(\theta,x) \cdot w \, dy \]
\[ - \int_{\Omega_0} (\lambda_{(x,\theta)} (I + \rho_0) + \alpha_x \rho_0) \cdot \overline{\nabla}(0,x) \cdot w \, dy \]
\[ + \int_{\Omega_0} |L(\theta,x)|^{p-2} \cdot L(\theta,x) \cdot w \, dy \]
\[ - \int_{\Omega_0} (\lambda_{(x,\theta)} (I + \rho_0) + \alpha_x \rho_0) \cdot \overline{\nabla}(0,x) \cdot w \, dy \]
\[ + \int_{\Omega_0} |L(\theta,x)|^{p-2} \cdot L(\theta,x) \cdot w \, dy \]

\[ = \int_{\Omega_0} |\overline{\nabla}(0,x) + L(\theta,x)) \cdot w \cdot \text{div}(\theta) \, dy \]
\[ - \int_{\Omega_0} \text{div}(\lambda_{(x,\theta)} (I + \rho_0) + \alpha_x \rho_0) \cdot \overline{\nabla}(0,x) \cdot w \, dy \]
\[ - \int_{\Omega_0} |L(\theta,x)|^{p-2} \cdot L(\theta,x) \cdot w \, dy \]

We note $G_{\theta} = \nabla g \theta + g \cdot (\text{div } \theta - \nabla \theta n \cdot n) + (\lambda_{(x,\theta)} (I + \rho_0) + \alpha_x \rho_0) \cdot \overline{\nabla}(0,x)$

So the transported derivative of $\overline{u}(\theta, x)$ at 0 in the direction $\theta$ is the Lagrangian $L = \langle u'(0), \theta \rangle$ which is the solution of the following equation:

\[ - \Delta_p L + |\nabla L(\theta,x)|^{p-2} \nabla L(\theta,x) = F \]
\[ + \text{div}(\lambda_{(x,\theta)} (I + \rho_0) + \alpha_x \rho_0) \cdot \nabla L(\theta,x) \]
\[ + \text{div} \left( |\nabla L(\theta,x)|^{p-2} (I + \rho_0) \cdot \overline{\nabla}(0,x) \right) \]
\[ - |L(\theta,x)|^{p-2} \cdot L(\theta,x) \]
\[ - (|\overline{\nabla}(0,x)|^{p-2} + |\nabla L(\theta,x)|^{p-2}) \cdot |\overline{\nabla}(0,x)|^{p-2} + L(\theta,x) \cdot \text{div } \theta \text{ in } \Omega \]
\[ C_0 - (\alpha_x - \lambda_{x,\theta}) \frac{\partial L(\theta, x)}{\partial n} - |\nabla L(\theta, x)|^{p-2} \]

\[ \cdot (I + \rho_\theta) \frac{\partial \mathbf{F}(0, x)}{\partial n} - |L(\theta, x)|^{p-2} \frac{\partial L(\theta, x)}{\partial n} = 0 \]

on \( \partial \Omega \)

where \( F = -\text{div}((\lambda_{x,\theta})(I + \rho_\theta) + \alpha_x \rho_\theta) \nabla \mathbf{F}(0, x) - f \theta \). \( \square \)

### 3. Optimality Condition

To calculate the optimality conditions of the following problem
\[
\inf_{\Omega \in \mathcal{U}_d} J(\Omega) \quad \text{with} \quad U_{ad} = \{ \Omega \in C(\Omega_0) \quad \text{and} \quad \int_\Omega dx = V_0 \},
\]
where \( C(\Omega_0) \) is the set of admissible forms obtained by

\[ \text{diffeomorphism}, \quad \text{the cost function} \quad J(\Omega) \quad \text{is the compliance defined by} \]

\[ J(\Omega) = \int_\Omega |u(\Omega) - u(\Omega_0)|^p \, dx \]  

(47)

to reach a target displacement \( u(\Omega_0) \in L^p(\mathbb{R}^N) \) where the function \( u(\Omega) \) is solution of the boundary problem posed (resp., Dirichlet or Neumann boundary conditions).

We consider the following boundary value problems.

**Dirichlet Boundary Condition**

\[ -\Delta_p (u) = f \quad \text{in} \quad \Omega \]

\[ u = 0 \quad \text{on} \quad \partial \Omega \]  

(48)

where \( f \in W^{1,p}(\mathbb{R}^N) \).

**Neumann Boundary Condition**

\[ -\Delta_p (u) + |u|^{p-2} u = f \quad \text{in} \quad \Omega \]

\[ |\nabla u|^{p-2} \frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial \Omega \]  

(49)

where \( f \in W^{1,p}(\mathbb{R}^N) \) and \( g \in W^{2,p}(\mathbb{R}^N) \).

The problem admits a unique solution \( u(\Omega) \).

**Theorem 13.** Let \( \Omega_0 \) be a regular bounded open set. The cost function \( J(\Omega) \) is differentiable and its derivative is

\[ J'(\Omega_0)(\theta) = \int_{\Omega_0} |u(\Omega_0) - u_0|^{p-2} \, \text{div} \, (\theta) \]

\[ + p |u(\Omega_0) - u_0|^{p-2} \, \text{div} \, (\theta) \]

\[ \cdot (L - \theta \cdot \nabla u_0) \, dx \]

\[ L \text{ is the Lagrangian derivative and also solution of} \]

\[ -\Delta_p L - \text{div} \, [\alpha_x \nabla L(\theta, x)] \]

\[ = \text{div} \left( |\nabla L(\theta, x)|^{p-2} \nabla u(0, x) \right) = g' + \text{div} \, (f \theta) \]

\[ \text{in} \quad \Omega \]

\[ L = 0 \quad \text{on} \quad \partial \Omega \]

with \( g' = -\text{div}((\lambda_{x,\theta} + \alpha_x \rho_\theta) \nabla \mathbf{F}(0, x)) \).

**Proof.** By applying Proposition 4 for the compliance we obtain

\[ J'(\Omega_0)(\theta) = \int_{\Omega_0} |u(\Omega_0) - u_0|^{p-2} \, \text{div} \, (\theta) \]

\[ + \int_{\Omega_0} |u(\Omega_0) - u_0|^p \, \theta \]

\[ J'(\Omega_0)(\theta) = \int_{\Omega_0} |u(\Omega_0) - u_0|^{p-2} \, (u(\Omega_0) - u_0) \]

\[ + p |u(\Omega_0) - u_0|^{p-2} \, (u(\Omega_0) - u_0) \]

\[ \cdot (L - \theta \cdot \nabla u_0) \, dx \]

(52)

where \( L \) is the Lagrangian derivative, solution of the PDE. \( \square \)

**Remark 14.** From extensive literature which deals with optimum condition calculus for problems as \( \inf_{\Omega \in \mathcal{U}_d} J(\Omega) \) we can cite

(i) \( I_1(\Omega) = \int_{\Omega_0} |u(\Omega_0) - u_0|^p \, dS \)

(ii) \( I_2(\Omega) = \int_{\Omega_0} |\nabla u(\Omega_0) - \nabla u_0|^p \, dS + \int_{\Omega_0} |u(\Omega_0) - u_0|^p \, dS \)

So, to calculate the gradient of each compliance we use the same argument by the propositions [13].

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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