CHARACTERIZATION PROBLEMS FOR LINEAR FORMS WITH FREE SUMMANDS.

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Abstract. Let \( T_1, \ldots, T_n \) denote free random variables. For two linear forms \( L_1 = \sum_{j=1}^{n} a_j T_j \) and \( L_2 = \sum_{j=1}^{n} b_j T_j \) with real coefficients \( a_j \) and \( b_j \) we shall describe all distributions of \( T_1, \ldots, T_n \) such that \( L_1 \) and \( L_2 \) are free. For identically distributed free random variables \( T_1, \ldots, T_n \) with distribution \( \mu \) we establish necessary and sufficient conditions on the coefficients \( a_j, b_j, j = 1, \ldots, n \), such that the statements:

(i) \( \mu \) is a centered semicircular distribution; and
(ii) \( L_1 \) and \( L_2 \) are identically distributed \((L_1 \overset{D}{=} L_2)\); are equivalent. In the proof we give a complete characterization of all sequences of free cumulants of measures with compact support and with a finite number of non-zero entries. The characterization is based on topological properties of regions defined by means of the Voiculescu transform \( \phi \) of such sequences.

1. Introduction

The intensive research on the asymptotic behaviour of random matrices induced more research on their infinitely dimensional limiting models as well. Free convolution of probability measures, introduced by D. Voiculescu, may be regarded as such a model [28], [29].

The key concept of this definition is the notion of freeness, which can be interpreted as a kind of independence for non-commutative random variables. As in the classical probability the concept of independence gives rise to the classical convolution, the concept of freeness leads to a binary operation on the probability measures on the real line, the free convolution. Many classical results in the theory of addition of independent random variables have their counterpart in this theory, such as the law of large numbers, the central limit theorem, the Lévy-Khintchine formula and others. We refer to Voiculescu, Dykema and Nica [30], Hiai and Petz [11], and Nica and Speicher [24] for an introduction to these topics.

In many problems of mathematical statistics, conclusions are based on the fact that certain special distributions have important properties which permit the reduction of
the original problem to a substantially simpler one, for instance via the notion of sufficiency.

The simplest type of statistics of independent observations, admitting a fairly complete description of the mutual independence and identical distribution, are linear statistics.

Consider independent scalar random variables $X_1, \ldots, X_n$ (not necessary identically distributed) and two linear statistics

$$L_1 := \sum_{j=1}^{n} \alpha_j X_j \quad \text{and} \quad L_2 := \sum_{j=1}^{n} \beta_j X_j,$$

where $\alpha_j, \beta_j$ are real constant coefficients. It turns out that the independence of the two linear statistics $L_1$ and $L_2$ essentially characterizes the normality of the variables $X_j$. To be precise, the following assertion, due to Darmois \[10\] and Skitovich \[26\], \[27\], holds.

Let $L_1$ and $L_2$ given by (1.1) be independent. Then the random variables $X_j$ such that $\alpha_j \beta_j \neq 0$, i.e., which enter in both $L_1$ and $L_2$, have normal distributions.

Note that the converse proposition holds in the following form: if $\sum_{j=1}^{n} \alpha_j \beta_j \text{Var}(X_j) = 0$ and all $X_j$ such that $\alpha_j \beta_j \neq 0$ are normal, then $L_1$ and $L_2$ are independent.

Polya \[25\] was the first who established that only the normal distribution leads to identically distributed linear statistics $X_1$ and $a_1 X_1 + a_2 X_2$, where $X_1$ and $X_2$ are independent and identically distributed. Marcinkiewicz \[22\] proved that distributions having moments of all orders and admitting the existence of a nontrivial pair of identically distributed linear statistics based on a random sample are normal. Yu. V. Linnik \[18\], \[19\] described the class of symmetric distributions admitting identically distributed linear statistics and studied in detail the problem of characterizing the normal distribution via properties of such statistics.

In this paper we give a complete description of those free random variables $T_1, \ldots, T_n$ such that the linear statistics $a_1 T_1 + \ldots + a_n T_n$ and $b_1 T_1 + \ldots + b_n T_n$ are free.

In addition we prove an analogue of Yu. V. Linnik’s results \[18\], \[19\], \[13\], and give the solution of the problem of characterization of the semicircular distribution via identical distribution of linear statistics $a_1 T_1 + \ldots + a_n T_n$ and $b_1 T_1 + \ldots + b_n T_n$, where $T_1, \ldots, T_n$ are identically distributed free random variables.

2. Results

Assume that $A$ is a finite von Neumann algebra with normal faithful trace state $\tau$. The pair $(A, \tau)$ will be called a tracial $W^*$-probability space. Assume that $A$ is acting on a Hilbert space $H$. We will denote by $\hat{A}$ the set of all operators on $H$ which are affiliated with $A$ and by $\hat{A}_{sa}$ the set of selfadjoint operators affiliated with $A$. Recall that a (generally unbounded) selfadjoint operator $X$ on $H$ is called affiliated with $A$ if all the spectral projections of $X$ belong to $A$. The elements of $\hat{A}_{sa}$ will be regarded as (possibly) unbounded random variables. Let $T \in \hat{A}_{sa}$. The distribution $\mu_T$ of $T$ is
characterization problems for linear forms with free summands.

the unique probability measure on \( \mathbb{R} \) satisfying the equality

\[
\tau(u(T)) = \int_{\mathbb{R}} u(\lambda) \mu_T(d\lambda)
\]

for every bounded Borel function \( u \) on \( \mathbb{R} \).

Recall that a family \( \{T_j\}_{j=1}^k \) of elements of \( T \in \tilde{A}_{sa} \) is said to be free if for all bounded continuous functions \( u_1, u_2, \ldots, u_n \) on \( \mathbb{R} \) we have

\[
\tau(u_1(T_{j_1})u_2(T_{j_2})\cdots u_n(T_{j_n})) = 0
\]

whenever \( \tau(u_l(T_{j_l})) = 0 \) for all \( l = 1, \ldots, n \) and all alternating sequences \( j_1, j_2, \ldots, j_n \) of 1’s, 2’s, and \( k \)’s, i.e., \( j_1 \neq j_2 \neq \cdots \neq j_n \).

Bercovici and Voiculescu [4] proved that if \( T_j \in \tilde{A}_{sa} \) are free random variables for \( j = 1, \ldots, n \), and \( Q \) is a selfadjoint polynomial in \( n \) non-commuting variables, then the distribution of the random variable \( Q(T_1, T_2, \ldots, T_n) \) depends only on the distributions of \( T_1, T_2, \ldots, T_n \).

If \( Q(T_1, \ldots, T_n) = T_1 + \cdots + T_n \), then the distribution of \( T_1 + \cdots + T_n \) only depends on the \( \mu_T \), and is called the additive free convolution of \( \mu_{T_1}, \ldots, \mu_{T_n} \). Denote this distribution by \( \mu_T \odot \cdots \odot \mu_T \).

Let \( T_1, \ldots, T_n \) denote free random variables with distributions \( \mu_1, \ldots, \mu_n \), respectively. Consider two linear statistics

\[
L_1 := a_1 T_1 + \cdots + a_n T_n \quad \text{and} \quad L_2 := b_1 T_1 + \cdots + b_n T_n
\]

with real coefficients \( a_j \) and \( b_j \). In the sequel we assume without loss of generality that \( |a_j| \leq 1 \) and \( |b_j| \leq 1 \) for all \( j = 1, \ldots, n \).

Denote by \( \mu_w \) the standard semicircular measure, i.e., the measure with the density

\[
p_w(x) = \frac{1}{2\pi} \sqrt{(4-x^2)_+}, \quad \text{where} \quad a_+ = \max\{0, a\}.
\]

We shall call measures with densities

\[
p(x) = \frac{1}{2\pi a^2} \sqrt{(4a^2 - (x-b)^2)_+}
\]

with some \( a > 0 \) and \( b \in \mathbb{R} \) semicircular.

Nica [23] established that the stability of freeness under rotations characterizes semicircular random variables. Lehner [17] proved that there are free random variables \( T_1, T_2, T_3 \) which are not semicircular and such that \( L_1 := a_1 T_1 + a_2 T_2 + a_3 T_3 \) and \( L_2 := b_1 T_1 + b_2 T_2 + b_3 T_3 \) are free. Hence the analogue of the Darmois–Skitovich theorem fails in the free case if there are at least three random variables involved. We can nevertheless describe all free random variables \( T_1, \ldots, T_n \) for which the linear statistics \( L_1 \) and \( L_2 \) in (2.1) are free. Our result extends the results of Nica and Lehner considerably. In order to formulate our first result we need the following notation.

Let \( T \in A_{as} \) be a given random variable with distribution \( \mu \) such that \( \beta_n(\mu) := \int_{\mathbb{R}} |x|^n \mu(dx) < \infty \) for some \( n \in \mathbb{N} \) and let \( T^{(k)}, k = 1, \ldots, n \), be its free copies. Let \( \omega \) be \( n \)-th primitive root of unity (e.g., \( \omega = e^{2\pi i/n} \)) and set

\[
T^\omega = \omega T^{(1)} + \omega^2 T^{(2)} + \cdots + \omega^n T^{(n)}.
\]

Following Lehner [17], we define the \( n \)th free cumulant of the random variable \( T \) to be

\[
\kappa_n(T) = \frac{1}{n} \tau(\langle T^\omega \rangle^n)
\]
in a short way. For a detailed definition of free cumulants see the monograph of Nica and Speicher \[24\] as well. Since the free cumulant \( \kappa_n \) depends on \( n \) and the distribution \( \mu \) of \( T \) only, we will denote this cumulant by \( \kappa_n(\mu) \) as well.

**Theorem 2.1.** Consider free random variables \( T_1, \ldots, T_n, n \geq 2 \), and let \( a_j, b_j \) be real numbers such that \( a_j b_j \neq 0 \) and \( \frac{b_j}{a_j} \neq \frac{b_s}{a_s} \) for \( j, s = 1, \ldots, m \), where \( m \leq n \), and \( a_j b_j = 0 \) for \( j = m + 1, \ldots, n \). The linear statistics \( L_1 \) and \( L_2 \) are free if and only if the distributions \( \mu_1, \ldots, \mu_m \) have compact supports and the free cumulants \( \kappa_s(T_j), j = 1, \ldots, m \), satisfy the relations:

\[
\sum_{j=1}^{m} a_j b_j \kappa_s(T_j) = 0 \quad (2.4)
\]

for all \( s = 2, \ldots, m \) and \( (l, t) \in \mathbb{N}^2 \) such that \( l + t = s \), and \( \kappa_s(T_j) = 0 \) for \( s \geq m + 1 \).

The following result describes all distributions \( \mu_1, \ldots, \mu_m \) in the previous theorem. Let \( \kappa_1, \ldots, \kappa_m \) be real numbers. Introduce the function

\[
\varphi(z) := \kappa_1 + \frac{\kappa_2}{z} + \cdots + \frac{\kappa_m}{z^{m-1}}, \quad z \in \mathbb{C} \setminus \{0\}.
\]

Denote by \( \Omega_\varphi \) the component of \( \{z \in \mathbb{C}^+ : \text{Im}(z + \varphi(z)) > 0\} \) which contains \( \infty \).

**Theorem 2.2.** A sequence \( \{\kappa_n\}_{n=1}^{\infty} \) of real numbers such that \( \kappa_n = 0 \) for \( n \geq m + 1 \), \( m \geq 2 \), is a sequence of free cumulants of some probability measure with compact support if and only if every Jordan curve, contained in \( \mathbb{C}^+ \cup \mathbb{R} \) and connecting \( 0 \) and \( \infty \), contains a point of the boundary of \( \Omega_\varphi \).

**Remark 2.3.** Consider the set \( S \) of free cumulant sequences of the form \( \{\kappa_1, \kappa_2, \ldots, \kappa_m, 0, 0 \ldots\} \). Then the set \( \{\kappa_1, \kappa_2, \ldots, \kappa_m\} \) is closed in the space \( \mathbb{R}^m \).

We easily obtain from Theorem 2.2 the following consequence.

**Corollary 2.4.** A sequence \( \{\kappa_n\}_{n=1}^{\infty} \) of real numbers such that \( \kappa_2 > 0, \kappa_n = 0 \) for \( n \geq m + 1 \), \( m \geq 2 \), and \( |\kappa_n| \leq \varepsilon, n = 3, \ldots, m \), with sufficiently small \( \varepsilon > 0 \), is a sequence of free cumulants of some probability measure with compact support.

Note that Bercovici and Voiculescu \[5\] proved a more general result than Corollary 2.4 and showed the failure of the well-known Cramér and Marcinkiewicz theorems in free probability theory. To illustrate these results in a low dimensional example we consider the case \( m = 4 \).

**Corollary 2.5.** A sequence \( 0, 1, \kappa_3, \kappa_4, 0, 0, \ldots \) is a free cumulant sequence of some probability measure if and only if \( \{\kappa_3, \kappa_4\} \in D \), where

\[
D := \{(x, y) \in \mathbb{R}^2 : |x| \leq f_1(y), -\frac{1}{12} \leq y \leq \frac{1}{36}\} \cup \{(x, y) \in \mathbb{R}^2 : |x| \leq f_2(y), \frac{1}{36} < y \leq \frac{1}{4}\}
\]

with \( f_1(y) := \frac{1}{3\sqrt{6}} \sqrt{1 + \sqrt{1 - 36y}(2 - \sqrt{1 - 36y})} \) for \( -\frac{1}{12} \leq y \leq \frac{1}{36} \), and

\[\quad\]
\[ f_2(y) := \frac{\sqrt{2\sqrt{y}(1 + \sqrt{1 - 12\sqrt{y} + 36y})(2 - \sqrt{1 - 12\sqrt{y} + 36y})}}{3\sqrt{3}} \sqrt{1 - 2\sqrt{y}} \quad \text{for } \frac{1}{36} < y < \frac{1}{4}; \quad f_2\left(\frac{1}{4}\right) := 0. \]

Figure 1. Region D of realized cumulants \((\kappa_3, \kappa_4)\).

We see from Corollary 2.5 that a sequence \(0, 1, \kappa_3, 0, \ldots\) is a free cumulant sequence of some probability measure if and only if \(|\kappa_3| \leq \frac{1}{3\sqrt{3}}\). This assertion was obtained by Lehner (oral communication) by other means.

Now we consider the problem of the description of identically distributed free random variables \(T_1, \ldots, T_n\) with distribution \(\mu\) such that the statistics \(L_1\) and \(L_2\) are identically distributed as well (\(L_1 \overset{D}{=} L_2\)).

Following Linnik [18], we introduce two entire functions of a complex variable \(z\):

\[ \Lambda_1(z) = |a_1|^z + \cdots + |a_n|^z - |b_1|^z - \cdots - |b_n|^z \]

and

\[ \Lambda_2(z) = a_1^z + \cdots + a_n^z - b_1^z - \cdots - b_n^z, \]

where \(a_1, \ldots, a_n\) and \(b_1, \ldots, b_n\) are restricted as in (2.1). It is easy to see that all zeros of the functions \(\Lambda_1(z)\) lie in a strip \(b_1 < \text{Re} z < b_2\) with some \(b_1, b_2 \in \mathbb{R}\).

We prove the following characterization of semicircular measures which is an analogue of Linnik's result [18] about a characterization of Gaussian probability measures. Recall that a probability measure \(\mu\) is called degenerate if \(\mu = \delta_a\), where \(\delta_a\) is the Dirac measure concentrated at the point \(a \in \mathbb{R}\).

**Theorem 2.6.** Let \(\Lambda_1(z) \neq 0\). In order that, for some non-degenerate probability measure \(\mu\), the statement (1) \(L_1 \overset{D}{=} L_2\) implies the statement (2) \(\mu\) is a semicircular measure, it is necessary and sufficient that the following conditions are satisfied:

a) \(2\) is a simple and unique positive zero of the function \(\Lambda_1(z)\),

b) \(\Lambda_2(2m + 1) \neq 0\) for all \(m = 1, \ldots\).

Note that (2) implies (1) if \(\Lambda_2(1) = \Lambda_1(2) = 0\). Let \(\mu\) be a semicircular measure with mean zero, then (2) implies (1) iff \(\Lambda_1(2) = 0\).

From Theorem 2.6 we obviously obtain the following consequences.
Corollary 2.7. Let $\Lambda_1(z) \not\equiv 0$, $\Lambda_2(1) = 0$ and let $\Lambda_1(1) \neq \Lambda_2(1)$. In order that, for some non-degenerate probability measure $\mu$, the statements (1) and (2) are equivalent, it is necessary and sufficient that the conditions a) and b) are satisfied.

Corollary 2.8. Let $\Lambda_1(z) \not\equiv 0$. In order that, for some non-degenerate probability measure $\mu$ with a median equal to 0, the statements (1) and (2) are equivalent, it is necessary and sufficient that the conditions a) and b) are satisfied.

Moreover Theorem 2.6 implies the following assertion.

Corollary 2.9. Assume that $a_1^2 + \cdots + a_n^2 = 1$, and $b_1 = 1$, $b_2 = \cdots = b_n = 0$. Furthermore, assume $L_1 \overset{D}{=} L_2$. Then either $\mu$ is a semicircular measure or $\mu$ is a degenerate probability measure.

Corollary 2.9 is an analogue of a result by Polya [25]. For analogues of the Polya result in non-commutative probability theory see Lehner [16].

We prove as well the following result for symmetric probability measures.

Theorem 2.10. Let $\Lambda_1(z) \not\equiv 0$. In order that, for some non-degenerate probability measure $\mu$, which is symmetric with respect to 0, the statements (1) and (2) are equivalent, it is necessary and sufficient that the condition a) is satisfied.

The following result for probability measures $\mu$ with moments of finite order is an analogue of a result by Linnik [18] in classical probability theory.

Theorem 2.11. Assume that $\Lambda_1(z) \not\equiv 0$ and that $\Lambda_1(z)$ has zeros in $-i\mathbb{C}^+$. Let $\gamma$ denote the maximum of the real parts of such zeros. In order that, for some non-degenerate probability measure $\mu$ such that $\int_R u^{2s} \mu(du) < \infty$ with $s = [\gamma/2 + 1]$, the statement (1) implies the statement (2) it is necessary and sufficient that $\Lambda_2(2) = 0$ and $\Lambda_2(m) \neq 0$ for all positive integers $m > 2$.

We prove in Lemma 8.1 that if $L_1 \overset{D}{=} L_2$, then the function $\Lambda_1(z)$ has a real root $\gamma$ such that $0 < \gamma \leq 2$.

In the case where all moments of $\mu$ exist we obtain from Theorem 2.11 the following result.

Theorem 2.12. Assume that $\Lambda_1(z) \not\equiv 0$. In order that, for some non-degenerate probability measure $\mu$ such that $\int_R u^{2m} \mu(du) < \infty$ for all $m \in \mathbb{N}$, the statement (1) implies the statement (2) it is necessary and sufficient that $\Lambda_2(2) = 0$ and $\Lambda_2(m) \neq 0$ for all positive integers $m > 2$.

Theorem 2.12 is an analogue of a result by Marcinkiewicz [22] in classical probability theory.
3. An analytic approach to a solution of the considered problems.

Auxiliary results.

Denote by $\mathcal{M}$ the family of all Borel probability measures defined on the real line $\mathbb{R}$. On the set $\mathcal{M}$ define two associative composition laws denoted $\ast$ and $\boxplus$. Let $\mu_1, \mu_2 \in \mathcal{M}$. The measure $\mu_1 \ast \mu_2$ will denote the classical convolution of $\mu_1$ and $\mu_2$. In probabilistic terms, $\mu_1 \ast \mu_2$ is the probability distribution of $X + Y$, where $X$ and $Y$ are (commuting) independent random variables with distributions $\mu_1$ and $\mu_2$ respectively. The measure $\mu_1 \boxplus \mu_2$ is the free (additive) convolution of $\mu_1$ and $\mu_2$ introduced by Voiculescu [28] for compactly supported measures. The free convolution was extended by Maassen [21] to measures with finite variance and by Bercovici and Voiculescu [4] to the whole class $\mathcal{M}$. Thus, $\mu_1 \boxplus \mu_2$ is the distribution of $X + Y$, where $X$ and $Y$ are free random variables with distributions $\mu_1$ and $\mu_2$, respectively.

Let $\mathbb{C}^+ (\mathbb{C}^-)$ denote the open upper (lower) half of the complex plane. If $\mu \in \mathcal{M}$, denote its Cauchy transform by

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{\mu(dt)}{z-t}, \quad z \in \mathbb{C}^+. \quad (3.1)$$

Following Maassen [21] and Bercovici and Voiculescu [4], we introduce the reciprocal Cauchy transform

$$F_\mu(z) = \frac{1}{G_\mu(z)}. \quad (3.2)$$

The corresponding class of reciprocal Cauchy transforms of all $\mu \in \mathcal{M}$ we denote by $\mathcal{F}$. This class admits a simple description. Recall that the Nevanlinna class $\mathcal{N}$ is the class of analytic functions $F : \mathbb{C}^+ \to \mathbb{C}^+$. The class $\mathcal{F}$ is the subclass of Nevanlinna’s functions $F_\mu$ such that $F_\mu(z)/z \to 1$ as $z \to \infty$ non-tangentially to $\mathbb{R}$ (i.e., such that $\text{Re } z/\text{Im } z$ stays bounded), and this implies that $F_\mu$ has certain invertibility properties. (For details see Akhiezer and Glazman [2], Akhiezer [1]). To be precise, for two numbers $\alpha > 0, \beta > 0$ we set

$$\Gamma_\alpha = \{z = x + iy \in \mathbb{C}^+ : |x| < \alpha y\} \quad \text{and} \quad \Gamma_{\alpha,\beta} = \{z = x + iy \in \Gamma_\alpha : y > \beta\}.$$ 

Then for every $\alpha > 0$ there exists $\beta = \beta(\mu, \alpha)$ such that $F_\mu$ has the right inverse $F^{(-1)}_\mu$ defined on $\Gamma_{\alpha,\beta}$. The function $\phi_\mu(z) = F^{(-1)}_\mu(z) - z$ will be called the Voiculescu transform of $\mu$. It is not hard to show that $\text{Im } \phi_\mu(z) \leq 0$ for $z \in \Gamma_{\alpha,\beta}$ where $\phi_\mu$ is defined. Note that $\phi_\mu(z) = o(z)$ as $|z| \to \infty$, $z \in \Gamma_\eta$. In the sequel we will denote $\phi_\mu(z)$ by $\phi_T(z)$ for a random variable $T$ with a distribution $\mu$ as well. It is easy to verify that $\phi_{uT}(z) = u\phi_T(z/u)$ for fixed $u \in \mathbb{R}$ and $z \in \Gamma_{\alpha,\beta}$, where $\phi_{uT}(z)$ and $\phi_T(z/u)$ are defined.

In the domain $\Gamma_{\alpha,\beta}$, where the functions $\phi_{\mu_1}(z), \phi_{\mu_2}(z)$, and $\phi_{\mu_1 \boxplus \mu_2}(z)$ are defined, we have

$$\phi_{\mu_1 \boxplus \mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z). \quad (3.3)$$
This characterization for the distribution $\mu_1 \boxplus \mu_2$ of $X + Y$, where $X$ and $Y$ are free random variables, is due to Voiculescu [28]. He considered compactly supported measures $\mu$. The result was extended by Maassen [21] to measures with finite variance; the general case was proved by Bercovici and Voiculescu [4].

We need the following auxiliary results.

**Proposition 3.1.** For every probability measure $\mu$ we have
\[
\phi_\mu(z) = z^2 \left( G_\mu(z) - \frac{1}{z} \right) \left( 1 + q_\mu(z) \right), \quad z \in \Gamma_{\alpha,\beta},
\]
where $q_\mu(z) = o(1)$ as $z \to \infty$.

The following lemma is well-known, see [1].

**Lemma 3.2.** Let $\mu$ be a probability measure such that
\[
m_k = m_k(\mu) := \int_R u^k \mu(du) < \infty, \quad k = 1, \ldots, 2n. \tag{3.4}
\]
Then the following relation holds
\[
\lim_{z \to \infty} z^{2n+1} \left( G_\mu(z) - \frac{1}{z} - \frac{m_1}{z^2} - \cdots - \frac{m_{2n-1}}{z^{2n-1}} \right) = m_{2n} \tag{3.5}
\]
uniformly in the angle $\delta \leq \arg z \leq \pi - \delta$, where $0 < \delta < \pi/2$.

Conversely, if for some function $G(z) \in \mathcal{N}$ the relation (3.5) holds with real numbers $m_k$ for $z = iy, y \to \infty$, then $G(z)$ admits the representation (3.1), where $\mu$ is a probability measure with moments (3.4).

By Lemma 3.2 and the Cartier–Good formula for free random variables (see Lehner [17]), we easily obtain the expansion of the function $\phi_\mu(z)$. See [14] as well.

**Proposition 3.3.** For every probability measure $\mu$ such that $m_{2n}(\mu) < \infty$ for a non-negative integer $n$ we have
\[
\phi_\mu(z) = \kappa_1 + \frac{\kappa_2}{z} + \cdots + \frac{\kappa_{2n}}{z^{2n-1}} + o(1), \quad z \in \Gamma_{\alpha,\beta}, \quad z \to \infty, \tag{3.6}
\]
where $\kappa_j = \kappa_j(\mu), j = 2, \ldots, 2n$, are the free cumulants of the probability measure $\mu$.

Conversely, if for some function $\phi_\mu(z)$ the relation (3.6) holds with real coefficients $\kappa_j$, then $\mu$ has a finite moment $m_{2n}(\mu) < \infty$ and $\kappa_j = \kappa_j(\mu), j = 1, \ldots, 2n$.

We also need the following well-known result (see for example [24]).

**Proposition 3.4.** In order that a probability measure $\mu$ has a compact support it is necessary and sufficient that the sequence $\{\kappa_s(\mu)\}_{s=1}^{\infty}$ of free cumulants of this measure satisfies the inequality
\[
|\kappa_s(\mu)| \leq c^s, \quad s \in \mathbb{N},
\]
with some constant $c > 0$. 

We introduced the definition of the free cumulants in Section 2. Let us recall the definition of mixed free cumulants as well. Let \( T_1, \ldots, T_n \in \tilde{A}_{sa} \) be random variables and let \( T_j^{(k)} \), \( k = 1, \ldots, n \), denote free copies. Set \( T_j^\omega \) as in (2.2). Following Lehner [17], the nth mixed cumulant may be defined via
\[
\kappa_n(T_1, \ldots, T_n) = \frac{1}{n} \tau(T_1^\omega T_2^\omega \cdots T_n^\omega)
\]
in a short way. We will use the following known results, see [17] and [24].

**Theorem 3.5.** Consider a non-commutative probability space \( (A, \tau) \) and let \( (\kappa_n)_{n \in \mathbb{N}} \) be the corresponding free cumulant functionals. Consider random variables \( (T_j)_{j \in I} \) in \( \tilde{A}_{sa} \). Then the following two statements are equivalent.

(i) \( (T_j)_{j \in I} \) are freely independent.

(ii) We have \( \kappa_n(T_{j_1}, \ldots, T_{j_n}) = 0 \) for all \( n \geq 2 \) and \( j_1, \ldots, j_n \in I \) such that at least two of these \( n \) indices \( j_h \) are different.

**Proposition 3.6.** Let \( L_k = \sum_{j=1}^{n} a_{kj} T_j \), \( k = 1, \ldots, m \), be an affine transformation of \( T_1, \ldots, T_n \), then we have, for \( m \geq 2 \),
\[
\kappa_m(L_1, \ldots, L_m) = \sum_{j_1, \ldots, j_m} a_{1j_1} \cdots a_{mj_m} \kappa_m(T_{j_1}, \ldots, T_{j_m}).
\]

**Proposition 3.7.** Mixed cumulants vanish. That is, if there is a nontrivial subset \( I \subset [n] \) \( (\text{i.e., } I \neq \emptyset \text{ and } I \neq [n], [n] := \{1, \ldots, n\}) \) such that \( \{T_j\}_{j \in I} \) and \( \{T_j\}_{j \in [n] \setminus I} \) are free, then \( \kappa_n(T_1, \ldots, T_n) = 0 \).

Now we prove an analogue of a known lemma of Linnik for the characteristic functions (see [12], Ch. 1, §6).

Recall that a probability measure \( \mu \) is symmetric if \( \mu(S) = \mu(-S) \) for any real Borel set \( S \). It is not difficult to verify (see [9]) that \( \mu \) is symmetric if and only if \( \phi_\mu(ay) \) takes imaginary values for \( y > 0 \), where \( \phi_\mu(ay) \) is defined.

**Lemma 3.8.** Let \( \mu \) be a symmetric probability measure and \( \{y_k\} \) be a sequence of positive numbers such that \( \lim y_k \to \infty \). If, for all \( k \), \( \phi_\mu(iy_k) = \phi_\nu(iy_k) \), where \( \nu \) is a symmetric probability measure with compact support, then \( \mu = \nu \).

**Proof.** We shall show that \( \mu \) has moments \( m_n(\mu) := \int_{\mathbb{R}} u^n \mu(du) \) of all orders and that \( m_n(\mu) = m_n(\nu) \) for all \( n = 1, \ldots, \). The proof proceeds by induction for even \( n \).

From the assumptions of the lemma we see
\[
G_\mu(it_k) = G_\nu(it_k), \quad k \geq k_0, \tag{3.7}
\]
where \( t_k := -iF_\nu(iy_k) \to \infty \) as \( y_k \to \infty \), and \( k_0 \) is sufficiently large positive integer. By (3.7), we obtain the following equation, using the symmetry of the measures \( \mu \) and \( \nu \),
\[
(it_k)^3 \left( G_\mu(it_k) - \frac{1}{it_k} \right) = \int_{\mathbb{R}} \frac{t_k^2u^2}{u^2 + t_k^2} \mu(du) = \int_{\mathbb{R}} \frac{t_k^2u^2}{u^2 + t_k^2} \nu(du). \tag{3.8}
\]
Lemma 3.10. Assume that \( m_n(\mu) = m_n(\nu) \) for all \( n = 0, 1, \ldots \). Letting \( t_k \to \infty \), we conclude from (3.8) that \( m_2(\mu) < \infty \) and \( m_2(\mu) = m_2(\nu) \). Now suppose that, for all \( p < n \), \( m_{2p}(\mu) \) exists and \( m_{2p}(\mu) = m_{2p}(\nu) \). Using (3.7) and the formula
\[
(i t_k)^{2n+1} \left( G_\mu(it_k) - \frac{1}{i t_k} - \frac{m_2(\mu)}{(i t_k)^3} - \cdots - \frac{m_{2n-2}(\mu)}{(i t_k)^{2n-1}} \right) = \int_\mathbb{R} \frac{t_k^2 u^{2n}}{u^2 + t_k^2} \mu(du)
\]
we arrive at the relation
\[
\int_\mathbb{R} \frac{t_k^2 u^{2n}}{u^2 + t_k^2} \mu(du) = \int_\mathbb{R} \frac{t_k^2 u^{2n}}{u^2 + t_k^2} \nu(du).
\]
Letting here \( t_k \to \infty \), we obtain \( m_{2n}(\mu) < \infty \) and \( m_{2n}(\mu) = m_{2n}(\nu) \) that was to be proved.

It remains to note that since \( m_n(\mu) = m_n(\nu) \) for all \( n = 0, 1, \ldots \) and the measure \( \nu \) has compact support, we have \( \mu = \nu \). Hence the lemma is proved. \( \square \)

Voiculescu [30], Maassen [21] and in the general case Biane [7] proved that there exist unique functions \( Z_1(z) \) and \( Z_1(z) \) from the class \( \mathcal{F} \) such that
\[
z = Z_1(z) + Z_2(z) - F_{H_1}(Z_1(z)) \quad \text{and} \quad F_{H_1}(Z_1(z)) = F_{H_2}(Z_1(z)) \quad z \in \mathbb{C}^+.
\]
In addition \( F_{H_1} \circ F_{H_2}(z) = F_{H_1}(Z_1(z)) \). The relation (3.9) was proved by purely analytic methods by Chistyakov and Götze [8] and Belinschi and Bercovici [3].

Introduce the class \( \mathcal{K}[a,b] \) in the following way. A function \( F(z) \) is in class \( \mathcal{K}[a,b] \) if
1) \( F(z) \) is in class \( \mathcal{N} \), and
2) \( F(z) \) is holomorphic and positive in the interval \((-\infty, a)\), and holomorphic and negative in the interval \((b, +\infty)\). The following theorem is due to Krein [15].

**Theorem 3.9.** A function \( F(z) \) is in class \( \mathcal{K}[a,b] \) if and only if it admits a representation
\[
F(z) = \int_{[a,b]} \frac{\sigma(dt)}{t - z},
\]
where \( \sigma \) is a finite non-negative measure.

We now prove a free analogue of result by Wintner, see [20], Ch. 3, §2.

**Lemma 3.10.** Assume that \( \mu = \mu_1 \boxdot \mu_2 \), where \( \mu \) has compact support. Then \( \mu_1 \) and \( \mu_2 \) have compact support as well.

**Proof.** It suffices to prove the lemma for the measure \( \mu_1 \). The proof for the measure \( \mu_2 \) is similar. By (3.9), there exists \( Z(z) \in \mathcal{F} \) such that \( F_\mu(z) = F_{H_1}(Z_1(z)) \), \( z \in \mathbb{C}^+ \). Hence we obtain the relation
\[
\int_{(-d,0]} \frac{\mu(du)}{z - u} = \int_\mathbb{R} \frac{\mu_1(du)}{Z(z) - u}, \quad z \in \mathbb{C}^+,
\]
where \( \mu \) is a finite non-negative measure.
Lemma 4.1. Consider the following equation, for independence of linear statistics and identical distribution of linear statistics.

$$\frac{1}{u - Z(z)} = \int_{\mathbb{R}} \frac{\sigma(u, ds)}{s - z}, \quad z \in \mathbb{C}^+, \quad \sigma(u, ds)$$

where $\sigma(u, ds)$ is a probability measure for every $u \in \mathbb{R}$ and $\sigma(u, S)$ is a measurable function for every Borel $S$ set in $\mathbb{R}$. Using this representation we deduce from (3.10)

$$\mu(S) = \int_{\mathbb{R}} \sigma(u, S) \mu_1(du)$$

(3.12)

for every Borel $S$ set in $\mathbb{R}$. Let $S_0 := (-\infty, -d) \cup (d, \infty)$. We see from (3.12) that $\mu_1(A) = 1$, where $A := \{u \in \mathbb{R} : \sigma(u, S_0) = 0\}$. Therefore, for every point $u_0 \in A$, the measure $\sigma(u_0, ds)$ has support contained in $[-d, d]$.

It remains to show that $\mu_1$ has a bounded support. Return to (3.11) with $u = u_0 \in A$. By Theorem 3.9, the function $Z(z) - u_0$ is holomorphic and real for $z = x < -d$ and for $z = x > d$. Since $Z(z)$ admits the representation

$$Z(z) = \alpha + z + \int_{\mathbb{R}} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right)(1 + t^2) \nu(dt),$$

where $\alpha \in \mathbb{R}$ and $\nu$ is a finite nonnegative measure, it follows from the Stieltjes–Perron inversion formula that the measure $\nu$ has bounded support contained in $[-d, d]$. Thus

$$Z(z) - u_0 = \gamma - u_0 + z + \int_{[-d,d]} \frac{(1 + t^2) \nu(dt)}{t - z}, \quad (3.13)$$

where $\gamma \in \mathbb{R}$. The parameter $\gamma$ and the measure $\nu$ depend on $Z$ only and do not depend on $u_0$. Let $u_0 > 0$ and be sufficiently large, i.e., $u_0 > c(Z) > 0$. Then, by (3.13), $Z(x) - u_0 < 0$ for $x = u_0/2 > 2d$, a contradiction with (3.11) for $u = u_0$ and $x = u_0/2$. An analogous argument holds for $u_0 < 0$. Hence there exists $c(Z) > 0$ such that the points $u_0 \in A$ satisfy the inequality $|u_0| \leq c(Z)$. The lemma is proved. \hfill \square

4. Auxiliary results on special functional equations.

In this section we first describe some results (see Kagan, Linnik, Rao [13]) on continuous solutions of special equations which were used to characterize distributions via independence of linear statistics and identical distribution of linear statistics.

Lemma 4.1. Consider the following equation, for $|u| < \delta_0$, $|v| < \delta_0$,

$$\psi_1(u + b_1v) + \cdots + \psi_r(u + b_r v) = A(u) + B(v) + P_k(u, v),$$

where $P_k$ is a polynomial of degree $k$; $\psi_j$, $A$ and $B$ are complex valued functions of two real variables $u$ and $v$. We assume that

(i) the numbers $b_j$ are all distinct
(ii) the functions $A$, $B$, and $\psi_j$ are continuous.

Then, in some neighborhood of the origin, all the functions $A$, $B$, and $\psi_j$ are polynomials of degree at most $\leq \max(r, k)$.

Consider the equation
\[
\int_0^1 v(st) dQ_1(s) = \int_0^1 v(st) dQ_2(s), \quad \text{for all } 0 < t < 1, \quad (4.1)
\]
for a bounded continuous function $v(t)$ defined on $(0, 1)$. Here $Q_1(s)$ and $Q_2(s)$ are nondecreasing functions satisfying the condition
\[
\int_0^1 s^{-b} d(Q_1(s) + Q_2(s)) < \infty \quad (4.2)
\]
for some $b > 0$. We assume that the relation (4.1) is nondegenerate, i.e., $Q_2(s) - Q_1(s) \neq \text{const}$.

Applying a Mellin transform to (4.1) we easily obtain, for $Q(s) := Q_1(s) - Q_2(s)$,
\[
\int_0^1 t^{z-1} dt \int_0^1 v(st) dQ(s) = \int_0^1 s^{-z} dQ(s) \int_0^s t^{z-1} v(t) dt = 0 \quad \text{for all } 0 < \text{Re } z < b. \quad (4.3)
\]
In view of (4.2) we deduce from (4.3), for $0 < \text{Re } z < b$,
\[
\Lambda(z) X(z; v) - K(z; v) = 0, \quad (4.4)
\]
where
\[
\Lambda(z) := \int_0^1 s^{-z} dQ(s), \quad X(z; v) := \int_0^1 t^{z-1} v(t) dt, \quad K(z; v) := \int_0^1 s^{-z} dQ(s) \int_0^s t^{z-1} v(t) dt. \quad (4.5)
\]
The functions $\Lambda(z)$ and $K(z; v)$ are analytic in the half-plane $\text{Re } z < b$, and the function $X(z; v)$ is analytic in the half-plane $\text{Re } z > 0$. We use the relation (4.4) for analytic continuation of $X(z; v)$ into the half-plane $\text{Re } z \leq 0$ as a meromorphic function. Keeping the same notation, we have
\[
X(z; v) = K(z; v) / \Lambda(z), \quad \text{Re } z < b. \quad (4.6)
\]
The singularities of $X(z; v)$ in the half-plane $\text{Re } z \leq 0$ happen to be poles distributed among the zeros of $\Lambda(z)$. 
Taking an arbitrary \( \lambda > 0 \), the inversion formula for the Mellin transform yields
\[
\int_0^t u^{\lambda - 1} \log(t/u)v(u) \, du = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} t^{\lambda - z} \frac{K(z; v)}{(z - \lambda)^2 \Lambda(z)} \, dz, \quad 0 < x < \min(b, \lambda). \tag{4.7}
\]

Let \( z_0 \) be some zero of \( \Lambda(z) \) of multiplicity \( m_0 \) in the half-plane \( \text{Re} \, z \leq 0 \). We see that
\[
\text{Res} \left( t^{\lambda - z} \frac{K(z; v)}{(z - \lambda)^2 \Lambda(z)} \right) = P_{z_0}(\log t) t^{\lambda - z_0}, \quad 0 < t < 1, \tag{4.8}
\]
where \( P_{z_0}(t) \) is a polynomial of degree at most \( m_0 - 1 \). These residues may depend on \( \lambda \), but it easily seen that the degree of the polynomial \( P_{z_0}(t) \) does not depend on \( \lambda \).

If \( P_{z_0}(t) \not\equiv 0 \), we call the number \( -z_0 \) an active exponent of the solution \( v(t) \) and the number \( \text{deg} P_{z_0}(t) + 1 \) will be called the multiplicity of the active exponent \( \xi = -z_0 \). All the active exponents of \( v(t) \) are located in the half-plane \( \text{Re} \, z \geq 0 \).

If the number of the active exponents \( \{z_k\}_{k=1}^d \) of \( v(t) \) is finite, then, by Jordan’s lemma on residues, it follows from (4.7) that
\[
\int_0^t u^{\lambda - 1} \log(t/u)v(u) \, du = t^\lambda \sum_{k=1}^d P_{z_k}(\log t) t^{-z_k}, \quad 0 < t < 1. \tag{4.9}
\]

We shall introduce the notation
\[
\sigma_1(v) := \inf \{\text{Re} \, \xi, \xi \text{ active exponents of } v(t)\}. \tag{4.10}
\]

We need the following results on active exponents and differentiable solutions \( v(t) \).

**Lemma 4.2.** If \( v(t) \geq 0 \) is a continuous solution of (4.1) such that \( v(t) \to 0 \) as \( t \to 0^+ \), then \( \sigma_1(v) \) is an active exponent and
\[
\sigma_1(v) > 0 \quad \text{and} \quad a_{\sigma_1(v)} > 0. \tag{4.11}
\]

**Lemma 4.3.** If under the hypothesis of Lemma 4.2 the function \( v(t) \) has a continuous derivative \( v^{(n)}(t) \) for some \( n \geq 1 \) and for all \( 0 < t < 1 \) and the following limit exists and is finite
\[
\lim_{t \to 0^+} v^{(n)}(t) = v^{(n)}_+(0),
\]
then all the active exponents \( \xi \) of \( v(t) \) which are not simultaneously integers and simple active exponents satisfy the condition \( \text{Re} \, \xi > n \).

5. **Proof of an analogue of the Darmois–Skitovich theorem.**

In this section we prove Theorem 2.1.

In order to prove Theorem 2.1 we follow the proof of the classical Darmois-Skitovich theorem (see [13]).
Proof. Necessity. Assume that free random variables \( T_1, \ldots, T_n \) with distributions \( \mu_1, \ldots, \mu_n \), respectively, are such that the linear statistics \( L_1 \) and \( L_2 \) (see (2.1)) are free and the coefficients \( a_j, b_j \) of these statistics satisfy the assumptions of Theorem 2.1. Then for every pair of real numbers \((u, v)\) the linear statistics \( uL_1 \) and \( vL_2 \) are free and we have the relation

\[
L := uL_1 + vL_2 = (ua_1 + vb_1)T_1 + \cdots + (ua_m + vb_m)T_m
\]

\[
+ (ua_{m+1} + vb_{m+1})T_{m+1} + \cdots + (ua_n + vb_n)T_n. \quad (5.1)
\]

Using (3.3), we deduce from (5.1) that

\[
\phi_{(ua_1+vb_1)T_1}(z) + \cdots + \phi_{(ua_n+vb_n)T_n}(z)
\]

\[
= \phi_{uL_1}(z) + \phi_{vL_2}(z) - \phi_{(ua_{m+1}+vb_{m+1})T_{m+1}}(z) - \cdots - \phi_{(ua_n+vb_n)T_n}(z) \quad (5.2)
\]

for \( z \in \Gamma_{\alpha,\beta} \) with some \( \alpha > 0 \) and \( \beta > 0 \), where all functions \( \phi_{(ua_j+vb_j)T_j}(z), j = 1, \ldots, n, \) and \( \phi_{uL_1}(z), \phi_{vL_2}(z) \) are defined. Hence (5.2) holds for \( z = i \) and for \( |u| \leq \delta \) and \( |v| \leq \delta \) with sufficiently small \( \delta > 0 \). Note that the functions \( \phi_{(ua_{m+1}+vb_{m+1})T_{m+1}}(z), \ldots, \phi_{(ua_n+vb_n)T_n}(z) \) depend on \( u \) and \( v \) only. Consider the functions \( \psi_j(w) := w\phi_{T_j}(i/w), j = 1, \ldots, n, \) for \( w \in \mathbb{R} \) and \( |w| \leq \delta' \) with sufficiently small \( \delta' > 0 \). Since \( \phi_{wT_j}(i) = w\phi_{T_j}(i/w), \) and \( w\phi_{T_j}(i/w) \to 0 \) as \( w \to 0 \), we see that (5.2) with \( z = i \) has the form

\[
\psi_1(ua_1 + vb_1) + \cdots + \psi_m(ua_m + vb_m) = A(u) + B(v), \quad |u| < \delta, \ |v| < \delta, \quad (5.3)
\]

where \( \psi_j, j = 1, \ldots, m, \) and \( A, B \) are complex-valued continuous functions, and \( b_j/a_j, j = 1, \ldots, m \) are all distinct. Then, by Lemma 4.1, the functions \( \psi_j, j = 1, \ldots, m, \) are polynomials of degree \( \leq m \). Therefore we have the representation

\[
\phi_{T_j}(z) = z\sum_{s=0}^{m} \frac{d_{sj}}{z^s}, \quad j = 1, \ldots, m, \quad (5.4)
\]

for \( z \in \Gamma_{\alpha',\beta'} \) with some \( \alpha' > 0 \) and \( \beta' > 0 \), where \( d_{sj} \) are complex valued coefficients. Since \( \phi_{T_j}(iy) = o(y) \) as \( y \to \infty \), all \( d_{0j} = 0 \). In view of the relations

\[
\phi_{-T_j}(iy) = -\text{Re} \phi_{T_j}(iy) + i \text{Im} \phi_{T_j}(iy), \quad y \geq \beta',
\]

we see that

\[
2\text{Im} \phi_{T_j}(iy) = \phi_{T_j}(iy) + \phi_{-T_j}(iy) = \sum_{l=1}^{[m/2]} (-1)^l \frac{d_{2lj}}{y^{2l-1}}
\]

and

\[
2\text{Re} \phi_{T_j}(iy) = \phi_{T_j}(iy) - \phi_{-T_j}(iy) = \sum_{l=0}^{[(m-1)/2]} (-1)^l \frac{d_{2l+1,j}}{y^{2l}}
\]

for \( y \geq \beta' \). We easily deduce from the last two relations that the coefficients \( d_{sj} \) are real-valued. Then, by Proposition 3.3, all moments \( m_k(\mu_j) \) exist and \( d_{sj} = \kappa_{s+1}(\mu_j), s = 1, \ldots, m \) and \( \kappa_s(\mu_j) = 0 \) for \( s > m \). By Proposition 3.4 the measures \( \mu_j, j = 1, \ldots, m, \) have compact supports. We now return to the relation (5.3). By (5.4), the functions on
both sides of \([5.3]\) are differentiable. Differentiating sequentially both sides of \((5.3)\) with respect to \(u\) and \(v\) we obtain a relation from which \((2.4)\) follows immediately.

**Sufficiency.** Assume that free random variables \(T_1, \ldots, T_n\) satisfy the assumptions of Theorem 2.1. Consider mixed cumulants \(\kappa_s(L_{j_1}, L_{j_2}, \ldots, L_{j_s})\) such that \(q\) indices \(j_k\) are equal 1 and \(s - q\) indices \(j_h\) are equal 2. Then, by Propositions 3.6, 3.7, and by \((2.4)\), \(L_1\) and \(L_2\) have vanishing mixed cumulants

\[
\kappa_s(L_{j_1}, L_{j_2}, \ldots, L_{j_s}) = \sum_{j=1}^{m} a_j^q b_j^{s-q} \kappa_s(T_j, \ldots, T_j) = \sum_{j=1}^{m} a_j^q b_j^{s-q} \kappa_s(T_j) = 0
\]

for \(s = 2, \ldots, m\) and \(q = 1, \ldots, s - 1\). In addition we clearly have \(\kappa_s(L_{j_1}, L_{j_2}, \ldots, L_{j_s}) = 0\) for all \(s \geq m + 1\) and \(q = 1, \ldots, s - 1\). Hence, by Theorem 3.5, the linear forms \(L_1\) and \(L_2\) are free independent.

The theorem is completely proved. \(\Box\)

### 6. Characterization of free cumulants

First we shall prove Theorem 2.2.

**Proof.** Sufficiency. We conclude from the definition of the domain \(\Omega_\varphi\) that its boundary is a Jordan curve consisting of a finite number of curves parametrized by algebraic functions. Moreover, by the assumptions of the theorem, there exist real numbers \(a < 0\) and \(b > 0\) such that \(\gamma := \gamma_1 \cup \gamma_2 \cup \gamma_3\), where \(\gamma_1\) is the half-line \(z = x\) with \(x \leq a\), \(\gamma_2\) is a Jordan curve laying in \(\mathbb{C}^+\) and connecting the point \(a\) and \(b\), \(\gamma_3\) is the half-line \(z = x\) with \(x \geq b\).

Note that the function \(\varphi(z) : \Omega_\varphi \to \mathbb{C}\) is analytic such that

\[
\lim_{R \to +\infty} \max_{|z|=R} |\varphi(z)| = 0.
\]

We shall show that the function \(f : \Omega_\varphi \to \mathbb{C}\) defined via \(z \mapsto z + \varphi(z)\) takes every value in \(\mathbb{C}^+\) precisely once. The inverse \(f^{-1} : \mathbb{C}^+ \to \mathbb{C}^+\) thus defined is in the class \(\mathcal{F}\).

Let \(R\) be a sufficiently large positive number. For every fixed \(w \in \mathbb{C}^+\) we consider a closed rectifiable curve \(\gamma_4\) consisting of a curve \(\gamma_{4,1}\), which is a part of the curve \(\gamma\), connecting \(-R\) to \(R\), and the circular arc \(\gamma_{4,2} : z = Re^{i\theta}\) with \(0 < \theta < \pi\) connecting \(R\) to \(-R\). The curve \(\gamma_4 = \gamma_4(R)\) depends on \(R\).

We see from the construction of the curve \(\gamma_{4,1}\) that if \(z\) runs through \(\gamma_{4,1}\) the image \(\zeta = f(z)\) lies on the interval \([-A_{-R}, A_R]\), where \(f(-R) = -A_{-R}\) and \(f(R) = A_R\). Here \(A_{\pm R} \to \infty\) as \(R \to \infty\). We note as well that if \(z\) runs through \(\gamma_{4,2}\) the image \(\zeta = f(z)\) lies in the domain \(|\zeta| \geq \min\{A_{-R}, A_R\}/2\), \(\text{Im} \zeta > 0\).

Hence \(f(z)\) winds around \(w \in \mathbb{C}^+\) once when \(z\) runs through \(\gamma_4\), and it follows from the argument principle that there is a unique point \(z_0\) in the interior of the curve \(\gamma_4\) such that \(f(z_0) = w\). Since this relation holds for all curves \(\gamma_4 = \gamma_4(R)\) with sufficiently large \(R > 0\), we deduce that the point \(z_0\) is unique in \(\Omega_\varphi\).

Therefore the inverse function \(f^{-1} : \mathbb{C}^+ \to \Omega_\varphi\) exists and is analytic on \(\mathbb{C}^+\). By condition \((6.1)\), \(z/f^{-1}(z) \to 1\) as \(z \to \infty\) non-tangentially to \(\mathbb{R}\) and therefore \(f^{-1}(z) \in\).
\[ \mathcal{F}. \] Hence there exists a probability measure \( \mu \) such that \( \varphi(z) = \phi_\mu(z) \) and the sufficiency of the assumptions of the theorem is proved.

**Necessity.** Let there exist a Jordan curve \( \gamma \), laying in \( \Omega_\varphi \) and connecting 0 and \( \infty \). We shall show that in this case it does not exist a probability measure \( \mu \) such that \( \varphi(z) = \phi_\mu(z) \). Assume to the contrary that there exists a probability measure \( \mu \) such that \( \varphi(z) = \phi_\mu(z) \) for \( z \in \Gamma_{\alpha,\beta} \) with some positive \( \alpha \) and \( \beta \). Then \( \varphi(z) = \phi_\mu(z) \) for \( z \in \mathbb{C}^+ \) and \( |z| \geq c \) with a sufficiently large constant \( c > 0 \). By Proposition 3.4 \( \mu \) has a compact support. It is obvious that the relation

\[ F_\mu(z + \varphi(z)) = z \tag{6.2} \]

holds for \( z \in \mathbb{C}^+, |z| \geq c \) and therefore it holds for \( z \in \Omega_\varphi \). Since \( \varphi(z) \to \infty \) as \( z \to 0 \), and \( F_\mu(z) = (1 + o(1))z \) as \( z \to \infty \), the relation (6.2) with \( z \in \gamma \) and \( z \to 0 \) leads to a contradiction. This proves the necessity of the assumptions of the theorem and completely proves the theorem. \( \square \)

**Proof of Remark 2.3** Since probability measures corresponding to the sequences of free cumulants of the set \( S \) have the uniformly bounded support, the conclusion of the remark follows immediately from the fact that \( S \) is conditionally compact. \( \square \)

**Proof of Corollary 2.5** Let \( 0 < \kappa_4 < \frac{1}{4} \). Without loss of generality we assume that \( \kappa_3 \leq 0 \). It follows from Theorem 2.2 that if the sequence \( 0, 1, \kappa_3, \kappa_4, 0, \ldots \) is a sequence of free cumulants of some probability measure then the polynomial \( P(r, x) := r^4 - r^2 - 2\kappa_3 x r + (1 - 4x^2)\kappa_4 \) has at the least one positive root for every fixed \( x \in [-1, 1] \). Denote \( r_{\max}(x) \) the maximum of such roots.

Assume that \( x \in [-1, -1/2] \). In this case \( P(r, x), r > 0, \) has one positive root \( r_{1,1}(x) = r_{\max}(x) \) and this root is a continuous function on \( [-1, -1/2] \). Let \( x \in (-1/2, 0) \). Then \( P(r, x) \) has two positive roots only, say \( r_{1,2}(x) \) and \( r_{2,2}(x) \) \( (r_{1,2}(x) < r_{2,2}(x) = r_{\max}(x)) \), and \( r_{1,2}(x) \to 0 \) as \( x \to -1/2 \). These roots are continuous functions on \( (-1/2, 0) \) and \( \lim_{x \to -1/2} r_{1,1}(x) = \lim_{x \to -1/2} r_{2,2}(x) \). In addition \( P(r, x) > 0 \) for \( r > r_{2,2}(x) \) and for \( 0 < r < r_{1,2}(x) \).

Consider the function

\[ \rho(x, \kappa_4) := r(x, \kappa_4)(1 - 2r^2(x, \kappa_4)), \quad \text{where} \quad r(x, \kappa_4) := \left\{ \frac{1}{2} \left( \frac{1}{3} + \sqrt{\frac{1}{9} - \frac{4}{3}(4x^2 - 1)\kappa_4} \right) \right\}^{1/2} \]

for \( 0 \leq x \leq x_1 = x_1(\kappa_4) := \min\{1, \frac{1}{2} \sqrt{1 + \frac{1}{12\kappa_4}} \} \). Note that \( x_1 = 1 \) for \( \kappa_4 \leq 1/36 \) and \( 1/\sqrt{3} < x_1 < 1 \) for \( 1/36 < \kappa_4 < 1/4 \).

It is easy to verify that the function \( \rho(x, \kappa_4)/x, 0 \leq x \leq x_1 \), is strictly monotone for \( 0 < x < x_2 \) and for \( x_2 \leq x \leq x_1 \) and has a unique minimum at the point \( x_2 = x_2(\kappa_4) := \min\{1, \frac{1}{2} \sqrt{\frac{1}{\kappa_4} - 2} \} \). Note that \( x_2 = \frac{1}{2} \sqrt{\frac{4}{3\kappa_4} - 2} \) if \( \kappa_4 \geq 1/36 \). In this case \( r(x_2, \kappa_4) = \sqrt{\kappa_4} \). Note as well that \( x_2 = 1 \) for \( \kappa_4 \leq 1/36 \) and \( x_2 < x_1 < 1 \) for \( 1/36 < \kappa_4 < 1/4 \).

Now let us show that if \( 0, 1, \kappa_3, \kappa_4, 0, 0, \ldots \) is a free cumulant sequence, then

\[ -\kappa_3 \leq \rho(x_2, \kappa_4)/x_2. \tag{6.3} \]
Let to the contrary $-\kappa_3 = \rho(x^*, \kappa_4)/x^*$ for some $x^* \in (0, x_2)$. Then we have the inequality

$$-\kappa_3 < \rho(x, \kappa_4)/x \quad \text{for} \quad 0 < x < x^* \quad \text{and} \quad -\kappa_3 > \rho(x, \kappa_4)/x \quad \text{for} \quad x^* < x < x_2.$$  

(6.4)

Fix a parameter $b := -(4x^2 - 1)\kappa_4$ with $x \in (0, x_1)$. The line $y = ar + b$ is a tangent to the curve $y = r^2(1 - r^2)$ at the point $r = r(x, \kappa_4)$ iff $a = 2\rho(x, \kappa_4)$.

Let $x \in (0, 1/2)$. Then $P(r, x)$ has a positive root iff $-\kappa_3 x \leq \rho(x)$. Assume that $x^* \in (0, 1/2)$ then, by (6.4), this inequality holds for $x \in (0, x^*)$ and there exists $x \in (x^*, 1/2)$ such that $P(r, x) > 0$ for $r > 0$, a contradiction with the assumptions of Theorem 2.2. Hence $x^* \notin (0, 1/2)$.

It is easy to see that, for every fixed $x \in (0, 1/2)$, $P(r, x)$ has a positive root iff $\lim_{x \to 0} P(r, x)$ has a positive root. Thus

$$\lim_{x \to 0} P(r, x) = \lim_{x \to 0} r_{1,3}(x) = \lim_{x \to 0} r_{3,4}(x) = \lim_{x \to 0} r_{2,3}(x).$$

In addition, $r_{1,3}(x) = \lim_{x \to 0} r_{2,3}(x) = \lim_{x \to 0} r_{3,4}(x)$ and $P(r, x)$ has a positive root iff $r_{1,3}(x) = r_{3,4}(x) = r_{2,3}(x)$. Hence we can conclude that $r_{1,3}(x)$ is a continuous function for $[-1, 1/2]$.

Now assume that $x^* \in [1/2, x_2)$. Let $x \in [1/2, x^*)$. It is easy to see in this case, in view of (6.4), that $P(r, x)$ has three positive roots, say $r_{1,4}(x) < r_{2,4}(x) < r_{3,4}(x) = r_{\max}(x)$. These functions are continuous for $x \in [1/2, x^*)$ and $\lim_{x \to 1/2+0} r_{2,4}(x) = \lim_{x \to 1/2-0} r_{1,3}(x)$, $\lim_{x \to 1/2+0} r_{3,4}(x) = \lim_{x \to 1/2-0} r_{2,3}(x)$. In addition, $r_{\max}(x)$ is a continuous function for $[1, x^*]$. By (6.4), we see that, for $x \in (x^*, x_2)$, $P(r, x)$ has only one positive root $r_{1,4}(x) = r_{\max}(x)$. Since three positive roots of $P(r, x)$, $r > 0$, with fixed $x \in (0, 1)$ coincide at the point $x = x_1$ only, we note that $\lim_{x \to x^*+0} r_{2,4}(x) = \lim_{x \to x^*-0} r_{3,4}(x)$ and $\lim_{x \to x^*-0} r_{\max}(x) = \lim_{x \to x^*-0} r_{\max}(x)$. Then there exists a Jordan curve in $\mathbb{C}^+$ containing $0$ and $\infty$ on which $\text{Im}(z + \varphi(z)) > 0$, a contradiction.

Thus $x^* = x_2$ and (6.3) is proved. We can rewrite this assumption in the following form.

If $0 < \kappa_4 \leq 1/36$, then $x_1 = x_2 = 1$ and we have the estimate $-\kappa_3 \leq \rho(1, \kappa_4)$.

If $1/36 < \kappa_4 < 1/4$, then $x_1 < 1$ and $x_2 \leq x_1$. In this case we have the estimate $-\kappa_3 \leq \rho(x_2, \kappa_4)/x_2$.

Now we assume that (6.3) holds. Let us show that $0, 1, \kappa_3, \kappa_4, 0, 0, \ldots$ is a free cumulant sequence.

Repeating the previous arguments we conclude that $r_{\max}(x)$ is a continuous function for $x \in [-1, x_1]$.

Finally note that if $x \in (x_1, 1]$ and $x_1 < 1$, then, for such fixed $x$ the polynomial $P(r, x)$ has one positive root $r_{1,5}(x)$ only. The function $r_{1,5}(x)$ is continuous on $x \in (x_1, 1]$ and such that $\lim_{x \to x_1-0} r_{3,4}(x) = \lim_{x \to x_1+0} r_{1,5}(x)$ and $r_{1,5}(1) > 0$. Therefore $r_{\max}(x)$ is a continuous function for all $x \in [-1, 1]$ and $r_{\max}(\pm 1) > 0$.

Hence the assumption (6.3) is necessary and sufficient in order that the sequence $0, 1, \kappa_3, \kappa_4, 0, \ldots$ is a sequence of free cumulants of some probability measure in the case $0 < \kappa_4 < 1/4$. 


Assume that $\kappa_4 > 1/4$. Then $P(r, 0) > 0$, $r > 0$, and, by Theorem 2.2, the sequence $0, 1, \kappa_3, \kappa_4, 0, \ldots$ is not a sequence of free cumulants of a probability measure.

We now assume that $-\frac{1}{12} < \kappa_4 < 0$. Note that $P(r, 1)$ has a positive root iff $-\kappa_3 \leq \rho(1, \kappa_4)$. But it is easy to see that under this condition $P(r, 1)$ has a positive root for every $x \in [-1, 1]$ and $r_{\text{max}}(x)$ is a continuous function on $[-1, 1]$. Hence the condition $-\kappa_3 \leq \rho(1, \kappa_4)$ is necessary and sufficient in order that the sequence $0, 1, \kappa_3, \kappa_4, 0, \ldots$ is a sequence of free cumulants of some probability measure.

Assume that $\kappa_4 < -\frac{1}{12}$ and $\kappa_3 \in \mathbb{R}$. In this case $P(r, \text{sign}(\kappa_3)) > 0$, $r > 0$, if $\kappa_3 \neq 0$ and $P(r, 1) > 0$, $r > 0$, if $\kappa_3 = 0$.

Assume that $\kappa_4 = 0$. We easily conclude from the previous arguments that the condition $|\kappa_3| \leq \frac{1}{3\sqrt{3}}$ is necessary and sufficient in order that the sequence $0, 1, \kappa_3, 0, 0, \ldots$ is a sequence of free cumulants of some probability measure.

It remains finally to note that the assertion of the corollary for $\kappa_4 = -1/12, 1/4$ follows immediately from Remark 2.3. □

7. Necessity of conditions for the characterization of semicircular measures. Auxiliary results

In order to prove Theorem 2.1 we need the following results.

The first of them is a description of $\boxplus$-stable distributions (see [6] and [4]).

Lemma 7.1. Every $\boxplus$-stable probability measure is equivalent to a unique probability measure whose Voiculescu transform is given by one of the following

(1) $\phi(z) = z^{-1}$;
(2) $\phi(z) = e^{i(\alpha-2)\rho \pi} z^{-\alpha+1}$ with $1 < \alpha < 2$, $0 \leq \rho \leq 1$;
(3) (i) $\phi(z) = 0$,
(ii) $\phi(z) = -2\rho i + 2(2\rho - 1)/\pi \log z$ with $0 \leq \rho \leq 1$;
(4) $\phi(z) = -e^{i\alpha \rho \pi} z^{-\alpha+1}$ with $0 < \alpha < 1$, $0 \leq \rho \leq 1$.

Here and in the sequel we choose the principal branch of the functions $z^{-\alpha+1}$ and $\log z$.

The stability index of a $\boxplus$-stable probability measure is equal to 2 in case (1), to $\alpha$ in cases (2) and (4), and to 1 in case (3). The parameter $\rho$ which appears in cases (2), (3) and (4) will be called the asymmetry coefficient, and one can see that the measure corresponding to the parameters $(\alpha, \rho)$ is the image of the measure with parameters $(\alpha, 1-\rho)$ by the map $t \mapsto -t$ on $\mathbb{R}$.

The next two lemmas are an analog of a result by Linnik (see [18], [19]).

Lemma 7.2. Let $\alpha > 1$ and $\alpha \neq 2m + 1$, where $m \in \mathbb{N}$. The function

$$\phi(z) = \frac{1}{z} - \varepsilon \cos \left( \frac{\alpha \pi}{2} \right) \frac{ie^{i\alpha \pi/2}}{z^{\alpha}}, \quad z \in \mathbb{C}^+,$$

with sufficiently small parameter $\varepsilon > 0$ is the Voiculescu transform of some symmetric probability measure.
Proof. Define the following region:

$$\Omega_{\alpha, \varepsilon} = \left\{ re^{i\theta} \in \mathbb{C}^+ : 0 < \theta < \pi, \ r^\alpha \left( r - \frac{1}{r} \right) > b \frac{\cos \left( \alpha \left( \theta - \pi/2 \right) \right)}{\sin \theta} \right\},$$

where $b := \varepsilon \cos(\alpha \pi/2)$. The region $\Omega_{\alpha, \varepsilon}$ is a Jordan domain with boundary curve $\gamma$ which is given by the equation $z = r(\theta)e^{i\theta}$, $0 < \theta < \pi$, where $r(\theta)$ is defined by the equation:

$$r^\alpha \left( r - \frac{1}{r} \right) = b \frac{\cos \left( \alpha \left( \theta - \pi/2 \right) \right)}{\sin \theta}.$$  \hfill (7.1)

We see that (7.1) has an unique solution $r(\theta)$ which is greater than 1 for $\theta$ such that $b \cos \left( \alpha \left( \theta - \pi/2 \right) \right) > 0$. If $b \cos \left( \alpha \left( \theta - \pi/2 \right) \right) < 0$, (7.1) has two solutions $r(\theta) < 1$. We choose the larger of them. If $\cos \left( \alpha \left( \theta - \pi/2 \right) \right) = 0$, then $r = 1$.

Note that the function $\phi(z) : \Omega_{\alpha, \varepsilon} \to \mathbb{C}$ is analytic with

$$\lim_{R \to +\infty} \max_{|z|=R, \Im z \geq 0} |\phi(z)| = 0.$$  \hfill (7.2)

We shall now show that the function $f : \Omega_{\alpha, \varepsilon} \to \mathbb{C}$ defined via $z \mapsto z + \phi(z)$ takes every value in $\mathbb{C}^+$ precisely once. The inverse $f^{(-1)} : \mathbb{C}^+ \to \mathbb{C}^+$ thus defined is in the class $\mathcal{F}$.

Denote by $a_R, \Re a_n > 0$, a point of an intersection of the curve $\gamma$ with the circle $|z| = R$ with sufficiently large $R \geq R_0$.

For every fixed $w \in \mathbb{C}^+$ we consider a closed rectifiable curve $\gamma_1$ consisting of some smooth curve $\gamma_{1,1}$, which is a part of the curve $\gamma$, connecting $-\bar{a}_R$ to $a_R$, the arc $\gamma_{1,2}$ of the semicircle $z = \Re e^{i\theta}$, $0 < \theta < \pi$, connecting $a_R$ to $-\bar{a}_R$. The curve $\gamma_1$ depend on $R$.

We see from the construction of the curve $\gamma_{1,1}$ that if $z$ runs through $\gamma_{1,1}$ the image $\zeta = f(z)$ lies on the interval $[-A_R, A_R]$, where $-A_R = f(-\bar{a}_R)$ and $A_R = f(a_R)$. Here $A_R \to \infty$ as $R \to \infty$. We note as well that if $z$ runs through $\gamma_{1,2}$ the image $\zeta = f(z)$ lies in the domain $|\zeta| \geq A_R/2, \Im \zeta > 0$.

Hence $f(z)$ winds around $w$ once when $z$ runs through $\gamma_1$, and it follows from the argument principle that inside the curve $\gamma_1$ there is a unique point $z_0$ such that $f(z_0) = w$. Since this relation holds for all sufficiently large $R > 0$, we deduce that the point $z_0$ is unique in $\Omega_{\alpha, \varepsilon}$.

Hence the inverse function $f^{(-1)} : \mathbb{C}^+ \to \mathbb{C}^+$ exists and is analytic in $\mathbb{C}^+$. By condition (7.2), $z/f^{(-1)}(z) \to 1$ as $z \to \infty, \Im z > 0$, non-tangentially to $\mathbb{R}$ and therefore $f^{(-1)}(z) \in \mathcal{F}$. This proves our assertion.

\hfill $\Box$

**Lemma 7.3.** The function

$$\phi(z) = 1 + \varepsilon \frac{(\log z - i\pi/2)}{z}, \quad z \in \mathbb{C}^+,$$

with sufficiently small parameter $\varepsilon > 0$ is the Voiculescu transform of some symmetric probability measure.
Proof. Denote \( z = re^{i\theta} \), \( r > 0 \), \( 0 < \theta < \pi \), and consider the function
\[
\psi(r, \theta) := r \sin \theta + \text{Im}(r^{i\theta}) = \left( r - \frac{1}{r} \right) \sin \theta - \frac{\varepsilon \sin \theta}{r} \log r + \frac{\varepsilon \cos \theta}{r} (\theta - \pi/2).
\]
We see from this formula that \( \psi(1, \theta) \leq 0 \) for \( 0 \leq \theta \leq \pi \). In addition, for every fixed \( \theta \in (0, \pi) \), \( \psi(r, \theta) \to +\infty \) as \( r \to +\infty \). Hence, for every fixed \( \theta \in (0, \pi) \), there exist points \( r_j \geq 1 \) such that \( \psi(r_j, \theta) = 0 \). Denote by \( r(\theta) \) maximum of them. Note that \( r(\theta) \to \infty \) as \( \theta \to 0 \) or \( \theta \to \pi \).

Introduce the curve \( \gamma \) by the equation \( z = r(\theta)e^{i\theta}, \ 0 < \theta < \pi \). Denote by \( \Omega_\varepsilon \) the domain \( \{z = r^{i\theta} : r > r(\theta), \ 0 < \theta < \pi\} \).

Note that the function \( \phi(z) : \Omega_\varepsilon \to \mathbb{C} \) is analytic with
\[
\lim_{R \to +\infty} \max_{|z| = R, \ \text{Im} z \geq 0} |\phi(z)| = 0. \tag{7.3}
\]
We shall now show that the function \( f : \Omega_\varepsilon \to \mathbb{C} \) defined via \( z \mapsto z + \phi(z) \) takes every value in \( \mathbb{C}^+ \) precisely once. The inverse \( f^{-1} : \mathbb{C}^+ \to \mathbb{C}^+ \) thus defined is in the class \( F \).

We define a closed rectifiable curve \( \gamma_1 \) in the same way as in the proof of Lemma 7.2. Repeating the argument of the proof of Lemma 7.2 we conclude that if \( z \) runs through \( \gamma_1 \) the image \( \zeta = f(z) \) winds around every fixed point \( w \in \mathbb{C}^+ \) once, and it follows from the argument principle that inside the curve \( \gamma_1 \) there is a unique point \( z_0 \) such that \( f(z_0) = w \). Since this relation holds for all sufficiently large \( R > 0 \), we deduce that the point \( z_0 \) is unique in \( \Omega_\varepsilon \).

Hence the inverse function \( f^{-1} : \mathbb{C}^+ \to \mathbb{C}^+ \) exists and is analytic in \( \mathbb{C}^+ \). By condition (7.3), \( z/f^{-1}(z) \to 1 \) as \( z \to \infty \) non-tangentially to \( \mathbb{R} \) and therefore \( f^{-1}(z) \in F \). This proves the lemma. \( \square \)

Remark 7.4. Using similar arguments as those in the proof of Lemma 7.3 one can prove a more general result.

Let \( m \) be a positive integer. The function
\[
\phi(z) = \frac{1 + \varepsilon (\log z - i\pi/2)^m}{z}, \quad z \in \mathbb{C}^+,
\]
with sufficiently small parameter \( \varepsilon > 0 \) is the Voiculescu transform of some symmetric probability measure.

8. Characterization of semicircular measures

In this section we shall prove Theorem 2.6, 2.10, 2.11 and Corollary 2.9. We use in the proof of these theorems some ideas of the papers [18], [19] and [32].

Proof of Theorem 2.6

Sufficiency. Let \( \mu \) be the non-degenerate distribution of free random variables \( T_1, \ldots, T_n \) which satisfy the relation \( L_1 \overset{D}{=} L_2 \). The Voiculescu transform \( \phi_\mu(z) \) of the probability measure \( \mu \) is defined in a domain \( \Gamma_{\alpha, \beta} \) with some \( \alpha > 0 \) and \( \beta > 0 \). Since one can extend the function \( G_\mu(z) \) on \( \mathbb{C}^- \) assuming \( G_\mu(z) = \overline{G_\mu(\bar{z})} \), we can extend the Voiculescu
transform $\phi_\mu(z)$ on the domain $-\Gamma_{\alpha,\beta}$ assuming $\phi_\mu(z) = \overline{\phi_\mu(\overline{z})}$. Now we note that the Voiculescu transform $\phi_\mu(z)$ satisfies the following equation

$$\sum_{j=1}^{n} a_j \phi_\mu(z/a_j) = \sum_{k=1}^{n} b_k \phi_\mu(z/b_k) \quad \text{for all } z \in \Gamma_{\alpha,\beta}. \tag{8.1}$$

Without loss of generality we assume that $\beta = 1$. As shown in Section 3, $\Im \phi_\mu(z) \leq 0$ for $z \in \Gamma_{\alpha,1}$. Denote

$$v(t) := t \Im \phi_\mu(i/t), \quad 0 < t < 1. \tag{8.2}$$

Note that the function $v(t), t \in (0, 1)$, is infinitely differentiable and $v(t) \to 0$ as $t \to 0+$. Moreover $v(at) = v(-at)$ for real $a$ and $t \in (0, 1)$, therefore it follows from (8.1) that

$$\sum_{j=1}^{n} v(|a_j|t) = \sum_{k=1}^{n} v(|b_k|t), \quad 0 < t < 1. \tag{8.3}$$

In the sequel we consider special solutions of these equations.

We shall apply the auxiliary results of Section 4 which describe solutions of the equation (8.3) in the case $Q(s) = Q_1(s) - Q_2(s)$, where $Q_1(s)$ and $Q_2(s)$ are distribution functions of the measures $\delta_{|a_1|} + \cdots + \delta_{|a_n|}$ and $\delta_{|b_1|} + \cdots + \delta_{|b_n|}$, respectively, and $v(t) := t \Im \phi_\mu(i/t)$. First we shall prove the following lemma.

**Lemma 8.1.** The parameter $\sigma_1(v)$, defined in (4.10), for a solution $v(t)$ of (8.3) is an active exponent and $0 < \sigma_1(v) \leq 2$, $a_{\sigma_1(v)} > 0$.

This lemma shows that if $v(t)$ from (8.2) is a solution of (8.3), then $\Lambda_1(z)$ has a root $\gamma$ such that $0 < \gamma \leq 2$.

**Proof.** By Lemma 4.2 we only need to prove the inequality $\sigma_1(v) \leq 2$. Let us assume to the contrary that $\sigma_1(v) = 2 + \eta$, $\eta > 0$. By the definition of $\sigma_1(v)$ we see that the function $X(z; v)$ is analytic for $\Re z > -2 - \eta$. Since $v(t) \geq 0$, $0 < t < 1$, we conclude by Lévy’s and Raikov’s theorem (see [20], Ch. 2, Theorem 2.2.1) that

$$\int_{0}^{1} t^{-3-\eta/2} v(t) \, dt < \infty.$$

It follows from this relation that there exists a sequence $\{t_l\}_{l=1}^\infty$ such that $t_l \to 0$ as $l \to \infty$ and for which

$$\lim_{l \to \infty} v(t_l)/t_l^2 = 0. \tag{8.4}$$

By Proposition 3.1

$$\phi_\mu(z) = z^2 \left( G_\mu(z) - \frac{1}{z} \right) (1 + q_\mu(z)), \quad z \in \Gamma_{\alpha_1,\beta_1}, \tag{8.5}$$

where $|q_\mu(z)| = o(1)$ as $z \to \infty$ non-tangentially to $\mathbb{R}$. Denote by $\overline{\mu}$ the probability measure such that $\overline{\mu}(S) := \mu(-S)$ for any Borel set $S$. It is easy to see that $\Im \phi_\mu(iy) =$
\[ \frac{1}{2} \text{Im} \phi_{\mu \mathcal{P}}(iy) \text{ for } y \geq y_0 > 0. \] In addition the measure \( \mu \boxplus \mathcal{P} \) is symmetric. Therefore it easily follows from (8.5) that the relation

\[
\text{Im} \phi_{\mu}(iy) = -\frac{y^2}{2} \text{Im} \left( G_{\mu \mathcal{P}}(iy) - \frac{1}{iy} \right) (1 + \text{Re} q_{\mu \mathcal{P}}(iy))
\]

\[
= -\frac{1}{2y} \int_{\mathbb{R}} \frac{u^2}{u^2 + y^2} (\mu \boxplus \mathcal{P})(du)(1 + \text{Re} q_{\mu \mathcal{P}}(iy)),
\]

holds, where \( \text{Re} q_{\mu \mathcal{P}}(iy) \to 0 \) as \( y \to \infty \). We conclude from (8.4) and (8.6) that

\[
\int_{\mathbb{R}} \frac{u^2}{u^2 + y^2} (\mu \boxplus \mathcal{P})(du) = o(1/y^2), \quad l \to \infty,
\]

for \( y_l := 1/t_i \). This relation implies \( \int_{\mathbb{R}} u^2 (\mu \boxplus \mathcal{P})(du) = 0 \) and therefore the measure \( \mu \boxplus \mathcal{P} = \delta_0 \). Since \( \phi_{\mu \mathcal{P}}(z) = \phi_{\mu}(z) + \phi_{\mathcal{P}}(z) = 0 \) for \( z \in \Gamma_{\alpha_1, \beta_1} \) with some \( \alpha_1, \beta_1 > 0 \), and \( \text{Im} \phi_{\mu}(z) \leq 0 \) and \( \text{Im} \phi_{\mathcal{P}}(z) \leq 0 \) for such \( z \), we easily conclude that \( \phi_{\mu}(z) = 0, z \in \Gamma_{\alpha_1, \beta_1} \), and \( \mu = \delta_a, a \in \mathbb{R} \), a contradiction. The lemma is proved. \( \square \)

From the definition of the active exponent \( \sigma_1(v) \) (see (4.8), where \( K(z; v) \) and \( \Lambda(z) \) are defined in (4.5) with \( Q(s) = Q_1(s) - Q_2(s) \), where \( Q_1(s) \) and \( Q_2(s) \) are distribution functions of the measures \( \delta_{[a_1]} + \cdots + \delta_{[a_n]} \) and \( \delta_{[b_1]} + \cdots + \delta_{[b_n]} \), respectively, and \( v(t) := t \text{Im} \phi_{\mu}(i/t) \)) we conclude that \( \sigma_1(v), 0 < \sigma_1(v) \leq 2, \) is a root of the function \( \Lambda_1(z) \).

By the assumptions of the theorem and Lemma 4.2 it follows that \( \sigma_1(v) = 2 \). Consider the function \( v_1(t) := v(t) - a_2 t^2 \), where we have chosen \( a_2 := a_2(v)(2 + \lambda)^2 \). The coefficient \( a_2(v) \) and the parameter \( \lambda \) were chosen in Section 4. It is clear that \( v_1(t) \) is a solution of equation (8.3). Moreover

\[
K(z; v_1) := K(z; v) - a_2 \int_0^1 \frac{s^{-z} - s^2}{z + 2} \, dQ(s)
\]

\[
= K(z; v) - a_2 \frac{\Lambda_1(-z) - \Lambda_1(2)}{z + 2} = K(z; v) - a_2 \frac{\Lambda_1(-z)}{z + 2}.
\]

Therefore

\[
\text{Res}_{z=-2} \left( t^{\lambda-z} \frac{K(z; v_1)}{(z - \lambda)^2 \Lambda_1(-z)} \right)
\]

\[
= \text{Res}_{z=-2} \left( t^{\lambda-z} \frac{K(z; v)}{(z - \lambda)^2 \Lambda_1(-z)} \right) - a_2 \text{Res}_{z=-2} \left( \frac{t^{\lambda-z}}{(z - \lambda)^2 (z + 2)} \right)
\]

\[
= t^{\lambda+2} \left( a_2(v) - \frac{a_2}{2 (2 + \lambda)^2} \right) = 0. \quad (8.7)
\]

Thus, we may choose \( a_2 \) in such a way that \( 2 \) is not an active exponent of the solution \( v_1(t) \). Hence \( v_1(t) \) has no active positive exponents.
Hence we arrive at two cases. In the first case \( v_1(t) \neq 0 \) in some interval \((0, t_0)\) with \(0 < t_0 \leq 1\). In the second case there exists a sequence \( \{t_k\} \), \(0 < t_k \leq 1\), \(\lim_{k \to \infty} t_k = 0\), such that \( v_1(t_k) = 0\).

In the first case, by Lemma 4.2 there exists a positive active exponent of the solution \(v_1(t)\), a contradiction. Hence, we may consider the second case only. In this case it is easy to see that the function \(\phi_{\mu \boxplus \overline{\mu}}(z)\) satisfies the assumptions of Lemma 3.8 and we obtain that \(\mu \boxplus \overline{\mu}\) is a semicircular measure.

In order to complete the proof of the sufficiency of the assumptions of the theorem it remains to apply the following lemma.

**Lemma 8.2.** Assume that the function \(\Lambda_2(z)\) satisfies the condition: \(\Lambda_2(2k - 1) \neq 0\) for all \(k = 2, 3, \ldots\). Let the statistics \(L_1\) and \(L_2\) be identically distributed and let \(\mu \boxplus \overline{\mu}\) be a semicircular measure. Then \(\mu\) is a semicircular measure as well.

*Proof.* By Lemma 3.10 the measure \(\mu\) has a compact support. Hence, the Voiculescu transform \(\phi_{\mu}(z)\) is an analytic function in the domain \(|z| > R\) with some parameter \(R > 0\) and it admits in this domain the following Laurent expansion

\[
\phi_{\mu}(z) = \kappa_1 + \frac{\kappa_2}{z} + \sum_{l=3}^{\infty} \frac{\kappa_l}{z^{l-1}}.
\]

Here \(\kappa_2 \geq 0\). Since \(\mu \boxplus \overline{\mu}\) is a semicircular measure, we have, using (2.3),

\[
\sum_{l=1}^{\infty} \frac{\kappa_{2l}}{z^{2l-1}} = \frac{b}{z}, \quad |z| > R,
\]

where \(b > 0\). From this formula we deduce that \(\kappa_2 = b\) and \(\kappa_{2l} = 0\) for \(l = 2, 3, \ldots\). Since the function \(\phi_{\mu}(z)\) satisfies the equation (8.1), we obtain the relation

\[
\frac{\kappa_2 \Lambda_2(2)}{z} + \sum_{l=1}^{\infty} \frac{\kappa_{2l-1} \Lambda_2(2l-1)}{z^{2l-2}} = 0, \quad |z| > R.
\]

By the assumptions of the lemma \(\Lambda_2(2l - 1) \neq 0\) for \(l = 2, 3, \ldots\), we conclude that \(\kappa_{2l-1} = 0\) for \(l = 2, 3, \ldots\).

Thus, the lemma is proved. \(\square\)

*Necessity.* We note that in order that the statement (1) of the theorem implies the statement (2) it is necessary that \(\Lambda_1(2) = 0\).

We shall first assume that the function \(\Lambda_1(z)\) has a root \(\gamma_1\) such that \(0 < \gamma_1 < 2\). Let \(0 < \gamma_1 < 1\) or \(1 < \gamma_1 < 2\). By Lemma 7.1 there exist a symmetric probability measure \(\mu\) whose the Voiculescu transform has the form \(\phi_{\mu}(z) = -e^{i\gamma_1 \pi/2} z^{-\gamma_1 + 1}\). We conclude for
this function that
\[ \sum_{j=1}^{n} a_j \phi_{\mu}(z/a_j) - \sum_{k=1}^{n} b_k \phi_{\mu}(z/b_k) = \sum_{j=1}^{n} |a_j| \phi_{\mu}(z/|a_j|) - \sum_{k=1}^{n} |b_k| \phi_{\mu}(z/|b_k|) \]
\[ = -e^{i\gamma_1 \pi/2} z^{-\gamma_1+1} \Lambda_1(\gamma_1) = 0, \quad z \in \mathbb{C}^+. \]

Let \( \gamma_1 = 1 \). By Lemma 7.1, there exist symmetric probability measure \( \mu \) whose the Voiculescu transform has the form \( \phi_{\mu}(z) = -i \). We obtain for this function
\[ \sum_{j=1}^{n} a_j \phi_{\mu}(z/a_j) - \sum_{k=1}^{n} b_k \phi_{\mu}(z/b_k) = -i \Lambda_1(1) = 0, \quad z \in \mathbb{C}^+. \]

We shall now assume that \( \gamma_1 = 2 \) and 2 is not a simple root of the function \( \Lambda_1(z) \). By Lemma 7.3, there exist a symmetric probability measure \( \mu \) whose the Voiculescu transform has the form
\[ \phi_{\mu}(z) = 1 + \varepsilon \left( \log z - i\pi/2 \right), \quad z \in \mathbb{C}^+, \]
with sufficiently small parameter \( \varepsilon > 0 \). It is easy to see that
\[ \sum_{j=1}^{n} a_j \phi_{\mu}(z/a_j) - \sum_{k=1}^{n} b_k \phi_{\mu}(z/b_k) = \Lambda_1(2) \frac{1}{z} + \varepsilon \frac{1}{z} \sum_{s=0}^{1} (-1)^s \left( \log z - i\pi/2 \right)^s \Lambda_1^{(1-s)}(2) = 0. \]

Assume that \( \gamma_1 > 2 \) and \( \gamma_1 \) is not even. By Lemma 7.2, there exist a symmetric probability measure \( \mu \) whose the Voiculescu transform \( \phi_{\mu}(z) \) has the form
\[ \phi_{\mu}(z) = \frac{1}{z} - \varepsilon \cos \left( (\gamma_1 - 1)\pi/2 \right) \frac{ie^{i(\gamma_1-1)\pi/2}}{z^{\gamma_1-1}}, \quad z \in \mathbb{C}^+, \]
with sufficiently small parameter \( \varepsilon > 0 \). We deduce as above that
\[ \sum_{j=1}^{n} a_j \phi_{\mu}(z/a_j) - \sum_{k=1}^{n} b_k \phi_{\mu}(z/b_k) = \Lambda_1(2) \frac{1}{z} - \varepsilon \cos \left( (\gamma_1 - 1)\pi/2 \right) \frac{ie^{i(\gamma_1-1)\pi/2}}{z^{\gamma_1-1}} \Lambda_1(\gamma_1) = 0 \]
for \( z \in \mathbb{C}^+ \).

We shall now show that if there exists a positive integer \( m > 2 \) such that \( \Lambda_2(m) = 0 \), then the statement (1) of the theorem does not imply the statement (2). Using Corollary 2.4 (see [5] as well), consider a probability measure \( \mu \) with the Voiculescu transform
\[ \phi_{\mu}(z) := \frac{1}{z} + \frac{\varepsilon}{z^{m-1}}, \]
where \( \varepsilon \in \mathbb{R} \) and is sufficiently small by modulus. We easily see that the function \( \phi_{\mu}(z) \) satisfies the equation (8.1). Moreover, the probability measure has a compact support.

Thus, we have established that if 2 is not unique simple positive zero of the function \( \Lambda_1(z) \) or there exist odd positive numbers \( 2l+1 \geq 3 \) such that \( \Lambda_2(2l+1) = 0 \) the statement (1) does not imply the statement (2) of the theorem.

The theorem is completely proved. \( \square \)
**Proof of Theorem 2.10.** The proof of this theorem easily follows from the arguments that we used in the proof of Theorem 2.6. Therefore we omit it. □

**Proof of Theorem 2.11.** We keep all previous notations. We assume that the statistics $L_1$ and $L_2$ are identically distributed. By the assumptions of the theorem, $m_{2s}(\mu) < \infty$ with $s := [\gamma/2 + 1]$, where $\gamma$ is maximum of the real parts of zeros of the function $\Lambda_1(z)$. By Proposition 3.3 and (8.2), we have

$$v(t) := t \Im \phi_\mu(i/t) = -\kappa_2(\mu)t^2 + \cdots + (-1)^s\kappa_{2s}(\mu)t^{2s} + o(t^{2s}), \quad t \to +0.$$  

Therefore \( \lim_{t \to +0} v(2s)(t) = (-1)^s(2s)!\kappa_{2s}(\mu). \) We now conclude from Lemmas 4.2 and 4.3 that all active exponents of \( v(t) \) are positive integers and simultaneously simple exponents. Since the number of active exponents of \( v(t) \) is finite we can use the formula (4.9). Using this identity we easily obtain the relation

$$\frac{1}{2} \Im \phi_{\mu \boxplus \mu}(i/t) - \Im \phi_\mu(i/t) = \sum_{l=1}^{2s} b_l t^{l-1}, \quad 0 < t < 1,$$  

(8.8)

where \( b_l, l = 1, \ldots, 2s, \) are real coefficients. We deduce from (8.8) that \( b_l = \frac{1}{2}\kappa_l(\mu \boxplus \mu), l = 1, \ldots, 2s, \) and \( \kappa_l(\mu \boxplus \mu) = 0 \) for \( l \geq 2s + 1. \) The function $\phi_\mu(z)$ satisfies the equation (8.1). Therefore, using (8.8), we get

$$\sum_{l=1}^{2s} \Lambda_2(l) \frac{\kappa_l(\mu \boxplus \mu)}{z^{l-1}} = 0, \quad z \in \mathbb{C}^+.$$

(8.9)

We conclude from (8.9) that $\kappa_l(\mu \boxplus \mu) = 0$ for $l = 2, \ldots$. Thus, $\mu \boxplus \mu$ is a semicircular measure.

From the assumption of the theorem and Lemma 8.2 it follows that $\mu$ is a semicircular measure as well.

One can prove the necessity of the assumptions of Theorem 2.11 in the same way as in the proof of the necessity of the assumptions of Theorem 2.6.

Thus, the theorem is completely proved. □

**Proof of Corollary 2.9.** Since \( |a_j| \leq 1, j = 1, \ldots, n, \) and $\Lambda_1(2) = 0$, we note that

$$|a_1|^x + \cdots + |a_n|^x > 1 \quad \text{for} \quad 0 < x < 2 \quad \text{and} \quad |a_1|^x + \cdots + |a_n|^x < 1 \quad \text{for} \quad x > 2.$$  

Thus, $\Lambda_1(x) \neq 0$ for all positive $x \neq 2$. In addition, it is easy to see that $\Lambda'_1(2) \neq 0$. By the inequality

$$|a_1|^m + \cdots + |a_n|^m \leq |a_1|^m + \cdots + |a_n|^m < 1 \quad \text{for} \quad m = 3, \ldots,$$

we have $\Lambda_2(m) \neq 0, m = 3, \ldots$. By Theorem 2.6, if $\mu$ is a non-degenerate probability measure, then $\mu$ is a semicircular measure. If $\mu = \delta_a$ with some $a \in \mathbb{R}$, then (8.1) holds if $\Lambda_2(1) = 0$. If $\Lambda_2(1) \neq 0$, then (8.1) holds for $\mu = \delta_0$ only. □
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