A set of multi-homogeneous equations for the Jacobian of a genus two curve is given. The approach used is to write down affine equations for the Jacobian minus various translations of the Theta-divisor by [2]-division points, and then to write down affine glue equations for the overlaps. Taking multi-projective completions for all of these then yields a complete multi-projective two-dimensional variety whose points are in one-to-one correspondence with degree zero divisor classes on the curve (i.e. the Picard group). This multi-projective variety then becomes a complete projective two-dimensional variety under the Segre imbedding.

Introduction and Motivation

The Jacobian variety associated with an algebraic curve is well known and widely studied. For such an important mathematical object, it is remarkable how difficult it is to write out defining equations. Of course, in genus one, which are elliptic curves, there is no problem. However, for anything beyond the genus one case, the situation is vastly more difficult. Even for genus two, where the Jacobian is an abelian surface, a set of defining equations is daunting.

From a fairly elementary viewpoint, the group of divisor classes of degree 0 on $C$ is the Picard group $\text{Pic}^0(C)$. This is defined as an abstract group and a priori does not have the structure of an algebraic variety. The realization of $\text{Pic}^0(C)$ as a projective algebraic variety is simply $\text{Jac}(C)$.

Few authors even try to write equations for abelian varieties (see siteMi1, [Mi2], and [Mu1], for example) and even when they do, (see [Mu2]), the approaches are far from concrete. Previous works by Flynn (see [Fl1], [Fl2], and [CF]), Grant (see [Gr]), and Wamellen ([Wa]) have yielded sets of such defining equations, but they tend to be long and complicated. Their approach has been to write down a basis for the linear system of a very ample divisor on the Jacobian and then take all the relations between functions in this linear system. Another approach by Anderson (see [An]) also yields a complicated set of equations based on invariant theory.

Key words and phrases. hyperelliptic curve, Jacobian variety, mult-projective space.

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There are many reasons for the difficulty of writing out such a set of equations. As has been noted by Mumford and Cantor (see [Ca] and [Mu3]), building on an approach going back to Jacobi, writing out a set of defining equations for an affine variety for a large piece of the Jacobian of a hyperelliptic curve $C$ is relatively straightforward. In general, if $C$ is an algebraic curve of genus $g$, then $\text{Jac}(C)$ is a projective variety (and therefore complete) of dimension $g$.

However, $\text{Jac}(C) - \Theta$ is an affine variety, where $\Theta$ (the theta divisor) is a subvariety of codimension 1, and in the case of $g = 2$, the variety $\Theta$ is simply a copy of the curve $C$ imbedded into $\text{Jac}(C)$. It is relatively straightforward to show that $\text{Jac}(C) - \Theta$ can be written as a system of $g$ equations in $2g$ variables. The problem is that the projective completion of $\text{Jac}(C) - \Theta$ does not yield $\text{Jac}(C)$. In fact, for $g > 1$, $\text{Jac}(C)$ is not a complete intersection, so there will always be an overdetermined system of defining equations for $\text{Jac}(C)$. (One way of seeing this problem is that the middle dimension of the cohomology of $\text{Jac}(C)$ is non-zero, while for a complete intersection, only the highest dimension cohomology is non-zero.)

The goal of this note is to show how to write out a set of equations for $\text{Jac}(C)$ that are not particularly complicated. The idea is to realize $\text{Jac}(C)$ as a multi-projective variety. Then the Segre imbedding of multi-projective space into ordinary projective space will give $\text{Jac}(C)$ as a projective variety.

This approach is implicit in Mumford’s work, although it is not explicitly worked out there. The idea is that multiple copies of the affine variety $\text{Jac}(C) - \Theta$ can be glued together to yield a projective variety whose points are in one-to-one correspondence with the elements of $\text{Pic}^0(C)$. These glue equations are also fairly simple and simply represent the results of adding points of order 2 in $\text{Pic}^0(C)$. More specifically, what really gets glued together are multiple copies of the projectivization of $\text{Jac}(C) - \Theta$, and what needs to be carefully checked is that there are no “extraneous” points on this glued together variety, i.e. that every geometric point corresponds to an actual element of $\text{Pic}^0(C)$.

One drawback of this approach is that the projective space in which $\text{Jac}(C)$ is imbedded is of high dimension, much larger than what others get. One advantage is that all the variables have a natural understanding and interpretation in the current approach. This approach also does not require the Riemann-Roch theorem on $\text{Jac}(C)$ in order to construct the imbedding. (There is a very ample divisor lurking in the background, but it is not at all explicit.)

Another advantage of this approach is its elementary nature, since it only requires rudimentary facts about multi-projective space and the Segre imbedding. One further drawback of this approach is that it requires that the base field be extended so that at least three of the Weierstrass points on $C$ are rational. Over algebraically closed fields this is not a problem, but some of the other constructions are not so constrained.

There are some other things that are not addressed in this note. First, it is not shown that $\text{Jac}(C)$ is an irreducible variety, i.e. that it does not have multiple components. It is also not shown that the equations given for $\text{Jac}(C)$ generate a prime ideal. From a geometric perspective, it is also not shown here that the defining equations for $\text{Jac}(C)$ yield a smooth (i.e. non-singular) variety, nor is it shown that the addition formulas are smooth.

It is however, relatively straightforward (using most computer algebra systems) to show that $\text{Jac}(C) - \Theta$ is an irreducible affine variety (i.e. that the defining equations for $\text{Jac}(C) - \Theta$ give a prime ideal) and that it is smooth (i.e. there are
not singular points).

This paper has not been written in the most compact way possible, and to some it may seem to ramble. However the author feels that it is important to give an exposition with adequate background for the construction of genus 2 Jacobians that might be accessible to non-experts, particularly those with a limited background in modern algebraic geometry.

The general organization of this paper is as follows. We begin with a number of remarks about multi-projective varieties, first reviewing the well known constructions of biprojective varieties and how they can be turned into projective varieties via the Segre map. This is then extended in a rather obvious way to multi-projective varieties. Then there is a short general discussion on divisor classes on hyperelliptic curves with an explanation of what the $\Theta$-divisor is and how $\text{Jac}(C) - \Theta$ is readily realizable as an affine variety. This is applied to the case of genus 2 curves in a very explicit way to yield a pair of equations in 4 variables that determine this affine variety. This is followed by a discussion of how to add a point of order two, beginning with a special case that yields a very simple set of formulas for addition. This leads to an analysis of a biprojective variety that comes fairly close to being the desired Jacobian, but which still has problems at the projective closure. In order to patch this problem up, the next idea is to work in a product of four projective spaces with associated glue equations, but in order to accomplish this, it is preceded by a discussion of how to modify the addition equations for more general Weierstrass points, and introduces some simplifying notation. This is followed by an analysis of the infinity types of the associated quadri-projective variety, which almost gets the right answer, however, there are still some additional points on this variety that should not be there. We finally succeed in solving all these problems by going to 8 copies of the affine variety that are all glued together by 3 different affine Weierstrass points on the original genus 2 curve.

This results is a rather large set of equations in a high dimensional multi-projective space, that can be made projective via the Segre embedding. These equations have a very systematic structure consisting of a pair of homogeneous equations at each corner of a cube and a set 9 bihomogeneous equations along each edge of the cube, referred to as glue equations. In the end, it comes out that the points on this multi-projective variety are in one-to-one correspondence with divisor classes on the original genus 2 curve.

Multi-Projective Space

Projective Space.

Working over a fixed base field, $K$, $n$-dimensional projective space, denoted $\mathbb{P}^n$, is defined by a set of $n+1$ projective coordinates, $S = \{x_0, x_1, \ldots, x_n\}$, which cannot all simultaneously be 0. Two sets of coordinates define the same point in $\mathbb{P}^n$ if all the coordinates are in a common ratio. Thus $(x_0, x_1, \ldots, x_n)$ and $(x'_0, x'_1, \ldots, x'_n)$ define the same point in $\mathbb{P}^n$ if there is some non-zero $\alpha$ such that $x'_i = \alpha x_i$ for $i = 0, 1, \ldots, n$.

Affine $n$-space, denoted $\mathbb{A}^n$ corresponds to projective points with $x_0 \neq 0$, so $\mathbb{A}^n$ can be understood as just $K^n$, which is a vector space. In general, if $V$ is an $n$-dimensional vector space over $K$, $\mathbb{P}(V)$, which is isomorphic to $\mathbb{P}^n$, can be understood as the set of lines in $V$.

Projective Varieties.
A polynomial, all of whose monomial terms are of degree $d$ in the variables $S$ is said to be a homogeneous polynomial of degree $d$ in the variables $S$. If $F(x_0, x_1, \ldots, x_n)$ is homogeneous of degree $d$, then

$$F(\alpha x_0, \alpha x_1, \ldots, \alpha x_n) = \alpha^n F(x_0, x_1, \ldots, x_n)$$

identically, and therefore the set of zeros of $F$ give rises to a set of well defined points in $\mathbb{P}^n$. Such a zero set is a projective variety.

If $f(x_1, \ldots, x_n)$ is any (not necessarily homogeneous) polynomial of degree $d$ in the variables $x_1, \ldots, x_n$, there is a corresponding homogeneous polynomial defined by

$$\tilde{f}(x_0, x_1, \ldots, x_n) = F(x_0, x_1, \ldots, x_n) = x_0^d f(x_1/x_0, \ldots, x_n/x_0).$$

The affine variety defined by $f(x_1, \ldots, x_n) = 0$ can be viewed as a subset of the projective variety defined by $\tilde{f}(x_0, x_1, \ldots, x_n) = 0$ corresponding to the affine points, which are defined to be where $x_0 \neq 0$.

Homogeneous polynomials give rise to projective varieties (actually, projective algebraic sets), and there is a corresponding notion of projective ideals, which are generated by projective polynomials. If $I$ is an ideal in $K[x_1, \ldots, x_n]$, then its projective closure is the projective ideal

$$\tilde{I} = \{ \tilde{f}(x_0, x_1, \ldots, x_n) | f(x_1, \ldots, x_n) \in I \}$$

in $K[x_0, x_1, \ldots, x_n]$ and if $A$ is the affine algebraic set in $\mathbb{A}^n$ determined by $I$, then its projective closure $\tilde{A}$ is the projective algebraic set in $\mathbb{P}^n$ determined by $\tilde{I}$. Computationally, a basis for $\tilde{I}$ can be determined by homogenizing a Grobner basis for $I$. If $A \subset \mathbb{A}^n$ is an affine variety, so that $I$ is a prime ideal, then $\tilde{I}$ is also a prime ideal, and $\tilde{A}$ is a projective variety.

**Hyperplanes in Projective Space.**

A hyperplane in $\mathbb{A}^n$ is defined by a linear equation, i.e. of the form

$$b = \sum_{i=1}^{n} a_i x_i$$

with the $a_i$’s not all zero. Any such hyperplane is isomorphic to a translation of $\mathbb{A}^{n-1}$.

Correspondingly, a hyperplane $\mathcal{H}$ in $\mathbb{P}^n$ is also defined by a homogeneous linear equation

$$0 = \sum_{i=0}^{n} a_i x_i$$

with the $a_i$’s not all zero. Any such hyperplane is isomorphic to $\mathbb{P}^{n-1}$. The intersection of a set of hyperplanes $\cap \mathcal{H}_j$ with each $\mathcal{H}_j$ defined by a homogeneous linear equation $0 = \sum_{i=0}^{n} a_{ji} x_i$ is isomorphic to $\mathbb{P}^k$ for some $k$. The dimension $k$ of such a system corresponds to the rank of the matrix $(a_{ji})$, i.e. the number of independent linear equations.
Biprojective Space.

As a set, biprojective \((m,n)\)-space is just \(\mathbb{P}^m \times \mathbb{P}^n\). The Segre map allows \(\mathbb{P}^m \times \mathbb{P}^n\) to be defined as a projective variety in \(\mathbb{P}^{(m+1)(n+1)−1}\). If \(V_1\) and \(V_2\) are vector spaces over \(K\) of dimensions \(n\) and \(m\) respectively, the Segre map can be viewed as a natural imbedding of \(\mathbb{P}(V_1) \times \mathbb{P}(V_2)\) into \(\mathbb{P}(V_1 \otimes V_2)\).

The Segre map

\[
\sigma_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \rightarrow \mathbb{P}^{(m+1)(n+1)−1}
\]

is defined as follows. If \(P_1 = (x_0, x_1, \ldots, x_m) \in \mathbb{P}^m\) and \(P_2 = (y_0, y_1, \ldots, y_n) \in \mathbb{P}^n\) are given, then by definition, the Segre map associates the point

\[
(x_0 y_0, x_1 y_0, \ldots, x_m y_0, x_0 y_1, x_1 y_1, \ldots, x_m y_1, x_0 y_2, \ldots, x_m y_m) \in \mathbb{P}^{(m+1)(n+1)−1}
\]

with \((P_1, P_2) \in \mathbb{P}^m \times \mathbb{P}^n\). The image of the Segre map in \(\mathbb{P}^{(m+1)(n+1)−1}\) is called the Segre variety \(\Sigma_{m,n}\). If the coordinates for \(\mathbb{P}^{(m+1)(n+1)−1}\) are \(z_k\) for \(k = 0, 1, \ldots, (m+1)(n+1)−1\), then

\[
z_{j(n+1)+i} = x_i y_j
\]

defines the Segre map. It is often easier to write \(z_{j(n+1)+i} = z_{i,j}\) for the coordinates of \(\mathbb{P}^{(m+1)(n+1)−1}\) where \(i\) ranges from 0 through \(m\) and \(j\) ranges from 0 through \(n\).

An important set of relations that hold are

\[
z_{i,j} z_{k,l} = z_{i,l} z_{k,j}
\]

for \(0 \leq i, k \leq m\) and \(0 \leq j, l \leq n\), and these relations define the ideal that determines \(\mathbb{P}^m \times \mathbb{P}^n\) as a projective variety in \(\mathbb{P}^{(m+1)(n+1)−1}\). Thus the homogeneous ideal determined by these quadratic relations determine the Segre variety \(\Sigma_{m,n}\).

Note that \(i = k\) or \(j = l\) give trivial relations and that by exchanging the order of the terms it may be assumed that \(i < k\) and then by exchanging the left and right sides of the equation, it may also be assumed that \(j < l\), and therefore it is possible to take \(0 \leq i < k \leq m\) and \(0 \leq j < l \leq n\) and therefore there are \(\binom{m+1}{2} \binom{n+1}{2}\) such equations that define \(\mathbb{P}^m \times \mathbb{P}^n\) as a projective variety in \(\mathbb{P}^{(m+1)(n+1)−1}\). However, the difference in dimensions is only \(m+n\). This turns out to be a good example of a projective variety that is not a complete intersection.

Biprojective Varieties.

Suppose that there are now two sets of variables \(S_1 = \{x_0, x_1, \ldots, x_m\}\) and \(S_2 = \{y_0, y_1, \ldots, y_n\}\). A monomial has bidegree \((a,b)\) if the sum of the degrees of the \(x\)-variables is \(a\) and the sum of the degrees of the \(y\)-variables is \(b\). Now, a polynomial \(P(x_0, \ldots, x_m, y_0, \ldots, y_n)\) in the \(S_1\) and \(S_2\) variables is homogeneous of bidegree \((a,b)\) if all the monomials in \(P\) are of bidegree \((a,b)\). In this case \(P(\alpha x_0, \ldots, \alpha x_m, \beta y_0, \ldots, \beta y_n) = \alpha^a \beta^b P(x_0, \ldots, x_m, y_0, \ldots, y_n)\) identically. The zero set of such a polynomial gives a well-defined subset of \(\mathbb{P}^m \times \mathbb{P}^n\), and is called a biprojective variety. The intersection of several such zero sets, each defined by a bihomogeneous polynomial also gives a biprojective variety. The Segre map turns any biprojective variety in \(\mathbb{P}^m \times \mathbb{P}^n\) into a projective variety in \(\mathbb{P}^{(m+1)(n+1)−1}\).
Mapping Bihomogeneous Equations to Homogeneous Equations.

It is often convenient to write \( \tilde{x} = (x_0, x_1, \ldots, x_m) \) for the projective \( x \) variables and \( \tilde{y} = (y_0, y_1, \ldots, y_m) \) for the projective \( y \) variables. Now suppose that \( P(\tilde{x}, \tilde{y}) \) is biprojective of bidegree \((a, b)\), so that \( P(\lambda x, \lambda y) = \lambda^a \lambda^b P(\tilde{x}, \tilde{y}) \) for all \( \lambda x, \lambda y \in F \). Suppose that \( a < b \) and let \( \bar{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_m) \) be an \( m + 1 \)-long vector of non-negative integers, and define the monomial

\[
x^{\bar{\alpha}} = x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_m^{\alpha_m}
\]

of degree \( |\alpha| = \sum_{i=0}^{m} \alpha_i \). Note that there are \( \binom{m+d}{d-1} \) monomials of degree \( d = |\alpha| \) in the \( m + 1 \) variables \( x_0, x_1, \ldots, x_m \). With this definition, \( \tilde{x}^{\bar{\alpha}} P(\lambda x, \lambda y) \) is of bidegree \((|\alpha| + a, b)\), so in particular, if \( |\alpha| = b - a \), then \( \tilde{x}^{\bar{\alpha}} P(\lambda x, \lambda y) \) is of bidegree \((b, b)\), i.e. every monomial appearing in \( \tilde{x}^{\bar{\alpha}} P(\lambda x, \lambda y) \) is a product of \( b \) \( x \)-variables and \( b \) \( y \)-variables. Now under the Segre map \( \sigma: x_i y_j \to z_{i,j} \)

the polynomial \( \tilde{x}^{\bar{\alpha}} P(\lambda x, \lambda y) \) maps to a homogeneous polynomial \( P^\alpha(\tilde{z}) \) of degree \( b \) in the \( z \)-variables \( z_{i,j} \). There are lot of different choices in how the \( x \)-variables and the \( y \)-variables pair up, so strictly speaking, this is not a completely well-defined map if \( b > 1 \). However, if this is viewed as a subvariety of the Segre variety, all the relations in the defining equations of the Segre variety make these different choices irrelevant. In the absence of the Segre defining equations, it is better to view \( P^\alpha(\tilde{z}) \) as a whole set of homogeneous equations in the \( z \)-variables, all of degree \( b \).

Another thing that can make a difference is the choice of \( \bar{\alpha} \). Here different choices of \( \bar{\alpha} \) with \( |\bar{\alpha}| = b - a \) can give different \( x \)-variables to be combined with the \( y \)-variables, and thereby give different \( z \)-variables. In mapping a bihomogenous equation in two set of variables to a homogeneous equation, what needs to be done is to take all possible choices of \( \bar{\alpha} \) with \( |\bar{\alpha}| = b - a \), which again gives multiple homogeneous equations associated to each bihomogeneous equation (unless \( b = a \)). Of course, if \( b < a \) instead of \( a < b \), just reverse the roles of the two sets of variables, and now multiply \( P(\lambda x, \lambda y) \) by \( \tilde{x}^{\bar{\beta}} \) where \( \bar{\beta} = (\beta_0, \beta_1, \ldots, \beta_m) \) is an \( n + 1 \)-long vector of non-negative integers with \( |\bar{\beta}| = \sum_{j=0}^{n} \beta_j \).

In any case, the set of homogeneous equations that define the projective variety that is the image of a biprojective variety under the Segre map, can be determined.

Biprojective Hyperplanes.

A biprojective hyperplane \( \mathcal{H} \) is now defined by a homogeneous bilinear equation

\[
0 = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{i,j} x_i y_j
\]

where the \( a_{i,j} \) are not all zero. Under the Segre map this becomes the linear equation

\[
0 = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{i,j} z_{i,j}.
\]

If \( 0 = \sum_{i=0}^{m} b_i x_i \) is a hyperplane in only one of the sets of variables, then this cuts out a subset of \( \mathbb{P}^n \) that is isomorphic to \( \mathbb{P}^{m-1} \). There is a corresponding set of hyperplanes in \( \mathbb{P}(m+1)(n+1)-1 \) whose intersection give the image of this hyperplane in the Segre map. The set of bihomogeneous equations is \( 0 = \sum_{i=0}^{m} b_i x_i y_j \) for \( j = 0, 1, \ldots, n \) and the corresponding set of linear hyperplanes in \( \mathbb{P}(m+1)(n+1)-1 \) is given by the equations \( 0 = \sum_{i=0}^{m} b_i z_{i,j} \) for \( j = 0, 1, \ldots, n \). Note that if \( 0 = \sum_{i=0}^{m} b_i x_i y_j \) for \( j = 0, 1, \ldots, n \), then since at least one of the \( y_j \)’s must be nonzero, this forces \( \sum_{i=0}^{m} b_i x_i \) to be zero.
Infinity Types for Biprojective Varieties.

If \( V \subset \mathbb{P}^m \times \mathbb{P}^n \) is a biprojective variety, it is useful to classify some of the points particularly if \( V \subset \mathbb{A} \times \mathbb{B} \) where \( \mathbb{A} \subset \mathbb{A}^m \) and \( \mathbb{B} \subset \mathbb{A}^n \) are affine varieties with projective closures \( \overline{\mathbb{A}} \subset \mathbb{P}^m \) and \( \overline{\mathbb{B}} \subset \mathbb{P}^n \), respectively. If \( (x_1, \ldots, x_m) \) are affine coordinates corresponding to projective coordinates \( (x_0, x_1, \ldots, x_m) \) in \( \mathbb{P}^m \), then the affine coordinates for a point in \( \mathbb{A}^m \) come from the projective coordinates by simply taking \( x_0 = 1 \), while the projective coordinates that don’t come from an affine point are the result of taking \( x_0 = 0 \). Similar ideas apply to taking the \( (y_0, y_1, \ldots, y_n) \) projective coordinates and taking \( y_0 = 1 \) for affine points in \( \mathbb{A}^n \) and \( y_0 = 0 \) for projective coordinates in \( \mathbb{P}^n \) that don’t come from an affine point in \( \mathbb{A}^n \). Thus points in \( \mathbb{P}^m \times \mathbb{P}^n \) come in four different flavors, depending on whether \( x_0 \) is zero or non-zero and on whether \( y_0 \) is zero or non-zero. This will be referred to as the \textit{infinity type} of a point in \( \mathbb{P}^m \times \mathbb{P}^n \).

Multi-projective Varieties.

All of this extends very naturally to more than two sets of variables and to products of more than two projective spaces. If each \( S_i \) is a set of \( m_i \) variables and there are \( L \) such sets, then a monomial in the union \( S \) of all the \( S_i \) can be assigned a multi-degree as an \( L \)-tuple of integers \( \vec{d} = (d_1, \ldots, d_L) \) and a polynomial is homogenous of multi-degree \( \vec{d} \) if each monomial is of multi-degree \( \vec{d} \). The zero-set of such a homogenous polynomial gives a well defined subset of the product of projective spaces \( \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_L} \). The intersection of several such zero sets of different multi-homogeneous polynomials \( f_j \) in \( S \), the set of all variables, is the set of points annihilated by all elements of the multi-graded ideal generated by the \( f_j \).

There is a natural generalization of the Segre map denoted

\[
\sigma_{\vec{d}} : \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_L} \to \mathbb{P}^D
\]

where \( D = \prod_{i=1}^L (d_i+1) - 1 \), and the image of \( \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_L} \) in \( \mathbb{P}^D \) is the generalized Segre variety, denoted \( \Sigma_{\vec{d}} \).

If \( S_i = \{ z_{i,0}, z_{i,1}, \ldots, z_{i,m_i} \} \) is the \( i \)-th set of variables, representing a point in \( \mathbb{P}^{d_i} \), and if \( \vec{z}_i = (z_{i,0}, z_{i,1}, \ldots, z_{i,m_i}) \) is the corresponding vector of variables, then

\[
\sigma_{\vec{d}} : x_{1,k_1} x_{2,k_2} \cdots x_{d,k_d} \to z_{k_1,k_2,\ldots,k_d}
\]

is the mapping from a product of projective spaces to a new projective space in the \( z_k \)-coordinates, with corresponding maps in the \( z_k \)-variables. The way to go from multi-homogeneous defining equations to homogeneous defining equations is by multiplying by appropriate “slack” monomials, just as in the case of bihomogeneous polynomials. Analogous concepts apply to multi-projective hyperplanes.

Infinity types for points in multi-projective spaces are defined in analogy to their concept in biprojective space. Starting with affine varieties \( \mathbb{A}_i \subset \mathbb{A}^{d_i} \) with projective closures \( \overline{\mathbb{A}}_i \subset \mathbb{P}^{d_i} \), points in \( \mathbb{A}_1 \times \mathbb{A}_2 \times \cdots \times \mathbb{A}_L \subset \mathbb{P}^{d_1} \times \mathbb{P}^{d_2} \times \cdots \mathbb{P}^{d_L} \) are of infinity type \( (x_{1,0} = c_1, x_{2,0} = c_2, \ldots, x_{L,0} = c_L) \) with \( c_1, c_2, \ldots, c_L \in \{0, 1\} \) depending on whether the coordinates are in the affine or non-affine part of the corresponding component. This is a very useful idea when glueing different parts of projective varieties together along common affine subsets, and then looking at what happens on the different parts of the projective closures.
Hyperelliptic Curves and the Picard Group

Let $F$ be a field that is not of characteristic 2. Let $f(x)$ be a monic polynomial over the base field $F$ of degree $2g+1$ with no repeated roots. The curve $C_{\text{aff}}$ defined by the affine equation

$$C_{\text{aff}} : y^2 = f(x)$$

is hyperelliptic of genus $g$ with hyperelliptic involution $\iota : C_{\text{aff}} \rightarrow C_{\text{aff}}$ defined by $\iota : (x, y) \mapsto (x, -y)$. Its projective closure $C$ is defined by the homogenous equation

$$C : Y^2 Z^{2g-1} = Z^{2g+1} f(X/Z)$$

and is obtained by adding the point $P_\infty$ at infinity with projective coordinates $(1, 0, 0)$ to $C_{\text{aff}}$ where the standard inclusion $C_{\text{aff}} \rightarrow C$ is given by $(x, y) \mapsto (x, y, 1)$ in projective coordinates. The hyperelliptic involution is extended by defining $\iota(P_\infty) = P_\infty$.

A Weierstrass point on $C$ is a point $P \in C$ such that $\iota(P) = P$. In addition to $P_\infty$, the affine Weierstrass points of $C$ are of the form $(\rho, 0)$ where $\rho$ is a root of $f(x)$. Thus, there are exactly $2g + 2$ Weierstrass points on $C$ if $C$ is of genus $g$.

Divisors and Divisor Classes on Hyperelliptic Curves.

A divisor $D$ on $C$ is just a finite formal sum of points with multiplicities, i.e. $D = \sum_{i=1}^{k} n_i P_i$ where $P_i \in C$. The degree of $D$ is defined by $\deg(D) = \sum_{i=1}^{k} n_i$. Every non-zero function $h$ on $C$ has an associated divisor

$$\text{div}(h) = \sum_{P \in C} \deg_P(h) \cdot P$$

and since every function on $C$ has only finitely many zeros and poles, this is actually a finite sum. The degree of the divisor of any non-zero function on $C$ is 0.

Two divisors $D$ and $D'$ are equivalent if their difference is the divisor of a function, i.e. $D \equiv D'$ if there is some function $h$ on $C$ such that $D = D' + \text{div}(h)$. The group of equivalence classes of divisors on $C$ is the Picard group $\text{Pic}(C)$ and the group of equivalence classes of degree 0 is denoted $\text{Pic}^0(C)$.

The geometric realization of $\text{Pic}^0(C)$ is the Jacobian variety $\text{Jac}(C)$, which is a projective algebraic variety of dimension $g$.

A divisor $D$ of degree 0 is semi-reduced if $D$ is of the form

$$D = \sum_{i=1}^{k} P_i - k \cdot P_\infty$$

where the $P_i$ are all on $C_{\text{aff}}$ and are such that if $P_i = \iota(P_j)$ then $i = j$, i.e. no affine point and its hyperelliptic involute and appear in the support of $D$, and if an affine Weierstrass point is in the support of $D$ then it appears with multiplicity 1. A semi-reduced divisor is reduced if $k \leq g$.

Theorem. Every divisor class of degree 0 on $C$ contains exactly one reduced divisor.

This follows from the Riemann-Roch theorem, but can also be proved by extending the theory of reduction of quadratic forms to polynomials over the base field $F$. The composition of quadratic forms corresponds to addition of divisor classes.
**Polynomial Representation of Reduced and Semi-Reduced Divisors.**

If \( D = \sum_{i=1}^{k} P_i - k \cdot P_{\infty} \) is a semi-reduced divisor there is a polynomial \( U(x; D) \) of degree \( k \) whose roots are the \( x \)-coordinates of the \( P_i \), i.e.

\[
U(x; D) = \prod_{i=1}^{k} (x - x(P_i))
\]

and there is an associated polynomial \( V(x; D) \) of degree at most \( k - 1 \) that interpolates the \( y \)-coordinates, i.e.

\[
V(x(P_i); D) = y(P_i)
\]

for \( i = 1, \ldots, k \). Note that if all the points in the support of \( D \) are distinct, then \( f(x) - V(x; D)^2 \) is 0 for \( x = x(P_i) \), and therefore \( U(x; D) \) divides \( f(x) - V(x; D)^2 \). If \( D \) has points of multiplicity greater than 1, then this is the defining property of \( V(x; D) \) along with the requirement \( \deg V(x; D) < k \). Thus there is a unique polynomial \( W(x; D) \) such that

\[
f(x) - V(x; D)^2 = U(x; D)W(x; D).
\]

Furthermore, and any such triple \( (U(x), V(x), W(x)) \) with \( f = V^2 + UW \) defines a unique semi-reduced divisor. The representation theorem above simply states that every divisor class of degree 0 has a unique \( (U(x), V(x), W(x)) \) with \( \deg U(x) \leq g \) and \( \deg V(x) < g \).

The set of divisor classes represented by a reduced divisor \( D = \sum_{i=1}^{k} P_i - k \cdot P_{\infty} \) with \( k < g \) is denoted \( \Theta \). Geometrically \( \Theta \) is a subvariety of \( \text{Jac}(C) \) of codimension 1, and is referred to as the \( \Theta \)-divisor. (This can be confusing, since \( \Theta \) is not a divisor on \( C \), but rather on \( \text{Jac}(C) \).)

**\( \text{Jac}(C) - \Theta \) as an affine variety.**

\( \text{Jac}(C) - \Theta \) can be given the structure of an affine variety (actually, just an affine algebraic set) as follows: Write

\[
U(x) = x^9 + \sum_{i=0}^{g-1} u_i x^i
\]

\[
V(x) = \sum_{i=0}^{g-1} u_i x^i
\]

\[
W(x) = x^{g+1} + \sum_{i=0}^{g} v_i x^i
\]

for variables \( s = \{u_0, \ldots, u_{g-1}, v_0, \ldots, v_{g-1}, w_0, \ldots, w_g\} \) and write

\[
f(x) = x^{2g+1} + \sum_{i=0}^{2g} a_i x^i
\]

and now simply equate coefficients in \( f = V(x)^2 + U(x)W(x) \). These equations on the coefficients define \( \text{Jac}(C) - \Theta \) as an affine variety. Alternatively, the \( w_i \)'s can be eliminated by simply reducing \( f(x) - V(x)^2 \) modulo \( U(x) \) generically, obtaining a polynomial of degree \( g - 1 \) in \( x \) with coefficients that are polynomials in the \( u_i \)'s and \( v_i \)'s, and the variety now follows by requiring that these coefficients all be 0.
Genus 2 curves

Let \( f(x) \) be a monic quintic polynomial over the base field \( F \) having no multiple roots and write
\[
    f(x) = x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0
    = (x - \rho^{(1)}) (x - \rho^{(2)}) (x - \rho^{(3)}) (x - \rho^{(4)}) (x - \rho^{(5)})
\]
with \( \rho^{(i)} \neq \rho^{(j)} \) for \( i \neq j \). These roots \( \rho^{(i)} \) may be in finite extension of the base field \( F \). It is also be useful to expand \( f(x) \) around the \( \rho^{(i)} \)'s as
\[
    f(x) = (x - \rho^{(1)})^5 + a_4^{(i)} (x - \rho^{(1)})^4 + a_3^{(i)} (x - \rho^{(1)})^3 + a_2^{(i)} (x - \rho^{(1)})^2 + a_1^{(i)} (x - \rho^{(1)})
\]
with \( a_1^{(i)} \neq 0 \) for \( i = 1, \ldots, 5 \). For compactness of notation, it is useful to write
\[
    \mathbf{a} = (a_0, a_1, a_2, a_3, a_4, 1)
\]
\[
    \mathbf{x} = (1, x, x^2, x^3, x^4, x^5)
\]
so \( f(x) = \mathbf{a} \cdot \mathbf{x} \). It is also useful to write
\[
    \mathbf{a}^{(i)} = (0, a_1^{(i)}, a_2^{(i)}, a_3^{(i)}, a_4^{(i)}, 1)
\]
\[
    \mathbf{x}^{(i)} = (1, (x - \rho^{(i)}), (x - \rho^{(i)})^2, (x - \rho^{(i)})^3, (x - \rho^{(i)})^4, (x - \rho^{(i)})^5)
\]
so \( f(x) = \mathbf{a}^{(i)} \cdot \mathbf{x}^{(i)} \), as well for \( i = 1, \ldots, 5 \).

Divisor classes on genus 2 curves.

In the particular case of genus 2, a general degree 0 divisor class will be represented by a reduced divisor of the form
\[
    D = P_1 + P_2 - 2 P_\infty
\]
where \( P_1 \) and \( P_2 \) are affine points on \( C \) and \( \iota(P_1) \neq P_2 \) (although \( P_1 = P_2 \) is allowed if \( P_1 \) is not a Weierstrass point). In this case the \( \Theta \)-divisor consists of those divisor classes on \( C \) represented by reduced divisors of the form
\[
    D = P - P_\infty
\]
for \( P \) an affine point on \( C \), along with the divisor \( 0 \). Alternatively, \( \Theta \) consists of those divisor classes with reduced divisor \( P - P_\infty \) as \( P \) ranges over all of \( C \). Thus \( \Theta \) is essentially an image of \( C \) imbedded in \( \text{Jac}(C) \) under the Abel-Jacobi mapping.

There are basically three types of degree 0 divisor classes; they are represented by reduced divisors of one of the three following types:

(i) \( P_1 + P_2 - 2 P_\infty \), where \( P_1, P_2 \in C_{\text{aff}} \) with \( P_1 \neq \iota(P_2) \);
(ii) \( P_1 - P_\infty \) where \( P_1 \in C_{\text{aff}} \);
(iii) 0.

The divisors of type (ii) and (iii) constitute the set \( \Theta \) known as the theta-divisor, and represent a copy of \( C \) inside of \( \text{Jac}(C) \). Notionally, \( \Theta \) is a one-dimensional object, while \( \text{Jac}(C) \) is a two-dimensional object (and more generally, on hyperelliptic curves of genus \( g \), \( \Theta \) is a codimension one subvariety of the \( g \)-dimensional Jacobian variety).
Points of order 2 on $\text{Jac}(C)$. 
In general, an abelian variety of dimension $g$ has $2^{2g}$ points of order two, so in the present case there are 16 points of order 2 on $\text{Jac}(C)$. If $P_i = (\rho_i, 0)$ is an affine Weierstrass point on $C$, then $X_i = P_i - P_\infty$ is a reduced divisor representing a point of order 2 on $\text{Jac}(C)$ since the divisor of the function $x - \rho_i$ is $2P_i - 2P_\infty$. There are 5 such points, all on $\Theta$, and also there is the divisor 0 on $\Theta$, making a total of 6 points on $\Theta$ of order 2. Furthermore, if $P_i$ and $P_j$ are distinct affine Weierstrass points on $C$, then $X_i + X_j = P_i + P_j - 2P_\infty$ is also a reduced divisor representing a point of order 2 on $\text{Jac}(C)$. This accounts for all 16 points on $\text{Jac}(C)$ of order 2. It is also worth noting that the divisor of the function $y$ on $C$ is $P_1 + P_2 + P_3 + P_4 + P_5 - 5P_\infty$, i.e $X_1 + X_2 + X_3 + X_4 + X_5 = 0$ on $\text{Jac}(C)$.

Defining equations for $\text{Jac}(C) - \Theta$ as an affine variety in genus 2.
Let $s = \{u_0, u_1, v_0, v_1\}$ be a set of (affine) variables and let

$$U(x) = x^2 + u_1 x + u_0$$
$$V(x) = v_1 x + v_0$$

be a pair of polynomials that represents a reduced divisor on $\text{Jac}(C) - \Theta$, so in particular they satisfy the relationship

$$U(x)|f(x) - V(x)|^2$$

and it is instructive to write

$$f(x) - V(x)|^2 = e_1 x + e_0 \text{ mod } U(x)$$

with $e_1 = e_1(u_0, u_1, v_0, v_1)$ and $e_0 = e_0(u_0, u_1, v_0, v_1)$ for a pair of polynomials $e_0, e_1 \in F[u_0, u_1, v_0, v_1]$. Specifically, these polynomials are

$$e_0(v_1, v_0, u_1, u_0) = v_1^2 u_0 - v_0^2 + a_3 v_0 - a_4 u_1^2 u_0 - 2u_1 u_0^2 + a_3 u_0 + a_4 u_0^2 - a_2 u_0 + a_0$$
$$e_1(v_1, v_0, u_1, u_0) = v_1^2 u_0 - v_0^2 + a_3 v_0 - a_4 u_1^2 u_0 - 2 u_1 u_0^2 + a_3 u_0 + a_4 u_0^2 - a_2 u_0 + a_0$$

Any point $(u_0, u_1, v_0, v_1) \in F^4$ where these two polynomials vanish simultaneously gives a pair of polynomials $(U(x), V(x)) = (U(x; u_0, u_1, v_0, v_1), V(x; u_0, u_1, v_0, v_1))$ with $U(x)$ monic, such that $U(x)|f(x) - V(x)|^2$.

Proposition. Let $s = (u_0, u_1, v_0, v_1)$ be a sequence of coordinates in $F^4$, and let $A(s) \in K^4$ be the zero set of the following pair of polynomials:

$$e_0 : a_0 - a_2 u_0 + a_4 u_0^2 + a_3 u_0 u_1 - 2u_0^2 u_1 - a_4 u_0 u_1^2 + u_0 u_1^3 - v_0^2 + v_0^2$$
$$e_1 : a_1 - a_3 u_0 + u_0^2 - a_2 u_1 + a_4 u_0 u_1 + a_3 u_1^2 - 3u_0 u_1^2 - a_4 u_1^2 + u_1^2 - v_0 u_1 + v_1 u_1^2$$

Then the points on the affine algebraic set $A(s)$ are in one-to-one correspondence with elements of $\text{Jac}(C) - \Theta$.

These two polynomials define a prime ideal in $F[u_0, u_1, v_0, v_1]$ whose associated affine variety is $\text{Jac}(C) - \Theta$. 

GENUS 2 JACOBIANS 11
Addition of a Point of Order 2: The Case of a Distinguished Weierstrass Point

It is useful to consider the case of \( a_0 = 0 \) in the equation for \( C \), which corresponds to \( C \) having a Weierstrass point at \( P_0 = (0, 0) \). The equations for \( A(s) \) are modified by setting \( a_0 = 0 \) in the polynomials \( e_0 \) and \( e_1 \). However, \( a_0 \) appears only in equation \( e_0 \), which now becomes

\[
e_0 = -a_2u_0 + a_4u_0^2 + a_3u_0u_1 - 2u_0^2u_1 - a_4u_0u_1^2 + u_0u_1^2 - v_0^2 + u_0v_1^2.
\]

Setting \( u_0 = 0 \) in the equations for \( e_0 \) and \( e_1 \) gives the pair of equations

\[
e_0 : 0 = v_0^2 + u_0v_1^2
\]
\[
e_1 : 0 = a_1 - a_2u_1 + a_3u_1^2 - a_4u_1^3 + u_1^4 - 2v_0v_1 + u_1v_1^2
\]

so \( v_0 = 0 \) and the second equation becomes

\[
0 = a_1 - a_2u_1 + a_3u_1^2 - a_4u_1^3 + u_1^4 - 2v_0v_1 + u_1v_1^2
\]

just as before.

**Proposition.** Suppose that 0 is a root of \( f(x) \) so \( P_0 = (0, 0) \) is an affine Weierstrass point on \( C \). Let \( P = (u_0, u_1, v_0, v_1) \) be the coordinates of a point on \( A(s) \). If \( u_0 = 0 \) then \( v_0 = 0 \) and \( P = (-u_1, u_1v_1) \) is an affine point on \( C \) with \( P \neq P_0 \), and \( P \) corresponds to the reduced divisor \( P + P_0 - 2P_\infty \) as an element of \( \text{Jac}(C) - \Theta \).

Adding a point of order 2.

The general procedure for adding two elements of \( \text{Pic}^0(C) \) (but which are assumed to all have distinct points in their support) that are represented by polynomial pairs \( (U^{(1)}(x), V^{(1)}(x)) \) and \( (U^{(2)}(x), V^{(2)}(x)) \) (which need not lie outside of \( \Theta \)) is as follows:

1. Find a polynomial \( \tilde{V}(x) \) that is of degree at most 3 and such that

\[
\tilde{V}(x) \equiv V^{(i)}(x) \mod U^{(i)}(x)
\]

for \( i = 1, 2 \).

2. Find the points where \( y = \tilde{V}(x) \) intersects the curve \( y^2 = f(x) \). This amounts to solving \( f(x) - \tilde{V}(x)^2 = 0 \). Some of these will be points already represented by the original polynomial pairs. The points that are not are the ones whose involutes are in the sum of the divisors. Thus

\[
U^{(3)}(x) = \left( f(x) - \tilde{V}(x)^2 \right) / U^{(1)}(x)U^{(2)}(x)
\]

where \( U^{(3)}(x) \) has as roots the \( x \)-coordinates of the desired points.

3. To find the interpolating polynomial for the \( y \)-coordinates, just note that

\[
\tilde{V}(x) \equiv -V^{(3)}(x) \mod U^{(3)}(x)
\]

\[1\] In this section only, the equations are modified from the more general equations above.
since the points on the intersection must be involuted to get the points in the sum. This procedure works perfectly well as long as there are no common factors between the $U^{(i)}(x)$’s.

This procedure is particularly simple to work out in one very interesting case, namely, when one of the points on Pic$^0(C)$ being added is the distinguished 2-division point

$$X_0 = \text{Cl}(P_0 - P_\infty).$$

For this point the representing polynomial pair is $(U(x; X_0) \ V(x; X_0)) = (x, 0)$. For simplicity of notation, let $X \in \text{Pic}^0(C)$ be arbitrary except for the requirement that $X, X + X_0 \notin \Theta$, and set

$$U(x) = U(x; X)$$
$$V(x) = V(x; X)$$
$$\hat{U}(x) = U(x; X + X_0)$$
$$\hat{V}(x) = V(x; X + X_0)$$

and in terms of explicit coefficients, write

$$U(x) = x^2 + u_1 x + u_0$$
$$V(x) = v_1 x + v_0$$
$$\hat{U}(x) = x^2 + \hat{u}_1 x + \hat{u}_0$$
$$\hat{V}(x) = \hat{v}_1 x + \hat{v}_0$$

with a goal of finding relations between the two sets of coordinates $s = (u_0, u_1, v_0, v_1)$ and $\hat{s} = (\hat{u}_0, \hat{u}_1, \hat{v}_0, \hat{v}_1)$.

In the case at hand, the congruence relation on $\hat{V}(x)$ imposed by $X_0$ is that $x | \hat{V}$. This means that it is possible to write

$$\hat{V}(x) = \hat{V}(x) x$$

for some polynomial $\hat{V}(x)$. Then

$$f(x) - \hat{V}(x)^2 = U(x) \hat{U}(x) x$$

so dividing by $x$ gives

$$x^4 + a_4 x^3 + a_3 x^2 + a_2 x + a_1 = \hat{V}(x)^2 x + U(x) \hat{U}(x)$$

and then setting $x = 0$ gives the relation

$$a_1 = u_0 \hat{u}_0.$$

Since $a_1 \neq 0$, this shows that neither $u_0$ nor $\hat{u}_0$ can be 0 if neither $X$ nor $X + X_0$ is on $\Theta$. This corresponds to what can be seen by looking at reduced divisors directly, and gives an indication of how the algebraic equations reflect the group law.

The other congruence relations on $\hat{V}$ are then satisfied by writing

$$\hat{V}(x) = V(x) - \frac{v_0}{u_0} U(x)$$
$$\hat{V}(x) = -\hat{V}(x) + \frac{\hat{v}_0}{u_0} \hat{U}(x)$$
which now gives

\[ V(x) + \hat{V}(x) = \frac{\hat{v}_0}{u_0} \hat{U}(x) + \frac{v_0}{u_0} U(x) = a_1^{-1} \left( \hat{v}_0 u_0 \hat{U}(x) + v_0 \hat{u}_0 U(x) \right) \]

in view of the relation between \( u_0 \) and \( \hat{u}_0 \). Equating the coefficients of \( x^2 \) and \( x \) and using the fact that \( a_1 \neq 0 \) now gives

\[
0 = \hat{v}_0 u_0 + v_0 \hat{u}_0 \\
a_1 (v_1 + \hat{v}_1) = \hat{v}_0 u_0 \hat{u}_1 + v_0 \hat{u}_0 u_1
\]
as further relations between the variables. Substituting the first equation into the second gives

\[
a_1 (v_1 + \hat{v}_1) = \hat{v}_0 u_0 \hat{u}_1 - \hat{v}_0 u_0 u_1
\]
and multiplying by \( \hat{u}_0 \) and dividing by \( a_1 \) then gives

\[
\hat{u}_0 (v_1 + \hat{v}_1) = \hat{v}_0 (\hat{u}_1 - u_1)
\]
and there is also the equation

\[
u_0 (v_1 + \hat{v}_1) = v_0 (u_1 - \hat{u}_1)
\]
where the roles of the hatted and the unhatted variables are reversed.

The above pair of equations for \( \hat{V}(x) \) also shows that \( \hat{V}(x) \) is of degree at most 2, which means that \( V(x) \) is of degree at most 1. The coefficient of \( x^2 \) in \( \hat{V}(x) \) is seen to be \( -v_0 / \hat{u}_0 \) and also \( \hat{v}_0 / u_0 + \) and therefore the coefficient of \( x^3 \) in \( \hat{V}(x)^2 \) is \( -v_0 \hat{v}_0 / a_1 \). Therefore equating coefficients of \( x^3 \) in the equation above gives

\[
a_1 a_4 = -v_0 \hat{v}_0 + a_1 u_1 + a_1 \hat{u}_1
\]
as another equation.

To summarize so far, the following set of equations holds between the two sets of coordinates:

\[
0 = g_1(s, \hat{s}) = u_0 \hat{u}_0 - a_1 \\
0 = g_2(s, \hat{s}) = \hat{v}_0 u_0 + v_0 \hat{u}_0 \\
0 = g_3(s, \hat{s}) = v_0 \hat{v}_0 + a_1 (a_4 - u_1 - \hat{u}_1) \\
0 = g_4(s, \hat{s}) = \hat{u}_0 (v_1 + \hat{v}_1) + \hat{v}_0 (u_1 - \hat{u}_1) \\
0 = g_5(s, \hat{s}) = u_0 (v_1 + \hat{v}_1) + v_0 (\hat{u}_1 - u_1)
\]
where for convenience, the polynomials have been labelled \( g_1, g_2, g_3, g_4, g_5 \) for ease of reference. These equations suffice to determine the values of \( \hat{s} = (\hat{u}_0, \hat{u}_1, \hat{v}_0, \hat{v}_1) \) in terms of the values of \( s = (u_0, u_1, v_0, v_1) \) as long as \( u_0 \neq 0 \), and vice versa.

**Additional relations for adding a distinguished point of order 2.**

These are not the only relations that are of interest, however. There are other interesting coordinate equations that can be seen by looking at the projective closure of the graph of the two affine varieties that are related by the addition of a point of order 2. One of the concerns that naturally arise from looking at the equations for \( e_0 \) and \( e_1 \) is that they are fairly high degree.
The goal will now be to derive some relations between hatted and unhatted variables that are of lower degree. The possibility of doing this is suggested by some of the degree lowering relations that are apparent in some of the glue equations (e.g. $u_0 \hat{u}_0 = a_1$), applying this relation to the $e_0$ and $e_1$ polynomials. For now, just note that these new relations depend on $u_0$ and $\hat{u}_0$ both being non-zero.

Start by noting that

$$ e_0(s, \hat{s}) \hat{u}_0 + g_2(s, \hat{s}) v_0 $$

$$ = u_0 (a_4 \hat{u}_0 u_0 - a_2 \hat{u}_0 + a_3 \hat{u}_0 u_1 - 2 \hat{u}_0 u_0 u_1 - a_4 \hat{u}_0 u_1^2 + \hat{u}_0 u_1^3 + \hat{v}_0 v_0 + \hat{u}_0 v_1^2) $$

and therefore, assuming $u_0 \neq 0$ this gives the relation

$$ 0 = h_0(s, \hat{s}) = -a_2 \hat{u}_0 + a_4 \hat{u}_0 u_0 + a_3 \hat{u}_0 u_1 - 2 \hat{u}_0 u_0 u_1 - a_4 \hat{u}_0 u_1^2 + \hat{u}_0 u_1^3 + \hat{v}_0 v_0 + \hat{u}_0 v_1^2 $$

where the polynomial $h_0$ has been defined here for convenience. Next define

$$ h_1(s, \hat{s}) = e_1(s, \hat{s}) \hat{u}_0 - h_0(s, \hat{s}) u_1 $$

$$ = a_1 \hat{u}_0 - a_3 \hat{u}_0 u_0 + \hat{u}_0 u_0^2 + a_4 \hat{u}_0 u_0 u_1 - \hat{u}_0 u_0 u_1^2 - u_1 \hat{v}_0 v_0 - 2 \hat{u}_0 v_0 v_1 $$

and now use the relation $u_0 \hat{u}_0 = a_1$ to get that $h_2(s, \hat{s}) = 0$ where

$$ h_2(s, \hat{s}) = -a_1 a_3 + a_1 \hat{u}_0 + a_1 u_0 + a_1 a_4 u_1 - a_1 u_1^2 - u_1 \hat{v}_0 v_0 - 2 \hat{u}_0 v_0 v_1 $$

and finally

$$ g_0(s, \hat{s}) = h_2(s, \hat{s}) - g_3(s, \hat{s}) u_1 $$

$$ = -a_1 a_3 + a_1 \hat{u}_0 + a_1 u_0 + a_1 \hat{u}_1 u_1 - 2 u_1 \hat{v}_0 v_0 - 2 \hat{u}_0 v_0 v_1 $$

must also be a relation between the unhatted and the hatted variables. There must also be a relation with the roles of the hatted and unhatted variables reversed, i.e.

$$ g_7(s, \hat{s}) = -a_1 a_3 + a_1 \hat{u}_0 + a_1 u_0 + a_1 \hat{u}_1 u_1 - 2 u_1 \hat{v}_0 v_0 - 2 u_0 \hat{v}_0 \hat{v}_1 $$

must also be 0.

Next, note that

$$ 0 = g_3(s, \hat{s}) u_0^2 - v_0 u_0 g_2(s, \hat{s}) + v_0^2 g_1(s, \hat{s}) = a_1 (a_4 u_0^3 - \hat{u}_1 u_0^2 - u_0^3 u_1 - v_0^2) $$

and therefore

$$ 0 = i_0(s, \hat{s}) = a_4 u_0^3 - \hat{u}_1 u_0^2 - u_0^3 u_1 - v_0^2 $$

since $a_1 \neq 0$. Then,

$$ 0 = e_0(s, \hat{s}) - i_0(s, \hat{s}) = u_0 (-a_2 + \hat{u}_1 u_0 + a_3 u_1 - u_0 u_1 - a_4 u_1^2 + u_1^3 + v_1^2) $$

and so

$$ 0 = i_1(s, \hat{s}) = -a_2 + \hat{u}_1 u_0 + a_3 u_1 - u_0 u_1 - a_4 u_1^2 + u_1^3 + v_1^2 $$

since $u_0 \neq 0$. Next, set

$$ 0 = i_2(s, \hat{s}) $$

$$ = \hat{v}_0 i_1(s, \hat{s}) + (u_1 - \hat{u}_1) g_2(s, \hat{s}) + g_4 \hat{u}_0 - (v_1 + \hat{v}_1) g_1(s, \hat{s}) $$

$$ = -a_2 \hat{v}_0 + a_3 u_1 \hat{v}_0 - a_4 u_1^2 \hat{v}_0 + u_1^3 \hat{v}_0 + a_1 \hat{v}_1 + a_1 v_1 + \hat{v}_0 v_1^2 $$
and finally set

\[ 0 = g_8(s, \hat{s}) = i_4(s, \hat{s})u_1 - \hat{v}_0 e_1(s, \hat{s}) \]

\[ = -a_1 \hat{v}_0 + a_3 u_0 \hat{v}_0 - u_0^2 \hat{v}_0 - 2a_4 u_0 u_1 \hat{v}_0 + 3u_0 u_1^2 \hat{v}_0 + a_1 u_1 v_1 + a_1 u_1 v_1 + 2\hat{v}_0 v_0 v_1 \]

as another equation relating the hatted and unhatted variables. In addition, there is equation

\[ 0 = g_9(s, \hat{s}) \]

\[ = -a_1 v_0 + a_3 u_0 v_0 - u_0^2 v_0 - 2a_4 u_0 u_1 v_0 + 3u_0 u_1^2 v_0 + a_1 u_1 v_1 + a_1 u_1 v_1 + 2v_0 v_0 \hat{v}_1 \]

obtained by reversing the roles of the hatted and the unhatted variables.

To summarize, the following equations hold between the coordinates for \( X \) and \( X + X_0 \):

\[ 0 = g_1(s, \hat{s}) = u_0 \hat{u}_0 - a_1 \]
\[ 0 = g_2(s, \hat{s}) = \hat{v}_0 u_0 + v_0 \hat{u}_0 \]
\[ 0 = g_3(s, \hat{s}) = v_0 \hat{v}_0 + a_1 (a_4 - u_1 - \hat{u}_1) \]
\[ 0 = g_4(s, \hat{s}) = \hat{u}_0 (v_1 + \hat{v}_1) + \hat{v}_0 (u_1 - \hat{u}_1) \]
\[ 0 = g_5(s, \hat{s}) = u_0 (v_1 + \hat{v}_1) + v_0 (u_1 - \hat{u}_1) \]
\[ 0 = g_6(s, \hat{s}) = -a_1 a_3 + a_1 \hat{u}_0 + a_1 u_0 + a_1 \hat{u}_1 u_1 - 2 a_1 v_0 v_0 - 2 \hat{u}_0 v_0 v_1 \]
\[ 0 = g_7(s, \hat{s}) = -a_1 a_3 + a_1 \hat{u}_0 + a_1 u_0 + a_1 \hat{u}_1 u_1 - 2 \hat{u}_1 \hat{v}_0 v_0 - 2 u_0 v_0 \hat{v}_1 \]
\[ 0 = g_8(s, \hat{s}) = -a_1 v_0 + a_3 u_0 \hat{v}_0 - u_0^2 \hat{v}_0 - 2a_4 u_0 u_1 \hat{v}_0 + 3u_0 u_1^2 \hat{v}_0 + a_1 u_1 \hat{v}_1 + a_1 u_1 v_1 + 2v_0 v_0 \hat{v}_1 \]
\[ 0 = g_9(s, \hat{s}) = -a_1 v_0 + a_3 u_0 v_0 - u_0^2 v_0 - 2a_4 u_0 u_1 v_0 + 3u_0 u_1^2 v_0 + a_1 u_1 v_1 + a_1 u_1 v_1 + 2v_0 \hat{v}_0 v_1 \]

and it is worth noting that \( g_1, g_2, \) and \( g_3 \) are symmetric, i.e. that \( g_1(s, \hat{s}) = g_1(\hat{s}, s) \), \( g_2(s, \hat{s}) = g_2(\hat{s}, s) \), and \( g_3(s, \hat{s}) = g_3(\hat{s}, s) \), while \( (g_4, g_5), (g_6, g_7) \), and \( (g_8, g_9) \) are complementary pairs, i.e. \( g_5(s, \hat{s}) = g_4(\hat{s}, s) \), \( g_7(s, \hat{s}) = g_6(\hat{s}, s) \), and \( g_9(s, \hat{s}) = g_8(\hat{s}, s) \). It is also worth noting that the bidegrees of \( g_1 \) and \( g_2 \) are \((1, 1)\), the bidegrees of \( g_4 \) and \( g_7 \) are \((1, 2)\), the bidegrees of \( g_5 \) and \( g_6 \) are \((2, 1)\), and the bidegrees of \( g_8 \) and \( g_9 \) are \((3, 1)\) and \((1, 3)\), respectively.

**Proposition.** Suppose \( f(x) = x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x \) has no repeated roots and let \( C \) be the curve \( y^2 = f(x) \). Let \( X_0 \) be the point on \( \text{Jac}(C) \) corresponding to the divisor class \((0, 0) - P_\infty\). Suppose that \( X \in \text{Jac}(C) - \Theta \) and that \( X + X_0 \in \text{Jac}(C) - \Theta \) also. Let \( X \) be represented by a quadruple \( s = (u_0, u_1, v_0, v_1) \) corresponding to polynomials \( U(x) = x^2 + u_1 x + u_0 \) and \( V(x) = v_1 x + v_0 \) and such that \( U(x)f(x) - V(x)^2 \) and let \( X + X_0 \) be represented by a quadruple \( \hat{s} = (\hat{u}_0, \hat{u}_1, \hat{v}_0, \hat{v}_1) \) corresponding to polynomials \( \hat{U}(x) = x^2 + \hat{u}_1 x + \hat{u}_0 \) and \( \hat{V}(x) = \hat{v}_1 x + \hat{v}_0 \) and such that \( \hat{U}(x)f(x) - \hat{V}(x)^2 \). Then for \( i = 1 \ldots, 9 \), the equations \( g_i(s, \hat{s}) = 0 \) are satisfied.
A Biprojective Variety

In general, a mapping of $C$ into $\text{Pic}^0(C)$ can be constructed for any $Q \in C$, not just $Q = \infty$. Thus for any $Q \in C$, there is a mapping $\phi_Q$ that imbeds $C$ into $\text{Pic}^0(C)$ given by

$$\phi_Q : P \mapsto \begin{cases} 
\text{Cl}(P - Q) & \text{if } P \neq Q; \\
0 & \text{if } P = Q;
\end{cases}$$

where $\text{Cl}(D)$ denotes the divisor class of the divisor $D$. The image of $C$ under $\phi_Q$ is just the translation of $\Theta$ by $\text{Cl}(P_\infty - Q)$, and is denoted $\Theta_Q$. In particular $\Theta = \Theta_{P_\infty}$. For notational convenience, it is useful to write $\Theta_{X_0}$ in place of $\Theta_{P_0}$.

For points $Q_1, Q_2 \in C$ with $Q_1 \neq Q_2$, it is interesting to compute the intersection $\Theta_{Q_1} \cap \Theta_{Q_2}$. This consists of those divisor classes $G \in \text{Pic}^0(C)$ that contain two divisors $D_1, D_2 \in G$ of the forms $D_1 = P_1 - Q_1$ and $D_2 = P_2 - Q_2$ for some pair of points $P_1, P_2 \in C$. Since $D_1$ and $D_2$ are in the same class, they are equivalent, $D_1 \sim D_2$, and therefore $P_1 - P_2 \sim Q_1 - Q_2$. Since $P_2 + \iota(P_2) - 2 \cdot P_\infty \sim 0$ and $Q_2 + \iota(Q_2) - 2 \cdot P_\infty \sim 0$, this gives $P_1 + \iota(P_2) - 2 \cdot P_\infty \sim Q_1 + \iota(Q_2) - 2 \cdot P_\infty$ and the assumption that $Q_2 \neq Q_2$ shows that the divisor on the right is not in $\Theta$. Now both of these divisors are reduced, which means that the points must be the same, i.e. that either $P_1 = Q_1$ and $P_2 = Q_2$ or $P_1 = \iota(Q_2)$ and $Q_1 = \iota(P_2)$. The first of these possibilities gives $D_1 = D_2 = 0$, while the second case gives $\iota(Q_2) - Q_1 = D_1 \sim D_2 = \iota(Q_1) - Q_2$. Therefore,

$$\Theta_{Q_1} \cap \Theta_{Q_2} = \{ 0, \text{Cl}(\iota(Q_1) - Q_2) \}$$

and there are just two points in $\Theta_{Q_1} \cap \Theta_{Q_2}$.

In particular, this can be applied to find

$$\Theta \cap \Theta_{X_0} = \{ 0, \text{Cl}(P_0 - P_\infty) \}$$

since $P_0$ is a Weierstrass point and $P_0 = \iota(P_0)$. Note incidentally, that $X_0 = \text{Cl}(P_0 - P_\infty)$ is a point of order 2 in $\text{Pic}^0(C)$.

Based on this, it is useful to have a slightly finer classification of divisor classes than the one above. Every divisor class $X \in \text{Pic}^0(C)$ contains a unique reduced divisor $D_X$ of one of the following five forms:

(i) $D_X = P_1 + P_2 - 2 \cdot P_\infty$ with $P_1, P_2 \in C_{\text{aff}}$ and $P_1 \neq \iota(P_2)$ and $P_1, P_2 \neq P_0$;

(ii) $D_X = P + P_0 - 2 \cdot P_\infty$ with $P \in C_{\text{aff}}$ and $P \neq P_0$;

(iia) $D_X = P - P_\infty$ with $P \in C_{\text{aff}}$ and $P \neq P_0$;

(iib) $D_X = P_0 - P_\infty$;

(iii) $D_X = 0$.

Now cases (i) and (ii) combine to form case (i) above and cases (iia) and (iib) combine to form cases (ii) above. With this finer classification, elements of $\text{Pic}^0(C)$ that are neither in $\Theta$ nor in $\Theta_{X_0}$ have reduced divisors of type (ia), elements of $\Theta_{X_0}$ that are not in $\Theta$ have reduced divisors of type (ib), elements of $\Theta$ that are not in $\Theta_{X_0}$ have reduced divisors of type (iia), and the two elements of $\text{Pic}^0(C)$ that are in both $\Theta$ and $\Theta_{X_0}$ have reduced divisors of types (iib) and (iii).

Glue Equations in Biprojective Space.

The idea now is to consider the “graph” of the map between the affine varieties that parametrize $\text{Jac}(C) - \Theta$ and $\text{Jac}(C) - \Theta_{X_0}$ and look at this as a biprojective
variety. The hope here is that except for two points corresponding to cases (iib) and (iii) above, every point will be accounted for. In particular, the two varieties just need to be projectivized with different homogenizing variables and then the “glue” needs to be applied which consists of bihomogenizing the equations that relate the hatted and the unhatted variables.

Thus, there are projective coordinates $s = (u_0, u_1, v_0, v_1, z)$ that satisfy the pair of equations gotten by homogenizing the affine equations for $\text{Jac}(C) - \Theta$ with respect to $z$. Thus, the equations are:

$$0 = E_0(S) = -a_2 u_0 z^3 + a_4 u_0^2 z^3 + a_3 u_0 u_1 z^2 - 2 u_0 u_1 z - a_4 u_0 u_1^2 z + u_0 u_1^3$$

$$0 = E_1(S) = z^4 - a_3 u_0 z^3 + u_0^2 z^2 - a_2 u_1 z^3 + 2 a_4 u_0 u_1 z^2 + a_3 u_1^2 z^2$$

and it won’t be necessary to go to the full projective closure. In a similar fashion, there are projective coordinates $\hat{S} = (\hat{u}_0, \hat{u}_1, \hat{v}_0, \hat{v}_1, \hat{z})$ for $\text{Jac}(C) - \Theta_0$. In this case, the equations are now the same as above but with all the variables “hatted”. Thus, the equations are:

$$0 = \hat{E}_0(\hat{S}) = -a_2 \hat{u}_0 \hat{z}^3 + a_2 \hat{u}_0^2 \hat{z}^3 + a_3 \hat{u}_0 \hat{u}_1 \hat{z}^2 - 2 \hat{u}_0 \hat{u}_1 \hat{z}$$

$$0 = \hat{E}_1(\hat{S}) = \hat{z}^4 - a_3 \hat{u}_0 \hat{z}^3 + \hat{u}_0^2 \hat{z}^2 - a_2 \hat{u}_1 \hat{z}^3 + 2 a_4 \hat{u}_0 \hat{u}_1 \hat{z}^2 + a_3 \hat{u}_1^2 \hat{z}^2 - 3 \hat{u}_0 \hat{u}_1^2 \hat{z}$$

and again, going to the full projective closure is unnecessary. Note that setting $z = 1$ in the first pair of equations recovers the affine variety that they came from and that setting $\hat{z} = 1$ in the second pair of equations recovers the affine variety that they came from.

The glue equations become the bihomogenization of the equations that relate the unhatted variables to the hatted ones. These equations are

$$0 = G_1(S, \hat{S}) = u_0 - a_1 z \hat{z}$$

$$0 = G_2(S, \hat{S}) = \hat{v}_0 u_0 + v_0 \hat{u}_0$$

$$0 = G_3(S, \hat{S}) = \hat{v}_0 \hat{v}_0 + a_1 (a_4 z \hat{z} - u_1 \hat{z} - \hat{u}_1 z)$$

$$0 = G_4(S, \hat{S}) = \hat{u}_0 (v_1 \hat{z} + \hat{v}_1 z) + \hat{v}_0 (u_1 \hat{z} - \hat{u}_1 z)$$

$$0 = G_5(S, \hat{S}) = u_0 (v_1 \hat{z} + \hat{v}_1 z) + v_0 (u_1 z - \hat{u}_1 \hat{z})$$

$$0 = G_6(S, \hat{S}) = a_1 \hat{u}_0 \hat{z}^2 - a_1 a_3 \hat{z}^2 + a_1 u_0 \hat{z} z + a_1 \hat{u}_1 u_1 z - 2 u_1 \hat{v}_0 v_0 - 2 u_0 v_0 v_1$$

$$0 = G_7(S, \hat{S}) = a_1 \hat{u}_0 \hat{z} - a_1 a_3 \hat{z}^2 + a_1 u_0 \hat{z}^2 + a_1 \hat{u}_1 u_1 \hat{z} - 2 \hat{u}_1 \hat{v}_0 v_0 - 2 u_0 \hat{v}_0 \hat{v}_1$$

$$0 = G_8(S, \hat{S}) = -a_1 \hat{v}_0 \hat{z}^3 + a_3 u_0 \hat{z}^2 - u_0^2 \hat{v}_0 z - 2 a_4 u_0 u_1 \hat{v}_0 z + 3 u_0 u_1^2 \hat{v}_0$$

$$+ a_1 u_1 \hat{v}_1 \hat{z}^2 + a_1 u_1 v_1 z \hat{z} + 2 \hat{v}_0 v_0 v_1 z$$

$$0 = G_9(S, \hat{S}) = -a_1 \hat{v}_0 \hat{z}^3 + a_3 u_0 \hat{z}^2 - \hat{u}_0^2 \hat{v}_0 \hat{z} - 2 a_4 \hat{u}_0 \hat{u}_1 \hat{v}_0 \hat{z} + 3 \hat{u}_0 \hat{u}_1^2 \hat{v}_0$$

$$+ a_1 \hat{u}_1 \hat{v}_1 \hat{z}^2 + a_1 \hat{u}_1 \hat{v}_1 \hat{z} + 2 \hat{v}_0 \hat{v}_0 \hat{v}_1 \hat{z}$$

and it is worth noting that $G_1$, $G_2$, and $G_3$ are symmetric, i.e. that $G_1(S, \hat{S}) = G_1(\hat{S}, S)$, $G_2(S, \hat{S}) = G_2(\hat{S}, S)$, and $G_3(S, \hat{S}) = G_3(\hat{S}, S)$, while $(G_4, G_5, (G_6, G_7),$
and \((G_8, G_9)\) are complementary pairs, i.e. \(G_5(S, \hat{S}) = G_4(S, \hat{S}), \quad G_7(S, \hat{S}) = G_6(S, S), \quad \text{and} \quad G_9(S, S) = G_8(S, \hat{S}).\) It is also worth noting that the homogeneous bidegrees of \(G_1, G_2\) and \(G_3\) are \((1, 1)\), the homogeneous bidegrees of \(G_4\) and \(G_7\) are \((1, 2)\), the homogeneous bidegrees of \(G_5\) and \(G_6\) are \((2, 1)\), and homogeneous the bidegrees of \(G_8\) and \(G_9\) are \((3, 1)\) and \((1, 3)\), respectively, all of which follow from the inhomogeneous case.

Now consider what happens at the different types of divisor classes and how they correspond to \(z\) and \(\hat{z}\) being nonzero or zero, there being four cases. The idea is that they should correspond to the refined classification of divisor classes.

If neither \(z\) nor \(\hat{z}\) is zero, then any biprojective point on this whole set of equations corresponds to a unique point in \(\text{Pic}^b(C)\) of type (ia), and every divisor class of type (ia) gives rise to a unique point on this biprojective variety with \(z, \hat{z} \neq 0\).

Things get more interesting if either \(z = 0\) or \(\hat{z} = 0\), but not both. This should correspond to divisor classes of type (ib) and (iia), and in fact this will be seen from looking at the equations, but for the moment this correspondence is not assumed.

**The case \((z, \hat{z}) = (1, 1)\).**

In this case, the biprojective equations simply become the affine equations that relate the coordinates of \(X\) to the coordinates of \(X + X_0\), as long as neither \(X\) nor \(X + X_0\) are on \(\Theta\). Alternatively, these can be thought of as equations on the affine variety \((Jac(C) - \Theta) \times (Jac(C) - \Theta_X)\) where \(\Theta_X\) is just the translation of \(\Theta\) by the point \(X_0\). Geometrically, \(Jac(C) - \Theta \simeq Jac(C) - \Theta_{X_0}\).

**The case \((z, \hat{z}) = (1, 0)\).**

Points of infinity type \((z, \hat{z}) = (0, 1)\) should correspond to divisors of type \(P + P_0 - 2P_\infty\) for some affine \(P \in C_{\text{aff}}\) with \(P \neq P_0\). Similarly, points of infinity type \((z, \hat{z}) = (0, 1)\), should correspond to divisors of type \(P - P_\infty\) for some affine \(P \in C_{\text{aff}}\) with \(P \neq P_0\).

So suppose \(\hat{z} = 0\) and \(z \neq 0\), in which case there is no loss of generality in taking \(z = 1\). Then it follows that \(u_1 = 0\) as well. The connecting equations after setting \(z = 1\) and \(\hat{z} = 0\) then become (in order)

\[
\begin{align*}
G_1 & : 0 = u_0 \hat{u}_0 \\
G_2 & : 0 = \hat{v}_0 u_0 + v_0 \hat{u}_0 \\
G_3 & : 0 = v_0 \hat{u}_0 + a_1 \hat{u}_1 \\
G_4 & : 0 = \hat{u}_0 \hat{v}_1 - \hat{v}_0 \hat{u}_1 \\
G_5 & : 0 = u_0 \hat{v}_1 + v_0 \hat{u}_1 \\
G_6 & : 0 = a_1 \hat{u}_0 + a_1 \hat{u}_1 u_1 - 2u_1 \hat{v}_0 v_0 - 2 \hat{u}_0 v_0 v_1 \\
G_7 & : 0 = -2 \hat{u}_1 \hat{v}_0 v_0 - 2u_0 \hat{v}_0 \hat{v}_1 \\
G_8 & : 0 = a_3 u_0 \hat{v}_0 - a_1 \hat{v}_0 - u_0^2 \hat{v}_0 - 2a_4 u_0 u_1 \hat{v}_0 + 3u_0 u_1^2 \hat{v}_0 + a_1 u_1 \hat{v}_1 + 2 \hat{v}_0 v_0 v_1 \\
G_9 & : 0 = 3 \hat{u}_0 \hat{u}_1^2 v_0
\end{align*}
\]

from which some interesting implications will follow. Setting \(\hat{z} = 0\) in equations \(\hat{E}_0\) and \(\hat{E}_1\) gives

\[
\begin{align*}
E_0 & : 0 = \hat{u}_0 \hat{u}_1^3 \\
E_1 & : 0 = \hat{u}_1^4
\end{align*}
\]
from which it follows that \( \hat{z} = 0 \) implies \( \hat{u}_1 = 0 \). Upon setting \( \hat{u}_1 = 0 \), the glue equations now become (in order)

\[
\begin{align*}
G_1 : & \quad 0 = u_0 \hat{u}_0 \\
G_2 : & \quad 0 = \hat{v}_0 u_0 + v_0 \hat{u}_0 \\
G_3 : & \quad 0 = v_0 \hat{v}_0 \\
G_4 : & \quad 0 = \hat{u}_0 \hat{v}_1 \\
G_5 : & \quad 0 = u_0 \hat{v}_1 \\
G_6 : & \quad 0 = a_1 \hat{u}_0 - 2 u_1 \hat{v}_0 v_0 - 2 \hat{u}_0 v_0 v_1 \\
G_7 : & \quad 0 = -2 u_0 \hat{v}_0 \hat{v}_1 \\
G_8 : & \quad 0 = a_3 u_0 \hat{v}_0 - a_1 \hat{v}_0 - u_0^2 \hat{v}_0 - 2 a_4 u_0 u_1 \hat{v}_0 + 3 u_0 u_1^2 \hat{v}_0 + a_1 u_1 \hat{v}_1 + 2 \hat{v}_0 v_0 v_1
\end{align*}
\]

Now suppose that \( u_0 \neq 0 \). Then the first of these equations gives \( \hat{u}_0 = 0 \) and the second equation becomes \( 0 = \hat{v}_0 u_0 \) from which it follows that \( \hat{v}_0 = 0 \). The fifth equation now gives \( \hat{v}_1 = 0 \). So the assumption \( u_0 \neq 0 \) requires all the hatted variables to be 0, but this is not a point in projective space, and so does not correspond to a point on the biprojective variety. Therefore \( u_0 = 0 \). Now look at the above glue equations and note that (in order) they become

\[
\begin{align*}
G_2 : & \quad 0 = v_0 \hat{u}_0 \\
G_3 : & \quad 0 = v_0 \hat{v}_0 \\
G_4 : & \quad 0 = \hat{u}_0 \hat{v}_1 \\
G_5 : & \quad 0 = a_1 \hat{u}_0 - 2 u_1 \hat{v}_0 v_0 - 2 \hat{u}_0 v_0 v_1 \\
G_6 : & \quad 0 = -a_1 \hat{v}_0 + a_1 u_1 \hat{v}_1 + 2 \hat{v}_0 v_0 v_1
\end{align*}
\]

and setting \( z = 1 \) and \( u_0 = 0 \) in equations \( E_0 \) and \( E_1 \) gives

\[
\begin{align*}
E_0 : & \quad 0 = -v_0^2 \\
E_1 : & \quad 0 = 1 - a_2 u_1 + a_3 u_1^2 - a_4 u_1^3 + u_4 - 2 v_0 v_1 + u_1 v_1^2
\end{align*}
\]

so that \( v_0 = 0 \) from the first of these and then the second equation now becomes

\[
E_1 : \quad 0 = 1 - a_2 u_1 + a_3 u_1^2 - a_4 u_1^3 + u_4 + u_1 v_1^2
\]

and note that this implies \( u_1 \neq 0 \). It is interesting to multiply by \( -u_1 \) and rewrite this as

\[
(u_1 v_1)^2 = (-u_1)^5 + a_4 (-u_1)^4 + a_3 (-u_1)^3 + a_2 (-u_1)^2 + (-u_1)
\]

so that the pair \( (-u_1, u_1 v_1) \) actually defines a point on the punctured affine curve \( C_{\text{aff}} - P_0 \).

Also setting \( v_0 = 0 \) in the glue equations gives

\[
\begin{align*}
G_4 : & \quad 0 = \hat{u}_0 \hat{v}_1 \\
G_6 : & \quad 0 = a_1 \hat{u}_0 \\
G_8 : & \quad 0 = -a_1 \hat{v}_0 + a_1 u_1 \hat{v}_1
\end{align*}
\]
and since $a_1 \neq 0$ it follows that $\hat{u}_0 = 0$ and the only remaining glue equation is now
\[ 0 = -\hat{v}_0 + u_1 \hat{v}_1 \]
and since $u_1 \neq 0$ this gives a single point in the projective space defined by the hatted variables. Thus
\[ \frac{\hat{v}_0}{\hat{v}_1} = u_1 \]
so the projected point on the hatted side of things is just (minus) the $x$ coordinate of an affine point on $C$ (excluding $P_0$). This corresponds precisely to divisor classes of type (ib), i.e. to classes having canonical representatives of the type $P + P_0 - 2P_\infty$ where $P \in C_{aff}$ with $p \neq P_0$.

**Proposition.** Suppose that 0 is a root of $f(x)$ so $P_0 = (0, 0)$ is an affine Weierstrass point on $C$. Let $P = (u_0, u_1, v_0, v_1, z)$ be the coordinates of a point on $\hat{A}(\mathcal{S})$, the projective closure of $A(\mathcal{S})$, and let $\hat{P} = (\hat{u}_0, \hat{u}_1, \hat{v}_0, \hat{v}_1, \hat{z})$ be the coordinates of a point on $\hat{A}(\hat{\mathcal{S}})$, the projective closure of $A(\hat{\mathcal{S}})$. Furthermore also assume that these coordinates also satisfy the bihomogenous glue equations $G(S, \hat{S}, a) = 0$. Suppose that $z = 1$ and $\hat{z} = 0$. Then $u_0 = v_0 = \hat{u}_0 = \hat{v}_0$ as well and $\hat{v}_0 = u_1 \hat{v}_1$. Furthermore $(-u_1, u_1 v_1)$ is an affine point on $C$ with $u_1 \neq 0$.

Thus, points on the biprojective variety with infinity type $(z, \hat{z}) = (1, 0)$ are in one-to-one correspondence with divisor classes of type (ib), as expected.

In a completely similar fashion, taking $z = 0$ and $\hat{z} \neq 0$ gives divisor classes of type (iiia). In particular, in this case it will be found that $u_0 = \hat{u}_0 = \hat{v}_0 = u_1 = 0$ and that $(-\hat{u}_1, \hat{u}_1 \hat{v}_1)$ are the coordinates of a point on the punctured affine curve $C_{aff} - P_0$ (so that $u_1 \neq 0$ and that on the unhatted side of things, the remaining nonzero variables $v_0$ and $v_1$ are in the fixed ratio
\[ \frac{v_0}{v_1} = \hat{u}_1. \]

**The case** $(z, \hat{z}) = (0, 0)$.

There are only two types of divisor classes left, namely (iib) and (iii), which each consist of a single point, with representatives $P_0 - P_\infty$ and 0, respectively. These should correspond to the case $z = \hat{z} = 0$. Unfortunately, here is where there are some problems. Setting $z = \hat{z} = 0$ in equations $E_1$ and $\hat{E}_1$ gives
\[ E_1: \quad 0 = u_1^4 \]
\[ \hat{E}_1: \quad 0 = \hat{u}_1^4 \]
so that here $u_1 = \hat{u}_1 = 0$. Now setting $z = \hat{z} = u_1 = \hat{u}_1 = 0$ in the glue equations gives (in order)
\[ G_1: \quad 0 = u_0 \hat{u}_0 \]
\[ G_2: \quad 0 = \hat{v}_0 u_0 + v_0 \hat{u}_0 \]
\[ G_3: \quad 0 = v_0 \hat{v}_0 \]
\[ G_6: \quad 0 = -2 \hat{u}_0 v_0 v_1 \]
\[ G_7: \quad 0 = -2 u_0 \hat{v}_0 \hat{v}_1 \]
Also going through the same process of closure as in the affine case would give
\[ 0 = u_0^2 v_1^2 \]
and
\[ 0 = \hat{u}_0^2 \hat{v}_1^2 \]
which could also be of use if needed.

Now look what happens if \( u_0 \neq 0 \). Then \( \hat{u}_0 = 0 \) from the first equation and then \( \hat{v}_0 = 0 \) from the second equation. Therefore in order that not all the hatted variables be zero this gives \( \hat{v}_1 \neq 0 \), so the hatted variables all define a single projective point. For the unhatted variables \( v_0 \) is unconstrained, but \( v_1 \) is forced to be 0. Therefore there is a projective line in the unhatted variables. Similarly if \( \hat{u}_0 \neq 0 \), the unhatted variables all reduce to a single point, but the hatted variables give a projective line.

If \( u_0 = \hat{u}_0 = 0 \), then the only equation left is \( 0 = v_0 \hat{v}_0 \) so either \( v_0 = 0 \) or \( \hat{v}_0 = 0 \).

If \( v_0 = 0 \) then \( v_1 \neq 0 \) or else all the unhatted variables would be 0, so that the unhatted variables give a single projective point, and the hatted variables \( \hat{v}_0 \) and \( \hat{v}_1 \) are unconstrained, so there is a projective line in the hatted variables. Similarly, if \( \hat{v}_0 = 0 \), then there is a projective point in the hatted variables and a projective line in the unhatted variables.

**Proposition.** Suppose that 0 is a root of \( f(x) \) so \( P_0 = (0, 0) \) is an affine Weierstrass point on \( C \). Let \( P = (u_0, u_1, v_0, v_1, z) \) be the coordinates of a point on \( \hat{A}(S) \), the projective closure of \( A(S) \), and let \( \hat{P} = (\hat{u}_0, \hat{u}_1, \hat{v}_0, \hat{v}_1, \hat{z}) \) be the coordinates of a point on \( \hat{A}(\hat{S}) \), the projective closure of \( A(\hat{S}) \). Furthermore also assume that these coordinates also satisfy the bihomogenous glue equations \( G(S, S, a) = 0 \). Suppose that \( z = 0 \) and \( \hat{z} = 0 \). Then \( v_1 = \hat{u}_1 = 0 \) and exactly one of the following occurs:

(i) \( v_1 = \hat{u}_0 = \hat{v}_0 = 0 \) and \( \hat{v}_1 \neq 0 \) and \( (u_0, v_0) \neq (0, 0) \);
(ii) \( \hat{v}_1 = u_0 = v_0 = 0 \) and \( v_1 \neq 0 \) and \( (\hat{u}_0, \hat{v}_0) \neq (0, 0) \);
(iii) \( u_0 = \hat{u}_0 = v_0 = 0 \) and \( v_1 \neq 0 \) and \( (\hat{v}_0, \hat{v}_1) \neq (0, 0) \);
(iv) \( u_0 = \hat{u}_0 = v_0 = 0 \) and \( \hat{v}_1 \neq 0 \) and \( (v_0, v_1) \neq (0, 0) \);

Each of these cases determines a biprojective line.

Thus the case \( z = \hat{z} = 0 \) gives a union of four projective lines rather that a pair of projective points. The unfortunate result of all of this is that for the infinity type \( (z, \hat{z}) = (0, 0) \), what results is not just a pair of points. The problem is now to somehow fix this up.

**More General Weierstrass Points: Adding a 2-Division Point in General.**

It seems worthwhile to slightly rewrite the formulas so far in a bit more generality to allow for an affine branch point on \( C \) other than \( (0, 0) \). Eventually, it will be important to allow multiple affine branch points and these formulas will be necessary. The actual modifications to the formulas are really fairly simple. Assume that \( \rho \) is a root of \( f(x) \), the monic quintic polynomial, and write the affine equation for the curve in the form
\[ y^2 = f(x) = (x - \rho)^5 + a_4'(x - \rho)^4 + a_5'(x - \rho)^3 + a_6'(x - \rho)^2 + a_7'(x - \rho) \]
and the affine branch point in question is \( P_\rho = (\rho, 0) \).
The $U$ and $V$ polynomials now can be written as

$$U(x; X) = (x - \rho)^2 + u'_1 (x - \rho) + u'_0$$
$$V(x; X) = v'_1 (x - \rho) + v'_0$$

and the condition to be satisfied is still

$$U(x; X)|f(x) - V(x; X)^2$$

which can be fulfilled by writing

$$f(x) - V(x; X)^2 \equiv e'_1 (x - \rho) + e'_0 \mod U(x; X)$$

and then insisting that

$$0 = e'_0$$
$$0 = e'_1$$

so that in these new coordinates, any quadruple $(u'_1, u'_0, v'_1, v'_0)$ for which both equations $e'_0 (u'_1, u'_0, v'_1, v'_0) = 0$ and $e'_1 (u'_1, u'_0, v'_1, v'_0) = 0$ hold gives rise to a pair of polynomials $U(x) = (x - \rho)^2 + u'_1 (x - \rho) + u'_0$ and $V(x) = v'_1 (x - \rho) + v'_0$ such that $U(x)|f(x) - V(x)^2$. These two modified equations then define the same affine variety as before, $\text{Pic}^0(C) - \Theta$.

Of course it is quite straightforward to convert these equations to the old coordinates. Just write

$$U(x; X) = (x - \rho)^2 + u'_1 (x - \rho) + u'_0$$
$$= x^2 + (u'_1 - 2 \rho) x + (u'_0 - u'_1 \rho + \rho^2)$$
$$= x^2 + u_1 x + u_0$$
$$V(x; X) = v'_1 (x - \rho) + v'_0$$
$$= v'_1 x + (v'_0 - v'_1 \rho)$$
$$= v_1 x + v_0$$

so that

$$u'_1 = u_1 + 2 \rho$$
$$u'_0 = u_0 + u_1 \rho + \rho^2$$
$$v'_1 = v_1$$
$$v'_0 = v_0 + v_1 \rho$$

which can be back substituted into equations $e'_0 = 0$ and $e'_1 = 0$.

It should be noted that with the substitutions

$$f(x) = x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$
$$= (x - \rho)^5 + a'_4 (x - \rho)^4 + a'_3 (x - \rho)^3 + a'_2 (x - \rho)^2 + a'_1 (x - \rho) + a'_0$$
it follows that
\[
\begin{align*}
    a_4 &= a'_4 - 5 \rho \\
    a_3 &= a'_3 - 4 a'_4 \rho + 10 \rho^2 \\
    a_2 &= a'_2 - 3 a'_4 \rho + a'_3 + 10 \rho^3 \\
    a_1 &= a'_1 - 2 a'_2 \rho + 3 a'_3 + 4 a'_4 + 5 \rho^4 \\
    a_0 &= a'_0 - a'_1 \rho + a'_2 \rho^2 - a'_4 \rho^3 + a'_4 \rho^4 - \rho^5
\end{align*}
\]
which is a substitution that must be made when using Weierstrass points other than \((0,0)\) (in which case \(a'_0 = 0\)). In matrix form this is
\[
\begin{pmatrix}
    a_0 \\
    a_1 \\
    a_2 \\
    a_3 \\
    a_4
\end{pmatrix} = \begin{pmatrix}
    1 & -\rho & \rho^2 & -\rho^3 & \rho^4 \\
    0 & 1 & -2 \rho & 3 \rho^2 & -4 \rho^3 \\
    0 & 0 & 1 & -3 \rho & 6 \rho^2 \\
    0 & 0 & 0 & 1 & -4 \rho \\
    0 & 0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
    a'_0 \\
    a'_1 \\
    a'_2 \\
    a'_3 \\
    a'_4
\end{pmatrix} + \begin{pmatrix}
    -\rho^5 \\
    5 \rho^4 \\
    -10 \rho^3 \\
    10 \rho^2 \\
    -5 \rho
\end{pmatrix}
\]
and the inverse transformation is
\[
\begin{pmatrix}
    a'_0 \\
    a'_1 \\
    a'_2 \\
    a'_3 \\
    a'_4
\end{pmatrix} = \begin{pmatrix}
    1 & \rho & \rho^2 & \rho^3 & \rho^4 \\
    0 & 1 & 2 \rho & 3 \rho^2 & 4 \rho^3 \\
    0 & 0 & 1 & \rho & 6 \rho^2 \\
    0 & 0 & 0 & 1 & 4 \rho \\
    0 & 0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
    a_0 \\
    a_1 \\
    a_2 \\
    a_3 \\
    a_4
\end{pmatrix} + \begin{pmatrix}
    \rho^5 \\
    5 \rho^4 \\
    10 \rho^3 \\
    10 \rho^2 \\
    5 \rho
\end{pmatrix}
\]
It is also interesting to note that the relationship between the defining polynomials \((e'_1, e'_0)\) and \((e_1, e_0)\) is also given by
\[
\begin{align*}
    e'_1(u'_1, u'_0, v'_1, v'_0) &= e_1(u_1, u_0, v_1, v_0) \\
    e'_0(u'_1, u'_0, v'_1, v'_0) &= e_0(u_1, u_0, v_1, v_0) + \rho e_1(u_1, u_0, v_1, v_0)
\end{align*}
\]
which follows directly from substituting \(x \mapsto x + \rho\) in
\[
f(x) - V(x)^2 \equiv e_1 x + e_0 \mod U(x)
\]
so that
\[
f(x + \rho) - V(x + \rho) \equiv e_1 (x + \rho) + e_0 = e'_1 x + e'_0 \mod U(x + \rho).
\]

A QUADRI-PROJECTIVE VARIETY

The next question is whether the problem at infinity with the biprojective variety (which was obtained by working with a single affine Weierstrass point) (which was obtained by working with a single affine Weierstrass point) can be cleared up by working with a second affine Weierstrass point.

To that end, let \(\rho^{(1)}\) and \(\rho^{(2)}\) be distinct roots of the quintic \(f(x)\), so that \(P_1 = (\rho^{(1)}, 0)\) and \(P_2 = (\rho^{(2)}, 0)\) are distinct Weierstrass points on the projective curve \(C\) defined by \(y^2 = f(x)\). The divisors \(P_1 - P_\infty\) and \(P_2 - P_\infty\) determine divisor classes \(X_1\) and \(X_2\) in \(\text{Pic}^0(C)\). Furthermore, the divisor of the function \(x - \rho^{(1)}\) is \(2P_1 - 2P_\infty\), so the divisor class \(X_1\) is of order 2 in \(\text{Pic}^0(C)\), and similarly the divisor class \(X_2\) is also of order 2.
With these two distinguished affine points, there is an even finer classification of divisor classes than before. The new classification is as follows: Every divisor class \( X \in \text{Pic}^0(C) \) contains a unique reduced divisor \( D_X \) of one of the following eight forms:

1. \( D_X = Q_1 + Q_2 - 2 \cdot P_{\infty} \) with \( Q_1, Q_2 \in C_{\text{aff}} \) and \( Q_1 \neq i(Q_2) \) and \( Q_1, Q_2 \notin \{ P_1, P_2 \} \);
2. \( D_X = Q + P_1 - 2 \cdot P_{\infty} \) with \( Q \in C_{\text{aff}} \) and \( Q \notin \{ P_1, P_2 \} \);
3. \( D_X = Q + P_2 - 2 \cdot P_{\infty} \) with \( Q \in C_{\text{aff}} \) and \( Q \notin \{ P_1, P_2 \} \);
4. \( D_X = P_1 - P_{\infty} \);
5. \( D_X = P_2 - P_{\infty} \);
6. \( D_X = 0 \).

Type (1a) is by far the largest of these types and comprises a two-dimensional affine variety. Types (1b), (1c), and (2a) each correspond to one-dimensional affine varieties that are isomorphic to \( C_{\text{aff}} \setminus \{ P_1, P_2 \} \). Types (1d), (2b), (2c), and (3) each correspond to a single point.

Every divisor class is one of these types, and these types don’t overlap. Furthermore, as will be seen in the following analysis, each of these types of divisor class corresponds to a distinct infinity type in the quadri-homogeneous variety constructed in this session, with the exception of type (1a) which includes two different infinity types. What needs to be checked is that there are no other infinity types that contain actual geometric points. Unfortunately, this will not turn out to be the case. However, where this goes awry is with the infinity type \((0, 0, 0, 0)\), and the way that it goes wrong is much less dramatic than the case of the bihomogeneous variety described above.

**Multi-homogeneous Coordinates.**

In an effort to fix up the problems at infinity that occurred in the bihomogeneous analysis, the next idea is to consider two rational affine Weierstrass points on the genus 2 curve \( C \). Thus \( C \) is still defined by \( y^2 = f(x) \) where \( f(x) \) is a monic polynomial of degree 5 with no multiple roots. Now suppose that \( \rho^{(1)} \) and \( \rho^{(2)} \) are two distinct roots of \( f(x) \), i.e. \( f(\rho^{(1)}) = f(\rho^{(2)}) = 0 \) with \( \rho^{(1)} \neq \rho^{(2)} \) and in general write

\[
\begin{align*}
f(x) &= (x - \rho^{(1)}) (x - \rho^{(2)}) (x - \rho^{(3)}) (x - \rho^{(4)}) (x - \rho^{(5)}) \\
&= a_0 \cdot x = a_{\rho(i)} \cdot x_{\rho(i)} \\
&= x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \\
&= (x - \rho^{(1)})^5 + a_4^{(i)} (x - \rho^{(i)})^4 + a_3^{(i)} (x - \rho^{(i)})^3 + a_2^{(i)} (x - \rho^{(i)})^2 + a_1^{(i)} (x - \rho^{(i)})
\end{align*}
\]

for \( i = 1, \ldots, 5 \), where

\[
\begin{align*}
a_0 &= (a_0, a_1, a_2, a_3, a_4, 1) \\
x &= (1, x, x^2, x^3, x^4, x^5) \\
a_{\rho(i)} &= (0, a_1^{(i)}, a_2^{(i)}, a_3^{(i)}, a_4^{(i)}, 1) \\
x_{\rho(i)} &= (1, (x - \rho^{(i)}), (x - \rho^{(i)})^2, (x - \rho^{(i)})^3, (x - \rho^{(i)})^4, (x - \rho^{(i)})^5)
\end{align*}
\]
and let $$s = (u_0, u_1, v_0, v_1)$$
$$\hat{s} = (\hat{u}_0, \hat{u}_1, \hat{v}_0, \hat{v}_1)$$
be sets of variables, and let $$S = (u_0, u_1, v_0, v_1, z)$$
$$\hat{S} = (\hat{u}_0, \hat{u}_1, \hat{v}_0, \hat{v}_1, \hat{z})$$
be the corresponding sets of homogeneous variables.

Now let
$$e_0(v_1, v_0, u_1, u_0)$$
$$= a_0 - a_2 u_0 + a_4 u_0^2 + a_3 u_0 u_1 - 2 u_0^2 u_1 - 4 u_0 u_1^2 + u_0 u_1^2 - v_0^2 + u_0 v_1^2$$
$$e_1(v_1, v_0, u_1, u_0)$$
$$= a_1 - a_3 u_0 + a_2 u_2 - 2 a_4 u_0 u_1 + a_3 u_1^2 - 3 u_0 u_1^2 - a_4 u_1^3 + u_1^4$$
$$- 2 v_0 v_1 + u_1 v_1^2$$
so the affine variety defined by $$e_0(v_1, v_0, u_1, u_0) = e_1(v_1, v_0, u_1, u_0) = 0$$ is $$\text{Jac}(C) - \Theta$$. It is useful to write $$e(s, a_0) = (e_0(s, a_0), e_1(s, a_0))$$ so the equations defining the affine variety are $$e(s, a_0) = 0$$. The homogenized versions of these polynomials are
$$E_0(v_1, v_0, u_1, u_0, z) = E_0(S, a_0)$$
$$= a_0 z^4 - a_2 u_0 z^3 + a_4 u_0^2 z^2 + a_3 u_0 u_1 z - 2 u_0^2 u_1 z - a_4 u_0 u_1^2 z + u_0 u_1^3$$
$$- v_0^2 z^2 + u_0 v_1^2 z$$
$$E_1(v_1, v_0, u_1, u_0, z) = E_1(S, a_0)$$
$$= a_1 z^4 - a_3 u_0 z^3 + a_2 u_2 z^2 - 2 a_4 u_0 u_1 z^3 + a_3 u_1^2 z^2 - 3 u_0 u_1^2 z$$
$$- a_4 u_1^3 z + u_1^4 - 2 v_0 v_1 z^2 + u_1 v_1^2 z$$
and the projective closure of the equations $$E_0(S, a_0) = E_1(S, a_0) = 0$$ is just the projective closure of $$\text{Jac}(C) - \Theta$$.

In an effort to fix up the problems at infinity that occurred in the bihomogeneous analysis, the next idea is to consider two rational affine Weierstrass points on the genus 2 curve $$C$$. Thus $$C$$ is still defined by $$y^2 = f(x)$$ where $$f(x)$$ is a monic polynomial of degree 5 with no multiple roots. Now suppose that $$\rho^{(1)}$$ and $$\rho^{(2)}$$ are two distinct roots of $$f(x)$$, i.e. $$f(\rho^{(1)}) = f(\rho^{(2)}) = 0$$ with $$\rho^{(1)} \neq \rho^{(2)}$$ and in general write
$$f(x) = (x - \rho^{(1)}) (x - \rho^{(2)}) (x - \rho^{(3)}) (x - \rho^{(4)}) (x - \rho^{(5)})$$
$$= a_0 \cdot x = a_{\rho^{(i)}} \cdot x_{\rho^{(i)}}$$
$$= x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$
$$= (x - \rho^{(i)})^5 + a_4^{(i)} (x - \rho^{(i)})^4 + a_3^{(i)} (x - \rho^{(i)})^3 + a_2^{(i)} (x - \rho^{(i)})^2 + a_1^{(i)} (x - \rho^{(i)})$$
for $$i = 1, \ldots, 5$$, where
$$a_0 = (a_0, a_1, a_2, a_3, a_4, 1)$$
$$x = (1, x, x^2, x^3, x^4, x^5)$$
$$a_{\rho^{(i)}} = (0, a_1^{(i)}, a_2^{(i)}, a_3^{(i)}, a_4^{(i)}, 1)$$
$$x_{\rho^{(i)}} = (1, x - \rho^{(i)}, (x - \rho^{(i)})^2, (x - \rho^{(i)})^3, (x - \rho^{(i)})^4, (x - \rho^{(i)})^5)$$
and let
\[
\mathbf{s} = (u_0, u_1, v_0, v_1)
\]
\[
\hat{\mathbf{s}} = (\hat{u}_0, \hat{u}_1, \hat{v}_0, \hat{v}_1)
\]
be sets of variables, and let
\[
\mathbf{S} = (u_0, u_1, v_0, v_1, z)
\]
\[
\hat{\mathbf{S}} = (\hat{u}_0, \hat{u}_1, \hat{v}_0, \hat{v}_1, \hat{z})
\]
be the corresponding sets of homogeneous variables.

Now let
\[
e_0(v_1, v_0, u_1, u_0) = a_0 - a_2 u_0 + a_4 u_0^2 + a_3 u_0 u_1 - 2 u_0^2 u_1 - a_4 u_0 u_1^2 + u_0 u_1^3 - v_0^2 + u_0 v_1^2
\]
\[
e_1(v_1, v_0, u_1, u_0) = a_1 - a_3 u_0 + u_0^2 - a_2 u_1 + 2 a_4 u_0 u_1 + a_3 u_1^2 - 3 u_0 u_1^2 - a_4 u_1^3 + u_1^4
\]
\[- 2 v_0 v_1 + u_1 v_1^2 \]
so the affine variety defined by \(e_0(v_1, v_0, u_1, u_0) = e_1(v_1, v_0, u_1, u_0) = 0\) is \(\text{Jac}(C) - \Theta\). It is useful to write \(e(\mathbf{s}, \mathbf{a}_0) = (e_0(\mathbf{s}, \mathbf{a}_0), e_1(\mathbf{s}, \mathbf{a}_0))\) so the equations defining the affine variety are \(e(\mathbf{s}, \mathbf{a}_0) = 0\). The homogenized versions of these polynomials are
\[
E_0(v_1, v_0, u_1, u_0, z) = E_0(\mathbf{S}, \mathbf{a}_0)
\]
\[
= a_0 z^4 - a_2 u_0 z^3 + a_4 u_0^2 z^2 + a_3 u_0 u_1 z - 2 u_0^2 u_1 z - a_4 u_0 u_1^2 z + u_0 u_1^3 - v_0^2 z^2 + u_0 v_1^2 z
\]
\[
E_1(v_1, v_0, u_1, u_0, z) = E_1(\mathbf{S}, \mathbf{a}_0)
\]
\[
= a_1 z^4 - a_3 u_0 z^3 + u_0^2 z^2 - a_2 u_1 z^3 + 2 a_4 u_0 u_1 z^2 + a_3 u_1^2 z^2 - 3 u_0 u_1^2 z - a_4 u_1^3 z + u_1^4 - 2 v_0 v_1 z^2 + u_1 v_1^2 z^2
\]
and the projective closure of the equations \(E_0(\mathbf{S}, \mathbf{a}_0) = E_1(\mathbf{S}, \mathbf{a}_0) = 0\) is just the projective closure of \(\text{Jac}(C) - \Theta\), which (unfortunately) is not \(\text{Jac}(C)\).

**Glue for Other Weierstrass Points.**

Let
\[
M_{\rho(i)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
\rho^{(i)} & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & \rho^{(i)} & 1 & 0 \\
\rho^{(i)^2} & 2 \rho^{(i)} & 0 & 0 & 1
\end{pmatrix}
\]
which effects the projective transformation
\[
u_0 \mapsto u_0 + u_1 \rho^{(i)} + z \rho^{(i)^2}
\]
\[
u_1 \mapsto u_1 + 2 z \rho^{(i)}
\]
\[
v_0 \mapsto v_0 + v_1 \rho^{(i)}
\]
\[
v_1 \mapsto v_1
\]
\[z \mapsto z\]
when written as $S \mapsto S M_{p(i)}$. The sets of equations $E(S M_{p(i)}, a^{(1)}) = 0$ for $i = 1, \ldots, 5$ are equivalent to the set of equations $E(S, a_0) = 0$.

In terms of the inhomogeneous equations, let

\[
m_{p(i)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \rho^{(i)} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \rho^{(i)} & 1 \end{pmatrix}
\]

\[
n_{p(i)} = \begin{pmatrix} \rho^{(i)} & 2 \rho^{(i)} & 0 & 0 \end{pmatrix}
\]

so that $s \mapsto s m_{p(i)} + n_{p(i)}$ effects the corresponding inhomogeneous transformation

\[
\begin{align*}
u_0 &\mapsto u_0 + u_1 \rho^{(i)} + \rho^{(i)}^2 \\
u_1 &\mapsto u_1 + 2 \rho^{(i)} \\
v_0 &\mapsto v_0 + v_1 \rho^{(i)} \\
v_1 &\mapsto v_1
\end{align*}
\]

which will prove useful.

**Quadrihomogeneous Coordinates.**

Let $P_{p(1)} = (\rho^{(1)}, 0) \in C$ and $P_{p(2)} = (\rho^{(2)}, 0) \in C$, and set $X_1 = \text{Cl}(P_{p(1)} - P_\infty) \in \text{Jac}(C)$ and $X_2 = \text{Cl}(P_{p(2)} - P_\infty) \in \text{Jac}(C)$, so that $X_1$ and $X_2$ are distinct and nontrivial [2]-division points in $\text{Jac}(C)$. If $\Theta_X$ denotes the translation of the $\Theta$ divisor on $\text{Jac}(C)$ by the point $X \in \text{Jac}(C)$ then there is a simple correspondence between affine varieties

\[
\begin{array}{c}
\text{Jac}(C) - \Theta \xrightarrow{(+X_1)} \text{Jac}(C) - \Theta_{X_1} \\
\text{Jac}(C) - \Theta_{X_2} \xleftarrow{(+X_2)} \text{Jac}(C) - \Theta_{X_1+X_2}
\end{array}
\]

and the reason that the arrows are bidirectional is that adding $X_1$ maps $\text{Jac}(C) - \Theta$ to $\text{Jac}(C) - \Theta_{X_1}$ and also $\text{Jac}(C) - \Theta_{X_1}$ back to $\text{Jac}(C) - \Theta$, and similarly for all the other arrows. The corresponding inhomogeneous equations are

\[
\begin{align*}
e(s_{00}, a_0) &\xleftarrow{g(s_{00} M_{p(1)}, s_{01} M_{p(1)}, a_{p(1)})} e(s_{01}, a_0) \\
e(s_{10}, a_0) &\xleftarrow{g(s_{10} M_{p(2)}, s_{11} M_{p(2)}, a_{p(2)})} e(s_{11}, a_0)
\end{align*}
\]

where

\[
s_{ij} = (u_{ij,0}, u_{ij,1}, v_{ij,0}, v_{ij,1})
\]

for $ij \in \{00, 01, 10, 11\}$. Here

\[
s_{ij} M_{p(i)} = (u_{ij,0} + u_{ij,1} \rho^{(1)} + \rho^{(1)}^2, u_{ij,1} + 2 \rho^{(1)}, v_{ij,0} + v_{ij,1} \rho^{(1)}, v_{ij,1})
\]

with a similar formula for $s_{ij} M_{p(2)}$. 

\[\]
The corresponding homogeneous polynomials fit into the diagram

\[
\begin{array}{ccc}
\text{E}(S_{00}, a_0) & \xrightarrow{G(S_{00}M_{\rho(1)}, S_{01}M_{\rho(1)}, a_{\rho(1)})} & \text{E}(S_{01}, a_0) \\
G(S_{00}M_{\rho(2)}, S_{10}M_{\rho(2)}, a_{\rho(2)}) & \xrightarrow{G(S_{00}M_{\rho(2)}, S_{11}M_{\rho(2)}, a_{\rho(2)})} & \text{E}(S_{10}, a_0) \\
\end{array}
\]

where \( S_{ij} = \{u_{ij,0}, u_{ij,1}, v_{ij,0}, v_{ij,1}, z_{ij}\} \)

for \( ij \in \{00, 01, 10, 11\} \). Here

\[
S_{ij}M_{\rho(1)} = (u_{ij,0} + u_{ij,1}\rho(1) + z_{ij}\rho(1)^2, u_{ij,1} + 2z_{ij}\rho(1), v_{ij,0} + v_{ij,1}\rho(1), v_{ij,1}, z_{ij})
\]

with a similar formula for \( S_{ij}M_{\rho(2)} \).

**Quadrihomogeneous Coordinates: Analysis of Infinity Types.**

In order to see what the multi-projective variety determined by these equations is, it is really necessary to understand what happens at infinity in each of these components given the glue that holds them all together. This can be done most easily by starting with the case \( \rho(1) = 0 \) and then translating to arbitrary \( \rho(1) \), and noting that these same equations apply for \( \rho(2) \).

Let \( i,j \) and \( i',j' \) be two different indices taken from the set \( \{00, 01, 10, 11\} \) Now with \( z_{ij} = 1 \) and \( z_{i'j'} = 0 \), (which also implies \( u_{i'j',1} = 0 \), \( U_{ij} \mapsto S_{ij}M_{\rho(1)} \) and \( U_{i'j'} \mapsto S_{i'j'}M_{\rho(1)} \) at \( z_{ij} = 1 \), and \( z_{i'j'} = u_{i'j',1} = 0 \) effects the transformation

\[
\begin{align*}
\hat{u}_0 &\mapsto u_{ij,0} + \rho(1) u_{ij,1} + \rho(1)^2 \\
\hat{u}_1 &\mapsto u_{ij,1} + 2 \rho(1) \\
\hat{v}_0 &\mapsto v_{ij,0} + \rho(1) v_{ij,1} \\
\hat{v}_1 &\mapsto v_{ij,1} \\
\hat{\hat{u}}_0 &\mapsto u_{i'j',0} + \rho(1) u_{i'j',1} \\
\hat{\hat{u}}_1 &\mapsto u_{i'j',1} \\
\hat{\hat{v}}_0 &\mapsto v_{i'j',0} + \rho(1) v_{i'j',1} \\
\hat{\hat{v}}_1 &\mapsto v_{i'j',1}
\end{align*}
\]

and gives rise to the following relations for \( G(S_{ij}M_{\rho(1)}, S_{i'j'}M_{\rho(1)}, a_{\rho(1)}) \)

\[
\begin{align*}
0 &= u_{ij,0} + \rho(1) u_{ij,1} + \rho(1)^2 \\
0 &= v_{ij,0} + \rho(1) v_{ij,1} \\
0 &= u_{i'j',0} + \rho(1) u_{i'j',1} = u_{i'j',0} \\
0 &= u_{i'j',1} \\
0 &= - (v_{i'j',0} + \rho(1) v_{i'j',1}) + (u_{ij,1} + 2 \rho(1)) v_{i'j',1} \\
&= - v_{i'j',0} + (u_{ij,1} + \rho(1)) v_{i'j',1}
\end{align*}
\]

with similar relations where \( \rho(1) \) is replaced by \( \rho(2) \) (when adding the order 2 point \( X_2 \) instead of \( X_1 \)).
Now in the variable sets $S_{00}$ and $S_{01}$ with $z_{00} = z_{01} = 0$, (which also imply $u_{00,1} = u_{01,1} = 0$), the three equations that result are

\[
\begin{align*}
0 &= u_{00,0} u_{01,0} \\
0 &= -(v_{01,0} + \rho^{(1)} v_{01,1}) u_{00,0} + (v_{00,0} + \rho^{(1)} v_{00,1}) u_{01,0} \\
0 &= (v_{00,0} + \rho^{(1)} v_{00,1})(v_{01,0} + \rho^{(1)} v_{01,1})
\end{align*}
\]

which are the only relations that can be inferred from these glue equations.

The goal here is that every point of $\text{Jac}(C)$ corresponds to a unique point in the multi-projective variety defined by these equations, and conversely that every point of the multi-projective variety corresponds to a unique point of $\text{Jac}(C)$. There are several cases to consider depending on the different infinity types. There are a lot of symmetries present which will reduce the number of cases that need to be considered.

**The Case** $z = (1, 1, 1, 1)$.

The simplest case is where all the homogenizing variables $z_{00}$, $z_{01}$, $z_{10}$, $z_{11}$ are non-zero. This corresponds to divisor classes $Z$ such that none of $Z$, $Z + X_1$, $Z + X_2$, $Z + X_1 + X_2$ are in $\Theta$, and it is therefore possible to take $z_{00} = z_{01} = z_{10} = z_{11} = 1$. This case presents no problems at all. Pictorially, this is

\[
\begin{align*}
\text{z}_{00} &= 1 \quad \text{z}_{01} = 1 \\
\text{z}_{10} &= 1 \quad \text{z}_{11} = 1
\end{align*}
\]

with all the the corners corresponding to affine points on (translated) copies of $\text{Jac}(C)(C) - \Theta$, the the glue equations are all affine.

**Proposition.** Solutions to the system of equations $B_4$ with $z = (1, 1, 1, 1)$ are in one-to-one correspondence with the set of division classes in $\text{Pic}^{(0)}(C) - (\Theta \cup \Theta_{X_1} \cup \Theta_{X_2} \cup \Theta_{X_1 + X_2})$.

**The Case** $z = (0, 1, 1, 1)$.

The next simplest case is where three of the homogenizing variables are non-zero and the fourth is zero. This corresponds to divisor classes $Z$ such that exactly one of $Z$, $Z + X_1$, $Z + X_2$, $Z + X_1 + X_2$ is in $\Theta$, the others being in $\text{Jac}(C) - \Theta$. This also presents no problems. Pictorially, the four cases are

\[
\begin{align*}
\text{z}_{00} &= 0 \quad \text{z}_{01} = 1 & \text{z}_{00} &= 1 \quad \text{z}_{01} = 0 \\
\text{z}_{10} &= 1 \quad \text{z}_{11} = 1 & \text{z}_{10} &= 1 \quad \text{z}_{11} = 1
\end{align*}
\]

\[
\begin{align*}
\text{z}_{00} &= 1 \quad \text{z}_{01} = 1 & \text{z}_{00} &= 1 \quad \text{z}_{01} = 1 \\
\text{z}_{10} &= 0 \quad \text{z}_{11} = 1 & \text{z}_{10} &= 1 \quad \text{z}_{11} = 0
\end{align*}
\]
and now by symmetry it is suffices to consider the case \( z_{00} = 0 \) and \( z_{01} = z_{10} = z_{11} = 1 \). By viewing each of the arrows as a simple bihomogeneous case, and referring to the previous analysis, it is readily apparent that the \( z_{00} = 0 \) refers to an element of \( \Theta \), and in particular \( u_{00,0} = v_{00,0} = 0 \) and \((-u_{00,1}, u_{00,1}v_{00,1})\) are the coordinates of an affine point \( P \) on \( C \) that is neither \( P_1 = (\rho^{(1)}, 0) \) nor \( P_2 = (\rho^{(2)}, 0) \). Then the affine points on the corners \( z_{01} = 1, z_{10} = 1, \) and \( z_{11} = 1 \) represent the divisor classes of \( P + P_1 - 2P_\infty, P + P_2 - 2P_\infty, \) and \( P + P_1 + P_2 - 3P_\infty \), respectively.

**Proposition.** Solutions to the system of equations \( B_4 \) with \( z = (0,1,1,1) \) are in one-to-one correspondence with the set of divisor classes in \( \Theta - \{0, X_1, X_2 \} \).

By symmetry there are completely analogous results for the other three corners, i.e. the cases \( z = (1,0,1,1) \), \( z = (1,1,0,1) \), and \( z = (1,1,1,0) \) correspond to divisor classes in the sets \( \Theta_{X_1} - \{X_1, 0, X_1 + X_2 \}, \Theta_{X_2} - \{X_2, 0, X_1 + X_2 \} \), and \( \Theta_{X_1+X_2} - \{X_1 + X_2, X_1, X_2 \} \), respectively.

The remainder of the infinity types require more explicit analysis. There should be no multi-projective points where exactly two of the \( z_{ij} \) are non-zero, nor should there be any multi-projective points where all the \( z_{ij} \) are zero. Where three of the \( z_{ij} \) are zero and one is non-zero, should correspond to a unique multi-projective point. By symmetry, it suffices to consider the cases \( z = (1,1,0,0), z = (1,0,0,1), \) \( z = (1,0,0,0), \) and \( z = (0,0,0,0) \), where \( z = (z_{00}, z_{01}, z_{10}, z_{11}) \).

**The Case \( z = (1,0,0,0) \).**

This really corresponds to four different cases, which can be pictorially viewed as

\[
\begin{align*}
    z_{00} = 1 & \quad \equiv \quad (+X_1) & \quad z_{01} = 0 & \quad \equiv \quad (+X_2) & \quad z_{10} = 0 & \quad \equiv \quad (+X_1) & \quad z_{11} = 1 & \quad \equiv \quad (+X_2) \\
    z_{00} = 0 & \quad \equiv \quad (+X_1) & \quad z_{01} = 0 & \quad \equiv \quad (+X_2) & \quad z_{10} = 0 & \quad \equiv \quad (+X_1) & \quad z_{11} = 1 & \quad \equiv \quad (+X_2) \\
    z_{00} = 0 & \quad \equiv \quad (+X_1) & \quad z_{01} = 0 & \quad \equiv \quad (+X_2) & \quad z_{10} = 0 & \quad \equiv \quad (+X_1) & \quad z_{11} = 1 & \quad \equiv \quad (+X_2) \\
    z_{00} = 0 & \quad \equiv \quad (+X_1) & \quad z_{01} = 0 & \quad \equiv \quad (+X_2) & \quad z_{10} = 0 & \quad \equiv \quad (+X_1) & \quad z_{11} = 1 & \quad \equiv \quad (+X_2)
\end{align*}
\]

and now by symmetry it is sufficient to consider just the case \( z_{10} = z_{01} = z_{00} = 0 \) and \( z_{11} = 1 \), and as before each of the arrows refers to a simple bihomogeneous case.

Intuitively, for this to be possible, this would correspond to points \( Z \in \text{Jac}(C) - \Theta \) such that \( Z + X_1 \in \Theta \) and \( Z + X_2 \in \Theta \) and \( Z + X_1 + X_2 \in \text{Jac}(C) - \Theta \). Now if \( Z \in \text{Jac}(C) - \Theta \) such that \( Z + X_1 \in \Theta \) then \( Z \) is represented by a reduced divisor of the form \( Q_1 + P_{\rho^{(1)}} - 2P_\infty \) for some affine point \( Q_1 \in C \). Similarly, if \( Z \in \text{Jac}(C) - \Theta \) such that \( Z + X_2 \in \Theta \) then \( Z \) is represented by a reduced divisor of the form \( Q_2 + P_{\rho^{(2)}} - 2P_\infty \) for some affine point \( Q_2 \in C \). This would mean that \( Q_1 = P_{\rho^{(2)}} \) and \( Q_2 = P_{\rho^{(1)}} \), so that \( Z \) is represented by the unique reduced divisor \( P_{\rho^{(1)}} + P_{\rho^{(2)}} - 2P_\infty \). But then \( Z = X_1 + X_2 \) (a 2-division point on \( \text{Jac}(C) \)), and \( Z + X_1 + X_2 = 0 \) so \( Z + X_1 + X_2 \in \Theta \). Thus there should be a unique point with this infinity type.
The analysis here starts out similar to the previous case. The conditions \( z_{00} = 1 \) and \( z_{01} = 0 \) imply that adding \( X_1 \) gives

\[
\begin{align*}
0 &= u_{00,0} + \rho^{(1)} u_{00,1} + \rho^{(1)2}
0 &= -v_{01,0} + (u_{00,1} + \rho^{(1)}) v_{01,1}
0 &= u_{01,0}
0 &= u_{01,1}
\end{align*}
\]

while the conditions \( z_{00} = 1 \) and \( z_{10} = 0 \) imply that adding \( X_2 \) gives

\[
\begin{align*}
0 &= u_{00,0} + \rho^{(2)} u_{00,1} + \rho^{(2)2}
0 &= -v_{10,0} + (u_{00,1} + \rho^{(2)}) v_{10,1}
0 &= u_{10,0}
0 &= u_{10,1}
\end{align*}
\]

and since \( \rho^{(1)} \neq \rho^{(2)} \), this allows the values of \( u_{00,0} \) and \( u_{00,1} \) to be determined as

\[
\begin{align*}
\rho^{(1)} \rho^{(2)} &= u_{00,0} \\
-\rho^{(1)} - \rho^{(2)} &= u_{00,1}
\end{align*}
\]

and therefore

\[
\begin{align*}
0 &= -v_{01,0} - \rho^{(2)} v_{01,1}
0 &= -v_{10,0} - \rho^{(1)} v_{10,1}
\end{align*}
\]

and now note that this forces

\[
\begin{align*}
0 &\neq -v_{01,0} - \rho^{(1)} v_{01,1}
0 &\neq -v_{10,0} - \rho^{(2)} v_{10,1}
\end{align*}
\]

since \( \rho^{(1)} \neq \rho^{(2)} \) since otherwise either all the \( s_{01} \) variables would all be 0 or all the \( S_{10} \) variables would all be 0.

Now note that the conditions \( z_{11} = 0 \) and \( z_{10} = 0 \) imply that adding \( X_1 \) gives

\[
\begin{align*}
0 &= u_{11,1}
0 &= u_{10,0} u_{11,0}
0 &= (v_{11,0} + \rho^{(1)} v_{11,1}) (v_{10,0} + \rho^{(1)} v_{10,1})
\end{align*}
\]

and similarly, the conditions \( z_{11} = 0 \) and \( z_{01} = 0 \) imply that adding \( X_2 \) gives

\[
\begin{align*}
0 &= u_{11,1}
0 &= u_{01,0} u_{11,0}
0 &= (v_{11,0} + \rho^{(2)} v_{11,1}) (v_{01,0} + \rho^{(2)} v_{01,1})
\end{align*}
\]

and since

\[
\begin{align*}
0 &\neq v_{01,0} + \rho^{(1)} v_{01,1}
0 &\neq v_{10,0} + \rho^{(2)} v_{10,1}
\end{align*}
\]
this implies that
\[ 0 = v_{11,0} + \rho^{(1)} v_{11,1} \]
\[ 0 = v_{11,0} + \rho^{(2)} v_{11,1} \]

Again, using the fact that \( \rho^{(1)} \neq \rho^{(2)} \) gives
\[ 0 = z_{11} \]
\[ 0 = u_{11,1} \]
\[ 0 = u_{11,0} \]
\[ 0 = v_{11,1} \]

and therefore \( u_{11,0} \neq 0 \) giving rise to single projective point in the \( S_{11} \) coordinates, as desired. Note that this is consistent with \( 0 = u_{01,0} u_{11,0} = u_{10,0} u_{11,0} \) since it already has been established that \( 0 = u_{01,0} = u_{10,0} \).

**Lemma.** There is exactly one solution to the system of equations \( B_4 \) with \( z = (1, 0, 0, 0) \). This corresponds to the point \( 0 \in \text{Pic}^0(C) \).

By symmetry, the cases \( z = (0, 1, 0, 0) \), \( z = (0, 0, 1, 0) \), and \( z = (0, 0, 0, 1) \) also correspond to single points, none of which have been included in any of the other cases. These correspond to the points \( X_1, X_2, \) and \( X_1 + X_2 \), respectively.

This exhausts all the divisor classes in \( \text{Pic}^0(C) \). However what has to be checked is whether there are any “extra” points on the quadri-homogeneous variety.

**The Case** \( z = (1, 0, 1, 0) \).

This really corresponds to two different cases which can be pictorially viewed as
\[
\begin{align*}
z_{00} &= 1 \quad (\longrightarrow) \quad z_{01} = 0 \quad z_{00} = 0 \quad (\longrightarrow) \quad z_{01} = 1 \\
(\longrightarrow) \quad z_{10} &= 0 \quad (\longrightarrow) \quad z_{11} = 1 \quad (\longrightarrow) \quad z_{10} = 1 \quad (\longrightarrow) \quad z_{11} = 0
\end{align*}
\]

and now by symmetry it is sufficient to consider just the case \( z_{00} = z_{01} = 1 \) and \( z_{10} = z_{11} = 0 \), and as before each of the arrows refers to a simple bihomogeneous case.

Intuitively, if this were possible, there would be some point \( Z \in \text{Jac}(C) \) such that \( Z + X_1 \in \Theta \) and \( Z + X_2 \in \text{Jac}(C) - \Theta \) and \( Z + X_1 + X_2 \in \Theta \). This is impossible because if \( Z \in \text{Jac}(C) - \Theta \) and \( Z + X_1 \in \Theta \), then \( Z \) is represented by a divisor of the form \( Q + P^{\rho(1)} - 2P_{\infty} \) for some affine point \( Q \in C \). Similarly if \( Z + X_2 \in \text{Jac}(C) - \Theta \) and \( Z + X_1 + X_2 \in \Theta \), then \( Z + X_2 \) is represented by a divisor of the form \( Q' + P^{\rho(1)} - 2P_{\infty} \) for some affine point \( Q' \in C \). However \( Z + X_2 \) is also represented by \( Q + P^{\rho(1)} + P^{\rho(2)} - 3P_{\infty} \). Thus
\[
Q + P^{\rho(1)} + P^{\rho(2)} - 3P_{\infty} \sim Q' + P^{\rho(1)} - 2P_{\infty}
\]
so
\[
Q + P^{\rho(2)} - 2P_{\infty} \sim Q' - P_{\infty}
\]
In particular, there is the homogeneous equation

\[ x_{10} = \rho^{(1)} v_{10,1} + \rho^{(1)} v_{10,1}^2 \]

\[ x_{11} = \rho^{(1)} v_{11,1} + \rho^{(1)} v_{11,1}^2 \]

\[ x_{01} = \rho^{(1)} v_{01,1} + \rho^{(1)} v_{01,1}^2 \]

\[ x_{00} = \rho^{(1)} v_{00,1} + \rho^{(1)} v_{00,1}^2 \]

\[ x_{00} = 0 \]

\[ x_{01} = 0 \]

\[ x_{11} = 0 \]

\[ x_{10} = 0 \]

since \( z_{00} = 1 \) and \( z_{01} = 0 \), and similarly, the relations between the \( S_{10} \) and \( S_{11} \) variables give

\[ 0 = u_{10,0} + \rho^{(1)} u_{10,1} + \rho^{(1)} u_{10,1}^2 \]

\[ 0 = v_{10,0} + \rho^{(1)} v_{10,1} \]

\[ 0 = u_{11,0} \]

\[ 0 = u_{11,1} \]

\[ 0 = v_{11,0} + (u_{10,1} \rho^{(1)}) v_{11,1} \]

\[ 0 = v_{11,0} + (u_{10,1} \rho^{(1)}) v_{11,1} \]

since \( z_{10} = 1 \) and \( z_{11} = 0 \), while the relations between the \( S_{01} \) and \( S_{11} \) variables give

\[ 0 = u_{01,0} u_{11,0} \]

\[ 0 = -(u_{01,0} + \rho^{(2)} v_{01,1}) u_{11,0} + (v_{11,0} + \rho^{(2)} v_{11,1}) u_{01,0} \]

\[ 0 = (v_{11,0} + \rho^{(2)} v_{11,1} + (v_{01,0} + \rho^{(2)} v_{01,1}) \]

since \( z_{01} = 1 \) and \( z_{11} = 0 \). Either \( v_{11,0} + \rho^{(2)} v_{11,1} = 0 \) or \( v_{01,0} + \rho^{(2)} v_{01,1} = 0 \). Now if \( 0 = v_{11,0} + \rho^{(2)} v_{11,1} \), then combined with \( v_{11,0} = (u_{10,1} + \rho^{(1)}) v_{11,1} \) gives \( 0 = (u_{10,1} + \rho^{(1)} v_{11,1} \) and noting that \( v_{11,1} \) can’t be 0 (if \( v_{11,1} = 0 \), then \( v_{11,0} = 0 \), which would mean that all the \( S_{11} \) variables were 0). Therefore \( u_{10,1} = -\rho^{(1)} - \rho^{(2)} \). However, \( u_{10,0} + \rho^{(1)} u_{10,1} + \rho^{(1)} u_{10,1}^2 = 0 \) which then implies that \( u_{10,0} = \rho^{(1)} \rho^{(2)} \). Thus the \( U \)-polynomial in the \( S_{10} \) variables is

\[ U(x) = x^2 + u_{10,1} x + u_{10,0} = x^2 - (\rho^{(1)} + \rho^{(2)}) + \rho^{(1)} \rho^{(2)} = (x - \rho^{(1)})(x - \rho^{(2)}) \]

and since \( \rho^{(1)} \) and \( \rho^{(2)} \) are both roots of \( f(x) \), this means that the \( V \) polynomial in the \( S_{10} \) variables, \( V(x) = v_{10,1} x + v_{10,0} \), must be zero when evaluated at \( x = \rho^{(1)} \) and at \( x = \rho^{(2)} \). Therefore \( v_{10,1} = v_{10,0} = 0 \) since \( \rho^{(1)} \neq \rho^{(2)} \).

Now consider the glue equations between the \( s_{00} \) variables and the \( S_{10} \) variables. In particular, there is the homogeneous equation

\[ 0 = G_1(s_{00} M_{\rho^{(1)}}, S_{10} M_{\rho^{(2)}}, a_{\rho^{(2)}}) \]

\[ = (u_{00,0} + \rho^{(2)} u_{00,1} + \rho^{(2)} z_{00})(u_{10,0} + \rho^{(2)} u_{10,1} + \rho^{(2)} z_{10}) - a_1 z_{00} z_{10} \]

which for \( z_{00} = z_{10} = 1 \) gives

\[ a_1 = (u_{00,0} + \rho^{(2)} u_{00,1} + \rho^{(2)} z_{00})(u_{10,0} + \rho^{(2)} u_{10,1} + \rho^{(2)} z_{10}) \]
However, $U(x) = x^2 + u_{10,1} x + u_{10,0} = (x - \rho^{(1)})(x - \rho^{(2)})$, so $\rho^{(1)} + \rho^{(2)} u_{10,1} + u_{10,0} = 0$, and therefore $a_1 = 0$, which is impossible.

In a completely similar fashion, $v_{01,0} + \rho^{(2)} v_{01,1} = 0$ implies $a_1 = 0$, as well. Therefore the case $z = (z_{00}, z_{01}, z_{10}, z_{11}) = (1, 0, 1, 0)$ has no solutions (i.e. there are no such points with this infinity type).

**Lemma.** There are no solutions to the system of equations $B_4$ with $z = (1, 0, 1, 0)$.

By symmetry, the cases $z = (1, 1, 0, 0)$, $z = (0, 1, 0, 1)$, and $z = (0, 0, 1, 1)$ also have no solutions.

**The Case** $z = (1, 0, 0, 1)$.

This really corresponds to four different cases, which can be pictorially viewed as

```
z_{00} = 1 (+X_1) z_{01} = 1
(+X_2) z_{10} = 0 (+X_2) z_{11} = 0
```

```
z_{00} = 0 (+X_1) z_{01} = 0
(+X_2) z_{10} = 0 (+X_2) z_{11} = 1
```

and now by symmetry it is sufficient to consider just the case $z_{01} = z_{00} = 0$ and $z_{10} = z_{11} = 1$, and as before each of the arrows refers to a simple bihomogeneous case.

Intuitively, if this were possible, there would be some point $Z \in \text{Jac}(C) - \Theta$ such that $Z + X_1 \in \Theta$ and $Z + X_2 \in \Theta$ and $Z + X_1 + X_2 \in \text{Jac}(C) - \Theta$. This is impossible because if $Z \in \text{Jac}(C) - \Theta$ such that $Z + X_1 \in \Theta$ then $Z$ is represented by a reduced divisor of the form $P_1 + P_{\rho^{(1)}} - 2P_{\infty}$ for some affine point $P_1 \in C$. Similarly, if $Z \in \text{Jac}(C) - \Theta$ such that $Z + X_2 \in \Theta$ then $Z$ is represented by a reduced divisor of the form $P_2 + P_{\rho^{(2)}} - 2P_{\infty}$ for some affine point $P_2 \in C$. This would mean that $P_1 = P_{\rho^{(1)}}$ and $P_2 = P_{\rho^{(2)}}$, so that $Z$ is represented by the unique reduced divisor $P_{\rho^{(1)}} + P_{\rho^{(2)}} - 2P_{\infty}$. But then $Z = X_1 + X_2$ (a 2-division point on $\text{Jac}(C)$), and $Z + X_1 + X_2 = 0$ so $Z + X_1 + X_2 \in \Theta$.

The conditions $z_{00} = 1$ and $z_{01} = 0$ imply that adding $X_1$ gives

$$
0 = u_{00,0} + \rho^{(1)} u_{00,1} + \rho^{(1)}
0 = -v_{01,0} + (u_{00,1} + \rho^{(1)}) v_{01,1}
0 = u_{01,0}
0 = u_{01,1}
$$

while the conditions $z_{00} = 1$ and $z_{10} = 0$ imply that adding $X_2$ gives

$$
0 = u_{00,0} + \rho^{(2)} u_{00,1} + \rho^{(2)}
0 = -v_{10,0} + (u_{00,1} + \rho^{(2)}) v_{10,1}
0 = u_{10,0}
0 = u_{10,1}
$$

...
and since $\rho^{(1)} \neq \rho^{(2)}$, this allows the values of $u_{00,0}$ and $u_{00,1}$ to be determined as
\[
\begin{align*}
\rho^{(1)} \rho^{(2)} &= u_{00,0} \\
-\rho^{(1)} - \rho^{(2)} &= u_{00,1}
\end{align*}
\]
and therefore
\[
\begin{align*}
0 = -v_{01,0} - \rho^{(2)} v_{01,1} \\
0 = -v_{10,0} - \rho^{(1)} v_{10,1}.
\end{align*}
\]

In a completely similar fashion, the conditions $z_{11} = 1$ and $z_{10} = 0$ imply that adding $X_1$ gives
\[
\begin{align*}
0 &= u_{11,0} + \rho^{(1)} u_{11,1} + \rho^{(1)^2} \\
0 &= -v_{10,0} + (u_{11,1} + \rho^{(1)}) v_{10,1} \\
0 &= u_{10,0} \\
0 &= u_{10,1}
\end{align*}
\]
while the conditions $z_{11} = 1$ and $z_{01} = 0$ imply that adding $X_2$ gives
\[
\begin{align*}
0 &= u_{11,0} + \rho^{(2)} u_{11,1} + \rho^{(2)^2} \\
0 &= -v_{01,0} + (u_{11,1} + \rho^{(2)}) v_{01,1} \\
0 &= u_{01,0} \\
0 &= u_{01,1}
\end{align*}
\]
and since $\rho^{(1)} \neq \rho^{(2)}$, this allows the values of $u_{11,0}$ and $u_{11,1}$ to be determined as
\[
\begin{align*}
\rho^{(1)} \rho^{(2)} &= u_{11,0} \\
-\rho^{(1)} - \rho^{(2)} &= u_{11,1}
\end{align*}
\]
and therefore
\[
\begin{align*}
0 = -v_{10,0} - \rho^{(2)} v_{10,1} \\
0 = -v_{01,0} - \rho^{(1)} v_{01,1}.
\end{align*}
\]

However, the conditions
\[
\begin{align*}
0 &= -v_{01,0} - \rho^{(2)} v_{01,1} \\
0 &= -v_{01,0} - \rho^{(1)} v_{01,1}
\end{align*}
\]
now have the unique solution
\[
\begin{align*}
0 &= v_{01,0} \\
0 &= v_{01,1}
\end{align*}
\]
since $\rho^{(1)} \neq \rho^{(2)}$. This now forces all the $s_{01}$ variables to be 0, which is impossible. A similar argument would force all the $s_{01}$ variables to be 0, as well (though this is not needed). Therefore there are no solutions of $B_4$ with $z = (z_{00}, z_{01}, z_{10}, z_{11}) = (1, 0, 0, 1)$, i.e. there are no such points with this infinity type.
Lemma. There are no solutions to the system of equations $B_4$ with $z = (1, 0, 0, 1)$.

By symmetry, the case $z = (0, 1, 1, 0)$ also has no solutions.

The Case $z = (0, 0, 0, 0)$.

This final case refers to the picture

\[
\begin{align*}
z_{00} &= 0 \quad (+X_1) \quad z_{01} = 0 \\
(+X_2) \quad (z_{10} = 0 \quad (+X_2) \quad z_{11} = 0
\end{align*}
\]

where all the points at all the corners refer to points at infinity in the projective completions of the affine components.

First note that $z_{00} = z_{01} = z_{10} = z_{11} = 0$ also imply $u_{00,1} = u_{01,1} = u_{10,1} = u_{11,1} = 0$ and the four sets of glue equations imply only the following twelve equations:

\[
\begin{align*}
0 &= u_{00,0} u_{01,0} \\
0 &= (v_{01,0} + \rho^{(1)} v_{01,1}) u_{00,0} + (v_{00,0} + \rho^{(1)} v_{00,1}) u_{01,0} \\
0 &= (v_{00,0} + \rho^{(1)} v_{00,1})(v_{01,0} + \rho^{(1)} v_{01,1}) \\
0 &= u_{00,0} u_{10,0} \\
0 &= (v_{10,0} + \rho^{(2)} v_{10,1}) u_{00,0} + (v_{00,0} + \rho^{(2)} v_{00,1}) u_{10,0} \\
0 &= (v_{00,0} + \rho^{(2)} v_{00,1})(v_{10,0} + \rho^{(2)} v_{10,1}) \\
0 &= u_{10,0} u_{11,0} \\
0 &= (v_{11,0} + \rho^{(1)} v_{11,1}) u_{10,0} + (v_{10,0} + \rho^{(1)} v_{10,1}) u_{11,0} \\
0 &= (v_{10,0} + \rho^{(1)} v_{10,1})(v_{11,0} + \rho^{(1)} v_{11,1}) \\
0 &= u_{01,0} u_{11,0} \\
0 &= (v_{11,0} + \rho^{(2)} v_{11,1}) u_{01,0} + (v_{01,0} + \rho^{(2)} v_{01,1}) u_{11,0} \\
0 &= (v_{01,0} + \rho^{(2)} v_{01,1})(v_{11,0} + \rho^{(2)} v_{11,1})
\end{align*}
\]

which are all that follow directly from the above set of glue equations. In fact all these equations can be derived from just the bilinear part of the glue.

Now suppose that one of the $u_{ij,0} \neq 0$, say $u_{00,0} \neq 0$ (with the other cases all being essentially the same). Then it follows that $u_{01,0} = u_{10,0} = 0$, and from this it follows that $(v_{01,0} + \rho^{(1)} v_{01,1}) u_{00,0} = (v_{10,0} + \rho^{(2)} v_{10,1}) u_{00,0} = 0$ and therefore $v_{01,0} + \rho^{(1)} v_{01,1} = v_{10,0} + \rho^{(2)} v_{10,1} = 0$. In order that all the $s_{01}$ variables not all be zero, it is necessary that $v_{01,0} + \rho^{(2)} v_{01,1} \neq 0$ since $\rho^{(1)} \neq \rho^{(2)}$. Similarly in order that all the $s_{10}$ variables not all be zero, it is necessary that $v_{10,0} + \rho^{(1)} v_{10,1} \neq 0$. This implies (in two different ways) that $u_{11,0} = 0$, and also that $v_{11,0} + \rho^{(1)} v_{11,1} = 0$ and $v_{11,0} + \rho^{(2)} v_{11,1} = 0$, which then imply that $v_{11,0} = v_{11,1} = 0$ since $\rho^{(1)} \neq \rho^{(2)}$. This gives that all the $s_{11}$ variables are zero, which is impossible. Therefore $u_{00,0} = 0$. Similarly $u_{01,0} = u_{10,0} = u_{11,0} = 0$, as well.
With all the $u_{ij,0} = 0$, the twelve equations reduce to the following four:

\[
\begin{align*}
0 &= (v_{00,0} + \rho^{(1)} v_{00,1})(v_{01,0} + \rho^{(1)} v_{01,1}) \\
0 &= (v_{00,0} + \rho^{(2)} v_{00,1})(v_{10,0} + \rho^{(2)} v_{10,1}) \\
0 &= (v_{10,0} + \rho^{(1)} v_{10,1})(v_{11,0} + \rho^{(1)} v_{11,1}) \\
0 &= (v_{01,0} + \rho^{(2)} v_{01,1})(v_{11,0} + \rho^{(2)} v_{11,1}).
\end{align*}
\]

Now suppose that $v_{00,0} + \rho^{(1)} v_{00,1} = 0$. In order that the $s_{00}$ variables not all be zero it follows that $v_{00,0} + \rho^{(2)} v_{00,1} \neq 0$ and therefore $v_{10,0} + \rho^{(2)} v_{10,1} = 0$. In order that the $S_{10}$ variables not all be zero it follows that $v_{10,0} + \rho^{(1)} v_{10,1} \neq 0$ and therefore $v_{01,0} + \rho^{(2)} v_{01,1} = 0$. In order that the $S_{01}$ variables not all be zero it follows that $v_{01,0} + \rho^{(1)} v_{01,1} \neq 0$. This then gives rise to a point on the multi-projective variety.

Alternatively, suppose that $v_{00,0} + \rho^{(1)} v_{00,1} \neq 0$. Then $v_{01,0} + \rho^{(1)} v_{01,1} = 0$ and in order that $s_{01}$ variables not all be zero it follows that $v_{01,0} + \rho^{(2)} v_{01,1} \neq 0$. Then $v_{11,0} + \rho^{(2)} v_{11,1} = 0$ and in order that $S_{11}$ variables not all be zero it follows that $v_{11,0} + \rho^{(1)} v_{11,1} \neq 0$. Then $v_{10,0} + \rho^{(1)} v_{10,1} \neq 0$. Then $v_{00,0} + \rho^{(2)} v_{00,1} = 0$. This then gives rise to a second point on the multi-projective variety.

To summarize, if $z_{00} = z_{01} = z_{10} = z_{11} = 0$, then the glue equations still allow the existence of two points. This is a big problem, since these two points should not exist.

**Proposition.** There are exactly two solutions to the system of equations $B_4$ with $\mathbf{z} = (0, 0, 0, 0)$.

Thus working only with two affine Weierstrass points does not lead to a set of equations for all of $\text{Jac}(C)$. To proceed further additional equations are needed to eliminate these two points.

**INTRODUCING ANOTHER WEIERSTRASS POINT**

One approach to resolving this whole problem is to introduce a third rational Weierstrass point into the analysis. Each rational Weierstrass point $P_{\rho^{(i)}} = (\rho^{(i)}, 0) \in C_{\text{aff}}$ is associated with a point $X_{\rho^{(i)}} = \text{Cl}(P_{\rho^{(i)}} - P_{\infty}) \in \text{Jac}(C)$ of order 2, which is also on $\Theta$. Now instead of four copies of the affine closure of $\text{Jac}(C) - \Theta$, there are eight such copies, and instead of four sets of inhomogeneous and homogeneous variables, there are now eight such sets. It is convenient to index them as $s_{ijk} = (u_{ijk,0}, u_{ijk,1}, v_{ijk,0}, v_{ijk,1})$ and $S_{ijk} = (u_{ijk,0}, u_{ijk,1}, v_{ijk,0}, v_{ijk,1}, z_{ijk})$, respectively for $ijk \in \mathcal{I}$ with index set $\mathcal{I} = \{000, 001, 010, 011, 100, 101, 110, 111\}$. There are also eight copies of the affine variety $A(s)$ that represents $\text{Jac}(C) - \Theta$ denoted $A(s_{ijk})$ for $ijk \in \mathcal{I}$. The product of the eight projective closures $\prod_{ijk \in \mathcal{I}} A(s_{ijk})$ is a multi-projective variety in $(\mathbb{P}^1)^8$ and under the Segre imbedding is a projective variety in $\mathbb{P}^{5^8-1} = \mathbb{P}^{390624}$, a very large dimensional ambient space.

These projective varieties $A(S_{ijk})$ may be thought of as corners of a cube, with adjacent corners being pairs of indices of Hamming distance 1. The edges of the cube are determined by glue relations and represent the addition of a point of order
2 in Jac(C) determined by \(X_{\rho(1)}, X_{\rho(2)},\) or \(X_{\rho(3)}\). These three points determine a subgroup of Jac(C) of order eight. Elements of this group will be written as \(X_{ijk}\) for \(ijk \in I\). Here \(i, j,\) and \(k\) can take the values 0 or 1

\[X_{ijk} = k \cdot X_{\rho(1)} + j \cdot X_{\rho(2)} + i \cdot X_{\rho(3)}\]

with a slight abuse of notation. Thus \(X_{ijk} \in \Theta\) if and only if \(ijk\) has Hamming weight 0 or 1. Note that the divisor of the function \(y\) on \(C\) is given by

\[\text{div}(y) = P_{\rho(1)} + P_{\rho(2)} + P_{\rho(3)} + P_{\rho(4)} + P_{\rho(5)} - 5 \cdot P_{\infty}\]

so \(X_{\rho(1)} + X_{\rho(2)} + X_{\rho(3)} = X_{\rho(4)} + X_{\rho(5)}\). Each corner of the cube also has the associated affine equations \(A(s_{ijk})\) which should be thought of as representing \(\text{Jac}(C) - \Theta_{X_{ijk}}\) where \(\Theta_{X_{ijk}}\) is the translate of the \(\Theta\)-divisor by the point \(X_{ijk}\) of order 2. Each edge of the cube is associated with the set of bihomogeneous equations between the variable sets on the two corners at the ends of the edge. These twelve sets of bihomogeneous equations corresponding to the twelve edges of the cube define a subvariety of the multi-projective variety \(\prod_{ijk \in I} \tilde{A}(s_{ijk})\). It is this subvariety, which is itself a multi-projective variety, that will be identified as \(\text{Jac}(C)(C)\).

An infinity type is an association of a value of 1 or 0 to each \(z_{ijk}\) on the eight corners of the cube, which determines whether or not a point in \(\text{Jac}(C)\) is an affine point on \(A(s_{ijk}) = \text{Jac}(C) - \Theta_{X_{ijk}}\). Every point in \(\text{Jac}(C)\) has some one or more indices where \(z_{ijk} = 1\). The question is whether every point in multi-projective space defined by the product of the closures \(\tilde{A}(s_{ijk})\) along with the glue equations corresponds to a point in \(\text{Jac}(C)\). In principle, there are \(2^8 = 256\) such infinity types, however, there are a lot of symmetries that reduce the number of cases that need to be considered. Pictorially, an infinity type is an assignment of 0 or 1 to each corner of the cube

![Diagram of a cube with infinity types]

and it will be important to determine exactly which infinity types are possible.

It is important to realize that if \(ijk\) and \(i'j'k'\) are two indices that are Hamming distance 1 apart from each other (i.e. they are connected by an edge of glue relations) and if \(z_{ijk} = 0\) while \(z_{i'j'k'} = 1\), then the previous analysis (between the hatted and the unhatted variables with \(z = 1\) and \(\hat{z} = 0\)) shows that the \(u_0, u_1, v_0,\) and \(v_1\) values for \(i'j'k'\) completely determine the \(u_0, u_1, v_0,\) and \(v_1\) values for \(ijk\).

It is useful to look at parts of the cube, where some of the vertices and their corresponding edges are considered. All the discussions that can be restricted to faces of the cube are derivable from the quadri-projective analysis. There are three cases of high interest that are parts of the cube that live in three dimensions.
The first case of interest is an analysis of a corner and its three edges where the corner will have $z = 1$ but the vertices at the opposite ends of each of the three edges will have $z = 0$. Thus the case is $z_{000} = 1$ and $z_{001} = z_{010} = z_{100} = 0$, i.e.

$$
\begin{array}{ccc}
  z_{100} & \\
  \downarrow & \downarrow & \\
  z_{000} & z_{010} & 0 \\
  \downarrow & \downarrow & \\
  z_{001} & 
\end{array}
$$

with the goal of showing that there are no points with this partial infinity type. Consider the three faces that intersect at this corner. Each of these faces has at least two 0’s and at least one 1. However, the square analysis shows that it is impossible to have two 1’s and two 0’s, therefore the corners need to be 1 and the infinity type of the cube then must look like

$$
\begin{array}{ccc}
  0 & \\
  \downarrow & \downarrow & \\
  1 & 0 \\
  \downarrow & \\
  0 & z_{111} & 0 \\
  \downarrow & \\
  0 & 
\end{array}
$$

with $z_{111}$ undetermined. Now each face is a square of type $(z_{00}, z_{10}, z_{01}, z_{11}) = (1, 0, 0, 0)$, which was analyzed above. Having three 0’s in a square turns out to be very restricting. If the order 2 points being added in the glue along the edges of the square are $X_{\rho(1)}$ and $X_{\rho(2)}$, then $U_{00}(x) = (x - \rho(1))(x - \rho(2))$ and $V_{00}(x) = 0$. Similarly, if the two points are $X_{\rho(1)}$ and $X_{\rho(3)}$ then $U_{00}(x) = (x - \rho(1))(x - \rho(3))$ and if the two points are $X_{\rho(2)}$ and $X_{\rho(3)}$ then $U_{00}(x) = (x - \rho(2))(x - \rho(3))$. However, $\rho(1)$, $\rho(2)$, and $\rho(3)$ are all distinct values. This shows that this infinity type cannot exist. This same analysis will apply to any corner of the cube to show that the analogous partial infinity type cannot exist.

The second case of interest is an analysis of a corner and its three edges where the corner will have $z = 0$ and the vertices at the opposite ends of each of the three edges will also have $z = 0$. Thus the case is $z_{000} = z_{001} = z_{010} = z_{100} = 0$, i.e.

$$
\begin{array}{ccc}
  z_{100} & \\
  \downarrow & \downarrow & \\
  z_{000} & z_{010} & 0 \\
  \downarrow & \downarrow & \\
  z_{001} & 
\end{array}
$$

and the goal now is to show that for the three faces that come together at this corner, the opposite vertices must all have $z = 1$, i.e. that $z_{110} \neq 0$, $z_{101} \neq 0$, and $z_{011} \neq 0$. Consider first what happens if one of the opposite vertices has a 1
and one of the other opposite vertices has a 0. Thus the picture (omitting the $z_{111}$ corner) is

$$
\begin{array}{c}
\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
\end{array}
\begin{array}{c}
0 \\
z_{110} \\
z_{010} \\
z_{000} \\
z_{101} \\
z_{100} \\
0 \\
0 \\
0 \\
0 \\
\end{array}
= 
\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
\end{array}
\end{array}
$$

where the assignments $z_{011} = 0$ and $z_{110} = 1$ have been made (and the other possibilities are equivalent by symmetry, so the same analysis will apply). Now consider the all zero face \{000, 001, 010, 011\}. The quadri-projective (two-dimensional) analysis showed that if this were to occur then the $u_0$ variable at the 000 vertex would have to be $u_{000,0} = 0$. Now the face with a single 1 (the \{000, 100, 011\} face) was analyzed in the quadri-projective case, and there it was shown that the $u_0$ variable at the 000 vertex would have to be $u_{000,0} \neq 0$. This is obviously a contradiction, so the assignments $z_{011} = 0$ and $z_{110} = 1$ are invalid, and this analysis applies regardless of whether, $z_{101} = 0$ or $z_{101} = 1$.

Next consider what happens if all three of the opposite vertices have $z = 0$, i.e.

$$
\begin{array}{c}
\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
\end{array}
\begin{array}{c}
0 \\
z_{110} \\
z_{010} \\
z_{000} \\
z_{101} \\
z_{100} \\
0 \\
0 \\
0 \\
0 \\
\end{array}
= 
\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
\end{array}
\end{array}
$$

again omitting the $z_{111}$ corner from the picture. Now the prior quadri-projective (two-dimensional) analysis of ($z_{001}, z_{101}, z_{110}$) = (0, 0, 0) applies to each of the three faces. In particular, it must be the case that at the 000 vertex, $u_{000,0} = 0$. Furthermore, the top face \{000, 001, 010, 011\}, which involves adding $X_{\rho(1)}$ and $X_{\rho(2)}$, requires that either $v_{000,0} + \rho(1) v_{000,1} = 0$ or $v_{000,0} + \rho(2) v_{000,1} = 0$, but not both (or else all the $S_{000}$ will be 0). Similarly, the \{000, 001, 100, 101\} face, which involves adding $X_{\rho(1)}$ and $X_{\rho(3)}$, requires that either $v_{000,0} + \rho(1) v_{000,1} = 0$ or $v_{000,0} + \rho(3) v_{000,1} = 0$, but not both, and the \{000, 010, 100, 110\} face, which involves adding $X_{\rho(2)}$ and $X_{\rho(3)}$, requires that either $v_{000,0} + \rho(2) v_{000,1} = 0$ or $v_{000,0} + \rho(3) v_{000,1} = 0$, but not both. These conditions cannot all be simultaneously satisfied. If $A$, $B$, and $C$ are logical propositions, and if ((A OR B) AND (A OR C) AND (B OR C)) is to be true, then at least two out the three logical propositions \{A, B, C\} must hold. This then shows that $z_{011} = z_{101} = z_{110} = 0$ cannot hold. Therefore it must be the case that

$$
\begin{array}{c}
\begin{array}{c}
0 \\
z_{101} \\
z_{110} \\
z_{010} \\
z_{000} \\
z_{100} \\
0 \\
0 \\
0 \\
0 \\
\end{array}
\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
\end{array}
\end{array}
$$

implies

$$
\begin{array}{c}
\begin{array}{c}
0 \\
1 \\
1 \\
1 \\
\end{array}
\begin{array}{c}
0 \\
0 \\
0 \\
0 \\
\end{array}
\end{array}
$$
and to finish off the \( z_{111} \) corner, note that there cannot be two 0’s and two 1’s on any face, and therefore \( z_{111} = 1 \), i.e. the full infinity type is

\[
\begin{align*}
\text{In this case it is possible to be more specific and state precisely what is at each corner.}
\end{align*}
\]

Note that another way of saying this is that any ball of Hamming radius 2 must contain at least one \( z = 1 \) value. In particular, this approach shows that in three dimensions (i.e. by using three affine Weierstrass points), no face can have \( z = 0 \) at all its corners. This result could not be shown by staying only in two dimensions by using only two affine Weierstrass points.

It is interesting that this infinity type corresponds to a unique point on \( \text{Jac}(C) \). This follows from the two-dimensional case for infinity type \( \{ z_{00}, z_{01}, z_{10}, z_{01} \} = \{ 0, 0, 0, 1 \} \) where now three of the four 1’s on the cube are determined to correspond to the points \( X_{\rho(1)} + X_{\rho(2)}, X_{\rho(1)} + X_{\rho(3)}, X_{\rho(2)} + X_{\rho(3)}, \) all of which are in \( \text{Jac}(C) - \Theta \). The fourth point at the 111 vertex corresponds to \( X_{\rho(1)} + X_{\rho(2)} + X_{\rho(3)} = X_{\rho(4)} + X_{\rho(5)} \) which is also in \( \text{Jac}(C) - \Theta \).

The third case of interest is an analysis of a corner and its three edges where the corner will have \( z = 0 \) but the vertices at the opposite ends of each of the three edges will have \( z = 1 \). Thus the case is \( z_{000} = 0 \) and \( z_{001} = z_{010} = z_{100} = 1 \), i.e.

\[
\begin{align*}
\text{and now note that each of the three faces that contain this corner has two 1’s and one 0. This means that the far corners on these three faces must all be 1’s with the final 111 vertex to be determined, i.e.}
\end{align*}
\]
with $z_{111}$ to be determined. There are thus two possible infinity types

![Diagram](attachment:image.png)

which can be analyzed more completely.

In the first case, where there is only one $z = 0$ component, these are points on $\text{Pic}^0(C)$ of the form $P - P_\infty$, where $P$ is an affine point that is not a Weierstrass point on the curve $C$. In the second case, the situation corresponds to $P$ being either $P_4$ or $P_5$, i.e. one of the two Weierstrass points on $C$ that have not been part of the analysis up to now. Since $P_1 + P_2 + P_3 + P_4 + P_5 - 5P_\infty$ is the divisors of the function $y$ on $C$, if $P = P_1$ or $P = P_5$, the points at the $z_{000}$ corner will be on $\Theta$ and the points at the $z_{111}$ corner will also be on $\Theta$ (actually on $\Theta_{X_1+X_2+X_3}$). The way to see this is as follows. Suppose the reduced divisor at $z_{000} = 0$ is $P - P_\infty$ and the reduced divisor at $z_{111} = 0$ is $P' - P_\infty$, adding $X_1 + X_2 + X_3$ moves from one to the other, i.e. the divisor class represented by the non-reduced divisor $P' + P_1 + P_2 + P_3 - 4P_\infty$ is the same as the divisor class represented by $P - P_\infty$, so these divisors are equivalent. This means that the divisors $P' + \iota(P) - 2P_\infty$ and $P_1 + P_2 + P_3 - 3P_\infty$ are equivalent. However $P_1 + P_2 + P_3 - 3P_\infty$ is equivalent to $P_4 + P_5 - 2P_\infty$, and therefore the divisors $P' + \iota(P) - 2P_\infty$ and $P_4 + P_5 - 2P_\infty$ are equivalent, but these are both reduced, unless it happens that $P' = P$. However if $P' = P$ then the reduction of the divisor $P' + \iota(P) - 2P_\infty$ is 0, which certainly not the same as $P_4 + P_5 - 2P_\infty$. Therefore there is actual equality as divisors, i.e. $P' + \iota(P) - 2P_\infty = P_4 + P_5 - 2P_\infty$, which means that either $P = P_4$ and $P' = P_5$ or else $P = P_5$ and $P' = P_4$.

**Proposition.** For the infinity type where two antipodal corners (say (000) and (111)) of the cube have $z_{000} = z_{111} = 0$ and all other cube corners have $z_{ijk} = 1$, there are exactly two multi-projective points. These correspond to the points $X_4$ and $X_5$ of order 2 in $\text{Pic}^0(C)$.

If this analysis had included all the Weierstrass points, these two points would not look special. However, that would have corresponded to a 4-dimensional cube (a tesseralact), rather than a 3-dimensional cube, in which the 16 corners would be in one-to-one correspondence with the 16 points of order 2 on $\text{Jac}(C)$. Fortunately, there is no need to go to these lengths to construct the Jacobian.

It is interesting to consider in three dimensions what happens if $ijk$ and $i'j'k'$ are two indices that are Hamming distance 1 apart with both $z_{ijk} = 0$ and $z_{i'j'k'} = 0$. Now such an edge is common to two different faces of the cube. Each face of the cube is then subject to the analysis for the quadri-projective case. It was shown there that on any square face, it is impossible to have an infinity type consisting of two 1’s and two 0’s. Now suppose that $z_{000} = z_{100} = 0$. The two faces to consider are $\{000, 100, 101, 001\}$ and $\{000, 100, 110, 010\}$, and there has to be at least one
more zero on each of these two faces. Up to symmetry, there are two possibilities:

\[ \begin{array}{c}
0 & \rightarrow & z_{101} \\
\downarrow & & \downarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
\downarrow & & \downarrow \\
0 & \rightarrow & z_{111} \\
\end{array} \quad \text{or} \quad \begin{array}{c}
0 & \rightarrow & z_{001} \\
\downarrow & & \downarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
\downarrow & & \downarrow \\
0 & \rightarrow & z_{011} \\
\end{array} \]

i.e. either \( z_{010} = z_{101} = 0 \) or \( z_{001} = z_{010} = 0 \). The first possibility has already been studied above and leads to a unique completion and a unique point on \( \Jac(C) \).

Now consider the second possibility. There are now two faces \{000, 001, 011, 010\} and \{100, 101, 110, 111\} that have two zeros, so now there has to be another zero on each of these two faces. The analysis above shows that it is necessary that \( z_{001} = z_{110} = 1 \) or else the 000 corner would have it and all its adjacent vertices and at least one vertex of Hamming distance 2 from 000 with \( z = 1 \). Thus there is the partial infinity type

\[ \begin{array}{c}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
\downarrow & & \downarrow \\
0 & \rightarrow & 1 \\
\end{array} \]

and now note that the infinity type completes to

\[ \begin{array}{c}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
\downarrow & & \downarrow \\
0 & \rightarrow & 1 \\
\end{array} \]

since no face can have two 0’s and two 1’s. However, this partial infinity type is also impossible. To see this, Note that the \{000, 001, 011, 010\} face requires the \( u_0 \) variable at the 010 vertex to be \( u_{010} \neq 0 \), while the \{000, 010, 110, 100\} face requires the \( u_0 \) variable at the 010 vertex to be \( u_{010,1} = 0 \).

Therefore, if some edge of the cube, say the \{000, 100\} edge, has partial infinity type \((0,0)\), i.e. \( z_{000} = z_{100} = 0 \), there are only two possible valid ways to complete the infinity type, either

\[ \begin{array}{c}
0 & \rightarrow & 1 \\
\downarrow & & \downarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
\downarrow & & \downarrow \\
1 & \rightarrow & 0 \\
\end{array} \quad \text{or} \quad \begin{array}{c}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
\downarrow & & \downarrow \\
1 & \rightarrow & 1 \\
\end{array} \]

which are equivalent (up to symmetry), each of which determines a single point in \( \Jac(C) \).
Proposition. If some edge of the cube, say the \(\{000,100\}\) edge, has partial infinity type \((0,0)\), i.e. \(z_{000} = z_{100} = 0\) then one of the two following two cases holds, each of which corresponds to a single point in \(\text{Jac}(C)\).

(i) \((000, 001, 010, 011, 100, 101, 110, 111) = (0, 0, 0, 1, 0, 1, 1, 1)\) corresponding to the divisor class 0, represented by the reduced divisor 0;

(ii) \((000, 001, 010, 011, 100, 101, 110, 111) = (0, 1, 1, 0, 0, 1)\) corresponding to the divisor class \(X_1\), represented by the reduced divisor \(P_1 - P_\infty\).

Basically, if there are two corners of the cube with \(z = 0\), the situation is very constrained, and there are only finitely many possibilities for which points on \(\text{Jac}(C)\) they correspond to. More importantly, there are no anomalous points. All the divisor classes in \(\text{Pic}^0(C)\) correspond to unique points on the multi-projective variety and every multi-projective point on the variety corresponds to a unique reduced divisor. The \(\Theta\)-divisor and all its translates by \([2]\)-division points that are some of the corners of the cube in the projective closures that are at each of the affine varieties on the corners.

Summary of the Octo-Homogeneous Case.

This completes the construction of \(\text{Jac}(C)\). It requires that only three of the affine Weierstrass points be rational over the base field. The Jacobian is constructed as a multi-projective variety sitting inside of the multi-projective space \((\mathbb{P}^4)^6\), and becomes a projective variety under the Segre map. The points on \(\text{Jac}(C)\) are in one-to-one correspondence with divisor classes in \(\text{Pic}^0(C)\) that are rational over the base field. It is useful to summarize this as follows.

Let \(K\) be a field of characteristic other than 2. Let \(f(x)\) be a monic quintic polynomial over \(K\) having no repeated roots and having at least three roots in \(K\), i.e. write

\[
f(x) = x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = (x - \rho^{(1)})(x - \rho^{(2)})(x - \rho^{(3)})(x - \rho^{(4)})(x - \rho^{(5)})
\]

with \(a_0, a_1, a_2, a_3, a_4, \rho^{(1)}, \rho^{(2)}, \rho^{(3)} \in K\) and write

\[
a = (a_0, a_1, a_2, a_3, a_4).
\]

For any \(\rho \in K\) define the matrices

\[
M(\rho) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \rho & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \rho & 1 & 0 \\ \rho^2 & 2 \rho & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A(\rho) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \rho^2 & 2 \rho & 1 & 0 & 0 \\ \rho^3 & 3 \rho^2 & 3 \rho & 1 & 0 \\ \rho^4 & 4 \rho^3 & 6 \rho^2 & 4 \rho & 1 \end{pmatrix}
\]

and the vector

\[
b(\rho) = (\rho^5, 5 \rho^4, 10 \rho^3, 10 \rho^2, 5 \rho).
\]

For a vector of variables \(S = (u_0, u_1, v_0, v_1, z)\) let

\[
E_0(S, a) = a_0 z^4 - a_2 u_0 z^3 + a_4 u_0^2 z^2 + a_3 u_0 u_1 z^2 - 2 u_0^2 u_1 z - a_4 u_0 u_1^2 z + u_0 u_1^3 - u_0^3 z^2 + u_0 v_1^2 z
\]

\[
E_1(S, a) = a_1 z^4 - a_3 u_0 z^3 + a_2 u_1 z^2 - a_2 u_1 z^3 + 2 a_4 u_0 u_1 z^3 + a_3 u_1^2 z^2 - 3 u_0 u_1^2 z - a_4 u_1^3 z + u_1^2 z^2 - 2 v_0 v_1 z^2 + u_1 v_1^2 z^2
\]
and for convenience write \( E(S, a) = (E_0(S, a), E_1(S, a)) \). For a second vector of variables \( \hat{S} = (\hat{u}_0, \hat{u}_1, \hat{v}_0, \hat{v}_1, \hat{z}) \) let

\[
G_1(S, \hat{S}, a) = u_0 \hat{u}_0 - a_1 z \hat{z} \\
G_2(S, \hat{S}, a) = \hat{v}_0 u_0 + v_0 \hat{u}_0 \\
G_3(S, \hat{S}, a) = v_0 \hat{v}_0 + \{a_4 z \hat{z} - u_1 \hat{z} - \hat{u}_1 z\} \\
G_4(S, \hat{S}, a) = \hat{u}_0 (v_1 \hat{z} + \hat{v}_1 z) + \hat{v}_0 (u_1 \hat{z} - \hat{u}_1 z) \\
G_5(S, \hat{S}, a) = u_0 (v_1 \hat{z} + \hat{v}_1 z) + v_0 (\hat{u}_1 z - u_1 \hat{z})
\]

\[
G_6(S, \hat{S}, a) = a_1 \hat{v}_0 z^2 - a_1 a_3 z^2 \hat{z} + a_1 u_0 \hat{z} z + a_1 \hat{u}_1 u_1 z - 2 u_1 \hat{v}_0 v_0 - 2 \hat{u}_0 v_0 v_1 \\
G_7(S, \hat{S}, a) = a_1 \hat{u}_0 z \hat{z} - a_1 a_3 z^2 \hat{z} + a_1 u_0 \hat{z} z + a_1 \hat{u}_1 u_1 z - 2 u_1 \hat{v}_0 v_0 - 2 u_0 \hat{v}_0 \hat{v}_1 \\
G_8(S, \hat{S}, a) = -a_1 \hat{v}_0 z^2 + a_3 u_0 \hat{v}_0 z^2 - u_0 \hat{v}_0 \hat{z} z - 2 a_4 u_0 u_1 \hat{v}_0 \hat{z} + 2 u_0 \hat{v}_0 ^2 \hat{v}_0
\]

\[
+ a_1 \hat{u}_1 v_1 z + v_0 \hat{v}_0 \hat{v}_1 v_1 z
\]

\[
G_9(S, \hat{S}, a) = -a_1 v_0 z^3 + a_3 \hat{u}_0 v_0 \hat{z} z^2 - \hat{u}_0 \hat{v}_0 \hat{z} z + 2 a_4 \hat{u}_0 \hat{u}_1 v_0 \hat{z} + 3 \hat{u}_0 \hat{u}_1 \hat{v}_0
\]

\[
+ a_1 \hat{u}_1 v_1 z + a_1 \hat{u}_1 \hat{v}_1 z + 2 \hat{v}_0 \hat{v}_0 \hat{v}_1 z
\]

and for convenience write \( G(S, \hat{S}, a) = \{G_1(S, \hat{S}, a), \ldots, G_9(S, \hat{S}, a)\} \).

Let \( \mathcal{I} = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \) be an index set, and let

\[
d_1 = (1, 0, 0) \\
d_2 = (0, 1, 0) \\
d_3 = (0, 0, 1)
\]

be a basis (mod 2) of \( \mathcal{I} \). For each \( c = (i, j, k) \in \mathcal{I} \) let

\[
S_c = (u_{c,0}, u_{c,1}, v_{c,0}, v_{c,1}, z_c)
\]

be a vector of projective variables. The elements of \( \mathcal{I} \) correspond to the corners of a cube, and each edge of the cube is \( (c, c + d_l) \) for some \( c \in \mathcal{I} \) and \( l \in \{1, 2, 3\} \) (but note that in this notation the edges \( (c, c + d_l) \) and \( (c + d_l, c) \) are the same).

Then for each \( c \in \mathcal{I} \) (i.e., at each edge of the cube) there are the defining homogeneous equations

\[
E(c, a) = \{E_0(S_c, a), E_1(S_c, a)\}
\]

and for each \( c \in \mathcal{I} \) and each \( l \in \{1, 2, 3\} \) (i.e., at each edge of the cube) there are the defining bihomogeneous equations

\[
G(c, l; a) = \left\{G_i(M(\rho^{(l)})S_c, M(\rho^{(l)})S_{c+d_l}, A(\rho^{(l)})a + b(\rho^{(l)}))\right\}_{i=1}^{1=9}
\]

and note that each edge has been counted twice in this notation, so each glue equation appears twice. There are a total 16 homogeneous equations for the corners and 108 bihomogeneous equations for the edges.
Theorem. Let $C$ be the genus 2 hyperelliptic curve

$$y^2 = f(x)$$

with

$$f(x) = x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

with coefficients in a field $K$ not of characteristic 2, and such that the $f(x)$ has no multiple roots and at least three roots in $K$. Let $J$ be the multi-homogeneous variety defined by the polynomials $E(c; a)$ as $c$ ranges over $I$ and also by the polynomials $G(c, l; a)$ as $c$ ranges over $I$ and $l$ ranges over $\{1, 2, 3\}$. Then $J$ is a

multi-homogeneous model of $\text{Jac}(C)$.

REFERENCES

[An] Anderson, Greg, Abeliants and their Applications to an Elementary Construction of Jacobians, Advances in Mathematics Volume 172 Issue 2 (December 2002), 169-205, Available at arxiv.org/pdf/math/0112321.pdf.

[Can] Cantor, David G., Computing in the Jacobian Of a Hyperelliptic Curve, Mathematics of Computation 48, Number 177 (1987), 95-101.

[Cas] Cassels, J.W.S., Jacobian in genus 2, Math, Proc. Cambridge Philos. Soc. Volume 114 Issue 1 (July 1993), 1-8.

[CF] Cassels, J. W. S. and Flynn, E. V., Prolegomena to a middlebrow arithmetic of curves of genus 2, London Mathematical Society Lecture Note Series 230 (1996), Cambridge University Press, Cambridge.

[Fl1] Flynn, E.V., The Jacobian and formal group of a curve of genus 2 over an arbitrary ground field, Math. Proc. Cambridge Philos. Soc. 107 (1990), 425-441.

[Fl2] Flynn, E.V., The group law on the Jacobian of a curve of genus 2, Journal f"ur die reine und angewandte Mathematik 439 (1993), 45-69.

[Gr] Grant, David, Formal groups in genus two, Journal f"ur die reine und angewandte Mathematik 411 (1990), 96-121.

[Mi1] Milne, James S., Abelian Varieties, 2008, pp. 166+vi pages, Version 2.001 available at www.milne.org/math.

[Mi2] Milne, James S., Jacobian Varieties, Arithmetic Geometry (Gary Cornell and Joseph Silverman, eds.), Springer-Verlag, New York, 1986, pp. 167-212, 2012 update available at www.milne.org/math/xnotes/JVs.pdf.

[Mu1] Mumford, David, Abelian Varieties, Oxford University Press India, 1974 (2nd edition).

[Mu2] Mumford, David, On the Equations Defining Abelian Varieties, I, II, III, Inventionnes Mathematicae 1, 3 (1966, 1967), 287-384, 75-135, 215-244.

[Mu3] Mumford, David, Tata Lectures on Theta I, II, III, Birkhauser-Boston, 1982, 1983, 1991.

[MK] Mumford, David (with an appendix by George Kempf), Varieties Defined by Quadratic Equations, in Questions on Algebraic Varieties, C.I.M.E. (Centro Internazionale Matematico Estivo) (1969).

[Wa] van Wamelen, Paul, Equations for the Jacobian of a Hyperelliptic Curve, Transactions of the American Mathematical Society Volume 350 Number 8 (August 1998), 3083-3106.