ON WELL-POSEDNESS OF GENERALIZED KORTEweg-de VRIES EQUATION IN SCALE CRITICAL $\hat{L}^{\alpha}$ SPACE

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Abstract. The purpose of this paper is to study local and global well-posedness of initial value problem for generalized Korteweg-de Vries (gKdV) equation in $\hat{L}^{\alpha}$ = \{f ∈ S'(R)| ∥f∥_{\hat{L}^{\alpha}} = ∥\hat{f}∥_{L^{r'}} < \infty\}. We show (large data) local well-posedness, small data global well-posedness, and small data scattering for gKdV equation in the scale critical $\hat{L}^{\alpha}$ space. A key ingredient is a Stein-Tomas type inequality for the Airy equation, which generalizes usual Strichartz’ estimates for $\hat{L}^{r}$-framework.

1. Introduction

We consider initial value problem for the generalized Korteweg-de Vries (gKdV) equation

\begin{equation}
\begin{cases}
\partial_t u + \partial_x^3 u = \mu \partial_x (|u|^{\alpha-1} u), & t, x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R},
\end{cases}
\tag{1.1}
\end{equation}

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an unknown function, $u_0 : \mathbb{R} \to \mathbb{R}$ is a given function, and $\mu \in \mathbb{R}\setminus\{0\}$ and $\alpha > 1$ are constants. We call that (1.1) is defocusing if $\mu > 0$ and focusing if $\mu < 0$.

The class of equations (1.1) arises in several fields of physics. Eq. (1.1) with $\alpha = 2$ is notable Korteweg-de Vries equation which models long waves propagating in a channel [22]. Eq. (1.1) with $\alpha = 3$ is also well known as the modified Korteweg-de Vries equation which describes a time evolution for the curvature of certain types of helical space curves [23].

The equation (1.1) has the following scale invariance: if $u(t, x)$ is a solution to (1.1), then

$$u_\lambda(t, x) := \lambda^{\frac{2}{\alpha-1}} u(\lambda^3 t, \lambda x)$$

is also a solution to (1.1) with an initial data $u_\lambda(0, x) = \lambda^{\frac{2}{\alpha-1}} u_0(\lambda x)$ for any $\lambda > 0$. In what follows, a Banach space for initial data is referred to as a scale critical space if its norm is invariant under $u_0(x) \mapsto \lambda^{\frac{2}{\alpha-1}} u_0(\lambda x)$.

The purpose of this paper is to study (large data) local well-posedness, small data global well-posedness and scattering for (1.1) in a scale critical space $\hat{L}^{(\alpha-1)/2}$. For $r \in [1, \infty]$, the function space $\hat{L}^r$ is defined by

$$\hat{L}^r = \hat{L}^r(\mathbb{R}) := \{f \in S'(\mathbb{R})| ||f||_{\hat{L}^r} = ||\hat{f}||_{L^{r'}} < \infty\},$$

where $\hat{f}$ stands for Fourier transform of $f$ with respect to space variable and $r'$ denotes the Hölder conjugate of $r$. We use the conventions $1' = \infty$ and $\infty' = 1$. Our notion of well-posedness contains of existence, uniqueness, and continuity of the data-to-solution map. We also consider persistent
property of the solution, that is, the solution describes a continuous curve in the function space $X$ whenever $u_0 \in X$.

Local well-posedness of the initial value problem \((1.1)\) in a scale subcritical Sobolev space $H^s(\mathbb{R})$, $s > s_\alpha := 1/2 - 2/(\alpha - 1)$, has been studied by many authors \([1, 8, 12, 15, 17, 18, 20, 24]\), where $s_\alpha$, a scale critical exponent, is unique number such that $H^{s_\alpha}$ becomes scale critical. A fundamental work on local well-posedness is due to Kenig-Ponce-Vega \([17]\). They proved that \((1.1)\) is locally well-posed in $H^s(\mathbb{R})$ with $s > 3/4$ ($\alpha = 2$, $s_2 = -3/2$), $s \geq 1/4$ ($\alpha = 3$, $s_3 = -1/2$), $s \geq 1/12$ ($\alpha = 4$, $s_4 = -1/6$) and $s \geq s_\alpha$ ($\alpha \geq 5$). Introducing Fourier restriction norms, Bourgain \([1]\) obtained local (and global) well-posedness of the KdV equation (i.e., \((1.1)\) with $\alpha = 2$) in $L^2(\mathbb{R})$. In \([18]\), Kenig-Ponce-Vega improved the previous results for the KdV equation to $H^s(\mathbb{R})$ with $s > -3/4$. Further, Guo \([12]\) and Kishimoto \([20]\) extended Kenig-Ponce-Vega’s result in $H^{-3/4}(\mathbb{R})$ (See also Buckmaster-Koch \([2]\) on the existence of weak solution to the KdV equation at $H^{-1}$.).

Grünerock \([5]\) has shown local well-posedness of the quartic KdV equation \((1.1)\) with $\alpha = 4$) in $H^s$ with $s > s_4$. Notice that all of the above results are based on contraction mapping principle for corresponding integral equation. Hence, a data-solution map associated with \((1.1)\) is Lipschitz continuous \([6]\).

Concerning the well-posedness of \((1.1)\) in the scale critical $H^{s_\alpha}$ space, Kenig-Ponce-Vega \([17]\) proved local well-posedness and global well-posedness for small data in the scale critical space $H^{s_\alpha}$ when $\alpha \geq 5$. Since the scale critical exponent $s_\alpha$ is negative in the mass-subcritical case $\alpha < 5$, well-posedness of \((1.1)\) in $H^{s_\alpha}$ becomes rather a difficult problem. Tao \([30]\) proved local well-posedness and global well-posedness for small data for \((1.1)\) with the quartic nonlinearity $\alpha = 4$ in $H^{s_4}$. Later on, the above results are extended to a homogeneous Besov space $B_{2,\infty}^{s_\alpha}$ by Koch-Marzuola \([21]\) ($\alpha = 4$) and Strunk ($\alpha \geq 5$). As far as we know, local well-posedness and small data global well-posedness of \((1.1)\) in $H^{s_\alpha}$ for the mass-subcritical case $\alpha < 5$ was open except for the case $\alpha = 4$.

Local and global well-posedness for a class of nonlinear dispersive equation is currently being intensively investigated also in the framework of $L^r$ space. For one dimensional nonlinear Schrödinger equation,

\begin{equation}
\begin{aligned}
\label{eq:1.2}
-\mu_0 v - \partial_x^2 v = \mu|v|^{\alpha-1}v, & \quad t, x \in \mathbb{R}, \\
v(0, x) = v_0(x), & \quad x \in \mathbb{R},
\end{aligned}
\end{equation}

where $\mu \in \mathbb{R}\setminus\{0\}$, Grünerock \([10]\) has shown local and global existence of solution to \((1.2)\) with $\alpha = 3$ in $L^r$. Hyakuna-Tsutsumi \([11]\) extended Grünerock’s result in $L^r$ to all mass-subcritical case $1 < \alpha < 5$. Grünerock \([9]\) and Grünerock-Vega \([11]\) proved local and global existence result for the modified KdV equation (i.e., \((1.1)\) with $\alpha = 3$) in $H^s_x$, where $H^s_x = \{ f \in S'; \| f \|_{H^s_x} = \|(1 + \xi^2)^{s/2} \hat{f}(\xi)\|_{L^r_x} < \infty\}$). However, the above results are not in scale critical settings.

\footnote{Since the equation \((1.1)\) preserves $L^2$ norm of solution in $t$, local well-posedness in $L^2$ yields global well-posedness in $L^2$ if $\alpha < 5$.}

\footnote{In fact, if the nonlinear term is analytic, then the data-solution map associated with \((1.1)\) is analytic.
It would be interesting to compare the scale critical space \( \dot{L}^{\frac{2\alpha}{23}}_{x,t} \) with some other scale critical spaces in view of symmetries. Other than the scaling, the \( \dot{L}^{\frac{2\alpha}{23}}_{x,t} \)-norm is invariant under the following three group operations

(i) Translation in physical space: \((T_a f)(x) = f(x - a), a \in \mathbb{R},\)

(ii) Translation in Fourier space: \((P_{\xi} f)(x) = e^{-it\xi} f(x), \xi \in \mathbb{R},\)

(iii) Airy flow: \((\text{Ai}(t) f)(x) = e^{-10t^2} f(x), t \in \mathbb{R}.\)

The critical Lebesgue space \( L^{\frac{2\alpha}{23}}_{x,t} \) is invariant under the former two symmetries but not under the Airy flow. The critical Sobolev space \( H^{s_\alpha} \) (or homogeneous Triebel-Lizorkin and homogeneous Besov spaces \( \dot{A}^{s_\alpha}_{2,q} \) \( (1 \leq q \leq \infty), \) more generally) is not invariant with respect to \( P_{\xi} \) if \( s_\alpha \neq 0 \). The critical weighted Lebesgue space \( \dot{H}^{0,-s_\alpha} := L^2(\mathbb{R}, |x|^{-2s_\alpha} dx) \) is not invariant with respect to \( T_a \) and \( \text{Ai}(t) \). Further, when \( \alpha = 5 \) these four spaces coincide with \( L^2 \), which is invariant under the above three symmetries. Thus, among the above four critical spaces, \( L^{\frac{2\alpha}{23}}_{x,t} \) possesses the most rich symmetries, and, in some sense, \( L^{\frac{2\alpha}{23}}_{x,t} \) is close to \( L^2 \) space. Inclusion relations between these spaces are summarized in Appendix B.

1.1. Local well-posedness. Before we state our main results, we introduce several notation.

**Definition 1.1.** Let \((s, r) \in \mathbb{R} \times [1, \infty].\) A pair \((s, r)\) is said to be acceptable if \(1/r \in [0, 3/4]\) and

\[
s \in \begin{cases} 
[-\frac{1}{2r}, \frac{2}{r}] & 0 \leq \frac{1}{r} \leq \frac{1}{2}, \\
\left(\frac{2}{r} - \frac{5}{4}, \frac{5}{2} - \frac{3}{r}\right) & \frac{1}{2} < \frac{1}{r} < \frac{3}{4}.
\end{cases}
\]

For an interval \(I \subset \mathbb{R}\) and an acceptable pair \((s, r)\), we define a function space \(X(I; s, r)\) of space-time functions with the following norm

\[
\|f\|_{X(I; s, r)} = \|D_x^a f\|_{L^p_x([s, \infty); L^q_t(I))},
\]

where the exponents in the above norm are given by

\[
\frac{2}{p(s, r)} + \frac{1}{q(s, r)} = \frac{1}{r}, \quad -\frac{1}{p(s, r)} + \frac{2}{q(s, r)} = s,
\]

or equivalently,

\[
\left(\frac{1}{p(s, r)}, \frac{1}{q(s, r)}\right) = \left(\frac{-1/5}{2/5}, \frac{2/5}{1/5}\right) \left(\frac{s}{1/r}\right).
\]

We refer \(X(I; s, r)\) to as an \(\dot{L}^r\)-admissible space.

Our main theorems are as follows.

**Theorem 1.2** (local well-posedness in \(\dot{L}^{\frac{2\alpha}{23}}_{x,t}\)). For \(21/5 < \alpha < 23/3\), the problem \((\text{I.3})\) is locally well-posed in \(\dot{L}^{\frac{2\alpha}{23}}_{x,t}\). Namely, for any \(u_0 \in \dot{L}^{\frac{2\alpha}{23}}_{x,t}(\mathbb{R})\), there exists an interval \(I = I(u_0)\) such that a unique solution

\[
u \in C(I; \dot{L}^{\frac{2\alpha}{23}}_{x,t}(\mathbb{R})) \cap \bigcap_{(s, \frac{2\alpha}{23}); \text{acceptable}} X(I; s, \alpha - \frac{1}{2})
\]
to (1.1) exists. Furthermore, for any given subinterval $I' \subset I$, there exists a neighborhood $V$ of $u_0$ in $\hat{L}^{\alpha-1}_x(\mathbb{R})$ such that the map $u_0 \mapsto u$ from $V$ into the class defined by (1.4) with $I'$ instead of $I$ is Lipschitz continuous.

Remark 1.3. Theorem 1.2 (and all results below) holds for more general nonlinearity of the form $\partial_x G(u)$ with $G \in \text{Lip}^\alpha$. For precise condition on $G$, see Remark 3.5.

The proof of Theorem 1.2 is based on a contraction argument, with a help of a space-time estimate for the Airy equation in $\hat{L}^r$. A key ingredient is Stein-Tomas type inequality for the Airy equation, a special case of [9, Corollary 3.6]:

$$\left\| D_x |^{1/r} e^{-t\partial_x^3} f \right\|_{L^r_t(I \times \mathbb{R})} \leq C \left\| f \right\|_{\hat{L}^{r/3}},$$

where $r \in (4, \infty]$. This inequality is a generalization of a well-known Strichartz estimate

$$\left\| D_x |^{1/6} e^{-t\partial_x^3} f \right\|_{L^6_t(I \times \mathbb{R})} \leq C \left\| f \right\|_{L^2}.$$ 

Moreover, interpolations between the above Stein-Tomas type inequality (1.5) and Kenig-Ruiz estimate or Kato’s local smoothing effect give us the following generalized Strichartz’ estimate for the Airy equation in $\hat{L}^r$-framework (Proposition 2.1): If $(s, r)$ is an acceptable pair then there exists $C$ such that

$$\left\| e^{-t\partial_x^3} f \right\|_{X(\mathbb{R}; s, r)} \leq C \left\| f \right\|_{\hat{L}^r}$$

for $f \in \hat{L}^r$. Furthermore, combining the homogeneous estimate and Christ-Kiselev lemma (Lemma 2.6), we also obtain a generalized version of inhomogeneous Strichartz’ estimates. The estimate (1.5) can be regarded as a kind of restriction estimate of Fourier transform, which goes back to Stein and Tomas [31] (for more information on the restriction theorem, see e.g. [31]). It is worth mentioning that the $\hat{L}^r$ spaces have naturally come out in this context.

We set $S(I; r) := X(I; 0, r)$. The $S(I; r)$ norm is so-called scattering norm. It is understood that a key for obtaining a closed estimate for the corresponding integral equation, from which local well-posedness immediately follows, is to bound the scattering norm $S(I; \frac{\alpha - 1}{2})$. In the proof of Theorem 1.2, the scattering norm is handled by means of the above generalized Strichartz’ estimate (1.6). Notice that the pair $(0, \frac{\alpha - 1}{2})$ is acceptable only if $\alpha > 21/5$. Our restriction $\alpha > 21/5$ comes from this fact. For the upper bound on $\alpha$, see Remark 1.4 below. Alternatively, Sobolev’s embedding also yields a bound on the scattering norm, provided $\alpha \geq 5$. In such case, we obtain local well-posedness in $\dot{H}^{5\alpha}$ as in [17] (see Remark 4.5).

1.2. Persistence of regularity. We establish two persistence-of-regularity type results for $\hat{L}^{\alpha-1}_x$-solutions given in Theorem 1.2. More specifically, we consider persistence of $\hat{L}^r$-regularity for $r \neq \frac{\alpha - 1}{2}$ and $\dot{H}^s$ regularity for
−1 < s < α. These results yield local well-posedness in other \( \hat{L}^r \) like space such as \( \hat{L}^{r_1} \cap \hat{L}^{r_2}, r_1 \leq \frac{\alpha - 1}{2} \leq r_2, \) and \( \dot{H}^s \cap \hat{L}^{\alpha-1} \).

**Theorem 1.4** (persistence of \( \dot{L}^r \)-regularity). Assume \( 21/5 < \alpha < 23/3 \). Let \( u_0 \in \hat{L}^{\frac{\alpha}{\alpha - 1} r} (\mathbb{R}) \) and let \( u \in C(I; \dot{L}^{\frac{\alpha}{\alpha - 1} r} (\mathbb{R})) \) be a corresponding solution given in Theorem 1.2. If \( u_0 \in \hat{L}^{\frac{\alpha}{\alpha - 1} r} (\mathbb{R}) \) for some \( 21/5 < \alpha_0 < 23/3, \alpha_0 \neq \alpha, \) then

\[
\begin{align*}
\dot{u} & \in C(I; \hat{L}^{\frac{\alpha_0}{\alpha_0 - 1} r} (\mathbb{R})) \cap \bigcap_{(s, \frac{\alpha_0}{\alpha_0 - 1} r): \text{acceptable}} X(I; s, \frac{\alpha_0 - 1}{2}).
\end{align*}
\]

**Theorem 1.5** (persistence of \( \dot{H}^s \)-regularity). Assume \( 21/5 < \alpha < 23/3 \). Let \( u_0 \in \hat{L}^{\alpha} (\mathbb{R}) \) and let \( u \in C(I, \dot{L}^{\alpha} (\mathbb{R})) \) be a corresponding solution given in Theorem 1.2. If \( u_0 \in \dot{H}^s (\mathbb{R}) \) for some \( -1 < s < \alpha, \) then

\[
|D_x|^{\sigma} u \in C(I; L^2 (\mathbb{R})) \cap \bigcap_{(s, 2): \text{acceptable}} X(I; s, 2).
\]

As a corollary, we obtain the following well-posedness results.

**Corollary 1.6.** We have the following.

(i) If \( 21/5 < \alpha < 23/3 \) then \((1.1)\) is locally well-posed in \( \hat{L}^{r_1} \cap \hat{L}^{r_2} \) as long as \( 8/5 < r_1 \leq \frac{\alpha - 1}{2} \leq r_2 < 10/3. \)

(ii) If \( 21/5 < \alpha < 5 \) then \((1.1)\) is locally well-posed in \( \dot{H}^{s_0} \cap \dot{L}^{\alpha-1} \), where \( s_0 = \frac{1}{2} - \frac{2}{\alpha - 1}. \)

Since \( \dot{L}^{\frac{\alpha}{\alpha - 1} r} \subset \dot{H}^{s_0} \) does not hold (see Lemma 3.2), the second is weaker than well-posedness in \( \dot{H}^{s_0} \).

Here we remark that an \( \dot{L}^{\frac{\alpha}{\alpha - 1} r} \)-solution has conserved quantities, provided the solution has appropriate regularity. More precisely, when \( u_0 \in \dot{L}^{\frac{\alpha}{\alpha - 1} r} \cap L^2 \), a solution \( u(t) \) has a conserved mass

\[
M[u(t)] := \|u(t)\|_{L^2}^2.
\]

Similarly, if \( u_0 \in \dot{L}^{\frac{\alpha}{\alpha - 1} r} \cap \dot{H}^1 \) then energy

\[
E[u(t)] := \frac{1}{2} \|\partial_x u(t)\|_{L^2}^2 + \frac{\mu}{\alpha + 1} \|u(t)\|_{L^{\alpha + 1}}^{\alpha + 1}
\]

is invariant.

**1.3. Blowup and scattering.** We next consider long time behavior of solutions given in Theorem 1.2. To this end, we give the definitions of blow up and scattering of \((1.1)\) for the initial data \( u_0 \in \dot{L}^r_x \). Set

\[
T_{\text{max}} := \sup \{ T > 0; \exists u \in C([0, T]; \dot{L}^r_x (\mathbb{R})) : \text{solution to } (1.1) \},
\]

\[
T_{\text{min}} := \sup \{ T > 0; \exists u \in C([-T, 0]; \dot{L}^r_x (\mathbb{R})) : \text{solution to } (1.1) \}.
\]

Denote the lifespan of \( u(t) \) as \((-T_{\text{min}}, T_{\text{max}})\). We say a solution \( u(t) \) blows up in finite time for positive (resp. negative) time direction if \( T_{\text{max}} < +\infty \) (resp.
$T_{\text{min}} < +\infty)$. We say a solution $u(t)$ scatters for positive time direction if $T_{\text{max}} = +\infty$ and there exists a unique function $u_+ \in \hat{L}_x^r$ such that
\[
\lim_{t \to +\infty} \|u(t) - e^{-i0\partial_x^2} u_+\|_{\hat{L}_x^r} = 0,
\]
where $e^{-i0\partial_x^2} u_+$ is a solution to the Airy equation $\partial_t v + \partial_x^3 v = 0$ with a initial condition $v(0, x) = u_+$. The scattering of $u$ for negative time direction is defined by a similar fashion.

Roughly speaking, a solution scatters if linear dispersion effect dominates the nonlinear interaction. A typical case is when the data (and the corresponding solution) is small. Here, we state this small data scattering for Eq. (1.1).

**Theorem 1.7** (Small data scattering). Let $21/5 < \alpha < 23/3$. There exists $\varepsilon_0 > 0$ such that if $u_0 \in \hat{L}_{x}^{\frac{\alpha - 1}{2}}(\mathbb{R})$ satisfies $\|u_0\|_{\hat{L}_{x}^{\frac{\alpha - 1}{2}}} \leq \varepsilon_0$, then the solution $u(t)$ to Eq. (1.1) given in Theorem 1.2 is global in time and scatters for both time directions. Moreover,
\[
\|u\|_{\hat{L}_{x}^{\alpha}(\mathbb{R}; \hat{L}_{x}^{\frac{\alpha - 1}{2}})} + \|u\|_{S(\mathbb{R}; \hat{L}_{x}^{\frac{\alpha - 1}{2}})} \leq 2\|u_0\|_{\hat{L}_{x}^{\frac{\alpha - 1}{2}}}.
\]

We now give criterion for blowup and scattering.

**Theorem 1.8** (Blowup criterion). Assume $21/5 < \alpha < 23/3$. Let $u_0 \in \hat{L}_{x}^{\frac{\alpha - 1}{2}}$ and let $u(t)$ be a corresponding unique solution of Eq. (1.1) given in Theorem 1.2. If $T_{\text{max}} < \infty$ then
\[
\|u\|_{S([0,T); \hat{L}_{x}^{\frac{\alpha - 1}{2}})} \to \infty
\]
as $T \uparrow T_{\text{max}}$. A similar statement is true for negative time direction.

**Theorem 1.9** (Scattering criterion). Assume $21/5 < \alpha < 23/3$. Let $u_0 \in \hat{L}_{x}^{\frac{\alpha - 1}{2}}$ and let $u(t)$ be a corresponding unique solution of Eq. (1.1) given in Theorem 1.2. The solution $u(t)$ scatters forward in time if and only if $T_{\text{max}} = +\infty$ and $\|u\|_{S([0,\infty); \hat{L}_{x}^{\frac{\alpha - 1}{2}})} < \infty$. A similar statement is true for negative time direction.

Finally, we give a criteria for scattering in terms of the energy. We note that if an $\hat{L}_{x}^{\frac{\alpha - 1}{2}}$-solution $u(t)$ scatters (in $\hat{L}_{x}^{\frac{\alpha - 1}{2}}$ sense) as $t \to \pm \infty$ and if $u_0 \in \hat{L}_{x}^{\frac{\alpha - 1}{2}}$ (resp. $u_0 \in \dot{H}^{\sigma}$) then $u(t)$ scatters as $t \to \pm \infty$ also in $\hat{L}_{x}^{\frac{\alpha - 1}{2}}$ sense (resp. $\dot{H}^{\sigma}$ sense).

**Theorem 1.10.** Let $21/5 < \alpha < 23/3$. If $u_0 \in \hat{L}_{x}^{\frac{\alpha - 1}{2}} \cap H^1$ satisfies $u_0 \neq 0$ and $E[u_0] \leq 0$ then $u(t)$ does not scatter as $t \to \pm \infty$.

The rest of the paper is organized as follows. In Section 2, we prove some linear space-time estimates for solutions to the Airy equation, in $\hat{L}_{x}^{r}$-framework. The generalized Strichartz estimates are established in Propositions 2.1 and 2.3. Section 3 is devoted to several nonlinear estimates. We also introduce several function spaces to work with in this section. Then, in Section 4, we prove our theorems. In Appendix A, we prove a fractional
chain rule in space-time function space (Lemma 3.7). Finally in Appendix B, we briefly collect some inclusion relation for \( \hat{L}^r \).

The following notation will be used throughout this paper: \(|D_x|^s = (-\partial_x^2)^{s/2}\) and \((D_x)^s = (I - \partial_x^2)^{s/2}\) denote the Riesz and Bessel potentials of order \(-s\), respectively. For \(1 \leq p, q \leq \infty\) and \(I \subset \mathbb{R}\), let us define a space-time norm

\[
\|f\|_{L^p_tL^q_x(I)} = \|f(t, \cdot)\|_{L^p_x(\mathbb{R})}L^q_t(I),
\]

\[
\|f\|_{L^p_xL^q_t(I)} = \|f(\cdot, x)\|_{L^p_x(I)}L^q_t(\mathbb{R}).
\]

2. Linear Estimates for Airy Equation

In this section we consider the space-time estimates of solution to the Airy equation

\[
\begin{aligned}
\partial_t u + \partial_x^3 u &= F(t, x), \quad t \in I, x \in \mathbb{R}, \\
\quad u(0, x) &= f(x), \quad x \in \mathbb{R},
\end{aligned}
\]

(2.1)

where \(I \subset \mathbb{R}\) is an interval, \(F : I \times \mathbb{R} \to \mathbb{R}\) and \(f : \mathbb{R} \to \mathbb{R}\) are given functions.

Let \(\{e^{-it\partial_x^3}\}_{t \in \mathbb{R}}\) be an isometric isomorphism group in \(\hat{L}^r\) defined by \(e^{-it\partial_x^3}F = \mathcal{F}^{-1}e^{it\xi^3}\mathcal{F}\), or more precisely by

\[
(e^{-it\partial_x^3}f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi + it\xi^3} \hat{f}(\xi) d\xi.
\]

Using the group, the solution to (2.1) can be written as

\[
u(t) = e^{-it\partial_x^3}f + \int_0^t e^{-(t-t')\partial_x^3}F(t') dt'.
\]

We first show a homogeneous estimates associated with (2.1).

**Proposition 2.1.** Let \(I\) be an interval. Let \((p, q)\) satisfy

\[
0 \leq \frac{1}{p} < \frac{1}{4}, \quad 0 \leq \frac{1}{q} < \frac{1}{2} - \frac{1}{p},
\]

Then, for any \(f \in \hat{L}^r\),

\[
\|(|D_x|^s e^{-it\partial_x^3}f)\|_{L^p_tL^q_x(I)} \leq C \|f\|_{\hat{L}^r},
\]

(2.2)

where

\[
\frac{1}{r} = \frac{2}{p} + \frac{1}{q}, \quad s = -\frac{1}{p} + \frac{2}{q},
\]

and positive constant \(C\) depends only on \(r\) and \(s\).

Figure 1 shows the range of \((p, q)\) satisfying the assumption of Proposition 2.1 where \(A = (1/4, 0), B = (1/4, 1/4),\) and \(C = (0, 1/2)\). The line segments \(OA\) and \(OC\) is included, but the other parts of border are excluded.
To prove Proposition 2.1, we show three lemmas. The first one is a Stein-Tomas type estimate.

**Lemma 2.2** (Stein-Tomas type estimate). For any $r \in (4, \infty]$, there exists a positive constant $C$ depending only on $r$ such that for any $f \in \dot{L}^{r/3}$

$$
\left\| D_x^{1/r} e^{-t\partial_x^3} f \right\|_{L_t^r(I)} \leq C \| f \|_{\dot{L}^{r/3}}.
$$

**Proof of Lemma 2.2.** Although a more general version is proved in [9, Corollary 3.6], here we give a direct proof which is based on the fact that the exponents for space-variable and time-variable in the left hand side coincide.

It suffices to prove (2.3) for the case $I = \mathbb{R}$. For notational simplicity, we omit $\mathbb{R}$. The case $r = \infty$ follows from the Hausdorff-Young inequality. Let $r < \infty$. Squaring both sides, we may show that

$$
\left\| D_x^{1/r} e^{-t\partial_x^3} f \right\|_{L_t^r(I)}^2 \leq C \| f \|_{\dot{L}^{r/3}}^2.
$$

The left hand side of (2.4) is equal to

$$
\left\| \int_{\mathbb{R}^2} e^{ix(\xi - \eta) + it(\xi^3 - \eta^3)} |\xi\eta|^{1/r} \hat{f}(\xi) \overline{\hat{f}(\eta)} d\xi d\eta \right\|_{L_t^{r/2}}.
$$

Changing variables by $a = \xi - \eta$ and $b = \xi^3 - \eta^3$, we have

$$
\left\| \int_{\mathbb{R}^2} e^{ixa + itb} |\xi\eta|^{1/r} \hat{f}(\xi) \overline{\hat{f}(\eta)} \frac{1}{3|\xi^2 - \eta^2|} dadb \right\|_{L_t^{r/2}}.
$$

We now use the Hausdorff-Young inequality to deduce that

$$
\left\| \int_{\mathbb{R}^2} e^{ixa + itb} |\xi\eta|^{1/r} \hat{f}(\xi) \overline{\hat{f}(\eta)} |\xi^2 - \eta^2|^{-1} dadb \right\|_{L_t^{r/2}} \leq C \left\| |\xi\eta|^{1/r} \hat{f}(\xi) \overline{\hat{f}(\eta)} |\xi^2 - \eta^2|^{-1} \right\|_{L_{a,b}^{r/2}}.
$$
By the Hölder and the Hardy-Littlewood-Sobolev inequality, we have
\[ C \left\{ \int_{\mathbb{R}^2} \frac{|\xi \eta|^{1/2} |\hat{f}(\xi)|^{2/3} |\hat{f}(\eta)|^{2/3}}{|\xi - \eta|^{2/3} |\xi + \eta|^{2/3}} \, d\xi d\eta \right\}^{1/2}. \]

Notice that \( r/2 \geq 2 \). We now split the integral region \( \mathbb{R}^2 \) into \( \{\xi \eta \geq 0\} \) and \( \{\xi \eta < 0\} \). We only consider the first case, since the other can be treated essentially in the same way. For \((\xi, \eta)\) with \( \xi \eta \geq 0 \), we have \( \xi \eta \leq (\xi + \eta)^2/4 \), and so
\[ \int_{\xi \eta \geq 0} \frac{|\xi \eta|^{1/2} |\hat{f}(\xi)|^{2/3} |\hat{f}(\eta)|^{2/3}}{|\xi - \eta|^{2/3} |\xi + \eta|^{2/3}} \, d\xi d\eta \leq C \int_{\xi \eta \geq 0} |\hat{f}(\xi)|^{2/3} |\hat{f}(\eta)|^{2/3} \, d\xi d\eta. \]

By the Hölder and the Hardy-Littlewood-Sobolev inequality, we have
\[ \int_{\xi \eta \geq 0} \frac{|\hat{f}(\xi)|^{2/3} |\hat{f}(\eta)|^{2/3}}{|\xi - \eta|^{2/3}} \, d\xi d\eta \leq \left\| \frac{|\hat{f}|^{2/3}}{|\xi - \eta|^{2/3}} \right\|_{L^{12/5}} \left\| \frac{|\hat{f}|^{2/3}}{|\xi - \eta|^{2/3}} \right\|_{L^{12/5}} \leq C \left\| f \right\|_{L^{2/3}} \]
as long as \( 2/(r - 2) < 1 \), that is, \( r > 4 \). Combining (2.5), (2.6) and (2.7), we obtain the result. \( \square \)

The second is Kenig-Ruiz type estimate \[19].

**Lemma 2.3** (Kenig-Ruiz type estimate). There exists a universal constant \( C \) such that for any interval \( I \) and any \( f \in L^2 \)
\[ \left\| \left| D_x \right|^\frac{1}{2} e^{-it\partial_x^3} f \right\|_{L^2_{\infty} L^r(I)} \leq C \left\| f \right\|_{L^2}. \]

**Proof of Lemma 2.3** See [16] Theorem 2.5]. \( \square \)

The last estimate is an \( \dot{L}^q \) version of the Kato’s local smoothing effect \[15].

**Lemma 2.4** (Kato’s smoothing effect). For any \( q \in [2, \infty] \), there exists a positive constant \( C \) depending only on \( q \) such that any interval \( I \) and for any \( f \in \dot{L}^q \)
\[ \left\| \left| D_x \right|^\frac{2}{q} e^{-it\partial_x^3} f \right\|_{L^\infty_{\partial_x^3} L^q(I)} \leq C \left\| f \right\|_{L^q}. \]

**Proof of Lemma 2.4** We show (2.9) by slightly modifying the argument due to Kenig-Ponce-Vega [16] Theorem 2.5]. We prove (2.9) for the case \( I = \mathbb{R} \) only.

The case \( q = \infty \) is treated in Lemma 2.2. Hence, we may suppose that \( q < \infty \). A direct computation shows
\[ \left| D_x \right|^\frac{2}{q} e^{-it\partial_x^3} f = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi + it\xi^3} |\xi|^{\frac{2}{q}} \hat{f}(\xi) \, d\xi \]
where we have used a change of variable $\eta = \xi^3$ to yield the last line. Take $L^q_t$ norm and apply the Hausdorff-Young inequality to obtain
\[ \left\| D_x \frac{2}{3} e^{-t\partial_x^2} f \right\|_{L^q_t} \leq C \left\| e^{ix\eta^{1/3}} |\eta|^{\frac{2}{3} - \frac{2}{q}} \hat{f}(\eta^\frac{1}{3}) \right\|_{L^p_{\eta}} \leq C \left\| \hat{f} \right\|_{L^p} = C \left\| f \right\|_{L^r}.
\]
Since the right hand side is independent of $x$, we obtain (2.9).

\[ \square \]

**Proof of Proposition 2.1** Interpolating (2.8), (2.9), and (2.10), we obtain (2.2).

Next we show an inhomogeneous estimates associated with (2.1).

**Proposition 2.5.** Let $4/3 < r < 4$ and let $(p_j, q_j)$ $(j = 1, 2)$ satisfy
\[ 0 \leq \frac{1}{p_j} < \frac{1}{4}, \quad 0 \leq \frac{1}{q_j} < \frac{1}{2} - \frac{1}{p_j}. \]

Then, the inequalities
\[ (2.10) \quad \left\| \int_0^t e^{-(t-t')\partial_x^2} F(t') dt' \right\|_{L^p_t L^q_x(I)} \leq C_1 \left\| |D_x|^{-s_2} F \right\|_{L^p_t L^q_x(I)}, \]
and
\[ (2.11) \quad \left\| D_x^{s_1} \int_0^t e^{-(t-t')\partial_x^2} F(t') dt' \right\|_{L^p_t L^q_x(I)} \leq C_2 \left\| |D_x|^{-s_2} F \right\|_{L^p_t L^q_x(I)} \]
hold for any $F$ satisfying $|D_x|^{-s_2} F \in L^p_t L^q_x$, where
\[ \frac{1}{r} = \frac{2}{p_1} + \frac{1}{q_1}, \quad s_1 = -\frac{1}{p_1} + \frac{2}{q_1} \]
and
\[ \frac{1}{r} = \frac{2}{p_2} + \frac{1}{q_2}, \quad s_2 = -\frac{1}{p_2} + \frac{2}{q_2}, \]
where the constant $C_1$ depends on $r$, $s_1$ and $I$, and the constant $C_2$ depends on $r$, $s_2$, $s_1$ and $I$.

To prove Theorem 2.5, we employ the following lemma which is essentially due to Christ-Kiselev [3]. The version of this lemma that we use is the one presented in Molinet-Ribaud [25].

**Lemma 2.6.** Let $I \subset \mathbb{R}$ be an interval and let $K : \mathcal{S}(I \times \mathbb{R}) \to C(\mathbb{R}^3)$. Assume that
\[ \left\| \int_I K(t, t') F(t') dt' \right\|_{L^p_t L^q_x(I)} \leq C \left\| F \right\|_{L^p_t L^q_x(I)} \]
for some $1 \leq p_1, p_2, q_1, q_2 \leq \infty$ with $\min(p_1, q_1) > \max(p_2, q_2)$. Then
\[ \left\| \int_0^t K(t, t') F(t') dt' \right\|_{L^p_t L^q_x(I)} \leq C \left\| F \right\|_{L^p_t L^q_x(I)}. \]

Moreover the case $q_1 = \infty$ and $p_2, q_2 < \infty$ is allowed.
Proof of Lemma 2.6 See [25, Lemma 2]. □

Proof of Proposition 2.5 We first prove the inequality (2.10). By the \( L^r \)-unitarity of the group \( \{ e^{-t\partial_x^3} \}_{t\in\mathbb{R}} \), the duality argument and Proposition 2.4, we have

\[
(2.12) \quad \left\| \int_0^t e^{-(t-t')\partial_x^3} F(t') dt' \right\|_{L_x^r} = \left\| \int_0^t e^{t'\partial_x^3} F(t') dt' \right\|_{L_x^r} = \sup_{\|g\|_{L_x^r}=1} \left[ \int_{-\infty}^{\infty} \left\{ \int_0^t e^{t'\partial_x^3} F(t', x) dt' \right\} g(x) dx \right]
\]

\[
= \sup_{\|g\|_{L_x^r}=1} \left[ \int_{-\infty}^{\infty} \left\{ \int_0^t D_x^{-s_2} F(t', x) D_x^{s_2} e^{-t'\partial_x^3} g(x) dt' dx \right\} \right] \leq \sup_{\|g\|_{L_x^r}=1} \left\| D_x^{-s_2} F \right\|_{L_x^{p'_2} L_t^{q'_2}(I)} \left\| D_x^{s_2} e^{-t'\partial_x^3} g \right\|_{L_x^{p_2} L_t^{q_2}(I)} \leq C \sup_{\|g\|_{L_x^r}=1} \left\| D_x^{-s_2} F \right\|_{L_x^{p'_2} L_t^{q'_2}(I)} \|g\|_{L_x^r} = C \| D_x^{-s_2} F \|_{L_x^{p'_2} L_t^{q'_2}(I)}
\]

where the constant \( C \) is independent of \( t \). Hence we have (2.10).

Next we prove the inequality (2.11). Since the case \( r = 2 \) has already proved in [17], we consider the case where \( r \neq 2 \). To prove (2.11), it suffices to prove

\[
(2.13) \quad \left\| D_x^{s_1} \int_I e^{-(t-t')\partial_x^3} F(t') dt' \right\|_{L_x^{p_1} L_t^{q_1}(I)} \leq C \| D_x^{-s_2} F \|_{L_x^{p'_2} L_t^{q'_2}(I)}.
\]

Indeed, since

\[
\min(p_1, q_1) = \left\{ \begin{array}{ll} \frac{r}{2} & (\frac{4}{3} < r < 2), \\ 2 & (2 < r < 4) \end{array} \right. > \max(p'_2, q'_2) = \left\{ \begin{array}{ll} \frac{r}{2} & (\frac{4}{3} < r < 2), \\ 2 & (2 < r < 4) \end{array} \right.
\]

we see that the combination of the Christ-Kiselev lemma (Lemma 2.6) with (2.13) implies (2.11). Therefore we concentrate our attention on prove (2.13). By Proposition 2.4

\[
(2.14) \quad \left\| D_x^{s_1} \int_I e^{-(t-t')\partial_x^3} F(t') dt' \right\|_{L_x^{p_1} L_t^{q_1}(I)} = \left\| D_x^{s_1} e^{-t\partial_x^3} \int_I e^{t'\partial_x^3} F(t') dt' \right\|_{L_x^{p_1} L_t^{q_1}(I)} \leq C \left\| \int_I e^{t'\partial_x^3} F(t') dt' \right\|_{L_x^{p_1} L_t^{q_1}(I)}
\]

By the duality argument similar to (2.12), we obtain

\[
(2.15) \quad \left\| \int_I e^{t\partial_x^3} F(t') dt' \right\|_{L_x^r} \leq C \| D_x^{-s_2} F \|_{L_x^{p'_2} L_t^{q'_2}(I)}.
\]
Combining (2.14) and (2.15), we obtain (2.13). □

3. NONLINEAR ESTIMATES

In this section, we prove several nonlinear estimates which are used to prove main theorems. We introduce several function spaces. Let us recall that a pair \((s, r)\) \(\in \mathbb{R} \times [1, \infty]\) is said to be acceptable if \(1/r \in [0, 3/4)\) and
\[
0 \leq \frac{1}{r} < \frac{1}{4}, \quad \frac{1}{r} > \frac{3}{4}.
\]

**Definition 3.1.** Let \((s, r)\) \(\in \mathbb{R} \times [1, \infty]\). A pair \((s, r)\) is said to be conjugate-acceptable if \((1 - s, r')\) is acceptable, where \(1/r' = 1 - 1/r \in [0, 1]\).

![Figure 2](image)

Figure 2 shows the ranges of acceptable pairs (quadrangle OABC) and conjugate-acceptable pairs (quadrangle DEFG). Here, \(O = (0, 0)\), \(A = (1/2, -1/4)\), \(B = (3/4, 1/4)\), \(C = (1/2, 1)\), \(D = (1, 1)\), \(E = (1/2, 5/4)\), \(F = (1/4, 3/4)\), and \(G = (1/2, 0)\).

For an interval \(I \subset \mathbb{R}\) and a conjugate-acceptable pair \((s, r)\), we define a function space \(Y(I; s, r)\) by
\[
\|f\|_{Y(I; s, r)} = \|D_x^s f\|_{L^{\tilde{p}(s, r)}_{\tilde{q}(s, r)}(I)},
\]
where the exponents are given by
\[
(3.1) \quad \frac{2}{\tilde{p}(s, r)} + \frac{1}{\tilde{q}(s, r)} = 2 + \frac{1}{r}, \quad -\frac{1}{p(s, r)} + \frac{2}{q(s, r)} = s,
\]
or equivalently,
\[
\begin{pmatrix}
\frac{1}{\tilde{p}(s, r)} \\
\frac{1}{\tilde{q}(s, r)}
\end{pmatrix} = \begin{pmatrix}
-1/5 & 2/5 \\
2/5 & 1/5
\end{pmatrix} \begin{pmatrix}
s \\
2 + 1/r
\end{pmatrix} = \begin{pmatrix}
1/p(s, r) \\
1/q(s, r)
\end{pmatrix} + \begin{pmatrix}
4/5 \\
2/5
\end{pmatrix}.
\]

With this terminology, Propositions 2.1 and 2.5 can be reformulated as follows:
Proposition 3.2. Let $I$ be an interval.

(i) Let $(s, r)$ be an acceptable pair. Then, there exists a positive constant $C$ depending only on $s$ and $r$ such that
\[
\left\| e^{-t\partial_x^3} f \right\|_{L^\infty(\mathbb{R}; L^r)} + \left\| e^{-t\partial_x^3} f \right\|_{X(\mathbb{R}; s, r)} \leq C_{s, r} \| f \|_{L^r}
\]
for any $f \in \hat{L}^r$.

(ii) Let $(s_1, r)$ be an acceptable pair and let $(s_2, r)$ be a conjugate-acceptable pair. Then, there exists a positive constant depending only on $s_1$ and $r$ such that for any $t_0 \in I \subset \mathbb{R}$ and any $F \in Y(I; s_2, r)$,
\[
\left\| \int_{t_0}^{t} e^{-(t-t')\partial_x^3} \partial_x F(t') dt' \right\|_{L^\infty(I; L^r') \cap X(I; s_1, r)} \leq C \| F \|_{Y(I; s_2, r)}.
\]

To handle $X(I; s, r)$ and $Y(I; s, r)$ spaces, the following lemma is useful.

Lemma 3.3. Let $1 < p_i, q_i < \infty$ and $s_i \in \mathbb{R}$ for $i = 1, 2$. Let $p, q, s$ be
\[
\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}, \quad s = \theta s_1 + (1-\theta)s_2
\]
for some $\theta \in (0, 1)$. Then, there exists a positive constant $C$ depending on $p_1, p_2, q_1, q_2, s_1, s_2$ and $\theta$ such that
\[
\| D_x^{|s|} f \|_{L^p L^q} \leq C \| D_x^{|s_1|} f \|_{L^{p_1} L^{q_1}}^\theta \| D_x^{|s_2|} f \|_{L^{p_2} L^{q_2}}^{1-\theta}
\]
holds for any $f$ such that $|D_x|^{|s|} f \in L^p L^q$ and $|D_x|^{|s_2|} f \in L^{p_2} L^{q_2}$.

Proof of Lemma 3.3. For $z \in \mathbb{C}$, define an operator $T_z = |D_x|^{z_1 + (1-z)s_2}$.

Let $g(t)$ and $h(x)$ be $\mathbb{R}$-valued simple functions and $G_z(t)$ and $H_z(x)$ be extensions of these functions defined by
\[
G_z(t) := \left| g(t) \right|^{\frac{1-(z/q_1 + (1-z)/q_2)}{1-1/q}} \text{sign} g(t)
\]
and
\[
H_z(x) := \left| h(x) \right|^{\frac{1-(z/p_1 + (1-z)/p_2)}{1-1/p}} \text{sign} h(x),
\]
respectively, for $z \in \mathbb{C}$ with $0 \leq \text{Re} z \leq 1$. Put
\[
\Psi(z) := \int_{\mathbb{R}^2} T_z f(t, x) G_z(t) H_z(x) dt dx.
\]

By density and duality, it suffices to show
\[
|\Psi(\theta)| \leq C \| D_x^{|s_1|} f \|_{L^{p_1} L^{q_1}}^\theta \| D_x^{|s_2|} f \|_{L^{p_2} L^{q_2}}^{1-\theta}
\]
for any $f \in S(\mathbb{R}^2)$ with compact Fourier support and any simple functions $g(t)$ and $h(x)$ such that $\| g \|_{L^q} = \| h \|_{L^p} = 1$.

Let us prove (3.2). It is easy to see that $\Psi(z)$ is analytic in $0 < \text{Re} z < 1$ and continuous in $0 \leq \text{Re} z \leq 1$. By a variant of multiplier theorem by Fernandez [4, Theorem 6.4], we see that $|D_x|^{it}$ is a bounded operator in $L^{p_1} L^{q_1}$ with norm $C(1 + |t|)$. Therefore, for any $y \in \mathbb{R}$,
\[
|\Psi(1 + iy)| \leq \left\| |D_x|^{iy(s_1-s_2)} (|D_x|^{s_1} f) \right\|_{L^{p_1} L^{q_1}} \left\| G_{1+iy} H_{1+iy} \right\|_{L^{p_1} L^{q_1}} \| g \|_{L^q} \| h \|_{L^p}
\]
\[
\leq C(1 + |y(s_1 - s_2)|) \| D_x^{|s_1|} f \|_{L^{p_1} L^{q_1}} \| g \|_{L^{q_1}} \| h \|_{L^{p_1}}
\]
\[ \lesssim C(1 + |y(s_1 - s_2)|) \|D_x|^{s_1} f\|_{L^p_x L^q_t}. \]

The same argument yields
\[ (3.4) \quad |\Psi(iy)| \lesssim C(1 + |y(s_1 - s_2)|) \|D_x|^{s_2} f\|_{L^p_x L^q_t}. \]

From (3.3), (3.4) and Hirschmann’s Lemma [13], we obtain (3.2) (see also [28]). □

3.1. Estimates on nonlinearity. In this subsection, we establish an estimate on nonlinearity. For this, we introduce a Lipschitz \( \mu \)

\[ 3.1. \text{Estimates on nonlinearity.} \quad \text{In this subsection, we establish an estimate on nonlinearity.} \quad \text{For this, we introduce a Lipschitz } \mu \text{ norm } (\mu > 0) \text{ as follows. Write } \mu = N + \beta \text{ with } N \in \mathbb{Z} \text{ and } \beta \in (0, 1]. \quad \text{For a function } G : \mathbb{C} \to \mathbb{C}, \text{ we define} \]

\[ \|G\|_{\text{Lipd}} := \sum_{j=0}^{N} \sup_{z \in [0,1]} \frac{|G^{(j)}(z)|}{|z|^{N-j}} + \sup_{x \neq y} \frac{|G^{(N)}(x) - G^{(N)}(y)|}{|x-y|^{\beta}}. \]

where \( G^{(j)} \) is \( j \)-th derivative of \( G \). We say \( G \in \text{Lipd} \) if \( G \in C^N(\mathbb{R}) \) and \( \|G\|_{\text{Lipd}} < \infty \).

The main estimates of this subsection is as follows:

**Lemma 3.4.** Suppose that \( G(z) \in \text{Lip}_{\alpha} \) for some 21/5 < \( \alpha < 23/3 \). Let \((s, r)\) be a pair which is acceptable and conjugate-acceptable. Then, the following two assertions hold:

(i) If \( u \in S(I; \frac{a-1}{2}) \cap X(I; s, r) \) then \( G(u) \in Y(I; s, r) \). Moreover, there exists a constant \( C \) such that

\[ \|G(u)\|_{Y(I; s, r)} \lesssim C \|u\|_{S(I; \frac{a-1}{2})} \|u\|_{X(I; s, r)} \]

for any \( u \in S(I; \frac{a-1}{2}) \cap X(I; s, r) \).

(ii) There exists a constant \( C \) such that

\[ \|G(u) - G(v)\|_{Y(I; s, r)} \lesssim C \|u\|_{X(I; s, r)} \]

\[ \lesssim C(\|u\|_{X(I; s, r)} + \|v\|_{X(I; s, r)}) \]

\[ \times \|u\|_{S(I; \frac{a-1}{2})} + \|v\|_{S(I; \frac{a-1}{2})} \|u-v\|_{S(I; \frac{a-1}{2})} \]

\[ + C(\|u\|_{S(I; \frac{a-1}{2})} + \|v\|_{S(I; \frac{a-1}{2})})^{\alpha-1} \|u-v\|_{X(I; s, r)} \]

for any \( u, v \in S(I; \frac{a-1}{2}) \cap X(I; s, r) \).

**Remark 3.5.** It is easy to see that \( |z|^\alpha z \in \text{Lip}_{\alpha} \). The validity of the above lemma is all assumption on the nonlinearity that we need. Hence, the all results of this article hold for an equation with generalized nonlinearity \( \partial_t u + \partial_x^3 u = \partial_x(G(u)) \), provided \( G(z) \in \text{Lip}_{\alpha} \).

To prove the above lemma, we recall the following two lemmas.

**Lemma 3.6.** Let \( I \) be an interval. Assume that \( s \geq 0 \). Let \( p, q, p_i, q_i, \in (1, \infty) \) \( (i = 1, 2, 3, 4) \). Then, we have

\[ \|D_x|^{s} (fg)\|_{L^p_x L^q_t(I)} \lesssim \]

\[ C(\|D_x|^{s} f\|_{L^p_x L^{p_1}_t(I)} \|g\|_{L^{q_2}_x L^{q_2}_t(I)} + \|f\|_{L^p_x L^{q_3}_t(I)} \|D_x|^{s} g\|_{L^p_x L^{q_4}_t(I)}) \]
provided that
\[ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}, \]
where the constant $C$ is independent of $I$ and $f$.

**Proof of Lemma 3.6.** If $s \in \mathbb{Z}$ then (classical) Leibniz’ rule, Hölder’s inequality, and Lemma 3.3 give us the result. By a similar argument, it suffices to consider the case $0 < s < 1$ to handle the general case. However, that case follows from [17, Theorem A.8] and Lemma 3.3. □

**Lemma 3.7.** Suppose that $\mu > 1$ and $s \in (0, \mu)$. Let $G \in \text{Lip}_\mu$. If $p, p_1, p_2, q, q_1, q_2 \in (1, \infty)$ satisfies
\[ \frac{1}{p} = \frac{\mu - 1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{\mu - 1}{q_1} + \frac{1}{q_2}, \]
then there exists a positive constant $C$ depending on $\mu, s, p_1, p_2, q_1, q_2$ and $I$ such that
\[ \|D_x|^s G(f)\|_{L^p L^q(I)} \leq C \|G\|_{\text{Lip}_\mu} \|f\|_{L^{p_1}_t L^{q_1}(I)}^\mu \|D_x|^s f\|_{L^{p_2}_t L^{q_2}(I)}^\mu \]
holds for any $f$ satisfying $f \in L^{p_1}_t L^{q_1}(I)$ and $|D_x|^s f \in L^{p_2}_t L^{q_2}(I)$.

Although Lemma 3.7 is essentially the same as [17, Theorem A.6], we give the proof of this lemma in Appendix A for self-containedness and in order to clarify the necessity of the assumption $G \in \text{Lip}_\mu$.

**Proof of Lemma 3.4.** We prove the second assertion since the first immediately follows from the second by letting $v = 0$. For simplicity, we name $S = S(I; \frac{\alpha - 1}{2}), L = X(I; s, r)$, and $N = Y(I; s, r)$.

Let us write
\[ G(u) - G(v) = (u - v) \int_0^1 G'(\theta u + (1 - \theta)v) d\theta. \]

Lemma 3.6 implies that
\[ \|G(u) - G(v)\|_N \leq C \|u - v\|_S \int_0^1 \|D_x|^s \{G'((\theta u + (1 - \theta)v)\} \|_{L^{p_1}_t L^{q_1}(I)} d\theta \]
\[ + C \|u - v\|_L \int_0^1 \|\{G'(\theta u + (1 - \theta)v)\} \|_{L^{p_2}_t L^{q_2}(I)} d\theta \]
\[ =: I_1 + I_2, \]
where
\[ \left( \frac{1}{p_1}, \frac{1}{q_1} \right) = \left( \frac{1}{\tilde{p}(s, r)}, \frac{1}{\tilde{q}(s, r)} \right) - \left( \frac{1}{p(0, \frac{\alpha - 1}{2})}, \frac{1}{q(0, \frac{\alpha - 1}{2})} \right) = (\alpha - 2) \left( \frac{1}{p(0, \frac{\alpha - 1}{2})}, \frac{1}{q(0, \frac{\alpha - 1}{2})} \right) + \left( \frac{1}{p(s, r)}, \frac{1}{q(s, r)} \right) \]
and
\[ \left( \frac{1}{p_2}, \frac{1}{q_2} \right) = \left( \frac{1}{\tilde{p}(s, r)}, \frac{1}{\tilde{q}(s, r)} \right) - \left( \frac{1}{p(s, r)}, \frac{1}{q(s, r)} \right). \]
Collecting the above inequalities, we obtain the result. □

It is easy to see that \( \|G'\|_{\text{Lip}(\alpha-1)} \leq \|G\|_{\text{Lip}^\alpha} < +\infty \). By definition of \( \|\cdot\|_{\text{Lip}(\alpha-1)} \), we estimate \( I_2 \) as

\[
I_2 \leq C\|u - v\|_L \|G'\|_{\text{Lip}(\alpha-1)} \int_0^1 \|\theta u + (1 - \theta)v\|^{\alpha-1}_{L^p} L^{\alpha}\| \,d\theta
\]

\[
\leq C\|u - v\|_L \int_0^1 (\|u\|_S + \|v\|_S)^{\alpha-1} \,d\theta
\]

\[
\leq C(\|u\|_S + \|v\|_S)^{\alpha-1}\|u - v\|_L.
\]

On the other hand, we see from Lemma 3.7 that

\[
\|D_x^\alpha\{G'\theta u + (1 - \theta)v\}\|_{L^p\|L^\alpha}\]

\[
\leq C\|G'\|_{\text{Lip}(\alpha-1)} \|\theta u + (1 - \theta)v\|^{\alpha-2}_{L^p} \|\theta u + (1 - \theta)v\|_L
\]

for any \( \theta \in (0, 1) \). Hence, we find the following estimate on \( I_1 \);

\[
I_1 \leq C\|u - v\|_S \|G'\|_{\text{Lip}(\alpha-1)} (\|u\|_S + \|v\|_S)^{\alpha-2}(\|u\|_L + \|v\|_L).
\]

Collecting the above inequalities, we obtain the result. □

4. PROOF OF MAIN THEOREMS

In this section, we prove the main theorems. Recall the notation \( S(I; r) = X(I; 0, r) \). Now, take a number \( s_L(\alpha) \) so that a pair \( (s_L(\alpha), \frac{\alpha-1}{2}) \) is acceptable and conjugate-acceptable. We denote \( L(I; \frac{\alpha-1}{2}) = X(I; s_L(\alpha), \frac{\alpha-1}{2}) \) and \( N(I; \frac{\alpha-1}{2}) = Y(I; s_L(\alpha), \frac{\alpha-1}{2}) \).

Remark 4.1. If \( 27/7 < \alpha < 23/3 \) then \( s_L(\alpha) \) with the above property exists. Indeed, \( s_L(\alpha) = \frac{\alpha}{4} - \frac{1}{\alpha-1} \) works. Our upper bound on \( \alpha \) comes from this point.

4.1. Local well-posedness in a scale-critical space. Let us prove Theorem 1.2. To prove this theorem, we show the following lemma.

Lemma 4.2. Assume \( 21/5 < \alpha < 23/3 \) and \( u_0 \in \dot{L}^{\frac{\alpha-1}{2}}_x \). Let \( t_0 \in \mathbb{R} \) and \( I \) be an interval with \( t_0 \in I \). Then, there exists a universal constant \( \delta > 0 \) such that, if a tempered distribution \( u_0 \) and an interval \( I \ni t_0 \) satisfy

\[
\varepsilon = \varepsilon(I; u_0, t_0) := \left\|e^{-(t-t_0)\partial_x^2}u_0\right\|_{S(I; \frac{\alpha-1}{2})} + \left\|e^{-(t-t_0)\partial_x^2}u_0\right\|_{L(I; \frac{\alpha-1}{2})} \leq \delta,
\]

then there exists a unique solution \( u \in C(I; \dot{L}^{\frac{\alpha-1}{2}}_x) \) to the following initial value problem

\[
\begin{cases}
\partial_t u + \partial_x^2 u = \mu \partial_x (|u|^{\alpha-1} u), & t, x \in \mathbb{R}, \\
u(t_0, x) = u_0(x), & x \in \mathbb{R}
\end{cases}
\]

(in the sense of corresponding integral equation) and satisfies

\[
\|u\|_{S(I; \frac{\alpha-1}{2})} + \|u\|_{L(I; \frac{\alpha-1}{2})} \leq 2\varepsilon.
\]

\[
\|u\|_{S(I; \frac{\alpha-1}{2})} + \|u\|_{L(I; \frac{\alpha-1}{2})} \leq 2\varepsilon.
\]
If \( u_0 \in \dot{L}^{\frac{\alpha-1}{2}} \), in addition, then

\[
\|u\|_{L^\infty(I; \dot{L}^{\frac{\alpha-1}{2}})} \leq \|u_0\|_{\dot{L}^{\frac{\alpha-1}{2}}} + C\varepsilon^\alpha
\]

holds for some constant \( C > 0 \) and \( u \) belongs to all \( \dot{L}^{\frac{\alpha-1}{2}} \)-admissible space \( X(I; s, \frac{\alpha-1}{2}) \).

**Proof of Lemma 4.2.** For \( R > 0 \), define a complete metric space

\[
Z_R = \left\{ u \in L(I; \alpha-1 \leftarrow 2) \cap S\left(I; \alpha-1 \leftarrow 2\right); \|u\|_Z \leq R \right\},
\]

\[
\|u\|_Z := \|u\|_{L(I; \alpha-1 \leftarrow 2)} + \|u\|_{S(I; \alpha-1 \leftarrow 2)}, \quad d_Z(u, v) := \|u - v\|_Z.
\]

For given tempered distribution \( u_0 \) with \( e^{-(t-t_0)\partial_x^3}u_0 \in Z_\delta \) and \( v \in Z_R \), we denote

\[
\Phi(v)(t) := e^{-(t-t_0)\partial_x^3}u_0 + \mu \int_{t_0}^t e^{-(t-t')\partial_x^3}\partial_x(|v|^{\alpha-1}v)(t')dt'.
\]

We show that there exist \( \delta > 0 \) such that \( \Phi : Z_{2\varepsilon} \to Z_{2\varepsilon} \) is a contraction map for any \( 0 < \varepsilon \leq \delta \).

To this end, we prove that there exist constants \( C_1, C_2 > 0 \) such that for any \( u, v \in Z_R \),

\[
\|\Phi(u)\|_Z \leq \|e^{-(t-t_0)\partial_x^3}u_0\|_Z + C_1 R^{\alpha}, \quad (4.1)
\]

\[
d_Z(\Phi(u), \Phi(v)) \leq C_2 R^{\alpha-1} d_Z(u, v). \quad (4.2)
\]

Let \( u \in Z_R \). We infer from Proposition 3.2 (ii) that

\[
\|\Phi(u)\|_Z \leq \|e^{-(t-t_0)\partial_x^3}u_0\|_Z + C\||u|^{\alpha-1}u\|_{N(I; \alpha-1 \leftarrow 2)}.
\]

We then apply Lemma 3.3 (i) with \( r = \frac{\alpha-1}{2} \) and \( s = s_L(0) \) to obtain (4.1). A similar argument shows (4.2). We just employ Lemma 3.4 (ii) instead.

Now let us choose \( \delta > 0 \) so that

\[
C_1(2\delta)^{\alpha-1} \leq \frac{1}{2}, \quad C_2(2\delta)^{\alpha-1} \leq \frac{1}{2}.
\]

Then, we conclude from (4.1), (4.2), and the smallness assumption that \( \Phi \) is a contraction map on \( Z_{2\varepsilon} \). Therefore, the Banach fixed point theorem ensures that there exists a unique solution \( u \in Z_{2\varepsilon} \) to (4.1).

We now suppose that \( u_0 \in \dot{L}^{\frac{\alpha-1}{2}} \). By means of Proposition 3.2, we have

\[
\|u\|_{L^\infty(I; \dot{L}^{\frac{\alpha-1}{2}})} \leq \|u_0\|_{\dot{L}^{\frac{\alpha-1}{2}}} + C\varepsilon^\alpha
\]

as in (4.1). The same argument shows \( u \in X(I; s, \frac{\alpha-1}{2}) \) for any \( s \) such that \( (s, \frac{\alpha-1}{2}) \) is acceptable. \( \Box \)

**Proof of Theorem 1.2.** By Lemma 4.2, we obtain a unique solution

\[
u \in L_t^\infty([-T, T]; \dot{L}^{\frac{\alpha-1}{2}}) \cap S([-T, T]; \frac{\alpha-1}{2}) \cap L([-T, T]; \frac{\alpha-1}{2})
\]

for small \( T = T(u_0) > 0 \). We repeat the above argument to extend the solution, and then obtain a solution which has a maximal lifespan. The regularity property 1.1 and the continuous dependence of solution on the initial data are shown by a usual way. This completes Theorem 1.2. \( \Box \)
4.2. Blowup criterion and scattering criterion. In this subsection we prove Theorems 1.7, 1.8 and 1.9

Proof of Theorem 1.8 Assume for contradiction that $T_{\text{max}} < \infty$ and $\|u\|_{S((0,T_{\text{max}});\frac{a-1}{2})} < \infty$.

Step 1. We first show that the above assumption yields

$$\|u\|_{L((0,T_{\text{max}});\frac{a-1}{2})} < \infty.$$ 

Fix $T$ so that $0 < T < T_{\text{max}}$. Let $s_L(\alpha)$ be as in the previous section (see Remark 4.1). If we take $\theta \in (0,1)$ so that $(\theta s_L(\alpha), \frac{a-1}{2})$ is conjugate-acceptable then it follows from Proposition 3.4 that

$$\|u\|_{L((0,T];\frac{a-1}{2})} \leq C \|u_0\|_{L^{\frac{a-1}{2}}} + C \|u|^{\alpha-1} u\|_{Y((0,T];\theta s_L(\alpha), \frac{a-1}{2})}.$$ 

Then, Lemma 3.4(i) with $r = \frac{a-1}{2}$ and Lemma 3.3 give us

$$\|u\|_{L((0,T];\frac{a-1}{2})} \leq C \|u_0\|_{L^{\frac{a-1}{2}}} + C \|u\|_{S((0,T];\frac{a-1}{2})}^\theta \|u\|_{L((0,T];\frac{a-1}{2})}^\theta.$$ 

By assumption,

$$\|u\|_{S((0,T];\frac{a-1}{2})} \leq \|u\|_{S((0,T_{\text{max}});\frac{a-1}{2})} < +\infty$$

for any $T \in (0,T_{\text{max}})$. Plugging this to the previous estimate, we see that there exist constants $A, B > 0$ such that

$$\|u\|_{L((0,T];\frac{a-1}{2})} \leq A + B \|u\|_{L((0,T];\frac{a-1}{2})}^\theta$$

for any $T \in (0,T_{\text{max}})$, which gives us the desired bound since $\theta < 1$.

Step 2. Let $t_0 \in (0,T_{\text{max}})$. Since

$$u(t) = e^{-(t-t_0)\partial_x^3} u(t_0) + \mu \int_{t_0}^t e^{-(t-t')\partial_x^3} \partial_x (|u|^{\alpha-1} u)(t') dt'$$

for $t \in (0,T_{\text{max}})$, the above estimates yield the following bound on $e^{-(t-t_0)\partial_x^3} u_0$:

$$\left\| e^{-(t-t_0)\partial_x^3} u(t_0) \right\|_{S((t_0,T_{\text{max}}];\frac{a-1}{2})} \leq \left\| u \right\|_{S((t_0,T_{\text{max}}];\frac{a-1}{2})} \left\| u \right\|_{L((t_0,T_{\text{max}}];\frac{a-1}{2})} \leq \frac{\delta}{2}.$$ 

Hence, one can take $\tau > 0$ so that

$$\left\| e^{-(t-t_0)\partial_x^3} u(t_0) \right\|_{S((t_0,T_{\text{max}}+\tau];\frac{a-1}{2})} \leq \delta.$$ 

Then, just as in the proof of Theorem 1.7 (or Lemma 4.2), we can construct a solution $u(t)$ to (1.1) in the interval $(-T_{\text{min}}, T_{\text{max}}+\tau)$, which contradicts to the definition of $T_{\text{max}}$. □
Proof of Theorem 1.9. We first assume that $T_{\text{max}} = +\infty$ and $\|u\|_{S([0, \infty); \frac{a-1}{2})} < \infty$. Then, as in the first step of the proof of Proposition 1.8, one obtains $\|u\|_{L((0, \infty); \frac{a-1}{2})} < \infty$. Since $\{e^{-\frac{\alpha}{2}t}\}_{t \in \mathbb{R}}$ is isometry in $L^2$, it suffices to show that $\{e^{\frac{\alpha}{2}t}u(t)\}_{t \in \mathbb{R}}$ is a Cauchy sequence in $L^2$ as $t \to \infty$. Let $0 < t_1 < t_2$. By an argument similar to the proof of (4.2), we obtain

$$
\left\|e^{\frac{\alpha}{2}t_1}u(t_1) - e^{\frac{\alpha}{2}t_2}u(t_2)\right\|_{L^2} \leq C\|u\|_{N([t_1, \infty); \frac{a-1}{2})}
$$

where $u = \lim_{t \to \infty} e^{\frac{\alpha}{2}t}u(t) \in L^2$ and $\delta$ is the constant given in Lemma 4.2. Moreover, it holds for sufficiently large $t_0 \in [T, \infty)$ that

$$
\left\|e^{\frac{\alpha}{2}t_0}(e^{\frac{\alpha}{2}t_1}u(t_1) - u_+)\right\|_{L^2} + \left\|e^{\frac{\alpha}{2}t_0}(e^{\frac{\alpha}{2}t_2}u(t_2) - u_+)\right\|_{L^2} \leq C\left\|e^{\frac{\alpha}{2}t_0}u(t_0) - u_+\right\|_{L^2}
$$

by means of (2.2). We then see that

$$
\left\|e^{-(t-t_0)}\partial_x u(t_0)\right\|_{L^2} + \left\|e^{-(t-t_0)}\partial_x u(t_0)\right\|_{L^2} \leq \delta.
$$

Then, Lemma 4.2 implies that $\|u\|_{S([T, \infty); \frac{a-1}{2})} \leq 2\delta$.

□

Proof of Theorem 1.7. By (2.2), we have

$$
\left\|e^{-\frac{\alpha}{2}t_0}u_0\right\|_{L^2} + \left\|e^{-\frac{\alpha}{2}t_0}u_0\right\|_{S([0, \frac{a-1}{2})} \leq C\varepsilon.
$$

Then, in light of Lemma 4.2 we see that $u$ exists globally in time and satisfies $\|u\|_{S} \leq 2C\varepsilon$, provided $\varepsilon$ is small compared with the constant $\delta$ given in Lemma 4.2. Proposition 1.9 ensures that $u$ scatters for both time direction. □

4.3. Persistence of regularity. In this subsection, we prove Theorems 1.4 and then 1.10.

Proof of Theorem 1.4. Let us prove that $u \in L(I; \frac{a-1}{2})$. As in the proof of Lemma 4.2 one deduces from Proposition 3.2 and Lemma 3.4 (i) that

$$
\|u\|_{L(I; \frac{a-1}{2})} \leq C\|u_0\|_{L^2} + C\|u\|_{N(I; \frac{a-1}{2})} \leq C\|u_0\|_{L^2} + C\|u\|_{S(I; \frac{a-1}{2})} \cdot
$$

Since we already know $\|u\|_{S(I; \frac{a-1}{2})} < \infty$ by assumption, we have the desired bound

$$
\|u\|_{L(I; \frac{a-1}{2})} \leq 2C\|u_0\|_{L^2}.
$$
for sufficiently short interval $I$. Then, again by Proposition 3.2
\[ \|u\|_{L^\infty(I; L^\infty_x)} \leq C \|u\|_{L^\infty_x} + C \|u\|_{S(I; \frac{\alpha - 1}{2})} \|u\|_{L(I; \frac{\alpha - 1}{2})} < +\infty \]
for any acceptable pair $(s, \frac{\alpha - 1}{2})$. Finite time use of this argument yields the result. □

Proof of Theorem 1.5. Let $0 < \sigma < \alpha$. Take a number $\varepsilon$ so that $0 < \varepsilon < \min(1, \alpha - \sigma)$. Since $|D_x|^\sigma$ commutes with $e^{-t0_2}$ and since $(\varepsilon, 2)$ is acceptable and conjugate-acceptable, we see from Proposition 3.2 that
\[ \|D_x|^\sigma u(t)\|_{X(I; \varepsilon, 2)} \leq C \|D_x|^\sigma u_0\|_{L^2} + C \|D_x|^\sigma (|u|^{\alpha - 1} u)\|_{Y(I; \varepsilon, 2)} \cdot \]
Since $\sigma + \varepsilon < \alpha$, arguing as in the proof of Lemma 3.4, one sees that
\[ \|D_x|^\sigma (|u|^{\alpha - 1} u)\|_{Y(I; \varepsilon, 2)} = \|D_x|^\sigma (|u|^{\alpha - 1} u)\|_{L^p_x(I; L^q_x)} \leq C \|u\|_{S(I; \frac{\alpha - 1}{2})} \|D_x|^\sigma u\|_{X(I; \varepsilon, 2)} \cdot \]
Hence, we obtain an upper bound for $\|D_x|^\sigma u\|_{X(I; \varepsilon, 2)}$ for a small interval. Then, the result follows as in Proposition 1.4.

Next, let $-1 < \sigma < 0$. Set $\varepsilon = -\sigma \in (0, 1)$. As in the previous case, we have
\[ \|D_x|^\sigma u(t)\|_{X(I; \varepsilon, 2)} \leq C \|D_x|^\sigma u_0\|_{L^2} + C \|D_x|^\sigma (|u|^{\alpha - 1} u)\|_{Y(I; \varepsilon, 2)} \]
since $(\varepsilon, 2)$ is acceptable and conjugate-acceptable. Then,
\[ \|D_x|^\sigma (|u|^{\alpha - 1} u)\|_{Y(I; \varepsilon, 2)} = \|u|^{\alpha - 1} u\|_{L^p_x(I; L^q_x)} \leq C \|u\|_{S(I; \frac{\alpha - 1}{2})} \|D_x|^\sigma u\|_{X(I; \varepsilon, 2)} \cdot \]
by Hölder’s inequality. The rest of the argument is the same. □

Remark 4.3. In the above proposition, the upper bound $s < \alpha$ is natural in view of the regularity which the nonlinearity $|u|^{\alpha - 1} u$ possesses. When $\alpha$ is an odd integer, that is, if $\alpha = 5, 7$, then the nonlinearity $u^5$ or $u^7$ are analytic (in $u$) and so we can remove the upper bound and treat all $s > 0$. We omit the details.

Remark 4.4. By modifying the proof of Theorem 1.5, we easily reproduce the local well-posedness in $H^\infty$ for $\alpha \geq 5$. More precisely, by Lemma 5.5,
\[ \|u\|_{S(I; \frac{\alpha - 1}{2})} \leq \|D_x|^{s_\alpha} u\|_{X(I; -1/4, 2)} \leq C \|D_x|^{s_\alpha - \frac{5\alpha - 13}{2(\alpha - 1)} u\|_{L^\infty_x} \leq C \|D_x|^{s_\alpha} u\|_{X(I; \frac{5\alpha - 13}{2(\alpha - 1)} \frac{5\alpha - 13}{\alpha - 1}} \cdot \]
By Sobolev’s embedding in space and Minkowski’s inequality,
\[ \|D_x|^{2(\alpha - 1)} u\|_{L^\infty_x} \leq C \|D_x|^{s_\alpha - \frac{5\alpha - 33}{2(\alpha - 1)} u\|_{L^\infty_x} \leq C \|D_x|^{s_\alpha} u\|_{X(I; \frac{5\alpha - 13}{2(\alpha - 1)} \frac{5\alpha - 13}{\alpha - 1}} \cdot \]
Hence, estimating as in the proof of Theorem 3.5, we obtain a closed estimate in $|D_x|^{-\alpha_0} X(I; \varepsilon, 2) \cap |D_x|^{-\alpha_0} X(I; -\frac{1}{4} + \frac{5}{30 - 13\eta_2}, 2) \cap |D_x|^{-\alpha_0} X(I; -\frac{1}{4}, 2)$, which yields local well-posedness in $H^{\alpha_0}$.  

We finally prove Theorem 1.10.

**Proof of Theorem 1.10.** We suppose for contradiction that $u(t)$ scatters to $u_+ \in \dot{L}^{\frac{3}{2}}$ as $t \to \infty$. Since $u_0 \in H^1$, Theorems 1.3 and 1.5 imply that $u(t) \in C(\mathbb{R}; H^1)$. Further, $u(t)$ scatters also in $H^1$ and so we see that $\|\partial_x u(t)\|_{L^2} = \|\partial_x e^{i\alpha \partial_x^2} u(t)\|_{L^2} \to \|u_+\|_{H^1}$ as $t \to \infty$.

On the other hand, by the Gagliardo-Nirenberg inequality and mass conservation,

$$\|u(t)\|_{L^{p+1}} \leq C \|u_0\|_{L^2} \|D_x|^{\frac{2}{3(a-1)} - \frac{1}{2}} u(t)\|_{L^2}^{\frac{1}{2} + \frac{1}{a-1}}.$$  

Since $u(t)$ scatters as $t \to \infty$, we see that $u \in X([0, \infty); \frac{2}{3(a-1)} - \frac{1}{2})$ as in the proof of Theorem 1.9. Therefore, we can take a sequence $\{t_n\}_n$ with $t_n \to \infty$ as $n \to \infty$ so that $\|u(t_n)\|_{L^{p+1}} \to 0$ as $n \to \infty$. Thus, by conservation of energy,

$$0 \geq E[u_0] = E[u(t_n)] = \frac{1}{2} \|\partial_x u(t_n)\|_{L^2}^2 - \frac{\mu}{\alpha + 1} \|u(t_n)\|_{L^{p+1}}^{p+1} - \frac{1}{2} \|u_+\|_{H^1}^2,$$

as $n \to \infty$. Hence, $E[u_0] < 0$ yields a contradiction. If $E[u_0] = 0$ then we see that $u_+ = 0$, and so that $\|u_0\|_{L^2} = \|u_+\|_{L^2} = 0$. This contradicts to $u_0 \neq 0$.  

---

**APPENDIX A. PROOF OF LEMMA 3.7**

In this appendix we prove Lemma 3.7. To prove this lemma, we need the following space-time bounds of the maximal function

$$(M u)(x) = \sup_{R > 0} \frac{1}{2R} \int_{x-R}^{x+R} |u(y)| dy.$$  

**Lemma A.1.** Let $I$ be an interval. Assume $1 < p, q < \infty$.

(i) There exists a positive constant $C$ depending on $p, q$ and $I$ such that

$$(A.1) \quad \|M f\|_{L^p L^q(I)} \leq C \|f\|_{L^p L^q(I)}$$

for any $f \in L^p_x L^q_t(I)$.

(ii) There exists a positive constant $C$ depending on $p, q$ and $I$ such that

$$(A.2) \quad \|M f_k\|_{L^p_x L^q_t(I)} \leq C \|f_k\|_{L^p_x L^q_t(I)}$$

for any $\{f_k\}_k \in L^p_x L^q_t(I)$.

---

3 Strictly speaking, we should work with pairs $(-\frac{1}{4} + \eta_1, 2)$ and $(-\frac{1}{4} + \frac{5}{30 - 13\eta_2}, 2)$ for small $\eta_1 = \eta_1(\alpha) > 0$ because the critical case $q(-1/4, 2) = \infty$ is excluded in Lemma 3.5. However, the modification is obvious.
Proof of Lemma A.1. See [9] for (A.1) and [17] Lemma A.3 (c) for (A.2).

Proof of Lemma 3.7. We follow [27] (see also [26]). Let \( \{ \varphi_k(D_x) \} \) be a Littlewood-Paley decomposition with respect to \( x \) variable. From [17] Lemma A.3, we see

\[
\| D_x |^s f \|_{L^p_x L^q_t L^r} \sim \left\| 2^{sk} \varphi_k(D_x) f \right\|_{L^p_x L^q_t L^r}.
\]

**Step 1.** Write \( G(z) = \sum_{l=0}^{N-1} \frac{G^{(l)}(a)}{l!} (z-a)^l + \int_a^z \frac{(z-v)^{N-1}}{(N-1)!} G^{(N)}(v) dv \).

\[
= N \sum_{l=0}^{N-1} \frac{G^{(l)}(a)}{l!} (z-a)^l + \int_a^z \frac{(z-v)^{N-1}}{(N-1)!} (G^{(N)}(v) - G^{(N)}(a)) dv.
\]

Hence, applying the above expansion with \( z = f(y) \) and \( a = f(x) \),

\[
\mathcal{F}^{-1} [ \varphi_k \mathcal{F} G(f)](x)
\]

\[
= c \int_{\mathbb{R}^n} (\mathcal{F}^{-1} \varphi_k)(x-y) G(f(y)) dy
\]

\[
= c \sum_{l=0}^{N} \frac{1}{(l-j)!} \int_{\mathbb{R}^n} (\mathcal{F}^{-1} \varphi_k)(x-y) (f(y)^j) dy
\]

\[
+ c \int_{\mathbb{R}^n} (\mathcal{F}^{-1} \varphi_k)(x-y) \int_{f(x)}^{f(y)} \frac{(f(y)-v)^{N-1}}{(N-1)!} (G^{(N)}(v) - G^{(N)}(f(x))) dv dy
\]

\[
=: T_{1,k} + T_{2,k}.
\]

We first estimate \( T_{1,k} \). Since \( \int \mathcal{F}^{-1} \varphi_k(y) dy = \varphi_k(0) = 0 \), the summand in \( T_{1,k} \) vanishes if \( j = 0 \). By the estimate

\[
|G^{(l)}(f(x))| \leq \| G \|_{Lips} |f(x)|^{\mu - l},
\]

we have

\[
\left\| 2^{sk} T_{1,k} \right\|_{L^p_x L^q_t L^r_k} \leq C \| G \|_{Lips} \sum_{j=1}^{N} \left\| \int_{f(x)}^{f(y)} (f(y)-v)^{N-1} (G^{(N)}(v) - G^{(N)}(f(x))) dv \right\|_{L^p_x L^q_t L^r_k}
\]

\[
\leq C \| G \|_{Lips} \sum_{j=1}^{N} \| f \|_{L^p_x L^q_t L^r_k} \left\| 2^{sk} \varphi_k(D_x) f \right\|_{L^p_x L^q_t L^r_k}.
\]

where

\[
\frac{1}{p} = \frac{\mu - j}{p_1} + \frac{1}{p_{2,j}}, \quad \frac{1}{q} = \frac{\mu - j}{q_1} + \frac{1}{q_{2,j}}.
\]
Further, a recursive use of Lemma 3.6 yield
\[ \|D_x^j\mu(f_j^i)\|_{L^{p_j, q_j}_{t,x}} \leq C_j \|f\|_{L^{p_0, q_0}_{t,x}}^{j-1} \|D_x^j f\|_{L^{p_j, q_j}_{t,x}} \]
for \( j \geq 2 \), which completes the estimate of \( T_{1,k} \).

Next, we estimate \( T_{2,k} \). First note that
\[ \left| \int_{f(x)}^{f(y)} \frac{(f(y) - f(x))^{N-1}}{(N-1)!} (G(N)(v) - G(N)(f(x))) dv \right| \leq C \|G\|_{\text{Lip}} |f(x) - f(y)|^\mu \]
by definition of \( \|G\|_{\text{Lip}} \). Further, for any \( M > 0 \), there exists \( C_M \) such that
\[ |(\mathcal{F}^{-1} \varphi_k)(x - y)| = 2^k |(\mathcal{F}^{-1} \varphi_0)(2^k(x - y))| \leq C_M 2^k (1 + 2^k |x - y|)^{-M}. \]
Therefore,
\[ |T_{2,k}| \leq C 2^k \|G\|_{\text{Lip}} \mu \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^\mu}{(1 + 2^k |x - y|)^M} dy \]
\[ \leq C \sum_{l=0}^{\infty} 2^{k-lM} (I_k^l f)(x), \]
where
\[ I_k^l f(x) = \int_{|z| \leq 2^{-k}} |f(x + z) - f(x)|^\mu dz. \]

We now claim that
\[ (A.5) \quad \left\| 2^{k(s+1)} (I_k^l f) \right\|_{L^p_t L^q_x} \leq C \left\| D_x^{s/\mu} f \right\|_{L^p_t L^q_x}. \]
This claim completes the proof. Indeed, combining the above estimates, we see that
\[ \left\| 2^{k(s+1)} T_{2,k} \right\|_{L^p_t L^q_x} \leq C \sum_{l=0}^{\infty} 2^{l(s-M+1)} \left\| 2^{k(n+s)} (I_k^l f) \right\|_{L^p_t L^q_x} \leq C \left\| D_x^{s/\mu} f \right\|_{L^p_t L^q_x}, \]
provided we choose \( M > s + 1 \). By Lemma 3.3 we conclude that
\[ \left\| D_x^{s/\mu} f \right\|_{L^p_t L^q_x} \leq \left\| f \right\|_{L^p_t L^q_x} \left\| D_x^{s/\mu} f \right\|_{L^p_t L^q_x} \left\| D_x^{s/\mu} f \right\|_{L^p_t L^q_x}. \]

**Step 2.** We prove claim (A.5). Let \( \Delta_h \) be a difference operator \( \Delta_h f(x) = f(x + h) - f(x) \). Since \( f = \sum_{m \in \mathbb{Z}} \varphi_{k+m}(D_x) f \) for any \( k \in \mathbb{Z} \), one sees that
\[ \left\| 2^{k(s+1)} (I_k^l f)(x) \right\|_{L^p_t L^q_x} = \left\| 2^{k} \int_{|z| \leq 1} |\Delta_{2^{-k}z} f(x)|^\mu dz \right\|_{L^p_t L^q_x} \]
\[ \leq \left\| 2^{k} \int_{|z| \leq 1} |\Delta_{2^{-k}z} \sum_{m=-\infty}^{-1} \varphi_{k+m}(D) f(x)|^\mu dz \right\|_{L^p_t L^q_x} \]
\[ + \left\| 2^{k} \int_{|z| \leq 1} |\Delta_{2^{-k}z} \sum_{m=0}^{\infty} \varphi_{k+m}(D) f(x)|^\mu dz \right\|_{L^p_t L^q_x} =: A + B. \]
We estimate $A$. Take $a \in (1/\mu, 1)$. Let $k \in \mathbb{Z}$. If $m < 0$ and $|h| \leq 2^{-k}$ then we have
\[
|\Delta_h \mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f](x)| \lesssim |h| |\nabla(\mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f])(x + \theta h)|
\]
\[
\lesssim 2^m \sup_{|y| \leq 2^{-k}} |(\nabla \mathcal{F}^{-1}[\varphi_0 \mathcal{F}[f(\frac{x}{2^{k+m}})]]) (2^{k+m}(x - y))|
\]
\[
\lesssim C_a 2^m \sup_{y \in \mathbb{R}} \frac{|\nabla \mathcal{F}^{-1}[\varphi_{k+m}\mathcal{F}f](x - y)|}{1 + |2^{k+m}y|^a}
\]
for any $x \in \mathbb{R}$, where we have used the estimate
\[
\sup_{y \in \mathbb{R}} \frac{\nabla \mathcal{F}^{-1}[\varphi_0 \mathcal{F}f](x - y)}{1 + |y|^a} \lesssim C \sup_{y \in \mathbb{R}} \frac{|\mathcal{F}^{-1}[\varphi_0 \mathcal{F}f](x - y)|}{1 + |y|^a}
\]
(see [26] Proposition 2.1.6/2 (i)) to obtain the last line. We define the Peetre-Fefferman-Stein maximal function by
\[
\varphi_j^{s,a} f(x) := \sup_{y \in \mathbb{R}} \frac{|\mathcal{F}^{-1}[\varphi_j \mathcal{F}f](x - y)|}{1 + |2^j y|^a}.
\]
By the above estimates, we have
\[
A \lesssim C \left\| 2^{ks} \sum_{m=-\infty}^{-1} |\Delta_{2^{-k}} \varphi_{k+m}(D) f(x)|^\mu \right\|_{L^\mu L^\mu L^\mu_k}
\]
\[
\lesssim C \sum_{m=-\infty}^{-1} 2^{m \mu} \left\| 2^{k m} \varphi_{k+m} f \right\|_{L^\mu L^\mu L^\mu_k}^{\mu}
\]
\[
\lesssim C \sum_{m=-\infty}^{-1} 2^{m(\mu - s)} \left\| 2^{(k+m) \frac{\mu}{\mu + a}} \varphi_{k+m} f \right\|_{L^\mu L^\mu L^\mu_k}^{\mu}
\]
\[
\lesssim C \left\| 2^{k \frac{\mu}{a}} \varphi_{k}^{s,a} f \right\|_{L^\mu L^\mu L^\mu_k}^{\mu},
\]
where we used the fact that $s < \mu$. Since $(\varphi_k^{s,a} f)(x) = (\varphi_{0}^{s,a} (\tilde{\varphi}_k(D_x) f)(\frac{x}{2^k}))(2^k x)$, [33] Lemma 2.3.6] yields
\[
(\varphi_k^{s,a} f)(x) \lesssim C \{ \mathcal{M}[(\tilde{\varphi}_k(D_x) f)^{1/2}] \}^a(x),
\]
where $\tilde{\varphi}_k = \sum_{i=-1}^{1} \varphi_{k+i}$. Since $1/\mu < a < 1$, [A.2], the embedding $\ell^2 \hookrightarrow \ell^q$ ($2 < q \leq \infty$), and [A.3] lead us to
\[
\left\| 2^{k \frac{\mu}{a}} \varphi_k^{s,a} f \right\|_{L^\mu L^\mu L^\mu_k}^{\mu} \lesssim C \left\| 2^{k \frac{\mu}{a}} \mathcal{M}((\tilde{\varphi}_k(D_x) f)^{1/2}) \right\|_{L^\mu L^\mu L^\mu_k}^{a}
\]
\[
\lesssim C \left\| 2^{k \frac{\mu}{a}} (\tilde{\varphi}_k(D_x) f)^{1/2} \right\|_{L^\mu L^\mu L^\mu_k}^{a}
\]
\[
\lesssim C \left\| 2^{k \frac{\mu}{a}} \tilde{\varphi}_k(D_x) f \right\|_{L^\mu L^\mu L^\mu_k}^{a}
\]
\[
\lesssim C \left\| D_x |s/\mu f \right\|_{L^\mu L^\mu_k}^{a}.
\]
Let us proceed to the estimate of $B$. We first note that

$$
\int_{|z|<1} \left| \Delta_{-k} \sum_{m=0}^{\infty} \varphi_{k+m}(D_x) f(x) \right|^{\mu} dz
$$

$$
= \int_{|z|<1} \left| \sum_{m=0}^{\infty} 2^{-\frac{m}{2}k} \Delta_{-k} \varphi_{k+m}(D_x) f(x) \right|^{\mu} dz
$$

$$
\leq C \int_{|z|<1} \sum_{m=0}^{\infty} 2^{\epsilon m} \left| \Delta_{-k} \varphi_{k+m}(D_x) f(x) \right|^{\mu} dz
$$

$$
= C \int_{|z|<1} \sum_{m=0}^{\infty} 2^{\epsilon m} \int_{|x|<1} \left| \Delta_{-k} \varphi_{k+m}(D_x) f(x) \right|^{\mu} dz
$$

$$
\leq C \sum_{m=0}^{\infty} 2^{\epsilon m} \left( \sup_{|z|<1} \left| \Delta_{-k} \varphi_{k+m}(D_x) f(x) \right| \right)^{\mu(1-\lambda)}
$$

$$
\times \int_{|z|<1} \left| \Delta_{-k} \varphi_{k+m}(D_x) f(x) \right|^{\mu \lambda} dz,
$$

where $\lambda \in (0,1)$. For $m \geq 0$ and $|h| \leq 2^{-k}$, the triangle inequality gives us

$$
|\Delta_h F^{-1}[\varphi_{k+m}Ff](x)| \leq 2 \sup_{|y|\leq 2^{-k}} |F^{-1}[\varphi_{k+m}Ff](x-y)|
$$

$$
\leq C 2^{m} \varphi_{k+m}^{1-a} f(x),
$$

where $a \in (1/\mu, 1)$. Further,

$$
\int_{|z|<1} \left| \Delta_{-k} \varphi_{k+m}(D_x) f(x) \right|^{\mu \lambda} dz \leq C \mathcal{M} \left[ \left| \varphi_{k+m}(D_x) f(x) \right|^{\mu \lambda} \right](x).
$$

Plugging these inequality, one deduces from Hölder’s inequality, the embedding $\ell^2 \hookrightarrow \ell^q$ ($2 < q \leq \infty$), (A.2), and (A.3) that

$$
B \leq C \left\| 2^{k} \sum_{m=0}^{\infty} 2^{me} \mathcal{M} \left[ \left| \varphi_{k+m}(D_x) f \right|^{\mu \lambda} \right] 2^{m \mu_\lambda (1-\lambda)/ \mu} \left| \varphi_{k+m}^{s} f \right|^{s \mu \lambda} \right\|_{L^q_x L^q_t \ell^2_k}
$$

$$
\leq C \sum_{m=0}^{\infty} \mathcal{M} \left[ \left| \varphi_{k+m}(D_x) f \right|^{\mu \lambda} \right] \left\| \varphi_{k+m}^{s} f \right\|_{L^q_x L^q_t \ell^2_k}
$$

$$
\leq C \sum_{m=0}^{\infty} \mathcal{M} \left[ \left| \varphi_{k+m}(D_x) f \right|^{\mu \lambda} \right] \left\| \varphi_{k+m}^{s} f \right\|_{L^q_x L^q_t \ell^2_k}
$$

$$
\leq C \sum_{m=0}^{\infty} \mathcal{M} \left[ \left| \varphi_{k+m}(D_x) f \right|^{\mu \lambda} \right] \left\| \varphi_{k+m}^{s} f \right\|_{L^q_x L^q_t \ell^2_k}
$$

$$
\leq C \sum_{m=0}^{\infty} \mathcal{M} \left[ \left| \varphi_{k+m}(D_x) f \right|^{\mu \lambda} \right] \left\| \varphi_{k+m}^{s} f \right\|_{L^q_x L^q_t \ell^2_k}
$$

$$
\leq \left\| \varphi_{k+m}^{s} f \right\|_{L^q_x L^q_t \ell^2_k},
$$

as long as $\epsilon + a\mu(1-\lambda) - s < 0$. Since $a \in (1/\mu, 1)$, we are able to choose $\lambda \in (0,1)$ and $\epsilon > 0$ suitably. Thus, the proof is completed. □
In this appendix we briefly summarize some inclusion relations between $\tilde{L}^r$ and other frequently used spaces such as Lebesgue space or Sobolev space. Here, $\tilde{H}^{s,r} = H^{0,s}(\mathbb{R})$ stands for a weighted $L^2$ space with norm $\|f\|_{\tilde{H}^{s,r}} = \|x|^s f\|_{L^2}$.

**Lemma B.1.** We have the following.

(i) $L^r \hookrightarrow \tilde{L}^r$ if $1 \leq r \leq 2$ and $\tilde{L}^r \hookrightarrow L^r$ if $2 \leq r \leq \infty$.

(ii) $\tilde{H}^{0,\frac{1}{r}} \hookrightarrow \tilde{L}^r$ if $1 < r \leq 2$ and $\tilde{L}^r \hookrightarrow \tilde{H}^{0,\frac{1}{r}}$ if $2 \leq r < \infty$.

(iii) $\tilde{L}^r \hookrightarrow B^{\frac{1}{r}}_{2,\infty}$ if $1 \leq r \leq 2$ and $B^{\frac{1}{r'}}_{2,r'} \hookrightarrow \tilde{L}^r$ if $2 \leq r \leq \infty$.

**Proof of Lemma [B.1].** The first assertion follows from the Hausdorff-Young inequality. The Sobolev embedding (in Fourier side) yields the second. We omit details.

The third is also immediate from the Hölder inequality. Indeed, if $2 \leq r \leq \infty$ then

$$\|f\|_{L^r((2^n \leq |\xi| \leq 2^{n+1}))} \leq C 2^{n(\frac{1}{r} - \frac{1}{2})} \|f\|_{L^2((2^n \leq |\xi| \leq 2^{n+1}))}$$

for any $n \in \mathbb{Z}$. Taking $\ell^r_n$ norm, we obtain the desired embedding. The case $1 \leq r \leq 2$ follows in the same way. □

Let $\tilde{H}^s = \tilde{H}^s(\mathbb{R})$ be a homogeneous Sobolev space with norm $\|f\|_{\tilde{H}^s} = \|\langle\xi\rangle^s \hat{f}\|_{L^2}$. Notice that the above inclusion is the same as for $\tilde{H}^{0,\frac{1}{r}}$. Namely, we can replace $\tilde{L}^r$ with $\tilde{H}^{0,\frac{1}{r}}$ in Lemma [B.1]. However, there is no inclusion between $L^r$ and $\tilde{H}^{0,\frac{1}{r}}$ for $r \neq 2$.

**Lemma B.2.** For $1 \leq r \leq \infty$ ($r \neq 2$), $\tilde{L}^r \not\hookrightarrow \tilde{H}^{0,\frac{1}{r}}$ and $\tilde{H}^{0,\frac{1}{r}} \not\hookrightarrow \tilde{L}^r$.

**Proof of Lemma [B.2].** If $2 < r \leq \infty$, we have the following counter examples: Let us define $f_n(x)$ by $\hat{f}_n(\xi) = 1$ for $n \leq |\xi| \leq n + 1$ and $\hat{f}_n(\xi) = 0$ elsewhere. Then, $f_n(x)$ satisfies $\|f_n\|_{\tilde{H}^{0,\frac{1}{r}}} \rightarrow \infty$ as $n \rightarrow \infty$, while $\|f_n\|_{\tilde{L}^r} = 1$. Hence, $\tilde{L}^r \not\hookrightarrow \tilde{H}^{0,\frac{1}{r}}$. On the other hand, for some $p \in (1/2, 1/r')$, take $g_n(x)$ $(n > 3)$ so that $\hat{g}_n(\xi) = \xi^{-1/r'} |\log \xi|^{-p}$ for $1/n \leq \xi \leq 1/2$ and $\hat{g}_n(\xi) = 0$ elsewhere. Then, $\|g_n\|_{\tilde{H}^{0,\frac{1}{r}}} = 1$ is bounded but $\|g_n\|_{\tilde{L}^r} \rightarrow \infty$ as $n \rightarrow \infty$. This shows $\tilde{H}^{0,\frac{1}{r}} \not\hookrightarrow \tilde{L}^r$.

The case $1 < r < 2$ follows by duality.

Let us consider the case $r = 1$. We note that $\delta_0(x) \in \tilde{L}^1 \setminus \tilde{H}^{-\frac{1}{2}}$, where $\delta_0(x)$ is the Dirac delta function. Therefore, $\tilde{L}^1 \not\hookrightarrow \tilde{H}^{-\frac{1}{2}}$. On the other hand, $f_n(x) = (\log(1 + 1/n))^{-1} F^{-1} [\chi_{\{1 \leq |\xi| \leq 1/n\}}](x)$ is a counter example for $\tilde{H}^{-\frac{1}{2}} \not\hookrightarrow \tilde{L}^1$. □

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