A HIGH-ORDER LAGRANGE-GALERKIN SCHEME FOR A CLASS OF FOKKER-PAMCK EQUATIONS AND APPLICATIONS TO MEAN FIELD GAMES

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Abstract. In this paper we propose a high-order numerical scheme for linear Fokker-Planck equations with a constant diffusion term. The scheme, which is built by combining Lagrange-Galerkin and semi-Lagrangian techniques, is explicit, conservative, consistent, and stable for large time steps compared with the space steps. We provide a convergence analysis for the exactly integrated Lagrange-Galerkin scheme, and we propose an implementable version with inexact integration. Our main application is the construction of a high-order scheme to approximate solutions of time dependent mean field games systems.

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1. Introduction

The Fokker-Planck (FP) equation appears in many areas of interest, including biology [44] and physics [48]. We refer the reader to [5] for systematic account of theory of linear FP equations and their probabilistic interpretation. In this work, the main application that we have in mind concerns the theory of Mean Field Games (MFGs), introduced simultaneously by Lasry-Lions in [32, 33, 34] and by Huang-Caines-Malhamé in [26], which characterizes Nash equilibria of stochastic differential games with an infinite number of indistinguishable players. In some particular cases, the aforementioned equilibria are characterized by a system of parabolic Partial Differential Equations (PDEs) consisting of a Hamilton-Jacobi-Bellman (HJB) equation, with a terminal condition, coupled with a FP equation with an initial condition.

The numerical solution of FP equations has been widely studied. There are several methods (see e.g. [31, 18, 51, 7, 40, 43, 36] and the references therein) based on the popular finite difference scheme proposed by Chang and Cooper in [15], which, in order to be explicit and stable, requires a parabolic CFL condition on the discretization steps. In the framework of MFGs, in [1] and [13] the authors propose a semi-implicit finite difference scheme and a Semi-Lagrangian (SL) type scheme, respectively, to approximate the solutions to FP equations. The scheme proposed in [13], which does not impose a CFL condition and hence allows for large time steps compared to space steps, has been extended in [14] to deal with nonlinear FP equations and in [16] to approximate FP equations with non-local diffusion terms.

The main purpose of this article is to provide a new approximation scheme for FP equations with constant diffusion and smooth drift terms, which is explicit, conservative, consistent, convergent, high-order accurate, meaning an order of convergence larger than two, and stable without requiring a CFL condition between the time and space steps. Furthermore, in the context of MFGs, our aim is to obtain a scheme which, coupled with a high-order SL discretization for the HJB equation, approximates the solution to MFG systems at least second order of accuracy. Inspired by [11] and [14], the scheme that we propose for the FP equation combines SL techniques (see e.g. [8, 13]) to discretize in time the (stochastic) characteristic curves underlying the FP equation, and a Lagrange-Galerkin (LG) type discretization in the space variable (see e.g. [12, 14]). More precisely, the characteristic curves are discretized with a Crank-Nicolson method (see e.g. [28, 49]), as in high-order SL schemes for parabolic equations (see [6]), and we discretize in the space variable by using a LG scheme with a symmetric Lagrangian basis of odd order.

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This last choice is inspired by the results in [21, 22], where the equivalence between SL and LG schemes has been studied, and where symmetric odd basis have shown a good behavior in terms of stability. In order to obtain an implementable scheme, we use a quadrature formula to approximate the integral terms in the LG discretization. Based on the previous scheme, we also propose a high-order accurate scheme to approximate the solutions to a class of second-order MFG systems with constant diffusion. To the best of our knowledge, only a few works deal with this problem. Let us mention [16] and [35], where the authors propose finite difference based second-order accurate methods to approximate MFG systems. In our approach, we first introduce a high-order accurate SL method for the HJB equation, which can be viewed as a nonlinear extension of the scheme considered in [6], and we couple it with our LG scheme for the FP equation. Because of this coupling, the resulting approximation is not explicit and it is solved by fixed point iterations. We numerically show high-order accuracy of the scheme by considering two MFG systems. The first one is a linear-quadratic MFG with non-local couplings (see e.g. [3]), for which we are able to compute its analytical solution, and the second one, taken from [46], is a MFG with local couplings (see e.g. [10]) and no explicit solution.

The paper is organized as follows. In Section 2, we recall some basic results on the FP equation that we consider, we introduce our scheme and establish its main properties: mass conservation and $L^2$-stability (Theorem 2.2), consistency (Proposition 2.1), and convergence (Theorem 2.3). Section 3 introduces the MFG system and the SL scheme for the HJB equation, for which consistency is shown in Proposition 3.1. Finally, in Section 4 we provide an implementable scheme, which is derived by using a cubic basis and Simpson’s Rule in the LG approximation. The article concludes by showing the performance of the proposed scheme in three examples: a damped and noisy harmonic oscillator, a linear-quadratic MFG with non-local couplings, and a MFG with no explicit solutions and local couplings. In all the examples, an order of accuracy which is mostly between two and three is numerically observed.

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2. A Lagrange-Galerkin Type Scheme for a Fokker-Planck Equation

Let $T > 0$. In the following, given a function $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ and $(t, x) \in (0, T) \times \mathbb{R}^d$, the notation $\nabla u(t, x)$ and $\Delta u(t, x)$ refer to the (weak) gradient and Laplacian of $u$ with respect to the spatial variable $x$. Similarly, given $v : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, the notation $Dv$ and $\text{div}(v)$ refer to the Jacobian matrix and the divergence of $v$ with respect to the space variable, respectively.

In this section, we consider the following linear FP equation

\begin{equation}
\tag{FP}
\begin{aligned}
\partial_t m - \frac{\sigma^2}{2} \Delta m + \text{div}(bm) &= 0 \text{ in } (0, T) \times \mathbb{R}^d, \\
m(0, \cdot) &= m^*_0 \text{ in } \mathbb{R}^d,
\end{aligned}
\end{equation}

where $\sigma \in \mathbb{R} \setminus \{0\}$, $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, and $m^*_0 : \mathbb{R}^d \to \mathbb{R}$.

(H1) We assume that:

(i) $m^*_0$ is nonnegative, continuous, has compact support, and $\int_{\mathbb{R}^d} m^*_0(x) dx = 1$.

(ii) $b$ is bounded, $b \in C([0, T] \times \mathbb{R}^d)$, and there exists $C_b > 0$ such that

$$|b(t, x) - b(t, y)| \leq C_b |x - y|, \quad \text{for } t \in [0, T] \text{ and } x, y \in \mathbb{R}^d.$$ 

In the following result, we summarize some important properties of equation (FP).

**Theorem 2.1.** Assume (H1). Then the following hold:

(i) Equation (FP) admits a unique classical solution $m^* \in C^{1,2}([0, T] \times \mathbb{R}^d)$.

(ii) $m^* \geq 0$.

(iii) $\int_{\mathbb{R}^d} m^*(t, x) dx = 1$ for all $t \in [0, T]$.

(iv) $m^*$ is the unique solution in $L^2([0, T] \times \mathbb{R}^d)$ to (FP) in the distributional sense.
Proof. We refer the reader to [5, Theorem 6.6.1] for the existence result in (i) as well as for the nonnegativity property in (ii). The uniqueness result in (i) and the mass conservation property in (iii) follow from [5, Theorem 9.3.6] and [5, Corollary 6.6.6], respectively. Finally, the proof of (iv) is given in [23, Theorem 4.3].

Let us recall the probabilistic interpretation of the solution \( m^* \) to (FP), which will be useful in order to construct a LG scheme. Let \( W \) be a \( d \)-dimensional Brownian motion defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and let \( Y_0 : \Omega \to \mathbb{R}^d \) be a random variable, independent of \( W \), and whose distribution is absolutely continuous with respect to the Lebesgue measure in \( \mathbb{R}^d \), with density given by \( m_0^* \). Given \((t, x) \in [0, T] \times \mathbb{R}^d\), we define \( Y^{t, x} \) as the unique strong solution to the SDE

\[
dY(s) = b(s, Y(s)) ds + \sigma dW(s) \quad \text{for } s \in (t, T),
\]

\[
Y(t) = x.
\]

Denote by \( \mathbb{E}(X) \) the expectation of a random variable \( X : \Omega \to \mathbb{R} \). Assumption (H1) implies that \( Y^{0, \xi}(t) \) is well defined for all \( t \in [0, T] \) and its distribution is absolutely continuous with respect to the Lebesgue measure in \( \mathbb{R}^d \), with density given by \( m^*(t, \cdot) \) (see e.g. [22]). From the \( \mathbb{P}\)-a.s. equality \( Y^{0, \xi}(s) = Y^{t, Y^{0, \xi}(t)}(s) \) for every \( 0 \leq t \leq s \leq T \), we deduce that for every continuous and bounded function \( \phi : \mathbb{R}^d \to \mathbb{R} \), we have

\[
\int_{\mathbb{R}^d} \phi(x) m^*(s, x) dx = \int_{\mathbb{R}^d} \mathbb{E} \left( \phi(Y^{t, x}(s)) \right) m^*(t, x) dx.
\]

2.1. A space-time Lagrange-Galerkin approximation. Let focus on the numerical approximation of (FP). Notice that if \( b \) is differentiable with respect to the space variable, (FP) can be written as

\[
\partial_t m - \frac{\sigma^2}{2} \Delta m + \langle b, \nabla m \rangle + \text{div}(b)m = 0 \quad \text{in } (0, T) \times \mathbb{R}^d,
\]

\[
m(0, \cdot) = m_0^* \quad \text{in } \mathbb{R}^d.
\]

Using this formulation, a second-order accurate semi-Lagrangian scheme can be derived to approximate \( m^* \) (see e.g. [6]). However, such a scheme is not conservative, i.e. the discrete solution does not satisfy the discrete analogons of Theorem 2.1(iii). The scheme that we consider, which will be built from (2.2), will allow us to preserve this property (see Theorem 2.2(ii) below).

Let us fix \( N_{\Delta t} \in \mathbb{N} \), set \( \mathcal{I}_{\Delta t} = \{0, \ldots, N_{\Delta t}\} \), \( \mathcal{I}_{\Delta t}^* = \mathcal{I}_{\Delta t} \setminus \{N_{\Delta t}\} \), \( \Delta t = T/N_{\Delta t} \), and \( t_k = k\Delta t \ (k \in \mathcal{I}_{\Delta t}) \). For \( k \in \mathcal{I}_{\Delta t}^* \) and \( x \in \mathbb{R}^d \), we denote by \( y^{t_k, x} \) a one-step Crank-Nicolson approximation of \( Y^{t_k, x}(t_{k+1}) \) (see [28, Section 15.4] and also [20, Section 2]). More precisely, \( y^{t_k, x} \) is the unique solution to

\[
y = x + \frac{\Delta t}{2} (b(t_k, x) + b(t_{k+1}, y)) + \sqrt{\Delta t} \sigma \xi,
\]

where \( \xi \) is a \( \mathbb{R}^d \)-valued random variable with i.i.d. components such that

\[
\mathbb{P}(\xi_i = 0) = 2/3 \quad \text{and} \quad \mathbb{P}(\xi_i = \pm \sqrt{3}) = 1/6 \quad \text{for } i = 1, \ldots, d.
\]

Notice that, by (H1)(ii), \( y^{t_k, x} \) is well-defined for \( \Delta t \) small enough. Let \( \mathcal{I}_d = \{1, \ldots, 3^d\} \), define \( \{e^\ell | \ell \in \mathcal{I}_d\} \) as the set of possible values of \( \xi \), set \( \omega^\ell = \mathbb{P}(\xi = e^\ell) \), and denote by \( y_k^\ell(x) \) the unique solution to (2.3) for \( \xi = e^\ell \ (\ell \in \mathcal{I}_d) \). By standard estimates for the weak approximation of \( Y^{t_k, x}(t_{k+1}) \) (see e.g. [28, Theorem 14.5.2]), if \( b \) is smooth enough with respect to the space variable, for every smooth function \( \phi : \mathbb{R}^d \to \mathbb{R} \), we have

\[
\left| \sum_{\ell \in \mathcal{I}_d} \phi(y_k^\ell(x)) \omega^\ell - \mathbb{E} \left( \phi(Y^{t_k, x}(t_{k+1})) \right) \right| = O \left( (\Delta t)^{3/2} \right).
\]

Thus, a discrete in time approximation of (2.2) is given by

\[
\int_{\mathbb{R}^d} \phi(x) m_{k+1}(x) dx = \sum_{\ell \in \mathcal{I}_d} \omega^\ell \int_{\mathbb{R}^d} \phi(y_k^\ell(x)) m_k(x) dx \quad \text{for } \phi \text{ continuous and bounded}, \ k \in \mathcal{I}_{\Delta t},
\]

with \( m_0 = m_0^* \) and unknowns \( \{m_k : \mathbb{R}^d \to \mathbb{R} | k \in \mathcal{I}_{\Delta t} \setminus \{0\} \} \). Note that the boundedness of \( b \) and (2.3) imply the existence of \( L_{\Delta t} = O(1/\sqrt{\Delta t}) \) such that the solution \( m_{\Delta t} \) to (2.6) satisfies

\[
\text{supp}(m_{\Delta t, k}) \subset [-L_{\Delta t}, L_{\Delta t}]^d \quad \text{for } k \in \mathcal{I}_{\Delta t}.
\]
In order to construct a space discretization of (2.6), we consider a symmetric Lagrangian basis of odd order. More precisely, let us fix $p \in \mathbb{N}$, set $q := 2p + 1$, and let $\beta : \mathbb{R} \to \mathbb{R}$ be defined by

\[
(\forall i \in \{0, \ldots, 2p+1\}, \quad \beta_{i}(x) = \frac{\xi - i}{-k} \quad \text{if} \ \xi \in [0, 1],
\]

\[
= \frac{\xi - (p,q)}{-k} \quad \text{if} \ \xi \in (1, 2],
\]

\[
= 0 \quad \text{if} \ \xi \in (p+1, \infty),
\]

\[
\beta(-\xi) \quad \text{if} \ \xi \in (-\infty, 0).
\]

(2.8) \quad (\forall \xi \in [0, \infty)) \quad \tilde{\beta}(\xi) = \cdots

Following [21], for $\Delta x \in (0, \infty)$, we consider the symmetric Lagrange interpolation basis function $\{\tilde{\beta}_{i}\}_{i \in \mathbb{Z}^d}$ defined as

\[
(\forall z = (z_1, \ldots, z_d) \in \mathbb{R}^d, i = (i_1, \ldots, i_d) \in \mathbb{Z}^d) \quad \beta_i(z) = \prod_{j=1}^{d} \tilde{\beta} \left( \frac{z_j - i_j}{\Delta x} \right).
\]

For all $i \in \mathbb{Z}^d$, let us set $x_i = i \Delta x$. Notice that $\beta_i$ has compact support, $\beta_i(x_j) = 1$ if $i = j$ and $\beta_i(x_j) = 0$ otherwise, and, for all $x \in \mathbb{R}^d$, $\sum_{j \in \mathbb{Z}^d} \beta_j(x) = 1$. Given $f \in W^q_{\infty}(\mathbb{R}^d)$, we define the interpolant $[f] : \mathbb{R}^d \to \mathbb{R}$ by

\[
[f](x) = \sum_{i \in \mathbb{Z}^d} f(x_i) \beta_i(x) \quad \text{for} \ x \in \mathbb{R}^d.
\]

By [17] Theorem 16.1, the following estimate holds

\[
\sup_{x \in \mathbb{R}^d} |f(x) - [f](x)| \leq C_I(\Delta x)^{q+1} \|D^{q+1} f\|_{L^\infty},
\]

where $C_I > 0$ is independent of $f$ and $\Delta x$. Notice that in the one dimensional case ($d = 1$), $[f]$ restricted to a given interval $(x_i, x_{i+1})$ ($i \in \mathbb{Z}$) is the Lagrange interpolating polynomial of degree $q$ constructed on the symmetric stencil $x_{i-(q-1)/2}, \ldots, x_{i+1+(q-1)/2}$.

Let $L_{\Delta x} > 0$ be as in [27], let $N_{\Delta x} \in \mathbb{N}$, and set $I_{\Delta x} = \{-N_{\Delta x}, \ldots, N_{\Delta x}\}$. From now on, we assume that $\Delta x = L_{\Delta t}/N_{\Delta x}$, we set

\[
\Delta = (\Delta t, \Delta x) \quad \text{and} \quad \Omega_{\Delta} = [-L_{\Delta t}, -p \Delta x, L_{\Delta t} + p \Delta x]^d.
\]

We look for an approximation $m_{\Delta}$ of the solution $m_{\ast}$ to (FP) such that, for all $k \in I_{\Delta t}$,

\[
m_{\Delta}(t_k, x) = \sum_{i \in I_{\Delta x}} m_{k,i} \beta_i(x) \quad \text{for} \ x \in \Omega_{\Delta}, \quad m_{\Delta}(t_k, x) = 0 \quad \text{for} \ x \in \mathbb{R}^d \setminus \Omega_{\Delta},
\]

where $m_{k,i} \in \mathbb{R}$ ($k \in I_{\Delta t}, i \in I_{\Delta x}$) have to be determined. Notice that, by definition of $I_{\Delta x}$, for all $k \in I_{\Delta t}$ we have that $\text{supp}\{m_{\Delta}(t_k, \cdot)\} \subset \Omega_{\Delta}$. Replacing $m$ by $m_{\Delta}$ and taking $\phi = \beta_i$ ($i \in I_{\Delta x}$) in (2.6) yields the following explicit iterative scheme for the unknowns $m_{k,i} \in \mathbb{R}$ ($k \in I_{\Delta t}, i \in I_{\Delta x}$)

\[
\sum_{j \in I_{\Delta x}} m_{k+1,j} \int_{\Omega_{\Delta}} \beta_i(x) \beta_j(x)\,dx = \sum_{j \in I_{\Delta x}} m_{k,j} \omega_k \int_{\Omega_{\Delta}} \beta_i(y_k(x)) \beta_j(x)\,dx
\]

(2.12) \quad \sum_{j \in I_{\Delta x}} m_{0,j} \int_{\Omega_{\Delta}} \beta_i(x) \beta_j(x)\,dx = \int_{\Omega_{\Delta}} m_{0}(x) \beta_i(x)\,dx.

for $k \in I_{\Delta t}, i \in I_{\Delta x}$.
Let $A$ be the $(2N+1)^d \times (2N+1)^d$ real mass matrix with entries given by

\[(2.13) \quad A_{i,j} = \int_{\Omega_\Delta} \beta_1(x)\beta_j(x)dx \quad \text{for} \quad (i,j) \in \mathcal{I}_{\Delta} \times \mathcal{I}_{\Delta}.
\]

For $k \in \mathcal{I}_{\Delta t}$ and $\ell \in \mathcal{I}_{\Delta}$, let $B_{\ell k}^i$ be the $(2N+1)^d \times (2N+1)^d$ real matrix with entries given by

\[(2.14) \quad (B_{\ell k}^i)_{j} = \int_{\Omega_\Delta} \beta_1(y_{\ell k}^i(x))\beta_j(x)dx \quad \text{for} \quad (i,j) \in \mathcal{I}_{\Delta} \times \mathcal{I}_{\Delta}.
\]

Let $m_{0,\Delta x}$ be the $(2N+1)^d$ dimensional real vector with entries

\[(m_{0,\Delta x})_j = \int_{\Omega_\Delta} m_0(x)\beta_j(x)dx \quad \text{for} \quad i \in \mathcal{I}_{\Delta}.
\]

Calling $m_k = (m_k,i)_{i \in \mathcal{I}_{\Delta t}}$, scheme (2.12) can be rewritten in the following matrix form: find $m_k$ ($k \in \mathcal{I}_{\Delta t}$) such that

\[(2.15) \quad Am_{k+1} = \sum_{\ell \in \mathcal{I}_{\Delta}} \omega_\ell B_{\ell k}^i m_k \quad \text{for} \quad k \in \mathcal{I}_{\Delta t},
\]

\[A m_0 = (m_{0,\Delta x}).
\]

### 2.2. Properties of the space-time Lagrange-Galerkin scheme.

We show below some important properties of the scheme (2.12).

**Theorem 2.2.** Assume (H1). Then for fixed $\Delta$, there exists a unique solution $(m_{k,i})_{k \in \mathcal{I}_{\Delta t}, i \in \mathcal{I}_{\Delta x}}$ to (2.15) and, defining $m_{\Delta}$ as in (2.11), the following hold:

- (i) [Initial condition] $\|m_0^\bullet - m_{\Delta}(0,\cdot)\|_{L^2} = O((\Delta)^q+1)$ if $m_0^\bullet \in H^{q+1}(\mathbb{R}^d)$.
- (ii) [Mass conservation] $\int_{\mathbb{R}^d} m_{\Delta}(t_k,\cdot)dx = 1$ for $k \in \mathcal{I}_{\Delta t}$.
- (iii) [L²-stability] If $b(t,\cdot)$ is differentiable for all $t \in [0,T]$, then $\max_{k \in \mathcal{I}_{\Delta t}} \|m_{\Delta}(t_k,\cdot)\|_{L^2}$ is uniformly bounded with respect to $\Delta$ for $\Delta t$ small enough.

**Proof.** The well-posedness of (2.15) follows from the positive definiteness of $A$ (see e.g. [77 Proposition 6.3.1]) and assertion (i) is a consequence of Assumption (H1)(i) and [77 Section 3.5]. In order to prove (ii), fix $k \in \mathcal{I}_{\Delta t}$ and sum over $i \in \mathbb{Z}^d$ in the first equation of (2.12) to obtain

\[
\sum_{j \in \mathcal{I}_{\Delta x}} m_{k+1,j} \int_{\Omega_\Delta} \beta_j(x)\beta_i(x)dx = \sum_{j \in \mathcal{I}_{\Delta x}} m_{k,j} \sum_{\ell \in \mathcal{I}_{\Delta}} \omega_\ell \sum_{i \in \mathbb{Z}^d} \int_{\Omega_\Delta} \beta_j(x)\beta_i(y_{\ell}^k(x))dx.
\]

Recalling that, for every $y \in \mathbb{R}^d$, $\sum_{i \in \mathbb{Z}^d} \beta_i(y) = 1$, the cardinality \{i $\in \mathbb{Z}^d \mid \beta_i(y) \neq 0$\} is bounded uniformly in $y$, and $\sum_{i \in \mathbb{Z}^d} \omega_\ell = 1$, Fubini’s theorem yields

\[
\int_{\Omega_\Delta} m_{\Delta}(t_{k+1},x)dx = \sum_{j \in \mathcal{I}_{\Delta x}} \omega_\ell \sum_{i \in \mathbb{Z}^d} m_{k,j} \int_{\Omega_\Delta} \beta_j(x)dx
\]

\[
= \sum_{j \in \mathcal{I}_{\Delta x}} m_{k,j} \int_{\Omega_\Delta} \beta_j(x)dx
\]

\[
= \sum_{j \in \mathcal{I}_{\Delta x}} m_{0,j} \int_{\Omega_\Delta} \beta_j(x)dx
\]

\[
= \int_{\Omega_\Delta} m_{\Delta}(0,x)dx.
\]

Analogously, using the second equation in (2.12) and summing over $i \in \mathbb{Z}^d$, we get that

\[
\int_{\Omega_\Delta} m_{\Delta}(0,x)dx = \int_{\Omega_\Delta} m_0^\bullet(x)dx = 1.
\]

Assertion (ii) follows from (2.16), (2.17), and (2.11). Finally, let us show assertion (iii). For $k = 0$, (iii) follows from Assumption (H1)(i) and Theorem 2.2(ii). For $k \in \mathcal{I}_{\Delta t}$, (2.12) implies that

\[
\| m_{\Delta}(t_{k+1},\cdot) \|_{L^2}^2 = \sum_{\ell \in \mathcal{I}_{\Delta}} \omega_\ell \sum_{j \in \mathcal{I}_{\Delta x}} m_{k+1,j} \int_{\Omega_\Delta} \beta_j(x)\beta_j(y_{\ell}^k(x))dx
\]

\[
= \sum_{\ell \in \mathcal{I}_{\Delta}} \omega_\ell \int_{\Omega_\Delta} m_{\Delta}(t_k,y_{\ell}^k(x))m_{\Delta}(t_{k+1},x)dx,
\]

and hence, by the Cauchy-Schwarz inequality,

\[
\| m_{\Delta}(t_{k+1},\cdot) \|_{L^2} \leq \max_{\ell \in \mathcal{I}_{\Delta}} \left( \int_{\Omega_\Delta} | m_{\Delta}(t_k,y_{\ell}^k(x))|^2 dx \right)^{1/2}.
\]
In order to estimate the right-hand-side above, fix \( x \in \mathbb{R}^d, \ell \in \mathcal{I}_d \), and notice that
\[
(2.20) \quad D_y y_k(t) = I_d + \frac{\Delta t}{2} \left( Db(t_k, x) + Db(t_{k+1}, y_k(x)) D_y y_k(t) \right),
\]
where \( I_d \) denotes the \( d \times d \) identity matrix. Since \( Db(\cdot, \cdot) \) is bounded, there exists \( \Delta t > 0 \) such that for all \( k \in \mathcal{I}_d, \Delta t \in [0, \Delta t] \), \( y_k \) is one-to-one, and, for all \( z \in \mathbb{R}^d \), the matrix \( I_d - \frac{\Delta t}{2} Db(t_{k+1}, z) \) is invertible. Therefore, by (2.20),
\[
(2.21) \quad D_y y_k(t) = \left( I_d - \frac{\Delta t}{2} Db(t_{k+1}, y_k(x)) \right)^{-1} \left( I_d + \frac{\Delta t}{2} Db(t_k, x) \right),
\]
from which we deduce that \( D_y y_k(t) \) is invertible. Then, by the change of variable formula, we get that
\[
(2.22) \quad \int_{\Omega} |m_\Delta(t, y_k(x))|^2 dx = \int_{\Omega(x)} |m_\Delta(t, z)|^2 |\det(D_y y_k((y_k(z))^{-1}(z)))|^{-1} dz.
\]
On the other hand, by (2.21) and Jacobi’s formula, for all \( x \in \mathbb{R}^d \) we have
\[
(2.23) \quad \left[ \det(D_y y_k(x)) \right]^{-1} = \frac{\det(I_d - \frac{\Delta t}{2} Db(t_{k+1}, y_k(x)))}{\det(I_d + \frac{\Delta t}{2} Db(t_k, x))} = 1 - \frac{\Delta t}{2} \text{Tr}(Db(t_{k+1}, y_k(x))) + O((\Delta t)^2)
\]
\[
= 1 - \frac{\Delta t}{2} \text{div}(b(t_{k+1}, y_k(x))) + O((\Delta t)^2)
\]
Thus, there exists a constant \( C > 0 \), independent of \( x, k, \ell \), and \( \Delta t \), such that
\[
(2.24) \quad \left| \left[ \det(D_y y_k(t)) \right]^{-1} \right| \leq 1 + C \Delta t.
\]
Combining the previous inequality with (2.22) yields
\[
(2.25) \quad \int_{\Omega} |m_\Delta(t, y_k(x))|^2 dx \leq (1 + C \Delta t) \|m_\Delta(t, \cdot)\|_{L^2}^2,
\]
and hence, by (2.19),
\[
\|m_\Delta(t_{k+1}, \cdot)\|_{L^2} \leq (1 + C \Delta t)^{\frac{1}{2}} \|m_\Delta(t_k, \cdot)\|_{L^2}.
\]
Thus,
\[
\|m_\Delta(t_{k+1}, \cdot)\|_{L^2} \leq \left( 1 + \frac{CT}{N_{\Delta t}} \right)^{N_{\Delta t}/2} \|m_\Delta(0, \cdot)\|_{L^2} \leq e^{CT/2} \|m_\Delta(0, \cdot)\|_{L^2},
\]
from which assertion (iii) follows. \( \square \)

**Remark 2.1.** Notice that Proposition (2.2 iii) and the Cauchy-Schwarz inequality imply that, for any compact set \( K \subseteq \mathbb{R}^d \), there exists \( C_K > 0 \), independent of \( \Delta \) for \( \Delta t \) small enough, such that
\[
\max_{k \in I_d} \int_K |m_\Delta(t, x)| dx \leq C_K.
\]

In the following, we still denote by \( m_\Delta \) its extension to \([0, T] \times \mathbb{R}^d\), defined as
\[
(2.26) \quad m_\Delta(t, x) = \frac{t - t_k}{\Delta t} m_\Delta(t_{k+1}, x) + \frac{t_{k+1} - t}{\Delta t} m_\Delta(t_k, x) \quad \text{if} \ (t, x) \in [t_k, t_{k+1}] \times \mathbb{R}^d \ (k \in \mathcal{I}_d).
\]
Notice that (2.26) and Theorem (2.2 ii)-(iii) imply that
\[
(2.27) \quad \int_{\Omega} m_\Delta(t, x) dx = 1 \quad \text{for all} \ t \in [0, T] \quad \text{and} \ \max_{t \in [0, T]} \|m_\Delta(t, \cdot)\|_{L^2} \leq C,
\]
for some \( C > 0 \), independent of \( \Delta \) for \( \Delta t \) small enough.

For \( k \in \mathbb{N} \cup \{\infty\} \), we denote by \( C^k_0(\mathbb{R}^d) \) the set of functions of class \( C^k \) with compact support.
Proposition 2.1. Under (H1), the following hold:

(i) [Equicontinuity] Let $\phi \in C^{q+1}_0(\mathbb{R}^d)$. Then there exists $C_\phi > 0$ such that for all $\Delta$, with $\Delta t$ small enough and $(\Delta x)^{q+1} \leq \Delta t$, we have

$$
(2.28) \quad \left| \int_{\mathbb{R}^d} \phi(x)m_\Delta(t,x)dx - \int_{\mathbb{R}^d} \phi(x)m_\Delta(s,x)dx \right| \leq C_\phi \Delta t \quad \text{for all } s, t \in [0,T].
$$

(ii) [Consistency] Assume that $b(t,\cdot) \in C^{q+1}_0(\mathbb{R}^d)$ for all $t \in [0,T]$ and let $\phi \in C^\infty_0(\mathbb{R}^d)$. Then for any $k \in \mathbb{N}$ and $\Delta$, with $\Delta t$ small enough and $(\Delta x)^{q+1} \leq \Delta t$, we have

$$
(2.29) \quad \int_{\mathbb{R}^d} \phi(x)(m_\Delta(t_{k+1},x) - m_\Delta(t_k,x))dx = \int_{t_k}^{t_{k+1}} \int_{\mathbb{R}^d} \left( \frac{\sigma^2}{2} \Delta \phi(x) + \langle b(s,x),\nabla \phi(x) \rangle \right) m_\Delta(s,x)dxds
$$

$$
+ O((\Delta x)^{q+1} + (\Delta t)^2 + \Delta t \psi_\phi(\Delta t)),
$$

where $\psi_\phi : [0,\infty) \to \mathbb{R}$ is a modulus of continuity of $b$ on $[0,T] \times \text{supp}(\phi)$.

Proof. In the proof of both assertions, we fix $\phi \in C^0_0(\mathbb{R}^d)$ and we will denote by $C$ a positive real number which can depend on $\phi$ but not on $\Delta$. We will also use the estimate

$$
(2.30) \quad \left| \sum_{i \in I_d} \omega_i \phi(y_i^k(x)) - \left[ \phi(x) + \Delta t \left( \frac{\sigma^2}{2} \Delta \phi(x) + \langle b(x,t_k),\nabla \phi(x) \rangle \right) \right] \right| \leq C(\Delta t)^2 \quad \text{for } x \in \mathbb{R}^d,
$$

which follows from the definition of $y_i^k(x)$ and a Taylor expansion (see for instance [6]).

(i) Let us first show the assertion for $t = t_{k+1}$ and $s = t_k$ for some $k \in \mathbb{Z}_\Delta$. Set $\varepsilon := \phi - I[\phi]$ and fix $k \in \mathbb{Z}_\Delta$. Remark 2.1 yields the existence of $C > 0$ such that

$$
(2.31) \quad \left| \int_{\mathbb{R}^d} \phi(x)(m_\Delta(t_{k+1},x) - m_\Delta(t_k,x))dx \right| \leq \left| \int_{\mathbb{R}^d} I[\phi](x)(m_\Delta(t_{k+1},x) - m_\Delta(t_k,x))dx \right| + C\|\varepsilon\|_{L^\infty}.
$$

Recalling that supp$$\{m_\Delta(t_k,\cdot)\} \subset O_\Delta$$ and using the definition of the scheme in (2.12), we have that

$$
(2.32) \quad \int_{\mathbb{R}^d} I[\phi](x)(m_\Delta(t_{k+1},x) - m_\Delta(t_k,x))dx = \int_{O_\Delta} \sum_{i \in I_d} \phi(x_i) \beta_i(x) \left( \sum_{j \in I_\Delta} (m_{k+1,j} - m_{k,j}) \beta_j(x) \right)dx
$$

$$
= \sum_{i \in I_d} \phi(x_i) \left( \sum_{j \in I_\Delta} (m_{k+1,j} - m_{k,j}) \int_{O_\Delta} \beta_i(x) \beta_j(x)dx \right)
$$

$$
= \sum_{i \in I_d} \phi(x_i) \left[ \sum_{i \in I_d} \omega_i \sum_{j \in I_\Delta} m_{k,j} \left( \int_{O_\Delta} \beta_i(y_i^k(x)) \beta_j(x)dx \right) - \int_{O_\Delta} \beta_i(x) \beta_j(x)dx \right]
$$

$$
= \sum_{i \in I_d} \omega_i \sum_{j \in I_\Delta} m_{k,j} \int_{O_\Delta} \left[ I[\phi](y_i^k(x)) - I[\phi](x) \right] \beta_j(x)dx.
$$

On the other hand, since $\phi$ has a compact support, there exists $C > 0$ such that

$$
(2.33) \quad \left\| \sum_{i \in I_d} \omega_i \left( I[\phi](y_i^k(x)) - \phi(y_i^k(x)) \right) \right\|_{L^2} + \|\phi - I[\phi]\|_{L^2} \leq C\|\varepsilon\|_{L^\infty}
$$
and, by (2.30) and (H1)(ii), there exists $C > 0$ such that

$$(2.34) \quad \left\| \sum_{t \in I_d} \omega_t \left( \phi(y_k^t(\cdot)) - \phi \right) \right\|_{L^2} \leq C \Delta t.$$ 

Thus, by the triangular and the Cauchy-Schwarz inequalities, Theorem 2.2 (iii), (2.31), (2.32), (2.33), and (2.34), we get the existence of $C > 0$ such that

$$\left| \int_{\mathbb{R}^d} \phi(x) \left( m_\Delta(t_{k+1}, x) - m_\Delta(t_k, x) \right) dx \right| \leq C \left( \| \varepsilon \|_{L^\infty} + \Delta t \right).$$

It follows from (2.10), and the condition $(\Delta x)^q \leq \Delta t$, the existence of $C > 0$ such that (2.28) holds for $t = t_{k+1}$ and $s = t_k$. Using this relation and the triangular inequality, we deduce that (2.28) holds for every $s = t_k$ and $t = t_m$ with $k, m \in I_\Delta$.

Now, let us fix $s, t \in [0, T]$ and assume, without loss of generality, that $t > s$. Let $k_1, k_2 \in I_\Delta$ be such that $s \in [t_{k_1}, t_{k_1+1}]$ and $t \in [t_{k_2}, t_{k_2+1}]$. If $k_1 = k_2$, then it follows from (2.26) that (2.28) holds with $C_\phi = C$. Otherwise, $k_2 \geq k_1 + 1$ and (2.26) yield

$$\left| \int_{\mathbb{R}^d} \phi(x) (m_\Delta(t_{k+1}, x) - m_\Delta(s, x)) dx \right| \leq \frac{t_{k+1} - s}{\Delta t} \left| \int_{\mathbb{R}^d} \phi(x) (m_\Delta(t_{k+1}, x) - m_\Delta(t_k, x)) dx \right| \leq C(t_{k+1} - s).$$

Similarly,

$$(2.36) \quad \left| \int_{\mathbb{R}^d} \phi(x) (m_\Delta(t_k, x) - m_\Delta(t, x)) dx \right| \leq C(t - t_k).$$

Altogether, it follows from the triangular inequality, (2.35), (2.36), and (2.28), with $t = t_{k_2}$ and $s = t_{k_1+1}$, that (2.28) holds with $C_\phi = C$.

(ii) By (2.10), Remark 2.1, and the definition of the scheme (2.12), for each $k \in I_\Delta$, we have

$$\int_{\mathbb{R}^d} \phi(x) m_\Delta(t_{k+1}, x) dx = \int_{\mathbb{R}^d} I[\phi](x) m_\Delta(t_{k+1}, x) dx + O((\Delta x)^q+1)$$

$$= \sum_{i \in \mathbb{Z}^d} \phi(x_i) \sum_{j \in I_{\Delta x}} m_{k+1,j} \int_{\mathbb{R}^d} \beta_i(x) \beta_j(x) dx + O((\Delta x)^q+1)$$

$$= \sum_{i \in \mathbb{Z}^d} \phi(x_i) \sum_{j \in I_{\Delta x}} m_{k,j} \sum_{t \in I_d} \omega_t \int_{\mathbb{R}^d} \beta_i(y_k^t(x)) \beta_j(x) dx + O((\Delta x)^q+1)$$

$$= \sum_{j \in I_{\Delta x}} m_{k,j} \sum_{t \in I_d} \omega_t \int_{\mathbb{R}^d} I[\phi](y_k^t(x)) \beta_j(x) dx + O((\Delta x)^q+1)$$

$$= \int_{\mathbb{R}^d} \left( \sum_{t \in I_d} \omega_t \phi(y_k^t(x)) \right) m_\Delta(t_k, x) dx + O((\Delta x)^q+1).$$

Using (2.30) and Remark 2.1, we obtain

$$(2.38) \quad \int_{\mathbb{R}^d} \phi(x) (m_\Delta(t_{k+1}, x) - m_\Delta(t_k, x)) dx = \Delta t \int_{\mathbb{R}^d} \left( \frac{\sigma^2}{2} \Delta \phi(x) + (b(t_k, x), \nabla \phi(x)) \right) m_\Delta(t_k, x) dx + O((\Delta x)^q+1 + (\Delta t)^2).$$

Notice that, for any $s \in [t_k, t_{k+1}]$, Remark 2.1 implies that

$$(2.39) \quad \int_{\mathbb{R}^d} (b(t_k, x), \nabla \phi(x)) m_\Delta(t_k, x) dx = \int_{\mathbb{R}^d} (b(s, x), \nabla \phi(x)) m_\Delta(t_k, x) dx + O(\omega_\phi(\Delta t)).$$

By (2.26) and the fact that $b(s, \cdot) \in C^{q+1}_0(\mathbb{R}^d)$, together with assertion (i), we have

$$\int_{t_k}^{t_{k+1}} \int_{\mathbb{R}^d} \left( \frac{\sigma^2}{2} \Delta \phi(x) + (b(s, x), \nabla \phi(x)) \right) (m_\Delta(s, x) - m_\Delta(t_k, x)) dx ds = O((\Delta t)^2).$$

Thus, (2.28) follows from (2.39), (2.40), and (2.38). \qed
Let us denote by $\mathcal{D}'(\mathbb{R}^d)$ the space of distributions, which we endow with the weak* topology. In the following, for every $\Delta = (\Delta t, \Delta x) \in (0, \infty)^2$ and $t \in [0, T]$, we identify $m_\Delta(t, \cdot)$ with the map
\[
C^\infty_0(\mathbb{R}^d) \ni \phi \mapsto \int_{\mathbb{R}^d} \phi(x)m_\Delta(t,x)dx \in \mathbb{R},
\]
which, by Remark 2.1, is a regular distribution. For every $\Delta$, let us denote, with a slight abuse of notation, $m_\Delta$ the map $[0, T] \ni t \mapsto m_\Delta(t, \cdot) \in \mathcal{D}'(\mathbb{R}^d)$. Notice that Proposition 2.1(i) implies that $m_\Delta \in C([0, T]; \mathcal{D}'(\mathbb{R}^d))$.

**Lemma 2.1.** Suppose that (H1) holds. Then there exists $\Delta t_0 > 0$ such that the family $\mathcal{M} = \{ m_\Delta | \Delta t \leq \Delta t_0, (\Delta x)^{q+1} \leq \Delta t \}$ is relatively compact in $C([0, T]; \mathcal{D}'(\mathbb{R}^d))$.

**Proof.** In view of the Arzelà-Ascoli theorem [27, Chapter 7, Theorem 18] (see also [29, Section 4]) and Proposition 2.1(i), it suffices to show that the family $\mathcal{M}$ is pointwise relatively compact. Let us consider the absolutely convex set $U_0 := \{ \phi \in C^\infty_0(\mathbb{R}^d) \ | \ ||\phi||_{L^\infty} < 1, \sup \phi \subseteq \overline{B}(0,1) \}$. This set is a neighborhood of 0 in the standard topology of $C^\infty_0(\mathbb{R}^d)$ (see e.g. [39, Chapter 10]) and, for any $t \in [0, T]$,
\[
\sup_{\phi \in U_0} \left| \int_{\mathbb{R}^d} m_\Delta(t,x)\phi(x)dx \right| = \sup_{\phi \in U_0} \left| \int_{\mathbb{R}^d} \int_{B(0,1)} m_\Delta(t,x)\phi(x)dx \right| \leq \| m_\Delta(t,\cdot) \|_{L^1(B(0,1))} \leq r,
\]
where $r := \sup\{ \| m_\Delta(t,\cdot) \|_{L^1(B(0,1))} | \Delta \in (0, \infty)^2 \}$ belongs to $[0, +\infty)$ by (2.27). This proves that $\{ m_\Delta(t,\cdot) | \Delta \in (0, \infty)^2 \} \subset \{ T \in \mathcal{D}'(\mathbb{R}^d) | \sup_{\phi \in U_0} T(\phi) \leq r \}$ which, by the Banach- Alaoglu-Bourbaki theorem (see e.g. [37, Theorem 23.5]), is a compact subset of $\mathcal{D}'(\mathbb{R}^d)$. □

We show now a convergence result.

**Theorem 2.3.** Assume (H1), $m_0^* \in H^{q+1}(\mathbb{R}^d)$ and that $b(t,\cdot) \in C^q_0(\mathbb{R}^d)$ for all $t \in [0, T]$. Consider a sequence $(\Delta_n)_{n\in\mathbb{N}} = ((\Delta t_n, \Delta x_n))_{n\in\mathbb{N}} \subseteq (0, \infty)^2$ such that, as $n \to \infty$, $\Delta_n \to (0,0)$ and $(\Delta x_n)^{q+1}/\Delta t_n \to 0$. Setting $m^n := m_{\Delta_n}$, as $n \to \infty$ we have that $(m^n)_{n\in\mathbb{N}}$ converges to $m^*$ in $C([0, T]; \mathcal{D}'(\mathbb{R}^d))$ and weakly in $L^2([0, T] \times \mathbb{R}^d)$, where $m^*$ is the unique classical solution to (FP).

**Proof.** By Theorem 2.2(iii), the sequence $(m^n)_{n\in\mathbb{N}}$ is bounded in $L^2([0, T] \times \mathbb{R}^d)$. Thus, there exists $\widehat{m}$ in $L^2([0, T] \times \mathbb{R}^d)$ such that, as $n \to \infty$ and up to some subsequence, $m^n$ converges weakly to $\widehat{m}$ in $L^2([0, T] \times \mathbb{R}^d)$.

Let us first show that for any $\phi \in C^\infty_0((0, T) \times \mathbb{R}^d)$, we have
\[
\int_0^T \int_{\mathbb{R}^d} \left[ \partial_t \phi(t,x) - \frac{\sigma^2}{2} \Delta \phi(t,x) - \langle b(s,x), \nabla \phi(t,x) \rangle \right] \widehat{m}(t,x)dxdt = 0.
\]

Let $\eta \in C^\infty_0((0, T), \psi \in C^\infty_0(\mathbb{R}^d)$ and suppose that $\phi$ has the form $\phi = \eta \psi \in C^\infty_0((0, T) \times \mathbb{R}^d)$. Denote by $K \subset \mathbb{R}^d$ the support of $\psi$. By (2.26) and Proposition 2.1(i), we have
\[
\int_0^T \int_{\mathbb{R}^d} \partial_t \phi(t,x)m^n(t,x)dxdt = \sum_{k=0}^{N\Delta t_n-1} \int_k^{k+1} \int_K \partial_t \phi(t,x)m^n(t,x)dxdt
\]
\[+ \sum_{k=0}^{N\Delta t_n-1} \int_k^{k+1} \int_K \partial_t \phi(t,x)(m^n(t_{k+1},x) - m^n(t_k,x)) \frac{t-t_k}{\Delta t_n}dxdt \]
\[= \sum_{k=0}^{N\Delta t_n-1} \int_k^{k+1} \int_K \partial_t \phi(t,x)m^n(t_k,x)dxdt + O(\Delta t_n).
\]
On the other hand, by Remark 2.1 we have
\[
\sum_{k=0}^{N} \int_{t_k}^{t_{k+1}} \int_{K} \partial_t \phi(t, x) m^n(t, x) \, dx \, dt = \sum_{k=0}^{N} \Delta t_n \int_{K} \partial_t \phi(t_k, x) m^n(t_k, x) \, dx + O(\Delta t_n)
\]

which is a modulus of continuity of \( b \) on \([0, T] \times K \). Thus,
\[
\int_{0}^{T} \int_{\mathbb{R}^d} \left[ \partial_t \phi(t, x) - \frac{\sigma^2}{2} \Delta \phi(t, x) - \langle b(s, x), \nabla \phi(t, x) \rangle \right] m^n(t, x) \, dx \, dt = O((\Delta x_n)^{q+1}/\Delta t_n + \Delta t_n + \omega(\Delta t_n))
\]
and hence, passing to the weak limit in \( L^2([0, T] \times \mathbb{R}^d) \), we get
\[
\int_{0}^{T} \int_{\mathbb{R}^d} \left[ \partial_t \phi(t, x) - \frac{\sigma^2}{2} \Delta \phi(t, x) - \langle b(s, x), \nabla \phi(t, x) \rangle \right] \hat{m}(t, x) \, dx \, dt = 0.
\]

Since the vector space spanned by \( \{ \eta \psi \mid \eta \in C^0_0((0, T)), \psi \in C_0^\infty(\mathbb{R}^d) \} \) is dense in \( C_0^{1, 2}((0, T) \times \mathbb{R}^d) \) (as in Corollary 1.6.2 of the Weierstrass Approximation Theorem), we get that (2.41) holds for any \( \phi \in C_0^{1, 2}((0, T) \times \mathbb{R}^d) \).

Finally, let us show that for any \( \phi \in C_0(\mathbb{R}^d) \)
\[
\int_{\mathbb{R}^d} \phi(x) (\hat{m}(t, x) - m^n_0(x)) \, dx \to 0 \quad \text{as } t \to 0^+.
\]

By Lemma 2.1 we have that \( \hat{m} \in C([0, T]; D'(\mathbb{R}^d)) \). Moreover, by Lemma 2.1, for any \( t \in [0, T] \) and for every \( \phi \in C_0(\mathbb{R}^d) \), it holds that
\[
\lim_{s \to t, s \in [0, T]} \int_{\mathbb{R}^d} \phi(x) \hat{m}(s, x) \, dx = \int_{\mathbb{R}^d} \phi(x) \hat{m}(t, x) \, dx.
\]

Since Theorem 2.2 implies that \( \hat{m}(0, \cdot) = m^n_0(\cdot) \), (2.46) follows from (2.47) with \( t = 0 \). Thus, the result follows from (2.41), (2.46) and the uniqueness result in Theorem 2.1(iv). □
Remark 2.2. The convergence of the sequence \( (m^n)_{n \in \mathbb{N}} \) to \( m^* \) in the previous theorem is rather weak. On the other hand, to the best of our knowledge this is the first convergence result of a high-order LG scheme for equation (FP). Notice that our proof does not depend on the smoothness of \( m^* \) recalled in Theorem 2.1(i), and it can be easily adapted to deal with equations whose second-order term are not uniformly elliptic (see e.g. [34, Theorem 2.4] and the numerical test in Section 4.2 below).

3. Application to mean field games

Let \( (P_1(\mathbb{R}^d), d) \) be the metric space of Borel probability measures on \( \mathbb{R}^d \) with finite first order moment, endowed with the 1-Wasserstein distance \( d \) (see e.g. [2] Section 7.1 for the definition of \( d \)).

In this section, we focus on the numerical approximation of the following time-dependent second-order MFG with non-local couplings (see e.g. [33, 34]):

\[
\begin{align*}
- \partial_tm - \frac{d^2}{2} \Delta m + \text{div}(\partial_p H(x, \nabla v)m) &= F(x, m(t)) \quad \text{in } [0, T) \times \mathbb{R}^d, \\
\partial_t v - \frac{d^2}{2} \Delta v + H(x, \nabla v) &= 0 \quad \text{in } (0, T] \times \mathbb{R}^d,
\end{align*}
\]

\[\text{(MFG)}\]

where \( \sigma \in \mathbb{R} \setminus \{0\} \), \( \mathbb{R}^d \times \mathbb{R}^d \supset (x, p) \mapsto H(x, p) \in \mathbb{R} \) is convex and differentiable with respect to \( p \), \( F \), \( G : \mathbb{R}^d \times P_1(\mathbb{R}^d) \to \mathbb{R} \), and \( m_0 : \mathbb{R} \to \mathbb{R} \). Notice that \( \text{(MFG)} \) consists of a Hamilton-Jacobi-Bellman (HJB) equation, with a terminal condition, coupled with a FP equation with an initial condition.

For the sake of simplicity, in what follows we will suppose that the Hamiltonian \( H \) is purely quadratic, i.e. \( H(x, p) = |p|^2/2 \) for all \( x, p \in \mathbb{R}^d \).

(H2) We assume that:

(i) \( m_0 \) is Hölder continuous and satisfies (H1)(i).

(ii) \( F \) and \( G \) are bounded and Lipschitz continuous. Moreover, for every \( \mu \in P_1(\mathbb{R}^d) \), \( F(\cdot, \mu) \) is of class \( C^2 \) and

\[
\sup_{x \in \mathbb{R}^d, \mu \in P_1(\mathbb{R}^d)} \left\{ \|DF(x, \mu)\|_\infty + \|D^2F(x, \mu)\|_\infty \right\} < \infty.
\]

Under (H2) system \( \text{(MFG)} \) admits at least one classical solution (see e.g. [9, Theorem 3.1]). Moreover, if the coupling terms \( F \) and \( G \) satisfy a monotonicity condition with respect to \( m \), then the classical solution is unique (see [34, Theorem 2.4]).

In the following, in order to obtain a high-order scheme for \( \text{(MFG)} \), we consider a high-order SL scheme for the HJB equation, which will be combined with the scheme (2.12) for the FP equation.

3.1. A semi-Lagrangian scheme for the HJB equation.

Given \( \mu \in C([0, T]; P_1(\mathbb{R}^d)) \), we consider the HJB equation:

\[
\begin{align*}
- \partial_tv - \frac{d^2}{2} \Delta v + \frac{1}{2} |\nabla v|^2 &= F(x, \mu(t)) \quad \text{in } (0, T) \times \mathbb{R}^d, \\
v(T, \cdot) &= G(\cdot, \mu(T)) \quad \text{in } \mathbb{R}^d.
\end{align*}
\]

\[\text{(HJB)}\]

Standard results for quasilinear parabolic equations (see e.g. [24] Chapter IV and V) yield that \( \text{(HJB)} \) admits a unique classical solution \( v(\mu) \). Moreover, using that \( v(\mu) \) is the value function associated with a stochastic optimal control problem (see e.g. [24] Chapters IV and V), it is easy to check that (H2) yields the existence of \( R > 0 \) such that

\[ |\nabla v(\mu)(t, x)| \leq R \quad \text{for all } t \in [0, T], x \in \mathbb{R}^d, \mu \in C([0, T]; P_1(\mathbb{R}^d)). \]

We now describe a variation of the scheme in [6] to deal with the nonlinearity of the Hamiltonian in \( \text{(HJB)} \) with respect to \( \nabla v \) (see also [38, 45] for related constructions). For a given \( \mu \in C([0, T]; P_1(\mathbb{R}^d)) \), let us define \( \{v_{k,i} \mid k \in I_{\Delta t}, i \in I_{\Delta x} \} \subset \mathbb{R} \) as the solution to

\[
\begin{align*}
\frac{v_{k+1,i} - v_k,i}{\Delta t} &= S[\mu](v_{k+1,i}, k, i) \quad \text{for all } k \in I_{\Delta t}, i \in I_{\Delta x}, \\
v_{N_{\Delta t},i} &= G(x_i, \mu(t_{N_{\Delta t}})) \quad \text{for all } i \in I_{\Delta x},
\end{align*}
\]

(3.1)
where, for a given \( f = \{ f_i \}_{i \in \mathcal{I}_{\Delta t}} \subset \mathbb{R}, k \in \mathcal{I}_{\Delta t}, \) and \( i \in \mathcal{I}_{\Delta x}, \)

\[
S[\mu](f, k, i) = \inf_{\alpha \in A} \left[ \sum_{t \in \mathcal{I}_d} \omega_t \left( I[f](x_i - \Delta t \alpha + \sqrt{\Delta t} \sigma \epsilon^t) + \frac{\Delta t}{2} F(x_i - \Delta t \alpha + \sqrt{\Delta t} \sigma \epsilon^t, \mu(t_{k+1})) \right) \right. \\
\left. + \frac{\Delta t}{2} |\alpha|^2 \right] + \frac{\Delta t}{2} F(x_i, \mu(t_k))
\]  

(3.2)

with \( A = \{ \alpha \in \mathbb{R}^d | |\alpha| \leq R \} \) and \( I[f] \) being defined by (2.9). The following consistency result for \( S[\mu] \) follows from (3.2) and (H2).

**Proposition 3.1.** Let \((\Delta t_n, \Delta x_n)_{n \in \mathbb{N}} \subset (0, +\infty)^2, (k_n)_{n \in \mathbb{N}} \subset \mathbb{N}, (i_n)_{n \in \mathbb{N}} \subset \mathbb{Z}^d, (\mu_n)_{n \in \mathbb{N}} \subset C([0, T]; \mathcal{P}_1(\mathbb{R}^d)), \) and \( \mu \in C([0, T]; \mathcal{P}_1(\mathbb{R}^d)) \). Assume that (H2)(ii) holds and, as \( n \to \infty, (\Delta t_n, \Delta x_n) \to (0, 0), (\Delta x_n)^{9/4}/\Delta t_n \to 0, k_n \in \mathcal{I}_{\Delta t_n}, i_n \in \mathcal{I}_{\Delta x_n}, t_{k_n} \to t, x_{i_n} \to x, \) and \( \mu_n \to \mu. \) Then, for every \( \phi \in C_b^{1,3}([0, T] \times \mathbb{R}^d) \) satisfying \( \| \nabla \phi \|_{L^\infty([0, T] \times \mathbb{R}^d)} \leq R, \) we have

\[
\lim_{n \to \infty} \frac{1}{\Delta t_n} [\phi(t_{k_n}, x_{i_n}) - S[\mu_n](\phi_{k_n+1, k_n}, i_n)] = -\partial_t \phi(t, x) - \frac{\sigma^2}{2} \Delta \phi(t, x) + \frac{1}{2} |\nabla \phi(t, x)|^2 - F(x, \mu(t)),
\]

where \( \phi_k(\phi(t, x)) \in \mathcal{I}_{\Delta t} \).

**Proof.** Let \( \Delta t > 0, \Delta x > 0, \) and \( \alpha \in A. \) In the computations below, the big \( O \) terms are uniform with respect to \( \alpha \in A. \) Let us apply (2.30) to \( \phi(t_{k+1}, \cdot), \) with \( b(t, x) = -\alpha, \) to obtain

\[
\sum_{t \in \mathcal{I}_d} \omega_t \phi \left( t_{k+1, i} - \Delta t \alpha + \sqrt{\Delta t} \sigma \epsilon^t \right) = \phi \left( t_{k+1, i} \right) + \Delta t \left( \frac{\sigma^2}{2} \Delta \phi(t_{k+1, i}) - \langle \nabla \phi(t_{k+1, i}), \alpha \rangle \right)
\]

+ \( O((\Delta t)^2) \).

By (H2)(ii) and using the first-order Taylor expansion of \( F(\cdot, \mu(t_k)) \) around \( x_i, \) we get

\[
\frac{1}{2} \left( \sum_{t \in \mathcal{I}_d} \omega_t F(x_i - \Delta t \alpha + \sqrt{\Delta t} \sigma \epsilon^t, \mu(t_{k+1})) + F(x_i, \mu(t_k)) \right) = F(x_i, \mu(t_{k+1}))
\]

+ \( O(\Delta t + d(\mu(t_{k+1}), \mu(t_k))) \).

Thus, by (3.2), (3.3), (3.4), and (2.10), we obtain

\[
S[\mu](\phi_{k+1, k}, i) = \phi \left( t_{k+1, i} \right) - \Delta t \sup_{\alpha \in A} \left[ \langle \nabla \phi(t_{k+1, i}), \alpha \rangle - \frac{|\alpha|^2}{2} \right] + \Delta t \frac{\sigma^2}{2} \Delta \phi(t_{k+1, i})
\]

+ \( \Delta t F(x_i, \mu(t_{k+1})) + O((\Delta t)^2 + (\Delta x)^{9/4} + \Delta t d(\mu(t_{k+1}), \mu(t_k))) \)

= \( \phi \left( t_{k+1, i} \right) - \Delta t \frac{\sigma^2}{2} |\nabla \phi(t_{k+1, i})|^2 + \Delta t \frac{\sigma^2}{2} \Delta \phi(t_{k+1, i}) + \Delta t F(x_i, \mu(t_{k+1})) + O((\Delta t)^2 + (\Delta x)^{9/4} + \Delta t d(\mu(t_{k+1}), \mu(t_k))) \).

Finally, we get

\[
\frac{1}{\Delta t} [\phi(t_k, x_i) - S[\mu](\phi_{k+1, k}, i)] = -\partial_t \phi(t_k, x_i) - \frac{\sigma^2}{2} \Delta \phi(t_{k+1, i}) + \frac{1}{2} |\nabla \phi(t_{k+1, i})|^2 - F(x_i, \mu(t_{k+1}))
\]

+ \( O \left( \Delta t + \frac{(\Delta x)^{9/4}}{\Delta t} + d(\mu(t_{k+1}), \mu(t_k)) \right) \),

from which the result follows.

**3.2. The scheme for MFG.** For \( \mu \in C([0, T]; \mathcal{P}_1(\mathbb{R}^d)), \) let us define

\[
v_{\Delta}[\mu](t, x) := I[v_{\Delta}[\mu]](t, x) \quad \text{for all} \quad (t, x) \in [0, T] \times \mathcal{O}_\Delta,
\]

where \( v_{\Delta, i} \) is given by (3.1). In order to get a differentiable function with respect to \( x, \) given \( \epsilon > 0 \) and a non-negative function \( \phi \in C^\infty(\mathbb{R}^d) \) such that \( f_{\phi, x}(\phi(x))dx = 1, \) let us set \( \phi_\epsilon(\cdot) = \frac{1}{\epsilon^d} \phi(\cdot / \epsilon) \) and define

\[
v_{\Delta, \epsilon}[\mu](t, \cdot) = (\phi_\epsilon \ast v_{\Delta}[\mu])(t, \cdot) \quad \text{for} \quad t \in [0, T].
\]
For $\ell \in \mathcal{I}_d$ and $k \in \mathcal{I}_{dr}$, let us define $y_{k,\ell}^j[\mu](x)$ the unique solution to

$$y = x - \frac{\Delta t}{2}[\nabla v_{\Delta,\varepsilon}[\mu](t_k, x) + \nabla v_{\Delta,\varepsilon}[\mu](t_{k+1}, y)] + \sqrt{\Delta t} \sigma e^{\ell},$$

where $\nabla v_{\Delta,\varepsilon}[\mu](t, x)$ is the gradient of $v_{\Delta,\varepsilon}[\mu]$ with respect to $x$.

We propose the following scheme for $\text{MFG}$: find $\{v_{k,i}, m_{k,i}\} \in \mathbb{R}^2 \mid k \in \mathcal{I}_{dr}, i \in \mathcal{I}_{dr}$ such that, for all $k \in \mathcal{I}_{dr}$ and $i \in \mathcal{I}_{dr}$,

$$v_{k,i} = S_{\Delta}[m_{\Delta}](v_{k+1,i}, k, i),$$

$$v_{N_{k,i}} = G(x, m),$$

$$\sum_{j \in \mathcal{I}_{dr}} m_{k+1,j} \int_{\mathcal{O}_{\Delta}} \beta_i(x) \beta_j(x) dx = \sum_{j \in \mathcal{I}_{dr}} m_{k-j} \sum_{\ell \in \mathcal{I}_d} \omega_{\ell} \int_{\mathcal{O}_{\Delta}} \beta_i(y_{k,\ell}^j[m_{\Delta}](x)) \beta_j(x) dx,$$

$$\sum_{j \in \mathcal{I}_{dr}} m_{0,j} \int_{\mathcal{O}_{\Delta}} \beta_i(x) \beta_j(x) dx = \int_{\mathcal{O}_{\Delta}} m_{0}^j \beta_i(x) dx.$$

System (3.8) is solved by a fixed point method as in [22]. The iterations are stopped as soon as the $L^1$-norm, approximated by the Simpson’s Rule, of the difference between two consecutive approximations of $m$ is less than a given tolerance $\tau > 0$.

4. Numerical results

In this section, we show the performance of the proposed scheme on three different problems: a linear FP equation in two spatial dimensions, a MFG with non-local couplings and a explicit solution, and a MFG with local couplings and no explicit solutions. For each test, we measure the accuracy of the scheme by computing the following relative errors in the discrete uniform and $L^2$ norms

$$E_\infty = \max_{i \in \mathcal{I}_{dr}} \frac{|h_{\Delta}(T, x_i) - h(T, x_i)|}{\max_{i \in \mathcal{I}_{dr}} |h(T, x_i)|},$$

$$E_2 = \left( \frac{\text{Int}_{\mathcal{O}_\Delta}(|h_{\Delta}(T, x) - h(T, x)|^2)}{\text{Int}_{\mathcal{O}_\Delta}(|h(T, x)|^2)} \right)^{1/2},$$

where $h = m$, $v = m_{\Delta}$, $v_{\Delta}$, and $\text{Int}_{\mathcal{O}_\Delta}$ denotes the approximation of the Riemann integral on $\mathcal{O}_{\Delta}$ by using the Simpson’s Rule. We denote by $p_\infty$ and $p_2$ the rates of convergence for $E_\infty$ and $E_2$, respectively.

Notice that, for the exactly integrated scheme (2.12), the local truncation error is given by the contributions of (2.5) and (2.10), which yield a global truncation error of order $(\Delta x)^{q+1}/\Delta t + (\Delta t)^2$. As in [20], we get that the order of consistency is maximized by taking $\Delta t = O((\Delta x)^{(q+1)/3})$. With respect to the space discretization step, the previous choice suggests an order of convergence given by $2(q + 1)/3$. In all the simulations we take $q = 3$, which yields an heuristic optimal rate equal to 8/3, and Simpson’s Rule to approximate the integrals in (2.12). The resulting scheme is dual to a SL scheme applied to the dual equation to the FP equation, and its high-order accuracy is illustrated numerically in the examples below. Indeed, the tables in the tests show rates of convergence $p_\infty$ and $p_2$ greater than 2 in most of the cases.

4.1. An implementable version of the scheme (2.12). In order to obtain a implementable version of (2.12), an approximation of the integrals therein has to be introduced. For simplicity, we consider the one-dimensional case, we use Simpson’s Rule on each element $[x_j, x_j + 2\Delta x]$ ($j = 2m, m \in \mathbb{Z}$) and cubic symmetric Lagrange interpolation basis functions $\beta_j$ ($p = 1$ in (2.8)). Recalling that $\beta_j$ has support in $[x_{j-2}, x_{j+2}]$, letting $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise, the entries of the mass matrix $A$ (see (2.13)) are approximated by

$$\int_{\mathcal{O}_\Delta} \beta_i(x) \beta_j(x) dx = \int_{x_{j-2}}^{x_j} \beta_i(x) \beta_j(x) dx + \int_{x_{j+2}}^{x_{j+4}} \beta_i(x) \beta_j(x) dx$$

while the entries of $B_k^j$ (see (2.14)) are approximated by

$$B_k^j_{i,j} = \int_{x_{i-2}}^{x_{i+2}} \beta_i(y_{k}^j(x)) \beta_j(x) dx \simeq \frac{2\Delta x}{3} \beta_i(y_{k}^j(x_j)).$$

We observe that, as usual in LG methods, the integrands in (4.1) and (4.2) have not the necessary regularity in order to guarantee the standard accuracy order of the quadrature rule. This can lead to fluctuations in the order of convergence, as can be observed in some instances of the numerical tests.
sufficient to apply (4.3) to approximate (\(\tilde{\sigma}\)) for \(\tilde{\sigma}\) and \(\tilde{\sigma}_0\) where

\[
m_{k+1} = \sum_{\ell \in I_{\Delta t}} \omega_\ell \tilde{B}^\ell_k m_k \quad \text{for} \quad k \in I_{\Delta t},
\]

\[
m_0 = \tilde{m}_0,
\]

where \(\tilde{B}^\ell_k\) is a \((2N_{\Delta t} + 1) \times (2N_{\Delta x} + 1)\) matrix with entries given by

\[
(\tilde{B}^\ell_k)_{i,j} = \beta_i(y^\ell_k(x_j))
\]

and \(\tilde{m}_0\) is vector of length \(2N_{\Delta x} + 1\) given by

\[
\tilde{m}_{0,i} = m_0^*(x_i) \quad \text{for} \quad i \in I_{\Delta x}.
\]

**Remark 4.1.** Applied to a linearization of equation (HJB), scheme (4.3) is the dual of the semi-Lagrangian scheme [29] when a Crank-Nicolson method is used to discretize the characteristic curves, together with a cubic symmetric Lagrange interpolation to reconstruct the values in the space variable. Moreover, scheme (4.3) is also a natural higher-order extension of the scheme proposed in [13,14] to approximate second-order MFGs.

### 4.2. Linear case: damped noisy harmonic oscillator.

We consider the numerical approximation of a FP equation modeling a noisy harmonic oscillator with damping coefficient \(\gamma > 2\) and noise coefficient \(\sigma \neq 0\). For \(T > 0\) and an initial condition \(x_0 \in \mathbb{R}^2\), the dynamics is described by the following SDE in the interval \((0, T)\)

\[
dY_1(t) = Y_2(t)dt,
\]

\[
dY_2(t) = (-Y_1(t) - \gamma Y_2(t))dt + \sigma dW(t),
\]

\[
Y(0) = x_0.
\]

The associated (degenerated) FP equation is given by

\[
\partial_t m - \frac{\sigma^2}{2} \partial^2_{x_1,x_2} m + \partial_{x_1}(x_2m) - \partial_{x_2}((x_1 + \gamma x_2)m) = 0 \quad \text{in} \quad (0, T] \times \mathbb{R}^2,
\]

\[
m(0) = \delta_{x_0} \quad \text{in} \quad \mathbb{R}^2,
\]

where \(\delta_{x_0}\) denotes the Dirac measure at \(x_0\). It is shown in [51] that (4.5) has a unique solution \(m^*\) such that, for all \(t \in (0, T]\), \(m^*(t)\) is absolutely continuous with respect to the Lebesgue measure, with density \(m^*(t, \cdot)\) given by

\[
m^*(t, x) = \frac{\nu(t, x)}{\nu(t, x)dy}, \quad \text{for all} \quad x \in \mathbb{R}^2, \quad \text{where} \quad \nu(t, x) = \frac{e^{\gamma t - s_{x_0}(t, x)/2D(t)}}{2\pi \sqrt{D(t)}},
\]

with

\[
s_{x_0}(t, x) = a(t)(\psi(t, x) - \psi(0, x_0))^2 + 2H(t)[\psi(t, x) - \psi(0, x_0)] [\eta(t, x) - \eta(0, x_0)]
\]

\[
+ b(t)(\eta(t, x) - \eta(0, x_0))^2,
\]

\[
D(t) = a(t)b(t) - H(t)^2,
\]

and, setting

\[
\mu_1 = -\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - 1}, \quad \mu_2 = -\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - 1},
\]

\(a, \psi, H, \eta, \) and \(b\) are respectively given by

\[
a(t) = \frac{\sigma^2}{2\mu_1}(1 - e^{-2\mu_1 t}), \quad \psi(t, x) = (x_1\mu_1 - x_2)e^{-\mu_2 t}, \quad H(t) = -\frac{\sigma^2}{\mu_1 + \mu_2}(1 - e^{-(\mu_1 + \mu_2) t}),
\]

\[
\eta(t, x) = (x_1\mu_2 - x_2)e^{-\mu_1 t}, \quad \text{and} \quad b(t) = \frac{\sigma^2}{2\mu_2}(1 - e^{-2\mu_2 t}).
\]

We apply scheme (4.3) to approximate \(m^*(t, \cdot)\) for \(t \in [t_0, T] = [1, 5, 3]\). We take \(\gamma = 2.1\), two values for \(\sigma^2/2\) given by 0.1 and 0.05, respectively, and \(x_0 = (1, 1)\). Since the SDE (4.4) is autonomous, it is sufficient to apply (4.3) to approximate \(\text{FP}\) in \([0, 1.5]\) with initial condition given by \(m^*(1.5, \cdot)\), the
latter being computed by using \(4.6\). Since the diffusion term in \(4.4\) can be written as \((0, \sigma) dW(t)\), the scheme \(4.3\) cannot be directly applied, but, as in \([20]\), it can be simply modified by setting

\[
A_\gamma = \begin{pmatrix} 0 & 1 \\ -1 & -\gamma \end{pmatrix}, \quad e^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad e^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

and considering the discrete characteristics \(y^\ell_k(x) (\ell = 1, 2, 3)\), defined as the unique solutions to

\[
y = x + \frac{\Delta t}{2} A_\gamma (x + y) + \sqrt{\Delta t} \sigma e^\ell,
\]

with corresponding weights \(\omega_\ell (\ell = 1, 2, 3)\) given by \(\omega_1 = 1/6, \omega_2 = 2/3, \) and \(\omega_3 = 1/6\). Since most of the support of the exact solution \(m\) is contained in \(O_D = (-2, 2)^2\), we consider the solution of our scheme restricted to this domain in order to obtain an implementable method. We consider homogeneous Dirichlet boundary conditions, which are approximated by taking \((\bar{B}_{\ell}^k)_{i,j} = 0\) in \(4.3\) if the characteristic \(y^\ell_k(x_j)\) exits from \(O_D\). Tables 1 and 2 show the errors and convergence rates in both norms. We have performed the simulations by taking \(\Delta t = (\Delta x)^2/8\) in the case of \(\sigma^2/2 = 0.1\) (Table 1), and \(\Delta t = (\Delta x)^4/4\) in the case of \(\sigma^2/2 = 0.05\) (Table 2). As for semi-Lagrangian schemes, the scheme \(4.3\) performs better in the hyperbolic regime case (small diffusion). In some simulations, the optimal rate of convergence \(8/3\) is reached.

| \(\Delta x\)          | Errors for the approximation of the FP equation | \(E_\infty\) | \(E_2\) | \(p\infty\) | \(p_2\) |
|------------------------|-----------------------------------------------|--------------|--------|-------------|--------|
| \(2.00 \cdot 10^{-4}\) | 1.83 \cdot 10^{-1}                           | -            | -      | -           | -      |
| \(1.00 \cdot 10^{-4}\) | 5.57 \cdot 10^{-2}                           | -            | -      | -           | -      |
| \(5.00 \cdot 10^{-4}\) | 8.38 \cdot 10^{-2}                           | -            | -      | -           | -      |
| \(2.50 \cdot 10^{-4}\) | 1.14 \cdot 10^{-2}                           | -            | -      | -           | -      |
| \(1.25 \cdot 10^{-4}\) | 3.18 \cdot 10^{-3}                           | -            | -      | -           | -      |

Table 1. Errors and convergence rates for the approximation of \(4.5\) with \(\sigma^2/2 = 0.1\).

| \(\Delta x\)          | Errors for the approximation of the FP equation | \(E_\infty\) | \(E_2\) | \(p\infty\) | \(p_2\) |
|------------------------|-----------------------------------------------|--------------|--------|-------------|--------|
| \(2.00 \cdot 10^{-4}\) | 3.05 \cdot 10^{-1}                           | -            | -      | -           | -      |
| \(1.00 \cdot 10^{-4}\) | 1.21 \cdot 10^{-1}                           | -            | -      | -           | -      |
| \(5.00 \cdot 10^{-4}\) | 2.54 \cdot 10^{-2}                           | -            | -      | -           | -      |
| \(2.50 \cdot 10^{-4}\) | 3.07 \cdot 10^{-3}                           | -            | -      | -           | -      |
| \(1.25 \cdot 10^{-4}\) | 6.25 \cdot 10^{-4}                           | -            | -      | -           | -      |

Table 2. Errors and convergence rates for the approximation of \(4.5\) with \(\sigma^2/2 = 0.05\).

### 4.3. Non-local MFG with analytical solution.

Consider the non-local MFG system

\[
\begin{align*}
-\partial_t v - \frac{\sigma^2}{2} \Delta v + \frac{1}{2} |\nabla v|^2 &= \frac{1}{2} \left( x - \int_{\mathbb{R}^d} y m(t, y) dy \right)^2 \quad \text{in } [0, T) \times \mathbb{R}^d, \\
\partial_t m - \frac{\sigma^2}{2} \Delta m - \text{div} (\nabla v m) &= 0 \quad \text{in } [0, T] \times \mathbb{R}^d, \\
v(T, \cdot) &= 0, \quad m(0, \cdot) = m_0^* \quad \text{in } \mathbb{R}^d.
\end{align*}
\]

(4.7)

where \(m_0^*\) is the density of a Gaussian random vector with mean \(\mu_0 \in \mathbb{R}^d\) and covariance matrix \(\Sigma_0 \in \mathbb{R}^{d \times d}\). For simplicity, we will assume that \(\Sigma_0\) is a diagonal matrix.

In what follows, we compute explicitly the unique solution \((v^*, m^*)\) to \(4.7\) (see e.g. \([3]\)). Since \(v^*\) is the value function associated with a linear-quadratic optimal control problem, standard results (see e.g. \([30]\) Chapter 6) show that \(v^*\) has the form

\[
v^*(t, x) = \frac{1}{2} \langle \Pi(t)x, x \rangle + \langle s(t), x \rangle + c(t) \quad \text{for } (t, x) \in [0, T) \times \mathbb{R}^d,
\]

(4.8)
where, setting $\nabla \nu^*(t,x) dy$ for all $t \in [0,T]$, $\Pi, s$, and $c$ satisfy
\begin{align}
-\Pi(t) &= -\Pi^2(t) + I_d \quad \text{for } t \in (0,T), \\
-s(t) &= -\Pi(t)s(t) - \nabla \nu(t) \quad \text{for } t \in (0,T), \\
-\dot{c}(t) &= \frac{\sigma^2}{2} \text{Tr}(\Pi(t)) - \frac{1}{2}|s(t)|^2 + \frac{1}{2}|\nabla \nu(t)|^2 \quad \text{for } t \in (0,T), \\
\Pi(T) &= 0, \quad s(T) = 0, \quad c(T) = 0.
\end{align}
(4.9)

Notice that $\Pi$ satisfies a Riccati equation whose analytical solution is given by
\begin{align}
\Pi(t) &= \left(\frac{e^{2T-t} - e^t}{e^{2T-t} + e^t}\right) I_d \quad \text{for } t \in [0,T].
\end{align}
(4.10)

Since $\nabla \nu^*(t,x) = \Pi(t)x + s(t)$, the SDE underlying the FP equation in (4.7) (see (2.1)) is given by
\begin{align}
dY(t) &= [-\Pi(t)Y(t) - s(t)] dt + \sigma dW(t) \quad \text{for } t \in (0,T), \\
Y(0) &= Y_0,
\end{align}
where $Y_0$ is a Gaussian random variable, independent of the $d$-dimensional Brownian motion $W$, with mean $\mu_0$ and covariance matrix $\Sigma_0$. Since
\begin{align}
Y(t) = Y_0 + \int_0^t [-\Pi(r)Y(r) - s(r)] dr + \sigma W(t) \quad \text{for } t \in (0,T),
\end{align}
and the coordinates $Y_{0,i}$ ($i = 1, \ldots, d$) of $Y_0$ are independent Gaussian random variables with means $\mu_{0,i}$ and variance $\Sigma_{0,i,i}$, for every $t \in [0,T]$, $Y(t)$ is a vector of independent Gaussian random variables $Y_i(t)$ ($i = 1, \ldots, d$) with mean $\nabla \nu_i(t)$ and variance $\Sigma(t)_{i,i} = \mathbb{E}(Y_i^2(t)) = Y_i^2(t)$ to be determined. In other words,
\begin{align}
m^*(t,x) = \Pi_{i=1}^d m^i_*(t,x_i) \quad \text{for } t \in [0,T], \quad x \in \mathbb{R}^d,
\end{align}
where, for every $t \in [0,T]$ and $i = 1, \ldots, d$, $m^i_*(t, \cdot)$ is a univariate Gaussian density with parameters $\nabla \nu_i(t)$ and variance $(\Sigma(t))_{i,i}$. In order to compute these parameters, notice that (4.11) implies that
\begin{align}
\nabla \nu(t) = \mu_0 + \int_0^t (-\Pi(r)\nabla \nu(r) - s(r)) dr \quad \text{for } t \in [0,T],
\end{align}
i.e. $\nabla \nu$ solves
\begin{align}
\frac{d\nabla \nu(t)}{dt} &= -\Pi(t)\nabla \nu(t) - s(t) \quad \text{for } t \in (0,T), \\
\nabla \nu(0) &= \mu_0.
\end{align}
(4.13)

Thus, by (4.9) and (4.13), the couple $(\nabla \nu, s)$ solves the boundary value problem
\begin{align}
\frac{d\nabla \nu(t)}{dt} &= -\Pi(t)\nabla \nu(t) - s(t) \quad \text{for } t \in (0,T), \\
\frac{ds(t)}{dt} &= \Pi(t)s(t) + \nabla \nu(t) \quad \text{for } t \in (0,T), \\
\nabla \nu(0) &= \mu_0, \quad s(T) = 0,
\end{align}
whose unique solution is given by (see e.g. [25])
\begin{align}
\nabla \nu(t) &= \mu_0, \quad s(t) = -\Pi(t)\mu_0 \quad \text{for } t \in [0,T],
\end{align}
(4.14)
where we recall that $\Pi$ is given by (4.10).

On the other hand, by (4.11) and Itô’s lemma, for every $i = 1, \ldots, d$, we have
\begin{align}
Y_i^2(t) = Y_{0,i}^2 - \int_0^t 2Y_i(r) [\Pi_{i,i}(r)Y_i(r) + s_i(r)] dr + 2\sigma \int_0^t Y_i(r) dW_i(r) + \sigma^2 t \quad \text{for } t \in [0,T].
\end{align}

Thus, denoting by $m_{0,i}^*$ the $i$th marginal of $m_0^*$ ($i = 1, \ldots, d$), (4.14) yields
\begin{align}
\mathbb{E}(Y_i^2(t)) = \int_{\mathbb{R}} x^2 m_{0,i}^*(x) dx - 2 \int_0^t \mathbb{E}(Y_i^2(r)) \Pi_{i,i}(r) + \mu_{0,i}s_i(r) dr + \sigma^2 t \quad \text{for } t \in [0,T].
\end{align}
In particular, \([0, T] \ni t \mapsto \mathbb{E}(Y^2_t(t)) \in \mathbb{R}\) is the unique solution to

\[
\begin{align*}
\dot{M}(t) &= -2\Pi_{t}(t)M(t) - 2\mu_0 s_i(t) + \sigma^2 \quad t \in (0, T), \\
M(0) &= \int_\mathbb{R} x^2 m^*_0(x) \, dx,
\end{align*}
\]

which, for all \(t \in [0, T]\), is given by

\[
\mathbb{E}(Y^2_t(t)) = (e^{2T-t} + e^2)^2 \left[ \frac{2 \int_\mathbb{R} x^2 m^*_0(x) \, dx - 2\mu^2_0 + \sigma^2 (e^{2T} + 1)}{2(e^{2T} + 1)^2} \right] + \mu^2_0.
\]

Thus, for all \(i = 1, \ldots, d\) and \(t \in [0, T]\),

\[
(S(t))_{i,j} = (e^{2T-t} + e^2)^2 \left[ \frac{2 \int_\mathbb{R} x^2 m^*_0(x) \, dx - 2\mu^2_0 + \sigma^2 (e^{2T} + 1)}{2(e^{2T} + 1)^2} \right] - \frac{\sigma^2}{2(e^{2T} + e^2)}.
\]

Altogether, for all \(t \in [0, T]\), \(m^*(t, \cdot)\) is given by (4.12), where the parameters of the univariate Gaussian densities \(m^*(t, \cdot)\) are given by (4.14) and (4.16), and the value function \(v^*\) is given by (4.8), with \(\Pi\) and \(s\) given by (4.10) and (4.14), respectively, and \(c\), obtained by integrating the third equation of (4.9), is given by

\[
c(t) = \frac{1}{2} \ln \left( \frac{2e^T}{e^{2T-t} + 1} \right) \quad \text{for} \ t \in [0, T].
\]

Let us now solve system (4.7) on a bounded domain in dimension \(d = 1, 2\). We choose \([0, T] \times O_\Delta = [0, 0.25] \times (-2, 2)^d\), with Dirichlet boundary conditions on \(\partial O_\Delta\), the latter being equal to the exact solution of (4.7) for the HJB equation and homogeneous for the FP equation. The numerical approximation of the boundary conditions for the HJB equation is based on the technique proposed in [6], while for the FP equation we proceed as in the previous test. In this and in the following test, to compute (3.3) we have used a fourth-order finite difference approximation of the gradient of the value function \(v_\Delta[m]\), and we have not introduced the mollifier \(\phi_c\).

For \(d = 1\) we consider two cases, one with \(\sigma^2/2 = 0.005\) and the other one with \(\sigma^2/2 = 0.05\). In all the simulations we choose \(\Delta t = (\Delta x)^{4/3}/4\). Tables 3 and 4 show the errors and the convergence rates for the approximation of the HJB and FP equations. In Table 4 the convergence rate tends to be close to the theoretical optimal rate \(8/3\).

Table 5 shows errors and convergence rates for problem (4.7) with \(d = 2\) and \(\sigma^2/2 = 0.05\), and Table 6 shows errors and convergence rates for the approximated gradient of the value function in (4.7) with the same parameters. In both tables the order of convergence is most of the cases is much larger than 2. The tolerance \(\tau\) for the stopping criterion is \(10^{-3}\). In Figure 1 we show the solution to (4.7) on \([0, T] \times O_\Delta = [0, 0.25] \times (-2, 2)\) with \(\sigma^2/2 = 0.005\), computed with \(\Delta x = 1.25 \cdot 10^{-2}\) and \(\Delta t = (\Delta x)^{4/3}/4\). Figure 2 displays a zoom of the initial density \(m^*_0\), the exact solution \(m^*(T, \cdot)\) and its approximation \(m_\Delta(T, \cdot)\), computed with \(\Delta x = 6.25 \cdot 10^{-3}\) and \(\Delta t = (\Delta x)^{4/3}/4\).

| \(\Delta x\) | Errors for the approximation of \(v^*(0, \cdot)\) | Errors for the approximation of \(m^*(T, \cdot)\) |
|-------------|-----------------------------|-----------------------------|
| \(E\infty\) | \(E2\) | \(p\infty\) | \(p2\) | \(E\infty\) | \(E2\) | \(p\infty\) | \(p2\) |
| 2.00 \cdot 10^{-3} | 6.20 \cdot 10^{-5} | 7.40 \cdot 10^{-5} | - | - | 2.22 \cdot 10^{-2} | 2.32 \cdot 10^{-2} | - | - |
| 1.00 \cdot 10^{-3} | 1.09 \cdot 10^{-5} | 1.43 \cdot 10^{-5} | 2.51 | 2.37 | 5.43 \cdot 10^{-4} | 5.10 \cdot 10^{-4} | 2.03 | 2.19 |
| 5.00 \cdot 10^{-3} | 2.13 \cdot 10^{-6} | 3.41 \cdot 10^{-6} | 2.36 | 2.07 | 9.32 \cdot 10^{-4} | 8.90 \cdot 10^{-4} | 2.54 | 2.52 |
| 2.50 \cdot 10^{-3} | 5.42 \cdot 10^{-7} | 1.00 \cdot 10^{-6} | 1.97 | 1.77 | 1.33 \cdot 10^{-4} | 1.26 \cdot 10^{-4} | 2.81 | 2.82 |
| 1.25 \cdot 10^{-3} | 1.67 \cdot 10^{-7} | 3.20 \cdot 10^{-7} | 1.70 | 1.64 | 1.22 \cdot 10^{-5} | 1.17 \cdot 10^{-5} | 3.45 | 3.43 |
| 6.25 \cdot 10^{-4} | 6.45 \cdot 10^{-8} | 1.11 \cdot 10^{-7} | 1.37 | 1.53 | 6.04 \cdot 10^{-7} | 6.08 \cdot 10^{-7} | 4.33 | 4.27 |

**Table 3.** Errors and convergence rates for the approximation of the solution to problem (4.7) with \(d = 1\) and \(\sigma^2/2 = 0.05\).
∆x | Errors for the approximation of $v^*(0, \cdot)$ | Errors for the approximation of $m^*(T, \cdot)$
---|---|---|---|---|---|---|---|---|
2.00 · 10^{-1} | 1.68 · 10^{-4} | E_\infty | 1.70 · 10^{-4} | p_\infty | 2.24 | 2.29 | 8.81 · 10^{-3} | 1.01 · 10^{-2}
1.00 · 10^{-1} | 3.56 · 10^{-5} | E_2 | 3.48 · 10^{-5} | p_2 | 2.06 | 2.06 | 3.06 · 10^{-3} | 2.53 · 10^{-3}
5.00 · 10^{-2} | 5.86 · 10^{-6} | $E_\infty$ | 5.75 · 10^{-6} | $p_\infty$ | 1.93 | 1.93 | 5.65 · 10^{-4} | 2.19
2.50 · 10^{-2} | 1.06 · 10^{-6} | $E_2$ | 1.04 · 10^{-6} | $p_2$ | 2.47 | 2.47 | 1.14 · 10^{-4} | 2.15
1.25 · 10^{-2} | 1.80 · 10^{-7} | $E_\infty$ | 2.13 · 10^{-7} | $p_\infty$ | 2.62 | 2.62 | 2.05 · 10^{-5} | 2.48
6.25 · 10^{-3} | 3.75 · 10^{-8} | $E_2$ | 5.24 · 10^{-8} | $p_2$ | 2.26 | 2.02 | 3.27 · 10^{-6} | 2.65

Table 4. Errors and convergence rates for the approximation of the solution to problem (4.7) with $d = 1$ and $\sigma^2/2 = 0.005$.

Figure 1. Solution to (4.7) on $[0, T] \times \Omega_\Delta = [0, 0.25] \times (-2, 2)$ with $\sigma^2/2 = 0.005$. On the left, we display the exact value function $v^*$ at times $t = 0, t = 0.25$, and the numerical approximation $v_\Delta$ at time $t = 0$. On the right, we display the exact density $m^*$ at times $t = 0, t = 0.25$, and the numerical approximation $m_\Delta$ at time $t = 0.25$.

Figure 2. Zoom of the exact density $m^*$ at times $t = 0, t = 0.25$, and of the numerical approximation $m_\Delta$ at time $t = 0.25$.

4.4. Mean field games with local couplings. In this section, we approximate the solution of the second-order MFG system with local couplings studied in [46, Section 5.2]. Namely, we consider system (MFG) in the time-space domain $[0, T] \times \Omega_\Delta = [0, 0.05] \times (0, 1)$, with homogeneous Neumann boundary conditions at $x = 0$ and $x = 1$, $\sigma^2/2 = 0.05$,

$$m_0(x) = \begin{cases} 4 \sin^2(2\pi(x) - \frac{1}{4}) & \text{if } x \in \left[\frac{1}{2}, \frac{3}{4}\right], \\ 0 & \text{otherwise}, \end{cases}$$
Errors for the approximation of $v^*(0, \cdot)$

| $\Delta x$ | $E_{\infty}$ | $E_2$ | $p_{\infty}$ | $p_2$ |
|------------|--------------|-------|---------------|------|
| 2.00 · $10^{-1}$ | 3.93 · $10^{-2}$ | 4.85 · $10^{-2}$ | - | - |
| 1.00 · $10^{-1}$ | 8.62 · $10^{-3}$ | 1.01 · $10^{-3}$ | 5.51 | 5.59 |
| 5.00 · $10^{-2}$ | 2.30 · $10^{-4}$ | 3.37 · $10^{-5}$ | 5.23 | 4.91 |
| 2.50 · $10^{-2}$ | 3.76 · $10^{-6}$ | 7.67 · $10^{-6}$ | 2.61 | 2.14 |

Errors for the approximation of $m^*(T, \cdot)$

| $\Delta x$ | $E_{\infty}$ | $E_2$ | $p_{\infty}$ | $p_2$ |
|------------|--------------|-------|---------------|------|
| 1.99 | 1 | 4.81 | 1.40 · $10^{-1}$ | - |
| 1.00 · $10^{-1}$ | 5.51 | 5.59 | 4.27 · $10^{-2}$ | 3.53 · $10^{-2}$ |
| 5.00 · $10^{-2}$ | 2.30 | 3.37 · $10^{-3}$ | 5.23 | 4.91 |
| 2.50 · $10^{-2}$ | 3.76 · $10^{-6}$ | 7.67 · $10^{-6}$ | 2.61 | 2.14 |

Table 5. Errors and convergence rates for the approximation of the solution to problem (4.7) with $d = 2$ and $\sigma^2/2 = 0.05$.

Errors for the approximation of $\partial_x v^*(0, \cdot)$

| $\Delta x$ | $E_{\infty}$ | $E_2$ | $p_{\infty}$ | $p_2$ |
|------------|--------------|-------|---------------|------|
| 2.00 · $10^{-1}$ | 4.64 · $10^{-1}$ | 1.41 · $10^{-1}$ | - | - |
| 1.00 · $10^{-1}$ | 1.65 · $10^{-2}$ | 3.52 · $10^{-3}$ | 4.81 | 5.32 |
| 5.00 · $10^{-2}$ | 4.45 · $10^{-4}$ | 1.04 · $10^{-4}$ | 5.21 | 5.08 |
| 2.50 · $10^{-2}$ | 9.12 · $10^{-6}$ | 2.32 · $10^{-6}$ | 2.29 | 2.16 |

Errors for the approximation of $\partial_{xx} v^*(0, \cdot)$

| $\Delta x$ | $E_{\infty}$ | $E_2$ | $p_{\infty}$ | $p_2$ |
|------------|--------------|-------|---------------|------|
| 1.99 | 1 | 4.81 | 1.40 · $10^{-1}$ | - |
| 1.00 · $10^{-1}$ | 5.51 | 5.59 | 4.27 · $10^{-2}$ | 3.53 · $10^{-2}$ |
| 5.00 · $10^{-2}$ | 2.30 | 3.37 · $10^{-3}$ | 5.23 | 4.91 |
| 2.50 · $10^{-2}$ | 3.76 · $10^{-6}$ | 7.67 · $10^{-6}$ | 2.61 | 2.14 |

Table 6. Errors and convergence rates for the approximation of $\nabla v^*(0, \cdot)$ with $d = 2$ and $\sigma^2/2 = 0.05$.

and

$$F(x, m) = 3n_\delta^0(x) - \min(4, m), \quad G(x, m) = 0, \quad \text{for } x \in (0, 1).$$

Notice that the coupling term $F$ depends on the density $m$ in a pointwise (or local) manner. The homogeneous Neumann boundary conditions are approximated as in (11). In this example, we do not have an explicit expression for $(v^*, m^*)$. In order to compute the errors and rates of convergence, we compare our approximations $(\tilde{v}_\Delta, \tilde{m}_\Delta)$ with a reference solution, which is still denoted by $(v^*, m^*)$, computed with $\Delta x = 6.67 \cdot 10^{-4}$ and $\Delta t = (\Delta x)^{3/2}/3$. In Tables 7 and 8 we show the errors and convergence rates for $v_\Delta(0, \cdot), \partial_x v_\Delta(0, \cdot), \text{ and } m_\Delta(T, \cdot)$, which are computed by taking $\Delta t = (\Delta x)^{3/2}/3$ for different values of $\Delta x$. We observe an order of convergence greater than two in most of the cases. The mass conservation error is of order $10^{-13}$ in all the tests, numerically confirming that the scheme is mass preserving. Finally, Figure 3 shows the approximated density $\tilde{m}_\Delta$ at time $t = T$ and the approximated value function $v_\Delta$, together with its gradient $\nabla v_\Delta$, at time $t = 0$. These approximations are computed with $\Delta x = 3.13 \cdot 10^{-3}$.

Errors for the approximation of $v^*(0, \cdot)$

| $\Delta x$ | $E_{\infty}$ | $E_2$ | $p_{\infty}$ | $p_2$ |
|------------|--------------|-------|---------------|------|
| 5.00 · $10^{-2}$ | 5.58 · $10^{-2}$ | 3.80 · $10^{-2}$ | - | - |
| 2.50 · $10^{-2}$ | 1.43 · $10^{-2}$ | 1.29 · $10^{-2}$ | 1.91 | 1.55 |
| 1.25 · $10^{-2}$ | 4.25 · $10^{-3}$ | 3.24 · $10^{-3}$ | 1.74 | 1.99 |
| 6.25 · $10^{-3}$ | 8.84 · $10^{-4}$ | 7.94 · $10^{-4}$ | 2.27 | 2.01 |

Errors for the approximation of $\partial_x v^*(0, \cdot)$

| $\Delta x$ | $E_{\infty}$ | $E_2$ | $p_{\infty}$ | $p_2$ |
|------------|--------------|-------|---------------|------|
| 5.00 · $10^{-2}$ | 9.07 · $10^{-2}$ | 4.82 · $10^{-2}$ | - | - |
| 2.50 · $10^{-2}$ | 1.81 · $10^{-2}$ | 6.79 · $10^{-3}$ | 2.32 | 2.82 |
| 1.25 · $10^{-2}$ | 4.81 · $10^{-3}$ | 1.36 · $10^{-3}$ | 1.91 | 2.32 |
| 6.25 · $10^{-3}$ | 7.64 · $10^{-4}$ | 2.06 · $10^{-4}$ | 2.65 | 2.72 |

Table 7. Errors and convergence rates for the approximation of $v^*(0, \cdot)$ and $\partial_x v^*(0, \cdot)$.

Errors for the approximation of $m^*(T, \cdot)$

| $\Delta x$ | $E_{\infty}$ | $E_2$ | $p_{\infty}$ | $p_2$ |
|------------|--------------|-------|---------------|------|
| 1.99 | 1 | 4.81 | 1.40 · $10^{-1}$ | - |
| 1.00 · $10^{-1}$ | 5.51 | 5.59 | 4.27 · $10^{-2}$ | 3.53 · $10^{-2}$ |
| 5.00 · $10^{-2}$ | 2.30 | 3.37 · $10^{-3}$ | 5.23 | 4.91 |
| 2.50 · $10^{-2}$ | 3.76 · $10^{-6}$ | 7.67 · $10^{-6}$ | 2.61 | 2.14 |

Table 8. Errors and convergence rates for the approximation of $m^*(T, \cdot)$. 
Figure 3. Approximated value function $v_\Delta(0, \cdot)$ (left), its derivate $\partial_x v_\Delta(0, \cdot)$ (center), and approximated density $m_\Delta(T, \cdot)$ (right).

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