Matching Drivers to Riders: A Two-stage Robust Approach

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Abstract

Matching demand (riders) to supply (drivers) efficiently is a fundamental problem for ride-sharing platforms who need to match the riders (almost) as soon as the request arrives with only partial knowledge about future ride requests. A myopic approach that computes an optimal matching for current requests ignoring future uncertainty can be highly sub-optimal. In this paper, we consider a two-stage robust optimization framework for this matching problem where future demand uncertainty is modeled using a set of demand scenarios (specified explicitly or implicitly). The goal is to match the current request to drivers (in the first stage) so that the cost of first stage matching and the worst case cost over all scenarios for the second stage matching is minimized. We show that the two-stage robust matching is NP-hard under various cost functions and present constant approximation algorithms for different settings of our two-stage problem. Furthermore, we test our algorithms on real-life taxi data from the city of Shenzhen and show that they substantially improve upon myopic solutions and reduce the maximum wait time of the second-stage riders by an average of 30% in our experimental results.

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1 Introduction

Matching demand (riders) with supply (drivers) is a fundamental problem for ride-hailing platforms such as Uber, Lyft and DiDi, who continually need to match drivers to current riders efficiently with only partial knowledge of future ride requests. A common approach in practice is batched matching: instead of matching each request sequentially as it arrives, aggregate the requests for a small amount of time (typically one to two minutes) and match all the requests to available drivers in one batch. However, computing this batch matching myopically without considering future requests can lead to a highly sub-optimal outcome for some subsequent riders. Motivated by this shortcoming, and by the possibility of using historical data to hedge against future uncertainty, we study a two-stage framework for the matching problem where the future demand uncertainty is modeled as a set of scenarios that are specified explicitly or implicitly. The goal is to compute a matching between the available drivers and current batch of riders such that the total worst-case cost of first stage and second stage matching is minimized. More specifically, we consider an adversarial model of uncertainty where the adversary observes the first stage matching of our algorithms and presents a worst-case scenario from the list of specified scenarios in the second stage. We primarily focus on the case where the first stage cost is the average weight of the first stage matching, and the second stage cost is the highest edge weight in the second stage matching. This is motivated by the goal of computing a low-cost first stage matching while also minimizing the waiting time for any ride in the worst-case scenario in the second stage. We also consider other metrics for the total cost and present related results.

Two-stage robust optimization is a popular model for hedging against uncertainty [10, 16]. Several combinatorial optimization problems have been studied in this model, including Set Cover and Capacity Planning, [13, 14]. Facility Location [3] and Network flow [1]. Two-stage matching problems with uncertainty, however, have not been studied extensively. They have been considered in the stochastic setting with uncertainty over the edges [23, 11]. Matuschke et al. [27] considered a two-stage version of the uni-chromatic problem (where there is no distinction between servers and clients). Their model can be seen as online min-cost matching with recourse while our model focuses on the worst-case performance with respect to an uncertainty set.

In this paper, we initiate the study of a two-stage robust approach for matching problems. We study the hardness of approximation of our two-stage problem under different cost functions and present constant approximation algorithms in several settings for both the implicit and explicit models of uncertainty. Furthermore, we test our algorithms on real-life taxi data from the city of Shenzhen and show that they significantly improve upon classical greedy solutions.

1.1 Results and Contributions

Problem definition. We consider the following Two-stage Robust Matching Problem. We are given a set of drivers $D$, a set of first stage riders $R_1$, a universe of potential second stage riders $R_2$, and a set of second stage scenarios $S \subseteq \mathcal{P}(R_2)$ where $\mathcal{P}(R_2)$ is the power set of $R_2$, the set of all subsets of $R_2$. We are given a metric distance $d$ on $V = R_1 \cup R_2 \cup D$. The goal is to find a subset of drivers $D_1 \subseteq D$ ($|D_1| = |R_1|$) to match all the first stage riders $R_1$ such that the sum of cost of first stage matching and worst-case cost of second stage matching (between $D \setminus D_1$ and the riders in the second stage scenario) is minimized. More specifically,

$$\min_{D_1 \subseteq D} \left\{ \text{cost}_1(D_1, R_1) + \max_{S \in \mathcal{S}} \text{cost}_2(D \setminus D_1, S) \right\}.$$ 

The first stage decision is denoted $D_1$ and its cost is $\text{cost}_1(D_1, R_1)$. Similarly, $\text{cost}_2(D \setminus D_1, S)$
is the second stage cost for scenario $S$, and $\max\{\text{cost}_2(D \setminus D_1, S) \mid S \in S\}$ is the worst-case cost over all possible scenarios. Let $|R_1| = m$, $|R_2| = n$. We denote the objective function for a feasible solution $D_1$ by

$$f(D_1) = \text{cost}_1(D_1, R_1) + \max_{S \in S} \text{cost}_2(D \setminus D_1, S).$$

We assume that there are sufficiently many drivers to satisfy both first and second stage demand. Given an optimal first-stage solution $D_1^*$, we denote

$$OPT_1 = \text{cost}_1(D_1^*, R_1), \quad OPT_2 = \max\{\text{cost}_2(D \setminus D_1^*, S) \mid S \in S\}, \quad OPT = OPT_1 + OPT_2.$$

As we mention earlier, we primarily focus on the setting where the first stage cost is the average weight of matching between $D_1$ and $R_1$, and the second stage cost is the bottleneck matching cost between $D \setminus D_1$ and $S$. We refer to this variant as the Two-Stage Robust Matching Bottleneck Problem (TSRMB). We also consider several other cost variants and present results in Section 6. Formally, let $M_1$ be the minimum weight perfect matching between $R_1$ and $D_1$, and given a scenario $S$, let $M_2^S$ be the bottleneck matching between the scenario $S$ and the available drivers $D \setminus D_1$, then the cost functions for the TSRMB are:

$$\text{cost}_1(D_1, R_1) = \frac{1}{m} \sum_{(i,j) \in M_1} d(i,j), \quad \text{and} \quad \text{cost}_2(D \setminus D_1, S) = \max_{(i,j) \in M_2^S} d(i,j).$$

The difference between the first and second stage metric is motivated by the fact that the platform has access to the current requests and can exactly compute the cost of matching these first stage requests. On the other hand, to ensure the robustness of the solution over the second stage uncertainty, we require for all second stage assignments to have low waiting times by accounting for the maximum wait time in every scenario. Note that we choose the first stage cost to be the average matching weight instead of the total weight for homogeneity reasons, so that first and second stage costs have comparable magnitudes.

**Scenario model of uncertainty.** Two common approaches to model uncertainty in robust optimization problems are to either explicitly enumerate all the realizations of the uncertain parameters or to specify them implicitly by a set of constraints over the uncertain parameters. In this paper, we consider both models. In the **explicit model**, we are given a list of scenarios: $S = \{S_1, \ldots, S_p\}$. In the **implicit model**, we consider the setting where we are given a universe of second stage riders, $R_2$ and any subset of size less than $k$ can be a scenario. Therefore, the set of scenarios is $S = \{S \subset R_2 \text{ s.t. } |S| \leq k\}$ for a given $k$. Note that the total number of possible scenarios is exponential in $k$; however, they are specified implicitly. This model is widely used in the Robust Optimization literature \cite{5, 18} and is known as budget of uncertainty or cardinality constraint set.

**Hardness.** We show that TSRMB is NP-hard even for two scenarios and NP-hard to approximate within a factor better than 2 for three or more scenarios. For the case of implicit model of uncertainty, we show that even when the number of scenarios is small, specifically $k = 1$, the problem is NP-hard to approximate within a factor better than 2. Given these hardness results, we focus on designing approximation algorithms for the TSRMB problem.

A natural candidate to address two-stage problems is the greedy approach that minimizes only the first stage cost without considering the uncertainty in the second stage. However, we show that this myopic approach can be bad, namely $\Omega(m) \cdot OPT$.\footnote{The bottleneck matching problem is to find a maximum matching that minimizes the length of the longest edge.}
Approximations algorithms. We first consider the case of a small number of explicit scenarios. This model is motivated by the desire to use historical data from past riders as our list of explicit scenarios. Our main result in this case is a constant approximation algorithm for TSRMB with two scenarios (Theorem 3). We further generalize the ideas of this algorithm to show a constant approximation for TSRMB with a fixed number of scenarios (Theorem 4). Our approximation does not depend on the number of first stage riders or the size of scenarios but scales with the number of scenarios. In particular, we have the following theorems.

Theorem 3 (restated). There is an algorithm that yields a 5-approximation to the TSRMB problem with 2 scenarios.

Theorem 4 (restated). There is an algorithm that yields a $O(p^{1.59})$-approximation to the TSRMB with $p$ explicit scenarios.

The main idea in our algorithms is to reduce the TSRMB problem with multiple scenarios to an instance with a single representative scenario while losing only a small factor. We then solve the single scenario instance (which can be done exactly in polynomial time) and recover a constant-factor approximation for our original problem. The challenge in constructing a single representative scenario is to find the right trade-off between effectively capturing the demand of all second stage riders and keeping the cost of this scenario close to the optimal cost of the original instance.

For the implicit model of uncertainty, the scenarios can be exponentially many in $k$, which makes even the evaluation of the total cost of a feasible solution challenging and not necessarily achievable in polynomial time. Our analysis depends on the imbalance between supply and demand. In fact, when the number of drivers is very large compared to riders, the problem is less interesting in practice. However, the problem becomes interesting when the supply and demand are comparable. In this case, drivers might need to be shared between different scenarios. This leads us to define the notion of surplus $\ell = |D| - |R_1| - k$, which is the maximum number of drivers that we can afford not to use in a solution. As a warm-up, we first show that if the surplus is equal to zero, (in this case all the drivers need to be used), using any scenario as a representative scenario and solving the single scenario instance gives a 3-approximation to TSRMB. The problem becomes significantly more challenging even with a small surplus. We show that under a reasonable assumption on the size of scenarios, there is a constant approximation to the TSRMB in the regime when the surplus $\ell$ is smaller than the demand $k$ (Theorem 5). This result is quite involved and requires several different new ideas and techniques to overcome the exponential number of scenarios.

Theorem 5 (restated). There is an algorithm that yields a $\frac{17}{\ell} \times k$-approximation to the TSRMB problem with implicit scenarios, when $\ell < k$ and $k \leq \sqrt{n}$.

The algorithm in Theorem 5 finds a clustering of drivers and riders that yields a simplified instance of TSRMB which can be solved within a constant factor. We show that we can cluster the riders into a ball (riders close to each others) and a set of outliers (riders far from each other) and apply some of our ideas from the analysis of two scenario on these two sets. Finally, since the evaluation problem is challenging because of the exponentially many scenarios, our algorithm constructs a set of a polynomial number of proxy scenarios on which we can evaluate any feasible solution within a constant approximation. We also address the case of arbitrary surplus if each scenario has only a single rider ($k = 1$). While this case has only polynomially many scenarios of size 1 each, it is still NP-hard and we use different techniques to get a constant-factor approximation (Theorem 6). In particular, we establish a connection between our problem and $q$-supplier problem [17] which we use as a subroutine to design a constant approximation algorithm in this case.
Theorem 6 (restated). There is an algorithm that yields a 15-approximation to the TSRMB with implicit scenarios in the case of $k = 1$.

Extensions and variants. While the majority of the paper studies the TSRBM problem, we also initiate the study of several other cost functions for two-stage matching problems both for adversarial and stochastic second stage scenarios. In particular, we consider the Two-Stage Stochastic Matching Bottleneck (TSSMB), where the first stage cost is the average weight of the matching, and the second stage is the expectation of the bottleneck matching cost over all scenarios. We also consider the Two-Stage Robust Matching problem (TSRM), where the first and second stage costs correspond both to the total weight of the matchings. Finally, we consider the Two-Stage Robust Bottleneck Bottleneck problem (TSRBB), where the first and second stage costs both correspond to the bottleneck matching cost. We study the hardness of these variants, and make a first attempt to present approximation algorithms under specific settings.

Experimental study. We implement our algorithms and test them on real-life taxi data from the city of Shenzhen [8]. Our experimental results show that our two-scenarios algorithm improves significantly upon the greedy algorithm both in and out of sample. Furthermore, the experiments show that while the second stage bottleneck of our algorithm is significantly less than the bottleneck of the greedy algorithm, the total weight of the matchings provided by the two algorithms are roughly similar. This implies that our algorithm reduces the maximal second stage wait time, without adding to the overall average wait time. For example, we show for the instances we consider in our experiments, that our two-scenarios algorithm reduces the maximum wait time of the second-stage riders by an average of 30%. See Section 7 for more details.

Outline. The paper is organized as follows. We review relevant literature in Section 2. In Section 3, we introduce some preliminary results on the hardness of TSRMB. We study the performance of the greedy approach and finally present a subroutine to solve the deterministic TSRMB with one scenario. In Section 4, we study TSRMB with explicit scenarios. In Section 5, we consider the case of implicit scenarios. Section 6 explores other variants of the two-stage robust matching problem with different cost functions. We present our numerical experiments on a set of real-life taxi data from the city of Shenzhen in Section 7.

2 Related Work

Finding a bipartite matching of minimum weight, is one of the classical problems in algorithmic graph theory and combinatorial optimization. The online version of this problem has received a considerable amount of attention over the years [20, 22, 24, 30, 7]. In this setting, we are given a known set of servers while a set of clients arrive online. In the online bipartite metric matching problem servers and clients correspond to points from a metric space. Upon arrival, each client must be matched to a server irrevocably, at cost equal to their distance. For general metric spaces, there is a tight bound of $(2n - 1)$ on the competitiveness factor of deterministic online algorithms, where $n$ is the number of servers [24, 20]. In the random arrival model, a natural question is whether randomization could help obtain an exponential improvement for general metric spaces. Meyerson, et al. [28] and Bansal et al. [2] provided poly-logarithmic competitive randomized algorithms for the problem. Recently, Raghvendra [31] presented a $O(\log n)$-competitive algorithm in the random arrival model.

Within two-stage stochastic optimization, matching has been studied in [23, 26, 11]. Kong and Schaefer [26] introduce the stochastic two-stage maximum matching problem. They prove that the
problem is NP-hard when the number of scenarios is an input of the problem and provide 1/2-
approximation algorithm. Escoffier et al. [11] further study this problem, strengthen the hardness
results, and slightly improve the approximation ratio. More recently, Katriel et al. [23] study
two stochastic minimum weight maximum matching problems in bipartite graphs. In their two
variants, the uncertainty is respectively on the second stage edge cost of the edges and on the set
of vertices to be matched. The authors present approximation algorithms and inapproximability
results. Matuschke et al. proposed a two-stage robust model for minimum weight matching with
recourse [27]. In the first stage, a perfect matching between 2n given nodes must be selected; in
the second stage 2k new nodes are introduced. The goal is to produce α-competitive matchings at
the end of both stages, and such that the number of edges removed from the first stage matching
is at most βk. Our model for TSRMB is different in 3 main aspects: 1) In our model the second
stage vertices come from an uncertainty set whereas in their model the only information given is
the number of second stage vertices. 2) We do not allow any recourse and our first stage matching
is irrevocable. 3) Our second stage cost is the bottleneck weight instead of the total weight. In
general, a bottleneck optimization problem on a graph with edge costs is the problem of finding a
subgraph of a certain kind that minimizes the maximum edge cost in the subgraph. The bottleneck
objective contrasts with the more common objective of minimizing the sum of edge costs. Several
Bottleneck problems have been considered, e.g. Shortest Path Problem [19, 6], Spanning Tree and
Maximum Cardinality Matching [14], and TSP problems [15] (see [17] for a compilation of graph
bottleneck problems).

3 Preliminaries

In this section, we study the hardness of approximation for TSRMB. We also examine the challenges
with the natural greedy approach for solving TSRMB. We finally present a subroutine to solve the
single scenario case that we will use later on in our general algorithms.

NP-hardness. We show that TSRMB is NP-hard under both the implicit and explicit models
of uncertainty. In the explicit model, TSRMB is NP-hard even for two scenarios. Moreover, we
show that it is NP-hard to approximate within a factor better than 2 even for three scenarios. In
the explicit model with a polynomial number of scenarios, it is clear that the problem is in NP.
However, in the implicit model, even though the problem can be described with a polynomial size
input, it is not clear that we can compute the total cost function in polynomial time since there
could be exponentially many scenarios. We also show that it is NP-hard to approximate TSRMB
with an implicit model of uncertainty within a factor better than 2 even when k = 1. The details
appear in Appendix A.

Theorem 1. In the explicit model of uncertainty, TSRMB is NP-hard even when the number of
scenarios is equal to 2. Furthermore, when the number of scenarios is ≥ 3, there is no (2 − ϵ)-
approximation algorithm for any fixed ϵ > 0, unless P = NP.

Theorem 2. In the implicit model of uncertainty, even when k = 1, there is no (2 − ϵ)-
approximation algorithm for TSRMB for any fixed ϵ > 0, unless P = NP.

Greedy Approach. A natural greedy approach is to choose the optimal matching for the first
stage riders R1 without considering the uncertainty in second stage in any way. We show via a
counterexample that this greedy approach could lead to a bad solution for TSRMB with a total
Counterexample. Consider the line example depicted in Figure 1 where we have $m$ first stage riders and $m + 1$ drivers that alternate on a line with distances $1$ and $1 - \epsilon$. There is only one second stage rider at the right endpoint of the line. A greedy matching would minimize the first stage cost by matching the first stage riders using the dashed edges, with an average weight of $1 - \epsilon$. When the second stage scenario is revealed, the rider can only be matched with the farthest driver for a cost of $1 + (2 - \epsilon)m$. Therefore the total cost of the greedy approach is $(2 - \epsilon)(m + 1)$, while the optimal cost is clearly equal to $2$. This example shows that the cost of the greedy algorithm for TSRMB could be far away from the optimal cost with an approximation ratio that scales with the dimension of the problem. The same observation generalizes to any number of scenarios by simply duplicating the second stage rider. Therefore any attempt to have a good approximation to the TSRMB needs to consider the second stage riders. In particular, we have the following lemma.

**Lemma 1.** The cost of the Greedy algorithm can be $\Omega(m \cdot OPT)$.

**Single Scenario.** The deterministic version of the TSRMB problem, i.e., when there is only a single scenario in the second stage, can be solved exactly in polynomial time. This is a simple preliminary result which we need for the general case. Denote $S$ a single second stage scenario. The instance $(R_1, S, D)$ of TSRMB is then simply given by

$$\min_{D_1 \subseteq D} \left\{ \text{cost}_1(D_1, R_1) + \text{cost}_2(D \setminus D_1, S) \right\}.$$ 

Since the second stage problem is a bottleneck problem, the value of the optimal second stage cost $w$ is one of the edge weights between $D$ and $S$. We iterate over all possible values of $w$ (at most $|S| \cdot |D|$ values), delete all edges between $R_2$ and $D$ with weights strictly higher than $w$ and set the weight of the remaining edges between $S$ and $D$ to zero. This reduces the problem to finding a minimum weight maximum cardinality matching. Below, are presented the details of our algorithm. We refer to it as **TSRMB-1-Scenario** (or Algorithm 1) in the rest of this paper.

We define the bottleneck graph of $w$ to be $BOTTLENECKG(w) = (R_1 \cup S \cup D, E_1 \cup E_2)$ where $E_2 = \{(i, j) \in D \times S, d(i, j) \leq w\}$ and $E_1 = \{(i, j) \in D \times R_1\}$. Furthermore, we assume that there are $q$ edges $\{e_1, \ldots, e_q\}$ between $S$ and $D$ with weights $w_1 \leq w_2 \leq \ldots \leq w_q$.

Note that the arg min in the last step of Algorithm 1 is only taken over values of $i$ for which there was no certificate of failure.

**Lemma 2.** **TSRMB-1-Scenario** (Algorithm 1) provides an exact solution to TSRMB with a single scenario.

**Proof.** Let $OPT_1$ and $OPT_2$ be the first and second stage cost of an optimal solution, and $i \in \{1, \ldots, q\}$ such that $w_i = OPT_2$. In this case, $G_i$ contains all the edges of this optimal solution. By
Algorithm 1: TSRMB-1-Scenario$(R_1, S, D)$

**Input:** First stage riders $R_1$, scenario $S$ and drivers $D$.

**Output:** First stage decision $D_1$.

1: for $i \in \{1, \ldots, q\}$ do
2: \ $G_i := BOTTLENECKG(w_i)$. \ 
3: \ Set all weights between $D$ and $S$ in $G_i$ to be 0. \ 
4: \ $M_i :=$ minimum weight maximum cardinality matching on $G_i$. \ 
5: \ if $R_1 \cup S$ is not completely matched in $M_i$ then \ 
6: \ output certificate of failure. \ 
7: \ else \ 
8: \ $D_i^1 :=$ first stage drivers in $M_i$. \ 
9: \ return $D_1 = \arg \min_{D_1^i: 1 \leq i \leq q} \left\{ \text{cost}_1(D_1^i, R_1) + \text{cost}_2(D \setminus D_i^1, S) \right\}$. \ 

setting all the edges in $E_2$ to 0, we are able to compute a minimum weight maximum cardinality matching between $R_1 \cup S$ and $D$ that matches both $R_1$ and $S$ and minimizes the weight of the edges matching $R_1$. The first stage cost of this matching is less than $OPT_1$, the second stage cost is clearly less than $OPT_2$ because we only allowed edges with weight less than $OPT_2$ in $G_i$. \ 

We also observe that we can use binary search in Algorithm 1 to iterate over the edge weights. For an iteration $i$, a failure to find a minimum weight maximum cardinality matching on $G_i$ that matches both $R_1$ and $S$ implies that we need to try an edge weight higher than $w_i$. On the other hand, if $M_i$ matches $R_1$ and $S$ such that $D_i^1$ gives a smaller total cost, then the optimal bottleneck value is lower than $w_i$. \ 

4 Explicit Scenarios

In this section, we consider TSRMB under the explicit model of uncertainty where we have an explicit list of scenarios for the second-stage and we optimize over the worst case scenario realization. We first present a constant factor approximation for TSRMB for the case of two scenarios. We then extend our result to the case of any fixed number of scenarios. However, the approximation factor scales with the number of scenarios $p$ as $O(p^{1.59})$. The idea of our algorithm is to reduce the instance of TSRMB with $p$ scenarios to an instance with only a single representative scenario by losing a small factor and then use Algorithm 1 to solve the single scenario instance. To illustrate the core ideas of our algorithm, we focus on the case of two scenarios first and then extend it to a constant number of scenarios.

4.1 Two scenarios

Consider two scenarios $S = \{S_1, S_2\}$. First, we can assume without loss of generality that we know the exact value of $OPT_2$ which corresponds to one of the edges connecting second stage riders $R_2$ to drivers $D$ (we can iterate over all the weights of second stage edges). We construct a representative scenario that serves as a proxy for $S_1$ and $S_2$ as follows. In the second stage, if a pair of riders $i \in S_1$ and $j \in S_2$ are served by the same driver in the optimal solution, then they should be close to each other. Therefore, we can consider a single representative rider for each such pair. While it is not easy to guess all such pairs, we can approximately compute the representative riders by
solving a maximum matching on \( S_1 \cup S_2 \) with edges less than \( 2OPT_2 \). More formally, let \( G_I \) be the induced bipartite subgraph of \( G \) on \( S_1 \cup S_2 \) containing only edges between \( S_1 \) and \( S_2 \) with weight less than or equal to \( 2OPT_2 \). We compute a maximum cardinality matching \( M \) between \( S_1 \) and \( S_2 \) in \( G_I \), and construct a representative scenario containing \( S_1 \) as well as the unmatched riders of \( S_2 \). We solve the single scenario problem on this representative scenario using Algorithm 1 and return its optimal first stage solution. We show in Theorem 3 that this solution leads to 5-approximation for our problem. Our algorithm is described below.

**Algorithm 2:** Two explicit scenarios.

**Input:** First stage riders \( R_1 \), two scenarios \( S_1 \) and \( S_2 \), drivers \( D \) and value of \( OPT_2 \).

**Output:** First stage decision \( D_1 \).

1. Let \( G_I \) be the induced subgraph of \( G \) on \( S_1 \cup S_2 \) with only the edges between \( S_1 \) and \( S_2 \) of weights less than \( 2OPT_2 \).
2. Set \( M := \) maximum cardinality matching between \( S_1 \) and \( S_2 \) in \( G_I \).
3. Set \( S_2^{\text{Match}} := \{ r \in S_2 \mid \exists s \in S_1 \text{ s.t } (s, r) \in M \} \) and \( S_2^{\text{Unmatch}} = S_2 \setminus S_2^{\text{Match}} \).
4. return \( D_1 := \) TSRMB-1-Scenario\((R_1, S_1 \cup S_2^{\text{Unmatch}}, D)\).

**Theorem 3.** Algorithm 2 yields a solution with total cost less than \( OPT_1 + 5OPT_2 \) for TSRMB with 2 scenarios.

Recall that \( OPT_1 \) and \( OPT_2 \) are respectively the first-stage and second-stage cost of an optimal solution for our TSRMB problem with two scenarios. The proof of Theorem 3 relies on the following structural lemma where we show that the set \( D_1 \) returned by Algorithm 2 yields a total cost at most \( (OPT_1 + 3OPT_2) \) when evaluated only on the single representative scenario \( S_1 \cup S_2^{\text{Unmatch}} \).

**Lemma 3.** Let \( D_1 \) be the set of first stage drivers returned by Algorithm 2. Then,

\[
\text{cost}_1(D_1, R_1) + \text{cost}_2(D \setminus D_1, S_1 \cup S_2^{\text{Unmatch}}) \leq OPT_1 + 3OPT_2.
\]

**Proof.** To prove the lemma, it is sufficient to show the existence of a matching \( M_a \) between \( R_1 \cup S_1 \cup S_2^{\text{Unmatch}} \) and \( D \) with a total cost less than \( OPT_1 + 3OPT_2 \). This would imply that the optimal solution \( D_1 \) of TSRMB-1-Scenario\((R_1, S_1 \cup S_2^{\text{Unmatch}}, D)\) has a total cost less than \( OPT_1 + 3OPT_2 \) and concludes the proof of the lemma. We show the existence of \( M_a \) by construction.

- **Step 1.** We first match \( R_1 \) with their mates in the optimal solution of TSRMB. Hence, the first stage cost of our constructed matching \( M_a \) is \( OPT_1 \).

- **Step 2.** Now, we focus on \( S_2^{\text{Unmatch}} \). Let \( S_2^{\text{Unmatch}} = S_{12} \cup S_{22} \) be a partition of \( S_2^{\text{Unmatch}} \) where \( S_{12} \) contains riders with a distance less than \( 2OPT_2 \) from \( S_1 \) and \( S_{22} \) contains riders with a distance strictly bigger than \( 2OPT_2 \) from \( S_1 \), where the distance from a set is the minimum distance to any element of the set. A rider in \( S_{22} \) cannot share any driver with a rider from \( S_1 \) in the optimal solution of TSRMB, because otherwise, the distance between these riders will be less than \( 2OPT_2 \) by using the triangle inequality. Therefore we can match \( S_{22} \) to their mates in the optimal solution and add them to \( M_a \), without using the optimal drivers of \( S_1 \). We pay less than \( OPT_2 \) for matching \( S_{22} \).

- **Step 3.** We still need to simultaneously match riders in \( S_1 \) and \( S_{12} \) to finish the construction of \( M_a \). Notice that some riders in \( S_{12} \) might share their optimal drivers with riders in \( S_1 \). We can assume without loss of generality that all riders in \( S_{12} \) share their optimal drivers
with $S_1$ (otherwise we can match them to their optimal drivers without affecting $S_1$). Denote $S_{12} = \{r_1, \ldots, r_q\}$ and $S_1 = \{s_1, \ldots, s_k\}$. For each $i \in [q]$ let’s say $s_i \in S_1$ is the rider that shares its optimal driver with $r_i$. We show that $q \leq |M|$. In fact, every rider in $S_{12}$ shares its optimal driver with a different rider in $S_1$, and is therefore within a distance $2OPT_2$ from $S_1$ by the triangle inequality. But since $S_{12}$ is not covered by the maximum cardinality matching $M$, this implies by the maximality of $M$ that there are $q$ other riders from $S_2^\text{Match}$ that are covered by $M$. Hence $q \leq |M|$. Finally, let $\{t_1, \ldots, t_q\} \subset S_2^\text{Match}$ be the mates of $\{s_1, \ldots, s_q\}$ in $M$, i.e., $(s_i, t_i) \in M$ for all $i \in [q]$. Recall that $d(s_i, t_i) \leq 2OPT_2$ for all $i \in [q]$. In what follows, we describe how to match $S_{12}$ and $S_1$:

- For $i \in [q]$, we match $r_i$ to its optimal driver and $s_i$ to the optimal driver of $t_i$. This is possible because the optimal driver of $t_i$ cannot be the same as the optimal driver of $r_i$ since both $r_i$ and $t_i$ are part of the same scenario $S_2$. Therefore, we pay a cost $OPT_2$ for the riders $r_i$ and a cost $3OPT_2$ (follows from the triangle inequality) for the riders $s_i$ where $i \in [q]$.

- We still need to match $\{s_{q+1}, \ldots, s_k\}$. Consider a rider $s_j$ with $j \in \{q + 1, \ldots, k\}$. If the optimal driver of $s_j$ is not shared with any $t_i \in \{t_1, \ldots, t_q\}$, then this optimal driver is still available and can be matched to $s_j$ with a cost less than $OPT_2$. If the optimal driver of $s_j$ is shared with some $t_i \in \{t_1, \ldots, t_q\}$, then $s_j$ is also covered by $M$. Otherwise $M$ can be augmented by deleting $(s_i, t_i)$ and adding $(r_i, s_i)$ and $(s_j, t_i)$. Therefore $s_j$ is covered by $M$ and has a mate $\tilde{t}_j \in S_2^\text{Match} \setminus \{t_1, \ldots, t_q\}$. Furthermore, the driver assigned to $\tilde{t}_j$ is still available. We can then match $s_j$ to the optimal driver of $\tilde{t}_j$. Similarly if the optimal driver of some $s_{j'} \in \{s_{q+1}, \ldots, s_k\} \setminus \{s_j\}$ is shared with $\tilde{t}_j$, then $s_{j'}$ is covered by $M$. Otherwise $(r_i, s_i, t_i, s_j, \tilde{t}_j, s_{j'})$ is an augmenting path in $M$. Therefore $s_{j'}$ has a mate in $M$ and we can match $s_{j'}$ to the optimal driver of its mate. We keep extending these augmenting paths until all the riders in $\{s_{q+1}, \ldots, s_k\}$ are matched. Furthermore, the augmenting paths $(r_i, s_i, t_i, s_j, \tilde{t}_j, s_{j'} \ldots)$ starting from two different riders $r_i \in S_{12}$ are vertex disjoint. This ensures that every driver is used at most once. Again, by the triangle inequality, the edges that match $\{s_{q+1}, \ldots, s_k\}$ in our solution have weights less then $3OPT_2$.

Putting it all together, we have constructed a matching $M_a$ where the first stage cost is exactly $OPT_1$ and the second-stage cost is less than $3OPT_2$ since the edges used for matching $S_1 \cup S_2^{\text{Unmatch}}$ in $M_a$ have a weight less than $3OPT_2$. Therefore, the total cost of $M_a$ is less than $OPT_1 + 3OPT_2$.

Proof of Theorem 3. Let $D_1$ be the set of drivers returned by Algorithm 2. Lemma 3 implies that

$$\text{cost}_1(D_1, R_1) + \text{cost}_2(D \setminus D_1, S_1) \leq OPT_1 + 3OPT_2$$

and

$$\text{cost}_3(D_1, R_1) + \text{cost}_2(D \setminus D_1, S_2^{\text{Unmatch}}) \leq OPT_1 + 3OPT_2.$$

We have $S_2 = S_2^{\text{Match}} \cup S_2^{\text{Unmatch}}$. If the scenario $S_2$ is realized, we use the drivers that were assigned to $S_1$ in the matching constructed in Lemma 3 to match $S_2^{\text{Match}}$. This is possible with edges of weights less than $\text{cost}_2(D \setminus D_1, S_1) + 2OPT_2$ because by definition $S_2^{\text{Match}}$ are connected to $S_1$ within edges of weight less than $2OPT_2$. Therefore,

$$\text{cost}_2(D \setminus D_1, S_2) \leq \max \{ \text{cost}_2(D \setminus D_1, S_2^{\text{Unmatch}}), \text{cost}_2(D \setminus D_1, S_1) + 2OPT_2 \}.$$
and therefore
\[ cost_1(D_1, R_1) + cost_2(D \setminus D_1, S_2) \leq OPT_1 + 5OPT_2. \]  

From (1) and (2), we conclude that
\[ cost_1(D_1, R_1) + \max_{S \in \{S_1, S_2\}} cost_2(D \setminus D_1, S) \leq OPT_1 + 5OPT_2. \]

4.2 Constant number of scenarios

We now consider the case of explicit list of scenarios, i.e., \( S = \{S_1, S_2, \ldots, S_p\} \). Building upon the ideas from Algorithm 2, we present \( O(p^{1.59}) \)-approximation to TSRMB with \( p \) scenarios. The idea of our algorithm is to construct the representative scenario recursively by processing pairs of “scenarios” at each step. Hence, we need \( O(\log_2 p) \) iterations to reduce the problem to an instance of a single scenario. At each iteration, we show that we only lose a multiplicative factor of 3 so that the final approximation ratio is \( O(3^{\log_2 p}) = O(p^{1.59}) \). We present details in Algorithm 3. Theorem 4 states the theoretical guarantee. Note that the approximation guarantee of our algorithm grows in a sub-quadratic manner with the number of scenarios \( p \) and it is an interesting question if there exists an algorithm for TSRMB with an approximation guarantee that does not depend on the number of scenarios.

Algorithm 3: \( p \) explicit scenarios.

**Input:** First-stage riders \( R_1 \), scenarios \( \{S_1, S_2, \ldots, S_p\} \), drivers \( D \) and value of \( OPT_2 \).

**Output:** First stage decision \( D_1 \).

1. Initialize \( \hat{S}_j := S_j \) for \( j = 1, \ldots, p \).
2. **for** \( i = 1, \ldots, \log_2 p \) **do**
3.   **for** \( j = 1, 2, \ldots, \frac{p}{2^i} \) **do**
4.     \( \sigma(j) = j + \frac{p}{2^i} \)
5.     \( M_j := \text{maximum cardinality matching between } \hat{S}_j \text{ and } \hat{S}_{\sigma(j)} \text{ with edges of weight less} \)
6.     \( \text{than } 2 \cdot 3^{i-1} \cdot OPT_2. \)
7.     \( \hat{S}_{\sigma(j)}^\text{Match} := \{ r \in \hat{S}_{\sigma(j)} | \exists s \in \hat{S}_j \text{ s.t } (s, r) \in M_j \}. \)
8.     \( \hat{S}_{\sigma(j)}^\text{Unmatch} := \hat{S}_{\sigma(j)} \setminus \hat{S}_{\sigma(j)}^\text{Match} \)
9.     \( \hat{S}_j = \hat{S}_j \cup \hat{S}_{\sigma(j)}^\text{Unmatch}. \)

**return** \( D_1 := \text{TSRMB-1-Scenario}(R_1, \hat{S}_1, D). \)

**Theorem 4.** Algorithm 3 yields a solution with total cost of \( O(p^{1.59}) \cdot OPT \) for TSRMB with an explicit list of \( p \) scenarios.

**Proof of Theorem 4.** The algorithm reduces the number of considered “scenarios” by half in every iteration, until only one scenario remains. In iteration \( i \), we have \( \frac{p}{2^i} \) scenarios that we aggregate in \( \frac{p}{2^i} \) pairs, namely \( (\hat{S}_j, \hat{S}_{\sigma(j)}) \) for \( j \in \{1, 2, \ldots, \frac{p}{2^i}\} \). For each pair, we construct a single representative scenario which plays the role of the new \( \hat{S}_j \) at the start of the next iteration \( i + 1 \).

**Claim 1.** There exists a first stage decision \( D_1^* \), such that at every iteration \( i \in \{1, \ldots, \log_2 p\} \), we have for all \( j \in \{1, 2, \ldots, \frac{p}{2^i}\} \):

1. \( R_1 \) can be matched to \( D_1^* \) with a first stage cost of \( OPT_1 \).
(2) \( \hat{S}_j \cup \tilde{S}_{\sigma(j)} \) can be matched to \( D \setminus D_1^* \) with a second stage cost less than \( 3^i \cdot OPT_2 \).

(3) There exists a matching between \( \hat{S}_{\sigma(j)}^{Match} \) and \( \tilde{S}_j \) with all edge weights less than \( 2 \cdot 3^{i-1} \cdot OPT_2 \).

Proof of Claim. Statement (3) follows from the definition of \( \hat{S}_{\sigma(j)}^{Match} \) in Algorithm. Let’s show (1) and (2) by induction over \( i \).

- Initialization: for \( i = 1 \), let’s take any two scenarios \( \hat{S}_j = S_j \) and \( \tilde{S}_{\sigma(j)} = S_{\sigma(j)} \). We know that these two scenarios can be matched to drivers of the optimal solution in the original problem with a cost less than \( OPT_2 \). In the proof of Lemma we show that if we use the optimal first stage decision \( D_1^* \) of the original problem, then we can match \( \hat{S}_j \) and \( \tilde{S}_{\sigma(j)} \) simultaneously to \( D \setminus D_1^* \) with a cost less than \( 3OPT_2 \).

- Maintenance. Assume the claim is true for all values less than \( i \leq \log_2 p - 1 \). We show it is true for \( i + 1 \). Since the claim is true for iteration \( i \), we know that at the start of iteration \( i + 1 \), for \( j \in \{1, \ldots, \frac{p}{2^i} \} \), \( \hat{S}_j \) can be matched to \( D \setminus D_1^* \) with a cost less than \( 3^i \cdot OPT_2 \). We can therefore consider a new TSRMB problem with \( \frac{p}{2^i} \) scenarios, where using \( D_1^* \) as a first stage decision ensures a second stage optimal value less than \( OPT_2 = 3^i \cdot OPT_2 \). By the proof of Lemma and by using \( D_1^* \) as a first stage decision in this problem, we ensure that for \( j \in \{1, \ldots, \frac{p}{2^i} \} \), \( \hat{S}_j \) and \( S_{\sigma(j)} \) can be simultaneously matched to \( D \setminus D_1^* \) with a cost less than \( 3OPT_2 \).

From Claim we have in the last iteration \( i = \log_2 p \),

- \( \hat{S}_1 \) can be matched to \( D_1^* \) with a first stage cost of \( OPT_1 \).
- \( \hat{S}_1 \) can be matched to \( D \setminus D_1^* \) with a second stage cost less than \( 3^{\log_2 p} \cdot OPT_2 \).

Computing the single scenario solution for \( \hat{S}_1 \) will therefore yield a first stage decision \( D_1 \) that gives a total cost less than \( OPT_1 + 3^{\log_2 p} \cdot OPT_2 \) when the second stage is evaluated on the scenario \( \hat{S}_1 \). We now bound the cost of \( D_1 \) on the original scenarios \( \{S_1, \ldots, S_p\} \). Consider a scenario \( S \in \{S_1, \ldots, S_p\} \). The riders in \( S \cap \hat{S}_1 \) can be matched to some drivers in \( D \setminus D_1 \) with a cost less than \( OPT_1 + 3^{\log_2 p} \cdot OPT_2 \). As for other riders of \( S \setminus \hat{S}_1 \), they are not part of \( \hat{S}_1 \) because they have been matched and deleted at some iteration \( i < \log_2 p \). Consider riders \( r \in S \setminus \hat{S}_1 \) that were matched and deleted from a representative scenario at some iteration, then by statement (3) in Claim, each \( r \) can be connected to a different rider in \( \hat{S}_1 \setminus (\hat{S}_1 \cap S) \) within a path of length at most

\[
\sum_{t=1}^{\log_2 p} 2 \cdot 3^{t-1} \cdot OPT_2 = (3^{\log_2 p} - 1) \cdot OPT_2.
\]

We know that \( R_1 \) and \( \hat{S}_1 \) can be matched respectively to \( D_1 \) and \( D \setminus D_1 \) with a total cost less than \( OPT_1 + 3^{\log_2 p} \cdot OPT_2 \). Therefore, we can match \( R_1 \) and \( S \) respectively to \( D_1 \) and \( D \setminus D_1 \) with a total cost less than

\[
OPT_1 + 3^{\log_2 p} \cdot OPT_2 + (3^{\log_2 p} - 1) \cdot OPT_2 = O(3^{\log_2 p}) \cdot OPT = O(p^{\ln 3/\ln 2}) \cdot OPT = O(p^{1.59}) \cdot OPT.
\]

Therefore, the worst-case total cost of the solution returned by Algorithm is \( O(p^{1.59}) \cdot OPT \). \( \square \)
5 Implicit Scenarios

In this section, we consider TSRMB under the implicit description of scenarios $S = \{S \subset R_2 \text{ s.t. } |S| \leq k\}$. This uncertainty model is widely used, however, it poses a challenge because the number of scenarios can be exponential. Therefore, even computing the worst case second stage cost, for a given first stage solution, might not be possible in polynomial time and we can no longer assume that we can guess $OPT_2$. We can show that the worst case occurs at scenarios of size exactly $k$, and hence we will focus on the implicit model of uncertainty where $S \subset R_2$ and $|S| = k$. Our analysis for this model of uncertainty will depend on the balance between supply (drivers) and demand (riders). In particular, we introduce the notion of surplus $\ell$ defined as the excess in the number of available drivers for matching first-stage riders and a second-stage scenario:

$$\ell = |D| - |R_1| - k.$$ 

As a warm-up, we start by studying the case of no surplus ($\ell = 0$). Then, we address the more general case with a small surplus of drivers. We end this section by examining the case of scenarios with a single rider ($k = 1$) and arbitrary surplus.

5.1 Warm-up: no surplus

When the number of drivers is equal to the number of first stage riders plus the size of scenarios (i.e., $\ell = 0$), we show a 3-approximation to TSRMB by simply solving a single scenario TSRMB with any of the scenarios. In fact, because there is no surplus, all scenarios are matched to the same drivers in the optimal solution. Hence, between any two scenarios, there exists a matching where all edge weights are less than $2OPT_2$. So by solving TSRMB with only one of these scenarios, we can recover a solution and bound the cost of the other scenarios within $OPT_1 + 3OPT_2$ using the triangular inequality. For completeness, we present below our algorithm for the no surplus case. We show its worst-case guarantee in Lemma 4.

**Algorithm 4:** Implicit scenarios with no surplus.

**Input:** First stage riders $R_1$, second stage riders $R_2$, size $k$ and drivers $D$.

**Output:** First stage decision $D_1$.

1. $S_1 :=$ a second stage scenario of size $k$.
2. $D_1 :=$ TSRMB-1-Scenario($R_1, S_1, D$).
3. return $D_1$.

**Lemma 4.** Algorithm 4 yields a solution with total cost less than $OPT_1 + 3OPT_2$ for TSRMB with implicit scenarios and no surplus.

**Proof.** Let $OPT_1$ and $OPT_2$ be the first and second stage cost of the optimal solution. Let $f(D_1)$ be the total cost of the solution returned by the algorithm. We claim that $f(D_1) \leq OPT_1 + 3OPT_2$. It is clear that $cost_1(D_1, R_1) + cost_2(D \setminus D_1, S_1) \leq OPT_1 + OPT_2$. Let $S \in S$ be another scenario. Because $|D| = |R_1| + k$, the optimal solution uses exactly the same $k$ drivers to match all the second stage scenarios. This implies that we can use the triangular inequality to find a matching between $S$ and $S_1$ of bottleneck cost less than $2OPT_2$. Hence for any scenario $S$,

$$cost_1(D_1, R_1) + cost_2(D \setminus D_1, S) \leq cost_1(D_1, R_1) + cost_2(D \setminus D_1, S_1) + 2OPT_2 \leq OPT_1 + 3OPT_2.$$

$\square$
We note that when the surplus is strictly greater than 0, Algorithm 4 no longer yields a constant approximation and its worst case performance can be as bad as \( \Omega(m) \). In particular, consider the example in Figure 2 with \( k = 1 \) and two second stage riders. The single scenario solution for \( S_1 \) uses only dashed edges in the first stage, and therefore uses the optimal second stage driver of \( S_2 \). Hence, if \( S_2 \) is realized, the cost of matching \( S_2 \) to the closest available driver is \( \Omega(m) \). By symmetry, solving the single scenario problem for \( S_2 \) yields a \( \Omega(m) \) bottleneck cost for \( S_1 \).

![Figure 2: Example instance with a surplus of one. Riders in first stage are depicted by black dots and drivers are indicated as black triangles. The two second stage riders are depicted as blue crosses. First and second stage optimum are depicted as solid green edges. \( S = \{S_1, S_2\} \) and \( k = 1 \).](image)

### 5.2 Small surplus

As observed in the example of Figure 2, the TSRMB problem becomes challenging even with a unit surplus of drivers and Algorithm 4 could be arbitrarily bad. Motivated by this, we focus on the case of a small surplus \( \ell \). In particular, we assume that \( \ell < k \), i.e., the excess in the total available drivers is smaller than the size of any scenario. We present a constant approximation algorithm in this regime for the implicit model of uncertainty where the size of scenarios is relatively small with respect to the size of the universe \( (k = O(\sqrt{n})) \). This technical assumption is needed for our analysis but it is not too restrictive and still captures the regime where the number of scenarios can be exponential. Our algorithm attempts to cluster the second stage riders in different groups (a ball and a set of outliers) in order to reduce the number of possible worst-case configurations. We then solve a sequence of instances with representative riders from each group. In what follows, we present our construction for these groups of riders.

**Our construction.** First, we show that many of the riders are contained in a ball with radius \( 3OPT_2 \). The center of this ball \( \delta \) can be found by enumerating over all drivers and selecting the one with the least maximum distance to its closest \( k \) second-stage riders, i.e.,

\[
\delta = \arg\min_{\delta' \in D} \max_{r \in R_k(\delta')} d(\delta', r) \tag{3}
\]

where \( R_k(\delta') \) is the set of the \( k \) closest second stage riders to \( \delta' \). Formally, we have the following lemma for which we defer the proof to Appendix 13.

**Lemma 5.** Suppose \( k \leq \sqrt{\frac{n}{2}} \) and \( \ell < k \) and let \( \delta \) be the driver given by (3). Then, the ball \( B \) centered at \( \delta \) with radius \( 3OPT_2 \) contains at least \( n - \ell \) second stage riders. Moreover, the distance between any of these riders and any rider in \( R_k(\delta) \) is less than \( 4OPT_2 \).

**Proof of Lemma 5.** Let \( \delta \) be the driver given by (3). We claim that the \( k \) closest riders to \( \delta \) are all within a distance less than \( OPT_2 \) from \( \delta \). Consider \( D_2^* \) to be the \( k + \ell \) drivers left for the
second stage in the optimal solution. Every driver in $D_2^*$ can be matched to a set of different second stage riders over different scenarios. Let us rank the drivers in $D_2^*$ according to how many different second stage riders they are matched to over all scenarios, in descending order. Formally, let $D_2^* = \{\delta_1, \delta_2, \ldots, \delta_{k+\ell}\}$ and let $R^*(\delta_i)$ be the second stage riders that are matched to $\delta_i$ in the optimal solution in some scenario. Let say

$$|R^*(\delta_1)| \geq \ldots \geq |R^*(\delta_{k+\ell})|. $$

We claim that $|R^*(\delta_1)| \geq k$. In fact, we have $\sum_{i=1}^{k+\ell} |R^*(\delta_i)| \geq n$ because every second stage rider is matched to at least one driver in some scenario. Therefore

$$|R^*(\delta_1)| \geq \frac{n}{k+\ell} \geq \frac{n}{2k} \geq k. $$

We know that all the second stage riders in $R^*(\delta_1)$ are within a distance less than $OPT_2$ from $\delta_1$. Therefore $\max_{r \in R_k(\delta_1)} d(\delta_1, r) \leq OPT_2$. But we know that by definition of $\delta_i$

$$\max_{r \in R_k(\delta)} d(\delta, r) \leq \max_{r \in R_k(\delta_1)} d(\delta_1, r) \leq OPT_2 $$

This proves that the $k$ closest second stage riders to $\delta$ are within a distance less than $OPT_2$. Let $R(\delta)$ be the set of all second stage riders that are within a distance less than $OPT_2$ from $\delta$. Recall that $R_k(\delta)$ is the set of the $k$ closest second stage riders to $\delta$. In the optimal solution, the scenario $R_k(\delta)$ is matched to a set of at least new $k-1$ drivers $\{\delta_{i_1}, \ldots, \delta_{i_{k-1}}\} \subset D_2^* \setminus \{\delta\}$. We show a lower bound on the size of $R(\delta)$ and the number of riders matched to $\{\delta_{i_1}, \ldots, \delta_{i_{k-1}}\}$ over all scenarios in the optimal solution.

**Claim 2.** $|R(\delta) \bigcup_{j=1}^{k-1} R^*(\delta_j)| \geq n - \ell$

**Proof.** Suppose the opposite, suppose that at least $\ell + 1$ riders from $R_2$ are not in the union. Let $F$ be the set of these $\ell + 1$ riders. Since $\ell + 1 \leq k$, we can construct a scenario $S$ that includes $F$. In the optimal solution, and in particular, in the second stage matching of $S$, at least one rider from $F$ needs to be matched to a driver from $\{\delta, \delta_{i_1}, \ldots, \delta_{i_{k-1}}\}$. Otherwise there are only $\ell$ second stage drivers left to match all of $F$. Therefore there exists $r \in F$ such that either $r \in R(\delta)$ or there exists $j \in \{1, \ldots, k-1\}$ such that $r \in R^*(\delta_j)$. This shows that $r \in R(\delta) \bigcup_{j=1}^{k-1} R^*(\delta_j)$, which is a contradiction. Therefore, at most $\ell$ second stage riders are not in the union.

**Claim 3.** For any rider $r \in R(\delta) \bigcup_{j=1}^{k-1} R^*(\delta_j)$, we have

$$d(r, \delta) \leq 3OPT_2 $$

**Proof.** If $r \in R(\delta)$ then by definition we have $d(r, \delta) \leq OPT_2$. Now suppose $r \in R^*(\delta_j)$ for $j \in [k-1]$. Let $r'$ be the rider from scenario $R_k(\delta)$ that was matched to $\delta_j$ in the optimal solution.

$$d(r, \delta) \leq d(r, \delta_j) + d(\delta_j, r') + d(r', \delta) \leq 3OPT_2.$$
From Claim 3, we see that the ball centered at \( \delta \), with radius \( 3OPT_2 \), contains at least \( n - \ell \) second stage riders in \( R(\delta) \bigcup_{j=1}^{k-1} R^*(\delta_{ij}) \). This proves the first part of the lemma. The second part is proved in the next claim.

**Claim 4.** For \( r_1 \in R_k(\delta) \) and \( r_2 \in R_k(\delta) \bigcup_{j=1}^{k-1} R^*(\delta_{ij}) \), we have

\[
d(r_1, r_2) \leq 4OPT_2
\]

**Proof.** Let \( r_1 \in R_k(\delta) \). If \( r_2 \in R_k(\delta) \) then \( d(r_1, r_2) \leq d(r_1, \delta) + d(\delta, r_2) \leq 2OPT_2 \). If \( r_2 \in R^*(\delta_{ij}) \) for some \( j \), and \( \delta' \) is the rider from scenario \( R_k(\delta) \) that was matched to \( \delta_{ij} \),

\[
d(r_1, r_2) \leq d(r_1, \delta) + d(\delta, \delta') + d(\delta', \delta_{ij}) + d(\delta_{ij}, r_2) \leq 4OPT_2.
\]

\[ \square \]

Now, let us focus on the rest of second stage riders. We introduce the following definition. We say that a rider \( r \in R_2 \) is an outlier if \( d(\delta, r) > 3OPT_2 \). Denote \( \{o_1, o_2, \ldots, o_\ell\} \) the farthest \( \ell \) riders from \( \delta \) with \( d(\delta, o_1) \geq d(\delta, o_2) \geq \ldots \geq d(\delta, o_{\ell}) \). Note that by Lemma 5, the \( n - \ell \) riders in \( B \) are not outliers and the only potential outliers could be in \( \{o_1, o_2, \ldots, o_\ell\} \). Let \( j^* \) be the threshold such that \( o_1, o_2, \ldots, o_{j^*} \) are outliers and \( o_{j^*+1}, \ldots, o_\ell \) are not, with the convention that \( j^* = 0 \) if there is no outlier. There are \( \ell + 1 \) possible values for \( j^* \). We call each of these possibilities a configuration. For \( j = 0, \ldots, \ell \), let \( C_j \) be the configuration corresponding to threshold candidate \( j \). Note that \( C_0 \) is the configuration where there is no outlier and \( C_{j^*} \) is the correct configuration (See Figure 3).

![Figure 3: Configuration \( C_{j^*} \)](image)

Now, we are ready to describe our algorithm. Recall that \( R_k(\delta) \) are the closest \( k \) second-stage riders to \( \delta \). For the sake of simplicity, we denote \( S_1 = R_k(\delta) \) and \( S_2 = \{o_1 \ldots o_\ell\} \). Note that \( S_2 \) is a feasible scenario since \( \ell < k \). For every configuration \( C_j \), we form a representative scenario using \( S_1 \) and \( \{o_1 \ldots o_\ell\} \). We solve TSRMB with this single representative scenario \( S_1 \cup \{o_1 \ldots o_j\} \) and denote \( D_1(j) \) the corresponding optimal solution, i.e.,

\[
D_1(j) = \text{TSRMB-1-Scenario}(R_1, S_1 \cup \{o_1 \ldots o_j\}, D).
\]

Since we cannot evaluate the cost of \( D_1(j) \) on all scenarios because we can have exponentially many, we evaluate the cost of \( D_1(j) \) on the two proxy scenarios \( S_1 \) and \( S_2 \). We finally show that the candidate \( D_1(j) \) with minimum cost over these two scenarios, gives a constant approximation to our original problem (See Theorem 5). The details of our algorithm are summarized below.
We present here a sketch of the proof. The complete details and proofs of the
Proof of Theorem 5.
Recall that TSRMB is NP-hard to approximate within a factor better than 2 even when
scenarios has a single rider (\(k = 1\)). Claim 5 establishes a bound on the cost of
\(f(\bar{D}_1(j)) \leq \max\{\beta_j + 4OPT_2, 3\beta_j + 2OPT_2\}\).
Suppose Algorithm 5 returns \(D_1(j)\) for some \(j\). From Claim 6 and the minimality of \(\beta_j\):
Recall \(f\) the objective function of TSRMB. In particular,
\[ f(D_1(j)) = cost_1(D_1(j), R_1) + \max_{S \in S} cost_2(D \setminus D_1(j), S) \]
Our proof is based on the following two claims. Claim 5 establishes a bound on the cost of
\(D_1(j^*)\) when evaluated on the proxy scenarios \(S_1\) and \(S_2\) and on all the scenarios in \(S\). Recall that
\(j^*\) is the threshold index for the outliers as defined earlier in our construction. Claim 6 bounds the
cost of \(f(D_1(j))\) for any \(j\). The proofs of both claims are presented in Appendix B.
Claim 5. \(\Omega_j + \Delta_j \leq OPT_1 + OPT_2\). and \(f(D_1(j^*)) \leq OPT_1 + 5OPT_2\).
Claim 6. For all \(j \in \{0, \ldots, \ell\}\) we have, \(\beta_j \leq f(D_1(j)) \leq \max\{\beta_j + 4OPT_2, 3\beta_j + 2OPT_2\}\).
Suppose Algorithm 5 returns \(D_1(j)\) for some \(j\). From Claim 6 and the minimality of \(\beta_j\):
\[ f(D_1(j)) \leq \max\{\beta_j + 4OPT_2, 3\beta_j + 2OPT_2\} \leq \max\{\beta_j + 4OPT_2, 3\beta_j + 2OPT_2\}. \]
From Claim 5 and Claim 6, we have \(\beta_{j^*} \leq f(D_1(j^*)) \leq OPT_1 + 5OPT_2\). We conclude that,
\[ f(D_1(\bar{j})) \leq \max\{OPT_1 + 9OPT_2, 3OPT_1 + 17OPT_2\} = 3OPT_1 + 17OPT_2. \]
5.3 Arbitrary surplus with \(k = 1\)
In this part, we consider TSRMB when the surplus can be arbitrary and each of the second stage
scenarios has a single rider (\(k = 1\)). We present a constant approximation algorithm for this case.
Recall that TSRMB is NP-hard to approximate within a factor better than 2 even when \(k = 1\).
In this case, the second stage objective function aims to minimize the maximum distance from the remaining drivers to the second stage riders. We show that our problem is closely related to an instance of the p-supplier problem \cite{17,29}. This is a a variant of the p-center problem on a bipartite graph where centers can only belong to one side of the graph. The idea of our algorithm is to save a set of drivers to the second-stage by solving a p-supplier problem for the second stage riders (using the 3-approximation algorithm in \cite{17}). Moreover, we reduce this set by pruning drivers that are close to each others within a threshold distance that depends on \(OPT\). Note that we can assume that we know \(OPT\) since the number of scenarios is exactly \(n\) and therefore we can evaluate any feasible solution in polynomial time. We show in Theorem \ref{thm:main} that the solution returned by Algorithm \ref{alg:main} gives a constant approximation.

**Algorithm 6:** Implicit scenarios with arbitrary surplus and \(k = 1\).

\begin{algorithm}
\begin{algorithmic}[1]
\State \textbf{Input:} First stage riders \(R_1\), second stage riders \(R_2\), drivers \(D\) and value of \(OPT\).
\State \textbf{Output:} First stage decision \(D_1\).
\State 1: Set \(p := |D| - |R_1|\).
\State 2: Solve the p-supplier problem on the bipartite graph \(D \cup R_2\), with centers in \(D\) using the 3-approximation algorithm in \cite{17}.
\State 3: Set \(D_2 := \) set of centers in the solution of the above p-supplier problem.
\State 4: \textbf{for} \(\delta \in D_2\) \textbf{do}
\State 5: \textbf{if} there exists another driver \(\delta' \in D_2\) such that \(d(\delta, \delta') \leq 8OPT\). \textbf{then}
\State 6: \(D_2 := D_2 \setminus \{\delta'\}\).
\State 7: \(M := \) minimum weight maximum cardinality matching \(R_1\) and \(D \setminus D_2\).
\State 8: \textbf{return} \(D_1 := \) drivers used in \(M\).
\end{algorithmic}
\end{algorithm}

**Theorem 6.** Algorithm \ref{alg:main} yields a solution with total cost less than \(OPT_1 + 15OPT_2\) for TSRMB with implicit scenarios and \(k = 1\).

Before presenting the proof of Theorem \ref{thm:main} let us introduce the p-supplier problem. The problem consists of \(n\) points in a metric space, that are partitioned into a client set \(C\) and a set of facilities \(F\). Additionally, we are given a bound \(p \leq |F|\). The objective is to open a set \(S \subset F\) of \(p\) facilities that minimizes the maximum distance of a client to its closest open facility. The p-supplier problem is a generalization of the p-center problem, where the client and facility sets are identical (see \cite{17,25} for more details). We use the 3-approximation algorithm presented in \cite{17} as a subroutine in Step 2 of Algorithm \ref{alg:main}.

**Proof of Theorem \ref{thm:main}**. To prove the theorem, we present the following two claims:

**Claim 7.** For all \(r \in R_2\), there exists \(\delta \in D_2\) s.t. \(d(r, \delta) \leq 11OPT_2\).

**Proof of Claim \ref{claim:7}**. Let \(D_1^*\) be an optimal solution for the TSRMB problem with \(k = 1\). \(D \setminus D_1^*\) is a feasible solution for the p-supplier problem on the bipartite graph \(D \cup R_2\), with centers in \(D\). The p-supplier cost for \(D \setminus D_1^*\) is equal to \(OPT_2\). Therefore, the 3-approximation computed in Step 2 of the algorithm has a cost less than \(3OPT_2\). This implies that initially, and before Step 4 of the algorithm, for every \(r \in R_2\) there exists \(\delta \in D_2\) s.t. \(d(r, \delta) \leq 3OPT_2\). If a driver \(\delta'\) is deleted in the loop of Step 4, then it is because there exists \(\delta \in D_2\) s.t. \(d(\delta, \delta') \leq 8OPT_2\). This implies that all the riders that were within distance \(3OPT_2\) from \(\delta'\) are now within a distance less than \(3OPT_2 + d(\delta, \delta') \leq 11OPT_2\) from \(\delta\). \(\square\)

**Claim 8.** For \(\delta \in D_2\), there exists \(r \in R_2\) s.t. \(d(r, \delta) \leq 3OPT_2\).

17
Proof of Claim 8. Initially, every driver in \( D_2 \) has at least one rider in \( R_2 \) that is within a distance less than \( 3OPT_2 \). Since the drivers that are not deleted from \( D_2 \) can only increase the set of riders to which they are the closest, every driver in \( D_2 \) always keeps at least one rider within a distance less than \( 3OPT_2 \). \( \square \)

We now show that we can match \( R_1 \) to \( D \setminus D_2 \) with a first stage cost less than \( OPT_1 + 4OPT_2 \). Let \( r \in R_1 \). If the optimal first stage driver of \( r \) is not in \( D_2 \), we simply match \( r \) to this optimal driver. On the other hand, suppose there exist \( r_i, r_j \in R_1 \) such that the optimal first stage drivers of \( r_i \) and \( r_j \), respectively \( \delta_i \) and \( \delta_j \), are both used in \( D_2 \). We show that we can match \( r_i \) and \( r_j \) to two different drivers in \( D \setminus D_2 \) within a distance less than \( OPT_1 + 4OPT_2 \). Since \( \delta_i, \delta_j \in D_2 \), there exists two different second stage riders \( s_i \in R_2 \) and \( s_j \in R_2 \) such that

\[
\begin{align*}
    d(\delta_i, s_i) &\leq 3OPT_2, \\
    d(\delta_j, s_j) &\leq 3OPT_2.
\end{align*}
\]

It is clear that \( s_i \neq s_j \), because otherwise \( d(\delta_i, \delta_j) \leq 6OPT_2 \) and either \( \delta_i \) or \( \delta_j \) would have been deleted from \( D_2 \). Let \( \delta_i^2 \) and \( \delta_j^2 \) be the optimal second stage drivers for \( s_i \) and \( s_j \) respectively. We argue that \( \delta_i^2 \neq \delta_j^2 \). Suppose \( \delta_i^2 = \delta_j^2 \), then

\[
    d(s_i, s_j) \leq d(\delta_i^2, s_i) + d(\delta_j^2, s_j) \leq 2OPT_2,
\]

and

\[
    d(\delta_i, \delta_j) \leq d(\delta_i, s_i) + d(s_i, s_j) + d(s_j, \delta_j) \leq 8OPT_2,
\]

but (4) implies that either \( \delta_i \) or \( \delta_j \) would have been deleted in step 3 of the algorithm. Therefore \( \delta_i^2 \neq \delta_j^2 \), and \( r_i \) (resp. \( r_j \)) can be matched to \( \delta_i^2 \) (resp. \( \delta_j^2 \)) within a distance less than

\[
    d(r_i, \delta_i^2) \leq d(r_i, \delta_i) + d(\delta_i, s_i) + d(s_i, \delta_i^2) \leq OPT_1 + 4OPT_2.
\]

We showed the existence of a matching between \( R_1 \) and \( D_1 \) with an average weight less than \( OPT_1 + 4OPT_2 \). We know from Claim 7 that the second stage cost is less than \( 11OPT_2 \). Therefore the total cost of the first stage decision \( D_1 \) is less than \( OPT_1 + 15OPT_2 \). \( \square \)

6 Other cost metrics

In this section, we initiate the study of other variants of two-stage matching problems, under both robust and stochastic models of uncertainty and for different cost functions. We define these problems, study their hardness of approximation and design approximation algorithms in some specific cases. We summarize our results below and defer all the details to Appendix C and Appendix D.

1. **Two-Stage Stochastic Matching Bottleneck Problem** (TSSMB). In this problem, the first stage cost is the same as the TSRMB (e.g. the average matching weight). However, we assume that we have an explicit list of scenarios \( S = \{S_1, \ldots, S_p\} \). Scenario \( S_i \) is realized with probability \( p_i \). The objective is to minimize the function

\[
    \min_{D_1 \subset D} \left\{ \text{cost}_1(D_1, R_1) + \sum_{i=1}^p p_i \cdot \text{cost}_2(D \setminus D_1, S_i) \right\},
\]

where \( \text{cost}_2(D \setminus D_1, S_i) \) is the bottleneck matching cost between \( D \setminus D_1 \) and scenario \( S_i \). In Appendix C we show this problem is NP-hard to approximate within a factor better than \( 4/3 \). We also provide an algorithm that yields a 3-approximation when there is no surplus.
2. **Two-Stage Robust Matching Problem** (TSRM). In this problem, the cost of the first stage is the total weight of the first stage matching, and the second stage cost is the total weight of the worst case matching over scenarios. We present the formal definition of this problem in Appendix D and show it is NP-hard even with two scenarios. Kalyanasundaram and Pruhs [20] consider the online version of this problem, and show that the greedy algorithm is 3-competitive for two stages and therefore yields a 3-approximation in the worst-case as well. We further improve this result and show a 7/3-approximation when there is no surplus.

3. **Two-Stage Robust Bottleneck Bottleneck Problem** (TSRBB). The only difference from the TSRMB is that the first stage cost is the bottleneck of the first stage matching. All our hardness and approximation algorithms from the TSRMB easily carry to this problem.

7 Numerical Experiments

In this section, we present an empirical comparison of Algorithm 2 with the greedy algorithm. We use a taxi data set from the city of Shenzhen to create realistic instances of the TSRMB problem.

7.1 Data

The data is collected for a month in the city of Shenzhen [8]. This data contains the GPS records of taxis in Shenzhen. The details of the data set are summarized in Table 1 where the sample rate means the interval between two adjacent GPS records. A trajectory is constructed by following one taxi between a pick-up (“Occupied” value change from 0 to 1) and a drop-off. A snapshot of the data is presented in Table 2.

| Size    | # Taxis | # Trajectories | Sample rate | Avg trip time |
|---------|---------|----------------|-------------|---------------|
| 32.7 GB | 9,475   | 6,068,516      | 10-30 s     | 863s          |

Table 1: Details of the taxi trajectory data

| Taxi ID | Time            | Longitude | Latitude | Speed | Direction | Occupied |
|---------|-----------------|-----------|----------|-------|-----------|----------|
| B97U79  | 2009-09-23 21:30:00 | 113.80275 | 22.66913 | 66    | 157       | 0        |
| B97U79  | 2009-09-23 21:30:20 | 113.80137 | 22.67106 | 18    | 157       | 1        |

Table 2: Example of the taxi trajectory data

7.2 Experiment Setup

We focus on the GPS records of downtown Shenzhen, with $|\text{longitude} - 114.075| \leq 0.075$ and $|\text{latitude} - 22.54| \leq 0.03$ (See Figure 4). In a specific time range, we locate the riders by following taxis and observing when the occupied entry changes from 0 to 1. This change means that a pickup occurred and the rider’s location is estimated to be the same as the taxi location at the time of pickup. For different days $d$ of the month, and different times $t$ of the day, we consider the pickups that were made in $[t, t + 1\text{min}]$ to be the first stage riders $R_1$. For the second stage riders, we construct two scenarios $S_1, S_2$ using the pickups that occurred in $[t + 1\text{min}, t + 2\text{min}]$ in $d - 7$ and $d - 14$ respectively, which represent the same day as $d$ in the two previous weeks. We also construct

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*The raw trajectory record can be found in [https://github.com/cbdog94/STL](https://github.com/cbdog94/STL).*
the realized scenario $S^*$, which contains the pickups in $[t + 1 \text{ min}, t + 2 \text{ min}]$ of day $d$. We use the taxis of day $d$ that were not occupied in the past 5 minutes before $t$ to sample the set of drivers $D$. The edge weights correspond to the distances between drivers and riders. Note that in the data set, the number of all available drivers in the past 5 minutes is considerably higher than the number of pickups. Hence, to simulate a busier time, we randomly sample $2.5 \times |R_1|$ drivers in every instance. For every instance, we preform 10 random driver samples, solve the problem for every sample, and report the average. We report results from different times of the 17th day of the month, with $S_1$ and $S_2$ constructed from the 3rd and the 10th day respectively.

7.3 Evaluation metrics and experimental results

We use Algorithm 2 to solve the TSRMB problem with second stage scenarios $S_1$ and $S_2$. We denote $\text{Alg}(S_1, S_2)$ the total cost of the solution returned by Algorithm 2 where the second stage cost is the worst-case cost over the scenarios $\{S_1, S_2\}$. We call this case In-sample. We denote $\text{Gr}(S_1, S_2)$ the total worst-case cost of the greedy solution that myopically solves the first stage and uses the remaining drivers to match $S_1$ or $S_2$. We compare the in-sample performance of Algorithm 2 with the greedy algorithm by computing the ratio $\text{Gr}(S_1, S_2)/\text{Alg}(S_1, S_2)$.

We also evaluate Algorithm 2 on out-of-sample data. In particular, we consider the solution (first stage drivers $D_1$) returned by Algorithm 2 and use $D \setminus D_1$ to satisfy the realized scenario $S^*$. We call this case Out-of-Sample. The idea is to use $S_1$ and $S_2$ as a prediction for $S^*$. $\text{Alg}(S^*)$ denotes the total cost of our solution on the realized scenario $S^*$. $\text{Gr}(S^*)$ denotes the total cost of the greedy solution that myopically solves the first stage and uses the remaining drivers to match $S^*$. Finally, $\text{OPT}(S^*)$ denotes the cost of the optimal solution that knows offline the scenario $S^*$, i.e., $\text{OPT}(S^*)$ is the cost of TSRMB problem with a single scenario $S^*$ which we compute using Algorithm 1. We compare the out-of-sample performance of Greedy and Algorithm 2 by computing the ratios $\text{Gr}(S^*)/\text{OPT}(S^*)$ and $\text{Alg}(S^*)/\text{OPT}(S^*)$. The in-sample and out-of-sample performances for different times on 09/17 are presented in Table 3. The columns “1st Stage” and “2nd Stage” denote the time range of the first and second stage respectively.
| 1st Stage | 2nd Stage | |D| |R1| |S*| |Out-of-sample| |In-sample|
|---|---|---|---|---|---|---|---|---|---|---|
| | | |Gr(S*)|OPT(S*)|Alg2(S*)|OPT(S*)|Gr(S1,S2)|Alg2(S1,S2)|
| 09:00-01 | 09:01-02 | 215 | 86 | 97 | 1.52 | 1.34 | 1.52 |
| 10:00-01 | 10:01-02 | 187 | 75 | 54 | 1.73 | 1.31 | 1.42 |
| 11:00-01 | 11:01-02 | 210 | 84 | 78 | 1.50 | 1.34 | 1.30 |
| 12:00-01 | 12:01-02 | 215 | 86 | 91 | 1.51 | 1.44 | 1.31 |
| 13:00-01 | 13:01-02 | 205 | 82 | 93 | 1.52 | 1.34 | 1.30 |
| 14:00-01 | 14:01-02 | 342 | 137 | 138 | 1.68 | 1.59 | 1.74 |
| 15:00-01 | 15:01-02 | 355 | 142 | 120 | 1.59 | 1.29 | 1.55 |
| 16:00-01 | 16:01-02 | 345 | 138 | 132 | 1.48 | 1.30 | 1.41 |
| 17:00-01 | 17:01-02 | 295 | 118 | 113 | 1.36 | 1.26 | 1.18 |
| 18:00-01 | 18:01-02 | 287 | 115 | 102 | 1.46 | 1.36 | 1.38 |
| 19:00-01 | 19:01-02 | 300 | 120 | 112 | 1.33 | 1.20 | 1.37 |
| 20:00-01 | 20:01-02 | 307 | 123 | 134 | 1.94 | 1.64 | 1.60 |
| 21:00-01 | 21:01-02 | 370 | 143 | 147 | 1.77 | 1.38 | 1.40 |

Table 3: In-sample and out-of-sample comparison between Greedy, Algorithm 2.

| 1st Stage | 2nd Stage | |D| |R1| |S*| |Gr2(S*)/Alg2(S*)| |Total Matching Ratio|
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | |
| 09:00-01 | 09:01-02 | 215 | 86 | 97 | 1.20 | 0.99 |
| 10:00-01 | 10:01-02 | 187 | 75 | 54 | 1.60 | 0.95 |
| 11:00-01 | 11:01-02 | 210 | 84 | 78 | 1.28 | 0.97 |
| 12:00-01 | 12:01-02 | 215 | 86 | 91 | 1.11 | 0.99 |
| 13:00-01 | 13:01-02 | 205 | 82 | 93 | 1.44 | 0.98 |
| 14:00-01 | 14:01-02 | 342 | 137 | 138 | 1.11 | 0.97 |
| 15:00-01 | 15:01-02 | 355 | 142 | 120 | 1.45 | 0.96 |
| 16:00-01 | 16:01-02 | 345 | 138 | 132 | 1.27 | 0.97 |
| 17:00-01 | 17:01-02 | 295 | 118 | 113 | 1.11 | 0.99 |
| 18:00-01 | 18:01-02 | 287 | 115 | 102 | 1.14 | 0.97 |
| 19:00-01 | 19:01-02 | 300 | 120 | 112 | 1.28 | 0.97 |
| 20:00-01 | 20:01-02 | 307 | 123 | 134 | 1.39 | 1.00 |
| 21:00-01 | 21:01-02 | 370 | 143 | 147 | 1.40 | 0.98 |

Table 4: Comparison of Algorithm 2 and Greedy on out-of-sample w.r.t the total matching weight and the second stage bottleneck.

Furthermore, in Table 4 we compare the second stage cost (bottleneck cost) of Greedy and of our solution on the out-of-sample scenario $S^*$. In particular, we report the ratio $Gr_2(S^*)/Alg_2(S^*)$, where $Gr_2(S^*)$ is the second stage cost if we use Greedy and the second stage scenario is $S^*$ and $Alg_2(S^*)$ is our second stage cost for scenario $S^*$ after we have used the solution returned by Algorithm 2. We also compute the ratio between the total weight of the greedy solution on $S^*$, and the total weight of the solution given by our algorithm when evaluated on $S^*$. This ratio is presented in the column “Total Matching Ratio”.

21
7.4 Discussion

We observe from Table 3 that our two-scenarios algorithm improves significantly upon the greedy algorithm both in-sample and out-of-sample. In-sample, our algorithm improves the total cost by an average of 42%. Out-of-sample, Table 3 shows that the greedy algorithm can be sub-optimal within 58% on average as compared to the optimal that knows offline the realization of the second-stage. Our two-scenarios algorithm performs significantly better and is only 36% higher than the optimal on average. Since Greedy, by definition, returns the best first stage cost, the two-scenarios algorithm can only improve upon Greedy by reducing the second stage bottleneck without considerably increasing the first stage cost. In particular, we observe from Table 4 that the second stage bottleneck of our algorithm on the realized scenario \( S^* \) is significantly less than the bottleneck of the greedy algorithm (by 30% on average), while the total weight of the matching provided by the two algorithms is roughly similar. If we think of the edge weights between drivers and riders as the wait times, then our results show that we substantially reduce the maximal second stage wait time, while the average wait time over the two stages is almost unchanged (only 2.4% higher in average). Our algorithm introduces more fairness in the distribution of the wait time between first and second stage, by reducing the maximum wait time, without materially affecting the overall average wait time.

8 Conclusion

In this paper, we present a new two-stage robust optimization framework for matching problems under both explicit and implicit models of uncertainty. Our problem is motivated by real-life applications in the ride-hailing industry. We consider different cost functions under this model, and study their theoretical hardness. We particularly focus on the Two-Stage Robust Matching Bottleneck variant, and design approximation algorithms for implicit and explicit scenarios under different settings. Our algorithms give a constant approximation if the number of scenarios is fixed, but require additional assumptions when there are polynomially or exponentially many scenarios to get a constant approximation. It is an interesting question if there exists a constant approximation algorithm in the most general case that does not depend on the number of scenarios. Furthermore, we have tested our algorithms on a taxi data set and showed that they improve significantly over the greedy approach, which results in reducing the maximum wait time for taxi riders.
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A NP-Hardness proofs for TSRMB

We start by presenting the 3-Dimensional Matching and Set Cover problems, that we use in our reductions to show Theorem 1 and Theorem 2. Both problems are known to be strongly NP-hard [12, 21].

3-Dimensional Matching (3-DM): Given three sets $U, V, W$ of equal cardinality $n$, and a subset $T$ of $U \times V \times W$, is there a subset $M$ of $T$ with $|M| = n$ such that whenever $(u, v, w)$ and $(u', v', w')$ are distinct triples in $M$, $u \neq u'$, $v \neq v'$, and $w \neq w'$?

Set Cover Problem: Given a set of elements $U = \{1, 2, ..., n\}$ (called the universe), a collection $S_1, \ldots, S_m$ of $m$ sets whose union equals the universe and an integer $p$. Question: Is there a set $C \subseteq \{1, \ldots, m\}$ such that $|C| \leq p$ and $\bigcup_{i \in C} S_i = U$?

Proof of Theorem 4: Consider an instance of the 3-Dimensional Matching Problem. We can use it to construct (in polynomial time) an instance of TSRMB with 2 scenarios as follows:

- Create two scenarios of size $n$: $S_1 = U$ and $S_2 = V$.
- Set $D = T$, every driver corresponds to a triple in $T$.
- For every $w \in W$, let $d_T(w)$ be the number of sets in $T$ that contain $w$. We create $d_T(w) - 1$ first stage riders, that are all copies of $w$. The total number of first stage riders is therefore $|R_1| = |T| - n$.
- For $(w, e) \in R_1 \times D$, $d(w, e) = \begin{cases} 1 & \text{if } w \in e \\ 3 & \text{otherwise.} \end{cases}$
- For $(u, e) \in S_1 \cup S_2 \times D$, $d(u, e) = \begin{cases} 1 & \text{if } u \in e \\ 3 & \text{otherwise.} \end{cases}$
- For $u, v \in R_1 \cup S_1 \cup S_2$, $d(u, v) = \min_{e \in D} d(u, e) + d(v, e)$.
- For $e, f \in D$, $d(e, f) = \min_{u \in R_1 \cup S_1 \cup S_2} d(u, e) + d(u, f)$.

This choice of distances induces a metric graph. We claim that there exists a 3-dimensional matching if and only if there exists a solution to this TSRMB instance with total cost equal to 2. Suppose that $M = \{e_1, \ldots, e_n\} \subseteq T$ is a 3-Dimensional matching. Let $e_1, \ldots, e_n$ be the drivers that correspond to $M$ in the TSRMB instance. We show that by using $D_1 = D \setminus \{e_1, \ldots, e_n\}$ as a first stage decision, we ensure that the total cost for the TSRMB instance is equal to 2. For any rider $u$ in scenario $S_1$, by definition of $M$, there exits a unique edge $e_i \in M$ that covers $u$. The corresponding driver $e_i \not\in D_1$ can be matched to $u$ with a distance equal to 1. Furthermore, $e_i$ cannot be matched to any other rider in $S_1$ with a cost less than 1. Similarly, for any rider $v$ in scenario $S_2$, since there exits a unique edge $e_j \in M$ that covers $v$, the corresponding driver can be matched to $v$ with a cost of 1. The second stage cost is therefore equal to 1. As for the first stage cost, we know by definition of $M$, that every element $w \in W$ is covered exactly once. Therefore, for every $w \in W$, there exists $d_T(w) - 1$ edges that contain $w$ in $T \setminus M$. This means that every 1st stage rider can be matched to a driver in $D_1$ with a cost equal to 1. Hence the total cost of this two-stage matching is equal to 2.

Suppose now that there exists a solution to the TSRMB instance with a cost equal to 2. This means that the first and second stage costs are both equal to 1. Let $M = \{e_1, \ldots, e_n\}$ be the set
of drivers used in the second stage of this solution. We show that \( M \) is a 3-dimensional matching.

Let \( e_i = (u, v, w) \) and \( e_j = (u', v', w') \) be distinct triples in \( M \). Since the second stage cost is equal to 1, the driver \( e_i \) (resp. \( e_j \)) must be matched to \( u \) (resp. \( u' \)) in \( S_1 \). Since we have exactly \( n \) second stage drivers and \( n \) riders in \( S_1 \), this means that \( e_i \) and \( e_j \) have to be matched to different second stage riders in \( S_1 \). Therefore we get \( u' \neq u \). Similarly we see that \( v' \neq v \). Assume now that \( w = w' \), this means that the TSRMB solution has used two drivers (triples) \( e_i \) and \( e_j \) that contain \( w \) in the second stage. It is therefore impossible to match all the \( d_T(w) - 1 \) copies of \( w \) in the first stage with a cost equal to 1. Therefore \( w \neq w' \). The above construction can be performed in polynomial time of the 3-DM input, and therefore shows that TSRMB with two scenarios is NP-hard.

Now, to show that TSRMB is hard to approximate within a factor better than 2, we consider three scenarios. Consider an instance of 3-DM. We can use it to construct an instance of TSRMB with 3 scenarios as follows:

- Create 3 scenarios of size \( n \): \( S_1 = U \), \( S_2 = V \) and \( S_3 = W \).
- Set \( D = T \).
- Create \( |R_1| = |T| - n \) first stage riders.
- For \((w, e) \in R_1 \times D\), \( d(w, e) = 1 \).
- For \((u, e) \in S_1 \cup S_2 \cup S_3 \times D\), \( d(u, e) = \begin{cases} 1 & \text{if } u \in e \\ 3 & \text{otherwise.} \end{cases} \)
- For \( u, v \in R_1 \cup S_1 \cup S_2 \cup S_3\), \( d(u, v) = \min_{e \in D} d(u, e) + d(v, e) \).
- For \( e, f \in D\), \( d(e, f) = \min_{u \in R_1 \cup S_1 \cup S_2 \cup S_3} d(u, e) + d(u, f) \).

This choice of distances induces a metric graph. Similarly to the proof of 2 scenarios, we can show that there exists a 3-dimensional matching if and only if there exists a TSRMB solution with cost equal to 2. Furthermore, any solution for this TSRMB instance have either total cost of 2 or 4 (the first stage cost is always equal to 1). We show that if a \((2 - \epsilon)\)-approximation (for some \( \epsilon > 0 \)) to the TSRMB exists then 3-Dimensional Matching is decidable. We know that this instance of TSRMB has a solution with total cost equal to 2 if and only if there is a 3-dimensional matching. Furthermore, if there is no 3-dimensional matching, the cost of the optimal solution to TSRMB must be 4. Therefore, if an algorithm guarantees a ratio of \((2 - \epsilon)\) and a 3-dimensional matching exists, the algorithm delivers a solution with total cost equal to 2. If there is no 3-dimensional matching, then the solution produced by the algorithm has a total cost of 4.

**Proof of Theorem**

We prove the theorem for \( k = 1 \). We start from an instance of the Set Cover problem and construct an instance of the TSRMB problem. Consider an instance of the decision problem of set cover. We can use it to construct the following TSRMB instance:

- Create \( m \) drivers \( D = \{1, \ldots, m\} \). For each \( j \in \{1, \ldots, m\} \), driver \( j \) corresponds to set \( S_j \).
- Create \( m - p \) first stage riders, \( R_1 = \{1, \ldots, m - p\} \).
- Create \( n \) second stage riders, \( R_2 = \{1, \ldots, n\} \).
- Set \( S = \{\{1\}, \ldots, \{n\}\} \). Every scenario is of size 1.
As for the distances between riders and drivers, we define them as follows:

- For \((i, j) \in R_1 \times D\), \(d(i, j) = 1\).
- For \((i, j) \in R_2 \times D\), \(d(i, j) = \begin{cases} 1 & \text{if } i \in S_j \\ 3 & \text{otherwise.} \end{cases}\)
- For \(i, i' \in R_1 \cup R_2\), \(d(i, i') = \min_{j \in D} d(i, j) + d(i', j)\).
- For \(j, j' \in D\), \(d(j, j') = \min_{i \in R_1 \cup R_2} d(i, j) + d(i, j')\).

This choice of distances induces a metric graph. Moreover, every feasible solution to this TSRMB instance has a first stage cost of exactly 1. We show that a set cover of size \(\leq p\) exists if and only if there is a TSRMB solution with total cost equal to 2. Suppose without loss of generality that \(S_1, \ldots, S_p\) is a set cover. Then by using the drivers \(\{1, \ldots, p\}\) in the second stage, we ensure that every scenario is matched with a cost of 1. This implies the existence of a solution with total cost equal to 2. Now suppose there is a solution to the TSRMB problem with cost equal to 2. Let \(D_2\) be the set of second stage drivers of this solution, then we have \(|D_2| = p\). We claim that the sets corresponding to drivers in \(D_2\) form a set cover. In fact, since the total cost of the TSRMB solution is equal to 2, the second stage cost is equal to 1. This means that for every scenario \(i \in \{1, \ldots, n\}\), there is a driver \(j \in D_2\) within a distance 1 from \(i\). Therefore \(i \in S_j\) and \(\{S_j : j \in D_2\}\) is a set cover.

Next we show that if \((2 - \epsilon)\)-approximation (for some \(\epsilon > 0\)) to the TSRMB exists then Set Cover is decidable. We know that the TSRMB problem has a solution of cost 2 if and only if there is a set cover of size less than \(p\). Furthermore, if there is no such set cover, the cost of the optimal solution must be 4. Therefore, if the algorithm guarantees a ratio of \((2 - \epsilon)\) and there is a set cover of size less than \(p\), the algorithm delivers a solution with a total cost of 2. If there is no set cover, then clearly the solution produced by the algorithm has a cost of 4.

**Remark 1.** For \(k \geq 2\), we can use a generalization of Set Cover to show that the problem is hard for any \(k\). We use a reduction from the Set MultiCover Problem (\cite{432}) defined below.

**Set MultiCover Problem:** Given a set of elements \(U = \{1, 2, \ldots, n\}\) (called the universe) and a collection \(S_1, \ldots, S_m\) of \(m\) sets whose union equals the universe. A "coverage factor" (positive integer) \(k\) and an integer \(p\). Is there a set \(C \subset \{1, \ldots, m\}\) such that \(|C| \leq p\) and for each element \(x \in U\), \(|j \in C : x \in S_j| \geq k\) ?

We can create an instance of TSRMB from a Set MultiCover instance similarly to Set Cover with the exception that \(S = \{S \subset R_2 \text{ s.t. } |S| = k\}\). The hardness result follows similarly.

**B Proofs of Section 5.2**

**Proof of Claim 3.**

1. In the optimal solution of the original problem, \(R_1\) is matched to a subset \(D_1^*\) of drivers. The scenario \(S_1\) is matched to a set of drivers \(D_{S_1}\) where \(D_1^* \cap D_{S_1} = \emptyset\). Let \(D_0\) be the set of drivers that are matched to \(o_1, \ldots, o_j^*\) in a scenario that contains \(o_1, \ldots, o_j^*\). It is clear that \(D_1^* \cap D_0 = \emptyset\). We claim that \(D_0 \cap D_{S_1} = \emptyset\). In fact, suppose there is a driver \(\rho \in D_0 \cap D_{S_1}\). This implies the existence of some \(o_j\) with \(j \leq j^*\) and some rider \(r \in S_1\) such that \(d(\rho, o_j) \leq OPT_2\) and \(d(\rho, r) \leq OPT_2\). But then \(d(\delta, o_j) \leq d(\delta, r) + d(\rho, r) + d(\rho, o_j) \leq 3OPT_2\) which contradicts
the fact the \( o_j \) is an outlier. Therefore \( D_o \cap D_{S_1} = \emptyset \). We show that \( D_1^* \) is a feasible first stage solution to the single scenario problem of \( S_1 \cup \{ o_1, \ldots, o_j^* \} \) with a cost less than \( OPT_1 + OPT_2 \). In fact, \( D_1^* \) can be matched to \( R_1 \) with a cost less than \( OPT_1, D_{S_1} \) to \( S_1 \) and \( D_o \) to \( \{ o_1, \ldots, o_j^* \} \) with a cost less than \( OPT_2 \). Therefore \( \Omega_j^* + \Delta_j^* \leq OPT_1 + OPT_2 \).

2. Recall that \( \text{cost}_1(D_1(j^*), R_1) = \Omega_j^* \). Consider a scenario \( S \) and a rider \( r \in S \). Let \( B' \) be the set of the \( n - \ell \) closest second stage riders to \( \delta \). Let \( D_{S_1}(j^*) \) be set of second stage drivers matched to \( S_1 \) in the single scenario problem for scenario \( S_1 \cup \{ o_1, \ldots, o_j^* \} \). Let \( D_o(j^*) \) be the set of second stage drivers matched to \( \{ o_1, \ldots, o_j^* \} \) in the single scenario problem for scenario \( S_1 \cup \{ o_1, \ldots, o_j^* \} \). Recall that the second stage cost for this single scenario problem is \( \Delta_j^* \). We distinguish three cases:

(a) If \( r \in B' \), then by Lemma 5, \( r \) is connected to every driver in \( D_{S_1}(j^*) \) within a distance less than \( \Delta_j^* + 4OPT_2 \).

(b) If \( r \in \{ o_{j^*+1}, \ldots, o_\ell \} \), then \( r \) is connected to every driver in \( D_{S_1}(j^*) \) within a distance less than \( 3OPT_2 + OPT_2 + \Delta_j^* \).

(c) If \( r \in \{ o_1, \ldots, o_{j^*} \} \) (i.e., \( r \) an outlier), then \( r \) can be matched to a different driver in \( D_o(j^*) \) within a distance less than \( OPT_2 \).

This means that in every case, we can match \( r \) to a driver in \( D \setminus D_1(j^*) \) with a cost less than \( 4OPT_2 + \Delta_j^* \). This implies that

\[
\max_{S \in \mathcal{S}} \text{cost}_2(D \setminus D_1(j^*), S) \leq 4OPT_2 + \Delta_j^*
\]

and therefore

\[
\Omega_j^* + \max_{S \in \mathcal{S}} \text{cost}_2(D \setminus D_1(j^*), S) \leq \Omega_j^* + \Delta_j^* + 4OPT_2 \leq OPT_1 + 5OPT_2.
\]

\[\square\]

**Proof of Claim 6.** Let \( \alpha_j \) be the second stage cost of \( D_1(j) \) on the TSRBM instance with scenarios \( S_1 \) and \( S_2 \). Formally, \( \alpha_j = \max_{S \in \{S_1, S_2\}} \text{cost}_2(D \setminus D_1(j), S) \). Therefore \( \beta_j = \Omega_j + \alpha_j \). Let’s consider the two sets

\[
O_1 = \{ r \in \{ o_1, \ldots, o_\ell \} \mid d(r, \delta) > 2\alpha_j + OPT_2 \},
\]

\[
O_2 = \{ o_1, \ldots, o_\ell \} \setminus O_1.
\]

Consider \( D_1(j) \) as a first stage decision to TSRMB with scenarios \( S_1 \) and \( S_2 \). Let \( \tilde{D}_1 \subset D \setminus D_1(j) \) be the set of drivers that are matched to \( O_1 \) when the scenario \( S_2 = \{ o_1, \ldots, o_\ell \} \) is realized. Similarly, let \( \tilde{D}_2 \subset D \setminus D_1(j) \) be the drivers matched to scenario \( S_1 \). We claim that \( \tilde{D}_1 \cap \tilde{D}_2 = \emptyset \). Suppose that there exists some driver \( \rho \in \tilde{D}_1 \cap \tilde{D}_2 \), this implies the existence of some \( o \in O_1 \) and \( r \in S_1 \) such that \( d(\rho, o) \leq \alpha_j \) and \( d(\rho, r) \leq \alpha_j \). And since \( d(r, \delta) \leq OPT_2 \) by definition of \( \delta \) we would have

\[
d(\rho, \delta) \leq d(\rho, o) + d(\rho, r) + d(r, \delta) \leq 2\alpha_j + OPT_2,
\]

which contradicts the definition of \( O_1 \). Therefore \( \tilde{D}_1 \cap \tilde{D}_2 = \emptyset \).

Now consider a scenario \( S \in \mathcal{S} \). The riders of \( S \cap O_1 \) can be matched to \( \tilde{D}_1 \) with a bottleneck cost less than \( \alpha_j \). Recall that by Lemma 5, any rider in \( R_2 \setminus \{ o_1, \ldots, o_\ell \} \) is within a distance less
than $4OPT_2$ from any rider in $S_1$. The riders $r \in S \setminus \{o_1, \ldots, o_\ell\}$ can therefore be matched to any driver $\rho \in \hat{D}_2$ within a distance less than

$$d(r, \rho) \leq d(r, S_1) + d(S_1, \rho) \leq 4OPT_2 + \alpha_j.$$  

As for riders $r \in S \cap O_2$, they can also be matched to any driver $\rho$ of $\hat{D}_2$ within a distance less than

$$d(r, \rho) \leq d(r, \delta) + d(\delta, S_1) + d(S_1, \rho) \leq 2\alpha_j + OPT_2 + OPT_2 + \alpha_j = 3\alpha_j + 2OPT_2.$$  

Therefore we can bound the second stage cost

$$\max_{S \in \mathcal{S}} cost_2(D \setminus D_1(j), S) \leq \max \{\alpha_j + 4OPT_2, 3\alpha_j + 2OPT_2\}$$

and we get that

$$cost_1(D_1(j), R_1) + \max_{S \in \mathcal{S}} cost_2(D \setminus D_1(j), S) \leq \max \{\beta_j + 4OPT_2, 3\beta_j + 2OPT_2\}$$

The other inequality $\beta_j \leq cost_1(D_1(j), R_1) + \max_{S \in \mathcal{S}} cost_2(D \setminus D_1(j))$ is trivial.  

\section{Two-Stage Stochastic Bottleneck Matching Problem (TSSMB)}

\subsection{Problem formulation}

In this section, we consider a variant of the TSRMB problem with an expected second stage cost over scenarios instead of a worst-case cost. In particular, we consider a set $R_1$ of first stage riders which is given first, and must immediately and irrevocably be matched to a subset of drivers $D_1$ ($D_1 \subset D$). Once $R_1$ is matched, a scenario $S_i \subset R_2$ is revealed from a list $S = \{S_1, \ldots, S_q\}$ with probability $p_i$ and need to be matched using the remaining drivers. The expected second stage cost is $\sum_{i=1}^q p_i \cdot cost_2(D \setminus D_1, S_i)$. The objective function is given by

$$\min_{D_1 \subset D} \left\{ cost_1(D_1, R_1) + \sum_{i=1}^q p_i \cdot cost_2(D \setminus D_1, S_i) \right\},$$

where $cost_1(D_1, R_1)$ and $cost_2(D \setminus D_1, S_i)$ are defined similarly to the TSRMB problem. For brevity of notation, we set $f(D_1) = cost_1(D_1, R_1) + \sum_{i=1}^q p_i \cdot cost_2(D \setminus D_1, S_i)$. Given an optimal first-stage solution $D_1^*$, we denote $OPT_1 = cost_1(D_1^*, R_1)$, $OPT_2 = \sum_{i=1}^q p_i \cdot cost_2(D \setminus D_1^*, S_i)$ and $OPT = OPT_1 + OPT_2$.

\subsection{NP-hardness}

\textbf{Corollary 1.} TSSMB is NP-hard to approximate within a factor better than $\frac{4}{3}$.

\textit{Proof.} Similar to the proof of Theorem \ref{thm:hardness} with $S_1 = U$, $S_2 = V$, $S_3 = W$, and equal probabilities $p_1 = p_2 = p_3 = \frac{1}{3}$. If there is a valid 3-DM then the total cost is equal to 2, and if there is no 3-DM then the total cost is at least $1 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 3 = \frac{8}{3}$. Therefore any algorithm with an approximation ratio strictly less than $\frac{8}{3} = \frac{4}{3}$ implies that 3-Dimensional Matching is decidable.  

\hfill $\square$
C.3 No surplus

Consider the case where there is no surplus of drivers, i.e., the total number of drivers is equal to $|R_1|$ plus the size of the maximum scenario. We assume for the sake of simplicity that all scenarios have the same size. The proof follows as well if the sizes are different. We show in this case that we can have a 3-approximation by considering every scenario independently to get different first stage decisions, and then picking the best first stage decision among them. For every scenario $S_i$, we solve the following problem

$$\min_{D_1 \subset D} \left\{ \text{cost}_1(D_1, R_1) + \text{cost}_2(D \setminus D_1, S_i) \right\}$$

When there is no surplus, we know that in the optimal solution, the same set of drivers is matched to every scenario. Therefore if we have a solution for one single scenario, we can use the triangular inequality to bound the cost of this solution for any scenario.

Algorithm 7:

1: for $i \in \{1, \ldots, q\}$ do
2: $D_1^i := \text{TSRMB-1-Scenario}(R_1, S_i, D)$.
3: return $D_1 = \arg \min_{D_1^i} f(D_1^i)$

Theorem 7. Algorithm 7 yields a solution with total cost less than $OPT_1 + 3OPT_2$ for TSSMB with no surplus.

Proof. Let $OPT_1$ and $OPT_2$ be the first and second stage cost of the optimal solution, and $b_1, \ldots, b_q$ be the bottleneck edge weights in the optimal second stage matchings for $S_1, \ldots, S_q$. Therefore $OPT_2 = \sum_{i=1}^{q} p_i \cdot b_i$. We claim that $\min_{i} f(D_1^i) \leq OPT_1 + 3OPT_2$. For every $i$, let $\alpha_i$ and $\beta_i$ be the first and second stage cost of $D_1^i$ when we consider only scenario $S_i$, that is

$$\text{cost}_1(D_1^i, R_1) + \text{cost}_2(D \setminus D_1^i, S_i) = \alpha_i + \beta_i.$$ 

It is clear that $\alpha_i + \beta_i \leq OPT_1 + b_i$. Furthermore, when a scenario $S_j$ ($j \neq i$) is realized, we can bound the cost of matching $S_j$ to $D \setminus D_1^i$ by using the triangular inequality,

$$\text{cost}_2(D \setminus D_1^i, S_j) \leq \beta_i + b_i + b_j.$$

Therefore we get that,

$$f(D_1^i) \leq \alpha_i + p_i \cdot \beta_i + \sum_{j \neq i} p_j (\beta_i + b_j + b_i)$$

$$= \alpha_i + \beta_i + \sum_{j \neq i} p_j (b_j + b_i)$$

$$\leq OPT_1 + p_i \cdot b_i + \sum_{j \neq i} p_j (b_j + 2b_i)$$

$$\leq OPT_1 + OPT_2 + 2(1 - p_i)b_i.$$
Next we show that \( \min_{1 \leq i \leq q} (1 - p_i) \cdot b_i \leq OPT_2 \). Suppose, in the contrary, that for all \( i \in \{1, \ldots, q\} \), we have \((1 - p_i) \cdot b_i > OPT_2\), then \( p_i \cdot b_i > \frac{p_i}{1 - p_i} OPT_2 \) and by summing we get that

\[
OPT_2 > OPT_2 \sum_{i=1}^{q} \frac{p_i}{1 - p_i}.
\]

We can assume without loss of generality that \( OPT_2 > 0 \) and therefore we get that

\[
1 > \sum_{i=1}^{q} \frac{p_i}{1 - p_i} = \sum_{i=1}^{q} \frac{p_i + 1 - 1}{1 - p_i} = \sum_{i=1}^{q} \frac{1}{1 - p_i} - q,
\]

which implies that

\[
q \geq \sum_{i=1}^{q} \frac{1}{1 - p_i} \tag{5}
\]

By the Cauchy-Schwarz inequality, we know that

\[
(q - 1) \cdot \sum_{i=1}^{q} \frac{1}{1 - p_i} = \sum_{i=1}^{q} (1 - p_i) \cdot \sum_{i=1}^{q} \frac{1}{1 - p_i} \geq q^2.
\]

Therefore,

\[
\sum_{i=1}^{q} \frac{1}{1 - p_i} \geq \frac{q^2}{q - 1} > q,
\]

which contradicts \( \text{(5)} \). Hence \( \min_{1 \leq i \leq q} (1 - p_i) \cdot b_i \leq OPT_2 \) and \( f(D_1) = \min_i f(D_1^i) \leq OPT_1 + 3OPT_2 \). \( \square \)

## D Two-Stage Robust Matching Problem (TSRM)

In this section, we consider a variant of the TSRMB problem with the total weight instead of the bottleneck as a second stage cost. Similarly, we consider a set \( R_1 \) of first stage riders which is given first, and must immediately and irrevocably be matched to a subset of drivers \( D_1 (D_1 \subset D) \). Once \( R_1 \) is matched, a scenario \( S_i \subset R_2 \) is revealed from a list \( S = \{S_1, \ldots, S_q\} \). The first stage cost, \( cost_1(D_1, R_1) \), is the total weight of the matching between \( D_1 \) and \( R_1 \) the second stage cost \( cost_2(D \setminus D_1, S) \) is the cost of the minimum weight matching between the scenario \( S \) and the available drivers \((D \setminus D_1)\). Formally, let \( M_1 \) be the minimum weight perfect matching between \( R_1 \) and \( D_1 \), and given a scenario \( S \), let \( M_2^S \) be the minimum weight perfect matching between the scenario \( S \) and the available drivers \((D \setminus D_1)\), then the cost functions for the TSRM are:

\[
cost_1(D_1, R_1) = \sum_{(i,j) \in M_1} d(i,j), \quad \text{and} \quad cost_2(D \setminus D_1, S) = \sum_{(i,j) \in M_2^S} d(i,j).
\]

Given an optimal first-stage solution \( D^*_1 \), we denote \( OPT_1 = cost_1(D^*_1, R_1) \), \( OPT_2 = \max \{cost_2(D \setminus D^*_1, S) | S \in S\} \) and \( OPT = OPT_1 + OPT_2 \). In this variant we consider an explicit model of scenarios and optimize over the worst case scenario. We show that the problem is NP-hard even with two scenarios. We restate a result from the literature that gives a 3-approximation using a greedy approach. We further improve over this approximation in the specific case of no surplus. We assume for the sake of simplicity that all the scenarios have the same size \( k \).
Theorem 8. TSRM is NP-hard even when the number of scenarios is equal to 2.

Proof. We construct (in polynomial time) a reduction from the 2-partition problem. Let \((I, (s_i)_{i \in I})\) be an instance of the 2-partition problem.

The 2-partition problem:
Instance: Finite set \(I\) and number \(s_i \in \mathbb{Z}^+\) for \(i \in I\).
Question: Is there a subset \(I' \subset I\) such that \(\sum_{i \in I'} s_i = \sum_{i \in I \setminus I'} s_i\)?

It is well known that the 2-partition problem is weakly NP-hard even when \(|I'| = |I|/2\).

Without loss of generality, suppose that \(I = \{1, \ldots, n\}\) for some integer \(n\). We construct the following instance of TSRM with two scenarios. Let \(R_1 = \{r_1, \ldots, r_n\}\), \(S_1 = \{r_{n+1}, \ldots, r_{2n}\}\) and \(S_2 = \{r_{2n+1}, \ldots, r_{3n}\}\). Note that every scenario is of size \(n\). Let \(D = \{\delta_1, \ldots, \delta_{2n}\}\). Let \(P\) be a sufficiently big constant such that \(P \geq \sum_{i \in I} s_i\). We define the distances between drivers and riders as follows:

- For \(j \in \{1, \ldots, n\}\): \(d(\delta_j, r_j) = P\), \(d(\delta_{n+j}, r_j) = P\) and \(d(\delta_i, r_j) = \infty\) otherwise.
- For \(j \in \{n+1, \ldots, 2n\}\): \(d(\delta_{j-n}, r_j) = P\), \(d(\delta_j, r_j) = s_{j-n}\) and \(d(\delta_i, r_j) = \infty\) otherwise.
- For \(j \in \{2n+1, \ldots, 3n\}\): \(d(\delta_{j-2n}, r_j) = s_{j-2n}\), \(d(\delta_{j-n}, r_j) = P\) and \(d(\delta_i, r_j) = \infty\) otherwise.

This choice of distances induces a metric bipartite graph on \(R_1\) and \(S_1 \cup S_2\). A feasible solution to this TSRM instance with bounded cost has two possibilities to match rider \(r_j \in R_1\) \((j \leq n)\): either to driver \(\delta_j\) or driver \(\delta_{n+j}\). Consider a feasible bounded cost first stage solution \(D_1\), and let \(I'\) be the set of indices \(j \leq n\) such that the first stage rider \(r_j\) is matched to driver \(\delta_j\) in the first stage. Then \(I \setminus I'\) is the set of elements \(j \leq n\) such that the first stage rider \(r_j\) is matched to driver \(\delta_{j+n}\).

In both cases, the cost of matching \(r_j \in R_1\) is equal to \(P\). When the scenario \(S_1\) is realized, the driver \(r_{n+j} \in S_1\) \((j \leq n)\) needs to be matched to \(\delta_{j+n}\) if \(j \in I'\), with a cost \(P\) and to \(\delta_j\) if \(j \in I \setminus I'\), with a cost \(s_j\). Similarly, when the scenario \(S_2\) is realized, the driver \(r_{2n+j} \in S_2\) \((j \leq n)\) needs to be matched to \(\delta_{j+n}\) if \(j \in I'\), with a cost \(s_j\) and to \(\delta_j\) if \(j \in I \setminus I'\), with a cost \(P\). The first and second stage costs are therefore:

\[
\begin{align*}
\text{cost}_1(D_1, R_1) &= P|I|, \\
\text{cost}_2(D \setminus D_1, S_1) &= P|I'| + \sum_{j \in I \setminus I'} s_j, \\
\text{cost}_2(D \setminus D_1, S_2) &= P|I'| + \sum_{j \in I'} s_j.
\end{align*}
\]

We claim that there exists a 2-partition \(I'\) such that \(|I'| = |I \setminus I'|\) if and only if there is a solution with total cost equal to \(\frac{1}{2}(3P|I| + \sum_{j \in I} s_j)\).

Suppose there exist a 2-partition \(I'\) with \(|I'| = |I \setminus I'|\). This implies that

\[
\sum_{j \in I'} s_j + P|I \setminus I'| = \sum_{j \in I \setminus I'} s_j + P|I'| = \frac{1}{2}(P|I| + \sum_{j \in I} s_j) \quad (6)
\]
Let $D_1$ be the first stage decision that for every $j \leq n$, matches $r_j$ to $\delta_j$ if $j \in I'$, and $r_j$ to $\delta_{n+j}$ otherwise. The costs of this first stage decision on scenarios $S_1$ and $S_2$ are:

$$\text{cost}_1(D_1, R_1) + \text{cost}_2(D \setminus D_1, S_1) = |P| + \sum_{j \in I \setminus I'} s_j = \frac{1}{2}(3|P| + \sum_{j \in I} s_j),$$

$$\text{cost}_1(D_1, R_1) + \text{cost}_2(D \setminus D_1, S_2) = |P| + |P| \setminus I' + \sum_{j \in I \setminus I'} s_j = \frac{1}{2}(3|P| + \sum_{j \in I} s_j).$$

Therefore the total cost of $D_1$ is equal to

$$\text{cost}_1(D_1, R_1) + \max_{S \subseteq \{S_1, S_2\}} \text{cost}_2(D \setminus D_1, S) = \frac{1}{2}(3|P| + \sum_{j \in I} s_j).$$

Suppose now that there is a first stage decision $D_1$ with bounded total cost equal to $\frac{1}{2}(3|P| + \sum_{j \in I} s_j)$. Let $I'$ be the set of indices $j \leq n$ such that, in the first stage matching of $D_1$, $r_j$ is matched to driver $\delta_j$ for $j \leq n$. We know that

$$\text{cost}_1(D_1, R_1) + \text{cost}_2(D \setminus D_1, S_1) = |P| + |P| \setminus I' + \sum_{j \in I \setminus I'} s_j \leq \frac{1}{2}(3|P| + \sum_{j \in I} s_j)$$

$$\text{cost}_1(D_1, R_1) + \text{cost}_2(D \setminus D_1, S_2) = |P| + |P| \setminus I' + \sum_{j \in I \setminus I'} s_j \leq \frac{1}{2}(3|P| + \sum_{j \in I} s_j)$$

This implies the following inequalities

$$P|I'| + \sum_{j \in I \setminus I'} s_j \leq \frac{1}{2}(P|I| + \sum_{j \in I} s_j) \tag{7}$$

$$P|I \setminus I'| + \sum_{j \in I'} s_j \leq \frac{1}{2}(P|I| + \sum_{j \in I} s_j) \tag{8}$$

The only way (7) and (8) can hold is if we have

$$P|I'| + \sum_{j \in I \setminus I'} s_j = P|I \setminus I'| + \sum_{j \in I'} s_j = \frac{1}{2}(P|I| + \sum_{j \in I} s_j) \tag{9}$$

Now suppose that $|I \setminus I'| > |I'|$, since we can make $P$ as big as needed, then equation (9) cannot hold. Therefore $|I \setminus I'| \leq |I'|$. Similarly, we get that $|I'| \leq |I \setminus I|$, Therefore, $|I'| = |I \setminus I'|$ and equation (9) becomes

$$\sum_{j \in I \setminus I'} s_j = \sum_{j \in I'} s_j. \tag{10}$$

This shows that $I'$ is a 2-Partition with $|I'| = |I \setminus I'|$.

**Lemma 6.** The greedy algorithm that minimizes only the first stage cost yields a solution with total cost less than $3\text{OPT}_1 + \text{OPT}_2$ to the TSRM.

**Proof.** Special case of Theorem 2.4 in [20].

**Lemma 7.** If the surplus $\ell = |D| - |R_1| - k$ is equal to zero, Algorithm 6 yields a solution with a total cost less than $\text{OPT}_1 + 5\text{OPT}_2$ to the TSRM.

33
Proof. Consider Algorithm 8. In the remaining of the proof, we will refer to the total cost of the solution given by Algorithm 8 as ALG, and to its first (resp. second) stage cost as ALG1 (resp. ALG2). The proof of the lemma follows immediately by combining the following two claims.

Claim 9. ALG2 ≤ 3OPT2.

Proof. We use the notation \( M(A,B) \) to refer to the total weight of the minimum weight perfect matching between a set of drivers \( A \) and a set of riders \( B \). If the scenario \( S \) that was picked by the algorithm is realized, then in this case we know that its second stage cost is less than \( OPT2 \). Now, suppose a different scenario \( S' \neq S \) is realized. Let \( D^*_2 \) be the set of \( k \) drivers that the optimal solution saves for the second stage. We use the triangular inequality to establish that:

\[
M(D_2, S') \leq M(D_2, S) + M(D^*_2, S) + M(D^*_2, S')
\]

Let’s bound the right hand side terms of equation (11). \( M(D_2, S) \leq OPT2 \) because by definition, the matching between \( D_2 \) and \( S \) is the best possible between \( S \) and any subset of drivers. Now since \( |D| = |R_1| + k \), \( s \) means that the optimal solution saves exactly \( k \) drivers to be matched with any scenario realization. This implies that \( M(D_2^*, S) \leq OPT2 \) and \( M(D^*_2, S') \leq OPT2 \). The claim follows immediately.

Claim 10. ALG1 ≤ OPT1 + 2OPT2.

Proof of claim. We construct a matching that between \( D \setminus D_2 \) and \( R_1 \) with a total weight less than \( OPT1 + 2OPT2 \). Let \( r_1 \in R_1 \), and \( \delta_1(r_1) \) be the driver matched to \( r_1 \) in the optimal solution. If \( \delta_1(r_1) \notin D_2 \), then just match \( \delta_1(r_1) \) with \( r_1 \). Therefore we can assume without loss of generality that all the drivers \( \delta_1(r_1) \) are used in \( D_2 \). This means that exactly \( |R_1| \) drivers of \( D \setminus D_2 \) are used in second stage of the optimal solution. We can match \( R_1 \) with \( D \setminus D_2 \) and bound the cost of this matching as follows:

\[
M(D \setminus D_2, R_1) \leq M(D \setminus D_2, S) + M(D_2, S) + M(D_2, R_1)
\]

\( M(D \setminus D_2, S) \leq OPT2 \) because exactly \( |R_1| \) drivers from \( D \setminus D_2 \) are used in the second stage of the optimal solution and \( |R_1| = |D \setminus D_2| \). \( M(D_2, S) \leq OPT2 \) by definition of \( D_2 \). Finally, \( M(D_2, R_1) \leq OPT1 \) because \( D_2 \) includes all the drivers that were used in the first stage of the optimal matching. Therefore, we get

\[
ALG1 = M(D \setminus D_2, R_1) \leq OPT1 + 2OPT2.
\]

Theorem 9. If the surplus \( \ell = |D| - |R_1| - k \) is equal to zero, there exists a polynomial time algorithm with a \( \frac{7}{3} \)-approximation to the TSRM problem.
Proof. We show the theorem by balancing between the results of Lemma [6] and Lemma [7]. Let $\text{Greedy}$ denote the total cost of the greedy algorithm. From Lemma [6] and Lemma [7] we have that $\text{Greedy} \leq 3OPT_1 + OPT_2$ and $\text{ALG} \leq OPT_1 + 5OPT_2$. By taking the minimum of the two algorithms we get:

$$\min\{\text{Greedy}, \text{ALG}\} = \min\{(3OPT_1 + OPT_2, OPT_1 + 5OPT_2)\}$$

$$= OPT \cdot \min\{(\frac{3OPT_1 + OPT_2}{OPT}, \frac{OPT_1 + 5OPT_2}{OPT})\}$$

Therefore, and since $OPT = OPT_1 + OPT_2$, we get that:

$$\min\{\text{Greedy}, \text{ALG}\} \leq OPT \cdot \max_{x \in [0,1]} \min\{3x + (1 - x), x + 5(1 - x)\} \leq \frac{7}{3} \cdot OPT.$$