Scattering of solutions to the nonlinear Schrödinger equations with regular potentials

Xing Cheng, Ze Li and Lifeng Zhao

Abstract

In this paper, we prove the scattering of radial solutions to high dimensional energy-critical nonlinear Schrödinger equations with regular potentials in the defocusing case.

1 Introduction

In this paper, we consider the nonlinear Schrödinger equation with a potential:

\[
\begin{align*}
&i\partial_t u + \Delta V u + \lambda |u|^{p-1}u = 0, \\
&u(0,x) = u_0(x).
\end{align*}
\]

where \( u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C} \), \( \Delta V = \Delta - V \), \( V : \mathbb{R}^d \to \mathbb{R} \), \( \lambda = \pm 1 \) and \( 1 < p < \infty \). If \( \lambda = -1 \), the equation is called defocusing; otherwise, it is called focusing if \( \lambda = 1 \).

There are many important areas of application which motivate the study of nonlinear Schrödinger equations with potentials (Gross-Pitaevskii equation). In the most fundamental level, it arises as a mean field limit model governing the interaction of a plenty large number of weakly interacting bosons \([22, 32, 40]\). In a macroscopic level, it arises as the equation governing the evolution of the envelope of the electric field of a light pulse propagating in a medium with defects, see for instance, \([19, 20]\).

First, we recall some history on the scattering of solutions to (1.1) for small initial data. When \( V = 0 \), it has been shown that for \( d \geq 1 \), \( p = 1 + \frac{2}{d} \) is the critical exponent for scattering. In fact, for \( 1 + \frac{2}{d} < p < 1 + \frac{4}{d} \), \( d \geq 1 \), decay and scattering of the solution in the small data case is proved by McKean, Shatah \([33]\). For \( 1 + \frac{4}{d} \leq p \leq 1 + \frac{4}{d-2} \), \( d \geq 3 \) and \( 1 + \frac{4}{d} \leq p < \infty \), \( d = 1, 2 \), local wellposedness and small data scattering was proved by Strauss \([42]\). Moreover, Strauss \([41]\) showed when \( 1 < p \leq 1 + \frac{2}{d} \) for \( d \geq 2 \) and \( 1 < p \leq 2 \) for \( d = 1, 2 \), the only scattering solution is zero. For all energy subcritical \( p \), Visciglia \([47]\) proved the \( L^r \) norm of the solution decays provided \( 2 < r < \frac{2d}{d-2} \) when \( d \geq 3 \) and \( 2 < r < \infty \) when \( d = 1, 2 \). When \( V \neq 0 \), the situation is much more involved. In \([9]\), Cuccagna, Georgiev, Visciglia proved decay and scattering for small initial data for \( p > 3 \) in one dimension for some Schwartz potentials.
Second, let us review known results on the scattering of solutions to \((1.1)\) for general data. There are a lot of works devoted to the case \(V = 0\). Ginibre, Velo \cite{18} proved the scattering when \(1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}, \quad d \geq 3\) and \(1 + \frac{4}{d} < p < \infty, \quad d = 1, 2\) by exploiting the Morawetz estimate in the defocusing case. We also mention the works of Nakanishi \cite{34}, Planchon, Vega \cite{35} for scattering of subcritical Schrödinger equations. Global well-posedness and scattering in the energy space for radial data in the energy critical defocusing case was proved by Bourgain \cite{5} by means of induction on energy. This result was extended to non-radial data by Colliander, Keel, Staffilani, Takaoka, Tao \cite{8} and high dimensions by Ryckman, Visan \cite{37, 48}. For energy critical focusing case, Kenig, Merle \cite{28} showed global wellposedness and scattering versus blow-up dichotomy below the ground state energy for radial data when \(d = 3, 4, 5\) by using the concentration-compactness/rigidity method. The radial assumption was removed in higher dimensions when \(d \geq 4\) by \cite{16, 29}. In the mass critical case, Killip, Visan, Tao, Zhang \cite{30, 44, 45} for radial data and Dodson \cite{12, 13, 14, 15} for non-radial data proved scattering for initial data of finite mass in the defocusing case and the dichotomy below the ground state mass in the focusing case.

When \(V \neq 0\), the long time behavior is strongly affected by the potential. For harmonic potential, it is widely conjectured that the solution will not scatter in energy space. For partial harmonic confinement, scattering for some \(p\) was proved in Antonelli, Carles, Drumond, Silva \cite{2}. When \(p = 3, 2 \leq d \leq 5\), Hani, Thomann \cite{21} showed the only scattering solution is zero if there is one direction which is not trapped. For regular potentials \(V\), scattering is affected by the discrete spectrum of the Schrödinger operator. Generally, if there is no discrete spectrum, the solution scatters in the defocusing case for any initial data or in the focusing case for the initial data with energy below the ground state. There are a lot of works on this topic, for instance, Colliander, Czubak, Lee \cite{7} proved scattering for the cubic NLS with electric and magnetic potentials by interaction Morawetz estimate. Concerning the scattering theory with a potential in the subcritical case, we also mention the works of Hong \cite{23}, Lafontaine \cite{31}, and Banica, Visciglia \cite{3}.

In the article, we will consider the potentials satisfying the following assumptions:

**Regular Potential Hypothesis**

Suppose that \(V\) is a real-valued potential satisfying

\[(i) \quad \langle x \rangle^N (|V(x)| + |\nabla V(x)|) \in L^\infty(\mathbb{R}^d), \text{ for some } N > d;\]

\[(ii) \quad \text{the spectrum of } -\Delta V \text{ is continuous, and } 0 \text{ is neither a resonance nor an eigenvalue of } -\Delta V;\]

\[(iii) \quad \langle x \rangle^\alpha V(x) \text{ is a bounded operator from } H^\eta \text{ to } H^\eta \text{ for some } \alpha > d + 4, \eta > 0 \text{ with } \mathcal{F}V \in L^1;\]

**Remark 1.1.** The continuous spectrum assumption in \((ii)\) is reasonable. If \(-\Delta V\) has discrete spectrum, it seems that the solution may not scatter in the energy space even in the small data case. This is supported in some sense by Soffer, Weinstein \cite{39}. They proved that the solution to nonlinear Klein-Gordon equation with a potential (NLKG) having small initial data decays to
zero as time goes to infinity. However, the linear Klein-Gordon equation with a potential (LKG) admits a family of periodic solutions with \( H^1 \) norms tending to zero. Thus solutions to NLKG with small data cannot scatter to those periodic solutions to LKG, i.e. the wave operator is not complete.

Remark 1.2. The hypothesis (iii) is assumed to provide a dispersive estimate of \( e^{it \Delta_V} \). The condition given here is due to Journe, Soffer, Sogge [25]. There are many related works in this direction such as Rodnianskii, Schlag [30], Schlag [38]. When \( d = 3 \), weaker assumption on \( V \) is available for the dispersive estimates, see for instance [4, 10].

In our case, for \( 1 + \frac{4}{d} < p < 1 + \frac{4}{d-2} \), global well-posedness and scattering can be proved by interacting Morawetz identity, see for instance [7]. Thus we only need to consider the energy-critical case. In the following, we prove global well-posedness and scattering for radial data in high dimensions (\( d \geq 7 \)).

Theorem 1.1. Assume that \( V \) is radial, nonnegative, \( \partial_r V \leq 0 \), and \( V \) satisfies Regular Potential Hypothesis. For \( d \geq 7 \), \( p = 1 + \frac{4}{d-2} \), \( \lambda = -1 \), \( u_0 \in \dot{H}^1_{rad}(\mathbb{R}^d) \), (1.1) is globally wellposed and moreover, there exists \( u_+ \in \dot{H}^1 \) such that

\[
\lim_{t \to \infty} \| u(t) - e^{it \Delta_V} u_+ \|_{\dot{H}^1} = 0.
\]

Remark 1.3. A similar theorem is possible if \( V \) has a small negative part. The radial assumption for \( V \) is to ensure that every radial initial data evolves into a radial solution. If one considers non-radial data, the assumption \( \partial_r V \leq 0 \) can be replaced by \( x \cdot \nabla V \leq 0 \).

Remark 1.4. Since \( V \geq 0 \), the spectrum of \( -\Delta_V \) is included in \([0, \infty)\). Since \( V \in L^2 \), by Weyl’s criterion, the essential spectrum of \( -\Delta_V \) is \((0, \infty)\). The decay of \( V \) guarantees that there are no positive eigenvalues by Kato’s theory. Moreover, it is known that there is no resonance for \( d \geq 5 \). Therefore, for \( V \) in Theorem 1.1, (ii) in Regular Potential Hypothesis is equivalent to that \( 0 \) is not an eigenvalue of \( -\Delta_V \). But this is true if \( V \) is non-negative. Therefore, (ii) is not needed in the presentation of Theorem 1.1.

Remark 1.5. The potentials satisfying the assumptions in Theorem 1.1 do exist. In fact, the Gaussian function \( e^{-|x|^2} \) satisfies all the assumptions in the theorem.

The facts that the equation is not scaling invariant and the energy space is homogeneous bring some new difficulties. As is known, the scaling invariance makes the bounded set in a homogeneous space noncompact. If the energy is not in the homogenous space, we can rule out one of the direction of the possible scaling such as what has been done in the study of scattering to nonlinear Klein-Gordon equations. If the equation is scaling invariant, the scaling will disappear when one does some estimates in the homogeneous space, which makes the analysis
of limits of scaling not so important. In our case, because the energy lies in $\dot{H}^1$ level, we have to handle two directions of the scaling. Meanwhile, the lack of scaling invariance makes the estimates sensitive to the varying of scaling. For instance, in the linear profile decomposition for the linear Schrödinger equation, the remainder term governed by the linear Schrödinger equation is asymptotically zero in Strichartz norms. However, if the scaling goes to infinity or zero, the remainder term tends to be a solution of free Schrödinger equation, for which whether it is asymptotically zero or not is not obvious. In order to overcome the difficulty, we prove two convergence results concerning the scaled Schrödinger operator and the free Schrödinger operator, namely Proposition 3.3 and 3.4. Proposition 3.3 gives the convergence of scaled Schrödinger operator to free Schrödinger operator in the strong operator topology. Proposition 3.4 proves the convergence in operator norm in a finite time interval. Although the strong operator topology convergence is weak, it is useful in proving profile decomposition since it is uniform in time. The operator norm convergence is essential in proving that the remainder is still asymptotically zero in Strichartz norms after taking a limit of scaling.

We assume $d \geq 7$ because the Strichartz norm in $\dot{H}^1$ level agrees with $\|(-\Delta)^{\frac{s}{2}} u\|_{S^0}$, where $S^0$ is the $L^2$ level Strichartz norm. However, for $d \leq 4$, the two norms are not equivalent in general. The equivalence relation can compensate the loss of Leibnitz rule for $(-\Delta)^{\frac{s}{2}}$ and the non-commutativity between $\nabla$ and $e^{it\Delta V}$. In principle, the scattering for (1.1) when $d = 5, 6$ can be proved similarly, we rule out the two cases for technical problems. The focusing case can be dealt with similarly, in the subcritical case, see for instance [23].

The article is organized as follows. In Section 2, we give some estimates on Schrödinger operators and prove local well-posedness and stability theorem. In Section 3, we prove some important convergence lemmas concerning scaled Schrödinger operators and free Schrödinger operators, as an application, we give the linear profile decomposition. In section 4, Theorem 1.1 is proved by the compactness-contradiction arguments.

**Notation and Preliminaries.** We denote $F_V$ as the distorted Fourier transformation defined in Section 3. For $s \in \mathbb{R}$, the fractional differential operator $|\nabla|^s$ is defined by $\mathcal{F}(|\nabla|^s f)(\xi) = |\xi|^s \mathcal{F}(f)(\xi)$. We also define $\langle \nabla \rangle^s$ by $\mathcal{F}(\langle \nabla \rangle^s f)(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}(f)(\xi)$.

We define the homogeneous Sobolev norms by

$$\|f\|_{H^s(\mathbb{R}^d)} = \| |\nabla|^s f\|_{L^2(\mathbb{R}^d)},$$

and inhomogeneous Sobolev norms by

$$\|f\|_{H^s(\mathbb{R}^d)} = \| \langle \nabla \rangle^s f\|_{L^2(\mathbb{R}^d)}.$$
The $\dot{H}_V^1$ norm is defined by
\[
\|u\|_{\dot{H}_V^1}^2 = \int_{\mathbb{R}^d} |\nabla u|^2 + V|u|^2 \, dx.
\]

The Besov norms are defined as follows: Let $\varphi \in C^\infty_c(\mathbb{R}^d)$ be such that $\varphi(\xi) = 1$ for $|\xi| \leq 1$ and $\text{supp} \varphi(\xi) \subset \{ \xi : |\xi| \leq 2 \}$. Then we define $\psi_k(\xi) = \varphi\left(\frac{\xi}{2^k}\right) - \varphi\left(\frac{\xi}{2^{k-1}}\right)$, $\forall k \in \mathbb{Z}$. For $1 \leq r, p \leq \infty$, $s \in \mathbb{R}$, we define for $u \in S'(\mathbb{R}^d)$,
\[
\|u\|_{\dot{B}^s_{r,p}} = \begin{cases} 
\left( \sum_{k \in \mathbb{Z}} 2^{ksp} \|\mathcal{F}^{-1}(\psi_k \mathcal{F} u)\|_{L^p_x}^p \right)^{\frac{1}{p}}, & p < \infty; \\
\sup_{k \in \mathbb{Z}} 2^{ks} \|\mathcal{F}^{-1}(\psi_k \mathcal{F} u)\|_{L^p_x}, & p = \infty. 
\end{cases}
\]

Denote $\phi = \mathcal{F}^{-1}\psi$.

For a linear operator $A$ from Banach space $X$ to Banach space $Y$, we denote its operator norm by $\|A\|_{L(X,Y)}$. All the constants are denoted by $C$ and they can change from line to line.

We use $\varepsilon$ to denote some sufficiently small constant and it may vary from line to line. We use the notation $b^+$ and $b^-$ to stand for a number slightly less than $b$ and a number slightly bigger than $b$ respectively.

**Proposition 1.1** (Dispersive estimate of $e^{it\Delta V}$, [25]). Let $d \geq 3$, $\langle x \rangle^\alpha V(x)$ is a bounded operator from $H^\eta$ to $H^\eta$ for some $\alpha > d + 4$, $\eta > 0$, with $\mathcal{F}V \in L^1$. Assume also that 0 is neither an eigenvalue nor a resonance of $-\Delta V$. Then
\[
\|e^{it\Delta V} P_c(\Delta V)\|_{p' \to p} \leq C|t|^{-\frac{d}{4} \left(1 - \frac{2}{p} \right)},
\]
where $\frac{1}{p'} + \frac{1}{p} = 1$, $2 \leq p \leq \infty$.

By the abstract theorem in Keel, Tao [26], one can prove:

**Proposition 1.2** (Strichartz estimate). Suppose that $V$ is the potential in Theorem 1.1. And assume that $(p, q)$ and $(\overline{p}, \overline{q})$ are Strichartz admissible with $2 \leq p, q, \overline{p}, \overline{q} \leq \infty$ except the endpoint $(p, q, d) = (2, \infty, 2)$, namely
\[
\frac{2}{p} + \frac{d}{q} = \frac{d}{2},
\]
then we have
\[
\left\| e^{it\Delta V} f \right\|_{L^p_t L^q_x(I \times \mathbb{R}^d)} \leq C\|f\|_{L^2},
\]
\[
\left\| \int_0^t e^{i(t-\tau)\Delta V} F(\tau) \, d\tau \right\|_{L^p_t L^q_x(I \times \mathbb{R}^d)} \leq C\|F\|_{L^p_t L^q_x'(I \times \mathbb{R}^d)},
\]
where $I$ is any interval containing $t = 0$, $C$ is some constant depending only on $V, d, p, q,$
In addition, we say \((p, q)\) is a \(\dot{H}^1\) level Strichartz pair if
\[
\frac{2}{p} + \frac{d}{q} = \frac{d}{2} - 1.
\]
We define the Strichartz norms to be
\[
\|u\|_{S^0(I \times \mathbb{R}^d)} \overset{\Delta}{=} \sup_{(q, r) \text{ admissible}} \|u\|_{L^q_t L^r_x(I \times \mathbb{R}^d)},
\]
\[
\|u\|_{S^1(I \times \mathbb{R}^d)} \overset{\Delta}{=} \|u\|_{S^0(I \times \mathbb{R}^d)} + \|\nabla u\|_{S^0(I \times \mathbb{R}^d)}.
\]
We also define \(\forall s \geq 0\),
\[
\|u\|_{S^s} = \|
abla|^s u\|_{S^0}.
\]

2 Preliminaries on Schrödinger operators, local theory and stability theorem

We consider the defocusing energy-critical NLS, namely
\[
\begin{cases}
 i\partial_t u + \Delta V u - |u|^{\frac{d}{d-2}} u = 0, \\
u(0, x) = u_0(x),
\end{cases}
\]
where \(u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}\).

Before going to the well-posedness theory, we recall some preliminaries on Schrödinger operators. Remark 5.3 in Chen, Magniez and Ouhabaz [6] proved the following result which implies the equivalence of \(\|(-\Delta V)^{\frac{s}{2}} u\|_p\) and \(\|\nabla u\|_p\) for some \(p\).

**Lemma 2.1.** Suppose that \(V \geq 0\) and \(V \in L^{\frac{d}{2} - \eta}(\mathbb{R}^d) \cap L^{\frac{d}{2} + \eta}(\mathbb{R}^d)\) for some \(\eta > 0\), then for \(p \in (1, d)\),
\[
\|\nabla (-\Delta V)^{-\frac{s}{2}} u\|_p \leq C\|u\|_p.
\]

From Lemma 2.1 and the complex interpolation, see for instance [10], we immediately deduce the following result.

**Corollary 2.1** (Norm equivalence). For \(V\) in Theorem 1.1, \(0 \leq s \leq 1, 1 < p < \frac{d}{s}\), we have
\[
\|(-\Delta V)^{\frac{s}{2}} u\|_p \sim \|(-\Delta)^{\frac{s}{2}} u\|_p.
\]

**Remark 2.1.** Although Lemma 2.1 only gives one direction of Corollary 2.1, the other direction of Corollary 2.1 can be as well proved by complex interpolation with Hölder and Sobolev inequality due to the fact \(V\) is regular. For \(d \geq 5\), Corollary 2.1 implies \(\|u\|_{S^s} \sim \|(-\Delta V)^{\frac{s}{2}} u\|_{S^0}\). However, the two norms are not equivalent for \(d \leq 4\).
Lemma 2.2. For $V$ satisfying the assumptions in Theorem 1.1, we have $\forall f \in \dot{H}^2$, 

$$\|\Delta f\|_2 \sim \|\Delta V f\|_2. \quad (2.2)$$

Proof. The Sobolev embedding $\|f\|_{\frac{2d}{d-4}} \leq C\|\Delta f\|_2$ and Hölder inequality yield 

$$\|\Delta V f\|_2 \leq C\|\Delta f\|_2. \quad (2.3)$$

Thus it suffices to prove the inverse direction 

$$\|\Delta f\|_2 \leq C\|\Delta V f\|_2. \quad (2.4)$$

We prove it by contradiction. Suppose that (2.4) is false, then there exists $\{f_n\} \subset \dot{H}^2$ such that 

$$\|\Delta f_n\|_2 \geq n\|\Delta V f_n\|_2.$$

Without loss of generality, we assume $\|\Delta f_n\|_2 = 1$. Then $\lim_{n \to \infty} \|\Delta V f_n\|_2 = 0$, i.e., 

$$\lim_{n \to \infty} \left(\|\Delta f_n\|_2^2 - \langle \Delta f_n, V f_n \rangle - \langle V f_n, \Delta f_n \rangle + \|V f_n\|_2^2\right) = 0. \quad (2.5)$$

Since $\|f_n\|_{\dot{H}^2}$ is bounded, after extracting a subsequence, we may assume $f_n \rightharpoonup f_*$ weakly in $\dot{H}^2$. We claim 

$$\lim_{n \to \infty} \langle \Delta f_n, V f_n \rangle = \langle \Delta f_*, V f_* \rangle, \quad \lim_{n \to \infty} \|V f_n\|_2^2 = \|V f_*\|_2^2. \quad (2.6)$$

Indeed, by integrating by parts, one has 

$$\int_{\mathbb{R}^d} \Delta f_n V f_n \, dx = -\int_{\mathbb{R}^d} \nabla f_n \cdot \nabla V f_n \, dx - \int_{\mathbb{R}^d} V \nabla f_n \cdot \nabla f_n \, dx.$$

For any $\varepsilon > 0$, choosing $R > 0$ sufficiently large, Hölder’s inequality and Sobolev embedding give 

$$\left| \int_{|x| \geq R} \nabla f_n \nabla V f_n \, dx \right| \leq \frac{1}{R} \int_{|x| \geq R} |\nabla f_n| \cdot |x| |\nabla V||f_n| \, dx \leq \frac{1}{R} \|\nabla f_n\|_2 \|f_n\|_{2d} \|x| \nabla V\|_{\frac{2d}{2d-4}} \leq \frac{1}{R} \|\Delta f_n\|_2 \|x| \nabla V\|_{\frac{2d}{4}} \lesssim \frac{1}{R} < \varepsilon. \quad (2.7)$$
Similarly we have
\[
\int_{|x| \geq R} V^2 |f_n|^2 \, dx \leq \frac{1}{R} \|\Delta f_n\|_2^2 \|x| V^2\|_2^\frac{d}{2} \lesssim \frac{1}{R} < \epsilon, \\
\int_{|x| \geq R} V |\nabla f_n|^2 \, dx \leq \frac{1}{R} \|\Delta f_n\|_2^2 \|x| V\|_2^\frac{d}{2} \lesssim \frac{1}{R} < \epsilon.
\]
(2.8)

(2.9)

Since the Sobolev embedding is compact on bounded domains, by extracting a subsequence, together with (2.7), (2.8) and (2.9), we obtain
\[
\int \nabla f_n \cdot \nabla V f_n \, dx \to \int \nabla f_* \cdot \nabla V f_* \, dx, \\
\int V \nabla f_n \cdot \nabla f_n \, dx \to \int V \nabla f_* \cdot \nabla f_* \, dx, \\
\|V f_n\|_{L^2}^2 \to \|V f_*\|_{L^2}^2, \quad \text{as } n \to \infty.
\]

Then (2.6) follows. Therefore, we have proved
\[
\lim_{n \to \infty} \left( - \langle \Delta f_n, V f_n \rangle - \langle V f_n, \Delta f_n \rangle + \|V f_n\|_2^2 \right) \\
= - \langle \Delta f_*, V f_* \rangle - \langle V f_*, \Delta f_* \rangle + \|V f_*\|_2^2. 
\]
(2.10)

Combining (2.5) and (2.10), with \(\liminf_{n \to \infty} \|\Delta f_n\|_2^2 \geq \|\Delta f_*\|_2^2\), we have
\[
\|\Delta f_*\|_2^2 - \langle \Delta f_*, V f_* \rangle - \langle V f_*, \Delta f_* \rangle + \|V f_*\|_2^2 \leq 0.
\]
Hence we have \(\|\Delta V f_*\|_2 = 0\). By Hölder inequality and \(f_* \in L^\frac{2d}{d-2}\), there exists \(\sigma > 0\) sufficiently large such that \(f_* \in L^2((x)^{-\sigma})\). If \(f_* \neq 0\), then we see \(f_*\) is an eigenfunction of \(-\Delta V\) at zero when \(f_* \in L^2\) or a resonance when \(f_* \not\in L^2\). Both of these two cases contradict with the assumption (ii) in the regular potential hypothesis. Hence \(f_* = 0\). Then (2.5) and (2.10) give
\[
\lim_{n \to \infty} \|\Delta f_n\|_2 = 0,
\]
which contradicts with \(\|\Delta f_n\|_2 = 1\). Therefore we have shown (2.4). Thus (2.2) follows from (2.3) and (2.4). \(\square\)

Now we give the local wellposedness theorem, the existence of wave operator and stability theorem without proofs, since they are standard.

**Lemma 2.3** (Local wellposedness). For any \(u_0 \in \dot{H}^1\), there exists a unique maximal lifespan solution \(u\) to (2.1), with \((T_{\min},T_{\max})\) be the maximal existence time interval such that \(u \in C^0_\text{b} \dot{H}^1((T_{\min},T_{\max}) \times \mathbb{R}^d) \cap \dot{S}^1(T_{\min},T_{\max})\). Moreover if \(\|u_0\|_{\dot{H}^1}\) is sufficiently small, then (2.1)
is globally well-defined with
\[ \|u\|_{\dot{H}^1([R^d])} \leq C\|u_0\|_{\dot{H}^1}. \]

Suppose that \((T_{\text{min}}, T_{\text{max}})\) is the lifespan of \(u(t)\), then the energy
\[ \mathcal{E}(u(t)) = \|u(t)\|^2_{\dot{H}^1} + \frac{d-2}{2d} \int_{R^d} |u|^{\frac{2d}{d-2}} \, dx, \]
is conserved in \((T_{\text{min}}, T_{\text{max}})\).

Lemma 2.4 (Existence of the wave operator). For any \(\varphi \in \dot{H}^1\), there exist positive constants \(T_1, T_2 > 0\) and solution to (2.1) \(u_1(t)\) defined on \([T_1, \infty)\), \(u_2(t)\) defined on \((-\infty, -T_2]\), such that
\[ \lim_{t \to \infty} \|u_1(t) - e^{it\Delta_{\varphi}} \varphi\|_{\dot{H}^1} = 0, \quad \lim_{t \to -\infty} \|u_2(t) - e^{it\Delta_{\varphi}} \varphi\|_{\dot{H}^1} = 0. \]

Lemma 2.5 (Scattering norm). If \(\|u\|_{L_{t,x}^{2(d+2)}([T_{\text{min}}, T_{\text{max}}] \times R^d)} < \infty\), then \((T_{\text{min}}, T_{\text{max}}) = R\) and \(u\) scatters to \(e^{it\Delta_{\varphi}} u_+\) for some \(u_+ \in \dot{H}^1\). If \(T_{\text{max}} < \infty\), then \(\|u\|_{L_{t,x}^{2(d+2)}([0, T_{\text{max}}] \times R^d)} = \infty\), a corresponding result holds if \(T_{\text{min}} < \infty\).

Lemma 2.6 (Stability theorem). Let \(I \subseteq R\) be an interval and let \(t_0 \in I\). Suppose that \(\tilde{u}\) is defined on \(I \times R^d\) and satisfies \(\sup_{t \in I} \|\tilde{u}\|_{\dot{H}^1} \leq A\) and \(\|\tilde{u}\|_{L_{t,x}^{2(d+2)}(I \times R)} \leq M\) for constants \(M, A > 0\). Assume that
\[ i\partial_t \tilde{u} + \Delta_{\varphi} \tilde{u} - |\tilde{u}|^{\frac{4}{d-2}} \tilde{u} = e, \]
for some function \(e\). If
\[ \|u_0 - \tilde{u}(t_0)\|_{\dot{H}^1} \leq A', \quad \|\nabla e\|_{L_{t,x}^{\frac{2d}{d-2}}} \leq \varepsilon, \quad \left\| e^{i(t-t_0)\Delta_{\varphi}} (u_0 - \tilde{u}(t_0)) \right\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \leq \varepsilon, \]
then there exists \(\varepsilon_0\) depending on \(M, A, A'\) and \(d\) such that there exists a solution \(u\) to (2.1) with \(u(t_0) = u_0\), for \(0 < \varepsilon < \varepsilon_0\), with \(\|u\|_{L_{t,x}^{2(d+2)}(I \times R^d)} < C(M, A, A', d)\).

3 Convergence lemmas and Linear profile decomposition

In order to establish the linear profile decomposition, we need to give some estimates. First, we will recall the spectral multiplier theorem and the distorted Fourier transformation.

The following spectral multiplier theorem is proved in Proposition 5.2 in [11].

**Proposition 3.1.** Assume that \(V \geq 0\) and \(\sup_x \int_{R^d} \frac{V(y)}{|x-y|^d} \, dy < \infty\). Then for any \(g \in C_c^\infty(R)\), \(\theta > 0\), the operator \(g(-\theta \Delta_{\varphi})\) is bounded on \(L^p(R^d), 1 \leq p \leq \infty\), with norm independent of \(\theta\):
\[ \|g(-\theta \Delta_{\varphi})\|_{L(L^p, L^p)} \leq C(p, d, g, V). \]
In [1], Alsholm and Schmidt proved the existence of distorted Fourier transformation. We briefly describe their results.

**Proposition 3.2** (Distorted Fourier transformation). Assume that \( V \) is the potential in Theorem 1.1, then there exists a function \( \varphi(x,k) \) and a unitary operator \( \mathcal{F}_V \) in \( L^2 \) defined by

\[
(\mathcal{F}_V u)(k) = \int_{\mathbb{R}^d} u(x) \varphi(x,k) \, dx.
\]

Moreover, \( \|\mathcal{F}_V f\|_2 = \|f\|_2 \), \( (\mathcal{F}_V g(-\Delta_V)f)(k) = g(k^2)(\mathcal{F}_V f)(k) \), where \( g \) is some Borel function in \( \mathbb{R} \).

**Lemma 3.1.** For \( V \) in Theorem 1.1, \( f \in \dot{H}^1 \), we have \( \forall \gamma > d \),

\[
\|\langle x \rangle^{-\gamma} \nabla e^{it\Delta_V} f\|_{L^2_tL^2_x} \leq C \|e^{it\Delta_V} f\|_{L_t^{\frac{2}{1+d}}L_x^\infty} \|f\|_{\dot{H}^1}^{\frac{1}{2}}. \tag{3.1}
\]

**Proof.** We claim that for \( f \in H^1 \),

\[
\|\langle x \rangle^{-\frac{2}{3}} \nabla e^{it\Delta_V} f\|_{L^2_tL^2_x} \leq C \|f\|_{\dot{H}^\frac{1}{2}}. \tag{3.2}
\]

To verify (3.2), recall the Morawetz identity. Let \( u \) be a solution to \( i\partial_t u + \Delta_V u = 0 \), for \( a(x) \) sufficiently smooth, one has

\[
\partial_t \Im \int \nabla a \nabla u \bar{u} \, dx = 2\Re \int a_{jk} u_j \bar{u}_k \, dx - \frac{1}{2} \int |u|^2 \Delta^2 a \, dx - \int |u|^2 \nabla a \cdot \nabla V \, dx.
\]

Taking \( a(x) = \langle x \rangle \), it is easy to see

\[
a_{jk} = \frac{\delta_{jk}}{\langle x \rangle} - \frac{x_j x_k}{\langle x \rangle^3}, \quad \Delta^2 a \leq 0, \quad \nabla a \cdot \nabla V \leq 0,
\]

where we have used \( V \) is radial and \( \partial_r V \leq 0 \). Hence

\[
\partial_t \Im \int \nabla a \cdot \nabla u \bar{u} \, dx \geq \int \langle x \rangle^{-3} |\nabla u(x)|^2 \, dx.
\]

Therefore, integrating in time, by Hardy’s inequality and complex interpolation (see for instance Lemma A.10 of [13]), we obtain (3.2). Now we prove (3.1). Take a cutoff function \( g \in C^\infty_c(\mathbb{R}) \) such that \( g(x) \) vanishes when \( |x| > 2 \), and \( g(x) \) equals one for \( |x| < 1 \). For \( \rho > 0 \), Hölder inequality, Corollary 2.1 and Proposition 3.1 yield

\[
\|\langle x \rangle^{-\gamma} \nabla g \left( \rho^{-1} \sqrt{-\Delta_V} \right) e^{it\Delta_V} f\|_{L^2_tL^2_x} \lesssim \|\langle x \rangle^{-\gamma} \|_{L^2_tL^2_x} \|\nabla g \left( \rho^{-1} \sqrt{-\Delta_V} \right) e^{it\Delta_V} f\|_{L_t^{\frac{2}{1+d}}L_x^\infty}.
\]
\[ \| (x)^{-\gamma} \|_{L_x^\infty} \| (\Delta V)^{\frac{1}{2}} g \left( \rho^{-1} \sqrt{-\Delta V} \right) e^{it \Delta V} f \|_{L_t^2 L_x^{\frac{2d}{d-4}}} \lesssim \rho \| (x)^{-\gamma} \|_{L_x^\infty} \| e^{i t \Delta V} f \|_{L_t^2 L_x^{\frac{2d}{d-4}}} \]

Meanwhile, Proposition 3.2 and 3.3 indicate

\[ \| (x)^{-\gamma} \nabla e^{it \Delta V} \left[ 1 - g \left( \rho^{-1} \sqrt{-\Delta V} \right) \right] f \|_{L_x^2} \lesssim \| (x)^{-\frac{3}{2}} \nabla e^{it \Delta V} \left[ 1 - g \left( \rho^{-1} \sqrt{-\Delta V} \right) \right] f \|_{L_x^2} \lesssim \| 1 - g \left( \rho^{-1} \sqrt{-\Delta V} \right) \|_{\dot{H}_x^{\frac{1}{2}}} \lesssim \| \left[ 1 - g \left( \rho^{-1} \sqrt{-\Delta V} \right) \right] (-\Delta V)^{\frac{1}{2}} f \|_{L_x^2} \lesssim \| \mathcal{F}_V \left[ 1 - g \left( \rho^{-1} \sqrt{-\Delta V} \right) \right] (-\Delta V)^{\frac{1}{2}} f \|_{L_x^2} \lesssim \| [1 - g (\rho^{-1}) k^\frac{1}{2} \mathcal{F}_V f (k)] \|_{L_x^2} \lesssim \rho^{-\frac{1}{2}} \| \mathcal{F}_V \left( \sqrt{-\Delta V} f \right) \|_{L_x^2} \lesssim \rho^{-\frac{1}{2}} \| \sqrt{-\Delta V} f \|_{L_x^2} \lesssim \rho^{-\frac{1}{2}} \| f \|_{\dot{H}_x^1}. \]

Therefore (3.1) follows by choosing \( \rho \) appropriately. \( \square \)

Lemma 3.1 can be used to prove the following corollary, which is important in proving the existence of the critical element.

**Corollary 3.1.** If \( f_n \) is bounded in \( \dot{H}_x^1 \), \( \lim_{n \to \infty} \| e^{i t \Delta V} f_n \|_{L_t^2 L_x^{\frac{2d}{d-4}}} = 0 \), then for \( V \) in Theorem 1.1, we have

\[ \lim_{n \to \infty} \| e^{i t \Delta} f_n \|_{L_t^2 L_x^{\frac{2d}{d-4}}} = 0. \]

**Proof.** It suffices to prove

\[ \lim_{n \to \infty} \| e^{i t \Delta} f_n - e^{i t \Delta V} f_n \|_{L_t^2 L_x^{\frac{2d}{d-4}}} = 0. \]

Let \( h_n = e^{i t \Delta V} f_n \), \( g_n = e^{i t \Delta} f_n - e^{i t \Delta V} f_n \), then

\[ g_n(t, x) = i \int_0^t e^{i(t-s)\Delta} V(x) h_n(s, x) \, ds. \]

Strichartz estimate, Hölder inequality and Lemma 3.1 give

\[ \| g_n \|_{L_t^2 L_x^{\frac{2d}{d-4}}} \leq C \| \nabla V h_n \|_{L_t^2 L_x^{\frac{2d}{d-4}}} + C \| V \nabla h_n \|_{L_t^2 L_x^{\frac{2d}{d-4}}} \]

11
Proof. Since we have Strichartz estimates, it is direct to verify Proposition 3.3. For \( g \) a test function \( u \).

Denote \( v \), then Corollary 2.1 with \( v \).

Thus \( L \) in Lemma 4.3. Let \( v \) thus finishing our proof.

The following approximate results are essential in proving the existence of the critical element in Lemma 4.3. Let \( L(\lambda) = \Delta - \lambda^2 V(\lambda x) \), and \( V_\lambda(x) = \lambda^2 V(\lambda x) \).

**Proposition 3.3.** For \( f \in \dot{H}^1 \), it holds

\[
\lim_{\lambda \to 0} \| e^{itL(\lambda)} f - e^{it\Delta} f \|_{\dot{H}^1} = 0. \tag{3.3}
\]

\[
\lim_{\lambda \to \infty} \| e^{itL(\lambda)} f - e^{it\Delta} f \|_{\dot{H}^1} = 0. \tag{3.4}
\]

**Proof.** Since we have

\[
e^{itL(\lambda)} f(x) = \left( e^{it\lambda^2 V}\left( \frac{x}{\lambda} \right) \right)(\lambda x), \tag{3.5}
\]

then Corollary 2.1 with \( s = 1, p = 2 \) gives

\[
\| e^{itL(\lambda)} f \|_{\dot{H}^1} = \lambda^{1 - \frac{d}{2}} \left\| e^{it\lambda^2 V} f\left( \frac{x}{\lambda} \right) \right\|_{\dot{H}^1} \\
\leq C \lambda^{1 - \frac{d}{2}} \left\| f\left( \frac{x}{\lambda} \right) \right\|_{\dot{H}^1} \leq C \| f \|_{\dot{H}^1}. \tag{3.6}
\]

Similarly, by Lemma 2.2 and (3.5), for \( f \in \dot{H}^2 \), we have

\[
\| e^{itL(\lambda)} f \|_{\dot{H}^2} \leq C \| f \|_{\dot{H}^2}.
\]

Denote \( u_1(t, x) = e^{itL(\lambda)} f(x) \) thus we have proved \( \| u_1 \|_{L_t^\infty \dot{H}_x^1} \lesssim \| f \|_{\dot{H}^1} \). For each \( \varepsilon > 0 \), take a test function \( g \in C_c^\infty \) such that \( \| f - g \|_{\dot{H}^1} < \varepsilon \). Denote \( u_2 = e^{itL(\lambda)} g \). Then by (3.5) and Strichartz estimates, it is direct to verify

\[
\| u_1 - u_2 \|_{\dot{H}^1} < C \varepsilon, \| u_1 - u_2 \|_{L_t^\infty L_x^{\frac{2d}{d-2}}} < C \varepsilon, \| \nabla (u_1 - u_2) \|_{L_t^\infty L_x^{\frac{2d}{d-2}}} < C \varepsilon, \\
\| u_2 \|_{L_t^\infty L_x^{\frac{2d}{d-2}}} \leq C \| g \|_2, \| \nabla u_2 \|_{L_t^\infty L_x^{\frac{2d}{d-2}}} \leq C \| \nabla g \|_2, \| \Delta u_2 \|_{L_t^\infty L_x^{\frac{2d}{d-2}}} \leq C \| \Delta g \|_2. \tag{3.7}
\]

Let \( v(t, x) = e^{itL(\lambda)} f - e^{it\Delta} f \), then \( v \) satisfies

\[
v(t, x) = i \int_0^t e^{i(t-s)\Delta} V_\lambda u_1(s, x) \, ds. \tag{3.8}
\]
Hence by Strichartz estimates, (3.7) and Hölder inequality, we deduce

\begin{align}
\|v\|_{S^1} &\leq C\|\nabla (V_\lambda u_1)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\
&\leq C\|\lambda^3 (\nabla V)(\lambda x)u_1\|_{L_t^2 L_x^{\frac{2d}{d+2}}} + C\|V_\lambda \nabla u_1\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\
&\leq C\|\lambda^3 (\nabla V)(\lambda x)(u_1 - u_2)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} + C\|V_\lambda (\nabla u_1 - \nabla u_2)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\
&\quad + C\|\lambda^3 (\nabla V)(\lambda x)u_2\|_{L_t^2 L_x^{\frac{2d}{d+2}}} + C\|V_\lambda \nabla u_2\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\
&\leq C\varepsilon\|\nabla V\|_{L_t^{\frac{d}{4}}} + C\varepsilon\|V\|_{L_t^{\frac{d}{2}}} + C\|\lambda^3 (\nabla V)(\lambda x)u_2\|_{L_t^2 L_x^{\frac{2d}{d+2}}} + C\|V_\lambda \nabla u_2\|_{L_t^2 L_x^{\frac{2d}{d+2}}}. \tag{3.10}
\end{align}

First, we consider \(\lambda \to 0\). (3.7) and Hölder inequality yield

\[\|\lambda^3 (\nabla V)(\lambda x)u_2\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \leq C\lambda^3 \|\nabla V\|_{L_t^{\frac{d}{4}}} \|u_2\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \leq C\lambda \|\nabla V\|_{L_t^{\frac{d}{4}}} \|g\|_{L^2}.\]

Hence it suffices to show

\[\lim_{\lambda \to 0} \|V_\lambda \nabla u_2\|_{L_t^2 L_x^{\frac{2d}{d+2}}} = 0. \tag{3.11}\]

Splitting the time interval \(\mathbb{R}\) into two parts, by Hölder inequality, we have

\[\|V_\lambda \nabla u_2\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \leq \|V_\lambda \nabla u_2\|_{L_t^2(|t| \leq 1) L_x^{\frac{2d}{d+2}}} + \|V_\lambda \nabla u_2\|_{L_t^2(|t| \geq 1) L_x^{\frac{2d}{d+2}}} \leq \|V_\lambda \nabla u_2\|_{L_t^\infty L_x^{\frac{2d}{d+2}}} + \|V_\lambda\|_{L^q} \|\nabla u_2\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \equiv I + II,
\]

where \(\frac{1}{q} + \frac{1}{\tilde{q}} = \frac{d+2}{2d}\) and \(\tilde{q} \in (\frac{2d}{d-1}, \frac{2d}{d-2})\). \(I\) is easy to handle:

\[I \leq \|V_\lambda\|_{L^q} \|\nabla u_2\|_{L_t^\infty L_x^2} \leq \lambda \|f\|_{H^1}.
\]

By Proposition 1.2, Corollary 2.1 and (3.5), we obtain

\begin{align}
\|\nabla u_2\|_{L_t^{\tilde{q}}} &\leq \lambda^{1-\frac{d}{\tilde{q}}} \|\nabla \left[ e^{it\lambda^2 \Delta V}\frac{g(T(x))}{\lambda}\right]\|_{L_t^{\tilde{q}}} \\
&\leq C\lambda^{1-\frac{d}{\tilde{q}}} \|e^{it\lambda^2 \Delta V}(-\Delta V)^{\frac{1}{2}} g(T(x))\|_{L_t^{\tilde{q}}} \\
&\leq C\lambda^{1-\frac{d}{\tilde{q}}} (t\lambda^2)^{-\frac{d}{2}} (1 - \frac{\lambda^2}{2}) \|(-\Delta V)^{\frac{1}{2}} g(T(x))\|_{\tilde{q}}.
\end{align}
\[ \leq Ct^{-\frac{d}{2} \left(1 - \frac{2}{q} \right)} \| \nabla g \|_{\tilde{q}'} \].

Therefore II can be estimated as follows:

\[ II \leq C \| V \|_{L^q} \left( \int_1^\infty t^{-d\left(\frac{1}{q'} - \frac{1}{q} \right)} dt \right)^{\frac{1}{2}} \| \nabla g \|_{\tilde{q}'} \.
\]

Since \( \tilde{q} \in (\frac{2q}{d-1}, \frac{2q}{d-2}) \), it is easy to see II = o(\lambda). Hence the proof of (3.3) is accomplished.

Second, we consider \( \lambda \to \infty \). Back to (3.10), for \( d \geq 7 \), from Hölder inequality and Sobolev embedding, we obtain

\[ \| V \|_{L^q} \| \nabla u_2 \|_{L^q L^\infty} + \| \lambda^3 (\nabla V)(\lambda x) u_2 \|_{L^q L^\infty} \leq \lambda^{-1} \| V \|_{L^q} \| \Delta u_2 \|_{L^q L^\infty} + \| \nabla V \|_{L^q} \| \Delta u_2 \|_{L^q L^\infty} \leq C \| \Delta g \|_{H^{\tilde{q}}_s} \left( \lambda^{-1} \| V \|_{L^q} + \lambda^{-1} \| \nabla V \|_{L^q} \right). \]

Let \( \lambda \to \infty \), (3.4) follows. \( \square \)

We give a local but uniform version of Proposition 3.3. As a preparation, we introduce an inhomogeneous Strichartz pair. It is elementary to verify that if \( \tilde{r} = \left(\frac{2d}{d+2}\right)^{-1}, 2 < q < \infty \), then for \( r \in (2, \infty) \) defined by

\[ \frac{1}{2} - \frac{1}{q} + \frac{d}{2} \left(\frac{1}{r} - \frac{1}{\tilde{r}}\right) = 1, \]

we have

\[ \frac{1}{r} - \frac{2}{d} < \frac{1}{r} \leq \frac{d}{d-2} \left(1 - \frac{1}{r}\right), 1 - \frac{2}{d} \left(\frac{1}{r} - \frac{1}{\tilde{r}}\right) > \frac{1}{2}. \]

Then by Theorem 2.4 in [46],

\[ \left\| \int_0^t e^{i(t-s)\Delta} f(s) \, ds \right\|_{L^q L^\infty} \leq C \| f \|_{L^q L^\infty}. \quad (3.12) \]

To avoid confusions, for \( (q, r) \) introduced above, we denote \( \| \nabla u \|_{L^q_t L^\infty_x} \) by \( u_{IH} \).

**Proposition 3.4.** For fixed \( T > 0, f \in \dot{H}^1 \), we have

\[ \lim_{\lambda \to 0} \| e^{itL(\lambda)} - e^{it\Delta} \|_{L(\dot{H}^1, S^1[-T, T])} = 0; \quad (3.13) \]

\[ \lim_{\lambda \to \infty} \| e^{itL(\lambda)} - e^{it\Delta} \|_{L(\dot{H}^1, IH)} = 0. \quad (3.14) \]

**Proof.** As before, denote \( u_1 = e^{itL(\lambda)} f, v = e^{itL(\lambda)} f - e^{it\Delta} f \). Then by (3.5) and Strichartz...
estimate, we have \( \|u_1\|_{\dot{H}^1} \leq C\|f\|_{\dot{H}^1} \). Strichartz estimates, (3.8) and H"older inequality show

\[
\|v\|_{\dot{H}^1([-T,T] \times \mathbb{R}^d)} \leq C\|\nabla (V\alpha u_1)\|_{L_t^2 L_x^\infty([-T,T] \times \mathbb{R}^d)} \\
\leq CT^{1\over 2}\|\lambda^3(\nabla V)(\lambda x)\|_{L_x^2}^{1\over 4}\|u_1\|_{L_t^\infty L_x^{2d}}^{2d\over 4} + CT^{1\over 2}\|\lambda^2 V(\lambda x)\|_{L_t^\infty L_x^2} \|\nabla u_1\|_{L_t^\infty L_x^2}, \\
\leq CT^{1\over 2}\|\lambda^3(\nabla V)(\lambda x)\|_{L_x^2} \|f\|_{\dot{H}^1} + CT^{1\over 2}\|\lambda^2 V(\lambda x)\|_{L_t^2} \|f\|_{\dot{H}^1},
\]

by which (3.13) follows. If \( \lambda \to \infty \), (3.8), (3.12) and H"older inequality give

\[
\|v\|_{\dot{H}^1} \leq C\|\nabla (V\alpha u_1)\|_{L_t^2 L_x^\infty} \\
\leq C\|\lambda^3(\nabla V)(\lambda x)\|_{L_x^2}^{1\over 4}\|u_1\|_{L_t^\infty L_x^{2d}}^{2d\over 4} + C\|\lambda^2 V(\lambda x)\|_{L_t^\infty L_x^2} \|\nabla u_1\|_{L_t^\infty L_x^2} \\
\leq C\lambda^{-\beta}\|\nabla V\|_{L_x^{4\beta}} \|f\|_{\dot{H}^1} + C\lambda^{-\alpha}\|V\|_{L_x^{4\beta}} \|f\|_{\dot{H}^1}
\]

where \( \alpha, \beta > 0 \). Thus (3.14) is proved.

We now come to the last preparation, after which we will give the linear profile decomposition. Suppose that \( h_n, h_n^j \in (0, \infty) \), define the transformation \( T_n, T_n^j \) as

\[
T_n u(x) = (h_n)^{-\frac{d-2}{2}} u \left( \frac{x}{h_n} \right), \quad T_n^j u(x) = (h_n^j)^{-\frac{d-2}{2}} u \left( \frac{x}{h_n^j} \right),
\]

with the inverse transform of \( T_n^j \) being

\[
(T_n^j)^{-1} u(x) = (h_n^j)^{\frac{d-2}{2}} u \left( h_n^j x \right).
\]

**Lemma 3.2.** If \( h_n \to 0 \) or \( \infty \), \( g_n \to 0 \) in \( \dot{H}^1 \), then for \( \psi \in \dot{H}^1 \),

\[
\lim_{n \to \infty} \langle T_n \psi, T_n g_n \rangle_{\dot{H}^1} = 0.
\]

**Proof.** It is easy to verify

\[
\langle T_n \psi, T_n g_n \rangle_{\dot{H}^1} = \langle \nabla \psi, \nabla g_n \rangle_{L^2} + \langle h_n^2 V(h_n x) \psi, g_n \rangle_{L^2}.
\]

For \( \forall \varepsilon, \psi \in \dot{H}^1 \), take a function \( \tilde{\psi} \in C_0^\infty \) such that \( \|\tilde{\psi} - \psi\|_{\dot{H}^1} < \varepsilon \), then the lemma follows from

\[
\langle h_n^2 V(h_n x) \psi, g_n \rangle_{L^2} = \langle h_n^2 V(h_n x)(\psi - \tilde{\psi}), g_n \rangle_{L^2} + \langle h_n^2 V(h_n x) \tilde{\psi}, g_n \rangle_{L^2}
\]
Proof.

Denote \( \forall \) and for \( h \) if (3.16) gives our proposition. If \( h \to \infty \), instead of (3.16), we use

\[
\| h_n^2 V(h_n x) \|_{L_x^2} \| \psi - \bar{\psi} \|_{H^1} \| g_n \|_{L_x^\infty}^2 + \| \bar{\psi} \|_{L_x^2(\frac{\partial f}{\partial x})} \| h_n^2 V(h_n x) \|_{L_x^2} \| g_n \|_{L_x^\infty}^2, \tag{3.16}
\]

If \( h_n \to 0 \), (3.16) gives our proposition. If \( h_n \to \infty \), instead of (3.16), we use

\[
\| h_n^2 V(h_n x) \|_{L_x^2} \| \psi - \bar{\psi} \|_{H^1} \| g_n \|_{L_x^\infty}^2 + \| \bar{\psi} \|_{L_x^2(\frac{\partial f}{\partial x})} \| h_n^2 V(h_n x) \|_{L_x^2} \| g_n \|_{L_x^\infty}^2.
\]

\( \square \)

The linear profile decomposition is given below and we follow arguments in [24].

**Proposition 3.5** (Linear profile decomposition in \( \dot{H}^1_{rad} \)). Suppose \( v_n = e^{i \Delta x} v_n(0) \) is a sequence of solutions to linear Schrödinger equations and \( \{ v_n(0) \} \) are bounded in \( \dot{H}^1_{rad} \). Then up to extracting a subsequence there exists \( K \in \mathbb{N} \) such that for each \( j \leq K \), there exist \( \phi^j \in \dot{H}^1(\mathbb{R}^d) \), \( \{(t^j_n, h^j_n)\} \subset \mathbb{R} \times (0, \infty) \) satisfying: If we define \( v^j_n, w^k_n \) for \( j < k \leq K \) by

\[
v^j_n = e^{i(t^j_n - t^j_n)} \Delta x T^j_n \phi^j, \quad v_n = \sum_{j=0}^{k-1} v^j_n + w^k_n,
\]

then

\[
\lim_{k \to K} \lim_{n \to \infty} \sup_{t \to \infty} \| \nabla w^k_n \|_{L^\infty(\mathbb{R} ; B^\frac{d}{2}_{1,\infty})} = 0; \tag{3.17}
\]

for \( l < j < k \leq K \), it holds

\[
\lim_{n \to \infty} \left( \frac{h^j_n}{h^l_n} + \frac{h^l_n}{h^k_n} + \frac{|t^j_n - t^l_n|}{(h^l_n)^2} \right) = \infty; \tag{3.18}
\]

and for \( \forall t \geq 0 \),

\[
\| v_n(t) \|_{H^1_{rad}}^2 = \sum_{j=0}^{k-1} \| v^j_n(t) \|_{H^1_{rad}}^2 + \| w^k_n(t) \|_{H^1_{rad}}^2 + o(n) \tag{1}. \tag{3.19}
\]

**Proof.** Denote \( v \equiv \limsup_{n \to \infty} \| \nabla v_n \|_{L_x^\infty B^\frac{d}{2}_{1,\infty}} \). If \( v = 0 \), take \( K = 0 \). Otherwise for \( n \) large enough, there exists \((t_n, x_n) \in \mathbb{R} \times \mathbb{R}^d \) and nonnegative integer \( k_n \) such that

\[
[2^{-\frac{d n}{2}} \phi_{k_n} \ast \nabla v_n(t_n)](x_n) \geq \frac{v}{2}. \tag{3.20}
\]
By radial Gagliardo-Nirenberg inequality and Bernstein inequality,

\[ \left\| 2^\frac{d}{2} \phi_{k_n} * \nabla v_n(t_n) \right\|_{L^\infty(|x| \geq R_n)} \lesssim R_n \frac{d-1}{2} 2^\frac{d}{2} \left\| \nabla \phi_{k_n} * \nabla v_n(t_n) \right\|_2 \frac{1}{2} \left\| \phi_{k_n} * \nabla v_n(t_n) \right\|_2 \]

\[ \lesssim R_n \frac{d-1}{2} 2^\frac{d}{2} \left\| \nabla \phi_{k_n} \right\|_2 \frac{1}{2} \left\| \phi_{k_n} \right\|_2 \left\| \nabla v_n(t_n) \right\|_{L^2} \]

\[ \lesssim R_n \frac{d-1}{2} 2^{\frac{d-1}{2}} \phi_{k_n}. \]

Take \( R_n = R_0 2^{-k_n} \) and let \( R_0 \) be sufficiently large such that

\[ \left\| 2^\frac{d}{2} \phi_{k_n} * \nabla v_n(t_n)(x) \right\|_{L^\infty(|x| \geq R_n)} < \frac{v}{4}. \]

Then by (3.20), \( x_n \) satisfies

\[ |x_n| \leq R_n \quad \text{and} \quad \left[ 2^\frac{d}{2} \phi_{k_n} * \nabla v_n(t_n)(x_n) \right] \geq \frac{v}{4}. \]  

(3.21)

Define \( h_n = 2^{-k_n} \), and let \( \psi_n(x) = h_n^{-\frac{d-2}{2}} v_n(t_n, h_n x) \). By (3.21),

\[ \int_{\mathbb{R}^d} \nabla \psi_n(y) \phi(h_n^{-1} x_n - y) \, dy > \frac{v}{4}. \]  

(3.22)

Because \( |x_n| \leq R_0 h_n \), up to extracting a subsequence, we can assume \( h_n^{-1} x_n \to x^* \) for some constant vector \( x^* \in \mathbb{R}^d \). Since \( \psi_n \) is bounded in \( \dot{H}^1 \), we can postulate \( \psi_n \to \psi \) in \( \dot{H}^1 \), then by \( h_n^{-1} x_n \to x^* \), (3.22) indicates

\[ \left\| \nabla \psi \right\|_2 \gtrsim \langle \nabla \psi(y), \phi(x^* - y) \rangle \geq \frac{v}{4}. \]

If \( h_n \to 0 \) or \( \infty \), we take \((t_0^n, h_0^n) = (t_n, h_n), \varphi^0 = \psi \). If \( h_n \to h_\infty > 0 \), then let

\[ (t_0^n, h_0^n) = (t_n, 1), \varphi^0(x) = h_\infty^{-\frac{d-2}{2}} \psi \left( h_\infty^{-1} x \right). \]

Then \( T_n \psi - T_n^0 \varphi^0 \to 0 \) in \( \dot{H}^1 \), as \( n \to \infty \). Now define

\[ v_n^0 = e^{i(t\cdot \nabla) T_n^0 \varphi_0^0}, \]

\[ w_n^1 = v_n - v_n^0, \]

then one has

\[ (T_n^0)^{-1} w_n^1(t_0^n) \to 0 \quad \text{in} \quad \dot{H}^1, \quad \text{as} \quad n \to \infty. \]  

(3.23)
We claim
\[ \lim_{n \to \infty} \langle v^0_n(t^0_n), w^1_n(t^0_n) \rangle_{\dot{H}^1} = 0. \] (3.24)

Indeed, when \( h_n \to h_{\infty} \),
\[ \langle v^0_n(t^0_n), w^1_n(t^0_n) \rangle_{\dot{H}^1} = \langle T^0_n \varphi^0, w^1_n(t^0_n) \rangle_{\dot{H}^1} = \left\langle \left( h^{-2}_{\infty} \psi(h^{-1}_{\infty}x), (T^0_n)^{-1} \right) w^1_n(t^0_n) \right\rangle_{\dot{H}^1} \to 0, \]
due to (3.23). When \( h_n \to 0 \) or \( \infty \), as a consequence of Lemma 3.2 and the fact \( \psi_n \to \psi \) in \( \dot{H}^1 \),
\[ \langle v^0_n(t^0_n), w^1_n(t^0_n) \rangle_{\dot{H}^1} = \langle T^0_n \psi, T^0_n \psi_n - T^0_n \psi \rangle_{\dot{H}^1} \to 0. \]

Therefore we have proved (3.24). Since the inner product is preserved with respect to \( t \), thus
\[ \lim_{n \to \infty} \langle v^0_n(t), w^1_n(t) \rangle_{\dot{H}^1} = 0. \]

Until now, we have accomplished the first step. Next, we treat \( w^1_n \) as \( v_n \) and do the same work. If \( \limsup_{n \to \infty} \| \nabla w^1_n \|_{L^\infty B_{\infty, \infty}^{-\frac{3}{4}}} = 0 \), take \( K = 1 \). Otherwise we can find \( v^1_n \) and \( w^2_n \) such that there exist \( (t^1_n, h^1_n) \in \mathbb{R} \times (0, \infty) \) and \( \varphi^1 \in \dot{H}^1(\mathbb{R}^d) \) for which
\[ w^1_n = v^1_n + w^2_n, \quad v^1_n = e^{i(t-t^1_n)\Delta} T^1_n \varphi^1; \]
\[ \langle v^1_n(t), w^2_n(t) \rangle_{\dot{H}^1} \to 0 \]
\[ (T^1_n)^{-1} w^2_n(t^1_n) \to 0 \] in \( \dot{H}^1 \), as \( n \to \infty \),
and
\[ \limsup_{n \to \infty} \| \nabla w^1_n \|_{L^\infty B_{\infty, \infty}^{-\frac{3}{4}}} \leq \| \varphi^1 \|_{\dot{H}^1}. \]

Iteration for times gives the desired decomposition, the remaining work is to verify (3.17), (3.18) and (3.19). Firstly, (3.17) is a direct corollary of (3.19) and the fact
\[ \limsup_{n \to \infty} \| \nabla w^k_n \|_{L^\infty B_{\infty, \infty}^{-\frac{3}{4}}} \leq \| \varphi^{k-1} \|_{\dot{H}^1}. \]

Secondly, we prove (3.19) under (3.18). We claim for \( l < j \),
\[ \langle v^l_n(0), v^l_n(0) \rangle_{\dot{H}^1} \to 0, \] as \( n \to \infty \). (3.25)
It is easy to verify
\[ \langle T_n^t f, g \rangle_{H^1_v} = \left\langle f, \left( h_n \right)^{\frac{d+2}{2}} \left( \Delta_v g \right) \left( h_n x \right) \right\rangle_{L^2}, \tag{3.26} \]
\[ \left( e^{i\eta \Delta_v a \left( \frac{x}{\lambda} \right)} \right) (\lambda x) = \left( e^{i\frac{\eta a(\lambda)}{\lambda^2}} a \right) (x). \tag{3.27} \]

Careful calculations with the help of (3.26) and (3.27) imply
\[
\begin{align*}
\left\langle \psi_n^i(0), \psi_n^j(0) \right\rangle_{H^1_v} &= \left\langle e^{-i t_n \Delta_v T_n^t \phi}, e^{-i t_n \Delta_v T_n^t \phi} \right\rangle_{H^1_v} = \left\langle T_n^t \phi, e^{i(t_n - t_0) \Delta_v T_n^t \phi} \right\rangle_{H^1_v} \\
&= \left( \frac{h_n}{h_n} \right)^{\frac{d+2}{2}} \left\langle \phi, (h_n^2) V(h_n^2)e^{i(t_n - t_0) L(h_n)} \phi \right\rangle_{L^2} - \left( \frac{h_n}{h_n} \right)^{\frac{d+2}{2}} \left\langle \nabla \phi, \nabla e^{i(t_n - t_0) L(h_n)} \phi \right\rangle_{L^2}.
\end{align*}
\]

When $\frac{h_n}{h_n} \to 0$, (3.25) follows from
\[
\begin{align*}
\left\langle \phi, (h_n^2) V(h_n^2)e^{i(t_n - t_0) L(h_n)} \phi \right\rangle_{L^2} &
\leq \left\| h_n^2 V(h_n x) \right\|_{L^{\frac{d}{2}}} \left\| \phi \right\|_{L^{d+2}} \left\| e^{i(t_n - t_0) L(h_n)} \phi \right\|_{\dot{H}^1} \\
&\leq \left\| V \right\|_{L^{\frac{d}{2}}} \left\| \phi \right\|_{\dot{H}^1},
\end{align*}
\]

and
\[
\left\langle \nabla \phi, \nabla e^{i(t_n - t_0) L(h_n)} \phi \right\rangle_{L^2} \leq \left\| \phi \right\|_{\dot{H}^1} \left\| \phi \right\|_{\dot{H}^1},
\]

where we have used (3.6). If $\log \left( \frac{h_n}{h_n} \right) \to c \in \mathbb{R}$, due to (3.18), we have $\frac{d + 2}{(h_n)^2} \to \infty$. In this case, note that by density arguments, it suffices to prove (3.25) for $\phi^i, \phi^j \in C_c^\infty$. From Proposition 1.1 and (3.5),
\[
\begin{align*}
\left\| \nabla \phi, \nabla e^{i(t_n - t_0) L(h_n)} \phi \right\|_{L^2} &\leq \left\| \Delta \phi \right\|_{L^{\frac{d}{2}}} \left\| e^{i(t_n - t_0) L(h_n)} \phi \right\|_{L^{d+2}} \to 0, \\
\left\| \phi, (h_n^2) V(h_n^2)e^{i(t_n - t_0) L(h_n)} \phi \right\|_{L^2} &\leq \left\| \phi \right\|_{L^{d+2}} \left\| e^{i(t_n - t_0) L(h_n)} \phi \right\|_{L^{d+2}} \left\| V \right\|_{L^{\frac{d}{2}}} \to 0.
\end{align*}
\]
Hence we have obtained (3.25). Since the inner product is preserved with respect to $t$, then
\[
\lim_{n \to \infty} \left\langle v^l_n(t), v^j_n(t) \right\rangle_{\dot{H}^1_V} = 0.
\]
(3.28)

By (3.25) and the procedure of construction,
\[
\left\langle v^j_n(t), w^k_n(t) \right\rangle_{\dot{H}^1_V} = \left\langle v^j_n(t), w^{j+1}_n(t) \right\rangle_{\dot{H}^1_V} - \sum_{m=j+1}^{k-1} \left\langle v^j_n(t), v^m_n(t) \right\rangle_{\dot{H}^1_V} \to 0.
\]
(3.29)

Then (3.19) follows easily from (3.28) and (3.29). Thirdly, we prove (3.18) by induction. Assume that (3.18) holds for $(n_1, n_2) < (l, j)$, we prove it holds for $(l, j)$. Suppose that (3.18) is false for $(l, j)$, then up to extracting a subsequence, we can assume
\[
h^l_n \to h^l_\infty \in \{0, \infty\} \cup \mathbb{R}, \quad \frac{(t^l_n - t^l_m)^2}{h^l_n} \to c \in \mathbb{R}, \quad \log \left(\frac{h^l_m}{h^l_n} \right) \to a \in \mathbb{R}, \quad \text{as} \quad n \to \infty.
\]
(3.30)

Notice that the process of constructing profiles $\{\varphi^m\}$ yields
\[
(T^l_n)^{-1} e^{i(t^l_n - t^m_n) \Delta V} T^m_n \varphi^m = (T^l_n)^{-1} e^{i(t^l_n - t^m_n) \Delta V} T^m_n \varphi^m + (T^l_n)^{-1} w^{j+1}_n(t^l_n),
\]
(3.31)
and
\[
(T^j_n)^{-1} w^{j+1}_n(t^j_n) \to 0 \text{ weakly in } \dot{H}^1, \quad (T^l_n)^{-1} w^{j+1}_n(t^l_n) \to 0 \text{ weakly in } \dot{H}^1.
\]
(3.32)

Meanwhile (3.27) gives,
\[
(T^l_n)^{-1} e^{i(t^l_n - t^m_n) \Delta V} T^m_n \varphi^m = \left(\frac{h^l_m}{h^l_n} \right)^{\frac{d-2}{2}} e^{i \frac{t^l_n - t^m_n}{h^l_n} L(h^m_n)} \left(\frac{h^l_m}{h^l_n} \right)^{\frac{d-2}{2}} \varphi^m = S^{l,m}_n \varphi^m.
\]

From our hypothesis, $S^{l,m}_n \varphi^m \to 0$ in $\dot{H}^1$, as $n \to \infty$, for $m < j$. Hence we deduce from (3.32), (3.30) and Proposition 3.3 that
\[
(T^l_n)^{-1} w^{j+1}_n(t^l_n) \to 0 \text{ weakly in } \dot{H}^1.
\]
Combining this with $S^{l,m}_n \varphi^m \to 0$ and (3.31), (3.32) gives
\[
\varphi^j \equiv 0,
\]
which is a contradiction.

The linear profile decomposition enjoys more properties than addressed in Proposition 3.5.
We collect them below.

**Proposition 3.6.** Suppose that \( v_n, v_j^n, w_k^n, h_j^n \) are the components of the profile decomposition in Proposition 3.5. Then there are only three cases for \( h_j^n \), namely, \( \lim_{n \to \infty} h_j^n = 0 \), or \( \lim_{n \to \infty} h_j^n = \infty \) or \( h_j^n = 1 \) for all \( n \). For any fixed \( t \), the following energy decoupling property holds:

\[
\mathcal{E}(v_n) = \sum_{j=0}^{k-1} \mathcal{E}(v_j^n) + \mathcal{E}(w_k^n) + o_n(1). \tag{3.33}
\]

And

\[
\lim_{k \to K} \limsup_{n \to \infty} \| u_n^k \|_{L_t^\infty L_x^{2d-4}} = 0. \tag{3.34}
\]

\[
\lim_{k \to K} \limsup_{n \to \infty} \| u_n^k \|_{L_t^m L_x^n} = 0. \tag{3.35}
\]

**Proof.** The proof of (3.33) is standard except some modifications, see for instance [27]. In fact, the linear part of \( \mathcal{E}(v_n) \) has been proved in (3.17). The nonlinear part can be proved with the help of Proposition 3.3 and (3.26). It remains to prove (3.34) and (3.35). The refined Sobolev embedding theorem gives

\[
\lim_{k \to K} \limsup_{n \to \infty} \| w_n^k \|_{L_t^\infty L_x^{2d-4}} = 0.
\]

Moreover, by interpolation we have

\[
\lim_{k \to K} \limsup_{n \to \infty} \| u_n^k \|_{L_t^m L_x^n} = 0, \tag{3.36}
\]

where \((m, n)\) is an \( \dot{H}^1 \)-admissible pair and \( m > 2 \), which implies (3.35). The Gagliardo-Nirenberg implies

\[
\| u_n^k \|_{L_t^\infty L_x^{2d-4}} \leq C \| \nabla w_n^k \|_{L_t^{\eta} L_x^{\gamma}} \| u_n^k \|_{L_t^{\gamma} L_x^{p}}^{1-\theta}, \tag{3.37}
\]

where \( p = (\frac{2d}{d-2})^+, \, r = (\frac{2d}{d-2})^+, \, \theta = \frac{2dr - (d-4)p}{2(d-r)p + 2pr} \). Using Hölder inequality, we conclude that

\[
\| u_n^k \|_{L_t^\infty L_x^{2d-4}} \leq C \| \nabla w_n^k \|_{L_t^\eta L_x^{\gamma}} \| u_n^k \|_{L_t^\gamma L_x^{p}}^{1-\theta},
\]

where \((\eta, p)\) is \( \dot{H}^1 \)-admissible pair, \((\gamma, r)\) is \( L^2 \)-admissible pair, and

\[
\frac{1 - \theta}{\eta} + \frac{\theta}{\gamma} = \frac{1}{2}. \tag{3.38}
\]

Direct calculation shows (3.38) coincides with the choice of \( \theta \), thus (3.35) follows from (3.36). \( \square \)
As a direct consequence of (3.35) and Corollary 3.1, we have

**Corollary 3.2.** For \( w^k_n \) in Proposition 3.5 and a fixed \( j \),

\[
\lim_{k \to K} \limsup_{n \to \infty} \left\| e^{it \Delta} e^{it_i \Delta V} w^k_n(0) \right\|_{L^2_{t,x} L^{\infty}_{x}} = 0.
\]

4 Proof of Theorem 1.1

4.1 The existence of critical elements

In this subsection, we will show if uniform global scattering norm bound fails for any finite energy solution to (2.1), then there exists a critical element, which is a global solution with infinite scattering norm and minimal energy.

Define

\[
\mathcal{E} = \left\{ m : \forall u_0 \in \dot{H}^1, \mathcal{E}(u_0) < E, \text{the solution to (2.1) is globally wellposed and } \|u(t,x)\|_{L^{2(d+2)}_{t,x}} < \infty \right\}.
\]

Denote \( E_* = \sup\{E : E \in \mathcal{E}\} \). We aim to prove \( E_* = \infty \) by contradiction. Suppose that \( E_* < \infty \), then there exists a sequence of solution (up to time translations) to (2.1), such that \( \mathcal{E}(u_n) \nRightarrow E_* \), as \( n \to \infty \), and

\[
\lim_{n \to \infty} \left\| u_n \right\|_{L^{2(d+2)}_{t,x}([0,\sup I_n] \times \mathbb{R}^d)} = \lim_{n \to \infty} \left\| u_n \right\|_{L^{2(d+2)}_{t,x}((\inf I_n,0] \times \mathbb{R}^d)} = \infty, \tag{4.1}
\]

where \( I_n \) denotes the maximal interval of \( u_n \) including 0.

Apply the linear profile decomposition to \( u_n(0) \), we get \( \varphi^i \), \( \{ (h_n^j, t_n^j) \} \) for which (3.17), (3.18), (3.19) hold and

\[
e^{it \Delta V} u_n(0) = \sum_{j=0}^{k-1} e^{i(t-t_n^j) \Delta V} T_n^j \varphi^j + w^k_n(t). \tag{4.2}
\]

Now we construct the corresponding nonlinear profiles. Suppose that \( U_n^j \) is a solution to (2.1) with initial data \( U_n^j(0) = e^{-it \Delta V} T_n^j \varphi^j \), then \( U_n^j(t) \) satisfies

\[
U_n^j(t) = e^{i(t-t_n^j) \Delta V} T_n^j \varphi^j - i \int_{0}^{t} e^{i(t-\tau) \Delta V} \left( [U_n^j] \frac{1}{t-\tau} U_n^j \right)(\tau) \, d\tau.
\]

22
Let \( U_n^j(t) = (h_n^j)^{-\frac{d-2}{2}} u_n^j \left( \frac{t-t_n^j}{(h_n^j)^2}, \frac{x}{h_n^j} \right) \). Then \( u_n^j(t, x) \) satisfies

\[
v_n^j(t, x) = e^{iL(h_n^j)} \varphi^j - i \int_0^t \frac{t_n^j}{(h_n^j)^2} e^{i(t-s)L(h_n^j)} \left( |v_n^j| \frac{d}{dx} \varphi^j \right)(s) \, ds.
\]

If \( h_n^j \to 0 \) or \( h_n^j \to \infty \), let \( u^j(t, x) \) be a solution to

\[
u^j = e^{i\Delta \varphi^j} - i \int_{\tau^j_{\infty}}^t e^{i(t-\tau)\Delta} \left( |u^j| \frac{d}{dx} u^j \right)(\tau) \, d\tau,
\]

where \( \tau^j_{\infty} = \lim_{n \to \infty} \frac{-t_n^j}{(h_n^j)^2} \). If \( \tau^j_{\infty} = \pm \infty \), then \( u^j \) is given by the wave operator. If \( \tau^j_{\infty} \in \mathbb{R} \), then \( u^j \) is given by the global well-posedness and scattering theorem in [48], and we have \( \|u^j\|_{S^1(\mathbb{R} \times \mathbb{R}^d)} < \infty \).

If \( h_n^j = 1 \), let \( u^j \) be a solution to

\[
u^j = e^{i\Delta \varphi^j} - i \int_{\tau^j_{\infty}}^t e^{i(t-\tau)\Delta \varphi^j} \left( |u^j| \frac{d}{dx} u^j \right)(s) \, ds.
\]

Again for \( \tau^j_{\infty} = \pm \infty \), Lemma 2.3 gives the existence of \( u^j \). For \( \tau^j_{\infty} \in \mathbb{R} \), local Cauchy theory namely Lemma 2.3 provides the existence of \( u^j \) at least in a small interval. We call \( u^j \) nonlinear profile. Suppose \( I^j = (T^j_{\min}, T^j_{\max}) \) is the lifespan of \( u^j \), then by the definition of \( u^j \), we have \( u^j \in C^0_t H^1_x(I^j \times \mathbb{R}^d) \) and

\[
\lim_{n \to \infty} \left\| u^j \left( - \frac{t_n^j}{(h_n^j)^2} \right) - e^{-i \frac{t_n^j}{(h_n^j)^2} \Delta} \varphi^j \right\|_{H^1} \to 0, \text{ if } h_n^j \to 0, \text{ or } h_n^j \to \infty, \quad (4.3)
\]

\[
\lim_{n \to \infty} \left\| u^j \left( - \frac{t_n^j}{(h_n^j)^2} \right) - e^{-i \frac{t_n^j}{(h_n^j)^2} \Delta \varphi^j} \right\|_{H^1} \to 0, \text{ if } h_n^j = 1. \quad (4.4)
\]

Define \( u_n^j(t, x) = (h_n^j)^{-\frac{d-2}{2}} u^j \left( \frac{t-t_n^j}{(h_n^j)^2}, \frac{x}{h_n^j} \right) \), then \( u_n^j \) has the lifespan \( I_n^j = ((h_n^j)^2 T^j_{\min} + t_n^j, (h_n^j)^2 T^j_{\max} + t_n^j) \). Define

\[
u_n^{< k}(t, x) = \sum_{j=0}^{k-1} u_n^j(t, x).
\]

The following two lemmas are standard, which can be easily obtained by using the well-posedness and scattering theory in Lemma 2.3 and Lemma 2.5 as well as Proposition 3.5.

**Lemma 4.1.** There exists \( j_0 \in \mathbb{N} \) such that \( T^j_{\min} = -\infty, T^j_{\max} = \infty \) for \( j > j_0 \) and

\[
\sum_{j > j_0} \|u^j\|_{S^2(\mathbb{R} \times \mathbb{R}^d)}^2 \lesssim \sum_{j > j_0} \|\varphi^j\|_{H^1(\mathbb{R}^d)}^2 < \infty.
\]
Lemma 4.2. In the nonlinear profile decomposition \([4,5]\), if

\[
\|u_j\| \frac{2(d+2)}{L_{i,x}^{d-2}} \left( (T_{\min}^j, T_{\max}^j) \times \mathbb{R}^d \right) < \infty, \quad 1 \leq j \leq j_0,
\]

then \(T_{\min}^j = -\infty, \ T_{\max}^j = \infty\) and for \(1 \leq j \leq j_0\), there exists \(B, B_1 > 0\) such that

\[
\lim_{n \to \infty} \|u_n^j\| \frac{2(d+2)}{L_{i,x}^{d-2}} \left( \mathbb{R} \times \mathbb{R}^d \right) \leq B, \quad \lim_{n \to \infty} \|u_n^j\|_{H^1(\mathbb{R}^d)} \leq B_1.
\]

Lemma 4.3. Let \(j_0\) be the integer in Lemma \([4.1]\) then there exists \(1 \leq j \leq j_0\) such that

\[
\|u_j\| \frac{2(d+2)}{L_{i,x}^{d-2}} \left( I \times \mathbb{R}^d \right) = \infty.
\]

Proof. We prove it by contradiction. Suppose that for \(1 \leq j \leq j_0\), \(\|u_j\| \frac{2(d+2)}{L_{i,x}^{d-2}} \left( I \times \mathbb{R}^d \right) < \infty\), then together with Lemma \([4.2]\), we have \(u_j^j\) exists globally for \(j \geq 1\). Thus \(u_n^j + w_n^j\) exists globally. If we have verified that for \(n, k\) sufficiently large, \(u_n^j + w_n^j\) is a perturbation of \(u_n\), then by the stability theorem, we can derive a contradiction. From Proposition \([3.5]\) and Lemma \([4.2]\), there exist positive constants \(B\) and \(B_1\) such that

\[
\limsup_{n \to \infty} \left\| u_n^j + w_n^j \right\|_{L^n(H^1(\mathbb{R} \times \mathbb{R}^d))} \leq B \tag{4.6}
\]

\[
\limsup_{n \to \infty} \left\| u_n^j + w_n^j \right\|_{L^n(H^1(\mathbb{R} \times \mathbb{R}^d))} \leq B_1 \tag{4.7}
\]

Denote \(\tau_n^j = -\frac{t_j}{(h_n^j)^2}\). When \(t = 0\), by (3.2), we can easily see

\[
\begin{aligned}
\left\| u_n^j(0) + u_n^k(0) - u_n(0) \right\|_{H^1}
&= \sum_{j=0}^{k-1} \left\| (h_n^j)^{-\frac{d+2}{2}} u_j^{- \frac{1}{(h_n^j)^2} \frac{x}{h_n^j}} - (h_n^j)^{-\frac{d+2}{2}} \left( e^{-it_n^j \Delta \varphi^j \left( \frac{x}{h_n^j} \right) \left( x \right)} \right) \right\|_{H^1}
&\leq \sum_{j=0}^{k-1} \left\| u_j^{\tau_n^j} - e^{-it_n^j \Delta \varphi^j \left( \frac{x}{h_n^j} \right) \left( x \right)} \right\|_{H^1}
&= \sum_{j=0}^{k-1} \left\| u_j^{\tau_n^j} - e^{i\tau_n^j L(h_n^j) \varphi^j} \right\|_{H^1}.
\end{aligned}
\]

Combining \([4.3]\), \([4.4]\) with Proposition \([3.3]\), we get

\[
\lim_{n \to \infty} \left\| u_n(0) - u_n^k(0) - w_n^k(0) \right\|_{H^1} = 0. \tag{4.8}
\]
We claim that
\[
\lim_{k \to \infty} \lim_{n \to \infty} \left\| \nabla [ (i \partial_t + \Delta_V)(u_n^{<k} + w_n^k) - F(u_n^{<k} + w_n^k) \right\|_{L_t^2 L_x^{\frac{2d}{d-2}}} = 0.
\] (4.9)

where \( F(u) = |u|^{\frac{4}{d-2}} u \). Suppose that the claim holds, then from (4.6)-(4.9) and the stability theorem, we will obtain for \( n \) sufficiently large,
\[
\| u_n \|_{L_t^2 L_x^{\frac{2(d+2)}{d-2}}} < \infty,
\]
which contradicts with (4.11). Thus Lemma 4.3 follows. Therefore we only need to prove (4.9).

Note that
\[
(i \partial_t + \Delta_V)(u_n^{<k} + w_n^k) - F(u_n^{<k} + w_n^k) = \sum_{j=0}^{k-1} (i \partial_t + \Delta_V)u_n^j - F(u_n^{<k}) - F(u_n^{<k}) + F(u_n^{<k}),
\] (4.10)

it suffices to verify
\[
\lim_{k \to \infty} \lim_{n \to \infty} \left\| \nabla \left( \sum_{j=0}^{k-1} (i \partial_t + \Delta_V)u_n^j - F(u_n^{<k}) \right) \right\|_{L_t^2 L_x^{\frac{2d}{d-2}}} = 0,
\] (4.12)

and
\[
\lim_{k \to \infty} \lim_{n \to \infty} \left\| \nabla (F(u_n^{<k} + w_n^k) - F(u_n^{<k})) \right\|_{L_t^2 L_x^{\frac{2d}{d-2}}} = 0.
\] (4.13)

First, we prove (4.12). When \( \lim_{n \to \infty} h_n^j = 0 \) or \( \lim_{n \to \infty} h_n^j = \infty \), \( u_n^j \) satisfies
\[
(i \partial_t + \Delta)u_n^j = F(u_n^j).
\]
From the scattering theorem in [48], we have
\[
\| u_n^j \|_{L_t^2 L_x^{\frac{2(d+2)}{d-2}}} < \infty.
\]

Direct calculations show
\[
\left\| \nabla ((i \partial_t + \Delta_V)u_n^j - F(u_n^j)) \right\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \leq \left\| \nabla \left( V(x)(h_n^j)^{-\frac{d-2}{2}} u^j \left( \frac{t - h_n^j}{(h_n^j)^2}, \frac{x}{h_n^j} \right) \right) \right\|_{L_t^2 L_x^{\frac{2d}{d-2}}}
\]
\[
\leq (h_n^{\gamma})^{-\frac{d+2}{2}} \left\| \nabla V(x) u^j \left( \frac{t - t_n^j}{(h_n^{\gamma})^2} \right) \right\|_{L^2_t L^\infty_x} + \langle x \rangle \gamma (h_n^{\gamma})^{-\frac{d+2}{2}} \left\| \langle x \rangle^{-\gamma} \nabla u_n^{k,j} \right\|_{L^2_t L^\infty_x}.
\]

For any \( \varepsilon > 0 \), take a function \( \tilde{u} \in C_c^\infty (\mathbb{R} \times \mathbb{R}^d) \) for which \( \| \tilde{u} - u \|_{L^2_t L^\infty_x} < \varepsilon \). Then Hölder inequality implies
\[
(h_n^{\gamma})^3 \left\| (\nabla V)(h_n^{\gamma} x) u^j (t, x) \right\|_{L^2_t L^\infty_x} \leq (h_n^{\gamma})^3 \left\| (\nabla V)(h_n^{\gamma} x)(u^j - \tilde{u})(t, x) \right\|_{L^2_t L^\infty_x} + (h_n^{\gamma})^3 \left\| (\nabla V)(h_n^{\gamma} x)\tilde{u}^j (t, x) \right\|_{L^2_t L^\infty_x}.
\]

Letting \( n \to \infty \), if \( h_n^{\gamma} \to 0 \), we get \( (h_n^{\gamma})^3 \left\| (\nabla V)(h_n^{\gamma} x) u^j (t, x) \right\|_{L^2_t L^\infty_x} \to 0 \), as \( n \to \infty \), and the same arguments show \( (h_n^{\gamma})^2 \left\| V(h_n^{\gamma} x)(\nabla u^j)(t, x) \right\|_{L^2_t L^\infty_x} \to 0 \), as \( n \to \infty \). When \( h_n^{\gamma} \to 0 \), similar arguments work. Hence for \( h_n^{\gamma} \to \infty \) and \( h_n^{\gamma} \to 0 \), we have proved
\[
\lim_{n \to \infty} \left\| \nabla \left( (i\partial_t + \Delta \nu)(u^j_n) - F(u_n^j) \right) \right\|_{L^2_t L^\infty_x} = 0. \tag{4.14}
\]

When \( h_n^{\gamma} = 1 \), (4.14) is obvious. By \( (4.14) \) and triangle inequality, (4.15) can be reduced to
\[
\lim_{k \to \infty} \limsup_{n \to \infty} \left\| \nabla \left( F(u_n^k) - \sum_{j=0}^{k-1} F(u_n^j) \right) \right\|_{L^2_t L^\infty_x} = 0. \tag{4.15}
\]

Following the same arguments in \([27, 29]\), (4.15) and (4.13) can be further reduced to
\[
\lim_{k \to \infty} \limsup_{n \to \infty} \left\| u_n^j \nabla u_n^k \right\|_{L^4_t L^\infty_x} = 0, \text{ for fixed } j. \tag{4.16}
\]

By density arguments, we can assume \( u^j \in C_c^\infty (\mathbb{R} \times \mathbb{R}^d) \) with \( \text{supp } u^j \subset [-T, T] \times [-R, R]^d \). **Case 1.** If \( h_n^{\gamma} = 1 \), then \( u_n^j(t, x) = u^j(t - t_n^j, x) \), Hölder’s inequality and Lemma \( 3.1 \) give
\[
\left\| u_n^j \nabla u_n^k \right\|_{L^4_t L^\infty_x} \leq \left\| \langle x \rangle^\gamma u^j(t - t_n^j, x) \right\|_{L^{\frac{2(d+2)}{d-2}}_t L^\infty_x} \left\| \langle x \rangle^{-\gamma} \nabla u_n^k \right\|_{L^2_t L^\infty_x} \leq C \left\| u_n^k \right\|_{L^2_t L^{\frac{2d}{d-2}}_x} \left\| u_n^j \right\|_{L^\infty H^1}.
\]
Thus Corollary 3.2 yields (4.16).

**Case 2.** If $h_n^j \to \infty$, by (3.13), Hölder’s inequality and smoothing effect of the free Schrödinger equation, we get

$$
\| u_n^j \nabla w_n^k \|_{L_{t,x}^{d+2}}^2 \leq (h_n^j)^{d-2} \| \hat{u}^j \nabla \left( e^{it\Delta} w_n^k(0) \right) \|_{L_{t,x}^{d+2}}^2 \leq (h_n^j)^{d-2} \| \hat{u}^j \nabla \left( e^{it\Delta} e^{i\tau h_n^j} w_n^k(0, h_n^j x) \right) \|_{L_{t,x}^{d+2}}^2
$$

Thus Corollary 3.2 yields (4.16).

**Case 3.** If $h_n^j \to 0$, replacing IH by $\hat{S}^1(-T, T)$ in case 2, we can similarly prove (4.16) by (3.13) and Corollary 3.2. Thus Lemma 4.3 follows.

By Lemma 4.2 and Lemma 4.3 we can derive the critical element by using the standard argument in the compactness-contradiction argument.

**Proposition 4.1** (Existence and compactness of a critical element). Suppose that $m_* < \infty$, then there exists a global solution $u_c \in C^0_T \dot{H}^1_x(\mathbb{R} \times \mathbb{R}^d)$ to (2.1) such that

$$
\mathcal{E}(u_c(t)) = E_*, \quad \text{for } \forall t \in \mathbb{R},
$$
\[ \| u_c \|_{L^2_t (\mathbb{R}^d \times (0, \infty))}^{2(d+2)} = \| u_c \|_{L^2_t (\mathbb{R}^d \times (-\infty, 0])}^{2(d+2)} = \infty. \]

Moreover, \{u_c(t) : t \in \mathbb{R}\} is pre-compact in \( H^1_{rad}(\mathbb{R}^d) \). Consequently, we have for any \( \varepsilon > 0 \), there exists a constant \( R_\varepsilon > 0 \), such that for all \( t \in \mathbb{R} \),
\[
\int_{|x| \geq R_\varepsilon} |\nabla u_c|^2 + \frac{|u_c|^2}{|x|^2} + |u_c|^\frac{2d}{2d-2} \, dx < \varepsilon. \tag{4.17}
\]

### 4.2 Proof of Theorem 1.1

Define a nonnegative radial function \( \phi \in C^\infty_c(\mathbb{R}) \) with
\[
\phi(x) = \begin{cases} 
|x|^2, & |x| \leq 1, \\
0, & |x| \geq 2.
\end{cases}
\]

Let \( \phi_R(x) = R^2 \phi(\frac{|x|}{R}) \), and
\[
V_R(t) = \int_{\mathbb{R}^d} \phi_R(x)|u(t, x)|^2 \, dx,
\]
where \( u(t, x) \) is a solution to (2.1). Then direct calculations give
\[
\frac{d}{dt} V_R(t) = 2\Re \int_{\mathbb{R}^d} \bar{u} \nabla u \cdot \nabla \phi_R \, dx, \tag{4.18}
\]
\[
\frac{d^2}{dt^2} V_R(t) = 4\Re \int_{\mathbb{R}^d} \partial_j \bar{u} \partial_k u \partial_j \partial_k \phi_R \, dx - 2 \int_{\mathbb{R}^d} \nabla V \cdot \nabla \phi_R |u|^2 \, dx - \int_{\mathbb{R}^d} \Delta^2 \phi_R |u|^2 \, dx + \frac{4}{d} \int_{\mathbb{R}^d} \Delta \phi_R |u|^\frac{2d}{2d-2} \, dx. \tag{4.19}
\]

By the virial identity above, we can prove the nonexistence of the critical element thus yielding a contradiction, from which Theorem 1.1 follows.

**Proposition 4.2.** The critical element \( u_c \) in Proposition 4.1 does not exist.

**Proof.** From Hardy’s inequality and (4.18), it is easy to see
\[
\left| \frac{d}{dt} V_R(t) \right| \leq CR^2 \| \nabla u_c(t) \|_2^2. \tag{4.20}
\]

(4.19) gives
\[
\frac{d^2}{dt^2} V_R(t) = 4\Re \int_{\mathbb{R}^d} \partial_j \bar{u} \partial_k u \partial_j \partial_k \phi_R \, dx - 2 \int_{\mathbb{R}^d} \nabla V \cdot \nabla \phi_R |u|^2 \, dx - \int_{\mathbb{R}^d} \Delta^2 \phi_R |u|^2 \, dx + \frac{4}{d} \int_{\mathbb{R}^d} \Delta \phi_R |u|^\frac{2d}{2d-2} \, dx
\geq 8 \int_{|x| \leq R} |\nabla u_c|^2 + |u_c|^\frac{2d}{2d-2} \, dx - 4 \int_{|x| \leq R} \partial_i V |x| |u_c|^2 \, dx
\]

28
\[-C_d \int_{R \leq |x| \leq 2R} |\nabla u_c|^2 + \frac{|u_c|^2}{|x|^2} + |u_c|^2 \frac{2d}{d^2} + \partial_x V |x|^3 \frac{|u_c|^2}{|x|^2} \, dx \]
\[ \geq 8 \int_{|x| \leq R} |\nabla u_c|^2 \, dx - C_d \int_{R \leq |x| \leq 2R} |\nabla u_c|^2 + \frac{|u_c|^2}{|x|^2} + |u_c|^2 \frac{2d}{d^2} \, dx. \quad (4.21) \]

By energy conservation and Sobolev embedding, we obtain
\[ \delta \|u_c(0)\|_{\dot{H}^1} \leq \|u_c(t)\|_{\dot{H}^1} \leq C\|u_c(0)\|_{\dot{H}^1}, \]
for some \( C, \delta > 0 \). Hence by choosing \( R \) sufficiently large, \((1.17)\) and \((1.21)\) imply for some \( \delta_1 > 0 \),
\[ \frac{d^2}{dt^2} V_R(t) \geq \delta_1 \|u_c(0)\|_{\dot{H}^1}, \]
which combined with \((4.20)\) yields
\[ \delta_1 t \|u_c(0)\|_{\dot{H}^1} \leq \int_0^t \frac{d^2}{ds^2} V_R(s) \, ds = \frac{d}{dt} V_R(t) - \frac{d}{dt} V_R(0) \leq C R^2 \|u_c(0)\|_{\dot{H}^1}^2. \]

Letting \( t \to \infty \), we get a contradiction since \( \|u_c(0)\|_{\dot{H}^1} \neq 0 \), thus finishing our proof of Proposition 4.2, from which Theorem 1.1 follows.

\[ \square \]

Acknowledgments

Ze Li thanks Shanlin Huang for helpful discussions. The authors would like to acknowledge the anonymous referee for helpful comments and improvements.

References

[1] P. Alsholm and G. Schmidt, Spectral and scattering theory for Schrödinger operators, Arch. Rational Mech. Anal., 40 (1971), 281–311.

[2] P. Antonelli, R. Carles and J. D. Silva, Scattering for nonlinear Schrödinger equation under partial harmonic confinement, Comm. Math. Phys., 334 (2015), 367–396.

[3] V. Banica and N. Visciglia, Scattering for nonlinear Schrödinger equation with a delta potential, J. Differential Equations, 260 (2016), 4410–4439.

[4] M. Beceanu and M. Goldberg, Schrödinger dispersive estimates for a scaling-critical class of potentials, Comm. Math. Phys., 314 (2012), 471–481.

[5] J. Bourgain, Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case, J. Amer. Math. Soc., 12 (1999), 145–171.

[6] P. Chen, J. Magniez and E. M. Ouhabaz, Riesz transforms on non-compact manifolds, arXiv1411.0137.
[7] J. Colliander, M. Czubak and J. Lee, Interaction Morawetz estimate for the magnetic Schrödinger equation and applications, *Adv. Differential Equations*, **19** (2014), 805–832.

[8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$, *Ann. of Math.*, **167** (2008), 767–865.

[9] S. Cuccagna, V. Georgiev and N. Visciglia, Decay and scattering of small solutions of pure power NLS in $\mathbb{R}$ with $p > 3$ and with a potential, *Comm. Pure Appl. Math.*, **67** (2014), 957–981.

[10] P. D’ancona, L. Fanelli, L. Vega and N. Visciglia, Endpoint Strichartz estimates for the magnetic Schrödinger equation, *J. Funct. Anal.*, **258** (2010), 3227–3240.

[11] P. D’ancona and V. Pierfelice, On the wave equation with a large rough potential, *J. Funct. Anal.*, **227** (2005), 30–77.

[12] B. Dodson, Global well-posedness and scattering for the defocusing, $L^2$-critical, nonlinear Schrödinger equation when $d \geq 3$, *J. Amer. Math. Soc.*, **25** (2012), 429–463.

[13] B. Dodson, Global well-posedness and scattering for the defocusing, $L^2$-critical, nonlinear Schrödinger equation when $d = 2$, *Duke Math. J.*, **165** (2016), no. 18, 3435–3516.

[14] B. Dodson, Global well-posedness and scattering for the defocusing, $L^2$-critical, nonlinear Schrödinger equation when $d = 1$, to appear in *Amer. J. Math.*, arXiv1010.0040.

[15] B. Dodson, Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state, *Advances in Mathematics*, **285** (2015), 1589–1618.

[16] B. Dodson, Global well-posedness and scattering for the focusing, energy-critical nonlinear Schrödinger problem in dimension $d = 4$ for initial data below a ground state threshold, arXiv1409.1950.

[17] J. Ginibre, T. Ozawa and G. Velo, On the existence of the wave operators for a class of nonlinear Schrödinger equations, *Ann. Inst. H. Poincare Phys. Theor.*, **60** (1994), 211–239.

[18] J. Ginibre and G. Velo, Scattering theory in the energy space for a class of nonlinear Schrödinger equations, *J. Math. Pures Appl.(9)*, **64** (1985), 363–401.

[19] R. H. Goodman, R. E. Slusher and M. I. Weinstein, Stopping light on a defect, *J. Opt. Soc. Am. B*, **19** (2002), 1635–1652.
[20] R. H. Goodman, M. I. Weinstein and P. J. Holmes, Nonlinear propagation of light in one-dimensional periodic structures, *J. Nonlinear Sci.*, **11** (2001), 123–168.

[21] Z. Hani and L. Thomann. Asymptotic behavior of the nonlinear Schrödinger equation with harmonic trapping, *Comm. Pure Appl. Math.*, **69** (2016), 1727–1776.

[22] K. Hepp, The classical limit for quantum mechanical correlation functions, *Comm. Math. Phys.*, **35** (1974), 265–277.

[23] Y. Hong, Scattering for a nonlinear Schrödinger equation with a potential, *Comm. Pure Appl. Anal.*, **15** (5) (2016), 1571–1601.

[24] S. Ibrahim, N. Masmoudi and K. Nakanishi, Scattering threshold for the focusing nonlinear Klein-Gordon equation, *Anal. PDE*, **4** (2011), 405–460.

[25] J. L. Journe, A. Soffer and C. D. Sogge, Decay estimates for Schrödinger operators, *Comm. Pure Appl. Math.*, **44** (1991), 573–604.

[26] M. Keel and T. Tao, Endpoint Strichartz estimates, *Amer. J. Math.*, **120** (1998), 955–980.

[27] S. Keraani, On the defect of compactness for the Strichartz estimates for the Schrödinger equations, *J. Differential Equations*, **175** (2001), 353–392.

[28] C. E. Kenig and F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case, *Invent. Math.*, **166** (2006), 645–675.

[29] R. Killip and M. Visan, The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher, *Amer. J. Math.*, **132** (2010), 361–424.

[30] R. Killip, M. Visan and X. Zhang. The mass-critical nonlinear Schrödinger equation with radial data in dimensions three and higher, *Analysis and PDE*, **1** (2009), 229–266.

[31] D. Lafontaine, Scattering for NLS with a potential on the line, *Asymptotic Analysis*, **100** (2016), 21–39.

[32] E. H. Lieb, R. Seiringer and J. Yngvason, A rigorous derivation of the Gross-Pitaevskii energy functional for a two-dimensional Bose gas, *Comm. Math. Phys.*, **224** (2001), 17–31.

[33] H. P. McKean and J. Shatah, The nonlinear Schrödinger equation and the nonlinear heat equation reduction to linear form, *Comm. Pure Appl. Math.*, **44** (1991), 1067–1080.

[34] K. Nakanishi, Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions 1 and 2, *Journal of Functional Analysis*, **169** (1999), 201–225.
[35] F. Planchon and L. Vega, Bilinear virial identities and applications, *Ann. Sci. Ec. Norm. Super.*, 42 (2009), 261–290.

[36] I. Rodnianski and W. Schlag, Time decay for solutions of Schrödinger equations with rough and time-dependent potentials, *Invent. Math.*, 155 (2004), 451–513.

[37] E. Ryckman and M. Visan, Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$, *Amer. J. Math.*, 129 (2007), 1–60.

[38] W. Schlag, Dispersive estimates for Schrödinger operators: A survey, *Ann. of Math. Stud.*, 163 (2007), 255–285.

[39] A. Soffer and M. I. Weinstein, Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations, *Invent. Math.*, 136 (1999), 9–74.

[40] H. Spohn, Kinetic equations from Hamiltonian dynamics, *Rev. Mod. Phys.*, 52 (1980), 569–615.

[41] W. Strauss, *Nonlinear scattering theory. Scattering theory in mathematical physics*, Proceedings of the NATO Advanced Study Institute, (Denver, 1973), 53–78. NATO Advanced Science Institutes, Volume C9. Reidel, Dordrecht, 1974.

[42] W. Strauss, Nonlinear scattering theory at low energy: Sequel, *J. Funct. Anal.*, 43 (1981), 281–293.

[43] T. Tao, *Nonlinear Dispersive Equations: Local and Global Analysis*, American Mathematical Society, 2006.

[44] T. Tao, M. Visan and X. Zhang, Minimal-mass blowup solutions of the mass-critical NLS, *Forum Mathematicum*, 20 (2008), 881–919.

[45] T. Tao, M. Visan and X. Zhang, Global well-posedness and scattering for the defocusing mass-critical nonlinear Schrödinger equation for radial data in high dimensions, *Duke Math. J.*, 140 (2007), 165–202.

[46] M. C. Vilela, Inhomogeneous Strichartz estimates for the Schrödinger equation, *Trans. Amer. Math. Soc.*, 359 (2007), 2123–2136.

[47] N. Visciglia, On the decay of solutions to a class of defocusing NLS, *Math. Res. Lett.*, 16 (2009), 919–926.

[48] M. Visan, The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions, *Duke Math. J.*, 138 (2007), 281–374.