The $BC_1$ quantum elliptic model: algebraic forms, hidden algebra $sl(2)$, polynomial eigenfunctions

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Abstract

The potential of the $BC_1$ quantum elliptic model is a superposition of two Weierstrass functions with a doubling of both periods (two coupling constants). The $BC_1$ elliptic model degenerates to an $A_1$ elliptic model characterized by the Lamé Hamiltonian. It is shown that in the space of the $BC_1$ elliptic invariant, the potential becomes a rational function, while the flat space metric becomes a polynomial. The model possesses the hidden $sl(2)$ algebra for arbitrary coupling constants: it is equivalent to the $sl(2)$ quantum top in three different magnetic fields. It is shown that three one-parametric families of coupling constants exist, for which a finite number of polynomial eigenfunctions (up to a factor) occur.

Keywords: elliptic potential, hidden algebra, polynomial eigenfunctions, particular integral
Following the formal definition, any one-dimensional dynamics is integrable. Among Calogero–Moser–Sutherland models, there exist only two models, $A_1$ and $BC_1$, which describe one-dimensional dynamics; in the case of $A_1$, it is the dynamics of the two-body relative motion. It is natural to ask what distinguishes these two models from all other integrable one-dimensional models. The goal of this paper is to show that both $A_1$ and $BC_1$ elliptic quantum systems are equivalent to the $sl(2)$ quantum top in a constant magnetic field. They are QES. The spectra of the $BC_1$ elliptic model is also studied.

The $BC_1$ quantum elliptic model, as it was introduced by Olshanetsky and Perelomov [8], is described by the Hamiltonian

$$H_{BC_1} = \frac{1}{2} \frac{d^2}{dx^2} + \kappa_2 \wp(2x) + \kappa_3 \wp(x) \equiv -\frac{1}{2} \Delta^{(1)} + V,$$

(1)

where $\Delta^{(1)}$ is the one-dimensional Laplace operator and $\kappa_{2,3}$ are coupling constants. The Weierstrass function, $\wp(x) \equiv \wp(x; g_2, g_3)$ [9], is defined as

$$(\wp'(x))^2 = 4 \wp^3(x) - g_2 \wp(x) - g_3 = 4((\wp(x) - e_1)(\wp(x) - e_2)(\wp(x) - e_3),$$

(2)

where $g_{2,3}$ are its invariants and $e_{1,2,3}$ are roots, $e_1 + e_2 + e_3 = 0$. If one of the coupling constants vanishes, $\kappa_2(\kappa_3) \equiv 0$, the Hamiltonian becomes $(A_1)$-Lamé Hamiltonian (see [10] and references therein). If the trigonometric limit is taken in the elliptic potential of equation (1) (one of periods tends to infinity, which implies that the condition $\Delta \equiv g_2^3 + 27g_3^3 = 0$ holds), the Hamiltonian of the $BC_1$ trigonometric/hyperbolic or generalized Pöschl–Teller model emerges.

Since we are interested in the general properties of the operator $H_{BC_1}^{(1)}$, without a loss of generality, we assume that the operator (1) is defined on the real line, $x \in \mathbb{R}$, and for the sake of convenience the fundamental domain of the Weierstrass function is rectangular with real period 1, and imaginary period $\tau_i$. The discrete symmetry of the Hamiltonian (1) is $\mathbb{Z}_2 \oplus T_\tau \oplus T_c$. It consists of the reflection $\mathbb{Z}_2(x \to -x)$ which is a $BC_1$ Weyl group, and two translations $T_c$: $x \to x + 1$ and $T_c$: $x \to x + i \tau$ (periodicity). Perhaps, $\mathbb{Z}_2 \oplus T_\tau \oplus T_c$ can make sense as a double-affine $BC_1$ Weyl group.

We will consider a formal eigenvalue problem

$$H_{BC_1}^{(1)} \Psi = E \Psi,$$

(3)

without posing concrete boundary conditions. It can be immediately checked that (3) has the exact solution

$$\Psi_0 = (\wp'(x))^\mu,$$

(4)

for coupling constants

$$\kappa_2 = 2\mu(\mu - 1), \quad \kappa_3 = 2\mu(1 + 2\mu),$$

(5)

and $\mu$ is an arbitrary parameter, for which the eigenvalue

$$E_0 = 0.$$

It implies that for parameters (5) the Hamiltonian $H_{BC_1}^{(1)}$ has a one-dimensional invariant subspace, $\Psi_0$ has the meaning of zero mode, and if $x \in [0, 1]$, the function $\Psi_0$ (4) is the ground state function (no nodes).

Now let us introduce a new variable,

$$\tau = \wp(x)$$

(6)

(cf [10] and references therein). It is evident that $\tau$ is invariant with respect to the action of the group $\mathbb{Z}_2 \oplus T_\tau \oplus T_c$, which is a double affine $BC_1$ Weyl group. The first observation is that
the potential (1) being written in the $\tau$-variable is a rational function,
\[
V(\tau) = \frac{\kappa_2 + 4\kappa_3}{4} \tau^4 + \frac{\kappa_2}{16} \frac{12g_5 \tau^2 + 36g_3 \tau + g_2^2}{4\tau^3 - g_2 \tau - g_3}
\]
and the ground state function (4) becomes
\[
\Psi_0(\tau) \equiv \left( 4\tau^3 - g_2 \tau - g_3 \right)^{\frac{\mu}{2}}, \tag{7}
\]
(cf (1)), which is the determinant of the metric with upper indices (see below) to the power $\frac{\mu}{2}$.

Making the gauge rotation
\[
h^{(e)} = -2(\Psi_0)^{-1} H^{(e)} \Psi_0
\]
and changing variable to $\tau$, we arrive at the algebraic operator
\[
h^{(e)}(\tau) = \Delta_g(\tau) + \mu \left( 12\tau^2 - g_2 \right) \partial_\tau - \kappa_3 \tau \tag{8}
\]
where $\Delta_g$ is the one-dimensional Laplace–Beltrami operator
\[
\Delta_g(\tau) = g^{-1/2} \frac{\partial}{\partial \tau} g^{1/2} g^{11} \frac{\partial}{\partial \tau} = g^{11} \frac{\partial^2}{\partial \tau^2} + \frac{g^{11}}{2} \frac{\partial}{\partial \tau}
\]
with flat metric
\[
g^{11} = \left( 4\tau^3 - g_2 \tau - g_3 \right) = \frac{1}{g}.
\]
Here, $g$ is its determinant with upper indices, and
\[
\kappa_3 = 2\kappa_3 - 4\mu(1 + 2\mu) \equiv 2n(2n + 1 + 6\mu).
\]

In the explicit form, the gauge-rotated operator (8) looks like
\[
h^{(e)}(\tau) = \left( 4\tau^3 - g_2 \tau - g_3 \right) \partial_\tau^2 + (1 + 2\mu) \left( 6\tau^2 - \frac{g_2}{2} \right) \partial_\tau - 2n(2n + 1 + 6\mu) \tau. \tag{9}
\]
It can be easily checked that if parameter $n$ is a non-negative integer the operator $h^{(e)}(\tau)$ (9) has the invariant subspace
\[
P_n = \left\{ \tau^p \mid 0 \leq p \leq n \right\},
\]
of dimension
\[
\dim P_n = (n + 1),
\]

namely,
\[
h^{(e)} : P_n \mapsto P_n.
\]

The space $P_n$ is invariant with regard to 1D projective (Möbius) transformation
\[
\tau \mapsto \frac{a \tau + b}{c \tau + d}.
\]
Furthermore, the space $P_n$ is the finite-dimensional representation space of the algebra $sl(2)$ of the first-order differential operators, realized as
\[
J^+(n) = \tau^2 \partial_\tau - n\tau, \quad J^0(n) = \tau \partial_\tau - n, \quad J^-(n) = \partial_\tau.
\]
Hence, the operator (8) can be rewritten in terms of $sl(2)$-generators
\[
\hat{h}^{(c)} = 4 \mathcal{J}^+(n) \mathcal{J}^0(n) - g_2 \mathcal{J}^0(n) \mathcal{J}^- - g_3 \mathcal{J}^- \mathcal{J}^- + 2(4n + 1 + 6\mu) \mathcal{J}^+(n) - g_2 \left(n + \frac{1}{2} + \mu\right) \mathcal{J}^-.
\] (11)

Thus, it is an $sl(2)$ quantum top in a constant magnetic field. This representation holds for any value of $n$. Thus, the algebra $sl(2)$ is the hidden algebra of the $BC_1$ elliptic model, with arbitrary coupling constants $\kappa_{2,3}$ parametrized as follows
\[
\kappa_2 = 2(\mu - 1), \quad \kappa_3 = (n + 2\mu)(n + 2\mu + 1).
\] (12)

If $n$ takes an integer value, the hidden algebra, $sl(2)$, appears in finite-dimensional representation, and the operator (9) has a finite-dimensional invariant subspace and possesses a number of polynomial eigenfunctions $P_{n,i}(\tau; \mu), \quad i = 1, \ldots (n+1)$. These polynomials can be called $BC_1$ Lamé polynomials (of the first kind). If $\mu = 0$, these polynomials degenerate to Lamé polynomials of the first (fourth) kind, respectively. For example, for $n = 0$ at coupling constants (5) or (12) at $n = 0$,
\[
E_{0,1} = 0, \quad R_{0,1} = 1.
\]

For $n = 1$ at coupling constants
\[
\kappa_2 = 2\mu (\mu - 1), \quad \kappa_3 = 2(1 + 2\mu)(1 + \mu),
\]
the eigenstates are
\[
E_{\varphi} = \pm(1 + 2\mu)\sqrt{3g_2}, \quad R_{\varphi,\tau} = \tau \mp \frac{1}{2} \sqrt{\frac{g_2}{3}}.
\]

As a function of $g_2$, both eigenvalues (eigenfunctions) are branches of a double-sheeted Riemann surface. Note that if $\mu = -\frac{1}{2}$, degeneracy occurs: both eigenvalues coincide, they are equal to zero, and any linear function is an eigenfunction. If $g_2 = 0$ but $\mu \neq -\frac{1}{2}$, the Jordan cell occurs: both eigenvalues are equal to zero but there exists a single eigenfunction, $P = \tau$. In general, for $n > 1$, polynomial eigenfunctions have the form of a polynomial in $\tau$ of degree $n$; they (as well as the eigenvalues) are branches of $(n + 1)$-sheeted Riemann surfaces in the parameter $g_2$. To summarize, it can be stated that for coupling constants (12) at integer $n$, the Hamiltonian (1) has $(n + 1)$ eigenfunctions of the form
\[
\Psi_{n,i} = P_{n,i}(\tau; \mu) \Psi_0, \quad i = 1, \ldots (n+1),
\] (13)

where $\Psi_0$ is given by (4).

It can be checked that the eigenvalue problem (3) has an exact solution other than (4),
\[
\Psi_{0,k} = \left[\mathcal{E}^{(1)}(x)\right]^\mu \left(\mathcal{E}(x) - e_k\right)^{\frac{1}{2} - \mu},
\] (14)

for coupling constants
\[
\kappa_2 = 2\mu (\mu - 1), \quad \kappa_3 = (1 + 2\mu)(1 - \mu),
\] (15)

where $\mu$ is an arbitrary parameter, for which the eigenvalue is
\[
E_{0,k} = \frac{4\mu^2 - 1}{2} e_k.
\]

Here, $e_k$ is the $k$th root of the Weierstrass function (2). It implies that for parameters (15) the Hamiltonian $H^{(c)}_{BC_1}$ has a one-dimensional invariant subspace.
Making a gauge rotation of the Hamiltonian (1) with subtracted $E_{0,k}$,

$$h_k^{(e)} = -2\left( \Psi_{0,k}^{-1} \left( H_{BC_1}^{(e)} - E_{0,k} \right) \Psi_{0,k} \right)$$

and changing the variable to $\tau$, we arrive at the algebraic operator

$$h_k^{(c)}(\tau) = (\tau - e_k)^{-\mu} \left( h^{(c)}(\tau) - 2E_{0,k} \right) (\tau - e_k)^{-\mu}$$

$$= \left( 4\tau^3 - g_2\tau - g_3 \right) \partial_\tau^2 + 2 \left( (5 + 2\mu)\tau^2 + 2(1 - 2\mu)e_k(\tau + e_k) - (3 - 2\mu) \frac{g_2}{4} \right) \partial_\tau - 2\kappa_3 \tau, \quad (16)$$

(cf (9)), where $e_k$ is the $k$th root of the Weierstrass function (see (2)), and

$$\kappa_3 = \kappa_3 - (1 - \mu)(1 + 2\mu).$$

It can be checked that if $\kappa_3 = 2n(n-1) + n(2\mu + 5)$ and the parameter $n$ takes non-negative integer values, the operator $h_k^{(c)}(\tau)$ has the invariant subspace $P_\nu$. Furthermore, the operator (16) can be rewritten in terms of $sl(2)$-generators (10) for any value of $n$ (cf (11)),

$$h_k^{(c)} = 4 J^+(n)J^0(n) - g_2 J^0(n) J^- - g_3 J^- J^-$$

$$+ 2(4n + 3 + 2\mu)J^+(n) + 4(1 - 2\mu)e_k \left( J^0(n) + n \right)$$

$$+ 2 \left( 2(1 - 2\mu)e_k^2 - (2n + 3 - 2\mu) \frac{g_2}{4} \right) J^- . \quad (17)$$

Thus, it is an $sl(2)$ quantum top in a constant magnetic field.

Hence, the algebra $sl(2)$ is the hidden algebra of the $BC_1$ elliptic model with arbitrary coupling constants $\kappa_2,3$ parametrized as follows

$$\kappa_2 = \mu(\mu - 1), \quad \kappa_3 = 2n^2 + n(3 + 2\mu) + (1 + 2\mu)(1 - \mu), \quad (18)$$

(cf (12)). If $n$ takes an integer value, the hidden algebra $sl(2)$ appears in a finite-dimensional representation, and the operator (16) has a finite-dimensional invariant subspace and possesses a number of polynomial eigenfunctions $P_{\nu,i}(\tau; \mu, e_k)$, $i = 1, \ldots (n + 1)$ and $k = 1, 2, 3$. These polynomials can be called $BC_1$ Lamé polynomials (of the second kind). If $\mu = 0, 1$, these polynomials degenerate to Lamé polynomials of the second (third) kind, respectively.

For example, for $n = 0$ at couplings (18),

$$E_{0,1} = \left( \frac{4\mu^2 - 1}{2} \right) e_k, \quad R_{0,1} = 1.$$
where \( \hat{k} \) is complement to \((i, j)\), for coupling constants
\[
\kappa_2 = 2\nu(\nu - 1), \quad \kappa_3 = \nu(1 - \nu),
\]
where \( \nu \) is an arbitrary parameter, for which the eigenvalue is
\[
E_{0,\hat{k}} = \frac{(1 - 2\nu)(3 - 2\nu)}{2} e_\hat{k}.
\]

Here, \( e_\hat{k} \) is the \( \hat{k} \)th root of the Weierstrass function (2). It implies that for parameters (21), the Hamiltonian \( H_{BC_1}^{(6)} \) has a one-dimensional invariant subspace. If in (20) \( \nu = 1 - \mu \), the solution (14) occurs.

Making a gauge rotation of the Hamiltonian (1) with subtracted \( E_{0,\hat{k}} \),
\[
h^{(c)}_\hat{k} = -2 \left( \Psi_{0,\hat{k}} \right)^{-1} \left( H_{BC_1}^{(c)} - E_{0,\hat{k}} \right) \Psi_{0,\hat{k}}
\]
and changing the variable to \( \tau \), we arrive at the algebraic operator
\[
h^{(c)}_\hat{k}(\tau) = \left[ (\tau - e_i)(\tau - e_j) \right]^{1+\mu} \left( h^{(c)}(\tau) - 2E_{0,\hat{k}} \right) \left[ (\tau - e_j)(\tau - e_i) \right]^{1-\mu}
\]
\[
= \left( 4\tau^3 - g_2\tau - g_3 \right) \partial_\tau^2 + 2 \left( 7 - 2\nu \right) \tau^2
\]
\[
+ 2(2\nu - 1)e_k(\tau + e_k) - (5 + 2\nu) \frac{g_2}{4} \partial_\tau - 2\kappa_3 \tau,
\]  
(cf (9)), where \( e_k \) is \( k \)th root of the Weierstrass function, see (2) and
\[\kappa_3 = \kappa_3 - \nu(3 - 2\nu).\]

It can be checked that if \( \kappa_3 = 2n(n - 1) + n(7 - 2\nu) \), and the parameter \( n \) takes a non-negative integer value, the operator \( h^{(c)}_\hat{k}(\tau) \) has the invariant subspace \( P_n \). Furthermore, the operator (22) can be rewritten in terms of \( sl(2) \)-generators (10) for any value of \( n \) (cf (11)),
\[
h^{(c)}_\hat{k} = 4J^+(n) J^0(n) - g_2 J^0(n) J^- - g_3 J^- J^-
\]
\[
+ 2(4n + 5 - 2\nu) J^+(n) + 4(2\nu - 1)e_k \left( J^0(n) + n \right)
\]
\[
+ 2 \left( 2(2\nu - 1)e_k^2 - (2n + 1 + 2\nu) \frac{g_2}{4} \right) J^-.
\]  
(23)

Thus, it is an \( sl(2) \) quantum top in a constant magnetic field.

Hence, the algebra \( sl(2) \) is the hidden algebra of \( BC_1 \) elliptic model with arbitrary coupling constants \( \kappa_{2,3} \) parametrized as follows
\[
\kappa_2 = \nu(\nu - 1), \quad \kappa_3 = 2n^2 + n(5 - 2\nu) + \nu(1 - 2\nu),
\]  
(cf (12), (18)). If \( n \) takes an integer value, the hidden algebra \( sl(2) \) appears in finite-dimensional representation, and the operator (22) has a finite-dimensional invariant subspace \( P_\nu \) and possesses a number of polynomial eigenfunctions \( P_{n,\nu}(\tau; e_k) \), \( i = 1, \ldots(n + 1) \) and \( k = 1, 2, 3 \). These polynomials can be called \( BC_1 \) Lamé polynomials (of the third kind). If \( \nu = 0, 1 \), these polynomials degenerate to Lamé polynomials of the third (second) kind, respectively. For example, for \( n = 0 \) at couplings (24),
\[
E_{0,1} = \frac{(1 - 2\nu)(3 - 2\nu)}{2} e_k, \quad P_{0,1} = 1.
\]

In general, for \( n > 1 \), the polynomial eigenfunctions have the form of a polynomial in \( \tau \) of degree \( n \), and they (as well as the eigenvalues) are branches of \((n + 1)\)-sheeted Riemann surfaces in \( g_2 \). To summarize, it can be stated that for coupling constants (24) at integer \( n \), the
Hamiltonian (1) has \((n + 1)\) eigenfunctions of the form
\[
\Psi_{n,k} = \hat{P}_{k1}(r; \nu, \epsilon_k) \Psi_{0,k}, \quad i = 1, \ldots(n + 1), \quad k = 1, 2, 3, \tag{25}
\]
where \(\Psi_{0,k}\) is given by (20).

**Observation:** Let us construct the operator
\[
i^{(n)}_{\text{par}}(r) = \prod_{j=0}^{n} \left( \mathcal{J}^0(n) + j \right),
\]
where \(\mathcal{J}^0(n)\) is the Euler-Cartan generator of the algebra \(sl(2)\) (10). It can be shown that any algebraic operator \(h^{(\nu)}(11), (17), (23)\) at integer \(n\) commutes with \(i^{(n)}_{\text{par}}(r)\),
\[
\left[ h^{(\nu)}(r), i^{(n)}_{\text{par}}(r) \right], \quad \mathcal{P}_n \mapsto 0.
\]
Hence, \(i^{(n)}_{\text{par}}(r)\) is the particular integral [11] of the \(BC_1\) elliptic model (1).

In this paper, we demonstrate that the \(BC_1\) elliptic model belongs to one-dimensional QES problems [12]. However, it is not in the list of known QES problems (see, e.g., [13]). We show the existence of three different algebraic forms of the \(BC_1\) Hamiltonian; all of them are the second-order polynomial elements of the universal enveloping algebra \(U_{sl(2)}\). If this algebra appears in a finite-dimensional representation, those elements possess a finite-dimensional invariant subspace. This phenomenon occurs for any of three one-parametric subfamilies of coupling constants for which polynomial eigenfunctions may occur. It is worth noting that a certain algebraic form for a general \(BC_n\) elliptic model was found some time ago in [5, 6].

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