Pulsating flow of Casson nanofluid through a channel with thermal radiation and applied magnetic field

N Thamaraikannan* and S Karthikeyan
Department of Mathematics, Erode Arts and Science College,
Erode 638 009, Tamil Nadu, India
Email: *kthamkar@gmail.com

Abstract: The pulsating flow of MHD Casson nanofluid flow through a channel with the influence of thermal radiation is studied, considering the nanofluid flow in the vertically upward direction, maintaining the same velocity at the lower wall and upper wall. The perturbation technique was employed to solve the dimensionless expressions of velocity and temperature. The effect of various parameters such as Prandtl number and Casson fluid parameter on flow variables has been discussed.

Keywords: pulsating flow, Casson nanofluid, thermal radiation, perturbation technique.

1. Introduction
The Casson fluid is a non-Newtonian fluid with yield pseudo plastic behaviour. It displays an infinite shear rate with low apparent viscosity and, zero shear rates at infinite apparent viscosity. The flow of MHD Casson fluid in a channel with free convection investigated by Sheikh et al. [1], the characteristics of Casson fluid on a permeable Riga-plate was analysed by Loganathan et al. [2]. The term ‘nano fluid’ is a fluid containing a suspension of solid nanoparticles which enhances or reduces the thermal conductivity of base fluid. Pulsating flow of Casson nano-fluid in a channel has enormous application in the process of flow of blood in heart, transpiration cooling. Combined forces and a flow free convection past a porous channel with a pulsating pressure is discussed by Bestman [3]. The emerging applications and advanced studies are reported in [4-6].

Magnetohydrodynamic fluid flow has enormous application in various engineering and industrial processes. Shehzad et al. [7] studied the effects of mass transfer on MHD flow of Casson fluid over a stretching surface. The study related to mass and heat transfer with thermal radiation has abundant appliance in much industrial and engineering process. This work deals with the effect of various parameters on the pulsating flow of MHD Casson nanofluid. The dimensionless expressions of velocity and temperature are solved by applying perturbation techniques and discussed the influence of some parameters on velocity and temperature.

2. Mathematical formulation:
Consider the uniform traverse applied magnetic field on the pulsatile flow of an incompressible Casson nano fluid with a time dependent pressure gradient in a channel between two fixed plates. [9, 10, 11]

\[
\frac{1}{\rho_{nf}} \frac{\partial \bar{p}}{\partial x} = -A\{1 + \xi e^{i\omega t}\}
\]
where, $\varepsilon < 1$ is suitably chosen positive quantity, $A$ is a constant and $\omega$ is the frequency. We take a Cartesian coordinate system with $X$-axis taken along the lower wall and $Y$-axis perpendicular to it. The lower wall maintaining the temperature $T_0$ and the upper wall maintains the temperature $T_1$ ($T_0 < T_1$). The nano fluid is injected and sucked out from the lower wall and upper wall respectively with the same velocity $v_0$. The magnetic field is taken along the normal direction. The rheological equation for the Casson nano fluid is defined as follows [10].

$$T_{ij} = \begin{cases} 
2\mu_{Bnf} + \frac{p_y}{\sqrt{\pi \tau}} & \pi_{cr} > \pi \\
2\mu_{Bnf} + \frac{p_y}{\sqrt{\pi \tau}} & \pi_{cr} < \pi 
\end{cases}$$ (1)

where $\pi_{cr}$ - critical value of $\pi$ based on the Casson model. $T_{mn}$ - $(m,n)^{th}$ stress tensor component, $\pi = e_{mn}$ with $e_{mn}$ being the $(m,n)^{th}$deformation rate. $p_y$ is the yield stress of the fluid.

The governing equation are defined below

$$\frac{\partial \bar{u}}{\partial t} + v_0 \frac{\partial \bar{u}}{\partial \bar{y}} + v_{nf} \left( 1 + \frac{1}{\beta} \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - \frac{1}{\rho_{nf}} \frac{\partial \bar{p}}{\partial \bar{x}} - \frac{\sigma B_0^2}{\rho_{nf}} \bar{u} \right),$$ (2)

$$0 = -\frac{1}{\rho_{nf}} \frac{\partial \bar{p}}{\partial \bar{y}},$$ (3)

$$\frac{\partial T}{\partial t} + v_0 \frac{\partial T}{\partial \bar{y}} = -\frac{\kappa_{nf}}{(\rho c_p)_{nf}} \frac{\partial^2 T}{\partial \bar{y}^2} + \frac{\mu_{nf}}{(\rho c_p)_{nf}} \left( 1 + \frac{1}{\beta} \right) \left( \frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 + \frac{\sigma B_0^2}{(\rho c_p)_{nf}} \bar{u} - \frac{1}{(\rho c_p)_{nf}} \frac{\partial q_r}{\partial \bar{y}} + \frac{Q_0}{(\rho c_p)_{nf}} (\bar{T} - T_0)$$ (4)

The boundary conditions for the present analysis are

$$\bar{u} = 0, \quad \bar{T} = T_0, \quad \text{at} \quad \bar{y} = 0$$
$$\bar{u} = 0, \quad \bar{T} = T_1, \quad \text{at} \quad \bar{y} = L$$

Where $\bar{u}$ is the dimensional velocity in $X$-direction [7]

$$\rho_{nf} = (1 - \varphi)\rho_f + \varphi \rho_p, \quad \mu_{nf} = \frac{\mu_f}{(1 - \varphi)\rho_f}$$

$$\nu_{nf} = \frac{\nu_{nf}}{\rho_{nf}}, \quad \frac{\kappa_{nf}}{\kappa_f} = \frac{2k_f + k_p - 2\varphi(k_f + k_p)}{2k_f + k_p + \varphi(k_f + k_p)}$$ (5)

Here $\frac{\kappa_{nf}}{\kappa_f}$ is restricted to spherical nano particles only, it does not account for other shapes of nano particles.

$Q_0$ is the heat source / sink, $\beta = \frac{\mu_{Bnf}}{\sqrt{2\pi c}} / p_y$ is Casson parameter, $T^*$ is the temperature of the nanofluid. Applying Rosseland approximation for radioactive heat flux, $q_r$ is defined as:

$$q_r = -\frac{4\lambda}{3K} \frac{\partial T^4}{\partial \bar{y}}$$ (6)

Where $K$ is the Rosseland mean absorption co-efficient.

We expand $T^{*4}$ in a Taylor’s series about $T_0$ and neglecting higher order, we obtain
Substituting Eqns. (6) and (7) in (4), we obtain
\[ \frac{\partial T}{\partial t} + \nu_0 \frac{\partial \bar{T}}{\partial y} = - \frac{k_{nf}}{\rho c_p}_{nf} \frac{\partial^2 \bar{T}}{\partial y^2} + \frac{\mu_{nf}}{\rho c_p}_{nf} \left( 1 + \frac{1}{\beta} \right) \left( \frac{\partial \bar{u}}{\partial y} \right)^2 + \frac{\sigma B_n^2}{\rho c_p}_{nf} \bar{u}^2 \\
- \frac{1}{\rho c_p}_{nf} \frac{16\nu_3 \sigma}{3k} \frac{\partial^2 \bar{T}}{\partial y^2} + \frac{Q_0}{\rho c_p}_{nf} (\bar{T} - T_0) \] \tag{8}

We introduce the following dimensionless variables
\[ x = \frac{x}{L}, \quad y = \frac{y}{L}, \quad t = t^* \omega, \]
\[ u = \frac{\bar{u} \omega}{A}, \quad p = \frac{\bar{p}}{\rho A L}, \quad \Theta = \frac{\bar{T} - T_0}{T_1 - T_0} \]

By using these above dimensionless variables, Eqns. (2), (8) become
\[ \frac{(1 + 2\beta)}{(1 - \beta)^{1.5} \nu_0^2 \rho_f} \frac{\partial^2 u}{\partial y^2} - \frac{Re}{(1 - \beta)^{2.5}} \frac{\partial u}{\partial y} - \frac{1}{(1 - \beta)^{2.5}} M^2 u - H^2 (\frac{\partial u}{\partial t} - \frac{\partial p}{\partial x}) = 0 \] \tag{9}
\[ \left( \frac{k_{nf}}{\nu_f} \left( \frac{A_f^2}{1 - \beta} \right) \right) \frac{\partial^2 \Theta}{\partial y^2} - \frac{Re}{(1 - \beta)^{2.5}} \frac{\partial \Theta}{\partial y} + Q \Theta = - \frac{(1 + 2\beta)}{(1 - \beta)^{2.5}} Ec \left( \frac{\partial \bar{u}}{\partial y} \right)^2 - Ec M^2 u^2 \\
+ H^2 \left( 1 - \beta + \frac{\phi \rho c_p}{\rho c_p}_f \right) \frac{\partial \Theta}{\partial t} \] \tag{10}

The new boundary conditions are
\[ u = 0, \quad \Theta = 0, \quad \text{at} \quad y = 0 \]
\[ u = 0, \quad \Theta = 1, \quad \text{at} \quad y = 1 \]

Where \( H = L \sqrt{\alpha} / \sqrt{\nu_f} \) is the ‘frequency parameter’, \( Re = V_0 L / \nu_f \) is the ‘cross flow Reynolds number’, \( M = B_0 L \sqrt{\sigma / \mu_f} \) is the Hartmann number, \( Rd = \frac{4\nu_3 \sigma}{k_f \nu} \) is radiation parameter, \( Ec = \frac{(\alpha)}{(cp)_f (T_1 - T_0)} \) is the ‘Eckert number’, \( Q = Q_0 L^2 / ((\rho c_p)_f \nu_f) \) is heat source/ sink parameter, \( Pr = \mu_f (cp)_f / k_f \) is Prandtl number.

**3. Method of solution**

The velocity \( u \) and temperature \( \Theta \) can be assumed to have the form
\[ u = u_0 (y) + \xi u_1 (y) e^{it} + \xi^2 u_2 (y) e^{2it} \] \tag{11}
\[ \Theta = \Theta_0 (y) + \xi \Theta_1 (y) e^{it} + \xi^2 \Theta_2 (y) e^{2it} \] \tag{12}

Now substitute equations (11) and (12) into the equations (9) and (10) then equating the coefficients of various powers of \( \xi \), we get
\[ u_0 = B_1 e^{m_1 y} + B_2 e^{m_2 y} + B_3 \]
\[ u_1 = B_4 e^{m_3 y} + B_5 e^{m_4 y} + B_6 \]

\[ u_2 = B_7 e^{m_5 y} + B_8 e^{m_6 y} + B_9 \]

\[ \theta_1 = B_{17} e^{m_7 y} + B_{16} e^{m_8 y} + B_{10} e^{m_1 y} + B_{11} e^{m_2 y} + B_{12} e^{m_1 y} + B_{13} e^{m_2 y} + B_{14} e^{(m_1 + m_2) y} + B_{15} \]

\[ \theta_2 = B_{27} e^{m_9 y} + B_{20} e^{m_{10} y} + B_{18} e^{(m_1 + m_3) y} + B_{19} e^{(m_1 + m_4) y} + B_{20} e^{(m_2 + m_3) y} + B_{21} e^{(m_2 + m_4) y} + B_{22} e^{m_3 y} + B_{23} e^{m_2 y} + B_{24} e^{m_3 y} + B_{25} e^{m_4 y} + B_{26} \]

\[ \theta_3 = B_{46} e^{m_{11} y} + B_{45} e^{m_{12} y} + B_{28} e^{m_2 y} + B_{30} e^{m_4 y} + B_{31} e^{(m_1 + m_4) y} + B_{32} e^{(m_1 + m_3) y} + B_{33} e^{(m_1 + m_4) y} + B_{34} e^{(m_2 + m_3) y} + B_{35} e^{(m_2 + m_4) y} + B_{36} e^{(m_1 + m_3) y} + B_{37} e^{(m_1 + m_4) y} + B_{38} e^{(m_2 + m_4) y} + B_{39} e^{(m_2 + m_3) y} + B_{40} e^{m_3 y} + B_{41} e^{m_4 y} + B_{42} e^{m_3 y} + B_{43} e^{m_4 y} + B_{44} \]

here m’s and B’s are given in Appendix.

The dimensionless Nusselt number at the walls is

\[ Nu = -\frac{k_{nf}}{k_f} \left( \frac{\partial \theta}{\partial y} \right)_{y=0,1} = -\frac{k_{nf}}{k_f} (\theta_0'(y) + \varepsilon \theta_1'(y)e^{it} + \varepsilon^2 \theta_2'(y)e^{2it})_{y=0,1} \]

4. Results and discussion

The influence of various parameters such as frequency parameter (H), Casson fluid parameter (\(\beta\)), (\(\varphi\)) and (M) on the heat transfer and velocity has been discussed in this section. Throughout the computation the parameters are taken as \(t = \pi/4\), \(\varepsilon = 0.01\), \(Re = 1\), \(\varphi = 0.1\), \(\frac{k_{nf}}{k_f} = 2\), \(M = 0.5\), \(\beta = 2\), \(Da = 0.1\), \(Ec = 0.1\), \(Q = 0.5\) \(Rd = 2\), \(Pr = 21\) and \(H = 3\) unless otherwise stated.

From Figure. 1(a & c) one can notice that the velocity enhances with a larger estimation of frequency parameter (H) and nanoparticle volume fraction (\(\varphi\)), however, from Figure. 1(b & d) it is seen that as (M) and (\(\beta\)) increases reduction in velocity occurs. From Figure. 2(a & b) it is observed that temperature distribution diminishing with a raise in cross flow Reynolds number (Re) and Prandtl number (Pr), but opposite effects are noticed for Radiation parameter (Rd) and Eckert number (Ec) in Figure. 3(a & b). From Figure.4(a) it is noticed that the unsteady temperature oscillating with declining Radiation parameter (Rd) and attaining maximum near the walls. From Figure. 4(b) one can notice that the steady temperature declines for enhancement in Rd.

From Figure. 5(a & b) it is clear that the unsteady and steady temperatures enhancing function of nanoparticle volume fraction (\(\varphi\)). From Figure.6 (a & b) It is noticed that larger estimation of nanoparticle volume fraction (\(\varphi\)) and Heat source, results with the decrease of Nusselt number (\(Nu\)) at the lower wall, however the results are reverse at the upper wall. From Figure.7 (a & b) It is evident that given raise in Hartmann number (M) and Radiation parameter (Rd), results with the uplift of Nusselt number (\(Nu\)) at the lower wall, however, it diminished at the upper wall.

The variations of Nusselt number for both base and nano fluid are presented in Table. 1. The nano fluid is taken as \(\varphi = 0.1\). One can notice from the table that for the base fluid and nano fluid, the Nusselt number distribution enhances at the lower wall as increasing M, Re and Rd, but the opposite effect can
be seen in $Q$. However, this behaviour is reversed at the upper wall. It is also observed that with the larger estimation of nano particle volume fraction ($\varphi$) the Nusselt ($Nu$) increases at the upper wall, while it is diminishing at the lower wall.

**Figure. 1.** Velocity distribution for $t = \pi/4, \varepsilon = 0.01$ and $Re = 1$: (a) effect of $M$ when $\beta = 2, H = 3$ and $\varphi = 0.1$ (b) effect of $H$ when $\varphi = 0.1, M = 0.5$, and $\beta = 2$; (c) effect of $\varphi$ when $M = 0.5, \beta = 2$ and $H = 3$; (d) effect of $\beta$ when $\varphi = 0.1, M = 0.5$, and $H = 3$.

**Figure. 2.** Temperature distribution for, $\varepsilon = 0.01, H = 3, \beta = 2, Ec = 0.1, Q = 0.5$ $Rd = 2$. 


$M = 0.5, Pr = 21$ and $t = \pi/4$ : (a) effect of Re when $Pr = 21$; (b) effect of $Pr$ when Re = 1

Figure 3. Temperature distribution for $t = \pi/4, \varepsilon = 0.01, H = 3, \beta = 2, Ec = 0.1, Q = 0.5$,

$M = 0.5, Re = 1$ and $\varphi = 0.1$ (a) Effect of Rd when $Ec = 0.1$, (b) Effect $Ec$ when Rd = 2.

Figure 4. Effect of Rd on unsteady and steady temperature distributions when $t = \pi/4, \varepsilon = 0.01, H = 3, \beta = 2, Ec = 1, Q = 0.5$, $M = 0.5, Re = 1$ Pr = 21 and $\varphi = 0.1$. 
Figure 5. Effect of $\phi$ on unsteady and steady temperature distributions when $t = \frac{\pi}{4}, \varepsilon = 0.01, H = 3, \beta = 2, Ec = 1, Q = 0.5$, $Re = 1, M = 2$ $Pr = 21$ and $Rd = 2$.

Figure 6. Nusselt number distribution for $t = \frac{\pi}{4}, \varepsilon = 0.01, H = 3, \beta = 2, Ec = 1, Rd = 2, Re = 1$ and $Pr = 21, M = 0.5$ (a) Effect of $\phi$ when $Q = 0.5$, (b) Effect of $Q$ when $\phi = 0.1$.

Table 1. Comparison of Nusselt number for Base fluid and nano-fluid, when $\varepsilon = 0.01, H = 3, \beta = 2, Ec = 1, Pr = 21$ and $t = \frac{\pi}{4}$

| Parameter | Values | $Nu = - \left( \frac{\partial \theta}{\partial y} \right)_{y=0}$ | $Nu = - \left( \frac{\partial \theta}{\partial y} \right)_{y=1}$ |
|-----------|--------|-------------------------------------------------|-------------------------------------------------|
|           |        | Base fluid ($\phi=0$) | Nano-fluid ($\phi=0.1$) | Base fluid ($\phi=0$) | Nano-fluid ($\phi=0.1$) |
| $M$       | 0      | -4.6639 | -8.1803 | 6.7795 | 18.8469 |
|           | 2      | -3.5881 | -6.5800 | 5.1058 | 15.5537 |
|           | 4      | -2.1744 | -4.1655 | 2.3940 | 9.8275 |
| $Re$      | 0      | -6.9373 | -14.4291 | 5.1818 | 12.6736 |
| (M=2)     | 1      | -3.5881 | -6.5800 | 5.1058 | 15.5537 |
|           | 2      | -2.0074 | -3.1039 | 3.2430 | 13.5129 |
| $Q$       | 0      | -3.3531 | -6.2537 | 4.0101 | 13.6935 |
|           | 0.5    | -3.5881 | -6.5800 | 5.1058 | 15.5537 |
|           | 1      | -3.8939 | -6.9867 | 6.5005 | 17.8981 |
| $Rd$      | 0      | -5.6501 | -9.8287 | 29.0248 | 76.2854 |
|           | 1      | -4.3592 | -7.8815 | 9.7147 | 27.7064 |
|           | 2      | -3.5881 | -6.5800 | 5.1058 | 15.5537 |
5. Conclusion
The pulsating flow of Casson nanofluid with thermal radiation and applied magnetic field in a channel between two fixed plates in the presence of heat sink/source has been studied. The analytic perturbation technique was applied to solve the flow equations. The major results of the present investigation are as follows:

- The velocity of the nano fluid increases with larger approximation of frequency parameter (H), Casson parameter (B), nano particle volume fraction (ϕ), but the opposite effect can be seen in M.
- The temperature of the nano fluid increases with a rise in nano particle volume fraction (ϕ) and Prandtl number (Pr). While the results are reversed for the rise in cross Reynolds number (Re) and the radiation parameter (Rd).
- The steady and unsteady temperature distributions of the nanofluid is an increasing function of (ϕ), while steady temperature is decreasing, and unsteady temperature oscillates for given rise in Rd.
- The Nusselt number (Nu) decrease at the lower wall and enhances at the upper wall with an increase in the nanoparticle volume fraction (ϕ), and the heat source.

Appendix:

\[
A_1 = \frac{(\frac{\varphi}{\varphi_2} + 1)}{(1 - \varphi + \frac{\varphi_2}{\varphi_1}) (1 - \varphi)^{2.5}}; \quad A_2 = \frac{1}{(1 - \varphi + \varphi_2 \varphi_1 \varphi_{2.5})} (M^2); \quad B_3 = \frac{\mu^2}{A_2}
\]

\[
m_{4,2} = \frac{Re}{(1 - \varphi + \varphi_2 \varphi_1 \varphi_{2.5})}; \quad B_1 = \frac{B_3 (e^{m_2 - 1})}{(e^{m_1 - e^{m_2}})}; \quad B_2 = \frac{B_3 (e^{m_1 - 1})}{(e^{m_1 - e^{m_2}})}
\]

\[
A_3 = \frac{1}{(1 - \varphi + \varphi_2 \varphi_1 \varphi_{2.5})} (M^2) + iH^2; \quad B_6 = \frac{\mu^2}{A_3}
\]

\[
m_{3,4} = \frac{Re}{(1 - \varphi + \varphi_2 \varphi_1 \varphi_{2.5})}; \quad B_4 = \frac{B_6 (e^{m_3 - 1})}{(e^{m_4 - e^{m_3}})}; \quad B_5 = \frac{B_6 (e^{m_4 - 1})}{(e^{m_3 - e^{m_4}})}
\]

\[
A_4 = \frac{1}{(1 - \varphi + \varphi_2 \varphi_1 \varphi_{2.5})} (M^2) + 2iH^2; \quad B_9 = \frac{\mu^2}{A_4}
\]

\[
m_{5,6} = \frac{Re}{(1 - \varphi + \varphi_2 \varphi_1 \varphi_{2.5})}; \quad B_7 = \frac{B_9 (e^{m_5 - 1})}{(e^{m_6 - e^{m_5}})}; \quad B_8 = \frac{B_9 (e^{m_6 - 1})}{(e^{m_5 - e^{m_6}})}
\]

\[
A_5 = \frac{(k_{nf} \varphi p_{nf} \varphi_{2.5})}{p_r}; \quad A_6 = \frac{Re (1 - \varphi + \varphi_2 \varphi_1 \varphi_{2.5})}{(1 - \varphi)^{2.5}}; \quad m_{7,8} = \frac{A_6 \varphi \varphi_{2.5} + 4A_5 Q}{2A_5}
\]

\[
A_7 = \frac{(1 + \varphi)}{(1 - \varphi)^{2.5}} Ec; \quad A_8 = M Ec; \quad B_{10} = -\frac{A_9 B_1 B_2}{4m_1^2 A_5^2 - 2m_1 A_0 + Q}
\]

\[
B_{11} = -\frac{A_9 B_1 B_2}{4m_2^2 A_5 - 2m_2 A_0 + Q}; \quad B_{12} = \frac{2A_9 B_1 B_2}{m_2^2 B_1 - m_1 A_0 + Q}; \quad B_{13} = \frac{2A_9 B_2 B_2}{m_2^2 A_5 - m_2 A_0 + Q}
\]
\[ B_{14} = -\frac{2A_2m_1m_2B_1B_2 + 2A_4B_2}{(m_1 + m_2)^2A_5 - (m_1 + m_2)A_6 + Q}; \quad B_{15} = \frac{A_8B_3^2}{Q}; \]

\[ B_{16} = \frac{1}{(e_{m_0} - e_{m_1})}(1 + B_{10}(e_{m_1} - e_{m_2}) - B_{11}(e_{m_1} - e_{m_2}^2) + B_{12}(e_{m_1} - e_{m_1}^2) + B_{13}(e_{m_1} - e_{m_2}) + B_{14}(e_{m_1} - e_{(m_1 + m_2)}) + B_{15}(e_{m_1} - 1)) \]

\[ B_{17} = -(B_{16} + B_{15} + B_{14} + B_{13} + B_{12} + B_{11} + B_{10}) \]

\[ A_9 = Q - t \left( 1 - \phi + \frac{\varphi(\rho_{CP})}{\rho_{CP}} \right) H^2; \quad m_{9,10} = \frac{A_2^2}{2A_5}; \quad B_{18} = -\frac{2A_2m_1m_2B_2B_5 + 2A_4B_2B_5}{(m_1 + m_2)^2A_5 - (m_1 + m_2)A_6 + A_9}; \]

\[ B_{20} = -\frac{2A_2m_2m_3B_2B_5 + 2A_4B_2B_5}{(m_2 + m_3)^2A_5 - (m_2 + m_3)A_6 + A_9}; \quad B_{21} = -\frac{2A_2m_1m_3B_2B_5 + 2A_4B_2B_5}{(m_1 + m_3)^2A_5 - (m_1 + m_3)A_6 + A_9}; \quad B_{22} = \frac{A_8B_3B_5}{A_9}; \]

\[ B_{23} = \frac{m_2^2A_5 - m_2A_6 + A_9}{m_2^2A_5 - m_2A_6 + A_9}; \quad B_{24} = \frac{m_2^2A_5 - m_2A_6 + A_9}{m_2^2A_5 - m_2A_6 + A_9}; \quad B_{25} = \frac{m_2^2A_5 - m_2A_6 + A_9}{m_2^2A_5 - m_2A_6 + A_9}; \quad B_{26} = -\frac{A_8B_3B_5}{A_9}; \]

\[ B_{27} = -(B_{20} + B_{18} + B_{19} + B_{21} + B_{22} + B_{23} + B_{24} + B_{26} + B_{25}) \]

\[ A_{10} = Q - 2i \left( 1 - \phi + \frac{\varphi(\rho_{CP})}{\rho_{CP}} \right) H^2; \quad m_{11,12} = \frac{B_{2} + \frac{B_{2}^2 + 2B_1B_5}{2A_5}}{2A_5}; \quad B_{29} = -\frac{A_7m_1B_2B_5 + A_8B_3}{m_2^2A_5 - 2m_3A_6 + A_{10}}; \]

\[ B_{30} = \frac{m_2^2A_5 - 2m_3A_6 + A_{10}}{m_2^2A_5 - 2m_3A_6 + A_{10}}; \quad B_{31} = \frac{A_7m_1m_3B_2}{m_2^2A_5 - 2m_3A_6 + A_{10}}; \]

\[ B_{32} = \frac{A_7m_1m_3B_2}{m_2^2A_5 - 2m_3A_6 + A_{10}}; \quad B_{33} = \frac{A_7m_1m_3B_2}{m_2^2A_5 - 2m_3A_6 + A_{10}}; \]

\[ B_{34} = \frac{A_7m_1m_3B_2}{m_2^2A_5 - 2m_3A_6 + A_{10}}; \quad B_{35} = \frac{A_7m_1m_3B_2}{m_2^2A_5 - 2m_3A_6 + A_{10}}; \]

\[ B_{36} = \frac{A_7m_1m_3B_2}{m_2^2A_5 - 2m_3A_6 + A_{10}}; \quad B_{37} = \frac{A_7m_1m_3B_2}{m_2^2A_5 - 2m_3A_6 + A_{10}}; \]

\[ B_{38} = \frac{A_7m_1m_3B_2}{m_2^2A_5 - 2m_3A_6 + A_{10}}; \quad B_{39} = \frac{A_7m_1m_3B_2}{m_2^2A_5 - 2m_3A_6 + A_{10}}; \]

\[ B_{40} = \frac{A_7m_1m_3B_2}{m_2^2A_5 - 2m_3A_6 + A_{10}}; \]

\[ B_{41} = \frac{A_7m_1m_3B_2}{m_2^2A_5 - 2m_3A_6 + A_{10}}; \quad B_{42} = \frac{A_7m_1m_3B_2}{m_2^2A_5 - 2m_3A_6 + A_{10}}; \]

\[ B_{43} = \frac{A_7m_1m_3B_2}{m_2^2A_5 - 2m_3A_6 + A_{10}}; \quad B_{44} = \frac{A_7m_1m_3B_2}{m_2^2A_5 - 2m_3A_6 + A_{10}}; \]

\[ B_{45} = \frac{1}{(e_{m_1} - e_{m_2})}(B_{29}(e_{m_1} - e_{m_2}) + B_{30}(e_{m_1} - e_{m_2}) + B_{31}(e_{m_1} - e_{m_2}) + B_{32}(e_{m_1} - e_{m_2}) + B_{33}(e_{m_1} - e_{m_2}) + B_{34}(e_{m_1} - e_{m_2}) + B_{35}(e_{m_1} - e_{m_2}) + B_{36}(e_{m_1} - e_{m_2}) + B_{37}(e_{m_1} - e_{m_2}) + B_{38}(e_{m_1} - e_{m_2}) + B_{39}(e_{m_1} - e_{m_2}) + B_{40}(e_{m_1} - e_{m_2}) + B_{41}(e_{m_1} - e_{m_2}) + B_{42}(e_{m_1} - e_{m_2}) + B_{43}(e_{m_1} - e_{m_2}) + B_{44}(e_{m_1} - 1))} \]
\[ B_{46} = -(B_{41} + B_{42} + B_{43} + B_{44} + B_{45} + B_{29} + B_{31} + B_{32} + B_{34} + B_{33} + B_{35} + B_{36} + B_{38} + B_{37} + B_{39} + B_{40}) \]

**Nomenclature**

| Symbol | Description |
|--------|-------------|
| \(A\) | constant defined in Eq. (1) |
| \(B_0\) | uniform magnetic field |
| \(C_{p_{nf}}\) | specific heat at constant pressure of nanofluid |
| \(D_a\) | Darcy number of the porous media |
| \(E_c\) | Eckert number |
| \(H\) | frequency parameter |
| \(k\) | permeability of the porous media |
| \(\kappa_{nf}\) | thermal conductivity of the nano fluid |
| \(M\) | Hartmann number |
| \(N_u\) | Nusselt number |
| \(P_r\) | Prandtl number |
| \(\bar{p}\) | pressure |
| \(Q\) | heat source/ sink parameter |
| \(R_e\) | cross flow Reynolds number |
| \(R_d\) | radiation parameter |
| \(T_0\) | temperature at the lower wall |
| \(T_1\) | temperature at the upper wall |

| Symbol | Description |
|--------|-------------|
| \(\phi\) | porosity of the porous medium |
| \(\sigma\) | electrical conductivity of the fluid |
| \(\mu_{nf}\) | dynamic viscosity of the nanofluid |
| \(\rho_{nf}\) | density of the nanofluid |
| \(\nu_{nf}\) | kinematic viscosity of the nanofluid |
| \(\varphi\) | nanoparticle volume fraction |
| \((\rho C_p)_{nf}\) | heat capacity of the nano fluid |
| \(\omega\) | frequency |
| \(\mu_B_{nf}\) | Plastic dynamic viscosity of the nanofluid |

**Subscripts**

| Symbol | Description |
|--------|-------------|
| \(f\) | fluid fraction |
| \(p\) | nanoparticle fraction |
| \(nf\) | nanofluid |

**References**

1. Sheikh NA, Ching DLC, Khan I, Kumar D, Nisar KS, (2020) Alex. Eng. 59 (5) 2865-2876.
2. Loganathan P, Deepa K (2020) Ind.J. Pure & Applied Physics, 58 79-86.
3. Bestman AR, (1982) Int.J. Heat Mass Transfer, 25 675-682.
4. Loganathan K, Alessa N, Tamilvanan K, Alshammari FS. (2021) Eur. Phys. J. Spec. Top. https://doi.org/10.1140/epjs/s11734-021-00056-6.
5. Loganathan K, Rajan S, (2020) J Therm Anal Calorim, DOI: 10.1007/s10973-020-09414-3
6. Loganathan K, Mohana K, Mohanraj M, Sakthivel P, and Rajan S (2020) J Therm Anal Calorim. DOI:10.1007/s10973-020-09751-3.
7. Shehad SA, Hayat T, Qasim M and Asghar S, (2013) Braz.J. Chemical Engineering, 30, 187-195.
8. Nadeem S, Ul.Haq R, Lee C, (2012) Sci, Iran. 19(6) 1550-1553.
9. Srinivas S, Kumar CK, Bhuvaneswari M, Reddy AS, (2018) Nonlinear Analusys: Modaling and Control. 23(2) 213-233.
10. Abu-Nada E, (2008) Int.J.Heat Fluid Flow 29 242-249.
11. Oztork HF and Abu-Nada E, (2008) Int.J.Heat Fluid Flow, 29(5) 1326-1336