A BACKWARD ERGODIC THEOREM ALONG TREES AND ITS CONSEQUENCES FOR FREE GROUP ACTIONS

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ABSTRACT. We prove a new pointwise ergodic theorem for probability-measure-preserving (pmp) actions of free groups, where the ergodic averages are taken over arbitrary finite subtrees of the standard Cayley graph rooted at the identity. This result is a significant strengthening of a theorem of Grigorchuk (1987) and Nevo and Stein (1994), and a version of it was conjectured by Bufetov in 2002.

Our theorem for free groups arises from a new – backward – ergodic theorem for a countable-to-one pmp transformation, where the averages are taken over arbitrary trees of finite height in the backward orbit of the point (i.e. trees of possible pasts). We also discuss other applications of this backward theorem, in particular to the shift map with Markov measures, which yields a pointwise ergodic theorem along trees for the boundary actions of free groups.

CONTENTS

1. INTRODUCTION ................................................................. 2
   1.A. Results ........................................................................ 2
   1.B. Context and history ...................................................... 5
   1.C. A word on the proof of the backward ergodic theorem ......... 6

2. PRELIMINARIES .................................................................. 7
   2.A. The Radon–Nikodym cocycle .......................................... 8
   2.B. Null-preserving orbit equivalence relations ....................... 8
   2.C. Right-inverses .............................................................. 9
   2.D. Ergodic decomposition and conditional expectation .......... 10
   2.E. Limits as weight goes to infinity ..................................... 11

3. THE LOCAL–GLOBAL BRIDGE .............................................. 11
   3.A. Markov operators ......................................................... 11
   3.B. Local-global bridge for null-preserving $T$ ....................... 11
   3.C. Local-global bridge for pmp $T$ ..................................... 13

4. THE TILING PROPERTY AND THE BACKWARD ERGODIC THEOREM ............................... 14
   4.A. The tiling property ....................................................... 15
   4.B. Poincaré recurrence ..................................................... 16
   4.C. Backward ergodic theorem along trees .......................... 16
   4.D. Convergence in $L^p$ along special sequences of trees ..... 19

5. APPLICATIONS TO SHIFT MAPS ......................................... 21
1. Introduction

The classical pointwise ergodic theorem, whose first instance dates back to Birkhoff [Bir31], states that for any measure-preserving (i.e., $T_*\mu = \mu$) transformation $T : X \to X$ on a standard probability space $(X, \mu)$ and $f \in L^1(X, \mu)$, $\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} f(T^k x) = \overline{f}$ for a.e. $x \in X$, where $\overline{f}$ is the conditional expectation of $f$ with respect to the $\sigma$-algebra of $T$-invariant Borel sets.

More generally, pointwise ergodic theorems have been proven for probability-measure-preserving (pmp) actions of countable (semi)groups perhaps with weighted ergodic averages (see Section 1.B for a survey). The first such results for nonamenable groups are due to Grigorchuk [Gri87, Gri99, Gri00], Nevo [Nev94], and Nevo and Stein [NS94], which apply to pmp actions of finitely generated free groups $\mathbb{F}_r$ (see Theorem 1.5). These results state the convergence (to the conditional expectation) of weighted ergodic averages taken over the balls in the standard Cayley graph of $\mathbb{F}_r$, where the weights are uniform over each sphere. This was later generalized by Bufetov [Buf00, Theorem 1] to finitely generated free semigroups and to a larger class of weight assignments.

1.A. Results

We prove a pointwise ergodic theorem for pmp actions of free groups where the ergodic averages are taken over arbitrary finite subtrees of the standard (left) Cayley graph; see Fig. 1 for an example of such a tree. A version of this was conjectured by Bufetov in 2002\(^1\), and it vastly strengthens the aforementioned theorem of Grigorchuk, Nevo, and Nevo–Stein.

**Theorem 1.1** (Pointwise ergodic for pmp actions of free groups). Let $\mathbb{F}_r$ be the free group on $2 \leq r < \infty$ generators and let $\mathbb{F}_r \rtimes^\alpha (X, \mu)$ be a (not necessarily free) pmp action of $\mathbb{F}_r$. Let $S_r$ be the standard symmetric set of generators of $\mathbb{F}_r$ and let $m_u$ be the uniform\(^2\) Markov measure on $\mathbb{F}_r$. Let $(\tau_n)$ be an arbitrary sequence of finite subtrees of the (left) Cayley graph of $\mathbb{F}_r$ containing the identity (see Fig. 1) and such that $\lim_n m_u(\tau_n) = \infty$. Then for every $f \in L^1(X, \mu)$, for a.e. $x \in X$,

$$\lim_{n \to \infty} \frac{1}{m_u(\tau_n)} \sum_{w \in \tau_n} f(w \cdot x) m_u(w) = \overline{f}(x),$$

where $\overline{f}$ is the conditional expectation of $f$ with respect to the $\sigma$-algebra of $\alpha$-invariant Borel sets.

---

\(^1\)Bufetov has kindly allowed us to state this, and he has also informed us that he asked closely related questions at ESI, Vienna in 2002.

\(^2\)is the nonbacktracking simple symmetric random walk on the standard (left) Cayley graph of $\mathbb{F}_r$, i.e. $m_u(w) := \frac{1}{2^r(2^r-1)}$ for each reduced word $w \in \mathbb{F}_r$ of length $\ell \geq 1$, and $m_u(1_{\mathbb{F}_r}) := 1$. 

A more general version of this theorem for a wider class of Markov measures is stated later as Theorem 6.2. Taking $\tau_n$ to be the ball of radius $n$ in $\Gamma_r$ gives the conclusion of [Gri87, Gri99, Gri00, NS94].

We also prove a pointwise ergodic theorem for the (non-pmp) action of the free group $F_r$ on $1 \leq r \leq \infty$ generators on its boundary $\partial F_r$, where we identify $\partial F_r$ with the space of infinite reduced words on the standard symmetric set of generators of $F_r$. We denote by $w^{-1}v$ the concatenation of the words $u, v \in F_r$.

**Theorem 1.2** (Pointwise ergodic for boundary actions of free groups). For $1 \leq r \leq \infty$, let $S_r$ be the standard symmetric set of generators of the free group $F_r$, and let $F_r \curvearrowright \beta \partial F_r$ be the natural action of $F_r$ on its boundary $\partial F_r \subseteq S_r^\mathbb{N}$. Let $m$ be a stationary Markov measure on the set of finite words in $S_r$ whose support is contained in $F_r$ (the set of reduced words) and let $\mathbb{P}_m$ denote the induced Markov (probability) measure on $\partial F_r$. For every $f \in L^1(\partial F_r, \mathbb{P}_m)$, for a.e. $x \in \partial F_r$,

$$
\frac{1}{m(\tau^{-1}x(0))} \sum_{w \in \tau} f(w \cdot x)m(w^{-1}x(0)) \to \overline{f}(x) \text{ as } m(\tau) \to \infty,
$$

where $\tau \subseteq F_r$ ranges over all finite subtrees of the (left) Cayley graph of $\Gamma_r$ containing the identity but not $x(0)^{-1}$ (see Fig. 1 for $x(0) = a$), and where $\tau^{-1}x(0) := \{w^{-1}x(0) : w \in \tau\}$ and $\overline{f}$ is the conditional expectation of $f$ with respect to the $\sigma$-algebra of $\beta$-invariant Borel sets.

We restate this theorem later as Corollary 5.5, which also includes the corresponding version of the maximal ergodic theorem, as well as convergence in $L^p$ for a specific sequence of trees. Bowen and Nevo in [BN13, Theorem 4.1] also provide a pointwise ergodic theorem for the boundary action of the free group, more precisely, for the diagonal action of the free group on the product of its boundary and a pmp action, and their averages are taken over horospherical balls. We also prove an ergodic theorem for this diagonal action (see Section 6) but we sample the averages over subtrees of the Cayley graph of $\Gamma_r$ as in Theorem 1.2.

The main result underlying Theorem 1.1 and Theorem 1.2 is a backward pointwise ergodic theorem for a pmp Borel transformation $T$ (Theorem 1.3, restated later as Theorem 4.7), where the averages are taken along trees of possible pasts (in the direction of $T^{-1}$). Although $T$ is pmp, the induced orbit equivalence relation $E_T$ is not pmp, unless $T$ is one-to-one, so the averages are weighted by the Radon–Nikodym cocycle of $E_T$ with respect to the measure.

**Theorem 1.3** (Backward pointwise ergodic along trees). Let $T$ be an aperiodic\footnote{\textit{T} is called aperiodic if for all $x \in X$ and $n \in \mathbb{N} \setminus \{0\}$, $T^n(x) \neq x$.} countable-to-one measure-preserving transformation on a standard probability space $(X, \mu)$. Let $E_T$ denote the induced orbit equivalence relation and let $(x, y) \mapsto \rho_x(y) : E_T \to \mathbb{R}^+$ be the Radon–Nikodym cocycle of $E_T$ with respect to $\mu$. For every $f \in L^1(X, \mu)$, for a.e. $x \in X$,

$$
\frac{1}{\rho_x(\tau_x)} \sum_{y \in \tau_x} f(y)\rho_x(y) \to \overline{f}(x) \text{ as } \rho_x(\tau_x) \to \infty,
$$

where $\tau_x$ ranges over all (possibly infinite) subtrees of the graph of $T$ of finite height rooted at $x$, directed towards $x$ (see Fig. 2), and where $\rho_x(\tau_x) := \sum_{y \in \tau_x} \rho_x(y)$ and $\overline{f}$ is the conditional expectation of $f$ with respect to the $\sigma$-algebra of $T$-invariant Borel sets.
Thus, while the classical pointwise ergodic theorem for $T$ says that to approximate $\bar{f}$, we can start almost anywhere in the space and walk forward in time (in the direction of $T$), Theorem 1.3 allows us to walk back in time (in the direction of $T^{-1}$) scanning sufficiently heavy trees of possible pasts. Note that Theorem 1.3, in particular, implies the classical (forward) pointwise ergodic theorem for one-to-one transformations $T$ when applied to $T^{-1}$.

We also prove a \textbf{backward maximal ergodic theorem} over arbitrary subtrees (see Theorem 4.10), but we do not use it in our proof of Theorem 1.3.

We obtain Theorem 1.1 and Theorem 1.2 by applying Theorem 1.3 to specific choices of $T$. In addition to Theorem 1.1 and Theorem 1.2, we also explore other applications of Theorem 1.3 to the shift map on spaces of infinite words in Section 5. We recall the class of Markov measures on these spaces that are shift-invariant (namely, stationary Markov measures), and point out that for each such measure, Theorem 1.3 allows us to calculate the expectation of an $L^1$ function by looking at trees of past trajectories of the Markov process.

Theorem 1.3 implies, in particular, convergence of the $\rho$-weighted averages along any sequence $(\tau_n)$ of subtrees with $\rho(\tau_n) \to \infty$.

We also obtain convergence in $L^p$, for all $p \geq 1$, for sequences of so-called \textit{fat} trees, see Corollary 4.15. The authors do not know whether the averages over all trees converge in $L^p$, see Question 4.16. An obvious example of a sequence of fat trees is that of complete trees of height $n$, i.e. $\bigcup_{i=0}^n T^{-i}(x)$, and we now state this important special case (later restated as Corollary 4.11).

\textbf{Corollary 1.4} (Backward ergodic along complete trees). Let $T$ be an aperiodic countable-to-one measure-preserving transformation on a standard probability space $(X, \mu)$. Let $E_T$ denote the induced orbit equivalence relation and let $(x, y) \to \rho_x(y) : E_T \to \mathbb{R}^+$ be the Radon–Nikodym cocycle of $E_T$ with respect to $\mu$. For any $1 \leq p < \infty$ and $f \in L^p(X, \mu)$,

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{i=0}^n \sum_{y \in T^{-i}(x)} f(y) \rho_x(y) = \overline{f}(x) \text{ a.e. and in } L^p,$$

where $\overline{f}$ is the conditional expectation of $f$ with respect to the $\sigma$-algebra of $T$-invariant Borel sets.

In the case where $f$ is bounded, Corollary 1.4 can be deduced directly from the classical (forward) pointwise ergodic theorem, so the new content of Corollary 1.4 is for unbounded $f$.

Unlike Theorem 1.3, Corollary 1.4 has an operator-theoretic formulation. The operator $P_T : L^1 \to L^1$ defined by $P_T(f)(x) := \sum_{y \in T^{-i}(x)} f(y) \rho_x(y)$ is nothing but the adjoint $K_T^*$ of the Koopman representation $K_T$ of $T$ (Proposition 3.5), so Corollary 1.4 simply states the convergence to $\overline{f}$ of $\frac{1}{n+1} \sum_{i=0}^n (K_T^*)^i(f)$, while the classical pointwise ergodic theorem states the same but for $K_T$. The mere convergence of $\frac{1}{n+1} \sum_{i=0}^n (K_T^*)^i(f)$, not specifically to $\overline{f}$, is already implied by [DS56] (see Remark 4.13), so the contribution of Corollary 1.4 is the fact that this limit is $T$-invariant (and hence equal to $\overline{f}$) and not just $K_T^*$-invariant.
1.B. Context and history

In general, a measure-preserving action of a countable (discrete) semigroup $G$ on a standard probability space $(X, \mu)$ is said to have the **pointwise ergodic property** along a sequence $(F_n)$ of finite subsets of $G$, if for every $f \in L^1(X, \mu)$, for a.e. $x \in X$,

$$\lim_{n \to \infty} \frac{1}{|F_n|} \sum_{g \in F_n} f(g \cdot x) = \bar{f},$$

where $\bar{f}$ is the conditional expectation of $f$ with respect to the $\sigma$-algebra of $G$-invariant Borel sets, so in case the action is ergodic, the limit is just $\int_X f \, d\mu$.

**1.B.i. For pmp actions.** It is a celebrated theorem of Lindenstrauss [Lin01] that the pointwise ergodic property is true for the pmp actions of all countable amenable groups along tempered Følner sequences and this was extended by Butkevich in [But00] to all countable left-cancellative amenable semigroups.

Amenability, or rather the fact that the Følner sets $F_n$ are almost invariant, is essential for the pointwise ergodic property as it ensures that the limit of averages is an invariant function. This is why, to obtain a version of the pointwise ergodic property for nonamenable (semi)groups, e.g. for the nonabelian free groups $F_r$, one has to imitate the almost invariance of finite sets by taking weighted averages instead, so that the weight of the boundary is small. The first instance of this was proven by Grigorchuk [Gri87, Gri99, Gri00], and independently by Nevo (for $L^2$ functions [Nev94]), and by Nevo and Stein [NS94]:

**Theorem 1.5** (Grigorchuk 1987; Nevo–Stein 1994). Let $r < \infty$ and let $\mathbb{F}_r \curvearrowright^\alpha (X, \mu)$ be a (not necessarily free) pmp action of the free group $\mathbb{F}_r$. For any $f \in L^1(X, \mu)$, for a.e. $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{w \in B_n} f(w \cdot x) m_w(w) = \bar{f}(x),$$

where $B_n$ is the (closed) ball of radius $n$ in the standard symmetric (left) Cayley graph of $\mathbb{F}_r$, $m_w$ is the uniform Markov measure\(^2\) on $\mathbb{F}_r$, and $\bar{f}$ is the conditional expectation of $f$ with respect to the $\sigma$-algebra of $\alpha$-invariant Borel sets.

This theorem of Grigorchuk and of Nevo and Stein is a special case of our Theorem 1.1. Indeed, one simply applies Theorem 1.1 to the sequence of balls $B_n$ of radius $n$ (in the standard Cayley graph of $\mathbb{F}_r$), observing that $m(B_n) = n+1$. It is worth noting that our proof of Theorem 1.1 does not use Theorem 1.5; in fact, it provides a new proof of this result.

Theorem 1.5 was vastly generalized by Grigorchuk in [Gri99] and Bufetov in [Buf00, Theorem 1] to finitely generated free semigroups (and to a large class of stationary Markov measures in Bufetov’s theorem).

The aforementioned theorems of Grigorchuk, Nevo–Stein, and Bufetov are proven using the Dunford–Schwartz theorem [DS56], an ergodic theorem for Markov operators. Our proof of Theorem 1.1 follows a similar sort of reduction, but to our Theorem 1.3 (instead of Dunford–Schwartz) applied to an auxiliary transformation on $X \times \partial \mathbb{F}_r$, which is referred to as the backward system in [BQ16, Section 1.5]. That is, our proof technique is self-contained. Furthermore, the proofs in [Gri87, Gri99, Gri00], [NS94], and [Buf00] use Markov operators, but this technique cannot be extended to yield our results with averages over trees, since trees do not correspond to iterates of operators.

Other instances of pointwise ergodic theorems are known for pmp actions of finitely generated groups where the averages are taken over sets that are not trees. For example, [NS94] and [Buf02] include that for pmp actions of free groups of finite rank, the averages over spheres of even radius in the Cayley graph converge a.e. if the function is in $L^2$ (Nevo–Stein) or even in $L \log L$ (Bufetov), however this is not true for functions in $L^1$ in general as shown by Tao [Tao15]. A more general
treatment of pointwise ergodic theorems for groups is given in [BN13] and [BN15, Theorems 6.2 and 6.3]. We refer to [BK12] for a survey of pointwise ergodic theorems for groups.

1.B.ii. For nonsingular (null-preserving) actions. There is a suitable analogue of the pointwise ergodic property for merely nonsingular\(^4\) actions of (semi)groups on a standard probability space: the averages have to be weighted by the corresponding Radon–Nikodym cocycle [KM04, Section 8]. Much less is known for such actions: a pointwise ergodic theorem for \(\mathbb{Z}^d\) was first proven by Feldman [Fel07] and then generalized in two different directions by Hochman [Hoc10] and by Dooley and Jarrett [DJ21]. For general groups of polynomial growth, Hochman obtained [Hoc13, Theorem 1.4], a slightly weaker form of a nonsingular ergodic theorem where the a.e. convergence is replaced with a.e. convergence in density.

On the negative side, Hochman proved [Hoc13, Theorem 1.1] that the pointwise ergodic theorem for null-preserving actions holds only along sequences of subsets of the group satisfying the so-called Besicovitch covering property. He then infers [Hoc13, Theorems 1.2 and 1.3] that the null-preserving pointwise ergodic theorem fails for any sequence of subsets of \(\bigoplus_{n \in \mathbb{N}} \mathbb{Z}\) and any subsequence of balls in nonabelian free groups as well as in the Heisenberg group.

In [Tse22], Tserunyan obtained a pointwise ergodic theorem for locally countable null-preserving Borel graphs (equivalently, for the Schreier graphs of null-preserving actions of countable groups). Given the negative results mentioned above, this is, perhaps, the most general result in this vein.

1.B.iii. Hybrid setting. Our main result (Theorem 1.3) is a contribution to the ergodic theory of actions of finitely generated (semi)groups in both the pmp and null-preserving settings. As mentioned above, although Theorem 1.3 is about a pmp transformation \(T\), its orbit equivalence relation \(E_T\) is generally not pmp, and assuming, as we may, that \(E_T\) is null-preserving, the ergodic averages are weighted with the corresponding Radon–Nikodym cocycle. Furthermore, Theorem 1.3 implies an ergodic theorem for the natural action of the free group \(\mathbb{F}\) on its boundary (Theorem 1.2), which is merely null-preserving. On the other hand, Theorem 1.3 also implies Theorem 1.1, which is about pmp actions.

1.C. A word on the proof of the backward ergodic theorem

The pointwise ergodic property equates the global condition of ergodicity with the local (pointwise) statistics of the action. The key in connecting the global analysis to the local statistics is that one can often replace integrals of functions with integrals of local averages of these functions over certain shapes. For example, for a pmp group action, it easily follows from the change of variable formula that \(\int f d\mu = \int \frac{1}{|\gamma|} \sum_{y \in F} f(\gamma \cdot x) d\mu(x)\) for any finite subset \(F\) of the group. In our case, we have \(\int f d\mu = \int \frac{1}{\rho_x(\mathcal{F}_x \cdot x)} \sum_{y \in \mathcal{F}_y \cdot x} f(y) \rho_x(y) d\mu(x)\) for \(n \in \mathbb{N}\), where \(\mathcal{F}_x \cdot x := \bigcup_{i \leq n} T^{-i}x\) (see Corollary 3.11). We call such statements local-global bridges. In operator-theoretic language, these are assertions that certain local averaging operators are Markov.

Local-global bridges reduce pointwise ergodic theorems to proving finitary tiling properties. These reductions are done as follows: we assume for a contradiction that the pointwise ergodic theorem fails, which gives us “bad” tiles for each point. Then, we tile the shapes from the local-global bridge with bad tiles, thus making the integral of the function over the whole space incorrect, a contradiction.

This scheme of proving ergodic theorems first appears implicitly in [KP06], more explicitly in [Tse18], and even more explicitly in [BZ20] and in [Tse22]. In the proof of Theorem 1.3, we tile sets of the form \(\mathcal{F}_x \cdot x\) with tiles of the form \(\tau_y\), where \(\tau_y\) is an arbitrary subtree of the graph of \(T\) of finite height rooted at \(y \in \mathcal{F}_x \cdot x\) and directed towards \(y\) (see Fig. 3 for when \(T\) is the shift map \(\sigma\) on \(2^\mathbb{N}\)).

\(^4\)A measurable action of a countable semigroup \(G\) on a probability space \((X, \mu)\) is called nonsingular (or null-preserving, or quasi-pmp) if for each \(g \in G, g_\ast \mu \sim \mu\); equivalently, the \(g\)-preimage of a null set is null.
For the above scheme to work, we need the limit (or rather, the \( \limsup \)) of local averages to be invariant (i.e. constant on each orbit). Unlike the classical ergodic theorem, this is not clear a priori in Theorem 1.3, or especially in its special case Corollary 1.4. Comparing the averages over the trees \( \tau_x \) and \( \tau_x \cup \{ T(x) \} \) only gives that the \( \limsup \) is nondecreasing in the direction of \( T \), so one has to apply Poincaré recurrence to deduce that the \( \limsup \) is constant on the orbit of \( x \).

**Organization.** In Section 2, we give the necessary notation and definitions that are used throughout the paper, and we prove some preliminary lemmas about countable-to-one pmp Borel transformations \( T \). In Section 3, we state and prove the local-global bridge lemmas (Lemma 3.8 and Corollary 3.11). In Section 4, we explicitly state and prove the suitable tiling property (Lemma 4.1) and deduce Theorem 1.3 from it. In Section 5, we provide examples of countable-to-one pmp ergodic Borel transformations (to which Theorem 1.3 applies), and deduce the corresponding ergodic theorems, in particular Theorem 1.2 for the boundary action of the free groups. In Section 6, we deduce Theorem 1.1 for pmp actions of free groups.

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## 2. Preliminaries

Our set \( \mathbb{N} \) of natural numbers includes 0 and our \( L^p \) spaces are real.
Throughout, let \((X, \mu)\) be a standard probability space. A **countable Borel equivalence relation** on \((X, \mu)\) is an equivalence relation that is a Borel subset of \(X^2\) whose every equivalence class is countable. By the Luzin–Novikov uniformization theorem [Kec95, Theorem 18.10], if \(B \subseteq X\) is Borel, then so is its **E-saturation** \(\left[ B \right]_E := \bigcup_{x \in B} [x]_E\).

2.A. The Radon–Nikodym cocycle

We say that a Borel transformation on \((X, \mu)\) is **\(\mu\)-preserving** (resp. **null-preserving**) if for any partial Borel injection \(\gamma : X \rightarrow X\) with graph(\(\gamma\)) \(\subseteq E\), \(\mu(\text{dom}(\gamma)) = \mu(\text{im}(\gamma))\) (resp. \(\text{dom}(\gamma)\) is \(\mu\)-null if and only if \(\text{im}(\gamma)\) is \(\mu\)-null).

Note that \(E\) is null-preserving if and only if the \(E\)-saturations of null sets are null. By [KM04, Section 8], a null-preserving \(E\) admits an a.e. unique **Radon–Nikodym cocycle** \(\rho : E \rightarrow \mathbb{R}^+\) with respect to \(\mu\). Being a **cocycle** for a function \((x, y) \mapsto \rho_x(y) : E \rightarrow \mathbb{R}^+\) means that it satisfies the cocycle identity:

\[
\rho_x(y)\rho_y(z) = \rho_x(z),
\]

for all \(E\)-equivalent \(x, y, z \in X\). We say that \(\rho\) is the **Radon–Nikodym cocycle** with respect to \(\mu\) (or that \(\mu\) is **\(\rho\)-invariant**) if it is Borel (as a real-valued function on the standard Borel space \(E\)) and for any partial Borel injection \(\gamma : X \rightarrow X\) with graph(\(\gamma\)) \(\subseteq E\) and \(f \in L^1(X, \mu)\),

\[
\int_{\text{im}(\gamma)} f(x) \, d\mu(x) = \int_{\text{dom}(\gamma)} f(\gamma(x))\rho_x(\gamma(x)) \, d\mu(x).
\]

We call a subset \(J\) of an \(E\)-equivalence class \(C\) **\(\rho\)-finite** if \(\rho_x(J) := \sum_{y \in J} \rho_x(y) < \infty\) for some \(x \in C\) (although the value \(\rho_x(J)\) depends on the choice of \(x\), its finiteness does not, by the cocycle identity). Further, for a nonempty \(\rho\)-finite \(J \subseteq C\) and a non-negative function \(f : X \rightarrow [0, \infty)\), we define the **\(\rho\)-weighted average** of \(f\) over \(J\) by

\[
A_f^\rho[J] := \frac{\sum_{y \in J} f(y)\rho_x(y)}{\rho_x(J)}
\]

for some \(x \in C\). Again, this value does not depend on the choice of \(x \in C\) by the cocycle identity.

We also use the same notation for a general real-valued function \(f\), provided \(\sum_{y \in J} |f(y)|\rho_x(y) < \infty\).

2.B. Null-preserving orbit equivalence relations

We say that a Borel transformation \(T : X \rightarrow X\) is **\(\mu\)-preserving** (resp. **null-preserving**) if \(T_*\mu = \mu\) (resp. \(T_*\mu \sim \mu\)). Let \(E_T\) denote the induced **orbit equivalence relation** on \(X\), that is:

\[
x \in E_T y \iff \exists n, m, T^n(x) = T^m(y).
\]

Note that if \(T\) is countable-to-one, \(E_T\) is countable (i.e. each \(E\)-class is countable).

Even when \(T\) is \(\mu\)-preserving, \(E_T\) may not be \(\mu\)-preserving since \(T\) may not be injective. In fact, \(E_T\) may not even be null-preserving (it is possible for the \(T\)-image of a null set to have positive measure). However, the following result (originally proven by Kechris using Woodin’s argument for the analogous statement for Baire category) shows that we may neglect this issue. Three different proofs of this are given in [Mil04, Proposition 2.1] and [Mil20, 1.3], and we give a fourth one here, which is a measure-exhaustion argument.

**Lemma 2.2** (Kechris–Woodin). Let \(E\) be a countable Borel equivalence relation on a standard probability space \((X, \mu)\). Then \(E\) is null-preserving when restricted to some conull set.

**Proof.** Using the Feldman–Moore theorem [FM77], fix a countable set \(\Delta\) of Borel involutions \(\delta : X \rightarrow X\) such that \(E = \bigcup_{\delta \in \Delta} \text{graph}(\delta)\).

**Claim.** For a Borel set \(Y \subseteq X\), and \(\delta \in \Delta\), let \(\delta_Y := \delta \cap Y^2\), i.e., the restriction of \(\delta\) to the set \(\{y \in Y \cap \text{dom}(\delta) : \delta(y) \in Y\}\). Then \(E|_Y\) is null-preserving if for each \(\delta \in \Delta\) and each Borel \(B \subseteq \text{dom}(\delta_Y), \mu(B) > 0\) implies \(\mu(\delta_Y(B)) > 0\).
Proof of Claim. Let $B \subseteq Y$ be a set whose saturation $[B]_{E^1_Y}$ has positive measure. Then since $[B]_{E^1_Y} = \bigcup_{\delta \in \Delta} \delta_Y(B \cap \text{dom}(\delta_Y))$, we must have $\mu(\delta_Y(B \cap \text{dom}(\delta_Y))) > 0$ for some $\delta \in \Delta$, which implies $\mu(B) \geq \mu(\delta_Y^* (B \cap \text{dom}(\delta_Y))) > 0$, by the assumption.

To construct a conull set $Y$ satisfying the hypothesis of the claim, it is enough to fix $\delta \in \Delta$ and find a conull set $Y_\delta$ such that for each Borel $B \subseteq \text{dom}(\delta_Y)$, $\mu(B) > 0$ implies $\mu(\delta_Y(B)) > 0$ (because then $Y := \bigcap_{\delta \in \Delta} Y_\delta$ is as desired).

To this end, fix $\delta \in \Delta$. We recursively construct a decreasing sequence $(X_n)$ of conull sets and a pairwise disjoint sequence $(B_n)$ of Borel subsets $B_n \subseteq X_n$ as follows. Let $X_0 := X$ and suppose $X_n$ has been constructed. If there is a Borel set $B \subseteq \text{dom}(\delta_{X_n})$ with $\mu(B) > 0$ and $\mu(\delta_{X_n}(B)) = 0$, let $B_n$ be one such set with

$$\mu(B_n) > \frac{1}{2} \sup \{\mu(B) : B \subseteq \text{dom}(\delta_{X_n}) \text{ Borel with } \mu(\delta_{X_n}(B)) = 0\}.$$  \hspace{1cm} (2.3)

Then put $X_{n+1} := X_n \setminus \delta_{X_n}(B_n)$. Having constructed these sequences of sets, we check that the conull set $X_\infty := \bigcap_{n \in \mathbb{N}} X_n$ is as desired. Indeed, let $B \subseteq \text{dom}(\delta_{X_\infty})$ be a Borel set with $\mu(\delta_{X_\infty}(B)) = 0$. Then for all $n$, (2.3) implies $\mu(B_n) > \frac{1}{2} \mu(B)$, so $\mu(B) = 0$ because $\mu(B_n) \to 0$, since the $B_n$ are pairwise disjoint and $\mu(X) < \infty$. □

**Proposition 2.4.** For any countable-to-one null-preserving Borel transformation $T$ on a standard probability space $(X, \mu)$, there is a conull set $X' \subseteq X$ such that $T(X') = X'$ and $E_T | X'$ is null-preserving.

**Proof.** Let $X_0 \subseteq X$ be a conull set given by Lemma 2.2, i.e. $E_T | X_0$ is null-preserving. Furthermore, because $T$ is null-preserving, the set $X_1 := \bigcap_{n \in \mathbb{N}} T^{-n}(X_0)$ is still conull, but we now have $T(X_1) \subseteq X_1$. Lastly, again because $T$ is null-preserving, the $T$-image of a conull set is conull, so $Z := X_1 \setminus T(X_1)$ is null. Hence, $[Z]_{E_T}$ is also null (because $E_T | X_1$ is null-preserving), and therefore $X' := X_1 \setminus [Z]_{E_T}$ is still conull, but now we finally have $T(X') = X'$. □

**Assumption 2.5.** Since all statements in the current paper are modulo null sets, without loss of generality (by Proposition 2.4), we assume that all countable-to-one null-preserving Borel transformations $T$ on $(X, \mu)$ are surjective and the induced equivalence relations $E_T$ are null-preserving. Thus, we let $\rho$ be the Radon–Nikodym cocycle of $E_T$ with respect to $\mu$.

**2.C. Right-inverses**

Call a set $\Gamma$ of Borel partial functions $\gamma : X \to X$ a complete set of Borel partial right-inverses of $T$ if the graphs of the $\gamma \in \Gamma$ are pairwise disjoint and for each $x \in X$,

$$T^{-1}(x) = \{\gamma(x) : x \in \text{dom}(\gamma) \text{ and } \gamma \in \Gamma\}.$$  

Because $T$ is countable-to-one, the Luzin–Novikov Uniformization theorem [Kec95, Theorem 18.10] ensures that $T$ admits a countable complete set $\Gamma$ of Borel partial right-inverses, and we fix such a (countable) $\Gamma$ for the remainder of this subsection.

For each $n \in \mathbb{N}$, we think of $\Gamma^n$ as words in $\Gamma$ of length $n$. For each word $t \in \Gamma^n$, we define the partial function $t : X \to X$ by recursion on the length of $t$ as follows: if $t = \emptyset$, put $t(x) := x$ for all $x \in X$. If $t = \gamma \cdot t'$, $x \in \text{dom}(t')$, and $t'(x) \in \text{dom}(\gamma)$, define $t(x) := \gamma(t'(x))$. Otherwise, leave $t(x)$ undefined.

**Observation 2.6.** If $J \subseteq \Gamma^n$ for some $n \in \mathbb{N}$, then the sets $t(X)$, $t \in J$, are pairwise disjoint.

For $x \in X$ and $J \subseteq \Gamma^{<\mathbb{N}}$, define

$$J \cdot x := \{t(x) : t \in J \text{ and } x \in \text{dom}(t)\}.$$  

Also, if $n \in \mathbb{N}$, define

$$\triangleright_n^T x := \bigcup_{i \leq n} T^{-1}(x) = \Gamma^\leq_n \cdot x.$$  

9
Notice that $\triangleright_{\Gamma}^\mu \cdot x$ does not depend on the choice of $\Gamma$.

For any nonempty $J \subseteq \Gamma \leq \mathbb{N}$ and for all $f \in L^1(X, \mu)$, let $\overline{J}f(x)$ be the weighted-average over the sets $J \cdot x$, for $x \in X$, i.e.

$$\overline{J}f(x) := A_J^\mu[J \cdot x].$$

### 2.D. Ergodic decomposition and conditional expectation

This subsection is only used in Section 6. Let $E$ be a null-preserving countable Borel equivalence relation on a standard probability space $(X, \mu)$ and let $\rho$ be the Radon–Nikodym cocycle of $E$ with respect to $\mu$. Let $P(X)$ denote the standard Borel space of all Borel probability measures on $X$.

**Definition 2.7.** A Borel map $\epsilon : X \rightarrow P(X)$, $x \mapsto \epsilon_x$, is called an $E$-**ergodic decomposition** of $\mu$ if for each $x \in X$, the measure $\epsilon_x$ is $E$-ergodic, $\rho$-invariant, supported on $\epsilon^{-1}(\epsilon_x)$, and for all Borel sets $A \subseteq X$, 

$$\mu(A) = \int_X \epsilon_x(A) \, d\mu(x).$$

It follows from this definition that an $E$-ergodic decomposition, if it exists, is unique modulo a $\mu$-null set. As for the existence, for pmp countable Borel equivalence relations this is due to Farrel and Varadarajan [Far62, Var63] (see also [KM04, Theorem 3.3]), and more generally, it is a theorem of Ditzen for null-preserving equivalence relations [Dit92] (see also [Mil08, Theorem 5.2]). We will only use the existence of an ergodic decomposition for pmp equivalence relations, but we will use the following connection with conditional expectation for null-preserving equivalence relations.

**Proposition 2.8** (Conditional expectation via ergodic decomposition). Let $\epsilon : X \rightarrow P(X)$ be an $E$-ergodic decomposition of $\mu$. For any $f \in L^1(X, \mu)$ and $E$-invariant Borel set $B \subseteq X$,

$$\int_B f \, d\mu = \int_B \int_X f(z) \, d\epsilon_x(z) \, d\mu(x).$$

(2.9)

In particular, for $\mu$-a.e. $x \in X$,

$$\overline{J}f(x) = \int_X f \, d\epsilon_x,$$

where $\overline{J}$ is the $\mu$-conditional expectation with respect to the $\sigma$-algebra of $E$-invariant Borel sets.

**Proof.** The “in particular” part follows from the definition and uniqueness of the conditional expectation and the fact that the map $x \mapsto \int_X f(z) \, d\epsilon_x(z)$ is $E$-invariant.

As for the main part, a standard approximation argument gives that for each $h \in L^1(X, \mu)$,

$$\int_X h \, d\mu = \int_X \int_X h(z) \, d\epsilon_x(z) \, d\mu(x).$$

(2.10)

**Claim.** For every $E$-invariant Borel set $B \subseteq X$, $\epsilon_x(B) = 1_B(x)$ for $\mu$-a.e. $x \in X$.

**Proof of Claim.** Indeed, letting $X_B := \{x \in X : \epsilon_x(B) = 1\}$, we see that

$$\mu(X_B \setminus B) = \int_X \epsilon_x(X_B \setminus B) \, d\mu(x) = \int_{X_B} \epsilon_x(X_B \setminus B) \, d\mu(x) + \int_{X\setminus X_B} \epsilon_x(X_B \setminus B) \, d\mu(x) = 0,$$

because for each $x \in X_B$, $\epsilon_x(B) = 1$, so $\epsilon_x(X_B \setminus B) = 0$, and for each $x \notin X_B$, $X_B \cap \epsilon^{-1}(\epsilon_x) = \emptyset$, so $\epsilon_x(X_B) = 0$, in particular, $\epsilon_x(X_B \setminus B) = 0$. The same argument, but with $X \setminus B$ in place of $B$, shows that $\mu(B \setminus X_B) = 0$. 

\[\square\]
Throughout this subsection suppose that $T$.

Proof. For each non-negative linear operator $\|\|$ implies (2.9):
\[
\int_X f \cdot \mathbb{1}_B \, d\mu = \int_X \int_X f(z) \cdot \mathbb{1}_B(z) \, d\epsilon(z) \, d\mu(x)
\]
\[
= \int_B \int_X f(z) \cdot \mathbb{1}_B(z) \, d\epsilon(z) \, d\mu(x) + \int_{X \setminus B} \int_X f(z) \cdot \mathbb{1}_B(z) \, d\epsilon(z) \, d\mu(x)
\]
\[
= \int_B \int_X f(z) \, d\epsilon(z) \, d\mu(x) + 0. \quad \square
\]

2.E. Limits as weight goes to infinity

For a set $T$ of objects (for us, it would be trees of various kinds), a weight-function $m : T \to [0, \infty)$, a function $g : T \to \mathbb{R}$, and $L \in [-\infty, \infty]$, we write
\[
g(\tau) \to L \text{ as } m(\tau) \to \infty, \text{ where } \tau \text{ ranges over } T
\]
to mean that for every $\varepsilon > 0$, there is $M > 0$ such that for all $\tau \in T$ with $m(\tau) \geq M$ we have $|g(\tau) - L| < \varepsilon$. We refer to $L$ as $\lim_{m(\tau) \to \infty} g(\tau)$, where $\tau$ ranges over $T$. We also write
\[
\lim \sup_{m(\tau) \to \infty} g(\tau) = L, \text{ where } \tau \text{ ranges over } T
\]
if $L$ is the limit of $\sup \{g(\tau) : \tau \in T \text{ and } m(\tau) \geq M\}$ as $M \to \infty$; and similarly, for the $\lim \inf$. We omit writing “where $\tau$ ranges over $T$” when it is clear from the context.

3. THE LOCAL-GLOBAL BRIDGE

Let $T : X \to X$ be a countable-to-one Borel transformation on a standard probability space $(X, \mu)$, and fix a countable complete set $\Gamma$ of Borel partial right-inverses of $T$.

3.A. Markov operators

Before proving local-global bridge lemmas, we define and recall convenient operator-theoretic terminology. We call a linear operator $P : L^1(X, \mu) \to L^1(X, \mu)$

- **non-negative**, denoted $P \geq 0$, if $Pf \geq 0$ for each non-negative $f \in L^1(X, \mu)$;
- **mean-preserving** if $\int_X Pf \, d\mu = \int_X f \, d\mu$ for each $f \in L^1(X, \mu)$;
- **Markov** if it is non-negative, mean-preserving, and $P1 = 1$;
- an $L^p$-contraction, for $1 \leq p \leq \infty$, if $\|Pf\|_p \leq \|f\|_p$ for each $f \in L^p(X, \mu)$.

**Proposition 3.1.** Every non-negative mean-preserving linear operator $P$ is an $L^1$-contraction. In fact, for $1 \leq p < \infty$, if $(P|f|^p)^p \leq P(|f|^p)$ for all $f \in L^p(X, \mu)$, then $P$ is an $L^p$-contraction.

**Proof.** We only prove the second assertion as it subsumes the first. Fix $f \in L^p(X, \mu)$. Because $P(|f| \pm f) \geq 0$, linearity implies $|Pf| \leq Pf$. Then $\|Pf\|_p = \|Pf|^p\|_1 \leq \|(P|f|^p)^p\|_1 \leq \|P(|f|^p)\|_1 = \|\|f|^p\|_1 = \|f\|_p$, where the penultimate equality is by mean-preservation. \hfill \square

**Proposition 3.2.** Every non-negative linear operator $P$ with $P1 = 1$ is an $L^\infty$-contraction.

**Proof.** For each $f \in L^\infty(X, \mu)$, $P(\|f\|_\infty \pm f) \geq 0$, so $|Pf| \leq P\|f\|_\infty = \|f\|_\infty$. \hfill \square

3.B. Local-global bridge for null-preserving $T$

Throughout this subsection suppose that $T$ is $\mu$-null-preserving. In addition, we assume without loss of generality that $T$ satisfies Assumption 2.5, and we let $\rho$ be the Radon–Nikodym cocycle of $E_T$ with respect to $\mu$. 

11
For $f \in L^1(X, \mu)$ and $x \in X$, let
\[
P_T f(x) := \sum_{y \in T^{-1}x} f(y)\rho_x(y).
\] (3.3)

**Lemma 3.4.** For each $n \in \mathbb{N}$, $J \subseteq \Gamma^n$, and $f \in L^1(X, \mu)$,
\[
\int_X \sum_{y \in J \times x} f(y)\rho_x(y) \, d\mu(x) = \int_{\text{im}(J)} f \, d\mu,
\]
where $\text{im}(J) = \bigsqcup_{t \in J} t(X)$ ($\bigsqcup$ denotes a disjoint union). In particular, $P_T$ is a non-negative mean-preserving operator, and hence an $L^1$-contraction.

**Proof.** That $P_T$ is mean-preserving (and hence an $L^1$-contraction by Proposition 3.1) is a special case of the main part with $J := \Gamma$ because $\Gamma \cdot x = T^{-1}x$. As for the main part, it is enough to prove it for non-negative functions, since then the statement for an arbitrary $f \in L^1(X, \mu)$ follows by the decomposition $f = f^+ - f^-$ into positive and negative parts.

\[
\int_X \sum_{y \in J \times x} f(y)\rho_x(y) \, d\mu(x) = \int_X \sum_{t \in J} \mathbb{1}_{\text{dom}(t)}(x) f(t(x))\rho_x(t(x)) \, d\mu(x)
\]

because $f \geq 0$

\[
= \sum_{t \in J} \int_X \mathbb{1}_{\text{dom}(t)}(x) f(t(x))\rho_x(t(x)) \, d\mu(x)
\]

by Eq. (2.1)

\[
= \sum_{t \in J} \int_{\text{im}(t)} f(x) \, d\mu(x)
\]

= $\int_{\text{im}(J)} f(x) \, d\mu(x)$.

Let $K_T$ denote the Koopman operator on $L^1(X, \mu)$ induced by $T$, i.e. $K_T f := f \circ T$ for $f \in L^1(X, \mu)$. We explicitly calculate its adjoint $K_T^*$.

**Proposition 3.5.** The operator $P_T$ is the adjoint $K_T^*$ of the Koopman operator $K_T$; more precisely, for all $f, g \in L^1(X, \mu)$ such that $K_T f \cdot g \in L^1(X, \mu),$
\[
\int_X K_T f \cdot g \, d\mu = \int_X f \cdot P_T g \, d\mu.
\]

**Proof.** We compute:
\[
\int_X K_T(f)(x) \cdot g(x) \, d\mu(x) = \int_X (f \circ T)(x) g(x) \, d\mu(x)
\]

[Lemma 3.4]

\[
= \int_X \sum_{y \in T^{-1}(x)} (f \circ T)(y) g(y)\rho_x(y) \, d\mu(x)
\]

= $\int_X f(x) \sum_{y \in T^{-1}(x)} g(y)\rho_x(y) \, d\mu(x)$

= $\int_X f(x) P_T(g)(x) \, d\mu(x)$.

The following observation follows easily by induction on $n$ and the cocycle identity.

**Observation 3.6.** $P_T^n = P_T^m$ for each $n \in \mathbb{N}$.

**Corollary 3.7.** For any $f \in L^1(X, \mu)$, for a.e. $x \in X$, and for all $n \in \mathbb{N},$
\[
\sum_{y \in T^n \cdot x} |f(y)|\rho_x(y) < \infty.
\]
Proof. This is just because $\sum_{y \in \mathbb{N} \cdot x} |f(y)\rho_x(y) = \sum_{n=0}^{\infty} P_T^n |f| (x)$ and each $P_T^n$ maps $L^1(X, \mu)$ to $L^1(X, \mu)$ (Lemma 3.4), so $\left\| \sum_{n=0}^{\infty} P_T^n |f| \right\|_1 < \infty$. □

**Lemma 3.8** (Local-global bridge for null-preserving $T$). Let $T$ be a countable-to-one null-preserving Borel transformation on $(X, \mu)$. For any $N \in \mathbb{N}$ and $f \in L^1(X, \mu)$,

$$\int_X f \, d\mu = \int_X \frac{1}{N+1} \sum_{y \in \mathbb{N} \cdot x} f(y)\rho_x(y) \, d\mu(x).$$

Proof. Observing that $\sum_{y \in \mathbb{N} \cdot x} f(y)\rho_x(y) = \sum_{n=0}^{N} P_T^n f(x)$, the statement follows from the fact that $P_T^n$ is mean-preserving (Lemma 3.4). □

Recall that for any nonempty $J \subseteq \Gamma^\leq \mathbb{N}$ and for all $f \in L^1(X, \mu)$, $\hat{J} f(x)$ is the weighted-average over the sets $J \cdot x$, for $x \in X$, i.e.

$$\hat{J} f(x) = A^p_f [J \cdot x].$$

Corollary 3.7 shows that $\hat{J} f$ is well-defined. Moreover:

**Corollary 3.9.** Let $n \in \mathbb{N}$ and $J \subseteq \Gamma^\leq \mathbb{N}$ be such that the function $x \mapsto \rho_x(J \cdot x)$ is bounded below by some $w > 0$. For any $1 \leq p \leq \infty$, $\hat{J}$ is a bounded operator on $L^p(X, \mu)$ with operator norm $\| \hat{J} \|_p \leq \left( \frac{n+1}{w} \right)^{1/p}$.

Proof. For $p = \infty$, the statement is obvious, so suppose $p < \infty$. For any $f \in L^p(X, \mu)$,

$$\| \hat{J} f \|_p^p = \int_X |A^p_f [J \cdot x]|^p \, d\mu(x) \leq \int_X A^p_{|f|^p} [J \cdot x] \, d\mu(x) \leq \int_X \frac{1}{\rho_x(J \cdot x)} \sum_{y \in J \cdot x} |f(y)|^p \rho_x(y) \, d\mu(x) \leq \frac{n+1}{w} \int_X \frac{1}{n+1} \sum_{y \in \mathbb{N} \cdot x} |f(y)|^p \rho_x(y) \, d\mu(x)$$

$$\leq \left[ \text{Jensen’s inequality} \right] \leq \int_X A^p_{|f|^p} [J \cdot x] \, d\mu(x) \leq \int_X \frac{1}{\rho_x(J \cdot x)} \sum_{y \in J \cdot x} |f(y)|^p \rho_x(y) \, d\mu(x) \leq \frac{n+1}{w} \int_X \frac{1}{n+1} \sum_{y \in \mathbb{N} \cdot x} |f(y)|^p \rho_x(y) \, d\mu(x)$$

$$\leq \frac{n+1}{w} \| f \|_p^p.$$ □

### 3.C. Local-global bridge for pmp $T$

Throughout this subsection, we assume in addition that $T$ preserves the measure $\mu$.

**Lemma 3.10.** For a.e. $x \in X$, $\rho_x(T^{-n}x) = 1$ for each $n \in \mathbb{N}$; in other words, $P_T^n 1 = 1$. In particular, $\rho_x(T^N \cdot x) = N + 1$ for each $N \in \mathbb{N}$.

Proof. The second statement is immediate from the first. For the first statement, we may switch the quantifiers, i.e. prove that for each $n \in \mathbb{N}$, the formula holds a.e. By Observation 3.6, it is enough to prove the statement for $n = 1$.

To this end, we show that for each $\varepsilon > 0$, the set $Z_\varepsilon := \{ x \in X : \rho_x(T^{-1}(x)) > 1 + \varepsilon \}$ is null. This implies that $\{ x \in X : \rho_x(T^{-1}(x)) > 1 \}$ is null and an analogous argument shows that $\{ x \in X : \rho_x(T^{-1}(x)) < 1 \}$
Throughout, let\( (X, \mu) \) be a standard probability space and let \( T : X \to X \) be a countable-to-one \( \mu \)-preserving Borel transformation, so by Assumption 2.5, \( T \) is surjective and \( E_T \) is null-preserving. Let \( \rho : E_T \to \mathbb{R}^+ \) be the Radon–Nikodym cocycle with respect to \( \mu \). Finally, let \( \Gamma \) be a complete set of Borel partial right-inverses of \( T \).

**Remark 3.12**

This implies that \( Z_\varepsilon \) is null, as desired.

Corollary 3.11 and Lemmas 3.8 and 3.10 together immediately yield:

**Corollary 3.11** *(Local-global bridge for pmp \( T \)).* Let \( T \) be a countable-to-one pmp Borel transformation on \( (X, \mu) \). For any \( N \in \mathbb{N} \) and \( f \in L^1(X, \mu) \), \( A_f^0[\mathbb{R}_+^N \cdot x] < \infty \) a.e. and

\[
\int_X f(x) \, d\mu(x) = \int_X A_f^0[\mathbb{R}_+^N \cdot x] \, d\mu(x).
\]

**Remark 3.12.** Corollary 3.11 fails when we replace \( \mathbb{R}_+^N \cdot x \) with arbitrary subsets of the back-orbit of \( x \), even trees (as in Section 4.A). We may observe this by looking at indicator functions of the images of right-inverses \( \gamma \) of \( T \). For example, if \( T \) is the shift map on \((2^\mathbb{N}, \{\{1, 2\}\}^\mathbb{N})\), and \( f := 1_{\{x(0) = 0\}} \), then \( \int f \, d\mu = \frac{1}{2} \), but \( \int A_f^0[\{x, 0^\infty\}] \, d\mu = \frac{2}{3} \).

**Corollary 3.13.** The operator \( P_T \) is Markov and an \( L^p \)-contraction for all \( 1 \leq p \leq \infty \).

**Proof.** \( P_T \) is Markov by Lemmas 3.4 and 3.10, so it is an \( L^\infty \)-contraction (Proposition 3.2). Furthermore, Lemma 3.10 makes Jensen’s inequality applicable for \( 1 \leq p < \infty \), yielding \( (P_T|f|)^p \leq P_T|f|^p \), so Proposition 3.1 applies. \( \square \)

It is convenient to define the weighted averages over the sets \( \mathbb{R}_+^N \) as operators: for each \( N \in \mathbb{N} \), \( f \in L^1(X, \mu) \), and \( x \in X \), define

\[
\triangle_T,N f(x) := A_f^0[\mathbb{R}_+^N \cdot x].
\]

**Lemma 3.10** and Observation 3.6 immediately imply:

**Corollary 3.15.** \( \triangle_T,N = \frac{1}{N+1} \sum_{n=0}^N P_T^n = \frac{1}{N+1} \sum_{n=0}^N P_T^n \) for each \( N \in \mathbb{N} \).

This and Corollary 3.13 imply:

**Corollary 3.16.** For each \( N \in \mathbb{N} \), the operator \( \triangle_T,N \) is Markov and an \( L^p \)-contraction for all \( 1 \leq p \leq \infty \).

4. **The Tiling Property and the Backward Ergodic Theorem**

Throughout, let \( (X, \mu) \) be a standard probability space and let \( T : X \to X \) be a countable-to-one \( \mu \)-preserving Borel transformation, so by Assumption 2.5, \( T \) is surjective and \( E_T \) is null-preserving. Let \( \rho : E_T \to \mathbb{R}^+ \) be the Radon–Nikodym cocycle with respect to \( \mu \). Finally, let \( \Gamma \) be a complete set of Borel partial right-inverses of \( T \).
4.A. The tiling property

We now prove the needed tiling property and deduce our backward pointwise ergodic theorem (Theorem 4.7) from it.

For a set $S$ (which will typically be a countable complete set of Borel right-inverses of a transformation $T$), let $T_S \subseteq \mathcal{P}(S^{<\mathbb{N}})$ be the set of nonempty set-theoretic (but right-rooted) trees on $S$ of finite height, where $S^{<\mathbb{N}}$ is the set of all finite sequences of elements of $S$. More precisely, for each $\tau \subseteq S^{<\mathbb{N}},$

$$\tau \in T_S :\iff \tau \text{ is of finite height, i.e. } \tau \subseteq \mathcal{P}(S^{\leq n}) \text{ for some } n,$$

$$\tau \text{ contains the empty word } \emptyset,$$

and for each $t_1, t_2 \in S^{<\mathbb{N}},$ if $t_1 t_2 \in \tau,$ then $t_2$ is also in $\tau.$

For each $\tau \in T_S$, denote by $h(\tau)$ the height of the tree $\tau$, i.e. the least $n \in \mathbb{N}$ such that $\tau \subseteq S^{\leq n}.$

**Lemma 4.1** (Tiling property). Let $T$, $\rho$, and $\Gamma$ be as above. Then for any measurable function $x \mapsto \tau_x : X \to T$ and $\varepsilon > 0$ there is $N \in \mathbb{N}$ and a set $X' \subseteq X$ of measure $\geq 1 - \varepsilon$ such that for all $x \in X'$, the complete tree $\triangledown_N^T \cdot x$ can be covered, up to $\varepsilon$-fraction of its $\rho_x$-weight, by disjoint tiles of the form $\tau_y \cdot y$.

More precisely, for every $x \in X'$ there is a subset $S_x$ of $\triangledown_N^T \cdot x$ with $\rho_x(S_x) \geq (1 - \varepsilon)\rho_x(\triangledown_N^T \cdot x)$ that is partitioned into sets of the form $\tau_y \cdot y$ for $y \in X$.

**Proof.** Let $L$ be large enough so that the set $B := \{x \in X : h(\tau_x) \geq L\}$ has measure less than $\frac{\varepsilon^2}{2}.$ Fix $N$ large enough so that $\frac{L}{N} < \frac{\varepsilon}{2}.$ By Lemma 3.10, we may assume that for each $x \in X$ and $n \in \mathbb{N}, \sum_{y \in T^n(x)} \rho_x(y) = 1,$ so

$$\rho_x(\triangledown_N^{T-L} \cdot x) = N - L > (1 - \frac{\varepsilon}{2})N = (1 - \frac{\varepsilon}{2})\rho_x(\triangledown_N^T \cdot x).$$

Thus, there is no harm in leaving $\triangledown_N^T \cdot x \setminus \triangledown_N^{T-L}(x)$ untiled.

We claim that for all but less than $\varepsilon$-measured set of $x \in X$, less than $\frac{\varepsilon}{2}$ $\rho_x$-fraction of $y \in \triangledown_N^T \cdot x$ are in $B,$ i.e. the set $C := \{x \in X : A_{LB}^\rho[\triangledown_N^T \cdot x] \geq \frac{\varepsilon}{2}\}$, has measure less than $\varepsilon.$ Indeed:

$$\frac{\varepsilon^2}{2} > \mu(B) = \int_X 1_B(x) \, d\mu(x)$$

$$\left[\text{by Corollary 3.11}\right] = \int_X A_{LB}^\rho[\triangledown_N^T \cdot x] \, d\mu(x)$$

$$\geq \int_C A_{LB}^\rho[\triangledown_N^T \cdot x] \, d\mu(x)$$

$$\geq \frac{\varepsilon}{2} \mu(C).$$

So we just need to fix $x \in X \setminus C$ and tile the set $\triangledown_N^T \cdot x$ up to an $\varepsilon \rho_x$-fraction. We do this by the following straightforward algorithm (see Fig. 3 in Section 1.C): if there is $n \leq N$ with a $y \in T^{-n}(x)$ that is not covered by a tile yet and $\tau_y \cdot y \subseteq \triangledown_N^T \cdot x,$ take the least such $n$ and for each such $y \in T^{-n}(x)$, place the tiles $\tau_y \cdot y$; repeat this until there is no such $n.$ Once this process terminates, the only points that are not covered by a tile must belong to either $B$ or $\triangledown_N^T \cdot x \setminus \triangledown_N^{T-L} \cdot x,$ so they comprise at most $\varepsilon \rho_x$-fraction of $\triangledown_N^T \cdot x.$

$\square$
4.B. Poincaré recurrence

Here, we recall some ergodic-theoretic terminology and basic facts, which are used in Section 4.C. A set \( W \subseteq X \) is called \( T \)-wandering if the sets \( T^{-n}(W), n \in \mathbb{N} \), are pairwise disjoint. Because the measures of the sets \( T^{-n}(W) \) are all equal and \( \mu \) is a probability measure, we have:

**Observation 4.2.** \( T \) is conservative, i.e., every \( T \)-wandering measurable set is null.

For a set \( U \subseteq X \), let
\[
[U]_T^+ := \bigcup_{n \in \mathbb{N}^+} T^n(U), \quad [U]_{T^{-1}}^+ := \bigcup_{n \in \mathbb{N}^+} T^{-n}(U), \quad [U]_{T^{-1}} := \bigcup_{n \in \mathbb{N}} T^{-n}(U).
\]

Abusing notation, we write \([x]_T^+\) when \( U = \{x\}\). Note that for any set \( U \), the set \( V := X \setminus [U]_{T^{-1}} \) is closed under \( T \), i.e. \( T(V) \subseteq V \).

Call a set \( U \subseteq X \) \( T \)-recurrent if for every \( x \in U \), \([x]_T^+ \cap U \neq \emptyset \); equivalently, \( U \setminus [U]_T^+ = \emptyset \). Consequently, we say that a set \( U \subseteq X \) is \( \mu \)-nowhere \( T \)-recurrent if it does not admit a \( T \)-recurrent subset of positive measure.

**Lemma 4.3** (Poincaré recurrence). Every Borel set \( U \subseteq X \) is \( T \)-recurrent a.e. In fact, there is a subset \( U' \subseteq U \) that is conull in \( U \) such that for every \( x \in [U']_{E_T} \), \([x]_T^+ \cap U' \neq \emptyset \).

**Proof.** The set \( U'' := \{x \in U : [x]_T^+ \cap U = \emptyset \} \) is \( T \)-wandering and hence null. Then \([U'']_{T^{-1}} \) is also null, and it is easy to check that \( U' := U \setminus [U'']_{T^{-1}} \) is as desired. \(\square\)

In light of Lemma 4.3, we may assume that all positively measured sets that come up are \( T \)-recurrent.

**Lemma 4.4.** Every Borel set \( U \subseteq X \) with the property that \( T(U) \subseteq U \) is such that \([U]_{E_T} = U \) off of a \( T \)-invariant null set.

**Proof.** Put \( V := [U]_{E_T} \setminus U \). Then since \( T(U) \subseteq U \), \( V \) is \( \mu \)-nowhere \( T \)-recurrent because for all \( x \in V \), there are only finitely many \( n \) with \( T^n(x) \in V \). Hence, by Lemma 4.3, \( V \) is null, and since \( E_T \) is null-preserving, so is \([V]_{E_T} \). Therefore, \( U \setminus [V]_{E_T} \) is \( E_T \)-invariant off of the invariant null set \([V]_{E_T} \). \(\square\)

We say that the periodic part of \( T \) is the subset \( \{x \in X : \exists n, m \in \mathbb{N} : T^n(x) = T^m(x)\} \).

**Lemma 4.5.** \( T \) is bijective on its periodic part off of a null set.

**Proof.** Let \( V \) be the periodic part of \( T \), and let \( U := \{x \in X : \exists n \in \mathbb{N} \setminus \{0\} : T^n(x) = x\} \). Then \([U]_{E_T} = V \). Notice that \( V \setminus U \) is nowhere \( T \)-recurrent, hence null, and that \( T|_U \) is bijective. \(\square\)

4.C. Backward ergodic theorem along trees

We think of \( \text{graph}(T) \) as a directed graph on \( X \), where \( X \) is the set of vertices and \( \text{graph}(T) \) is the set of directed edges. For \( x \in X \), let \( \mathcal{T}_x \) denote the collection of subtrees of \( \text{graph}(T) \) of finite height rooted at \( x \) and directed towards \( x \) (see Fig. 2 in Section 1.A). More precisely, \( \tau_x \in \mathcal{T}_x \) exactly when the following three conditions hold:

(i) \( \tau_x \subseteq \bigcup_{i=0}^{n} T^{-i}(x) \) for some \( n \in \mathbb{N} \);
(ii) \( x \in \tau_x \);
(iii) if \( y \in \tau_x \) and \( y \neq x \) then \( T(y) \in \tau_x \).

Notice that if \( \Gamma \) is a complete set of Borel partial right-inverses of \( T \), then \( \tau_x \in \mathcal{T}_x \) exactly when \( \tau_x = \tau \cdot x \) for some \( \tau \in \mathcal{T}_{\Gamma} \). With this in mind, we use graph-theoretic trees \( (\tau_x \subseteq \text{graph}(T)) \) and set-theoretic trees \( (\tau \subseteq S^\mathbb{N}) \) interchangeably in the rest of the paper.
Lemma 4.6. Let $T$ and $\rho$ be as above, and additionally assume that $T$ is aperiodic. Then for any $f \in L^1(X, \mu)$, for a.e. $x \in X$, for all $\tau_x \in \mathcal{T}_x$, we have that $A^\rho_{f\mid \tau_x} < \infty$ and the functions

$$f^*(x) := \limsup_{\rho_x(\tau_x) \to \infty} A^\rho_{f\mid \tau_x} \text{ and } f_*(x) := \liminf_{\rho_x(\tau_x) \to \infty} A^\rho_{f\mid \tau_x},$$

where $\tau_x$ ranges over $\mathcal{T}_x$,

are $T$-invariant a.e. (i.e. off of an invariant null set).

Proof. That $A^\rho_{f\mid \tau_x} < \infty$ for a.e. $x \in X$ and each $\tau_x \in \mathcal{T}_x$, is by Corollary 3.7 because $f \in L^1(X, \mu)$. Thus, we may assume without loss of generality that this holds for all $x \in X$, and in particular, $A^\rho_{f\mid \tau_x}$ is well-defined for all $x \in X$.

As for invariance, it is enough to show that $f^*$ is $T$-invariant as $f_* = -(−f)^*$. For that, it is enough to show that for each $a \in \mathbb{Q}$, the set

$$X_{\geq a} := \{x \in X : f^*(x) \geq a\}$$

is $T$-invariant, modulo a null set. Fix $a \in \mathbb{Q}$. By Lemma 4.4, we just need to show $T(X_{\geq a}) \subseteq X_{\geq a}$.

Intuitively, since $y$ is only one point, we can add it to heavy trees $\tau \in \mathcal{T}_x$ without having much impact on the weighted average, hence $f^*(x) \leq f^*(y)$. To see this more formally, recall that $f^*(x) \geq a > −\infty$ and fix an arbitrary real $S < f^*(x)$, weight $w > 0$, and error $\varepsilon > 0$. It is enough to find $\tau_y \in \mathcal{T}_y$ such that $\rho_x(\tau_y) \geq \rho_x(y) \cdot w$ (equivalently, $\rho_y(\tau_y) > w$) and $A^\rho_{f\mid \tau_y} \geq S − \varepsilon$.

To this end, take $\tau_x \in \mathcal{T}_x$ of large enough $\rho_x$-weight so that $\rho_x(\tau_x) \geq \rho_x(y) \cdot w$ and

$$|S| + |f(y)| \leq \frac{\rho_x(\tau_x)}{\rho_x(y)} \cdot \varepsilon$$

and $A^\rho_{f\mid \tau_x} \geq S$. Putting $\tau_y := \tau_x \cup \{y\}$ (hence, $\tau_y \in \mathcal{T}_y$), we have:

$$A^\rho_{f\mid \tau_y} = \left(1 − \frac{\rho_x(y)}{\rho_x(\tau_y)}\right) \cdot A^\rho_{f\mid \tau_x} + \frac{\rho_x(y)}{\rho_x(\tau_y)} \cdot f(y)$$

$$\geq S − \frac{\rho_x(y)}{\rho_x(\tau_y)} \cdot |S| − \frac{\rho_x(y)}{\rho_x(\tau_y)} \cdot |f(y)|$$

$$= S − \frac{\rho_x(y)}{\rho_x(\tau_y)} \cdot (|S| + |f(y)|)$$

$$\geq S − \frac{\rho_x(y)}{\rho_x(\tau_y)} \cdot \frac{\rho_x(\tau_x)}{\rho_x(y)} \cdot \varepsilon$$

$$\geq S − \varepsilon. \quad \square$$

Theorem 4.7 (Backward pointwise ergodic along trees). Let $T$ be an aperiodic countable-to-one pmp Borel transformation on a standard probability space $(X, \mu)$, so by Assumption 2.5, $T$ is surjective and $E_T$ is null-preserving. Let $(x, y) \mapsto \rho_x(y) : E_T \to \mathbb{R}^+$ be the Radon–Nikodym cocycle of $E_T$ with respect to $\mu$. For every $f \in L^1(X, \mu)$ and for a.e. $x \in X$, we have $A^\rho_{f\mid \tau_x} < \infty$ for all $\tau_x \in \mathcal{T}_x$, and

$$A^\rho_{f\mid \tau_x} \to \overline{f}(x) \text{ as } \rho_x(\tau_x) \to \infty,$$

where $\tau_x$ ranges over $\mathcal{T}_x$, and $\overline{f}$ is the conditional expectation of $f$ with respect to the $\sigma$-algebra of $T$-invariant Borel sets.

Proof. By Lemma 4.6, for a.e. $x \in X$ and each $\tau_x \in \mathcal{T}_x$, $A^\rho_{f\mid \tau_x} < \infty$, and $f^*$ and $f_*$ are $T$-invariant. By replacing $f$ with $f − \overline{f}$, we may assume without loss of generality that $\overline{f} = 0$. We will show that $f^* \leq 0$ a.e., and an analogous argument shows $f_* \geq 0$ a.e.

Assume by way of contradiction that $f^* > 0$ on a positively measured (necessarily $T$-invariant) set, restricting to which we might as well assume that $f^*(x) > 0$ for all $x \in X$. Put $g := \min \left\{ \frac{f^*}{2}, 1 \right\}$,
so $0 < g \leq 1$, and $g \in L^1(X, \mu)$. Put $c := \int g \, d\mu > 0$. Fix a complete set $\Gamma$ of Borel partial right-inverses of $T$, and let $\mathcal{T}_\Gamma \subseteq \mathcal{T}_T$ denote the collection of finite trees in $\mathcal{T}_T$. Notice that for each $x \in X$, the quantity $\limsup_{\tau \to \infty} A^\rho_T[\tau \cdot x]$ does not change if we restrict the range of $\tau$ to only $\mathcal{T}_\Gamma$ since $\sum_{y \in \tau_x} f(y) \rho_x(y)$ converges absolutely for each $\tau_x \in \mathcal{T}_x$.

Fix an enumeration $\{\tau_n\}$ of $\mathcal{T}_\Gamma$, and define $\ell : X \to \mathcal{T}_\Gamma$ by $x \mapsto$ the least (according to the enumeration) $\tau \in \mathcal{T}_\Gamma$ such that $A^\rho_T[\tau \cdot x] > g(x)$ (equivalently, $A^\rho_T[\tau \cdot x] > 0$).

Fix $\delta > 0$ small enough so that for any measurable $Y \subseteq X$, $\mu(Y) < \delta$ implies $\int_Y (f - g) \, d\mu > -\frac{1}{\delta}$, and let $M \in \mathbb{N}$ be large enough so that the set $Y := f^{-1}(-M, \infty)$ has measure at least $1 - \delta$.

The tiling property (Lemma 4.1) applied to the function $\ell$ with $\varepsilon := \frac{1}{2(M + 1)}$ gives $N \in \mathbb{N}$ such that $\mu(Z) \geq 1 - \varepsilon$, where $Z$ is the set of all $x \in X$ such that at least $1 - \varepsilon \rho_x$-fraction of $v^n_T \cdot x$ is partitioned into sets of the form $\ell(y) \cdot y$.

Claim. $A^\rho_{\mathbb{I}_Y(f-g)}[v^n_T \cdot x] \geq -(M + 1)\varepsilon$ for each $x \in Z$.

Proof of Claim. By the definition of $Z$, on a subset $B \subseteq v^n_T \cdot x$ that occupies at least $1 - \varepsilon \rho_x$-fraction of $v^n_T \cdot x$, the $\rho$-average of $f - g$ is positive, and hence that of $\mathbb{I}_Y(f - g)|_B$ is non-negative. On the remaining set $v^n_T \cdot x \setminus B$, the function $\mathbb{I}_Y(f - g)$ is at least $-(M + 1)$, by the definition of $Y$, and hence so is its $\rho$-weighted average. Thus, the $\rho$-weighted average of $\mathbb{I}_Y(f - g)$ on the entire $v^n_T \cdot x$ is at least $-(M + 1)\varepsilon$. $\blacksquare$

Now we compute using this claim and Corollary 3.11:

$$\int_Y (f - g) \, d\mu = \int_X A^\rho_{\mathbb{I}_Y(f-g)}[v^n_T \cdot x] \, d\mu(x)$$
$$= \int_Z A^\rho_{\mathbb{I}_Y(f-g)}[v^n_T \cdot x] \, d\mu(x) + \int_{X \setminus Z} A^\rho_{\mathbb{I}_Y(f-g)}[v^n_T \cdot x] \, d\mu(x)$$
$$\geq - (M + 1)\varepsilon - (M + 1)\varepsilon = -2(M + 1)\varepsilon = -\frac{c}{3}.$$

This gives a contradiction:

$$0 = \int_X f \, d\mu = \int_X (f - g) \, d\mu + \int_X (f - g) \, d\mu$$
$$= c + \int_Y (f - g) \, d\mu + \int_{X \setminus Y} (f - g) \, d\mu$$
$$> c - \frac{c}{3} - \frac{c}{3} > 0. \quad \Box$$

Remark 4.8. It is worth explicitly pointing out particular sequences $\{\tau_n\}$ of trees in $\mathcal{T}_\Gamma$ such that $\rho_x(\tau_n \cdot x) \to \infty$ regardless of the base point $x \in X$. Such is the sequence $v^n_T$; indeed, by Lemma 3.10, $\rho_x(v^n_T \cdot x) = n + 1$ for a.e. $x \in X$. More generally, this is true for sequences $\{\tau_n\}$ of trees that contain shifted complete trees whose heights tend to infinity. By this we mean that there is a fixed word $t \in S^{<\mathbb{N}}$ such that $\tau_n$ contains the shifted complete tree $v^n_T t$, where $h_n \to \infty$.

Remark 4.9. By Lemma 4.5, $T$ is bijective on its periodic part $Y$ (mod null). By Assumption 2.5, we can assume $E_T|_Y$ is null-preserving and $T|_Y$ is bijective. Consequently, $E_T|_Y$ is measure-preserving, so the backward averages are unweighted (as in the standard pointwise ergodic theorem for $Z$). However, the only trees in each orbit of $T|_Y$ are paths of bounded length, so their weights do not tend to infinity. We can still reformulate the statement of our theorem with set theoretic trees instead (i.e. over $\tau \cdot x$ where $\tau \in \mathcal{T}_T$ for some complete set $\Gamma$ of Borel partial right-inverses of $T$) and the statement would still be true without the aperiodicity assumption.

If $T$ and $f$ are as in Theorem 4.7, then for a.e. $x \in X$, the set $\left\{ \frac{1}{n} \sum_{i<n} f(T^i(x)) : n \in \mathbb{N} \right\}$ of forward averages is bounded (since this is a convergent sequence). Looking backward, our Theorem 4.7
also says that the averages $A_f^p[\tau_x]$ over $\tau_x \in T_x$ converge as $\rho_x(\tau_x) \to \infty$, but there are infinitely-many trees $\tau_x \in T_x$ of bounded weight, so the mere convergence does not imply that the set 
\[ \left\{ A_f^p[\tau_x] : \tau_x \in T_x \right\} \] is bounded. Nevertheless, we show it is indeed bounded; in fact, the maximal ergodic theorem holds along backward trees (see [KP06] for the classical forward version). We use the boundedness in the proof of Theorem 1.1, but of course, the maximal ergodic theorem is interesting in its own right.

**Theorem 4.10** (Backward maximal ergodic theorem along trees). Let $T$ be as in Theorem 4.7. Let $f \in L^1(X, \mu)$, and define $f^*(x) := \sup_{\tau_x \in T_x} A_f^p[\tau_x]$ for each $x \in X$. Then for any $\lambda \in \mathbb{R}$,
\[ \int_{f^* > \lambda} f \, d\mu \geq \lambda \mu \left\{ f^* > \lambda \right\}. \]

In particular, $f^* < \infty$ a.e.

**Proof.** Fix $\lambda \in \mathbb{R}$ and let $Y := \{ x \in X : f^* > \lambda \}$. We will show $\int_Y f \, d\mu \geq \lambda \mu(Y)$. First note that this is equivalent to showing $\int_Y (f - \lambda) \, d\mu \geq -\varepsilon$ for arbitrary $\varepsilon > 0$.

For each $x \in Y$, let $\tau_x \in T_x$ be a minimal witness to $x$ being in $Y$, i.e., $A_f^p[\tau_x] > \lambda$, and no proper subtree of $\tau_x$ has this property. Hence, for any $y \in \tau_x$, $y$ is also in $Y$ (since $\tau_x \cap \bigcup_{i \in \mathbb{N}} T^{-i}(y)$ has average greater than $\lambda$ by the minimality of $\tau_x$). Then $A_{\Pi_Y(f - \lambda)}^p[\tau_x] = A_{(f - \lambda)}^p[\tau_x] > 0$ for each $x \in Y$. For $x \notin Y$, set $\tau_x := \{ x \}$.

The rest of the proof is morally the same as that of Theorem 4.7, so we will not provide all of the details. We may assume without loss of generality (by the same argument as in Theorem 4.7) that $f$ is bounded from below. We apply the tiling property (Lemma 4.1) to get $N$ large enough so that for each point $x$ in a set $Z$ of large measure (which will depend on the lower bound of $f$), we can tile most of $\mathbf{b}_T^N \cdot x$ with tiles of the form $\tau_y$, which, together with the lower bound for $f$ guarantees that $A_{\Pi_Y(f - \lambda)}^p[\mathbf{b}_T^N \cdot x] \geq -\frac{\varepsilon}{2}$. Hence, by the local-global bridge (Corollary 3.11),
\[ \int_Y (f - \lambda) \, d\mu = \int Z A_{\Pi_Y(f - \lambda)}^p[\mathbf{b}_T^N \cdot x] \, d\mu \]
\[ = \int Z A_{\Pi_Y(f - \lambda)}^p[\mathbf{b}_T^N \cdot x] \, d\mu + \int_{X \setminus Z} A_{\Pi_Y(f - \lambda)}^p[\mathbf{b}_T^N \cdot x] \, d\mu \]
\[ \geq -\frac{\varepsilon}{2} + \int_{X \setminus Z} A_{\Pi_Y(f - \lambda)}^p[\mathbf{b}_T^N \cdot x] \, d\mu. \]

By taking $Z$ to have arbitrarily large measure, we get $\int_Y (f - \lambda) \, d\mu \geq -\varepsilon$. \qed

**4.D. Convergence in $L^p$ along special sequences of trees**

Besides pointwise convergence, we also get convergence in $L^p$ along the sequence of complete trees $\mathbf{b}^n_T$. This is the content of Corollary 1.4, which we restate and prove here. We also remark afterwards that the theorem holds for other special sequences of trees as well.

Recall (Corollary 3.16) that for each $n$, the operator $\Delta_{T,n}$ on $L^1(X, \mu)$ defined by $\Delta_{T,n}f(x) := A_f^p[\mathbf{b}^n_T \cdot x]$ is a Markov operator, which is an $L^p$-contraction for all $1 \leq p < \infty$.

**Corollary 4.11** (Backward ergodic along complete trees). Let $T$ and $\rho$ be as in Theorem 4.7. For any $1 \leq p < \infty$ and $f \in L^p(X, \mu)$,
\[ \lim_{n \to \infty} \Delta_{T,n}f = \mathcal{F} \text{ a.e. and in } L^p, \]
where $\mathcal{F}$ is the conditional expectation of $f$ with respect to the $\sigma$-algebra of $T$-invariant Borel sets.

**Proof.** The pointwise convergence follows immediately from Theorem 4.7 by considering, for $x \in X$, the sequence $\mathbf{b}^n_T \cdot x$ of complete trees, recalling that by Lemma 3.10, $\rho_x(\mathbf{b}^n_T \cdot x) = n + 1 \to \infty$. 

19
If \( f \in L^\infty(X, \mu) \), then \( |\Delta_{T,n} f| \leq \|f\|_\infty \) for every \( n \in \mathbb{N} \), so by the dominated convergence theorem, \( \Delta_{T,n} f \) converges in \( L^p \) to \( \bar{f} \) as \( n \to \infty \).

For a general \( f \in L^p(X, \mu) \), let \( (f_k) \) be a sequence of bounded functions converging to \( f \) in \( L^p \). Fix \( \varepsilon > 0 \), and let \( k \) be large enough so that \( \|f - f_k\|_p < \frac{\varepsilon}{3} \). This implies, for all \( n \in \mathbb{N} \), that 

\[
\|\Delta_{T.n} f - \Delta_{T.n} f_k\|_p < \frac{\varepsilon}{3} \text{ and } \|\bar{f} - \bar{f}_k\|_p < \frac{\varepsilon}{3}
\]

because both \( \Delta_{T,n} \) and conditional expectation are \( L^p \)-contractions (Corollary 3.16 and [Dur19, Theorem 4.1.11]). Thus,

\[
\|\Delta_{T,n} f - \bar{f}\|_p \leq \|\Delta_{T,n} f - \Delta_{T,n} f_k\|_p + \|\Delta_{T,n} f_k - \bar{f}_k\|_p + \|\bar{f}_k - \bar{f}\|_p < \frac{\varepsilon}{3} + \|\Delta_{T,n} f_k - \bar{f}_k\|_p + \frac{\varepsilon}{3} < \varepsilon,
\]

for large enough \( n \) because we already know that \( \lim_{n \to \infty} \|\Delta_{T,n} f_k - \bar{f}_k\|_p = 0 \). \( \square \)

Recalling Proposition 3.5 and Corollary 3.15, we now restate Corollary 4.11 in terms of the adjoint \( K_T^* \) of the Koopman representation \( K_T \) of \( T \).

**Corollary 4.12.** Let \( T \) be as in Theorem 4.7. For any \( 1 \leq p < \infty \) and \( f \in L^p(X, \mu) \),

\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{i=0}^{n} (K_T^*)^i(f) = \bar{f} \text{ a.e. and in } L^p,
\]

where \( \bar{f} \) is the conditional expectation of \( f \) with respect to the \( \sigma \)-algebra of \( T \)-invariant Borel sets.

**Remark 4.13.** The mere convergence of the sequence of averages in Corollary 4.12 is implied by the Dunford–Schwartz ergodic theorem [DS56]:

**Theorem 4.14 (Dunford–Schwartz 1956).** If \( Q : L^1(X, \mu) \to L^1(X, \mu) \) is a non-negative \( L^1 \)-\( L^\infty \)-contraction\(^5\) on a probability space \((X, \mu)\), then for every \( f \in L^1(X, \mu) \), there exists a \( Q \)-invariant \( \bar{f} \in L^1(X, \mu) \) such that \( \frac{1}{n+1} \sum_{i=0}^{n} Q^i f \to \bar{f} \) as \( n \to \infty \) both a.e. and in \( L^1 \).

Indeed, by Corollary 3.13 and Proposition 3.5 \( Q := K_T^* \) is an \( L^1 \)-\( L^\infty \)-contraction, so the sequence \( \frac{1}{n} \sum_{i=0}^{n} (K_T^*)^i(f) \) converges a.e. and in \( L^1 \) to a \( K_T^* \)-invariant function \( \bar{f} \in L^1(X, \mu) \). However, the \( K_T^* \)-invariance of \( \bar{f} \) does not directly imply that \( \bar{f} \) is \( T \)-invariant (and hence one cannot conclude that \( \bar{f} \) is the conditional expectation \( \bar{f} \) with respect to the \( T \)-invariant \( \sigma \)-algebra of Borel sets). The main new content of Corollary 4.12 is that \( \bar{f} \) is indeed \( T \)-invariant. To prove this, we rephrased it in terms of averages over complete backward trees (Corollary 4.11) and proved the stronger statement of Theorem 4.7 that the averages over arbitrary backward trees converge.

Lastly, we discuss a more general version of Corollary 4.11, replacing the sequence of complete backward trees with that of “fat” backward trees. Let \( T \) and \( \rho \) be as in Theorem 4.7 and let \( \Gamma \) be a complete set of Borel partial right-inverses of \( T \). For \( c > 0 \), call a tree \( \tau \subseteq T \) \( c \)-fat for the cocycle \( \rho \) if \( \rho_x(\tau) \geq c \) for a.e. \( x \in X \). In particular, \( \Gamma^{\leq n} \) is \( 1 \)-fat and Corollary 4.11 is about the averaging operators \( \Gamma^{\leq n} \), defined in Section 2.C.

**Corollary 4.15 (Backward ergodic theorem along fat trees).** Let \( T \), \( \rho \), and \( \Gamma \) be as above. Let \( c > 0 \) and let \( (\tau_n) \) be any sequence of trees in \( T \) that are \( c \)-fat for \( \rho \) and such that \( h(\tau_n) \to \infty \) as \( n \to \infty \). Then for any \( 1 \leq p < \infty \) and \( f \in L^p(X, \mu) \),

\[
\lim_{n \to \infty} \tilde{\tau}_n f = \bar{f} \text{ a.e. and in } L^p,
\]

where \( \bar{f} \) is the conditional expectation of \( f \) with respect to the \( \sigma \)-algebra of \( T \)-invariant Borel sets.

**Proof.** Replacing the operators \( \Delta_{T,n} \) with \( \tilde{\tau}_n \), the argument is word-for-word the same as for Corollary 4.11, except that the operators \( \tilde{\tau}_n \) may not be \( L^p \)-contractions, but by Corollary 3.9, their norms are uniformly bounded by \( c \) (independent of \( n \)), which is all the argument needs. \( \square \)

---

\(^5\)This means both an \( L^1 \)-contraction and an \( L^\infty \)-contraction.
Question 4.16. For a complete set $\Gamma$ of Borel partial right-inverses of $T$, for which sequences $(\tau_n)$ of trees in $\mathcal{T}_T$ do we have convergence in $L^p$ of the $\rho$-weighted averages over $\tau_n \cdot x$?

5. Applications to Shift Maps

An example of a countable-to-one Borel transformation is the shift map $\sigma : S^N \rightarrow S^N$, for some countable set $S$, where $\sigma(x) := (x(1 + n))_{n \in \mathbb{N}}$. See Fig. 4 for the depiction of $\sigma$ for $S := 2 := \{0, 1\}$.

![Figure 4. Shift on $2^N$](image)

With the exception of the first example, the measures on $S^N$ discussed in this section will be Markov measures. Indeed, Markov measures on the state space $S$ provide a large class of Borel probability measures on $S^N$, including shift-invariant ones (which are exactly those with stationary initial distribution). Thus, Theorem 4.7 says that averaging a function while walking backward in the directions according to a tree $\tau \in \mathcal{T}_S$ approximates the conditional expectation of the function with respect to the $\sigma$-algebra of shift-invariant Borel sets (Fig. 4 depicts all backward walks of length 3 for $S := \{0, 1\}$).

5.A. Example: the Gauss map

Let $X := (0, 1]$ and $\mu$ be the Gauss measure, i.e., $d\mu := \frac{1}{\log 2(1 + x)} d\lambda$, where $\lambda$ is Lebesgue measure. Let $T : X \rightarrow X$ be the Gauss map $x \mapsto \frac{1}{2} x \mod 1$. For $x \in \mathbb{N}$, we denote by $[x]$ the real in $(0, 1]$ whose continued fraction expansion is $x$. The map $x \mapsto [x] : \mathbb{N} \rightarrow (0, 1]$ is an equivariant isomorphism of the shift $\sigma$ on $\mathbb{N}$ and the Gauss map $T$ on $(0, 1]$. In particular, $T$ is countable-to-one. Moreover, $T$ is $\mu$-preserving and ergodic (see [Kea95]), so Theorems 4.7 and 4.10 and Corollary 4.11 apply. For concrete applications of this theorem, it is useful to have an explicit formula for the Radon–Nikodym cocycle of $E_T$ with respect to the Gauss measure $\mu$. For each $j \in \mathbb{N}^+$ and $x \in \mathbb{N}$, let

$$\rho_{[x]}([j, x]) := \frac{1 + [x]}{([x] + j)([x] + j + 1)} = (1 + [x])[j, x][j + 1, x].$$

This induces a cocycle on $E_T$ via the cocycle identity. One can simply check that this fits the definition of the Radon–Nikodym cocycle of $E_T$ with respect to $\mu$. In particular, the adjoint $K_T^*$ of the Koopman representation is given by the formula:

$$K_T^* f(x) = \sum_{j \in \mathbb{N}^+} f([j, x])(1 + [x])[j, x][j + 1, x]$$

---

6This formula was obtained by Federico Rodriguez-Hertz by change of variable trickery.
for $f \in L^1(X, \mu)$ and $x \in X$ (see Proposition 3.5). Furthermore, the averages of iterates of $K_T^*$ converge to the expectation a.e. and in $L^p$ for all $p \geq 1$ (see Corollary 4.12).

5.B. Preliminaries on Markov measures

Here we give some basics of Markov measures, referring the reader to [Dur19] for a more comprehensive exposition of this subject.

Let $S$ be a countable discrete state space. For $w \in S^{<\mathbb{N}}$, let $|w|$ denote the length of $w$, and for a finite or infinite word $w' \in S^{<\mathbb{N}} \cup S^{\mathbb{N}}$, let $w \cdot w'$ denote the concatenation of $w$ and $w'$ (i.e. the word $w$ followed by $w'$), and for $J \subseteq S^{<\mathbb{N}}$, let $J^{-} := \{ v^{-} w : v \in J \}$.

An $S \times S$ (row) stochastic matrix $P$ is called irreducible if for each $i, j \in S$ there is $n \geq 1$ such that the $(i, j)$ entry of $P^n$ is positive. Call a probability distribution $\pi$ on $S$ stationary for the matrix $P$ if treating $\pi$ as a row-vector, we have $\pi P = \pi$.

The Markov measure on $S^{<\mathbb{N}}$ with transition matrix $P$ and initial distribution $\pi$ is the measure $m$ on $S^{<\mathbb{N}}$ defined by

$$m(w) := \pi(w(0)) \cdot P(w(0), w(1)) \cdot \cdots \cdot P(w(\ell - 2), w(\ell - 1)),$$

for each nonempty word $w \in S^{<\mathbb{N}}$, and $m(\emptyset) := 1$. Note that $m$ is a probability distribution on $S^n$ for each $n \geq 0$. We say that $m$ is irreducible (resp. stationary) if $P$ is irreducible (resp. $\pi$ is a stationary distribution for $P$).

Assumption 5.1. We assume throughout that the initial distribution $\pi$ of every Markov measure is positive (i.e. all of its entries are positive).

Moving to infinite words, we equip $S^{\mathbb{N}}$ with the standard Borel structure induced by the product topology, where $S$ is discrete. In particular, the cylindrical sets

$$[w] := \{ x \in S^{\mathbb{N}} : w \text{ is an initial subword of } x \},$$

$w \in S^{<\mathbb{N}}$, are clopen and form a basis for the topology. Any Markov measure $m$ on $S^{<\mathbb{N}}$ induces a probability measure $P_m$ on $S^{\mathbb{N}}$ uniquely defined by $P_m[w] := m(w)$ for each $w \in S^{<\mathbb{N}}$. We also refer to $P_m$ as a Markov measure on $S^{\mathbb{N}}$.

The following proposition records the basic connections between the properties of $m$ and $P_m$ that we use in our arguments; the proofs of these connections are standard, see [Dur19, 5.5].

Proposition 5.2. Let $m$ be a Markov chain on $S$ and let $\sigma$ denote the shift map on $S^{\mathbb{N}}$.

(a) If the initial distribution of $m$ is positive (Assumption 5.1), then $\sigma$ is $P_m$-null-preserving if and only if the transition matrix of $m$ does not have a zero column.

(b) $\sigma$ is $P_m$-preserving if and only if $m$ is stationary.

(c) $\sigma$ is $P_m$-ergodic if $m$ is irreducible and its transition matrix admits a stationary distribution.

5.C. Backward pointwise ergodic theorem for Markov measures

If a Markov measure $m$ has positive initial distribution (Assumption 5.1) and its transition matrix has no zero columns, then the shift map $\sigma$ is $P_m$-null-preserving (Proposition 5.2(a)). In fact, $E_\sigma$ is null-preserving on the conull set

$$Y := \{ x \in S^{\mathbb{N}} : P(x(n), x(n + 1)) > 0 \text{ for all } n \in \mathbb{N} \}.$$

On this conull set $Y$, we explicitly calculate the Radon–Nikodym cocycle of $E_\sigma$ with respect to $P_m$.

\footnote{A square matrix with nonnegative entries whose rows add up to 1.}
**Proposition 5.3.** Let \( m \) be a Markov chain on \( S \) whose initial distribution is positive (Assumption 5.1) and whose transition matrix has no zero columns. Then the Radon–Nikodym cocycle of \( E_\sigma \) with respect to \( \mathbb{P}_m \) is given by \( \rho_{m(x)}(x) = \frac{m(x(0) \cdot x(1) \cdot \ldots \cdot x(n))}{m(x(n))} \) for all \( n \in \mathbb{N} \) and a.e. \( x \in S^\mathbb{N} \).

**Proof.** To see that \( \rho \) is the Radon–Nikodym cocycle with respect to \( \mathbb{P}_m \), it suffices to check that for all \( w \in S^\mathbb{N} \) and \( i \in S \) with \( m(i \cdot w) > 0 \), we have

\[
\mathbb{P}_m(i \cdot w) = \int_{[w]} \rho_x(i \cdot x) \, d\mathbb{P}_m(x).
\]

Let \( P \) and \( \pi \) be the transition matrix and the initial distribution of \( m \). If \( w \neq \emptyset \), then \( \rho_x(i \cdot x) = \frac{\pi(i)}{\pi(w(0))} P(i, w(0)) \) a.e. \( x \in [w] \). Hence,

\[
\int_{[w]} \rho_x(i \cdot x) \, d\mathbb{P}_m(x) = \frac{\pi(i)}{\pi(w(0))} \cdot \int_{[w]} P(i, w(0)) \, d\mathbb{P}_m(x) = \mathbb{P}_m([i \cdot w]).
\]

Lastly, if \( w = \emptyset \), then

\[
\int_{[w]} \rho_x(i \cdot x) \, d\mathbb{P}_m(x) = \int_{S^\mathbb{N}} \rho_x(i \cdot x) \, d\mathbb{P}_m(x) = \sum_{j \in S} \int_{[j]} \rho_x(i \cdot x) \, d\mathbb{P}_m(x) = \sum_{j \in S} \frac{\pi(i)}{\pi(j)} P(i, j) \mathbb{P}_m([j]) = \pi(i) \sum_{j \in S} P(i, j) = \pi(i) = \mathbb{P}_m([i]) = \mathbb{P}_m([i \cdot w]).
\]

We now state Theorems 4.7 and 4.10 and Corollary 4.11 for the shift map on \( S^\mathbb{N} \) with a Markov measure, using Proposition 5.2(b) and Proposition 5.3. Recall that \( T_S \) denotes the set of right-rooted set-theoretic trees on \( S \) (see Section 4.A).

**Corollary 5.4 (Pointwise ergodic property for Markov measures).** Let \( m \) be a stationary Markov measure on \( S^\mathbb{N} \). For every \( 1 \leq p < \infty \) and \( f \in L^p(S^\mathbb{N}, \mathbb{P}_m) \), we have the following.

(a) **Pointwise convergence along arbitrary trees:**

For a.e. \( x \in S^\mathbb{N} \),

\[
\frac{1}{m(x(0))} \sum_{w \in T} f(w \cdot x) m(w \cdot x(0)) \rightarrow \mathcal{I} \text{ as } m(x(0)) \rightarrow \infty,
\]

where \( T \) ranges over \( T_S \), and \( \mathcal{I} \) is the conditional expectation of \( f \) with respect to the \( \sigma \)-algebra of shift-invariant Borel sets.

(b) **\( L^p \) convergence along complete trees:**

The functions \( x \mapsto \frac{1}{m(x(0))} \sum_{w \in S \leq n} f(w \cdot x) m(w \cdot x(0)) \) converge to \( \mathcal{I} \) both a.e. and in \( L^p \).

(c) **Maximal ergodic theorem along arbitrary trees:**

Letting \( f^*(x) := \sup_{\tau \in T_S} \frac{1}{m(x(0))} \sum_{w \in T} f(w \cdot x) m(w \cdot x(0)) \), we have

\[
\int_{f^* > \lambda} f \, d\mu \geq \lambda \mu \{ f^* > \lambda \},
\]

for any \( \lambda \in \mathbb{R} \). In particular, \( f^* < \infty \) a.e.
5.D. Bernoulli shifts

For a finite state space $S$, the simplest example of a Markov measure $P$ on $S^{<\mathbb{N}}$ to which Corollary 5.4 applies and for which the shift map $\sigma$ is $P$-ergodic is the one whose initial distribution $\pi$ is uniform and the transition matrix is constant, i.e. all entries are equal to $\frac{1}{|S|}$. Indeed, in this case $P$ is just the product measure $\pi^\mathbb{N}$ and the Radon–Nikodym cocycle of $E_\sigma$ with respect to $P$ is given by $\rho(x) := \frac{1}{|S|}$ for all $x \in S^\mathbb{N}$.

We can also view the sequences $x \in S^\mathbb{N}$ as $|S|$-ary representations of $x \in [0,1)$. Thus, $\sigma$ is the same as the so-called baker’s map $T : [0,1) \to [0,1)$ given by $x \mapsto |S| \cdot x \bmod 1$, with Lebesgue measure on $[0,1)$.

5.E. Boundary actions of free groups

For $1 \leq r \leq \infty$, let $\mathbb{F}_r$ be the free group on $r$ generators $\{b_i\}_{i < r}$, and let $S_r := \{a_i\}_{i < 2r}$, where $a_{2i} := b_i$ and $a_{2i+1} := b_i^{-1}$ for each $i < r$. We recall that the boundary $\partial \mathbb{F}_r$ can be viewed as the set of all infinite reduced\(^8\) words in $S_r$. This makes $\partial \mathbb{F}_r$ a closed subset of $S^\mathbb{N}_r$.

The group $\mathbb{F}_r$ has a natural (boundary) action $\mathbb{F}_r \curvearrowright \partial \mathbb{F}_r$ by concatenation and cancellation: for $w \in \mathbb{F}_r$ and $x \in \partial \mathbb{F}_r$, $w \cdot x := (w^\frown x)^*$, where the latter denotes the reduction of the word $w^\frown x$. This action is free on the (councoverable) set of aperiodic words\(^9\).

The main relevant fact about this action is that its orbit equivalence relation is the same as that of the shift $\sigma: \partial \mathbb{F}_r \to \partial \mathbb{F}_r$, in fact, for $x \in \partial \mathbb{F}_r$, $\sigma(x) = x(0)^{-1} \cdot x$, and conversely, for any $a \in S_r$ and $x \in \partial \mathbb{F}_r$, $x(0)^{-1}$, $a \cdot x = a^\frown x \in \sigma^{-1}(x)$. Thus, for $x \in \partial \mathbb{F}_r$ and $n \in \mathbb{N}$, we have $\nabla^n : x = B_n x(0)^{-1} \cdot x$, where $B_n x(0)$ is the set of all reduced words of length at most $n$ that do not end with $x(0)^{-1}$. Therefore, applying Corollary 5.4 to an appropriate class of Markov measures on $S^\mathbb{N}_r$ yields a pointwise ergodic theorem (Theorem 1.2, restated below as Corollary 5.5) for the boundary action $\mathbb{F}_r \curvearrowright \partial \mathbb{F}_r$.

To translate the conclusion of Corollary 5.4 into a statement about the boundary action $\mathbb{F}_r \curvearrowright \partial \mathbb{F}_r$, we need the support of the Markov measure $P_m$ on $S^\mathbb{N}_r$ to be contained in $\partial \mathbb{F}_r$. This is the same as requiring that the support of $m$ is contained in $\mathbb{F}_r$, which is equivalent to the transition matrix $P$ of $m$ satisfying

$$P(a, a^{-1}) = 0 \text{ for all } a \in S_r.$$  

Lastly, we denote by $T'_r$, the set of all finite height subtrees of the (left) Cayley graph of $\mathbb{F}_r$, containing the identity.

**Corollary 5.5** (Pointwise ergodic for boundary actions of free groups). Let $1 \leq r \leq \infty$ and let $S_r$ be the standard symmetric set of generators of $\mathbb{F}_r$. Let $w \cdot x$ denote the boundary action of $w \in \mathbb{F}_r$ on $x \in \partial \mathbb{F}_r$. Let $m$ be a stationary Markov measure on $S^\mathbb{N}_r$ whose support is contained in $\mathbb{F}_r$. For every $1 \leq p < \infty$ and $f \in L^p(\partial \mathbb{F}_r, m^\mathbb{F}_r)$, we have the following.

(a) Pointwise convergence along arbitrary trees:

For a.e. $x \in \partial \mathbb{F}_r$,

$$\frac{1}{m(\tau^\frown x(0))} \sum_{w \in \tau} f(w \cdot x)m(w^\frown x(0)) \to f(x) \text{ as } m(\tau) \to \infty,$$

where $\tau$ ranges over all finite height subtrees of the (left) Cayley graph of $\mathbb{F}_r$, containing the identity but not $x(0)^{-1}$, and $f$ is the conditional expectation of $f$ with respect to the $\sigma$-algebra of $\beta$-invariant Borel sets.

---

\(^8\)A finite or infinite word $w$ on the set $S_r$ is called reduced if a generator and its inverse do not appear side-by-side in $w$.

\(^9\)A word on a set $S$ of symbols is called periodic if it is of the form $w v w v v \ldots$ for some $w, v \in S^\mathbb{N}_r$. 24
(b) $L^p$ convergence along complete trees:

The functions $x \mapsto \frac{1}{(n+1)\mu(x(0))} \sum_{w \in B_n(x(0))} f(w \cdot x)\mu(w \cdot x(0))$ converge to $\overline{f}$ both a.e. and in $L^p$.

(c) Maximal ergodic theorem along arbitrary trees:

Letting

$$f^*(x) := \sup \left\{ \frac{1}{\mu(\tau^x(0))} \sum_{w \in \tau} f(w \cdot x)\mu(w \cdot x(0)) : \tau \in \mathcal{T}_S, \text{ and } x(0)^{-1} \notin \tau \right\},$$

we have

$$\int_{f^* > \lambda} f \, d\mu \geq \lambda \mu \{ f^* > \lambda \},$$

for any $\lambda \in \mathbb{R}$. In particular, $f^* < \infty$ a.e.

We now provide explicit examples of Markov measures $\mu$ on $S_r^{<\infty}$ to which Corollary 5.5 applies and for which the boundary action $\mathcal{F}_r \curvearrowright \partial \mathcal{F}_r$ is $\mathbb{P}_m$-ergodic.

5.E.i. An example for $r < \infty$. For $r < \infty$, let $\mu_u$ denote the uniform Markov measure on $\mathcal{F}_r$ (the nonbacktracking simple symmetric random walk on $\mathcal{F}_r$). That is, the initial distribution $\pi$ is uniform (constant $\frac{1}{N}$), and the transition matrix $P$ is defined by setting $P(a, a^{-1}) := 0$ and $P(a, b) := \frac{1}{2r-1}$ for all $a, b \in S_r$ with $b \neq a^{-1}$. It is easy to check that $\pi$ is a stationary distribution for $P$ and $P$ is irreducible, hence $\mu_u$ satisfies the hypothesis of Corollary 5.5 and the shift map $\sigma$ is $\mathbb{P}_m$-ergodic.

5.E.ii. An example for $r = \infty$. Recall that $\mathcal{F}_\infty = \langle b_i \rangle_{i < \infty}$ and $S_\infty = \langle a_i \rangle_{i < \infty}$, where $a_{2i} := b_i$ and $a_{2i+1} := b_i^{-1}$ for each $i < \infty$. Define an $S_\infty \times S_\infty$ matrix $P$ by setting its $a_i^{th}$ row to be the sequence $(\frac{1}{2r-1})_{j < \infty}$ with a zero inserted for the entry corresponding to $a_i^{-1}$. More precisely, for all $i, j < \infty$,

$$P(a_{2i}, a_j) := \begin{cases} \frac{1}{2r-1} & \text{ if } j < 2i + 1 \\ 0 & \text{ if } j = 2i + 1 \\ \frac{1}{2r} & \text{ if } j > 2i + 1 \end{cases} \quad \text{and} \quad P(a_{2i+1}, a_j) := \begin{cases} \frac{1}{2r-1} & \text{ if } j < 2i \\ 0 & \text{ if } j = 2i \\ \frac{1}{2r} & \text{ if } j > 2i. \end{cases}$$

Again, $P$ is irreducible, and one can check that every $a \in S_\infty$ is a positive recurrent state (see [Dur19, paragraph above Theorem 5.5.12] for the definition), so $P$ admits a positive stationary distribution $\pi$ by [Dur19, Theorems 5.5.11 and 5.5.12]. Thus, the Markov measure $\mu$ on $S_\infty^{<\infty}$ with transition matrix $P$ and initial distribution $\pi$ satisfies the hypothesis of Corollary 5.5 and the shift map $\sigma$ is $\mathbb{P}_m$-ergodic.

6. Application to pmp actions of free groups

Let $(X, \mu)$ be a standard probability space, and let $\mathcal{F}_r$ be the free group on $r$ generators, where $2 \leq r < \infty$. As in Section 5.E, let $S_r := \{a_i\}_{i < 2r}$ be the standard symmetric set of generators and let $\partial \mathcal{F}_r$ be the boundary of $\mathcal{F}_r$ (i.e. the set of all infinite reduced words in $S_r$). Let $\mu_u$ be the uniform Markov measure as in Section 5.E.i whose initial distribution $\pi$ is the constant $\frac{1}{N}$ vector and whose transition matrix $P$ is such that for all $i, j < r$, $P(a_i, a_j) = \frac{1}{2r-1}$ if $a_i \neq a_j^{-1}$, and $P(a_i, a_i^{-1}) = 0$ otherwise. In particular, for a word $w \in \mathcal{F}_r$ of length $n \geq 1$, $\mu_u(w) = \frac{1}{(2r)(2r-1)^{n-1}}$.

Theorem 1.1 is stated for $\mu_u$ but we prove it here more generally for all stationary Markov measures $\mu$ on $\mathcal{F}_r$ so long as the boundary action $\mathcal{F}_r \curvearrowright (\partial \mathcal{F}_r, \mathbb{P}_m)$ is weakly mixing, i.e., the product of $\beta$ with any ergodic pmp action is weakly mixing.

Due to Kaimanovich, and Glasner and Weiss, this holds for the uniform Markov measure $\mu_u$, i.e. the boundary action of $\mathcal{F}_r$ on $(\partial \mathcal{F}_r, \mathbb{P}_m)$ is weakly mixing. Indeed, the measure $\mathbb{P}_m$ coincides with the measure induced by the simple symmetric random walk on $\mathcal{F}_r$, so by [Kai95, Corollary following Theorem 2.4.6] the boundary action of $\mathcal{F}_r$ (for $r \geq 2$) on $(\partial \mathcal{F}_r, \mathbb{P}_m)$ is doubly ergodic. This implies that it is weakly mixing, by [GW16, Theorem 1.1]. See also [GW16, Example 5.1].
Remark 6.1. The upcoming work of the authors is devoted to characterizing all Markov measures \( m \) on \( \mathbb{F} \), that make the boundary action of \( \mathbb{F} \) on \( (\partial \mathbb{F}, \mathbb{P}_m) \) weakly mixing.

**Theorem 6.2.** Let \( 2 \leq r < \infty \) and let \( \mathbb{F} \) be a (not necessarily free) pmp action of \( \mathbb{F} \). Let \( m \) be a stationary Markov measure on \( S_{\leq r}^{< \mathbb{N}} \) whose support is contained in \( \mathbb{F} \). Suppose that the boundary action \( \mathbb{F} \) is weakly mixing. Then for every \( f \in L^1(X, \mu) \), for \( \mu \)-a.e. \( x \in X \),

\[
\frac{1}{m(\tau)} \sum_{w \in \tau} f(w \cdot x) m(w) \rightarrow \mathcal{f}(x) \text{ as } m(\tau) \rightarrow \infty,
\]

where \( \tau \) ranges over all finite subtrees of the (left) Cayley graph of \( \mathbb{F} \) containing the identity and \( \mathcal{f} \) is the conditional expectation of \( f \) with respect to the \( \sigma \)-algebra of \( \alpha \)-invariant Borel sets.

Remark 6.3. Note that because we defined Markov measures \( m \) via a row stochastic matrix \( P \), we may think that \( m \) assigns weights to words in \( S_{\leq r}^{< \mathbb{N}} \) from left to right (algorithmically), i.e., \( m(w^{-1}a) = m(w)P(w_{n-1}, a) \) for each \( w \in S_{\leq r}, a \in S_r \), and \( n \geq 1 \). On the other hand, in the left Cayley graph of \( \mathbb{F} \), the words grow from right to left, i.e., the edges are between the words \( w \) and \( a^{-1}w \). In light of this, the setup of Theorem 6.2 may seem unnatural. However, because \( m \) is assumed to be stationary, it can be also expressed by a right-to-left recursive formula \( m(a^{-1}w) = \tilde{P}(a, w_0) m(w) \), where \( \tilde{P} \) is a column stochastic matrix defined by \( \tilde{P}(a, b) := \frac{m(a)}{m(b)} P(a, b) \), for which the column-vector \( \pi := (m(a))_{a \in S_r} \) is stationary, i.e., \( \tilde{P} \pi = \pi \).

The rest of this section is devoted to the proof of Theorem 6.2. Let \( \pi \) and \( P \) be the initial distribution (row-vector) and the (row stochastic) transition matrix of \( m \), respectively.

We will obtain Theorem 6.2 by applying Theorem 4.7 to the transformation (so-called **backward system**) \( T : X \times \partial \mathbb{F} \to X \times \partial \mathbb{F} \), defined by

\[
(x, y) \mapsto (y(0)^{-1} \cdot x, \sigma(y))
\]

where \( \sigma : \partial \mathbb{F} \to \partial \mathbb{F} \) is the shift map. We equip \( X \times \partial \mathbb{F} \) with the measure \( \nu := \mu \times \mathbb{P}_m \). Intuitively, we direct the standard Cayley graph of \( \mathbb{F} \) towards every end (or rather its inverse) and for each such directing, consider the shift map towards that end. This transforms every tree in the Cayley graph into a union of \( 2r \) backward trees of the shift map, and each of these trees either is negligible or has the correct average.

For \( i_0, \ldots, i_n < 2r \), we will also abuse notation and write \([i_0, \ldots, i_n]\) for \([a_{i_0}, \ldots, a_{i_n}], \pi(i_0)\) for \(\pi(a_{i_0})\), and \(P(i_0, i_1)\) for \(P(a_{i_0}, a_{i_1})\).

**Lemma 6.4.** \( T \) is measure-preserving.

**Proof.** Let \( U \subseteq X \) be Borel and \( i_1, \ldots, i_n < 2r \). Then for any \( x \in X \), \( i_0 < 2r \) and \( y \in [i_0] \),

\[
(x, y) \in T^{-1}(U \times [i_1, \ldots, i_n]) \iff (a_{i_0}^{-1} \cdot x , \sigma(y)) \in U \times [i_1, \ldots, i_n] \\
\iff (x, y) \in (a_{i_0} \cdot U) \times [i_0, i_1, \ldots, i_n],
\]

so \( T^{-1}(U \times [i_1, \ldots, i_n]) = \bigcup_{i_0 < 2r} (a_{i_0} \cdot U) \times [i_0, i_1, \ldots, i_n] \). Hence,

\[
\nu(T^{-1}(U \times [i_1, \ldots, i_n])) = \sum_{i_0 < 2r} \nu((a_{i_0} \cdot U) \times [i_0, i_1, \ldots, i_n])
\]

\[
= \mu(U) \cdot \sum_{i_0 < 2r} \mathbb{P}_m([i_0, i_1, \ldots, i_n])
\]

\[
= \mu(U) \cdot \mathbb{P}_m(\sigma^{-1}[i_1, \ldots, i_n])
\]

\[
[\text{m is stationary, so \( \sigma \) is pmp}]
\]

\[
= \nu(U \times [i_1, \ldots, i_n]). \quad \Box
\]
Lemma 6.5. Let $\mathbb{F}_r \ltimes \partial \mathbb{F}_r$ be the boundary action of $\mathbb{F}_r$ as in Section 5.6. Let $\mathbb{F}_r \ltimes \mathbb{F}_r \times \partial \mathbb{F}_r$ be the diagonal action $g \cdot (x, y) = (g \cdot x, g \cdot y)$. Then the induced equivalence relation $E_{\alpha \times \beta}$ is the same as $E_T$ (the equivalence relation induced by $T$).

Proof. Fix $(x, y) \in X \times \partial \mathbb{F}_r$. Then for $a \in S_r$ if $a \neq y(0)^{-1}$, then $T(a \cdot (x, y)) = T(a \cdot x, a^{-1}y) = (x, y)$. If $a = y(0)^{-1}$, then $T(x, y) = (a \cdot x, \sigma(y)) = a \cdot (x, y)$. On the other hand, $T(x, y) = (y(0)^{-1}x, \sigma(y)) = y(0)^{-1} \cdot (x, y)$. □

Given that $E_T$ is the orbit equivalence relation of the diagonal action and the action of $\mathbb{F}_r$ on $X$ is pmp, the following lemma is clear, but we state it with an explicit formula for the Radon–Nikodym cocycle.

Lemma 6.6. $E_T$ is null-preserving on the conull set $X \times Y$, where $Y$ is the set of all infinite words $y \in S_r$ in which each pair $y(n), y(n + 1)$ of consequent letters has $P(y(n), y(n + 1)) > 0$. The Radon–Nikodym cocycle of $E_T \mid X \times Y$ with respect to $\nu$ only depends on the $Y$-coordinates and is equal to the Radon–Nikodym derivative of $E_{\sigma \mid Y}$ with respect to $\mathbb{P}_m$; more explicitly (by Proposition 5.3): 

$$\rho_{T^n(x, y)}((x, y)) = \rho_{\sigma^n(y)}(y) = \frac{m(y(0) \cap y(1) \cap \ldots \cap y(n))}{m(y(n))}.$$ 

Proof. By the cocycle identity, it is enough to prove

$$\rho_{T(x, y)}((x, y)) = \frac{\pi(y(0))}{\pi(y(1))} P(y(0), y(1)).$$

For this it suffices to check that for Borel sets $U \subseteq X$ and $[i_0, \ldots, i_n] \subseteq \partial \mathbb{F}_r$,

$$\nu(T(U \times [i_0, \ldots, i_n])) = \int_{U \times [i_0, \ldots, i_n]} \frac{\pi(i_1)}{\pi(i_0) P(i_0, i_1)} d\nu.$$

To see this, observe that

$$\int_{U \times [i_0, \ldots, i_n]} \frac{\pi(i_1)}{\pi(i_0) P(i_0, i_1)} d\nu = \frac{\pi(i_1)}{\pi(i_0) P(i_0, i_1)} \cdot \nu(U \times [i_0, \ldots, i_n])$$

$$= \frac{\pi(i_1)}{\pi(i_0) P(i_0, i_1)} \cdot \mu(U) \cdot \frac{\pi(i_0) P(i_0, i_1)}{\pi(i_1)} \cdot \mathbb{P}[i_1, \ldots, i_n]$$

$$= \mu(U) \cdot \mathbb{P}[i_1, \ldots, i_n]$$

$$= \mu(a_{i_0} \cdot U) \cdot \mathbb{P}[i_1, \ldots, i_n]$$

$$= \nu(T(U \times [i_0, \ldots, i_n])).$$ □

Lemma 6.7. If $\mathbb{F}_r \ltimes (X, \mu)$ is ergodic, then $T$ is ergodic.

Proof. By Lemma 6.5, it is enough to show $E_{\alpha \times \beta}$ is ergodic, but this follows from the hypothesis that the boundary action of $\mathbb{F}_r$ on $(\partial \mathbb{F}_r, \mathbb{P}_m)$ is weakly mixing. □

Lemma 6.8. For $f \in L^1(X, \mu)$, define $F \in L^1(X \times \partial \mathbb{F}_r, \mu \times \mathbb{P}_m)$ by $F(x, y) := f(x)$. Then for $\mu$-a.e. $x \in X$ and $\mathbb{P}_m$-a.e. $y \in \partial \mathbb{F}_r$, $\overline{F}(x, y) = \overline{f}(x)$, where $\overline{f}$ and $\overline{F}$ are the conditional expectations of $f$ and $F$ with respect to the $\sigma$-algebras of $\alpha$ and $T$-invariant Borel sets, respectively.

Proof. If the action $\mathbb{F}_r \ltimes (X, \mu)$ is ergodic, then so is $T$ by Lemma 6.7, and hence $\overline{F}(x, y) = \int_{X \times \partial \mathbb{F}_r} F d\nu = \int_X f(x) d\mu(x) = \overline{f}(x)$.

In the general case, we use the Ergodic Decomposition theorem for pmp countable Borel equivalence relations (see Definition 2.7 and the following paragraph), which gives us an $E_\alpha$-ergodic decomposition $e : X \to P(X).$ By Proposition 2.8, $\overline{f}(x) = \int_X f(x) d\nu$ for $\mu$-a.e. $x \in X$.

But $\int_X f d\nu = \int_{X \times \partial \mathbb{F}_r} F d(\epsilon \times \mathbb{P}_m)$ by the definition of $F$, so it remains to show that $\overline{F}(x, y) = \int_{X \times \partial \mathbb{F}_r} F d(\epsilon \times \mathbb{P}_m)$ for a.e. $x \in X$ and $\mathbb{P}_m$-a.e. $y \in \partial \mathbb{F}_r$. □
To this end, for any $H \in L^1(X \times \partial \mathcal{F}_r, \mu \times \mathbb{P})$, Fubini’s theorem gives
\[
\int_{X \times \partial \mathcal{F}_r} H \, d(\mu \times \mathbb{P}) = \int_X \int_{\partial \mathcal{F}_r} H(x, y) \, d\mathbb{P}_m(y) \, d\mu(x)
\]

**Proposition 2.8**
\[
\int_X \int_{\partial \mathcal{F}_r} H(z, y) \, d\mathbb{P}_m(y) \, d\epsilon_x(z) \, d\mu(x)
\]

**Fubini**
\[
\int_X \int_{X \times \partial \mathcal{F}_r} H(z, y) \, d(\epsilon_x \times \mathbb{P}_m)(z, y) \, d\mu(x)
\]

**dummy integration**
\[
\int_{X \times \partial \mathcal{F}_r} \int_{X \times \partial \mathcal{F}_r} H(z, y) \, d(\epsilon_x \times \mathbb{P}_m)(z, y) \, d(\mu \times \mathbb{P}_m)(x, y').
\]

Recalling, in addition, that each measure $\epsilon_x \times \mathbb{P}_m$ is $T$-ergodic, by Lemma 6.7, we see that the map $(x, y) \mapsto \epsilon_x \times \mathbb{P}_m$ is the $E_T$-ergodic decomposition of $\mu \times \mathbb{P}_m$ (Definition 2.7). Thus, again by Proposition 2.8, $\bar{F}(x, y) = \int_{X \times \partial \mathcal{F}_r} F \, d(\epsilon_x \times \mathbb{P}_m)$ for $(\mu \times \mathbb{P}_m)$-a.e. $(x, y) \in X \times \partial \mathcal{F}_r$, finishing the proof.

We now show that Theorem 4.7 applied to $T$ yields Theorem 6.2:

**Proof of Theorem 6.2.** For each $a \in S_r$, define the partial function $\widehat{\alpha} : X \times \partial \mathcal{F}_r \rightarrow X \times \partial \mathcal{F}_r$ by $\widehat{\alpha}(x, y) := (a \cdot x, a \cdot y)$ for $(x, y) \in \text{dom}(\widehat{\alpha}) := X \times (\partial \mathcal{F}_r \setminus \bigcup_{b \in S_r, \rho \in \mathcal{P}(a,b) = 0} b)$. Let $\hat{S}_r := \{\widehat{\alpha} : a \in S_r\}$, and notice that for each $(x, y) \in X \times \partial \mathcal{F}_r$,
\[
T^{-1}(x, y) = \left\{(a_1 \cdot x, a_2 \cdot y) : i < 2r \text{ and } a_i \neq y(0) = 0 \right\} = \hat{S}_r \cdot (x, y),
\]

where $\hat{S}_r \cdot (x, y)$ is defined as in Section 2.C. Thus, $\hat{S}_r$ is a complete set of Borel partial right-inverses of $T$. For a reduced word $w := s_1 s_2 \ldots s_n$ with $s_i \in S_r$, we let $\hat{w} := \hat{s}_1 \circ \hat{s}_2 \circ \cdots \circ \hat{s}_n$ be the induced partial function on $X \times \partial \mathcal{F}_r$. Lastly, we denote by $T^r_{\hat{S}_r}$ the set of all finite subtrees of the (left) Cayley graph of $F_r$ containing the identity, and we put $\hat{T} := \{\hat{w} : w \in \tau\}$ for each $\tau \in T^r_{\hat{S}_r}$.

Fix $f \in L^1(X, \mu)$. Define $\bar{F} \in L^1(X \times \partial \mathcal{F}_r, \nu)$ by $F(x, y) := \hat{f}(x)$. For $\tau \in T^r_{\hat{S}_r}$ and $(x, y) \in X \times \partial \mathcal{F}_r$, we will abuse notation and write $\rho_{y_1}(y_1)$ for $\rho_{(x_2, y_2)}((x_1, y_1))$, as well as $\rho_y(\hat{T} \cdot y)$ for $\rho_{(x, y)}(\hat{T} \cdot (x, y))$, since $\rho$ does not depend on $x$.

**Claim 1.** For each $\tau \in T^r_{\hat{S}_r}$, $x \in X$, and $y_i \in [i]$ where $i$ ranges in $\{0, \ldots, 2r - 1\}$,

(a) $m(\tau) = \sum_{i < 2r} \pi(i) \rho_{y_i}(\hat{T} \cdot y_i)$,

(b) $\sum_{w \in \tau} f(w \cdot x) m(w) = \sum_{i < 2r} A_{\hat{F}}(\tau \cdot (x, y_i)) \pi(i) \rho_{y_i}(\hat{T} \cdot y_i)$.

**Proof of Claim.** Part (a) follows from (b) by plugging-in $f = 1$. As for (b), we compute:
\[
\sum_{i < 2r} \pi(i) \rho_{y_i}(\hat{T} \cdot y_i) A_{\hat{F}}(\tau \cdot (x, y_i)) = \sum_{i < 2r} \pi(i) \sum_{w \in \tau, P(w|w| - 1, i) > 0} f(w \cdot x) \rho_{y_i}(w \cdot y_i)
\]

\[
\left[ m(w \cdot y_i) = 0 \text{ when } P(w|w| - 1, i) = 0 \right] = \sum_{i < 2r} \pi(i) \sum_{w \in \tau} f(w \cdot x) \frac{m(w \cdot y_i)}{\pi(i)}
\]

**Fubini**
\[
= \sum_{w \in \tau} f(w \cdot x) \sum_{i < 2r} m(w \cdot y_i)
\]

\[
= \sum_{w \in \tau} f(w \cdot x) m(w).
\]

Applying Theorems 4.7 and 4.10 to the transformation $T$, we get that the conclusions of these theorems hold for $\nu$-a.e. $(x, y) \in X \times \partial \mathcal{F}_r$. Thus, for each point $x$ in a $\mu$-conull set $X' \subseteq X$ and for each $i < 2r$, there is $y_i \in \partial \mathcal{F}_r \cap [i]$ such that the conclusions of Theorems 4.7 and 4.10 hold at $(x, y_i)$. Fix $x \in X'$ and $(y_i)_{i < 2r}$ as above, as well as $\epsilon > 0$. Then the choices of $x$ and $(y_i)_{i < 2r}$ (namely,
Theorems 4.7 and 4.10), and the fact that $r < \infty$, yield an $M > 0$ large enough so that for each $i < 2r$ and $\tau_{(x, y_i)} \in T_{(x, y_i)}$:

(i) $|\bar{f}(x)| \leq M$ and $|A^\rho_F[\tau_{(x, y_i)}]| \leq M$,

(ii) $A^\rho_F[\tau_{(x, y_i)}] \approx_{\pi(i)\epsilon} \bar{F}(x, y_i)$ whenever $\rho_{y_i}(\tau_{(x, y_i)}) \geq M$.

It remains to show $\frac{1}{m(\tau)} \sum_{w \in \tau} f(w \cdot x)m(w) \approx_{\pi(i)\epsilon} \bar{f}(x)$ for each $\tau \in T_{S_i}$, with $m(\tau) > \frac{2M^2}{\epsilon}$.

**Claim 2.** $A^\rho_F[\tau \cdot (x, y_i)] \approx_{\pi(i)\epsilon} \bar{F}(x, y_i)$ for each $i < 2r$.

**Proof of Claim.** For each $i < 2r$, there are two cases.

If $\rho_{y_i}(\bar{\tau} \cdot y_i) < M$, then $\frac{\pi(i)\rho_{y_i}(\bar{\tau} \cdot y_i)}{m(\tau)} < \frac{\pi(i)M\epsilon}{2M^2} = \frac{\pi(i)\epsilon}{2M}$, and both $|A^\rho_F[\bar{\tau} \cdot (x, y_i)]|$ and $|\bar{F}(x, y_i)|$ are at most $M$, so both quantities in question are less than $\frac{\pi(i)\epsilon}{2M}$ and hence within $\pi(i)\epsilon$ of each other.

If $\rho_{y_i}(\bar{\tau} \cdot y_i) \geq M$, then $A^\rho_F[\bar{\tau} \cdot (x, y_i)] \approx_{\pi(i)\epsilon} \bar{F}(x, y_i)$, so we are done because $\frac{\pi(i)\rho_{y_i}(\bar{\tau} \cdot y_i)}{m(\tau)} \leq 1$ by Claim 1(a). \[\Box\]

It remains to compute, using Claim 1(b) on the first line:

$$
\frac{1}{m(\tau)} \sum_{w \in \tau} f(w \cdot x)m(w) = \sum_{i < 2r} A^\rho_F[\bar{\tau} \cdot (x, y_i)] \frac{\pi(i)\rho_{y_i}(\bar{\tau} \cdot y_i)}{m(\tau)}
$$

$$
\text{[Claim 2]} \approx_{\epsilon} \sum_{i < 2r} \bar{F}(x, y_i) \frac{\pi(i)\rho_{y_i}(\bar{\tau} \cdot y_i)}{m(\tau)}
$$

$$
\text{[Lemma 6.8]} = \bar{f}(x) \sum_{i < 2r} \frac{\pi(i)\rho_{y_i}(\bar{\tau} \cdot y_i)}{m(\tau)}
$$

$$
\text{[Claim 1(a)]} = \bar{f}(x).
$$

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30