Precise Numerical Evaluation of the Scalar One-Loop Integrals with the Infrared Divergence

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We present a new approach for obtaining very precise integration results for infrared vertex and box diagrams, where the integration is carried out directly without performing any analytic integration of Feynman parameters. Using an appropriate numerical integration routine with an extrapolation method, together with a multi-precision library, we have obtained integration results which agree with the analytic results to 10 digits even for such a very small photon mass as $10^{-150}$ GeV in the infrared vertex diagram.

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1. Introduction

In the field of the particle physics more and more precise computation is required for the progress of the accuracy of the future collider experiments. For this reason, the computation of one- and higher loop corrections is mandatory.

General solution has been known for one-loop integrals[1] and there are several excellent programs developed in this decade such as FF[2], FormCalc-LoopTools[3], XLOOPS-Ginac[4]. In parallel, there have been continual efforts to carry out loop integrals by fully or semi-fully numerical methods. In early 1990 Fujimoto et al.[5, 6, 7, 8] have proposed several excellent methods and have shown numerical results. In early 2000 Kurihara et al.[9] have developed the numerical contour method and have shown numerical results.

Since 2003 we have been developing a method which enables us to carry out the loop integrals in a completely numerical way. In this numerical method, to prevent the integral diverging, we put $i\varepsilon$ in the denominator of the integrand. Here $\varepsilon$ is a real positive constant and it is not necessary to be infinitesimal. Thanks to this $i\varepsilon$, the denominator does not vanish and we get the numerical result of $I(\varepsilon)$ for a given $\varepsilon$. Calculating a sequence of $I(\varepsilon_l)$ varying $\varepsilon_l = \varepsilon_0 \times (\text{const.})^{-l} (l = 0, 1, 2, \cdots)$ and extrapolating them, we can get the final result of the integration in the limit of $\varepsilon \rightarrow 0$. So far in this method we have calculated several one-loop and two-loop integrals and reported the numerical
results in [10, 11, 12, 13]. In these demonstrations it has been clearly shown that the extrapolation method is very efficient for not only one-loop but also two-loop diagrams.

In the numerical calculation of the loop integrals, it is also very important to confirm whether the method can handle the infrared singularities. In [12], we have calculated the loop integrals with the infrared singularities with a photon mass of up to $10^{-15}$ GeV. In the calculation we used the double precision arithmetic for the photon mass of up to $10^{-6}$ GeV. However for the photon mass of below $10^{-8}$ GeV the double precision is not enough and we use the quadruple precision arithmetic. In the quadruple precision calculation the relative error is given the order of $10^{-7}$.

In this paper, we propose a new approach for our numerical method. In this approach we include a precision control technique in addition to the extrapolation method for the loop integral. We show that a precision control is mandatory to get the stable results of one-loop diagrams which have infrared singularities with smaller photon mass below $10^{-30}$ GeV.

The layout of this paper is as follows. In §2 we give formulae of the loop integrals we consider in this paper. In §3 we give an explanation of a new approach of our numerical method. Numerical results are shown in §4. Finally in §5 we will summary this paper.

2. Loop Integrals

2.1 One-Loop Vertex

For one-loop vertex diagram in Fig. 1, the loop integral we consider in this paper is

$$I = \int_0^1 dx \int_0^{1-x} dy \frac{1}{D},$$

where

$$D = -xys + (x+y)^2 m^2 + (1-x-y)\lambda^2.$$ 

Here $s$ denotes squared central mass energy and $m$ and $\lambda$ are a mass of external particles and a fictitious photon mass respectively. We introduce $\lambda$ so as to regularize the infrared singularities.

Replacing $s$ by $s + i\epsilon$ in (2.2), the real part of the integral (2.1) becomes
\[ \Re = \int_S \frac{D}{D^2 + \varepsilon^2 x^2 y^2} dx dy, \quad (2.3) \]

and the imaginary part becomes
\[ \Im = \int_S \frac{\varepsilon x y}{D^2 + \varepsilon^2 x^2 y^2} dx dy. \quad (2.4) \]

Here \( S \) is the triangular region as \( 0 < x, 0 < y \) and \( x + y < 1 \). Corresponding analytic formulae for one-loop vertex diagrams are given in \([5, 6, 8]\).

### 2.2 One-Loop Box

For one-loop box diagram in Fig. 2, the loop integral we consider in this paper is
\[
I = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \frac{1}{D^2},
\]
where
\[
D = -xys - tz(1-x-y-z) + (x+y)\lambda^2 + (1-x-y-z)(1-x-y)m^2 + z(1-x-y)M^2. \quad (2.6)
\]

In (2.6) \( s \) denotes squared central mass energy and \( m \) and \( M \) are external masses of particles.

![One-loop box diagram](produced by GRACE)

**Figure 2:** One-loop box diagram

When we replace \( s \) by \( s + i\varepsilon \) in (2.5), the real part of the integral becomes
\[ \Re = \int_S \frac{D^2 + \varepsilon^2 x^2 y^2}{(D^2 + \varepsilon^2 x^2 y^2)^2} dx dy dz, \quad (2.7) \]

and the imaginary part becomes
\[ \Im = \int_S \frac{2\varepsilon x y D}{(D^2 + \varepsilon^2 x^2 y^2)^2} dx dy dz. \quad (2.8) \]

Here \( S \) is the tetrahedron region as \( 0 < x, 0 < y, 0 < z \) and \( x + y + z < 1 \). Corresponding analytic formulae for one-loop box diagrams are given in \([6]\). When we put \( M = \mu \), the scalar integral for one-loop box diagram becomes
$I_{\text{analytic}} = -\frac{1}{s} \sqrt{\frac{1}{s - (t + \mu^2 - m^2)^2 - 4m^2t}}$

\[\times \ln \left( -\frac{s}{\lambda^2} \right) \ln \left( \frac{-t + \mu^2 + m^2 + \sqrt{(-t + \mu^2 - m^2)^2 - 4m^2t}}{4m^2\mu^2} \right)^2, \quad (2.9)\]

3. Numerical Method

The basic idea of our numerical method is the combination of an efficient multi-dimensional integration routine and an extrapolation method. Using this method we have shown several numerical results of loop integrals and that they show good agreements with analytical results. In the calculation so far we have used double precision arithmetic or quadruple precision arithmetic if necessary for one-loop and two-loop integrals. However for the infrared divergent diagram even with the quadruple precision arithmetic it becomes harder to get a result with an enough accuracy as the value of the $\lambda$ becomes much smaller such as $10^{-15}$ GeV\[12\]. Therefore in our new approach we include the concept of a precision control.

In the following subsections, we give a very simple explanation of the multi-dimensional integration routine, an extrapolation method and a multi-precision library we used in our new approach respectively.

3.1 Multi-dimensional Integration Routine

We use DQAGE routine in the multi-dimensional integration. It is included in the package QUADPACK \[14\] and it is a globally adaptive integration routine. Though it is a routine for one-dimensional integration, we use it iteratedly \[15\].

3.2 Extrapolation Method

Although there are several ways to accelerate the conversion of the sequences. Of them, the $\varepsilon$ algorithm by P.Wynn\[16\] is recommended as the best all-purpose method for slowly converging sequences. It is an implementation of Shanks’ transformation \[17\] and is represented in a recursive formulae.

3.3 Multi-Precision Library

When we know correctly whether the precision arithmetic used in the calculation is sufficient or not, we can select the necessary precision arithmetic and the results become reliable. J.Fujimoto et. al have discussed the importance of a precision control in \[18\]. We use HMlib\[19\] as the multi-precision library in our new approach because it gives an information of the lost-bits during the calculation and we can guarantee the precision of the results. Taking the calculation of one-loop vertex diagram as an example, we show the information of the lost-bits supplied by HMlib in Table \[1\].
Table 1: In $P$-precision presentation implemented in HMLib, a sign bit is 1 bit and an exponent bit is 15 bits and a mantissa is $(32 \times P - 16)$ bits. When $P = 4$ (quadruple precision), the mantissa becomes 112 bits. In the Table when $\lambda$ is $10^{-23}$ GeV, the maximum number of lost-bits is the same as the number of bits of the mantissa.

| $\lambda$ [GeV] | Average lost-bits | Maximum lost-bits |
|----------------|------------------|------------------|
| $10^{-20}$     | 88               | 92               |
| $10^{-21}$     | 98               | 102              |
| $10^{-22}$     | 108              | 112              |

Table 2: Numerical results of one-loop vertex diagram with $\sqrt{s} = 500$ GeV, $t = -150^2$ GeV$^2$, $m = m_e = 0.5 \times 10^{-3}$ GeV. The upper is the result of Real part and the lower is one of Imaginary part.

| $\lambda$ [GeV] | Numerical Results | Precision | Analytical Results [quadruple precision] | Agreement |
|----------------|-------------------|-----------|-----------------------------------------|-----------|
| $10^{-20}$     | -0.150899286980769753D-01 ± 0.771D-26 | 8         | -0.15089928698048229151911707856807798E-01 | 12        |
|                | 0.189229839615898822D-02 ± 0.124D-25 | 8         | 0.18922983961525893435366770832051E-02 | 12        |
| $10^{-21}$     | -0.3035935256254951D-01 ± 0.178D-16 | 16        | -0.30359352562601542232930705958645E-01 | 12        |
|                | 0.362840665514227622D-02 ± 0.896D-13 | 16        | 0.36284066551349654484509779501460548E-02 | 11        |
| $10^{-22}$     | -0.405390396284235075D-01 ± 0.580D-15 | 16        | -0.40539039628344359602607706522058889E-01 | 11        |
|                | 0.478581216125478532D-02 ± 0.401D-12 | 16        | 0.478581216112143981844493853981879066E-02 | 10        |
| $10^{-23}$     | -0.608983238726997427D-01 ± 0.556D-15 | 16        | -0.60898323872581587963164505454259761E-01 | 11        |
|                | 0.710062173257286666D-02 ± 0.415D-12 | 32        | 0.7100621732709433885588462002942716100E-02 | 10        |
| $10^{-24}$     | -0.761677949309069452D-01 ± 0.931D-15 | 32        | -0.76167794930759374233582104516222189E-01 | 10        |
|                | 0.883673143185801414D-02 ± 0.260D-13 | 32        | 0.883673143207410011344838114507342543E-02 | 7         |
| $10^{-25}$     | -0.812576170752810666D-01 ± 0.549D-10 | 32        | -0.812576178347269926993757808016127325E-01 | 7         |
|                | 0.941543434672244271406905030553399E-02 | 32        | 0.941543434672244271406905030553399E-02 | 7         |

4. Numerical Results

4.1 One-Loop Vertex

The numerical results are shown in Table 2. Results are compared to analytic results evaluated by the formulae in [8]. As an example, in Table 3 we show the parameters we used in the calculation which are key and limit used in DQAGE, starting $\epsilon$ and ending $\epsilon$ for an extrapolation method.

4.2 One-Loop Box

The numerical results of one-loop box diagram shown are shown in Table 4. The results are compared to the analytic results evaluated by (2.9) in a quadruple precision. As an example, in Table 5 we show the parameters we used in the calculation which are key and limit used in DQAGE, starting $\epsilon$ and ending $\epsilon$ for an extrapolation method.

4.3 Elapsed Time

In Table 6 and 7, the elapsed times required in the several typical calculations are shown.
Table 3: Parameters used in DQAGE and in an extrapolation method for the one-loop vertex diagram with \( \sqrt{s} = 500 \text{ GeV}, m = m_e = 0.5 \times 10^{-3} \text{ GeV}, \lambda = 10^{-150} \text{ GeV}. \) The upper is the parameters for the Real part and the lower is ones for Imaginary part.

| key_x | limit_x | \( \varepsilon_{\text{start}} \) | \( \varepsilon_{\text{end}} \) |
|-------|---------|-----------------|-----------------|
| 0     | 600     | 0.36572620D+04  | 0.28485158D+03  |
| 0     | 600     | 0.36572620D+04  | 0.28485158D+03  |

Table 4: Numerical results of the loop integral of one-loop box diagram with \( \sqrt{s} = 500 \text{ GeV}, t = -150^2 \text{GeV}^2, m = 0.5 \times 10^{-5} \text{ GeV}, M = 150 \text{ GeV}. \) This is results of Real part.

| \( \lambda \) [GeV] | Numerical Results | Precision | Numerical Results [quadruple precision] | Agreement |
|--------------------|------------------|-----------|------------------------------------------|-----------|
| \( 10^{-15} \)     | -0.1927861122439641160689577177195708E-06 | 4,4,2     | -0.1927861122439641160689577177195708E-06 | 8         |
| \( 10^{-20} \)     | -0.247248635259915789993796838555422E-06 | 4,4,2     | -0.247248635259915789993796838555422E-06 | 9         |
| \( 10^{-25} \)     | -0.301711125376463D-06 ± 0.111D-13     | 4,4,2     | -0.301711125376463D-06 ± 0.111D-13     | 7         |
| \( 10^{-30} \)     | -0.3562028823831722D-06 ± 0.867D-10    | 4,4,2     | -0.3562028823831722D-06 ± 0.867D-10    | 4         |

Table 5: Parameters used in DQAGE and in an extrapolation method for the one-loop box diagram with \( \sqrt{s} = 500 \text{ GeV}, t = -150^2 \text{GeV}^2, m = 0.5 \times 10^{-3} \text{ GeV}, M = 150 \text{ GeV} \) and \( \lambda = 10^{-25} \text{ GeV}. \)

| key_x | limit_x | \( \varepsilon_{\text{start}} \) | \( \varepsilon_{\text{end}} \) |
|-------|---------|-----------------|-----------------|
| 0     | 400     | 0.36572620D+04  | 0.28485158D+03  |
| 0     | 400     | 0.36572620D+04  | 0.28485158D+03  |
| 0     | 100     | 0.36572620D+04  | 0.28485158D+03  |

Table 6: Elapsed time required in the calculation of the one-loop vertex diagram

| \( \lambda \) [GeV] | Real Part | Imaginary Part | Precision | CPU          |
|--------------------|-----------|----------------|-----------|--------------|
| \( 10^{-30} \)     | 1.8days   | 1.8days        | 8,8       | Xeon 3.06GHz |
| \( 10^{-150} \)    | 6.8days   | 6.7days        | 32,32     | Xeon 3.06GHz |

Table 7: Elapsed time required in the calculation of the one-loop box diagram

| \( \lambda \) [GeV] | Real Part | Precision | CPU          |
|--------------------|-----------|-----------|--------------|
| \( 10^{-15} \)     | 1.5days   | 4,4,2     | Opteron 2.2GHz |
| \( 10^{-25} \)     | 3.0days   | 4,4,2     | Opteron 2.2GHz |
5. Summary

We have shown that a new numerical approach presented in this paper is applicable to the scalar one-loop vertex and box diagram with infrared singularities. The new approach consists of an integration routine, an extrapolation method and a precision control. Several numerical results are shown and they agree with analytic ones. All the calculation of the one-loop vertex diagram are carried out in at least quadruple precision arithmetic and some are in 8-, 16- and 32-precision arithmetic. From these demonstrations, it is confirmed that a precision control plays an important role in the calculation of the loop integrals with the infrared singularities. Although higher precision arithmetic costs a lot of CPU time, it supplies the information of the lost-bits occurred during the calculation and this is extremely valuable in getting the reliable results. To reduce CPU time we will be able to use the parallel computing technique.

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