Spherically symmetric stationary flows of a gas suspension

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Abstract. The dynamics of an isothermal gas suspension is studied. A spherically symmetric stationary submodel is derived and partially integrated. In a barochronous case, this submodel is fully integrated and results are discussed. Different modes of motion are analytically described in the barochronous case. The interpretation of the solutions is proposed. We highlight three essentially different modes of motion. In the first mode, we have a dispersion of gas suspension from the origin with a relaxation of velocities and leading to the uniform motion. In the second mode, phases focus in the origin with finite nonzero velocities and infinitely increasing densities. In the third mode, phases are interpenetrating to each other and decelerating due to the friction force, that leads to an accumulation of particles and an unlimited increase of densities in different nonzero radiiuses with zero velocities.

1. Introduction

Consider a system of differential equations

\[
\begin{align*}
\frac{\partial \rho_1}{\partial t} + \vec{u}_1 \cdot \nabla \rho_1 + \rho_1 \text{div} \vec{u}_1 &= 0, \\
\frac{\partial \rho_2}{\partial t} + \vec{u}_2 \cdot \nabla \rho_2 + \rho_2 \text{div} \vec{u}_2 &= 0, \\
\rho_1 \left( \frac{\partial \vec{u}_1}{\partial t} + \vec{u}_1 \cdot \nabla \vec{u}_1 \right) + m_1 \nabla P(\rho_1, \rho_2) &= -\frac{\rho_2}{\tau} (\vec{u}_1 - \vec{u}_2), \\
\rho_2 \left( \frac{\partial \vec{u}_2}{\partial t} + \vec{u}_2 \cdot \nabla \vec{u}_2 \right) + m_2 \nabla P(\rho_1, \rho_2) &= \frac{\rho_2}{\tau} (\vec{u}_1 - \vec{u}_2),
\end{align*}
\]

(1)

that describes motions of a two-phase isothermal fluid. Here \(\vec{u}_1 = (u_1, v_1, w_1), \vec{u}_2 = (u_2, v_2, w_2)\) are velocity vectors of the first and the second phase, \(\rho_1, \rho_2\) are phase densities, \(P(\rho_1, \rho_2)\) is a pressure of a mixture, it is a functional parameter, \(m_2 = \rho_2/\rho_2^2\) is a volume concentration of the second phase, \(\rho_2\) is an absolute density of the second phase, it is a constant parameter, \(m_1 = 1 - m_2\) is a volume concentration of the first phase and \(\tau\) is a time of a relaxation.

System (1) was introduced by Kh.A. Rakhmatulin [1]. There are a lot of studies of this system [2-4]. The formation and the structure of shock waves in two-phase fluids were studied, for example, in articles [5,6]. There are more complicated systems of differential equations which take to account the chaotic pressure of the particles (the second phase) [5,7]. Many exact solutions of system (1) were found in the paper [8]. In this work, we study the smooth motions of gas suspension dynamics described by system (1) with two additional constraints: spherical symmetry and stationarity.
2. Spherically symmetric stationary flows

If we focus on isotropic and stationary motions, then the solution of system (1) has a form

\[ \vec{u}_i = \frac{\vec{v}}{r} q_i(r), \quad \rho_i = \rho_i(r), \quad r = \sqrt{x^2 + y^2 + z^2}, \quad i = 1, 2. \]

Here \( q_i(r) \) is a radial velocity and \( \rho_i(r) \) is a density of the phase on a sphere of radius \( r \). Substitution of the last formulas to system (1) leads to a system of ordinary differential equations

\[
\begin{aligned}
q_1 \rho'_1 + (2 \frac{\rho_1}{r} + q'_1) \rho_1 &= 0, \\
q_2 \rho'_2 + (2 \frac{\rho_2}{r} + q'_2) \rho_2 &= 0, \\
\rho_1 q_1 q'_1 + m_1 P' &= -\frac{\rho_2}{r}(q_1 - q_2), \\
\rho_2 q_2 q'_2 + m_2 P' &= \frac{\rho_2}{r}(q_1 - q_2),
\end{aligned}
\]

with differentiation with respect to \( r \). The first two equations are quickly integrated and we get conservation laws of a phase flow through a sphere of radius \( r \)

\[ \rho_1 q_1 r^2 = c_1, \quad \rho_2 q_2 r^2 = c_2. \]

If we get \( P = \frac{\rho_1 a^2}{q_1 r^2} \) (phases are ideal), then the last two equations in (2) after the substitutions \( \rho_1 = \frac{c_1}{q_1}, \rho_2 = \frac{c_2}{q_2 r^2} \) are reduced to a system of a form

\[ A(q, r) \begin{pmatrix} q'_1 \\ q'_2 \end{pmatrix} = \begin{pmatrix} b_1(q, r) \\ b_2(q, r) \end{pmatrix}. \]

After multiplication on the matrix \( A^{-1}(q, r) \), we get the system

\[
\begin{aligned}
q'_1 &= \frac{q_1 (2 a^2 c_1 c_7 r^2 (\rho_{22} r^2 q_2 - c_2) q_2 q_1^2 r + c_2 (q_1 - q_2) (a^2 c_1 r^2 \rho_{22} - q_1 (\rho_{22} r^2 q_2 - c_2)^2))}{\tau c_1 (q_2 (q_1^2 - a^2) (\rho_{22} r^2 q_2 - c_2)^2 - a^2 c_1 c_2 q_1)}, \\
q'_2 &= \frac{2 a^2 c_1^2 c_7 \rho_{22} r^2 q_2 q_1^2 r + c_1 (q_1 - q_2) (\rho_{22} r^2 q_2 - c_2) (-a^2 \rho_{22} q_2 r^2 + q_2 (\rho_{22} r^2 q_2 - c_2))}{\tau c_1 (q_2 (q_1^2 - a^2) (\rho_{22} r^2 q_2 - c_2)^2 - a^2 c_1 c_2 q_1)}. \\
\end{aligned}
\]

At this moment, we have no idea how to solve this system in an analytic form. Intention to construct a phase portrait for this dynamical system leads to difficulties with singular points. Therefore, firstly, consider the simplest case when the pressure \( P \) depends on time only (barochronous flows), then the derivative \( P' \) with respect to \( r \) equals zero in (2).

3. Barochronous flows

In this case, system (2) has a form

\[
\begin{aligned}
q_1 \rho'_1 + (2 \frac{\rho_1}{r} + q'_1) \rho_1 &= 0, \\
q_2 \rho'_2 + (2 \frac{\rho_2}{r} + q'_2) \rho_2 &= 0, \\
\rho_1 q_1 q'_1 &= -\frac{\rho_2}{r}(q_1 - q_2), \\
\rho_2 q_2 q'_2 &= \frac{\rho_2}{r}(q_1 - q_2).
\end{aligned}
\]

The solution of this system is

\[
\begin{aligned}
\rho_1 q_1 r^2 &= c_1, \quad \rho_2 q_2 r^2 = c_2, \\
c_1 q_1 + c_2 q_2 &= c_3, \quad -q_2 - \frac{c_4}{q} \ln |c_3 - dq_2| = br + c_4.
\end{aligned}
\]
where \( d = c_1 + c_2 \neq 0, b = \frac{c_1 + c_2}{c_1} \), and the constants \( c_1, c_2, c_3, c_4 \) are uniquely determined by the initial conditions

\[
\rho_1(r_0) = \rho_{10}, \quad \rho_2(r_0) = \rho_{20}, \quad q_1(r_0) = q_{10}, \quad q_2(r_0) = q_{20},
\]

and the equalities

\[
c_1 = \rho_{10} q_{10} r_0^2, \quad c_2 = \rho_{20} q_{20} r_0^2,
\]

\[
c_3 = c_1 q_{10} + c_2 q_{20}, \quad c_4 = -q_{20} - \frac{c_3}{c_1 + c_2} \ln |c_3 - (c_1 + c_2) q_{20}| - \frac{c_1 + c_2}{c_1} r_0.
\]

The first two equalities in (3) determine a phase flow through a sphere of radius \( r \). The third equality in (3) defines a relationship between velocities of phases, and the last equality determines a distribution of the second velocity along the radius \( r \).

There are two different cases \( c_1 c_2 > 0 \) and \( c_1 c_2 < 0 \). Consider the first case when \( c_1 c_2 > 0 \). We can not explicitly represent \( q_2 \) as a function of \( r \), so consider the function

\[
r(q_2) = \frac{1}{b} \left( -q_2 - \frac{c_3}{d} \ln |c_3 - dq_2| - c_4 \right).
\]

The graphs of the functions \( r(q_2) \), \( r(q_1) \) are presented in figures 1 and 2.

![Figure 1](image1.png)  
**Figure 1.** Case \( c_1, c_2 > 0, q^* = \frac{c_3}{d}, \tilde{q} = \frac{c_4}{c_1} \).  

![Figure 2](image2.png)  
**Figure 2.** Case \( c_1, c_2 < 0, q^* = \frac{c_3}{d}, \tilde{q} = \frac{c_4}{c_1} \).

In all figures we suppose \( c_4 \) such that \( r(q_2)|_{q_2=0} > 0 \). This assumption does not change a qualitative picture. A dashed line on figures indicates a prohibited mode of a phase motion. This mode is associated with the first three integrals in (3) that say velocities \( q_1, q_2 \) to be positive for all \( r \), if they are positive for \( r_0 \). Also some dashed lines are a consequence of the condition \( r > 0 \). The arrows illustrate the fact that if \( q_1, q_2 \) are positive then the distance to the origin should increase.

If the initial velocities are positive, then \( c_1, c_2, c_3 \) are positive, and the dynamics is determined by figure 1. In this case, there are two possibilities. If \( q_{20} > q_{10} \), then \( q^* > q_{10} \) and \( q^* < q_{20} \), so for all next \( r \) will be \( q_2 > q_1 \). When \( q_{20} < q_{10} \), then \( q^* < q_{10} \) and \( q^* > q_{20} \), so for all next \( r \) will be \( q_2 < q_1 \). Thus, for positive \( q_{10}, q_{20} \), the distance \( r \) unlimited increase, the velocities \( q_1, q_2 \) tend to \( q^* \) and the densities \( \rho_1, \rho_2 \) tend to zero. Here, we have a dispersion of a gas suspension from the origin with a relaxation of velocities and leading to the uniform motion.

If the initial velocities are negative, then \( c_1, c_2, c_3 \) are negative, and figure 2 presents the dynamics. There are also two cases. When \( q_{20} > q_{10} \), then for all next \( r > r_0 \) will be \( q_2 > q_1 \). If \( q_{20} < q_{10} \), then for all next \( r > r_0 \) will be \( q_2 < q_1 \). For the negative initial velocities \( q_{10}, q_{20} \), we have the collapse of densities when \( r = 0 \) with some finite velocities.

Consider the second case \( c_1 c_2 < 0 \). If \( q_{10} > 0 \) and \( q_{20} < 0 \), then \( c_1 > 0 \) and \( c_2 < 0 \). The dynamics of phases is presented in figures 3, 4 according to \( d > 0 \) or \( d < 0 \). In figures 3 and 4 the dynamics of the first phase is determined only in the interval \((0, \tilde{q})\), because
Case $c_1 > 0, c_2 < 0, d > 0, q^* = \frac{c_1}{d}, \tilde{q} = \frac{c_1}{c_1}$. 

Case $c_1 > 0, c_2 < 0, d < 0, q^* = \frac{c_1}{d}, \tilde{q} = \frac{c_1}{c_1}$. 

The rest of the graph when $q_1 > \tilde{q}$ is a dashed line. In figure 4, it is easy to see that $q_20 > q^*$, so the rest of the graph when $q_2 < q^*$ is a dashed line. In figures 3 and 4 we see the first and the second phase, which flow to each other and, due to the friction force, they are stopping. Therefore there are two radii $r_{1c}$ and $r_{2c}$ for the first and the second phase where the particles of the phase build up and the phase density equals infinity. So, in these radii, there are collapses of densities with zero velocities of phases.

If $q_{10} < 0$ and $q_{20} > 0$ then $c_1 < 0$ and $c_2 > 0$. The dynamics of phases is presented in figures 5, 6 according to $d < 0$ or $d > 0$. The motion of the first phase is determined only in the interval $(\tilde{q}, 0)$ for the same reasons as in the previous case. Therefore, the rest of the graph when $q_1 < \tilde{q}$ is a dashed line. In figures 5, 6 we see the same picture as in figures 3 and 4: phases are interpenetrating to each other and decelerating due to the friction force, that leads to an accumulation of particles and an unlimit increase of densities. So, in some two radii there are collapses of densities with zero velocities.

Note that in figures 3, 4 for $q_2$ and in figures 5, 6 for $q_1$, there may exists $c_4$ such that $\rho_2$ or $\rho_1$ collapses when $r_{ic}$ equals zero, but velocities do not.

Finally, we need to see the last case $d = 0$. Here we have a solution

$$
\rho_1 q_1 r^2 = c_1, \quad \rho_2 q_2 r^2 = c_2,
$$

$$
c_1 q_1 + c_2 q_2 = c_3, \quad \frac{q_2^2}{2} = \frac{c_3}{c_1} r + c_5,
$$

In this case we have next graphs of solutions, see figures 7 and 8. The interpretation of the graphs in figures 7 and 8 is the same one, as in the previous four cases.
Figure 7. Case $c_1 > 0, c_2 < 0, \tilde{q} = \frac{c_2}{c_1}, r^* = \frac{\tau c_1 c_2}{c_3}$.

Figure 8. Case $c_1 < 0, c_2 > 0, \tilde{q} = \frac{c_2}{c_1}, r^* = \frac{\tau c_1 c_2}{c_3}$.

Acknowledgments

The reported study was funded by RFBR according to the research projects No. 18-31-00226 (paragraph 3) and No. 18-08-00156 (paragraph 2).

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