LIFTING COARSE HOMOTOPIES

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Abstract. Coarse geometry, and in particular coarse homotopy theory, has proven to be a powerful tool for approaching problems in geometric group theory and higher index theory. In this paper, we continue to develop theory in this area by proving a Coarse Lifting Lemma with respect to a certain class of bornologous surjective maps. This class is wide enough to include quotients by coarsely discontinuous group actions, which allows us to obtain results concerning the coarse fundamental group of quotients which are analogous to classical topological results for the fundamental group. As an application, we compute the fundamental group of metric cones over negatively curved compact Riemannian manifolds.

1. Introduction

Coarse geometry studies those properties of spaces which are invariant at large scale – under quasi-isometry or, more generally, coarse equivalence. The notion of coarse equivalence arises naturally in geometric group theory, for example, because the word-metric on a finitely generated group does not depend – up to coarse equivalence – on the choice of generating set. Moreover, if a group acts geometrically on a proper geodesic metric space, then it is coarsely equivalent to that space (this is the Milnor-Svárč Lemma). This allows one to talk freely about large-scale properties such as the number of ends or Gromov hyperbolicity as properties of both the group (independent of presentation) and any space it acts on geometrically. Another consequence of the Milnor-Svárč Lemma is that the fundamental group of a closed Riemannian manifold is coarsely equivalent to the universal cover of that manifold. Roe was led to study the coarse geometry of complete Riemannian manifolds (for example, universal covers of compact manifolds) in order to formulate a notion of “coarse index” [13]. This coarse index is a generalization of the index of a differential operator on a compact manifold (the subject of the celebrated Atiyah-Singer Index Theorem) to the case where the manifold is no longer compact. Roe's ideas led to the formulation of the coarse Baum-Connes Conjecture (see e.g. [17] for a statement), which was later shown to be false. However, when the conjecture holds for a finitely generated group then it implies the Novikov Conjecture for all compact manifolds with $\pi_1(M) = G$. Guoliang Yu proved in 1998 that the coarse Baum-Connes Conjecture holds for spaces of finite asymptotic dimension [18], later improving his result to include all spaces which coarsely embed into Hilbert space [19].

Many of the successes of coarse geometry have been the result of translating important ideas and techniques from topology to the world of coarse geometry. Asymptotic dimension, mentioned above, was introduced by Gromov in [17] as a natural coarse version of Lebesgue covering dimension. Much
of Roe’s work in index theory revolves around coarse cohomology, a coarse version of Alexander-Spanier cohomology. Coarse versions of homotopy have also played an important role. Among the very first proofs of the coarse Baum-Connes Conjecture for certain spaces were a proof for manifolds with “Lipschitz good covers” by Guoliang Yu [17] and a proof by Higson and Roe for Gromov hyperbolic spaces [9], both of which used some notion of coarse homotopy equivalence. More recently, coarse homotopy theory is being developed by Bunke, Engel and others using the context of $\infty$-categories [2].

In [3], Dranishnikov proposed that it would be interesting to define the coarse fundamental group using the upper half-plane, but the coarse fundamental group was only recently formally defined in a paper by Mitchener-Norouzizadeh-Schick (hereafter MNS) [10] using ideas from Mitchener’s coarse homology theory. In that paper, MNS also introduce a notion of coarse homotopy very similar to one used by Bunke and Engel in [2], a definition which we will use with only a small modification (Definition 2.3). Another large-scale version of the fundamental group is the fundamental group at infinity studied in geometric group theory (see e.g. Chapter 16 of [6]), which is central to the long-standing Semistability Conjecture for finitely presented groups.

The goal of this paper is to contribute to the development of coarse homotopy theory with an eye towards applications in geometric group theory and higher index theory. In particular, we prove a Coarse Lifting Lemma and build out some related theory around it. Recall that the general Lifting Lemma from topology states that given any covering map $\pi : E \to B$, any homotopy $f : X \times I \to B$ and any continuous map $f_0 : X \to E$ such that the diagram of solid arrows below commutes, there is a homotopy $\tilde{f} : X \times I \to E$ making the diagram commute.

$$
\begin{array}{ccc}
X & \xrightarrow{f_0} & E \\
\downarrow{1_X \times 0} & & \downarrow{\pi} \\
X \times I & \xrightarrow{f} & B \\
\end{array}
$$

Our main result (Theorem 3.4) is a Coarse Lifting Lemma, replacing continuous maps with their coarse analogues, bornologous maps, and replacing $X \times I$ by a coarse cylinder object $I_p X$. It states that for a surjective bornologous map $\pi$ satisfying certain conditions and for any bornologous map $f_0$ and coarse homotopy $f$ such that the diagram of solid arrows below commutes, there is a coarse homotopy $\tilde{f}$ making the diagram commute.

$$
\begin{array}{ccc}
X & \xrightarrow{f_0} & E \\
\downarrow{i_0} & & \downarrow{\pi} \\
I_p X & \xrightarrow{f} & B \\
\end{array}
$$

The class of surjective bornologous maps $\pi$ for which this result holds includes quotients by group actions satisfying certain conditions (see Lemma 5.2). This allows us to mimic the application of covering space theory to fundamental groups found in topology (see for example Chapters 9 and 13 of [11]) in the coarse setting. In particular, we obtain a short exact sequence

$$
0 \to \pi_{1,\text{coarse}}(X) \to \pi_{1,\text{coarse}}(X/G) \to G \to 0.
$$
for certain kinds of group actions (Theorem 5.3).

A useful way to construct spaces with interesting large scale behaviour is to take a compact space \( M \) and construct the metric cone \( O M \) over it. In Section 3 we show how to apply Theorem 5.3 to obtain an isomorphism

\[
\pi_1^\text{coarse}(OM) \cong \pi_1(M),
\]

between the coarse fundamental group of \( OM \) and the (topological) fundamental group of \( M \) when \( M \) is a compact Riemannian manifold of non-positive sectional curvature (Theorem 6.1).

2. Coarse homotopies

We will work in the setting of metric spaces rather than the abstract setting of coarse spaces introduced by Roe [14] (see also [2] for a similar definition). Denote the space \([0,\infty)\) (see also the large scale spaces of Dydak-Hoffland [5]), but all the definitions and results in this paper can be easily generalized to that setting. Since a (classical) homotopy is a continuous map \( X \times I \to Y \), it is natural to define a coarse homotopy via a cylinder object (to replace \( X \times I \)) and a large-scale notion of continuous map. The latter notion is usually called a bornologous map.

Definition 2.1. A map \( f \) from a metric space \( X \) to a metric space \( Y \) is called bornologous if there is some function \( \rho \) (which we will call a control function for \( f \)) such that for any \( x, x' \in X \),

\[
d(f(x), f(x')) \leq \rho(d(x, x')).
\]

A map is called coarse if it is bornologous and the inverse image of a bounded set is a bounded set.

We will use a mildly adapted version of the definition of coarse homotopy introduced by MNS in [10] (see also [2] for a similar definition). Denote the space \([0,\infty)\) \(\subseteq\mathbb{R} \) by \( \mathbb{R}_+ \).

Definition 2.2. Let \( X \) be a metric space and \( p : X \to [-1,\infty) \) be a function. Then the \( p \)-cylinder is defined as

\[
I_pX = \{(x,t) \in X \times \mathbb{R}_+ | t \leq p(x) + 1 \},
\]

with the \( \ell^2 \) metric.

In other words, a coarse cylinder is a cylinder which “opens out” in a controlled way as you go to infinity. A canonical example we will need later is \( I_p(\mathbb{R}_+) \) where \( p \) is the map \( x \mapsto x - 1 \), which is just the subset of the plane \( \{(x,t) | t \leq x \} \). We will call this space the metric cone over \([0,1] \) and denote it by \( c([0,1]) \) (see Figure 1).

When the map \( p \) is coarse (as it will be when defining coarse homotopies), there are the evident inclusions \( i_0 : X \to I_pX \) and \( i_1 : X \to I_pX \) which send \( x \) to \((x,0)\) and \((x,p(x) + 1)\) respectively. In the case of \( c([0,1]) \), these are the maps \( i_0(x) = (x,0) \) and \( i_1(x) = (x,x) \).

Definition 2.3. Let \( X \) and \( Y \) be metric spaces. Then a coarse homotopy is a coarse map \( H : I_pX \to Y \) for some coarse map \( p : X \to [-1,\infty) \). We say that the coarse homotopy \( H \) is from \( f \) to \( g \) if \( H \circ i_0 = f \) and \( H \circ i_1 = g \).

To get an intuitive picture of this definition, notice that for every \( x \in X \), \( p(x,\cdot) \) gives a coarse map from \([0,p(x) + 1]\) to \( Y \), which we can think of as a coarse path associated to \( x \). Just as for a classical homotopy, the definition requires the collection of all such paths to fit together in some sense.
Example 2.4. Recall that two coarse maps \( f, g : X \to Y \) are called close if \( \sup_x d(f(x), g(x)) < \infty \).

It is easy to check that any two close maps from a metric space are coarsely homotopic.

Finally, paralleling the classical case, we say that a coarse homotopy \( H : I_p X \to Y \) is a coarse homotopy relative to \( A \subseteq X \) if \( H(a, t) = H(a, 0) \) for all \( a \in A \) and all \( 0 \leq t \leq p(a) + 1 \).

Remark 2.5. Note that the original definition in [10] requires \( p \) to have domain \( \mathbb{R}_+ \) when defining a \( p \)-cylinder. This is a disadvantage in that we will eventually want to view the cone \( c([0, 1]) \) over the interval as a \( p \)-cylinder, which requires us to define \( p(x) = x - 1 \). Clearly any coarse homotopy from \( I_p X \) where \( p \) takes values only in \( \mathbb{R}_+ \) is also a coarse homotopy in our sense. Conversely, any coarse homotopy from \( I_p X \) in our sense gives rise naturally to a coarse homotopy in the sense of [10] via extending by a constant coarse homotopy.

3. Coarse Lifting lemma

Now that we have a definition of coarse homotopy, we proceed to state and prove the Coarse Lifting Lemma. The classical Lifting Lemma is stated for covering maps, so we introduce a coarse analog here.

Definition 3.1. Let \( f : X \to Y \) be a bornologous map between metric spaces. Then \( f \) is called a soft quotient map if it is surjective and for every \( R > 0 \) there is an \( S > 0 \) such that if \( d(f(x), y) \leq R \) for some \( x \in X \), \( y \in Y \), then there is an \( x' \in f^{-1}(y) \) with \( d(x, x') \leq S \).

The terminology is based on [4], in which the authors define (weakly) soft maps in the context of balleans. The following example is also the main example for the applications in this paper.

Example 3.2. Let \( G \) be a group acting on a metric space \( X \) by isometries, and let \( X/G \) be the orbit space equipped with the metric

\[
d([x], [y]) = \inf \{ d(x, y') \mid y' \in [y] \}.
\]

Then the quotient map \( q : X \to X/G \) is a weakly soft quotient map.

We now introduce the following new definition.
Definition 3.3. Let \( f : X \to Y \) be a map between metric spaces. We say that \( f \) has **scattered fibres** if for every \( R > 0 \), there is a bounded set \( K \) in \( Y \) such that if \( f(x) = f(x') \notin K \) for \( x, x' \in X \) and \( d(x, x') \leq R \), then \( x = x' \).

Recall that a surjective map \( f : X \to Y \) between metric spaces was called **asymptotically faithful** in [10] if for every \( R > 0 \) there is a bounded set \( K \subset Y \) such that \( f \) is an isometry on every \( R \)-ball not intersecting \( f^{-1}(K) \). It is easy to check that any such map has scattered fibres, though it need not be a soft quotient map. We are now ready to state and prove the Coarse Lifting Lemma.

**Theorem 3.4 (Coarse Lifting Lemma).** Let \( \pi : A \to B \) be a soft quotient map with scattered fibres, let \( f : I_p X \to B \) be a coarse homotopy and let \( f_0 : X \to A \) be a coarse map such that the diagram of solid arrows below commutes. Then there is a coarse homotopy \( \tilde{f} : I_p X \to A \) making the diagram commute.

\[
\begin{array}{ccc}
X & \xrightarrow{f_0} & A \\
\downarrow{\pi} & & \downarrow{f} \\
I_p X & \xrightarrow{f} & B
\end{array}
\]

Moreover, if \( \tilde{f} \) and \( \tilde{f}' \) are two coarse homotopies making the diagram commute, then there is a bounded set \( K \) in \( I_p X \) such that \( \tilde{f}|_{I_p X \setminus K} = \tilde{f}'|_{I_p X \setminus K} \).

**Proof.** The main idea, based on the topological situation, is to lift each "coarse path" \( f(x, \cdot) \) individually and prove that this gives the right map. Let \( \rho \) be a control function for \( f \). Since \( \pi \) is a soft quotient map, there is a \( T > 0 \) so that if \( \pi(a) \) is within \( \rho(1) \) of \( \pi(a') \) then there is an \( a'' \in \pi^{-1}(\pi(a')) \) such that \( d(a, a'') \leq T \). In particular, we can define \( \tilde{f}(x, \varepsilon) \) for any \( 0 < \varepsilon \leq 1 \) so that \( \pi(\tilde{f}(x, \varepsilon)) = f(x, \varepsilon) \) and \( d(\tilde{f}(x, \varepsilon), f_0(x)) \leq T \). Continuing by induction and varying \( x \) we can define \( \tilde{f} \) in such a way that \( \pi \circ \tilde{f} = f \) and \( f(x, n) \) is within \( T \) of \( \tilde{f}(x, n + \varepsilon) \) whenever \( n \in \mathbb{N} \) and \( \varepsilon \leq 1 \). In particular, the maps \( \tilde{f}(x, \cdot) \) are coarse maps which share a control function, so if we can show the same for the maps \( \tilde{f}(\cdot, t) \), we will be done with the first part of the theorem. Let \( \rho_0 \) be the control function for \( f_0 \), let \( \varepsilon \leq 1 \) and let \( R > 0 \). Since \( \pi \) is a soft quotient map, there is an \( S > \rho_0(R) \) such that if \( d(\pi(a), b) \leq \rho(R) \) then there is an \( a' \in \pi^{-1}(b) \) with \( d(a, a') \leq S \). Since \( \pi \) has scattered fibres, there is a bounded set \( K \) in \( B \) such that if \( \pi(a) = \pi(a') \) and \( a \) is within \( 2S + 2T \) of \( a' \), then \( a = a' \). Because \( f \) and \( \rho \) are coarse we can choose a bounded set \( L \) in \( X \) such that \( L \times \mathbb{R}_+ \cap I_p X \) contains \( f^{-1}(K) \). We claim that \( \tilde{f}(x, \varepsilon) \) and \( \tilde{f}(x', \varepsilon) \) are at most \( S \) apart whenever \( x, x' \notin L \) and \( d(x, x') \leq R \). For \( x, x' \notin L \), let \( a \) be a point in \( \pi^{-1}(f(x', \varepsilon)) \) which is at most \( S \) away from \( \tilde{f}(x, \varepsilon) \). Then \( a \) and \( x' \) are at most \( 2T + 2\rho_0(R) < 2T + 2S \) apart, and thus \( a = x' \) which proves the claim. An inductive argument now shows that \( \tilde{f}(x, t) \) and \( \tilde{f}(x', t) \) are at most \( S \) apart for any \( t \) whenever \( x, x' \notin L \) and \( d(x, x') \leq R \). Since \( \tilde{f}(L \times \mathbb{R}_+ \cap I_p X) \) is clearly bounded, we get the required result. The final part of the statement is easy to prove using similar arguments.

4. **Coarse fundamental group**

Along with their definition of coarse homotopy, MNS also introduce the notion of coarse fundamental groups in [10]. Since the classical Lifting Lemma can be used to compute topological fundamental groups, it is no surprise that we have an analogous situation in the coarse setting.
If \( f, g : X \to Y \) are two coarse maps between metric spaces, then they are **coarsely homotopic** if there exists a coarse homotopy between them, that is, there is a coarse map \( H : I_p X \to Y \) such that \( H \circ i_0 = f \) and \( H \circ i_1 = g \). Given a metric space \( X \), an \( \mathbb{R}_+ \)-**basepoint** is a coarse map \( b : \mathbb{R}_+ \to X \). Recall that \( c([0,1]) \), the metric cone over the interval, is defined as \( \{(x,t) \mid t \leq x\} \subseteq \mathbb{R}^2 \) with the inherited metric. We denote the boundary \( \{(x,t) \in c([0,1]) \mid t = x \text{ or } t = 0\} \) by \( \partial c([0,1]) \).

**Remark** 4.1. It appears that the construction of \( c([0,1]) \) above differs from the representation of \( c([0,1]) \) in [10] (where \( x \geq y \) is replaced by \( x \leq y \)), but ours seems more natural if one is inclined to view \( c([0,1]) \) also as a kind of \( p \)-cylinder, and the two constructions yield the same space up to isometry.

**Definition 4.2.** Let \( X \) be a metric space with a chosen \( \mathbb{R}_+ \)-basepoint \( b : \mathbb{R}_+ \to X \). Then the 1st **coarse homotopy group** \( \pi_1^\text{coarse}(X, b) \) is the set of all relative coarse homotopy (relative to \( \partial c([0,1]) \)) classes of coarse maps \( \alpha : c([0,1]) \to X \) such that \( \alpha \circ i_0 = \alpha \circ i_1 = b \).

Before describing the group structure, we should record the following “pasting lemmas” which are commonly used in coarse geometry, and which are easy to prove.

**Lemma 4.3.** Let \( X \) be a geodesic space, and let \( X = \cup_i A_i \) be a decomposition of \( X \) into a finite number of closed subsets. If \( f : X \to Y \) is a map which is coarse when restricted to each \( A_i \), then \( f \) is coarse.

**Lemma 4.4.** Let \( X \) be a geodesic space, and let \( X = \cup_i A_i \) be a decomposition of \( X \) so that every compact set in \( X \) intersects only finitely many of the \( A_i \). If \( f : X \to Y \) is a map which is \( C \)-Lipschitz when restricted to each \( A_i \), then \( f \) is \( C \)-Lipschitz.

The group structure is described by MNS and is constructed in an analogous way to the topological situation. Generalizing that construction, given two coarse maps \( \alpha, \beta : c([0,1]) \to X \) for which \( \sup_x d(\alpha(i_1(x)), \beta(i_0(x))) < \infty \), one can construct the concatenation \( \alpha \ast \beta : c([0,1]) \to X \) via

\[
\alpha \ast \beta(x,t) = \begin{cases} 
\alpha(x,2t) & t \leq x/2 \\
\beta(x,2t-x) & x/2 < t \leq x 
\end{cases}
\]

One easily checks that \( \alpha \ast \beta \) indeed defines a coarse map using Lemma 4.3. The group structure on \( \pi_1^\text{coarse}(X, b) \) is then given by applying \( \ast \) to representatives: \([\alpha] \cdot [\beta] = [\alpha \ast \beta] \).

We will be interested in results for general bornologous maps, not just for coarse maps. An obvious obstacle here is that a bornologous map \( f : X \to Y \) does not in general induce a map on coarse fundamental groups since it may not send coarse homotopies to coarse homotopies. We will thus need a relative version of coarseness. This is just a special case of the notion of coarse map between bornologous coarse spaces as introduced in [2], but we will not need the general theory here.

**Definition 4.5.** Let \( f : X \to Y \) be a bornologous map. A map \( g : W \to X \) is called \( f \)-coarse if \( f \circ g \) is a coarse map.

Note that any \( f \)-coarse map is necessarily a coarse map. For a bornologous map \( f : X \to Y \), there is then a corresponding notion of \( f \)-coarse homotopy between two maps \( g_0, g_1 : W \to X \), namely that there is a \( f \)-coarse map \( H : I_p W \to X \) such that \( H \circ i_0 = g_0 \) and \( H \circ i_1 = g_1 \).
Definition 4.6. Let $X$ be a metric space with a chosen $\mathbb{R}_+$-basepoint $b : \mathbb{R}_+ \rightarrow X$ and a bornologous map $f : X \rightarrow Y$. Then the 1st $f$-coarse homotopy group $\pi_{1,f}^\text{coarse}(X, b)$ is the set of all relative $f$-coarse homotopy (relative to $\partial c([0,1])$) classes of $f$-coarse maps $\alpha : c([0,1]) \rightarrow X$ such that $\alpha \circ i_0 = \alpha \circ i_1 = b$.

It is easy to check that the same group operations work for the $f$-coarse homotopy group. Any bornologous map $f : X \rightarrow Y$ now induces a group homomorphism

$$f_* : \pi_{1,f}^\text{coarse}(X, b) \rightarrow \pi_{1}^\text{coarse}(Y, f \circ b)$$

for any $\mathbb{R}_+$-basepoint $b$ in $X$. Our next theorem is a coarse lifting correspondence similar to the lifting correspondence in topological covering space theory (see e.g. Chapter 9 of [11]), and the proof is based on the topological one.

Proposition 4.7 (Lifting Correspondence). Let $\pi : X \rightarrow Y$ be a soft quotient map with scattered fibres. Let $b : \mathbb{R}_+ \rightarrow Y$ be an $\mathbb{R}_+$-basepoint in $Y$ and $b'$ a lift of $b$ to $X$. Suppose that for any other lift $b'' : \mathbb{R}_+ \rightarrow X$ of $b$, there is a $\pi$-coarse homotopy $H$ from $b'$ to $b''$. Then there is a canonical (once $b'$ is chosen) bijection between the right cosets of $\pi_*(\pi_{1,\pi}(X, b'))$ in $\pi_1^\text{coarse}(Y, b)$ and equivalence classes of liftings $b'' : \mathbb{R}_+ \rightarrow X$ of $b$ under the equivalence relation $b'' \sim b'''$ if

$$\{t \in \mathbb{R} \mid b''(t) \neq b'''(t)\}$$

is bounded.

Proof. If $\alpha : c([0,1]) \rightarrow Y$ represents a class in $\pi_{1}^\text{coarse}(Y, b)$, then we can lift it by Theorem 3.4 to $\hat{\alpha} : c([0,1]) \rightarrow X$ such that $\hat{\alpha} \circ i_0 = b'$. Define $\Phi(\alpha)$ to be the map $\hat{\alpha} \circ i_1$. We will show that $\Phi$ gives the required bijection. It is easy to show using the uniqueness part of Theorem 3.4 that $[\Phi(\alpha)]_\sim$ is well-defined, and moreover that $[\alpha] = [\beta] \in \pi_1^\text{coarse}(Y, b)$ implies $\Phi(\alpha) \sim \Phi(\beta)$. If $\Phi(\alpha) \sim \Phi(\beta)$, with $\hat{\alpha}$ and $\hat{\beta}$ lifts of $\alpha$ and $\beta$ respectively, then $\hat{\alpha} \ast \hat{\beta}^*$ (the concatenation of $\alpha$ with the reverse of $\hat{\beta}$) represents an element of $\pi_{1,\pi}(X, b')$ whose image under $\pi$ is $\alpha \ast \beta^*$. It follows that $\alpha$ and $\beta$ are in the same right coset of $\pi_*(\pi_{1,\pi}(X, b'))$ in $\pi_1^\text{coarse}(Y, b)$. An easy argument shows the converse, so that $\Phi$ descends to an injection from cosets of $\pi_*(\pi_{1,\pi}^\text{coarse}(X, b'))$ in $\pi_1^\text{coarse}(Y, b)$ to ~-equivalence classes of liftings. To show surjectivity, note that by assumption any lift $b''$ of $b$ is connected to $b'$ by a $\pi$-coarse homotopy $H : I_p\mathbb{R}_+ \rightarrow X$. We may adapt $b''$ so that $b''(0) = b'(0)$ without changing its ~-class. An easy adaptation of Lemma 2.6 in [10] now shows that $H$ gives rise to a map $H' : c([0,1]) \rightarrow X$ with $H' \circ i_0 = b'$ and $H' \circ i_1 = b''$. Since $H'$ is a lift of the coarse homotopy $\pi \circ H$ from $b$ to $b$, we have that $\Phi(\pi \circ H) = [b'']$ as required. \hfill $\square$

In the setting of the Proposition 4.7 if $\pi$ is actually a coarse map then the $\pi$-coarse homotopy groups coincide with the coarse homotopy groups, so we get the following.

Corollary 4.8. Let $\pi : X \rightarrow Y$ be a coarse soft quotient map with scattered fibres. Let $b : \mathbb{R}_+ \rightarrow Y$ be an $\mathbb{R}_+$-basepoint in $Y$ and $b'$ a lift of $b$ to $X$. Suppose that for any other lift $b'' : \mathbb{R}_+ \rightarrow X$ of $b$, there is a coarse homotopy $H$ from $b'$ to $b''$. Then there is a canonical bijection between the cosets of $\pi_*(\pi_{1,\pi}^\text{coarse}(X, b'))$ in $\pi_1^\text{coarse}(Y, b)$ and ~-equivalence classes of liftings $b' : \mathbb{R}_+ \rightarrow X$ of $b$. In particular, if $\pi_{1,\pi}^\text{coarse}(X, b')$ is trivial, then there is a canonical (once $b'$ has been chosen) bijection between $\pi_1^\text{coarse}(Y, b)$ and ~-equivalence classes of lifts of $b$. 


5. Group actions

We now turn our attention to quotients by group actions. For simplicity, we will consider only actions by isometries, but the results apply slightly more generally to uniformly bornologous actions. Given a group $G$ acting on a metric space $X$ by isometries, we will assume that the quotient space $X/G$ is given the metric as in Example 5.2.

**Definition 5.1.** Let $G$ be a group acting on a metric space $X$. We say that $G$ acts uniformly coarsely discontinuously if for every $R > 0$, there is a bounded set $K$ such that if $x \notin \cup_{g \in G} g(K)$ and $g$ is not the identity, then $d(x, g \cdot x) > R$.

Note that the term “uniformly coarsely discontinuously” is based on (but different from) the definition of “coarsely discontinuously” in [8], which is more suited to the study of warped spaces than orbit spaces.

**Lemma 5.2.** Let $G$ be a group acting on a metric space $X$ by isometries. Let $X/G$ be the orbit space and $q : X \to X/G$ the natural quotient map. Suppose further that the action of $G$ is uniformly coarsely discontinuous. Then $q$ has scattered fibres.

**Proof.** This is obvious from the definition, once one notices that $q^{-1}(q(K)) = \cup_{g \in G} g(K)$. \qed

We are now ready to prove a result concerning the first coarse fundamental group of $X/G$.

**Theorem 5.3.** Let $G$ be a group acting on a metric space $X$ by isometries. Let $X/G$ be the orbit space and $q : X \to X/G$ the natural quotient map. Suppose further that the action of $G$ is uniformly coarsely discontinuous. Let $b : \mathbb{R}_+ \to X/G$ be an $\mathbb{R}_+$-basepoint such that for any two lifts $b', b'' : \mathbb{R}_+ \to X$, there is a $q$-coarse homotopy $H$ from $b'$ to $b''$. Then for any lift $b'$ of $b$, we have a canonical short exact sequence

$$
0 \longrightarrow \pi_1^{\text{coarse}}(X, b') \xrightarrow{q_*} \pi_1^{\text{coarse}}(X/G, b) \longrightarrow G \longrightarrow 0.
$$

**Proof.** The map $q$ is a soft quotient map with scattered fibres, so we can apply Proposition 4.7 to conclude that the right cosets of $q_*(\pi_1^{\text{coarse}}(X, b'))$ in $\pi_1^{\text{coarse}}(X/G, b)$ are in bijection with $\sim$-equivalence classes of lifts of $b$. Note that we have a lift $g \circ b'$ of $b$ for every $g \in G$. It is easy to check that if $g \circ b' \sim h \circ b'$ then $g = h$ using the fact that the action is uniformly coarsely discontinuous. On the other hand, if $b''$ is some other lift of $b$ then for every $t$, $b''(t) = g_t \cdot b'(t)$ for some $g_t \in G$. If $|t - t'| \leq 1$, then $g_t'(t)$ and $g_t^{-1} g_t(b'(t))$ are at most $2\rho(1)$ apart, where $\rho$ is the control function for $b$. But then $g_t = g_t'$ whenever $|t - t'| < 1$ outside of a bounded set. We have thus shown that for any lift $b''$ of $b$, we have $b'' \sim g \circ b'$ for a unique $g \in G$. This bijection between $G$ and $\sim$-classes of lifts of $b$ gives rise to a bijection from cosets of $q_*(\pi_1^{\text{coarse}}(X, b'))$ in $\pi_1^{\text{coarse}}(X/G, b)$ to elements of $G$ via Proposition 4.7. All that remains is to check that this bijection respects the group operations and that the map $q_*$ is injective, both of which are easy to show. \qed

**Corollary 5.4.** Let $G$ be a group acting on a metric space $X$ by isometries such that for every bounded set $K$, the set $\cup_{g \in G} g(K)$ is also bounded. Let $X/G$ be the orbit space and $q : X \to X/G$ the natural quotient map. Suppose further that the action of $G$ is uniformly coarsely discontinuous.
Let $b : \mathbb{R}_+ \to X/G$ be an $\mathbb{R}_+$-basepoint such that for any two lifts $b', b'' : \mathbb{R}_+ \to X$, there is a coarse homotopy $H$ from $b'$ to $b''$. Then for any lift $b'$ of $b$, we have a canonical short exact sequence

$$0 \to \pi_1^\text{coarse}(X, b') \xrightarrow{q^*} \pi_1^\text{coarse}(X/G, b) \to G \to 0.$$ 

**Proof.** Easy once we notice that $q$ is coarse. \qed

Notice that the condition $K$ bounded $\implies \cup_{g \in G} g(K)$ bounded from the above corollary is always satisfied if $G$ is finite.

### 6. Application to metric cones

In order to demonstrate how the results of the previous section can be used to compute coarse fundamental groups of spaces, we consider metric cones over compact Riemannian manifolds of non-positive sectional curvature (for convenience, all manifolds in this section will be assumed to be connected unless stated otherwise). Specifically, we prove the following theorem.

**Theorem 6.1.** Let $M$ be a compact Riemannian manifold of non-positive sectional curvature, and let $\mathcal{O}M$ be the metric cone over $M$. Let $b$ be any $\mathbb{R}_+$-basepoint in $\mathcal{O}M$. Then there is an isomorphism

$$\pi_1^\text{coarse}(\mathcal{O}M, b) \cong \pi_1(M),$$

where $\pi_1(M)$ is the fundamental group of $M$.

The main idea of the proof is to write such a cone as the quotient of the cone over the universal cover of the manifold. This works because the cone over the universal cover is sufficiently “trivial” at the level of coarse fundamental groups. Unfortunately, we require a number of technical lemmas before we can execute this idea.

Given a Riemannian manifold $(M, g_M)$, we define the **metric cone** $\mathcal{O}M$ over $M$ to be the manifold with boundary $M \times [1, \infty)$ with the Riemannian metric $t^2 g_M + \mathbb{R}$. When $M$ is compact, this is known to coincide, up to coarse equivalence, with other common constructions of the metric cone (see e.g. [15]). The following lemma is easy to prove, but will be useful to record for later calculations.

**Lemma 6.2.** Let $\mathcal{O}M$ be the metric cone over a Riemannian manifold $M$. If $(x, t)$ and $(x', t')$ are two points in $\mathcal{O}M$, then we have the following inequalities.

$$d_{\mathcal{O}M}((x, t), (x', t')) \leq |t - t'| + d_M(x, x') : t$$

$$|t - t'| \leq d_{\mathcal{O}M}((x, t), (x', t'))$$

$$d_M(x, x') \leq d_{\mathcal{O}M}((x, t), (x', t'))$$

The main fact about Riemannian manifolds of non-positive sectional curvature we will need is the following lemma. Recall that a **Hadamard manifold** is a complete simply connected Riemannian manifold of non-positive sectional curvature.
Lemma 6.3. Let $\gamma_1, \gamma_2 : [0, \infty) \to M$ be unit speed geodesics on a Hadamard manifold. Then for any $\theta \in [0, 1]$ and $q, q' > 0$,
\[ d(\gamma_1(\theta q), \gamma_2(\theta q')) \leq \max(d(\gamma_1(0), \gamma_2(0)), d(\gamma_1(q), \gamma_2(q'))). \]

Proof. This follows from the fact that any Hadamard manifold is a CAT(0) space (this is a consequence of the Cartan-Hadamard theorem and a result of Alexandrov; see Theorem 1A.6 and Theorem 4.1 in [1]).

The following two lemmas allow us to move from locally Lipschitz to globally Lipschitz via coarse homotopy (note the similarity to Lemma 4.2 in [9] which concerns a different notion of coarse homotopy, and which allows one to move from continuous maps to coarse ones).

Lemma 6.4. Let $f : c([0,1]) \to Y$ be a map to a metric space $Y$ such that for every $K > 0$ there is an $L_K > 0$ so that
\[ d(f(x,t), f(x',t')) \leq L_K \cdot d((x,t), (x',t')) \]
for all $x, x', t, t' \in [0, K]$. Then there is a coarse map $g : c([0,1]) \to c([0,1])$ which is coarsely homotopic to the identity such that $f \circ g$ is a 1-Lipschitz map. In particular, $f$ is coarsely homotopic to a 1-Lipschitz map.

Proof. We may assume that the $L_K$ are increasing, are integer valued and are all at least 1. The map $g$ will have the form $g(x,t) = (\rho(x), t\rho(x)/x)$ for a monotone map $\rho : \mathbb{R}_+ \to \mathbb{R}_+$. Define $\rho$ on $[0, 2L_2]$ to be the linear map from $[0, 2L_2]$ to $[0, 1]$, and proceed by induction as follows: if $\rho$ has been defined for $x \in [0, n]$, $n \geq 1$, with $\rho(n) = k$, define $\rho$ on $[n, n + L_{k+2}(k+2)]$ to be an affine map with image $[k, k+1]$. One easily checks that the map $f \circ g$ is 1-Lipschitz using Lemma 6.4.

Moreover, $g$ is coarsely homotopic to the identity via the obvious straight line homotopy.

Lemma 6.5. Let $f : c([0,1]) \to OM$ be a $p_2$-coarse map where $M$ is a Hadamard manifold with $p_2 : OM \to [1, \infty)$ the projection onto the second coordinate. Then $f$ is $p_2$-coarsely homotopic to a 1-Lipschitz map $f'$. Moreover, if $f \circ i_0 = f \circ i_1$, then we can choose the $p_2$-coarse homotopy $H$ so that $H((x,0), s) = H((x,x), s)$ for all $s$.

Proof. Let $A = c([0,1]) \cap \mathbb{Z}^2$ be the set of points in $c([0,1])$ with integer coordinates, and let $f_A$ be the restriction of $f$ to $A$. Each of the maps $p_1 \circ f_A : A \to M$ and $p_2 \circ f_A : A \to [1, \infty)$, where $p_1$ is the projection onto the first coordinate, can be extended to Lipschitz maps by interpolating with geodesics. Taking the product of these extensions, we obtain a map $g : c([0,1]) \to OM$ which is close to $f$ (and hence coarsely homotopic to it), but which may be only locally Lipschitz on each square $c([0,1]) \cap [k,k+1]^2$. However, by Lemma 6.4, $g$ is $p_2$-coarsely homotopic to a 1-Lipschitz map $f'$ on all of $c([0,1])$, so we obtain the required result (after checking that both coarse homotopies satisfy the additional condition at the end of the lemma).

The proof of the following Proposition is based on the proof of Proposition 5.2 in [10].

Proposition 6.6. Let $\alpha : c([0,1]) \to OM$ be a $p_2$-coarse map where $M$ is a Hadamard manifold with $p_2$ the projection onto the second coordinate. Suppose that $\alpha \circ i_0 = \alpha \circ i_1$. Then $\alpha$ is $p_2$-coarsely homotopic to a map of the form $\beta(x, t) = b(x)$ via a $p_2$-coarse homotopy $H$ satisfying $H((x,0), s) = H((x,x), s)$ for all $s$. 

Proof. Pick any $p \in M$. By adjusting $\alpha$ at one point if necessary, we may assume that $\alpha(0, 0) = (p, 1)$. By Lemma 6.5 we may also suppose that $\alpha$ is 1-Lipschitz. We start by noticing that the map $\alpha$ is coarsely homotopic to the map $\alpha' : c([0, 1]) \to OM$ given by

$$\alpha'(x, t) = \alpha(x/\max(1, \sqrt{x}), t/\max(1, \sqrt{x})).$$

via a homotopy $H_1((x, t), s) = (x - s, t(x - s)/x)$ for $s \leq x - \max(1, \sqrt{x})$ composed with $\alpha$. Note that $\alpha \circ H_1((x, 0), s) = \alpha \circ H_1(\alpha(x, s), s)$ for all $s$. Now consider the map $H : IqOM \to OM$ defined by $H((y, u), s) = (f(y, s/u), u)$, with $f$ is defined by

$$f(x, t) = \begin{cases} \gamma_x(t) & t \leq d(p, x) \\ p & t > d(p, x) \end{cases}$$

where $\gamma_x$ is the unique unit speed geodesic from $x$ to $p$ in $M$, and $q(x, t) = d(x, p)t - 1$. This naturally leads to a map

$$H' : I_{qoa}c([0, 1]) \to OM$$

given by $H'((x, t), s) = H(\alpha'(x, t), s)$ for $s \leq q(\alpha'(x, t)))$. Note that $H' \circ i_1$ has image contained in $p \times [1, \infty)$. We now show that $H'$ is a $p_2$-coarse homotopy.

Let $(x, t)$ and $(x', t')$ be two points of distance at most 1 apart in $c([0, 1])$, with $x, x' > 1$. Using the Mean Value Theorem, we have

$$d((\sqrt{x}/\sqrt{x}), (\sqrt{x'}/\sqrt{x'})) \leq |\sqrt{x} - \sqrt{x'}| + |t/\sqrt{x} - t/\sqrt{x'}| + |t/\sqrt{x'} - t'/\sqrt{x'}|$$

$$\leq \frac{1}{2\sqrt{x}} + \frac{t}{2x\sqrt{x}} + \frac{|t - t'|}{\sqrt{x}} \leq \frac{2}{\sqrt{x}}.$$

Let $\alpha'(x, t) = (y, u)$ and $\alpha'(x', t') = (y', u')$. By the above and Lemma 6.2 we have $|u - u'| \leq 2/\sqrt{x}$ and $d(y, y') \leq 2/\sqrt{x}$. Since $\alpha'(0, 0) = (p, 1)$, we have that $d(y, p)$ and $u$ are both bounded above by $(t/\sqrt{x} + \sqrt{x}) \leq 2/\sqrt{x}$ by Lemma 6.2. Thus,

$$|q(y, u) - q(y', u')| \leq d(y', p) \cdot |u' - u| + u \cdot |d(y, p) - d(y', p)|$$

$$\leq d(y', p) \cdot |u' - u| + u \cdot d(y, y')$$

$$\leq 2\sqrt{x} \cdot \frac{2}{\sqrt{x}} + 2\sqrt{x} \cdot \frac{2}{\sqrt{x}} = 8$$

which, using the fact that $c([0, 1])$ is a geodesic space, shows that $q \circ \alpha'$ is coarse for the region where $x, x' > 1$. Since the image of the region where $x, x' \leq 1$ under $q \circ \alpha'$ is clearly bounded, we get coarseness on all of $c([0, 1])$. With a view to showing coarseness of $H'$, suppose further that $s \leq \min(q(\alpha'(x, t))), q(\alpha'(x', t')))$. The distance between $H'((x, t), s)$ and $H'((x', t'), s))$ is bounded above by

$$d(H((y, u), s), H'(y', u', s)) + d(H((y', u', s)), H((y', u'), s)).$$

Looking at the triangle $(p, y, y')$ and using the fact that $M$ is a CAT(0) space, we have that $d(f(y, s/u), f(y', s/u)) \leq d(y, y')$, so the first term is bounded by

$$u \cdot d(y, y') \leq (2\sqrt{x})(2/\sqrt{x}) = 4.$$
For the second term, assume without loss of generality that \( u \leq u' \). Then the second term is bounded above by (using the Mean Value theorem again),

\[
    u \cdot d(f(y', s/u), f(y', s/u')) + |u - u'| \leq u \cdot s \cdot (1/u - 1/u') + |u - u'| \\
    \leq u \cdot u' \cdot d(y, p) \cdot \frac{2}{\sqrt{x}} \cdot \frac{1}{u^2} + \frac{2}{\sqrt{x}} \\
    \leq d(y, p) \cdot \frac{2}{\sqrt{x}} + \frac{2}{\sqrt{x}} \leq 4 + \frac{2}{\sqrt{x}}
\]

which is bounded. It is easy to check that \( H' \) is coarse in the third coordinate; indeed,

\[
    d(H((y, u), s), H((y, u), s')) \leq |s/u - s'/u| + u \leq |s - s'|.
\]

We have thus shown that \( H' \) is coarse, and it is moreover clearly \( p_2 \)-coarse if \( \alpha \) is \( p_2 \)-coarse. The image of the map \( \beta' = H' \circ i_1 \) is completely contained in \( p \times [1, \infty) \), and so is \( p_2 \)-coarsely homotopic relative to the boundary \( \partial c([0, 1]) \) to the map \( \beta(x, t) = \beta'(x, 0) \) by a linear homotopy (or by invoking Theorem 5.6 of [10] for the space \([1, \infty)\)). It is easy to check that \( H'(x, 0, s) = H'(x, x, s) \) for all \( s \), from which the last condition in the statement follows. \( \square \)

We are almost done with technical proofs; the following result brings together the previous lemmas to show that that the metric cone over \( \tilde{M} \) is trivial at the level of coarse homotopy.

**Corollary 6.7.** Let \( M \) be a compact Riemannian manifold of non-positive sectional curvature and let \( \tilde{M} \) be its universal cover equipped with the metric lifted from \( M \). Let \( \sigma : \Omega \tilde{M} \to \Omega M \) be the map on cones induced by the covering map \( \tilde{M} \to M \). Then

1. any two \( \sigma \)-coarse \( \mathbb{R}_+ \)-basepoints \( b \) and \( b' \) are \( \sigma \)-coarsely homotopic.

2. for any \( \sigma \)-coarse \( \mathbb{R}_+ \)-basepoint \( b \) in \( \Omega \tilde{M} \), \( \pi^{\coarse}_{1, \sigma}(\Omega \tilde{M}, b) \) is trivial.

**Proof.** Note that since \( M \) is compact, a map to \( \Omega \tilde{M} \) is \( \sigma \)-coarse if and only if it is \( p_2 \)-coarse where \( p_2 \) is projection onto the second coordinate. Pick any \( p \in M \). If we consider \( b \) as a map \( \mathcal{c}([0, 1]) \to \tilde{M} \) sending \((x, t)\) to \( b(x) \), then the proof of Proposition 6.6 shows that \( b \) is \( \sigma \)-coarsely homotopic to a map whose image is contained in \( p \times [1, \infty) \). Any two \( \mathbb{R}_+ \)-basepoints in \( p \times [1, \infty) \) are \( \sigma \)-coarsely homotopic by the obvious linear homotopy (one can also appeal to Theorem 5.6 of [10] for this fact), so we have shown (1).

Let \( \alpha \) represent a class in \( \pi^{\coarse}_{1, \sigma}(\Omega \tilde{M}, b) \). We may replace \( \alpha \) up to closeness (and hence up to relative \( p_2 \)-coarse homotopy) by a map which is constant in \( t \) on \([0, 1]^2 \cap c([0, 1])\). Proposition 6.7 shows that \( \alpha \) is \( \sigma \)-coarsely homotopic via \( H \) to a “trivial” map \( H \circ i_1(x, t) = b'(x) \). Again, make the following small adjustment for technical reasons which does not affect the coarseness of \( H \): redefine \( H \) so that \( H((x, t), s) = H((x, t), 0) \) for \((x, t) \in [0, 1]^2\). Having done this, we can use an easy adaptation of Lemma 2.6 in [10] to assume that \( H \) is of the form \( H : I_p \mathcal{c}([0, 1]) \to \Omega M \) with \( p(x, t) = x - 1 \). The coarse homotopy \( H \) is still not a homotopy relative to the boundary, though, so we have to use
a trick which is familiar from topology. Construct a new map $H' : I_{p(x)}c([0,1]) \to OM$ as follows:

$$H'((x,t),s) = \begin{cases} b(x) & t \leq x/4 - s/4 \\ H((x,0),4t-x+s) & x/4 - s/4 \leq t \leq x/4 \\ H((x,2t-x/2),s) & x/4 \leq t \leq 3x/4 \\ H((x,0),3x-4t+s) & 3x/4 \leq t \leq 3x/4 + s/4 \\ b(x) & 3x/4 + s/4 \leq t \leq x \end{cases}$$

It is straightforward to show that this is a $p_2$-coarse homotopy relative to the boundary from $\alpha$ to a concatenation of a map $\lambda$ from $c([0,1])$, followed by map constant in $t$, followed by the reverse of $\lambda$. This concatenation is easily shown to be $p_2$-coarsely homotopic to the trivial element of $\pi_{1,\sigma}^{\text{coarse}}(OM,b)$, which completes the proof of (2).

Lemma 6.8. Let $M$ be a Riemannian manifold and let $G$ be a group acting on $M$ properly discontinuously and cocompactly by isometries. Then the induced action of $G$ on $OM$ is uniformly coarsely discontinuous.

Proof. Since $G$ acts properly discontinuously by isometries, there is a global $C > 0$ so that $d(x,g\cdot x) \geq C$ for all $x \in M$ and all $g \in G \setminus \{e\}$. Let $R > 0$ and let $g \in G \setminus \{e\}$. We claim that for any $(x,t) \in OM$ with $t > R + R/C$, the distance between $(x,t)$ and $(g\cdot x,t)$ is at least $R$. Indeed, suppose the distance is less than $R$. If $\gamma$ is a length-minimizing unit speed geodesic from $(x,t)$ to $(g\cdot x,t)$, then it must be contained in $OM \cap M \times [R/C, \infty)$ since it has length at most $R$. But then the length of $\gamma$ is at least $R/C \cdot d(x,g\cdot x) \geq R$ as required. Since the action is cocompact, $OM \cap M \times [0,R/C + R]$ is contained in $\bigcup_{g \in G} K$ for some compact $K$, which proves the lemma.

Proof of Theorem 6.7. Consider the map $\sigma : OM \to OM$ induced by the covering map from the universal cover $\tilde{M}$. Since the metric on $OM$ is lifted from $OM$, $OM$ is isometric to the quotient of $\tilde{OM}$ by the action of $\pi_1(M)$ induced by the action of $\pi_1(M)$ on $\tilde{M}$ (with the metric given as in Example 5.2). By Lemma 6.8 the action of $\pi_1(M)$ is uniformly coarsely discontinuous, so the result follows from Theorem 5.3 and Corollary 6.7.

In Theorem 5.6 of [10], MNS prove a similar result for cones over finite simplicial complexes. This suggests that the curvature condition could possibly be relaxed in Theorem 6.1 above. This question, as well as the question of whether Theorem 6.1 can be recovered from the result in [10] via a triangulation argument, is left for a future paper. Even if Theorem 6.1 is a corollary to the result for simplicial complexes, it still serves as an illustrative example of computing coarse fundamental groups using the Coarse Lifting Lemma

References

[1] M. R. Bridson, and A. Haefliger, Metric spaces of non-positive curvature, Vol. 319, Springer Science & Business Media, 2013.
[2] U. Bunke and A. Engel, Homotopy theory with bornological coarse spaces, preprint, arXiv:1607.03057
[3] A. N. Dranishnikov, Asymptotic topology, Russian Mathematical Surveys 55.6, 2000, 1085.
[4] D. Dikranjan and N. Zava, Some categorical aspects of coarse spaces and balleans, Topology and its Applications 225, 2017, 164–194.
[5] J. Dydak and C.S. Hoffland, An Alternative Definition of Coarse Structures, Topology and its Applications 155 (9), 2008, 1013-1021.
[6] R. Geoghegan, *Topological methods in group theory*, Springer Science & Business Media, 2007.
[7] M. Gromov, Asymptotic invariants of infinite groups, Geometric group theory vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser. vol. 182, Cambridge University Press, Cambridge, 1993, 1–295.
[8] L. Higginbotham and T. Weighill, *Coarse quotients by group actions and the maximal Roe algebra*, Journal of Topology and Analysis (online ready), https://doi.org/10.1142/S1793525319500341.
[9] N. Higson and J. Roe, *On the coarse Baum-Connes conjecture*, Novikov conjectures, index theorems and rigidity 2, 1995, 227–254.
[10] P. Mitchener, B. Norouzizadeh and T. Schick, *Coarse homotopy groups*, arXiv preprint, arXiv:1811.10090.
[11] J. Munkres, *Topology*, Prentice Hall, 2000, 2nd edition.
[12] J. Roe, Coarse cohomology and index theory on complete Riemannian manifolds, Memoirs of the American Mathematical Society 497, 1993.
[13] J. Roe. Index theory, coarse geometry, and topology of manifolds, CBMS Regional Conference Series in Mathematics Volume 90, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996.
[14] J. Roe, *Lectures in Coarse Geometry*, University Lecture Series 31, American Mathematical Society, Providence, RI, 2003.
[15] J. Roe, *Warped cones and property A*, Geometry & Topology 9.1, 2005, 163–178.
[16] R. Willett and G. Yu, *Higher index theory for certain expanders and Gromov monster groups, I*, Advances in Mathematics 229, 2012, 1380–1416.
[17] G. Yu, *The coarse Baum-Connes conjecture*, K-theory 9, 1995.
[18] G. Yu, *The Novikov conjecture for groups with finite asymptotic dimension*, Annals of Mathematics 147, 1998, 325–355.
[19] G. Yu, *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Inventiones mathematicae 139.1, 2000, 201-240.

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