A family of combinatorial identities arising from quantum affine algebras

Alexey Sevostyanov *
Institute of Theoretical Physics, Uppsala University, and Steklov Mathematical Institute, St.Petersburg

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Abstract

We obtain a family of new combinatorial identities for symmetric formal power series.

Introduction

This present paper is motivated mainly by the needs of the Drinfeld–Sokolov reduction for quantum groups (see [2] for the motivations). However the identities which we derive are purely combinatorial and may be considered in a more general situation. Our main observation is that some solutions of a simple system of functional equations satisfy a family of complicated combinatorial identities. This phenomenon may lead to new interesting effects in the theory of symmetric functions [2].

* e-mail seva@teorfys.uu.se
1 A family of combinatorial identities

Let $a_{ij}, i, j = 1, \ldots, l$ be a generalized Cartan matrix: $a_{ii} = 2$, $a_{ij}$ are nonpositive integers for $i \neq j$ and $a_{ij} = 0$ implies $a_{ji} = 0$. Suppose also that $a_{ij}$ is symmetrizable, i.e., there are coprime positive integers $d_1, \ldots, d_l$ such that the matrix $b_{ij} = a_{ij}d_i$ is symmetric.

We shall make use of formal power series (f.p.s.) which are infinite in both directions. The space of such series is denoted by $\mathbb{C}((z))$. The product of two f.p.s. $f(z) = \sum_{n=-\infty}^{\infty} f_n z^n, g(z) = \sum_{n=-\infty}^{\infty} g_n z^n$ is said to exist if the coefficients of the series

$$\sum_{p=-\infty}^{\infty} \sum_{k+n=p} f_n g_k$$

are well defined. That is the series

$$\sum_{k+n=p} f_n g_k$$

converges for every $p$.

Similarly, the product of three series $f(z) = \sum_{n=-\infty}^{\infty} f_n z^n, g(z) = \sum_{n=-\infty}^{\infty} g_n z^n, h(z) = \sum_{n=-\infty}^{\infty} h_n z^n$ exists if the series

$$\sum_{k+n+l=p} f_n g_k h_l$$

converges for every $p$ and its sum does not depend on the ordering of the terms. Clearly, in this case the products $g(z)h(z), g(z)f(z)$ and $f(z)h(z)$ are well-defined.

For instance, if two or more series have a common domain of convergence their product is well-defined.

Denote by $\frac{1}{1-x}$ the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Let $\Sigma_n$ be the symmetric group of order $n$. For $q \in \mathbb{C}$ we shall consider the following system of equations for f.p.s:
\[(u - vq^{b_{ij}})F_{ji}(z^{-1}) = (q^{b_{ij}}u - v)F_{ij}(z), \quad a_{ij} \neq 0, \quad (1)\]

\[
\sum_{\pi \in \Sigma_{1-a_{ij}}} \sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1 - a_{ij}}{k} \right] \prod_{p < q} F_{ii}(\frac{z_{\pi(p)}}{z_{\pi(q)}}) \prod_{r=1}^{k} F_{ji}(\frac{w}{z_{\pi(r)}}) \times \\
\prod_{s=k+1}^{1-a_{ij}} F_{ij}(\frac{z_{\pi(s)}}{w}) = 0, \quad a_{ij} \neq 0, \quad i \neq j, \quad q_i = q_j, \quad (2)\]

where \[\left[ \begin{array}{c} m \\ n \end{array} \right]_q = \frac{[m]_q!}{[n]_q! [m-n]_q!}, \quad [n]_q! = [n]_q [n-1]_q \ldots [1]_q, \quad [n]_q = \frac{q^n - q^{-n}}{q-q^{-1}}.\]

Denote the l.h.s of (2) by \(P_{ij}(z_1, \ldots, z_{1-a_{ij}}, w).\) Let \(|q| < 1.\) Put

\[
s_{ij}(z_1, \ldots, z_{1-a_{ij}}) = \prod_{p \neq q} \frac{1}{1 - q^{b_{ij}} \frac{z_p}{z_q}},
\]

\[
r_{ij}(z_1, \ldots, z_{1-a_{ij}}, w) = \prod_{s=1}^{1-a_{ij}} \frac{1}{1 - q^{b_{ij}} \frac{z_s}{w}}.
\]

Our main goal is to prove the following theorem:

**Theorem 1** Let \(F_{kl}(z), k, l = 1, \ldots, l\) be a solution of system (1). Suppose that for some \(i\) and \(j\) the product

\[
s_{ij} \cdot r_{ij} \cdot \prod_{p \neq q} (1 - q^{b_{ij}} \frac{z_p}{z_q}) \prod_{s=1}^{1-a_{ij}} (1 - q^{b_{ij}} \frac{z_s}{w}) P_{ij}, \quad (3)
\]

is well-defined in the sense of f.p.s.

Then

\[P_{ij} = 0.\]

in the sense of f.p.s. Thus the solution \(F_{kl}(z)\) satisfies to the identity (3).

We divide the proof of the theorem into several lemmas.
Lemma 2 Let $F_{kl}(z), k, l = 1, \ldots, l$ be a solution of system (1). Then

$$\prod_{p \neq q} (1 - q^{b_{ij} z_q w}) \prod_{s=1}^{1-a_{ij}} (1 - q^{b_{ij} z_s w}) F_{ij} = 0, i \neq j, a_{ij} \neq 0. \quad (4)$$

Proof. Let $\pi \in \Sigma_{1-a_{ij}}$. Consider the product:

$$\prod_{s=1}^{1-a_{ij}} (1 - q^{b_{ij} z_s w}) \prod_{r=1}^{k} F_{ji} \left( \frac{w}{z_{\pi(r)}} \right) \prod_{s=k+1}^{1-a_{ij}} F_{ij} \left( \frac{z_{\pi(s)} w}{w} \right).$$

The f.p.s. $\prod_{s=1}^{1-a_{ij}} (1 - q^{b_{ij} z_s w})$ is symmetric with respect to permutations of the formal variables $z_s$. Therefore

$$\prod_{s=1}^{1-a_{ij}} (1 - q^{b_{ij} z_s w}) \prod_{r=1}^{k} F_{ji} \left( \frac{w}{z_{\pi(r)}} \right) \prod_{s=k+1}^{1-a_{ij}} F_{ij} \left( \frac{z_{\pi(s)} w}{w} \right) =$$

$$\prod_{s=1}^{1-a_{ij}} (1 - q^{b_{ij} z_{\pi(s)} w}) \prod_{r=1}^{k} F_{ji} \left( \frac{w}{z_{\pi(r)}} \right) \prod_{s=k+1}^{1-a_{ij}} F_{ij} \left( \frac{z_{\pi(s)} w}{w} \right).$$

Now using equations (1) for $F_{ij}$ we obtain:

$$\prod_{s=1}^{1-a_{ij}} (1 - q^{b_{ij} z_{\pi(s)} w}) \prod_{r=1}^{k} F_{ji} \left( \frac{w}{z_{\pi(r)}} \right) \prod_{s=k+1}^{1-a_{ij}} F_{ij} \left( \frac{z_{\pi(s)} w}{w} \right) =$$

$$\prod_{s=1}^{k} (1 - q^{b_{ij} z_{\pi(s)} w}) \prod_{r=1}^{1-a_{ij}} \left( q^{b_{ij}} - \frac{z_{\pi(s)} w}{w} \right) \prod_{r=1}^{1-a_{ij}} F_{ji} \left( \frac{w}{z_{\pi(r)}} \right) =$$

$$\prod_{s=1}^{k} (1 - q^{b_{ij} z_{\pi(s)} w}) \prod_{s=k+1}^{1-a_{ij}} \left( q^{b_{ij}} - \frac{z_{\pi(s)} w}{w} \right) \prod_{r=1}^{1-a_{ij}} F_{ji} \left( \frac{w}{z_{\pi(r)}} \right), \quad (5)$$

since $\prod_{r=1}^{1-a_{ij}} F_{ji} \left( \frac{w}{z_{\pi(r)}} \right)$ is also a symmetric f.p.s.

Similarly,
\[
\prod_{p \neq q} (1 - q^{b_{hi}} \frac{z_q}{z_p}) \prod_{p < q} F_{ii}(\frac{z_{\pi(q)}}{z_{\pi(p)}}) = 
\prod_{p \neq q} (1 - q^{b_{hi}} \frac{z_{\pi(q)}}{z_{\pi(p)}}) \prod_{p < q} F_{ii}(\frac{z_{\pi(q)}}{z_{\pi(p)}}) = 
\prod_{p < q} \left( \prod_{\pi(q) > p} F_{ii}(\frac{z_{\pi(q)}}{z_{\pi(p)}}) \prod_{\pi(q) < p} (1 - q^{b_{hi}} \frac{z_{\pi(q)}}{z_{\pi(p)}}) \right) \times 
\prod_{p < q} \left( \prod_{\pi(q) > p} (1 - q^{b_{hi}} \frac{z_{\pi(q)}}{z_{\pi(p)}}) \right) \times 
\prod_{p > q} \left( \prod_{\pi(q) > p} F_{ii}(\frac{z_{\pi(q)}}{z_{\pi(p)}}) \prod_{\pi(q) < p} (1 - q^{b_{hi}} \frac{z_{\pi(q)}}{z_{\pi(p)}}) \right) \times 
\prod_{p > q} (1 - q^{b_{hi}} \frac{z_{\pi(q)}}{z_{\pi(p)}}) \prod_{p < q, \pi(q) > p} (q^{b_{hi}} - q^{\frac{z_{\pi(q)}}{z_{\pi(p)}}}).
\]

Substituting (3) and (8) into (4) we get:

\[
\prod_{p \neq q} (1 - q^{b_{hi}} \frac{z_q}{z_p}) \prod_{s=1}^{1-a_{ij}} (1 - q^{b_{hi}} \frac{z_q}{w}) P_{ij} = 
\prod_{s=1}^{1-a_{ij}} F_{ji}(\frac{w}{z_{x_i}}) \prod_{p > q} \left( 1 - q^{b_{hi}} \frac{z_q}{z_p} \right) \prod_{s=1}^{1-a_{ij}} (1 - q^{b_{hi}} \frac{z_q}{w}) \times 
\sum_{\pi \in \Sigma_{1-a_{ij}}} \sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_{q_i} \prod_{p > q, \pi(q) > p} (1 - q^{b_{hi}} \frac{z_{\pi(q)}}{z_{\pi(p)}}) \prod_{p < q, \pi(q) > p} (q^{b_{hi}} - \frac{z_{\pi(q)}}{z_{\pi(p)}}) \times 
\prod_{s=1}^{1-a_{ij}} (1 - q^{b_{hi}} \frac{z_x}{w}) \prod_{s=k+1}^{1-a_{ij}} (q^{b_{hi}} - \frac{z_x}{w}).
\]

Put \( q_i = t \). Lemma 2 follows from

**Lemma 3** For any \( m \in \mathbb{Z}, m \leq 0 \) the following identity holds:
\[
\sum_{\pi \in \Sigma_{1-m}} \sum_{k=0}^{1-m} (-1)^k \left[ \frac{1 - a_{ij}}{k} \right] t^k \times 
\prod_{p > q, \pi(p) > \pi(q)} \left( 1 - t^2 \frac{z_{\pi(p)}}{z_{\pi(q)}} \right) \prod_{p < q, \pi(q) > \pi(p)} \left( t^2 - \frac{z_{\pi(q)}}{z_{\pi(p)}} \right) \times 
\prod_{s=1}^{k} (1 - t^m \frac{z_{\pi(s)}}{w}) \prod_{s=k+1}^{1} (t^m - \frac{z_{\pi(s)}}{w}) = 0.
\]

(11)

Proof. (see example I) below

Proof of the theorem. The conditions of the theorem imply that

\[s_{ij} \cdot r_{ij} \cdot \prod_{p \neq q} \left( 1 - q^{b_{ij}} \frac{z_q}{z_p} \right) \prod_{s=1}^{1-a_{ij}} \left( 1 - q^{b_{ij}} \frac{z_s}{w} \right) P_{ij} = P_{ij}.
\]

But by lemma 2

\[ (1 - q^{b_{ij}} \frac{z_q}{z_p}) \prod_{s=1}^{1-a_{ij}} \left( 1 - q^{b_{ij}} \frac{z_s}{w} \right) P_{ij} = 0.
\]

Therefore

\[ P_{ij} = 0.
\]

This concludes the proof.

One can formulate several versions of theorem I. For example, put

\[ r'_{ij}(z_1, \ldots, z_{1-a_{ij}}, w) = \prod_{s=1}^{1-a_{ij}} \frac{w}{1 - q^{b_{ij}} \frac{z_s}{w}}.
\]

Theorem 4 Let \(F_{kl}(z), k, l = 1, \ldots, l\) be a solution of system (4). Suppose that for some \(i\) and \(j\) the product

\[s_{ij} \cdot r'_{ij} \cdot \prod_{p \neq q} \left( 1 - q^{b_{ij}} \frac{z_q}{z_p} \right) \prod_{s=1}^{1-a_{ij}} \left( \frac{z_s}{w} - q^{b_{ij}} \right) P_{ij}
\]

is well-defined in the sense of f.p.s.

Then

\[ P_{ij} = 0.
\]

in the sense of f.p.s.. Thus the solution \(F_{kl}(z)\) satisfies the identity (2).

Similar statements exist for \(|q| > 1\).
2 Examples of combinatorial identities

Example 1 Polynomial identities

First consider the single equation

\[(z - c)F(z^{-1}, c) = (cz - 1)F(z, c)\] (13)

Remark 1 If \(F(z, c)\) is a solution of the equation then \(\Phi(z, c) = F(z^{-1}, c^{-1})\) is also a solution. Thus the transformation \((z, c) \mapsto (z^{-1}, c^{-1})\) is a symmetry of the equation.

Lemma 5 The elements

\[P_n(z, c) = z^n + cz^{-n} + \sum_{p=1-n}^{-1} e^{p+n-1}(c^2 - 1)z^p - e^{n-1}(c + 1), n \geq 1\] (14)

form a basis in the space of Laurent polynomial solutions of equation (13).

By remark 1 the set

\[Q_n(z, c) = P_n(z^{-1}, c^{-1}), n \geq 1\]

is another basis in the space of Laurent polynomial solutions.

Proof. Let \(P(z) = \sum_{-N}^{M} C_n z^n, N, M > 0\) be a solution of (13). Using the equation one can express the coefficients \(C_n, n \leq 0\) via \(C_n, n > 0\). Simple calculation shows that actually \(M = N\) and

\[P(z) = \sum_{n=1}^{N} C_n P_n(z, c).\]

The proof follows.

Put \(F_{ij} = P_n(z, q^{b_i})\). This gives a solution of system (1). The conditions of theorem 1 are satisfied for every \(a_{ij}\) since \(P_n(z, q^{b_i})\) are Laurent polynomials. Thus we obtain a family of combinatorial identities (2) for the elements \(P_n(z, q^{b_i})\).
Similarly one derives a family of identities for the elements $Q_n(z, q^{b_{ij}})$.
Consider the solution $P_1(z, q^{b_{ij}})$ in more detail. By (14) we have:

$$P_1(z, q^{b_{ij}}) = z + q^{b_{ij}}z^{-1} - (q^{b_{ij}} + 1) = -(1 - z)(1 - q^{b_{ij}}z^{-1}).$$

The identities (2) for $P_1(z, q^{b_{ij}})$ amount to the relations:

$$\sum_{\pi \in \Sigma_{1-a_{ij}}} \sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1 - a_{ij}}{k} \right] \prod_{p<q} \left( 1 - \frac{z_{\pi(p)}}{z_{\pi(q)}} \right) \left( 1 - q^{b_{ij}} \frac{z_{\pi(p)}}{z_{\pi(q)}} \right) \times$$

$$\prod_{r=1}^{k} \left( 1 - \frac{w}{z_{\pi(r)}} \right) \left( 1 - q^{b_{ij}} \frac{w}{z_{\pi(r)}} \right) = 0, \quad a_{ij} \neq 0, \; i \neq j. \quad (15)$$

Pulling out the symmetric factor $\prod_{r=1}^{1-a_{ij}} \left( 1 - \frac{w}{z_{\pi(r)}} \right)$ and the antisymmetric one $\prod_{p<q} \left( \frac{1}{z_{\pi(p)}} - \frac{1}{z_{\pi(q)}} \right)$ we find that the identities (15) are equivalent to the following relations:

$$\sum_{\pi \in \Sigma_{1-m}} (-1)^{t(\pi)} \sum_{k=0}^{1-m} (-1)^k \left[ \frac{1 - m}{k} \right] \prod_{p<q} \left( z_{\pi(p)} - t^2 z_{\pi(q)} \right) \times$$

$$\prod_{r=1}^{k} \left( 1 - t^m \frac{z_{\pi(r)}}{w} \right) \prod_{s=k+1}^{1-m} \left( \frac{z_{\pi(s)}}{w} - t^m \right) = 0, \quad m \in \mathbb{Z}, \; m \leq 0, \quad (16)$$

which coincide with the identities obtained by Jing in [1]. Since the solution $P_1(z, q^{b_{ij}})$ is formed by Laurent polynomials the identity (16) is equivalent to the relation (11). In particular, this proves lemma 3.

Example 2 Identities for Taylor series

Now let $F_{ij} \in \mathbb{C}[[z]]$.

**Lemma 6** The system of equations (17) has a unique nontrivial solution in $\mathbb{C}[[z]]$ with the asymptotics $F_{ij}(0) = q_{ij}^{n_{ij}}$. The solution has the form:

$$F_{ij}(z) = \frac{q_{ij}^{n_{ij}} - zq_{ji}^{n_{ji}}}{1 - zq_{ij}^{b_{ij}}}, \quad a_{ij} \neq 0, \quad (17)$$
Proof. Put

\[ F_{ij}(z) = \sum_{n=0}^{\infty} c_{n}^{ij} z^{n}. \]

The l.h.s. of (1) must belong to \( \mathbb{C}[[z]] \). This allows us to determine \( F_{ij}(z) \) up to a constant:

\[ F_{ij}(z) = c_{0}^{ij} + c_{1}^{ij} \frac{z}{1 - z q^{b_{ij}}}, c_{0}^{ij} = q_{n}^{ij}. \]

Substituting this ansatz into (1) we get the following relation for the coefficients \( c_{0}^{ij}, c_{1}^{ij} \):

\[ c_{1}^{ij} = -c_{1}^{ji} + q^{b_{ij}} c_{0}^{ij}. \]

This yields (17).

Theorem 7 For every \( n_{ij} \) such that \( d_{i} n_{ji} - d_{j} n_{ij} = \varepsilon_{ij} b_{ij}, \varepsilon_{ij} = -\varepsilon_{ji}, \varepsilon_{ij} = \pm 1 \) the solution (17) satisfies the identities (2).

Proof. An important property of the solution (17) subject to the conditions of the theorem is that either \( F_{ji} = q_{i}^{n_{ji}} \) or \( F_{ij} = q_{j}^{n_{ij}} \). Using this fact one can show that either the series in (3) or the series in (12) have a common domain of convergence. This allows us to apply theorem 1 or theorem 4, respectively, to obtain identities (2) for the solution (17).

Example 3 Identities for general formal power series

Lemma 8 The elements

\[ \varphi_{n}(z, c) = z^{-n} + z^{n} \frac{c - z}{1 - cz}, n \geq 1 \]

\[ \varphi_{0}(z, c) = \frac{1 - z}{1 - cz} \]

are solutions of equation (13). Every solution of equation (13) may be uniquely represented as a Taylor series \( \sum_{n=0}^{\infty} C_{n} \varphi_{n}(z, c) \).
Proof is quite similar to that of lemma 3.

We shall say that the set \( \{ \varphi_n(z, c) \}_{n \geq 0} \) is a basis in the space of f.p.s. solutions of (13). As a consequence we obtain that the series

\[
\psi_n(z, c) = z^n + z^{-n} \frac{c^{-1} - z^{-1}}{1 - c^{-1}z^{-1}}, n \geq 1
\]

\[
\psi_0(z, c) = 1 - z^{-1} \frac{1 - c^{-1}}{1 - c^{-1}z^{-1}}
\]

form another basis in the same space.

Thus the sets \( \{ \varphi_n(z, q^{bi}) \}_{n \geq 0} \) and \( \{ \psi_n(z, q^{bi}) \}_{n \geq 0} \) are two bases in the space of f.p.s. solutions of equations (1) for \( i = j \).

Let \( i \neq j \).

**Lemma 9** Let \( F_{ij}(z) = \sum_{n=-\infty}^{\infty} c_{ij}^n z^n, i < j \) be arbitrary f.p.s.. Put

\[
F_{ji}(z) = C \delta(q^{bij} z) - C_{ij}^0 \left( \frac{z}{1-q^{bij} z} + \frac{z^{-1}}{1-q^{bij} z^{-1}} \right) + \sum_{n=1}^{\infty} \left( C_{ij}^{-n} z^n q^{bij} - z - C_{ij}^n z^{-n} q^{bij} - z^{-1} \right).
\]

Then the set \( \{ F_{ij}(z), F_{ji}(z), i < j \} \) is a solution of equations (1) for \( i \neq j \).

Moreover, every solution may be uniquely represented in this form for some \( C, C_{ij}^n, i < j, n \in \mathbb{Z} \).

The lemma can be proved similarly to lemma 3.

Thus we have described the space of solutions of the system (1) completely. This yields several examples of identities of the type (2).

For instance, put

\[
F_{ii}(z) = \varphi_n(z, q^{bii}),
\]

\[
F_{ij}(z) = \sum_{n=-M}^{-N} C_{ij}^n z^n, i < j, N, M > 0, N > M,
\]

\( C = 0 \).

Then

\[
F_{ji}(z) = \sum_{n=M}^{N} C_{ij}^{-n} z^n q^{bij} - z \frac{1 - q^{bij} z}{1 - q^{bij} z}.
\]

The conditions of theorem 4 are satisfied for \( i < j \) since in that case the series in (12) have a common domain of convergence. Hence we obtain a family of identities (3) for the solution.
References

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