Quantum kinematics

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Abstract

The FRT quantum group and space theory is reformulated from the standard mathematical basis to an arbitrary one. The $N$-dimensional quantum vector Cayley-Klein spaces are described in Cartesian basis and the quantum analogs of $(N−1)$-dimensional constant curvature spaces are introduced. Part of the 4-dimensional constant curvature spaces are interpreted as the non-commutative analogs of $(1 + 3)$ kinematics. A different unifications of Cayley-Klein and Hopf structures in a kinematics are described with the help of permutations. All permutations which lead to the physically nonequivalent kinematics are found and the corresponding non-commutative $(1 + 3)$ kinematics are investigated. As a result the quantum (anti) de Sitter, Minkowski, Newton, Galilei kinematics with the fundamental length, the fundamental mass and the fundamental velocity are obtained.

1 Introduction

Space-time is a fundamental conception which underline the most significant physical theories. Therefore the analysis of a possible space-time models (or kinematics) has the fundamental meaning for physics. Space and time in non-relativistic physics were regarded as independent what mathematically is connected with fiber property of Galilei kinematics. In special relativity was determined that space and time are depend on each other and must be regarded as integrated object, namely flat Minkowski space-time with pseudo-Euclidean metric. The notion of curvature was introduced in physics by general relativity. Anti de Sitter and de Sitter kinematics with constant
positive respectively negative curvature are the simplest relativistic space-time models with curvature. Possible kinematics, which satisfy the natural physical postulates: space is isotropic and rotations in space-time planes form non-compact subgroup, were described in [1] on the level of Lie algebras.

The Snyder quantized space-time coordinates [2] or, respectively, the curved momentum space is the oldest example of using the non-commutative geometry in physics. The simplest curved de Sitter geometry with constant curvature were used instead of flat Minkowski space in different generalizations of quantum field theory [3]–[8] as a momentum space model. The universal constant, the fundamental length $l$, or fundamental mass $M$, related to $l$ by $l = \frac{\hbar}{Mc}$, where $\hbar$ is Plank constant and $c$ velocity of light enters necessarily into the theory [3], [6], [8].

New possibility for construction of the non-commutative space-time models is provided by quantum groups and quantum vector spaces [9]. Space-time coordinates commutation relations of the $\kappa$-Minkowski kinematics [10], [11], [12] was obtained from Lie algebra quantum deformation of Poincaré group

$$[x_\mu, x_\nu] = \frac{i}{\kappa} (a_\mu x_\nu - x_\mu a_\nu),$$  \hspace{1cm} (1)$$

where $a_\mu$ is four-vector in Minkowski space, determining the direction of quantum deformation $y = a^\mu x_\mu$. The arbitrary choice of $a_\mu$ is equivalent to the description of standard $\kappa$-deformation in space-time with an arbitrary basis [11]. If we put $\hbar = c = 1$ the deformation parameter $\Lambda = \kappa^{-1} = [\text{length}]$ may be treated as the fundamental length parameter and $\kappa$ may be regarded as the fundamental mass $[\kappa] = [\text{mass}]$.

The standard $\kappa$-deformation obtained for $a_\mu = (1, 0, 0, 0)$, $a_\mu a^\mu = 1$ leads to relations ($i, k = 1, 2, 3$)

$$[x_0, x_i] = \frac{i}{\kappa} x_i, \; \; [x_i, x_k] = 0.$$ \hspace{1cm} (2)$$

Space-like vector $a_\mu = (0, 1, 0, 0)$, $a_\mu a^\mu = -1$ define tachyonic $\kappa$-deformation ($p = 2, 3$)

$$[x_1, x_0] = \frac{i}{\kappa} x_0, \; \; [x_p, x_0] = 0, \; \; [x_1, x_p] = \frac{i}{\kappa} x_p, \; \; [x_2, x_3] = 0.$$ \hspace{1cm} (3)$$

Finally light-like vector $a_\mu = (1, 1, 0, 0)$, $a_\mu a^\mu = 0$ provided light-cone $\kappa$-deformation

$$[x_0, x_1] = \frac{i}{\kappa} (x_1 - x_0), \; \; [x_0, x_p] = \frac{i}{\kappa} x_p, \; \; [x_1, x_p] = \frac{i}{\kappa} x_p, \; \; [x_2, x_3] = 0,$$ \hspace{1cm} (4)$$
which was suggested in [13] under the name of the null-plane Poincaré algebra.

Possible commutative kinematics [1] are realized [14] as a constant curvature spaces, which may be obtained from the spherical space by contractions and analytical continuations known as Cayley-Klein scheme or Cayley-Klein structure [15]. The standard quantum group theory [9] was reformulated to the Cartesian basis and the non-commutative analogs of constant curvature spaces (CCS) including fiber (or flag) spaces and their motion groups were investigated in [16]–[20]. Quantum algebras corresponding to the flat Cayley-Klein spaces were described in [21], [22].

It was shown [23] that Hopf algebra and Cayley-Klein structures for the quantum orthogonal algebra may be combined in a different way. An arbitrary permutations of indices of its generators in Cartesian basis, which transform a set of primitive generators to a new set of primitive generators, generally give in result the isomorphic quantum algebra. But such isomorphism may be destroyed if a physical interpretation of the generators is introduced or a contraction of quantum algebra is realized. Constructions of the quantum orthogonal groups for arbitrary permutations were investigated in [24]–[27].

In this paper $N$-dimensional quantum vector Cayley-Klein spaces are regarded and quantum analogs of $(N - 1)$-dimensional constant curvature spaces are obtained. For $N = 5$ some of the quantum CCS are interpreted as non-commutative $(1 + 3)$ kinematics: (anti) de Sitter, Minkowski, Newton, Galilei, as well as exotic Carroll ones. All permutations which corresponding to physically different kinematics are found.

The paper is organized as follows. In section 2, the unified description of the commutative CCS and their interpretation as kinematics are briefly recalled. Section 3 is devoted to the reformulation of the standard quantum group and space theory for an arbitrary basis and the description of the $N$-dimensional quantum vector Cayley-Klein spaces in Cartesian basis. Non-commutative quantum $(1 + 3)$ kinematics for different permutations are discussed in section 4. The obtained results are summarized in Conclusion.

2 Commutative kinematics

Classical four-dimensional space-time models may be obtained [14], [15] by the physical interpretation of the orthogonal coordinates of the most sym-
metric spaces, namely constant curvature spaces. All $3^N$ $N$-dimensional CCS are realized on the spheres

$$S_N(j) = \{\xi_1^2 + j_1^2 \xi_2^2 + \ldots + (1, N + 1)^2 \xi_{N+1}^2 = 1\},$$

where

$$\max(i,k)-1 \prod_{l=\min(i,k)}^{(i,k)} j_l, \quad (k,k) \equiv 1,$$

each of parameters $j_k = 1, \iota_k, i, k = 1, \ldots, N$. Here $\iota_k$ are nilpotent generators $\iota_k^2 = 0$, with commutative law of multiplication $\iota_k \iota_m = \iota_m \iota_k \neq 0, k \neq m$.

Division of complex numbers on the nilpotent generators $a/\iota_k, a \in \mathbb{C}$, as well as division of nilpotent generators with different indices $\iota_k/\iota_p, k \neq p$ are not defined. But it is possible consistently define division of nilpotent generators on itself, namely $\iota_k/\iota_k = 1$ means that an equation $a \iota_k = b \iota_k$ has the single solution $a = b$ for $a, b \in \mathbb{R}$ or $\mathbb{C}$.

The intrinsic Beltrami coordinates $x_k = \xi_{k+1} \xi_1^{-1}, k = 1, 2, \ldots, N$ present the coordinate system on CCS, which coordinate lines $x_k = const$ are geodesic. CCS has positive curvature for $j_1 = 1, \iota_1, i, j_2 = \iota_2, i, j_3 = j_4 = 1$ if one interpret $x_1$ as the time axis $t = \xi_2 \xi_1^{-1}$ and the rest as the space axes $r_k = \xi_{k+2} \xi_1^{-1}, k = 1, 2, 3$. These kinematics are represented on Fig.1, where coordinate lines are drawn.

Except of standard (anti) de Sitter, Minkowski, Newton, Galilei kinematics theoretically are possible non-standard exotic Carroll kinematics [1], [28], where space and time properties are changed as compared with those of Galilei kinematics. Time is absolute in Galilei kinematics: two simultaneous events in some inertial reference frame remain simultaneous in any other one. But in Carroll kinematics space is absolute: two events which take place at some space point of inertial reference frame will take place at the same space point in any other one.

Carroll kinematics as well are realized [14], [15] as CCS for $N = 4, j_4 = \iota_4, j_1 = 1, \iota_1, i, j_2 = j_3 = 1$, but with different physical interpretation of Beltrami coordinates, namely $t = \xi_5 \xi_1^{-1}$ is time axis and $r_k = \xi_{k+1} \xi_1^{-1}, k = 1, 2, 3$ are space axes. Carroll kinematics with flat proper space firstly described in [28] correspond to $j_1 = \iota_1$. 


Figure 1: Classical (1+3) kinematics. Light cone is drawn by dotted lines. Proper space of the non-relativistic kinematics is represented by thick lines.

3 Quantum orthogonal groups and quantum Cayley-Klein vector spaces

According with FRT theory [9] the algebra function on quantum orthogonal group $\text{Fun}(SO_q(N))$ (or simply quantum orthogonal group $SO_q(N)$) is the algebra of non-commutative polynomials of $n^2$ variables $t_{ij}, i, j = 1, \ldots, n$, which are subject of commutation relations

$$R_q T_1 T_2 = T_2 T_1 R_q$$

and additional relations of $q$-orthogonality

$$TCT^d = C, \quad T^d C^{-1} T = C^{-1}. \quad (8)$$

Here $T_1 = T \otimes I$, $T_2 = I \otimes T \in M_{n^2} (\mathbb{C} \langle t_{ij} \rangle)$, $T = (t_{ij})_{i,j=1}^n \in M_n (\mathbb{C} \langle t_{ij} \rangle)$, $I$ is unit matrix in $M_n (\mathbb{C})$, $C = C_0 q^\rho$, $\rho = \text{diag}(\rho_1, \ldots, \rho_N)$, $(C_0)_{ij} = \delta_{ij}$, $i'$ =
\[ N + 1 - i, \ i, j = 1, \ldots, N, \text{ that is } (C)_{ij} = q^{\rho_i \rho_j} \delta_{ij} \text{ and } C^{-1} = C, \]
\[
(p_1, \ldots, p_N) = \begin{cases} 
(n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, -n + \frac{1}{2}), & N = 2n + 1 \\
(n - 1, n - 2, \ldots, 1, 0, 0, -1, \ldots, -n + 1), & N = 2n.
\end{cases}
\]

(9)

The numerical matrix \( R_q \) is the well-known solution \[9\] of Yang-Baxter equation and its elements serve as the structure constant of quantum group generators.

Quantum orthogonal group \( SO_q(N) \) is a Hopf algebra with the following coproduct \( \Delta \), counit \( \epsilon \) and antipode \( S \)
\[
\Delta T = T \otimes T, \quad \Delta t_{ij} = \sum_{k=1}^{n} t_{ik} \otimes t_{kj}, \quad \epsilon(T) = I, \quad \epsilon(t_{ij}) = \delta_{ij},
\]
\[
S(T) = C T^a C^{-1}, \quad S(t_{ij}) = q^{\rho_i - \rho_j} t_{ji}, \quad i, j = 1, \ldots, N.
\]

(10)

Let us remind the definition of the quantum vector space \[9\]. An algebra \( O_q^N(C) \) with generators \( x_1, \ldots, x_N \) and commutation relations
\[
\hat{R}_q(x \otimes x) = qx \otimes x - \frac{q - q^{-1}}{1 + q^{-2}} x^t C x W_q,
\]
where \( \hat{R}_q = PR_q \), \( P u \otimes v = v \otimes u, \forall u, v \in C^n \), \( W_q = \sum_{i=1}^{N} q^{\rho_i} e_i \otimes e_{i'} \),
\[
x^t C x = \sum_{i,j=1}^{N} x_i C_{ij} x_j = \epsilon x_{n+1}^2 + \sum_{k=1}^{n} \left( q^{-\rho_k} x_k x_{k'} + q^{\rho_k} x_{k'} x_k \right),
\]
(12)

\( \epsilon = 1 \) for \( N = 2n+1 \), \( \epsilon = 0 \) for \( N = 2n \) and vector \( (e_i)_k = \delta_{ik} \), \( i, k = 1, \ldots, N \) is called the algebra of functions on \( N \)-dimensional quantum Euclidean space (or simply the quantum vector space) \( O_q^N(C) \).

Co-action of the quantum group \( SO_q(N) \) on the non-commutative vector space \( O_q^N(C) \) is given by
\[
\delta(x) = T \otimes x, \quad \delta(x_i) = \sum_{k=1}^{n} t_{ik} \otimes x_{k}, \quad i = 1, \ldots, n
\]
(13)

and quadratic form \[12\] is invariant \( inv = x^t C x \) with respect to this co-action:
\[
\delta(x^t C x) = I \otimes x^t C x.
\]
(14)
The matrix $C$ has non-zero elements only on the secondary diagonal which are equal to unit in commutative limit $q = 1$. Therefore the quantum group $SO_q(N)$ and the quantum vector space $O_q^N(C)$ are described by equations (7), (8), (11), (12) in mathematical (or “symplectic”) basis, where for $q = 1$ the invariant form $inv = x^tC_0x$ is given by the matrix $C_0$ with the only non-zero elements on the secondary diagonal which are equal to real units.

New generators $y = D^{-1}x$ of the vector space $O_q^N(C)$ in arbitrary basis are obtained \[16\]–\[20\] with the help of non-degenerate matrix $D \in M_N$ and are subject of the commutation relations

$$\hat{R}(y \otimes y) = qy \otimes y - \frac{\lambda}{1 + q^{N-2}} y^t C'yW,$$

where $\hat{R} = (D \otimes D)^{-1} \hat{R}_q(D \otimes D)$, $W = (D \otimes D)^{-1}W_q$, $C' = D^tCD$. The corresponding quantum group $SO_q(N)$ is generated in arbitrary basis by $U = (u_{ij})_{i,j=1}^N$, where $U = D^{-1}TD$. The commutation relations of the new generators are

$$\hat{R}U_1U_2 = U_2U_1\hat{R},$$

and $q$-orthogonality relations looks as follows

$$U \tilde{C}U^t = \tilde{C}, \quad U^t(\tilde{C})^{-1}U = (\tilde{C})^{-1},$$

where $\tilde{R} = (D \otimes D)^{-1}R_q(D \otimes D)$, $\tilde{C} = D^{-1}C(D^{-1})^t$.

In the case of kinematics most natural is the Cartesian basis where the invariant form $inv = y^t y$ is given by the unit matrix $I$. The transformation from the mathematical (or “symplectic”) basis $x$ to the Cartesian basis $y$ is described by the matrix $D$ which is a solution of the following equation

$$D^t C_0 D = I.$$

This equation has many solutions. Take one of these, namely

$$D = \frac{1}{\sqrt{2}} \begin{pmatrix} I & 0 & -i\tilde{C}_0 \\ 0 & \sqrt{2} & 0 \\ \tilde{C}_0 & 0 & iI \end{pmatrix}, \quad N = 2n + 1,$$  \hspace{1cm} (19)

where $\tilde{C}_0$ is the $n \times n$ matrix with real units on the secondary diagonal. For $N = 2n$ the matrix $D$ is given by (19) without the middle column and row. All solutions of (18) are given by the matrices $D_\sigma = DV_\sigma$, which are obtained
from \([19]\) by the right multiplication on the matrix \(V_\sigma \in M_N\) with elements 
\((V_\sigma)_{ik} = \delta_{\sigma, k}\), where \(\sigma \in S(N)\) is a permutation of \(N\)-th order \([24]–[27]\).

We derive the quantum Cayley-Klein spaces with the same transformation of Cartesian generators 
\(y = \psi \xi, \, \psi = \text{diag}(1, (1, 2), \ldots, (1, N)) \in M_N\), as in commutative case \([15], [17]\). The transformation 
\(z = Jv\) of the deformation parameter \(q = e^z\) need be added in quantum case. The commutation
relations of the Cartesian generators of the quantum \(N\)-dimensional Cayley-Klein space are given by equations

\[
\hat{R}_\sigma(j) \xi \otimes \xi = e^{Jv} \xi \otimes \xi - \frac{2 \sinh(Jv)}{1 + e^{Jv(N-2)}} \xi^t C_\sigma(j) \xi W_\sigma(j),
\]

\[
\hat{R}_\sigma(j) = \Psi^{-1} \hat{R}_\sigma \Psi, \quad W_\sigma(j) = \Psi^{-1} W_\sigma,
\]

\[
C_\sigma(j) = \psi D_\sigma^t CD_\sigma \psi = \psi V_\sigma^t D_\sigma^t CDV_\sigma \psi, \quad \Psi = \psi \otimes \psi \tag{20}
\]

and in explicit form are

\[
\xi_{\sigma_k} \xi_{\sigma_m} = \xi_{\sigma_m} \xi_{\sigma_k} \cosh Jv - i \xi_{\sigma_m} \xi_{\sigma_k'} (1, \sigma_k) (1, \sigma_k')^{-1} \sinh Jv, \quad k < m < k', \quad k \neq m',
\]

\[
\xi_{\sigma_k} \xi_{\sigma_m} = \xi_{\sigma_m} \xi_{\sigma_k} \cosh Jv - i \xi_{\sigma_m} \xi_{\sigma_k} (1, \sigma_m) (1, \sigma_m')^{-1} \sinh Jv, \quad m' < k < m, \quad k \neq m',
\]

\[
[\xi_{\sigma_k}, \xi_{\sigma_k'}] = 2i \epsilon \sinh \left( \frac{Jv}{2} \right) \cosh(Jv)^{n-k} \xi_{\sigma_n+1}^2 \left( 1, \sigma_k \right) \left( 1, \sigma_k' \right) + 
\]

\[
+ i \frac{\sinh(Jv)}{(\cosh(Jv))^{k+1} (1, \sigma_k) (1, \sigma_k')} \sum_{m=k+1}^n \left( \cosh(Jv) \right)^m \left( 1, \sigma_m \right)^2 \xi_{\sigma_m}^2 + \left( 1, \sigma_{m'} \right)^2 \xi_{\sigma_{m'}}^2, \tag{21}
\]

where \(k, m = 1, 2, \ldots, n\). The invariant form of the Cayley-Klein space \(O^N_v(j; \sigma; C)\) is written as

\[
\text{inv}(j) = \cosh(Jv \rho_1) \left( \epsilon \left( 1, \sigma_{n+1} \right)^2 \xi_{\sigma_{n+1}}^2 \frac{(\cosh(Jv))^n}{\cosh(Jv/2)} + 
\]

\[
+ \sum_{k=1}^n \left( 1, \sigma_k \right)^2 \xi_{\sigma_k}^2 + \left( 1, \sigma_{k'} \right)^2 \xi_{\sigma_{k'}}^2 \right) \cosh(Jv)^{k-1}. \tag{22}
\]

The multiplier \(J\) in the transformation \(z = Jv\) of the deformation parameter
is chosen as \(J = \bigcup_{k=1}^n (\sigma_k, \sigma_{k'})\). This is the minimal multiplier, which guarantees
\([27]\) the existence of the Hopf algebra structure for the associated quantum

\(SO_v(N; j; \sigma)\). The “union” \((\sigma_k, \sigma_{p}) \cup (\sigma_m, \sigma_{r})\) is understood as the first
power multiplication of all parameters \( j_k \), which are appear at least in one multiplier \((\sigma_k, \sigma_p)\) or \((\sigma_m, \sigma_r)\), for example, \((j_1 j_2) \cup (j_2 j_3) = j_1 j_2 j_3\).

In the case of Euclidean vector spaces \( O^N_v(\mathbb{C}) \) the use of different \( D_\sigma \) for \( \sigma \in S(N) \) has no sense because all quantum vector spaces are isomorphic. But the matter is radically different for the quantum Cayley-Klein spaces. In this case Cartesian generators are multiplied by \((1,k)\) and for nilpotent values of all or some parameters \( j_k \) such isomorphism of quantum vector spaces is destroyed. The necessity of using different \( D_\sigma \) is arisen as well if there is some physical interpretation of generators. In this case a physically different generators may be confused by permutations \( \sigma \), for example, time and space generators of kinematics. Mathematically isomorphic kinematics may be physically non-equivalent.

4 Quantum vector spaces \( O^5_v(j; \sigma) \)

Quantum vector spaces \( O^5_v(j; \sigma) \) are generated by \( \xi_{\sigma_l}, \ l = 1, 2, 3, 4, 5 \), with commutation relations \((k = 2, 3, 4)\)

\[
\xi_{\sigma_1} \xi_{\sigma_k} = \xi_{\sigma_k} \xi_{\sigma_1} \cosh(Jv) - i \xi_{\sigma_k} \xi_{\sigma_5} \left( \frac{1, \sigma_5}{1, \sigma_1} \right) \sinh(Jv),
\]

\[
\xi_{\sigma_1} \xi_{\sigma_5} = \xi_{\sigma_5} \xi_{\sigma_1} \cosh(Jv) - i \xi_{\sigma_5} \xi_{\sigma_3} \left( \frac{1, \sigma_3}{1, \sigma_5} \right) \sinh(Jv),
\]

\[
\xi_{\sigma_2} \xi_{\sigma_5} = \xi_{\sigma_5} \xi_{\sigma_2} \cosh(Jv) - i \xi_{\sigma_3} \xi_{\sigma_5} \left( \frac{1, \sigma_5}{1, \sigma_2} \right) \sinh(Jv),
\]

\[
\xi_{\sigma_3} \xi_{\sigma_4} = \xi_{\sigma_4} \xi_{\sigma_3} \cosh(Jv) - i \xi_{\sigma_2} \xi_{\sigma_3} \left( \frac{1, \sigma_3}{1, \sigma_4} \right) \sinh(Jv),
\]

\[
[\xi_{\sigma_2}, \xi_{\sigma_4}] = 2i \xi_{\sigma_3} \left( \frac{1, \sigma_2}{1, \sigma_1} \right) \sinh(Jv/2),
\]

\[
[\xi_{\sigma_1}, \xi_{\sigma_3}] = 2i \left( \xi_{\sigma_1}^2 (1, \sigma_3)^2 \cosh(Jv) + (\xi_{\sigma_1}^2 (1, \sigma_2)^2 + \xi_{\sigma_1}^2 (1, \sigma_4)^2) \cosh(Jv/2) \right) \left( \frac{1, \sigma_3}{1, \sigma_5} \right) \sinh(Jv/2). \tag{23}
\]

Coaction of \( SO_v(5; j; \sigma) \) on \( O^5_v(j; \sigma) \) is given by

\[
\delta(\xi(j; \sigma)) = U(j; \sigma) \hat{\xi}(j; \sigma) \tag{24}
\]
and the following form

\[
\text{inv}(j) = \left( \xi_{\sigma_1}(1, \sigma_3)^2 \frac{\cosh(Jv)^2}{\cosh(Jv/2)} + \xi_{\sigma_1}(1, \sigma_1)^2 + \xi_{\sigma_5}(1, \sigma_5)^2 + \right.
\]
\[\left. + (\xi_{\sigma_2}(1, \sigma_2)^2 + \xi_{\sigma_4}(1, \sigma_4)^2) \cosh(Jv) \cosh(3Jv/2) \right)
\]

(25)

is invariant under this coaction.

Quantum orthogonal Cayley-Klein sphere \(S^4_v(j; \sigma)\) is obtained as the quotient of \(O^5_v(j; \sigma)\) by \(\text{inv}(j) = 1\). Generators

\[
\zeta_{\sigma_1} = Au_{\sigma_1 \sigma_k}, \quad \zeta_{\sigma_2} = Au_{\sigma_2 \sigma_k}, \quad \zeta_{\sigma_3} = Au_{\sigma_3 \sigma_k},
\]
\[
\zeta_{\sigma_4} = Au_{\sigma_4 \sigma_k}, \quad \zeta_{\sigma_5} = Au_{\sigma_5 \sigma_k}, \quad \sigma_k = 1
\]

(26)

of the quantum orthogonal sphere \(S^4_v(j, \sigma)\), forming the vector

\[
\zeta^t(j; \sigma) = ((1, \sigma_1)\zeta_{\sigma_1}, (1, \sigma_2)\zeta_{\sigma_2}, (1, \sigma_3)\zeta_{\sigma_3})^t, (1, \sigma_4)\zeta_{\sigma_4}, (1, \sigma_5)\zeta_{\sigma_5}),
\]

are proportional to the first column elements of the matrix \(U(j; \sigma)\). It follows from the \((v, j)\)-orthogonality of \(U(j, \sigma)\), that the additional relation

\[
\zeta^t(j; \sigma)C(j)\zeta(j; \sigma) = 1
\]

(27)

is held for generators therefore they belong to \(S^4_v(j; \sigma)\). The quantum analogs of the intrinsic Beltrami coordinates on this sphere are given by the set of independent generators

\[
x_{\sigma_{i-1}} = \zeta_{\sigma_i} \cdot \zeta_{\sigma_1}^{-1}, \quad i = 1, 2, 3, 4, 5, \quad i \neq k.
\]

(28)

The systematic investigation of the quantum vector spaces and the quantum orthogonal spheres for different \(\sigma\) and different contractions give in result a quantum analogs of \((1 + 3)\) kinematics. It is easily to find out that the commutation relations are invariant relative to the replacement of \(\sigma_1\) on \(\sigma_5\), \(\sigma_2\) on \(\sigma_4\) and vice-versa, therefore it is sufficient to consider only 30 permutations instead of \(5! = 120\). Moreover only 3 permutations \(\sigma_0 = (1, 2, 3, 4, 5), \quad \sigma' = (1, 4, 3, 5, 2), \quad \bar{\sigma} = (2, 3, 1, 4, 5)\) give in result non-isomorphic quantum kinematics.
4.1 Quantum vector space $O^5_0(j; \sigma_0)$

For the identical permutation $\sigma_0 = (1, 2, 3, 4, 5)$ deformation parameter is multiplied by $J = (1, 5) = j_1 j_2 j_3 j_4$ and commutation relations of the generators are $(m = 2, 3, 4)$

$$\xi_1 \xi_m = \xi_m \xi_1 \cosh Jv - i \xi_m \xi_5 J \sinh Jv, \quad \xi_5 \xi_m = \xi_5 \xi_m \cosh Jv - i \xi_1 \xi_m \frac{1}{J} \sinh Jv,$$

$$\xi_2 \xi_3 = \xi_3 \xi_2 \cosh Jv - i \xi_3 \xi_4 j_2 j_3 \sinh Jv, \quad \xi_3 \xi_4 = \xi_4 \xi_3 \cosh Jv - i \xi_2 \xi_3 j_2 j_3 \frac{1}{j_2 j_3} \sinh Jv,$$

$$[\xi_1, \xi_5] = 2i j_1^2 \left( j_2^2 \xi_3 \cosh(Jv) + (\xi_2 \xi_3 + j_2 j_3 \xi_4^2) \cosh \frac{Jv}{2} \right) \frac{1}{J} \sinh \frac{Jv}{2},$$

$$[\xi_2, \xi_4] = 2i j_3 \frac{j_2}{j_3} \sinh \frac{Jv}{2}. \tag{29}$$

The following quadratic form

$$inv(j) = \left( \xi_1^2 + J^2 \xi_5^2 + j_1^2 (\xi_2^2 + j_2 j_3 \xi_4^2) \cosh(Jv) + \right.$$

$$+ j_1 j_2 j_3 \xi_4^2 \left( \frac{\cosh(Jv)}{\cosh(Jv/2)} \right) \cosh \frac{3Jv}{2} \right) \tag{30}$$

is invariant under the coaction of $SO_v(5; j; \sigma_0)$.

For the standard interpretation of the independent generators $t = \xi_2 \xi_1^{-1}$,

$$r_k = \xi_k^{-1} \xi_k, \quad k = 1, 2, 3,$$

which give rise to the (anti) de Sitter, Minkowski, Newton and Galilei kinematics [14], [15], the mathematical parameter $j_1$ is replaced by the physical one $\tilde{j}_1 T^{-1}$, and the parameter $j_2$ is replaced by the $ic^{-1}$, where $\tilde{j}_1 = 1, i$. The limit $T \to \infty$ correspond to the contraction $j_1 = \iota_1$ and the limit $c \to \infty$ correspond to $j_2 = \iota_2$. In result the commutation relations (29) are rewritten as follows $(m = 2, 3, 4)$

$$\xi_1 \xi_m = \xi_m \xi_1 \cos \frac{\tilde{j}_1 v}{c T} + i \xi_m \xi_5 \frac{\tilde{j}_1}{c T} \sin \frac{\tilde{j}_1 v}{c T},$$

$$\xi_m \xi_5 = \xi_5 \xi_m \cos \frac{\tilde{j}_1 v}{c T} - i \xi_1 \xi_m \frac{c T}{\tilde{j}_1} \sin \frac{\tilde{j}_1 v}{c T},$$

$$\xi_2 \xi_3 = \xi_3 \xi_2 \cos \frac{\tilde{j}_1 v}{c T} + i \xi_3 \xi_4 \frac{1}{c} \sin \frac{\tilde{j}_1 v}{c T},$$

$$\xi_3 \xi_4 = \xi_4 \xi_3 \cos \frac{\tilde{j}_1 v}{c T} - i \xi_2 \xi_3 \frac{c}{\tilde{j}_1} \sin \frac{\tilde{j}_1 v}{c T},$$

$$\xi_4 \xi_2 = \xi_2 \xi_4 \cos \frac{\tilde{j}_1 v}{c T} + i \xi_2 \xi_3 \frac{1}{c} \sin \frac{\tilde{j}_1 v}{c T},$$

$$\xi_5 \xi_1 = \xi_1 \xi_5 \cos \frac{\tilde{j}_1 v}{c T} - i \xi_5 \xi_3 \frac{c T}{\tilde{j}_1} \sin \frac{\tilde{j}_1 v}{c T},$$

$$\xi_5 \xi_3 = \xi_3 \xi_5 \cos \frac{\tilde{j}_1 v}{c T} + i \xi_5 \xi_4 \frac{1}{c} \sin \frac{\tilde{j}_1 v}{c T}.$$
\[
\xi_3 \xi_4 = \xi_4 \xi_3 \cos \frac{\dot{J}_1 v}{cT} - i \xi_2 \xi_3 c \sin \frac{\dot{J}_1 v}{cT},
\]

\[
[\xi_1, \xi_5] = 2i \frac{\dot{J}_1 c}{T} \left( (\xi_2^2 - \frac{1}{c^2} \xi_1^2) \cos \frac{\dot{J}_1 v}{2cT} - \frac{1}{c^2} \xi_3 \sin \frac{\dot{J}_1 v}{2cT} \right) \sin \frac{\dot{J}_1 v}{cT},
\]

\[
[\xi_2, \xi_4] = -2i \xi_3^2 \frac{1}{c} \sin \frac{\dot{J}_1 v}{2cT}.
\]

As far as the generator \( \xi_1 \) do not commute with \( \xi_s, s = 2, 3, 4, 5 \) it is convenient to introduce right and left time \( t = \xi_3 \xi_1^{-1}, \hat{t} = \xi_1^{-1} \xi_2 \) and space \( r_k = \xi_{k+2} \xi_1^{-1}, \hat{r}_k = \xi_1^{-1} \xi_{k+2}, k = 1, 2, 3 \) generators. The reason for such definition is the simplification of expressions for commutation relations of the (anti) de Sitter quantum kinematics. It is possible to use only say right generators, but its commutators are cumbersome. The commutation relations of the independent generators are obtained from (31) in the form

\[
S^{q(\pm)}(\sigma_0) = \{t, r\} \quad \hat{t} r_1 = \hat{r}_1 t \cos \frac{\dot{J}_1 v}{cT} + i \hat{r}_1 \frac{c}{J_1} \sin \frac{\dot{J}_1 v}{cT},
\]

\[
\hat{t} r_2 - \hat{r}_2 t = -2i \hat{r}_1 \hat{r}_1 \frac{1}{c} \sin \frac{\dot{J}_1 v}{2cT}, \quad \hat{t} r_3 = \hat{r}_3 t \cos \frac{\dot{J}_1 v}{cT} - i \hat{t} \frac{cT}{J_1} \sin \frac{\dot{J}_1 v}{cT},
\]

\[
\hat{r}_1 r_2 = \hat{r}_2 r_1 \cos \frac{\dot{J}_1 v}{cT} + i \hat{r}_1 \frac{c}{J_1} \sin \frac{\dot{J}_1 v}{cT},
\]

\[
\hat{r}_p r_3 = \hat{r}_3 r_p \cos \frac{\dot{J}_1 v}{cT} - i \hat{r}_p \frac{cT}{J_1} \sin \frac{\dot{J}_1 v}{cT} \}.
\]

The right and left generators are connected as follows

\[
r_3 - \hat{r}_3 = 2i \frac{\dot{J}_1}{cT} \left( \left( \hat{t} - \frac{1}{c^2} \hat{r}_2 r_2 \right) \cos \frac{\dot{J}_1 v}{2cT} - \left( i \frac{1}{c^2} \hat{r}_1 r_1 \cos \frac{\dot{J}_1 v}{cT} \right) \sin \frac{\dot{J}_1 v}{2cT} \right) \sin \frac{\dot{J}_1 v}{cT},
\]

\[
\hat{r}_p = r_p \cos \frac{\dot{J}_1 v}{cT} - i \hat{r}_3 r_p \frac{\dot{J}_1}{cT} \sin \frac{\dot{J}_1 v}{cT}, \quad p = 1, 2,
\]

\[
\hat{t} = t \cos \frac{\dot{J}_1 v}{cT} - i \hat{r}_2 t \frac{\dot{J}_1}{cT} \sin \frac{\dot{J}_1 v}{cT}.
\]

The commutation relations of the time \( t, \hat{t} \) and space \( r_k, \hat{r}_k \) generators of the \((1 + 3)\) non-commutative quantum de Sitter \( S^{q(\pm)}_v(\sigma_0) \) and anti de Sitter \( S^{q(\pm)}_v(\sigma_0) \) kinematics are given by (32) for \( \dot{J}_1 = i \) and \( \dot{J}_1 = 1 \), respectively.
The parameter $T$ is interpreted as the curvature radius and has the time physical dimension $[T] = \text{[time]}$, the parameter $c$ is the light velocity $[c] = [\text{length}][\text{time}]^{-1}$, deformation parameter $v$ for the system units, where $\hbar = 1$, has the physical dimension of length $[v] = [cT] = \text{[length]} = \text{[momentum]}^{-1}$ and may be interpreted as the fundamental length.

The quantum $(1 + 3)$ Minkowski kinematics $M^4_v(\sigma_0)$ is obtained from the quantum (anti) de Sitter kinematics $S^{4(\pm)}(\sigma_0)$ in the zero curvature limit $T \to \infty$. The left and right generators are the same $\hat{t} = t, \hat{r}_k = r_k$ in this limit and we have

$$M^4_v(\sigma_0) = \{t, r\} \ [t, r_p] = 0, \ [r_3, t] = ivt,$$

$$[r_1, r_2] = 0, \ [r_3, r_p] = ivr_p, \ p = 1, 2 \}.$$  \hspace{1cm} (34)

This kinematics is isomorphic to the tachyonic $\kappa$-deformation of the Minkowski kinematics \[3\], where $v = \Lambda = \kappa^{-1}$.

In the non-relativistic limit $c \to \infty$ the quantum kinematics $S^{4(\pm)}(\sigma_0)$ are contracted to the non-commutative analogs of $(1 + 3)$ Newton kinematics $N^{4(\pm)}(\sigma_0)$ with non-zero curvature. In this limit $\hat{t} = t$, $\hat{r}_p = r_p$, $\hat{r}_3 = r_3 - ivj_1t^2/T^2$ and the commutation relations of the right space and time generators are as follows

$$N^{4(\pm)}_v(\sigma_0) = \{t, r\} \ [t, r_p] = 0, \ [r_3, t] = ivt(1 + j_1^2 t^2/T^2),$$

$$[r_1, r_2] = 0, \ [r_3, r_p] = ivr_p(1 + j_1^2 t^2/T^2), \ p = 1, 2 \}.$$  \hspace{1cm} (35)

In the zero curvature limit $T \to \infty$ both quantum Newton kinematics are passed into the quantum Galilei kinematics

$$G^4_v(\sigma_0) = \{t, r\} \ [t, r_p] = 0, \ [r_3, t] = ivt,$$

$$[r_1, r_2] = 0, \ [r_3, r_p] = ivr_p, \ p = 1, 2 \}.$$  \hspace{1cm} (36)

which commutation relations are identical with those of the quantum Minkowski kinematics \[34\].

Carroll kinematics \[1\], \[28\] are also realized as constant curvature spaces, but with different interpretation of the Beltrami coordinates, namely $r_k = \xi_k + \xi_1^{-1}$, $k = 1, 2, 3$ are the space generators and $t = \xi_5 \xi_1^{-1}$ is the time generator \[14\], \[15\]. Due to this interpretation the new physical dimensions...
of the contraction parameters are appeared: the parameter \( j_1 \) is replaced by \( \tilde{j}_1 R^{-1} \), where \( R \to \infty \) correspond to \( j_1 = \lambda_1 \) and \([R] = \) [length]; the parameter \( j_4 \) is replaced by \( c \), where \( c \to 0 \) correspond to \( j_4 = \frac{\xi}{\lambda_4} \) and \([c] = \) [velocity]. The deformation parameter \([\nu] = [R][c]^{-1} = \) [time] = [energy]^{-1} is interpreted as the fundamental time. The commutation relations (29) are rewritten in the form \((m = 2, 3, 4)\)

\[
\begin{align*}
\xi_1 \xi_m &= \xi_m \xi_1 \cosh \frac{j_1 \nu}{R} - i \xi_m \xi_5 \frac{j_1 c}{j_1 c} \sinh \frac{j_1 \nu}{R}, \\
\xi_m \xi_5 &= \xi_5 \xi_m \cosh \frac{j_1 \nu}{R} - i \xi_m \xi_3 \frac{j_1 c}{j_1 c} \sinh \frac{j_1 \nu}{R}, \\
\xi_2 \xi_3 &= \xi_3 \xi_2 \cosh \frac{j_1 \nu}{R} - i \xi_3 \xi_4 \sinh \frac{j_1 \nu}{R}, \\
\xi_3 \xi_4 &= \xi_4 \xi_3 \cosh \frac{j_1 \nu}{R} - i \xi_2 \xi_3 \sinh \frac{j_1 \nu}{R}, \\
[\xi_1, \xi_5] &= 2i \frac{j_1 \nu}{cR} \left( (\xi_2 + \xi_4) \cosh \frac{j_1 \nu}{2R} + \xi_3 \cosh \frac{j_1 \nu}{R} \right) \sinh \frac{j_1 \nu}{2R}, \\
[\xi_2, \xi_4] &= 2i \xi_3 \sinh \frac{j_1 \nu}{2R} \tag{37}
\end{align*}
\]

and in the limit \( c \to 0 \) are

\[
\begin{align*}
[\xi_1, \xi_m] &= 0, \quad [\xi_2, \xi_3] = 0, \quad [\xi_2, \xi_4] = 0, \quad [\xi_3, \xi_4] = 0, \\
\xi_m \xi_5 &= \xi_5 \xi_m - iv \xi_1 \xi_m, \quad [\xi_1, \xi_5] = iv \frac{j_1^2}{R^2} (\xi_2^2 + \xi_3^2 + \xi_4^2). \tag{38}
\end{align*}
\]

Introducing space \( r_k = \xi_{k+1} \xi_1^{-1}, \ k = 1, 2, 3 \) and time \( t = \xi_5 \xi_1^{-1}, \ \tilde{t} = \xi_1^{-1} \xi_5 \) generators and taking into account that \( \tilde{t} = t - i v \frac{j_1^2}{R^2} r^2 \), where \( r^2 = r_1^2 + r_2^2 + r_3^2 \), one obtain the commutation relations for the quantum analogs of Carroll kinematics \( C^{4(\pm)}(\sigma_0) \) with positive \((j_1 = 1)\) and negative \((j_1 = i)\) space curvature \((i, k = 1, 2, 3)\)

\[
C^{4(\pm)}(\sigma_0) = \{ t, r | \ [t, r_k] = iv r_k (1 + \frac{j_1^2}{R^2} r^2), \ [r_i, r_k] = 0 \}. \tag{39}
\]

The quantum Carroll kinematics with zero curvature is achieved in the limit \( R \to \infty \) and is as follows

\[
C^{4(0)}(\sigma_0) = \{ t, r | \ [t, r_k] = iv r_k, \ [r_i, r_k] = 0, \ i, k = 1, 2, 3 \}. \tag{40}
\]
This kinematics is the non-commutative analog of the Carroll kinematics first introduced in \[28\].

### 4.2 Quantum vector space $O_5^v(j; \sigma')$

For the permutation $\sigma' = (1, 4, 3, 5, 2)$ the deformation parameter is multiplied by $J = (1, 2) \cup (4, 5) = j_1 j_4$ and commutation relations of the generators are $(m = 3, 4, 5)$

\[
\xi_1 \xi_m = \xi_m \xi_1 \cosh J v - i \xi_m \xi_2 j_1 \sinh J v, \quad \xi_m \xi_2 = \xi_2 \xi_m \cosh J v - i \xi_1 \xi_m \frac{1}{j_1} \sinh J v,
\]

\[
\xi_4 \xi_3 = \xi_3 \xi_4 \cosh J v - i \xi_3 \xi_4 j_1 \sinh J v, \quad \xi_3 \xi_5 = \xi_5 \xi_3 \cosh J v - i \xi_2 \xi_3 \frac{1}{j_4} \sinh J v,
\]

\[
\left[\xi_1, \xi_2\right] = 2i j_1 \left( j_2^2 \xi_3^2 \cosh J v + (\xi_2^2 + j_2^2 j_3^2) \cosh \frac{J v}{2}\right) \sinh \frac{J v}{2},
\]

\[
\left[\xi_4, \xi_5\right] = 2i \xi_3^2 \frac{1}{j_3 j_4} \sinh \frac{J v}{2}. \quad (41)
\]

The quadratic form

\[
\text{inv}(j) = \left( \xi_1^2 + j_1^2 \xi_2^2 + j_1^2 j_3^2 (\xi_3^2 + j_4^2 \xi_5^2) \cosh J v + \right.
\]

\[
\left. + j_1^2 j_2 j_3^2 \frac{(\cosh J v)^2}{\cosh(\frac{J v}{2})} \right) \cosh \frac{3 J v}{2}, \quad (42)
\]

is invariant under the coaction of $SO_v(5; j; \sigma')$.

For the left and right time and space generators and the physical contraction parameters the commutation relations of the (anti) de Sitter kinematics $S_v^{4(\pm)}(\sigma')$ are written in the form

\[
S_v^{4(\pm)}(\sigma') = \{t, r | \hat{r}_k t = \hat{t} r_k \cosh \frac{j_1 v}{T} - i r_k \frac{T}{j_1} \sinh \frac{j_1 v}{T}, \]

\[
\hat{r}_2 r_1 = \hat{r}_1 r_2 \cosh \frac{j_1 v}{T} - i \hat{r}_1 r_3 \sinh \frac{j_1 v}{T}, \quad \hat{r}_1 r_3 = \hat{r}_3 r_1 \cosh \frac{j_1 v}{T} - i \hat{r}_2 r_1 \sinh \frac{j_1 v}{T}, \]

\[
\hat{r}_2 r_3 - \hat{r}_3 r_2 = 2i \hat{r}_1 r_1 \sinh \frac{j_1 v}{2T}. \quad (43)
\]
The right and left generators are connected as follows

\[ \hat{r}_k = r_k \cosh \frac{\tilde{j}_1 v}{T} + i \hat{r}_k \frac{\tilde{j}_1}{T} \sinh \frac{\tilde{j}_1 v}{T}, \]

\[ \hat{t} = t + 2i \frac{\tilde{j}_1}{e^2 T} \left( \hat{r}_1 r_1 \cosh \frac{\tilde{j}_1 v}{T} + (\hat{r}_2 r_2 + \hat{r}_3 r_3) \cosh \frac{\tilde{j}_1 v}{2T} \right) \sinh \frac{\tilde{j}_1 v}{2T}. \] (44)

The deformation parameter \( v \) has time dimension \([v] = [T] = \text{[time]} = \text{[energy]}^{-1}\), therefore kinematics \( S_4^{(\pm)}(\sigma') \) are not isomorphic to kinematics \( S_4^{(\pm)}(\sigma_0) \). The quantum (anti) de Sitter kinematics \( S_4^{(\pm)}(\sigma') \) may be regarded as a kinematics with the fundamental time.

The quantum (1 + 3) Minkowski kinematics \( M_4(\sigma') \) is obtained from the quantum (anti) de Sitter kinematics \( S_4^{(\pm)}(\sigma') \) in the zero curvature limit \( T \to \infty \). If the left and right generators are coincided \( \hat{t} = t, \hat{r}_k = r_k \) in this limit and we have

\[ M_4(\sigma') = \{ t, r | [t, r_k] =ivr_k, [r_i, r_k] = 0, i, k = 1, 2, 3 \}. \] (45)

This kinematics is isomorphic to the standard \( \kappa \)-Minkowski kinematics \( \mathcal{M}_4(\sigma') \).

In the non-relativistic limit \( c \to \infty \) the quantum kinematics \( S_4^{(\pm)}(\sigma') \) are contracted to the non-commutative analogs of (1 + 3) non-relativistic Newton kinematics \( N_4^{(\pm)}(\sigma') \) with non-zero curvature. In this limit the deformation parameter remain untouched, the right and left time generators are the same \( \hat{t} = t, \hat{r}_k = r_k \) in this limit and we have

\[ M_4(\sigma') = \{ t, r | [t, r_k] =ivr_k, [r_i, r_k] = 0, i, k = 1, 2, 3 \}. \] (45)

The quantum Newton kinematics are given by the commutation relations

\[ N_4^{(\pm)}(\sigma') = \{ t, r | [t, r_k] =ivr_k, [r_i, r_k] = 0, i, k = 1, 2, 3 \}. \] (47)

In the zero curvature limit \( T \to \infty \) both quantum Newton kinematics are contracted to the quantum Galilei kinematics

\[ G_4(\sigma') = \{ t, r | [t, r_k] =ivr_k, [r_i, r_k] = 0, i, k = 1, 2, 3 \}. \] (48)
which commutation relations are identical with those of the Minkowski kinetics (45).

To obtain Carroll kinematics let us replace the parameter $j_1$ by $\tilde{j}_1 R^{-1}$, where $R \to \infty$ correspond to $j_1 = \iota_1$ and $[R] = [\text{length}]$. The parameter $j_4$ is replaced by $c$, where $[c] = [\text{velocity}]$ and $c \to 0$ correspond to $j_4 = \iota_4$. Then the deformation parameter receive the physical time dimension $[v] = [R][c]^{-1} = [\text{time}] = [\text{energy}]^{-1}$, that is the same dimension as for the standard kinematics. Commutation relations (41) are rewritten in the form ($m = 3, 4, 5$)

$$\xi_1 \xi_m = \xi_m \xi_1 \cosh \frac{\tilde{j}_1 c v}{R} - i \xi_m \xi_2 \frac{\tilde{j}_1}{R} \sinh \frac{\tilde{j}_1 c v}{R},$$

$$\xi_m \xi_2 = \xi_2 \xi_m \cosh \frac{\tilde{j}_1 c v}{R} - i \xi_2 \xi_m \frac{R}{j_1} \sinh \frac{\tilde{j}_1 c v}{R},$$

$$\xi_4 \xi_3 = \xi_3 \xi_4 \cosh \frac{\tilde{j}_1 c v}{R} - i c \xi_3 \xi_5 \sinh \frac{\tilde{j}_1 c v}{R},$$

$$\xi_3 \xi_5 = \xi_5 \xi_3 \cosh \frac{\tilde{j}_1 c v}{R} - i \xi_3 \xi_5 \frac{1}{c} \sinh \frac{\tilde{j}_1 c v}{R},$$

$$[\xi_1, \xi_2] = 2i \frac{\tilde{j}_1}{R} \left( \xi_3^2 \cosh \frac{\tilde{j}_1 c v}{R} + (\xi_4^2 + c^2 \xi_5^2) \cosh \frac{\tilde{j}_1 c v}{2R} \right) \sinh \frac{\tilde{j}_1 c v}{2R},$$

$$[\xi_4, \xi_5] = 2 i \xi_3^2 \frac{1}{c} \cosh \frac{\tilde{j}_1 c v}{2R},$$

and in the limit $c \to 0$ are as follows

$$[\xi_1, \xi_m] = 0, \quad [\xi_2, \xi_m] = 0, \quad [\xi_1, \xi_2] = 0, \quad [\xi_3, \xi_4] = 0,$$

$$\xi_3 \xi_5 = \xi_5 \xi_3 - i v \frac{\tilde{j}_1}{R} \xi_4 \xi_3, \quad [\xi_4, \xi_5] = i v \frac{\tilde{j}_1}{R} \xi_3^2.$$  

(50)

The generator $\xi_1$ commute with $\xi_s$, $s = 2, 3, 4, 5$, therefore the left and right generators are coincided $\hat{r}_k = r_k = \xi_{k+1} \xi_1^{-1}$, $k = 1, 2, 3$, $\hat{t} = t = \xi_4 \xi_1^{-1}$ and commutation relations of the quantum Carroll kinematics $C_v^{4(\pm)}(\sigma')$ are easily obtained ($i, k = 1, 2, 3$)

$$C_v^{4(\pm)}(\sigma') = \{t, r \} \quad [t, r_1] = 0, \quad [t, r_2] = i v \frac{\tilde{j}_1}{R} r_3 r_2,$$

$$[r_3, t] = i v \frac{\tilde{j}_1}{R} r_1^2, \quad [r_i, r_k] = 0.$$  

(51)
As before the quantum Carroll kinematics with zero curvature is achieved in the limit \( R \to \infty \)

\[
C_4^{(0)}(\sigma') = \{ t, r \mid [t, r_k] = 0, \ [r_i, r_k] = 0, \ i, k = 1, 2, 3 \}. \tag{52}
\]

It is remarkable that all commutators are equal to zero and the quantum Carroll kinematics is identical with the commutative Carroll kinematics \cite{28}.

### 4.3 Quantum vector space \( O^5_v(j; 1) \)

For the permutation \( \tilde{\sigma} = (2, 3, 1, 4, 5) \) the deformation parameter is multiplied by \( J = (2, 5) \cup (3, 4) = j_2 j_3 j_4 \) and commutation relations of the generators are \((m = 1, 3, 4)\)

\[
\xi_2 \xi_m = \xi_m \xi_2 \cosh J v - i \xi_m \xi_5 J \sinh J v, \quad \xi_5 \xi_m = \xi_m \xi_5 \cosh J v + i \xi_m \xi_2 \frac{1}{J} \sinh J v,
\]

\[
\xi_3 \xi_1 = \xi_1 \xi_3 \cosh J v - i \xi_1 \xi_4 j_3 \sinh J v, \quad \xi_4 \xi_1 = \xi_1 \xi_4 \cosh J v + i \xi_1 \xi_3 \frac{1}{j_3} \sinh J v,
\]

\[
\left[ \xi_2, \xi_3 \right] = 2 i \frac{1}{j_3} \left( \xi_1^2 \cosh J v + j_1^2 j_2 \left( \xi_3 + j_3^2 \xi_4 \right) \cosh \frac{J v}{2} \right) \frac{1}{J} \sinh \frac{J v}{2},
\]

\[
\left[ \xi_3, \xi_4 \right] = 2 i \xi_4^2 \frac{1}{j_3 j_2 j_3} \sinh \frac{J v}{2}. \tag{53}
\]

The quadratic form

\[
\text{inv}(j) = \left( \xi_1^2 \frac{(\cosh J v)^2}{\cosh J v/2} + j_1^2 \xi_2^2 + J_5^2 \xi_5^2 + j_1 j_2 \left( \xi_3 + j_3^2 \xi_4 \right) \cosh J v \right) \cosh \frac{3 J v}{2} \tag{54}
\]

is invariant under the coaction \( (13) \) of \( SO_v(5; j; \tilde{\sigma}) \).

For the left and right time and space generators and the physical contraction parameters the commutation relations of the (anti) de Sitter kinematics \( S_v^{(\pm)}(\tilde{\sigma}) \) are written in the form \((p = 1, 2)\)

\[
S_v^{(\pm)}(\tilde{\sigma}) = \{ t, r \mid \dot{t} r_p = \dot{r}_p t \cos \frac{v}{c} + i \dot{r}_p r_3 \frac{1}{c} \sin \frac{v}{c},
\]

\[
\dot{r}_3 - \dot{r}_3 = 2 i \frac{c T^2}{j_1} \left( \cos \frac{v}{c} - \frac{j_1^2 \dot{r}_1 r_3}{c^2 T^2} + \frac{j_1}{2 c T^2} \dot{r}_2 r_1 \cos \frac{v}{2 c} \right) \sin \frac{v}{2 c}.
\]
\[\hat{r}_p r_3 = \hat{r}_3 r_p \cos \frac{v}{c} - i \hat{r}_p c \sin \frac{v}{c}, \quad \hat{r}_2 r_1 = \hat{r}_1 r_2 = 2 \frac{e^{2T^2}}{j_1^2} \sin \frac{v}{2c}. \] (55)

The left and right generators are connected by the following relations

\[\hat{t} = t \cos \frac{v}{c} - i r_3 \frac{1}{c} \sin \frac{v}{c}, \quad \hat{r}_1 = r_1 \cos \frac{v}{c} + r_3 \sin \frac{v}{c},\]
\[\hat{r}_2 = r_2 \cos \frac{v}{c} - r_1 \sin \frac{v}{c}, \quad \hat{r}_3 = r_3 \cos \frac{v}{c} + i t c \sin \frac{v}{c}. \] (56)

The deformation parameter has the velocity dimension \([v] = [c] = \text{[velocity]},\) therefore kinematics \(S_v^4(\pm)(\tilde{\sigma})\) are not isomorphic to kinematics \(S_v^4(\pm)(\sigma_0),\) \(S_v^4(\pm)(\sigma'),\) and may be regarded as a kinematics with the fundamental velocity. As it follows from (56), both contractions \(T \to \infty, c \to \infty\) are not permitted, therefore the quantum (anti) de Sitter kinematics \(S_v^4(\pm)(\tilde{\sigma})\) have not Minkowski, Newton, Galilei kinematics as a limiting cases.

To obtain Carroll kinematics let us replace the parameter \(j_1\) by \(\tilde{j}_1 R^{-1}\), where \(R \to \infty\) correspond to \(j_1 = t_1\) and \([R] = \text{[length]}\). The parameter \(j_4\) is replaced by \(c\), where \([c] = \text{[velocity]}\) and \(c \to 0\) correspond to \(j_4 = t_4\). Then the deformation parameter receive the physical dimension \([v] = [c]^{-1} = \text{[velocity]}^{-1}\), that is the inverse dimension as compare with the standard kinematics. Commutation relations (53) are rewritten in the form \((m = 1, 3, 4)\)

\[\xi_2 \xi_m = \xi_m \xi_2 \cosh(cv) - i c \xi_m \xi_5 \sinh(cv), \quad \xi_5 \xi_m = \xi_m \xi_5 \cosh(cv) + i \xi_m \xi_2 \frac{1}{c} \sinh(cv),\]
\[\xi_3 \xi_1 = \xi_1 \xi_3 \cosh(cv) - i \xi_1 \xi_4 \sinh(cv), \quad \xi_4 \xi_1 = \xi_1 \xi_4 \cosh(cv) + i \xi_1 \xi_3 \sinh(cv),\]
\[\left[\xi_2, \xi_3\right] = 2i \frac{R^2}{e j_1^2} \left(\xi_1^2 \cosh(cv) + \frac{j_1^2}{R^2} (\xi_3^2 + \xi_4^2) \cosh \frac{cv}{2}\right) \sinh \frac{cv}{2},\]
\[\left[\xi_3, \xi_4\right] = 2i \frac{R^2}{j_1^2} \sinh \frac{cv}{2}. \] (57)

and in the limit \(c \to 0\) are

\[\left[\xi_2, \xi_m\right] = 0, \quad \left[\xi_1, \xi_3\right] = 0, \quad \left[\xi_1, \xi_4\right] = 0, \quad \left[\xi_3, \xi_4\right] = 0,\]
\[\xi_5 \xi_m = \xi_m \xi_5 + iv \xi_m \xi_2, \quad \left[\xi_2, \xi_5\right] = iv \frac{R^2}{j_1^2} \left(\xi_1^2 + \frac{j_1^2}{R^2} (\xi_3^2 + \xi_4^2)\right). \] (58)
As far as $\xi_1$ commute with $\xi_2, \xi_3, \xi_4$, the left and right space generators are equal $\hat{r}_k = r_k = \xi_{k+1}\xi_1^{-1}, k = 1, 2, 3$, but the left and right time generators are related as $\hat{t} = t + ivr_1$ and commutation relations of the quantum Carroll kinematics $C^4(\pm)(\tilde{\sigma})$ with curvature are as follows $(i, k = 1, 2, 3, p = 2, 3)$

$$C^4(\pm)(\tilde{\sigma}) = \{t, r | [t, r_p] = 0, [r_i, r_k] = 0, [r_1, t] = iv\left(\frac{R^2}{j^2} + r^2\right)\}. \quad (59)$$

There is no quantum analogs of the flat Carroll kinematics [28], since the limit $R \to \infty$ is forbidden.

5 Conclusion

The narrow analysis of the multiplier $J = (\sigma_1, \sigma_5) \cup (\sigma_2, \sigma_4)$, which is appeared in the transformation of the deformation parameter $z = Jv$, and commutation relations [23] of the quantum vector space generators for different permutations $\sigma$ made possible to find three permutations giving a different $J$ and a physically nonequivalent kinematics. For the identical permutation $\sigma_0$ the quantum (anti) de Sitter kinematics [32] are characterized by the fundamental length $[v] = [\text{length}]$, for the permutation $\sigma'$ are characterized by the fundamental time $[v] = [\text{time}]$ [43] and for the permutation $\tilde{\sigma} —$ by the fundamental velocity $[v] = [\text{velocity}]$ [55]. Recall that the same physical dimensions of the deformation parameter have been received for the quantum algebras $so_0(3; j; \sigma)$ and corresponding $(1 + 1)$ kinematics for a different permutations [23].

In the zero curvature limit $T \to \infty$ two quantum Minkowski kinematics [34] and [45] have been obtained

$$M^4_v(\sigma_0) = \{t, r | [t, r_p] = 0, [r_3, t] = ivt,$$

$$[r_2, r_1] = 0, [r_3, r_p] = ivr_p, p = 1, 2, \},$$

$$M^4_v(\sigma') = \{t, r | [t, r_k] = ivr_k, [r_i, r_k] = 0, i, k = 1, 2, 3\}. \quad (60)$$

The first one is isomorphic to the tachyonic $\kappa$-deformation [3], the second one to the standard $\kappa$-deformation [2]. For both $\kappa$-Minkowski kinematics [2], [3] in the system units $\hbar = c = 1$ the deformation parameter $\Lambda = \kappa^{-1}$ has the physical dimension of length and is interpreted as the fundamental length. But in the system units $\hbar = 1$ the deformation parameter has a different
dimensions, namely $v$ is the fundamental length for $M^4_v(\sigma_0)$ kinematics and $v$ is the fundamental time for $M^4_v(\sigma')$.

As far as the commutation relations (60) do not depend on $c$ they are not changed in the limit $c \to \infty$, therefore the generators of the quantum Galilei kinematics $G^4_v(\sigma_0)$ (36) and $G^4_v(\sigma')$ (47) are subject of the same commutation relations. The only difference consist in following: for the Galilei kinematics there are two invariants $\text{inv}_1 = t^2$, $\text{inv}_2 = r_1^2 + r_2^2 + r_3^2$ with respect to the coaction of the corresponding quantum groups, whereas for the Minkowski kinematics there is only one invariant $\text{inv} = t^2 - (r_1^2 + r_2^2 + r_3^2)$. Thereby the quantum deformations of the flat kinematics are identical up to the coaction of the corresponding quantum groups for both relativistic and non-relativistic one. It is remarkable that the same invariants are in the commutative case.

There are two non-commutative analogs of the non-relativistic Newton kinematics (35), (47)

\[
N^4_v(\sigma_0) = \{ t, r \mid [t, r_p] = 0, \quad [r_3, t] = ivt(1 + \frac{j_1^2 t^2}{T^2}),
\]

\[
[r_1, r_2] = 0, \quad [r_3, r_p] = ivr_p(1 + \frac{j_1^2 t^2}{T^2}), \quad p = 1, 2\},
\]

\[
N^4_v(\sigma') = \{ t, r \mid [t, r_k] = i(r_k + \frac{j_1^2}{T^2} tr_k t) \frac{T}{j_1} \tanh \frac{j_1 v}{T},
\]

\[
r_2 r_1 = r_1 r_2 \cosh \frac{j_1 v}{T} - ir_1 r_3 \sinh \frac{j_1 v}{T}, \quad r_1 r_3 = r_3 r_1 \cosh \frac{j_1 v}{T} - ir_2 r_1 \sinh \frac{j_1 v}{T},
\]

\[
[r_2, r_3] = 2ir_1^2 \sinh \frac{j_1 v}{2T}\},
\]

where in the last case the deformation parameter do not transformed under contraction. The multiplier $T^{-1}$ is appeared as the result of the physical interpretations of the quantum space generators.

There are three non-commutative analogs of the exotic non-zero curvature Carroll kinematics (39), (51), (59)

\[
C^4_v(\sigma_0) = \{ t, r \mid [t, r_k] = ivr_k(1 + \frac{j_1^2 r^2}{R^2}), \quad [r_i, r_k] = 0, \quad [r_i, r_k] = 0, \}
\]

\[
C^4_v(\sigma') = \{ t, r \mid [t, r_1] = 0, \quad [t, r_2] = iv \frac{j_1}{R} r_3 r_2, \}
\]
\[ [r_3, t] = i\nu \frac{\hat{j}_1}{R} r_1^2, \\
[r_i, r_k] = 0, \]

\[ C^4_v(\pm) (\hat{\sigma}) = \{ t, r | [t, r_p] = 0, [r_i, r_k] = 0, [r_1, t] = i\nu \left(\frac{R^2}{j_1^2} + r^2\right) \} \] (62)

and two quantum analogs of the zero curvature Carroll kinematics (40), (52)

\[ C^4_v(0) (\sigma_0) = \{ t, r | [t, r_k] =ivr_k, [r_i, r_k] = 0, i, k = 1, 2, 3 \}, \]

\[ C^4_v(0) (\sigma') = \{ t, r | [t, r_k] = 0, [r_i, r_k] = 0, i, k = 1, 2, 3 \}. \] (63)

The deformation parameter has the physical dimension of time \([v] = \text{[time]}\) for permutations \(\sigma_0, \sigma'\) and the dimension of inverse velocity \([v] = \text{[velocity]}^{-1}\) for permutation \(\hat{\sigma}\).

In spite of the fact that the commutation relations of generators of \(C^4_v(0) (\sigma_0)\) and \(M^4_v (\sigma')\) are identical, both kinematics are physically different. Mathematically isomorphic kinematics may be physically non-equivalent.

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