FACETS OF SECONDARY POLYTOPES AND CHOW STABILITY

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Abstract. Chow stability is one of notions of Mumford’s Geometric Invariant Theory to study the moduli space of polarized varieties. Kapranov, Sturmfels and Zelevinsky detected that Chow stability of polarized toric varieties is determined by its inherent secondary polytope, which is a polytope whose vertices are corresponding to regular triangulations of the associated (Delzant) polytope [6]. In this paper, we give a purely convex-geometrical proof that the Chow form of a smooth polarized toric variety is $H$-semistable if and only if it is $H$-polystable for the standard complex torus $H$.

1. INTRODUCTION

Let $(X,L)$ be an $n$-dimensional polarized variety, that is $X$ is an $n$-dimensional complex variety with $\deg X \geq 2$ and $L$ is a very ample line bundle over $X$. By a complete linear system of $L$, one can embed $X$ into a certain projective space $\mathbb{P}^N$. This makes $X$ to be a projective variety.

Chow stability is the one of notions of Geometric Invariant Theory (GIT) invested by many researchers. In the present paper, we study poly(semi)stability on a projective toric variety for the standard complex torus action. To state our result more precisely, let us briefly recall the fundamental knowledge of toric varieties. See [1, 5, 11] for more details.

Let $A$ be a point configuration in $\mathbb{Z}^n$. We assume that the convex hull $Q := \text{Conv}(A)$ of $A$ satisfies the following conditions:

a) For each vertex $a_j \in Q$, there are exactly $n$ edges $\ell_{j,1}, \ldots, \ell_{j,n}$ of $Q$ which is incident to $a_j$ (simplicity).

b) Let $e_{j,i}$ be the primitive generators of the edges $\ell_{j,i}$ for $i = 1, \ldots, n$. Then the vectors $e_{j,i}$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ (smoothness).

Then the $n$-dimensional integral polytope $Q$ is said to be the integral Delzant polytope. The closure of the $A$-monomial embedding of a complex torus $(\mathbb{C}^\times)^n$ to the projective space defines the $n$-dimensional projective algebraic variety $X_A$ with the very ample $(\mathbb{C}^\times)^n$-equivariant line bundle $L_A$. In particular, if the associated projective varieties are Fano varieties (i.e., $K_X^{-1}$ is ample) with the anticanonical polarization, then the integral polytopes $Q$ correspond to reflexive Delzant polytopes. Recall that a fully dimensional integral polytope $Q$ containing the origin in its interior is called the reflexive polytope if vertices are primitive lattice points and whose polar dual polytope is again an integral polytope.

Next we quick review some related results on Chow stability of polarized varieties which will be the source of our motivation. One of the reasons why Chow stability is important in Kähler geometry is that this notion is closely related to the existence problem of canonical metrics on a certain Kähler manifold. A breakthrough result has been achieved by Donaldson in [3]. He showed that the existence of a constant scalar curvature Kähler (cscK)
metric representing the first Chern class \( c_1(L) \) implies asymptotically Chow stability of a polarized variety \((X, L)\) whenever \( X \) has no holomorphic vector fields. This result has been extended by Mabuchi in the case where the automorphism group is not discrete. In [8], Mabuchi proved that if \((X, L)\) admits a cscK in \( c_1(L) \) then \((X, L)\) is asymptotically Chow polystable whenever \((X, L)\) satisfies the hypothesis of the obstruction for asymptotically Chow semistability. Eventually, Futaki has detected that Mabuchi’s hypothesis is equivalent to the vanishing of a collection of integral invariants \( F_{Td^i}, \ldots, F_{Td^n} \) defined in [4], where \( Td^i \) denotes the \( i \)-th Todd polynomial. The reader should bear in mind that \( F_{Td^1} \) equals the classical Futaki invariant which is an obstruction to the existence of cscK metric up to a multiplicative constant. Since these integral invariants are a generalization of the classical Futaki invariants, we call them higher Futaki invariants. Combining Mabuchi’s result and Futaki’s statement, we have the following

**Theorem 1.1** (Mabuchi-Futaki [8, 4]). Let \((X, L)\) be an \( n \)-dimensional smooth polarized variety. Assume that the higher Futaki invariants \( F_{Td^i} \) vanishes for each \( i = 1, \ldots, n \). If \((X, L)\) admits cscK metric in \( c_1(L) \) then \((X, L)\) is asymptotically Chow polystable.

One of the best possible result on the canonical metrics of toric Fano varieties, due to X.J. Wang and X. Zhu, is the following:

**Theorem 1.2** (Wang-Zhu [14]). Let \( X \) be a smooth toric Fano variety with the anticanonical polarization. Then \((X, K_X^{-1})\) admits a Kähler-Einstein metric in \( c_1(K_X^{-1}) \) if and only if the classical Futaki invariant vanishes.

Note that all cscK metrics in \( c_1(K_X^{-1}) \) are Kähler-Einstein metrics on smooth Fano varieties. Summing up these results, one can see that in the case of smooth toric Fano varieties with the anticanonical polarization, asymptotically Chow semistability implies asymptotically Chow polystability. Hence it is natural to ask whether a direct proof of this result exists in the framework of combinatorial argument. Moreover, we provide the following solution to this problem by a purely convex-geometrical proof.

**Theorem 1.3.** Let \( A \subset \mathbb{Z}^n \) be a point configuration which affinely spans \( \mathbb{Z}^n \) over \( \mathbb{Z} \). Let \((X_A, L_A)\) be the associated smooth polarized toric variety. Define the subtorus of \((\mathbb{C}^\times)^A\) by

\[
H = \{ (t_0, \ldots, t_N) \in (\mathbb{C}^\times)^A \mid \prod_{j=0}^N t_j = 1 \}
\]

where \( N = \sharp A - 1 \). Then the Chow point of \((X_A, L_A)\) is \( H \)-semistable if and only if it is \( H \)-polystable.

We shall show the assertion by the contradiction. Remark that \( H \)-polystability always implies \( H \)-semistability by its definition (see Definition [1]). The main idea of our proof comes from the following observation. Let \( G \) be a reductive algebraic group. Suppose \( G \) acts a finite dimensional vector space \( V \) linearly. The well-known Hilbert-Mumford numerical criterion of GIT (Proposition 2.1) gives the necessary and sufficient condition for a nonzero vector \( v \in V \) being poly(semi)stable. In the special case when the reductive group \( G \) is isomorphic to the algebraic torus, this criterion can be restated in terms of combinatorial description (Proposition 2.2). See [2, 12, 13] for more details. Roughly speaking, the condition of polystability of \( X \) is equivalent to the fact that the corresponding weight polytope \( N_X \) (the Chow polytope) containing the origin in its interior. On the other hand,
the condition of semistability is equivalent to the fact that $\mathcal{N}_X$ containing the origin. In order to show our main result, we prove that if $\mathcal{N}_X$ contains the origin then the origin is in the interior of $\mathcal{N}_X$;

$$0 \in \mathcal{N}_X \implies 0 \in \text{Int}(\mathcal{N}_X).$$

Here $\text{Int}(P)$ denotes the interior of an integral polytope $P$. Let $\partial P$ denote the boundary of $P$. Assuming that the boundary $\partial \mathcal{N}_X$ contains the origin, we prove that this contradicts the boundary condition of $\mathcal{N}_X$ and this contradiction shows the assertion.

This paper is organized as follows. Section 2 is a brief review of Geometric Invariant Theory and Chow stability. In Section 3, we first define the secondary polytope and discuss about its fundamental property due to the work of Gelfand, Kapranov and Zelevinsky. The structure of the secondary polytope is well-discussed in [5, 7]. Section 3.2 collects a combinatorial framework of these arguments. We give a proof of the main theorem in Section 4.

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2. Preliminaries

2.1. Weight polytope. Let $G$ be a reductive algebraic group and $V$ be a finite dimensional complex vector space. Suppose $G$ acts linearly on $V$. Let us denote a point $v^*$ in $V$ which is a representative of $v \in \mathbb{P}(V)$.

**Definition 1.** Let $v^*$ be as above and let $\mathcal{O}_G(v^*)$ be the $G$-orbit.

i) $v^*$ is called $G$-semistable if the Zariski closure of $\mathcal{O}_G(v^*)$ does not contain the origin; $0 \notin \overline{\mathcal{O}_G(v^*)}$.

ii) $v^*$ is called $G$-polystable if $\mathcal{O}_G(v^*)$ is closed.

Analogously, $v \in \mathbb{P}(V)$ is said to be $G$-polystable (resp. semistable) if any representative of $v$ in $V \setminus \{0\}$ is $G$-polystable(resp. semistable).

**Remark 2.** The closure of $\mathcal{O}_G(v^*)$ in the Euclidean topology coincides with the Zariski closure $\overline{\mathcal{O}_G(v^*)}$ (see, [9, Theorem 2.33]).

From Definition 1, one can see that polystability implies semistability as $G$-orbit itself never contain the origin. The following Hilbert-Mumford criterion is well-known in Geometric Invariant Theory.

**Proposition 2.1** (Hilbert-Mumford criterion [10]). $v \in \mathbb{P}(V)$ is $G$-polystable (resp. semistable) if and only if $v$ is $H$-polystable (resp. semistable) for all maximal algebraic tori $H \leq G$.

Now we assume that the reductive group $G$ is isomorphic to an algebraic torus $(\mathbb{G}_m)^{N+1}$ where $\mathbb{G}_m = \text{Spm}(\mathbb{C}[T, T^{-1}]) \simeq \mathbb{C}^\times$. It is well known that $V$ splits into weight spaces

$$V = \bigoplus_{\chi \in \chi(G)} V_{\chi}, \quad V_{\chi} := \{ v^* \in V \mid t \cdot v^* = \chi(t) \cdot v^*, \ t \in G \}$$
where $\chi(G)$ is the character group of the algebraic torus $G$. Since any rational character $\chi : G \to \mathbb{C}^\times$ is given by a Laurent monomial
\[
t^a = t_0^{a_0} \cdot \cdots \cdot t_N^{a_N} \quad \text{with} \quad \mathbf{a} = (a_0, \ldots, a_N) \in \mathbb{Z}^{N+1},
\]
we have an isomorphism $\chi(G) \simeq \mathbb{Z}^{N+1}$.

**Definition 2 (Weight polytope).** Let $v^* \in V \setminus \{0\}$ be a nonzero vector in $V$ with

\[
v^* = \sum_{\chi \in \chi(G)} v_\chi, \quad v_\chi \in V_\chi.
\]

The **weight polytope** of $v^*$ (with respect to $G$) is the integral convex polytope in $\chi(G) \otimes \mathbb{R} \simeq \mathbb{R}^{N+1}$ defined by

\[
\mathcal{N}_G(v^*) := \text{Conv} \{ \chi \in \chi(G) \mid v_\chi \neq 0 \} \subset \mathbb{R}^{N+1},
\]

where $\text{Conv} \{ A \}$ denotes the convex hull of a finite set of points $A$.

The fundamental property of this weight polytope is the following fact.

**Proposition 2.2 (The numerical criterion):** [2] Theorem 9.2, [13] Theorem 1.5.1. Let $G$ be isomorphic to an algebraic torus and let $v^*$ be a nonzero vector in $V$. Then

i) $v^*$ is $G$-semistable if and only if $\mathcal{N}_G(v^*)$ contains the origin.

ii) $v^*$ is $G$-polystable if and only if $\mathcal{N}_G(v^*)$ contains the origin in its interior.

2.2. **Chow form.** Now we recall the definition of the Chow form of irreducible complex projective varieties. See [5] for more details.

Let $X \to \mathbb{P}^N$ be an $n$-dimensional irreducible complex projective variety of degree $d \geq 2$. Recall that the Grassmann variety $\mathbb{G}(k, \mathbb{P}^N)$ parameterizes $k$-dimensional projective linear subspaces of $\mathbb{P}^N$.

**Definition 3 (Associated hypersurface).** The **associated hypersurface** of $X \to \mathbb{P}^N$ is the subvariety in $\mathbb{G}(N-n-1, \mathbb{P}^N)$ which is given by

\[
Z_X := \{ L \in \mathbb{G}(N-n-1, \mathbb{P}^N) \mid L \cap X \neq \emptyset \}.
\]

The fundamental property of $Z_X$ can be summarized as follows (see [5, pg. 99]):

1. $Z_X$ is irreducible.
2. $\text{Codim} Z_X = 1$ (that is $Z_X$ is a divisor in $\mathbb{G}(N-n-1, \mathbb{P}^N)$).
3. $\text{deg} Z_X = d$ in the Plücker coordinates.
4. $Z_X$ is given by the vanishing of a section $R^*_X \in H^0(\mathbb{G}(N-n-1, \mathbb{P}^N), \mathcal{O}(d))$ such that $Z_X = \{ R^*_X = 0 \}$.

We call $R^*_X$ the **Chow form** of $X$. Note that $R^*_X$ can be determined up to a multiplicative constant. Setting $V := H^0(\mathbb{G}(N-n-1, \mathbb{P}^N), \mathcal{O}(d))$ and $R_X \in \mathbb{P}(V)$ which is the projection of $R^*_X$, we call $R_X$ the **Chow point** of $X$. Since we have the natural action of $G = \text{SL}(N+1, \mathbb{C})$ on $\mathbb{P}(V)$, we can define $\text{SL}(N+1)$-polystability (semistability) of $R_X$.

**Definition 4 (Chow stability).** Let $X \to \mathbb{P}^N$ be an irreducible, $n$-dimensional complex projective variety. Then $X$ is said to be **Chow polystable** (resp. **semistable**) if the Chow point $R_X$ of $X$ is $\text{SL}(N+1, \mathbb{C})$-polystable (resp. semistable).
Let \((X, L)\) be a polarized variety. We denote the Kodaira image by \(\Phi_i(X)\) where

\[
\Phi_i : X \longrightarrow \mathbb{P}((H^0(X, L^i))^*)
\]
is the Kodaira embedding.

**Definition 5** (Asymptotically Chow stability). \((X, L)\) is Chow polystable (resp. semistable) if the Kodaira image \(\Phi(X)\) of \(\Phi : X \longrightarrow \mathbb{P}((H^0(X, L))^*)\) is Chow polystable (resp. semistable). In particular, \((X, L)\) is asymptotically Chow polystable (resp. semistable) if \(\Phi_i(X)\) in (2.1) is Chow polystable (resp. semistable) for all \(i \gg 0\).

### 2.3. Toric case

In order to finish this section, let us investigate Chow stability in the case of projective toric varieties. Recall that a toric variety is a complex irreducible algebraic variety with a complex torus action having an open dense orbit. A construction of toric varieties of projective toric varieties. Recall that a toric variety is a complex irreducible algebraic variety with a complex torus action having an open dense orbit. A construction of toric varieties of projective toric varieties. Recall that a toric variety is a complex irreducible algebraic variety with a complex torus action having an open dense orbit. A construction of toric varieties of projective toric varieties. Recall that a toric variety is a complex irreducible algebraic variety with a complex torus action having an open dense orbit.

Let us fix a point configuration \(a_0, \ldots, a_N\) be a point configuration in \(\mathbb{Z}^n\). Suppose that \(A\) affinely spans \(\mathbb{R}^n\). Setting

\[
X_A^0 = \{ x^{a_0} : \cdots : x^{a_N} \in \mathbb{P}^N \mid x = (x_1, \ldots, x_n) \in (\mathbb{C}^*)^n \},
\]

we define the variety \(X_A \subset \mathbb{P}^N\) to be the closure of \(X_A^0\) in \(\mathbb{P}^N\). Then \(X_A\) is an \(n\)-dimensional equivariantly embedded subvariety in \(\mathbb{P}^N\) with the very ample line bundle \(L_A\), where \(N = \dim(H^0(X_A, L_A^*)) - 1\). We denote the associated \(n\)-dimensional polarized toric variety for a point configuration \(A\) by \(X_A \overset{L_A}{\longrightarrow} \mathbb{P}^N\). Let \((X_A, L_A)\) be the \(n\)-dimensional polarized toric variety. For any positive integer \(i > 0\), \((X_A, L_A)\) is Chow polystable (resp. semistable) if and only if the Chow point \(R_{X,A}^i\) is \(\text{SL}(N_i, \mathbb{C})\)-polystable (resp. semistable) where \(N_i = \sharp(iQ \cap \mathbb{Z}^n)\) is the number of integral lattice points in \(i\)-fold dilation of \(Q\). Then the Hilbert-Mumford criterion implies that this is equivalent to the condition that \(R_{X,A}^i\) is \(H\)-polystable (resp. semistable) for every maximal algebraic torus \(H \leq \text{SL}(N_i, \mathbb{C})\).

### 3. Secondary polytopes and regular triangulations

#### 3.1. A construction of secondary polytopes

In this section we recall the definition of the secondary polytope and its fundamental property. For more details, see [5, 7].

Let \(A = \{ a_0, \ldots, a_N \}\) be a point configuration in \(\mathbb{R}^n\) and \(J = \{ 0, \ldots, N \}\) be an index set of labels. Unless otherwise stated, we always assume that each \(a_i\) is an integer point in \(\mathbb{Z}^n\) and \(A\) affinely generates the lattice \(\mathbb{Z}^n\) over \(\mathbb{Z}\). Fix a subset \(I \subset J\). For a linear functional \(\psi \in (\mathbb{R}^n)^*\), we define the *face of \(I\) in direction \(\psi\)* by (see Figure 1)

\[
\text{Face}_A(I, \psi) := \{ i \in I \mid \psi(a_i) = \max_{j \in I}(\psi(a_j)) \}.
\]

\[\begin{array}{cccc}
3 & 4 & I = \{ 0, 1, 2, 4 \} \\
5 & 4 & \\
0 & 1 & 2 & \psi = (0, -1) \\
\end{array}\]

**Figure 1.** Face of \(I = \{ 0, 1, 2, 4 \}\) in direction \(\psi = (0, -1)\).

Now we give a process of subdivisions of \(A\) which will be a central role in our argument. Let us fix a point configuration \(A = \{ a_0, \ldots, a_N \}\) in \(\mathbb{R}^n\) and let denote the convex hull of \(A\) by \(Q := \text{Conv}(A)\).
Step 1. (Lifting): Pick a height function $\omega : A \to \mathbb{R}$ which can be regarded as a vector $\omega = (\omega_0, \ldots, \omega_N) \in \mathbb{R}^A$ by setting $\omega(a_i) = \omega_i$. Then the lifted point configuration in $\mathbb{R}^{n+1}$ is defined by

$$A^\omega := (a_0 \cdots a_N \omega_0 \cdots \omega_N).$$

Step 2. (Lower Face): Setting $Q^\omega := \text{Conv}(A^\omega) \subset \mathbb{R}^{n+1}$, we consider a lower face of $(Q^\omega, A^\omega)$ which is given by any face $F = \text{Face}_{A^\omega}(J, \psi)$ of $J$ in direction of some functional $\psi$ with positive last coordinates (that is, a face which is “visible from below”).

Step 3. (Projection): The canonical projection $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ bijectively project lower faces of $(Q^\omega, A^\omega)$ to subsets of $(Q, A)$ because lower faces are not vertical. Then the set of collection of projected lower faces

$$C := \{ p(F) \mid F \text{ all the lower faces of } (Q^\omega, A^\omega) \}$$

forms a subdivision of $(Q, A)$.

**Definition 6** (Regular Subdivision, Regular Triangulation). A subdivision $S$ of $(Q, A)$ is called regular if and only if it can be obtained by the process of Step 1 $\sim$ Step 3. Let denote $\mathcal{S}(A, \omega)$ the regular subdivision of $(Q, A)$ produced by $\omega$. Recall that a subdivision $S$ of $(Q, A)$ is called triangulation if each cell $C \in S$ is a simplex with vertices in $A$. Analogously, a regular subdivision $\mathcal{S}(A, \omega)$ is said to be regular triangulation if each cell $C \in \mathcal{S}(A, \omega)$ is a simplex with vertices in $A$.

**Definition 7** (Characteristic Section). Let $T$ be a triangulation of $(Q, A)$ and let $\omega \in \mathbb{R}^A$ be a height function. The characteristic section of $T$ with respect to $\omega$ is a piecewise linear function which is defined by

$$g_{\omega, T} : Q \to \mathbb{R} \quad a_i \mapsto g_{\omega, T}(a_i) = \omega_i$$

and extended affinely on $C := \text{Conv}(C)$ for each cell $C \in T$.

**Remark 3.** In the definition of the characteristic section, we do not require $\omega$ to be the height function that induces the triangulation $T$. We often abuse notation slightly by identifying each cell $C \in T$ a subset of the index set $J$ with its convex hull $C = \text{Conv}(C)$.

**Definition 8** (GKZ-vector). Let $T$ be a triangulation of $(Q, A)$. The Gelfand-Kapranov-Zelevinsky (GKZ) vector of $T$ is

$$\phi_A(T) := \sum_{j \in J} \sum_{C \in T \cap C} n! \text{Vol}(C) e_j \in \mathbb{R}^A,$$

where $e_j$ for $j \in J$ is the standard basis of $\mathbb{R}^A$. In particular, $\phi_A(T)_j = 0$ for $j \in J$ if and only if $a_j \in A$ is not a vertex of any simplex of $T$.!
**Definition 9** (Secondary polytope). The *secondary polytope* \( \Sigma_{\text{sec}}(A) \) is the polytope in \( \mathbb{R}^A \) defined by

\[
\Sigma_{\text{sec}}(A) := \text{Conv} \left\{ \phi_A(T) \mid T \text{ all the triangulations of } (Q, A) \right\}.
\]

The following properties are well known in the geometry of secondary polytopes.

**Theorem 3.1** ([5] pg. 221, Theorem 1.7). For a point configuration \( A = \{a_0, \ldots, a_N\} \) in \( \mathbb{Z}^n \), we have

i) \( \dim \Sigma_{\text{sec}}(A) = N - n \).

ii) There is a one to one correspondence between the regular triangulations of \((Q, A)\) and vertices of \( \Sigma_{\text{sec}}(A) \). In particular, the GKZ-vector \( \phi_A(T) \) for a triangulation \( T \) of \((Q, A)\) will be the vertex of \( \Sigma_{\text{sec}}(A) \) if and only if \( T \) is regular.

**Example 3.1** (The twisted cubic curve) Let \( \mathbb{P}^1 \overset{\nu_3}{\hookrightarrow} \mathbb{P}^3 \) be the third Veronese embedding of \( \mathbb{P}^1 \) in \( \mathbb{P}^3 \). Then the Veronese image \( \nu_3(\mathbb{P}^1) \) is isomorphic to \( \mathbb{P}^1 \) and is called the *twisted cubic curve*. This algebraic curve is a toric variety and the corresponding point configuration is \( A = \{a_0, a_1, a_2, a_3\} = \{0, 1, 2, 3\} \subset \mathbb{Z} \). Thus the associated integral polytope is \( Q = \text{Conv} \{0, 3\} \subset \mathbb{R} \). Let us compute the secondary polytope \( \Sigma_{\text{sec}}(A) \) in \( \mathbb{R}^4 \approx \mathbb{R}^4 \).

From Theorem 3.1, \( \dim(\Sigma_{\text{sec}}(A)) = 3 - 1 = 2 \). We readily get the following list of regular triangulations and GKZ-vectors:

| Triangulations | GKZ-vectors |
|----------------|-------------|
| \( T_0 = \{03\} \)  | \( \phi(T_0) = (3, 0, 0, 3) \) |
| \( T_1 = \{01, 13\} \) | \( \phi(T_1) = (1, 3, 0, 2) \) |
| \( T_2 = \{02, 23\} \) | \( \phi(T_2) = (2, 0, 3, 1) \) |
| \( T_2 = \{01, 12, 23\} \) | \( \phi(T_3) = (1, 2, 2, 1) \) |

Thus, the secondary polytope is

\[
\Sigma_{\text{sec}}(A) = \text{Conv} \left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} \right\}.
\]

Note that all these \( T_i \) are regular triangulations because any one-dimensional triangulations are regular.

**Proposition 3.1.** Let \( T \) be any triangulation of \((Q, A)\) and let \( \omega : A \to \mathbb{R} \) be a height function. For each simplex \( C \in T \), the characteristic section \( g_{\omega,T} \) of \( T \) with respect to \( \omega \) satisfies

\[
\int_C g_{\omega,T}(x)dv = \frac{\text{Vol}(C)}{n + 1} \sum_{j \in C} \omega_j.
\]
Proof. This follows from a straightforward computation. From the definition of $g_{\omega,T}$ (see, Definition 7), we have $g_{\omega,T}(a_j) = \omega_j$. Furthermore, if we represent this piecewise linear function $g_{\omega,T}$ as a row vector, we have

$$\omega = g_{\omega,T} \cdot A.$$ 

Note that the integral of a linear function on a domain is equal to the multiplication of the volume of a domain with the value of a linear function at the centroid. In our case, this implies

$$\int_C g_{\omega,T}(x) dv = \text{Vol}(C) g_{\omega,T}(b_C)$$

where $b_C$ is the centroid of a simplex $C$. Now we use the fact that the centroid of a simplex is given by the average of its vertices:

$$b_C := \int_C x dv = \frac{1}{n+1} \sum_{a \in \mathcal{V}(C)} a.$$ 

Here $\mathcal{V}(C)$ denotes the set of vertices of $C$. By the linearity of $g_{\omega,T}$, (3.1) becomes (see [7], pg. 219)

$$(3.2) \quad \int_C g_{\omega,T}(x) dv = \frac{\text{Vol}(C)}{n+1} \sum_{j \in C} \omega_j.$$ 

The following proposition is a consequence of Proposition 3.1.

**Proposition 3.2** ([5] pg. 221, Lemma 1.8, [7] Theorem 5.1.11). Let $\omega \in \mathbb{R}^A$ be a height function and let $T$ be any triangulation of $(Q,A)$. For the characteristic section $g_{\omega,T}$ and the GKZ-vector $\phi_A(T)$, we have

$$\langle \omega, \phi_A(T) \rangle = (n+1)! \int_Q g_{\omega,T}(x) dv \\ = n! \sum_{C \in T} \text{Vol}(C) \sum_{j \in C} \omega_j.$$ 

In our case, following result is the main motivation to discuss about secondary polytopes.

**Theorem 3.2** (Kapranov-Sturmfels-Zelevinsky [6]). Let $X_A \longrightarrow \mathbb{P}^N$ be the n-dimensional projective toric variety associated with a point configuration $A \subset \mathbb{Z}^n$. Let $R_X$ be the Chow form of $X_A$. Then the weight polytope $N_{(\mathbb{C}^*)^A}(R_X)$ of $R_X$ with respect to the torus action $(\mathbb{C}^*)^A$ coincides with the secondary polytope $\Sigma_{sec}(A)$.

3.2. **Facets of the secondary polytope.** In this subsection we describe the structure of the faces of secondary polytopes. A combinatorial framework of $(Q,A)$ is called *coarsest subdivisions* which is corresponding to the facets (i.e., codimension 1 faces) of $\Sigma_{sec}(A)$. We shall give the definition of this subdivision and the defining equation of the facet of $\Sigma_{sec}(A)$ corresponding to a certain coarsest subdivision. To begin with, we shall define a *refinement* of a polyhedral subdivision.
**Definition 10** (Refinement). Let $S$ and $S'$ be two subdivisions of $(Q, A)$. Then $S$ is said to be a refinement of $S'$ if for any cell $C \in S$, there is a cell $C' \in S'$ with $C \subseteq C'$ and is denoted by $S \preceq S'$.

The following Theorem due to Gelfand, Kapranov and Zelevinsky is crucial, in particular it gives a combinatorial description to deal with the faces of secondary polytopes. (cf. Theorem 3.1).

**Theorem 3.3** ([5] pg. 228, Theorem 2.4, [7] Theorem 5.1.9). Let $S$ be any regular subdivision of $(Q, A)$ and let $F(S)$ denote the convex hull in $\mathbb{R}^A$ of the GKZ-vectors for all triangulations $T$ which is obtained by refining $S$:  

$$F(S) := \text{Conv} \{ \phi_A(T) \mid T \text{ all triangulations refining } S \}.$$  

Then two faces of $\Sigma_{\text{sec}}(A)$ satisfy $F(S) \subset F(S')$ if and only if $S \preceq S'$.

From Theorem 3.3, the facets of the secondary polytope $\Sigma_{\text{sec}}(A)$ correspond to regular subdivisions of $(Q, A)$ which only refine the trivial subdivision and no other. We call these subdivisions the coarsest subdivisions. Note that the trivial subdivision always exists and is given by the zero height function $\omega = (0, \ldots, 0)$. The following proposition states the explicit linear equality defining the facet of $\Sigma_{\text{sec}}(A)$ corresponding to the coarsest subdivision of $(Q, A)$.

**Proposition 3.3** ([7] Exercise 5.11). Let $\omega \in \mathbb{R}^A$ be a height function which produces the coarsest subdivision $\mathcal{S}(A, \omega)$ of $(Q, A)$. The defining linear equality of the facet of $\Sigma_{\text{sec}}(A)$ corresponding to $\mathcal{S}(A, \omega)$ is given by

$$\sum_{j \in J} \omega(a_j) \varphi_j = n! \sum_{C \in T} \text{Vol}(C) \sum_{j \in C} \omega_j \quad \text{for} \quad \varphi = (\varphi_0, \ldots, \varphi_N) \in \mathbb{R}^A,$$

where $T \preceq \mathcal{S}(A, \omega)$ is a certain triangulation which is obtained by refining $\mathcal{S}(A, \omega)$.

**Example 3.2** (Example 3.1 continued) Again we consider the twisted cubic. To begin with, we recall the following list of regular triangulations:

| Triangulations | Height functions |
|----------------|------------------|
| $T_0 = \{03\}$ | $\omega_0 = (1, 2, 2, 1)$ |
| $T_1 = \{01, 13\}$ | $\omega_1 = (2, 1, 2, 2)$ |
| $T_2 = \{02, 23\}$ | $\omega_2 = (2, 2, 1, 2)$ |
| $T_2 = \{01, 12, 23\}$ | $\omega_3 = (2, 1, 1, 2)$ |

Next we consider the following list of regular subdivisions:
Here the indices denote which triangulations refine them. The following Picture 1 shows the Hasse diagram of subdivisions of $Q$.

Let us fix a height function $\omega = \omega_{23} = (0, 0, 0, 1)$. Then
\[
\sum_{j \in J} \omega(a_j) \varphi_j = 0 \cdot \varphi_0 + 0 \cdot \varphi_1 + 0 \cdot \varphi_2 + 1 \cdot \varphi_3
\]

\[= \varphi_3.
\]

The Hasse-diagram (Picture 1) tells us that a triangulation $T \preceq \mathcal{I}(A, \omega) = \mathcal{T}_{23}$ can be taken as $T_2$ or $T_3$. In both cases, we conclude that
\[
\sum_{C \in T} \text{Vol}(C) g_{\omega,T}(C) = 1.
\]

It is well known that the affine hull of $\Sigma \text{sec}(A)$ is given by equalities
\[
\sum_{j \in J} \varphi_j = (n + 1)! \text{Vol}(Q), \quad \sum_{j \in J} \varphi_j a_j = (n + 1)! \int_Q x dv.
\]

See [5] pg. 222, Proposition 1.11. In our case, this implies
\[
\varphi_0 + \varphi_1 + \varphi_2 + \varphi_3 = 6, \quad \varphi_1 + 2\varphi_2 + 3\varphi_3 = 9.
\]

Consequently, the affine hull of the facet $F$ of $\Sigma \text{sec}(A)$ corresponding to the coarsest subdivision $\mathcal{T}_{23}$ is given by
\[
\text{Aff}_R(F) = \{(\varphi_0, \varphi_1, \varphi_2, \varphi_3) \in \mathbb{R}^4 \mid \varphi_0 + \varphi_1 + \varphi_2 = 5, \varphi_1 + 2\varphi_2 = 6, \varphi_3 = 1\}
\]
\[= \text{Aff}_R(\text{Conv} \{ \phi_A(T_2), \phi_A(T_3) \}).
\]

Since these and only these regular triangulations $T_2, T_3$ refine $\mathcal{T}_{23}$, two vertices $\phi_A(T_2)$ and $\phi_A(T_3)$ are extremal in the facet $F$. This result is consistent with Theorem 3.3.
4. Proof of the main theorem

Proof. Let \( A = \{a_0, \ldots, a_N\} \) be a point configuration in \( \mathbb{Z}^n \) and let \( J \) be an index set of labels. Let \( X_A \to \mathbb{P}^N \) be the associated smooth, linearly normal projective toric variety of degree \( d = \deg X \geq 2 \). We denote the Chow point of \( X_A \) by \( R_{X_A} \). Considering the complex torus \((\mathbb{C}^\times)^A \simeq (\mathbb{C}^\times)^{N+1}\), we define the subtorus of \((\mathbb{C}^\times)^A \) by

\[
H = \{ (t_0, \ldots, t_N) \in (\mathbb{C}^\times)^A \mid \prod_{j \in J} t_j = 1 \} \simeq (\mathbb{C}^\times)^N.
\]

Suppose that \( R_{X_A} \) is \( H \)-semistable but not \( H \)-polystable. Setting \( G = (\mathbb{C}^\times)^A \), we consider the projection

\[
\pi_H : \chi(G) \otimes \mathbb{R} \simeq \mathbb{R}^{N+1} \to \chi(H) \otimes \mathbb{R} \simeq \mathbb{R}^N,
\]

\[
(\varphi_0, \ldots, \varphi_N) \mapsto (\varphi_0 - \varphi_N, \ldots, \varphi_{N-1} - \varphi_N).
\]

Then observe that

\[
\pi_H(\Sigma_{sec}(A)) = \mathcal{N}_H(R_{X_A}) \quad \text{and} \quad \pi_H^{-1}(\partial\mathcal{N}_H(R_{X_A})) \subset \partial\Sigma_{sec}(A).
\]

Recall that \( \partial P \) denotes the boundary of an integral lattice polytope \( P \). Thus, the numerical criterion (Proposition 2.2) implies that there is an element \( \varphi = (\varphi_0, \ldots, \varphi_N) \) in \( \partial\Sigma_{sec}(A) \) such that \( \pi_H(\varphi) = 0 \). In particular, there exists \( t \in \mathbb{R}^\times \) such that

\[
(t, \ldots, t) \in \partial\Sigma_{sec}(A)
\]

from (4.1). Hence we can take the facet \( F \) of \( \Sigma_{sec}(A) \) which contains this point. As discussed in Section 3 there is a height function \( \omega \in \mathbb{R}^A \) which produces the coarsest subdivision \( \mathcal{S}(A, \omega) \) corresponding to this facet \( F \). Then Proposition 3.3 implies that there exists \( t \in \mathbb{R}^\times \) such that

\[
t \sum_{j \in J} \omega_j = n! \sum_{C \in \mathcal{S}} \text{Vol}(C) \sum_{j \in C} \omega_j
\]
for a certain triangulation $\mathcal{T} \leq \mathcal{S}(A, \omega)$ of $(Q, A)$. Also, Proposition \ref{prop:3.2} gives

$$\langle \omega, \phi_A(T) \rangle = n! \sum_{C \in \mathcal{T}} \text{Vol}(C) \sum_{j \in C} \omega_j$$

for any triangulation $T$ of $(Q, A)$. Taking $T = \mathcal{T} \leq \mathcal{S}(A, \omega)$, we have

$$t \sum_{j \in J} \omega_j = \langle \omega, \phi_A(T) \rangle$$

Therefore, we conclude that

(4.2) \hspace{1cm} \phi_A(T) = (t, \ldots, t) \in \mathbb{R}^A.

For any GKZ-vector, (4.2) holds if and only if $Q = \text{Conv}(A)$ is the standard $n$-dimensional simplex $\Delta_n = \text{Conv} \{ e_i \mid 1 \leq i \leq n \}$ and $T$ is the trivial triangulation of $(\Delta_n, A)$. Then the associated toric variety is exactly $\mathbb{P}^n$ which is not linearly normal. This contradiction shows the assertion. \hfill \square

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