Two-body scattering in a trap and a special periodic phenomenon sensitive to the interaction

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The determination of interactions among microscopic particles is an important topic in physics. Historically, the measurement of the cross sections of 2-body scattering provides an important source of information. For neutral atoms and molecules the determination is more difficult because the interaction is in general weak and the initial momentum of the incident particle is difficult to be precisely controlled. However, if the scattering occurs in a trap, new phenomena previously unknown might emerge. Due to the recent progress in the techniques of trapping atoms (molecules) by using optical traps [1–3], trapped scattering might be eventually experimentally realized. In a previous paper a model was proposed to study the trapped 2-body scattering theoretically [4]. Instead of a single collision, numerous repeated collisions have been found. Thereby the effect of interaction can be accumulated and enlarged, and might be eventually detected. This favors the determination of interactions, the observation of the special periodic phenomenon might be a valid source of information on weak interactions among neutral particles.

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Each trap contains a particle in the lowest harmonic oscillator (h.o.) state. The two particles are assumed to be identical bosons (the generalization to fermions is straightforward), the interaction is assumed to be spin-independent, and \( \omega_p \) is large enough so that the particles are well localized initially and the overlap of their wave functions is negligible. Suddenly the two deep traps are cancelled. Instead, a broader new trap located at the origin \( U_{\text{evol}}(r) = \frac{1}{2} M \omega^2 r^2 \) is created, \( \omega < \omega_p \). Since the initial state is not an eigenstate of the new Hamiltonian, the system begins to evolve. The evolution is affected not only by \( U_{\text{evol}}(r) \) but also by the interaction \( V(|r_1 - r_2|) \). In what follows the details of the evolution is studied, two-body collisions occurring repeatedly are found, and the effect of interaction is demonstrated.

Let \( \hbar \omega \) and \( \sqrt{\hbar/M \omega} \) be used as units of energy and length. The normalized initial state is

\[
\Psi_I = \frac{1}{\sqrt{2}}(1 + P_{1,2}) \left( \frac{n}{\pi} \right)^{\frac{3}{4}} \exp\left[-\frac{n}{2}(|r_1 - a|^2 + |r_2 - b|^2)\right]
\]

where \( P_{1,2} \) implies an interchange of 1 and 2, and \( \eta = \omega_p/\omega \). We consider the case that \( a \) is lying along the \( X \)-axis, while \( b \) along the negative \( Z \)-axis. When \( R = (r_1 + r_2)/2 \) and \( r = r_2 - r_1 \) for the c.m. and relative motions, respectively, are introduced, and the h.o. states of the new trap are selected as base functions, the initial state can be expanded as

\[
\Psi_I = \sum_{NLnlJM} B_{NLnlJM} \varphi_{NL}^{(2)}(R) \varphi_{nl}^{(1/2)}(r) \]

where \( \varphi_{nl}^{(\mu)}(r) \) is a normalized eigenstates of the Hamiltonian \( -\frac{\hbar^2}{2M} \Delta_r + \frac{\eta}{2} \mu r^2 \) with the eigenenergy \( 2n + l + 3/2 \). \( L \) and \( l \) are coupled to the total orbital angular momentum,

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The analytical form of the coefficients can be found in [5]. From Eqs. (6) and (8), making use of the orthonormality of the base functions, the integration in Eq. (7) is easy to carry out, and it is straightforward to obtain the analytical expression of \( \rho(\mathbf{r}_1, \tau) \). Incidentally, since the wave function is symmetrized, the behaviors of the two particles are exactly the same. The observation of only one particle is sufficient.

To obtain numerical results as examples, it is first assumed that \( a = 2, b = 0, \eta = 2, \) and the interaction contains a stronger repulsive core and a weaker attractive tail as

\[
V(r) = \begin{cases} 
V_0, & \text{if } r < 0.2 \\
-C_6/r^6, & \text{else}
\end{cases}
\]

where \( V_0 \) and \( C_6 \) are positive numbers, and \( V_0 \gg C_6 \). We define \( K = 2(N+n)+L+l \) to control the dimension of the base. Mostly, the base functions with \( K \leq K_{\text{max}} = 20 \) are adopted in the following calculation.

![FIG. 1: (Color online) \( \rho(\mathbf{r}_1, \tau) \) plotted on the \( X - Z \) plane. \( \tau = 0, \pi/6, \pi/3, \pi/2, 5\pi/6 \) and \( \pi \), respectively, for (a) to (f). The parameters are \( \eta = 2, a = 2, b = 0, V_0 = 10, r_0 = 0.2 \) and \( c_6 = -0.3r_0^6 \). The units of energy and length in this paper are \( \hbar \omega \) and \( \sqrt{\hbar/M\omega} \), where \( \omega \) has not yet been specified. Every maximum in the panels is marked by a cross. The values associated with the inmost contours of (a) to (f) are 0.224, 0.098, 0.073, 0.063, 0.098 and 0.224. Thus the peaks in (a) and (f) are much higher. Therefore, the particles are better localized when they separate from each other. The values of the outmost contours are from 0.01 to 0.03.](image_url)
behavior originating from the harmonic trap. If the interaction is neglected, the recovery would be exact. However, due to the interaction, $\rho(r_1, \tau)$ is not exactly equal to $\rho(r_1, \tau + 2\pi)$. With the above parameters, the deviation is very small. Say, when $r_1 = a$, we have $\rho(a, 0) = 0.2553$ and $\rho(a, 2\pi) = 0.2544$. Obviously, in the period $(0, 2\pi)$ the system undergoes a pair of collisions. When the time goes on, a series of head-on collisions occur repeatedly in the trap. Although the effect of interaction on a round goes on, a series of head-on collisions occur repeatedly in the trap. If the interaction is extremely weak, the effect of many rounds might be accumulated and therefore might become strong. This is shown below.

![Graph](image)

**FIG. 2:** (Color online) $\rho(a, \tau)$ plotted against $\tau$, where $a$ denotes the initial position of the incident particle and $\tau$ is given only in discrete values $2k\pi$, where $k$ is an integer from 0 are 60. Three choices of $V_0$, namely, 20, 10, and 2 are adopted, and the associated $\rho$ are marked with circles, diamonds, and triangles, respectively (however the values of $\rho$ associated with an odd $k$ have been neglected just for simplicity). The other parameters are the same as in Fig. 1.

In Fig. 2 $\rho(a, \tau)$ is given at $\tau = 2k\pi$, where $k$ is an integer. It implies that $\rho$ is observed at the initial position of the incident particle repeatedly. If the interaction is removed, all the symbols in the figure would lie along a horizontal line (implying an exact periodicity with a period $2\pi$). However, the interaction causes a deviation. The deviation would become larger if the interaction is stronger (the black circles to be compared with the triangles) and/or if $\tau$ is larger. It is found that, when the time goes on, $\rho(a, 2k\pi)$ against increasing $k$ will first arrive at a minimum, then arrive at the second maximum which is a little lower than the first maximum at $\tau = 0$, then again a minimum, and afterward arrive at the third maximum with a height close to the first maximum. This behavior will repeat again and again. E.g., for $V_0 = 20$ (black circles), the first, second and third maxima appear at $\tau = 0$, $54\pi$, and $110\pi$, respectively. Whereas for $V_0 = 10$ (red diamonds), they appear at $0, 104\pi$ and $210\pi$.

On the other hand, making use of the expansion Eq. (6), we calculate the overlap $|\langle \Psi(0) | \Psi(2k\pi) \rangle|$ which varies with $k$. When $k$ leads to a minimum (maximum) of $\rho(a, 2k\pi)$, the overlap is small (close to one). For examples, with the parameters for Fig. 1 when $\tau = 48\pi$, 104$\pi$ and 210$\pi$ associated with the first minimum, the second maximum, and the third maximum, respectively, we have $|\langle \Psi(0) | \Psi(\tau) \rangle|$ = 0.103, 0.918 and 0.994. It is further noted that the imaginary part of $\langle \Psi(0) | \Psi(210\pi) \rangle$ is very small. Thus, $\Psi(210\pi)$ is extremely close to the initial state. Therefore, from the time-dependent Schrödinger equation, we know that what happens during the interval $(0, 210\pi)$ will nearly exactly repeat again in the next interval $(210\pi, 420\pi)$, and so on. It implies the existence of another nearly periodic behavior. Thus, there are two distinct periodic behaviors. One has a period $2\pi$, and the other one has a much longer period (say, for the above case, the period is $210\pi$).

![Graph](image)

**FIG. 3:** (Color online) $\rho(r_1, \tau)$ plotted along the X-axis. The curves "1" to "5" in 3a have $\tau$ from 0 to $\pi$ with a step $\pi/4$. Those of 3b have $\tau$ from $48\pi$ to $49\pi$ with the same step. The parameters are the same as in Fig. 1.

![Graph](image)

**FIG. 4:** (Color online) The same as Fig. 3, but the domain of $\tau$ is $(104\pi, 105\pi)$ in 4a and $(210\pi, 211\pi)$ in 4b.
The periodic behaviors can be shown in more detail by observing directly the densities. Let $r_i = x_i\mathbf{i}$, where $\mathbf{i}$ is a unit vector along the $X$–axis. The distribution of $\rho(x_i, \tau)$ along the $X$–axis is plotted in Figs. 3 and 4, where $\tau$ is given at a number of values. Fig. 3 describes the evolution in the interval $(0, \pi)$, where the curve “1” is for the initial state. From “1” to “5” $\tau$ goes from 0 to $\pi$ with a step $\pi/4$. One can see how the two particles undergo a round of collision. In fact, Fig. 4 and Fig. 5 describe the same thing except that $\rho$ is plotted on the $X–Z$ plane in the former but only along the $X$–axis in the latter. When $\tau$ goes from $\pi$ to $2\pi$, the process occurring in $(0, \pi)$ will repeat again but in reverse direction. Therefore the cycle with the $2\pi$ period is completed. Fig. 3 describes the evolution in the interval $(48\pi, 49\pi)$ associated with the first minimum of the diamonds in Fig. 2. The peaks in 3a are much lower than those of 3b, and these peaks are located at different places. Therefore the system behaves differently in the two intervals, and the previous clear picture of a head-on collision becomes ambiguous. Fig. 4a is associated with the second maximum. The collision can be roughly seen but is not as clear as in Fig. 3a. Fig. 4b is associated with the third maximum, and is nearly identical to 3a. Therefore the cycle with the longer period is completed.

Fig. 5: (Color online) Similar to Fig. 2 but $V_0$ is fixed at 10 and $c_0$ has three choices: $-0.3r_0^6$, $-1.5r_0^6$ and $-3r_0^6$. The values of $\rho(a, \tau)$ are, respectively, marked by black circles, red diamonds, and triangles. The other parameters are the same as in Fig. 2. This figure has a longer range than that in Fig. 4, and only the values of $\rho$ with $\tau = 8k\pi$ are shown.

In Fig. 5 the strength of the attractive tail has been given at three values. In each case a slightly lower peak followed by a higher peak (the third maximum) appears again. With the three choices of $c_0$, the third maximum appears at $\tau = 210\pi$, $236\pi$, and $282\pi$, respectively. At these three instants, $|\langle \Psi(0)|\Psi(\tau) \rangle|$ are all equal to 0.994 and the associated imaginary parts are very small. It implies a nearly exact recovery. Thus the nearly periodic behavior with the much longer period appears again. When $b \neq 0$ (i.e., the target particle is not at the center initially), the above qualitative features remain. In particular, for a specific interaction, the longer period is not changed with $b$. Thus we conclude that the periodic phenomenon is common to trapping 2-body scatterings. It is emphasized that this phenomenon is sensitive to the interactions. A stronger repulsive (attractive) force would lead to a shorter (longer) long-period.

The accuracy of the above numerical results depends on $K_{\text{max}}$. As an example selected values of $\rho(a, \tau)$ are listed in Tab. I to show the dependence.

| $K_{\text{max}}$ | 12   | 16   | 20   |
|-----------------|------|------|------|
| $\rho(a, 0)$    | 0.253| 0.255| 0.255|
| $\rho(a, 50\pi)$| 0.067| 0.068| 0.068|
| $\rho(a, 100\pi)$| 0.229| 0.230| 0.229|

The convergency appears to be satisfying. Thus the numerical results obtained by using $K_{\text{max}} = 20$ are accurate enough in qualitative sense.

In this paper the traditional 2-body scattering is considered under a new environment, namely, in a trap. Due to the trap, the two particles collide with each other repeatedly. Therefore, even the interaction is weak; the effect of interaction can be accumulated via the repeated collisions. Besides, comparing with the case of charged particles, the initial scattering states of neutral particles are more difficult to control. This disadvantage can be overcome by using optical traps. Furthermore, two periodic phenomena are found. They are essentially caused by the trap and by the interaction, respectively. The period of the latter is much longer and is sensitive to the interaction. Instead of measuring the differential cross section as usually does, the observation of the longer period and the details of the time-dependent density might be a valid source of information on weak interactions among neutral particles.

The above approach can be easily generalized to the cases with various interactions, and/or to the case of Fermion systems. It is reasonable to expect that the above trapped scattering could be experimentally realized via the progress of techniques in trapping neutral particles by optical traps.

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