Abstract

The objective of this paper is to study the characteristics (geometric and otherwise) of very large attribute based undirected networks. Real-world networks are often very large and fast evolving. Their analysis and understanding present a great challenge. An Attribute based network is a graph in which the edges depend on certain properties of the vertices on which they are incident. In context of a social network, the existence of links between two individuals may depend on certain attributes of the two of them. We use the Lovasz type sampling strategy of observing a certain random process on a graph "locally", i.e., in the neighborhood of a node, and deriving information about "global" properties of the graph. The corresponding adjacency matrix is our primary object of interest. We study the efficiency of recently proposed sampling strategies, modified to our set up, to estimate the degree distribution, centrality measures, planarity etc. The limiting distributions are derived using recently developed probabilistic techniques for random matrices and hence we devise relevant test statistics and confidence intervals for different parameters / hypotheses of interest. We hope that our work will be useful for social and computer scientists for designing sampling strategies and computational algorithms appropriate to their respective domains of inquiry. Extensive simulations studies are done to empirically verify the probabilistic statements made in the paper.

1 Introduction

1.1 Need for Sampling and Common Strategies

Real-world networks are often very large and fast evolving. Their analysis and understanding present a great challenge. In the past few years, a number of different techniques have been proposed for sampling large networks to allow for their faster and more efficient analysis. Several studies on network sampling analyze the match between the original networks and their sampled variants [1, 2, 3], as well as comparing the performance of different sampling techniques [4, 5, 6].

Sampling techniques can be roughly divided into two categories: random selection and network exploration techniques. In the first category, nodes or links are included in the sample uniformly at random or proportional to some particular characteristic like degree. In the second category, the sample is constructed by retrieving a neighborhood of a randomly selected seed node using different strategies like breadth-first search, random walk and forest-fire. On these basis, the following algorithms have been proposed in the literature.

- random node selection [4] (RNS), where the sample consists of nodes selected uniformly at random and all their mutual links
- random node selection by degree [4] (RND), the nodes are selected randomly with probability proportional to their degrees and all their mutual links are included in the sample
- random link selection [4] (RLS), where the sample consists of links selected uniformly at random
- random link selection with subgraph induction [7] (RLI), the sample consists of links selected uniformly at random and any additional links between their endpoints
- random walk sampling [4] (RWS), where the random walk is simulated on the network, starting at a randomly selected seed node
- forest-fire sampling [4] (FFS). Here, a broad neighborhood of a randomly selected seed node is retrieved from partial breadth-first search
random walk sampling with subgraph induction (R WI) and
forest-fire sampling with subgraph induction (FFI)

Recall the Lovasz sampling strategy of observing a certain random process on a graph "locally", i.e., in the neighborhood of a node, and deriving information about "global" properties of the graph. For example, what do you know about a graph based on observing the returns of a random walk to a given node? Almost all sampling strategies uses this philosophy or some close variant. We also aim to use this as an ingredient in our sampling strategy and derive a test for such returns.

1.2 Why Attribute based Network?
An Attribute based network is a graph in which the edges depend on certain properties of the vertices on which they are incident. In context of a social network, the existence of links between two individuals may depend on certain attributes of the two of them, for example their geographic location or socioeconomic status. We work with the underlying assumption that similar people connect to each other with higher probability. In the context of a social or a neural network, the connection between individual vertices depend on certain intrinsic qualities of the vertices themselves. It makes sense to consider the connection probabilities as a function of the vertex attributes. In earlier work (Sarkar, Ray and Mukherjee, 2015) we have shown that in the context of predictive modeling, attribute based networks are indeed worthwhile to study.

1.3 Plan of the Paper
The objective of this paper is to study the characteristics (geometric and otherwise) of very large attribute based undirected networks. The corresponding adjacency matrix is our primary object of interest. We study the efficiency of recently proposed sampling strategies, modified to our set up, to estimate the degree distribution, centrality measures, planarity etc. The limiting distributions are derived using recently developed probabilistic techniques for random matrices and hence we devise relevant test statistics and confidence intervals for different parameters / hypotheses of interest. We hope that our work will be useful for social and computer scientists for designing sampling strategies and computational algorithms appropriate to their respective domains of inquiry. Extensive simulations studies are done to empirically verify the probabilistic statements made in the paper.

2 Preliminaries
Let the network be represented by a simple undirected graph $G = (V; E)$, where $V$ denotes the set of nodes ($n = |V|$) and $E$ is the set of links ($m = |E|$). The goal of network sampling is to create a sampled network $G' = (V'; E')$, where $V' \subset V$, $E' \subset E$ and $n' = |V'| << n$, $m' = |E'| << m$. The sample $G'$ is obtained in two steps. In the first step, nodes or links are sampled using a particular strategy like random selection and network exploration sampling. In the second step, the sampled nodes and links are retrieved from the original network.

2.1 Notions of Centrality
A network can be characterized by various notions of centrality, whose relevance and utility are context-specific. A complex network with a heterogeneous topology might not have the same optimality properties for a single measure of centrality throughout the graph.

However, for a sampling based approach without any prior idea of the graph type, it may be difficult to know which centrality measure is best suited for the study of the graph. If we operate under the simplified assumptions about the attributes-based network then our graph structure is simplified. Particularly, if the attribute random variable $\{X_i\}$ take values in a finite set, then the set of possible connecting probabilities is also finite. Then we can break a graph into different classes which are expected to have similar behavior.

Here we intend to develop a sampling analogue for finding the Eigenvector centrality, which is the solution of the Eigenvector Equation $Ax = \lambda x$. According to the Perron Frobenius theorem due to strict positivity of the eigen values we only require the largest eigenvalue.
A possible generalization of Eigenvector centrality as well as Degree Centrality is the Katz centrality. It measures the number of all nodes that can be connected via a path to the vertex in question, while the contributions to distant nodes are devalued. It is mathematically written as \( x_i = \sum_k \sum_j \alpha^k A^{k}_{ji} \).

Katz centrality can be viewed as a variant of eigenvector centrality. Another form of Katz centrality is \( x_i = \alpha \sum_{j=1}^{N} a_{ij} (x_j + 1) \). Compared to the expression of eigenvector centrality, \( x_j \) is replaced by \( x_j + 1 \).

It is shown that the principal eigenvector (associated with the largest eigenvalue of \( A \), the adjacency matrix) is the limit of Katz centrality as \( \alpha \) approaches \( 1/\lambda \) from below.

### 2.2 Assumptions on the Network

**Basic Assumptions:** We denote the variable \( X_i \) as an attribute of the class \( i \), where \( i \) is assumed to take only finitely many values. In a population, there can be infinitely many people with same attribute. Let us consider all of them members of the class \( i \). Call it \( c_i \).

- Looking at degree distribution is not very meaningful as even if we know that the degrees are distributed by power law or Normally, we still don’t know what the degree should be for an individual node.

- The degree needs to be a specific property of a node for us to meaningfully select a node. In context of social networks, it makes sense to consider vertices appended with attributes. An edge or connection can be considered to be dependent on the attributes of the involved nodes.

- We consider the accessory variable \( X_i \) appended to each vertex \( i \).

- We need to look at \( p_{ij} = P(\delta_{ij} = 1|X_i, X_j) \). If Erdos Renyi Graph, then this is the unconditional probability same for all \( i,j \).

- We make some assumptions on \( p_{ij} \). Even if \( i \) and \( j \) belong to the same class and hence share the same attributes, \( p_{ij} \neq 1 \). \( p_{ij} \) is assumed to be bounded away from 1 and 0.

- Fix an \( i \). Consider \( \arg\max \sum_j P(\delta_{ij} = 1|X_i, X_j) \). This can be approximated by \( \arg\max \mathbb{E}(d|X_i) \) where \( d \) is the degree

- The \( \{X_n\} \) sequence, if stochastic, is assumed to form a Markov Random Field. we have a sort of dependence structure within a neighborhood; distant nodes can be assumed to be more or less independent.

In particular, if \( \{X_n\} \) is finite, then \( p_{ij} \) also takes finitely many values.

### 3 Probabilistic Formulation

Instead of the Adjacency Matrix, we can consider the matrix \( P = (p_{ij}) \), the matrix of the probabilities. If we consider a random motion on the graph, we consider \( p_{ij} \) to be a transition probability on the graph. A possible notion of centrality in this context is whether the vertex is recurrent or transient. A recurrent vertex indicates that there are multiple paths leading back to the graph. This is somewhat analogous to Betweenness centrality. We need to however note that in a large graph modeled on a social network which would be mostly sparse with intermediate densely connected cliques, the actual betweenness for all but few vertices would be vanishingly small. On a global scale these vertices may not be important; but their local influence cannot be dismissed.

We look at an irreducible aperiodic subset of the graph. If the motion is considered to be Markov, then noting that transience and recurrence are solidarity properties, we attempt to verify that using our model.

If \( d \) is a metric defined on the \( \sigma - Field \) generated by the random variables \( \{X_n\} \), then consider \( p_{ij} = f(d(X_i, X_j)) \), where \( f \) is a decreasing function of \( d \) and bounded in \([0, 1]\). An easy example is \( d'(x, y) = 1 - \min(1, d(x, y)) \).

Again, if \( \{X_n\} \) is finite (or countable), \( d \) only takes finitely (countably) many values and consequently the set of values of \( p \) is also finite (countable.)
If the \( \{X_n\} \) is known, then \( P \) is also completely known, as is \( P^k \). In principal, we can also calculate if \( \sum f_{ij}^n < 1 \), where \( f_{ij}^{(n)} = \Pr(T_i = n) \). If this holds, then the vertex is transient, else recurrent.

### 3.1 Degree Distribution

We are estimating \( p_{ij} = P(\delta_{ij} = 1|X_i, X_j) \). Let \( \mathbb{P} = (p_{ij}) \) and \( \mathbb{E} = (\delta_{ij}) \), symmetric with \( \delta_{ii} = 0 \), \( \forall i \). So \( \delta_{ij} \sim \text{Bernoulli}(p_{ij}) \). Symbolically \( \mathbb{E} \sim \text{Bernoulli}(\mathbb{P}) \).

Let degree, \( d_i = \sum_j \delta_{ij} \). What is the distribution of \( \delta_{ij}|d_i \), should be "hypergeometric" type?

So can we use \textit{iterated expectation} in the following way?

First condition on row total \( d_i \) to use the "hypergeometric type" calculation for \( (\delta_{ij}|X_i, X_j) \).

Then take expectation on \( d_i \sim F(.)|X_i \) following from the ERG model. If we can show that the distribution of \( d_i \) is of "Binomial type", then for fixed \(|V| = n\) we can do the calculation and then take limit \( n \to \infty \).

Assuming \( \{X_i\} \) to be non-stochastic, if degree of the \( i^{th} \) vertex is \( d_i \), we have:

\[
d_i = \sum_{j=1 \atop j \neq i}^n \delta_{ij}, \text{ where } \delta_{ij} \text{ is the indicator variable which is } 1 \text{ if there is a connection between the } i^{th} \text{ and the } j^{th} \text{vertices.}
\]

If \( p_{ij} \) is the connection probability of the \( i \) and the \( j^{th} \) vertex, then \( p_{ij} = f(x_i, x_j) \) is completely known.

Consider the degree proportion:

\[
\frac{d_i}{n-1} = \frac{1}{n-1} \sum_{j=1 \atop j \neq i}^n \delta_{ij}.
\]

We also assume that the connections depend entirely on the two involved vertices and not on other factors. So, \( \delta'_{ij} \) s are independent.

The distribution of \( d_i \) can be explicitly obtained by the results from Wang’s Paper\(^2\)

**Proposition:** By an easy application of the Lyapunoff condition regarding to the Central limit theorem for independent but not identically distributed random variables, we have the large sample distribution of \( \frac{d_i}{\sqrt{n}} \) as

\[
\sqrt{n} \left( \frac{\sum_{j=1}^n (\delta_{ij} - p_{ij})}{\sqrt{p_{ij}(1 - p_{ij})/n}} \right) \xrightarrow{LAW} N(0, 1), \text{ as long as } p_{ij} \text{ is bounded away from } 0 \text{ and } 1.
\]

\[
\sqrt{n} \left( \frac{\sum_{j=1}^n (\delta_{ij} - p_{ij})}{\sqrt{p_{ij}(1 - p_{ij})/n}} \right) = \frac{d_i}{\sqrt{n}}, \text{ which is a scaled degree-density.}
\]

Then

Also note, under this structure

\[
\text{Cov}(d_i, d_k) = \text{Cov}(\sum_{j=1 \atop j \neq i}^n \delta_{ij}, \sum_{j=1, j \neq k}^n \delta_{kj}, \ldots) = \sum_{j \neq i} \text{Cov}(\delta_{ij}, \delta_{ik})
\]

Under the condition of independence, we have

\[
\text{Cov}(\delta_{ik}, \delta_{ik}) = \text{Var}(\delta_{ik}) = p_{ik}(1 - p_{ik})
\]

Consider the vector \( d = (d_1, d_2, \ldots, d_n) \)

Then the \( \text{Cov}(\frac{d_i}{\sqrt{n}}, \frac{d_k}{\sqrt{n}}) = \frac{p_{ik}(1 - p_{ik})}{n} \to 0 \text{ asymptotically.} \)

\[
\text{Var}(d_i) = \text{Var}(\sum_{j=1 \atop j \neq i}^n \delta_{ij}) = \sum_{j \neq i} \text{Var}(\delta_{ij}) \underbrace{\text{under independence}}_{\text{as long as } p_{ij} \text{ is bounded away from } 0 \text{ and } 1}.
\]

\[
\text{Var}(p_{ij}) = (1 - p_{ij}) = \sum_{j \neq i} \text{Var}(\delta_{ij}) \underbrace{\text{under independence}}_{\text{as long as } p_{ij} \text{ is bounded away from } 0 \text{ and } 1}.
\]

Then Centrality can be tackled as an eigenvalue \( \mathbb{E} = Q \Lambda Q' \). Limiting arguments are not easily available but see the references below.

The precision for estimation of \( d_i \) is \( \frac{1}{\sum p_{ij}(1 - p_{ij})} \) \{I cannot use this to formulate the sampling argument\}

### 3.2 Planarity etc.

One important question: \( 1 - P(\text{planarity}) \leq P(K_5) + P(K_{3,3}) \), need to compute these. So if \( P(K_5) \leq \alpha_5 \) and \( P(K_{3,3}) \leq \alpha_{3,3} \) s.t. \( \alpha_5 + \alpha_{3,3} \leq \alpha \) then we have one definition of "planar with \((1 - \alpha)\) confidence."

#### 3.2.1 Limiting Distribution of Adjacency Matrix

The concept of a limiting distribution of the adjacency matrix, when \( n \to \infty \) will be very helpful for \( s \). In this context we recall the following:
Bose and Sen (2008) deal with real symmetric matrices. If $\lambda$ is an eigenvalue of multiplicity $m$ of an $n \times n$ matrix $A_n$, then the Empirical Spectral Measure puts mass $m/n$ at $\lambda$. Note that if the entries of $A_n$ are random, then this is a random probability measure. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all the eigenvalues, then the empirical spectral distribution function (ESD) $F^{A_n}$ of $A_n$ is given by

$$F^{A_n}(x) = \frac{1}{n} \sum_{i=1}^{n} I(\lambda_i \leq x)$$

Let $\{A_n\}$ be a sequence of square matrices with the corresponding ESD $\{F^{A_n}\}$. The Limiting Spectral Distribution (or measure) (LSD) of the sequence is defined as the weak limit of the sequence $\{F^{A_n}\}$, if it exists. If $\{A_n\}$ are random, the limit is in the “almost sure” or “in probability” sense.

The relevant example for us is the Wigner matrix. In its simplest form, a Wigner matrix $W_n$ of order $n$ is an $n \times n$ symmetric matrix whose entries on and above the diagonal are i.i.d. random variables with zero mean and variance one. Denoting those i.i.d. random variables by $\{x_{ij}: 1 \leq i \leq j\}$, we can visualize the Wigner matrix as

$$W_n = \begin{bmatrix}
\begin{array}{cccccc}
x_{11} & x_{12} & x_{13} & \cdots & x_{1(n-1)} & x_{1n} \\
x_{12} & x_{22} & x_{23} & \cdots & x_{2(n-1)} & x_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
x_{1n} & x_{2n} & x_{3n} & \cdots & x_{n(n-1)} & x_{nn}
\end{array}
\end{bmatrix}$$

The semi-circular law $W$ arises as the LSD of $n^{-1/2} W_n$. It has the density function

$$p_W(s) = \begin{cases} 
\frac{1}{2\pi} \sqrt{4-s^2} & \text{if } |s| \leq 2 \\
0 & \text{otherwise}
\end{cases}$$

All its odd moments are zero. The even moments are given by

$$\beta_{2k}(W) = (2k)!/k!(k+1)!$$

**Theorem:** Let $\{w_{ij}: 1 \leq i \leq j, j \geq 1\}$ be a double sequence of independent random variables with $E(w_{ij}) = 0$ for all $i \leq j$ and $E(w_{ij}^2) = 1$ which are either (i) uniformly bounded or (ii) identically distributed. Let $W_n$ be an $n \times n$ Wigner matrix with the entries $w_{ij}$. Then with probability one, $F^{n^{-1/2}W_n}$ converges weakly to the semicircular law.

We will use this result...

Using Bose and Sen, 2008: the eigenvalues of $K_5$ are $(4, -1, -1, -1, -1)$ and that of $K_{3,3}$ are $(3, 0, 0, 0, 0, -3)$.

If we are able to work out the ESD for $\left(\frac{\delta_{ij} - p_{ij}}{\sqrt{p_{ij}(1-p_{ij})}}\right)$ then if we can probabilistically bound it (i) above by 4, then $K_5$ is ruled out; (ii) below by -3 to rule out $K_{3,3}$. Then we have a test for planarity.

### 4 Sampling Strategy

If we use attribute information for nodes, then one approach could be as follows:

1. Use Dirichlet process on attributes (as in Sethuraman, 1994).
   Prior sampling for computing posterior distribution.

2. Now we can use either Markov Random Fields model (as in Jordan and Wairight, 2008) or Exponential Random Graph model (as in Christakis et. al 2011) for edges: $\delta_{ij} | X_i, X_j$.

   Now $\delta_i \sim \sum_j \delta_{ij} | X_i, X_j$

   So we use sample estimate for $d_i, \hat{d}_i$ (use Lovasz type strategy).
3. Now we extrapolate for $|V| \to \infty$
   - this is where the limit theorems will need to be formulated. Target is to establish weak laws and Central Limit Theorems.
   
   Assumptions on similarity as $n \to \infty$.
   Approximate finite basis with dimension $k(n)$ s.t. $\frac{k(n)}{n} \to 0$ as $n \to \infty$. In fact target $\frac{k(n)}{N} \to 0$ with sample size $N$.
   So need to bound two approximations.

Lovasz type sampling strategy

- Start with any vertex, call it 1, observation available is $\{X_1\}$.
- Crawl all connections (neighbours) of 1, observation available $\{d_1, X_i\}$ for all neighbours
- Randomly select some of the neighbours of 1. Crawl all connections of them ...
- After these two layers, we will have data on $\{d_i, X_i\}$ for $i$ belonging to sampled vertices of these two layers (say first N) and $\{X_j\}$ for connections $j$ of them (say N+1 to N+M).

This can be visualised in terms of the data structure given below. Here the first N rows & columns will be completely known. For the next M rows & columns, $X_j$ will be known and some of the $d_{ij}$’s will be known (loops back). Now from this data the analysis will begin.

4.1 Algorithm

Assuming $X_n$ to be a discrete valued random variable, say taking values from the set $S = \{s_1, s_2, ..., s_k\}$. There are multiple iid copies of vertices with the attribute value $s_i$. Since by our assumption the behavior of the vertex is completely determined by its attribute value, then the centrality of all vertices with same attribute value should be same. Thus $p_{ij}$ also takes finitely many values, and it is enough to look at the matrix consisting of $p_{ij} = P(\delta_{ij} = 1|X_i = s_k, X_j = s_l)$ for all $k, l = 1, ..., n$

We attempt to apply a strategy similar to Lovasz. We randomly sample a vertex and consider its depth-2 neighborhood. By the proportion of his connections to different vertices with different attribute values, we get an estimate of $p_{ij}$. If $\exists m, n$ such that $p_{mn}$ is not estimated from the sample, but $p_{mr}$ and $p_{nr}$ are for some r, we note that $d$ being a metric we have the triangle inequality $d(s_m, s_n) \leq d(s_m, s_r) + d(s_r, s_n)$

So, we have $p_{mn} \geq p_{mr} + p_{nr}$ which provides a lower bound for $p_{mn}$. We can have $p_{mn} \geq \sup_r \{p_{mr} + p_{nr}\}$.

So an iterative updation may be done of the lower bound. If in our final sample it is still not estimated we can take $p_{mn} \sim U(\sup_r \{p_{mr} + p_{nr}\}, 1)$

A. Sampling Algorithm:

Step 1. Select at random $n$ vertices from all the vertices in the graph.
   Step 2. Consider the proportion of vertices that are from the $i^{th}$ class, $i = 1, ..., k$. Call it $s_i^0$.
   Step 3. For the connection probability of elements of Class $i$ and Class $j$, use the first measure:
   $$\hat{p}_{ij} = \frac{\# \text{connection present among elements of class } i \text{ and } j} {\text{total possible connections from } i \text{ to } j} = \frac{n_{ij}}{s_i^0 s_j^0 n^2}$$

   Step 4: For each of the vertices chosen at Step 1, say the selected vertex is from class $k$ and the neighbours are from classes $\alpha_{1k}, ..., \alpha_{nk}$. Pick one of these neighbours, say the $j^{th}$ one from class $i$ at random with probability $p_{k\alpha_{1k}}$.

1See Saad, Basar et. al. for a different but interesting application of such technique in modelling sharing of information for more efficient estimation.
Step 5: Using the information from the arbitrarily chosen neighbours, repeat Step 2 to get $s_i^1$, and calculate $\hat{p}_{ij}^1$ similarly.

Step 6. Calculate $q_{ij} = \beta \hat{p}_{ij}^0 + (1 - \beta) \hat{p}_{ij}^1$ where $\beta \in (0,1)$. Report it as the probability.

Step 7: If $\exists m, n$ such that $p_{mn}$ is not estimated from the sample, but $p_{mr}$ and $p_{nr}$ are for some $r$, we note that $d$ being a metric we have the triangle inequality $d(s_m, s_n) \leq d(s_m, s_r) + d(s_r, s_n)$.

So, we have $p_{mn} \geq p_{mr} + p_{nr}$ which provides a lower bound for $p_{mn}$. We can have $p_{mn} \geq \sup_r \{p_{mr} + p_{nr}\}$.

We take $q_{mn} \sim U(\sup_r \{p_{mr} + p_{nr}\}, 1)$.

B. Generation of the Adjacency Matrix from our Sampling Scheme:

Assume that the graph size is unknown.

First assign the vertex $i$ to a class $C_i$ by generating the random variable $X$ from the discrete distribution of the standardized $s_i$.

Then, if vertex $i$ is from $C_k$ and if vertex $j$ is from $C_r$, then $((a_{ij})) \sim \text{Bernoulli}(q_{kr})$.

Note:

$\text{Prob}(a_{ij} = 1) = \sum_{k,r} \text{Prob}(a_{ij} = 1; i \in C_k, j \in C_r) = \sum_{k,r} \text{Prob}(a_{ij} = 1; i \in C_k, j \in C_r) \text{Prob}(i \in C_k) \text{Prob}(j \in C_r) = \sum_{k,r} q_{kr} s_i^k s_j^r$.

C. The resulting data structure

The above sampling scheme will give rise to data which is a finite subgraph of the original graph in the following structure:

| $X_1$ | $X_2$ | $X_3$ | ... | $X_k$ | ... | $d$ |
|-------|-------|-------|-----|-------|-----|-----|
| 1     | 0     | $\delta_{12}$ | $\delta_{13}$ | ... | $\delta_{1k}$ | ... | $d_1$ |
| 2     | $\delta_{12}$ | 0     | $\delta_{23}$ | ... | $\delta_{2k}$ | ... | $d_2$ |
| 3     | $\delta_{13}$ | $\delta_{23}$ | 0     | ... | $\delta_{3k}$ | ... | $d_3$ |
| ...   | ...   | ...   | ... | ... | ... | ... |
| $k$   | ...   | ...   | 0   | ... | ... | $d_k$ |
| ...   | ...   | ...   | ... | ... | ... | ... |

5 Results

- Simulation Results
- Comparison with existing methods (degree centrality etc.)
- Test results for Planarity etc.

6 Discussion and Conclusions

6.1: On Infinite Graph Spectrums:

In Bose and Sen\cite{1}, the questions about the spectral decomposition of infinite dimensional matrices are tackled, with results derived for the Winger Matrix. However, the realised Adjacency Matrix for our model does not have an iid structure of rows, as the value on the $(i, j)^{th}$ row depends totally on the class of the $i^{th}$ element and the $j^{th}$ element. The submatrices of the form given below are Winger (ie, the rows are generated from an iid process) and for the individual blocks we can obtain the limiting spectral distribution (LSD), which is the limit of $F^{A_n}(x) = n^{-1} \sum 1(\lambda_i \leq x)$. 

7
\[ A_n = \begin{pmatrix} c_{j1} & c_{j2} & c_{j3} & \ldots \\ c_{i1} \\ c_{i2} \\ c_{i3} \\ \vdots \end{pmatrix} \]

This, while can give an idea about the large sample centrality of the classes, seems to fail to generalize to give overall graph spectra.

Our overall graph has finitely many blocks of the above form, with each block of infinite size. If we can prove the result for an adjacency matrix with two classes, then we can extend the result to finitely many blocks.

6.2: Weighted Graph

If instead of considering the actual adjacency matrix we consider the weighted graph adjacency matrix, where link weights are the probabilities of connection between the two vertices, the underlying weighted graph is connected.

However, with such a notion of weighted graph the degree of a vertex of an infinite graph \( d_i = \sum_j p_{ij} \) is always going to be infinite, as it will connect to all other vertices with some nonzero probability, and is basically a sum of infinitely many values of \( p_{ij} \neq 0 \) for atleast one \( j \) (the graph being infinite) and hence converges to infinity.

Another issue with such a setup would be that all vertices of the same class should theoretically have the same centrality, as connections are dependent only on class properties and not the individual vertices themselves. Thus, we may end up characterizing an “influential” group of people rather than identifying any one individual- which makes sense from a marketing/ SNA perspective. If our attributes are fine enough, then the number of classes will be high with low class size for most, leading to zeroing in on one influential person compared to the rest.

The notion of \( p_{ij} \) bounded away from zero arises from the notion that we may have incomplete information about the attributes, so we cannot with certainty say who will connect to/avoid whom.

6.3: Spectrum of the Weighted Graph

In such a case, we may again note that any finite column-truncated (respectively row-truncated) subgraph has repeated rows (respectively column truncated), and finding the eigenvalues of any finite dimensional subgraph with repeated rows is equivalent to finding the eigenvalue of a transformed lower dimensional matrix, as follows.

In general, if \( I \) is a set of rows which are identical, then let \( v_I \) be the vector which is \( 1/\sqrt{|I|} \) on the coordinates in \( I \) and 0 elsewhere. The \( v_I \) are orthonormal, complete them to an orthonormal basis by adding vectors \( w_j \). Then \( A \) will annihilate the \( w_j \) and will take \( \text{Span}(v_I) \) to itself. The matrix of endomorphism of \( \text{Span}(v_I) \) will have entries that look like \( \sqrt{|I|} a_{ij} \), with \( i \in I \) and \( j \in J \). So it suffices to compute the eigenvalues of this finite matrix, and the rest are 0.

For every finite subgraph of the matrix say of dimension \( n \times n \), calculating the subgraph spectra is thus equivalent to calculating the graph spectra of a smaller transformed matrix, i.e, if we have \( k \) classes, we can simply compute the eigenvalues of the finite matrix of dimension \( \binom{k}{2} \times \binom{k}{2} \).
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