Universal Polarization
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Abstract—A method to polarize channels universally is introduced. The method is based on combining two distinct channels in each polarization step as opposed to Arıkan’s original method of combining identical channels. This creates an equal number of only two types of channels, one of which becomes progressively better as the other becomes worse. The locations of the good polarized channels are independent of the underlying channel, guaranteeing universality. Polarizing the good channels further with Arıkan’s method results in universal polar codes of rate $1/2$. The method is generalized to construct codes of arbitrary rates.

It is also shown that the less noisy ordering of channels is preserved under polarization, and thus a good polar code for a given channel will perform well over a less noisy one.

Index Terms—Universal polar codes, universal polarization, compound channels, less noisy ordering.

I. INTRODUCTION

The compound channel models communication without perfect knowledge of the physical channel. The channel is assumed to belong to a certain class, and a code needs to be designed to perform well over all members of the class. The problem is relevant from a practical standpoint since one can rarely estimate the channel perfectly, and it is undesirable for small variations in the channel to change the code performance dramatically.

Let $W$ be a class of binary-input memoryless channels, and let $I(W)$ denote the symmetric compound capacity of $W$, which is the highest achievable rate over all $W \in W$ by codes with an equal frequency of zeros and ones. It is known [1] that

$$I(W) = \inf_{W \in W} I(W)$$

where $I(W)$ denotes the symmetric-capacity of $W$. A code sequence of rate $R$ is said to be universal if its error probability vanishes over the class of all channels with $I(W) > R$.

A problem of practical interest is to design universal codes with low encoding and decoding complexities. To this end, Kudekar et. al. [2] recently showed that spatially-coupled LDPC codes are universal (for symmetric channels) under low-complexity message-passing decoders. Here, we investigate whether universality can be attained by Arıkan’s polarization methods [3]. As in [2], we consider the setting where the channel is unknown only to the transmitter. This is an idealized version of the practical scenario where the receiver may estimate the channel prior to data transmission, e.g., through the use of training symbols. Polar coding for this setting was first considered by Hassani et. al. [4], who concluded that Arıkan’s original codes are not universal under successive cancellation (SC) decoding. It is worth noting, however, that polar codes are universal under the optimal but computationally unfeasible maximum likelihood decoding [5, pp. 87–89].

There are cases in which designing a polar code for multiple channels is easy. The most prominent of these is the degraded case: A polar code tailor to a given channel will also perform well over all upgraded versions of that channel [6]. In Appendix A, we show that a similar statement holds for the more general class of less noisy comparable channels.

Our aim here is to show a method to polarize channels universally. We will first discuss how to achieve rate $1/2$, and later show constructions that achieve arbitrary rates.

II. METHOD

As in Arıkan’s original method, we will polarize channels recursively. The construction will have two stages, which we call the slow polarization and the fast polarization stages. Slow polarization will create only two types of channels after each recursion: Almost half of the polarized channels will be of the first type and become increasingly good, the other half will become increasingly bad. The indices of the good channels will be independent of the underlying channel, and thus universality will be attained at this stage. However, this type of polarization will be too slow to allow reliable SC decoding. In order to improve reliability, we will switch to Arıkan’s regular (fast) polarization method once sufficient universality is achieved.

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Given two binary-input memoryless channels $W: \{0,1\} \rightarrow \mathcal{Y}$ and $V: \{0,1\} \rightarrow \mathcal{Z}$, define the binary-input channels

$$(W, V)^-(y, z | x) = \sum_{u \in \{0,1\}} \frac{1}{2} W(y | u + x)V(z | u)$$

and

$$(W, V)^+(y, z, u | x) = \frac{1}{2} W(y | u + x)V(z | x).$$

Note that if $W = V$, then these are equivalent to the regular polarized channels $W^-$ and $W^+$ in [3]. We will let $L_n$ and $R_n$ denote the two channels that will emerge in the $n$th level of slow polarization. These are defined recursively through

$$L_0 = R_0 = W$$
$$L_{n+1} = (L_n, R_n)^-$$
$$R_{n+1} = (L_n, R_n)^+ \quad n = 0, 1, \ldots$$

Observe that each recursion except the first combines two different channels to produce the channels of the next level. This is in contrast with the original polarization method, which combines identical channels to create $2^n$ polarized channels at the $n$th recursion,

$$W^s_- = (W^s, W^s)^-, \quad s \in \{-, +\}^{n-1}.$$

$$W^s_+ = (W^s, W^s)^+, \quad s \in \{-, +\}^{n-1}.$$

It is readily seen that for all $n$ we have

$$I(L_n) + I(R_n) = 2I(W).$$

Standard arguments also show that $I(L_n)$ is decreasing and $I(R_n)$ is increasing:

$$I(L_{n+1}) \leq I(L_n) \leq I(R_n) \leq I(R_{n+1}).$$

Since both $I(L_n)$ and $I(R_n)$ are monotone and bounded by 0 and 1, they have $[0,1]$-valued limits, which we respectively call $I(L_\infty)$ and $I(R_\infty)$. Further, it follows from [5, Lemma 2.1] that the inequalities above are strict unless $I(L_n) \in \{0,1\}$ or $I(R_n) \in \{0,1\}$. This implies the following polarization result.

**Proposition 1.**

(i) If $I(W) \geq 1/2$, then

$$I(L_\infty) = 2I(W) - 1, \quad I(R_\infty) = 1.$$  

(ii) If $I(W) \leq 1/2$, then

$$I(L_\infty) = 0, \quad I(R_\infty) = 2I(W).$$

We now describe a transform that recursively produces the channels $L_n$ and $R_n$. This is best done graphically; the claims will be evident from the figures. Note first that $L_1$ and $R_1$ are identical to $W^-$ and $W^+$, and thus can be obtained in the regular manner (Figure 1). In order to create $L_2$ and $R_2$ from these, one can take two independent $(L_1, R_1)$ pairs, and combine the $L_1$ from one pair with the $R_1$ from the other, as in Figure 2.

Inspecting the figure, one may be tempted to combine the second $L_1$ and $R_1$ to obtain another $(L_2, R_2)$ pair, but some thought reveals that doing so fails to create the desired effect.

Instead, more $(L_2, R_2)$ pairs can be obtained by combining more than two $(L_1, R_1)$ pairs in a chain. This is shown in Figure 3 where four $(L_1, R_1)$ pairs are chained. The resulting transform creates three $(L_2, R_2)$ pairs. One can more generally chain $K$ $(L_1, R_1)$ pairs to produce $K - 1$ $(L_2, R_2)$ pairs. Thus, the fraction of $(L_2, R_2)$ pairs can be made as close to 1 as desired by taking $K$ sufficiently large. Observe also that the channels $U_i \rightarrow Y_i^{-1}U_i^{-1}$ obtained by such a chain are equivalent to $U_i \rightarrow Y_i^{-1}U_i^{-1}$. That is, not all channel outputs are relevant to $U_i$.

There are several ways to continue this construction to attain further levels of polarization. We describe here perhaps the simplest one, where chaining as in Figure 3 is used only in the second recursion. Each subsequent recursion combines only two blocks. The third level of this construction with $K = 4$ is shown in Figure 4. Here, only the level-2 channels $L_2$ and $R_2$ are combined in the third recursion, $L_1$ and $G_1$ are not. Further, the first $L_2$ in the first block and the last $R_2$ in the second are also left unconnected, in order to ensure that the remaining channels polarize to the third level to produce $L_3$ and $R_3$. This idea is easily extended to further levels: To obtain $L_{n+1}$ and $R_{n+1}$ in the $(n + 1)$-th recursion, one only combines the $L_n$s from the first block with the $R_n$s from the second, and vice versa. The first $L_n$ from the

![Figure 1](image1.png)

Fig. 1. When $X_1$ and $X_2$ are uniform and independent, the channel $U_1 \rightarrow Y_1^2$ is equivalent to $L_1 = W^-$ and $U_2 \rightarrow Y_2^2U_1$ is equivalent to $R_1 = W^+$.

![Figure 2](image2.png)

Fig. 2. The channels $U_i \rightarrow Y_i^{-1}U_i^{-1}$ are equivalent to those inside the parentheses.
Recall that our initial goal was to ensure that all channels after the \( n \)th recursion become either \( L_n \) or \( R_n \), but the procedure described above leaves some channels in lower levels of polarization. The number of these less polarized channels in fact increases with each recursion, although fortunately, this loss is limited. One can indeed check that the blocklength is \( N = 2^{n-1}K \) after the \( n \)th recursion, and the number of channels at level \( 1 \leq i \leq n-1 \) is \( 2^{n-i} \). Therefore the fraction of level-\( n \) channels can be lower bounded as

\[
1 - \frac{\sum_{i=1}^{n-1} 2^{n-i}}{2^{n-1}K} \geq 1 - \frac{2^n}{2^{n-1}K} = 1 - \frac{2}{K},
\]

which can be made arbitrarily close to 1 by picking a large \( K \).

Observe that the construction described above is universal: The positions of the good channels (that is, \( R_n \)s) after the transformation is independent of the underlying channel \( W \). One may therefore hope to use these channels to achieve rate 1/2 over any \( W \) with \( I(W) \geq 1/2 \). Unfortunately, however, the speed of polarization is too slow for an SC decoder to succeed. This is most easily seen by noting that the Bhattacharyya parameter of \( R_{n+1} \) is given by

\[
Z(R_{n+1}) = Z(R_n)Z(L_n).
\]

Since \( Z(L_n) \) approaches a non-zero constant (in particular, 1 if \( I(W) = 1/2 \) as \( n \) grows, the multiplicative improvement in \( Z(R_n) \) gradually slows down (to a halt

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**Fig. 3.** Four pairs of level-1 channels are chained to create six level-2 and two level-1 channels. The channels \( U_i \rightarrow Y_{1}^{S} U_{i-1} \) are equivalent to the ones on the left.

**Fig. 4.** A 3-level transform with \( K = 4 \). Each of the two blocks represents the transform in Figure 3. The channels written inside the respective blocks correspond to \( S_i \rightarrow Y_{1}^{S} S_{i-1} \) and \( T_i \rightarrow Y_{1}^{T} T_{i-1} \). If one labels \( U_1 \) to \( U_{16} \) as above, then the channels \( U_i \rightarrow Y_{1}^{S} U_{i-1} \) are equivalent to the ones on the left.

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**Fig. 5.** The \( (n+1) \)-level construction. Here, \( \ell = 2^{n-1} - 1 \).
if $I(W) = 1/2$). Compare this with Arikan’s original method, in which identical channels are combined in each step, and thus the improvement in the Bhattacharyya parameters speeds up as channels become better. The speed of universal polarization can be bounded as follows:

**Proposition 2.** Let $h : [0, 1/2] \to [0, 1]$ denote the binary entropy function, let $a * b = a(1 - b) + (1 - a)b$, and define

$$f(t, x) = t(2x - t)$$
$$g(t, x) = 2x - h\left(h^{-1}(t) * h^{-1}(2x - t)\right),$$

over $x \in [0, 1]$ and $t \in \max\{0, 2x - 1\}, x$. Finally define

$$F_0(x) = G_0(x) = x$$
$$F_n(x) = f(F_{n-1}(x), x)$$
$$G_n(x) = g(G_{n-1}(x), x), \quad n = 1, 2, \ldots.$$ 

We have

$$1 - G_n(1 - I(W)) \leq I(R_n) \leq 1 - F_n(1 - I(W)).$$

**Proof:** Let $H(W)$ denote the entropy of $W$’s uniformly distributed input conditioned on its output, that is, $H(W) = 1 - I(W)$. The claim is equivalent to

$$F_n(H(W)) \leq H(R_n) \leq G_n(H(W)), \quad (2)$$

which holds trivially for $n = 0$. Suppose now that (2) holds for some $n \geq 1$. Recall that among all pairs of channels $V$ and $W$ with given entropies, $H((V, W)^+) \leq \frac{1}{2}H(V) + \frac{1}{2}H(W)$, and this is minimized when $V$ and $W$ are both binary erasure channels (BECs) and maximized when both are binary symmetric channels (BSCs) [5, Lemma 2.1]. This implies that

$$f(H(R_n), H(W)) \leq H(R_{n+1}) \leq g(H(R_n), H(W)). \quad (3)$$

On the other hand, $f(t, x)$ and $g(t, x)$ are increasing in $t$, a proof for the latter is given in Appendix B. It then follows from (2) that

$$F_{n+1}(H(W)) = f(F_n(H(W)), H(W)) \leq f(H(R_n), H(W))$$

and

$$g(H(R_n), H(W)) \leq g(G_n(H(W)), H(W)) = G_{n+1}(H(W)).$$

Combining these with (3) implies the claim for $n + 1$, concluding the proof.

Observe that the above upper bound on $I(R_n)$ is obtained by replacing $R_n$ and $L_n$ with BECs with symmetric capacities $I(R_n)$ and $I(L_n)$ respectively before each polarization step. Similarly, the lower bound is obtained by replacing these channels with BSCs. Recall that the descendants of BECs remain BECs during polarization, whereas those of BSCs do not remain BSCs. This implies that while the upper bound is achieved by the BEC, the lower bound is loose. Tables I and II list the bounds for $I(W) = 0.5$ and $I(W) = 0.8$.

| $n$ | lower bound | upper bound |
|-----|-------------|-------------|
| 0   | 0.5         | 0.5         |
| 1   | 0.713       | 0.750       |
| 2   | 0.771       | 0.812       |
| 3   | 0.805       | 0.847       |
| 4   | 0.829       | 0.870       |
| 5   | 0.846       | 0.887       |
| 10  | 0.895       | 0.931       |
| 20  | 0.932       | 0.960       |
| 30  | 0.949       | 0.972       |
| 40  | 0.958       | 0.978       |

TABLE I

| $n$ | lower bound | upper bound |
|-----|-------------|-------------|
| 0   | 0.8         | 0.8         |
| 1   | 0.928       | 0.960       |
| 2   | 0.957       | 0.986       |
| 3   | 0.972       | 0.994       |
| 4   | 0.981       | 0.997       |
| 5   | 0.986       | 0.999       |
| 10  | 0.996       | 0.9999991   |
| 15  | 0.9999000001 | 0.9999999991 |
| 20  | 0.9996      | 0.9999999991 |

TABLE II

**A. Universal polar coding**

To obtain a good code, we can append Arikan’s fast (but not universal) polarizing transform to the universal (but slow) polarizing transform described above. That is, once $n$ is sufficiently large so that $I(R_n) > 1 - \epsilon$ for all $W$, we may start polarizing $R_n$ fast. This can be done by taking $M = 2^m$ copies of the slow polarization transform and passing the $M$ copies of each $R_n$ through the standard length-$M$ transform. Inputs to the remaining channels are frozen and the resulting code blocks are decoded in succession.

One may tailor the polar codes in the second stage to the channel that is least degraded with respect to all channels with $I(W) \geq 1 - \epsilon$. How to find such
channels is shown in [7]. A computationally simpler alternative is to find a universal upper bound $Z(R_n) \leq \delta$ (as in Proposition 2) and tailor the second-layer code to a BEC with erasure probability $\delta$. This method is motivated by the fact that among all channels with a fixed $Z(W)$, the BEC’s polarized descendants have the highest Bhattacharyya parameters, and the latter can be computed in linear time [3].

B. Rate

Since $I(R_n)$ is close to 1, both approaches mentioned in the paragraph above will induce a negligible rate loss in the fast polarization stage. Recall also that the loss in the slow polarization stage is $O(1/K)$. Hence the rate of the code can be made as close to 1/2 as desired.

C. Error probability

Recall that the reliabilities of the good channels after fast polarization is $o(2^{-M^\beta})$ for all $\beta < 1/2$ [8], and thus the block error probability of this code of length $NM$ is upper bounded all $W \in W(1/2)$ by

$$N o(2^{-M^\beta}),$$

which for fixed $N$ vanishes as $M$ grows.

D. Complexity

To estimate the decoding complexity, it is useful to explain the decoding scheme in some detail: The decoder begins by computing the likelihood ratios for the frozen bits that see $L_1, \ldots, L_{n-1}$, followed by the first $L_n$ in each block. Then, the likelihood ratio for the first $R_n$ in each block is computed and passed to the SC decoder for the first block of the fast polarization stage. Once this block is decoded, the bit values are passed back to the slow polarization stage and are used to compute the likelihood ratios for the next $L_n$ and $R_n$ in each block. These steps are repeated until all $R_n$ blocks are decoded. A straightforward computation shows that the total complexity of this decoder is

$$O(N)k_f(M) + M k_s(N),$$

where $k_f(M)$ and $k_s(N)$ respectively are the decoding complexities of the fast polar transform of length $M$ and a slow polar transform of length $N$. It is known [3] that $k_f(M) = O(M \log M)$. Now, observe that the slow polar transform is almost identical to the fast one; it only differs in the chaining operation in the second level and in the combination of non-identical channels at each step. It is easy to see that neither of these differences affects the complexity of computing the likelihood ratios of the polarized channels. That is, $k_s(N) = O(N \log N)$. This implies that the total decoding complexity at blocklength $MN$ is $O(MN \log MN)$, similar to regular polar codes. Similar arguments show that the encoding complexity is also $O(MN \log MN)$. Note also that the chain length $K$ affects encoding/decoding complexities only insofar as it appears as a linear factor in the blocklength.

III. CODES WITH ARBITRARY RATES

We now discuss how to obtain universal polar codes with rates other than 1/2. Recall that in the previous section we fixed the rate of the code by using only the universally good channel $R_n$ for coding. When $I(W)$ is greater than 1/2, the code rate can be increased by considering coding over $L_n$ also, since Proposition 1 then implies $I(L_n) > 0$. For example, when $I(R_n)$ becomes sufficiently close to 1, one may obtain more universally good channels by only slow-polarizing $L_n$ further. When $I(W)$ is less than 1/2, the same method can be used by slow-polarizing $R_n$ further once $I(R_n)$ becomes sufficiently close to $2I(W)$. Each stage of this polarization method turns half of the remaining nonextremal channels to extremal ones. The resulting good channels can then be fast-polarized for coding. However, the blocklengths of such constructions can be very large, since even a single stage of slow polarization requires a large number of recursions (recall Table I).

Instead, here we generalize the ideas in Section III to construct codes with rates $g/(b+g)$ for given positive integers $g$ and $b$. Following the rate-1/2 case, this can be done if one can (i) combine $b+g$ channels at a time to create only $b+g$ channel types after each level of slow polarization, and (ii) ensure that $g$ of these become better in each step and the remaining $b$ become worse. As before, once the good channels become nearly perfect, one can boost their reliabilities through fast polarization.

It thus suffices to describe a construction that has properties (i) and (ii). Again, the simplest description is through figures. Figure 6 shows an example of the type of transforms we will consider. In particular, the transform circuit consists of $b+g$ horizontal lines, each of which has a single modulo-2 addition that connects it to the line below. Starting at the second line from the top, one can place this connection to the right or to the left of the connection above.

The channels $U_i \rightarrow Y_1^{bg} U_{i-1}$ produced by the transform are defined as usual, where the inputs and outputs are numbered in increasing order from top to bottom. We label the channels as follows (see Figure 6):

If a line’s connection to the bottom is on the left side of its connection to the top, then the corresponding channel
is called an $L$-channel. The $i$th such channel from the top is called $L^{(i)}$. Similarly, a channel whose connection to the bottom is on the right side of its connection to the top is called a $R$-channel. In addition, the top channel is an $L$-channel and the bottom channel is an $R$-channel. Observe that the fraction of $L$- and $R$-channels can be adjusted to arbitrary non-zero values by an appropriate choice of transform.

We will restrict our attention to two types of transforms for which the claims will be easy to verify. For $g \leq b$ (that is, when the target rate is less than $1/2$), we will use the transform that produces the channels in the order

$$LL \ldots L LRLR \ldots LR$$

That is, the top $b-g$ channels will be of type $L$, followed by an alternating sequence of $L$- and $R$-channels. In order to define a recursion, we need to specify the order in which the $b+g$ channels enter the transform in the next level. In the present case, the input order is obtained by cyclically down-shifting (4) by one:

$$R LL \ldots LLRLR \ldots RL$$

For the $g \geq b$ case, the top channels produced by the transform will be of alternating types, followed by a sequence of $R$-channels:

$$LRLR \ldots LR RR \ldots R$$

(6)

These channels will be input to the next recursion after up-shifting the order (6) by one:

$$RL \ldots RL RR \ldots RL$$

(7)

Examples of both recursions are shown in Figure 7.

We will label the channels produced by these recursions as in the previous section: If $g \leq b$, then the channels $L_n^{(1)} , \ldots , L_n^{(b)}$ and $R_n^{(1)} \ldots R_n^{(g)}$ after the $n$th recursion are transformed through (4) and (5) to produce $L_{n+1}^{(1)} , \ldots , L_{n+1}^{(b)}$ and $R_{n+1}^{(1)} \ldots R_{n+1}^{(g)}$. The first recursion takes $b+g$ copies of $W$ as input. For the case $g \geq b$, the recursions are defined through (6) and (7).

The reason for the labeling above is the analogy between the $(L,R)$-channels and the channels $(L_1,R_1)$ of the previous section. Indeed, suppose that we combine $b+g$ copies of $W$ through a transform that produces the channels $L_1^{(1)} , \ldots , L_1^{(b)}$ and $R_1^{(1)} \ldots R_1^{(g)}$. We clearly have

$$\sum_{i=1}^{b} I(L_n^{(i)}) + \sum_{i=1}^{g} I(R_n^{(i)}) = (b+g)I(W).$$

Moreover, the $L$-channels are worse than $W$ and the $R$-channels are better:

**Proposition 3.** For all $i = 1, \ldots , b$ and $j = 1, \ldots , g$ we have

$$I(L_1^{(i)}) \leq I(W) \leq I(R_1^{(j)}).$$

*Both inequalities are strict unless $I(W) \in \{0,1\}$."

**Proof:** We prove the statement for the case $g \leq b$. The case $g > b$ can be analyzed similarly. By construction, we have from top to bottom the following sequence of channels

$$L_1^{(1)} , \ldots , L_1^{(b-g)}, L_1^{(b-g+1)}, R_1^{(1)}, \ldots , L_1^{(b)}, R_1^{(g)} .$$

(8)

For $g \geq b$, the proofs will be similar.
Define $Q_1 = W$ and $Q_{i+1} = (Q_i, W)^+$ for $i = 1, 2, \ldots$. If $g = 1$, we have $L_1^{(i)} = (Q_i, W)^-$ for $1 \leq i \leq b$ and $R_1^{(i)} = (Q_b, W)^+$. If $g > 1$, we have

$$L_1^{(i)} = \begin{cases} (Q_i, W)^- & 1 \leq i \leq b - g \\ (Q_{b-g+1}, W)^- & i = b - g + 1 \\ (W^+, W^-) & b - g + 1 < i < b \\ (W^+, W^-) & i = b \end{cases}$$

and

$$R_1^{(j)} = \begin{cases} (Q_{b-g+1}, W)^+ & j = 1 \\ (W^+, W^-)^+ & 1 < j < g . \\ (W^+, W^-)^+ & j = g \end{cases}$$

The claim then follows by noting that

$$I((W, V)^- \leq \min\{I(W), I(V)\} \
\leq \max\{I(W), I(V)\} \
\leq I((W, V)^+)$$

for any two channels $W$ and $V$. Strict inequalities follow again from [5] Lemma 2.1.

Having created $b$ bad and $g$ good channels out of $W$, we wish to enhance polarization by making the bad channels worse and the good channels better. The main result of this section is that these recursions indeed polarize channels universally:

**Proposition 4.**

(i) If $I(W) \geq g/(b + g)$, then for all $1 \leq i \leq g$

$$\lim_{n \to \infty} I(R_n^{(i)}) = 1.$$  

(ii) If $I(W) \leq g/(b + g)$, then for all $1 \leq i \leq b$

$$\lim_{n \to \infty} I(L_n^{(i)}) = 0.$$  

**Proof:** We prove (i) for the case $g \leq b$. The arguments for the remaining three cases are similar. Recall that the recursions for the case $g \leq b$ are defined through [4] and [5]. Define $Q_1 = R_n^{(g)}$ and $Q_{i+1} = (Q_i, L_n^{(i)})^+$ for $1 \leq i \leq b - g$. When $g = 1$, we have

$$L_n^{(i)} = (Q_i, L_n^{(i)})^-$$

and

$$R_n^{(1)} = (Q_b, L_n^{(b)})^+.$$  

Therefore, $I(L_n^{(i)})$ is decreasing while in $n$ $I(R_n^{(1)})$ is increasing. When $g > 1$, define

$$P_i^+ = (L_n^{(b-g+i)}, R_n^{(i)})^+$$

$$P_i^- = (L_n^{(b-g+i)}, R_n^{(i)})^-$$

for $1 \leq i < g$. One can check that the order of inputs to the recursion implies (see Figure 7 for reference)

$$L_{n+1}^{(i)} = \begin{cases} (Q_i, L_n^{(i)})^- & 1 \leq i \leq b - g \\ (Q_{b-g+1}, P_n^-)^- & i = b - g + 1 \\ (P_{g-b+i-1}^+, P_{b-g+i}^-) & b - g + 1 < i < b \\ (P_{g-1}^+, L_n^{(b)})^- & i = b \end{cases}$$

Note that $I(P_{b-g+i}^-) = I((L_n^{(i)}, R_n^{(g-b+i)})) \leq I(L_n^{(i)})$ for $b - g + 1 \leq i < b$. It follows that $I(L_n^{(i)}) \leq I(L_n^{(i)})$ for all $1 \leq i \leq b$. Similarly, one can check that

$$R_{n+1}^{(j)} = \begin{cases} (Q_{b-g+1}, P_n^-)^+ & j = 1 \\ (P_{j-1}^+, P_j^-)^+ & 1 < j < g \\ (P_{j-1}^+, L_n^{(b)})^- & j = g \end{cases}$$

Note that $I(Q_{b-g+1}) \geq I(R_n^{(g)})$ and $I(P_{j-1}^+ = I((L_n^{(j-1)}, R_n^{(b-j+1)})) \geq I(R_n^{(j-1)})$ for $1 \leq j \leq g$. It follows that $I(R_n^{(j-1)} \geq I(R_n^{(j-1)})$ for $1 \leq j \leq g$. That is the $R$-channels at level $n + 1$ are better than the ones at level $n$, with a shift in indices.

To show that the improvement in $I(R_n^{(i)})$ is strict unless all the $R$-channels are perfect, one needs to rule out the following possibility: If at a point in the polarization process some, but not all, $R$-channels become perfect, then the perfect channels entering subsequent recursions may stall the polarization of the non-perfect ones. We now argue that the structure of the channel combinations does not allow this. Suppose that all but one $R$-channels polarize to perfect ones. Then, there must be at least one unpolarized $L$-channel, since otherwise the inequality $I(W) \geq g/(b + g)$ would be violated. Suppose that there is only one such $L$-channel $L^{(k)}$ and all others are polarized to useless ones. One can then check that either the unpolarized $L$- and $R$-channels will be combined in the next recursion (which will further polarize the $R$-channel), or their positions will change. In particular, the $L$-channel index $k$ will remain unchanged after each recursion, while the $R$-channel index will be cyclically shifted by one. If $1 \leq k \leq b - g + 1$, then $L^{(k)}$ will be combined with the unpolarized $R$-channel when the $R$-channel index is shifted to $g$. On the other hand, if $b - g + 1 < k \leq b$, then $L^{(k)}$ will be combined with the unpolarized $R$-channel when the $R$-channel index is shifted to neighboring positions $k - b + g - 1$ or $k - b + g$. Therefore, regardless of the unpolarized $R$-channel’s position, strict polarization will take place in at most $g$ recursions. (See Figure 8 for an example of strict polarization of period two over the rate-2/6 recursion.) Therefore the $R$-channel will polarize further, eventually.
becoming perfect. The same reasoning can be used when there is more than one unpolarized $R$-channel and $L$-channel.

A. Polar coding

Fix a transform of rate $g/(b + g)$. The code construction is identical to the one in Section II. In the first level, channels are combined in the usual fashion. This is followed by a single step of chaining $K$ transforms that combines channels of different types. Then, each subsequent step combines $b + g$ transform blocks in the same fashion. Once sufficient universal polarization is attained, the good channels $R_{n(i)}$ are fast-polarized further using the Arikan transform.

B. Rate

As in the rate-1/2 case, the slow polarization stage involves leaving some channels unconnected. Similar arguments to those in Section II show that the fraction of unpolarized channels is upper bounded by

$$\frac{(g + b)^2}{K},$$

which can be made as small as desired by picking a large $K$.

C. Error probability

Since the reliability of the good channels are determined essentially by the fast polarization stage, the error probability of the SC decoder can again be upper bounded for all $W \in \mathcal{W}(g/b + g)$ by

$$N_o(2^{-M^3}),$$

where $N$ and $M$ respectively are the lengths of the slow and the fast polarization stages.

D. Complexity

The present construction differs from the one in Section II only in the size $b + g$ of the basic transform, and it is easily seen that the basic transforms we discussed can be encoded and decoded in linear time. Hence, $b + g$ does not affect the encoding and decoding complexities, which are both $O(MN \log MN)$ for a blocklength-$MN$ code.

IV. DISCUSSION

In independent work [9], Hassani and Urbanke propose two polarization-based methods to construct universal codes. On close inspection, one of these methods and the one presented here are seen to be complementary. In particular, whereas the method here guarantees universality in the first stage and reliability in the second, the construction in [9] reverses this order by combining identical channels in the first stage (i.e., fast polarization) and distinct channels in the second (i.e., slow polarization). It is evident from both works that many other variations are possible for constructing universal polar codes, such as interleaving the fast and slow polarization stages. Such alternatives may help reduce the impractically large blocklengths that the present paper’s methods require (see Table I) to simultaneously achieve universality and reliability. For this purpose one may also consider using larger $(b + g)$-type constructions for simple fractional rates such as 1/2, or mixing the unconnected channels into the process to increase the speed of slow polarization. The investigation of these are left for future study.

In addition to providing robustness to point-to-point channel coding, universal polarization is also of interest from a theoretical perspective. Recall that one of the many appeals of polarization methods is the ease with which they have been extended to other communication settings. Polar codes’ optimality have already been
established for multiple-access channels [10], degraded wiretap channels [11], lossless [12], lossy [13], and distributed source coding [10]. However, the theory is more difficult to extend to settings with two or more receivers, and the main bottleneck appears to be the incompatibility of polar code designs for different receivers. With universal polarization schemes, it may be possible to implement polar coding to achieve the best known rates over broadcast channels, interference channels, etc.

It is worth mentioning that the methods discussed here also yield universal source codes, and can be extended to non-binary alphabets using standard arguments [5, Chapter 3].

V. APPENDIX A: POLARIZATION PRESERVES LESS NOISY ORDERING

Recall that designing a polar code of length $2^n$ for a channel $W$ consists in finding a set of $n$ good channels among $W^s$, $s \in \{-, +\}^n$, which are defined recursively through

$$W^-(y_2^2 | u_1) = \sum_{u_2 \in \{0, 1\}} \frac{1}{2} W(y_1 | u_1 + u_2)W(y_2 | u_2),$$

$$W^+(y_2^2, u_1 | u_2) = \frac{1}{2} W(y_1 | u_1 + u_2)W(y_2 | u_2).$$

A good code of rate $R < I(W)$ can be obtained by picking an $R$ fraction of these channels whose symmetric-capacities $I(W^s)$ are largest. Here, we show that a polar code designed in this manner for a channel is also good for all less noisy versions of this channel. This result is also established independently in [14]. Here we show it by proving the stronger statement that the less noisy ordering of channels is preserved under polarization. Recall that a channel $V$ is said to be less noisy than $W$ if $I(T; Y) \leq I(T; Z)$ for all distributions of the form

$$p(t, x, y, z) = p(x, t)W(y|x)V(z|x),$$

that is, for all distributions for which $T-X-YZ$ is a Markov chain [15]. Observe that this implies $I(W) \leq I(V)$, and thus will also imply that $I(W^s) \leq I(V^s)$ for all $s$ once we show that polarization preserves the less noisy order. Due to the recursive description of the polarized channels, it suffices to prove the latter claim for a single-step:

**Proposition 5.** Let $W$ and $V$ be binary-input channels. If $W$ is less noisy than $V$, then

(i) $V^+$ is less noisy than $W^+$,

(ii) $V^-$ is less noisy than $W^-.$

**Proof:** To prove (i), we need to show that $I(T; Y_1 Y_2 U_1) \leq I(T; Z_1 Z_2 U_1)$ for all random variables $(T, U_1^2, Y_1^2, Z_1^2)$ that are jointly distributed as

$$p(t, u_2^2, y_1^2, z_2^2) = p(t, u_2)W^+(y_1^2 | u_1 | u_2)W^+(z_2^2 | u_1 | u_2).$$

Note that the channels $W^+$ and $V^+$ here share an output, namely $U_1$, but this does not affect the mutual informations in question. This assumption will simplify the proof. Define $X_1 = U_1 + U_2$ and $X_2 = U_2$ (see Figure 9). We have

$$I(T; Y_1 Y_2 U_1) = I(T; Y_1 Z_1 | U_1)$$

$$= I(T; Y_1 | U_1) + I(T; Z_2 | Y_1 U_1)$$

$$\leq I(T; Y_1 | U_1) + I(T; Z_2 | Y_1 U_1)$$

$$= I(T; Y_1 Z_2 | U_1)$$

$$= I(T; Z_2 | Y_1 Z_2 U_1)$$

$$\leq I(T; Z_2 | Y_2 U_1) + I(T; Z_1 | Z_2 U_1)$$

$$= I(T; Z_1 Z_2 | U_1).$$

To see the first inequality, note that

$$TU_1 U_2 X_1 Y_1 Z_1 — X_2 — Y_2 Z_2$$

is a Markov chain. Therefore we have

$$p(t, u_2, x_1, x_2, y_2, z_1, z_2 | y_1, u_1)$$

$$= p(t, u_2) p(y_2 | z_2 | x_2) \frac{p(t, u_1, u_2, x_1, y_1, z_1 | x_2)}{p(y_1, u_1)}.$$ 

That is, conditioned on $Y_1 = y_1$ and $U_1 = u_1$,

$$TU_2 X_1 Z_1 — X_2 — Y_2 Z_2$$

is a Markov chain, and therefore so is $T—X_2—Y_2 Z_2$. This and the less noisiness of $V$ imply $I(T; Y_2 | Y_1 = y_1, U_1 = u_1) \leq I(T; Z_2 | Y_1 = y_1, U_1 = u_1).$ Averaging over $(y_1, u_1)$ yields the first inequality. Similarly, for the second inequality, note that

$$TU_1 U_2 Y_2 Z_2 — X_1 — Y_1 Z_1$$

is a Markov chain, and therefore so is $T—X_1—Y_1 Z_1$ for every $Z_2 = z_2$ and $U_1 = u_1$. The less noisy relation then implies $I(T; Y_1 | Z_2 = z_2, U_1 = u_1) \leq I(T; Z_1 | Z_2 = z_2, U_1 = u_1).$ Averaging over $(z_2, u_1)$ yields the inequality.
To prove (ii), we need to show that \( I(T; Y_1 Y_2) \leq I(T; Z_1 Z_2) \) for all \((T, U_1, Y_1^2, Z_1^2)\) for which

\[
\begin{align*}
    p(t, u_1, y_1^2) &= q(t, u_1)W^-(y_1^2|u_1) \\
    p(t, u_1, z_1^2) &= q(t, u_1)V^-(z_1^2|u_1)
\end{align*}
\]  

We will also define a random variable \( U_2 \) such that \((T, U_1^2, Y_1^2, Z_1^2)\) is jointly distributed as

\[
p(t, u_1^2, y_1^2, z_1^2) = \frac{1}{2}q(t, u_1)W(y_1|u_1 + u_2)W(y_2|u_2)
\cdot V(z_1|u_1 + u_2)V(z_2|u_2).
\]  

Observe that this definition is consistent with (10), it will simplify the proof. Defining again \( X_1 = U_1 + U_2 \) and \( X_2 = U_2 \) (see Figure 10), we can write

\[
I(T; Y_1 Y_2) = I(T; Y_1) + I(T; Y_2|Y_1) \\
\leq I(T; Y_1) + I(T; Z_2|Y_1) \\
= I(T; Z_2) + I(T; Y_1|Z_2) \\
\leq I(T; Z_2) + I(T; Z_1|Z_2) \\
= I(T; Z_1 Z_2).
\]

To see the first inequality, note that the distribution in (11) implies that

\[
TU_1 U_2 X_1 Y_1 Z_1 — X_2 — Y_2 Z_2
\]

is a Markov chain. Therefore we have

\[
p(t, u_1, u_2, x_1, x_2, y_2, z_1, z_2|y_1) = p(x_1)p(y_2|x_2)p(t, u_1, u_2, x_1, y_1) / p(y_1).
\]

That is, for any fixed value of \( Y_1 \),

\[
TU_1 U_2 X_1 Z_1 — X_2 — Y_2 Z_2
\]

is a Markov chain, and therefore so is \( T — X_2 — Y_2 Z_2 \). This and the less noisiness of \( V \) imply \( I(T; Y_2|Y_1 = y_1) \leq I(T; Z_2|Y_1 = y_1) \). Averaging over \( y_1 \) yields the first inequality. The proof of the second inequality follows by similar arguments.

Note that the choice of the polarization transform and the alphabet size are immaterial to the proof above, and thus the result holds in more generality as long as the polarized channels are appropriately defined.

It is of interest to characterize the weakest relations that are preserved under polarization. For example, the more capable relation [15], which is weaker than the less noisy relation, is not preserved. A channel \( V(z|x) \) is said to be more capable than \( W(y|x) \) if \( I(X; Y) \leq I(X; Z) \) for all \( p(x, y, z) = p(x)W(y|x)V(z|x) \). It is shown in [5] Lemma 7.1] that in the class of symmetric binary-input channels with a given capacity, the binary symmetric channel is the least capable. However, the ‘minus’ version of the binary symmetric channel has a larger capacity than the minus versions of all other channels in this class, and therefore is not less capable than any such channel. See, for example, [5] Lemma 2.1 for a proof.

VI. APPENDIX B

Lemma 1. For every \( x \in [0, 1] \), the function

\[
g_x(t) = h(h^{-1}(t) + h^{-1}(2x - t))
\]

defined over \( t \in [\max\{0, 2x - 1\}, x] \) is decreasing.

Proof: The function \( h \) is monotonically increasing over \([0, 1/2] \). So it suffices to show \( k_x(t) := h^{-1}(t) + h^{-1}(2x - t) \) is decreasing in \( t \). Defining \( f = h^{-1} \), some algebra yields

\[
\frac{d}{dt}k_x(t) = f'(t)[1 - 2f(2x - t)] - f'(2x - t)[1 - 2f(t)].
\]

The right-hand-side of the above is non-positive since

\[
f'(t) \leq f'(2x - t) \quad \text{and} \quad f(t) \leq f(2x - t) \leq 1/2,
\]

which follow from \( f \) being convex, increasing, and \([0, 1/2]-valued. \)

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