EXAMPLES OF SELF-DUAL, EINSTEIN METRICS OF 
(2, 2)-SIGNATURE

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Abstract. In this paper we construct a family of examples of self-dual Einstein metrics of neutral signature, which are not Ricci flat, nor locally homogeneous. Curvature of these manifolds is studied in details. These are obtained by the para-quaternionic reduction. We compare our examples with the orbifolds \( O_{p,q}(1) \) given by Galicki and Lawson, for which some new properties are also established. Particularly, the sign and the pinching of their sectional curvatures are studied.

1. Introduction

A self-dual manifolds are important in the Riemannian geometry (of the positive definite \((+++++)\) signature), as well in the Kleinian geometry (of the neutral \((-+-++)\) signature). Some examples of Einstein, self-dual metrics of neutral signature are studied in the recent times \([8]\), but non-trivial examples of non-Ricci flat, Einstein self-dual manifolds were not known. In the positive definite case a family of such examples is constructed in \([6]\) and \([7]\). Motivated by these examples, here we construct a family \( M_{p,q}, p, q \in \mathbb{N}, (p, q) = 1, p \neq q \) of Einstein, self-dual manifolds of neutral signature, which are even not locally homogenous. As remarked in \([9]\), the quaternionic reduction can be generalized to the pseudo-Riemannian category. Particularly, we apply the reduction procedure in the setting of \( C(1,1) \) Clifford algebra (para-quaternionic numbers) instead of \( C(2) \) Clifford algebra (quaternions). This approach is appropriate because of the following reasons. Curvature of the self-dual manifolds share some properties with general quaternionic and para-quaternionic manifolds (see \([1, 6, 7]\)). The other reason is that the Clifford algebra \( C(1,1) \) provides a natural language for describing metrics of the neutral signatures. It is interesting to compare this construction with example given in \([4]\) obtained by quaternionic reduction. In the both cases an algebraic submanifold \( K_0 \) of projective plane \((\mathbb{H}P^2 \text{ or } \tilde{\mathbb{H}}P^2)\) is constructed. In the quaternionic (positive definite) case \( K = K_0 \) is a differentiable, complete, submanifold but in the other case (para-quaternionic, indefinite) \( K_0 \) is not a differentiable submanifold. Its singular set \( S \) lies on a real hyperquadric in \( \mathbb{H}P^2 \). Let \( K = K_0 \setminus S \). In the both cases the family of isometric actions \( \phi_{p,q}, \; p, q \in \mathbb{N}, \; (p, q) = 1 \) of a group \( G \) \((S^1 \text{ or } \mathbb{R})\) on five-dimensional submanifold \( K \) is defined. In the quaternionic case this is the locally free action and in the para-quaternionic case the actions \( \phi_{p,q} \) are always free. In the quaternionic case \( K/G \) is a Riemannian orbifold and in para-quaternionic case \( K/G \) is a manifold with incomplete metric of neutral signature. Their "singularity sets" are of different character. Here is the brief outline to this

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paper. In §2, we shall discuss the elementary properties of the Clifford algebra \( \tilde{\mathbb{H}} = C(1, 1) = C(0, 2) \) (para-quaternions) and the projective plane which it defines. In §3 we shall define the action of the group \( G = \mathbb{R} \) on the para-quaternionic projective plane \( \tilde{\mathbb{H}}P^2 \) and the algebraic submanifold \( K_0 \) in that plane. Moreover, we shall study the basic geometry of the orbit space \( K_0/G \), equipped with the metric \( g_{p,q} \) induced by the submersion \( K_0 \to K_0/G \). In §4 we shall compute the sectional curvature and the Jacoby curvature operator of the metric \( g_{p,q} \) in terms of the para-quaternions. This shall give us possibility to prove in §5 that \( g_{p,q} \) is the self-dual, Einstein metric (pointwise Osserman) which is not even locally homogeneous. We conclude in §6 by applying this ideas to study in more details the curvature of the orbifolds \( O_{p,q}(1) \), defined by Galicki and Lawson (\( \tilde{\mathbb{H}} \tilde{\mathbb{H}} \)). For some known results new proofs and generalizations are given.

2. Preliminaries

Denote by \( \tilde{\mathbb{H}} = C(1, 1) = C(0, 2) \) the real Clifford algebra with unity 1 of para-quaternionic numbers generated by elements \( i, j, k \) satisfying
\[
i^2 = -\epsilon_1 = -1, \quad j^2 = -\epsilon_2 = 1, \quad k^2 = -\epsilon_3 = 1, \quad ij = k = -ji.
\]
The conjugate \( \bar{q} \) of para-quaternionic number \( q := x + yi + zj + wk \) is defined by
\[
\bar{q} := x - yi - zj - wk
\]
and real and imaginary part, respectively, by
\[
\Re q := x, \quad \Im q := yi + zj + wk.
\]
The square norm of the para-quaternion \( q \)
\[
|q|^2 := q\bar{q} = x^2 + y^2 - z^2 - w^2,
\]
is multiplicative, i.e. satisfies
\[
|q_1q_2|^2 = |q_1|^2|q_2|^2.
\]
Notice that the pseudosphere of signature (2, 1) of all unit para-quaternions
\[
S^{2,1} = \tilde{\mathbb{H}}_1 := \{ q \in \tilde{\mathbb{H}} \mid |q|^2 = 1 \}
\]
is a Lie group \( \tilde{\mathbb{H}}_1 = SU(1, 1) \). On the vector space \( \tilde{\mathbb{H}}^3 \) we have the pseudo-Riemannian metric of signature (6, 6) given by
\[
g(u, v) := \Re(\bar{u}v_1 + \bar{u}_2v_2 + \bar{u}_3v_3) = \Re(\bar{u} \cdot v)
\]
for all \( u = (u_1, u_2, u_3), \quad v = (v_1, v_2, v_3) \) in \( \tilde{\mathbb{H}}^3 \). Denote by \( S^{6,5} \) pseudosphere of signature (6, 5) in \( \tilde{\mathbb{H}}^3 \). The para-quaternionic projective plane \( \tilde{\mathbb{H}}P^2 \) is defined as a set of equivalence classes
\[
\tilde{\mathbb{H}}P^2 := \{ [u] \mid u \in S^{6,5} \}
\]
where the equivalence \( \sim \) is defined by
\[
u = (u_0, u_1, u_2) \sim (u_0h, u_1h, u_2h) = uh, \quad h \in \tilde{\mathbb{H}}_1.
\]
Denote by \( \pi : S^{6,5} \to \tilde{\mathbb{H}}P^2 \) the natural projection \( \pi(u) := [u] \). The vertical subspace \( V_u \subset T_uS^{6,5} \subset T_u\tilde{\mathbb{H}}^3 \cong \tilde{\mathbb{H}}^3 \) of the submersion \( \pi \) in the point \( u \in S^{6,5} \) is generated by the vectors \( ui, uj, uk \) i.e.
\[
V_u = \Re(\bar{u}i, \bar{u}j, \bar{u}k).
\]
Moreover, the orthogonal decomposition with respect to the metric $g$
\[ T_u S^{6.5} = V_u \oplus H_u \]
onto vertical and horizontal subspace is $\mathbb{H}_1$ invariant. We usually fix a horizontal local section $u \subset S^{6.5}$ over $\mathbb{H}P^2$. This allows us to indentify
\[ T_{[u]} \mathbb{H}P^2 \cong H_u \quad \text{by} \quad T_{[u]} \mathbb{H}P^2 \ni X \leftrightarrow \tilde{X} \in H_u \subset \mathbb{H}^3, \]
where $\tilde{X}$ is the unique horizontal lift of the vector $X$. Right multiplication of vectors form $\mathbb{H}^3$ by $i, j, k$ preserves horizontal subspace $H_u$ so we can locally define the endomorphisms
\[ J_1(X) := \pi_*(\tilde{X}i), \quad J_2(X) := \pi_*(\tilde{X}j), \quad J_3(X) := \pi_*(\tilde{X}k) \]
where the vectors $X$ and $\tilde{X}$ are related as above. They satisfy the same relations as $i, j$ and $k$, i.e.
\[ J^2 = -\epsilon_\alpha, \quad J_\alpha J_\beta = -\epsilon_\gamma J_\gamma, \]
where $(\alpha, \beta, \gamma)$ is any cyclic permutation of $(1, 2, 3)$. Although the definition of $J_1, J_2$ and $J_3$ depends on the section $u$ over $\mathbb{H}P^2$ we have globally defined subbundle $\mathbb{R}\langle J_1, J_2, J_3 \rangle$ of $\text{End}(T\mathbb{H}P^2)$ which is a pseudo-Riemannian analogue of quaternionic Kähler structure on quaternionic projective plane $\mathbb{H}P^2$. The pseudo-Riemannian metric $g$ on $S^{6.5}$ induces the pseudo-Riemannian metric
\[ \langle X, Y \rangle = g(\tilde{X}, \tilde{Y}) \]
of signature $(4, 4)$ on $\mathbb{H}P^2$ by the submersion. One can easily check that the endomorphism $J_1$ is an isometry and $J_2, J_3$ are anti-isometries. We denote the Levi-Civita connection on $S^{6.5}$ by $\tilde{\nabla}$ and by $\nabla$ the induced connection on $\mathbb{H}P^2$ with a curvature tensor $\tilde{R}$ and of the constant para-quaternionic sectional curvature $K$ (see [3]).

3. The orbit space $\mathcal{K}/G$ and its metric

Let $p, q \in \mathbb{N}, (p, q) = 1, p \neq q$ be relatively prime natural numbers. We define an isometric action $\phi$ of a group $G := \{ e^{jt} \mid t \in \mathbb{R} \} \cong (\mathbb{R}, +)$ on $\mathbb{H}P^2$ by
\[ \phi_t(u_0, u_1, u_2) := (e^{jt}u_0, e^{jt}u_1, e^{jt}u_2), \]
where $e^{jt} := \cosh t + j \sinh t$, $t \in \mathbb{R}$. The action preserves para-quaternionic structure on $\mathbb{H}P^2$. The induced Killing vector field at the point $[u] \in \mathbb{H}P^2$ is
\[ V_u := \pi_*(j(qu_0, pu_1, pu_2)). \]
Consider an algebraic submanifold $\mathcal{K}_0$ of $\mathbb{H}P^2$ defined by equation
\[ \mathcal{K}_0 := \{ [u_0, u_1, u_2] \in \mathbb{H}P^2 \mid qu_0ju_0 + pu_1ju_1 + pu_2ju_2 = 0 \}. \]
Notice that its real codimension in $\mathbb{H}P^2$ is 3 and $\dim_{\mathbb{R}} \mathcal{K}_0 = 5$.

**Lemma 3.0.1.** The action $\phi_t$ of the group $G$ preserves the submanifold $\mathcal{K}_0$. It acts freely on $\mathcal{K}_0$.

**Proof:** The manifold $\mathcal{K}_0$ is invariant under the the action of $G$ because of the definition of the definition of $\phi_{p,q}$. Suppose that the action of $G$ is not free, i.e. that for some $h \in \mathbb{H}_1$, $t \in \mathbb{R}$, $t \neq 0$ and some $u \in \mathcal{K}_0 \subset S^{6.5}$ we have
\[ (e^{jt}u_0, e^{jt}u_1, e^{jt}u_2) = \phi_t(u) = uh = (u_0h, u_1h, u_2h). \]
Using the equation of the submanifold $K_0$ we obtain

\[ 0 = q\tilde{u}_0ju_0 + p\tilde{u}_1ju_1h + p\tilde{u}_2ju_2h = q\tilde{u}_0je^{ipt}u_0 + p\tilde{u}_1je^{ipt}u_1 + p\tilde{u}_1je^{ipt}u_2 = |u_0|^2(q\sinh pt - p\sinh pt) + p\sinh pt + q(cosh pt - cosh pt)\tilde{u}_0ju_0. \]

Hence, we have $q(cosh pt - cosh pt)\tilde{u}_0ju_0 \in \mathbb{R}$. If $cosh pt = cosh pt$ then, since $t \neq 0$, we have $p = \pm q$, what is a contradiction. Otherwise, we have $\tilde{u}_0ju_0 = a \in \mathbb{R}$. If $|u_0|^2 \neq 0$ then $j = a|u_0|^{-2} \in \mathbb{R}$. It is a contradiction again. If $|u_0|^2 = 0$ then $a \neq 0$, since $p\sinh pt \neq 0$, so we have

\[ 0 = |u_0|^2|j|^2|u_0|^2 = |a|^2 \neq 0 \]

and the action of $G$ is free.

Since the action $\phi_t$ of the group $G$ preserves the algebraic submanifold $K_0$, the vector $V_u$ is tangent to $K_0$ at the point $[u] \in K_0$. One can check that the vectors

\[ J_1V_u, J_2V_u, J_3V_u \in T_u\mathbb{H}P^2 \]

are normal to the submanifold $K_0$ at the point $[u]$. In points $[u] \in K_0$ such that $|V_u|^2 \neq 0$ vectors $V_u, J_1V_u, J_2V_u, J_3V_u$ are linearly independent. Hence, in order to proceed with calculations we restrict to the subset $K$ of $K_0$ consisting of such points, i.e.

\[ K := \{[u] = [u_0, u_1, u_2] \in K_0 | n^2 := |V_u|^2 = q^2|u_0|^2 + p^2|u_1|^2 + p^2|u_2|^2 \neq 0 \}. \]

On $K$ we have the metric $\langle \cdot, \cdot \rangle$ induced from $\mathbb{H}P^2$ and the connection, curvature and sectional curvature which we denote by $\nabla, R, \kappa$, respectively. The following lemma is immediate consequence of Lemma 3.0.1.

**Lemma 3.0.2.** Set $K$ is a differentiable manifold of real dimension five. The isometric action $\phi_t$ of the group $G$ preserves $K$ and it is free on $K$. Then $K/G$ is a pseudo-Riemannian manifold of real signature $(2, 2)$.

Denote by $g_{p,q}$ the metric induced on $K/G$ by Riemannian submersion $\xi : K \to K/G$. The construction is represented by the following diagram.

\[
\begin{array}{ccccccc}
\mathbb{H}_{\text{I}} & \rightarrow & G = \mathbb{R} & \rightarrow & G = \mathbb{R} & \rightarrow & G = \mathbb{R} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^{2,1} & \supset & S^6,5 & \xrightarrow{\eta} & \mathbb{H}P^2 & \supset & K_0 & \supset & K = K_0 \setminus S & \xrightarrow{\xi} & M_{p,q} = K/G \\
\end{array}
\]

It is not difficult to check that it can be expressed in terms of global coordinates $u \in S^6,5$ in the following way

\[(\xi \circ \pi)^* (g_{p,q}) = d\tilde{u} \otimes_{\mathbb{R}} du + (d\tilde{u} \cdot u) \otimes_{\mathbb{R}} (\tilde{u} \cdot du) + \frac{(d\tilde{u} \cdot V_u) \otimes_{\mathbb{R}} (\tilde{V}_u \cdot du)}{|V_u|^2}.
\]

Notice, that the first two terms determine the metric of the para-quaternionic projective plane. The endomorphisms $J_i, i = 1, 2, 3$ of $T_u\mathbb{H}P^n$ induce the endomorphisms $\tilde{J}_i, i = 1, 2, 3$ of $T_{\xi([u])}(K/G)$, satisfying the relations (1), by

\[ \tilde{J}_i(\tilde{X}, \tilde{Y}) = J_i(X, Y), \quad \tilde{X}, \tilde{Y} \in T_{\xi([u])}(K/G), \]

where $X, Y$ are horizontal lifts of vectors $\tilde{X}, \tilde{Y}$. Moreover, $\tilde{J}_1$ is the isometry and $\tilde{J}_2, \tilde{J}_3$ are the anti-isometries with respect to the metric $g_{p,q}$. Denote the induced
connection, curvature and the sectional curvature on $K/G$ by $\nabla$, $R$ and $K$, respectively.

4. THE CURVATURE OF $K/G$

In order to study the local geometry of $K/G$ we will compute its curvature tensor in this section. The sectional curvature of the quaternionic and the para-quaternionic projective plane can be expressed as

$$\bar{K}(X,Y) = 1 + 3|Pr(Y,J_1Y,J_2Y,J_3Y)X|^2/\epsilon(X),$$

for arbitrary orthonormal tangent vectors $X$ and $Y$, where $\epsilon(X)$ denotes the sign of square length of the vector $X$ (see [3]). Now, we will compute the connection $\nabla$ and the curvature $R$ of the submanifold $K_0$ of $\tilde{H}P^2$. Let $i : K \rightarrow \tilde{H}P^2$ denotes the inclusion. For $[u] \in K$ we will denote the vectors $X \in T_uK$ and $i_*X \in T_u\tilde{H}P^2$ by the same letter $X$.

The following relation holds

$$\nabla_X Y - \nabla_Y X = B(X,Y) = \sum_{i=1}^3 \alpha_i J_i V_u,$$

where

$$\alpha_i = \frac{\epsilon_i}{n^2} \langle \nabla_X Y, J_i V_u \rangle, \quad i = 1, 2, 3.$$

For $i = 1, 2, 3$ we have

$$\langle \nabla_X Y, J_i V_u \rangle = -\langle Y, \nabla_X J_i V_u \rangle = -\langle Y, J_i AX \rangle,$$

where the skew-symmetric operator $\Lambda : T_u\tilde{H}P^2 \rightarrow T_u\tilde{H}P^2$ is defined by

$$\Lambda X := \pi_* (j(qx_0, px_1, px_2))$$

and $(x_0, x_1, x_2)$ is horizontal lift of $X \in T_u\tilde{H}P^2$ to $T_uS^{6,5}$. Using equations (3), (4) and (5) we finally obtain

$$B(X,Y) = -\frac{1}{n^2} \sum_{i=1}^3 \epsilon_i \langle Y, J_i AX \rangle J_i V_u.$$

Using the relation

$$\bar{K'}(X,Y) = \bar{K}(X,Y) - \frac{1}{Q}(\langle B(X,Y) \rangle^2 - \langle B(X,Y), B(Y,Y) \rangle),$$

$$Q := Q(X,Y) = |X|^2 |Y| - \langle X, Y \rangle^2,$$

between second fundamental form and sectional curvature we obtain

$$\bar{K'}(X,Y) = \bar{K}(X,Y) + \frac{1}{n^2 Q} \sum_{i=1}^3 \epsilon_i \left( -\langle Y, J_i AX \rangle^2 + \langle X, J_i AX \rangle \langle Y, J_i AY \rangle \right),$$
where the sectional curvature $\bar{K}$ is constant. As the second step, we are going to compute the sectional curvature $\bar{K}$ on $\mathcal{K}/G$ using O'Neill’s formula for submersion

$$K(\hat{X}, \hat{Y}) = \bar{K}(X, Y) + \frac{3}{4} \frac{\langle v[X, Y], v[X, Y] \rangle}{Q(X, Y)}, \quad \hat{X}, \hat{Y} \in T_{\xi([u])}(\mathcal{K}/G)$$

where $v[X, Y]$ denotes the vertical component of the commutator $[X, Y]$ and $X, Y$ are the unique horizontal lifts of the vectors $\hat{X}, \hat{Y}$ to $T_{[u]}\mathcal{K}$. Since the vertical space of the submersion is generated by the vector $V_u$, we have

$$v[X, Y] = \frac{1}{n^2} \langle [X, Y], V_u \rangle V_u = \frac{1}{n^2} (\langle \nabla_X Y, V_u \rangle - \langle \nabla_Y X, V_u \rangle) V_u = \frac{1}{n^2} (-\langle Y, AX \rangle + \langle X, AY \rangle) V_u = \frac{2}{n^2} \langle X, AY \rangle V_u.$$

Using (6) the sectional curvature of $\mathcal{K}/G$ is

$$K(\hat{X}, \hat{Y}) = \bar{K}(X, Y) + \frac{1}{n^2 Q} \left[ \sum_{i=1}^{3} c_i \left( \langle X, J_i \Lambda X \rangle \langle Y, J_i \Lambda Y \rangle - \langle Y, J_i \Lambda X \rangle^2 \right) + 3 \langle X, AY \rangle^2 \right]$$

and $X, Y$ are related to $\hat{X}, \hat{Y}$ as above. Then, the Jacoby operator is

$$K_X(\hat{Y}) := R(\hat{X}, \hat{Y}) \hat{X} = h(\bar{R}(X, Y) X) + \frac{1}{n} \left[ \sum_{i=1}^{3} c_i \langle X, J_i \Lambda X \rangle h(J_i \Lambda Y) + \langle X, J_i \Lambda Y \rangle h(J_i \Lambda X) - 3 \langle Y, AX \rangle h(\Lambda X) \right],$$

where $h(X)$ denotes the horizontal part of the vector $X \in T_{[u]}\mathcal{H}P^2$, i.e. the orthogonal projection of the vector $X$ onto the space $R\langle V_u, J_1 V_u, J_2 V_u, J_3 V_u \rangle^\perp$.

5. The local geometry of $\mathcal{K}/G$

In this section we shall use the curvature of $\mathcal{K}/G$ to describe its interesting geometric properties. First, we recall the definition of the Osserman manifold $(M, g)$. Let $m \in M$ and $X \in T_m M$ be a vector such that $|X|^2 = 1$. Jacoby operator $K_X$ at the point $m \in M$ and in the direction $X$ is defined by

$$K_X(Y) := R(X, Y) X,$$

where $R$ is the curvature tensor of $(M, g)$. Jacoby operator is a self-adjoint operator on $T_m M$. Manifold $M$ is called pointwise Osserman if the Jordan form of the Jacoby operator $K_X$ does not depend on the unit direction $X$ and (globally) Osserman if the Jordan form of $K_X$ does not depend both on the point $m \in M$ and the unit direction $X \in T_m M$.

**Lemma 5.0.3.** The manifold $\mathcal{K}/G$ is pointwise Osserman of the neutral signature.

Proof: Let $X \in T_{[u]}\mathcal{K}$ denote the horizontal lift of a vector $\hat{X} \in T_{\xi([u])}(\mathcal{K}/G)$. For any unit vector $\hat{X} \in T_{\xi([u])}(\mathcal{K}/G)$ (i.e. with square norm equals 1) the basis $(\hat{X}, J_1 \hat{X}, J_2 \hat{X}, J_3 \hat{X})$ is pseudo-orthonormal. The vector $\hat{X}$ is the eigenvector corresponding to the eigenvalue 0 of the self-adjoint operator $K_{\hat{X}}$. Thus, we are interested only in the restriction of the operator $K_{\hat{X}}$ to the orthogonal complement
The restriction of $K_{\tilde{X}}$ in the basis $(\tilde{J}_1 \tilde{X}, \tilde{J}_2 \tilde{X}, \tilde{J}_3 \tilde{X})$ is represented by the following matrix

$$K = \frac{2}{n^2} \begin{pmatrix}
2a^2 + b^2 + c^2 - \frac{2n^2}{2} & -3ab & -3ac \\
3ab & -a^2 - 2b^2 + c^2 - \frac{2n^2}{2} & -3bc \\
3ac & -3bc & -a^2 + b^2 - 2c^2 - \frac{2n^2}{2}
\end{pmatrix},$$

where $a, b, c$ are the coordinates of the horizontal component $h(\Lambda X)$ of the vector $\Lambda X$ in the basis $(J_1 X, J_2 X, J_3 X)$, i.e.

$$a := \langle X, J_1 \Lambda X \rangle, \quad b := \langle X, J_2 \Lambda X \rangle, \quad c := \langle X, J_3 \Lambda X \rangle,$$

and $\bar{c}$ is the constant para-quaternionic sectional curvature of the corresponding projective plane. By a direct computation we find that the eigenvalues of the matrix $K$ are

$$\lambda_1 = \frac{2}{n^2}(-a^2 + b^2 + c^2) + \bar{c}, \quad \lambda_2 = -\frac{2}{n^2}(a^2 + b^2 + c^2) + \bar{c}, \quad \lambda_3 = \frac{4}{n^2}(-a^2 + b^2 + c^2) + \bar{c}.$$

To show that the space $\mathcal{K}/G$ is the pointwise Osserman we have to show that the eigenvalues $\lambda_i$ are independent of the chosen unit direction $\tilde{X}$, i.e. of its horizontal unit lift $X \in T_{[\bar{c}]K}$. Notice that $-a^2 + b^2 + c^2 = |h(\Lambda X)|^2$. Let $e \in T_{[\bar{c}]K}, |e|^2 = 1$ be a fixed unit vector and $(e, J_1 e, J_2 e, J_3 e)$ the corresponding pseudo-orthonormal basis. Any horizontal unit vector $X$ can be written in the form

$$X = X_0 e + X_1 J_1 e + X_2 J_2 e + X_3 J_3 e, \quad X_0^2 + X_1^2 - X_2^2 - X_3^2 = 1.$$ 

Now one can check directly that

$$-a^2 + b^2 + c^2 = |h(\Lambda X)|^2 = |h(e)|^2 = const$$

and hence the eigenvalues of the Jacobi operator $K_{\tilde{X}}$ are constant at a given point. Moreover, one can check that the Jacobi operator is diagonalizable what completes the proof of the lemma.

**Theorem 5.1.** The manifold $\mathcal{K}/G$ is Einstein, self-dual and not locally homogeneous of the signature $(- - + +)$.

**Proof:** In [2], the self-dual, Einstein manifold of neutral signature are characterized as the pointwise-Osserman manifolds. Then Lemma 5.0.3 implies that $\mathcal{K}/G$ is Einstein and self-dual. A pointwise-Osserman, locally homogenous manifold is Osserman, i.e. the Jordan normal form of its Jacobi operator is also independent of the point. But for our examples, we can check that the eigenvalues of the Jacoby operator, $\lambda_1$, $\lambda_2$ and $\lambda_3$ are not constant on $\mathcal{K}/G$. After some computations, where a few unexpected cancelations happened, we obtain

$$\lambda_1 = \lambda_2 = \frac{2p^4q^2}{n^6} - 4, \quad \lambda_3 = \frac{4p^4q^2}{n^6} - 4.$$ 

Clearly, $n^2$ is not a constant along $\mathcal{K}/G$ ($\mathcal{K}/G$ is not globally Osserman). Thus, the manifold $\mathcal{K}/G$ is not locally homogenous, and hence not locally symmetric. □

**Remark:** It is interesting that there exist a three-dimensional submanifold of $\mathcal{K}/G$ along which the eigenvalues of the Jacoby operator are constant.

**Remark:** In the case $p = 1 = q$ there are no singular points, so $\mathcal{K} = \mathcal{K}_0$, and the action of the group $G$ on $\mathcal{K}$ is free. However, since $\mathcal{K}/G$ is a complete manifold of neutral signature, one can check that vector $\Lambda X$, $X \in T_{[\bar{c}]K}$, is horizontal and $\bar{c} = 4$, so $\lambda_1 = 2 = \lambda_2$, $\lambda_3 = 8$, i.e. the eigenvalues of the Jacoby operator are constant over $\mathcal{K}/G$. This implies that $M_{1,1} = \mathcal{K}/G$ is Einstein, self-dual, globally
Osserman neutral manifold. Moreover, it is isometric to para-complex projective plane, where para-complex structure is induced by the left multiplication by the generator \( j \) (see [4]).

6. Study of the Galicki and Lawson’s example

We are coming back to the starting point for our work, the compact Riemannian orbifolds

\[ \mathcal{O}_{q,p} = \mathcal{L}_H / S^1, \]

constructed and studied by Galicki and Lawson [7]. In this section we will discuss some additional properties of \( \mathcal{O}_{p,q} \) concerning to its local geometry and curvature. Particularly, we will see that the orbifolds \( \mathcal{O}_{p,q} \) are not locally homogeneous for any \( p, q \) and the orbifolds are of positive sectional curvature for some values of \( p, q \). Also, an estimate of the pinching constant, ratio of the minimal and maximal sectional curvature, is given. In [7] was shown that \( \mathcal{O}_{q,p}(n - 1) \) is not locally symmetric for \( 0 < q/p < 1 \). Here, for \( n = 2 \) we will give another proof of this result based on the study of its curvature operators (as in Section §5). This approach provides an opportunity to see that \( \mathcal{O}_{p,q}(1) \) is not even locally homogeneous.

**Theorem 6.1** ([7]). The orbifold \( \mathcal{O}_{q,p} = \mathcal{L}_H / S^1 \) is an Einstein, self-dual manifolds which is not locally homogeneous for any \( p, q \), \( (p, q) \neq 1 \).

Proof: The proof is based on the characterization of the Einstein, self-dual manifolds as manifolds with constant eigenvalues of the Jacobi operators (pointwise Osserman manifolds), obtained by Vanhecke and Sekigawa [10]. Repeating the long and nice computations from the neutral signature case (under some minor modifications), we express the eigenvalues of the Jacobi operator, \( K_{\hat{X}} : Y \to R(\hat{X}, Y)\hat{X} \), in very simply way

\[
\lambda_1 = \lambda_2 = \frac{2p^4q^2}{|\Lambda u|^2} - 4, \quad \lambda_3 = -\frac{4p^4q^2}{|\Lambda u|^2} - 4,
\]

at the orbit determined by \( [u] \in \mathcal{L}_H \) and the arbitrary unit tangent vector \( \hat{X} \). In this computations operator \( \Lambda : T_{[u]}\mathbb{H}P^2 \to T_{[u]}\mathbb{H}P^2 \) plays an important role. It is important to notice that we express the eigenvalues in the terms of the quaternionic notation. By the argument as in the proof of Theorem 5.1 proof is completed. □

Now, we will study the cases when the sectional curvature of the orbifold \( \mathcal{O}_{p,q}(1) \) is positive.

**Lemma 6.1.1.** Let \( \hat{X} \) and \( \hat{Y} \) be the unit orthogonal tangent vectors. For the sectional curvature \( K(\hat{X}, \hat{Y}) \) of the orbifold \( \mathcal{O}_{p,q}(1) \), for \( p \leq q \), holds

\[
4 - \frac{2q^2}{p^2} \leq K(\hat{X}, \hat{Y}) \leq 4 + \frac{4q^2}{p^2},
\]

and for \( p \geq q \)

\[
4 - \frac{2p^4}{q^4} \leq K(\hat{X}, \hat{Y}) \leq 4 + \frac{4p^4}{q^4}.
\]

Proof: For \( p \leq q \) we have \( p^2 \leq |\Lambda u|^2 \leq q^2 \) and \( -\lambda_1 \geq 4 - \frac{2q^2}{p^2} \), \( -\lambda_3 \leq 4 + \frac{4q^2}{p^2} \).
Since $-\lambda_1 \leq K(x,y) \leq -\lambda_3$, the relation (5) holds. Similarly, for $p \geq q$ we have $q^2 \leq |\Delta u|^2 \leq p^2$ and

$$-\lambda_1 \geq 4 - \frac{2p^4}{q^4}, \quad -\lambda_3 \leq 4 + \frac{4p^4}{q^4}.$$ 

As in the previous case this completes the proof the Lemma. \(\square\)

The direct consequences are the following results.

**Theorem 6.2.** The orbifold $O_{p,q}(1)$ is of the positive sectional curvature for $p^2 < \sqrt{2}q^2 < 2\sqrt{2}p^2$.

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**Lemma 6.2.1.** For $p^2 < \sqrt{2}q^2 < 2\sqrt{2}p^2$, let the sectional curvature $K(\tilde{X}, \tilde{Y})$ of the orbifold $O_{p,q}(1)$ is $k$-pinched, $0 < k < 1$. Then, for $p \leq q$, in arbitrary point

$$\frac{1}{4} - \frac{3q^2 - p^2}{4p^2 + q^2} \leq k \leq \frac{1}{4} + \frac{3q^4 - p^4}{4p^4 + q^4},$$

and, for $p \geq q$

$$\frac{1}{4} - \frac{3p^4 - q^4}{4p^4 + q^4} \leq k \leq \frac{1}{4} + \frac{3p^2 - q^2}{4p^2 + q^2}.$$ 

For $p \to q$, $k \to 1/4$. Moreover, $k = 1/4$ on $O_{p,q}(1)$ if and only if $p = q = 1$. \(\square\)

**Remark:** Orbifold $O_{1,1}(1)$ is globally Osserman and of constant holomorphic sectional curvature with respect to the complex structure induced by the left multiplication by $j$. (see [3]).

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