Metastable dynamics of internal interfaces for a convection-reaction-diffusion equation

Marta Strani

Université Paris Diderot, Institut de Mathématiques de Jussieu, 75013 Paris, France
E-mail: martastri@gmail.com

Received 27 September 2014, revised 26 September 2015
Accepted for publication 12 October 2015
Published 6 November 2015

Abstract
We study the one-dimensional metastable dynamics of internal interfaces for the initial boundary value problem for the following convection-reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \varepsilon \Delta^2 u - \partial f(u) + f'(u).$$

Metastable behaviour appears when the time-dependent solution develops into a layered function in a relatively short time, and subsequently approaches its steady state in a very long time interval. A rigorous analysis is used to study such behaviour by means of the construction of a one-parameter family \(\{U(x; \xi)\}_{\xi}\) of approximate stationary solutions and of a linearisation of the original system around an element of this family. We obtain a system consisting of an ODE for the parameter \(\xi\), describing the position of the interface coupled with a PDE for the perturbation \(v\) and defined as the difference \(v := u - U\). The key of our analysis are the spectral properties of the linearised operator around an element of the family \(\{U\}_{\xi}\): the presence of a first eigenvalue, small with respect to \(\varepsilon\), leads to metastable behaviour when \(\varepsilon \ll 1\).

Keywords: metastability, slow motion of interfaces, internal layers, spectral analysis
Mathematics Subject Classification numbers: 35K20, 35B36, 35B40, 35P15

(Some figures may appear in colour only in the online journal)
1. Introduction

The slow motion of internal shock layers has been widely studied for a large class of evolutive PDEs in the form

$$\partial_t u = \mathcal{P}[u],$$

where $\mathcal{P}[u]$ is a nonlinear differential operator that depends singularly on the parameter $\varepsilon$. This phenomenon is known as metastability. The qualitative features of metastable dynamics are the following: through a transient process a pattern of internal layers is formed from initial data over a $O(1)$ time interval; once this pattern is formed, the subsequent motion of these interfaces is exponentially slow, converging to their asymptotic limit. As a consequence, two different time scales emerge: for short times, the solutions are close to a non-stationary state; subsequently, they drift towards the equilibrium solution with a rate that is exponentially small.

In other words, the equation exhibits in finite time metastable shock profiles (called interfaces) that persist during an exponentially (with respect to a small parameter) long time period and that move at an exponentially slow speed.

Many fundamental partial differential equations concerning different areas exhibit such behaviour. Among others, we include viscous shock problems (see, for example [18, 19, 22] and [29] for viscous conservation laws, and [6, 33] for Burgers-type equations), relaxation models such as the Jin-Xin system [31], phase transition problems described by the Allen-Cahn equation with the fundamental contributions [8, 11] and the most recent references [26, 32], and the Cahn-Hilliard equation studied in [1] and [27].

In this paper we study the slow motion of internal interfaces generated by the evolution of the solution to a convection-reaction-diffusion equation. Given $\ell > 0$, we consider the initial-boundary value problem

\begin{align*}
\frac{\partial u}{\partial t} &= \varepsilon \frac{\partial^2 u}{\partial x^2} - \partial_t f(u) + f'(u), & x \in I, t > 0, \\
u(0, t) &= u(\ell, t) = 0, & t > 0, \\
u(x, 0) &= u_0(x), & x \in I,
\end{align*}

(1.1)

where $I = [0, \ell]$ is a bounded interval of the real line, and the unknown $u \in C^0([0, \ell]; H^1(I))$. Here $\varepsilon$ is a small and positive parameter that can be seen as a viscosity coefficient, while $f$ satisfies

$$f''(u) > 0, \quad f(0) = f'(0) = 0, \quad f'(-u) = -f'(u).$$

(1.2)

The main example we have in mind is the initial-boundary value problem for the generalised Burgers equation, also known as the Burgers–Sivashinsky equation, which is

$$\partial_t u = \varepsilon \frac{\partial^2 u}{\partial x^2} - u \partial_t u + u,$$

(1.3)

together with the boundary conditions and initial datum $u_0(x)$. This equation arises from the study of the dynamics of an upwardly propagating flame-front in a vertical channel (see [28]): precisely, setting $u(x, t) = -\partial_t y(x, t)$, the dimensionless shape $y = y(x, t)$ of the flame front interface satisfies

\begin{align*}
\partial_t y &= \frac{1}{2}(\partial_t y)^2 + \varepsilon \partial_{xx} y + y - \int_0^\ell y \, dx, & x \in (0, \ell), & t > 0, \\
\partial_t y(0, t) &= \partial_t y(\ell, t) = 0, & t > 0, \\
y(x, 0) &= y_0(x), & x \in (0, \ell).
\end{align*}

(1.4)
We mean to analyse the behaviour of solutions $u^\varepsilon$ to (1.1) in the vanishing viscosity limit, i.e. $\varepsilon \to 0$. In particular, we address the question as to whether a phenomenon of metastability occurs for this kind of problem, and how this special dynamics is related to the size of the viscosity coefficient and to the choice of the initial datum $u_0$.

In the limit $\varepsilon \to 0$, equation (1.1) formally reduces to the first order hyperbolic equation

$$\partial_t u = -\partial_x f(u) + f'(u), \quad u(x, 0) = u_0(x). \quad (1.5)$$

The set of solutions for this equation is the one given by the entropy formulation, in the sense of Kruzkov (see [16]). In this case the existence and uniqueness of the solution $U_{\text{visc}}(x, t)$ to the Cauchy problem for (1.5) (i.e. $x \in \mathbb{R}$) is well known. Moreover, denoting by $u^\varepsilon$ the solution to (1.1), it is possible to prove that

$$u^\varepsilon(x, t) \to U_{\text{visc}}(x, t) \quad \text{as} \quad \varepsilon \to 0,$$

(for more detail, see [17, 20, 25]). Finally, this convergence is in $L^1_{\text{loc}}$, and it is uniform away from shock waves, as proven in [24].

When Dirichlet boundary conditions are taken into consideration, the case of a bounded domain is more delicate than the Cauchy problem. First of all, the boundary conditions $u(0, t) = u(\ell, t) = 0$ have to be interpreted in a non-classical way, in the sense of [3].

Concerning the existence results and asymptotic behaviour for the solutions to the boundary-value-problem for (1.5), a complete analysis was performed by Mascia and Terracina in [21]; here the authors deal with a general reaction-convection equation of the form

$$\begin{cases}
\partial_t u = -\partial_x f(u) + g(x, u), & x \in (0, \ell), t \geq 0, \\
u(0, t) = u_-, \quad u(\ell, t) = u_+, & t \geq 0, \\
u(x, 0) = u_0(x), & x \in (0, \ell),
\end{cases} \quad (1.6)$$

showing how the boundary conditions influence the large-time behaviour of solutions. Moreover, they prove the presence of discontinuous stationary solutions for (1.6) that correspond to stationary solutions with internal layers for the viscous problem when $\varepsilon > 0$.

When $\varepsilon > 0$, the presence of a diffusive term has a smoothing effect on solutions. Indeed, all the discontinuities turn into smooth internal layers that are less sharp as $\varepsilon$ increases.

Numerical computations show that, for a certain class of initial data, equation (1.3) exhibits a metastable behaviour (see [33]). In [5] it was proven that a sufficient condition for the appearance of metastable dynamics is that the initial datum satisfies

$$u_0(x) < 0 \text{ in } (0, a_0), \quad u_0(x) > 0 \text{ in } (a_0, \ell) \quad \text{for some} \quad a_0 \in (0, \ell), \quad (1.7)$$

where $u_0(x)$ has to be considered in $C^0_0(I)$.

Starting from this initial configuration, one observes that an interface is formed in an $O(1)$ time scale. Once this interface is formed, it starts to move towards one of the walls $x = 0$ or $x = \ell$, but this motion is extremely slow (see figure 1). Hence, two different time scales emerge: a first transient phase of order $O(1)$ in time where the interface is formed, and an exponentially long time phase, which can be extremely long provided $\varepsilon$ is very small, where the interface drifts towards its equilibrium configuration.

In terms of the shape $y(x, t)$, at the first stage of its dynamics the solution assumes a somewhat asymmetric parabolic shape. In particular, the tip of this parabola corresponds to the point where $u$ vanishes. The subsequent motion of the tip of the parabolic flame-front interface towards one of the walls $x = 0$ or $x = \ell$ can be extremely slow with respect to the parameter $\varepsilon$.

There are several papers concerning the dynamics of solutions to equation (1.1), which is a special case of a more general class of convection-reaction-diffusion equations in the form
The initial-boundary value problem in different dimensional sets for (1.8) has been investigated in a number of works under different assumptions for the functions \( f \) and \( g \). To name some of these papers, we recall here [7, 9, 13–15].

A pioneering article in the study of the metastable dynamics of solutions for the convection-reaction-diffusion equation (1.1) is [6]. Here the authors analyse the specific case \( f(u) = u^2/2 \) (Burgers–Sivashinsky equation), proving that there exist different types of stationary solutions: more precisely, there exist one positive stationary solution \( \varepsilon^+ \) and one negative stationary solution \( \varepsilon^- \) that are linearly stable, and two other unstable equilibrium solutions \( \varepsilon^\pm \) that have exactly one zero inside the interval.

Moreover, it turns out that the stationary solution \( U_{\varepsilon,1}^- \) (that, in terms of the shape \( y \), corresponds to a parabolic-shaped flame-front interface) is metastable. Indeed, concerning the time-dependent problem in [6], the authors rigorously prove that, for the particular class of initial data considered in [5], the solutions generated by such initial configurations exhibit metastable behaviour in the sense that they remain close to the initial configuration for a time of order \( T \approx e^{\varepsilon c} \), \( c > 0 \) before converging to one of the stable steady states \( U_{\varepsilon}^\pm \).

This slow motion is a consequence of the presence of a first small eigenvalue associated with the linearisation around the unstable stationary solution \( U_{\varepsilon,1}^- \); in particular, \( \lambda_1^- \) is positive but exponentially small in \( \varepsilon \). Therefore, starting from initial data that are a small perturbation of such an unstable steady state, the corresponding time-dependent solution starts to move towards one of the stable equilibrium configurations \( U_{\varepsilon}^\pm \), but this motion is extremely slow, since it is described in terms of order \( e^{\lambda_1^- t} \).

\[
\partial_t u = \varepsilon \partial_x^2 u - f(x, u) \partial_x u + g(x, u). \tag{1.8}
\]
Hence, different from the cases considered in [22, 23, 31], here the metastable behaviour is characterised by the fact that the steady state $U_{\epsilon,1}$ is unstable, so that the solutions starting from an initial configuration close to $U_{\epsilon,1}$ are pushed away towards their asymptotic limit, but the time of convergence can be extremely long.

The problem of slow motion for equation (1.1) with a generic flux function $f(u)$ satisfying hypotheses (1.2) was examined in [33]. Here the authors showed that the first eigenvalue associated with the linearised problem around the equilibrium solution $U_{\epsilon,1}$ is exponentially small with respect to $\epsilon$; more precisely, they provided an asymptotic expression for such a principal eigenvalue. The presence of a first small eigenvalue leads to metastable behaviour for the time-dependent problem, studied by using the so-called projection method: the authors were able to derive an (asymptotic) ordinary differential equation describing the motion of the unique zero of the solution $u$ that corresponds, in terms of the shape $y$, to the tip location of a parabolic-shaped interface.

To the best of our knowledge, it is in [33] that the study of metastability for the general problem (1.1) is first addressed.

The existence and the stability properties of stationary solutions to (1.1) that have more than one zero inside the interval and the corresponding time-dependent problem were conversely studied in [12].

The aim of this paper is to prove the existence results for the stationary solutions to (1.1) in the case of a generic flux function $f(u)$, and to rigorously study the subsequent motion of the solutions starting from the initial data in form (1.7).

The first part of our study is thus concerned with the stationary problem

$$
\varepsilon \partial^2_t u = \partial_x f(u) - f'(u),
$$

complemented by boundary conditions $u(0) = u(\ell) = 0$. Our first main result gives a description of the solutions to (1.9).

**Theorem 1.1.** There exist one positive solution $U_{\epsilon,+}$ and one negative solution $U_{\epsilon,-}$ to (1.9). Additionally,

- For $\varepsilon \to 0$, $U_{\epsilon,+}$ converges pointwise to $x$ in $(0, \ell)$.
- For $\varepsilon \to 0$, $U_{\epsilon,-}$ converges pointwise to $x - \ell$ in $(0, \ell)$.

Moreover, there exist two other solutions $U^{r}_{m}$ and $U^{r}_{ns}$ that have one zero inside the interval $(0, \ell)$ and such that

- For $\varepsilon \to 0$, $U^{r}_{m}$ converges pointwise to $x - \ell/2$ in $(0, \ell)$.
- For $\varepsilon \to 0$, $U^{r}_{ns}$ converges pointwise to the function

$$
U^{r}_{ns}(x) := \begin{cases} 
  x & x \in (0, \ell/2), \\
  x - \ell & x \in (\ell/2, \ell),
\end{cases}
$$

for $x \neq \ell/2$.

Concerning the long-time dynamics of the solutions to the time-dependent problem (1.1), the strategy we mean to use here is analogous to the one first performed in [22] to study the slow motion of internal layers for parabolic evolutive systems, under appropriate assumptions on the spectrum of the linearised operator around the steady state. In particular, in [22] it was required that the spectrum of such a linearised operator was composed of real and negative eigenvalues (for more detail see [22, section 2]). For the problem studied in this paper, we know that there exists a first positive eigenvalue that is exponentially small in $\epsilon$, so that the
stationary solution is unstable but it is metastable, as already stressed. Hence, we need to slightly modify the assumptions we have to require. This is the strategy we mean to follow.

- We ask for the existence of a one-parameter family of functions \( \{ U^\varepsilon(x; \xi) \}_{\xi \in I} \), where the parameter \( \xi \) represents the unique zero of the function \( U^\varepsilon \), i.e. the location of the interface, and such that each element of the family can be seen as an approximation of the unstable steady state \( U_m^\varepsilon \) in a sense that we will specify later.
- We linearise around an element of the family by looking for a solution to (1.1) in the form \( u(x, t) = U^\varepsilon(x; \xi(t)) + v(x, t) \).
- We use a modified version of the projection method in order to obtain a coupled system for the perturbation \( v \) and for the interface location \( \xi \).
- We prove that, under appropriate assumptions on the spectrum of the linearised operator, metastable behaviour occurs.

This strategy dates back to the work of Carr and Pego [8]: here the authors considered a function \( u^\varepsilon(x) \) (with \( \xi = (\xi_1, \ldots, \xi_N) \) describing the position of the interfaces) which approximates a metastable state with \( N \) transition layers. The admissible layer positions lie in a set \( \Omega_p \), where \( \xi_j - \xi_{j-1} \geq \varepsilon/p \), so that the set of states \( u^\varepsilon \) forms an \( N \)-dimensional manifold \( \mathcal{M} = \{ u^\varepsilon : \xi \in \Omega_p \} \). To study the dynamics of the solutions located near \( \mathcal{M} \), the authors linearised around an element of the manifold and studied spectral properties of the linearised operator.

Starting from there, the strategy of constructing invariant manifolds of approximate steady states were widely used to describe the slow motion of solutions to different PDEs. We quote here [4], as well as the more recent contributions from [22, 31, 32].

The projection method was also employed in [33]: here the authors utilised this approach in order to give an explicit asymptotic characterisation of the metastable motion of the interface, by deriving an asymptotic ordinary differential equation describing its slow motion (see [33, section 4]). However, let us point out that although the analysis in [33] is not rigorous, the ODE derived therein characterises the metastable dynamics and is favourably compared with numerical simulations.

In our framework, the main difference with respect to [33] is that here we derive a rigorous ODE for the position of the interface, where the nonlinear terms are also taken into account; these terms keep track of the nonlinear evolution of the variable \( \xi \) when starting far away from its equilibrium configuration.

Even though this ODE is rigorous, in the specific example proposed in section 5 it leads to a non-optimal rate of speed for the interface location, since it shows an algebraic in the epsilon speed of the layer, in contrast to the expected exponentially slow speed in epsilon that was anticipated in [33] (for more detail see remark 5.1 and the subsequent discussion); hence, to provide rigorous proof of the result derived in [33] is still an open problem.

Finally, the projection method allows us to derive an explicit PDE for the perturbation \( v \); in order to use a spectral decomposition for \( v \), we exploit the spectral properties of the linearised operator and we achieve a rigorous result by using in depth the structure of the equation.

Going into greater detail, these are the objects we will use in the following and the hypotheses we need to state.

**H1. Hypotheses on the family of approximate steady states**

There exists a family \( \{ U^\varepsilon(x; \xi) \}_{\xi \in I} \) such that

1. There exists a value \( \bar{\xi} \in I \) such that the element \( U^\varepsilon(x; \bar{\xi}) \) corresponds to a stable steady state for the original equation.
(ii) For every $\xi \in I$, each element of the family satisfies
$$U^e(x; \xi) > 0 \quad \text{for} \quad x \in (0, \xi) \quad \text{and} \quad U^e(x; \xi) < 0 \quad \text{for} \quad x \in (\xi, \ell).$$

(iii) There exists a family of smooth and positive functions $\Omega^e(\xi)$, uniformly converging to zero as $\varepsilon \to 0$, such that there holds
$$|\langle \psi(\cdot), \mathcal{P}^e[U^e(\cdot; \xi)] \rangle| \leq \Omega^e(\xi) \| \psi \|_{L^\infty}, \quad \forall \, \psi \in C(I), \; \forall \, \xi \in I.$$

(iv) There exists a family of smooth positive functions $\omega^e = \varepsilon \Omega^e(\xi)$, uniformly converging to zero as $\varepsilon \to 0$, such that
$$\Omega^e(\xi) \leq \omega^e(\xi) |\xi - \xi^e|.$$

**H2. Hypothesis on the eigenvalues and on the eigenfunctions of the linearised operator**

Let $L^e_\xi$ be the linearised operator obtained from the linearisation of equation (1.1) around an element of the family $\{U^e(x; \xi)\}$. Also, let $\{\lambda^e_\xi(\xi)\}_{k \in \mathbb{N}}$ be the sequence of the eigenvalues of $L^e_\xi$, and let $\varphi^e_k(\cdot; \xi)$ and $\psi^e_k(\cdot; \xi)$ be the eigenfunctions of $L^e_\xi$ and its adjoint $L^e_\xi^*$, respectively.

(i) The sequence of eigenvalues $\{\lambda^e_\xi(\xi)\}_{k \in \mathbb{N}}$ is such that
- $\lambda^e_\xi(\xi) \to 0$ as $\varepsilon \to 0$ uniformly with respect to $\xi$.
- All the eigenvalues $\{\lambda^e_k\}_{k \geq 2}$ are negative and there exist constants $C, C'$ such that
  $$\lambda^e_1(\xi) - \lambda^e_2(\xi) \geq C' \quad \forall \, \xi \in I, \quad \lambda^e_k(\xi) \leq - Ck^2, \quad \text{if} \quad k \geq 2.$$

(ii) The eigenfunctions $\varphi^e_k(\cdot; \xi)$ and $\psi^e_k(\cdot; \xi)$ are normalised so that
$$\langle \psi^e_j(\cdot; \xi), \partial U^e(\cdot; \xi) \rangle = 1 \quad \text{and} \quad \langle \psi^e_j, \varphi^e_k \rangle = \begin{cases} 1 & \text{if} \ j = k, \\ 0 & \text{if} \ j \neq k, \end{cases}$$

and we assume
$$\sum_j \langle \partial \psi^e_k, \varphi^e_j \rangle^2 = \sum_j \langle \psi^e_j, \partial \varphi^e_k \rangle^2 \leq C \quad \forall \, k.$$
Under these hypotheses, as stated above, we are able to prove that the solution to the initial-boundary-value problem (1.1) experiences metastable behaviour. Precisely, we describe the slow motion of the solution by describing the behaviour of the perturbation $v$ and of the interface location $\xi(t)$; first, we consider a simplified partial differential equation for the perturbation $v$, where the higher order terms arising from the linearisation are cancelled out.

Our first main contribution is the following

**Theorem 1.3.** Let $u(x, t) = v(x, t) + U'(x; \xi(t))$ be the solution of the initial-boundary value problem (1.1). Let us assume that hypotheses $H1–H2$ are satisfied. Hence, for sufficiently small $\varepsilon$, there exists a time $T^*$ of order $\varepsilon^{\delta}$, $\delta > 0$, such that, for $t \leq T^*$, the following bounds hold

$$
|v|_{L^2} \leq c_1 |\nu_0|_{L^2} e^{-ct} + c_2 |\Omega^*|_{L^\infty} 
$$

and

$$
|\xi(t) - \bar{\xi}| \leq |\xi_0|_{L^\infty} e^{-\beta^* t}, \quad \beta^* \to 0 \text{ as } \varepsilon \to 0.
$$

**Remark 1.4.** Theorem 1.3 states that the perturbation $v$ has a very fast decay in time up to a reminder that is small in $\varepsilon$, so that the solution $u$ to (1.1) is drifting to its equilibrium configuration at a rate of speed dictated by $\beta^*$; hence, the convergence towards the steady state is much slower as $\varepsilon$ becomes smaller.

Subsequently, we consider the complete system for the perturbation $v$, where the higher order terms are also taken into account. This leads to the second main contribution of this paper

**Theorem 1.5.** Let $u(x, t) = v(x, t) + U'(x; \xi(t))$ be the solution of the initial-boundary value problem (1.1). Let us assume that hypotheses $H1–H2$ are satisfied. Then, for sufficiently small $\varepsilon$, there exists a time $T^*$ of order $1/\varepsilon^\alpha$ for some $\alpha \in (0, 1)$ such that, for $t \leq T^*$, the following bounds hold

$$
|v|_{H^1} \leq c_1 |\nu_0|_{H^1} e^{-ct} + c_2 e^{\delta t}, \quad \text{and} \quad |\xi(t) - \bar{\xi}| \leq |\xi_0|_{H^1} e^{-\beta^* t}, \quad -\beta^* \to 0 \text{ as } \varepsilon \to 0,
$$

where $\delta \in (0, 1)$.

**Remark 1.6.** As we will see in detail in the following sections, the nonlinear terms in the equation for the perturbation $v$ also depend on the first space derivative, so that, in order to prove theorem 1.5, an additional bound for the $H^1$-norm of $v$ is needed. This is the reason the final bound for $v$ is weaker than the corresponding formula stated in theorem 1.3. Also, the final time $T^*$ is diverging to infinity as $\varepsilon^\alpha$, $\alpha \in (0, 1)$, rather than $e^{-\varepsilon^\delta}$.

As a direct consequence of theorems 1.3 and 1.5, we can state the following corollary concerning the solution to the initial-boundary-value problem (1.1).

**Corollary 1.7.** Let $u(x, t)$ be the solution to (1.1), with initial datum $u_0$ on the form (1.7). If $a_0 \in (0, \epsilon/2)$, then $u(x, t)$ converges to $U_{-\varphi}(x)$ for $t \to +\infty$. Conversely, if $a_0 \in (\epsilon/2, \epsilon)$, then $u(x, t)$ converges to $U_{-\varphi}(x)$ for $t \to +\infty$. In both cases, the rate of speed of convergence is given by $\beta^*$, and the time of convergence is proportional to $T^*$.

The main difference with respect to the other papers that have considered the problem of metastability for equation (1.1) is that here we are dealing with a generic flux function $f$ that satisfies hypotheses (1.2); also, we develop a general theory to rigorously prove the slow motion of the internal interfaces, which could also be applicable to other types of convection-reaction-diffusion equations and, hopefully, to the case of systems, provided that the
assumptions $H1-H2$ are satisfied. As we shall see in detail in section 4, the proofs of theorems 1.3 and 1.5 can be extended to the case of an unknown $v \in [L^2(I)]^n$, $n \geq 2$, with only minor changes (see also [22, theorem 2.1]). In this direction, we quote here the recent contributions [23] and [31], where the the isentropic Navier-Stokes system and hyperbolic-parabolic Jin-Xin system have been considered. In principle, when rigorous results are not achievable, it could be possible to obtain numerical evidence of the spectrum of the linearised operator.

Moreover, in this paper we are able to give explicit expressions for the speed and for the size of the interface location $\xi$, as well as for the time of convergence of such an interface towards its equilibrium configuration; this is a direct consequence of the strategy we used, which is the description of the solution to (1.1) as the sum of two functions, each of them satisfying an explicit equation. As a consequence, the two phases of the dynamics are explicitly described and separated.

We close this introduction with an overview of the paper.

In section 2 we prove the existence of four different types of stationary solutions for equation (1.1), and we discuss the stability properties of these steady states.

In section 3 we develop a general approach to describing the dynamics of solutions belonging to a neighbourhood of a one-parameter family $\{U^0(x; \xi)\}_{\xi \in \xi}$ of approximate steady states, where we use as coordinates the parameter $\xi$, describing the location of the internal interface, and the perturbation $v$, describing the distance between the solution $u$ and an element of the family. By linearising the original equation around an element of the family, we end up with a coupled system for the variables $(\xi, v)$, whose analysis is performed in the subsequent section 4. In particular, here we deal with an approximation of the system, obtained by linearising with respect to $v$ and by disregarding the $o(v)$ -terms. Specifically, we state and prove theorem 4.1, providing, under appropriate assumptions on the spectrum of the linearised operator around $U^0$ as well as on the behaviour of $U^0$ as $\varepsilon \to 0$, an explicit estimate for the perturbation $v$. Such estimate will be subsequently used to decoupled the system for $(\xi, v)$, in order to obtain a reduced equation for the interface location $\xi(t)$, as analysed in proposition 4.3. In particular, these results characterising the couple $(\xi, v)$ give a good qualitative explanation of the transition from the metastable state to the finale stable state.

The last part of this section is devoted to the analysis of the complete system for the couple $(\xi, v)$, where the higher order terms in $v$ are also considered: the main contribution of this section is theorem 4.4, where we prove an estimate for the difference $\|v - z(t)\|_{H^1}$, where $z$ is a function with a very fast decay in time. This result, together with theorem 4.1, makes the theory complete.

Finally, in section 5, we consider, as an example, the Burgers–Sivashinsky equation: in this case we are able to provide an explicit expression for the approximated family $\{U^0\}$. In order to apply the general theory developed in the previous sections, we give a measure of how far an element of the family $\{U^0\}$ is from being an exact steady state, as well as an explicit expression for the speed of convergence of the interface. It turns out that all these terms are small with respect to $\varepsilon$. Subsequently, we analyse the spectral properties of the linear operator arising from the linearisation around the approximate steady state $U^0$, showing that the spectrum can be decomposed as follows: the first eigenvalue $\lambda_1^\varepsilon$ is positive and of order $\varepsilon^{-1/2}$; all the remaining eigenvalues $\{\lambda_k^\varepsilon\}_{k \geq 2}$ are negative and behave like $-C\sqrt{\varepsilon}$. These estimates will translate into one-dimensional dynamics, since all of the components of the perturbation relative of all the eigenvectors except the first one will have a very fast decay for a small $\varepsilon$, and in slow motion for the interface as a consequence of the size estimate for the first eigenvalue. This analysis is needed to give evidence of the validity of the assumptions of theorems 4.1 and 4.4.
2. The stationary problem

In this section we deal with the stationary problem for (1.1), that is

\[
\begin{align*}
\varepsilon \partial_t^2 u &= \partial_x f(u) - f'(u) \\
u(0) &= u(\varepsilon) = 0
\end{align*}
\]  

(2.1)

for \(x \in (0, \varepsilon]\). This problem has been extensively studied in the case \(f(u) = u^2/2\) in the work of Berestycki et al [6]. Here the authors proved the existence and uniqueness of four types of solutions to (2.1): they proved that there exists a unique positive solution \(U_+^\varepsilon\), a unique negative solution \(U_-^\varepsilon\), and two other stationary solutions \(U_{1,\varepsilon}^\varepsilon\) and \(U_{1,\varepsilon}^\varepsilon\), which have one zero inside the interval (for more detail see [6, theorem 1]). Additionally, the authors proved that the solutions \(U_+^\varepsilon\) are stable, while \(U_{1,\varepsilon}^\varepsilon\), \(U_{1,\varepsilon}^\varepsilon\), are unstable with respect to perturbations of initial data (see [6, theorem 6.4]).

2.1. The existence of stationary solutions

Here we mean to use techniques analogous to those used in [6] in order to prove the existence of stationary solutions in the case of a generic flux function that satisfies hypotheses (1.2). In particular, we are interested in studying the existence of the stationary solution, named here \(U_{\varepsilon}^\varepsilon\), that gives rise to metastable behaviour.

Let us stress again that, in this case, \(U_{\varepsilon}^\varepsilon\) is said to be metastable because, starting from an initial datum located near \(U_{\varepsilon}^\varepsilon\), the solution drifts apart the unstable steady state towards one of the stable equilibrium configurations, and this motion is extremely slow. This behaviour is different from other cases (see, for example, [22, 31]) where the unique steady state is metastable in the sense that, starting from an initial configuration located far from the equilibrium, the time-dependent solution starts to drift in an exponentially long time towards the asymptotic limit.

To prove the existence of the metastable steady state, we first prove the existence of a positive steady state, named here \(U_{\varepsilon}^\varepsilon\), and then we deduce the existence and properties of the steady state \(U_{\varepsilon}^\varepsilon\) by making use of the symmetries and scalings in the problem.

Before stating our result, let us define the tools we will use in the following.

**Definition 2.1.** A function \(v \in \mathcal{H}_1^1[0, \varepsilon]\) is a sub-solution (respectively a super-solution) to (2.1) if \(v(0), \varepsilon) \leq 0\) (respectively \(v(0), \varepsilon) \geq 0\) and

\[
\int_0^{\varepsilon} (v' \varphi' - f(v) \varphi' - f'(v) \varphi) \, dx \leq 0 \quad \text{(respectively } \geq 0),
\]

for all \(\varphi \in \mathcal{C}_0[0, \varepsilon]\) such that \(\varphi(0) = \varphi(\varepsilon) = 0\).

**Remark 2.2.** In the following, we will prove the existence of the solution starting from the existence of a sub-solution and a super-solution and by making use of a standard monotone iteration technique (see [10, 30]).

**Proposition 2.3.** There exists a unique solution \(U_{\varepsilon, \varepsilon}(x)\) to (2.1), which is positive in the interval \((0, \varepsilon]\) and such that

\[
U_{\varepsilon, \varepsilon}^\varepsilon \leq 1, \quad 0 < U_{\varepsilon, \varepsilon}^\varepsilon \leq x \quad \text{and} \quad U_{\varepsilon, \varepsilon}^{\varepsilon, \varepsilon} \leq 0 \quad \text{for } x \in (0, \varepsilon].
\]
Proof. Denoting by \( N\mathbf{u} := -\varepsilon \partial_x^2 \mathbf{u} + f'(\mathbf{u}) \partial_x \mathbf{u} - f(\mathbf{u}) \), the function \( v(x) = x \) is such that \( N v \geq 0 \), that is, \( \varepsilon \) is a super-solution. On the other side, given \( \alpha \in \mathbb{R}^+ \), we consider the function \( v(x) = \alpha \sin \left( \frac{\pi}{\ell} x \right) \). We get

\[
N v = \varepsilon \alpha \left( \frac{\pi}{\ell} \right)^2 \sin \left( \frac{\pi}{\ell} x \right) + f'(v) \left[ \frac{\alpha \pi}{\ell} \cos \left( \frac{\pi}{\ell} x \right) - 1 \right] \leq \varepsilon \alpha \left( \frac{\pi}{\ell} \right)^2 + f'(v) \left[ \frac{\alpha \pi}{\ell} - 1 \right].
\] (2.2)

Since \( f'(v) \) is positive inside the interval \((0, \ell)\), if we denote by \( m = \max_{v \in (0, \ell)} f'(v(x)) \), in order to have \( N v \leq 0 \), we have to require \( \frac{\alpha \pi}{\ell} \leq 1 \). Hence, we can choose \( \alpha = \alpha(m) \) such that (2.2) is non-positive, that is, \( \varepsilon \) is a sub-solution to (2.1) in the interval \((0, \ell)\). Finally, since \( \frac{\alpha \pi}{\ell} \leq 1 \), we have

\[
\alpha \sin \left( \frac{\pi}{\ell} x \right) \leq x,
\]

so that there exists a positive solution \( u_+(x) \) to (2.1) in \((0, \ell)\).

To prove the uniqueness of the positive steady state, we only give a sketch of the proof, since the computations are similar to the ones used in [6, section 4] in the case \( f(u) = u^2/2 \). The idea is to rescale the problem by performing the change of variable \( u(x) = \varepsilon^{\beta} u \left( \frac{x}{\varepsilon} \right) \) for some \( \beta > 0 \) chosen such that we get the following equation

\[
\partial_x^2 v - f'(v) \partial_x v + f'(v) = 0, \quad v(0) = v(\varepsilon^{\beta} \ell) = 0.
\] (2.3)

For example, if \( f(u) = u^{\gamma} \), then \( \beta = 1/\gamma \). The uniqueness for the solution to (2.1) corresponds to the uniqueness for the solution to (2.3). We then introduce the following initial value problem

\[
\partial_x^2 v - f'(v) \partial_x v + f'(v) = 0, \quad v(0) = v'(0) = \alpha,
\] (2.4)

for some \( \alpha \in (0, 1) \) and one can see that the solution \( \zeta_0 \) to (2.4) has a first zero, denoted here by \( x = \zeta(\alpha) \), such that \( \zeta_0 > 0 \) in \((0, \zeta(\alpha))\) and \( \zeta_0(0) = \zeta_0(\zeta(\alpha)) = 0 \). To prove the uniqueness for (2.3) it is then sufficient to prove that \( \zeta(\alpha) \) is strictly increasing with respect to \( \alpha \). For the proof of this statement and for further detail we refer to [6], proposition 4.2 and appendix A.

For the proof of the second part of the proposition, we know that \( u_+(x) \leq x \) for all \( x \in (0, \ell) \), so that \( u'_+(0) \leq 1 \); moreover, since \( u_+ > 0 \) and \( u_+(\ell) = 0 \), there follows \( u'_+(\ell) < 0 \). Hence, there exists a maximum \( x_1 \in (0, \ell) \) for the function \( u_+ \). Now let us suppose that there exists a value \( x_2 > x_1 \) such that \( u_+(x_2) \) is an internal minimum for \( u_+ \). From the equation we have

\[
\varepsilon \partial_x^2 u_+(x_2) = -f'(u_+(x_2)) < 0,
\]

which is impossible since \( x_2 \) is a minimum for \( u_+ \). Hence, \( u''_+(x) \) is negative for all \( x \in (0, \ell) \). Finally, from the equation

\[
0 > \varepsilon \partial_x^2 u_+ = f'(u_+)(\partial_x u_+ - 1),
\]

that is, \( f'(u_+) > 0, \partial_x u_+ < 1 \). \( \square \)
Starting from the existence of the function $U_{\varepsilon^+}$, the following results concerning $U^\varepsilon_M$ can be proven.

**Proposition 2.4.** There exists a unique $U^\varepsilon_M(x)$ solution to (2.1), and there exists $x_0 \in (0, \varepsilon)$ such that $U^\varepsilon_M(x_0) = 0$ and

$$U^\varepsilon_M(x) < 0 \text{ for } x < x_0, \quad U^\varepsilon_M(x) > 0 \text{ for } x > x_0.$$ 

Moreover, $U^\varepsilon_M'(x) \leq 1$ for $x \in (0, \varepsilon)$ and

$$U^\varepsilon_M''(x) > 0 \text{ for } x < x_0 \text{ and } U^\varepsilon_M'''(x) < 0 \text{ for } x > x_0.$$

**Remark 2.5.** Because of the assumption (1.2) on the symmetry of the flux function $f$, it turns out that $x_0 \equiv \varepsilon/2$.

**Proof.** Let $x_0 \in (0, \varepsilon)$, and let us consider the interval $(x_0, \varepsilon)$. The proof of the statement follows by using the same arguments as in the proof of proposition 2.3, and by choosing $v_1(x) = x - x_0$ and $v_2(x) = \alpha \sin(\frac{x}{\varepsilon - x_0}(x - x_0))$ as a super-solution and sub-solution, respectively.

In particular, there exists a positive solution $u_+(x)$ to (2.1) in $(x_0, \varepsilon)$ such that $u_+(x) = 0$. A symmetric argument can be used inside the interval $(0, x_0)$ to prove the existence of a negative solution $u_-(x)$, so that

$$U^\varepsilon_M(x) = \begin{cases} u_-(x) & \text{for } x < x_0, \\ u_+(x) & \text{for } x > x_0. \end{cases} \tag{2.5}$$

and $U^\varepsilon_M(x_0) = 0$. Because of the assumption (1.2) on the symmetry of the flux function $f$, then $x_0 \equiv \varepsilon/2$. In particular, the steady state $U^\varepsilon_M$ is a $C^1$-matched function. The unicity follows immediately from the unicity of $U_{\varepsilon^+}$.

For the proof of the second part of the proposition, it is enough to give a description of the positive solution $u_+(x)$ for $x \in (x_0, \varepsilon)$. The same arguments can be used for the symmetric case of $u_-(x)$ in the interval $(0, x_0)$.

Again, the proof is identical to the one of proposition 2.3, by considering the interval $(x_0, \varepsilon)$ instead of $(0, \varepsilon)$. $\square$

**Remark 2.6.** The end of the proof of proposition 2.4 is justified by the symmetry properties of the solutions to (2.1); indeed, if we consider the interval $(a, b) \subset (0, \varepsilon)$, and if we define $\bar{U}^+(x; a, b)$ as the unique positive solution to

$$\begin{cases} \varepsilon \bar{u}'' + \bar{u} = f(u) - f'(u) \\ u(a) = u(b) = 0, \quad x \in (a, b), \end{cases}$$

the solutions to (2.1) can be defined starting from $\bar{U}^+(x; a, b)$. For example

$$U^\varepsilon_M(x) := \begin{cases} \bar{U}^+(\varepsilon/2 - x; 0, \varepsilon/2) & \text{for } x \in (0, \varepsilon/2) \\ \bar{U}^+(x; \varepsilon/2, \varepsilon) & \text{for } x \in (\varepsilon/2, \varepsilon). \end{cases}$$

The following results characterise the behaviour of $U_{\varepsilon^+}$ and $U^\varepsilon_M$ with respect to the parameter $\varepsilon$. 4342
Proposition 2.7. The solution $U_{\varepsilon, +}(x)$ converges pointwise to the function $x$ in $(0, \ell)$ when $\varepsilon \to 0$.

Proposition 2.8. The solution $U^\varepsilon_\mu(x)$ converges pointwise to the function $x - \ell/2$ in $(0, \ell)$ when $\varepsilon \to 0$.

We only prove the convergence property of the metastable steady state $U^\varepsilon_\mu$, the one we are interested the most. Again, a minor modification to the argument can be used in the other cases. In order to prove proposition 2.8, we need to state and prove the following lemma.

Lemma 2.9. Let $\varepsilon < \varepsilon_0$ and let $U_{\varepsilon, +}$ and $U_{\varepsilon, -}$ the (unique) positive solutions to (2.1) with $\varepsilon$ and $\varepsilon_0$, respectively. Then $U_{\varepsilon, +}(x) < U_{\varepsilon, -}(x)$ for all $x \in (0, \ell)$.

Proof. Let $x \in (0, \ell)$. Since $U''_{\varepsilon, +}(x) < 0$ we note that, if $\varepsilon < \varepsilon_0$, then $U_{\varepsilon, +}$ is a sub-solution for (2.1); moreover, there exists a larger super-solution that is $h(x) = x$. Hence, by uniqueness, there follows $U_{\varepsilon, +} < U_{\varepsilon, -}$.

Proof of proposition 2.8. Because of lemma 2.9 and because of the symmetry properties of the problem, if $\varepsilon < \varepsilon_0$, then $U''_{\varepsilon} > U''_\mu$ for $x > \ell/2$, while $U''_{\varepsilon} < U''_\mu$ for $x < \ell/2$. Hence, since the function $v = x - \ell/2$ is a sub-solution (super-solution, respectively) in the interval $(0, \ell/2)$ (in the interval $(\ell/2, \ell)$, respectively), we have

$$\lim_{x \to 0} U''_{\varepsilon}(x) = L(x) > x - \ell/2 \quad \text{for} \quad x \in (0, \ell/2),$$
$$\lim_{x \to \ell} U''_{\varepsilon}(x) = l(x) < x - \ell/2 \quad \text{for} \quad x \in (\ell/2, \ell).$$

Now we show that $l(x) = L(x) = x - \ell/2$. To this aim, let $a \in (0, \ell/2)$, $c \in (\ell/2, \ell)$, $b = (\ell/2 + a)/2$, $d = (\ell + c)/2$. Given $\lambda \in (0, 1)$, let us consider the following function

$$h^\lambda_{a,c}(x) = \begin{cases} 
\frac{\lambda}{a} (b - \ell/2) x & \text{for} \quad x \in (0, a) \\
\omega(x) & \text{for} \quad x \in (a, b) \\
\frac{\lambda}{\ell - d} (c - \ell/2) & \text{for} \quad x \in (b, c) \\
\theta(x) & \text{for} \quad x \in (c, d) \\
\frac{\lambda}{\ell - d} (c - \ell/2) (\ell - x) & \text{for} \quad x \in (d, \ell) 
\end{cases}$$

where $\omega(x) < 0$ and $\theta(x) > 0$ are $C^2$ functions such that $h^\lambda_{a,c}$ is a continuous function with a continuous derivative in $x = a, b, c, d$. More precisely we require

$$\omega(a) = \omega(b), \quad \omega'(a) = \frac{\lambda(b - \ell/2)}{a}, \quad \omega'(b) = \lambda, \quad \omega''(a) = \omega''(b) = 0,$$
$$\theta(c) = \theta(d), \quad \theta'(c) = \lambda, \quad \theta'(d) = \frac{\lambda(\ell/2 - c)}{\ell - d}, \quad \theta''(c) = \theta''(d) = 0.$$

Under these hypotheses, it is easy to check that $h^\lambda_{a,c}$ is a super-solution to (2.1) for $x \in (0, \ell/2)$, and a sub-solution for $x \in (\ell/2, \ell)$. Hence, for small enough $\varepsilon$, we deduce

$$x - \ell/2 \leq U^\varepsilon_\mu(x) \leq h^\lambda_{a,c}(x) \quad \text{for} \quad x \in (0, \ell/2),$$
$$h^\lambda_{a,c}(x) \leq U^\varepsilon_\mu(x) \leq x - \ell/2 \quad \text{for} \quad x \in (\ell/2, \ell).$$
Since $\lambda$ can be chosen arbitrarily close to 1, while $a$ and $c$ can be chosen arbitrarily close to 0 and $\ell'$, respectively, it follows that $U'_\varepsilon(x)$ converges pointwise to $x - \ell'/2$ as $\varepsilon \to 0$ for all $x \in (0, \ell')$.

Starting from the properties of $U_{\varepsilon, +}$ it is possible to prove similar results for the other solutions to (2.1) by making use of the symmetries and scalings in the problem. Furthermore, equilibrium solutions with more than one zero crossing are also possible (see [12] for more detail).

**Proposition 2.10.** Concerning the solutions to (2.1), there exists a unique negative solution $U_{\varepsilon, -}(x) := -U_{\varepsilon, +}(\ell' - x)$ such that

1. $U'_{\varepsilon, -} \leq 1$, $x - \ell < U_{\varepsilon, -} \leq 0$ and $U''_{\varepsilon, -} \geq 0$ for $x \in (0, \ell')$.
2. $U_{\varepsilon, -}$ converges to $x - \ell$ in $(0, \ell')$ pointwise when $\varepsilon \to 0$.

Additionally, there exists a unique stationary solution $U^\varepsilon_{ss}$ that has one zero inside the interval $(0, \ell')$ and such that

1. $0 < U^\varepsilon_{ss}(x) < x$ for $x < \ell/2$, and $x - \ell < U^\varepsilon_{ss}(x) < 0$ for $x > \ell/2$.
2. $U^\varepsilon_{ss}$ converges to the function

$$U^\varepsilon_{ss}(x) := \begin{cases} x & x \in (0, \ell/2) \\ x - \ell & x \in (\ell/2, \ell) \end{cases}$$

pointwise for $x = \ell/2$ when $\varepsilon \to 0$.

### 2.2. The stability of stationary solutions

**Definition 2.11.** A stationary solution $v$ to (1.1) is stable if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that, if $|u_0(x) - v(x)|_{L^\infty} < \delta$, then there exists a time $T > 0$ such that $|u(x, t) - v(x)|_{L^\infty} < \varepsilon$ for all $t \geq T$. Finally, $v$ is unstable if it is not stable.

Concerning the stability properties of the stationary solutions to (1.1), the following proposition holds.

**Proposition 2.12.** The stationary solutions $U_{\varepsilon, \pm}(x)$ are stable, while $U^\varepsilon_{ss}(x)$ and $U^\varepsilon_{ss}(x)$ are unstable.

The proof of proposition 2.12 is based on the following well-known result concerning the evolution problem (1.1) with a sub or super-solution as the initial datum (see [2, 30] and [6, theorem 6.4]).

**Proposition 2.13.** Let $v(x)$ be a weak sub-solution (respectively a super-solution) to (2.1). Let $u(x, t)$ be the solution to (1.1) with the initial datum $u_0(x) = v(x)$. Then, for $t \to +\infty$, $u(x, t)$ converges monotonically to a stationary solution $U(x)$ to (1.1), i.e. $u(x, t) \nearrow U(x)$ (or $u(x, t) \searrow U(x)$).

**Proof of proposition 2.12.** We prove only the instability property of $U^\varepsilon_{ss}$. To prove the results stated for $U^\varepsilon_{ss}$ and $U^\varepsilon_{ss}$, the basic idea is the same, and follows the lines of [6, theorem 6.4].

Given $(a, b) \subset (0, \ell')$, let $\tilde{U}^{ab}(x; a, b)$ the unique positive (respectively negative) solution to

$$\begin{cases} \varepsilon \frac{\partial^2 u}{\partial x^2} = \partial_x f(u) + f'(u) & x \in (a, b) \\ u(a) = u(b) = 0 \end{cases}$$
Moreover, given \( 0 < \alpha < \beta < \gamma < \ell \), let us define

\[
v(x) = \begin{cases} 
\bar{U}^- (x; \alpha, \beta) & \text{for } x \in [\alpha, \beta] \\
\bar{U}^+ (x; \beta, \gamma) & \text{for } x \in [\beta, \gamma] \\
0 & \text{for } x \in [\gamma, \ell]
\end{cases}
\]

Hence, if \( \alpha = 0, 0 < \beta < \ell/2 \) and \( \gamma = \ell \), we know that \( U^\varepsilon(x) \leq v(x) \). Now let \( v(x, t) \) be the solution to (1.1) with the initial datum \( v_0(x) = v(x) \). We have \( v(x, t) \xrightarrow{t \to +\infty} U(x) \), since there are no other stationary solutions \( U(x) \) such that \( v \leq U \). This proves the instability of \( U^\varepsilon \), since \( v(x) \to U^\varepsilon \) as \( \beta \to \ell/2 \).

As a consequence of the instability of \( U^\varepsilon \), if we start from an initial datum \( u_0(x) \) close to such an unstable configuration, we will see in finite time that the solution \( u(x, t) \) to (1.1) ‘runs away’ from \( U^\varepsilon \). Precisely, there exists \( \delta > 0 \) and a time \( T > 0 \) such that

\[
|u(x, t) - U^\varepsilon(x)| > \delta, \quad \forall x \in I, \quad \forall t \geq T.
\]

However, as already stressed, solutions to (1.1) generated by initial data close to \( U^\varepsilon \) exhibit metastable behaviour, i.e. the convergence to one of the equilibrium configurations \( U_{\varepsilon, \pm} \) is exponentially slow in time. To rigorously describe this behaviour, from now on we will consider equation (1.1) together with continuous initial data of the form

\[
u_0(x) < 0 \text{ in } (0, a_0), \quad u_0(x) > 0 \text{ in } (a_0, \ell), \text{ for some } a_0 \in (0, \ell).
\]

### 3. The metastable dynamics and the linearised problem

In this section we analyse the solution \( u \) to (1.1) in the vanishing viscosity limit, i.e. \( \varepsilon \to 0 \). In particular, we address the question as to whether a phenomenon of metastability occurs for this kind of problem, and how these special dynamics are related to the viscosity coefficient and to the initial datum \( u_0 \).

Let us define the nonlinear differential operator

\[
P^\varepsilon [u] := \varepsilon \partial_x^2 u - \partial_x f(u) + f'(u),
\]

that depends singularly on the parameter \( \varepsilon \), meaning that \( P^0[u] \) is of a lower order.

Our primary assumption is the following: we suppose that there exists a one-parameter family of functions \( \{ U^\varepsilon(x; \xi) \}_{\xi \in I} \) such that

- The nonlinear term \( P^\varepsilon [U^\varepsilon] \) is small in \( \varepsilon \) in a sense that we will specify later.
- Each element of the family is such that

\[
\begin{cases} 
U^\varepsilon(x; \xi) < 0 & \text{for } 0 < x < \xi, \\
U^\varepsilon(x; \xi) > 0 & \text{for } \xi < x < \ell.
\end{cases}
\]

- There exists a value \( \xi \in I \) such that the element of the family \( U^\varepsilon(x; \xi) \) corresponds to a stable steady state to (1.1).

The parameter \( \xi \) describes the unique zero of \( U^\varepsilon \), corresponding to the location of the interface; under these hypotheses, starting from an initial configuration close to \( U^\varepsilon \), metastable behaviour for the time-dependent solution is expected.

The family \( \{ U^\varepsilon(x; \xi) \}_{\xi \in I} \) can be seen as a family of approximate steady states for (1.1), in the sense that each element satisfies the stationary equation up to an error that is small in \( \varepsilon \).
More precisely, we ask for the existence of a family of smooth positive functions \( \Omega^\varepsilon(\xi) \), that converge to zero as \( \varepsilon \to 0 \), uniformly with respect to \( \xi \), and such that
\[
|\psi(\cdot), \mathcal{P}[U^\varepsilon(\cdot; \xi)]| \leq \Omega^\varepsilon(\xi)|\psi|_{L^\infty}, \quad \forall \psi \in C(I), \forall \xi \in I.
\]
We also consider the following additional assumption specifying the structure of the term \( \Omega^\varepsilon \): we require that there exists a family of smooth positive functions \( \omega^\varepsilon = \omega^\varepsilon(\xi) \), uniformly convergent to zero as \( \varepsilon \to 0 \), such that
\[
\Omega^\varepsilon(\xi) \leq \omega^\varepsilon(\xi)|\xi - \bar{\xi}|.
\]
This hypothesis incorporates the fact that, for a distinct value \( \bar{\xi} \in I \), the element \( U^\varepsilon(\cdot, \bar{\xi}) \) solves the stationary equation. The dependence of \( \Omega^\varepsilon \) and \( \omega^\varepsilon \) on \( \varepsilon \) is crucial here since they measure how far an element of the family \( U^\varepsilon \) is from being an exact stationary solution.

Let us stress that, different to the construction in [33], where the approximate stationary solutions satisfy exactly the equation and the boundary condition to within exponentially small terms, here we assume that the generic element \( U^\varepsilon \) satisfies the boundary conditions exactly and the equation approximately.

Once the one-parameter family \( \{U^\varepsilon(x; \xi)\}_{\xi \in I} \) is chosen, we look for a solution to (1.1) in the form
\[
u(x, t) = U^\varepsilon(x; \xi(t)) + v(x, t), \quad (3.1)
\]
where the perturbation \( v \in L^2(I) \) is determined by the difference between the solution \( u \) and an element of the family of approximate steady states.

The idea of a linearisation around \( U^\varepsilon \) is developed in order to separate the two distinct phases of the dynamics of the solution. First, we mean to understand what happens far from the stable equilibrium solution when the interface is formed; subsequently, we want to follow its evolution towards the asymptotic limit. To this aim, we suppose that the parameter \( \xi \), describing the unique zero of the ‘quasi-stationary’ solution \( U^\varepsilon \) depends on time, so that its evolution towards one of the walls \( x = 0 \) or \( x = \ell \) (corresponding to the equilibrium solutions \( U_{\varepsilon,+} \) and \( U_{\varepsilon,-} \), respectively) describes the asymptotic convergence of the interface towards the equilibrium. Hence, our purpose is to determine an equation for the value \( \xi(t) \), characterising the metastable behaviour.

By substituting (3.1) into (1.1), we obtain
\[
\partial_t v = L^{\varepsilon}_{U^\varepsilon}v + \mathcal{P}[U^\varepsilon(\cdot; \xi)] - \partial_t U^\varepsilon(\cdot; \xi) \frac{d\xi}{dt} + \mathcal{Q}[v, \xi], \quad (3.2)
\]
where
\[
L^{\varepsilon}_{U^\varepsilon}v := \varepsilon \partial^2_t v - \partial_x(f'(U^\varepsilon)v) + f''(U^\varepsilon)v
\]
is the linearised operator arising from the linearisation around \( U^\varepsilon \), while \( \mathcal{Q}[v, \xi] \) collects quadratic terms in \( v \), and, since we are assuming \( v \) to be small, it is obtained by disregarding higher order terms in the variable. Precisely,
\[
\mathcal{Q}[v, \xi] := \frac{1}{2} \{ -\partial_x(f''(U^\varepsilon)v^2) + f'''(U^\varepsilon)v^3 \}. 
\]
3.1. Spectral hypotheses and the projection method

We begin by analysing the spectrum of the linearised operator $\mathcal{L}_\epsilon^\xi$. The eigenvalue problem reads

$$\epsilon \partial_\xi^2 \varphi - \partial_t (f'(U^\epsilon) \varphi) + f''(U^\epsilon) \varphi = \lambda \varphi, \quad \varphi(0) = \varphi(\xi) = 0.$$ 

First, we show that the eigenvalues of $\mathcal{L}_\epsilon^\xi$ are real. To this aim, let us introduce the self-adjoint operator

$$\mathcal{M}(t, \psi) := \epsilon^2 \partial_\xi^2 \psi - a^\epsilon(x; \xi(t)) \psi + \epsilon f''(U^\epsilon) \psi, \quad a^\epsilon(x; \xi(t)) := \left( \frac{f'(U^\epsilon)}{2} \right)^2 + \frac{1}{2} \epsilon \partial_t f''(U^\epsilon).$$

It is easy to check that $\varphi^\epsilon$ is an eigenfunction for $\mathcal{L}_\epsilon^\xi$ relative to the eigenvalue $\lambda^\epsilon$ if and only if

$$\psi^\epsilon(x; \xi) = \exp \left( -\frac{1}{\epsilon} \int_{t_0}^t f''(U^\epsilon)(t; \xi) dt \right) \varphi^\epsilon(x; \xi)$$

is an eigenfunction for the operator $\mathcal{M}_\epsilon^\xi$ relative to the eigenvalue $\mu^\epsilon = \epsilon \lambda^\epsilon$. Hence

$$\epsilon \sigma(\mathcal{M}_\epsilon^\xi) \equiv \sigma(\mathcal{M}_\epsilon),$$

so that, since $\mathcal{M}_\epsilon^\xi$ is self-adjoint, we can state the spectrum of $\mathcal{L}_\epsilon^\xi$ is composed of real eigenvalues.

Moreover, we assume the spectrum of $\mathcal{L}_\epsilon^\xi$ to be composed of a decreasing sequence $\{ \lambda^\epsilon_k \}_{k \in \mathbb{N}}$ of real eigenvalues such that

- $\lambda^\epsilon_1(\xi) \to 0$ as $\epsilon \to 0$, uniformly with respect to $\xi$.
- All the eigenvalues $\{ \lambda^\epsilon_k \}_{k \geq 2}$ are negative and there exist constants $C, C'$ such that

$$\lambda^\epsilon(\xi) - \lambda^\epsilon_1(\xi) \geq C' \quad \forall \xi \in I, \quad \lambda^\epsilon_k(\xi) \leq -Ck^2, \quad \text{for } k \geq 2.$$ 

Hence, we assume that there is a spectral gap between the first and the second eigenvalues. Moreover, we ask for $\lambda^\epsilon_1$ to be small in $\epsilon$ (uniformly with respect to $\xi$) and we assume the sequence $\{ \lambda^\epsilon_1 \}_{k \geq 2}$ to diverge to $-\infty$ as $-k^2$.

**Remark 3.1.** We note that there are no requests on the sign of the first eigenvalue $\lambda^\epsilon_1$: in section 2 we have proven the instability of $U^\epsilon_{\mu}$ so that, since $U^\epsilon$ well approximates the exact steady state $U^\mu$, we can state that the first eigenvalue $\lambda^\epsilon_1$ is positive. The metastable behaviour is indeed a consequence of the smallness, with respect to $\epsilon$, of this first eigenvalue.

In order to obtain a differential equation for the parameter $\xi$, we use an adapted version of the projection method: since we have supposed the first eigenvalue of the linearised operator to be small in $\epsilon$, i.e. $\lambda^\epsilon_1 \to 0$ as $\epsilon \to 0$, a necessary condition for the solvability of (3.2) is that the first component of the solution has to be zero. More precisely, in order to remove the singular part of the operator $\mathcal{L}_\epsilon^\xi$ in the limit $\epsilon \to 0$, we set an algebraic condition ensuring orthogonality between $\psi^\epsilon_1$ and $v$, so that the equation for the parameter $\xi(t)$ is chosen in such a way that the unique growing terms in the perturbation $v$ are cancelled out. Setting $v_2 = v_2(\xi; t) := \langle \psi^\epsilon_1(\cdot; \xi), v(\cdot, t) \rangle$, we thus impose

$$\frac{d}{dt} \langle \psi^\epsilon_1(\cdot; \xi(t)), v(\cdot, t) \rangle = 0 \quad \text{and} \quad \langle \psi^\epsilon_1(\cdot; \xi_0), v_0(\cdot) \rangle = 0.$$ 

Using equation (3.2), we have
\((\psi^*_1(\xi, \cdot), \mathcal{L}^* v + \mathcal{P}^*[U^*(\cdot; \xi)] - \partial_t U^*(\cdot; \xi) \frac{dc}{dt} + \mathcal{Q}[v, \xi]) + \langle \partial_t \psi^*_1(\xi, \cdot), \frac{dc}{dt}, v \rangle = 0.\)

Since \(\langle \psi^*_1, \mathcal{L}^* v \rangle = \lambda^*_1(\psi^*_1, v) = 0\), we obtain a scalar nonlinear differential equation for the variable \(\xi\), that is

\[
\frac{d\xi}{dt} = \frac{\langle \psi^*_1(\cdot; \xi), \mathcal{P}^*[U^*(\cdot; \xi)] + \mathcal{Q}[v, \xi] \rangle}{\langle \psi^*_1(\cdot; \xi), \partial_t U^*(\cdot; \xi) \rangle - \langle \partial_t \psi^*_1(\cdot; \xi), v \rangle}.
\] (3.3)

We notice that if \(U^*(\cdot; \xi)\) is the exact stationary solution, then

\[
\mathcal{P}[U^*(\cdot; \xi)] = \mathcal{P}[U^*(\cdot; \xi)] - \mathcal{P}[U^*(\cdot; \xi)] \approx \mathcal{L}^*_\xi \partial_t U^*(\cdot; \xi)(\xi - \xi).
\]

The fact that \(\mathcal{L}^*_\xi(\partial_t U^*)\) is uniformly small suggests that the first eigenfunction \(\psi^*_1\) is proportional to \(\partial_t U^*\) (at least for small \(\epsilon\)), so that we can renormalise the first adjoint eigenfunction in such a way

\[
\langle \psi^*_1(\cdot; \xi), \partial_t U^*(\cdot; \xi) \rangle = 1, \quad \forall \xi \in I.
\]

Since we consider a small perturbation, in the regime \(v \to 0\) we have

\[
\frac{1}{1 - \langle \partial_t \psi^*_1(\cdot; \xi), v \rangle} = 1 + \langle \partial_t \psi^*_1, v \rangle + R[v],
\]

where the remainder \(R\) is of order \(o(|v|)\), and it is defined as

\[
R[v] := \frac{\langle \partial_t \psi^*_1(\cdot; \xi), v \rangle^2}{1 - \langle \partial_t \psi^*_1(\cdot; \xi), v \rangle}.
\]

Inserting in (3.3), we end up with the following nonlinear equation for \(\xi\)

\[
\frac{d\xi}{dt} = \theta^*(\xi)(1 + \langle \partial_t \psi^*_1, v \rangle) + \rho^*[\xi, v],
\] (3.4)

where

\[
\theta^*(\xi) := \langle \psi^*_1, \mathcal{P}^*[U^*] \rangle,
\rho^*[\xi, v] := \langle \psi^*_1, \mathcal{Q}[v, \xi](1 + \langle \partial_t \psi^*_1, v \rangle) + \langle \psi^*_1, \mathcal{P}^*[U^*] + \mathcal{Q}[v, \xi] \rangle R[v]\]

Moreover, plugging (3.4) into (3.2), we obtain a partial differential equation for the perturbation \(v\)

\[
\partial_t v = H^v(v; \xi) + (\mathcal{L}^*_\xi + \mathcal{M}^*_\xi) v + \mathcal{R}[v, \xi],
\] (3.5)

where

\[
H^v(\cdot; \xi) := \mathcal{P}^*[U^*(\cdot; \xi)] - \partial_t U^*(\cdot; \xi) \theta^*(\xi),
\mathcal{M}^*_\xi v := -\partial_t U^*(\cdot; \xi) \theta^*(\xi) \langle \partial_t \psi^*_1, v \rangle,
\mathcal{R}[v, \xi] := \mathcal{Q}[v, \xi] - \partial_t U^*(\cdot; \xi) \rho^*[\xi, v].
\]
4. The slow motion of the interface location

4.1. Analysis of the linearised system

Equations (3.4) and (3.5) form a coupled system for the couple \((\xi, v)\). This system is obtained by linearising with respect to \(v\), and by keeping the nonlinear dependence on \(\xi\) in order to describe the evolution of the interface when it is localised far from the equilibrium location. Hence, the terms arising from the linearisation around \(\sim v_0\) are asymptotically smaller than the other terms and can be neglected, so that in the following we will consider the following reduced system

\[
\begin{aligned}
\frac{d\xi}{dt} &= \theta'(\xi)(1 + \langle \partial_\xi \psi_0, v \rangle), \\
\partial_\xi v &= H'(x; \xi) + (L_\xi + M_\xi) v,
\end{aligned}
\]  

(4.1)

where the \(o(v)\) order terms have been cancelled out, together with initial data

\[
\langle \psi_0(; \xi_0), v_0 \rangle = 0, \quad v_0 = u_0 - U'(\cdot; \xi_0).
\]  

(4.2)

We mean to analyse the behaviour of the solution to (4.1) in the limit of small \(\varepsilon\). In order to state our first result, let us recall the hypotheses we assumed on the terms of such system.

**H1.** There exists a family of approximate steady states \(\{ U^\varepsilon(x; \xi) \}_{\xi \in I} \) such that

- There exists a value \(\bar{\xi} \in I\) such that the element \(U^\varepsilon(x; \bar{\xi})\) corresponds to a stable steady state for the original equation.
- For every \(\xi \in I\), each element of the family satisfies \(U^\varepsilon(x; \xi) > 0\) for \(x \in (0, \xi)\) and \(U^\varepsilon(x; \xi) < 0\) for \(x \in (\xi, \ell)\).
- There exists a family of smooth and positive functions \(\Omega^\varepsilon(\xi)\), uniformly converging to zero as \(\varepsilon \to 0\), such that there holds
  \[
  |(\psi(\cdot), \mathcal{P}[U^\varepsilon(\cdot; \xi)])| \leq \Omega^\varepsilon(\xi) |\psi|_{L^\infty}, \quad \forall \psi \in C(\bar{U}), \quad \forall \xi \in I.
  \]
- There exists a family of smooth positive functions \(\omega^\varepsilon = \omega^\varepsilon(\xi)\), uniformly convergent to zero as \(\varepsilon \to 0\), such that
  \[
  \Omega^\varepsilon(\xi) \leq \omega^\varepsilon(\xi) |\xi - \bar{\xi}|.
  \]

**H2.** The sequence of eigenvalues \(\{ \lambda_k(\xi) \}_{k \in \mathbb{N}}\) of the linearised operator \(L_\xi\) is such that

- \(\lambda_k(\xi) \to 0\) as \(\varepsilon \to 0\) uniformly with respect to \(\xi\).
- All the eigenvalues \(\{ \lambda_k \}_{k \geq 2}\) are negative and there exist constants \(C, C'\) such that
  \[
  \lambda_3(\xi) - \lambda_2(\xi) \geq C' \quad \forall \xi \in I, \quad \lambda_2(\xi) \leq - Ck^2, \quad \text{if} \quad k \geq 2.
  \]

Finally, The eigenfunctions \(\varphi_k^\varepsilon(\cdot; \xi)\) and \(\psi_k^\varepsilon(\cdot; \xi)\) are normalized so that

\[
\langle \psi_k^\varepsilon(\cdot; \xi), \partial_\xi U^\varepsilon(\cdot; \xi) \rangle = 1 \quad \text{and} \quad \langle \psi_k^\varepsilon, \varphi_j^\varepsilon \rangle = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}
\]

and we assume

\[
\sum_k (\partial_\xi \psi_k^\varepsilon, \varphi_k^\varepsilon) = \sum_k (\psi_k^\varepsilon, \partial_\xi \varphi_k^\varepsilon)^2 \leq C \quad \forall k,
\]  

(4.3)
for some constant $C$ that does not depend on $\xi$.

We recall that we denoted by $\varphi^\ast_k = \varphi^\ast_k(\cdot; \xi)$ the right eigenfunctions of $\mathcal{L}^\ast_\xi$ and by $\psi^\ast_k = \psi^\ast_k(\cdot; \xi)$ the eigenfunctions of the corresponding adjoint operator $\mathcal{L}^{\ast\ast}_\xi$. In the following, we will use the notation $\Lambda^\ast_k := \sup_{\xi \in I} \lambda^\ast_k(\xi)$.

**Theorem 4.1.** Let hypotheses H1–H2 be satisfied. Then, denoted by $(\xi, v)$ the solution to the initial-value problem (4.1), for any sufficiently small $\varepsilon$, there exists a time $T^\varepsilon$ such that for any $t \leq T^\varepsilon$ the solution $v$ can be represented as

$$v = z + R,$$

where $z$ is defined by

$$z(x, t) := \sum_{k \geq 2} v_k(0) \exp\left(\int_0^t \lambda^\ast_k(\xi(\sigma)) \, d\sigma\right) \varphi^\ast_k(x; \xi(t)),$$

and the remainder $R$ satisfies the estimate

$$|R|_{L^2} \leq C |\Omega^\ast|_{L^\infty} (|v_0|_{L^2} + 1),$$

for some constant $C > 0$.

Moreover, for sufficiently small $v_0$ in $L^2$, the final time $T^\varepsilon$ can be chosen of the order $C|\Lambda^\ast_k|^{-1}$, hence diverging to $+\infty$ as $\varepsilon \to 0$.

**Proof.** The idea of the proof is analogous to the proof of [22, theorem 2.1]. Setting

$$v(x, t) = \sum_j v_j(t) \varphi^\ast_j(x, \xi(t)),$$

we obtain an infinite-dimensional differential system for the coefficients $v_j$

$$\frac{dv_j}{dt} = \lambda^\ast_k(\xi_k) v_k + \langle \psi^\ast_k, F \rangle,$$  

where

$$F := H^\varepsilon + \sum_j v_j \left\{ \mathcal{M}_\xi \varphi^\ast_j - \partial_\xi \varphi^\ast_j \frac{d\xi}{dt} \right\} = H^\varepsilon - \theta^\varepsilon \sum_j \left( a_j + \sum_{\ell} b_{j\ell} v_\ell \right) v_j,$$

and the coefficients $a_j, b_{j\ell}$ are given by

$$a_j := \langle \partial_\xi \psi^\ast_j, \varphi^\ast_j \rangle \partial_\xi U^\varepsilon + \partial_\xi \varphi^\ast_j, \quad b_{j\ell} := \langle \partial_\xi \psi^\ast_j, \varphi^\ast_j \partial_\xi \varphi^\ast_k \rangle.$$

The convergence of the series is guaranteed by the assumption (4.3).

Now let us set

$$E_\varepsilon(s, t) := \exp\left(\int_s^t \lambda^\ast_k(\xi(\sigma)) \, d\sigma\right).$$

Note that, for $0 \leq s < t$, there holds

$$|E_\varepsilon(s, t)| \leq \exp\left(\int_s^t |\lambda^\ast_k(\xi(\sigma))| \, d\sigma\right)$$

for some constant $C > 0$. 

4350
\[ E_q(s, t) = \frac{E_q(0, t)}{E_q(0, s)} \quad \text{and} \quad 0 \leq E_q(s, t) \leq e^{\lambda q(t-s)}. \]

By differentiating (4.3), we deduce
\[ \langle \partial_t \psi^k_j, \phi^k_j \rangle + \langle \psi^k_j, \partial_t \phi^k_j \rangle = 0. \quad (4.6) \]

Hence, for the coefficients \( a_j \) there holds
\[ \langle \psi^k_j, a_j \rangle = \langle \partial_t \psi^k_j, \phi^k_j \rangle \left( \langle \psi^k_j, \partial_t U^k \rangle - 1 \right), \]
so that, since \( \langle \psi^k_j, a_j \rangle = 0 \) for any \( j \), equation (4.5) for \( k = 1 \) simplifies to
\[ \frac{dv_1}{dt} = \lambda_1^1(\xi) v_1 - \theta^1(\xi) \sum_{\ell,j} \langle \psi^j_\ell, b_{\ell j} \rangle v_\ell v_j. \quad (4.7) \]

Choosing \( v_1(0) = 0 \), it follows
\[ v_1(t) = -\int_0^t \theta^1(\xi) \sum_{\ell,j} \langle \psi^j_\ell, b_{\ell j} \rangle v_\ell v_j, E_1(s, t) \, ds \]
\[ v_2(t) = v_2(0) E_2(0, t) \]
\[ + \int_0^t \left\{ \langle \psi^j_\ell, H^\ell \rangle - \theta^\ell(\xi) \sum_{\ell,j} \langle \psi^j_\ell, b_{\ell j} \rangle v_\ell v_j \right\} E_k(s, t) \, ds, \quad (4.8) \]

for \( k \geq 2 \). Let us introduce the function
\[ z(x, t) := \sum_{k \geq 2} v_2(0) E_2(0, t) \varphi^k_\ell(x; \xi(t)), \]
which satisfies the estimate \(|z|_{L^2} \leq |v_0|_{L^2} e^{\lambda q t} \). Since there holds
\[ |\theta^\ell(\xi)| \leq C \Omega^\ell (\xi) \quad \text{and} \quad |\langle \psi^k_j, H^\ell \rangle| \leq C \Omega^\ell (\xi) \{1 + |\langle \psi^j_\ell, \partial_t U^\ell \rangle|\}, \]
after some computations we end up with
\[ |v - z|_{L^2(t)} \leq C \int_0^t \Omega^\ell (\xi) \left| v_{L^2}^j(s) E_1(s, t) \right| ds + C \sum_{k \geq 2} \int_0^t \Omega^\ell (\xi) (1 + |v_{L^2}^j(s)|) E_k(s, t) \, ds \]
\[ \leq C \int_0^t \Omega^\ell (\xi) \left\{ |v_{L^2}^j(s) E_1(s, t)| + \sum_{k \geq 2} E_k(s, t) \right\} ds. \]

for some constant \( C > 0 \) depending on the \( L^\infty \)-norm of \( \psi^k_\ell \). The assumption on the asymptotic behaviour of the eigenvalues \( \lambda_1^k, k \geq 2 \), can now be used to bound the series. Indeed, there holds
\[ \sum_{k \geq 2} E_k(s, t) \leq E_2(s, t) \sum_{k \geq 2} \frac{E_k(s, t)}{E_2(s, t)} \leq C (t-s)^{-1/2} E_2(s, t) \]
so that
\[ E_t(t, 0) |v - z|_{L^2} \leq C \int_0^t \Omega^\varepsilon(\xi) \left[ |v - z|_{L^2}^2(s) E_t(s, 0) + |z|_{L^2}^2(s) E_t(s, 0) + (t - s)^{-1/2} E_z(s, t) E_t(s, 0) \right] \, ds. \]

Setting
\[ N(t) := \sup_{s \in [0, t]} |v - z|_{L^2} E_t(s, 0), \]

then, since \( \Lambda_2^2 \leq \Lambda_1^2 \), we obtain
\[ E_t(t, 0) |v - z|_{L^2} \leq C \int_0^t \Omega^\varepsilon(\xi) N^2(s) E_t(0, s) \, ds + C \int_0^t \Omega^\varepsilon(\xi) \left[ |v|_{L^2}^2 e^{2\Lambda_2^2 s} E_t(s, 0) + (t - s)^{-1/2} E_z(s, t) E_t(s, 0) \right] \, ds. \]

Moreover
\[ \int_0^t e^{2\Lambda_2^2 s} E_t(s, 0) \, ds \leq E_t(t, 0) \int_0^t e^{\Lambda_2^2 s} \, ds = E_t(t, 0) \frac{1}{\Lambda_2^2} (e^{\Lambda_2^2 t} - 1) \leq \frac{1}{|\Lambda_2^2|} E_t(t, 0), \]
\[ \int_0^t (t - s)^{-1/2} E_z(s, t) \, ds \leq \int_0^t (t - s)^{-1/2} e^{\Lambda_2^2 (t - s)} \, ds \leq \frac{1}{|\Lambda_2^2|^{1/2}}, \]

so that, recalling that \( \Lambda_2^2 \) is bounded away from 0, we deduce
\[ E_t(t, 0) |v - z|_{L^2} \leq C \left\{ N^2(t) \left[ \int_0^t \Omega^\varepsilon(\xi) E_t(0, s) \, ds \right] + C |\Omega^\varepsilon|_{L^\infty} (|v|_{L^2}^2 E_t(0, t) + E(t, 0)) \right\}, \]

that is
\[ N(t) \leq A N^2(t) + B \quad \text{with} \quad \begin{cases} A := C \left\{ \int_0^t \Omega^\varepsilon(\xi) E_t(0, s) \, ds \right\}, \\ B := C |\Omega^\varepsilon|_{L^\infty} E_t(0, t) (|v|_{L^2}^2 + 1) \end{cases}. \]

Hence, as soon as
\[ 4AB = 4C |\Omega^\varepsilon|_{L^\infty} E_t(0, t) (|v|_{L^2}^2 + 1) \left( \int_0^t \Omega^\varepsilon(\xi) E_t(0, s) \, ds \right) \leq 1, \quad (4.9) \]

there holds
\[ N(t) \leq \frac{2B}{1 + \sqrt{4AB}} \leq 2B = C |\Omega^\varepsilon|_{L^\infty} E_t(0, t) (|v|_{L^2}^2 + 1), \]

that means, in term of the difference \( v - z \),
Finally, condition (4.9) gives a constraint on the final time $T^\varepsilon$. Indeed we ask for

$$4C^2 |\Omega| \leq E_0(t, 0)(|v_0|^2 + 1) < 1 \quad (4.10)$$

and

$$\int_0^T \Omega^\varepsilon(\xi) E_0(0, s) \, ds < 1 \quad (4.11)$$

to assure condition (4.9) to be satisfied. Constraint (4.10) can be rewritten as

$$\exp\left(- \int_0^t \lambda_\varepsilon(\xi) \, d\sigma\right) = E_0(t, 0) \leq \frac{C}{|\Omega| \leq (|v_0|^2 + 1)},$$

that is, we can choose $T^{\varepsilon, 1}$ of the form

$$T^{\varepsilon} := \frac{1}{|\Lambda^\varepsilon|} \ln \left( \frac{C}{|\Omega| \leq (|v_0|^2 + 1)} \right) \sim C |\Lambda^\varepsilon|^{-1} \ln |\Omega|^{-1}.$$ 

On the other hand, from (4.11), we have

$$\int_0^T \Omega^\varepsilon(\xi) E_0(0, s) \, ds \leq |\Omega| \int_0^t e^{N^\varepsilon} < 1,$$

that is

$$|\Omega| \leq |\Lambda^\varepsilon|^{-1}(e^{N^\varepsilon} - 1) < 1 \quad \Rightarrow \quad T^{\varepsilon, 2} := \frac{1}{|\Lambda^\varepsilon|} \ln \left( \frac{\Lambda^\varepsilon}{|\Omega| \leq} + 1 \right) \sim C |\Lambda^\varepsilon|^{-1}.$$ 

Now the proof is completed. \qed

Theorem 4.1 gives a very precise estimate for the perturbation $v$. Indeed, we have proven that $v$ has decay properties similar to those of the function $z(x, t)$, hence it converges to zero very fast for $t \to +\infty$; moreover, the difference $|v - z|$, is bounded by $|\Omega|$, meaning that it is small with respect to $\varepsilon$.

**Remark 4.2.** The previous proof can be easily extended to the case $\varepsilon \in [L^2(\Omega)]^n$ (see also [22, theorem 2.1]). This is meaningful in light of a possible application of this result in the case of systems.

Since the final time $T^\varepsilon$ is diverging to $+\infty$ for $\varepsilon \to 0$, estimate (4.4) holds globally in time, and the precise decomposition for the perturbation $v$ can be used in the equation for $\xi(t)$ in order to decouple the system (4.1). Indeed

$$|\langle \partial_\xi \xi^\varepsilon, v \rangle| \leq C(|v_0| \leq e^{N^\varepsilon} + |\Omega|),$$

so that, for small $\varepsilon$ and $|v_0|$, the function $\xi(t)$ behaves like the solution $\zeta(t)$ to the following problem.
Proposition 4.3. Let hypotheses H1–H2 be satisfied. Let us also assume that 
\[ \theta'(\xi) < 0 \text{ for any } \xi \in I, \quad \text{and} \quad \theta''(\bar{\xi}) < 0, \] 
(4.12)
Then, for sufficiently small \( \varepsilon \) and \( |\nu_0| \), \( \xi(t) \) converges to its equilibrium location \( \bar{\xi} \) as \( t \to +\infty \).

Proof. For any initial datum \( \xi_0 \), the variable \( \xi(t) \) solves an equation of the form
\[
\frac{d\xi}{dt} = \theta'(\xi)(1 + r(t)) \quad \text{with} \quad |r(t)| \leq C(|\nu_0| \varepsilon e^{\Lambda t} + |\Omega'|_{\infty}).
\]
Therefore, by means of the standard method of separation of variables and since \( \theta'(\xi) \sim \theta'(\bar{\xi})(\xi - \bar{\xi}) \), we get
\[
\int_{\xi_0}^{\xi(t)} \frac{d\eta}{\eta - \bar{\xi}} \sim \theta'(\bar{\xi}) t,
\]
that is, \( \xi \) converges to \( \bar{\xi} \) as \( t \to +\infty \) and the convergence is exponential, in the sense that there exists \( \beta > 0 \) such that
\[
\xi(t) \sim \bar{\xi} + \xi_0 e^{-\beta t}, \quad \beta = -\theta'(\bar{\xi})
\]
(4.13)
for any \( t \) under consideration.

Estimate (4.13) shows the exponentially slow motion of the interface for small \( \varepsilon \). Indeed, its evolution towards the equilibrium position is much slower as \( \varepsilon \) becomes smaller, since \( \beta \to 0 \) as \( \varepsilon \to 0 \).

More precisely, the location of the interface \( \xi(t) \) remains close to some non-equilibrium value for time \( T^\varepsilon \), which can be extremely long when \( \varepsilon \) is small, before converging to its stable configuration, corresponding to one of the wall \( x = 0 \) or \( x = \ell \).

4.2. Nonlinear metastability for a convection-reaction-diffusion equation

The general result presented in theorem 4.1 is concerned with the reduced system (4.1), obtained by disregarding higher order terms in the variable \( v \) but still keeping the nonlinear dependence on \( \xi \). In the last part of this section we mean to analyse the complete system for the couple \( (\xi, v) \), that is
\[
\begin{aligned}
\frac{d\xi}{dt} &= \theta'(\xi)(1 + \langle \partial_\xi \psi_1, v \rangle + \rho'[\xi, v]), \\
\partial_t v &= H^\varepsilon(x; \xi) + (L_\xi^\varepsilon + M_\xi^\varepsilon)v + R^\varepsilon[v, \xi],
\end{aligned}
\]
(4.14)
where
\[
\rho'[\xi, v] := \langle \psi_1^*, Q[v, \xi] \rangle(1 + \langle \partial_\xi \psi_1, v \rangle + \langle \psi_1^*, P'[U'] \rangle + Q'[v, \xi])R(v),
\]
\[
R^\varepsilon[v, \xi] := Q'[v, \xi] - \partial_\xi U^\varepsilon(\cdot; \xi) \rho'[\xi, v].
\]
In particular, recalling that
\[ Q^e[v, \xi] := \frac{1}{2} \left\{-\partial_x(f''(U)v^2) + f''(U)v^2\right\}. \]
we have
\[ |Q^e[v, \xi]|_{L^2} \leq C \int_0^T |v|^2_{L^2} dt. \]
Thus, in order to proceed with the same methodology implemented in the proof of theorem 4.1, we also need an estimate for the \( L^2 \) norm of the space derivative of the component \( v \).

**Theorem 4.4.** Let hypotheses H1–H2 be satisfied and let us denote by \((\xi, v)\) the solution to the initial-value problem (4.14), with
\[ \xi(0) = \xi_0 \in (0, \varepsilon) \quad \text{and} \quad v(x, 0) = v_0(x) \in H^1(I), \]
Then, for sufficiently small \( \varepsilon \), there exists a time \( T^\varepsilon \geq 0 \), such that, for any \( t \leq T^\varepsilon \), the solution \( v \) can be represented as
\[ v = z + R, \]
where \( z \) is defined by
\[ z(x, t) := \sum_{k \geq 2} \int_{0}^{t} \lambda_k(\xi(\sigma)) v_k(\sigma) \phi_k(x; \xi(t)). \]
and the remainder \( R \) satisfies the estimate
\[ |R|_{L^p} \leq C \left\{ \epsilon^\delta \exp \left( \int_{\lambda_0}^{\lambda_{max}} \lambda_k(\xi(\sigma)) d\sigma \right) \|v_0\|^2_{L^2} + \epsilon^{\gamma - \delta} \right\}, \quad (4.15) \]
for some constant \( C > 0 \) and for some \( \delta \in (0, \gamma), \gamma > 0 \). Furthermore, the final time \( T^\varepsilon \) can be chosen of order \( 1/\varepsilon^\alpha \), for some \( \alpha > 0 \).

We note that the estimate (4.15) is weaker than the corresponding formula (4.4) obtained for the reduced system, since it states that the remainder \( R \) tends to 0 as \( \varepsilon^\delta \) instead of \( |\Omega|/\varepsilon \).

This deterioration of the estimate is a consequence of the necessity of also estimating the first order derivative and it is probably related to the specific strategy we use at such stage. We suspect that this bound is not optimal. Anyway, let us stress that this nonlinear result makes the theory much more complete.

**Proof.** Since the plan of the proof closely resembles the one used for proving theorem 4.1, here we propose only the major modifications to the argument. In particular, the key point is how to handle the nonlinear terms.

Setting as usual
\[ v(x, t) = \sum_j v_j(t) \phi_j(x, \xi(t)), \]
we obtain an infinite-dimensional differential system for the coefficients \( v_j \)
\[ \frac{dv_j}{dr} = \lambda_j(\xi) v_k + \psi_k, F + \psi_k, G, \quad (4.16) \]
where $F$ is defined as before as

$$F := H^\epsilon - \theta^\epsilon \sum_j a_j + \sum_{j' \neq j} b_{j'j} v_j,$$

with

$$a_j := \langle \partial_x \psi_{1j}, \varphi_j^\epsilon \rangle \partial_x U^\epsilon + \partial_{xx} \psi_{1j}, \quad b_{j'j} := \langle \partial_x \psi_{1j'}, \varphi_j^\epsilon \rangle \partial_{xx} \psi_{1j'}.$$

The term $G$ comes out from the higher order terms $\rho^\epsilon$ and $\mathcal{R}^\epsilon$ and has the following expression

$$G := Q^\epsilon - \left\{ \sum_j \partial_x \varphi_j v_j + \partial_{xx} U^\epsilon \right\} \left\{ \frac{\langle \psi_{1j}, Q^\epsilon \rangle}{1 - \langle \partial_x \psi_{1j}, v \rangle} - \theta^\epsilon \frac{\langle \partial_x \psi_{1j}, v \rangle^2}{1 - \langle \partial_x \psi_{1j}, v \rangle} \right\}.$$

Moreover, we have

$$|\langle \psi_{kj}, G \rangle| \leq (1 + |\Omega^\epsilon|_{\infty}) |v|^2_{L^2} + C |v|^2_{\mathcal{R}}$$

so that, setting

$$E_k(s, t) := \exp \left( \int_s^t \lambda_k^\epsilon(\xi(\sigma)) d\sigma \right)$$

and since $v_1 = 0$, we have the following expression for the coefficients $v_k$, $k \geq 2$

$$v_k(t) = v_k(0) E_k(0, t) + \int_0^t \{ \langle \psi_{k}, F \rangle + \langle \psi_{k}, G \rangle \} E_k(s, t) d s.$$

By introducing the function

$$z(x, t) := \sum_{k \geq 2} v_k(0) E_k(0, t) \varphi_{kj}(x, \xi(t)),$$

we end up with the following estimate for the $L^2$-norm of the difference $v - z$

$$|v - z|_{L^2} \leq \sum_{k \geq 2} \int_0^t (\Omega^\epsilon(\xi(1 + |\Omega^\epsilon|_{L^2} + |v|^2_{\mathcal{R}})) E_k(s, t) d s,$$

that is

$$|v - z|_{L^2} \leq C \int_0^t \Omega^\epsilon(\xi(t - s))^{-1/2} E_2(s, t) d s + \int_0^t (|v|^2_{\mathcal{R}} + |z|^2_{\mathcal{R}})(t - s)^{-1/2} E_2(s, t) d s.$$
where we used
\[ \sum_{k \geq 2} E_k(s, t) \leq C (t - s)^{-1/2} E(s, t). \]

Now we need to differentiate with respect to \( x \) the equation for \( v \) in order to obtain an estimate for \( |\partial_x (v - z)| \). By setting \( y = \partial_x v \), we obtain
\[ \partial_y = L \partial_y + \tilde{M}_x y - \partial_x (\partial_x U^x v) + \partial_x H^x(x, \xi) + \partial_x R^x[\xi, v], \]
where
\[ \tilde{M}_x y := -\partial_x U^x(\cdot; \xi) \theta^x(\xi) \langle \partial_x \psi_1^x, v \rangle. \]

Hence, by setting as usual
\[ y(x, t) = \sum_j y_j(t) \varphi_j^x(x, \xi(t)), \]
we have
\[ \frac{dy_j}{dt} = \lambda_j^x(\xi) y_j + \langle \psi_1^x, F^x \rangle - \langle \psi_1^x, \partial_x (\partial_x U^x v) \rangle + \langle \psi_1^x, \partial_x R^x \rangle, \]
where
\[ F^x := \partial_x H^x - \sum_j v_j \left\{ \theta^x \left( \partial_x U^x \langle \partial_x \psi_1^x, \varphi_j^x \rangle + \partial_x \varphi_j^x \left( 1 + \sum_{x'} v_{x'} (\partial_x \psi_1^x, \varphi_j) \right) - \partial_x \varphi_j^x \rho^{x'} \right\}. \]

Moreover, for some \( m > 0 \), there holds
\[ |\langle \psi_1^x, \partial_x R^x \rangle| \leq C |v|_{L^2}^2, \quad |\langle \psi_1^x, \partial_x (\partial_x U^x v) \rangle| \leq C |v|_{L^2}^2 + \frac{1}{\varepsilon^m} |v|_{L^2}^2. \]

Again, by integrating in time and by summing on \( k \), we end up with
\[ |y - \partial_x z| \leq C \int_0^t \left\{ \Omega^x(\xi)(1 + |v|_{L^2}^2) + \left( 1 + \frac{1}{\varepsilon^m} \right) |v|_{L^2}^2 + \varepsilon^m |U|_{L^\infty}^2 \right\} E_k(s, t) ds \]
\[ + C \int_0^t \left\{ \Omega^x(\xi)(1 + |v|_{L^2}^2) + \left( 1 + \frac{1}{\varepsilon^m} \right) |v|_{L^2}^2 + \varepsilon^m |U|_{L^\infty}^2 \right\} \sum_{k \geq 2} E_k(s, t) ds. \]

Now, given \( n > 0 \), let us set
\[ N(t) := \frac{1}{\varepsilon^n} \sup_{s \in [0, t]} |v - z|_{L^p} E(s, 0), \]
so that we have

\[
\frac{1}{\varepsilon} E_I(t, 0) |v - z|_p^2 \leq C \int_0^t \frac{Q_T(E_I)}{\varepsilon^n}(t - s)^{-1/2} E_2(s, t) E_i(s, 0) ds \\
+ \int_0^t \frac{1}{\varepsilon^n} \left[ |v - z|^2_{p'} + |z|^2_{p'} \right] (t - s)^{-1/2} E_2(s, t) E_i(s, 0) ds
\]

(4.17)

and

\[
\frac{1}{\varepsilon} E_I(t, 0) |y - \partial_t z|_p^2 \leq C \int_0^t \frac{Q_T(E_I)}{\varepsilon^n} \{ E_i(s, 0) + (t - s)^{-1/2} E_2(s, t) E_i(s, 0) \} ds \\
+ C \int_0^t \left\{ \frac{1}{\varepsilon^n} \left[ |v - z|^2_{p'} + |z|^2_{p'} \right] + \frac{1}{\varepsilon^{n+m}} \left| U^T \right|^2_{L^2} \right\} E_i(s, 0) ds \\
+ C \int_0^t \left\{ \frac{1}{\varepsilon^n} \left[ |v - z|^2_{p'} + |z|^2_{p'} \right] + \frac{1}{\varepsilon^{n+m}} \left| U^T \right|^2_{L^2} \right\} (t - s)^{-1/2} E_2(s, t) E_i(s, 0) ds.
\]

(4.18)

By summing (4.17) and (4.18) and since there holds

\[
\int_0^t e^{2\lambda_2^{-1} - \lambda_2^{-m}} ds \leq \int_0^t e^{\lambda_2^{-m}} ds = \frac{1}{\lambda_2^2} (e^{\lambda_2^{-m}} - 1) \leq \frac{1}{|\lambda_2^2|},
\]

\[
\int_0^t (t - s)^{-1/2} E_2(s, t) ds \leq \int_0^t (t - s)^{-1/2} e^{\lambda_2^{-m}(t-s)} ds \leq \frac{1}{|\lambda_2^2|^{1/2}},
\]

we end up with the estimate \( N(t) \leq AN^2(t) + B \), with

\[
\begin{align*}
A & := \varepsilon^{n-m} E_i(0, t) (t + |\lambda_2^{-1}|), \\
B & := C |\Omega^e|_{L^2} E_i(0, t) (t + |\lambda_2^{-1}|) \\
& \quad + \varepsilon^{n-m} |\lambda_2^{-1}| |v_0|^2_{p'} + \varepsilon^{m-n} |U^T|^2_{L^2} E_i(0, t) (t + |\lambda_2^{-1}|).
\end{align*}
\]

Supposing that \(|\lambda_2^{-1}| \sim \varepsilon^{-\gamma} \) for some \( \gamma > 0 \), if we require \( m < \gamma \), for all \( n > 0 \) there holds

\[
N(t) < 2B \text{ that is}
\]

\[
|v - z|_{p'} \leq C |\Omega^e|_{L^2} \varepsilon^\gamma + (\varepsilon^{\gamma-m}) |v_0|^2_{p'} E_i(0, t) + \varepsilon^m.
\]

(4.19)

Precisely, we can choose \( m = \gamma - \delta \), for some \( \delta \in (0, \gamma) \). Finally, providing \( m > n \), we can choose the final time \( T^* \) of order \( O(\varepsilon^{-\alpha}) \) for some \( 0 < \alpha < 1 \). Now the proof is completed. \( \square \)

Estimate (4.19) can now be used to decouple the complete system (4.14) more precisely, under the hypotheses of proposition 4.3 on the function \( \theta^\xi(\xi) \), we get

\[
\frac{d\xi}{dt} = \theta^\xi(\xi)(1 + r) + \rho^\xi,
\]

where

\[
|r| \leq |v_0|^2_{p'} e^{-\varepsilon t} + (e^\delta + e^{\gamma-\delta}) \quad \text{and} \quad |\rho^\xi| \leq C |v^2_{p'} \leq |v_0|^2_{p'} e^{-\varepsilon t} + e^\delta + e^{\gamma-\delta}.
\]

4358
Hence, there exists $\beta^c > 0$, $\beta^e \to 0$ as $\varepsilon \to 0$, such that

$$|\xi - \bar{\xi}| \leq |\xi_0|e^{-\beta^c t} + \{w_0^2 + e^{-\beta^e t} + e^\gamma t\}e^{-\beta^e t},$$

showing the exponentially (slow) motion of the position of the interface towards its equilibrium location $\bar{\xi}$.

5. Application to the Burgers–Sivashinsky equation

The aim of this section is to apply the general theory developed in the previous sections to the specific example of the so-called Burgers–Sivashinsky equation, that is

$$\partial_t u = \varepsilon \partial_{xx}^2 u - u \partial_x u + u,$$  \hspace{1cm} (5.1)

complemented by boundary conditions and initial datum

$$u(0, t) = u(\ell, t) = 0 \quad t \geq 0, \quad \text{and} \quad u(x, 0) = u_0(x) \quad x \in I.$$  \hspace{1cm} (5.2)

More precisely, we mean to show that the hypotheses of theorem 4.1 and theorem 4.4 are satisfied in this specific case.

To begin with, let us consider the case $\varepsilon = 0$: equation (5.1) formally reduces to the first-order hyperbolic equation

$$\partial_t u = -u \partial_x u + u,$$ \hspace{1cm} (5.3)

together with boundary and initial conditions. In this case, stationary solutions solve the first-order equation

$$\frac{d}{dx}\left(\frac{u^2}{2}\right) = u \implies \frac{du}{dx} = 1,$$

where the boundary conditions have to be interpreted in the sense of [3]. We concentrate on entropy stationary solutions with at most one internal jump. Hence, we can construct a two-parameter family of steady states, solutions to (5.3), defined as

$$U_{\text{hyp}}(x) := \begin{cases} x - \xi_1 & 0 \leq x < (\xi_1 + \xi_2)/2 \\ x - \xi_2 & (\xi_1 + \xi_2)/2 < x \leq \ell \end{cases}$$

where the parameters $(\xi_1, \xi_2)$ belong to the triangle $T := \{(\xi_1, \xi_2) : 0 \leq \xi_1 \leq \xi_2 \leq \ell\}$. In particular, it is possible to distinguish several different regions of the triangle, corresponding to different types of steady states with different properties.

1. When $(\xi_1, \xi_2)$ belongs to the interior of the triangle $\tilde{T} := \{(\xi_1, \xi_2) : 0 < \xi_1 < \xi_2 < \ell\}$, the corresponding solution has two boundary layers and a single jump inside the interval $(0, \ell)$.
2. When $(\xi_1, \xi_2)$ belongs to one of the sides $\Gamma_1 := \{(\xi_1, \xi_2) : \xi_1 = 0, 0 < \xi_2 < \ell\}$ and $\Gamma_2 := \{(\xi_1, \xi_2) : 0 < \xi_1 < \ell, \xi_2 = \ell\}$, the corresponding steady state has one internal jump and one boundary layer.
3. When $(\xi_1, \xi_2)$ belongs to the diagonal side $D := \{(\xi_1, \xi_2) : \xi_1 = \xi_2\}$, the steady $U_{\text{hyp}}$ has two boundary layers and no jump discontinuities. It is the hyperbolic version of $U_{\text{w}}$ and we named it here $U_{\text{hyp}}^0$.
4. When $(\xi_1, \xi_2) = (0, \ell)$, the steady state has a single internal jump in the middle point of the interval, and no boundary layers. We call it here $U_{\text{hyp}}^0$, the rougher version of $U_{\text{w}}$. 

4359
5. Finally, when $(\xi_1, \xi_2)$ coincides with one of the vertices $(0, 0)$ or $(\ell, \ell)$, the corresponding solutions have one boundary layer (on the right-hand side and the left-hand side of the interval, respectively), and no jump discontinuities. These solutions are the hyperbolic versions of $U_{+, \ell}$ and $U_{-, \ell}$, respectively, and we refer to them as $\pm U_0$.

Hence, at the level $\varepsilon = 0$, we have infinitely many steady states with different properties (see figures 2 and 3).

When $\varepsilon > 0$, the presence of the Laplace operator has a smoothing effect on stationary solutions: all the jump discontinuities turn into a viscous shock. It is possible to obtain an implicit expression for the stationary solutions to (5.1), which solve

$$\frac{d}{dx} \left( \frac{u^2}{2} - \frac{\varepsilon du}{dx} \right) = u \iff \begin{cases} \frac{\varepsilon du}{dx} = \frac{u^2}{2} - v \\ \frac{dv}{dx} = u. \end{cases}$$

If we look for a solution of the form $u = \sigma(v)$, we get the following Bernoulli equation

$$\frac{d\sigma}{dv} = \frac{du}{dx} \frac{dx}{dv} = \frac{1}{2\varepsilon} \sigma - \frac{1}{\varepsilon} \sigma v,$$

that can be solved by means of the standard change of variable $\omega = \sigma^2$. We deduce the following equation for $\omega$

$$\frac{d\omega}{dv} - \frac{\omega}{\varepsilon} = -\frac{2v(x)}{\varepsilon} \iff \frac{d}{dv} (e^{-\varepsilon/\omega}) = \frac{d}{dv} (2(\varepsilon + v)e^{-\varepsilon/\omega}),$$

(5.4)

whose solution is given by

$$\frac{1}{2} u^2 = \varepsilon + v + \kappa e^{\varepsilon/\omega}, \quad \frac{dv}{dx} = u.$$
for some $\kappa \in \mathbb{R}$. We are interested in stationary solutions corresponding, when $\varepsilon > 0$, to the steady states $U_{\varepsilon U x}^{\pm}$ (case 5), $U_{\varepsilon U x}^{NS}$ (case 4) and, specially, $U_{\varepsilon U x}^{0}$ (case 3), the one that gives rise to metastable behaviour. In these cases, stationary solutions to (5.1) can be seen as a smoothed version of the states $\pm U_{0}$, $U_{0}^{NS}$ and of the state $U_{0}^{M}$ with the choice $\kappa = \xi = \ell / 2$.

5.1. The family of approximate steady states and the remainder $P^r[U]$

Following the general approach introduced in section 3, the first step is the construction of a one-parameter family of functions $\{U^{\varepsilon}(x; \xi)\}_{\xi \in I}$ that approximate the metastable steady state $U_{\varepsilon U x}^{M}(x)$. There are several possible choices to build up such family. One idea is to define $U^{\varepsilon}(x; \xi)$ as a smoothed version of the hyperbolic steady state $U_{\varepsilon U x}^{0}(x)$. The smoothing consists in substituting a viscous shock-like version of the layers in a small neighbourhood of the transition point, based on the explicit formula

$$U^{\varepsilon}_{\text{ve}}(x) := -u_0 \tanh \left( \frac{u_0(x - a_0)}{2\varepsilon} \right).$$
that describes a viscous shock wave centered in \( a_0 \) connecting the left state \( u^* \) and the right state \(-u^*\). Hence, we define

\[
U^\varepsilon(x; \xi) = \max \left\{ \min \left\{ x - \xi, -(\ell - \xi) \tanh \left( \frac{\xi - \xi}{2\varepsilon} (x - \ell) \right) \right\}, -\xi \tanh \left( \frac{\xi}{2\varepsilon} \right) \right\},
\]

where the parameter \( \xi \in I \) represents the unique zero of the function \( U^\varepsilon \) (see figure 3).

With such a construction, \( U^\varepsilon \) matches exactly the boundary conditions and satisfies the stationary equation up to an error, denoted here as \( \varepsilon \mathcal{P}[U^\varepsilon] \).

Moreover, for fixed \( \xi \in I \), each element of the family \( \{ U^\varepsilon(x; \xi) \} \) converges, as \( \varepsilon \to 0 \), to \( U^0_{d}(x) \), defined as

\[
U^0_{d}(x) = x - \xi, \quad \xi \in (0, \ell), \quad U^0_{d}(0) = U^0_{d}(\ell) = 0.
\]

Once the family of approximate steady states is explicitly given, the next step is to compute the error \( \varepsilon \mathcal{P}[U^\varepsilon] \). We recall here that, in order to compute the remainder \( \varepsilon \mathcal{P}[U^\varepsilon] \), we need an explicit expression for the term \( \Omega^\varepsilon(\xi) \), defined as

\[
|\langle \psi(\cdot), \varepsilon \mathcal{P}[U^\varepsilon(\cdot; \xi)] \rangle | \leq \Omega^\varepsilon(\xi) \| \psi \|_{L^\infty}, \quad \forall \, \psi \in C(I), \forall \, \xi \in I.
\]

Given \( U^\varepsilon \) as in (5.5), we check that \( U^\varepsilon \) is an exact stationary solution for (5.1), outside the intervals \((0, u_1^*)\) and \((u_2^*, \ell)\), where \( u_1^* \) and \( u_2^* \) are implicitly defined as

\[
-\xi \tanh \left( \frac{\xi}{2\varepsilon} u_1^* \right) = u_1^* - \xi, \quad -(\ell - \xi) \tanh \left( \frac{\ell - \xi}{2\varepsilon} (u_2^* - \ell) \right) = u_2^* - \xi.
\]

From (5.6), since \( u_1^* \sim 0 \) and \( u_2^* \sim \ell \), we get

\[
u_1^* \sim 0 \xi \quad \text{and} \quad u_2^* \sim \ell - \varepsilon \xi \quad \text{as} \quad \varepsilon \to 0,
\]

where we used the Taylor expansion of \( \tanh(x) \) for \( x \sim 0 \). Now we can derive an asymptotic expression for the term \( \Omega^\varepsilon(\xi) \). Indeed we get

---

**Figure 4.** The element \( U^\varepsilon \) of the family of approximate stationary solutions obtained as a smoothed version of the hyperbolic steady state \( U^0_{d}(x) \).
\|
\psi(t), P^\ell [U^e (\cdot; \xi)] \| \leq \| \psi \|_{L^2} \left( \int_0^L \left| \xi \tan \left( \frac{\xi}{2\varepsilon} \right) \right| dx + \int_{u_2^L}^{\ell} \left( \ell - \xi \right) \tan \left( \frac{\ell - \xi}{2\varepsilon} \right) (x - \ell) \right) dx \right)
\leq \| \psi \|_{L^2} \left( \int_{u_2^L}^{\ell} (\ell - \xi) (x - \ell) \right) dx,
\leq \| \psi \|_{L^2} [\xi u_2^L + (\ell - \xi)(\ell - u_2^L)],

and we deduce
\Omega^e (\xi) \sim \xi^2 \varepsilon + (\ell - \xi) \xi \varepsilon,

showing that \Omega^e is null at \xi = 0 and small with respect to \varepsilon. We can also define
\omega^0 (\xi) \sim \xi \varepsilon + (\ell - \xi) \varepsilon,

where \Omega^e (\xi) \leq \omega^0 (\xi) \xi, meaning that, when \xi = 0, the element \overline{U^e (x; 0)} solves the stationary equation. Indeed \overline{U^e (x; 0)}, defined as
\overline{U^e (x; 0)} := \min \left\{ x - \ell \tan \left( \frac{\xi}{2\varepsilon} \right), 0 \right\}

is an approximation (up to an error that is small in \varepsilon) for the stable stationary solution \overline{U_{\varepsilon, x}(x)}.

It is also possible to derive an explicit expression for the leading order term in the equation of motion for \xi(t), defined as
\theta^e (\xi) = \frac{\langle \psi_1^e, P^\ell [U^e] \rangle}{\langle \psi_1^e, \partial_\xi U^e \rangle}.

Indeed, with the approximation \overline{U^e (x, \xi)} \sim \overline{U_{\varepsilon, x}(x; \xi)} = x - \xi, we get \partial_\xi \overline{U^e} = -1, so that
\langle \psi_1^e, P^\ell [U^e] \rangle \sim \xi^2 \varepsilon + (\ell - \xi) \xi \varepsilon \int \psi_1^e (x; \xi) dx \quad \text{and} \quad \langle \psi_1^e, \partial_\xi U^e \rangle \sim - \int \psi_1^e (x; \xi) dx,

and
\theta^e (\xi) \sim -\xi^2 \varepsilon - (\ell - \xi) \xi \varepsilon. \quad (5.7)

Since, formally, for small \varepsilon and small \ell, the dynamics of the parameter \xi is approximately given by
\frac{d\xi}{dt} = \theta^e (\xi),

formula (5.7) shows that the speed of the interface is small with respect to \varepsilon. Finally, \theta^e (\xi) < 0 for all \xi \in I, and it is easy to check that
\theta^e (0) = -\varepsilon \ell < 0.

Hence, from (4.13) (see proposition 4.3), we get
\xi (t) \sim \xi_0 e^{-\beta t}, \quad \text{with} \quad \beta^e \sim \varepsilon,

showing the exponentially slow convergence of the interface location toward its equilibrium \xi = 0. More precisely, the speed rate of convergence is proportional to \varepsilon, and the convergence is slower as \varepsilon becomes smaller.
Remark 5.1. The previous construction for the family of functions \( \{ U^\varepsilon(x; \xi) \} \) is useful since we can derive an explicit expression for an element of the family, so that we can explicitly develop computations; also, it is possible to generalise this construction for a general nonlinearity \( f \) by using in (5.5) the solution to \( \varepsilon \partial_x^2 u = \partial_x f(u) \) (instead of \( U^\varepsilon_{sc} \), solution to \( \varepsilon \partial_x^2 u = u \partial_x u \)). However, this construction does not lead to an optimal estimate for the terms \( \Omega^\varepsilon \) and \( \theta^\varepsilon \), since we expect they behave like \( \varepsilon^{-1/2} \). The reason is that the construction given in (5.5) is a very crude approximation of the exact steady state of the problem; indeed, in a \( O(\varepsilon) \) neighbourhood of the transition points, we replace the real solution with a shock version of the layers, and this procedure will then produce errors of order \( \varepsilon \) rather then of order \( \varepsilon^{-1/2} \).

Following the line of [22], another possible way to build up the family \( \{ U^\varepsilon(x; \xi) \} \) consists of matching two exact steady states in the intervals \((0, \xi)\) and \((\xi, \ell)\) at \( x = \xi \) under appropriate boundary conditions. Precisely, denoted by \( U^\pm(x, a, b) \) the unique positive (negative, respectively) stationary solution to

\[
\begin{cases}
\partial_x u = \varepsilon \partial_x^2 u - \partial_x f(u) + f'(u) & x \in (a, b), t \geq 0 \\
u(a, t) = u(b, t) = 0 & t \geq 0
\end{cases}
\]

\( U^\varepsilon(x; \xi) \) is obtained by matching the functions \( U^+(x, 0, \xi) \) and \( U^-(x, \xi, \ell) \) (satisfying the left and the right boundary conditions respectively, together with the request \( U^\varepsilon(\xi) = 0 \)). In formulas

\[
U^\varepsilon(x; \xi) := \begin{cases} U^+(x; 0, \xi) & \text{for } x < \xi \\ U^-(x, \xi, \ell) & \text{for } x > \xi. \end{cases}
\]

With such a construction \( U^\varepsilon \in C^0(I) \) but it is not a \( C^1 \)-matched function. In this case we have no explicit expression for an element of the family of approximate steady states, and we can only state that the error \( \mathcal{P}^\varepsilon[U^\varepsilon] \) is concentrated in \( x = \xi \), and it is defined as

\[
\mathcal{P}^\varepsilon[U^\varepsilon(x; \xi)] = \| \partial_x U^\varepsilon \|_{x=\xi} \delta_{x=\xi}.
\]

so that \( \Omega^\varepsilon(\xi) \sim \| \partial_x U^\varepsilon \|_{x=\xi} \). Hence, in the limit \( \varepsilon \to 0 \), the nonlinear term \( \theta^\varepsilon(\xi) \) behaves like

\[
\theta^\varepsilon(\xi) \sim \| \partial_x U^\varepsilon \|_{x=\xi},
\]

and the (slow) speed of the interface depends only on the residual of \( U^\varepsilon \); as already pointed out in [22, 23, 31]. Motivated by these previous papers, we expect both \( \Omega^\varepsilon \) and \( \theta^\varepsilon \) to be exponentially small in \( \varepsilon \), even if we are not able to prove it in this present paper.

This possible construction is meaningful since it proposes a way to build up a family of approximate steady states for any choice of \( f \) that should lead to optimal estimates for the error terms, in contrast with the one given in (5.5); indeed, with such a construction, the specific form of the nonlinearity \( f \) is really taken into account when solving the viscous stationary equation in order to define \( U^\varepsilon(x; \xi) \). The main difficulty here is to be able to have an explicit expression for the exact steady state: in principle, this could be done numerically, and the computation of the error \( \Omega^\varepsilon \) should follow easily.

5.2. The spectrum of the linearised operator

We analyse the spectrum of the linearised operator \( \mathcal{L}_\Omega \) and we give a precise distribution of its (real) eigenvalues, in order to show that the general technique developed in section 3 is indeed applicable in the case of the Burgers–Sivashinsky equations.
Having chosen an approximate steady state $U^c(x; \xi)$ satisfying the boundary conditions $U^c(0) = U^c(\ell) = 0$, and linearising equation (5.1) around $U^c$, we end up with the following linearised problem

$$\partial_t v = \mathcal{L}_\xi v := \varepsilon \partial_t^2 v - \partial_\xi (U^c v) + v, \quad v(0) = v(\ell) = 0.$$  \hfill (5.8)

First of all, let us notice that $\lambda^c_1$ is an eigenvalue for $\mathcal{L}_\xi$ if and only if $\lambda^c - 1$ is an eigenvalue for the differential linear diffusion-transport operator

$$\mathcal{L}_\xi^{\text{vsc}} v := \varepsilon \partial_t^2 v - \partial_\xi (U^c v),$$  \hfill (5.9)

so that we reduce to the study of the eigenvalue problem for $\mathcal{L}_\xi^{\text{vsc}}$. We recall here that $U^c(x) \to U_{\text{hyp}}(x)$ for $\varepsilon \to 0$, where $U_{\text{hyp}}$ is defined as $U_{\text{hyp}}(x) = x - \xi$. More precisely there exists $C > 0$ such that

$$|U^c - U_{\text{hyp}}| \leq C\varepsilon.$$

5.2.1. Estimate for the first eigenvalue. An asymptotic expression for the first eigenvalue of the linearised operator $\mathcal{L}_\xi$ was provided by Sun and Ward in [33]. Here the authors linearised equation (5.1) around an approximate steady state obtained by using the so-called method of matched asymptotic expansion; moreover, they show that the principal eigenvalue associated with this linearisation is positive and has the following asymptotic expression (for more details, see [33, section 3])

$$\lambda^c_1(\xi) \sim \frac{1}{\varepsilon} \left[ \xi - (\ell - \xi) + (\ell - \xi)(\ell - \xi) e^{-\varepsilon^{1/2}} \right].$$

In particular, the equivalence holds true in the limit $\varepsilon \to 0$. The positivity of $\lambda^c_1$ is strictly related to the instability property of the stationary solution $U^c(x)$. However, even if $\lambda^c_1$ is positive, it is exponentially small with respect to $\varepsilon$, leading to a metastable behaviour for solutions starting from an initial datum close to $U^c$. Indeed the dynamics take place within a time scale of order $\varepsilon^{1/2}$, $c > 0$, since the long-time behaviour of solutions is described by terms of order $e^{-\varepsilon^{1/2}}$.

5.2.2. Estimate for the second eigenvalue. We mean to control from above the location of the second (and subsequent) eigenvalue of $\mathcal{L}_\xi^{\text{vsc}}$, in order to control the location of the eigenvalues of $\mathcal{L}_\xi$. Let us introduce the self-adjoint operator

$$\mathcal{M}_\xi v := \varepsilon^2 \partial_\xi^2 v + a^c(x; \xi(t))v, \quad a^c(x; \xi(t)) := \left( \frac{U^c}{2} \right)^2 + \frac{1}{2} \varepsilon \partial_\xi U^c.$$  \hfill (5.10)

It is easy to check that if $\varphi^c$ is an eigenfunction of (5.9) of eigenvalue $\lambda^c$, then the function $\psi^c$ defined as

$$\psi^c(x; \xi) = \exp\left(-\frac{1}{2\varepsilon} \int_{x_0}^x U^c(y)dy\right) \varphi^c(x; \xi)$$  \hfill (5.11)

is an eigenfunction of (5.10) relative to the eigenvalue $\mu^c = \varepsilon \lambda^c$.

Now let us consider the couples $(\varphi^c_1, \lambda^c_1)$ and $(\psi^c_2, \mu^c_2)$. Since $\lambda^c_2$ is the second eigenvalue, then there exists $x_0 \in (0, \ell)$ such that $\varphi^c_2(x_0) = 0$. Hence, $\varphi^c_2$ restricted to $(x_0, \ell)$ can be seen as the first eigenfunction (relative to the first eigenvalue) of the linearised operator $\mathcal{L}_\xi^{\text{vsc}}$ in the interval $(x_0, \ell)$ and with Dirichlet boundary conditions. In other words, $\varphi^c_2(x)$ can be seen as the first
eigenfunction of the linearised operator obtained by linearising (5.1) around $U^{x,\xi}(x, x_0, \xi)$, that is an approximation of the unique positive stationary solution to (5.1) in the interval $(x_0, \ell)$. The same argument holds for $\psi_2^{x}(x)$.

From now on, without loss of generality, we restrict our analysis to the interval $J = (x_0, \ell)$ and we renormalise $\psi_2^{x}$ so that $\psi_2^{\xi}(x_0) = 1$. Integrating over $J$ we obtain

$$\lambda_2^{x} \int_{x_0}^{\ell} \phi_2^{x}(x) dx = \varepsilon (\phi_2^{x}(\ell) - \phi_2^{x}(x_0)) < -\varepsilon \phi_2^{x}(x_0). \quad (5.12)$$

Moreover, because of (5.11) we have

$$\phi_2^{x}(x) = \psi_2^{x}(x) \exp \left( \frac{1}{2\varepsilon} \int_{x_0}^{x} U^{x}(y) dy \right) + \psi_2^{x}(x) \exp \left( -\frac{1}{2\varepsilon} \int_{x_0}^{x} U^{x}(y) dy \right) \frac{U^{x}(x)}{2\varepsilon},$$

so that, since $\psi_2^{x}(x_0) = 0$, we get $\phi_2^{x}(x_0) = \psi_2^{x}(x_0)$. Hence, from (5.12) we can state that

$$\left| \lambda_2^{x} \right| > \varepsilon I_{\varepsilon}^{-1} \psi_2^{x}(x_0), \quad I_{\varepsilon} := \int_{x_0}^{\ell} \exp \left( \frac{1}{2\varepsilon} \int_{x_0}^{x} U^{x}(y) dy \right) dx.$$

Now we need an estimate from above for $I_{\varepsilon}$. We get

$$I_{\varepsilon} \leq \exp \left( \frac{1}{2\varepsilon} \int_{x_0}^{\ell} y - \xi dy \right) dx \leq \exp \left( \frac{1}{2\varepsilon} \int_{x_0}^{\ell} e^{(x - \xi)^{2}/4\varepsilon} dx, \right.$$

where we used $|U^{x} - U^{y}|_{L_{\varepsilon}} \leq C$. Now, recalling the definition of the imaginary error function

$$\text{erfi}(x) := \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{t^2} dt,$$

we state that

$$\int_{x_0}^{\ell} \frac{e^{(x - \xi)^{2}/4\varepsilon}}{4\varepsilon} dx = \sqrt{\pi \varepsilon} \left[ \text{erfi} \left( \frac{\ell - \xi}{2\sqrt{\varepsilon}} \right) - \text{erfi} \left( \frac{x_0 - \xi}{2\sqrt{\varepsilon}} \right) \right].$$

Moreover, from the asymptotic expansion of $\text{erfi}(x)$ for $x \to +\infty$, we have, in the limit $\varepsilon \to 0$

$$\text{erfi} \left( \frac{\ell - \xi}{2\sqrt{\varepsilon}} \right) \sim \frac{e^{-\frac{(\ell - \xi)^{2}}{4\varepsilon}}}{2\sqrt{\varepsilon}} \quad \text{and} \quad \text{erfi} \left( \frac{x_0 - \xi}{2\sqrt{\varepsilon}} \right) \sim e^{-\frac{(x_0 - \xi)^{2}}{4\varepsilon}}.$$

From (5.13) and (5.14), we deduce

$$I_{\varepsilon} \sim C \sqrt{\varepsilon} e^{-\frac{(x_0 - \xi)^{2}}{4\varepsilon}} \left[ e^{i(x_0 - \xi)^{2}/4\varepsilon} - e^{i\xi^{2}/4\varepsilon} - 1 \right].$$

Hence

$$I_{\varepsilon}^{-1} \sim \frac{1}{\sqrt{\varepsilon}} \frac{1}{\exp \left( \frac{(x_0 - \xi)^{2}}{4\varepsilon} \right) - 1}.$$
so that, provided \( x_0, \ell \) and \( \xi \) such that \( 2\xi > x_0 + \ell \), we obtain
\[
|\lambda_2^\varepsilon| > \sqrt{\varepsilon} \psi_2^\varepsilon(x_0). \tag{5.15}
\]
To complete our computations, we need an estimate from below for \( \psi_2^\varepsilon(x_0) \). To this aim, let us denote by \( x_\mu \in (x_0, \ell) \) the \( x \) value such that \( \psi_2^\varepsilon(x_\mu) = \max \psi_2^\varepsilon \) and let us renormalise \( \psi_2^\varepsilon \) such that \( \psi_2^\varepsilon(x_\mu) = 1 \). It is possible to prove that \( x_\mu \to \ell \) as \( \varepsilon \to 0 \); hence, \( |x_\mu - \ell| \leq c \varepsilon \), that is \( |x_0 - x_\mu| \geq c \varepsilon \). Therefore, for each \( x \in (x_0, x_\mu) \), we get
\[
\psi_2^\varepsilon(x) = \frac{1}{x_\mu - x_0} \geq \frac{1}{c \varepsilon},
\]
so that, since \( \psi_2^\varepsilon \) is concave
\[
\psi_2^\varepsilon(x_0) \geq \frac{1}{c \varepsilon}. \tag{5.16}
\]
Plugging (5.16) into (5.15), we end up with
\[
|\lambda_2^\varepsilon| \geq C \sqrt{\varepsilon} \iff \lambda_2^\varepsilon \leq - C \sqrt{\varepsilon},
\]
proving that all the eigenvalues of \( L_\xi^{\varepsilon, \text{vsc}} \) (and of \( L_\xi^\varepsilon \) as well) for \( k \geq 2 \) are negative, bounded away from zero, and behave like \(-1/\sqrt{\varepsilon}\).

**Remark 5.2.** The spectral analysis performed above can be adapted to the case of a general function \( f \) satisfying hypotheses (1.2), provided to substitute \( \psi^{\varepsilon}(U) \) instead of \( U^\varepsilon \) in (5.8), and so on in the following computations. This is important, especially when trying to find an estimate from above for \( I^\varepsilon \); we also point out that in the general case one should verify the stronger condition \(|f'(U^\varepsilon) - f'(U_{\varepsilon,0})| \leq C \varepsilon \), and should construct \( U^\varepsilon \) accordingly.

**References**

[1] Alikakos N, Bates P W and Fusco G 1991 Slow motion for the Cahn–Hilliard equation in one space dimension *J. Differ. Equ.* 90 81–135

[2] Aronson D G, Crandall M G and Peletier L A 1982 Stabilization of solutions of a degenerate nonlinear diffusion problem *Nonlinear Anal. Theory Methods Appl.* 6 1001–22

[3] Bardos C, le Roux A Y and Nédélec J-C 1979 First order quasilinear equations with boundary conditions *Commun. PD* 4 1017–34

[4] Beck M and Wayne C E 2009 Using global invariant manifolds to understand metastability in the Burgers equation with small viscosity *SIAM J. Appl. Dyn. Syst.* 8 1043–65

[5] Berestycki H, Kamin S and Sivashinsky G 1995 Nonlinear dynamics and metastability in a Burgers type equation *C. R. Acad. Sci., Paris* 321 185–90

[6] Berestycki H, Kamin S and Sivashinsky G 2001 Metastability in a flame front evolution equation *Interfaces Free Bound.* 3 361–92

[7] Bobidus L E, O’Regan D and Royalty W D 1987 Steady-state reaction-diffusion-convection equations: dead cores and singular perturbations *Nonlinear Anal. Theory Methods Appl.* 11 527–38

[8] Carr J and Pego R L 1989 Metastable patterns in solutions of \( u_t = \varepsilon^2 u_{xx} + f(u) \) *Commun. Pure Appl. Math.* 42 523–76

[9] Chen T, Levine H A and Sacks P E 1988 Analysis of a convective reaction-diffusion equation *Nonlinear Anal. Theory Methods Appl.* 12 1349–70

[10] Evans L C 2010 *Partial Differential Equations (Graduate Studies in Mathematics* vol 19) 2nd edn (Providence, RI: American Mathematical Society) p xxii + 749

[11] Fusco G and Hale J K 1989 Slow-motion manifolds, dormant instability, and singular perturbations *J. Dyn. Differ. Equ.* 1 75–94
[12] Goodman J 1994 Stability of the Kuramoto–Sivashinsky and related systems Commun. Pure Appl. Math. 47 293–306
[13] Hill A T and Suli E 1995 Dynamics of a nonlinear convection-diffusion equation in multidimensional bounded domains Proc. R. Soc. Ed. 125A 439–48
[14] Howes F A 1988 The asymptotic stability of steady solutions of reaction-convection-diffusion equations J. Reine Angew. Math. 388 212–20
[15] Howes F A and Whitaker S 1988 Asymptotic stability in the presence of convection Nonlinear Anal. Theory Methods Appl. 12 1451–9
[16] Kruzkov S N 1970 First order quasilinear equations in several independent variables Mat. Sb. 81 228–55
Kruzkov S N 1970 First order quasilinear equations in several independent variables Math. USSR Sb. 10 217–43
[17] Lax P D 1954 Weak solutions of nonlinear hyperbolic equations and Their numerical computations Commun. Pure Appl. Math. 7 159–93
[18] Laforgue J G L and O’Malley R E Jr 1994 On the motion of viscous shocks and the supersensitivity of their steady-state limits Methods Appl. Anal. 1 465–87
[19] Laforgue J G L and O’Malley R E Jr 1995 Shock layer movement for Burgers equation SIAM J. Appl. Math. 55 332–47 (Perturbations Methods in Physical Mathematics (Troy, NY, 1993))
[20] Lyberopoulos N 1991 Asymptotic oscillations of solutions to scalar conservation laws with convexity under the action of a linear excitation Q. Appl. Math. 48 755–65
[21] Mascia C and Terracina A 1999 Large-time behavior for conservation laws with source in a bounded domain J. Differ. Equ. 159 485–514
[22] Mascia C and Strani M 2013 Metastability for nonlinear parabolic equations with application to scalar conservation laws SIAM J. Math. Anal. 45 3084–113
[23] Mascia C and Strani M 2014 Slow motion for compressible isentropic Navier–Stokes equations preprint
[24] Nessyahu H 1996 Convergence rate of approximate solutions to weakly coupled nonlinear system Math. Comput. 65 575–86
[25] Oleinik O A 1957 Discontinuous solutions of nonlinear differential equations Am. Math. Soc. Transl. 26 95–172
[26] Otto F and Reznikoff M G 2006 Slow motion of gradient flows J. Differ. Equ. 237 372–420
[27] Pego R L 1989 Front migration in the nonlinear Cahn–Hilliard equation Proc. R. Soc. Lond. A 422 261–78
[28] Rakib Z and Sivashinsky G I 1987 Instabilities in upward propagating flames Combust. Sci. Technol. 54 69–84
[29] Reyna L G and Ward M J 1995 On the exponentially slow motion of a viscous shock Commun. Pure Appl. Math. 48 79–120
[30] Sattinger D H 1972 Monotone methods in nonlinear elliptic and parabolic boundary value problems Indiana Univ. Math. J. 21 979–1000
[31] Strani M 2015 Slow motion of internal shock layers for the Jin–Xin system in one space dimension J. Dyn. Differ. Equ. 27 1–27
[32] Strani M 2014 Nonlinear metastability for a parabolic system of reaction-diffusion equations submitted
[33] Sun X and Ward M J 1999 Metastability for a generalized Burgers equation with application to propagating flame fronts Eur. J. Appl. Math. 10 27–53