Abstract

In the previous paper [Yamada, Chaos, Solitons & Fractals, 109,99(2018)], we investigated localization properties of one-dimensional disordered electronic system with long-range correlation generated by modified Bernoulli (MB) map. In the present paper, we report localization properties of phonon in disordered harmonic chains generated by the MB map. Here we show that Lyapunov exponent becomes positive definite for almost all frequencies except $\omega = 0$, and the $B$-dependence changes to exponential decrease for $B > 2$, where $B$ is a correlation parameter of the MB map. The distribution of the Lyapunov exponent of the phonon amplitude has a slow convergence, different from that of uncorrelated disordered systems obeying a normal central-limit theorem. Moreover, we calculate the phonon dynamics in the MB chains. We show that the time-dependence of spread in the phonon amplitude and energy wave packet changes from that in the disordered chain to that in the periodic one, as the correlation parameter $B$ increases.

Keywords: Phonon, Acoustic wave, Localization, Bernoulli map, Delocalization, Long-range, Correlation

1. Introduction

It has been established that one-dimensional disordered system (1DDS) has a pure point energy spectrum and its eigenfunctions are exponentially localized in an infinite system [1,2]. As a result, the ensemble averaged transmission coefficient of a large enough system decreases exponentially with respect to the system size $N$ [3]. This statement is established for standard 1DDS without particular reference to electronic or phonon system [1]. On the other hand, it has been reported that in binary disordered systems special delocalized states exist. For example, the diagonally disordered dimer model corresponding to a one-dimensional tight-binding binary alloy should have extended states the number of which is proportional to $\sqrt{N}$ for a finite system size [4]. In addition to this, a set of extended modes close to the critical frequency has been confirmed in the disordered one-dimensional dimer harmonic chain [5,6].

It is difficult to experimentally investigate the effect of structural correlation on the localization of one-dimensional the electronic systems due to the electron-electron interaction. The correlation effect have been experimentally realized by using a one-dimensional optical systems [7,8,9]. Indeed, in recent years, there have been experimental studies on photonic localization in disordered glasses in which light waves perform a Lévy flight [7] and anomalous localization in microwave waveguide with long-range correlated disorder [8].

In our earlier papers, we have also numerically investigated the localization phenomena of electronic systems with long-range correlations generated by the modified Bernoulli map (MB) with stationary-nonstationary chaotic transition (SNCT) [10]. The detailed property of the MB map is given in Refs. [12,13,14,15,16,17]. In particular, wave packet dynamics in the nonstationary potential has been investigated [18,19]. The sequence exhibits asymptotic non-stationary chaos characterized by the power spectrum $S(f) \sim 1/\alpha^2$ ($f << 1$), where $f$ denotes frequency and $\alpha$ is spectrum index, for $\alpha > 1$. We shall refer to such a system MB chain in the following [12,15]. In the MB chain, it is possible to create the potential sequence that changes its property from short-range correlation (SRC) including $\delta$-correlations to the long-range correlation (LRC), if we regulate the correlation parameter $B$. The relation between the spectrum index $\alpha$ and the correlation parameter $B$ of the MB map is given by Eq.(8) in the text. However, the studies on the properties of the phonon system with LRC, as compared with those in the electron system, are rather scarce [20,21,22,23,24]. In particular, there are many unclear points as to phonon dynamics of the disordered harmonic chains with LRC [25,26,27,28,29,30,31,32,33,34,35,36].

In this paper, disordered phonon systems with LRC generated by MB map are studied numerically. We aim at reporting the characteristic $B$-dependences of the Lyapunov exponent ($L$-exponent), and the phonon dynamics. Although the $L$-exponent is positive throughout the entire set of $B$ regions studied here, the $B$-dependence decreases linearly for $B < 2$, and it decays exponentially for $B > 2$. The distribution of the $L$-exponent over the ensemble for the disorder configuration exhibits some slow convergence as $N \to \infty$, unlike that of uncorrelated disordered systems because of its LRC. In addition, we investigate the time-dependence of the initially localized wave packet in the MB chain due to the displacement excitation by changing the correlation parameter. We confirm that the spread of the energy wave packet exhibits subdiffusive behaviour, compared to ballistic one as $B$ increases.
This paper is organized as follows. In the next section, we shall briefly introduce the phonon model and the modified Bernoulli map. In Sect. [3] we report about the behaviour of the B-dependence of Lyapunov exponent at some frequencies by the numerical calculation. We show the correlation of the mass sequence effects on the convergence property of the distribution as the system size increases. In Sect. [4] phonon dynamics in the MB chains is investigated by changing the correlation parameter. The summary and discussion are presented in the last section. Appendix [Appendix A] shows that the anomalous distribution of phonon transmission coefficient over ensemble have a non-universal form, which is different from that in uncorrelated disordered chains.

2. Model

Here we consider the harmonic chain model represented by the following equation of motion:

\[ m_n \frac{\partial^2 u_n(t)}{\partial t^2} = -K_n(u_n - u_{n-1}) + K_{n+1}(u_{n+1} - u_n), \]

(1)

where \( u_n \) is the displacement from its equilibrium position of the \( n \)–the atom and \( m_n \)'s and \( K_n \)'s are sequences of masses and force constants of nearest neighbour atoms, respectively. We deal with two types of phonon model, the one is with disordered masses but constant force constants \( K_n = K(= 1) \) (mass model), and the other is spring disordered model with constant mass \( m_n = m(= 1) \) (spring model). The one model is transformed into the other model by a dual transformation \([3,7]\).

In the long wavelength approximation of the mass model, we obtain a scalar wave equation, using continuous variables \( x \),

\[ \frac{\partial^2 u(x, t)}{\partial t^2} = \frac{K}{m(x)} \frac{\partial u(x, t)}{\partial x}, \]

(2)

and for the continuous variable version of the spring model we get the following wave equation:

\[ \frac{\partial^2 u(x, t)}{\partial t^2} = \frac{\partial}{\partial x} \left[ e(x) \frac{\partial u(x, t)}{\partial x} \right], \]

(3)

where \( e(x) \) is the \( x \)–dependent elastic stiffness and \( m_0 \) is the mass density of the medium.

These phonon systems start to be equivalent to that of off-diagonal tightly binding electronic system \(-t_{n+1}v_{n+1} - t_{n-1}v_n + \beta v_n = E v_n \) with constant diagonal element \( \beta \) by a transformation using mass-reduction as follows:

\[ \begin{cases} \sqrt{m_n} u_n \to v_n, \\ \sqrt{m_{n+1}} \to t_{n+1}. \end{cases} \]

(4)

Furthermore, the particle-hole symmetry for \( E = 0 \) in the off-diagonal electronic system corresponds to the translation mode of \( \omega = 0 \) in the phonon system. For weak correlations, all the eigenmodes with \( \omega > 0 \) are localized. The uniform mode (\( \omega = 0 \)) remains extended in the thermodynamic limit.

In this paper we deal with mass model that the mass sequences \( m_n \)'s is generated by a modified Bernoulli (MB) map with LRC. The MB map is one-dimensional map proposed in order to reveal the statistical properties of an intermittent chaos \([12]\).

\[ X_{n+1} = \begin{cases} X_n + 2^{\alpha - 1} X_n^{\beta} & (0 \leq X_n < 1/2) \\ X_n - 2^{\alpha - 1}(1 - X_n)^\beta & (1/2 \leq X_n \leq 1), \end{cases} \]

(5)

where \( B \) is a non-negative bifurcation parameter which controls the strength of correlation among the sequence \( X_n \)'s. We use the symbolized sequence according to the following rule:

\[ m_n = \begin{cases} m_a & 0 \leq X_n < 1/2 \\ m_b & 1/2 \leq X_n \leq 1. \end{cases} \]

(6)

as a sequence of the masses. The mass ratio \( R = m_b/m_a \) stands for the parameter controlling the strength of the disorder. The power spectrum in the low frequency limit \( f \ll 1 \) and thermodynamic limit \( N \to \infty \)

\[ S(f) = \frac{1}{N} \left| \sum_{n=0}^{N} m_n e^{-2\pi i fnN} \right|^2 \]

\( f = 0, 1, 2, ..., N = 1 \) should behave as

\[ S(f) \sim \begin{cases} f^0 & 1 \leq B < 3/2 \\ f^{-\alpha} & 3/2 \leq B \leq \infty, \end{cases} \]

(7)

where

\[ \alpha = \frac{2B - 3}{B - 1}. \]

(8)

The bifurcation parameter \( B = 1 \) exhibits an exponential damping of the correlation. In the range of \( 1 < B < 3/2 \), the resulting white power spectrum proves that the sequence has only SRC. The theoretical interpretation of the power spectral density has been given by renewal process analysis for the semi-markovian symbolic dynamics \([11]\). And it has been reported that the numerical result also supports the theoretical result \([12]\).

Here, it is important to mention the difference between the MB chain and the model with Lévy-type disorder in Ref.\([36]\).

There is a common point in that the inverse power-law distributions are used in order to characterize the correlation, but there is a significant difference in the nonstationary characterized by the power spectrum \( S(f) \sim f^{-\alpha} \) with \( \alpha \geq 1 \) \([36]\). The Lévy-type disorder model generates the nonstationary sequence with \( \alpha = 1 \) at a point of the control parameter of the correlation (Lévy exponent), while MB chain with \( B \geq 2 \) can generate various nonstationary sequence with \( 1 \leq \alpha < 2 \).

Assuming the monochromatic time dependence \( u_n(t) = e^{-i\omega t} u_n(t = 0) \) for Eq.\([1]\) we obtain the stationary equation of motion,

\[ -m_n\omega^2 u_n = -K_n(u_n - u_{n-1}) + K_{n+1}(u_{n+1} - u_n), \]

(9)

characterized by a frequency \( \omega \). The phonon spectrum \( G(\omega^2) \) of the mass model \( (m_a = 1.0, m_b = 2.0) \) is shown in Fig\([1]\). This spectrum for the case of \( B = 1.01 \) is strongly resembling that in the uncorrelated chains. This spectra is characterized by a singular-peak structure and infinitesimally small gaps dubbed the special frequencies \([33,39]\). A complicated structure in phonon amplitude and phase is related to this singular structure of the phonon spectra. We can also observe a structure like the van-Hove singularity in a periodic lattice in the case of \( B = 1.8 \).
as a result of LRC. Naturally the same features are observed. The details will be reported in elsewhere. We keep our eyes on the localization properties of phonon amplitude in the next section.

3. Localization of phonon amplitude

In this section, we study the Lyapunov exponent depending on the correlation parameter $B$ by using the transfer matrix method \[40, 41\].

3.1. Transfer matrix and Lyapunov exponent

The equation (9) can be written in terms of the product of the transfer matrix $T_n$ as

$$
\begin{align*}
\begin{pmatrix}
\mu_{n+1} \\ \mu_n
\end{pmatrix} &=
T_n
\begin{pmatrix}
\mu_{n-1} \\ \mu_{n-2}
\end{pmatrix} =
\Pi_{i=1}^n T_i
\begin{pmatrix}
\mu_1 \\ \mu_0
\end{pmatrix},
\end{align*}
$$

where

$$
T_n =
\begin{pmatrix}
(K_{n+1} + K_{n+1}^{-1} - m_n) & -K_{n+1}^{-1} \\ -1 & 0
\end{pmatrix}.
$$

We are interested in the asymptotic property of the amplitude $\mu_n$ for $n \to \infty$ or the corresponding limit theorem for the product of the matrices. The asymptotic behaviour of Eq. (10) with respect to the system size is characterized by the L-exponent of the phonon amplitude of finite size $N$ as follows:

$$
\gamma_N = \frac{\ln||M(N)\mu_0||}{2N},
$$

where $M(n) = \Pi_{i=1}^n T_i$, $\mu_0 = (\mu_1, \mu_0)^T = (1, 0)^T$ when the set of the initial values $\mu_0$ and $\mu_1$ is given. The frequency dependence of the localization length $\xi_N \approx 1/\gamma_N$ for acoustic and electromagnetic waves in a one-dimensional randomly layered media is also studied analytically \[40\].

Figure 1 shows the $\omega^2$ dependence of the averaged L-exponents $\langle \gamma_N \rangle$ in the MB chains. The zero mode $\omega = 0$ corresponds to the extended state in the translational motion mode. In the case of SRC ($B = 1.1$), it can be confirmed that $\gamma_N \propto \omega^3 (\omega << 1, N >> 1)$. Looking at the smaller side of $\omega$ that works for dynamics, we see that L-exponent decreases as $B$ increases. The $\omega$ dependence of the L-exponent has been investigated for the anomalous localization in one-dimensional chains with Lévy-type disorder and the power-law dependence for the Lévy exponent is found \[36\].

We investigate whether the state is delocalized by increasing $B$ or not. The $B$ dependence of the averaged Lyapunov

![Figure 1](image1.png)

![Figure 2](image2.png)
is positive even for $B \geq 2$, at least in the investigated range ($2 \leq B \leq 3$). This result is consistent with many claims that there is no transition to delocalization in the regime $\alpha < 2$, in many electronic systems which are well studied numerically by using the finite-size scaling\cite{18, 43, 44, 45}. Correspondingly, in the phonon system examined in this paper, for $B \geq 2$ and increasing $N$, the averaged L-exponent does not seem to become zero except for $\omega = 0$. However, as shown in in the Fig\ref{fig:3}(a), it is suggested that the dependence on $B$ changes with $B \approx 2$ from linear decay to exponential decay, i.e. $\langle \gamma_N \rangle \sim e^{-cB^{−2}}$. This change of the $B$–dependence corresponds to the chaos-chaos transition at $B = 2$ of the MB map.

We define the normalized localization length (NLL) to characterize the tail of the wavefunction\cite{19, 42, 43, 46}.

$$\Lambda_N \equiv \frac{1}{\langle \gamma_N \rangle N}. \quad (13)$$

It is useful to study the localization and delocalization property that $\Lambda_N$ decreases (increases) with the system size $N$ for localized (extended) states, and it becomes constant for the critical states. The $B$–dependence of the NLL are shown in Fig\ref{fig:3}(b). It is found that the even for $2 < B(\leq 3)$ the NLL decreases and the localization length $1/\langle \gamma_N \rangle$ is less than the system size $N$ in the thermodynamic limit $N \to \infty$. As a result, we can say that the states are exponentially localized for the case of $\omega^2 = 1.01$.

It is worth noting that Furstenberg’s theorem can be applied to the product of matrices for at least the stationary regime $B < 2$, because the sequence is a renewal process with a finite average residence time in the MB chain\cite{1}. As a result, L-exponent in the infinite system is positive-definite and sample independent with probability 1 for any non-vanishing and finite initial vector\cite{47, 48}.

More detailed investigation is necessary in the nonstationary regime with the power spectrum $S(f) \sim f^{-\alpha}$ with $1 \leq \alpha < 2$ because the nonstationary sequence is not the sufficient condition for the delocalization.

### 3.2. Convergence of the distribution

Previous section demonstrates, the $B$–dependence of the average value of L-exponent is similar to that of the electronic system. The correlation effect is also expected to show up in the form at the anomalous distribution and variance, as in the case of electronic system. Therefore, we will confirm the point.

Figures\ref{fig:4} and\ref{fig:5} show the distribution of L-exponent of phonon amplitude over $2^{15}$ samples. We have performed the calculations for the mass model with a mass ratio ($R = 2$) at some values of squared frequency $\omega^2$. Distribution of almost Gaussian type are observed in the case of $B = 1.3$, in which the structural correlation is of short range. The behaviour of the distribution is quite similar to that of the uncorrelated disordered system, while the distribution obeys normal central-limit theorem (CLT). The multi-peak structure is observed in the distributions for the case of $B = 2.2$ at the squared frequency $\omega^2 = 2.1$. Two sharp peaks on the both sides of the main distribution in Fig\ref{fig:5}(c) come from the LRC, which proves a certain amount of very long pure and almost pure subsystems.

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![Figure 3: (Color online) (a)The ensemble averaged Lyapunov exponents ($\gamma_N$) of the phonon amplitude for $\omega^2 = 1.01$ as a function of the bifurcation parameter. The cases of $N = 2^{16}, 2^{18}, 2^{20}$ are plotted. The ensemble size is $2^{15}$. The inset shows the logarithmic plot. (b)The normalized localization length of the phonon amplitude for $\omega^2 = 1.01$ as a function of the bifurcation parameter.](image-url)
Figure 4: (Color online) Histograms of the distribution of $L$-exponent of phonon amplitude in the MB chains with some system size $N = 2^9 - 2^{16}$ for squared frequency $\omega^2 = 1.01$ in the cases of (a) $B = 1.3$, (b) $B = 1.8$ and $B = 2.2$. We have used mass model with $m_a = 1$, $m_b = 2$ and a mesh of histogram in a horizontal line is 0.001. The ensemble size is $2^{15}$.

Figure 5: (Color online) Histograms of the distribution of $L$-exponent of phonon amplitude in the MB chains with some system size $N = 2^9 - 2^{16}$ for squared frequency $\omega^2 = 2.1$ in the cases of (a) $B = 1.3$, (b) $B = 1.8$ and $B = 2.2$. We have used mass model with $m_a = 1$, $m_b = 2$ and a mesh of histogram in a horizontal line is 0.001. The ensemble size is $2^{15}$. 
We consider the fluctuation of L-exponent distribution using the scaling form,
\[ \sqrt{\langle (\Delta \gamma)^2 \rangle} \propto N^{-\kappa(B)}, \]  
(14)
to fit the numerical data. The estimated value of \( \kappa(B) \) is plotted in Fig.6. For \( 1 < B < 3/2 \), the value of \( \kappa \) is roughly 1/2, implying that the convergent property of the distribution with respect to \( N \) obeys or approximately obeys CLT. However, for 3/2 < \( B < 2 \), the distribution converges more slowly than that obeying the CLT. This property is a remarkable feature as the correlation increases. Moreover, convergence of distribution is hardly observed for \( B \geq 2 \). In particular, in the case of \( \omega^2 = 2.01 \), it accumulates to \( \langle \gamma_N \rangle \approx 0 \), so variance of distribution increases. As a result, the \( B \)-dependence of the L-exponent and the convergence property of distribution form change around SNCT \( B = 2 \) of the MB map.

![Figure 6](image-url)

Figure 6: (Color online) The exponent \( \kappa(B) \) of the power for the standard deviation of the L-exponent of the phonon amplitude as a function of the bifurcation parameter. The cases of \( \omega^2 = 1.01, 2.1, 3.1 \) are plotted.

The apparently anomalous distribution of the L-exponent is also reflected in the distribution of the transmission coefficient which is closer to the physical quantity. The data showing the abnormality of the distribution of the phonon transmission coefficient different from the usual uncorrelated disordered ones are in the Appendix.

4. Phonon dynamics

In this section, we study the time-evolution of vibrational wave packet described by the equation of motion in Eq.(1) on the MB chains with the long-range correlated random masses by using the leap-frog integration scheme [49] with the time mesh \( \Delta t = 0.0001 \).

As initial conditions \( t = 0 \), we chose a delta function at middle site of the chain, and the zero velocities at all the sites as follows;
\[
\begin{align*}
\mu_n(t = 0) &= U_0 \delta_{n,N/2} \\
\nu_n(t = 0) &= 0.
\end{align*}
\]  
(15)

Further we chose \( U_0 = 1 \). Although other choices are possible, the qualitative conclusions of this report are independent of the initial conditions. To qualitatively measure the degree of localization of the phonon, we evaluate the spreading of the wave packet by the mean square displacement of the phonon amplitude \( m_2(t) \) [24], the participation ratio of the amplitude \( P(t) \) [20, 21], and the energy spread of the wave packet \( E(t) \) [22, 24, 35] as a function of time. Below, we see how these three quantities change when \( B \) increases, compared with a case of binary periodic chain (BP).

4.1. Mean square displacement \( m_2(t) \)

First, we compute the time-dependence of the second order moment;
\[ m_2(t) = \frac{1}{N} \sum_{n=1}^{N} \left( \frac{n - N/2}{N/2} \right)^2 \langle |\mu_n(t)|^2 \rangle, \]  
(16)
where \( \langle .. \rangle \) denotes the average over the initial values of the MB map.

Figure 7 shows the numerical result of the root mean square displacement \( \sqrt{m_2(t)} \) in the MB chains with \( B = 1.1, 1.7, 3.0 \). The log-log plot of the data reveals \( \sqrt{m_2(t)} \sim t^\mu \), and numerically estimated \( \mu \) is shown in the inset. It occurs that for weak correlation cases it is subdiffusive \( 0.7 \lesssim \mu < 1 \), and the index \( \mu \) gradually approaches 1 as the \( B \) increases, which proves the ballistic spreading. This tendency agrees with that reported by Naumis et al [24]. Accordingly, to sum up, it can be said that the difference due to the change in \( B \) is small.

4.2. Participation ratio \( P(t) \)

We define the time-dependent participation ratio \( (PR) \) of the displacement \( \mu_n(t) \) by the following equation,
\[ P(t) = \frac{\sum_{n=1}^{N} \langle |\mu_n(t)|^2 \rangle}{\sum_{n=1}^{N} \langle |\mu_n(t)|^4 \rangle}. \]  
(17)
It is clear that \( P(t) \approx \xi_p^2 \) for an exponentially localized case, where \( \xi_p \) is the localization length of the wave packet. On the other hand, if the wave packet is extended \( P(t) \) will be of order of unity, \( P(t) \sim O(1) \) [29].

Figure 8(a) shows time-dependence of the PR in the logarithmic scale. It follows that the localization length \( \xi_p \) increases as \( B \) increases, and it is very different from the behavior of the binary periodic system. In Fig.8(b), the squared localization length \( \xi_p^2 \) of the phonon amplitude is shown, which is numerically estimated by \( P(t) \approx \xi_p^2 \) for \( t \gg 1 \). It can be seen that \( \xi_p^2 \) increases gradually for \( B < 2 \) and rapidly increases for \( B > 2 \) as \( B \) increases.
The energy of the disordered chain is distributed in a time-varying fashion between their kinetic and potential energies. Here, we can define the actual mid-value of the energy of the pulse as follows:

\[ R(t) = \frac{\sum_{n=1}^{N} n \langle E_n(t) \rangle}{\sum_{n=1}^{N} \langle E_n(t) \rangle}. \]  

(18)

where \( E_n(t) \) denotes the local energy at site \( n \). Furthermore, the spread of the energy for the initial displacement excitation can be defined by the second order moment as follows:

\[ E(t) = \frac{\sum_{n=1}^{N} (n - R(t))^2 \langle E_n(t) \rangle}{\sum_{n=1}^{N} \langle E_n(t) \rangle}. \]  

(19)

Although the fluctuation of the position \( R(t) \) of the centre of the energy exists, the energy spreading Eq.\( (19) \) is essentially the same as in the case of \( \nu = 2 \) in the Eq.(47) in the reference [31]. It has been reported that in the uncorrelated random chains while the participation number remains finite, i.e. localized state, the energy spread is shown to be way \( E(t) \sim t^{0.5} \) (short-wavelength limit) after displacement excitation owing to the unscattered states of the order \( O(\sqrt{N}) \) around \( \omega = 0 \) [22]. In periodic chains, \( E(t) \) exhibits the ballistic spread as \( E(t) \sim t^2 \).

It is expected, as shown in Fig.\( 9 \), that the energy spread asymptotically approaches the behaviour \( E(t) \sim t^{0.5} \) at infinite time for \( B = 1.1 \). This corresponds to the case of uncorrelated 1DDS. Numerically, it is consistent with the result of the references [36, 35]. Furthermore, it seems that the transition from \( E(t) \sim t^{0.5} \) behavior to \( E(t) \sim t^2 \) is occurring at \( B \).
large. The $B$-dependence of the index $\delta$ evaluated by fitting for $E(t) \sim t^\delta$ is shown in the inset in Fig.9. It turns out that $\delta$ rapidly increases from $\delta \approx 0.5$ and gradually increases toward $\delta \approx 2$ in the case of the binary periodic systems, as $B$ becomes large. At least in the SRC regime ($B < 3/2$) it is $\delta \approx 0.5$, but more detailed numerical investigation is needed on how to increase $\delta$ for $B > 3/2$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9.png}
\caption{(Color online) (a) The time-dependence of the energy spread $E(t)$ in the MB chains with the initial condition \cite{15} for $B = 1.1, 1.7, 3.0$. Result of the binary periodic chain (BP) is also plotted as a reference. Note that the data are plotted in double logarithmic scales. The ensemble size is 100. Three bold lines correspond to $B=1.1$, 1.3 and 1.5, respectively. The inset shows the power law index $\delta$ as a function of the bifurcation parameter $B$ which is numerically estimated by the time-dependence of the energy spread $E(t)$.}
\end{figure}

5. Summary and discussion

We have studied here some statistical properties of L-exponent of phonon amplitude in a one-dimensional disordered harmonic chain with LRC, which is generated by MB map. The Lyapunov exponents are positive-definite except for the zero mode $\omega = 0$. the $B$-dependence of the L-exponent and the convergence property of the distribution clearly change around the SNCT $B \approx 2$ for the MB map. The convergence properties of the distribution of those quantities with system size $N$ do not obey the central-limit theorem at least for $B > 3/2$. As $B$ increases, the convergence becomes more slow. The slow convergence corresponds to the anomalous large deviation property of the symbolic sequence \cite{13}.

Moreover, here we have investigated the phonon dynamics of the initially localized displacement excitation in the correlated disordered chains. There is a tendency that the participation ratio of the phonon amplitude is maintained at its finite value even if the correlation parameter $B$ increases, i.e., localized state persists. On the other hand, it has been found that the spread of the local phonon energy changes from the behavior, $\delta \sim B^{0.5}$, to the ballistic one, $\delta \sim B^{2}$, along with the increase of $B$. The diffusion index $\delta$ rapidly increases around SNCT $B \approx 2$ of the MB map. Still, for the phonon dynamics it is necessary to obtain a more detailed numerical result.

In this report, we dealt with the harmonic phonon system, but, if the anharmonic terms are introduced in the phonon dynamics, localization effect due to the existence of the breathers modes should also be expected to occur. Finally, the effect of the long-ranged correlation on thermal conduction should also be an interesting feature to investigate \cite{50,51}.

The localization phenomenon in the disordered system with strong correlation appears in various natural phenomena regardless of the electron and phonon systems. For example, seismic wave propagation in a heterogeneous rock can also be localized due to the multiple scattering and interference of the wave \cite{52}.

We expect that the present work would stimulate further studies of the localization in the diverse systems.

Appendix A. Anomalous distribution of the phonon transmission coefficient

In the same way as in the case of electronic system \cite{53,54,55,56,57}, this appendix should present investigations of the correlation effect on the statistical property of the phonon transmission coefficient (PTC) of a finite chain. We consider a finite chain embedded into an infinite perfect lattice with a constant mass ($m = 1$), as compared with those in the uncorrelated cases ($B = 1.1$). It depends only on the transfer matrix $M(N)$ itself and is independent of the boundary condition. The PTC $T(N)$ of a finite system $N$ is given as,

\begin{equation}
T(N) = \frac{4 \sin^2 K}{| -M_{11}e^{iK} + M_{21} - M_{12} + M_{22}e^{iK}|^2},
\end{equation}

where $M_{i,j}$ denotes the $i, j$ matrix element of the transfer matrix $M(N)$ in Eq.(6) and the $K$ means wavevector of incident wave from semi-infinite perfect lattice with the lattice constant. The dispersion relation in the perfect lattice is $\omega^2 = 2(1 - \cos K)$ \cite{53}.

For the sake of understanding the anomalous feature of the distribution over ensemble in detail, we study the relation between the cumulants of distribution for PTC. Some typical relations between the cumulants for the uncorrelated case $B = 1.1$ are shown in Fig.A.10 \cite{56}. It is well observed that some data are plotted on universal curve regardless of the mass ratio. Though these data is only an example of the case of an for a incident wave with a squared frequency $\omega^2 \approx 2$, we can confirm that it is true also for the other cases when the bifurcation parameter is in white-power-spectrum regime ($1 < B < 3/2$). This kind of universality has been strongly suggested to exist in electronic random 1DDS by using several methods \cite{54}.

On the other hand, some remarkable deviations are observed in the relations between stimulants for some values of the bifurcation parameter ($3/2 < B < 2$) in Fig.A.11 compared with the universal one. Here, we can say that the distribution of the PTC in MB system does depend on the bifurcation parameter $B$ controlling the structural correlation and it also depends on the mass ratio $R = m_b/m_a$ in 1DDS with LRC.
Recently, it has been theoretically and experimentally investigated that the anomalous localizations on the transmission of electron and the distribution are caused by the correlated disorder [58].

The cases of \( m_a = 1 \) and \( m_b = 0.9, 0.8, 0.7 \) are plotted. The data of \( B = 1.1 \) in Fig.A.10 are added as a reference.

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Author contribution statement

The sole author had responsibility for all parts of the manuscript.

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