A loss discounting framework for model averaging and selection in time series models

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Abstract

We introduce a Loss Discounting Framework for model and forecast combination which generalises and combines Bayesian model synthesis and generalized Bayes methodologies. We use a loss function to score the performance of different models and introduce a multilevel discounting scheme which allows a flexible specification of the dynamics of the model weights. This novel and simple model combination approach can be easily applied to large scale model averaging/selection, can handle unusual features such as sudden regime changes, and can be tailored to different forecasting problems. We compare our method to both established methodologies and state of the art methods for a number of macroeconomic forecasting examples. We find that the proposed method offers an attractive, computationally efficient alternative to the benchmark methodologies and often outperforms more complex techniques.

Keywords: Bayesian model synthesis; Density forecasting; Forecast combination; Forecast averaging; Multilevel discounting

1. Introduction

In recent years, the development and popularisation of econometric modelling and machine learning techniques as well as an increasingly easy access to vast computational resources and data has lead to a proliferation of forecasting models. These models yield either point forecasts or full forecast density functions and originate from both established subject experts as well...
as newcomers armed with powerful software tools. This trend has been met with a renewed interest in tools that can effectively use these different forecasts such as model selection, and combining, pooling or synthesising forecast densities or point forecasts. e.g., Stock and Watson (2004), Hendry and Clements (2004), Hall and Mitchell (2007), Raftery et al. (2010), Geweke and Amisano (2011), Waggoner and Zha (2012), Koop and Korobilis (2012), Billio et al. (2013), Del Negro et al. (2016), Yao et al. (2018), McAlinn and West (2019), Diebold et al. (2022) to mention just a few. For economic use cases Aastveit et al. (2018) provide a useful survey of methods in the literature.

Combining forecasts from different models is intuitively appealing. For example, Hendry and Clements (2004) suggested that forecast combination provides an insurance against poor performance due to the individual models which are misspecified, poorly estimated or non-stationary, and may provide superior results to any single model. In point forecasting, Bates and Granger (1969) and Stock and Watson (2004) showed that a combination or pooling of competing individual point forecasts can yield forecasts with a lower RMSE than a single model.

In density forecasting, the superiority of a combination over single models is less clear. Bayesian model averaging (BMA) (Leamer, 1978) provides model weights which can be used to weight forecasts in the combination but, as shown in the seminal work of Diebold (1991), BMA may not be optimal under logarithmic scoring when the set of models to be combined is misspecified. Since sets of models will usually not include the true data generating mechanism, this result has driven a substantial literature proposing alternatives to BMA. Hall and Mitchell (2007) proposed to combine forecast densities by minimising the Kullback-Leibler divergence between the forecast density combination and the true but unknown density leading to the logarithmic scoring rule. Their method leads to a time-invariant linear pool with weights chosen on the simplex to maximise the average logarithmic scoring rule. This idea of optimizing the logarithmic scoring rule of the forecast combination has been developed in several directions. Geweke and Amisano (2011) introduced a Bayesian approach to estimating the weights of a time-invariant linear pool. Waggoner and Zha (2012) (Markov switching weights) and Del Negro et al. (2016) (dynamic linear pools) showed that allowing time-varying weights can lead to better forecasting performance. These approaches were combined and extended by Billio et al. (2013) who used sequential Monte Carlo to derive the forecast combinations. These methods can be computationally expensive. An alternative approach, which
is often much computationally cheaper, directly adjusts the model weights from BMA to allow time-variation. Raftery et al. (2005) used BMA applied to a sliding window of \( N \) observations. This led to a more general approach, Dynamic Model Averaging (DMA) (Raftery et al., 2010), which uses an exponential discounting of Bayes factors with a discount/forgetting/decay\(^1\) to achieve time-varying model weights. Del Negro et al. (2016) argued that this can be seen as an approximation to the method of Waggoner and Zha (2012). Koop and Korobilis (2012) used the idea of logarithmic score maximisation to find an optimal discount factor for DMA. Beckmann et al. (2020) applied this idea to model selection and developed Dynamic Model Learning (DML) method with an application to foreign exchange forecasting. Through the lens of the research of Hall and Mitchell (2007), this method could be seen as an optimisation problem with respect to the forgetting factor where one tries to minimise the Kullback-Leibler divergence between the forecasts combined via DMA and the true forecast density.

Recently, McAlinn and West (2019) and McAlinn et al. (2020) proposed a broad theoretical framework called Bayesian Predictive Synthesis (BPS). In their work they used the latent factor regression model, cast as a Bayesian seemingly unrelated regression (SUR), to update latent agent states. In both works the BPS model outperformed the BMA benchmark as well as the optimal linear pool. They also showed that the majority of proposed Bayesian techniques can be classified as special cases of the BPS framework. A recent strand of the literature (for example, Tallman and West (2022) expands the BPS framework to work with general losses through entropic tilting. While this framework, in theory, offers general dynamics for model averaging it can prove computationally demanding in practice. Outside the formal Bayesian framework, Diebold et al. (2022) suggested using a simple average of the forecasts from a team of \( N \) (or less) forecasters, where \( N \) is set \textit{a priori}. The team is selected by choosing the forecasters with the highest average logarithmic scores in the previous \( rw \)-periods. This can be seen as a localised and simplified version of the method of Hall and Mitchell (2007).

This paper describes our Loss Discounting Framework (LDF) which extends DMA and DML to general loss function (to reflect goals other than accuracy of predictive densities such as optimising Sharpe ratios in portfolio selection), and more general discounting dynamics. We interpret DMA or

\(^1\)The terms discount/forgetting/decay factor are used interchangeably in this paper.
DML as the application of a softmax or argmax function to logarithmic scores of each model in a pool. This allows the use of a generalized DMA or DML by replacing the logarithm scores by other loss functions. A time-varying discounting scheme is constructed through layers of meta-models, where the initial layer is just the models in our pool. Meta-models at other layers are constructed by applying our generalized DMA or DML with different discount factors to the meta-models at the previous layer. The use of DMA and DML at each layer leads to a sequence of predictive densities with non-decreasing predictive performance (according to the chosen loss function) whilst maintaining the computational efficiency of DMA and DML. In a simulation study we show that model averaging using LDF outperforms other benchmark methods and is more robust to the choice of the hyperparameters than DMA and DML. We also consider two empirical studies. In the first study, which looks at the foreign exchange forecasting based on econometric fundamentals, LDF based models outperform competing methods in problems with between 32 and 2048 forecasts per period. We also show how the score can be tailored to the construction of long-short foreign exchange portfolios and illustrate the corresponding economic gains from using the LDF. A second example considers US inflation forecasts in presence of only four models and illustrates the limitations of our methodology.

The paper is organised as follows. Section 2 presents some background which leads into a description of the proposed methodology in Section 3. In Section 4, the performance of the LDF approach is examined in a simulated example and foreign exchange, and US inflation using time series models. We discuss the limitations of our approach and set out directions for further research in Section 5. The code to reproduce our study is freely available via the provided link.

2. Background

It is common in Bayesian analysis (Bernardo and Smith, 2009; Yao et al., 2018 and references therein) to distinguish three types of model pool \( \mathcal{M} = \{ M_1, M_2, \ldots, M_K \} \): \( \mathcal{M} \)-closed – the true data generating process is described by one of the models in \( \mathcal{M} \) but is unknown to researchers; \( \mathcal{M} \)-complete – the model for the true data generating process exists but is not in \( \mathcal{M} \), which

\[\text{https://github.com/dbernaciak/ldf}\]
is viewed as a set of useful approximating models; \( \mathcal{M} \)-open – the model for the true data generating process is not in \( \mathcal{M} \) and the true model cannot be constructed either in principle or due to a lack of resources, expertise etc.\(^\text{5}\) Model selection based on BMA only converges to the true model in the \( \mathcal{M} \)-closed case (see e.g. [Diebold, 1991]) and can perform poorly otherwise.

There are several reasons to believe that econometric problems are outside the \( \mathcal{M} \)-closed setting. Firstly, real-world forecasting applications often involve complex systems and the model pool will only include approximations at best. In fact, one might argue that econometric modellers have an inherent belief that the models they propose provide reasonable approximation to the data generating process even if certain process features escape the capabilities of the supplied methodologies. Secondly, in many applications, the data generating process is not constant in time ([Del Negro et al., 2016]) and may involve regime changes and considerable model uncertainty. For example, in the foreign exchange context, [Bacchetta and Van Wincoop (2004)] proposed the scapegoat theory suggesting that investors display a rational confusion about the true source of exchange rate fluctuations. If an exchange rate movement is affected by a factor which is unobservable or unknown, investors may attribute this movement to some other observable macroeconomic fundamental variable. This induces regimes where different market observables might be more or less important.

These concerns motivate a model averaging framework that is both, suitable for \( \mathcal{M} \)-complete (or even \( \mathcal{M} \)-open) situations and incorporates time-varying model weights. We use \( \pi_t|s,k \) to represent the weight of model \( k \) at time \( t \) using information to time \( s \) and use the forecast combination density

\[
p(y_{t+1}|y_s) = \sum_{k=1}^{K} \pi_{t+1|s,k} p_k(y_{t+1}|y_s)
\]

where \( p_k(y_{t+1}|y_s) \) represents the forecast density of model \( k \) at time \( t + 1 \) using information \( y_1, \ldots, y_s \), which we call the predictive likelihood. DMA ([Raftery et al., 2010]), assumes that \( s = t - 1 \) and updates \( \pi_{t|t-1,k} \) using

\[\text{Clarke et al. (2013)}\] give, a slightly unusual, example of works of William Shakespeare as an \( \mathcal{M} \)-open problem. The works (data) has a true data generating process (William Shakespeare) but one can argue that it makes no sense to model the mechanism by which the data was generated.
the observation at time $t$, $y_t$, and a forgetting factor, denoted by $\alpha$, by the recursion

$$
\pi_{t|t,k} = \frac{\pi_{t-1|t,k} P_j(y_t|y_{t-1})}{\sum_{l=1}^K \pi_{t-1|t,l} P_l(y_t|y_{t-1})},
$$

(2.1)

$$
\pi_{t+1|t,k} = \frac{\pi_{t|t,k} + c}{\sum_{l=1}^K \pi_{t|t,l} + c},
$$

(2.2)

c is a small positive number introduced to avoid model probability being brought to machine zero by aberrant observations. The log-sum-exp trick is an alternative way of handling this numerical instability which would, at least in part, eliminate the need for the constant $c$. We leave the role of this parameter to further research.

The recursions in (2.1) and (2.2) amount to a closed form algorithm to update the probability that model $k$ is the best predictive model given information up to time $t$, for forecasting at time $t$. A model receives a higher weight if it performed well in the recent past. The discount factor $\alpha$ controls the importance that one attaches to the recent past. For example, if $\alpha = 0.7$, the forecast performance 12 periods prior to the last one receives approximately 2% of the importance of the most recent observation. However, if $\alpha = 0.9$, this importance is as high as 31%. Therefore, lower values of $\alpha$ lead to large changes in the model weights. In particular, $\alpha \to 0$ would lead to equal model weights and $\alpha = 1$ recovers the standard BMA.

DMA has been shown to perform well in econometric applications whilst avoiding the computational burden of calculating large scale Markov Chain Monte Carlo (MCMC) or sequential Monte Carlo associated with methods such as Waggoner and Zha (2012). Del Negro et al. (2016) showed that DMA performed comparably to their novel dynamic prediction pooling method in forecasting inflation and output growth. It was subsequently expanded and successfully used in econometric applications by Koop and Korobilis (2012), Koop and Korobilis (2013) and Beckmann et al. (2020). In the first two papers the authors test DMA for a few values of discount factors $\alpha$, whereas,

\footnote{Yusupova et al. (2019) note that this constant is only present in the original work by Raftery et al. (2010) and then in the implementation by Koop and Korobilis (2012) but then subsequently dropped in further works, software packages and citations. They also notice that this constant has a non-trivial and often critical effect of the dynamics of weight changes. We comment on this aspect in Appendix C.}
in the latest paper the authors follow the recommendation of Raftery et al. (2010) to estimate the forgetting factor online in the context of Bayesian model selection. We find that estimating the forgetting factor is key to performance of DMA as we will show in our simulation study and empirical examples. Our LDF provides a general approach by combining multiple layers of discounting with time-varying discount factors to provide better performance and robustness to the hyperparameter choice.

3. Methodology

3.1. Loss Discounting Framework

In this section we outline our proposed loss discounting framework (LDF) which can be used for both dynamic model averaging and dynamic model selection. This framework formalizes a method for flexible discounting of a general measure of model performance. Popular discounting based techniques such DMA and DML are special cases of the framework. The framework builds a time-varying discounting scheme by defining layers of meta-models with discount factors.

Under LDF one scores layers of meta-model(s). We first describe how a score can be used to generalize DMA and then describe our discounting scheme using meta-models. The score or loss (we will use these terms interchangeably) is defined for the prediction of an observation with predictive distribution \( p \) and observed value \( y \) and denoted \( S(p, y) \). This measures the quality of the predictive distribution if the corresponding observed value is \( y \).

For a set of \( K \) models, we assume that the (one-step ahead) predictive distribution for model \( k \) at time \( t \) is \( p_{k,t} = p_k(y_t|y_{t-1}) \) we define the log-discounted predictive likelihood for the \( k \)-th model at time \( t \) using discount factor \( \alpha \) to be

\[
\text{LDPL}_{t,k}(\alpha) = \sum_{i=1}^{t-1} \alpha^{i-1} S(p_{k,t_i}, y_{t-i}) .
\]

This can be used to define a model averaged predictive distribution

\[
\sum_{k=1}^{K} w_{t|t-1,k}(\alpha) p_k(y_t|y_{t-1})
\]
where

\[
(w_{t|t-1,1}(m), \ldots, w_{t|t-1,K}(m)) = \text{softmax}(\text{LDPL}_{t,1}(m), \ldots, \text{LDPL}_{t,K}(m))
\]

This generalizes the use of the logarithmic scoring in DMA. The use of scoring rules for Bayesian updating for parameters was pioneered by [Bissiri et al. (2016)](Bissiri2016) (rather than inference about models in forecast combination) and is justified in a \(\mathcal{M}\)-open or misspecified setting. [Loaiza-Maya et al. (2021)](Loaiza2021) extend this approach to econometric forecasting. They both consider sums which are equally weighted (i.e. \(\alpha = 1\) for layer 2). [Miller and Dunson (2019)](Miller2019) provide a justification for using a powered version of the likelihood of misspecified models.

Each meta-model is defined using a recipe for model or meta-model averaging/selection. In this paper we consider a specific type of such recipe which is based on exponential discounting of the scores with different discount factors from a set of possible values \(S_\alpha = \{\alpha_1, \ldots, \alpha_M\}\). In the first layer, each model in the model pool is scored and the \(i\)-th meta-model is defined by applying either DMA or DML with discounting \(\alpha_i\) and the weights defined above.

Then, to construct the second layer, the meta-models in the first layer are scored and the \(i\)-th meta-model is again defined by applying either DMA or DML with discounting \(\alpha_i\) to these scores. This iterative process can be easily extended to an arbitrary number of layers. We highlight two parallels between the methods used in LDF for time series models and concepts in Bayesian modelling. The first parallel is between the layers of meta-models in LDF and the use of hyperpriors in the Bayesian hierarchical models. Similarly to making a decision on the set up of hyperpriors in the hierarchical models LDF allows for varying depth and type of meta-model layers appropriate for the use case in question. We also draw analogy between the model selection versus the maximum a posteriori probability (MAP) estimate of the quantity, and model weights in model averaging versus full posterior distribution.

To provide a full description of the approach, we will write these as \(p^{(0)}_1(y_t|y_{t-1}), \ldots, p^{(0)}_K(y_t|y_{t-1})\) to make notation consistent. At every other layer, we define predictive meta-models which are an average of (meta-)models at the previous layers and have the form

\[
p^{(1)}_m(y_t|y_{t-1}) = \sum_{k=1}^{K} w_{t|t-1,k}^{(1)}(m) p^{(0)}_k(y_t|y_{t-1})
\]
or, for \( n \geq 2 \), we sum over the \( M \) meta-models specified by a \( S_\alpha \),

\[
p_m^{(n)}(y_t|y_{t-1}) = \sum_{k=1}^{M} w_{t|t-1,k}^{(n)}(m) p_k^{(n-1)}(y_t|y_{t-1}).
\]

To define the weights \( w_{t|t-1,k}^{(n)} \), we extend the log-discounted predictive likelihood for the \( k \)-th (meta-)model at the \( n \)-th layer at time \( t \) using discount factor \( \alpha_m \) to be

\[
LDPL_{t,k}^{(n)}(m) = \sum_{i=1}^{t-1} \alpha_m^{i-1} S \left( p_k^{(n)}(y_t|y_{t-i}) \right).
\]

(3.1)

The weights in layer \( n \) are constructed using either a softmax (to give a form of (meta-)model averaging) or an argmax (to give a form of (meta-)model selection). We use the notation \( L_n \) to represent this operation in the \( n \)-th layer which can either take the value \( s \) (softmax) or \( a \) (argmax). If \( L_n = s \),

\[
\left( w_{t|t-1,1}^{(n)}(m), \ldots, w_{t|t-1,K}^{(n)}(m) \right) = \text{softmax} \left( LDPL_{t,1}^{(n-1)}(m), \ldots, LDPL_{t,K}^{(n-1)}(m) \right) \quad \text{if } n = 1
\]
or

\[
\left( w_{t|t-1,1}^{(n)}(m), \ldots, w_{t|t-1,M}^{(n)}(m) \right) = \text{softmax} \left( LDPL_{t,1}^{(n-1)}(m), \ldots, LDPL_{t,M}^{(n-1)}(m) \right) \quad \text{if } n \geq 2.
\]

If \( L_n = a \),

\[
w_{t|t-1,k}^{(n)}(m) = \begin{cases} 
1 & k = k^*(m) \\
0 & k \neq k^*(m)
\end{cases}
\]

where

\[
k^*(m) = \text{argmax} \left( LDPL_{t,1}^{(r-1)}(m), \ldots, LDPL_{t,K}^{(r-1)}(m) \right)
\]

if \( n = 1 \) or, if \( n \geq 2 \),

\[
k^*(m) = \text{argmax} \left( LDPL_{t,1}^{(r-1)}(m), \ldots, LDPL_{t,M}^{(r-1)}(m) \right).
\]

The \( N \)-layer LDF with score \( S \) and with choice \( L_n \) (equal to \( s \) or \( a \)) at layer \( n \) will be written \( \text{LDF}^{N}_{L_1L_2\ldots L_N}(S) \).

The scheme only needs a a single discount factor to be chosen in the final
meta-model layer. This parameter might be set by an expert or calculated on a calibration sample if the data sample is sufficiently large to permit a robust estimation. In LDF, we refer to the discount factor in the final meta-model layer as $\alpha$.

As well as defining a model combination at each layer, LDF also leads to a discount model averaging of the initial model set for any $N$ since

$$p^{(N)}_m(y_t|y_{t-1}) = \sum_{k_N=1}^{M} w_{t[t-1],k_N}^{(N)}(m) p_{k_N}^{(N-1)}(y_t|y_{t-1})$$

$$= \sum_{k_1=1}^{K} \left[ \sum_{k_2=1}^{M} \cdots \sum_{k_N=1}^{M} w_{t[t-1],k_N}^{(N)}(m) \prod_{p=1}^{N-1} w_{t[t-1],k_p}^{(p)}(k_{p+1}) \right] p_{k_1}^{(0)}(y_t|y_{t-1}).$$

Given this set up the models and meta-models are either averaged by using the softmax function or selected by using the argmax function applied to the log-discounted predictive likelihood.

3.2. Special cases

3.2.1. Dynamic Model Averaging

The updates of the Dynamic Model Averaging weights in (2.2) correspond to passing LDPL$^{(0)}_{1,\ldots, K}$ with the logarithmic scoring function through the softmax function. In DMA we only have one level of discounting where $p_k(y_t|y_{t-1})$ are the different forecaster densities. Therefore, we could denote DMA as LDF$^1_S$ where the superscript indicates a single level of loss discounting and the $s$ subscript indicates the use of the softmax function.

3.2.2. Dynamic Model Learning

Dynamic Model Learning (DML) (Beckmann et al., 2020) provides a way to optimally choose a single discount factor for the purposes of model selection. In DML the logarithmic scoring function $S \left( p_k^{(0)}, y_{t-i} \right)$ are passed through an argmax function to select the best model. We could refer to

$$\text{softmax}(a_1, \ldots, a_J) = \left( \frac{\exp(a_1)}{\sum_{j=1}^{J} \exp(a_j)}, \ldots, \frac{\exp(a_J)}{\sum_{j=1}^{J} \exp(a_j)} \right)$$

5
DML as LDF_{α,a} with the second layer of meta-models prior restricted to a single point on the grid, namely \( S_α = \{1\} \) for \( n = 2 \).

A similar idea for model averaging using the softmax function for selection an ensemble of parameters \( α \), was developed in Zhao et al. (2016).

**3.2.3. Two-Layer Model Averaging/Selection within Loss Discounting Framework**

The Loss Discounting Framework allows us to describe more general set-ups for discounting in forecast combination, such as these models with two or more meta-model levels. In the rest of this research paper we focus and the benefits of types of models with two layers of meta-models such as LDF^2_{s,a}, LDF^2_{a,s}, LDF^2_{a,a} and LDF^2_{s,s}, as well as, the limiting cases such as LDF^∞_{s,s}.

In contrast to DMA and DML having two (with \( α \neq 1 \)) or more layers of meta-models makes the discount factors in the other layers time dependent which, as we show in the next sections, leads to an improved performance of model averaging and selection.

In terms of computation time our proposed algorithm is very fast as it just relies on simple addition and multiplication. This is an advantage over more sophisticated forecasts combination methods when the time series is long and/or we would like to incorporate a large (usually greater than 10) number of forecasters.

As mentioned before, LDF^2_{a,a} is a generalised version of DML presented in Beckmann et al. (2020) where implicitly the authors suggest \( α = 1 \), i.e, all past performances of the forgetting factors are equally weighted. In the limit \( α \to 0 \) we would choose the discount factor \( α \) which performed best in the latest run, disregarding any other history. The LDF^2_{a,s} specification is a hybrid between model selection and model averaging. The first layer performs the model selection for each discount factor, the second layer performs the model averaging for the discount factors. Therefore, for each discount factor we select a single model but then we take a mixture of discount factors which results in a mixture of models.

**3.3. Properties of LDF as \( N \to \infty \)**

It is natural to consider the impact of additional layers in an LDF model. If we use either the softmax or the argmax at all layers, the weights for each model converge as \( N \to \infty \) and so adding more layers has a diminishing effect on the sequence of predictive distributions. Intuitively, for the softmax functions, we have a diminishing impact on the final result as we take weighted
averages of the weighted averages of the models, and for the argmax/model selection the LDF approach settles on a single model for any discount factor in the final layer. The detailed and rigorous proofs are provided in the technical Appendix A. We demonstrate in the empirical sections that the sequence converges to a predictive distribution which is often the best or nearly best performing set up of the LDF framework.

3.4. Interpretation

3.4.1. Momentum

The time varying nature of the discount factors in the first meta-model layer might be intuitively associated with the momentum strategy where an agent/investor places a bet on the highest performing assets. Based on this interpretation, the discounted score plays the role of the performance metric for optimal model and meta-model averaging/selection.

The two-layer configuration boils down to a two-step procedure. In the first step find the best model average (or best model) based on the past exponentially discounted performance. In the second step find the best meta-model average (or best meta-model) based on the past exponentially discounted performance.

Note that this method effectively averages the best performing models which is in contrast with, for example, linear pools or Diebold et al. (2022) where the methodology aims to find the best performing average. Linear pools might choose models to either boost to overall combined volatility or fatten the tails of the combination when the agent forecasts are, for example, all Gaussian but the true DGP is Student-t, say. That however, might create problems when the models that provided a needed boost in tails misbehave at a future time period. Our proposed technique might help in some cases, for example, when pruning the outlier models before any averaging takes place is desirable. This induced model sparsity might also be useful when the model pool is large.

An underlying assumption in the BMA technique is that the true model is included in the model set and weights are constant over time. Following these assumption it can be shown that, in the limit, BMA converges (i.e. the weight goes to 1) to the right model. However, if the model set does not include the true model, i.e., the model set is misspecified, then BMA fails to average as pointed out in Diebold (1991). Our set up can also be interpreted in terms of time smoothed BMA selecting the right model but only locally in time. Even if no model is correct across all times BMA might
be able to indicate the right model or set of models in the set up where the process characteristics change in time. I.e., if no model is right across all times it is possible that at least some models are close to the right one at certain times. Our method also provides a remedy for the constant weight assumption assuming the optimal weights being constant only locally in time.

3.4.2. Markov switching model with time-varying transition matrix

Following the argument in Del Negro et al. (2016) to interpret DMA in terms of a Markov switching model, our extension allows a time-varying transition matrix, i.e. $Q_t = (q(t)_{kl})$. The gradual forgetting of the performance of the discount factor $\alpha$ allows for a change of optimal discount factor when the underlying changes in transition matrix are required. However, we also show that our model outperforms the standard DMA model even when the transition matrix is non-time-varying. This point will be further illustrated in Appendix B.

4. Examples

Our methodology is best suited for data with multiple regime switches with a potentially time-varying transition matrix. As such, it is particularly useful for modelling data such as inflation levels, interest or foreign exchange rates. We illustrate our model on a simulated example and three real data examples. The supplementary materials for our examples are given in Appendix B, Appendix C, Appendix D and Appendix E.

We compare examples of our LDF to several popular model averaging methodologies. The approaches used are

- Two-layer LDF - 2 hyperparameters, i.e., $\alpha$, $c$;
- BMA - 0 hyperparameters;
- DMA - 2 hyperparameters, i.e., $\alpha$, $c$;
- BPS (McAlinn and West, 2019) - 5 hyperparameters, i.e., $\beta$ discount factor for state evolution matrices, $\delta$ discount factor for residual volatility, $n_0$ prior number of degrees of freedom, $s_0$ prior on BPS observation variance, $R_0$ prior covariance matrix of BPS coefficients;
- best N-average (Diebold et al., 2022) - 2 hyperparameters, i.e., $N$ number of models, rolling window length $rw$. 
• DeCo ([Billio et al., 2013] - 5 hyperparameters (but defaults and online estimation options are available).

We evaluate the performance of the models by calculating the out-of-sample mean log predictive score (MLS) and log predictive density ratios (LPDR)

\[
MLS = \frac{1}{T - s} \sum_{t=s+1}^{T} \log p(y_t|y_1, \ldots, y_{t-1}),
\]

\[
LPDR(\tau) = \sum_{t=s+1}^{\tau} \log \{ p(y_t|y_1, \ldots, y_{t-1})/p_{LDF}(y_t|y_1, \ldots, y_{t-1}) \},
\]

where \(y_1, \ldots, y_s\) are the observations for a calibration period and \(T\) is the total number of observations and \(p_{LDF}\) correspond to the selected LDF model.

4.1. Simulation study

The data generating process (DGP) of Diebold et al. (2022) is

\[
y_t = \mu_t + x_t + \sigma_y \epsilon_t, \quad \epsilon_t \sim N(0, 1), \quad (4.1)
\]

\[
x_t = \phi_x x_{t-1} + \sigma_x v_t, \quad v_t \sim N(0, 1), \quad (4.2)
\]

where \(y_t\) is the variable to be forecast, \(x_t\) is the long-run component of \(y_t\), \(\mu_t\) is the time-varying level (in Diebold et al. (2022) set to 0). The error terms are all i.i.d and uncorrelated. It is assumed that the data generating process is known to each forecaster apart from the level component \(\mu_t\). Each individual forecaster \(k\) models \(x_t\) with noise and applies different level \(\eta_k\) to \(y_t\):

\[
z_{kt} = x_t + \sigma_{tk} \nu_{kt}, \quad \nu_{kt} \sim N(0, 1), \quad (4.3)
\]

\[
\tilde{y}_{kt} = \eta_k + z_{kt} + \sigma_y \epsilon_t, \quad \epsilon_t \sim N(0, 1). \quad (4.4)
\]

Notice that the individual forecasters’ levels are not time varying. This emulates a situation where forecasters have access to different sets of information and/or models which might guide a different choice of level. It emulates an \(\mathcal{M}\)-complete or even \(\mathcal{M}\)-open setting where no forecaster is right at all times.

In contrast to Diebold et al. (2022), we allow the variable \(y_t\) to have multiple regime switches. The settings are as follows: \(\phi_x = 0.9, \sigma_x = 0.3, \sigma_y = 0.3, \sigma_{tk} = 0.1 \forall k, K = 20, T = 2001, \eta_k = -2 + 0.2105(k - 1), k = \ldots \)
1, ..., K and finally:

\[ \mu_t = \begin{cases} 
0, & \text{for } t \in [0, 49] \cup [200, 399] \cup [800, 849] \cup [970, 979] \\
 & \cup [1000, 1049] \cup [1600, 1650] \cup [1700, 2001] \\
1, & \text{for } t \in [100, 150] \cup [900, 949] \cup [960, 969] \cup [990, 999] \\
 & \cup [1050, 1099] \cup [1200, 1599] \cup [1700, 1749] \\
-1, & \text{otherwise}.
\end{cases} \]

More examples are discussed in Appendix B, where we draw the levels from a Markov switching models 10 times. For LDF we set \(S = \{1.0, 0.99, 0.95, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 0.001\}\) and \(c = 10^{-20}\) similarly to Koop and Korobilis (2012).

In Figure 1 we present how synthesised agent forecast level of LDF\(^2\) adjusts to the mean levels implied by the DGP. We can see that the model is very reactive to the mean predicted level following the true DGP mean closely with only a small time lag.

All results\(^6\) are based on 10 runs, where the levels were fixed but the random numbers regenerated. The standard Bayesian model averaging (MLS = -4.34) fared poorly since it quickly converged to the wrong model. DeCo (MLS = -0.57) was adapted to output 39 quantiles from which we calculated the log scores\(^7\) did not manage to cope well with abrupt level changes in our

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\(^6\)Except BPS for which we performed only 1 run due to computational cost.

\(^7\)39 quantiles in increments of 0.025. We used the default setting in DeCo package with \(\Sigma = 0.09\) (matching our DGP), and with learning and parameter estimation. The
numerical example, overestimated the variance which lead to poorer scores. BPS (MLS = -0.73) with normal agent predictive densities performed better than BMA but struggled to quickly adjust to the regime changes which resulted in low log-scores at the change points. The N-average method performed better (we chose rolling-window of 5 observations which performed best), with an MLS of -0.52 for $N = 3$ and $N = 4$, than BMA and BPS and similarly to the standard DMA method of Raftery et al. (2010).

Crucially, we note that DMA’s performance varies significantly depending on the hyperparameter choice, whereas the multilayered LDF methods’ performance does not. This is clearly illustrated in Figure 2a. One could adopt various strategies in trying to find the hyperparameters. The most basic one would be based on tuning the hyperparameter on the calibration period and keeping the parameter constant thereafter. In this case, for example, if we set the calibration period to 250 the methods choose discounts as: DMA 0.5 (MLS = -0.50); LDF$_{S,S}^2$ 0.6 (MLS = -0.42); LDF$_{S,A}^2$ 0.7 (MLS = -0.49). For comparison the stable state LDF$_{S,...,S}^\infty$ achieves MLS = -0.41. The non-LDF model averaging models, namely BPS, DeCO and best N-average, were tuned to achieve best performance to the entire sample a posteriori in contrast to LDF models where we select a single configuration based on the initial sample of 250 observations. Another strategy could be based on selecting the best discount factor at each time step (online) based on the expanding window, potentially exponentially weighted - this boils down to an LDF approach with an additional argmax layer. In this case DMA simply becomes LDF$_{S,A}^2$ and as shown can lead to better results. Even better results and more robustness can be achieved using LDF$_{S,S}^2$ where a mix of discount factors is being used.

In Figure 2b we present LPDR for the tested models against LDF$_{S,...,S}^\infty$. The LDF models (including DMA) performed generally better, however, the results suggest that the 2-layer LDF which can weight multiple discount factors model is more robust to abrupt changes in the level than the other models.

Figure 3 show how the average parameter $\alpha$ in the first meta-model layer quantiles indicated that the output can be well approximated by the normal distribution.\footnote{We used the original set of parameters (adjusted $\beta = 0.95$ and $\delta = 0.95$ to get better results) as proposed by the authors of the paper but adjusted the prior variance to match the $\sigma_y$ parameter. The model was run for 5000 MCMC paths with 3000 paths burnin period.}
dynamically changes using LDF$^2_{S,A}$ with $\alpha = 0.95$ and LDF$^2_{S,S}$ with $\alpha = 0.8$. It is close to 1 in periods of stability and closer to 0 in times of abrupt changes. In comparison, for $\alpha = 1$ the average parameter $\alpha$ is the first meta-model layer is rather stable, oscillating around 0.6. As mentioned before, this variation in parameter $\alpha$ might be beneficial since the lower the $\alpha$ parameter more models will be taken into consideration and the final outcome might show more uncertainty. Additionally, a lower parameter $\alpha$ facilitates the ability to quickly re-weight the models to adapt to the new regime. Whereas, in the times of stability it might be better to narrow down the meaningful forecasts to a smaller group by increasing the parameter $\alpha$. This illustrates how the two layer model provides useful flexibility in the discount factors in
the first meta-model layer. Another observation from Fig. 3 is concerning the average values of discount parameters $\alpha$ across time. For $\text{LDF}_{5,a}^2$, the average $\alpha$ for $\alpha = 0.95$ is 0.75 and for $\text{LDF}_{5,s}^2$ with $\alpha = 0.8$ it is 0.71. Whereas the average $\alpha$ for both LDF models with $\alpha = 0.1$ is 0.61.

4.2. Foreign Exchange Forecasts

Exchange rate forecasting is a popular, important and actively researched topic in economics and finance (see Rossi, 2013, for a comprehensive review). The random walk is a typical benchmark in the literature as it corresponds to the claim that the exchange rates are unpredictable. In the review, Rossi (2013) argues that there is an evidence suggesting that the predictive ability of the economic variables might be time-varying. In this light, Beckmann et al. (2020) consider forecast combination using DML for a pool of Time-Varying Parameter Bayesian Vector Autoregressive (TVP-BVAR) models with different subsets of economic fundamentals.

We conduct a comprehensive analysis of the benefits of the LDF framework. In the first part, we compare various competing model averaging techniques in terms of log-scores and we show that LDF offers superior performance. Then we shift our attention to LDF framework used in the context of model selection and there we not only show that the double discounting configuration achieves better log-scores but we also demonstrate how these differences in scores manifest themselves in an economic evaluation. In the economic evaluation we construct a long-short currency portfolio based on the outputs of each model. In this analysis we first conduct the model selection based on models’ log-scores, then, we use the ideas inspired by the work of Loaiza-Maya et al. (2021) and Lavine et al. (2021) to construct a scoring function focused to optimise the model performance as measured by Sharpe ratio to show that our framework also works well with generalised scoring or loss function.

We will follow closely the set up of Beckmann et al. (2020). An outline of the model is given in Appendix D.1, additional details regarding the model parameters are provided in Appendix D.2 and the full information can be found in Beckmann et al. (2020).9

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9Following Koop and Korobilis (2013) we adopt an Exponentially Weighted Moving Average (EWMA) estimator for the measurement covariance matrix to avoid the need for the posterior simulation for multivariate stochastic volatility. This is different than Beckmann et al. (2020) who use the approximation derived by Triantafyllopoulos (2011).
We use a set of G10 currencies to evaluate the model performance, namely: Australian dollar (AUD), the Canadian dollar (CAD), the euro (EUR), the Japanese yen (JPY), the New Zealand dollar (NZD), the Norwegian krone (NOK), the Swedish krona (SEK), the Swiss franc (CHF), pound (GBP) and the US dollar (USD). All currencies are expressed in terms of the amount of dollars per unit of a foreign currency, i.e. the domestic price of a foreign currency.

The data sample runs from November 1989 to July 2020. This is a more up-to-date data set than the one used in similar studies, but similar in length. Some studies, like Della Corte and Tsiakas (2012) or Kouwenberg et al. (2017), consider the data beginning from 1973. However, we note that the data samples from the 1970s and 1980s can vary between data providers and the available quotes are of lower quality than the newer data.

In terms of the macroeconomic fundamentals we consider the following:

- Uncovered Interest Rate Parity (UIP) which postulates that, given the spot rate \( S_t \), the expected rate of appreciation (or depreciation) is approximately:

\[
\frac{E(S_{t+h} - S_t)}{S_t} = i_t - i^*_t,
\]

where \( i_t \) is the domestic and \( i^*_t \) is the foreign interest rate corresponding to the time horizon \( h \) of the return\(^{10}\).

- Long-short interest rate difference - the difference between 10 year benchmark government yield and 1 month deposit rate.

- Stock growth - monthly return on the main stock index of each of the G10 currencies/countries.

- Gold price - monthly change in the gold price.

The interest rate difference and stock growth factors were previously used by Wright (2008), although, there the author used the annual stock growth as opposed to monthly. In this research we also introduce the change in

\(^{10}\) In this context we use the 1 month deposit rates. Theoretically, one should use the 1 month rates from the appropriate cross-currency curves. However, we assume that the difference between the deposit rates in two countries provides a good proxy for the interest rate differential.
gold price as a predictor. The gold prices act as exogenous non-asset specific factors, whereas UIP and INT DIFF are asset specific.

All collected and analysed economic data as well as the forecasts are monthly (as of month ends). In line with issues highlighted by Rossi (2013), the data underwent scrutiny in terms of any adjustments, revisions and timing issues that could introduce information not available to an investor at the time of decision making. At this point we note that the data between 1989 and 1990 is of worse quality. The reason of this is twofold. First, some of the financial instruments were not well-developed or liquid enough to be reliably sourced from data providers (for example, short terms deposit or 10 year Government bond benchmarks are not available for all currencies in question at the beginning of the sample). Secondly, certain currencies and stock indices started being available in the 1990s (e.g. EUR exchange rate). The details concerning the data sources and any proxies used are presented in the data Appendix. The data is standardised based on the mean and standard deviation calibrated to an initial training period of 10 years.

To understand effects of the pool size, we consider a small pool (which consists of the 32 models based on UIP only and time-constant parameters), and a larger pool (which consists of 2048 models including all possible subsets of the fundamentals). An exhaustive list of model parameter settings is outlined in Appendix. The smaller pool allows a comparison to the N-average method of Diebold et al. (2022) and BPS method of McAlinn and West (2019) which are computationally costly when the number of models is large.

4.2.1. Analysis of model scores

Setting the parameter $\alpha < 1$ improves both the model averaging as well as model selection procedures for both model pools for two layer LDF models. The best scores in model averaging/selection were achieved for $LDF_{s,s}^2/LDF_{a,s}^2$ specification with $\alpha = 0.8$ for the large model pool and $\alpha = 0.9$ for the small model pool. Model averaging scored better than model selection and the benefit of using a two layer LDF over a single layer LDF method appears to be greater for the bigger model pool.

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Footnote: If month end data was not available, it was substituted with the beginning of the month data or monthly average. These substitutions were unavoidable for some of the data in the 1980s. See Appendix for more details.
Figure 4: FX – MLS versus values of $\alpha$ for LDF and $\alpha$ for DMA in the x-axis for the small and large model pool. The upper plots show the cases of model averaging whereas the lower plots show model selection.

Interestingly, in the larger pool, the EWMA Random Walk\textsuperscript{12} (RW), decay factor 0.97, model was not the best model of all models considered (MLS = 21.77) but it performed almost on par with the \textit{a posteriori} best model (MLS = 21.78) which indicates that even from a big pool of models it is hard to find a model that outperforms the random walk. However, both the single layer as well as two layer LDF based model selection methods can provide a better performance versus using the random walk. We note

\textsuperscript{12}I.e. we estimate the volatility of the random walk based on the exponentially weighted moving average.
that the proposed $LDF_{\alpha,a}^{2}$ methodology improved upon the DML method Beckmann et al. (2020), which as we recall is $LDF_{\alpha,a}^{2}$ with $\alpha = 1$, in model selection for both the large and the small model pool.

The model averaging/selection using $LDF_{s,s}^{2}/LDF_{\alpha,s}^{2}$ outperforms all other benchmark methods for both the large as well as the small model pools. This indicates that LDF can perform well for quick and simple model averaging/selection when a large pool of models is available as well demonstrate competitive performance against more sophisticated methods with medium sized model pools.

For the small universe of 32 models we benchmark LDF model averaging methods against 4-model average with 20-period rolling window (that was the best performing N-day method), BPS, DeCo, and simple average.

For multivariate normal BPS we have set the prior for joint covariance matrix $s_0$ to a diagonal matrix with 7.7% annual volatility for all currencies and matrix $R_0$ to diagonal with entries equal to 0.001 and $\delta = 0.95$, $\beta = 0.99$, the remaining settings for BPS are as in McAlinn et al. (2020).13 We see that for model averaging the $LDF_{s,s}^{2}$ method with $\alpha = 0.9$ performs best (MLS=22.16), followed by other two layer LDF specifications and the 4-model average (MLS=22.10). BPS method (MLS = 21.6) did not perform well here. Similarly, DeCo (MLS = 18.31) method using multivari-

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13 We tried different values for $R_0$ and $\beta$ and we report the best result achieved.
In terms of model performance out of sample, LDF$_{\alpha}^2$ calibrated only on initial 10 years of data (to select $\alpha$) - $\alpha = 0.8$ (MLS=22.15) - still outperforms the other non-LDF models which were calibrated in-sample. The detailed results are presented in Table D.3 in Appendix D.

The stable state LDF models performed similarly to the two layer specification, LDF$_{\alpha}^\infty$,···, achieves MLS=22.13 and MLS=22.35 and LDF$_{\alpha}^\infty$,···,a scores MLS=22.07 and MLS=21.74 for the small and the large model pools respectively.

In Figure 4 we show how the average log-score changes with the choice of hyperparameters for LDF based model averaging and DMA methods with both the small and the large pool of models. We show that LDF provides not only a better performance in terms of optimal choice of the hyperparameter but also more robustness with respect to that choice which is visible as a relatively smaller gradient around the maximum value of the mean score, whereas DMA is more sensitive to the choice of $\alpha$ which has a single, narrow peak.

In order to illustrate the time-dependent performance of the models we calibrated the models’ hyperparameters to the first 10 years of data. In Figure 5 we present the LPDR across time as compared to LDF$_{\alpha}^2$,···. We note that the competing models perform worse and the sudden drops in performance of LDF$_{\alpha}^2$,··· and Best-4 average models correspond to the period of big FX volatility increases as measured by Barclays G10 FX index.

The average value of discount parameters $\alpha$ in the first meta-model layer across time, for the small universe of models for LDF$_{\alpha}^2$,··· with $\alpha = 0.9$ is 0.77 which was very similar for $\alpha = 1$. For the large pool of models the results are similar with LDF$_{\alpha}^2$,··· with $\alpha = 0.8$ having the average $\alpha$ in the first meta-model layer of 0.75 and for $\alpha = 1$ the average was 0.7. However, the variability of $\alpha$ in the first meta-model layer for $\alpha < 1$ was much larger, i.e. $\alpha$ being closer to 0 during times of increased volatility and closer to 1 during calmer times (same observation of either pool of models).

\textsuperscript{14}For DeCo, we checked that the marginal distributions are well described by the normal distribution. We then output the covariance matrix from the DeCo source code to complete the multivariate normal approximation.

\textsuperscript{15}In Appendix D.3 we show that with a dense grid of allowable values for $\alpha$ the points in figure 4 become smooth curves.
4.2.2. Economic evaluation of model selection

For the economic evaluation we used the smaller model pool of 32 models. Based on the outputs of each model we construct a portfolio of long and short currency positions targeting 10% annual volatility with 8bps transaction costs. We measure the performance by looking at the cumulative wealth over time as well as the Sharpe ratio, which captures the risk adjusted performance.

For the focused scoring we used a function which aims to optimise the Sharpe ratio. To that end we define this score as the portfolio returns divided by the the portfolio standard deviation based on a rolling twelve months window. The goal is to compare whether the model selection based on this focused prediction score yields better performance, and whether these scores (or the log scores) are closely correlated to the final Sharpe ratios. Recently, Lavine et al. (2021) have also focusing predictions in model synthesis and introduce an annealing factor as discussed further in Bissiri et al. (2016). In our analysis we do not incorporate the annealing factor in the focused scores but it would be an easy extension to do so.

In this analysis we only consider LDF configurations that select a single model at a time, that is LDF$_2^{2a}$ and LDF$_1^{1a}$. This is because the portfolio construction is maximizing the returns subject to a fixed risk per model, as in Beckmann et al. (2020), and any averaging effects would make the comparison invalid due to the correlation effects between the investment strategies which would inevitably change the target risk level of the portfolio. An alternative
A approach to portfolio construction was presented in Tallman and West (2022) who use the multivariate focused prediction score in the context of model averaging where each model aims to minimise the risk subject to a fixed return target.

As shown in Figure 4, the outperformance of $LDF^2_{\alpha,\alpha}$ over $LDF^1_{\alpha}$ is not large in terms of log scores but it does translate to higher Sharpe ratio as shown in Figure 6 which is in line with the observations in Beckmann et al. (2020) who note that small differences in the log scores can translate to noteworthy economic differences.

When using the mean focused scores (MFS) in the analysis instead of the log scores the direct linear relationship between the scores and the final Sharpe ratio becomes clearer as shown in the right panel of Figure 6. That means, that by using focused scores the selection of well-performing model, as measured by the scores, is strongly associated with the higher performance in terms of the final goal. This is in contrast with the situation where the log-scores were used, where this association was less clear. The double discounting version of LDF achieves better scores which directly translates into higher Sharpe ratios and higher final wealth as seen in Figure 7, where we compare the models that achieved a posteriori best mean focused scores. In the same figure we also note the behaviour of the discount factors for the three models illustrated. The double discounting of $LDF^2_{\alpha,\alpha}$ allows for the discount factor changes which drive the performance. The discount factor drops in the times of higher volatility such as during the great financial crisis.
or the Chinese crash or the Brexit referendum. For $\alpha = 0.7$ the time average value of $\alpha$ is 0.71 and with $\alpha = 1$ it is 0.80. This is in contrast to DML (LDF$_{a,a}^2$ with $\alpha = 1$) and LDF$_a^1$ specifications where in the former the discount factor settles at 0.9 and does not move and in the latter it is just fixed to a predetermined constant value.

In Figure 8 we show the portfolio composition though the time. We note that the weights display stability in the times when the portfolio value experiences periods of growth and the sudden weight changes correspond to periods of growth plateau. The weights generally follow the carry trade strategy which is well documented in the literature, see Della Corte and Tsiakas (2012) and references therein.

4.3. US Inflation Forecasts

The final study considers an example of McAlinn and West (2019), which involves forecasting the quarterly US inflation rate between 1961/Q1 and 2014/Q4. Here, the inflation rate corresponds to the annual percentage change in a chain-weighted GDP price index. There are four competing models: $M_1$ includes one period lagged inflation rate, $M_2$ includes period one, two and three lagged inflation interest and unemployment rates, $M_3$ includes period one, two and three lagged inflation rate only and $M_4$ includes period one lagged inflation interest and unemployment rates. All four models provide Student-t distributed forecasts with around 20 degrees of freedom.
The distinguishing features of this example are the small number of models and the existence of time periods when none of the models or model combinations lying on simplex provide an accurate mean forecast. In this example we will see the limitation of the LDF and other simplex based methodologies which are unable to correct for forecasting biases if bias corrected models are not explicitly available in the pool.

The BPS method (MLS = 0.06) dominates all other methodologies since it allows for model combinations not adhering to simplex. In fact, there were six dates in the evaluation period where the mean of BPS synthesised model was greater than the maximum of the underlying models. The feature to go beyond simplex proved to be one of the key factors in the superior performance.

The next most effective method was N-model average of Diebold et al. [2022] which for N = 2 and N = 3 models had a MLS equal to -0.01 and provided better performance than the best single model (M2, MLS = -0.02). For N = 2, out of the 100 evaluation points, the algorithm selected the pair (M0, M1) 35 times, the pair (M2, M3) 49 times and the pair (M1, M3) 16 times. On the other hand, both 2-level LDF model averaging and DMA methods did not work very well in this example but improved upon picking just a single model. The poor performance of 2-level LDF and DMA could mostly be attributed to the highly dynamic nature of these methods which sometimes attached too much weight to a single model that would score poorly.

![Figure 9: US inflation – The MLS versus values of α for LDF and α for DML in the x-axis.](image-url)
5. Discussion

This paper contributes to the model averaging and selection literature by introducing a Loss Discounting Framework which encompasses Dynamic Model Averaging, first presented by Raftery et al. (2010), generalises Dynamic Model Learning (Beckmann et al., 2020) and introduces additional model averaging or selection specifications. The framework allows for general dynamics for model weights, and works well with focused scores for goal-oriented decision making. The methodology offers extra flexibility which can lead to better forecast scores and yield results which are less sensitive to the choice of hyperparameters. This is particularly important in a more realistic online forecasting setting where selection of the globally optimal hyperparameters is often unattainable. It also empowers users to choose the model specification in terms of number of levels of discounting layers which is suitable for the problem at hand.

We show that our proposed methodology performs well in both the simulation study as well as in the empirical examples based on the exchange rate forecasts where we show the superiority of our approach both for model averaging as well as model selection, where for the latter we also demonstrate how the differences in the scores translate to noteworthy economic gains. We find that the LDF can be a good choice when: the number of forecasters is fairly large and sophisticated methods become burdensome; if we want to have only a small number of hyperparameters to calibrate; we suspect that we are in the $\mathcal{M}$-complete/open setting and different models might be optimal at different times but there is no consistent bias to be eliminated across all models; if we believe that scoring forecasters on the joint predictive density or joint utility basis is reasonable.

The LDF is by no means the panacea for model synthesis and the performance of different model synthesis methods depends on the problem (as seen in the empirical studies). However, LDF is often able to achieve competitive performance with a low computational overhead by using flexible dynamics and general model scores in an easy-to-implement and compute framework.

There are multiple open avenues to explore. Many current forecast combination methods described in the literature assume that the pool of forecasters does not change over time (see e.g. Raftery et al., 2010; Diebold et al., 2022; McAllin and West, 2019). In some situations this is a substantial limitation, for example, if the forecasts are provided by a pool of experts.

Let us first consider the situation of a new agent being added to the
existing pool of forecasters. The existing forecasters already have a track record of forecasts and corresponding scores. A new forecaster could be included with an initial weight. This could be fairly easily achieved in the LDF by considering a few initial scores. It is not clear what this weight should be, especially in more formal methodologies which relax the simplex restriction like McAlinn and West (2019). Similarly, forecasters may drop out completely or for some quarters before providing new forecasts. Again, in general, it is hard to know how to weight these forecasters. The LDF provides a rationale, we should be using an estimate of that forecaster’s score when a forecast is made. This is a time series prediction problem and can be approached using standard methods.

We noted in the empirical sections that the best performing discount factor in the second layer is larger than the average across time discount factor in the first layer. We showed that as one keeps adding more and more layers of meta-models the weights converge to an equilibrium. I.e. adding more layers does not change the scores any more and any choice of the discount factor in the final layer leads to the same score and same discount factors in all other layers.

As mentioned before, in most examples, we use joint predictive log-likelihood as a statistical measure of out-of-sample forecasting performance. It gives an indication of how likely the realisation of the modelled variable was conditional on the model parameters. The logarithmic scoring rule is strictly proper but it severely penalises low probability events and hence it is sensitive to tail or extreme cases, see Gneiting and Raftery (2007). A different proper scoring rule could be used when needed or if a decision is to be made based on the outcomes of model averaging/selection then a focused score (or utility), aligned with the final goal, can be used as successfully demonstrated in one of our examples.

Furthermore, since the scoring function is often based on the joint forecast probability density function, our methodology is not best suited to take strength from forecasters who might be good at forecasting one or more variables but not the others. This is partially due to the fact that our methodology does not consider any dependency structure between expert models and the weighting is solely performance based. An extension introducing a way to take the agent inter-dependencies into consideration would be of considerable interest.

More broadly, the exponential discounting recipe could be generalised and expanded by any forecast of the scores which could involve more parameters.
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Appendix A. Asymptotic properties of Loss Discounting Framework

**Theorem 1** (Convergence for softmax). For LDF\(_{ss...s}^n(S)\) and LDF\(_{ss...s}^{n+1}(S)\) models, then let \(X_1^{(n)}, X_2^{(n)}, \ldots \sim p_m^{(n)}\) and \(X_1^{(n+1)}, X_2^{(n+1)}, \ldots \sim p_m^{(n+1)}\) then \(n \to \infty \ X^{(n)} \xrightarrow{d} X^{(n+1)} \xrightarrow{d} X\) for any \(k = 1, \ldots, K\), where \(X \sim p^*\) with \(p^*\) being some probability distribution function.

**Proof.** First let us recall that for \(n \geq 2\) a model \(p_m^{(n)}(y_t | y_{t-1})\) is a weighted average of the models in the previous layer and each weight \(w \in [0, 1]\). In effect in each layer we have some average of models \(p_m^{(n)}(y_t | y_{t-1}) = \sum_{k=1}^{K} \omega_{t-1,k}^{(n)}(m) p_k^{(0)}\), as shown in Eq. 3.3 This implies that \(\omega_{t-1,k}^{(n)}(m) \in \left[ \min(\{\omega_{t-1,k}^{(n-1)}(m)\}_{m=1}^{M}), \max(\{\omega_{t-1,k}^{(n-1)}(m)\}_{m=1}^{M}) \right]\) which we call \(\mathcal{S}_{\omega_k}^{(n)}(m)\) for \(m = 1, \ldots, M\) and \(k = 1, \ldots, K\). From this can see that \([0, 1] \supseteq \mathcal{S}_{\omega_k}^{(1)}(m) \supseteq \mathcal{S}_{\omega_k}^{(2)}(m) \cdots \) so that \(\lim_{n \to \infty} \mu\left(\mathcal{S}_{\omega_k}^{(n)}(m)\right) = 0\), where \(\mu\) is the Lebesgue measure on \(\mathbb{R}\) as long as the weights \(w\) are not all \(\{0, 1\}\) which for the softmax function is satisfied almost surely. Since this is true for any \(m\) we have that in the limit \(n \to \infty \ \omega_k^{(\infty)}(1) = \omega_k^{(\infty)}(2) = \ldots = \omega_k^{(\infty)}(M)\) \(\forall k\).

**Corollary 1.1** (Convergence for argmax). For LDF\(_{aa...a}^n(S)\) and LDF\(_{aa...a}^{n+1}(S)\) models, then let \(X_1^{(n)}, X_2^{(n)}, \ldots \sim p_k^{(n)}\) and \(X_1^{(n+1)}, X_2^{(n+1)}, \ldots \sim p_k^{(n+1)}\) then \(n \to \infty \ X^{(n)} \xrightarrow{d} X^{(n+1)} \xrightarrow{d} X\) for any \(k = 1, \ldots, K\), where \(X \sim p^*\) with \(p^*\) being some probability distribution function.

**Proof.** The proof of this result is harder than the previous one since all weights are always either 0 or 1 and, ultimately, only a single model will be selected. This means that for \(\mathcal{S}_{p^{(1)}} = \{p_k^{(1)}(y_t | y_{t-1})\}_{m=1}^{M}\) we can have at most \(\min(K, M)\) distinct models. At each next layer we will have \(\min(K, M)\) or less distinct models so that \(\mathcal{S}_{p^{(1)}} \supseteq \mathcal{S}_{p^{(2)}} \cdots\). Therefore, \(\lim_{n \to \infty} |\mathcal{S}_{p^{(n)}}| = 1\) almost surely.

Appendix B. Simulation study - supplementary material

In this appendix we provide additional details corresponding to the simulation example from section 4.1 as well as the results of additional experiments performed in order to check the robustness and persistence of the results presented.
Table B.1: Predictive log-likelihood for our simulated example averaged over $R = 10$ runs and the associated standard deviation. We denote the average of the mean log score as $\bar{\text{MLS}} = \frac{1}{TR} \sum_r \sum_t \log(p)$, the average of the cumulative log score as $\sum \log(p) = \frac{1}{R} \sum_r \sum_t \log(p)$ and by $\sigma(.)$ the corresponding standard deviation of the quantities in question. We can see that our proposed model outperforms all other methods.

- time constant Markov switching levels
- time varying Markov switching levels

**Appendix B.1. Simulation study - additional results**

In Table B.1 we present the full set of results from the simulation study in section 4.1. Apart from MSE we also provide the sums of log-scores which also show that the two-level LDF models also dominate in this performance metric.

**Appendix B.2. Time-constant Markov switching model**

In this experiment we adopt the same set up as in Section 4.1 but we set the Markov transition matrix for $\mu_t$ to $Q = \begin{pmatrix} 0.990 & 0.005 & 0.005 \\ 0.005 & 0.990 & 0.005 \\ 0.005 & 0.005 & 0.990 \end{pmatrix}$ for three states $\{-1, 0, 1\}$ and the rest of parameters we set to the same values as in the Section 4.1, namely, $\phi_x = 0.9$, $\sigma_x = 0.3$, $\sigma_y = 0.3$, $\sigma_{tk} = 0.1 \forall k$, $K = 20$, $T = 2001$. We compare our 2-level method to the plain DMA of Raftery et al. (2010). We run the experiment 10 times. The results are
Table B.2: Predictive log-likelihoods for our additional simulation study averaged over 10 runs. We can see that LDF based model outperforms the standard DMA and static online learning.

Interestingly our method outperforms the standard DMA for all \( \alpha \) parameters reported here which suggest a good degree of robustness of the method.

**Appendix B.3. Time-varying Markov switching model**

In this simulation study we changed the definition of the transition matrix to be:

\[
Q_t = \begin{pmatrix}
0.990 & 0.005 & 0.005 \\
0.005 & 0.990 & 0.005 \\
0.005 & 0.005 & 0.990
\end{pmatrix}
\]

for \( t < 1000 \) and

\[
Q_t = \begin{pmatrix}
0.980 & 0.010 & 0.010 \\
0.010 & 0.980 & 0.010 \\
0.010 & 0.010 & 0.980
\end{pmatrix}
\]

for \( t \geq 1000 \). The rest of parameters remains as defined before. The results are reported in first two columns from the right in Table [B.2]. In this case, similarly to the time-constant transition matrix example, our model shows a superior performance over the benchmarks.

**Appendix C. Parameter c**

As described in detail in [Yusupova et al. (2019)] the parameter \( c \) introduced by [Raftery et al. (2010)] to “avoid a model probability being brought to machine zero by aberrant observations” can have a sizeable impact on the
model averaging algorithm performance. This parameter gives a small extra weight to all models which means that the final model density combination has fatter tails.

This can be beneficial when we are dealing with a regime switching Gaussian process. In this situation our model averaging will necessarily lag behind the observations (since it will take a few observations to realise that regime switch occurred) which in turn will cause a number of predictive posterior distributions to miss the realizations by a potentially large margin. If the model is scored using logarithmic rule the penalty for such misses will be quite high. However, a fatter-tailed predictive posterior distribution will incur less severe losses in this case.

However, if the regime switches are infrequent the fatter-tailed predictive posterior distribution will on average receive a lower log-score as opposed to a distribution mixture with thinner tails.

In our research we set the parameter $c = 10^{-20}$, just to avoid the machine zero probabilities of models but to avoid introducing fatter-tailed predictive distributions. This was done to aim for a fair comparison between various model averaging algorithms. The tuning of this parameter is outside of the scope of this paper.

**Appendix D. Foreign Exchange Study - supplementary materials**

**Appendix D.1. TVP VAR model overview**

Vector Autoregressive (VAR) model is a generalisation of a univariate autoregressive model to multiple variables. All variables enter the model the same way and each variable has an equation involving its own lags, cross-lags and, potentially, exogenous variables. The time-varying version allows the parameters of the vector autoregression to be functions of time. The TVP-VAR model used for this research is as specified in Beckmann et al. (2020):

\[
y_t = X_t \beta_t + \epsilon_t, \quad \epsilon_t \sim N(0, \Sigma_t), \quad \text{(D.1)}
\]

\[
\beta_{t+1} = \beta_t + u_t, \quad u_t \sim N(0, Q_t), \quad \text{(D.2)}
\]

where $y_t$ is an $m \times 1$ vector containing observations of $m$ variables (in this case, discrete exchange rate returns), $X_t$ is a model matrix where each row contains variables of a single VAR equation. That is, $X_t$ contains: intercept, $p$ lags of endogenous variables (own lags and cross-lags) and a single lag of exogenous variables.
variables (economic factors). The set of exogenous variables is divided into two classes: asset specific, with $n_x$ elements and non-asset specific, with $n_{xx}$ variables. Therefore, the coefficient vector $\beta_t$ has $k = m(1 + p \cdot m + n_x + n_{xx})$ elements.

We utilise a variant of the Minnesota prior:

$$\beta_0 \sim N(0, \Omega_0).$$

(D.3)

This setup means that the expected values of the coefficient vector $\beta_0$ are initialised as a vector of zeroes with the covariance matrix $\Omega_0$. The Minnesota prior assumes a diagonal structure of the covariance matrix where the size of the elements determine the strength of shrinkage of the respective coefficients. The smaller the diagonal elements the stronger the shrinkage towards 0.

Appendix D.2. TVP VAR model parameters

The model settings are as follows:

- **Discount factor** $\kappa$ - we follow [Reuters (1996)] and set $S_\kappa = \{0.97\}$.
- **Number of lags** $p$ - we follow [Beckmann et al. (2020)] and set number of lags to $S_p = \{6\}$.
- **Discount factor** $\lambda$ - we verify the findings by [Abbate and Marcellino (2018)], suggesting that the time variation of VAR coefficients $\beta_t$ is not desirable from the forecasting performance perspective by comparing the forecast performances between $S_\lambda = \{1\}$ and $S_\lambda = \{0.5, 0.7, 0.9, 1\}$.
- **Discount factor** $\alpha$
- **Intercept shrinkage parameter** $\gamma_1$ - we consider a grid $S_{\gamma_1} = \{0, 10\}$.
- **Endogenous variable shrinkage parameters** $\gamma_2, \gamma_3$ - the models are based on the grid of possible values $S_{\gamma_2} = S_{\gamma_3} = \{0, 0.1, 0.5, 0.9\}$.
- **Exogenous variable shrinkage parameters** $\gamma_4, \ldots, \gamma_{n_x+n_{xx}+3}$ - we consider a binary grid $S_{\gamma_4} = \ldots = S_{\gamma_{n_x+n_{xx}+3}} = \{0, 1\}$, i.e., a variable is either included or not.
- **Target volatility** - we follow [Beckmann et al. (2020)] and [Della Corte and Tsiakas (2012)] and set the target annual volatility to $\sigma = 0.1$ (10%).
• **Transaction costs** - we follow [Beckmann et al. (2020)](beckmann2020) and set the transaction costs to be the fixed, symmetric value \( \tau = 0.0008 \) (8bps).

To evaluate the models we first specify the allowable BVAR model space/universe by defining the model parameter restrictions, data sample, regressors, and covariance matrix estimator. Those definitions translate into a set of possible models that an investor can choose at any point in time by taking all possible permutations of the model parameter restrictions. For example, if we restrict the economic variables to the UIP only and impose further restrictions on \( \lambda \) and \( \alpha \), i.e. \( S_\lambda = \{1\} \), \( S_\alpha = \{1\} \), then a model universe comprises of a set of 64 possible models \( |S_{Y_1}| \cdot |S_{Y_2}| \cdot |S_{Y_3}| \cdot |S_{Y_4}| \cdot |S_\lambda| \cdot |S_\alpha| = 2 \cdot 4 \cdot 4 \cdot 2 \cdot 1 \cdot 1 = 64 \), where \( |S| \) denotes the number of elements in set \( S \) and \( S_{Y_4} \) is the set of shrinkage parameters for the exogenous variable UIP).

**Appendix D.3. Dense grid results**

In Figure [D.10](figure) we show that once we make the grid of allowable values dense (both between 0.2 and 1 with a space of 0.01) the mean log-scores become smooth functions of the hyperparameters. Naturally a denser grid leads to a slower computation and a such a user can priorities between the speed of calculation and model specification.
Table D.3: Predictive log-likelihood for our FX empirical study for a large pool of 2048 models and a small pool of 32 models. We denote the mean log score as MLS and the cumulative log score as $\sum \log(p)$. “Best” is the single model which performed best and was selected a posteriori after we got the results.

Appendix D.4. Results table

In Table D.3 we show the average and the sum of predictive log-scores for the large and small universes of models for model averaging and selection using LDF methodology, DMA, BMA, BPS, best-N models and random walk benchmark. We see that $LDF^2_{S,S}$ outperforms in model averaging and $LDF^2_{a,S}$ is best in model selection.

Appendix D.5. GBP/USD case study

In Figure D.11 we show that, for example, for GBP/USD currency rate BPS method showed little directionality as opposed to the LDF$_S$ method. We see that the BPS method “over-smoothed” the individual forecasts by giving both the less responsive estimates of mean returns as well as volatility.

Appendix D.6. Data description

The data which was used for the empirical model evaluation was sourced mostly from Refinitiv Eikon, using tickers outlined in Table E.5. There we also show if the data was downloaded from any other sources due to limited history available from our primary source. For EUR we backfilled the history pre 1st January 1999 with German mark exchange rates re-indexed with..
Figure D.11: FX: Return forecasts (with 95% credible intervals) for GBP/USD using LDF and BPS with the small model pool. We can spot that BPS is over-smoothing in this example, whereas LDF model averaging is capable of dynamic adjustments in modelled volatility and returns. LDF model averaging with $\alpha = 0.8$ modelled returns and 95% credible intervals versus the realised returns for GBP/USD currency pair. BPS modelled returns and 95% credible intervals versus the realised returns for GBP/USD currency pair.

Figure E.12: US inflation: Forecast means for each model and realised year-on-year

the conversion rate of 1.95583. We acknowledge and thank UCL for making the data licence available to support this research. We publish detailed description of the data sources in hope that they can make the results of this research easily replicable.

Appendix E. US inflation study - supplementary material

In Figure E.12 we show the forecast means of each of four models considered vs realised year-on-year inflation.

In Table E.4 we show the full set of results for this study.
Table E.4: Predictive log-likelihood for US inflation example from [McAlinn and West (2019)]. We denote the mean log score as MLS, the cumulative log score as $\sum \log(p)$. We note that BPS model provides the best synthesis in this example.
| Currency | FX | IR Deposit | IR 10Y Government Bond benchmark | Stock |
|----------|----|------------|---------------------------------|-------|
| DEM      | Bank Of England £EINI fixing | From 09/1988 AUD1MD, before Australia Overnight Cash rate from archives of Royal Bank of Australia | From 06/1990 AU10Y=RR, Before Fed St Luis | S&P 200 AXJO From 05/1992 before All ordinaries Index |
| AUD      | Eislon | From 09/1988 AUD1MD, before Australia Overnight Cash rate from archives of Royal Bank of Australia | From 06/1990 CA10Y=RR, Before Fed St Luis | S&P Comp Index GSPTSE |
| CAD      | Eislon | CA1MD | From 05/1986 CA10Y=RR, Before Fed St Luis | DAX GDAX From 12/1987 |
| EUR      | Eislon | From 01/1999 onwards EUR1MD, From 02/1994 to 12/1998 EUR1MD Eislon, Before Bundesbank Monthly average | From 01/1991 EU10Y=RR, From 08/1992 to 12/1998 DE10Y, before Germany 10Y Benchmark FED St Luis | |
| JPY      | Eislon | JPY1MD | From 03/1988 JPY10Y=RR | NIKKEI 225 NIKKEI |
| NZD      | Eislon | NZ1MD, before New Zealand central bank New Zealand state wholesale 30d bills | From 01/1994 NZ10Y=RR, Before Fed St Luis | NZX 50 from 12/2000, Before NZX All rebased to NZX 50 |
| NOK      | Eislon | NOK1MD, Bank of Norway overnight monthly rate | From 01/1994 NO10Y=RR, Before Fed St Luis | OMXS 30 from 09/1999, before OS-EAX rebased |
| SEK      | Eislon | SEK1MD, Fed Bank St Luis average monthly rate | From 01/1994 SEK10Y=RR, Before Fed St Luis | OMXS 30 from 09/1999, before OS-EAX rebased |
| CHF      | Eislon | CHF1MD | From 01/1992 CHF10Y=RR, Before Fed St Luis | SMI SSMI From 01/1988 |
| GBP      | Eislon | GBP1MD | From 03/1988 GBP10Y=RR, Before Fed St Luis | FTSE100 FTSE |
| USD      | Eislon | USD1MD | From 06/1983 USD10Y=RR, Before Fed St Luis | S&P 500 SPX |

Table E.5: Sources of data used for FX study.