Toric Landau–Ginzburg models

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Abstract. This review of the theory of toric Landau–Ginzburg models describes an effective approach to mirror symmetry for Fano varieties. It focuses mainly on the cases of dimensions 2 and 3, as well as on the case of complete intersections in weighted projective spaces and Grassmannians. Conjectures that relate invariants of Fano varieties and their Landau–Ginzburg models, such as the Katzarkov–Kontsevich–Pantev conjectures, are also studied.

Bibliography: 89 titles.

Keywords: toric Landau–Ginzburg models, mirror symmetry, toric geometry, Fano varieties.

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1. Introduction

One of the most brilliant ideas in mathematics in the last three decades is mirror symmetry; as often happens, this came to mathematics from physics. To wit, Calabi–Yau threefolds (that is, varieties of complex dimension 3 with a holomorphic 3-form that vanishes nowhere) play a central role in the description of elementary particles in string theory. These varieties, enhanced by symplectic forms and complex structures, can be considered as symplectic or algebraic manifolds. Physicists noticed that these varieties come in (non-uniquely defined) pairs such that the symplectic properties of a Calabi–Yau manifold $X$ (so-called branes of type $A$) correspond to the algebraic properties of its partner $Y$ (so-called branes of type $B$) and, conversely, the symplectic properties of $Y$ correspond to the algebraic properties of $X$. One numerical consequence of the correspondence is the mirror symmetry of Hodge numbers, which states that $h^{i,j}(X) = h^{3-i,3-j}(Y)$. One can say that by putting a mirror to the Hodge diamond for $X$ one can see the Hodge diamond for $Y$. This justifies the term ‘mirror symmetry’.

Just after this breakthrough was made it was straightforwardly generalized to higher-dimensional Calabi–Yau varieties. Some numerical consequences of the discovery were also formulated, which made it possible to formulate the idea of mirror symmetry mathematically. The first example of the phenomenon was given in the famous paper [13], where the general quintic threefold in $\mathbb{P}^4$ was considered. A certain series for the hypersurface was considered, which was constructed from the
expected numbers of rational curves of given degrees lying on the quintic (Clemens’
conjecture states that for a very general quintic these numbers are finite). A certain
one-dimensional family was considered whose period, that is, the function given by
the integrals of fibrewise forms over fibrewise cycles, after a certain transformation
coincides with the series for the quintic. This principle of correspondence of the
series constructed from the numbers of rational curves lying on the manifold and
the periods of the dual one-parameter family is the basis of the mirror symmetry
conjecture for the variations of Hodge structures.

A subsequent generalization of the mirror symmetry conjecture was its formul-
ation for Fano varieties, that is, varieties with ample anticanonical class. Such
varieties play an important role in algebraic geometry: for instance, they are the
main ‘building blocks’ in the minimal model programme. Moreover, they have a
rich geometry; for example, many rational curves lie on them. In contrast
to the Calabi–Yau case, mirror partners for Fano varieties are not varieties of
the same kind but certain varieties together with complex-valued functions called
superpotentials. Such varieties are called Landau–Ginzburg models and can be
described as one-dimensional families of fibres of superpotentials. In particular,
fibres of the families are Calabi–Yau varieties mirror dual to anticanonical sec-
tions of the Fano varieties. The mirror symmetry conjecture for the variations of
Hodge structures asserts a correspondence between \( I \)-series that are constructed
from Gromov–Witten invariants, in other words, from the expected numbers of
rational curves of given degree lying on the manifold (it is important here that it is
Fano or ‘close to’ Fano, so that it has enough rational curves), and periods of the
dual family. That is, it asserts the coincidence of the second Dubrovin connection
for the Fano manifold and the Gauss–Manin connection for the dual Landau–Ginzburg
model, or the coincidence of the regularized quantum differential equation of the
variety and the Picard–Fuchs differential equation of the dual model.

The first and main example of when the mirror symmetry conjecture for the
variations of Hodge structures holds was given by Givental (see [39], and also [44]).
He constructed Landau–Ginzburg models for complete intersections in smooth toric
varieties. This construction can be generalized to complete intersections in singular
toric varieties and, more generally, to varieties admitting ‘good’ toric degenera-
tions, such as Grassmannians or partial flag manifolds (see [9] and [10]). Moreover,
Givental’s model for a toric variety \( T \) can be simplified by expressing some vari-
ables monomially in terms of others such that the superpotential becomes a Laurent
polynomial in \( \dim T \) variables. The Newton polytope of the Laurent polynomial
coincides with the fan polytope for \( T \), that is, with the convex hull of the integral
generators of rays of a fan for \( T \). For a complete intersection it is often possible
to make one more birational change of variables after which the superpotential
remains representable by a Laurent polynomial. Moreover, this change of variables
transforms the Givental integral that expresses the periods of the Landau–Ginzburg
model correctly.

Let \( T \) be a Gorenstein toric variety. Its fan polytope is reflexive, which means
that the dual polytope is integral. Consider the toric variety \( T^\vee \) dual to \( T \); in
other words, the varieties \( T \) and \( T^\vee \) are defined by dual polytopes. Let \( X \) be
a Calabi–Yau complete intersection in \( T \) of dimension \( n \), which is defined by some
nef-partition. Batyrev and Borisov [8] defined the dual nef-partition, which gives the
dual Calabi–Yau variety \( Y \). According to Givental, the mirror symmetry conjecture for the variations of Hodge structures holds for \( X \) and \( Y \). In the same paper Batyrev and Borisov showed that

\[
h_{st}^{p,q}(X) = h_{st}^{p,n-q}(Y),
\]

where the \( h_{st}^{p,q}(Y) \) are stringy Hodge numbers. (In particular, in our case they coincide with the Hodge numbers of a crepant resolution of \( Y \), which, by Batyrev’s theorem (see [6]) do not depend on the particular resolution.) Thus, in our case the mirror symmetry conjecture for Hodge numbers follows from the mirror symmetry conjecture for the variations of Hodge structures. In the Fano case one cannot assert the correspondence of Hodge numbers directly, because the dual objects are not varieties but families of varieties. In [56] the analogues of Hodge numbers for ‘tame compactified Landau–Ginzburg models’ were defined (in three ways) and a conjecture about mirror correspondence for them was made. In particular, in this paper we study these conjectures, correct them slightly and present schemes for their proofs for del Pezzo surfaces and Fano threefolds.

The next step was Kontsevich’s homological mirror symmetry conjecture, which formulates mirror correspondence in terms of derived categories. Thus, by considering a Fano manifold \( X \) as an algebraic variety one can construct the derived category of coherent sheaves \( D^b(\text{coh} \, X) \), and by considering \( X \) as a symplectic variety (with chosen symplectic form) one can construct the Fukaya category \( \text{Fuk}(X) \), whose objects are Lagrangian submanifolds for the symplectic form and whose morphisms are defined by Floer homology. On the other hand, similar categories can be defined for a Landau–Ginzburg model \( w: Y \to \mathbb{C} \). The analogue of the derived category of coherent sheaves for the Landau–Ginzburg model is the derived category of singularities \( D^b_{\text{sing}}(Y, w) \), that is, a product over all singular fibres of quotients of derived categories of coherent sheaves by subcategories of perfect complexes. The analogue of the Fukaya category is the Fukaya–Seidel category \( \text{FS}(Y, w) \), whose objects are Lagrangian cycles (for the chosen symplectic form on the Landau–Ginzburg model) vanishing towards singularities. The homological mirror symmetry conjecture asserts the equivalences

\[
\text{Fuk}(X) \cong D^b_{\text{sing}}(Y, w) \quad \text{and} \quad D^b(\text{coh} \, X) \cong \text{FS}(Y, w).
\]

The homological mirror symmetry conjecture is very powerful. For instance, the Bondal–Orlov theorem states that a Fano variety can be reconstructed from its derived category of coherent sheaves. However, because of the deepness of the conjecture, it is hard to prove even in the simplest cases. Positive examples include the partial proofs of the conjecture (that is, the proof of one of equivalences in the conjecture) for del Pezzo surfaces [4], toric varieties [1], and some hypersurfaces [89]. We mention that the mirror symmetry conjecture for the variations of Hodge structures is reckoned to be a numerical consequence of the homological mirror symmetry conjecture, since the equivalence of categories implies the isomorphism of their Hochschild cohomology, which in our case correspond to quantum cohomology and variations of Hodge structures.

The different versions of mirror symmetry conjectures are expected to agree with one another. This means that Givental’s Landau–Ginzburg models satisfy the homological mirror symmetry conjecture. More precisely, the following compactification principle should hold: there should exist a fibrewise (log-) compactification of
a Landau–Ginzburg model which, after the choice of a symplectic form, satisfies the homological mirror symmetry conjecture. In particular, fibres of the compactification should be Calabi–Yau varieties mirror dual to anticanonical sections of the Fano variety. These three properties (correspondence of Gromov–Witten invariants to periods, the existence of a compactification to a family of Calabi–Yau varieties, and a connection with toric degenerations) justify the notion of toric Landau–Ginzburg model which is central in this paper. As in the case of Givental models of smooth toric varieties (but not complete intersections in them!), a toric Landau–Ginzburg model is an algebraic torus together with a non-constant complex-valued function satisfying the properties discussed above. Since the function on the torus (after choice of a basis) is nothing but a Laurent polynomial, we call such a polynomial (satisfying certain properties) a toric Landau–Ginzburg model. See §3 for the precise definition.

A strong version of the mirror symmetry conjecture for the variations of Hodge structures asserts the existence of a toric Landau–Ginzburg model for each smooth Fano variety.

The notion of toric Landau–Ginzburg model turned out to be an effective tool for studying mirror symmetry conjectures. This paper is a review of the theory of toric Landau–Ginzburg models. In particular, we construct them for a large class of Fano varieties such as del Pezzo surfaces, Fano threefolds, complete intersections in (weighted) projective spaces, and Grassmannians. We also construct their compactifications and study their properties, invariants, and related conjectures.

We present only sketches of the proofs for many of the results in this paper (one can find details in the references). Our paper is organized as follows. Section 2 contains definitions and preliminaries needed for what follows. Section 3 is devoted to the notion of toric Landau–Ginzburg models. Del Pezzo surfaces are discussed in §4. We present there an explicit construction of toric Landau–Ginzburg models depending on the choice of a divisor on a del Pezzo surface.

Section 5, which is central to the paper, is devoted to the Fano threefold case. Section 5.1 contains a construction for weak Landau–Ginzburg models. In §5.2 (log-) Calabi–Yau compactifications are constructed. In §5.3 we discuss toric degenerations of Fano threefolds that correspond to their weak Landau–Ginzburg models. We also present an explicit construction for the Picard–rank-1 case. In §5.4 we compute Picard lattices of fibres of Landau–Ginzburg models for the Picard–rank-1 case and show that these fibres are Dolgachev–Nikulin mirror partners to anticanonical sections of Fano varieties.

In §6 we study Katzarkov–Kontsevich–Pantev conjectures on Hodge numbers of Landau–Ginzburg models. In §6.1, following [56], we define and discuss the Hodge numbers of Landau–Ginzburg models and Katzarkov–Kontsevich–Pantev conjectures. In §6.2 we prove the conjectures for del Pezzo surfaces. Finally, in §6.3 we present a scheme for the proof of the conjectures in the threefold case.

Section 7 is devoted to the higher-dimensional case, that is, the case of (weighted) complete intersections and Grassmannians. A general Givental construction of Landau–Ginzburg models for complete intersections in smooth toric varieties is presented in §7.1. Most of the results in the rest of the section are related to the question of the existence of generalizations of such models and to the question of whether they are birational to weak Landau–Ginzburg models. In §7.2 we consider
the case of weighted complete intersections and present results on the existence of nef-partitions that guarantee the existence of weak Landau–Ginzburg models. In the case of complete intersections in the usual projective spaces we show the existence of Calabi–Yau compactifications and toric degenerations. The rest of the section contains boundedness results for families of smooth complete intersections. One can find more details on this part in the review [87] (in preparation). Finally, in §7.3 we consider the case of complete intersections in Grassmannians. For each such complete intersection we show the existence of a Batyrev–Ciocan-Fontanine–Kim–van Straten construction that is birationally equivalent to weak Landau–Ginzburg models.

Notation and conventions. All varieties are considered over the field of complex numbers $\mathbb{C}$.

We consider only genus zero Gromov–Witten invariants.

We denote the homology $H_*(X,\mathbb{Z})$ and the cohomology $H^*(X,\mathbb{Z})$ by $H_*(X)$ and $H^*(X)$, respectively, and the cohomology with compact support (of a variety $X$, with coefficients in the constant sheaf $\mathbb{C}_X$) by $H^c_*(X)$. We denote the Poincaré dual class to $\gamma \in H^*(X)$ by $\gamma^\vee$. The space $\text{Pic}(X) \otimes \mathbb{C}$ is denoted by $\text{Pic}(X)_\mathbb{C}$.

For any two numbers $n_1$ and $n_2$ we denote the set $\{i \mid n_1 \leq i \leq n_2\}$ by $[n_1,n_2]$.

For a Calabi–Yau variety is a projective variety with trivial canonical class.

We often denote a Cartier divisor on a variety $X$ and its class in $\text{Pic}(X)$ by the same symbol.

A smooth del Pezzo surface of degree $d$ (except for the quadric surface) is denoted by $S_d$.

A smooth Fano variety (considered as an element of a family of varieties of its type) of Picard rank $k$ and number $m$ in the lists in [51] is denoted by $X_{k-m}$.

We use the notation $\mathbb{P}(w_0,\ldots,w_n)$ for a weighted projective space with weights $w_0,\ldots,w_n$. We denote (weighted) projective spaces with coordinates $x_0,\ldots,x_n$ by $\mathbb{P}[x_0 : \cdots : x_n]$ and an affine space with coordinates $x_0,\ldots,x_n$ by $\mathbb{A}[x_0,\ldots,x_n]$.

We denote the ring $\mathbb{C}[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$ by $\mathbb{T}[x_1,\ldots,x_n]$.

We denote the torus $\text{Spec}\,\mathbb{T}[x_1,\ldots,x_n]$ by $\mathcal{T}[x_1,\ldots,x_n]$.

For us an integral polytope $\Delta \in \mathbb{Z}^n \otimes \mathbb{R}$ is a polytope with integral vertices, that is, with vertices lying in $\mathbb{Z}^n$. The integral length of an integral segment is the number of integral points on it minus 1.

We consider pencils in the birational sense: that is, for us a pencil is a family birational to a family of fibres of a map to $\mathbb{P}^1$.

2. Preliminaries

2.1. Gromov–Witten invariants and $I$-series. In this part we introduce the notions and notation of the Gromov–Witten theory which we need. One can find details, for instance, in [64].

Definition 2.1 (see [64], Chap. V, §3.3.2). The moduli space of stable maps to $X$ of rational curves of class $\beta \in H_2(X)$ with $n$ marked points (we denote it by $\overline{M}_n(X,\beta)$) is the Deligne–Mumford stack (see [64], Chap. V, §5.5) of stable maps $f : C \to X$ of curves of genus 0 with $n$ marked points such that $f_*(C) = \beta$.
We consider the evaluation maps

\[ \text{ev}_i : \overline{M}_n(X, \beta) \to X, \quad \text{ev}_i(C; p_1, \ldots, p_n, f) = f(p_i). \]

Let \( \pi_{n+1} : \overline{M}_{n+1}(X, \beta) \to \overline{M}_n(X, \beta) \) be the forgetful map at the point \( p_{n+1} \) that forgets this point and contracts the components which then become unstable. Consider the section \( \sigma_i : \overline{M}_n(X, \beta) \to \overline{M}_{n+1}(X, \beta) \) corresponding to the marked point \( p_i \) and defined as follows: the image of a curve \( (C; p_1, \ldots, p_n, f) \) under the map \( \sigma_i \) is a curve \( (C'; p_1, \ldots, p_{n+1}, f') \), where \( C' = C \cup C_0, C_0 \simeq \mathbb{P}^1, C_0 \) and \( C \) intersect in the unmarked point \( p_i \) on \( C' \), and \( p_{n+1} \) and \( p_i \) lie on \( C_0 \).

The map \( f' \) contracts the curve \( C_0 \) to a point and \( f'|_C = f \).

Consider the sheaf \( L_i \) given by \( L_i = \sigma_i^* \omega_{\pi_{n+1}} \), where \( \omega_{\pi_{n+1}} \) is the relative dualizing sheaf of \( \pi_{n+1} \). Its fibre over the point \( (C; p_1, \ldots, p_n, f) \) is \( T_{p_i}^* C \).

**Definition 2.2** (see [64], Chap. VI, §2.1). The cotangent line class is the class

\[ \psi_i = c_1(L_i) \in H^2(\overline{M}_n(X, \beta)). \]

**Definition 2.3** (see [64], Chap. VI, §2.1). Consider

\[ \gamma_1, \ldots, \gamma_n \in H^*(X); \]

let \( a_1, \ldots, a_n \) be non-negative integers, and let \( \beta \in H_2(X) \). Then the Gromov–Witten invariant with descendants is the number given by

\[ \langle \tau_{a_1} \gamma_1, \ldots, \tau_{a_n} \gamma_n \rangle_\beta = \psi_1^{a_1} \text{ev}_1^*(\gamma_1) \cdots \psi_n^{a_n} \text{ev}_n^*(\gamma_n) \cdot [\overline{M}_n(X, \beta)]^\text{virt} \]

if \( \sum \text{codim } \gamma_i + \sum a_i = \text{vdim } \overline{M}_n(X, \beta) \), and 0 otherwise. The invariants with \( a_i = 0, i = 1, \ldots, n \), are said to be prime and are equal to the expected number of rational curves in the class \( \beta \) on \( X \) that intersect cycles dual to \( \gamma_1, \ldots, \gamma_n \). We omit the symbols \( \tau_0 \) in this case.

Gromov–Witten invariants are usually ‘packed’ into different structures for convenience. The simplest ones are for one point \( (n = 1) \); they are usually packed into \( I \)-series.

Let \( X \) be a smooth Fano variety of dimension \( N \) and Picard rank \( \rho \). Choose a basis

\[ \{H_1, \ldots, H_\rho\} \]

in \( H^2(X) \) so that for any \( i \in [1, \rho] \) and any curve \( \beta \) in the Kähler cone \( K \) of \( X \) one has \( H_i \cdot \beta \geq 0 \). We introduce the formal variables \( q \) and \( \sigma_i, i \in [1, \rho] \), and let \( q_i = q^{\sigma_i} \). For any \( \beta \in H_2(X) \) let

\[ q^\beta = q^{\sum \sigma_i(H_i \cdot \beta)}. \]

Consider the Novikov ring \( \mathbb{C}_q \), the group ring for \( H_2(X) \). We treat it as a ring of polynomials over \( \mathbb{C} \) in the formal variables \( q^\beta \), with the relations

\[ q^\beta_1 q^\beta_2 = q^{\beta_1 + \beta_2}. \]

Note that for any \( \beta \in K \) the monomial \( q^\beta \) has a non-negative degree with respect to the \( q_i \).
Definition 2.4 (see details in [37] and [72]). Let $\mu_1, \ldots, \mu_N$ be a basis in $H^*(X)$ and let $\hat{\mu}_1, \ldots, \hat{\mu}_N$ be the dual basis. The $I$-series (or Givental $J$-series) for $X$ is given by the following:

$$I^X_\beta = \text{ev}_*\left(\frac{1}{1 - \psi} [\overline{\mathcal{M}}_1(X, \beta)]^\text{virt}\right) = \sum_{i,j \geq 0} \langle \tau_i \mu_j \rangle_{\beta} \hat{\mu}_j,$$

$$I^X(q_1, \ldots, q_\rho) = 1 + \sum_{\beta \in \mathcal{K}} I^X_\beta \cdot q^\beta.$$

The constant term $I^X_0$ of the $I$-series is

$$I^X_0(q_1, \ldots, q_\rho) = 1 + \sum_{\beta \in \mathcal{K}} \langle \tau(-K_X \cdot \beta - 21) \rangle_{\beta} \cdot q^\beta,$$

where 1 is the fundamental class. (The map $\text{ev}$ and the cotangent line class are unique for one-point invariants, so we omit indices.) The series

$$\tilde{I}^X_0(q_1, \ldots, q_\rho) = 1 + \sum_{\beta \in \mathcal{K}} (-K_X \cdot \beta)! \langle \tau(-K_X \cdot \beta - 21) \rangle_{\beta} \cdot q^\beta$$

is called the constant term of the regularized $I$-series for $X$.

Consider the class of divisors

$$H = \sum \alpha_i H_i.$$

One can restrict the $I$-series (both the usual and the regularized ones), to the direction corresponding to the divisor class by putting $\sigma_i = \alpha_i \sigma$ and $t = q^\sigma$. Fix a divisor class $D$. We are interested in the restrictions of the $I$-series to the anticanonical direction corresponding to $D$. To this end we change $q^\beta$ to $e^{-D \cdot \beta - K_X \cdot \beta}$. In particular, one can define a restriction of the constant term of the regularized $I$-series to the anticanonical direction (so that $D = 0$). It has the form

$$\tilde{I}^X_0(t) = 1 + a_1 t + a_2 t^2 + \cdots, \quad a_i \in \mathbb{C}.$$

2.2. Toric geometry. See the definition and main properties of toric varieties in [24] or [35]. We just recall that a toric variety is a variety with an action of a torus $\text{Spec}(\mathbb{C}^*)^N$ such that one of its orbits is a Zariski open set. A toric variety is determined by its fan, that is, some collection of cones with vertices at points of the lattice that is dual to the lattice of torus characters. Moreover, the algebro-geometric properties of a toric variety can be formulated in terms of the properties of this fan. We recall some of these.

Every cone in the fan $\kappa \subset \mathcal{M}_Q = \mathcal{N} \otimes \mathbb{R}$, $\mathcal{N} \simeq \mathbb{Z}^N$, of dimension $r$ corresponds to an orbit of the torus of dimension $N - r$. Thus, each edge (one-dimensional cone) corresponds to an (equivariant) Weil divisor. That is, let $\Sigma \in \mathcal{N}$ be a fan of the toric variety $X_\Sigma$ and let $\sigma \in \Sigma$ be any cone. Let $\mathcal{M}$ be the lattice dual to $\mathcal{N}$ with respect to some non-degenerate pairing $\langle \cdot, \cdot \rangle$, and let $\sigma^\vee$ be the dual cone to $\sigma$ (that is, $\sigma^\vee = \{l \in \mathcal{M} \mid \langle l, k \rangle \geq 0 \text{ for all } k \in \sigma\}$). Let $U_\sigma = \text{Spec} \mathbb{C}[\sigma^\vee]$ correspond
to $\sigma$. The variety $X_\sigma$ is obtained from the affine varieties $U_\sigma$, $\sigma \in \Sigma$, by gluing together $U_\sigma$ and $U_\tau$ along $U_{\sigma \cap \tau}$, $\sigma, \tau \in \Sigma$. The divisors which correspond to edges of the fan generate the divisor class group. A Weil divisor $M_n$ if the toric variety is $\sigma$ to $\tau$ together $U$. A Weil divisor is a Cartier divisor. If such a vector is the same for all cones, then the divisor is principal. Hence if the toric variety is $N$-dimensional and the number of edges is $N + \rho$, then the rank of the divisor class group is $\rho$.

**Definition 2.5.** A variety is said to be $\mathbb{Q}$-factorial if some multiple of each Weil divisor is a Cartier divisor.

In particular, there exists an intersection theory for Weil divisors on a $\mathbb{Q}$-factorial variety. A toric variety is $\mathbb{Q}$-factorial if and only if any cone in the fan that corresponds to this variety is simplicial. In this case the Picard group is generated (over $\mathbb{Q}$) by the divisors which correspond to edges of the fan.

Consider a weighted projective space $\mathbb{P} = \mathbb{P}(w_0, \ldots, w_N)$. The fan which corresponds to it is generated by integer vectors $m_0, \ldots, m_N \in \mathbb{R}^N$ such that $\sum w_i m_i = 0$. If $w_0 = 1$, then one can put $m_0 = (-w_1, \ldots, -w_N)$, $m_i = e_i$, where the $e_i$ form a basis of $\mathbb{R}^N$. The collection $\{m_i\}$ corresponds to the collection of standard strata divisors $\{D_i \in H^0(\mathcal{O}_\mathbb{P}(w_i))\}$.

A toric variety is smooth if for any cone $\sigma$ in the fan that corresponds to this variety the subgroup $\sigma \cap \mathbb{Z}^N$ is generated by a subset $m_1^\sigma, \ldots, m_k^\sigma$ of the basis of the lattice. Adding the edge $a = a_1 m_1^\sigma + \cdots + a_k m_k^\sigma$, $a_i \in \mathbb{Q}$, to the cone (and connecting it with ‘neighbouring’ edges by faces) corresponds to a weighted blowup (along the subvariety which corresponds to $\sigma$) with weights $(1/r)(\alpha_1, \ldots, \alpha_k)$, where $\alpha_i \in \mathbb{Z}$ and $a_i = \alpha_i/r$. Thus, we can obtain a toric resolution of a toric variety by adding successive edges to the fan in this way.

In particular, the singular locus of $\mathbb{P}$ is a union of strata given by equations of the form $x_{i_1} = \cdots = x_{i_j} = 0$, where $x_{i_j}$ is the coordinate of weight $w_{i_j}$ and $\{i_1, \ldots, i_j\}$ is the maximal set of indices such that greatest common factor of the other indices is greater than 1 (see Lemma 7.12).

Let $X$ be a factorial $N$-dimensional toric Fano variety of Picard rank $\rho$ corresponding to a fan $\Sigma_X$ in the lattice $\mathcal{N}$. Let $D_1, \ldots, D_{N+\rho}$ be its prime invariant divisors. Let $\mathcal{M} = \mathcal{N}^\vee$, and let $\mathcal{D} \simeq \mathbb{Z}^{N+\rho}$ be a lattice with the basis $\{D_1, \ldots, D_{N+\rho}\}$ (so that one has a natural identification $\mathcal{D} \simeq \mathcal{N}^\vee$). By Theorem 4.2.1 in [23], one has an exact sequence

$$0 \to \mathcal{M} \to \mathcal{D} \to \mathcal{A}_{N-1}(X) = \text{Pic}(X) \simeq \mathbb{Z}^\rho \to 0.$$

We use factoriality of $X$ here to identify the class group $A_{N-1}(X)$ and the Picard group $\text{Pic}(X)$. Dualizing this exact sequence, we obtain an exact sequence

$$0 \to \text{Pic}(X)^\vee \to \mathcal{D} \to \mathcal{N} \to 0. \quad (2.1)$$

Thus, $\text{Pic}(X)^\vee$ can be identified with the lattice of relations on primitive vectors on the rays of $\Sigma_X$, considered as Laurent monomials in the variables $u_i$. On the other hand, once the basis in $\text{Pic}(X)$ is chosen, we can identify $\text{Pic}(X)^\vee$ and $\text{Pic}(X) = H^2(X)$. Hence, we can choose a basis in the lattice of relations on the
primitive vectors of the rays of \( \Sigma_X \) corresponding to the basis \( \{ H_i \} \), and thus to the variables \( \{ q_i \} \). We denote these relations by \( R_i \) and interpret them as monomials in the variables \( u_1, \ldots, u_{N+\rho} \). We will also then use the \( D_i \) to denote the images of elements \( D_i \in \mathcal{D} \) in \( \text{Pic}(X) \).

Consider a toric variety \( T \). A fan (or spanning) polytope \( F(T) \) is the convex hull of the integral generators of rays in the fan for \( T \). Let

\[
\Delta = F(T) \subset \mathcal{M}_\mathbb{R},
\]

and let

\[
\nabla = \{ x \mid \langle x, y \rangle \geq -1 \text{ for all } y \in \Delta \} \subset \mathcal{M}_\mathbb{R} = \mathcal{N}^\vee \otimes \mathbb{R}
\]

be the dual polytope of \( \Delta \).

With an integral polytope \( \Delta \) we associate a (singular) toric Fano variety \( T_\Delta \) defined by a fan whose cones are cones over faces of \( \Delta \). We also associate with it a (not uniquely defined) toric variety \( \tilde{T}_\Delta \) with \( F(\tilde{T}_\Delta) = \Delta \) such that for any toric variety \( T' \) with \( F(T') = \Delta \) and for any morphism \( T' \to \tilde{T}_\Delta \) one has \( T' \simeq \tilde{T}_\Delta \). In other words, \( \tilde{T}_\Delta \) is given by a ‘maximal triangulation’ of \( \Delta \).

**Definition 2.6.** The variety \( T_\Delta \) and the polytope \( \Delta \) are said to be reflexive if \( \nabla \) is integral.

Let \( T \) be reflexive. Denote \( T^\vee \) by \( T_\Delta \) and \( \tilde{T}^\vee \) by \( \tilde{T}_\Delta \).

Finally, we summarize some facts relating to toric varieties and their anticanonical sections. One can consult, for instance, \([24]\) for details. For the following it is more convenient to start from the toric variety \( T^\vee \).

**Fact 2.7.** Let the anticanonical class \( -K_{T^\vee} \) be very ample (in particular, this holds in the reflexive threefold case; see \([53]\) and \([80]\)). One can embed \( T^\vee \) into a projective space in the following way. Consider a set \( A \subset M \) of integral points in the polytope \( \Delta \) dual to \( F(T^\vee) \), and consider a projective space \( \mathbb{P} \) whose coordinates \( x_i \) correspond to elements \( a_i \) of \( A \). Associate a homogenous equation \( \prod x_i^{\alpha_i} = \prod x_j^{\beta_j} \) with any homogenous relation \( \sum \alpha_i a_i = \sum \beta_j a_j, \alpha_i, \beta_j \in \mathbb{Z}_+ \). The variety \( T^\vee \) is cut out in \( \mathbb{P} \) by equations associated with all the homogenous relations on \( a_i \).

**Fact 2.8.** The anticanonical linear system of \( T^\vee \) is a restriction of \( \mathcal{O}_\mathbb{P}(1) \). In particular, it can be described as (the projectivization of) a linear system of Laurent polynomials whose Newton polytopes are contained in \( \Delta \).

**Fact 2.9.** Toric strata of \( T^\vee \) of dimension \( k \) correspond to \( k \)-dimensional faces of \( \Delta \). Denote by \( R_f \) the anticanonical section corresponding to a Laurent polynomial \( f \in \mathbb{C}[N] \) and by \( F_Q \) the stratum corresponding to a face \( Q \) of \( \Delta \). Denote by \( f \mid Q \) the sum of those monomials in \( f \) whose supports lie in \( Q \), and denote by \( \mathbb{P}_Q \) the projective space whose coordinates correspond to \( Q \cap N \). (In particular, \( Q \) is cut out in \( \mathbb{P}_Q \) by homogenous relations for the integral points of \( Q \cap N \).) Then

\[
R_{Q,f} = R_f \big|_{F_Q} = \{ f \mid Q = 0 \} \subset \mathbb{P}_Q.
\]

**Fact 2.10.** In particular, \( R_f \) does not pass through a toric point corresponding to a vertex of \( \Delta \) if and only if its coefficient at this vertex is non-zero. The constant Laurent polynomial corresponds to the boundary divisor of \( T^\vee \).
3. Toric Landau–Ginzburg models

Consider a smooth Fano manifold $X$ of dimension $n$ and a divisor $D$ on it, and consider the restriction

$$
\tilde{I}_0^{X,D}(t) = 1 + \sum_{\beta \in \mathcal{K}, \ a \in \mathbb{Z}_{>0}} (-K_X \beta)! \langle \tau_a \rangle_\beta \cdot e^{-\beta \cdot D} t^{-K_X \cdot \beta}
$$

of the constant term of the regularized $I$-series corresponding to $D$.

Let $f$ be a function on the torus $G_n^m = \prod_{i=1}^n \mathbb{T}[x_i]$. This function can be represented by a Laurent polynomial: $f = f(x_1^{\pm 1}, \ldots, x_n^{\pm 1})$. Denote the constant term (that is, the coefficient of $x_1^0 \cdots x_n^0$) in the polynomial $f$ by $[f]_0$ and put

$$
\Phi_f = \sum_{i=0}^\infty [f^i]_0 t^i \in \mathbb{C}[[t]].
$$

**Definition 3.1.** The series $\Phi_f$ is called the *constant terms series* for $f$.

**Definition 3.2.** Let $f$ be a Laurent polynomial in the $n$ variables $x_1, \ldots, x_n$. The integral

$$
I_f(t) = \frac{1}{(2\pi i)^n} \int_{|x_i| = \varepsilon_i} \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \frac{1}{1 - tf}
$$

is called the *main period* for $f$, where the $\varepsilon_i$ are some positive real numbers.

**Remark 3.3.** One has $I_f(t) = \Phi_f$.

The following theorem justifies this definition.

**Theorem 3.4** (see [75], Proposition 2.3). Let $f$ be a Laurent polynomial in $n$ variables. Let $PF_f = PF_f(t, \partial/\partial t)$ be a Picard–Fuchs differential operator for the pencil of hypersurfaces in the torus given by $f$. Then $PF_f[I_f(t)] = 0$.

Now we give the central definition of the paper.

**Definition 3.5** (see [77], §6). A *toric Landau–Ginzburg model* for a pair comprising a smooth Fano variety $X$ of dimension $n$ and a divisor $D$ on it is a Laurent polynomial $f \in \mathbb{T}[x_1, \ldots, x_n]$ which satisfies the following.

**Period condition.** One has $I_f = \tilde{I}_0^{X,D}$.

**Calabi–Yau condition.** There exists a relative compactification of a family

$$
f: (\mathbb{C}^*)^n \to \mathbb{C}
$$

whose total space is a (non-compact) smooth Calabi–Yau variety $Y$. Such a compactification is called a *Calabi–Yau compactification*.

**Toric condition.** There is a degeneration $X \rightsquigarrow T_X$ to a toric variety $T_X$ such that $F(T_X) = N(f)$, where $N(f)$ is the Newton polytope for $f$.

A Laurent polynomial satisfying the period condition is called a *weak Landau–Ginzburg model*. 
Definition 3.6 (see [78]). A compactification of the family \( f : (\mathbb{C}^*)^n \to \mathbb{C} \) to a family \( f : Z \to \mathbb{P}^1 \), where \( Z \) is smooth and \(-K_Z = f^{-1}(\infty)\), is called a log-Calabi–Yau compactification (see Definition 6.3).

Now we discuss why the notion of toric Landau–Ginzburg model is natural.

The period condition is nothing but the mirror symmetry conjecture for the variations of Hodge structures in the case when the ambient space is an algebraic torus. This condition relates Gromov–Witten invariants to periods of the dual model. Periods are integrals of fibrewise forms over fibrewise cycles. This means that they are preserved under birational transformations which are biregular in a neighbourhood of the cycles we integrate over. For instance, Givental constructed Landau–Ginzburg models for complete intersections in smooth toric varieties as quasi-affine varieties with superpotentials (see §7.1 and §7.3.1). However, in many cases these models are birational to algebraic tori, and the main periods are preserved by the corresponding birational isomorphisms (see §7.2.2, and also [31] and [21]).

The Calabi–Yau condition goes back to the following principle.

Principle 3.7 (Compactification principle; see [77], Principle 32). There exists a fibrewise compactification of the family of fibres of a ‘good’ toric Landau–Ginzburg model (defined up to flops) satisfying (the B-side of) the homological mirror symmetry conjecture.

In particular, this means that there should exist a fibrewise compactification to an (open) smooth Calabi–Yau variety that is a family of smooth compact Calabi–Yau varieties. This condition is strong enough: for example, if \( f(x_1, \ldots, x_n) \) is a toric Landau–Ginzburg model for a variety \( X \), then for \( k > 1 \) the Laurent polynomial \( f(x_1^k, \ldots, x_n) \) satisfies the period condition for \( X \) but not the Calabi–Yau condition, and thus it does not satisfy the compactification principle. However, the period condition and the Calabi–Yau condition are not sufficient for the compactification principle to hold.

Example 3.8. The polynomials

\[
\frac{(x + y + 1)^3}{xyzw} + z + w \quad \text{and} \quad \left( x_1 + x_2 + \frac{1}{x_1 x_2} \right) \left( y_1 + y_2 + \frac{1}{y_1 y_2} \right)
\]

satisfy the period condition and the Calabi–Yau condition for a cubic fourfold (see [57]). However, they are not fibrewise birational (they have different numbers of components of central fibres; see §5.4). It is expected that the first polynomial should satisfy the compactification principle.

One can easily see that the second Laurent polynomial in Example 3.8 is not a toric Landau–Ginzburg model for a cubic fourfold: the degree of the corresponding toric variety differs from the degree of a cubic.

Finally, the toric condition goes back to the Batyrev–Borisov construction of mirror duality for Calabi–Yau complete intersections in toric varieties via the duality of toric varieties (see [5]).

We consider mirror symmetry as a correspondence between Fano varieties and Laurent polynomials. That is, the strong version of the mirror symmetry conjecture for the variations of Hodge structures states the following.
Conjecture 3.9. Any pair comprising a smooth Fano manifold and a divisor on it has a toric Landau–Ginzburg model.

Corollary 3.10. Any smooth Fano variety has a toric degeneration.

This gives us hope to have the following picture.

Optimistic picture 3.11 (see [77], Optimistic Picture 38). The toric degenerations of smooth Fano varieties are in a one-to-one correspondence with the toric Landau–Ginzburg models. They satisfy the compactification principle.

Question 3.12. Does the converse of the second part of the compactification principle hold? That is, is it true that all Landau–Ginzburg models (in the sense of homological mirror symmetry) of the same dimension as the initial Fano variety are compactifications of toric ones? In particular, is it true that all of them are rational?

Question 3.13. Does the Optimistic Picture 3.11 need any extra conditions on toric varieties?

4. Del Pezzo surfaces

Let us begin this section by recalling some well-known facts about del Pezzo surfaces. We refer, for example, to [30] as one of a huge number of references on del Pezzo surfaces.

The original definition of a del Pezzo surface is the following one given by del Pezzo himself.

Definition 4.1 (see [25]). A del Pezzo surface is a non-degenerate (that is, not lying in a linear subspace) irreducible linear normal (that is, it is not a projection of a degree-\(d\) surface in \(\mathbb{P}^{d+1}\)) surface in \(\mathbb{P}^d\) of degree \(d\) which is not a cone.

In modern words this means that a del Pezzo surface is an (anticanonically embedded) surface with very ample anticanonical class and canonical singularities (in other words, du Val singularities, simple surface singularities, Kleinian singularities, or rational double points). (Classes of canonical and Gorenstein singularities for surfaces coincide.) So we use the following more general definition.

Definition 4.2. A del Pezzo surface is a complete surface with very ample anticanonical class and canonical singularities. A weak del Pezzo surface is a complete surface with nef and big anticanonical class and canonical singularities.

Remark 4.3. Weak del Pezzo surfaces are (partial) minimal resolutions of singularities of del Pezzo surfaces. Exceptional divisors of the resolutions are \((-2)\)-curves.

The degree of a del Pezzo surface \(S\) is the integer \(d = (-K_S)^2\). One has \(1 \leq d \leq 9\). If \(d > 2\), then the anticanonical class of \(S\) is very ample and gives the embedding \(S \hookrightarrow \mathbb{P}^d\), so both definitions coincide. In this section we assume that \(d > 2\).

Obviously, in projecting a degree-\(d\) surface in \(\mathbb{P}^d\) from a smooth point on it one obtains a degree \(d - 1\) surface in \(\mathbb{P}^{d-1}\). This projection is nothing but a blowup of the centre of the projection and a blowdown of all lines passing through this point. (By the adjunction formula these lines are \((-2)\)-curves.) If we choose general (for example, not lying on lines) centres of projections, then we get a classical description
of a smooth del Pezzo surface of degree $d$ as a quadric surface (with $d = 8$) or a blowup of $\mathbb{P}^2$ at $9 - d$ points. They degenerate to singular surfaces which are projections from non-general points (including infinitely close ones). Moreover, all del Pezzo surfaces of given degree lie in the same irreducible deformation space except for degree 8, when there are two components (one for a quadric surface and one for the blowup $F_1$ of $\mathbb{P}^2$ at a point). General elements of these families are smooth, and all singular del Pezzo surfaces are degenerations of smooth ones in these families. This description enables us to construct toric degenerations of del Pezzo surfaces. In particular, $\mathbb{P}^2$ is itself toric. Projecting from toric points one obtains (possibly singular) toric del Pezzo surfaces.

**Remark 4.4.** The approach to description of del Pezzo surfaces via their toric degenerations and the connection of degenerations by elementary transformations (projections) can be generalized to the threefold case. That is, smooth Fano threefolds can be connected via toric degenerations and *toric basic links*. For details see [14].

**Remark 4.5.** Del Pezzo surfaces of degree 1 or 2 also have toric degenerations. Indeed, these surfaces can be described as hypersurfaces in weighted projective spaces, that is, ones of degree 4 in $\mathbb{P}(1, 1, 1, 2)$ and of degree 6 in $\mathbb{P}(1, 1, 2, 3)$, respectively, so they can be degenerated to binomial hypersurfaces (see Example 7.28). However, their singularities are worse then canonical.

Let $T_S$ be a Gorenstein toric degeneration of a del Pezzo surface $S$ of degree $d$, and let

$$\Delta = F(T_S) \subset M_\mathbb{R} = \mathbb{Z}^2 \otimes \mathbb{R}$$

be a fan polygon of $T_S$. Let $f$ be a Laurent polynomial such that $N(f) = \Delta$.

Our goal now is to describe in detail a way to construct a Calabi–Yau compactification for $f$. More precisely, we construct a commutative diagram

$$\begin{array}{ccc}
(C^*)^2 & \longrightarrow & Y \\
\downarrow f & & \downarrow \ \\
A^1 & \longrightarrow & Z
\end{array}$$

where $Y$ and $Z$ are smooth, fibres of the maps $Y \to A^1$ and $Z \to \mathbb{P}^1$ are compact, and $-K_Z = f^{-1}(\infty)$; for simplicity we denote all ‘vertical’ maps in the diagram by $f$.

The strategy is the following. First we consider a natural compactification of the pencil $\{f = \lambda\}$ to an elliptic pencil in a toric del Pezzo surface $T^\vee$. Then we resolve singularities of $T^\vee$ and obtain a pencil in a smooth toric weak del Pezzo surface $\tilde{T}^\vee$. Finally, we resolve a base locus of the pencil to obtain $Z$. We obtain $Y$ by cutting out the strict transform of the boundary divisor of $\tilde{T}^\vee$.

The polygon $\Delta$ has integral vertices in $M_\mathbb{R}$ and it has the origin as a unique strictly interior integral point. The dual polygon $\nabla = \Delta^\vee \subset M = M^\vee$ has integral vertices and a unique strictly interior integral point as well. Geometrically this means that singularities of $T$ and $T^\vee$ are canonical.

**Remark 4.6.** The normalized volume of $\nabla$ is given by

$$\text{vol} \nabla = |\text{integral points in } \nabla| - 1 = (-K_S)^2 = d.$$
It is easy to see that

$$|\text{integral points on the boundary of } \Delta| + |\text{integral points on the boundary of } \nabla| = 12.$$ 

In particular, \( \text{vol } \Delta = 12 - d \).

**Compactification construction 4.7** (see [78]). By Fact 2.8, the anticanonical linear system on \( T^\vee \) can be described as the projectivization of the linear space of Laurent polynomials whose Newton polygons are contained in \( \nabla^\vee = \Delta \). Thus the natural way to compactify the family is to do it using the embedding \((\mathbb{C}^*)^2 \hookrightarrow T^\vee\). Fibres of the family are anticanonical divisors in this (possibly singular) toric variety. Two anticanonical sections intersect in \((-K_{T^\vee})^2 = \text{vol } \Delta = 12 - d\) points (counted with multiplicities), so the compactification of the pencil in \( T^\vee \) has \( 12 - d \) base points (possibly with multiplicities). The pencil \( \{ \lambda_0 f = \lambda_1 \}, (\lambda_0 : \lambda_1) \in \mathbb{P} \), is generated by its general member and a divisor corresponding to the constant Laurent polynomial, that is, the boundary divisor of \( T^\vee \). We note that the torus-invariant points of \( T^\vee \) do not lie in the base locus of the family by Fact 2.10.

Let \( \tilde{T}^\vee \to T^\vee \) be a minimal resolution of singularities of \( T^\vee \). Pull back the pencil under consideration. We get an elliptic pencil with \( 12 - d \) base points (with multiplicities), which are smooth points of the boundary divisor \( D \) of the toric surface \( \tilde{T}^\vee \); this divisor is a wheel of \( d \) smooth rational curves. Blow up these base points and get an elliptic surface \( Z \). Let \( E_1, \ldots, E_{12-d} \) be the exceptional curves of the blowup \( \pi: Z \to \tilde{T}^\vee \); in particular, \( Z \) is not toric. For simplicity we denote the strict transform of \( D \) by \( \tilde{D} \). Then one has

$$-K_Z = \pi^*(-K_{\tilde{T}^\vee}) - \sum E_i = D + \sum E_i - \sum E_i = D.$$ 

Thus, the anticanonical class \(-K_Z\) contains \( D \) and consists of fibres of \( Z \). This, in particular, means that an open variety \( Y = Z \setminus D \) is a Calabi–Yau compactification of the pencil provided by \( f \). This variety has \( e > 0 \) sections, where \( e \) is the number of base points of the pencil in \( \tilde{T}^\vee \) counted without multiplicities.

Summarizing, we obtain an elliptic surface \( f: Z \to \mathbb{P}^1 \) with smooth total space \( Z \) and a wheel \( D \) of \( d \) smooth rational curves over infinity.

**Remark 4.8.** Let the polynomial \( f \) be general among those with the same Newton polygon. Then singular fibres of \( Z \to \mathbb{P}^1 \) are either curves with a single node or a wheel of \( d \) rational curves over infinity. By the Noether formula

$$12\chi(\mathcal{O}_Z) = (-K_Z)^2 + e(Z) = e(Z),$$ 

where \( e(Z) \) is the topological Euler characteristic. Therefore, the singular fibres for \( Z \to \mathbb{P}^1 \) are \( d \) curves with one node and a wheel of \( d \) curves over infinity. This description was given in [4].

**Remark 4.9.** One can simultaneously compactify all toric Landau–Ginzburg models for all del Pezzo surfaces of degree at least three. That is, all reflexive polygons are contained in the biggest polygon \( B \) that has vertices \((2, -1), (-1, 2)\) and \((-1, -1)\). Thus, fibres of all toric Landau–Ginzburg models can be simultaneously compactified to (possibly singular) anticanonical curves on \( T_{B^\vee} = \mathbb{P}^2 \). Blow up the base locus.
to construct a base-point free family. However, in this case a general member of the family can pass through toric points, since it can happen that $N(f) \subset B$. This means that some exceptional divisors of the minimal resolution are extra curves in a wheel over infinity.

In other words, consider a triangle of lines and an elliptic curve on $\mathbb{P}^2$. A general member of the pencil given by $f$ is an elliptic curve on $\mathbb{P}^2$. The total space of the log-Calabi–Yau compactification is a blowup of nine intersection points (counted with multiplicities) of the elliptic curve and the triangle of lines. Exceptional divisors for points lying over vertices of the triangle are components of the wheel over infinity for the log-Calabi–Yau compactification; the others are either sections of the pencil or components of fibres over finite points.

Now following [78] we describe toric Landau–Ginzburg models for del Pezzo surfaces and toric weak del Pezzo surfaces. That is, for a del Pezzo surface $S$, its Gorenstein toric degeneration $T$ with a fan polygon $\Delta$, its crepant resolution $e_T$ with the same fan polygon, and a divisor $D \in \text{Pic}(S)_C \simeq \text{Pic}(T)_C$, we construct by induction two Laurent polynomials $f_{S,D}$ and $f_{e_T,D}$ that are toric Landau–Ginzburg models for $S$ and $e_T$, respectively. To do this we use, in particular, Givental’s construction of Landau–Ginzburg models for smooth toric varieties (see §7.1).

Let $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$ be a quadric surface, and let $D_S$ be an $(a,b)$-divisor on it. Let $T_1 = S$, and let $T_2$ be a quadratic cone; $T_1$ and $T_2$ are the only Gorenstein toric degenerations of $S$. The crepant resolution of $T_2$ is a Hirzebruch surface $F_2$, so let $D_{F_2} = \alpha s + \beta f$, where $s$ is a section of $F_2$, so that $s^2 = -2$, and $f$ is a fibre of the map $F_2 \to \mathbb{P}^1$. Define

$$f_{S,D_S} = f_{T_1,D_S} = x + \frac{e^{-a}}{x} + y + \frac{e^{-b}}{y},$$

for the first toric degeneration and

$$f_{S,D_S} = y + e^{-a} \frac{1}{xy} + (e^{-a} + e^{-b}) \frac{1}{y} + e^{-b} \frac{x}{y},$$

$$f_{T_2,D_{F_2}} = y + \frac{e^{-\beta}}{xy} + \frac{e^{-\alpha}}{y} + \frac{x}{y}$$

for the second.

Now assume that $S$ is a blowup of $\mathbb{P}^2$. First let $S = T = \tilde{T} = \mathbb{P}^2$, let $l$ be the class of a line on $S$, and let $D = a_0 l$. Then up to a toric change of variables

$$f_{(\mathbb{P}^2,D)} = x + y + \frac{e^{-a_0}}{xy}.$$  

Now let $S'$ be a blowup of $\mathbb{P}^2$ at $k$ points with exceptional divisors $e_1, \ldots, e_k$, let $S$ be a blowup of $S'$ at a point, and let $e_{k+1}$ be an exceptional divisor for the blowup. We identify divisors on $S'$ and their strict transforms on $S$, so

$$\text{Pic}(S') = \text{Pic}(\tilde{T}') = \mathbb{Z}l + Ze_1 + \cdots + Ze_k$$

and

$$\text{Pic}(S) = \mathbb{Z}l + Ze_1 + \cdots + Ze_k + Ze_{k+1}.$$
Let

\[ D' = a_0 l + a_1 e_1 + \cdots + a_k e_k \in \text{Pic}(S')_C \]

and \( D = D' + a_{k+1} e_{k+1} \in \text{Pic}(S)_C \). First we describe the polynomial \( f_{\bar{T},D} \). Combinatorially \( \Delta = F(\bar{T}) \) is obtained from a polygon \( \Delta' = F(\bar{T}') \) by adding one integral point \( K \) that corresponds to the exceptional divisor \( e_{k+1} \), and taking the convex hull. Let \( L \) and \( R \) be the boundary points of \( \Delta \) neighbouring with \( K \) on the left and right with respect to the clockwise order. Let \( c_L \) and \( c_R \) be the coefficients of the monomials corresponding to \( L \) and \( R \) in \( f_{\bar{T}',D'} \). Let \( M \in \mathbb{T}[x,y] \) be the monomial corresponding to \( K \). Then from Givental’s description of Landau–Ginzburg models for toric varieties (see §7.1) one has

\[ f_{\bar{T},D} = f_{\bar{T}',D'} + c_L c_R e^{-a_{k+1}} M. \]

The polynomial \( f_{S,D} \) differs from \( f_{\bar{T},D} \) by coefficients at non-vertex boundary points. For any boundary point \( K \in \Delta \) define the marking \( m_K \) to be the coefficient of \( f_{\bar{T},D} \) at \( K \). Consider a facet of \( \Delta \) and let \( K_0, \ldots, K_r \) be integral points of this facet in clockwise order. Then the coefficient of \( f_{S,D} \) at \( K_i \) is the coefficient of \( s^i \) in the polynomial

\[ m_{K_0} \left( 1 + \frac{m_{K_1}}{m_{K_0}} s \right) \cdots \left( 1 + \frac{m_{K_r}}{m_{K_{r-1}}} s \right). \]

Remark 4.10. One has \( \text{Pic}(S) \simeq \text{Pic}(\bar{T}) \). That is, if \( S \) is not a quadric, then both \( S \) and \( \bar{T} \) are obtained by a sequence of blowups at points (the only difference is that for \( \bar{T} \) the points can lie on the exceptional divisors of previous blowups). Thus, in both cases Picard groups are generated by a class of a line on \( \mathbb{P}^2 \) and exceptional divisors \( e_1, \ldots, e_k \). However, an image of \( e_i \) under the map of Picard groups given by the degeneration of \( S \) to \( \bar{T} \) can be equal not to \( e_i \) itself but to some linear combination of the exceptional divisors. In other words, these bases do not agree with degenerations.

Remark 4.11. The spaces parameterizing toric Landau–Ginzburg models for \( S \) and for \( \bar{T} \) are the same — they are the spaces of Laurent polynomials with Newton polygon \( \Delta \) modulo toric rescaling. Therefore, any Laurent polynomial corresponds to different elements of \( \text{Pic}(S)_C \simeq \text{Pic}(\bar{T})_C \). This gives a map \( \text{Pic}(S)_C \rightarrow \text{Pic}(\bar{T})_C \). However, this map is transcendental because of the exponential nature of the parametrization.

Proposition 4.12 (see [78], Proposition 21). The Laurent polynomial \( f_{S,D} \) is a toric Landau–Ginzburg model for \((S,D)\).

Proof. It is well known that \( S \) is either a smooth toric variety or a complete intersection in a smooth toric variety. This enables one to compute the series \( \bar{I}^S \), and therefore the series \( \bar{I}^{S,D} \), following [39] (see Theorem 7.8). By using this it is straightforward to check that the period condition for \( f_{S,D} \) holds. The Calabi–Yau condition holds by the compactification construction 4.7. Finally, the toric condition holds by construction (see Example 4.15). \( \square \)

Proposition 4.13 ([78], Proposition 22). Let \( T_1 \) and \( T_2 \) be two different Gorenstein toric degenerations of a del Pezzo surface \( S \), and let \( \Delta_1 = F(T_1) \) and \( \Delta_2 = F(T_2) \).
Consider the families of Calabi–Yau compactifications of Laurent polynomials with Newton polygons $\Delta_1$ and $\Delta_2$. Then there is a birational isomorphism of these families. In other words, there is a birational isomorphism between affine spaces of Laurent polynomials with supports in $\Delta_1$ and $\Delta_2$ modulo a toric change of variables that preserves Calabi–Yau compactifications.

Proof. One can check that the polygons $\Delta_1$ and $\Delta_2$ differ by (a sequence of) mutations (for instance, see [3] and also [58]). These mutations agree, by construction, with fibrewise birational isomorphisms of toric Landau–Ginzburg models modulo a change of basis in $H^2(S, \mathbb{Z})$. The statement follows from the fact that birational elliptic curves are isomorphic. □

Remark 4.14. Let $D = 0$. Then the polynomial $f_{S,0}$ has coefficients 1 at vertices of its Newton polygon and $\binom{n}{k}$ at the $k$th integral point of an edge of integral length $n$. In other words, $f_{S,0}$ is binomial (see §5.1).

Example 4.15. Let $S = S_7$. This surface has two Gorenstein toric degenerations: it is toric itself, and also it can be degenerated to a singular surface which is obtained by a blowup of $\mathbb{P}^2$, a blowup of a point on the exceptional curve, and a blowdown of the first exceptional curve to a point of type $A_1$.

Let $\Delta_1$ be the polygon with vertices

$$(1,0), \quad (1,1), \quad (0,1), \quad (-1,-1), \quad (0,-1),$$

and let $D = a_0l + a_1e_1 + a_2e_2$. Then

$$f_{\tilde{T}_{\Delta_1},D} = f_{S,D} = x + y + e^{-a_0} \frac{1}{xy} + e^{-(a_0+a_1)} \frac{1}{y} + e^{-a_2} xy.$$

Let $\Delta_2$ be the polygon with vertices

$$(1,0), \quad (0,1), \quad (-1,-1), \quad (1,1),$$

and let $D = a_0l + a_1e_1 + a_2e_2$. Then

$$f_{\tilde{T}_{\Delta_2},D} = x + y + e^{-a_0} \frac{1}{xy} + e^{-(a_0+a_1)} \frac{1}{y} + e^{-(a_0+a_1+a_2)} \frac{x}{y},$$

and

$$f'_{S,D} = x + y + e^{-a_0} \frac{1}{xy} + (e^{-(a_0+a_1)} + e^{-(a_0+a_2)}) \frac{1}{y} + e^{-(a_0+a_1+a_2)} \frac{x}{y}.$$

(Here $f_{S,D}$ and $f'_{S,D}$ are toric Landau–Ginzburg models for $(S, D)$ in different bases in $(\mathbb{C}^*)^2$.) One can easily check that the mutation

$$x \rightarrow x, \quad y \rightarrow \frac{y}{1 + e^{-a_2} x}$$

sends $f_{S,D}$ to $f'_{S,D}$.

The surface $S$ is toric, so by Givental

$$\tilde{I}_0^{S,D} = \sum_{k,l,m} \frac{(2k + 3l + 2m)! \exp(-a_0(k + l + m) - a_1 k - a_2 m) t^{2k+3l+2m}}{(k + l)! (l + m)! k! l! m!}$$

(see [19]). One can check that $\tilde{I}_0^{S,D} = I_{f_{S,D}} = I_{f'_{S,D}}$. 
This part is devoted to the most studied case of toric Landau–Ginzburg models, that is, models for Fano threefolds. We focus mainly on the Picard-rank-1 case.

5. Fano threefolds

5.1. Weak Landau–Ginzburg models. Consider a smooth Fano threefold $X$ of Picard rank $\rho$ and with a divisor $D$ on it. Recall that we associate with this pair the regularized series $\widetilde{I}^{X,D}$ and, in particular, the constant term $\widetilde{I}_0^{X,D}$ of this series. These series are given by the intersection theory (for curves and divisors) on $X$ and the series $\widetilde{I}^X = \widetilde{I}^{X,0}$ (see the beginning of §3), or even the series $I_0^X$ (see [74]). The coefficients of the series depend on the even part of the cohomology of $X$, which is quite simple:

$$H^0(X,\mathbb{Z}) = H^6(X,\mathbb{Z}) = \mathbb{Z} \quad \text{and} \quad H^2(X,\mathbb{Z}) = H^4(X,\mathbb{Z}) = \mathbb{Z}^\rho.$$  

The relations on Gromov–Witten invariants show that the coefficients are given by a finite (small) number of three-point Gromov–Witten invariants (see details, for example, in [74]). In the case of Picard-rank-1 Fano threefolds these three-point invariants were found, using results due to Givental, Fulton–Woodward, and others (see [72] and [73] and references therein). Theorems 7.8 and 7.44 enable one to compute $\widetilde{I}_0^X$ for complete intersections in smooth toric varieties and Grassmannians. Fano threefolds with $\rho > 1$ have descriptions of this type, and the corresponding series $\widetilde{I}_0^X$ are computed in [20] (see also [19]). We need the series $\widetilde{I}_0^X$ unless otherwise stated, so we do not need the intersection theory on $X$. From now on we assume that the series $\widetilde{I}_0^X$ is known.

We assume that $D = 0$. Recall that a Laurent polynomial $f_X$ is called a weak Landau–Ginzburg model for $X$ if it satisfies the period condition, that is, its principal period (see Definition 3.2) coincides with $I_0^X$. There are 105 families of smooth Fano threefolds (see, for instance, [51] and [67]). The anticanonical classes are very ample for 98 of these. Weak Landau–Ginzburg models are known for each of them (they are usually not unique); see [19] for the case of very ample anticanonical class and Proposition 5.1 for the other seven cases.

Smooth Fano threefolds with anticanonical classes that are not very ample can be described as complete intersections in smooth toric varieties or weighted projective spaces, so one can construct Givental’s Landau–Ginzburg models (see Definition 7.3) satisfying the period condition.

Proposition 5.1 (see Proposition 5.11). The Fano threefolds $X_{1-1}$, $X_{1-11}$, $X_{2-1}$, $X_{2-2}$, $X_{2-3}$, $X_{9-1}$, and $X_{10-1}$ have weak Landau–Ginzburg models.

Proof. The Fano threefold $X_{1-1}$ is a hypersurface of degree 6 in $\mathbb{P}(1, 1, 1, 1, 3)$, and $X_{1-11}$ is a hypersurface of degree 6 in $\mathbb{P}(1, 1, 1, 2, 3)$. The Fano variety $X_{2-1}$ is a hypersurface section of type $(1, 1)$ in $\mathbb{P}^1 \times X_{1-11}$ in an anticanonical embedding; in other words, it is a complete intersection of hypersurfaces of types $(1, 1)$ and $(0, 6)$ in $\mathbb{P}^1 \times \mathbb{P}(1, 1, 1, 2, 3)$. The Fano variety $X_{2-2}$ is a hypersurface in a certain toric variety (see [20]). The Fano variety $X_{2-3}$ is a hyperplane section of type $(1, 1)$ in $\mathbb{P}^1 \times X_{1-12}$ in an anticanonical embedding; in other words, it is a complete intersection of hypersurfaces of types $(1, 1)$ and $(0, 4)$ in $\mathbb{P}^1 \times \mathbb{P}(1, 1, 1, 1, 2)$. Finally, $X_{9-1} = \mathbb{P}^1 \times S_2$ and $X_{10-1} = \mathbb{P}^1 \times S_1$.  

This part is devoted to the most studied case of toric Landau–Ginzburg models, that is, models for Fano threefolds. We focus mainly on the Picard-rank-1 case.
For a variety $X_{i-j}$ we construct its Givental’s Landau–Ginzburg model and then represent it by a Laurent polynomial $f_{i-j}$ (see, for instance, the formula (7.8)). It satisfies the period condition by [19]. We consider these cases one by one.

Givental’s Landau–Ginzburg model for $X_{2-1}$ is the complete intersection

$$
\begin{aligned}
&u + v_0 = 0, \\
v_1 + v_2 + v_3 = 0
\end{aligned}
$$

in $\mathcal{T}[u, v_0, v_1, v_2, v_3]$ with a function

$$
u + \frac{1}{u} + v_0 + v_1 + v_2 + v_3 + \frac{1}{v_1 v_2 v_3}$$

(see Definition 7.3). After the birational change of variables

$$
v_1 = \frac{x}{x + y + 1}, \quad v_2 = \frac{y}{x + y + 1}, \quad v_3 = \frac{1}{x + y + 1}, \quad u = \frac{z}{z + 1}, \quad v_0 = \frac{1}{z + 1},
$$

up to an additive shift, one obtains the function

$$
f_{2-1} = \frac{(x + y + 1)^6(z + 1)}{xy^2} + \frac{1}{z}
$$

on a torus $\mathcal{T}[x, y, z]$.

In a similar way one obtains weak Calabi–Yau compactifications for the other varieties. One has

$$
\begin{aligned}
f_{1-1} &= \frac{(x + y + z + 1)^6}{xyz}, \\
f_{1-11} &= \frac{(x + y + 1)^6}{xy^2z} + z, \\
f_{2-2} &= \frac{(x + y + z + 1)^2}{x} + \frac{(x + y + z + 1)^4}{yz}, \\
f_{2-3} &= \frac{(x + y + 1)^4(z + 1)}{xyz} + z + 1, \\
f_9 &= x + \frac{1}{x} + \frac{(y + z + 1)^4}{yz}, \\
f_{10-1} &= \frac{(x + y + 1)^6}{xy^2} + z + \frac{1}{z}.
\end{aligned}
$$

Thus, one can assume that the anticanonical class of $X$ is very ample. To find a weak Landau–Ginzburg model for $X$ one can, as in Proposition 5.1, construct Givental’s Landau–Ginzburg model and try to find birational isomorphisms of total spaces of such models with an algebraic torus (see Theorem 7.34). However, we use another approach. That is, we consider ‘good’ three-dimensional polytopes and study ‘valid’ Laurent polynomials supported on them (in particular, their coefficients are symmetric enough). At the moment, the most appropriate method for constructing weak Landau–Ginzburg models seems to be a generalization of that described below (see Remark 5.4). However, we do not need it here.

Weak Landau–Ginzburg models, ‘guessed’ via the period condition (see [75]) or obtained from Landau–Ginzburg models for complete intersections, first, often have reflexive Newton polytopes and, second, often satisfy the binomial principle...
This latter prescribes a way of specifying the coefficients of Laurent polynomials with fixed Newton polytopes. Namely, one needs to put 1s at the vertices of such a polytope, and \( \binom{n}{i} \) at the \( i \)th (from any end) integral point of an edge of integral length \( n \). This principle can be applied in many cases (namely, to Newton polytopes of toric varieties with cDV singularities, that is, to polytopes whose integral points, except for the origin, lie on edges). Most smooth Fano threefolds have degenerations to toric varieties with cDV singularities, but unfortunately not all of them. Thus, we use the following generalization of the binomial principle.

**Definition 5.2** (see [20]). An integral polygon is said to be of type \( A_n, n \geq 0 \), if it is a triangle such that two of its edges have integral length 1 and the remaining one has integral length \( n \). (In other words, its integral points are the 3 vertices and \( n-1 \) points lying on the same edge.) In particular, \( A_0 \) is a segment of integral length 1.

An integral three-dimensional polytope is said to be Minkowski if it is reflexive and if all its facets are Minkowski polygons.

**Definition 5.3** (see [20]). Let \( P \in \mathbb{Z}^2 \otimes \mathbb{R} \) be an integral polygon of type \( A_n \). Let \( v_0, \ldots, v_n \) be the consecutive integral points on an edge of \( P \) of integral length \( n \) and let \( u \) be the remaining integral point of \( P \). Let \( x = (x_1, x_2) \) be a multivariable that corresponds to the integral lattice \( \mathbb{Z}^2 \subset \mathbb{R}^2 \). Put
\[
 f_P = x^u + \sum \binom{n}{k} x^{v_k}.
\]
(In particular, \( f_P = x^u + x^{v_0} \) for \( n = 0 \).)

Let \( Q = Q_1 + \cdots + Q_s \) be an admissible lattice Minkowski decomposition of an integral polygon \( Q \subset \mathbb{R}^2 \), and let
\[
 f_{Q_1,\ldots,Q_s} = f_{Q_1} \cdots f_{Q_s}.
\]

A Laurent polynomial \( f \subset \mathbb{T}[x_1, x_2, x_3] \) is said to be Minkowski if \( N(f) \) is Minkowski and for any facet \( Q \subset N(f) \), as for an integral polygon, there exists an admissible lattice Minkowski decomposition \( Q = Q_1 + \cdots + Q_s \) such that \( f|_Q = f_{Q_1,\ldots,Q_s} \).

All the 98 families of smooth Fano threefolds that have very ample anticanonical classes have weak Landau–Ginzburg models of Minkowski type (see [20]).

**Remark 5.4.** There exists the following notion of maximally mutable polynomials (see, for instance, [54]). A birational isomorphism
\[
 \phi: \mathbb{T}[x_1, \ldots, x_n] \to \mathbb{T}[y_1, \ldots, y_n]
\]
is called an elementary mutation of polynomials \( f \) and \( g \) if it is given by \( y_1 = r(x_1, \ldots, x_n), \ y_i = x_i \) for \( 2 \leq i \leq n \), and \( \phi^*(f) = g \). The Laurent polynomials \( f \) and \( g \) in \( n \) variables are said to be mutationally equivalent if there exists a sequence of elementary mutations transforming one to the other. On the other hand, if we have a polytope \( \Delta \) and a vector \( v \) in the dual space, one can define a mutation of \( \Delta \) in \( v \) (if it exists) by multiplying the \( k \)-slice for \( \Delta \) by the \( k \)th power of some fixed polytope \( \Delta_v \) (or dividing by it for \( k < 0 \)). (Mutations of polytopes correspond to deformations of the toric Fano varieties whose fan polytopes they are; see [49].) It is clear that mutations of polynomials induce mutations of their Newton polytopes. However, the converse is not true in general. Strong restrictions are needed to make the converse true.

A Laurent polynomial is said to be maximally mutational if any mutation of its Newton polytope is given by a mutation of the polynomial and if the same is true for all mutations of the polynomial. Rigid (without parameters) maximally mutational Laurent polynomials form a class of weak Landau–Ginzburg models which now fits in very well with the list of Fano varieties. In dimension 2 there are exactly 10 such classes, and elements of each of these are weak Landau–Ginzburg models for all of the 10 families of del Pezzo surfaces. In dimension 3 there are 105 classes, and each of these corresponds to one of the 105 families of Fano threefolds (private communication from A. Kasprzyk).

**Remark 5.5.** Minkowski decompositions of facets of Newton polytopes of Laurent polynomials of Minkowski type naturally give mutations of these polynomials. It turns out (see [20]) that Minkowski type polynomials are mutationally equivalent if and only if they have the same series of constant terms (and thus are weak Landau–Ginzburg models of the same Fano variety if such a variety exists). Classes of Laurent polynomials of Minkowski type which do not correspond to smooth Fano threefolds are expected to correspond to smooth orbifolds.

### 5.2. Calabi–Yau compactifications

Let \( f \) be a weak Landau–Ginzburg model for a smooth Fano threefold \( X \) and a divisor \( D \) on it. Recall the notation from §2.2. Let

\[
\Delta = N(f), \quad \nabla = \Delta^\vee, \quad T = T_\Delta, \quad \text{and} \quad T^\vee = T_\nabla.
\]

In many cases polynomials satisfying the period and toric conditions satisfy the Calabi–Yau condition as well. However, it is not easy to check this condition: unlike for the first two conditions, there are no sufficiently general approaches for the third condition; usually one needs to check the Calabi–Yau condition ‘by hand’. The natural idea is to compactify the fibres of the map \( f: (\mathbb{C}^*)^3 \to \mathbb{C} \) using the embedding \((\mathbb{C}^*)^3 \to T^\vee\). Indeed, the fibres compactify to anticanonical sections in \( T^\vee \), and therefore have trivial canonical classes. However, first, \( T^\vee \) is usually singular, and, even if we resolve it (if it has a crepant resolution!), we can only conclude that its general anticanonical section is a smooth Calabi–Yau variety, but it is hard to say anything about the particular sections we need. Second, the family of anticanonical sections we are interested in has a base locus which we need to resolve to construct a Calabi–Yau compactification; and this resolution can be non-crepant.

The coefficients of polynomials corresponding to Fano threefolds tend to be symmetric, at least for the simplest Newton polytopes. In this case the base loci are
simpler and enable us to construct Calabi–Yau compactifications. We will assume $f$ to be of Minkowski type. In particular, $\nabla$ is integral, in other words, $\Delta$ is reflexive, and integral points of both $\Delta$ and $\nabla$ are either boundary points or the origins.

**Lemma 5.6.** Let $T$ be a threefold reflexive toric variety. Then $\tilde{T}^\vee$ is smooth.

*Proof.* Let $C$ be a two-dimensional cone of the fan of $\tilde{T}^\vee$. It is a cone over an integral triangle $R$ without strictly interior integral points such that $R$ lies in the affine plane $L = \{x : \langle x, y \rangle \geq -1\}$ for some $y \in \mathcal{M}$. This means that in some basis $e_1, e_2, e_3$ in $\mathcal{M}$ one has $L = \{a_1e_1 + a_2e_2 + e_3\}$. Let $P$ be a pyramid over $R$ whose vertex is an origin. Then by Pick’s formula $\text{vol} R = 1/2$ and $\text{vol} P = 1/6$, which means that the vertices of $R$ form a basis in $\mathcal{M}$, so $\tilde{T}^\vee$ is smooth. □

Unfortunately, Lemma 5.6 does not hold for higher dimensions in general, because there are $n$-dimensional simplices whose only integral points are vertices, and with volumes greater then $1/n!$.

**Lemma 5.7** (see [78], Lemma 25). Let $f$ be a Laurent polynomial of Minkowski type. Then for any facet $Q$ of $\Delta$ the curve $R_{Q,f}$ is the union of (transversally intersecting) smooth rational curves (possibly with multiplicities).

*Idea of proof.* For the non-Minkowski decomposable case this follows from Facts 2.7 and 2.9. In the decomposable case the curve $R_{Q,f}$ is a union of curves that correspond to Minkowski summands of $Q$. □

**Proposition 5.8.** Let $W$ be a smooth threefold. Let $F$ be a one-dimensional anticanonical linear system on $W$ with reduced fibre $D = F_\infty$. Let the base locus $B \subset D$ be a union of smooth curves (possibly with multiplicities) such that for any two components $D_1$ and $D_2$ of $D$ one has $D_1 \cap D_2 \not\subset B$. Then there exists a resolution of the base locus $f : Z \to \mathbb{P}^1$ with a smooth total space $Z$ such that $-K_Z = f^{-1}(\infty)$.

*Proof* (see the compactification construction 4.7). Let $\pi : W' \to W$ be a blowup of one component $C$ of $B$ on $W$. Since $\pi$ is a blowup of a smooth curve on a smooth variety, $W'$ is smooth. Let $E$ be the exceptional divisor of the blowup. Let $D' = \bigcup D'_i$ be the strict transform of $D = \bigcup D_i$. Since the multiplicity of $C$ in $D$ is 1, one has

$$-K_{W'} = \pi^*(-K_W) - E = D' + E - E = D'.$$

Moreover, the base locus of the family on $W'$ is the same as $B$ or $B \setminus C$, possibly together with a smooth curve $C'$ which is isomorphic to $E \cap D'_i$; in particular, $C$ is isomorphic to $\mathbb{P}^1$. (There are no isolated base points because the base locus is an intersection of two divisors on a smooth variety.) Thus, all the conditions of the proposition hold for $W'$. Since $(W, F)$ is a canonical pair, the base locus $B$ can be resolved in a finite number of blowups. This gives the required resolution. □

**Theorem 5.9.** Any Minkowski Laurent polynomial in three variables admits a log-Calabi–Yau compactification.

*Proof.* Let $f$ be a Minkowski Laurent polynomial. Recall that the Newton polytope $\Delta$ of $f$ is reflexive, and the (singular Fano) toric variety whose fan polytope is $\nabla = \Delta^\vee$ is denoted by $T^\vee$. The family of fibres of the map given by $f$ is a pencil $\{f = \lambda, \lambda \in \mathbb{C}\}$. Members of this family have natural compactifications to
anticanonical sections of $T^\vee$. This family (more precisely, its compactification to a family $\{\lambda_0 f = \lambda_1\}$ over $\mathbb{P}[\lambda_0 : \lambda_1]$) is generated by a general member and the member that corresponds to the constant Laurent polynomial. The latter is nothing but the boundary divisor $D$ of $T^\vee$ by Fact 2.10. Denote the pencil obtained on $T^\vee$ by $f : Z_{T^\vee} \rightarrow \mathbb{P}^1$ (for simplicity we use the same notation $f$ for the Laurent polynomial, the corresponding pencil, and resolutions of this pencil). By Lemma 5.7, the base locus of $f$ on $Z_{T^\vee}$ is a union of smooth (rational) curves (possibly with multiplicities). By Lemma 5.6, the variety $e_{T^\vee}$ is a crepant resolution of $T^\vee$. By the definition of a Newton polytope, the coefficients of the Minkowski Laurent polynomial at vertices of $\Delta$ are non-zero. This means that the base locus does not contain any torus-invariant strata of $T^\vee$ since it does not contain torus-invariant points by Fact 2.10. Thus we obtain a pencil $f : Z_{e_{T^\vee}} \rightarrow \mathbb{P}^1$, whose total space is smooth and the base locus is again a union of (transversally intersecting) smooth curves (possibly with multiplicities). By Proposition 5.8, there exists a resolution $f : Z \rightarrow \mathbb{P}^1$ of the base locus on $Z_{e_{T^\vee}}$ such that $Z$ is smooth and $-K_Z = f^{-1}(\infty)$. Therefore, $Z$ is the required log-Calabi–Yau compactification, and $Y = Z \setminus f^{-1}(\infty)$ is a Calabi–Yau compactification. □

Remark 5.10. The construction of a Calabi–Yau compactification is not canonical: it depends on the order of blowups of base-locus components. However, all log-Calabi–Yau compactifications are isomorphic in codimension one.

Proposition 5.11 (see Proposition 5.1). The Fano threefolds $X_{1,-1}$, $X_{1,-11}$, $X_{2,-1}$, $X_{2,-2}$, $X_{2,-3}$, $X_{9,-1}$, and $X_{10,-1}$ have toric Landau–Ginzburg models.

Proof. By Proposition 5.1 these varieties have weak Landau–Ginzburg models. By [47] and [32] they satisfy the period condition. In the spirit of [77] we compactify the family given by $f_{i-j}$ to a family of (singular) anticanonical hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^2$ or $\mathbb{P}^3$ and then crepantly resolve the singularities of the total space of the family. We consider these cases one by one.

- A weak Landau–Ginzburg model for $X_{2,-1}$ is the polynomial

$$f_{2,-1} = \frac{(x+y+1)^{6}(z+1)}{xy^2} + \frac{1}{z},$$

that is, a function on $\mathcal{F}[x, y, z]$.

Consider the family $\{f_{2,-1} = \lambda, \ \lambda \in \mathbb{C}\}$, make the birational change of variables

$$x = \frac{1}{b_1} - \frac{1}{b_1^2 b_2} - 1, \quad y = \frac{1}{b_1^2 b_2}, \quad z = \frac{1}{a_1} - 1,$$

and multiply the expression obtained by the denominator. We see that the family is birational to

$$\{(1-a_1)b_2^3 = ((1-a_1)\lambda - a_1)a_1(b_1 b_2 - b_1^2 b_2 - 1)\} \subset \mathbb{A}[a_1, b_1, b_2] \times \mathbb{A}[\lambda].$$

Now this family can be compactified to a family of hypersurfaces of bidegree $(2,3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$ using the embedding

$$\mathcal{F}[a_1, b_1, b_2] \hookrightarrow \mathbb{P}[a_0 : a_1] \times \mathbb{P}[b_0 : b_1 : b_2].$$
The (non-compact) total space of the family has trivial canonical class, and its singularities are a union of (possibly) ordinary double points and rational curves which are du Val along lines at general points. Blow up any of these curves. We again obtain singularities of the same type. After several crepant blowups one arrives at a threefold with just ordinary double points; these points admit a small resolution. This resolution completes the construction of the Calabi–Yau compactification. Note that the total space \((\mathbb{C}^*)^3\) of the initial family is embedded into the resolution.

In a similar way one obtains Calabi–Yau compactifications for the other varieties.

- One has
  \[
  f_{1-1} = \frac{(x + y + z + 1)^6}{xyz}.
  \]

The change of variables
  \[
  x = ab, \quad y = ac, \quad z = a - ab - ac - 1,
  \]

applied to the family \(\{f_{1-1} = \lambda\}\), and multiplication by the denominator give the family of quartics
  \[
  a^4 = \lambda bc(a - ab - ac - 1).
  \]

The embedding \(\mathcal{T}[a, b, c] \hookrightarrow \mathbb{P}[a : b : c : d]\) gives a compactification to a family of singular quartics over \(\mathbb{A}^1\).

- One has
  \[
  f_{1-11} = \frac{(x + y + 1)^6}{xy^2z} + z.
  \]

The change of variables
  \[
  x = a - ab - \frac{c}{b} - 1, \quad y = ab, \quad z = \frac{c}{b},
  \]

applied to the family \(\{f_{1-11} = \lambda\}\), and multiplication by the denominator give the family of quartics
  \[
  a^4 = (\lambda b - c)(a - ab - 1)c.
  \]

The embedding \(\mathcal{T}[a, b, c] \hookrightarrow \mathbb{P}[a : b : c : d]\) gives a compactification to a family of singular quartics over \(\mathbb{A}^1\).

- One has
  \[
  f_{2-2} = \frac{(x + y + z + 1)^2}{x} + \frac{(x + y + z + 1)^4}{yz}.
  \]

The change of variables
  \[
  x = ab, \quad y = bc, \quad z = c - ab - bc - 1,
  \]

applied to the family \(\{f_{2-2} = \lambda\}\) and multiplication by the denominator give the family of quartics
  \[
  ac^3 = (c - ab - bc - 1)(\lambda ab - c^2).
  \]

The embedding \(\mathcal{T}[a, b, c] \hookrightarrow \mathbb{P}[a : b : c : d]\) gives a compactification to a family of singular quartics over \(\mathbb{A}^1\).
• One has
\[ f_{2-3} = \frac{(x + y + 1)^4(z + 1)}{xyz} + z + 1. \]

The change of variables
\[ x = ac, \quad y = a - ac - 1, \quad z = \frac{b}{c} - 1, \]
applied to the family \( \{ f_{2-3} = \lambda \} \), and multiplication by the denominator give the family
\[ a^3b = (\lambda c - b)(b - c)(a - ac - 1). \]

The embedding \( \mathcal{F}[a, b, c] \hookrightarrow \mathbb{P}[a : b : c : d] \) gives a compactification to a family of singular quartics over \( \mathbb{A}^1 \).

• One has
\[ f_{9-1} = x + \frac{1}{x} + \frac{(y + z + 1)^4}{yz}. \]

The change of variables
\[ x = \frac{c}{b}, \quad y = ac, \quad z = a - ac - 1, \]
applied to the family \( \{ f_{9-1} = \lambda \} \), and multiplication by the denominator give the family
\[ a^3b = (\lambda bc - b^2 - c^2)(a - ac - 1). \]

The embedding \( \mathcal{F}[a, b, c] \hookrightarrow \mathbb{P}[a : b : c : d] \) gives a compactification to a family of singular quartics over \( \mathbb{A}^1 \).

• One has
\[ f_{10-1} = \frac{(x + y + 1)^6}{xy^2} + z + \frac{1}{z}. \]

The change of variables
\[ x = \frac{1}{b_1} - \frac{1}{b_1^2b_2} - 1, \quad y = \frac{1}{b_1^2b_2}, \quad z = a_1, \]
applied to the family \( \{ f_{10-1} = \lambda \} \), and multiplication by the denominator give the family
\[ a_1b_3^3 = (\lambda a_1 - a_1^2 - 1)(b_1b_2 - b_1^2b_2 - 1). \]

The embedding \( \mathcal{F}[a_1, b_1, b_2] \hookrightarrow \mathbb{P}[a_0 : a_1] \times \mathbb{P}[b_0 : b_1 : b_2] \) gives a compactification to a family of singular hypersurfaces of bidegree \((2, 3)\) in \( \mathbb{P}^1 \times \mathbb{P}^2 \) over \( \mathbb{A}^1 \).

In all the cases the total spaces of the families have crepant resolutions. The proposition is proved. \( \square \)

In [32] and [47] it is shown that all Fano threefolds with very ample anticanonical classes have weak Landau–Ginzburg models of Minkowski type satisfying the toric condition. Thus, summarizing Theorem 5.9, Proposition 5.1, Proposition 5.11, and the results of [32] and [47], one obtains the following assertion.
Corollary 5.12. A pair of a smooth Fano threefold $X$ and a trivial divisor on it has a toric Landau–Ginzburg model. Moreover, if $-K_X$ is very ample, then any Minkowski Laurent polynomial satisfying the period condition for $(X,0)$ is a toric Landau–Ginzburg model.

Remark 5.13. The construction of the compactification implies that $h^{i,0}(Z) = 0$ for all $i > 0$.

Remark 5.14. Recall that $\widetilde{T}$ is a smooth toric variety with $F(\widetilde{T}) = \Delta$. Let $f$ be a general Laurent polynomial with $N(f) = \Delta$. The Laurent polynomial $f$ is a toric Landau–Ginzburg model for a pair $(\widetilde{T},D)$, where $D$ is a general $\mathbb{C}$-divisor on $\widetilde{T}$. Indeed, the period condition for it is satisfied by [39]. Following the compactification procedure described in the proof of Theorem 5.9, one can see that the base locus $B$ is a union of smooth transversally intersecting curves (not necessary rational). This means that in the same way as above, the statement of Theorem 5.9 holds for $f$, so that $f$ satisfies the Calabi–Yau condition (see [42]). Finally, the toric condition holds for $f$ tautologically. Thus $f$ is a toric Landau–Ginzburg model for $(\widetilde{T},D)$.

Problem 5.15. Prove this for smooth Fano threefolds and any divisor. (A description of Laurent polynomials for all Fano threefolds and any divisor is contained in [32].)

Question 5.16. Is it true that the Calabi–Yau condition follows from the period condition and the toric condition? If not, then what conditions should be imposed on a Laurent polynomial for the implication to hold?

Another advantage of the compactification procedure described in Theorem 5.9 is that it enables one to describe ‘fibres of compactified toric Landau–Ginzburg models over infinity’. These fibres play an important role, for instance, in computing Landau–Ginzburg Hodge numbers (see §6). We summarize these considerations in the following assertion.

Corollary 5.17 (see [42], Conjecture 2.3.13). Let $f$ be a Minkowski Laurent polynomial. There is a log-Calabi–Yau compactification $f: Z \to \mathbb{P}^1$ with $-K_Z = f^{-1}(\infty) = D$, where $D$ consists of $(-K_{T_{N(f)}})^3/2 + 2$ components combinatorially given by a triangulation of a sphere. (This means that vertices of the triangulation correspond to components of $D$, edges correspond to intersections of components, and triangles correspond to triple intersection points of components.)

Proof. Let $\widetilde{T}^\vee$ be a (smooth) maximally triangulated toric variety such that $F(\widetilde{T}^\vee) = N(f)$, and let $D$ be a boundary divisor of $\widetilde{T}^\vee$. The numbers of components of $D$ and $D'$ coincide. Let $v$ be the number of vertices in a triangulation of $\nabla$; in other words, $v$ is the number of integral points on the boundary of $\nabla$, or, what amounts to the same thing, the number of components of $D$. Let $e$ be the number of edges in the triangulation of $\nabla$, and let $f$ be the number of triangles in the triangulation. Since the triangulation is a triangulation of a sphere, one has $v - e + f = 2$. On the other hand $2e = 3f$. This means that $v = f/2 + 2$. The assertion of the corollary follows from the fact that both $(-K_{T_{N(f)}})^3$ and $f$ are equal to a normalized volume of $\nabla$. □
Remark 5.18. Let \( g = (-K_X)^3/2 + 1 \) be the genus of the Fano threefold \( X \); in particular, \( D \) consists of \( g + 1 \) components. Then \( g + 1 = \dim |-K_X| \).

General fibres of compactified toric Landau–Ginzburg models are smooth K3 surfaces. However, some of them can be singular or even reducible. Our observations give almost no information about them. However, singular fibres are of special interest: they contain information about the derived category of singularities. There is a lack of examples of descriptions of singular fibres. A more computable invariant is the number of components of fibres (see Theorems 6.37 and 7.26).

5.3. Toric Landau–Ginzburg models. As we have mentioned, in [32] and [47] the toric condition was proved for a huge number of smooth Fano threefolds (in particular, for those we need). The methods used there include the theory of toric degenerations and an analysis of tangent bundles to deformation spaces at the points in these spaces that we need. In this section we study toric degenerations of Picard-rank-1 Fano threefolds in detail.

Let us give examples of toric Landau–Ginzburg models (of Minkowski type) and prove the toric condition for them.

Consider a projective variety \( X \subseteq \mathbb{P}^n \). Let it be defined by some homogeneous ideal \( I \subseteq S = \mathbb{C}[x_0, \ldots, x_n] \). If \( \prec \) is some monomial order for \( S \), then there is a flat family degenerating \( X \) to \( X_\prec = V(\text{init}_\prec(I)) \), where \( \text{init}_\prec(I) \) is the initial ideal of \( I \) with respect to the monomial order \( \prec \). This is not of immediate help in finding toric degenerations of \( X \), since in general \( X_\prec \) is highly singular with multiple components and thus cannot be equal to or degenerate to a toric variety.

Instead, the point is to consider toric varieties embedded in \( \mathbb{P}^n \) which also degenerate to \( X_\prec \). Consider such a toric variety \( Z \), and let \( \mathcal{H} \) be the Hilbert scheme of subvarieties of \( \mathbb{P}^n \) with Hilbert polynomial equal to that of \( X \). If \( X \) corresponds to a sufficiently general point of a component of \( \mathcal{H} \) and \( X_\prec \) lies only on this component, then \( X \) must degenerate to \( Z \). This is the geometric background for the following theorem; the triangulations which appear correspond to degenerations of toric varieties to certain special monomial ideals with unobstructed deformations.

Theorem 5.19 ([17], Corollary 3.4). Consider a three-dimensional reflexive polytope \( \nabla \) with \( m \) lattice points, \( 7 \leq m \leq 11 \), which admits a regular unimodular triangulation with the origin contained in every full-dimensional simplex, and every other vertex having valency 5 or 6. Then the smooth Fano threefold of index 1 and degree \( 2m - 6 \) admits a degeneration to \( T_\Delta \), where \( \Delta = \nabla^\vee \).

Example 5.20 \((X_{1-6})\). Let \( f \) be the Laurent polynomial from Table 1 for the Fano threefold \( X_{1-6} \). The dual of the Newton polytope \( \nabla = \Delta_f^\vee \) is the convex hull of the vectors \( \pm e_1, \pm e_2, e_3, -e_1 - e_2, e_2 + e_3, \) and \( -e_1 - e_2 - e_3 \) (see Fig. 1). The polytope \( \nabla \) has only one non-simplicial facet, a parallelogram. Subdividing this facet by either one of its diagonals gives a triangulation of \( \partial \nabla \), which naturally induces a triangulation of \( \nabla \) with the origin contained in every full-dimensional simplex. It is not difficult to check that this triangulation is in fact regular and unimodular; furthermore, all vertices (with the exception of the origin) have valency 5 or 6. Thus, by Theorem 5.19, the variety \( X_{1-6} \) degenerates to \( T_\Delta_f \).
Table 1. Toric Landau–Ginzburg models for smooth Fano threefolds of Picard rank 1.

| Index | Degree | Description | Toric LG model |
|-------|--------|-------------|----------------|
| $X_{1-1}$ | 1 2 | A hypersurface of degree 6 in $\mathbb{P}(1, 1, 1, 1, 3)$ | $\frac{(x+y+z+1)^6}{xyz}$ |
| $X_{1-2}$ | 1 4 | A general element is a quartic | $\frac{(x+y+z+1)^4}{xyz}$ |
| $X_{1-3}$ | 1 6 | A complete intersection of a quadric and a cubic | $\frac{(x+1)^2(y+z+1)^3}{xyz}$ |
| $X_{1-4}$ | 1 8 | A complete intersection of three quadrics | $\frac{(x+1)^2(y+1)(z+1)^2}{xyz}$ |
| $X_{1-5}$ | 1 10 | A general element is a section of $\text{Gr}(2, 5)$ by two hyperplanes and a quadric | $\frac{(1+x+y+z+xz+xy+yz)^2}{xyz}$ |
| $X_{1-6}$ | 1 12 | A linear section of the orthogonal Grassmannian $\text{OG}(5, 10)$ of codimension 7 | $\frac{(x+z+1)(x+y+z+1)(z+1)(y+z)}{xyz}$ |
| $X_{1-7}$ | 1 14 | A linear section of $\text{Gr}(2, 6)$ of codimension 5 | $\frac{(x+y+z+1)^2}{x} + \frac{(x+y+z+1)(y+z+1)(z+1)}{xyz}$ |
| $X_{1-8}$ | 1 16 | A linear section of symplectic Grassmannian $\text{SGr}(3, 6)$ of codimension 3 | $\frac{(x+y+z+1)(x+1)(y+1)(z+1)}{xyz}$ |
| $X_{1-9}$ | 1 18 | A linear section of the Grassmannian of the group $G_2$ of codimension 2 | $\frac{(x+y+z)(x+x+y+xy+xz+y+z+y+z)}{xyz}$ |
| $X_{1-10}$ | 1 22 | A section of the vector bundle $\Lambda^2 U^* \oplus \Lambda^2 U^* \oplus \Lambda^2 U^*$ on the Grassmannian $\text{Gr}(3, 7)$, where $U$ is the tautological bundle | $\frac{(z+1)(x+y+1)(x+y+z)}{xyz} + \frac{xy}{z} + z + 3$ |
| $X_{1-11}$ | 2 8 1 | A hypersurface of degree 6 in $\mathbb{P}(1, 1, 1, 2, 3)$ | $\frac{(x+y+1)^6}{xyz} + z$ |
| $X_{1-12}$ | 2 8 2 | A hypersurface of degree 4 in $\mathbb{P}(1, 1, 1, 1, 2)$ | $\frac{(x+y+1)^4}{xyz} + z$ |
| $X_{1-13}$ | 2 8 3 | A cubic | $\frac{(x+y+1)^3}{xyz} + z$ |
| $X_{1-14}$ | 2 8 4 | An intersection of two quadrics | $\frac{(x+1)^2(y+1)^2}{xyz} + z$ |
| $X_{1-15}$ | 2 8 5 | A linear section of $\text{Gr}(2, 5)$ of codimension 3 | $x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + xyz$ |
| $X_{1-16}$ | 3 27 2 | A quadric | $\frac{(x+1)^2}{xyz} + y + z$ |
| $X_{1-17}$ | 4 64 | $\mathbb{P}^3$ | $x + y + z + \frac{1}{xyz}$ |
Example 5.21 ($X_{1-4}$, $X_{1-5}$, $X_{1-7}$, and $X_{1-8}$). Let $f$ be the Laurent polynomial from Table 1 for $X_{1-i}$, $i \in \{4, 5, 7, 8\}$. As in the above example for $i = 6$, one can check ‘by hand’ that the polytope $\Delta_f^\vee$ satisfies the conditions of Theorem 5.19. Therefore, there is a degeneration of $X_{1-i}$ to the toric variety $T_\Delta$ corresponding to the Landau–Ginzburg model given by $f$.

Example 5.22 ($X_{1-9}$). Let $f$ be the Laurent polynomial from Table 1 for $X_{1-9}$. Here $\nabla = \Delta_f^\vee$ has 12 lattice points, so we cannot apply Theorem 5.19, but similar techniques may be used to show the existence of the desired degeneration. Indeed, the dimension of the component $U$ corresponding to $X_{1-9}$ in the Hilbert scheme $\mathcal{H}_{X_{1-9}}$ of its anticanonical embedding is 153 (see [17], Proposition 4.1). The variety $T_\Delta$, where $\Delta = N(f)$, corresponds to a point $[T_\Delta]$ in $\mathcal{H}_{X_{1-9}}$ since its Hilbert polynomial agrees with that of $X_{1-9}$. A standard deformation-theoretic calculation shows that $[T_\Delta]$ is a smooth point on a component of dimension 153. It remains to show that this component is in fact $U$.

Now, $\nabla = \Delta_f^\vee$ admits a regular unimodular triangulation such that the origin is contained in every full-dimensional simplex, one boundary vertex has valency 6, and every other vertex has valency 4 or 5. The boundary of this triangulation is in fact the unique triangulation of the sphere with these properties. In this case, $T_\Delta$ degenerates to the Stanley–Reisner scheme $R$ corresponding to this triangulation, and $X_{1-9}$ does as well (see [17], Corollary 3.3). Furthermore, a standard deformation-theoretic calculation shows that at the point $[R]$, $\mathcal{H}_{X_{1-9}}$ has only one 153-dimensional component. Thus, $[T_\Delta]$ must lie on $U$, and $X_{1-9}$ must degenerate to $T_\Delta$.

Therefore, independently of [32] and [47] we have proved the following theorem (see Corollary 5.12).

**Theorem 5.23.** Each Fano threefold of rank 1 has a toric Landau–Ginzburg model.

**Proof.** According to [20], the Laurent polynomials in Table 1 are weak Landau–Ginzburg models of the corresponding Fano varieties. And according to Theorem 5.9, they satisfy the Calabi–Yau condition. Thus, the last thing one needs
to check is the toric condition. The varieties $X_{1-i}$ for $i \in \{1, 2, 3, 4, 11, 12, 13, 14\}$ are complete intersections in weighted projective spaces, so the toric condition for them follows from Theorem 7.27. The varieties $X_{1-10}$ and $X_{1-15}$ have small toric degenerations (that is, degenerations to terminal Gorenstein toric varieties), so the toric condition for them follows from [36]. The toric condition for $X_{1-i}$ with $i \in \{5, 6, 7, 8\}$ follows from Examples 5.20 and 5.21. The toric condition for $X_{1-9}$ follows from Example 5.22. Finally, $X_{1-17} = \mathbb{P}^3$ is toric.

5.4. Modularity. In this section we present results from [32] (see also [48]). Mirror symmetry predicts that the fibres of a Landau–Ginzburg model for a Fano variety are Calabi–Yau varieties. More precisely, these fibres are expected to be mirror dual to anticanonical sections of the Fano variety. In the threefold case this duality is nothing but Dolgachev–Nikulin duality of K3 surfaces.

Let $H$ be a hyperbolic lattice, that is, $\mathbb{Z} \oplus \mathbb{Z}$ with intersection form

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
$$

The intersection lattice on the second cohomology of any K3 surface is $N = H \oplus H \oplus H \oplus E_8(-1) \oplus E_8(-1)$. Consider a family $U_K$ of K3 surfaces whose lattice of algebraic cycles contains the lattice $K \subset N$ (and coincides with $K$ for a general K3 surface), and consider the lattice $L' = K^\perp$ orthogonal to $K$ in $N$. Let $L' = H \oplus L$.

**Definition 5.24** (see [29]). The family of K3 surfaces $U_L$ is called the *Dolgachev–Nikulin dual family to* $U_K$.

Consider a principally polarized family of anticanonical sections of a Fano threefold $X$ of index $i$ and degree $(-K_X)^3 = i^3k$. This is nothing but $U_{(2n)}$, $2n = ik$, where $\langle r \rangle$ is a rank-1 lattice generated by the vector whose square is $r$. The lattice $U_{(2n)}$ is a sublattice of $H$. Using this embedding into one of the $H$-summands of $N$, we can see that the Dolgachev–Nikulin dual lattice to $U_{(2n)}$ is the lattice

$$M_n = H \oplus E_8(-1) \oplus E_8(-1) + \langle -2n \rangle.$$ 

Surfaces with Picard lattices $M_n$ are *Shioda–Inose* surfaces. They are resolutions of quotients of specific K3 surfaces $S$ by the *Nikulin involution*, which preserves the transcendental lattice $T_S$; it interchanges the two copies of $E_8(-1)$. Another description of Shioda–Inose surfaces is as Kummer surfaces, going back to products of elliptic curves with $n$-isogenic ones. $M_n$-polarized Shioda–Inose surfaces form a one-dimensional irreducible family.

It turns out that the fibres of the toric Landau–Ginzburg models in Table 1 can be compactified to Shioda–Inose surfaces dual to anticanonical sections of Fano threefolds. In this section we prove the following.

**Theorem 5.25** (see [32]). Let $X$ be a Fano threefold of Picard rank 1 and index $i$ and let $(-K_X)^3 = i^3k$. Then a general fibre of a toric weak Landau–Ginzburg model from Table 1 is a Shioda–Inose surface with Picard lattice $M_{ik/2}$.

The toric Landau–Ginzburg model for $X$ is said to satisfy the Dolgachev–Nikulin condition if the assertion of Theorem 5.25 holds for it. We call such toric Landau–Ginzburg models *good*. 

Thus, compactifications of good Landau–Ginzburg models are, modulo coverings and the standard action of PSL(2, C) on the base, the unique families of the corresponding Shioda–Inose surfaces. More precisely, they are index-to-one coverings of the moduli spaces.

Corollary 5.26 (see [33]). A Calabi–Yau compactification of a good toric Landau–Ginzburg model is unique up to flops.

Thus, if the homological mirror symmetry conjecture holds for Picard-rank-1 Fano threefolds, then, up to flops, their Landau–Ginzburg models (in the homological mirror symmetry sense) are compactifications of toric ones from Table 1. Moreover, all other good toric Landau–Ginzburg models are birational (over $\mathbb{A}^1$) to these.

To prove Theorem 5.25 we study all 17 cases one by one and compute the Neron–Severi lattices of the compactified toric Landau–Ginzburg models.

Remark 5.27. In [40] Golyshev described Landau–Ginzburg models for Fano threefolds of Picard rank 1 as universal families over $X_0(n)/\tau$, where $\tau$ is an Atkin–Lehner involution, with fibres that are Kummer surfaces associated with products of elliptic curves by $n$-isogenic ones. Golyshev’s description of the periods of these dual families as modular forms seems to be natural to expect from this point of view. The variations of Hodge structures of our families of Shioda–Inose surfaces are the same (over $\mathbb{Q}$) as the variations for products of elliptic curves and the same over $\mathbb{Z}$ as for Kummer surfaces; this follows from the description of Shioda–Inose surfaces given above.

Remark 5.28. Fibres of Landau–Ginzburg models are expected to be Dolgachev–Nikulin dual to anticanonical sections of Fano varieties of any Picard rank. As the Picard rank of the Fano varieties increases, the mirror K3 fibres will no longer be Shioda–Inose. However, they are still K3 surfaces of high Picard rank, so we can hope to find analogous modular-type (or, automorphic) properties in these cases as well. For example, fibres of Landau–Ginzburg models are Kummer surfaces given by products of elliptic curves for the Picard-rank-2 case and by Abelian surfaces for the Picard-rank-3 case. These lattices are computed over $\mathbb{Q}$ in [15]; however, the computations over $\mathbb{Z}$ require deeper methods.

5.4.1. Lattice facts. If $L$ is a lattice and $k$ is a field, then we will write $L_k$ for $L \otimes \mathbb{Z} k$. We will use $\mathcal{N}$ and $\mathcal{M}$ to denote two dual rank-three lattices. Let $f_{1-i}$ denote the Laurent polynomial defining the Landau–Ginzburg model from Table 1 that corresponds to $X_{1-i}$, let $\Delta_{f_{1-i}} \subset \mathcal{M}_\mathbb{R}$ be its Newton polytope, and let $\nabla_{f_{1-i}} \subset \mathcal{N}_\mathbb{R}$ be its polar (dual) polytope.

We denote by $A_n$, $D_n$, and $E_n$ the root lattices of the corresponding Dynkin diagrams. By $M$ we denote the rank-18 lattice $H \oplus E_8(-1) \oplus E_8(-1)$, and by $M_n$ the rank-19 lattice $M \oplus (-2n)$.

We will use $(x : y : z : w)$ as homogeneous coordinates on $\mathbb{P}^3$. For distinct, non-empty subsets $I, J, K \subset \{1, 2, 3, 4\}$, we will write $H_I$ for the hyperplane defined by setting the sum of coordinates in $I$ equal to zero. Thus, for example, $H_{\{1\}}$ is the coordinate hyperplane $x = 0$, while $H_{\{2,4\}}$ is the hyperplane defined by $y + w = 0$. We write

\[ L_{I,J} = H_I \cap H_J \quad \text{and} \quad p_{I,J,K} = H_I \cap H_J \cap H_K. \]
In many cases we will use Calabi–Yau compactifications that are different from those of §5.2. That is, we use compactifications given by
\[(\mathbb{C}^*)^3 \hookrightarrow \mathbb{P}[x : y : z : w]\]
(see §6.3). This gives precise descriptions of fibres of compactifications as quartics in \(\mathbb{P}^3\) with ordinary double points. In these cases we will identify some curves on the minimal resolutions of these singular quartics (which will be fibres of compactified Landau–Ginzburg models) and compute the intersection matrix of the identified curves, with a check that this matrix has rank 19. To avoid boring the reader, we will omit the details of these computations. In other cases we will use elliptic fibrations as described below.

Because we will use them later, we recall a few (perhaps not very well-known) facts about lattices. Most are contained in [68]; a very readable reference is [11]. Let \(L\) be a lattice and \(\langle \cdot, \cdot \rangle\) the bilinear pairing on \(L\). Denote by \(L^*\) the dual lattice \(\text{Hom}(L, \mathbb{Z})\). Since the pairing induces an isomorphism \(L \cong \text{Hom}(L, \mathbb{Q})\), we may suppose that \(L^* \subset L \otimes \mathbb{Q}\). The pairing \(\langle \cdot, \cdot \rangle\) induces a quadratic form \(q_L\) on the discriminant group \(D(L) = L^*/L\) by
\[q_L(\phi) = \langle \phi, \phi \rangle.\]

\(A \text{ priori, } q_L\) takes values in \(\mathbb{Q}/\mathbb{Z}\), but if \(L\) is an even lattice, it will take values in \(\mathbb{Q}/(2\mathbb{Z})\).

Fixing a basis \(e_1, \ldots, e_r\) for \(L\), we may form the Gram matrix \(I_L\) whose \((i, j)\)th entry is \(\langle e_i, e_j \rangle\). We call \(d(L) = \det(I_L)\) the determinant of \(L\).

**Fact 5.29.** Let \(L\) be an even lattice of rank \(r\) and signature \((s, r-s)\), and let \(d\) be the minimal number of generators of \(L^*/L\). If \(r > d + 2\), then \(q_L\) and \(s\) determine \(L\) uniquely.

**Fact 5.30.** Let \(L \subset M\) be even lattices of the same rank. Then
\[\left[ M : L \right]^2 = \frac{d(L)}{d(M)}.\]

**Fact 5.31.** Let \(L \subset M\) be even lattices of the same rank, and let \(G = M/L \subset L^*/L = D\). Note that since \(L \subset M \subset M^* \subset L^*\), we have
\[G \subset M^*/L \subset D \quad \text{and} \quad (M^*/L)/G \cong M^*/M.\]

Now let
\[G^\perp = \{ a \in D \mid q_L(a + H) = q_L(a) \}.\]
Since \(M\) is even, \(q_L|_G = 0\), and hence \(G \subset G^\perp\). Moreover, given \(a \in D\), choose \(\tilde{a} \in L^*\) such that \(a = \tilde{a} + L\). Then \(a \in G^\perp\) if and only if \(\langle \tilde{a}, M \rangle \subset \mathbb{Z}\), that is, \(G^\perp = M^*/L\). Thus we see that the quadratic form \(q_M\) is nothing but \(q_L|_{G^\perp}\) inherited from \(G^\perp/G\).

Conversely, given a subgroup \(G \subset D\) such that \(q_L(G) = 0\), there exists a lattice \(M\) containing \(L\) such that \(M/L = G\).

**Fact 5.32.** Let \(L\) be a sublattice of a unimodular lattice \(\Lambda\). Then \(D(L) \cong D(L^\perp)\) and \(q_L = -q_{L^\perp}\).
For convenience, we include in Table 2 the discriminant groups and forms of some of the lattices that play a role in the present study. We present the discriminant forms by giving their values on the generators of the discriminant group. Note that this description is not unique. For example, if the discriminant group is $\mathbb{Z}/(8)$ and the form is listed as $1/8$, this means that a generator $g$ of the group has $q(g) = 1/8$. Of course, $3g$ is also a generator, and it has $q(3g) = 9/8$.

| Lattice $L$ | Group $D(L)$ | Form $q_L$ |
|-----------|-------------|----------|
| $H$       | $\{1\}$    | 0        |
| $\langle-2n\rangle$ | $\mathbb{Z}/(2n)$ | $-1/(2n)$ |
| $A_1$     | $\mathbb{Z}/(2)$ | $-1/2$   |
| $A_2$     | $\mathbb{Z}/(3)$ | $4/3$    |
| $A_3$     | $\mathbb{Z}/(4)$ | $5/4$    |
| $A_4$     | $\mathbb{Z}/(5)$ | $4/5$    |
| $A_5$     | $\mathbb{Z}/(6)$ | $-5/6$   |
| $A_6$     | $\mathbb{Z}/(7)$ | $2/7$    |
| $A_7$     | $\mathbb{Z}/(8)$ | $1/8$    |
| $A_8$     | $\mathbb{Z}/(9)$ | $4/9$    |
| $A_9$     | $\mathbb{Z}/(10)$ | $-9/10$ |
| $A_{10}$  | $\mathbb{Z}/(11)$ | $4/11$   |
| $A_{11}$  | $\mathbb{Z}/(12)$ | $-11/12$ |
| $D_5$     | $\mathbb{Z}/(4)$ | $-5/4$   |
| $D_8$     | $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$ | $0, 1$   |
| $D_{10}$  | $\mathbb{Z}/(2) \oplus \mathbb{Z}/(2)$ | $1, 1$   |
| $E_6$     | $\mathbb{Z}/(3)$ | $2/3$    |
| $E_7$     | $\mathbb{Z}/(2)$ | $1/2$    |
| $E_8$     | $\{1\}$    | 0        |

5.4.2. Elliptic fibrations on K3 surfaces. We briefly recall a few of facts about elliptic fibrations with section on K3 surfaces.

**Definition 5.33.** An elliptic K3 surface with section is a triple $(X, \pi, \sigma)$, where $X$ is a K3 surface and $\pi: X \to \mathbb{P}^1$ and $\sigma: \mathbb{P}^1 \to X$ are morphisms such that the general fibre of $\pi$ is an elliptic curve and $\pi \circ \sigma = \text{id}_{\mathbb{P}^1}$.

Any elliptic curve over the complex numbers can be realized as a smooth cubic curve in $\mathbb{P}^2$ in the *Weierstrass normal form*

$$y^2z = 4x^3 - g_2xz^2 - g_3z^3.$$  \hspace{1cm} (5.1)

Conversely, the equation (5.1) defines a smooth elliptic curve if $\Delta = g_3^3 - 27g_2^2 \neq 0$.

Similarly, an elliptic K3 surface with section can be embedded into the $\mathbb{P}^2$ bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(6))$ as a subvariety defined by equation (5.1), where now $g_2$ and $g_3$ are global sections of $\mathcal{O}_{\mathbb{P}^1}(8)$ and $\mathcal{O}_{\mathbb{P}^1}(12)$, respectively (that is, they are homogeneous polynomials of degrees 8 and 12). The singular fibres of $\pi$ are the roots of the degree-24 homogeneous polynomial

$$\Delta = g_2^3 - 27g_3^2 \in H^0(\mathcal{O}_{\mathbb{P}^1}(24)).$$
Tate’s algorithm can be used to determine the type of the singular fibre over a root $p$ of $\Delta$ from the orders of vanishing of $g_2$, $g_3$, and $\Delta$ at $p$.

**Proposition 5.34** (see [18], Lemma 3.9). A general fibre of $\pi$ and the image of $\sigma$ span a copy of $H$ in $\text{Pic}(X)$. Furthermore, the components of singular fibres of $\pi$ that do not intersect $\sigma$ span a sublattice $S$ of $\text{Pic}(X)$ orthogonal to this $H$, and $\text{Pic}(X)/(H \oplus S)$ is isomorphic to the Mordell–Weil group $\text{MW}(X, \pi)$ of sections of $\pi$.

**Proposition 5.35** ([66], Corollary VII.3.1). The torsion subgroup of $\text{MW}(X, \pi)$ embeds in $D(S)$.

When K3 surfaces are realized as hypersurfaces in toric varieties, one can construct elliptic fibrations combinatorially. As before, let $\Delta \subset \mathcal{N}_\mathbb{Q}$ be a reflexive polytope, and suppose $P \subset \mathcal{N}$ is a plane such that $Q = \Delta \cap P$ is a reflexive polygon. Let $m \in \mathcal{M} = \mathcal{N}^\vee$ be a normal vector to $P$. Then $P$ induces a torus-invariant map $\mathbb{P}(\Delta^\vee) \to \mathbb{P}^1$ with general fibre $\mathbb{P}_Q$, which is given in homogeneous coordinates by

$$\pi_m: (z_1, \ldots, z_r) \mapsto \prod_{\langle v_i, m \rangle > 0} z_i^{\langle v_i, m \rangle}, \prod_{\langle v_i, m \rangle < 0} z_i^{-\langle v_i, m \rangle}.$$ 

Restricting $\pi_m$ to an anticanonical K3 surface, we obtain an elliptic fibration. If $Q$ has an edge without interior points, then this fibration will have a section as well. See [60] for more details.

### 5.4.3. Picard lattices of fibres of Landau–Ginzburg models.

- **$X_{1-1}$**. Recall that a Landau–Ginzburg model of Givental type for $X_{1-1}$ is

$$\begin{cases} y_0 y_1 y_2 y_3 y_4^3 = 1, \\ y_1 + y_2 + y_3 + y_4 = 1 \end{cases}$$

with superpotential

$$w = y_0.$$ 

Consider the change of variables

$$y_1 = \frac{x}{x + y + z + t}, \quad y_2 = \frac{y}{x + y + z + t},$$ 

$$y_3 = \frac{z}{x + y + z + t}, \quad y_4 = \frac{t}{x + y + z + t},$$

where $x$, $y$, $z$, and $t$ are projective coordinates. We obtain the Landau–Ginzburg model

$$y_0 x y z t^3 = (x + y + z + t)^6, \quad w = y_0.$$ 

Thus in the local chart, say, $t \neq 0$, we get the toric Landau–Ginzburg model from Table 1:

$$f_{1-1} = \frac{(x + y + z + 1)^6}{x y z}.$$
A general element of the pencil that corresponds to \(f_{1-1}\) is birational to the general element of the initial Landau–Ginzburg model. Invert the superpotential: \(u = 1/w\). We obtain the pencil given by

\[ y_1y_2y_3y_4^3 = u, \quad y_1 + y_2 + y_3 + y_4 = 1. \]

This is a Landau–Ginzburg model for the weighted projective space \(\mathbb{P}(1 : 1 : 1 : 3)\) (see [22], (2)). (In particular, by Theorem 1.15 in [22], its general element is birational to a K3 surface.) However, we make another change of variables in Givental’s Landau–Ginzburg model by putting \(x = y_1, y = y_2, \) and \(z = y_4\). Then we obtain the family given by

\[ \tilde{f}_{1-1} = x + y + z + \frac{w}{xyz} - 1 = 0. \]

Let \(\tilde{\Delta}_{f_{1-1}}\) be the Newton polytope of the polynomial \(\tilde{f}_{1-1}\) and let \(\tilde{\nabla}_{f_{1-1}} = \tilde{\Delta}_{f_{1-1}}^\vee\). Then fibres of the pencil \(\{\tilde{f}_{1-1} = 0\}\) can be compactified inside \(T_{\tilde{\nabla}_{f_{1-1}}}\) (see §5.2). The normal vector \((1,2,3)\) induces an elliptic fibration with a section. The Weierstrass normal form of the fibres of the elliptic fibration is

\[-\frac{t^4u}{48} + \frac{1}{864}t^5(864t^2 + 1728tw - t + 864w^2) + u^3 + v^2 = 0.\]

Hence by Tate’s algorithm there are singular fibres of type \(II^*\) at \(t = 0, \infty,\) and \(I_2\) at \(t = -w\). Therefore, the K3 surfaces in question are polarized by

\[ H \oplus E_8(-1) \oplus E_8(-1) \oplus A_1(-1) = M_1. \]

There is also another fibration induced by the normal \((1,0,1)\), which gives a polarization by

\[ H \oplus E_7(-1) \oplus D_{10}(-1). \]

- \(X_{1-2}\). Compactify this family to the family of quartics

\[ \{(x + y + z + w)^4 - \lambda xyzw = 0\} \]

in \(\mathbb{P}^3\). Intersecting the quartic with the pencil of planes containing one of lines lying on it gives a family of divisors with this line as base locus. Subtracting the line gives a pencil of cubics. Blowing up the base points of this pencil gives an elliptic fibration with a section, which gives a polarization of the K3 surfaces by \(H \oplus E_6(-1) \oplus A_{11}(-1)\). This fibration has a 3-torsion section, and it can have no other torsion sections by Proposition 5.35. Thus, by Fact 5.30 the general fibre \(X\) of \(f_{1-2}\) has \(d(\text{NS}(X)) = 4\). As we shall see, fibres of the Landau–Ginzburg model \(X_{1-17} = \mathbb{P}^3\) have fibrations of this type as well, and comparing the parameters of the two Weierstrass equations, we see that the fibres of the compactified toric Landau–Ginzburg models for \(X_{1-1}\) and \(X_{1-17}\) are the same. Because general fibres of compactifications for \(f_{1-17}\) are \(M_2\)-polarized (as we will see soon), \(\text{NS}(X) \simeq M_2\).

- \(X_{1-3}\). Compactify the fibres of \(f_{1-3}\) as a family of anticanonical divisors in \(\mathbb{P}^1 \times \mathbb{P}^2\) via \((x, y, z) \mapsto ((x : 1) \times (y : z : 1))\). Explicitly, \(f_{1-3}^{-1}(\lambda)\) compactifies to the K3 surface

\[ Y_\lambda = \{((x : x_0), (y : z : w)) \in \mathbb{P}^1 \times \mathbb{P}^2 \mid (x + x_0)^2(y + z + w)^3 - \lambda xx_0yzw = 0\}. \]
The projection $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ induces an elliptic fibration on $Y_\lambda$ for general $\lambda$. The map $(x : x_0) \mapsto ((x : x_0), (1 : -1 : 0))$ gives a section of this elliptic fibration. Putting the fibre over $(1 : a)$ into the Weierstrass form

$$\frac{a^3\lambda^3(24(1 + a)^2 - a\lambda)}{48}X - \frac{a^4\lambda^4(36(1 + a)^2 - 2as + a^2s^2)}{864} + X^3 + Y^2 = 0$$

and using Tate’s algorithm, we see singular fibres of Kodaira type $IV^*$ at $a = 0, \infty$, of type $I_6$ at $a = -1$, and of type $I_1$ for $27(a + 1)^2 - \lambda a = 0$. Hence, the rank-19 lattice $H \oplus E_6(-1) \oplus E_6(-1) \oplus A_5(-1)$ embeds in the Picard lattice of $Y_\lambda$.

As we will see later, fibres of $f_{1-16}$ also have fibrations of this type and are $M_3$-polarized. Matching the Weierstrass equations, we conclude that fibres for $f_{1-3}$ are isomorphic to fibres for $f_{1-16}$, and hence fibres for $f_{1-3}$ must also be $M_3$-polarized.

- $X_{1-4}$. As in the case above, we compactify the family as anticanonical K3 surfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The projection onto one of the $\mathbb{P}^1$ factors gives the general K3 fibre an elliptic fibration with a section. Putting this into the Weierstrass form and running Tate’s algorithm, we obtain an embedding of the rank-19 lattice $H \oplus A_7(-1) \oplus D_5(-1) \oplus D_5(-1)$ into the Picard group of the general fibre. Moreover, the Mordell–Weil group is isomorphic to $\mathbb{Z}/(4)$. Applying the lattice facts above, with

$$L = H \oplus A_7(-1) \oplus D_5(-1) \oplus D_5(-1), \quad M = \text{NS}(X),$$

$$G = \text{MW}(X) \simeq \mathbb{Z}/(4), \quad D = L^*/L \simeq \mathbb{Z}/(8) \oplus \mathbb{Z}/(4) \oplus \mathbb{Z}/(4),$$

we have that $d(M) = 8$. Examining the possibilities for $G \subset D$, we conclude that $M^*/M \simeq \mathbb{Z}/(8)$, and that $q_M$ for a generator element is 7/8.

Now we claim that $M \simeq M_4$. Let $e$ be a generator of the $\langle -8 \rangle$ direct summand of $M_4$. Since $H$ and $E_8(-1)$ are unimodular, the group $M_4^*/M_4 \simeq \mathbb{Z}/(8)$ is generated by $\epsilon = e/8$, and $q_{M_4}(\epsilon) = -1/8$. Note that the element $3\epsilon$ also generates $M_4^*/M_4$, and $q_{M_4}(3\epsilon) = -9/8 \equiv 7/8 \pmod{2\mathbb{Z}}$. Thus, we see that $M$ and $M_4$ have the same discriminant form, and their ranks are sufficiently large relative to the number of generators of the discriminant groups. Hence $M$ and $M_4$ must be isomorphic by Fact 5.29.

- $X_{1-5}$. Compactify fibres to singular quartics. There are singularities at $p_{i,j,k}$ for $1 \leq i \neq j \leq 3$ of type $D_4$ and at $p_{i,j,k}$, where $\{i,j,k\} = \{1,2,3\}$, of type $A_1$. Thus, the exceptional curves generate a sublattice of rank 15. The quartics also contain lines $L_{i,j}$ and conics $C_{i,j}$ for $1 \leq i \neq j \leq 3$, subject to the relations arising from

$$H_{\{1\}} = 2L_{\{2\}} + 2L_{\{3\}} + 3L_{\{4\}},$$

$$H_{\{2\}} = 2L_{\{1\}} + 2L_{\{3\}} + 3L_{\{4\}},$$

$$H_{\{3\}} = 2L_{\{1\}} + 2L_{\{2\}} + 3L_{\{4\}},$$

$$H_{\{1,2\}} = L_{\{1\}} + L_{\{2\}} + L_{\{3\}} + L_{\{4\}} + C_{\{1,2,3\}},$$

$$H_{\{1,3\}} = L_{\{1\}} + L_{\{3\}} + L_{\{2\}} + L_{\{4\}} + C_{\{1,3,4\}},$$

$$H_{\{2,3\}} = L_{\{2\}} + L_{\{3\}} + L_{\{1\}} + L_{\{4\}} + C_{\{2,3,4\}},$$

which leave a lattice of rank 19.
Explicit computation of the intersection matrix for the identified curves (one needs to blow up singular curves for this and figure out how the strict transforms of the curves intersect exceptional curves; see Proposition 6.35) shows that they generate a lattice with determinant 10 and discriminant group $\mathbb{Z}/(10)$ with a generator $q(\alpha) = 11/10$. Choosing instead the generator $\beta = 3\alpha$, we have

$$q(\beta) = \frac{99}{10} \equiv -\frac{1}{10} \pmod{2\mathbb{Z}}.$$ 

Hence this lattice is isomorphic to $M_5$.

We have just shown that for a general element $Y_\lambda$ in this pencil, $\text{NS}(X)$ contains $M_5$. To see that $\text{NS}(X)$ actually equals $M_5$, we note that since $M$ is unimodular and contained in $\text{NS}(X)$, it must be a direct summand. Because $\text{NS}(X)$ is an even lattice of signature $(1,18)$, the orthogonal complement of $M \subset \text{NS}(X)$ must be even, negative definite, and of rank 1, and hence must be equal to $(-2n)$ for some $n$. From Fact 5.30, $10/(\det \text{NS}(X)) = 5/n$ must be a square, and hence $n = 5$.

Alternatively, the intersection of one of the singular quartics with a plane containing $L_{\{1,2,4\}}$ consists of $L_{\{1,2,4\}}$ and a (generally) smooth cubic. The pencil of these cubics, with base points blown up, gives an elliptic fibration on the minimal resolution of the quartic. This fibration has singular fibres of types $I_2^*, I_1^*, I_6$, and 3 fibres of type $I_1$. It also has a section of infinite order and a 2-torsion section. Hence, the Picard lattice of the general member of this family is a rank-19 lattice containing

$$H \oplus D_6(-1) \oplus D_5(-1) \oplus A_5(-1)$$

with quotient $\mathbb{Z} \oplus \mathbb{Z}/(2)$.

- $X_{1-6}$. Again, we can compactify the fibres for $f_{1-6}$ to singular quartics in the standard way. There are $A_1$ singularities at $(1 : -1 : 0 : 0)$, $(1 : 0 : -1 : 0)$, and $(0 : 1 : -1 : 0)$; $A_2$ singularities at $(1 : 0 : 0 : 0)$ and $(0 : 0 : 1 : -1)$, and $A_3$ singularities at $(0 : 1 : 0 : 0)$ and $(1 : 0 : 0 : -1)$. These quartics also include twelve lines:

$$L_{\{1,2,3\}}, L_{\{1,3,4\}}, L_{\{1,2,3,4\}}, L_{\{2,3\}}, L_{\{2,3,4\}}, L_{\{2,1,3,4\}}, L_{\{3,4\}}, L_{\{3,1,4\}}, L_{\{3,1,2,4\}}, L_{\{4,1,3\}}, L_{\{4,2,3\}}, L_{\{4,1,2,3\}}$$

subject to the relations arising from equating the hyperplane sections $H_{\{1\}}, H_{\{2\}}, H_{\{3\}}, H_{\{4\}}, H_{\{1,3,4\}}, H_{\{1,2,3,4\}}, H_{\{3,4\}}, \text{ and } H_{\{2,3\}}$. These relations show that only six of these twelve lines are linearly independent. Hence, the exceptional locus and strict transforms of lines generate a sublattice of the Picard lattice of the minimal resolutions of K3 surfaces of rank $13 + 6 = 19$.

By explicitly computing the intersection matrix for the 25 rational curves identified, we conclude that the lattice they generate has determinant $\pm 12$, discriminant group $\mathbb{Z}/(12)$, and discriminant form $23/12 \equiv -1/12 \pmod{2\mathbb{Z}}$. Therefore, this lattice is isomorphic to $M_6$. As in the argument in the case of $X_{1-5}$, Fact 5.30 shows that the Picard lattice must be equal to $M_6$.

- $X_{1-7}$. Again, we compactify fibres to singular quartics. These quartics are defined by

$$(x + y + z + w)(yz(x + y + z + w) + (y + z + w)(z + w)^2) - \lambda xyzw = 0.$$
The singularities are: type $A_1$ at $(0 : 1 : 0 : -1)$, type $A_2$ at $(1 : 0 : 0 : 0)$, $(0 : 1 : -1 : 0)$, and $(\lambda : 0 : -1 : 1)$, type $A_3$ at $(0 : 0 : 1 : -1)$, and type $A_4$ at $(1 : -1 : 0 : 0)$. The quartics include eight lines:

\[
L_{(i),\{1,2,3,4\}} \quad (1 \leq i \leq 4), \quad L_{\{2\},\{3,4\}}, \quad L_{\{3\},\{2,4\}},
\]

\[
L_{\{3\},\{4\}}, \quad L_{\{234\},*} = \{y + z + w = x - \lambda w = 0\}
\]

and two conics:

\[
C_1 = \{x = yz + (z + w)^2 = 0\} \quad \text{and} \quad C_4 = \{w = xy + (y + z)^2 = 0\},
\]

subject to the relations arising from equating the hyperplane sections $H_{\{1\}}, H_{\{2\}}, H_{\{3\}}, H_{\{4\}}, H_{\{2,3,4\}}$, and $H_{\{1,2,3,4\}}$. These relations show that these 10 rational curves on the quartic generate a sublattice of rank 5 in the Picard lattice. Hence, the exceptional loci generate a lattice of rank 19. The quartics include eight lines:

\[
(−1 : 0 : 0 : 1), \quad (0 : -1 : 0 : 1), \quad (0 : 0 : -1 : 1), \quad (1 : -1 : 0 : 0), \quad (1 : 0 : -1 : 0), \quad (0 : 1 : -1 : 0)
\]

and of type $A_2$ at

\[
(1 : 0 : 0 : 0), \quad (0 : 1 : 0 : 0), \quad (0 : 0 : 1 : 0).
\]

There are also 13 lines:

\[
L_{(i),\{1,2,3,4\}}, \quad L_{\{j\},\{4\}}, \quad L_{\{j\},\{k,4\}} \quad \text{for} \ 1 \leq i \leq 4 \ \text{and} \ 1 \leq j \neq k \leq 3,
\]

subject to relations arising from equating the hyperplane sections $H_{\{i\}}, H_{\{j,4\}}$, and $H_{\{1,2,3,4\}}$ for $1 \leq i \leq 4$, $1 \leq j \leq 3$. These relations show that the lattice generated by the 13 lines has rank 7. Hence, the strict transforms of the lines and the exceptional loci generate a lattice of rank 19.

By explicitly computing the intersection matrix for the 25 rational curves identified, we conclude that the lattice they generate has determinant $\pm 16$ and discriminant group $\mathbb{Z}/(16)$ with a generator $\alpha$ such that $q(\alpha) = 23/16$. Taking $\beta = 5\alpha$ as generator, we have

\[
q(\beta) = \frac{575}{16} \equiv -\frac{1}{16} \pmod{2\mathbb{Z}}.
\]

Therefore, this lattice is isomorphic to $M_8$. In this case, Fact 5.30 shows that the Picard lattice of the general K3 in the pencil is either $M_8$ or $M_2$. We now use the results of [40], which implies that this pencil exhibits the same variations of Hodge structure as the $M_8$ pencil, and hence variations different from the $M_2$ variations. Thus we conclude that this pencil must be $M_8$-polarized.

- $X_{1-9}$. We compactify fibres to quartics in $\mathbb{P}^3$ in the standard way.
These quartics have an elliptic fibration with a section arising from intersections with planes containing $L_{\{4\},\{1,2,3\}}$, which gives a polarization of the Picard lattice of the minimal resolution by the rank-19 lattice

$$H \oplus A_8(-1) \oplus A_2(-1) \oplus A_1(-1) \oplus E_6(-1).$$

By Proposition 5.35 there can be no sections of this fibration other than the zero section, and so the Picard lattice must be equal to

$$H \oplus A_8(-1) \oplus A_2(-1) \oplus A_1(-1) \oplus E_6(-1) = M_9.$$

- $X_{1-10}$. The quartic compactification contains the lines

$$L_{\{1\},\{3\}}, L_{\{1\},\{4\}}, L_{\{1\},\{2,4\}}, L_{\{1\},\{3,4\}}, L_{\{2\},\{3\}}, L_{\{2\},\{4\}},$$

$$L_{\{2\},\{1,4\}}, L_{\{2\},\{3,4\}}, L_{\{3\},\{1,4\}}, L_{\{3\},\{2,4\}}, L_{\{2\},\{3,4\}}, L_{\{2\},\{3\}}, L_{\{2\},\{4\}},$$

$$L_{\{1,4\},*} = \{x + w = (s - 2)x + y = 0\},$$

$$L_{\{2,4\},*} = \{y + w = (s - 2)y + x = 0\},$$

and the conics

$$C_{\{3,4\}} = \{z + w = xy + (\lambda - 2)z^2 = 0\},$$

$$C_{\{1,2,4\}} = \{x + y + w = xy + (\lambda - 3)(x + y)z + z^2 = 0\},$$

$$C = \{z = (\lambda + 1)w, (\lambda + 1)w^2 + xy = 0\},$$

$$C' = \{z = (\lambda + 1)w, 2w(w + x + y) + \lambda w(x + y) + xy = 0\},$$

which are subject to the relations arising from the $H_{\{i\}}$, and singularities of type $A_3$ at $(1 : 0 : 0 : 0)$ and $(0 : 1 : 0 : 0)$, type $A_2$ at $(0 : 0 : 1 : 0)$, and type $A_1$ at $(-1 : 0 : 0 : 1)$ and $(0 : -1 : 0 : 1)$.

The lines are subject to the relations from equating $H_{\{1\}}, H_{\{2\}}, H_{\{3\}}, H_{\{4\}}, H_{\{1,3\}}, H_{\{2,3\}}, H_{\{1,4\}}, H_{\{2,4\}},$ and $H_{\{3,4\}}$.

Computing the intersection matrix shows that the Picard lattice is $M_{10}$.

- $X_{1-11}$. By Proposition 5.11, fibres of the compactification are compactifications for the family

$$\{x^4 - (\lambda y - z)(xw - xy - w^2)z = 0\}.$$

We may consider the elliptic fibration on the fibres of the compactification to quartics given by intersection with planes containing $L_{\{1\},\{3\}}$. Putting this fibration into the Weierstrass form and applying Tate’s algorithm, one has a polarization by $H \oplus E_7(-1) \oplus D_{10}(-1)$. Comparing the Weierstrass form of this fibration with the Weierstrass form of the analogous fibration for $f_{1-1}$, we conclude that the Picard lattice must be $M_1$.

- $X_{1-12}$. Compactify fibres of the pencil for $f_{1-12}$ to quartics in $\mathbb{P}^3$. Intersecting these quartics with the pencils of planes containing $L_{\{1\},\{2,4\}}$, subtracting this line, and blowing up base points, one obtains an elliptic fibration with a section. The induced polarization is given by the rank-19 lattice $H \oplus E_6(-1) \oplus A_{11}(-1)$. Comparing with the analogous fibration for $X_{1-17}$, we see that the general lattice must be $M_2$. 
\( X_{1-13} \). Compactify fibres of the pencil for \( f_{1-13} \) to quartics in \( \mathbb{P}^3 \). Intersecting these quartics with planes containing the line \( L_{\{1\}, \{4\}} \) gives an elliptic fibration that results in a polarization by

\[
H \oplus E_6(-1) \oplus E_6(-1) \oplus A_5(-1).
\]

The Mordell–Weil group of this fibration is \( \mathbb{Z}/(3) \). Hence, applying Fact 5.30, we have \( d(\text{NS}(X)) = \pm 6 \). In fact, by matching parameters with the similar fibrations for \( X_{1-3} \) and \( X_{1-16} \), we conclude that \( \text{NS}(X) \simeq M_3 \).

\( X_{1-14} \). We can compactify the pencil in the toric variety \( T_{\nabla f_{1-14}} \) and consider the elliptic fibration with section induced by \((0,0,1)\). This yields a fibration with fibres of type \( I_8 \) at \( \infty \) and \( I^*_8 \) at \( t = (\lambda \pm \sqrt{\lambda^2 + 16})/2 \). Hence, the fibres carry a polarization by \( H \oplus A_7(-1) \oplus D_5(-1) \oplus D_5(-1) \). Moreover, the Mordell–Weil group is isomorphic to \( \mathbb{Z}/(4) \). So, as for \( X_{1-4} \), these K3 surfaces are \( M_4 \)-polarized.

\( X_{1-15} \). Compactify the fibres for \( f_{1-15} \) in \( T_{\nabla f_{1-15}} \). The vector \( m = (1,1,0) \) induces an elliptic fibration on the general compactified fibre \( Y_\lambda \). The Weierstrass form of this fibration is

\[
-\frac{1}{48} t^2 P(s,t) u + \frac{1}{864} t^3 (s^2(-t) + 4t^2 + 12t + 8) (P(s,t) + 24(1+t)^2) + u^3 + v^2 = 0,
\]

where \( P(s,t) = s^4 t^2 - 8s^2 t^3 - 24s^2 t^2 - 16s^2 t + 16t^4 + 24t^3 - 8t^2 - 24t - 8 \). This fibration has a section of infinite order given by

\[
t \mapsto \left( -\frac{1}{12} t(s^2 t + 8t^2 + 12t + 4), -\frac{1}{2} st^2(t + 1)^2 \right) = (u,v),
\]

and a 2-torsion section given by

\[
t \mapsto \left( \frac{1}{12} (-s^2 t + 4t^2 + 12t + 8), 0 \right) = (u,v).
\]

Hence by Proposition 5.34, the lattice \( \text{NS}(Y_\lambda) \) is a rank-19 lattice containing \( H \oplus D_6(-1) \oplus D_5(-1) \oplus A_5(-1) \) with the quotient \( \mathbb{Z} \oplus \mathbb{Z}/(2) \). Matching this elliptic fibration with the one for \( X_{1-5} \), we conclude that fibres for \( f_{1-15} \) are isomorphic to fibres of \( f_{1-5} \), and hence these K3 surfaces are \( M_5 \)-polarized.

\( X_{1-16} \). The vector \( m = (1,2,1) \) defines an elliptic fibration with a section on the general fibre \( Y_\lambda \) of the Landau–Ginzburg model. The Weierstrass form of this fibration is

\[
-\frac{1}{48} st^3 u(s^3 t + 48t + 48) + \frac{1}{864} t^5 (s^6(-t) - 72s^3 t - 72s^3 + 864 t^2 + 1728 t + 864) + u^3 + v^2 = 0,
\]

and there are singular fibres of types \( \Pi^* \) at \( t = 0 \), \( \Pi^* \) at \( t = \infty \), and \( I_3 \) at \( t = -1 \). Hence, the K3 fibre is polarized by the rank-19 lattice \( N \oplus A_2(-1) \). By Proposition 5.35, there can be no torsion sections (the discriminant groups of the two singular fibres have coprime orders), so

\[
\text{NS}(Y_\lambda) = K \oplus A_2(-1), \quad \text{where } K = H \oplus E_8(-1) \oplus E_7(-1).
\]
Now note that $D(K \oplus A_2(-1)) = \mathbb{Z}/(2) \oplus \mathbb{Z}/(3) \simeq \mathbb{Z}/(6)$. If we write the isomorphism $\mathbb{Z}/(6) \to \mathbb{Z}/(2) \oplus \mathbb{Z}/(3)$ as $1 \mapsto (1, 1)$, then we can express the form

$$q_{K \oplus A_2(-1)}: \mathbb{Z}/(6) \to \mathbb{Q}/(2),$$

by specifying

$$q_{K \oplus A_2(-1)}(1) = \frac{1}{2} + \frac{4}{3} = \frac{11}{6} \equiv -\frac{1}{6} \pmod{2\mathbb{Z}}.$$  

Thus $q_{N \oplus A_2} \simeq q_{(-6)} \simeq q_{M_3}$, and hence $\text{NS}(Y_\lambda) \simeq M_3$ by Fact 5.29.

For the cases $X_{1-3}$ and $X_{1-13}$, we note that $m = (1, 0, 0)$ gives a fibration with lattice $H \oplus E_6(-1) \oplus E_6(-1) \oplus A_5(-1)$ plus additional sections.

- $X_{17}$. Anticanonical K3 surfaces in $\mathbb{P}^3$ have general Picard lattice $\langle 4 \rangle$ generated by the hyperplane section. We claim that fibres of the compactified toric Landau–Ginzburg model are $M_2$-polarized. We can see this explicitly from the toric fibration on $T_Vf_{17}$ defined by the normal vector $m = (1, -1, -2)$. Restricting this fibration to the general fibre of the Landau–Ginzburg model gives the fibre the structure of an elliptic surface with Weierstrass equation

$$-\frac{1}{48}(s^4 + 144)t^4u + \frac{1}{864}t^5(s^6(-t) + 648s^2t + 864t^2 + 864) + u^3 + v^2 = 0.$$

Applying Tate’s algorithm, we see singular fibres of Kodaira type $\Pi^*$ at $t = 0, \infty$, and hence the K3 surfaces are $M$-polarized. Moreover,

$$(u, v) = \left( -\frac{4s^4 + 120s^2 + 108}{12s^2}, \frac{3(4s^4 + 30s^2 + 18)}{2s^3} \right)$$

gives a section of infinite order in $\text{MW}(\pi_m)$, enhancing the polarization to rank 19. Since these are $M$-polarized rank-19 K3 surfaces, they must be $M_n$ polarized for some $n$, and as in the case for $X_{1-8}$, we now appeal to [40] to conclude that the Picard lattice must be $M_2$.

Remark 5.36 (see Remark 5.27). It was shown in [40] that in the cases under consideration the Landau–Ginzburg models have the same variations of Hodge structures (up to pullbacks) as the modular variations associated with products of elliptic curves with isogeny. Explicitly, for $X$ equal to one of the Fano threefolds under consideration, let

$$(N, d) = \left( \frac{(-K_X)^3}{2 \cdot i(X)^2}, i(X) \right),$$

where $i(X)$ is the index of $X$. Let $X_0(N) + N$ denote the modular curve $(\Gamma_0(N) + N) \setminus \mathbb{H}$, and let $t_N$ be a Hauptmodul for $X_0(N) + N$ such that $t_N = 0$ at the image of the cusp ioc. The Picard–Fuchs equation for the Landau–Ginzburg model of $X$ is now the pullback of the symmetric square of the uniformizing differential equation for $X_0(N) + N$ by $\lambda = t_N^4$.

We can check that the pullback part of Golysh’evo’s theorem follows in a straightforward way from the geometry of fibres of the Landau–Ginzburg model.

- Cases $X_{1-1}$ and $X_{1-11}$. Both have polarizations by

$$H \oplus E_7(-1) \oplus D_{10}(-1).$$
Clearly, since the moduli space of $H \oplus E_7(-1) \oplus D_{10}(-1)$-polarized K3 surfaces is one dimensional, we see \emph{a posteriori} that the Landau–Ginzburg models $f_{1-1}$ and $f_{1-11}$ have isomorphic K3-compactified fibres.

- \textbf{The cases $X_{1-2}$, $X_{1-12}$, and $X_{1-17}$.} Similarly, since the moduli space of K3 surfaces polarized by $H \oplus E_6(-1) \oplus A_{11}(-1)$ is one dimensional, we see \emph{a posteriori} that the Landau–Ginzburg models $f_{1-2}$, $f_{1-12}$, and $f_{1-17}$ have isomorphic fibres. If we write down the Weierstrass forms for the elliptic fibrations that give this polarization in each case, then we can match the fibrations fibrewise to check that indeed fibres for $X_{1-12}$ are obtained from those for $X_{1-2}$ by the pullback $\lambda \mapsto \lambda^2$, and similarly the compactification for $X_{1-17}$ is the pullback $\lambda \mapsto \lambda^4$ of the compactification for $X_{1-2}$.

- $X_{1-3}$, $X_{1-13}$, and $X_{1-16}$ are treated as in the previous cases, using the polarizations by $H \oplus E_6(-1) \oplus E_6(-1) \oplus A_5(-1)$.

- \textbf{Cases $X_{1-4}$ and $X_{1-14}$ are treated as in the previous cases, using the polarizations by $H \oplus A_7(-1) \oplus D_5(-1) \oplus D_5(-1)$.}

- \textbf{Cases $X_{1-5}$ and $X_{1-15}$.} In this case, a pullback can be used to derive the polynomial $f_{1-15}$.

6. Katzarkov–Kontsevich–Pantev conjectures

This section is based on the papers [56], [63], and [15]. We study here the Katzarkov–Kontsevich–Pantev conjectures about the Hodge numbers of Landau–Ginzburg models and prove them in the cases of dimension 2 and 3.

6.1. Formulation. We recall some numerical conjectures from [56] which would follow from the conjectured homological mirror symmetry between Fano manifolds and Landau–Ginzburg models.

\textbf{Definition 6.1.} A \emph{Landau–Ginzburg model} is a pair $(Y, w)$, where

(i) $Y$ is a smooth complex quasi-projective variety with trivial canonical bundle $K_Y$;

(ii) $w: Y \to \mathbb{A}^1$ is a morphism with a compact critical locus $\text{crit}(w) \subset Y$.

\textbf{Remark 6.2.} Note that there are no conditions on singularities of fibres.

Following [56], we assume that there exists a \emph{tame} compactification of the Landau–Ginzburg model as defined below (see Definition 3.6).

\textbf{Definition 6.3.} A \emph{tame compactified Landau–Ginzburg model} is data $((Z, f), D_Z)$, where:

(i) $Z$ is a smooth projective variety and $f: Z \to \mathbb{P}^1$ is a flat morphism;

(ii) $D_Z = (\bigcup_j D^v_i) \cup (\bigcup_j D^v_j)$ is a reduced normal-crossings divisor such that

(ii.1) $D^v = \bigcup_j D^v_j$ is a scheme-theoretic pole divisor of $f$, that is, $f^{-1}(\infty) = D^v$, and in particular, $\text{ord}_{D^v_j}(f) = -1$ for all $j$,
(ii.2) each component $D^h_i$ of $D^h = \bigcup_i D^h_i$ is smooth and horizontal for $f$, that is, $f|_{D^h_i}$ is a flat morphism,

(ii.3) the critical locus $\text{crit}(f) \subset Z$ does not intersect $D^h$;

(iii) $D_Z$ is an anticanonical divisor on $Z$.

One says that $((Z, f), D_Z)$ is a compactification of the Landau–Ginzburg model $(Y, w)$ if, in addition, the following holds:

(iv) $Y = Z \setminus D_Z$, $f|_Y = w$ (and in this case we let $j: Y \hookrightarrow Z$ denote the open embedding).

Remark 6.4. In [56] the authors required an additional choice of compatible holomorphic volume forms on $Z$ and $Y$ in the above definitions. Since these forms will play no role in this paper, we omit them.

Assume that we are given a Landau–Ginzburg model $(Y, w)$ with a tame compactification $((Z, f), D_Z)$ as above. We denote by $n = \dim Y = \dim Z$ the (complex) dimension of $Y$ and $Z$. Choose a point $b \in \mathbb{A}^1$ which is near $\infty$ and such that the fibre $Y_b = w^{-1}(b) \subset Y$ is smooth. In [56] the authors define geometrically three sets of what they call ‘Hodge numbers’ $i^{p,q}(Y, w)$, $h^{p,q}(Y, w)$, and $f^{p,q}(Y, w)$. We recall these definitions.

6.1.1. The numbers $f^{p,q}(Y, w)$. We recall the definition of the logarithmic de Rham complex $\Omega^*_Z(\log D_Z)$. Namely,

$$\Omega^*_Z(\log D_Z) = \wedge^* \Omega^1_{Z}(\log D_Z),$$

where $\Omega^1_{Z}(\log D_Z)$ is the locally free $\mathcal{O}_Z$-module generated locally by

$$\frac{dz_1}{z_1}, \ldots, \frac{dz_k}{z_k}, dz_{k+1}, \ldots, dz_n,$$

where $z_1 \cdots z_k = 0$ is a local equation for the divisor $D_Z$. Hence, in particular,

$$\Omega^0_{Z}(\log D_Z) = \mathcal{O}_Z.$$

The numbers $f^{p,q}(Y, w)$ are defined using the subcomplex

$$\Omega^*_Z(\log D_Z, f) \subset \Omega^*_Z(\log D_Z)$$

of $f$-adapted forms, which we recall next.

Definition 6.5 (see [56], Definition 2.11). For each $a \geq 0$ define the sheaf $\Omega^a_Z(\log D_Z, f)$ of $f$-adapted logarithmic forms as a subsheaf of $\Omega^a_Z(\log D_Z, f)$ consisting of forms which stay logarithmic after multiplication by $df$. Thus,

$$\Omega^a_Z(\log D_Z, f) = \{ \alpha \in \Omega^a_Z(\log D_Z) \mid df \wedge \alpha \in \Omega^{a+1}_Z(\log D_Z) \},$$

where one regards $f$ as a meromorphic function on $Z$ and $df$ as a meromorphic 1-form.

Definition 6.6 (see [56], Definition 3.1). The Landau–Ginzburg Hodge numbers $f^{p,q}(Y, w)$ are defined as follows:

$$f^{p,q}(Y, w) = \dim H^p(Z, \Omega^q_Z(\log D_Z, f)).$$
6.1.2. The numbers $h^{p,q}(Y, w)$. Let $N: V \to V$ be a nilpotent operator on a finite-dimensional vector space $V$ such that $N^{m+1} = 0$. Recall that this data defines a canonical (monodromy) weight filtration $W = W_s(N, m)$, centred at $m$, of the space $V$,

$$0 \subset W_0(N, m) \subset W_1(N, m) \subset \cdots \subset W_{2m-1}(N, m) \subset W_{2m}(N, m) = V$$

with the properties

(i) $N(W_i) \subset W_{i-2}$,

(ii) the map $N^l: \text{gr}^{W,m}_{m+l} V \to \text{gr}^{W,m}_{m-l} V$ is an isomorphism for all $l \geq 0$.

Let $S^1 \simeq C \subset \mathbb{P}^1$ be a smooth loop passing through the point $b$ that goes once around $\infty$ in the counter-clockwise direction in such a way that there are no singular points of $w$ on or inside $C$. This gives the monodromy transformation

$$T: H^*(Y_b) \to H^*(Y_b),$$

and also the corresponding monodromy transformation on the relative cohomology

$$T: H^*(Y, Y_b) \to H^*(Y, Y_b)$$

in such a way that the sequence

$$\cdots \to H^m(Y, Y_b) \to H^m(Y) \to H^m(Y_b) \to H^{m+1}(Y, Y_b) \to \cdots$$

is $T$-equivariant, where $T$ acts trivially on $H^*(Y)$. (See §6.2.1 for the construction and a discussion of the monodromy transformation $T: H^*(Y_b) \to H^*(Y, Y_b)$.) Since we assume that the infinite fibre $f^{-1}(\infty) \subset Z$ is a reduced divisor with normal crossings, by the Griffiths–Landman–Grothendieck Theorem (see [55]) the operator $T: H^m(Y_b) \to H^m(Y_b)$ is unipotent and $(T - \text{id})^{m+1} = 0$. It follows that the transformation (6.1) is also unipotent. Denote by $N$ the logarithm of the transformation (6.1), which is therefore a nilpotent operator on $H^*(Y, Y_b)$. One has $N^{m+1} = 0$.

**Definition 6.7** (see [63], Definition 7). We say that the Landau–Ginzburg model $(Y, w)$ is of **Fano type** if the operator $N$ on the relative cohomology $H^{n+a}(Y, Y_b)$ has the following properties:

(i) $N^{n-a} \neq 0$ and (ii) $N^{n-a+1} = 0$.

This definition is motivated by the expectation that the Landau–Ginzburg model of Fano type usually arises as a mirror of a projective Fano manifold $X$ (see §6.1.4).

**Definition 6.8** (see [56], Definition 3.2, and [63], Definition 8). Assume that $(Y, w)$ is a Landau–Ginzburg model of Fano type. Consider the relative cohomology $H^*(Y, Y_b)$ with the nilpotent operator $N$ and the induced canonical filtration $W$. The **Landau–Ginzburg Hodge numbers** $h^{p,q}(Y, w)$ are defined as follows:

$$h^{p,n-a}(Y, w) = \dim \text{gr}^{W(N, n-a)}_{2(n-p)} H^{n+p-a}(Y, Y_b) \quad \text{if } a = p-q > 0,$$

$$h^{p,n-a}(Y, w) = \dim \text{gr}^{W(N, n+a)}_{2(n-q)} H^{n+p-a}(Y, Y_b) \quad \text{if } a = p-q < 0.$$
Remark 6.9. Our Definition 6.8 differs from Definition 3.2 in [56],

\[ h^{p,q}(Y, w) = \dim \text{gr}_{p}^{W,p+q} H^{p+q}(Y, Y_{b}), \]  

(6.2)

by the indices of the grading. The equality (6.2) seems not to be what the authors had in mind. For example, according to (6.2) the index \( p \) is allowed to vary from 0 to \( 2n \) and \( q \) is allowed to be negative (see §6.1.4 for details).

6.1.3. The numbers \( i^{p,q}(Y, w) \). Recall that for each \( \lambda \in \mathbb{A}^{1} \) one has the corresponding sheaf \( \phi_{w-\lambda}C_{Y} \) of vanishing cycles for the fibre \( Y_{\lambda} \). The sheaf \( \phi_{w-\lambda}C_{Y} \) is supported on the fibre \( Y_{\lambda} \) and is equal to zero if \( \lambda \) is not a critical value of \( w \).

From the works of Schmid, Steenbrink, and Saito it is classically known that the constructible complex \( \phi_{w-\lambda}C_{Y} \) carries the structure of a mixed Hodge module and so its hypercohomology inherits a mixed Hodge structure. For a mixed Hodge module \( S \) we will denote by \( i^{p,q}S \) the \((p, q)\)-Hodge numbers of the \((p+q)\)th weight graded piece \( \text{gr}_{p}^{W,p+q}S \).

Definition 6.10 ([56], Definition 3.4). (i) Assume that the horizontal divisor \( D^{h} \subset Z \) is empty, that is, the map \( w: Y \to \mathbb{A}^{1} \) is proper. Then the Landau–Ginzburg Hodge numbers \( i^{p,q}(Y, w) \) are defined as follows:

\[ i^{p,q}(Y, w) = \sum_{\lambda \in \mathbb{A}^{1}} \sum_{k} i^{p,q+k}H^{p+q-1}(Y_{\lambda}, \phi_{w-\lambda}C_{Y}). \]

(ii) In the general case denote by \( j: Y \hookrightarrow Z \) the open embedding and define similarly

\[ i^{p,q}(Y, w) = \sum_{\lambda \in \mathbb{A}^{1}} \sum_{k} i^{p,q+k}H^{p+q-1}(Y_{\lambda}, \phi_{w-\lambda}Rj_{*}C_{Y}). \]

6.1.4. Conjectures. It was proved in [56] that for every \( m \) the above numbers satisfy the equalities

\[ \dim H^{m}(Y, Y_{b}; \mathbb{C}) = \sum_{p+q=m} i^{p,q}(Y, w) = \sum_{p+q=m} h^{p,q}(Y, w) = \sum_{p+q=m} f^{p,q}(Y, w). \]  

(6.3)

The authors stated several conjectures which together refine the equalities (6.3).

The following is a modification of Conjecture 3.6 in [56] (see Remark 6.9).

Conjecture 6.11. Assume that \( (Y, w) \) is a Landau–Ginzburg model of Fano type. Then for every \( p \) and \( q \)

\[ h^{p,q}(Y, w) = f^{p,q}(Y, w) = i^{p,q}(Y, w). \]

The Landau–Ginzburg model \((Y, w)\) of Fano type (together with a tame compactification) typically arises as a mirror of a projective Fano manifold \( X \), \( \dim X = \dim Y \).

Conjecture 6.12 ([56], Conjecture 3.7; see Remark 6.9). In the above mirror situation, for all \( p \) and \( q \)

\[ f^{p,q}(Y, w) = h^{p,n+q}(X), \]

where the \( h^{p,q}(X) \) are the usual Hodge numbers for \( X \).
We refer the interested reader to [56] for a detailed description of the motivation for Conjectures 6.11 and 6.12. Basically, the motivation comes from homological mirror symmetry, the Hochschild homology identifications, and the identification of the monodromy operator with the Serre functor. Namely, assume that the Landau–Ginzburg model \((Y, w)\) as above (together with a tame compactification) is of Fano type and is a mirror of a projective Fano manifold \(X\), \(\dim X = \dim Y\).

Then by the homological mirror symmetry conjecture one expects an equivalence of categories
\[
D^b(\text{coh} X) \cong \text{FS}(Y, w, \omega_Y),
\]
where \(\text{FS}((Y, w), \omega_Y)\) is the Fukaya–Seidel category of the Landau–Ginzburg model \((Y, w)\) with an appropriate symplectic form \(\omega_Y\). This equivalence induces for each \(a\) an isomorphism of the Hochschild homology spaces
\[
HH_a(D^b(\text{coh} X)) \cong HH_a(\text{FS}(Y, w, \omega_Y)).
\]

It is known that
\[
HH_a(D^b(\text{coh} X)) \cong \bigoplus_{p-q=a} H^p(X, \Omega^q_X),
\]
and it is expected that
\[
HH_a(\text{FS}(Y, w, \omega_Y)) \cong H^{n+a}(Y, Y_b).
\]

The equivalence (6.4) and the isomorphisms (6.5) and (6.6) suggest an isomorphism
\[
H^{n+a}(Y, Y_b) \cong \bigoplus_{p-q=a} H^p(X, \Omega^q_X).
\]
Moreover, the equivalence (6.4) identifies the Serre functors \(S_X\) and \(S_Y\) on the two categories. The functor \(S_X\) acts on the cohomology \(H^\ast(X)\), and the logarithm of this operator is equal (up to sign) to the cup-product with \(c_1(K_X)\). Since \(X\) is Fano, the operator \(c_1(K_X) \cup (\cdot)\) is a Lefschetz operator on the space
\[
\bigoplus_{p-q=a} H^p(X, \Omega^q_X)
\]
for each \(a\). On the other hand, the Serre functor \(S_Y\) induces an operator on the space \(H^{n+a}(Y, Y_b)\) which is the inverse of the monodromy transformation \(T\). This suggests that the weight filtration for the nilpotent operator \(c_1(K_X) \cup (\cdot)\) on the space \(\bigoplus_{p-q=a} H^p(X, \Omega^q_X)\) should coincide with the analogous filtration for the logarithm \(N\) of the operator \(T\) on \(H^{n+a}(Y, Y_b)\). First, note that the operator \(c_1(K_X) \cup (\cdot)\) on the space \(\bigoplus_{p-q=a} H^p(X, \Omega^q_X)\) satisfies
\[
(c_1(K_X) \cup (\cdot))^{n-|a|} \neq 0
\]
by the Hard Lefschetz theorem, and
\[
(c_1(K_X) \cup (\cdot))^{n-|a|+1} = 0.
\]
This explains our Definition 6.7. Moreover, the induced filtration $W$ on $\bigoplus_{p-q=a} H^p(X, \Omega^q_X)$ has the properties

$$h^{p,q}(X) = \text{gr}^W_{2(n-p)} \left[ \bigoplus_{p-q=a} H^p(X, \Omega^q_X) \right]$$

if $a \geq 0$ and

$$h^{p,q}(X) = \text{gr}^W_{2(n-q)} \left[ \bigoplus_{p-q=a} H^p(X, \Omega^q_X) \right]$$

if $a < 0$.

Thus one expects the equality of Hodge numbers

$$h^{p,n-q}(Y, w) = h^{p,q}(X),$$

which is a combination of the above conjectures.

### 6.2. Del Pezzo surfaces

The mirror symmetry conjecture we are interested in here is the homological mirror symmetry conjecture. It (more precisely, half of it) was proved for del Pezzo surfaces in [4]. A Landau–Ginzburg model for a del Pezzo surface of degree $d$ is constructed there as a pencil of elliptic curves whose fibre over infinity is a wheel of $12 - d$ curves, while the other singular fibres are $d$ fibres having a single ordinary double point (node). Such a pencil is a Landau–Ginzburg model for the del Pezzo surface with a general symplectic form on the model. However, a Fukaya–Seidel category is invariant under deformations of pencils, so to study mirror symmetry it is enough to consider the case of a general form. Moreover, the results of this section do not depend on singular fibres away from infinity. Finally, note that the Landau–Ginzburg models studied here correspond to all del Pezzo surfaces, not only to those of degree greater than 2 as in §4.

Following [63], we revise Conjectures 6.11 and 6.12 slightly and prove them for del Pezzo surfaces. Consider a tame compactified Landau–Ginzburg model $(Z, f)$ of dimension 2. More precisely, consider a rational elliptic surface $f: Z \to \mathbb{P}^1$ with $f^{-1}(\infty)$ being a reduced divisor which is a wheel of $d$ rational curves, $1 \leq d \leq 9$ (it is a nodal rational curve if $d = 1$). In this case the horizontal divisor $D^h$ is empty, so $D = D^v$. It was proved in [4] that the corresponding Landau–Ginzburg model $(Y, w)$ appears as a (homological) mirror of a del Pezzo surface $S_d$ of degree $d$.

The authors also established Homological Mirror Symmetry for the case $d = 0$: in this case $f^{-1}(\infty)$ is a smooth elliptic curve and $(Y, w)$ is a mirror to the blowup $S_0$ of $\mathbb{P}^2$ at nine points of intersection of two cubic curves. Note that this $S_0$ is not Fano, hence one expects that the corresponding Landau–Ginzburg model $(Y, w)$ is not of Fano type. We confirm this prediction. The next theorem summarizes the main results of this section.

**Theorem 6.13** (see [63], Theorem 11). Let $f: Z \to \mathbb{P}^1$ be an elliptic surface with the reduced infinite fibre $D = f^{-1}(\infty)$ which is a wheel of $d$ rational curves for $1 \leq d \leq 9$ or a smooth elliptic curve for $d = 0$. Assume that $f$ has a section. As before, put $(Y, w) = (Z \setminus D, f|_{Z\setminus D})$.

(i) If $1 \leq d \leq 9$, then the Landau–Ginzburg model $(Y, w)$ is of Fano type and there is the equality of Hodge numbers

$$f^{p,q}(Y, w) = h^{p,q}(Y, w).$$
(ii) Let \( 1 \leq d \leq 9 \) and let \( S_d \) be a del Pezzo surface which is a mirror in the sense of [4] to the Landau–Ginzburg model \((Y, w)\). Then there is the equality of Hodge numbers
\[
f^{p,q}(Y, w) = h^{p,2-q}(S_d).
\]

(iii) If \( d = 0 \), then \( (Y, w) \) is not of Fano type.

The proof of Theorem 6.13 is contained in Propositions 6.25 and 6.30 and Remark 6.31.

Thus, Conjecture 6.11 about the numbers \( f^{p,q}(Y, w) \) and \( h^{p,q}(Y, w) \) and Conjecture 6.12 hold when \((Y, w)\) is of Fano type \((1 \leq d \leq 9)\). We will also show that in the context of Theorem 6.13 the numbers \( i^{p,q}(Y, w) \) (or to \( h^{p,q}(Y, w) \) or \( h^{p,2-q}(X) \)), and therefore provide a counterexample to Conjecture 6.11 (see Remark 6.32). We do not know how to define the ‘correct’ numbers \( i^{p,q}(Y, w) \) which would make Conjecture 6.11 true.

6.2.1. Monodromy action on relative cohomology. Let \( V \) be a smooth complex algebraic variety of dimension \( n \) with a proper morphism \( w: V \to \mathbb{C} \). Let \( b \in \mathbb{C} \) be a regular value of \( w \). In this section we construct the monodromy action on the relative homology \( H_\ast(V, V_b) \), which by duality will induce the desired action on \( H^\ast(V, V_b) \).

Let \( C \simeq S^1 \subset \mathbb{P}^1 \) be a smooth loop passing through the point \( b \) that goes once around the \( \infty \) in the counter-clockwise direction in such a way that there are no singular values of \( w \) on or inside \( C \). Denote by \( M \) the pre-image \( w^{-1}(C) \subset Y \). Then \( M \) is a compact oriented smooth manifold which contains the fibre \( V_b \). The (real) dimensions of \( M \) and \( V_b \) are \( 2n-1 \) and \( 2n-2 \) respectively. By Ehresmann’s Lemma the map \( w: M \to C \) is a locally trivial fibration of smooth manifolds with fibres diffeomorphic to \( V_b \). Therefore, there exists a diffeomorphism \( T: V_b \to V_b \) such that \( M \) is diffeomorphic to the quotient
\[
M = V_b \times [0,1]/\{(a,0) = (T(a),1) \text{ for all } a \in V_b\}.
\]

For the pair \((M, V_b)\) we have the corresponding long exact homology sequence
\[
\cdots \to H_i(V_b) \overset{\alpha_i}{\to} H_i(M) \overset{\beta_i}{\to} H_i(M, V_b) \overset{\partial_i}{\to} H_{i-1}(V_b) \to \cdots
\]
(6.7)

The diffeomorphism \( T: V_b \to V_b \) induces an automorphism
\[
T: H_i(V_b) \to H_i(V_b)
\]
for each \( i \).

**Lemma 6.14.** For each \( i \geq 0 \), there exists a homomorphism
\[
L_i: H_i(V_b) \to H_{i+1}(M, V_b)
\]
such that for all \( x \in H_i(V_b) \)
\[
\partial_{i+1}L_i(x) = T(x) - x.
\]
Proof. Let \( z \) be an \( i \)-dimensional cycle in \( V_b \). Consider the \((i + 1)\)-dimensional relative cycle \( z \times [0, 1] \) in \((V_b \times [0, 1], V_b \times \{0\} \cup V_b \times \{1\})\) with boundary \( z \times \{1\} - z \times \{0\}\). Its image \( L_i(z) \) in \( M \) is a relative \((i + 1)\)-cycle with boundary \( T(z) - z \) in \( V_b \). This construction yields the required homomorphism \( L_i: H_i(V_b) \to H_{i+1}(M, V_b) \). For a given \( x \in H_i(V_b) \) the equality

\[ \partial_{i+1} L_i(x) = T(x) - x \]

is clear from the construction. \( \square \)

Proposition 6.15 (see [63], Proposition 13). The map \( L_i: H_i(V_b) \to H_{i+1}(M, V_b) \) is injective for each \( i \geq 0 \).

Definition 6.16. For each \( i \) define the endomorphism

\[ T: H_i(M, V_b) \to H_i(M, V_b) \quad \text{by} \quad T = \text{id} + L_{i-1} \partial_i \]

and the endomorphism

\[ T: H_i(M) \to H_i(M) \quad \text{by} \quad T = \text{id} \]

(In particular, \( T = \text{id} \) on \( H_0(M, V_b) \).)

The inclusion of pairs \((M, V_b) \subset (V, V_b)\) induces a morphism of the homology sequences

\[
\begin{array}{cccccccc}
\cdots & \longrightarrow & H_i(M) & \longrightarrow & H_i(M, V_b) & \longrightarrow & H_{i-1}(V_b) & \longrightarrow & \cdots \\
& & \downarrow \gamma_i & & \downarrow & & \downarrow & & \\
& & \cdots & \longrightarrow & H_i(V) & \longrightarrow & H_i(V, V_b) & \longrightarrow & \cdots \\
\end{array}
\]

Definition 6.17. For each \( i \geq 0 \) define the endomorphism

\[ T: H_i(V, V_b) \to H_i(V, V_b) \]

to be the composition

\[ T(y) = y + \gamma_i L_{i-1} \partial_i(y) \]

for \( y \in H_i(V, V_b) \). In particular, \( T = \text{id} \) on \( H_0(V, V_b) \). We also define \( T: H_i(V) \to H_i(V) \) to be the identity.

By duality this defines the operators \( T \) on the cohomologies \( H^i(V_b), H^i(V, V_b), \) and \( H^i(V) \).

Corollary 6.18. The sequence

\[ \cdots \to H_i(V) \to H_i(V, V_b) \to H_{i-1}(V_b) \to \cdots \]

is compatible with the endomorphisms \( T \). Hence, also the dual cohomology sequence

\[ \cdots \to H^{i-1}(V_b) \to H^i(V, V_b) \to H^i(V) \to \cdots \]

is compatible with \( T \).
Proof. This follows directly from the definition of the operators $T$ together with the formula in Lemma 6.14. □

**Proposition 6.19** (see [63], Proposition 18). (i) Assume that the morphism

$$\gamma_i: H_i(M, V_b) \to H_i(V, V_b)$$

is injective. Then the image of the morphism $H_i(V) \to H_i(V, V_b)$ is the space $H_i(V, V_b)^T$ of $T$-invariants.

(ii) If $H^{2n-i-1}(V) = 0$, then the map $H_i(M, V_b) \to H_i(V, V_b)$ is injective. Therefore, by (i) the image of the morphism $H_i(V) \to H_i(V, V_b)$ is the space $H_i(V, V_b)^T$ of $T$-invariants.

6.2.2. Topology of rational elliptic surfaces. Now we use the notation from the beginning of §6.2 for the special case considered in the rest of this subsection. Fix a number $0 \leq d \leq 9$ and let $f: Z \to \mathbb{P}^1$ be a rational elliptic surface such that $D = D^v = f^{-1}(\infty)$ is a wheel $I_d$ of $d$ smooth rational curves for $d \geq 2$, a rational curve with one node $I_1$ for $d = 1$, or a smooth elliptic curve $I_0$ for $d = 0$. Assume in addition that there exists a section $\mathbb{P}^1 \to E \subset Z$. Recall that $Y = Z \setminus D$.

Since $Z$ is rational, it follows that $\chi(\mathcal{O}_Z) = 1$. One has $-K_Z = D$ (see, for instance, [52], §10.2). Hence $c_2^1(Z) = 0$, so by Noether’s formula the topological Euler characteristic of $Z$ is equal to 12. This means that

$$h^i(Z) = \begin{cases} 1 & \text{for } i = 0, 4, \\ 10 & \text{for } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

By the adjunction formula $(K_Z + E) \cdot E = 2g(E) - 2 = -2$, so $E^2 = -1$.

**Lemma 6.20.** (i) If $d = 0$, then

$$h^i(D) = \begin{cases} 1 & \text{for } i = 0, 2, \\ 2 & \text{for } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If $d > 0$, then

$$h^i(D) = \begin{cases} 1 & \text{for } i = 0, 1, \\ d & \text{for } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Part (i) is clear. We prove part (ii). Let $p_1, \ldots, p_d$ be the intersection points of components of $D$, and let $\pi: \tilde{D} \to D$ be the normalization. Then $\tilde{D}$ is a disjoint union of $d$ copies of $\mathbb{P}^1$. Consider an exact sequence of sheaves on $D$:

$$0 \to \mathbb{C}_D \to \pi_*\pi^*\mathbb{C}_D \to \bigoplus_{i=1}^d \mathbb{C}_{p_i} \to 0,$$  \hspace{1cm} (6.8)

where $\mathbb{C}_{p_i}$ is a skyscraper sheaf supported at $p_i$. Note that

$$\dim H^i(D, \pi_*\pi^*\mathbb{C}_D) = \dim H^i(\tilde{D}) = \begin{cases} d & \text{for } i = 0, 2, \\ 0 & \text{for } i = 1. \end{cases}$$
Note also that $H^0(D, \mathbb{C}_D) = \mathbb{C}$ and the map

$$H^0(D, \mathbb{C}_D) \to H^0(D, \pi_*\pi^*\mathbb{C}_D)$$

is injective. The lemma now follows from the long exact cohomology sequence, applied to the short exact sequence (6.8). □

**Lemma 6.21** (see [63], Lemma 20). The restriction map

$$s: H^2(Z) \to H^2(D)$$

is surjective.

Next we compute the cohomology $H_c^i(Y)$ of $Y$ with compact support.

**Lemma 6.22** (see [63], Lemma 21). The following equalities hold:

$$h^i_c(Y) = h^i(Z, j_!\mathbb{C}_Y) = \begin{cases} 0 & \text{for } i = 0, 1, 3, \\ 11 - d & \text{for } i = 2, \\ 1 & \text{for } i = 4. \end{cases}$$

**Idea of proof.** This follows from the long exact cohomology sequence $H^*(Z, -)$ for the short exact sequence

$$0 \to j_!\mathbb{C}_Y \to \mathbb{C}_Z \to \mathbb{C}_D \to 0.$$

**Corollary 6.23.** By Poincaré duality for $Y$

$$h^i(Y) = \begin{cases} 1 & \text{for } i = 0, \\ 11 - d & \text{for } i = 2, \\ 0 & \text{for } i = 1, 3, 4. \end{cases}$$

6.2.3. Landau–Ginzburg Hodge numbers for rational elliptic surfaces.

6.2.3.1. The numbers $h^{p,q}(Y, w)$. We keep the notation of §6.2.2.

Consider the long exact homology sequence

$$\cdots \to H_2(Y) \to H_2(Y, Y_b) \to H_1(Y_b) \to \cdots.$$

Recall that there is a compatible action of the monodromy $T$ on each term of this sequence as explained in §6.2.1.

**Corollary 6.24.** The image of the map $H_2(Y) \to H_2(Y, Y_b)$ coincides with the space $H^*_2(Y, Y_b)^T$ of $T$-invariants.

**Proof.** In the notation of Proposition 6.19 we have $n = 2$ and $i = 2$, and by Corollary 6.23 we have $H^{2n-i-1}(Y) = H^1(Y) = 0$. Hence, the assertion follows from Proposition 6.19, (ii). □

**Proposition 6.25.** (i)

$$H^k(Y, Y_b) = \begin{cases} \mathbb{C}^{12-d} & \text{for } k = 2, \\ 0 & \text{otherwise}. \end{cases}$$ (6.9)
(ii) For \(d > 0\) the Landau–Ginzburg model \((Y, w)\) is of Fano type and
\[
h^{p,q}(Y, w) = \begin{cases} 
1 & \text{for } (p, q) = (0, 2), (2, 0), \\
10 - d & \text{for } (p, q) = (1, 1), \\
0 & \text{otherwise.}
\end{cases}
\]
(6.10)

(iii) For \(d = 0\) the Landau–Ginzburg model \((Y, w)\) is not of Fano type.

This proposition proves Theorem 6.13, (iii) and computes the right-hand side of
the equality of Theorem 6.13, (i).

The proof of Proposition 6.25 will occupy the rest of this subsection.

**Lemma 6.26.** The restriction map \(H^2(Y) \to H^2(Y_b)\) is surjective. Hence, the
map \(H_2(Y_b) \to H_2(Y)\) is injective.

**Proof.** Since \(Y_b\) is a smooth projective curve, \(H^2(Y_b)\) has dimension one and is
spanned by the first Chern class \(c_1(L)\) of any ample line bundle \(L\) on \(Y_b\). It suffices
to take any ample line bundle \(M\) on \(Y\), so that its restriction \(L = M|_{Y_b}\) is also ample
and \(c_1(M) \in H^2(Y)\) restricts to \(c_1(L) \in H^2(Y_b)\). □

The equality (6.9) now follows from the long exact cohomology sequence
\[
\cdots \to H^i(Y, Y_b) \to H^i(Y) \to H^i(Y_b) \to \cdots ,
\]
using Corollary 6.23, the fact that \(Y_b\) is an elliptic curve, and Lemma 6.26. This
proves part (i) of the proposition.

To prove parts (ii) and (iii) it remains to understand the action of the mono-
odromy \(T\) on \(H_2(Y, Y_b)\).

We consider a part of the long exact homology sequence:
\[
H_3(Y, Y_b) \to H_2(Y_b) \to H_2(Y) \to H_2(Y, Y_b) \to H_1(Y_b) \to H_1(Y).
\]
We know that the map \(H_2(Y_b) \to H_2(Y)\) is injective and \(H_1(Y) = H^1(Y)^\vee = 0\).
Therefore, the sequence
\[
0 \to H_2(Y_b) \to H_2(Y) \to H_2(Y, Y_b) \to H_1(Y_b) \to 0
\]
(6.11)
is also exact. We have \(H_2(Y_b) = \mathbb{C}\), \(H_1(Y_b) = \mathbb{C}^2\) and \(H_2(Y) = \mathbb{C}^{11-d}\), hence the
sequence (6.11) is isomorphic to
\[
0 \to \mathbb{C} \to \mathbb{C}^{11-d} \to \mathbb{C}^{12-d} \to \mathbb{C}^2 \to 0.
\]

These sequences are \(T\)-equivariant, where \(T\) acts trivially on \(H_2(Y_b)\) and \(H_2(Y)\).
By Landman’s theorem \(T\) acts unipotently on \(H_1(Y_b)\).

For \(d = 0\) the fibre \(f^{-1}(\infty)\) is smooth, hence the action of \(T\) on \(H_1(Y_b)\) is trivial.
Therefore, the exact sequence (6.11) and Corollary 6.24 imply that the \(T\)-action
on \(H_2(Y, Y_b)\) is unipotent, with two Jordan blocks of size 2 and eight blocks of
size 1. This means that the Landau–Ginzburg model \((Y, w)\) is not of Fano type,
which proves (iii).

For \(d > 0\) the fibre \(f^{-1}(\infty)\) is singular, so the \(T\)-action on \(H_1(Y_b)\) is non-trivial
(see [59], Table 1). Therefore, the exact sequence (6.11) and Corollary 6.24 imply
that the $T$-action on $H_2(Y, Y_b)$ is unipotent, with one Jordan block of size 3 and $9 - d$ blocks of size 1. Therefore, $(Y, w)$ is of Fano type and (6.10) holds. This completes the proof of Proposition 6.25.

6.2.3.2. The numbers $f^{p,q}(Y, w)$. Recall that we have the open embedding $j: Y \hookrightarrow Z$.

Lemma 6.27.

$$\Omega^0_Z(\log D) = \mathcal{O}_Z \quad \text{and} \quad \Omega^2_Z(\log D) = \mathcal{O}_Z.$$  

Hence

$$\Omega^0_Z(\log D)(-D) = \Omega^2_Z(\log D)(-D) = \omega_Z.$$  

Proof. This follows from the definition of the logarithmic complex in §6.1.1 and the fact that $D$ is the anticanonical divisor. □

Proposition 6.28 (see [63], Proposition 27). The following equalities hold:

$$h^i(Z, \Omega^0_Z(\log D)(-D)) = h^i(Z, \Omega^2_Z(\log D)(-D)) = \begin{cases} 0 & \text{for } i = 0, 1, \\ 1 & \text{for } i = 2; \end{cases} \quad (6.12)$$

$$h^i(Z, \Omega^2_Z(\log D)(-D)) = \begin{cases} 0 & \text{for } i = 0, 2, \\ 10 - d & \text{for } i = 1. \end{cases} \quad (6.13)$$

Idea of proof. The equalities (6.12) follows from Serre duality and Lemma 6.27. The equality (6.13) follows from analysis of the complex

$$\Omega^2_Z(\log D)(-D) \rightarrow \Omega^2_Z(\log D)(-D) \rightarrow \Omega^2_Z(\log D)(-D) \rightarrow 0,$$

which is a resolvent of the sheaf $j_! \mathcal{C}_Y$ (see, for instance, [26], p.268). This complex gives the spectral sequence

$$E_1^{pq} = H^p(Z, \Omega^q_Z(\log D)(-D)),$$

which converges to $H^{p+q}(Z, j_! \mathcal{C}_Y)$. □

Proposition 6.29 (see [63], Proposition 28). (i) There is an isomorphism

$$\Omega^0_Z(\log D, f) = \mathcal{O}_Z(-D) = \omega_Z.$$  

(ii) There is an isomorphism

$$\Omega^2_Z(\log D, f) = \Omega^2_Z(\log D) = \mathcal{O}_Z.$$  

(iii) There exists a short exact sequence of sheaves on $Z$

$$0 \rightarrow \Omega^1_Z(\log D)(-D) \rightarrow \Omega^1_Z(\log D, f) \rightarrow \mathcal{O}_D \rightarrow 0.$$

Proposition 6.30.

$$f^{p,q}(Y, w) = \begin{cases} 1 & \text{for } (p, q) = (0, 2), (2, 0), \\ 10 - d & \text{for } (p, q) = (1, 1), \\ 0 & \text{otherwise.} \end{cases}$$
Proof. Proposition 6.28 and Lemma 6.27 give

\[ f_{p,0}(Y, w) = h^p(Z, \Omega^0_Z(\log D, f)) = h^p(Z, \omega_Z) = \begin{cases} 0 & \text{for } p = 0, 1, \\ 1 & \text{for } p = 2, \end{cases} \]

\[ f_{p,1}(Y, w) = h^p(Z, \Omega^1_Z(\log D, f)) = h^p(Z, \Omega^1_Z(\log D)(-D)) \]

\[ = \begin{cases} 0 & \text{for } p = 0, 2, \\ 10 - d & \text{for } p = 1, \end{cases} \]

and

\[ f_{p,2}(Y, w) = h^p(Z, \Omega^2_Z(\log D, f)) = h^p(Z, \mathcal{O}_Z) = \begin{cases} 1 & \text{for } p = 0, \\ 0 & \text{for } p = 1, 2. \end{cases} \]

6.2.4. End of the proof of Theorem 6.13 and discussion. The study of elliptic surfaces in §6.2.3 is motivated by mirror symmetry constructions for del Pezzo surfaces in [4]. There, the authors proved ‘half’ of the homological mirror symmetry conjecture for del Pezzo surfaces. More precisely, they proved that for a general del Pezzo surface \( S_d \) of degree \( d \), \( 1 \leq d \leq 9 \), obtained by a blowup of \( \mathbb{P}^2 \) at \( 9 - d \) general points there exists a complexified symplectic form \( \omega_Y \) on \( (Y, w) \), where \( (Y, w) \) has \( 12 - d \) nodal singular fibres, and that \( Y \) can be compactified to \( Z \) for which \( D \) is a wheel of \( d \) curves, such that

\[ D^b(\text{coh } S_d) \cong \text{FS}(Y, w, \omega_Y). \] (6.14)

We call \((Y, w)\) a Landau–Ginzburg model for \( S_d \). We allow the case \( d = 0 \) as well; in this case \((Y, w)\) is a Landau–Ginzburg model for \( \mathbb{P}^2 \) blown up at nine intersection points of two elliptic curves (see [4]). The equivalence (6.14) holds in this case as well.

Remark 6.31. Description of the del Pezzo surface \( X \) of degree \( d \) as a blowup of \( \mathbb{P}^2 \) gives the following equalities:

\[ h^{p,q}(X) = \begin{cases} 1 & \text{for } (p, q) = (0, 2), (2, 0), \\ 10 - d & \text{for } (p, q) = (1, 1), \\ 0 & \text{otherwise}. \end{cases} \]

This remark, together with Proposition 6.30, provides a proof of part (ii) of Theorem 6.13 and thus, in combination with Proposition 6.25, completes the proof of this theorem. In other words, Conjecture 6.12 and a ‘half’ of Conjecture 6.11 hold for (mirrors of) del Pezzo surfaces.

Remark 6.32. The second part of Conjecture 6.11 does not hold even for the Landau–Ginzburg model \((Y, w)\) for \( \mathbb{P}^2 \). Indeed, one has

\[ h^{0,0}(Y, w) = h^{1,1}(Y, w) = h^{2,2}(Y, w) = 1. \]

However, the Landau–Ginzburg model \((Y, w)\) has exactly three singular fibres, and the singular set of these fibres is a single node. Hence, the numbers \( i^{p,q}(Y, w) \) are integers divisible by 3.
Remark 6.33. Del Pezzo surfaces are blowups of $\mathbb{P}^2$ with one exception, which is a quadric surface. However, by §4 a toric Landau–Ginzburg model for a quadric is an elliptic pencil with a reduced fibre over infinity which is a wheel of eight curves. Thus the assertion of Theorem 6.13 holds for a quadric as well.

6.3. Fano threefolds. In this section, following [15], we study Conjecture 6.12 in the three-dimensional case. The important ingredient of the proof is the following result of Harder that treats this conjecture in terms of the geometry of Landau–Ginzburg models. That is, consider a tame compactified Landau–Ginzburg model $(Y, w)$, where $w: Y \to \mathbb{C}$ and $\dim Y = 3$. Denote its compactification by $(Z, f)$. Let the divisor over infinity $f^{-1}(\infty)$ be combinatorially a triangulation of a sphere. Assume that $h^{i,0}(Z) = 0$ for $i > 0$. Let the general fibre $f^{-1}(\lambda)$ be a K3 surface.

Theorem 6.34 (see [43], Theorem 10). The Hodge diamond for the numbers $f^{p,q}(Y, w)$ is

$$
\begin{array}{c c c}
0 & 0 & 0 \\
0 & k_Y & 0 \\
1 & ph - 2 + h^{1,2}(Z) & ph - 2 + h^{2,1}(Z) \\
0 & k_Y & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}
$$

where

$$
ph = \dim \operatorname{coker}(H^2(Z, \mathbb{R}) \to H^2(V, \mathbb{R}))
$$

is the corank of the restriction of the second cohomology of the ambient space to a general fibre $V$, and $k_Y$ is given by

$$
k_Y = \sum_{s \in \Sigma} (\rho_s - 1),
$$

where $\Sigma$ is the set of critical values of $w$ and $\rho_s$ is the number of irreducible components of $w^{-1}(s)$.

In particular, the assumptions of this theorem hold for toric Landau–Ginzburg models by Theorem 5.9 and Remark 5.13. Moreover, in that case $h^{2,1}(Z) = 0$.

Note that birational smooth Calabi–Yau varieties are isomorphic in codimension 1, so $k_Y$ and $ph$ do not depend on the particular Calabi–Yau compactification $Y$ of the toric Landau–Ginzburg model for $X$. Moreover, in accordance with Remark 5.5 they do not even depend on the particular toric model of Minkowski type.

We need the following statements on the intersection theory for du Val surfaces for the proof of the Katzarkov–Kontsevich–Pantev conjectures.

Proposition 6.35 (see [15], Proposition A.1.2). Suppose that $O$ is a du Val singular point of the surface $S$, both curves $C$ and $Z$ are smooth at $O$, and $C$ intersects $Z$...
transversally at \( O \). Then for the local intersection indices \((C \cdot Z)_O\) the following assertions hold.

(i) The point \( O \) is a singular point of \( S \) of type \( \mathbb{A}_n \) or \( \mathbb{D}_n \).

(ii) If \( O \) is a singular point of type \( \mathbb{A}_n \) and the strict transforms of \( C \) and \( Z \) on the minimal resolution \( \tilde{S} \) of \( O \) intersect the \( k \)th and \( r \)th exceptional curves in the chain of exceptional curves of the minimal resolution of \( O \), then

\[
(C \cdot Z)_O = \begin{cases} 
\frac{r(n+1-k)}{n+1} & \text{for } r \leq k, \\
\frac{k(n+1-r)}{n+1} & \text{for } r > k.
\end{cases}
\]

(iii) If \( O \) is of type \( \mathbb{D}_n \), then \((C \cdot Z)_O = 1/2\).

**Proposition 6.36** (see [15], Proposition A.1.3). Suppose that \( O \) is a du Val singular point of the surface \( S \), and the curve \( C \) is smooth at the point \( O \). Then the following hold.

(i) The point \( O \) is a singular point of the surface \( S \) of type \( \mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \) or \( \mathbb{E}_7 \).

(ii) If \( O \) is a singular point of type \( \mathbb{A}_n \), and the strict transform \( \tilde{C} \) of \( C \) intersects the \( k \)th exceptional curve in the chain of exceptional curves of the minimal resolution of \( O \), then

\[
C^2 = \tilde{C}^2 + \frac{k(n+1-k)}{n+1}.
\]

(iii) If \( O \) is a singular point of type \( \mathbb{D}_n \), then \( C^2 = \tilde{C}^2 + n/4 \) or \( C^2 = \tilde{C}^2 + 1 \).

(The second case occurs when the strict transform of \( C \) does not contain the singular point of the blowup of \( O \).)

(iv) If \( O \) is a singular point of type \( \mathbb{E}_6 \), then \( C^2 = \tilde{C}^2 + 4/3 \).

(v) If \( O \) is a singular point of type \( \mathbb{E}_7 \), then \( C^2 = \tilde{C}^2 + 3/2 \).

**Theorem 6.37** (see [15], Main Theorem). Conjecture 6.12 holds for smooth Fano threefolds.

**Idea of proof.** Consider a smooth Fano threefold \( X \). By Corollary 5.12 it has a toric Landau–Ginzburg model. If \(-K_X\) is very ample, then choose a model \( f(x, y, z)\) such that after a multiplication by \( xyz \) and the compactification given by the natural embedding \( \mathbb{A}[x, y, z] \hookrightarrow \mathbb{P}[x : y : z : t] \) we obtain a family of quartics \( \mathcal{S} \) defined by

\[
f_4(x, y, z, t) = \lambda xyzt, \quad \lambda \in \mathbb{C} \cup \{\infty\}.
\]

One can check that this is always possible. If \(-K_X\) is not very ample, then we compactify a toric Landau–Ginzburg model for \( X \) to a family of quartics \( \mathcal{S} \) using Proposition 5.11.

Now we resolve these families by blowing up the base locus and keeping track of the number of exceptional divisors lying in fibres. To do this we study singularities of fibres along the base locus. For instance, a ‘floating’ singularity (whose coordinates change as elements of the family vary) or an isolated du Val singularity for each fibre does not give a component of a fibre of the resolution. In the general case for each fibre of the family \( \mathcal{S}_\lambda \) one can define the defect \( D^\lambda_k \) of the singular point \( P \) as the number of exceptional divisors of the resolution lying in the fibre.
over the point, and the defect $C^\lambda$ of a base curve $C$ of the pencil in the fibre $\mathcal{S}_\lambda$. In particular, the defect of an isolated du Val singularity is 0.

The defects of curves can be computed in terms of the multiplicities of these curves in fibres. To compute the defects of points one needs a deeper analysis, involving counting the base curves lying over the point. For more details see [15], §1.

Denote the number of irreducible components of a variety $V$ by $[V]$. For a resolution $f: Y \to \mathbb{P}^1$ of the pencil $\mathcal{S}_\lambda$ we have

$$[f^{-1}(\lambda)] = [\mathcal{S}_\lambda] + \sum_{i=1}^r C_i^\lambda + \sum_{P \in \Sigma} D_P^\lambda,$$

where $\{C_1, \ldots, C_r\}$ is the set of base curves and $\Sigma$ is the set of points over which exceptional divisors lie. We denote the total space of the resolution by $Y$ since by Remark 5.10 it is isomorphic in codimension 1 to the log-Calabi–Yau compactification from Corollary 5.12. Taking the sum of the defects over all fibres, we find $k_Y$ and compare it with the number $h^1,2(\mathbb{X})$, which can be found, for instance, in [51].

Let $M$ be the $r \times r$ matrix with entries $M_{ij} \in \mathbb{Q}$ that are given by

$$M_{ij} = C_i \cdot C_j,$$

where $C_i \cdot C_j$ is the intersection of the curves $C_i$ and $C_j$ on the surface $S_\lambda$. One can easily show that for general $\lambda$

$$\dim \text{coker}(H^2(\mathbb{Z}, \mathbb{R}) \to H^2(\mathbb{V}, \mathbb{R})) - 2 = 22 - \text{rk} \text{Pic}(\mathcal{S}_\lambda / \mathcal{S}) - \text{rk}(M),$$

where $\mathcal{S}_\lambda$ is a minimal resolution. Since for a general $\lambda$ the surface $\tilde{S}_\lambda$ has du Val singularities, to find the relative Picard rank it suffices to find the types of these singularities.

The theorem can be proved by direct computations for each Fano threefold in the way outlined above. $\square$

Harder’s results and Conjecture 6.12 motivate the following. Consider a smooth Fano variety $X$ of dimension $N$ and let $Y$ be its $N$-dimensional Landau–Ginzburg model. Define $k_Y$ as before, as the difference between the number of irreducible components of reducible fibres of $Y$ and the number of reducible fibres.

**Conjecture 6.38** (see [77], Problem 27 and [82], Conjecture 1.1; see [41]). For a smooth Fano variety $X$ of dimension $N \geq 3$

$$h^{1,N-1}(\mathbb{X}) = k_Y.$$

Thus, Theorem 6.37 implies Conjecture 6.38 for threefolds. A proof of Conjecture 6.38 for complete intersections is given by Theorem 7.26.

Finally, by homological mirror symmetry the number of reducible fibres of a threefold Landau–Ginzburg model is expected not to be greater than the Picard rank of the corresponding Fano variety. In particular, the proof of Theorem 6.37 implies that in the Picard-rank-1 case one has at most one reducible fibre. It turns out that one can obtain important information from the monodromy at the reducible fibre. Namely, by comparing results of Iskovskikh [50] and Golyshev [40], and the compactified toric Landau–Ginzburg models constructed above, one can obtain the following.
Theorem 6.39 (see [57], Theorem 3.3). Let $X$ be a smooth Picard-rank-1 Fano threefold whose compactified Landau–Ginzburg model has a fibre with non-isolated singularities. Then the monodromy (in the second cohomology) at this fibre is unipotent if and only if $X$ is rational.

For another approach to (non-)rationality of Fano varieties via their Landau–Ginzburg models see [46].

7. Complete intersections in (weighted) projective spaces and Grassmannians

In this section we study (toric) Landau–Ginzburg models of smooth complete intersections in weighted projective spaces and Grassmannians.

We focus mainly on complete intersections in Grassmannians. Weighted complete intersections were studied in the preprint [88]. Here we just briefly present the main results.

First we describe Givental’s construction in [39] for Landau–Ginzburg models of Fano complete intersections in smooth toric varieties. We also describe their period integrals. We apply this construction to complete intersections, and its generalization to ‘good’ toric degenerations to del Pezzo surfaces (see §4) and, following [9], to complete intersections in Grassmannians (see §7.3).

7.1. Givental’s construction. Let $X$ be a factorial $N$-dimensional toric Fano variety of Picard rank $\rho$ corresponding to a fan $\Sigma_X$ in a lattice $\mathcal{N} \simeq \mathbb{Z}^N$. Let $D_1, \ldots, D_{N+\rho}$ be its prime invariant divisors. Let $Y_1, \ldots, Y_l$ be ample divisors in $X$ cutting out a smooth Fano complete intersection $Y$. Put

$$Y_0 = -K_X - Y_1 - \cdots - Y_k.$$ 

We choose a basis

$$\{H_1, \ldots, H_\rho\} \subset H^2(X)$$

consisting of nef divisors, and we introduce formal variables $q_1, \ldots, q_\rho$ as in §2.2. Define $\kappa_i$ by $-K_Y = \sum \kappa_i H_i$.

The following theorem is a particular case of the Quantum Lefschetz Hyperplane Theorem.

Theorem 7.1 (see [39], Theorem 0.1). Suppose that $\dim Y \geq 3$. Then the constant term of the regularized $I$-series for $Y$ is given by

$$\tilde{I}_0^Y(q_1, \ldots, q_\rho) = \exp(\mu(q)) \sum_{\beta \in K} q^{\beta} \prod_{j=1}^l \frac{\prod_{i=0}^{N+\rho} |\beta \cdot Y_i|!}{\prod_{j=1}^l (|\beta \cdot D_j|)!} |\beta \cdot D_j|^{-1},$$

(7.1)

where $\mu(q)$ is a correction term which is linear in the $q_i$ (and, in particular, is zero in case of index greater than 1). For $\dim Y = 2$ the same formula holds after replacing $H^2(Y)$ in the definition of $\tilde{I}_0^Y$ by the restriction of $H^2(X)$ to $Y$.

Remark 7.2. Note that the summands of the series (7.1) have non-negative degrees with respect to the $q_i$. 

Now we describe Givental’s construction of a Landau–Ginzburg model dual to $Y$ and compute its periods. We introduce $N + \rho$ formal variables $u_1, \ldots, u_{N + \rho}$ corresponding to the divisors $D_1, \ldots, D_{N + \rho}$.

Recall that the short exact sequence (2.1) identifies $\text{Pic}(X)^\vee$ with the lattice of relations on primitive vectors on the rays of $\Sigma_X$ considered as Laurent monomials in the variables $u_i$. On the other hand, since we have chosen the basis in $\text{Pic}(X)$, we can identify $\text{Pic}(X)^\vee$ and $\text{Pic}(X) = H^2(X)$. Hence, we can choose a basis in the lattice of relations on primitive vectors on the rays of $\Sigma_X$ corresponding to $\{H_i\}$ and thus to $\{q_i\}$. We denote these relations by $R_i$, and interpret them as monomials in the variables $u_1, \ldots, u_{N + \rho}$. We also denote the image of $D_i \in \mathcal{D}$ in $\text{Pic}X$ by $D_i$.

We choose a nef-partition, that is, a partition of the set $[1, N + \rho]$ into disjoint sets $E_0, \ldots, E_k$ such that for any $i \in [1, k]$ the divisor $\sum_{j \in E_i} D_j$ is linearly equivalent to $Y_i$ (which also implies that $\sum_{j \in E_0} D_j$ is linearly equivalent to $Y_0$).

The following definition is well known (see the discussion after Corollary 0.4 in [39], and also [44], §7.2).

**Definition 7.3.** *Givental’s Landau–Ginzburg model* for $Y$ is a variety $\text{LG}_0(Y)$ in a torus $\text{Spec} \mathbb{C}_q[u_1^{\pm 1}, \ldots, u_{N + \rho}^{\pm 1}]$ and is given by equations

$$R_i = q_i, \quad i \in [1, \rho],$$

and

$$\sum_{s \in E_j} u_s = 1, \quad j \in [1, k],$$

with superpotential $w = \sum_{s \in E_0} u_s$. Given a divisor $D \sim \sum r_i H_i \in \text{Pic}(Y)_{\mathbb{C}}$, we define a *Landau–Ginzburg model of Givental type* $\text{LG}(Y, D)$ corresponding to $(Y, D)$ by putting $q_i = \exp(r_i)$. Let $\text{LG}(Y) = \text{LG}(Y, 0)$.

**Remark 7.4.** The superpotential of Givental’s Landau–Ginzburg model can be defined by $w' = u_1 + \cdots + u_{N + \rho}$. However, we do not make a distinction between the two superpotentials $w$ and $w'$ since $w' = w + k$, and so both these functions define the same family over $\mathbb{C}_q$.

Given variables $x_1, \ldots, x_r$, we define the *standard logarithmic form in these variables* by

$$\Omega(x_1, \ldots, x_r) = \frac{1}{(2\pi i)^r} \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_r}{x_r}.$$

The following definition is well known (see the discussion after Corollary 0.4 in [39], and also [38]).

**Definition 7.5.** Fix $N + \rho$ positive real numbers $\varepsilon_1, \ldots, \varepsilon_{N + \rho}$ and define the $(N + \rho)$-cycle

$$\delta = \{|u_i = \varepsilon_i|\} \subset \mathbb{C}[u_1^{\pm 1}, \ldots, u_{N + \rho}^{\pm 1}].$$

*Givental’s integral* for $Y$ or $\text{LG}_0(Y)$ is the integral

$$I_Y^0 = \int_{\delta} \frac{\Omega(u_1, \ldots, u_{N + \rho})}{\prod_{i=1}^{\rho} (1 - q_i/R_i) \cdot \prod_{j=0}^{k} (1 - \sum_{s \in E_j} u_s)} \in \mathbb{C}[[q_1, \ldots, q_\rho]].$$

(7.3)
Given a divisor $D = \sum r_i H_i$, one can specialize Givental’s integral to the anticanonical direction and the divisor $D$ by putting $q_i = e^{r_i t^{e_i}}$ in the integral (7.3) (see Definition 2.4). We denote the result of the specialization by $I_{(Y,D)}$. Let $I_{(Y,0)} = I_Y$.

**Remark 7.6.** The integral (7.3) does not depend on the numbers $\varepsilon_i$ if they are small enough.

**Remark 7.7.** The integral (7.3) is defined up to sign, since we do not specify the order of the variables.

The following assertion is well known to experts (see [39], Theorem 0.1, and also the discussion after Corollary 0.4 in [39]).

**Theorem 7.8.** Up to sign (see Remark 7.7),

$$I_Y^0 = I_Y^0.$$ 

The recipe for Givental’s Landau–Ginzburg model and integral can be written down in another, simpler, way. That is, we make a suitable monomial change of variables $u_1, \ldots, u_{N+\rho}$ and eliminate some of them using (7.2). More precisely, since $\mathcal{N}$ is a free group, by using the exact sequence (2.1) one obtains an isomorphism

$$\mathcal{D} \simeq \text{Pic}(X)^{\vee} \oplus \mathcal{N}.$$ 

Thus, we can find a monomial change of variables $u_1, \ldots, u_{N+\rho}$ to some new variables $x_1, \ldots, x_N, y_1, \ldots, y_\rho$ such that

$$u_i = \bar{X}_i(x_1, \ldots, x_N, y_1, \ldots, y_\rho, q_1, \ldots, q_\rho)$$

and for any $i \in [1, \rho]$

$$\frac{R_i(u_1, \ldots, u_{N+\rho})}{q_i} = \frac{1}{y_i}.$$ 

Let

$$X_i = \bar{X}_i(x_1, \ldots, x_N, 1, \ldots, 1, q_1, \ldots, q_\rho).$$

Then LG$_0(Y)$ is given in the torus Spec $\mathbb{C}_q[x_1^\pm 1, \ldots, x_N^\pm 1]$ by the equations

$$\sum_{s \in E_j} \alpha_s X_s = 1, \quad j \in [1, k],$$

with superpotential $w = \sum_{s \in E_0} \alpha_s X_s$, where $\alpha_i = \prod q_j^{r_{i,j}}$ for some integers $r_{i,j}$.

We remark that for a Laurent monomial $U_i$ in the variables $u_j$, $j \in [1, N+\rho]$, that does not depend on a variable $u_i$ we have the equality

$$\Omega(u_1, \ldots, u_i^{\pm 1}, U_i, \ldots, u_{N+\rho}) = \pm \Omega(u_1, \ldots, u_i, \ldots, u_{N+\rho}).$$

This means that

$$I_Y^0 = \int_{\delta'} \pm \frac{\Omega(y_1, \ldots, y_\rho) \wedge \Omega(x_1, \ldots, x_N)}{\prod_{i=1}^\rho (1 - y_i) \cdot \prod_{j=0}^k (1 - \sum_{s \in E_j} \alpha_s \bar{X}_s)}$$ (7.4)

for some $(N+\rho)$-cycle $\delta'$. 
Consider an integral
\[ \int_{\sigma} \frac{dU}{U} \wedge \Omega_0 \]
for some form \( \Omega_0 \) and a cycle \( \sigma = \sigma' \cap \{|U| = \varepsilon \} \) for some cycle \( \sigma' \subset \{U = 0\} \). It is well known (see, for instance, [2], Theorem 1.1) that
\[ \frac{1}{2\pi i} \int_{\sigma} \frac{dU}{U} \wedge \Omega_0 = \int_{\sigma'} \Omega_0|_{U=0}, \]
if both integrals are well defined (in particular, if the form \( \Omega_0 \) does not have a pole along \( \{U = 0\} \)).

Let
\[ \Omega_0|_{U=0} = \text{Res}_U \left( \frac{dU}{U} \wedge \Omega_0 \right). \]

Taking the residues with respect to \( y_i \) of the form on the right-hand side of (7.4), we have
\[ \mathcal{I}^0_Y = \int_{\delta''} \frac{\pm \Omega(x_1, \ldots, x_N)}{\prod_{j=0}^{k-1} (1 - \sum_{s \in E_j} \alpha_s X_s)} = I_{(Y,D)} \]
for some \( N \)-cycle \( \delta'' \).

Moreover, we can introduce a new parameter \( t \) and the scale \( u_i \rightarrow tu_i \) for \( i \in E_0 \). Fix a divisor class \( D = \sum r_i H_i \). One can check that after a change of coordinates \( q_i = e^{r_i t} \kappa_i \) the original integral restricts to the integral
\[ \int_{\delta_1} \frac{\pm \Omega(x_1, \ldots, x_N)}{\prod_{j=1}^{k} (1 - \sum_{s \in E_j} \gamma_s X_s) \cdot (1 - t \sum_{i \in E_0} \gamma_i X_i)} = I_{(Y,D)} \]
for some monomials \( \gamma_i \) and an \( N \)-cycle \( \delta_1 \) homologous to a cycle
\[ \delta_1^0 = \{ |x_i| = \varepsilon_i \mid i \in [1, N] \}. \]

In particular, for \( D = 0 \) we have \( \gamma_i = 1 \). The same specialization defines the Landau–Ginzburg model \( \text{LG}(Y) \):
\[ \sum_{s \in E_j} X_s = 1, \quad j \in [1, k], \tag{7.6} \]
with superpotential \( w = \sum_{s \in E_0} X_s \).

We consider a non-toric variety \( X \) that has a small (that is, terminal Gorenstein) toric degeneration \( T \). Let \( Y \) be a Fano complete intersection in \( X \). Consider a nef-partition for the set of rays of the fan of \( T \) corresponding to (degenerations of) hypersurfaces cutting out \( Y \). Let \( \text{LG}(Y) \) be the result of applying the Givental construction for \( T \) and the nef-partition in the same way as in the case of complete intersections in smooth toric varieties. Batyrev in [7] proposed \( \text{LG}(Y) \) as a Landau–Ginzburg model for \( Y \). Moreover, at least in some cases such as complete intersections in Grassmannians (see §7.3.2), Givental’s integral and Landau–Ginzburg model can be simplified further by making birational changes of variables and taking residues. This gives weak Landau–Ginzburg models for
complete intersections in projective spaces (see §7.2.2) and, more generally, Grassmannians (see §7.3).

We also generalize the model (7.6) for smooth complete intersections in weighted projective spaces (see §7.2.2). Such a complete intersection can be described as a complete intersection in a smooth toric variety after a resolution of singularities that are far away from the complete intersection. However, this description is equivalent to applying the construction (7.6) directly (see [73]).

7.2. Weighted complete intersections. In this section we apply constructions in §7.1 to complete intersections in weighted projective spaces. See [88] for more details.

7.2.1. Nef-partitions. A crucial ingredient of Givental’s construction from §7.1 for complete intersections in toric varieties and its generalization is the existence of nef-partitions for such complete intersections. Obviously, such nef-partitions exist for complete intersections in projective spaces. However, in general the existence of such nef-partitions is not guaranteed. From the classification point of view the most interesting Fano varieties are those with Picard group \( \mathbb{Z} \). If a complete intersection in a toric variety admits a nef-partition, then the ambient toric variety is a weighted projective space (or its quotient if the complete intersection is singular). In general the existence of a nef-partition for a complete intersection in a weighted projective space is expected but not proven.

**Conjecture 7.9.** A smooth well-formed weighted complete intersection has a good nef-partition and a toric Landau–Ginzburg model (for definitions see below).

**Remark 7.10.** The existence of a good (see Definition 7.15) nef-partition implies the existence of a weak Landau–Ginzburg model (see §7.2.2) satisfying the toric condition (see §7.2.4). By analogy with Theorem 7.22, in many cases one can verify the Calabi–Yau condition, which will show that the Landau–Ginzburg model is toric. The main problem in showing this is that in general the Newton polytope of the weak Landau–Ginzburg model is not reflexive.

Let \((a_1, \ldots, a_r)\) denote the greatest common divisor of the numbers \(a_1, \ldots, a_r \in \mathbb{N}\).

We recall some facts about weighted projective spaces (see [28] for more details). Consider a weighted projective space \(\mathbb{P} = \mathbb{P}(w_0, \ldots, w_N)\).

**Definition 7.11** (see [45], Definition 5.11). The weighted projective space \(\mathbb{P}\) is said to be well formed if the greatest common divisor of any \(N\) of the weights \(w_i\) is 1.

Any weighted projective space is isomorphic to a well-formed one (see [28], 1.3.1).

**Lemma 7.12** (see [45], 5.15). The singular locus of \(\mathbb{P}\) is a union of strata

\[ \Lambda_J = \{(x_0 : \cdots : x_N) \mid x_j = 0 \text{ for all } j \notin J\} \]

for all subsets \(J \subset [0, N]\) such that the greatest common divisor of the weights \(a_j\) for \(j \in J\) is greater than 1.

**Definition 7.13** (see [45], Definition 6.9). A subvariety \(X \subset \mathbb{P}\) of codimension \(k\) is said to be well formed if \(\mathbb{P}\) is well formed and

\[ \text{codim}_X(X \cap \text{Sing} \mathbb{P}) \geq 2. \]
Definition 7.14. The zeros of a (weighted) homogenous polynomial

\[ f \in \mathbb{C}[x_0, \ldots, x_N] \]

of weighted degree \(d\), where \(\text{wt}(x_i) = w_i\), are called a degree-\(d\) hypersurface in \(\mathbb{P}\).

The rank of the divisor class group of a weighted projective space is 1, so some multiple of any effective Weil divisor is equal to the zeros of some weighted homogenous polynomial. This enables us to define the degree of any Weil divisor. It is easy to see that a Weil divisor of degree \(d\) is Cartier if and only if all weights \(w_i\) divide \(d\).

The singularities of a general complete intersection \(X = X_1 \cap \cdots \cap X_k\) of Cartier divisors \(X_1, \ldots, X_k\) are the intersection of \(X\) with the singularities of \(\mathbb{P}\). Thus, \(X\) is smooth if and only if the greatest codimension of strata of singularities of \(\mathbb{P}\) is less than \(k\). This means that \((w_{i_1}, \ldots, w_{i_{k+1}}) = 1\) for any collection of weights \(w_{i_1}, \ldots, w_{i_{k+1}}\) (see [27]).

Let \(\deg X_i = d_i\). A canonical sheaf of \(X\) is

\[ \mathcal{O}(d_1 + \cdots + d_k - w_0 - \cdots - w_N)|_X. \]

Therefore, \(X\) is Fano if and only if \(\sum d_i < \sum w_j\).

Definition 7.15. Let \(X\) be a smooth complete intersection of divisors of degrees \(d_1, \ldots, d_k\) in a well-formed weighted projective space \(\mathbb{P}(w_0, \ldots, w_N)\). Recall that a partition of \([0, N]\) into \(k\) non-intersecting subsects \(E_0, \ldots, E_k \subset [0, n]\) such that \(d_i = \sum_{j \in E_i} w_j\) for all \(i > 0\), is called a nef-partition. A nef-partition is said to be good if there is an index \(j \in E_0 = [0, N] \setminus (E_1 \cup \cdots \cup E_k)\) such that \(w_j = 1\). A good nef-partition is said to be very good if \(w_j = 1\) for all \(j \in E_0\).

Proposition 7.16 (see [76], Theorem 9 and Remark 14). Let \(X\) be a smooth complete intersection of Cartier divisors in a well-formed weighted projective space. Let \(X\) be Fano. Then it admits a very good nef-partition.

Remark 7.17. Denote the Fano index of a variety \(X\) by \(d_0 = \sum w_i - \sum d_j\). The proof of Proposition 7.16 shows that at least \(d_0 + 1\) weights are equal to 1. This bound is strict: an example is a hypersurface of degree 6 in \(\mathbb{P}(1, 1, 2, 3)\).

Conjecture 7.9 does not only hold for complete intersections of Cartier divisors.

Theorem 7.18 (see [85], Theorem 1.3). A smooth well-formed Fano complete intersection of codimension 2 admits a very good nef-partition.

Idea of proof. One needs to study the so-called weighted projective graphs, that is, graphs whose vertices are marked by weights of the weighted projective space, and whose edges connect precisely those vertices whose marks have a non-trivial common divisor. □

If \(X\) is a smooth well-formed Calabi–Yau weighted complete intersection of codimension 1 or 2, then we can argue in the same way as in the proof of Proposition 7.16 and Theorem 7.18 to show that there exists a nef-partition for \(X\) such that \(E_0 = \emptyset\) in the notation of Definition 7.15. Constructing the dual nef-partition, we obtain a Calabi–Yau variety \(Y\) that is mirror dual to \(X\) (see [8]). In the same paper it
is proved that the Hodge mirror symmetry holds for $X$ and $Y$. That is, given a variety $V$ one can define the string-theoretic Hodge numbers $h^{p,q}_{st}(V)$ to be the Hodge numbers of a crepant resolution of $V$ if such a resolution exists. Then $h^{p,q}_{st}(X) = h^{n-p,q}_{st}(Y)$ for $n = \dim X = \dim Y$, provided that the ambient toric variety (weighted projective space in our case) is Gorenstein.

Finally, we would like to point out a possible approach to a proof of Conjecture 7.9 along the lines of Theorem 7.18. If $X$ is a smooth well-formed Fano weighted complete intersection of codimension 3 or more in a weighted projective space $\mathbb{P} = \mathbb{P}(w_0, \ldots, w_N)$, then it is possible that some three weights $w_{i_1}$, $w_{i_2}$, and $w_{i_3}$ are not coprime. Thus, the weighted projective graph constructed in the proof of Theorem 7.18 does not provide an adequate description of the singularities of the weighted projective space $\mathbb{P}$. An obvious way to (try to) cope with this is to replace the graph by a simplicial complex that would remember the greatest common divisors of arbitrary subsets of weights. However, this leads to combinatorial difficulties that we cannot overcome at the moment. Apart from the most straightforward problems, such as the effects on weak vertices (which would not be that easy to control) and a possibly larger number of exceptions, there is also a less obvious one (which is in fact easy to deal with). Namely, we need finer information about weights and degrees than that provided by Lemma 2.15 in [84].

Example 7.19. Let $X$ be a weighted complete intersection of hypersurfaces of degrees 2, 3, 5, and 30 in $\mathbb{P}(1^k, 6, 10, 15)$, where $1^k$ stands for 1 repeated $k$ times. Then $X$ is a well-formed Fano weighted complete intersection if $k$ is large and $X$ is general. Note that the conclusion of Lemma 2.15 in [84] holds for $X$. However, it is easy to see that $X$ is not smooth. Moreover, there is no nef-partition for $X$.

In any case, it is easy to see that the actual information one can deduce from the fact that a weighted complete intersection is smooth is much stronger than that provided by Lemma 2.15 in [84]. We also expect that the combinatorial difficulties that one has to face on the way to the proof of Conjecture 7.9 proposed in the proof of Theorem 7.18 are possible to overcome.

7.2.2. Weak Landau–Ginzburg models. Consider a general complete intersection $Y \subset \mathbb{P}[w_0, \ldots, w_N]$ of hypersurfaces of degrees $d_1, \ldots, d_k$.

Let

$$d_0 = \sum w_i - \sum d_j,$$

and let $d_0 \geq 1$, that is, let $Y$ be Fano. Assume the existence of a nef-partition $E_0, \ldots, E_k$ for $Y$. Let $a_{i,1}, \ldots, a_{i,r_i}$ be variables that correspond to the indices in $E_i$. Givental’s Landau–Ginzburg model for $Y$ and the trivial divisor is given in the torus

$$(\mathbb{C}^*)^N \simeq \mathcal{T}[a_{i,j}], \quad i \in [1, k], \quad j \in [1, i_r],$$

by the equations

$$a_{i,1} + \cdots + a_{i,r_i} = 1, \quad i \in [1, k], \quad (7.7)$$

and the superpotential $w = \sum a_{0,j}$. The variety given by (7.7), after the change of variables

$$x_{i,j} = \frac{a_{i,j}}{\sum_s a_{i,s}}, \quad a_{i,r_i} = 1, \quad i \in [1, k],$$

Givental’s Landau–Ginzburg model for $Y$ and the trivial divisor is given in the torus

$$(\mathbb{C}^*)^N \simeq \mathcal{T}[a_{i,j}], \quad i \in [1, k], \quad j \in [1, i_r],$$

by the equations

$$a_{i,1} + \cdots + a_{i,r_i} = 1, \quad i \in [1, k], \quad (7.7)$$

and the superpotential $w = \sum a_{0,j}$. The variety given by (7.7), after the change of variables

$$x_{i,j} = \frac{a_{i,j}}{\sum_s a_{i,s}}, \quad a_{i,r_i} = 1, \quad i \in [1, k],$$
is birational to the torus

$$(\mathbb{C}^*)^m \simeq \mathcal{T}[x_{i,j}], \quad i \in [0, k], \quad j \in [1, r_i - 1].$$

The superpotential $w$ in the new variables is given by the Laurent polynomial

$$f_Y = \frac{\prod_{i=1}^{k} (x_{i,1} + \cdots + x_{i,r_i-1} + 1)^{d_i}}{\prod_{i=0}^{k} \prod_{j=1}^{d_i-1} x_{i,j}} + x_{0,1} + \cdots + x_{0,r_0-1}.$$  \hspace{1cm} (7.8)

The formula (7.1) enables one to find the constant term of the regularized $I$-series for $Y$ easily and to compare it with the constant term series for $f_Y$. The formula for this series can easily be found combinatorially. One can check that the period condition holds for $f_Y$, that is, it is a weak Landau–Ginzburg model for $Y$. However, one can prove that these series coincide with Givental’s integral.

**Proposition 7.20** (see [86], Proposition 10.4). The following holds:

$$I_Y = \int_{|x_{i,j}|=\varepsilon_{i,j}} \frac{\Omega(x_{0,1}, \ldots, x_{k,d_k-1})}{1 - tf_Y}.$$  

**Idea of proof.** Use changes of variables and the residue theorem. \square

Thus, smooth Fano complete intersections with a good nef-partition have a weak Landau–Ginzburg model as well.

**Remark 7.21.** It seems natural to consider Givental’s Landau–Ginzburg models for quasi-smooth Fano complete intersections. However, even a quasi-smooth Cartier hypersurface does not always admit such a model. An example is a hypersurface of degree 30 in $\mathbb{P}(1, 6, 10, 15)$. Moreover, even if such a hypersurface has a Givental type Landau–Ginzburg model, it is not always presentable by a weak Landau–Ginzburg model as above. An example is the hypersurface of degree 30 in $\mathbb{P}(1, 1, 1, 1, 1, 6, 10, 15)$.

7.2.3. **Calabi–Yau compactifications.** The method of constructing log-Calabi–Yau compactifications used in Theorem 5.9 can be generalized to higher dimensions. That is, this can be done if the coefficients of the weak Landau–Ginzburg models of Givental type guarantee that the base locus of the pencil of hypersurfaces in the toric variety we compactify in is a union of components corresponding to linear sections. These components can be singular in the case of complete intersections, however, singularities ‘come from the ambient space’ and can be resolved under a crepant resolution of the toric variety we compactify in. This proves that the Calabi–Yau principle holds for weighted complete intersections. However, this works only if the Newton polytope of the weak Landau–Ginzburg model is reflexive. This always holds for the usual complete intersections but rarely holds for weighted ones.
Consider the matrix

$$
M_{d_1, \ldots, d_k; i_Y} =
\begin{pmatrix}
  i_Y & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & -1 & \ldots & -1 \\
  0 & i_Y & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & -1 & \ldots & -1 \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots \\
  0 & 0 & \ldots & i_Y & \ldots & 0 & 0 & \ldots & 0 & -1 & \ldots & -1 \\
  -i_Y & -i_Y & \ldots & -i_Y & \ldots & -i_Y & 0 & \ldots & 0 & -1 & \ldots & -1 \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots \\
  0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & i_Y & -1 & \ldots & -1 \\
  0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & i_Y -1 \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots \\
  0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\end{pmatrix},
$$

depending on positive integers $d_1, \ldots, d_k, i_Y$, which is formed from $k$ blocks of sizes $(d_i - 1) \times d_i$ and one last block of size $i_Y \times i_Y$. Define $k_{d_1, \ldots, d_k; i_Y}$ to be the number one less than the number of integral points in the convex hull of rays of rows of the matrix.

**Theorem 7.22** (see [79], Theorem 1). Let $Y \subset \mathbb{P}^N$ be a Fano complete intersection of hypersurfaces of degrees $d_1, \ldots, d_k$. Let

$$
i_Y = N + 1 - \sum d_i.
$$

Let $f_Y$ be a toric Landau–Ginzburg model of Givental type for $Y$. Then $f_Y$ admits a log-Calabi–Yau compactification $f_Y : Z \to \mathbb{P}^1$ such that $f_Y^{-1}(\infty)$ is a reduced divisor which is a union of smooth rational varieties. It consists of $k_{d_1, \ldots, d_k; i_Y}$ components and is combinatorially given by a triangulation of a sphere.

**Idea of proof.** This is similar to the proof of Theorem 5.9. □

**Problem 7.23.** Find a formula for $k_{d_1, \ldots, d_k; i_Y}$ in terms of $d_1, \ldots, d_k, i_Y$.

**Question 7.24.** By Corollary 5.17 and Theorem 7.22, the fibres over infinity of log-Calabi–Yau compactifications of toric Landau–Ginzburg models for Fano threefolds and complete intersections are reduced and are combinatorially given by triangulations of spheres. Does this hold in the general case?

**Remark 7.25** (see Remark 5.14). Let $T$ be a smooth toric variety with $F(T) = \Delta$. Let $f$ be a general Laurent polynomial with $N(f) = \Delta$. The Laurent polynomial $f$ is a toric Landau–Ginzburg model for a pair $(T, D)$, where $D$ is a general divisor on $\widetilde{T}$. Indeed, the period condition for it is satisfied by [39]. Following the compactification procedure from Theorem 7.22, one can see that the base locus $B$ is a union of smooth transversally intersecting subvarieties of codimension 2 (not necessarily rational). This means that in the same way as above, $f$ satisfies the Calabi–Yau condition. Finally, the toric condition holds for $f$ tautologically. Thus, $f$ is a toric Landau–Ginzburg model for $(T, D)$.

In [82] Calabi–Yau compactifications for Fano complete intersections in the usual projective spaces were constructed in an approach different from that in Theorem 7.22. The method used in [82] enables one to follow the number of reducible components of fibres of the compactification.
Theorem 7.26 (see [82], Theorem 1.2). Conjecture 6.38 holds for Fano complete intersections.

7.2.4. Toric Landau–Ginzburg models. The toric variety given by a polytope dual to a Newton polytope of a toric Landau–Ginzburg model enables one to show that Landau–Ginzburg models for weighted complete intersections satisfy the toric condition. Here, unlike in Theorem 7.22, we do not need integrality of the polytope. Recall that Facts 2.7 and 2.9 enable one to define by equations the toric variety whose fan polytope is the Newton polytope for the polynomial (7.8). Recall also that these equations are homogenous relations on the integral points of the Newton polytope. The shape of the polynomial shows that the polytope is given by ‘triangles’, so the relations are of Veronese type. In other words, the toric degeneration corresponding to the polynomial (7.8) is the image by a Veronese map of the complete intersection

\[
\begin{align*}
&z_{1,1} \cdots z_{1,r_1} = z_{0,1}^{d_1}, \\
&\cdots \cdots \cdots \\
&z_{k,1} \cdots z_{k,r_k} = z_{0,1}^{d_k}
\end{align*}
\]

in \( \mathbb{P}[z_{i,j}] \), \( i \in [0,k] \), \( j \in [1,r_i] \), where the weights of the \( z_{i,j} \) correspond to elements of \( E_i \) and the weight of \( z_{0,1} \) is 1.

Thus, the following theorem holds.

Theorem 7.27 (see [48], Theorem 2.2). There exists a flat degeneration of \( X \) to the toric variety \( T_{N(f_X)} \).

Example 7.28 (the del Pezzo surface \( S_2 \) of degree 2). We now consider the example of a del Pezzo surface of degree 2 and a description of its degeneration via generators and relations (see Remark 4.5). This is a hypersurface of degree 4 in \( \mathbb{P}(1,1,1,2) \). Its weak Landau–Ginzburg model is

\[
f_{S_2} = \frac{(x + y + 1)^4}{xy}.
\]

The corresponding Newton polytope \( \Delta_{f_{S_2}} \) has vertices equal to the columns of the matrix

\[
\begin{pmatrix}
3 & -1 & -1 \\
-1 & 3 & -1
\end{pmatrix}.
\]

The dual polytope \( \nabla_{f_{S_2}} = \Delta_{f_{S_2}}^\vee \) thus has vertices equal to the columns of the matrix

\[
\begin{pmatrix}
1 & 0 & -1/2 \\
0 & 1 & -1/2
\end{pmatrix}.
\]

This is not a lattice polytope (so the polygon \( \Delta_{f_{S_2}} \) is not reflexive). However, its double dilation \( \nabla_{f_{S_2}}^2 = 2\nabla_{f_{S_2}} \) is in fact integral. The integral points of \( \nabla \) are

\[
u = (-1, -1) \quad \text{and} \quad v_{ab} = (a, b) \quad \text{for} \ a, b \geq 0, a + b \leq 2.
\]

These correspond to generators for the homogeneous coordinate ring of the toric degeneration \( T \) in this (double-anticanonical) embedding.
Affine homogeneous relations between these lattice points correspond to binomial relations in the ideal of $T$. In that case these relations are generated by the five 2-Veronese type relations

$$v_{20} + v_{02} = 2v_{11}, \quad v_{20} + v_{01} = v_{10} + v_{11},$$
$$v_{20} + v_{00} = 2v_{10}, \quad v_{02} + v_{10} = v_{01} + v_{11},$$
$$v_{02} + v_{00} = 2v_{01}$$

together with the relation

$$u + v_{11} = 2v_{00}.$$

On the other hand, consider the 2-Veronese embedding of $\{x_0 x_1 x_2 = y_0^4\} \subset \mathbb{P}(1, 1, 2, 1)$. In the coordinates

$$z_{02} = x_0^2, \quad z_{20} = x_1^2, \quad w = x_2, \quad z_{00} = y_0^2,$$
$$z_{11} = x_0 x_1, \quad z_{01} = x_0 y_0, \quad z_{10} = x_1 y_0$$

this hypersurface is given by

$$w z_{11} = z_{00}^2$$

together with the five 2-Veronese-type equations

$$z_{20} z_{02} = z_{11}^2, \quad z_{20} z_{01} = z_{10} z_{11},$$
$$z_{20} z_{00} = z_{10}^2, \quad z_{02} z_{10} = z_{01} z_{11},$$
$$z_{02} z_{00} = z_{01}^2.$$

These correspond to the affine homogeneous relations above, so we can in fact realize our $T$ as the hypersurface $\{x_0 x_1 x_2 = y_0^4\} \subset \mathbb{P}(1, 1, 2, 1)$. Thus, by degenerating the equation defining $S_2$, we obtain a degeneration of the del Pezzo surface of degree 2 to $T$.

Proposition 7.20, Theorem 7.22, and Theorem 7.27 imply the following.

**Corollary 7.29.** Smooth Fano complete intersections have toric Landau–Ginzburg models.

7.2.5. **Boundedness of complete intersections.** In the previous subsections we discussed toric Landau–Ginzburg models for weighted complete intersections. One can easily bound the number of usual complete intersections of given dimension. It turns out that the number of weighted complete intersections is also bounded.

That is, the following statement is a combination of [84], Theorem 1.1, [16], Theorem 1.3, and [69], Corollary 5.3, (i).

**Theorem 7.30** (see [87], Theorem 2.4). Let $X$ be a smooth well-formed Fano complete intersection in the weighted projective space $\mathbb{P} = \mathbb{P}(w_0, \ldots, w_N)$ which is not a section of a linear cone (in other words, all degrees defining the complete intersection differ from the weights of $\mathbb{P}$). Let $k$ be the codimension of $X$ in $\mathbb{P}$, let $n = N - k = \dim X$, and let $l$ be the number of weights equal to 1 among the $w_i$. Then

(i) $w_N \leq N$; \quad (ii) $k \leq n$; \quad (iii) $l \geq k$. 

In particular, this theorem implies the following.

**Proposition 7.31** (see [85], §5). Smooth Fano weighted complete intersections of dimension at most 5 have very good nef-partitions. In particular, they have weak Landau–Ginzburg models satisfying the toric condition.

Thus, the discussion above implies the following.

**Theorem 7.32.** Let $X$ be a smooth complete intersection in a well-formed weighted projective space such that either $X$ is a complete intersection of Cartier divisors, or it is of codimension 2, or its dimension is not greater than 5. Then $X$ has a weak Landau–Ginzburg model satisfying the toric condition.

**Proof.** By Proposition 7.16, Theorem 7.18, or Proposition 7.31, the variety $X$ has a very good nef-partition. Therefore, applying the change of variables in §7.2.2, we obtain a Laurent polynomial of type (7.8). A standard combinatorial count (or a straightforward generalization of Proposition 7.20 to weighted projective spaces) shows that these polynomials satisfy the period condition. Moreover, by Theorem 7.27 they satisfy the toric condition as well. $\square$

**Question 7.33.** Many varieties have several different nef-partitions. In [62] (see also [70]) it is shown that under some mild conditions Givental’s Landau–Ginzburg models for complete intersections in Gorenstein toric varieties corresponding to different nef-partitions are birational. Is this true for smooth weighted complete intersections?

### 7.3. Complete intersections in Grassmannians.

It turns out that Givental’s constructions can be applied not only to complete intersections in smooth toric varieties, but also to complete intersections in varieties admitting ‘good’ toric degenerations. In this section, following [83], we use such degenerations for Grassmannians $\text{Gr}(n, k+n)$, $k, n \geq 2$, and construct weak Landau–Ginzburg models for complete intersections in them. We will use the constructions in [9] and [10] for Landau–Ginzburg models that are analogous to those of Givental (see also (B.25) in [34]). Following [83], we show that they can be represented by weak Landau–Ginzburg models. For other methods of representing them as weak Landau–Ginzburg models, see [86] (see also [81] and [71]).

**7.3.1. Construction.** We define a quiver $\mathcal{Q}$ as a set of vertices

$$\text{Ver}(\mathcal{Q}) = \{(i, j) \mid i \in [1, k], j \in [1, n]\} \cup \{(0, 1), (k, n+1)\}$$

and a set of arrows $\text{Ar}(\mathcal{Q})$ described as follows. All arrows are either *vertical* or *horizontal*. For any $i \in [1, k-1]$ and $j \in [1, n]$ there is one vertical arrow

$$v_{i,j} = (i, j) \rightarrow (i+1, j),$$

that goes from the vertex $(i, j)$ down to the vertex $(i+1, j)$. For any $i \in [1, k]$ and any $j \in [1, n-1]$ there is one horizontal arrow

$$h_{i,j} = (i, j) \rightarrow (i, j+1),$$

that goes from the vertex $(i, j)$ to the vertex $(i, j+1)$.
that goes from the vertex \((i, j)\) to the right to the vertex \((i, j + 1)\). We also add an extra vertical arrow
\[
\mathbf{v}_{0,1} = \langle (0, 1) \to (1, 1) \rangle
\]
and an extra horizontal arrow
\[
\mathbf{h}_{k,n} = \langle (k, n) \to (k, n + 1) \rangle
\]
to \(\text{Ar}(\mathcal{Q})\) (see Fig. 2 for an example of such a quiver).

Figure 2. Quiver \(\mathcal{Q}\) for the Grassmannian \(\text{Gr}(3, 6)\).

For any arrow
\[
\alpha = \langle (i, j) \to (i', j') \rangle \in \text{Ar}(\mathcal{Q})
\]
we define its tail \(t(\alpha)\) and its head \(h(\alpha)\) to be the vertices \((i, j)\) and \((i', j')\), respectively.

For \(r, s \in [0, k]\), \(r < s\), we define the horizontal block \(\text{HB}(r, s)\) to be the set of all vertical arrows \(\mathbf{v}_{i,j}\) with \(i \in [r, s - 1]\). For example, the horizontal block \(\text{HB}(0, 1)\) consists of the single arrow \(\mathbf{v}_{0,1}\), while the horizontal block \(\text{HB}(1, 3)\) consists of all arrows \(\mathbf{v}_{1,j}\) and \(\mathbf{v}_{2,j}\), \(j \in [1, n]\). Similarly, for \(r, s \in [1, n + 1]\), \(r < s\), we define the vertical block \(\text{VB}(r, s)\) to be the set of all horizontal arrows \(\mathbf{h}_{i,j}\) with \(j \in [r, s - 1]\). Finally, for \(r \in [0, k]\) and \(s \in [1, n + 1]\) we define the mixed block
\[
\text{MB}(r, s) = \text{HB}(r, k) \cup \text{VB}(1, s).
\]

For example, the mixed block \(\text{MB}(0, n)\) consists of all arrows in \(\text{Ar}(\mathcal{Q})\) except for \(\mathbf{h}_{k,n}\). When we speak about a block, we mean either a horizontal, or a vertical, or a mixed block. We say that the size of a horizontal block \(\text{HB}(r, s)\) or a vertical block \(\text{VB}(r, s)\) is \(s - r\), and the size of a mixed block \(\text{MB}(r, s)\) is \(s + k - r\).

Let \(B_1, \ldots, B_l\) be blocks. We say that they are consecutive if the arrow \(\mathbf{v}_{0,1}\) is contained in \(B_1\), and for any \(p \in [1, l]\) the union \(B_1 \cup \cdots \cup B_p\) is a block. This happens in only one of the following two situations: either there exist an index \(p_0 \in [1, l]\) and sequences of integers
\[
0 < r_1 < \cdots < r_{p_0} = k \quad \text{and} \quad 0 < r'_1 < \cdots < r'_{l-p_0} \leq n + 1
\]
such that
\[ B_1 = HB(0, r_1), \quad B_2 = HB(r_1, r_2), \quad \ldots, \quad B_{p_0} = HB(r_{p_0-1}, r_{p_0}), \]
\[ B_{p_0+1} = VB(0, r'_1), \quad \ldots, \quad B_l = VB(r'_l-p_0-1, r'_l-p_0), \]
or there exist an index \( p_0 \in [1, l] \) and sequences of integers
\[ 0 < r_1 < \cdots < r_{p_0-1} < k \quad \text{and} \quad 0 < r'_1 < \cdots < r'_{l-p_0-1} \leq n+1 \]
such that
\[ B_1 = HB(0, r_1), \quad B_2 = HB(r_1, r_2), \quad \ldots, \quad B_{p_0-1} = HB(r_{p_0-2}, r_{p_0-1}), \]
\[ B_{p_0} = MB(r_{p_0}, r'_1), \quad B_{p_0+1} = VB(r'_1, r'_2), \quad \ldots, \quad B_l = VB(r'_l-p_0-2, r'_l-p_0-1). \]

The first case occurs when there are no mixed blocks among \( B_1, \ldots, B_l \), and the second case occurs when one of the blocks is mixed.

Let \( S = \{x_1, \ldots, x_N\} \) be a finite set. We introduce a set of variables \( \tilde{V} = \{\tilde{a}_{i,j} \mid i \in [1, k], j \in [1, n]\} \). It is convenient to think that the variable \( \tilde{a}_{i,j} \) is associated with a vertex \((i, j)\) of the quiver \( \mathcal{D} \). Laurent polynomials in the variables \( \tilde{a}_{i,j} \) are regular functions on the torus \( \mathcal{T}(\tilde{V}) \). We also put \( \tilde{a}_{0,1} = \tilde{a}_{k,n+1} = 1 \).

For any subset \( A \subset \text{Ar}(\mathcal{D}) \) we define the regular function
\[ \tilde{F}_A = \sum_{\alpha \in A} \frac{\tilde{a}_{h(\alpha)}}{\tilde{a}_{t(\alpha)}} \]
on the torus \( \mathcal{T}(\tilde{V}) \).

Let \( Y \) be a complete intersection of hypersurfaces of degrees \( d_1, \ldots, d_l \) in \( \text{Gr}(k, n+k) \), \( \sum d_i < n+k \). Consider consecutive blocks \( B_1, \ldots, B_l \) of sizes \( d_1, \ldots, d_l \), respectively, and put
\[ B_0 = \text{Ar}(\mathcal{D}) \setminus (B_1 \cup \cdots \cup B_l). \]

Let \( \tilde{L} \subset \mathcal{T}(\tilde{V}) \) be the subvariety defined by the equations
\[ \tilde{F}_{B_1} = \cdots = \tilde{F}_{B_l} = 1. \]

In [9] and [10] it was conjectured that a Landau–Ginzburg model for \( Y \) is given by the variety \( \tilde{L} \) with superpotential given by the function \( \tilde{F}_{B_0} \). We call it a model of type BCFKS.

The main result of this subsection is the following.

**Theorem 7.34** (see [83], Theorem 2.2). The subvariety \( \tilde{L} \) is birational to a torus \( \mathcal{Y} \sim (\mathbb{C}^\ast)^{nk-l} \), and the birational equivalence \( \tilde{\tau} : \mathcal{Y} \dashrightarrow \tilde{L} \) can be chosen so that \( \tilde{\tau}^* (\tilde{F}_{B_0}) \) is a regular function on \( \mathcal{Y} \). In particular, this function is given by a Laurent polynomial.

**Remark 7.35.** The Laurent polynomial provided by Theorem 7.34 may change significantly if one takes the degrees \( d_1, \ldots, d_l \) in a different order (see Examples 7.48 and 7.49).
To prove Theorem 7.34 we will use slightly more convenient coordinates than \( \tilde{a}_{i,j} \).

We make a monomial change of variables \( \psi: \mathcal{V}(V) \to \mathcal{V}(V) \) defined by

\[
a_{i,j} = \tilde{a}_{i,j} \tilde{a}_{k,n}, \quad a = \tilde{a}_{k,n}
\]

and put

\[
V = \{ a_{i,j} \mid i \in [1, k], j \in [1, n], (i, j) \neq (k, n) \} \cup \{ a \}.
\]

Let \( a_{k,n} = 1 \) and \( a_{0,1} = a_{k,n+1} = a \) for convenience. As above, for any subset \( A \subset \text{Ar}(Q) \) we define the regular function

\[
F_A = \sum_{\alpha \in A} \frac{a_{h(\alpha)}}{a_{t(\alpha)}}
\]

on the torus \( \mathcal{V}(V) \). Let \( L \subset \mathcal{V}(V) \) be the subvariety defined by the equations

\[
F_{B_1} = \cdots = F_{B_l} = 1.
\]

We are going to verify that the subvariety \( L \) is birational to a torus \( \mathcal{Y} \simeq (\mathbb{C}^*)^{nk-l} \), and that the birational equivalence \( \tau: \mathcal{Y} \dashrightarrow L \) can be chosen so that the pull-back of \( F_{B_0} \) is a regular function on \( \mathcal{Y} \). Obviously, the latter assertion is equivalent to Theorem 7.34.

The following assertion is well known and easy to check.

**Lemma 7.36.** Let \( X \) be a variety with a free action of a torus \( \mathcal{T} \). Put \( Y = X/\mathcal{T} \), and let \( \varphi: X \to Y \) be the natural projection. Suppose that \( \varphi \) has a section \( \sigma: Y \to X \). Then there is an isomorphism

\[
\xi: X \xrightarrow{\sim} \mathcal{T} \times Y.
\]

Moreover, suppose that a function \( F \in H^0(X, \mathcal{O}_X) \) is semi-invariant with respect to the \( \mathcal{T} \)-action, that is, there is a character \( \chi \) of \( \mathcal{T} \) such that \( F(tx) = \chi(t)F(x) \) for any \( x \in X \) and \( t \in \mathcal{T} \). Then there is a function \( \overline{F} \in H^0(Y, \mathcal{O}_Y) \) such that \( F = \xi^* (\chi \cdot \overline{F}) \).

Recall that \( B_1, \ldots, B_l \) are consecutive blocks. In particular, the arrow \( v_{0,1} \) is contained in \( B_1 \).

We are going to define the weights \( w_{t_1}, \ldots, w_t \) of vertices of \( Q \) so that the following properties are satisfied. Consider an arrow \( \alpha \in \text{Ar}(Q) \). Then

\[
\text{wt}_p(h(\alpha)) - \text{wt}_p(t(\alpha)) = \begin{cases} -1 & \text{if } \alpha \in B_p, \\ 0 & \text{if } \alpha \notin B_p \text{ and } \alpha \neq h_{k,n}. \end{cases}
\]

Also, for any \( p \in [1, l] \) we require the following properties:

(a) \( \text{wt}_p(i, j) \geq 0 \) for all \( (i, j) \);

(b) \( \text{wt}_p(k, n) = 0 \), so that

\[
\text{wt}_p(k, n + 1) - \text{wt}_p(k, n) = \text{wt}_p(k, n + 1) \geq 0;
\]

(c) \( \text{wt}_p(0, 1) = \text{wt}_p(k, n + 1) \).
Actually, there is only one way to assign weights so that the above requirements are met. Choose an index \( p \in [1, l] \). If \( B_p = \text{HB}(r, s) \) is a horizontal block, then we put

\[
\text{wt}_p(i, j) = \begin{cases} 
  s - i & \text{if } i \in [r, s], j \in [1, n], \\
  0 & \text{if } i \in [s + 1, k], j \in [1, n], \\
  s - r & \text{if } i \in [1, r - 1], j \in [1, n], \text{ or } (i, j) = (0, 1).
\end{cases}
\]

In particular, this gives \( \text{wt}_p(0, 1) = s - r \). If \( B_p = \text{MB}(r, s) \) is a mixed block, then we put

\[
\text{wt}_p(i, j) = \begin{cases} 
  (k - i) + (s - j) & \text{if } i \in [r, k], j \in [1, s], \\
  k - i & \text{if } i \in [r, k], j \in [s + 1, n], \\
  (k - r) + (s - j) & \text{if } i \in [1, r - 1], j \in [1, s], \text{ or } (i, j) = (0, 1), \\
  k - r & \text{if } i \in [1, r - 1], j \in [s + 1, n].
\end{cases}
\]

If \( B_p = \text{VB}(r, s) \) is a vertical block, then we put

\[
\text{wt}_p(i, j) = \begin{cases} 
  s - j & \text{if } i \in [1, k], j \in [r, s], \\
  s - r & \text{if } i \in [1, k], j \in [1, r - 1], \text{ or } (i, j) = (0, 1), \\
  0 & \text{if } i \in [1, k], j \in [s + 1, n].
\end{cases}
\]

Finally, we always put \( \text{wt}_p(k, n + 1) = \text{wt}_p(0, 1) \).

With any block \( B \) we associate a weight vertex of the quiver \( \mathcal{Q} \) as follows. If \( B = \text{HB}(r, s) \) is a horizontal block, then its weight vertex is \((s - 1, 1)\). If \( B \) is a mixed block \( \text{MB}(r, s) \) or a vertical block \( \text{VB}(r, s) \), then its weight vertex is \((k, s - 1)\). If \( B \) is a block and \((i, j)\) is its weight vertex, then we define the weight variable of \( B \) to be \( a_{i,j} \) if \((i, j) \neq (0, 1)\), and to be a otherwise.

An example of weights assignment corresponding to the Grassmannian \( \text{Gr}(3, 6) \) and the mixed block \( B = \text{MB}(2, 2) \) is given in Fig. 3. Solid arrows are ones that are contained in \( B \), while dashed arrows are those of \( \text{Ar}(\mathcal{Q}) \setminus B \). The weight vertex \((3, 1)\) of \( B \) is marked by a white circle.

**Example 7.37.** Let \( \mathcal{Q} \) be the quiver corresponding to the Grassmannian \( \text{Gr}(3, 6) \) (see Fig. 2). Suppose that \( l = 4 \) and

\[
B_1 = \text{HB}(0, 1), \quad B_2 = \text{HB}(1, 2), \quad B_3 = \text{MB}(2, 2), \quad B_4 = \text{VB}(2, 3).
\]

Then the weight vertices of the blocks \( B_1, B_2, B_3, \) and \( B_4 \) are \((0, 1)\), \((1, 1)\), \((3, 1)\), and \((3, 2)\), respectively, and the weight variables are \( a, a_{1,1}, a_{3,1}, \) and \( a_{3,2} \).

Consider the torus

\[
\mathcal{T} = \mathcal{T}(V) \simeq (\mathbb{C}^*)^{nk},
\]

and the torus \( \mathcal{T} \simeq (\mathbb{C}^*)^l \) with coordinates \( w_1, \ldots, w_l \). Define an action of \( \mathcal{T} \) on \( \mathcal{T} \) by

\[
(w_1, \ldots, w_l) \cdot a_{i,j} = w_1^{\text{wt}_1(i,j)} \cdots w_l^{\text{wt}_l(i,j)} \cdot a_{i,j}
\]

for all \( i \in [1, k], j \in [1, n], (i, j) \neq (k, n) \), and

\[
(w_1, \ldots, w_l) \cdot a = w_1^{\text{wt}_1(0,1)} \cdots w_l^{\text{wt}_l(0,1)} \cdot a.
\]
Using nothing but the basic properties of weights, we obtain the following lemmas.

**Lemma 7.38.** Fix \( p \in [1, l] \). Then \( F_{B_p} \) is a semi-invariant function on \( \mathcal{X}^- \) with respect to the action of \( \mathcal{T} \) with weight \( w_p^{-1} \).

Recall that

\[
B_0 = \text{Ar}(\mathcal{D}) \setminus (B_1 \cup \cdots \cup B_l).
\]

Let \( A = B_0 \setminus \{h_{k,n}\} \), and note that \( F_{B_0} = F_A + a \).

**Lemma 7.39.** The function \( F_A \) is invariant with respect to the action of \( \mathcal{T} \). On the other hand, the function \( a \) is semi-invariant with weight

\[
\mu(w) = w_1^{d_1} \cdots w_l^{d_l}.
\]

Consider the quotient \( \mathcal{Y} = \mathcal{X}^- \div \mathcal{T} \), and let \( \varphi : \mathcal{X}^- \to \mathcal{Y} \) be the natural projection. Let \( x_1, \ldots, x_l \) be weight variables of the blocks \( B_1, \ldots, B_l \), respectively, and let \( \Sigma \subset \mathcal{X}^- \) be the subvariety defined by the equations

\[
\{x_i = 1 \mid i \in [1, l]\} \subset \mathcal{X}^-.
\]

Note that \( \mathcal{T} \) acts on a coordinate \( x_i \) by multiplying it by \( w_i \cdot N_i \), where \( N_i \) is a monomial in \( w_{i+1}, \ldots, w_l \). In other words, define a matrix \( M \) by

\[
(w_1, \ldots, w_l) \cdot x_i = \prod w_j^{M_{i,j}} x_i.
\]

Then \( M \) is an integral upper-triangular matrix with 1s on the diagonal. Thus, \( \Sigma \) has a unique common point with any fibre of \( \varphi \). Therefore, there exists a section \( \sigma : \mathcal{Y} \to \mathcal{X}^- \) of the projection \( \varphi \) whose image is \( \Sigma \). Also, we see that the action of \( \mathcal{T} \) on \( \mathcal{X}^- \) is free. From Lemma 7.36 we conclude that \( \mathcal{X}^- \cong \mathcal{T} \times \mathcal{Y} \). In particular,

\[
\mathcal{Y} \cong (\mathbb{C}^*)^{nk-l}.
\]

Let \( V' \) be the set of all variables of \( V \) except for \( x_1, \ldots, x_l \). We regard the variables of \( V \) as coordinates on \( \mathcal{X}^- \) and those of \( V' \) as coordinates on \( \mathcal{Y} \cong \mathcal{T}(V') \).
In these coordinates the morphism $\sigma$ is given in a particularly simple way. Namely, for any point $y \in \mathcal{Y}$ the point $\sigma(y)$ has all weight coordinates equal to 1, and the other coordinates equal to the corresponding coordinates of $y$.

**Example 7.40.** In the notation of Example 7.37 one has

$$\mathcal{X} = \mathcal{T}(\{a, a_{1,1}, a_{1,2}, a_{1,3}, a_{2,1}, a_{2,2}, a_{2,3}, a_{3,1}, a_{3,2}\})$$

and

$$\mathcal{Y} = \mathcal{T}(\{a_{1,2}, a_{1,3}, a_{2,1}, a_{2,2}, a_{2,3}\}).$$

The action of the torus $\mathcal{T} \simeq (\mathbb{C}^*)^4$ is defined by the matrix

$$M = \begin{pmatrix}
1 & 1 & 2 & 1 \\
0 & 1 & 2 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

by

$$(w_1, w_2, w_3, w_4) \rightarrow (a, a_{1,1}, a_{1,2}, a_{1,3}, a_{2,1}, a_{2,2}, a_{2,3}, a_{3,1}, a_{3,2})$$

$$\quad \quad \quad \quad \quad \quad \rightarrow (w_1 w_2 w_3^2 w_4 \cdot a, w_2 w_3^2 w_4 \cdot a_{1,1}, w_2 w_3 w_4 \cdot a_{1,2}, w_2 w_3 \cdot a_{1,3}, w_3^2 w_4 \cdot a_{2,1}, w_3 w_4 \cdot a_{2,2}, w_3 \cdot a_{2,3}, w_3 w_4 \cdot a_{3,1}, w_4 \cdot a_{3,2}).$$

(Note that the weights corresponding to the block $B_3$ can be seen in Fig. 3.) The matrix

$$M^{-1} = \begin{pmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & -2 & 1 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

gives $w_1^{-1} = a/a_{1,1}$, $w_2^{-1} = a_{1,1}a_{3,2}/a_{3,1}^2$, $w_3^{-1} = a_{3,1}/a_{3,2}$, and $w_4^{-1} = a_{3,2}$, so the projection $\varphi: \mathcal{X} \rightarrow \mathcal{Y}$ is given by

$$\varphi: (a, a_{1,1}, a_{1,2}, a_{1,3}, a_{2,1}, a_{2,2}, a_{2,3}, a_{3,1}, a_{3,2})$$

$$\quad \quad \quad \quad \quad \quad \rightarrow \left(\frac{a_{3,1}}{a_{1,1}a_{3,2}}, \frac{a_{3,1}}{a_{1,2}}, \frac{a_{3,1}}{a_{1,3}}, \frac{a_{3,2}}{a_{3,1}^2} a_{2,1}, \frac{1}{a_{3,1}} a_{2,2}, \frac{a_{3,2}}{a_{3,1}} a_{2,3}\right),$$

and the section $\sigma: \mathcal{Y} \rightarrow \mathcal{X}$ is given by

$$\sigma: (a_{1,2}, a_{1,3}, a_{2,1}, a_{2,2}, a_{2,3}) \rightarrow (1, 1, a_{1,2}, a_{1,3}, a_{2,1}, a_{2,2}, a_{2,3}, 1, 1).$$

Applying Lemma 7.38 together with Lemma 7.36, we see that there exist regular functions $F_p$, $p \in [1, l]$, on $\mathcal{Y}$ such that under the identification $\mathcal{X} \simeq \mathcal{T} \times \mathcal{Y}$ one has

$$F_p = w_p^{-1} \cdot \varphi^* F_p.$$
Consider a rational map
\[ y \mapsto (\overline{F}_1(y), \ldots, \overline{F}_l(y)) \]
from \( \mathcal{Y} \) to \( \mathcal{X} \). Define a rational map \( \tau : \mathcal{Y} \rightarrow \mathcal{X} \) by
\[ y \mapsto (\overline{F}_1(y), \ldots, \overline{F}_l(y)) \cdot \sigma(y). \]
It is easy to see that the closure of the image of \( \mathcal{Y} \) under the map \( \tau \) is a subvariety \( L \subset \mathcal{X} \). In particular, \( \tau \) gives a birational equivalence between \( \mathcal{Y} \) and \( L \).

Now it remains to note that
\[ \tau^* F_A = \tau^* \varphi^* \overline{F}_A = \overline{F}_A. \]

On the other hand,
\[ \tau^* a \mu(\overline{F}_1(y), \ldots, \overline{F}_l(y)) \sigma^* \overline{a} = \mu(\overline{F}_1(y), \ldots, \overline{F}_l(y)) \overline{a}. \]

This means that the map \( \overline{\tau} = \tau \varphi \psi \), where \( \psi \) is given by (7.9), provides a birational map as required in Theorem 7.34.

Remark 7.41. The above proof of Theorem 7.34 provides a very explicit way to write down the Laurent polynomial \( \tau^* F_{B_0} \). Namely, consider a complete intersection \( Y \subset \text{Gr}(n, n + k) \) of hypersurfaces of degrees \( d_i, i \in [1, l] \). The following cases are possible.

(i) One has \( d_1 + \cdots + d_l \leq k \). Let \( u_i = d_1 + \cdots + d_i \) for \( i \in [1, l] \). Then the BCFKS Landau–Ginzburg model for \( Y \) is birational to \((\mathbb{C}^*)^{n_k-l}\) with superpotential
\[ \sum_{i=1}^{k} \sum_{j=1}^{n} \frac{a_{i,j}}{a_{i-1,j}} + a \left( \frac{a_{1,1}}{a} + \sum_{i=2}^{d_1} \sum_{j=1}^{n} \frac{a_{i,j}}{a_{i-1,j}} \right) d_1 \prod_{p=2}^{l} \left( \sum_{i=1}^{u_{p-1}} \sum_{j=1}^{n} \frac{a_{i,j}}{a_{i-1,j}} \right) d_p, \]
where we put \( a_{1,u_1-1} = 1 \) if \( u_1 > 1 \) and \( a = 1 \) otherwise, \( a_{1,u_i-1} = 1 \) for \( i \in [2, l] \), and \( a_{k,n} = 1 \).

(ii) One has \( d_1 + \cdots + d_l > k \). Let \( m \in [0, l-1] \) be the maximal index such that \( d_1 + \cdots + d_m \leq k \). Put \( u_i = d_1 + \cdots + d_i \) for \( i \in [1, m] \) and \( u_i = d_1 + \cdots + d_i - k \) for \( i \in [m+1, l] \).

(ii.1) If \( m = 0 \), then the BCFKS Landau–Ginzburg model for \( Y \) is birational to \((\mathbb{C}^*)^{n_k-l}\) with superpotential
\[ \sum_{i=1}^{k} \sum_{j=1}^{n} \frac{a_{i,j}}{a_{i-1,j}} + a \left( \frac{a_{1,1}}{a} + \sum_{i=2}^{k} \sum_{j=1}^{n} \frac{a_{i,j}}{a_{i-1,j}} + \sum_{i=1}^{k} \sum_{j=1}^{u_1} \frac{a_{i,j}}{a_{i-1,j}} \right) d_1 \prod_{p=2}^{l} \left( \sum_{i=1}^{u_{p-1}} \sum_{j=1}^{n} \frac{a_{i,j}}{a_{i-1,j}} \right) d_p. \]
(ii.2) If \( m > 1 \), then the BCFKS Landau–Ginzburg model for \( Y \) is birational to \((\mathbb{C}^*)^{nk−l}\) with superpotential
\[
\sum_{i=1}^{k} \sum_{j=1}^{n} \frac{a_{i,j}}{a_{i,j-1}} + a \left( \frac{a_{1,1}}{a} + \sum_{i=2}^{d_1} \sum_{j=1}^{n} \frac{a_{i,j}}{a_{i-1,j}} \right) \prod_{p=2}^{m} \left( \sum_{i=u_{p-1}}^{n} \sum_{j=1}^{a_{i,j}} \frac{a_{i,j}}{a_{i,j-1}} \right)^{d_p}
\times \left( \sum_{i=1}^{k} \sum_{j=1}^{n} \frac{a_{i,j}}{a_{i,j-1}} + \sum_{i=1}^{k} \sum_{j=2}^{a_{m+1}} \frac{a_{i,j}}{a_{i,j-1}} \right) \prod_{p=m+2}^{l} \left( \sum_{i=1}^{k} \sum_{j=u_{p-1}}^{a_{i,j}} \frac{a_{i,j}}{a_{i,j-1}} \right)^{d_p}.
\]

In both cases (ii.1) and (ii.2) we put \( a_{1,u_1-1} = 1 \) if \( u_1 > 1 \) and \( a = 1 \) otherwise, \( a_{1,u_p-1} = 1 \) for \( p \in [2, m] \), \( a_{k,u_p-1} \) for \( p \in [m + 1, l] \), and \( a_k,n = 1 \).

**Example 7.42** (see [81]). Consider a smooth Fano fourfold \( Y \) of index 2 that is a section of the Grassmannian \( \text{Gr}(2, 6) \) by four hyperplanes. A very weak Landau–Ginzburg model of \( Y \) is given by
\[
f_Y = \frac{(a_4 + a_3)(a_4 + a_3 + a_2)}{a_3 a_2 a_1} + \frac{a_4 + a_3}{a_3 a_2} + \frac{1}{a_3} + \frac{1}{a_4} + a_3 + a_2 + a_1.
\]

Let \( \mathcal{T} = \mathcal{T}[a_1, a_2, a_3, a_4] \). Consider the relative compactification of the family \( f_Y : \mathcal{T} \rightarrow \mathbb{A}^1 \) which is given by an embedding of \( \mathcal{T} \) into the projective space \( \mathbb{P}^4 \) with homogeneous coordinates \( a_0, \ldots, a_4 \). It is a family of compact singular Calabi–Yau threefolds. The total space of this family admits a crepant resolution of singularities \( \text{LG}_Y \). Moreover, one can check that \( \text{LG}_Y \) is a family of Calabi–Yau threefolds such that its general fibre is smooth, and \( \text{LG}_Y \) has exactly twelve singular fibres. Furthermore, each of these singular fibres has exactly one singular point, and this point is an ordinary double singularity. In accordance with the compactification principle 3.7, we expect that \( \text{LG}_Y \) satisfies the homological mirror symmetry conjecture. The structure of singular fibres of \( \text{LG}_Y \) confirms this expectation. Indeed, by Corollary 10.3 in [61] there is a full exceptional collection of length 12 on \( Y \). On the other hand, by the homological mirror symmetry conjecture the category \( D^b(\text{coh} Y) \) is equivalent to the Fukaya–Seidel category for a dual Landau–Ginzburg model.

**7.3.2. Periods.** In this subsection we discuss period integrals for the Laurent polynomials obtained in Theorem 7.34.

Recall the definition of Givental’s integral in our case.

Given a torus \( \mathcal{T}(\{x_1, \ldots, x_r\}) \) we call a cycle \( \{ |x_i| = \varepsilon_i \mid i \in [1, r] \} \) depending on some real numbers \( \varepsilon_i \) standard.

**Definition 7.43** (see [9]). An \( \text{(anticanonical)} \) Givental’s integral for \( Y \) is an integral
\[
I^0_Y = \int_{\delta} \frac{\Omega(\{\tilde{a}_{i,j}\})}{\prod_{j=1}^{l} (1 - F_j) \cdot (1 - i F_0)} \in \mathbb{C}[[t]]
\]
for a standard cycle \( \delta = \{ |\tilde{a}_{i,j}| = \varepsilon_{i,j} \mid i \in [1, k], j \in [1, n], \varepsilon_{i,j} \in \mathbb{R}_+ \} \), whose orientation is chosen such that \( I^0_Y|_{t=0} = 1 \).

In Conjecture 5.2.3 of [9] it was conjectured that \( \overline{I}^0_G = I^0_C \), and a formula for \( \overline{I}^0_G \) was provided. This conjecture was proved for \( n = 2 \) in Proposition 3.5 of [12],
and for any \( n \geq 2 \) in [65]. In the discussion after Conjecture 5.2.1 in [10] it was explained that the above theorems and the Quantum Lefschetz Theorem imply that Givental’s integral \( I_Y^0 \) equals \( \overline{I}_0^Y \). We summarize the results above as follows.

**Theorem 7.44.** Let

\[ Y = Y_1 \cap \cdots \cap Y_l \]

be a smooth Fano complete intersection in \( \text{Gr}(n, k + n) \). Let \( d_i = \deg Y_i \) and \( d_0 = k + n - \sum d_i \). Then

\[
\overline{I}_0^Y = I_Y^0 = \sum_{d \geq 0} \sum_{s, j \geq 0} \prod_{i=0}^l (d_i d)^{k+n} \prod_{i=1}^{k-1} \prod_{j=1}^{n-1} \left( \begin{array}{c} s_{i+1, j} + 1 \\ s_{i, j} \\ s_{i, j} \end{array} \right) t^{d_0 d},
\]

where \( s_{k, j} = s_{i, n} \).

It turns out that the changes of variables in Theorem 7.34 preserve this period.

**Proposition 7.45.** The period condition holds for Laurent polynomials given by Theorem 7.34. In other words, Theorem 7.34 provides weak Landau–Ginzburg models for Fano complete intersections in Grassmannians.

**Proof.** We follow the notation from Theorem 7.34. A toric change of variables \( \varphi \psi \) changes the coordinates \( \{a_{i, j}\} \) to coordinates \( \{w_i\} \cup V' \). One obtains

\[
I_Y^0 = \int_{\delta} \frac{\Omega(\{a_{i, j}\})}{\prod_{j=1}^{l} (1 - F_j) \cdot (1 - t F_0)}
= \int_{\delta'} \Omega(V') \wedge \left( \prod_{j=1}^{l} \frac{1}{2\pi\sqrt{-1}} \frac{dw_j}{w_j \cdot (1 - F_j/w_j)} \right) \cdot \frac{1}{1 - t F}
\]

for an appropriate choice of an orientation on \( \delta' \), where \( \overline{F} = F_A + \mu(w) \cdot \nu \). Following the birational isomorphism \( \tau \), we consider the variables \( u_i = w_i - F_i \) instead of \( w_i \). Then, after an appropriate choice of cycle \( \Delta' \) (see the proof of Proposition 10.5 in [86]), we get that

\[
I_Y^0 \int_{\delta'} \Omega(V') \wedge \left( \prod_{j=1}^{l} \frac{1}{2\pi\sqrt{-1}} \frac{du_j}{u_j} \right) \cdot \frac{1}{1 - t F}
= \int_{\Delta'} \Omega(V') \wedge \left( \prod_{j=1}^{l} \frac{1}{2\pi\sqrt{-1}} \frac{du_j}{u_j} \right) \cdot \frac{1}{1 - t F} = \sum [f] t^i,
\]

where \( \Delta \) is the projection of \( \Delta' \) on \( \mathcal{T}(V) \) and \( F_u \) is the result of replacing \( w_i \) by \( u_i + F_{B_i} \) in \( \overline{F} \).

**Problem 7.46** (see [77], Problem 17). Let \( Y \) be a Fano complete intersection in \( \text{Gr}(n, k + n) \), and let \( f_Y \) be the Laurent polynomial for \( Y \) given by Theorem 7.34. Prove that \( f_Y \) is a toric Landau–Ginzburg model. Prove that the number of components of a central fibre of a Calabi–Yau compactification for \( f_Y \) is equal to \( h^{l, \dim Y - 1}(Y) + 1 \) (see Conjecture 6.38).
Remark 7.47. In [31] it was shown using other methods that BCFKS Landau–Ginzburg models are birational to algebraic tori. Moreover, it follows from [31] that the Laurent polynomials representing the superpotentials are recovered from toric degenerations. Thus if one shows that these Laurent polynomials satisfy the period condition, then they are weak Landau–Ginzburg models for complete intersections in Grassmannians which satisfy the toric condition.

Example 7.48. Let $Y$ be a smooth intersection of the Grassmannian $\text{Gr}(3, 6)$ with a quadric and three hyperplanes. Put $l = 4$, $d_1 = d_2 = d_4 = 1$, and $d_3 = 2$. The BCFKS Landau–Ginzburg model in this case is birational to a torus

$$\mathcal{Y} \simeq \mathcal{T}(\{a_{1,2}, a_{1,3}, a_{2,1}, a_{2,2}, a_{2,3}\})$$

with the superpotential

$$f_Y = \left( a_{2,1} + \frac{a_{2,2}}{a_{1,2}} + \frac{a_{2,3}}{a_{1,3}} \right) \left( \frac{1}{a_{2,1}} + \frac{a_{3,2}}{a_{2,2}} + \frac{1}{a_{2,3}} + a_{1,2} + \frac{a_{2,2}}{a_{2,1}} + 1 \right)^2$$

$$\times \left( \frac{a_{1,3}}{a_{1,2}} + \frac{a_{2,3}}{a_{2,2}} + 1 \right)$$

given by Remark 7.41. By Theorem 7.44 (see also [9], Example 5.2.2) one has

$$I_Y^0 = \sum_{d,b_1,b_2,b_3,b_4} \frac{(2d)!}{(d!)^2} \left( \frac{b_2}{b_1} \right) \left( \frac{b_3}{b_1} \right) \left( \frac{d}{b_2} \right) \left( \frac{b_4}{b_2} \right) \left( \frac{d}{b_3} \right) \left( \frac{d}{b_4} \right) \left( \frac{d}{d} \right)^2 t^d$$

$$= 1 + 12t + 756t^2 + 78960t^3 + 10451700t^4 + 1587790512t^5$$

$$+ 263964176784t^6 + 46763681545152t^7 + 8685492699286260t^8 + \cdots. \quad (7.10)$$

One can check that the first few constant terms we write down on the right-hand side of (7.10) are equal to the first few terms of the series $\sum [f_Y^i] t^i$.

Example 7.49. Let $Y$ be a smooth intersection of the Grassmannian $\text{Gr}(3, 6)$ with a quadric and three hyperplanes, that is, the variety that was already considered in Example 7.48.

Let $l = 4$, $d_1 = d_2 = d_3 = 1$, and $d_4 = 2$. Then

$$\mathcal{Y} = \mathcal{T}(\{a_{1,2}, a_{1,3}, a_{2,2}, a_{2,3}, a_{3,1}\}).$$

By Remark 7.41,

$$f_Y = \left( 1 + \frac{a_{2,2}}{a_{1,2}} + \frac{a_{2,3}}{a_{1,3}} \right) \left( a_{3,1} + \frac{1}{a_{2,2}} + \frac{1}{a_{2,3}} \right) \times \left( a_{1,2} + a_{2,2} + \frac{1}{a_{3,1}} + \frac{a_{1,3}}{a_{1,2}} + \frac{a_{2,3}}{a_{2,2}} + 1 \right)^2.$$

One can check that the first few constant terms $[f_Y^i]$ coincide with the first few terms of the series on the right-hand side of (7.10). We note that the Laurent polynomial $f_Y$ cannot be obtained from the polynomial in Example 7.48 by a monomial change of variables (see Remark 7.35). It would be interesting to find out if these two Laurent polynomials are mutationally equivalent (see [31], Theorem 2.24).
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