On the effective theory of neutrino-electron and neutrino-quark interactions

RICHARD J. HILL AND OLEKSANDR TOMALAK

Department of Physics and Astronomy, University of Kentucky, Lexington, KY 40506, USA

Theoretical Physics Department, Fermi National Accelerator Laboratory, Batavia, IL 60510, USA

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Abstract

We determine the four-fermion effective theory of neutrino interactions within the Standard Model including one-loop electroweak radiative corrections, in combination with the measured muon lifetime and precision electroweak data. Including two-loop matching and three-loop running corrections, we determine lepton coefficients accounting for all large logarithms through relative order $\mathcal{O}(\alpha \alpha_s)$ and quark coefficients accounting for all large logarithms through $\mathcal{O}(\alpha)$. We present four-fermion coefficients valid in $n_f = 3$ and $n_f = 4$ flavor quark theories, as well as in the extreme low-energy limit. We relate the coefficients in this limit to neutrino charge radii governing matter effects via forward neutrino scattering on charged particles.

Experimental studies of unmeasured or precisely predicted quantities advance our knowledge of interaction mechanisms on short distances and shed light on potential new physics. Besides the Higgs boson couplings of heavy particles in the Standard Model, the parameters of the neutrino oscillation matrix [1] are poised to be measured more accurately in the coming years. Improved precision on the neutrino mixing angles and phase(s) [2–5], requires improved knowledge of neutrino-nucleus cross sections and improved methods to determine neutrino flux [6–8]. Future neutrino experiments [9–12] will reach sub-percent statistical precision even for such a rare process as neutrino-electron scattering. This process plays the role of “standard candle” to constrain neutrino beam flux at near detectors. As a proof of principle, the MINERvA Collaboration reduced uncertainties of the NUMI beam flux normalization from 7.5% to 3.9% [13, 14] by exploiting the neutrino-electron scattering channel.

Achieving the percent level precision target of next generation experiments requires that one-loop radiative corrections, and higher order corrections enhanced by large renormalization-group logarithms, are properly treated. Leading radiative corrections to purely leptonic processes can be readily evaluated within the complete Standard Model. For the simplest process of elastic neutrino-electron scattering, the next-to-leading order electroweak result was presented by Marciano and Sirlin in Refs. [15, 16], exploiting the current algebra formalism and the onshell renormalization scheme (see also Refs. [17–20] describing a direct evaluation of Feynman diagrams). The authors of Refs. [15, 16] also calculated the electron energy spectrum accounting for radiation of one real photon in the massless electron limit [21]. Different aspects of radiative corrections in elastic neutrino-electron scattering were also discussed in Refs. [22–39].

However, even in leptonic processes, hadronic effects contribute a sizable component of the radiative correction, and a dominant component of the error budget. Computations using the complete Standard Model [15, 16, 40, 41] cause electroweak scale physics and hadronic physics to be intermingled. Four-fermion effective theory [42–44] systematically separates contributions from electroweak and hadronic

\footnote{References [15, 16] resort to quark-model evaluation of the hadronic contribution.}
scales. Electroweak scale physics is computed perturbatively and is represented by four-fermion operator coefficients. The effective operator matrix elements provide a rigorous starting point for nonperturbative (e.g. lattice QCD) evaluation and/or experimental analysis of the relevant hadronic amplitudes.

In this work, we construct the low-energy effective field theory of neutrino-lepton and neutrino-quark interactions suitable for predictions with sub-percent accuracy. Predictions for low-energy processes including bremsstrahlung and virtual QED corrections can be evaluated using this theory as starting point (see e.g. Ref. [45] for an application to neutrino-electron scattering). We match the effective theory to the Standard Model (SM) at the electroweak (EW) energy scale, evaluating all effective couplings and determining scale-independent combinations. We then solve renormalization group equations (RGEs) and heavy quark threshold matching conditions, and present four-fermion coefficients at the GeV energy scale in $n_f = 3$ and $n_f = 4$ flavor QCD. We also present coefficients in the extreme low-energy QED limit and determine neutrino charge radii.

1 Effective theory and tree level matching

Neglecting corrections of order $\alpha m_f^2/m_W^2$ and $|q^2|/m_W^2$, where $m_f$ denotes a fermion mass, and $q^2$ denotes an invariant momentum transfer, the structure of neutrino interactions with quarks and charged leptons in the Standard Model is severely constrained.\(^2\) Besides the usual kinetic, mass, and QED+QCD gauge coupling terms for neutrinos, charged leptons, and quarks (including the mass mixing matrix of neutrinos), the effective Lagrangian consists of dimension 6 four-fermion operators and neutrino-photon couplings,

$$
\mathcal{L}_{\text{eff}} = -\sum_{\ell,\ell'}\overline{\nu}_\ell\gamma^\mu P_L\nu_\ell\bar{\nu}_{\ell'}\gamma_\mu(c_{\ell\ell'}^{qL}P_L + c_{\ell\ell'}^{qR}P_R)\ell' - \sum_{\ell,q}\overline{\nu}_\ell\gamma^\mu P_L\nu_\ell\bar{q}\gamma_\mu(c_{\ellL}^q P_L + c_{\ellR}^q P_R)q - c\sum_{\ell\neq\ell'}\overline{\nu}_\ell\gamma^\mu P_L\nu_\ell\bar{\nu}_{\ell'}\gamma_\mu P_L\ell' - \sum_{q,q'}\left(\epsilon_{q'q}(\bar{\nu}_\ell\gamma^\mu P_L\nu_\ell\bar{q}\gamma_\mu P_Lq' + \text{h.c.}) - \frac{1}{e}\sum_{\ell}\epsilon'_{\ell\ell'}\gamma^\mu\partial_\mu F_{\ell\ell'} \bar{\nu}_\ell\nu_\ell\right),
$$

where $e$ denotes the positron charge. The sums in Eq. (1) run over 3 lepton flavors ($\ell = e, \mu, \tau$) and $n_f$ active quark flavors ($q = u, c$ and $q' = d, s, b$ for $n_f = 5$). $P_L = (1 - \gamma_5)/2$ and $P_R = (1 + \gamma_5)/2$ are projection operators onto left- and right-handed states, respectively. For neutrino scattering applications, it is convenient to replace the neutrino-photon operator by an equivalent combination of four-fermion operators in the effective theory, obtained by the field redefinition

$$
A^\mu = A'^\mu + \frac{1}{e}\sum_{\ell}\epsilon'_{\ell\ell'}\gamma^\mu P_L\nu_\ell.
$$

Under Eq. (2), charged current coefficients $c$ and $\epsilon_{q'q}$ remain unchanged while in the neutral current sector

$$
c_{\ellL}^q \rightarrow \frac{1}{e}\epsilon'_{\ell\ell'}\gamma^\mu P_L\nu_\ell, \quad c_{\ellR}^q \rightarrow \frac{1}{e}\epsilon'_{\ell\ell'}\gamma^\mu P_L\nu_\ell + c_{\ellL}^q, \quad c_{\ellR}^q \rightarrow \frac{1}{e}\epsilon'_{\ell\ell'}\gamma^\mu P_L\nu_\ell + c_{\ellL}^q, \quad c_{\ellL}^q \rightarrow c_{\ellL}^q - Q_q c_{\ellR}^q, \quad c_{\ellR}^q \rightarrow c_{\ellR}^q - Q_q c_{\ellL}^q.
$$

The four-fermion coefficients can be determined order-by-order in perturbation theory by matching amplitudes in the full (SM) theory and the effective theory. In the MS renormalization scheme [46], the lepton coefficients may be written

$$
c(\mu) = \frac{2\pi\alpha(\mu)}{M_W^2(\mu)}g(\mu), \quad c_{\ellL}^{\nu\ell'}(\mu) = [g_L(\mu) + \delta_{\ell\ell'}g(\mu)]\frac{c(\mu)}{g(\mu)}, \quad c_{\ellR}^{\nu\ell'}(\mu) = g_R(\mu)\frac{c(\mu)}{g(\mu)}.
$$

\(^2\)Power corrections to four-fermion theory can be readily estimated for specific processes. For neutrino-electron scattering, these corrections are negligible, at the $10^{-6}$ level for $E_\nu \lesssim 10$ GeV [45]. For neutrino-nucleon scattering, leading corrections to four-fermion theory scale as $|q^2|/M_W^2 \lesssim 2m_NE_\nu/M_W^4$, where $m_N \approx 1$ GeV is the nucleon mass and $E_\nu$ is the neutrino beam energy. For $E_\nu \sim \text{few} \times \text{GeV}$, these corrections can amount to few-permille contributions. However, form factors in hadronic amplitudes typically suppress the region of large $|q^2|$ making these corrections further subdominant.
For the quark coefficients, we have

\[ c^{\nu \gamma}(\mu) = \frac{2\pi\alpha(\mu)}{M_W(\mu)s_W^2(\mu)} V_{qq'} [g(\mu) + \delta g(\mu)], \quad c_L^q(\mu) = g_L^q(\mu) \frac{c(\mu)}{g(\mu)}, \quad c_R^q(\mu) = g_R^q(\mu) \frac{c(\mu)}{g(\mu)}, \quad (5) \]

where for up-type quarks \( q \), and down type quarks \( q' \), \( V_{qq'} \) is the corresponding element of the Cabibbo-Kobayashi-Maskawa (CKM) matrix [47,48]. We write \( M_W \) and \( M_Z \) for the mass of the \( W^\pm \) and \( Z^0 \) bosons, related as \( M_W = M_Z \cos \theta_W \), using the notation \( s_W = \sin \theta_W \) and \( c_W = \cos \theta_W \). The neutrino-photon coupling is expressed as

\[ c^{\nu \gamma}(\mu) = g^{\nu \gamma}(\mu) \frac{c(\mu)}{g(\mu)}. \quad (6) \]

At tree level, we have the well-known expressions [40,41] for lepton couplings,

\[ g = 1, \quad g_L^0 = s_W^2 - \frac{1}{2}, \quad g_R^0 = c_W, \quad (7) \]

and for quark couplings,

\[ \delta g = 0, \quad g_L^{q0} = T_q^3 - Q_q s_W^2, \quad g_R^{q0} = -Q_q c_W. \quad (8) \]

The neutrino-photon coupling vanishes at tree level, \( g^{\nu \gamma} = 0 \) and \( c^{\nu \gamma} = \mathcal{O}(\alpha G_F) \). Here \( Q_q \) is the quark electric charge in units of the positron charge \( (Q_u = 2/3, Q_d = -1/3, T_q^3 = +1/2, \ T_u^3 = -1/2) \), and \( \alpha(\mu) \) is the electromagnetic coupling constant.

2 One-loop electroweak matching

We perform the matching onto the effective theory (1) by first integrating out heavy vector and scalar bosons, \( W^\pm, Z^0 \) and \( H \), and the top quark, \( t \), in the Standard Model. Before the field redefinition (2), the neutrino-photon coupling is determined by computing the neutrino scattering process \( \nu_e(p) \to \nu_e(p') \) in a background electromagnetic field.\(^3\) After \( \overline{\text{MS}} \) renormalization, the one-loop contribution in Feynman-t’Hooft gauge is

\[ g^{\nu \gamma}(\mu) = -\frac{\alpha}{8\pi} (9 - 10s_W^2) \ln \frac{\mu^2}{M_W^2} - \frac{\alpha}{12\pi} (5 - 2s_W^2) + \frac{\alpha}{18\pi} (3 - 8s_W^2) \ln \frac{\mu^2}{m_t^2}. \quad (9) \]

After the field redefinition (2), this coefficient vanishes \( g^{\nu \gamma} \to 0 \) and all other couplings change according to Eq. (3). Reproducing one-loop EW radiative corrections in the Standard Model [15,16,18–20] in Feynman-t’Hooft gauge with massless leptons and quarks, besides the top, and subtracting the corresponding diagrams in the effective theory, we obtain the following reduced couplings for neutrino-lepton interactions after the field redefinition (3):

\[ g_L(\mu) = g_L^0 \left[ 1 - \frac{\alpha}{16\pi^2 s_W^2} \left[ 6r_L \ln \frac{\mu^2}{m_t^2} - \ln \frac{\mu^2}{M_W^2} + \frac{3}{1 - r_H} \left( r_H \ln \frac{\mu^2}{M_H^2} - \ln \frac{\mu^2}{M_Z^2} \right) + \frac{r_H - 7}{2} + 6g_L^0 \right] \right] 
+ \frac{\alpha}{2\pi} \ln \frac{\mu^2}{M_W^2} - \frac{\alpha}{4\pi} \frac{c_W^2}{s_W^2} \ln \frac{\mu^2}{M_W^2} - \frac{\alpha}{2\pi} \frac{c_W^2}{s_W^2} + c^{\nu \gamma}, \]

\[ g_R(\mu) = g_R^0 \left[ 1 - \frac{\alpha}{16\pi^2 s_W^2} \left[ 6r_L \ln \frac{\mu^2}{m_t^2} - \ln \frac{\mu^2}{M_W^2} + \frac{3}{1 - r_H} \left( r_H \ln \frac{\mu^2}{M_H^2} - \ln \frac{\mu^2}{M_Z^2} \right) + \frac{r_H - 7}{2} - 6g_R^0 \right] \right]. \]

\(^3\)The coefficient \( g^{\nu \gamma}(\mu) \) is determined by \( \gamma Z \)-mixing diagrams, penguin-type diagrams with \( W \), and the closed top-loop contribution on the Standard Model side subtracting appropriate contributions in the effective theory.
\[
g(\mu) = 1 - \frac{\alpha}{16\pi c_W s_W} \left[ 6r_t \left( \frac{\mu^2}{m_t^2} + \frac{1}{2} \right) - c_W^2 \ln \frac{\mu^2}{M_W^2} + \frac{3c_W^2}{e_W^2 - r_H} \left( r_H \ln \frac{\mu^2}{M_H^2} - c_W^2 \ln \frac{\mu^2}{M_W^2} \right) \right] + \frac{r_H - \frac{7}{2}}{2} \right] + \frac{\alpha}{4\pi s_W^2 c_W} \left( 1 + c_W^2 \right) \ln \frac{\mu^2}{M_W^2} + \frac{1 + 3q^2}{16\pi s_W^2 c_W} \left( 7s_W^2 - 3 \right) \ln \frac{M_W^2}{M_Z^2} + \frac{7\alpha}{16\pi s_W^2}, \tag{10}
\]

where \(r_t = m_t^2/M_Z^2\) and \(r_H = M_H^2/M_Z^2\) with masses of the top quark \(m_t\) and the Higgs boson \(M_H\). Tree level couplings \(g_{L,R}^{0}\) have been specified above and depend on renormalization scale \(\mu\) through \(s_W^2\).

To perform the matching, we have enforced the vanishing of the Higgs tadpole (the amputated and renormalized Higgs field one-point function) as a renormalization condition, but use MS renormalization for non-tadpole counterterms. This hybrid renormalization scheme leads to compact, but gauge-dependent, expressions for the reduced couplings (10) and for \(\overline{\text{MS}}\) masses. The prefactor, \(\alpha M_W^{-2} s_W^{-2}\), in Eq. (4), and the tree level reduced couplings in (7) must be evaluated in this scheme in order to employ the expressions (10). Here \(\alpha(\mu)\), as well as \(M_W, M_Z\) and \(s_W\), refer to the full SM particle content. We have verified that the complete effective couplings (5) are gauge independent and have obtained the same one-loop matching onto the effective Lagrangian of Eq. (1) in an arbitrary \(R_\xi\) gauge.\(^4\) We have also obtained the one-loop coefficients using on-shell renormalization, first expressing the effective couplings (5) in terms of on-shell (pole) \(M_W, M_Z\), and low-energy (Thomson limit) \(\alpha\), and then expressing these on-shell quantities in terms of \(M_W(\mu), M_Z(\mu)\) and \(\alpha(\mu)\). In performing the matching, we have regulated infrared divergences in both full and effective theories with small photon and fermion masses. The matching could be performed in the exact massless limit but would require consideration of new operator structures in the effective theory to account for Fierz rearrangements in \(d \neq 4\).

Similar to Eq. (10), we can determine neutrino-quark couplings as

\[
g_{L}^{q}(\mu) = g_{L}^{0} \left( 1 - \frac{\alpha}{16\pi s_W^2 c_W} \left[ 6r_t \ln \frac{\mu^2}{m_t^2} - \ln \frac{\mu^2}{M_Z^2} + \frac{3}{1 - r_H} \right] \right) + \frac{r_H - \frac{7}{2}}{2} + 6g_{L}^{0},
\]

\[
g_{R}^{q}(\mu) = g_{R}^{0} \left( 1 - \frac{\alpha}{16\pi s_W^2 c_W} \left[ 6r_t \ln \frac{\mu^2}{m_t^2} - \ln \frac{\mu^2}{M_Z^2} + \frac{3}{1 - r_H} \right] \right) + \frac{r_H - \frac{7}{2}}{2} - 6g_{R}^{0},
\]

\[
d\delta\mu(\mu) = -\frac{\alpha}{2\pi} \ln \frac{\mu^2}{M_Z^2} - (1 - a) \frac{\alpha}{2\pi}, \tag{11}
\]

where \(\omega_t = m_t^2/M_Z^2\). Here \(a\) stands for the scheme parameter defining the behavior of \(\gamma_5\) in \(d \neq 4\). It appears in the reduction of Dirac matrices resulting from two-boson exchange diagrams, e.g.

\[
\gamma^\alpha \gamma^\beta \gamma^\mu P_L \otimes \gamma_\mu \gamma_\beta \gamma_\alpha P_L = 4 \left[ 1 + a \left( 4 - d \right) \right] \gamma^\mu P_L \otimes \gamma_\mu P_L + E(a), \tag{12}
\]

where \(E\) is an evanescent operator with vanishing matrix element in \(d = 4\). We choose \(a = -1\) in the following, appropriate for anticommuting \(\gamma_5\) in the basis \(\gamma^\mu \otimes \gamma_\mu, \gamma^\mu \otimes \gamma_\mu \gamma_5, \gamma_5 \gamma^\mu \otimes \gamma_\mu, \gamma_5 \gamma^\mu \otimes \gamma_\mu \gamma_5\) [52–54]. We remark that the difference between left-handed couplings to \(s\) and \(d\) quarks in Eq. (11) is negligibly

\(^4\)See Refs. [49–51] for the discussion of one-loop EW corrections in \(R_\xi\) gauges.
small, using $|V_{ub}|^2 = 1 - |V_{td}|^2 - |V_{ts}|^2 \approx 1$. However, the top quark contributes significantly to the left-handed coupling of the $b$ quark.

The starting point for our renormalization analysis below is $n_f = 5$ flavor QCD, with four-fermion coefficients determined at $\mu = M_Z$ through $\mathcal{O}(\alpha G_F)$ in Eqs. (10) and (11). For lepton coefficients (10), we also include complete corrections through $\mathcal{O}(\alpha \alpha_s G_F)$ [55–61], arising from gluons attached to closed quark loops. Details about this correction, and the determination of appropriate $\overline{\text{MS}}$ masses, are discussed in the Appendix A. Uncomputed corrections of $\mathcal{O}(\alpha \alpha_s G_F)$ to quark coefficients are left to future work; these corrections are small compared to hadronic uncertainties in current neutrino scattering analyses.

Note that in Eqs. (10) and (11) we have performed the field redefinition (2) and we maintain the condition $\nu_\ell \gamma = 0$ for renormalized coefficients.

### 3 Standard Model inputs

Having computed expressions for the four-fermion operator coefficients, let us define the numerical inputs to these expressions. We begin by isolating coefficient combinations that are independent of renormalization scale, and combinations that vanish up to neglected $\mathcal{O}(\alpha m_f^2/m_W^2)$ corrections.

As a consequence of Ward-Takahashi identities [62, 63] and the partial conservation of vector and axial-vector currents in QED+QCD gauge theory, the evolution equation for couplings in the neutral current sector is governed by vacuum polarization diagrams as in Fig. 1,

$$\frac{d}{d \ln \mu^2} \frac{c^f_S}{Q_f} = \frac{\alpha}{6\pi} \left[ \sum_{f',S'} N_{c,f} Q_{f} c^f_{S'} \right] + \ldots , \quad (13)$$

where $f$ denotes an active lepton or quark flavor in the theory, $N_{c,f}$ is the number of color degrees of freedom ($N_{c,f} = 1$ for leptons, $N_{c,f} = 3$ for quarks), and $S = L, R$. In particular, the running of $c^f_S/Q_f$ does not depend on $f$ or $S$. This property of the anomalous dimension matrix holds also at higher orders so that

$$\frac{c^f_S(\mu)}{Q_f} - \frac{c^{f'}_{S'}(\mu)}{Q_{f'}} = \text{constant} . \quad (14)$$

All couplings in the neutral current sector may thus be written as linear combinations of scale-invariant quantities and a single scale-dependent coupling. Neglecting fermion mass corrections of order $\alpha m_f^2/m_Z^2$.

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5For definiteness, we employ the mass of $Z$ boson, $M_Z^{\text{PDG}}$ from Eq. (19), as the scale $\mu$ for all $\overline{\text{MS}}$ quantities at $\mu = M_Z$.

6Diagrams as in Fig. 1 with the exchanged photon replaced by a gluon vanish by color symmetry. Diagrams with two photons exchanged between the loop in Fig. 1 and exterior fermion legs vanish in the sum of direct and crossed contributions according to Furry’s theorem. The sum of three-photon exchange contributions is UV finite [64, 65] and does not affect the running. Similar arguments apply to two- and three-gluon exchange diagrams. While not phenomenologically important, it may be formally interesting to investigate this structure to higher orders in QED+QCD couplings.
we also have (at $\mu = M_Z$): $c_{L,R}^{\nu \ell} = c_{L,R}^{\nu e}$ ($\ell = \ell' = e, \mu, \tau$), $c_{L,R}^{\nu e} = c_{L,R}^{\nu \ell}$ ($\ell \neq \ell'$), $c_{R}^{\nu e} = c_{R}^{\nu \ell}$, $c_{R}^{b} = c_{R}^{d}$, $c_{L}^{c} = c_{L}^{d}$, $c_{R}^{c} = c_{R}^{d}$.

In the charged current leptonic sector described by coefficient $c(\mu)$, the operator anomalous dimension vanishes, and we may define the scale-independent Fermi constant [66, 67]

$$c(\mu) = c \equiv 2\sqrt{2}G_F.$$  (15)

In the charged current semileptonic sector, the presence of three external charged fermion lines results in scale dependence of $c^{\nu q}(\mu)$, cf. Eq. (23) below. The remaining independent coefficients are the scale-dependent quantity $c_R(\mu) \equiv c_R^{\nu e}(\mu)$, and the scale-independent combinations

$$c_L^{\nu e} - c_L^{\nu u} = 2\sqrt{2}G_F,$$

$$3c_{L}^{u} + 2c_{L}^{u e} = \sqrt{2}G_u,$$

$$-3c_{L}^{d} + c_{L}^{u e} = 2\sqrt{2}G_d,$$

$$c_{R}^{\nu e} - c_{R} = -\sqrt{2}G_e,$$

$$c_{L}^{u} - c_{R}^{u} = \sqrt{2}G_u,$$

$$c_{L}^{d} - c_{R}^{d} = -\sqrt{2}G_d,$$

$$c_{L}^{b} - c_{R}^{b} = -\sqrt{2}G_b.$$  (16)

The quantities $G_q$ ($q = u, d$) are differences of left-handed couplings $c_q^L$ and $c_q^L$, while tilded quantities $\tilde{G}_f$ ($f = e, u, d, b$) are differences of $c_f^L$ and $c_f^L$.

Having reduced the scale dependence to $c_R(\mu)$ in the neutral current sector and $c^{\nu q}(\mu)$ in the charged current sector, we proceed to specify numerical inputs to the initial conditions of RGE. For numerical evaluation, we employ high order running and threshold matching corrections for $\overline{\text{MS}}$ QCD and QED couplings 7 with input values [1]

$$\alpha_s^{(5)}(\mu = M_Z) = 0.1187(16), \quad \alpha^{(5)}(\mu = M_Z)^{-1} = 127.955(10).$$  (17)

The $\overline{\text{MS}}$ quantities $\alpha_s^{(5)}$ and $\alpha_s^{(5)}$ in Eq. (17) have been defined in the theory containing $W^\pm$, $Z^0$ and $H$, i.e., the complete SM particle content except the top quark. For values at lower scales and in the theory with top quark, see Appendix B. We use the Fermi constant determined from muon decay,

$$G_F = 1.1663787(6) \times 10^{-5} \text{GeV}^{-2},$$  (18)

and the following mass parameters [1]

$$M_Z^{\text{PDG}} = 91.1876(21) \text{ GeV}, \quad M_H = 125.10(16) \text{ GeV}, \quad m_t^{\text{PDG}} = 172.9(4) \text{ GeV}.$$  (19)

The quantity $M_Z^{\text{PDG}}$ represents a fit parameter in $Z^0$ lineshape analysis; its relation to the corresponding pole mass and $\overline{\text{MS}}$ mass is discussed in the Appendix A. The quantity $m_t^{\text{PDG}}$ represents the top quark pole mass. The translation to the corresponding $\overline{\text{MS}}$ mass, relevant for $\mathcal{O}(\alpha_s)$ corrections to four-fermion coefficients, is discussed in the Appendix A. The quantities (17), (18), and (19), together with CKM elements for semileptonic charged current processes, fully determine the electroweak scale matching coefficients.

Leptonic and semileptonic four-fermion coefficients at $\mu = M_Z$ are displayed in Tables 1 and 2, respectively. For reference, we also display the determination of $s_W^2(\mu = M_Z)$ in the tables.8 9 To

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7Evolution of $\alpha_s(\mu)$ is computed with five-loop QCD running and four-loop threshold matching corrections, ignoring QED. Evolution of $\alpha(\mu)$ is computed with two-loop QED and $\mathcal{O}(\alpha_s^4)$ corrections to running, and one-loop QED and $\mathcal{O}(\alpha_s^2)$ corrections to threshold matching.

8Note that we employ the $\overline{\text{MS}}$ scheme for $s_W^2$, in contrast to the quantity $s_W^2$ of Ref. [1] which includes also a finite subtraction [56]. Our convention corresponds to the quantity $s_W^2$ of Ref. [1].

9Choosing the right-handed coefficient $c_R(\mu)$ as the relevant scale-dependent coupling avoids difficulties in defining $s_W^2$ at low energies [68–70].
Table 1: Neutrino-lepton four-fermion coefficients. \( G_F \) and \( \tilde{G}_e \) are in units \( 10^{-5} \text{ GeV}^{-2} \). The second and third lines of the table provide our evaluation using inputs \( (M_W, M_Z, \alpha^{(5)}) \) and \( (G_F, M_Z, \alpha^{(5)}) \) respectively. Uncertainty due to Standard Model input parameters is denoted with index \( i \). Perturbative uncertainty is denoted with index \( p \).

| inputs                        | \( G_F \)                  | \( \tilde{G}_e \) | \( \frac{c_{\tau}(M_Z)}{2\sqrt{2}G_F} \) | \( s_W^2(M_Z) \) |
|-------------------------------|-----------------------------|-------------------|------------------------------------------|------------------|
| tree level                    | \( 1.12508(85)_i \)        | \( 1.12508(85)_i \) | \( 0.22301(23)_i \)                     | \( 0.22301(23)_i \) |
| \( M_W, M_Z, \alpha^{(5)} \) | \( 1.16713(83)_i(12)_p \) | \( 1.18161(85)_i(33)_p \) | \( 0.22998(7)_i \)                     | \( 0.23123(23)_i \) |
| \( G_F, M_Z, \alpha^{(5)} \) | \( 1.1663787(6)_i \)       | \( 1.18083(4)_i(21)_p \) | \( 0.23004(3)_i \)                     | \( 0.23144(3)_i \) |

Table 2: Same as Table 1 but for neutrino-quark four-fermion coefficients, in units \( 10^{-5} \text{ GeV}^{-2} \).

| inputs                        | \( G_u \)                  | \( G_d \) | \( G_b \) | \( \frac{c_{\tau}'(M_Z)}{2\sqrt{2}V_{ub}'} \) |
|-------------------------------|-----------------------------|-------------------|-------------------|------------------------------------------|
| tree level                    | \( 1.12508(85)_i \)        | \( 1.12508(85)_i \) | \( 1.12508(85)_i \) | \( 1.12508(85)_i \) |
| \( M_W, M_Z, \alpha^{(5)} \) | \( 1.16917(83)_i(32)_p \) | \( 1.18231(85)_i(33)_p \) | \( 1.16325(83)_i(80)_p \) | \( 1.16250(2)_i(69)_p \) |
| \( G_F, M_Z, \alpha^{(5)} \) | \( 1.16841(4)_i(20)_p \)  | \( 1.18154(4)_i(21)_p \) | \( 1.16622(83)_p \)       | \( 1.165468(4)_i \) |

illustrate the overall size of electroweak corrections at \( \mu = M_Z \), the first line of the tables shows the tree level evaluation of these quantities. For this purpose, we extract the weak mixing angle from pole masses \( c_W = M_W/M_Z \) and take the electromagnetic coupling constant \( \alpha \) in the Thomson limit \[1\], \( \alpha_0^{-1} = 137.035999139(31) \). Our final results for four-fermion coefficients employ the inputs \( (17), (18), \) and \( (19) \), and are given by the third line of the tables. For comparison, the second line of the tables presents coefficients determined without the muon lifetime constraint, employing \( M_W^{PDG} = 80.379(12) \text{ GeV} \) in place of \( G_F \).

The uncertainties displayed in Tables 1 and 2 correspond to Standard Model input parameters (denoted by index “\( i \)”), and to uncalculated higher-order perturbative corrections (denoted by index “\( p \)”). The leading unaccounted corrections appear at \( \mathcal{O}(\alpha \alpha_s^2, \alpha^2) \) for leptonic coefficients and at \( \mathcal{O}(\alpha \alpha_s) \) for quark coefficients. \[10\] The perturbative uncertainty, at the level \( 10^{-4} \), is estimated by varying the matching scale in the range \( M_Z^2/2 \leq \mu^2 \leq 2M_Z^2 \) for scale-invariant quantities. Note that the scale dependence is larger for \( \tilde{G}_b \) due to the large CKM matrix element \( V_{tb} \). This uncertainty is subdominant to SM parameter input uncertainty when using EW inputs, represented by the second line of Tables 1 and 2. With the \( G_F \) constraint, represented by the third line of Tables 1 and 2, perturbative uncertainty dominates but is well below current hadronic uncertainties for neutrino scattering applications.

As an immediate by-product of our analysis, we may compare the electroweak scale determination of \( G_F \) to the low-scale muon lifetime measurement. Such an analysis was discussed in detail in Refs. [78, 79] with last update in Ref. [80]. The precision of EW parameters at high energies \[1\] and of muon lifetime measurements \[81–84\] have significantly increased after the publication of Refs. [78–80]. The comparison of \( G_F = 1.16713(83)(12) \times 10^{-5} \text{ GeV}^2 \) at \( \mu = M_Z \) versus \( G_F = 1.1663787(6) \times 10^{-5} \text{ GeV}^2 \) at \( \mu = m_{\mu} \) shows only 0.9\( \sigma \) tension with 6 \( \times \) 10\(^{-4} \) relative difference.

\[10\] For a discussion of higher-order corrections, see Refs. [71–77].
4 Running

The RGE in the neutral current sector mixes all quark and lepton operators involving a given neutrino flavor:

$$\frac{d}{d \ln \mu^2} c^f_{Qf} = \frac{1}{2} \sum f', s' \beta_f c^f_{s' Qf'}.$$  \hspace{1cm} (20)

Here $\beta_f$ denotes the contribution of fermion $f$ to the QED beta function:

$$\frac{d \ln \alpha}{d \ln \mu^2} = \sum_f \beta_f,$$  \hspace{1cm} (21)

where for leptons and quarks we have

$$\beta_\ell = \frac{\alpha}{\pi} \left( \frac{1}{3} + \frac{\alpha}{4\pi} \right), \quad \beta_q = Q^2 N_c \frac{\alpha}{\pi} \left[ \frac{1}{3} + \frac{\alpha_s}{3\pi} + \left( \frac{125}{48} - \frac{11n_f}{72} \right) \frac{\alpha_s^2}{3\pi^2} + \frac{\alpha}{4\pi} Q^2 \right],$$  \hspace{1cm} (22)

with $N_c = 3$. The $O(\alpha^3)$ contribution does not change results of this work within significant digits. In the charged current sector, the coefficient $c$ is scale independent while the anomalous dimension of $c_{qq'}$ is not zero [52, 85–90].

$$\frac{d \ln c_{qq'}}{d \ln \mu^2} = -\frac{\alpha}{2\pi} \left( 1 - \frac{\alpha_s}{4\pi} \right).$$  \hspace{1cm} (23)

Accounting for the scale-dependence of $\alpha$ and MS vector-boson masses $M_W$ and $M_Z$, it is readily verified that the $\mu$-dependence of the one-loop matching coefficients, (4) and (5), obeys the evolution equations derived from the UV structure of effective theory operators, Eqs. (20) and (23).

We solve the RGE between particle thresholds numerically for the coefficients $c_R (\mu)$, $c_{qq'} (\mu)$, using high order running of $\alpha (\mu)$ and $\alpha_s (\mu)$. Integrating out heavy $b$ and $c$ quarks, we account for small threshold matching effects through $O(\alpha\alpha_s)$. We use the following values for bottom and charm quark masses [1],

$$m_b(m_b) = 4.18(3) \text{ GeV}, \quad m_c(m_c) = 1.28(2) \text{ GeV}.$$  \hspace{1cm} (24)

The running from EW scale to GeV energies yields the effective couplings in Table 3. We have employed the weak scale couplings from the last row of each of Tables 1 and 2. The uncertainty from higher-order perturbative corrections is estimated by varying matching scales $\mu_Z^2 = M_Z^2$, $\mu_b^2 = m_b^2$, $\mu_c^2 = m_c^2$ and $\mu_\tau^2 = m_\tau^2$ up and down by a factor of 2. The $\mu_Z^2$ and $\mu_{\ell}^2$ variations are numerically small. This perturbative uncertainty is added in quadrature to the propagated error of SM inputs. The total error is dominated by weak scale matching as estimated by $\mu_Z$ variation.

5 Low-energy leptonic theory

For certain processes, the momentum transfer entering the loop of Fig. 1 is below the scale where perturbative QCD is applicable. An important example is the neutrino-electron scattering process where momentum transfer is bounded by $Q^2 \leq 2m_\ell E_\nu \lesssim 0.01 \text{ GeV}^2$ for incoming neutrino energy $E_\nu \lesssim 10 \text{ GeV}$. This process is described by an effective theory in which hadrons and heavy leptons are integrated out and only neutrinos and electrons remain as dynamical fields. Here we determine the four-fermion couplings in this low-energy leptonic theory. We perform the matching in a series of steps, first integrating...

\footnote{The contribution of order $\alpha\alpha_s$ to anomalous dimension in Eq. (23) is based on results of Ref. [52] obtained within $a = -1$ scheme in Feynman-'t Hooft gauge. However, it is expected to be gauge independent [91].}
Table 3: Effective couplings in the Fermi theory of neutrino-fermion scattering with \( n_f \) quark flavors, at different renormalization scales, in units \( 10^{-5} \text{ GeV}^{-2} \). \( \tau \) is not present in the theory described by last three rows. The final row gives couplings to \( \nu_\tau \).

| \( \mu \) | \( n_f \) | \( c_{\ell \ell}^{\nu e} \cdot \ell = \ell' \) | \( -c_{\ell \ell}^{\nu e} \cdot \ell \neq \ell' \) | \( c_{\ell \ell} \) | \( c_{R}^{e\ell} / V_{R00} \) |
|---|---|---|---|---|---|
| \( M_Z \) | 5 | 2.38798(32) | 0.91103(32) | 0.75891(60) | 3.29644(1) |
| \( m_{\mu} \) | 4 | 2.39676(33) | 0.90226(33) | 0.76769(60) | 3.32110(6) |
| 2 GeV | 4 | 2.39818(33) | 0.90084(33) | 0.76911(60) | 3.32688(8) |
| \( m_{\tau} \) | 4 | 2.39841(33) | 0.90060(33) | 0.76935(60) | 3.32776(8) |
| \( m_{e} \) | 3 | 2.39912(33) | 0.89989(35) | 0.77006(61) | 3.33029(9) |

| \( \mu \) | \( n_f \) | \( c_{L}^{\nu e} \) | \( -c_{R}^{\nu e} \) | \( -c_{L}^{\nu e} \) | \( c_{R}^{\nu e} \) |
|---|---|---|---|---|---|
| \( M_Z \) | 5 | 1.14745(13) | 0.50494(38) | 1.41181(12) | 0.25277(20) |
| \( m_{\mu} \) | 4 | 1.14160(13) | 0.51079(38) | 1.41525(12) | 0.25570(20) |
| 2 GeV | 4 | 1.14065(13) | 0.51173(38) | 1.41478(12) | 0.25617(20) |
| \( m_{\tau} \) | 4 | 1.14049(14) | 0.51189(38) | 1.41470(12) | 0.25625(20) |
| \( m_{e} \) | 3 | 1.14002(16) | 0.51236(39) | 1.41447(13) | 0.25648(20) |
| | | 1.13945(16) | 0.51294(39) | 1.41418(13) | 0.25677(20) |

Table 4: Effective couplings (in units \( 10^{-5} \text{ GeV}^{-2} \)) in the Fermi theory of neutrino-lepton scattering, at renormalization scale \( \mu = m_{\mu} \) (in the theory with neutrinos, \( e \) and \( \mu \)) and in the low-energy limit at \( \mu = m_{e} \) (in the theory with neutrinos and \( e \)). The error is dominated by the light-quark contribution.

| \( \mu = m_{\mu} \) | \( c_{\ell \ell}^{\nu e} \cdot \ell = \ell' \) | \( c_{\ell \ell}^{\nu e} \cdot \ell \neq \ell' \) | \( c_{\ell}^{\nu e} \) | \( c_{R}^{\nu e} \) | \( c_{L}^{\nu e} \) | \( c_{L}^{\nu e} \) |
|---|---|---|---|---|---|---|
| \( \mu = m_{e} \) | 2.3897(29) | -0.8994(29) | -0.8921(29) | 0.7706(29) | 0.7706(29) | 0.7779(29) |

Out hadrons, then muons. For the hadronic contribution to effective couplings, we employ results from Ref. 45. In a final step, we also describe the extreme low-energy limit applicable to forward scattering on nonrelativistic electrons. Note that integrating out a heavy charged lepton violates the equality of couplings \( c_{\ell \ell}^{\nu e} \) for any \( \ell \) and \( \ell' \), and of \( c_{L}^{\nu e} \) for any \( \ell \neq \ell' \). Consequently, the conventional definition of Weinberg angle in terms of effective couplings is flavor-independent only above the \( \tau \)-lepton mass scale.

In the theory below the hadron mass scale, i.e., \( \mu \lesssim m_{\pi} \), only neutrino couplings to muons and electrons contribute to running. The renormalization group equations in the theory above the muon mass scale, i.e., \( m_{\mu} \leq \mu \leq m_{\tau} \), are given by

\[
\frac{d}{d \ln \mu^2} (c_{\ell \ell}^{\nu e} + c_{\ell \ell}) = \frac{d}{d \ln \mu^2} (c_{\ell \ell}^{\nu e} + c_{R}) = \beta_{\ell} (c_{\ell \ell}^{\nu e} + c_{\ell \ell}^{\nu e} + 2c_{R}) , \quad \frac{d \ln \alpha}{d \ln \mu^2} = 2\beta_{\ell} , \quad (25)
\]

where \( c_{R}^{\nu e} = c_{R}^{\nu e} = c_{R} \). The solution is

\[
c_{R} (\mu) = \frac{\alpha (\mu)}{\alpha (\mu_0)} c_{R} (\mu_0) + \left( 1 - \frac{\alpha (\mu)}{\alpha (\mu_0)} \right) \frac{G_{\nu e} - G_{\nu F}}{\sqrt{2}} . \quad (26)
\]

In the theory below the muon mass scale and above the electron mass scale, i.e., \( m_{e} \leq \mu \leq m_{\mu} \), RGEs for different neutrino flavors decouple:

\[
\frac{d}{d \ln \mu^2} (c_{\ell \ell}^{\nu e} + c_{\ell \ell}^{\nu e}) = \beta_{\ell} (c_{\ell \ell}^{\nu e} + c_{\ell \ell}^{\nu e}) , \quad \frac{d}{d \ln \mu^2} (c_{\ell \ell}^{\nu e} + c_{R}^{\nu e}) = \beta_{\ell} (c_{\ell \ell}^{\nu e} + c_{R}^{\nu e}) , \quad \frac{d \ln \alpha}{d \ln \mu^2} = \beta_{\ell} . \quad (27)
\]
resulting in distinct right-handed couplings in the scattering of electron- and muon-type neutrinos.\(^ {12} \)

\[
\begin{align*}
\nu^e_{\mu e} (\mu) &= \frac{\alpha (\mu)}{\alpha (m_\mu)} c_R (m_\mu) + \left( 1 - \frac{\alpha (\mu)}{\alpha (m_\mu)} \right) \tilde{G}_e \frac{1}{\sqrt{2}} \\
\nu^e_{\mu e} (\mu) &= \frac{\alpha (\mu)}{\alpha (m_\mu)} c_R (m_\mu) + \left( 1 - \frac{\alpha (\mu)}{\alpha (m_\mu)} \right) \tilde{G}_e - 2G_F. \quad (28)
\end{align*}
\]

For momentum transfers below the electron mass scale, the effective theory in the one-electron sector describes the interaction of neutrinos with a static electron source.\(^ {13} \) In the theory valid for \( \mu \leq m_e \), the electron is no longer involved in dynamics and the four-fermion couplings are scale invariant:

\[
\nu_{\ell e}^e (\mu) = c_{\ell R}^e (m_e), \quad \ell = e, \mu, \tau. \quad (30)
\]

These effective couplings may be interpreted as effective radii, analogous to the description of low-momentum neutron scattering on charged particles [92, 93]. In detail, we may define the effective radius of interaction as

\[
-Q \tilde{e}_0^2 \tilde{e}_0^2 e_{\nu e} = \frac{1}{2} \left[ \nu_{\ell e}^e (m_e) + \nu_{\mu e}^e (m_e) \right], \quad (31)
\]

where \( e_0^2 = 4\pi\alpha_0 \) denotes the QED coupling in the low-energy limit. The matching from the \( n_f = 3 \) or \( n_f = 4 \) quark level theory to the leptonic theory involves nonperturbative QCD. Taking the relevant photon vacuum polarization tensor \( \Pi(0) \) from experimental data, the result for electron neutrino effective radius is

\[
r_{\nu e}^2 = \left[ 40.05 + 0.36 \left( \frac{\Pi(0)}{\Pi(0)} - 1 \right) \right] \times 10^{-6} \text{fm}^2 = (40.05 \pm 0.08) \times 10^{-6} \text{fm}^2. \quad (32)
\]

The uncertainty 0.08 fm\(^2 \) results from \( \Pi(0)/\Pi(0) = 1 \pm 0.2 \) as in Ref. [45].

The difference between radii for different neutrino flavors, \( \nu_\ell \) and \( \nu_\nu \), is independent of the target particle \( T \), when \( T \) is distinct from the charged lepton partner of the incident neutrinos. Examples include \( \nu_\ell = \nu_\mu \) and \( \nu_\nu = \nu_\tau \) scattering on electrons, \( T = e \); or any flavors \( \nu_\ell \) and \( \nu_\nu \) scattering on protons or neutrons. In these cases, the difference \( r_{\nu e}^2 - r_{\nu e}^2 \equiv r_{\nu e}^2 - r_{\nu e}^2 \) represents a radius that is intrinsic to the neutrino species.\(^ {14} \) Starting from the theory with three active charged leptons (e.g. the \( n_f = 4 \) flavor theory renormalized at \( \mu = 2 \text{GeV} \) in Table 3), the differences in radii are induced by loops of charged leptons. Summing over leptons in the loop and using the relations (4), we obtain\(^ {15} \)

\[
\frac{e_0^2}{6} (r_{\nu_\mu}^2 - r_{\nu_\nu}^2) = -\frac{\alpha_0}{2\pi} [\Pi(0, m_\ell, \mu) - \Pi(0, m_\mu, \mu)] c(\mu) = \frac{\sqrt{2} \alpha_0 G_F}{3\pi} \ln \frac{m_\ell^2}{m_\mu^2} \left( 1 + \frac{3\alpha(\mu)}{4\pi} + \ldots \right). \quad (33)
\]

In particular,

\[
r_{\nu_\mu}^2 - r_{\nu_\nu}^2 = 3.476 \times 10^{-7} \text{fm}^2,
\]

\(^ {12} \)It is convenient to perform the matching between effective theories with and without the muon degree of freedom at exactly \( \mu = m_\mu \). In this case, threshold matching corrections to effective couplings and running \( \alpha \) vanish up to neglected corrections of relative order \( \alpha^2 \).

\(^ {13} \)Corrections to the static limit may be described by nonrelativistic EFT.

\(^ {14} \)Our definition of radii, in terms of four Fermi operators of left-handed neutrinos, is independent of the neutrino mass sector, in particular whether right-handed neutrinos are introduced to form a Dirac mass, or a Majorana mass term is included. For a general discussion of electromagnetic interactions of Majorana neutrinos, see Ref. [94, 95].

\(^ {15} \)Our normalization for the lepton vacuum polarization function is as in Ref. [45]: \( \Pi(0, m, \mu) = N_c \left[ \frac{1}{2} \ln \frac{m_e^2 + C_F}{m_e^2} \right] \times \ln \frac{m_e^2 + C_F}{m_e^2} \), in terms of pole mass \( m \). For QED, we take \( N_c \to 1, C_F \to 1, \alpha_s \to \alpha \). We also have replaced \( \alpha(\mu) \) by \( \alpha_0 \) including vacuum polarization, electron vertex and field renormalization corrections.
\[ r^2_{\nu_\tau} - r^2_{\nu_\mu} = 1.840 \times 10^{-7} \text{fm}^2. \]  

For the special case of \( \ell = \tau \) and \( \ell' = \mu \), the same result is obtained from Eq. (31) upon substitution of low-energy coefficients from Table 4:  
\[ r^2_{\nu_\tau} - r^2_{\nu_\mu} = (3/4\pi a_0)(c_{L}(m_\nu) + c_{R}(m_\nu) - c_{L}(m_\mu) - c_{R}(m_\mu)). \]

For the evaluation using Table 4, hadronic corrections cancel in the difference. The evaluations differ by matching corrections of \( O(\alpha^2) \), present in Eq. (33) but omitted in the RG analysis; this difference impacts digits not displayed in Table 4.

The effective potential in matter can be expressed in terms of the effective radii as a sum over all target particles
\[ V^{\nu_\ell} = -\frac{e^2_0}{6} \sum_T n_T Q_T r^2_{\nu_\ell T}, \]

with particle charge \( Q_T \) and number density \( n_T \). In a charge-neutral medium consisting of protons, neutrons and electrons, differences \( r^2_{\nu_\tau T} - r^2_{\nu_\mu T} \) enter with an opposite sign for positively and negatively charged particles resulting in \( V^{\nu_\tau} = V^{\nu_\mu} \) up to corrections suppressed by powers \( m^2_\tau/M^2_W \) [96]. Such differences would appear as corrections to the weak scale matching coefficients in (1), and as \( O(G^2_F) \) corrections to forward scattering computed using (1).

### 6 Summary

We have determined the low-energy neutrino-fermion effective field theory in \( \overline{\text{MS}} \) renormalization scheme as the basis for neutrino scattering on electrons [45], nucleons and nuclei at sub-percent level. Electroweak scale coupling constants were determined by matching to the Standard Model including complete one-loop electroweak corrections, two-loop mixed QCD-electroweak corrections in the lepton sector, and the \( \gamma_5 \) scheme dependence of effective operators. Among the eleven independent effective couplings, only two depend on the scale. Solving the renormalization group equation for these couplings, we have determined all parameters in the quark level effective Lagrangian with \( n_f = 3 \) or \( n_f = 4 \) quark flavors. A complete error budget due to parametric inputs and higher-order perturbative corrections is presented. Using experimental data and SU(3) flavor symmetry constraints, we have evaluated hadronic contributions to determine the matching onto the low-energy theory involving leptons and the extreme low-energy theory describing neutrino interactions with static electric charge distributions. The hadronic correction provides the dominant source of uncertainty in low-energy neutral-current interactions. As a byproduct of our analysis, we revisited the comparison of the Fermi coupling constant \( G_F \) evaluated at the electroweak scale to extractions from muon lifetime measurements; the comparison shows only 0.9σ tension with \( 6 \times 10^{-4} \) relative difference.

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A \ MS \ masses \ of \ vector \ bosons \ and \ \mathcal{O}(\alpha \alpha_s G_F) \ corrections

To relate pole and \MS masses of vector bosons at some scale \(\mu\), we consider renormalization of the vector boson propagator \(D_{\mu \nu}\), using the example of the Z boson as illustration:

\[
D^{\mu \nu} = i \frac{-g^{\mu \nu} + (1 - \xi_Z) \frac{\delta^{\mu \nu}}{q^2 - \xi_Z M^2_Z}}{q^2 - M^2_Z} + i \frac{(1 - \xi_Z) \frac{\delta^{\mu \nu}}{q^2 - \xi_Z M^2_Z}}{q^2 - M^2_Z}, \tag{36}
\]

with vector boson self-energy

\[
\Sigma_{\rho \sigma}^Z = \left( g_{\rho \sigma} \frac{-q_{\rho} q_{\sigma}}{q^2} \right) \Sigma_{T}^Z + \frac{q_{\rho} q_{\sigma}}{q^2} \Sigma_{L}^Z. \tag{37}
\]

The pole mass is determined by the inverse propagator:

\[
D_{\mu \nu}^{-1} = i g_{\mu \nu} \left( q^2 - M^2_Z \right) + i \frac{1 - \xi_Z}{\xi_Z} q_{\mu} q_{\nu} + i \left( g_{\mu \nu} \frac{-q_{\rho} q_{\sigma}}{q^2} \right) \Sigma_{T}^Z + i \frac{q_{\rho} q_{\sigma}}{q^2} \Sigma_{L}^Z. \tag{38}
\]

We have cross checked expressions in terms of scalar integrals from Refs. \[20,102\] in Feynman-'t Hooft gauge and have verified the gauge independence of mass renormalization at one loop when including Higgs tadpoles. For numerical evaluations, we exploit analytical expressions for scalar integrals obtained following Refs. \[103,104\], work in Feynman-'t Hooft gauge and do not include Higgs tadpoles. We account for the leading QCD corrections of Refs. \[56-61\] representing the fermionic contribution to the following \MS differences (cf. Ref. \[56\]) at renormalization scale \(\mu\), in terms of \MS mass of top quark:

\[
\frac{\hat{\Sigma}_T^W (\mu, q^2 = M^2_W)}{M^2_W} - \frac{\hat{\Sigma}_T^W (\mu, q^2 = 0)}{M^2_W} = \frac{\alpha}{2 \pi s_W^2} B^{(f)}_0 + \frac{\alpha_s}{\pi} B^{(f)}_{\text{QCD}}, \tag{39}
\]

\[
\frac{\hat{\Sigma}_T^Z (\mu, q^2 = M^2_Z)}{M^2_Z} - \frac{\hat{\Sigma}_T^Z (\mu, q^2 = M^2_Z)}{M^2_Z} = \frac{\alpha}{2 \pi s_W^2} C^{(f)}_0 + \frac{\alpha_s}{\pi} C^{(f)}_{\text{QCD}}. \tag{40}
\]

The normalization at \(q^2 = 0\) is given by \[56,58,60,61\]

\[
\hat{\Sigma}_T^W (\mu, q^2 = 0) = \frac{3 \alpha m^2_t}{4 \pi s_W^2} \left[ -\frac{1}{4} - \frac{1}{2} \ln \frac{\mu^2}{m^2_t} + \frac{\alpha_s}{3 \pi} \left( -\frac{13}{8} + \zeta(2) - \ln \frac{\mu^2}{m^2_t} - \frac{3}{2} \ln^2 \frac{\mu^2}{m^2_t} \right) \right]. \tag{41}
\]

The \(\mathcal{O}(\alpha)\) correction is given by

\[
B^{(f)}_0 = 2 \left( \ln \frac{\mu^2}{M^2_W} + \frac{5}{3} \right) - \frac{\ln \omega_t}{2} - \omega_t \left( 1 + 2 \omega_t \right) \frac{1 + \omega_t}{2} \ln \frac{1}{1 - \omega_t}, \]

\[
C^{(f)}_0 = B^{(f)}_0 - \frac{\omega_t}{2 \pi c^2_W} \ln \frac{\mu^2}{M^2_W} - \frac{5}{3 c^2_W} \left( \frac{7}{4} - \frac{10}{3} s^2_W + \frac{40}{9} s_W^4 \right) + \frac{3 r_t}{2 c^2_W} \left( \frac{3}{4} - \Lambda (r_t) \right) - \frac{1}{8 c^2_W} \left[ 1 + \left( 1 - \frac{8}{3} s_W^2 \right)^2 \right] 6 \ln \frac{\mu^2}{M^2_Z} - \ln r_t - \frac{1}{3} + 2 (1 + 2 r_t) \left( 1 - \Lambda (r_t) \right), \tag{42}
\]

with \(\Lambda (r_t) = \sqrt{4 r_t - 1} \sin^{-1} (4 r_t)^{-1/2}\). The \(\mathcal{O}(\alpha \alpha_s)\) contribution is expressed in terms of \(B^{(f)}_{\text{QCD}}\) and \(C^{(f)}_{\text{QCD}}\):

\[
B^{(f)}_{\text{QCD}} = \ln \frac{\mu^2}{M^2_W} - 4 \zeta(3) + \frac{55}{12} + \frac{1}{2} \left[ \ln \frac{\mu^2}{m^2_t} - 4 \zeta(3) + \frac{55}{12} + 4 \omega_t \left( F_1 \left( \frac{1}{\omega_t} \right) - F_1 \left( 0 \right) \right) \right]

- \frac{1}{4} \left[ 1 + 2 \omega_t - 2 (1 - \omega_t^2) \ln \left( 1 - \frac{1}{\omega_t} \right) \right] \left[ 3 \ln \frac{\mu^2}{m^2_t} + 1 \right], \tag{43}
\]
\[ C_{\text{QCD}}^{(f)} = \frac{3}{2} \left[ \ln \frac{\mu^2}{M_W^2} + \left( -4\zeta(3) + \frac{55}{12} \right) \left( 1 - \frac{20}{9} s_W^2 \right) s_W^2 \right] - \frac{\ln 2}{r_t} + 2 \omega \ln \left( \frac{1}{\omega_t} \right) - \frac{\ln 2}{r_t} A_1 \left( \frac{1}{4 r_t} \right) - \ln \frac{\mu^2}{m_t^2} \]

\[ - \frac{1}{2 c_W^2} \left[ \frac{1}{4} \ln \frac{\mu^2}{m_t^2} + r_t V_1 \left( \frac{1}{4 r_t} \right) \right] + \frac{1}{c_W^2} \left( \frac{7 s_W^2}{3} - \frac{22}{9} s_W^4 - \frac{5}{4} \right) \ln \frac{\mu^2}{M_Z^2} \]

\[ + 2 \omega_t \left[ \frac{16}{9} s_W^2 (4 c_W^2 - 1) - \omega_t + \left( 1 - \omega_t^2 \right) \ln \left( 1 - \frac{1}{\omega_t} \right) \right] \left( \frac{3}{4} \ln \frac{\mu^2}{m_t^2} + 1 \right) \]

\[ + \frac{4 \omega_t A(r_t)}{1 - 4 r_t} \left[ 2 r_t - \frac{32}{9} r_t s_W^2 (1 - 4 c_W^2) \right] \left( \frac{3}{4} \ln \frac{\mu^2}{m_t^2} + 1 \right), \]

(44)

with functions \( F_1(x), V_1(x) \) and \( A_1(x) \) from Ref. [57] for an argument \( 0 \leq x < 1 \). QCD corrections to \( \gamma Z \) mixing may be treated similarly; the result is equivalent to replacing \( \ln \mu^2/m_t^2 \) in Eq. (9) as

\[ \ln \frac{\mu^2}{m_t^2} \rightarrow \ln \frac{\mu^2}{m_t^2} + \frac{\alpha_s}{\pi} \left( \frac{13}{12} - \ln \frac{\mu^2}{m_t^2} \right). \]

(45)

In the \( \overline{\text{MS}} \) renormalization scheme, masses of vector bosons are related by \( M_W(\mu) = M_Z(\mu) c_W(\mu) \) which we exploit for the evaluation of \( s_W(\mu) \). To determine \( M_W(\mu) \) and \( M_Z(\mu) \), we solve a system of equations:

\[ G_F = \frac{\pi \alpha(\mu)}{\sqrt{2} M_W^2(\mu) s_W^2(\mu)} g(\mu), \]

(46)

\[ (M_W^p)^2 = M_Z^2(\mu) - \Sigma_T^Z(\mu), \]

(47)

with \( \overline{\text{MS}} \) masses entering \( g(\mu) \) and \( \Sigma_T^Z(\mu) \). The pole mass of \( Z \) boson \( M_W^p \) is given in terms of the center of the \( Z \) peak as reported in PDG, \( M_Z^{\text{PDG}} \), and the inclusive width \( \Gamma_Z \) as

\[ (M_Z^p)^2 + \Gamma_Z^2 = (M_Z^{\text{PDG}})^2. \]

(48)

We use the top quark pole mass, \( m_t^p = m_t^{\text{PDG}} \), as input to determine the \( \overline{\text{MS}} \) mass as a function of renormalization scale \( \mu \), \( m_t(\mu) \), according to Refs. [105–110]. We do not consider renormalon effects which can cause an ambiguity in top quark \( \overline{\text{MS}} \) mass around 110 MeV [111]. After this determination, we evaluate \( M_W(\mu) \) and \( M_Z(\mu) \) solving Eqs. (46) and (47).

For our numerical analysis, we consider the top quark mass, Higgs boson mass, electromagnetic coupling constant, strong coupling constant, Fermi constant and \( Z \)-boson mass as independent inputs and propagate errors of these parameters. From the inputs of Eqs. (17), (18) and (19), we obtain the following \( \overline{\text{MS}} \) masses in our renormalization scheme:

\[ M_Z(M_Z) = 92.3499(82) \text{ GeV}, \]
\[ M_W(M_Z) = 80.961(8) \text{ GeV}, \]
\[ m_t(M_Z) = 170.9(4) \text{ GeV}. \]

(49)

The Weinberg angle is determined from the ratio of \( \overline{\text{MS}} \) masses, \( c_W(M_Z) = M_W(M_Z)/M_Z(M_Z) \), as

\[ s_W^2(M_Z) = 1 - \frac{M_W^2(M_Z)}{M_Z^2(M_Z)} = 0.23144 \pm 0.00003. \]

(50)

In expressions (49, 50) above, we present only the error propagated from the uncertainty of the Standard Model parameters. The error of unaccounted high order perturbation theory is discussed in the main text.

\[ ^{16} \text{For the second row in Tables 1 and 2, we exploit } (M_W^p)^2 = M_W^2(\mu) - \Sigma_T^W(\mu) \text{ instead of Eq. (46).} \]
B QED and QCD couplings

In order to evaluate the expressions (4) and (5) at \( \mu = M_Z \), we require values for \( M_W(\mu) \), \( M_Z(\mu) \), \( s_W(\mu) \) and \( \alpha(\mu) \), evaluated in the \( \overline{\text{MS}} \) scheme without tadpoles, with the full SM particle content. We begin by deriving the necessary QED and QCD couplings from Ref. [1], which were defined in the theory without top quark (cf. Refs. [55, 68]). We use the same precision of running and matching as described in Section 3.7 The translation from the inputs (17) results in

\[
\alpha^{\text{SM}}(M_Z) - 1 = 128.120(10), \quad \alpha_s^{\text{SM}}(M_Z) = 0.1176(16). \tag{51}
\]

For comparison, our inputs together with solution of RGEs and threshold matching conditions yield \([\alpha^{(4)}(\mu = m_\tau)]^{-1} = 133.476(7)\), and \(\alpha_s^{(4)}(\mu = m_\tau) = 0.325(15)\).

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