On the vacua in the massless Thirring model

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Abstract

We calculate the most general effective potential for the massless Thirring model in dependence on all local fields of collective fermion–antifermion excitations. We analyse the minima of this potential describing different vacua of the quantum system. We confirm the existence of the absolute minimum found in EPJC 20, 723 (2001) corresponding to a chirally broken phase of the massless Thirring model. As has been shown in EPJC 20, 723 (2001) this minimum is stable under quantum fluctuations.

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1 Introduction

The massless Thirring model \[1\] is a theory of a self–coupled Dirac field \(\psi(x)\)

\[ \mathcal{L}_{\text{Th}}(x) =: \bar{\psi}(x)i\gamma^\mu \partial_\mu \psi(x) - \frac{1}{2} g :\bar{\psi}(x)\gamma^\mu \psi(x)\gamma_\mu \psi(x) :_\mu, \] (1.1)

where \(g\) is a dimensionless coupling constant that can be both positive and negative as well and \(\psi(x)\) is a spinor field with two components \(\psi_1(x)\) and \(\psi_2(x)\). The products of fermion operators in the Lagrangian (1.1) are taken in the normal–ordered form at the scale \(\mu \to 0\).

The massless Thirring model describes a system of self–coupled fermions with a non–trivial four–fermion interaction. In quantum field theoretic models, defined in 3+1–dimensional space–time \[2\]–\[6\], the four–fermion interaction is responsible for a non–trivial vacuum with the properties of a BCS–type superconducting ground state \[7\].

Recently we have shown \[8\]–\[18\] that the massless Thirring model possesses a non–perturbative phase with the BCS–type wave function \[8\].

It is well–known that the massless Thirring model is a solvable model. All solutions of the massless Thirring model have been found in the chiral symmetric phase \[19\]–\[22\]. The most general solution of the massless Thirring model has been obtained by Hagen \[21, 23\]. Hagen found a one–parameter family of solutions of the massless Thirring model parameterized by the parameter \(\xi\). For \(\xi = 1/2\) Hagen’s solution coincides with Johnson’s solution \[19\].

The massless Thirring model possesses four collective fermion–antifermion excitations: \(\psi_1^\dagger(x)\psi_1(x), \psi_2^\dagger(x)\psi_2(x), \psi_1^\dagger(x)\psi_2(x)\) and \(\psi_2^\dagger(x)\psi_1(x)\). In a Lorentz covariant form they can be written as

\[
\begin{align*}
A^0(x) & \sim \bar{\psi}(x)\gamma^0 \psi(x) = \psi_1^\dagger(x)\psi_1(x) + \psi_2^\dagger(x)\psi_2(x), \\
A^1(x) & \sim \bar{\psi}(x)\gamma^1 \psi(x) = \psi_1^\dagger(x)\psi_1(x) - \psi_2^\dagger(x)\psi_2(x), \\
\sigma(x) & \sim \bar{\psi}(x)\psi(x) = \psi_2^\dagger(x)\psi_1(x) + \psi_1^\dagger(x)\psi_2(x), \\
\varphi(x) & \sim \bar{\psi}(x)\gamma^5 \psi(x) = \psi_2^\dagger(x)\psi_1(x) - \psi_1^\dagger(x)\psi_2(x),
\end{align*}
\] (1.2)

where \(A^0(x)\) and \(A^1(x)\) are the time and spatial components of the 2–vector \(A^\mu(x) = (A^0(x), A^1(x))\); \(\sigma(x)\) and \(\varphi(x)\) are chiral partners under chiral \(U(1) \times U(1)\) rotations \[8\].

In polar representation the collective excitations \(\sigma(x)\) and \(\varphi(x)\) are defined by \(\sigma(x) = \rho(x) \cos \vartheta(x)\) and \(\varphi(x) = \rho(x) \sin \vartheta(x)\). In the chirally broken phase of the massless Thirring model the vacuum expectation value of the field \(\rho(x)\) does not vanish and is equal to \(\langle \rho(x) \rangle = M\), where \(M\) is the dynamical mass of Thirring fermion fields quantized in the chirally broken phase \[8\]. \(^3\) As has been shown in \[8\]–\[18\], the evolution of fermions in the massless Thirring model obeys the constant of motion

\[ :[\bar{\psi}(x)\psi(x)]^2 + [\bar{\psi}(x)i\gamma^5 \psi(x)]^2 := C, \] (1.3)
where \( C = M^2/g^2 \). 

In terms of the components of the fermion fields the four-fermion interaction reads

\[
- \frac{1}{2} : \bar{\psi}(x) \gamma_{\mu} \psi(x) \bar{\psi}(x) \gamma^{\mu} \psi(x) := -2 : \psi_{1}(x) \psi_{1}(x) \psi_{2}(x) \psi_{2}(x) :. \tag{1.4}
\]

The chiral invariant four-fermion interaction, containing the scalar and pseudoscalar four-fermion vertices, is equal to

\[
\frac{1}{2} : (\bar{\psi}(x) \psi(x))^2 + (\bar{\psi}(x) i \gamma^5 \psi(x))^2 : = : \psi_{1}^\dagger(x) \psi_{2}(x) \psi_{1}(x) \psi_{2}(x) : + : \psi_{2}^\dagger(x) \psi_{1}(x) \psi_{1}^\dagger(x) \psi_{2}(x) :. \tag{1.5}
\]

Using equal-time canonical anti-commutation relations

\[
\{ \psi_{i}(x^0, x^1), \psi_{j}^\dagger(x^0, y^1) \} = \delta_{ij} \delta(x^1 - y^1), \tag{1.6}
\]

where \( i(j) = 1, 2 \), we reduce the r.h.s. of (1.5) to the form

\[
\frac{1}{2} : (\bar{\psi}(x) \psi(x))^2 + (\bar{\psi}(x) i \gamma^5 \psi(x))^2 : = -2 : \psi_{1}^\dagger(x) \psi_{1}(x) \psi_{2}(x) \psi_{2}(x) : + \delta(0) : \psi_{1}^\dagger(x) \psi_{1}(x) + \psi_{2}^\dagger(x) \psi_{2}(x) :. \tag{1.7}
\]

Using (1.7) the four-fermion interaction of Thirring fermions can be transcribed into the form

\[
- \frac{1}{2} : \bar{\psi}(x) \gamma_{\mu} \psi(x) \bar{\psi}(x) \gamma^{\mu} \psi(x) : = \frac{1}{2} : (\bar{\psi}(x) \psi(x))^2 + (\bar{\psi}(x) i \gamma^5 \psi(x))^2 : - \delta(0) : \bar{\psi}(x) \gamma^0 \psi(x) :. \tag{1.8}
\]

Hence, one can conclude that the four-fermion interaction of Thirring fermions is responsible for the description of vector, scalar and pseudoscalar fermion–antifermion collective excitations.

Following Nambu and Jona–Lasino we introduce these excitations with coupling constants \( g_1 \) and \( g_2 \). However, in our case, unlike the Nambu–Jona–Lasino approach, the coupling constant \( g_1 \) and \( g_2 \) obey the constraint \( g_1 + g_2 = g \). The four-fermion interactions can then be rewritten in the following form

\[
- \frac{1}{2} g : \bar{\psi}(x) \gamma_{\mu} \psi(x) \bar{\psi}(x) \gamma^{\mu} \psi(x) := - \frac{1}{2} g_1 : \bar{\psi}(x) \gamma_{\mu} \psi(x) \bar{\psi}(x) \gamma^{\mu} \psi(x) : + \frac{1}{2} g_2 : (\bar{\psi}(x) \psi(x))^2 + (\bar{\psi}(x) i \gamma^5 \psi(x))^2 : - g_2 \delta(0) : \bar{\psi}(x) \gamma^0 \psi(x) :. \tag{1.9}
\]

Due to (1.9) the Lagrangian (1.1) reads

\[
\mathcal{L}_{\text{Th}}(x) = : \bar{\psi}(x) (i \gamma^\mu \partial_\mu - g_2 \delta(0) \gamma^0) \psi(x) : + \frac{1}{2} g_1 : \bar{\psi}(x) \gamma_{\mu} \psi(x) \bar{\psi}(x) \gamma^\mu \psi(x) : + \frac{1}{2} g_2 : (\bar{\psi}(x) \psi(x))^2 + (\bar{\psi}(x) i \gamma^5 \psi(x))^2 :. \tag{1.10}
\]
The term $g_2 \delta(0) \gamma^0$ can be removed by a time-dependent phase transformation $\psi(x) \rightarrow \psi'(x) = e^{-i g_2 \delta(0) x^0} \psi(x)$. This gives

$$
\mathcal{L}_{\text{Th}}(x) = : \bar{\psi}(x) i \gamma^\mu \partial_\mu \psi(x) : - \frac{1}{2} g_1 : \bar{\psi}(x) \gamma^\mu \psi(x) \bar{\psi}(x) \gamma_\mu \psi(x) : + \frac{1}{2} g_2 : (\bar{\psi}(x) \psi(x))^2 + (\bar{\psi}(x) i \gamma^5 \psi(x))^2 : .
$$

(1.11)

The massless Thirring model can be represented in terms of local fields of fermion–antifermion collective excitations within the path–integral approach. In accordance with (1.11), the generating functional of correlation functions we define as

$$
Z_{\text{Th}}[s, p, a_\mu] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp i \int d^2 x \left\{ \bar{\psi}(x) i \hat{\partial} \psi(x) - \frac{1}{2} g_1 \bar{\psi}(x) \gamma_\mu \psi(x) \bar{\psi}(x) \gamma^\mu \psi(x) \\
+ \frac{1}{2} g_2 (\bar{\psi}(x) \psi(x))^2 + \frac{1}{2} g_2 (\bar{\psi}(x) i \gamma^5 \psi(x))^2 + \bar{\psi}(x) \psi(x) s(x) \\
+ \bar{\psi}(x) i \gamma^5 \psi(x) p(x) + \bar{\psi}(x) \gamma^\mu \psi(x) a_\mu(x) \right\},
$$

(1.12)

where $s(x), p(x)$ and $a_\mu(x)$ are external sources of scalar, pseudoscalar and vector fermion–antiferom densities. The generating functional of correlation functions (1.12) is normalized by $Z_{\text{Th}}[0, 0, 0] = 1$.

In terms of $Z_{\text{Th}}[s, p, a_\mu]$ the fermion condensate is defined by

$$
\frac{1}{i} \frac{\delta Z_{\text{Th}}[s, p, a_\mu]}{\delta s(x)} \bigg|_{s=p=a_\mu=0} = \langle \bar{\psi}(x) \psi(x) \rangle.
$$

(1.13)

The linearization of the four–fermion interactions we carry out in the usual way

$$
Z_{\text{Th}}[s, p, a_\mu] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp i \int d^2 x \left\{ \bar{\psi}(x) (i \hat{\partial} + \hat{A}(x) + a(x) - \sigma(x) + s(x) \\
- i \gamma^5 \varphi(x) + i \gamma^5 p(x)) \psi(x) + \frac{1}{2 g_1} A_\mu(x) A_\mu^*(x) - \frac{1}{2 g_2} (\sigma^2(x) + \varphi^2(x)) \right\}.
$$

(1.14)

Making a change of variables $A_\mu + a_\mu \rightarrow A_\mu, \sigma - \sigma \rightarrow \sigma$ and $\varphi - p \rightarrow \varphi$ we get

$$
Z_{\text{Th}}[s, p, a_\mu] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp i \int d^2 x \left\{ \bar{\psi}(x) (i \hat{\partial} + \hat{A}(x) - \sigma(x) - i \gamma^5 \varphi(x)) \psi(x) \\
+ \frac{1}{2 g_1} A_\mu(x) A_\mu^*(x) - \frac{1}{2 g_2} (\sigma^2(x) + \varphi^2(x)) - \frac{1}{g_1} A_\mu(x) a_\mu^*(x) + \frac{1}{2 g_1} a_\mu(x) a_\mu^*(x) \\
- \frac{1}{2 g_2} \sigma(x) s(x) - \frac{1}{2 g_2} \varphi(x) p(x) - \frac{1}{2 g_2} (s^2(x) + p^2(x)) \right\}.
$$

(1.15)

Integrating over the fermion fields we arrive at the expression

$$
Z_{\text{Th}}[s, p, a_\mu] = \int \mathcal{D}\sigma \mathcal{D}\varphi \mathcal{D}^2 A \exp i \int d^2 x \left\{ \mathcal{L}_{\text{eff}}[\sigma(x), \varphi(x), A_\mu(x)] - \frac{1}{g_1} A_\mu(x) a_\mu^*(x) \\
- \frac{1}{2 g_2} \sigma(x) s(x) - \frac{1}{2 g_2} \varphi(x) p(x) + \frac{1}{2 g_1} a_\mu(x) a_\mu^*(x) - \frac{1}{2 g_2} (s^2(x) + p^2(x)) \right\}.
$$

(1.16)
The effective Lagrangian $\mathcal{L}_{\text{eff}}[\sigma(x), \varphi(x), A^\mu(x)]$ is defined by

$$
\mathcal{L}_{\text{eff}}[\sigma(x), \varphi(x), A^\mu(x)] = -i \text{tr}\langle x | \ln (i \hat{\partial} + \hat{A} - \sigma - i\gamma^5 \varphi) | x \rangle
+ \frac{1}{2g_1} A_\mu(x) A^\mu(x) - \frac{1}{2g_2} (\sigma^2(x) + \varphi^2(x)),
$$

(1.17)

where the first term is caused by the contribution of the fermion determinant with the trace calculated over the Dirac matrices.

According to (1.14), we can write down the bosonization rules [8]

$$
\bar{\psi}(x) \gamma^\mu \psi(x) = - \frac{A^\mu(x)}{g_1}, \quad \bar{\psi}(x) \psi(x) = - \frac{\sigma(x)}{g_2}, \quad \bar{\psi}(x) i\gamma^5 \psi(x) = - \frac{\varphi(x)}{g_2}.
$$

(1.18)

In terms of the generating functional (1.16) the fermion condensate $\langle \bar{\psi}(x) \psi(x) \rangle$ is defined by

$$
\langle \bar{\psi}(x) \psi(x) \rangle = \frac{1}{i} \frac{\delta Z_{\text{Th}}[s, p, a_\mu]}{\delta s(x)} \bigg|_{s=p=a_\mu=0} = - \frac{1}{g_2} \langle \sigma(x) \rangle.
$$

(1.19)

This agrees with the bosonization rules (1.18).

The main question, which we would like to clarify below, is what kind of collective excitations is the most important for the solution of the massless Thirring model $A^\mu$ or $\sigma$ and $\varphi$? The reply on this question can be obtained by investigating the effective potential of the massless Thirring model in terms of the fields of the collective excitations.

The paper is organized as follows. In Section 2 we calculate the effective potential of the massless Thirring model in terms of the local fields of fermion–antifermion collective excitations. We investigate the minima of this potential and show that the absolute minimum corresponds only to the existence of $\sigma$ and $\varphi$ collective excitations in agreement with our analysis of the massless Thirring model carried out in [8]–[17]. In Section 3 we discuss the wave function of the ground state of the massless Thirring model around the absolute minimum of the effective potential of fermion–antifermion collective excitations. In the Conclusion we discuss the obtained results.

2 Effective potential of collective excitations in the massless Thirring model

It is well–known that the minima of the effective potential of collective excitations define the ground states of the quantum system – the vacua. Therefore, for the understanding of the problem of the vacua of the massless Thirring model we have to calculate the effective potential as a functional of $\sigma(x)$, $\varphi(x)$ and $A^\mu(x)$ fields, i.e. $V_{\text{eff}}[\sigma, \varphi, A]$.

For the calculation of the effective potential $V_{\text{eff}}[\sigma, \varphi, A]$ of the quantum field theory defined by the effective Lagrangian (1.17) we drop the contributions of derivatives of the collective fields $\sigma$, $\varphi$ and $A^\mu$. The general expression for the effective potential reads [8]

$$
V_{\text{eff}}[\sigma, \varphi, A] = - \tilde{\mathcal{L}}_{\text{eff}}[\sigma, \varphi, A]|_{\partial \sigma=\partial \varphi=\partial A=0} - \frac{1}{2g_1} A_\mu(x) A^\mu(x) + \frac{1}{2g_2} (\sigma^2(x) + \varphi^2(x)).
$$

(2.1)
The effective Lagrangian $\tilde{\mathcal{L}}_{\text{eff}}[\sigma, \varphi, A]|_{\partial\sigma = \partial\varphi = \partial A = 0}$ is determined by

$$
\tilde{\mathcal{L}}_{\text{eff}}[\sigma, \varphi, A]|_{\partial\sigma = \partial\varphi = \partial A = 0} = \frac{1}{2i} \text{tr}(x|\ln((-i \partial + A)^2 + \sigma^2 + \varphi^2)|x) = \\
= \int \frac{d^2k}{(2\pi)^2} \frac{\ln((-k + A(x))^2 + \sigma^2 + \varphi^2))}{\partial k} = \\
= \int \frac{d^2k}{(2\pi)^2} \exp(A(x) \cdot \frac{\partial}{\partial k}) \ln(-k^2 + \sigma^2 + \varphi^2). \quad (2.2)
$$

In the polar field variables $\sigma(x) = \rho(x) \cos \vartheta(x)$ and $\varphi(x) = \rho(x) \sin \vartheta(x)$ the effective potential depends only on $\rho(x)$ and $A_\mu(x)$

$$
V_{\text{eff}}[\rho, A] = - \int \frac{d^2k}{(2\pi)^2} \exp(A(x) \cdot \frac{\partial}{\partial k}) \ln(-k^2 + \rho^2) - \frac{1}{2g_1} A^2(x) + \frac{1}{2g_2} \rho^2(x), \quad (2.3)
$$

where we have denoted $A^2(x) = A_\mu(x) A^\mu(x)$.

By a Wick rotation we obtain

$$
V_{\text{eff}}[\rho, A_E] = - \int \frac{d^2k_E}{(2\pi)^2} \exp(A_E(x) \cdot \frac{\partial}{\partial k_E}) \ln(k_E^2 + \rho^2(x)) + \frac{1}{2g_1} A^2_E(x) + \frac{1}{2g_2} \rho^2(x) = \\
= - \int \frac{d^2k_E}{(2\pi)^2} \ln((k_E + A_E)^2 + \rho^2(x)) + \frac{1}{2g_1} A^2_E(x) + \frac{1}{2g_2} \rho^2(x), \quad (2.4)
$$

where $A^2_E = -A_\mu(x) A^\mu(x)$.

The calculation of the integral over $k$ reads

$$
\int \frac{d^2k}{(2\pi)^2} \ln((k + A)^2 + \rho^2) = \frac{1}{4\pi^2} \int_0^\Lambda dk \int_0^{2\pi} d\alpha \ln(k^2 + \rho^2 + A^2 + 2kA \cos \alpha) = \\
= \frac{1}{8\pi^2} \int_0^{\Lambda^2} du \int_0^{2\pi} d\alpha \ln(u + \rho^2 + A^2 + 2\sqrt{u} A \cos \alpha) = \\
= \frac{\Lambda^2}{8\pi^2} \int_0^{2\pi} d\alpha \ln(\Lambda^2 + \rho^2 + A^2 + 2\Lambda A \cos \alpha) - \frac{\Lambda^2}{8\pi} \\
- \frac{1}{16\pi^2} \int_0^{\Lambda^2} du \int_0^{2\pi} d\alpha \frac{u + \sqrt{u} A \cos \alpha}{u + \rho^2 + A^2 + 2\sqrt{u} A \cos \alpha}. \quad (2.5)
$$

In this equation $A = \sqrt{A^2_E}$ and $\Lambda$ is the ultra–violet cut–off.

In the last integral over $\alpha$ it is convenient to make a change of variables $e^{i\alpha} = z$ and recast it into the contour integral

$$
\int_0^{2\pi} d\alpha \frac{1}{u + \rho^2 + A^2 + 2\sqrt{u} A \cos \alpha} = \frac{2\pi}{A\sqrt{u}} \oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{z^2 + z \frac{u + \rho^2 + A^2}{A\sqrt{u}} + 1} =
$$
\[ \int_{|z|=1} \frac{dz}{2\pi i} \frac{1}{(z-z_1)(z-z_2)} = \frac{2\pi}{z_1-z_2} = \frac{2\pi}{\sqrt{u+\rho^2+A^2}^2 - 4uA^2} \]

\[ \frac{2\pi}{\sqrt{u^2+2(\rho^2-A^2)u+(\rho^2+A^2)^2}} \] (2.6)

where \( z_1 \) and \( z_2 \) are two roots of the denominator

\[ z_1 = -\frac{u+\rho^2+A^2}{2\Lambda\sqrt{u}} + \frac{\sqrt{(u+\rho^2+A^2)^2 - 4uA^2}}{2a\sqrt{u}} \]

\[ z_2 = -\frac{u+\rho^2+A^2}{2\Lambda\sqrt{u}} - \frac{\sqrt{(u+\rho^2+A^2)^2 - 4uA^2}}{2a\sqrt{u}} \] (2.7)

The effective potential is now defined by

\[ V_{\text{eff}}[\rho, A] = \frac{\Lambda^2}{8\pi} - \frac{\Lambda^2}{8\pi^2} \int_0^{2\pi} d\alpha \ln(\Lambda^2 + \rho^2 + A^2_E + 2\Lambda A_E \cos \alpha) \]

\[ + \frac{1}{8\pi} \int_0^{\Lambda^2} du \frac{u-\rho^2-A^2_E}{\sqrt{u^2+2(\rho^2-A^2_E)u+(\rho^2+A^2_E)^2}} + \frac{1}{2g_1} A^2_E + \frac{1}{2g_2} \rho^2, \] (2.8)

where \( A_E = \sqrt{-A_eA^e} \).

Integrating over \( u \) we obtain

\[ \frac{1}{8\pi} \int_0^{\Lambda^2} du \frac{u-\rho^2-A^2_E}{\sqrt{u^2+2(\rho^2-A^2_E)u+(\rho^2+A^2_E)^2}} = \]

\[ = \frac{1}{8\pi} \int_0^{\Lambda^2} du \frac{(u+\rho^2-A^2_E) - 2\rho^2}{\sqrt{(u^2+\rho^2-A^2_E)^2 + 4\rho^2A^2_E}} = \frac{1}{8\pi} \sqrt{(u^2+\rho^2-A^2_E)^2 + 4\rho^2A^2_E} \bigg|_0^{\Lambda^2} \]

\[ - \frac{\rho^2}{4\pi} \int_0^{\Lambda^2} du \frac{1}{\sqrt{(u^2+\rho^2-A^2_E)^2 + 4\rho^2A^2_E}} = \]

\[ = \frac{1}{8\pi} \left( \sqrt{(A^2 + \rho^2 - A^2_E)^2 + 4\rho^2A^2_E} - (\rho^2 + A^2_E) \right) \]

\[ - \frac{\rho^2}{4\pi} \ln \left[ \frac{A^2 + \rho^2 - A^2_E + \sqrt{(A^2 + \rho^2 - A^2_E)^2 + 4\rho^2A^2_E}}{2\rho^2} \right]. \] (2.9)

The integral over \( \alpha \) can be calculated explicitly and the result reads

\[ \int_0^{2\pi} d\alpha \ln(\Lambda^2 + \rho^2 + A^2_E + 2\Lambda A_E \cos \alpha) = \]

\[ = 2\pi \ln \left[ \frac{\Lambda^2 + \rho^2 + A^2_E + \sqrt{(\Lambda^2 + \rho^2 + A^2_E)^2 - 4\Lambda^2A^2_E}}{2} \right]. \] (2.10)

Substituting (2.9) and (2.10) in (2.8) and imposing the condition \( V_{\text{eff}}[0,0] = 0 \), we end up with the expression for the effective potential

\[ V_{\text{eff}}[\rho, A_E] = \frac{\Lambda^2}{4\pi} \left\{ - \ln \left[ \frac{1}{2} \left( 1 + \frac{\rho^2}{\Lambda^2} + \frac{A^2_E}{\Lambda^2} + \sqrt{1 + \frac{\rho^2}{\Lambda^2} + \frac{A^2_E}{\Lambda^2}} \right)^2 - 4 \frac{A^2_E}{\Lambda^2} \right] \right\} \]
\[- \frac{1}{2} \left( 1 + \frac{\rho^2}{\Lambda^2} + \frac{A_{E}^2}{\Lambda^2} \right) - \sqrt{\left( 1 + \frac{\rho^2}{\Lambda^2} - \frac{\Lambda_{E}^2}{\Lambda^2} \right)^2 + 4 \frac{\rho^2}{\Lambda^2} \frac{A_{E}^2}{\Lambda^2}} + \frac{\rho^2}{\Lambda^2} \ln \frac{\rho^2}{\Lambda^2} \]

\[- \frac{\rho^2}{\Lambda^2} \ln \left[ \frac{1}{2} \left( 1 + \frac{\rho^2}{\Lambda^2} + \frac{A_{E}^2}{\Lambda^2} \right) - \sqrt{\left( 1 + \frac{\rho^2}{\Lambda^2} - \frac{\Lambda_{E}^2}{\Lambda^2} \right)^2 + 4 \frac{\rho^2}{\Lambda^2} \frac{A_{E}^2}{\Lambda^2}} \right] \]

\[+ \frac{2\pi}{g_1} \frac{A_{E}^2}{\Lambda^2} + \frac{2\pi}{g_2} \frac{\rho^2}{\Lambda^2} \right). \tag{2.11}\]

In the limit \( g_1 \to 0 \) corresponding \( A_{E} \to 0 \) we get the effective potential \( V_{\text{eff}}[\rho, 0] \)

\[V_{\text{eff}}[\rho, 0] = \frac{\Lambda^2}{4\pi} \left\{ \ln \frac{\rho^2}{\Lambda^2} - \left( 1 + \frac{\rho^2}{\Lambda^2} \right) \ln \left( 1 - \frac{\rho^2}{\Lambda^2} \right) + \frac{2\pi}{g} \frac{\rho^2}{\Lambda^2} \right\} \tag{2.12}\]

which coincides with the effective potential evaluated in \( \text{[8]} \) (see Eq. (3.20)). As we have already found in \( \text{[8]} \), the minimum of the effective potential (2.12) is at \( \rho = M \):

\[V_{\text{eff}}[M, 0] = -\frac{\Lambda^2}{4\pi} \ln \left( 1 + \frac{M^2}{\Lambda^2} \right) = \frac{\Lambda^2}{4\pi} \ln \left( 1 - e^{-2\pi/g} \right). \tag{2.13}\]

The shape of the potential \( V_{\text{eff}}[\rho, 0] \) is depicted in Fig. 1 as a function of \( \rho/\Lambda \) for \( 2\pi/g = 2/3, 1, 3/2 \).

![Figure 1: The effective potential \( V_{\text{eff}}[\rho, 0] \) of Eq. (2.12) as a function of \( \rho/\Lambda \) for \( 2\pi/g = 2/3, 1, 3/2 \). With increasing \( g \) the minimum gets lower and is shifted to larger values of \( \rho \).](image)

In turn in the limit \( g_2 \to 0 \) leading to \( \rho = 0 \) we reduce the effective potential (2.11) to the form

\[V_{\text{eff}}[0, A_{E}] = \frac{\Lambda^2}{4\pi} \left\{ - \ln \left[ \frac{1}{2} \left( 1 + \frac{A_{E}^2}{\Lambda^2} + \left| 1 - \frac{A_{E}^2}{\Lambda^2} \right| \right) \right] \right. \]

\[\left. - \frac{1}{2} \left( 1 + \frac{A_{E}^2}{\Lambda^2} - \left| 1 - \frac{A_{E}^2}{\Lambda^2} \right| \right) + \frac{2\pi}{g} \frac{A_{E}^2}{\Lambda^2} \right\}. \tag{2.14}\]

The shape of this potential can be seen in Fig. 2 as Extrema of the effective potential (2.14)
are defined by the first derivative with respect to $A_E$

$$
\frac{\delta V_{\text{eff}}[0, A_E]}{\delta A_E} = \frac{1}{2\pi} A_E^2 - \Lambda^2 \theta(A_E^2 - \Lambda^2) - \frac{1}{2\pi} \left(1 - \frac{2\pi}{g}\right) A_E = 0,
$$

(2.15)

where $\theta(A_E^2 - \Lambda^2)$ is the Heaviside function. The roots of the equation (2.15) are equal to

$$
A_E^{\text{min}} = \begin{cases} 
0, & A_E^2 < \Lambda^2, \\
\sqrt{\frac{g}{2\pi}} \Lambda, & A_E^2 > \Lambda^2.
\end{cases}
$$

(2.16)

A non–trivial solution of the equation (2.15) exists only for $g > 2\pi$.

Analysing the second derivative of the effective potential with respect to $A_E$ we can obtain the conditions for the existence of the minima. We get

$$
\frac{\delta^2 V_{\text{eff}}[0, A_E]}{\delta A_E^2} = \frac{1}{2\pi} A_E^2 + \Lambda^2 \theta(A_E^2 - \Lambda^2) - \frac{1}{2\pi} \left(1 - \frac{2\pi}{g}\right).
$$

(2.17)

We have used the relation $(A_E^2 - \Lambda^2)\delta(A_E^2 - \Lambda^2) = 0$, where $\delta(A_E^2 - \Lambda^2)$ is the Dirac $\delta$–function. For $A_E$ given by (2.16) the second derivative is equal to

$$
\frac{\delta^2 V_{\text{eff}}[0, A_E^{\text{min}}]}{\delta A_E^2} = \begin{cases} 
-\frac{1}{2\pi} + \frac{1}{g}, & A_E^2 < \Lambda^2, \\
\frac{2}{g}, & A_E^2 > \Lambda^2.
\end{cases}
$$

(2.18)

Hence, a minimum of the effective potential at $\rho = 0$ and $g_1 = g$ can exist only either for coupling constants $g < 2\pi$ at $A_E^{\text{min}} = 0$ or for $g > 2\pi$ at $A_E^{\text{min}} = \sqrt{g/2\pi} \Lambda$.

We would like to emphasize that $A_E = 0$ is not a real minimum of the effective potential for $\rho = 0$ but a saddle–point. Indeed, according to the criterion for the saddle–point, it is a minimum in $A_E$ but a maximum in $\rho$.

The effective potential (2.14) at $A_E^{\text{min}}$, given by (2.16), is equal to

$$
V_{\text{eff}}[0, A_E^{\text{min}}] = \begin{cases} 
0, & A_E^{\text{min}} = 0, \\
-\frac{\Lambda^2}{4\pi} \ell_n\left(\frac{g}{2\pi}\right), & A_E^{\text{min}} = \sqrt{\frac{g}{2\pi}} \Lambda.
\end{cases}
$$

(2.19)
This could testify that the chiral symmetric phase of the massless Thirring model can exist for coupling constants \( g < 2\pi \) only. But \( \rho = A_E = 0 \) is a saddle-point of the effective potential for any \( 0 < g < 2\pi \). This is clearly seen from Eqs. (2.12)–(2.19) and also Figs. 1 and 2. Since the point \( \rho = A_E = 0 \) is not a minimum of the effective potential (2.11), the chiral symmetric phase is not stable, and it should make a transition to the chirally broken phase in the region \( 0 < g < 2\pi \). This agrees with our results obtained in [8, 10].

The difference \( \Delta V_{\text{eff}} \) of the effective potentials \( V_{\text{eff}}[M, 0] \) and \( V_{\text{eff}}[0, A_E^{\text{min}}] \) is equal to

\[
\Delta V_{\text{eff}} = V_{\text{eff}}[M, 0] - V_{\text{eff}}[0, A_E^{\text{min}}] = \begin{cases} 
\frac{\Lambda^2}{4\pi} \ln \left( 1 - e^{-2\pi g} \right), & 0 < g < 2\pi, \\
\frac{\Lambda^2}{4\pi} \ln \left[ \frac{g}{2\pi} \left( 1 - e^{-2\pi/g} \right) \right], & g > 2\pi.
\end{cases}
\]

It is seen that for \( g > 0 \) the minimum of the effective potential \( V_{\text{eff}}[\rho, 0] \) is always deeper than the minimum of \( V_{\text{eff}}[0, A_E] \). Therefore, below we call a metastable minimum and an absolute minimum the values of the effective potential (2.11) at \((\rho = 0, A_E = \sqrt{g/2\pi} \Lambda)\) and \((\rho = M, A_E = 0)\), respectively. These minima are separated by a saddle-point in the \((\rho, A_E, \arctan(g_1/g_2))\) space for \( g > 2\pi \). The values of these minima are depicted in Fig. 3.

![Figure 3: Depth of the minima \( V_{\text{eff}}[M, 0] \) and \( V_{\text{eff}}[0, A_E^{\text{min}}] \) of the effective potentials \( V_{\text{eff}}[\rho, 0] \) and \( V_{\text{eff}}[0, A_E] \). The minimum of \( V_{\text{eff}}[0, A_E] \) is non-trivial for \( g > 2\pi \) only. For all couplings \( g > 0 \) the absolute minimum is the chirally broken minimum \( V_{\text{eff}}[M, 0] \).](image-url)
The non–zero value $A_E = \sqrt{g/2\pi} \Lambda$ for $g > 2\pi$ should correspond to a chirally broken phase. This can be proved by calculating the vacuum expectation value of the product of the vector currents

$$A_E^2 = -\langle A_\mu(x) A^\mu(x) \rangle = - g^2 \langle \bar{\psi}(x) \gamma_\mu \psi(x) \bar{\psi}(x) \gamma_\mu \psi(x) \rangle = - g^2 \text{tr} \{ S_F(0) \gamma_\mu S_F(0) \gamma^\mu \}.$$ (2.21)

Substituting (2.22) in (2.21), using the identity $\gamma_\mu \gamma^\alpha \gamma^\mu = 0$ and integrating over $k$ we end up with the expression

$$A_E^2 = \left[ \frac{g}{2\pi} \int_0^\infty dm^2 \rho_2(m^2) \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right]^2.$$ (2.23)

Figure 4: Effective potential for $g = 3\pi$, depicted as a function of $\rho/\Lambda$, $A/E/\Lambda$ and $\arctan(g_1/g_2)$ by two of the most characteristic equipotential surfaces above the absolute minimum $V_{\text{eff}}[M, 0]$. For the lower value we have chosen $V_{\text{eff}}[\rho, A] = V_{\text{eff}}[M, 0] + \Lambda^2/12\pi$ which corresponds to the smaller surface. The higher value $V_{\text{eff}}[M, 0] + \Lambda^2/6\pi$ leads to an equipotential surface connecting the two minima by a tunnel. By a tunnel. At the most narrow point of the tunnel there is a saddle–point of the effective potential. For the decreasing values of $g$ the saddle–point moves towards $(0, 0, \pi/2)$. The non–zero value $A_E = \sqrt{g/2\pi} \Lambda$ for $g > 2\pi$ should correspond to a chirally broken phase. This can be proved by calculating the vacuum expectation value of the product of the vector currents

$$A_E^2 = -\langle A_\mu(x) A^\mu(x) \rangle = - g^2 \langle \bar{\psi}(x) \gamma_\mu \psi(x) \bar{\psi}(x) \gamma_\mu \psi(x) \rangle = - g^2 \text{tr} \{ S_F(0) \gamma_\mu S_F(0) \gamma^\mu \}.$$ (2.21)

where we have used the bosonization rules (1.18) and $S_F(x)$ is the two–point causal Green function of the Thirring fermion fields, which we define using the Källen–Lehmann representation [17, 18]

$$S_F(x)_{ab} = i\langle T(\psi_a(x) \bar{\psi}_b(0)) \rangle = \int \frac{d^2k}{(2\pi)^2} e^{-ik \cdot x} \int_0^\infty dm^2 \frac{(k^2 + m^2 - i0)}{m^2 - k^2 - m^2}.$$ (2.22)

Here $\rho_1(m^2)$ and $\rho_2(m^2)$ are Källen–Lehmann spectral functions. In the chiral symmetric phase the spectral function $\rho_2(m^2)$ should vanish, $\rho_2(m^2) = 0$. Let us show that the non–zero value of $A_E$ is defined only by $\rho_2(m^2) \neq 0$.

Substituting (2.22) in (2.21), using the identity $\gamma_\mu \gamma^\alpha \gamma^\mu = 0$ and integrating over $k$ we end up with the expression [17]

$$A_E^2 = \left[ \frac{g}{2\pi} \int_0^\infty dm^2 \rho_2(m^2) \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right]^2.$$ (2.23)
Hence, $A_E \neq 0$ only for $\rho_2(m^2) \neq 0$.

For the comparison with $A_E$ we suggest to calculate $\rho$ defined by

$$
\rho^2 \equiv \langle g^2(x) + \varphi^2(x) \rangle = g^2 \langle [\bar{\psi}(x)\psi(x)]^2 + [\bar{\psi}(x)i\gamma^5\psi(x)]^2 \rangle = g^2 \text{tr} \{S_F(0)\}^2 + g^2 \text{tr} \{S_F(0)S_F(0) - \gamma^5 S_F(0)\gamma^5 S_F(0)\} = 
$$

$$
= \left[ \frac{g}{2\pi} \int_0^\infty \rho_2(m^2) \ell n \left( 1 + \frac{A^2}{m^2} \right) \right]^2,
$$

(2.24)

where we have also used the bosonization rules [1,18].

This testifies that only the saddle–point of the effective potential ($\rho = 0$, $A_E = 0$) corresponds to the chiral symmetric phase of the massless Thirring model. It can be realized only for $\rho_2(m^2) = 0$. In turn, the minima ($\rho = 0$, $A_E = \sqrt{g/2\pi}\Lambda$) and ($\rho = M$, $A_E = 0$) correspond to the chirally broken phases of the massless Thirring model, which demands $\rho_2(m^2) \neq 0$. The stability of the minimum ($\rho = M$, $A_E = 0$) under quantum fluctuations $\rho(x) = M + \tilde{\rho}(x)$, where $\tilde{\rho}(x)$ is a fluctuating field has been shown in [8].

3 Ground state of the massless Thirring model at the minimum ($\rho = M$, $A_E = 0$)

Placing massless Thirring fermions into a finite volume $L$ the BCS wave function of the ground state of the massless Thirring model is defined by [8,13]

$$
|\Omega(0)\rangle = \prod_{p^1} \left[ u_{p^1} + v_{p^1} a^\dagger(p^1)b^\dagger(-p^1) \right] |\Psi_0\rangle,
$$

(3.1)

where $|\Psi_0\rangle$ is the wave function of the chiral symmetric vacuum, the coefficients $u_{p^1}$ and $v_{p^1}$ have the properties: (i) $u_{p^1}^2 + v_{p^1}^2 = 1$ and (ii) $u_{-p^1} = u_{p^1}$ and $v_{-p^1} = -v_{p^1}$ [8,13]. $a^\dagger(p^1)$ and $b^\dagger(p^1)$ are creation operators of fermions and antifermions with momentum $p^1$.

According to Nambu and Jona–Lasinio [24], the wave function (3.1) should be a linear superposition of the eigenfunctions $|\Omega_{2n}\rangle$ of the chirality operator $X$, defined by

$$
X(x^0) = \lim_{L \to \infty} \int_{-L/2}^{+L/2} dx^1 \psi^\dagger(x^0,x^1)\gamma^5\psi(x^0,x^1),
$$

(3.2)

with eigenvalues $X_n = 2n$, $n \in \mathbb{Z}$, i.e.

$$
|\Omega(0)\rangle = \sum_{n \in \mathbb{Z}} C_{2n}|\Omega_{2n}\rangle = \sum_{n \in \mathbb{Z}} |\Phi_{2n}\rangle.
$$

(3.3)

A chiral rotation of a fermion field $\psi(x)$ is defined by [24] (see also [8])

$$
e^{-i\alpha_A X(x^0)} \psi(x) e^{+i\alpha_A X(x^0)} = e^{+i\gamma^5\alpha_A \psi(x)},
$$

(3.4)

where we have used canonical anticommutation relations

$$
\{\psi(x^0,x^1),\psi^\dagger(x^0,y^1)\} = \delta(x^1 - y^1).
$$

(3.5)
For chiral rotations of fermion fields with a chiral phase $\alpha_A$ the wave function (3.1) changes as follows [8]

$$|\Omega(\alpha_A)\rangle = \prod_p [u_p e^{-2i\varepsilon(p^1)\alpha_A} a^+(p^1) b^+( -p^1 )] |\Psi_0\rangle,$$

(3.6)

where $\varepsilon(p^1)$ is a sign function. In terms of $|\Omega(\alpha_A)\rangle$ the wave functions $|\Phi_{2n}\rangle$ are defined by [24]

$$|\Phi_{2n}\rangle = C_{2n} |\Omega_{2n}\rangle = \int_0^{2\pi} \frac{d\alpha_A}{2\pi} e^{+2i n \alpha_A} |\Omega(\alpha_A)\rangle.$$

(3.7)

Substituting (3.7) in (3.6) and using the identity [25]

$$\sum_{n=\pm \infty} e^{2i n \alpha_A} = \frac{1}{\pi} \sum_{k\in\mathbb{Z}} \delta(\alpha_A - k\pi)$$

(3.8)

one arrives at the BCS wave function (3.1). Defining the $\delta$–functions as [25]

$$\delta(\alpha_A - k\pi) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \frac{\varepsilon}{(\alpha_A - k\pi)^2 + \varepsilon^2}$$

(3.9)

we obtain

$$\sum_{n=\pm \infty} |\Phi_{2n}\rangle = \int_0^{2\pi} \frac{d\alpha_A}{2\pi} \sum_{n=\pm \infty} e^{+2i n \alpha_A} |\Omega(\alpha_A)\rangle =$$

$$= \frac{1}{2} \int_0^{2\pi} \frac{d\alpha_A}{2\pi} \sum_{k=\pm \infty} \delta(\alpha_A - k\pi) |\Omega(\alpha_A)\rangle = \frac{1}{2} \int_0^{2\pi} \frac{d\alpha_A}{2\pi} \delta(\alpha_A) |\Omega(\alpha_A)\rangle$$

$$+ \frac{1}{2} \int_0^{2\pi} \frac{d\alpha_A}{2\pi} \delta(\alpha_A - \pi) |\Omega(\alpha_A)\rangle + \frac{1}{2} \int_0^{2\pi} \frac{d\alpha_A}{2\pi} \delta(\alpha_A - 2\pi) |\Omega(\alpha_A)\rangle =$$

$$= \frac{1}{4} |\Omega(\epsilon)\rangle + \frac{1}{2} |\Omega(\pi)\rangle + \frac{1}{4} |\Omega(2\pi)\rangle = |\Omega(\epsilon)\rangle.$$

(3.10)

Since $|\Phi_{2n}\rangle$ are eigenfunctions of the chirality operator, $X|\Phi_{2n}\rangle = 2n|\Phi_{2n}\rangle$, they transform under chiral rotations as follows

$$e^{-i\alpha X(0)} |\Phi_{2n}\rangle = \int_0^{2\pi} \frac{d\alpha_A}{2\pi} e^{+2i n \alpha_A} e^{-i\alpha X(0)} |\Omega(\alpha_A)\rangle =$$

$$= \int_0^{2\pi} \frac{d\alpha_A}{2\pi} e^{+2i n \alpha_A} |\Omega(\alpha_A + \alpha)\rangle = e^{-i2n\alpha} |\Phi_{2n}\rangle.$$

(3.11)

This agrees with Nambu and Jona–Lasinio [24].

It is seen that only the wave function $|\Phi_0\rangle$ with chirality zero is invariant under chiral rotations. Hence, it can describe a non–trivial chiral symmetric ground state of the massless Thirring model in the vicinity of the minimum ($\rho = M, A_E = 0$).

As has been shown in [8] the effective potential, calculated at the minimum ($\rho = M, A_E = 0$), coincides with the energy density of the massless Thirring model with the
ground state described by the wave function (3.1). The Hamiltonian of the massless Thirring model is equal to

$$H(x^0, x^1) = -\frac{\partial}{\partial x^1} \psi(x^0, x^1) : \mu \psi(x^0, x^1) : + \frac{1}{2} g : \bar{\psi}(x^0, x^1) \gamma_\mu \psi(x^0, x^1) \bar{\psi}(x^0, x^1) \gamma_\mu \psi(x^0, x^1) : \mu,$$

(3.12)

where \( : \ldots : \) indicates normal ordering at an infrared scale \( \mu \), which should be finally set to zero \( \mu \to 0 \).

The energy density of massless Thirring fermions in the ground state, described by the wave function (3.1), is defined by

$$E(M) = \lim_{L \to \infty} \frac{1}{L} \int_{-L/2}^{+L/2} dx^1 \langle \Omega(0)|H(x^0, x^1)|\Omega(0) \rangle.$$

(3.13)

As has been shown in [8] it is equal to

$$E(M) = V[M, 0],$$

(3.14)

where \( V[M, 0] \) is the effective potential of collective fermion–antifermion excitations calculated at \( \rho = M_A = 0 \).

Since the Hamiltonian (3.12) is invariant under chiral rotations, the same energy density (3.14) corresponds to the ground state described by the wave function \( |\Omega(\alpha_A)\rangle \)

$$E(M)_{\alpha_A} = \lim_{L \to \infty} \frac{1}{L} \int_{-L/2}^{+L/2} dx^1 \langle \Omega(\alpha_A)|H(x^0, x^1)|\Omega(\alpha_A) \rangle =$$

$$= \lim_{L \to \infty} \frac{1}{L} \int_{-L/2}^{+L/2} dx^1 \langle \Omega(0)|e^{+i\alpha_A X} H(x^0, x^1) e^{-i\alpha_A X}|\Omega(0) \rangle =$$

$$= \lim_{L \to \infty} \frac{1}{L} \int_{-L/2}^{+L/2} dx^1 \langle \Omega(0)|H(x^0, x^1)|\Omega(0) \rangle = E(M),$$

(3.15)

where we have used that

$$e^{+i\alpha_A X} H(x^0, x^1) e^{-i\alpha_A X} = H(x^0, x^1).$$

(3.16)

The fact that the BCS wave function is not an eigenstate of chirality guarantees that the properties of the ground state do not depend on the exact number of fermion pairs.

In order to clarify this assertion we would like to draw a similarity between chirality in the massless Thirring model with triality in QCD [26]. In QCD there exist no triality changing transitions, this means a dynamical change of triality is impossible. It is well–known that the confined phase in QCD is \( Z(3) \) symmetric. Triality zero states are screened, and triality non–zero states are confined. This means that states with different triality behave differently. Whereas in the high–temperature phase of QCD all triality states behave in the same way, they get screened. This is guaranteed by the spontaneous breaking of \( Z(3) \) symmetry. In our case the situation is similar to the deconfined phase. In the massless Thirring model there are no chirality changing transitions. The ground state is of BCS-type, defining a condensate of fermion-antifermion pairs with different
chiralities. In order to get a ground state with properties independent on the exact value of total chirality of all fermion–antifermion pairs, we need spontaneous breaking of chiral symmetry similar to the the spontaneous breaking of $Z(3)$ symmetry in QCD. Such a spontaneous breaking of chiral symmetry is realized by the BCS wave function.

However, for the description of the ground state of massless Thirring fermions around the minimum of the effective potential ($\rho = M, A_{E} = 0$) one can use $|\Phi_{2n}\rangle$, the eigenfunction of chirality operator $X$. One can show that the energy density of massless Thirring fermions in the ground state, described by the wave function $|\Phi_{2n}\rangle$, is equal to $\mathcal{E}(M) = V[M, 0]$. The proof runs in the way

\[
\mathcal{E}(M)_{2n} = \lim_{L \to \infty} \frac{1}{L} \int_{-L/2}^{+L/2} dx \langle \Phi_{2n} | \mathcal{H}(x^{0}, x^{1}) | \Phi_{2n} \rangle = \lim_{L \to \infty} \int_{0}^{2\pi} \frac{d\alpha_{A}'}{2\pi} \int_{0}^{2\pi} \frac{d\alpha_{A}}{2\pi} \\
\times e^{-i2\pi(\alpha_{A}' - \alpha_{A})} \frac{1}{L} \int_{-L/2}^{+L/2} dx \langle \Omega(\alpha_{A}') | \mathcal{H}(x^{0}, x^{1}) | \Omega(\alpha_{A}) \rangle.
\]

(3.17)

The wave functions $|\Omega(\alpha_{A}')\rangle$ and $|\Omega(\alpha_{A})\rangle$ are orthogonal for $\alpha_{A}' \neq \alpha_{A}$. Indeed, the scalar product $\langle \Omega(\alpha_{A}') | \Omega(\alpha_{A}) \rangle$ of the wave function for $\alpha_{A}' \neq \alpha_{A}$ is equal to $[8, 24]$

\[
\langle \Omega(\alpha_{A}') | \Omega(\alpha_{A}) \rangle = \lim_{L \to \infty} \exp \left\{ \frac{L}{2\pi} \int_{0}^{2\pi} dk \ln \left[ 1 - \frac{\sin^{2}(\alpha_{A}' - \alpha_{A})}{M^{2} + (k^{1})^{2}} \right] \right\} = \delta_{\alpha_{A}', \alpha_{A}}.
\]

(3.18)

It is obvious that the matrix element of the Hamiltonian should be diagonal

\[
\langle \Omega(\alpha_{A}') | \mathcal{H}(x^{0}, x^{1}) | \Omega(\alpha_{A}) \rangle = \mathcal{E}(M) \delta_{\alpha_{A}', \alpha_{A}}.
\]

(3.19)

This agrees with Nambu and Jona–Lasinio [24]. Insertion of (3.19) into (3.17) gives

\[
\mathcal{E}(M)_{2n} = \mathcal{E}(M).
\]

(3.20)

The order parameter of the superconducting BCS ground state is defined by the vacuum expectation value [8]

\[
\langle O_{+} \rangle = \lim_{L \to \infty} \frac{2}{L} \sum_{p_{1}} \varepsilon(p_{1}) \langle \Omega(0) | b^{\dagger}(-p_{1}) a^{\dagger}(p_{1}) | \Omega(0) \rangle = -\lim_{L \to \infty} \frac{2}{L} \sum_{p_{1}} \varepsilon(p_{1}) u_{p_{1}} v_{p_{1}} = -\frac{M}{g}.
\]

(3.21)

Using the procedure developed in [15] one can show that in the limit $L \to \infty$ the operator $O_{+}$ coincides with the scalar fermion density operator $\bar{\psi}(0)\psi(0)$.

It is important to emphasize that the operator $O_{+}$ satisfies the selection rules $\Delta X = 2$. This means that a non–vanishing value of the order parameter can be obtained only for the ground state of the massless Thirring fermions described by the BCS–type wave functions (3.1) and (3.6), which are not eigenfunctions of the chirality operator (3.2).

We would like to notice that spontaneous breaking of chiral symmetry in the massless Thirring model is not due to the fermion condensation but caused by the wave functions of the ground state of the fermion system. Indeed, the vacuum expectation value of the
operator \( \mathcal{O}_+ \) is zero for any wave function \( |\Phi_{2n}\rangle \) accepted as the wave function of the ground state of the massless Thirring model. The main peculiarity of the wave function \( |\Phi_{2n}\rangle \) is the fixed chirality corresponding the fixed number of fermion–antifermion pairs. As we have shown above the wave functions of fermion system with a fixed number of fermion–antifermion pairs describe the ground state with a broken chiral symmetry and Thirring fermions with a non–vanishing dynamical mass \( M \) but the fermion condensate is zero.

However, the ground state (or the vacuum state) of quantum field theories of fermions in 2D–dimensional space–time does not have a fixed number of fermion–antifermion pairs but the number of them is infinite. Therefore, the correct wave function of the vacuum (the ground state) should be a linear superposition over all states with a fixed chirality or the fixed number of fermion–antifermion pairs. This leads to the BCS–type wave function. For the BCS–type wave function the fermion condensate does not vanish [8].

4 Conclusion

We have analysed the massless Thirring model in terms of local fields of collective fermion–antifermion excitations, which can be excited by the four–fermion interaction. We have calculated the effective potential \( V_{\text{eff}}[\rho, A_E] \), where we have denoted \( A_E = \sqrt{-A_\mu(x)A^\mu(x)} \) and \( \rho = \sqrt{\sigma^2(x) + \varphi^2(x)} \). The fields \( A^\mu(x) \), \( \sigma(x) \) and \( \varphi(x) \) are the local fields of fermion–antifermion collective excitations, related to Thirring fermion fields through the bosonization rules given by [11,15].

Setting \( \rho = 0 \) and \( g_2 = 0 \), we have shown that for coupling constants \( g < 2\pi \) the effective potential \( V_{\text{eff}}[0, A_E] \) possesses a chiral symmetric minimum at \( A_E = 0 \). Allowing for \( \rho \neq 0 \) this state turns out to be only a saddle–point. Hence, this state cannot define the real ground state of the massless Thirring model which can be found at \( \rho = M, A_E = 0 \).

For \( g > 2\pi \) there are two minima \( (\rho = M, A_E = 0) \) and \( (\rho = 0, A_E = \sqrt{g/2\pi} \Lambda) \) corresponding to a broken chiral symmetry. The first minimum \( (\rho = M, A_E = 0) \) is deeper than the second one \( (\rho = 0, A_E = \sqrt{g/2\pi} \Lambda) \) and is, therefore, energetically preferable.

Thus, one can conclude that the evolution of the massless Thirring model goes through the formation of scalar and pseudoscalar fermion–antifermion collective excitations with fermion fields quantized relative to the minimum of the effective potential \( V_{\text{eff}}[\rho, A_E] \) at \( (\rho = M, A_E = 0) \).

We have shown that the ground state of the massless Thirring model quantized around the absolute minimum \( (\rho = M, A_E = 0) \) is described by a BCS–type wave function, which is not invariant under chiral rotations and provides spontaneous breaking of chiral symmetry. The fermion condensate of Thirring fermions described by this wave function does not vanish because the BCS–type wave function is not an eigenfunction of the chirality operator. For eigenfunctions of the chirality operator the fermion condensate is zero. Nevertheless, such a choice of the wave function of the ground state does not destroy the position of the absolute minimum of the effective potential coinciding with the energy density of the ground state, the dynamical mass \( M \). This means that the chiral symmetry breaking in the ground state implies an independence of the ground state on the exact value of chirality.

In polar degrees of freedom the effective quantum field theory of collective fermion–
antifermion excitations with fermion fields quantized around this minimum is the quantum field theory of a free massless (pseudo)scalar field $\vartheta(x)$, which can be constructed without infrared divergences [9]. The former can be reached by removing the collective zero–mode from the observable modes. Indeed, since the massless Thirring model is well–defined and does not suffer from infrared divergences (see also [21]) a formulation of the free massless (pseudo)scalar field $\vartheta(x)$, bosonizing the massless Thirring model in the chirally broken phase, should not also suffer from an infrared problem. The ground state of a free massless (pseudo)scalar field $\vartheta(x)$ is described by the bosonized BCS–type wave function of the ground state of the massless Thirring model in the chirally broken phase [15, 16].

In this connection we would like to remind that, according to Wightman [27], a quantum field theory of a free massless (pseudo)scalar field $\vartheta(x)$, suffering from infrared divergences, leads to a violation of Wightman’s positive definiteness condition if Wightman’s observables defined on the test functions $h(x)$ from the Schwartz class $S(\mathbb{R}^2)$. Due to this problem Wightman [27] and then Coleman [28] prohibited the existence of quantum field theories of free massless (pseudo)scalar fields in 1+1–dimensional space–time. In Coleman’s formulation such a suppression was expressed as a non–existence of Goldstone bosons and spontaneously broken continuous symmetry in quantum field theories defined in 1+1–dimensional space–time.

It is interesting to remind that Wightman [27] admitted a possibility for the existence of a quantum field theory of a free massless (pseudo)scalar field in 1+1–dimensional space. He pointed out that for Wightman’s observables defined on the test functions from the Schwartz class $S_0(\mathbb{R}^2) = \{h(x) \in S(\mathbb{R}^2); \hat{h}(0) = 0\}$, where $\hat{h}(0)$ is the Fourier transform of $h(x)$ given by

$$\tilde{h}(0) = \lim_{k \to 0} \int d^2x \: h(x) \: e^{-ik \cdot x} = 0,$$

Wightman’s positive definiteness condition should be fulfilled. As a result there are no objections for the existence of such a quantum field theory of a free massless (pseudo)scalar field. As we have shown in [11] for test functions taken from the Schwartz class $S_0(\mathbb{R}^2) = \{h(x) \in S(\mathbb{R}^2); \hat{h}(0) = 0\}$ Coleman’s theorem tells nothing about the non–existence of Goldstone bosons and spontaneously broken continuous symmetry in 1+1–dimensional space–time [4]. The removal of the collective zero–mode from the observable vibrational modes of the free massless (pseudo)scalar field $\vartheta(x)$ corresponds to Wightman’s assumption to formulate the quantum field theory of a free massless (pseudo)scalar field $\vartheta(x)$ on the Schwartz class of test functions $S_0(\mathbb{R}^2)$.

We have reformulated this assertion in terms of the generating functional of Green functions [9]. In [11] we have shown that the collective zero–mode is a classical degree of freedom. Therefore, it cannot be quantized and used as a fluctuating degree of freedom in the path–integral describing correlation functions of massless Thirring fermion fields in the bosonized form. Thus, the use of the free massless (pseudo)scalar field $\vartheta(x)$ with the classical collective zero–mode removed describes well the bosonized version of the massless Thirring model in the chirally broken phase. We would like to mention that the necessity to remove the collective zero–mode for the non–linear $\sigma$–model in one and two–dimensional space has been pointed out by Hasenfratz [30].

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[4] The irrelevance of the Mermin–Wagner–Hohenberg theorem [29] to the analysis of spontaneously broken chiral symmetry in the massless Thirring model and a free massless (pseudo)scalar field theory in 1+1–dimensional space–time we discussed in details in [9].
References

[1] W. Thirring, Ann. Phys. (N.Y.) 3, 91 (1958).

[2] Y. Nambu and G. Jona–Lasinio, Phys. Rev. 122, 345 (1961).

[3] D. Ebert and H. Reinhardt, Nucl. Phys. B 271, 188 (1986); Phys. Lett. B 173, 453 (1986).

[4] A. N. Ivanov, M. Nagy, and N. I. Troitskaya, Int. J. Mod. Phys. A 7, 7305 (1992); A. N. Ivanov, Int. J. Mod. Phys. A 8, 865 (1993).

[5] M. Faber, A. N. Ivanov, W. Kainz, and N. I. Troitskaya, Phys. Lett. B 386, 198 (1996); Z. Phys. C 74, 721 (1997); M. Faber, A. N. Ivanov, A. Müller, N. I. Troitskaya, and M. Zach, Eur. Phys. J. C 7, 685 (1999); Eur. Phys. J. C 10, 537 (1999).

[6] A. N. Ivanov, H. Oberhummer, N. I. Troitskaya, and M. Faber, Eur. Phys. J. A 7, 519 (2000); A. N. Ivanov, V. A. Ivanova, H. Oberhummer, N. I. Troitskaya and M. Faber, Eur. Phys. J. A 12, 87 (2001).

[7] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. 106, 162 (1957); Phys. Rev. 108, 1175 (1957).

[8] M. Faber and A. N. Ivanov, Eur. Phys. J. C 20, 723 (2001), hep-th/0105057.

[9] M. Faber and A. N. Ivanov, Eur. Phys. J. C 24, 653 (2002).

[10] M. Faber and A. N. Ivanov, On the solution of the massless Thirring model in the chiral symmetric phase, hep-th/0112183.

[11] M. Faber and A. N. Ivanov, On spontaneous breaking of continuous symmetry in 1+1-dimensional space–time, hep-th/0204237.

[12] M. Faber and A. N. Ivanov, Is the energy density of the ground state of the sine–Gordon model unbounded from below for $\beta^2 > 8\pi$?, hep-th/0205249 (to appear in Journal of Physics A).

[13] M. Faber and A. N. Ivanov, Massless Thirring fermion fields in the boson field representation, hep-th/0206034.

[14] M. Faber and A. N. Ivanov, Quantum field theory of a free massless (pseudo)scalar field in 1+1-dimensional space–time as a test for the massless Thirring model, hep-th/0206244.

[15] M. Faber and A. N. Ivanov, Phys. Lett. B 563, 231 (2003).

[16] M. Faber and A. N. Ivanov, On the ground state of the massless (pseudo)scalar field in two dimensions, hep-th/0212226.

[17] M. Faber and A. N. Ivanov, Goldstone bosons in the massless Thirring model. Witten’s criterion, hep-th/0305174.
[18] M. Faber and A. N. Ivanov, *Dynamical breaking of conformal symmetry in the massless Thirring model*, hep-th/0305203.

[19] K. Johnson, Nuovo Cim. **20**, 773 (1961).

[20] B. Klaiber, in *LECTURES IN THEORETICAL PHYSICS*, Lectures delivered at the Summer Institute for Theoretical Physics, University of Colorado, Boulder, 1967, edited by A. Barut and W. Brittin, Gordon and Breach, New York, 1968, Vol. X, part A, pp.141–176.

[21] C. R. Hagen, Nuovo Cim. **51**, 169 (1967).

[22] K. Furuya, Re. E. Gamboa Saravi and F. A. Schaposnik, Nucl. Phys. B **208**, 159 (1982); C. M. Naón, Phys. Rev. D **31**, 2035 (1985).

[23] M. Faber, C. R. Hagen, and A. N. Ivanov, *Correlation functions of left–right fermion densities in Hagen’s approach to the massless Thirring model*, 2003 (unpublished).

[24] Y. Nambu and G. Jona–Lasinio, Phys. Rev. **122**, 345 (1960).

[25] I. M. Gel’fand and G. E. Shilov, in *GENERALIZED FUNCTIONS, Properties and Operations*, Vol.1, Academic Press, New York, 1964, p.332.

[26] M. Faber, O. Borisenko, and G. Zinovev, Nucl. Phys. B **444**, 563 (1995); hep-ph/9504264.

[27] A. S. Wightman, *Introduction to Some Aspects of the Relativistic Dynamics of Quantized Fields*, in Cargèse Lectures in Theoretical Physics, edited by M. Levy, 1964, Gordon and Breach, 1967, pp.171–291; R. F. Streater and A. S. Wightman, in *PCT, SPIN AND STATISTICS, AND ALL THAT*, Princeton University Press, Princeton and Oxford, Third Edition, 1980.

[28] S. Coleman, Comm. Math. Phys. **31**, 259 (1973).

[29] N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966); P. C. Hohenberg, Phys. Rev. **158**, 383 (1967); N. D. Mermin, J. Math. Phys. **8**, 1061 (1967).

[30] P. Hasenfratz, Phys. Lett. B **141**, 385 (1984).