Self-correspondences of K3 surfaces
via moduli of sheaves and arithmetic hyperbolic
reflection groups

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Abstract

In series of our papers with Carlo Madonna (2002–2008) we described
self-correspondences of K3 surfaces over \( \mathbb{C} \) via moduli of sheaves with
primitive isotropic Mukai vector for Picard number one or two of the K3
surfaces.

Here we give a natural and functorial answer to the same problem for
arbitrary Picard number.

As an application, we characterise in terms of self-correspondences via
moduli of sheaves K3 surfaces with reflective Picard lattices, when the
automorphism group of the lattice is generated by reflections up to finite
index. It is known since 1981 that the number of reflective hyperbolic
lattices is, in essential, finite. We also formulate several natural unsolved
related problems.

To the memory of Vasily Alexeevich Iskovskikh

1 Introduction

In series of our papers with Carlo Madonna [2] – [4] and [16] – [18], we considered
self-correspondences of K3 surfaces (over \( \mathbb{C} \)) via moduli of sheaves.

They are determined by primitive isotropic Mukai vectors \( v = (r, H, s) \) with
\( r \in \mathbb{N}, \ s \in \mathbb{Z} \) and \( H \in N(X) \) where \( N(X) \) is the Picard lattice of \( X \), and
\( H^2 = 2rs \). Due to Mukai [3], [4] (see also Yoshioka [26]), the moduli space
\( Y = M_X(v) \) of stable (with respect to some ample element \( H' \in N(X) \)) coherent
sheaves on \( X \) of rank \( r \) with first Chern class \( H \) and Euler characteristic \( r + s \)
is again a K3-surface. Chern class of the quasi-universal sheaf \( E \) on \( X \times Y \)
gives some 2-dimensional algebraic cycle on \( X \times Y \) and can be considered as

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a correspondence between $X$ and $Y$. If $Y \cong X$, it can be considered as a self-correspondence of $X$ via moduli of sheaves with Mukai vector $v$.

In our papers above, we studied when $Y \cong X$, and we gave a complete answer if the Picard number of $X$ is 1 or 2, and $X$ is general for its Picard lattice. We showed that all of them can be reduced to so called Tyurin’s isomorphisms $Y = M_X(v) \cong X$ when $v = (\pm H^2/2, H, \pm 1)$ where $H \in N(X)$ has $\pm H^2 > 0$.

When $Y \cong X$, using Mukai’s results, in [18] we considered the action of the self-correspondences of $X$ via moduli of sheaves on $H^2(X, \mathbb{Q})$. According to [18] and our considerations in this paper (see Sec. 3), the action of Tyurin’s isomorphism is naturally given by the reflection $s_H$ with respect to $H$ (see (11)), up to the action of the group $\{\pm 1\}W^{-2}(N(X))$ where $W^{-2}(N(X))$ is generated by reflections in elements $\delta \in N(X)$ with $\delta^2 = -2$ (they are called $(-2)$ roots of $N(X)$). We call such self-correspondence of $X$ as Tyurin’s self-correspondence $Tyu(H)$ of $X$.

One can associate to $(-2)$ root $\delta \in N(X)$ a self-correspondences of $X$ which is given by 2-dimensional cycle $\Delta + E \times E$ on $X \times X$ where $\Delta$ is the diagonal, and $E$ is an effective curve from $|\pm \delta|$, and the action of this self-correspondence is given by the reflection $s_\delta$ with respect to $\delta$.

Using well-known fact that the automorphism group of a rational quadratic form is generated by reflections, we obtain from here the main result of the paper: any self-correspondence of $X$ defined by a primitive isotropic Mukai vector, and any their composition is numerically equivalent to composition of self-correspondences of $X$ defined by $(-2)$ roots, and Tyurin’s self-correspondences up to finite index (see Theorem 2.1 and Corollary 2.2 below for exact formulation).

It is also interesting to consider self-correspondences of $X$ with integral action in Picard lattice $N(X)$. One can consider them as analogous to automorphisms. See Sect. 3.

We obtain that the action in $N(X)$ of self-correspondences of $X$ which are given by all $(-2)$ roots of $N(X)$ and by all Tyurin’s self-correspondences $Tyu(H)$ with $H^2 < 0$ and integral action in $N(X)$ (then $H \in N(X)$ is proportional to a primitive root of $N(X)$) generate the group $\{\pm 1\}W(N(X))$ up to $\{\pm 1\}$. Here $W(N(X))$ is the reflection group of the hyperbolic lattice $N(X)$; it is generated by reflections in all roots of $N(X)$ with negative square. In particular, actions of these self-correspondences in $N(X)$ generate a subgroup of finite index in $O(N(X))$ if and only if $[O(N(X)) : W(N(X))] < \infty$. Such hyperbolic lattices $N(X)$ are called reflective. Thus, we characterize K3 surfaces with reflective Picard lattices in terms of their self-correspondences.

Classification of reflective hyperbolic lattices is the subject of the theory of arithmetic hyperbolic reflection groups. In [12, 13] and [24, 25], it was shown that the number of similarity classes of reflective hyperbolic lattices of rank at least 3 is finite.

Thus, for $\text{rk} N(X) \geq 3$, action in $N(X)$ of compositions of correspondences of $X$ defined by $(-2)$ roots of $X$ and by Tyurin’s self-correspondences $Tyu(H)$ with negative $H^2$ and integral action in $N(X)$ generate a subgroup of infinite
index in \( O(N(X)) \), if \( N(X) \) is different from finite number of similarity classes of hyperbolic lattices.

This results relate two different topics: Self-correspondences of K3 surfaces and Arithmetic hyperbolic reflection groups.

## 2 Self-correspondences of K3 surfaces via moduli of sheaves

In series of our papers \([2] - [4]\) (with Carlo Madonna) and \([16] - [18]\), we considered self-correspondences of K3 surfaces (over \( \mathbb{C} \)) via moduli of sheaves with primitive isotropic Mukai vectors. For Picard numbers one or two of K3 surface we described these self-correspondences in big details.

Here we want to give a natural and functorial answer to this problem for arbitrary Picard number. Perhaps, the natural answer will be to give some natural generators for these self-correspondences such that all other self-correspondences of this type will be compositions of these generators. Below, we follow this idea.

We consider primitive isotropic Mukai vectors \( v = (r, H, s) \) where \( r \in \mathbb{N} \), \( s \in \mathbb{Z} \) and \( H \in N(X) \) where \( N(X) \) is the Picard lattice of \( X \), and \( H^2 = 2rs \). Due to Mukai \([5, 6]\) (see also Yoshioka \([26]\)), the moduli space \( Y = M_X(v) \) of stable (with respect to some ample element \( H' \in N(X) \)) coherent sheaves on \( X \) of rank \( r \) with first Chern class \( H \) and Euler characteristic \( r + s \) is again a K3-surface.

Write \( \pi_X \), \( \pi_Y \) for the projections of \( X \times Y \) to \( X \) and \( Y \). By Mukai \([6]\) Theorem 1.5, the algebraic cycle

\[
Z_\mathcal{E} = (\pi_X^* \sqrt{td_X}) \cdot ch(\mathcal{E}) \cdot (\pi_Y^* \sqrt{td_Y})/\sigma(\mathcal{E})
\]

arising from the quasi-universal sheaf \( \mathcal{E} \) on \( X \times Y \) defines an isomorphism of the full cohomology groups

\[
f_{Z_\mathcal{E}} : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q}), \quad t \mapsto \pi_Y^* (Z_\mathcal{E} \cdot \pi_X^* t)
\]

with their Hodge structures (see \([6]\) Theorem 1.5, for details). Moreover, according to Mukai, it defines an isomorphism of lattices (an isometry)

\[
f_{Z_\mathcal{E}} : v^+ \to H^1(Y, \mathbb{Z}) \oplus H^2(Y, \mathbb{Z})
\]

where \( f_{Z_\mathcal{E}} (v) = w \in H^1(Y, \mathbb{Z}) \) is the fundamental cocycle, and the orthogonal complement \( v^+ \) is taken in the Mukai lattice \( \bar{H}(X, \mathbb{Z}) \) (with Mukai pairing \( v^2 = -2rs + H^2 \)). The Mukai’s cycle \([1]\) and this construction can be viewed as a correspondence of \( X \) and \( Y \) via moduli of sheaves defined by the primitive isotropic Mukai vector \( v \). If \( Y \) is isomorphic to \( X \), this defines a self-correspondence of \( X \).

Using this Mukai’s construction, in \([18]\), we considered the action of the self-correspondence on \( H^2(X, \mathbb{Q}) \). Below we recall this construction. See \([18]\) for details.
Composing $f_{Z_E}$ with the projection $\pi: H^4(Y, \mathbb{Z}) \oplus H^2(Y, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ gives an embedding of lattices

$$\pi \cdot f_{Z_E}: H^4(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$$

that extends to an isometry

$$\tilde{f}_{Z_E}: H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})$$

of quadratic forms over $\mathbb{Q}$ by Witt’s Theorem. If $H^2 = 0$, this extension is unique.

If $H^2 \neq 0$, there are two such extensions, differing by the reflection

$$s_H: x \to x - \frac{2(x \cdot H)H}{H^2}, \quad x \in H^2(X, \mathbb{Q}).$$

where $s_H$ is identity on the orthogonal complement $H^\perp$, and $s_H(H) = -H$. We agree to take

$$\tilde{f}_{Z_E}(H) = (-r, 0, s) \mod \mathbb{Z}v$$

(see [18] for details). Another possibility is to consider both such extensions, their difference will be by the reflection $s_H$.

The Hodge isometry (5) can be viewed as a minor modification of Mukai’s algebraic cycle (4) to obtain an isometry in second cohomology. Clearly, it is defined by some 2-dimensional algebraic cycle on $X \times Y$, because it only changes the Mukai isomorphism (2) in the algebraic part.

Now let us assume that $Y = X$, that is the moduli of sheaves over $X$ are parametrized by $X$ itself. Then the Hodge isometry (5) defines the Hodge automorphism

$$\tilde{f}_{Z_E}: H^2(X, \mathbb{Q}) \to H^2(X, \mathbb{Q})$$

of quadratic forms over $\mathbb{Q}$ which gives an automorphism of the transcendental periods $(T(X), H^2(X))$ of $X$. Here $T(X) = (N(X))_{H^2(X, \mathbb{Z})}$ is the transcendental lattice of the K3 surface $X$. The automorphism (8) is called the action of the self-correspondence of $X$ via moduli of sheaves $E$ with the primitive isotropic Mukai vector $v = (r, H, s)$. Changing the parametrization of sheaves by an automorphism of $X$, one changes the action by the action of the automorphism on $H^2(X, \mathbb{Z})$ (and the correspondence by its composition with the graph of the automorphism). Since non-trivial automorphisms of a K3 surface $X$ act non-trivially on $H^2(X, \mathbb{Z})$, the exact choice of the parametrization and the self-correspondence of $X$ via moduli of sheaves with Mukai vector $v$ is defined by its action. We always choose the simplest action that arrives at the most general K3 surfaces with the given type of Mukai vector. See [18] and also [2]–[4], [16], [17] about this subject. We discuss this choice below for Tyurin’s isomorphisms.

We note that by Global Torelli Theorem for K3 surfaces [20], the K3 surfaces $Y$ and $X$ are isomorphic if and only if two sublattices $H^2(X, \mathbb{Z})$ and $\tilde{f}_{Z_E}^{-1}(H^2(Y, \mathbb{Z}))$ for the defined action $\tilde{f}_{Z_E}$ are conjugate by a rational automorphism $f \in O(H^2(X, \mathbb{Q}))$, that is $f^{-1}(H^2(X, \mathbb{Z})) = \tilde{f}_{Z_E}^{-1}(H^2(Y, \mathbb{Z}))$, such that
(f \otimes \mathbb{C})(H^{2,0}(X)) = H^{2,0}(X). \) Thus, the defined action \( \tilde{f}_{Z_e} \) shows when \( Y \cong X \), and it is natural to consider.

It is known that \( Y \cong X \) for the Mukai vectors

\[
v = (\pm H^2/2, H, \pm 1)
\]

where \( H \in N(X) \) and \( \pm H^2 > 0 \). We call this isomorphism as Tyurin’s isomorphism. Tyurin \cite{21, 22} described it geometrically for general K3 surfaces \( X \). In \cite{18} we calculated the action of the Tyurin’s isomorphism, and the result is that \( \text{there exists a unique choice of the identification } Y = M_X(v) = X \text{ such that the action } (8) \) satisfies

\[
\tilde{f}_{Z_e} = s_H \mod \{ \pm 1 \} W(\mathbb{Z})(N(X)).
\]

Here \( s_H \) is the reflection \cite{6} with respect to \( H \), that is \( s_H \) is given by the formula

\[
s_H : x \rightarrow x - \frac{2(x \cdot H)H}{H^2}, \quad x \in H^2(X, \mathbb{Q}),
\]

and \( W(\mathbb{Z})(N(X)) \subset O(H^2(X, \mathbb{Z})) \) is generated by reflections

\[
s_\delta : x \rightarrow x + (x \cdot \delta)\delta, \quad x \in H^2(X, \mathbb{Z}),
\]

with respect to elements \( \delta \in N(X) \) with square \( \delta^2 = -2 \) (they are called reflections in \((-2)\) roots).

More exactly, this is the action if \( X \) is general with the Mukai vector \( v \), that is \( N(X) \otimes \mathbb{Q} = \mathbb{Q}H \) and \( \text{Aut}(T(X), H^{2,0}(X)) = \pm 1 \). Actually, one of \( \pm s_H \) is the only possible action if \( X \) is general. Thus, it coincides with Tyurin’s geometric definition. For an arbitrary \( X \), we can take the action from the coset \( \{ \pm 1 \} W(\mathbb{Z})(N(X))s_H \), and the choice is unique. Here we use Global Torelli Theorem for K3 surfaces, \cite{20}. Geometrically this means that we choose the action which is a specialization of the action from a general K3 surface with the Mukai vector of the type \( v \). The self-correspondence of \( X \) defined by moduli of sheaves on \( X \) with Mukai vector \( v \) of Tyurin’s type \cite{9} and with action \cite{10} we call as Tyurin’s self-correspondence Tyu(\( H \)) of \( X \) via moduli of sheaves defined by an element \( H \in N(X) \) with \( H^2 \neq 0 \).

Assume that \( \delta \in N(X) \) has \( \delta^2 = -2 \). By the Riemann–Roch theorem for K3 surfaces, \( \pm \delta \) contains an effective curve \( E \). If \( \Delta \subset X \times X \) is the diagonal, the effective 2-dimensional algebraic cycle \( \Delta + E \times E \subset X \times X \) acts as the reflection \( s_\delta \) in \( H^2(X, \mathbb{Z}) \) (I learnt this from Mukai \cite{9}). We call this self-correspondences of \( X \) as defined by \((-2)\) roots of \( N(X) \). Actions of their compositions give the group \( W(\mathbb{Z})(N(X)) \).

Using \cite{10}, we obtain the following fundamental relation between self-correspondences of \( X \) via moduli of sheaves with primitive isotropic Mukai vectors and Tyurin’s self-correspondences.

We say that two correspondences of K3 surfaces \( X \) and \( Y_1 \), and \( X \) and \( Y_2 \) (that is 2-dimensional algebraic cycles on \( X \times Y_1 \) and \( X \times Y_2 \)) are \textit{numerically equivalent} if their actions \( f_1 : H^2(X, \mathbb{Q}) \rightarrow H^2(Y_1, \mathbb{Q}) \) and \( f_2 : H^2(X, \mathbb{Q}) \rightarrow \)
Theorem 2.1. Let \( X \) be a K3 surface over \( \mathbb{C} \). It is known (see [10] and [14]) that the automorphism group \( \text{Aut}(T(X), H^{2,0}(X)) \) of the transcendental periods of \( X \) is a finite cyclic group of an order \( n \) where \( \phi(n) \) is the Euler function. Then, obviously, there exists a finite number \(< n/2 \) of primitive isotropic Mukai vectors \( v_1, \ldots, v_k \) giving self-correspondences of \( X \) such that compositions of these self-correspondences give all possible actions on the transcendental periods \( (T(X), H^{2,0}(X)) \) up to \( \pm 1 \).

Let \( v = (r,H,s) \) be a primitive isotropic Mukai vector on \( X \). Then \( Y = M_X(v) \) is isomorphic to \( X \) if and only if the correspondence between \( X \) and \( Y \) defined by \( v \) with the action
\[
\tilde{f}_Z^E : H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})
\]
given in (4) is numerically equivalent, up to \( \pm 1 \), to a composition of self-correspondences defined by \((-2)\) roots, Tyurin’s self-correspondences, and self-correspondences defined by \( v_1, \ldots, v_k \).

In particular, any self-correspondence of \( X \) via moduli of sheaves with primitive isotropic Mukai vector and any their composition is numerically equivalent up to \( \{\pm 1\} \) to composition of self-correspondences defined by \((-2)\) roots, Tyurin’s self-correspondences and self-correspondences defined by \( v_1, \ldots, v_k \).

For a general (for its Picard lattice) K3 surface \( X \) we, obviously, don’t need self-correspondences defined by \( v_1, \ldots, v_k \), and we obtain the Corollary below.

Corollary 2.2. Let \( X \) be a K3 surface over \( \mathbb{C} \) which is general for its Picard lattice that is \( \text{Aut}(T(X), H^{2,0}(X)) = \{\pm 1\} \). Let \( v = (r,H,s) \) be a primitive isotropic Mukai vector on \( X \).

Then \( Y = M_X(v) \) is isomorphic to \( X \) if and only if the correspondence between \( X \) and \( Y \) defined by \( v \) with the action
\[
\tilde{f}_Z^E : H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})
\]
given in (4) is numerically equivalent, up to \( \pm 1 \), to a composition of correspondences defined by \((-2)\) roots, and Tyurin’s self-correspondences.

In particular, any self-correspondence of \( X \) via moduli of sheaves with primitive isotropic Mukai vector and any their composition is numerically equivalent up to \( \{\pm 1\} \) to composition of self-correspondences defined by \((-2)\) roots, and Tyurin’s self-correspondences.

Proof. Using \( \tilde{f}_Z^E \), let us identify \( H^2(Y, \mathbb{Z}) \) with a sublattice \((\tilde{f}_Z^E)^{-1}(H^2(Y, \mathbb{Z}))\) in \( H^2(X, \mathbb{Q}) \). Then \( \tilde{f}_Z^E \) will be identified with the identity of \( H^2(X, \mathbb{Q}) \).

By the isomorphism (4) and the embedding (1) of Hodge structures, we obtain an embedding of the transcendental periods \((T(X), H^{2,0}(X)) \subset (T(Y), H^{2,0}(Y))\).
\( H^{2,0}(Y) \). If \( Y \cong X \), we must then have
\[
(T(X), H^{2,0}(X)) = (T(Y), H^{2,0}(Y)).
\number{(12)}

This identification of transcendental periods of \( X \) and \( Y \) is unique up to multiplication by \( \{ \pm 1 \} \) and by composition of self-correspondences defined by \( v_1, \ldots, v_k \).

By Global Torelli Theorem for K3 surfaces \( [20] \), \( Y \cong X \) if and only if the identification \( (12) \) can be extended to an isomorphism of the sublattices \( H^2(X, \mathbb{Z}) \) and \( H^2(Y, \mathbb{Z}) \) of \( H^2(X, \mathbb{Q}) \). Equivalently, there must exist \( f \in O(H^2(X, \mathbb{Q})) \) such that \( f(H^2(X, \mathbb{Z})) = H^2(Y, \mathbb{Z}) \) and \( f(T(X)) = \pm 1 \). Changing \( f \) to \(-f\) if necessary, we can assume that \( f(T(X)) = 1 \). Then \( f \) is defined by its action in \( N(X) \otimes \mathbb{Q} \). By the well-known result about rational quadratic forms (e.g., see \( [1] \)), such \( f \) is a composition of reflections \( f = s_{H_m} \cdots s_{H_1} \) with respect to elements \( H_i \in N(X) \) with \( (H_i)^2 \neq 0 \). Note that these reflections give identity in \( T(X) \). Thus, our considerations are reversible.

If \( Y = X \), we can consider \( f \) as an identification of \( H^2(X, \mathbb{Z}) \) with another copy of \( H^2(X, \mathbb{Z}) \). Then the action is \( f^{-1} = s_{H_1} \cdots s_{H_m} \). It follows the statement.

Theorem \( 2.1 \) and Corollary \( 2.2 \) show that \( (-2) \) roots and Tyurin’s self-correspondences of \( X \) are fundamental for all self-correspondences of \( X \) via moduli of sheaves with primitive isotropic Mukai vector.

The proof of these statements demonstrates an application of the purely arithmetic fact that the group of automorphisms of a rational quadratic form is generated by reflections. The same is valid for quadratic forms over fields of characteristic different from 2.

These results can be also considered as some general (for any Picard number) alternative to our results with Carlo Madonna \( [2] \)–\( [4] \), \( [15] \)–\( [17] \) about self-correspondences of K3 surfaces via moduli of sheaves with primitive isotropic Mukai vector which were valid for K3 surfaces with Picard number one or two only.

\textbf{Problem 2.3.} Can one replace in Theorem \( 2.1 \) and Corollary \( 2.2 \) the numerical equivalence by some algebraic equivalence of self-correspondences?

\section{K3 surfaces and arithmetic hyperbolic reflection groups}

\subsection{Reflections and reflective hyperbolic lattices}

We use the notation and terminology of \( [11] \) for lattices, and their discriminant groups and forms. A \textit{lattice} \( L \) is a nondegenerate integral symmetric bilinear form. That is, \( L \) is a free \( \mathbb{Z} \)-module of a finite rank with a symmetric pairing \( x \cdot y \in \mathbb{Z} \) for \( x, y \in L \), assumed to be nondegenerate. We write \( x^2 = x \cdot x \). The \textit{signature} of \( L \) is the signature of the corresponding real form \( L \otimes \mathbb{R} \). The lattice \( L \) is called \textit{even} if \( x^2 \) is even for any \( x \in L \). Otherwise, \( L \) is called
The determinant of $L$ is defined to be $\det L = \det(e_i \cdot e_j)$ where $\{e_i\}$ is some basis of $L$. The lattice $L$ is unimodular if $\det L = \pm 1$. The dual lattice of $L$ is $L^* = \text{Hom}(L, \mathbb{Z}) \subset L \otimes \mathbb{Q}$. The discriminant group of $L$ is $A_L = L^*/L$; it has order $|\det L|$, and is equipped with a discriminant bilinear form $b_L : A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$ and, if $L$ is even, with a discriminant quadratic form $q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$. To define these, we extend the form on $L$ to a form on the dual lattice $L^*$ with values in $\mathbb{Q}$.

An embedding $M \subset L$ of lattices is called primitive if $L/M$ has no torsion. Similarly, a non-zero element $x \in L$ is called primitive if $\mathbb{Z}x \subset L$ is a primitive sublattice.

A non-zero element $\delta \in L$ is called positive, negative, and isotropic if $\delta^2 > 0$, $\delta^2 < 0$, and $\delta^2 = 0$ respectively. An element $\delta$ of a lattice $L$ is is called root if $\delta^2 \neq 0$ and $\delta^2|2(\delta \cdot L)$. For example, $\delta \in L$ with $\delta^2 = \pm 2$ is root. It is called $(\pm 2)$ root.

Further $O(L)$ denotes the full automorphism group of the lattice $L$. Each root $\delta \in L$ gives a reflection $s_\delta \in O(L)$ with respect to $\delta$. It is given by the formula

$$s_\delta : x \to x - \frac{2(x \cdot \delta)\delta}{\delta^2}, \quad x \in L,$$

(13)

and it is uniquely determined by the properties that $s_\delta(\delta) = -\delta$ and $s_\delta$ gives identity on the orthogonal complement $(\delta)^\perp_L$ to $\delta$ in $L$. Two proportional roots define the same reflection. Thus, considering reflections $s_\delta$, we can restrict to primitive roots. For a primitive $\delta \in L$ with $\delta^2 \neq 0$, the formula (13) defines an automorphism $s_\delta \in O(L)$ of $L$ if and only if $\delta$ is root. Otherwise, $s_\delta \in O(L \otimes \mathbb{Q})$ is a rational automorphism of the rational quadratic form $L \otimes \mathbb{Q}$. The reflection $s_\delta$ is called positive, negative if $\delta$ is respectively positive, negative.

We denote by $W^\pm(L)$, $W^+(L)$, and $W(L) = W^-(L)$ normal subgroups of $O(L)$ generated by reflections in all positive and negative roots, all positive roots, and all negative roots of $L$ respectively.

We denote by $W^{(-2)}(L)$ the normal subgroup of $O(L)$ generated by reflections in all elements of $L$ with square $-2$ that is all $(-2)$ roots of $L$.

A lattice $L$ is called hyperbolic if it has signature $(1, \rho - 1)$ where $\rho = \text{rk} L$. A hyperbolic lattice $L$ is called (classically) reflective, equivalently $(\pm)$ reflective if $W^-(L)$ has finite index in $O(L)$, that is the automorphism group $O(L)$ is generated by all negative reflections of $L$, up to finite index. Similarly, we call a hyperbolic lattice $L$ as $(\pm)$ reflective, and $(\pm)$ reflective if $W^\pm(L)$, and $W^+(L)$ has finite index in $O(L)$ respectively.

Further $L(m)$ denotes a lattice obtained by the multiplication of the form of a lattice $L$ by positive $m \in \mathbb{Q}$. The lattices $L$ and $L(m)$ are called similar. The automorphism groups of similar lattices are naturally identified, and the lattices are reflective of any type simultaneously.

Any hyperbolic lattice $L$ of rank one is reflective since $O(L) = \{\pm 1\}$. Any hyperbolic lattice $L$ of rank two is reflective if and only if either $L$ has an isotropic element (then $O(L)$ is finite), or $L$ has a negative root (equivalently, a negative reflection). It is known since 1981 that there exist only finite number of reflective hyperbolic lattices $L$ of $\text{rk} L \geq 3$, up to similarity. We had shown
this in [12], [13] for a fixed \( \text{rk} L \geq 3 \). Vinberg [24], [25] had shown that \( \text{rk} L < 30 \)
for reflective hyperbolic lattices \( L \).

We note that recently similar finiteness results were completed for reflective hyperbolic lattices over all totally real algebraic number fields together. For example, see [19] and references there.

Unfortunately, we don’t know similar finiteness results for \((\pm)\) reflective and \((+)\) reflective hyperbolic lattices.

3.2 K3 surfaces with reflective Picard lattices

Further we consider Tyurin’s self-correspondences \( \text{Tyu}(H) \) of \( X \) with integral action in \( N(X) \), that is \( s_H | N(X) \in O(N(X)) \). By our discussion in Sec. 3.1 this is equivalent for \( H \) to be multiple of a primitive root in \( N(X) \), equivalently, \( 2(H \cdot N(X))H/H^2 \in N(X) \). Self-correspondences of \( X \) with integral action in \( N(X) \) can be viewed as similar to ones defined by graphs of automorphisms of \( X \) and by \((\pm 2)\) roots of \( N(X) \). We call self-correspondence \( \text{Tyu}(H) \) positive (respectively negative) if \( H^2 > 0 \) (respectively \( H^2 < 0 \)). From (10) and our considerations in Section 2, we get the following result.

**Theorem 3.1.** Let \( X \) be a K3 surface over \( \mathbb{C} \).

The action of compositions of all self-correspondences of \( X \) defined by \((\pm 2)\) roots of \( N(X) \) and by Tyurin’s self-correspondences \( \text{Tyu}(H) \) with integral action in \( N(X) \) generate the group \( \{\pm 1\} W^\pm(N(X)) \) up to \( \{\pm 1\} \). They generate a subgroup of finite index in \( O(N(X)) \) if and only if \( N(X) \) is (classically) reflective.

The action of compositions of all self-correspondences of \( X \) defined by \((\pm 2)\) roots of \( N(X) \) and by Tyurin’s self-correspondences \( \text{Tyu}(H) \) with negative \( H^2 \) and integral action in \( N(X) \) generate the group

\[
\{\pm 1\} W(N(X)) = \{\pm 1\} W^-(N(X))
\]

up to \( \{\pm 1\} \). They generate a subgroup of finite index in \( O(N(X)) \) if and only if \( N(X) \) is (classically) reflective.

By Piatetsky-Shapiro and Shafarevich [20] (this is an important corollary of Global Torelli Theorem for K3 surfaces), we have

\[
[O(N(X)) : W^{(-2)}(N(X)) \text{Aut} X] < \infty,
\]

that is \( W^{(-2)}(N(X)) \text{Aut} X \) is a subgroup of finite index in \( O(N(X)) \). Here we identify Aut \( X \) with its action in \( N(X) \). It has a finite kernel. Moreover, \( W^{(-2)}(N(X)) \text{Aut} X \) is the semi-direct product of groups where \( W^{(-2)}(N(X)) \) is a normal subgroup.

Since by [12] and [13], the number of similarity classes of reflective hyperbolic lattices of a fixed rank at least 3 is finite, and \( \text{rk} N(X) \leq 20 \), applying Theorem 3.1 and the result (14) by Piatetsky-Shapiro and Shafarevich, we obtain the following result.
Theorem 3.2. Let $X$ be a K3 surface over $\mathbb{C}$.

By the result \[\text{[14]}\] by Piatetsky-Shapiro and Shafarevich \[\text{[20]}\], the action in Picard lattice $N(X)$ of compositions of self-correspondences of $X$ defined by all $(-2)$ roots of $N(X)$ and by graphs of automorphisms of $X$ give a subgroup of finite index in $O(N(X))$.

In contrary, the action in Picard lattice $N(X)$ of compositions of self-correspondences of $X$ defined by all $(-2)$ roots of $N(X)$ and by graphs of automorphisms of $X$ give a subgroup of infinite index in $O(N(X))$ if and only if $N(X)$ is not reflective. This is the case if $\text{rk} \, N(X) \geq 3$ and $N(X)$ is different from a finite number of similarity classes of reflective hyperbolic lattices of rank $\geq 3$.

If $N(X)$ is reflective, then the action of a finite number of $(-2)$ roots in $N(X)$ and a finite number of Tyurin’s self-correspondences $\text{Tyu}(H)$ with $H^2 < 0$ and integral action in $N(X)$ generate a subgroup of finite index of $O(N(X))$.

In the last statement, we use the well-known fact that the group $W(N(X))$ is generated by a finite number of reflections in negative roots of $N(X)$ if $N(X)$ is reflective. This follows from the fact that the fundamental chamber for $W(N(X))$ is a finite polyhedron in the hyperbolic space determined by $N(X)$, the group $W(N(X))$ is generated by reflections in negative roots which are perpendicular to codimension one faces of this chamber. Their number is finite.

Unfortunately, we don’t know finiteness for $(\pm)$ reflective hyperbolic lattices of rank at least 3. This raises an interesting question.

Problem 3.3. Is the number of similarity classes of $(\pm)$ reflective hyperbolic lattices of rank at least 3 finite?

Here a hyperbolic lattice $S$ is $(\pm)$ reflective if the group $W^\pm(S)$ generated by reflections in all positive and negative roots of $S$ has finite index in $O(S)$.

Theorem \[\text{[52]}\] shows that to obtain a sufficient number (with sufficiently arbitrary action in $N(X)$) of self-correspondences of $X$ with integral action, one has to consider all self-correspondences of $X$ generated by $(-2)$ roots, Tyurin’s self-correspondences $\text{Tyu}(H)$ with integral action, and graphs of automorphisms together. This raises the following question which is interesting for K3 surfaces with reflective and not reflective Picard lattices.

Problem 3.4. What is the kernel of the action in $N(X)$ of the group of self-correspondences of $X$ generated by all $(-2)$ roots of $N(X)$, $\text{Tyu}(H)$ with integral (or arbitrary) action in $N(X)$, and graphs of automorphisms of $X$?

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