Functional Integral Approach to the N-Flavor Schwinger Model

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Abstract

We study massless QED₂ with N flavors using path integrals. We identify the sector that is generated by the \( N^2 \) classically conserved vector currents. One of them (the \( U(1) \) current) creates a massive particle, while the others create massless ones. We show that the mass spectrum obeys a Witten-Veneziano type formula. Two theorems on \( n \)-point functions clarify the structure of the Hilbert space. Evaluation of the Fredenhagen-Marcu order parameter indicates that a confining force exists only between charges that are integer multiples of \( \pm Ne \), whereas charges that are nonzero \( \text{mod}(N) \) screen their confining forces and lead to non-vacuum sectors. Finally we identify operators that violate clustering, and decompose the theory into clustering \( \theta \) vacua.

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1 Introduction

The Schwinger model is a laboratory to study many important aspects of more realistic models of particle physics such as QCD. It shows confinement, mass generation of its would-be Goldstone particle via the axial anomaly – thereby solving its $U(1)$ problem, it also has topological sectors and a vacuum angle $\theta$ dual to them. While these features are well known and understood in the case of Schwinger’s original one-flavor model, in a model with several flavors the situation is much less trivial and has not been investigated in much detail so far. This $N$-flavor Schwinger model has in fact more features in common with four dimensional QCD: On the classical level it has a symmetry group $U(N)_L \times U(N)_R$ that is broken down by the anomaly to $SU(N)_L \times SU(N)_R \times U(1)_V$ just like in QCD.

In this paper we study this model using the euclidean functional integral. The boundary conditions we use (vector potentials decaying at infinity sufficiently fast to be square integrable) correspond to zero topological charge for the gauge field. This leads to an infinite volume vacuum state enjoying chiral invariance, but violating the cluster decomposition property. It turns out that by explicit construction this vacuum state can be decomposed into a mixture of clustering states labeled by an angle $\theta$, the so-called vacuum angle, which is conventionally introduced as a dual variable to the topological charge (first Chern number) of the gauge field.

We compute the correlation functions of the vector/pseudovector currents as well as the scalar/pseudoscalar densities and find that the theory describes a multiplet of one massive and $N^2 - 1$ massless pseudoscalar ‘mesons’. There is a relation between the masses that looks exactly like the so-called Witten-Veneziano formula for the pseudoscalar meson masses in QCD; in fact it allows to give a precise meaning to this formula with proper attention to the short distance fluctuations of the topological density that play an essential role.

The model also reveals an interesting superselection structure. Computation of the so-called Fredenhagen-Marcu confinement parameter suggests that there are superselection sectors labeled by a charge that is defined as an integer mod($N$). This means that the confining Coulomb potential does not prevent operators of fermion number $\neq 0$ mod($N$) from creating new charged sectors; their electric charge seems to be ‘screened’. Of course QCD is not expected to share this property.

The model is still trivial in the sense that there is no real interaction between its particles, but it is not Gaussian like the one-flavor model. Its vacuum sector turns out to be a tensor product of the Fock space of a free massive scalar particle with the representation space of the $SU(N)_L \times SU(N)_R$ current (Kac-Moody) algebra induced by $N$ massless Dirac fermions.

We would like to make a remark about the level of rigor of this paper: Our aim is to elucidate the physical content by explicit calculation. Therefore we
do not spend time to justify all the manipulations involved in deriving the results with full mathematical rigor, even though it would not be too difficult to do so. In [1] it is sketched how this is to be done for the one-flavor model; the $N-$flavor models does not pose any fundamentally new problems in this respect.

There are some open problems left: For instance it should be expected that the subspace created form the vacuum by the scalar/pseudovector currents is identical to the one created by the vector/pseudoscalar densities; we did not attempt to prove that. Likewise the structure of superselection sectors suggested by the Fredenhagen-Marcu parameter has not been established rigorously, because that parameter by its nature has only suggestive value.

The paper is organized as follows: In the next section we define the model and introduce our notations. Furthermore we derive some basic expressions for generating functionals and Green’s functions. In Section 3 we investigate the sector that is generated by vector currents. The mass spectrum that we obtain fits a Witten-Veneziano type formula, which we discuss in Section 4. To get more insight into the subspace generated by the vector currents we derive general expressions for their $n$-point functions in section 5. The next section is dedicated to the problem of confinement. We evaluate the Fredenhagen-Marcu order parameter for a $e^+e^-$ system and higher products of fermion fields. Finally we decompose the model into clustering theories and thereby obtain the vacuum angle.

2 Setup

We use the following representation of the 2-d Euclidean $\gamma$-matrices.

\[
\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_5 = i\gamma_2\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\] (1)

They obey the commutation relations $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ ; $\mu, \nu = 1, 2, 5$. The Green's function $G(x, y; A)$ for fermions in a background field $A$ obeys the equation

\[
\gamma_\mu(\partial_\mu - ieA_\mu)G(x, y; A) = \delta(x, y).
\] (2)

In two dimensions it can be solved explicitely [4], provided the background field satisfies some mild regularity and falloff conditions; the solution is

\[
G(x, y; A) = e^{ie[\Phi(x)-\Phi(y)]} G^0(x - y)
\] (3)

where

\[
\Phi(x) = -\int d^2z D(x - z)\left[\partial_\mu A_\mu(z) + i\gamma_5\varepsilon_{\mu\nu}\partial_\mu A_\nu(z)\right].
\] (4)

$G^0(x)$ is the Green’s function for zero background field

\[
G^0(x) = \frac{1}{2\pi} \frac{\gamma_\mu x_\mu}{x^2}.
\] (5)
$D(x)$ denotes the Green’s function for free massless bosons ($-\Delta D(x) = \delta(x)$).

The action for the $N$-flavor Schwinger model reads

$$S[\psi, \bar{\psi}, A] = S_g[A] + S_f[\psi, \bar{\psi}, A],$$

with

$$S_g[A] = \int d^2x \left[ \frac{1}{4} F_{\mu\nu}(x) F_{\mu\nu}(x) + \frac{1}{2} \lambda \left( \partial_\mu A_\mu(x) \right)^2 \right] =: \frac{1}{2} (A, QA).$$

The kernel $Q$ of the quadratic form we introduced is given by

$$Q = (1 - \lambda) \partial \otimes \partial - \Delta.$$ We work in Landau gauge, which means $\lambda \to \infty$ after the integration over the gauge field.

The fermionic part of the action is a generalization of the one flavor model

$$S_f[\psi, \bar{\psi}, A] = \sum_{a=1}^{N} \int d^2x \bar{\psi}^{(a)}(x) \gamma_\mu \left( \partial_\mu - i e A_\mu(x) \right) \psi^{(a)}(x).$$

$\bar{\psi}^{(a)}$, $\psi^{(a)}$ are 2-component spinors with a flavor index $a = 1, 2, \ldots N$. It is useful to introduce sources for the fermions and to define the generating functional

$$Z[\eta, \bar{\eta}] := \int [d\psi][d\bar{\psi}][dA] e^{-S[\psi, \bar{\psi}, A] + \sum_a \left\{ (\bar{\psi}^{(a)}, \psi^{(a)}) + (\bar{\eta}^{(a)}(x), \eta^{(a)}(y)) \right\}}.$$ (9)

Vacuum expectation values can be obtained as functional derivatives with respect to $\eta, \bar{\eta}$. Performing formally the Berezin integral over the Grassmann variables leads to

$$Z[\eta, \bar{\eta}] = \int [dA] e^{-S_g[A]} \det[\partial - i e A] \sum_a \int d^2x d^2y \bar{\eta}^{(a)}(x) G(x, y; A) \eta^{(a)}(y).$$ (10)

Note that the formal expression $\det[\partial - i e A]$ is only defined when an ultraviolet and infrared cutoff (for instance a finite space-time lattice) is introduced. The determinant can then be normalized to 1 for $e = 0$, by replacing it with $\det[1 - e K(A)]$ where $K(A) = i A \partial^{-1}$. In two dimensions this determinant can be computed explicitly [2], using the idea of regularized fermion determinants (for a review see e.g. [3], for the QED$_2$ case [4]). If we assume that the vector potential $A$ satisfies some mild regularity and falloff conditions at infinity to make it square integrable, the answer is

$$\det[1 - e K(A)] = e^{-\frac{e^2}{\pi} \|A^T\|^2},$$ (11)

where $A^T_\mu$ is the transverse part of the gauge field

$$A^T_\mu = A_\mu - \frac{\partial_\mu \partial_\nu}{\Delta} A_\nu.$$ (12)
We combine the logarithm of the determinant and the gauge field action in one common quadratic form for $A_\mu$. This gives rise to a Gaussian measure $d\mu_C[A] = \frac{1}{Z}[dA] \exp(-\frac{1}{2}(A, C^{-1}A))$ for the gauge field with covariance $C$ defined by

$$
(A, C^{-1}A) := (A, QA) + e^2 \frac{N}{\pi} \|A^T\|^2_2 .
$$

A few lines of algebra lead to

$$
C^{-1}_{\mu\nu} = \left(- \Delta + e^2 \frac{N}{\pi} \right) \left( \delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Delta} \right) - \partial_\mu \partial_\nu \lambda .
$$

Inverting and taking the limit $\lambda \to \infty$ gives

$$
C_{\mu\nu} = \frac{1}{-\Delta + e^2 \frac{N}{\pi} \left( \delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Delta} \right) } .
$$

Besides this form of the measure for the gauge field, we compute the measure for a field $\varphi$ which is a linear functional of the gauge field. This leads to a very simple dependence of the propagator on the gauge field. $G(x, y; A)$ depends on $\Phi[A]$ via the exponential $\exp(ie[\Phi(x) - \Phi(y)])$ (see equation (3)). $\Phi$ in general depends on the gauge field through the terms $\partial_\mu A_\mu$ and $\varepsilon_{\mu\nu} \partial_\mu A_\nu$, but the first term vanishes (with probability 1) in Landau gauge so that $\Phi$ becomes a function of $\varepsilon_{\mu\nu} \partial_\mu A_\nu$ alone. To get the spinor structure of the propagator explicitly, we define (after dropping the $\partial_\mu A_\mu$ term in $\Phi$)

$$
\Phi(x) =: i\gamma_5 \varphi(x) ,
$$

where

$$
\varphi(x) = - \int d^2 z D(x - z) \varepsilon_{\mu\nu} \partial_\mu A_\nu(z) .
$$

This implies

$$
\varphi = \frac{\varepsilon_{\mu\nu} \partial_\mu}{\Delta} A_\nu .
$$

Using this equation one obtains the Gaussian measure $d\mu_{\tilde{C}}[\varphi]$ for the field $\varphi$, induced by the measure for the gauge field with covariance $\tilde{C}$ given by

$$
\tilde{C} = \frac{-1}{(-\Delta + N \varepsilon^2 \frac{e^2}{\pi}) \Delta} .
$$

Expressed in terms of $\varphi$ the propagator reads

$$
G(x, y; \varphi) = e^{-\varepsilon_5 \varphi(x) - \varphi(y)} G^0(x - y) =

\frac{1}{2\pi} \left( \begin{array}{cc} 0 & e^{-\varepsilon_5 \varphi(x) - \varphi(y)} \left( \tilde{x} - \tilde{y} \right) \\ e^{+\varepsilon_5 \varphi(x) - \varphi(y)} \left( \tilde{x} - \tilde{y} \right) & 0 \end{array} \right) ,
$$

where we introduced the complex coordinate

$$
\tilde{x} = x_1 + ix_2 .
$$
3 Vector currents

In this section we derive the 2-point functions of the vector/pseudovector currents and obtain thereby information about the particle spectrum of the theory. Note that the axial vector currents are simply linear combinations of the vector currents due to the fact that in two dimensions $\gamma_5 \gamma_{\mu\nu} = i \varepsilon_{\mu\nu} \gamma_{\nu}$.

3.1 Bosonizing a Cartan subalgebra of $U(N)$

It was noticed in [4] that the $U(1)$ current together with the other elements of a Cartan subalgebra of $U(N)_{\text{flavor}}$ can be bosonized in terms of free scalar fields. This result is the first step in our analysis of the particle structure of the model and therefore we reproduce this result in terms of the functional integral formalism.

Define vector currents

$$j_{\mu}^{(b)}(x) := \overline{\psi}^{(b)}(x) \gamma_{\mu} \psi^{(b)}(x) , \quad b = 1, 2, \ldots N .$$

In principle one would have to regularize these currents by e.g. a point splitting prescription, but we can use the fact that the fermion current couples to an external field $a_{\mu}$ the same way as to the quantized field $A_{\mu}$ and thereby avoid all problems due to short distance singularities. We couple the currents to source fields $a_{\mu}^{(b)}(x)$ obeying $\partial_{\mu} a_{\mu}^{(b)} = 0$, and evaluate the characteristic functional using the explicit form of the determinant

$$\langle e^{ie \sum_{b=1}^{N} (j_{\mu}^{(b)}, a_{\mu}^{(b)})} \rangle = \frac{1}{Z} \int [dA] e^{-S_{b}[A]} \prod_{b=1}^{N} \det \left[ 1 - eK(A + a^{(b)}) \right] =$$

$$e^{-\frac{e^2}{2\pi} \sum_{b} \|a^{(b)}\|^2_2} \int d\mu_{C}[A] e^{-(A, A)} = e^{-\frac{e^2}{2\pi} \sum_{b} \|a^{(b)}\|^2_2} e^{\frac{1}{2}(\Lambda, CA)} ,$$

$$\Lambda_{\mu} = \frac{e^2}{\pi} \sum_{b=1}^{N} a_{\mu}^{(b)} .$$

After some reordering of the terms in the last equation one ends up with the following expression for the characteristic functional

$$\langle e^{-ie \sum_{b} (j_{\mu}^{(b)}, a_{\mu}^{(b)})} \rangle = e^{-\frac{e^2}{2\pi} \sum_{b,c} (\varepsilon_{\mu\nu} \partial_{\mu} a_{\nu}^{(b)}, M^{(b,c)} \varepsilon_{\rho\sigma} \partial_{\rho} a_{\sigma}^{(c)})} ,$$

where $M$ is given by (using matrix notation in flavor space)

$$M = \frac{-e^2/\pi}{(-\Delta + e^2 \frac{N}{\pi})\Delta} R + \frac{1}{-\Delta + e^2 \frac{N}{\pi}} \mathbf{1} ,$$
The numerical matrix $R$ can be diagonalized by an orthogonal matrix $U$ given by $U^T = (r^{(1)}, r^{(2)}, ..., r^{(N)})$ where the $r^{(l)}$ are the normalized eigenvectors of $R$. The corresponding eigenvalues are 0, $N$, $N$, ..., $N$, implying $URU^T = \text{diag}(0, N, ..., N)$. This allows to diagonalize the covariance $M$. Define

$$K := UMU^T = \begin{pmatrix} \frac{1}{-\Delta + e^2 \pi} & 1 \\ \frac{1}{-\Delta} & \frac{1}{-\Delta} \end{pmatrix}.$$

We now choose the sources $a^{(l)}_{\mu}$ in flavor space proportional to one of the eigenvectors $r^{(l)}$. This corresponds to a change of the basis in flavor space and allows to express expectation values of the new vector currents in terms of free bosons. Define

$$J^{(l)}_{\mu} := \bar{\psi} \gamma_{\mu} H^{(l)} \psi ,$$

where the $N \times N$ matrices $H^{(l)}$ are generators of a Cartan subalgebra of $U(N)_{\text{flavor}}$

$$H^{(1)} = \frac{1}{\sqrt{N}} 1 ,$$

$$H^{(2)} = \frac{1}{\sqrt{N-1 + (N-1)^2}} \text{diag}(1, 1, ..., 1, -N+1) ,$$

$$H^{(3)} = \frac{1}{\sqrt{N-2 + (N-2)^2}} \text{diag}(1, 1, ..., 1, -N+2, 0) , ... ,$$

$$H^{(N)} = \frac{1}{\sqrt{2}} \text{diag}(1, -1, 0, 0, ..., 0) .$$

The diagonals of these generators are the eigenvectors of $M$. Using equation (24) one easily finds the generating functional for the $J^{(l)}_{\mu}$

$$\langle e^{ie \left( J^{(l)}_{\mu} h^{(l)}_{\mu} \right)} \rangle = e^{\frac{-\varepsilon^2}{2\pi} \left( r^{(l)} \varepsilon_{\mu\nu} \partial_{\mu} b^{(l)}_{\nu} , M r^{(l)} \varepsilon_{\rho\sigma} \partial_{\rho} b^{(l)}_{\sigma} \right)} =$$

$$e^{\frac{-\varepsilon^2}{2\pi} \left( \varepsilon_{\mu\nu} \partial_{\mu} b^{(l)}_{\nu} , K_{ll} \varepsilon_{\rho\sigma} \partial_{\rho} b^{(l)}_{\sigma} \right)} = \langle e^{ie \frac{1}{\sqrt{N}} \left( \varepsilon_{\mu\nu} \partial_{\nu} b^{(l)}_{\mu}, b^{(l)}_{\mu} \right)} \rangle_B ,$$

where $h^{(l)}_{\mu}$ is a source for the currents $J^{(l)}_{\mu}$. The index $l$ now labels the ‘meson’ fields (27). $\langle \cdot \rangle_B$ denotes the Gaussian expectation values of an $N$-tuple of scalar
fields $\varphi^{(l)}$, $l = 1, 2, ... N$ with covariance $K$ given by equation (26). Hence we can identify
\[ J^{(l)}_{\mu} = \frac{1}{\sqrt{\pi}} \varepsilon_{\mu\nu} \partial_{\nu} \varphi^{(l)}. \] (30)

The last equation is the bosonization formula of a Cartan subalgebra of $U(N)$.

### 3.2 More meson fields

In analogy to the construction of meson states in QCD, one can define vector currents for all the generators of $U(N)$. A convenient basis of the Lie algebra of $U(N)$ is given by the $N(N - 1)/2$ generators $H^{(l)}$ of the form
\[ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \] (31)

and $N(N - 1)/2$ of the form
\[ \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ i \\ i \end{pmatrix}. \] (32)

The corresponding vector currents are
\[ J^{(l)}_{\mu}(x) := \bar{\psi}(x) H^{(l)} \psi(x). \] (33)

Again no point splitting has to be taken into account, since only different flavors (which cannot contract) sit at one space-time point and therefore there is no short distance singularity.

First we notice that all $N^2$ vector currents generate orthogonal states
\[ \langle J^{(l)}_{\mu}(x)J^{(l')}_{\nu}(y) \rangle = \delta_{ll'} \mathcal{F}^{(l)}_{\mu\nu}(x, y) \quad l = 1, 2, ... N^2, \] (34)

where $\mathcal{F}$ is the two point function. For the Cartan currents (34) follows directly from the bosonization. To prove it for the set of all $N^2$ currents one has to take functional derivatives of the generating functional (10) with respect to the fermion sources $\eta, \bar{\eta}$. If $l \neq l'$, either the different flavors do not contract entirely, or terms with opposite sign cancel.
In the case \( l = l', l = N + 1, N + 2, \ldots N^2 \), functional derivation leads to (take e.g. \( \mu = \nu = 1 \))

\[
\mathcal{F}_{11} = - \int d\mu \tilde{C} \{ G_{21}(x, y; \varphi) G_{21}(y, x; \varphi) + G_{12}(x, y; \varphi) G_{12}(y, x; \varphi) \}. \tag{35}
\]

Using the explicit form (20) for \( G \), one immediately sees that the exponentials involving \( \varphi \) cancel. Integration over \( \varphi \) simply gives a factor 1. The same is true for arbitrary \( \mu, \nu \). Hence the two point function is the same as for free, massless fermions. It can be expressed in terms of derivatives of the propagator of a free massless boson, giving rise to the same expression as was obtained for the Cartan subalgebra, i.e. for \( l = 2, 3, \ldots N \). Putting things together we conclude for the two-point functions of the vector currents

\[
\langle J^{(l)}_{\mu}(x) J^{(l')}_{\nu}(y) \rangle = \delta_{ll'} \frac{1}{\pi} \varepsilon_{\mu\rho} \partial \varepsilon_{\nu\sigma} \partial y \langle \varphi^{(l)}(x) \varphi^{(l)}(y) \rangle_B, \tag{36}
\]

where the scalar field \( \varphi^{(l)} \) has mass \( m = e \sqrt{\frac{N}{\pi}} \) and \( \varphi^{(l)}, l = 2, 3, \ldots N^2 \) are massless. In addition we obtained explicit expressions for arbitrary \( n \)-point functions of the Cartan currents \( (l = 1, 2, \ldots N) \) via the bosonization prescription (30). In Section 5 we will obtain the \( n \)-point functions for all the currents.

4 Mass spectrum and Witten-Veneziano type formulas

In 1979 Witten [5] proposed a formula that relates the mass of the \( \eta' \) meson to the topological susceptibility of quarkless QCD. The purpose of this was to show how QCD solves its \( U(1) \) problem, without having to bring instantons into the discussion. The derivation given by Witten has some problems (terms of equal sign are cancelled against each other) as was pointed out by Seiler and Stamatescu [6]. But it was also noted there that for the one flavor Schwinger model the formula is nevertheless true, if the topological susceptibility is interpreted appropriately, with proper attention to contact terms (the fact that therefore the correct treatment of short distance fluctuations of the topological charge density is essential makes of course the application to lattice gauge theories cumbersome, to say the least).

The \( N \) flavor Schwinger model is a convenient laboratory to study the \( U(1) \) problem in a more general setting, much more reminiscent of QCD, because it has both flavor changing axial currents that are conserved and a flavor non-changing one whose conservation is violated by the axial anomaly. The mass spectrum we obtained in the last section contains both massless Goldstone particles related to the conserved axial currents and a massive particle related to the anomalous \( U(1) \) axial current. So it has the features expected to be
present in the meson sector of QCD in the chiral limit. Hence it is natural to check if our model satisfies a formula of the Witten-Veneziano type.

Here we cite the more general formulation of Veneziano ([7],[8]) which reduces to Witten’s form when setting all meson masses (except \(m_{\eta'}\)) equal zero. It is given by

\[
m_{\eta'}^2 - \frac{1}{2}m_\eta^2 - \frac{1}{2}m_{\pi^0}^2 = \frac{12}{f_{\pi}^2} \chi .
\]  

(37)

\(f_{\pi}\) denotes the meson decay constant. \(\chi\) has a more subtle meaning. The interpretation as topological susceptibility faces some problems, as was pointed out by Seiler and Stamatescu [6]. But a formula of this type can be derived if \(\chi\) is interpreted as the contact term that occurs in the spectral representation of the two point function of the topological charge density (compare also the appendix in [6], and the derivation given by Smit and Vink [8]). Here we use the form that was obtained by Seiler and Stamatescu for an arbitrary number of flavors \(N\) and vanishing quark masses, which implies \(m_\eta = m_{\pi^0} = 0\). It reads

\[
m_{\eta'}^2 = \frac{4N}{|f_{\eta'}|^2} P(0) .
\]  

(38)

\(P(0)\) denotes the announced contact term. (38) contains the \(\eta'\) decay constant \(f_{\eta'}\), which shows up also in Witten’s derivation, and is replaced later by \(f_{\pi}\) to which it is approximately equal.

In QED\(_2\) the topological charge density \(q(x)\) is given by

\[
q(x) = \frac{e}{2\pi} F_{12}(x) .
\]  

(39)

The topological susceptibility is defined to be

\[
\chi_{top} = \int \langle q(x)q(0) \rangle dx = \frac{e^2}{(2\pi)^2} \int \langle F_{12}(x)F_{12}(0) \rangle dx = \frac{e^2}{2\pi} \hat{G}_{FF}(0) ,
\]  

(40)

where \(G_{FF}\) denotes the \(F_{12}\) propagator, and \(\hat{G}_{FF}(0)\) it’s Fourier transform at zero momentum. Since

\[
F_{12}(x) = \varepsilon_{\mu\nu} \partial_\mu A_\nu(x) ,
\]  

(41)

the propagator \(G_{FF}\) is given by

\[
G_{FF} = -\varepsilon_{\mu\nu} \partial_\nu C_{\mu\rho} \varepsilon_{\rho\sigma} \partial_\sigma ,
\]  

(42)

where \(C\) is the gauge field propagator (15). Inserting \(C\) and transforming to momentum space we obtain

\[
\hat{G}_{FF}(p) = \frac{p^2}{p^2 + e^2 N_{\pi}} = 1 - \frac{e^2 N_{\pi}}{p^2 + e^2 N_{\pi}} = 1 - \int_0^\infty \frac{d\rho(\mu^2)}{p^2 + \mu^2} .
\]  

(43)
In the last step we made the spectral integral explicit. One nicely sees that the spectral measure
\[ d\rho(\mu^2) = \delta(\mu^2 - m^2)d\mu^2 \] (44)
is ‘dominated’ by the contribution of the \( \eta' \), which should of course be identified with our massive particle. Hence we have
\[ m_{\eta'}^2 = m^2 = \frac{e^2 N}{\pi} . \] (45)

From (43) one immediately reads off the contact term \( P(0) \) in the spectral decomposition of \( \chi_{\text{top}} \)
\[ P(0) = \frac{e^2}{4\pi^2} . \] (46)
The last missing ingredient is the decay constant \( f_{\eta'} \). It is defined by
\[ f_{\eta'} = m^{-2}\langle 0 | \partial_\mu J^{(1)} | \eta' \rangle . \] (47)
The anomaly equation for the current \( j^{(a)}_\mu = \bar{\psi}^{(a)}(a)\gamma_\mu \psi^{(a)}(a) \) reads (see eg. [1])
\[ \partial_\mu j^{(a)}_\mu = 2q . \] (48)
Using the definition of the \( U(1) \) current (27),(28) one obtains the anomaly equation
\[ \partial_\mu J^{(1)} = 2\sqrt{N}q , \] (49)
which we insert in (47) to end up with
\[ f_{\eta'} = m^{-2} 2\sqrt{N}\langle 0 | q | \eta' \rangle = m^{-2} 2\sqrt{N}\langle 0 | q \frac{i}{m} F_{12} | 0 \rangle . \] (50)

In the last step we generated the \( |\eta'\rangle \) state as \( Z\frac{i}{p^2 + m^2} F_{12}|0\rangle \) with the normalization condition
\[ \hat{G}_{\eta'\eta'} \frac{1}{p^2 + m^2} + \text{contact term} , \] (51)
giving rise to \( Z = -\frac{1}{m^2} \). We end up with
\[ f_{\eta'} = i\frac{1}{\sqrt{\pi}} . \] (52)

Insertion in (38) of the quantities obtained gives an identity. This explicit computation shows that eq. (38) holds in massless QED_2 with \( N \) flavors. But one can also interprete the result as a verification of the original form of the Witten-Veneziano formula, because the topological susceptibility of the quenched theory reduces to the contact term. It is not true, however, that the topological susceptibility appearing in the formula expresses only a property of the long distance fluctuations of the topological density.

1Note that in our definition of the \( U(1) \) current there is an extra factor \( 1/\sqrt{N} \) which modifies (47),(49) by this factor, compared to the usual notation.
5 N-point functions

To gain more insight into the sector generated by the vector currents we compute explicit expressions for the connected \( n \)-point functions \( C_n \). Considering fully connected correlations has the advantage of avoiding contractions of fermions at the same point. Therefore we can avoid using test functions as would be required in principle by the distributional nature of the fields. Let

\[
C_{n+1} := \left\langle J^{(l_0)}_{\mu_0}(x_0), J^{(l_1)}_{\mu_1}(x_1), \ldots, J^{(l_n)}_{\mu_n}(x_n) \right\rangle_c =
\sum_{a_i, b_i} H^{(l_0)}_{a_0 b_0} H^{(l_1)}_{a_1 b_1} \cdots H^{(l_n)}_{a_n b_n} \sum_{\alpha_i, \beta_i} (\gamma_{\mu_0})_{\alpha_0 \beta_0} (\gamma_{\mu_1})_{\alpha_1 \beta_1} \cdots (\gamma_{\mu_n})_{\alpha_n \beta_n}
\left\langle \overline{\psi}^{(a_0)}(x_0) \psi^{(b_0)}(x_0) \overline{\psi}^{(a_1)}(x_1) \psi^{(b_1)}(x_1) \cdots \overline{\psi}^{(a_n)}(x_n) \psi^{(b_n)}(x_n) \right\rangle_c .
\tag{53}
\]

Nonvanishing contributions occur only if the color indices can form a closed chain (e.g. \( b_0 = a_1, b_1 = a_2, \ldots b_{n-1} = a_n, b_n = a_0 \)). The corresponding factor is simply the trace over the flavor matrices \( H^{(l)} \). To find all possible contributions one has to sum over all permutations \( \pi \), keeping the first term fixed.

\[
C_{n+1} = - \sum_{\pi(1,2,\ldots,n)} \text{Tr} [ H^{(l_0)} H^{(l_{\pi(1)})} \cdots H^{(l_{\pi(n)})} ] \sum_{\alpha_i, \beta_i} (\gamma_{\mu_0})_{\alpha_0 \beta_0} (\gamma_{\mu_1})_{\alpha_1 \beta_1} \cdots (\gamma_{\mu_n})_{\alpha_n \beta_n}
\int d\mu_C[\varphi] G_{\beta_0 \alpha_{\pi(1)}}(x_0, x_{\pi(1)}; \varphi) G_{\beta_{\pi(1)} \alpha_{\pi(2)}}(x_{\pi(1)}, x_{\pi(2)}; \varphi) \cdots G_{\beta_{\pi(n)} \alpha_0}(x_{\pi(n)}, x_0; \varphi)
\tag{54}
\]

Since in the chosen representation the \( \gamma \) matrices and the propagator \( G(x, y; \varphi) \) have only off-diagonal entries (cf. equation(20)), we find the following chain of implications for e.g. \( \beta_0 = 1 \)

\[
\beta_0 = 1 \Rightarrow \alpha_{\pi(1)} = 2 \Rightarrow \beta_{\pi(1)} = 1 \Rightarrow \alpha_{\pi(2)} = 2 \ldots \Rightarrow \beta_{\pi(n)} = 1 \Rightarrow \alpha_0 = 2 .
\tag{55}
\]

When starting with \( \beta_0 = 2 \) one ends up with the opposite result of all \( \beta_i = 2 \) and all \( \alpha_i = 1 \). Besides those two no other nonvanishing terms contribute. Again one finds that all dependence on \( \varphi \) cancels, and only free propagators \( G^0 \) remain. Using \( G^0(x) = \frac{1}{2\pi} \frac{2e^{i\pi}}{x^2} \) and making use of the complex notation (21) we obtain

\[
C_{n+1} = - \frac{1}{(2\pi)^{n+1}} \sum_{\pi(1,2,\ldots,n)} \text{Tr} [ H^{(l_0)} H^{(l_{\pi(1)})} \cdots H^{(l_{\pi(n)})} ]
\left\{ \prod_{i=0}^{n} (\gamma_{\mu_i})_{21} \frac{1}{x_0 - x_{\pi(1)}} \frac{1}{x_{\pi(1)} - x_{\pi(2)}} \cdots \frac{1}{x_{\pi(n)} - x_0} + \text{c.c.} \right\} ,
\tag{56}
\]

where c.c. denotes complex conjugation.

Using this basic formula, we construct for \( N \geq 2 \) and arbitrary \( n \) fully connected \( n \)-point functions that do not vanish; in other words we show that
the currents are not Gaussian. For simplicity we give the construction for the case of \( N = 2 \). Since for arbitrary \( N > 2 \) generators with the same commutation relations exist, it is obvious how to generalize the construction to general \( N \).

The generators we are using are the Pauli matrices up to a normalization factor. To distinguish them from arbitrary generators \( H^{(l)} \) we denote the special set of matrices needed in the construction

\[
\tau^{(1)} := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^{(2)} := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^{(3)} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(57)

Define

\[
F_{n+1}(\tilde{x}_0, \tilde{x}_1, ..., \tilde{x}_n) := \sum_{\pi(1,2,...,n)} \text{Tr} \left[ \tau^{(l_0)} \tau^{(l_1)} ... \tau^{(l_n)} \right] \frac{1}{\tilde{x}_0 - \tilde{x}_{\pi(1)}} \frac{1}{\tilde{x}_{\pi(1)} - \tilde{x}_{\pi(2)}} ... \frac{1}{\tilde{x}_{\pi(n)} - \tilde{x}_0},
\]

(58)

where the special set of \( \tau \) matrices is given by

\[
\{ \tau^{(l_0)}, \tau^{(l_1)}, ..., \tau^{(l_n)} \} := \\
\{ \tau^{(2)}, \tau^{(3)}, ..., \tau^{(3)} \} \quad \text{for } n + 1 = 2m + 1 \ (n + 1 \text{ odd}) \\
\{ \tau^{(1)}, \tau^{(3)}, ..., \tau^{(3)} \} \quad \text{for } n + 1 = 2m + 2 \ (n + 1 \text{ even})
\]

(59)

**Theorem 1:**

For arbitrary \( m \geq 1 \):

\[
F_{2m+1} \neq 0, \ F_{2m+2} \neq 0.
\]

(60)

**Proof:** We prove the statement by induction.

1) \( F_3 \neq 0 \) and \( F_4 \neq 0 \) can be checked easily.

2) Assume \( F_{2m+1} \neq 0 \) and \( F_{2m+2} \neq 0 \).

3) We have to show \( F_{2m+3} \neq 0 \) and \( F_{2m+4} \neq 0 \). First we check \( F_{2m+3} \). Again we abbreviate \( 2m + 3 := n + 1 \). The trick is to consider \( F_{n+1}(\tilde{x}_0, \tilde{x}_1, ..., \tilde{x}_n) \) as a function of \( \tilde{x}_0 \), and to compute the corresponding residues

\[
\text{Res}_{\tilde{x}_0} [F_{n+1}] = \sum_{\pi(1,2,...,n), \pi(1)=1} \text{Tr} \left[ \tau^{(2)} \tau^{(3)} \tau^{(\pi(2))} ... \tau^{(\pi(n))} \right] \frac{1}{\tilde{x}_1 - \tilde{x}_{\pi(2)}} \frac{1}{\tilde{x}_{\pi(2)} - \tilde{x}_{\pi(3)}} ... \frac{1}{\tilde{x}_{\pi(n)} - \tilde{x}_1}
\]

\[
- \sum_{\pi'(1,2,...,n), \pi'(n)=1} \text{Tr} \left[ \tau^{(2)} \tau^{(\pi'(1))} ... \tau^{(\pi'(n-1))} \tau^{(3)} \right] \frac{1}{\tilde{x}_1 - \tilde{x}_{\pi'(1)}} \frac{1}{\tilde{x}_{\pi'(1)} - \tilde{x}_{\pi'(2)}} ... \frac{1}{\tilde{x}_{\pi'(n-1)} - \tilde{x}_1}
\]

(61)

For a given permutation \( \pi \) choose the permutation \( \pi' \) such that \( \pi'(1) = \pi(2), \pi'(2) = \pi(3), ... \pi'(n-1) = \pi(n), \pi'(n) = \pi(1) = 1 \). This gives for the residue

\[
\sum_{\pi(1,2,...,n), \pi(1)=1} \frac{1}{\tilde{x}_1 - \tilde{x}_{\pi(2)}} \frac{1}{\tilde{x}_{\pi(2)} - \tilde{x}_{\pi(3)}} ... \frac{1}{\tilde{x}_{\pi(n)} - \tilde{x}_1}
\]

We acknowledge a useful discussion with Peter Breitenlohner on this point.
\[
\left\{ \text{Tr}\left[ \tau^{(2)} \tau^{(3)} \tau^{(\pi(2))} \tau^{(\pi(3))} \ldots \tau^{(\pi(n))} \right] - \text{Tr}\left[ \tau^{(2)} \tau^{(\pi(2))} \tau^{(\pi(3))} \ldots \tau^{(\pi(n))} \tau^{(3)} \right] \right\}. \tag{62}
\]

Using the cyclicity of the trace and \( \{\tau^{(2)}, \tau^{(3)}\} = 0 \), one finds that the second trace is the negative of the first one. Furthermore \( \tau^{(2)} \tau^{(3)} = \frac{i}{\sqrt{2}} \tau^{(1)} \). Looking at the definition \( F_n \) for even \( n \) we conclude

\[
\text{Res}_{\tilde{x}_0}[F_{n+1}, \tilde{x}_0 = \tilde{x}_1] = i\sqrt{2} F_n(\tilde{x}_1, \ldots, \tilde{x}_n) = i\sqrt{2} F_{2m+2}(\tilde{x}_1, \ldots, \tilde{x}_{2m+2}). \tag{63}
\]

Since by assumption \( F_{2m+2} \neq 0 \), \( \text{Res}[F_{2m+3}] \neq 0 \) and hence \( F_{2m+3} \) does not vanish. The same trick can be applied to prove that this implies \( F_{2m+4} \neq 0 \).

A \( n + 1 \)-point function with the flavor content given by (59) is simply the real part of a multiple of \( F_{n+1} \) (compare (56)). Hence there exist nonvanishing, fully connected \( n \)-point functions for arbitrary \( n \).

This result implies that it is not possible to bosonize the whole set of all \( N^2 \) vector currents \( J^{(l)}_\mu \) in terms of free bosons (this was the reason why Witten introduced his nonabelian bosonization). This can be done only for a Cartan subalgebra where all generators commute, and the two traces in equation (62) cancel. The last observation allows to prove a second theorem.

**Theorem 2:**

Any fully connected \( n \)-point function

\[
\langle J^{(1)}_{\mu_0}(x_0) J^{(l_1)}_{\mu_1}(x_1) J^{(l_2)}_{\mu_2}(x_2) \ldots J^{(l_n)}_{\mu_n}(x_n) \rangle_c
\]

containing the \( U(1) \) current vanishes, with the exception of the two point function \( \langle J^{(l)}_\mu(x) J^{(1)}_\nu(y) \rangle_c \) (without loss of generality we shifted the \( U(1) \) current in equation (64) to the first position).

**Proof:** To prove the statement we use the same trick as for theorem 1. Again we consider the residues of functions \( F_{n+1} \) defined by

\[
F_{n+1}(\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_n) := \sum_{\pi(1, 2, \ldots, n)} \text{Tr}\left[ H^{(1)} H^{(l_{\pi(1)})} \ldots H^{(l_{\pi(n)})} \right] \frac{1}{\tilde{x}_0 - \tilde{x}_{\pi(1)}} \frac{1}{\tilde{x}_{\pi(1)} - \tilde{x}_{\pi(2)}} \ldots \frac{1}{\tilde{x}_{\pi(n)} - \tilde{x}_0}. \tag{65}
\]

For the residue at \( \tilde{x}_0 = \tilde{x}_1 \) we obtain an equation equivalent to (61). After making the same choice for \( \pi' \) one finds

\[
\text{Res}_{\tilde{x}_0}[F_{n+1}, \tilde{x}_0 = \tilde{x}_1] = \sum_{\pi(1, 2, \ldots, n), \pi(1) = 1} \frac{1}{\tilde{x}_1 - \tilde{x}_{\pi(2)}} \frac{1}{\tilde{x}_{\pi(2)} - \tilde{x}_{\pi(3)}} \ldots \frac{1}{\tilde{x}_{\pi(n)} - \tilde{x}_1}

\left\{ \text{Tr}\left[ H^{(1)} H^{(l_1)} H^{(l_{\pi(2)})} \ldots H^{(l_{\pi(n)})} H^{(l_1)} \right] - \text{Tr}\left[ H^{(1)} H^{(l_2)} \ldots H^{(l_{\pi(n)})} H^{(l_1)} \right] \right\}. \tag{66}
\]

Since \( H^{(1)} \) commutes with all generators \( H^{(l_i)} \), the two traces are the same and cancel. Using the same argument one can show that all residues at \( \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n \) vanish. So \( F_{n+1} \) is analytic and bounded in the entire \( \tilde{x}_0 \) plane. By Liouville’s theorem \( F_{n+1} \) is a constant, and the limit \( \tilde{x}_0 \to \infty \) shows that this constant
is zero. Since the $n + 1$-point function defined in (64) is proportional to $F_{n+1}$, it has to vanish. \(\square\)

Theorem 2 allows to prove the following proposition about the structure of the Hilbert space.

**Proposition:**

The Hilbert space $\mathcal{H}$ generated by the vector currents from the vacuum is the tensor product

$$\mathcal{H} = \mathcal{H}_{U(1)} \otimes \mathcal{H}_{\text{mass}=0}.$$  \hspace{1cm} (67)

To prove this we need the connection between untruncated and fully connected $n$-point functions (see e.g. [14]).

$$\langle \phi_1 \ldots \phi_n \rangle = \sum_{\pi \in \mathcal{P}_n} \prod_{p \in \pi} \langle \phi_{i_p} \rangle_c,$$  \hspace{1cm} (68)

where $\mathcal{P}_n$ is the set of all partitions of $\{1, 2, \ldots n\}$, $\pi = \{p_1, p_2, \ldots p_{|\pi|}\}$ denotes an element of $\mathcal{P}_n$, and $\{i_1, i_2, \ldots i_{|\pi|}\}$ is an element $p$ of $\pi$. An arbitrary $n + k$-point function (without loss of generality we write the $U(1)$ currents first; $l_i \neq 1, i = 1, 2, \ldots k$) factorizes due to Theorem 2

$$\sum_{\pi \in \mathcal{P}_{n+k}} \prod_{p \in \pi} \langle J^{(1)}_{\mu_1}(x_{i_1}) J^{(1)}_{\mu_2}(x_{i_2}) \ldots J^{(1)}_{\mu_n}(x_{i_n}) J^{(l_1)}_{\nu_1}(y_1) J^{(l_2)}_{\nu_2}(y_2) \ldots J^{(l_k)}_{\nu_k}(y_k) \rangle_c \hspace{1cm}$$

$$= \left[ \sum_{\pi \in \mathcal{P}_n} \prod_{p \in \pi} \langle J^{(1)}_{\mu_1}(x_{i_1}) J^{(1)}_{\mu_2}(x_{i_2}) \ldots J^{(1)}_{\mu_n}(x_{i_n}) \rangle_c \right]$$

$$\quad \times \left[ \sum_{\pi' \in \mathcal{P}_k} \prod_{p' \in \pi'} \langle J^{(l_1)}_{\nu_1}(y_{j_1}) J^{(l_2)}_{\nu_2}(y_{j_2}) \ldots J^{(l_k)}_{\nu_k}(y_{j_k}) \rangle_c \right].$$

(69)

From this the tensor product structure of the Hilbert space follows easily. \(\square\)

We observed in equations (54)-(56) that the dependence of the fully connected correlations on the gauge field cancels entirely. Hence the vector currents in the 'massless' sector of the Hilbert space obey the same algebra as in a system of uncoupled fermions; this is the well-known level 1 representation of the $SU(N)_L \times SU(N)_R$ current (Kac-Moody) algebra (see for instance [14]).

### 6 Confinement

In order to study confinement, we evaluate the Fredenhagen Marcu order parameter [12]. It is defined by

$$\rho := \lim_{L \to \infty} \rho(L) := \lim_{L \to \infty} \sum_{\alpha, \beta = 1}^2 \frac{N_{\alpha, \beta}^{(L)} N_{\beta, \alpha}^{(L)}}{\langle W(L) \rangle},$$  \hspace{1cm} (70)
where
\[ N_{\alpha\beta}^{(L)} := \langle \psi_\alpha^{(a)}(-L,0)U(C^{(L)})\overline{\psi}_\beta^{(a)}(L,0) \rangle \] for arbitrary flavor \( a = 1, 2, \ldots N \),

and
\[ U(C^{(L)}) = e^{ie \int_{c^{(L)}} A_\mu(x)dx_\mu}. \]

For the contour \( C^{(L)} \) see Fig. 1a. \( W^{(L)} \) denotes the Wilson loop
\[ W^{(L)} = e^{ie \int_{C_W^{(L)}} A_\mu(x)dx_\mu}, \]
along the contour \( C_W^{(L)} \) given in Fig. 1b.

The physical interpretation is simple: if there is no confinement and free ‘quarks’ can be isolated, the free quark lives in a sector orthogonal to the vacuum sector (obtained by applying gauge invariant operators to the vacuum); all ‘mesonic’ states belong to the vacuum sector. In particular \( \rho = 0 \) follows in this case. In the case of confinement, however, one expects ‘fragmentation’. If one tries to separate a quark-antiquark pair, one is producing another quark-antiquark pair from the vacuum which binds to the separated pair and forms two mesons, i.e. a state in the vacuum sector. In this case one expects therefore \( \rho \neq 0 \).

The computational trick is to rewrite the contour integral over \( A_\mu \) as a scalar product with a current having its support on the contour. e.g.
\[ \int_{C_W^{(L)}} A_\mu(x)dx_\mu := \int A_\mu(x)j_{W}^{(L)}(x)d^2x, \]

with
\[ j_{W}^{(L)}(x) = \left( \begin{array}{c}
\theta(x_1 + L)\theta(L - x_1) [\delta(x_2 - L) - \delta(x_2 + L)] \\
\theta(x_2 + L)\theta(L - x_2) [\delta(x_1 + L) - \delta(x_1 - L)]
\end{array} \right). \]
(strictly speaking a limiting procedure is required here, which we leave as an exercise). Performing the functional integration one ends up with

$$
\langle W^{(L)} \rangle = \int d\mu_C[A] e^{i\langle A^\mu j^W_\mu^{(L)} \rangle} = e^{\frac{e^2}{2}(j^W_\mu^{(L)},C_{\mu\nu}j^W_\nu^{(L)})}.
$$

(76)

For $N_{\alpha\beta}$ in addition a propagator appears in the gauge field integral

$$
N^{(L)}_{\mu\nu} = \int d\mu_C[A] e^{i\langle A^\mu j^{\alpha\beta}^{(L)} \rangle} G_{\mu\nu}\left((-L,0),(L,0);A\right).
$$

(77)

It is useful to rewrite the gauge field integration in terms of the field $\varphi$ introduced in (16)-(18). In the chosen representation of the $\gamma$ matrices we have $N^{(L)}_{\mu\mu} = 0$ and $N^{(L)}_{21} = \overline{N}_{12}$. Inspecting the quadratic forms that remains after the functional integration over $\varphi$ one finds $N^{(L)}_{21} = N^{(L)}_{12} \in \mathbb{R}$. Putting things together gives

$$
\rho^{(L)} = \frac{2}{(2\pi)^2 (2L)^2} e^{I^{(L)}},
$$

(78)

where the integral $I^{(L)} := I_s^{(L)} + I_c^{(L)}$ contains two parts $I_s^{(L)}$ and $I_c^{(L)}$. $I_s^{(L)}$ will be solved explicitly, and $I_c^{(L)}$ can be shown to go to a nonvanishing constant for $L \to \infty$. $I_s^{(L)}$ is given by ($m^2 = e^2 N/\pi$)

$$
I_s^{(L)} = \frac{2e^2}{\pi^2} \int d^2p \frac{1}{(p^2 + m^2)p^2} \sin^2(p_1L).
$$

(79)

To solve it, we transform it to polar coordinates and integrate over the angle. This leads to an integral over a Bessel function, which can (after some partial integration) be found in integral tables. The result is

$$
I_s^{(L)} = \frac{1}{N} \ln(2L) + \frac{1}{N} K_0\left(2Le\sqrt{\frac{N}{\pi}}\right) + \frac{1}{N} \ln \left(\frac{e}{2} \sqrt{\frac{N}{\pi}}\right) + \gamma + \frac{1}{N},
$$

(80)

where $\gamma$ denotes Euler’s constant. The modified Bessel function $K_0$ vanishes for $L \to \infty$. The second integral $I_c^{(L)}$ is given by

$$
I_c^{(L)} = \frac{e^2}{\pi^2} \int d^2p \frac{1}{(p^2 + m^2)^2} \left\{ -\frac{\sin^2(p_1L)}{(p_1)^2} + \right.
$$

$$
4 \sin^2(p_1L) \frac{1}{(p_2)^2} \left[ \sin^2(p_2L) - \sin^2(p_2L/2) \right] \right\} =
$$

$$
\frac{e^2}{\pi^2} \int d^2p \frac{1}{(p^2 + m^2)(p_2)^2} \left\{ \sin^2(p_2L) - 2 \sin^2(p_2L/2) \right\} -
$$

$$
\frac{2e^2}{\pi^2} \int d^2p \frac{1}{(p^2 + m^2)(p_2)^2} \cos(p_12L) \left\{ \sin^2(p_2L) - \sin^2(p_2L) \right\}.
$$

(81)
Performing the $p_1$ integration in the second term, one sees that this integral is exponentially decaying for $L \to \infty$. The first term we integrate over $p_2$ first and find the cancellation of two terms linear in $L$. The remaining part gives an exponentially suppressed term, and the constant $1/N$.

Putting things together, we obtain for $\rho(L)$

$$\rho(L) = \frac{1}{2\pi^2} \frac{e^2}{4} \left( \frac{N}{\pi} \right)^\frac{1}{N} e^{\frac{N}{2N} (2\gamma+1)} \left( \frac{1}{2L} \right)^{\frac{2N-1}{N}}. \quad (82)$$

(In the exponent we dropped terms that vanish for $L \to \infty$.) For the confinement parameter this gives two different results depending on the number of flavors

$$\rho = \lim_{L \to \infty} \rho(L) = \begin{cases} \frac{1}{2\pi} \frac{e^2}{4} e^{2\gamma+1} & \text{for } N = 1 \\ 0 & \text{for } N > 1 \end{cases}. \quad (83)$$

This result may appear surprising. Confinement of a quark-antiquark system depends on the number of flavors! (Up to a factor the $N = 1$ result can be found in [13].)

Having solved the above case one can easily generalize the formula to products of $n$ spinors with pairwise distinct flavor quantum numbers. Define for $n \leq N$

$$N^{(L)}_{\alpha_1...\alpha_n\beta_1...\beta_n} := \left\langle \psi^{(1)}_{\alpha_1} (-L,0)...\psi^{(n)}_{\alpha_n} (-L,0) U(C^{(L)})^{N^{(1)}_{\beta_1}} (L,0)...\psi^{(n)}_{\beta_n} (L,0) \right\rangle. \quad (84)$$

The corresponding $\rho^{(n)}(L)$ is given by

$$\rho^{(n)}(L) := \frac{\sum_{\alpha_i,\beta_j} N^{(L)^\dagger}_{\alpha_1...\alpha_n\beta_1...\beta_n} N^{(L)}_{\beta_1...\beta_n\alpha_1...\alpha_n}}{\langle W(L)^n \rangle} = \quad (85)$$

$$\frac{2}{(4\pi L)^{2n}} e^{n^2/2} + O(1/L) = \frac{2}{(2\pi)^{2n}} \left( \frac{e^2 N}{4\pi} \right)^\frac{n^2}{N} e^{\frac{n^2}{2N} (2\gamma+1)} \left( \frac{1}{2L} \right)^{\frac{2N-n^2}{N}} + O(1/L). \quad (86)$$

Performing the limit $L \to \infty$ we obtain

$$\rho^{(n)} = \lim_{L \to \infty} \rho^{(n)}(L) = \begin{cases} \frac{2}{(2\pi)^{2N}} \left( \frac{e^2 N}{4\pi} \right)^N e^{N(2\gamma+1)} & \text{for } n = N \\ 0 & \text{for } n < N \end{cases}. \quad (87)$$

Thus in the $N$-flavor model an arrangement of $N$ 'quarks' is bound by a confining force to an arrangement of $N$ 'antiquarks'.

A further generalization shows that an operator of $N + n, n \leq N$ quarks behaves like the product of only $n$ quarks. The corresponding $N$ is defined by

$$N^{(L)}_{\alpha_1...\alpha_N\alpha'_{1}...\alpha'_{n}\beta_1...\beta_n} := \left\langle \psi^{(1)}_{\alpha_1} ... \psi^{(N)}_{\alpha_N} \psi^{(1)}_{\alpha'_{1}} ... \psi^{(n)}_{\alpha'_{n}} (-L,0) U(C^{(L)})^{N+n} \overline{\psi}^{(1)}_{\beta_1} ... \overline{\psi}^{(N)}_{\beta_N} \overline{\psi}^{(1)}_{\beta'_{1}} ... \overline{\psi}^{(n)}_{\beta'_{n}} (L,0) \right\rangle. \quad (88)$$
For the fields with flavors 1, 2, ... n extra contractions are possible. Some of these cancel each other. The remaining terms have no dependence on φ and contribute only to the 1/L behaviour of the free propagator. The flavors $n + 1, n + 2, ... N$ are seen to contribute a factor

$$e^{(N-n)^2 I(L)},$$

after performing the path integral. Collecting all terms one obtains

$$\rho^{(N+n)} = \lim_{L \to \infty} \rho^{(N+n)}(L) = \begin{cases} \text{const} & \neq 0 \text{ for } n = 0, N \\ 0 & \text{for } 0 < n < N \end{cases}. \quad (90)$$

The physical interpretation suggested by this behavior of the Fredenhagen-Marcu order parameters is the following: The model has $N$ distinct superselection sectors labeled by a charge $Q$ that is defined only modulo $N$. To obtain a state in the the sector of charge $Q = n, n < N$, one applies an operator consisting of $n$ ‘quarks’ and $n$ antiquarks, separated by distance $L$ and takes the limit $L \to \infty$.

7 Decomposition into clustering states and the vacuum angle

The state (expectation functional) which we have constructed so far violates clustering, as we will show in this section. We will also construct an integral decomposition of this state into 'pure phases' labeled by a vacuum angle $\theta$ and satisfying the cluster decomposition property. Thus we recover the structure that is usually associated with the existence of topological sectors of the gauge field. The words ‘pure phase’ should not be taken literally, because the different states are not related by a symmetry operation as in the case of spontaneous symmetry breaking. The transformations that intertwine the different $\theta$-states – the axial $U(1)$ transformations – are not symmetry transformations because the corresponding current is not conserved (due to the presence of the Adler-Bardeen anomaly), and therefore the corresponding Ward identities contain an anomaly term.

7.1 Identification of operators that violate clustering

To identify the operators that violate clustering, we start with an ansatz containing only the chiral densities $\bar{\psi}^{(a)} P_\pm \psi^{(a)}$ ($P_\pm := (1 \pm \gamma_5)/2$) and discuss the effect of adding vector currents later. Define

$$C(\tau) = \langle \prod_{a=1}^{N} \prod_{i=1}^{n_a} \bar{\psi}^{(a)}(x_i^{(a)} + \hat{\tau}) P_+ \psi^{(a)}(x_i^{(a)} + \hat{\tau}) \prod_{i=1}^{m_a} \bar{\psi}^{(a)}(y_i^{(a)} + \hat{\tau}) P_- \psi^{(a)}(y_i^{(a)} + \hat{\tau}) \rangle$$
\[
\begin{align*}
\frac{n'_a}{n'_a} \prod_{i=1}^{n'_a} \overline{\psi}^{(a)}(x_i^{(a)}) P_+ \psi^{(a)}(x_i^{(a)}) \Bigg\langle \prod_{i=1}^{m'_a} \overline{\psi}^{(a)}(y_i^{(a)}) P_- \psi^{(a)}(y_i^{(a)}) \Bigg\rangle - \\
\Bigg\langle \prod_{a=1}^{N} \prod_{i=1}^{n_a} \overline{\psi}^{(a)}(x_i^{(a)}) P_+ \psi^{(a)}(x_i^{(a)}) \prod_{i=1}^{m_a} \overline{\psi}^{(a)}(y_i^{(a)}) P_- \psi^{(a)}(y_i^{(a)}) \Bigg\rangle \\
\Bigg\langle \prod_{a=1}^{N} \prod_{i=1}^{n'_a} \overline{\psi}^{(a)}(x_i^{(r(a))}) P_+ \psi^{(a)}(x_i^{(r(a))}) \prod_{i=1}^{m'_a} \overline{\psi}^{(a)}(y_i^{(r(a))}) P_- \psi^{(a)}(y_i^{(r(a))}) \Bigg\rangle,
\end{align*}
\] where \( \tilde{\tau} \) is the vector of length \( \tau \) in 2-direction. Violation of the cluster property now manifests itself in a nonvanishing limit

\[\lim_{\tau \to \infty} C(\tau) =: C \neq 0.\]

Because \( G_{a,a} = 0 \), the second term contributes only if \( m_a = n_a, m'_a = n'_a \) for all \( a = 1, \ldots, N \), and the first one only if

\[n_a + n'_a = m_a + m'_a, \quad a = 1, \ldots, N.\]

After some reordering the first term reads

\[\Bigg\langle \prod_{a=1}^{N} \prod_{i=1}^{n_a} \psi_1^{(a)}(x_i^{(a)} + \tilde{\tau}) \prod_{i=1}^{m_a} \overline{\psi}_2^{(a)}(y_i^{(a)}) \prod_{i=1}^{n'_a} \psi_1^{(a)}(x_i^{(r(a))}) \prod_{i=1}^{m'_a} \overline{\psi}_2^{(a)}(y_i^{(r(a))}) \Bigg\rangle.\]

We dropped an overall sign depending on \( n_a, m_a, n'_a, m'_a \) which is not relevant yet. Using the explicit form (20) of the propagator, one finds that the gauge field integral can be factorized off:

\[\langle \cdot \rangle = I \langle \cdot \rangle_0,\]

where \( \langle \cdot \rangle \) stands for the expectation value (94), and \( \langle \cdot \rangle_0 \) denotes the same expectation value for \( e = 0 \). The factor \( I \) is given by

\[I = \int d\mu_C[\varphi] e^{2e \sum_{a=1}^{N} \left[ \sum_{i=1}^{m_a} \varphi(y_i^{(a)} + \tilde{\tau}) + \sum_{i=1}^{m'_a} \varphi(x_i^{(a)}) - \sum_{i=1}^{n_a} \varphi(x_i^{(a)}) - \sum_{i=1}^{n'_a} \varphi(y_i^{(a)}) \right]}.
\]

It has the general structure

\[\int d\mu_C[\varphi] e^{2e \sum_{i=1}^{m} \varphi(w_i) - \varphi(z_i)} = e^{\sum_{i,j} V(w_i - z_j) - \frac{1}{2} \sum_{i \neq j} V(w_i - w_j) - \frac{1}{2} \sum_{i \neq j} V(z_i - z_j)},\]

\[19\]
where we performed the functional integral and used (19) for the covariance $C$, to obtain after some reordering of the arguments

$$ V(x) = 4e^2 \int \frac{d^2p}{(2\pi)^2} \frac{1 - \cos(px)}{p^2(p^2 + m^2)} . $$

(98)

Up to a trivial factor this is the integral defined in equation (79), which we already solved in the last section. The result is

$$ V(x) = \frac{1}{N} \ln(x^2) + V'(x), \quad V'(x) = \frac{2}{N} \left[ K_0 \left( e \sqrt{\frac{N}{\pi}} |x| \right) + \ln \left( \frac{e}{2} \sqrt{\frac{N}{\pi}} \right) + \gamma \right] . $$

(99)

where we split $V(x)$ into the logarithmic part and a part $V'(x)$ which approaches the constant $\frac{2}{N} \ln(\frac{e}{2} \sqrt{\frac{N}{\pi}}) + \gamma$ for $x \to \infty$.

When applying (97) to (96), the sets $\{w_i\}$ and $\{z_i\}$ are given by

$$ w_i^{(a)} = \{y_i^{(b)} + \hat{\tau}, y_k^{(b')} \mid l = 1, ...m_a; k = 1, ...m_a'; b = 1, ...N \} , $$

$$ z_i^{(a)} = \{x_i^{(b)} + \hat{\tau}, x_k^{(b')} \mid l = 1, ...n_a; k = 1, ...n_a'; b = 1, ...N \} . $$

(100)

Inserting these arguments in (97) and collecting the terms with nontrivial $\tau$ dependence, we obtain for the large-$\tau$ behaviour of the gauge field integral a factor

$$ \left( \tau^2 \right)^{\frac{1}{2}} \left[ (\Sigma_a n_a)(\Sigma_a m_a) + (\Sigma_a m_a)(\Sigma_a m_a') - (\Sigma_a n_a)(\Sigma_a n_a') - (\Sigma_a n_a)(\Sigma_a m_a') \right] . $$

(101)

The $\langle \cdot \rangle_0$ factor is

$$ \langle \cdot \rangle_0 = \prod_{a=1}^{N} \left( \prod_{i=1}^{n_a} \psi^{(a)}_1(x_i^{(a)} + \hat{\tau}) \prod_{i=1}^{m_a} \psi^{(a)}_2(y_i^{(a)} + \hat{\tau}) \prod_{i=1}^{n_a'} \psi^{(a)}_1(x_i^{(a)'}) \prod_{i=1}^{m_a'} \psi^{(a)}_2(y_i^{(a)'} + \hat{\tau}) \right) \prod_{i=1}^{m_a'} \psi^{(a)'}_2(y_i^{(a)'}) \prod_{i=1}^{n_a'} \psi^{(a)'}_1(x_i^{(a)'}). $$

(102)

It can be expressed in terms of determinants. The general structure is

$$ C_0 := \left\langle \prod_{i=1}^{n} \psi_1(z_i) \psi_2(w_i) \prod_{i=1}^{n} \psi_1(z_i) \psi_2(w_i) \right\rangle_0 = $$

$$ \left( \frac{1}{2\pi} \right)^{2n} \sum_{\pi(n)} \text{sign} \frac{1}{z_1 - \hat{w}_{\pi(1)}} ... \frac{1}{z_n - \hat{w}_{\pi(n)}} \text{sign} \frac{1}{\hat{z}_1 - \hat{w}_{\pi'(1)}} ... \frac{1}{\hat{z}_n - \hat{w}_{\pi'(n)}} = $$

$$ \left( \frac{1}{2\pi} \right)^{2n} \left| \det_{(i,j)} \left( \frac{1}{z_i - \hat{w}_j} \right) \right|^2 . $$

(103)
Determinants of this type can be rewritten using Cauchy’s identity (see e.g. [14])

\[
\det_{(i,j)} \left( \frac{1}{z_i - w_j} \right) = (-1)^{\frac{n(n-1)}{2}} \frac{\prod_{1 \leq i < j \leq n} (\tilde{z}_i - \tilde{z}_j) (\tilde{w}_i - \tilde{w}_j)}{\prod_{i,j=1}^n (\tilde{z}_i - \tilde{w}_j)} .
\]  

(104)

Hence we obtain

\[
C_0 = \frac{\prod_{1 \leq i < j \leq n} (z_i - z_j)^2 (w_i - w_j)^2}{\prod_{i,j=1}^n (z_i - w_j)^2} .
\]  

(105)

Inserting the sets (100) for \( w_i, z_i \) and collecting the \( \tau \) dependent powers gives the large \( \tau \) behaviour

\[
\prod_{a=1}^N \left( \tau^2 \right)^{n_a n_a' + m_a m_a' - n_a m_a' - m_a n_a'} = \left( \tau^2 \right)^{\sum_{a=1}^n (n_a - m_a)(n_a' - m_a')} .
\]  

(106)

Combining this with (101) gives the total large-\( \tau \) dependence for the first term of \( C(\tau) \)

\[
\left( \frac{1}{\tau^2} \right)^{\frac{1}{2} \sum_{a,b=1}^N (n_a - m_a)(n_b' - m_b')} - \sum_{a=1}^N (n_a - m_a)(n_a' - m_a') =: \left( \frac{1}{\tau^2} \right)^E .
\]  

(107)

Violation of clustering can only appear if the exponent vanishes and \( C(\tau) \) cannot approach zero. Inserting the condition (93) we can rewrite the exponent

\[
E = \sum_{a=1}^N (n_a - m_a)(n_a - m_a) - \frac{1}{N} \sum_{a,b=1}^N (n_a - m_a)(n_b - m_b) = \frac{1}{N} \sum_{a,b=1}^N (n_a - m_a) R_{ab}(n_b - m_b) ,
\]  

(108)

where the matrix \( R \) was already defined in (25), section 3. There we solved the corresponding eigenvalue problem. We found one eigenvalue 0, and \( N - 1 \) eigenvalues \( N \). The eigenvector \( x_0 \) to the eigenvalue 0 is given by \( x_0 = 1/\sqrt{N}(1,1,...,1)^T \). Hence the quadratic form \( x^T Rx \) is positive semidefinite, and vanishes only if \( x \) is a multiple of \( x_0 \). This implies that the exponent \( E \) is nonnegative and vanishes only for

\[
n_a - m_a = m_a' - n_a' = n , \quad n \in \mathbb{Z} .
\]  

(109)

In the special case \( n = 0 \) the second term in (91) does not vanish. It cancels the contribution of the first term. Hence we find no violation of clustering for \( n = 0 \). As already mentioned, for \( n \neq 0 \) the second term vanishes due to \( G_{\alpha \alpha} = 0 \), and no such cancellation is possible.

How does this picture change when we allow vector currents as well? First we notice that vector currents do not contribute to the gauge field integral.
Consider e.g. the term $\bar{\psi}_1^{(1)}(x)\psi_2^{(2)}(x)$. The $\bar{\psi}_1^{(1)}(x)$ generates a propagator $G_{21}(\cdot, x)$, the $\psi_2^{(2)}(x)$ a propagator $G_{21}(x, \cdot)$. Inspecting (20) immediately shows the cancellation of the $\varphi$ dependence in the product of the propagators. Hence each vector current can only contribute a $1/\tau$ from the free propagator.

Nonvanishing results remain only if the flavors that occur in the vector currents can contract entirely. Thus we have to consider only ‘closed cycles’ like e.g.

$$
\bar{\psi}^{(1)}(x)\gamma_\mu\psi^{(2)}(x) \bar{\psi}^{(2)}(y)\gamma_\nu\psi^{(3)}(y) \bar{\psi}^{(3)}(z)\gamma_\omega\psi^{(1)}(z).
$$

(110)

In principle there are two possibilities to distribute the space-time arguments $x, y, z$. If they are all in one cluster they do not bring in any $\tau$ dependence. They do not modify the clustering, only the constant $C$. If one distributes the closed cycle over both clusters then the situation changes. Each vector current with a partner in the other cluster contributes a $1/\tau$ from the free propagator. This implies that any combination of vector currents alone clusters. Nevertheless a combination of vector currents together with the ansatz (91) could violate clustering. But we saw that the gauge field integral contributions from the chiral charges $\bar{\psi}P_\pm\psi$ can at most compensate the $1/\tau$ of these charges, nothing else ($x^TRx$ is positive semidefinite, compare (107)). Hence adding vector currents that can only contract between the clusters can at most enforce operators to cluster, never create extra powers of $\tau$ that lead to violation of clustering. The same is true when inserting currents of the Cartan subalgebra where one has to introduce a point splitting regulator. The gauge field transporter that connects $\bar{\psi}(x - \varepsilon)$ and $\psi(x + \varepsilon)$ only is a modification within a cluster that does not change the clustering behaviour.

We now want to analyze the symmetry properties of the nonclustering operators. To make our notation more convenient we introduce

$$
O_\pm(\{x\}) := \prod_{a=1}^{N} \bar{\psi}^{(a)}(x^{(a)}) P_\pm \psi^{(a)}(x^{(a)}) .
$$

(111)

It will turn out that the lack of clustering of $O_\pm$ is related to the fact that they are singlets under the conserved symmetry group $U(1)_V \times SU(N)_L \times SU(N)_R$, but transform nontrivially under the explicitly broken $U(1)_A$. To obtain operators that transform under a single irreducible representation of the symmetry group, we antisymmetrize $O_\pm$ with respect to the flavor indices and call the result $O_\pm^a$.

$$
O_\pm^a(\{x\}) := (-1)^{N(N+1)/2} \left[ \frac{1}{N!} \sum_{\pi} \text{sign}(\pi) \prod_{a=1}^{N} \bar{\psi}^{(\pi(a))}(x^{(a)}) \right]

\left[ \frac{1}{N!} \sum_{\pi'} \text{sign}(\pi') \prod_{a'=1}^{N} \psi^{(\pi'(a'))}(x^{(a')}) \right].
$$

(112)
The global sign comes from shifting all $\bar{\psi}$ to the left. Using
\[
\prod_{a=1}^{N} \overline{\psi}^{(a)}_{\tau}(x^{(\pi(a))}) = \text{sign}(\pi^{-1}) \prod_{a=1}^{N} \overline{\psi}^{(\pi^{-1}(a))}_{\tau}(x^{(a)}) ,
\]
(113)
one can express $\mathcal{O}_{\pm}^{a}$ as well in terms of symmetrized space time arguments,
\[
\mathcal{O}_{\pm}^{a}(\{x\}) := \frac{1}{(N!)^{2}} \sum_{\pi,\pi'}^{N} \prod_{a=1}^{N} \overline{\psi}^{(a)}_{\tau}(x^{(\pi(a))}) P_{\pm} \psi^{(a)}_{\tau}(x^{(\pi(a))}) .
\]
(114)
Since the constant $C := \lim_{\tau \to \infty} C(\tau)$ is invariant under the permutation of arguments within a cluster (compare (97),(104)), the latter expression shows explicitly that the constant $C$ is the same for $\mathcal{O}_{\pm}$ and $\mathcal{O}_{\pm}^{a}$. From (112) one easily reads off the invariance of $\mathcal{O}_{\pm}^{a}$ under
\[
U(1)_{V} \times SU(N)_{L} \times SU(N)_{R} .
\]
(115)
We end up with the following picture of the clustering problem: The prototype of a correlation function that violates clustering is given by
\[
C(\tau) = \left\langle \prod_{i=1}^{n} \mathcal{O}_{+}^{a}(\{x_{i} + \tau\}) \prod_{i=1}^{m} \mathcal{O}_{-}^{a}(\{y_{i} + \tau\}) \prod_{i=1}^{n'} \mathcal{O}_{-}^{a}(\{x'_{i}\}) \prod_{i=1}^{m'} \mathcal{O}_{+}^{a}(\{y'_{i}\}) \rightangle - \left\langle \prod_{i=1}^{n} \mathcal{O}_{+}^{a}(\{x_{i}\}) \prod_{i=1}^{m} \mathcal{O}_{-}^{a}(\{y_{i}\}) \rightangle \left\langle \prod_{i=1}^{n'} \mathcal{O}_{-}^{a}(\{x'_{i}\}) \prod_{i=1}^{m'} \mathcal{O}_{+}^{a}(\{y'_{i}\}) \rightangle ,
\]
(116)
with the condition
\[
n - m = -n' + m' \in \mathbb{Z} \setminus \{0\} .
\]
(117)
Insertion of closed cycles of vector currents into a cluster does not change the clustering behaviour. If one inserts vector currents that can contract only to a partner in the other cluster, operators that violated clustering before now become operators obeying the cluster property. Of course it is possible to generalize the operators $\mathcal{O}_{\pm}$ further by e.g. splitting the arguments and connect them with a parallel transporter. Since this is a modification within a cluster, the extra terms in the functional integral will not depend on $\tau$ and only modify the constant.

For completeness we quote the constant $C = \lim_{\tau \to \infty} C(\tau)$ with $C(\tau)$ defined in (116)
\[
C = \mathcal{F}(\{x_{i}\}, \{y_{i}\}) \mathcal{F}(\{x'_{i}\}, \{y'_{i}\}) .
\]
(118)
The dependence on the space time arguments factorizes into two parts that depend on the arguments in the two clusters. This function $\mathcal{F}$ is unique for an operator, if only the other operator has the right quantum numbers to form a pair that violates clustering. It is given by
\[
\mathcal{F}(\{x_{i}\}, \{y_{i}\}) = \prod_{a=1}^{N} \prod_{1 \leq i < j \leq n} (x_{i}^{(a)} - x_{j}^{(a)})^{2} \prod_{1 \leq i < j \leq m} (y_{i}^{(a)} - y_{j}^{(a)})^{2} / \prod_{i=1}^{n} \prod_{j=1}^{m} (x_{i}^{(a)} - y_{j}^{(a)})^{2} .
\]
(119)
\[ e^{\sum V(x_i^{(a)} - y_j^{(b)}) - \frac{1}{2} \sum V(x_i^{(a)} - x_j^{(a)}) - \frac{1}{2} \sum V(y_i^{(a)} - y_j^{(b)})} \left( \frac{1}{2\pi} \right)^{N(n+m)} \left[ \frac{e^2 N}{4\pi} e^{2\gamma} \right]^{\frac{N(n-m)^2}{2}}. \]  

(119)

\[ \sum \] denotes summation over all possible values of the indices \( a, b, i, j \). This factorization property remains valid when vector currents are inserted.

### 7.2 Decomposition into clustering states

To decompose the vacuum state of the theory in terms of clustering \( \theta \) vacua we use the charge that is associated to the axial \( U(1) \) transformation

\[ \psi^{(a)}(x) \rightarrow e^{i\varepsilon_5 \gamma_5} \psi^{(a)}(x). \]  

(120)

An arbitrary product \( B \) of \( \psi^{(a)}(x), \psi^{(a')}(x') \) transforms under \( U(1)_A \)

\[ B(x) \rightarrow e^{imB} B(x), \ m \in \mathbb{Z}. \]  

(121)

(More generally we can consider observables that are sums of operators with definite transformation properties under \( U(1)_A \)). Define the corresponding charge \( Q_5(B) \) as

\[ Q_5(B) = \pm 2N, \ \text{for} \ Q_5(B) = \pm 2N, \ \text{n} \geq 1, \]  

\[ 1 \ \text{otherwise}. \]  

(123)

The set of ‘test operators’ \( U_r(B) \) is defined by

\[ U_r(B) := \left\{ \begin{array}{ll} \mathcal{N}(n)(\{x\}) \prod_{i=1}^{n} \mathcal{O}_+^{a_i}(\{x_i\}) & \text{for } Q_5(B) = \pm 2nN, \ n \geq 1, \\ 1 & \text{otherwise}. \end{array} \right. \]  

(123)

Up to the requirement of being nondegenerate, the arguments \( \{x\} \) are arbitrary. The normalizing factor \( \mathcal{N}(n)(\{x\}) \) is defined such that

\[ \lim_{\tau \to \infty} \langle U_r(B) \rangle \mathcal{B}(\{x\}) = 1. \]  

It can be read off from (119)

\[ \mathcal{N}(n)(\{x\}) = \left( \frac{1}{2\pi} \right)^{-Nn} \left[ \frac{e^2 N}{4\pi} e^{2\gamma} \right]^{-\frac{N^2}{2}} \prod_{a=1}^{N} \prod_{i<j}^{n} (x_i^{(a)} - x_j^{(a)})^{2} e^{V(x_i^{(a)} - x_j^{(b)})}. \]  

(124)

The expectation functionals (states) \( \langle \cdot \rangle_\theta \) have the following properties:

**Theorem 3** :

i) The state \( \langle \cdot \rangle \) constructed initially is recovered by averaging over \( \theta \)

\[ \langle \cdot \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \langle \cdot \rangle_\theta d\theta. \]  

(125)

ii) The cluster decomposition property holds.
Proof:
i): The averaging procedure leaves a nonvanishing result only for operators $B$ with vanishing charge $Q_5(B) = 0$

$$
\langle B \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \langle B \rangle d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\theta Q_5(B)} d\theta \lim_{\tau \to \infty} \langle U_\tau(B) \rangle = \delta_{Q_5(B),0} \langle B \rangle .
$$

(126)

In the last step we used $U_\tau(B) = 1$ if $Q_5(B) \neq 0$. To complete the proof, one has to check that in the state $\langle \cdot \rangle$ constructed initially the vacuum expectation values of operators $B$ with $Q_5(B) \neq 0$ vanish (cf. also [14]). In our representation of the $\gamma$ matrices this can be seen immediately. $Q_5(B) \neq 0$ means that the number of $\bar{\psi}_1, \psi_1$ is not equal to the number $\bar{\psi}_2, \psi_2$. Since $G_{\alpha\alpha} = 0$, for each $\psi_1$ there has to be a $\bar{\psi}_2$ and for each $\psi_2$ a $\bar{\psi}_1$ to give a nonvanishing contribution. But this implies that the number of fields $\bar{\psi}_1, \psi_1$ in $B$ is equal to the number of fields $\bar{\psi}_2, \psi_2$. Hence

$$\langle B \rangle = 0 \text{ for } Q_5(B) \neq 0 .$$

(127)

ii): Let $A$ and $B$ be arbitrary operators. Define

$$C_\theta(\tau) := \lim_{\tau \to \infty} [\langle A(\tau)B(0) \rangle - \langle A(0) \rangle \langle B(0) \rangle] .$$

(128)

We do not display the dependence on the space time arguments $\{x\}$ explicitly, only the dependence on the shift variable $\tau$. Depending on the axial charge $Q_5$ of the operators $A, B, AB$, we have to insert the different definitions of $\langle \cdot \rangle_\theta$. We introduce the following convenient notation: An operator for which the first alternative in equation (123) holds we call a ‘type I’ operator, the operators where the second alternative holds is called ‘type II’. We remark that all operators of type II cannot have the problem of violating the clustering condition. Even among the type I operators there are examples that are not able to violate clustering (like e.g. $(\bar{\psi}^{(a)} P_+ \psi^{(a)})^N$ which does not contain a $SU(N)_L \times SU(N)_R$ singlet part). The operators with the structure $\Pi^m O_+ \Pi^m O_- \Pi J^{(l)}_\mu$, $n - m \neq 0$ we call ‘type V’ for violating.

We have to distinguish the following cases

| case # | $Q_5(A)$ | $Q_5(B)$ | $Q_5(AB)$ |
|--------|---------|---------|-----------|
| 1      | II      | II      | II        |
| 2      | II ($\neq 0$) | II ($\neq 0$) | I |
| 3      | II ($\neq 0$) | I | II |
| 4      | II ($= 0$) | I | I |
| 5      | I ($= q$) | I ($= -q$) | II ($= 0$) |
| 6      | I | I | I |

In brackets ($\cdot$) we denoted facts that necessarily follow. For example in the case 2 the requirement $Q_5(A) \in \Pi, Q_5(B) \in \Pi, Q_5(AB) \in I$ implies $Q_5(A) \neq 0$
and $Q_5(b) \neq 0$. If one of these two had charge zero, the other operator would be of type $\text{I}$, since $Q_5(ab) = Q_5(a) + Q_5(b)$.

**Case 1:**

$$C_\theta(\tau) = \langle A(\tau)B \rangle - \langle A \rangle \langle B \rangle \xrightarrow{\tau \to \infty} 0 ,$$

(129)

since if $A$ and $B$ are of type $\text{II}$, they do not form a pair that violates clustering.

**Case 2:**

$$C_\theta(\tau) = \lim_{\tau' \to \infty} \langle U_{\tau'}(ab)A(\tau)B \rangle - \langle A \rangle \langle B \rangle \xrightarrow{\tau \to \infty} 0 ,$$

(130)

Case 3:

$$C_\theta(\tau) = \langle A(\tau)B \rangle - \langle A \rangle \lim_{\tau' \to \infty} \langle U_{\tau'}(b)B \rangle \xrightarrow{\tau \to \infty} 0 ,$$

(131)

for the same reasons as in the last case.

**Case 4:**

$$C_\theta(\tau) = \lim_{\tau' \to \infty} \langle U_{\tau'}(ab)A(\tau)B \rangle - \langle A \rangle \lim_{\tau' \to \infty} \langle U_{\tau'}(b)B \rangle \xrightarrow{\tau \to \infty} 0 ,$$

(132)

Again $\langle A \rangle$ factorizes, and $U_{\tau'}(ab) = U_{\tau'}(b)$ since $Q_5(ab) = Q_5(b)$.

**Case 5:**

$$C_\theta(\tau) = \langle A(\tau)B \rangle - \lim_{\tau' \to \infty} \langle U_{\tau'}(a)A \rangle \lim_{\tau'' \to \infty} \langle U_{\tau''}(b)B \rangle \xrightarrow{\tau \to \infty} 0 ,$$

(133)

In the first case we used the factorization of the argument function $F_{AB}$ introduced in (118).

**Case 6:**

$$C_\theta(\tau) = \lim_{\tau' \to \infty} \langle U_{\tau'}(ab)A(\tau)B \rangle - \lim_{\tau' \to \infty} \langle U_{\tau'}(a)A \rangle \lim_{\tau'' \to \infty} \langle U_{\tau''}(b)B \rangle \xrightarrow{\tau \to \infty} 0 ,$$

(134)

To justify the first case, one has to show that $\lim_{\tau' \to \infty} F_{U_{\tau'}(b)} = F_B$. This can be seen immediately from (119). Shifting $\tau'$ in $F_{U_{\tau'}(b)}$ corresponds to shifting
the arguments of the test operator $\mathcal{U}_{\tau'}(\mathcal{B})$. For example this could be the set (refering to (119) for $\mathcal{F}$)

$$\left\{ x_i^{(a)} \mid n' < i \leq n; a = 1, 2, \ldots N \right\},$$

(135)

where $n' < n$ depends on the charge $Q_5(\mathcal{B})$. The $\tau'$ terms cancel for $\tau' \to \infty$, and what remains is $\mathcal{F}_B$, since the normalization of $\mathcal{U}_{\tau'}(\mathcal{B})$ cancels exactly the extra terms. $\Box$

This concludes our analysis of the vacuum structure of the $N$ flavor Schwinger model. The result is in full agreement with the picture that is conventionally deduced from the discussions of topological sectors [16], [17].
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