Longitudinal permeability of collisional plasmas under arbitrary degree of
degeneration of electron gas

A. V. Latyshev¹ and A. A. Yushkanov²

¹Department of Mathematical Analysis and Geometry, Moscow State Regional University, 105005,
Moscow, Radio st., 10–A

²Department of Theoretical Physics, Moscow State Regional University, 105005,
Moscow, Radio st., 10–A

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Electric conductivity and dielectric permeability of the non–degenerate electronic gas for the collisional plasmas under arbitrary degree of degeneration of electron gas is found. The kinetic equation of Wigner — Vlasov — Boltzmann with collision integral in relaxation form BGK (Bhatnagar, Gross and Krook) in coordinate space is used. Dielectric permeability with using of the relaxation equation in the momentum space has been received by Mermin. Comparison with Mermin's formula has been realized. It is shown, that in the limit when Planck's constant tends to zero expression for dielectric permeability passes in the classical.

Key words: Dielectric Permeability and Conductivity, Collision Integral, Non–degenerate Electron Gas, Lindhard’s Function, Wigner’s Function, Mermin’s formula.

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I. INTRODUCTION

In the present work formulas for electric conductivity and for dielectric permeability of quantum electronic plasma under arbitrary degree of degeneration of electron gas are deduced.

Dielectric permeability is one of the major plasma characteristics. This quantity is necessary for description of process of propagation and attenuation of the plasma oscillations, skin effect, the mechanism of electromagnetic waves penetration in plasma [1] – [3], and for analysis of other problems in plasma physics.

Dielectric permeability in the collisionless quantum gaseous plasma was studied by many authors (see, for example, [4]-[30]). In work [7], where the one–dimensional case of the quantum plasma is investigated, importance of derivation of dielectric permeability with use of the quantum kinetic equation with collision integral in the form of BGK – model

*Electronic address: avlatyshev@mail.ru
†Electronic address: yushkanov@inbox.ru
(Bhatnagar, Gross, Krook) was marked. The present work is devoted to performance of this problem.

In the present work for a derivation of dielectric permeability the quantum kinetic Wigner — Vlasov — Boltzmann equation (WVB-equation) with collision integral in the form of $\tau$–models is applied. Such collision integral is named BGK–collision integral.

The WVB–equation is written for Wigner function, which is analogue of a distribution function of electrons for quantum plasma (see [11], [12] and [31]).

The most widespread method of investigation of quantum plasmas is the method of Hartree — Fock or a method equivalent to it, namely, the method of Random Phase Approximation [17], [18]. In work [22] this method has been applied to receive expression for dielectric permeability of quantum plasma in $\tau$–approach. However, in work [24] it is shown, that expression received in [22] is noncorrect, as does not turn into classical expression under a condition, when quantum amendments can be neglected. Thus in work [24] empirically corrected expression for dielectric permeability of quantum plasma, free from the specified lack has been offered. By means of this expression the authors investigated quantum amendments to optical properties of metal [25], [26].

Dielectric permeability of quantum plasma is widely used also for studying the screening of the electric field and Friedel oscillations (see, for example, [19] - [21]). In the work [27] screening of the Coulomb fields in magnetised electronic gas has been is studied.

In the theory of quantum plasma there exist two essentially different possibilities of construction of the relaxation kinetic equation in $\tau$ — approximation: in the space of momentum (in the space of Fourier images of the distribution function) and in the space of coordinates. On the basis of the relaxation kinetic equations in the space of momentum Mermin [23] has carried out consistent derivation of the dielectric permeability for quantum collisional plasma in 1970 for the first time.

In the present work expression for the longitudinal dielectric permeability with use of the relaxation equations in space of coordinates is deduced. If in the obtained expression we make Planck constant converges to zero ($\hbar \to 0$), we will receive exactly classical expression of dielectric permeability of non–degenerate plasma. Various limiting cases of the dielectric permeability are investigated. Comparison with Mermin’s result is carried out also.

II. SOLUTION OF THE KINETIC EQUATION

We consider the kinetic Wigner — Vlasov — Boltzmann equation with collisional integral in the form BGK–model

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial r} = \frac{ie}{\hbar} W[U] + \nu[f_{eq} - f].$$

(2.1)

This equation describes evolution of the Wigner function for electrons in quantum plasma.
Here $e$ is the charge of electron, $\nu$ is the effective collision frequency of electrons with ions and neutral atoms, $f(r, p, t)$ is the Wigner function for electrons. The function

$$f_{eq}(r, v, t) = \frac{1}{1 + \exp \left[ \frac{mv^2}{2k_B T} - \frac{\mu(r, t)}{k_B T} \right]}$$

is the equilibrium distribution Fermi — Dirac function for electrons, $W[f]$ is the Wigner — Vlasov functional for the scalar potential $U = U(r, t)$

$$W[f] = \frac{1}{(2\pi)^3} \int \left[ U(r - \frac{\hbar b}{2}, t) - U(r + \frac{\hbar b}{2}, t) \right] \times$$

$$\times f(r, p', t) e^{i\hbar (p' - p)} d^3 b d^3 p', \quad (2.2)$$

$b = \{b_x, b_y, b_z\}$ is the vector, $\hbar$ is the Planck constant, $k_B$ is the Boltzmann constant, $\mu(r, t)$ is the chemical potential, $p$ is the momentum of electrons, $m$ and $v$ are their mass and velocity.

The Wigner function is analogue of distribution function for quantum systems. It is widely used in the various physics problems. Wigner’s function was investigated, for example, in works [9] and [31].

We introduce the Fourier transformation of the Wigner function

$$F(r, b, t) = \frac{1}{(2\pi)^3} \int f(r, p', t) e^{i\hbar p} d^3 p.$$

The Wigner — Vlasov functional with the Fourier transformation equals to

$$W[f] = \int \left[ U(r - \frac{\hbar b}{2}, t) - U(r + \frac{\hbar b}{2}, t) \right] F(r, b, t) e^{-i\hbar p} d^3 p.$$

Let’s consider, that electron distribution function depends on one spatial coordinate $x$, time $t$ and momentum $p$, and the electric scalar potential depends on one spatial coordinate $x$ and time $t$. We take the scalar potential in the form

$$U(x, t) = U_0 e^{i(kx - \omega t)}. \quad (2.3)$$

We will calculate the Wigner — Vlasov functional (2.2). It is easy to see, that

$$U \left( x - \frac{\hbar b_x}{2}, t \right) - U \left( x + \frac{\hbar b_x}{2}, t \right) =$$

$$= U(x, t) \left[ \exp \left( -i \frac{\hbar k b_x}{2} \right) - \exp \left( i \frac{\hbar k b_x}{2} \right) \right]$$

Let’s calculate internal integral in (2.2). We receive, that

$$\frac{1}{(2\pi)^3} \int \left[ U \left( x - \frac{\hbar b_x}{2}, t \right) - U \left( x + \frac{\hbar b_x}{2}, t \right) \right] \exp(i b (p' - p)) d^3 b =$$

$$= \frac{U(x, t)}{(2\pi)^3} \int \left[ \exp \left( -i \frac{\hbar k b_x}{2} \right) - \exp \left( i \frac{\hbar k b_x}{2} \right) \right] \exp(i b (p' - p)) d^3 b =$$
\[ \delta(p_y' - p_y)\delta(p_z' - p_z) \left[ \delta \left( p_x' - p_x - \frac{\hbar k}{2} \right) - \delta \left( p_x' - p_x + \frac{\hbar k}{2} \right) \right]. \]

By means of last equality the Wigner—Vlasov functional (2.2) it is possible to present in the form

\[ W[f] = U(x, t) \int \delta(p_y - p_y')\delta(p_z - p_z') \times \]

\[ \times \left[ \delta \left( p_x - p_x' + \frac{\hbar k}{2} \right) - \delta \left( p_x - p_x' - \frac{\hbar k}{2} \right) \right] f(x, p, t) \, d^3p'. \]

Now it becomes clear, that

\[ W[f] = U(x, t) \left[ f^+(x, p, t) - f^-(x, p, t) \right], \quad (2.4) \]

where

\[ f^\pm = f(x, p_x \pm \frac{\hbar k}{2}, p_y, p_z). \]

The Fermi—Dirac locally equilibrium distribution \( f_{eq} \) we will be linearize about absolute distribution of Fermi—Dirac

\[ f_F(v) \equiv f_0(v) = \frac{1}{1 + \exp \left[ \frac{mv^2}{2k_BT} - \frac{\mu}{k_BT} \right]}, \quad \mu = \text{const}. \]

We will enter dimensionless electron velocity and chemical potential

\[ c = \frac{v}{v_0}, \quad v_0 = \frac{1}{\sqrt{\beta}} = \sqrt{\frac{2k_BT}{m}}, \quad \alpha(x) = \frac{\mu(x)}{k_BT}. \]

Expressions for \( f_{eq} \) and \( f_0 \) thus become simpler

\[ f_{eq} = \frac{1}{1 + e^{c^2 - \alpha}}, \quad f_0(c) = \frac{1}{1 + e^{c^2 - \alpha}}. \]

Here

\[ \alpha(x) = \alpha + \delta\alpha(x), \quad \alpha = \text{const}. \]

Linearization of \( f_{eq} \) leads us to expression

\[ f_{eq} = f_0(c) + g(c)\delta\alpha(x), \quad g(c) = \frac{e^{c^2 - \alpha}}{(1 + e^{c^2 - \alpha})^2}. \quad (2.5) \]

Now we search the Wigner function in the form

\[ f = f_0(c) + U(x, t)g(c)h(c). \quad (2.6) \]

From the law of number of particles conservation

\[ \int (f_{eq} - f) \, d\Omega_F = 0, \quad d\Omega = \frac{2d^3p}{(2\pi\hbar)^3}; \]
we seek that
\[
\delta \alpha(x) = \frac{U(x, t)}{2\pi \varphi_0(\alpha)} \int h(c)g(c) \, d^3c,
\]
where
\[
\varphi_0(\alpha) = \int_0^\infty f_0(c) \, dc = 2 \int_0^\infty g(c) c^2 \, dc.
\]

We will replace the Wigner function \( f \) in Wigner – Vlasov functional in linear approximation on Fermi – Dirac absolute distribution \( f_0(c) \). We will substitute in the equation (2.1) linear expressions (2.5), (2.6) and \( W[f] = W[f_0] = U[f_0^+ - f_0^-] \). As a result we receive the equation
\[
\int h(c)g(c)[1 - i\omega \tau + ik_1c_x] = \frac{i e}{\hbar \nu} [f_0^+(c) - f_0^-(c)] + \frac{g(c)}{2\pi \varphi_0(\alpha)} \int h(c)g(c) \, d^3c.
\]

From this equation we find
\[
h(c)g(c) = \left[ \frac{g(c)}{2\pi \varphi_0(\alpha)} \frac{A}{1 - i\omega \tau + ik_1c_x} + \frac{i e}{\hbar \nu} \frac{f_0^+(c) - f_0^-(c)}{1 - i\omega \tau + ik_1c_x} \right],
\]
where
\[
A = \int h(c)g(c) \, d^3c,
\]
\[
f_0^{\pm}(c) = \frac{1}{1 + \exp \left[ (c_x \pm q/2)^2 + c_y^2 + c_z^2 - \alpha \right]}, \quad q = \frac{\hbar k}{mv_0} = \frac{k}{k_0}.
\]
k_0 \equiv mv_0/\hbar \text{ is the thermal wave number.}

For finding the constant \( A \) we will substitute (2.7) in (2.8). As result it is received, that
\[
A = \frac{i e}{\hbar \nu} \int \frac{f_0^+(c) - f_0^-(c)}{1 - i\omega \tau + ik_1c_x} \, d^3c + \frac{A}{2\pi \varphi_0(\alpha)} \int \frac{g(c) \, d^3c}{1 - i\omega \tau + ik_1c_x}.
\]

Let’s calculate two internal integrals
\[
\int_0^\infty \int \frac{e^{c^2 - \alpha} dc_y dc_z}{(1 + e^{c^2 - \alpha})^2} = 2\pi \int_0^\infty \frac{e^{c^2 + r^2 - \alpha} r \, dr}{(1 + e^{c^2 + r^2 - \alpha})^2} = \frac{\pi}{1 + e^{c^2 - \alpha}} = \pi f_0(c_x),
\]
\[
\int_0^\infty \int [f_0^+(c) - f_0^-(c)] dc_y dc_z =
\]
\[
= 2\pi \int_0^\infty \left[ \frac{1}{1 + e^{(c_x + q/2)^2 + r^2 - \alpha}} - \frac{1}{1 + e^{(c_x - q/2)^2 + r^2 - \alpha}} \right] r \, dr.
\]
We notice that
\[
\int_0^\infty \frac{r dr}{1 + e^{(c_x+q/2)^2-r^2}} = \frac{1}{2} \ln[1 + e^{\alpha-(c_x+q/2)^2}].
\]

We find now, that
\[
\int_0^\infty \int_{-\infty}^\infty \left[f_0^+(c) - f_0^-(c)\right] dc_y dc_z = \pi \ln[1 + e^{\alpha-(c_x+q/2)^2}] - \ln(1 + e^{\alpha-(c_x-q/2)^2}) = \pi \ln \frac{1 + e^{\alpha-(c_x+q/2)^2}}{1 + e^{\alpha-(c_x-q/2)^2}}.
\]

Hence, the quantity \( A \) is equal
\[
A = \frac{ie\pi}{\hbar \nu} \frac{J^+ - J^-}{1 - T/2\varphi_0(\alpha)}, (2.9)
\]
where
\[
T \equiv T(\omega \tau, k_1, \alpha) = \int_{-\infty}^\infty \frac{f_0(t) dt}{1 - i\omega \tau + ik_1 t} = \int_{-\infty}^\infty \frac{dt}{(1 + e^{t^2-\alpha})(1 - i\omega \tau + ik_1 t)},
\]
\[
J^\pm \equiv J^\pm(\omega \tau, k_1, q, \alpha) = \int_{-\infty}^\infty \frac{\ln[1 + e^{\alpha-(t\pm q/2)^2}] dt}{1 - i\omega \tau + ik_1 t} = \int_{-\infty}^\infty \frac{\ln(1 + e^{\alpha-t^2})}{1 - i\omega \tau + ik_1 t \mp ik_1 q/2}.
\]

III. CONDUCTIVITY AND PERMEABILITY

Let’s consider a relationship between electric field and potential
\[
E(x, t) = -\text{grad} \ U(x, t),
\]
or
\[
E(x, t) = -\left\{ \frac{\partial U(x, t)}{\partial x}, 0, 0 \right\},
\]
and a continuity equation for current and charge densities
\[
\frac{\partial \rho}{\partial t} + \frac{\partial j_x}{\partial x} = 0.
\]

Here according to definition of electric conductivity we may represent the current density in the form
\[
j_x = \sigma_i E_x = -\sigma_i \frac{\partial U}{\partial x} = -\sigma_i U_0 ike^{i(kx - \omega t)} = -\sigma_i ikU(x, t).
\]
Hence,

\[ \frac{\partial j_x}{\partial x} = \sigma_l k^2 U(x, t). \]

Taking into account obvious equality for charge density

\[ \rho = e \int f d\Omega_F = e \int [f_0(c) + U(x, t)g(c)h(c)] \frac{2p_0^3 d^3c}{(2\pi\hbar)^3}, \]

we obtain

\[ \frac{\partial \rho}{\partial t} = -i\omega e U(x, t) \int h(c) \frac{2p_0^3 d^3c}{(2\pi\hbar)^3} = -iU(x, t) \frac{\omega e p_0^3}{(2\pi\hbar)^3} A. \]

From the continuity equation and the expressions for derivative of current and charge density, we find

\[ \sigma_l k^2 U(x, t) = -\frac{\partial \rho}{\partial t} = iU(x, t) \frac{\omega e p_0^3}{(2\pi\hbar)^3} A, \]

whence we receive expression for longitudinal dielectric conductivity

\[ \sigma_l = i \frac{\omega e p_0^3}{(2\pi\hbar)^3 k^2} A, \]

or, with using (2.9), we receive

\[ \sigma_l = - \frac{2\pi e^2 \omega p_0^3}{k^2(2\pi\hbar)^3 \hbar \nu} \cdot \frac{J^+ - J^-}{1 - \frac{T}{2\varphi_0(\alpha)}}. \] (3.1)

The number of particles is equal in the equilibrium condition to

\[ N^{(0)} = \int f_0(c) d\Omega_F = \frac{2p_0^3}{(2\pi\hbar)^3} \int f_0(c) d^3c = \frac{p_0^3 \varphi_2(\alpha)}{2\pi^2 \hbar^3}, \] (3.2)

where

\[ \varphi_2(\alpha) = \int_0^\infty c^2 f_0(c) dc. \]

Using the equality (3.2), we may transform the expression for longitudinal conductivity (3.1)

\[ \frac{\sigma_l}{\sigma_0} = - \frac{\omega}{\nu q k_1 4 \varphi_2(\alpha)} \cdot \frac{J^+ - J^-}{1 - \frac{T}{2\varphi_0(\alpha)}}. \] (3.3)

Let’s overwrite the formula (3.3) in the obvious form

\[ \frac{\sigma_l}{\sigma_0} = \frac{\omega}{\nu q k_1 4 \varphi_2(\alpha)} \cdot \frac{\int_0^\infty \ln \frac{1 + e^{\alpha-(t-q/2)^2}}{1 + e^{\alpha-(t+q/2)^2}} \frac{dt}{1 - i\omega \tau + ik_1 t} \int_{-\infty}^\infty f_0(t) dt}{1 - \frac{1}{2\varphi_0(\alpha)} \int_{-\infty}^\infty \frac{f_0(t) dt}{1 - i\omega \tau + ik_1 t}}. \] (3.4)
After some manipulations the last expression may be transformed to

$$\frac{\sigma_l}{\sigma_0} = -i \frac{\omega}{\nu 4 \varphi_2(\alpha)} \frac{\int_{-\infty}^{\infty} \frac{\ln(1 + e^{\alpha/2^2}) dt}{(1 - i\omega \tau + ik_1 t)^2 + k_1^2 q^2/4}}{1 - \frac{1}{2 \varphi_0(\alpha)} \int_{-\infty}^{\infty} \frac{f_0(t) dt}{1 - i\omega \tau + ik_1 t}}. \quad (3.5)$$

Let’s enter the plasma (Langmuir) frequency

$$\omega_p = \frac{4\pi e^2 N(0)}{m}.$$  

Using the formula for standard conductivity \(\sigma_0 = e^2 N(0)/m \nu = \omega_p^2/4\nu\), we present the dielectric permeability in the form: \(\varepsilon_l = 4\pi i\sigma_l/\omega\). If we take conductivity according to (3.4) we will receive

$$\varepsilon_l = 1 + \frac{i\omega_p^2}{\nu^2 q k_1 4 \varphi_2(\alpha)} \frac{\int_{-\infty}^{\infty} \frac{\ln(1 + e^{\alpha/2^2 - (t+q/2)^2}) dt}{1 + e^{\alpha/2^2 - (t-q/2)^2}}}{1 + e^{\alpha/2^2 - (t+q/2)^2}} \frac{\int_{-\infty}^{\infty} \frac{f_0(t) dt}{1 - i\omega \tau + ik_1 t}}{1 - \frac{1}{2 \varphi_0(\alpha)} \int_{-\infty}^{\infty} \frac{f_0(t) dt}{1 - i\omega \tau + ik_1 t}}. \quad (3.6)$$

If to take conductivity under the formula (3.5), for permeability we receive the following expression

$$\varepsilon_l = 1 + \frac{\omega_p^2}{\nu^2 4 \varphi_2(\alpha)} \frac{\int_{-\infty}^{\infty} \frac{\ln(1 + e^{\alpha/2^2 - (t+q/2)^2}) dt}{1 + e^{\alpha/2^2 - (t-q/2)^2}}}{1 + e^{\alpha/2^2 - (t+q/2)^2}} \frac{\int_{-\infty}^{\infty} \frac{f_0(t) dt}{1 - i\omega \tau + ik_1 t}}{1 - \frac{1}{2 \varphi_0(\alpha)} \int_{-\infty}^{\infty} \frac{f_0(t) dt}{1 - i\omega \tau + ik_1 t}}. \quad (3.7)$$

Let’s enter dimensionless parameters

$$z = x + iy = \frac{\omega + i\nu}{k_0 v_0}, \quad x = \frac{\omega}{k_0 v_0}, \quad y = \frac{\nu}{k_0 v_0} = \frac{1}{k_0} = \frac{1}{k_{01}}.$$  

Here \(k_{01} = k_0 l\) is the thermal dimensionless wave number. By means of these parameters we have

$$1 - i\omega \tau = \frac{\nu - i\omega}{\nu} = -i \frac{z}{y}, \quad k_{01} = \frac{1}{y}, \quad k_1 = \frac{q}{y}.$$  

Besides, we receive

$$1 - i\omega \tau + ik_1 t = \frac{i}{y}(qt - z), \quad (1 - i\omega \tau + ik_1 t)^2 + k_1^2 q^2/4 = -\frac{1}{y^2}[(qt - z)^2 - q^4/4].$$
Let’s overwrite last four formulas in dimensionless parameters. We receive for electric conductivity accordingly two formulas

\[
\frac{\sigma_l}{\sigma_0} = -i \frac{xy}{q^2 4 \varphi_2(\alpha)} \int_{-\infty}^{\infty} \frac{\ln \left(1 + \frac{e^{\alpha - (t-q/2)^2}}{1 + e^{\alpha - (t+q/2)^2}} \right) dt}{qt - z}
\]

(3.8)

\[
\frac{\sigma_l}{\sigma_0} = i \frac{xy}{4 \varphi_2(\alpha)} \int_{-\infty}^{\infty} \frac{\frac{i y}{2 \varphi_0(\alpha)} \int_{-\infty}^{\infty} f_0(t)dt}{qt - z} \frac{(qt - z)^2 - q^4/4}{(qt - z)^2 - q^4/4}
\]

(3.9)

Similarly for dielectric permeability we have

\[
\varepsilon_t = 1 + \frac{x_p^2}{q^2 4 \varphi_2(\alpha)} \int_{-\infty}^{\infty} \frac{\ln \left(1 + \frac{e^{\alpha - (t-q/2)^2}}{1 + e^{\alpha - (t+q/2)^2}} \right) dt}{qt - z}
\]

(3.10)

\[
\varepsilon_t = 1 - \frac{x_p^2}{4 \varphi_2(\alpha)} \int_{-\infty}^{\infty} \frac{\frac{i y}{2 \varphi_0(\alpha)} \int_{-\infty}^{\infty} f_0(t)dt}{qt - z} \frac{(qt - z)^2 - q^4/4}{(qt - z)^2 - q^4/4}
\]

(3.11)

In formulas (3.10) and (3.11) \(x_p\) is the dimensionless plasma frequency, \(x_p = \omega_p/k_0v_0\).

IV. SPECIAL CASES OF CONDUCTIVITY AND PERMEABILITY

Let’s consider the limit of conductivity and permeability at \(\nu \to 0\), i.e. when collisional plasma passes in non–collisional. For this purpose we take the formula (3.4) and let’s transform it in appropriate manner

\[
\sigma_l = \frac{e^2 N(0) \omega}{4 \varphi_2(\alpha) \hbar k^2} \int_{-\infty}^{\infty} \frac{\ln \left(1 + \frac{e^{\alpha - (t-q/2)^2}}{1 + e^{\alpha - (t+q/2)^2}} \right) dt}{\nu - i\omega + i k v_0 t}
\]

(4.1)
Passing in the formula (4.1) to a limit at \( \nu \to 0 \), we receive the expression for conductivity in non–collisional plasma:

\[
\sigma_i^\circ = i \frac{e^2 N(0) \omega}{4\varphi_2(\alpha) \hbar k^2} \int_{-\infty}^{\infty} \ln \frac{1 + e^{-i(\alpha-\nu/2)^2}}{1 + e^{-i(\alpha+\nu/2)^2}} \frac{dt}{\omega - kv_0 t}.
\] (4.2)

Arguing in the same way, from the formula (3.5) in the limit at \( \nu \to 0 \) we receive the expression for conductivity in non–collisional plasma

\[
\sigma_i^\circ = i \frac{e^2 N(0) \omega}{m4\varphi_2(\alpha)} \int_{-\infty}^{\infty} \ln \left(1 + e^{-i\alpha^2} \right) \frac{dt}{(\omega - kv_0 t)^2 - k^2v_0^2q^2/4}.
\] (4.3)

The formula (4.3) may be obtained by the another way. For this purpose the logarithm in (4.2) we will write down in the form of difference, and we receive the difference of two integrals. In each of these integrals we will make suitable replacement variable. Then after simple transformations we come to (4.3).

On the basis of (4.2) and (4.3) we will receive two expressions for dielectric permeability in non–collisional plasma

\[
\varepsilon_i^\circ = 1 + \frac{\omega_p^2 m}{\hbar k^2 4\varphi_2(\alpha)} \int_{-\infty}^{\infty} \ln \left(1 + e^{-i\alpha^2} \right) \frac{dt}{\omega - kv_0 t}.
\] (4.4)

and

\[
\varepsilon_i^\circ = 1 - \frac{\omega_p^2}{4\varphi_2(\alpha)} \int_{-\infty}^{\infty} \ln \left(1 + e^{-i\alpha^2} \right) \frac{dt}{(\omega - kv_0 t)^2 - (k^2\hbar/2m)^2}.
\] (4.5)

Formulas for calculation of the dielectric permeability in non–collisional plasma is called in the literature by Lindhard’s dielectric functions.

Let’s transform the formula (3.6) (or (3.7)) to a form convenient for researches. We will give some representations of the dielectric functions, convenient for research in various problems. Let’s enter dimensionless parameter \( w = \frac{\omega + i\nu}{kv_0} = \frac{z}{q} \). It is obvious, that

\[1 - i\omega \tau + ik_1 t = ik_1(t - w).
\]

Hence,

\[
T = \int_{-\infty}^{\infty} \frac{f_0(t)dt}{1 - i\omega \tau + ik_1 t} = \frac{1}{ik_1} \int_{-\infty}^{\infty} \frac{f_0(t)dt}{t - w} = -\frac{i}{kl} \int_{-\infty}^{\infty} \frac{f_0(t)dt}{t - w} =
\]

\[-\frac{i\nu}{kv_0} F_0(w),
\]

where

\[
F_0(w) = \int_{-\infty}^{\infty} \frac{f_0(t)dt}{t - w}.
\]
Therefore we obtain the denominator from the formula (3.10) in the following form

\[
1 - \frac{1}{2\varphi_0(\alpha)} \int_{-\infty}^{\infty} \frac{f_0(t)dt}{1 - i\omega t + ik_1 t} = 1 + \frac{i\nu}{kv_0^2\varphi_0(\alpha)} F_0(w).
\]

In the same way we will transform numerator from (3.6)

\[
\int_{-\infty}^{\infty} \frac{\ln(1 + e^{\alpha-(t-q/2)^2}) - \ln(1 + e^{\alpha-(t+q/2)^2})}{1 - i\omega t + ik_1 t} dt = \]

\[
= \frac{1}{k^2} \int_{-\infty}^{\infty} \frac{\ln(1 + e^{\alpha-t^2})dt}{(t - w)^2 - q^2/4}.
\]

Therefore for dielectric permeability we obtain

\[
\varepsilon_l = 1 - \frac{\omega_p^2}{k^2v_0^24\varphi_0(\alpha)} \int_{-\infty}^{\infty} \frac{\ln(1 + e^{\alpha-t^2})dt}{(t - w)^2 - q^2/4} (4.6)
\]

where we have put temporarily \( v = \frac{\nu}{kv_0}, w = u + iv, u = \frac{\omega}{kv_0} \).

Let’s consider the denominator of the formula (4.6) and we will transform it in the following way

\[
1 + \frac{ivF_0(w)}{2\varphi_0(\alpha)} = 1 + \frac{ivwF_0(w)}{(u + iv)2\varphi_0(\alpha)} = 1 + \frac{ivwF_0(w)}{(\omega + iv)2\varphi_0(\alpha)} = \]

\[
= \frac{\omega + iv[1 + wF_0(w)/2\varphi_0(\alpha)]}{\omega + iv}.
\]

We will enter dispersion function which we name dispersion Fermi — Dirac function

\[
\lambda_0(w, \alpha) = 1 + \frac{w}{2\varphi_0(\alpha)} F_0(w) = 1 + \frac{w}{2\varphi_0(\alpha)} \int_{-\infty}^{\infty} \frac{f_0(t)dt}{t - w}. (4.7)
\]

Let’s notice, that at \( \alpha \to -\infty \) dispersion Fermi — Dirac function passes in dispersion function of classical plasma

\[
\lambda_c(w) = 1 + \frac{w}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2} dt}{t - w}.
\]

This function was introduced for the first time, apparently, by Van Kampen [32].

To establish this limiting transition, it is enough to notice, that at \( \alpha \to -\infty \) we have

\[
f_0(t) = \frac{1}{1 + e^{\alpha-t^2}} \approx e^{\alpha} \cdot e^{-t^2},
\]

\[
2\varphi_0(\alpha) \int_{0}^{\infty} \frac{dt}{1 + e^{\alpha-t^2}} \approx 2e^{\alpha} \int_{0}^{\infty} e^{-t^2} dt = \sqrt{\pi} e^{\alpha}.
\]
Let’s notice, that Fermi — Dirac and Van Kampen dispersion functions can be written down accordingly in the form

$$\lambda_0(w, \alpha) = \frac{1}{2\varphi_0(\alpha)} \int_{-\infty}^{\infty} \frac{tf_0(t)dt}{t-w},$$

and

$$\lambda_c(w) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{te^{-t^2}dt}{t-w}.$$

Thus, from the formula (4.6) it is possible to present the denominator in the form

$$1 + \frac{ivF_0(w)}{2\varphi_0(\alpha)} = \frac{\omega + iv\lambda_0(w, \alpha)}{\omega + iv} = \frac{u + iv\lambda_0(w, \alpha)}{u + iv}.$$

By means of this formula we will write down the formula for the longitudinal Permeability of quantum collisional plasma in the form

$$\varepsilon_l = 1 - \frac{u_p^2}{4\varphi_2(\alpha)} \cdot \frac{(u + iv)L(w, q, \alpha)}{u + iv\lambda_0(w, \alpha)}. \tag{4.7}$$

Here we was entered the function

$$L(w, q, \alpha) = \int_{-\infty}^{\infty} \frac{\ln(1 + e^{\alpha-t^2})dt}{(t-w)^2 - q^2/4}, \tag{4.8}$$

and dimensionless plasma frequency $u_p = \frac{\omega_p}{k\nu_0}$.

Let’s consider the case when quantum plasma passes in classical, i.e. $\hbar \to 0$ or $q \to 0$.

In this case the formula (4.7) passes in the following form

$$\varepsilon_l^{\text{classic}} = 1 - \frac{u_p^2}{4\varphi_2(\alpha)} \cdot \frac{u + iv}{u + iv\lambda_0(w, \alpha)} \int_{-\infty}^{\infty} \frac{\ln(1 + e^{\alpha-t^2})}{(t-w)^2} dt. \tag{4.8}$$

Calculating integral from (4.8) in parts, we receive, that

$$\int_{-\infty}^{\infty} \frac{\ln(1 + e^{\alpha-t^2})}{(t-w)^2} dt = -2 \int_{-\infty}^{\infty} \frac{tf_0(t)dt}{t-w} = -4\varphi_0(\alpha)\lambda_0(w, \alpha).$$

Hence, the formula (4.8) will be transformed to the form

$$\varepsilon_l^{\text{classic}} = 1 + u_p^2 \cdot \frac{\varphi_0(\alpha)}{\varphi_2(\alpha)} \cdot \frac{(u + iv)\lambda_0(w, \alpha)}{u + iv\lambda_0(w, \alpha)}. \tag{4.9}$$

The formula (4.9) in accuracy coincides with the known formula for longitudinal permeability of classical plasma of the arbitrary degeneration degree of electron gas. We will write down this formula in dimensionless parametres $z = x + iy, q$:

$$\varepsilon_l^{\text{classic}} = 1 + u_p^2 \frac{x^2\varphi_0(\alpha)}{q^2\varphi_2(\alpha)} \cdot \frac{(x + iy)\lambda_0(z/q, \alpha)}{x + iy\lambda_0(z/q, \alpha)}. \tag{4.10}$$
where
\[ \lambda_0(z/q, \alpha) = \frac{q}{2\varphi_0(\alpha)} \int_{-\infty}^{\infty} \frac{tf_0(t)dt}{qt - z}, \quad z = \frac{\omega + i\nu}{k_0v_0}, \quad q = \frac{k}{k_0}, \]
or
\[ \varepsilon_l^{\text{classic}} = 1 + \frac{\omega^2\varphi_0(\alpha)}{k^2v_0^2\varphi_2(\alpha)} \frac{(\omega + i\nu)\lambda_0(w, \alpha)}{\omega + i\nu\lambda_0(w, \alpha)}, \quad w = \frac{\omega + i\nu}{kv_0}. \]

Let’s give some more representations of dielectric permeability. Let’s enter auxiliary functions
\[ l(w \mp q/2) = \int_{-\infty}^{\infty} \ln(1 + e^{\alpha -(t+q/2)^2}) \frac{dt}{t - w} = \int_{-\infty}^{\infty} \frac{\ln(1 + e^{\alpha - t^2})}{t - (w \mp q/2)} dt. \]

We can present the Formula (3.6) in the form
\[ \varepsilon_l = 1 + \frac{\omega^2(\omega + i\nu)}{qk^2v_0^2\varphi_2(\alpha)} \frac{l(w - q/2) - l(w + q/2)}{\omega + i\nu\lambda_0(w, \alpha)}, \quad \text{(4.11)} \]
or
\[ \varepsilon_l = 1 + \frac{\omega^2w^2}{q4\varphi_2(\alpha)} \frac{l(w - q/2) - l(w + q/2)}{(\omega + i\nu)(\omega + i\nu\lambda_0(w, \alpha))}. \quad \text{(4.12)} \]

We will notice, that
\[ l(w - q/2) - l(w + q/2) = -q \int_{-\infty}^{\infty} \frac{\ln(1 + e^{\alpha - t^2})dt}{(t - w)^2 - q^2/4} \equiv -qL(w, q, \alpha). \]

Hence, the formula (4.12) can be presented in the form
\[ \varepsilon_l = 1 - \frac{\omega^2(\omega + i\nu)L(w, q, \alpha)}{k^2v_0^24\varphi_2(\alpha)(\omega + i\nu\lambda_0(w, \alpha))}. \quad \text{(4.13)} \]

V. COMPARISON WITH MERMIN’S RESULT

Mermin [23] considered the kinetic relaxation equation in \( \tau \) – approach in momentum space for finding the general expression of dielectric permeability.

Mermin (see Mermin N.D. [23]) has been received the general expression of dielectric function
\[ \varepsilon^M(\omega, k) = 1 + \frac{(\omega + i\nu)[\varepsilon^0(\omega + i\nu, k) - 1]}{\omega + i\nu} - \varepsilon^0(0, k) \]. \quad \text{(5.1)}

In the formula (5.1) the designation is entered: \( \varepsilon^0(\omega, k) \) is the so-called Lindhard’s dielectric function, i.e. the dielectric function received for non-collisional plasma, expression \( \varepsilon^0(\omega + i\nu, k) \) means, that argument of Lindhard dielectric function \( \omega \) is replaced formally on \( \omega + i\nu \). According to (4.11) Lindhard function looks like
\[ \varepsilon^0_l(\omega, k) = 1 + \frac{u^2k_0}{4k\varphi_2(\alpha)}[l(\frac{\omega}{kv_0} - \frac{k}{2k_0}) - l(\frac{\omega}{kv_0} + \frac{k}{2k_0})], \]

\[ \text{where } \lambda_0(z,q,\alpha) = \frac{q}{2\varphi_0(\alpha)} \int_{-\infty}^{\infty} \frac{tf_0(t)dt}{qt - z}, \quad z = \frac{\omega + i\nu}{k_0v_0}, \quad q = \frac{k}{k_0}. \]
or

\[ \varepsilon_i^0(\omega, q) = 1 + \frac{u_p^2}{q^4\varphi_2(\alpha)} \left[ l\left( \frac{\omega}{kv_0} - \frac{q}{2} \right) - l\left( \frac{\omega}{kv_0} + \frac{q}{2} \right) \right], \quad u_p = \frac{\omega_p}{kv_0}. \]

From last equality we deduce the following two formulas

\[ \varepsilon_i^0(\omega + i\nu, q) - 1 = \frac{u_p^2}{q^4\varphi_2(\alpha)} \left[ l\left( \frac{\omega + i\nu}{kv_0} - \frac{q}{2} \right) - l\left( \frac{\omega + i\nu}{kv_0} + \frac{q}{2} \right) \right], \quad (5.2) \]

\[ \varepsilon_i^0(0, q) - 1 = \frac{u_p^2}{q^4\varphi_2(\alpha)} \left[ l(-q/2) - l(q/2) \right]. \quad (5.3) \]

Let’s make the relation of the left parts of equalities (5.2) and (5.3). We have

\[ \varepsilon_i^0(\omega + i\nu, q) - 1 = \frac{l(\omega + i\nu - q/2) - l(\omega + i\nu + q/2)}{l(-q/2) - l(q/2)}. \quad (5.4) \]

By means of equalities (5.2) and (5.4) we can write down Mermin’s formula (5.1) in the form

\[ \varepsilon_i = 1 + \frac{v^2(\omega + i\nu)}{q^4\varphi_2(\alpha)} \cdot \frac{l(\omega + i\nu - q/2) - l(\omega + i\nu + q/2)}{\omega + i\nu d}, \quad (5.5) \]

where

\[ d = d(\omega + i\nu, q) = \frac{\varepsilon_i^0(\omega + i\nu, q) - 1}{\varepsilon_i^0(0, q) - 1} = \frac{l(\omega + i\nu - q/2) - l(\omega + i\nu + q/2)}{l(-q/2) - l(q/2)}. \]

The formula (5.5) gives representation of dielectric function obtained with the use of kinetic equation in the form of relaxation \( \tau \) – models in momentum space.

Let’s notice, that at \( \omega = 0 \) (a low-frequency limit) the Mermin’s formula gives representation of dielectric function, which not depends on collisional frequency of electrons and looks like:

\[ \varepsilon_i^N = 1 + \frac{\omega_p^2k_0}{k^3v_0^2\varphi_2(\alpha)} \left[ l(-\frac{k}{2k_0}) - l(\frac{k}{2k_0}) \right]. \]

At the same time from our formula (4.11) we receive expression for the dielectric function, depending on collisional frequency of electrons

\[ \varepsilon_i = 1 + \frac{\omega_p^2}{q^2k^2v_0^2\varphi_2(\alpha)} \cdot \frac{l\left( \frac{i\nu}{kv_0} - \frac{k}{2k_0} \right) - l\left( \frac{i\nu}{kv_0} + \frac{k}{2k_0} \right)}{1 + \frac{i\nu}{kv_0\varphi_2(\alpha)}F_0(\frac{i\nu}{kv_0})}. \]

For comparison we will present the formula (4.11) for dielectric function:

\[ \varepsilon_i = 1 + \frac{u_p^2(\omega + i\nu)}{q^4\varphi_2(\alpha)} \cdot \frac{l(\omega + i\nu - \alpha)}{\omega + i\nu \lambda_0(\omega, \alpha)}. \quad (5.6) \]

The formula (5.6) gives representation of dielectric function that obtained with the use of BGK–equation.

The difference between formulas (5.6) and (5.5) consists of the replacement the function \( d \) by function \( \lambda_0 \).
It is possible to show, that at $\hbar \to 0$ both formulas give the same results. It means, that at transition from quantum plasma to classical dielectric function, received on the basis of the kinetic equation as in momentum space, and in coordinates space, passes in the same dielectric function.

For this purpose it is necessary to prove, that
\[
\lim_{\hbar \to 0} d = \lambda_0. \tag{5.7}
\]

Let’s present expression for function $d$ in the following form
\[
d = \frac{l(w - q/2) - l(w + q/2)}{q} \cdot \frac{q}{l(-q/2) - l(q/2)}.
\]

Earlier it has been shown, that
\[
\lim_{q \to 0} \frac{l(w - q/2) - l(w + q/2)}{q} = -\lim_{q \to 0} \int_{-\infty}^{\infty} \frac{\ln(1 + e^{\alpha - t^2})}{(t - w)^2 - q^2/4} dt = \int_{-\infty}^{\infty} \frac{\ln(1 + e^{\alpha - t^2})}{(t - w)^2} dt = 2 \int_{-\infty}^{\infty} \frac{tf_0(t)dt}{t - w} = 4\varphi_0(\alpha)\lambda_0(w, \alpha). \tag{5.8}
\]

From this equality at $w = 0$ it is received
\[
\lim_{q \to 0} \frac{l(-q/2) - l(q/2)}{q} = 4\varphi_0(\alpha)\lambda_0(0, q) = 4\varphi_0(\alpha). \tag{5.9}
\]

From equalities (5.8) and (5.9) we find, that
\[
\lim_{q \to 0} \frac{l(w - q/2) - l(w + q/2)}{l(-q/2) - l(q/2)} \equiv \lim_{q \to 0} d = \lambda_0(w, \alpha).
\]

As it was required to show.

Thus, both relaxation BGK – equations and in coordinates space, and in momentum space at $q \to 0$ lead to the same dielectric function.

VI. CONCLUSION

In the present work the correct formula for calculation of longitudinal electric conductivity and dielectric permeability in the quantum collisinal plasma under arbitrary degeneration degree of electron gas is deduced. For this purpose the Wigner — Vlasov — Boltzmann kinetic equation with collisional integral in the form of BGK–model (Bhatnagar, Gross and Krook) in coordinate space is used. Comparison with Lindhard’s formula has been realized.

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