On asymptotic structure of the critical
Galton-Watson Branching Processes with
infinite variance and Immigration

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Abstract. We observe the Galton-Watson Branching Processes. Limit properties of
transition functions and their convergence to invariant measures are investigated.
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Invariant measures.

1 Introduction

Let \( \{X_n, n \in \mathbb{N}_0\} \) be the Galton-Watson Branching Process allowing Immigration
(GWPI), where \( \mathbb{N}_0 = \{0\} \cup \mathbb{N} \) and \( \mathbb{N} = \{1, 2, \ldots\} \). This is a homogeneous
Markov chain with state space \( S \subset \mathbb{N}_0 \) and whose transition probabilities are

\[
p_{ij} = \text{coefficient of } s^j \text{ in } h(s)(f(s))^i, \quad s \in [0, 1),
\]

where \( h(s) = \sum_{j \in S} h_j s^j \) and \( f(s) = \sum_{j \in S} p_j s^j \) are probability generating
functions (PGF’s). The variable \( X_n \) is interpreted as the population size in
GWPI at the moment \( n \). An evolution of the process will occurs by following
scheme. An initial state is empty that is \( X_0 = 0 \) and the process starts owing to
immigrants. Each individual at time \( n \) produces \( j \) progeny with probability \( p_j \)
independently of each other so that \( p_0 > 0 \). Simultaneously in the population \( i \)
immigrants arrive with probability \( h_i \) in each moment \( n \in \mathbb{N} \). These individuals
undergo further transformation obeying the reproduction law \( \{p_j\} \) and \( n \)-step
transition probabilities \( p_{ij}^{(n)} := P \{X_{n+k} = j | X_k = i\} \) for any \( k \in \mathbb{N} \) are given by

\[
P_{ij}^{(n)}(s) := \sum_{j \in S} p_{ij}^{(n)} s^j = (f_n(s)) \prod_{k=0}^{n-1} h_k(f_k(s)) \quad \text{for any } i \in S,
\]

where \( f_n(s) \) is \( n \)-fold iteration of PGF \( f(s) \); see for example [6]. Thus the transition probabilities \( \{p_{ij}^{(n)}\} \) are completely defined by the probabilities \( \{p_j\} \)
and \( \{h_j\} \).

Classification of states of the chain \( \{X_n\} \) is one of fundamental problems
in theory of GWPI. Direct differentiation of (1) gives

\[
\mathbb{E}[X_n | X_0 = i] = \begin{cases} 
  an + i, & \text{when } m = 1, \\
  \left( \frac{a}{m-1} + i \right) m^n - \frac{a}{m-1}, & \text{when } m \neq 1,
\end{cases}
\]
where $m = f'(1-) \text{ is mean per-capita offspring number and } a = h'(1-)$. The received formula for $E[X_n | X_0 = i]$ shows that classification of states of GWPI depends on a value of the parameter $m$. Process $\{X_n\}$ is classified as sub-critical, critical and supercritical if $m < 1$, $m = 1$ and $m > 1$ accordingly.

The above described population process was considered first by Heathcote [3] in 1965. Further long-term properties of $S$ and a problem of existence and uniqueness of invariant measures of GWPI were investigated by Seneta [12], Pakes [8], [9] and by many other authors. Therein some moment conditions for PGF $f(s)$ and $h(s)$ was required to be satisfied. In aforementioned works of Seneta the ergodic properties of $\{X_n\}$ were investigated. He has proved that when $m \leq 1$ the process $\{X_n\}$ has an invariant measure $\{\mu_k, k \in S\}$ which is unique up to multiplicative constant. Pakes [9] have shown that in supercritical case $S$ is transient. In the critical case $S$ can be transient, null-recurrent or ergodic. In this case, if in addition to assume that $2b := f''(1-) < \infty$, properties of $S$ depend on value of parameter $\lambda = a/b$: if $\lambda > 1$ or $\lambda < 1$, then $S$ is transient or null-recurrent accordingly. In the case when $\lambda = 1$, Pakes [8] studied necessary and sufficient conditions for a null-recurrence property. Limiting distribution law for critical process $\{X_n\}$ was found first by Seneta [11]. He has proved that when $m \leq 1$ the process $\{X_n\}$ has an invariant measure $\{\mu_k, k \in S\}$ which is unique up to multiplicative constant. Pakes [9] have shown that in supercritical case $S$ is transient. In the critical case $S$ can be transient, null-recurrent or ergodic. In this case, if in addition to assume that $2b := f''(1-) < \infty$, properties of $S$ depend on value of parameter $\lambda = a/b$: if $\lambda > 1$ or $\lambda < 1$, then $S$ is transient or null-recurrent accordingly. In the case when $\lambda = 1$, Pakes [8] studied necessary and sufficient conditions for a null-recurrence property. Limiting distribution law for critical process $\{X_n\}$ was found first by Seneta [11]. He has proved that the normalized process $X_n/(bn)$ has limiting Gamma distribution with density function $\Gamma^{-1}(\lambda) x^{\lambda-1}e^{-x}$ provided that $0 < \lambda < \infty$, where $x > 0$ and $\Gamma(*)$ is Euler’s Gamma function. This result has been established also by Pakes [8] without reference to Seneta. Afterwards Pakes [6], [7], has obtained principally new results for all cases $m < \infty$ and $b = \infty$.

Throughout the paper we keep on the critical case only and $b = \infty$. Our reasoning will bound up with elements of slow variation theory in sense of Karamata; see [10]. Remind that real-valued, positive and measurable function $L(x)$ is said to be slowly varying (SV) at infinity if $L(\lambda x)/L(x) \to 1$ as $x \to \infty$ for each $\lambda > 0$. We refer the reader to [1], [2] and [10] for more information.

In second section we study invariant measures of the simple Galton-Watson (GW) Process. In third section the invariant properties of GWPI will be investigated.

### 2 Invariant measures of GW Process

Let $\{Z_n, n \in \mathbb{N}_0\}$ be the simple GW Branching Process without immigration given by offspring PGF $f(s)$. Discussing this case we will assume that the offspring PGF $f(s)$ has the following representation:

$$f(s) = s + (1 - s)^{1+\nu} \mathcal{L}\left(\frac{1}{1-s}\right),$$

where $0 < \nu \leq 1$ and $\mathcal{L}(x)$ is SV at infinity. By the criticality of the process the condition $[f_{\nu}]$ implies that $b = \infty$. This includes the case $b < \infty$ when $\nu = 1$ and $\mathcal{L}(t) \to b$ as $t \to \infty$.

Consider PGF $f_n(s) := \mathbb{E}\left[s^{Z_n} | Z_0 = 1\right]$ and write $R_n(s) := 1 - f_n(s)$. Evidently $Q_n := R_n(0)$ is the survival probability of the process. By arguments
of Slack [13] one can be shown that if the condition \([f_\nu]\) holds then
\[
Q_\nu \cdot \mathcal{L}\left(\frac{1}{Q_n}\right) \sim \frac{1}{\nu n} \quad \text{as} \quad n \to \infty.
\] (2)

Slack [13] also has shown that
\[
U_n(s) := \frac{f_n(s) - f_n(0)}{f_n(0) - f_{n-1}(0)} \to U(s)
\] (3)
for \(s \in [0, 1)\), where the limit function \(U(s)\) satisfies the Abel equation
\[
U(f(s)) = U(s) + 1,
\] (4)
so that \(U(s)\) is PGF of invariant measure for the GW process \(\{Z_n\}\). Combining \([f_\nu]\), (2) and (3) and considering properties of the process \(\{Z_n\}\) we have
\[
U_n(s) \sim \frac{1 - R_n(s)}{Q_n} \nu n \quad \text{as} \quad n \to \infty.
\]

So we proved the following lemma.

**Lemma 1.** If the condition \([f_\nu]\) holds then
\[
R_n(s) = \frac{N(n)}{(\nu n)^{1/\nu}} \cdot \left[1 - \frac{U_n(s)}{\nu n}\right],
\] (5)
where the function \(N(x)\) is SV at infinity and
\[
N(n) \cdot \mathcal{L}^{1/\nu}\left(\frac{(\nu n)^{1/\nu}}{N(n)}\right) \to 1 \quad \text{as} \quad n \to \infty,
\] (6)
and the function \(U_n(s)\) enjoys following properties:

- \(U_n(s) \to U(s)\) as \(n \to \infty\) so that the equation (4) holds;
- \(\lim_{s \to 1} U_n(s) = \nu n\) for each fixed \(n \in \mathbb{N}\);
- \(U_n(0) = 0\) for each fixed \(n \in \mathbb{N}\).

Evidently that this lemma is generalization of (2) and herein it established by more simple proof rather than as shown in [4].

Further writing \(\Lambda(y) = y^{\nu} \mathcal{L}(1/y)\) we consider the function
\[
\mathcal{M}_n(s) := 1 - \frac{A(R_n(s))}{A(Q_n)}.
\] (7)
It follows from (6) and from the properties of SV-function that
\[
\mathcal{M}_n(s) = 1 - \left(\frac{R_n(s)}{Q_n}\right)^{\nu} \frac{\mathcal{L}(1/R_n(s))}{\mathcal{L}(1/Q_n)}
\sim 1 - \left(1 - \frac{U_n(s)}{\nu n}\right)^{\nu} \frac{U_n(s)}{n} (1 + \rho_n(s)) \quad \text{as} \quad n \to \infty,
\]
where \(\rho_n(s) = \mathcal{O}(1/n)\) uniformly for all \(s \in [0, 1)\).

Thus we obtain the following assertion.
Lemma 2. If the condition $[f_{\nu}]$ holds then
$$n \cdot M_n(s) \to U(s) \quad \text{as} \quad n \to \infty,$$
where $U(s)$ is PGF of invariant measure of GW Process.

In the following Lemma we find out an explicit form of PGF of $U(s)$. Write
$$\mathcal{V}(s) = \frac{1}{\nu A (1-s)}.$$

Lemma 3. If the condition $[f_{\nu}]$ holds then
$$U(s) = \mathcal{V}(s) - \mathcal{V}(0).$$

Proof. In pursuance of reasoning from [2, p. 401] we obtain the following relation:
$$\mathcal{V}(f_{n+1}(s)) - \mathcal{V}(f_n(s)) \to 1 \quad \text{as} \quad n \to \infty.$$
Thence summing by $n$ we find
$$\mathcal{V}(f_n(s)) - \mathcal{V}(s) = n \cdot \left(1 + o(1)\right) \quad \text{as} \quad n \to \infty.$$
Keeping our designation we easily will transform last equality to a form of
$$A(R_n(s)) = \frac{A(1-s)}{A(1-s)\nu + 1} \left(1 + o(1)\right) \quad \text{as} \quad n \to \infty.$$  (10)
Combining (7), (8) and (10) we reach (9).

3 Invariant measures of GWPI

Consider GWPI. Pakes [7] has proved the following theorem.

Theorem P1 [7]. If $m = 1$ then
$$p_{00}^{(n)} \sim K \exp \left\{ \int_{1}^{c_n} \frac{\ln h(1-\varphi(y))}{y} \, dy \right\} \quad \text{as} \quad n \to \infty,$$
where $\varphi(y)$ is decreasing SV-function. If
$$\sum_{m=0}^{\infty} \left[ (1-h(f_m(0))(1-f'(f_m(0))) \right] < \infty,$$
then
$$p_{00}^{(n)} \sim K_1 \exp \left\{ \int_{0}^{f_{n}(0)} \frac{\ln h(y)}{f(y) - y} \, dy \right\} \quad \text{as} \quad n \to \infty.$$  
Herein $K$ and $K_1$ are some constants.
Since this point we everywhere will consider the case that immigration PGF $h(s)$ has the following form:

$$1 - h(s) = (1 - s)^\delta \ell \left( \frac{1}{1 - s} \right),$$

where $0 < \delta \leq 1$ and $\ell(x)$ is SV at infinity.

Our results appear provided that conditions $[f_{\nu}]$ and $[h_\delta]$ hold and $\delta > \nu$. As it has been shown in [7] that in this case $S$ is ergodic. Namely we improve statements of Theorem P1. Herewith we put forward an additional requirement concerning $L(x)$ and $\ell(x)$. So since $L(x)$ is SV we can write

$$\frac{L(\lambda x)}{L(x)} = 1 + \alpha(x)$$

for each $\lambda > 0$, where $\alpha(x) \to 0$ as $x \to \infty$. Henceforth we suppose that some positive function $g(x)$ is given so that $g(x) \to 0$ and $\alpha(x) = o(g(x))$ as $x \to \infty$. In this case $L(x)$ is called SV with remainder $\alpha(x)$; see [2, p. 185, condition SR3]. Wherever we exploit the condition $[L_\alpha]$ we will suppose that

$$\alpha(x) = o \left( \frac{L(x)}{x^\nu} \right) \quad \text{as} \quad x \to \infty.$$  (11)

And also by perforce we suppose the condition

$$\frac{\ell(\lambda x)}{\ell(x)} = 1 + \beta(x)$$

for each $\lambda > 0$, where

$$\beta(x) = o \left( \frac{\ell(x)}{x^\delta} \right) \quad \text{as} \quad x \to \infty.$$  

Since $f_n(s) \uparrow 1$ for all $s \in [0, 1)$ in virtue of [11] it sufficiently to observe the case $i = 0$ as $n \to \infty$. Write

$$P_n(s) = P_n^{(0)}(s).$$

The following theorem is generalization of the Theorem P1.

**Theorem 1.** Let conditions $[f_{\nu}]$, $[h_\delta]$ hold. If $\delta > \nu$ then

$$P_n(s) \sim K(s) \exp \left\{ - \int_s^{f_n(s)} \frac{1 - h(y)}{f(y) - y} \left[ 1 \delta(1 - y) \right] dy \right\}$$

as $n \to \infty$, where $K(s)$ is a bounded function for $s \in [0, 1)$ and $\delta(x) \to 0$ as $x \downarrow 0$. If in addition, the conditions $[L_\alpha]$ and (10) are satisfied then

$$P_n(s) \sim K(s) \exp \left\{ - \int_s^{f_n(s)} \frac{1 - h(y)}{f(y) - y} \left[ 1 + o(A(1 - y)) \right] dy \right\} \quad \text{as} \quad n \to \infty.$$
Corollary 1. Let conditions \([f_\nu], [h_\delta]\) hold. If \(\delta > \nu\) then
\[
p_{00}^{(n)} \sim A \exp \left\{ -N^\nu(n) \cdot \ell \left( \frac{(\nu n)^{1/\nu}}{N(n)} \right) \right\} \quad \text{as } n \to \infty,
\]
where \(A\) is a positive constant and \(N(x)\) is SV at infinity defined in (6).

We make sure that at the conditions of second part of Theorem 1 PGF \(\mathcal{P}_n(s)\) converges to a limit \(\pi(s)\) which we denote by the power series representation
\[
\pi(s) = \sum_{j \in S} \pi_j s^j.
\]

In our conditions we can establish a speed rate of this convergence.

Theorem 2. Let conditions \([f_\nu], [h_\delta]\) hold and \(\delta > \nu\). Then \(\mathcal{P}_n(s)\) converges to \(\pi(s)\) which generates the invariant measures \(\{\pi_j\}\) for GWPI. The convergence is uniform over compact subsets of the open unit disc. If in addition, the conditions \([\mathcal{L}_\alpha], [10]\) and \([\ell_\beta]\) are fulfilled then
\[
\mathcal{P}_n(s) = \pi(s) \left( 1 + \Delta_n(s) \mathcal{N}_\delta \left( \frac{1}{R_n(s)} \right) \right),
\]
where \(\mathcal{N}_\delta(x) = \mathcal{N}^\delta(x)\ell(x)\), the function \(\mathcal{N}(x)\) is defined in (6) and
\[
\Delta_n(s) = \frac{1}{\delta - \nu} \left( \frac{1}{\nu_n(s)} \right)^{\delta/\nu - 1} - \frac{1 + \nu}{2\nu} \ln \frac{\nu_n(s)}{(\nu_n(s))^{\delta/\nu}} (1 + o(1))
\]
as \(n \to \infty\) and \(\nu_n(s) = \nu n + A^{-1}(1 - s)\).

The following result is direct consequence of Theorem 2

Corollary 2. If conditions of Theorem 2 hold then
\[
p_{00}^{(n)} = \pi_0 \cdot \left( 1 + \Delta_n \mathcal{N}_\delta(n) \right),
\]
where \(\mathcal{N}_\delta(n)\) is SV at infinity and
\[
\Delta_n = \frac{1}{\delta - \nu} \left( \frac{1}{\nu n} \right)^{\delta/\nu - 1} - \frac{1 + \nu}{2\nu} \ln \frac{n}{(\nu n)^{\delta/\nu}} (1 + o(1)) \quad \text{as } n \to \infty.
\]

Remark 1. The analogous result as in Theorem 2 has been proved in [5] provided that \(\delta = 1\) and \(f''(1-) < \infty\).

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