A Constant-factor Approximation for Weighted Bond Cover

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Abstract
The Weighted $\mathcal{F}$-Vertex Deletion for a class $\mathcal{F}$ of graphs asks, weighted graph $G$, for a minimum weight vertex set $S$ such that $G - S \in \mathcal{F}$. The case when $\mathcal{F}$ is minor-closed and excludes some graph as a minor has received particular attention but a constant-factor approximation remained elusive for Weighted $\mathcal{F}$-Vertex Deletion. Only three cases of minor-closed $\mathcal{F}$ are known to admit constant-factor approximations, namely Vertex Cover, Feedback Vertex Set and Diamond Hitting Set. We study the problem for the class $\mathcal{F}$ of $\theta_c$-minor-free graphs, under the equivalent setting of the Weighted $\rho$-Bond Cover problem, and present a constant-factor approximation algorithm using the primal-dual method. For this, we leverage a structure theorem implicit in [Joret et al., SIDMA’14] which states the following: any graph $G$ containing a $\theta_c$-minor-model either contains a large two-terminal protrusion, or contains a constant-size $\theta_c$-minor-model, or a collection of pairwise disjoint constant-sized connected sets that can be contracted simultaneously to yield a dense graph. In the first case, we tame the graph by replacing the protrusion with a special-purpose weighted gadget. For the second and third case, we provide a weighting scheme which guarantees a local approximation ratio. Besides making an important step in the quest of (dis)proving a constant-factor approximation for Weighted $\mathcal{F}$-Vertex Deletion, our result may be useful as a template for algorithms for other minor-closed families.

keywords: Constant-factor approximation algorithms, Primal-dual method, Bonds in graphs, Graph minors, Graph modification problems.
1 Introduction

For a class $\mathcal{F}$ of graphs, the problem Weighted $\mathcal{F}$-Vertex Deletion asks, given weighted graph $G = (V, E, w)$, for a vertex set $S \subseteq V$ of minimum weight such that $G - S$ belongs to the class $\mathcal{F}$. The Weighted $\mathcal{F}$-Vertex Deletion captures classic graph problems such as Weighted Vertex Cover and Weighted Feedback Vertex Set, which corresponds to $\mathcal{F}$ being the classes of edgeless and acyclic graphs, respectively. A vast literature is devoted to the study of (Weighted) $\mathcal{F}$-Vertex Deletion for various instantiations of $\mathcal{F}$, both in approximation algorithms and in parameterized complexity. Much of the work considers a class $\mathcal{F}$ that is characterized by a set of forbidden (induced) subgraphs [1,3–6,17,31,35,39,43] or that is minor-closed [2,8,11,19,21–26,33,34,46], thus characterized by a (finite) set of forbidden minors.

Lewis and Yannakakis [37] showed that $\mathcal{F}$-Vertex Deletion, the unweighted version of Weighted $\mathcal{F}$-Vertex Deletion, is NP-hard whenever $\mathcal{F}$ is nontrivial (there are infinitely many graphs in and outside of $\mathcal{F}$) and hereditary (is closed under taking induced subgraphs). It was also long known that $\mathcal{F}$-Vertex Deletion is APX-hard for every nontrivial hereditary class $\mathcal{F}$ [41]. So, the natural question is for which class $\mathcal{F}$, $\mathcal{F}$-Vertex Deletion and Weighted $\mathcal{F}$-Vertex Deletion admit constant-factor approximation algorithms.

When $\mathcal{F}$ is characterized by a finite set of forbidden induced subgraphs, a constant-factor approximation for Weighted $\mathcal{F}$-Vertex Deletion is readily derived with LP-
rounding technique. Lund and Yannakakis [41] conjectured that for $F$ characterized by a set of minimal forbidden induced subgraphs, the finiteness of $F$ defines the borderline between approximability and inapproximability with constant ratio of $F$-VERTEX DELETION. This conjecture was refuted due to the existence of 2-approximation for WEIGHTED FEEDBACK VERTEX SET [7, 13, 19]. Since then, a few more classes with an infinite set of forbidden induced subgraphs are known to allow constant-factor approximations for $F$-VERTEX DELETION, such as block graphs [1], 3-leaf power graphs [5], interval graphs [17], ptolemaic graphs [6], and bounded treewidth graphs [23, 26]. That is, we are only in the nascent stage when it comes to charting the landscape of (WEIGHTED) $F$-VERTEX DELETION as to constant-factor approximability. In the remainder of this section, we focus on the case where $F$ is a minor-closed class.

**Known results on (Weighted) $F$-Vertex Deletion.** According to Robertson and Seymour theorem, every non-trivial minor-closed graph class $F$ is characterized by a finite set, called (minor) obstruction set, of minimal forbidden minors, called (minor) obstructions [45]. It is also well-known that $F$ has bounded treewidth if and only if one of the obstructions is planar [44]. Therefore, the $F$-VERTEX DELETION for $F$ excluding at least one planar graph as a minor can be deemed a natural extension of FEEDBACK VERTEX SET. In this context, it is not surprising that $F$-VERTEX DELETION, for minor-closed $F$, attracted particular attention in parameterized complexity, where Feedback Vertex Set was considered the flagship problem serving as an igniter and a testbed for new techniques.

For every minor-closed $F$, the class of yes-instances to the decision version of $F$-VERTEX DELETION is minor-closed again (for every fixed size of a solution), thus there exists a finite obstruction set for the set of its yes-instances. With a minor-membership test algorithm [29], this implies that $F$-VERTEX DELETION is fixed-parameter tractable. The caveat is, such a fixed-parameter algorithm is non-uniform and non-constructive, and the exponential term in the running time is gigantic. Much endeavour was made to reduce the parametric dependence of such algorithms for $F$-VERTEX DELETION. The case when $F$ has bounded treewidth is now understood well. The corresponding $F$-VERTEX DELETION is known to be solvable in time $2^{O(k)} \cdot n^{O(1)}$ [23, 34] and the single-exponential dependency on $k$ is asymptotically optimal under the Exponential Time Hypothesis\(^1\) [34]. (See also [46] for recent parameterized algorithms for general minor-closed $F$’s).

Turning to approximability, the (unweighted) $F$-VERTEX DELETION can be approximated within a constant-factor when $F$ has bounded treewidth, or equivalently, when the obstruction set of $F$ contains some planar graph. The first general result in this direction was the randomized $f(t)$-approximation of Fomin et al. [23]. Gupta et al. [26] made a further progress with an $O(\log t)$-approximation algorithm. Unfortunately, such approximation algorithms whose approximation ratio depends only on $F$ are not known when the input is weighted. A principal reason for this is that most of the techniques

\(^1\)The ETH states that 3-SAT on $n$ variables cannot be solved in time $2^{o(n)}$, see [30] for more details.
developed for the unweighted case do not extend to the weighted setting. In this direction, Agrawal et al. [2] presented a randomized \( O(\log^{1.5} n) \)-approximation algorithm and a deterministic \( O(\log^2 n) \)-approximation algorithm which run in time \( n^{O(t)} \) when \( \mathcal{F} \) has treewidth at most \( t \). It is reported in [2] that an \( O(\log n \cdot \log \log n) \)-approximation can be deduced from the approximation algorithm of Bansal et al. [8] for the edge deletion variant of \textsc{Weighted }\mathcal{F}-\textsc{Vertex Deletion}. For the class \( \mathcal{F} \) of planar graphs, Kawarabayashi and Sidiropoulos [33] presented an approximation algorithm for \( \mathcal{F} \)-\textsc{Vertex Deletion} with polylogarithmic approximation ratio running in quasi-polynomial time. Beyond this work, no nontrivial approximation algorithm is known for \( \mathcal{F} \) of unbounded treewidth.

Regarding constant-factor approximability for \textsc{Weighted }\mathcal{F}-\textsc{Vertex Deletion} with minor-closed \( \mathcal{F} \), only three results are known till now. For the \textsc{Weighted Vertex Cover}, it was observed early that a 2-approximation can be instantly derived from the half-integrality of LP [42]. The local-ratio algorithm by Bar-Yehuda and Even [10] was presumably the first primal-dual algorithm and laid the groundwork for subsequent development of the primal-dual method.\(^2\) For the \textsc{Weighted Feedback Vertex Set}, 2-approximation algorithms were proposed using the primal-dual method [7,13,19]. Furthermore, a constant-factor approximation algorithm was given for \textsc{Weighted Diamond Hitting Set} by Fiorini, Joret, and Pietropaoli [21] in 2010. To the best of our knowledge, since then, no progress is done on approximation with constant ratio for minor-closed \( \mathcal{F} \).

For minor-closed \( \mathcal{F} \) with graphs of bounded treewidth, the known approximation algorithms for (\textsc{Weighted}) \( \mathcal{F} \)-\textsc{Vertex Deletion} take one of the following two avenues. First, the algorithms in [2,8,26] draw on the fact that a graph of constant treewidth has a constant-size separator which breaks down the graph into smaller pieces. The measure for smallness is an important design feature of these algorithms. Regardless of the design specification, however, it seems there is an inherent bottleneck to extend these algorithmic strategy to handle weights while achieving a constant approximation ratio; the above results either use an algorithm for the \textsc{Balanced Separator} problem that does not admit a constant-factor approximation ratio, under the Small Set Expansion Hypothesis [40], or use a relationship between the size of the separator and the size of resulting pieces that do not hold for weighted graphs.

The second direction is the primal-dual method [7,10,13,19,21]. The constant-factor approximation of [23] for \( \mathcal{F} \)-\textsc{Vertex Deletion} is also based on the same core observation of the primal-dual algorithm such as [13]. The 2-approximation for \textsc{Weighted Feedback Vertex Set} became available by introducing a new LP formulation which translates the property ‘\( G - X \) is a forest’ in terms of the sum of degree contribution of \( X \). The idea of expressing the sparsity condition of \( G - X \) in terms of the degree contribution of \( X \)

\(^2\)In this paper, we consider local-ratio and primal-dual as the same algorithms design paradigm and use the word primal-dual throughout the paper even when the underlying LP is not explicitly given. We refer the reader to the classic survey of Bar-Yehuda et al. [9] for the equivalence.
again played the key role in [21] for \textsc{Weighted Diamond Hitting Set}. However, the (extended) sparsity inequality of [21] is highly intricate as the LP constraint describes the precise structure of diamond-minor-free graphs (after taming the graph via some special protrusion replacer). Therefore, expressing the sparsity condition for other classes \( \mathcal{F} \) with tailor-made LP constraints is likely to be prohibitively convoluted. This implies that a radical simplification of the known algorithm for, say, \textsc{Weighted Diamond Hitting Set} will be necessary if one intends to apply the primal-dual method for broader classes.

Our result and the key ideas. The central problem we study is the \textsc{Weighted \( \mathcal{F} \)-Vertex Deletion} where \( \mathcal{F} \) is the class of \( \theta_c \)-minor-free graphs: a weighted graph \( G = (V,E,w) \) is given as input, and the goal is to find a vertex set \( S \) of minimum weight such that \( G - S \) is \( \theta_c \)-minor-free.\(^3\) We call this particular problem the \textsc{Weighted \( c \)-Bond Cover} problem, as we believe that this nomenclature is more adequate for reasons to be clear later (see Observation 3 in Section 2). Our main result is the following.

\textbf{Theorem 1.} There is a constant-factor approximation algorithm for \textsc{Weighted \( c \)-Bond Cover} which runs in time\(^4\) \( O_c(n^{O(1)}) \), for every positive \( c \).

Let us briefly recall the classic 2-approximation algorithms for \textsc{Weighted Feedback Vertex Set} [7,13]. These algorithms repeatedly delineate a vertex subset \( S \) on which the induced subgraph contains an obstruction (a cycle), and “peel off” a weighted graph on \( S \) from the current weighted graph so that the weight of at least one vertex of the current graph drops to zero. The crux of this approach is to create a weighted graph to peel off (or design a weighting scheme) on which every (minimal) feasible solution is consistently an \( \alpha \)-approximate solution. We remark that peeling-off of a weighted graph on \( S \) can be viewed as increasing the dual variable (from zero) corresponding to \( S \) until some dual constraint becomes tight, as articulated in [19].

If one aims to capitalize on the power of the primal-dual method for other minor-closed classes and ultimately for arbitrary \( \mathcal{F} \) with graphs of bounded treewidth, more sophisticated weighting scheme is needed. As we already mentioned, this was successfully done by Fiorini, Joret and Pietropaoli [21] for \textsc{Weighted Diamond Hitting Set}, where their primal-dual algorithm is based on an intricate LP formulation. Our primal-dual algorithm diverges from such tactics, and instead use the following technical theorem as a guide for the weighting scheme. The formal definitions of \( c \)-outgrowth and cluster collection are given in Section 2. Intuitively, one may see a \( c \)-outgrowth as a \( \theta_c \)-minor free subgraph of \( G \) with two vertices in common with the rest of the graph. Also, a cluster collection as a collection collection of pairwise disjoint connected sets and the capacity of this collection is the maximum size of its elements.

\(^3\)The graph \( \theta_c \) is the graph on two vertices joined by \( c \) parallel edges.

\(^4\)We use notation \( O_c(f) \) in order to denote \( g(c) \cdot O(f) \), for some computable function \( g : \mathbb{N} \rightarrow \mathbb{N} \).
Theorem 2. There is a function $f_1 : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that, for every two positive integers $c$ and $t$, there is a uniformly polynomial time algorithm that, given as input a graph $G$, outputs one of the following:

1. a $c$-outgrowth of size at least $c$, or
2. a $\theta_c$-model $M$ of $G$ of size at most $f_1(c,t)$, or
3. a cluster collection $C$ of $G$ of capacity at most $f_1(c,t)$ such that $\delta(G/C) \geq t$, or
4. a report that $G$ is $\theta_c$-minor free.

(By $\delta(G)$ we denote the minimum edge-degree of a vertex in $G$.)

A variant of Theorem 2 was originally proved by Joret et al. [32] without the capacity condition on a cluster collection in Case 3. With a slight modification of their proof, it is not difficult to excavate the above statement and we provide a proof in Section 3. It turns out that imposing the capacity condition of Case 3 is crucial for designing a weighting scheme.

At each iteration, our primal-dual algorithm invokes Theorem 2. Depending on the outcome, the algorithm either runs a replacer (defined in Section 5), that is used in order to reduce the size of a $c$-outgrowth, or computes a suitable weighted graph which we call $\alpha$-thin layer (defined in Section 6), using a suitable weighting scheme, thus reducing the current weight. In both cases, we convert the current weighted graph $G = (V, E, w)$ into a new weighted graph $G' = (V', E', w')$ on a strictly smaller number of vertices so that an $\alpha$-approximate solution for $G'$ implies an $\alpha$-approximate solution for $G$ for some particular value of $\alpha$.

We stress that the replacer is compatible with any approximation ratio in the sense that the optimal weight of a solution is unchanged and every solution after the replacement can be transformed to a solution that is at least as good. When Theorem 2 reports a constant-sized $\theta_c$-model, it is easy to see that a uniformly weighted $\alpha$-thin layer suffices. The gist of Theorem 2 is that in the third case that promises a collection of pairwise disjoint constant-sized connected sets.

Let us first consider the simplest such case where all connected sets are singletons, namely when $\delta(G) \geq t$. It is not difficult to see that, if we consider $t := 6c$ and under the edge-degree-proportional weight function, that is for every $v \in V(G)$, $w(v) := \text{edeg}_G(v)$, any feasible solution to Weighted $c$-Bond Cover is a 4-approximate solution. For completeness, we include the proof of this observation for the the general Weighted $\mathcal{F}$-Vertex Deletion problem in Appendix A.

In the general case where we have a collection of pairwise disjoint connected sets, each of size at most $r$, the critical observation (Lemma 10) is that if the contraction of these sets

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5 The edge-degree of a vertex $v$ of $G$, denoted by $\text{edeg}_G(v)$, is the number of edge that are incident to $v$. 

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yields a graph of minimum edge-degree at least \( t := 8c \), then a weighting scheme akin to the simple case also works. That is, any feasible solution to Weighted \( c \)-Bond Cover is a \( 4r \)-approximate solution (Section 4). The overall primal-dual framework is summarized in Section 6.

2 Basic definitions and preliminary results

We use \( \mathbb{N} \) for the set of non-negative integers and \( \mathbb{R}_{\geq 0} \) for the set of non-negative reals. Given a set \( X \) and two functions \( w_1, w_2 : X \to \mathbb{R}_{\geq 0} \), we denote by \( w_1 \pm w_2 : X \to \mathbb{R}_{\geq 0} \) the function where for each \( x \in X \), \( (w_1 \pm w_2)(x) = w_1(x) \pm w_2(x) \). Given some \( r \in \mathbb{N} \), we define \([r] = \{1, \ldots, r\}\). Given some collection \( \mathcal{A} \) of objects on which the union operation can be defined, we define \( \bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A \).

All graphs we consider are multigraphs without loops. We denote a graph by \( G = (V, E) \) where \( V \) and \( E \) is its vertex and edge set respectively. When we deal with vertex-weighted graphs, we denote them by \( G = (V, E, w) \) where \( w : V(G) \to \mathbb{R}_{\geq 0} \) and we say that \( G \) is a \( w \)-weighted graph. When considering edge contractions we sum up edge multiplicities of multiple edges that are created during the contraction. However, when a loop appears after a contraction, then we suppress it. We use \( V(G) \) and \( E(G) \) for the vertex set and the edge multiset of \( G \). We also refer to \( |V(G)| \) as the size of \( G \). If \( X \subseteq V(G) \), we denote by \( G[X] \) the subgraph of \( G \) induced by \( X \) and by \( G \setminus X \) the graph \( G[V(G) \setminus X] \). We say that \( X \) is connected in \( G \) if \( G[X] \) is connected. A graph \( H \) is a minor of a graph \( G \) if \( H \) can be obtained from a subgraph of \( G \) after contracting edges. Given a graph \( H \), we say that \( G \) is \( H \)-minor free if \( G \) does not contain \( H \) as a minor.

Given an edge \( e \) of a graph \( G \), we denote by \( m(e) \) its multiplicity and we define \( \mu(G) = \max\{m(e) \mid e \in E(G)\} \). Given a vertex \( v \in V(G) \), we define the edge-degree of \( v \) in \( G \), denoted by \( \deg_G(v) \), as the number of edges of \( G \) that are incident to \( v \). We denote by \( N_G(v) \) the set of all neighbors of \( v \) in \( G \) and we call \( |N_G(v)| \) vertex-degree of \( v \). We denote by \( \delta(G) \) the minimum edge-degree of a vertex in \( G \).

**Bonds and pumpkins.** Let \( G \) be a graph. Given two disjoint subsets \( X, Y \) of \( V(G) \), the edges crossing \( X \) and \( Y \) is the set of edges with one endpoint in \( X \) and the other in \( Y \). The graph \( \theta_c \) is the graph on two vertices joined by \( c \) parallel edges (\( \theta_c \) is also known as the \( c \)-pumpkin). Notice that \( \theta_c \) is a minor of \( G \) iff \( G \) contains two disjoint connected sets \( X \) and \( Y \) crossed by \( c \) edges of \( G \). We call the union \( M := X \cup Y \theta_c \)-model of \( G \).

Given a bipartition \( \{V_1, V_2\} \) of \( V(G) \), the set of edges crossing \( V_1 \) and \( V_2 \) is called the cut of \( \{V_1, V_2\} \) and an edge set is a cut if it is a cut of some vertex bipartition. For disjoint vertex subsets \( S, T \subseteq V \), an \( (S, T) \)-cut is a cut of a bipartition \( \{V_1, V_2\} \) such that \( S \subseteq V_1 \) and \( T \subseteq V_2 \). Note that, for a connected graph, cuts are precisely the edge sets whose deletion
strictly increases the number of connected components. A block of a graph \( G \) is either an isolated vertex (a vertex of vertex-degree 0) or a bridge or a biconnected component of \( G \).

A minimal non-empty cut is known as a bond in the literature. We remark that the bonds of \( G \) are precisely the circuits of the cographic matroid of \( G \). Given a positive integer \( c \), a \( c \)-bond of a graph \( G \) is any minimal cut of \( G \) of size at least \( c \). Notice that if a \((X, Y)\)-cut of a graph \( G \) is a bond and \( X \cup Y = V(G) \), then both \( X \) and \( Y \) are connected sets of \( G \). The problem of finding the maximum \( c \) for which a graph \( G \) contains a \( c \)-bond has been examined both from the approximation [18, 28] and the parameterized point of view [20].

We next see how a \( c \)-bond is related to a \( \theta_c \)-model.

**Observation 3.** For every \( c \in \mathbb{N} \), a graph \( G \) contains \( \theta_c \) as a minor iff it has a \( c \)-bond.

**Proof.** Consider a \( \theta_c \)-model \( M = X \cup Y \) where \( X \) and \( Y \) are two disjoint connected vertex sets of \( G \), with at least \( c \) edges crossing \( X \) and \( Y \). For every vertex \( v \) in the same connected component \( C \) as \( X \cup Y \), add \( v \) to either of \( X \) or \( Y \) in which \( v \) has a neighbor. Let \( X' \) and \( Y' \) be the resulting sets with \( X \subseteq X' \) and \( Y \subseteq Y' \), and let \( F \) be the edge set crossing \( X' \) and \( Y' \). Clearly \( F \) is the cut of \((X', V \setminus X')\).

To show that \( F \) is a \( c \)-bond, suppose that the cut \( F' \) of a bipartition \((A, B)\) of \( V \) is a subset of \( F \). If \( \emptyset \neq A \cap X' \cap C \subseteq X' \cap C \), then the cut of \((A, B)\) contains all the edges crossing \( X' \cap C \cap A \) and \((X' \cap C) \setminus A \). Since \( X' \cap C \) is connected, there is at least one edge \( e \) crossing \( X' \cap C \cap A \) and \((X' \cap C) \setminus A \). Because \( e \) is an edge whose both endpoints belong to \( X' \cap C \), it follows \( e \in F' \setminus F \), a contradiction. Therefore, either \( A \cap X' \cap C = \emptyset \) or \( A \cap X' \cap C = X' \cap C \) holds, and likewise either \( B \cap Y' \cap C = \emptyset \) or \( B \cap Y' \cap C = Y' \cap C \) holds. It follows that \( F' \) is either an empty set or equals \( F \), thus proving that \( F \) is a \( c \)-bond. The backward direction is straightforward from that \( F \) takes all its edges from a single connected component of \( G \). \( \Box \)

**Treewidth.** A tree decomposition of a graph \( G \) is a pair \((T, \chi)\), where \( T \) is a tree and \( \chi : V(T) \to 2^{V(G)} \) such that:

1. \( \bigcup_{q \in V(T)} \chi(q) = V(G) \),
2. for every edge \( \{u, v\} \in E \), there is a \( q \in V(T) \) such that \( \{u, v\} \subseteq \chi(q) \), and
3. for every vertex \( v \in V(G) \), the set \( \{t \in V(T) \mid v \in \chi(t)\} \) is connected in \( T \).

The width of a tree decomposition \((T, \chi)\) is \( \max\{|\chi(q)| \mid q \in V(T)\} - 1 \). The treewidth of \( G \) is the minimum width over all tree decompositions of \( G \).

We need the following two observations. Observations (i) and (ii) follow from the combinatorial results of [38] and [16] respectively.

**Proposition 4.** For every positive integer \( c \) and every \( \theta_c \)-minor free multigraph \( G \), it holds that (i) \(|E(G)| \leq 2c \cdot |V(G)|\) and (ii) \( G \) has treewidth less than \( 2c \).
Covering bonds. Given a set $S \subseteq V(G)$, we say that $S$ is a $c$-bond cover of $G$ if $G - S$ is $\theta_c$-minor free. Notice that $S$ is a $c$-bond cover iff $G \setminus S$ does not contain a $c$-bond. Given a weighted graph $G = (V, E, w)$ with $w : V(G) \to \mathbb{R}_{\geq 0}$, a minimum weight $c$-bond cover of $G$ is a $c$-bond cover $S$ where the weight of $S$, defined as $w(S) := \sum_{v \in S} w(v)$, is minimized.

We consider the following optimization problem for every positive integer $c$.

\begin{center}
\textbf{Weighted $c$-Bond Cover}
\end{center}

\textit{Input:} a vertex weighted graph $G = (V, E, w)$.

\textit{Solution:} a minimum weight $c$-bond cover of $G$.

The next proposition follows by applying the general algorithmic results of [12] on the graph $\theta_c$, taking into account Observation 3.

**Proposition 5.** For every two positive integers $c, t$ there exists a uniformly linear-time algorithm that, given a positive integer $c$ and a $w$-weighted $n$-vertex graph $G$ of treewidth at most $t$, outputs a minimum weight $c$-bond cover of $G$.

**Cluster collections.** Let $G$ be a graph. A cluster collection of $G$ is a non-empty collection $\mathcal{C} = \{C_1, \ldots, C_r\}$ of pairwise disjoint non-empty connected subsets of $V(G)$. In case $\bigcup \mathcal{C} = V(G)$ we say that $\mathcal{C}$ is a cluster partition of $G$. The capacity of a cluster collection $\mathcal{C}$ is the maximum number of vertices of a cluster in $\mathcal{C}$. We use the notation $G[\mathcal{C}]$ for the multigraph obtained from $G[\bigcup \mathcal{C}]$ by contracting all edges in $G[C_i]$ for each $i \in \{1, \ldots, r\}$. Given a cluster $C \in \mathcal{C}$ we denote by $\text{ext}_C(C)$ (or simply $\text{ext}(C)$) the set of edges with one endpoint in $C$ and the other not in $C$.

**Lemma 6.** Let $Z$ be a multigraph where $\mu(Z) < c$ and let $\{A, B\}$ be a partition of $V(Z)$ such that each vertex in $A$ has edge-degree at least $k$, $B$ is an independent set of $Z$, and each vertex in $B$ has vertex-degree at least $2$. Then $Z$ has a cluster collection $\mathcal{C}$ where each vertex in $A$ belongs in exactly one cluster, no cluster has more than $k + 1$ vertices, and each vertex of $\mathcal{C}$ has edge-degree at least $k/c$ in $Z[\bigcup \mathcal{C}]$.

**Proof.** Let $a \in A$ and let $b \in N_G(a) \cap B$. Observe that $G$ has less than $c$ edges between $a$ and $b$ and at least one edge between $b$ and some vertex in $A \setminus \{a\}$. Using this fact, observe that we can pick a subset $B_a$ of $N_G(a) \cap B$ of at most $k$ vertices such that at least $k/c$ edges of $G$ have exactly one endpoint in $\{a\} \cup B_a$ and the other point in $A \setminus a$. In fact we can greedily pick such a set $B_a$ for every $a \in A$ in a way that if $a \neq a'$ then $B_a \cap B_{a'} = \emptyset$, and for each $a \in A$, there are at least $k/c$ edges between $a \cup B_a$ and $\bigcup \mathcal{C} \setminus (\{a\} \cup B_a)$, where $\mathcal{C} := \{a \cup B_a \mid a \in A\}$. Let us begin with $B_a = \emptyset$ for every $a \in A$. Suppose there exists a vertex $a \in A$ with less than $k/c$ edges with one endpoint in $a \cup B_a$ and another in
Let $p$ be the number of edges between $a$ and $\mathcal{U} \setminus \{a\}$. Let $p$ be the number of edges between $a$ and $\mathcal{U} \backslash \{a\}$. Due to the edge-degrees lower bound on $a$, there are at least $k - p > 0$ edges between $a$ and $\mathcal{U} \setminus \{a\}$. Update $B_a$ to a minimal subset of $B \setminus \mathcal{U}$ so that there are at least $k - p$ edges between $a$ and $B_a$. Now there are at least $p + (k - p)\geq k/c$ edges with exactly one endpoint in $\{a\} \cup B_a$ and another in $\mathcal{U} \setminus \{a\}$. Therefore, we can continue the procedure until the cluster collection $\mathcal{C} = \{\{a\} \cup B_a \mid a \in A\}$ at hand has the desired property.

Detecting and covering $c$-outgrowths. Given a graph $G$, a $c$-outgrowth of $G$ is a triple $K = (K, u, v)$ where $u, v$ are distinct vertices of $G$, $K$ is a component of $G \setminus \{u, v\}$, $N_G(V(K)) = \{u, v\}$, and the graph, denoted by $K^{(x,y)}$, obtained from $G[V(K) \cup \{u, v\}]$ if we remove all edges with endpoints $u$ and $v$ is $\theta_c$-minor free. The size of a $c$-outgrowth of $G$ is the size of $K$.

We say that a graph $G$ is $c$-reduced if every $c$-outgrowth $(K, u, v)$ of $G$ has size at most $c - 1$. Clearly, if $G$ is $1$-reduced then $G$ has no $c$-outgrowth for any $c \geq 1$.

**Lemma 7.** For every two positive integers $c, c'$, there exists a uniformly polynomial-time algorithm that given and an $n$-vertex graph $G$, outputs, if exists, a $c$-outgrowth $(K, u, v)$ of size $c'$.

**Proof.** The algorithm considers, for every two distinct vertices $x, y \in V(G)$, all the components of $G - \{x, y\}$ of size $c'$. For each such triple $(K, x, y)$ the algorithm checks whether $(K, x, y)$ is a $c$-outgrowth, that is whether the graph $K^{(x,y)}$ is $\theta_c$-minor-free. This can be checked using the algorithm of [14], which runs in time linear in $n$. Notice that if $K^{(x,y)}$ has treewidth $> 2c$, then it follows from Proposition 4.ii that $K^{(x,y)}$ is not $\theta_c$-minor-free. In case $K^{(x,y)}$ has treewidth at most $2c$, then one may verify whether $(K, x, y)$ is a $c$-outgrowth by using the algorithm Proposition 5: just check whether the minimum $c$-bond cover of $K^{(x,y)}$ has size zero. \hfill $\square$

**Lemma 8.** For every positive integer $c$, there exists a uniformly linear-time algorithm that given a $c$-outgrowth $K = (K, u, v)$, and integer $i \in \{0, \ldots, c - 1\}$, and a vertex weighting $w : V(K) \rightarrow \mathbb{R}$, outputs a minimum weight subset $S$ of $V(K)$ such that $K^{(u,v)} \setminus S$ does not contain any $\theta_{i+1}$-model $M = X \cup Y$ with $u$ and $v$ in different sets in $\{X, Y\}$.

**Proof.** In order to compute the required set $S$ consider the graph $G$ obtained by $K^{(u,v)}$ after assigning $u$ and $v$ with $c$ parallel edges. Then extend the weighting of $w$ to one on $V(G)$ by assigning sufficiently large weights to $u$ and $v$, e.g. $w(u) := w(v) := \sum_{x \in V(K)} w(x)$. Notice that $G$ is $\theta_{2c}$-minor free, therefore, by Proposition 4.ii, it has treewidth at most $4c$. Then run the algorithm of Proposition 5 for $c := c + i + 1$ and $t := 2c$ and output a minimum weight set $S$ such that $G \setminus S$ is $\theta_{c+i+1}$-minor free. Notice that, because $K^{(u,v)}$ is already $\theta_c$-minor free and because we added $c$ parallel edges, it follows that $K^{(u,v)} \setminus S$ does not contain a $\theta_{i+1}$-model $M = X \cup Y$ with $u$ and $v$ in different sets in $\{X, Y\}$ if and only if $G \setminus S$ is $\theta_{c+i+1}$-minor free. Therefore the output $S$ is indeed the required set. \hfill $\square$
In the statements of Proposition 5, Lemma 7, and Lemma 8, the term uniformly linear/polynomial indicates that it is possible to construct an algorithm of running time $O_q(n^z)$ for some universal constant $z$ (where $z = 1$ in case of a linear algorithm) and where $q$ constructively depends only on the other constants (that is $c, c', t$). We stress that in the statements of our results we did not make any effort to optimize the running of the claimed polynomial algorithm.

3 A structural theorem

In this section we prove the following structural result.

**Theorem 2.** There is a function $f_1(c, t) : \mathbb{N}^2 \to \mathbb{N}$ such that, for every two positive integers $c, t$, there is a uniformly polynomial time algorithm that, given as input a graph $G$, outputs one of the following:

1. a $c$-outgrowth of size at least $c$, or
2. a $\theta_c$-model $M$ of $G$ of size at most $f_1(c, t)$, or
3. a cluster collection $C$ of $G$ of capacity at most $f_1(c, t)$ such that $\delta(G/C) \geq t$, or
4. a report that $G$ is $\theta_c$-minor free.

The proof of Theorem 2 is based on the following lemma, whose proof follows closely the construction and some of the arguments of [32, Lemma 4.10]. In the proof we present the points that differentiate our proof from the one of [32, Lemma 4.10].

**Lemma 9.** There is a function $f_2 : \mathbb{N}^2 \to \mathbb{N}$ such that for every two positive integers $c, t$, there is a uniformly polynomial time algorithm that, given as input a 1-reduced graph $G$, outputs either a $\theta_c$-model $M$ of $G$ of size at most $f_2(c, t)$, or some cluster collection $C$ of $G$ of capacity at most $f_2(c, t)$ and such that $\delta(G/C) \geq t$, or a report that $G$ is $\theta_c$-minor free.

**Proof.** Let $f_2(c, t) := \min\{k^r + k + 1, r + r(k-1)k^r\}$, where $k := c \cdot t$ and $r := (2c)^{2c} \cdot k$. We also use $\ell := f_2(c, t)$.

First of all, we may assume that all edges have multiplicity less than $c$, as otherwise the endpoints of such an edge would be the desired $\theta_c$-model of $G$. Also we may assume that all blocks of $G$ contain some $\theta_c$-model. If not, we apply the proof to the updated $G$ taken by removing all vertices of these blocks that are not cut-vertices (take into account that the union of $\theta_c$-free blocks is also $\theta_c$-free). We refer to this last assumption as the block assumption.

Let $W$ be the set of vertices of $G$ with edge-degree at least $k$. We next consider a maximal collection $\mathcal{P}$ of pairwise vertex-disjoint induced subgraphs of $G - W$, each isomorphic to
a multipath on exactly $r$ vertices (a multipath is a connected graph $P$ that, when we suppress the multiplicity of its edges, gives a path). Let $\mathcal{C}$ be the set of components of $G - (W \cup V(\mathcal{P}))$. Observe that each graph in $\mathcal{C}$ has diameter at most $r - 1$ and maximum edge-degree at most $k - 1$, therefore each graph in $\mathcal{C}$ has at most $k^r$ vertices. We refer to the subgraphs of $G$ that belong in $\mathcal{C}$ (resp. $\mathcal{P}$) as the $\mathcal{C}$-clusters (resp. $\mathcal{P}$-clusters) of $G$ (see Figure 1).

Let now $K = G/\left(\mathcal{C} \cup \mathcal{P}\right)$, i.e., we contract in $G$ the $\mathcal{C}$-clusters and the $\mathcal{P}$-clusters. Let $v_C$ (resp. $v_P$) be the result of the contraction of $\mathcal{C}$-cluster $C \in \mathcal{C}$ (resp. $\mathcal{P}$-cluster $P \in \mathcal{P}$) in $K$. We define $V_c = \{v_C \mid C \in \mathcal{C}\}$, $V_p = \{v_P \mid P \in \mathcal{P}\}$. Observe that $\{W, V_c, V_p\}$ is a partition of $V(K)$ and that $V_c$ is an independent set of $K$. Also, according to [32], either it is possible to find in polynomial time some $\theta_c$-model of $G$ of size at most $\max\{k^r + 1, r + 1, 2r, k^r + r\} \leq \ell$ or we may assume that $\mu(K) < c$. We assume the later. Moreover, we may also assume that $\forall C \in \mathcal{C} \ vdeg_K(v_C) \geq 1$, otherwise, by the block assumption, a $\theta_c$-model of size $k^r \leq \ell$ can be found in $C$.

We further partition the $\mathcal{C}$-clusters as follows: Let $C \in \mathcal{C}$. If the vertex-degree of $v_C$ is one in $K$, then we say that $C$ is a bad $\mathcal{C}$-cluster of $G$, otherwise we say that $C$ is a good $\mathcal{C}$-cluster of $G$. This gives rise to the partition $\{\mathcal{C}_{\text{good}}, \mathcal{C}_{\text{bad}}\}$ of $\mathcal{C}$ by and the corresponding partition $\{V_c^{\text{good}}, V_c^{\text{bad}}\}$ of $V_c$. Recall that every vertex in $V_c^{\text{bad}}$ has exactly one neighbor in $K$. Moreover, this neighbor should be some vertex of $V_p$. Indeed, if this neighbor is a vertex in $W$ then by the block assumption we may find some some $\theta_c$-model in $G$ of size $k^r \leq \ell$ and we are done.

Let $v$ be a vertex of some $\mathcal{P}$-cluster of $G$. We say that $v$ is a black vertex of $P$ if, in $G$, all its neighbors outside $P$ are vertices of bad $\mathcal{C}$-clusters, i.e., $N_G(v) \setminus V(P) \subseteq V(\mathcal{C}_c^{\text{bad}})$. If a vertex of $P$ is not black then it is white. According to [32, Claim 4], either one can find, in polynomial time, a $\theta_c$-model of $G$ of size at most $r + r(k - 1)k^r \leq \ell$ or there is no $\mathcal{P}$-cluster that, being a multipath, contains $(2c)^2c$ consecutive black vertices. This implies that we
may assume that every $\mathcal{P}$-cluster contains at least $k = \frac{r}{(2c)^2}$ white vertices. Observe that

$$\mathcal{C}' = \{w \mid w \in W \} \cup \{V(P) \mid v_P \in \mathcal{V}_p \} \cup \{V(C) \mid v_C \in \mathcal{V}_c^{good}\}$$

forms a cluster collection of $G$ of capacity $\leq k^r$ where we can see $v_P$ (resp. $v_C$) as the result of the contraction of the $\mathcal{P}$-cluster $V(P)$ (resp. the good $\mathcal{C}$-cluster $V(C)$). As we have seen above, every vertex $v_P$ has edge-degree at least $k$ in $G/\mathcal{C}'$. Moreover, less than $c$ edges may exist between each pair of the clusters of $\mathcal{C}'$, otherwise we obtain a $\theta_c$-model in $G$ of size at most $2k^r \leq \ell$. This means that $\mu(G/\mathcal{C}') < c$. As $Z := G/\mathcal{C}'$ satisfies the conditions of Lemma 6 for $A := W \cup V_P$ and $B := \mathcal{V}_C^{good}$, we have that there exists a cluster collection $\mathcal{C}''$ of $Z$ where each vertex $w \in W \cup V_P$ belongs in exactly one cluster $C_w$, no cluster has more than $k + 1$ vertices, and each vertex of $Z/\mathcal{C}''$ has edge-degree at least $k/c$. As a last step, we further merge the clusters in $\mathcal{C}'$ as indicated by the cluster collection $\mathcal{C}''$ of $G/\mathcal{C}'$ (during this merging, the clusters of $\mathcal{C}'$ that do not correspond to vertices in $G/\mathcal{C}'$ are discarded). That way we obtain a cluster collection $\mathcal{C}'''$ of $G$ of capacity $\leq k^r + k + 1 \leq \ell$ and where $\delta(G/\mathcal{C}'') \geq k/c = t$ as required. 

We are now in position to give the proof of Theorem 2.

**Proof of Theorem 2.** Let $f_1(c,t) := c \cdot f_2(c,t)$, where $f_2$ is the function of Lemma 9. By applying Lemma 7, we may assume that every $c$-outgrowth of $G$ has size smaller than $c$. We call such an $c$-outgrowth $(K,u,v)$, of size $< c$, *maximal* if there is no other $c$-outgrowth $(K',u',v')$, of size $< c$, where $V(K) \cup \{u',v'\}$ is a proper subset of $V(K) \cup \{u,v\}$. Also we call a collection $\mathcal{K}$ of $c$-outgrowths of size $< c$ *complete* if for every two distinct $(K,u,v), (K',u',v') \in \mathcal{K}$, $V(K) \cap V(K') = \emptyset$. Let now $\mathcal{K}$ be some complete collection of maximal outgrowths of $G$. Notice that $\mathcal{K}$ can be computed in polynomial time by greedily including in it maximal $c$-outgrowths $(K',u',v')$ of $G$, of size $< c$ as long as the completeness condition is not violated.

Starting from $G$, we construct an auxiliary graph $G'$ as follows: for each $c$-outgrowth $K = (K,u,v) \in \mathcal{K}$, we remove $V(K)$ and introduce a new edge whose multiplicity $i_K$ is the maximum $i$ such that $K^{(u,v)}$ contains a $\theta_i$-model $M = X \cup Y$ with $u$ and $v$ in different sets in $\{X,Y\}$ (the multiplicity $i_K$ is summed up to the so far multiplicity). The value of $i_K$ can be computed using the the algorithm of Lemma 8 because it is equal to the minimum $i \in \{0, \ldots , c-1\}$ for which this algorithm returns the empty set. Notice that $\mu(G') < c$, otherwise we can find in $G$ a $\theta_c$-model of at most $c^2 \leq c \cdot f_2(c,t)$ vertices (recall that each edge in $G'$ is either an edge of $G$ or may correspond to an $c$-outgrowth of size $< c$). By the maximality and the completeness of the $c$-outgrowths of $\mathcal{K}$ it follows that $G'$ is an 1-reduced graph. Therefore, we can apply Lemma 9 on $G'$ and obtain either a $\theta_c$-model $M'$ of $G'$ of size at most $f_2(c,t)$ or some cluster collection $\mathcal{C}'$ of $G'$ of capacity at most $f_2(c,t)$ and where $\delta(G'/\mathcal{C}') \geq t$.
If the output of Lemma 9 is a $\theta_c$-model $M'$ of $G'$ of size at most $f_2(c,t)$, then this gives rise to a $\theta_c$-model $M$ of $G$ of size at most $c \cdot f_2(c,t)$. In case Lemma 9 gives some cluster collection $C'$ of $G'$ of capacity at most $f_2(c,t)$ then $C'$ can be straightforwardly transformed to a cluster collection $C$ of $G$ of capacity at most $c \cdot f_2(c,t)$ where $\delta(G/C) \geq \delta(G'/C') \geq t$, as required. \qed

4 The weighting scheme

Let $G$ be a graph and let $C$ be a cluster partition of $G$ of capacity at most $r$. Let also $G$ be an instance of $c$-BOND COVER for some positive integer $c$. We define the vertex weighting function $w_C : V(G) \rightarrow \mathbb{R}_{\geq 0}$ so that if $v \in C \in C$, then

$$w_C(v) = \frac{|\text{ext}(C)|}{|C|}.$$  \hspace{1cm} (1)

When $C$ is clear from the context, we simply write $w$ instead $w_C$. The main result of this section is that, with respect to the weight function $w$ in Equation 1, every $c$-bond covering of $G$ is a $4r$-approximation.

Lemma 10. Let $c$ be a non negative integer, $G$ be a graph, $r$ be a positive integer, $C$ be a connected partition of $G$ of capacity at most $r$ and such that $\delta(G/C) \geq 8c$, and $w : V(G) \rightarrow \mathbb{R}_{\geq 0}$ be a vertex weighting function as in Equation 1. Then for every $c$-bond covering $X$ of $G$, it holds that

$$\frac{1}{2r} \cdot |E(G/C)| \leq \sum_{v \in X} w(v) \leq 2 \cdot |E(G/C)|.$$ 

Proof. For the upper bound, note that $\sum_{v \in X} w(v) = \sum_{v \in X} \frac{|\text{ext}(C)|}{|C|} \leq \sum_{C \in C} \sum_{v \in C} \frac{|\text{ext}(C)|}{|C|} = \sum_{C \in C} |\text{ext}(C)| = \sum_{x \in V(G/C)} \text{edeg}_{G/C}(x) = 2 \cdot |E(G/C)|.$$

For the lower bound, let $X$ be a $c$-bond covering of $G$ and let $F = V(G) \setminus X$, $C_X = \{C \in C \mid C \cap X \neq \emptyset\}$, and $C_F = C \setminus C_X$. Since $\delta(G/C) \geq 8c$, we obtain that $|E(G/C)}/2 \geq 2c \cdot |V(G/C)| = 2c \cdot |C|$. We claim that $\sum_{C \in C_X} |\text{ext}(C)| \geq |E(G/C)|/2$. Indeed, if this is not the case then, by the fact that $|E(G/C)| = |E(G[F]/C_F)| + \sum_{C \in C_X} |\text{ext}(C)|$, we know that $|E(G[F]/C_F)| > |E(G/C)|/2 \geq 2c \cdot |C| \geq 2c \cdot |C_F|$ and this last inequality, from Proposition 4.i, gives that $\theta_c$ is a minor of $G/C_F$ which is a minor of $G[F]$, a contradiction. Therefore, since each set in $C_X$ contains at least one vertex of $X$,

$$\sum_{v \in X} w(v) \geq \sum_{C \in C_X} \frac{|\text{ext}(C)|}{|C|} \geq \frac{1}{r} \sum_{C \in C_X} |\text{ext}(C)| \geq \frac{|E(G/C)|}{2r},$$

which proves the lower bound. \qed
5 Replacing outgrowths

In this section, we present our $c$-outgrowth replacer. Given a $w$-weighted graph $G$, we denote by $\text{opt}_c(G)$ (or, for simplicity, $\text{opt}(G)$) the weight of an optimal solution for WEIGHTED $c$-BOND COVER on $G$. The main result of the section is as follows.

**Lemma 11.** For every positive integer $c$, there is a uniformly polynomial time algorithm that, given a $w$-weighted graph $G$ and a $c$-outgrowth $K = (K, u, v)$ of $G$, outputs a weighted graph $G'$ where $K$ is replaced by another $c$-outgrowth $K' = (K', u, v)$ of size at most $c - 1$ such that

1. $\text{opt}(G) = \text{opt}(G')$.

2. Given a $c$-bond cover $S' \subseteq V(G')$ of $G'$, one can construct in polynomial time a $c$-bond cover $S \subseteq V(G)$ such that $w(S) \leq w(S')$.

In particular, an $\alpha$-approximate solution for $G'$ implies an $\alpha$-approximate solution for $G$.

**Proof.** For $i \in \{0, \ldots, c - 1\}$, let $K_i^{(u,v)}$ be the graph obtained from $K^{(u,v)}$ by adding $i$ edges connecting $u$ and $v$. Obviously $K_0^{(u,v)}$ equals $K^{(u,v)}$. Let also $T_i \subseteq V(K)$ be a minimum weight set contained in $V(K)$ such that $K_i^{(u,v)} - T_i$ is $\theta_c$-minor-free, and $w_i = w(T_i)$. Note that $T_i \subseteq V(K)$ implies that $T_i$ contains neither $u$ nor $v$. For example, $T_{c-1}$ is a minimum (internal) vertex cut separating $u$ and $v$ in $K^{(u,v)}$, and $w_{c-1} = w(T_{c-1})$ is finite since there is no edge between $u$ and $v$ in $K^{(u,v)}$. By definition, it holds that

$$0 = w_0 \leq w_1 \leq \cdots \leq w_{c-1} < \infty,$$

and by Lemma 8, these values can be computed in polynomial time. We also remark that $T_i$ is a $c$-bond cover of $K_i^{(u,v)}$ for all $i \leq j$.

We construct the $c$-outgrowth $K' = (K', u, v)$ so that $K^{(u,v)}$ is as follows (see Figure 2).

- $V(K^{(u,v)}) = \{u, v, x_1, \ldots, x_{c-1}\}$ where $K' = \{x_1, \ldots, x_{c-1}\}$. For each $1 \leq i \leq c - 1$, the weight of $x_i$ is $w_i$.

- There are edges $(u, x_1), (x_1, x_2), \ldots, (x_{c-2}, x_{c-1}), (x_{c-1}, v)$. Additionally for each $2 \leq i \leq c - 1$, there is an edge $(x_i, u)$.

Figure 2: The construction of the replacement $c$-outgrowth $K' = (K', u, v)$.

We observe that for each $i \in \{0, \ldots, c - 1\}$, the set $\{x_i\}$ is the minimum weight $c$-bond cover of $K_i^{(u,v)}$. The next claims are handy.
Claim 12. Let \((K, u, v)\) be a \(c\)-outgrowth in \(G\) and let \(M = (X, Y)\) be a minimal \(\theta_c\)-model in \(G\). If \(M\) does not contain \(u\), then we have \((X \cup Y) \cap V(K) = \emptyset\). Furthermore, if \(S\) is a minimal \(c\)-bond cover of \(G\) and if \(S\) contains \(u\) or \(v\), say \(u\), then \(S \cap V(K) = \emptyset\).

Proof of Claim. Consider a minimal \(\theta_c\)-models \(M = X \cup Y\) of \(G\), and suppose that \(M\) does not contain \(u\). If \(X \cup Y\) contains \(v\) (say \(v \in X\)), then \(Y\) is disjoint from \(V(K) \cup \{u, v\}\) because otherwise \(Y \subseteq K\) and \(M' = (X \cap (V(K) \cup \{v\}), Y)\) is a \(\theta_c\)-model in \(K^{(u,v)}\), a contradiction. From \(Y\) being disjoint from \(V(K) \cup \{u, v\}\), we deduce that \(M' = (X \setminus V(K)) \cup Y\) is a \(\theta_c\)-model disjoint from \(V(K)\). If \(X \cup Y\) contains neither \(u\) nor \(v\), it implies that both \(X\) and \(Y\) are disjoint from \(V(K) \cup \{u, v\}\). This proves the first claim.

To see the second claim, suppose the contrary, i.e. \(S \cap V(K) \neq \emptyset\). Then the minimality of \(S\) implies that \(S \setminus V(K)\) is not a \(c\)-bond cover of \(G\) and especially there is a minimal \(\theta_c\)-model \(M = (X, Y)\) in \(G - S \setminus V(K)\). However, the first claim implies that \(M\) is disjoint from \(K\), therefore it is already present in \(G - S\). This contradicts that \(S\) is a \(c\)-bond cover of \(G\), completing the proof.

Claim 13. Let \(Z\) be a \(c\)-bond cover of \(G - V(K)\) and let \(\ell\) be the maximum integer\(^6\) such that \(G - (K \cup Z)\) contains a \(\theta_{\ell}\)-model with \(u\) and \(v\) in different sets. Then \(Z' = Z \cup T_{\ell}\) is a \(c\)-bond cover of \(G\).

Proof of Claim. Suppose not, and let \(M = (X, Y)\) be a minimal \(\theta_c\)-model in \(G - Z'\) and note that \((X \cup Y) \cap V(K) \neq \emptyset\) since otherwise \(M\) is also a \(\theta_c\)-model in \(G - (V(K) \cup Z)\), a contradiction. Due to Claim 12 and the minimality of \(M\), we know that \(X \cup Y\) must contain both \(u\) and \(v\). Without loss of generality, let \(u \in X\). There are two possible cases.

Case A. \(u, v \in X\) and \(Y \subseteq K\): Note that \(G[X \setminus V(K)]\) is connected since otherwise we have the \(\theta_c\)-model \(M\) entirely contained in \(K^{(u,v)}\), a contradiction. This also implies that \(\ell \geq 1\). Let \(P\) be a \((u, v)\)-path in \(G[X \setminus V(K)]\). Because \(K^{(u,v)}_{1} - T_{\ell}\) is a minor of \(G[X \cup Y \cup (V(K) \setminus T_{\ell})]\) obtained by replacing \(P\) by an edge connecting \(u\) and \(v\), the graph obtained from \(M = (X, Y)\) by replacing \(P\) by an edge connecting \(u\) and \(v\) is a \(\theta_c\)-model in \(K^{(u,v)}_{1} - T_{\ell}\). However, \(K^{(u,v)}_{1} - T_{\ell}\) is a minor of \(K^{(u,v)}_{\ell} - T_{\ell}\), and \(K^{(u,v)}_{1} - T_{\ell}\) is \(\theta_c\)-minor-free by the construction of \(T_{\ell}\), a contradiction.

Case B. \(u \in X\) and \(v \in Y\): Observe that both \((X \cap V(K^{(u,v)}), Y \cap V(K^{(u,v)}))\) and \((X \setminus V(K), Y \setminus V(K))\) are a \(\theta_i\)-model and a \(\theta_j\)-model for some \(i, j \geq 1\) such that \(i + j \geq c\) and \(j \leq \ell < c\) by the choice of \(\ell\). This means that there exists a \(\theta_c\)-model in \(K^{(u,v)}_{j} - T_{\ell}\) because \(K^{(u,v)}_{j} - T_{\ell}\) is a minor of \(G[X \cup Y \cup (V(K) \setminus T_{\ell})]\) obtained by contracting \(X \setminus V(K)\) and \(Y \setminus V(K)\), and contracting these sets in \(M\) preserves the property of being a \(\theta_c\)-model.

\(^6\)If \(u\) and \(v\) are not connected in \(G - (K \cup Z)\), we let \(\ell = 0\).
However, $K_{j}^{(u,v)} - T_\ell$ is a minor of $K_{\ell}^{(u,v)} - T_\ell$ due to $j \leq \ell$, which contradict the construction of $T_\ell$.

So, in both cases we reach a desired contradiction. That is, $Z'$ is a $c$-bond cover of $G$. This proves the claim.

We begin with proving the second part of the statement. Let $G'$ be the graph where $K^{(u,v)}$ is replaced by $K^{(u,v)}$. It suffices to prove the second statement for an arbitrary minimal $c$-bond cover $S' \subseteq V(G')$ of $G'$.

First, assume that $S'$ contains $u$ or $v$, say $u$. Claim 12 is applied to $G'$ verbatim with $G \leftarrow G'$, $K \leftarrow K'$, $K^{(u,v)} \leftarrow K^{(u,v)}$, and we deduce that $S' \cap V(K') = \emptyset$. Now we take $S \leftarrow S'$, and let us argue that $S$ is a $c$-bond cover of $G$. Again Claim 12 implies that if $G - S$ contains a $\theta_c$-model, then one can find one disjoint from $V(K)$. This is not possible because $S = S'$ is a $c$-bond cover of $G - K = G' - K'$.

Secondly, let us assume that $S' \cap \{u, v\} = \emptyset$. Let $\ell$ be the maximum integer such that $G' - (K' \cup S')$ contains a $\theta_\ell$-model $M = (X,Y)$ with $u$ and $v$ in different sets, say $u \in X$ and $v \in Y$. Clearly $\ell$ is strictly smaller than $c$ because $S'$ is a $c$-bond cover of $G' - K'$. Note that $K^{(u,v)}_\ell$ is obtained from $G'[X \cup Y \cup V(K')]$ by contracting $X$ and $Y$. Because $S' \cap V(K')$ is a $c$-bond cover of $G'[X \cup Y \cup V(K')]$, it is also a $c$-bond cover of $K^{(u,v)}_\ell$. Therefore we have $w_\ell \leq w(S' \cap V(K'))$.

Let $S = (S' \setminus V(K')) \cup T_\ell$ be a vertex set of $G$ and note that $w(S) \leq w(S')$. Now applying Claim 13 to $G$ with $Z \leftarrow S' \setminus V(K')$ (as a vertex set of $G$), we conclude that $S$ is a $c$-bond cover of $G$. This proves the second part of the statement, which also establishes $\text{opt}(G) \leq \text{opt}(G')$ in the first part of the statement.

It remains to show $\text{opt}(G) \geq \text{opt}(G')$. Consider an optimal $c$-bond cover $S$ of $G$, and let $p$ be the maximum integer such that $G - (K \cup S)$ contains a $\theta_p$-model $M = (X,Y)$ with $u$ and $v$ in different sets. Again we apply Claim 13 with $G \leftarrow G'$, $Z \leftarrow S \setminus V(K)$ (as a vertex set of $G'$), $K \leftarrow K'$ and $T_\ell \leftarrow \{x_p\}$, and derive that $(S \setminus V(K)) \cup \{x_p\}$ is a $c$-bond cover of $G'$. Lastly, observe that $K^{(u,v)}_p$ is a minor of $G[X \cup Y \cup V(K)]$, and because $S \cap V(K)$ is a $c$-bond cover of the latter, it is also an $\alpha$-bond cover of the former. Therefore, we have $w(S \cap V(K)) \geq w_p$, from which we have $\text{opt}(G) = w(S) \geq w(S \cap V(K)) + w_p = w((S \setminus V(K)) \cup \{x_p\}) \geq \text{opt}(G')$. This finishes the proof.

\section{The primal-dual approach}

We begin the section by formalizing the notions of replacer and $\alpha$-thin layer. A $c$-\textit{outgrowth replacer} (hereinafter replacer) is a polynomial-time algorithm which, given a weighted graph $G = (V,E,w)$ and a $c$-outgrowth $K = (K,u,v)$ of size at least $c$, outputs a weighted graph $G' = (V',E',w')$ with the following property.
1. \( K \) is replaced by another \( c \)-outgrowth \( K' = (K', u, v) \) of size at most \( c - 1 \).
2. \( \text{opt}(G) = \text{opt}(G') \).
3. Given a \( c \)-bond cover \( S' \subseteq V(G') \), one can construct in polynomial time a \( c \)-bond cover \( S \subseteq V(G) \) such that \( w(S) \leq w(S') \).

An \( \alpha \)-thin layer of a weighted graph \( G = (V, E, w) \) is a weighted graph \( H = (V, E, w') \) such that the following holds.

- \( w'(v) \leq w(v) \) for every \( v \in V \),
- \( w'(v) = w(v) \) for some \( v \in V \), and
- \( w'(S) \geq (1/\alpha) \cdot w'(V) \) for any \( c \)-bond cover \( S \subseteq V \) of \( H \).

We are now ready to prove our main approximation result.

**Theorem 1.** There is a uniformly polynomial-time algorithm which, given a positive integer \( c \) and a weighted graph \( G = (V, E, w) \), computes a \( c \)-bond cover of weight at most \( \alpha \cdot \text{opt}(G) \) for some \( \alpha = \alpha(c) \).

**Proof.** The algorithm initially sets \( G_1 = G \), and iteratively constructs a sequence of weighted graphs \( G_i = (V_i, E_i, w_i) \) for \( i = 0, 1, \ldots \). At \( i \)-th iteration, we run the algorithm \( A \) of Theorem 2 for \( t = 8c \). Recall that one of the following is the output:

1. a \( c \)-outgrowth of size at least \( c \) or
2. a \( \theta_c \)-model \( M \) of \( G \) of size at most \( f_1(c, 8c) \) or
3. a cluster collection \( C \) of \( G \) of capacity at most \( f_1(c, 8c) \) where \( \delta(G/C) \geq 8c \), or
4. a report that \( G \) is \( \theta_c \)-minor-free.

If \( A \) detects a \( c \)-outgrowth of size at least \( c \), then we call the algorithm of Lemma 11, which is clearly a replacer. We run the replacer on \( G_i \) and set \( G_{i+1} \) to be the output of the replacer. If \( \theta_c \)-model \( M \) of \( G_i \) of size at most \( f_1(c, t) \) is detected by \( A \), then let \( \epsilon := \min\{w_i(v) : v \in M\} \) and consider the weighted graph \( H_i = (V_i, E_i, w_i') \) with the weight function \( w \) as follows:

\[
w_i'(v) = \begin{cases} 
\epsilon & \text{if } v \in M \\
0 & \text{otherwise} 
\end{cases}
\]

It is obvious that \( H_i \) is an \( \alpha \)-thin layer with \( \alpha = f_1(c, t) \).
In the third case, note that the cluster collection $\mathcal{C}$ forms a cluster partition of $G_i[\mathcal{U}\mathcal{C}]$. Consider the weight function $w : \mathcal{U}\mathcal{C} \to \mathbb{R}_{\geq 0}$ as in Equation 1 of $G_i[\mathcal{U}\mathcal{C}]$. Let $\epsilon := \min\{w_i(v)/w(v) : v \in \mathcal{U}\mathcal{C}\}$ and $H_i = (V_i, E_i, w_i^\epsilon)$ be the weighted graph, where

$$ w_i^\epsilon(v) = \begin{cases} 
\epsilon \cdot w(v) & \text{if } v \in \mathcal{U}\mathcal{C} \\
0 & \text{otherwise.}
\end{cases} $$

Let us verify that $H_i$ is an $\alpha$-thin layer of $G_i$ for $\alpha = 4r$, where $r = f_1(c,t)$. It is straightforward to see that the first two requirement of $\alpha$-thin layer are met due to the choice of $\epsilon$. To check the last requirement, consider an arbitrary $c$-bond cover $S \subseteq V_i$ of $H_i$. By Lemma 10, it holds that

$$ \frac{\epsilon}{2r} \cdot |E(G_i[\mathcal{U}\mathcal{C}]/\mathcal{C})| \leq \sum_{v \in S} w_i^\epsilon(v) \leq \sum_{v \in V_i} w_i^\epsilon(v) \leq 2\epsilon \cdot |E(G_i[\mathcal{U}\mathcal{C}]/\mathcal{C})|, $$

and therefore,

$$ \sum_{v \in S} w_i^\epsilon(v) \geq \frac{\epsilon}{2r} \cdot |E(G_i[\mathcal{U}\mathcal{C}]/\mathcal{C})| \geq \frac{1}{4r} \cdot \sum_{v \in V_i} w_i^\epsilon(v). $$

Now we set $G_{i+1}$ to be the weighted graph $(V_i, E_i, w_i^\epsilon)$ after removing all vertices of weight zero.

Finally, if $A$ reports that $G_i$ is $\theta_c$-minor free, then we terminate the iteration. Let $G = G_1, G_2, \ldots, G_\ell$ be the constructed sequence of weighted graph at the end, with $G_\ell$ being a $\theta_c$-minor-free graph. Observe that our algorithm strictly decrease the number of vertices before the $\ell$-th iteration, and thus $\ell \leq n$.

To establish the main statement, it suffices show that there is a polynomial-time algorithm which produces an $4r$-approximate solution for $G_i$ given an $4r$-approximate solution $T_{i+1}$ for $G_{i+1}$, where $r = f_1(c,t)$ and $t = 8c$. This trivially holds if the execution of $A$ at $i$-th iteration calls the replacer.

Suppose that $i$-th iteration produces an $\alpha$-thin layer $H_i = (V_i, E_i, w_i^\epsilon)$, and recall that every $\alpha$-thin layer produced in our algorithm satisfies $\alpha \leq 4r$. As $T_{i+1}$ is an $4r$-approximate solution for $G_{i+1}$, we have

$$ \text{opt}(G_{i+1}) \geq (1/4r) \cdot w_{i+1}(T_{i+1}), \quad (2) $$

Claim 14. $T_i := T_{i+1} \cup (V_i \setminus V_{i+1})$ is an $4r$-approximate solution for $G_i$.

Proof of Claim. Let $D_i = V_i \setminus V_{i+1}$, namely the vertices deleted from $G_i$ to obtain $G_{i+1}$. It is obvious that $T_{i+1} \cup D_i$ is a feasible solution for $G_i$, that is, a $c$-bond cover of $G_i$ because $G_{i+1} - T_{i+1} = G_i - (T_{i+1} \cup D_i)$ and $T_{i+1}$ is a $c$-bond cover of $G_{i+1}$. Let $Q \subseteq V_i$ be an
optimal solution for \( G_i \). Then \( Q \) is a feasible solution for \( H_i \) and \( Q \cap V_{i+1} \) is a feasible solution for \( G_{i+1} \), therefore

\[
\begin{align*}
w_i^Q(Q) & \geq (1/4r) \cdot w_i^Q(V_i) \quad \text{and} \\
w_{i+1}(Q \cap V_{i+1}) & \geq \text{opt}(G_{i+1}),
\end{align*}
\]

where the inequality 3 is due to the third requirement of \( \alpha \)-thin layer. Furthermore, it holds that

\[
\begin{align*}
w_i(v) &= w_i^Q(v) + w_{i+1}(v) \quad \text{for each } v \in V_{i+1} \quad \text{and} \\
w_i(v) &= w_i^Q(v) \quad \text{for each } v \in D_i.
\end{align*}
\]

Therefore,

\[
\begin{align*}
w_i(Q) &= w_i^Q(Q) + w_{i+1}(Q \cap V_{i+1}) \quad \because (5), (6) \\
& \geq (1/4r) \cdot w_i^Q(V_i) + \text{opt}(G_{i+1}) \quad \because (3), (4) \\
& \geq (1/4r) \cdot w_i^Q(T_{i+1} \cup D_i) + (1/4r) \cdot w_{i+1}(T_{i+1}) \\
& = (1/4r) \cdot (w_i^Q(T_{i+1}) + w_{i+1}(T_{i+1})) + (1/4r) \cdot w_i^Q(D_i) \\
& = (1/4r) \cdot w_i(T_{i+1} \cup D_i) \quad \because (5), (6)
\end{align*}
\]

and the claim follows. ♦

We inductively obtain a \( 4r \)-approximate solution for \( G_i \), and finally for the graph \( G_1 = G \). This finishes the proof.

7 Discussion

In this paper we construct a polynomial constant-factor approximation algorithm for the \textsc{Weighted } \( F \)-\textsc{Vertex Deletion} problem in the case \( F \) is the class of graphs not containing a \( c \)-bond or, alternatively, the \( \theta_c \)-minor free graphs. The constant-factor of our approximation algorithm is a (constructible) function of \( c \) and the running time is uniformly polynomial, that is it runs in time \( O_c(n^{O(1)}) \). Our results, in case \( c = 2 \), yield a constant-factor approximation for the \textsc{Vertex Weighted Feedback Set}. Also, a constant-factor approximation for \textsc{Diamond Hitting Set} can easily be derived for the case where \( c = 3 \). For this we apply our results on simple graphs and observe that each time a \( \theta_3 \)-minor-model appears, this model, under the absence of multiple edges, should contain 4 vertices and therefore is a minor-model of the diamond \( K_4^- \) (that is \( K_4 \) without an edge).

Certainly the general open question is whether \textsc{Weighted } \( F \)-\textsc{Vertex Deletion} admits a constant-factor approximation for more general instantiations of the minor-closed
class $\mathcal{F}$. In this direction, the challenge is to use our approach when the graphs in $\mathcal{F}$ have bounded treewidth (or, equivalently, if the minor obstruction of $\mathcal{F}$ contains some planar graph). For this, one needs to extend the structural result of Theorem 2 and, based on this to build a replacer as in Lemma 11.

Given an $r \in \mathbb{N}$, an $r$-protrusion of a graph $G$ is a set $X \subseteq V(G)$ such that $G[X]$ has treewidth at most $t$ and $|\partial_G(X)| \leq t$, were $\partial_G(X)$ is the set of vertices of $X$ that are incident to edges not in $G[X]$. We conjecture that a possible extension of Theorem 2 might be the following.

**Conjecture 15.** There are functions $f_3 : \mathbb{N}^2 \to \mathbb{N}$ and $f_4 : \mathbb{N}^3 \to \mathbb{N}$ such that, for every $h$-vertex planar graph $H$ and every two positive integers $t, p$, there is a uniformly polynomial time algorithm that, given as input a graph $G$, outputs one of the following:

1. an $f_3(h, t)$-protrusion $X$ of size at least $p$, or
2. a minor-model of $H$ of of size at most $f_4(h, t, p)$, or
3. a cluster collection $\mathcal{C}$ of $G$ of capacity at most $f_4(h, t, p)$ such that $\delta(G/\mathcal{C}) \geq t$, or
4. a report that $G$ is $H$-minor free.

Given a proof of some suitable version of Conjecture 15 at hand, cases 1, 2, and 3 above can be treated using the method proposed in this paper. In the first case, we need to find a weighted protrusion replacer that can replace, in the weighed graph $G = (V, E, w)$, the subgraph $G[X]$ by another one (glued on the same boundary) and create a new weighted graph $G' = (V', E', w')$ so that an optimal solution has the same weight in both instances. In our case, the role of a protrusion is played by the $c$-outgrowth, where $X$ is the vertex set of $K^{(u,v)}$ that has treewidth at most $2c$ and $|\partial_G(X)| \leq 2$, i.e., $V(K^{(u,v)})$ is a $2c$-protrusion of $G$. In the case of $\theta_c$, the the replacer is given in Lemma 11. The existence of such a replacer in the general case is wide open, first because the boundary $\partial_G(X)$ has bigger size (depending on $h$ but perhaps also on $t$) and second, and most important, because we now must deal with weights which does not permit us to use any protrusion replacement machinery such as the one used in $[22, 23]$ unweighted version of the problem (based on the, so called, FII-property [15] for more details).

We believe that a possible way to prove Conjecture 15 is to use as departure the proof of the main combinatorial result in [48]. However, in our opinion, the most challenging step is to design a weighted protrusion replacer (or, on the negative side, to provide instantiations of $H$ where such a replacer does not exist). As such a replacer needs to work on the presence of weights, we suggest that its design might use techniques related to mimicking networks technology (see [27, 36]).

Finally, since our algorithm is based on the primal-dual framework and proceeds by constructing suitable weights for the second and third case where every feasible solution
is $O(1)$-approximate, one can ask whether it is possible to bypass the need for a replacer and construct suitable weights for the first case. Indeed, the previous approximation algorithms for Weighted Feedback Vertex Set \cite{7,13,19} designed suitable weights even for the case 1 where every minimal solution is $O(1)$-approximate. (And used the additional “reverse delete” step at the end to ensure that the final solution remains minimal, for every weighted graph constructed.) In Appendix B, we show that such weights cannot exist for a simple planar graph $H$, which suggests that replacers are inherently needed for this class of algorithms for Weighted $F$-Vertex Deletion.

References

\cite{1} A. Agrawal, S. Kolay, D. Lokshtanov, and S. Saurabh. A faster FPT algorithm and a smaller kernel for block graph vertex deletion. In LATIN 2016: theoretical informatics, volume 9644 of Lecture Notes in Comput. Sci., pages 1–13. Springer, Berlin, 2016.

\cite{2} A. Agrawal, D. Lokshtanov, P. Misra, S. Saurabh, and M. Zehavi. Polylogarithmic approximation algorithms for weighted-$F$-deletion problems. In Approximation, randomization, and combinatorial optimization. Algorithms and techniques, volume 116 of LIPIcs. Leibniz Int. Proc. Inform., pages Art. No. 1, 15. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018.

\cite{3} A. Agrawal, D. Lokshtanov, P. Misra, S. Saurabh, and M. Zehavi. Feedback vertex set inspired kernel for chordal vertex deletion. ACM Trans. Algorithms, 15(1):Art. 11, 28, 2019.

\cite{4} A. Agrawal, P. Misra, S. Saurabh, and M. Zehavi. Interval vertex deletion admits a polynomial kernel. In Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 1711–1730. SIAM, Philadelphia, PA, 2019.

\cite{5} J. Ahn, E. Eiben, O.-j. Kwon, and S.-i. Oum. A polynomial kernel for 3-leaf power deletion. In 45th International Symposium on Mathematical Foundations of Computer Science, volume 170 of LIPIcs. Leibniz Int. Proc. Inform., pages 5:1–5:14. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2020.

\cite{6} J. Ahn, E. J. Kim, and E. Lee. Towards constant-factor approximation for chordal / distance-hereditary vertex deletion. In Y. Cao, S. Cheng, and M. Li, editors, 31st International Symposium on Algorithms and Computation, ISAAC 2020, December 14-18, 2020, Hong Kong, China (Virtual Conference), volume 181 of LIPIcs, pages 62:1–62:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020.

\cite{7} V. Bafna, P. Berman, and T. Fujito. A 2-approximation algorithm for the undirected feedback vertex set problem. SIAM J. Discret. Math., 12(3):289–297, 1999.
[8] N. Bansal, D. Reichman, and S. W. Umboh. LP-based robust algorithms for noisy minor-free and bounded treewidth graphs. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1964–1979. SIAM, Philadelphia, PA, 2017.

[9] R. Bar-Yehuda, K. Bendel, A. Freund, and D. Rawitz. Local ratio: A unified framework for approximation algorithms. in memoriam: Shimon even 1935-2004. *ACM Computing Surveys (CSUR)*, 36(4):422–463, 2004.

[10] R. Bar-Yehuda and S. Even. A local-ratio theorem for approximating the weighted vertex cover problem. In G. Ausiello and M. Lucertini, editors, *Analysis and Design of Algorithms for Combinatorial Problems*, volume 109 of *North-Holland Mathematics Studies*, pages 27–45. North-Holland, 1985.

[11] J. Baste, I. Sau, and D. M. Thilikos. A complexity dichotomy for hitting connected minors on bounded treewidth graphs: the chair and the banner draw the boundary. In S. Chawla, editor, *Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms, SODA 2020, Salt Lake City, UT, USA, January 5-8, 2020*, pages 951–970. SIAM, 2020.

[12] J. Baste, I. Sau, and D. M. Thilikos. Hitting minors on bounded treewidth graphs. I. general upper bounds. *SIAM J. Discret. Math.*, 34(3):1623–1648, 2020.

[13] A. Becker and D. Geiger. Optimization of pearl’s method of conditioning and greedy-like approximation algorithms for the vertex feedback set problem. *Artificial Intelligence*, 83(1):167–188, 1996.

[14] H. L. Bodlaender. A linear-time algorithm for finding tree-decompositions of small treewidth. *SIAM J. Comput.*, 25(6):1305–1317, 1996.

[15] H. L. Bodlaender, F. V. Fomin, D. Lokshtanov, E. Penninkx, S. Saurabh, and D. M. Thilikos. (meta) kernelization. *J. ACM*, 63(5):44:1–44:69, 2016.

[16] H. L. Bodlaender, J. van Leeuwen, R. B. Tan, and D. M. Thilikos. On interval routing schemes and treewidth. *Inf. Comput.*, 139(1):92–109, 1997.

[17] Y. Cao. Linear recognition of almost interval graphs. In *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1096–1115. ACM, New York, 2016.

[18] R. D. Carr, L. Fleischer, V. J. Leung, and C. A. Phillips. Strengthening integrality gaps for capacitated network design and covering problems. In D. B. Shmoys, editor, *Proceedings of the Eleventh Annual ACM-SIAM Symposium on Discrete Algorithms, January 9-11, 2000, San Francisco, CA, USA*, pages 106–115. ACM/SIAM, 2000.
[19] F. A. Chudak, M. X. Goemans, D. S. Hochbaum, and D. P. Williamson. A primal-dual interpretation of two 2-approximation algorithms for the feedback vertex set problem in undirected graphs. *Oper. Res. Lett.*, 22(4-5):111–118, 1998.

[20] H. Eto, T. Hanaka, Y. Kobayashi, and Y. Kobayashi. Parameterized algorithms for maximum cut with connectivity constraints. In B. M. P. Jansen and J. A. Telle, editors, *14th International Symposium on Parameterized and Exact Computation, IPEC 2019, September 11-13, 2019, Munich, Germany*, volume 148 of *LIPIcs*, pages 13:1–13:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.

[21] S. Fiorini, G. Joret, and U. Pietropaoli. Hitting diamonds and growing cacti. In *Integer Programming and Combinatorial Optimization – IPCO 2010*, volume 6080 of *Lecture Notes in Comput. Sci.*, pages 191–204. Springer, Berlin, 2010.

[22] F. V. Fomin, D. Lokshtanov, N. Misra, G. Philip, and S. Saurabh. Hitting forbidden minors: Approximation and kernelization. *SIAM J. Discret. Math.*, 30(1):383–410, 2016.

[23] F. V. Fomin, D. Lokshtanov, N. Misra, and S. Saurabh. Planar $F$-Deletion: approximation, kernelization and optimal FPT algorithms. In *2012 IEEE 53rd Annual Symposium on Foundations of Computer Science – FOCS 2012*, pages 470–479. IEEE Computer Soc., Los Alamitos, CA, 2012.

[24] F. V. Fomin, D. Lokshtanov, F. Panolan, S. Saurabh, and M. Zehavi. Hitting topological minors is FPT. In K. Makarychev, Y. Makarychev, M. Tulsiani, G. Kamath, and J. Chuzhoy, editors, *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing, STOC 2020*, Chicago, IL, USA, June 22-26, 2020, pages 1317–1326. ACM, 2020.

[25] F. V. Fomin, D. Lokshtanov, S. Saurabh, and M. Zehavi. Approximation schemes via width/weight trade-offs on minor-free graphs. In S. Chawla, editor, *Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms, SODA 2020*, Salt Lake City, UT, USA, January 5-8, 2020, pages 2299–2318. SIAM, 2020.

[26] A. Gupta, E. Lee, J. Li, P. Manurangsi, and M. Włodarczyk. Losing treewidth by separating subsets. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1731–1749. SIAM, Philadelphia, PA, 2019.

[27] T. Hagerup, J. Katajainen, N. Nishimura, and P. Ragde. Characterizing multiterminal flow networks and computing flows in networks of small treewidth. *J. Comput. Syst. Sci.*, 57(3):366–375, 1998.
[28] D. J. Haglin and S. M. Venkatesan. Approximation and intractability results for the maximum cut problem and its variants. *IEEE Trans. Computers*, 40(1):110–113, 1991.

[29] K. ichi Kawarabayashi, Y. Kobayashi, and B. Reed. The disjoint paths problem in quadratic time. *Journal of Combinatorial Theory, Series B*, 102(2):424–435, 2012.

[30] R. Impagliazzo, R. Paturi, and F. Zane. Which problems have strongly exponential complexity? *J. Comput. Syst. Sci.*., 63(4):512–530, 2001.

[31] B. M. P. Jansen and M. Pilipczuk. Approximation and kernelization for chordal vertex deletion. *SIAM J. Discrete Math.*, 32(3):2258–2301, 2018.

[32] G. Joret, C. Paul, I. Sau, S. Saurabh, and S. Thomassé. Hitting and harvesting pumpkins. *SIAM J. Discret. Math.*, 28(3):1363–1390, 2014.

[33] K. Kawarabayashi and A. Sidiropoulos. Polylogarithmic approximation for minimum planarization (almost). In C. Umans, editor, *58th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2017, Berkeley, CA, USA, October 15-17, 2017*, pages 779–788. IEEE Computer Society, 2017.

[34] E. J. Kim, A. Langer, C. Paul, F. Reidl, P. Rossmanith, I. Sau, and S. Sikdar. Linear kernels and single-exponential algorithms via protrusion decompositions. *ACM Trans. Algorithms*, 12(2):21:1–21:41, 2016.

[35] S. Kratsch and M. Wahlström. Compression via matroids: a randomized polynomial kernel for odd cycle transversal. *ACM Trans. Algorithms*, 10(4):Art. 20, 15, 2014.

[36] R. Krauthgamer and I. Rika. Mimicking networks and succinct representations of terminal cuts. In S. Khanna, editor, *Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2013, New Orleans, Louisiana, USA, January 6-8, 2013*, pages 1789–1799. SIAM, 2013.

[37] J. M. Lewis and M. Yannakakis. The node-deletion problem for hereditary properties is NP-complete. *J. Comput. System Sci.*, 20(2):219–230, 1980. ACM-SIGACT Symposium on the Theory of Computing (San Diego, Calif., 1978).

[38] S. Limnios, C. Paul, J. Perret, and D. M. Thilikos. Edge degeneracy: Algorithmic and structural results. *Theor. Comput. Sci.*, 839:164–175, 2020.

[39] D. Lokshhtanov, P. Misra, F. Panolan, G. Philip, and S. Saurabh. A $(2 + \epsilon)$-factor approximation algorithm for split vertex deletion. In A. Czumaj, A. Dawar, and E. Merelli, editors, *47th International Colloquium on Automata, Languages, and Programming, ICALP 2020, July 8-11, 2020, Saarbrücken, Germany (Virtual Conference)*, volume 168 of *LIPIcs*, pages 80:1–80:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020.
[40] A. Louis, P. Raghavendra, and S. Vempala. The complexity of approximating vertex expansion. In *2013 IEEE 54th Annual Symposium on Foundations of Computer Science*, pages 360–369. IEEE, 2013.

[41] C. Lund and M. Yannakakis. The approximation of maximum subgraph problems. In A. Lingas, R. G. Karlsson, and S. Carlsson, editors, *Automata, Languages and Programming, 20nd International Colloquium, ICALP93, Lund, Sweden, July 5-9, 1993, Proceedings*, volume 700 of *Lecture Notes in Computer Science*, pages 40–51. Springer, 1993.

[42] G. L. Nemhauser and L. E. T. Jr. Properties of vertex packing and independence system polyhedra. *Math. Program.*, 6(1):48–61, 1974.

[43] B. Reed, K. Smith, and A. Vetta. Finding odd cycle transversals. *Oper. Res. Lett.*, 32(4):299–301, 2004.

[44] N. Robertson and P. D. Seymour. Graph minors. V. excluding a planar graph. *J. Comb. Theory, Ser. B*, 41(1):92–114, 1986.

[45] N. Robertson and P. D. Seymour. Graph minors. XX. Wagner’s conjecture. *J. Comb. Theory, Ser. B*, 92(2):325–357, 2004.

[46] I. Sau, G. Stamoulis, and D. M. Thilikos. An fpt-algorithm for recognizing k-apices of minor-closed graph classes. In A. Czumaj, A. Dawar, and E. Merelli, editors, *47th International Colloquium on Automata, Languages, and Programming, ICALP 2020, July 8-11, 2020, Saarbrücken, Germany (Virtual Conference)*, volume 168 of *LIPIcs*, pages 95:1–95:20. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020.

[47] A. Thomason. The extremal function for complete minors. *J. Comb. Theory Ser. B*, 81(2):318–338, 2001.

[48] W. C. van Batenburg, T. Huynh, G. Joret, and J. Raymond. A tight erdős-pósa function for planar minors. In T. M. Chan, editor, *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, January 6-9, 2019*, pages 1485–1500. SIAM, 2019.
A The case of big minimum edge-degree

It is known that if $\mathcal{F}$ is a non-trivial minor-closed graph class, then there is some $d_\mathcal{F}$ such that for every (multi) graph in $\mathcal{F}$ it holds that $|E(G)| \leq \mu(G) \cdot d_\mathcal{F} \cdot |V(G)|$ (see e.g. [47]).

**Lemma 16.** Let $G$ be an instance of $\mathcal{F}$-Vertex Deletion where $\mu(G) \leq m$ and $\delta(G) \geq 3m \cdot d_\mathcal{F}$. Then for every solution $X$ of $\mathcal{F}$-Vertex Deletion on $G$, it holds that

$$|E(G)| \leq 2 \sum_{v \in X} \text{edeg}_G(v) \leq 4|E(G)|.$$

**Proof.** We prove the first inequality as the second one is obvious. Let $F := G - X$, $d := d_\mathcal{F}$, and $b := \delta(G) \geq 3md$. Observe that

$$|E(X, F)| = \sum_{v \in F} \text{edeg}_G(v) - \sum_{v \in F} \text{edeg}_F(v)$$

$$\geq b \cdot |V(F)| - 2|E(F)|$$

$$\geq b \cdot |V(F)| - 2md \cdot |V(F)| \quad \therefore |E(F)| \leq md \cdot |V(F)|$$

$$\geq |E(F)| \quad \therefore b \geq 3md$$

Therefore, we deduce

$$\sum_{v \in X} \text{edeg}_G(v) = 2|E(X)| + |E(X, F)|$$

$$= \frac{1}{2} |E(X)| + \frac{1}{2} |E(X, F)|$$

$$\geq \frac{1}{2} \left( |E(X)| + |E(X, F)| + |E(F)| \right)$$

$$= \frac{1}{2} \cdot |E(G)|$$

$$= \frac{1}{4} \cdot \sum_{v \in V(G)} \text{edeg}_G(v)$$

$$= \frac{1}{2} |E(G)|$$

as claimed. \hfill \Box

In the special case where $\mathcal{F}$ is the class of $\theta_c$-free graphs, we know, from Proposition 4.i, that every simple $\theta_c$-minor free graph $G$ satisfies $E(G) \leq 2c \cdot |V(G)|$. As for such graphs $\mu(G) < c$, we can apply the above lemma for $d_\mathcal{F} = 2$. Therefore, given an instance $G$ of (unweighted) $c$-BOND COVER where $\delta(G) \geq 6c$, every feasible solution $X$ is 4-approximation. This implies that if we set $w(v) := \text{edeg}_G(v)$ for every $v \in V$, every feasible solution to WEIGHTED $c$-BOND COVER is a 4-approximate solution.
B  Necessity of a replacer

The previous approximation algorithms for Weighted Feedback Vertex Set, at least the ones stated in [7, 13, 19], did not use the notion of a replacer that we introduce here. Instead, these algorithms use the primal-dual framework by always finding suitable weightings \( w : V(G) \to \mathbb{R}_{\geq 0} \) of the current graph \( G \) such that any minimal feedback vertex set (i.e., cycle cover) for \( G \) is a 2-approximate solution with respect to \( w \). (In this section, minimality is defined with respect to set inclusion.) The existence of such a weighting \( w \) relies on the fact that, for any induced path \( P \) of \( G \), any minimal feedback vertex set has at most one internal vertex from \( P \). We prove the following lemma revealing that such weights cannot exist for \( H \)-minor cover for some planar graph \( H \). (In fact, every graph \( H \) that satisfies the mild technical conditions (1) and (2) in the proof.)

Lemma 17. There exists a fixed planar graph \( H \) such that the following holds. For infinitely many values of \( n \), there exists an \( n \)-vertex graph \( G \) such that for any weighting \( w : V(G) \to \mathbb{R}_{\geq 0} \), there exists a minimal \( H \)-minor cover \( S \subseteq V(G) \) of \( G \) such that \( w(S) = \Omega(n) \cdot \text{opt}(G) \).

Proof. Let \( H \) be any 2-connected planar graph with treewidth at least five (for this, one may just take the \((5 \times 5)\)-grid). Choose \( s, t \in V(H) \) such that \( \{s, t\} \in E(H) \). Let \( k \) be any parameter bigger than five. The graph \( G \) is constructed as follows.

- Let \( R = \{0, 1, 2, 3\} \times [k] \) and \( V(G) = V(H) \cup R \). So \( n = 4k + |V(H)| \).

- \( E(G) \) consists of the following edges (\( \oplus \) and \( \ominus \) denotes the addition and subtraction modulo 4 respectively).
  - the edges in \( E(H) \setminus \{\{s, t\}\} \).
  - \( s \) is adjacent to \( (i, 1) \) and \( t \) is adjacent to \( (i, k) \) for each \( i \in \{0, 1, 2, 3\} \).
  - \( G[R] \) is the \((4 \times k)\)-grid. Formally, for \( 1 \leq j \leq k \) and \( 0 \leq i < 3 \), \( E(G) \) contains the edge \( \{(i, j), (i \oplus 1, j)\} \). Moreover, for \( 1 \leq j < k \) and \( 0 \leq i < 3 \), \( E(G) \) contains the edges \( \{(i, j), (i, j + 1)\} \).

Let \( Z := G[\{s, t\} \cup R] \). We need the following two properties of \( H \) (we remark that the lemma is true for any \( H \) that satisfies them):

1. \( H \) is 2-vertex connected and
2. \( H \) is not a minor of \( Z \).

For the above, it is easy to verify that \( Z = G[\{s, t\} \cup R] \) has treewidth at most four, therefore it cannot contain as a minor the graph \( H \) that has treewidth at least five.
For $1 \leq j \leq k$, let $Y_j = \{(0, j), (1, j), (2, j), (3, j)\}$ and notice that $Y_i$ is a minimal $s$-$t$ separator of $R$ (that is $s$ and $t$ are in different connected components of $R - S$ and none of its subsets has this property). In the proof of the following claim, by $H$-minor-model we mean a cluster collection $\mathcal{M} = \{M_v \mid v \in V(H)\}$ of $G$ such that $G/\mathcal{M}$ is equal to $H$ with possibly some additional edges.

**Claim 18.** If $S$ is a minimal $s$-$t$ separator of $Z$, then $S$ is a $H$-minor cover of $G$.

**Proof of Claim.** Once $S$ is removed from $G$, there is no $s$-$t$ path in $Z$. Let $A \subseteq R$ be the vertices connected to $s$ and $B \subseteq R$ be the vertices connected to $t$ (so $(A, B, S)$ is a partition of $R$). Assume towards contradiction that there exists a $H$-minor-model $\mathcal{M} = \{M_v \mid v \in V(H)\}$ of $G' = G - S$. We consider the following cases.

- If there exists some $v \in V(H)$ such that $M_v \subseteq A$, the 2-vertex-connectivity of $H$ ensures that there is no $w \in V(H)$ such that $M_w \subseteq V(G - S) \setminus (A \cup \{s\})$, since $s$ is a singleton vertex cut separating $A$ and $V(G - S) \setminus (A \cup \{s\})$. By letting $M_v \gets M_v \cap (A \cup \{s\})$ for each $v \in V(H)$, this implies that there exists a $H$-model contained in $A \cup \{s\}$, which contradicts the fact $H$ is not a minor of $G[\{s\} \cup R]$. Similarly, there is no $v \in V(H)$ such that $M_v \subseteq B$.
- Then there is no $v \in V(H)$ such that $M_v \subseteq A$ or $M_v \subseteq B$. In that case, letting $M_v \gets M_v \setminus (A \cup B)$ for each $v \in V(H)$ gives a $H$-model of $G - S$ entirely contained in $V(G - S) \setminus (A \cup B)$. But note that $G - (S \cup A \cup B) = G - R$ is exactly $H$ minus one edge, so contradiction.

On the other hand, we define, for $0 \leq i < 3$, the sets

$$X_i = (\{i \oplus 1\} \times \{2, \ldots, k - 1\}) \cup (\{i \oplus 1\} \times \{2, \ldots, k - 1\}) \cup \{(i, k), (i \oplus 2, 1)\}.$$ 

By construction, in $Z$, each $X_i$ is a minimal $s$-$t$ separator. Therefore, $X_i$ is a minimal $H$-minor-cover of $G$.

For any $w : V(G) \to \mathbb{R}_{\geq 0}$, note that $\sum_{j=1}^{k} w(Y_j) \leq \frac{1}{4} \sum_{i=0}^{3} w(X_i)$. Indeed, for each $(i, j) \in R$, $(i, j)$ appears at least twice in $\{X_0, X_1, X_2, X_3\}$ and exactly once in $\{Y_j : 1 \leq j \leq k\}$. Since each $Y_j$ is a $H$-minor cover, it holds that

$$k \cdot \text{opt}^{(w)}(G) \leq \sum_{j=1}^{k} w(Y_j) \leq \frac{1}{2} \sum_{i \in \{0, 1, 2, 3\}} w(X_i).$$

(Here we denote by $\text{opt}^{(w)}(G)$ the optimum value attained with respect to the weighting $w$.) Therefore, for some $i$, $X_i$ is a minimal $H$-cover whose weight is at least $\frac{k}{2} \cdot \text{opt}^{(w)}(G)$. As $|V(G)| = 25 + 4k$, the lemma follows. □
We stress that in the above proof, the treewidth of $G$ is always equal to five (that is the treewidth of $H$). This means that one may not expect a better behaviour than the one testified by Lemma 17, even when restricted to graphs of fixed treewidth. Notice also that the graph $G$ is not planar (while the graph $Z$ is). Any construction of a planar $G$, as in Lemma 17, seems to be an interesting challenge.