Black hole solutions in $d = 5$ Chern-Simons gravity

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Abstract: The five dimensional Einstein-Gauss-Bonnet gravity with a negative cosmological constant becomes, for a special value of the Gauss-Bonnet coupling constant, a Chern-Simons (CS) theory of gravity. In this work we discuss the properties of several different types of black object solutions of this model. Special attention is paid to the case of spinning black holes with equal-magnitude angular momenta which possess a regular horizon of spherical topology. Closed form solutions are obtained in the small angular momentum limit. Nonperturbative solutions are constructed by solving numerically the equations of the model. Apart from that, new exact solutions describing static squashed black holes and black strings are also discussed. The action and global charges of all configurations studied in this work are obtained by using the quasilocal formalism with boundary counterterms generalized for the case of a $d = 5$ CS theory.

Keywords: Chern-Simons theory of gravity, black holes, numerical solutions
1. Introduction

It is known that in dimensions higher than four, the Einstein theory of gravity can be generalized by including in the action a linear combination of all lower dimensional Euler densities [1]. This gives rise to the so-called Lovelock models of gravity which possess some special features. For example, the field equations still obey generalized Bianchi identities and are linear in the second order derivatives of the metric (this being also the maximal order). Moreover, the Lovelock models are known to be free of ghosts when expanding around a flat space background, thus avoiding problems with unitarity [2], [3].
However, as first discussed in [4], in odd spacetime dimensions there exist a special combination of the terms which enter the corresponding Lovelock gravity model that allows to write the corresponding Lagrangeans as Chern-Simons (CS) densities. The resulting models retain all properties of the Lovelock gravity, exhibiting at the same time some new interesting features. For example, the CS theory possesses an enhanced local symmetry and can be reformulated as a gauge theory of gravity.

In this work we shall restrict ourselves to the case of a CS theory in \( d = 5 \) spacetime dimensions with a negative cosmological constant. In this case, the Lovelock model of gravity corresponds to the so-called Einstein-Gauss-Bonnet (EGB) theory, which contains quadratic powers of the curvature. As discussed by many authors, the inclusion of a GB term in the gravity action leads to a variety of new features (see [5], [6] for reviews of these models in the larger context of higher order gravity theories). In particular, the black holes of EGB theory do not in general obey the Bekenstein-Hawking area law, but the entropy formula includes a new contribution coming from the higher curvature terms in the action [7, 8].

The \( d = 5 \) solutions of the EGB equations with an Anti-de Sitter (AdS) spacetime background have been extensively studied in the recent years, mainly motivated by the conjectured AdS/CFT correspondence\(^1\) [12]. Then a set of the EGB-AdS solutions with special a value of the GB coupling constant \( \alpha \) fixed by the cosmological constant \( \Lambda \), corresponds also to solutions of the CS model. However, a search in the literature shows that the only solutions of the CS model which have been extensively studied are the counterparts of the Schwarzschild black holes (see e.g. [13], [14], [15]). Moreover, several classes of solutions which are known to exist in Einstein gravity (e.g. black strings and spinning black holes) are still missing in this case. Also, at a technical level, the solutions of the CS model are rather special, due to the fact that for the specific ratio between the \( \alpha \) and \( \Lambda \), the equations of motion of the Lovelock theory become somehow degenerate. As we shall see, this leads to the existence of exact solutions in a number of cases where no closed form solutions could be found in the EGB case. Moreover, some results found in the generic EGB model cannot be safely extrapolated to the particular case of a CS model.

The main purpose of this work is to provide a discussion of several different classes of solutions of the \( d = 5 \) CS model, looking for generic properties. In the static case, we consider generalizations of the squashed Schwarzschild-AdS black holes in [16], [17], as well as the CS counterparts of the asymptotically AdS black strings in [18], [19]. Different from the case of pure Einstein gravity, we are able to find in these case a simple analytic expression of the solutions.

Apart from these static solutions, we consider also rotating black holes with equal magnitude angular momenta and possessing a spherical horizon topology. Since in this case we could not find closed form solutions (except in the slowly rotating limit), these configurations are constructed numerically by matching the near horizon expansion of the metrics to their asymptotic form.

\(^1\)In this framework, the introduction of higher order terms in the gravity action corresponds to next to leading order corrections to the \( 1/N \) expansion of the boundary dual theory [4], [10], [11].
The paper is organized as follows. In the next Section we explain the CS model and describe the computation of the physical quantities of the solutions such as their action and mass-energy. Our proposal for the general counterterm expression in $d = 5$ CS theory is also presented there. The next two Sections contains applications of the general formalism. In Section 3 we give our results for the static black holes and black strings solutions. Then in Section 4 we discuss the basic properties of the spinning black hole solutions with an $S^3$ event horizon topology. We conclude in Section 5 with some further remarks. The Appendices contain technical details on the rotating black holes together with an exact solution describing a singular spinning configuration.

2. The model

2.1 The action

We start by considering the five dimensional Einstein-Hilbert action with a negative cosmological constant, supplemented by quadratic terms:

$$I = \frac{1}{16\pi G} \int_M \sqrt{-g} \left[ R - 2\Lambda + \frac{1}{4} \left( \alpha_1 R^2 + \alpha_3 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R_{\mu\sigma\kappa\tau} R^{\mu\sigma\kappa\tau} \right) \right] d^5 x, \quad (2.1)$$

where $g$ is the determinant of the metric tensor $g_{\mu\nu}$, $R$ is the Ricci scalar, $R_{\mu\nu}$ is the Ricci tensor, $R_{\mu\sigma\kappa\tau}$ is the Riemann tensor and $\Lambda = -6/\ell^2$ is the cosmological constant.

The case

$$\alpha_1 = -\frac{1}{4} \alpha_2 = \alpha_1 = \alpha \quad (2.2)$$

is special, the quadratic part in (2.1) becoming the Lagrangean of the Gauss-Bonnet gravity,

$$L_{GB} = R^2 - 4R_{\mu\nu} R^{\mu\nu} + R_{\mu\sigma\kappa\tau} R^{\mu\sigma\kappa\tau}. \quad (2.3)$$

The constant $\alpha$ in (2.2) is the GB coefficient with dimension $(length)^2$ and is positive in the string theory. For the case of a CS theory in this work, its value is fixed by the cosmological constant,

$$\alpha = -\frac{3}{\Lambda} = \ell^2/2. \quad (2.4)$$

Then, as first discussed in [4], the resulting model can be thought as a higher-dimensional generalization of the well-known CS formulation of the three-dimensional Einstein gravity [20]. Without entering into details, we mention that to show this construction explicitly, it is necessary to employ the first order formalism in terms of the spin connection $w^{ab} = w^{ab}_{\mu} dx^\mu$ with the veilbeins $e^a_\mu$ which define the metric $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$ and the one forms $e^a = e^a_\mu dx^\mu$. Then $R^{ab} = dw^{ab} + w^a_\sigma w^\sigma b$ is the curvature two form (where the wedge product between forms is understood) and (2.1) can be written as a CS form for the AdS group [4]

$$I = \frac{1}{16\pi G} \int_M Tr \left\{ AdAdA + \frac{3}{2} dAA^3 + \frac{3}{5} A^5 \right\}, \quad (2.5)$$

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where

\[ A^{ab} = \begin{pmatrix} \omega^{ab} & e^a/l \\ -e^b/l & 0 \end{pmatrix}, \tag{2.6} \]

with \(a, b\) running here from 1 to 6 (thus \(A^{ab}\) is a six-dimensional one form). One should mention that different from the three dimensional CS gravity theory \([20]\) which has no propagating degrees of freedom, the model (2.5) is a fully interacting theory. Also, it is clear that this theory does not present a limit where the standard general relativity is recovered. The corresponding supergravity generalizations have been constructed in \([21]\), \([22]\), \([23]\) (see \([24]\) for an extensive introduction to the subject of CS gravity and supergravity).

### 2.2 Field equations and boundary terms

For the purposes of this work, however, it is more convenient to use the usual formulation of the model in a coordinate basis. Then the variation of the (2.1) with respect to the metric tensor \(g_{\mu\nu}\) (with the choices (2.2), (2.4) of the coupling constants) results in the field equations

\[ E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{6}{\ell^2} g_{\mu\nu} \]

\[ + \frac{\ell^2}{8} \left( 2(R_{\mu\sigma\kappa\tau} R_{\nu}^{\sigma\kappa\tau} - 2R_{\mu\nu\sigma} R^{\sigma\nu} - 2R_{\mu\sigma} R^{\sigma\nu} + RR_{\mu\nu} - \frac{1}{2} L_{GB} g_{\mu\nu} \right) = 0. \tag{2.7} \]

In this work we are interested in solutions of (2.7) approaching asymptotically a spacetime of negative constant curvature. This implies the asymptotic expression of the Riemann tensor\(^2\):

\[ R_{\mu\nu}^{\lambda\sigma} = -\left( \delta_{\mu}^{\lambda} \delta_{\nu}^{\sigma} - \delta_{\mu}^{\sigma} \delta_{\nu}^{\lambda} \right)/\ell_{\text{eff}}^2, \]

where \(\ell_{\text{eff}} = \ell/\sqrt{2}\) is the new effective radius of the AdS space in CS theory. It is worth mentioning that this asymptotically locally AdS condition reflects a local property at the boundary, but it does not restrict the global topology of the spacetime manifold. In fact, there is a wide class of solutions that satisfy this condition, including the (squashed) black holes and black strings in this work. Also, similar to the pure Einstein gravity case, the solutions can be written in Gauss-normal coordinates and admit a Fefferman-Graham–like asymptotic expansion \([24]\).

Returning to the issue of model’s action, we mention that for a well-defined variational principle with a fixed metric on the boundary, one has to supplement (2.1) with a boundary term which is the sum of the the Gibbons-Hawking surface term \([27]\) and its counterpart for GB gravity \([28]\):

\[ I_b = -\frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^4x \sqrt{-\gamma} \left( K + \frac{\ell^2}{4} (J - 2 G_{ab} K^{ab}) \right). \tag{2.8} \]

In the above relation, \(\gamma_{ab}\) is the induced metric on the boundary, \(K_{ab}\) is the extrinsic curvature tensor of the boundary and \(K\) the trace of this tensor, \(G_{ab}\) is the Einstein tensor of the metric \(\gamma_{ab}\) and \(J\) is the trace of the tensor

\[ J_{ab} = \frac{1}{3} (2K K_{ac} K^c_b + K_{cd} K^{cd} K_{ab} - 2K_{ac} K^{cd} K_{db} - K^2 K_{ab}) \]. \tag{2.9} \]

\(^2\)A precise definition of asymptotically AdS spacetime in higher curvature gravitational theories is given e.g. in \([25]\).
2.3 The counterterms and boundary stress tensor

The action and global charges of the solutions in this work are computed by using the procedure proposed by Balasubramanian and Kraus [29] for the case of the Einstein gravity with negative cosmological constant. This technique was inspired by the AdS/CFT correspondence (since quantum field theories in general contain counterterms) and consists of adding to the action suitable boundary counterterms $I_{ct}$, which are functionals only of curvature invariants of the induced metric on the boundary. These counterterms remove all power-law divergencies from the on-shell action, without being necessary to specify a background metric.

In this approach, we supplement the general action (which consist in (2.1) together with the boundary term (2.8)) with the following boundary counterterm

$$I_{ct} = \frac{1}{8\pi G} \int_{\partial \mathcal{M}} d^4x \sqrt{-\gamma} \left( \frac{2\sqrt{2}}{\ell} - \frac{\ell}{2\sqrt{2}} R \right),$$

(2.10)

where $R$ and $R^{ab}$ are the curvature and the Ricci tensor associated with the induced metric $\gamma$. The expression (2.10) is obtained by taking the limit $\alpha \to \ell^2/2$ in the general EGB counterterm expression proposed in [32], [33].

Then one can define a boundary stress-energy tensor which is the variation of the total action with respect to the boundary metric, $T_{ab} = \frac{2}{\sqrt{-\gamma}} \frac{\delta}{\delta \gamma_{ab}} (I + I_b + I_{ct})$. Its explicit expression is

$$T_{ab} = \frac{1}{8\pi G} \left[ K_{ab} - \gamma_{ab} K + \frac{\ell^2}{4} (Q_{ab} - \frac{1}{3} Q \gamma_{ab}) - \frac{2\sqrt{2}}{\ell} \gamma_{ab} + \frac{\ell}{\sqrt{2}} \left( R_{ab} - \frac{1}{2} \gamma_{ab} R \right) \right],$$

(2.11)

where we define

$$Q_{ab} = 2K_{ac}K^c_b - 2K_{ac}K^{cd}K_{db} + K_{ab}(K_{cd}K^{cd} - K^2) + 2KR_{ab} + RK_{ab} - 2K^{cd}R_{cadb} - 4R_{ac}K^c_b.$$

(2.12)

The computation of the global charges associated with the Killing symmetries of the boundary metric is done in a similar way to the well-known case of Einstein gravity [29]. The boundary submanifold of all solutions in this work can be foliated in a standard ADM form, with $\gamma_{ab} dx^a dx^b = -N^2 dt^2 + \sigma_{ij}(dy^i + N^i dt)(dy^j + N^j dt)$, where $N$ and $N^i$ are the lapse function, respectively the shift vector, and $y^i$ are the intrinsic coordinates on a closed surface $\Sigma$ of constant time $t$ on the boundary. Then a conserved charge

$$\Omega_\xi = \oint_{\Sigma} d^3y \sqrt{\sigma} u^a \xi^b T_{ab},$$

(2.13)

can be associated with the closed surface $\Sigma$ (with normal $u^a$), provided the boundary geometry has an isometry generated by a Killing vector $\xi^a$. For example, the conserved mass/energy $M$

\footnote{A different construction of the holographic stress tensor in CS gravity can be found in [30], [31], together with a discussion of the holographic anomalies.}
is the charge associated with the time translation symmetry (with \( \xi = \partial/\partial t \)), being computed from the \( tt \)-component of the boundary stress tensor.

Once the global charges are computed, the thermodynamics of the vacuum black objects in this work is formulated by using the standard Euclidean approach \[^s2\]. In the semiclassical approximation, the partition function is given by \( Z \simeq e^{-I_{cl}} \), where \( I_{cl} \) is the classical action (i.e. the sum of \( I \), \( I_b \) and \( I_{ct} \)) evaluated on the equations of motion. As usual, the Hawking temperature \( T_H \) of a black object is found by demanding regularity of the Euclideanized manifold, or equivalently, by evaluating the surface gravity. Upon application of the Gibbs-Duhem relation to the partition function (see e.g. \[^s3\]), this yields an expression for the entropy (with \( \beta = 1/T_H \))

\[
S = \beta (\mathcal{M} - \mu_i \mathfrak{C}_i) - I_{cl},
\]

with chemical potentials \( \mathfrak{C}_i \) and conserved charges\(^4\) \( \mu_i \). The entropy can also be written in Wald’s form \[^s4\] as an integral over the event horizon

\[
S = \frac{1}{4G} \int_{\Sigma_h} d^3x \sqrt{-\tilde{h}} (1 + \frac{\ell^2}{4} \tilde{R}),
\]

(where \( \tilde{h} \) is the determinant of the induced metric on the horizon and \( \tilde{R} \) is the event horizon curvature). As usual, the first law of thermodynamics

\[
dS = \beta (d\mathcal{M} - \mu_i d\mathfrak{C}_i). \tag{2.16}
\]

provides an important test of the consistency of results.

The background metric upon which the dual field theory resides is defined as \( h_{ab} = \lim_{r \to \infty} \frac{\ell^2}{2\pi r} \gamma_{ab} \). The expectation value of the stress tensor of the dual theory can be computed using the relation \[^s5\]:

\[
\sqrt{-h} h^{ab} <\tau_{bc}> = \lim_{r \to \infty} \sqrt{-\gamma} \gamma^{ab} T_{bc}. \tag{2.17}
\]

We close this part by mentioning the existence of a complementary regularization scheme, the so-called ‘Kounterterm’ approach \[^s6\]. There are some important differences in this case. First, the ‘Kounterterm’ approach is more naturally associated with a variational principle where the extrinsic curvature \( K_{ij} \) is kept fixed on the boundary. Also, the boundary counterterms are constructed in terms of both the extrinsic and intrinsic curvature tensors. The case of a CS-AdS\(_5\) gravity is discussed in \[^s7\], the corresponding ‘Kounterterm’ expression being

\[
I_{Kt} = \frac{1}{8\pi G} \frac{\ell^2}{8} \int_{\partial\mathcal{M}} d^4x \sqrt{-\gamma} \left[ \delta^{c_1a_1b_1}_{b_2a_2a_3} K^{b_1}_{a_1} (R^{b_2b_3}_{a_2a_3} - K^{b_2}_{a_2} K^{b_3}_{a_3} + \frac{2}{3\ell^2} \gamma^{b_2a_2} \gamma^{b_3a_3} \right]. \tag{2.18}
\]

The action of the model consists in this case the sum of (2.1) together with the boundary term (2.18) (note that the usual boundary term (2.8) is absent due to the chosen set of

\(^4\)The expression of \( \mathfrak{C}_i, \mu_i \) depends on the physical situation. For example, the spinning black holes in Section 4 have \( \mathfrak{C} = \Omega_{1,2} = \Omega_H \) and \( \mu = J_{1,2} = J \).
boundary conditions). Given a solution of the field equations, once the action is computed
in this way, the global charges can be computed using the Noether theorem, or by using the
Euclidean black hole thermodynamics. Further details on this formalism together with some
examples and the generalization to the case \( d = 2n + 1 \) \((n \geq 2)\) can be found in [39], [40].

3. Static black objects in \( d = 5 \) CS theory

3.1 The Schwarzschild-CS solution

The counterparts of the Schwarzschild solution for a \( d = 5 \) CS theory of gravity have been
discussed already in the literature. For completeness, we review here their basic properties,
together with a derivation of their mass and entropy by using the general formalism described
above.

Here one starts by considering the following line element

\[
d s^2 = \frac{d r^2}{f(r)} + r^2 d \Sigma^2_{k,3} - b(r) d t^2
\]

where \( d \Sigma^2_{k,3} \) is the line element of a three-dimensional manifold \( \Sigma_{k,3} \)

\[
d \Sigma^2_{k,3} = \begin{cases} d \Omega^2_3 & \text{for } k = +1 \\ \sum_{i=1}^3 dx_i^2 & \text{for } k = 0 \\ d \Xi^2_3 & \text{for } k = -1 \end{cases}
\]

\( d \Omega^2_3 \) denoting the unit metric on \( S^3 \); by \( H^3 \) we will understand the three–dimensional hyperbolic space.

The solution with

\[
b(r) = f(r) = \frac{2 r^2}{\ell^2} + k,
\]

corresponds to the AdS backgrounds in CS theory. These configurations share all properties of
their Einstein gravity counterparts (for example they are also maximally symmetric). Similar
to that case [29], these backgrounds have a nonvanishing mass

\[
\mathcal{M}^{(k)} = - \frac{V_{k,3} 3 k^2 \ell^2}{8 \pi G} \frac{1}{8},
\]

(with \( V_{k,3} \) is the (dimensionless) volume associated with the metric \( d \Sigma^2_{k,3} \)) which, within the
AdS/CFT correspondence, is interpreted as a Casimir term.

The Schwarzschild-CS black hole solutions have a very simple form which resembles the
\( d = 3 \) BTZ metric [41] and have been studied by many authors [13], [14], [15]. Within the
metric ansatz (3.1), they are found for

\[
b(r) = f(r) = \frac{2 r^2}{\ell^2} (r^2 - r_H^2),
\]
with \( r_H > 0 \) a constant corresponding to the event horizon radius. A straightforward computation shows the absence of singularities for \( r > r_H \). However, different from the Einstein gravity case, the limit \( r_H \to 0 \) is singular for \( k \neq 0 \).

The mass \( \mathcal{M} \), temperature \( T_H \) and entropy \( S \) of this solution are

\[
\mathcal{M} = \frac{V_{k,3}}{8\pi G} \frac{3(1 + k \ell^2/2r_H^2)}{8}, \quad T_H = \frac{r_H}{\pi \ell^2}, \quad S = \frac{V_{k,3}}{4G} \frac{r_H^3(1 + \frac{3k \ell^2}{2r_H^2})}{8r_H^2}.
\]

Based on these relations, one can write the relatively simple equation of state (analogous to \( f(p, V, T) \), for, say, a gas at pressure \( p \) and volume \( V \))

\[
\mathcal{M} = \frac{3}{4} T_H S \left( 1 - \frac{2k}{3} \frac{1}{1 + k \sqrt{1 + c_0 T_H S}} \right),
\]

with \( c_0 = 64\pi G/(9\ell^2 V_{k,3}) \). Note that, as usual, one can isolate the Casimir contribution to the total mass \( \mathcal{M} \) by writing

\[
\mathcal{M} = \mathcal{M}_0^{(k)} + \mathcal{M}_c^{(k)}, \quad \text{where} \quad \mathcal{M}_0^{(k)} = \frac{3V_{k,3} \ell^2}{8\pi G} \left( 1 + \frac{2r_H^2}{k \ell^2} \right)^2.
\]

These black holes have some special properties and have been extensively studied in the literature. For example, different from the Einstein gravity case, the \( k = 0, 1 \) solutions have a strictly positive specific heat, since

\[
C = T_H \frac{\partial S}{\partial T_H} = \frac{3V_{k,3}}{4Gr_H} \left( 1 + k \frac{\ell^2}{2r_H^2} \right) = \frac{3\pi V_{k,3} \ell^4}{8G} \left( 1 + \frac{2\pi^2/2T_H^2}{k \ell^2} \right) T_H,
\]

while the \( k = -1 \) black holes become thermally unstable for small enough temperatures. Note also that the \( k = -1 \) background is a special case of the general solution (3.5) with \( r_H = \ell/\sqrt{2} \) and possesses a nonvanishing temperature and entropy.

It is easy to see that for these solutions, the boundary metric upon which the dual field theory resides corresponds to a static Einstein universe in four dimensions, with a line element

\[
h_{ab} dx^a dx^b = \frac{\ell^2}{8\ell^2} d\Sigma_{k,3}^2 - dt^2.
\]

The stress tensor for the boundary dual theory, as computed according to (2.17) has a vanishing trace, with

\[
8\pi G < \tau^a_0 > = U \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix},
\]

with \( U = \sqrt{\frac{\ell^2}{8\ell^2} (r_H^2 + k \ell^2)} \) for black holes and \( U = -\frac{k^2}{2\sqrt{\ell}} \) for the AdS backgrounds (here \( x^4 = t \)).

We close this part by remarking that, as discussed in [42], the solutions (3.3), (3.5) do not exhaust all possibilities allowed by the metric ansatz (3.1). For example, more complicated solutions describing wormholes and 'spacetime horns' do also exist. Moreover, rather unusual, due to a degeneration of the field equations, the choice \( f(r) = \frac{2r^2}{\ell^2} + k \) provides a solution of the field equation for any expression of the redshift function \( b(r) \).
3.2 Squashed black hole solutions

Interestingly, we have found that the black hole solutions (3.1) admit ‘squashed’ generalizations. Although the topology of the horizon remains the same, the shape of the horizon is changed, as well as the boundary metric.

Squashed black hole solutions have been originally proposed in the context of \( d = 4 + 1 \) Kaluza-Klein theory with a vanishing cosmological constant. Such configurations enjoyed considerable interest following the discovery of an exact solution in the five dimensional Einstein-Maxwell theory \[43\]. The horizon of a squashed black hole in Kaluza-Klein theory has \( S^3 \) topology, while its spacelike infinity is a squashed sphere or a \( S^1 \) bundle over \( S^2 \).

Of interest here are squashed black holes in AdS spacetime, considered in \[16\], \[17\] within the Einstein gravity framework. These solutions have a number of interesting properties; in particular they provide the gravity dual for a \( \mathcal{N} = 4 \) super Yang-Mills theory on a background whose spatial part is a squashed three sphere. They are also relevant in connection to the so-called ‘fragility’ of the AdS black holes. As described in \[44\] it turns out that AdS black holes can become unstable to stringy effects when their horizon geometries are sufficiently distorted.

However, different from the case of a Kaluza-Klein theory \[43\], no exact solutions could be constructed in the presence of a negative cosmological constant. The configurations in \[16\], \[17\] have been constructed numerically, by matching the near-horizon expansion of the metric to their asymptotic Fefferman-Graham form \[26\].

The squashed black hole solutions in CS-AdS theory are found within the same metric ansatz as in the Einstein gravity case \[17\]:

\[
ds^2 = \frac{dr^2}{f(r)} + r^2 \left( d\theta^2 + F_k^2(\theta) d\varphi^2 \right) + a(r) \left( dz + 4n F_k^2(\theta) d\varphi \right)^2 - b(r) dt^2, \tag{3.10} \]

with \( F_k(\theta) \) a function which is fixed by the discrete parameter \( k = 0, \pm 1 \):

\[
F_k(\theta) = \begin{cases} 
\sin \theta, & \text{for } k = 1 \\
\theta, & \text{for } k = 0 \\
\sinh \theta, & \text{for } k = -1.
\end{cases} \tag{3.11}
\]

As usual, \( r \) and \( t \) in the line element (3.10) are radial and time coordinates, respectively. For \( k = 1 \), \( \theta \) and \( \varphi \) are the spherical coordinates with the usual range \( 0 \leq \theta \leq \pi, \ 0 \leq \varphi \leq 2\pi \); for \( k = 0 \) and \( k = -1 \), the range of \( \theta \) is not restricted, while \( \varphi \) is still a periodic coordinate with \( 0 \leq \varphi \leq 2\pi \). Then \( d\Omega_{k,2}^2 = d\theta^2 + F_k^2(\theta) d\varphi^2 \) is the metric on a two-dimensional surface of constant curvature \( 2k \); when \( k = 0 \), a constant \((r, t)\) slice is a flat surface, while for \( k = -1 \), this sector is a space with constant negative curvature, also known as a hyperbolic plane.

The periodicity \( L \) of the coordinate \( z \) is fixed for \( k = 1 \) only, in which case \( z \) becomes essentially an Euler angle, with \( L = 8\pi n \), as imposed by the absence of conical singularities. Then a surface of constant \((t, r)\) is a squashed, topologically \( S^3 \) sphere.
Given this ansatz, we have found that the CS equations (2.7) admit the following exact solution
\[ f(r) = b(r) = \frac{2 \ell^2}{k^3} + \frac{2 n^2}{3 \ell^2} \quad \text{and} \quad a(r) = \frac{2 r^2}{\ell^2}. \quad (3.12) \]
One can easily verify that the curvature invariants are finite everywhere, in particular at \( r = 0 \). Thus it is natural to interpret (3.12) as the squashed deformation of the general AdS background (3.3). These solutions possess a nonvanishing mass
\[ M_c^{(k)} = - \frac{V_{k,2} L}{96 \sqrt{2} \pi G} \left( \frac{2 n^2}{\ell^2} - k \right)^2, \quad (3.13) \]
(with \( V_{k,2} \) the total area of the \((\theta, \varphi)\) surface) which is again interpreted as a Casimir term.

The squashed black holes have
\[ f(r) = b(r) = \frac{2 \ell^2}{k^3} - \frac{2 r^2}{\ell^2} \quad \text{and} \quad a(r) = \frac{2 r^2}{\ell^2}, \quad (3.14) \]
possessing an event horizon located at \( r = r_H > 0 \). One can easily verify the absence of singularities for any \( r \geq r_H \) (although the limit \( r_H \to 0 \) is again pathological).

The parameter \( n \) is an input constant of the model, which is not fixed \textit{apriori}. For \( k = 1 \) a value of interest is
\[ n = \ell \sqrt{\frac{2}{\pi}}. \quad (3.15) \]
in which case the surface of constant \( r, t \) is a round sphere \( S^3 \) and the Schwarzschild-CS black hole (3.5) is recovered\(^6\). This configuration separates prolate metrics from the oblate case \((n > \ell \sqrt{\frac{2}{\pi}})\).

A straightforward computation based on the general formalism in Section 2 leads to the following expressions for the mass, temperature and entropy of these black holes\(^7\)
\[ M = \frac{1}{8 \pi G} \frac{V_{k,2} L}{2} \frac{r^2}{r_H} \left( k - \frac{2 n^2}{\ell^2} + \frac{3 r_H^2}{\ell^2} \right), \quad (3.16) \]
\[ T_H = \frac{r_H}{\pi \ell^2}, \quad S = \frac{r^3}{2} V_{k,2} \frac{L}{3 \sqrt{2} G} \frac{r_H^2}{\ell} \left( 1 + \frac{k \ell^2}{2 r_H^2} - \frac{n^2}{r_H^2} \right), \]
the first law of thermodynamics being satisfied. Again, one can isolate the Casimir term in the expression of the mass by writing
\[ M = M_c^{(k)} + M_0^{(k)}, \quad \text{where} \quad M_0^{(k)} = \frac{L \ell^2 V_{k,2}}{96 \sqrt{2} \pi G} \left( \frac{6 r_H^2}{\ell^2} + k - \frac{2 n^2}{\ell^2} \right)^2, \quad (3.17) \]

\(^5\)We have found that the metric ansatz (3.10) allows for other solutions than (3.12), (3.14). In fact, it seems that all configurations discussed in [42] possess squashed generalizations.

\(^6\)Note that the normalization of the metric on \( S^3 \) in (3.10) differs from that used in (3.1). The radial and the time coordinates are also different.

\(^7\)We have verified that similar results are obtained when using instead the 'Kounterterm' regularization procedure. This holds also for the black string limit of the solutions in this work.
with $\mathcal{M}_c^{(k)}$ given by (3.13).

The specific heat of the squashed black holes is

$$C = \frac{\pi \ell V k}{4 \sqrt{2} G} T_H \left( 6 \pi^2 t^4 T_H^2 + k t^2 - 2n^2 \right),$$

(3.18)

and does not possess a fixed sign. Also, the above relation implies that squashing a black hole may render the configuration thermally unstable (note also that, for any $k$, the entropy $S$ becomes negative for large enough values of $n$, which appear to be a signal of pathological behaviour).

The boundary metric upon which the dual field theory resides corresponds to a squashed static Einstein universe in four dimensions

$$h_{ab} dx^a dx^b = \frac{\ell^2}{2} (d\theta^2 + F_k^2(\theta) d\phi^2) + (dz + 4n F_k^2(\frac{\theta}{2}) d\phi)^2 - dt^2.$$  

(3.19)

Also, the stress tensor for the boundary dual theory is trace less (where $x^1 = z$, $x^2 = \theta$, $x^3 = \phi$, $x^4 = t$)

$$8\pi G < \tau_{ab} > = U_1 \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{array} \right) + U_2 \left( \begin{array}{cccc} 1 & 0 & 4n F_k^2(\frac{\theta}{2}) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right),$$

(3.20)

with

$$U_1 = \sqrt{2} \nu H (r_H^2 + 2n^2), \quad U_2 = -\frac{8n^2 - k \ell^2}{2n^2 + r_H^2}$$

for squashed black holes and

$$U_1 = \frac{(2n^2 - k \ell^2)(14n^2 - k \ell^2)}{18 \sqrt{2} \ell^2}, \quad U_2 = -\frac{6(8n^2 - k \ell^2)}{(14n^2 - k \ell^2)}$$

for the squashed AdS backgrounds.

### 3.3 Uniform black strings

Taking the limit $n = 0$ in (3.10), (3.14), we find the following exact solution

$$ds^2 = \frac{dr^2}{r^2 (r^2 - r_H^2)} + r^2 (d\theta^2 + F_k^2(\theta) d\phi^2) + \frac{2r^2}{\ell^2} dz^2 - \frac{2}{\ell^2} (r^2 - r_H^2) dt^2,$$

(3.21)

describing an uniform AdS black string in $d = 5$ CS gravity. The properties of these solutions are rather special as compared to the squashed case. For example, while the range of $\theta$ and $\phi$ is similar to the squashed black hole case, for black strings with any $k$, the period $L$ of the compact 'extra'-direction $z$ is an arbitrary positive constant which plays no role in our results\(^8\).

This kind of asymptotically locally-AdS solutions have been considered for the first time by Horowitz and Copsey in [18], for configurations with $k = 1$ only. These black strings have an event horizon topology $S^2 \times S^1$, their conformal boundary being the product of time and $S^2 \times S^1$. These solutions have been generalized in [19] to the case of an event horizon

\(^8\)However, similar to the $\Lambda = 0$ Kaluza-Klein theory, the value of $L$ becomes relevant when discussing the issue of Gregory-Laflamme instability of these objects [45, 46].
topology\(^9\) \(H^2 \times S^1\), black strings in \(d \geq 5\) dimensions being considered as well. AdS black strings with gauge fields are studied in [47], [48], [49], [50]. The Ref. [48] discussed also the properties of a special set of spinning black string solutions. The issue of Gregory-Laflamme instability [51] for AdS black strings in Einstein gravity was addressed in [45], nonuniform solutions (i.e. with dependence on the compact ‘extra’-dimension \(z\)) being constructed in [46], [52].

As argued in [18], [19], such solutions in \(d - 3\) dimensions (with \(d \geq 5\)) provide the gravity dual of a field theory on a \(S^{d-3} \times S^1 \times S^1\) (or \(H^{d-3} \times S^1 \times S^1\)) background. Different from the \(\Lambda = 0\) limit, it was found in [18], [19] that the AdS black string solutions with an event horizon topology \(S^{d-3} \times S^1\) have a nontrivial, vortex-like globally regular limit with zero event horizon radius.

However, a general feature of all these AdS black strings/vortices is the absence of exact solutions (see, however, the extremal Einstein-Maxwell configurations in [47]). Remarkably, as shown by (3.21), the extension of the gravity to a CS model allows for exact solutions.

Let us start by discussing first another set of solutions, which provide a natural background for the black strings (3.21). The corresponding expression of the line element, as resulting from (3.12), reads

\[
ds^2 = \frac{dr^2}{2r} + r^2 (d\theta^2 + F_2^2(\theta) d\varphi^2) + \frac{2r^2}{\ell^2} dz^2 - \left(\frac{2r}{\ell^2} + \frac{k}{3}\right) dt^2.\tag{3.22}
\]

One can see that the \(k = 1\) solution describe the CS counterparts of the Einstein-gravity vortices in [19]; the \(k = 0\) solution is already contained in (3.1), (3.3), while the \(k = -1\) case corresponds to a special black string in (3.21).

The computation of the boundary stress tensor \(T_{ab}\) based on the relations in Section 2 is again straightforward. Apart from the mass-energy \(\mathcal{M}\), the solutions possess this time a second charge associated with the compact \(z\) direction, corresponding to a tension \(T\). Then we find the following expressions for the mass and tension of the background (3.22)

\[
\mathcal{M}_{(k)}^c = - \frac{L\ell V_{k,2}}{96\pi G\sqrt{2}} k^2, \quad T_{(k)}^c = - \frac{5L\ell V_{k,2}}{288\pi G\sqrt{2}} k^2.
\tag{3.23}
\]

Returning to the general black string solution (3.21), the corresponding expressions for conserved global charges read

\[
\mathcal{M} = \frac{V_{k,2} L^2 r_H^2 (3r_H^2 + k\ell^2)}{8\pi G \sqrt{2\ell^3}}, \quad T = \frac{V_{k,2} r_H^2 (r_H^2 + k\ell^2)}{8\pi G \sqrt{2\ell^3}},
\tag{3.24}
\]

which can also be written as

\[
\mathcal{M} = \mathcal{M}^{(k)}_0 + \mathcal{M}^{(k)}_c, \quad \text{where} \quad \mathcal{M}^{(k)}_0 = \frac{L\ell V_{k,2}}{96\sqrt{2\pi G}} \left(\frac{6r_H^2}{\ell^2} + k\right),
\tag{3.25}
\]

\[
T = T^{(k)}_0 + T^{(k)}_c, \quad \text{where} \quad T^{(k)}_0 = \frac{L\ell V_{k,2}}{288\sqrt{2\pi G}} \left(\frac{6r_H^2}{\ell^2} + k\right) \left(\frac{6r_H^2}{\ell^2} + 5k\right).
\]

\(^9\)The \(k = 0\) AdS black strings correspond to planar black holes.
with $\mathcal{M}_c^{(k)}$, $\mathcal{T}_0^{(k)}$ given by (3.23).

The Hawking temperature and entropy of these solutions are given by

$$T_H = \frac{r_H}{\pi \ell^2}, \quad S = \frac{V_4}{4\sqrt{2G}\ell} r_H (2r_H^2 + k\ell^2). \quad (3.26)$$

Similar to the Einstein gravity case [19], the black strings satisfy also a simple Smarr-type formula, relating quantities defined at infinity to quantities defined at the event horizon:

$$\mathcal{M} + \mathcal{T}L = T_H S, \quad (3.27)$$

the first law of thermodynamics

$$d\mathcal{M} = T_H dS - T dL, \quad (3.28)$$

being also fulfilled. Also, one can easily see that the thermodynamics of these solutions is rather similar to the (un-squashed) black hole case (for example the $k = 1$ solutions have a positive specific heat).

Let us also mention that the boundary metric upon which the dual field theory resides is

$$h_{ab} dx^a dx^b = \ell^2 \left( d\theta^2 + F_k^2 \theta^2 d\varphi_1^2 \right) + dz^2 - dt^2. \quad (4.1)$$

The expression of the stress tensor for the boundary dual theory is be found by taking the limit $n = 0$ in (3.20).

4. Rotating black holes with spherical horizon topology and two equal angular momenta

On physical grounds, it is natural to expect the existence of rotating generalizations of the static black holes discussed in Section 3.1. However, the presence of the GB term in the action makes highly non-trivial the task of finding a closed form of these solutions\(^\text{10}\).

While rotating black holes will generically possess two independent angular momenta and a more general topology of the event horizon, restricting the study to the special case of configurations with equal-magnitude angular momenta and a spherical horizon topology leads to a dramatic simplification of the problem. As first observed in [54], these configurations have a symmetry enhancement which allows to factorize the angular dependence. This results in a system of ordinary differential equations which can be easily studied.

The suitable metric ansatz in this case is similar to that used in the previous work [32], [53], with

$$ds^2 = \frac{dr^2}{f(r)} + g(r) d\theta^2 + h(r) \sin^2 \theta (d\varphi_1 - w(r) dt)^2 + h(r) \cos^2 \theta (d\varphi_2 - w(r) dt)^2 \quad (4.1)$$

$$+ \left( g(r) - h(r) \right) \sin^2 \theta \cos^2 \theta (d\varphi_1 - d\varphi_2)^2 - b(r) dt^2,$$

\(^\text{10}\)Some results in this direction are reported in [63]. However, the solution there has rather special properties (for example the line element is not circular) and does not describe a black object.
where $\theta \in [0, \pi/2]$, $(\varphi_1, \varphi_2) \in [0, 2\pi]$, and $r$ and $t$ denote the radial and time coordinate, respectively. Note that this metric ansatz admits a simpler expression in terms of the left-invariant 1-forms $\sigma_1$ on $S^3$, with

$$ds^2 = \frac{dr^2}{f(r)} + \frac{1}{4}g(r)(\sigma_1^2 + \sigma_2^2) + \frac{1}{4}h(r)(\sigma_3 - 2w(r)dt)^2 - b(r)dt^2,$$  \hspace{1cm} (4.2)

where $\sigma_1 = \cos \psi d\bar{\theta} + \sin \psi \sin \theta d\phi$, $\sigma_2 = -\cos \psi d\bar{\theta} + \cos \psi \sin \theta d\phi$, $\sigma_3 = d\psi + \cos \theta d\phi$ and $2\theta = \theta, \varphi_2 - \varphi_1 = \phi, \varphi_1 + \varphi_2 = \psi$. For such solutions the isometry group is enhanced from $R \times U(1)^2$ to $R \times U(2)$, where $R$ denotes the time translation$^{11}$. In the following we fix the metric gauge by taking $g(r) = r^2$. Then the field equations (2.7) reduce to a set of four ordinary differential equations for the functions $b, f, h$ and $w$ plus one extra constraint equation. The explicit form of these equations is given in Appendix A, together with a discussion of the issue of constraint.

We close this part by giving some useful relations which follow directly from the eqs. (A.2)-(A.6) and which are relevant for computing the action of solutions:

$$\frac{1}{\sin^2 \theta} (R_{\varphi_1}^t + \frac{\ell^2}{8}H_{\varphi_1}^t) = \frac{1}{\cos^2 \theta} (R_{\varphi_2}^t + \frac{\ell^2}{8}H_{\varphi_2}^t) = \frac{1}{2r^2} \sqrt{\frac{f}{bh}} \frac{d}{dr} \left[ \sqrt{\frac{fh}{b}} h w' \left( -r^2 + \frac{\ell^2}{2}(f - 4 + \frac{3h}{r^2}) \right) \right],$$

$$R_{t}^t + \frac{\ell^2}{8}(H_{t}^t + \frac{1}{2}L_{GB}) = \frac{1}{2r^2} \sqrt{\frac{f}{bh}} \frac{d}{dr} \left[ \sqrt{\frac{fh}{b}} \left( r^2(hw' - b') + \frac{\ell^2}{2} \left( f - 4 + \frac{h}{r^2} + \frac{rfh'}{h} b' + (4 - \frac{3h}{r^2} - f)hw' + rfhw'^2 \right) \right) \right],$$

where $H_{\mu\nu} = 2(R_{\mu\nu\sigma\tau}R_{\sigma\tau} - 2R_{\mu\nu\sigma}R_{\sigma\tau} - 2R_{\mu\sigma}R_{\nu} + RR_{\mu\nu}) - \frac{1}{2}L_{GB}g_{\mu\nu}$ is the Lanczos (or the Gauss-Bonnet) tensor.

4.1 Asymptotic form of the solutions

The horizon of these spinning black hole resides at the constant value of the radial coordinate $r = r_H > 0$, and is characterized by the condition

$$f(r_H) = b(r_H) = 0.$$  \hspace{1cm} (4.5)

Restricting to nonextremal solutions, the following expansion holds near the event horizon:

$$f(r) = f_1(r - r_H) + O(r - r_H)^2, \hspace{0.5cm} h(r) = h_H + O(r - r_H),$$

$$b(r) = b_1(r - r_H) + O(r - r_H)^2, \hspace{0.5cm} w(r) = \Omega_H + w_1(r - r_H) + O(r - r_H)^2.$$

$^{11}$The Myers-Perry-AdS (MP-AdS) solution with two angular momenta can also be written in this form, with $f(r) = 1 + r^2/\ell^2 - 2M\Sigma/r^2 + 2Ma^2/r^4$, $h(r) = r^2(1 + 2Ma^2/r^4)$, $w(r) = 2Ma/(r^2 h(r))$, $b(r) = r^2 f(r)/h(r)$ and $g(r) = r^2$. Here $M$ and $a$ are two constants while $\Sigma = 1 - a^2/\ell^2$.
For a given event horizon radius, the essential parameters characterizing the event horizon are \( f_1, b_1, \Omega_H \) and \( w_1 \) (with \( f_1 > 0, b_1 > 0 \)), which fix all other coefficients in (4.6), including those of the higher order terms not displayed there. The explicit form of these coefficients is very complicated, except for \( h_H \) which has the expression\(^{12}\)

\[
h_H = \sqrt{\frac{b_1}{f_1}} \frac{r_H}{\ell^2 w_1} (4r_H - f_1 \ell^2),\tag{4.7}
\]

The metric of a spatial cross section of the horizon is

\[
ds^2_H = \frac{1}{4} r_H^2 \left( \sigma_1^2 + \sigma_2^2 + \frac{h_H}{r_H^2} \sigma_3^2 \right),\tag{4.8}
\]

which corresponds to a squashed \( S^3 \) sphere.

The (constant) horizon angular velocity \( \Omega_H \) is defined in terms of the Killing vector \( \chi = \partial/\partial t + \Omega_1 \partial/\partial \varphi_1 + \Omega_2 \partial/\partial \varphi_2 \) which is null at the horizon. For the solutions within the ansatz (4.1), the horizon angular velocities are equal, \( \Omega_1 = \Omega_2 = \Omega_H \).

One can also write an asymptotic form of the solutions, involving three free parameters \( U, V \) and \( W \), with

\[
f(r) = \frac{2r^2}{\ell^2} + 1 + V + f_2^{(as)} \frac{\ell^2}{2r^2} + O(1/r^4), \quad b(r) = \frac{2r^2}{\ell^2} + 1 + U + b_2^{(as)} \frac{\ell^2}{2r^2} + O(1/r^4),
\]

\[
h(r) = r^2 (1 + h_2^{(as)} \frac{2\ell^2}{r^2}) + O(1/r^2), \quad w(r) = W \frac{\ell^2}{2r^2} + \frac{\ell^4}{4r^4} w_4^{(as)} + O(1/r^6).\tag{4.9}
\]

All coefficients in these series (including the higher order ones) can be expressed as combinations of \( U, V \) and \( W \). One finds e.g.

\[
f_2^{(as)} = \frac{-18(U - V)^3 V + (2U^2 + 3UV - 6V^2)\ell^2 W^2 - \ell^4 W^4}{(2U - 3V)(2(U - V) - \ell^2 W^2)},
\]

\[
b_2^{(as)} = \frac{1}{8} \left( 2U(6 + U) - 2(U + 7)V - \frac{(4 + 3V)}{V} \ell^2 W^2 - \frac{4(2U - 3V)^2(U - V)}{2U(V - U) + \ell^2 W^2} \right),
\]

\[
h_2^{(as)} = \frac{3(U - V) V - \ell^2 W^2}{2U - 3V},\tag{4.10}
\]

\[
w_4^{(as)} = \frac{1}{4} \frac{W}{2U - 3V} \left( 2(U - 4)V + 3V - 7UV + 6V^2 + \frac{(4 + 3V)}{V} \ell^2 W^2 - \frac{4(2U - 3V)^2(U - V)}{2U(V - U) + \ell^2 W^2} \right),
\]

the expression of other terms being too complicated to display it here. Also, one can see that for these asymptotics, the boundary metric is not rotating.

\(^{12}\)Note the unusual dependence of \( h(r_H) \) on \( w'(r_H) \). The corresponding expression for the Einstein gravity solution reads \( h_H = r_H^2 (4 - f_1 r_H + 4r_H^2/\ell^2)/2 \).
4.2 Physical quantities

The Hawking temperature of these configurations is given by

\[ T_H = \frac{1}{4\pi} \sqrt{b'(r_H)f'(r_H)} . \] (4.11)

The conserved charges of the rotating black holes are obtained by using again the counterterm method in conjunction with the quasilocal formalism, as described in Section 2. These are given by the following complicated expressions

\[ \mathcal{M} = \mathcal{M}_0 + \mathcal{M}_c^{(1)}, \quad \text{with} \quad \mathcal{M}_0 = \frac{V_{1.3}}{8\pi G} \left[ \frac{\ell^2 \mathcal{V}}{8(2\mathcal{U} - 3\mathcal{V})} (3\mathcal{V}(\mathcal{V} - 2\mathcal{U}) + 4\ell^2 \mathcal{W}^2) \right], \] (4.12)

\[ J_1 = J_2 = \frac{V_{1.3}}{64\pi G} (2 - \mathcal{V}) \ell^4 \mathcal{W}, \] (4.13)

where \( V_{1.3} = 2\pi^2 \) and \( \mathcal{M}_c^{(1)} \) given by (3.4). Thus in constrast to the case of spinning solutions in Einstein gravity [56] or in EGB model with \( \alpha \neq \ell^2/2 \) [32], the constant \( \mathcal{W} \) which enters the asymptotics of the metric functions \( w(r) \) associated with rotation \( (w(r) = \mathcal{W}\ell^2/(2r^2) + \ldots) \) does not fix alone the angular momentum. Moreover, \( \mathcal{W} \) enters the expression of mass.

The computation of solutions’ action is standard. The bulk action evaluated on the equations of motion can easily be computed by replacing the \( R + \frac{12}{r^2} + \frac{\ell^2}{8} L_{GB} \) volume term with \( 2(R^t_t + \frac{\ell^2}{8}(H^t_t + \frac{1}{2}L_{GB})) \). Then one makes use of (4.4) to express the volume integral of this quantity as the difference of two boundary integrals. The boundary integral on the event horizon is simplified by using the identity (4.3). Then the divergencies of the boundary integral at infinity, together with the contribution from (2.8) are regularized by the counterterm (2.10).

Upon application of the Gibbs-Duhem relation to the partition function, one finds the entropy

\[ S = \beta (\mathcal{M} - 2\Omega H J) - I_{el} = \frac{V_{1.3}}{4G} \ell^2 H H \left( 1 + \frac{\ell^2}{2r_H^2} (4 - \frac{h_H}{r_H^2}) \right). \] (4.14)

With these quantities, the solutions should satisfy the first law of thermodynamics (2.16), which, for this particular case reads

\[ d\mathcal{M} = T_H dS + 2\Omega H dJ. \] (4.15)

For completness, let us mention that boundary metric upon which the dual field theory resides corresponds to a static Einstein universe with a line element \( h_{ab}dx^a dx^b = \ell^2 (d\theta^2 + \sin^2 \theta d\varphi_1^2 + \cos^2 \theta d\varphi_2^2) - dt^2 \). The stress tensor for the boundary dual theory is again traceless
with a rather complicated expression (here \(x^1 = \theta, x^2 = \varphi_1, x^3 = \varphi_3\) and \(x^4 = t\):

\[
8\pi G < \tau^a_b > = U_1 \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -3
\end{pmatrix} + U_2 \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \sin^2 \theta \cos^2 \theta & 0 & 0 \\
0 & \sin^2 \theta \cos^2 \theta & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} + U_3 \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 \sin^2 \theta \cos^2 \theta & 0 & 0 & 0
\end{pmatrix} + U_4 \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}, \tag{4.16}
\]

where

\[
U_1 = \frac{-4U^2V + U(6V^2 - 2) + V(3 - 3V^2 + 2\ell^2W^2)}{2\sqrt{2}(2U - 3V)}, \quad U_2 = \frac{-6(U - V)^2V + \ell^2VW^2}{\sqrt{2}(2U - 3V)},
\]

\[
U_3 = -\frac{\ell}{\sqrt{2}}(V - 2)W, \quad U_4 = \frac{\sqrt{2}}{\ell}(2 + V)W. \tag{4.17}
\]

### 4.3 Slowly rotating solutions

We strongly suspect that, in contrast to the generic EGB case, the spinning black hole solutions could be constructed in closed form for the CS model. However, despite our effort, we could not find this solution so far (a spinning, singular configuration is reported in Appendix C).

Nevertheless, some progress in this direction can be achieved in the slowly rotating limit. Such solutions can be found by considering perturbation theory around the static solution (3.1) in terms of a small parameter \(a\) which corresponds to the constant \(W\) entering the asymptotic expansion (4.3). The second parameter of the solution is the horizon radius \(r_H\).

Our results up to order seven in perturbation theory lead to conjecture that the spinning black solutions admit an expression on the form\(^{13}\)

\[
b(r) = b_0(r) \left(1 + \sum_{k \geq 1} \frac{(-1)^k a_k}{\beta^k(1 + 2\beta^2)^{k-1}} \sum_{j=1}^k b_{kj} \beta^{2j} \left(\frac{\ell}{r}\right)^{2j}\right),
\]

\[
f(r) = f_0(r) \left(1 + \sum_{k \geq 1} \frac{(-1)^k a_k}{\beta^k(1 + 2\beta^2)^{k-1}} \sum_{j=1}^k f_{kj} \beta^{2j} \left(\frac{\ell}{r}\right)^{2j}\right), \tag{4.18}
\]

\[
h(r) = h_0(r) \left(1 + \sum_{k \geq 1} \frac{(-1)^k a_k}{\beta^k(1 + 2\beta^2)^{k-1}} \sum_{j=1}^k h_{kj} \beta^{2j} \left(\frac{\ell}{r}\right)^{2j}\right),
\]

\[
w(r) = w_0(r) \left(\beta \frac{\ell}{\ell} \sum_{k \geq 1} \frac{(k+1)a_k}{\beta^k(1 + 2\beta^2)^{k-1}} \sum_{j=1}^k w_{kj} \beta^{2j} \left(\frac{\ell}{r}\right)^{2j}\right),
\]

\(^{13}\)This corresponds to the most general (perturbative) solution of the equations of motion compatible with both the near horizon expansion (4.1) and the asymptotic form (4.9). However, a more general solution can be written when relaxing these assumptions.
where
\[ b_0(r) = f_0(r) = \frac{2}{\ell^2}(r^2 - r_H^2), \quad h_0(r) = r^2, \]  
(4.19)
are the metric functions of the static black hole solution and
\[ a = \frac{1}{2}\Omega W \]  
(4.20)
is the expansion parameter. Also, to simplify the relations we define the dimensionless parameter
\[ \beta = \frac{r_H}{\ell}. \]  
(4.21)
The parameters \( b_{kj}, f_{kj}, h_{kj} \) and \( w_{kj} \) in the expression (4.18) are polynomials in \( \beta^2 \); their explicit form up to order four is given in Appendix C. Here we give only the first order expression of the solution14:
\[ b(r) = \frac{2}{\ell^2}(r^2 - \beta^2 \ell^2), \quad f(r) = \frac{2}{\ell^2}(r^2 - \beta^2 \ell^2)\left(1 - (-1)^P \frac{a\ell^2}{2\beta r^2}\right), \]  
(4.22)
\[ h(r) = r^2\left(1 - (-1)^P \frac{3a\ell^2}{2\beta r^2}\right), \quad w(r) = \frac{a\ell}{r^2}. \]

The relations (4.18), (4.22) contain an extra-parameter \( P \), with \( P = 0 \) or \( P = 1 \), which shows the existence of two different families of solutions. The difference between these two solutions becomes more transparent when writing the expression of some relevant global quantities.

After replacing the expressions (4.18) in (4.11), (4.12), (4.13), (4.14) one finds the general expressions15:
\[ M = M^{(0)} + \frac{\pi\ell^2}{G} \sum_{k \geq 1} \frac{(-1)^k a^k}{\beta^{3k-3}(1 + 2\beta^2)^{k-1}} M_k(\beta), \quad J = \frac{\pi\ell^3}{16G} \sum_{k \geq 1} \frac{(-1)^{k+1} a^k}{\beta^{3k-3}(1 + 2\beta^2)^{k-1}} f_k(\beta), \]
\[ \Omega_H = \frac{1}{\ell} \sum_{k \geq 1} \frac{(-1)^k a^k}{\beta^{3k-1}(1 + 2\beta^2)^{k-1}} w_k(\beta), \quad S = S^{(0)} + \frac{\pi^2\ell^3}{16G} \sum_{k \geq 1} \frac{(-1)^k a^k}{\beta^{3k-1}(1 + 2\beta^2)^{k-2}} s_k(\beta), \]
\[ T_H = T_H^{(0)} + \frac{1}{4\pi\ell} \sum_{k \geq 1} \frac{(-1)^k a^k}{\beta^{3k-1}(1 + 2\beta^2)^{k-1}} t_k(\beta), \]  
(4.23)

14The corresponding expression in a general EGB theory with \( \ell^2 > 2\alpha \) has \( b_1(r) = f_1(r) = h_1(r) = 0 \) and
\[ w_1(r) = \frac{\ell^4}{4\sqrt{\ell^2 - 2\alpha}} \left(\sqrt{1 - \frac{2\alpha}{\ell^2}} + \sqrt{1 - \frac{2\alpha}{\ell^2} + \frac{\alpha}{\ell^4}(2\beta^2(1 + \beta^2)\ell^2 + \alpha)}\right)^{-1}, \]
such that \( w_1(r) \sim \frac{\ell^4}{4\sqrt{\ell^2 - 2\alpha}} \) asymptotically (note the different power of \( r \) as compared to (4.19)). However, in this case we could not construct a general perturbative solution similar to (4.18).

15We have verified that these quantities satisfy the first law of thermodynamics up to order seven. Also the horizon remain regular up to that order.
where $M^{(0)}$, $S^{(0)}$ and $T_H^{(0)}$ are the mass, entropy and temperature of the static black hole.

Here we give again only the first order expressions (these quantities up to order four are given in Appendix B; we only mention that $M_k$, $j_k$, $s_k$, $w_k$ and $t_k$ are polynomials in $\beta^2$):

\[
M = \frac{2\pi}{8G} \ell^2 \beta^2 (1 + \beta^2) - (-1)^p \frac{\pi \ell^2}{16G} 3(1 + 2\beta^2), \quad J = a \frac{\pi \ell^3}{16G} (3 + 2\beta^2),
\]
\[
\Omega_H = a \frac{1}{\ell \beta^2}, \quad S = \frac{\beta (3 + 2\beta^2) \pi^2 \ell^3}{4G} - (-1)^p a \frac{\pi^2 \ell^3}{16G} \frac{3(1 + 2\beta^2)}{\beta^2}, \quad T_H = \frac{\beta}{\pi \ell} - (-1)^p \frac{1}{\pi \ell} \frac{1}{4\beta^2}.
\]

Note that, different from the case of Myers-Perry solution, all physical relevant quantities receive corrections already in the first order of perturbation theory. On a technical level, this can be attributed to the special form of the perturbative solution (4.22) and is anticipated already by the expressions (4.7), (4.13).

Supposing $a > 0$, one can see that for $P = 0$ the mass, entropy and temperature of solutions decrease with $a$ (at least to lowest order). Rather unusual, these quantities increase with $a$ for the second solution ($P = 1$). When changing the sign of $a$, the two solutions interchange.

### 4.4 Nonperturbative solutions: numerical results

The non-perturbative solutions are constructed numerically, by integrating the system of coupled non-linear ordinary differential equations given in Appendix A, with appropriate boundary conditions which follow from (4.6), (4.9).

To simplify the problem, we restrict our integration to the region outside the event horizon, $r \geq r_H$. In our approach, we use a standard solver [57], which involves a Newton-Raphson method for boundary-value ordinary differential equations, equipped with an adaptive mesh selection procedure. Typical mesh sizes include around 400 points. Also, the solutions in this work have a typical relative accuracy of $10^{-6}$.

Somehow unexpected, we have found the numerical study of the spinning solutions in the $d = 5$ CS theory more complicated than in the case of rotating solutions of the generic EGB model in [32], [33], [55]. For example, for the solutions in this work we did not manage to fix the total angular momentum as an input parameter and thus we could not study in a systematic way spinning black holes in a canonical ensemble (this can be attributed to the complicated expression (4.13) of $J$, with a dependence on both $\mathcal{V}$ and $\mathcal{W}$). Therefore in our numerical scheme, for a given value of $\ell$, we have chosen $r_H$ and $\Omega_H$ as input parameters. The numerical output provides the profiles of the functions $b, f, h$ and $w$ and their first derivative on some mesh. By interpolating these profiles, we could compute all quantities of interest.

Our numerical results confirm the existence of spinning generalizations of the spherically symmetric black hole solutions also at the non-perturbative level\(^{16}\). The profiles of a typical solution are presented of Figure 1 as a function of the radial coordinate.

\(^{16}\)We have found that the perturbative solutions provide a good approximation for a large part of the numerical configurations.
One can see that the term $2r^2/\ell^2$ starts dominating the profiles of $b, f$ and $h$ very rapidly. Also, typically, the metric functions interpolate monotonically between the corresponding values at $r = r_H$ and the asymptotic values at infinity, without presenting any local extrema.

We mention also that similar to other rotating solutions, these black present also an ergoregion inside of which the observers cannot remain stationary, and will move in the direction of rotation. The ergoregion is the region bounded by the event horizon, and the stationary limit surface, or the ergosurface, $r = r_e$. The Killing vector $\partial/\partial t$ becomes null on the ergosurface, i.e. $g_{tt}(r_e) = -b(r_e) + h(r_e)w^2(r_e) = 0$. The ergosurface does not intersect the horizon.

A feature of these solutions which can already be anticipated based on the perturbative results is their nonuniqueness in terms of $(r_H, \Omega_H)$ (this contrasts with the case of MP-AdS).
Figure 2: The mass $M$, angular momentum $J$, entropy $S$ and Hawking temperature $T_H$ are shown vs. the angular velocity of the horizon for 1st and 2nd branch solutions with $r_H = 1$ and $\ell = 0.707$ (here and in Figure 3 we set $G = 1$).

Figure 3: The mass $M$, angular momentum $J$ and entropy $S$ are shown as a function of the Hawking temperature $T_H$ for 1st branch solutions with $\ell = 2$ and several fixed values of the angular velocity of the horizon.

black holes). That is, given a static black hole solution with some $r_H > 0$, we have found numerical evidence for the occurrence of two different branches of spinning solutions with the same value of the event horizon velocity $\Omega_H$. The profiles in Figure 1 correspond to a 1st branch solution. The insets there show the difference between and first and second branch solutions. The two branches can be distinguished, namely, by the value $k(r_H)$ (where we define $k(r) \equiv h(r)/r^2$). On one branch, the solutions have $k(r_H) < 1, k'(r_H) > 0$ (we will refer to it as the first branch); the second branch has $k(r_H) > 1, k'(r_H) < 0$. In relation
with the perturbative expansion (4.18), the first (resp. second) branch corresponds to \( P = 0 \) (resp. \( P = 1 \)).

The solutions of the first branch obey a standard pattern: for a fixed value of the event horizon radius \( r_H \), their temperature decreases when \( \Omega_H \) increases. Accordingly, a configuration with an extremal horizon at \( r = r_H \) is approached for a maximal event horizon velocity, say \( \Omega_H = \Omega_{H,ex} \). This value, of course, depends on \( r_H, \ell \). Setting \( r_H = 1 \), we find for example \( \Omega_{H,ex} \approx 2.35 \) for \( \ell^2 = 1/2 \) (case of Figure 2) and \( \Omega_{H,ex} \approx 0.935 \) for \( \ell^2 = 2 \).

The solutions of the second branch exhibit a very different picture. For example, rather unusual, their temperature increase with \( \Omega_H \). Also, we have found that they have a relatively small extension in \( \Omega_H \). In fact, increasing of the event horizon velocity result in a very rapid increases of the parameter \( |k'(r_H)| \), suggesting that a critical configuration is approached for some \( \Omega_H^{max} \). Although the numerical accuracy is lost as \( \Omega_H \rightarrow \Omega_H^{max} \), our results indicate that the horizon becomes singular for the critical configuration. Some of these features are illustrated in Figure 2, where we show a number physical quantities for 1st and 2nd branch solutions as a function of \( \Omega_H \) (there we take \( r_H = 1 \) and \( \ell = 0.707 \); however, similar results have been found for other choices of these parameters).

Returning on 1st branch solutions (which are likely to be physically more interesting), we show in Figure 3 the mass, angular momentum and entropy of solutions as functions of temperature for several fixed values of \( \Omega_H \) (thus we consider configurations in a grand canonical ensemble). The control parameter there is the event horizon radius, which varies between a small value close to zero (although this region is difficult to explore numerically) up to a maximal value which depends on \( \Omega_H \). The results in Figure 3 suggest the following picture. Similar to the static case, the spinning solutions appear to possess a \( r_H = 0 \) limit, which has a vanishing temperature\(^{17} \). When increasing the event horizon radius, a branch of spinning black holes -regular on and outside the horizon- occurs, the temperature, mass and the absolute value of \( J \) increase along the branch. However, different from the static case, for any \( \Omega_H \neq 0 \), there is a maximal value for the temperature which is approached for some critical value of \( r_H \). At that point, a secondary branch of solutions emerges\(^{18} \), which extends backwards in \( T_H \) (note that the event horizon radius, the black hole entropy and the global charges still increase along this branch). The end point of this branch is an extremal configuration with vanishing temperature, finite (and nonvanishing) horizon size and finite global charges.

These limiting extremal configurations, however, satisfy a different set of boundary conditions at \( r = r_H \) than the nonextremal ones (for example \( f|_{r-r_H} = O(r-r_H)^2 \), \( b|_{r-r_H} = O(r-r_H)^2 \)). (Note that the extremal solutions keep the asymptotic form (4.9), with a single essential parameter in the expansion there.) Finding such solutions explicitly or numerically is beyond the scope of this paper.

\(^{17}\) This limiting configuration could not be studied numerically. However, based on the perturbative results, we expect it to be singular.

\(^{18}\) For the data plotted in Figure 3, this can be seen for \( \Omega_H = 0.8 \). However, the backbending occurs for other values of \( \Omega_H \) there as well, although for larger values of \( T_H, M \) and \(|J|\).
4.5 Extremal solutions: near horizon geometry and the entropy function

However, as usual, some information about the properties of the extremal black holes can be obtained by studying the near horizon solution together with the corresponding entropy function.

Therefore we consider the following metric form, describing a rotating squashed $AdS_2 \times S^3$ spacetime

\[ ds^2 = v_1 \left( \frac{dr^2}{r^2} - r^2 dt^2 \right) + \frac{v_2}{4} \left( \sigma_1^2 + \sigma_2^2 \right) + \frac{v_2 v_3}{4} (\sigma_3 + 2 kr dt)^2 \] (4.24)

(i.e. for $b(r) = v_1 r^2$, $f(r) = r^2/v_1$, $g(r) = v_2$, $h(r) = v_2 v_3$, $w(r) = kr$ within the parametrization (4.2)), such that the horizon is located\(^{19}\) at $r = 0$. This geometry describes a fibration of $AdS_2$ over the homogeneously squashed $S^3$ with symmetry group $SO(2,1) \times SU(2) \times U(1)$ \[58\]. Also, (4.24) would correspond to the neighbourhood of the event horizon of an extremal limit of the spinning black holes discussed above.

Given this ansatz, the equations (2.7) of the model reduce to a set of algebraic relations for the parameters $v_i, k$. In what follows we choose to determine these relations by using the formalism proposed in \[59\], thus by extremizing an entropy function. This allows us also to compute the entropy of the extremal black holes and to show that these CS solutions exhibit attractor behaviour.

Therefore, as usual in the literature, let us denote by $f(k, \vec{v})$ the lagrangian density $\sqrt{-g} \mathcal{L}$ (as read from (2.1) together with (2.2), (2.4)) evaluated for the near horizon geometry (4.24), and integrated over the angular coordinates,

\[ f(k, \vec{v}) = \int d\theta d\phi d\psi \sqrt{-g} \mathcal{L} = \frac{1}{16 \pi G} \int d\theta d\phi d\psi \sqrt{-g} (R + \frac{12}{\ell^2} + \frac{\ell^2}{8} L_{GB}). \] (4.25)

The field equations (2.7) for the near horizon geometry (4.24) now correspond to $\frac{\partial f}{\partial k} = J$, $\frac{\partial f}{\partial v_i} = 0$, with $J$ the angular momenta of the solutions.

Then, following \[59\], we define the entropy function by taking the Legendre transform of the above integral with respect to the parameter $k$,

\[ \mathcal{E}(J, k, \vec{v}) = 2\pi (Jk - f(k, \vec{v})). \] (4.26)

The entropy and the near horizon geometry of the spinning black holes are obtained by extremizing this entropy function. This leads to the algebraic equations of motion

\[ \frac{\partial \mathcal{E}}{\partial k} = 0, \quad \frac{\partial \mathcal{E}}{\partial v_i} = 0, \] (4.27)

the entropy associated with the black hole being $S_{\text{extremal}} = \mathcal{E}(J, \vec{v})$ evaluated at the extremum (4.27).

\(^{19}\)This position of the horizon can always be obtained by taking $r \rightarrow r - r_H$. 

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For the metric ansatz [4.24], a straightforward calculation gives
\[ \mathcal{E}(J, v) = 2\pi \left[ Jk - \frac{\pi \sqrt{v_2 v_3}}{16 G v_1} \left( \frac{24 v_1^2 v_2}{\ell^2} + k^2 v_2^2 v_3 - 4 v_1^2 (v_3 - 4) - 4 v_1 v_2 \right) \right]. \]

Then the explicit form of the equations [1.27] is
\[ -16 v_1^2 + 4 v_1^2 v_3 + k^2 v_2^2 v_3 + \frac{\ell^2}{2} k^2 v_2 v_3 (4 - 3 v_3) - \frac{24}{\ell^2} v_1^2 v_2 = 0, \tag{4.29} \]
\[ -16 v_1^2 + 12 v_1 v_2 + 4 v_1^2 v_3 - 5 k^2 v_2^2 v_3 + \frac{\ell^2}{2} (-4 v_1 (v_3 - 4) + 3 k^2 v_2 v_3 (3 v_3 - 4)) - \frac{72}{\ell^2} v_1^2 v_2 = 0, \]
\[ -16 v_1^2 + 4 v_1 v_2 + 12 v_1^2 v_3 - 3 k^2 v_2^2 v_3 + \frac{\ell^2}{2} (-4 v_1 (3 v_3 - 4) + 3 k^2 v_2 v_3 (5 v_3 - 4)) - \frac{24}{\ell^2} v_1^2 v_2 = 0, \]

together with
\[ J = \frac{\pi k (v_2 v_3)^{3/2}}{8 G v_1} (v_2 + \frac{\ell^2}{2} (4 - 3 v_3)). \tag{4.30} \]

A solution of the above equations takes a relatively simple form when expressed in terms of the 'relative squashing' parameter \( v_3 \). One finds
\[ v_1 = \frac{\ell^2}{8} \left( \frac{6 v_2 + \ell^2 (4 - v_3) (2 v_2 + \ell^2 (4 - v_3))}{24 v_2^2 + \ell^4 (4 - v_3) (4 - 3 v_3) - 6 \ell^2 v_2 (5 v_3 - 8)} \right), \tag{4.31} \]
\[ J = \frac{\pi v_2 v_3}{4 G} \left( 4 - v_3 + \frac{6 v_2}{\ell^2} (v_2 + \ell^2 (2 - \frac{3 v_3}{2})) \right), \tag{4.32} \]
and
\[ k = \frac{16 G J v_1}{\pi (v_2 v_3)^{3/2} (4 \ell^2 + 2 v_2 - 3 \ell^2 v_3)}. \tag{4.33} \]

The radius \( v_2 \) of the round \( S^2 \) sphere is also a function of \( v_3 \), with\(^{20}\)
\[ v_2 = \frac{\ell^2}{2} \left( 3 v_3 - 2 + \sqrt{4 + 8 (v_3 - 1) v_3} \right). \tag{4.34} \]

Finally, inserting these expressions into Eq. [4.28] we obtain for the entropy function of the extremal black hole:
\[ \mathcal{E} = S_{\text{extremal}} = \frac{\pi^2}{2 G} \sqrt{v_2 v_3} \left( v_2 - \frac{\ell^2}{2} (v_3 - 4) \right), \tag{4.35} \]
(with \( v_2(v_3) \) as implied by [4.34]). We have verified that the above result agrees with Wald's form [2.13] evaluated for the near horizon geometry [4.24]. We also note that, in principle, \( \mathcal{E} \) can be expressed in terms of the conserved charge \( J \) by inverting the relation [4.32] together with [4.34].

We interpret these results as an indication for the existence of corresponding bulk extremal solutions, in which case, for the parametrization in this work, \( v_2 = r_H^2 \) and \( v_3 = h_H/r_H^2 \).

\(^{20}\)One can notice that the relations [4.33]–[4.34] are invariant under the scaling \( v_1 \to \lambda v_1, \ v_2 \to \lambda v_2, \ \mathcal{E} \to \lambda^{3/2} \mathcal{E}, \ J \to \lambda^{3/2} J, \ k \to k, \) and \( \ell \to \lambda^{1/2} \ell \), which shows that the solutions exist for any \( \ell \).
5. Conclusions

The main purpose of this paper was to discuss the basic properties of several types of black object solutions in $d = 5$ Chern-Simons-AdS theory of gravity. Apart from the known Schwarschild black holes, we have considered also squashed black holes and uniform black strings. Remarkably, different from the case of pure Einstein gravity, the expression of the squashed black holes and uniform black strings could be found in closed form. Our results show that the properties of the known Schwarschild black holes in CS theory are somehow generic.

Apart from that, we have given arguments for the existence of rotating black holes in $d = 5$ Chern-Simons theory of gravity. These configurations posses a regular horizon of spherical topology and have two equal-magnitude angular momenta, representing generalizations of a particular class of Myers-Perry black holes. The structure of the asymptotic series suggests the existence of an exact solution also in that case, but so far we could not find it. Therefore we have studied the properties of the spinning solutions using both analytical and numerical methods. Exact solutions were constructed in the slowly rotation limit, by considering them as a perturbations around the Schwarzschild black hole. The nonperturbative solutions were found by solving numerically the field equations. We hope that these results will prove useful in constructing the solutions in a closed form.

We also proposed to adapt the boundary counterterm formalism of [29] to the CS-AdS theory, computing in this way the global charges of all solutions in this work. This formalism can also be generalized to the situation when matter fields are added to the bulk action (2.1). For example, we have verified verify that the counterterms proposed in Section 2 regularize the action and mass of the Reissner-Nordstrom generalizations of the black holes (3.1).

As avenues for further research, it would be interesting to consider other classes of solutions apart from those in this work. Here let us mention that we have also found a class of exact solutions that can be viewed as the $d = 4$ Taub-NUT-AdS solutions uplifted to five dimensions, in the presence of a negative cosmological constant (the corresponding Einstein gravity configurations are discussed in [17]). Moreover, the present work could also be extended by supplementing the model (2.1) with matter fields, for instance gauged scalar fields. This would lead to charged and/or hairy generalisations of the black holes solutions that we have presented. Such solutions were obtained e.g. in [60] for the generic values of the Gauss-Bonnet coupling constant. Finally, let us mention that we expect the results in this work to admit direct generalizations to the higher dimensional case $d = 2n + 1 \geq 7$.

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A. Rotating solutions: the system of differential equations

Given the metric ansatz (A.3), one can show that the tensor $E_{\mu}^\nu$ as defined by (2.7) has only five linearly independent components: $E_r^r$, $E_\theta^\theta$, $E_\varphi^\varphi_1$, $E_\varphi^\varphi_2$ and $E_t^t$. All other equations are identically zero or are linear combinations of these components. One finds e.g.

$$
\cos^2 \theta E_{\varphi_1}^r = \sin^2 \theta E_{\varphi_2}^t, \quad E_{\varphi_1}^\varphi - E_{\varphi_2}^\varphi + \frac{\cos 2\theta}{\sin^2 \theta} E_{\varphi_1}^\varphi = 0, \quad E_{\theta}^\varphi - E_{\varphi_1}^\varphi + E_{\varphi_2}^\varphi = 0. \quad (A.1)
$$

The explicit expression of the essential components of $E_{\mu}^\nu$ is

$$
\begin{align*}
E_r^r &= - \left(1 + \frac{\ell^2}{3r^2} - \frac{\ell^2 h}{6r^4} - \frac{\ell^2 f}{6r^4}\right) \frac{6}{r^2} + \left(1 + \frac{\ell^2 h}{2r^4}\right) \frac{f b'}{rb} + \frac{f h'}{r h}, \quad (A.2) \\
E_\theta^\varphi &= - \left(1 + \frac{\ell^2}{3r^2} - \frac{\ell^2 h}{2r^4} - \frac{\ell^2 f}{2r^4}\right) \left(h' - \frac{h^2}{2h} + \frac{f h'}{2f}ight) f \frac{w^2}{4b}, \quad (A.3) \\
\frac{1}{\sin^2 \theta} E_{\varphi_1}^\varphi &= \left(f' + \frac{b'}{b} \frac{f}{2r} - \left(1 - \frac{f}{4} - \frac{b'}{r^2}\right) \left(1 + \frac{\ell^2 f b'^2}{8b^2} \right) \frac{4}{r^2} - \left(1 + \frac{r' f}{2f} - \frac{rh'}{2h}\right) \frac{fh'}{2r h}ight) \left(\frac{r f h'}{r b} - \frac{fh'}{2b} - \frac{3f h'}{2r} w' w'' \right) \frac{1}{r^2}, \quad (A.4) \\
\frac{1}{\sin^2 \theta} E_{\varphi_2}^\varphi &= \left(- \frac{2f r}{r^2} - \frac{\ell^2 f}{2r^4} - \frac{3\ell^2 h}{2r^4}\right) \left(- \frac{f'}{b} + \frac{f'}{f} + \frac{3h'}{h} w' + w'' \left(\frac{w f b}{2b} + \frac{\ell^2 f h}{4br} w' w'' \right) \frac{1}{r^2}. \quad (A.5)
\end{align*}
$$
\[ E_t^\ell = -\frac{4}{\ell^2} - \left(1 + \frac{2\ell^2}{r^2} - \frac{\ell^2 f}{2r^2} - \frac{\ell^2 h}{2r^2}\right) \frac{f'}{r} - \left(1 - r\left(\frac{f'}{6f} + \frac{h'}{h}\right)\right) \frac{3\ell^2 fh}{r^4} + \frac{fh'}{r} \]

\[
\left(1 + \frac{3\ell^2 h}{2r^4}\right) \frac{fh}{rb} w' + \left(1 + \frac{2\ell^2}{r^2} - \frac{\ell^2 f}{2r^2} - \frac{3\ell^2 h}{2r^4}\right) \left(\frac{h'}{h} + \frac{hw'}{b}\right) \frac{f'}{4} \quad \text{(A.6)}
\]

\[- \left(1 + \frac{2\ell^2}{r^2} - \frac{\ell^2 f}{2r^2} + \frac{3\ell^2 h}{2r^4}\right) \frac{fh'^2}{4h^2} + \left(1 + \frac{2\ell^2}{r^2} - \frac{\ell^2 f}{2r^2} - \frac{5\ell^2 h}{2r^4}\right) \frac{3fh'}{4b} w' + \left(1 + \frac{2\ell^2}{r^2} - \frac{\ell^2 f}{2r^2} - \frac{3\ell^2 h}{2r^4}\right) \left(\frac{h'}{h} - \frac{wb'w'}{2b^2} + \frac{w'^2}{2b} + \frac{ww''}{b}\right) \frac{fh'}{2} .
\]

However, the following relation holds\(^{21}\):

\[
dE^\gamma_r = \left(\frac{-2}{r} - \frac{1}{2b} - \frac{h'}{2h}\right) E^\gamma_r + \left(\frac{2}{r} + \frac{h'}{2h}\right) E^0_y + \frac{h'}{2h} \sin^2 \theta \frac{E^0_{\phi^2}}{\sin^2 \theta} + \frac{b'}{2b} E^t_t .
\quad \text{(A.7)}
\]

Thus we are left with a set of four essential equations for the metric functions \(f, b, h, w\). Note also that the equation \(E^t_{\phi_1}\) implies the existence of the first integral

\[
r^2 h \sqrt{\frac{fh}{b}} \left(1 + \frac{2\ell^2}{r^2} - \frac{\ell^2 f}{2r^2} - \frac{3\ell^2 h}{2r^4}\right) w' = \text{const.} \quad \text{(A.8)}
\]

In the numerical construction of the solutions, we have chosen to solve a suitable linear combination of the equations

\[
E^0_0 = 0, \quad E^\phi_{\phi^2} = 0, \quad E^t_{\phi_1} = 0 \quad \text{and} \quad E^t_t = 0 . \quad \text{(A.9)}
\]

The remaining equation \(E^r_r = 0\) becomes a constraint, which is implicitly satisfied for the chosen set of boundary conditions. Here we use the observation that (A.7) implies that (A.9) can be written as \(\frac{d}{dr}(r^2 \sqrt{b h} E^r_r) = 0\), i.e., \(r^2 \sqrt{b h} E^r_r = \text{const.}\) However, since \(b(r_H) = 0\), we find that \(E^r_r \equiv 0\). In practice, we have used the \(E^r_r\) equation together with the first integral (A.8) to monitor the accuracy of the numerical results.

For completeness, we mention that the same set of equations are found when considering instead an effective Lagrangean approach. Without fixing a metric gauge, a straightforward computation leads to the following reduced action for the system

\[
A_{\text{eff}} = \int dr \left[ \sqrt{\frac{fh}{b}} \left(b'g' + \frac{g}{2h}b'h' + \frac{b}{2f}g'' + \frac{b}{h}g'h' + \frac{1}{2}ghw'^2 + \frac{2b}{f}(4 - \frac{h}{g} + \frac{12bg}{r^2 f})\right) + \frac{\ell^2}{8} \left(\frac{4h}{g}b'g' + 2(4g - 3h)\left(\frac{b'h'}{h} + hw'^2\right) - \frac{f}{2b}b'h'g'' - \frac{1}{2}fhh''w'^2\right) \right] .
\]

Then, when taking the variation of \(A_{\text{eff}}\) with respect to \(a, b, f, g\) and \(w\) and fixing afterwards the metric gauge \(g(r) = r^2\) one finds a linear combination of the Eqs. (A.2)–(A.4).

\(^{21}\)The existence of (A.7) is a consequence of the generalized Bianchi identities \(E^\mu_{\nu,\mu} = 0\).
B. The coefficients of the perturbative rotating black hole solution

The coefficients which enters the perturbative solution (4.18) have the following expression, up to order four:

\[
\begin{align*}
  b_{11} &= 0, \\  b_{21} &= -\frac{3}{16} + \frac{\beta^2}{8} - \beta^4, \\  b_{22} &= 0, \\  b_{31} &= \frac{23}{8} - \frac{59}{36} \beta^2 + \frac{7}{18} \beta^4 - 3 \beta^6, \\  b_{32} &= \frac{1}{576} (387 - 4 \beta^2 (305 - 323 \beta^2 + 432 \beta^4)), \\  b_{33} &= 0, \\  b_{41} &= \frac{1}{18432} (-751203 + 2 \beta^2 (190027 + 2 \beta^2 (-56497 + 55106 \beta^2 - 57600 \beta^4 + 9216 \beta^6))), \\  b_{42} &= \frac{1}{18432} (-235989 + 2 \beta^2 (195277 + 2 \beta^2 (-90391 + 80462 \beta^2 - 63360 \beta^4 + 9216 \beta^6))), \\  b_{43} &= \frac{1}{18432} (-54081 + 228786 \beta^2 - 305132 \beta^4 + 21644 \beta^6 - 142848 \beta^8), \\  b_{44} &= 0,
\end{align*}
\]

\[
\begin{align*}
  f_{11} &= -\frac{1}{2}, \\  f_{21} &= \frac{1}{16} (21 - 22 \beta^2), \\  f_{22} &= \frac{1}{16} (9 + 2 \beta^2), \\  f_{31} &= \frac{1}{576} (-6039 + 5020 \beta^2 - 3292 \beta^4 + 288 \beta^6), \\  f_{32} &= \frac{1}{288} (-1683 + 308 \beta^2 + 52 \beta^4), \\  f_{33} &= \frac{1}{576} (-2043 + 20 \beta^2 (17 + \beta^2)), \\  f_{41} &= \frac{1}{6144} (724131 - 598870 \beta^2 + 391492 \beta^4 - 191496 \beta^6 + 35328 \beta^8), \\  f_{42} &= \frac{1}{18432} (1277973 - 2 \beta^2 (182029 - 83630 \beta^2 + 5020 \beta^4 + 2304 \beta^6)), \\  f_{43} &= \frac{1}{18432} (898299 - 285446 \beta^2 + 136420 \beta^4 - 2120 \beta^6), \\  f_{44} &= \frac{1}{18432} (406575 - 67502 \beta^2 - 300 \beta^4 - 40 \beta^6),
\end{align*}
\]

\[
\begin{align*}
  h_{11} &= -\frac{3}{2}, \\  h_{21} &= \frac{9}{2} - \frac{3 \beta^2}{2} + \beta^4, \\  h_{22} &= \frac{69}{16} + \frac{5 \beta^2}{8}, \\  h_{31} &= \frac{1}{64} (-2565 + 836 \beta^2 - 596 \beta^4 + 288 \beta^6), \\  h_{32} &= \frac{1}{64} (-1863 + 4 \beta^2 - 284 \beta^4), \\  h_{33} &= \frac{1}{192} (-4311 - 316 \beta^2 + 100 \beta^4), \\  h_{41} &= \frac{1}{1536} (730899 - 256454 \beta^2 + 199012 \beta^4 - 97736 \beta^6 + 37248 \beta^8 - 3072 \beta^{10}), \\  h_{42} &= \frac{1}{1024} (325629 - 21066 \beta^2 + 52476 \beta^4 - 17528 \beta^6 + 1280 \beta^8), \\  h_{43} &= \frac{1}{1536} (352845 + 694 \beta^2 + 60668 \beta^4 + 2952 \beta^6), \\  h_{44} &= \frac{1}{6144} (874665 + 21310 \beta^2 - 27316 \beta^4 + 2280 \beta^6),
\end{align*}
\]
\[ w_{11} = 1, \ w_{21} = 0, \ w_{22} = -1 + 3\beta^2, \ w_{31} = 0, \ w_{32} = 3 - \frac{27}{4}\beta^2 + \frac{9}{2}\beta^4 - 2\beta^6, \]

\[ w_{33} = \frac{1}{16}(107 + 124\beta^2(-1 + 2\beta^2)), \ w_{41} = 0, \]

\[ w_{42} = \frac{1}{96}(-2565 + 6335\beta^2 - 2844\beta^4 + 2068\beta^6 - 864\beta^8), \]

\[ w_{43} = \frac{1}{576}(-23733 + 24798\beta^2 - 31276\beta^4 + 16648\beta^6 - 6912\beta^8), \]

\[ w_{44} = \frac{1}{1152}(-47835 + 62600\beta^2 - 39548\beta^4 + 21266\beta^6). \]

To the same order, the coefficients which enter the expressions \([1,23]\) of the global quantities are

\[ M^{(0)} = \frac{3\pi\ell^2}{8G}\beta^2(1 + \beta^2), \quad S^{(0)} = \frac{\pi^2\ell^3}{4G}\beta(3 + 2\beta)^2, \quad T^{(0)}_H = \frac{\beta}{\pi\ell}, \quad (B.1) \]

and

\[ M_1 = -\frac{3}{16}, \quad M_2 = \frac{1}{128}(81 - 18\beta^2 + 32\beta^4), \quad M_3 = \frac{1}{1536}(-9351 + 4\beta^2(620 - 413\beta^2 + 456\beta^4)), \]

\[ M_4 = \frac{1}{49152}(3674799 - 1054150\beta^2 + 872692\beta^4 - 475272\beta^6 + 290304\beta^8 - 24576\beta^{10}), \]

\[ j_1 = 3 + 2\beta^2, \quad j_2 = \beta^2(1 + 2\beta^2), \quad j_3 = \frac{1}{8}\beta^2(1 + 2\beta^2)(22\beta^2 - 21), \]

\[ j_4 = -\frac{1}{288}\beta^2(1 + 2\beta^2)(-6039 + 5020\beta^2 - 3292\beta^4 + 288\beta^6), \]

\[ w_1 = 1, \quad w_2 = -1 + 3\beta^2, \quad w_3 = \frac{1}{16}(155 - 4\beta^2(58 - 49\beta^2 + 8\beta^4)), \]

\[ w_4 = \frac{1}{1152}(-126081 + 4\beta^2(47054 - 34057\beta^2 + 19844\beta^4 - 6048\beta^6)), \]

\[ s_1 = -3, \quad s_2 = \frac{39}{4} - 4\beta^2 + 2\beta^4, \quad s_3 = \frac{1}{96}(-91117 + 4\beta^2(722 - 527\beta^2 + 288\beta^4)), \]

\[ s_4 = \frac{1}{3072}(3589695 - 2\beta^2(599607 - 472522\beta^2 + 235924\beta^4 - 106752\beta^6 + 7680\beta^8)), \]

\[ t_1 = -1, \quad t_2 = \frac{13}{4} - \frac{5}{2}\beta^2 - 2\beta^4, \quad t_3 = \frac{1}{288}(-9117 + 4172\beta^2 - 1940\beta^4 - 2880\beta^6), \]

\[ t_4 = \frac{1}{3072}(1196565 + 2\beta^2(-277901 + 2\beta^2(58871 + 60002\beta^2 - 36864\beta^4 + 5376\beta^6))). \]

We close this part by remarking that similar expressions have been found also for orders five, six and seven, without being possible to identify a general pattern for the coefficients.
C. An exact solution with non-vanishing rotation

By using an 'educated guess' approach,\textsuperscript{22} we have found the following exact solution of the CS equations \[(A.2)-(A.6)\]:

\[
\begin{align*}
f(r) &= 1 + \frac{2r^2}{\ell^2} + c_1 F(r), \\
b(r) &= \left(1 + \frac{2r^2}{\ell^2} \frac{1}{1+c_1 F(r)}\right)(1 + c_1), \\
h(r) &= r^2(1 + c_1 F(r)), \\
w(r) &= \frac{\sqrt{2c_1}}{\ell \sqrt{1+c_1}} \frac{F(r)-1}{1+c_1 F(r)},
\end{align*}
\]

where

\[
F(r) = \sqrt{1 + \frac{c_2}{r^2}},
\]

\(c_1\) and \(c_2 > 0\) being constants of integration. As \(r \to \infty\), this solution becomes

\[
\begin{align*}
f(r) &= \frac{2r^2}{\ell^2} + 1 + c_1 + O(1/r^2), \\
b(r) &= \frac{2r^2}{\ell^2} + (1 + c_1)(1 - \frac{c_1 c_2}{(1 + c_1)^2 \ell^2}) + O(1/r^2), \\
h(r) &= r^2(1 + c_1) + \frac{c_1 c_2}{2}, \\
w(r) &= \frac{c_1 c_2}{\sqrt{2}(1 + c_1)^{3/2} \ell r^2},
\end{align*}
\]

such that the boundary \(S^3\) sphere is squashed in this case. Then the formalism described in the Section 2 leads to the following global charges of this solution

\[
\mathcal{M} = \frac{\pi}{32G\sqrt{1+c_1}} (-16c_1^2 c_2 + (1 + c_1)^2 (5c_1 - 3) \ell^2), \\
J = \frac{\pi \ell}{8\sqrt{2}G} (1 - c_1) c_1 c_2.
\]

Unfortunately, one can see that this configuration possess some unphysical properties for any non-vanishing \(c_1, c_2\). First, if we take \(c_1 > 0\), then \(r = 0\) corresponds to a naked singularity\textsuperscript{23} (e.g. \(R \to -3c_1 \sqrt{c_2}/r^3\) in that limit). If we choose instead to take \(c_1 < 0\) such that the condition \(f(r_0) = b(r_0) = 0\) is satisfied (with \(r_0 > 0\)), one finds that \(r = r_0\) is a singular point, with \(h(r_0) < 0\).

Finally, let us mention the existence of another solution, which is find by taking an appropriate limit in (C.1):

\[
\begin{align*}
f(r) &= 1 + \frac{2r^2}{\ell^2} + \frac{c_1}{r}, \\
b(r) &= \left(1 + \frac{2r^2}{\ell^2} \frac{1}{1+\frac{c_1}{r}}\right)(1 + c_1), \\
h(r) &= r^2(1 + \frac{c_1}{r}), \\
w(r) &= \frac{\sqrt{2c_1 \sqrt{1+c_1}}}{\ell r(1+\frac{c_1}{r})}.
\end{align*}
\]

However, one can easily show that this solution inherits all pathologies of (C.1). Moreover, its asymptotics are nonstandard in this case, with a different form than (C.2).

\textsuperscript{22}In deriving this solution, we have started by imposed the 'ad hoc' condition \(b(r) = \frac{r^2 f(r)}{h(r)}\), (which holds for the Myer-Perry-AdS\textsubscript{2} black hole). In the next step, we have set \(h(r) = r^2 U(r)\) and looked for a suitable combination of \[(A.3)-(A.6)\] which could be solved analytically. The last step is to rescale \(b, w\) to make them compatible with the asymptotics \[(A.4)\].

\textsuperscript{23}Note that we have failed to find rotating generalizations of the \(k = 1\) AdS background \[(3.1)\], \[(3.3)\], both numerically and also when looking for a perturbative solution.
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