Maxwell-Chern-Simons Q-balls

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Abstract: We examine the energetics of Q-balls in Maxwell-Chern-Simons theory in two space dimensions. Whereas gauged Q-balls are unallowed in this dimension in the absence of a Chern-Simons term due to a divergent electromagnetic energy, the addition of a Chern-Simons term introduces a gauge field mass and renders finite the otherwise-divergent electromagnetic energy of the Q-ball. Similar to the case of gauged Q-balls, Maxwell-Chern-Simons Q-balls have a maximal charge. The properties of these solitons are studied as a function of the parameters of the model considered, using a numerical technique known as relaxation. The results are compared to expectations based on qualitative arguments.

Résumé: Nous examinons l’énergétique de solitons non topologiques nommés Q-balls en théorie Maxwell-Chern-Simons en deux dimensions d’espace. Tandis que les Q-balls jaugés ne peuvent exister en cette dimension en raison d’une divergence de l’énergie électromagnétique, l’ajout du terme de Chern-Simons rend massif le champ de jauge et élimine cette divergence. Comme dans le cas jaugé, les Q-balls Maxwell-Chern-Simons ont une charge maximale. Les propriétés de ces solitons sont étudiées en fonction des paramètres du modèle, utilisant une technique numérique nommée relaxation. Les résultats ainsi obtenus sont comparés aux attentes basées sur des arguments qualitatifs.

1. Introduction

A class of non-topological solitons (see [1] for a comprehensive review) dubbed Q-balls were examined some time ago by Coleman [2]. These objects owe their existence to a conserved global charge. Under certain circumstances, a localized configuration of charge \( Q \) can be created which has a lower energy than the “naive" lowest-energy configuration, namely, \( Q \) widely-separated ordinary particles (each of unit charge) at zero momentum. This latter state obviously has energy \( Qm \), where \( m \) is the mass of the quanta of the theory. If another configuration of charge \( Q \) can be constructed whose energy is lower, then that state cannot decay into ordinary matter: either it is stable or some other “non-naive" configuration of the same charge and still lower energy is stable.

A number of variations have been studied since, including non-abelian Q-balls [3, 4], gauged Q-balls [5, 6], Q-balls in other dimensions [7, 8], higher-dimensional Q-objects [9, 10], spinning Q-balls [11, 12], and so on.

Of particular interest here is the paper of Lee, et al. [5], who discussed the case of gauged Q-balls in three dimensions, using a combination of analytical and numerical techniques.

In two dimensions, the gauged Q-ball’s existence is problematic, for a fairly straightforward reason: the electric field goes like \( 1/r \) and the electric field energy diverges logarithmically. Such a divergence...
is sufficiently mild that one could still contemplate a configuration of several positively- and negatively-charged $Q$-balls with total charge neutrality, in the spirit of global cosmic strings and vortices in liquid helium which also have logarithmically divergent energies. Nonetheless, strictly speaking, an isolated gauged $Q$-ball in two dimensions has divergent energy and therefore cannot hope to compete energetically with ordinary matter.

However, a new possibility exists in two dimensions: one can consider a model with a Chern-Simons term, either on its own or in addition to the usual Maxwell term. Among the well-known physical effects of the Chern-Simons term (to say nothing of its profound mathematical properties) are a greater interplay between electric and magnetic phenomena (for example, a static charge distribution gives rise to both electric and magnetic fields) [13, 14], parity and time reversal violation [15], fractional spin and statistics [16, 17], and mass generation for the gauge field [18]. The latter property is particularly pertinent here since the electric field of a Maxwell-Chern-Simons (MCS) $Q$-ball decays exponentially, and the argument given above leading to the conclusion that the electric field energy diverges no longer applies. Thus, we can address the question of whether gauged $Q$-balls exist (that is to say, whether they can compete energetically with ordinary matter) in 2 space dimensions if the Chern-Simons term is present.

In this paper we do such an analysis, numerically. We begin by briefly reviewing refs. [2, 5], which discuss the ungauged and gauged cases, respectively. We then move on to the MCS case, describing the model studied and the $Q$-ball ansatz generalized to the MCS case. Next, we make some qualitative observations to indicate what we might expect. In the remainder of the paper, we describe the numerical approach used and the results obtained.

Before getting our hands dirty, it is perhaps useful to address the question: Why should we study MCS $Q$-balls in the first place? The motivation lies mainly in the intrinsic interest of the model studied, and of the Chern-Simons term in particular. This term is as interesting mathematically as physically. On the mathematical side, one can mention topological field theory and applications to knot theory; on the physical side, at least two concrete physical applications have been advanced: the fractional quantum Hall effect (see [19] for a thorough discussion), and a proposed mechanism of superconductivity based on anyons [20]. Thus, it is worth studying any effect of this term. Furthermore, the model studied below (see (1)) is a relatively simple and uncontrived one, and whenever a fairly simple model gives rise to interesting phenomena such as solitons, it is worth examining in detail, without the need to evoke concrete physical applications to justify the work.

2. Ungauged Q-balls

The simplest model in which $Q$-balls exist is a complex scalar field theory. A wide variety of potentials give rise to $Q$-balls; the precise requirements of the potential are that it be minimized at the origin (so that the symmetry is unbroken), and that a parabola passing through the origin exists which, firstly, is wider than the potential at the origin, and, secondly, intersects the potential at some nonzero field value.

The $Q$-ball ansatz is $\phi(x) = f(r) \exp(-i\omega t)$, where $f(r)$ is a real function interpolating between some nonzero value (determined dynamically) at the origin and zero at infinity. Since the equation of motion for $f$ is a single second-order differential equation, a simple but revealing mechanical analogy exists, wherein $f$ is the position and $r$ the time; with this, one can see that $Q$-balls have the following properties: 1. They exist with any charge; 2. As $\omega$ increases, the charge of the $Q$-ball decreases; 3. For large $Q$, $f$ is approximately constant and independent of $Q$ inside the $Q$-ball, and the energy is proportional to $Q$. Furthermore, there is no significant difference in 2+1 dimensions (that is to say, all these properties remain true, though the details of the solution will, of course, differ).

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3. Gauged Q-balls

The gauged case is considerably more difficult to analyze, essentially because the mechanical analogy mentioned in the previous section is no longer useful. Nonetheless, Lee, et al. [5] studied the case of gauged \( Q \)-balls in three dimensions. They argued that when the charge exceeds a critical value the \( Q \)-ball’s energy exceeds \( Qm \), so the \( Q \)-ball is at best metastable. This is intuitively reasonable, since a ball of electric charge will have a Coulomb energy which grows roughly as the square of the charge, so eventually the \( Q \)-ball will be unable to compete with ordinary matter. On the other hand, as the charge decreases the \( Q \)-ball gets smaller and smaller; surface effects become important and eventually destabilize the \( Q \)-ball. These two observations indicate that there may or may not be a range of charges for which \( Q \)-balls exist, depending on the point at which each of these effects becomes significant.

The gauged \( Q \)-ball ansatz is that of the ungauged case to which is added a scalar potential which describes the electric charge of the \( Q \)-ball. In a way, the gauged \( Q \)-ball is the compliment of a superconducting vortex: whereas the latter has \( |\phi| \) interpolating between zero at the origin and the non-zero vev at infinity (so that the core is normal as opposed to superconducting), the gauged \( Q \)-ball is superconducting in the core and normal outside.

As indicated above, while there is no significant difference between \( Q \)-balls in ungauged models in 2+1 and 3+1 dimensions, this is not the case with gauged \( Q \)-balls: they have infinite energy in 2+1 dimensions. This leads us to consider the addition of a Chern-Simons term, with which the energy is rendered finite.

4. Maxwell-Chern-Simons Q-balls: Generalities

The model we consider has a complex scalar field \( \phi(x) \) with gauged U(1) symmetry, described by the following Lagrangian (in 2+1 dimensions):

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \frac{\kappa}{2} \epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma + |D_\mu \phi|^2 - V(\phi),
\]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( D_\mu \phi = (\partial_\mu + ieA_\mu)\phi \). The potential must satisfy the same requirements mentioned in Section 2. These are satisfied with the potential

\[
V(\phi) = \phi^* \phi - \frac{1}{2} (\phi^* \phi)^2 + \frac{g}{3} (\phi^* \phi)^3,
\]

if \( g > 3/16 \).

Since the Chern-Simons term renders the gauge field massive, the electric field decays exponentially rather than like \( 1/r \), and its contribution to the \( Q \)-ball energy is finite. By rescaling the fields and position, the mass of the \( \phi \) field can be set to unity, setting the standard to which \( Q \)-balls must be compared. If we can construct a field configuration for which \( E/Q < 1 \) (where \( Q \) is the particle number, henceforth referred to simply as charge), then it cannot decay into ordinary matter, and either it or some other lower-energy configuration of the same charge is stable. To look for such a configuration, we use an ansatz much like that of the gauged \( Q \)-ball, along with an appropriate form for the vector potential (recall that, with the Chern-Simons term, electric charge is a source for both electric and magnetic fields), namely:

\[
\phi(x) = e^{-i\omega t} f(r), \quad A_0^\alpha(x) = \alpha(r), \quad A^i(x) = \frac{\epsilon^{ij} r \beta_j}{r} \beta(r).
\]

The fields \( f \) and \( \alpha \) are nonzero at the origin and tend to zero exponentially as \( r \to \infty \), while \( \beta \) is zero at the origin, rises, and then returns to zero (as \( 1/r \)) as \( r \to \infty \).

There is a restriction on the frequency \( \omega \) for which \( Q \)-balls may exist. The \( \omega \)-dependent term can be thought of as being a part of an effective potential, and (as explained in [2]) for \( Q \)-balls to exist,

\[
\sqrt{1 - \frac{3}{16g}} < \omega < 1.
\]

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For ungauged $Q$-balls, values of $\omega$ near the upper end of this range correspond to small $Q$-balls, while the lower end of the range corresponds to large $Q$-balls. Things are slightly more complicated in the gauged case (whether with or without a Chern-Simons term), as will be described below.

The equations of motion have four parameters: $e, g, \kappa, \omega$. Of these, the first three are parameters of the model itself (appearing in (1)), while the fourth is a parameter of the ansatz.

Once a solution is found, the energy and charge are evaluated numerically in a straightforward manner, and the ratio of the two indicates whether decay to ordinary matter is energetically possible or not. Of particular interest is the lowest value of $E/Q$ for given parameters of the model, since this indicates the maximal energy savings gained in forming a $Q$-ball; it is thus an indicator of the most stable configuration (at least, among $Q$-balls).

What can be said qualitatively, before attacking the problem numerically? Even in a gauged model without Chern-Simons term [5], a qualitative analysis is much more difficult than in the ungauged case because, with the addition of the gauge field, Coleman’s mechanical analogy no longer applies. This is all the more so in the MCS case, where the ansatz has three fields. However, a couple of general observations can easily be made.

Suppose we held $Q$ fixed and “turned on” $e$. Clearly, this would create an additional contribution to the $Q$-ball energy coming from the electromagnetic field. Thus, we would expect $E/Q$ to increase with $e$ for fixed $Q$; furthermore, we would expect the maximal charge at which $Q$-balls occur to decrease with increasing $e$, very similar to the gauged case [5].

As for the Chern-Simons term, as mentioned above, its presence is essential (in two dimensions) since without it the $Q$-ball’s energy diverges, so we cannot turn its coefficient $\kappa$ on, since it cannot be zero. How do we expect the energetics to vary as the coefficient of the Chern-Simons term varies? Since the gauge field’s mass is proportional to $\kappa$, the electromagnetic fields have a decay length of $1/\kappa$, so as $\kappa$ increases, the electromagnetic contribution to the $Q$-ball energy should decrease, and $Q$-balls should be stable over a wider range of the other parameters (or, equivalently, over a wider range of charges).

These two qualitative behaviours are indeed seen in our numerical work, as will now be discussed.

5. Maxwell-Chern-Simons Q-balls: Numerical analysis

The search for $Q$-balls was performed using an iterative method known as relaxation, as described in detail in [21]. A discretized configuration is provided as an initial guess; the algorithm estimates an error by determining to what extent this configuration is not a solution, adds a correction to the configuration in an intelligently-chosen “direction” in configuration space, re-estimates the error, and so on, until the error is sufficiently small.

Let us describe in detail our results for typical values of the parameters of the model: $(e, g, \kappa) = (0.1, 0.5, 2.0)$. Varying $\omega$ within the allowed range given in (4) (for $g = 0.5$, this is $0.7906 \leq \omega \leq 1$) reveals two classes of $Q$-balls, which can be described as small and large $Q$-balls. Small $Q$-balls, with charges ranging from about 20 to roughly 2000, exist in the range $0.8111 \leq \omega < 1.0$, while large $Q$-balls, with charges ranging from about 2000 to 43000, exist for $0.8111 \leq \omega \leq 0.8995$. For $\omega$ in the region $0.7906 \leq \omega \leq 0.8111$, no $Q$-ball solutions were found.

Since $E/Q \geq 1$ for ordinary matter, the energetic advantage of forming a $Q$-ball can be seen by comparing $E$ and $Q$. Fig. 1 shows $E/Q$ as a function of $\omega$ and of $Q$. The tiny gap in these figures is the numerical artifact just mentioned, due to the numerically delicate transition between small and large $Q$-balls.

Also displayed in Fig. 1 are typical small and large $Q$-balls. Note that, as was the case with gauged $Q$-balls [5], the charge density of large $Q$-balls increases from the centre to the exterior before dropping off, a behaviour which can be attributed to the repulsive gauge force. Note also that the prominent “tail” of the field $\beta$ in Fig. 1d is a pure gauge, as mentioned above; all physical quantities tend to zero exponentially as $r \to \infty$, as expected.
Fig. 1. $E/Q$ vs (a) $\omega$ and (b) $Q$, at parameter values $(e, g, \kappa) = (0.1, 0.5, 2.0)$. Note the tiny gap between the upper and lower branches of (a) (corresponding to small and large $Q$-balls, respectively), and the left and right branches of (b) (idem). Typical (c) small and (d) large $Q$-ball profiles, at $\omega = 0.85$. Their charges are 99.998 and 19557, respectively.

Two features of our results are unexpected. The first concerns the fact that, while $Q$ is a monotonic decreasing function of $\omega$ in the ungauged case, this is not the case of MCS $Q$-balls: small $Q$-balls act in this way, but the charge of large $Q$-balls is a monotonic increasing function of $\omega$. Indeed, the very existence of two $Q$-balls at the same value of $\omega$ is unlike the ungauged case.

The second unexpected feature of our results concerns the maximum $Q$-ball charge. As explained above, while there is no upper limit to the ungauged $Q$-ball’s charge, we anticipate a maximal charge in the gauged case (with or without the Chern-Simons term), due to the electromagnetic contribution to the $Q$-ball’s energy. This contribution, one would think, should give rise to $E/Q > 1$, at which point $Q$-balls are no longer energetically advantageous compared with ordinary matter. Indeed there is a maximal charge. However, as can be seen from Fig. 1a, this occurs when $E/Q$ is increasing but nonetheless considerably less than one. (At the maximal charge, $E/Q = 0.8540$). This is also the case with gauged $Q$-balls in three dimensions, as can be seen from Fig. 3 of [5]. Perhaps there is another explanation for this maximal charge (other than the inability to compete energetically with ordinary matter), but we have not come up with one.

Varying $e$ and $\kappa$ had the anticipated effect on $Q$-ball energetics, as described in the previous section. Details can be found in Ref. [22].
6. Conclusions

To summarize, we have argued that in two space dimensions, a Chern-Simons term (or some other mass generation mechanism for the gauge field) is necessary in order for $Q$-balls to have finite energy. Considering the simplest model which might then give rise to $Q$-balls, we have performed a numerical search for these objects. Two types were found for a wide range of parameters, large and small $Q$-balls, the former being more stable in that they typically have lower values of $E/Q$. While many aspects of the behaviour of these objects are largely as expected, the fact that large $Q$-balls cease to exist when they do (in particular, with $E/Q$ well below unity) is surprising and merits further study.

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