Balancing non-rectangular tables

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Abstract

Balancing square and rectangular tables by rotation has been an interesting way to illustrate the intermediate value theorem. The aim of this note is to show that the balancing act but with non-rectangular tables can be a nice application of the ergodic theorem (or more generally, invariant measures).

1 Introduction

If you have tried to set up a pick-nick table or used a stepladder, then you are certainly aware how of the wobbly table problem. How can you make this object stop to wobble (without using extra tools)? Well, if you have a square or even rectangular table, then you are in luck. It turns out that rotating the table along some axis will suffice to eventually stabilise it. Unfortunately, many offices are now full of tables which have a trapezoidal shape. What now?

Unfortunately, the arguments for square or rectangular table hinge crucially on the symmetry of the object. This note will hopefully convince you, that you should have no fear of buying tables which are less symmetric.

Here is a more detailed description of this problem, which was originally made public in [5] and [6]. You have some table (or other object) with 4 legs and you find yourself on some terrain. Find conditions on the legs and quite unrestricted conditions on the terrain, so that turning the table around some axis can make it stop to wobble. Note that the table may not be level, it will just stop to wobble.

The original arguments of [5] and [6] cover square tables on a continuous terrain, but is presented in an abstract setting which overlooks the rigidity of the problem. A more realistic setup is covered in [2] and [9]. The terrain needs to be Lipschitz (with a Lipschitz constant bounded above). One of the very first work which relates to this problem is [4]; the reader is directed to [2] for an extensive historical overview.

The main result of this note is to discuss an extension to cover cyclic quadrilaterals (a conjecture raised in [9]). The main narrative of the proof is presented in §2. However, much like the first presentations of this problem in [5] and [6], this narrative overlooks some technicalities. A closer look at the problem is then discussed in §3. The complete hypothesis are as follows:

- the quadrilateral formed by the ends of legs is cyclic
- its diagonals are of equal length.
- the angle (from the centre of the excircle) which support the diagonal is rational
the angle which brings one diagonal on the other is also rational.

Then the table can be stabilised by rotating. (As in [9] or [2], the terrain needs to be Lipschitz.)

A typical example of a non-rectangular table satisfying the conditions of the theorem which the reader might have in his office has the following shape: take an hexagon and cut along two opposite vertices. The resulting quadrilateral is a symmetric trapezoid (with 3 equal sides).

2 Main steps of the proof

2.1 Preliminaries

Let us start by some simple results, reductions, assumptions and notations:

• First off, it seems the very least to demand that your table be stable if the terrain is perfectly flat. Hence the ends of the legs must be on the same plane \( P \). The end of the legs will be denoted \( A, B, C \) and \( D \).

• The axis around which you rotate should be perpendicular to the plane \( P \) (you don’t want to turn your table upside down!).

• Also the legs should all run along the same circle (i.e. the quadrilateral \( ABCD \) is cyclic) otherwise stabilisation by rotation is not possible (the terrain could be high along the circle described by one leg and low along the circle described by another leg). This is a necessary condition raised in [9]. \( Z \) will denote the centre of this circle.

• This note stays in a fairly idealised setup, Physical problems, such as “the terrain is going through the [legs of the] table”, “the table will turn over since its centre of mass is ill-placed” or “the legs have a thickness”, will be ignored.

• The whole problem boils is about the four feet of the table. The actual shape of the table is not really important.

• The table starts in the air. There is some basic position corresponding to the angle \( 0^\circ \). This means that, in this basic position, the leg \( A \) makes an angle of \( 0^\circ \) as seem from the excentre \( Z \). The angles \( \theta_B, \theta_C \) and \( \theta_D \) are the angles which make the other legs when in this basic position. (For convenience \( \theta_A = 0 \).)

• As you turn your table, all the angles of the legs are changed by the same number. (The table is rigid after all.)

2.2 Putting the table down

The next step is to look at possible touchdowns for your table. As you rotate the table consider the following way to put it down. First, you could put the legs \( A \) and \( C \) on the ground (and completely ignore where \( B \) and \( D \) are). If \( B \) and \( D \) are above the terrain, then the table wobbles, if not, then this was not completely legal, but it is still useful to
think about it (as some sort of negative wobbling). \cite{2} discusses the touchdown much more carefully.

Next, you could do the same thing with \( B \) and \( D \) touching down first. Note that these touchdowns should happen on the same curve (some distorted circle), otherwise there is no hope to conclude. Assume for now that this is the case, see \cite{2} and \S3.1 for a correct description of this process.

Consider \( X \) to be the intersection point of the diagonals \( AC \) and \( BD \). If we make a touchdown with \( AC \), the coordinates of \( X \) do not change when the table wobbles. Likewise with \( BD \). This gives us two height functions: \( h_{AC}(\theta) \) is the height \((z\text{-coordinate})\) of \( X \) after an \( AC \)-touchdown, where \( \theta \) is an angle of rotation (of the table). And likewise for \( h_{BD} \).

**Lemma 2.1.** \textit{If the table wobbles then} \( h_{BD}(\theta) \neq h_{AC}(\theta) \).

**Proof.** If the table wobbles then one of the pair \( BD \) or \( AC \) can be pushed further down. As a consequence, the height of the centre will differ. \qed

The coordinates of \( X \) are a convex combinations of the coordinates of the legs. More precisely, it’s a small exercise with vectors that if \( \tau \in ]0,1[ \) and \( \mu \in ]0,1[ \), then
\[
\overrightarrow{OX} = \tau \overrightarrow{OA} + (1 - \tau) \overrightarrow{OC} = \mu \overrightarrow{OB} + (1 - \mu) \overrightarrow{OD}
\]

In particular, this holds for the height coordinate:

\[
h_{AC}(\theta) = \tau h_A(\theta) + (1 - \tau)h_C(\theta) \quad \text{and} \quad h_{BD}(\theta) = \mu h_B(\theta) + (1 - \mu)h_D(\theta)
\]

where \( h_E(\theta) \) gives the height of (the end of) leg \( E \) after a touchdown after a rotation [of the table] by the angle \( \theta \) and \( E \in \{A,B,C,D\} \). Since every leg can be brought at the position of the other leg by a rotation, the functions \( h_A, h_B, h_C \) and \( h_D \) are all identical up to a translation (see \S3.1 and \S3.2 for details).

### 2.3 Table turning

**Theorem 2.2.** \textit{Assume} \( h_E \) \textit{are continuous, then there is an angle} \( \theta \) \textit{so that the table does not wobble.}

**Proof.** Since \( h_E \) is measurable, let \( \int h_E = H \) (the average height). By assumption \( h_E \) is continuous, hence so are \( h_{BD} \) and \( h_{AC} \). Let \( h_\Delta = h_{BD} - h_{AC} \).

Assume the table cannot be stabilised, then, by Lemma 2.1 \( h_\Delta \) is never 0. Since \( h_\Delta \) is continuous, there is an \( \varepsilon > 0 \) so that,

\[
\forall \theta, \quad h_\Delta(\theta) > \varepsilon \quad \text{or} \quad \forall \theta, \quad h_\Delta(\theta) < -\varepsilon
\]

Without loss of generality, we may assume the first holds. Let \( \theta_0 \) be an irrational angle. Then

\[
\forall N, \quad \frac{1}{N} \sum_{i=1}^N h(i \cdot \theta_0) > \varepsilon
\]
On the other hand

\[ \frac{1}{N} \sum_{i=1}^{N} h(i \cdot \theta_0) = \frac{1}{N} \sum_{i=1}^{N} \left( \mu h_B(i \cdot \theta_0) + (1 - \mu) h_D(i \cdot \theta_0) - \tau h_A(i \cdot \theta_0) - (1 - \tau) h_C(i \cdot \theta_0) \right) \]

By the equidistribution theorem or the ergodic theorem (see, among many possibilities [1, Appendix 1], [5, Section 23.10], [10, Exercise 2.2.12] and [11])

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} h(i \cdot \theta_0) = \int h_\Delta = \mu \int h_B + (1 - \mu) \int h_D - \tau \int h_A - (1 - \tau) \int h_C \]

But the right-hand side is just

\[ \mu H + (1 - \mu) H - \tau H + (1 - \tau) H = H - H = 0. \]

Hence

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} h(i \cdot \theta_0) = 0 > \varepsilon \]

a contradiction.

\[ \square \]

3 A closer look

3.1 Equal hovering position

A tool from [2, §3] is to look only at the touchdown with respect to $AC$ and then consider how far the vertices $BD$ are in the air. This is done so that both vertices $B$ and $D$ are equally far from the ground and is called the “equal hovering position”.

Let us sketch how to apply Theorem 2.2 in this context. Consider the average (over all angles) of $h_{AC}$, $\int h_{AC}$. Since the average of $h_{BD}$ is equal to that of $h_{AC}$, the equal hovering position is on average 0. But the equal hovering position is continuously defined, hence if the average is 0, it should be 0 somewhere. When the equal hovering position is 0, then the table does not wobble.

However there is a hidden hypothesis in this argument. Namely, if the two diagonals do not have the same length, then it could be that $h_{BD}$ takes on different value than $h_{AC}$. Variations in the steepness of terrain could make the pair of vertices $B$ and $D$ linger different time in different regions. This leads to the following

Assumption 3.1. The length of the diagonals of $ABCD$ are equal.

3.2 Further assumptions

As mentioned before, [2] contains a detailed description how to realise the touchdown of two opposite vertices ($AC$ or $BD$). The focus of this section is to point out what the proof of Theorem 2.2 requires. The various height functions $h_A$, $h_B$, $h_C$ and $h_D$ need:
(H1). to be well-defined

(H2). to be continuous

(H3). to differ only by a translation, i.e. \( h_E(\theta) = h_A(\theta + \theta_E) \)

Before looking closer at this, let’s point the problem out: while letting the table down, you will need to turn the table according to other axes. This means there is \textit{a priori} an uncertainty in how you put the table down. As mentioned in [9] these constructions are highly non-unique.

For example, this plays an implicit role in Lemma 2.1. Indeed, on needs to assume that, for any given angle \( \theta \), the touchdown according to \( AC \) and \( BD \) are done in one go. So you cannot let the table down and see where \( AC \) end up, and then let the table down and see where \( BD \) end up. You need to let the table down, see which of the two first come up and then go on to the second.

In short, for the proof of Lemma 2.1, a transversality condition is probably necessary, namely that the \( z \)-coordinate of \( X \) is decreasing while letting the table down. This means that for a given rotation angle \( \theta \) the touchdown of both diagonals must be taken into account simultaneously. To make sure this is possible requires most certainly three additional assumptions.

The first of these assumptions, is that for transversality, the terrain needs to be a \( C^1 \)-function with some upper bound on the first derivative. Lipschitz-continuity (with an upper bound on the Lipschitz constant) is however sufficient, see [2] and [9] for details.

The second assumption comes from the fact that the height functions \( h_E \) should only depend on the angle (and on \( \theta_E \)). The touchdown \( h_{BD}(0) \) needs to be defined simultaneously with the touchdown \( h_{AC}(0) \). But then, \( h_{AC}(\theta_B), h_{AC}(\theta_C) \) and \( h_{AC}(\theta_D) \) are also defined (since the leg \( A \) will land where the legs \( B, C \) and \( D \) did).

Repeating this process, one sees that all angles \( k_1\theta_B + k_2\theta_C + k_3\theta_D \) (where \( k_1, k_2 \) and \( k_3 \in \mathbb{Z}_{\geq 0} \)) need to be defined at once. In order to avoid an infinite number of choices, this leads to the following two assumptions:

\textbf{Assumption 3.2.} The rotation bringing one diagonal on the other diagonal is rational.

\textbf{Assumption 3.3.} The rotation bringing one end of a diagonal on the other end is rational (i.e. the angle supporting the diagonals from the excircle is rational).

Lastly, note that Assumption 3.1 is particularly important for condition (H3) mentioned above.

Note that Assumptions 3.1 and 3.3 are both automatically verified in the case of a rectangle (diagonals of equal length is a characteristic feature of rectangles among parallelogram; the rotation is 180° since the excentre lies on the diagonal). Assumption 3.2 might however not hold

\textbf{Remark 3.4.} One might be able to get rid of the rationality assumptions. A standard way to do so would be to consider a sequence of tables with rational angles which tend to the desired (irrational) table. Since everything is happening over a compact part of \( \mathbb{R}^2 \), the stabilising positions of the rational tables will converge to (at least) one position. This position should stabilise the (irrational) table.  

\( \diamond \)
3.3 Invariant measures

Note that the proof relied on a very specific invariant measure: the uniform measure, which is invariant under translations.

Perhaps one way to get rid of Assumption 3.1 would be to consider a sequence \( \theta_i \) which tends to some well-chosen measure, rather than just taking this sequence of angles to be multiples of an irrational angle (\( i \cdot \theta_0 \) in the proof).

More concretely, let \( h_B(\theta) = h_A(\theta + f_B(\theta)) \), \( h_C(\theta) = h_A(\theta + f_C(\theta)) \) and \( h_D(\theta) = h_A(\theta + f_D(\theta)) \). The functions \( x \mapsto x + f_E(x) \) (where \( E \in \{B, C, D\} \)) generate a monoid (of circle maps). If the action of this monoid is amenable, then there is an invariant measure \( \mu \) (see [7] for generic background on amenability and [10] §2.3 and §3.2 for more specific details in the case of groups acting on the circle). Since The convex hull of Dirac masses is weak* dense in the space of means, one can choose the sequence of angles \( \theta_i \) so that \( \frac{1}{N} \sum_{i=1}^{N} \delta_{\theta_i} \) tends (weak*) to \( \mu \).

Consequently, the assumptions presented in §3.1 and §3.2 can perhaps be relaxed.

A simple way of satisfying the amenability assumption is to assume that the monoid is contained in the monoid given by some rational rotations (e.g. coming from Assumptions 3.3 and 3.2) as well as some map (which may not be a rational rotation). Since the monoid is then a finite extension of an amenable one (the single map which is not a rational yields a monoid isomorphic to \( \mathbb{N} \)), it is amenable as well.

The author would like to thank Yves de Cornulier [3] for pointing out that [10] essentially shows there are not other examples where this monoid is amenable (beside the given example where two maps are rational).

References

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