NONCOMMUTATIVE KHINTCHINE
AND PALEY INEQUALITIES
VIA GENERIC FACTORIZATION

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Abstract. We reprove an inequality for Rademacher series with coefficients in the Schatten class $S_1$. Our method yields the same estimate for coefficients after suitable gaps in $S_1$-valued trigonometric series; this was known for scalar-valued functions. A very similar method gives a new proof of the extension to $S_1$-valued $H^1$ functions of Paley’s theorem about lacunary coefficients.

1. Introduction

Given a function $f$ in $L^1((-\pi, \pi])$, form its Fourier coefficients

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} \, dt.$$  

Identify the interval $(-\pi, \pi]$ with the unit circle group $\mathbb{T}$ in the usual way. Extend formula (1.1) to Bochner integrable functions $f$ mapping $\mathbb{T}$ into the Schatten classes $S_p$ with $1 \leq p < \infty$. Let

$$\|f\|_{L^p(\mathbb{T}; S_p)} = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \|f(t)\|_{S_p} \right)^p \, dt \right\}^{1/p}.$$  

In Section 2 we discuss the properties of $S_p$ and $L^p(\mathbb{T}; S_p)$ that we use.

Let $(k_j)_{j=0}^\infty$ be a sequence of nonnegative integers for which

$$k_{j+1} > 2k_j$$  

for all $j$. Denote the range of such a sequence by $K$, and call $K$ strongly lacunary. In Theorem 1.2 below, we present two cases where, if $f \in L^1(\mathbb{T}; S_1)$ and if $\hat{f}$ vanishes on a suitable subset of the complement of such a set $K$, then the restriction of $\hat{f}$ to $K$ has special properties. This is known for scalar-valued functions, as is one of the cases for functions with values in $S_1$, but our proof is new in that context.

2010 Mathematics Subject Classification. Primary 46L52; Secondary 42A55, 46N30.

Announced at the 6th Conference on Function Spaces in May 2010.

Research partially supported by NSERC grant 4822.
We also consider functions in $L^1([0,1); S_1)$ whose Walsh coefficients, in the Paley ordering, vanish except at the powers of 2; see Section 3 for more details. The Walsh series for such functions reduce to Rademacher series $\sum_{j=0}^{\infty} d_j r_j(\cdot)$ with $S_1$-valued coefficients. We give a new proof that these coefficients also have special properties.

Those involve the following norm on some sequences, $(c_j)$ say, of compact operators on a Hilbert space, $\mathcal{H}$ say. Let

$$(1.3) \quad \| (c_j) \|_{c_E} = \left\| \sqrt{\sum_j c_j^* c_j} \right\|_{S_1} = \text{tr} \sqrt{\sum_j c_j^* c_j}.$$ 

This is the norm in $S_1(\ell^2(\mathcal{H}))$ of any operator-valued square matrix in which one column is the sequence $(c_j)$ and the other columns are trivial. Note that

$$(1.4) \quad \| (c_j^*) \|_{c_E} = \left\| \sqrt{\sum_j c_j c_j^*} \right\|_{S_1} = \text{tr} \sqrt{\sum_j c_j c_j^*}.$$ 

Finally, follow [10] and let

$$(1.5) \quad \| (c_j) \| = \inf \left\{ \| (a_j) \|_{c_E} + \| (b_j^*) \|_{c_E} : (c_j) = (a_j) + (b_j) \right\}.$$ 

In [7] this functional is denoted instead by $\| \cdot \|^*$, and a dual norm is denoted by $\| \cdot \|$.

We prove the assertions below, in which $C$ is an absolute constant, in Section 3.

**Theorem 1.1.** If a Rademacher series with coefficients $(d_j)$ represents a function $f$ in $L^1([0,1); S_1)$, then

$$(1.6) \quad \| (d_j) \| \leq C \| f \|_{L^1([0,1); S_1)}.$$ 

**Theorem 1.2.** Let $K$ be a strongly lacunary set, and let $f \in L^1(\mathbb{T}; S_1)$. Then the estimate

$$(1.7) \quad \| (\hat{f}(k_j)) \| \leq C \| f \|_{L^1(\mathbb{T}, S_1)}$$ 

holds in each of the following cases:

1. $\hat{f}$ vanishes on the set of negative integers.
2. $\hat{f}$ vanishes at all positive integers in the complement of $K$.

**Corollary 1.3.** The same estimate holds when $\hat{f}$ vanishes off $K$.

The corollary is a counterpart for trigonometric series of Theorem 1.1. In [10], Lust and Pisier proved Case 1 of Theorem 1.2 first, and deduced Theorem 1.1 from the corollary. Case 1 is an extension of Paley’s theorem about lacunary coefficients of scalar-valued $H^1$ functions [14]. The
proof in [10], like Paley’s, used a suitable factorization of $H^1$ functions as products of $H^2$ functions. That analytic factorization does not apply as readily in Case 2 and it is not available in Theorem 1.1. Instead, we give direct proofs of Theorem 1.1 and both parts of Theorem 1.2 using the generic factorization method introduced in [3, Section 2] and modified in [5, Section 2]. See Remark 3.2 below for further comparison of methods.

I thank Christian Le Merdy and Fedor Sukochev for pointing out an error in a earlier version of this paper. For more details, see Remark 3.4.

Remark 1.4. The methods used here work when $C = 2$. Dual methods in [7] show that Corollary 1.3 and Theorem 1.1 hold with $C$ equal to $\sqrt{2}$ and $\sqrt{3}$ respectively, and that $\sqrt{2}$ is the best constant in the corollary. For scalar valued functions, dual methods in [4] and [11] yield the two cases of Theorem 1.2 with constants $\sqrt{2}$ and $\sqrt{e}$ respectively; again, the former is best possible in that context. It is not known to what extent those dual methods extend to operator-valued functions. As indicated in Remark 3.3 below, further analysis of the methods used here supports the possibility of such extensions.

Remark 1.5. Case 2 of Theorem 1.2 for scalar-valued functions was rediscovered several times. That instance follows easily from another theorem of Paley [13], but this may not have been noticed until [3, Theorem 10] and [6]. Meanwhile, an equivalent dual construction had been found [11] by Clunie, and Meyer [11, pp. 532-533] had had used other methods to prove pointwise estimates that imply inequality (1.7) in that instance. Related proofs of Case 2 for such functions appeared in [17], [3, Theorem 11] and [9]. The hypotheses about $K$ and $f$ were significantly weakened in [18].

2. Properties of these operator spaces

See [16] for much more about the spaces $S_p$, which are often denoted by $c_p$ or $C_p$. The members of $S_1$ are the compact operators, on the fixed Hilbert space $\mathcal{H}$, whose sequence of singular values belongs to $\ell^1$. Define $\|A\|_{S_1}$ to be the sum of that sequence, that is the sum of the eigenvalues of $|A| := \sqrt{A^*A}$. Also denote that sum is by $\text{tr}(|A|)$, thereby defining the functional $\text{tr}$ on the set of positive operators in $S_1$. It extends to become a linear map from $S_1$ into the complex numbers.

For each positive real number $p$, the space $S_p$ consists of all operators $A$ for which $|A|^p \in S_1$; then

$$\|A\|_{S_p} := \left\{ \text{tr}\left[ \left(\sqrt{A^*A}\right)^p \right] \right\}^{1/p} = \left\{ \left\| \left(\sqrt{A^*A}\right)^p \right\|_{S_1} \right\}^{1/p}.$$
There is a counterpart of Hölder’s inequality, stating that

\[(2.1) \quad \|AB\|_{S_r} \leq \|A\|_{S_p} \|B\|_{S_q} \quad \text{when} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.
\]

Conversely, for such indices, operators the unit ball of \(S_r\) factor as products of operators in the unit balls of \(S_p\) and \(S_q\). There also are such factorizations in the form \(B^*A\), since \(\|B^*\|_{S_q} = \|B\|_{S_q}\).

The sets \(S_p\) are Banach spaces when \(1 \leq p < \infty\). Moreover, \(S_2\) is a Hilbert space, with inner product \(\langle A, B \rangle := \text{tr}(B^*A)\). It follows that \(L^2(\mathbb{T}; S_2)\) is also a Hilbert space with inner product

\[
\langle f, g \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{tr}[g(t)^*f(t)] \, dt.
\]

This is also equal to the trace of

\[(2.2) \quad \langle f, g \rangle_p := \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)^*f(t) \, dt.
\]

Call the operator-valued expression above a *partial inner product*.

Finally \(L^1(\mathbb{T}; S_1)\) is a Banach space, and a function belongs to its unit ball if and only that function is the product of two functions in the unit ball of \(L^2(\mathbb{T}; S_2)\). Call that a *generic factorization*. The proof of Case 1 in [10] also uses such factors, but adds the requirement [15, 12, 8] that their Fourier coefficients vanish at all negative integers. This is called *analytic factorization*.

**Remark 2.1.** As noted in [10, p. 250], a weaker form of analytic factorization suffices for that proof of Case 1. In Remark 3.1 we explain how our method works with a weaker form of generic factorization.

### 3. Two Orthogonality Steps

Recall that the Rademacher function \(r_0\) has period 1 on the real line \(\mathbb{R}\), and takes the values 1 and \(-1\) in the intervals \([0, 1/2)\) and \([1/2, 1)\) respectively; then \(r_j(t) := r_0(2^j t)\) for all \(t\) in \(\mathbb{R}\) and each positive integer \(j\). The Walsh functions \((w_n)_{n=0}^\infty\) are the products of finitely-many Rademacher functions, including the empty product 1. We use the Paley enumeration of this system, where \(w_n\) is the product of the distinct functions \(r_{jm}\) for which \(n = \sum_m 2^m\); in particular, \(w_0 = 1\) and \(w_{2^j} = r_j\). Our method transfers easily to other standard enumerations.

**Proof of Theorem [14]** Rescale to make \(\|f\|_{L^1([0,1); S_1)} = 1\). Then factor \(f\) as \(h^*g\) where \(g\) and \(h\) belong to the unit ball of the Hilbert
space $L^2([0, 1); S_2)$.

Rewrite the Walsh coefficients of $f$ in the form

(3.1) \[ \hat{f}(w_n) = \int_{0}^{1} w_n(t) h(t) g(t) \, dt; \]

in this context, use the notation $\langle g, w_n h \rangle_p$ for the integral above. The hypothesis in the theorem is that this partial inner product vanishes when $n$ is not a power of 2.

Let $A_j$ be the unitary operator on $L^2([0, 1); S_2)$ that multiplies each function by $w_{2^j}$. Matters reduce to splitting the sequence $(\hat{f}(w_{2^j})) = (\langle g, A_j h \rangle_p)$ as a sum of two sequences $(a_j)$ and $(b_j)$ for which

(3.2) \[ \|(a_j)\|_{C_E} \leq \frac{C}{2} \quad \text{and} \quad \|(b_j)^*\|_{C_E} \leq \frac{C}{2}. \]

This will be done using nested closed subspaces of $L^2([0, 1); S_2)$ and the orthogonal projections onto them.

Denote the set of functions orthogonal to a closed subspace, $M$ say, of $L^2([0, 1); S_2)$ by $M^\perp$. Since $\langle u, v \rangle = 0$ when $u \in M$ and $v \in M^\perp$, the trace of $\langle u, v \rangle_p$ vanishes in that case.

For some subspaces $M$, the stronger condition that $\langle u, v \rangle_p = 0$ holds when $u \in M$ and $v \in M^\perp$. We claim that this happens if $ub \in M$ whenever $u \in M$ and $b$ belongs to the space $B(\mathcal{H})$ of bounded operators on $\mathcal{H}$. Indeed, suppose that $M$ has the latter property, and observe that

(3.3) \[ \langle ub, v \rangle_p = (\langle u, v \rangle_p)b \]

for all functions $u$ and $v$ in $L^2([0, 1); S_2)$ and all bounded operators $b$. If $u \in M$ and $v \in M^\perp$, then $v \perp ub$ since $MB(\mathcal{H}) \subset M$; that is,

$$0 = \langle ub, v \rangle = \text{tr}(ub, v)_p.$$ By equation (3.3), $\text{tr}[(\langle u, v \rangle_p)b]$ then vanishes for all $b$ in $B(\mathcal{H})$. Therefore $\langle u, v \rangle_p = 0$, as claimed.

In that situation, denote the orthogonal projection with range equal to the closed subspace $M$ by $Q$, and denote the orthogonal projection with range $M^\perp$ by $Q^\perp$. Split any two members $F$ and $G$ of $L^2([0, 1); S_2)$ as $QF + Q^\perp F$ and $QG + Q^\perp G$. Since $\langle Q^\perp F, QG \rangle_p$ and $\langle QF, Q^\perp G \rangle_p$ both vanish,

(3.4) \[ \langle F, QG \rangle_p = \langle QF, QG \rangle_p = \langle QF, G \rangle_p. \]

Let $M_j$ be the smallest closed subspace of $L^2([0, 1); S_2)$ that contains all the products $w_n h b$ in which $0 < n < 2^{j+1}$ and $b \in B(\mathcal{H})$;
those factors $w_n$ are the nonempty products of distinct Rademacher functions $r_j$ with $j' \leq j$. Clearly,

$$M_0 \subset M_1 \subset \cdots \subset M_J \subset \cdots. \quad (3.5)$$

Then $A_j M_j$ is the smallest closed subspace containing all the products $w_n h b$ with $n \neq 2^j$ and $0 \leq n < 2^{j+1}$, and with $b \in B(\mathcal{H})$; these factors $w_n$ are the products of distinct Rademacher functions $r_j$ with $j' \leq j$ except for the singleton product that gives $r_j$. Again,

$$A_0 M_0 \subset A_1 M_1 \subset \cdots \subset A_j M_j \subset \cdots. \quad (3.6)$$

Moreover, $A_j h \in A_{j+1} M_{j+1}$. Finally, $A_{j+1} M_j$ is the smallest closed subspace containing all the products $w_n h b$ with $2^{j+1} < n < 2^{j+2}$ and $b$ in $B(\mathcal{H})$. It is assumed in Theorem 1.1 that $\hat{f}(w_n)$ for these indices $n$, that is $\langle g, w_n h b \rangle_p = 0$. Equation (3.3) then makes $\langle g, w_n h b \rangle_p = 0$ for all $b$ in $B(\mathcal{H})$. Hence $\langle g, v \rangle_p = 0$ for all functions $v$ in $A_{j+1} M_j$. Write this as $g \perp_p A_{j+1} M_j$.

Denote the orthogonal projection onto $A_j M_j$ by $Q_j$. The subspaces $M_j$ were chosen so that $M_j B(\mathcal{H}) \subset M_{j+1}$, and their images $A_j M_j$ also have this property. Use the fact that $A_j h \in A_{j+1} M_{j+1}$, and apply equation (3.4) with $M = A_{j+1} M_{j+1}$ to write

$$\langle g, A_j h \rangle_p - \langle g, Q_{j+1} A_j h \rangle_p = \langle Q_{j+1} g, A_j h \rangle_p \quad (3.7)$$

$$= \langle Q_j g, A_j h \rangle_p + \langle (Q_{j+1} - Q_j) g, A_j h \rangle_p = a_j + b_j \quad \text{say.} \quad (3.8)$$

Let $g_j = (Q_{j+1} - Q_j) g$. Since the orthogonal projections $Q_j$ nest,

$$\sum_j \|g_j\|^2_{L^2([0,1);L^2)} \leq \|g\|^2_{L^2([0,1);L^2)} = 1.$$ 

Now argue as in [10] p. 250]. Write $b_j^*$ as $\int_0^1 w_{2j}(t) g_j^*(t) h(t) \, dt$, and regard this as the average of the operators $w_{2j}(t) g_j^*(t) h(t)$. Since $\| \cdot \|_{C_E}$ is a norm,

$$\|b_j^*\|_{C_E} \leq \int_0^1 \|(w_{2j}(t) g_j^*(t) h(t))\|_{C_E} \, dt = \int_0^1 \|(g_j^*(t) h(t))\|_{C_E} \, dt.$$ 

Fix $t$, and expand the last inner norm above as

$$\sqrt{\left( h(t)^* \left( \sum_j g_j(t) g_j^*(t) \right) h(t) \right)}_{S_1}.$$ 

Let $G(t)$ be the operator $\sum_j g_j(t) g_j^*(t)$. The quantity above is equal to the square root of $\|h(t)^* G(t) h(t)\|_{S_1/2}$. By the H"older inequality (2.1),

$$\|h(t)^* G(t) h(t)\|_{S_1/2} \leq \|h(t)^*\|_{S_2} \|G(t) h(t)\|_{S_2/3}.$$
and \( \|G(t)h(t)\|_{S_{2/3}} \leq \|G(t)\|_{S_1} \|h(t)\|_{S_2} \).

Hence
\[
\|(b_j^*)\|_{c_E} \leq \int_0^1 \sqrt{\|h(t)^*\|_{S_2}} \|G(t)\|_{S_1} \|h(t)\|_{S_2} \, dt
= \int_0^1 \|h(t)\|_{S_2} \sqrt{\|G(t)\|_{S_1}} \, dt.
\]

Cauchy-Schwarz then gives the upper bound
\[
\|h\|_{L^2([0,1]; S_2)} \sqrt{\|G\|_{L^1([0,1]; S_1)}}.
\]

The first factor is equal to 1. The square of the second factor is
\[
\int_0^1 \text{tr} \sum_j g_j(t)g_j^*(t) \, dt = \sum_j \|g_j^*\|_{L^2([0,1]; S_2)}^2 = \sum_j \|g_j\|_{L^2([0,1]; S_2)}^2 \leq 1.
\]

This gives the second inequality in line (3.2) with \( C = 2 \).

Apply equation (3.4) with \( M = A_j M_j \) to rewrite \( a_j \) as \( \langle g, Q_j A_j h \rangle_p \).

Let \( P_j \) be the orthogonal projection with range \( M_j \). Since \( A_j \) is unitary,
\[
A_j P_j = Q_j A_j \quad \text{and} \quad a_j = \langle g, A_j P_j h \rangle_p.
\]

Recall that \( g \perp_p A_{j+1} M_j \) for all \( j \). So \( \langle g, A_j P_{j-1} h \rangle_p = 0 \) for all \( j \geq 1 \).

Make this true when \( j = 0 \) by letting \( P_{-1} = 0 \). Then
\[
a_j = \langle g, A_j (P_j - P_{j-1}) h \rangle_p \quad \text{for all} \quad j.
\]

Use the method applied to \( (b_j^*) \) to confirm that \( \|(a_j)\|_{c_E} \leq 1 \). \qedhere

**Proof of Case 2 of Theorem 1.2** Transfer the argument above as follows. Replace \( L^2([0,1]; S_2) \) with \( L^2(\mathbb{T}; S_2) \), and use the partial inner product given in formula (2.2). Factor a function \( f \) in the unit ball of \( L^1(\mathbb{T}; S_1) \) as \( h^*g \) where \( g \) and \( h \) belong to the unit ball of \( L^2(\mathbb{T}; S_2) \).

Let \( z \) be the function mapping each number \( t \) in the interval \( (-\pi, \pi] \) to \( e^{it} \). Then
\[
(3.9) \quad \hat{f}(n) = \langle g, z^n h \rangle_p \quad \text{for all} \quad n.
\]

Let \( A_j \) be the unitary operator on \( L^2(\mathbb{T}; S_2) \) that multiplies each function by \( z^{k_j} \). Then \( \hat{f}(k_j) = \langle g, A_j h \rangle \).

Let \( M_j \) be the closure of the subspace of \( L^2(\mathbb{T}; S_2) \) spanned by the products \( z^n h b \) for which \( -k_j \leq n < 0 \) and \( b \in B(\mathcal{H}) \). Then the inclusions (3.5) hold for these subspaces, as do the inclusions (3.6) for their images \( A_j M_j \). Again,
\[
A_j h \in A_{j+1} M_{j+1} \quad \text{and} \quad (A_j M_j)B(\mathcal{H}) \subset A_j M_j.
\]

Now \( A_{j+1} M_j \) is the closure of the subspace spanned by the products \( z^n h b \) in which \( b \in B(\mathcal{H}) \) and \( n \in [k_{j+1} - k_j, k_{j+1}) \).
lacunarity, that interval is included in \((k_j, k_{j+1})\). The gap hypothesis on \(\hat{f}(n)\) and formulas (3.9) and (3.3) then imply that \(g \perp_p A_{j+1}M_j\).

Define the projections \(Q_j\) and \(P_j\) as before, and split \(\langle g, A_jh \rangle_p\) in the same way. Estimate \(||(b'_j)\||_{C_E}\) and \(||(a_j)\||_{C_E}\) as above.

**Proof of Case 1 of Theorem 1.2.** Use the same factorization and the same operators \(A_j\) as in Case 2, but replace the subspaces \(M_j\) with the closures, \(L_j\) say, in \(L^2(T; S_2)\) of the span of the products \(z^n h b\) for which \(n < -k_j\) and \(b \in B(H)\). These subspaces nest in the opposite way, that is

\[
L_0 \supset L_1 \supset \cdots \supset L_J \supset \cdots
\]

In every case, the subspace \(A_jL_j\) is the same, namely the closure of the span of the products \(z^n h b\) with \(n < 0\). Formula (3.9) and the hypothesis that \(\hat{f}(n)\) for all \(n < 0\) now makes \(g \perp_p A_jL_j\) for all \(j\). The lacunarity hypothesis implies that \(A_jh \in A_{j+1}L_j\) for all \(j\), and that

\[
A_1L_0 \subset A_2L_1 \subset \cdots \subset A_{j+1}L_j \subset \cdots.
\]

Finally, \((A_{j+1}L_j)B(H) \subset A_{j+1}L_j\) in all cases.

Now denote the orthogonal projection onto \(A_{j+1}L_j\) by \(Q_{j+1}\), and let \(Q_0 = 0\). Then it is again true that

\[
\langle g, A_jh \rangle_p = \langle g, Q_{j+1}A_jh \rangle_p = \langle Q_{j+1}g, A_jh \rangle_p
\]

\[
= \langle Q_jg, A_jh \rangle_p + \langle (Q_{j+1} - Q_j)g, A_jh \rangle_p = a_j + b_j \quad \text{say}.
\]

Estimate \(||(b'_j)\||_{C_E}\) as before.

Note that \(a_0 = 0\) since \(Q_0 = 0\). When \(j > 0\), rewrite \(a_j\) as \(\langle g, Q_jA_jh \rangle_p\).

For those indices \(j\), let \(P_{j-1}\) be the orthogonal projection with range \(L_{j-1}\).

This time,

\[
Q_jA_j = A_jP_{j-1} \quad \text{and} \quad a_j = \langle g, A_jP_{j-1}h \rangle_p.
\]

Now \(\langle g, A_jP_jh \rangle_p = 0\), since \(g \perp_p A_jL_j\) for all \(j\). Write

\[
a_j = \langle g, A_j(P_{j-1} - P_j)h \rangle_p.
\]

The desired estimate \(||(a_j)\||_{C_E} \leq 1\) follows as before. □

**Remark 3.1.** A weaker form of generic factorization suffices. In Theorem 1.1, for instance, it is enough to prove that

\[
\|\hat{f}(w_j)\| \leq 2 \quad \text{when} \quad \|f\|_{L^1(D; S_1)} < 1.
\]

By the definition of Bochner integration, \(f\) can then be represented as the sum of a series

\[
\sum_{m=1}^{\infty} f_m
\]
where the terms \( f_m \) are simple functions, and \( \sum_{m=1}^{\infty} \| f_m \|_{L^1(D; S_1)} \leq 1 \). Generic factorization with simple factors is easy to check for simple functions. Use this to write \( f_m = h_m^* g_m \), where \( \sum_{m=1}^{\infty} \| g_m \|_{L^2(D; S_2)}^2 \leq 1 \) and the same is true for the factors \( h_m \).

The sequences \( g = (g_m) \) and \( h = (h_m) \) are members of the unit ball of the Hilbert space \( \ell^2(L^2(D; S_2)) \); their inner product is equal to \( \text{tr}(\langle g, h \rangle_p) \) for the partial inner product given by

\[
\langle g, h \rangle_p = \sum_{m=1}^{\infty} \int_D h_m^* g_m.
\]

The methods of this section work in this setting with \( L^2(D; S_2) \) replaced by \( \ell^2(L^2(D; S_2)) \), and \( A_j \) redefined to act on \( (h_m) \) by termwise multiplication.

**Remark 3.2.** The subspaces \( L_j \) used in Case 1 are invariant under multiplication by \( z \), and their adjoints are invariant under multiplication by \( z \). In the discussion of scalar-valued functions in [2], it is observed that the latter subspaces must be simply invariant if \( f \in H^1(\mathbb{T}) \), and that one can apply the characterization of simply invariant subspaces of \( L^2(\mathbb{T}) \) to show that both factors \( h^* \) and \( g \) can be chosen to belong to \( H^2(\mathbb{T}) \). One proof [12] of analytic factorization in \( H^1(\mathbb{T}; S_1) \) uses those ideas and more.

**Remark 3.3.** The generic factorization methods work equally well with the hypotheses in Theorem 1.2 weakened to only require that \( \hat{f} \) vanish on suitable smaller sets of integers. The same smaller sets arise in analyses of the dual methods for scalar-valued functions. See [3, Sections 4 and 5] for more details.

**Remark 3.4.** In this version of the paper, we worked with closed subspaces \( M \) for which the inclusion \( MB(\mathcal{H}) \subset M \) holds. Of course, this inclusion is really an equality, since the identity operator belongs to \( B(\mathcal{H}) \). We showed in Section 3 that the inclusion implies for the orthogonal projection \( Q \) onto \( M \) that

\[
\langle F, QG \rangle_p = \langle QF, G \rangle_p \quad \text{for all } F \text{ and } G.
\]

The converse is also true. Indeed, using equation (3.10) with \( F \) in \( M \) and \( G \perp M \) yields that \( \langle F, G \rangle_p = 0 \) in that case. It then follows from equation (3.3) that \( Fb \perp G \) for all \( b \) in \( B(\mathcal{H}) \) and all \( G \) in \( M^\perp \), that is \( Fb \in (M^\perp)^\perp = M \).

The error in the first version of this paper was the use of property (3.10), in six places like equation (3.7) in this version, for subspaces that do not have that property.
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