GLOBAL WELL-POSEDNESS OF THE ENERGY SUBCRITICAL NONLINEAR WAVE EQUATION WITH INITIAL DATA IN A CRITICAL SPACE

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ABSTRACT. In this paper we prove global well-posedness for the defocusing, energy-subcritical, nonlinear wave equation on $\mathbb{R}^{1+3}$ with initial data in a critical Besov space. No radial symmetry assumption is needed.

1. Introduction

In this paper, we continue the study of the defocusing, cubic nonlinear wave equation,

$$u_{tt} - \Delta u + |u|^{p-1} u = 0, \quad u(0, x) = u_0, \quad u_t(0, x) = u_1,$$

with initial data in a critical space. A critical space is a space that is invariant under the scaling symmetry. Observe that (1.1) is invariant under the scaling symmetry

$$u(t, x) \mapsto \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x), \quad \lambda > 0.$$

Under the above scaling symmetry, the size of the initial data changes by a factor of

$$\|\lambda u_0(\lambda x)\|_{\dot{H}^s(\mathbb{R}^3)} = \lambda^{s + \frac{2}{p-1} - \frac{3}{2}} \|u_0\|_{\dot{H}^s(\mathbb{R}^3)}, \quad \|\lambda^2 u_1(\lambda x)\|_{\dot{H}^{s-1}(\mathbb{R}^3)} = \lambda^{s + \frac{2}{p-1} - \frac{3}{2}} \|u_1\|_{\dot{H}^{s-1}(\mathbb{R}^3)}.$$

Thus, (1.1) is called $\dot{H}^s \times \dot{H}^{s-1}$-critical when

$$s_c = \frac{3}{2} - \frac{2}{p-1},$$

because this norm is invariant under (1.2).

The scaling symmetry (1.2) completely determines the local well-posedness theory for (1.1).

Theorem 1. Equation (1.1) is locally well-posed for initial data in $(u_0, u_1) \in \dot{H}^{s_c}(\mathbb{R}^3) \times \dot{H}^{s_c-1}(\mathbb{R}^3)$ on some interval $[-T(u_0, u_1), T(u_0, u_1)]$. The time of well-posedness $T(u_0, u_1)$ depends on the profile of the initial data $(u_0, u_1)$, not just its size. For data sufficiently small in $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$, global well-posedness and scattering hold.

Additional regularity is enough to give a lower bound on the time of well-posedness. Therefore, there exists some $T(\|u_0\|_{\dot{H}^s}, \|u_1\|_{\dot{H}^{s-1}}) > 0$ for any $s_c < s < \frac{3}{2}$.

Negatively, equation (1.1) is ill-posed for $u_0 \in \dot{H}^s(\mathbb{R}^3)$ and $u_1 \in \dot{H}^{s-1}(\mathbb{R}^3)$ when $s < s_c$.

Proof. See [LS95]. □

Local well-posedness is defined in the usual way.

Definition 1 (Locally well-posed). The initial value problem (1.1) is said to be locally well-posed if there exists an open interval $I \subset \mathbb{R}$ containing 0 such that:

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Theorem 4. Equation (1.9) space defined by the norm $u$ posed and scattering for radially symmetric initial data (1.7) $\lim$ and (1.8) $u$. That is, if (1.5) $\|u\|_{L^{2(p-1)}(\mathbb{R}^{3})}$, $u_{1} \in L^{\infty}_{t,loc} \dot{H}^{s-1}(I \times \mathbb{R}^{3})$ exists.

(2) The solution $u$ is continuous in time, $u \in C(I; \dot{H}^{s}(\mathbb{R}^{3}))$, $u_{1} \in C(I; \dot{H}^{s-1}(\mathbb{R}^{3}))$.

(3) The solution $u$ depends continuously on the initial data in the topology of item one.

The scaling symmetry (1.2) also completely determines the long time behavior of (1.1) with radially symmetric initial data.

Theorem 2. For $3 \leq p < 5$, the initial value problem (1.1) is globally well-posed and scattering for radial initial data $(u_{0}, u_{1}) \in \dot{H}^{s}(\mathbb{R}^{3}) \times \dot{H}^{s-1}(\mathbb{R}^{3})$. Moreover, there exists a function $f : [3, 5) \times [0, \infty) \to [0, \infty)$ such that if $u$ solves (1.1) with initial data $(u_{0}, u_{1}) \in \dot{H}^{s} \times \dot{H}^{s-1}$, then (1.5) $\|u\|_{L^{2(p-1)}(\mathbb{R} \times \mathbb{R}^{3})} \leq f(p, \|u_{0}\|_{\dot{H}^{s}(\mathbb{R}^{3})} + \|u_{1}\|_{\dot{H}^{s-1}(\mathbb{R}^{3})})$.

Proof. This was proved in [Dod18a] when $p = 3$ and in [Dod18c] when $3 < p < 5$. □

Remark 1. The argument in [LS95] may be used to show that (1.5) is equivalent to scattering in the critical Sobolev norm.

Definition 2 (Scattering). A solution to (1.1) with initial data $(u_{0}, u_{1})$ is said to be scattering in some $\dot{H}^{s}(\mathbb{R}^{3}) \times \dot{H}^{s-1}(\mathbb{R}^{3})$ if there exist $(u_{0}^{+}, u_{1}^{+}), (u_{0}^{-}, u_{1}^{-}) \in \dot{H}^{s} \times \dot{H}^{s-1}$ such that (1.6) $\lim_{t \to +\infty} \|(u(t), u_{1}(t)) - S(t)(u_{0}^{+}, u_{1}^{+})\|_{\dot{H}^{s} \times \dot{H}^{s-1}} = 0$ and (1.7) $\lim_{t \to -\infty} \|(u(t), u_{1}(t)) - S(t)(u_{0}^{-}, u_{1}^{-})\|_{\dot{H}^{s} \times \dot{H}^{s-1}} = 0$.

where $u$ is the solution to (1.1) with initial data $(u_{0}, u_{1})$ and $S(t)(f, g)$ is the solution operator to the linear wave equation. That is, if $(u(t), u_{1}(t)) = S(t)(f, g)$, then (1.8) $u_{tt} - \Delta u = 0$, $u(0, x) = f$, $u_{t}(0, x) = g$.

Equation (1.1) is called scattering for data in a certain subset $X$ if the solution to (1.1) with initial data in $X$ is globally well-posed, the solution scatters both forward and backward in time, and the scattering states $(u_{0}^{+}, u_{1}^{+})$ and $(u_{0}^{-}, u_{1}^{-})$ depend continuously on the initial data.

An important stepping stone in the proof of Theorem 2 was the result of [Dod18a] for radially symmetric initial data in a critical Besov space.

Theorem 3. The defocusing, cubic nonlinear wave equation (1.1) when $p = 3$ is globally well-posed and scattering for radially symmetric initial data $u_{0} \in B_{1,1}^{2}$ and $u_{1} \in B_{1,1}^{1}$. $B_{p,q}^{s}$ is the Besov space defined by the norm (1.9) $\|u\|_{B_{p,q}^{s}(\mathbb{R}^{3})} = \left(\sum_{j} 2^{jsp} \|P_{j}u\|_{L^{p}(\mathbb{R}^{3})}^{p}\right)^{1/p}$.

The operator $P_{j}$ is the usual Littlewood–Paley projection operator.

In this paper we generalize Theorem 3 to the case when $3 < p < 5$ with nonradial initial data.

Theorem 4. Equation (1.1) is globally well-posed when $3 < p < 5$ initial data $u_{0} \in B_{1,1}^{2} + \frac{2}{p}$ and $u_{1} \in B_{1,1}^{1} + \frac{2}{p}$. Furthermore, there exists $f : [3, 5) \times [0, \infty) \to [0, \infty)$ such that (1.10) $\|u\|_{L^{2(p-1)}_{t,loc}(\mathbb{R} \times \mathbb{R}^{3})} \leq f(p, \|u_{0}\|_{B_{1,1}^{2} + \frac{2}{p}(\mathbb{R}^{3})} + \|u_{1}\|_{B_{1,1}^{1} + \frac{2}{p}(\mathbb{R}^{3})})$. 
The $B^{s+\frac{2}{p}}_{1,1} \times B^{s+\frac{2}{p}}_{1,1}$ norm is invariant under the scaling symmetry (1.2). By the Sobolev embedding theorem, $B^s_{1,1} \subset \dot{H}^s$ and $B^{s+\frac{2}{p}}_{1,1} \subset \dot{H}^{s+1}$. The main advantage that $B^{s+\frac{2}{p}}_{1,1} \times B^{s+\frac{2}{p}}_{1,1}$ provides is the dispersive estimate for the wave equation

\[(1.11) \quad \|S(t)(u_0, u_1)\|_{L^\infty} \lesssim \frac{1}{t} \|\nabla^2 u_0, \nabla u_1\|_{L^1 \times L^1},\]

which implies good behavior for the solution to the linear wave equation with initial data $(u_0, u_1)$ for $t \neq 0$. Therefore, a helpful heuristic in thinking about Theorem 3 is that blowup of a solution to (1.1) with initial data in $B^s_{1,1}$ must occur when $t = 0$ if it occurs at all. Radial symmetry further implies that the blowup must occur at the origin in space and time.

The results in Theorem 2 addressed initial data that was merely radially symmetric, but not in $B^s_{1,1} \times B^s_{1,1}$, so the blowup could occur at any time, but only at the origin in space, $x = 0$. Theorem 4 approaches this problem from the other direction. The fact that $(u_0, u_1) \in B^{s+\frac{2}{p}}_{1,1} \times B^{s+\frac{2}{p}}_{1,1}$ means that, heuristically, the blowup may occur anywhere in $\mathbb{R}^3$, but only at time $t = 0$, if it occurs at all.

1.1. Outline of the proof. The only obstacle to proving Theorem 4 is that the $\dot{H}^s \times \dot{H}^{s-1}$ norm of $(u, u_t)$ may blow up either forward or backward in time.

**Theorem 5.** Suppose $(u_0, u_1) \in \dot{H}^s(\mathbb{R}^3) \times \dot{H}^{s-1}(\mathbb{R}^3)$ and $u$ solves (1.1) on a maximal interval $0 \in I \subset \mathbb{R}$, with $3 < p < 5$ and

\[(1.12) \quad \sup_{t \in I} \|u(t)\|_{\dot{H}^s(\mathbb{R}^3)} + \|u_t(t)\|_{\dot{H}^{s-1}(\mathbb{R}^3)} < \infty.\]

Then $I = \mathbb{R}$ and the solution $u$ scatters both forward and backward in time.

**Proof.** This theorem was proved in [DLMM20].

While there is no known conserved quantity that controls the $\dot{H}^s \times \dot{H}^{s-1}$ norm of $(u(t), u_t(t))$ for a solution to (1.1) with generic initial data $(u_0, u_1) \in \dot{H}^s \times \dot{H}^{s-1}$, a solution to (1.1) does have the conserved energy

\[(1.13) \quad E(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 dx + \frac{1}{2} \int u_t(t, x)^2 dx + \frac{1}{p+1} \int |u(t, x)|^{p+1} dx = E(u(0)).\]

For $u_0 \in \dot{H}^1 \cap \dot{H}^s$ and $u_1 \in L^2 \cap \dot{H}^{s-1}$, the Sobolev embedding theorem implies

\[(1.14) \quad \|u(0)\|_{L^{p+1}(\mathbb{R}^3)}^{p+1} \lesssim \|u_0\|_{\dot{H}^s(\mathbb{R}^3)}^{p-1} \|u_0\|_{\dot{H}^1(\mathbb{R}^3)}^2,\]

so

\[(1.15) \quad E(u(0)) \lesssim \|u_0\|_{\dot{H}^s}^2 + \|u_1\|_{L^2}^2.\]

Conservation of energy then implies a uniform bound on the $(\|u(t), u_t(t)\|_{\dot{H}^1 \times L^2}$ norm for the entire time of existence of $u$, which by Theorem 3 implies that the solution to (1.1) with initial data $u_0 \in \dot{H}^1 \cap \dot{H}^s$ and $u_1 \in L^2 \cap \dot{H}^{s-1}$ is global.

For generic initial data $u_0 \in B^{s+\frac{2}{p}}_{1,1}$ and $u_1 \in B^{s+\frac{2}{p}}_{1,1}$, there is no reason to think that the initial data lies in $\dot{H}^1 \times L^2$. However, using the dispersive estimate (1.11), we can split a solution $u(t)$ into a piece lying in $\dot{H}^1 \times L^2$ and a piece with good decay estimates as $t$ becomes large. A similar computation was used in [Dod18a] to prove Theorem 3.
The local well-posedness result of Theorem 1 implies that there exists an open neighborhood $I$ of 0 for which (1.1) has a solution, and
\begin{equation}
\|u\|_{L^2_t L^{p-1}(I \times \mathbb{R}^3)} \leq \epsilon,
\end{equation}
for some $\epsilon > 0$ small. Rescaling by (1.2),
\begin{equation}
\|u\|_{L^2_t L^{p-1}([-1,1] \times \mathbb{R}^3)} \leq \epsilon.
\end{equation}
This solution satisfies Duhamel's principle
\begin{equation}
u(t) = S(t)(u_0,v_1) - \int_0^t S(t - \tau)(0,|u|^{p-1}u)d\tau.
\end{equation}

Next, combining the dispersive estimate (1.11) and local well-posedness theory, it is possible to prove that
\begin{equation}t^{\frac{2-p}{p}} \|u(t)\|_{L^2_x},
\end{equation}
is uniformly bounded for all $t \in [-1,1]$. Therefore, by standard energy estimates,
\begin{equation}\| \int_{1/2}^1 S(1 - \tau)(0,|u|^{p-1}u)d\tau\|_{H^1_x L^2} \lesssim 1,
\end{equation}
with implicit constant bounded by the norm of the initial data in $B_{1,1}^{s_c + \frac{1}{2}} \times B_{1,1}^{s_c + \frac{1}{2}}$.

Let
\begin{equation}v(1) = \int_{1/2}^1 S(1 - \tau)(0,|u|^{p-1}u)d\tau, \quad v_t(1) = \partial_t \int_{1/2}^1 S(t - \tau)(0,|u|^{p-1}u)d\tau|_{t=1},
\end{equation}
and let
\begin{equation}w(1) = u(1) - v(1), \quad w_t(1) = u_t(1) - v_t(1).
\end{equation}
It follows from (1.17) and Theorem 1 that (1.1) has a local solution on $[1,T]$ for some $T > 1$. Decompose this solution $u = v + w$, which solve
\begin{equation}
w_{tt} - \Delta w = 0, \quad w(1,x) = w(1), \quad w_t(1,x) = w_t(1),
\end{equation}
\begin{equation}v_{tt} - \Delta v + u^3 = 0, \quad v(1,x) = v(1), \quad v_t(1,x) = v_t(1).
\end{equation}
To prove that $T$ may be extended to $T = \infty$, it is enough to prove that $E(v(t))$, where $E$ is given by (1.13), is uniformly bounded on any compact subset of $[1,\infty)$. To see why, first note that $w_{tt} - \Delta w = 0$ has a global solution. Next, the rescaling used to obtain (1.17) will be used to show that for any $T \geq 0$,
\begin{equation}\|w\|_{L^2_t L^{p-1}([T,T+1] \times \mathbb{R}^3)} \leq \frac{\epsilon}{2}.
\end{equation}
Therefore, using standard perturbative arguments,
\begin{equation}v_{tt} - \Delta v + |u|^{p-1}u = 0,
\end{equation}
may be treated as a perturbation of
\begin{equation}v_{tt} - \Delta v + |v|^{p-1}v = 0,
\end{equation}
on short time intervals. Therefore, if $E(v(t_0)) < \infty$, (1.1) is locally well-posed on the interval $[t_0,t_0 + \frac{1}{E(v(t_0))}]$, so it is enough to prove that $E(v(t))$ is uniformly bounded on any compact subset of $[1,\infty)$. 


To prove the uniform bound, standard calculations imply
\begin{equation}
\frac{d}{dt} E(v(t)) = -\langle v_t, |u|^{p-1}u - |v|^{p-1}v \rangle.
\end{equation}
The most difficult component of (1.27) is a term of the form
\begin{equation}
-\langle v_t, v^{p-1}w \rangle \lesssim \|\nabla|^{s_c-\frac{1}{2}}w\|_{L^\infty} E(v(t)).
\end{equation}
Using the dispersive estimate (1.11) it is possible to prove \( \|\nabla|^{s_c-\frac{1}{2}}w\|_{L^\infty} \lesssim \frac{1}{t} \). Plugging this estimate into (1.28) and using Gronwall’s inequality then proves a uniform bound on \( E(v(t)) \) on any compact subset, completing the proof of global well-posedness.

The above computations are not enough to prove scattering. In fact, even if one assumed initial data \( u_0 \in H^1 \cap H^{s_c} \) and \( u_1 \in L^2 \cap H^{s_c-1} \), conservation of energy would not guarantee a uniform bound on \( \|u(t), u_t(t)\|_{H^{s_c} \times H^{s_c-1}} \). Indeed, recall that [Str68] assumed sufficient decay on the initial data.

However, the Lebesgue dominated convergence theorem implies that outside a compact set, the initial data has small \( H^{s_c} \times H^{s_c-1} \) norm. By finite propagation speed, this implies scattering outside a light cone. Inside the light cone, we follow [She14], [She17], [Dod18b], [Dod18a], and make a conformal change of coordinates to prove that this solution scatters.

We obtain the bound (1.10) using the profile decomposition argument in [Ram12].

2. Local behavior of the solution to (1.1)

Using (1.2), it is possible to rescale equation (1.1) so that (1.1) is locally well-posed on \([-1,1]\) and the solution satisfies
\begin{equation}
\|u\|_{L_t^2 L_x^{(p-1)}([-1,1] \times \mathbb{R}^3)} \leq \epsilon.
\end{equation}

**Proof of 2.1.** Recall the Strichartz estimates for the wave equation.

**Theorem 6.** Let \( I \) be a time interval and let \( u : I \times \mathbb{R}^3 \to \mathbb{R} \) be a Schwartz solution to the wave equation
\begin{equation}
\partial_{tt} u - \Delta u = F, \quad u(0) = u_0, \quad \partial_t u(0) = u_1,
\end{equation}
where \( 0 \in I \). Then we have the estimates,
\begin{equation}
\|u\|_{L_t^q L_x^r(I \times \mathbb{R}^3)} + \|u\|_{C_{t,x}^0 H_x^s(I \times \mathbb{R}^3)} + \|\partial_t u\|_{C_{t,x}^0 H_x^s(I \times \mathbb{R}^3)} \lesssim_q r \left( \|u_0\|_{\dot{H}^s(\mathbb{R}^3)} + \|u_1\|_{\dot{H}^{s-1}(\mathbb{R}^3)} + \|F\|_{L_t^q L_x^r(I \times \mathbb{R}^3)} \right),
\end{equation}
for any \( s \geq 0, 2 < q, \tilde{q} \leq \infty, \) and \( 2 \leq r, \tilde{r} < \infty \) obey the scaling condition,
\begin{equation}
\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - s = \frac{1}{q'} + \frac{3}{\tilde{r}} = 2,
\end{equation}
and satisfy the wave admissibility conditions
\begin{equation}
\frac{1}{q} + \frac{1}{r}, \quad \frac{1}{\tilde{q}} + \frac{1}{\tilde{r}} \leq \frac{1}{2}.
\end{equation}

**Proof.** This theorem is copied from [Tao06]. See [Str77], [Kat94], [GV95], [Kap89], [LS95], [Sog95], [SS00], [KT98] for the proof of this theorem.
By Theorem 6 if \( u \) solves (2.11), then
\[
\|u\|_{L^q_t L^r_x([-1,1] \times \mathbb{R}^3)} \lesssim_p \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} + \|F\|_{L^q_t L^r_x([-1,1] \times \mathbb{R}^3)},
\]
where
\[
\frac{1}{q} = \frac{1}{2} s_c, \quad \frac{1}{r} = \frac{1}{2} - \frac{1}{2} s_c, \quad s_c = \frac{3}{2} - \frac{2}{p-1}, \quad \frac{1}{q'} = \frac{1}{q} + \frac{1}{2}, \quad \frac{1}{r'} = \frac{1}{r} + \frac{1}{2}.
\]
When \( 3 < p < 5 \), \((q, r)\) is an admissible pair that satisfies (2.5), and \( q' \) and \( r' \) satisfies (2.4).

Since \((u_0, u_1) \in B^{s_c+\frac{1}{2}}_{1,1} \times B^{s_c+\frac{1}{2}}_{1,1}\), there exists some \( j_0 \in \mathbb{Z} \) such that
\[
\sum_{j \geq j_0} 2^{j(s_c+\frac{1}{2})} \|P_j u_0\|_{L^1} + 2^{j(s_c+\frac{1}{2})} \|P_j u_1\|_{L^1} \leq c \epsilon,
\]
for some \( c > 0 \) that is determined by the implicit constant in (2.3). Using (1.2), rescale so that
\[
2^{j_0(1-s_c)} \cdot \|(u_0, u_1)\|_{B^{s_c+\frac{1}{2}}_{1,1} \times B^{s_c+\frac{1}{2}}_{1,1}} \leq c \epsilon.
\]

Theorem 6 and (2.8) imply
\[
\|S(t)(P_{\geq j_0} u_0, P_{\geq j_0} u_1)\|_{L^q_t L^r_x([-1,1] \times \mathbb{R}^3)} \leq \frac{\epsilon}{4},
\]
Also, by the Sobolev embedding theorem, (2.8), and the fact that \( S(t) \) is a unitary operator on \( \dot{H}^s \times \dot{H}^{s-1} \),
\[
\|S(t)(P_{\leq j_0} u_0, P_{\leq j_0} u_1)\|_{L^q_t L^r_x([-1,1] \times \mathbb{R}^3)} \leq \frac{\epsilon}{4},
\]
so by Hölder’s inequality,
\[
\|S(t)(u_0, u_1)\|_{L^q_t L^r_x([-1,1] \times \mathbb{R}^3)} \leq \frac{3\epsilon}{4}.
\]
A similar calculation also implies
\[
\|S(t)(u_0, u_1)\|_{L^q_t L^r_x([-1,1] \times \mathbb{R}^3)} \leq \frac{3\epsilon}{4}.
\]

Plugging (2.12) into (1.15) and using (2.3) and Picard iteration implies that for \( \epsilon > 0 \) sufficiently small, (1.1) is locally well-posed on \([-1,1] \times \mathbb{R}^3 \), and the solution satisfies
\[
\|u\|_{L^q_t L^r_x([-1,1] \times \mathbb{R}^3)} \leq \epsilon.
\]

See [LS95] for a detailed proof. ☐

The constant \( \epsilon > 0 \) will eventually be chosen to depend on \( \|u_0\|_{B^{s_c+\frac{1}{2}}_{1,1}} + \|u_1\|_{B^{s_c+\frac{1}{2}}_{1,1}} \). Under (2.11), the behavior of \( u \) on the interval \([-1,1] \) is approximately linear.

**Theorem 7.** If \( u \) is a solution to (1.1) on \([-1,1] \) with \( \|u\|_{L^2([-1,1] \times \mathbb{R}^3)} \leq \epsilon(A) \), where \((u_0, u_1) \in B^{3/2+s_c}_{1,1} \times B^{1/2+s_c}_{1,1}\) with \( A = \|u_0\|_{B^{3/2+s_c}_{1,1}} + \|u_1\|_{B^{1/2+s_c}_{1,1}} \), then
\[
\sum_{j} 2^{j s_c} \|P_j u_0\|_{L^p_x L^q_t([-1,1] \times \mathbb{R}^3)} \lesssim A.
\]
Proof. Using the Strichartz estimates in Theorem 6 if \((q, r)\) and \((\tilde{q}, \tilde{r})\) are given by (2.17),
\[
2^{s_c} \| P_j u \|_{L_t^q L_x^r([-1,1] \times \mathbb{R}^3)} + \| P_j u \|_{L_t^{q(\tilde{r}-1)} L_x^{\tilde{r}([-1,1] \times \mathbb{R}^3)} + 2^{j(s_c - \frac{4}{n})} \| P_j u \|_{L_t^q L_x^r([-1,1] \times \mathbb{R}^3)}
\]
\[
+ 2^{-j(1-s_c)/2} \| P_j u \|_{L_t^{2q} L_x^{2r}([-1,1] \times \mathbb{R}^3)} \lesssim 2^{s_c} \| P_j u_0 \|_{L^2} + 2^{j(s_c - 1)} \| P_j u_1 \|_{L^2}
\]
\[
+ 2^{-j(1-s_c)/2} \| P_j F_1 \|_{L_t^{2q} L_x^{2r}([-1,1] \times \mathbb{R}^3)} + 2^{j(s_c - \frac{4}{n})} \| P_j F_2 \|_{L_t^{\frac{8}{3}} L_x^{\frac{8}{7}}([-1,1] \times \mathbb{R}^3)},
\]
where \(P_j F_1 + P_j F_2 = P_j(|u|^{p-1} u)\) is a decomposition of the nonlinearity. Using Taylor’s theorem, decompose
\[
F_1 = |P_{\leq j} u|^{p-1} (P_{\leq j} u), \quad F_2 = |u|^{p-1} u - |P_{\leq j} u|^{p-1} (P_{\leq j} u) = O(|u|^{p-1} |P_{\geq j} u|).
\]
Theorem 7 follows directly from (2.16) and \(u_0 \in B^s_{1,1}+1/2\), \(u_1 \in B^{s_c+1/2}_{1,1}+1/2\). Indeed,
\[
\| F_1 \|_{L_t^{q} L_x^{r}([-1,1] \times \mathbb{R}^3)} \lesssim \| P_{\leq j} u \|_{L_t^{\frac{p}{p-1}} L_x^{\frac{r}{p-1}}([-1,1] \times \mathbb{R}^3)} \| P_{\leq j} u \|_{L_t^{q} L_x^{r}([-1,1] \times \mathbb{R}^3)}
\]
\[
\| F_2 \|_{L_t^{q} L_x^{r}([-1,1] \times \mathbb{R}^3)} \lesssim \| P_{\geq j} u \|_{L_t^{\frac{p}{p-1}} L_x^{\frac{r}{p-1}}([-1,1] \times \mathbb{R}^3)} \| u \|_{L_t^{\frac{p}{p-1}} L_x^{\frac{r}{p-1}}([-1,1] \times \mathbb{R}^3)}
\]
so by Young’s inequality and (2.16), the proof of Theorem 7 is complete. Indeed, letting \(X_j\) denote the left hand side of (2.16),
\[
X_j \lesssim 2^{j s_c} \| P_j u_0 \|_{L^2} + 2^{j(s_c - 1)} \| P_j u_1 \|_{L^2} + e^{p-1} \sum_{k \geq j} 2^{(j-k)(s_c - \frac{4}{n})} X_k + e^{p-1} \sum_{k \leq j} 2^{(k-j)(s_c - \frac{4}{n})} X_k,
\]
which implies (2.15).

The dispersive estimates (1.11) also give additional \(L_t^q L_x^r\) bounds on the solution \(u\) in \([-1,1]\) that lie outside the admissible pairs in Theorem 6.

**Theorem 8.** For \(3 < p < 5\), if \(\frac{1}{q} = \frac{3}{2} - s_c = \frac{2}{p-1}\),
\[
\| u \|_{L_t^q L_x^{\infty}([-1,1] \times \mathbb{R}^3)} + \sum_{j \in \mathbb{Z}} 2^{j(s_c - \frac{4}{n})} \sup_{t \in [-1,1]} \| P_j u \|_{L^\infty} \lesssim \epsilon.
\]

Proof. Using the dispersive estimate
\[
\| S(t)(u_0, u_1) \|_{L^\infty} \lesssim \frac{1}{t} \| (u_0, u_1) \|_{B_t^{s} \times B_t^{s}},
\]
for any \(j \in \mathbb{Z},\)
\[
\| S(t)(P_j u_0, P_j u_1) \|_{L^\infty} \lesssim \frac{1}{t} 2^{-j(s_c - \frac{4}{n})} \| |2ji(\frac{4}{n}+s_c)| P_j u_0 \|_{L^1} + 2^{j(i+s_c)} \| P_j u_1 \|_{L^1}.
\]
Interpolating (2.23) with
\[
\| S(t)(P_j u_0, P_j u_1) \|_{L^\infty} \lesssim 2^{j(s_c - \frac{4}{n})} \| (P_j u_0, P_j u_1) \|_{H^{s_c} \times H^{s_c-1}},
\]
and making use of (2.23) and (2.10), we have proved
\[
\sum_j \sup_{t \in [-1,1]} t^{\frac{2}{n}-s_c} \| S(t)(P_j u_0, P_j u_1) \|_{L^\infty} + \sum_j \| S(t)(P_j u_0, P_j u_1) \|_{L_t^q L_x^{\infty}(\mathbb{R} \times \mathbb{R}^3)}
\]
\[
+ \sum_j 2^{j(s_c - \frac{4}{n})} \sup_{t \in [-1,1]} t^{\| S(t)(P_j u_0, P_j u_1) \|_{L^\infty} \lesssim \epsilon.
\]
Meanwhile, since Bernstein’s inequality, \(2.33\) and \(2.35\), for any \(x \in \mathbb{R}^3\),
\[
|S(t-\tau)(0, |u|^{p-1}u)(x)| \lesssim \frac{1}{|t-\tau|} \int_{\partial B(x,t-\tau)} |u(y,\tau)|^p d\sigma(y).
\]

Once again, split
\[
2.27 \quad P_j(|u|^{p-1}u) = P_j F_1 + P_j F_2, \quad F_1 = |P_{\leq j} u|^{p-1}(P_{\leq j} u), \quad F_2 = O(|P_{\geq j} u| u)^{p-1}).
\]

Plugging \(F_2\) into \(2.26\), for any \(t \in [-1,1], \ x \in \mathbb{R}^3\),
\[
| \int_{0}^{+} S(t-\tau)(0, P_j F_2)(t, x)d\tau | \lesssim \frac{1}{t} \| \| u \|_{L_t^{\infty} L_x^\infty([0,\frac{1}{2}] \times \mathbb{R}^3)} \| |u|^{p-1} \|_{L_t^p L_x^\infty} \cdot \sup_{\tau \in [0,\frac{1}{2}]} \left( \int_{\partial B(x,t-\tau)} |u(\tau, y)|^p d\sigma(y) \right)^{1/2}.
\]

By an argument similar to the Sobolev embedding theorem, for any \(k \in \mathbb{Z}\),
\[
2.29 \quad \int_{\partial B(x,t-\tau)} |P_k u(y,\tau)|^{p-1} d\sigma(y) \lesssim 2^k \| P_k u \|_{L_p}.
\]

**Remark 2.** To see why this is so, recall that the Littlewood-Paley kernel for \(P_k\) may be approximated by \(2^{k}\) multiplied by the characteristic function of a ball of radius \(2^{-k}\). Then consider the cases when \(2^{-k} \leq |t-\tau|\) and \(2^{-k} > |t-\tau|\) separately. Indeed, for \(|t-\tau| \lesssim 2^{-k}\), there exists some \(C\) such that
\[
2.30 \quad \int_{\partial B(x,t-\tau)} |P_k u(y,\tau)|^{p-1} d\sigma(y) \lesssim 2^{3k} |t-\tau|^{2k} \int_{B(x,2^{-k})} |P_k u(\tau, y)|^{p-1} dy \lesssim 2^k \| P_k u(\tau) \|_{p-1}.
\]

Meanwhile, for \(|t-\tau| \gg 2^{-k}\),
\[
2.31 \quad \int_{\partial B(x,t-\tau)} |P_k u(y,\tau)|^{p-1} d\sigma(y) \lesssim 2^k \int_{\text{dist}(B(x,t-\tau), y) \leq 2^{-k}} |P_k u(\tau, y)|^{p-1} dy \lesssim 2^k \| P_k u(\tau) \|_{p-1}.
\]

Now, then, since the Littlewood-Paley kernel obeys the bounds
\[
2.32 \quad \mathcal{F}(P_k y) \lesssim N 2^{3k} (1 + 2^k |y|)^{-N},
\]
for any \(N\), calculations similar to \(2.30\) and \(2.31\) imply \(2.29\).

Plugging \(2.29\) into \(2.28\), by Young’s inequality,
\[
2.33 \quad \sum_j 2^j(s_{j-\frac{1}{2}}) \sup_{\tau \in [-1,1]} \| \int_{0}^{\frac{1}{2}} S(t-\tau)(0, P_j F_2) d\tau \|_{L_t^\infty} \lesssim \| u \|_{L_t^p L_x^\infty([-1,1] \times \mathbb{R}^3)} A^{s_{j-\frac{1}{2}}}.\]

Meanwhile, since by Bernstein’s inequality,
\[
2.34 \quad P_j(F_1) \sim 2^{-j} \nabla P_j F_1 \sim 2^{-j} |P_{\leq j} u|^{p-1} |\nabla P_{\leq j} u|,
\]
\[
| \int_{0}^{\frac{1}{2}} S(t-\tau)(0, P_j F_1) d\tau | \lesssim \frac{2^{-j}}{t} \| u \|_{L_t^p L_x^\infty([0,\frac{1}{2}] \times \mathbb{R}^3)} \lesssim \sup_{\tau \in [0,\frac{1}{2}]} \left( \int_{B(x,t-\tau)} |u(\tau, y)|^p d\sigma(y) \right)^{1/2} \cdot \sup_{\tau \in [0,\frac{1}{2}]} \left( \int_{B(x,t-\tau)} |\nabla P_{\leq j} u(\tau, y)|^2 dy \right)^{1/2},
\]
and therefore,

\[ (3.2) \quad \sum_j 2^{j(s_p - \frac{1}{2})} \sup_{t \in [-1,1]} \int_0^t |S(t - \tau)(0, P_j |u|^{p-1} u)|_2^2 dt \lesssim \frac{1}{t^{1-s_p}} \left( \sum_j 2^{j(s_p - \frac{1}{2})} \right) \sup_{t \in [-1,1]} A^{\frac{p-1}{2}} u^2. \]

For \( \tau \in \left[ \frac{t}{2}, t \right] \), energy estimates and the Sobolev embedding theorem imply,

\[ (3.3) \quad \| S(t - \tau)(0, P_j |u|^{p-1} u) \|_{L^2} \lesssim 1 \left( \sum_j \sup_{\tau \in \left[ \frac{t}{2}, t \right]} \tau \cdot \| u(\tau) \|_2^{p-1} \| P_j u \|_{L^2} \right) \left( \sum_j \sup_{\tau \in \left[ \frac{t}{2}, t \right]} \tau \cdot \| u(\tau) \|_2^{p-1} \| P_j u \|_{L^2} \right)^{2} \| P_j \| \lesssim u \|_{L^2}. \]

Therefore, by Young’s inequality, the Sobolev embedding theorem, and Theorem 4,

\[ (3.4) \quad \sum_j 2^{j(s_p - \frac{1}{2})} \sup_{t \in [-1,1]} t \int_0^t |S(t - \tau)(0, |u|^{p-1} u)|_2^2 dt \lesssim ( \sup_{t \in [-1,1]} t^{\frac{2}{p} - s_p} \| u(t) \|_{L^2} )^{p-1} A. \]

Combining (2.23), (3.2), and (3.3),

\[ (3.5) \quad \sum_j 2^{j(s_p - \frac{1}{2})} \sup_{t \in [-1,1]} t \| P_j u \|_{L^2} \lesssim \epsilon + ( \sup_{t \in [-1,1]} t^{\frac{2}{p} - s_p} \| u(t) \|_{L^2} )^{p-1} A. \]

Now then, for any \( 3 < p < 5 \), Theorem 4, and the Sobolev embedding theorem imply

\[ (3.6) \quad \sum_j \| P_j u \|_{L^2} \lesssim \epsilon + ( \sup_{t \in [-1,1]} t^{\frac{2}{p} - s_p} \| u(t) \|_{L^2} )^{p-1} A. \]

Combining (3.3) with (3.6) proves the Theorem. \( \square \)

Theorem 5 implies finite energy for a piece of the Duhamel term.

**Corollary 1.** For any \( t \in [-1,1] \),

\[ (3.7) \quad \int_{t/2}^t \| u^p(\tau) \|_{L^2} d\tau \lesssim \frac{A^p}{t^{1-s_p}}. \]

**Proof.** Use the energy estimate in (2.24). \( \square \)

3. **Proof of Global well-posedness**

By time reversal symmetry and local well-posedness on the interval \([-1,1]\), to prove Theorem 4 it suffices to prove global well-posedness in the positive time direction, \( t > 1 \) for \( \| \| \) with initial data \((u(1,x), u_t(1,x))\). The local well-posedness arguments used to prove Theorem 4 imply that \( (1.1) \) has a solution on some open interval \([0, T)\) for some \( T > 1 \), so to prove global well-posedness it suffices to show that \( T \) can be taken to go to infinity.

Split

\[ (3.8) \quad \begin{align*}
    (u(1,x), u_t(1,x)) &= S(1)(u_0, u_1) + \int_0^{1/2} S(1 - \tau)(0, |u|^{p-1} u) d\tau + \int_{1/2}^1 S(1 - \tau)(0, |u|^{p-1} u) d\tau.
\end{align*} \]

By Corollary 1 the second Duhamel term has finite energy.

\[ (3.9) \quad \| (v(1,x), v_t(1,x)) \|_{H^1 \times L^2} = \| \int_{1/2}^1 S(1 - \tau)(0, |u|^{p-1} u) d\tau \|_{H^1 \times L^2} \lesssim A. \]
Now let $u$ be the solution to (1.14) on $[1, T)$. Split $u = v + w$, where $v$ solves
\begin{equation}
 v_{tt} - \Delta v + |v|^{p-1}u = 0, \tag{3.3}
\end{equation}
on $[1, T)$ with initial data given by (3.2), and
\begin{equation}
 w_{tt} - \Delta w = 0, \quad w(1, x) = u(1, x) - v(1, x), \quad w_t(1, x) = u_t(1, x) - v_t(1, x). \tag{3.4}
\end{equation}
Set
\begin{equation}
 E(v) = \int \left[ \frac{1}{2} |v_t|^2 + \frac{1}{2} |\nabla v|^2 + \frac{1}{p+1} |v|^{p+1} \right] dx, \tag{3.5}
\end{equation}
and compute
\begin{equation}
 \frac{d}{dt} E(v) = \langle v_t, -|u|^{p-1}u + |v|^{p-1}v \rangle. \tag{3.6}
\end{equation}
By Taylor’s theorem,
\begin{equation}
 |u|^{p-1}u - |v|^{p-1}v = p|v|^{p-1}w + O(|w|^2|v|^{p-2}) + O(|w|^p). \tag{3.7}
\end{equation}
By Hölder’s inequality,
\begin{equation}
 \langle O(|w|^2|v|^{p-2}), v_t \rangle \lesssim \|v_t\|_{L^p} \|v\|_{L^{p+1}} \|w\|_{L^{2p+1}} \lesssim E(v(t))^{\frac{1}{2} + \frac{p-2}{2p+1}} \|w\|_{L^\infty} \|w\|_{L^{2p+1}}. \tag{3.8}
\end{equation}
Interpolating (2.21) with $\|w\|_{L^{2p+1}} \lesssim \|w\|_{H^s_x} \lesssim A_1$, proves $\|w\|_{L^{p+1}} \lesssim A_1$. Also,
\begin{equation}
 \langle |w|^p, v_t \rangle \lesssim \|v_t\|_{L^p} \|w\|_{L^{p+1}} \|w\|_{L^{2p+1}} \lesssim E(v(t))^{1/2} \|w\|_{L^\infty} \|w\|_{L^{2p+1}}. \tag{3.9}
\end{equation}
If we could ignore the term
\begin{equation}
 \langle v_t, p|v|^{p-1}w \rangle, \tag{3.10}
\end{equation}
then $E(v(t))$ would be uniformly bounded on $\mathbb{R}$ by Gronwall’s inequality. Indeed, by (2.21),
\begin{equation}
 \int_1^T E(v(t))^{\frac{1}{2} + \frac{p-2}{2p+1}} \|w\|_{L^{p+1}} \|w\|_{L^{2p+1}} dt + \int E(v(t))^{1/2} \|w\|_{L^{p+1}} \|w\|_{L^{2p+1}} dt 
\lesssim \sup_{t \in [1, T)} \epsilon E(v(t))^{\frac{1}{2} + \frac{p-2}{2p+1}} + \sup_{t \in [1, T)} \epsilon E(v(t))^{1/2}, \tag{3.11}
\end{equation}
which implies a uniform bound on $E(v(t))$.
To deal with the contribution of (3.10), take the modified energy
\begin{equation}
 \mathcal{E}(v(t)) = E(v(t)) + \langle |v|^{p-1}v, w \rangle. \tag{3.12}
\end{equation}
Then (3.6) and (3.7) imply
\begin{equation}
 \frac{d}{dt} \mathcal{E}(v(t)) = \langle v_t, -|u|^{p-1}u + |v|^{p-1}v \rangle + \langle p|v|^{p-1}w, v_t \rangle + \langle |v|^{p-1}v, w_t \rangle 
= \langle |v|^{p-1}v, w_t \rangle + O(E(v(t))^{\frac{1}{2} + \frac{p-2}{2p+1}} \|w\|_{L^\infty} \|w\|_{L^{2p+1}}) + O(E(v(t))^{1/2} \|w\|_{L^\infty} \|w\|_{L^{2p+1}}). \tag{3.13}
\end{equation}
Also,
\begin{equation}
 \langle |v|^{p-1}v, w \rangle \lesssim \|v\|_{L^{p+1}} \|w\|_{L^{p+1}} \lesssim E(v(t))^{\frac{1}{p+1}}, \tag{3.14}
\end{equation}
so when $E(v(t))$ is large,
\begin{equation}
 E(v(t)) \sim \mathcal{E}(v(t)), \tag{3.15}
\end{equation}
and
\begin{equation}
\frac{d}{dt} \mathcal{E}(v(t)) = \langle |v|^{p-1}v, w_t \rangle + O(\mathcal{E}(v(t))^{3/2} \| |v|^{2} \|_{L_{x}^{\infty}}^{1/2} \| |v|^{2} \|_{L_{x}^{p+1}}^{1/2}) + O(\mathcal{E}(v(t))^{1/2} \| |v|^{p+1} \|_{L_{x}^{\infty}} \| |v|^{p+1} \|_{L_{x}^{p+1}}).
\end{equation}

Splitting $w_t = \sum_{j} P_{j} w_{t}$,
\begin{equation}
\langle |v|^{p-1}v, w_{t} \rangle = \sum_{j} \langle P_{j} (|v|^{p-1}v), P_{j} w_{t} \rangle.
\end{equation}

Now by Bernstein’s inequality and (2.21),
\begin{equation}
\sum_{j} \langle P_{j} (|v|^{p-1}v - |P_{\leq j} v|^{p-1}(P_{\leq j} v)), P_{j} w_{t} \rangle \lesssim \sum_{j} \| P_{j} w_{t} \|_{L^{\infty}} \| P_{\geq j} v \|_{L^{\frac{p}{p-1}}} \| v |^{p-1} \lesssim \frac{\epsilon}{t} E(v(t)).
\end{equation}

Indeed, by (2.21),
\begin{equation}
\sum_{j} 2^{j(s_{c} - \frac{3}{2})} 2^{-j} \| P_{j} w_{t} \|_{L^{\infty}} \lesssim \frac{\epsilon}{t}.
\end{equation}

Meanwhile, by Bernstein’s inequality, for any fixed $j \in \mathbb{Z}$,
\begin{equation}
\frac{\epsilon}{t} 2^{j(\frac{3}{2} - s_{c})} \| P_{\geq j} v \|_{L^{\frac{p}{p-1}}} \| v |^{p-1} \lesssim \frac{\epsilon}{t} 2^{j(\frac{3}{2} - s_{c})} \| P_{\geq j} v \|_{L^{\frac{p}{p-1}}} \| v |^{p-1} \lesssim \frac{\epsilon}{t} \| \nabla v \|_{L^{\frac{p}{p-1}}} \| v |^{p-1} \lesssim \frac{\epsilon}{t} E(v(t)).
\end{equation}

Also, by Bernstein’s inequality,
\begin{equation}
\sum_{j} \langle P_{j} (|P_{\leq j} v|^{p-1}(P_{\leq j} v)), P_{j} w_{t} \rangle \lesssim \sum_{j} 2^{-j(\frac{3}{2} - s_{c})} \| \nabla v \|_{L^{\frac{p}{p-1}}} \| P_{j} w_{t} \|_{L^{\infty}} \| v |^{p-1} \lesssim \frac{\epsilon}{t} E(v(t)).
\end{equation}

Therefore, by Gronwall’s inequality,
\begin{equation}
\mathcal{E}(v(t)) < \infty, \quad \text{and} \quad E(v(t)) < \infty,
\end{equation}
for any $t \in \mathbb{R}$. This proves global well-posedness.

4. Proof of scattering

Now we prove that the global solution in the previous section scatters. By time reversal symmetry, to prove
\begin{equation}
\| u \|_{L_{x}^{2p-1}(\mathbb{R} \times \mathbb{R}^{3})} < \infty,
\end{equation}
if $u$ is a solution to (1.1) with initial data $(u_{0}, u_{1}) \in B^{\frac{3}{2} + s_{c}}_{1,1} \times B^{\frac{5}{4} + s_{c}}_{1,1}$, it is enough to prove that
\begin{equation}
\| u \|_{L_{x}^{2p-1}([0, \infty) \times \mathbb{R}^{3})} < \infty.
\end{equation}

Recall from (2.8) that
\begin{equation}
\| P_{\geq j_{0}} u_{0} \|_{H^{s_{c}}(\mathbb{R}^{3})} + \| P_{\geq j_{0}} u_{1} \|_{H^{s_{c}-1}(\mathbb{R}^{3})} \lesssim \epsilon.
\end{equation}

Also, let $\chi \in C_{0}^{\infty}(\mathbb{R}^{3})$ be a smooth, compactly supported function, and suppose $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x)$ is supported on $|x| \leq 2$. By the dominated convergence theorem there exists $100 \leq R(u_{0}, u_{1}, \epsilon) < \infty$ such that
\begin{equation}
\| (1 - \chi(\frac{x}{R})) P_{\leq j_{0}} u_{0} \|_{H^{s_{c}}(\mathbb{R}^{3})} + \| (1 - \chi(\frac{x}{R})) P_{\leq j_{0}} u_{1} \|_{H^{s_{c}-1}(\mathbb{R}^{3})} \lesssim \epsilon.
\end{equation}
Rescaling using (1.2) and translating the initial data in time, (1.2) is equivalent to proving
\begin{equation}
\|u\|_{L^{2(p-1)}([1 - \frac{4}{4R}]x, \infty) \times \mathbb{R}^3} < \infty,
\end{equation}
where
\begin{equation}
u(1 - \frac{1}{4R}, x) = (4R)^{\frac{3}{2-\tau}}u_0(4Rx), \quad u_t(1 - \frac{1}{4R}, x) = (4R)^{1+\frac{2}{\tau}}u_1(4Rx).
\end{equation}
Then decompose
\begin{equation}
v(1 - \frac{1}{4R}, x) = \chi(\frac{x}{4})(4R)^{\frac{3}{2-\tau}}(P \leq_j u_0)(4Rx), \quad u_t(1 - \frac{1}{4R}, x) = \chi(\frac{x}{4})(4R)^{\frac{3}{2-\tau}}(P \leq_j u_1)(4Rx),
\end{equation}
\begin{equation}
u(1 - \frac{1}{4R}, x) = v(1 - \frac{1}{4R}, x) + w(1 - \frac{1}{4R}, x), \quad u_t(1 - \frac{1}{4R}, x) = v_t(1 - \frac{1}{4R}, x) + w_t(1 - \frac{1}{4R}, x),
\end{equation}
and let \( v \) and \( w \) solve the system of equations:
\begin{equation}
v_{tt} - \Delta v = 0, \quad \text{for} \quad 1 - \frac{1}{4R} \leq t \leq 1,
\end{equation}
\begin{equation}
v_{tt} - \Delta v + (|v + w|^{p-1}(v + w) - |w|^{p-1}w) = 0, \quad \text{for} \quad t \geq 1,
\end{equation}
and
\begin{equation}
w_{tt} - \Delta w + |w + v|^{p-1}(w + v) = 0, \quad \text{for} \quad 1 - \frac{1}{4R} \leq t \leq 1,
\end{equation}
\begin{equation}
w_{tt} - \Delta w + |w|^{p-1}w = 0, \quad \text{for} \quad t \geq 1.
\end{equation}
Using the small data arguments in (LS95) combined with (2.13), (4.3), and (4.4),
\begin{equation}
\|w\|_{L^{2(p-1)}([1 - \frac{4}{4R}]\times \mathbb{R}^3)} \lesssim \epsilon.
\end{equation}
Therefore, (4.5) is equivalent to
\begin{equation}
\|v\|_{L^{2(p-1)}([1 - \frac{4}{4R}]\times \mathbb{R}^3)} < \epsilon.
\end{equation}
The proof of (4.11) will make use of some additional estimates on \( w \).

**Lemma 1.** There exists a sequence \( a_j \in l^1(\mathbb{Z}) \) such that
\begin{equation}
w(t, x) = \sum_{j \in \mathbb{Z}} w_j(t, x), \quad \text{for any} \quad t \in [1, \infty), \quad x \in \mathbb{R}^3,
\end{equation}
\begin{equation}|w_j(t, x)| \leq a_j 2^{-j\frac{3}{2-\tau}}(t - 1 + \frac{1}{4R})^{-1}, \quad \text{for any} \quad t \in [1, \infty), \quad x \in \mathbb{R}^3,
\end{equation}
\begin{equation}|
abla w_j(t, x)| + |\partial_t w_j(t, x)| \leq a_j 2^{j\frac{3}{2-\tau}}(t - 1 + \frac{1}{4R})^{-1}, \quad \text{for any} \quad t \in [1, \infty), \quad x \in \mathbb{R}^3,
\end{equation}
and
\begin{equation}|w_j(t, x)| \leq a_j 2^{j\frac{3}{2-\tau}}, \quad \text{for any} \quad t \in [1, \infty), \quad x \in \mathbb{R}^3,
\end{equation}
where
\begin{equation}\sum_j a_j \lesssim \epsilon.
\end{equation}
Proof. Using the scaling symmetry in (1.2), Lemma 1 is equivalent to proving the bounds in (1.3)–
(4.15) with \( t - \frac{1}{4\pi} \) replaced by \( t \), for \( w \) solving
\[
\begin{align*}
\frac{\partial w}{\partial t} - \Delta w + |w|^{p-1}w &= 0, & \text{for} & & 0 \leq t \leq 1, \\
\frac{\partial w}{\partial t} - \Delta w + |w|^{p-1}w &= 0, & \text{for} & & t \geq 1,
\end{align*}
\]
with initial data satisfying (2.39). First, by Theorem 8 and the scaling symmetry (1.2), (4.3)–(4.16)
hold for the Littlewood–Paley decomposition of
\[
S(t) = \int_0^1 S(t - \tau)(0, |u|^{p-1}u)d\tau.
\]
Also, by the dispersive estimates in (2.22), the bounds in (4.13)–(4.16) also hold for
\( S(t)(P_{\geq j_0} u_0, P_{\geq j_0} u_1) \).
Meanwhile, by the dominated convergence theorem, for \( R \) sufficiently large,
\[
\| (1 - \chi(\frac{x}{R})) P_j u_0 \|_{L^1} \leq a_j 2^{-(\frac{3}{2} - s_c)j},
\]
and
\[
\| \nabla^3 (1 - \chi(\frac{x}{R})) P_j u_0 \|_{L^1} \leq a_j 2^{(\frac{3}{2} - s_c)j} + \frac{1}{R^3} \| \chi'''(x) \|_{L^1} \leq a_j 2^{(\frac{3}{2} - s_c)j}.
\]
Similar computations also hold for \( u_1 \), so the bounds in (4.13)–(4.15) hold for
\[
S(t)(P_{\leq j_0} u_0, P_{\leq j_0} u_1) + \int_0^1 S(t - \tau)(0, |u|^{p-1}u)d\tau.
\]
Combining the bounds for (4.21) with (4.10) and the proofs of Theorems 7 and 8 proves (4.13)–
(4.15).

Returning to the solutions to (4.8) and (4.9) with initial data given by (4.7), to prove (4.11)
we will use the conformal change of coordinates, similar to the computations in [She17], [Dod18a]
and [Dod18c]. First observe that by the finite propagation speed, \( v \) is supported on the set
\[
\{(t, x) : t \geq 1, \ |x| \leq t - \frac{1}{2} + \frac{1}{400}\}.
\]
Since \( \frac{3}{2} + \frac{1}{400} < \sqrt{3} \),
\[
\{(t, x) : |x| \geq t - \frac{1}{2} + \frac{1}{400} \} \cap \{(t, x) : t \geq 2\} \subset \|(t, x) : t^2 - |x|^2 \geq 1\} \cap \{(t, x) : t \geq 2\}.
\]
In fact, there exists some \( \delta_0 > 0 \) such that for any \( 0 \leq \delta \leq \delta_0 \),
\[
\{(t, x) : |x| \geq t - \frac{1}{2} + \frac{1}{400} \} \cap \{(t, x) : t \geq 2\} \subset \{(t, x) : t^2 - |x|^2 \geq e^{2\delta}\} \cap \{(t, x) : t \geq 2\}.
\]
Let
\[
\hat{u}(\tau, y) = \frac{e^{\tau} \sinh y}{|y|} u(e^{\tau} \cosh |y|, e^{\tau} \sinh |y|, \frac{y}{|y|}).
\]
By direct computation (see [She17]) for more information, if \( u \) solves
\[
\frac{\partial u}{\partial t} - \Delta u = F,
\]
inside (4.23), then
\[
(\partial_{\tau} - \Delta_y) \hat{u} = e^{3\tau} \frac{\sinh y}{|y|} F(e^{\tau} \cosh |y|, e^{\tau} \sinh |y|, \frac{y}{|y|}).
\]
when $\tau > 0$. In particular, for

\begin{equation}
\tilde{v}(\tau, y) = \frac{e^\tau \sinh |y|}{|y|} v(e^\tau \cosh |y|, e^\tau \sinh |y| \frac{y}{|y|}), \quad \tilde{w}(\tau, y) = \frac{e^\tau \sinh |y|}{|y|} w(e^\tau \cosh |y|, e^\tau \sinh |y| \frac{y}{|y|}),
\end{equation}

\begin{equation}
(\partial_{\tau^2} - \Delta_y) \tilde{v} + e^{-(p-3)\tau} (\frac{|y|}{\sinh |y|})^{p-1} |\tilde{v}|^{p-1} |\tilde{v}| = 0,
\end{equation}

and

\begin{equation}
(\partial_{\tau^2} - \Delta_y) \tilde{w} + e^{-(p-3)\tau} (\frac{|y|}{\sinh |y|})^{p-1} |\tilde{w}|^{p-1} |\tilde{w}| = 0.
\end{equation}

Let $E(\tilde{v})$ denote the hyperbolic energy of $\tilde{v}$,

\begin{equation}
E(\tilde{v}) = \frac{1}{2} \|\tilde{v}_\tau\|_{L^2}^2 + \frac{1}{2} \|\nabla \tilde{v}\|_{L^2}^2 + \frac{1}{p+1} \int e^{-(p-3)\tau} |\tilde{v}|^p \frac{|y|}{\sinh |y|} dy.
\end{equation}

**Lemma 2.** There exists some $0 \leq \tau_0 \leq \delta_0$ such that (4.31) is finite.

**Proof.** To prove Lemma 2 it suffices to prove

\begin{equation}
\int_0^{\delta_0} \int (\tilde{v}_\tau(\tau, y))^2 + |\nabla \tilde{v}|^2 + e^{-(p-3)\tau} (\frac{|y|}{\sinh |y|})^{p-1} |\tilde{v}|^p dy dt < \infty.
\end{equation}

By direct computation,

\begin{equation}
\tilde{v}_\tau(\tau, y) = \frac{e^{2\tau} \sinh |y| \cosh |y|}{|y|} v_\tau(e^{\tau} \cosh |y|, e^{\tau} \sinh |y| \frac{y}{|y|}) + \frac{e^{2\tau} \sinh^2 |y|}{|y|^2} v_\tau(e^{\tau} \cosh |y|, e^{\tau} \sinh |y| \frac{y}{|y|})
\end{equation}

\begin{equation}
\ + \frac{e^{\tau} \sinh |y|}{|y|} v(e^{\tau} \cosh |y|, e^{\tau} \sinh |y| \frac{y}{|y|}).
\end{equation}

By the support properties in (4.23) and (4.24) and the change of variables formula in [She17], and the proof of Theorem 3 in section three,

\begin{equation}
\int_0^{\delta_0} \int \frac{e^{4\tau} \sin^2 |y|}{|y|^2} v_\tau^2(e^{\tau} \cosh |y|, e^{\tau} \sinh |y|)^2 dy dt \lesssim \int_1^2 \int v_\tau(t, y)^2 dy dt \lesssim R 1,
\end{equation}

\begin{equation}
\int_0^{\delta_0} \int \frac{e^{4\tau} \sinh^4 |y|}{|y|^2} v_\tau^2(e^{\tau} \cosh |y|, e^{\tau} \sinh |y|)^2 dy dt \lesssim \int_1^2 \int v_\tau(t, y)^2 dy dt \lesssim R 1,
\end{equation}

and by Hardy’s inequality,

\begin{equation}
\int_0^{\delta_0} \int \frac{e^{2\tau} \sin^2 |y|}{|y|^2} \nu^2(e^{\tau} \cosh |y|, e^{\tau} \sinh |y| \frac{y}{|y|}) \lesssim \int_1^2 \int \frac{1}{|y|^2} \nu(t, y)^2 dy dt \lesssim R 1.
\end{equation}

Therefore, $\int_0^{\delta_0} \int \tilde{v}_\tau(\tau, y)^2 dy dt \lesssim R 1$. A similar computation proves $\int_0^{\delta_0} \int |\nabla \tilde{v}(\tau, y)|^2 dy dt \lesssim R 1$.

Finally,

\begin{equation}
\int_0^{\delta_0} \int e^{(p+1)\tau} \sinh^{p+1} |y| |v(e^{\tau} \cosh |y|, e^{\tau} \sinh |y| \frac{y}{|y|})|^{p+1} e^{-(p-3)\tau} (\frac{|y|}{\sinh |y|})^{p-1} dy dt
\end{equation}

\begin{equation}
\lesssim \int_1^2 \int |v(t, y)|^{p+1} dy dt \lesssim R 1.
\end{equation}

This proves (4.32), which proves the Lemma.  \qed
The finite energy at $\tau_0$ grows very slowly.

**Theorem 9.** For $\tau_0 \leq \tau \leq 1$,

$$E(\tilde{v}) \lesssim_R E(\tilde{v}(\delta_0)).$$

**Proof.** Computing the change of the hyperbolic energy,

$$\frac{d}{dt}E(\tilde{v}) = -\frac{p-3}{p-1} \int e^{-(p-3)\tau} |\tilde{v}|^{p+1} \frac{|y|}{\sinh|y|} dy$$

$$- \frac{1}{p+1} \int e^{-(p-3)\tau} \left( \frac{|y|}{\sinh|y|} \right)^{p-1} \tilde{v}_r[|\tilde{u}|^{p-1} \tilde{u} - |\tilde{v}|^{p-1} \tilde{v} - |\tilde{w}|^{p-1} \tilde{w}] dy.$$

By Hölder’s inequality,

$$\int e^{-(p-3)\tau} \left( \frac{|y|}{\sinh|y|} \right)^{p-1} |\tilde{v}_r| |\tilde{v}|^\frac{p+1}{2} \tilde{v}_r dy \lesssim \|\tilde{v}_r\|_{L^\infty} \left( \int e^{-(p-3)\tau} \left( \frac{|y|}{\sinh|y|} \right)^{p-1} |\tilde{v}|^{p+1} dy \right)^{1/2} \tilde{v}_r \|_{L^2}$$

$$\lesssim E(\tilde{v}) \|e^\tau |w(e^\tau \cosh|y|, e^\tau \sinh|y|)|y|/|y|\|^{p+1}/2 \|_{L^\infty}.$$

By Hardy’s inequality and the Sobolev embedding theorem,

$$\int e^{-(p-3)\tau} \left( \frac{|y|}{\sinh|y|} \right)^{p-1} |\tilde{v}_r| |\tilde{v}| |\tilde{w}| dy \lesssim \|\tilde{v}_r\|_{L^2} \frac{1}{|y|} \|\tilde{v}_r\|_{L^2} \|e^\tau |w(e^\tau \cosh|y|, e^\tau \sinh|y|)|y|/|y|\|^{p-1} \|_{L^\infty}$$

$$\lesssim E(\tilde{v}) \|e^\tau |w(e^\tau \cosh|y|, e^\tau \sinh|y|)|y|/|y|\|^{p+1}/2 \|_{L^\infty} \|e^\tau |w(e^\tau \cosh|y|, e^\tau \sinh|y|)|y|/|y|\| \|_{L^\infty}.$$

By Lemma 11 for any $j \in \mathbb{Z}$,

$$|w_j(e^\tau \cosh|y|, e^\tau \sinh|y|)\|y|/|y|\| \lesssim 2^{-j/4} (e^\tau \cosh|y| - 1 + \frac{1}{4R})^{-1} \lesssim 2^{-j/2} (e^\tau - 1 + \frac{1}{4R})^{-1} a_j,$$

$$|w_j(e^\tau \cosh|y|, e^\tau \sinh|y|)\|y|/|y|\| \lesssim 2^{-j} a_j.$$

When $j \geq 0$,

$$\int_0^{2^{-j}} 2^j e^\tau d\tau \lesssim 1,$$

and

$$\int_{2^{-j}}^{\infty} e^\tau 2^{-j/2} (e^\tau - 1 + \frac{1}{4R})^{-1/2} d\tau \lesssim \int_1^{1} \tau^{-\frac{1}{8}} 2^{-j/8} d\tau + 2^{-j/8} \int_{2^{-j}}^{\infty} e^{-\frac{1}{8} \tau} d\tau \lesssim 1.$$
Also observe that for any $k > (4.55) \sim (4.57)$, so for almost every $\tau$ on $\psi \in \mathbb{R}$.

Therefore, the contribution of (4.52) to (4.51) may be absorbed into the left hand side of (4.39), proving

\begin{equation}
\sup_{\tau_0 \leq \tau \leq T_0} E(\hat{\psi}(\tau)) \lesssim E(\hat{\psi}(\delta_0)) + \sup_{\tau_0 \leq \tau \leq T_0} \int_{\tau_0}^{\tau} \int p|\hat{v}|^{p-1} \hat{\psi} \hat{\psi}_j \left( \frac{|y|}{\sinh |y|} \right)^{p-1} e^{-(p-3)\tau} dy d\tau.
\end{equation}

Lemma 1 implies that to bound the second term on the right hand side of (4.52), it suffices to obtain a bound of

\begin{equation}
\sup_{\tau_0 \leq \tau \leq T} \int_{\tau_0}^{\tau} \int p|\hat{v}|^{p-1} \hat{\psi} \hat{\psi}_j \left( \frac{|y|}{\sinh |y|} \right)^{p-1} e^{-(p-3)\tau} dy d\tau,
\end{equation}

with a bound summable in $j$.

To prove this bound we will use a modification of the Littlewood–Paley decomposition. Let $\psi \in C^\infty_0(\mathbb{R})$ be a smooth function satisfying $\psi(x) \geq 0$ on $\mathbb{R}$, $\int \psi(x) dx = 1$, and $\psi(x)$ is supported on $|x| \leq 1$. Then for $f \in L^1_w$, set

\begin{equation}
\hat{P}_0 f = \int \psi(\tau - s) f(s), \quad \text{and for} \quad k > 0,
\end{equation}

\begin{equation}
\hat{P}_k f = \int 2^k \psi(2^k (\tau - s)) f(s) ds - \int 2^{k-1} \psi(2^{k-1} (\tau - s)) f(s) ds.
\end{equation}

Also observe that for any $k > 0$, summing up the telescoping sum in (4.54),

\begin{equation}
\hat{P}_{\leq k} f = \int 2^k \psi(2^k (\tau - s)) f(s) ds.
\end{equation}

Suppose $E(\hat{v})$ is bounded on the interval $[\tau_0, T]$. Then by local well-posedness arguments,

\begin{equation}
1_{[\tau_0, T]} \partial_\tau (|\hat{v}|^{p-1} \hat{v}) \in L^1_{\tau, y},
\end{equation}

so

\begin{equation}
\hat{P}_0 (1_{[\tau_0, T]} |\hat{v}|^{p-1} \hat{v}) + \sum_{j \geq 1} \hat{P}_j (1_{[\tau_0, T]} |\hat{v}|^{p-1} \hat{v})
\end{equation}

\begin{equation}
= \partial_\tau \hat{P}_0 (1_{[\tau_0, T]} |\hat{v}|^{p-1} \hat{v}) + \sum_{j \geq 1} \hat{P}_j \partial_\tau (1_{[\tau_0, T]} |\hat{v}|^{p-1} \hat{v}) - (|\hat{v}|^{p-1} \hat{v})|_{\tau_0}^T,
\end{equation}

for almost every $\tau \in \mathbb{R}$, where $1_{[a,b]}$ is the indicator function of the interval $[a,b]$. 

The second term on the right hand side of (4.57) can be computed using Hardy’s inequality and Lemma \[1\]
(4.58)
\[
\int \left( \left| \frac{y}{\sinh |y|} \right|^{p-1} \frac{|y|}{\sinh |y|} \right) e^{-(p-3)\frac{2}{p-1}r |e^{r \tau} (e^\tau \cosh |y|, e^\tau \sinh |y|)\right| T\int dy
\]
\[
\lesssim \left( \int e^{-(p-3)\frac{2}{p-1}r |e^{r \tau} (e^\tau \cosh |y|, e^\tau \sinh |y|)\right| T\int dy
\]
\[
\lesssim E(\tilde{v}) \left( \frac{y}{\sinh |y|} \right) \left( |e^{r \tau} w_j(e^\tau \cosh |y|, e^\tau \sinh |y|)\right| T\int dy
\]
\[
\lesssim a_j \sup_{\tau \in [\tau_0, T]} E(\tilde{v}).
\]

The sum of these terms in \( j \) can be absorbed into the left hand side of (4.59).

To handle the first term on the right hand side of (4.57), it is useful to consider a number of cases separately.

**Case 1**, \( 2^j |e| \leq 1 \): By Lemma \[1\]
(4.59)
\[
\int_{\tau_0}^T \partial_\tau \tilde{P}_{\partial, \partial} (1_{[\tau_0,\tau]} \|\tilde{v}\|_{p-1} \tilde{v}) \left( \frac{|y|}{\sinh |y|} \right) e^{-(p-3)\frac{2}{p-1}r |e^{r \tau} w_j(e^\tau \cosh |y|, e^\tau \sinh |y|)\right| T\int dy
\]
\[
\lesssim \int_{\tau_0}^T \frac{1}{|y|} e^{-(p-3)\frac{2}{p-1}r |e^{r \tau} w_j(e^\tau \cosh |y|, e^\tau \sinh |y|)\right| T\int dy
\]
\[
\lesssim \sup_{\tau \in \tau_0, T} \left( \frac{|y|}{\sinh |y|} \right) \left( |e^{r \tau} w_j(e^\tau \cosh |y|, e^\tau \sinh |y|)\right| T\int dy
\]
\[
\lesssim a_j \int_{\tau_0}^T \frac{1}{|y|} e^{-(p-3)\frac{2}{p-1}r |e^{r \tau} w_j(e^\tau \cosh |y|, e^\tau \sinh |y|)\right| T\int dy
\]

Next, by the fundamental theorem of calculus, for any \( k \geq 0 \),
(4.60)
\[
\tilde{P} f(\tau) = f(\tau) - 2^k \int \psi(2^k(\tau-s)) f(s) ds = 2^k \int \psi(2^k(\tau-s)) [f(\tau) - f(s)] ds = 2^k \int \psi(2^k(\tau-s)) [f(\tau)] ds.
\]

Integrating by parts and following (4.58) for the third term, (4.60) with \( j = 0 \) for the second, and (4.59) for the first,
(4.61)
\[
\int_{\tau_0}^T \partial_\tau \tilde{P}_{\partial, \partial} (1_{[\tau_0,\tau]} \|\tilde{v}\|_{p-1} \tilde{v}) \left( \frac{|y|}{\sinh |y|} \right) e^{-(p-3)\frac{2}{p-1}r |e^{r \tau} w_j(e^\tau \cosh |y|, e^\tau \sinh |y|)\right| T\int dy
\]
\[
(4.62)
= (p-4) \int_{\tau_0}^T \tilde{P}_{\partial, \partial} (1_{[\tau_0,\tau]} \|\tilde{v}\|_{p-1} \tilde{v}) \left( \frac{|y|}{\sinh |y|} \right) e^{-(p-3)\frac{2}{p-1}r |e^{r \tau} w_j(e^\tau \cosh |y|, e^\tau \sinh |y|)\right| T\int dy
\]
\[
- \int_{\tau_0}^T \tilde{P}_{\partial, \partial} (1_{[\tau_0,\tau]} \|\tilde{v}\|_{p-1} \tilde{v}) \left( \frac{|y|}{\sinh |y|} \right) e^{-(p-3)\frac{2}{p-1}r |e^{r \tau} w_j(e^\tau \cosh |y|, e^\tau \sinh |y|)\right| T\int dy
\]
\[
\times [e^\tau \cosh |y| (\partial_\tau w_j) + e^\tau \sinh |y| (\partial_\tau w_j)] (e^\tau \cosh |y|, e^\tau \sinh |y|) \right| T\int dy
\]
\[
+ \tilde{P}_{\partial, \partial} (1_{[\tau_0,\tau]} \|\tilde{v}\|_{p-1} \tilde{v}) \left( \frac{|y|}{\sinh |y|} \right) e^{-(p-3)\frac{2}{p-1}r |e^{r \tau} w_j(e^\tau \cosh |y|, e^\tau \sinh |y|)\right| T\int dy
\]
\[
\lesssim a_j \int_{\tau_0}^{T} \frac{1}{|y|} \tilde{v}^{\frac{2-\tau}{p-3}} (e^{-(p-3)\tau} (\frac{|y|}{\sinh |y|} p^{-1} |\tilde{v}|^{p+1}) \frac{\tilde{v}^2}{p-3}) d\tau \\
+ a_j \int_{\tau_0}^{T} |\tilde{v}|^{\frac{2-\tau}{p-3}} (e^{-(p-3)\tau} (\frac{|y|}{\sinh |y|} p^{-1} |\tilde{v}|^{p+1}) \frac{\tilde{v}^2}{p-3}) d\tau + a_j \sup_{\tau \in [\tau_0, T]} E(\tilde{v}).
\]

These estimates are acceptable for our purposes.

**Case 2**, \(2e|y| \sim 2^k \geq 1\): In this case, the contribution of \(\tilde{P}_{>k}\) may be handled in a manner very similar to (4.61) and (4.62). Indeed,

\[
\int_{\tau_0}^{T} \partial_\tau \tilde{P}_{>k}(1_{[\delta_0, T]} |\tilde{v}|^{p-1}) \left(\frac{|y|}{\sinh |y|} p^{-2} e^{-(p-3)\frac{\tilde{v}^2}{p-3}} [e^{\frac{\tilde{v}^2}{p-3}} w_j (e^\tau \cosh |y|, e^\tau \sinh |y| \frac{y}{|y|})] d\tau
\]

\[
= (p-4) \int_{\tau_0}^{T} \tilde{P}_{>k}(1_{[\delta_0, T]} |\tilde{v}|^{p-1}) \left(\frac{|y|}{\sinh |y|} p^{-2} e^{-(p-3)\frac{\tilde{v}^2}{p-3}} [e^{\frac{\tilde{v}^2}{p-3}} w_j (e^\tau \cosh |y|, e^\tau \sinh |y| \frac{y}{|y|})] d\tau
\]

Following the computations in (4.60) and using the fact that Lemma 1 implies

\[
||e^\tau \cosh |y|(\partial_t w_j) + e^\tau \sinh |y|[(\partial_n w_j)] (e^\tau \cosh |y|, e^\tau \sinh |y| \frac{y}{|y|})||_{L^\infty} \lesssim Re^{\frac{\tilde{v}^2}{p-3}}.
\]

The computations in (4.59) may be copied over in this case. Finally, take \(l \in \mathbb{Z}, 0 < l \leq k\). In this case, by Lemma 1

\[
\int_{\tau_0}^{T} \partial_\tau \tilde{P}(1_{[\delta_0, T]} |\tilde{v}|^{p-1}) \left(\frac{|y|}{\sinh |y|} p^{-2} e^{-(p-3)\frac{\tilde{v}^2}{p-3}} [e^{\frac{\tilde{v}^2}{p-3}} w_j (e^\tau \cosh |y|, e^\tau \sinh |y| \frac{y}{|y|})] d\tau
\]

Summing up in \(j\) and \(l\), and integrating in \(y\), we have therefore proved

\[
\sup_{\tau_0 \leq \tau \leq T_0} E(\tilde{v}(\tau)) \lesssim E(\tilde{v}(\delta_0)) + \epsilon R \int_{\tau_0}^{T_0} \left(\int \frac{1}{|y|^2} |\tilde{v}|^2 + |\tilde{v}|^2 |dy| \right)^{\frac{1}{p-3}} (e^{-(p-3)\tau} (\frac{|y|}{\sinh |y|} p^{-1} |\tilde{v}|^{p+1}) \frac{\tilde{v}^2}{p-3}) d\tau.
\]

Taking \(T_0 = \tau_0 + \frac{1}{R}\), and making a standard bootstrap argument, it is possible to absorb the second term on the right hand side of (4.68) into the left hand side. Iterating this argument \(O_R(1)\) times proves the Theorem.

We can upgrade this to a global integral result.

**Theorem 10.**

\[
\int_{1}^{\infty} \int_{1}^{\infty} |\tilde{v}|^{p+1} e^{-(p-3)\tau} (\frac{|y|}{\sinh |y|} p^{-1} dy d\tau) \lesssim_R 1.
\]
Proof. When \( \tau \geq 1 \), by Lemma [1]
\[
e^{\tau \frac{p}{2}} |w_j(e^\tau \cosh |y|, e^\tau \sinh |y| \frac{y}{|y|})| \lesssim e^{-\tau \frac{p}{2}} 2^{-j} \frac{p}{2} \cosh(|y|)^{-1},
\]
and
\[
e^{\tau \frac{p}{2}} e^{\tau} \cosh |y| |(\partial_y w_j)(e^\tau \cosh |y|, e^\tau \sinh |y| \frac{y}{|y|})|
\]
\[
+ e^{\tau \frac{p}{2}} e^{\tau} \cosh |y| |(\partial_y w_j)(e^\tau \cosh |y|, e^\tau \sinh |y| \frac{y}{|y|})| \lesssim e^{-\tau \frac{p}{2}} 2^{j} \frac{p}{2} \tau.
\]

Revisiting (4.39) and (4.42),
\[
\int_1^{T_0} \int_1 e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} |\tilde{v}|^{p+1} dy d\tau + \sup_{1 \leq \tau \leq T_0} E(\tilde{v}(\tau)) \lesssim e^{-\tau \frac{p}{2}} 2^{j} \frac{p}{2} \tau.
\]
\[
\int_1^{T_0} \int_1 \int_1 e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} |\tilde{v}|^{p+1} dy d\tau + \sup_{1 \leq \tau \leq T_0} \int_1^{T_0} \int_1 \int_1 e^{-(p-3)\tau} dy d\tau.
\]

Now take the partition of unity
\[
1 = \sum_{m \in \mathbb{Z}} \chi(\tau - m), \quad \text{which satisfies} \quad \sum_{m \in \mathbb{Z}} \left| \chi'(\tau - m) \right| \lesssim 1.
\]

Let \( k(m, y) = \sup\{0, j + \frac{|y|}{m(2)} + \frac{m}{m(2)} \} \). Integrating by parts in \( \tau \), as in (4.61) and (4.62),
\[
\sum_m \int_1^T \partial_\tau \tilde{P}_{k(m, y)} \left( 1_{[1, T]} |\tilde{v}|^{p-1} \right) \left( \frac{|y|}{\sinh |y|} \right)^{p-2} e^{-\frac{p}{2} \tau} \text{e}^{\tau \frac{p}{2}} \chi(\tau - m) \left| e^{\tau \frac{p}{2}} \cosh |y|, e^{\tau \sinh |y|} \frac{y}{|y|} \right| d\tau
\]
\[
\lesssim \sum_m \int_1^T \tilde{P}_{k(m, y)} \left( 1_{[1, T]} |\tilde{v}|^{p-1} \right) \left( \frac{|y|}{\sinh |y|} \right)^{p-2} e^{-\frac{p}{2} \tau} \text{e}^{\tau \frac{p}{2}} \chi(\tau - m) \left| e^{\tau \frac{p}{2}} \cosh |y|, e^{\tau \sinh |y|} \frac{y}{|y|} \right| d\tau
\]
\[
\times (|\chi(\tau - m)| + |\chi'(\tau - m)|) |e^{\tau \frac{p}{2}} \cosh |y|, e^{\tau \sinh |y|} \frac{y}{|y|}| d\tau
\]
\[
- \sum_m \int_1^T \tilde{P}_{k(m, y)} \left( 1_{[1, T]} |\tilde{v}|^{p-1} \right) \left( \frac{|y|}{\sinh |y|} \right)^{p-2} e^{-\frac{p}{2} \tau} \text{e}^{\tau \frac{p}{2}} \chi(\tau - m) \left| e^{\tau \frac{p}{2}} \cosh |y|, e^{\tau \sinh |y|} \frac{y}{|y|} \right| d\tau
\]
\[
+ \sum_m \tilde{P}_{k(m, y)} \chi(\tau - m) \left( 1_{[1, T]} |\tilde{v}|^{p-1} \right) \left( \frac{|y|}{\sinh |y|} \right)^{p-2} e^{-\frac{p}{2} \tau} \text{e}^{\tau \frac{p}{2}} \chi(\tau - m) \left| e^{\tau \frac{p}{2}} \cosh |y|, e^{\tau \sinh |y|} \frac{y}{|y|} \right| d\tau.
\]

Therefore, using the computations leading up to (4.68), by (4.71),
\[
\int \left( \frac{e^{-\tau \frac{p}{2}} 2^{-j} \frac{p}{2} \cosh(|y|)^{-1}}{|y|^2 \cosh^2 |y|} |\tilde{v}|^{2} + \frac{1}{\cosh^2 |y|} |\tilde{v}|^{2} dy \right) \lesssim e^{-\tau \frac{p}{2}} 2^{j} \frac{p}{2} \tau \int \left( e^{\tau \frac{p}{2}} e^{\tau} \cosh |y| |(\partial_y w_j)(e^\tau \cosh |y|, e^\tau \sinh |y| \frac{y}{|y|})| \right)
\]
\[
\int \left( \frac{e^{-\tau \frac{p}{2}} 2^{-j} \frac{p}{2} \cosh(|y|)^{-1}}{|y|^2 \cosh^2 |y|} |\tilde{v}|^{2} + \frac{1}{\cosh^2 |y|} |\tilde{v}|^{2} dy \right) \lesssim e^{-\tau \frac{p}{2}} 2^{j} \frac{p}{2} \tau \int \left( e^{\tau \frac{p}{2}} e^{\tau} \cosh |y| |(\partial_y w_j)(e^\tau \cosh |y|, e^\tau \sinh |y| \frac{y}{|y|})| \right).
Meanwhile, using the computations in (4.39) and (4.67),

(4.77)

\[ \int \sum_{m} \int_{1}^{T} \partial_{\tau} \tilde{P}_{k(m,y)}(1_{[1,T]} ||\tilde{v}||^{p-1} \tilde{v}) \left( \frac{|y|}{\sinh |y|} \right)^{p-2} e^{-(p-3)\frac{y^2}{1+y^2}} \times \chi(\tau - m) \| e^\tau \cosh |y|, e^\tau \sinh |y| \frac{y}{|y|} \| dr dy \]

\[ \lesssim a_j \int_{1}^{T} \left( \int \frac{1}{|y|^2 \cosh |y|} ||\tilde{\varphi}||^2 + \frac{1}{\cosh |y|} ||\tilde{v}||^2 dy \right) \frac{r+1}{|y|} \cdot (e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} ||\tilde{v}||^{p+1} dy) \frac{r+\frac{2}{3}}{\tau} d\tau. \]

Summing in \( j \),

(4.78)

\[ \sum_{j} (4.76) + (4.77) \lesssim \epsilon \left( \int_{1}^{T} \int e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} ||\tilde{v}||^{p+1} dy d\tau \right) + \epsilon \left( \int_{1}^{T} \int \frac{1}{|y|^2 \cosh |y|} ||\tilde{\varphi}||^2 + \frac{1}{\cosh |y|} ||\tilde{v}||^2 dy \right). \]

The first term on the right hand side of (4.78) may be absorbed into the left hand side of (4.72). The second term on the right hand side of (4.78) can be controlled by a local energy decay estimate.

**Theorem 11** (Local energy decay).

(4.79)

\[ \int_{1}^{T} \int \frac{1}{(1+|y|^2)^{3/2}} \left( ||\tilde{v}||^2 + ||\nabla \tilde{v}||^2 \right) dy d\tau + \int_{1}^{T} \int \frac{1}{(1+|y|^2)^{3/2}} \tilde{v} \cdot \nabla \tilde{v} dy d\tau \]

\[ \lesssim \sup_{\tau \in [1,T]} E(\tilde{v}) + \epsilon \left( \int_{1}^{T} \int e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} ||\tilde{v}||^{p+1} dy d\tau \right). \]

Postponing the proof of Theorem 11, Theorem 10 follows. \( \square \)

The bounds in Theorem 10 imply bounds on \( ||\tilde{v}||_{L^2_y([0,\infty) \times \mathbb{R}^3)} < \infty \). Since \( E(\tilde{v}) \) is uniformly bounded, and (4.69) is finite, partition \([0,\infty)\) into finitely many subintervals such that

(4.80)

\[ \int_{1}^{T} \int e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} ||\tilde{v}||^{p+1} dy d\tau < \epsilon. \]

For any \( 3 < p < 5 \), by (4.12) and (4.30), there exists \( \theta(p) \) such that for \( \epsilon > 0 \) sufficiently small,

(4.81)

\[ ||\tilde{v}||_{\tilde{S}^{\frac{1}{2}}(I_j \times \mathbb{R}^3)} \lesssim \sup_{\tau} E(\tilde{v})^{\frac{1}{2}} + e^{\theta(p-1) \frac{1}{2}} ||\tilde{v}||^{1-(\frac{1}{2}-\theta)(p-1)}_{\tilde{S}^{\frac{1}{2}}(I_j \times \mathbb{R}^3)} + ||\tilde{v}||^{p-1} e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} \|_{L^2_{t,y}} \lesssim \sup_{\tau} E(\tilde{v})^{\frac{1}{2}}. \]

Therefore,

(4.82)

\[ ||\tilde{v}||_{L^2_{t,y}([\delta_0,\infty) \times \mathbb{R}^3)} < \infty. \]

Interpolating this bound with (4.38) and (4.69) then implies

(4.83)

\[ \int_{\tau_0}^{\infty} \int e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} ||\tilde{v}||^{2(p-1)} dy d\tau < \infty. \]
Using the change of variables formula, since $p - 1 > 2$,

$$
\int_{\tau_0}^{\infty} \int e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} |\tilde{v}(\tau, y)|^{2(p-1)} dy d\tau
= \int_{\tau_0}^{\infty} \int e^{2\tau} |v(e^\tau \cosh |y|, e^\tau \sinh |y|)|^{p-1} \left( \frac{e^\tau \sinh |y|}{|y|} \right)^{p-1} |v(e^\tau \cosh |y|, e^\tau \sinh |y|)|^{p-1} dy d\tau
\geq \int_{\tau^2 - |x|^2 \geq e^{2\tau_0}} |v(t, x)|^{2(p-1)} dx dt.
$$

Since $v$ is supported in $[4, 24]$ with $\delta = \tau_0$ and $t \geq 2$,

$$
\|v\|_{L^2((2, \infty) \times \mathbb{R}^3)} < \infty.
$$

The global well-posedness results of the previous section combined with (4.85) implies (4.11).

5. Local energy decay

Theorem [11] is proved using a virial identity. Let

$$
M(\tau) = \int \frac{y}{(1 + |y|^2)^{1/2}} \tilde{v}_\tau \cdot \nabla \tilde{v} dy + \int \frac{1}{(1 + |y|^2)^{1/2}} \tilde{v}_\tau \tilde{v} dy.
$$

By Hardy’s inequality, $\sup_{\tau \in [1, T]} M(\tau) \lesssim \sup_{\tau \in [1, T]} E(\tilde{v})$. By direct computation,

$$
\frac{d}{d\tau} M(t) = \int \frac{y}{(1 + |y|^2)^{1/2}} \tilde{v}_\tau \cdot \nabla \tilde{v} dy + \int \frac{1}{(1 + |y|^2)^{1/2}} \tilde{v}_\tau \cdot \nabla \tilde{v} \tilde{v} dy + \int \frac{1}{(1 + |y|^2)^{1/2}} \tilde{v}_\tau \tilde{v} \tilde{v} dy.
$$

Integrating by parts,

$$
\frac{1}{2} \int \frac{y}{(1 + |y|^2)^{1/2}} \cdot \nabla (\tilde{v}^2) dy + \int \frac{1}{(1 + |y|^2)^{1/2}} \tilde{v}_\tau^2 dy = -\frac{1}{2} \int \frac{1}{(1 + |y|^2)^{3/2}} \tilde{v}^2 dy.
$$

Substituting (4.30),

$$
\tilde{v}_\tau = \Delta \tilde{v} - e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} |\tilde{v}|^{p-1} \tilde{v} - e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} |\tilde{v}|^{p-1} \tilde{v} - |\tilde{v}|^{p-1} \tilde{v}.
$$

Integrating by parts,

$$
= -\frac{1}{2} \int \frac{1}{(1 + |y|^2)^{1/2}} \nabla \tilde{v}^2 - \frac{1}{2} \int \frac{|y|^2}{(1 + |y|^2)^{3/2}} \nabla \tilde{v}^2 + \int \frac{|y|^2}{(1 + |y|^2)^{1/2}} |\partial_\tau \tilde{v}|^2 - \frac{1}{2} \int \frac{1}{(1 + |y|^2)^{5/2}} \tilde{v}^2 dy \leq -\frac{1}{2} \int \frac{1}{(1 + |y|^2)^{3/2}} |\nabla \tilde{v}|^2 - \frac{1}{2} \int \frac{1}{(1 + |y|^2)^{5/2}} \tilde{v}^2 dy.
$$

By direct computation,
Next, integrating by parts,

\[ e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} |\tilde{v}|^{p-1} \tilde{v} \cdot \nabla \tilde{v} - \int \frac{1}{(1 + |y|^2)^{1/2}} e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} |\tilde{v}|^{p+1} dy \]

\[ = - \frac{1}{p+1} \int \frac{y}{(1 + |y|^2)^{1/2}} e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} \nabla (|\tilde{v}|^{p+1}) dy \]

\[ = (\frac{3}{p+1} - 1) \int \frac{1}{(1 + |y|^2)^{1/2}} e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} |\tilde{v}|^{p+1} dy \]

\[ + \frac{(p-1)}{p+1} \int \frac{y}{(1 + |y|^2)^{1/2}} e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-2} \left( \left| \nabla \left( \frac{|y|}{\sinh |y|} \right) \right| \right) |\tilde{v}|^{p+1} dy \]

\[ \leq \frac{p-2}{p+1} \int \frac{1}{(1 + |y|^2)^{1/2}} e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} |\tilde{v}|^{p+1} dy. \]

The error terms arising from

\[ e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} [|\tilde{u}|^{p-1} \tilde{u} - |\tilde{v}|^{p-1} \tilde{v} - |\tilde{\omega}|^{p-1} \tilde{\omega}] \]

can be handled similar to the error terms in the previous section. Recalling (5.4), by (5.5), (5.6), and Hardy's inequality,

\[ \int T \int |y| |\nabla \tilde{v}| + |\tilde{v}| \left( \frac{|y|}{1 + |y|^2} \right)^{1/2} e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} |\tilde{v}|^2 \frac{d\tau}{\sinh |y|} \]

\[ \lesssim \| e^7 w(e^\tau \cosh |y|, e^7 \sinh |y|) \frac{y}{|y|} \|_{L^p_y L^\infty_T} \| \nabla \tilde{v} \|_{L^p_y L^\infty_T} \sup_{\tau \in [1, T]} \left( \int e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} |\tilde{v}|^{p+1} dy \right)^{1/2} \]

\[ \lesssim \epsilon \sup_{\tau \in [1, T]} E(\tilde{v}(\tau)), \]

and

\[ \int T \int |y| |\nabla \tilde{v}| + |\tilde{v}| \left( \frac{|y|}{1 + |y|^2} \right)^{1/2} e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} |\tilde{v}|^2 \frac{d\tau}{\sinh |y|} \]

\[ \lesssim \| e^7 w(e^\tau \cosh |y|, e^7 \sinh |y|) \frac{y}{|y|} \|_{L^p_y L^\infty_T} \| \nabla \tilde{v} \|_{L^p_y L^\infty_T} \| e^7 \|_{L^2_y L^\infty_T} \| w(e^\tau \cosh |y|, e^7 \sinh |y|) \frac{y}{|y|} \|^{1/2} \|

\[ \lesssim \epsilon \sup_{\tau \in [1, T]} E(\tilde{v}(\tau)). \]
Next, by Hölder’s inequality,
\begin{align}
\int_1^T \int \frac{1}{(1 + |y|^2)^1/2} e^{-(p-3)\tau} \frac{|y|}{\sinh |y|} e^{-\frac{p}{p-1} |\tilde{v}|^p} d\tau dy \\
\lesssim \int_1^T \int e^{-(p-3)\tau} \frac{|y|}{\sinh |y|} e^{-\frac{p}{p-1} |\tilde{v}|^p} dy d\tau \\
\lesssim \int_1^T \int e^{-(p-3)\tau} \frac{|y|}{\sinh |y|} e^{-\frac{p}{p-1} |\tilde{v}|^p} dy d\tau + \epsilon \int_1^T \int \frac{1}{(1 + |y|^2)^{1/2}} |\tilde{v}|^2 d\tau dy.
\end{align}

Turning to
\begin{equation}
\int \frac{y}{(1 + |y|^2)^{1/2}} e^{-(p-3)\tau} \frac{|y|}{\sinh |y|} e^{-\frac{p}{p-1} |\tilde{v}|^p} \cdot \nabla (|\tilde{v}|^{p-1} \tilde{v}) dy,
\end{equation}
consider
\begin{equation}
\int \frac{y}{(1 + |y|^2)^{1/2}} e^{-(p-3)\tau} \frac{|y|}{\sinh |y|} e^{-\frac{p}{p-1} |\tilde{v}|^p} \tilde{w}_j \cdot \nabla (|\tilde{v}|^{p-1} \tilde{v}) dy,
\end{equation}
for a fixed $j \in \mathbb{Z}$. Define a modified Littlewood–Paley function, this time in space. This function is similar to \textbf{(4.12)}. Let
\begin{equation}
\tilde{P}_0 f = \int \psi(y-z) f(z), \quad \text{when} \quad k > 0, \quad \tilde{P}_k f = 2^{3k} \int \psi(2^k(y-z)) f(z) dz - 2^{3(k-1)} \int \psi(2^{k-1}(y-z)) f(z),
\end{equation}
where $\psi \in C_0^\infty(\mathbb{R}^3)$ is supported on $|y| \leq 1$ and $\int \psi(y) dy = 1$. Now make a partition of unity
\begin{equation}
1 = \sum_{m \geq 0} \chi (|y| - m).
\end{equation}
Define $k(m, \tau) = \sup\{0, \frac{m}{m+2} + \frac{\tau}{m+2} + j\}$. Since $\frac{|y|}{\sinh |y|} \sim \frac{|z|}{\sinh |z|}$ when $|y - z| \leq 1$,
\begin{equation}
\sum_{m \geq 0} \int \frac{y}{(1 + |y|^2)^{1/2}} e^{-(p-3)\tau} \frac{|y|}{\sinh |y|} e^{-\frac{p}{p-1} |\tilde{v}|^p} \chi (|y| - m) \tilde{w}_j (\tau, y) \cdot \nabla (|P_{\leq k(m, \tau)}|^{p-1} (P_{\leq k(m, \tau)} \tilde{v})) dy
\lesssim \sum_{m \geq 0} 2^{k(m, \tau)} \frac{1}{\sinh |y|} \int_{m-2 \leq |y| \leq m+2} \frac{1}{|y|} e^{-(p-3)\tau} |\tilde{v}|^p dy \\
\times \left( \int_{m-2 \leq |y| \leq m+2} \frac{1}{|y| \cosh^2 |y|} |\nabla \tilde{v}|^2 + \frac{1}{|y| \cosh^2 |y|} |\tilde{v}|^2 dy \right)^{1/2} \\
\times \left( \sup_{m-2 \leq |y| \leq m+2} e^{\frac{3\tau}{2}} \cosh^2 |y| \tilde{w}_j (e^{\tau} \cosh |y|, e^{\tau} \sinh |y|, \frac{y}{|y|}) \right)
\lesssim a_j \left( \int \frac{1}{|y|} e^{-(p-3)\tau} |\tilde{v}|^p dy \right)^{1/2} \left( \int \frac{1}{|y| \cosh^2 |y|} |\nabla \tilde{v}|^2 + \frac{1}{|y| \cosh^2 |y|} |\tilde{v}|^2 dy \right)^{1/2}.
Integrating by parts, (5.16)
\[
\sum_{m \geq 0} \int \frac{y}{(1 + |y|^2)^{1/2}} \, e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} \chi(|y| - m) \hat{w}_j(\tau, y) \cdot \nabla (|\tilde{v}|^{p-1}|P_{\geq k(m, \tau)}\tilde{v}|) dy
\]
\[= \sum_{m \geq 0} \int \frac{y}{(1 + |y|^2)^{1/2}} \, e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-2} \chi(|y| - m) e^{\frac{2}{\tau} \tau} w_j(\tilde{v}^+ \cosh |y|, \tilde{v}^+ \sinh |y| \frac{y}{|y|})
\]
\[
\cdot \nabla (|\tilde{v}|^{p-1}|P_{\geq k(m, \tau)}\tilde{v}|) dy
\]
\[= - \sum_{m \geq 0} \int \frac{y}{(1 + |y|^2)^{1/2}} \, e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-2} \chi(|y| - m) (|\tilde{v}|^{p-1}|P_{\geq k(m, \tau)}\tilde{v}|)
\]
\[
eq e^{\frac{2}{\tau} \tau} e^\tau \sinh |y| \frac{y}{|y|} \partial_j w_j + e^\tau \cosh |y| \frac{y}{|y|} (\partial_j w_j) \left( e^\tau \cosh |y|, e^\tau \sinh |y| \frac{y}{|y|} \right) dy
\]
\[= \sum_{m \geq 0} \int \nabla \cdot \left( \frac{y}{(1 + |y|^2)^{1/2}} \, e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-2} \chi(|y| - m) e^{\frac{2}{\tau} \tau} w_j \left( e^\tau \cosh |y|, e^\tau \sinh |y| \frac{y}{|y|} \right) \right) \nabla \tilde{v} dy
\]
\[\lesssim a_j \left( \int \frac{|y|}{\sinh |y|} \, |\tilde{v}|^{p-1} \left( \int \frac{\tau}{\cos^2 |y|} \frac{\cos^2 |y|}{|y|^2} dy \right)^{p-1} \right) \frac{\tau}{\cos^2 |y|} \left( \int \frac{\tau}{\cos^2 |y|} \frac{\cos^2 |y|}{|y|^2} dy \right)^{p-1} \nabla \tilde{v}^2.
\]

Also, (5.18)
\[
- \sum_{m \geq 0} \int \frac{|y|}{(1 + |y|^2)^{1/2}} \, e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-2} \chi(|y| - m) e^{\frac{2}{\tau} \tau} w_j \left( e^\tau \cosh |y|, e^\tau \sinh |y| \frac{y}{|y|} \right) \nabla \tilde{v} dy
\]
\[= \sum_{m \geq 0} \int \nabla \cdot \left( \frac{y}{(1 + |y|^2)^{1/2}} \, e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-2} \chi(|y| - m) e^{\frac{2}{\tau} \tau} w_j \left( e^\tau \cosh |y|, e^\tau \sinh |y| \frac{y}{|y|} \right) \right) \nabla \tilde{v} dy
\]
\[\lesssim a_j \left( \int \frac{|y|}{\sinh |y|} \, |\tilde{v}|^{p-1} \left( \int \frac{\tau}{\cos^2 |y|} \frac{\cos^2 |y|}{|y|^2} dy \right)^{p-1} \right) \frac{\tau}{\cos^2 |y|} \left( \int \frac{\tau}{\cos^2 |y|} \frac{\cos^2 |y|}{|y|^2} dy \right)^{p-1} \nabla \tilde{v}^2.
\]
Therefore, we have proved
\[
\int_1^T \int \frac{y}{(1 + |y|^2)^{1/2}} e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} \tilde{w}_j(\tau, y) \cdot \nabla(|\tilde{v}|^{p-1}\tilde{v}) \, dy \, d\tau \\
\lesssim \epsilon \int_1^T \int e^{-(p-3)\tau} \left( \frac{|y|}{\sinh |y|} \right)^{p-1} |\tilde{v}|^{p+1} \, dy \, d\tau + \epsilon \int_1^T \int \frac{1}{(1 + |y|^2)^{5/2}} |\tilde{v}|^2 \, dy \, d\tau,
\]
which completes the proof of Theorem \textbf{11}.

6. Profile decomposition argument

Having obtained a scattering result for any \( u_0 \in B_{1,1}^{3-p,s} \), \( u_1 \in B_{1,1}^{3-p,s} \), it only remains to show that this bound is uniform over all \( (u_0, u_1) \) satisfying
\[
\|(u_0, u_1)\|_{B_{1,1}^{3-p,s} \times B_{1,1}^{3-p,s}} \leq A,
\]
for some \( A < \infty \). The proof argument is exactly parallel to the arguments in [Dod18a], [Dod18c], and especially in [Dod18b]. Here we are in the nonradial setting, however, we are aided by the fact that the nonlinearity is not the Lorentz invariant nonlinearity.

Let \( (u_0^n, u_1^n) \) be a bounded sequence in \( B_{1,1}^{3-p,s} \times B_{1,1}^{3-p,s} \). Since this sequence is bounded in \( H^s \times H^{s-1} \), then by Theorem 3.1 in [Ram12], we may make the profile decomposition
\[
S(t)(u_0, u_1) = \sum_{j=1}^N \Gamma_j^n S(t)(\phi_j^n, \phi_j^n) + S(t)(R_0^n, R_1^n),
\]
where
\[
\lim_{N \to \infty} \limsup_{n \to \infty} \|S(t)(R_0^n, R_1^n)\|_{L_t^2(\mathbb{R}^{n+1})} = 0.
\]
The group \( \Gamma_j^n \) is the group of operators generated by translation in space and in time, and also by the scaling symmetry. That is, there exist \( x_j^n \in \mathbb{R}^3, t_j^n \in \mathbb{R}, \) and \( \lambda_j^n \in (0, \infty) \) such that
\[
\Gamma_j^n \psi(t, x) = (\lambda_j^n)^{2/p} \psi(x_j^n (t - t_j^n), \lambda_j^n (x - x_j^n)).
\]
Furthermore, the \( \Gamma_j^n \)'s have the asymptotic orthogonality property that when \( j \neq k \),
\[
\lim_{n \to \infty} \left| \ln \left( \frac{\lambda_j^n}{\lambda_k^n} \right) \right| + (\lambda_j^n)^{1/2} (\lambda_k^n)^{1/2} (|x_j^n - x_k^n| + |t_j^n - t_k^n|) = \infty.
\]
Using the dispersive estimate in (1.11), \( \frac{|\psi^n_j|}{\lambda_j^n} \) is uniformly bounded for any \( j \).

**Lemma 3.** If \( \frac{|\psi^n_j|}{\lambda_j^n} \to \infty \) then \( \phi_j^n = 0 \) and \( \phi_j^n = 0 \).

**Proof.** Indeed, from [Ram12], for any fixed \( j \),
\[
\lim_{n \to \infty} (\Gamma_j^n)^{-1} S(t)(u_0^n, u_1^n) \to S(t)(\phi_j^n, \phi_j^n)
\]
weakly in \( L_t^{2(p-1)} \). Rewriting \( (\Gamma_j^n)^{-1} \),
\[
(\Gamma_j^n)^{-1} S(t)(u_0^n, u_1^n) = S(t - \frac{t_j^n}{\lambda_j^n})(\frac{\lambda_j^n}{\lambda_j^n} t_j^n u_0^n (\frac{x - x_j^n}{\lambda_j^n}), (\lambda_j^n)^{\frac{p+1}{p-1}} u_1^n (\frac{x - x_j^n}{\lambda_j^n})),
\]
and then by the dispersive estimate \((1.1)\), for any fixed Littlewood–Paley projection, if \(\frac{t^n_j}{\lambda^n_j} \to \pm \infty\),
\[
S(t - \frac{t^n_j}{\lambda^n_j})((\lambda^n_j)^{-\frac{2}{p-1}}u^n_0(x - \frac{x^n_j}{\lambda^n_j}), (\lambda^n_j)^{-\frac{2}{p-1}}u^n_1(x - \frac{x^n_j}{\lambda^n_j})) \to 0,
\]
weakly in \(L^{2(p-1)}_{t,x}\), which proves the lemma. \(\square\)

Since \(t^n_j\) is bounded for any \(j\), after passing to a subsequence, \(t^n_j \to t_j\). Absorbing the remainder into \(R_N\), we may rewrite \((2.2)\) with \(\Gamma^n_j\) having no translation in time, that is,
\[
\Gamma^n_j v(t, x) = (\lambda^n_j)^{\frac{2}{p-1}}(\lambda^n_j^n t, \lambda^n_j^n(x - x^n_j)).
\]
Furthermore, since
\[
(\lambda^n_j)^{\frac{2}{p-1}}u^n_j(\lambda^n_j^n t, x) \to \phi^n_0, \quad \text{and} \quad (\lambda^n_j)^{\frac{2}{p-1}+1}u^n_j(\lambda^n_j^n x) \to \phi^n_1,
\]
we have the bounds
\[
\|\phi^n_0\|_{B^{\frac{1}{2}+s}_t} + \|\phi^n_1\|_{B^{\frac{1}{2}+s}_t} \leq A.
\]
Therefore, the solution to \((1.1)\) with initial data equal to \((\phi^n_0, \phi^n_1)\) has a finite \(L^{2(p-1)}_{t,x}\) norm. Furthermore,
\[
\lim_{N \to \infty} \sum_{j=1}^N \|(\phi^n_0, \phi^n_1)\|^2_{H^{s+} \times H^{-s-1}} \leq \sup_{n \to \infty} \|(u_{0,n}, u_{1,n})\|^2_{H^{s+} \times H^{-s-1}},
\]
so for only finitely many \(j\), \(\|(\phi^n_0, \phi^n_1)\|_{H^{s+} \times H^{-s-1}} \geq \epsilon\). If \(\|(\phi^n_0, \phi^n_1)\|_{H^{s+} \times H^{-s-1}} \leq \epsilon\), then the solution to \((1.1)\) with initial data \((\phi^n_0, \phi^n_1)\) has the bound
\[
\|u\|_{L^{2(p-1)}_{t,x}} \lesssim \|(\phi^n_0, \phi^n_1)\|_{H^{s+} \times H^{-s-1}}.
\]
Therefore, by standard perturbative arguments combined with the asymptotic orthogonality in \((1.5)\), if \(u_n\) is the solution to \((1.1)\) with initial data \((u_{0,n}, u_{1,n})\),
\[
\lim_{n \to \infty} \|u_n\|_{L^{2(p-1)}_{t,x}} < \infty.
\]
Thus, there must exist a uniform upper bound on the \(L^{2(p-1)}_{t,x}\) norm of a solution \(u\) to \((1.1)\) whose initial data has bounded Besov norm.

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