A direct product theorem for quantum communication complexity with applications to device-independent QKD*

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Abstract

We give a direct product theorem for the entanglement-assisted interactive quantum communication complexity of an \( l \)-player predicate \( V \). In particular we show that for a distribution \( p \) that is product across the input sets of the \( l \) players, the success probability of any entanglement-assisted quantum communication protocol for computing \( n \) copies of \( V \), whose communication is \( o(\log(\text{eff}\,(V,p)) \cdot n) \), goes down exponentially in \( n \). Here \( \text{eff}\,(V,p) \) is a distributional version of the quantum efficiency or partition bound introduced by Laplante, Lerays and Roland (2012), which is a lower bound on the distributional quantum communication complexity of computing a single copy of \( V \) with respect to \( p \). For a two-input boolean function \( f \), the best result for interactive quantum communication complexity known previously was due to Sherstov (2018), who showed a direct product theorem in terms of the generalized discrepancy, which is a lower bound on communication. Our lower bound on non-distributional communication complexity is in terms of \( \max_\text{product}\, \text{eff}\,(V,p) \), and there is no known relationship between this and the generalized discrepancy. But we define a distributional version of the generalized discrepancy bound and can show that for a given \( p \), \( \text{eff}\,(V,p) \) upper bounds it. Moreover, unlike Sherstov’s result, our result works for two-input functions or relations whose outputs are non-boolean as well, and is a strong direct product theorem for functions or relations whose quantum communication complexity is characterized by \( \text{eff}\,(V,f,p) \) for a product \( p \).

As an application of our result, we show that it is possible to do device-independent quantum key distribution (DIQKD) without the assumption that devices do not leak any information after inputs are provided to them. We analyze the DIQKD protocol given by Jain, Miller and Shi (2020), and show that when the protocol is carried out with devices that are compatible with \( n \) copies of the Magic Square game, it is possible to extract \( \Omega(n) \) bits of key from it, even in the presence of \( O(n) \) bits of leakage. Our security proof is parallel, i.e., the honest parties can enter all their inputs into their devices at once, and works for a leakage model that is arbitrarily interactive, i.e., the devices of the honest parties Alice and Bob can exchange information with each other and with the eavesdropper Eve in any number of rounds, as long as the total number of bits or qubits communicated is bounded.

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1 Introduction

Communication complexity is an important model of computation with connections to many parts of theoretical computer science [KN96]. In this paper, we consider the communication complexity of computing a predicate $V$ on $(\mathcal{A}^1 \times \ldots \times \mathcal{A}^l) \times (\mathcal{X}^1 \times \ldots \times \mathcal{X}^l)$ by $l (\geq 2)$ players who receive inputs $x^1 \ldots x^l \in \mathcal{X}^1 \times \ldots \times \mathcal{X}^l$, and after communicating interactively, are required to produce outputs $a^1 \ldots a^l$ such that $V(a^1 \ldots a^l, x^1 \ldots x^l)$ is satisfied. The $l$ players cooperate and wish to minimize the total number of bits (in the classical model) or qubits (in the quantum model) communicated. The communication complexity of predicates generalizes the communication complexity of (total or partial) functions and relations that are most often considered in the literature.

In any model of computation, a fundamental question is: if we know how to do one copy of a task, what is the best way to do $n$ independent copies of it? One possible way is to simply each copy independently; if we have an algorithm that successfully does a single copy of the task with probability $1 - \varepsilon$, the success probability of this product strategy is $(1 - \varepsilon)^n$ and its cost is $n$ times the cost of doing a single copy. For many tasks, this is the best one can do, and a direct product theorem for the task proves so. That is, a direct product theorem proves that any protocol for doing $n$ copies of the task that has cost at most $cn$, where $c$ is some lower bound on the cost of doing one copy with success probability less than 1, has success probability exponentially small in $n$. When $c$ is the exact cost of doing a single copy of the task, we call such a result a strong direct product theorem.

Direct product theorems are known in a number of computational models. In classical communication complexity, there is a long line of works showing direct product and weaker direct sum theorems (which show that the success probability of a protocol that uses $cn$ resources is at most constant, instead of exponentially small) in the two-party setting [Raz92, CSWY01, BYJKS02, JRS05, KSdW07, VW08, LS08, HJMR10, BR11, JY12, BBCR13, BRWY13a, BRWY13b, JPY16].

For quantum communication, a direct sum theorem for one-way quantum communication for general functions was shown by [JRS05]. [BRdW08] showed a direct product theorem for functions whose one-way quantum communication complexity is characterized by VC dimension, and [JK21] showed a direct product theorem for one-way quantum communication complexity of general functions. In the interactive quantum setting however, direct product theorems are known only for special classes of functions, for example [KSdW07] showed a direct product theorem for symmetric functions. [She18] showed a direct product theorem for the generalized discrepancy method, which is one of the strongest lower bound techniques on quantum communication complexity — this gives a strong direct product theorem for functions whose quantum communication complexity is exactly characterized by the generalized discrepancy method.

Direct product theorems in communication are related to parallel repetition theorems for non-local games. A non-local game with $l$ players is defined by a predicate $V$ and a distribution $p$. The players are given inputs $x^1 \ldots x^l$ from distribution $p$ on $\mathcal{X}^1 \times \ldots \times \mathcal{X}^l$, and they are required to produce outputs $a^1 \ldots a^l$ in $\mathcal{A}^1 \times \ldots \times \mathcal{A}^l$ so that $V(a^1 \ldots a^l, x^1 \ldots x^l)$ is satisfied, without communicating. In the classical model, the players are allowed to share randomness, and in the quantum model they are allowed to share entanglement. The maximum winning probability of the game over all strategies is called the value of the game, which may be quantum or classical. A parallel repetition theorem shows that the value of $n$ independent instances of a non-local game is $(1 - \varepsilon)^\Omega(n)$, if the value a single instance is $(1 - \varepsilon)$.

A parallel repetition theorem for the classical value of general two-player non-local games was
first shown by Raz [Raz95], and the proof was subsequently simplified by Holenstein [Hol07]. A strong parallel repetition theorem for the quantum value of a general two-player non-local game is not known. Parallel repetition theorems were shown for special classes of two-player games such as XOR games [CSUU08], unique games [KRT10] and projection games [DSV15]. When the type of game is not restricted but the input distribution is, parallel repetition theorems have been shown under product distributions [JPY14] and anchored distributions [BVY17] — both of these results can be extended to $l$ players. For general two-player games, the best current result is due to Yuen [Yue16], which shows that the quantum value of $n$ parallel instances of a general game goes down polynomially in $n$, if the quantum value of the original game is strictly less than 1. The situation for more than two players is much less understood.

**Device-independent cryptography.** Quantum cryptography lets us do a number of tasks with information theoretic security, i.e., security without any computational assumptions, that are not possible classically. One such example is quantum key distribution (QKD) [BB84]. In a key distribution scenario, two honest parties Alice and Bob want to share a key, i.e., a uniformly random string of a given length, which is secret from a third party eavesdropper Eve. If Alice and Bob have access to secure private randomness and an authenticated classical channel, it is possible to do the key distribution task quantumly with information theoretic security, but not classically. In a conventional security proof for QKD (or any other quantum cryptographic protocol), one needs to have a complete description of the quantum devices, i.e., the states and measurements used by Alice and Bob. However, in practice quantum devices are often not fully characterized, and protocols that rely on complete characterization of quantum devices often have loopholes.

A way around this problem is the framework of device-independent cryptography, which tries to give quantum protocols for cryptographic tasks that are secure even when the devices used by the honest parties are not fully characterized, and in fact can be arbitrarily manipulated by dishonest parties. All known device-independent protocols with information theoretic security use non-local games and rely on the property of self-testing or rigidity displayed by some non-local games. Suppose we play a non-local game with devices implementing some unknown state and measurements, and in fact even the dimension of the systems are unspecified. If these state and measurements regardless achieve a winning probability for the game that is close to its optimal winning probability, then self-testing tells us that the state and measurements are close to the ideal state and measurements for that game, up to trivial isometries. For device-independent QKD (DIQKD), this means in particular that the measurement outputs of the devices given the inputs are random, i.e., they cannot be predicted by a third party even if they have access to the inputs used. This lets us use the outputs of the devices to produce a secret key.

A number of protocols and security proofs for DIQKD have been given over the years, in the sequential [PAB+09, AFDF+18, VV19] as well as parallel setting [JMS20, Vid17]. Aside from assuming that Alice and Bob’s devices are modelled by quantum mechanics however, all these proofs require the assumption that Alice and Bob’s devices do not leak any information, i.e., do not communicate with each other or with Eve, unbeknownst to Alice and Bob. Although there have been some works studying non-local games in the presence of communication [TZCBB+20, TZCWP20], and an argument showing device-independent QKD may be possible in the presence of a specific model of information leakage in [SPM13], none of these approaches have been developed into a full-fledged proof of security when there is leakage.
1.1  Our results

1.1.1 Direct product theorem

Let \( V(a_1 \ldots a_i, x_1, \ldots, x_l) \) be a predicate on \( (A^1 \times \ldots \times A^l) \times (\mathcal{X}^1 \times \ldots \times \mathcal{X}^l) \). We shall use \( V^n(a_1 \ldots a_i \ldots a_n, x_1^1 \ldots x_1^l \ldots x_n^1 \ldots x_n^l) \) to denote \( n \) independent copies of \( V \), i.e., the predicate which is satisfied when all \( n \) \((a_1^1 \ldots a_i^1 \ldots a_n^1, x_1^1 \ldots x_1^l \ldots x_n^1)\)s satisfy \( V \).

For a probability distribution \( p \) on \( \mathcal{X}^1 \times \ldots \times \mathcal{X}^l \), a (quantum) communication protocol \( P \) between \( l \) parties that takes inputs from \( \mathcal{X}^1 \times \ldots \times \mathcal{X}^l \) and produces outputs in \( A_1 \times \ldots \times A_n \), produces a conditional probability distribution on \( A^1 \times \ldots \times A^l \) conditioned on \( \mathcal{X}^1 \times \ldots \times \mathcal{X}^l \), and along with \( p \) there is an induced distribution on \( (A^1 \times \ldots \times A^l) \times (\mathcal{X}^1 \times \ldots \times \mathcal{X}^l) \). Let \( \text{succ}(p, V, P) \) be the probability that the predicate \( V \) is satisfied according to this distribution.

Let \( \text{eff}_{\epsilon}^*(V, p) \) denote the distributional quantum partition bound with error \( \epsilon \) for \( V \) with respect to input distribution \( p \), which we shall define formally in Section 4. \( \text{eff}_{\epsilon}^*(V, p) \) is a lower bound on the quantum communication complexity of \( V \). Let \( \omega^*(G(p, V)) \) denote the quantum value of the non-local game \( G = (p, \mathcal{X}^1 \times \ldots \times \mathcal{X}^l, A^1 \times \ldots \times A^l, V) \).

With this notation, our direct product theorem is stated below.

**Theorem 1.** For any \( \epsilon, \zeta > 0 \), any predicate \( V \) on \( (A^1 \times \ldots \times A^l) \times (\mathcal{X}^1 \times \ldots \times \mathcal{X}^l) \) and any product probability distribution \( p \) on \( \mathcal{X}^1 \times \ldots \times \mathcal{X}^l \), if \( P \) is an interactive entanglement-assisted quantum communication protocol between \( l \) parties which has total communication \( c n \).

1. If \( c < 1 \), then
   \[
   \text{succ}(p^n, V^n, P) \leq \left( 1 - \frac{\nu}{2} + \sqrt{2lc} \right) ^{\Omega(n/\log(|A^1| \ldots |A^l|))}
   \]
   where \( \nu = 1 - \omega^*(G(p, V)) \).

2. If \( 1 \leq c = \delta \cdot \frac{\epsilon^2}{\zeta} \text{eff}_{\epsilon+\zeta}^*(V, p) \) for small enough \( \delta \), then
   \[
   \text{succ}(p^n, V^n, P) \leq (1 - \epsilon) ^{\Omega(n/\log(|A^1| \ldots |A^l|))}
   \]

The two cases in Theorem 1 should be interpreted as follows: \( c < 1 \) means there is less than one qubit of communication per copy of \( V \), and we are close to the non-local game situation where there is no communication. Therefore we get an upper bound on the success probability for computing \( V^n \) in terms of the winning probability of the corresponding game. The theorem in this case is essentially saying that parallel-repeated non-local games under product distributions are resistant to communication, i.e., if the winning probability of \( n \) copies of the game goes down exponentially in \( n \), then it also goes down exponentially in \( n \) if there is a small amount of communication. We also remark that in case (i), a corresponding theorem can also be proved if \( p \) is an anchored distribution, which was introduced in [BVY17], instead of a product distribution. We expand on this more in Section 2.

The case \( c \geq 1 \) means on average at least one qubit is communicated per copy of \( V \). This corresponds to the true communication scenario, and thus if \( c \) is less than a lower bound on the per copy communication complexity of \( V \), we get that the probability of success for computing \( V^n \) goes down exponentially in \( n \). By Yao’s Lemma, case (ii) of Theorem 1 has the following corollary for communication complexity.
Corollary 2. For a predicate $V$ on $(\mathcal{A}^1 \times \ldots \times \mathcal{A}^l) \times (\mathcal{X}^1 \times \ldots \times \mathcal{X}^l)$, let $Q_{\epsilon}(V)$ denote the interactive entanglement-assisted quantum communication complexity of computing it, and $\text{eff}_{\epsilon+\zeta}(V, p)$ be its distributional quantum partition bound for distribution $p$, and any $\epsilon, \zeta > 0$. Then,

$$Q_{1-(1-\epsilon)^{n/(\log(|\mathcal{A}| \cdots |\mathcal{A}|))}}(V^p) = \Omega \left( \frac{\zeta^2 n}{f^3} \left( \max_{\text{product } p} \log \text{eff}_{\epsilon+\zeta}(V, p) \right) \right).$$

Corollary 2 is a strong direct product theorem for predicates whose interactive entanglement-assisted communication complexity is characterized by $\max_{\text{product } p} \text{eff}_{\epsilon}(V, p)$.

1.1.2 Applications in two-party communication complexity of functions

In the communication complexity setting for a two-input function or relation $f \subseteq \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$, we normally require that only one party gives an output. Nevertheless, we can define a predicate $V_f$ for it in which one party has a singleton output set, say $\{\top\}$, and the other party’s output set is $\mathcal{Z}$. We define

$$V_f(\top z, xy) = 1 \iff z \in f(x, y).$$

It is clear then that the two-party communication complexity of $f$ is equal to the communication complexity of $V_f$.

In [ABJO21], it is shown that a large class of functions exists, whose quantum communication complexity is characterized by $\text{eff}_{\epsilon}(V_f, p)$ for a product $p$. In particular, they show that a class of functions known as two-wise independent functions, $\text{eff}_{\epsilon}(V_f, p)$ takes the maximum possible value of the uniform distribution, which is product.

Fact 1 ([ABJO21]). Let $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ be a two-wise independent function with $|\mathcal{X}| = |\mathcal{Y}|$, and let $p_U$ be the uniform distribution on $\mathcal{X} \times \mathcal{Y}$. Then for any $\epsilon > 0$,

$$\text{eff}_{\epsilon}(V_f, p_U) \geq \frac{|\mathcal{X}|}{|\mathcal{Z}|} \left( 1 - \gamma - \frac{1}{|\mathcal{Z}|} \right)^2.$$

An example of a two-wise independent function is the generalized inner product $\text{IP}_q^n : \mathbb{F}_q^n \times \mathbb{F}_q^n \to \mathbb{F}_q$ defined by:

$$\text{IP}_q^n(x, y) = \sum_{i=1}^n x_i y_i \mod q.$$

This makes our result the first strong direct product theorem for generalized inner product that we are aware of. The direct product theorem in terms of the generalized discrepancy method by Sherstov [She18] works only for boolean-output functions, and gives a strong direct product theorem for quantum communication of $\text{IP}_q^n$.

For further comparison between our direct product theorem and Sherstov’s, we prove Theorem 3. For a total function $f : \mathcal{X} \times \mathcal{Y} \to \{-1, +1\}$, let $F$ denote the $|\mathcal{X}| \times |\mathcal{Y}|$ matrix whose $[x, y]$-th entry is given by $f(x, y)$. The generalized discrepancy method lower bounds communication in terms of $\log \gamma_2^a(F)$, where $\gamma_2^a(M)$ is the $a$-approximate factorization norm of a matrix $M$. For a function $f$, $\gamma_2^a(F)$ can be expressed as $\max_p \gamma_2^a(F, p)$ where $\gamma_2^a(F, p)$ is a distributional version of $\gamma_2^a(F)$ with respect to $p$ over $\mathcal{X} \times \mathcal{Y}$.
**Theorem 3.** For a total function \( f : \mathcal{X} \times \mathcal{Y} \rightarrow \{-1, +1\} \), let \( V_f \) denote the predicate on \((\{-1, +1\})^2 \times (\mathcal{X} \times \mathcal{Y})\) given by

\[
V(ab, xy) = 1 \iff a \cdot b = f(x, y).
\]

Then for any distribution \( p \) on \( \mathcal{X} \times \mathcal{Y} \),

\[
\text{eff}^*_{\epsilon}(V_f, p) \geq (1 - 2\epsilon)\gamma_2^2(F, p)
\]

with \( \alpha = \frac{1 + 2\epsilon}{1 - 2\epsilon} \).

This shows that \( \text{eff}^*_{\epsilon}(V_f, p) \) is a stronger lower bound technique than \( \gamma_2(F, p) \) for boolean \( f \). However, since our direct product theorem is in terms of \( \text{max}_p \log \text{eff}^*_{\epsilon}(V_f, p) \), and Sherstov’s in terms of \( \text{max}_p \log \gamma_2(F, p) \), the two results cannot be directly compared.

### 1.1.3 DIQKD secure against leakage

**Leakage model.** In the device-independent setting, each honest party’s device is modelled as a black box, into which the party provides inputs and from which they get outputs to play a non-local game. Ideally the boxes play \( n \) independent copies of the non-local game, although they may do so noisily, i.e., each game is won with probability \( \delta \)-close to its optimal quantum value.

For DIQKD, the honest parties are Alice and Bob and we assume their boxes are supplied by the eavesdropper Eve. The states and measurements implemented by these boxes may be very far from those corresponding to the two-player non-local game that each of Alice and Bob’s boxes ideally play. In fact, instead of Alice and Bob sharing an entangled state that is uncorrelated with anything else, Eve may hold a purification of Alice and Bob’s state, which we also model as a box.

As mentioned before, known DIQKD protocols rely on the assumption that Alice and Bob and Eve’s boxes do not communicate with each other. We relax the assumption in a strong way: we assume Alice, Bob and Eve’s boxes can all send classical messages to each other (since they share entanglement, this means they can also effectively exchange quantum states via teleportation) after Alice and Bob have entered their inputs into their boxes and before they receive their outputs. The communication between Alice, Bob and Eve’s boxes may be arbitrarily interactive: we do not put any bound on the number of rounds of communication, only on the total number of bits communicated.

For the sake of concreteness, we analyze the parallel DIQKD protocol given by [JMS20] under this leakage model, but in principle the same analysis could be applied to any DIQKD protocol that is based on a non-local game that has: (i) a product input distribution, and (ii) a common bit that Alice and Bob can ideally both know given their outputs \( a \) and \( b \), and both parties’ inputs \( x \) and \( y \) (and this bit is their shared key). Using case (i) of Theorem 1, we prove the following theorem.

**Theorem 4.** There are universal constants \( 0 < \delta_0 < 1 \) and \( 0 < c_0 < 1 \) such that for any \( 0 \leq \delta \leq \delta_0 \), and \( 0 \leq c \leq c_0 \), if the [JMS20] DIQKD protocol (given in Protocol 1) is carried out with boxes that play \( n \) copies of the Magic Square game \( \delta \)-noisily, it is possible to extract \( r(\delta, c)n \) bits of secret key in the interactive leakage model, with the total communication between Alice, Bob and Eve’s boxes being \( cn \) bits, for some \( r(\delta, c) > 0 \).

**Remark 1.** In practice Alice and Bob’s boxes can also continue sending messages to Eve after their outputs are produced (there can also be communication to Alice and Bob’s boxes, but the key rate depends on Eve’s
probability of guessing Alice and Bob’s outputs, which cannot change due to communication to Alice and Bob’s boxes after they have produced their outputs, so we ignore these at this time). But as far as security analysis is concerned, this communication is equivalent to Eve gaining some information about Alice and Bob’s outputs after they have been produced, e.g. from communication between Alice and Bob over a public channel, which is a standard part of QKD protocols and can be handled by standard DIQKD proof techniques. Using standard techniques, the amount of communication after the outputs are produced would just be subtracted from the key rate, and after a certain threshold of communication, key rate would just be zero. Communication before Alice and Bob’s outputs are produced cannot be handled by standard techniques, however, and hence we focus on this in the above theorem.

We also note that though we give a specific proof only for DIQKD with leakage, our proof technique can be seen as a general framework for making device-independent protocols that use parallel repetition theorems in their security proofs, secure against leakage. For example, this technique can also be applied to the device-independent protocol for encryption with certified deletion given by [KT20]. The security proof for that protocol uses a parallel repetition theorem for an anchored two-round game (where players receive two rounds of inputs and give two rounds of outputs). As we have already said, a version of Theorem 1 in case (i) also applies to anchored distributions for one-round games, and it is not difficult to generalize to two-round games by considering an appropriate round-by-round leakage model.

1.2 Organization of the paper

In Section 2 we give an overview of our proofs. In Section 3 we provide definitions and known results about the quantities used in our proofs. In Section 4, we introduce variants of the quantum partition bound, prove that they lower bound communication and also Theorem 3. In Section 5, we prove a lemma called the Substate Perturbation Lemma, which is a main tool for our direct product theorem. In Section 6, we give the proof of our main direct product theorem. Finally, in Section 7 we show the application of our direct product theorem to prove security of DIQKD with leakage.

2 Proof overview

2.1 Direct product theorem

We follow the information-theoretic framework for parallel repetition and direct product theorems introduced by [Raz95] and [Hol07]. The idea is this: take a protocol \( P \) for \( V \) that is “too good”. We condition on the success in some \( t \) coordinates in this protocol, and show that either the probability of success in these coordinates is already small, or there is an \( i \) in the other \( n - t \) coordinates such that the probability of success of \( i \) conditioned on success event \( E \) is bounded away from 1. This is done by showing that if the probability of \( E \) and the probability of success in \( i \) conditioned on \( E \) are both large, we can give a protocol \( P' \) for \( V \) that is “too efficient”. Now our lower bound in the \( c \geq 1 \) case is in terms of \( \text{eff}_c^\ell(V, p) \), which intuitively speaking, corresponds to the inverse of the maximum probability of not aborting in a zero-communication protocol in which the \( l \) parties either abort, or produce outputs that satisfy \( V \) with probability at least \( 1 - \epsilon \) (conditioned on not aborting). Therefore, \( P' \) for us will be a zero-communication protocol with
aborts that computes \( V \) with high probability conditioned on not aborting, whose probability of not aborting is too high.

For simplicity, we shall give an overview of the proof with only two parties Alice and Bob; the proof for \( l \) parties follows similarly. When Alice and Bob’s inputs are \( x_i \) and \( y_i \) respectively at the \( i \)-th coordinates in \( \mathcal{P} \), we define a state \( |\varphi\rangle_{x_iy_i} \) that represents the state at the end of \( \mathcal{P} \) conditioned on \( \mathcal{E} \). Considering the state at the end instead of round by round is the same approach as that taken in [JRS05], who use it to show a direct sum theorem. On input \((x_i, y_i)\) in \( \mathcal{P}' \), Alice and Bob will try to either abort, or get a shared state close to \( |\varphi\rangle_{x_iy_i} \). Once they have this state, they can perform measurements on the \( i \)-th output registers to give their outputs \((a_i, b_i)\). Their output distribution will be close to the output distribution in the \( i \)-th coordinate of \( \mathcal{P} \) conditioned on \( \mathcal{E} \); hence if the probability of success on \( i \) conditioned on \( \mathcal{E} \) is too large, the probability of Alice and Bob correctly computing \( V \) in \( \mathcal{P}' \) conditioned on not aborting is also large. Hence our proof mainly consists of showing how Alice and Bob can get the shared state close to \( |\varphi\rangle_{x_iy_i} \) with probability of aborting \( 2^{-\mathcal{O}(c)} \), where \( cn \) is the communication in \( \mathcal{P} \). Since the probability of aborting in \( \mathcal{P}' \) cannot be smaller than \( \text{eff}^n \), this gives the desired lower bound on the communication of \( \mathcal{P} \) in terms of \( \text{eff}^n \).

In the \( c < 1 \) case, our proof is very similar to the proof of a parallel repetition theorem for non-local games with product distributions due to [JPY14]. The main difference between that \( c \geq 1 \) case and the parallel repetition of \( c < 1 \) case is that in the latter, we need to show that Alice and Bob can get the shared state \( |\varphi\rangle \) by local unitaries (without aborting). We briefly describe their proof below.

**Parallel repetition for games under product distribution.** Let \( |\varphi\rangle_{x_i} \) be the superposition of \( |\varphi\rangle_{x_iy_i} \) over the distribution of \( Y_i \), \( |\varphi\rangle_{y_i} \) be the superposition over the distribution of \( X_i \), and \( |\varphi\rangle \) be the superposition over both. If the probability of \( \mathcal{E} \) is large, then conditioning on it, the following can be shown:

1. By chain rule of mutual information, there is an \( X_i \) whose mutual information with Bob’s registers in \( |\varphi\rangle \) is small. Hence by Uhlmann’s theorem, there exist unitaries \( U_{x_i} \) acting on Alice’s registers that take \( |\varphi\rangle \) close to \( |\varphi\rangle_{x_i} \).

2. Similarly, the mutual information between \( Y_i \) and Alice’s registers in \( |\varphi\rangle \) is small, and hence there exist unitaries \( V_{y_i} \) acting on Bob’s registers that take \( |\varphi\rangle \) close to \( |\varphi\rangle_{y_i} \).

3. By applying the quantum operation that measures the \( X_i \) register and records the outcome, it can be shown that \( V_{y_i} \) also takes \( |\varphi\rangle_{x_i} \) to \( |\varphi\rangle_{x_iy_i} \).

4. Since \( U_{x_i} \) and \( V_{y_i} \) act on disjoint registers, \( U_{x_i} \otimes V_{y_i} \) then takes \( |\varphi\rangle \) close to \( |\varphi\rangle_{x_iy_i} \).

Alice and Bob can thus share \( |\varphi\rangle \) as entanglement, and get close to \( |\varphi\rangle_{x_iy_i} \) by local unitariess \( U_{x_i} \) and \( V_{y_i} \). In case (i) of our proof, everything is similar to this, except that the distance between \( |\varphi\rangle \) and \( |\varphi\rangle_{x_i} \) also accounts for \( c^A \), \( c^A n \) being Alice’s total communication to Bob, and the distance between \( |\varphi\rangle \) and \( |\varphi\rangle_{y_i} \) also accounts for \( c^B \), \( c^B n \) being Bob’s communication.

If we wish a give a proof for case (i) with anchored distributions instead of product distributions, we would need to follow the equivalent steps in the proof of the parallel repetition theorem for anchored games given in [BVY17] or the alternative proof given in [JK21] instead, and account for communication there.
Direct product for communication under product distribution. In case $c \geq 1$, we cannot use Uhlmann unitaries to go from $|\psi\rangle$ to $|\psi\rangle_{X_i}$ and $|\psi\rangle_{Y_i}$, as there is a lot of dependence between Alice’s registers and Bob’s registers due to communication. But we can use a compression scheme due to [JRS02, JRS05] which says that if the mutual information between $X_i$ and Bob’s registers is $c$, then there exist measurement operators $M_{X_i}$ acting on Alice’s registers which succeed on $|\psi\rangle$ with probability $2^{-c}$, and on success take it close to $|\psi\rangle_{X_i}$. Following parallel repetition proof we can show:

1. If the total communication from Alice to Bob in $P$ is $cA n$, then the mutual information between $X_1 \ldots X_n$ and Bob’s registers in $|\psi\rangle$ is $O(cA n)$. By chain rule of mutual information, there exists an $i$ such that the mutual information between $X_i$ and Bob’s registers is $O(cA)$, and hence there exist measurement operators $M_{X_i}$ acting on Alice’s registers which succeed with probability $2^{-O(cA)}$ on $|\psi\rangle$ and on success take $|\psi\rangle$ close to $|\psi\rangle_{X_i}$.

2. Similarly, if the total communication from Bob to Alice in $P$ is $cB n$, then there exist measurement operators $N_{Y_i}$ acting on Bob’s registers which succeed with probability $2^{-O(cB)}$ on $|\psi\rangle$ and on success take $|\psi\rangle$ close to $|\psi\rangle_{Y_i}$.

3. By applying the same argument with the operation measuring the $X_i$ register and recording the outcome, it can be shown that $N_{Y_i}$ succeeds on $|\psi\rangle_{X_i}$ with probability $2^{-O(cB)}$ and on success takes it close to $|\psi\rangle_{X_i}$. However, unlike in the case of unitaries, even though $M_{X_i}$ and $N_{Y_i}$ commute, there is a problem in combining items 2 and 3 above to say that $M_{X_i} \otimes N_{Y_i}$ succeed on $|\psi\rangle$ with probability $2^{-O(cA + cB)}$ and on success take it close to $|\psi\rangle_{X_i Y_i}$. Since $\sqrt{\frac{1}{2^{-O(cA)} M_{X_i} |\psi\rangle}}$ (i.e., the normalized state on success of $M_{X_i}$ on $|\psi\rangle$) is only close to $|\psi\rangle_{X_i}$ rather than exactly equal to it, acting $N_{Y_i}$ on this state cannot take it close to $|\psi\rangle_{X_i Y_i}$, unless the distance between $\sqrt{\frac{1}{2^{-O(cA)} M_{X_i} |\psi\rangle}}$ and $|\psi\rangle_{X_i}$ is of the same order as the success probability of $M_{Y_i}$ on $|\psi\rangle_{X_i}$. This distance figures in the exponent in the success probability $2^{-O(cA)}$, so we cannot afford to make it that small.

Instead we shall directly try to get projectors $N'_{Y_i}$ that succeed with high probability on $|\rho\rangle$, which is what we call the superposition over $X_i$ of $\sqrt{\frac{1}{2^{-O(cA)} M_{X_i} |\psi\rangle}}$, and on success take it close to $|\psi\rangle_{Y_i}$ (these will also take $|\rho\rangle_{X_i}$ close to $|\psi\rangle_{X_i Y_i}$). Since we do not have a bound on the mutual information between $Y_i$ and Alice’s registers in $\rho$, we prove what we call the Substate Perturbation Lemma in order to do this. The quantity that is actually of relevance in the [JRS05] compression scheme is the smoothed relative min-entropy $D_{\alpha}^{c}$ between $\varphi_{Y_i A}$ and $\varphi_{Y_i} \otimes \varphi_{A}$ ($A$ being Alice’s registers), which is $O(cB / \epsilon^2)$ if the mutual information between $Y_i$ and $A$ is $O(cB)$, due to the Quantum Substate Theorem [JRS02, JRS09, JN12]. In the Substate Perturbation Lemma, which is one of our main technical contributions, we show that if $D_{\alpha}^{c}(\varphi_{Y_i A} \|| \varphi_{Y_i} \otimes \varphi_{A})$ is $c'$ and $\rho_{A}$ and $\varphi_{A}$ are $\delta$-close, then $D_{\alpha\delta}^{c} + \delta(\varphi_{Y_i A} \|| \varphi_{Y_i} \otimes \rho_{A})$ is $O(c')$. Using the [JRS05] compression scheme, this lets us get projectors $N'_{Y_i}$ on Bob’s registers that succeed with probability $2^{-O(cB)}$ on $|\rho\rangle$ and on success take it close to $|\psi\rangle_{Y_i}$.

The protocol $P'$ will thus involve the following: Alice and Bob share $|\psi\rangle$ as entanglement and on inputs $(x_i, y_i)$, apply the measurements $\{M_{X_i}, 1 - M_{X_i}\}$ and $\{N'_{Y_i}, 1 - N'_{Y_i}\}$ on it. They abort if the $M_{X_i}$ or $N'_{Y_i}$ projector does not succeed. Since $M_{X_i} \otimes N'_{Y_i}$ succeeds on $|\psi\rangle$ with probability
$2^{-O(c^a+c^b)} = 2^{-O(c)}$, $P'$ does not abort with probability $2^{-O(c)}$ and on not aborting, gets a state close to $|\psi\rangle_{x_iy_i}$.

2.2 Security of DIQKD with leakage

The [JMS20] protocol is based on the Magic Square non-local game. In a single copy of the Magic Square game, henceforth denoted by MS, Alice and Bob receive trits $x$ and $y$ and are required to output 3-bit strings $a$ and $b$ which respectively have even and odd parity; they win the game if their outputs satisfy the condition $a[y] = b[x]$. In the [JMS20] protocol, Alice and Bob have boxes which are compatible with $n$ copies of MS. Using trusted private randomness, Alice and Bob generate i.i.d. inputs $x_i, y_i$ for each game and generate outputs $a_i, b_i$. The inputs $x_i, y_i$ are then publicly communicated. Alice and Bob select a small random subset of $|n|$ to test the MS winning condition on, i.e., they check if $a_i[y_i] = b_i[x_i]$ for $i$ in that subset (up to error tolerance). If the test passes, they select $K^A = (a_i[y_i])_i$, and $K^B = (b_i[x_i])_i$ as their raw secret keys; otherwise the protocol aborts. Due to error correction and privacy amplification, we can get a linear amount of secret key from this scheme if we can show [Ren05]

$$H^\varepsilon_0(K^A|\tilde{E'})_\rho - H^\varepsilon_0(K^A|K^B)_\rho = \Omega(n),$$

where $H^\varepsilon_0$ is the $\varepsilon$-smoothed conditional min-entropy and $H^\varepsilon_0$ is the $\varepsilon$-smoothed conditional Hartley entropy, $\rho$ is the shared state of Alice, Bob and Eve conditioned on not aborting, and $\tilde{E'}$ is everything Eve holds at the end of the protocol, including a quantum purification of Alice and Bob’s systems and also the classical information $X_iY_i$ that Alice and Bob have communicated publicly.

**Challenges in a sequential security proof.** Most security proofs for DIQKD work in the sequential setting, where Alice and Bob have to enter their inputs into their boxes and get their outputs one by one; in particular the sequential security proofs require the assumption that the $(i-1)$-th output is recorded before the box receives the $i$-th input. Sequential security proofs generally give better parameters than parallel ones, but we do not how to apply techniques for sequential proofs in the setting with leakage without fairly unnatural assumptions.

For example, one tool widely used in sequential security proofs is the Entropy Accumulation Theorem [DFR20, AFRV19]. Suppose the information released to Eve in the $i$-th round (in the sequential setting we call each time Alice and Bob enter inputs $x_i, y_i$ into their box, a round) is $T_i$, and Eve’s quantum register is $\tilde{E}$. Then in order to apply the Entropy Accumulation Theorem to bound $H^\varepsilon_0(K^A|T_1\ldots T_n\tilde{E})_\rho$, we require the Markov condition $(A_1\ldots A_{i-1}) - (T_1\ldots T_{i-1}\tilde{E}) - T_i$ for all $i$, i.e., the information leaked in the $i$-th round is independent of the Alice’s outputs of the rounds before $i$, given Eve’s side information before the $i$-th round. In the setting without leakage, $T_i$ is just Alice and Bob’s inputs $X_iY_i$ for the $i$-th round, which are picked with trusted private randomness, and thus can be made independent of everything else. In the setting with leakage however, $T_i$ would include the information leaked by Alice and Bob’s boxes in the $i$-th round as well. Once we allow the boxes to leak information, there is nothing stopping them from leaking information about the outputs of the $(i-1)$-th round in the $i$-th round. Thus imposing the Markov condition here feels fairly unnatural, and closes off the possibility of using Entropy Accumulation in the model with leakage.

**Parallel security proof.** Instead we closely follow the approach of [JMS20] in giving a parallel security proof for their protocol. Here “parallel” means that their security proof works when Alice
and Bob enter all their inputs into their boxes at once, and no Markov condition is required. The security proof of [JMS20] is based on the parallel repetition theorem for non-local games under product distributions [JPY14]. Since we are working in the setting with leakage, instead of a parallel repetition theorem for games, we use our direct product theorem for communication. The communication setting with 3 players exactly corresponds to the leakage model between the parties Alice, Bob and Eve in QKD. Case (i) of our direct product theorem says that if communication is cn for sufficiently small $c < 1$, then the probability of computing $n$ copies of a non-local game’s predicate correctly goes down exponentially in $n$.

The game we consider is a three-player version of MS, which is a hybrid of the games considered by [JMS20] and [Vid17], and this gives a simplified version of the [JMS20] proof. In this game, which we call MSE, Alice and Bob play MS between them, and in addition Eve, who has no input, has to guess both their inputs $x, y$, and Alice’s output bit $a[y]$. The winning probability of this game is strictly smaller than $\frac{1}{2}$ (which is Eve’s probability of correctly guessing $x, y$). Due to our direct product result, in the presence of a bounded amount of communication before the outputs are produced, the winning probability of $n$ copies of this game is $\left(\frac{1}{2}(1 - \nu)\right)^{\Omega(n)}$ for some $\nu > 0$.

Since Alice and Bob have performed the test to see that $a_i[y_i] = b_i[x_i]$ on a random subset, this condition is satisfied in most locations with high probability conditioned on not aborting. Therefore, MSE is won if Eve can correctly guess $x_i, y_i, a_i[y_i]$. Now, suppose $\phi_{K^A K^B X_1 \ldots X_n Y_1 \ldots Y_n \tilde{E}}$ is the shared quantum state before $x_1 \ldots x_n, y_1 \ldots y_n$ are communicated, conditioned on not aborting, with $\tilde{E}$ being Eve’s quantum register. Operationally $H_{\phi_n}(X_1 \ldots X_n Y_1 \ldots Y_n K^A | \tilde{E})_{\phi}$ is the negative logarithm of Eve’s probability of guessing $x_1 \ldots x_n y_1 \ldots y_n k^A$, which is the probability of winning $n$ instances of MSE, since Alice and Bob’s winning condition is satisfied with high probability. Hence by the direct product theorem, in the presence of a bounded amount of communication, $H_{\phi_n}(X_1 \ldots X_n Y_1 \ldots Y_n K^A | \tilde{E})_{\phi} = \Omega(n(\log 9 + \log(1/(1 - \nu))))$. By the chain rule of conditional min-entropy, this means that $H_{\phi_n}(K^A | XY \tilde{E})_{\phi} = \Omega(n \log(1/(1 - \nu)))$. We remark that since our direct product theorem is not “perfect”, i.e., the exponent we have is $\Omega(n)$ instead of $n$, we can only have Alice and Bob communicate a subset of $x_1 \ldots x_n y_1 \ldots y_n$ here instead of all of them (and $XY$ in the notation refers to the subset), and use those for key generation, so as not to make $H_{\phi_n}(K^A | XY \tilde{E})_{\phi}$ negative.

In the actual state $\rho$ after $XY$ is released, Eve can do some local operations on $XY \tilde{E}$, but these do not change $H_{\phi_n}(K^A | XY \tilde{E})_{\phi}$, and hence we have the same lower bound for $H_{\phi_n}(K^A | XY \tilde{E})_{\phi}$. In order to upper bound $H_{\phi_n}(K^B | K^A)_{\rho}$, we use the operational interpretation of $H_{\phi_n}(K^B | K^A)_{\rho}$ as the maximum number of possible values of $K^B$ given $K^A$. As mentioned before, conditioned on not aborting, $K^A$ and $K^B$ differ in very few locations with high probability, and hence we can bound this quantity.

**Remark 2.** An alternate security proof of the [JMS20] protocol was given in [Vid17] by using the parallel repetition of anchored games instead of product games. A version of case (i) of Theorem 1 with anchored games could also be used to follow this proof instead, to prove security against leakage.

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1Due to technical reasons, we also need to include the following feature in the game: Alice gets an additional input bit $z$, and Alice and Bob’s winning condition $a[y] = b[x]$ not being satisfied is forgiven if Eve is able to guess $z$.

2Alice and Bob cannot actually check the abort condition before $x_1 \ldots x_n y_1 \ldots y_n$ are communicated, but the aborting condition is a well-defined event on $K^A K^B XY$ and thus can be conditioned on before this.
3 Preliminaries

3.1 Probability theory

We shall denote the probability distribution of a random variable $X$ on some set $\mathcal{X}$ by $P_X$. For any event $E$ on $\mathcal{X}$, the distribution of $X$ conditioned on $E$ will be denoted by $P_{X|E}$. For joint random variables $XY$, $P_{X|Y=y}(x)$ is the conditional distribution of $X$ given $Y=y$; when it is clear from context which variable’s value is being conditioned on, we shall often shorten this to $P_{X|y}$. We shall use $P_{XY}P_{Z|X}$ to refer to the distribution

$$(P_{XY}P_{Z|X})(x, y, z) = P_{XY}(x, y) \cdot P_{Z|X=x}(z).$$

For two distributions $P_X$ and $P_{X'}$ on the same set $\mathcal{X}$, the $\ell_1$ distance between them is defined as

$$\|P_X - P_{X'}\|_1 = \sum_{x \in \mathcal{X}} |P_X(x) - P_{X'}(x)|.$$

**Fact 2.** For joint distributions $P_{XY}$ and $P_{X'Y'}$ on the same sets,

$$\|P_X - P_{X'}\|_1 \leq \|P_{XY} - P_{X'Y'}\|_1.$$

**Fact 3.** For two distributions $P_X$ and $P_{X'}$ on the same set and an event $E$ on the set,

$$|P_X(E) - P_{X'}(E)| \leq \frac{1}{2}\|P_X - P_{X'}\|_1.$$

The following result is a consequence of the well-known Serfing bound.

**Fact 4 ([TL17]).** Let $Z = Z_1 \ldots Z_n$ be $n$ binary random variables with an arbitrary joint distribution, and let $T$ be a random subset of size $\gamma n$ for $0 \leq \gamma \leq 1$, picked uniformly among all such subsets of $[n]$ and independently of $Z$. Then,

$$\Pr \left[ \left( \sum_{i \in T} Z_i \geq (1 - \epsilon)\gamma n \right) \land \left( \sum_{i \in [n]} Z_i < (1 - 2\epsilon)n \right) \right] \leq 2^{-2\epsilon^2 n}.$$

3.2 Quantum information

The $\ell_1$ distance between two quantum states $\rho$ and $\sigma$ is given by

$$\|\rho - \sigma\|_1 = \text{Tr}(\sqrt{\rho - \sigma})^\dagger(\rho - \sigma) = \text{Tr}|\rho - \sigma|.$$

The fidelity between two quantum states is given by

$$F(\rho, \sigma) = \|\sqrt{\rho} \sqrt{\sigma}\|_1 = \max_U \text{Tr}(U \sqrt{\rho} \sqrt{\sigma}).$$

The purified distance based on fidelity is given by

$$\Delta(\rho, \sigma) = \sqrt{1 - F(\rho, \sigma)^2}.$$

The Bures distance which is also based on fidelity is given by

$$B(\rho, \sigma) = \sqrt{1 - F(\rho, \sigma)}.$$

$\ell_1$ distance, $\Delta$ and $B$ are all metrics that satisfy the triangle inequality.
Fact 5 (Uhlmann’s theorem). Suppose $\rho$ and $\sigma$ are states on register $X$ which are purified to $|\rho\rangle_{XY}$ and $|\sigma\rangle_{XY'}$, with $Y$ ad $Y'$ not necessarily being of the same dimension, then it holds that

$$F(\rho, \sigma) = \max_U |\langle 1_X \otimes U | \sigma \rangle|$$

where the maximization is over isometries taking $Y'$ to $Y$.

Fact 6 (Fuchs-van de Graaf inequality). For any pair of quantum states $\rho$ and $\sigma$,

$$2(1 - F(\rho, \sigma)) \leq \|\rho - \sigma\|_1 \leq 2\sqrt{1 - F(\rho, \sigma)^2}.$$  

Consequently,

$$2B(\rho, \sigma)^2 \leq \|\rho - \sigma\|_1 \leq 2\sqrt{2} \cdot B(\rho, \sigma).$$

For two pure states $|\psi\rangle$ and $|\phi\rangle$, we have

$$\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_1 = \sqrt{1 - F(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|)^2} = \sqrt{1 - |\langle\psi|\phi\rangle|^2}.$$  

Fact 7 ([Tom16]). The square of the fidelity is jointly concave in both arguments, i.e.,

$$F(\epsilon \rho + (1 - \epsilon) \rho', \epsilon \sigma + (1 - \epsilon) \sigma')^2 \geq \epsilon F(\rho, \sigma)^2 + (1 - \epsilon) F(\rho', \sigma')^2.$$  

Fact 8 (Data-processing inequality). For a quantum channel $O$ and states $\rho$ and $\sigma$,

$$\|O(\rho) - O(\sigma)\|_1 \leq \|\rho - \sigma\|_1 \quad \text{and} \quad F(O(\rho), O(\sigma)) \geq F(\rho, \sigma).$$

The entropy of a quantum state $\rho$ on a register $Z$ is given by

$$H(\rho) = -\text{Tr}(\rho \log \rho).$$

We shall also denote this by $H(Z)_\rho$. For a state $\rho_{YZ}$ on registers $YZ$, the entropy of $Y$ conditioned on $Z$ is given by

$$H(Y|Z)_\rho = H(YZ)_\rho - H(Z)_\rho$$

where $H(Z)_\rho$ is calculated w.r.t. the reduced state $\rho_Z$. The relative entropy between two states $\rho$ and $\sigma$ of the same dimensions is given by

$$D(\rho||\sigma) = \text{Tr}(\rho \log \rho) - \text{Tr}(\rho \log \sigma).$$

The relative min-entropy between $\rho$ and $\sigma$ is defined as

$$D_\infty(\rho||\sigma) = \min \{ \lambda : \rho \leq 2^\lambda \sigma \}.$$  

It is easy to see that for all $\rho$ and $\sigma$,

$$0 \leq D(\rho||\sigma) \leq D_\infty(\rho||\sigma).$$

Fact 9 (Pinsker’s inequality). For any two states $\rho$ and $\sigma$,

$$\|\rho - \sigma\|_1^2 \leq 2 \ln 2 \cdot D(\rho||\sigma) \quad \text{and} \quad B(\rho, \sigma)^2 \leq \ln 2 \cdot D(\rho||\sigma).$$
Fact 10. For any unitary $U$, and states $\rho, \sigma$, $D(U\rho U^\dagger \| U\sigma U^\dagger) = D(\rho \| \sigma)$, and $D_\infty(U\rho U^\dagger \| U\sigma U^\dagger) = D_\infty(\rho \| \sigma)$.

Fact 11. If $\sigma = \epsilon\rho + (1 - \epsilon)\rho'$, then $D_\infty(\rho \| \sigma) \leq \log(1/\epsilon)$.

Fact 12. For any three quantum states $\rho, \sigma, \varphi$ such that $\text{supp}(\rho) \subseteq \text{supp}(\varphi) \subseteq \text{supp}(\sigma)$,

$$D_\infty(\rho \| \sigma) \leq D_\infty(\rho \| \varphi) + D_\infty(\varphi \| \sigma).$$

The conditional min-entropy of $Y$ given $Z$ is defined as

$$H_\infty(Y|Z)_\rho = \inf\{\lambda : \exists \sigma_Z \text{ s.t. } \rho_{YZ} \leq 2^{-\lambda} \mathbb{1}_Y \otimes \sigma_Z\}.$$

The conditional Hartley entropy of $Y$ given $Z$ is defined as

$$H_0(Y|Z)_\rho = \log \left( \sup_{\sigma_Z} \text{Tr}(\text{supp}(\rho_{YZ})(\mathbb{1}_Y \otimes \sigma_Z)) \right)$$

where $\text{supp}(\rho_{YZ})$ is the projector on to the support of $\rho_{YZ}$. For a classical distribution $P_{YZ}$, this reduces to

$$H_0(Y|Z)_{P_{YZ}} = \log \left( \sup_z |\{y : P_{YZ}(y, z) > 0\}| \right).$$

Fact 13. If $\rho_{YZ} = (\mathbb{1}_Y \otimes U)\sigma_{YZ}(\mathbb{1}_Y \otimes U)$, then

$$H_\infty(Y|Z)_\sigma = H_\infty(Y|Z)_\rho \quad \text{and} \quad H_0(Y|Z)_\sigma = H_0(Y|Z)_\rho.$$

For any distance measure (not necessarily a metric) $d$ between states, the $\epsilon$-smoothed relative min-entropy between $\rho$ and $\sigma$ w.r.t. $d$ is defined as

$$D_\infty^{d, \epsilon}(\rho \| \sigma) = \inf_{\rho' : d(\rho, \rho') \leq \epsilon} D_\infty(\rho' \| \sigma).$$

When $d$ is the $\ell_1$ distance, we often omit the superscript.

Fact 14 (Quantum Substate Theorem, [JRS02, JRS09, JN12]). For any two states $\rho$ and $\sigma$ such that the support of $\rho$ is contained in the support of $\sigma$, and any $\epsilon > 0$,

$$D_\infty^{F, \epsilon}(\rho \| \sigma) \leq \frac{D(\rho \| \sigma) + 1}{\epsilon} + \log \left( \frac{1}{1 - \epsilon} \right).$$

Consequently,

$$D_\epsilon(\rho \| \sigma) \leq \frac{4D(\rho \| \sigma) + 1}{\epsilon^2} + \log \left( \frac{1}{1 - \epsilon^2/4} \right).$$

Fact 15 ([JRS02]). For two states $\rho_X$ and $\sigma_X$, if $D_\infty^{\Delta}(\rho_X \| \sigma_X) = c$, then for any purifications $|\rho\rangle_{XY}$ and $|\sigma\rangle_{XY}$, there exists a measurement operator $M$ taking $Y'$ to $Y$, such that $\mathbb{1} \otimes M$ succeeds on $|\sigma\rangle_{XY}$, with probability $2^{-c}$, and

$$\Delta \left( 2^c (\mathbb{1} \otimes M) |\sigma\rangle\langle\sigma|_{XY'} (\mathbb{1} \otimes M^\dagger), |\rho\rangle\langle\rho|_{XY} \right) \leq \epsilon.$$

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3Since $1 - F$ is the distance measure rather than $F$ itself, the closeness condition for $D_\infty^{F, \epsilon}(\rho \| \sigma)$ is $F(\rho, \rho') \geq 1 - \epsilon$. 

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Fact 16. For any quantum state \( \rho_{YZ} \),
\[
\inf_{\sigma_Z} D_{\infty}(\rho_{YZ} \| \rho_Y \otimes \sigma_Z) \leq 2 \min \{ \log |Y|, \log |Z| \}.
\]

The \( \epsilon \)-smoothed versions of the conditional entropies are defined as
\[
H^\epsilon_{\infty}(Y|Z)_\rho = \sup_{\rho': \| \rho - \rho' \|_1 \leq \epsilon} H_{\infty}(Y|Z)_{\rho'} \quad \text{and} \quad H^\epsilon_{0}(Y|Z)_{\rho_{YZ}} = \inf_{\rho': \| \rho - \rho' \|_1 \leq \epsilon} H_{0}(Y|Z)_{\rho'}.
\]

Fact 17. For any state \( \rho_{XYZ} \),
\[
H^\epsilon_{\infty}(Y|Z)_\rho \geq H^\epsilon_{\infty}(Y|XZ)_\rho \geq H^\epsilon_{\infty}(YX|Z)_\rho - \log |X|.
\]

The mutual information between \( Y \) and \( Z \) with respect to a state \( \rho \) on \( YZ \) can be defined in the following equivalent ways:
\[
I(Y : Z)_\rho = D(\rho_{YZ} \| \rho_Y \otimes \rho_Z) = H(Y)_{\rho} - H(Y|Z)_{\rho} = H(Z)_{\rho} - H(Z|Y)_{\rho}.
\]

The conditional mutual information between \( Y \) and \( Z \) conditioned on \( X \) is defined as
\[
I(Y : Z|X)_\rho = H(Y|X)_{\rho} - H(Y|XZ)_{\rho} = H(Z|X)_{\rho} - H(Z|XY)_{\rho}.
\]

Mutual information can be seen to satisfy the chain rule
\[
I(XY : Z)_\rho = I(X : Z)_\rho + I(Y : Z|X)_\rho.
\]

Fact 18 (Quantum Gibbs’ inequality, see e.g. [BVY17]). For any three states \( \rho_{XY}, \sigma_X, \varphi_Y \),
\[
D(\rho_{XY} || \sigma_X \otimes \varphi_Y) \geq D(\rho_{XY} || \sigma_X \otimes \rho_Y) \geq I(X : Y)_{\rho}.
\]

A state of the form
\[
\rho_{XY} = \sum_x P_X(x) |x\rangle \langle x|_X \otimes \rho_{Y|x}
\]
is called a CQ (classical-quantum) state, with \( X \) being the classical register and \( Y \) being quantum. We shall use \( X \) to refer to both the classical register and the classical random variable with the associated distribution. As in the classical case, here we are using \( \rho_{Y|x} \) to denote the state of the register \( Y \) conditioned on \( X = x \), or in other words the state of the register \( Y \) when a measurement is done on the \( X \) register and the outcome is \( x \). Hence \( \rho_{XY|x} = |x\rangle \langle x|_X \otimes \rho_{Y|x} \). When the registers are clear from context we shall often write simply \( \rho_x \).

For CQ states where \( X \) is the classical register, relative entropy has the chain rule
\[
D(\rho_{XY} || \sigma_{XY}) = D(\rho_X || \sigma_X) + \mathbb{E}_{\rho_X} D(\rho_{Y|x} || \sigma_{Y|x}).
\]

Using this, the following fact follows by expanding out the relative entropies.

Fact 19. For CQ states \( \rho_{XY} \) and \( \sigma_{XY} \),
\[
\mathbb{E}_{\rho_X} D(\rho_{Y|x} || \sigma_Y) - D(\rho_Y || \sigma_Y) = \mathbb{E}_{\rho_X} D(\rho_{Y|x} || \rho_Y) \geq 0.
\]

Fact 20 ([KRS09]). For a CQ state \( \rho_{XY} \) where \( X \) is the classical register, \( H_{\infty}(X|Y)_{\rho} \) is equal to the negative logarithm of the maximum probability of guessing \( X \) from the quantum system \( \rho_{Y|x} \), i.e.,
\[
H_{\infty}(X|Y)_{\rho} = - \log \left( \sup_{\{M_x\}_x} \sum_x P_X(x) \text{Tr}(M_x \rho_{Y|x}) \right)
\]
where the maximization is over the set of POVMs with elements indexed by \( x \).
3.3 Quantum communication & non-local games

An interactive entanglement-assisted quantum communication protocol \( \mathcal{P} \) between \( l \) parties goes as follows: before the start of the protocol, the \( l \) parties share a joint entangled state, and at the start parties 1 through \( l \) receive inputs \( x_1^l, \ldots, x_l^l \) respectively from \( \mathcal{X}^1 \times \cdots \times \mathcal{X}^l \). We assume that only the \( j \)-th party communicates in rounds \( \{j, j + l, j + 2l, \ldots\} \), and sends messages to all the other parties. For \( i \in \{j, j + l, \ldots\} \), in the \( i \)-th round the \( j \)-th party has a memory register \( E_i \) from the previous round in which they communicated (when \( i = j \), this is just the \( j \)-th party’s part of the initial shared entangled state), as well as message registers \( M_{i-1}^j, \ldots, M_{i+1}^j \) that they have received from all the other parties in the \( (i - l + 1) \)-th to \( (i - 1) \)-th rounds. The \( j \)-th party applies a unitary depending on their input \( x_j^l \) on all these registers, to generate a register \( E_i \) that they keep as memory, and a message \( M_i^j = M_i^j \cdots M_{i+1}^j \cdot M_{i-1}^j \cdots M_i^j \), where \( M_i^j \) is sent to the \( j' \)-th party in this round. After all the communication rounds are done, the \( j \)-th party applies a final unitary on the memory and message registers they currently have, and then measures in the computational basis to produce their answer \( a^j \in \mathcal{A}^j \). We shall denote the outputs of \( \mathcal{P} \) on inputs \( x_1^l \ldots x_l^l \) to \( \mathcal{P} \) by \( \mathcal{P}(x_1^l \ldots x_l^l) \) — this is a random variable, as \( \mathcal{P} \)'s outputs are not necessarily deterministic.

The following lemma about the final state of a quantum communication protocol is proved in Appendix B.

**Lemma 5.** Let \( |\sigma\rangle_{\mathcal{A}^1 \ldots \mathcal{A}^l|x_1^l \ldots x_l^l} \) be the pure state shared by the \( l \) parties at the end of a quantum communication protocol, on inputs \( x_1^l, \ldots, x_l^l \), with party \( j \) holding register \( \mathcal{A}^j \). For any product input distribution \( \mathcal{P}_{x_1^l \ldots x_l^l} \) on \( \mathcal{X}^1 \times \cdots \times \mathcal{X}^l \), define

\[
|\sigma\rangle_{x_1^l \bar{x}_1^j, \ldots, x_l^l \bar{x}_l^j; A^1 \ldots A^l} = \sum_{x_1^l \ldots x_l^l} \sqrt{\mathcal{P}_{x_1^l \ldots x_l^l}(x_1^l \ldots x_l^l)} |x_1^l x_1^l \ldots x_l^l x_l^l\rangle_{x_1^l \bar{x}_1^j, \ldots, x_l^l \bar{x}_l^j} \sigma_{A^1 \ldots A^l|x_1^l \ldots x_l^l}.
\]

If \( c^j \) is the total communication from the \( j \)-th party in the protocol, then there for all \( j \in [l] \), there exists a state \( \rho_{x_1^l \bar{x}_1^j; A^1} \) such that

\[
D_\infty (\sigma_{x_1^l; x_1^j; A^1 \cdots A^j} | \sigma_{x_1^l} \otimes \rho_{x_1^l; A^1 \cdots A^j}) \leq 2c^j
\]

where \( X^{-j} \) denotes \( X^1 \ldots X^{i-1} X^{i+1} \ldots X^l \), and \( \bar{x}_j \) and \( \bar{A} \) are defined analogously.

**Definition 1.** For a predicate \( \mathcal{V} \) on \( (\mathcal{A}^1 \times \cdots \times \mathcal{A}^l) \times (\mathcal{X}^1 \times \cdots \times \mathcal{X}^l) \), its entanglement-assisted \( l \)-party quantum communication complexity with error \( 0 < \varepsilon < 1 \), denoted by \( \mathcal{Q}_\varepsilon(\mathcal{V}) \), is the minimum total communication in an interactive entanglement-assisted quantum protocol such that for all \( x_1^l \ldots x_l^l \in X^1 \times \cdots \times X^l \),

\[
\Pr \left[ \mathcal{V} (\mathcal{P}(x_1^l \ldots x_l^l), x_1^l \ldots x_l^l) = 1 \right] \geq 1 - \varepsilon.
\]

**Definition 2.** For a predicate \( \mathcal{V} \) on \( (\mathcal{A}^1 \times \cdots \times \mathcal{A}^l) \times (\mathcal{X}^1 \times \cdots \times \mathcal{X}^l) \) and a distribution \( p \) on \( \mathcal{X}^1 \times \cdots \times \mathcal{X}^l \), the distributional entanglement-assisted \( l \)-party quantum communication complexity of \( \mathcal{V} \) with error \( 0 < \varepsilon < 1 \) w.r.t. distribution \( p \), denoted by \( \mathcal{Q}_\varepsilon(\mathcal{V}, p) \), is the minimum total communication in an interactive entanglement-assisted quantum protocol such that,

\[
\Pr \left[ \mathcal{V} (\mathcal{P}(x_1^l \ldots x_l^l), x_1^l \ldots x_l^l) = 1 \right] \geq 1 - \varepsilon
\]

where the probability is taken over the distribution \( p \) for \( x_1^l \ldots x_l^l \), as well as the internal randomness of \( \mathcal{P} \).

**Fact 21** (Yao’s Lemma, [Yao77]). For any \( 0 < \varepsilon < 1 \), and any predicate \( \mathcal{V} \), \( \mathcal{Q}_\varepsilon(\mathcal{V}) = \max_p \mathcal{Q}_\varepsilon(\mathcal{V}, p) \).

\footnote{Note that this statement of Yao’s lemma is not contradicted by the results in [dGdW02], which are about exact quantum communication protocols. For completeness, a proof of this lemma is provided in Appendix A.}
An $l$-player non-local game $G$ is described as $(p, \mathcal{X}^1 \times \ldots \times \mathcal{X}^l, \mathcal{A}^1 \times \ldots \times \mathcal{A}^l, V)$ where $p$ is a distribution over the input set $\mathcal{X}^1 \times \ldots \times \mathcal{X}^l$, $\mathcal{A}^1 \times \ldots \times \mathcal{A}^l$ is the output set, and $V$ is a predicate on the outputs and inputs. In an entangled strategy for a non-local game, the players are allowed to share an $l$-partite entangled state. Player $j$ gets input $x^j$ and performs a unitary and a measurement depending on their input on their part of the entangled state, to give their output $a^j$. The value achieved by a strategy on $G$ is the probability over $p$ and the internal randomness of the strategy that $V(a^1 \ldots a^l, x^1 \ldots x^l) = 1$.

**Definition 3.** The entangled value of a game $G = (p, \mathcal{X}^1 \times \ldots \times \mathcal{X}^l, \mathcal{A}^1 \times \ldots \times \mathcal{A}^l, V)$, denoted by $\omega^*(G)$, is the maximum value achieved by any strategy for $G$.

### 4 Quantum partition bound

For sets $\mathcal{X}^1, \ldots, \mathcal{X}^l$ and $\mathcal{A}^1, \ldots, \mathcal{A}^l$, let $Q(\mathcal{A}^1 \times \ldots \times \mathcal{A}^l, \mathcal{X}^1 \times \ldots \times \mathcal{X}^l)$ denote the set of conditional probability distributions $q(a^1 \ldots a^l | x^1 \ldots x^l)$ that can be obtained by $l$ parties who share an $l$-partite entangled state, receive inputs $x^j \in \mathcal{X}^j$ respectively, and perform measurements on their parts of the entangled state to obtain outputs $a^j$, without communicating. That is, $Q(\mathcal{A}^1 \times \ldots \times \mathcal{A}^l, \mathcal{X}^1 \times \ldots \times \mathcal{X}^l)$ is the following set:

$$\left\{ \left( \langle \psi | M_{a^1|x^1} \otimes \cdots \otimes M_{a^l|x^l} | \psi \rangle \right)_{a^1 \ldots a^l, x^1 \ldots x^l} \bigg| \text{is a state, } \forall a^j, x^j, \sum_{a^j \in \mathcal{A}^j} M_{a^j|x^j} = 1, M_{a^j|x^j} \geq 0 \right\}.$$ 

We state definitions for three variants of the quantum partition bound, the first of which is non-distributional and was given by [LLR12]. The second two are distributional modifications which we shall use.

**Definition 4.** For a predicate $\forall$ on $(\mathcal{A}^1 \times \ldots \times \mathcal{A}^l) \times (\mathcal{X}^1 \times \ldots \times \mathcal{X}^l)$, and $0 < \varepsilon < 1$, let $\bot$ be a special symbol not in any $\mathcal{A}^j$. The quantum partition bound for $\forall$ with $\varepsilon$ error, denoted by $\text{eff}_\varepsilon(\forall)$, is defined as the optimal value of the following optimization problem:

$$\min \frac{1}{\eta} \sum_{a^1 \ldots a^l : \forall(a^1 \ldots a^l, x^1 \ldots x^l) = 1} q(a^1 \ldots a^l | x^1 \ldots x^l) \geq (1 - \varepsilon) \eta \quad \forall x^1 \ldots x^l \in \mathcal{X}^1 \times \ldots \times \mathcal{X}^l$$

$$\sum_{a^1 \ldots a^l \in \mathcal{A}^1 \times \ldots \times \mathcal{A}^l} q(a^1 \ldots a^l | x^1 \ldots x^l) = \eta \quad \forall x^1 \ldots x^l \in \mathcal{X}^1 \times \ldots \times \mathcal{X}^l$$

$$q(a^1 \ldots a^l | x^1 \ldots x^l) \in Q((\mathcal{A}^1 \cup \{\bot\}) \times \ldots \times (\mathcal{A}^l \cup \{\bot\}), \mathcal{X}^1 \times \ldots \times \mathcal{X}^l).$$

**Definition 5.** For a predicate $\forall$ on $(\mathcal{A}^1 \times \ldots \times \mathcal{A}^l) \times (\mathcal{X}^1 \times \ldots \times \mathcal{X}^l)$, a distribution $p(x^1 \ldots x^l)$ on $\mathcal{X}^1 \times \ldots \times \mathcal{X}^l$, and $0 < \varepsilon < 1$, let $\bot$ be a special symbol not in any $\mathcal{A}^j$. The quantum partition bound for $\forall$ with $\varepsilon$ error with respect to $p$, denoted by $\text{eff}_\varepsilon^p(\forall, p)$, is defined as the optimal value of the following optimization problem:

$$\min \frac{1}{\eta} \sum_{x^1 \ldots x^l \in \mathcal{X}^1 \times \ldots \times \mathcal{X}^l} p(x^1 \ldots x^l) \sum_{a^1 \ldots a^l : \forall(a^1 \ldots a^l, x^1 \ldots x^l) = 1} q(a^1 \ldots a^l | x^1 \ldots x^l) \geq (1 - \varepsilon) \eta$$
\[
\sum_{a^1 \ldots a^l \in \mathcal{A}^1 \times \ldots \times \mathcal{A}^l} q(a^1 \ldots a^l | x^1 \ldots x^l) = \eta \quad \forall x^1 \ldots x^l \in \mathcal{X}^1 \times \ldots \times \mathcal{X}^l
\]

\[
q(a^1 \ldots a^l | x^1 \ldots x^l) \in \mathcal{Q} \left( (\mathcal{A}^l \cup \{\bot\}) \times \ldots \times (\mathcal{A}^l \cup \{\bot\}), \mathcal{X}^1 \times \ldots \times \mathcal{X}^l \right).
\]

**Definition 6.** For a predicate \( V \) on \( (\mathcal{A}^1 \times \ldots \times \mathcal{A}^l) \times (\mathcal{X}^1 \times \ldots \times \mathcal{X}^l) \), a distribution \( p(x^1 \ldots x^l) \) on \( \mathcal{X}^1 \times \ldots \times \mathcal{X}^l \), and \( 0 < \epsilon < 1 \), let \( \bot \) be a special symbol not in any \( \mathcal{A}^j \). The average quantum partition bound for \( V \) with \( \epsilon \) error with respect to \( p \), denoted by \( \text{eff}_\epsilon^* (V, p) \), is defined as the optimal value of the following optimization problem:

\[
\min \frac{1}{\eta} \quad \text{subject to } \\
\sum_{x^1 \ldots x^l \in \mathcal{X}^1 \times \ldots \times \mathcal{X}^l} p(x^1 \ldots x^l) \sum_{a^1 \ldots a^l : V(a^1 \ldots a^l, x^1 \ldots x^l) = 1} q(a^1 \ldots a^l | x^1 \ldots x^l) \geq (1 - \epsilon) \eta \\
\sum_{x^1 \ldots x^l \in \mathcal{X}^1 \times \ldots \times \mathcal{X}^l} p(x^1 \ldots x^l) \sum_{a^1 \ldots a^l \in \mathcal{A}^1 \times \ldots \times \mathcal{A}^l} q(a^1 \ldots a^l | x^1 \ldots x^l) = \eta \\
q(a^1 \ldots a^l | x^1 \ldots x^l) \in \mathcal{Q} \left( (\mathcal{A}^l \cup \{\bot\}) \times \ldots \times (\mathcal{A}^l \cup \{\bot\}), \mathcal{X}^1 \times \ldots \times \mathcal{X}^l \right).
\]

Operationally, \( \text{eff}_\epsilon^* (V) \), \( \text{eff}_\epsilon^\ast (V, p) \) and \( \text{eff}_\epsilon^\ast (V, p) \) are connected to zero-communication protocol (with aborts) to compute \( V \). A zero-communication protocol is one which any player is allowed to abort (indicated by them outputting the \( \bot \) symbol), but if nobody aborts they need to compute \( V \) correctly. A zero-communication protocol for \( V \) is basically a strategy for a non-local game version of \( V \), with the output alphabet extended to \( \mathcal{A}^1 \cup \{\bot\} \times \ldots \times (\mathcal{A}^l \cup \{\bot\}) \). Now we can have different conditions on the abort and success probability conditioned on not aborting for such protocols.

- Suppose the protocol is required to not abort on every input \( x^1 \ldots x^l \) with the same probability \( \eta \), and conditioned on not aborting, every input is required to compute \( V \) correctly with probability \( (1 - \epsilon) \). \( \text{eff}_\epsilon^* (V) \) corresponds to the efficiency, i.e., the inverse of the maximum probability of not aborting, in such a protocol.

- Suppose the protocol is required to not abort with the same probability \( \eta \) on every input \( x^1 \ldots x^l \), but conditioned on not aborting, the probability of computing \( V \) correctly, averaged over the inputs from \( p \), is at least \( (1 - \epsilon) \). \( \text{eff}_\epsilon^* (V) \) is the inverse of the maximum probability of not aborting in such a protocol.

- Suppose the protocol aborts on input \( x^1 \ldots x^l \) with probability \( \eta_{x^1 \ldots x^l} \), and we require that the average over \( x^1 \ldots x^l \) from \( p \) is \( \eta \). Moreover, we require that the average probability of computing \( V \) correctly is at least \( (1 - \epsilon) \eta \), i.e., the average probability of correctness conditioned on not aborting is at least \( (1 - \epsilon) \). \( \text{eff}_\epsilon^* (V, p) \) is the inverse of the maximum probability of not aborting in such a protocol.

Because the requirements from the protocols are successively relaxed, it is easy to see that for any \( p \),

\[
\text{eff}_\epsilon^* (V) \geq \text{eff}_\epsilon^\ast (V, p) \geq \text{eff}_\epsilon^\ast (V, p).
\]

The following lemma shows that \( \text{eff}_\epsilon^\ast (V, p) \), and hence \( \text{eff}_\epsilon^* (V, p) \) lower bounds communication. The proof of this is a slight modification the proof in [LLR12] which lower bounded \( Q_\epsilon(f) \) by \( \text{eff}_\epsilon^* (V) \). We provide the proof in Appendix C for completeness.
Lemma 6. For any predicate \( V \) on \((A^1 \times \ldots \times A^l) \times (X^1 \times \ldots \times X^l)\), any distribution \( p \) on \(X^1 \times \ldots \times X^l\) and error \( \varepsilon \),

\[
Q_\varepsilon(V, p) \geq \frac{1}{2} \log \tilde{\text{eff}}^*_\varepsilon(V, p).
\]

Yao’s Lemma and Lemma 6 imply that for any \( p \), \( \log \tilde{\text{eff}}^*_\varepsilon(V, p) \) and therefore \( \log \text{eff}^*_\varepsilon(V, p) \) are lower bounds on \( Q_\varepsilon(V) \).

4.1 Relationship between \( \text{eff}^*_\varepsilon \) and the generalized discrepancy method

In this section we shall prove Theorem 3, recalled below.

Theorem 3. For a total function \( f : \mathcal{X} \times \mathcal{Y} \to \{-1, +1\} \), let \( V_f \) denote the predicate on \((\{-1, +1\})^2 \times (\mathcal{X} \times \mathcal{Y})\) given by

\[
V(ab, xy) = 1 \iff a \cdot b = f(x, y).
\]

Then for any distribution \( p \) on \( \mathcal{X} \times \mathcal{Y} \),

\[
\text{eff}^*_\varepsilon(V_f, p) \geq (1 - 2\varepsilon)\gamma_2^*(F, p)
\]

with \( \alpha = \frac{1 + 2\varepsilon}{1 - 2\varepsilon} \).

We shall not define \( \gamma_2^\alpha \) and its dual norm \( \gamma_2^\alpha \) for general matrices. Instead, we shall use an exact characterization of \( \gamma_2^\alpha(F) \) for a boolean \( f \) in terms of non-local games given by Tsirelson, and then use a duality relation to express \( \gamma_2^\alpha \) in terms of \( \gamma_2^\alpha \).

Fact 22 ([Tsi87]). For total \( f : \mathcal{X} \times \mathcal{Y} \to \{-1, +1\} \), let \( V_f \) denote its corresponding predicate as given in the statement of Theorem 3, and let \( p \) be any distribution on \( \mathcal{X} \times \mathcal{Y} \). Then,

\[
\omega^*(G(p, V_f)) = \frac{1}{2}(1 + \gamma_2^*(F \circ p)),
\]

where \( F \circ p \) denotes the entry-wise product of \( F \) and \( p \).

Fact 23 (see e.g. - Theorem 64 in [LS09]). For any matrix \( A \), \( \alpha \geq 1 \), \( \gamma_2^\alpha(A) \) and \( \gamma_2^\alpha(A) \) are related as

\[
\gamma_2^\alpha(A) = \max_M \frac{(\alpha + 1) \langle A, M \rangle - (\alpha - 1)\|M\|_1}{2\gamma_2^\alpha(M)}.
\]

When \( A \) is the matrix corresponding to a boolean function \( f \), this can also be expressed as

\[
\gamma_2^\alpha(F) = \max_{F', p} \frac{(\alpha + 1) \langle F, F' \circ p \rangle - (\alpha - 1)}{2\gamma_2^\alpha(F' \circ p)}
\]

where the maximization is taken over matrices \( F' \) with \( \pm 1 \) entries, and distributions \( p \).

Using this characterization, we give the following definition of \( \gamma_2^\alpha(F, p) \).

Definition 7. For matrix \( F \) with \( \pm 1 \) entries, \( \gamma_2^\alpha(F, p) \) with respect to distribution \( p \) is defined as

\[
\gamma_2^\alpha(F, p) = \max_{F'} \frac{(\alpha + 1) \langle F, F' \circ p \rangle - (\alpha - 1)}{2\gamma_2^\alpha(F' \circ p)}.
\]
Proof of Theorem 3. Our proof closely follows the lower bound for $Q_2(f)$ in terms of $\log \gamma_2^a$ as described in Section 5.3.2 of [LS09], which is credited to Harry Buhrman.

Suppose $\text{eff}^*_\epsilon(V_f, p) = \frac{1}{\eta}$ for some $\eta$. Let $\mathcal{P}$ be a zero-communication protocol for $V_f$ with constraints as required in the definition of $\text{eff}^*_\epsilon(V_f, p)$. Let $\eta_{xy}$ denote the probability of the protocol aborts on input $(x, y)$. Let $O(x, y)$ denote the average (over internal randomness) output given by $\mathcal{P}$ conditioned on not aborting on inputs $x, y$. Here we are calling $a \cdot b$ the output of the protocol, if Alice outputs $a$ and Bob outputs $b$, which means $O(x, y)$ is some number in $[-1, 1]$. Note that $O(x, y)$ is defined conditioned on not aborting, so it is in fact normalized by the quantity $\eta$. From the definition of $\text{eff}^*_\epsilon(V_f, p)$, the following condition holds

$$\sum_{x,y} p(x,y) f(x,y) O(x,y) \geq 1 - 2\epsilon.$$

The above expression is actually the difference between the probability of computing $f$ correctly and the probability of computing it incorrectly, which is why we get $1 - 2\epsilon$.

Now let $f' : \mathcal{X} \times \mathcal{Y} \to \{-1, +1\}$ be an arbitrary boolean function, and define $V_{f'}$ the same way as $V_f$. We shall give a strategy $\mathcal{S}$ for the game $G(p, V_{f'})$ using the zero-communication protocol $\mathcal{P}$. $\mathcal{S}$ works as follows:

- On inputs $x, y$ for $G(p, V_{f'})$, Alice and Bob run the protocol $\mathcal{P}$ on $x, y$.
- If $\mathcal{P}$ gives output $\perp$ for either player, they output $\pm 1$ uniformly at random.
- If $\mathcal{P}$ does not abort, then Alice and Bob both output according to $\mathcal{P}$.

Note that conditioned on $\mathcal{P}$ not aborting, the average output produced by Alice and Bob on inputs $x, y$ is also $O(x, y)$. Strategy $\mathcal{S}$ thus wins with probability $\frac{1}{2}(1 + \delta)$ ($\delta$ may be negative), where

$$\delta = \eta \sum_{x,y} p(x,y) f'(x,y) O(x,y).$$

$f(x,y), f'(x,y)$ are in $\{-1, +1\}$, and $O(x,y)$ is in $[-1, 1]$. For three numbers $a, \beta \in \{-1, +1\}$, $\theta \in [-1, 1]$, the following condition is true, and can be checked by putting in the four possible values of $(a, \beta)$:

$$\beta \theta \geq a \beta + a \theta - 1.$$

Using the above on $f(x,y), f'(x,y), O(x,y)$ we get,

$$\sum_{x,y} p(x,y) f'(x,y) O(x,y) \geq \sum_{x,y} p(x,y) (f(x,y) f'(x,y) + f(x,y) O(x,y) - 1) \geq \sum_{x,y} p(x,y) f(x,y) f'(x,y) + (1 - 2\epsilon) - 1 = \langle F, F' \circ p \rangle - 2\epsilon.$$

By Fact 22 we have,

$$\gamma_2^2(F' \circ p) \geq \delta \geq \eta(\langle F, F' \circ p \rangle - 2\epsilon)$$

which gives us

$$\frac{1}{\eta} \geq \max_F \frac{\langle F, F' \circ p \rangle - 2\epsilon}{\gamma_2^2(F' \circ p)} = (1 - 2\epsilon) \gamma_2^a(F, p)$$

with $\alpha = \frac{1 + 2\epsilon}{1 - 2\epsilon}$. \qed
5 Substate Perturbation Lemma

To prove the Substate Perturbation Lemma, we use the following result due to [ABJT20]. This result is stated in terms of \( l_{\max} \) for general states in [ABJT20], where some of the states involved are optimized over. Moreover, the distance between \( \sigma'_{XB} \) and \( \sigma_B \) is considered, rather than just the distance between \( \sigma'_B \) and \( \sigma_B \). However, for the purposes of the proof the optimal state does not matter, and only the distance between \( \sigma'_B \) and \( \sigma_B \) is relevant; so we state in the form below. Our proof of the Substate Perturbation Lemma is also heavily inspired by their proof of this result.

Fact 24 ([ABJT20], Theorem 2). Suppose there are states \( \sigma'_{XB}, \sigma_B \) and \( \psi_X \) satisfying \( \Delta(\sigma_B, \sigma'_B) \leq \varepsilon \) and

\[
\sigma'_{XB} \leq 2^c (\psi_X \otimes \sigma_B).
\]

Then for any \( \delta > 0 \), there exists a state \( \sigma''_{XB} \) satisfying \( \Delta(\sigma'_{XB}, \sigma''_{XB}) \leq \varepsilon + \delta, \sigma''_B = \sigma_B, \) and

\[
\sigma''_{XB} \leq 2^c \left( 1 + \frac{8}{\delta} \right) \psi_X \otimes \sigma_B.
\]

Lemma 7 (Substate Perturbation Lemma). Suppose there are states \( \sigma'_{XB}, \sigma_B \) and \( \psi_X \) satisfying \( \Delta(\sigma_B, \sigma'_B) \leq \varepsilon \),

\[
\sigma'_{XB} \leq 2^c (\psi_X \otimes \sigma_B)
\]

and a state \( \rho_B \) satisfying \( \Delta(\sigma_B, \rho_B) \leq \delta_1 \). Then for any \( \delta_0 > 0 \), there exists state \( \rho'_{XB} \) satisfying \( \Delta(\rho'_{XB}, \sigma'_{XB}) \leq \varepsilon + \delta_0 + \delta_1 \), and

\[
\rho'_{XB} \leq 2^{c+1} \left( 1 + \frac{4}{\delta_0} \right) \psi_X \otimes \rho_B.
\]

Proof. First we use Fact 24 to get a state \( \sigma''_{XB} \) satisfying

\[
\sigma''_{XB} \leq 2^c \left( 1 + \frac{8}{\delta_0} \right) \psi_X \otimes \sigma_B
\]

such that \( \Delta(\sigma''_{XB}, \sigma''_{XB}) \leq \varepsilon + \delta_0 \) and \( \sigma''_B = \sigma_B \).

Let \( U \) be the unitary such that

\[
F(\rho_B, \sigma_B) = \text{Tr} \left( U \rho_B^{1/2} \sigma_B^{1/2} \right).
\]

Define

\[
\rho'_{XB} = \frac{(1 \otimes \rho_B^{1/2} U \sigma_B^{1/2}) \sigma''_{XB} (1 \otimes \sigma_B^{-1/2} U^\dagger \rho_B^{1/2})}{\psi_{XB}} + \frac{\sigma_X \otimes \rho_B^{1/2} \left( 1 - U \Pi U^\dagger \right) \rho_B^{1/2}}{\psi_{XB}}
\]

where all the inverses are generalized and \( \Pi \) is the projector onto the support of \( \sigma_B \). Note that

\[
(1 \otimes \rho_B^{1/2} U \sigma_B^{1/2}) \sigma''_{XB} (1 \otimes \sigma_B^{-1/2} U^\dagger \rho_B^{1/2}) \leq 2^c \left( 1 + \frac{8}{\delta_0} \right) \psi_X \otimes \rho_B^{1/2} U \sigma_B^{-1/2} \sigma_B \sigma_B^{-1/2} U^\dagger \rho_B^{1/2},
\]

and hence

\[
\rho'_{XB} \leq 2^c \left( 1 + \frac{8}{\delta_0} \right) \psi_X \otimes \rho_B^{1/2} U \Pi U^\dagger \rho_B^{1/2} + \psi_X \otimes \rho_B^{1/2} \left( 1 - U \Pi U^\dagger \right) \rho_B^{1/2}
\]

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\[ \leq 2^{c+1} \left( 1 + \frac{4}{\rho_B} \right) \psi_X \otimes \rho_B. \]

Now we only have to show that \( \Delta(\rho'_{XB}, \sigma'_{XB}) \leq 2\epsilon + \delta_0 + \delta_1 \). In order to do this, we note that
\[ \Delta(\rho'_{XB}, \sigma'_{XB}) \leq \Delta(\rho'_{XB}, \sigma''_{XB}) + \Delta(\sigma''_{XB}, \sigma'_{XB}). \] (1)

Using Fact 7,
\[
F(\rho'_{XB}, \sigma''_{XB})^2 \geq \text{Tr}(\phi_{XB}) \cdot F\left( \frac{\phi_{XB}}{\text{Tr}(\phi_{XB})}, \sigma''_{XB} \right)^2 + \text{Tr}(\psi_{XB}) \cdot F\left( \frac{\psi_{XB}}{\text{Tr}(\psi_{XB})}, \sigma''_{XB} \right)^2 \\
\geq \text{Tr}(\phi_{XB}) \cdot F\left( \frac{\phi_{XB}}{\text{Tr}(\phi_{XB})}, \sigma''_{XB} \right)^2 \\
\geq \text{Tr}(\phi_{XB}) \cdot \left( |\phi\rangle\langle\phi|_{XBC} + |\sigma''\rangle\langle\sigma''|_{XBC} \right)^2
\]

where in the last step \(|\phi\rangle_{XBC}\) and \(|\sigma''\rangle_{XBC}\) are arbitrary purifications of \(\phi_{XB} = \phi_{XB}/\text{Tr}(\phi_{XB})\) and \(\sigma''_{XB}\), and we have used Fact 8 with the tracing out operation. Note that \(\phi_{BC}\) is obtained from \(\sigma''_{BC}\) by doing an operation \(\mathcal{O}_B\) only on \(B\), which is akin to applying a measurement and conditioning on success. In particular this operation preserves purity of states. We let \(|\phi\rangle_{XBC}\) be the state we get by applying \(\mathcal{O}_B\) on \(|\sigma''\rangle_{XBC}\). Now let \(|\sigma_1''\rangle_{B\tilde{B}}\) be the canonical purification of \(\sigma''\) and \(|\phi_1\rangle_{B\tilde{B}}\) be the state we get by applying \(\mathcal{O}_B\) on \(|\sigma_1''\rangle_{B\tilde{B}}\). These are given by
\[
|\sigma_1''\rangle_{B\tilde{B}} = ((\sigma''_B)^{1/2} \otimes 1) \sum_i |i\rangle_B |i\rangle_{\tilde{B}} = (\sigma_B^{1/2} \otimes 1) \sum_i |i\rangle_B |i\rangle_{\tilde{B}} \\
|\phi_1\rangle_{B\tilde{B}} = \frac{\rho_B^{1/2} U \sigma_B^{-1/2} \otimes 1}{\text{Tr}(\phi_{XB})^{1/2}} |\sigma''\rangle_{B\tilde{B}} = \frac{\rho_B^{1/2} U \Pi \otimes 1}{\text{Tr}(\phi_{XB})^{1/2}} \sum_i |i\rangle_B |i\rangle_{\tilde{B}}.
\]

Since \(|\sigma''\rangle_{XBC}\) is also a purification of \(\sigma''\), there exists an isometry \(V\) acting only on \(\tilde{B}\) such that \(1_B \otimes V |\sigma''\rangle_{XBC} = |\sigma_1''\rangle_{B\tilde{B}}\). Hence,
\[
F(\langle\phi|_{XBC}, |\sigma''\rangle_{XBC}) = F(\mathcal{O}_B(|\sigma''\rangle_{XBC}), |\sigma''\rangle_{XBC}) \\
= F \left( 1_B \otimes V (\mathcal{O}_B(|\sigma''\rangle_{XBC})), 1_B \otimes V |\sigma''\rangle_{XBC} \right) \\
= F(\mathcal{O}_B \left( 1_B \otimes V |\sigma''\rangle_{XBC} \right), 1_B \otimes V |\sigma''\rangle_{XBC}) \\
= F(\mathcal{O}_B |\sigma_1''\rangle_{B\tilde{B}}, |\phi_1\rangle_{B\tilde{B}}).
\]

Putting this in (2) gives us
\[
F(\rho'_{XB}, \sigma''_{XB})^2 \geq \sum_i \sum_j \left| \langle ii | (\Pi U \rho_B^{1/2} \otimes 1) (\rho_B^{1/2} \otimes 1) | jj \rangle \right|^2 \\
= \sum_i |\langle i | \Pi U \rho_B^{1/2} \sigma_B^{1/2} | i \rangle|^2 \\
= \text{Tr}(\Pi U \rho_B^{1/2} \sigma_B^{1/2})^2 \\
= \text{Tr}(U \rho_B^{1/2} \sigma_B^{1/2})^2 = F(\rho_B, \sigma_B)^2
\]

where we have used the fact that \(\sigma_B^{1/2} \Pi = \sigma_B^{1/2}\), and the definition of \(U\). Putting this in (1) we get,
\[ \Delta(\rho'_{XB}, \sigma'_{XB}) \leq \Delta(\rho_B, \sigma_B) + \Delta(\sigma''_{XB}, \sigma'_{XB}) \leq \delta_1 + \epsilon + \delta_0. \]
6 Proof of the direct product theorem

In this section, we prove Theorem 1, whose statement is recalled below.

**Theorem 1.** For any $\epsilon, \zeta > 0$, any predicate $V$ on $(A^1 \times \ldots \times A^l) \times (X^1 \times \ldots \times X^l)$ and any product probability distribution $p$ on $X^1 \times \ldots \times X^l$, if $P$ is an interactive entanglement-assisted quantum communication protocol between $l$ parties which has total communication $cn$.

(i) If $c < 1$, then
\[
\text{suc}(p^n, V^n, P) \leq \left(1 - \frac{\nu}{2} + \sqrt{2l\epsilon}\right)^{\Omega((\log(|A^1|\ldots\cdot|A^l|) / \log(n))}
\]
where $\nu = 1 - \omega^*(G(p, V))$.

(ii) If $1 \leq c = \delta \cdot \frac{\epsilon^2 \cdot \text{eff}_{\epsilon+\zeta}^* (V, p)}{\epsilon}$ for some small enough $\delta$, then
\[
\text{suc}(p^n, V^n, P) \leq (1 - \epsilon)^{\Omega((\log(|A^1|\ldots\cdot|A^l|) / \log(n))}
\]

6.1 Setup

We consider an interactive quantum protocol $P$ for $n$ copies of $V$ with player $j$ having input registers $X^j = X^j_1 \ldots X^j_{\ell^j}$, and communicating $c\ell^j n$ bits. The total communication of the protocol is $cn$, where $c = \sum_{j=1}^{\ell} c^j$. In the case $c \geq 1$, we shall also assume each $c^j \geq 1$; if some $c^j$ is smaller than 1, we can pad extra bits to it, and this increases total communication by a factor of at most $\ell$. Hence we have, $\sum_{j=1}^{\ell} c^j \leq c\ell^j$.

We define the following pure state
\[
|\psi\rangle_{X^1X^2\ldots X^lX^\ell_1E^1\ldots E^\ell A^n} = \sum_{xy} \sqrt{P_{X^1X^2\ldots X^l}(x^1 \ldots x^l)} |x^1x^2\ldots x^l\rangle_{X^1X^2\ldots X^l} |\psi\rangle_{E^1\ldots E^\ell A^n} |x^1\ldots x^l\rangle
\]
where $P_{X^1X^2\ldots X^l}$ is the distribution $p^n$ on $(X^1 \times \ldots \times X^l)^n$, and $|\psi\rangle_{E^1\ldots E^\ell A^n} |x^1\ldots x^l\rangle$ being the state at the end of the protocol on inputs $x^1, \ldots, x^l$. In $|\psi\rangle_{E^1\ldots E^\ell A^n} |x^1\ldots x^l\rangle$, $A^j = A^j_1 \ldots A^j_{\ell^j}$ are the output registers of player $j$, and $E^j$ is some quantum register they have that they don’t measure. We use $P_{X^1X^2\ldots X^lA^n}$ to denote the distribution of $X^1 \ldots X^l A^1 \ldots A^l$ in $|\psi\rangle$. We shall use $X$ to denote $X^1 \ldots X^l$, $X_i$ to denote $X^i_1 \ldots X^i_l$, $X^{-i}$ to denote $X^1 \ldots X^{i-1}X^{i+1} \ldots X^l$, and $X_{\leq i}$ to denote $X^1 \ldots X^i$. Similar notation will be used for $\bar{X}^j, E^j, A^j$. Also for a subset $C \subseteq [n]$, we shall use use $X_C$ to denote $(X_i)_{i \in C}$.

We shall show the following lemma, which can be applied inductively to get Theorem 1.

**Lemma 8.** For $i \in [k]$, let $T_i = V(A^1 \ldots A^i, X^1 \ldots X^i)$ in $P$, and let $E$ denote the event $\prod_{i \in C} T_i = 1$ for some $C \subseteq [n]$ such that $|C| \leq n/2$.

(i) If $c < 1$,
\[
\mathbb{E}_{i \in C} \text{Pr}[T_i = 1|\bar{E}] \leq \omega^*(G(p, V)) + \sqrt{2l\epsilon} + 1\sqrt{2\delta},
\]

(ii) If $1 \leq c = \frac{\epsilon^2 \cdot \text{eff}_{\epsilon+\zeta}^* (V, p)}{\epsilon}$, and if $\delta < 1$, there exists an $i \in \bar{C}$ such that
\[
\text{Pr}[T_i = 1|E] \leq 1 - \epsilon,
\]
where

\[ \delta = \frac{|C| \log(|A^1| \cdot \ldots \cdot |A^l|)}{n} + \log(1/ \text{Pr}[\mathcal{E}]). \]

In order to get the statement of case (i) of Theorem 1 from case (i) of Lemma 8, we start with \( C = \emptyset \), and find some \( i \in [n] \) such that \( \text{Pr}[T_i = 1] \leq 1 - v + \sqrt{2lC + \frac{\gamma}{n}} \). As long as \( l \sqrt{2} \delta \) is at most \( \frac{\gamma}{n} \) we can do this. When we have built up a non-empty set \( C \) this way, if either \( |C| = \Omega \left( \frac{n^{2p}}{\log(|A^1| \cdot \ldots \cdot |A^l|)} \right) \), or \( \text{Pr}[^{\mathcal{E}} \prod_{i \in C} T_i = 1] \leq \exp \left( -\Omega \left( \frac{n^{2p}}{\log(|A^1| \cdot \ldots \cdot |A^l|)} \right) \right) \), we are already done. Otherwise, \( l \sqrt{2} \delta < \frac{\gamma}{n} \) and we can continue the process.

The bound on \( \text{Pr}[T_i = 1|\mathcal{E}] \) in case (ii) of Lemma 8 does not depend on \( \delta \), but it requires \( \delta < 1 \) as a precondition. Hence following the same process there, we can go up to \( C \) of size \( |C| = \Theta \left( \frac{n}{\log(|A^1| \cdot \ldots \cdot |A^l|)} \right) \), or \( \text{Pr}[^{\mathcal{E}} \prod_{i \in C} T_i = 1] = \exp \left( -\Theta \left( \frac{n}{\log(|A^1| \cdot \ldots \cdot |A^l|)} \right) \right) \).

Since in case (i) Lemma 8 gives us a bound on \( \mathbb{E}_{i \in C} \text{Pr}[T_i = 1|\mathcal{E}] \) rather than showing just that there exists an \( i \) for which \( \text{Pr}[T_i = 1|\mathcal{E}] \) is bounded, we can use it to show the following corollary, which we shall later use in our DIQKD application. See Appendix C of [JMS20] for a proof of how this follows from the lemma.

**Corollary 9.** Let \( V_{\text{rand}}^{t/n} \) be the randomized predicate which is satisfied if \( V \) is satisfied on a random subset of size \( t \) of \( [n] \). If the communication cost of \( \mathcal{P} \) is \( cn < n \), then\(^5\)

\[ \text{suc}(p^n, V_{\text{rand}}^{t/n}, \mathcal{P}) \leq \left( \omega^*(G(p, V)) + O \left( \sqrt{lC + l \sqrt{\frac{t \cdot \log(|A^1| \cdot \ldots \cdot |A^l|)}{n}} \right) \right)^t. \]

**6.2 Proof of Lemma 8**

We define the following state which is \( |\tilde{\psi}\rangle \) conditioned on success event \( \mathcal{E} \) in \( C \):

\[ |\tilde{\psi}\rangle_{X\bar{X}\bar{E}A} = \frac{1}{\sqrt{T}} \sum_{x_i c} \sqrt{p_{x}(x_i C)} |x_i C x_C x_C x_C \rangle_{X \bar{X}} \otimes \sum_{a_C \in V | C(a_C, x_C) = 1} \sum_{c} |a_C\rangle_{A_C} \langle \tilde{\psi}\rangle_{E A_c|x_C x_c a_C} \]

where \( |\tilde{\psi}\rangle_{E A_c|x_C x_c a_C} \) is a subnormalized state satisfying \( \| |\tilde{\psi}\rangle_{E A_c|x_C x_c a_C} \|_2^2 = p_{A_C|x_C x_c} (a_C) \), and \( \gamma = \text{Pr}[\mathcal{E}] \).

We shall use the following lemma, whose proof we give later.

**Lemma 10.** Letting \( R = X_C A_C \), the following conditions hold:

(i) If \( c < 1 \), for every \( i \in C \) and \( j \in [l] \), there exist isometries \( U_i^j \) taking registers \( X_C^i \bar{X}_C^i E A_C^i \) to \( \bar{X}_C^i E A_C^i \) such that

\[ \mathbb{E}_{i \in C} \left\| \left( \bigotimes_{j \in [l]} U_i^j \right) \left( |\psi\rangle_{X_C x_i} \otimes |\phi\rangle_{X_C \bar{X}_C E A_C R} \right) \left( \bigotimes_{j \in [l]} (U_i^j)^* \right) - |\phi\rangle_{X_C \bar{X}_C E A_C R} \right\|_1 \leq 2 \sqrt{2lC} + 2l \sqrt{2} \delta; \]

---

\(^5\)Note that \( \text{suc}(p^n, V_{\text{rand}}^{t/n}, \mathcal{P}) \) accounts for the randomness inherent in \( V_{\text{rand}}^{t/n} \) in addition to \( p^n \) and the protocol.
(ii) In case (ii): 1 ≤ \( c \leq \frac{2}{\sqrt{240p} \text{eff} + \zeta(V,p)} \) and \( \delta < 1 \), there exists an \( i \in \mathcal{C} \) such that for every \( j \in [l] \), there exist measurement operators \( M_j^i \) taking registers \( X_j^i \otimes X_j^i E_j \) to \( X_j^i \otimes X_j^i E_j \) (with \( M_j^i (M_j^i)^\dagger \) being the POVM element), such that each \( \bigotimes_{j \in [l]} M_j^i \) succeeds on \( |\psi\rangle X_j^i \otimes |\varphi\rangle \tilde{X}_c E_A C \) with probability \( \alpha_i \geq 2^{-\frac{20p}{\text{eff} + \zeta(V,p)}} \) and

\[
\left\| \frac{1}{\alpha_i} \left( \bigotimes_{j \in [l]} M_j^i \right) \left( |\psi\rangle \langle \psi| \otimes |\varphi\rangle \langle \varphi| \tilde{X}_c E_A C \right) \left( \bigotimes_{j \in [l]} (M_j^i)^\dagger \right) - |\varphi\rangle \langle \varphi| \tilde{X}_c E_A C \right\|_1 \leq 2\zeta
\]

where \( |\psi\rangle X_j^i \otimes |\varphi\rangle \tilde{X}_c E_A C \) is the same state as \( |\varphi\rangle X_j^i \tilde{X}_c E_A C \) with the \( X_j^i \) register replaced by the \( X_j^i \) register.

In order to prove Lemma 8 assuming Lemma 10, we shall view the inputs to a strategy for \( G(p,V) \) or a zero-communication protocol for \( V \) as being quantum, instead of coming from a classical distribution. If the players receive inputs from a distribution \( P_y = P_{Y^1 \ldots Y^l} \), we can think of them as receiving registers \( Y^1, \ldots, Y^l \) respectively of a pure state

\[
|\sigma\rangle_{Y^1} = \sum_y \sqrt{P_y(y)} |yy\rangle_{Y^1} \]

with say a referee holding the \( Y^i \) registers (which the players cannot touch). The players hold a shared entangled state \( |\rho\rangle_{E_A} = |\rho\rangle_{E_1 \ldots E_l} A_1 \ldots A_l \), with player \( j \) holding \( E_i A_i \), \( A_i \) being the answer register. Player \( j \) now applies some isometry or measurement on registers \( Y^j E_j A_j \) to determine their output. Strictly speaking, this isometry or measurement should only use \( Y^j \) as a control register, since it is classical. But player \( j \) can always copy over \( Y^j \) to a different register \( \tilde{Y}^j \) and apply a general isometry or measurement on \( \tilde{Y}^j E_j A_j \) — the effect of this will be the same as applying a general isometry or measurement on \( Y^j E_j A_j \) that does not use \( Y^j \) as a control register. So we shall assume that player \( j \) can in fact apply a general isometry or measurement on \( Y^j E_j A_j \). Note that in this case we are talking we are talking about isometries instead of unitaries because these need not preserve the \( Y^j \) (or other) registers.

For zero-communication protocol with aborts, we shall also assume the players first apply a measurement to decide whether they will abort or not abort, and conditioned on not aborting, do another measurement to give outputs in \( A_1 \times \ldots \times A_l \) (in general they can do a single measurement to decide their output, which may be abort, or some element of \( A_i \), but the protocol \( P' \) we describe will have two measurements). In fact they do not need to actually do this last measurement in order for us to determine the average success probability: we can assume that the state conditioned on not aborting already has the correlations they want between the registers \( Y^j \) and \( A \) (the \( Y^j \) registers may have been modified by the measurement), and the average success probability is determined by computing \( V \) on \( Y^j A \) of the state conditioned on not aborting. That is, suppose the measurement operator corresponding to not abort for player \( j \) is \( M_j^i \). Then the average probability of not aborting in the protocol is the success probability \( \alpha \) of \( \bigotimes_{j \in [l]} M_j^i \) on \( |\sigma\rangle_{Y^1} \otimes |\rho\rangle_{E_A} \). And the average success probability of the protocol conditioned on not aborting is determined by computing \( V \) on the \( Y^j A \) registers of \( \frac{1}{\sqrt{\alpha}} \left( \bigotimes_{j \in [l]} M_j^i \right) |\sigma\rangle_{Y^1} \otimes |\rho\rangle_{E_A} \). Similarly, for games, the average success probability will be determined by computing \( V \) on the \( Y^j A \) registers of \( \bigotimes_{j \in [l]} U_j |\sigma\rangle_{Y^1} \otimes |\rho\rangle_{E_A} \), where \( U_j \) is the isometry applied by the \( j \)-th player.
6.2.1 Case (i): $c < 1$

In this case, we shall use the isometries given by Lemma 10 to give a quantum strategy $S$ for $G(p, V)$, whose winning probability is at least

$$\mathbb{E}_{i \in \mathcal{C}} \Pr[T_i = 1 | \mathcal{E}] - \sqrt{2lc - l}\sqrt{2}\delta.$$ 

By the definition of $\omega^*(G(p, V))$, $S$ cannot have success probability more than $\omega^*(G(p, V))$. This gives the required upper bound on $\mathbb{E}_{i \in \mathcal{C}} \Pr[T_i = 1 | \mathcal{E}]$.

The strategy $S$ is as follows:

- The players share $\log |\mathcal{C}|$ uniformly random bits as randomness (which we shall treat as classical), and $|\psi\rangle_{X_C E A_C R}$ as shared entanglement, with player $j$ holding the registers $\tilde{X}_C^j E A_C^j$ (the extra $R$ register can go to any player, say the first, but they won’t need to do anything on it).

- The players receive inputs as the $X_j^i$ register of $|\psi\rangle_{X_i X_i}$ (note that the distribution in this state is the correct one, $\mu$).

- The players use their shared randomness to sample a uniformly random $i \in \mathcal{C}$.

- Player $j$ applies the isometry $U_j^i$ on the registers $X_j^i \tilde{X}_C^j E A_C^j$ according to the sampled $i$.

- Player $j$ provides $A_j^i$ of the resulting state as their answer register.

The state obtained by the players at the end of this strategy is obviously

$$\mathbb{E}_{i \in \mathcal{C}} \left( \bigotimes_{j \in [l]} U_j^i \right) \left( |\psi\rangle_{X_i X_i} \otimes |\psi\rangle_{X_C \tilde{X}_C E A_C R} \right) \left( \bigotimes_{j \in [l]} (U_j^i)^\dagger \right).$$

Note that if $i$ is sampled uniformly in $\mathcal{C}$, and $V$ is computed on the $X_j^i A_i$ registers of $|\psi\rangle_{X_i \tilde{X}_C E A_C R'}$, then the average success probability is $\mathbb{E}_{i \in \mathcal{C}} \Pr[T_i = 1 | \mathcal{E}]$. By condition (ii) of Lemma 10, then the average success probability of $S$ is as claimed.

6.2.2 Case (ii): $c \geq 1$

In this case, we shall use the measurement operators given by Lemma 10 to give a zero-communication protocol $P'$ for $V$ whose average probability of not aborting is at least $2^{-\frac{2m^2}{\epsilon^2}} > 1/\text{eff}^*_{\epsilon + \zeta}(V, p)$ (by the condition on $cn$), and conditioned on not aborting, is correct with probability at least

$$\Pr[T_i = 1 | \mathcal{E}] - \zeta$$

(with the $i$ provided by condition (ii)) averaged on inputs from $p$. By the definition of $\text{eff}^*_{\epsilon + \zeta}(V, p)$, $P'$ cannot be correct conditioned on not aborting with probability more than $1 - (\epsilon + \zeta)$ when inputs come from $p$. This gives the required upper bound on $\Pr[T_i = 1 | \mathcal{E}]$.

The protocol $P'$ is as follows:
• The players share $|\psi\rangle_{\bar{X_c}EAC^R}$ as shared entanglement, with player $j$ holding the registers $\bar{X}^j_i E/A^j_C$ (the extra $R$ register can go to any player, say the first, but they won’t need to do anything on it).

• The players receive inputs as the $X^j_i$ register of $|\psi\rangle_{X^j_i X_i}$ (note that the distribution in this state is the correct one, $p$).

• Player $j$ applies measurements $\{M^j_i (M^j_i)^\dagger, 1-M^j_i (M^j_i)^\dagger\}$ on the registers $X^j_i \bar{X}^j_i E/A^j_C$ and declares not abort if the $M^j_i$ measurement succeeds.

• Conditioned on not aborting, player $j$ provides $A^j_i$ as their answer register.

By our description above, and condition (ii), the average probability of not aborting in this protocol is $\alpha_i \geq 2^{-\frac{2\alpha_i}{c}} > \frac{1}{\text{ent}_{\text{ip}}(V,p)}$ by the condition on $c$. Now note that if $V$ is computed in the $X^j_i A_i$ register of $|\psi\rangle_{X^j_i \bar{X_c}EAC^R}$, the average success probability is by definition $\Pr[T_i = 1|\mathcal{E}]$. Since by condition (ii),

$$\left\|\frac{1}{\alpha_i} \left( \bigotimes_{j \in \llbracket l \rrbracket} M^j_i \right) \left( |\psi\rangle_{X^j_i X_i} \otimes |\psi\rangle_{\bar{X_c}EAC^R} \right) \left( \bigotimes_{j \in \llbracket l \rrbracket} (M^j_i)^\dagger \right) - |\psi\rangle_{X^j_i \bar{X_c}EAC^R} \right\|_1 \leq 2\zeta$$

the average success probability on $\frac{1}{\sqrt{\alpha_i}} \left( \bigotimes_{j \in \llbracket l \rrbracket} M^j_i \right) |\psi\rangle_{X^j_i X_i} \otimes |\psi\rangle_{\bar{X_c}EAC^R}$, that is, the average success probability of $\mathcal{P}'$ conditioned on not aborting, is at least $\Pr[T_i = 1|\mathcal{E}] - \zeta$.

### 6.3 Proof of Lemma 10

The first part of the proof goes the same way for both cases (i) and (ii). We shall proceed with a common proof and then diverge when required.

Since player $j$’s communication in $\mathcal{P}$ is $c'n$ bits, by Lemma 5 for the final state $|\psi\rangle$ of $\mathcal{P}$, there exists a state $\rho_{X^j_i \bar{X}^j_i E^{-j} A^{-j}}$ such that

$$D_{\infty} \left( |\psi\rangle_{X^j_i X^j_i \bar{X}^j_i E^{-j} A^{-j}} \| \rho_{X^j_i \bar{X}^j_i E^{-j} A^{-j}} \right) \leq 2c'n.$$

Using Facts 11 and 12, this gives us

$$\begin{align*}
\mathbb{E}_{\mathcal{P},\mathcal{E}} \mathbb{D} \left( |\psi\rangle_{X^j_i X^j_i \bar{X}^j_i E^{-j} A^{-j}} \| \rho_{X^j_i \bar{X}^j_i E^{-j} A^{-j}} \right) \\
= \mathbb{E}_{\mathcal{P}_{X^j_i}} \mathbb{D} \left( |\psi\rangle_{X^j_i X^j_i \bar{X}^j_i E^{-j} A^{-j}} \| \rho_{X^j_i \bar{X}^j_i E^{-j} A^{-j}} \right) \\
\leq \mathbb{E}_{\mathcal{P}_{X^j_i}} \mathbb{D} \left( |\psi\rangle_{X^j_i X^j_i \bar{X}^j_i E^{-j} A^{-j}} \| \rho_{X^j_i \bar{X}^j_i E^{-j} A^{-j}} \right) \\
\leq \mathbb{E}_{\mathcal{P}_{X^j_i}} \mathbb{D} \left( |\psi\rangle_{X^j_i X^j_i \bar{X}^j_i E^{-j} A^{-j}} \| \rho_{X^j_i \bar{X}^j_i E^{-j} A^{-j}} \right) \\
\leq \mathbb{E}_{\mathcal{P}_{X^j_i}} \left[ D_{\infty} \left( |\psi\rangle_{X^j_i X^j_i \bar{X}^j_i E^{-j} A^{-j}} \| \rho_{X^j_i \bar{X}^j_i E^{-j} A^{-j}} \right) \right]
\end{align*}$$
\[ + \mathcal{D}_\infty \left( \varphi_{X_i|X_{-i} \sim \rho^{X_{-i}|A_{-i}}} \| \varphi_{X_i|X_{-i} \sim \rho^{X_{-i}|A_{-i}}} \right) \]
\[ + \mathcal{D}_\infty \left( \varphi_{X_i|X_{-i} \sim \rho^{X_{-i}|A_{-i}}} \| \varphi_{X} \otimes \varphi_{A_{-i}} \right) \]
\[ \leq \mathbb{E}_{\mathcal{P}_{X \subset A \subset X}} \left[ \log \left( \frac{1}{\mathbb{P}_{A \subset X}(a_{\mathcal{P}})} \right) + \log \left( \frac{1}{\mathbb{P}(\mathcal{E})} \right) + 2c\mathbb{P} \right] \]
\[ \leq \mathbb{E}_{\mathcal{P}_{X \subset A \subset X}} \left[ \mathcal{C} \cdot \log \left( |\mathcal{A}^{1} \cdots |\mathcal{A}^{i}| \right) + \log \left( \frac{1}{\mathbb{P}(\mathcal{E})} \right) + 2c\mathbb{P} \right] = (\delta + 2c\mathbb{P})n. \] (3)

Similarly we also have,
\[ \mathcal{D} \left( \varphi_{X_{c}^{i}} \| \varphi_{X_{c}} \otimes \varphi_{R} \right) = \mathbb{E}_{\mathcal{P}_{R \in \mathcal{E}}} \left\| \varphi_{X_{c}^{i}} \right\| \leq \delta \mathbb{P}. \] (4)

Using the Quantum Gibb’s inequality on (3) we have,
\[ \mathbb{P}_{X \subset A \subset X} \mathbb{E} \left( \mathcal{D} \left( \varphi_{X_{c}^{i} \sim \rho_{X_{c}}^{i} \sim \rho_{A_{c}}^{i} \ | \ x_{c^{i}} \right) \right) = \mathbb{E}_{\mathcal{P}_{R \in \mathcal{E}}} \mathcal{D} \left( \varphi_{X_{c}^{i} \sim \rho_{X_{c}}^{i} \sim \rho_{A_{c}}^{i} \ | \ x_{c^{i}} \right) \right) \]
\[ \leq \mathbb{E}_{\mathcal{P}_{R \in \mathcal{E}}} \mathcal{D} \left( \varphi_{X_{c}^{i} \sim \rho_{X_{c}}^{i} \sim \rho_{A_{c}}^{i} \ | \ x_{c^{i}} \right) \right) \]
\[ \leq (2c + \delta) \mathbb{P}. \]

Hence by the chain rule of relative entropy on this and (4),
\[ \mathcal{D} \left( \varphi_{X_{c}^{i} \sim \rho_{X_{c}}^{i} \sim \rho_{A_{c}}^{i} \ | \ x_{c^{i}} \right) \right) \leq 2(c + \delta) \mathbb{P}. \]

Using \( \mathcal{C}_{<i} \) to denote the set of coordinates if \( \mathcal{C} \) less than \( i \), we have by the chain rule of relative entropy again,
\[ 4(c + \delta) \geq \mathbb{E}_{i \in \mathcal{C}} \mathbb{E}_{X_{c_{<i}}} \left( \mathcal{D} \left( \varphi_{X_{c_{<i}}^{i} \sim \rho_{X_{c_{<i}}^{i}} \sim \rho_{A_{c_{<i}}^{i}} \ | \ x_{c_{<i}} \right) \right) \]
\[ \geq \mathbb{E}_{i \in \mathcal{C}} \left( \mathcal{D} \left( \varphi_{X_{c_{<i}}^{i} \sim \rho_{X_{c_{<i}}^{i}} \sim \rho_{A_{c_{<i}}^{i}} \ | \ x_{c_{<i}} \right) \right) \] (5)

where we have used Fact 19.

### 6.3.1 Case (i): \( \epsilon < 1 \)

We shall apply Pinsker’s inequality (Fact 9) on (5) for all \( j \in [L] \), after tracing out \( X_{c_{<}}^{j} \). After also applying Jensen’s inequality with respect to the expectation over \( i \), this gives us
\[ \mathbb{E}_{i \in \mathcal{C}} \left( \mathbb{E} \left( \mathcal{B} \left( \varphi_{X_{c_{<i}}^{i} \sim \rho_{X_{c_{<i}}^{i}} \sim \rho_{A_{c_{<i}}^{i}} \ | \ x_{c_{<i}} \right) \right) \right) \right) \]
\[ \leq 2 \sqrt{c + \delta}. \]

Now since \( X_{c_{<i}}^{j} \) as used in \( |\psi\rangle_{X_{c_{<i}}^{j}} \) and \( \varphi_{X_{c_{<i}}^{j} \sim \rho_{X_{c_{<i}}^{j} \sim \rho_{A_{c_{<i}}^{j}}}} \) in Lemma 10, is identical to \( X_{i}^{j} \), we also have,
\[ \mathbb{E}_{i \in \mathcal{C}} \left( \mathbb{E} \left( \mathcal{B} \left( \varphi_{X_{c_{<i}}^{i} \sim \rho_{X_{c_{<i}}^{i}} \sim \rho_{A_{c_{<i}}^{i}} \ | \ x_{c_{<i}} \right) \right) \right) \right) \]
\[ \leq 2 \sqrt{c + \delta}. \] (6)
Note that $|\psi\rangle_{X'_j \bar{X}_c E A_c^R}$ is a purification of the state in the first argument of the Bures distance in (6), and $|\psi\rangle_{X'_j X'_i} \otimes |\varphi\rangle_{\bar{X}_c E A_c^R}$ is a purification of the state in the second argument (since $P_{X'_j \ldots X'_i}$ is a product distribution, each $|\psi\rangle_{X'_j X'_i}$ is pure). Therefore, by Uhlmann’s theorem, there exist isometries $\{U^j_i\}$, taking registers $X'_j \bar{X}_E A'_L$ to $\bar{X}_E A'_C$, such that
\[
\mathbb{E} \mathcal{B} \left( \bigotimes_{i \in \mathbb{C}} (U^j_i \otimes 1) \left( |\psi\rangle_{X'_j X'_i} \otimes |\varphi\rangle_{X'_j \bar{X}_c E A_c^R} \right) \left( (U^j_i)^\dagger \otimes 1 \right), |\varphi\rangle_{X'_j \bar{X}_c E A_c^R} \right) \leq 2 \sqrt{2c^j + \delta}.
\]

Since $U^j_i$ do not act on the $X^{-j}_i$ registers, we in fact have,
\[
\mathbb{E} \mathcal{B} \left( \bigotimes_{j \in [l]} \left( U^j_i \otimes 1 \right), |\psi\rangle_{X'_j X'_i} \otimes |\varphi\rangle_{X'_j \bar{X}_c E A_c^R} \right) \leq 2 \sqrt{2c^j + \delta}. \quad (7)
\]

Now we need to find the action of $\bigotimes_{j \in [l]} U^j_i$ on $|\psi\rangle_{X'_j X'_i} \otimes |\varphi\rangle_{\bar{X}_c E A_c^R}$. By the triangle inequality, we see that
\[
\mathbb{E} \mathcal{B} \left( \bigotimes_{j \in [l]} (U^j_i)^k \right), \left( |\psi\rangle_{X'_j X'_i} \otimes |\varphi\rangle_{X'_j \bar{X}_c E A_c^R} \right) \leq \mathbb{E} \mathcal{B} \left( \bigotimes_{j \in [l]} (U^j_i)^k \right), \left( |\psi\rangle_{X'_j X'_i} \otimes |\varphi\rangle_{X'_j \bar{X}_c E A_c^R} \right) \leq 2 \sum_{k=1}^l \sqrt{c^k + \delta} \leq 2 \sum_{k=1}^l \sqrt{c^k + \sqrt{\delta}} \leq 2 \sqrt{2l\sqrt{c^k + \sqrt{\delta}}}
\]

where in $a$ we have used (7) with $j = k$, and in $b$ we have used the Cauchy-Schwarz inequality. Finally, by applying the Fuchs-van de Graaf inequality, we have,
\[
\mathbb{E} \left\| \bigotimes_{j \in [l]} (U^j_i)^k \right\|_1 \leq 2 \sqrt{2l\sqrt{c^k + \sqrt{\delta}}},
\]

which proves condition (i) of Lemma 10.
6.3.2 Case (ii): $c \geq 1$

Using the Quantum Substate Theorem on (5) and tracing out $X^{−1}_c$, we get for all $j \in [l]$,  
\[
\mathbb{E}_{i \in \mathcal{C}} D\left(\rho^{X^{−1}_c | E^{−1}_c | R}, \rho^{X^{−1}_c | E^{−1}_c | R}_i\right) \leq \frac{4c_j + 4\delta + 1}{\zeta'} + \log \left(\frac{1}{1 - \zeta'}\right)
\]
for some $\zeta'$ to be fixed later. Now since $X^{ij}_c$ as used in $|\psi\rangle^{X^{ij}_c}_i$ and $|\phi\rangle^{X^{ij}_c}_c$ in condition (ii) of Lemma 10, is identical to $X^{ij}_i$, we also have,
\[
\mathbb{E}_{i \in \mathcal{C}} D\left(\rho^{X^{ij}_c | E^{−1}_c | R}, \rho^{X^{ij}_c | E^{−1}_c | R}_i\right) \leq \frac{4c_j + 4\delta + 1}{\zeta'} + \log \left(\frac{1}{1 - \zeta'}\right)
\]
(8)

To find the measurement operators $M^{ij}_i$, we shall do induction on the number of players. In particular we shall prove the following lemma.

**Lemma 11.** Suppose we have measurement operators $\{ M^{ij}_j \}_i$ for $j \in [k], i \in \mathcal{C}, 0 \leq k < l$, taking registers $X^{ij}_c E^{−1}_c$ to $X^{ij}_c E^{−1}_c$ respectively, such that $\otimes_{j \in [k]} M^{ij}_i$ succeeds on $\left(\otimes_{j \in [k]} |\psi\rangle^{X^{ij}_c}_j \right) \otimes |\phi\rangle^{X^{ij}_c}_c$ with probability $\gamma_i^{\leq k} = 2^{-\sum_{j=1}^k c_j}$ where
\[
\mathbb{E}_{i \in \mathcal{C}} c_i^j \leq \frac{15c_j}{\zeta'},
\]
and for all $i \in \mathcal{C}$,
\[
\Delta\left(\frac{1}{\gamma_i^{(k+1)}} \left(\otimes_{j \in [k+1]} M^{ij}_j \otimes 1\right) \left(\otimes_{j \in [k+1]} |\psi\rangle^{X^{ij}_c}_j \otimes |\phi\rangle^{X^{ij}_c}_c\right) \left(\otimes_{j \in [k+1]} (M^{ij}_j)^+ \otimes 1\right),
\]
\[
|\psi\rangle^{X^{ij}_c}_j \otimes |\phi\rangle^{X^{ij}_c}_c\right) \leq (3k - 2) \sqrt{2\zeta'}.
\]
(9)

Then there are measurement operators $\{ M^{ij+1}_j \}_j$ taking registers $X^{k+1}_c E^{−1}_c$ to $X^{k+1}_c E^{−1}_c$ such that $\otimes_{j \in [k+1]} M^{ij}_i$ succeeds on $\left(\otimes_{j \in [k+1]} |\psi\rangle^{X^{ij}_c}_j \right) \otimes |\phi\rangle^{X^{ij}_c}_c$ with probability $\gamma_i^{(k+1)} = \gamma_i^{(k+1)} \gamma_i^{\leq k}$ where $\gamma_i^{(k+1)} = 2^{-c_i^{k+1}}$, with
\[
\mathbb{E}_{i \in \mathcal{C}} c_i^{k+1} \leq \frac{15c_i^{k+1}}{\zeta'},
\]
and for all $i \in \mathcal{C}$
\[
\Delta\left(\frac{1}{\gamma_i^{(k+1)}} \left(\otimes_{j \in [k+1]} M^{ij}_j \otimes 1\right) \left(\otimes_{j \in [k+1]} |\psi\rangle^{X^{ij}_c}_j \otimes |\phi\rangle^{X^{ij}_c}_c\right) \left(\otimes_{j \in [k+1]} (M^{ij}_j)^+ \otimes 1\right),
\]
\[
|\psi\rangle^{X^{ij}_c}_j \otimes |\phi\rangle^{X^{ij}_c}_c\right) \leq (3k + 1) \sqrt{2\zeta'}.
\]

We also clarify that if the distance in (9) is $\Delta[k]$, then the way we pick our parameters in the proof of the lemma gives us $\Delta[k + 1] = \Delta[k] + 3\sqrt{2\zeta'}$. The expression $(3k - 2) \sqrt{2\zeta'}$ is obtained by setting $\Delta[1] = \sqrt{2\zeta'}$. 

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Proof of Lemma 11. Let

$$|\rho\rangle_{X_i'X_i^kX_cE_AER} = \frac{1}{\sqrt{\alpha_i^{k+1}}} \left( \bigotimes_{j \in [k]} M_j^i \otimes 1 \right) \left( \bigotimes_{j \in [k]} |\psi\rangle_{X'_iX_j} \otimes |\varphi\rangle_{X_cE_AER} \right).$$

Note that $|\rho\rangle$ has an $i$ dependence, but we are not writing it explicitly. By (9),

$$\mathbb{E} \Delta \left( \rho_{X_i'^{(k+1)}E^{-(k+1)}A_c^{-(k+1)}R} \varphi_{X_i'^{(k+1)}E^{-(k+1)}A_c^{-(k+1)}R} \right) \leq \Delta[k].$$

Moreover, since none of the operators $M_j^i$ for $j \in [k]$ act on the $X_i'^{k+1}$ register,

$$\rho_{X_i'^{(k+1)}X_c'^{(k+1)}E^{-(k+1)}A_c^{-(k+1)}R} = \rho_{X_i'^{(k+1)}X_c'^{(k+1)}E^{-(k+1)}A_c^{-(k+1)}R} = \rho_{X_i'^{(k+1)}X_c'^{(k+1)}E^{-(k+1)}A_c^{-(k+1)}R}.$$

Using the Substate Perturbation Lemma on the above and (8) with $j = k + 1$, picking parameters $\epsilon = \delta_0 = \sqrt{2\zeta'}, \delta_1 = \Delta[k]$ we get,

$$\mathbb{E} \Delta^{[k+1],\Delta} \left( \varphi_{X_i'^{(k+1)}X_c'^{(k+1)}E^{-(k+1)}A_c^{-(k+1)}R} \right) \leq \frac{4\epsilon^{k+1} + 4\delta + 1}{\zeta'} + \log \left( \frac{1}{1 - \zeta'} \right) + 1 + \log \left( 1 + \frac{1}{\zeta'} \right)$$

$$\leq \frac{4\epsilon^{k+1} + 4\delta + 1}{\zeta'} + 3\zeta' + 1 + \frac{2}{\zeta'} \leq \frac{15\epsilon^{k+1}}{\zeta'}.$$

Now note that $|\psi\rangle_{X_i'^{(k+1)}X_i^kX_cE_AER} \otimes |\varphi\rangle_{X_i'^{(k+1)}X_cE_AER}$ is a purification of the state in the first argument in the above smoothed entropy, and $|\rho\rangle_{X_i'^kX_cE_AER}$ is obviously a purification of the state in the second. Therefore, by Fact 15, there exist measurement operators $\left\{ M_i^{k+1} \right\}_i$ taking registers $X_i'^{k+1}X_c'^{k+1}E^{k+1}A_c^{k+1}$ to $X_c'^{k+1}E^{k+1}A_c^{k+1}$, that succeed on $|\rho\rangle_{X_i'^kX_cE_AER}$ with probability $\alpha_i^{k+1} = 2^{-\epsilon^{k+1}}$, where

$$\mathbb{E} \Delta_i^{k+1} \leq \frac{15\epsilon^{k+1}}{\zeta'},$$

and for all $i$,

$$\Delta \left( \frac{1}{\alpha_i^{k+1}} \bigotimes_{j \in [k+1]} M_j^i \otimes 1 \right) \left( \bigotimes_{j \in [k+1]} |\psi\rangle_{X_i'^{(k+1)}X_j} \otimes |\varphi\rangle_{X_cE_AER} \right) \left( \bigotimes_{j \in [k+1]} \left( M_j^i \right)^+ \otimes 1 \right),$$

$$|\psi\rangle_{X_i'^{(k+1)}X_i^kX_cE_AER} \otimes |\varphi\rangle_{X_i'^{(k+1)}X_cE_AER}$$

$$= \Delta \left( \frac{1}{\alpha_i^{k+1}} \bigotimes_{j \in [k+1]} M_j^i \otimes 1 \right) \left( |\rho\rangle_{X_i'^kX_cE_AER} \right) \left( \bigotimes_{j \in [k+1]} \left( M_j^i \right)^+ \otimes 1 \right),$$

$$|\psi\rangle_{X_i'^{(k+1)}X_i^kX_cE_AER} \otimes |\varphi\rangle_{X_i'^{(k+1)}X_cE_AER} \leq \Delta[k + 1].$$

This proves the lemma. □
After the induction process, we have measurement operators \( \{M_i^j\}_i \) for \( j \in [l] \) and the conditions in the statement of Lemma 11 hold with \( k = 1 \). Therefore, by the Fuchs-van de Graaf inequality,

\[
\left\| \frac{1}{\alpha_i} \left( \bigotimes_{j \in [l]} M_i^j \right) \left( \psi\langle\psi|_{X_i^j} \otimes |\phi\rangle\langle\phi|_{\tilde{X}_cEA_cR} \right) \left( \bigotimes_{j \in [l]} (M_i^j)^\dagger \right) - |\phi\rangle\langle\phi|_{X_i^j\tilde{X}_cEA_cR} \right\|_1 \leq 2(3l - 2) \sqrt{2\zeta_i}'.
\]

Setting \((3l - 2) \sqrt{2\zeta_i'} = \zeta\) we get, \(\zeta' \geq \frac{\zeta^2}{1672}\). This gives us

\[
\mathbb{E} \sum_{i \in C} c_i \leq \frac{270l^2}{\zeta^2} \sum_{j = 1}^l c_j \leq \frac{270l^3c}{\zeta^2}.
\]

Since \(2^{-x}\) is a convex function, by Jensen’s inequality we have,

\[
\mathbb{E} \sum_{i \in C} c_i = \mathbb{E} 2^{-\sum_{j = 1}^l c_i} \geq 2^{-\mathbb{E}_{i \in C} \sum_{j = 1}^l c_i} \geq 2^{-270l^3c/\zeta^2}.
\]

Therefore there exists an \( i \in \tilde{C} \) such that \( \alpha_i \geq 2^{-270l^3c/\zeta^2} \). This proves condition (ii) in Lemma 10.

7 DIQKD with leakage

In this section, we prove Theorem 4, whose statement is recalled below.

**Theorem 4.** There are universal constants \( 0 < \delta_0 < 1 \) and \( 0 < c_0 < 1 \) such that for any \( 0 \leq \delta \leq \delta_0 \), and \( 0 \leq c \leq c_0 \), if the [JMS20] DIQKD protocol (given in Protocol 1) is carried out with boxes that play \( n \) copies of the Magic Square game \( \delta \)-noisily, it is possible to extract \( r(\delta, c)n \) bits of secret key in the interactive leakage model, with the total communication between Alice, Bob and Eve’s boxes being \( cn \) bits, for some \( r(\delta, c) > 0 \).

Protocol 1 is given below. It makes use of the following equipment:

(i) Boxes \((B^A, B^B)\) with Alice and Bob respectively, whose honest behaviour is to play \( n \) i.i.d. instances of MS \( \delta \)-noisily, i.e., each copy of MS is won with probability \( 1 - \delta \);

(ii) Private sources of randomness for both Alice and Bob;

(iii) A public authenticated channel between Alice and Bob.

We shall prove the following theorem about Protocol 1, which implies Theorem 4.

**Theorem 12.** Let \( \rho_{K^A|K^B|X_5Y_5A_7ST\tilde{E}} \) be the state of Alice’s and Bob’s raw keys and Eve’s side information conditioned on not aborting in Protocol 1 carried out with parameters \( \alpha, \gamma, \delta \) (where \( \tilde{E} \) is Eve’s quantum register and \( X_5Y_5A_7ST \) is the communication through the public channel which she also has access to). If the total communication in the interactive leakage model is \( cn \) for some \( c < 1 \), then the state \( \rho \) satisfies

\[
H_\infty(K^A|X_5Y_5A_7ST\tilde{E})\rho - H_0(K^A|K^B)\rho \geq \alpha \left( \nu - \beta(\sqrt{c} + \sqrt{\alpha}) - 2\eta_2(4\delta) - \gamma \right) n - \log(1/\Pr[\mathcal{E}]),
\]

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**Protocol 1** DIQKD protocol (with parameters $\alpha, \gamma, \delta$)

1. Alice chooses $x_1 \ldots x_n \in \{0,1,2\}^n$ uniformly at random from private randomness, inputs it into her box $B^A$, and records the output $a_1 \ldots a_n$
2. Bob chooses $y_1 \ldots y_n \in \{0,1,2\}^n$ uniformly at random from private randomness, inputs it into his box $B^B$, and records the output $b_1 \ldots b_n$
3. Alice chooses $S \subseteq [n]$ of size $\alpha n$, $T \subseteq S$ of size $\gamma |S|$ uniformly at random from private randomness
4. Alice sends $(S, T, x_S, a_T)$ to Bob using the public channel
5. Bob sends $y_S$ to Alice using the public channel
6. Bob tests if $a_i[y_i] = b_i[x_i]$ for at least $(1 - 2\delta)|T|$ many $i$-s in $T$
7. if the test fails then
8. Bob aborts the protocol
9. else
10. Alice sets $(K^A)_{i \in S} = a_i[y_i]$ and Bob sets $(K^B)_{i \in S} = b_i[x_i]$ as their respective keys

where $\mathcal{E}$ is the event that the protocol does not abort, $\varepsilon' = \frac{2^{2 - 8\gamma \alpha n}}{\Pr[\mathcal{E}]}$, $\beta, \nu$ are constants in $(0, 1)$, and $h_2$ is the binary entropy function. Moreover, when $(B^A, B^B)$ have their honest $\delta$-noisy behaviour, then $\Pr[\mathcal{E}] \geq 1 - 2^{-2\beta \gamma \alpha n}$.

We are free to pick the parameters $\alpha, \gamma$ in Protocol 1; $\delta$ is also a parameter in the Protocol, which is picked according to the noise level expected in honest boxes. The constant $\nu$ is the one provided by Fact 25. For $c, \delta$ such that $\nu > \beta \sqrt{c} + 2h_2(4\delta)$, there exist choices of $\alpha, \gamma$ and values of $\Pr[\mathcal{E}]$ for which the above quantity is positive. Hence we get a positive key rate for $c, \delta$ in this region.

### 7.1 Properties of the Magic Square game

**Definition 8.** The 2-player Magic Square game, denoted by MS, is as follows:

- Alice and Bob receive respective inputs $x \in \{0,1,2\}$ and $y \in \{0,1,2\}$ independently and uniformly at random.
- Alice outputs $a \in \{0,1\}^3$ such that $a[0] \oplus a[1] \oplus a[2] = 0$ and Bob outputs $b \in \{0,1\}^3$ such that $b[0] \oplus b[1] \oplus b[2] = 1$.
- Alice and Bob win the game iff $a[y] = b[x]$.

The classical value of the magic square game is $\omega(MS) = 8/9$, whereas the quantum value is $\omega^*(MS) = 1$.

**Definition 9.** The 3-player variant of the Magic Square game, denoted by MSE, is as follows:

- Alice receives inputs $x \in \{0,1,2\}, z \in \{0,1\}$ and Bob receives input $y \in \{0,1,2\}$ independently and uniformly at random; Eve receives no input.
- Alice outputs $a \in \{0,1\}^3$ such that $a[0] \oplus a[1] \oplus a[2] = 0$, Bob outputs $b \in \{0,1\}^3$ such that $b[0] \oplus b[1] \oplus b[2] = 1$, and Eve outputs $x' \in \{0,1,2\}, y' \in \{0,1,2\}, z' \in \{0,1\}$ and $c \in \{0,1\}$. 

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• Alice, Bob and Eve win the game iff

\[(x = x') \land (y = y') \land (a[y] = c) \land ((a[y] = b[x]) \lor (z = z')).\]

**Fact 25 ([JMS20]).** There is a constant \(0 < v < 1\) such that \(\omega^v(\text{MSE}) = \frac{1}{2}(1 - v)\).

The above fact is a consequence of Proposition 4.1 in [JMS20]. The game considered in the statement of this proposition in [JMS20] is different: they consider a 6-player game between Alice, Bob, Alice', Bob', Charlie and Charlie'. Here we have given Charlie’s role to Alice, and merged Alice', Bob' and Charlie' into Eve (this is later done in the analysis in [JMS20] anyway). Doing this makes no difference in the proof of the game’s winning probability as given in [JMS20].\(^6\)

Alternatively, the fact can be seen as a consequence of Lemma 2 in [Vid17]. The game considered in [Vid17] does not include Eve having to produce guesses \(x', y', z'\) for \(x, y, z\). Suppose the probability of winning Vidick’s game is \((1 - v')\). Since by no-signalling Eve’s best probability of guessing \(z\) is \(\frac{1}{2}\), the probability of winning the version of the game where Eve has to produce \(z'\) but not \(x', y'\) is \((1 - \frac{v}{2})\). Further, since Eve’s probability of guessing \(x\) and \(y\) are both \(\frac{1}{3}\) the probability of winning MSE where she has to produce \(x', y'\) is \(\frac{1}{3}(1 - \frac{v}{2})\).

Now Corollary 9 has the following consequence for the parallel-repeated MSE game in the interactive leakage model.

**Corollary 13.** There exists a constant \(\beta > 0\) such that if the total communication in the interactive leakage model is at most \(cn\) for some \(c < 1\), with \(v\) being the constant from Fact 25, then the probability of winning MSE in a random subset of size \(t\) out of \(n\) instances is at most

\[
\left(1 - v + \beta(\sqrt{c} + \sqrt{t/n})\right)^t.
\]

### 7.2 Security proof with leakage

We introduce some notation for states. Note that we have defined \(\mathcal{E}\) to be the abort event, but we can equivalently define it to be the event that \(a_i[y_i] = b_i[x_i]\) for at least \((1 - 2\delta)|T|\) many \(i\)-s in \(T\). This way we can condition states of the protocol before Alice and Bob have communicated on \(\mathcal{E}\) as well, even though they cannot abort at this point. For the variable \(K^V\) that is defined in Lemma 14, we use:

\[
\begin{align*}
\rho_{K^A K^B X S A T ST E} & : \text{state conditioned on } \mathcal{E} \text{ at the end of Protocol 1} \\
\sigma_{K^A K^B K^V X S A T ST E} & : \text{state after step 3 in Protocol 1} \\
\varphi_{K^A K^B K^V X S A T ST E} & : \text{state after step 3 in Protocol 1 conditioned on } \mathcal{E}.
\end{align*}
\]

First we shall prove some lemmas about the states \(\sigma\) and \(\varphi\), and then use them to get the final min-entropy bound on \(\rho\).

**Lemma 14.** Define the variable

\[
(K^V)_{i \in S} = \begin{cases}
0 & \text{if } a_i[y_i] = b_i[x_i] \\
0/1 \text{ w.p. } \frac{1}{2} & \text{otherwise.}
\end{cases}
\]

\(^6\)In [JMS20], \(z'\) is the input of Charlie' rather than an output. For the application perspective, we think it makes more sense to make it an output, since we consider the probability of Eve guessing \(z\). However, due to no-signalling the probability that \(z = z'\) is \(\frac{1}{2}\) regardless, and this change makes no difference.
If the total communication in the interactive leakage model is at most \( cn \) for some \( c < 1 \), then

\[
H_\infty(K^A V | X S Y S E) \geq \alpha \left( \nu - \beta(\sqrt{c} + \sqrt{\alpha}) \right) n
\]

where \( \beta, \nu \) are the constants from Corollary 13.

The extra bit \( K^Y_i \) in the statement of this lemma takes nontrivial value when \( a_i[Y_i] \neq b_i[Y_i] \), and Eve can potentially guess this. This will take the role of Alice’s extra input bit \( z \) in the definition of MSE, so that it is possible to win MSE on all coordinates in \( S \), even if \( a_i[Y_i] \neq b_i[X_i] \). In order to use Corollary 13 for our security proof, it is important that it is possible to win MSE on all these coordinates. Using Corollary 13 on Protocol 1 will give us a min-entropy bound including the extra \( K^Y_i \) bits, but these can be taken away later as conditioned on the not-aborting event, \( K^Y_i \) takes non-trivial value on very few coordinates.

**Proof of Lemma 14.** Consider the \( \text{MSE}^{an/n} \) game being played on the state shared by Alice, Bob and Eve (with \( S \) being the random subset of size \( an \), and \( \text{MSE}^{an/n} \) being won if the instances in the random subset \( S \) are won) in Protocol 1. Here \( K^Y_i \) is being interpreted as Alice’s input \( Z_i \) when \( A_i[Y_i] \neq B_i[X_i] \); when \( A_i[Y_i] = B_i[X_i] \), \( Z_i \) is irrelevant to the winning condition of MSE, so it does not matter that \( K^Y_i \) is trivial here. Let \( U_i \) be the indicator variable of the event that Eve guesses \( X_i Y_i A_i[Y_i] \) correctly, \( V_i \) be the indicator variable for the event Eve guesses \( K^Y_i \) correctly and \( W_i \) be the indicator variable for the event that \( A_i[Y_i] = B_i[X_i] \) for \( i \in S \). From Fact 20,

\[
H_\infty(K^A V | X S Y S E) \geq \log \left( \frac{1}{\text{Pr}[\bigwedge_i U_i \land \neg W_i \implies V_i]} \right)
\]

\[
= \log \left( \frac{1}{\text{Pr}[\text{Win MSE}^{an/n}]} \right)
\]

\[
\geq an \cdot \log \left( \frac{9}{1 - \nu + \beta(\sqrt{c} + \sqrt{\alpha})} \right)
\]

where we have used Corollary 13 along with the upper bound on communication in the last line.

Since \( X_i Y_i \) are uniformly random on a set of support size 9 we then have by Fact 17,

\[
H_\infty(K^A V | X S Y S E) \geq an \cdot \log \left( \frac{9}{1 - \nu + \beta(\sqrt{c} + \sqrt{\alpha})} \right) - \log |X S Y S|
\]

\[
\geq an \cdot \log \left( \frac{1}{1 - \nu + \beta(\sqrt{c} + \sqrt{\alpha})} \right) - an \cdot \log 9
\]

\[
= an \cdot \log \left( \frac{1}{1 - \nu + \beta(\sqrt{c} + \sqrt{\alpha})} \right)
\]

\[
\geq \alpha \left( \nu - \beta(\sqrt{c} + \sqrt{\alpha}) \right) n.
\]
Proof. Firstly, since \( \varphi \) is \( \sigma \) conditioned on an event of probability \( \Pr[\mathcal{E}] \), by Fact 11 and the previous lemma we have,

\[
H_\infty(K^{\mathcal{A}}K^{\mathcal{V}}|X_SY_SST\bar{E})_{\varphi} \geq \alpha (v - \beta(\sqrt{\epsilon} + \sqrt{\alpha})) n - \log(1/ \Pr[\mathcal{E}]).
\]

Let \( W_i \) denote the indicator variable for the event \( A_i[Y_i] = B_i[X_i] \) and let \( \varphi' \) denote \( \sigma \) conditioned on the following event which we call \( \mathcal{E}' \):

\[
\left( \sum_{i \in T} W_i \geq (1 - 2\delta)|T| \right) \wedge \left( \sum_{i \in S} W_i \geq (1 - 4\delta)|S| \right).
\]

By Fact 4, \( \Pr[\mathcal{E}'] \geq 1 - 2^{-8\delta^2\epsilon\gamma n} \), which gives us \( \| \varphi - \varphi' \|_1 \leq 2 \cdot 2^{-8\delta^2\epsilon\gamma n}/\Pr[\mathcal{E}]. \) In \( \varphi' \), \( A_i[Y_i] \) and \( B_i[X_i] \) differ in at most \( 4\delta|S| \) many places in \( S \), and \( K^\mathcal{V} \) is a uniformly random bit only in these places. Hence by Fact 17,

\[
H_\infty(K^{\mathcal{A}}|X_SY_SST\bar{E})_{\varphi'} \geq \alpha (v - \beta(\sqrt{\epsilon} + \sqrt{\alpha})) n - \log(1/ \Pr[\mathcal{E}]) - 4\delta an,
\]

which gives us the \( \epsilon \)-smoothed bound for \( \varphi \) from \( \ell_1 \) bound between \( \varphi \) and \( \varphi' \).

Proof of Theorem 12. First we shall condition the conditional min-entropy bound from Lemma 15 further on \( (T, A_T) \). Among these, \( T \) is independent of \( K^\mathcal{A} \), so conditioning on it makes no difference. \( A_T \) is contained in \( K^\mathcal{A} \), and uniformly random in \( \{0, 1, 2\}^{|T|} \). Hence,

\[
H^\epsilon_\infty(K^{\mathcal{A}}|X_SY_SA_TST\bar{E})_{\varphi} \geq \alpha (v - \beta(\sqrt{\epsilon} + \sqrt{\alpha}) - 4\delta)n - \log(1/ \Pr[\mathcal{E}]) - \alpha \gamma n.
\]

Now notice that in \( \rho \), \( X_SY_SA_TST\bar{E} \) is revealed to Eve, so she may do some operations on her side depending on these. \( \rho \) is thus related to \( \varphi \) by some local operations on the registers \( X_SY_SA_TST\bar{E} \). Hence by Fact 13,

\[
H^\epsilon_\infty(K^{\mathcal{A}}|X_SY_SA_TST\bar{E})_{\rho} \geq \alpha (v - \beta(\sqrt{\epsilon} + \sqrt{\alpha}) - 4\delta - \gamma)n - \log(1/ \Pr[\mathcal{E}]).
\]

Finally, to bound \( H^\epsilon_0(K^{\mathcal{A}}|K^\mathcal{B})_{\rho} \), we consider the state \( \rho' \), which is conditioned on the event \( \mathcal{E}' \) as defined in the proof of Lemma 15 instead of \( \mathcal{E} \) like \( \rho \). They satisfy \( \| \rho - \rho' \|_1 \leq 2 \cdot 2^{-8\delta^2\epsilon\gamma n}/\Pr[\mathcal{E}]. \) The number of strings \( K^\mathcal{B} \) of length that can differ from a given value of \( K^{\mathcal{A}} \) in at most \( 4\delta|S| \) places is at most \( 2^{h_2(4\delta)|S|} \), which gives us \( H_0(K^\mathcal{B}|K^\mathcal{A})_{\rho'} \leq h_2(4\delta)\gamma n. \) Putting everything together we get,

\[
H^\epsilon_\infty(K^{\mathcal{A}}|X_SY_SA_TST\bar{E})_{\rho} - H^\epsilon_0(K^\mathcal{B}|K^\mathcal{A})_{\rho} \geq \alpha (v - 2\beta(\sqrt{\epsilon} + \sqrt{\alpha}) - 4\delta - \gamma - h_2(4\delta)) n - \log(1/ \Pr[\mathcal{E}])
\]

\[
\geq \alpha (v - \beta(\sqrt{\epsilon} + \sqrt{\alpha}) - 2h_2(4\delta) - \gamma)n - 1/ \Pr(\mathcal{E})
\]

for \( \delta \leq 1/2. \)

To lower bound \( \Pr[\mathcal{E}] \) in the honest case when each instance of MS is won with probability \( 1 - \delta \), we use the Chernoff bound. Letting \( W_i \) denote the indicator variable for \( A_i[Y_i] = B_i[X_i] \), the \( W_i \)-s are i.i.d. in this case, and the expected value of each \( W_i \) is \( 1 - \delta \). Hence

\[
\Pr[\neg \mathcal{E}] = \Pr \left[ \sum_{i \in T} W_i < (1 - 2\epsilon)|T| \right] \leq 2^{-2\delta^2\epsilon\gamma n}.
\]
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A Proof of Yao’s lemma

In order to prove this lemma, we shall use Sion’s minimax theorem [Sio58]. Sion’s minimax theorem is true for quasisaddle and semicontinuous functions in general. However, we shall only need to apply it to continuous and saddle functions, so we shall state it for such below. A function \( f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R} \) is called saddle if \( f(\cdot, y) \) is convex as a function of \( x \) for each fixed \( y \in \mathcal{Y} \), and \( f(x, \cdot) \) is concave as a function of \( y \) for each fixed \( x \in \mathcal{X} \).
Fact 26 ([Sio58]). Suppose $\mathcal{X}$ and $\mathcal{Y}$ are convex spaces, one of which is compact, and $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is a continuous (in both arguments) and saddle function. Then,
\[
\inf_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} f(x, y) = \sup_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} f(x, y).
\]

The set $\mathcal{Y}$ that we shall be considering a supremum over in our application of Sion’s minimax lemma is going to be the set of probability distributions over the inputs to a communication protocol. Since the set of inputs is finite, this set is compact. It is also obviously convex.

A quantum protocol $\mathcal{P}$ running on input $x = x^1 \ldots x^l$ gives rise to a distribution $P_{A|\mathcal{P},x}$ on the outputs. Let $P_{A|\mathcal{P}}$ denote the tuple $(P_{A|\mathcal{P},x})_{x \in \mathcal{X}^1 \times \ldots \times \mathcal{X}^l}$, which is the entire output distribution of $\mathcal{P}$. Let $\mathbb{P}_t^Q$ denote the set of $P_{A|\mathcal{P}}$ for quantum protocols $\mathcal{P}$ with at most $t$ communication. Given two protocols $\mathcal{P}_1$ and $\mathcal{P}_2$ between $l$ players, with at most $r$ rounds and $t$ communication, their convex combination can be implemented by the players sharing an appropriate superposition of $|0^l\rangle$ and $|1^l\rangle$ as entanglement, and implementing the steps of $\mathcal{P}_1$ or $\mathcal{P}_2$ controlled on their part of the shared entanglement being $|0\rangle$ or $|1\rangle$. The resultant quantum protocol $\mathcal{P}$ has at most $r$ rounds and $t$ communication. Therefore, the set $\mathbb{P}_t^Q$ is convex. This will be the set $\mathcal{X}$ we take supremum over in Sion’s minimax theorem.

For a protocol $\mathcal{P}$ and input $x$, let $\text{err}_V(\mathcal{P}, x)$ denote the error that $\mathcal{P}$ makes on input $x$ for predicate $V$. With some abuse of notation, we shall also use $\text{err}_V(\mathcal{P}, \mu)$ to denote the average error of $\mathcal{P}$ over an input distribution $\mu$. Note that $\text{err}_V(\mathcal{P}, x)$ and $\text{err}_V(\mathcal{P}, \mu)$ can be defined using just $P_{A|\mathcal{P}},$ and with abuse of notation we shall use $\text{err}_V(P_{A|\mathcal{P}}, x)$ and $\text{err}_V(P_{A|\mathcal{P}}, \mu)$ to denote these. The set of all quantum protocols which communicate at most $t$ qubits has a somewhat complicated structure, but the infimum of $\text{err}_V(\mathcal{P}, \mu)$ over this set is the same as the infimum of $\text{err}_V(P_{A|\mathcal{P}}, \mu)$ over $\mathbb{P}_t^Q$.

For a fixed $\mathcal{P}$, $\text{err}_V(\mathcal{P}, \cdot)$ (or $\text{err}_V(P_{A|\mathcal{P}}, \cdot)$) is linear\footnote{Here by ‘linear’ we mean a function that is both convex and concave.} in the second argument, because the error for a convex combination of distributions is just the convex combination of the errors for the individual distributions. Similarly, the error for a convex combination of algorithms is the convex combination of errors for the individual algorithms, i.e., $\text{err}_V(\cdot, \mu)$ is linear in the first argument. Therefore $\text{err}_x$ is saddle, and is also continuous in both arguments.

The function $\text{err}_V$, the set $\mathbb{P}_t^Q$ and the set of probability distributions satisfy the conditions of Sion’s minimax theorem. Therefore we can say,
\[
\inf_{P_{A|\mathcal{P}} \in \mathbb{P}_t^Q} \sup_{\mu} \text{err}_V(P_{A|\mathcal{P}}, \mu) = \sup_{\mu} \inf_{P_{A|\mathcal{P}} \in \mathbb{P}_t^Q} \text{err}_V(P_{A|\mathcal{P}}, \mu).
\]

It is clear that the supremum over $\mu$ on the left-hand side of the above equation is the same as the supremum over inputs $x$. Now consider $t = Q_\varepsilon(V) - 1$. By the definition of $Q_\varepsilon(V)$, the left-hand sides of the equation must then be strictly greater than $\varepsilon$. Moreover, the sup on the right-hand side is a max, since the set of distributions is compact. Hence, there is a distribution $\mu$ such that all algorithms making less than $Q_\varepsilon(V)$ queries must make more than $\varepsilon$ error over inputs from $\mu$, i.e.,
\[
\max_{\mu} Q_\varepsilon(V, \mu) \geq Q_\varepsilon(V).
\]
Moreover $Q_\varepsilon(V) \geq \max_{\mu} Q_\varepsilon(V, \mu)$ also holds, since an algorithm that makes $\varepsilon$ error in the worst case also makes at most $\varepsilon$ error averaged over $\mu$. This completes the proof of the lemma.
B Proof of Lemma 5

We shall do induction on the number of rounds. Let $c_i$ be the communication in the $i$-th round and $R^j = \{j, j + l, \ldots\}$ denote the set of rounds in which the $j$-th player communicates, so that $\sum_{i \in R^j} c_i = c^j$. Let $M_i$ be the message register of the $i$-th round, $E_i$ be the memory register the party who communicates in the $i$-th round holds after sending their message. For $i \in R^j$, the registers held by the $j$-th party at the beginning of the $i$-th round are messages $M_{i-l+1}^j \ldots M_{i-1}^j$ from other parties in the $(i-l+1)$-th to $(i-1)$-th rounds, which we shall jointly denote by $N_{i-1}^j$, and their memory register $E_{i-1}$ which they have retained from the $(i-l)$-th round. We shall denote all other (non-input) registers held by parties other than the $j$-th party at the beginning of the $i$-th round by $F_{i-1}^j$. Since $i \in R^j$, clearly $F_{i-1}^j = F_{i-1}^j M_i$. Using $X$ to denote $X^1 \ldots X^l$ and similar notation for $\bar{X}$, we shall call the shared state including the input purifications at the beginning of the the $i$-th round

$$|\sigma^j\rangle_{X\bar{X}N_{i-1}^jE_{i-1}F_{i-1}^j} = \sum_x \sqrt{P_X(x)} |xx\rangle_{X\bar{X}} |\sigma^j\rangle_{N_{i-1}^jE_{i-1}F_{i-1}^j}.$$ 

For the base case $i = 1$, communication is zero. Since $P_{X^1 \ldots X^l}$ is a product distribution, $\sigma^1_{X\bar{X}^{-i}X^{-i}F_{i-1}^j}$ is product between $X^j$ and the other registers, $F_{0}^{-j}$ being simply the other parties’ parts of the initial shared entangled state, which is independent of the inputs. So the condition trivially holds.

For the induction step, we shall assume the condition

$$D_\infty \left( |\sigma^i\rangle_{X\bar{X}^{-i}X^{-i}F_{i-1}^j} \otimes \rho^i_{X\bar{X}^{-i}X^{-i}F_{i-1}^j} \right) \leq 2 \sum_{i' \in R^j, i' < i} c_{i'}$$

holds at the beginning of the $i$-th round, where $i \in R^j$, for some state $\rho^i_{X\bar{X}^{-i}X^{-i}F_{i-1}^j}$, and see how it changes in the $i$-th to $(i+l-1)$-th rounds.

In the $i$-th round, the $j$-th party applies a unitary on the $X^jN_{i-1}^jE_{i-1}$ registers, getting registers $X^jM_iE_i$. By Fact 16, there exists a state $\tilde{\rho}^{i+1}_{M_i}^j$ such that

$$D_\infty \left( |\sigma^{i+1}\rangle_{X\bar{X}^{-i}X^{-i}F_{i-1}^j M_i} \otimes \tilde{\rho}^{i+1}_{M_i}^j \right) \leq 2c_i.$$ 

Now note that the marginal states $\sigma^i_{X\bar{X}^{-i}X^{-i}F_{i-1}^j}$ and $\sigma^{i+1}_{X\bar{X}^{-i}X^{-i}F_{i-1}^j}$ are exactly the same, since the unitary relating $|\sigma^j\rangle$ and $|\sigma^{j+1}\rangle$ does not act on $X^{-i}\bar{X}^{-i}F_{i-1}^j$ at all, and only uses $X^j$ as a control register. Hence we have,

$$D_\infty \left( |\sigma^{i+1}\rangle_{X\bar{X}^{-i}X^{-i}F_{i-1}^j} \otimes \tilde{\rho}^{i+1}_{M_i}^j \right) \leq 2 \sum_{i' \in R^j, i' < i} c_{i'}.$$ 

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Now using Fact 12 we can say,

\[
\begin{align*}
D_{\infty} \left( \sigma_{i+1}^{j+1} X^{j+1} X^{j+1} F_{i+1}^{j+1} M_{i+1} \right) & \leq \left\| \sigma_{i+1}^{j+1} X^{j+1} X^{j+1} F_{i+1}^{j+1} M_{i+1} \right\| \sigma_{i+1}^{j+1} X^{j+1} X^{j+1} F_{i+1}^{j+1} M_{i+1} \right\| \sigma_{i+1}^{j+1} X^{j+1} X^{j+1} F_{i+1}^{j+1} M_{i+1} \\
& \leq 2c_i + 2 \sum_{i' \in R^j, \ i' < i} c_{i'}, \\
& = 2 \sum_{i' \in R^j, \ i' < i} c_{i'}.
\end{align*}
\]

Hence the condition holds at the beginning of the \((i + 1)\)-th round with \(\rho_{i+1}^{j+1} X^{j+1} X^{j+1} F_{i+1}^{j+1} M_{i+1} = \rho_{i+1}^{j+1} X^{j+1} X^{j+1} F_{i+1}^{j+1} M_{i+1} \).

In the \((i + 1)\)-th round, the \((j + 1)\)-th player applies a unitary on the \(X^{j+1} N_i^{j+1} E_{i-1}^{j+1}\) registers, getting registers \(X^i M_{i+1} X^i M_{i+1} \ldots M_{i+1} X^i M_{i+1} X^j F_{i+1}^{j+1} M_{i+1}\), of which they send \(M_{i+1}^{j+1}\) to the \(j\)-th player. So after this round, the registers held by the \(j\)-th player are \(E_{i} M_{i+1}^j\) and \(F_{i+1}^{j-1}\) does not include \(M_{i+1}^j\). By Fact 10 we have that,

\[
\begin{align*}
D_{\infty} \left( \sigma_{i+2}^{j+2} X^{j+2} X^{j+2} F_{i+1}^{j+1} M_{i+1} \right) & \leq \left\| \sigma_{i+2}^{j+2} X^{j+2} X^{j+2} F_{i+1}^{j+1} M_{i+1} \right\| \sigma_{i+2}^{j+2} X^{j+2} X^{j+2} F_{i+1}^{j+1} M_{i+1} \right\| \sigma_{i+2}^{j+2} X^{j+2} X^{j+2} F_{i+1}^{j+1} M_{i+1} \\
& \leq \sum_{i' \in R^j, \ i' < i} c_{i'}, \\
& = \sum_{i' \in R^j, \ i' < i} c_{i'},
\end{align*}
\]

where \(\rho^{j+2}\) is the state obtained by applying the \((j + 1)\)-th player’s unitary in the \((i + 1)\)-th round to \(\rho^{j+1}\). From this we can trace out the \(M_{i+1}^j\)-th register to show that

\[
\begin{align*}
D_{\infty} \left( \sigma_{i+2}^{j+2} X^{j+2} X^{j+2} F_{i+1}^{j+1} M_{i+1} \right) & \leq 2 \sum_{i' \in R^j, \ i' < i} c_{i'}, \\
& = 2 \sum_{i' \in R^j, \ i' < i} c_{i'},
\end{align*}
\]

The bound is similarly unchanged in the rounds \(i + 2, \ldots, i + l - 1\). Hence we can say that at the beginning of the next round \(i + l\) in which the \(j\)-th party communicates, it holds that

\[
\begin{align*}
D_{\infty} \left( \sigma_{i+l}^{j+l} X^{j+l} X^{j+l} F_{i+l}^{j+l} M_{i+l} \right) & \leq 2 \sum_{i' \in R^j, \ i' < i+l} c_{i'}, \\
& = 2 \sum_{i' \in R^j, \ i' < i+l} c_{i'},
\end{align*}
\]

\section*{C \ Proof of Lemma 6}

We shall show that if there is a quantum interactive protocol \(\mathcal{P}\) for \(V\) with \(c\) qubits of communication and error probability at most \(\epsilon\), over input distribution \(p\), then there is a zero-communication
quantum protocol $\mathcal{P}''$ which does not abort with probability $2^{-2c}$ worst case over all inputs, and when it does not abort it computes $V$ with the same error probability over $p$.

Firstly, we can use entanglement and teleportation to get a protocol $\mathcal{P}'$ from $\mathcal{P}$, which only involves at most $2c$ bits of classical communication (with the players doing measurements according to the classical messages they receive and their inputs, on their parts of a shared entangled state). We assume that the number of bits communicated in $\mathcal{P}'$ is of some fixed length every round for every input, with the total communication being $2c$ (this can be done by padding dummy bits if necessary).

Now in the zero-communication protocol $\mathcal{P}''$, the players will share the same initial entangled state as in $\mathcal{P}'$, and also $2c$ uniformly random classical bits. If player $j$ communicates in the $i$-th round, let $r_i = r_{i-1}^j \ldots r_{i-l+1}^j$ denote the portion of the shared randomness that corresponds to the bits in the $i$-th round of communication in $\mathcal{P}'$, with $r_k^j$ corresponding to the message to the $k$-th player. On inputs $x_1^l$, the players do the following in $\mathcal{P}''$:

- For each round $i$, if player $j$ is the one communicating in that round, player $j$ assumes $r_{i-l+1}^j \ldots r_{i-1}^j$ are the classical messages she has received from the other $l-1$ players between the $(i-l)$-th and the $i$-th round. They do a measurement on their part of the entangled state as she does in the $i$-th round of $\mathcal{P}'$, depending on $x_i^j$, their previous measurement outcomes, and messages from the other players. If $r_i$ is not compatible with her input and these measurement outcomes and previous messages, then player $j$ outputs $\perp$.

- At the end, if a player has not output $\perp$ yet, they output according to $\mathcal{P}'$.

Once the outputs of the measurements are fixed, the protocol is deterministic. So a transcript that is separately compatible for all the players, is compatible for all of them, and there is exactly one such transcript. $\{r_i\}_i$ is equal to this transcript with probability $2^{-2c}$, and hence no player outputs $\perp$ with probability $2^{-2c}$. When they do not output $\perp$, the transcript is correct for input $x_1^l$, and hence $\mathcal{P}''$ is correct with probability at least $1 - \varepsilon$ over the distribution $p$ on $x_1^l$, due to the correctness of $\mathcal{P}'$. 

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