Research Article

Some Monotonicity Properties of Gamma and $q$-Gamma Functions

Peng Gao

Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore 637371

Correspondence should be addressed to Peng Gao, penggao@ntu.edu.sg

Received 1 December 2010; Accepted 20 December 2010

Academic Editors: O. Miyagaki, G. Olafsson, and S. Zhang

Copyright © 2011 Peng Gao. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We prove some monotonicity properties of functions involving gamma and $q$-gamma functions.

1. Introduction

The $q$-gamma function is defined for positive real numbers $x$ and $q \neq 1$ by

$$
\Gamma_q(x) = (1 - q)^{1-x} \prod_{n=0}^{\infty} \frac{1 - q^{n+1}}{1 - q^{n+x}}, \quad 0 < q < 1,
$$

$$
\Gamma_q(x) = (q - 1)^{1-x} q^{(1/2)x(x-1)} \prod_{n=0}^{\infty} \frac{1 - q^{n+1}}{1 - q^{n+x}}, \quad q > 1.
$$

We note here according to [1] that the limit of $\Gamma_q(x)$ as $q \to 1^-$ gives back the well-known Euler’s gamma function:

$$
\lim_{q \to 1^-} \Gamma_q(x) = \Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \frac{dt}{t}.
$$

Note also that from the definition we have for positive $x$ and $0 < q < 1$,

$$
\Gamma_{1/q}(x) = q^{(x-1)(1-x/2)} \Gamma_q(x),
$$

$$
\Gamma_{1/q}(x) = q^{(x-1)(1-x/2)} \Gamma_q(x),
$$
we see that \( \lim_{q \to 1} \Gamma_q(x) = \Gamma(x) \). For historical remarks on gamma and \( q \)-gamma functions, we refer the reader to [1–3].

There exists an extensive and rich literature on inequalities for the gamma and \( q \)-gamma functions. For some developments in this area, we refer the reader to the articles in [2–7] and the references therein. Many of these inequalities follow from the monotonicity properties of functions which are closely related to \( \Gamma \) (resp., \( \Gamma_q \)) and its logarithmic derivative \( \psi \) (resp. \( \psi_q \)). Here we recall that a function \( f(x) \) is said to be completely monotonic on \((a, b)\) if it has derivatives of all orders and \((-1)^k f^{(k)}(x) \geq 0, x \in (a, b), k \geq 0 \) and \( f(x) \) is said to be strictly completely monotonic on \((a, b)\) if \((-1)^k f^{(k)}(x) > 0, x \in (a, b), k \geq 0 \). Lemma 2.3 below asserts that \( f(x) = e^{-h(x)} \) is completely monotonic on an interval if \( h' \) is. Following [8], we call such functions \( f(x) \) logarithmically completely monotonic. For similar notions of logarithmically completely monotonic functions and the history of the notions, we refer the reader to the articles in [9–12].

We note here that \( \lim_{q \to 1} \psi_q(x) = \psi(x) \) (see [13]) and that \( \psi' \) and \( \psi_q' \) are completely monotonic functions on \((0, +\infty)\) (see [5, 14]). Thus, one expects to deduce results on gamma and \( q \)-gamma functions from properties of (logarithmically) completely monotonic functions, by applying them to functions related to \( \psi' \) or \( \psi_q' \). It is our goal in this paper to obtain some results on gamma and \( q \)-gamma functions via this approach. As an example, we recall the following result of Bustoz and Ismail (the case \( q = 1 \)) as well as Ismail and Muldoon.

**Theorem 1.1** (see [15, Theorem 3], [4, Theorem 2.5]). Let \( a + 1 \geq b > a, \alpha = \max(-a,-c) \) and for \( q > 0 \) and \( q \neq 1 \), define

\[
\varphi_q(x; a, b, c) = \left( 1 - \frac{q^{x+c}}{1 - q} \right)^{a-b} \frac{\Gamma_q(x + b)}{\Gamma_q(x + a)}, \quad x > \alpha. \tag{1.4}
\]

For \( q = 1 \), let \( \varphi_1(x; a, b, c) = \lim_{q \to 1} \varphi_q(x; a, b, c) \). Then \( \varphi_q(x; a, b, c) \) is logarithmically completely monotonic on \((a, +\infty)\) if \( c \leq (a + b - 1)/2 \) and \( 1/\varphi_q(x; a, b, c) \) is logarithmically completely monotonic on \((a, +\infty)\) if \( c \geq a \).

It follows immediately from the above theorem that for \( 0 < q < 1 \) and \( 0 < s < 1 \), one has [4] for \( x > 0 \):

\[
\left( \frac{1 - q^{x+s/2}}{1 - q} \right)^{1-s} \leq \frac{\Gamma_q(x + 1)}{\Gamma_q(x + s)} \leq \left( \frac{1 - q^{x+s}}{1 - q} \right)^{1-s}. \tag{1.5}
\]

Alzer later [3] determined the best values \( u(q, s), v(q, s) \) such that the inequalities

\[
\left( \frac{1 - q^{x+u(q,s)}}{1 - q} \right)^{1-s} \leq \frac{\Gamma_q(x + 1)}{\Gamma_q(x + s)} \leq \left( \frac{1 - q^{x+v(q,s)}}{1 - q} \right)^{1-s}. \tag{1.6}
\]
hold for \( q > 0, 0 < s < 1, x > 0 \) to be

\[
\begin{align*}
  u(q,s) &= \begin{cases} 
    \frac{\ln((q^s - q)/(1-s)(1-q))}{\ln q}, & 0 < q < 1, \\
    \frac{s}{2}, & q > 1,
  \end{cases} \\
  v(q,s) &= \frac{\ln\left(1 - (1-q)\Gamma_q^{1/(s-1)}(s)\right)}{\ln q}.
\end{align*}
\] (1.7)

Motivated by the above results, we will show in Section 3 that the function \( g_q(x; s, 1, u(q,s)) \) as defined in Theorem 1.1 is logarithmically completely monotonic on \((0, +\infty)\) for \( 0 < q < 1 \). This will enable us to deduce the left-hand side inequality of Alzer’s result above for \( 0 < q < 1 \). The derivatives of \( \psi(x) \) are known as polygamma functions and in Section 4 we will prove some inequalities involving the polygamma functions.

### 2. Lemmas

The following lemma lists some facts about \( \psi(x) \) and \( \psi_q(x) \). These can be found, for example, in [3, equation (2.7)] and [5, equations (1.2)–(1.5)].

**Lemma 2.1.** For \( x > 0 \),

\[
\psi_q(x) = -\ln(1-q) + \ln q \sum_{n=1}^{\infty} q^{nx} \frac{1}{1-q^n}, \quad 0 < q < 1,
\] (2.1)

\[
(-1)^n \psi^{(n)}(x) = \int_0^x e^{-t} \frac{t^n}{1-e^{-t}} dt = n! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{n+1}}, \quad n \geq 1,
\] (2.2)

\[
\phi^{(n)}(x+1) = \psi^{(n)}(x) + (-1)^n \frac{n!}{x^{n+1}}, \quad n \geq 0,
\] (2.3)

\[
\phi(x) = \ln x - \frac{1}{2x} - \frac{1}{12x^2} + O\left(\frac{1}{x^3}\right), \quad x \to +\infty,
\] (2.4)

\[
(-1)^n \phi^{(n)}(x) = \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + O\left(\frac{1}{x^{n+2}}\right), \quad n \geq 1, \quad x \to +\infty.
\] (2.5)

The set of all the completely monotonic functions on an interval is equipped with a ring structure with the usual addition and multiplication of functions. The next two simple lemmas can also be used to construct new (logarithmically) completely monotonic functions; part of Lemma 2.2 is contained in [4, Lemma 1.3].

**Lemma 2.2.** If \( f(x) \) is completely monotonic on some interval \((a,b)\), then so is \( f(x) - f(x + c) \) on \((a,b) \cap (a-c,b-c)\) for any \( c > 0 \). Consequently, if \( f(x) \) is logarithmically completely monotonic on some interval \((a,b)\), then so is \( f(x)/f(x + c) \) on \((a,b) \cap (a-c,b-c)\) for any \( c > 0 \).

**Lemma 2.3** (see [15, Lemma 2.1]). If \( f'(x) \) is completely monotonic on an interval, then \( \exp(-f(x)) \) is also completely monotonic on the same interval.
Lemma 2.4. Let \( a_i \) and \( b_i \) (\( i = 1, \ldots, n \)) be real numbers such that \( 0 < a_1 \leq \cdots \leq a_n, 0 < b_1 \leq \cdots \leq b_n \), and \( \sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i \) for \( k = 1, \ldots, n \). If the function \( f(x) \) is decreasing and convex on \((0, +\infty)\), then

\[
\sum_{i=1}^{n} f(b_i) \leq \sum_{i=1}^{n} f(a_i).
\] (2.6)

If \( \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i \), then one only needs \( f(x) \) to be convex for the above inequality to hold.

The above lemma, except the last part, is a special case of Lemma 2 in [2]. This lemma follows from the theory of majorization, for example, see the discussions in [16].

Lemma 2.5 (Hadamard’s inequality). Let \( f(x) \) be a convex function on \([a,b]\), then

\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_{a}^{b} f(x)dx \leq \frac{f(a) + f(b)}{2}.
\] (2.7)

Lemma 2.6. For real numbers \( 0 < s, q < 1 \) and any integer \( n \geq 1 \),

\[
\left(\frac{q^s - q}{(1-s)(1-q)}\right)^n \geq \frac{q^{ns} - q^n}{(1-s)(1-q^n)}.
\] (2.8)

Proof. We can recast the above inequality as

\[
E(-1,s-1; q,1) \geq E(-n,n(s-1); q,1),
\] (2.9)

where for positive numbers \( x, y \) and real numbers \( r, s \) with \( rs(r-s)(x-y) \neq 0 \), we define

\[
E(r,s;x,y) = \left(\frac{r}{s} \cdot \frac{y^s - x^s}{y^r - x^r}\right)^{(s-r)}/(s-r).
\] (2.10)

The assertion of the lemma now follows from the fact that \( E(r,s;x,y) \) is increasing in both \( x \) and \( y \) and in both \( r \) and \( s \) (see, e.g., Theorem 1 in [17] for a proof of this).

Lemma 2.7. Let \( m > n \geq 1 \) be two integers, then for any fixed constant \( 0 < c < 1 \), the function

\[
a(t; m, n, c) = t^{m-n} + t^n - c(1 + t^n)
\] (2.11)

has exactly one root when \( t \geq 1 \).

Proof. We have

\[
a'(t; m, n, c) = t^{m-1}(m-n)t^n + nt^{n-m} - cm.
\] (2.12)

The function \( t \mapsto (m-n)t^n + nt^{n-m} - cm \) is clearly decreasing when \( t \geq 1 \). By considering the cases \( t = 1 \) and \( t \to +\infty \) we conclude that \( a'(t; m, n, c) \) has exactly one root when
It follows from this and Cauchy’s mean value theorem that \( a(t;m,n,c) \) has at most two roots when \( t \geq 1 \). This combined with the observation that \( a(1;m,n,c) > 0 \) and \( \lim_{t \to +\infty} a(t;m,n,c) < 0 \) yields the desired conclusion.

**Lemma 2.8.** For \( t \geq s \geq 0 \), one has

\[
\frac{s}{1 - e^{-s}} \cdot \frac{t - s}{1 - e^{-(t-s)}} \geq \frac{t}{1 - e^{-t}}. \tag{2.13}
\]

**Proof.** We write

\[
f(x) = \frac{x}{1 - e^{-x}} \tag{2.14}
\]

and we observe that

\[
\left( \frac{f''(t)}{f(t)} \right)' = \frac{e^{-t} \left( t^2 - (e^{t/2} - e^{-t/2})^2 \right)}{t^2 (1 - e^{-t})^2} \leq 0, \tag{2.15}
\]

where the last inequality follows from

\[
e^{t/2} - e^{-t/2} \geq t, \quad t \geq 0. \tag{2.16}
\]

The above inequality follows from the fact that the derivative of \( \sinh(t) - t \) is \( \cosh(t) - 1 > 0 \) for \( t > 0 \) and that \( \sinh(0) - 0 = 0 \). We now deduce that

\[
\frac{f'(t-s)}{f(t-s)} - \frac{f'(t)}{f(t)} \geq 0, \tag{2.17}
\]

for \( t \geq s \geq 0 \). This implies that the function \( t \mapsto \ln f(t-s) - \ln f(t) \) is increasing for \( t > s \). Thus we get

\[
\ln f(s) + \ln f(t-s) - \ln f(t) \geq \lim_{t \to s^+} (\ln f(s) + \ln f(t-s) - \ln f(t)) = 0, \tag{2.18}
\]

which is the desired result.

**3. Main Results**

**Theorem 3.1.** Let \( a_i \) and \( b_i \) (\( i = 1, \ldots, n \)) be real numbers such that \( 0 \leq a_1 \leq \cdots \leq a_n \), \( 0 \leq b_1 \leq \cdots \leq b_n \), and \( \sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i \) for \( k = 1, \ldots, n \). If \( f''(x) \) is completely monotonic on \((0, +\infty)\), then

\[
\exp \left( \sum_{i=1}^{n} (f(x + a_i) - f(x + b_i)) \right) \tag{3.1}
\]

is logarithmically completely monotonic on \((0, +\infty)\).
Proof. It suffices to show that

\[
-\sum_{i=1}^{n} (f'(x + a_i) - f'(x + b_i))
\]

is completely monotonic on \((0, +\infty)\) or for \(k \geq 1\),

\[
(-1)^k \sum_{i=1}^{n} f^{(k)}(x + a_i) \geq (-1)^k \sum_{i=1}^{n} f^{(k)}(x + b_i).
\]

By Lemma 2.4, it suffices to show that \((-1)^k f^{(k)}(x)\) is decreasing and convex on \((0, +\infty)\) or equivalently, \((-1)^k f^{(k+1)}(x) \leq 0\) and \((-1)^k f^{(k+2)}(x) \geq 0\) for \(k \geq 1\). The last two inequalities hold since we assume that \(f''(x)\) is completely monotonic on \((0, +\infty)\). This completes the proof. \(\square\)

As a direct consequence of Theorem 3.1, we now generalize a result of Alzer [2, Theorem 10].

**Corollary 3.2.** Let \(a_i\) and \(b_i\) \((i = 1, \ldots, n)\) be real numbers such that \(0 \leq a_1 \leq \cdots \leq a_n, 0 \leq b_1 \leq \cdots \leq b_n,\) and \(\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i\) for \(k = 1, \ldots, n\). Then,

\[
x \mapsto \prod_{i=1}^{n} \frac{\Gamma_q(x + a_i)}{\Gamma_q(x + b_i)}
\]

is logarithmically completely monotonic on \((0, +\infty)\).

**Proof.** Apply Theorem 3.1 to \(f(x) = \ln \Gamma_q(x)\) and note that \(f''(x) = \psi'_q(x)\) is completely monotonic on \((0, +\infty)\) and this completes the proof. \(\square\)

**Theorem 3.3.** Let \(f''(x)\) be completely monotonic on \((0, +\infty)\), then for \(0 \leq s \leq 1\), the functions

\[
x \mapsto \exp \left( -\left( f(x + 1) - f(x + s) - (1-s)f' \left(x + \frac{1+s}{2} \right) \right) \right),
\]

\[
x \mapsto \exp \left( f(x + 1) - f(x + s) - \frac{(1-s)}{2} \left( f'(x + 1) + f'(x + s) \right) \right)
\]

are logarithmically completely monotonic on \((0, +\infty)\).

**Proof.** We may assume \(0 \leq s < 1\). We will prove the first assertion and the second one can be shown similarly. It suffices to show that

\[
f'(x + 1) - f'(x + s) - (1-s)f'' \left(x + \frac{1+s}{2} \right)
\]

(3.6)
Corollary 3.4. For $0 \leq s \leq 1$, the functions

$$
\begin{align*}
&x \mapsto \frac{\Gamma_q(x+s)}{\Gamma_q(x+1)} \exp\left( (1-s)\psi_q\left( x + \frac{1+s}{2} \right) \right), \\
&x \mapsto \frac{\Gamma_q(x+1)}{\Gamma_q(x+s)} \exp\left( -\frac{(1-s)}{2} (\psi_q(x+1) + \psi_q(x+s)) \right)
\end{align*}
$$

are logarithmically completely monotonic on $(0, +\infty)$.

Proof. Apply Theorem 3.3 to $f(x) = \ln \Gamma_q(x)$ and note that $f''(x) = \psi_q'(x)$ is completely monotonic on $(0, +\infty)$ and this completes the proof. □

By applying Lemma 2.5 to $f(x) = -\psi_q(x)$, we obtain the following theorem.

Theorem 3.5. For positive $x$ and $0 \leq s \leq 1$,

$$
\exp\left( \frac{(1-s)}{2} (\psi_q(x+1) + \psi_q(x+s)) \right) \leq \frac{\Gamma_q(x+1)}{\Gamma_q(x+s)} \leq \exp\left( (1-s)\psi_q\left( x + \frac{1+s}{2} \right) \right).
$$

The upper bound in Theorem 3.5 is due to Ismail and Muldoon [4]. Our proof here is similar to that of Corollary 3 in [18].

Our next result refines the left-hand side inequality of (1.6) for $0 < q < 1$.

Theorem 3.6. Let $0 < s < 1$ and $0 < q < 1$. Let $u(q,s)$ be defined by (1.7) and let the function $g_q(x; s, 1, u(q,s))$ be defined as in Theorem 1.1. Then $g_q(x; s, 1, u(q,s))$ is logarithmically completely monotonic on $(0, +\infty)$.

Proof. Define

$$
h_q(x) = -\ln g_q(x; s, 1, u(q,s)) = -\ln \Gamma_q(x+1) + \ln \Gamma_q(x+s) + (1-s) \ln \left( \frac{1-q^{x+u(q,s)}}{1-q} \right).
$$

(3.10)

It suffices to show that $h'_q(x)$ is completely monotonic on $(0, +\infty)$. We have

$$
h'_q(x) = -\psi_q(x+1) + \psi_q(x+s) - (1-s) \ln q \frac{q^{x+u(q,s)}}{1-q^{x+u(q,s)}}.
$$

(3.11)
Using the expression (2.1), we can rewrite $h'_q(x)$ as

$$h'_q(x) = -\ln q \left( \sum_{n=1}^{\infty} \frac{q^nx (q^n - q^{ns})}{1 - q^n} + (1 - s) \frac{q^{x+u(q,s)}}{1 - q^{x+u(q,s)}} \right), \quad 0 < q < 1. \quad (3.12)$$

Expanding $(1 - q^{x+u(q,s)})^{-1}$, we may further rewrite $h'_q(x)$ as

$$h'_q(x) = -\ln q \sum_{n=1}^{\infty} \frac{q^nx}{1 - q^n} w_{q,n}(s), \quad 0 < q < 1, \quad (3.13)$$

where

$$w_{q,n}(c) = q^n - q^{ns} + (1 - s) q^{nu(q,s)} (1 - q^n). \quad (3.14)$$

In order for $h'_q(x)$ to be completely monotonic on $(0, +\infty)$, it suffices to show $w_{q,n}(s) \geq 0$ for $0 < s < 1$. This is just Lemma 2.6 and this completes the proof.

Theorem 3.6 implies that $g_q(x; s, 1, u(q, s)) > \lim_{x \to +\infty} g_q(x; s, 1, u(q, s)) = 1$, where the limit can be easily evaluated using (1.1) and we recover the left-hand side inequality of (1.6) for $0 < q < 1$.

For $\alpha \geq 0$, let

$$f_\alpha(x) = -\ln \Gamma(x) + \left( x - \frac{1}{2} \right) \ln x - x + \frac{1}{12} q'\Gamma(x + \alpha). \quad (3.15)$$

Alzer [19, Theorem 1] showed that for fixed $0 < s < 1$, the function $\exp(-(f_\alpha(x+s) - f_\alpha(x+1)))$ is strictly completely monotonic on $(0, +\infty)$ if and only if $\alpha \geq 1/2$ and the function $\exp(f_\alpha(x+s) - f_\alpha(x+1))$ is strictly completely monotonic on $(0, +\infty)$ if and only if $\alpha = 0$. In view of Lemmas 2.2 and 2.3, Alzer’s result would follow if one can show that $f'_\alpha(x)$ is strictly completely monotonic on $(0, +\infty)$ if and only if $\alpha \geq 1/2$ and that $-f'_\alpha(x)$ is strictly completely monotonic on $(0, +\infty)$ if and only if $\alpha = 0$. We now establish the above assertions in the following.

**Proposition 3.7.** For $\alpha \geq 0$, let $f_\alpha(x)$ be defined as in (3.15). Then $f'_\alpha(x)$ is strictly completely monotonic on $(0, +\infty)$ if and only if $\alpha \geq 1/2$ and $-f'_\alpha(x)$ is strictly completely monotonic on $(0, +\infty)$ if and only if $\alpha = 0$.

**Proof.** As $q'(x)$ is completely monotonic on $(0, +\infty)$, we may just focus on the case $\alpha = 1/2$ or 0. We have

$$f'_\alpha(x) = -q'(x) + \ln x - \frac{1}{2x} + \frac{1}{12} q''(x + \alpha), \quad f''_\alpha(x) = -q''(x) + \frac{1}{x^2} + \frac{1}{12} q'''(x + \alpha). \quad (3.16)$$

Using the asymptotic expressions (2.4) and (2.5), we see that $\lim_{x \to +\infty} f''_\alpha(x) = 0$ for any integer $n \geq 0$. It is then easy to see that the “if” part of the assertions of the proposition will
follow if we can show that $f''_{1/2}(x + 1) - f''_{1/2}(x)$ is strictly completely monotonic on $(0, +\infty)$ and $f''_0(x) - f''_0(x + 1)$ is completely monotonic on $(0, +\infty)$. Using (2.3), it is easy to see that

\[
f''_{1/2}(x + 1) - f''_{1/2}(x) = \frac{x^2 + x + 1/8}{4x^2(1 + x)^2(x + 1/2)^4} = \frac{1}{4(1 + x)^2(x + 1/2)^4} \left(1 + \frac{1}{x} + \frac{1}{8x^2}\right). \tag{3.17}
\]

It is easy to see from this and the ring structure of completely monotonic functions on $(0, +\infty)$ that $f''_{1/2}(x + 1) - f''_{1/2}(x)$ is strictly completely monotonic on $(0, +\infty)$. Similarly, one shows that $f''_0(x) - f''_0(x + 1)$ is completely monotonic on $(0, +\infty)$.

To prove the “only if” part of the assertions of the proposition, we note that

\[
f''_\alpha(x + 1) - f''_\alpha(x) = \frac{(4\alpha - 2)x^3 + (6\alpha^2 - 1)x^2 + 4\alpha^3x + \alpha^4}{2x^2(1 + x)^2(x + \alpha)^4}, \tag{3.18}
\]

from which we see easily by considering the case $x \to +\infty$ that $f''_\alpha(x)$ fails to be increasing if $\alpha < 1/2$. Similarly, by considering the case $x \to 0^+$ that $f''_\alpha(x)$ fails to be decreasing if $\alpha > 0$ and this completes the proof. 

\section*{4. Some Inequalities Involving Polygamma Functions}

For integers $p \geq m \geq n \geq q \geq 0$ and any real number $c$, we define

\[
F_{p,m,n,q}(x; c) = (-1)^{m+n}q^{(m)}(x)\psi^{(n)}(x) - c(-1)^{p+q}q^{(p)}(x)\psi^{(q)}(x). \tag{4.1}
\]

Here we set $q^{(0)}(x) = -1$ for convenience.

For $n \geq 2$, a result of Alzer and Wells [20, Theorem 2.1] asserts that the function $F_{n+1,n,n-1}(x; c)$ is completely monotonic on $(0, +\infty)$ if and only if $c < (n - 1)/n$ and $-F_{n+1,n,n-1}(x; c)$ is completely monotonic on $(0, +\infty)$ if and only if $c \geq n/(n + 1)$.

We denote

\[
c_{p,m,n,q} = \frac{(m - 1)!(n-1)!}{(p-1)!(q-1)!}, \quad c_{p,m,n,0} = \frac{(m - 1)!(n-1)!}{(p-1)!}, \tag{4.2}
\]

and note that $0 < c_{p,m,n,q}, d_{p,m,n,q} < 1$ when $p + q = m + n, p > m$. We now extend the result of Alzer and Wells to the following.

\textbf{Theorem 4.1.} Let $p > m \geq n \geq q \geq 0$ be integers satisfying $m + n = p + q$. The function $F_{p,m,n,q}(x; c_{p,m,n,q})$ is completely monotonic on $(0, +\infty)$. The function $-F_{p,m,n,q}(x; d_{p,m,n,q})$ is also completely monotonic on $(0, +\infty)$ when $q > 0$. 

Proof. We first prove the assertion for $F_{p,m,n,q}(x;c_{p,m,n,q})$ with $q \geq 1$. The proof here uses the method in [20]. Using the integral representation (2.2) for $(-1)^{n+1}q^{(m)}(x)$ and using $*$ for the Laplace convolution, we get

$$F_{p,m,n,q}(x;c_{p,m,n,q}) = \int_{0}^{\infty} e^{-xt} g(t) dt,$$

(4.3)

where

$$g(t) = \frac{t^m}{1 - e^{-t}} * \frac{t^n}{1 - e^{-t}} - c_{p,m,n,q} \frac{t^p}{1 - e^{-t}} * \frac{t^q}{1 - e^{-t}}$$

(4.4)

$$= \int_{0}^{t} ((t - s)^m s^n - c_{p,m,n,q}(t - s)^p s^q) h(t - s) h(s) ds,$$

with

$$h(s) = \frac{1}{1 - e^{-s}}.$$

(4.5)

It suffices to show that $g(t) \geq 0$. By change of variables $s \to ts$ we can recast it as

$$g(t) = t^{m+n+1} \int_{0}^{1} ((1 - s)^m s^n - c_{p,m,n,q}(1 - s)^p s^q) h(1 - s) h(ts) ds.$$  

(4.6)

We now break the above integral into two integrals, one from 0 to $1/2$ and the other from $1/2$ to 1. We make a further change of variable $s \to (1 - s)/2$ for the first one and $s \to (1 + s)/2$ for the second one. We now combine them to get

$$g(t) = \left(\frac{t}{2}\right)^{m+n+1} \int_{0}^{1} a \left(\frac{1 + s}{1 - s}; p - q, n - q, c_{p,m,n,q}\right) (1 - s^2)^q (1 - s)^{p-q}$$

$$\times h\left(\frac{t(1 - s)}{2}\right) h\left(\frac{t(1 + s)}{2}\right) ds,$$

(4.7)

where the function $a(t;m,n,c)$ is defined as in Lemma 2.7. Note that $(1 + s)/(1 - s) \geq 1$ for $0 \leq s < 1$, hence by Lemma 2.7, there is a unique number $0 < s_0 < 1$ such that

$$a\left(\frac{1 + s_0}{1 - s_0}; p - q, n - q, c_{p,m,n,q}\right) = 0.$$  

(4.8)

We further note it is shown in the proof of [20, Lemma 2.2] that the function

$$s \mapsto (1 - s^2) h\left(\frac{t(1 - s)}{2}\right) h\left(\frac{t(1 + s)}{2}\right)$$

(4.9)
is a decreasing function on \((0,1)\) so that for \(0 \leq s \leq 1\),

\[
a\left(\frac{1+s}{1-s};p-q,n-q, c_{p,m,n,q}\right) \left(1-s^2\right)^q (1-s)^{p-q} h\left(\frac{t(1+s)}{2}\right) h\left(\frac{t(1-s)}{2}\right) \\
\geq a\left(\frac{1+s}{1-s};p-q,n-q, c_{p,m,n,q}\right) \left(1-s^2\right)^{q-1} (1-s)^{p-q} \left(1-s^2_0\right) \\
\times h\left(\frac{t(1-s_0)}{2}\right) h\left(\frac{t(1+s_0)}{2}\right).
\]

Hence

\[
g(t) \geq \left(\frac{1}{2}\right)^{m+n-1} \left(1-s_0^2\right) h\left(\frac{t(1-s_0)}{2}\right) h\left(\frac{t(1+s_0)}{2}\right) \\
\cdot \int_0^1 a\left(\frac{1+s}{1-s};p-q,n-q, c_{p,m,n,q}\right) \left(1-s^2\right)^{q-1} (1-s)^{p-q} ds.
\]

Note that the integral above is (by reversing the process above on changing variables)

\[
2^{m+n-1} \int_0^1 \left((1-s)^{m-1} s^{n-1} - c_{p,m,n,q}(1-s)^{p-q} s^{q-1}\right) ds = 0,
\]

where the last step follows from the well-known beta function identity

\[
B(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y > 0,
\]

and the well-known fact \(\Gamma(n) = (n-1)!\) for \(n \geq 1\).

Now we prove the assertion for \(F_{p,m,n,0}(x; c_{p,m,n,0})\). In this case \(p = m + n\) and we note that

\[
c_{m+n,m,n,0} = B(m,n) = \int_0^1 s^{m-1} (1-s)^{n-1} ds,
\]

and we use this to write

\[
c_{m+n,m,n,0} \frac{t^{m+n}}{1-e^{-t}} = \int_0^t s^{m-1} (t-s)^{n-1} \frac{dt}{1-e^{-t}} ds.
\]
It follows that

\[
F_{p,m,n}(x; c_{p,m,n,0}) = \int_{0}^{\infty} e^{-xt} \left( \frac{t^{m}}{1-e^{-t}} - c_{m+n,m,n,0} \frac{t^{m+n}}{1-e^{-t}} \right) dt
\]

\[
= \int_{0}^{\infty} e^{-xt} \left( \int_{0}^{s} \frac{(t-s)^{n-1}}{1-e^{-(t-s)}} ds \right) dt
\]

\[
\geq 0,
\]

(4.16)

where the last inequality follows from Lemma 2.8.

It remains to show the assertion for \(-F_{p,m,n,q}(x; d_{p,m,n,q})\). In this case we use the series representation in (2.2) for \((-1)^{n+1} \varphi(n)(x)\) to get

\[
-F_{p,m,n,q}(x; d_{p,m,n,q}) = m!n! \left( \sum_{i=0}^{\infty} \frac{1}{(x+i)^{p+1}} \right) \left( \sum_{j=0}^{\infty} \frac{1}{(x+j)^{q+1}} \right) - \left( \sum_{i=0}^{\infty} \frac{1}{(x+i)^{m+1}} \right) \left( \sum_{j=0}^{\infty} \frac{1}{(x+j)^{n+1}} \right).
\]

(4.17)

We note the following Binet-Cauchy identity:

\[
\left( \sum_{i=0}^{n} a_{i}c_{i} \right) \left( \sum_{i=0}^{n} b_{i}d_{i} \right) - \left( \sum_{i=0}^{n} a_{i}d_{i} \right) \left( \sum_{i=0}^{n} b_{i}c_{i} \right) = \sum_{0 \leq i < j \leq n} (a_{i}b_{j} - a_{j}b_{i})(c_{i}d_{j} - c_{j}d_{i}).
\]

(4.18)

We now apply the above identity with

\[
a_{i} = \frac{1}{(x+i)^{m+1}}, \quad b_{j} = \frac{1}{(x+j)^{q+1}}, \quad c_{i} = \frac{1}{(x+i)^{p+1}}, \quad d_{i} = 1
\]

(4.19)

to get

\[
-F_{p,m,n,q}(x; d_{p,m,n,q}) = m!n! \sum_{0 \leq i < j} \left( \frac{1}{(x+i)^{p+1}(x+j)^{q+1}} \right) - \left( \frac{1}{(x+i)^{m+1}(x+j)^{n+1}} \right) \left( \frac{1}{(x+i)^{p+1}} - \frac{1}{(x+j)^{p+1}} \right).
\]

(4.20)
We note that the second factor on the right-hand side above is completely monotonic on \((0, +\infty)\) and also that

\[
\frac{1}{(x+i)^{m+1}(x+j)^{q+1}} - \frac{1}{(x+j)^{m+1}(x+i)^{q+1}} = \frac{1}{(x+i)^{q+1}} \frac{1}{(x+j)^{q+1}} \left( \frac{1}{(x+i)^{m-q}} - \frac{1}{(x+j)^{m-q}} \right).
\]

(4.21)

Certainly each factor on the right-hand side above is completely monotonic on \((0, +\infty)\) and it follows from the ring structure of completely monotonic functions on \((0, +\infty)\) that the left-hand side expression in (4.21) is also completely monotonic on \((0, +\infty)\). Hence by the ring structure of completely monotonic functions on \((0, +\infty)\) again we deduce that \(-F_{p,m,n,q}(x; d_{p,m,n,q})\) is completely monotonic on \((0, +\infty)\).

We note here that when \(q = 0\), the function \(-F_{m+n,m,n,0}(x; d_{m+n,m,n,0})\) is not completely monotonic on \((0, +\infty)\) in general, as we observe for example that it follows from (2.5) that as 

\[x \to +\infty, -F_{2,1,1,0}(x; d_{2,1,1,0}) < 0.\]

Corresponding to \(m = n = 1, p = 2, q = 0\) in Theorem 4.1, it was shown in the proof of [5, equation (4.39)] and in [21, Lemma 1.1] the following special case (in fact with strict inequality):

\[(q'(x))^2 + q''(x) \geq 0, \quad x > 0.\]

(4.22)

We note that inequality (4.22) follows from the limiting case \(c \to 0\) of the following inequalities for any \(0 < c < 1\) and \(x > 0\):

\[
\frac{1}{c}(q'(x+c) - q'(x))^2 > q'(x) - q'(x+c) > (q(x+c) - q(x))^2,
\]

(4.23)

where the above inequalities reverse when \(c > 1\).

Inequalities (4.23) are special cases of Theorem 1 in [10] as one can relate the inequalities in (4.23) to the properties concerning \(\Delta_{0,c:1}(x)\) or \(\Delta_{0,c:1/c}(x)\) defined in [10]. The left-hand side inequality of (4.23) was also established in the proof of [22, Theorem 1.1] and the right-hand side inequality in (4.23) was proven in [23, Lemma 7].

To end this paper, we prove a \(q\)-analogue to (4.23).

**Theorem 4.2.** Let \(0 < q < 1\) and \(0 < c < 1\) be fixed. Then for any \(x > 0\),

\[
\frac{1-q}{1-q^c} (q'_q(x+c) - q'_q(x))^2 > q^c (q'_q(x) - q'_q(x+c)) > (q_q(x+c) - q_q(x))^2.
\]

(4.24)

The above inequalities reverse when \(c > 1\).
Proof. We first prove the left-hand side inequality of (4.24). For this, we define

\[ f(x) = q^x \left( q_x'(x + c) - q_x'(x) \right) + \frac{1 - q}{1 - q^c} \left( q_x(x + c) - q_x(x) \right)^2. \]  

(4.25)

Applying (2.1), we obtain

\[ f(x) = (\ln q)^2 \sum_{n=1}^{\infty} nq^{(n+1)x} \left( q^{nc} - 1 \right) + \frac{1 - q}{1 - q^c} \left( \ln q \sum_{n=1}^{\infty} q^{nx} \left( q^{nc} - 1 \right) \right)^2 \]

\[ = \frac{(1 - q)(\ln q)^2}{1 - q^c} \sum_{n=2}^{\infty} q^{nx} g_n(c; q), \]  

where

\[ g_n(c; q) = \sum_{k=1}^{n-1} \left( \frac{(1 - q^k)(1 - q^{(n-k)c})}{(1 - q^k)(1 - q^{n-k})} - \frac{(1 - q^c)(1 - q^{(n-1)c})}{(1 - q)(1 - q^{n-1})} \right). \]  

(4.27)

It suffices to show that \( g_n(c; q) > 0 \) for \( 0 < c < 1 \) and \( g_n(c; q) < 0 \) for \( c > 1 \) when \( n \geq 2 \). For this, we let \( y = q^c \) so that \( 0 < y < 1 \) and it suffices to show the function \( (1 - y^k)(1 - y^{n-k})/(1 - y)(1 - y^{n-1}) \) is increasing for \( 0 < y < 1 \) and \( 1 \leq k \leq n - 1 \). On taking the logarithmic derivative of the above function, we see that it suffices to show that \( h(k; y) + h(n - k; y) \leq h(1; y) + h(n - 1; y) \), where

\[ h(z; y) = \frac{z}{1 - y^z}. \]  

(4.28)

We now regard \( h(z; y) \) as a function of \( z \) and note that

\[ h''(z; y) = \frac{(\ln y)y^z}{(1 - y^z)^3} u(y^z), \quad u(t) = 2 - 2t + \ln t + t \ln t. \]  

(4.29)

It is easy to see that \( u'(t) > 0 \) for \( 0 < t < 1 \) so that \( u(t) < u(1) = 0 \) for \( 0 < t < 1 \). It follows that \( h''(z; y) > 0 \). We then deduce from this and Lemma 2.4 that \( h(k; y) + h(n - k; y) \leq h(1; y) + h(n - 1; y) \) holds and this completes the proof for the left-hand side inequality of (4.24). For the right-hand side inequality of (4.24), one proceeds similarly to the above argument to see that it suffices to show the function \( (1 - y^k)(1 - y^{n-k})/(1 - y^{n-1}) \) is decreasing for \( 0 < y < 1 \) and \( 1 \leq k \leq n - 1 \). This follows from the observation that both functions \( (1 - y^k)/(1 - y^{n-1}) \) and \( 1 - y^{n-k} \) are decreasing and this completes the proof. \( \square \)

Acknowledgments

This work was partially carried out while the author was visiting the American Institute of Mathematics in fall 2005 and the Centre de Recherches Mathématiques at the Université de Montréal in spring 2006. The author would like to thank both the American Institute of
Mathematics and the Centre de Recherches Mathématiques at the Université de Montréal for their generous support and hospitality. The author is also grateful to the referees for their helpful comments and suggestions.

References

[1] T. H. Koornwinder, “Jacobi functions as limit cases of q-ultraspherical polynomials,” Journal of Mathematical Analysis and Applications, vol. 148, no. 1, pp. 44–54, 1990.
[2] H. Alzer, “On some inequalities for the gamma and psi functions,” Mathematics of Computation, vol. 66, no. 217, pp. 373–389, 1997.
[3] H. Alzer, “Sharp bounds for the ratio of q-gamma functions,” Mathematische Nachrichten, vol. 222, pp. 5–14, 2001.
[4] M. E. H. Ismail and M. E. Muldoon, “Inequalities and monotonicity properties for gamma and q-gamma functions,” in Approximation and Computation, vol. 119 of International Series of Numerical Mathematics, pp. 309–323, Birkhäuser, Boston, Mass, USA, 1994.
[5] H. Alzer, “Sharp inequalities for the digamma and polygamma functions,” Forum Mathematicum, vol. 16, no. 2, pp. 181–221, 2004.
[6] S.-L. Qi and M. Vuorinen, “Some properties of the gamma and psi functions, with applications,” Mathematics of Computation, vol. 74, no. 250, pp. 723–742, 2005.
[7] F. Qi, “Bounds for the ratio of two gamma functions,” Journal of Inequalities and Applications, vol. 2010, Article ID 493058, 84 pages, 2010.
[8] A. Z. Grinshpan and M. E. H. Ismail, “Completely monotonic functions involving the gamma and q-gamma functions,” Proceedings of the American Mathematical Society, vol. 134, no. 4, pp. 1153–1160, 2006.
[9] F. Qi and B.-N. Guo, “Complete monotonicities of functions involving the gamma and digamma functions,” RGMIA Research Report Collection, vol. 7, no. 1, article 8, pp. 63–72, 2004.
[10] F. Qi and B.-N. Guo, “Necessary and sufficient conditions for a function involving divided differences of the di- and tri-gamma functions to be completely monotonic,” http://arxiv.org/abs/0903.3071v1.
[11] F. Qi, B.-N. Guo, and C.-P. Chen, “Some completely monotonic functions involving the gamma and polygamma functions,” RGMIA Research Report Collection, vol. 7, no. 1, article 5, pp. 31–36, 2004.
[12] F. Qi, S. Guo, and B.-N. Guo, “Complete monotonicity of some functions involving polygamma functions,” Journal of Computational and Applied Mathematics, vol. 233, no. 9, pp. 2149–2160, 2010.
[13] C. Krattenthaler and H. M. Srivastava, “Summations for basic hypergeometric series involving a q-analogue of the digamma function,” Computers & Mathematics with Applications, vol. 32, no. 3, pp. 73–91, 1996.
[14] H.-H. Kairies and M. E. Muldoon, “Some characterizations of q-factorial functions,” Aequationes Mathematicae, vol. 25, no. 1, pp. 67–76, 1982.
[15] I. Bustoz and M. E. H. Ismail, “On gamma function inequalities,” Mathematics of Computation, vol. 47, no. 176, pp. 659–667, 1986.
[16] A. W. Marshall and I. Olkin, Inequalities: Theory of Majorization and Its Applications, vol. 143 of Mathematics in Science and Engineering, Academic Press, New York, NY, USA, 1979.
[17] F. Qi and Q.-M. Luo, “A simple proof of monotonicity for extended mean values,” Journal of Mathematical Analysis and Applications, vol. 224, no. 2, pp. 356–359, 1998.
[18] M. Merkle, “Convexity, Schur-convexity and bounds for the gamma function involving the digamma function,” The Rocky Mountain Journal of Mathematics, vol. 28, no. 3, pp. 1053–1066, 1998.
[19] H. Alzer, “Some gamma function inequalities,” Mathematics of Computation, vol. 60, no. 201, pp. 337–346, 1993.
[20] H. Alzer and J. Wells, “Inequalities for the polygamma functions,” SIAM Journal on Mathematical Analysis, vol. 29, no. 6, pp. 1459–1466, 1998.
[21] N. Batir, “Some new inequalities for gamma and polygamma functions,” Journal of Inequalities in Pure and Applied Mathematics, vol. 6, no. 4, article 103, 9 pages, 2005.
[22] C.-P. Chen, “Monotonicity and convexity for the gamma function,” Journal of Inequalities in Pure and Applied Mathematics, vol. 6, no. 4, article 100, 6 pages, 2005.
[23] H. Alzer, “Sub- and superadditive properties of Euler’s gamma function,” Proceedings of the American Mathematical Society, vol. 135, no. 11, pp. 3641–3648, 2007.
