Higher dimensional nonclassical eigenvalue asymptotics

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March 11, 2014

Abstract
In this article we extend B. Simon’s construction and results [11] for leading order eigenvalue asymptotics to $n$-dimensional Schrödinger operators with non-confining potentials given by: $H_\alpha^n = -\Delta + \prod_{i=1}^n |x_i|^\alpha_i$ on $\mathbb{R}^n$ ($n > 2$), $\alpha := (\alpha_1, \cdots, \alpha_n) \in (\mathbb{R}^*_+)^n$. We apply the results to also derive the leading order spectral asymptotics in the case of the Dirichlet Laplacian $-\Delta^D$ on domains $\Omega_\alpha^n = \{x \in \mathbb{R}^n : \prod_{j=1}^n |x_j|^{\frac{n-1}{n}} < 1\}$.

keywords: Trace formulae; Schrödinger operators; Singular asymptotics.
1 Introduction and main results.

Since the seminal work of Weyl [14], and its generalizations the eigenvalue asymptotics of the Laplacian $-\Delta$ on compact domains $\Omega \subset \mathbb{R}^n$ with various boundary conditions have been understood to encode information about the geometry of the domain. Let $(\lambda_j)_{j \in \mathbb{N}}$ denote the sequence of eigenvalues of $-\Delta$ on such a domain endowed with Dirichlet boundary conditions. Let furthermore $N(E) = \{j \mid \lambda_j \leq E\}$ be the counting function of eigenvalues. Not even any regularity of the boundary $\partial \Omega$ is required for the Weyl law (see [3]):

$$N(E) = \frac{\text{vol}(S^{n-1})}{(2\pi)^n} \text{vol}(\Omega) E^{n/2} + o(E^{n/2}), \quad \text{as } E \to \infty,$$

This, and further research on additional terms in the asymptotic expansion led to the famous question of M. Kac: "Can one hear the shape of a drum?" [8]. However, despite the appealing simplicity of this leading order asymptotics, neither compactness nor finite volume of $\Omega$ are necessary conditions for purely discrete spectrum of the Dirichlet Laplacian, denoted from now on by $-\Delta^D$.

A class of 2-dimensional examples for infinite volume domains with discrete spectrum, as well as their leading order eigenvalue asymptotics was given by B. Simon in [11]. He considered domains of the form $\Omega_\alpha := \{(x, y) \mid |x|^{\alpha} < 1\} \subset \mathbb{R}^2$ with $\alpha > 0$ and derived the asymptotics:

$$N(E) = \begin{cases} 
\zeta(\alpha) (\frac{\pi}{2})^{-\alpha} \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi \Gamma\left(\frac{\alpha+1}{2}+\frac{1}{2}\right)}} E^{\alpha+1} + o(E^{\alpha+1}), & \alpha > 1 \\
\frac{1}{\pi} E \ln E + o(E \log(E)), & \alpha = 1 
\end{cases}$$

For $1 > \alpha > 0$ the first formula holds if one replaces $\alpha$ by $\alpha^{-1}$. This article is concerned with an extension of this example of eigenvalue asymptotics to higher dimensions. To this end we will determine the spectral asymptotics for (most members of) the class of Schrödinger operators given by:
\[ H_n^\alpha = -\Delta + \prod_{i=1}^{n} |x_i|^{\alpha_i}, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad \alpha_i \in \mathbb{R}_+^*. \]

If \( n \neq 1 \) their potentials are non-confining, yet their spectrum is purely discrete as we will see shortly. In turn this will enable us to find the eigenvalue asymptotics of the Dirichlet Laplacian \(-\Delta^D\) on domains of the form:

\[ \Omega_n^\alpha = \{ x \in \mathbb{R}^n : \prod_{j=1}^{n} |x_j|^{|\alpha_j|/\alpha_n} < 1 \} \]

**Remark 1** Define the classical energy surfaces:

\[ \Sigma_E = \{(x, \xi) \in T^*\mathbb{R}^n : ||\xi||^2 + V(x) = E\}, \]

equipped with the (classical flow invariant) Liouville measures \( d\text{LVol}_E \), and the classical areas:

\[ V(E) = \bigcup_{\Sigma \leq E} \Sigma = \{(x, \xi) \in T^*\mathbb{R}^n : ||\xi||^2 + V(x) \leq E\}, \]

for the usual Lebesgues measure \( d\mu \). Then, for any choice of the powers \( \alpha_i \), we deal with **non-compact energy surfaces of infinite volume**, i.e. \( \text{Lvol}_E(\Sigma_E) = +\infty \). Since we also have \( \mu(V(E)) = +\infty \) for \( E > 0 \), Weyl’s law does apply, neither in the classical nor the micro-local category.

Despite this fact, B. Simon determined the asymptotics of the Schrödinger operators above in the case \( n = 2 \). We will extend the calculation of these asymptotics to dimensions \( n > 2 \) covering both the generic case \( \alpha_i \neq \alpha_j, \forall i, j \) as well as the most singular case \( \alpha_i = \alpha_j \forall i, j \). The former case is joint work, while the latter is a result of the first author. Note that we can freely assume \( \alpha_1 \geq \cdots \geq \alpha_n > 0 \). Our main results then read as follows:

**Theorem 2** For \( n \geq 2 \), assume \( \alpha_1 > \cdots > \alpha_n \geq 0 \) define:

\[ d_n = d_n(\alpha) := \frac{\alpha_1 + \cdots + \alpha_n + 1/2}{2\alpha_n}. \]
Then, as $t$ tends to $0^+$, we have:

$$\lim_{t \to 0^+} t^{(d_n+1/2)} \text{Tr}(e^{-tH_n^\alpha}) = \frac{\text{Tr}((H_{n-1}^\alpha)^{-d_n})}{\pi^{n/2}} \Gamma(d_n + 1),$$

where:

$$\text{Tr}((H_{n-1}^\alpha)^{-d_n}) = \text{Tr}((-\Delta_{x_1...x_{n-1}} + \prod_{i=1}^{n-1} |x_i|^\alpha)^{-d_n}) < \infty,$$

is the spectral-zeta function, evaluated at $d_n$, of the $(n-1)$-dimensional Schrödinger operator obtained by removing the direction of smallest decay at infinity.

Assuming only that $\alpha_1 = \cdots = \alpha_n = \alpha_0 \in \mathbb{R}$ we prove:

**Theorem 3** If $\alpha_1 = \cdots = \alpha_n = \alpha_0$, and thus $d_n = \frac{n-1}{2} + \alpha_0^{-1}$, then as $t$ tends to $0^+$ we have:

$$\lim_{t \to 0^+} t^{n/2+\alpha_0^{-1}} \ln(t)^{(n-1)} \text{Tr}(e^{-tH_n^\alpha}) = \frac{(1 + \alpha_0^{-1})(n/2 + \alpha_0^{-1})^{(n-1)}}{\pi^{n/2}(n-1)!}.$$

Subsequently, using the Tauberian theorem of Karamata, cf. [11], we can prove that the eigenvalue counting functions $N_{H_n^\alpha}$ given by:

$$N_{H_n^\alpha}(E) = \#\{j \in \mathbb{N} : \lambda_j \in \sigma(H_n^\alpha) \leq E\}$$

satisfy the following asymptotic laws as $E \to \infty$:

**Theorem 4** For $n \geq 2$, and assuming $\alpha_1 > \cdots > \alpha_n > 0$ we have:

$$\lim_{E \to \infty} E^{-(d_n+1/2)} N_{H_n^\alpha}(E) = \frac{\text{Tr}((H_{n-1}^\alpha)^{-d_n})\Gamma(d_n + 1)}{\pi^{n/2}\Gamma(d_n + 3/2)},$$

where $d_n$ and $\text{Tr}((H_{n-1}^\alpha)^{-d_n})$ are as in Theorem 1.

**Theorem 5** If $\alpha_1 = \cdots = \alpha_n = \alpha_0$, then we have:

$$\lim_{E \to \infty} E^{-(n/2+\alpha_0^{-1})} \log(E)^{(n-1)} N_{H_n^\alpha}(E) = \frac{(1 + \alpha_0^{-1})(n/2 + \alpha_0^{-1})^{(n-1)}}{\Gamma(n/2 + \alpha_0^{-1})\pi^{n/2}(n-1)!}.$$
Finally, this will imply the following result for the spectrum of the Dirichlet Laplacian $-\Delta^D$ on the domains $\Omega^n_{\alpha}$:

**Theorem 6** For $\alpha_1 > \cdots > \alpha_n$ the Dirichlet-Laplacian $-\Delta^D = -\Delta_{D,n}^{\alpha}$ attached to the domain $\Omega^n_{\alpha}$, $n \geq 2$ has discrete spectrum and the counting function of eigenvalues satisfies:

$$\lim_{E \to \infty} E^{-(q(\alpha)+\frac{1}{2})} N_{\Delta_{D,n}^{\alpha}}(E) = \frac{\text{Tr}((-\Delta_{D,n}^{\alpha})^{-q(\alpha)})\Gamma(q(\alpha) + 1)}{\pi^{n/2} \Gamma(q(\alpha) + 3/2)}.$$  

where $-\Delta_{D,n-1}^{\alpha}$ is the Dirichlet Laplacian on the $n-1$-dimensional domain obtained by projecting $\Omega^n_{\alpha}$ onto the $x_n = 0$ hyperplane and

$$q(\alpha) = \frac{\alpha_1 + \ldots + \alpha_{n-1}}{2 \alpha_n}.$$  

In the case $\alpha_1 < \cdots < \alpha_n$, the same formula holds by replacing $q(\alpha)$ with $q(\alpha)^{-1}$.

**Theorem 7** The Dirichlet-Laplacian $-\Delta^D$ on $\Omega := \{x \in \mathbb{R}^n \mid \prod_{i=1}^{n} |x_i| < 1\}$, $n \geq 2$ has discrete spectrum and:

$$\lim_{E \to \infty} \log(E)^{-(n-1)} E^{-\frac{n}{2}} N_{\Delta^D}(E) = \frac{n^{n-1}}{\Gamma\left(\frac{n}{2}\right) \sqrt{\pi^{n-1}} 2^{n-1} (n-1)!}.$$  

Some extensions of the original example have already been investigated in other works. In [1], Aramaki and Nurmuhammad have considered the potentials:

$$V(z) = V(x, y) = ||x||^{2p} ||y||^{2q}, p, q > 0,$$

and derived the leading order asymptotics for these cases. The main contribution of our work is to show that more than two exponents can be considered in the definition of the domain and potentials and the asymptotics can still be computed.
Before we begin our analysis we recall that for a Schrödinger operator with a positively homogeneous potential of degree $p \neq -2$, i.e. $V(tx) = |t|^p V(x)$ for all $x$, we have the following scaling relations for all $c > 0$:

$$
\sigma(-\Delta + cV) = c^{\frac{2}{p+2}} \sigma(-\Delta + V),
$$

(1)

$$
\sigma(-c\Delta + V) = c^{\frac{p}{p+2}} \sigma(-\Delta + V).
$$

(2)

These are equalities between spectra and these scaling relations are independent of the dimension $n$.

Following the strategy of B. Simon, who approached the two-dimensional case with what he calls the 'sliced-bread inequalities', we recall now for the readers convenience the classical Tauberian theorem relating the small time behavior of the quantum partition function

$$
Z_Q(t) = \text{Tr}(e^{-tH}),
$$

and the large energy behavior of the counting function $N_H(E)$ as seen in [11]:

**Theorem 8 (Karamata’s Tauberian Theorem.)**

Let $H = -\Delta + V$ with $V$ continuous and non-negative. When $l$ and $d$ are positive, we have the equivalences:

$$
\lim_{E \to +\infty} E^{-l} N_H(E) = c \Leftrightarrow \lim_{t \to 0^+} t^l \text{Tr}(e^{-tH}) = c \Gamma(l + 1),
$$

$$
\lim_{E \to +\infty} \frac{E^{-l}}{\log E} d N_H(E) = c \Leftrightarrow \lim_{t \to 0^+} \frac{t^l}{|\log(t)|^d} \text{Tr}(e^{-tH}) = c \Gamma(l + 1).
$$

So we will concentrate on the short time asymptotics of $Z_Q$. As we already remarked the infinite volume of the energy surfaces provides a serious obstacle.

For example, in the case of the operators we are interested in the common trick to estimate the quantum partition function by a classical integral fails for precisely this reason. Indeed if we define:

$$
Z_{cl}(t) = \frac{1}{(2\pi)^n} \int_{T^*\mathbb{R}^n} e^{-t(|\xi|^2 + V(x))} dx d\xi,
$$
the inequality $Z_Q(t) \leq Z_{cl}(t)$ is valid (this can be viewed as a convexity property of the exponential function or a consequence of the abstract Golden-Thompson inequality), but for the product potential:

$$V_\alpha(x) := \prod_{i=1}^{n} |x_i|^{\alpha_i}$$

we get:

$$(2\pi)^n Z_{cl}(t) = \left(\frac{\pi}{t}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-tV_\alpha(x)} \, dx$$

$$= \left(\frac{4\pi}{t}\right)^{\frac{n}{2}} \Gamma\left(\frac{\alpha_n + 1}{\alpha_n}\right) \int_{\mathbb{R}^{n-1}} \left(\frac{\prod_{j=1}^{n-1} |x_j|^{\alpha_j}}{\alpha_n}\right)^{-\frac{1}{\alpha_n}} \, dx_1 \ldots dx_{n-1}$$

$$= \left(\frac{4\pi}{t}\right)^{\frac{n}{2}} \Gamma\left(\frac{\alpha_n + 1}{\alpha_n}\right) t^{\frac{1}{\alpha_n}} \prod_{j=1}^{n-1} \int_{0}^{\infty} x_j^{-\frac{\alpha_j}{\alpha_n}} \, dx_j = +\infty,$$

none of these integrals being convergent on $[0, \infty]$, for any choice of the $\alpha_j$ (the convergence at the origin implying the divergence at infinity and vice-versa). Despite the fact that the classical estimate does not yield any useful information, we can easily obtain:

**Lemma 9** For any $n$ and any $\alpha \in (\mathbb{R}^*_+)^n$, the spectrum of $H^\alpha_n$ is discrete.

Let us give first a rough proof that the $H^\alpha_n$’s have discrete spectrum. This approach will not even catch the right asymptotic power. In dimension 2, this was given by B. Simon in [10] as the most elementary proof between 6 different proofs of the discreteness of the spectrum.

**Proof of Lemma 9.** Using that the spectrum of the 1 dimensional operators:

$$-\Delta + |x|^{\nu}, \nu > 0,$$

has a strictly positive lowest eigenvalue $\lambda_0(\nu)$, by scaling (see Eq.(1)) we get the
following lower bound valid in the sense of quadratic forms on $H^1(\mathbb{R}^n)$:

$$-\Delta x_1, \ldots, x_n + |x_1|^{\alpha_1} \prod_{j=2}^{n} |x_j|^{\alpha_j} \geq -\Delta x_2, \ldots, x_n + \lambda_0(\alpha_1)(\prod_{j=2}^{n} |x_j|^{\alpha_j})^{\frac{2}{2n-1}}.$$  

By induction, and symmetrization w.r.t. $x_1, \ldots, x_n$, it is easy to show that there exists $n$ positive continuous functions $f_i : \mathbb{R} \to \mathbb{R}_+$ with $\lim_{t \to \pm \infty} f_i(t) = +\infty$ such that:

$$-\Delta + \prod_{j=1}^{n} |x_j|^{\alpha_j} \geq \sum_{j=1}^{n} (-\Delta x_j + f_j(x_j)) = \sum_{j=1}^{n} T_j.$$  

A fortiori, e.g. by a min-max argument, the spectrum of $H_\alpha^n$ is discrete each of the operators $T_j$ appearing in the r.h.s. being a 1-dimensional Schrödinger operator with confining potential.

\[ \blacksquare \]

**Remark 10** As it was observed in the 2 dimensional case, the inequality:

$$Z_Q(t) = \text{Tr} e^{-tH_\alpha^n} \leq \prod_{j=1}^{n} \text{Tr} e^{-tT_j},$$

can be exploited to get an upper bound, e.g. using Bohr-Sommerfeld quantization conditions for each $T_j$ (see [2], chapter 5). But these bounds are not good in view of the results stated in Theorem 2 and 3.

**Comments and perspectives:**

To our knowledge very few things are known about the interpretation of the results stated in Theorem 2 and 3 in terms of geometry, physics or dynamical systems. For example, a geometrical interpretation of the constants appearing in the short time expansion of $e^{-tH_\alpha^n}$ cannot have a 'classical' meaning (at least for the usual symplectic structure of the phase space $T^*\mathbb{R}^n$). At least these coefficients can be used to construct some (global) measures on eigenvectors:

$$w(a) = \lim_{t \to 0^+} \frac{\text{Tr} A e^{-tH_\alpha^n}}{\text{Tr} e^{-tH_\alpha^n}}, A = \text{Op}(a), a \in S^0(\mathbb{R}^{2n}),$$

\[ ^1 \text{One could also exponentiate the functional inequality and take the trace.} \]
i.e. the asymptotic results stated in theorems 2 and 3 provides some normalization factors (in terms of probability measures). Also, because of the strong singularities of the potential (and of the Liouville measure) on any hypersurface \( \{ x_j = 0 \} \), one could expect some concentration phenomena for the associated eigenfunctions estimates like it was observed in [4, 5].

Finally the approach based on the asymptotic behavior of the quantum partition function \( Z_Q(t) \) does not allow to see very much concerning the classical dynamics generated by the potentials \( V_\alpha \) (when the dynamics is globally defined in the usual way, meaning that the Hamiltonian vector field has the Lipschitz regularity). But, as a matter of honesty, the usual semi-classical methods and their underlying stationary-phase approximations seem to be inefficient because of the infinite volume of energy surfaces.

We will conclude the introduction with the proof of theorems 6 and 7. Of course, because of the scaling properties of our operators similar results are valid for Dirichlet-Laplacian in domains \( \Omega^n_\alpha(a) = \{ x \in \mathbb{R}^n : |x_1|^\alpha_1 \cdots |x_n|^\alpha_n < a \} \) for any positive \( a \).

**Proof of theorem 6 and 7:** The proof of these results is straightforward if we use a sequence of potentials with strictly increasing exponents \( (\alpha)_j = j.\alpha = (j\alpha_1, \cdots, j\alpha_n), j > 0 \). The ratios of theorem 2 satisfy:

\[
d_{n}(\alpha)_j = \frac{j\alpha_1 + \cdots + j\alpha_{n-1} + 2}{2j\alpha_n} \rightarrow \frac{\alpha_1 + \cdots + \alpha_{n-1}}{2\alpha_n} \text{ as } j \rightarrow +\infty.
\]

On the other side, if \( j \) tends to infinity by homogeneity we have:

\[
V_{(\alpha)_j} = (V_\alpha)^j \rightarrow \begin{cases} 
0 & \text{if } V_\alpha < 1, \\
1 & \text{if } V_\alpha = 1, \\
+\infty & \text{if } V_\alpha > 1.
\end{cases}
\]
By taking the exponential (the potentials are everywhere positive) we get the desired result since \( e^{-t(-\Delta + V(\alpha))} \) converges to \( e^{-t(-\Delta_{\alpha}^D)} \) as \( j \to \infty \) in the trace norm.

The rest of the paper is organized as follows: in section 2 we recall the sliced bread estimate used to get good upper bound on \( Z_Q(t) \). Section 3 then contains the proof of Theorem 2, while in section 4 we prove the technically more involved asymptotics of Theorem 3.

## 2 Slicing techniques for the partition function:

As was noted in the last section, the Tauberian theorem of Karamata allows us to focus on the small time divergence of the partition function \( Z_Q(t) \). When analyzing the trace of an operator of this type, it is useful to 'slice' the problem. That is, we write an operator \( A = -\Delta + V(x, y), (x, y) \in \mathbb{R}^a \times \mathbb{R}^b = \mathbb{R}^c \) on \( L^2(\mathbb{R}^c) \), with \( V(x, y) \) continuous and bounded from below as a sum:

\[
A = -\Delta_x + A_x
\]

Here \( A_x = -\Delta_y + V(x, y) \) as an operator on \( L^2(\mathbb{R}^b) \) depending on \( x \). Let \( \lambda_k(x) \) be the increasing sequence of eigenvalues of \( H_x \), repeated according to their multiplicities. Define:

\[
Z_{SB}(t) = \sum_k \text{Tr}_{L^2(\mathbb{R}^a)}(e^{-t(\Delta_x + \lambda_k(x))}),
\]

\[
Z_{SGT}(t) = \int \frac{e^{-tp^2}}{(2\pi)^a} \text{Tr}_{L^2(\mathbb{R}^b)}(e^{-t(H_x)})d^apd^a x,
\]

\[
Z_{CL}(t) = \int \frac{e^{-tp^2 + V(x,y)}}{(2\pi)^c}d^cpd^a xd^b y.
\]

Here SB stands for sliced bread, SGT is sliced Golden-Thompson. The 'sliced-bread' and 'sliced-Golden-Thompson' techniques are now centered around the
following theorem, which is due to B. Simon [11]:

**Theorem 11 (Barry Simon’s sliced bread inequalities.)**

For each $t > 0$ we have:

$$Z_Q(t) \leq Z_{SB}(t) \leq Z_{SGT}(t) \leq Z_{cl}(t).$$

The potentials we deal with in this paper provide examples where $Z_{cl}(t) = \infty$ and, depending on the choice of $\alpha_i$, even $Z_{SGT}(t) = \infty$, yet the traces $Z_{SB}(t)$ and $Z_Q(t)$ exist. This provides a set of examples where these estimates prove to be more powerful than the classical one. In the case of the potentials covered in this paper, working with either $Z_{SGT}(t)$ or $Z_{SB}(t)$ leads to studying functions of the type:

$$F(x_n, t) = \text{Tr}_{L^2(dx_1...dx_{n-1})} \left( \exp\left[ -t(\Delta x_1...x_{n-1} + \prod_{i=1}^{n} |x_i|^{\alpha_i}) \right] \right)$$

This function satisfies a remarkable functional equation:

**Lemma 12** The sliced bread function satisfies the following scaling relation:

$$F(x_n, t) = F(x_n t^{d_n}, 1) = F(1, t |x_n|^{1/d_n}).$$

**Proof.** The homogeneity of the potential is crucial in proving this. Use the scaling relations of Eqs. [12] and thus show:

$$F(x_n, t) = \text{Tr}_{L^2(dx_1...dx_{n-1})} \left( \exp\left[ -t(\Delta x_1...x_{n-1} + \prod_{i=1}^{n} |x_i|^{\alpha_i}) \right] \right)$$

$$= \text{Tr}_{L^2(dx_1...dx_{n-1})} \left( \exp\left[ -(\Delta x_1...x_{n-1} + t^{d_n}x_n |x_n|^{\alpha_n} \prod_{i=1}^{n-1} |x_i|^{\alpha_i}) \right] \right)$$

$$= \text{Tr}_{L^2(dx_1...dx_{n-1})} \left( \exp\left[ -t|x_n|^{(1/d_n)}(\Delta x_1...x_{n-1} + \prod_{i=1}^{n-1} |x_i|^{\alpha_i}) \right] \right).$$

For the first equality set $c = t$ and apply (1) then (2), for the second one set $c = |x_n|$ and work in reverse order. □
3 Eigenvalue asymptotics, $\alpha_i \neq \alpha_j, \ \forall i \neq j$

This case is the simpler due to the finiteness of $Z_{SGT}(t)$. When all indices are
different, up to permutation of coordinates we can assume that:

$$V_\alpha(x) = \prod_{i=1}^{n} |x_i|^{\alpha_i}, \text{ with } \alpha_1 > ... > \alpha_n.$$  

The proof will be by induction over dimension. The dimension 2 case was shown
by B. Simon in [11]. Suppose now that in dimension $n - 1$, we have:

$$\lim_{t \to 0} t^{(dn-1+1/2)} \text{Tr}(e^{-tH^\alpha}) = \text{Tr}((H^\alpha)_{n-2}^{d_{n-1}}) \pi^{-1/2} \Gamma(d_{n-1} + 1)$$

We will estimate $Z_Q(t)$ from above and below show that their difference asymptotically goes to zero and compute the asymptotics of the upper bound. The
lower bound will be found using the Feynman-Kac formula that gives a represen-
tation of the trace of the heat kernel as an expectation value of Brownian
motion. The upper bound will be found using the sliced bread inequalities. We
will slice in direction of the coordinate of smallest power in the potential, $x_n$.

Remark 13 The slicing for $Z_{SGT}(t)$ works only if one takes slices in the right
direction, that is, the one with smallest exponent in the potential. As we will
see one has no choice, as the integrals will not converge if one slices in a different
way.

From sliced bread inequalities, we get $Z_Q(t) \leq Z_{SGT}(t)$. Let us first rewrite
this upper bound a little. Doing the $p$-integral explicitly, we get:

$$Z_{SGT}(t) = (\pi t)^{-1/2} \int_0^\infty F(x_n, t) dx_n.$$  

Lower bound: Next we will prove a lower bound that is easy to compare to
the expression of $Z_{SGT}$ as an integral of $F$. Using the Feynman-Kac formula we
rewrite $Z_Q(t)$ just like Barry Simon did in the two dimensional case, see [9, 11]:

$$Z_Q(t) = (4\pi t)^{-n/2} \int_{x \in \mathbb{R}^n} E_{x,x;2t}[\exp\left(-\int_0^{2t} \frac{1}{2}|b_1(s)|^{\alpha_1} \ldots |b_{n-1}(s)|^{\alpha_{n-1}} |b_n(s)|^{\alpha_n} ds\right)] dx,$$

where $b_i$ denotes the 1-dimensional Brownian motion and $E_{x,x;2t}$ is the conditional expectation value w.r.t. the Brownian motion with conditions to start and end in $x$ in time $2t$. We proceed by cutting off all paths such that:

$$\sup_{0 \leq s \leq 2t} |b_n(s) - x_n| > 1,$$

and replacing $|b_n(s)|^{\alpha_n}$ by its upper bound $(|x_n|+1)^{\alpha_n}$. Using that our potential is a product function, and since the probability measure of the $n$-dimensional Brownian motion is a product measure, this gives a lower bound for $Z_Q(t)$, namely:

$$Z_Q(t) \geq (\pi t)^{-\frac{1}{2}} (1 - \rho(t)) \int_0^{-\infty} F(|x_n| + 1, t) dx_n$$

$$= (\pi t)^{-\frac{1}{2}} (1 - \rho(t)) \int_1^{\infty} F(x_n, t) dx_n.$$  

This inequality is valid by symmetry and by the following two facts. First, the probability that a path leaves a compact interval $[x_n - 1, x_n + 1]$ during a small time interval $[0, 2t]$ is small:

$$\rho(t) \geq \text{Prob} \sup_{0 \leq s \leq 2t} (|b_n(s) - x_n| > 1),$$

with $\rho(t) \to 0$ as $t \to 0^+$. In fact, a classical result concerning the 1-dimensional Brownian motion is that for each $\varepsilon > 0$, there exists a positive constant $C(\varepsilon)$ such that:

$$\rho(t) \leq C(\varepsilon) e^{-(1-\varepsilon)/4t}, \text{ as } t \to 0^+.$$  

Second, the monotony of the exponential, the positivity of the integral and the product structure of the potential gives a lower bound at $|x_n| + 1$.

\footnote{Of course, the fact that $E_{x,y;2t}$ is restricted to the diagonal $\{x = y\}$ corresponds to the fact that $Z_Q$ is the trace of the heat kernel.}
Upper bound: Let us now analyze the upper bound more closely. Theorem 2 follows from the following three statements:

\[
\lim_{t \to 0} t^{d_n} \int_0^1 F(x_n, t) dx_n = 0 \tag{3}
\]

\[
\lim_{t \to 0} t^{d_n} \int_0^\infty F(x_n, t) dx_n = \text{Tr}((H_{n-1}^\alpha)^{-d_n}) \Gamma(d_n + 1) \tag{4}
\]

\[
\text{Tr}((H_{n-1}^\alpha)^{-d_n}) < \infty \tag{5}
\]

This will finish the proof because equation (3) implies that our upper bound and lower bound agree asymptotically since their difference does not contribute a top order term, while equations (4,5) prove the asymptotic power and constant.

We treat now the result stated in equation (3). According to the hypothesis of induction we have:

\[
F(1, t) \sim C t^{\frac{1}{2} - d_{n-1}} \text{, as } t \to 0^+,
\]

for some \( C > 0 \). It follows that for \( |x| \) in a compact subset of \( \mathbb{R}_+ \) and \( t > 0 \) small:

\[
F(x, t) \leq C(|x|^\frac{1}{2n})^{-\frac{1}{2} - d_{n-1}} t^{-\frac{1}{2} - d_{n-1}}.
\]

Hence since:

\[
d_n - \frac{1}{2} - d_{n-1} = \frac{\alpha_1 + \cdots + \alpha_{n-1} + 2}{2\alpha_n} - \frac{1}{2} - \frac{\alpha_1 + \cdots + \alpha_{n-2} + 2}{2\alpha_{n-1}}
\]

\[
= \frac{\alpha_1 + \cdots + \alpha_{n-1} + 2}{2\alpha_n} - \frac{\alpha_1 + \cdots + \alpha_{n-1} + 2}{2\alpha_{n-1}} > 0.
\]

The decay w.r.t. \( t \) is achieved. It remains to show that the integral w.r.t. \( x \) is finite. For \( t \) small we have that:

\[
\int_0^1 F(x, t) dx \leq C(t) \int_0^1 |x|^\frac{1}{2n} \cdot t^{-\frac{1}{2} - d_{n-1}} dx.
\]

We see that this singularity in \( x = 0 \) is integrable if and only if:

\[
-\frac{1}{d_n} \left( \frac{1}{2} + d_{n-1} \right) > -1,
\]
since we have:

$$\frac{1}{d_n} \left( \frac{1}{2} + l_{d-1} \right) = \frac{2\alpha_n}{\alpha_1 + \cdots + \alpha_{n-1} + 2} \cdot \frac{\alpha_1 + \cdots + \alpha_{n-1} + 2}{2\alpha_{n-1}} = \frac{\alpha_n}{\alpha_{n-1}} < 1,$$

we obtain that the integral is finite.

**Remark 14** This result concerning the singularity in $x=0$ is asymptotically exact as $t \to 0^+$ so that the condition $\alpha_n < \alpha_{n-1}$ is necessary to get Eq.(4). Hence, like in dimension 2, there is a good way to slice imposed by the exponents of the potential. One slices always in direction of the smallest exponent. Also sliced Gordon Thompson is not working with $\alpha_n = \alpha_{n-1}$ (log singularity at the origin) and sliced bread is required.

Let us now show the result stated in equation (4). First, via the functional equation for $F$, we can scale $t$ out as:

$$\int_0^\infty F(x_n, t) dx_n = \int_0^\infty F(x_n t^{d_n}, 1) dx_n = t^{d_n} \int_0^\infty F(x_n, 1) dx_n.$$

Using again lemma 12, we can scale out the $n-1$ dimensional operator $H_{n-1}^{\alpha_n}$ via:

$$\int_0^\infty F(x_n, 1) dx_n = \int_0^\infty F(1, x_n^{1/d_n}) dx_n = d_n \Gamma \left( \int_0^\infty s^{d_n-1} e^{-s H_{n-1}^{\alpha_n}} ds \right)$$

$$= d_n \Gamma(d_n) \text{Tr}((H_{n-1}^{\alpha_n})^{-d_n}) = \Gamma(d_n + 1) \text{Tr}((H_{n-1}^{\alpha_n})^{-d_n}),$$

as required.

Finally, to achieve the proof of Theorem 2 it remains to establish the finiteness of $\text{Tr}((H_{n-1}^{\alpha_n})^{-d_n})$, stated in equation (5). As shown above, we have:
\[ \text{Tr}(H_{n-1}^{\alpha} - d_n) = \frac{1}{\Gamma(d_n + 1)} \int_0^{\infty} F(1, x_n^{1/d_n}) dx_n. \]

We now show that the right hand side is finite. As we just have seen, for small \( x_n \), the induction hypothesis tells us that:

\[ F(1, x_n^{1/d_n}) \sim C x_n^{-\frac{d}{\alpha} - d_n - 1}. \]

Since \( \frac{\alpha}{d_{n-1}} < 1 \), the singularity at zero is integrable. On the other hand, for \( x_n \) large we see from the induction hypothesis that \( F(1, x_n^{1/d_n}) \) decays as \( e^{-c x_n^{1/d_n}} \) and it follows that \( \text{Tr}(H_{n-1}^{\alpha} - d_n) < \infty. \)

\[ \blacksquare \]

\section{Eigenvalue asymptotics, \( \alpha_1 = \cdots = \alpha_n = \alpha_0 \).}

Consider now the singular case of equal exponents in all directions. As was pointed out in the last section, a change of tools is necessary. Instead of working with \( Z_{SGT} \), one now needs to analyze the asymptotics of \( Z_{SB} \). This time we cannot easily estimate the difference between the lower and upper bound and thus compute their leading order asymptotics separately. We will use again an induction argument, with dimension 2 solved by Simon. Recall that the main Theorems 3 and 5 read as follows:

\[ \lim_{t \to 0^+} t^{(n/2) + \alpha_0^{-1}} \ln(t)^{n-1} Z_Q(t)(E) = \frac{\Gamma(1 + \alpha_0^{-1})(n/2 + \alpha_0^{-1}(n-1))}{\pi^{n/2}(n-1)!}. \]

\[ \lim_{E \to \infty} E^{-(n/2) + \alpha_0^{-1}} \ln(E)^{-n-1} N_{H_2}(E) = \frac{\Gamma(1 + \alpha_0^{-1})(n/2 + \alpha_0^{-1}(n-1))}{\Gamma(n/2 + \alpha_0^{-1})\pi^{n/2}(n-1)!} =: a_n. \]
Upper bound. From the sliced bread inequalities, we know $Z_Q(t) \leq Z_{SB}(t)$. We compute the asymptotics of $Z_{SB}$ directly. We will make heavy use of scaling arguments again. Before we proceed, we prove the following lemma:

**Lemma 15** Let $A_g = -(\Delta x_n) + g|x_n|^\gamma$. Let $F_g(s) = \text{Tr}(-s A_g)$ and $N_g(E)$ be the eigenvalue counting function of $-\Delta x_1, \ldots, x_{n-1} + g \prod_{i=1}^{n-1} |x_i|^{\alpha_0}$. Omit the index $g$ whenever $g=1$. Then:

- **a)** $F_g(s) = F(s^\tau)$, where $\tau = 2/(\gamma + 2)$,
- **b)** $N_g(E) = N(s^\eta E)$, where $\eta = ((n-1)\alpha_0 + 2)/2$,
- **c)** $\lim_{s \to 0^+} s^\mu F(s) = \pi^{-1/2} \Gamma(1 + \gamma^{-1})$, where $\mu = (\gamma + 2)/2\gamma$,
- **d)** $\lim_{s \to 0^+} s^{\mu+1} F'(s) = -\mu \pi^{-1/2} \Gamma(1 + \gamma^{-1})$

**Proof:** Part a) and b) follow by the scaling relations (1) and (2). Part d) follows from c) by formally differentiating the leading order term. Part c) follows from the observation that since for the operator $A_g$ in one dimension the potential $|x_n|^\gamma$ is confining, we actually have $Z_{cl}(t) = 1$ as $t \to 0^+$, cf. [12]. Now:

$$Z_{cl}(t) = \frac{1}{2\pi} \int e^{-s(p^2 + |x|^{\gamma})} dx dp = s^{\gamma} \pi^{-1/2} \Gamma(1 + \gamma^{-1})$$

and the claim follows. \hfill \Box.

With the help of this lemma, we find a representation of $Z_{SB}(t)$ by an integral expression which is suitable for computation of the asymptotics. Let $\epsilon_j(x_n)$ be the $j$th eigenvalue of $-\Delta x_1, \ldots, x_{n-1} + |x_n|^{\alpha_0} \prod_{i=1}^{n-1} |x_i|^{\alpha_0}$ so with $d_n = (n-1)/2 + \alpha_0^{-1}$ we get by the scaling relation b): $\epsilon_j(x_n) = |x_n|^{1/d_n} \epsilon_j(1) =: |x|^{1/d_n} \epsilon_j$. Now compute:
\[ Z_{SB}(t) = \sum_j \text{Tr}(\exp[-t(-\Delta_x + \epsilon_j(x_n))]) \]

\[ = \sum_j \text{Tr}(\exp[-t(-\Delta_x + \epsilon_j|x_n|^{1/d_n})]) \]

\[ = \sum_j \text{Tr}(\exp[-te_j^{b_n}(-\Delta_x + |x_n|^{1/d_n})]) \]

where we have used scaling so: \( b_n = \frac{d_n}{d_n + (1/2)} \). We now represent \( Z_{SB} \) by an integral and use integration by parts:

\[ Z_{SB}(t) = \sum_j F(1/d_n)(te_j^{b_n}) \]

\[ = \int_0^\infty F(1/d_n)(tE^{b_n})dN(E) \]

\[ = -\int_0^\infty tE^{b_n-1}F'(1/d_n)(tE^{b_n})N(E)dE \]

The bracketed index \( F(1/d_n) \) is there to remind of the exponent in the corresponding potential. The integration by parts used \( N = 0 \) for small \( E \) and \( FN \to 0 \) for fixed \( t \) and \( E \to \infty \) to see that there are no boundary terms present. We proceed from here very similarly to Simon’s arguments for dimension 2 by analyzing this integral on several pieces. First, note that the integral goes from \( E_0 > 0 \) to \( \infty \), since \( N(E) = 0 \) for \( E \) small by the induction hypothesis. Now we pick values \( E_0 < E_1 < E_2 < \infty \) such that

\[ E_1^{b_n}t = |\ln t|^{-1}; \quad E_2^{b_n}t = 1 \]

We will now estimate the integral on \( (E_0, E_1), (E_1, E_2), (E_2, \infty) \) separately and see that only the integral on \( (E_0, E_1) \) will contribute to the leading order asymptotics.
On \((E_2, \infty)\), we note that \(N(E) \leq cE^{d_n} \ln(E)^{n-2}\) for all \(E\) because of the hypothesis of induction. In addition, in the region \(y \geq 1\), \(-F'(y) = \sum \bar{\xi}_j e^{-\bar{\xi}_j y} \leq De^{-cy}\), with \(D, C > 0\) and \(\bar{\xi}_j\) being the eigenvalues of \(-\Delta_{x_n} + |y|^{1/d_n}\). Thus we estimate:

\[
- \int_{E_2}^{\infty} \left( t b_n \right) E^{b_n - 1} N(E) F'^{(1/d_n)}(t E^{b_n}) dE \\
\leq c_1 \int_{E_2}^{\infty} t E^{b_n - 1 + d_n} \ln(E)^{n-2} \exp(-ctE^{b_n}) dE \\
= c_2 t^{-b_n/d_n} \int_{1}^{\infty} \left( \ln \frac{y}{t} \right)^{n-2} y^{d_n/b_n} e^{-cy} dy
\]

For some constants \(c_1, c_2\). We see that all terms in the integral are bounded by \(t^{-d_n/b_n} \ln(t^{-1})^{-(n-2)} = t^{-(d_n+1/2)} \ln(t^{-1})^{-(n-2)}\) which is small on the level of \(t^{-(d_n+1/2)} \ln(t^{-1})^{-(n-1)}\).

On \((E_1, E_2)\), we bound \(N(E)\) as above and find: \(-F'(y) \leq Cy^{-d_n - \frac{3}{2}}\), by result \(d)\) of the previous lemma. Then, where \(c\) is a constant which changes from equation to equation:

\[
- \int_{E_1}^{E_2} \left( t b_n \right) E^{b_n - 1} N(E) F'^{(1/d_n)}(t E^{b_n}) dE \\
\leq c \int_{E_1}^{E_2} t E^{b_n - 1 + d_n} \ln(E)^{n-2} (t E^{b_n})^{-d_n - 3/2} dE \\
\leq c t^{-d_n - 1/2} \int_{E_1}^{E_2} \ln(E)^{n-2} E^{-1} dE \\
\leq c t^{-d_n - 1/2} \ln(E_2/E_1)^{n-1}
\]

where we have used \(b_n - 1 + d_n - b_n(d_n + \frac{3}{2}) = -1\), since \(b_n(d_n + \frac{3}{2}) = d_n\). Since \(\ln(E_2/E_1) = b_n^{-1} \ln(|\ln t|)\), this integral is small compared to \(t^{-(d_n+1/2)} \ln(t^{-1})^{-(n-1)}\).
Finally, for the last piece we replace $F'$ by its asymptotic value making a multiplicative error of the form $1 + o(1)$, i.e., we can bound $F'$ above and below by $-(1 \pm \epsilon(t))(d_n + \frac{1}{2})\pi^{-1/2}\Gamma(1 + d_n)$ with $\epsilon(t) \to 0$. This is true because the arguments of $F'$, namely $tE_{b_n}$ are bounded from above by $|\ln(t)|^{-1}$. Thus, if $\sim$ means the ratio goes to 1, we see that:

$$\int_{E_0}^{E_1} (tb_nE_{b_n}^{-1}N(E)F'_{(1/d'_n)}(tE_{b_n}))dE$$

$$\sim \int_{E_0}^{E_1} tb_nE^{-1}\ln(E)^{n-2}t^{-d_n-3/2}(d_n + 1/2)\pi^{-1/2}\Gamma(1 + d_n)a_{n-1}\left[\frac{N(E)}{a_{n-1}E_{b_n}\ln(E)^{n-2}}\right]dE$$

$$= A$$

where $a_{n-1}$ is just the constant of the eigenvalue counting function in dimension $(n - 1)$. Since by the induction hypothesis $N(E)/a_{n-1}E_{b_n}\ln(E)^{n-2} \to 1$ as $E \to \infty$, we find:

$$A \sim a_{n-1}b_n(d_n + 1/2)\pi^{-1/2}\Gamma(1 + d_n)t^{-d_n-1/2}\int_{E_0}^{E_1} E^{-1}\ln(E)^{n-2}dE$$

$$= \frac{a_{n-1}d_n\Gamma(1 + d_n)t^{-d_n-1/2}\ln(1)_{b_n}\ln(n-1)}{(n-1)!}$$

Now we compute:

$$\ln(E_1/E_0) = \ln[ct^{-1/b_n}|\ln(t)|^{-1}] \sim \frac{1}{b_n}\ln(t^{-1})$$

where $c$ is once more some constant. We conclude that:

$$A\sim d_n^{-1/2}(\ln(t))^{-1/2}a_{n-1}d_n b_n^{-(1/2)}(1 + d_n)(n-1)^{-1} = \frac{\Gamma(d_n)d_n^{-1}}{\pi^{n/2}(n-1)!}$$

So we get for the upper bound:

$$\lim_{t \to 0} t^{n/2+1}a_0^{-1}|\ln(t)^{-1}|^{-1}\text{Tr}(e^{-tH_n^{n-1}}) \leq \frac{\Gamma(1 + a_0^{-1})(n/2 + 1)}{\pi^{n/2}(n-1)!}$$

**Lower bound.** For the lower bound, we will employ the Feynman-Kac formula as before. First however, we prove a general integral equality relevant to our computation:
Proposition 16 Let $f : \mathbb{R} \to \mathbb{R}$ be any integrable function. Let $P : \mathbb{R}^n \to \mathbb{R}$ denote the map given by $P(x_1, ..., x_n) = \prod_{i=1}^{n} |x_i|$. Let $Q_{2a}$ denote the cube of width $2a$ centered at the origin. Then we have:

$$\int_{\mathbb{R}^n \setminus Q_{2a}} f \circ P \, dx = 2^n \int_{a^n}^{\infty} \frac{f(p)}{(n-1)!} \log\left(\frac{p}{a^n}\right)^{n-1} \, dp$$

Proof: We prove this by induction. If $n=1$ or $n=2$, the equality is easily verified. Now:

$$\int_{\mathbb{R}^n \setminus Q_{2a}} f \circ P \, dx$$

$$= 2^n \int_{a}^{\infty} \left( \int_{a}^{\infty} \cdots \int_{a}^{\infty} f(x_1 \cdots x_n) \, dx_1 \cdots dx_{n-1} \right) \, dx_n$$

$$= 2^n \int_{a}^{\infty} \left( \int_{a^{n-1}}^{\infty} \frac{f(x) \cdot q}{(n-2)!} \log\left(\frac{q}{a^{n-1}}\right)^{n-2} \, dq \right) \, dx_n$$

$$= 2^n \int_{a^n}^{\infty} f(p) \left( \int_{a}^{\frac{1}{x_n a^{n-1}}} \frac{1}{(n-2)!} \left[ \log\left(\frac{p}{x_n a^{n-1}}\right) \right]^{n-2} \frac{1}{x_n} \, dx_n \right) \, dp$$

$$= 2^n \int_{a^n}^{\infty} f(p) \left( \frac{1}{(n-2)!} \left[ \sum_{k=0}^{n-2} \binom{n-2}{k} \log\left(\frac{p}{a^{n-1}}\right)^k \left[ -\log(x_n) \right]^{n-k-2} \frac{1}{x_n} \right] \right) \, dp$$

$$= 2^n \int_{a^n}^{\infty} f(p) \left( \frac{1}{(n-1)!} \left[ \sum_{k=0}^{n-2} \binom{n-1}{k} \log\left(\frac{p}{a^{n-1}}\right)^k (-1)^{n-k-2} \log\left(\frac{p}{a^{n-1}}\right)^{n-k-1} \right] \right) \, dp$$

$$= 2^n \int_{a^n}^{\infty} f(p) \left( \left\{ \sum_{k=0}^{n-2} \binom{n-1}{k} \log\left(\frac{p}{a^{n-1}}\right)^k (-1)^{n-k-2} \log\left(\frac{p}{a^{n-1}}\right)^{n-k-1} \right\} \right) \, dp$$

$$= 2^n \int_{a^n}^{\infty} f(p) \left( \left[ -\sum_{k=0}^{n-2} \binom{n-1}{k} \log\left(\frac{p}{a^{n-1}}\right)^k (-1)^{n-k-2} \log\left(\frac{p}{a^{n-1}}\right)^{n-k-1} \right] \right) \, dp$$

$$= 2^n \int_{a^n}^{\infty} f(p) \left[ -\log\left(\frac{p}{a^{n-1}}\right)^{n-1} + \log\left(\frac{p}{a^{n-1}}\right)^{n-1} \right] \, dp$$

$$= 2^n \int_{a^n}^{\infty} f(p) \log\left(\frac{p}{a^n}\right)^{n-1} \, dp$$

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From this, we compute the lower bound as follows:

Apply the Feynmann-Kac formula again, but remove some of the points \( x = (x_1, \ldots, x_n) \) in the integral, as well as some Brownian paths. Only keep those points \( x \) that satisfy \( x \notin Q_{2t^{1/2} \ln(t)} \). Only consider paths with \( \sup_{0 \leq s \leq 2t} |b_i(s) - x_i| \leq t^{1/2} \ln(t) \), \( \forall 1 \leq i \leq n \). The measure of such paths is \( 1 - \rho(t) \), where \( \rho(t) \to 0 \) (as \( e^{-D(\ln(t))^n} \)). Write \( p := \prod_{i=1}^{n} x_i \). We have \( \frac{\partial p}{\partial x_i} = \frac{1}{x_i}, \forall i \), so everywhere along the path we get \( |\ln(p(b(s))) - \ln(p(x))| \leq c/|\ln(t)| \). Thus, with \( \kappa(t) := \exp(n/\ln(t)) \) we have:

\[
\text{Tr}(e^{-tH_n}) \geq \frac{1 - \rho(t)}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n \setminus Q_{2t^{1/2} \ln(t)}^2} e^{-t\kappa^\alpha p(x_1, \ldots, x_n)^\alpha} dx_1 \cdots dx_n
\]

\[
= \frac{1 - \rho(t)}{t^{n/2} \ln(t)^{2n}} \int_{t^{n/2} \ln(t)^{2n}}^{\infty} \frac{e^{-t\kappa^\alpha p^\alpha}}{(n-1)!} \ln(t)^{2n-1} dp
\]

\[
= \frac{1 - \rho(t)}{\kappa t^{n/2} (\pi t)^{n/2}} \int_{t^{n/2 + \alpha^{-1} \ln(t)^{2n}}^{\infty}} \frac{e^{-w^\alpha}}{(n-1)!} \ln(t)^{2n-1} dw
\]

We have \( \rho \to 0 \) and \( \kappa \to 1 \). Thus:

\[
\liminf_{t \to 0} t^{n/2 + \alpha^{-1} \ln(t^{-n/2 - \alpha^{-1}})}(n-1) \text{Tr}(e^{-tH_n}) \geq \frac{1}{\pi^{n/2}(n-1)!} \int_{0}^{\infty} e^{-w^\alpha} dw
\]

The Integral is equal to \((\alpha^{-1}_0)\Gamma(\alpha^{-1}_0) = \Gamma(1 + \alpha^{-1}_0)\) and since \( \ln(t^{-n/2 - \alpha^{-1}}) = (n/2 + \alpha^{-1}) \ln(t^{-1}) \) we now conclude:

\[
\liminf_{t \to 0} t^{n/2 + \alpha^{-1}_0 \ln(t^{-1})} \text{Tr}(e^{-tH_n}) \geq \frac{\Gamma(1 + \alpha^{-1}_0)(n/2 + \alpha^{-1}_0)^{(n-1)}}{\pi^{n/2}(n-1)!}
\]
Thus lower and upper bound agree to first order and the theorem is proven. ■

Acknowledgments. The authors thank Werner Kirsch and Brice Franke for explaining us Feynman-Kac representations of heat-kernels/operators and the Ito-calculus. Both authors were supported by the project SFB-TR12, Symmetries and Universality in Mesoscopic Systems founded by the DFG.

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