WILLMORE SURFACES IN SPHERES VIA LOOP GROUPS II: A COARSE CLASSIFICATION OF WILLMORE TWO-SPHERES BY POTENTIALS

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Abstract

This paper is to provide a coarse classification of Willmore two-spheres in $S^{n+2}$, and to give explicit applications to the case of $S^6$. We use the theory developed by Burstall and Guest for harmonic maps of finite uniton type. More precisely, for this purpose, we apply the DPW version of Burstall-Guest theory presented in [12], and derive a coarse classification of Willmore two-spheres in $S^{n+2}$ in terms of the normalized potential of their (harmonic) conformal Gauss maps. Moreover, for the case of $S^6$, some geometric properties of the corresponding Willmore two-spheres are derived, and in particular some new Willmore two-spheres. The classical results concerning Willmore two-spheres in $S^4$ are also derived as a corollary.

Keywords: harmonic maps of finite uniton; Willmore two-spheres; normalized potential; S-Willmore surfaces; totally isotropic surfaces.

MSC(2010): 58E20; 53C43; 53A30; 53C35

1. INTRODUCTION

Willmore surfaces can be looked at as surfaces sharing a best conformal placement in $S^{n+2}$, since they are critical surfaces of the conformally invariant Willmore functional

$$\int_M (H^2 - K + 1) dM.$$ 

It is well known that minimal surfaces in the three space forms $\mathbb{R}^{n+2}$, $S^{n+2}$ and $H^{n+2}$ are all Willmore surfaces (see e.g. [14], [16]), providing a large class of examples and indicating the high variety and complexity of Willmore surfaces.

The classical theorem, due to Bryant, states that every Willmore two-sphere in $S^3$ is conformally equivalent to a minimal surface in $\mathbb{R}^3$ with embedded ends. This work has led to a systematical investigation of Willmore two-spheres as well as Willmore surfaces in $S^{n+2}$. In this context, many interesting geometric objects with “nice behavior” has been introduced, including minimal surfaces with embedded planer ends ([2], [3]) as well as holomorphic curves or anti-holomorphic curves in the twistor bundle of $S^4$ ([14], [23], [21]).

Contributions by Bryant’s work include proof of the harmonicity of the conformal Gauss map of a Willmore surface, the introduce of dual (Willmore) surfaces and holomorphic forms related with Willmore surfaces [2]. To study Willmore surfaces in higher dimensional spheres $S^{n+2}$, Ejiri generalized Bryant’s definition of conformal Gauss map and showed the equivalence of surfaces being Willmore and having a harmonic conformal Gauss map. However the duality properties of Willmore surfaces no longer hold for general Willmore surfaces in $S^{n+2}$ when $n > 1$ (for a generalization of duality we refer to [1], [22] and [21]). To use the duality properties, Ejiri introduced the notion of S-Willmore surfaces and proved that a Willmore surface has a dual
(Willmore) surface if and only if it is S-Willmore. Moreover, Ejiri provided a classification of S-Willmore (Willmore) two-spheres in $S^{n+2}$ by the construction of holomorphic forms for these surfaces [14]. Especially, for Willmore two-spheres in $S^4$, the existence of a certain holomorphic 4-form shows that they are S-Willmore automatically ([14], [24], [23], [4], [22]).

Since Ejiri’s work in 1988, it has been an open question that whether are there Willmore two-spheres in $S^{n+2}$ which are not S-Willmore. If such Willmore two-spheres exist, then one needs to show how to construct and to characterize them.

As stated in the title, as an application of [11] and [12], this paper aims to provide a coarse classification of Willmore two-spheres by the loop group method. And in [25] and [26], the construction of new Willmore two-spheres will be presented. Moreover, a concrete, new, isotropic Willmore two-sphere in $S^6$, without any dual surface, is constructed as an illustration of the theory [26].

The techniques we apply here are based on the classification theory of harmonic maps of finite uniton type [5], [13], and the loop group method for the construction of (conformally) harmonic maps [10], [11]. The vital idea of our work is that, since there has been a complete description of harmonic two-spheres in compact symmetric spaces ([29], [5], [13]), one should try to modify this method to give a classification as well as examples of the conformal Gauss map of Willmore two-spheres. This way one should be able to obtain the classification as well as new examples of Willmore two-spheres. There are several difficulties one has to overcome before one is able to implement this idea. The first one is that the target manifold of the conformal Gauss map is a non-compact symmetric space. The second one is how to read off the properties of Willmore surfaces from the harmonic conformal Gauss maps. The third one is how to modify the theory of Burstall and Guest to harmonic maps into non-compact Lie groups in the framework of DPW. And the last one is how to do the explicit Iwasawa decompositions required by the loop group method. The first two problems has been solved in [11] and the third problem has been solved in [12]. This leads to the possibility to solve the problem considered in this paper under the framework of [11], by using the technics introduced in [10] and [5] (see [12]).

According to the theory due to Uhlenbeck [29], Segal [27], Burstall and Guest [5], [13], any harmonic map from $S^2$ to $G/K$, $G/K$ a compact or non-compact inner symmetric space, is of finite uniton type (For the case of non-compact $G/K$, see [11], [12]). Moreover, for any harmonic map of finite uniton type, there exists a normalized potential which takes values in some nilpotent Lie subalgebra. And the corresponding normalized framing takes values in the corresponding Lie subgroup, which can be determined by the recipe of [5]). Therefore, a classification of Willmore two-spheres in $S^{n+2}$ is equivalent to classify the corresponding nilpotent Lie subalgebras related with the Grassmannian where the conformal Gauss map takes value in. Since in the Willmore case, $G/K = SO^+(1, n + 3)/(SO^+(1, 3) \times SO(n))$, we classify by the procedure stated in [5] all nilpotent Lie subalgebras which are associated with this symmetric space. See Theorem 2.6 for details. The proof forms the main content of this paper. Especially, as an application, we provide a classification of Willmore 2-spheres into $S^6$ in Theorem 2.8. To read off concrete inform about the harmonic maps of Willmore surfaces as well as the explicit expressions of these Willmore two-spheres, one needs to carry out Iwasawa decompositions in a concrete fashion. We leave this to [25], [26].

We also notice that there have been recently several publications on harmonic maps of finite uniton type into compact groups like $U(n)$ and $G_2$ (see [8], [15], and [28] and reference therein). Here we follow the spirits of Burstall-Guest and DPW, which is slightly different from the work just mentioned. Comparing these slightly different techniques will be an interesting topic.

This paper is organized as follows. Section 2 is a statement of the main results, including the classification of nilpotent normalized potentials, applications to Willmore two-spheres in $S^6$, and
examples of Willmore surfaces of finite uniton type. The proof of the classification theorem is the main content of Section 3. To be concrete, in this section, we first transform $SO^+(1, 2m - 1, \mathbb{C})$ into an isomorphic group $G(2m, \mathbb{C})$, such that the nilpotent Lie subalgebras can be transformed into subalgebras of the upper-triangular matrices in $\mathfrak{g}(2m, \mathbb{C})$. This plays an important role in the concrete Iwasawa decompositions in [25], [26]. Then, in this new picture, we derive the classification of nilpotent Lie subalgebras as well as the classification of normalized potentials corresponding to Willmore surfaces, finishing the proof of Theorem 2.6.

2. Willmore surfaces of finite uniton type

In this section, we will first review the connections between Willmore surfaces and the loop group method. Then we will give a coarse classification of Willmore surfaces of finite uniton type into $S^{n+2}$ in terms of the forms of their potentials. For such Willmore surfaces in $S^6$, their geometric properties are also discussed in Theorem 2.8, which provides a partial geometric classification of Willmore two-spheres in $S^6$. Then, several concrete examples of Willmore surfaces are discussed in the second part of this paper.

We note that these results also provide many insights into a geometric classification of Willmore two-spheres in $S^5$ [20].

2.1. Willmore surfaces and strongly conformally harmonic maps. Since all the loop group theory needed in this paper is discussed in much detail in [11], we will just recall some basic notation and results of [11]. Note the treatment in [11] is different from the ones applied in [16, 17] and [31], where a different kind of harmonic maps (first introduced by Hélein [16]) are mainly used.

Let $\mathbb{R}^{n+4}$ be the Lorentz-Minkowski space equipped with the Lorentzian metric
\[
\langle x, y \rangle = -x_0 y_0 + \sum_{j=1}^{n+3} x_j y_j = x^t I_{1,n+3} y, \quad I_{1,n+3} = \text{diag}\{-1, 1, \ldots, 1\}, \quad x, y \in \mathbb{R}^{n+4}.
\]
Let $C_+^{n+3} = \{ x \in \mathbb{R}^{n+4} | \langle x, x \rangle = 0, x_0 > 0 \}$ denote the forward light cone of $\mathbb{R}^{n+4}$ and $Q^{n+2} = C_+^{n+3} / \mathbb{R}^+ = S^{n+2}$ denote the projective light cone. For a conformal immersion $y : M \to S^{n+2}$ from a Riemann surface $M$, one has a canonical lift $Y = e^{-\omega}(1, y)$ into $C_+^{n+3}$ with respect to a local complex coordinate $z$ of $M$, where $e^{2\omega} = 2 \langle y_z, y_{\bar{z}} \rangle$. It is easily verified that there is a global bundle decomposition
\[
(2.1) \quad M \times \mathbb{R}_1^{n+4} = V \oplus V^\perp, \quad \text{with } V = \text{Span}\{Y, ReY_z, ImY_z, Y_{z\bar{z}}\},
\]
where $V^\perp$ denotes the orthogonal complement of $V$. The complexifications of $V$ and $V^\perp$ are denoted by $V_C$ and $V_C^\perp$, respectively. Choose the frame $\{Y, Y_z, Y_{\bar{z}}, N\}$ of $V_C$, where $N$ is the uniquely determined section of $V$ over $M$ satisfying $\langle N, Y_z \rangle = \langle N, Y_{\bar{z}} \rangle = \langle N, N \rangle = 0, \langle N, Y \rangle = -1$. Let $D$ denote the normal connection on $V_C^\perp$. For any section $\psi \in \Gamma(V_C^\perp)$ of the normal bundle and a canonical lift $Y$ w.r.t $z$, we obtain the structure equations:
\[
(2.2) \quad \begin{cases}
Y_{zz} = -\frac{1}{2} Y + \kappa, \\
Y_{\bar{z}z} = -\langle \kappa, \bar{\kappa} \rangle Y + \frac{1}{2} N, \\
N_z = -2 \langle \kappa, \bar{\kappa} \rangle Y_z - s Y_{\bar{z}} + 2 D_z \kappa, \\
\psi_z = D_z \psi + 2 \langle \psi, D_z \kappa \rangle Y - 2 \langle \psi, \kappa \rangle Y_{\bar{z}}.
\end{cases}
\]
Here $\kappa$ is the conformal Hopf differential of $y$, and $s$ is the Schwarzian of $y$ [6]. For the structure equations (2.2) the integrability conditions are the conformal Gauss, Codazzi and Ricci equations.
respectively:

\[
\begin{aligned}
\frac{1}{2}s_\bar{z} &= 3(\kappa, D_\bar{z}\bar{\kappa}) + (D_\bar{z}\kappa, \bar{\kappa}), \\
\text{Im}(D_\bar{z}D_\bar{z}\kappa + s\kappa) &= 0, \\
R_{\bar{z}\bar{z}}^D\psi &= D_\bar{z}D_\bar{z}\psi - D_\bar{z}D_\bar{z}\psi = 2(\psi, \kappa)\bar{\kappa} - 2(\psi, \bar{\kappa})\kappa.
\end{aligned}
\]

(2.3)

The Willmore functional of \( y \) is defined as:

\[
W(y) := 2i \int_M \langle \kappa, \bar{\kappa} \rangle dz \wedge d\bar{z}.
\]

(2.4)

An immersed surface \( y : M \to S^{n+2} \) is called a Willmore surface, if it is a critical point of the Willmore functional with respect to any variation (with compact support) of the map \( y : M \to S^{n+2} \).

The conformal Gauss map of \( y \) is defined as follow (See also \[2\] [6] [14] [22]): Set

\[
Gr := Y \wedge Y_u \wedge Y_v \wedge N = -2i \cdot Y \wedge Y_\bar{z} \wedge Y_\bar{z} \wedge N.
\]

It is easy to verify that \( Gr \) is well defined. We call \( Gr : M \to Gr_{1,3}(\mathbb{R}^{n+4}_1) = SO^+(1,n+3)/SO^+(1,3) \times SO(n) \)

the conformal Gauss map of \( y \). Here \( SO^+(1,n+3) \) is the orientation-preserving and timelike direction-preserving isometry group of \( \mathbb{R}^{n+4}_1 \):

\[
SO^+(1,n+3) = \{ T \in Mat(n+4,\mathbb{R}) \mid \langle Tx,Ty \rangle = \langle x,y \rangle, \forall x,y \in \mathbb{R}^{n+4}_1, \det T = 1, Tc_+^{n+3} \subset C_+^{n+3} \}.
\]

(2.5)

It is well-known that Willmore surfaces can be characterized as follows \[2\] [6] [14].

**Theorem 2.1.** For a conformal immersion \( y : M \to S^{n+2} \), the following three statements are equivalent

1. \( y \) is a Willmore surface.
2. Its conformal Gauss map \( Gr \) is a (conformally) harmonic map into \( G_{3,1}(\mathbb{R}^{n+3}_1) \).
3. The conformal Hopf differential \( \kappa \) satisfies the “Willmore equation”:

\[
D_\bar{z}D_\bar{z}\kappa + \frac{8}{7}\kappa = 0.
\]

(2.6)

By \[14\], a Willmore immersion \( y : M^2 \to S^n \) is called an S-Willmore surface if its conformal Hopf differential satisfies \( D_\bar{z}\kappa || \kappa \), i.e. there exists some function \( \mu \) on \( M \) such that \( D_\bar{z}\kappa + \mu \kappa = 0 \). S-Willmore surfaces can be characterized by

**Theorem 2.2.** (\[14\], [22]) A Willmore surface \( y \) is S-Willmore if and only if it has a dual (Willmore) surface.

**Remark 2.3.** Although there are restrictions for Willmore surfaces to have a dual surface, it is known that there exist special transforms for Willmore surfaces, called “adjoint transforms”, see \[21\] [22] for details. These transformations have a close relation with Helein’s work \[16\] (See \[11\] for some details).

There exist Willmore surfaces which fail to be immersions at some points or even curves. To include surfaces of this type, we introduce the notion of Willmore maps and strong Willmore maps as generalizations.

**Definition 2.4.** A smooth map \( y \) from a Riemann surface \( M \) to \( S^{n+2} \) is called a Willmore map if it is a conformal Willmore immersion on an open dense subset \( M \) of \( M \). The points in \( M_0 = M \setminus \tilde{M} \) are called branch points of \( y \).

Moreover, \( y \) is called a strong Willmore map if the conformal Gauss map \( Gr : \tilde{M} \to SO^+(1,n+3)/SO^+(1,3) \times SO(n) \) of \( y \) can be extended smoothly to \( M \).
As a consequence, one can go from a strong Willmore map to a harmonic conformal map. To describe such harmonic maps and to characterize those harmonic maps related to Willmore surfaces, we first recall from [11] that for any strong Willmore map $y$, locally on $U \subset M$ we can choose a frame

\begin{equation}
F := \left( \frac{1}{\sqrt{2}}(Y + N), \frac{1}{\sqrt{2}}(-Y + N), e_1, e_2, \psi_1, \ldots, \psi_n \right) \in SO^+(1, n + 3)
\end{equation}

with the Maurer-Cartan form

$$\alpha = F^{-1}dF = \begin{pmatrix} A_1 & B_1 \\ -B_1^tI_{1,3} & A_2 \end{pmatrix}dz + \begin{pmatrix} \tilde{A}_1 & \tilde{B}_1 \\ -\tilde{B}_1^tI_{1,3} & \tilde{A}_2 \end{pmatrix}d\bar{z},$$

where

\begin{equation}
B_1 = \begin{pmatrix} \sqrt{2}\beta_1 & \cdots & \sqrt{2}\beta_n \\ -\sqrt{2}\beta_1 & \cdots & -\sqrt{2}\beta_n \\ -k_1 & \cdots & -k_n \\ -ik_1 & \cdots & -ik_n \end{pmatrix}.
\end{equation}

Here $\{\psi_j\}$ is an orthonormal basis of $V^\perp$ on $U$ and $\kappa = \sum j \beta_j \psi_j$. And the entries of $A_1$ and $A_2$ are determined by $s$, $\kappa$ and the normal connection (See [11]).

It is easy to see that $B_1$ has a special form. Moreover, a direct computation shows

$$B_1^tI_{1,3}B_1 = 0.$$ 

In [11] this property of $B_1$ plays an important role in the characterization of harmonic maps related to Willmore surfaces. We also point out that $y$ is S-Willmore if and only if $\text{rank}(B_1) = 1$ on an open dense subset of $M$.

Conversely, assume we have the frame $F = (\phi_1, \cdots, \phi_4, \psi_1, \cdots, \psi_{n+4}) : U \to SO^+(1, n + 3)$ of some immersion on $U \subset M$, such that the Maurer-Cartan form $\alpha = F^{-1}dF$ of $F$ is of the above form, then

\begin{equation}
y = \pi_0(F) = [(\phi_1 - \phi_2)]
\end{equation}

is a conformal immersion from $U$ into $Q^{n+2} \cong S^{n+2}$ (with canonical lift $\frac{1}{\sqrt{2}}(\phi_1 - \phi_2)$).

Next, the symmetric space $SO^+(1, n + 3)/SO^+(1, 3) \times SO(n)$ is defined by the involution

\begin{equation}
\sigma : SO^+(1, n + 3)/SO^+(1, 3) \times SO(n) \to SO^+(1, n + 3), \quad \sigma(A) = DAD^{-1},
\end{equation}

with $D = \text{diag}\{-I_4, I_n\}$. Then

$$\mathfrak{k} = \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \right| I_{1,3}A_1 + A_1^tI_{1,3} = 0, A_2^2 + A_2 = 0 \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B_1 \\ -\tilde{B}_1^tI_{1,3} & 0 \end{pmatrix} \right| B_1 \in \text{Mat}(4 \times n, \mathbb{R}) \right\}.$$

Let $F : M \to SO^+(1, n + 3)/SO^+(1, 3) \times SO(n)$ be a harmonic map. Then it has a local lift $F : U \to SO^+(1, n + 3)$, and the Maurer-Cartan form $\alpha = F^{-1}dF$ of $F$ is of the form

$$\alpha = \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix}dz + \begin{pmatrix} \tilde{A}_1 & \tilde{B}_1 \\ \tilde{B}_2 & \tilde{A}_2 \end{pmatrix}d\bar{z}.$$

**Definition 2.5.** [11] $F$ is called a **strongly conformally harmonic map** if for any point $p \in M$, there exists a neighborhood $U_p$ of $p$ and a frame $F$ (with Maurer-Cartan forms $\alpha$) of $F$ on $U_p$ satisfying

\begin{equation}
B_1^tI_{1,3}B_1 = 0.
\end{equation}
We see that the conformal Gauss map of any Willmore surface is a strongly conformally harmonic map. Conversely, Theorem 3.10 of [11] shows that from a strongly conformally harmonic map, one either obtains a (branched) Willmore surface, or a constant map. In [25], we have classified the potentials of all strongly conformally harmonic maps yielding a constant map. The conformal Gauss maps of minimal surfaces in $\mathbb{R}^m$ are of this type. The results stated above altogether, provide a way to characterize Willmore surfaces by describing all strongly conformally harmonic maps.

2.2. Coarse classification of Willmore surfaces of finite uniton type. Now let us turn to strongly conformally harmonic maps of finite uniton type in $SO^+(1, n + 3)/SO^+(1, 3) \times SO(n)$. Due to the seminal work of [5] (see also [13]), and also Theorem 5.3 of [12], harmonic maps of finite uniton type into any inner symmetric space, compact or non-compact, can be described by some nilpotent Lie subalgebra valued meromorphic 1-forms. Applying this to strongly conformally harmonic maps of finite uniton type in $SO^+(1, n + 3)/SO^+(1, 3) \times SO(n)$, the nilpotent Lie subalgebras need to be related to the symmetric space $SO^+(1, n + 3)/SO^+(1, 3) \times SO(n)$ [5]. This will provide the forms of the (nilpotent) normalized potentials of strongly conformally harmonic maps of finite uniton type. Moreover, the conditions on $B_1$ will give further restrictions. Altogether, we will obtain a description of the normalized potentials of strongly conformally harmonic maps of finite uniton type, and hence the normalized potentials of Willmore surfaces of finite uniton type. Especially, we obtain a description of the normalized potentials of all Willmore two-spheres. To be concrete, we have the following theorem.

2.2.1. The general case. Let $\mathbb{D}$ be a contractible open subset of $\mathbb{C}$ with complex coordinate $z$.

Theorem 2.6. Let $\mathcal{F} : \mathbb{D} \to SO^+(1, n + 3)/SO^+(1, 3) \times SO(n)$ be a finite-uniton type strongly conformally harmonic map, with $n + 4 = 2m$. Then there exists a normalized potential

$$
\eta = \lambda^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{pmatrix} \, dz
$$

of $\mathcal{F}$ such that

$$
\hat{B}_1 = \begin{pmatrix} \hat{B}_{11} & \cdots & \hat{B}_{1,m-2} \end{pmatrix}, \quad \hat{B}_{1j} = (v_j, \hat{v}_j) \in \text{Mat}(4 \times 2, \mathbb{C}),
$$

and every $\hat{B}_{1j}$ of $\hat{B}_1$ has one of the following two forms:

$$
(i) \quad v_j = \begin{pmatrix} h_{1j} \\ h_{1j} \\ h_{3j} \\ ih_{3j} \end{pmatrix}, \quad \hat{v}_j = \begin{pmatrix} \hat{h}_{1j} \\ \hat{h}_{1j} \\ \hat{h}_{3j} \\ \hat{ih}_{3j} \end{pmatrix} ; \quad (ii) \quad v_j = \begin{pmatrix} h_{1j} \\ h_{2j} \\ h_{3j} \\ h_{4j} \end{pmatrix}, \quad \hat{v}_j = iv_j = \begin{pmatrix} \hat{i}h_{1j} \\ \hat{i}h_{2j} \\ \hat{i}h_{3j} \\ \hat{i}h_{4j} \end{pmatrix},
$$

with all of $\{v_j, \hat{v}_j\}$ satisfying the following conditions

$$
v_j^t I_{1,3} v_l = v_l^t I_{1,3} \hat{v}_l = \hat{v}_j^t I_{1,3} \hat{v}_l = 0, \quad j, l = 1, \cdots, m - 2.
$$

In other words, there are $m - 1$ types of normalized potentials with $\hat{B}_1$ satisfying $\hat{B}_1^t I_{1,3} \hat{B}_1 = 0$, namely those being of one of the following $m - 1$ forms (up to some conjugation):

1. (all pairs are of type (i))

$$
\hat{B}_1 = \begin{pmatrix} h_{11} & \hat{h}_{11} & h_{12} & \cdots & h_{1,m-2} & \hat{h}_{1,m-2} \\ h_{11} & \hat{h}_{11} & h_{12} & \cdots & h_{1,m-2} & \hat{h}_{1,m-2} \\ h_{31} & \hat{h}_{31} & h_{32} & \cdots & h_{3,m-2} & \hat{h}_{3,m-2} \\ ih_{31} & \hat{ih}_{31} & ih_{32} & \cdots & ih_{3,m-2} & \hat{ih}_{3,m-2} \end{pmatrix};
$$
(2) \((\text{the first pair is of type (ii), all others are of type (i)})\)

\[
\hat{B}_1 = \begin{pmatrix}
h_{11} & ih_{11} & h_{12} & \hat{h}_{12} & \cdots & h_{1,m-2} & \hat{h}_{1,m-2} \\
h_{21} & ih_{21} & h_{12} & \hat{h}_{12} & \cdots & h_{1,m-2} & \hat{h}_{1,m-2} \\
h_{31} & ih_{31} & h_{32} & \hat{h}_{32} & \cdots & h_{3,m-2} & \hat{h}_{3,m-2} \\
h_{41} & ih_{41} & ih_{32} & \hat{h}_{32} & \cdots & ih_{3,m-2} & \hat{h}_{3,m-2}
\end{pmatrix};
\]

Introducing consecutively more pairs of type (ii), one finally arrives at \((m - 1)\) (all pairs are of type (ii))

\[
\hat{B}_1 = \begin{pmatrix}
h_{11} & ih_{11} & h_{12} & ih_{12} & \cdots & h_{1,m-2} & ih_{1,m-2} \\
h_{21} & ih_{21} & h_{22} & ih_{22} & \cdots & h_{2,m-2} & ih_{2,m-2} \\
h_{31} & ih_{31} & h_{32} & ih_{32} & \cdots & h_{3,m-2} & ih_{3,m-2} \\
h_{41} & ih_{41} & ih_{42} & ih_{42} & \cdots & ih_{4,m-2} & ih_{4,m-2}
\end{pmatrix}.
\]

**Proof.** As the discussions above, the proof of Theorem 2.6 reduces to classifying nilpotent Lie sub-algebras related with \(SO^+(1, n + 3)/SO(1, 3) \times SO(n)\) and to find out the potentials taking values in these nilpotent Lie sub-algebras which are related with Willmore surfaces \([5, 12]\). Since the detailed computations are lengthy and technical, we will divide them into several lemmas in Section 3, and leave the concrete proof of this theorem to Section 3.3.

Note that both type \((1)\) and type \((m - 1)\) are of finite uniton number 2 (Lemma 3.5 of Section 3.2) and are hence \(S^1\)-invariant by Corollary 5.6 of \([5]\) (see also \([9]\) for \(S^1\)-invariant harmonic maps), which also provides a proof of Corollary 5.10, Corollary 5.11 and part of Theorem 5.3 of \([11]\). The remaining cases are in general of finite uniton number \(\geq 4\) and hence will be more complicated. Here we will use Theorem 2.6 to derive some geometric properties of these maps and list some new examples.

**Remark 2.7.**

1. If \(\text{rank}(\hat{B}_1) = 1\), the potentials of type \((1)\) provide no Willmore maps, or Willmore maps with a constant light-like vector, which turn out to be corresponding to minimal surfaces in \(\mathbb{R}^{n+2}\). When \(\text{rank}(\hat{B}_1) = 2\) the potentials of the first type will produce no Willmore surfaces at all \([25]\).

2. Potentials of type \((m - 1)\) are conjectured to correspond to totally isotropic Willmore surfaces. This has been proven in Theorem 3.1 of \([26]\) when \(m = 4\), i.e., for Willmore two-spheres in \(S^6\). Moreover, excluding the intersection with the first type, every normalized potential of this type produces a unique non S-Willmore surface when \(\text{rank}(\hat{B}_1) = 2\) and gives a pair of dual Willmore surfaces when \(\text{rank}(\hat{B}_1) = 1\).

3. For any potential of the other types, excluding the intersections with the first type and the last type, one obtains a unique non S-Willmore surface which has a non-isotropic conformal Hopf differential, since in such cases \(\text{rank}(\hat{B}_1) = 2\).

**2.2.2. Willmore two-spheres in \(S^6\).** Concerning the case of Willmore two-spheres in \(S^6\), we have a geometric description as follows:

**Theorem 2.8.** Let \(y : S^2 \to S^6\) be a strong Willmore map. Assume that the normalized potential of \((\text{the conformal Gauss map of})\) \(y\) is of the form

\[
\eta = \lambda^{-1} \left( \begin{array}{cc} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{array} \right) dz = \lambda^{-1} \eta_{-1} dz.
\]

Then \(y\), as well as \(\eta\), belongs to one of the three cases:
(1) $y$ is conformally equivalent to a complete minimal surface in $\mathbb{R}^6$ with planar ends. In this case,

\begin{equation}
\hat{B}_1 = \begin{pmatrix} h_{10}v_1 & h_{20}v_1 & h_{30}v_1 & h_{40}v_1 \end{pmatrix}
\end{equation}

with

\begin{equation}
v_1 = \begin{pmatrix} \tilde{h}_{11} \\ \tilde{h}_{13} \\ i\tilde{h}_{13} \\ i\tilde{h}_{13} \end{pmatrix}, \quad \text{and} \quad (|\tilde{h}_{11}|^2 + |\tilde{h}_{13}|^2) \neq 0.
\end{equation}

(2) $y$ is not $S$-Willmore and the Hopf differential of $y$ is not isotropic. In this case,

\begin{equation}
\hat{B}_1 = \begin{pmatrix} h_{10}v_1 & hv_{10}v_1 & h_{30}v_2 & h_{40}v_2 \end{pmatrix}
\end{equation}

with

\begin{equation}
v_1 = \begin{pmatrix} 1 + \tilde{h}_{11}\tilde{h}_{13} \\ -1 + \tilde{h}_{11}\tilde{h}_{13} \\ \tilde{h}_{11} + \tilde{h}_{13} \\ i(\tilde{h}_{11} - \tilde{h}_{13}) \end{pmatrix}, \quad v_2 = \begin{pmatrix} \tilde{h}_{11} \\ \tilde{h}_{11} \\ 1 \\ i \end{pmatrix}
\end{equation}

and

\begin{equation}
(\tilde{h}_{11}'^2 + |\tilde{h}_{13}'|^2)(h_{30}^2 + h_{40}^2)\tilde{h}_{11} \neq 0.
\end{equation}

(3) $y$ is totally isotropic in $S^6$, i.e., it comes from the twistor projection of some holomorphic or anti-holomorphic curve into the twistor bundle $\mathbb{T}S^6$ of $S^6$. In this case,

\begin{equation}
\hat{B}_1 = \begin{pmatrix} h_{10}v_1 & ih_{10}v_1 & h_{30}v_1 + h_{40}v_2 & h_{30}v_1 + h_{40}v_2 \end{pmatrix}
\end{equation}

with $v_1$ and $v_2$ of the form (2.20) and

\begin{equation}
(\tilde{h}_{11}'^2 + |\tilde{h}_{13}'|^2)(|h_{30}|^2 + |h_{40}|^2)\tilde{h}_{11} \neq 0.
\end{equation}

In all of above cases, $h_{10}$, $h_{20}$, $h_{30}$, $h_{40}$, $h_{11}$ and $\tilde{h}_{13}$ are assumed to be meromorphic functions on $S^2$ such that $\eta_-$ has a global meromorphic framing on $S^2$, i.e., there exists some meromorphic framing $F_- : S^2 \to \Lambda^{-1} G_\sigma^C$ satisfying $F_-^{-1}dF_- = \eta_-$. 

**Proof.** Setting $m = 4$ in Theorem 2.6, we see that there are three kinds of possible nilpotent normalized potentials from (2.14), (2.15) and (2.16):

\begin{equation}
\hat{B}_1 = \begin{pmatrix} h_{11} & h_{11} & h_{12} & \hat{h}_{12} \\ h_{11} & h_{11} & h_{12} & \hat{h}_{12} \\ h_{13} & h_{13} & h_{32} & h_{32} \\ ih_{31} & ih_{31} & ih_{32} & ih_{32} \end{pmatrix},
\end{equation}

\begin{equation}
\hat{B}_1 = \begin{pmatrix} h_{11} & ih_{11} & h_{12} & \hat{h}_{12} \\ h_{21} & ih_{21} & h_{12} & \hat{h}_{12} \\ h_{31} & ih_{31} & h_{32} & \hat{h}_{32} \\ h_{41} & ih_{41} & ih_{32} & ih_{32} \end{pmatrix},
\end{equation}

and

\begin{equation}
\hat{B}_1 = \begin{pmatrix} h_{11} & ih_{11} & h_{12} & ih_{12} \\ h_{21} & ih_{21} & h_{22} & ih_{22} \\ h_{31} & ih_{31} & h_{32} & ih_{32} \\ h_{41} & ih_{41} & h_{42} & ih_{42} \end{pmatrix}.
\end{equation}
We see that (2.17) comes from (2.24).

We need to show that (2.19) comes from (2.25) and (2.22) comes from (2.26). Assume that \( \hat{B}_1 = ( \hat{v}_1 \hat{v}_2 \hat{v}_3 \hat{v}_4 ) \). Notice that (2.11) requires

\[
v_j^I I_{1,3} \tilde{v}_k = 0, \quad \text{for all } j, k = 1, \cdots , 4.
\]

Especially, \( v_j^I I_{1,3} \tilde{v}_j = 0 \). So, we can assume that \( \tilde{v}_1 = h_{10} v_1 \) (see also [19]), with

\[
v_1 = \begin{pmatrix} 1 + \bar{h}_{11} \bar{h}_{13} \\ -1 + \bar{h}_{11} \bar{h}_{13} \\ \bar{h}_{11} + \bar{h}_{13} \\ i(\bar{h}_{11} - \bar{h}_{13}) \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} \bar{h}_{11} \\ \bar{h}_{11} \\ 1 \\ i \end{pmatrix}.
\]

Then the conditions \( v_j^I I_{1,3} \tilde{v}_1 = v_j^I I_{1,3} \tilde{v}_j = 0 \) and \( \tilde{v}_j \in \mathbb{R}^{1,3} \otimes \mathbb{C}, \ j = 3, 4 \), indicates that

\[
\tilde{v}_j = \tilde{h}_{j0} v_1 + \tilde{h}_{j0} v_2, \ j = 3, 4,
\]

for some functions \( \tilde{h}_{j0}, \tilde{h}_{j0} \). Together with the restrictions of (2.25) and (2.26), (2.19) and (2.22) follow straightforwardly.

As to the geometric descriptions, Case (1) follows from Theorem 2.1 of [25] and Case (3) follows from Theorem 3.1 of [26]. For Case (2), \( y \) is not S-Willmore since \( \text{rank}(\hat{B}_1) = 2 \). Assume \( \langle \kappa, \kappa \rangle \equiv 0 \), then \( \langle D_2 \kappa, \kappa \rangle \equiv 0 \) and \( \langle D_2 \kappa, D_2 \kappa \rangle = -\langle D_2 D_2 \kappa, \kappa \rangle \equiv 0 \), since \( \kappa \) satisfies the Willmore equation \( D_2 D_2 \kappa + \frac{2}{3} \kappa = 0 \). Therefore \( \kappa \) being isotropic is equivalent to \( B_1 B_1^T I_{1,3} \equiv 0 \), which is equivalent to \( \hat{B}_1 \hat{B}_1^T I_{1,3} \equiv 0 \) by Wu’s formula [30], [11]. On the other hand \( \hat{B}_1 \hat{B}_1^T I_{1,3} \equiv 0 \) is equivalent to \( h_{30}^2 + h_{40}^2 = 0 \), which is not allowed due to (2.24). Therefore \( \kappa \) is not isotropic.

\[
\Box
\]

**Corollary 2.9.** Let \( y : S^2 \to S^4 \) be a strong Willmore map with the normalized potential of its the conformal Gauss map

\[
\eta = \lambda^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^T I_{1,3} & 0 \end{pmatrix} dz = \lambda^{-1} \eta_{-1} dz.
\]

Then \( y \), as well as \( \eta \), belongs to one of the cases:

1. \( y \) is conformally equivalent to a complete minimal surface in \( \mathbb{R}^4 \) with planar ends. In this case, \( r(\xi) = 2 \) and

\[
(2.27) \quad \hat{B}_1 = \begin{pmatrix} h_{10} v_1 \\ h_{20} v_1 \end{pmatrix},
\]

with

\[
(2.28) \quad v_1^I = ( \bar{h}_{11} \bar{h}_{11} \bar{h}_{13} \bar{h}_{13} ) \quad \text{and} \quad |\bar{h}_{11}|^2 + |\bar{h}_{13}|^2 \neq 0.
\]

2. \( y \) is isotropic in \( S^4 \), therefore it comes from the twistor projection of a holomorphic or anti-holomorphic curve into the twistor bundle \( \mathbb{C}P^3 \) of \( S^4 \). In this case, \( r(\xi) = 2 \) and

\[
(2.29) \quad \hat{B}_1 = \begin{pmatrix} h_{10} v_1 \\ ih_{10} v_1 \end{pmatrix}
\]

with \( v_1 \) being of the form (2.20).

In all of the cases above, \( h_{10}, h_{20}, \bar{h}_{11} \) and \( \bar{h}_{13} \) are meromorphic functions on \( S^2 \).

In particular, both kinds of Willmore maps as above are S-Willmore.

**Proof.** Restricting to \( S^4 \), it is easy to see that Case (2) of Theorem 2.8 can not happen. The isotropic case is a corollary of Theorem 2.8. The last statement comes from the simple observation that in both cases, \( \text{rank} \hat{B}_1 = 1 \). \( \Box \)
Remark 2.10. It is usually not easy to check whether \( y \) will be an immersion. Fortunately we do have an example of type (3) which is a non-S-Willmore Willmore immersion. See the next subsection for details.

The maps of type (3) in Theorem 2.8 are full in some even dimensional spheres. And the maps of type (2) sometimes can reduce to maps into some \( S^5 \subset S^6 \). Therefore we obtain

Corollary 2.11. Let \( y : S^2 \to S^5 \) be a Willmore map, which is not S-Willmore. Then there exists a normalized potential of \( y \) being of the form

\[
\eta = \lambda^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{pmatrix} dz, \quad \text{with} \quad \hat{B}_1 = \begin{pmatrix} h_1 h_0 & i h_1 h_0 & h_2 & 0 \\ -h_1 h_0 & -i h_1 h_0 & h_2 & 0 \\ h_1 & i h_1 & h_2 h_0 & 0 \\ -i h_1 & h_1 & i h_2 h_0 & 0 \end{pmatrix}
\]

with \( h_j, j = 0, 1, 2 \), non-constant meromorphic functions on \( S^2 \). Note that \( \hat{B}_1 \) is of type (2) in Theorem 2.8 (excluding type (1) and type (3) in Theorem 2.8).

If one chooses the \( h_j \) as polynomials in \( z \), then one will obtain a possibly branched Willmore two-sphere in \( S^5 \) which is not S-Willmore.

In [20], a Willmore two-sphere in \( S^5 \) which is not S-Willmore is provided. This example is given by some adjoint transforming of a minimal surface in \( \mathbb{R}^5 \) with isotropic Hopf differential and special ends [20]. Moreover, applying the classification theorem in [20], one obtains

Corollary 2.12. The Willmore 2-spheres in \( S^5 \) obtained in the corollary above can be derived by some adjoint transform of a minimal surface in \( \mathbb{R}^5 \) with isotropic Hopf differential and special ends.

We refer to [20] for more details.

Corollary 2.13. Let \( \eta \) be a normalized potential with \( \hat{B}_1 \) of the form (type (3) in Theorem 2.8)

\[
\hat{B}_1 = \begin{pmatrix} h_1 h_0 & i h_1 h_0 & h_2 & i h_2 \\ -h_1 h_0 & -i h_1 h_0 & h_2 & i h_2 \\ h_1 & i h_1 & h_2 h_0 & i h_2 h_0 \\ -i h_1 & h_1 & i h_2 h_0 & -h_2 h_0 \end{pmatrix}
\]

with \( h_j, j = 0, 1, 2 \), non-constant meromorphic functions on \( T^2 \). Moreover, if the following six integrals

\[
\begin{align*}
h_1 &= \int h_1 dz, \quad h_2 = \int h_2 dz, \quad h_{10} = \int h_1 h_0 dz, \quad h_{20} = \int h_2 h_0 dz, \\
h_{31} &= -\int h_{10} h_2 dz + \int h_1 h_2 h_0 dz, \quad h_{32} = -\int h_2 h_1 h_0 dz + \int h_2 h_1 dz
\end{align*}
\]

are also meromorphic functions on \( T^2 \), one will obtain a totally isotropic strong Willmore map with trivial monodromy which is full in \( S^6 \). [2], [3] and [18].

It is not difficult to satisfy the conditions above. For example, let \( \hat{h}_1, \hat{h}_2 \) be meromorphic functions on \( T^2 \). Then \( h_1 = h_2 = \hat{h}_1, h_0 = \frac{\hat{h}_2}{\hat{h}_1} \) will provide a totally isotropic strong Willmore map full in \( S^6 \). To determine when such a Willmore map will be an immersion will be a highly non-trivial and interesting problem, see for example [1]. Note that solving similar problems is an important topic in the minimal surface theory.
2.3. A concrete example. To derive concrete examples, one needs to work out the Iwasawa decompositions in a concrete fashion. We will leave for [26]. Here we will just show one example of Willmore 2-sphere which is fully immersed in to $S^6$. This is the first Willmore two-sphere admitting no dual surface.

Theorem 2.14. [11, 26] Let

\[
\eta = \lambda^{-1} \left( \begin{array}{ccc} 0 & \hat{B}_1 & \hat{B}_1 \end{array} \right) dz, \quad \text{with} \quad \hat{B}_1 = \frac{1}{2} \begin{pmatrix} 2iz & -2z & -i & 1 \\ -2iz & 2z & -i & 1 \\ -2 & -2i & -z & -iz \\ 2i & -2 & -iz & z \end{pmatrix}.
\]

Then $\hat{B}_1$ is of type (3) in Theorem 2.8 and the associated family of unbranched Willmore two-spheres $x_\lambda, \lambda \in S^1$, corresponding to $\eta$, is

\[
x_\lambda = \frac{1}{1 + r^2 + \frac{5r^4}{4} + \frac{4r^6}{9} + \frac{r^8}{36}} \begin{pmatrix} (1 - r^2 - \frac{3r^4}{4} + \frac{4r^6}{9} - \frac{r^8}{36}) \\ -i(z - \bar{z})(1 + \frac{r^6}{9}) \\ (z + \bar{z})(1 + \frac{r^6}{9}) \\ -i\left(\lambda^{-1}z^2 - \lambda\bar{z}^2\right)(1 - \frac{r^4}{12}) \\ \left(\lambda^{-1}z^2 + \lambda\bar{z}^2\right)(1 - \frac{4r^2}{12}) \\ -i\frac{r^2}{2}\left(\lambda^{-1}z - \lambda\bar{z}\right)(1 + \frac{4r^2}{3}) \\ \frac{r^2}{2}\left(\lambda^{-1}z + \lambda\bar{z}\right)(1 + \frac{4r^2}{3}) \end{pmatrix},
\]

with $r = |z|$. Note that for every $\lambda \in S^1$, $x_\lambda$ is isometric to the other ones by some rotation in $SO(7)$. Moreover $x_\lambda : S^2 \to S^6$ is a Willmore immersion in $S^6$, which is full, not $S$-Willmore, and totally isotropic.

Remark 2.15. It has been shown in [11] (see also [23, 24]) that there exist $S$-Willmore two-spheres which are obtained by the twistor projection of holomorphic or anti-holomorphic curves in the twistor bundle $\Sigma S^{2n}$ of $S^{2n}$ (for a general theory about twistor geometry, we refer to [7]). Our example shows that Willmore two-spheres derived in this way can also be non-$S$-Willmore. And we also note that even in $S^6$, in general, the surfaces obtained by the twistor projection of holomorphic curves of the twistor bundle $\Sigma S^6$ will not be Willmore.

3. Classification of Nilpotent Lie subalgebras in $SO^+(1, n + 3, \mathbb{C})$

In this section we will give a proof of Theorem 2.6 by describing all nilpotent Lie subalgebras of $\mathfrak{so}(1, n + 3, \mathbb{C})$ which correspond to the symmetric space $SO^+(1, n + 3)/SO^+(1, 3) \times SO(n)$. We divide this proof into three steps:

(i). Realizing $SO^+(1, n + 3, \mathbb{C})$ by the group $G(n + 4, \mathbb{C})$, such that the nilpotent Lie subalgebras are conjugated into some upper-triangular Lie subalgebras. This will simplify computations considerably.

(ii). Classifying the corresponding Lie subalgebras in $\mathfrak{g}(n + 4, \mathbb{C})$.

(iii). Going back to $\mathfrak{so}(1, n + 3, \mathbb{C})$ and using the special condition $B_1^T I_{1,3} B_1 = 0$ to obtain the classification of normalized potentials of strongly conformally harmonic maps.

Correspondingly we have three subsections.

3.1. Realization of $SO^+(1, n + 4, \mathbb{C})$ by $G(n + 4, \mathbb{C})$. Set

\[
G(n + 4, \mathbb{C}) := \{ A \in Mat(n + 4, \mathbb{C}) | A^t J_{n+4} A = J_{n+4}, \det A = 1 \},
\]
with

\[ J_{n+4} = \begin{pmatrix} 1 & 1 & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} = (j_{k,l})_{(n+4) \times (n+4)} \] where \( j_{k,l} = \delta_{j+l,n+5} \).

We recall that \( SO^+(1, n + 3) = SO(1, n + 3)_0 \), the connected subgroup of

\[ SO(1, n + 3) := \{ A \in Mat(n + 4, \mathbb{C}) \mid A^t I_{1,n+3} A = I_{1,n+3}, \det A = 1, \ A = \bar{A} \}, \]

and the connected subgroup \( K = SO^+(1, 3) \times SO(n) \subset SO^+(1, n + 3) \) is defined by the involution

\[ \sigma: SO^+(1, n + 3) \rightarrow SO^+(1, n + 3) \]

\[ A \mapsto DAD^{-1}. \]

with \( D = \text{diag}\{-I_4, I_n\} \). For simplicity, we assume that \( n + 4 = 2m \) in the following computations.

**Lemma 3.1.** Set

\[ \tilde{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & & -1 \\ & 1 & & \\ -i & & i & \\ & 1 & & \\ & & \ddots & \\ -i & i & & \\ & & & 1 \end{pmatrix}. \]

Then we have

\[ J_{2m} = \tilde{P}^t I_{1,2m-1} \tilde{P}. \]

Let \( A, \tilde{A} \in Mat(2m, \mathbb{C}) \) be two matrices satisfying \( \tilde{A} = \tilde{P}^{-1} A \tilde{P} \). Then \( A \in SO^+(1, 2m - 1, \mathbb{C}) \) if and only if \( \tilde{A} \in G(2m, \mathbb{C}) \), i.e., \( A \) satisfies \( A^t I_{1,n+3} A = I_{1,n+3} \) if and only if \( \tilde{A} \) satisfies \( \tilde{A}^t J_{2m} \tilde{A} = J_{2m} \). This defines a Lie group isomorphism

\[ \mathcal{P}: SO^+(1, 2m - 1, \mathbb{C}) \rightarrow G(2m, \mathbb{C}) \]

\[ A \rightarrow \tilde{A} = \tilde{P}^{-1} A \tilde{P} \]

Moreover, we have

1. (Reality) The image of \( SO^+(1, 2m - 1) \) under \( \mathcal{P} \) is the connected components of the subgroup \( \{ F \in G(2m, \mathbb{C}) \mid F = S^{-1}_{2m} FS_{2m} \} \) containing \( I_{2m} \). Here \( "-" \) denotes complex conjugation, and

\[ S_{2m} = \tilde{P}^{-1} \tilde{P} = \begin{pmatrix} 1 & & \end{pmatrix} J_{2m-2} \begin{pmatrix} \end{pmatrix}. \]

This induces an involution of \( \Lambda G(2m, \mathbb{C}) \)

\[ \hat{\tau}: \Lambda G(2m, \mathbb{C}) \rightarrow \Lambda G(2m, \mathbb{C}) \]

\[ F \mapsto S^{-1}_{2m} FS_{2m} \]

with \( \mathcal{P}(\Lambda SO^+(1, 2m - 1)) \) as its fixed point set:

\[ \mathcal{P}(\Lambda SO^+(1, 2m - 1)) = \{ F \in \Lambda G(2m, \mathbb{C}) \mid \hat{\tau}(F) = F \}. \]
(2). (Symmetric space) The image of $SO^+(1, 3) \times SO(2m - 4)$ under the map $\mathcal{P}$ is of the form

$$\mathcal{P}\left((SO^+(1, 3) \times SO(2m - 4))^C\right) = \{ F \in G(2m, \mathbb{C}) \mid F = D_0^{-1} F D_0 \}$$

with

$$D_0 = \tilde{P}^{-1} D \tilde{P} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & I_{2m-4} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$  

(3). (Isomorphism of Lie algebras) $\mathcal{P}$ induces an isomorphism of Lie algebras from $\mathfrak{so}(1, 2m - 1, \mathbb{C})$ to $\mathfrak{g}(2m, \mathbb{C})$. Let $A = (a_{jk})_{2m \times 2m} \in \mathfrak{so}(1, 2m - 1, \mathbb{C})$. Then we have

$$a_{jj} = 0, \ j = 1, \cdots, 2m, \ \text{and} \ a_{1j} = a_{j1}, \ a_{jk} = -a_{kj}, \ j, k = 2, \cdots, 2m.$$  

Assume that

$$\mathcal{P}(A) = \tilde{P}^{-1} A \tilde{P} = \begin{pmatrix} \hat{a}_1 & \hat{f}_1 & \hat{a}_2 \\ -\hat{f}_2 & \hat{a}_3 & -\hat{f}_1 \\ \hat{a}_3 & \hat{f}_2 & \hat{a}_4 \end{pmatrix}.$$  

Then

$$\left\{ \begin{array}{l}
\hat{a}_1 = \frac{1}{2} \begin{pmatrix} 2a_{12} & -ia_{13} - ia_{23} + a_{14} + a_{24} \\ ia_{13} - ia_{23} + a_{14} - a_{24} & 2ia_{34} \end{pmatrix}, \\
\hat{a}_2 = \frac{1}{2} \begin{pmatrix} ia_{13} + ia_{23} + a_{14} + a_{24} & 0 \\ 0 & -ia_{13} - ia_{23} - a_{14} - a_{24} \end{pmatrix}, \\
\hat{a}_3 = \frac{1}{2} \begin{pmatrix} -ia_{13} + ia_{23} + a_{14} - a_{24} \\ 0 & ia_{13} - ia_{23} - a_{14} + a_{24} \end{pmatrix}, \\
\hat{a}_4 = -J_2 \hat{a}_1 J_2,
\end{array} \right.$$  

and

$$\hat{f}_1 = \begin{pmatrix} \hat{f}_{13} & \cdots & \hat{f}_{1,m} & \hat{f}_{1,m+1} & \cdots & \hat{f}_{1,2m-2} \\ \hat{f}_{23} & \cdots & \hat{f}_{2,m} & \hat{f}_{2,m+1} & \cdots & \hat{f}_{2,2m-2} \end{pmatrix},$$
$$\hat{f}_2 = \begin{pmatrix} \hat{f}_{2m-1,3} & \cdots & \hat{f}_{2m-1,m} & \hat{f}_{2m-1,m+1} & \cdots & \hat{f}_{2m-1,2m-2} \\ \hat{f}_{2m,3} & \cdots & \hat{f}_{2m,m} & \hat{f}_{2m,m+1} & \cdots & \hat{f}_{2m,2m-2} \end{pmatrix},$$

with

$$\left\{ \begin{array}{l}
\hat{f}_{1,j} = \frac{1}{2} (-ia_{1,2j-1} - ia_{2,2j-1} + a_{1,2j} + a_{2,2j}), \\
\hat{f}_{1,2m+1-j} = \frac{1}{2} (+ia_{1,2j-1} + ia_{2,2j-1} + a_{1,2j} + a_{2,2j}), \ j = 3, \cdots, m, \\
\hat{f}_{2,j} = \frac{1}{2} (+a_{3,2j-1} - ia_{4,2j-1} + ia_{3,2j} + a_{4,2j}), \\
\hat{f}_{2,2m+1-j} = \frac{1}{2} (-a_{3,2j-1} + ia_{4,2j-1} + ia_{3,2j} + a_{4,2j}),
\end{array} \right.$$
and
\begin{equation}
\begin{aligned}
\hat{f}_{2m-1,j} &= \frac{1}{2} (-a_{3,2j-1} - ia_{4,2j-1} - ia_{3,2j} + a_{4,2j}), \\
\hat{f}_{2m-1,2m+1-j} &= \frac{1}{2} (+a_{3,2j-1} + ia_{4,2j-1} - ia_{3,2j} + a_{4,2j}), \\
\hat{f}_{2m,j} &= \frac{1}{2} (+ia_{1,2j-1} - ia_{2,2j-1} - a_{1,2j} + a_{2,2j}), \\
\hat{f}_{2m,2m+1-j} &= \frac{1}{2} (-ia_{1,2j-1} + ia_{2,2j-1} - a_{1,2j} + a_{2,2j}),
\end{aligned}
\tag{3.12}
\end{equation}

Proof. (3.4) comes from a straightforward multiplication of matrices and the rest statements before (1) is a corollary of (3.4).

(1). Since \( S_{2m} = \tilde{P}^{-1} \tilde{P} \), we have
\[ \mathcal{P}(SO(1,2m-1)) = \{ \tilde{P}^{-1} A \tilde{P} | A \in SO(1,2m-1) \} \]
\[ = \{ \tilde{A} | \tilde{A} = \tilde{P}^{-1} A \tilde{P} \text{ for some } A \in SO(1,2m-1,\mathbb{C}), A = \tilde{A} \} \]
\[ = \{ \tilde{A} | \tilde{A} \in G(2m,\mathbb{C}), \tilde{P} \tilde{A} \tilde{P}^{-1} = \tilde{P} A \tilde{P}^{-1} \} \]
\[ = \{ \tilde{A} | \tilde{A} \in G(2m,\mathbb{C}), \tilde{A} = \tilde{P}^{-1} \tilde{P} A \tilde{P}^{-1} \tilde{P} \} \]
\[ = \{ \tilde{A} | \tilde{A} \in G(2m,\mathbb{C}), \tilde{A} = S_{2m}^{-1} \tilde{A} S_{2m} \}. \]
The rest of (1) follows directly.

(2). Similarly, we have
\[ \mathcal{P} \left( (SO^+(1,3) \times SO^+(2m-4))^{\mathbb{C}} \right) = \{ \tilde{P}^{-1} A \tilde{P} | A \in SO^+(1,2m-1,\mathbb{C}), AD = DA \} \]
\[ = \{ \tilde{A} | \tilde{A} = \tilde{P}^{-1} A \tilde{P}, A \in SO^+(1,2m-1,\mathbb{C}), AD = DA \} \]
\[ = \{ \tilde{A} \in G(2m,\mathbb{C}) | \tilde{P} \tilde{A} \tilde{P}^{-1} D = D \tilde{P} A \tilde{P}^{-1} \} \]
\[ = \{ F \in G(2m,\mathbb{C}) \mid F = D_0^{-1} F D_0 \}, \]
with \( D_0 = \tilde{P}^{-1} D \tilde{P} = \text{diag}\{-I_2,I_{2m-4},-I_2\} \).

(3). The proof is derived by straightforward multiplications of matrices. So we leave it to interested readers. Note that in fact one only needs to verify above formulas by setting \( m = 3 \).

The general case is essentially the same as the case \( m = 3 \). So we leave it to the interested reader to verify the formulas above. \( \square \)

3.2. The nilpotent Lie subalgebras in \( \mathfrak{g}(n+4,\mathbb{C}) \) associated with \( SO^+(1,n+3)/SO^+(1,3) \times SO(n) \). Now let us turn to the classification of nilpotent Lie subalgebras in \( \mathfrak{g}(n+4,\mathbb{C}) \) related with the symmetric space \( \mathcal{P} \left( SO^+(1,n+3)/SO^+(1,3) \times SO(n) \right) \).

According to Theorem 4.11 of \[12\], the nilpotent Lie subalgebras in question are in one to one relation with canonical elements \( \xi \). So the first step in our classification will be to derive all canonical elements \( \xi \) related with the symmetric space \( SO^+(1,n+3)/SO^+(1,3) \times SO(n) \). Then after computation of the spaces \( \sum_{j>0} g_j^\xi \), we will obtain the desired nilpotent Lie subalgebras.

Recall that \( \mathfrak{g}(2m,\mathbb{C}) = \{ A \in Mat(2m,\mathbb{C}) | A^t J_{2m} + J_{2m} A = 0 \} \). We choose a maximal torus
\[ t = \begin{pmatrix}
ia_{11} & & & \\
& \ddots & \ddots & \\
& & -ia_{mm} & \\
& & & \ddots & \\
& & & & -ia_{11}
\end{pmatrix}, \]
with \( t \subset \mathfrak{k} \subset \mathfrak{g}(2m, \mathbb{C}) \). Set
\[
(3.13) \quad \hat{\xi}_1 = \text{diag}\{i, 0, \ldots, 0, 0, \ldots, 0, -i\}, \quad \ldots, \quad \hat{\xi}_m = \text{diag}\{0, \ldots, 0, i, -i, 0, \ldots, 0\}.
\]
So any \( \xi \in t \) can be expressed as \( \xi = a_{11}\hat{\xi}_1 + \cdots + a_{mm}\hat{\xi}_m \). Moreover, for any \( \xi \in t \) we have
\[
(3.14) \quad g = g(2m, \mathbb{C}) = \sum_{j = -r(\xi)}^{r(\xi)} g_j^\xi, \quad \text{with} \quad g_j^\xi := \{X \in g(2m, \mathbb{C})| [\xi, X] = i \cdot j \cdot X\},
\]
and \( r(\xi) \) being the maximal \( j \) satisfying \( g_j^\xi \neq \{0\} \).

Denote by \( E_{jk} \) the \( 2m \times 2m \) matrix with the \((j, k)\)–entry being 1 and all other entries being zero. Set
\[
\tilde{E}_{jk} = E_{jk} - E_{2m+1-k,2m+1-j}, \quad \text{for all} \quad 1 \leq j, k \leq 2m, \quad j + k \leq 2m.
\]
So \( \tilde{E}_{jk} \in g(2m, \mathbb{C}) \), and it is straightforward to verify

**Lemma 3.2.** For all \( 1 \leq j, k \leq 2m, \quad j + k \leq 2m, \quad 1 \leq l \leq m \), we have
\[
(3.15) \quad [\hat{\xi}_l, \tilde{E}_{jk}] = \sqrt{-1}(\delta_{lj} - \delta_{lk} + \delta_{l,2m+1-k} - \delta_{l,2m+1-j})\tilde{E}_{jk}.
\]

Hence
\[
(3.16) \quad \left\{\begin{array}{l}
g_1^\hat{\xi} = \text{Span}_C \{\tilde{E}_{jk} | j = l, 1 \leq k \leq 2m - l, k \neq l \} \quad \text{or} \quad 1 \leq j \leq l - 1, k = 2m + 1 - l\}, \\
g_0^\hat{\xi} = \text{Span}_C \{\tilde{E}_{jk} | 1 \leq j, k, j + k \leq 2m, j, k \notin \{l, 2m + 1 - l\} \} \quad \text{or} \quad j = k = l\}, \\
g_{-1}^\hat{\xi} = \text{Span}_C \{\tilde{E}_{jk} | k = l, 1 \leq j \leq 2m - l, j \neq l \} \quad \text{or} \quad 1 \leq k \leq l - 1, j = 2m + 1 - l\}, \\
g_j^\hat{\xi} = \{0\}, \quad \text{for all} \quad |j| > 1.
\]

So \( r(\hat{\xi}_l) = 1 \) for all \( l, 1 \leq l \leq m \).

Next we turn to express the simple roots by use of \( \{\hat{\xi}_j\} \). Let \( \{\hat{\xi}_1, \ldots, \hat{\xi}_m\} \) be an arbitrary permutation of \( \{\hat{\xi}_1, \ldots, \hat{\xi}_m\} \). Let \( \hat{\theta}_j \) be the dual of \( \hat{\xi}_j \); i.e.,
\[
\hat{\theta}_j(\hat{\xi}_k) = \sqrt{-1}\delta_{jk}, \quad \text{for all} \quad j, k = 1, \ldots, m.
\]

By standard Lie group theory, the roots of \( \mathfrak{g}(2m, \mathbb{C}) \) are
\[
\{\pm(\hat{\theta}_j - \hat{\theta}_k), \quad \pm(\hat{\theta}_j + \hat{\theta}_k), \quad 1 \leq j < k \leq m\}.
\]
Let
\[
\{(\hat{\theta}_j - \hat{\theta}_k), \quad (\hat{\theta}_j + \hat{\theta}_k), \quad 1 \leq j < k \leq m\}
\]
be the set of positive roots. Then it is straightforward to obtain that the simple roots \( \{\theta_1, \ldots, \theta_m\} \) of \( \mathfrak{g}(2m, \mathbb{C}) \) can be expressed as
\[
\theta_j = \hat{\theta}_j - \hat{\theta}_{j+1}, \quad \text{for} \quad j = 1, \ldots, m - 1, \quad \theta_m = \hat{\theta}_{m-1} + \hat{\theta}_m.
\]

Let \( \{\xi_1, \ldots, \xi_m\} \) be the dual of the simple roots \( \{\theta_1, \ldots, \theta_m\} \), i.e., \( \{\xi_j\} \) satisfies
\[
\theta_j(\xi_k) = \sqrt{-1}\delta_{jk} \quad \text{for all} \quad j, k = 1, \ldots, m.
\]

Then we obtain
\[
(3.17) \quad \xi_j = \sum_{k=1}^{j} \hat{\xi}_k, \quad 1 \leq j \leq m - 2, \quad \xi_{m-1} = \frac{1}{2}\left(\sum_{j=1}^{m-1} \hat{\xi}_j - \hat{\xi}_m\right), \quad \xi_m = \frac{1}{2}\left(\sum_{j=1}^{m-1} \hat{\xi}_j + \hat{\xi}_m\right).
\]
Lemma 3.3. Let \( \xi = \xi_{j_1} + \cdots + \xi_{j_r} \) be a canonical element. Then \( \xi \) has the form
\[
\xi = n_1 \xi_1 + n_2 \xi_2 + \cdots + n_m \xi_m.
\]
Moreover, there exists \( A \in G(2m, \mathbb{C}) \) such that
\[
\text{Fix}_\xi = \{ g \in G(2m, \mathbb{C}), \exp(\pi \xi)g \exp(\pi \xi)^{-1} = g \} = A \cdot K^C \cdot A^{-1}
\]
if and only if \( n_1, \ldots, n_m \) satisfy the following conditions:
1. \( n_1, \ldots, n_m \in \mathbb{Z} \);
2. \( \max\{m-1,4\} \geq n_1 \geq \cdots \geq n_m = 0 \);
3. \( 1 \geq n_j - n_{j+1} \geq 0, j = 1, \ldots, m-1 \);
4. \( \sharp\{ n_j | n_j \text{ is odd} \} = 2, \text{ or } \sharp\{ n_j | n_j \text{ is even} \} = 2 \).

Proof. Recall that in our case, \( K^C = P((SO^+(1,3) \times SO(n))^C) \) is defined as the fixed point set of \( D_0 = \text{diag}\{-1,-1,I_{2m-4},-1,-1\} \).

It is straightforward to see that if \( n_1, \ldots, n_m \) satisfy (1)-(4), then there exists some permutation matrix \( A \in G(2m, \mathbb{C}) \) such that \( \exp(\pi \xi) = AD_0A^{-1} \) or \( \exp(\pi \xi) = -AD_0A^{-1} \) and then \( \text{Fix}_\xi = A \cdot K^C \cdot A^{-1} \).

On the other hand, let \( \xi \) be a canonical element such that \( \text{Fix}_\xi = A \cdot K^C \cdot A^{-1} \) for some \( A \in G(2m, \mathbb{C}) \). Then \( \exp(\pi \xi) = AD_0A^{-1} \) or \( \exp(\pi \xi) = -AD_0A^{-1} \). Therefore \( \exp(\pi \xi) \) has the same eigenvalues as \( D_0 \) or \( -D_0 \), i.e., \( \sharp\{ n_j | n_j \text{ is odd} \} = 2, \text{ or } \sharp\{ n_j | n_j \text{ is even} \} = 2 \). It has only the eigenvalues \( \pm 1 \), showing that it is the sum of some of the elements \( \xi_1, \ldots, \xi_{m-2}, \xi_{m-1} + \xi_m \). Then (1), (2) and (3) follow easily.

Applying Lemma 3.3, we obtain

Lemma 3.4. There are \( \frac{1}{2}m(m-1) \) types of nilpotent Lie subalgebras related with \( SO^+(1,n+3)/SO^+(1,3) \times SO(n) \) in \( G(n+4, \mathbb{C}) \), \( n+4 = 2m \). Up to conjugation, the corresponding canonical elements \( \xi \) is given by one of the following \( \frac{1}{2}m(m-1) \) elements:

\[
(3.18a) \quad 2\hat{\xi}_1 + \hat{\xi}_2,
\]
\[
(3.18b) \quad \hat{\xi}_1 + \hat{\xi}_2 + 2 \sum_{j=3}^{l} \hat{\xi}_j, \quad 3 \leq l \leq m-1;
\]
\[
(3.18c) \quad 3\hat{\xi}_1 + \hat{\xi}_2 + 2 \sum_{j=3}^{l} \hat{\xi}_j, \quad 3 \leq l \leq m-1;
\]
\[
(3.18d) \quad 3\hat{\xi}_1 + \hat{\xi}_2 + 4 \sum_{j=3}^{l} \hat{\xi}_j + 2 \sum_{j=l+1}^{t} \hat{\xi}_j, \quad 3 \leq l < t \leq m-1;
\]
\[
(3.18e) \quad \sum_{j=3}^{m} \hat{\xi}_j,
\]
\[
(3.18f) \quad 2\hat{\xi}_1 + \sum_{j=3}^{m} \hat{\xi}_j,
\]
\[
(3.18g) \quad 2\hat{\xi}_1 + 3 \sum_{j=3}^{k} \hat{\xi}_j + \sum_{j=k+1}^{m} \hat{\xi}_j, \quad 3 \leq k \leq m-1.
\]

Proof. From the proof of Lemma 3.3 we see that to obtain \( K^C \) as fixed point set of \( \text{Ad} \exp(\pi \xi) \), we need that
\[
\exp \pi \xi = D_0 \quad \text{or} \quad \exp \pi \xi = -D_0.
\]
We define the subsets of $p_{NI}$ (3.19)

If $\max\{n_j\} \leq 2$, there are $m - 2$ types of choices of $\xi$ (up to conjugation):

$$\hat{\xi}_1 + \hat{\xi}_2, \hat{\xi}_1 + \hat{\xi}_2 + 2 \sum_{k=1}^{j} \hat{\xi}_k, \ 3 \leq j \leq m - 1.$$ 

If $\max\{n_j\} = 3$, there are $m - 3$ types of choices of $\xi$:

$$3\hat{\xi}_1 + \hat{\xi}_2 + 2 \sum_{k=1}^{j} \hat{\xi}_k, \ 3 \leq j \leq m - 1.$$ 

If $\max\{n_j\} = 4$, let $t$ be the number of non-zero $n_j$ in $\xi$. We have that $4 \leq t \leq m - 1$. For every $t$, there are $t - 3$ types of choices of $\xi$:

$$3\hat{\xi}_1 + \hat{\xi}_2 + 4 \sum_{j=3}^{l} \hat{\xi}_j + 2 \sum_{j=l+1}^{m} \hat{\xi}_j, \ 3 \leq l < t \leq m - 1.$$ 

So altogether there are $\frac{1}{3}(m - 3)(m - 4)$ types of canonical elements when $\max\{n_j\} = 4$.

(ii). For the case $\exp \pi \xi = -D_0$, we see that there are $m - 1$ types of choices of $\xi$:

$$\sum_{j=3}^{m} \hat{\xi}_j, \ 2\hat{\xi}_1 + 3 \sum_{j=3}^{k} \hat{\xi}_j + 2 \sum_{j=k+1}^{m} \hat{\xi}_j, \ 3 \leq k \leq m - 1.$$ 

In a sum, we have that the number of these canonical elements mentioned above is $m - 2 + m - 3 + \frac{1}{2}(m - 3)(m - 4) + m - 1 = \frac{1}{2}m(m - 1)$. \hfill $\square$

From the definition of $\hat{\xi}_j$ in (3.13), for every $\xi$ of the form in (3.18) it is straightforward to see that $g_\xi^{2j+1} \subset p^C$, $g_\xi^{2j} \subset \mathfrak{t}^C$, for all $j \in \mathbb{Z}$.

Now let’s turn to the discussion of the nilpotent Lie algebra $\sum_{j>0} g_\xi^j$ for some canonical element $\xi$ in Lemma 3.4. Moreover, as stated in Theorem 4.13 of [12], to classify the normalized potentials, we need only an explicit description of

$$\sum_{j \geq 0} g_\xi^{2j+1} = \sum_{j>0} g_\xi^j \cap p^C.$$ 

To this end, we first consider matrices in $p^C$ in $\mathfrak{g}(2m, \mathbb{C})$, which have the form

$$\begin{pmatrix} 0 & \tilde{f}_1 & 0 \\ -\bar{f}_2^2 & 0 & -\bar{f}_1^2 \\ 0 & \tilde{f}_2 & 0 \end{pmatrix},$$

with

$$f_1 = \begin{pmatrix} \tilde{f}_{13} & \tilde{f}_{14} & \cdots & \tilde{f}_{1,2m-3} & \tilde{f}_{1,2m-2} \\ \tilde{f}_{23} & \tilde{f}_{24} & \cdots & \tilde{f}_{2,2m-3} & \tilde{f}_{2,2m-2} \end{pmatrix},$$

$$\bar{f}_2 = \begin{pmatrix} \tilde{f}_{2m-1,3} & \tilde{f}_{2m-1,4} & \cdots & \tilde{f}_{2m-1,2m-3} & \tilde{f}_{2m-1,2m-2} \\ \tilde{f}_{2m,3} & \tilde{f}_{2m,4} & \cdots & \tilde{f}_{2m,2m-3} & \tilde{f}_{2m,2m-2} \end{pmatrix}.$$ 

We define the subsets of $p^C$ as below.

(3.19) $NI_{22} := \left\{ X \in p^C | f_2 = 0 \right\}$. 
For $t$ satisfying $3 \leq t \leq m$, we define
\begin{equation}
NI_{2t} := \{X \in p^C | \tilde{f}_{2,j} = 0, \ 3 \leq j \leq t; \ \tilde{f}_{2m-1,j} = 0, \ 3 \leq j \leq 2m - 2. \}.
\end{equation}
(3.20)

For $l$ and $t$ satisfying $3 \leq l \leq t \leq m$, we define
\begin{equation}
NI_{lt} := \{X \in p^C | \tilde{f}_{1,j} = 0, \ 3 \leq j \leq l; \ \tilde{f}_{2,j} = 0, \ 3 \leq j \leq t, \\
\tilde{f}_{2m-1,j} = 0, \ 3 \leq j \leq 2m - t; \ \tilde{f}_{2m,j} = 0, \ 3 \leq j \leq 2m - l. \}.
\end{equation}
(3.21)

Note that
\[ \sharp\{NI_{lt}| \ 2 \leq l \leq t \leq m\} = \frac{1}{2}m(m - 1). \]

As shown in the following lemma, these subsets provide all the possible $\sum_{j \geq 0} g_{2j+1}^\xi$ related with some $\xi$ in Lemma 3.3.

\textbf{Lemma 3.5.} Let $g^\xi$ be the nilpotent Lie subalgebra given by a canonical element $\xi$ in Lemma 3.3. Then the subset $\sum_{j \geq 0} g_{2j+1}^\xi$ of $g^\xi$ is equal to one of the sets $NI_{lt}$, $2 \leq l \leq t \leq m$ defined in (3.19), (3.20) and (3.21). To be concrete, we have the following correspondence:

(3.22a) $\sum_{j \geq 0} g_{2j+1}^\xi = NI_{22}$, $r(g^\xi) = 2$ if $\xi = \hat{\xi}_1 + \hat{\xi}_2$;

(3.22b) $\sum_{j \geq 0} g_{2j+1}^\xi = NI_{lt}$, $r(g^\xi) \leq 4$ if $\xi = \hat{\xi}_1 + \hat{\xi}_2 + 2 \sum_{j = 3}^t \hat{\xi}_j$, $3 \leq l \leq m - 1$;

(3.22c) $\sum_{j \geq 0} g_{2j+1}^\xi = NI_{2t}$, $r(g^\xi) = 5$ if $\xi = 3\hat{\xi}_1 + \hat{\xi}_2 + 2 \sum_{j = 3}^t \hat{\xi}_j$, $3 \leq l \leq m - 1$;

(3.22d) $\sum_{j \geq 0} g_{2j+1}^\xi = NI_{lt}$, $r(g^\xi) \leq 8$ if $\xi = 3\hat{\xi}_1 + \hat{\xi}_2 + 4 \sum_{j = 3}^t \hat{\xi}_j + \sum_{j = l+1}^h 2\hat{\xi}_j$, $3 \leq l < h \leq m - 1$;

(3.22e) $\sum_{j \geq 0} g_{2j+1}^\xi = NI_{mm}$, $r(g^\xi) = 2$ if $\xi = \sum_{j = 3}^m \hat{\xi}_j$,

(3.22f) $\sum_{j \geq 0} g_{2j+1}^\xi = NI_{2m}$, $r(g^\xi) = 3$ if $\xi = 2\hat{\xi}_1 + \sum_{j = 3}^m \hat{\xi}_j$,

(3.22g) $\sum_{j \geq 0} g_{2j+1}^\xi = NI_{lm}$, $r(g^\xi) \leq 6$ if $\xi = 2\hat{\xi}_1 + 3 \sum_{j = 3}^l \hat{\xi}_j + \sum_{j = l+1}^m \hat{\xi}_j$, $3 \leq l \leq m - 1$.

Since the proof is a lengthy and elementary computation, we will leave it to Section 3.4.

\textbf{Example 3.6.} Here we provide an example to illustrate the classification of Lemma 3.5. When $m = 4$, there are 6 kinds of different nilpotent Lie subalgebras as below, where $\tilde{f}_1$ and $\tilde{f}_2$ are sub-matrices of $X$ as above:

(1) $\tilde{f}_1 = \begin{pmatrix} \tilde{f}_{13} & \tilde{f}_{14} & \tilde{f}_{15} & \tilde{f}_{16} \\ \tilde{f}_{23} & \tilde{f}_{24} & \tilde{f}_{25} & \tilde{f}_{26} \end{pmatrix}$, $\tilde{f}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$;

(2) $\tilde{f}_1 = \begin{pmatrix} \tilde{f}_{13} & \tilde{f}_{14} & \tilde{f}_{15} & \tilde{f}_{16} \\ 0 & \tilde{f}_{24} & \tilde{f}_{25} & \tilde{f}_{26} \end{pmatrix}$, $\tilde{f}_2 = \begin{pmatrix} 0 & 0 & 0 & \tilde{f}_{76} \\ 0 & 0 & 0 & 0 \end{pmatrix}$;
3.3. Classification of normalized potentials. To prove Theorem 2.6 we need one more technical lemma which is easy to verify.

Lemma 3.7. We retain the assumptions and the notation of Lemma 3.5 and transform $X \in \sum_{j \geq 0} \mathfrak{g}_{2j+1}^\xi$ back to $\mathfrak{so}(1, n + 3, \mathbb{C})$, by (3.11) and (3.12). Then the transformed matrix is of the form

$$
\tilde{X} = \begin{pmatrix}
0 & B_1 \\
-B_1^t I_{1,3} & 0
\end{pmatrix},
$$

where

$$
B_1 = (v_1, \hat{v}_1, \ldots, v_{m-2}, \hat{v}_{m-2}), v_j = (h_{kj}), \hat{v}_j = (\hat{h}_{kj}), \quad k = 1, 2, 3, 4, \quad j = 1, \ldots, m-2,
$$

with

$$
\begin{aligned}
   h_{1j} &= \frac{1}{2} \left( i(\tilde{f}_{1,j+2} - \tilde{f}_{2m,j+2}) - i(\tilde{f}_{1,2m-j-1} - \tilde{f}_{2m,2m-j-1}) \right), \\
   h_{2j} &= \frac{1}{2} \left( i(\tilde{f}_{1,j+2} + \tilde{f}_{2m,j+2}) - i(\tilde{f}_{1,2m-j-1} + \tilde{f}_{2m,2m-j-1}) \right), \\
   h_{3j} &= \frac{1}{2} \left( (\tilde{f}_{1,j+2} - \tilde{f}_{2m,1,j+2}) - (\tilde{f}_{2,2m-j-1} - \tilde{f}_{2m,2m-j-1}) \right), \\
   h_{4j} &= \frac{1}{2} \left( i(\tilde{f}_{2,j+2} + \tilde{f}_{2m,j+2}) - i(\tilde{f}_{2,2m-j-1} + \tilde{f}_{2m,2m-j-1}) \right), \\
   \hat{h}_{1j} &= \frac{1}{2} \left( (\tilde{f}_{1,j+2} - \tilde{f}_{2m,j+2}) + (\tilde{f}_{1,2m-j-1} - \tilde{f}_{2m,2m-j-1}) \right), \\
   \hat{h}_{2j} &= \frac{1}{2} \left( (\tilde{f}_{1,j+2} + \tilde{f}_{2m,j+2}) + (\tilde{f}_{1,2m-j-1} + \tilde{f}_{2m,2m-j-1}) \right), \\
   \hat{h}_{3j} &= \frac{1}{2} \left( -i(\tilde{f}_{2,j+2} - \tilde{f}_{2m,1,j+2}) - i(\tilde{f}_{2,2m-j-1} - \tilde{f}_{2m,2m-j-1}) \right), \\
   \hat{h}_{4j} &= \frac{1}{2} \left( (\tilde{f}_{2,j+2} + \tilde{f}_{2m,1,j+2}) + (\tilde{f}_{2,2m-j-1} + \tilde{f}_{2m,2m-j-1}) \right),
\end{aligned}
$$

for $j = 1, \ldots, m-2$.

Proof of Theorem 2.6. Combining Lemma 3.5 and Lemma 3.7 we see that for elements in the nilpotent Lie subalgebra related with $\xi$, the basic summands of (3.23)

$$
\begin{pmatrix}
\tilde{f}_{1,j+2} & \tilde{f}_{1,2m-j-1} & \tilde{f}_{2m-1,j+2} & \tilde{f}_{2m-1,2m-j-1} \\
\tilde{f}_{2,j+2} & \tilde{f}_{2,2m-j-1} & \tilde{f}_{2m,j+2} & \tilde{f}_{2m,2m-j-1}
\end{pmatrix},
$$

belong to one of the following three cases:

$$
\begin{pmatrix}
\tilde{f}_{1,j+2} & \tilde{f}_{1,2m-j-1} & 0 & 0 \\
\tilde{f}_{2,j+2} & \tilde{f}_{2,2m-j-1} & 0 & 0
\end{pmatrix},
$$
Therefore (3.22a) follows.

\[
\begin{pmatrix}
\hat{f}_{1,j+2} & \hat{f}_{1,2m-j-1} & 0 & \hat{f}_{2m-1,2m-j-1} \\
0 & \hat{f}_{2,2m-j-1} & 0 & 0 \\
0 & \hat{f}_{1,2m-j-1} & 0 & \hat{f}_{2m-1,2m-j-1} \\
0 & \hat{f}_{2,2m-j-1} & 0 & \hat{f}_{2m,2m-j-1}
\end{pmatrix},
\]

So the corresponding \((v_j, \tilde{v}_j)\) belong to one of the following three cases:

\[
\frac{1}{2}
\begin{pmatrix}
 i(\hat{f}_{1,j+2} - \hat{f}_{1,2m-j-1}) & (\hat{f}_{1,j+2} + \hat{f}_{1,2m-j-1}) \\
 i(\hat{f}_{1,j+2} - \hat{f}_{1,2m-j-1}) & (\hat{f}_{1,j+2} + \hat{f}_{1,2m-j-1}) \\
-(\hat{f}_{2,2m-j-1} - \hat{f}_{2m-1,2m-j-1}) & -i(\hat{f}_{2,2m-j-1} - \hat{f}_{2m-1,2m-j-1}) \\
-i(\hat{f}_{2,2m-j-1} + \hat{f}_{2m-1,2m-j-1}) & (\hat{f}_{2,2m-j-1} + \hat{f}_{2m-1,2m-j-1})
\end{pmatrix},
\]

Noticing the condition \(-\hat{B}_1^t I_{1,3} \hat{B}_1 = 0\), and using some gauge if necessary, the first two cases of \((v_j, \tilde{v}_j)\) are the same. So we obtain Theorem 2.6

3.4. Proof of Lemma 3.5

Proof. The proof is a straightforward and elementary computation by using the definitions of \(\hat{\xi}_j\) case by case.

(1). When \(\xi = \hat{\xi}_1 + \hat{\xi}_2\), we have

\[\xi = \text{diag}\{i, i, 0, \cdots, 0, i, -i\}.\]

By (3.15), it is direct to see that

\[\langle \xi, \hat{E}_{kt} \rangle = \sum_{j=1}^{2} \sqrt{-1} (\delta_{jk} - \delta_{jt} + \delta_{j,2m+1-t} - \delta_{j,2m+1-k}) \hat{E}_{kt} = \varsigma_{kt} \hat{E}_{kt}.\]

So \(\varsigma_{kt}\) takes values as the following table according to the values of \(k\) and \(t\):

| 1 \leq k \leq 2 | 1 \leq t \leq 2 | 3 \leq t \leq 2m-2 | 2m-1 \leq t \leq 2m |
|-----------------|-----------------|-----------------|-----------------|
| 1 \leq k \leq 2 | 0               | +1              | +2              |
| 3 \leq k \leq 2m-2 | -1             | 0               | +1              |
| 2m-1 \leq k \leq 2m | -2            | -1              | 0               |

From this we read \((k + t \leq 2m \text{ for all } k, t)\)

\[\begin{cases}
\mathfrak{g}_1^\xi = \text{Span}_C \left\{ \hat{E}_{kt} \mid k = 1, 2, \text{ and } 3 \leq t \leq 2m-2 \right\}, \\
\mathfrak{g}_2^\xi = \text{Span}_C \left\{ \hat{E}_{kt} \mid k = 1, 2, \text{ and } t = 2m-1 \right\}, \\
\mathfrak{g}_j^\xi = 0, j > 2.
\]

Therefore (3.22a) follows.
(2). When $\xi = \hat{\xi}_1 + \hat{\xi}_2 + 2 \sum_{j=3}^l \hat{\xi}_j$, we have by (3.15),

$$[\xi, \hat{E}_{kt}] = \sqrt{-1}\hat{E}_{kt} (\delta_{1k} - \delta_{1t} + \delta_{1,2m+1-t} - \delta_{1,2m+1-k} + \delta_{2k} - \delta_{2t} + \delta_{2,2m+1-t} - \delta_{2,2m+1-k}$$

$$+ 2 \sum_{j=3}^l (\delta_{jk} - \delta_{jt} + \delta_{j,2m+1-t} - \delta_{j,2m+1-k}))$$

$$= \sqrt{-1} \zeta_{kt} \hat{E}_{kt}.$$ 

It is straightforward to see that $\zeta_{kt}$ takes values as the following table according to the values of $k$ and $t$ (Here $l^* = 2m - l$):

|       | $1 \leq t \leq 2$ | $3 \leq t \leq l$ | $l < t \leq l^*$ | $l^* < t \leq 2m - 2$ | $2m - 1 \leq t \leq 2m$ |
|-------|--------------------|--------------------|------------------|----------------------|----------------------|
| $1 \leq k \leq 2$ | 0                  | -1                | +1               | +3                   | +2                   |
| $3 \leq k \leq l$ | +1                 | 0                 | +2               | +4                   | +3                   |
| $l < k \leq l^*$ | -1                | -2                | 0                | +2                   | +1                   |
| $l^* < k \leq 2m - 2$ | -3                | -4                | -2               | 0                    | -1                   |
| $2m - 1 \leq k \leq 2m$ | -2                | -3                | -1               | +1                   | 0                    |

From this we read $(k + t \leq 2m$ for all $k, t$)

$$g_1^\xi = \text{Span}_C \{ \hat{E}_{kt} | 1 \leq k \leq 2, l + 1 \leq t \leq l^*; \text{ or } 3 \leq k \leq l, 1 \leq t \leq 2 \},$$

$$g_2^\xi = \text{Span}_C \{ \hat{E}_{kt} | 1 \leq k \leq 2, 2m - 1 \leq t \leq 2m; \text{ or } 3 \leq k \leq l, l + 1 \leq t \leq l^* \},$$

$$g_3^\xi = \text{Span}_C \{ \hat{E}_{kt} | 1 \leq k \leq 2m, l^* + 1 \leq t \leq 2m - 2 \},$$

$$g_4^\xi = \text{Span}_C \{ \hat{E}_{kt} | 3 \leq k \leq l, l^* + 1 \leq t \leq 2m - 2 \},$$

$$g_4^\xi = 0, \ j > 4.$$ 

(3.24) Since $NI_{kt} = g_1^\xi \oplus g_2^\xi$, follows.

(3). When $\xi = 3\hat{\xi}_1 + \hat{\xi}_2 + 2 \sum_{j=3}^l \hat{\xi}_j$, we have by (3.15),

$$[\xi, \hat{E}_{kt}] = \sqrt{-1}\hat{E}_{kt} (3(\delta_{1k} - \delta_{1t} + \delta_{1,2m+1-t} - \delta_{1,2m+1-k}) + \delta_{2k} - \delta_{2t} + \delta_{2,2m+1-t} - \delta_{2,2m+1-k}$$

$$+ 2 \sum_{j=3}^l (\delta_{jk} - \delta_{jt} + \delta_{j,2m+1-t} - \delta_{j,2m+1-k}))$$

$$= \sqrt{-1} \zeta_{kt} \hat{E}_{kt}.$$ 

It is straightforward to see that $\zeta_{kt}$ takes values as the following table according to the values of $k$ and $t$ (Here $l^* = 2m - l$):

|       | $t = 1$ | $t = 2$ | $3 \leq t \leq l$ | $l < t \leq l^*$ | $l^* < t \leq 2m - 2$ | $2m - 1 \leq t \leq 2m$ |
|-------|---------|---------|--------------------|------------------|----------------------|----------------------|
| $k = 1$ | 0       | 0       | 0                  | 0                | 0                    | 0                    |
| $k = 2$ | 0       | 0       | 0                  | 0                | 0                    | 0                    |
| $3 \leq k \leq l$ | -1     | 1       | 0                  | 0                | 0                    | 0                    |
| $l < k \leq l^*$ | -3     | -1      | 0                  | 0                | 0                    | 0                    |
| $l^* < k \leq 2m - 2$ | -5     | -3      | -4                | -2               | 0                    | -1                   |
| $k = 2m - 1$ | -4     | -3      | -1                | 0                | 0                    | -2                   |
| $k = 2m$ | -4      | -5      | -3                | -1               | -2                   | 0                    |
From this we read that $g_0^j = 0$, $j > 5$, and $(k + t \leq 2m$ for all $k, t$)

\[(3.25) \quad g_1^\xi = \text{Span}_C \left\{ \hat{E}_{kl} | k = 1, 3 \leq t \leq l; \text{ or } k = 2, l + 1 \leq t \leq l^*; \text{ or } 3 \leq k \leq l, t = 2 \right\}, \]

\[(3.26) \quad g_2^\xi = \text{Span}_C \left\{ \hat{E}_{kl} | k = 1, t = 2; \text{ or } 3 \leq k \leq l, l + 1 \leq t \leq l^* \right\}, \]

\[(3.27) \quad g_3^\xi = \text{Span}_C \left\{ \hat{E}_{kl} | k = 1, l + 1 \leq t \leq l^*; \text{ or } k = 2, l^* + 1 \leq t \leq 2m - 2 \right\}, \]

\[(3.28) \quad g_4^\xi = \text{Span}_C \left\{ \hat{E}_{kl} | k = 1, t = 2m - 1; \text{ or } 3 \leq k \leq l, l^* + 1 \leq t \leq 2m - 2 \right\}, \]

\[(3.29) \quad g_5^\xi = \text{Span}_C \left\{ \hat{E}_{kl} | k = 1, l^* + 1 \leq t \leq 2m - 2 \right\}. \]

Since $NI_{2l} = g_1^\xi \oplus g_3^\xi \oplus g_5^\xi$, (3.22) follows.

(4). When $\xi = 3\hat{\epsilon}_1 + \hat{\epsilon}_2 + 4 \sum_{j=3}^l \hat{\epsilon}_j + 2 \sum_{j=l+1}^h \hat{\epsilon}_j$, we have by (3.15),

$$[\xi, \hat{E}_{kl}] = -\hat{\epsilon}_1 \hat{E}_{kl} (3(\delta_{lk} - \delta_{lk^*} + \delta_{1,2m+1-t} - \delta_{1,2m+1-k} + \delta_{2k} - \delta_{2t} + \delta_{2,2m+1-t} - \delta_{2,2m+1-k})$$

$$+ \sum_{j=3}^l 4(\delta_{jk} - \delta_{lt} + \delta_{j,2m+1-t} - \delta_{j,2m+1-k}) + \sum_{j=l+1}^h 2(\delta_{jk} - \delta_{lt} + \delta_{j,2m+1-t} - \delta_{j,2m+1-k}))$$

$$= -\hat{\epsilon}_{kl} \hat{E}_{kl}. \]

It is straightforward to see that $\hat{\epsilon}_{kl}$ takes values as the following table according to the values of $k$ and $t$ (Here $l^* = 2m - l$, $h^* = 2m - h$, $m^* = 2m - 2$):

|      | $t = 1$ | $t = 2$ | $3 < t \leq l$ | $l < t \leq h$ | $h < t \leq h^*$ | $h^* < t \leq l^*$ | $l^* < t \leq m^*$ | $m < t \leq 2m - 1$ | $t = 2m$ |
|------|---------|---------|----------------|----------------|----------------|----------------|----------------|---------------------|---------|
| $k = 1$ | 0       | +2      | -1            | +1            | +3            | +5            | +7            | +4                  | \       |
| $k = 2$ | -2      | 0       | -3            | -1            | +1            | +3            | +5            | \                   | +4      |
| $3 \leq k \leq l$ | +1      | +3      | 0             | +2            | +4            | +6            | +8            | +5                  | +7      |
| $t < k \leq h$ | -1      | +1      | -2            | 0             | +2            | +4            | +6            | +3                  | +5      |
| $h < k \leq h^*$ | -3      | -1      | -4            | -2            | 0             | +2            | +4            | +1                  | +3      |
| $h^* < k \leq l^*$ | -5      | -3      | -6            | -4            | -2            | 0             | +2            | +1                  | +1      |
| $l^* < k \leq m^*$ | -7      | -5      | -8            | -6            | -4            | -2            | 0             | -3                  | -1      |
| $k = 2m - 1$ | -4      | \       | -5            | -3            | -1            | +3            | 0             | +2                  |         |
| $k = 2m$ | \       | -4      | -7            | -5            | -3            | -1            | +2            | 0                   |         |

From this we read that $g_0^j = 0$, $j > 5$, and $(k + t \leq 2m$ for all $k, t$)

\[(3.30) \quad g_1^\xi = \text{Span}_C \left\{ \hat{E}_{kl} | k = 1, l \leq t \leq h; \text{ or } k = 2, h < t \leq h^*; \text{ or } 3 \leq k \leq l, t = 1; \text{ or } l < k \leq h, t = 2 \right\}, \]

\[(3.31) \quad g_2^\xi = \text{Span}_C \left\{ \hat{E}_{kl} | k = 1, t = 2; \text{ or } 3 \leq k \leq l, l < t \leq h; \text{ or } l < k \leq h, h < t \leq h^* \right\}, \]

\[(3.32) \quad g_3^\xi = \text{Span}_C \left\{ \hat{E}_{kl} | k = 1, h \leq t \leq h^*; \text{ or } k = 2, h^* < t \leq l^*; 3 \leq k \leq l, t = 2 \right\}, \]

\[(3.33) \quad g_4^\xi = \text{Span}_C \left\{ \hat{E}_{kl} | k = 1, t = 2m - 1; \text{ or } 3 \leq k \leq l, h + 1 \leq t \leq h^* \right\}. \]
(3.34) \[ g^c_k = \text{Span}_C \left\{ \hat{E}_{kt} | k = 1, h^* + 1 \leq t \leq l^*; \text{ or } k = 2, l^* + 1 \leq t \leq m^* \right\}. \]

(3.35) \[ g^c_k = \text{Span}_C \left\{ \hat{E}_{kt} | 3 \leq k \leq l, h^* + 1 \leq t \leq l^* \right\}. \]

(3.36) \[ g^c_k = \text{Span}_C \left\{ \hat{E}_{kt} | k = 1, l^* + 1 \leq t \leq 2m - 2 \right\}. \]

(3.37) \[ g^c_k = \text{Span}_C \left\{ \hat{E}_{kt} | 3 \leq k \leq l, l^* + 1 \leq t \leq 2m - 2 \right\}. \]

Since \( NI_{lh} = g^c_1 \oplus g^c_3 \oplus g^c_5 \oplus g^c_7 \), (3.22d) follows.

(5). When \( \xi = \sum_{j=3}^{m} \hat{\xi}_j \), we have by (3.15)

\[ [\xi, \hat{E}_{kt}] = \sqrt{-1} \sum_{j=3}^{m} (\delta_{jk} - \delta_{jt} + \delta_{j,2m+1-t} - \delta_{j,2m+1-k}) \hat{E}_{kt} = \varsigma_{kt} \hat{E}_{kt}. \]

So \( \varsigma_{kt} \) takes values as the following table according to the values of \( k \) and \( t \):

| \( 1 \leq k \leq 2 \) | \( 1 \leq t \leq 2 \) | \( 3 \leq t \leq m \) | \( m < t \leq 2m - 2 \) | \( 2m - 1 \leq t \leq 2m \) |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 | 1 |

From this we read \((k + t \leq 2m \text{ for all } k, t)\)

\( g^c_k = \text{Span}_C \left\{ \hat{E}_{kt} | 1 \leq k \leq 2, m < t \leq 2m - 2; \text{ or } 3 \leq k \leq m, 3 \leq t \leq m \right\}, \)

(3.38) \[ g^c_k = \text{Span}_C \left\{ \hat{E}_{kt} | 3 \leq k \leq m, m < t \leq 2m - 2 \right\}, \]

\( g^c_j = 0, \text{ } j > 2. \)

Since \( NI_{mm} = g^c_k \), (3.22d) follows.

(6). When \( \xi = 2\hat{\xi}_1 + \sum_{j=3}^{m} \hat{\xi}_j \), we have by (3.15),

\[ [\xi, \hat{E}_{kt}] = \sqrt{-1} \hat{E}_{kt} (\delta_{1k} - \delta_{1t} + \delta_{1,2m+1-t} - \delta_{1,2m+1-k} + 2 \sum_{j=3}^{m} (\delta_{jk} - \delta_{jt} + \delta_{j,2m+1-t} - \delta_{j,2m+1-k})) \]

\[ = \sqrt{-1} \varsigma_{kt} \hat{E}_{kt}. \]

So \( \varsigma_{kt} \) takes values as the following table according to the values of \( k \) and \( t \):

| \( k \) | \( t = 1 \) | \( t = 2 \) | \( 3 \leq t \leq m \) | \( m < t \leq 2m - 2 \) | \( t = 2m - 1 \) | \( t = 2m \) |
|---|---|---|---|---|---|---|
| 1 | 0 | +2 | +3 | +2 | \_ | \_ |
| 2 | 0 | 0 | +1 | +1 | \_ | \_ |
| 3 \leq k \leq m | -1 | +1 | 0 | 0 | +1 | +3 |
| m < k \leq 2m - 2 | -3 | -1 | -2 | 0 | -1 | +1 |
| k = 2m - 1 | -2 | \_ | -1 | 0 | +2 | \_ |
| k = 2m | \_ | -2 | -3 | -1 | -2 | 0 |

From this we read \((k + t \leq 2m \text{ for all } k, t)\)
\[
\begin{aligned}
\mathbf{g}_i^\xi = \text{Span}_C \left\{ \hat{E}_{kt} | k = 1, 3 \leq t \leq m; k = 2, m < t \leq 2m - 2; \text{ or } 3 \leq k \leq m, t = 2 \right\}, \\
\mathbf{g}_2^\xi = \text{Span}_C \left\{ \hat{E}_{kt} | k = 1, t = 2, 2m - 1; \text{ or } 3 \leq k \leq m, m < t \leq 2m - 2 \right\}, \\
\mathbf{g}_3^\xi = \text{Span}_C \left\{ \hat{E}_{kt} | k = 1, m < t \leq 2m - 2 \right\}, \\
\mathbf{g}_{j}^\xi = 0, \; j > 4.
\end{aligned}
\]

(3.39)

Since \( NI_{2m} = g_1^\xi \oplus g_3^\xi \), (3.22f) follows.

(7). When \( \xi = 2\zeta_1 + 3 \sum_{j=3}^3 \zeta_j + \sum_{j=4}^m \zeta_j \), we have by [3.15],

\[
\begin{aligned}
|\xi, \hat{E}_{kt}| &= \sqrt{-1} \hat{E}_{kt} \left( \sum_{j=3}^l 3(\delta_{jk} - \delta_{jt} + \delta_{j,2m+1-t} - \delta_{j,2m+1-k})
\right.
\\& + 2(\delta_{lk} - \delta_{lt} + \delta_{l,2m+1-t} - \delta_{l,2m+1-k}) + \sum_{j=l+1}^m (\delta_{jk} - \delta_{jt} + \delta_{j,2m+1-t} - \delta_{j,2m+1-k})
\\left. \right) \\
&= \sqrt{-1}_{\xi,kt} \hat{E}_{kt}.
\end{aligned}
\]

So \( \xi_{kt} \) takes values as the following table according to the values of \( k \) and \( t \) (Here \( l^* = 2m - l, m^* = 2m - 2 \)):

| \( k \) \( t \) \( 3 < t \leq l \) \( l < t \leq m \) \( m < t \leq t^* \) \( t^* < t \leq 2m - 1 \) | \( t = 1 \) | \( t = 2 \) |
|-----------------|----------------|----------------|----------------|----------------|----------------|
| \( k = 1 \)     | 0              | +2             | +1             | +3             | +5             | +2             |
| \( k = 2 \)     | -2             | 0              | -3             | -1             | +1             | +3             | 2              |
| \( 3 \leq k \leq l \) | +1             | +3             | 0              | +2             | +4             | +6             | +3             |
| \( l < k \leq m \) | -1             | +1             | -2             | 0              | +2             | +4             | +1             | +3             |
| \( m < k \leq t^* \) | -3             | -1             | -4             | -2             | 0              | +2             | -1             | +1             |
| \( t^* < k \leq m^* \) | -5             | -3             | -6             | -4             | -2             | 0              | -3             | -1             |
| \( k = 2m - 1 \) | -2             | \( \cdot \)    | -3             | -1             | +1             | +3             | 0              | +2             |
| \( k = 2m \)    | \( \cdot \)    | -2             | -5             | -3             | -1             | +1             | -2             | 0              |

From this we read that \( g_j^\xi = 0, \; j > 5 \), and \( k + t \leq 2m \) for all \( k, t \)

(3.40)

\[
\mathbf{g}_1^\xi = \text{Span}_C \left\{ \hat{E}_{kt} | k = 1, l \leq t \leq m; \; k = 2, m < t \leq l^*; \; \text{ or } 3 \leq k \leq l, t = 1; \; \text{ or } l < k \leq m, t = 2 \right\},
\]

(3.41)

\[
\mathbf{g}_2^\xi = \text{Span}_C \left\{ \hat{E}_{kt} | k = 1, t = 2; \; \text{ or } k = 1, 2m - 1; \; \text{ or } 3 \leq k \leq l, l + 1 \leq t \leq m \right\},
\]

(3.42)

\[
\mathbf{g}_3^\xi = \text{Span}_C \left\{ \hat{E}_{kt} | k = 1, m \leq t \leq l^*; \; \text{ or } k = 2, l^* < t \leq 2m - 2; \; \text{ or } 3 \leq k \leq l, t = 2 \right\},
\]

(3.43)

\[
\mathbf{g}_4^\xi = \text{Span}_C \left\{ \hat{E}_{kt} | 3 \leq k \leq l, l^* + 1 \leq t \leq 2m - 2 \right\},
\]

(3.44)

\[
\mathbf{g}_5^\xi = \text{Span}_C \left\{ \hat{E}_{kt} | k = 1, l^* < t \leq 2m - 2 \right\},
\]

(3.45)

\[
\mathbf{g}_6^\xi = \text{Span}_C \left\{ \hat{E}_{kt} | 3 \leq k \leq l, l^* < t \leq 2m - 2 \right\}.
\]

Since \( NI_{lm} = g_1^\xi \oplus g_3^\xi \oplus g_5^\xi \), (3.22f) follows. \[\square\]
As a byproduct, we obtain

**Corollary 3.8.** Formula $(3.24) - (3.45)$ provides all nilpotent Lie algebras of $\mathfrak{g}(2m, \mathbb{C})$ related with the symmetric space $\mathcal{P}(SO(1,2m - 1)/SO(1,3) \times SO(2m - 4))$.

**Acknowledgements** The author is thankful to Prof. Josef Dorfmeister, Prof. Changping Wang and Prof. Xiang Ma for their suggestions and encouragement. This work is supported by the Project 11201340 of NSFC and the Fundamental Research Funds for the Central Universities.

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