MORDELL-WEIL GROUPS AND AUTOMORPHISM GROUPS OF
ELLiptIC K3 surFACES

ICHIRO SHIMADA

Abstract. We present a method to calculate the action of the Mordell-Weil

group of an elliptic K3 surface on the numerical Néron-Severi lattice of the
K3 surface. As an application, we compute a finite generating set of the
automorphism group of a K3 surface birational to the double plane branched
along a 6-cuspidal sextic curve of torus type.

1. Introduction

We work over an algebraically closed field $k$.

Let $X$ be a projective K3 surface. We denote by $S_X$ the numerical Néron-Severi
lattice of $X$, that is, the group of numerical equivalence classes of divisors of $X$
with the intersection pairing

\[
\langle \cdot \rangle : S_X \times S_X \rightarrow \mathbb{Z}.
\]

Let $O(S_X)$ denote the group of isometries of the lattice $S_X$. We investigate the
automorphism group $\text{Aut}(X)$ of $X$ by means of the action

\[
\text{Aut}(X) \rightarrow O(S_X)
\]

of $\text{Aut}(X)$ on the lattice $S_X$.

Let $\phi: X \rightarrow \mathbb{P}^1$ be an elliptic fibration with a distinguished section $\zeta: \mathbb{P}^1 \rightarrow X$. In this case, we say that $(\phi, \zeta)$ is a Jacobian fibration. We denote by $\text{MW}(X, \phi, \zeta)$
the Mordell-Weil group of sections of $\phi$ with $\zeta$ being the zero element. An element
$\sigma \in \text{MW}(X, \phi, \zeta)$ acts on the generic fiber of $\phi$ by translation. Since $X$ is minimal,
this birational automorphism of $X$ is an automorphism of $X$, and hence we have an
embedding of $\text{MW}(X, \phi, \zeta)$ into $\text{Aut}(X)$. In this paper, we investigate the composite
homomorphism

\[
\text{MW}(X, \phi, \zeta) \rightarrow \text{Aut}(X) \rightarrow O(S_X).
\]

This homomorphism has been used in many situations in the study of automor-
phisms of K3 surfaces (see, for example, [25]). The purpose of this paper is to
present a general algorithm to calculate (1.1) explicitly and to give applications.

Borcherds’ method ([5, 6]) is a method to calculate a finite generating set of the
image of $\text{Aut}(X) \rightarrow O(S_X)$ by means of a certain decomposition of the nef-and-big
cone of $X$ into a union of polyhedral cones. The first application of this method
to the study of the automorphism group of a K3 surface was given by Kondo [15].
See also [29]. Since this method is based on lattice-theoretic computation, the

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geometric meaning of elements in the generating set obtained by this method is not clear in general. The homomorphism (1.1) helps us to express the generating set geometrically. See Remark 5.16.

As an application, we calculate the automorphism group of the complex $K3$ surface $X_{f,g}$ obtained as the minimal resolution of the double cover $\overline{X}_{f,g}$ of $\mathbb{P}^2$ defined by

\begin{equation}
 w^2 = f(x, y, z)^2 + g(x, y, z)^3,
\end{equation}

where $f$ and $g$ are very general homogeneous polynomials on $\mathbb{P}^2$ of degree 3 and 2, respectively. Here being very general means that there exist at most countably many analytic subsets of $H^0(\mathbb{P}^2, \mathcal{O}(3)) \times H^0(\mathbb{P}^2, \mathcal{O}(2))$ with codimension $\geq 1$ such that the pair $(f, g)$ does not belong to any of them. We prove the following:

**Theorem 1.1.** The automorphism group $\text{Aut}(X_{f,g})$ of $X_{f,g}$ is generated by 463 involutions associated with double coverings $X_{f,g} \to \mathbb{P}^2$ and 360 elements of infinite order in Mordell-Weil groups of Jacobian fibrations of $X_{f,g}$.

Here, by a double covering, we mean a generically finite morphism of degree 2.

**Theorem 1.2.** The automorphism group $\text{Aut}(X_{f,g})$ acts on the set of smooth rational curves on $X_{f,g}$ transitively.

The branch curve of the finite double cover $\overline{X}_{f,g} \to \mathbb{P}^2$ is defined by the equation $f^2 + g^3 = 0$. This plane curve is called a 6-cuspidal plane sextic of torus type, and was studied intensively from various points of view. See, for example, [22]. In fact, Zariski [37] observed that there exists a 6-cuspidal plane sextic of non-torus type, and the seminal notion of Zariski pairs emerged from this observation. See [1] and [2]. In [9] and [27], this classical example of Zariski pairs was studied in relation to the theory of $K3$ surfaces. It would be an interesting problem to calculate the automorphism group of the $K3$ surface obtained from the 6-cuspidal plane sextic of non-torus type.

The generating set in Theorem 1.1 is constructed in such a way that we can clearly see the geometric meaning of each element. See Section 6 for more precise descriptions of these automorphisms. Remark that this generating set is not minimal at all.

In fact, we give divisors of $X_{f,g}$ whose classes generate $S_X$. Hence we can calculate, in principle, the equations of the double coverings and the Jacobian fibrations by the method given in [28]. The actual computation of the equations, however, would be very hard.

**Theorem 1.1** is proved in the following three steps.

(a) We find many automorphisms of $X_{f,g}$ geometrically by the methods explained in Section 3 (especially Section 3.7) and Section 4.

(b) We find a finite generating set of $\text{Aut}(X_{f,g})$ by Borcehrs’ method, which will be explained in Section 5.

(c) We then show that the group generated by the automorphisms obtained in Step (a) contains the generating set obtained in Step (b).

See [17], [18] and [33] for general finiteness results of the automorphism group of a $K3$ surface and its action on the nef-and-big cone.

This paper is organized as follows. After fixing some notions and notation about lattices in Section 2, we summarize in Section 3 various computational tools that are useful in the study of the geometry of $K3$ surfaces. These tools are based
on an algorithm given in [28] to calculate \( \text{Sep}(v_1, v_2) \) of separating \((-2)\)-vectors in a hyperbolic lattice. In Section 4, we present an algorithm to calculate the homomorphism (1.1). In Section 5, we review Borcherds’ method. We employ a graph-theoretic formulation of Borcherds’ method given in [7, Section 4.1]. Sections 3–5 are intended to be summaries of computational methods in the study of \( K3 \) surfaces for future reference. In Section 6, we calculate \( \text{Aut}(X_{f,g}) \) by means of all these algorithms, and prove Theorems 1.1 and 1.2. We used GAP [11] for the actual computation. In the author’s webpage [31], we put detailed computation data about \( \text{Aut}(X_{f,g}) \).

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2. **Notation and Terminologies**

By a *lattice*, we mean a free \( \mathbb{Z} \)-module \( L \) of finite rank with a non-degenerate symmetric bilinear form
\[
\langle \; \rangle : L \times L \to \mathbb{Z},
\]
which we call the *intersection form* (or the *intersection pairing*) of \( L \). The group of isometries of a lattice \( L \) is denoted by \( \text{O}(L) \), which we let act on \( L \) from the right.

Let \( L \) be a lattice. Then the *dual lattice* \( L^{\vee} \) of \( L \) is defined to be
\[
\{ x \in L \otimes \mathbb{Q} \mid \langle x, v \rangle \in \mathbb{Z} \text{ for all } v \in L \}.
\]
The finite abelian group \( A(L) := L^{\vee}/L \) is called the *discriminant group of* \( L \). We say that \( L \) is *unimodular* if \( L = L^{\vee} \).

A lattice \( L \) is said to be *even* if \( \langle v, v \rangle \in 2\mathbb{Z} \) holds for all \( v \in L \). A *root* of an even lattice \( L \) is a vector \( r \in L \) such that \( \langle r, r \rangle \) is either 2 or \(-2\). A \((-2)\)-vector of \( L \) is a root \( r \in L \) such that \( \langle r, r \rangle = -2 \). Suppose that \( L \) is even and negative-definite. Then the set
\[
\text{Roots}(L) := \{ r \in L \mid \langle r, r \rangle = -2 \}
\]
is finite. An even negative-definite lattice \( L \) is called a *root lattice* if \( L \) is generated by \( \text{Roots}(L) \). A root lattice has a basis consisting of roots whose dual graph is a Dynkin diagram of type ADE. See, for example, [10, Section 1] for the definition of dual graphs, Dynkin diagrams, and their ADE-types.

A lattice \( L \) of rank \( n > 1 \) is said to be *hyperbolic* if the signature of the real quadratic space \( L \otimes \mathbb{R} \) is \((1, n-1)\). Let \( L \) be an even hyperbolic lattice. A *positive cone* of \( L \) is one of the two connected components of the space
\[
\{ x \in L \otimes \mathbb{R} \mid \langle x, x \rangle > 0 \}.
\]
Let \( \mathcal{P} \) be a positive cone of \( L \). We put
\[
\text{O}(L, \mathcal{P}) := \{ g \in \text{O}(L) \mid \mathcal{P}^g = \mathcal{P} \}.
\]
We have \( \text{O}(L) = \text{O}(L, \mathcal{P}) \times \{ \pm 1 \} \). For \( v \in L \otimes \mathbb{R} \), we put
\[
v^\perp := \{ x \in L \otimes \mathbb{R} \mid \langle x, v \rangle = 0 \}.
\]
When $v \in \mathcal{P} \cap L$, the intersection $v^+ \cap L$ is an even negative-definite sublattice of $L$, and hence we can effectively calculate the finite set
$$\text{Roots}(v^+ \cap L) = \{ r \in L \mid \langle r, v \rangle = 0, \langle r, r \rangle = -2 \}$$
of $(−2)$-vectors in $L$ perpendicular to $v$.
For $v \in L \otimes \mathbb{R}$ with $\langle v, v \rangle < 0$, we put
$$(v)^+ := v^+ \cap \mathcal{P} = \{ x \in \mathcal{P} \mid \langle x, v \rangle = 0 \},$$
which is a real hyperplane of $\mathcal{P}$. Let $v_1, v_2 \in L \otimes \mathbb{Q}$ be rational vectors in $\mathcal{P}$. Then we can calculate the finite set
$$\text{Sep}(v_1, v_2) := \{ r \in L \mid \langle r, v_1 \rangle > 0, \langle r, v_2 \rangle < 0, \langle r, r \rangle = -2 \}$$
of $(−2)$-vectors separating $v_1$ and $v_2$. See [28] for the algorithm. As will be explained in Section 3, this algorithm is very useful in the study of K3 surfaces.

**Definition 2.1.** By a chamber, we mean a closed subset $D$ of $\mathcal{P}$ such that
- $D$ contains a non-empty open subset of $\mathcal{P}$, and
- $D$ is defined by linear inequalities $\langle x, v_i \rangle \geq 0 (i \in I)$, where $v_i (i \in I)$ are vectors of $L \otimes \mathbb{R}$ with $\langle v_i, v_i \rangle < 0$ such that the family $\{(v_i)\perp \mid i \in I\}$ of hyperplanes is locally finite in $\mathcal{P}$.

**Definition 2.2.** Let $D$ be a chamber. A wall of $D$ is a closed subset of $D$ of the form $D \cap (v)^\perp$ such that the hyperplane $(v)^\perp$ is disjoint from the interior of $D$ and that $D \cap (v)^\perp$ contains a non-empty open subset of $(v)^\perp$. We say that a vector $v \in L \otimes \mathbb{R}$ defines a wall $w$ of $D$ if $w = D \cap (v)^\perp$ and $\langle x, v \rangle > 0$ for an interior point $x$ of $D$ (and hence $\langle x, v \rangle \geq 0$ for all $x \in D$). A defining vector of a wall of a chamber is unique up to positive multiplicative constant.

**Definition 2.3.** Let $\mathcal{F} := \{(v_\alpha)^\perp \mid \alpha \in F\}$ be a locally finite family of hyperplanes in $\mathcal{P}$. Then the closure in $\mathcal{P}$ of each connected component of
$$\mathcal{P} \setminus \bigcup_{\alpha \in F} (v_\alpha)^\perp$$
is a chamber. Let $\mathcal{C}_\mathcal{F}$ be the set of these chambers. In this situation, we say that $\mathcal{P}$ is tessellated by the chambers in $\mathcal{C}_\mathcal{F}$. If a subset $N$ of $\mathcal{P}$ is the union of chambers in a subset of $\mathcal{C}_\mathcal{F}$, we say that $N$ is tessellated by chambers in $\mathcal{C}_\mathcal{F}$.

Let $w$ be a wall of a chamber $D \in \mathcal{C}_\mathcal{F}$. Then there exists a unique chamber $D' \in \mathcal{C}_\mathcal{F}$ such that $D \neq D'$ and $w \subset D'$. This chamber $D'$ is called the chamber adjacent to $D$ across the wall $w$.

A $(−2)$-vector $r \in L$ defines a reflection
$$s_r : x \mapsto x + \langle x, r \rangle r$$
into the mirror $(r)^\perp$. We have $s_r \in \text{O}(L, \mathcal{P})$. Let $W(L)$ denote the subgroup of $\text{O}(L, \mathcal{P})$ generated by all the reflections $s_r$ with respect to $(−2)$-vectors $r$. We call $W(L)$ the Weyl group of $L$. Note that the family of hyperplanes $(r)^\perp$ defined by $(−2)$-vectors $r$ is locally finite in $\mathcal{P}$.

**Definition 2.4.** A standard fundamental domain of $W(L)$ is the closure of a connected component of
$$\mathcal{P} \setminus \bigcup_r (r)^\perp,$$
where $r$ runs through the set of $(−2)$-vectors.
3. The numerical Néron-Severi lattice of a K3 surface

Let $X$ be a K3 surface, and $S_X$ the lattice of numerical equivalence classes of divisors of $X$, which we call the numerical Néron-Severi lattice of $X$. For a divisor $D$ of $X$, we denote by $[D] \in S_X$ the class of $D$. Suppose that $S_X$ is of rank $n > 1$. Then $S_X$ is an even hyperbolic lattice. Let $\mathcal{P}_X$ be the positive cone of $S_X$ containing an ample class of $X$, and $\mathcal{P}_X$ the closure of $\mathcal{P}_X$ in $S_X \otimes \mathbb{R}$. We put

$$N_X := \{ x \in \mathcal{P}_X \mid \langle x, [C] \rangle \geq 0 \text{ for all curves } C \text{ on } X \},$$

$$N_X^\circ := \text{the interior of } N_X,$$

$$\overline{N}_X := \text{the closure of } N_X \text{ in } \mathcal{P}_X.$$

The cone $N_X$ is called the nef-and-big cone of $X$. If $C$ is a smooth rational curve on $X$, then its class $[C]$ is a $(-2)$-vector of $S_X$. We put

$$\text{Rats}(X) := \{ [C] \in S_X \mid C \text{ is a smooth rational curve on } X \}.$$

We have the following:

**Theorem 3.1.** The nef-and-big cone $N_X$ is a standard fundamental domain of the Weyl group $W(S_X)$ of $S_X$. A $(-2)$-vector $r \in S_X$ belongs to $\text{Rats}(X)$ if and only if $r$ defines a wall of the chamber $N_X$. \hfill \Box

Suppose that we have an ample class $a \in N_X^\circ \cap S_X$. Then Vinberg’s algorithm [35] enables us to enumerate, for a given positive integer $m$, all the walls $N_X \cap (r)^\perp$ of $N_X$ defined by $r \in \text{Rats}(X)$ with $\langle r, a \rangle \leq m$. (See (3.2) below.) Our algorithm [28] of calculating the set $\text{Sep}(v_1, v_2)$ of separating $(-2)$-vectors provides us with an alternative method to investigate the nef-and-big cone $N_X$. Below are some examples.

3.1. Finding an ample class. It is well-known that a class $v \in S_X$ is ample if and only if $v \in N_X^\circ$. Let $\overline{X}$ be a normal surface birational to $X$, and $h \in S_X$ the pull-back of an ample class of $\overline{X}$ by the minimal resolution $X \rightarrow \overline{X}$. Then we have $h \in N_X$. It is known [3] that $\overline{X}$ has only rational double points as its singularities, and hence the exceptional locus of the desingularization $X \rightarrow \overline{X}$ is a union of smooth rational curves whose dual graph is a Dynkin diagram of type ADE. Let $r_1, \ldots, r_\mu$ be the classes of smooth rational curves contracted by $X \rightarrow \overline{X}$. Then, *locally around* $h$, the chamber $N_X$ is defined by $\langle x, r_i \rangle \geq 0$ for $i = 1, \ldots, \mu$. Therefore a vector $v \in \mathcal{P}_X \cap S_X$ is ample if and only if

$$\text{Sep}(h, v) = \emptyset, \quad \text{Roots}(v^\perp \cap S_X) = \emptyset, \quad \text{and } \langle v, r_i \rangle > 0 \text{ for } i = 1, \ldots, \mu.$$

If $a' \in S_X$ satisfies $\langle a', r_i \rangle > 0$ for $i = 1, \ldots, \mu$, then $a := mh + a'$ is ample for sufficiently large integers $m$. 


3.2. Nefness and ampleness. Suppose that we have an ample class \( a \in S_X \). Then we can characterize \( N_X \) as the unique standard fundamental domain of \( W(S_X) \) containing \( a \). Let \( v \in S_X \) be a vector with \( \langle v, v \rangle > 0 \). Then we have
\[
v \in P_X \iff \langle a, v \rangle > 0.
\]
When these are the case, we have
\[
v \in N_X \iff \text{Sep}(a, v) = \emptyset.
\]
When these are the case, we have
\[
v \in N_X^\circ \iff \text{Roots}(v^\perp \cap S_X) = \emptyset.
\]

3.3. The group \( O(S_X, N_X) \). Recall that \( O(S_X, N_X) \) is the subgroup of \( O(S_X, P_X) \) consisting of all isometries \( g \) such that \( N_X^\circ = N_X \). Suppose again that we have an ample class \( a \in S_X \). Let \( g \) be an element of \( O(S_X) \). Then we have
\[
g \in O(S_X, P_X) \iff \langle a, a^g \rangle > 0.
\]
When these are the case, we have
\[
(3.1) \quad g \in O(S_X, N_X) \iff \text{Sep}(a, a^g) = \emptyset,
\]
because, for \( g \in O(S_X, P_X) \), the chamber \( N_X^\circ \) is also a standard fundamental domain of \( W(S_X) \).

3.4. The set \( \text{Rats}(X) \). Again we assume that we have an ample class \( a \in S_X \). Let \( r \in S_X \) be a \((-2)\)-vector such that \( \langle a, r \rangle > 0 \). Then there exists an effective divisor \( D \) of \( X \) such that \( r = [D] \). We have \( r \in \text{Rats}(X) \) if and only if \( D \) is irreducible.

Since \( D \) contains a smooth rational curve \( C \) such that \( \langle [C], [D] \rangle < 0 \) as an irreducible component, we have the following criterion, which is a geometric interpretation of Vinberg’s algorithm [35] applied to \((-2)\)-vectors:
\[
(3.2) \quad r \in \text{Rats}(X) \iff \langle r, r' \rangle \geq 0 \quad \text{for all } r' \in \text{Rats}(X) \text{ with } \langle r', a \rangle < \langle r, a \rangle.
\]

Thanks to the algorithm to calculate \( \text{Sep}(v_1, v_2) \), we obtain another criterion.

**Proposition 3.2.** Let \( r \in S_X \) be a \((-2)\)-vector with \( \langle a, r \rangle > 0 \). We put
\[
a'_r := a + \frac{\langle a, r \rangle}{2} r.
\]
Then \( r \in \text{Rats}(X) \) if and only if
\[
(3.3) \quad \text{Roots}(a'_r^\perp \cap S_X) = \{r, -r\} \quad \text{and} \quad \text{Sep}(a'_r, a) = \emptyset.
\]

**Proof.** Since \( \langle a'_r, r \rangle = 0 \) and \( \langle a'_r, a'_r \rangle > 0 \), we have \( a'_r \in (r)^\perp \subseteq P_X \), and hence the set \( \text{Sep}(a'_r, a) \) makes sense. In fact, the point \( a'_r \) in \( (r)^\perp \) is the image of \( a \) by the orthogonal projection to the hyperplane \( (r)^\perp \) in \( P \). In particular, we have \( \{r, -r\} \subseteq \text{Roots}(a'_r^\perp \cap S_X) \). Then Proposition 3.2 follows from [36, Proposition 2.2]. We present proof for the convenience of readers.

If (3.3) holds, then \( a'_r \in N_X \) and a small neighborhood of \( a'_r \) in \( (r)^\perp \) is contained in \( N_X \). In particular, \( r \) is a defining \((-2)\)-vector of a wall of \( N_X \) and hence \( r \in \text{Rats}(X) \). Conversely, suppose that \( r \in \text{Rats}(X) \). Then for any \( r' \in \text{Rats}(X) \) with \( r' \neq r \), we have \( \langle r, r' \rangle \geq 0 \) and \( \langle a, r' \rangle > 0 \), and hence
\[
\langle a'_r, r' \rangle = \langle a, r' \rangle + \frac{\langle a, r \rangle \langle r, r' \rangle}{2} > 0.
\]
Therefore (3.3) holds. \( \square \)
3.5. Nefness of a vector of norm 0. Suppose again that we have \( a \in N_X^0 \cap S_X \).

**Proposition 3.3.** Let \( f \) be a non-zero vector in \( \mathcal{P}_X \cap S_X \) with \( \langle f, f \rangle = 0 \). Then \( f \in \overline{N}_X \) if and only if \( \text{Sep}(a'_f, a) = \emptyset \), where \( a'_f := a + \langle a, f \rangle f \).

**Proof.** First note that, since \( f \notin \overline{N}_X \), we have \( \langle a, f \rangle > 0 \), \( a'_f \in \mathcal{P}_X \), and hence \( \text{Sep}(a'_f, a) \) makes sense.

Suppose that \( f \in \overline{N}_X \). Since \( a \in N_X^0 \), we have \( a'_f \in N_X^0 \) and hence \( \text{Sep}(a'_f, a) = \emptyset \). Suppose that \( f \notin \overline{N}_X \). Then there exists a smooth rational curve \( C \) such that \( \langle f, [C] \rangle < 0 \). We put \( r := [C] \). Then we have \( \langle f, r \rangle \leq -1 \). Since \( \langle f, f \rangle = 0 \) and \( \langle f, a \rangle > 0 \), there exists an ample divisor \( F \) on \( X \) such that \( f = [F] \). Then \( C \) is an irreducible component of \( F \) such that \( C \neq F \), and hence \( \langle a, r \rangle < \langle a, f \rangle \). The intersection point of \( (r) \) and the open line segment

\[
\langle a, f \rangle := \{ p(t) = a + tf \mid t \in \mathbb{R}_{>0} \} \subset \mathcal{P}_X
\]

is equal to \( p(t_0) \), where

\[
t_0 := -\frac{\langle a, r \rangle}{\langle f, r \rangle} \leq \langle a, r \rangle < \langle a, f \rangle.
\]

Since \( a'_f = p(\langle a, f \rangle) \), the intersection point \( p(t_0) \) is located on the open line segment \( \langle a, a'_f \rangle \subset \langle a, f \rangle \). Therefore \( r \) is a \((-2)\)-vector separating \( a'_f \) and \( a \). \( \square \)

3.6. **Singularities of a normal surface birational to \( X \).** Suppose again that we have \( a \in N_X^0 \cap S_X \). Let \( h \) be a vector in \( N_X \cap S_X \), and let \( \mathcal{L} \) be a line bundle whose class is \( h \). Then, for some large positive integer \( m \), the complete linear system \( |\mathcal{L}^\otimes m| \) gives a birational morphism \( X \rightarrow X' \) to a normal surface \( X' \). See Saint-Donat [24].

The surface \( X' \) is smooth if and only if \( h \notin N_X^0 \). Suppose that \( h \notin N_X^0 \). Then the singularities of \( X' \) consist of rational double points (see Artin [3]), and the set of classes of smooth rational curves contracted by the birational morphism \( X \rightarrow X' \) is equal to

\[
\{ r \in \text{Rats}(X) \mid \langle r, h \rangle = 0 \} = \text{Rats}(X) \cap \text{Roots}(h^\perp \cap S_X).
\]

3.7. **Finding automorphisms from nef vectors of norm 2.** Let \( a \in S_X \) be an ample class of \( X \). Let \( h \) be a vector in \( N_X \cap S_X \) with \( \langle h, h \rangle = 2 \). By a double covering, we mean a generically finite morphism of degree 2. By abuse of notation, we write \( |h| \) for the complete linear system of a line bundle whose class is \( h \). Then either one of the following holds (see Saint-Donat [24] or Nikulin [21]).

- The complete linear system \( |h| \) is base-point free and defines a double covering \( \pi(h) : X \rightarrow \mathbb{P}^2 \), or
- \( |h| \) has a fixed component \( Z \), which is a smooth rational curve, and every member of \( |h| \) is of the form \( Z + E_1 + E_2 \), where \( E_1 \) and \( E_2 \) are members of a pencil \( |E| \) of elliptic curves such that \( \langle [E], [Z] \rangle = 1 \).

These two cases can be distinguished by the following criterion. We put

\[
\mathcal{E} := \{ e \in S_X \mid \langle e, e \rangle = 0, \langle e, h \rangle = 1 \}.
\]

Since the quadratic part of the intersection form \( \langle , \rangle \) restricted to the affine hyperplane of \( S_X \otimes \mathbb{R} \) defined by \( \langle x, h \rangle = 1 \) is negative-definite, the set \( \mathcal{E} \) is finite and can be calculated effectively.
• If $E = \emptyset$, then $|h|$ is base-point free. In this case, we say that $h$ is a polarization of degree 2, and denote by $i(h) \in \text{Aut}(X)$ the involution associated with the double covering $\pi(h): X \to \mathbb{P}^2$ given by $|h|$. Let

$$X \to \overline{X} \to \mathbb{P}^2$$

be the Stein factorization of $\pi(h)$, and let $B(h) \subset \mathbb{P}^2$ be the branch curve of the finite double covering $\overline{X} \to \mathbb{P}^2$. We can calculate the set

$$\text{Rats}(X) \cap \text{Roots}(h^\perp \cap S_X)$$

of classes of smooth rational curves contracted by $\pi(h)$. Hence we obtain the ADE-type of $\text{Sing}(B(h))$, and the invariant part

$$\{ v \in S_X \otimes \mathbb{Q} \mid v^{i(h)} = v \}$$

of the action of $i(h)$ on $S_X \otimes \mathbb{Q}$. Indeed, applying to $\overline{X}$ the theory of canonical resolutions of rational double points due to Horikawa [12], we have a successive blowing up $Y : \to \mathbb{P}^2$ of $\mathbb{P}^2$ such that $X \to \mathbb{P}^2$ factors through a finite double covering $X \to Y$, and the invariant part is equal to the pull-back of the space $S_Y \otimes \mathbb{Q}$ of the numerical equivalence classes of curves on the rational surface $Y$. See [30] for detail. From this subspace, we can calculate the action of the involution $i(h)$ on $S_X$, because $i(h)$ acts on the orthogonal complement of the invariant subspace as the scalar multiplication by $-1$.

**Remark 3.4.** The equality $i(h) = i(h')$ of involutions does not imply $h = h'$ in general. See Remark 6.11, for example. The set of polarizations $h$ of degree 2 that induce the same involution $i(h)$ is in one-to-one correspondence with the set of blowing-downs of $Y$ to $\mathbb{P}^2$.

• Suppose that $E \neq \emptyset$. Then we have a unique element $f \in E$ such that

$$f \in \overline{X}, \quad \text{and} \quad z := h - 2f \in \text{Rats}(X).$$

We can find this $f$ by the methods in Sections 3.5 and 3.4. Then $f$ is the class of a fiber of a Jacobian fibration $\phi: X \to \mathbb{P}^1$ with $z$ being the class of the zero section $\zeta: \mathbb{P}^1 \to X$. From these vectors $f, z$, we can calculate the Mordell-Weil group $\text{MW}(X, \phi, \zeta)$ and its action on $S_X$ by the algorithm explained in Section 4.

4. The action of a Mordell-Weil group on $S_X$

In this section, we assume that the characteristic of the base field $k$ is $\neq 2, 3$ for simplicity. Let $X$ be a K3 surface, and $a \in S_X$ an ample class.

Let $\phi: X \to \mathbb{P}^1$ be a fibration whose general fiber is a curve of genus 1. Suppose that $\phi$ has a distinguished section $\zeta: \mathbb{P}^1 \to X$, that is, the pair $(\phi, \zeta)$ is a Jacobian fibration. Let $\eta = \text{Spec } k(\mathbb{P}^1)$ be the generic point of the base curve $\mathbb{P}^1$. Then the generic fiber $E_\eta := \phi^{-1}(\eta)$ of $\phi$ is an elliptic curve defined over $k(\mathbb{P}^1)$ with the zero element being the $k(\mathbb{P}^1)$-rational point corresponding to $\zeta$, and the set

$$\text{MW}_\phi := \text{MW}(X, \phi, \zeta)$$

of sections of $\phi$ has a structure of the abelian group with $\zeta = 0$. This group $\text{MW}_\phi$ is called the Mordell-Weil group. The group $\text{MW}_\phi$ acts on $E_\eta$ via the translation $x \mapsto x +_E \sigma$ on $E_\eta$, where $\sigma \in \text{MW}_\phi$ is a section and $+_E$ denotes the addition
in the elliptic curve $E_\eta$. Since $X$ is minimal, this automorphism of $E_\eta$ gives an automorphism of $X$. Hence $\text{MW}_\phi$ embeds in $\text{Aut}(X)$, and acts on the lattice $S_X$:

$$\text{MW}_\phi \rightarrow \text{Aut}(X) \rightarrow \text{O}(S_X, P_X).$$

Let $f \in S_X$ be the class of a fiber of $\phi$, and $z = [\zeta] \in S_X$ the class of the image of $\zeta$. Since the Jacobian fibration $(\phi, \zeta)$ is uniquely determined by the classes $f$ and $z$, we sometimes write $\text{MW}(X, f, z)$ for $\text{MW}(X, \phi, \zeta)$. The purpose of this section is to show that we can calculate the homomorphism (4.1) from the classes $f$, $z$ and an ample class $a$.

We review the theory of elliptic $K3$ surfaces, and fix some notation. Since $\langle f, f \rangle = 0$, $\langle f, z \rangle = 1$ and $\langle z, z \rangle = -2$, the classes $f$ and $z$ generate a unimodular hyperbolic sublattice $U_\phi$ in $S_X$ of rank 2. Let $W_\phi$ denote the orthogonal complement of $U_\phi$ in $S_X$. Since $U_\phi$ is unimodular, we have an orthogonal direct-sum decomposition

$$S_X = U_\phi \oplus W_\phi.$$

Since $W_\phi$ is negative-definite, we can calculate the set

$$\text{Roots}(W_\phi) = \{ r \in W_\phi \mid \langle r, r \rangle = -2 \}.$$

Hence we can compute

$$\Theta_\phi := \text{Roots}(W_\phi) \cap \text{Rats}(X)$$

by Proposition 3.2. Let $\Sigma_\phi$ denote the sublattice of $W_\phi$ generated by $\text{Roots}(W_\phi)$, and $\tau_\phi$ the ADE-type of the root lattice $\Sigma_\phi$. Here an ADE-type is a finite formal sum of the symbols $A_\ell$, $D_\ell$, and $E_\ell$. See, for example, [10, Section 1] for the definition of ADE-types of root lattices. Then we have the following proposition. The first part follows from the definition of $\text{Rats}(X)$, and the second part follows from the classification of singular fibers of elliptic surfaces due to Kodaira and Néron. See [26, Chapters 5 and 6].

**Proposition 4.1.** The set $\Theta_\phi$ defined by (4.2) is equal to the set of classes of smooth rational curves that are contracted to points by $\phi$ and are disjoint from the zero section $\zeta$. The vectors in $\Theta_\phi$ form a basis of the root lattice $\Sigma_\phi$, and their dual graph is the Dynkin diagram of type $\tau_\phi$.

**Definition 4.2.** The sublattice $U_\phi \oplus \Sigma_\phi$ of $S_X$ is called the trivial sublattice of the Jacobian fibration $(\phi, \zeta)$.

The following is of fundamental importance in the theory of Mordell-Weil groups. This holds, not only for $K3$ surfaces, but also for elliptic surfaces in general. See [26, Chapter 6].

**Theorem 4.3.** Let $\text{MW}_\phi (\cdot)$ denote the mapping that associates to each section $\sigma \in \text{MW}_\phi$ the class $[\sigma] \in \text{Rats}(X)$ of the image of $\sigma$. Then the composite

$$\text{MW}_\phi (\cdot) \rightarrow \text{Rats}(X) \hookrightarrow S_X \rightarrow S_X/(U_\phi \oplus \Sigma_\phi)$$

is an isomorphism of abelian groups.

**Remark 4.4.** By the isomorphism (4.3), Shioda [32] (see also [26]) introduced a structure of the positive-definite lattice (with a $\mathbb{Q}$-valued intersection form) on the free $\mathbb{Z}$-module $\text{MW}_\phi / (\text{torsion})$. This lattice is called the Mordell-Weil lattice. The norm of the Mordell-Weil lattice is very useful, for example, in finding good generators of $\text{MW}_\phi$. See Section 6.6.
For a vector \( v \in S_X \), we denote by \( s(v) \in MW_\phi \) the section that corresponds to \( v \mod (U_\phi \oplus \Sigma_\phi) \) by the isomorphism (4.3). First, we will explain a method to calculate \([s(v)] \in \text{Rats}(X)\) for a given \( v \in S_X \).

We review the Kodaira–Néron theory of singular fibers of an elliptic surface in more detail. See [26, Chapters 5 and 6], [14], [19], and [34, Table in page 46]. Recall that \( \Theta_\phi \) is the set of classes of smooth rational curves in fibers of \( \phi \) that is disjoint from the zero section \( \zeta \), and that the dual graph of \( \Theta_\phi \) is the Dynkin diagram of type \( \tau_\phi \). Let

\[
\Theta_\phi = \Theta_1 \sqcup \cdots \sqcup \Theta_n
\]

be the decomposition according to the decomposition of the Dynkin diagram into connected components. Then two elements \( r = [C] \) and \( r' = [C'] \) of \( \Theta_\phi \), where \( C \) and \( C' \) are smooth rational curves on \( X \), belong to the same \( \Theta_\nu \) if and only if \( \phi \) maps \( C \) and \( C' \) to the same point. Hence the set \( \{ \Theta_1, \ldots, \Theta_n \} \) is in one-to-one correspondence with the set

\[
\{ p \in \mathbb{P}^1 \mid \phi^{-1}(p) \text{ is reducible} \} = \{ p_1, \ldots, p_n \}
\]

in such a way that \( p_\nu \in \mathbb{P}^1 \) is the point \( \phi(C) \) for \( [C] \in \Theta_\nu \). We put

\[
\rho(\nu) := \text{Card}(\Theta_\nu), \quad \tau_\nu := \text{ADE-type of } \Theta_\nu.
\]

Recall that \( \Sigma_\phi \) is the root lattice generated by \( \Theta_\phi \). Let \( \Sigma_\nu \) be the sublattice of \( \Sigma_\phi \) generated by the elements of \( \Theta_\nu \). We have an orthogonal direct-sum decomposition

\[
\Sigma_\phi = \Sigma_1 \oplus \cdots \oplus \Sigma_n.
\]

The fiber \( \phi^{-1}(p_\nu) \) consists of \( \rho(\nu) + 1 \) smooth rational curves

\[ C_{\nu,0}, C_{\nu,1}, \ldots, C_{\nu,\rho(\nu)} \]

such that \( \Theta_\nu = \{ [C_{\nu,1}], \ldots, [C_{\nu,\rho(\nu)}] \} \) and that \( C_{\nu,0} \) intersects the zero section \( \zeta \).

The dual graph of

\[
\tilde{\Theta}_\nu := \{ [C_{\nu,0}] \} \cup \Theta_\nu
\]

is the affine Dynkin diagram of type \( \tau_\nu \). We number the smooth rational curves in \( \tilde{\Theta}_\nu \) as in Figure 4.1. The divisor \( \phi^*(p_\nu) \) is written as

\[
\phi^*(p_\nu) = \sum_{j=0}^{\rho(\nu)} m_{\nu,j} C_{\nu,j} \quad (m_{\nu,j} \in \mathbb{Z}_{>0}),
\]

where the coefficients \( m_{\nu,j} \) are given in Table 4.1. We put

\[
J_\nu := \{ j \mid m_{\nu,j} = 1 \}.
\]

We have \( 0 \in J_\nu \), and the class \([C_{\nu,0}]\) is calculated by

\[
[C_{\nu,0}] = f - \sum_{j=1}^{\rho(\nu)} m_{\nu,j} [C_{\nu,j}].
\]

(It is well known that \( m_{\nu,j} \) with \( j > 0 \) are the coefficients of the highest root of the root system \( \Theta_\nu \).) Let \( \phi^*(p_\nu)^\ddagger \) denote the smooth part of the divisor \( \phi^*(p_\nu) \):

\[
\phi^*(p_\nu)^\ddagger = \bigcup_{j \in J_\nu} C_{\nu,j}^\circ.
\]
where \( C_{\nu,j}^o \) is \( C_{\nu,j} \) minus the intersection points of \( C_{\nu,j} \) with other irreducible components of \( \phi^{-1}(p_\nu) \). By Kodaira–Néron theory, we can equip \( \phi^*(p_\nu)^J \) with the structure of an abelian Lie group. See [26, Section 5.6.1]. (When we work over \( \mathbb{C} \), this group structure is obtained as the limit of the group structures of general fibers of \( \phi \).) Then the set \( J_\nu \), which is regarded as the set of connected components \( C_{\nu,j}^o \) of \( \phi^*(p_\nu)^J \), also has a natural structure of an abelian group as a quotient group of \( \phi^*(p_\nu)^J \). The element \( 0 \in J_\nu \) is the zero element. See Table 4.2, which is copied from [34, Table in page 46], for the precise description of the group structure of \( J_\nu \).
\[ \tau_{\nu} \quad j = 0, 1, 2, \ldots, \rho(\nu) \]

| \( A_{\ell} \) | \{0, 1, \ldots, \ell\} | cyclic group \( \mathbb{Z}/(\ell + 1)\mathbb{Z} \): the sum of \( a, b \in J_{\nu} \) is \( c \in J_{\nu} \) such that \( a + b \equiv c \mod (\ell + 1) \) |
| \( D_{\ell} \) (\( \ell \) : even) | \{0, 1, 2, \ell\} | \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) |
| \( D_{\ell} \) (\( \ell \) : odd) | \{0, 1, 2, \ell\} | \( \mathbb{Z}/4\mathbb{Z} \) generated by 1 \( \in J_{\nu} \) with \( \ell \in J_{\nu} \) being of order 2 |
| \( E_{6} \) | \{0, 2, 6\} | \( \mathbb{Z}/3\mathbb{Z} \) |
| \( E_{7} \) | \{0, 7\} | \( \mathbb{Z}/2\mathbb{Z} \) |
| \( E_{8} \) | \{0\} | trivial |

Table 4.1. Coefficients \( m_{\nu,j} \)

Let \( \Sigma_{\nu}^\vee \) be the dual lattice of \( \Sigma_{\nu} \), and let \( \gamma_{\nu,1}, \ldots, \gamma_{\nu,\rho(\nu)} \) be the basis of \( \Sigma_{\nu}^\vee \) dual to the basis \([C_{\nu,1}], \ldots, [C_{\nu,\rho(\nu)}]\) of \( \Sigma_{\nu} \). We also put

\[ \gamma_{\nu,0} := 0 \in \Sigma_{\nu}^\vee. \]

For \( j = 0, 1, \ldots, \rho(\nu) \), we denote by \( \overline{\tau}_{\nu,j} \) the element \( \gamma_{\nu,j}(\mod \Sigma_{\nu}) \) of the discriminant group \( A(\Sigma_{\nu}) = \Sigma_{\nu}^\vee/\Sigma_{\nu} \) of \( \Sigma_{\nu} \). The following is the key observation for our method:

**Lemma 4.5.** The map \( j \mapsto \overline{\tau}_{\nu,j} \) gives an isomorphism \( J_{\nu} \cong \Sigma_{\nu}^\vee/\Sigma_{\nu} \) of abelian groups.

**Proof.** We compare Table 4.2 calculated in the Kodaira–Néron theory with the discriminant groups \( \Sigma_{\nu}^\vee/\Sigma_{\nu} \) of root lattices of type \( A_{\ell}, D_{\ell} \), and \( E_{\ell} \). The order of \( \Sigma_{\nu}^\vee/\Sigma_{\nu} \) is classically known, and coincides with \( |J_{\nu}| \). We equip the vector space \( \mathbb{R}^n \) with the standard basis \( e_1, \ldots, e_n \) and with the negative-definite intersection form \( \langle e_i, e_j \rangle := -\delta_{ij} \).

**The case** \( \tau_{\nu} = A_{\ell} \). We embed \( \Sigma_{\nu} \) into \( \mathbb{R}^{\ell+1} \) by \( [C_{\nu,j}] \mapsto e_j - e_{j+1} \) so that

\[ \Sigma_{\nu} = \{ (x_1, \ldots, x_{\ell+1}) \in \mathbb{Z}^{\ell+1} \mid x_1 + \cdots + x_{\ell+1} = 0 \}. \]
Then we have
\[ \gamma_{\nu,j} = \frac{1}{\ell+1} \left( - \sum_{k=1}^{j} (\ell + 1 - j) e_k + \sum_{k=j+1}^{\ell+1} j e_k \right) \in \Sigma_{\nu} \otimes \mathbb{Q}. \]

It is easy to check that \( j \gamma_{\nu,1} = \gamma_{\nu,j} \in \Sigma_{\nu} \). Hence \( j \mapsto \gamma_{\nu,j} \) gives an isomorphism \( \mathbb{Z} / (\ell + 1) \mathbb{Z} \cong \Sigma^\vee_{\nu} / \Sigma_{\nu} \).

The case \( \tau_\nu = D_\ell \). We embed \( \Sigma_{\nu} \) into \( \mathbb{R}^\ell \) by
\[ [C_{\nu,1}] \mapsto -e_1 - e_2, \quad [C_{\nu,2}] \mapsto e_1 - e_2, \quad [C_{\nu,j}] \mapsto e_{j-1} - e_j \quad (j = 3, \ldots, \ell), \]
so that we have
\[ \Sigma_{\nu} = \{ (x_1, \ldots, x_\ell) \in \mathbb{Z}^\ell \mid x_1 + \cdots + x_\ell \in 2\mathbb{Z} \}. \]

The vectors \( \gamma_{\nu,j} \in \Sigma_{\nu} \otimes \mathbb{Q} \) are given by
\[ \gamma_{\nu,1} = \frac{1}{2} \sum_{k=1}^{\ell} e_k, \quad \gamma_{\nu,2} = -\frac{1}{3} (3, 4, 5, 6, 4, 2), \quad \gamma_{\nu,6} = \frac{1}{3} (3, 2, 4, 6, 5, 4) = 2 \gamma_{\nu,2} \pmod{\Sigma_{\nu}}. \]

Hence we have \( \Sigma^\vee_{\nu} / \Sigma_{\nu} \cong \{ \bar{\tau}_{\nu,0}, \bar{\tau}_{\nu,2}, \bar{\tau}_{\nu,6} \} \cong \mathbb{Z} / 3\mathbb{Z} \). Note that we have \( \bar{\tau}_{\nu,5} = \bar{\tau}_{\nu,2}, \bar{\tau}_{\nu,3} = \bar{\tau}_{\nu,6}, \bar{\tau}_{\nu,1} = \bar{\tau}_{\nu,4} = \bar{\tau}_{\nu,0} \) in \( \Sigma^\vee_{\nu} / \Sigma_{\nu} \).

The case \( \tau_\nu = E_7 \). Using the basis \([C_{\nu,1}], \ldots, [C_{\nu,6}]\) of \( \Sigma_{\nu} \), we can write
\[ \gamma_{\nu,7} = \frac{1}{2} (3, 2, 4, 6, 5, 4). \]

Hence we have \( \Sigma^\vee_{\nu} / \Sigma_{\nu} \cong \{ \bar{\tau}_{\nu,0}, \bar{\tau}_{\nu,7} \} \cong \mathbb{Z} / 2\mathbb{Z} \). Note that we have \( \bar{\tau}_{\nu,1} = \bar{\tau}_{\nu,5} = \bar{\tau}_{\nu,7} \) and \( \bar{\tau}_{\nu,2} = \bar{\tau}_{\nu,3} = \bar{\tau}_{\nu,4} = \bar{\tau}_{\nu,6} = \bar{\tau}_{\nu,0} \) in \( \Sigma^\vee_{\nu} / \Sigma_{\nu} \).

The case \( \tau_\nu = E_8 \). Trivial. \( \square \)

A section \( \sigma \in \text{MW}_\phi \) intersects \( \phi^{-1}(p_\nu) \) at a single point \( \text{sp}_\nu(\sigma) \), and the intersection is transverse. Hence the intersection point \( \text{sp}_\nu(\sigma) \) is a smooth point of the fiber, that is, we have \( \text{sp}_\nu(\sigma) \in \phi^*(p_\nu)^g \). Thus we have the specialization map
\[ \text{sp}_\nu : \text{MW}_\phi \to \phi^*(p_\nu)^g. \]

By the definition of the group structure on \( \phi^*(p_\nu)^g \), the map \( \text{sp}_\nu \) is a group homomorphism. (See [26, Section 5.6.1].) The inclusion \( \Sigma_{\nu} \hookrightarrow S_X \) gives rise to the restriction homomorphism \( S_X \to \Sigma^\vee_{\nu} \), which we write as
\[ v \mapsto v|_{\nu}. \]

For \( \sigma \in \text{MW}_\phi \), we have
\[ [\sigma]|_{\nu} = \gamma_{\nu,j}[\sigma], \]
where \( j[\sigma] \in J_\nu \) is the index of the connected component of \( \phi^*(p_\nu)^g \) intersecting \( \sigma \), or equivalently, containing the point \( \text{sp}_\nu(\sigma) \). The kernel of the composite of
Step 1. Let \( \nu \) be a group homomorphism. By definition, the following diagram is commutative:

\[
\begin{array}{ccc}
\text{MW}_\phi & \xrightarrow{1} & S_X \\
\text{MW}_\phi & \xrightarrow{(4.6)} & \Sigma'_\nu / \Sigma_\nu
\end{array}
\]

is a group homomorphism. By definition, the following diagram is commutative:

\[
\begin{array}{ccc}
\text{MW}_\phi & \xrightarrow{(4.6)} & \Sigma'_\nu / \Sigma_\nu \\
\phi' & \downarrow & \downarrow \phi
\end{array}
\]

by Lemma 4.5

\[
\phi^*(p_\nu)^2 \rightarrow J_\nu,
\]

where the lower horizontal arrow is the natural quotient homomorphism.

Suppose that a vector \( v \in S_X \) is given. Then the class \([s(v)] \in S_X\) of the section \( s(v) \in \text{MW}_\phi \) corresponding to \( v \) mod \( (U_\phi \oplus \Sigma_\phi) \) by (4.3) satisfies the following:

(i) \([s(v)], [s(v)]\) = -2 and \([s(v)], f\) = 1. Hence, by the orthogonal direct-sum decomposition \( S_X = U_\phi \oplus W_\phi \), we have \([s(v)] = tf + z + w\), where \( w \in W_\phi \) and \( t = -(w, w)/2 \).

(ii) \([s(v)] \equiv v \mod U_\phi \oplus \Sigma_\phi\). In particular, for each \( \nu = 1, \ldots, n \), we have \(([s(v)] - v)|_\nu \in \Sigma_\nu\).

(iii) For each \( \nu = 1, \ldots, n \), there exists a unique index \( j(v) \in J_\nu \) such that \( [s(v)]|_\nu = \gamma_{\nu,j(v)} \). This \( j(v) \) is the index \( j \) of the connected component \( C_{\nu,j} \) that contains the intersection point \( sp_\nu(s(v)) \) of \( s(v) \) and \( \phi^{-1}(p_\nu) \), and hence \( j(v) \) is the image of \( v \) by \( S_X \rightarrow J_\nu \) in the diagrams (4.6) and (4.7).

Therefore the following calculations compute the class \([s(v)]\).

Step 1. Let \( v' \in W_\phi \) be the image of \( v \) by the projection to \( W_\phi \) under the orthogonal direct-sum decomposition \( S_X = U_\phi \oplus W_\phi \).

Step 2. For each \( \nu = 1, \ldots, n \), calculate the element \( \delta_\nu(v') := v'|_\nu \mod \Sigma_\nu \) of the discriminant group \( \Sigma'_\nu / \Sigma_\nu \), and find the index \( j(v) \in J_\nu \) such that \( \delta_\nu(v') \) is equal to \( \gamma_{\nu,j(v)} \). Then the element \( v'|_\nu - \gamma_{\nu,j(v)} \) of \( \Sigma'_\nu \) belongs to \( \Sigma_\nu \). We calculate the integers \( \alpha_{\nu,k} \) such that

\[
v'|_\nu - \gamma_{\nu,j(v)} = \sum_{k=1}^{\rho(\nu)} \alpha_{\nu,k} [C_{\nu,k}].
\]

Step 3. We put

\[
v'' := v' - \sum_{\nu=1}^{n} \sum_{k=1}^{\rho(\nu)} \alpha_{\nu,k} [C_{\nu,k}].
\]

Then we have \([s(v)] = tf + z + v''\), where \( t := -(v'', v'')/2 \).

Next, we explain how to calculate, for a given vector \( v \in S_X \), the isometry

\[
g(s(v)) \in O(S_X, \mathcal{P}_X)
\]

induced by the translation \( x \mapsto x + E s(v) \) on \( E_\eta \) by the section \( s(v) \in \text{MW}_\phi \), where \( +_E \) is the addition on the elliptic curve \( E_\eta \) over \( k(\mathbb{P}^1) \). Let \( m \) be the Mordell-Weil rank of \( \phi \):

\[
m := \dim(\text{MW}_\phi \otimes \mathbb{Q}) = \text{rank} S_X - 2 - \sum_{\nu=1}^{n} \rho(\nu),
\]
where the second equality follows from Theorem 4.3. We choose vectors \( u_1, \ldots, u_m \in S_X \) such that their images by

\[
S_X \to (S_X/(U_0 \oplus \Sigma_\phi)) \otimes \mathbb{Q}
\]

form a basis of \( MW_\phi \otimes \mathbb{Q} \). Then \( S_X \otimes \mathbb{Q} \) is spanned by

\[
f, z = [s(0)], [s(u_1)], \ldots, [s(u_m)], \quad \text{and the vectors } [C_{v,1}], \ldots, [C_{v,\rho(v)}]
\]

in \( \Theta_\nu \) for \( \nu = 1, \ldots, n \).

Therefore, to calculate \( g(s(v)) \), it is enough to calculate the images of vectors in (4.8) by \( g(s(v)) \). It is obvious that

\[
f g(s(v)) = f, \quad z g(s(v)) = [s(v)], \quad [s(u_\mu)] g(s(v)) = [s(u_\mu + v)] \quad \text{for } \mu = 1, \ldots, m.
\]

Hence it remains only to calculate the image by \( g(s(v)) \) of the classes in \( \Theta_\nu \). Note that \( g(s(v)) \) induces a permutation on the set \( \Theta_\nu = \{ [C_{v,0}] \} \cup \Theta_\nu \) that preserves the subset \( J_\nu \) of classes of reduced irreducible components. By the method described in Step 2 above, we calculate the index \( j(v) \in J_\nu \), which is the image of \( s(v) \in MW_\phi \) by the composite of \( sp_\nu \colon MW_\phi \to \phi^*(p_\nu)^* \) and \( \phi^*(p_\nu)^* \to J_\nu \). The translation of \( \phi^*(p_\nu)^* \) by \( sp_\nu(s(v)) \) induces the translation of \( J_\nu \) by \( j(v) \). Checking each Dynkin diagram of type \( A_\ell, D_\ell, E_\ell \), we see that this permutation of \( J_\nu \) extends uniquely to a permutation of \( \Theta_\nu \) that preserves the dual graph. See Table 4.3, in which we abbreviate \( \Theta_\nu = \{ [C_{v,0}] \ldots [C_{v,\rho(v)}] \} \) as \( \{0, 1, \ldots, \rho(v)\} \). Hence the image of each element of \( \Theta_\nu \) by \( g(s(v)) \) is computed. Using (4.5), we can calculate the action of \( g(s(v)) \) on the classes of \( \Theta_\nu \).

5. Borcherds’ method

5.1. An algorithm on a graph. We recall an algorithm introduced in [7]. Let \((V, E)\) be a simple non-oriented connected graph, where \( V \) is the set of vertices and \( E \) is the set of edges, which is a set of non-ordered pairs of distinct elements of \( V \):

\[
E \subset \binom{V}{2}.
\]

We say that \( v, v' \in V \) are adjacent if \( \{v, v'\} \in E \). The set \( V \) may be infinite. The assumption that \((V, E)\) be connected is important. Suppose that a group \( G \) acts on \((V, E)\) from the right. For vertices \( v, v' \in V \), we put

\[
T_G(v, v') := \{ g \in G \mid v^g = v' \},
\]

and define the \( G \)-equivalence relation \( \sim \) on \( V \) by

\[
v \sim v' \iff T_G(v, v') \neq \emptyset.
\]

Thus we have two relations on \( V \), the adjacency relation and the \( G \)-equivalence relation. Suppose that \( V_0 \) is a non-empty subset of \( V \) with the following properties.

(a) If \( v, v' \in V_0 \) are distinct, then \( v \) and \( v' \) are not \( G \)-equivalent.

(b) If a vertex \( v \in V \) is adjacent to a vertex in \( V_0 \), then \( v \) is \( G \)-equivalent to a vertex in \( V_0 \).
We put
\[ \tilde{V}_0 := \{ v \in V \mid v \text{ is adjacent to a vertex in } V_0 \}. \]

Then, for each \( v \in \tilde{V}_0 \), there exists a vertex \( u_0(v) \in V_0 \) such that \( T_G(v, u_0(v)) \neq \emptyset \).

Note that \( u_0(v) \in V_0 \) is unique by assumption (a). We choose an element \( h(v) \) from \( T_G(v, u_0(v)) \) for each \( v \in \tilde{V}_0 \), and put
\begin{equation}
H := \{ h(v) \mid v \in \tilde{V}_0 \}. \tag{5.1}
\end{equation}

**Proposition 5.1** (Proposition 4.1 of [7]). The subset \( V_0 \subset V \) is a complete set of representatives of the orbit decomposition of \( V \) by \( G \), and the group \( G \) is generated by the union of \( H \) and the stabilizer subgroup \( \text{Stab}_G(v_0) = T_G(v_0, v_0) \) of a vertex \( v_0 \in V_0 \).

In [7, Section 4.1], we presented an algorithm to obtain \( V_0 \) and \( H \) under the assumption that \( (V, E) \) and \( G \) have certain local effectiveness properties.

### 5.2. Period condition

In this subsection, we assume that the base field \( k \) is the complex number field \( \mathbb{C} \), and introduce period condition on elements of \( \text{O}(S_X) \). The period condition is, however, also defined when \( X \) is a supersingular \( K3 \) surface in positive characteristic. See, for example, [16].

Let \( L \) be an even lattice, and \( A(L) = L^\vee / L \) the discriminant group of \( L \). We define a quadratic form
\[ q(L) : A(L) \to \mathbb{Q}/2\mathbb{Z} \]
by \( q(x \mod L) := \langle x, x \rangle \mod 2\mathbb{Z} \). This finite quadratic form is called the discriminant form of \( L \), which was introduced by Nikulin [20]. Let \( M \) be a primitive
sublattice of $L$, and $N$ the orthogonal complement of $M$ in $L$. Then we have natural embeddings

$$M \oplus N \subset L \subset L^\vee \subset M^\vee \oplus N^\vee.$$ 

Suppose that $L$ is unimodular, that is, $L^\vee = L$. Then the submodule

$$L/(M \oplus N) \subset A(M) \times A(N)$$

is a graph of an isomorphism $A(M) \cong A(N)$, which induces an isomorphism

$$\iota_L: q(M) \cong -q(N).$$

Nikulin [20] proved the following.

**Proposition 5.2.** Suppose that $L$ is unimodular. Let $G_N$ be a subgroup of $O(N)$, and let $q(G_N) \subset \text{Aut}(q(N))$ be the image of $G_N$ by the natural homomorphism $O(N) \to \text{Aut}(q(N))$. Then an isometry $g_M$ of $M$ extends to an isometry $g_L$ of $L$ such that its restriction $g_L|N$ to $N$ is an element of $G_N$ if and only if the action of $g_M$ on $q(M)$ belongs to $q(G_N)$ via the isomorphism $\text{Aut}(q(M)) \cong \text{Aut}(q(N))$ induced by $\iota_L: q(M) \cong -q(N)$. 

We apply this result to the primitive embedding of $S_X$ into the even unimodular lattice $H^2(X, \mathbb{Z})$ of rank 22 defined by the cup product. Let $T_X$ denote the orthogonal complement of $S_X$ in $H^2(X, \mathbb{Z})$, which we call the transcendental lattice of $X$. Then $H^2(X, \mathbb{Z})$ induces an isomorphism

$$\iota_H: q(S_X) \cong -q(T_X).$$

Note that $T_X$ is the minimal primitive submodule of $H^2(X, \mathbb{Z})$ such that $T_X \otimes \mathbb{C}$ contains the period $H^{2,0}(X) = C\omega_X \subset H^2(X, \mathbb{C})$ of $X$, where $\omega_X$ is a nonzero holomorphic 2-form on $X$.

**Definition 5.3.** We put

$$O(T_X, \omega_X) := \{ g_T \in O(T_X) \mid g_T \otimes \mathbb{C} \text{ preserves } H^{2,0}(X) \}.$$ 

Then we say that $g_S \in O(S_X)$ satisfies the period condition if the action of $g_S$ on $q(S_X)$ is equal to the action on $q(T_X)$ of some of $g_T \in O(T_X, \omega_X)$ via the isomorphism $\iota_H: q(S_X) \cong -q(T_X)$ induced by $H^2(X, \mathbb{Z})$.

By Proposition 5.2, we see that an isometry $g_S \in O(S_X)$ extends to an isometry of $H^2(X, \mathbb{Z})$ preserving the period $H^{2,0}(X)$ if and only if $g_S$ satisfies the period condition. By Torelli theorem [23] (see also [4, Chapter VIII]), we obtain the following:

**Theorem 5.4.** We put

$$G := \text{Im}(\text{Aut}(X) \to O(S_X, \mathcal{P}_X)).$$

Then $g \in O(S_X, \mathcal{P}_X)$ belongs to $G$ if and only if $g$ preserves $N_X$ and satisfies the period condition. 

**Example 5.5.** Suppose that rank $T_X \geq 3$ and that $\omega_X$ is very general in the period domain $Q$ in $\mathbb{P}(T_X \otimes \mathbb{C})$. (See [4, Chapter VIII] for the definition of the period domain.) Then we have

$$O(T_X, \omega_X) = \{ \pm 1 \},$$

and hence $g_S \in O(S_X)$ satisfies the period condition if and only if the action of $g_S$ on the discriminant group $A(S_X)$ is $1$ or $-1$. 

We give a proof of (5.2). The period domain $Q$ is an open subset (in the classical topology) of a smooth quadratic hypersurface in $\mathbb{P}_+(T_X \otimes \mathbb{C})$, and hence we have $\dim Q = \text{rank} T_X - 2 > 0$. For $\gamma \in \text{O}(T_X)$, let $V_{\gamma,\lambda} \subset T_X \otimes \mathbb{C}$ denote the eigenspace of $\gamma$ with eigenvalue $\lambda \in \mathbb{C}$. If $\gamma \notin \{\pm 1\}$, then $\dim V_{\gamma,\lambda} < \text{rank} T_X$ and hence $\mathbb{P}_+(V_{\gamma,\lambda}) \cap Q$ is a proper analytic subspace of $Q$ for any $\lambda$. Since a countable union of proper analytic subspaces of a positive-dimensional connected complex manifold cannot cover the total space, we have (5.2) for $\omega_X$ very general in $Q$.

Suppose moreover that $-1 \in \text{O}(T_X, \omega_X)$ acts on $A(T_X)$ non-trivially (that is, the abelian group $A(T_X) \cong A(S_X)$ is not 2-elementary). By Proposition 5.2, there exists no isometry $g_H$ of the overlattice $H^2(X, \mathbb{Z})$ of $S_X \oplus T_X$ such that $g_H|S_X = 1$ and $g_H|T_X = -1$. Since $\text{Aut}(X)$ acts on $H^2(X, \mathbb{Z})$ faithfully, the natural homomorphism $\text{Aut}(X) \to O(S_X, \mathcal{P}_X)$ is injective.

**Remark 5.6.** For supersingular $K3$ surfaces, we have to prove (5.2) in a different method, because the period domain is a subvariety of codimension $> 1$ in a Grassmannian variety. See [16].

### 5.3. Tessellation by $L_{26}/S_X$-chambers

Let $L_{26}$ denote an even unimodular hyperbolic lattice of rank 26, which is unique up to isomorphism. We choose a positive cone $\mathcal{P}_{26}$ of $L_{26}$. A standard fundamental domain of $W(L_{26})$ was determined by Conway [8] by means of Vinberg’s algorithm [35].

**Definition 5.7.** A vector $w \in L_{26}$ is called a Weyl vector if $w$ is a non-zero primitive vector of $L_{26}$ contained in $\partial \mathcal{P}_{26}$ (in particular, we have $\langle w, w \rangle = 0$ and hence $Zw \subset (Zw)_{\perp}$) such that $(Zw)_{\perp}/Zw$ is isomorphic to the negative-definite Leech lattice.

**Definition 5.8.** Let $w$ be a Weyl vector. A $(-2)$-vector $r \in L_{26}$ is said to be a Leech root with respect to $w$ if $\langle w, r \rangle = 1$. We then put

$$C(w) := \{ x \in \mathcal{P}_{26} \mid \langle x, r \rangle \geq 0 \text{ for all Leech roots } r \text{ with respect to } w \}.$$  

**Theorem 5.9** (Conway [8]). (1) The mapping $w \mapsto C(w)$ gives a bijection from the set of Weyl vectors to the set of standard fundamental domains of $W(L_{26})$.

(2) Let $w$ be a Weyl vector. Then the mapping $r \mapsto C(w) \cap (r)_{\perp}$ gives a bijection from the set of Leech roots with respect to $w$ to the set of walls of the chamber $C(w)$. \hfill $\square$

**Definition 5.10.** We call a standard fundamental domain of $W(L_{26})$ a Conway chamber. Hence $\mathcal{P}_{26}$ is tessellated by the Conway chambers.

Suppose that we have a primitive embedding $\iota: S_X \hookrightarrow L_{26}$. Replacing $\iota$ by $-\iota$ if necessary, we assume that $\iota$ maps $\mathcal{P}_X$ into $\mathcal{P}_{26}$, and regard $\mathcal{P}_X$ as a subspace of $\mathcal{P}_{26}$:

$$\mathcal{P}_X = \iota^{-1}(\mathcal{P}_{26}) = (S_X \otimes \mathbb{R}) \cap \mathcal{P}_{26}.$$  

**Definition 5.11.** An $L_{26}/S_X$-chamber is a chamber $D$ of $\mathcal{P}_X$ that is obtained as the intersection $\mathcal{P}_X \cap C(w)$ of $\mathcal{P}_X$ with a Conway chamber $C(w)$.

The tessellation of $\mathcal{P}_{26}$ by the Conway chambers induces a tessellation of $\mathcal{P}_X$ by the $L_{26}/S_X$-chambers. By definition, the nef-and-big cone $N_X$, which is a standard fundamental domain of $W(S_X)$, is tessellated by $L_{26}/S_X$-chambers. In other words,
the tessellation of $\mathcal{P}_X$ by the $L_{26}/S_X$-chambers is a refinement of the tessellation by the standard fundamental domains of $W(S_X)$.

**Definition 5.12.** We define a graph $(V, E)$ by the following.

- The set $V$ of vertices is the set of $L_{26}/S_X$-chambers contained in $N_X$.
- The set $E$ of edges is the set of pairs of adjacent $L_{26}/S_X$-chambers.

Let $G$ be the image of the natural homomorphism $\text{Aut}(X) \to O(S_X, \mathcal{P}_X)$. Suppose that

\begin{equation}
\text{(5.3) the period condition for } g \in O(S_X) \text{ is that the action of } g \text{ on the discriminant group } A(S_X) \text{ be } 1 \text{ or } -1.
\end{equation}

See Example 5.5 for a case where this assumption is satisfied. Then, by Proposition 5.2, every element $g \in G$ extends to an isometry of $L_{26}$. In particular, the action of $G$ preserves the tessellation of $\mathcal{P}_X$ by the $L_{26}/S_X$-chambers. Since the action of $G$ preserves $N_X$, we obtain the following:

**Proposition 5.13.** If (5.3) holds, then $G$ acts on the graph $(V, E)$.  

**Definition 5.14.** Let $D = \mathcal{P}_X \cap C(w)$ be an $L_{26}/S_X$-chamber. For each wall $w$ of $D$, there exists a unique defining vector $v$ of $w$ in the dual lattice $S_X^\vee$ that is primitive in $S_X^\vee$. We call this vector $v \in S_X^\vee$ the **primitive defining vector** of the wall $w$.

Note that a Conway chamber has infinitely many walls. For the graph $(V, E)$ to have local effectiveness properties in [7], it needs that each $L_{26}/S_X$-chamber has only a finite number of walls. We consider the following assumption:

\begin{equation}
\text{(5.4) The orthogonal complement of } S_X \text{ in } L_{26} \text{ cannot be embedded in the negative-definite Leech lattice.}
\end{equation}

This holds, for example, if the orthogonal complement contains at least one $(-2)$-vector.

**Proposition 5.15 ([29]).** Suppose that (5.4) holds. Then each $L_{26}/S_X$-chamber has only a finite number of walls. If $D = \mathcal{P}_X \cap C(w)$ is an $L_{26}/S_X$-chamber obtained by the Conway chamber $C(w)$ associated with a Weyl vector $w$, then we can calculate the primitive defining vectors of walls of $D$ from $w$. Moreover, for each wall $w$ of $D$, we can calculate a Weyl vector $w'$ such that $D' = \mathcal{P}_X \cap C(w')$ is the $L_{26}/S_X$-chamber adjacent to $D$ across the wall $w$.  

Thus, under assumptions (5.3) and (5.4), the local effectiveness properties in [7] hold for $(V, E)$ and $G$, and we can apply the algorithm in [7, Section 4.1] to $(V, E)$ and $G$.

**Remark 5.16.** The amount of the computation of this method is estimated by $|V_0| = |V/G|$, that is, the number of the orbits of the action of $\text{Aut}(X)$ on the set of $L_{26}/S_X$-chambers contained in $N_X$.

In practice, it seems that Borcherds' method carried out without using computer (for example, [15]) can only deal with the case where $|V_0| = 1$. Some cases with $|V_0| > 1$ were treated in [29], where $V_0$ is of size about $10^3 \sim 10^4$. However, the geometric description of the generators of $\text{Aut}(X)$ was not given for these cases. We also have observed some cases where $|V_0|$ is too large for Borcherds' method to terminate in a reasonable time (for example, [13]).

In the case of the present article (see Section 6), we have $|V_0| = 7$. Since this is not so large, we have managed to obtain geometric generators.
Remark 5.17. It has been empirically observed that $|V_0|$ is small when the orthogonal complement of $i: S_X \hookrightarrow L_{26}$ contains a root lattice as a sublattice of finite index.

6. Computation of $\text{Aut}(X_{f,g})$

In this section, we prove Theorems 1.1 and 1.2. For simplicity, we write $X$ for the $K3$ surface $X_{f,g}$. Recall that the polynomials $f$ and $g$ in the defining equation (1.2) of $X_{f,g}$ are assumed to be very general. We use this assumption throughout this section.

6.1. The lattice $S_X$. First, we describe the lattice $S_X$ and the nef-and-big cone $N_X$. Let $H \subset X$ denote the pull-back of a line of $\mathbb{P}^2$, and we put

$$h := [H] \in S_X.$$ 

The singular locus of the branch curve $B(h) = \{f^2 + g^3 = 0\} \subset \mathbb{P}^2$ of the finite double covering $\overline{X}_{f,g} \to \mathbb{P}^2$ consists of six ordinary cusps $\bar{p}_1, \ldots, \bar{p}_6$, which are located at the locus defined by $f = g = 0$. Hence the singularities of $\overline{X}_{f,g}$ consist of six rational double points $p_1, \ldots, p_6$ of type $A_2$, where $p_i$ is located over $\bar{p}_i$. Let $E_i^{(+)}$ and $E_i^{(-)}$ denote the exceptional curves that are contracted to the point $p_i \in \text{Sing}(\overline{X}_{f,g})$ by the desingularization $X \to \overline{X}_{f,g}$. We put

$$e_i^{(+)} := [E_i^{(+)}] \in S_X, \quad e_i^{(-)} := [E_i^{(-)}] \in S_X.$$ 

Let $T \subset \mathbb{P}^2$ be the conic defined by $g = 0$. Then $T$ passes through the six cusps $\bar{p}_1, \ldots, \bar{p}_6$ of $B(h)$. Hence the strict transform of $T$ in $X$ is a disjoint union of two smooth rational curves $\Gamma^{(+)}$ and $\Gamma^{(-)}$. We put

$$\gamma^{(+)} := [\Gamma^{(+)}] \in S_X, \quad \gamma^{(-)} := [\Gamma^{(-)}] \in S_X.$$ 

For each $i \in \{1, \ldots, 6\}$, the curve $\Gamma^{(+)}$ intersects one of $E_i^{(+)}$ or $E_i^{(-)}$ and is disjoint from the other. Interchanging $E_i^{(+)}$ and $E_i^{(-)}$ if necessary, we can assume that

$$\langle \gamma^{(+)}, e_i^{(+)} \rangle = 1, \quad \langle \gamma^{(-)}, e_i^{(-)} \rangle = 0$$

holds for $i = 1, \ldots, 6$. Then we have the following. (See also [27].)

Proposition 6.1 (Degtyarev [9]). The $\mathbb{Q}$-vector space $S_X \otimes \mathbb{Q}$ is of dimension 13, and is generated by the classes

$$\{h, e_1^{(+)}, e_1^{(-)}, \ldots, e_6^{(+)}, e_6^{(-)}\}.$$ 

The sublattice $S_{X,0}$ of $S_X$ generated by the classes in (6.1) is of index 3 in $S_X$. The lattice $S_X$ is generated by $S_{X,0}$ and the class $\gamma^{(+)}$. \hfill $\Box$

By Proposition 6.1, a vector $v$ of $S_X \otimes \mathbb{Q}$ is uniquely determined by the list of intersection numbers

$$\langle v, h \rangle, \langle v, e_1^{(+)} \rangle, \langle v, e_1^{(-)} \rangle, \ldots, \langle v, e_5^{(+)} \rangle, \langle v, e_5^{(-)} \rangle.$$ 

Moreover, an isometry $g$ of $S_X$ is specified by the images of the classes in (6.1) by $g$. For example, the involution $i(h)$ associated with the double covering $\pi(h): X \to \mathbb{P}^2$ defined by $[h]$ is given by

$$h^{i(h)} = h, \quad (e_i^{(+)})^{i(h)} = e_i^{(-)}, \quad (e_i^{(-)})^{i(h)} = e_i^{(+)} \quad (i = 1, \ldots, 6).$$
The vector $a \in S_X \otimes \mathbb{Q}$ defined by
\begin{equation}
(a, h) = 8, \;
(a, e^+_i) = 1, \;
(a, e^-_i) = 1 \quad (i = 1, \ldots, 6)
\end{equation}
is a vector of $\mathcal{P}_X \cap S_X$, and satisfies
\[ (a, a) = 20, \;
\text{Roots}(a^+ \cap S_X) = \emptyset, \;
\text{Sep}(h, a) = \emptyset. \]
Hence $a$ is ample (see Section 3.1). By this ample class $a$, we can specify the nef-and-big cone $N_X$ in $\mathcal{P}_X$.

Next, we investigate the period condition of $X$. We consider the moduli space $\mathcal{M}$ of lattice-polarized $K3$ surfaces $(X', \eta')$, where $X'$ is a $K3$ surface and $\eta'$ is an isometry $H^2(X, \mathbb{Z}) \cong H^2(X', \mathbb{Z})$ that induces an embedding $S_X \hookrightarrow S_{X'}$. Then $\mathcal{M}$ is covered by the period domain $\mathcal{Q} \subset \mathbb{P}_*(T_X \otimes \mathbb{C})$. If $(X', \eta')$ is very general in $\mathcal{M}$, then we have $S_X = S_{X'}$. Looking at the lattice $S_X = S_{X'}$, we obtain the following:

**Proposition 6.2** (Degtyarev [9]). If $(X', \eta')$ is very general in $\mathcal{M}$, then there exist homogeneous polynomials $f'$ and $g'$ of degree 3 and 2, respectively, such that $X'$ is birational to the double plane defined by $w^2 = f'^2 + g'^3$. \(\square\)

**Remark 6.3.** The following naïve dimension count may help in understanding Proposition 6.2: the dimension of the parameter space of pairs $(f', g')$ of homogeneous polynomials of degree 3 and 2 modulo linear transformation is equal to
\[ \dim H^0(\mathbb{P}^2, \mathcal{O}(3)) + \dim H^0(\mathbb{P}^2, \mathcal{O}(2)) - \dim \text{GL}(3, \mathbb{C}) = 7 = \text{rank} T_X - 2 = \dim \mathcal{Q}. \]
See also [27] for the proof of Proposition 6.2.

Since $f$ and $g$ are very general, we see that $X$ is very general in $\mathcal{M}$, and hence we can assume that $\omega_X$ is very general in the period domain $\mathcal{Q}$. Therefore, by (5.2) in Example 5.5, we have
\begin{equation}
O(T_X, \omega_X) = \{ \pm 1 \}.
\end{equation}
The discriminant group $A(S_X)$ of $S_X$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^4$. Hence, by Example 5.5, we obtain the following:

**Proposition 6.4.** The natural representation of Aut$(X)$ on $S_X$ is faithful. \(\square\)

We will consider Aut$(X)$ as a subgroup of $O(S_X, \mathcal{P}_X)$ from now on. By Theorem 5.4 and (3.1), we have the following:

**Proposition 6.5.** An element $g \in O(S_X, \mathcal{P}_X)$ belongs to Aut$(X)$ if and only if $g$ acts on $A(S_X)$ as 1 or $-1$, and $\text{Sep}(a, a^g) = \emptyset$ holds. \(\square\)

We introduce an auxiliary group $M$, which makes the descriptions of $N_X$ and Aut$(X)$ much easier. Let $M$ be the subgroup of $O(S_X, \mathcal{P}_X)$ consisting of elements $g$ satisfying $h^g = h$ and
\[ \{ e_1^+, e_1^-, \ldots, e_6^+, e_6^- \}^g = \{ e_1^+, e_1^-, \ldots, e_6^+, e_6^- \}. \]
Then $M$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times S_6$, generated by the involution $i(h)$ and permutations $\sigma \in S_6$ given by
\[ h^\sigma = h, \quad e_i^+ = e_i^+, \quad e_i^- = e_i^- \].
For each $g \in M$, we have $a = a^g$, and hence $M \subset O(S_X, N_X)$. The discriminant form $q(S_X)$ of $S_X$ is isomorphic to
\[ \left( \left[ \frac{1}{2} \right], \mathbb{Z}/2\mathbb{Z} \right) \oplus \left( \left[ \frac{4}{3} \right], \mathbb{Z}/3\mathbb{Z} \right) \oplus \left( \left[ \frac{2}{3} \right], \mathbb{Z}/3\mathbb{Z} \right). \]
Here \( ([α], \mathbb{Z}/m\mathbb{Z}) \) denotes a cyclic group \( A = \langle γ \rangle \) of order \( m \) generated by \( γ \) equipped with the quadratic form \( q: A \to \mathbb{Q}/2\mathbb{Z} \) such that \( q(γ) = α \). The natural homomorphism \( O(S_X) \to \text{Aut}(q(S_X)) \) maps \( M \) to \( \text{Aut}(q(S_X)) \) isomorphically. Note that \( \iota(h) \) acts on \( A(S_X) \) as \(-1\). Hence we have

\[
M \cap \text{Aut}(X) = \{1, i(h)\}.
\]

**Remark 6.6.** By means of the methods in Section 3.4, we can make the list of classes of smooth rational curves \( C \) on \( X \) with \( \langle [C], h \rangle = m \) for each non-negative integer \( m \). The size \( ν(m) \) of this list is as follows: When \( m \) is odd, we have \( ν(m) = 0 \), whereas for \( m \) even, we have

| \( m \) | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
|-----|---|---|---|---|---|----|----|----|
| \( ν(m) \) | 12 | 17 | 0 | 492 | 720 | 492 | 8292 | 8730 |

For \( i, j \) with \( 1 ≤ i ≤ 6, 1 ≤ j ≤ 6, i ≠ j \), let \( ℓ_{ij} \subset \mathbb{P}^2 \) denote the line passing through the singular points \( \tilde{p}_i \) and \( \tilde{p}_j \) of the branch curve \( B(h) \), and let \( \tilde{ℓ}_{ij} \subset X \) be the strict transform of \( ℓ_{ij} \). The \( ν(2) = 17 \) smooth rational curves on \( X \) of degree 2 with respect to \( h \) are the lifts \( \Gamma^{(±)} \) of the conic \( \Gamma \subset \mathbb{P}^2 \) and the curves \( \tilde{ℓ}_{ij} \).

**6.2. Automorphisms of \( X \).** By the method in Section 3.7, we find many automorphisms of \( X \) from nef vectors of norm 2. Among them, we have the following automorphisms:

Type (a): the involution \( i(h) \),

Type (b): 90 involutions \( i(h_{IJ}) \) associated with polarizations \( h_{IJ} \) of degree 2 such that \( \langle h_{IJ}, h \rangle = 6 \) and that \( \text{Sing}(B(h_{IJ})) \) is of type \( A_3 + A_5 \),

Type (c): 12 involutions \( i(h_{IJ}^±) \) associated with polarizations \( h_{IJ}^± \) of degree 2 such that \( \langle h_{IJ}^±, h \rangle = 4 \) and that \( \text{Sing}(B(h_{IJ}^±)) \) is of type \( A_2 + 5A_1 \),

Type (d): 360 involutions \( i(h_{IJ}^±) \) associated with polarizations \( h_{IJ}^± \) of degree 2 such that \( \langle h_{IJ}^±, h \rangle = 14 \), and that \( \text{Sing}(B(h_{IJ}^±)) \) is of type \( D_4 + A_5 \), and

Type (e): 360 translations associated with sections \( e_j^{(±)} \) of infinite order of 120 Jacobian fibrations \( φ: X \to \mathbb{P}^1 \) defined by \( (f_φ, z_φ) = (f_{±,1}, e_j^{(±)}) \) with \( \langle f_{±,1}, h \rangle = 4 \) such that \( \text{MW}_φ \) is torsion-free of rank 4 and that the reducible fibers of \( φ \) are of type \( D_4 + A_4 \).

See subsections below for more precise descriptions of these automorphisms. We will show, by Borcherds’ method, that these automorphisms generate \( \text{Aut}(X) \).

**6.3. Primitive embedding \( S_X \to L_{26}. \)** To apply Borcherds’ method, we embed \( S_X \) into \( L_{26} \) primitively. Let \( R_0 \) be a negative-definite root lattice of type \( A_1 + 6A_2 \) with a basis

\[
(6.4) \quad \alpha, β_1^{(+)}, β_1^{(-)}, \ldots, β_6^{(+)}, β_6^{(-)}
\]

consisting of roots that form the dual graph as in Figure 6.1. Let

\[
α^{∨}, β_1^{(+)^∨}, β_1^{(-)^∨}, \ldots, β_6^{(+)^∨}, β_6^{(-)^∨}
\]

be the basis of the dual lattice \( R_0^{∨} \) that is dual to the basis (6.4). Then

\[
R := R_0 + \mathbb{Z} \left( β_1^{(+)^∨} + \cdots + β_6^{(+)^∨} \right) \subset R_0^{∨}
\]

is an even lattice whose discriminant form is isomorphic to \( -q(S_X) \). Recall that the natural homomorphism \( O(S_X) \to \text{Aut}(q(S_X)) \) maps \( M \) to \( \text{Aut}(q(S_X)) \) isomorphically, and hence is surjective. Therefore, by Nikulin [20], there exists a unique (up
to the action of $O(S_X)$ even unimodular overlattice of $S_X \oplus R$ in which $S_X$ and $R$ are both primitive. Taking this unimodular overlattice as $L_{26}$, we find a primitive embedding

$$\iota : S_X \hookrightarrow L_{26}.$$  

We consider the tessellation of $N_X \subset \mathcal{P}_X$ by the $L_{26}/S_X$-chambers associated with this primitive embedding. Let $(V, E)$ be the graph of $L_{26}/S_X$-chambers contained in $N_X$ (see Definition 5.12). By (6.3) and Propositions 5.13, 5.15, we see that the group $G = \text{Aut}(X) \subset O(S_X, \mathcal{P}_X)$ acts on the graph $(V, E)$, and we can apply the algorithm in [7, Section 4.1].

**Remark 6.7.** Primitive embeddings of $S_X$ into $L_{26}$ are not unique. In fact, the genus of negative-definite even lattices containing the isomorphism class of $R$ consists of 26 isomorphism classes.

The image $\iota(a) \in \mathcal{P}_{26} \cap L_{26}$ of the ample class $a \in S_X$ defined by (6.2) satisfies

$$\text{Roots}(\iota(a) \hookrightarrow L_{26})^\perp = \text{Sep}(\iota : S_X \hookrightarrow L_{26}) \cong \text{Roots}(R),$$

where $\iota(a)$ is the sublattice of $L_{26}$ generated by $\iota(a)$. Hence $a$ is an interior point of an $L_{26}/S_X$-chamber, which we denote by $D_0$. Moreover, we have

$$\text{Sep}_{26}(\iota(a), \iota(h)) = \emptyset,$$

where we denote by $\text{Sep}_{26}$ the set of separating $(-2)$-vectors in $L_{26}$. Hence the class $h$ is a point of $D_0$. We choose a vector $\tilde{a} \in \mathcal{P}_L \cap L_{26}$ that satisfies

$$\text{Roots}(\iota(\tilde{a}) \hookrightarrow L_{26})^\perp = \emptyset, \quad \text{Sep}_{26}(\iota(a), \tilde{a}) = \emptyset.$$

Then $\tilde{a}$ is an interior point of a Conway chamber $C_0$ such that $\iota^{-1}(C_0) = D_0$. We can calculate a subset of the set of roots $\tilde{r}$ of $L_{26}$ such that $C_0 \cap (\tilde{r})^\perp$ is a wall of $C_0$, either by Vinberg’s algorithm [35], or by calculating $\text{Sep}_{26}(\tilde{a}, v)$, where $v \in \mathcal{P}_{26} \cap L_{26}$ are randomly chosen vectors. If this subset is large enough, these roots $\tilde{r}$ span $L_{26} \otimes \mathbb{Q}$ and hence the Weyl vector $w_0$ of the Conway chamber $C_0$ is calculated by solving the equations $\langle w_0, \tilde{r} \rangle = 1$.

**Remark 6.8.** The ADE-type of the roots in (6.5) is $A_1 + 6A_2$. Hence the hyperplanes perpendicular to these roots decompose $R \otimes \mathbb{R}$ into $2 \times 6^6$ regions. Therefore there exist exactly $2 \times 6^6$ Conway chambers $C$ such that $\iota^{-1}(C) = D_0$.

Thus we prepared all the data necessary to start the algorithm of [7, Section 4.1] to calculate a complete set $V_0$ of the representatives of $V/G$ and a finite generating set of $G = \text{Aut}(X)$. We executed this algorithm. The computation terminated and yielded the following:

**Proposition 6.9.** The set $V_0$ consists of the following seven $L_{26}/S_X$-chambers:

$$D_0, D_1^{(1)}, D_1^{(2)}, D_1^{(3)}, D_1^{(4)}, D_1^{(5)}, D_1^{(6)}.$$
We will describe each of these $L_{26}/S_X$-chambers in $V_0$, and during the description, we present automorphisms in the set $H$ defined by (5.1).

We use the following convention. Let $D$ be an $L_{26}/S_X$-chamber, and let $C$ be a Conway chamber such that $\iota^{-1}(C) = D$. Let $w$ be the Weyl vector of $C$. For a wall $w$ of $D$, let $v \in S_X^w$ be the primitive defining vector of $w$ (see Definition 5.14), and we put

$$n(w) := \langle v, v \rangle, \quad a(w) := \langle w, \iota(v) \rangle, \quad h(w) := \langle h, v \rangle.$$ 

These rational numbers are useful in classifying walls.

6.4. The $L_{26}/S_X$-chamber $D_0$. The initial $L_{26}/S_X$-chamber $D_0$ contains the ample class $a$ in its interior. The stabilizer subgroup of $D_0$ in $G$ is $\{1, i(h)\}$. The group $M$ leaves $D_0$ invariant. The chamber $D_0$ has 110 walls, and the action of $M$ decomposes the walls of $D_0$ into four orbits $o_1, o_2, o_3, o_4$ of sizes 2, 12, 6, 90, respectively. The data of these orbits are given in Table 6.1.

| $o_i$ | size | $n$ | $a$ | $h$ |
|------|------|----|----|----|
| $o_1$ | 2    | -2 | 1  | 2  | $\gamma^{(\pm)}$ |
| $o_2$ | 12   | -2 | 1  | 0  | $e_i^{(\pm)}$ |
| $o_3$ | 6    | -3/2 | 3/2 | 1 | isom with $D_1^{(\alpha)}$ |
| $o_4$ | 90   | -2/3 | 3  | 2  | isom with $D_0$ |

Table 6.1. Walls of $D_0$

The orbit $o_1$ of size 2 consists of $(\gamma^{(\pm)})_{\perp} \cap D_0$. The orbit $o_2$ of size 12 consists of $(e_i^{(\pm)})_{\perp} \cap D_0$. Hence the $L_{26}/S_X$-chamber adjacent to $D_0$ across a wall in $o_1$ or $o_2$ is not contained in $N_X$.

The orbit $o_3$ of size 6 consists of the walls $(v_\alpha)_{\perp} \cap D_0$ whose primitive defining vectors $v_\alpha$ are given by

$$\langle v_\alpha, h \rangle = 1, \quad \langle v_\alpha, e_i^{(+)} \rangle = \langle v_\alpha, e_i^{(-)} \rangle = \begin{cases} 1 & \text{if } i = \alpha, \\ 0 & \text{if } i \neq \alpha. \end{cases}$$

(6.6)

Let $D_1^{(\alpha)}$ be the $L_{26}/S_X$-chamber adjacent to $D_0$ across the wall $(v_\alpha)_{\perp} \cap D_0$. Then $D_1^{(\alpha)}$ is contained in $N_X$, but is not $G$-equivalent to $D_0$, and any two of $D_1^{(1)}, \ldots, D_1^{(6)}$ are not $G$-equivalent to each other. Hence these chambers $D_1^{(\alpha)}$ ($\alpha = 1, \ldots, 6$) are added to $V_0$ as new representatives of $V/G$.

The walls $w_{IJ}$ in the orbit $o_4$ of size 90 are indexed by ordered pairs $(I, J)$, where $I$ and $J$ are subsets of $\{1, \ldots, 6\}$ satisfying $\vert I \vert = \vert J \vert = 2$ and $I \cap J = \emptyset$. The primitive defining vector $v_{IJ} \in S_X^w$ of $w_{IJ} \in o_4$ is given by

$$\langle v_{IJ}, h \rangle = 2, \quad \langle v_{IJ}, e_i^{(+)} \rangle = 0, \quad \langle v_{IJ}, e_i^{(-)} \rangle = 0, \quad \text{if } i \notin I \cup J,$$

$$\langle v_{IJ}, e_i^{(+)} \rangle = 1, \quad \langle v_{IJ}, e_i^{(-)} \rangle = 0, \quad \text{if } i \in I,$$

$$\langle v_{IJ}, e_i^{(+)} \rangle = 0, \quad \langle v_{IJ}, e_i^{(-)} \rangle = 1, \quad \text{if } i \in J.$$

The $L_{26}/S_X$-chamber $D_{IJ}$ adjacent to $D_0$ across the wall $w_{IJ}$ is $G$-equivalent to $D_0$. An automorphism $g_{IJ} \in G$ that maps $D_0$ to $D_{IJ}$ isomorphically is given as

$$g_{IJ}(v) = v_{IJ}.$$
follows. Let \( h_{IJ} \) be a vector of \( S_X \otimes \mathbb{Q} \) defined by

\[
\langle h_{IJ}, h \rangle = 6,
\]

\[
\langle h_{IJ}, e_i^{(+)} \rangle = 0, \quad \langle h_{IJ}, e_i^{(-)} \rangle = 0, \quad \text{if } i \notin I \cup J,
\]

\[
\langle h_{IJ}, e_i^{(+)} \rangle = 1, \quad \langle h_{IJ}, e_i^{(-)} \rangle = 1, \quad \text{if } i \in I,
\]

\[
\langle h_{IJ}, e_i^{(+)} \rangle = 0, \quad \langle h_{IJ}, e_i^{(-)} \rangle = 3, \quad \text{if } i \in J.
\]

Then \( h_{IJ} \in S_X \) and \( \langle h_{IJ}, h_{IJ} \rangle = 2 \). We confirm \( \text{Sep}(h_{IJ}, a) = \emptyset \), and hence \( h_{IJ} \in N_X \). The complete linear system \( |h_{IJ}| \) is proved to be fixed-component free by the criterion in Section 3.7. The involution \( i(h_{IJ}) \) associated with the double covering \( \pi(h_{IJ}) : X \to \mathbb{P}^2 \) given by \( |h_{IJ}| \) maps \( D_0 \) to \( D_{IJ} \) isomorphically. Therefore

\[
i(h_{IJ})^{-1} = i(h_{IJ}) \in T_G(D_{IJ}, u_0(D_{IJ}))
\]

in the notation of Section 5.1. These involutions \( i(h_{IJ}) \) are the involutions of type (b) in Section 6.2.

Remark 6.10. Suppose that \( I = \{i_1, i_2\} \), \( J = \{j_1, j_2\} \), and

\[
\{1, \ldots, 6\} - (I \cup J) = \{k_1, k_2\}.
\]

Then the smooth rational curves on \( X \) contracted to points by the double covering \( \pi(h_{IJ}) : X \to \mathbb{P}^2 \) are as in Figure 6.2, where \( \hat{\ell}_{j_1 j_2} \) is the curve given in Remark 6.6. In particular, the singular locus of the branch curve \( B(h_{IJ}) \) is of type \( A_3 + A_5 \).

Remark 6.11. We have \( v_{IJ} i(h) = v_{IJ}, \ h_{IJ} i(h) \neq h_{IJ} \), and can confirm that the involution \( i(h_{IJ} i(h)) = i(h) i(h_{IJ}) i(h) \) is equal to \( i(h_{IJ}) \).

6.5. **The \( L_{26}/S_X \)-chamber \( D_1^{(a)} \).** The stabilizer subgroup of \( D_1^{(a)} \) in \( G \) is \( \{1, i(h)\} \).

The group \( M \) acts on the set \( \{D_1^{(1)}, \ldots, D_1^{(6)}\} \) transitively. Let \( M_\alpha \) be the stabilizer subgroup of \( D_1^{(a)} \) in \( M \). Then \( M_\alpha \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \times S_5 \). The chamber \( D_1^{(a)} \) has 110 walls, and the action of \( M_\alpha \) decomposes the walls of \( D_1^{(a)} \) into seven orbits \( o_1', \ldots, o_7' \). The data of these orbits are given in Table 6.2.

The orbit \( o_1' \) consists of a single wall, and the adjacent \( L_{26}/S_X \)-chamber across this wall is \( D_0 \), which means that this wall is in the orbit \( o_3 \) of walls of \( D_0 \) viewed from the opposite side.

The orbit \( o_2' \) of size 2 consists of \( (\gamma^{(\pm)})^+ \cap D_1^{(a)} \), the orbit \( o_3' \) of size 5 consists of \( (\tilde{\ell}_{\alpha})^+ \cap D_1^{(a)} \) with \( \beta \neq \alpha \), and the orbit \( o_4' \) of size 10 consists of \( (e^{(\pm)}_3)^+ \cap D_1^{(a)} \) with \( \beta \neq \alpha \). The adjacent \( L_{26}/S_X \)-chambers across these walls are therefore not contained in \( N_X \).
The orbit $o_5'$ is of size 2. One of the walls in $o_5'$ is defined by a vector $v^+_\alpha \in S_X$ satisfying

\[
\langle v^+_\alpha, h \rangle = 1, \\
\langle v^+_\alpha, e_\alpha^{(+)} \rangle = 2, \\
\langle v^+_\alpha, e_\alpha^{(-)} \rangle = -1, \\
\langle v^+_\alpha, e_\beta^{(+)} \rangle = 0, \\
\langle v^+_\alpha, e_\beta^{(-)} \rangle = 0 \quad (\beta \neq \alpha),
\]

and the other wall in $o_5'$ is defined by the vector

\[
v^-_\alpha := (v^+_\alpha)^i(h).
\]

The adjacent $L_{26}/S_X$-chamber $D_1^+$ across the wall $(v^+_\alpha)^\perp \cap D_1^{(\alpha)}$ is $G$-equivalent to $D_1^{(\alpha)}$. Indeed, the following automorphism $i(h^+_\alpha) \in G$ maps $D_1^{(\alpha)}$ to $D_1^+$ isomorphically. Let $h^+_\alpha$ be the vector defined by

\[
\langle h^+_\alpha, h \rangle = 4, \\
\langle h^+_\alpha, e_\alpha^{(+)} \rangle = 2, \\
\langle h^+_\alpha, e_\alpha^{(-)} \rangle = 0, \\
\langle h^+_\alpha, e_\beta^{(+)} \rangle = 0, \\
\langle h^+_\alpha, e_\beta^{(-)} \rangle = 1 \quad (\beta \neq \alpha).
\]

Then we have $h^+_\alpha \in S_X$ and $(h^+_\alpha, h^+_\alpha) = 2$. We confirm $\text{Sep}(h^+_\alpha, a) = \emptyset$ and hence $h^+_\alpha \in N_X$. The complete linear system $|h^+_\alpha|$ is proved to be fixed-component free by the criterion in Section 3.7. Then we can confirm by direct computation that the involution $i(h^+_\alpha)$ associated with the double covering $\pi(h^+_\alpha) : X \to \mathbb{P}^2$ given by $|h^+_\alpha|$ induces $D_1^{(\alpha)} \cong D_1^+$. It is obvious that the automorphism $i(h^-_\alpha) := i(h)i(h^+_\alpha)i(h)$ maps $D_1^{(\alpha)}$ to the adjacent $L_{26}/S_X$-chamber $D^-_1$ across the wall $(v^-_\alpha)^\perp \cap D_1^{(\alpha)}$. Therefore we have

\[
i(h^+_\alpha) = (i(h^+_\alpha))^{-1} \in T_G(D^+_\alpha, u_0(D^+_\alpha))
\]

in the notation of Section 5.1. These involutions $i(h^\pm_\alpha)$ are the involutions of type (c) in Section 6.2.

Remark 6.12. The branch curve $B(h^+_\alpha)$ of the double covering $\pi(h^+_\alpha)$ has the singularities of type $A_2 + 5A_1$. The exceptional curves over the singular point of type $A_2$ are $\gamma^{(-)}$ and $e_\alpha^{(-)}$, whereas the exceptional curves over the singular points of type $A_1$ are $e_\beta^{(+)}$ for $\beta \neq \alpha$. In particular, the involution $i(h^+_\alpha)$ interchanges $\gamma^{(-)}$ and $e_\alpha^{(-)}$. 

| size | $n$ | $a$ | $h$ |
|------|-----|-----|-----|
| $o_1'$ | 1 | $-3/2$ | $3/2$ | $-1$ |
| $o_2'$ | 2 | $-2$ | 1 | 2 |
| $o_3'$ | 5 | $-2$ | 1 | 2 |
| $o_4'$ | 10 | $-2$ | 1 | 0 |
| $o_5'$ | 2 | $-3/2$ | $3/2$ | 1 |
| $o_6'$ | 30 | $-1/6$ | 7/2 | 3 |
| $o_7'$ | 60 | $-2/3$ | 3 | 2 |

Table 6.2. Walls of $D_1^{(\alpha)}$
The description of the orbit $o'_6$ is rather complicated, and hence is postponed to the next subsection.

We describe the orbit $o'_7$ of size 60. Suppose that $\beta \in \{1, \ldots, 6\}$ and $F = \{i_1, i_6\} \subset \{1, \ldots, 6\}$ satisfy $i_1 \neq i_6, \beta \neq \alpha$ and $\{\alpha, \beta\} \cap \{i_1, i_6\} = \emptyset$. Let $v^{(+)}_{\beta F} \in S_X$ be the vector defined by
\[ \langle v^{(+)}_{\beta F}, h \rangle = 2, \quad \langle v^{(+)}_{\beta F}, e_{i}^{(\pm)} \rangle = 1, \quad \langle v^{(+)}_{\beta F}, e_{i}^{(-)} \rangle = 0 \text{ if } i \in \{\alpha, \beta\}, \quad \langle v^{(+)}_{\beta F}, e_{i}^{(-)} \rangle = 1 \text{ if } i \in F, \quad \langle v^{(+)}_{\beta F}, e_{i}^{(\pm)} \rangle = 0, \quad \langle v^{(+)}_{\beta F}, e_{i}^{(-)} \rangle = 0 \text{ otherwise.} \]

We then put
\[ v^{(-)}_{\beta F} := \left(v^{(+)}_{\beta F}\right)^{i(h)}. \]

The orbit $o'_7$ consists of walls $(v^{(+)}_{\beta F})^\perp \cap D^{(\alpha)}_1$ and $(v^{(-)}_{\beta F})^\perp \cap D^{(\alpha)}_1$. The adjacent $L_{26}/S_X$-chamber $D^{(\alpha)}_1$ across the wall $(v^{(\pm)}_{\beta F})^\perp \cap D^{(\alpha)}_1$ is $G$-equivalent to $D^{(\beta)}_1$. We put $A := \{\alpha, \beta\}$, and consider the polarization $h_{AF}$ of degree 2 defined by (6.7) with $I = A$ and $J = F$. The involution $i(h_{AF})$, which is an involution of type (b) in Section 6.2, maps $D^{(\beta)}_1$ to $D^{(\beta)}_1$ isomorphically, whereas the involution $i(h_{FA})$ maps $D^{(\beta)}_1$ to $D^{(\beta)}_1$ isomorphically. (See Remark 6.11.) Therefore we have
\[ i(h_{AF}) = i(h_{AF})^{-1} \in T_G(D^{(\alpha)}_{\beta F}, u_0(D^{(\alpha)}_{\beta F})), \quad i(h_{FA}) = i(h_{FA})^{-1} \in T_G(D^{(\alpha)}_{\beta F}, u_0(D^{(\alpha)}_{\beta F})), \]
in the notation of Section 5.1.

### 6.6. The orbit $o'_6$. In the following, for a sign $\sigma \in \{+,-\}$, let $\bar{\sigma}$ denote the opposite sign: $\{\sigma, \bar{\sigma}\} = \{+,-\}$. First, we define automorphisms $g'_\sigma I_j$ and $g'_\sigma J$.

Let $I$ be the set of ordered triples
\[ I = \{(i_1), \{i_2, i_3, i_4\}, \{i_5, i_6\}\} \]
such that $\{i_1, \ldots, i_6\} = \{1, \ldots, 6\}$. We have $|I| = 60$. For a pair of $\sigma \in \{+,-\}$ and $I \in I$, we have the configuration of smooth rational curves as in Figure 6.3. Then
\[ f_\sigma := f_{\sigma I} := e_{i_1}^{(\sigma)} + e_{i_2}^{(\sigma)} + e_{i_3}^{(\sigma)} + e_{i_4}^{(\sigma)} + 2\gamma^{(\sigma)} \]
\[ = \gamma^{(\sigma)} + e_{i_5}^{(\sigma)} + e_{i_6}^{(\sigma)} + \tilde{t}_{i_5i_6} \]
is the class of a fiber of an elliptic fibration $\phi: X \to \mathbb{P}^1$ with
\[ z_\phi := z_{\sigma I} := e_{i_1}^{(\sigma)} \]
being the class of a section. Thus we obtain a Jacobian fibration $\phi$ with the zero section $z_\phi$, and its Mordell-Weil group
\[ \text{MW}_\phi := \text{MW}(X, f_\phi, z_\phi) \subset G = \text{Aut}(X). \]
Calculating the set $\Theta_\phi = \text{Roots}(W_\phi) \cap \text{Rats}(X)$, we see that the ADE-type of the reducible fibers of $\phi: X \to \mathbb{P}^1$ is $D_4 + A_3$. Hence the rank of $\text{MW}_\phi$ is 4. Since the trivial sublattice of $\phi$, which is of rank 9 generated by the classes of the ten curves
in Figure 6.3, is primitive in \( S_X \), we see that \( \text{MW}_\phi \) is torsion free. A Gram matrix of the Mordell-Weil lattice \( \text{MW}_\phi \) (see Remark 4.4) is

\[
\begin{bmatrix}
3 & -1 & -1 & -1 \\
-1 & 3 & -1 & -1 \\
-1 & -1 & 3 & 1 \\
-1 & -1 & 1 & 3
\end{bmatrix}.
\]

The numbers \( n(s) \) of elements with small Mordell-Weil norms \( s \) in \( \text{MW}_\phi \) are given as follows:

| \( s \) | 9/4 | 3 | 21/4 | 6 |
|-----|-----|-----|-----|-----|
| \( n(s) \) | 12 | 14 | 16 | 30 |

Among these, we have the following sections of \( \phi \):

- The six smooth rational curves \( \tilde{\ell}_{j_1,j_2} \), where \( j_1 \in \{i_2, i_3, i_4\} \) and \( j_2 \in \{i_5, i_6\} \), satisfy \( \langle \tilde{\ell}_{j_1,j_2}, f \rangle = 1 \), and hence they are sections of \( \phi \). Their Mordell-Weil norms are 9/4.
- The three smooth rational curves \( e^{(\sigma)}_j \), where \( j \in \{i_2, i_3, i_4\} \), also satisfy \( \langle e^{(\sigma)}_j, f \rangle = 1 \), and hence they are sections of \( \phi \). Their Mordell-Weil norms are equal to 3.

These 6 + 3 sections \( \tilde{\ell}_{j_1,j_2} \) and \( e^{(\sigma)}_j \) generate \( \text{MW}_\phi \).

**Definition 6.13.** For \( j \in \{i_2, i_3, i_4\} \), we denote by \( g'_{\sigma I_j} \) the automorphism of \( X \) obtained as the translation by the section \( e^{(\sigma)}_j \in \text{MW}_\phi \). This is the automorphism of type (c) in Section 6.2.

Let \( \mathcal{J} \) be the set of ordered 4-tuples

\[
J = (\{i_1\}, \{i_2, i_3\}, \{i_4, i_5\}, \{i_6\})
\]
such that \( \{i_1, \ldots, i_6\} = \{1, \ldots, 6\} \). We have \( |J| = 180 \). For a pair of \( \sigma \in \{+, -\} \) and \( J \in \mathcal{J} \), let \( h_{\sigma J} \) be the vector of \( S_X \otimes \mathbb{Q} \) defined by

\[
\begin{align*}
\langle h_{\sigma J}, h \rangle &= 14, \\
\langle h_{\sigma J}, e_{i_1}^{(\sigma)} \rangle &= 1 \quad \text{and} \quad \langle h_{\sigma J}, e_{i_3}^{(\sigma)} \rangle = 0, \\
\langle h_{\sigma J}, e_{i_2}^{(\sigma)} \rangle &= 4 \quad \text{and} \quad \langle h_{\sigma J}, e_{i_3}^{(\sigma)} \rangle = 0 \quad \text{for } i = i_2 \text{ and } i = i_3, \\
\langle h_{\sigma J}, e_{i_6}^{(\sigma)} \rangle &= 0 \quad \text{and} \quad \langle h_{\sigma J}, e_{i_6}^{(\sigma)} \rangle = 5 \quad \text{for } i = i_4 \text{ and } i = i_5, \\
\langle h_{\sigma J}, e_{i_6}^{(\sigma)} \rangle &= 5 \quad \text{and} \quad \langle h_{\sigma J}, e_{i_6}^{(\sigma)} \rangle = 4.
\end{align*}
\]

Then \( h_{\sigma J} \in S_X \) and \( \langle h_{\sigma J}, h_{\sigma J} \rangle = 2 \). We confirm \( \text{Sep}(h_{\sigma J}, a) = \emptyset \), and hence \( h_{\sigma J} \in N_X \). The complete linear system \( |h_{\sigma J}| \) is proved to be fixed-component free by the criterion in Section 3.7.

**Definition 6.14.** We denote by \( g_{\sigma J}^{\nu} \) the involution \( i(h_{\sigma J}) \). This is the involution of type (d) in Section 6.2.

**Remark 6.15.** The smooth rational curves on \( X \) contracted by the double covering \( \pi(h_{\sigma J}) : X \to \mathbb{P}^2 \) associated with \( |h_{\sigma J}| \) are as in Figure 6.4. In particular, \( \text{Sing}(B(h_{\sigma J})) \) is of type \( D_4 + A_5 \).

We now describe the orbit \( \mathcal{O}_6 \) of walls of \( D_1^{(\alpha)} \). The size of \( \mathcal{O}_6 \) is 30. Suppose that \( \beta \in \{1, \ldots, 6\} \) and \( F = \{i_1, i_2\} \subset \{1, \ldots, 6\} \) satisfy \( i_1 \neq i_2, \beta \neq \alpha \) and \( \{\alpha, \beta\} \cap \{i_1, i_2\} = \emptyset \). Let \( u := u_{\beta F} \in S_X^* \) be the vector defined by

\[
\begin{align*}
\langle u, h \rangle &= 3, \\
\langle u, e_{i_1}^{(+)} \rangle &= 1, \quad \langle u, e_{i_1}^{(-)} \rangle = 1, \\
\langle u, e_{i_2}^{(+)} \rangle &= 0, \quad \langle u, e_{i_2}^{(-)} \rangle = 0, \\
\langle u, e_{i_6}^{(+)} \rangle &= 0, \quad \langle u, e_{i_6}^{(-)} \rangle = 1 \quad \text{if } i \in F, \\
\langle u, e_{i_6}^{(+)} \rangle &= 1, \quad \langle u, e_{i_6}^{(-)} \rangle = 0 \quad \text{if } i \notin \{\alpha, \beta\} \cup F.
\end{align*}
\]

The orbit \( \mathcal{O}_6 \) consists of walls \( (u_{\beta F})^+ \cap D_1^{(\alpha)} \). The \( L_{26}/S_X \)-chamber \( D_{\alpha \beta F} \) adjacent to \( D_1^{(\alpha)} \) across the wall \( (u_{\beta F})^+ \cap D_1^{(\alpha)} \) is \( G \)-equivalent to \( D_1^{(\beta)} \). An automorphism \( g_{\alpha \beta F} \in G \) that maps \( D_1^{(\beta)} \) to \( D_{\alpha \beta F} \) isomorphically is given as follows. We put

\[ K := \{1, \ldots, 6\} \setminus \{\{\alpha, \beta\} \cup F\}. \]
Then we have
\[(6.10) \quad g_{\alpha \beta F} = g'_{\alpha} I_{\beta} \cdot g''_{\beta} J = g'_{\alpha} I_{\beta} \cdot g''_{\beta} J',\]
where
\[I = (\{\alpha\}, K \cup \{\beta\}, F) \in \mathcal{I}, \quad J = (\{\beta\}, K, \{\alpha\}) \in \mathcal{J},\]
\[I' = (\{\alpha\}, F \cup \{\beta\}, K) \in \mathcal{I}, \quad J' = (\{\beta\}, F, K, \{\alpha\}) \in \mathcal{J}.\]

Therefore we have
\[g_{\alpha \beta F}^{-1} \in T_G(D_{\alpha \beta F}, u_0(D_{\alpha \beta F}))\]
in the notation of Section 5.1.

**Remark 6.16.** The equality \((6.10)\) was found by trying small combinations of the automorphisms of type (a)-(e).

### 6.7. Proof of Theorem 1.1.

Any two distinct elements of \(V_0\) are not \(G\)-equivalent. Any \(L_{26}/S_X\)-chamber that is contained in \(N_X\) and is adjacent to an element of \(V_0\) is \(G\)-equivalent to an element of \(V_0\). Hence, by Proposition 5.1, the set \(V_0\) is a complete set of representatives of \(V/G\).

As the set \(\mathcal{H}\) defined by \((5.1)\), we can take the set consisting of the identity element 1, all involutions of type (b), (c), and the automorphisms \(g_{\alpha \beta F}^{-1}\), where \(g_{\alpha \beta F}\) is given by \((6.10)\) and is a product of automorphisms of type (d) and (e). The stabilizer subgroup \(\text{Stab}_G(D_0)\) of the initial element \(D_0 \in V_0\) is \(\{1, i(\mathbf{h})\}\). Hence, by Proposition 5.1, the group \(G = \text{Aut}(X)\) is generated by the automorphisms of type (a)-(e).

**Remark 6.17.** This generating set is very redundant.

### 6.8. Proof of Theorem 1.2.

We prove that \(G = \text{Aut}(X)\) acts on \(\text{Rats}(X)\) transitively. Let \(r\) be an arbitrary element of \(\text{Rats}(X)\). Since \(r\) defines a wall of \(N_X\), there exists an \(L_{26}/S_X\)-chamber \(D\) contained in \(N_X\) such that \(r\) defines a wall of \(D\). We have an automorphism \(g \in G\) such that \(D^g \in V_0\). By the description of walls of the representative \(L_{26}/S_X\)-chambers in \(V_0\), we see that \(r^g\) is one of the 12 + 2 + 15 smooth rational curves \(e_{\alpha}^{(+)}\), \(\gamma^{(+)}\), and \(\ell_{ij}\). The action of \(i(\mathbf{h})\) gives \(e_{\alpha}^{(+)} \leftrightarrow e_{\alpha}^{(-)}\) and \(\gamma^{(+)} \leftrightarrow \gamma^{(-)}\). By Remark 6.10, the involution \(i(h_{IJ})\) of type (b) interchanges \(e_{j1}^{(+)}\) and \(e_{j2}^{(+)}\) (see Figure 6.2). By Remark 6.12, the involution \(i(h_{h_{0}}^{+})\) of type (c) interchanges \(\gamma^{(-)}\) and \(e_{\alpha}^{(-)}\). As was shown in Section 6.6, the elliptic fibration \(X \to \mathbb{P}^1\) given by \(f_{IJ}\) has sections \(e_{j}^{(\sigma)}\) and \(\ell_{j1,j2}\), and hence they belong to the same \(G\)-orbit. Therefore these 12 + 2 + 15 smooth rational curves are in the same \(G\)-orbit.

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Department of Mathematics, Graduate School of Science, Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima, 739-8526 JAPAN

Email address: ichiro-shimada@hiroshima-u.ac.jp