DISCRETE LQR AND ILQR METHODS BASED ON HIGH ORDER RUNGE-KUTTA METHODS

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Abstract. In this paper, discrete linear quadratic regulator (DLQR) and iterative linear quadratic regulator (ILQR) methods based on high-order Runge-Kutta (RK) discretization are proposed for solving linear and nonlinear quadratic optimal control problems respectively. As discovered in [W. Hager, Runge-Kutta method in optimal control and the discrete adjoint system, Numer. Math., 2000, pp. 247-282], direct approach with RK discretization is equivalent with indirect approach based on symplectic partitioned Runge-Kutta (SPRK) integration. In this paper, we will reconstruct this equivalence by the analogue of continuous and discrete dynamic programming. Then, based on the equivalence, we discuss the issue that the internal-stage controls produced by direct approach may have lower order accuracy than the RK method used. We propose order conditions for internal-stage controls and then demonstrate that third or fourth order explicit RK discretization cannot avoid the order reduction phenomenon. To overcome this obstacle, we calculate node control instead of internal-stage controls in DLQR and ILQR methods. And numerical examples will illustrate the validity of our methods. Another advantage of our methods is high computational efficiency which comes from the usage of feedback technique. In this paper, we also demonstrate that ILQR is essentially a quasi-Newton method with linear convergence rate.

Key words. direct approach for optimal control, symplectic partitioned Runge-Kutta method, LQR, ILQR

AMS subject classifications. 93C15, 65L06, 49M25, 49N10

1. Introduction. The numerical approaches for solving optimal control problem (OCP) are generally divided into two branches: one is indirect approach which aims at solving the canonical equations derived from Pontryagin Maximum Principle (PMP); the other is direct approach which suggests discretizing the state equations and cost function directly and transform the original problem into a finite-dimensional optimization problem [3]. In this paper, we restrict the cost function as quadratic. In the framework of direct approach, this paper aims at two tasks:

Task (1): How to make sure the calculated control values have the same accuracy order as the RK discretization?

Task (2): How to employ linear feedback technique (specifically, DLQR and ILQR methods) to solve the optimization problem after high-order RK discretization?

During the last years, much work has emerged in studying the convergence properties of RK discretization in OCP ([4, 7, 10, 21, 24]). Though the convergence properties of RK methods applied in ordinary differential equations have been studied deeply, the analysis in OCP is usually more complicated. As exemplified in [7, 10], some RK discretization may not converge when solving OCP. In addition, when control or state is under constraints, stricter algebraic conditions must be satisfied by RK methods to ensure convergence or high accuracy (related work can be seen in [7, 24]). In the non-constraint case, there exists an equivalence between direct approach based on RK discretization and indirect approach with corresponding SPRK integration [7, 23]. Based on this fact, Hager [10] proposed the notation of RK method’s order in OCP and computed the order conditions up to four. Then, Bonnans and Laurent-Varin [4] extended the order conditions to 7. Under the smooth and coercivity assumptions, a convergence order analysis for RK method in OCP is given in [10].

When applying a s-stage RK discretization in direct approach, one will obtain s
internal-stage controls. The main obstacle to increase the accuracy by utilizing high-order RK method, as discovered in [10], is that order reduction phenomenon may happen in these internal control values. Though this problem has been referred in some papers (see [1, 2, 22]), deep analysis is still lack as we know. In this paper, we will give the convergence analysis of internal-stage controls and highlight calculating node control to overcome this obstacle.

After RK discretization, the OCP with quadratic objective function and linear state equations will be transformed into a discrete linear quadratic problem which can be solved by the linear feedback technique called DLQR method in this paper. When the state equations in OCP and the corresponding constraints in optimization problem are nonlinear, ILQR method, first proposed by Li and Todorov [14], can be employed. As a simple version of differential dynamic programming (DDP) method [17], ILQR uses iterative linearization of nonlinear system and searches for control modifications by solving discrete linear quadratic regulator problems [14]. The prominent advantage of ILQR over other optimization technique such as sequential quadratic programming (SQP) [13] is that the search direction for iteration is obtained by the linear feedback technique, which saves the computational cost [27]. Though ILQR method cannot achieve quadratic convergence as DDP, the formal only requires calculation of the first order derivative of dynamical system and the matrix operation in which takes less computational payment [9]. As an efficient method to solve discrete optimal control problem, ILQR has been successfully applied in the fields of robotic and stochastic control [8, 15, 26].

As remarked in [18], technique like ILQR, in which state equations are linearized, cannot attain quadratic convergence. In [9], the convergence rate of ILQR is claimed as linear but a detailed analysis is lack there.

1.1. Overview and notations. The rest of this paper is organized as following. In Section 2, we will reconstruct the equivalence between direct and indirect approaches within a dynamic programming viewpoint and discuss an important prerequisite to ensure this equivalence. In Section 3, we deduce order conditions for internal-stage controls and reveal that some common RK discretization cannot avoid the order reduction problem. In Section 4, we propose DLQR method based on high-order RK discretization for solving the linear quadratic OCP. Section 5 aims at solving the nonlinear quadratic OCP with ILQR method and gives a convergence analysis for the iterative progress of ILQR. The numerical examples will be given in Sections 4 and 5 to show the advantages of our methods.

In this paper, \( x \in \mathbb{R}^n \) is regarded as column vector and equipped with the standard Euclidean norm \( ||x|| = (x^T x)^{1/2} \). For several vectors \( x_1, ..., x_m \), we use the notation \((x_1, ..., x_m) = [x_1^T, ..., x_m^T]^T\) as the composed vector. The \( n \)-dimension identity matrix is denoted as \( I_n \). We introduce the following notation
\[
\begin{bmatrix}
A(1) & 0 & \cdots & 0 \\
0 & A(2) & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & A(m)
\end{bmatrix}
\]
to signify the diagonal of a quasi-diagonal matrix composed of \( m \in \mathbb{R} \) identical matrices \( A \).

Let \( n_i \) be any positive integer \((i = 1, 2, 3)\). For a differentiable function with two variables \( F : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_3} \), \( D_x F(x, y) \) (or \( D_y F(x, y) \)) denotes the partial derivative. The transpose of \( D_x F(x, y) \) is denoted as \( D_x^T F(x, y) \). For a differentiable
Table 1  

Coefficients of $s$-stage partitioned RK method

| $c_1$ | $a_{11}$  ...  $a_{1s}$ | $\bar{c}_1$ | $\bar{a}_{11}$  ...  $\bar{a}_{1s}$ |
|-------|-----------------|-------------|-----------------|
|       | $\vdots$       | $\vdots$   | $\vdots$       |
| $c_s$ | $a_{s1}$  ...  $a_{ss}$ | $\bar{c}_s$ | $\bar{a}_{s1}$  ...  $\bar{a}_{ss}$ |
|       | $b_1$  ...  $b_s$ | $\bar{b}_1$ | $\bar{b}_s$ |
(HJB) equation:
\begin{equation}
- V_t(t, x) = \min_u \{C(x, u) + D_x V(t, x) \cdot f(x, u)\}
\end{equation}

with boundary condition $V(t_f, x) = \Phi(x(t_f))$. As mentioned in Section 7.1.2 of [16], the characteristics of this partial differential equation are actually the canonical equations deduced from the maximum principle, namely,
\begin{align}
(2.6a) \quad & \quad \frac{dx}{dt} = f(x, u), \quad x(0) = a, \\
(2.6b) \quad & \quad \frac{dp}{dt} = -D_x^T f(x, u)p - D_x C(x, u), \quad p(t_f) = \Phi'(x(t_f)), \\
(2.6c) \quad & \quad D_x^T f(x, u)p + D_u C(x, u) = 0
\end{align}

where the costate variable $p(t)$, $t \in [0, t_f]$, satisfies
\begin{equation}
(2.7) \quad p(t) = D_x V(t, x).
\end{equation}

Suppose the equation (2.6c) implicitly determines a function $u = \hat{u}(x, p)$, then we can transform the canonical equations (2.6a)-(2.6c) into an autonomous form
\begin{align}
(2.8a) \quad & \quad \frac{dx}{dt} = f(x, \hat{u}(x, p)), \quad x(0) = a, \\
(2.8b) \quad & \quad \frac{dp}{dt} = -D_x^T f(x, \hat{u}(x, p))p - D_x C(x, \hat{u}(x, p)), \quad p(t_f) = \Phi'(x(t_f)).
\end{align}

To solve Problem 2.1 numerically in the framework of direct approach, we construct the discrete time sequence $\{t_k = kh, \quad k = 0, \ldots, t_f/h\}$ and utilize a $s$-stage RK method with coefficients $a_{ij}, b_i$ for discretization, then we can get the following optimization problem:

**Problem 2.2**

\begin{equation}
(2.9) \quad \min_{u_{k_i}} J_d = \min_{u_{k_i}} \left\{ \sum_{k=0}^{N-1} \sum_{i=1}^s b_i C(x_{ki}, u_{ki}) + \phi(x_N) \right\}
\end{equation}

subject to
\begin{align}
(2.10a) \quad & \quad x_{k+1} = x_k + h \sum_{i=1}^s b_i f(x_{ki}, u_{ki}), \quad k = 0, \ldots, N - 1, \\
(2.10b) \quad & \quad x_{ki} = x_k + h \sum_{j=1}^s a_{ij} f(x_{kj}, u_{kj}), \quad k = 0, \ldots, N - 1, \quad i = 1, \ldots, s.
\end{align}

Similar to the continuous case, we define the discrete value function for Problem 2.2 as following
\begin{equation}
(2.11) \quad V_k(x_k) = \min_{u_{k_i}} \left\{ h \sum_{i=1}^{N-1} \sum_{i=1}^s b_i C(x_{ii}, u_{ii}) + \phi(x_N) \right\}, \quad 0 \leq k \leq N - 1,
\end{equation}

and $V_N(x_N) = \Phi(x_N)$. Then the discrete HJB equation with internal-stage controls is given by [20]:
\begin{equation}
(2.12) \quad V_k(x_k) = \min_{u_{k_1}, \ldots, u_{k_s}} \left\{ h \sum_{i=1}^s b_i C(x_{ki}, u_{ki}) + V_{k+1}(x_{k+1}) \right\}
\end{equation}

\begin{equation}
= \min_{u_{k_1}, \ldots, u_{k_s}} \left\{ h \sum_{i=1}^s b_i C(x_{ki}, u_{ki}) + V_{k+1}(x_k + h \sum_{i=1}^s b_i f(x_{ki}, u_{ki})) \right\}, \quad 0 \leq k \leq N - 1,
\end{equation}

with boundary condition $V_N(x_N) = \Phi(x_N)$.

With all the preparations above, now we can introduce the main deductive result of this section and the proof is left in Appendix A.
Proposition 2.1. If \( b_i > 0 \) \( (1 \leq i \leq s) \) and the time step \( h \) is small enough, then the solution \( \{ u_{ki} \} : 0 \leq k \leq N - 1, \ 1 \leq i \leq s \) of Problem 2.2, can be determined by the following discrete system:

\[
\begin{align*}
(2.13a) \quad x_{k+1} &= x_k + h \sum_{i=1}^{s} b_i f(x_{ki}, u_{ki}), \\
(2.13b) \quad x_{ki} &= x_k + h \sum_{j=1}^{s} a_{ij} f(x_{kj}, u_{ki}), \\
(2.13c) \quad p_{k+1} &= p_k - h \sum_{i=1}^{s} b_i (D^T_f f(x_{ki}, u_{ki}) p_{ki} + D_x C(x_{ki}, u_{ki})), \\
(2.13d) \quad p_{ki} &= p_k - h \sum_{j=1}^{s} \tilde{a}_{ij} (D^T_f f(x_{kj}, u_{ki}) p_{kj} + D_x C(x_{kj}, u_{ki})),
\end{align*}
\]

\( 0 \leq k \leq N - 1, \ 1 \leq i \leq s \), with boundary conditions \( x_0 = a \) and \( p_N = \Phi'(x_N) \), where

\[
\begin{align*}
(2.14a) \quad u_{ki} &= \hat{u}(x_{ki}, p_{ki}), \\
(2.14b) \quad p_k &= (V_k)'(x_k), \\
(2.14c) \quad \tilde{a}_{ij} &= b_j - \frac{b_j a_{ji}}{b_i}.
\end{align*}
\]

Remark 2.2. The conclusion in Proposition 2.1 also can be obtained by introducing Lagrangian multiplies for state constraints (2.10a) and (2.10b) and then deducing the KKT conditions for Problem 2.2, as in [10]. In fact, according to (2.14b), the Lagrangian multiplies is equivalent to the gradient of discrete value function, which can be taken as the discrete version of (2.7) in continuous setting.

Note that the expression (2.14c) is actually the symplectic condition for partitioned RK methods (see Theorem 4.6 in chapter VI of [11]), we can conclude that the solution of direct approach based on RK discretization also can be reached by the corresponding SPRK integration of canonical systems (2.8a) and (2.8b). Consequently, it is natural to introduce the following definition which was first proposed in [10]:

Definition 2.3. A RK method with coefficients \( a_{ij}, b_i > 0 \) is of order \( r \) in OCP, if the corresponding SPRK method with coefficients \( a_{ij}, \tilde{a}_{ij} = b_j - \frac{b_j a_{ji}}{b_i} \), \( b_i = \tilde{b}_i \) has order \( r \).

In Table 2-4, we present three explicit RK methods (notated as A, B and C methods) and their respective adjoint part. One can verify that A, B and C methods are of order 2, 3 and 4 in OCP, respectively (refer to chapter III.2 in [11] or Table 1 in [10]).

| Table 2 | A Method (left) and its adjoint part (right) |
|---------|---------------------------------------------|
| 0       | 0/2 -1/2                                    |
| 1/2     | 1/2 1/2                                     |
| 1       | 1/2 1/2                                     |

| Table 3 | B Method (left) and its adjoint part (right) |
|---------|---------------------------------------------|
| 0       | 0/6 -4/3 7/6                                |
| 1/2     | 1/6 2/3                                     |
| 1       | 1/6 2/3                                     |
| 1/6     | 1/6 2/3 1/6                                 |


Though the equivalence revealed by Proposition 2.1 is indeed an elegant result, one restriction should be stated here. As is shown in Figure 1, the internal-stage control $u_{ki}$ is usually taken as the final result to approximate the optimal control at time $t_k + c_i h$. However, when the coefficients of the RK method satisfy $c_i = c_j, i \neq j$ (C method in Table 4, with coefficients $c_2 = c_3 = 1/2$, for example), there may be two different internal-stage controls for the same time $t_k + c_i h$. Therefore, it seems to be more reasonable to add an internal-stage overlap constraint $u_{ki} = u_{kj}$. However, under this kind of constraint, the equivalence stated in Proposition 2.1 will not hold any longer and two difficulties will arise in practice: on the one hand, more stringent algebraic conditions must be satisfied by the RK method to ensure the convergence or high-order accuracy (refer to [7]); on the other hand, the optimization problem after RK discretization will become more elaborate and the linear feedback technique proposed in Section 4 and 5 fails to work. Fortunately, analysis in the next section will show that we can achieve high-order accuracy without considering the internal-stage overlap constraint.

$$u_{k1} \approx u^*(t_k + c_1 h) \quad u_{k2} \approx u^*(t_k + c_2 h) \quad \ldots \quad u_{ks} \approx u^*(t_k + c_s h)$$

**Fig. 1.** Internal-stage controls $u_{k1}, ..., u_{ks}$ in the time interval $[t_k, t_{k+1}]$.

### 3. Error estimate and internal-stage control problem.

Because of the equivalence stated in Section 2, to analysis the convergence of a direct approach with RK discretization, it suffices to study the numerical integration for the canonical equations with corresponding SPRK method. By doing so, Hager proved the following theorem.

**Theorem 3.1** (Error estimate for node values [10]). Suppose a $s$-stage RK method of order $r$ in OCP is used in direct approach for solving problem (2.1), then there exists unique solution $x_k, p_k$ for discrete system (2.13a)-(2.13d) when time step $h$ is small enough, such that

$$\max_{0 \leq k \leq N} \left\| x_k - x^*(t_k) \right\| + \left\| p_k - p^*(t_k) \right\| + \left\| \hat{u}(x_k, p_k) - u^*(t_k) \right\| \leq O(h^r),$$

where $(x^*(t), p^*(t))$ is the exact solution of canonical equations (2.8a) and (2.8b), and $u^*(t) = \hat{u}(x^*(t), p^*(t))$ is the optimal control for Problem (2.1).

The error estimate above is not about internal-stage controls $\|u_{ki} - u^*(t_k + c_i h)\|$ $(1 \leq i \leq s)$ but about node control $\|\hat{u}(x_k, p_k) - u^*(t_k)\|$. As remarked and exemplified
in [10], the node control calculated by $u_k = \hat{u}(x_k, p_k)$ usually converges faster than the internal-stage controls “for Runge–Kutta schemes of third or fourth order”. Indeed, as we will clarify later, such internal-stage control problem is so common that some RK methods with second order cannot avoid it. Let us first give the following definition.

**Definition 3.2.** In the setting of Theorem 3.1, the $i$-th ($i \in \{1, ..., s\}$) internal-stage control $u_{ki}$ has order $n$, if for all sufficiently regular Problems (2.1), the error satisfies

$$
\max_{0 \leq k \leq N-1} \{ ||u_{ki} - u^*(t_k + c_i h)|| \} \leq O(h^n).
$$

As a main result of this section, we give the following order conditions for internal-stage controls and the proof is left to the end of this section.

**Theorem 3.3 (Order conditions for internal-stage controls).** For $i \in \{1, ..., s\}$, suppose the RK method in Theorem 3.1, with order $r$ in OCP, satisfies

\begin{align}
\sum_{j=1}^{s} a_{ij} c_j^{l-2} &= \frac{c_i^{l-1}}{l-1}, \quad l = 2, ..., q_1 \leq r, \tag{3.2a} \\
\sum_{j=1}^{s} \bar{a}_{ij} c_j^{l-2} &= \frac{\bar{c}_i^{l-1}}{l-1}, \quad l = 2, ..., q_2 \leq r, \tag{3.2b}
\end{align}

then the order of the $i$-th internal-stage control $u_{ki}$ is

$$
n = \begin{cases} 
1 & \text{if } c_i \neq \bar{c}_i, \\
\min(q_1, q_2) & \text{if } c_i = \bar{c}_i.
\end{cases} \tag{3.3}
$$

Conversely, to ensure $u_{ki}$ has order $n$ (2 $\leq$ $n$ $\leq$ $r$), the conditions $c_i = \bar{c}_i$ and $\min(q_1, q_2) = n$ must be satisfied for the RK method.

Some necessary explanations for Theorem 3.3 are given in the remarks below.

**Remark 3.4.** The orders of internal-stage controls are less than or equal to the order of RK method used, that is why we emphasize $n \leq r$ in Theorem 3.3.

**Remark 3.5.** The condition $c_i = \bar{c}_i$ is crucial for the $i$-th internal-stage control to attain high order. Note that A, B and C methods in Table 2-4 satisfy $c_i = \bar{c}_i$ for all $i \in \{1, ..., s\}$, consequently all the internal-stage controls produced by them have at least second order. Nevertheless, for the second-order implicit trapezoidal method, as is shown in Table 5 that $c_1 = 0 \neq \bar{c}_1 = 1/2$ and $c_2 = 1 \neq \bar{c}_2 = 1/2$, the two internal-stage controls $u_{k1}$ and $u_{k2}$ have just order 1.

**Remark 3.6.** The internal-stage controls may converge faster than the result in Theorem 3.3 when Problem (2.1) is in some special form. For example, when the function $u = \hat{u}(x, p)$ determined by the third of canonical equation (2.6c) is independent with the state variable $x$, namely $u = \hat{u}(p)$, then the accuracy order of $i$-th internal-stage control will be $q_2$ instead of $\min(q_1, q_2)$. The reason can be explained easily after we give the proof of Theorem 3.3. However, as we have stated in Definition 3.2, we discuss the orders of internal-stage controls in a general problem setting rather than some special cases.

To show the validity of Theorem 3.3, we give a simple example here.
Table 5
Implicit trapezoidal method (left) and its adjoint part (right).

|    |  0  |  1/2 |  1/2 | 1/2 |  1/2 |  0    |
|----|-----|------|------|-----|------|-------|
|  0 |     |  1   |  1/2 |  1/2|  1/2 |  0    |
|  1 |  1/2|  1/2 |  1/2 |  1/2|  1/2 |  0    |

Table 6
$q_1$ and $q_2$ of method A, B and C

| i | A | B | C |
|---|---|---|---|
| 1 | 2 | 1 | 2 | 3 |
| 2 | 2 | 3 | 2 | 2 |
| 3 | 2 | 2 | 3 | 4 |
| 4 | 3 | 2 | 2 | 4 |

Example 3.1. Consider the problem

$$
\min_u \int_0^1 \left( \frac{1}{2} x^2 + \frac{1}{2} xu + \frac{1}{2} u^2 \right) dt,
$$

subject to

$$
\dot{x} = u, \quad x(0) = 1.
$$

The optimal control of this problem is given by

$$
u^*(t) = \frac{0.5e^t - 1.5e^{2-t}}{0.5 + 1.5e^2}.
$$

The max errors of internal-stage controls generated by A, B and C methods (Table 2-4) are shown in Table 7. We just discuss the result of C method here. According to Theorem 3.3 and Table 6, the orders of four internal-stage controls $u_{k1}$, $u_{k2}$, $u_{k3}$, $u_{k4}$ should have the same numbers as $\min(q_1, q_2) : 3$, 2, 2, 3, which coincides with the result in Table 7.

High-order explicit RK methods are advantageous in practice for saving computational cost. Thus it is necessary to study whether there exists $s$-stage explicit RK method with $s$-order in OCP can avoid internal-stage control problem ($s = 2, 3, 4$). The reason why we restrict $s \leq 4$ is that according to the well-known result of Butcher (see Theorem 5.1 in [12]), explicit RK method of $s$-stage cannot attain order $s$ when $s > 4$. To save computational payment, we do not concern the explicit RK method with stage number greater than the order.

For the second order explicit RK method with coefficients $a_{21}$, $b_1$ and $b_2$, to ensure the first internal-stage control $u_{k1}$ attain order 2, the following conditions must be satisfied:

$$
\sum_{i=1}^2 b_i = 1, \quad \sum_{i=1}^2 b_ic_i = \frac{1}{2}, \quad c_1 = \tilde{c}_1.
$$

The first two conditions can ensure the RK method attain second order in OCP (see Table 1 in [10]) and the third condition can ensure $u_{k1}$ achieve second order. After some simple calculations, we can check that only A method (Table 2) with coefficients $a_{21} = 1, b_1 = b_2 = \frac{1}{2}$ satisfies these conditions. In addition, since $c_2 = \tilde{c}_2$
Table 7
Max error of internal controls \((\max_{k=0,\ldots,N-1} ||u^*(kh + c_i h) - u_{ki}||, \ i = 1, \ldots, s)\) in Example 3.1.

| \(h\) | \(A: u_{k0}\) | \(A: u_{k1}\) | \(B: u_{k0}\) | \(B: u_{k1}\) | \(C: u_{k0}\) | \(C: u_{k1}\) | \(C: u_{k2}\) | \(C: u_{k3}\) |
|------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 0.1  | 2.4E-03        | 2.4E-03        | 1.2E-03        | 9.23E-04       | 1.40E-04       | 2.67E-04       | 3.02E-04       | 1.11E-05       |
| 0.05 | 5.94E-04       | 6.56E-04       | 4.20E-04       | 2.60E-04       | 6.42E-04       | 1.76E-05       | 7.02E-05       | 7.45E-05       |
| 0.04 | 3.79E-04       | 4.25E-04       | 2.82E-04       | 1.70E-04       | 4.09E-04       | 9.05E-06       | 4.53E-05       | 4.75E-05       |
| 0.02 | 9.43E-05       | 1.09E-04       | 7.67E-05       | 4.41E-05       | 1.01E-04       | 1.13E-06       | 1.15E-05       | 1.18E-05       |
| 0.01 | 2.35E-05       | 2.74E-05       | 1.99E-05       | 1.42E-05       | 2.91E-06       | 2.94E-06       | 1.34E-08       |                |

Table 8
3-stage explicit RK method with order 3 in OCP

| \(i\) | \(c_i\) |
|------|--------|
| 0    | -      |
| 1    | 3c_2 - 1 - 3c_2^2 |
| 2    | 1 - c_2 |
| 3    | \(\frac{2 \cdot 3 \cdot c_2}{6} = \frac{2 - 3 \cdot c_2}{6(1 - c_2)}\) |

is also satisfied for A method, we can conclude that the second internal-stage control \(u_{k2}\) also has order 2.

For the third or fourth order explicit methods, we first verify the conditions \(\bar{c}_i = c_i, \ i = 1, \ldots, s\) are satisfied.

**Proposition 3.7.** For \(s\)-stage explicit RK methods with \(s\) order in OCP \((s = 3, 4)\), the conditions \(c_i = \bar{c}_i, \ (i = 1, \ldots, s)\) hold.

**Proof.** Since

\[
\bar{c}_i = \sum_{j=1}^{s} \bar{a}_{ij} = \sum_{j=1}^{s} (b_j - \frac{b_j a_{ji}}{b_i}), \ i = 1, \ldots, s,
\]

to prove \(\bar{c}_i = c_i\), it suffices to verify:

\[
(3.4) \quad b_i \sum_{j=1}^{s} b_j - \sum_{j=1}^{s} b_j a_{ji} = b_i - \sum_{j=1}^{s} b_j a_{ji} = b_i c_i, \ i = 1, \ldots, s,
\]

where we have applied the first order condition for RK method: \(\sum_{j=1}^{s} b_j = 1\).

To ensure a 3-stage explicit RK attain third order in OCP, the unique form is shown in Table 8 (refer [10]). By a direct computation, we can verify that the coefficients in Table 8 satisfy (3.4). For the fourth order explicit RK methods, it is presented in [5] (P178) that (3.4) is automatically satisfied.

As an immediate result of Remark 3.5 and Proposition 3.7, all internal-stage controls generated by explicit RK methods with third or fourth order in OCP have at least second order.

For a third order explicit method, to ensure the \(i\)-th internal-stage control \(u_{ki}\) \((i \in \{1, \ldots, s\})\) owns third order, two additional conditions \((l = 3\) in (3.2a) and (3.2b)) must be satisfied. However, there is only one independent coefficient \(c_2\), as shown in Table 8, thus it is impossible to avoid order reduction. A similar reasoning can be used for the fourth order explicit RK methods. Thus we can conclude that the \(s\)-order explicit RK methods with \(s\) stages cannot avoid the internal-stage control problem.

When we choose the internal-stage control as final result, the order reduction problem restricts the choice of RK methods. Of course, one can construct high-order implicit RK methods satisfying enough conditions in Theorem 3.3. But this issue is not concerned in this paper. Another way to solve the internal-stage control problem, as
suggested in the introduction of [10], is taking the node control $u_k = \hat{u}(x_k, p_k)$ as final result instead of the internal-stage controls. According to the convergence analysis in Theorem 3.1, $u_k$ have the same accuracy order as the RK method itself. This handling is suitable for more general RK discretization, and the additional computational cost, as we will show in the later sections concerning specific numerical algorithms, will not be huge.

We conclude this section with the proof of theorem 3.3. To begin with, we give two lemmas on error estimate of internal-stage values generated by RK method.

**Lemma 3.8.** Consider a differential equation $\dot{y} = f(y)$, $y(t_0) = y_0$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ smooth enough. Apply a s-stage Runge-Kutta method with step size $h$ to this equation, then for any $i \in \{1, \ldots, s\}$, the internal-stage value $y_i$, calculated by

$$y_i = y_0 + h \sum_{j=1}^s a_{ij} f(y_j),$$

satisfies $\|y(t_0 + c_i h) - y_i\| = O(h^n)$ if and only if

$$\sum_{j=1}^s a_{ij} c_j^{l-2} = \frac{c_i^{l-1}}{l-1}, \text{ for } l = 2, \ldots, n. \quad (3.5)$$

**Proof.** The details of root tree theory in this proof can be seen in Chapter III of [11], we just list some formulas when necessary. Regard the internal-stage value $y_i(h) = y_0 + h \sum_{j=1}^s f(y_j)$ as a function with variable $h$. We can express the $l$-th derivatives of $y(t)$ and $y_i(h)$ as following

$$y^{(l)}(t_0) = \sum_{|\tau|=l} \alpha(\tau) F(\tau)(y_0),$$

$$y_i^{(l)}(0) = \sum_{|\tau|=l} \gamma(\tau) u_i(\tau) \alpha(\tau) F(\tau)(y_0),$$

where $F(\tau)$ is the elementary differential about tree $\tau$, $\alpha(\tau)$ and $\gamma(\tau)$ are the integer coefficients, and $u_i(\tau)$ is the factor about $a_{ij}$. According to the Taylor expansion, $\|y(t_0 + c_i h) - y_i\| = O(h^n)$ if and only if $\gamma(\tau) u_i(\tau) = c_i^l$ for each $\tau$ with tree order $1 \leq |\tau| = l \leq n - 1$. Specially, when we choose the tree $\tau = [\tau_1, \ldots, \tau_{n-1}]$ where $\tau_1 = \cdots = \tau_{n-1} = \bullet$, according to the formulas (1.14) and (1.15) in Chapter III of [11]:

$$\gamma(\tau) = |\tau| \gamma(\tau_1) \cdots \gamma(\tau_{n-1}),$$

$$u_i(\tau) = \sum_{j=1}^s a_{ij} u_j(\tau_1) \cdots u_j(\tau_{n-1}),$$

and $\gamma(\bullet) = 1$, $u_j(\bullet) = \sum_{k=1}^s a_{jk} = c_j$, we have

$$\gamma(\tau) = l,$$

$$u_i(\tau) = \sum_{j=1}^s a_{ij} c_j^{l-1},$$

thus the condition

$$l \cdot \sum_{j=1}^s a_{ij} c_j^{l-1} = c_i^l$$

must be satisfied for $1 \leq l \leq n - 1$, which proves the necessity.

Sufficiency can be proved by induction. For $|\tau| = 1$, $\tau$ can only be $\bullet$, thus the equality $\gamma(\tau) u_i(\tau) = c_i$ is just the condition (3.5) with $l = 2$. Suppose $\gamma(\tau) u_i(\tau) = c_i^l$ for each $\tau$ with $1 \leq |\tau| = l \leq n - 1$, we have

$$l \cdot \sum_{j=1}^s a_{ij} c_j^{l-1} = c_i^l$$

must be satisfied for $1 \leq l \leq n - 1$, which proves the necessity.
is satisfied for $|\tau| = 1, 2, ..., N < n - 1$. For $\tau = [\tau_1, ..., \tau_m]$ with $|\tau| = 1 + |\tau_1| + ... + |\tau_m| = N + 1$, we have $|\tau_w| \leq N$ for $w = 1, ..., m$. Then

$$
\gamma(\tau)u_1(\tau) = |\tau| \sum_{j=1}^{s} a_{ij} \prod_{w=1}^{m} \gamma(\tau_w)u_j(\tau_w)
$$

$$
= |\tau| \sum_{j=1}^{s} a_{ij} e^{\tau_j} = (N+1) \sum_{j=1}^{s} a_{ij} e_j = c_i^{N+1},
$$

(3.9)

where the last equality is due to the condition (3.5) with $l = N + 2$. Now we have proven the sufficient part of the lemma.

The analogous result for partitioned Runge-Kutta methods requires only minor modifications. The proof is quite similar to Lemma 3.8 and so is omitted.

**Lemma 3.9.** Consider differential equations

$$
\dot{x} = f(x, y), \quad x(t_0) = x_0,
$$

$$
\dot{y} = g(x, y), \quad y(t_0) = y_0
$$

where $f$ and $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are smooth enough. When a partitioned RK method with coefficients $a_{ij}$, $b_i$, $\bar{a}_{ij}$, $\bar{b}_i$ satisfying $c_i = \sum_{j=1}^{s} a_{ij} = \bar{c}_i = \sum_{j=1}^{s} \bar{a}_{ij}$ is applied to numerical integration, the intermediate values

$$
x_j = x_0 + h \sum_{j=1}^{s} a_{ij} f(x_j, y_j),
$$

$$
y_j = y_0 + h \sum_{j=1}^{s} \bar{a}_{ij} g(x_j, y_j)
$$

satisfy $\|x(t_0 + c_i h) - x_i\| \leq O(h^{q_1})$ and $\|y(t_0 + c_i h) - y_i\| \leq O(h^{q_2})$ if and only if

$$
\sum_{j=1}^{s} a_{ij} c_j^{l-2} = \frac{c_i^{l-1}}{l-1}, \quad l = 2, ..., q_1,
$$

$$
\sum_{j=1}^{s} \bar{a}_{ij} c_j^{l-2} = \frac{\bar{c}_i^{l-1}}{l-1}, \quad l = 2, ..., q_2.
$$

**Remark 3.10.** When $c_i \neq \bar{c}_i$ in Lemma 3.9, we can express the intermediate value $y_i$ by Taylor expansion

$$
y_i = y_0 + h \sum_{i=1}^{s} \bar{a}_{ij} g(x_0, y_0) + O(h^2)
$$

$$
= y_0 + h c_i g(x_0, y_0) + O(h^2),
$$

then it is an immediate result that

$$
\|y(t_0 + c_i h) - y_i\| \leq h \|(c_i - \bar{c}_i) g(x_0, y_0)\| + O(h^2) = O(h).
$$

Now we are in a position to prove Theorem 3.3.

**Proof.** For simple notation, we denote the vector field determined by canonical equations (2.8a) and (2.8b) as $F(z)$ where $z(t) = (x(t), p(t))$ and the exact flow with any initial condition $z(t_0) = z_0$ as $\phi(t; t_0, z_0)$. We also denote $L$ and $L_0$ as the Lipschitz constants for $F(z)$ and $\tilde{u}(z)$ in a neighbourhood of the trajectory $\{z(t) = (x^*(t), p^*(t)) \in \mathbb{R}^{2n} : 0 \leq t \leq t_f\}$ respectively. Suppose the conditions in Theorem 3.3 are satisfied, then the difference between $z_{ki} = (x_{ki}, p_{ki})$ and $z^*(t_k + c_i h)$ satisfies

$$
\|z_{ki} - z^*(t_k + c_i h)\| \leq \|z_{ki} - \phi(t_k + c_i h; t_k, z_k)\|
$$

$$
+ \|\phi(t_k + c_i h; t_k, z_k) - \phi(t_k + c_i h; t_k, z^*(t_k))\|
$$

$$
\leq O(h^a) + \|\phi(t_k + c_i h; t_k, z_k) - \phi(t_k + c_i h; t_k, z^*(t_k))\|
$$

(3.12)
where \( n \) satisfies (3.3) according to the Lemma 3.9 and Remark 3.10. Since

\[
\left\| \phi(t; t_k, z_k) - \phi(t; t_k, z^*(t_k)) \right\| = \left\| z_k - z^*(t_k) + \int_{t_k}^{t} F(\phi(\tau; t_k, z_k)) - F(\phi(\tau; t_k, z^*(t_k))) d\tau \right\|
\]

\[
\leq \left\| z_k - z^*(t_k) \right\| + L \int_{t_k}^{t} \left\| \phi(\tau; t_k, z_k) - \phi(\tau; t_k, z^*(t_k)) \right\| d\tau
\]

for \( t_k \leq t \leq t_k + c_i h \), according to Gronwall’s inequality of integral form, we can obtain

\[
(3.13) \quad \left\| \phi(t_k + c_i h; t_k, z_k) - \phi(t_k + c_i h; t_k, z^*(t_k)) \right\| \leq (1 + c_i h L e^{c_i h L}) \left\| z_k - z^*(t_k) \right\|
\]

where \( \left\| z_k - z^*(t_k) \right\| \leq O(h^r) \) is due to the Theorem 3.1. Plugging (3.13) into (3.12), an error estimate for \( z_{ki} \) is given by

\[
\left\| z_{ki} - z^*(t_k + c_i h) \right\| \leq (1 + c_i h L e^{c_i h L}) O(h^r) + O(h^n).
\]

Finally, the convergence rate of \( u_{ki} \) is obtained by

\[
\left\| u_{ki} - u^*(t_k + c_i h) \right\| = \left\| \hat{u}(z_{ki}) - \hat{u}(z^*(t_k + c_i h)) \right\| \leq L_0 \left\| z_{ki} - z^*(t_k + c_i h) \right\|.
\]

Recall that, in Remark 3.6, we have referred a special case \( u = \hat{u}(p) \) where the accuracy order of internal-stage control is determined by \( q_2 \) instead of \( \min(q_1, q_2) \). This can be verified easily with variable \( z = (x, p) \) substituted by \( p \) in the proof above and note that the internal-stage costate has order \( n = q_2 \) according to Lemma 3.9.

4. DLQR method based on high-order RK discretization for linear quadratic problems. In this section, we restrict the Problem 2.1 as the following linear quadratic form

\[
\begin{align*}
\min_u J &= \min_u \int_0^{t_f} \left( \frac{1}{2} x^T Q x + \frac{1}{2} u^T R u \right) dt + \frac{1}{2} (x(t_f))^T M x(t_f) \\
\text{subject to} & \quad \dot{x} = Ax + Bu, \ x(0) = a, \ t \in [0, t_f],
\end{align*}
\]

where \( Q \geq 0, R > 0 \) and \( M \geq 0 \) are symmetric matrices.

Based on the discussion of Section 3, we will calculate the node control to avoid the order reduction problem of internal-stage control. To achieve this goal, it is necessary to deduce the function \( u = \hat{u}(x, p) \). In the present problem setting, the third canonical equation (2.6c) is given by

\[
B^T p + Ru = 0,
\]

and thus we can obtain the linear feedback law:

\[
(4.1) \quad \hat{u}(x, p) = -R^{-1} B^T p.
\]
4.1. Algorithm: DLQR method based on high-order RK discretization.

Our method is divided into four steps:

1. **Discretization and simplification.** As the starting step, we apply a $s$-stage RK method with coefficients $a_{ij}$, $b_i$ to discretization, and then obtain the following optimization problem with linear constraints:

   $$\min_{u_{ki}} J_d = \min_{u_{ki}} \frac{h}{2} \sum_{k=0}^{N-1} \sum_{i=1}^{s} b_i (x_{ki}^T Q x_{ki} + u_{ki}^T Q u_{ki}) + \frac{1}{2} x_N^T M x_N$$

   subject to

   $$x_{k+1} = x_k + h \sum_{i=1}^{s} b_i (A x_{ki} + B u_{ki}), \quad 0 \leq k \leq N - 1,$$

   $$x_{ki} = x_k + h \sum_{j=1}^{s} a_{ij} (A x_{kj} + B u_{kj}), \quad 0 \leq k \leq N - 1, \quad 1 \leq i \leq s.$$  

2. For convenience, let us introduce the notations $X_k = (x_{k1}, ..., x_{ks})$ and $U_k = (u_{k1}, ..., u_{ks})$ for the vectors composed by all the internal-stage state and control values during $[t_k, t_{k+1}]$ respectively, then we can transform the discrete cost function (4.2) into a concise form:

   $$J_d = \frac{1}{2} \sum_{k=0}^{N-1} (X_k^T Q h X_k + U_k^T R h U_k) + \frac{1}{2} x_N^T M x_N,$$

   where

   $$Q_h = h \begin{bmatrix} b_1 Q & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ 0 & \cdots & b_s Q \end{bmatrix}, \quad R_h = h \begin{bmatrix} b_1 R & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ 0 & \cdots & b_s R \end{bmatrix}.$$  

3. Further, by introducing the notations

   $$Z = \begin{bmatrix} I_n(1) \\ \vdots \\ I_n(s) \end{bmatrix}, \quad A = h \begin{bmatrix} a_{11} A & \cdots & a_{1s} A \\ \vdots & \ddots & \vdots \\ a_{s1} A & \cdots & a_{ss} A \end{bmatrix}, \quad B = h \begin{bmatrix} a_{11} B & \cdots & a_{1s} B \\ \vdots & \ddots & \vdots \\ a_{s1} B & \cdots & a_{ss} B \end{bmatrix},$$

   we can rewrite the constraints (4.3b) as follows

   $$X_k = Z x_k + A X_k + B U_k, \quad 0 \leq k \leq N - 1,$$

   or equivalently, denoting $E = (I_n - A)^{-1} Z$ and $F = (I_n - A)^{-1} B$, we have

   $$X_k = E x_k + F U_k, \quad 0 \leq k \leq N - 1.$$  

   Note that the inverse matrix of $I_n - A$ exists when time step $h$ is small enough. Substituting (4.6) into (4.3a), we can obtain the following formula after combing like terms

   $$x_{k+1} = G x_k + H U_k, \quad 0 \leq k \leq N - 1,$$

   where

   $$G = I_n + h \begin{bmatrix} b_1 A & \cdots & b_s A \end{bmatrix} E,$$

   $$H = h \begin{bmatrix} b_1 A & \cdots & b_s A \end{bmatrix} F + h \begin{bmatrix} b_1 B & \cdots & b_s B \end{bmatrix}.$$
(2) **Construction of feedback law.** Denoting the discrete value function as $V_k(x_k)$ ($0 \leq k \leq N$), then the discrete HJB equation is given by

$$
V_k(x_k) = \min_{U_k} \left\{ \frac{1}{2} x_k^T Q_h x_k + \frac{1}{2} U_k^T R_h U_k + V_{k+1}(x_{k+1}) \right\}
$$

(4.9)

$$
= \min_{U_k} \left\{ \frac{1}{2} (E x_k + F U_k)^T Q_h (E x_k + F U_k) + \frac{1}{2} U_k^T R_h U_k + V_{k+1}(G x_k + H U_k) \right\}, \quad 0 \leq k \leq N - 1,
$$

and the boundary condition is $V_N(x_N) = \frac{1}{2} x_N^T M x_N$. Suppose the value function has a quadratic form

(4.10)

$$
V_k(x_k) = \frac{1}{2} x_k^T M_k x_k, \quad 0 \leq k \leq N,
$$

where $M_k$ is symmetric, then according to the boundary condition, we have $M_N = M$. Plug (4.10) into the discrete HJB equation (4.9) and let the derivative of the right hand of (4.9) with respect to $U_k$ vanish, we can obtain

$$(F^T Q_h F + R_h + H^T M_{k+1} H) U_k + (F^T Q_h E + H^T M_{k+1} G) x_k = 0,$$

$0 \leq k \leq N - 1$, it follows that $U_k$ should satisfies a linear feedback law

(4.11)

$$
U_k = L_k x_k, \quad 0 \leq k \leq N - 1,
$$

where

$$
L_k = -(F^T Q_h F + R_h + H^T M_{k+1} H)^{-1} (F^T Q_h E + H^T M_{k+1} G).
$$

(3) **Calculation of the value function.** Plugging (4.11) into the discrete HJB equation (4.9) and comparing both sides of the equality, we obtain the iteration formula for matrix $M_k$:

$$
M_k = (E + F L_k)^T Q_h (E + F L_k) + L_k^T R_h L_k + (G + H L_k)^T M_{k+1} (G + H L_k),
$$

$k = N - 1, ..., 0$. With $M_N = M$ known, $M_{N-1}, ..., M_0$ can be calculated recursively.

(4) **Calculation of the node control.** Substituting (4.11) into (4.7), we can calculate the discrete state trajectory by

$$
x_{k+1} = (G + H L_k) x_k, \quad x_0 = a, \quad 0 \leq k \leq N - 1.
$$

With $M_k$ and $x_k$ known, we can obtain the discrete costates by (2.14b):

$$
p_k = (V_k)'(x_k) = M_k x_k, \quad 0 \leq k \leq N.
$$

Finally, the node controls $u_k$ ($0 \leq k \leq N$) can be calculated by plugging $p_k$ into the formula (4.1), namely,

$$
u_k = -R^{-1} B^T p_k, \quad 0 \leq k \leq N.$$
4.2. Numerical example: Linear spring oscillator. Consider a linear oscillator under control with following state equations and initial conditions:

\[
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2
\end{bmatrix} = \begin{bmatrix}
    0 & 1 \\
    -1 & 0
\end{bmatrix} \begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix} + \begin{bmatrix}
    1 \\
    0
\end{bmatrix} \begin{bmatrix}
    x_1(0) \\
    x_2(0)
\end{bmatrix} = \begin{bmatrix}
    1
\end{bmatrix}.
\]

Now search for control \( u(t) \), \( t \in [0, 40] \), to minimize the cost function:

\[
J = \int_0^{40} \left( \frac{1}{2} x^T x + 1.5u^2 \right) dt + 5x(1)^T x(1),
\]

where \( x = (x_1, x_2) \).

We apply explicit Euler, implicit trapezoidal and B methods in this numerical example. As Figure 2 shows, DLQR method based on high order RK discretization shows remarkable advantage in accuracy compared to the low-order case. In particular, the implicit trapezoidal method, as we have demonstrated in Remark 3.5, which produces the internal-stage controls with only first order, generates the node control with second order accuracy in this example.

![Comparison of different RK discretization: maximum control error (max||uk − u*(tk)|| : 0 ≤ k ≤ N) for linear spring oscillator problem.](image)

5. ILQR method based on high-order RK discretization for non-linear quadratic problems. Consider the following non-linear quadratic problem:

**Problem 5.1**

\[
\begin{align*}
\min_u J &= \min_u \int_0^{t_f} \left( \frac{1}{2} x^T Q x + \frac{1}{2} u^T R u \right) dt + \frac{1}{2} (x(t_f))^T M x(t_f) \\
\text{subject to} & \\
\dot{x} &= f(x, u), \quad x(0) = a,
\end{align*}
\]

where \( Q \succeq 0, \ R > 0, \ M \succeq 0 \) are symmetric matrices. Applying a \( s \)-stage RK method with coefficients \( a_{ij}, b_i \) to discretize **Problem 5.1**, we can get the following optimization problem:

**Problem 5.2**

\[
\begin{align*}
\min_{u_{ki}} J_d &= \min_{u_{ki}} \sum_{k=0}^{N-1} h \sum_{i=1}^s b_i \left( \frac{1}{2} x_{ki}^T Q x_{ki} + \frac{1}{2} u_{ki}^T R u_{ki} \right) + \frac{1}{2} x_N^T M x_N
\end{align*}
\]
subject to
\begin{align}
\text{(5.4a)} \quad x_{k+1} &= x_k + h \sum_j b_j f(x_{ki}, u_{ki}), \quad x_0 = a, \quad 0 \leq k \leq N - 1, \\
\text{(5.4b)} \quad x_{ki} &= x_k + h \sum_j a_{ij} f(x_{kj}, u_{kj}), \quad 0 \leq k \leq N - 1, \quad 1 \leq i \leq s.
\end{align}

We denote the solutions of Problem 5.2 as \( u_{k+1}^* \) and the corresponding discrete states calculated by (5.4a) and (5.4b) as \( x_{k+1}^* \) and \( x_k^* \). Different from the linear case in Section 4, the value function \( V_k(x_k) \) for discrete cost function (5.3) is not in a quadratic form in general, and consequently we cannot expect to get a linear feedback law as (4.11). Besides, note that the third canonical equation (2.6c) of Problem 5.1 is given by

\begin{equation}
D_x^T f(x,u)p + Ru = 0,
\end{equation}

we may be not able to obtain an explicit expression for \( u = \hat{u}(x,p) \).

There are many notations in this section, we list them in advance for convenience. Same as Section 4, we also introduce the notations \( U_k = (u_{k1}, ..., u_{ks}) \) and \( X_k = (x_{k1}, ..., x_{ks}) \) for the vectors combined by internal-stage state and control values during \([t_k, t_{k+1}]\) respectively. Further, we denote the vectors composed by all the internal-stage controls, internal-stage states and node states as \( U = (U_0, ..., U_{N-1}), X = (X_0, ..., X_{N-1}) \) and \( x_d = (x_0, ..., x_N) \), respectively. When we add a symbol on these new notations, it just means adding the same symbol to all the elements in brackets. For example, \( x_d^0 = (x_0^0, ..., x_N^0) \). In the rest of this section, we always suppose the time step \( h \) in (5.4a) and (5.4b) is small enough to ensure that, according to implicit function theorem, \( X \) and \( x_d \) can be uniquely determined by \( U \). Note that the total dimension of \( (U, X, x_d) \) is \( N_{\text{total}} = msN + nsN + n(N + 1) \), thus the constraints (5.4a) and (5.4b) determine a submanifold \( \mathcal{M} \) of Euclidean space \( \mathbb{R}^{N_{\text{total}}} \): \( \mathcal{M} = \{(U, X, x_d) \in \mathbb{R}^{N_{\text{total}}} : (5.4a), (5.4b)\} \).

5.1. Algorithm: ILQR method based on high-order RK discretization.

Before stating the algorithm formally, we give an inspiring example which is helpful to understand the iterative progress in ILQR method.

**Example 5.1.** Considering the following optimization problem,

\begin{equation}
\min_u \ j(x,u) = \min_u \left\{ \frac{1}{2} x^2 + \frac{1}{2} u^2 \right\}
\end{equation}

subject to \( x = g(u) \), where \( x, u \in \mathbb{R} \) and \( g: \mathbb{R} \rightarrow \mathbb{R} \) is twice differentiable.

As we can see in Figure 3, the function \( x = g(u) \) determines a curve, and the minimal for objective function (5.6) is just the point on the curve closest to the origin \((0,0)\). First, we choose an initial value \( u^0 \) and thus determine a point \((u^0, x^0) = (u^0, g(u^0))\) on the curve. Then we can draw the tangent of curve at point \((u^0, x^0)\) and search for minimal \( (\tilde{u}, \tilde{x}) \) of objective function \( j \) over this tangent. Once \( \tilde{u} \) is determined, we obtain a search direction \( \tilde{u} - u^0 \) (arrow direction in Figure 3). By mean of a linear search technique, we can obtain the next iterative point \((u^1, x^1) = (u^1, g(u^1))\) on the curve segment \( \gamma(\alpha) = \{(u(\alpha), x(\alpha)) : u(\alpha) = u^0 + \alpha(\tilde{u} - u^0), \ x(\alpha) = g(u(\alpha))\}, \ \alpha \geq 0.\)
Now we turn to the ILQR method which can be divided into two steps. The first step aims at solving Problem 5.2 in a similar way as Example 5.1, and the node control will be calculated in the second step.

1. **Iterative progress.** Choosing an initial value for internal-stage control $U^0$, then we can determine a point $(U^0, X^0, x^0_d)$ on $M$ where $X^0$ and $x^0_d$ are calculated by (5.4a) and (5.4b). The equations of tangent plane of $M$ at this point is given by:

\[
x_{k+1} - x_k = h \sum_{i=1}^{s} b_i (D_x f(x_{ki}, u_{ki}))(x_{ki} - x_k) + D_u f(x_{ki}, u_{ki})(u_{ki} - u^0_{ki}),
\]

(5.7a)

\[
x_{ki} - x_k = x_k - x^0_k + h \sum_{j=1}^{s} a_{ij} (D_x f(x_{kj}, u^0_{kj}))(x_{kj} - x^0_k) + D_u f(x_{kj}, u^0_{kj})(u_{kj} - u^0_{kj}),
\]

(5.7b)

for $0 \leq k \leq N - 1$, $1 \leq i \leq s$ and $x^0_0 = a$. Similar to Example 5.1, we search the minimum $(U, X, x_d)$ of cost function (5.3) over this tangent plane. The progress to obtain $U$ is given in Appendix B. With $U$ calculated, we get a search direction $U - U^0$ for Problem 5.2. Considering the curve segment on $M$:

\[
\gamma(\alpha) = \{(U, X, x_d) : U = U^0 + \alpha(U - U^0), X, x_d\}
\]

we can choose the step length $\alpha$ by some linear search technique and then let $\gamma(\alpha)$ be the next iterative point $(U^1, X^1, x^1_d)$ on $M$. Repeat this progress until a preset condition for interruption is satisfied, then we can take the final iterative point $U^L, X^L, x^L_d$ as the approximate solution of problem 5.2: $(U^L, X^L, x^L_d) \approx (U^*, X^*, x^*_d)$.

2. **Calculation of node controls.** According to Proposition 2.1, the solution of problem 5.2 $u^*_k$, $x^*_k$, and $x^*_d$ will satisfy the following equations:

\[
p_{k+1} = p_k - h \sum_{i=1}^{s} b_i (D_x^T f(x^*_{ki}, u^*_{ki}))(p^*_k + Q x^*_{ki}),
\]

(5.9a)

\[
p^*_k = p^*_k - h \sum_{i=1}^{s} a_{ij} (D_x^T f(x^*_{kj}, u^*_{kj}))(p^*_{kj} + Q x^*_{kj}),
\]

(5.9b)
0 \leq k \leq N - 1, 1 \leq i \leq s$, with boundary condition $p_N^* = M x_N^*$. Since we have obtained $u_{k,i}^*, x_{k,i}^*$ and $x_k^*$ in step (1), we can calculate the discrete costates $p_{N-1}^*,...,p_0^*$ recursively by solving (5.9a) and (5.9b). Plugging $x_k^*$ and $p_k^*$ into the third canonical equation (5.5), we can obtain the node control $u_k^*$ by solving

\[ D_u^T f(x_k^*, u_k^*) p_k^* + Ru_k^* = 0, \quad 0 \leq k \leq N. \]

5.2. Convergence analysis. In this subsection, we give a convergence analysis for the iterative progress of ILQR method. The main results contain:

(i) The iterative progress of ILQR technique belongs to the quasi-Newton type methods and produces descent search direction for Problem 5.2.

(ii) ILQR cannot attain super-linear convergence rate in general.

To begin with, let us rewritten the cost function (5.3) in the following form:

\[
J(U, X, x_d) = \min_U \frac{1}{2} X^T Q X + \frac{1}{2} U^T R U + \frac{1}{2} x_N^T M x_N,
\]

where

\[
Q = \begin{bmatrix} Q_h(1) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & Q_h(N) \end{bmatrix}, \quad R = \begin{bmatrix} R_h(1) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & R_h(N) \end{bmatrix},
\]

$Q_h$ and $R_h$ are the same as (4.5).

Since the internal-stage state $X$ and node states $x_k$ ($k = 0, ..., N$) are determined by the internal-stage control $U$ according to equations (5.4a) and (5.4b), it is convenient to introduce the notations $X = F(U)$ and $x_k = F_d^k(U)$ ($k = 0, ..., N$). Specially, since $x_0 = a$, we have $F_d^0(U) \equiv a$. Note that when we plug these functions in (5.10), we can express the discrete cost function with only variable $U$, namely,

\[
J_d(U) = \frac{1}{2} (F(U))^T Q F(U) + \frac{1}{2} U^T R U + \frac{1}{2} (F_d^N(U))^T M F_d^N(U).
\]

For convenience of the discussion latter, we give the gradient and Hesse matrix of $J_d(U)$ at $U^0$:

\[
(J_d)'(U^0) = (F'(U^0))^T Q F(U^0) + RU^0 + ((F_d^N)'(U^0))^T M F_d^N(U^0),
\]

\[
(J_d)''(U^0) = (F'(U^0))^T Q F'(U^0) + (F''(U^0))^T Q F(U^0) + R + ((F_d^N)'(U^0))^T M F_d^N(U^0) + ((F_d^N)''(U^0))^T M F_d^N(U^0).
\]

Next we will calculate the search direction generated by ILQR iterative progress. The tangent plane of manifold $M$ at the point $(U^0, X^0, x_1^0, ..., x_N^0) = (U^0, F(U^0), F_d^1(U^0), ..., F_d^N(U^0))$ is given by

\[
X - X^0 = F'(U^0)(U - U^0),
\]

\[
x_k - x_k^0 = (F_d^k)'(U^0)(U - U^0), \quad 0 \leq k \leq N.
\]

Particularly, the $(N + 1)$-th formula of (5.15) is

\[
x_N - x_N^0 = (F_d^N)'(U^0)(U - U^0).
\]
Substituting (5.14) and (5.16) into (5.10), we can obtain the following expression after combing similar terms:

$$\frac{1}{2}(U - U^0)^TW(U^0)(U - U^0) + (U - U^0)^TY(U^0) + C(U^0)$$

where

$$W(U^0) = (F'(U^0))^TQF'(U^0) + R + ((F_d^N)'(U^0))^TM(F_d^N)'(U^0),$$

$$Y(U^0) = (F'(U^0))^TQX^0 + RU^0 + ((F_d^N)'(U^0))^TMx_N^0,$$

$$C(U^0) = \frac{1}{2}(X^0)^TQX^0 + \frac{1}{2}(U^0)^TRU^0 + \frac{1}{2}(x_N^0)^TMx_N^0.$$ 

Since $W(U^0)$ is a symmetric matrix, the necessary condition for minimizing (5.17) is

$$W(U^0)(U - U^0) + Y(U^0) = 0,$$

then the search direction of ILQR can be expressed as

$$U - U^0 = -(W(U^0))^{-1}Y(U^0).$$

Comparing (5.12) with (5.18b), we have $Y(U^0) = (J_d)'(U^0)$. Consequently, we can rewrite the ILQR search direction (5.19) in a quasi-Newton type

$$U - U^0 = -(W(U^0))^{-1}(J_d)'(U^0),$$

According to (5.18a), $W(U^0)$ is positive-definite, thus the search direction generated by ILQR is a decrease one for **Problem 5.2**.

Now we study whether the convergence rate of ILQR is super-linear. Suppose the sequence $\{U^l\}$, generated by ILQR iteration $U^{l+1} = U^l - W(U^l)^{-1}(J_d)'(U^l)$, converges to the minimum of **Problem 5.2** (denoted by $U^*$). According to the necessary and sufficient condition for super-linear convergence (see Theorem 3.7 in [19]), ILQR can attain a super-linear convergence rate if and only if

$$\lim_{l \to +\infty} \frac{||W(U^l) - (J_d)'(U^l) \cdot (U^{l+1} - U^l)||}{||U^{l+1} - U^l||} = 0.$$ 

However, (5.21) cannot be achieved in general. To demonstrate this, let us go back to Example 5.1 where $W(u') = 1 + (g'(u'))^2$ and $j''(u') = 1 + (g'(u'))^2 + g''(u')g(u')$. We have

$$\lim_{l \to +\infty} \frac{|j''(u') - W(u') \cdot (u^{l+1} - u^l)|}{|u^{l+1} - u^l|} = |j''(u^*) - W(u^*)| = |g''(u^*)g(u^*)|.$$ 

It is easy to cite an example in which $|g''(u^*)g(u^*)| \neq 0$. When $g(u) = u^2 + 1$, we have $u^* = 0$, $g(u^*) = 1$ and $g''(u^*) = 2$.

**5.3. Numerical example: inverted pendulum.** Consider the state equations of inverted pendulum under control:

$$\dot{\theta} = \omega, \ \theta(0) = \frac{\pi}{3},$$

$$\dot{\omega} = \sin\theta + u, \ \omega(0) = 0.$$
The cost function of this problem is:

\[ J = \int_0^4 0.025u^2 dt + 2.5x^Tf, \]

where \( x_f = (\theta(4), \omega(4)) \).

For this non-linear problem, we explore the dependence of initial control \( u_0 \) (the first node control value calculated by ILQR method) on the node number for different RK discretization. As we can see in Figure 4, for the B method with third order in OCP, the calculated initial control \( u_0 \) converges faster than the other two produced by lower order discretization. When the node number is increased to nearly 800, the initial control \( u_0 \) generated by the second order implicit trapezoidal method can attain a satisfying accuracy, while explicit Euler method still behaves badly.

6. Conclusion. We propose DLQR and ILQR methods based on high-order RK discretization to solve linear and nonlinear quadratic optimal control problems respectively. The equivalence between direct approach based on RK discretization and SPRK integration for canonical equations, is reconstructed in the framework of dynamic programming and plays important roles in both theoretical analysis and algorithm implementation. Based on this equivalence, we deduce the order conditions for internal-stage controls in Theorem 3.3 and discover that some common RK methods with high order in OCP (third or fourth order explicit methods; implicit methods such as implicit trapezoidal method) cannot avoid the order reduction. Then, by mean of the the equivalence in Section 2, we calculate node control in DLQR and ILQR methods to overcome this obstacle. Numerical examples show the advantage of our methods on the accuracy increase. Because both DLQR and ILQR methods are achieved with the feedback technique, they have high computational efficiency. In the convergence analysis for ILQR iteration, we demonstrate that ILQR technique is essentially a quasi-Newton method and the convergence rate is linear in general.

Appendix A. Proof for proposition 2.1. Denoting the gradient of discrete value function \((V_k)'(x_k)\) as \(p_k\) \((0 \leq k \leq N)\), we have \(p_N = (V_N)'(x_N) = \Phi'(x_N)\) according to the boundary condition \(V_N(x_N) = \Phi(x_N)\). To achieve the minimization
of the right hand of discrete HJB equation (2.12), we first introduce a multiply $\varphi_{ki}$ for the $i$-th intermediate equation of (2.10a) $(1 \leq i \leq s)$ and define

$$
V_k(x_k, x_{k1}, ..., x_{ks}, u_{k1}, ..., u_{ks}, \varphi_{k1}, ..., \varphi_{ks})
= h \sum_{i=1}^{s} b_i C(x_{ki}, u_{ki}) + V_{k+1}(x_k + h \sum_{i=1}^{s} b_i f(x_{ki}, u_{ki}))
+ \sum_{i=1}^{s} \varphi_{ki}^T (x_k + h \sum_{j=1}^{s} a_{ij} f(x_{kj}, u_{kj}) - x_{ki}),
$$

then the necessary conditions for minimization state that the partial derivative of $V_k$ with respect to $x_{ki}$, $u_{ki}$ and $\varphi_{ki}$ $(i = 1, ..., s)$ should vanish. Specifically, we have,

$$(A.1a) \quad D_{x_k} V_k(x_k, x_{k1}, ..., x_{ks}, u_{k1}, ..., u_{ks}, \varphi_{k1}, ..., \varphi_{ks}) = h b_i [D_x C(x_{ki}, u_{ki}) + D_x^T f(x_{ki}, u_{ki}) (p_{k+1} + \sum_{j=1}^{s} a_{ij} \varphi_{kj})] - \varphi_{ki} = 0,$$

$$(A.1b) \quad D_{u_k} V_k(x_k, x_{k1}, ..., x_{ks}, u_{k1}, ..., u_{ks}, \varphi_{k1}, ..., \varphi_{ks}) = h b_i [D_u C(x_{ki}, u_{ki}) + D_u^T f(x_{ki}, u_{ki}) (p_{k+1} + \sum_{j=1}^{s} a_{ij} \varphi_{kj})] = 0,$$

$$(A.1c) \quad x_k + h \sum_{j=1}^{s} a_{ij} f(x_{kj}, u_{kj}) - x_{ki} = 0$$

for all $1 \leq i \leq s$. The discrete HJB equation (2.12) can now be rewritten as

$$
V_k(x_k) = V_k(x_k, x_{k1}, ..., x_{ks}, u_{k1}, ..., u_{ks}, \varphi_{k1}, ..., \varphi_{ks})
= h \sum_{i=1}^{s} b_i C(x_{ki}, u_{ki}) + V_{k+1}(x_k + h \sum_{i=1}^{s} b_i f(x_{ki}, u_{ki}))
+ \sum_{i=1}^{s} \varphi_{ki}^T (x_k + h \sum_{j=1}^{s} a_{ij} f(x_{kj}, u_{kj}) - x_{ki})
$$

in which $x_{ki}$, $u_{ki}$ and $\varphi_{ki}$ $(1 \leq i \leq s)$ are functions with variable $x_k$ implicitly determined by (A.1a), (A.1b) and (A.1c). Derivate both side of (A.2) with respect to $x_k$ and note that the partial derivatives of $V$ with respect to $x_{ki}$, $u_{ki}$, $\varphi_{ki}$ all vanish, we can obtain:

$$
p_k = p_{k+1} + \sum_{i=1}^{s} \varphi_{ki}.
$$

By introducing the variables

$$
p_{ki} \triangleq p_{k+1} + \sum_{j=1}^{s} \frac{a_{ij}}{b_i} \varphi_{kj}, \quad 1 \leq i \leq s,
$$

we can rewrite the conditions (A.1a) and (A.1b) as

$$(A.5a) \quad h b_i (D_x C(x_{ki}, u_{ki}) + D_x^T f(x_{ki}, u_{ki}) p_{ki}) = \varphi_{ki},$$

$$(A.5b) \quad h b_i (D_u C(x_{ki}, u_{ki}) + D_u^T f(x_{ki}, u_{ki}) p_{ki}) = 0.$$ 

Since $h > 0$, $b_i > 0$, (A.5b) is equivalent to the following formula

$$
D_u C(x_{ki}, u_{ki}) + D_u^T f(x_{ki}, u_{ki}) p_{ki} = 0
$$

which has the same form as the third canonical equation (2.6c), thus we have $u_{ki} = \hat{u}(x_{ki}, p_{ki})$. Substituting (A.5a) into (A.3), we can obtain

$$
p_k = p_{k+1} + h \sum_{k=1}^{s} b_i (D_x C(x_{ki}, u_{ki}) + D_x^T f(x_{ki}, u_{ki}) p_{ki}).$$
Plugging (A.5a) and (A.6) into (A.4), we can get the expression of \( p_{ki} \), namely,

\[
p_{ki} = p_k - h \sum_{j=1}^{s} a_{ij} [D_x f(x_{kj}, u_{kj}) p_{kj} + D_x C(x_{kj}, u_{kj})], \quad 1 \leq i \leq s,
\]

where \( a_{ij} = b_j - \frac{h a_{i2}}{b_i} \).

Now we have proven that the necessary condition for Problem 2.2 is given by the equations (2.13a)-(2.13d) in Proposition 2.1. The existence and uniqueness of solution can be ensured when time step \( h \) is small enough (more details can be seen in [10]).

**Appendix B. Calculation of search direction in ILQR method.** The search direction \( \hat{U} - U^0 \) of ILQR in Subsection 5.1 can be calculated by solving the following optimization problem in which the minimum is \( \hat{U} \).

\[
\min_{\hat{U}} \sum_{k=0}^{N-1} \left( \frac{1}{2} X_k^T Q_k X_k + \frac{1}{2} U_k^T R_k U_k \right) + \frac{1}{2} x_N^T M x_N
\]

subject to

\[
\begin{align*}
(B.1a) \quad x_{k+1} - x_k^0 &= x_k - x_k^0 + B_k (X_k - X_k^0) + C_k (U_k - U_k^0), \\
(B.1b) \quad X_k - X_k^0 &= Z(x_k - x_k^0) + A_k^1 (X_k - X_k^0) + A_k^2 (U_k - U_k^0),
\end{align*}
\]

where \( Q_k \) and \( R_k \) have the same definitions as in (4.4), and

\[
A_k^1 = h \begin{bmatrix} a_{11} D_x f(x_{k1}^0, u_{k1}^0) & \cdots & a_{1s} D_x f(x_{ks}^0, u_{ks}^0) \\ \vdots & \ddots & \vdots \\ a_{s1} D_x f(x_{k1}^0, u_{k1}^0) & \cdots & a_{ss} D_x f(x_{ks}^0, u_{ks}^0) \end{bmatrix}, \quad A_k^2 = h \begin{bmatrix} a_{11} D_u f(x_{k1}^0, u_{k1}^0) & \cdots & a_{1s} D_u f(x_{ks}^0, u_{ks}^0) \\ \vdots & \ddots & \vdots \\ a_{s1} D_u f(x_{k1}^0, u_{k1}^0) & \cdots & a_{ss} D_u f(x_{ks}^0, u_{ks}^0) \end{bmatrix}, \quad Z = \begin{bmatrix} I_n(1) \\ \vdots \\ I_n(s) \end{bmatrix},
\]

\[
B_k = h \begin{bmatrix} b_1 D_x f(x_{k1}^0, u_{k1}^0) & \cdots & b_s D_x f(x_{ks}^0, u_{ks}^0) \\ \vdots & \ddots & \vdots \\ b_1 D_u f(x_{k1}^0, u_{k1}^0) & \cdots & b_s D_u f(x_{ks}^0, u_{ks}^0) \end{bmatrix}, \quad C_k = h \begin{bmatrix} b_1 D_u f(x_{k1}^0, u_{k1}^0) & \cdots & b_s D_u f(x_{ks}^0, u_{ks}^0) \end{bmatrix}.
\]

We mention that the constraints (B.1a) and (B.1b) are the same as the tangent plane equations (5.7a) and (5.7b). After some simple operation, we can transform the constraints (B.1b) and (B.1a) into the form

\[
\begin{align*}
(B.3a) \quad X_k &= E_k x_k + F_k U_k + D_k^1, \\
(B.3b) \quad x_{k+1} &= G_k x_k + H_k U_k + D_k^2,
\end{align*}
\]

where

\[
\begin{align*}
E_k &= (I_n - A_k^1)^{-1} Z, & F_k &= (I_n - A_k^1)^{-1} A_k^2, \\
G_k &= I_n + B_k E_k, & H_k &= B_k F_k + C_k, \\
D_k^1 &= X_k^0 - E_k x_k - F_k U_k, & D_k^2 &= x_{k+1}^0 - G_k x_k - H_k U_k^0.
\end{align*}
\]
Now we can write the discrete HJB equation as following

\[
V_k(x_k) = \min_{U_k} \left\{ \frac{1}{2} X_k^T Q_h X_k + \frac{1}{2} U_k^T R_h U_k + V_{k+1}(x_{k+1}) \right\}
\]

\[
= \min_{U_k} \left\{ \frac{1}{2} (E_k x_k + F_k U_k + D_k^1)^T Q_h (E_k x_k + F_k U_k + D_k^1) 
+ \frac{1}{2} U_k^T R_h U_k + V_{k+1}(G_k x_k + H_k U_k + D_k^2) \right\}, \quad 0 \leq k \leq N - 1,
\]

with boundary condition \( V_N(x_N) = \frac{1}{2} x_N^T M x_N \). Then, we suppose \( V_k(x_k) \) has a quadratic form

\[
V_k(x_k) = \frac{1}{2} x_k^T M_k x_k + Y_k^T x_k + c_k, \quad 0 \leq k \leq N,
\]

where \( M_k \) is symmetric. According to the boundary condition, we have \( M_N = M \), \( Y_N = 0 \), \( c_N = 0 \). Substituting (B.6) into the discrete HJB equation (B.5) and let the derivative of the right hand of equality with respect to \( U_k \) vanish, we can get the following linear feedback law after combing like terms and matrix inversion

\[
U_k = U_k^1 x_k + U_k^2
\]

where

\[
U_k^1 = -(F_k^T Q_h F_k + R_k + H_k^T M_{k+1} H_k)^{-1} (F_k^T Q_h E_k + H_k^T M_{k+1} G_k),
\]

\[
U_k^2 = -(F_k^T Q_h F_k + R_k + H_k^T M_{k+1} H_k)^{-1} (F_k^T Q_h D_k^1 + H_k^T M_{k+1} D_k^2 + H_k^T Y_{k+1}),
\]

\( 0 \leq k \leq N - 1 \). Plugging (B.7) into the discrete HJB equation (B.5), we can obtain the following iterative formulas for \( M_k \) and \( Y_k \) by comparing both sides of equality:

\[
M_k = (E_k + F_k U_k^1)^T Q_h (E_k + F_k U_k^1) + (U_k^1)^T R_h U_k^1 
+ (G_k + H_k U_k^1)^T M_{k+1} (G_k + H_k U_k^1),
\]

\[
Y_k = (E_k + F_k U_k^1)^T Q_h (F_k U_k^2 + D_k^1) + (U_k^1)^T R_h U_k^2 
+ (G_k + H_k U_k^1)^T M_{k+1} (H_k U_k^2 + D_k^2) + (G_k + H_k U_k^1)^T Y_{k+1},
\]

\( 0 \leq k \leq N - 1 \). With \( M_N \) and \( Y_N \) known, we can calculate \( M_k \) and \( Y_k \) \( (k = N - 1, \ldots, 0) \) recursively. Then according to (B.8a) and (B.8b), we can obtain the coefficient matrices \( U_k^1 \) and \( U_k^2 \). Plugging the feedback law (B.7) into (B.3b), the discrete state trajectory can be obtained by

\[
x_{k+1} = (G_k + H_k U_k^1) x_k + H_k U_k^2 + D_k^2, \quad x_0 = a, \quad 0 \leq k \leq N - 1.
\]

At last, the internal stage control \( U_k \) can be calculated by substituting \( x_k \) into (B.7).

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