A RIGIDITY THEOREM FOR HOLOMORPHIC DISKS IN TEICHMÜLLER SPACE

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Abstract. In this paper, we discuss a rigidity property for holomorphic disks in Teichmüller space. In fact, we give an improvement of Tanigawa’s rigidity theorem. We will also treat the rigidity property of holomorphic disks for complex manifolds. We observe the rigidity property is valid for bounded strictly pseudoconvex domains with $C^2$-boundaries, but the rigidity property does not hold for product manifolds.

1. Introduction

1.1. Let $S$ be a compact orientable surface with negative Euler characteristic (possibly with boundary). Let $\mathcal{T}(S)$ be the Teichmüller space of $S$ and $d_T$ denote the Teichmüller distance on $\mathcal{T}(S)$. Fix $x_0 \in \mathcal{T}(S)$. The Gromov product with basepoint $x_0$ is defined by

$$\langle x \mid y \rangle_{x_0} = \frac{1}{2}(d_T(x_0, x) + d_T(x_0, y) - d_T(x, y))$$

for $x, y \in \mathcal{T}(S)$. The main purpose of this paper is to show the following.

Theorem 1.1 (Rigidity of holomorphic disks). Let $f_1$ and $f_2$ be holomorphic mappings from the unit disk $D$ to $\mathcal{T}(S)$. Suppose that there is a measurable set $E \subset \partial D$ of positive linear measure with the following property: For any $z_0 \in E$, there is a sequence $\{z_n\}_{n=1}^\infty \subset D$ such that $z_n \to z_0$ non-tangentially and $\langle f_1(z_n) \mid f_2(z_n) \rangle_{x_0} \to \infty$. Then, $f_1(z) = f_2(z)$ for all $z \in D$.

We say here that a sequence in $D$ converges to $z_0 \in \partial D$ non-tangentially if it tends to $z_0$ from the inside of any fixed Stolz region with the vertex at $z_0$ (cf. [28]).

Since $|\langle x \mid y \rangle_{x_0} - \langle x \mid y \rangle_{x_1}| \leq d_T(x_0, x_1)$ for $x, y, x_0, x_1 \in \mathcal{T}(S)$, the assumption in the theorem is independent of the choice of the basepoint. Furthermore, since $\langle x \mid y \rangle_{x_0} \leq d_T(x_0, x)$, each holomorphic mapping $f_i$ $(i = 1, 2)$ in the theorem satisfies $d_T(x_0, f_i(z_n)) \to \infty$ as $n \to \infty$.

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1.2. A typical example of a pair of holomorphic mappings satisfying the assumption in Theorem 1.1 is a pair consisting of \( f_1, f_2 : \mathbb{D} \to \mathcal{T}(S) \) which admits a measurable subset \( E \) of positive linear measure such that for any \( z_0 \in E \), there is a sequence \( \{z_n\}_n \subset \mathbb{D} \) such that \( z_n \to z_0 \) nontangentially and \( d_T(x_0, f_i(z_n)) \to \infty \) as \( n \to \infty \) \((i = 1, 2)\) but \( d_T(f_1(z_n), f_2(z_n)) \) remains bounded. Thus, Theorem 1.1 is recognized as an improvement of Tanigawa’s rigidity theorem of holomorphic families of holomorphic disks in Teichmüller space (cf. [20, Theorem 1]). The rigidity of holomorphic disks in Teichmüller space plays an important role for studying holomorphic families of Riemann surfaces over Riemann surfaces (cf. [12], [24] and [25]).

We will prove Theorem 1.1 in §3. Applying the rigidity theorem, we also obtain a uniqueness theorem of holomorphic disks (cf. Corollary 3.1).

1.3. We first sketch the proof of Theorem 1.1 in the case \( \dim_\mathbb{C} \mathcal{T}(S) = 1 \). Namely, \( S \) is assumed to be either a once holed torus or a fourth holed sphere: We realize \( \mathcal{T}(S) \) in \( \mathbb{C} \) via the Bers embedding. Then \( \mathcal{T}(S) \) is a bounded domain which is conformally equivalent to the unit disk \( \mathbb{D} \) and hence \( (T(S), d_T) \) is isometric to the Poincaré hyperbolic disk of curvature \(-4\). Since the closure of \( \mathcal{T}(S) \) is homeomorphic to a Jordan domain (cf. [18]), the Gromov boundary of \( (T(S), d_T) \) is canonically identified with the Euclidean boundary of \( T(S) \) in \( \mathbb{C} \) (cf. §2.4 below).

By Fatou’s theorem, we may assume that each \( f_i \) has nontangential limit \( f_i^* \) at any point of \( E \) for \( i = 1, 2 \) (cf. [23, Theorem IV.7]). Let \( z_0 \in E \) and take \( \{z_n\}_{n=1}^\infty \subset \mathbb{D} \) as in the theorem. The condition \( (f_1(z_n), f_2(z_n))_{x_0} \to \infty \) implies that \( \{f_1(z_n)\}_{n=1}^\infty \) and \( \{f_2(z_n)\}_{n=1}^\infty \) determine the same ideal boundary point in the Gromov boundary of \( T(S) \) and hence \( f_1^*(z_0) = f_2^*(z_0) \). Therefore, we conclude \( f_1(z) = f_2(z) \) for all \( z \in \mathbb{D} \) by Lusin-Priwaloff-Riesz’s theorem (cf. [15, §14, §15] and [23, Theorem IV. 9]).

The proof of the case \( \dim_\mathbb{C} \mathcal{T}(S) \geq 2 \) is established by a similar argument. Unfortunately, the situation drastically changes from the above case. Indeed, when \( \dim_\mathbb{C} \mathcal{T}(S) \geq 2 \), Teichmüller space is not Gromov hyperbolic, and less information is known about the geometry of the Bers boundary (to the author’s knowledge). To overcome these difficulties, we will apply the extremal length geometry of Teichmüller space and sophisticated technologies from the theory of Kleinian groups. We recall these briefly in §2.

1.4. The Teichmüller distance coincides with the Kobayashi distance on Teichmüller space (cf. [23]). Since the Kobayashi distances are biholomorphic invariants of complex manifolds, the rigidity of holomorphic disks stated in Theorem 1.1 is thought of as a property of complex manifolds. We will observe that the rigidity property in our sense is valid for complex manifolds which are biholomorphic to bounded strictly pseudoconvex domains with \( C^2 \)-boundaries. Meanwhile, Teichmüller space is not biholomorphic to such domains unless the complex dimension is one. The rigidity property does not hold for product manifolds. As a corollary, we conclude that Teichmüller space is not realized as the product of complex manifolds, which was already proven by H. Tanigawa (cf. [20, Corollary 3]).

2. Notation

2.1. **Teichmüller space.** A marked Riemann surface is a pair \((X, f)\) of a Riemann surface \( X \) of analytically finite type and an orientation preserving homeomorphism \( f : \text{Int}(S) \to X \), where \( \text{Int}(S) \) is the interior of \( S \). Two marked Riemann surfaces...
(X₁, f₁) and (X₂, f₂) are said to be Teichmüller equivalent if there is a conformal mapping h: X₁ → X₂ such that h ◦ f₁ is homotopic to f₂. The Teichmüller space \( \mathcal{T}(S) \) of S is the set of Teichmüller equivalence classes of marked Riemann surfaces. The Teichmüller distance is a distance on \( \mathcal{T}(S) \) defined by

\[
d_T(x, y) = \frac{1}{2} \inf_h \log K(h)
\]

for \( x = (X, f) \) and \( y = (Y, g) \) in \( \mathcal{T}(S) \), where h runs all quasiconformal mappings \( h: X \to Y \) which are homotopic to \( f_2 \circ f_1^{-1} \) and \( K(h) \) is the maximal dilatation of h.

2.2. Thurston theory.

2.2.1. Measured laminations. Let \( S \) be the set of homotopy classes of nontrivial and nonperipheral simple closed curves on S. Let \( WS \) be the set of weighted simple closed curves \( ta \) on S, where \( t \geq 0 \) and \( \alpha \in S \). The closure \( MF \) of the image of the embedding

\[
WS \ni ta \mapsto [S \ni \beta \mapsto t \cdot i(\alpha, \beta)] \in \mathcal{R} := [0, \infty]^S
\]

is called the space of measured foliations on S, where \( i(\alpha, \beta) \) is the geometric intersection number between \( \alpha \) and \( \beta \). When we fix a complete hyperbolic structure on Int(S) of finite area, measured foliations are canonically identified with measured geodesic laminations. A geodesic lamination is a compact set in Int(S) which is foliated by disjoint complete geodesics. A measured geodesic lamination is a geodesic lamination with transverse invariant measure (e.g. [6] and [22]). The underlying geodesic lamination is called the support.

By definition, \( MF \) contains \( WS \) as a dense subset. We define \( i(ta, s\beta) = ts \\cdot i(\alpha, \beta) \) for \( ta, s\beta \in WS \). It is known that the intersection number function on \( WS \times WS \) extends continuously to the product space \( MF \times MF \). The space \( \mathcal{R} \) admits a natural action of positive numbers by multiplication. The quotient space of \( \mathcal{R} - \{0\} \) under this action is denoted by \( \mathcal{PR} \). Let \( \text{proj}: \mathcal{R} - \{0\} \to \mathcal{PR} \) be the projection. The image \( \mathcal{PMF} \) of \( MF - \{0\} \) under the projection is called the space of projective measured foliations on S.

2.2.2. Kleinian groups. A Kleinian group is a discrete subgroup of \( \text{PSL}_2(\mathbb{C}) \). Any Kleinian group acts on the hyperbolic 3-space discontinuously. By a Kleinian surface group we mean a Kleinian group isomorphic to \( \pi_1(S) \) via a type-preserving representation (i.e. a representation which sends all peripheral loops to parabolic transformations). An accidental parabolic transformation (APT) in a Kleinian surface group is a parabolic element which corresponds to a nonperipheral loop on S.

Bonahon’s tameness theorem asserts that the quotient hyperbolic manifold of a Kleinian surface group is homeomorphic to Int(S) \( \times \mathbb{R} \) (cf. [5]). When a Kleinian surface group does not contain APT, the quotient manifold has two ends corresponding to Int(S) \( \times \{t > 0\} \) and Int(S) \( \times \{t < 0\} \). An end is said to be geometrically infinite or simply degenerate if any neighborhood of the end contains a closed geodesic which is homotopic to a simple closed curve on Int(S) \( \times \{0\} \). For a geometrically infinite end, we associate a unique geodesic lamination, which we call the ending lamination of the geometrically infinite end. The ending lamination is filling in the sense that it intersects transversely the support of every measured lamination except for itself (cf. [5], [8], §2.5) and [27].
A quasifuchsian group is, by definition, a Kleinian surface group which is obtained by a quasi-conformal deformation of a Fuchsian group. A Kleinian surface group is said to be a \textit{b-group} if it has a unique simply connected invariant component. A b-group is called a \textit{totally degenerate group} if its region of discontinuity is connected (cf. \cite{4} §2).

2.2.3. The Bers embedding and b-groups. The Teichmüller space \(\mathcal{T}(S)\) of \(S\) is embedded into a finite dimensional complex Banach space via the Bers embedding (cf. \cite{4} §1). The image of the Bers embedding is a bounded domain. By the Bers embedding, each point in \(\mathcal{T}(S)\) is associated with a quasifuchsian group. Every point in the boundary of \(\mathcal{T}(S)\), called the Bers boundary, corresponds to a b-group. If a totally degenerate group does not contain APT, the quotient hyperbolic manifold has a unique geometrically infinite end. The \textit{ending lamination theorem} asserts that two totally degenerate groups without APT in the Bers boundary agree if and only if they have the same ending lamination (cf. \cite{8}).

2.3. Extremal length geometry of Teichmüller space.

2.3.1. Gromov product of the Teichmüller distance. For \(\alpha \in S\) and \(y = (Y,f) \in \mathcal{T}(S)\) we denote by \(\text{Ext}_y(\alpha)\) the extremal length of the family of rectifiable simple closed curves on \(Y\) homotopic to \(f(\alpha)\). When we put \(\text{Ext}_y(t\alpha) = t^2\text{Ext}_y(\alpha)\), the extremal length extends continuously to \(\mathcal{MF}\) (cf. \cite{13} Proposition 3). The Gardiner-Masur embedding \(\Phi_{GM}\) is defined by

\[\Phi_{GM}: \mathcal{T}(S) \ni y \mapsto \text{proj}([S \ni \alpha \mapsto \text{Ext}_y(\alpha)^{1/2}]) \in \mathcal{PR}.\]

The closure \(\text{cl}_{GM}(\mathcal{T}(S))\) of the image is called the \textit{Gardiner-Masur closure} and the complement \(\partial_{GM}\mathcal{T}(S) = \text{cl}_{GM}(\mathcal{T}(S)) - \Phi_{GM}(\mathcal{T}(S))\) the \textit{Gardiner-Masur boundary}. F. Gardiner and H. Masur observed that the closure \(\text{cl}_{GM}(\mathcal{T}(S))\) is compact and \(\mathcal{PMF} \subset \partial_{GM}\mathcal{T}(S)\) (cf. \cite{9}).

In \cite{19}, the author proved the following theorem.

\textbf{Theorem 2.1} (Extension theorem). \textit{Fix} \(x_0 \in \mathcal{T}(S)\). \textit{The Gromov product} \(\langle \cdot | \cdot \rangle_{x_0} \) \textit{on} \(\mathcal{T}(S) \times \mathcal{T}(S)\) \textit{extends continuously to} \(\text{cl}_{GM}(\mathcal{T}(S)) \times \text{cl}_{GM}(\mathcal{T}(S))\) \textit{with values in the interval} \([0,\infty]\). \textit{Furthermore, for} \([F],[G] \in \mathcal{PMF} \subset \partial_{GM}\mathcal{T}(S)\), \textit{we have}

\[
\exp(-2\langle [F] | [G]\rangle_{x_0}) = \frac{i(F,G)}{\text{Ext}_{x_0}(F)^{1/2}\text{Ext}_{x_0}(G)^{1/2}}.
\]

2.3.2. Intersection number with basepoint. We define the \textit{intersection number with basepoint} \(x_0 \in \mathcal{T}(S)\) by

\[i_{x_0}(p_1,p_2) = \exp(-2\langle p_1 | p_2\rangle_{x_0})\]

for \(p_1,p_2 \in \text{cl}_{GM}(\mathcal{T}(S))\). It is known that

\[
i_{x_0}(y,[F]) = \frac{e^{-d_T(x_0,y)}\text{Ext}_y(F)^{1/2}}{\text{Ext}_{x_0}(F)^{1/2}}
\]

for \(y \in \mathcal{T}(S)\) and \([F] \in \mathcal{PMF}\) where we set \(\exp(-\infty) = 0\) (cf. \cite{19} §5.1). For \(p \in \text{cl}_{GM}(\mathcal{T}(S))\), we define

\[\mathcal{N}(p) = \{ q \in \text{cl}_{GM}(\mathcal{T}(S)) | i_{x_0}(p,q) = 0 \}.
\]

In \cite{20}, the author showed the following.

\textbf{Theorem 2.2} (Null set). \(\mathcal{N}(p) \neq \emptyset\) \textit{if and only if} \(p \in \partial_{GM}\mathcal{T}(S)\). \textit{In addition, for any} \(p \in \partial_{GM}\mathcal{T}(S)\), \textit{there is} \([F] \in \mathcal{PMF}\) \textit{such that} \(\mathcal{N}(p) = \mathcal{N}([F])\).
2.4. Gromov hyperbolic space. Let \((X, d_X)\) be a metric space. Let \(x_0 \in X\) be a basepoint. The Gromov product with reference point \(x_0\) is defined by
\[
\langle x \mid y \rangle_{x_0}^X = \frac{1}{2} (d_X(x_0, x) + d_X(x_0, y) - d_X(x, y)).
\] (2.3)

A Gromov hyperbolic space is a metric space \((X, d_X)\) with the property that there is \(\delta > 0\) such that
\[
\langle x \mid y \rangle_{x_0}^X \geq \min\{\langle x \mid z \rangle_{x_0}^X, \langle y \mid z \rangle_{x_0}^X\} - \delta
\]
for all \(x, y, z \in X\) ([11] §1.1).

Let \((X, d_X)\) be a Gromov hyperbolic space. A sequence \(\{x_n\}_{n=1}^\infty \subset X\) is said to be convergent at infinity if \(\langle x_n \mid x_m \rangle_{x_0} \to \infty\) as \(n, m \to \infty\). Two convergent sequences \(\{x_n\}_{n=1}^\infty\) and \(\{y_n\}_{n=1}^\infty\) at infinity are equivalent if \(\liminf_{n \to 0} \langle x_n \mid y_n \rangle_{x_0} = \infty\). The set of equivalence classes of convergent sequences at infinity is called the Gromov boundary and denoted by \(\partial_\infty X\) ([11] §1.8]). The Poincaré hyperbolic disk \((\mathbb{D}, d_\mathbb{D})\) is a typical example of Gromov hyperbolic space. The Gromov boundary \(\partial_\infty \mathbb{D}\) of \((\mathbb{D}, d_\mathbb{D})\) is canonically identified with the Euclidean boundary \(\partial \mathbb{D}\) ([11] §1.5]). However, when \(\dim_{\mathbb{C}} (T(S)) \geq 2\), \((T(S), d_T)\) is not Gromov hyperbolic (cf. [17] Theorem 3.1).

3. Proof of the theorem

3.1. Proof of the theorem. We identify \(T(S)\) with a bounded domain in a finite dimensional complex Banach space via the Bers embedding. By Fatou’s theorem, there is a measurable set \(E_0 \subset \partial \mathbb{D}\) of full measure such that \(f_1\) and \(f_2\) have nontangential limits at every \(z_0 \in E_0\). Furthermore, from Shiga’s theorem [21] Theorem 5], we may assume that the nontangential limit at any point in \(E_0\) corresponds to either a quasifuchsian group or a totally degenerate group without APT.

Let \(E_1 = E_0 \cap E\) and \(z_0 \in E_1\). By the assumption, there is a sequence \(\{z_n\}_{n=1}^\infty \subset \mathbb{D}\) such that \(z_n \to z_0\) nontangentially and \(\langle f_1(z_n) \mid f_2(z_n) \rangle_{x_0} \to \infty\) as \(n \to \infty\). Denote by \(f_i^*(z_0)\) the nontangential limit of \(f_i\) at \(z_0\). Since \(d_T(f_1(z_0), f_2(z_0)) \to \infty\), \(f_i^*(z_0)\) corresponds to a totally degenerate group for \(i = 1, 2\). Let \(\lambda_1\) and \(\lambda_2\) be the ending laminations of geometrically infinite ends of the hyperbolic manifolds associated with \(f_1^*(z_0)\) and \(f_2^*(z_0)\).

Fix \(i = 1, 2\). Take \(\alpha_i^i \in S\) with \(\operatorname{Ext}_{f_i(z_n)}(\alpha_i^i) \leq M\) for some constant \(M > 0\) independent of \(n\) (cf. [3] Theorem 1]). By taking a subsequence, there is a bounded sequence \(\{t_n^i\}_{n=1}^\infty\) such that \(t_n^i \alpha_n^i \to \mu_i\in \mathcal{MF} - \{0\}\). Since \(f_i(z_n)\) converges to a totally degenerate group without APT, from [11] Theorem 2], we can see that \(\operatorname{Ext}_{x_0}(\alpha_n^i) \to \infty\) as \(n \to \infty\). Hence, we have that \(t_n^i \to 0\) since \((t_n^i)^2 \operatorname{Ext}_{x_0}(\alpha_n^i) \to \operatorname{Ext}_{x_0}(\mu_i)\). By Bers’ inequality [4] Theorem 3] and Maskit’s comparison theorem 16, the hyperbolic length of \(t_n^i \alpha_n^i\) in the quasifuchsian manifold associated with \(f_i(z_n)\) tends to 0. From the continuity of the Thurston’s length function, any sublamination of the support of \(\mu_i\) is nonrealizable in the hyperbolic manifold associated with \(f_i^*(z_0)\) (cf. [21] and [7] Theorem 7.1, Corollary 7.3]). Hence, the support of \(\mu_i\) is contained in \(\lambda_i\) (cf. [5] and [21] §9]). Since \(\lambda_i\) is filling on \(S\), the support of \(\mu_i\) coincides with \(\lambda_i\) (cf. [22] 2.2).

By taking a subsequence if necessary, we may assume that \(\{\Phi_{GM}(f_i(z_n))\}_{n=1}^\infty\) converges to a point \(p_i \in \partial GM T(S)\). By Theorem 22, there is \(\nu_i \in \mathcal{MF}\) such that
\[ N(p_i) = N([\nu_i]). \] By Theorem 2.1 and (2.2), we have
\[
i_{x_0}(p_i, \mu_i) = \lim_{n \to \infty} i_{x_0}(f_i(z_n), t_n^i \alpha_n^i)
\]
\[
= \lim_{n \to \infty} e^{-d_T(x_0, f_i(z_n))} \frac{\text{Ext}_f(z_n) (t^i_n \alpha_n^i)^{1/2}}{\text{Ext}_{x_0} (t^i_n \alpha_n^i)^{1/2}}
\]
\[
\leq \lim_{n \to \infty} M^{1/2} t_n^i e^{-d_T(x_0, f_i(z_n))} = 0.
\]

Hence, we obtain \( i(\nu_1, \mu_2) = 0 \) from (2.1). Therefore, the support of \( \nu_i \) coincides with that of \( \mu_i \) since the ending lamination \( \lambda_i \) is filling (cf. [2.2.2]).

Our assumption \( \langle f_1(z_n) \rangle f_2(z_n) \rangle_{x_0} \to \infty \) implies that \( i_{x_0}(p_1, p_2) = 0 \) and hence \( i(\nu_1, \nu_2) = 0 \) from (2.1) again. Thus we obtain that \( \lambda_1 = \lambda_2 \) and \( f^*_1(0) = f^*_2(0) \) from the ending lamination theorem (cf. [2.2.3]). Since \( E_1 \) has positive linear measure, the coincidence between \( f_1 \) and \( f_2 \) on \( \mathbb{D} \) follows from Lusin-Priwaloff-Riesz’s theorem.

3.2. Uniqueness of holomorphic disks. From Theorem 1.1 we conclude the following uniqueness theorem.

Corollary 3.1 (Uniqueness theorem). Let \( f_1, f_2: \mathbb{D} \to \mathcal{T}(S) \) be holomorphic mappings. The following are equivalent:

1. \( f_1(z) = f_2(z) \) for all \( z \in \mathbb{D} \).
2. There is a measurable subset \( E \subset \partial \mathbb{D} \) of positive linear measure such that for any \( z_0 \in E \) there is a sequence \( \{z_n\}_{n=1}^\infty \subset \mathbb{D} \) converging nontangentially to \( z_0 \) which satisfies one of the following:
   a. \( \langle f_1(z_n) \rangle f_2(z_n) \rangle_{x_0} = O(1) \) and \( d_T(f_1(z_n), f_2(z_n)) \to 0 \) as \( n \to \infty \).
   b. \( \langle f_1(z_n) \rangle f_2(z_n) \rangle_{x_0} \to \infty \) as \( n \to \infty \).

Proof. We only check that (2) implies (1). Suppose the assertion (2). We realize \( \mathcal{T}(S) \) as a bounded domain via the Bers embedding. From Shiga’s theorem, we may assume that each \( f_i \) has the nontangential limit \( f^*_i \) at any point in \( E \) and the limit corresponds to either a quasifuchsian group or a totally degenerate group without APT.

Let \( z_0 \in E \) and take a sequence \( \{z_n\}_{n=1}^\infty \subset \mathbb{D} \) as in the assertion (2). Suppose (a) holds. Since
\[
d_T(x_0, f_i(z_n)) \leq 2 \langle f_1(z_n) \rangle f_2(z_n) \rangle_{x_0} + d_T(f_1(z_n), f_2(z_n)) = O(1)
\]as \( n \to \infty \) for \( i = 1, 2 \), the limits \( f^*_1(0) \) and \( f^*_2(0) \) are quasifuchsian groups. Since \( d_T(f_1(z_n), f_2(z_n)) \to 0 \), we have \( f^*_1(0) = f^*_2(0) \). If (b) holds, we also deduce the equality \( f^*_1(0) = f^*_2(0) \) by the same argument as that in Theorem 1.1. \( \square \)

4. Rigidity of holomorphic disks in complex manifolds

We shall discuss what kind of complex manifolds the rigidity theorem in our sense is valid. Henceforth, let \( \Omega \) be a complex manifold. Denote by \( d_\Omega \) the Kobayashi distance on \( \Omega \). Fix a point \( x_0 \in \Omega \) and set \( \langle x \rangle_{x_0}^\Omega \) to be the Gromov product on \( (\Omega, d_\Omega) \) with reference point \( x_0 \) (cf. (2.3)).
4.1. The rigidity theorem in our sense holds when \( \Omega \) is biholomorphic to a bounded strongly pseudoconvex domain with \( C^2 \)-boundary: The proof is established by the same argument as that for the case of Teichmüller space of dimension one (cf. [13]). However, we shall give a proof for completeness.

Notice that I. Graham showed that \( (\Omega, d_\Omega) \) is a complete metric space (cf [10] Proposition 5]). In addition, Z. Balogh and M. Bonk observed that \( (\Omega, d_\Omega) \) is Gromov hyperbolic and the Gromov boundary \( \partial_\infty \Omega \) of \( \Omega \) canonically coincides with the Euclidean boundary \( \partial \Omega \) (cf. [2] Theorem 1.4]).

Let \( f \) and \( g \) be holomorphic mappings from \( \mathbb{D} \) to \( \Omega \). Suppose that there is a measurable set \( E \subset \partial \mathbb{D} \) of positive linear measure such that for any \( z_0 \in E \), there is a sequence \( \{z_n\}_{n=1}^{\infty} \subset \mathbb{D} \) such that \( z_n \to z_0 \) non-tangentially and \( \langle f(z_n) \rangle g(z_n)_{x_0}^\Omega \to \infty \). Since \( \Omega \) is a bounded domain, we may assume that each of \( f \) and \( g \) admits the non-tangential limit at every point in \( E \). The condition \( \langle f(z_n) \rangle g(z_n)_{x_0}^\Omega \to \infty \) implies that sequences \( \{f(z_n)\}_{n=1}^{\infty} \) and \( \{g(z_n)\}_{n=1}^{\infty} \) converge the same ideal boundary point in \( \partial_\infty \Omega = \partial \Omega \) (cf. [24]). Hence \( f \) and \( g \) have the same non-tangential limits on \( E \). Since \( E \) has positive linear measure, \( f \) coincides with \( g \) on \( \mathbb{D} \) by Lusin-Priwaloff-Riesz’s theorem as in the previous section.

4.2. On the other hand, when \( \dim \mathbb{C} T(S) \geq 2 \), \( (T(S), d_T) \) is not Gromov hyperbolic. Hence the argument in the previous section does not work for Teichmüller spaces unless \( \dim \mathbb{C} T(S) = 1 \). As a consequence, the class of complex manifolds with the rigidity property in our sense is strictly larger than the class of bounded Gromov-hyperbolic pseudoconvex domains (in terms of the Kobayashi distances) whose Gromov boundaries coincide with the Euclidean boundaries. By applying the discussion in the previous section, we can easily see that any pseudoconvex domain in the latter class satisfies the rigidity property in our sense. One can also check that the uniqueness theorem in our sense also holds for domains in the latter class.

4.3. The rigidity theorem in our sense does not hold if \( \Omega \) is biholomorphic to the product manifold \( M_1 \times M_2 \) of some complex manifolds \( M_i \) \((i = 1, 2)\) which admits a holomorphic mapping \( f: \mathbb{D} \to \Omega \) with the property that there is a measurable set \( E \subset \partial \mathbb{D} \) of positive linear measure such that for any \( z_0 \in E \) there is a sequence \( \{z_n\}_{n=1}^{\infty} \subset \mathbb{D} \) such that \( z_n \to z_0 \) non-tangentially and \( d_\Omega(x_0, f(z_n)) \to \infty \). For instance, when \( M_2 = \mathbb{D} \), a product manifold \( M_1 \times M_2 \) has this property. However, when each \( M_i \) is a closed complex manifold, the product manifold \( M_1 \times M_2 \) does not have the property.

It is known that

\[
\max\{d_{M_1}(z^1, z^2), d_{M_2}(w^1, w^2)\} \leq d_\Omega((z^1, z^2), (w^1, w^2))
\]

\[
\leq d_{M_1}(z^1, z^2) + d_{M_2}(w^1, w^2)
\]

for \((z^1, z^2), (w^1, w^2) \in \Omega = M_1 \times M_2\) (cf. [14] Proposition 2.5]).

Let \( f = (f_1, f_2) \) and \( x_0 = (x^1_0, x^2_0) \). From (4.1), by taking a measurable subset in \( E \) of positive linear measure if necessary, we may assume that for any \( z_0 \in E \) there is a sequence \( \{z_n\}_{n=1}^{\infty} \subset \mathbb{D} \) such that \( z_n \to z_0 \) non-tangentially and \( d_{M_1}(x^1_0, f_1(z_n)) \) tends to \( \infty \). Let \( y^2_0 \in M_2 \) with \( y^2_0 \neq x^2_0 \). Define

\[
g_1(z) = (f_1(z), x^2_0),
\]

\[
g_2(z) = (f_1(z), y^2_0).
\]
Then, $g_1(z) \neq g_2(z)$ but $d_{\Omega}(g_1(z), g_2(z)) = d_{M_2}(x_0^2, y_0^2)$ for all $z \in \mathbb{D}$. For any $z_0 \in E$, there is a sequence $\{z_n\}_n \subset \mathbb{D}$ such that $z_n \to z_0$ nontangentially and

$$\langle g_1(z_n) \rangle_{g_2(z_n)}^{\Omega} = \frac{1}{2}(d_{\Omega}(x_0, g_1(z_n)) + d_{\Omega}(x_0, g_2(z_n)) - d_{\Omega}(g_1(z_1), g_2(z_n)))$$

$$\geq d_{M_1}(x_0^1, f_1(z_n)) - d_{M_2}(x_0^2, y_0^2) \to \infty$$

as $n \to \infty$.

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