Coefficients of Šapovalov elements for simple Lie algebras and contragredient Lie superalgebras.

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Abstract

We provide upper bounds on the degrees of the coefficients of Šapovalov elements for a simple Lie algebra. If \( \mathfrak{g} \) is a contragredient Lie superalgebra and \( \gamma \) is a positive isotropic root of \( \mathfrak{g} \), we prove the existence and uniqueness of the Šapovalov element for \( \gamma \) and we obtain upper bounds on the degrees of their coefficients. For type A Lie superalgebras we give a closed formula for Šapovalov elements. We also explore the behavior of Šapovalov elements when the Borel subalgebra is changed, and the survival of Šapovalov elements in factor modules of Verma modules.

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1 Introduction.

Throughout this paper we work over an algebraically closed field \( \mathbb{k} \) of characteristic zero. If \( \mathfrak{g} \) is a simple Lie algebra necessary and sufficient conditions for the existence of a non-zero homomorphism from \( M(\mu) \) to \( M(\lambda) \) can be obtained by combining work of Verma [Ver68] with work of Bernstein, Gelfand and Gelfand [BGG71],

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Such maps can be described explicitly in terms of certain elements introduced by N.N. Šapovalov in [Šap72]. Verma modules are fundamental objects in the study of category $\mathcal{O}$, a study that has blossomed into an extremely rich theory in the years since these early papers appeared. Highlights include the Kazhdan-Lusztig conjecture [KL79], [BB81], [BK81], the work of Beilinson, Ginzburg, and Soergel on Koszul duality [BGS96], and more recent results on categorification see [Hum08] and [Maz12] for more details. A refined version of the Kazhdan-Lusztig conjecture giving the composition multiplicities in successive quotients of the Jantzen filtration conjecture was proven in [BB93] by showing that the localization functor sends the Jantzen filtration to the weight filtration on perverse sheaves. This also established a conjecture of Jantzen on a compatibility property of the filtration from [Jan79] with Verma submodules.

Recently significant advances have been made in the study of the category $\mathcal{O}$ for classical simple Lie superalgebras using a variety of techniques. After the early work of Kac [Kac77], [Kac77b] the first major advance was made by Serganova who used geometric techniques to obtain a character formula for Kac modules over $\mathfrak{gl}(m, n)$, [Ser96]. The next development was Brundan’s approach to the same problem using a combination of algebraic and combinatorial techniques. In his seminal paper [Bru03] also introduced a Fock space representation $\mathcal{T}^{m|n} := \bigotimes^m \mathcal{V}^* \otimes \bigotimes^n \mathcal{V}$ of the quantized enveloping algebra $\mathcal{U} = U_q(\mathfrak{gl}_\infty)$ where $\mathcal{V}$ is the natural representation of $\mathcal{U}$, and $\mathcal{V}^*$ is its restricted dual. He then introduced monomial and canonical bases for $\mathcal{T}^{m|n}$, and using the transition matrices between these matrices defined polynomials which have become known as Brundan-Kazhdan-Lusztig polynomials.

Brundan then made the extraordinary conjecture that the values at $q = 1$ of these polynomials solve the multiplicity problem of for composition factors of Verma modules. For $\mathfrak{gl}(m)$ this is equivalent to the Kazhdan-Lusztig conjecture. Brundan’s conjecture was later confirmed by Cheng, Lam and Wang [CLW12], exploiting connections with super-duality. Super-duality connects the parabolic category $\mathcal{O}$ for $\mathfrak{gl}(m, n)$ to a corresponding parabolic category for $\mathfrak{gl}(m + n)$. The authors later extended this connection to the orthosymplectic case [CLW11], [CW12]. A new proof of the conjecture was provided by Brundan, Losev and Webster [BLW13], at the same time showing that any integral block of the category $\mathcal{O}$ for $\mathfrak{gl}(m, n)$ has a graded lift which is Koszul, see the recent survey article [Bru] by Brundan for these developments.

The Šapovalov determinant, also introduced in [Šap72] has been developed in a variety of contexts, such as Kac-Moody algebras [KK79], quantum groups [Jos95], and Lie superalgebras [Gor02], [Gor04], [Gor06]. However neither Šapovalov elements nor the Jantzen filtration have received much attention for classical simple Lie superalgebras. The purpose of this paper is to initiate the study of Šapovalov elements in the super case. New phenomena arise due to the presence of isotropic roots. A sequel will focus on the Jantzen filtration and sum formula [Mus].
Let \( \mathfrak{g} = \mathfrak{g}(A, \tau) \) be a finite dimensional contragredient Lie superalgebra with Cartan subalgebra \( \mathfrak{h} \), and set of simple roots \( \Pi \). The superalgebras \( \mathfrak{g}(A, \tau) \) coincide with the basic classical simple Lie superalgebras, except that instead of \( \mathfrak{psl}(n, n) \) we obtain \( \mathfrak{gl}(n, n) \). Implicit in the definition of the \( \mathfrak{g}(A, \tau) \) is a preferred Borel subalgebra.

\[
\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \tag{1.1}
\]

be the set of positive roots containing \( \Pi \), and the corresponding triangular decomposition of \( \mathfrak{g} \) respectively. We use the Borel subalgebras \( \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+ \) and \( \mathfrak{b}^- = \mathfrak{n}^- \oplus \mathfrak{h} \).

The Verma module \( M(\lambda) \) with highest weight \( \lambda \in \mathfrak{h}^* \), and highest weight vector \( v_\lambda \) is induced from \( \mathfrak{b} \). Suppose that \( \gamma \) is a positive root, and \( m \) is a positive integer.

\[\theta_{\gamma, m} = \sum_{\pi \in \mathbf{P}(m \gamma)} e^{-\pi} H_\pi, \tag{1.2}\]

where \( H_\pi \in U(\mathfrak{h}) \), and has the property that if \( \lambda \) lies on a certain hyperplane then \( \theta_{\gamma, m} v_\lambda \) is a highest weight vector in \( M(\lambda) \), see (1.4). In (1.2) the sum is indexed by the set \( \mathbf{P}(m \gamma) \) of partitions of \( m \gamma \), the \( e^{-\pi} \) with \( \pi \in \mathbf{P}(m \gamma) \) form a basis for the weight space \( U(\mathfrak{n}^-)^{-m \gamma} \), and the coefficients \( H_\pi \) are in \( U(\mathfrak{h}) \). These results appear to be new even for simple Lie algebras. We normalize \( \theta_{\gamma, m} \) so that for a certain \( \pi^0 \in \mathbf{P}(m \gamma) \), the coefficient \( H_{\pi^0} \) is equal to 1. This guarantees that \( \theta_{\gamma, m} v_\lambda \) is never zero.

The main results in this paper give bounds on the degrees of the coefficients \( H_\pi \) in (1.2). There is always a unique coefficient of highest degree, and we determine the leading term of this coefficient up to a scalar multiple. The exact form of the coefficients depends on the way the positive roots are ordered. Nevertheless they seem to have interesting properties both combinatorially and from the point of view of representation theory. For example with a suitable ordering the coefficients are often products of linear factors and the vanishing of these factors has an interpretation in terms of representation theory.

The existence of a unique coefficient of highest degree is useful in the construction of some new highest weight modules \( M_\gamma(\lambda) \), where \( \gamma \) is an isotropic root and \( (\lambda + \rho, \gamma) = 0 \). This results in an improvement in the Jantzen sum formula from [Mus12] Theorem 10.3.1. The module \( M_\gamma(\lambda) \) has character \( e^{\lambda} p_\gamma \), see (2.2) for notation, so this leads to a formula where both sides are sums of characters in the category \( \mathcal{O} \).

Šapovalov elements corresponding to non-isotropic roots for a basic classical simple Lie superalgebra were constructed in [Mus12] Chapter 9. This closely parallels the semisimple case. Properties of the coefficients of these elements were announced in [Mus12] Theorem 9.2.10. However the bounds on the degrees of the coefficients claimed in [Mus12] are incorrect if \( \Pi \) contains a non-isotropic odd root. They are corrected by Theorem 1.3.
I would like to thank Jon Brundan for suggesting the use of noncommutative determinants to write Šapovalov elements in Section 7, and raising the possibility of using Theorem 7.1 to prove Theorem 8.4. I also thank Kevin Coulembier for some helpful conversations.

1.1 Preliminaries.

We use the definition of partitions from [Mus12] Remark 8.4.3. Set $Q^+ = \sum_{\alpha \in \Pi} \mathbb{N}_0 \alpha$. If $\eta \in Q^+$, a partition of $\eta$ is a map $\pi : \Delta^+ \to \mathbb{N}$ such that $\pi(\alpha) = 0$ or 1 for all isotropic roots $\alpha$, $\pi(\alpha) = 0$ for all even roots $\alpha$ such that $\alpha/2$ is a root, and

$$\sum_{\alpha \in \Delta^+} \pi(\alpha) \alpha = \eta.$$ 

For $\eta \in Q^+$, we denote by $\mathcal{P}(\eta)$ the set of partitions of $\eta$. If $\pi \in \mathcal{P}(\eta)$ the degree of $\pi$ is defined to be $|\pi| = \sum_{\alpha \in \Delta^+} \pi(\alpha)$.

Fix a non-degenerate invariant symmetric bilinear form $(\ , \ )$ on $\mathfrak{h}^*$, and for all $\alpha \in \mathfrak{h}^*$, let $h_\alpha \in \mathfrak{h}$ be the unique element such that $(\alpha, \beta) = \beta(h_\alpha)$ for all $\beta \in \mathfrak{h}^*$. Then for all $\alpha \in \Delta^+$, choose elements $e_{\pm \alpha} \in \mathfrak{g}^{\pm \alpha}$ such that

$$[e_\alpha, e_{-\alpha}] = h_\alpha.$$ 

Fix an ordering on the set $\Delta^+$, and for $\pi$ a partition, set

$$e_{-\pi} = \prod_{\alpha \in \Delta^+} e_{\pi(\alpha)},$$

the product being taken with respect to this order. Then the elements $e_{-\pi}$, with $\pi \in \mathcal{P}(\eta)$ form a basis of $U(\mathfrak{n}^-)^{-\eta}$. For a non-isotropic root $\alpha$, we set $\alpha_\gamma = 2\alpha/(\alpha, \alpha)$, and denote the reflection corresponding to $\alpha$ by $s_\alpha$. As usual the Weyl group $W$ is the subgroup of $GL(\mathfrak{h}^*)$ generated by all reflections. For $u \in W$ set

$$N(u) = \{ \alpha \in \Delta^+_0 \mid u\alpha < 0 \}, \quad \ell(u) = |N(u)|.$$ 

We use the following well-known fact several times.

**Lemma 1.1.** If $w = s_\alpha u$ with $\ell(w) > \ell(u)$ and $\alpha$ is a simple non-isotropic root, then we have a disjoint union

$$N(w^{-1}) = s_\alpha N(u^{-1}) \cup \{ \alpha \}.$$ \hspace{1cm} (1.3)

**Proof.** See for example [Hum90] Chapter 1. \hfill \Box

Set

$$\rho_0(\mathfrak{b}) = \frac{1}{2} \sum_{\alpha \in \Delta^+_0} \alpha, \quad \rho_1(\mathfrak{b}) = \frac{1}{2} \sum_{\alpha \in \Delta^+_1} \alpha, \quad \rho(\mathfrak{b}) = \rho_0(\mathfrak{b}) - \rho(\mathfrak{b}_1).$$

Except in Section 9 we work with a fixed Borel subalgebra, and if this is the case we set $\rho_i = \rho_i(\mathfrak{b})$ for $i = 1, 2$ and $\rho = \rho(\mathfrak{b})$. 

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1.2 Main Results.

Fix a positive root $\gamma$ and a positive integer $m$. Let $\pi^0 \in \overline{P}(m\gamma)$ be the unique partition of $m\gamma$ such that $\pi^0(\alpha) = 0$ if $\alpha \in \Delta^+ \setminus \Pi$. The partition $m\pi_\gamma$ of $m\gamma$ is given by $m\pi_\gamma(\gamma) = m$, and $m\pi_\gamma(\alpha) = 0$ for all positive roots $\alpha$ different from $\gamma$. We say that $\theta \in U(b^-)^{m\gamma}$ is a ˇSapovalov element for the pair $(\gamma, m)$ if it has the form (1.2) with $H_{\pi^0} = 1$, and

$$e_\alpha \theta \in U(\mathfrak{g})(h_\gamma + \rho(h_\gamma) - m(\gamma, \gamma)/2) + U(\mathfrak{g})n^+, \text{ for all } \alpha \in \Delta^+. \quad (1.4)$$

For a semisimple Lie algebra, the existence of such elements was shown by ˇSapovalov, [ˇSap72] Lemma 1. Let

$$\mathcal{H}_{\gamma, m} = \{ \lambda \in \mathfrak{h}^* | (\lambda + \rho, \gamma) = m(\gamma, \gamma)/2 \},$$

and let $I(\mathcal{H}_{\gamma, m})$ be the ideal of $S(\mathfrak{h})$ consisting of functions vanishing on $\mathcal{H}_{\gamma, m}$. Thus the ring of regular functions on $\mathcal{H}_{\gamma, m}$ is

$$S(\mathfrak{h})/(h_\gamma + \rho(h_\gamma) - m(\gamma, \gamma)/2).$$

Note that if $\lambda \in \mathcal{H}_{\gamma, m}$, and $\theta$ satisfies (1.4) then $\theta v_\lambda$ is a highest weight vector of weight $\lambda - m\gamma$ in $M(\lambda)$. Now for $\mu, \lambda \in \mathfrak{h}^*$ we have

$$\dim \text{Hom}_g(M(\mu), M(\lambda)) \leq 1, \quad (1.5)$$

by [Dix96] Theorem 7.6.6, and it follows that the ˇSapovalov element $\theta = \theta_{\gamma, m}$ for the pair $(\gamma, m)$ is unique modulo the left ideal $U(b^-)I(\mathcal{H}_{\gamma, m})$.

In general if $\mathfrak{g}$ is a finite dimensional contragredient Lie superalgebra the representation theory of $\mathfrak{g}$ is complicated by the existence of several conjugacy classes of Borel subalgebras. This problem is partially resolved by at first fixing a Borel subalgebra (or equivalently a basis of simple roots for $\mathfrak{g}$) with special properties. Later we study the effect of changing the Borel subalgebra. The first explicit description of the system of roots and possible Dynkin-Kac diagrams, up to conjugacy by the Weyl group, was given by Kac in [Kac77] sections 2.5.4 and 2.5.5. We remark that there are some omissions on these lists. The corrected lists appear in [FSS00] (and elsewhere). Note that if $\mathfrak{g} = \mathfrak{osp}(2m, 2n)$ there are some diagrams that correspond to two sets of simple roots. The corresponding Borel subalgebras are conjugate under an outer (diagram) automorphism of $\mathfrak{g}$, [Mus12] Corollary 5.5.13.

In [Kac77] Table VI Kac gave a particular diagram in each case that we will call distinguished. The corresponding set of simple roots and Borel subalgebra are also called distinguished. The distinguished Borel subalgebra contains at most one simple isotropic root vector. Unless $\mathfrak{g} = \mathfrak{osp}(1, 2n)$, $\mathfrak{g} = \mathfrak{osp}(2, 2n)$ there is exactly one other Borel subalgebra with this property up to conjugacy in $\text{Aut} \mathfrak{g}$. A representative of this class (and its set of simple roots) will be called anti-distinguished. If $\mathfrak{g} = \mathfrak{osp}(1, 2n)$ there is only one conjugacy class, while if $\mathfrak{g} = \mathfrak{osp}(2, 2n)$ there is a basis containing exactly two isotropic roots $\delta_n \pm \epsilon_1$ (in the notation of Kac).
In the latter case we call this basis anti-distinguished. Suppose \( \Pi_{\text{nonisotropic}} \) (resp. \( \Pi_{\text{even}} \)) be the set of nonisotropic (resp. even) simple roots, and let \( W_{\text{nonisotropic}} \) (resp. \( W_{\text{even}} \)) be the subgroup of \( W \) generated by the reflections \( s_\alpha \), where \( \alpha \in \Pi_{\text{nonisotropic}} \) (resp. \( \alpha \in \Pi_{\text{even}} \)). Consider the following hypotheses.

The set of simple roots of \( \Pi \) is either distinguished or anti-distinguished. \( \text{(1.6)} \)

\[ \gamma = w\beta \text{ for a simple root } \beta \text{ and } w \in W_{\text{even}}. \] \( \text{(1.7)} \)

\[ \gamma = w\beta \text{ for a simple root } \beta \text{ and } w \in W_{\text{nonisotropic}}. \] \( \text{(1.8)} \)

When (1.6) and either of (1.7) or (1.8) holds we always assume that \( \ell(w) \) is minimal, and for \( \alpha \in N(w^{-1}) \), we define \( q(w, \alpha) = (w\beta, \alpha^\vee) \). If \( \Pi = \{ \alpha_i | i = 1, \ldots, t \} \) is the set of simple roots, and \( \gamma = \sum_{i=1}^t a_i \alpha_i \), then the height \( \text{ht} \gamma \) of \( \gamma \) is defined to be \( \sum_{i=1}^t a_i \).

**Theorem 1.2.** Suppose \( \mathfrak{g} \) is semisimple or a contragredient Lie superalgebra, and \( \gamma \) is a positive root such that (1.6) and (1.7) hold. If \( \gamma \) is isotropic assume that \( m = 1 \). Then there exists a Šapovalov element \( \theta_{\gamma,m} \in U(b^-)^{-m\gamma} \), which is unique modulo the left ideal \( U(b^-)I(H_{\gamma}) \), and the coefficients of \( \theta_{\gamma,m} \) satisfy

\[ |\pi| + \deg H_{\pi} \leq m \text{ht} \gamma, \] \( \text{(1.9)} \)

and

\[ H_{m\pi,\gamma} \text{ has leading term } \prod_{\alpha \in N(w^{-1})} h_{\alpha}^{m q(w, \alpha)}. \] \( \text{(1.10)} \)

If we assume hypothesis (1.8) instead of (1.7), it seems difficult to obtain the same estimates on Šapovalov elements as in Theorem 1.2. However it is still possible to obtain a reasonable estimate using a different definition of the degree of a partition, at least if \( m = 1 \). To simplify notation we set \( H_{\gamma} = H_{\gamma,1} \) and denote a Šapovalov element for the pair \( (\gamma,1) \) by \( \theta_{\gamma} \). In the Theorem below we assume that \( \Pi \) contains an odd non-isotropic root, since otherwise (1.7) holds and the situation is covered by Theorem 1.2. This assumption is essential for Lemma 5.1. Likewise \( \gamma \) is odd and non-isotropic, then again (1.7) holds, so we assume that \( \gamma = w\beta \) with \( w \in W_{\text{nonisotropic}} \) and \( \beta \in \Delta^0 \cup \Delta^1 \), (see (2.1) for notation).

For \( \alpha \) a positive root, and then for \( \pi \) a partition, we define the Clifford degree of \( \alpha, \pi \) by

\[ \text{Cdeg}(\alpha) = 2 - i, \text{ for } \alpha \in \Delta^+_i, \quad \text{Cdeg}(\pi) = \sum_{\alpha \in \Delta^+_i} \pi(\alpha)\text{Cdeg}(\alpha). \]

The reason for this terminology is that if we set \( U_n = \text{span} \{ e_{-\pi} | \text{Cdeg}(\pi) \leq n \} \), then \( \{ U_n \}_{n \geq 0} \) is the Clifford filtration on \( U(n^-) \) as in [Mus12] Section 6.5. The associated graded ring \( \text{gr} U(n^-) = \bigoplus_{n \geq 0} U_n/U_{n-1} \) is isomorphic to a Clifford algebra. In addition \( \text{gr} U(n^-) \) is isomorphic to an enveloping algebra \( U(\mathfrak{k}) \) where the Lie superalgebra \( \mathfrak{k} \) is equal to \( n^- \) as a graded vector space, and the product is modified so that \( \mathfrak{g}_0 \) is central in \( \mathfrak{k} \).
Theorem 1.3. Suppose that $g$ is a finite dimensional contragredient, and that $\Pi$ contains an odd non-isotropic root. Assume $\gamma$ is a positive root such that (1.6) and (1.8) hold. If $\gamma$ is isotropic assume that $m = 1$. Then there exists a ˇSapovalov element $\theta_{\gamma,m} \in U(b^-)^{-m\gamma}$, which is unique modulo the left ideal $U(b^-)\mathcal{I}(\mathcal{H}_\gamma)$. If $m = 1$, the coefficients of $\theta_{\gamma}$ satisfy

$$2 \deg H_\pi \leq 2\ell(w) + 1 - \text{Cdeg}(\pi),$$

and

$$H_\pi \text{ has leading term } \prod_{\alpha \in N(w^{-1})} h_\alpha.$$ 

Corollary 1.4. In Theorems 1.2 and 1.3 $H_{m\pi,\gamma}$ is the unique term of highest degree in $\theta_{\gamma,m}$.

Proof. This follows easily from the given degree estimates.

In the next Section we discuss the uniqueness of ˇSapovalov elements. Theorems 1.2 and 1.3 are proved in Sections 4 and 5 respectively. The proofs depend on a rather subtle cancelation property which is illustrated in Section 6. In Section 7 we give a closed formula for ˇSapovalov elements in Type A. By definition ˇSapovalov elements give rise to highest weight vectors in a Verma modules. The question of when the images of these highest weight vectors in various factor modules is non-zero is studied in Section 8.

Some of our results hold without assumption (1.6) above. However it seems more interesting to compare ˇSapovalov elements for an arbitrary Borel to those obtained using the distinguished or anti-distinguished Borel subalgebra as we do in Section 9. This pair of reference Borels lie at two extremes. To explain what this means suppose $b, b'$ is an arbitrary pair of Borel subalgebras (always with the same even part). Then there is a sequence

$$b = b^{(0)}, b^{(1)}, \ldots, b^{(r)}.$$  

of Borel subalgebras such that $b^{(i-1)}$ and $b^{(i)}$ are adjacent for $1 \leq i \leq r$, and $b^{(r)} = b'$. If there is no chain of adjacent Borel subalgebras connecting $b$ and $b'$ of shorter length than (1.12), we set $d(b, b') = r$. Then $d(b, b')$ is maximal if and only if $(b, b')$ is our reference pair, and if $b''$ is any other Borel we can find a chain as in (1.12) with $b, b'$ distinguished and antidistinguished respectively, and $b = b^{(i)}$ for some $i$. In addition if $g$ is of Type A, B or D and $\gamma$ a root of the form $\delta_i - \epsilon_j$, then $\gamma$ (resp. $-\gamma$) is a positive root for $b$ (resp. $b'$) and we can arrange that $\gamma$ (resp. $-\gamma$) is a simple root for $b^{(i-1)}$ (resp. $b^{(i)}$).

Remark 1.5. It is interesting to compare the inequalities (1.9) and (1.11). If $\Pi$ does not contain an odd non-isotropic root, then $\text{Cdeg}(\pi) = 2|\pi|$ for all $\pi \in \overline{\mathcal{P}}(\gamma)$. Also it follows by induction and (1.3), that $\text{ht}\gamma = 1 + \sum_{\alpha \in N(w^{-1})} q(w, \alpha)$. It follows from (1.9) that

$$\text{Cdeg}(\pi) + 2 \deg H_\pi \leq 2 + 2 \sum_{\alpha \in N(w^{-1})} q(w, \alpha).$$
Thus if $q(w, \alpha) = 1$ for all $\alpha \in N(w^{-1})$ we have
\[
\text{Cdeg}(\pi) + 2 \deg H_\pi \leq 2 + 2\ell(w).
\]
Under the hypotheses of Theorem 1.3, (1.11) sharpens this bound. On the other hand if $g = \mathfrak{so}(5)$ with simple roots $\alpha = \epsilon_1 - \epsilon_2, \beta = \epsilon_2$, and $\gamma = s_\alpha(\beta)$, then the inequality (1.11) does not hold.

2 Uniqueness of Šapovalov elements.

The uniqueness of Šapovalov elements is easily taken care of, and so we do so here. To do this we need a version of the Jantzen sum formula, which will also play an important role in the sequel.

First set
\[
\Delta^+_0 = \{ \alpha \in \Delta^+_0 | \alpha/2 \notin \Delta^+_1 \}, \quad \Delta^+_1 = \{ \alpha \in \Delta^+_1 | 2\alpha \notin \Delta^+_0 \}.
\]
(2.1)
Then for $\lambda \in \mathfrak{h}^*$ define
\[
A(\lambda)_0 = \{ \alpha \in \Delta^+_0 | (\lambda + \rho, \alpha^\vee) \in \mathbb{N}\} \setminus \{0\},
\]
\[
A(\lambda)_1 = \{ \alpha \in \Delta^+_1 \setminus \Delta^+_1 | (\lambda + \rho, \alpha^\vee) \in 2\mathbb{N} + 1 \},
\]
\[
A(\lambda) = A(\lambda)_0 \cup A(\lambda)_1,
\]
and
\[
B(\lambda) = \{ \alpha \in \Delta^+_1 | (\lambda + \rho, \alpha^\vee) = 0 \}.
\]
If $\alpha \in \Delta^+_1$, let $p_\alpha(\eta)$ be the number of partitions $\pi$ of $\eta$ such that $\pi(\alpha) = 0$, and then let $p_\alpha$ be the generating function given by $p_\alpha = \sum p_\alpha(\eta)e^{-\eta}$.

Then by [Mus12] Theorem 10.3.1, the Jantzen filtration $\{M_i(\lambda)\}_{i \geq 1}$ on $M(\lambda)$ satisfies the sum formula
\[
\sum_{i > 0} \text{ch} M_i(\lambda) = \sum_{\alpha \in A(\lambda)} \text{ch} M(s_\alpha \cdot \lambda) + \sum_{\alpha \in B(\lambda)} e^{\lambda - \alpha} p_\alpha.
\]
(2.2)

Lemma 2.1. Suppose $\theta_1, \theta_2$ are Šapovalov elements for the pair $(\gamma, m)$. Then

(a) for all $\lambda \in \mathcal{H}_{\gamma,m}$ we have $\theta_1v_\lambda = \theta_2v_\lambda$

(b) $\theta_1 - \theta_2 \in U(\mathfrak{g})\mathfrak{n}^+ + U(\mathfrak{g})\mathcal{I}(\mathcal{H})$.

Proof. Set
\[
\Lambda = \{ \lambda \in \mathcal{H}_{\gamma,m} | A(\lambda) = \{ \gamma \}, \quad B(\lambda) = \emptyset \},
\]
if $\gamma$ is non-isotropic, and
\[
\Lambda = \{ \lambda \in \mathcal{H}_{\gamma} | B(\lambda) = \{ \gamma \}, \quad A(\lambda) = \emptyset \},
\]
if $\gamma$ is isotropic. If $\lambda \in \Lambda$ it follows from the sum formula that $M_1(\lambda)^{\lambda-m\gamma}$ is one-dimensional. Because $M_1(\lambda)$ is the unique maximal submodule of $M(\lambda)$, $\theta_1 v_\lambda$ and $\theta_2 v_\lambda$ are proportional. Then from the requirement that $e_{-\mu^0}$ occurs with coefficient 1 in a Šapovalov element we have $\theta_1 v_\lambda = \theta_2 v_\lambda$. Since $\Lambda$ is Zariski dense in $H_{\gamma,m}$, (a) holds and (b) follows from (a) because by [Mus12] Lemma 9.4.1 we have

$$\bigcap_{\lambda \in \Lambda} \text{ann}_{U(g)} v_\lambda = U(g)n^+ + U(g)\mathcal{I}(H).$$

(2.3)

We remark that this proof does not resolve the issue of whether (1.5) holds in general, but we note that the analog of (1.5) fails for parabolic Verma modules over simple Lie algebras, [IS88], [IS88b].

### 3 Outline of the Proof and Preliminary Lemmas.

Theorems 1.2 and 1.3 are proved by looking at the proofs given in [Hum08] or [Mus12] and keeping track of the coefficients. Given $\lambda \in \mathfrak{h}^*$ we define the specialization at $\lambda$ to be the map

$$\varepsilon^\lambda : U(b^-) = U(n^-) \otimes S(\mathfrak{h}) \rightarrow M(\lambda), \quad \sum_i a_i \otimes b_i \rightarrow \sum_i a_i b_i(\lambda)v_\lambda.$$

Let $(\gamma, m)$ be as in the statement of the Theorems and set $\mathcal{H} = \mathcal{H}_{\gamma,m}$. If $\theta$ is as in the conclusion of the Theorem, then for any $\lambda \in \mathcal{H}$, $\theta(\lambda)v_\lambda$ is a highest weight vector in $M(\lambda)^{\lambda-m\gamma}$. Conversely suppose that $\Lambda$ is a dense subset of $\mathcal{H}$ and that for all $\lambda \in \Lambda$ we have constructed $\theta^\lambda \in U(n)^{-m\gamma}$ such that $\theta^\lambda v_\lambda$ is a highest weight vector in $M(\lambda)^{\lambda-m\gamma}$ and that

$$\theta^\lambda = \sum_{\pi \in \mathcal{P}(m\gamma)} a_{\pi,\lambda} e_{-\pi}.$$

where $a_{\pi,\lambda}$ is a polynomial function of $\lambda \in \Lambda$ satisfying suitable conditions. For $\pi \in \mathcal{P}(m\gamma)$, the assignment $\lambda \rightarrow a_{\pi,\lambda}$ for $\lambda \in \Lambda$ determines a polynomial map from $\mathcal{H}$ to $U(n)^{-\gamma}$, so there exists an element $H_\pi \in U(\mathfrak{h})$ uniquely determined modulo $\mathcal{I}(H)$ such that $H_\pi(\lambda) = a_{\pi,\lambda}$ for all $\lambda \in \Lambda$. We define the element $\theta \in U(b^-)$ by setting

$$\theta = \sum_{\pi \in \mathcal{P}(m\gamma)} e_{-\pi} H_\pi.$$

Note that $\theta$ is uniquely determined modulo the left ideal $U(b^-)\mathcal{I}(H)$, and that $\theta(\lambda) = \theta^\lambda$. Also, for $\alpha \in \Delta^+$ and $\lambda \in \Lambda$ we have $e_\alpha \theta v_\lambda = e_\alpha \theta^\lambda v_\lambda = 0$, because $\theta^\lambda v_\lambda = 0$ is a highest weight vector, so $e_\alpha \theta \in \bigcap_{\lambda \in \Lambda} \text{ann}_{U(g)} v_\lambda$. Thus (1.4) follows from (2.3).

We need to examine the polynomial nature of the coefficients of $\theta_{\gamma,m}$. The following easy observation (see [Dix96] Lemma 7.6.9), is the key to doing this. Let
A be a $\mathbb{Z}_2$-graded associative algebra, and suppose that $e$ is an even element of $A$. Then for $a \in A$ and all $r \in \mathbb{N}$,

$$e^r a = \sum_{i=0}^{r} \binom{r}{i} ((\text{ad} e)^i a) e^{r-i}. \quad (3.1)$$

The following consequence is well-known, [BR75]. We give the short proof for completeness.

**Corollary 3.1.** With the same hypothesis as above, suppose that $\text{ad} e$ is locally nilpotent. Then the set $\{ e^n | n \in \mathbb{N} \}$ is an Ore set in $A$.

**Proof.** Given $a \in A$ and $n \in \mathbb{N}$, suppose that $(\text{ad} e)^{k+1} a = 0$. Then $e^{k+n} a = a' e^n$, where

$$a' = \sum_{i=0}^{k} \binom{n+k}{i} ((\text{ad} e)^i a) e^{k-i}. \quad (3.2)$$

Now suppose $\alpha \in \Pi_{\text{nonisotropic}}$, and set $e = e_{-\alpha}$. Then $e$ is a nonzero divisor in $U = U(\mathfrak{n}^-)$, and the set $\{ e^n | n \in \mathbb{N} \}$ is an Ore set in $U$ by Corollary 3.1. We write $U_e$ for the corresponding Ore localization. The adjoint action of $\mathfrak{h}$ on $U$ extends to $U_e$, and in the next result we give a basis for the weight spaces of $U_e$. Let $\hat{P}(\eta)$ be the set of pairs $(k, \pi)$ such that $k \in \mathbb{Z}$, $\pi \in \mathfrak{p}(\eta - k\alpha)$ and $\pi(\alpha) = 0$. Then we have

**Lemma 3.2.**

(a) The set $\{ e^{-\pi} e^k | (k, \pi) \in \hat{P}(\eta) \}$ forms a $\mathbb{k}$-basis for the weight space $U^{-\eta}_e$.

(b) If $u = \sum_{(k, \pi) \in \hat{P}(\eta)} c_{(k, \pi)} e^{-\pi} e^k \in U^{-\eta}_e$ with $c_{(k, \pi)} \in \mathbb{k}$, then $u \in U$ if and only if $c_{(k, \pi)} \neq 0$ implies $k \geq 0$.

**Proof.** (a) Suppose $u \in U^{-\eta}_e$. We need to show that $u$ is uniquely expressible in the form

$$u = \sum_{(k, \pi) \in \hat{P}(\eta)} c_{(k, \pi)} e^{-\pi} e^k \quad (3.2)$$

We have $ue^N \in U^{- (N\alpha + \eta)}$ for some $N$. Hence by the PBW Theorem for $U$ we have a unique expression

$$ue^N = \sum_{\sigma \in \hat{P}(\eta + N\alpha)} a_{\sigma} e_{-\sigma}. \quad (3.3)$$

Now if $a_{\sigma} \neq 0$, then $e_{-\sigma} = e_{-\pi} e^\ell$ where $\sigma(\alpha) = \ell$ and where $\pi \in \hat{P}(\eta + N\alpha - \ell\alpha)$ satisfies $\pi(\alpha) = 0$. Then $\pi$ and $k = N - \ell$ are uniquely determined by $\sigma$, so we set $c_{(k, \pi)} = a_{\sigma}$. Then clearly (3.2) holds. Given (a), (b) follows from the PBW Theorem. \qed
We remark that if $\alpha$ is a non-isotropic odd root, then we can use $e^2$ in place of $e = e_{-\alpha}$ in the above Corollary and Lemma. However we will need a version of Equation (3.1) when $e$ is replaced by an odd element $x$ of a $\mathbb{Z}_2$-graded algebra $A$. Suppose that $z$ is homogeneous, and define $z^{[i]} = (ad x)z$. Set $e = x^2$ and apply (3.1) to $a = xz = [x, z] + (-1)^{\ell}z$, to obtain

$$x^{2\ell+1}z = \sum_{i=0}^{\ell} \binom{\ell}{i} z^{[2i+1]}x^{2(\ell-i)} + (-1)^{\ell} \sum_{i=0}^{\ell} \binom{\ell}{i} z^{[2i]}x^{2\ell-2i+1}. \quad (3.3)$$

The Šapovalov elements in Theorems 1.2 and 1.3 are constructed inductively using the next Lemma. Suppose that the pair $(\gamma, m)$ satisfies one of the following

- $m$ is an odd positive integer if $\gamma$ is an odd non-isotropic root, \hspace{1cm} (3.4)
- $m$ is a positive integer if $\gamma$ is an even root such that $\gamma/2$ is not a root. \hspace{1cm} (3.5)
- $\gamma$ is an odd isotropic root and $m = 1$. \hspace{1cm} (3.6)

**Lemma 3.3.** Suppose that $\alpha \in \Pi_{\text{nonisotropic}}$, and set

$$\mu = s_\alpha \cdot \lambda, \quad \gamma' = s_\alpha \gamma, \quad p = (\mu + \rho, \alpha^\vee), \quad q = (\gamma, \alpha^\vee).$$

Assume that $q \in \mathbb{N}\setminus\{0\}$, the pair $(\alpha, m)$ satisfies (3.4) or (3.5), and

(a) If (3.4) holds, then $q = 2$ and $p$ is odd.
(b) $\theta' \in U(n^-)^{-mq}$ is such that $v = \theta'v_\mu \in M(\mu)$ is a highest weight vector.

Then there is a unique $\theta \in U(n^-)^{-mq}$ such that

$$e^{p+mq}\theta' = \theta e^p. \quad (3.7)$$

**Proof.** This is well-known, see for example [Hum08] Section 4.13 or [Mus12] Theorem 9.4.3. \hfill \Box

In the proofs of Theorems 1.2 and 1.3 we write $\gamma = w\beta$ for $\beta \in \Pi$ and $w \in W$. We use the Zariski dense subset $\Lambda$ of $\mathcal{H}_{\beta, m}$ defined by

$$\Lambda = \left\{ \nu \in \mathfrak{h}^* \middle| (\nu + \rho, \beta^\vee) = m \text{ and } (\nu + \rho, \alpha^\vee) \in \mathbb{N}\setminus\{0\} \text{ for all } \alpha \in \Pi, \text{ with } (\nu + \rho, \alpha^\vee) \text{ odd if } \alpha \text{ is odd} \right\}. \quad (3.8)$$

**4 Proof of Theorem 1.2.**

In this section we assume $g$ is Contragredient and hypotheses (1.6) and (1.7) hold. If $\gamma$ is a simple root, then $\theta_{\gamma, m} = e_{-\gamma}^m$ satisfies the conditions of Theorem 1.2. Otherwise we have $\gamma = w\beta$ for some $w \in W_{\text{even}}, w \neq 1$. Write

$$w = s_\alpha u, \quad \gamma' = u\beta, \quad \gamma = w\beta = s_\alpha \gamma', \quad \ell(w) = \ell(u) + 1.$$ \hspace{1cm} (4.1)

with $\alpha \in \Pi_{\text{even}}$ and $\ell(w) = \ell(u) + 1$. Since the statement of Theorem 1.2 involves precise but somewhat lengthy conditions on the coefficients, we introduce the following definition as a shorthand.
Definition 4.1. We say that a family of elements \( \theta_{\gamma,m}^\lambda \in U(n)^{-m\gamma} \) is well posed for \( w \) if for all \( \lambda \in w \cdot \Lambda \) we have

\[
\theta_{\gamma,m}^\lambda = \sum_{\pi \in \Phi(m\gamma)} a_{\pi,\lambda} e_{-\pi},
\]

(4.2)

where the coefficients \( a_{\pi,\lambda} \in k \) depend polynomially on \( \lambda \in w \cdot \Lambda \), and

(a) \( \deg a_{\pi,\lambda} \leq m\text{ht}\gamma - |\pi| \)

(b) \( a_{m\pi,\lambda} \) is a polynomial function of \( \lambda \) of degree \( m(\text{ht}\gamma - 1) \) with highest term equal to \( c \prod_{\alpha \in \Delta^+} (\lambda, \alpha)^{mq(w,\alpha)} \) for a nonzero constant \( c \).

We show that the conditions on the coefficients in this definition are independent of the ordering on the positive roots \( \Delta^+ \) used to define the \( e_{-\pi} \). Consider two orderings on \( \Delta^+ \), and for \( \pi \in \Phi(m\gamma) \), set \( e_{-\pi} = \prod_{\alpha \in \Delta^+} e_{-\alpha}^{\pi(\alpha)} \) and \( \overline{e}_{-\pi} = \prod_{\alpha \in \Delta^+} \overline{e}_{-\alpha} \), the product being taken with respect to the given orderings.

Lemma 4.2. Fix a total order on the set \( \Phi(m\gamma) \) such that if \( \pi, \sigma \in \Phi(m\gamma) \) and \( |\pi| > |\sigma| \) then \( \pi \) precedes \( \sigma \), and use this order on partitions to induce orders on the bases \( B_1 = \{e_{-\pi} | \pi \in \Phi(m\gamma)\} \) and \( B_2 = \{\overline{e}_{-\pi} | \pi \in \Phi(m\gamma)\} \) for \( U(n)^{-m\gamma} \). Then the change of basis matrix from the basis \( B_1 \) to \( B_2 \) is upper triangular with all diagonal entries equal to 1.

Proof. Let \( \{U_n = U_n(n^-)\} \) be the standard filtration on \( U = U(n^-) \). Note that if \( \pi \in \Phi(m\gamma) \), then \( e_{-\pi}, \overline{e}_{-\pi} \in U_{|\pi|}(n^-)^{-m\gamma} \). Also the factors of \( e_{-\pi} \) commute modulo lower degree terms, so for all \( \pi \in \Phi(m\gamma) \), \( e_{-\pi} - \overline{e}_{-\pi} \in U_{|\pi| - 1}(n^-)^{-m\gamma} \). The result follows easily. \( \square \)

Lemma 4.3. For \( x \in U(n)^{-m\gamma} \otimes S(h) \), write

\[
x = \sum_{\pi \in \Phi(m\gamma)} e_{-\pi} f_{\pi} = \sum_{\pi \in \Phi(m\gamma)} \overline{e}_{-\pi} g_{\pi}.
\]

(4.3)

Suppose that \( f_{m\pi,\gamma} \) has degree \( m(\text{ht}\gamma - 1) \), and that for all \( \pi \in \Phi(m\gamma) \), we have \( \deg f_{\pi} \leq m\text{ht}\gamma - |\pi| \). Then \( g_{m\pi,\gamma} \) has the same degree and leading term as \( f_{m\pi,\gamma} \), and for all \( \pi \in \Phi(m\gamma) \), we have \( \deg g_{\pi} \leq m\text{ht}\gamma - |\pi| \).

Proof. By Lemma 4.2 we can write

\[
e_{-\pi} = \sum_{\zeta \in \Phi(m\gamma)} c_{\pi,\zeta} \overline{e}_{-\zeta},
\]

where \( c_{\pi,\zeta} \in k, c_{\pi,\pi} = 1 \) and if \( c_{\pi,\zeta} \neq 0 \) with \( \zeta \neq \pi \), then \( |\zeta| < |\pi| \). Thus (4.3) holds with

\[
g_{\zeta} = \sum_{\pi \in \Phi(m\gamma)} c_{\pi,\zeta} f_{\pi}.
\]
It follows that \( g_\zeta \) is a linear combination of polynomials of degree less than \( m \text{ht} \gamma - |\zeta| \). Also \( |m\pi_\gamma| = m \), and for \( \zeta \in \overline{P}(m\gamma), \zeta \neq m\pi_\gamma \), we have \( |\zeta| > m \). Therefore

\[
g_{m\pi_\gamma} = f_{m\pi_\gamma} + \text{a linear combination of polynomials of smaller degree.}
\]

The result follows easily from this.

Now recall the notation from Equation (4.1). Suppose \( \nu \in \Lambda \) and set

\[
\mu = u \cdot \nu, \quad \lambda = w \cdot \nu = s_\alpha \cdot \mu.
\]

The next Lemma is the key step in establishing the degree estimates in the proof of Theorem 1.2. The idea is to use Equation (3.7) and the fact that \( \theta \in U(\mathfrak{n}^-) \), rather than a localization of \( U(\mathfrak{n}^-) \), to show that certain coefficients cancel. Then using induction and (3.7) we obtain the required degree estimates. Since the proof of the Lemma is rather long we break it into a number of steps.

**Lemma 4.4.** Suppose that \( p, m, q \) are as in Lemma 3.3, \( \alpha \in \Pi_{\text{even}} \) and

\[
ed^{p+mq}_\alpha \theta^\mu_{\gamma',m} = \theta^\lambda_{\gamma,m} e^p_{\alpha}.
\]

Then the family \( \theta^\lambda_{\gamma,m} \) is well posed for \( w \) if the family \( \theta^\mu_{\gamma',m} \) is well posed for \( u \).

**Proof.** Step 1. Setting the stage.

Suppose that

\[
\theta^\mu_{\gamma',m} = \sum_{\pi' \in \overline{P}(m\gamma')} a'_{\pi',\mu} e_{-\pi'},
\]

and let

\[
e^{(j)}_{\pi'} = (\text{ad} e_{-\alpha})^j e_{-\pi'} \in U_{|\pi'|}(\mathfrak{n}^-)^{-m\gamma' + j\alpha},
\]

for all \( j \geq 0 \), and \( \pi' \in \overline{P}(m\gamma') \). Then by Equation (3.1)

\[
ed^{p+mq}_{-\alpha} e_{-\pi'} = \sum_{i \geq 0} \binom{p+mq}{j} e^{(j)}_{\pi'} e^{p+mq-j}_{-\alpha}.
\]

Choose \( N \) so that \( e^{(N+1)}_{\pi'} = 0 \), for all \( \pi' \in \overline{P}(m\gamma') \). Then for all such \( \pi' \) and \( j = 0, \ldots, N \) we can write

\[
e^{(j)}_{\pi'} e^{-N-j}_{-\alpha} = \sum_{\zeta \in \overline{P}(m\gamma' + N\alpha)} b^{\pi'}_{j,\zeta} e_{-\zeta},
\]

with \( b^{\pi'}_{j,\zeta} \in k \). Furthermore if \( b^{\pi'}_{j,\zeta} \neq 0 \), then since \( e^{(j)}_{\pi'} e^{-N-j}_{-\alpha} \in U_{|\pi'| + N - j} \), (4.7) gives

\[
|\zeta| \leq |\pi'| + N - j.
\]
Step 2. The cancelation step.

By Equations (4.6) and (4.8)
\[
e^{-\alpha}\theta_{\gamma',m}^{\mu} = \sum_{\pi' \in \mathcal{P}(m\gamma')} a'_{\pi',\mu} e^{-\alpha} \theta_{\pi',m}^{\mu} \tag{4.11}
\]
\[
= \sum_{\pi' \in \mathcal{P}(m\gamma'), j \geq 0} \left( \frac{p + mq}{j} \right) a'_{\pi',\mu} e^{-\alpha} \theta_{\pi',m}^{\mu} - \pi'(j). \tag{4.12}
\]

Now collecting coefficients, set
\[
c_{\zeta,\lambda} = \sum_{\pi' \in \mathcal{P}(m\gamma'), j \geq 0} \left( \frac{p + mq}{j} \right) a'_{\pi',\mu} b_{j,\zeta}^{\pi'}. \tag{4.13}
\]

Then using Equations (4.9) and (4.11), we have in $U_e$, where $e = e^{-\alpha},$
\[
e^{-\alpha}\theta_{\gamma',m}^{\mu} = \sum_{\zeta \in \mathcal{P}(m\gamma' + N\alpha)} c_{\zeta,\lambda} e^{-\zeta} e^{p + mq - N}. \tag{4.14}
\]

By (4.5) and Lemma 3.2, $c_{\zeta,\lambda} = 0$ unless $\zeta(\sigma) \geq N - mq$.

Step 3. The coefficients $a_{\pi,\lambda}$.

It remains to deal with the nonzero terms $c_{\zeta,\lambda}$. There is a bijection
\[
f : \mathcal{P}(m\gamma) \rightarrow \{ \zeta \in \mathcal{P}(m\gamma' + N\alpha) \mid \zeta(\alpha) \geq N - mq \},
\]
defined by
\[
(f\pi)(\sigma) = \begin{cases} 
\pi(\sigma) & \text{if } \sigma \neq \alpha, \\
\pi(\alpha) + N - mq & \text{if } \sigma = \alpha.
\end{cases} \tag{4.15}
\]

Moreover if $f\pi = \zeta$, then
\[
|\zeta| = |\pi| + N - mq
\]
and $e_{-\pi} = e_{-\zeta} e^{mq - N}$. Thus in Equation (4.5) the coefficients $a_{\pi,\lambda}$ of $\theta_{\gamma,m}^{\lambda}$ (see (4.2)) are given by
\[
a_{\pi,\lambda} = c_{f(\pi),\lambda}. \tag{4.16}
\]

Step 4. Completion of the proof.

We now show that the family $\theta_{\gamma,m}^{\lambda}$ is well posed for $w$. For this we use Equations (4.12) and (4.16), noting that $p = (s_\alpha \cdot \lambda + \rho, \alpha)$ depends linearly on $\lambda$. It is clear that the coefficients $a_{\pi,\lambda}$ are polynomials in $\lambda$. By induction $\deg a'_{\pi',\mu} \leq m h \gamma' - |\pi'|$. Thus using (4.12),
\[
\deg a_{\pi,\lambda} = \deg c_{\zeta,\lambda} \leq \max\{ j + \deg a'_{\pi',\mu} \mid b_{j,\zeta}^{\pi'} \neq 0 \}. \tag{4.17}
\]
Now if $b_{\lambda,\mu}^{\gamma} \neq 0$ then Equation (4.10) holds. Therefore by Equation (4.15)

$$\deg a_{\pi,\lambda} \leq \deg a_{\pi',\mu}^{\gamma} + \lvert \pi' \rvert + N - \lvert \zeta \rvert = \deg a_{\pi',\mu}^{\gamma} + \lvert \pi' \rvert - \lvert \pi \rvert + mq$$

Finally since $\gamma = \gamma' + q\alpha$, induction gives (a) in Definition 4.1.

Also, modulo terms of lower degree

$$a_{m\pi,\lambda} = \left( \frac{p + mq}{mq} \right) a_{m\pi,\mu}^{\gamma}.$$  \hspace{1cm} (4.18)

Note that the above binomial coefficient is a polynomial of degree $mq$ in $p$. By induction $a_{m\pi,\mu}^{\gamma}$ has highest term $c' \prod_{\tau \in N(\mu - 1)} (\mu, \tau)^{iq(u,\tau)}$ as a polynomial in $\mu$, for a nonzero constant $c'$. Now $(\mu + \rho, \tau) = (\lambda + \rho, s_{\alpha}\tau)$ and $(\mu + \rho, \tau) - (\mu, \tau)$, $(\lambda + \rho, s_{\alpha}\tau) - (\lambda, s_{\alpha}\tau)$ are constant. Therefore as a polynomial in $a_{m\pi,\mu}^{\gamma}$ has highest term

$$c' \prod_{\tau \in N(\mu - 1)} (\lambda, s_{\alpha}\tau)^{mq(u,\tau)} = c' \prod_{\tau \in N(\mu - 1)} (\lambda, s_{\alpha}\tau)^{mq(u,\tau)}.$$  \hspace{1cm} (4.19)

From the representation theory of $sl(2)$, it follows that $(ad e_{-\alpha})^q(g^{-\gamma}) = g^{-\gamma}$. Since $e_{-\gamma}$ is not used in the construction of $\theta_{\gamma,m}$, we can choose the notation so that $e_{-\gamma}^{mq} = e_{-\gamma}^{mq+1}$. Then $e_{-\gamma}^{(mq+1)} = 0$. Now $q(w, \alpha) = (\gamma, \alpha^\vee) = q$, and the degree of the binomial coefficient in (4.18) as a polynomial in $p$ is $mq$, so the claim about the leading term of $a_{m\pi,\lambda}$ in Definition 4.1 (b) follows from Equations (4.18) and (1.3).

**Theorem 4.5.** Suppose $\gamma = w\beta$ with $w \in W_{\text{even}}$ and $\beta$ simple. There exists a family of elements $\theta_{\gamma,m}^\lambda \in U(n^-)^{\mu-\gamma}$ for all $\lambda \in w \cdot \Lambda$ which is well posed for $w$ such that

$$\theta_{\gamma,m}^\lambda v_\lambda$$

is a highest weight vector in $M(\lambda)^{\mu-\gamma}$.

**Proof.** We use induction on the length of $w$. If $w = 1$, we take $\theta_{\gamma,m}^\lambda = e_{-\beta}^m$ for all $\lambda$. Now assume that $w \neq 1$, and use the notation of Equations (4.1) and (4.4). If $\lambda = s_{\alpha} \cdot \mu \in \Lambda$, then it is well known that $M(\lambda)$ is uniquely embedded in $M(\mu)$. Set

$$p = (\mu + \rho, \alpha^\vee) = (\nu + \rho, u^{-1} \alpha^\vee), \quad (\gamma, \alpha^\vee) = q.$$  \hspace{1cm} (4.20)

Then $p$ and $q$ are positive integers. Also $\lambda = \mu - p\alpha$ and $\gamma = \gamma' + q\alpha$. By induction there exist elements $\theta_{\gamma',m}^{\mu} \in U(n^-)^{-\gamma'}$ which are well posed for $u$ such that

$$v = \theta_{\gamma',m}^{\mu} v_\mu \in M(\mu)^{\mu-\gamma'}$$

is a highest weight vector.

By Lemma 3.3 there exists a unique element $\theta_{\gamma,m}^\lambda \in U(n^-)^{-\gamma}$ such that (4.5) holds and therefore

$$e_{-\alpha}^{p+mq} v = \theta_{\gamma,m}^\lambda e_{-\alpha} v_\mu \in U(n^-) e_{-\alpha} v_\mu = M(\lambda).$$

It follows from Lemma 4.4 that the family $\theta_{\gamma,m}^\lambda$ is well posed for $w$. \hspace{1cm} \Box
Proof of Theorem 1.2. Let \( \theta_{\gamma,m}^\lambda \) be the family of elements from Theorem 4.5. The existence of the elements

\[
\theta_{\gamma,m} = \sum_{\pi \in \mathfrak{P}(m\gamma)} e_{-\pi} H_{\pi} \in U(b^-),
\]

such that \( \theta_{\gamma,m}(\lambda) = \theta_{\gamma,m}^\lambda \) for all \( \lambda \in \Lambda \) follows since \( w \cdot \Lambda \) is Zariski dense in \( \mathcal{H}_{\gamma,m} \). The claims about the coefficients \( H_{\pi} \) follow from the fact that the family \( \theta_{\gamma,m}^\lambda \) is well posed for \( w \). □

5 Proof of Theorem 1.3.

Now suppose that \( g \) is contragredient, with \( \Pi \) as in (1.6). We assume that \( \Pi \) contains an odd non-isotropic root and \( \gamma = w\beta \) with \( w \in W_{\text{nonisotropic}} \), \( \beta \in \Xi_0^+ \cup \Xi_1^+ \). Our assumptions have the following consequence.

Lemma 5.1. Suppose that \( \gamma \) is an isotropic root and \( \alpha \in \Pi_{\text{nonisotropic}} \) is such that \( (\gamma,\alpha) \neq 0 \). Then

(a) If \( \alpha \) is even then \( (\gamma,\alpha^\vee) = \pm 1 \).

(b) If \( \alpha \) is odd then \( (\gamma,\alpha^\vee) = \pm 2 \).

Proof. Left to the reader. □

In this section \( \{U_n = U_n(n^-)\} \) is the Clifford filtration on \( U = U(n^-) \).

Lemma 5.2. The Clifford filtration on \( U(n^-) \) stable under the adjoint action of \( n_0^- \), and satisfies \( \text{ad } n_0^-(U_n) \subseteq U_{n+1} \).

Proof. Left to the reader. □

Fix a total order on the set \( \overline{\mathfrak{P}}(\gamma) \) such that if \( \pi, \sigma \in \overline{\mathfrak{P}}(\gamma) \) and \( |\pi| > |\sigma| \), or if \( |\pi| = |\sigma| \) and \( \text{Cdeg}(\pi) > \text{Cdeg}(\sigma) \), then \( \pi \) precedes \( \sigma \), and use this order to induce orders on the bases \( B_1 = \{e_{-\pi} | \pi \in \overline{\mathfrak{P}}(\gamma)\} \) and \( B_2 = \{e_{-\pi} | \pi \in \overline{\mathfrak{P}}(\gamma)\} \) for \( U(n^-)^{-\gamma} \). Consider two orderings on \( \Delta^+ \), and for \( \pi \in \overline{\mathfrak{P}}(\gamma) \), set \( e_{-\pi} = \prod_{\alpha \in \Delta^+} e_{-\pi(\alpha)} \), and \( e_{-\pi} = \prod_{\alpha \in \Delta^+} e_{-\alpha} \), the product being taken with respect to the given orderings.

Next we prove a Lemma relating \( |\pi| \) to \( \text{Cdeg}(\pi) \). Set

\[
\Xi = \Delta_0^+ \setminus \overline{\Delta}_0^+, \quad \text{and } a(\pi) = \sum_{2\delta \in \Xi} \pi(\delta).
\]

Lemma 5.3. Suppose \( \pi, \sigma \in \overline{\mathfrak{P}}(\gamma) \).

(a) We have \( 2|\pi| - \text{Cdeg}(\pi) = a(\pi) \).

(b) \( a(\pi) \leq 2 \).
(c) If \( \sigma \) precedes \( \pi \), then \( \text{Cdeg}(\pi) \leq \text{Cdeg}(\sigma) \).

\textbf{Proof.} (a) follows since
\[
|\pi| = a(\pi) + \sum_{\alpha \in \Delta^+} \pi(\alpha),
\]
and
\[
\text{Cdeg}(\pi) = a(\pi) + 2 \sum_{\alpha \in \Delta^0} \pi(\alpha).
\]

For (b) we note that the Lie superalgebras that have an odd non-isotropic root \( \delta \) are \( G(3) \) and the family \( \text{osp}(2m + 1, 2n) \). Define a group homomorphism \( f : \bigoplus_{\alpha \in \Pi} \mathbb{Z}\alpha \rightarrow \mathbb{Z} \) by setting \( f(\delta) = 1 \) and \( f(\alpha) = 0 \) for any \( \alpha \in \Pi, \alpha \neq \delta \). It can be checked on a case-by-case basis that if \( \delta \in \Pi \), then \( \delta \) occurs with coefficient at most two when a positive root \( \gamma \) is written as a linear combination of simple roots. Since \( a(\pi) = f(\gamma) \) for \( \pi \in \overline{\Pi}(\gamma) \), (b) follows. If (c) is false, then by definition of the order, we must have \( |\pi| < |\sigma| \) and \( \text{Cdeg}(\pi) > \text{Cdeg}(\sigma) \). But then by (a) this implies that
\[
a(\sigma) = 2|\sigma| - \text{Cdeg}(\sigma) \geq 2|\pi| - \text{Cdeg}(\pi) + 3 = a(\pi) + 3 \geq 3
\]
which contradicts (b). \qed

\textbf{Definition 5.4.} We say that a family of elements \( \theta^\lambda_\gamma \in U(n^-)^-\gamma \) is well posed for \( w \) if for all \( \lambda \in w \cdot \Lambda \) we have
\[
\theta^\lambda_\gamma = \sum_{\pi \in \overline{\Pi}(\gamma)} a_{\pi, \lambda} e_{-\pi},
\]
where the coefficients \( a_{\pi, \lambda} \in k \) depend polynomially on \( \lambda \in w \cdot \Lambda \), and

(a) \( 2 \deg a_{\pi, \lambda} \leq 2\ell(w) + 1 - \text{Cdeg}(\pi) \)

(b) \( a_{\pi, \lambda} \) is a polynomial function of \( \lambda \) of degree \( \ell(w) \) with highest term equal to \( c \prod_{\alpha \in N(w^{-1})(\lambda, \alpha)^{g(w, \alpha)}} \) for a nonzero constant \( c \).

\textbf{Lemma 5.5.} Write \( x \in U(n^-)^-\gamma \otimes S(\mathfrak{h}) \),
\[
x = \sum_{\pi \in \overline{\Pi}(m \gamma)} e_{-\pi} f_{\pi} = \sum_{\pi \in \overline{\Pi}(m \gamma)} \overline{e}_{-\pi} g_{\pi},
\]
as in Equation (4.3). If the coefficients \( f_{\pi} \), satisfy

(a) \( 2 \deg f_{\pi} \leq 2\ell(w) + 1 - \text{Cdeg}(\pi) \)

(b) \( f_{\pi, \gamma} \) is a polynomial of degree \( \ell(w) \) with highest term equal to \( c \prod_{\alpha \in N(w^{-1})} h^{(w, \alpha)}_{\alpha} \) for a nonzero constant \( c \).

then the coefficients \( g_{\pi} \) satisfy the same conditions.
Proof. Taking Lemma 5.3 into account, this is the same as the proof of Lemma 4.3. ■

**Lemma 5.6.** Suppose that $p, q$ and $\alpha$ are as in Lemma 3.3 and

$$e_{-\alpha}^{p+q} \theta_\gamma^\mu = \theta_\gamma^\lambda e_{-\alpha}^p. \tag{5.2}$$

Then the family $\theta_\gamma^\lambda$ is well posed for $w$ if the family $\theta_\gamma^\mu$ is well posed for $u$.

Proof. If $\alpha$ is odd, then $p = 2\ell - 1$ is odd by (3.4), and $q = 2$ by Lemma 5.1. Write $\theta_\gamma^\mu$ as in (4.6) and then define the $e_{-\pi'}^{(j)}$ as in (4.7). Set $\varepsilon(\gamma') = 1$ if $\gamma'$ is an even root and $\varepsilon(\gamma') = -1$ if $\gamma'$ is odd. Then instead of (4.8) we have, by (3.3)

$$e_{-\alpha}^{2\ell+1} e_{-\pi'} = \sum_{i=0}^{\ell} \binom{\ell}{i} e_{-\pi'}^{(2i+1)} e_{-\alpha}^{2(\ell-i)} + \varepsilon(\gamma') \sum_{i=0}^{\ell} \binom{\ell}{i} e_{-\pi'}^{2\ell-2i+1}. \tag{5.3}$$

Parallel to the definition of the $b_{\gamma',\lambda}$ in (4.9), we set for sufficiently large $N$

$$e_{-\pi'}^{(j)} e_{-\alpha}^{N-j} = \sum_{\zeta \in \mathcal{P}(\gamma'+NA)} b_{\gamma',\lambda}^{N-j} e_{-\zeta}. \tag{5.4}$$

For $x \in \mathbb{R}$ we denote the largest integer not greater than $x$ by $\lfloor x \rfloor$. Then if $b_{\gamma',\lambda}^{N-j} \neq 0$, we have

$$Cdeg(\zeta) \leq Cdeg(\pi') + N - 2 \left\lfloor \frac{j}{2} \right\rfloor. \tag{5.5}$$

Indeed this holds because by Lemma 5.2, we have for such $j$

$$e_{-\pi'}^{(j)} e_{-\alpha}^{N-j} \in U_{Cdeg(\pi')+N-2\left\lfloor \frac{j}{2} \right\rfloor}. \tag{5.6}$$

Replacing (4.12) we set,

$$c_{\gamma,\lambda} = \sum_{\pi' \in \mathcal{F}(\gamma'), i \geq 0} \binom{\ell}{i} a_{\pi',\mu} b_{\gamma'+1,\pi'}^{N-j} e_{-\alpha}^{\varepsilon(\gamma')} + \varepsilon(\gamma') \sum_{\pi' \in \mathcal{F}(\gamma'), i \geq 0} \binom{\ell}{i} a_{\pi',\mu} b_{\gamma'+1,\pi'}^{N-j} e_{-\alpha}^{\varepsilon(\gamma')} \tag{5.7}$$

Then we obtain the following variant of Equation (4.13)

$$e_{-\alpha}^{2\ell+1} \theta_\gamma^\mu = \sum_{\zeta \in \mathcal{F}(\gamma'+NA)} c_{\gamma,\lambda} e_{-\zeta} e_{-\alpha}^{2\ell+1-N}. \tag{5.8}$$

In the cancelation step we find that $c_{\gamma,\lambda} = 0$ unless $\zeta(\alpha) \geq N - 2$, and the bijection

$$f : \mathcal{F}(\gamma) \longrightarrow \{ \zeta \in \mathcal{F}(\gamma'+NA) | \zeta(\alpha) \geq N - 2 \},$$

is defined as in Equation (4.14) with $m = 1$ and $q = 2$. Then the coefficients $a_{\pi,\lambda}$ are defined as in (4.16). Instead of Equations (5.15) and (4.17) we have

$$Cdeg(\zeta) = Cdeg(\pi') + N - 2, \tag{5.9}$$

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and
\[ \deg a_{\pi,\lambda} \leq \max\{\lfloor j/2 \rfloor + \deg a'_{\pi',\mu} \mid b'_{j,\zeta} \neq 0\}. \]  \hfill (5.9)

Hence using (5.4) in place of (4.10), and then (5.8) we obtain,
\[ 2\deg a_{\pi,\lambda} \leq 2\deg a'_{\pi',\mu} + \text{Cdeg}(\pi') + N - \text{Cdeg}(\zeta) \]  \hfill (5.10)
\[ = 2\deg a'_{\pi',\mu} + \text{Cdeg}(\pi') - \text{Cdeg}(\pi) + 2. \]

Therefore by induction
\[ 2\deg a_{\pi,\lambda} \leq 2\ell(u) + 1 - \text{Cdeg}(\pi) + 2 \]  \hfill (5.10)
\[ = 2\ell(w) + 1 - \text{Cdeg}(\pi). \]

giving condition (a) in Definition 5.4.

The proof in the case where \( \alpha \) is an even root is the same as in Section 4 apart from the inequalities. If \( b'_{j,\zeta} \neq 0 \), then instead of (4.10), we have
\[ \text{Cdeg}(\zeta) \leq \text{Cdeg}(\pi') + 2(N - j). \]  \hfill (5.11)

Now condition (a) follows since in place of (4.15) we have, using \( m = q = 1 \),
\[ \text{Cdeg}(\zeta) = \text{Cdeg}(\pi) + 2(N - 1). \]  \hfill (5.12)

We leave the proof that (b) holds in Definition 5.4 to the reader. \( \square \)

6 An (ortho) symplectic example.

Example 6.1. A crucial step in the construction of Šapovalov elements was the observation in the proofs of Lemmas 4.4 and 5.6 that the term \( c_{\zeta,\lambda} \) defined in Equation (4.12) are zero unless \( \zeta(\alpha) \geq N - mq \), (using the notation of the Lemmas). We give an example where the individual terms on the right of Equation (4.12) are not identically zero, and verify directly that the sum itself is zero. This cannot happen in Type A. The key difference in the examples below seems to be that it is necessary to apply Equation (3.7) more than once with the same simple root \( \alpha \). Consider the Dynkin-Kac diagram below for the Lie superalgebra \( g = \mathfrak{osp}(2, 4) \).

\[ \begin{array}{c}
\otimes \\
\uparrow \\
\epsilon - \delta_1 \\
\downarrow \\
\delta_1 - \delta_2 \\
\downarrow \\
2\delta_2 \\
\end{array} \]

Let \( \beta = \epsilon - \delta_1, \alpha_1 = \delta_1 - \delta_2, \alpha_2 = 2\delta_2 \), be the corresponding simple roots. If we change the grey node to a white node we obtain the Dynkin diagram for \( \mathfrak{sp}(6) \). In this case the simple roots are \( \beta = \delta_0 - \delta_1, \alpha_1 = \delta_1 - \delta_2 \) and \( \alpha_2 = 2\delta_2 \). Let \( e_{-\beta}, e_{-\alpha_1}, e_{-\alpha_2} \) be the negative simple root vectors. The computation of the Šapovalov elements \( \theta_1, \theta_2, \theta_3 \) for the roots \( \beta + \alpha_1, \beta + \alpha_1 + \alpha_2 \) and \( \beta + 2\alpha_1 + \alpha_2 \) respectively, is the same.
for $\mathfrak{osp}(2, 4)$ and for $\mathfrak{sp}(6)$. Let $s_1, s_2$ be the reflections corresponding to the simple roots $\alpha_1, \alpha_2$. Then define the other negative root vectors by

$$e_{-\alpha_1-\alpha_2} = [e_{-\alpha_1}, e_{-\alpha_2}], \quad e_{-2\alpha_1-\alpha_2} = [e_{-\alpha_1}, e_{-\alpha_1-\alpha_2}],$$

$$e_{-\beta-\alpha_1} = [e_{-\alpha_1}, e_{-\beta}], \quad e_{-\beta-\alpha_2} = [e_{-\alpha_2}, e_{-\beta-\alpha_1}], \quad e_{-\beta-2\alpha_1-\alpha_2} = [e_{-\alpha_1}, e_{-\beta-\alpha_1-\alpha_2}].$$

It follows from the Jacobi identity that

$$[e_{-\beta}, e_{-\alpha_1-\alpha_2}] = e_{-\beta-\alpha_1-\alpha_2}, \quad [e_{-\alpha_1-\alpha_2}, e_{-\beta}] = e_{-\beta-2\alpha_1-\alpha_2},$$

and

$$[e_{-\beta}, e_{-2\alpha_1-\alpha_2}] = 2e_{-\beta-2\alpha_1-\alpha_2}.$$

We order the set of positive roots so that for any partition $\pi$, $e_{-\alpha_1}$ occurs first if at all in $e_{-\pi}$, and any root vector $e_{-\sigma}$ with $\sigma$ an odd root occurs last.

Suppose that $\lambda \in \mathfrak{h}^*$ and define $\lambda_1 = s_1 \cdot \lambda, \lambda_2 = s_2 \cdot \lambda, \mu = s_1 \cdot \lambda_2$. Let

$$(\lambda + \rho, \alpha_1^\vee) = p = -(\mu + \rho, (\alpha_1 + \alpha_2)^\vee)$$

and

$$(\lambda_1 + \rho, \alpha_1^\vee) = (\lambda + \rho, (2\alpha_1 + \alpha_2)^\vee) = q = -(\mu + \rho, (2\alpha_1 + \alpha_2)^\vee),$$

$$(\lambda_2 + \rho, \alpha_1^\vee) = (\lambda + \rho, (\alpha_1 + \alpha_2)^\vee) = r = -(\mu + \rho, \alpha_1^\vee).$$

Then $r = 2q - p$. Let $\gamma$ be any positive root that involves $\beta$ with non-zero coefficient when expressed as a linear combination of simple roots. We compute the Šapovalov elements $\theta_{\gamma, i}$ for $\mathfrak{sp}(6)$ and $\theta_{\gamma}$ for $\mathfrak{osp}(2, 4)$. To do this we use Equation (3.7). We can assume $\gamma \neq \beta$. Suppose that $p, q, r$ are nonnegative integers. Then

$$e_{-\alpha_1}^{p+1} e_{-\beta} = \theta_1 e_{-\alpha_1}^p,$$

$$e_{-\alpha_2}^q \theta_1 = \theta_2 e_{-\alpha_2}^q,$$

$$e_{-\alpha_1}^{r+1} \theta_2 = \theta_3 e_{-\alpha_1}^r.$$

In the computations below we write $e_{-\pi} v_\lambda$ for $\pi$ a partition (resp. $\theta_i$ for $i = 1, 2, 3$) in place of $e_{-\pi} v_\lambda$ (resp. $\theta_i v_\lambda$). First note that

$$[e_{-\alpha_1}^{p+1}, e_{-\beta}] = (p + 1)e_{-\beta-\alpha_1} e_{-\alpha_1}^p,$$

$$[e_{-\alpha_2}^q, e_{-\beta-\alpha_1}] = (q + 1)e_{-\beta-\alpha_1-\alpha_2} e_{-\alpha_2}^q,$$

$$[e_{-\alpha_2}^{q+1}, e_{-\alpha_1}] = -(q + 1)e_{-\alpha_1-\alpha_2} e_{-\alpha_2}^q.$$

This easily gives

$$\theta_1 = (p + 1)e_{-\beta-\alpha_1} + e_{-\beta} e_{-\alpha_1} = pe_{-\beta-\alpha_1} + e_{-\alpha_1} e_{-\beta}.$$

We order the set of positive roots so that for any partition $\pi$, $e_{-\alpha_2}$ occurs last if at all in $e_{-\pi}$, and any root vector $e_{-\sigma}$ with $\sigma$ an odd root occurs first.

$\theta_2 = (p + 1)[(q + 1)e_{-\beta-\alpha_1-\alpha_2} + e_{-\beta-\alpha_1} e_{-\alpha_2}] + e_{-\beta}[e_{-\alpha_1} e_{-\alpha_2} - (q + 1)e_{-\alpha_1-\alpha_2}].$
Next order the set of positive roots so that for any partition \( \pi, \ e_{-\alpha_1} \) occurs last if at all in \( e_{-\pi} \), and any root vector \( e_{-\sigma} \) with \( \sigma \) an odd root occurs first. To find \( \theta_3 \) we use

\[
[e_{-\alpha_1}^{r+1}, e_{-\beta-\alpha_1-\alpha_2}] = (r + 1)e_{-\beta-2\alpha_1-\alpha_2}e_{-\alpha_1}^{r},
\]

\[
[e_{-\alpha_1}^{r+1}, e_{-\beta-\alpha_1}e_{-\alpha_2}] = (r + 1)e_{-\beta-\alpha_1}e_{-\alpha_1-\alpha_2}e_{-\alpha_1}^{r} + \left( \frac{r + 1}{2} \right) e_{-\beta-\alpha_1}e_{-2\alpha_1-\alpha_2}e_{-\alpha_1}^{r-1},
\]

\[
[e_{-\alpha_1}^{r+1}, e_{-\beta}e_{-\alpha_1-\alpha_2}] = (r + 1)[e_{-\beta}e_{-2\alpha_1-\alpha_2}e_{-\alpha_1}^{r} + e_{-\beta}e_{-\alpha_1-\alpha_2}e_{-\alpha_1}^{r} + e_{-\beta}e_{-\alpha_1}e_{-2\alpha_1-\alpha_2}e_{-\alpha_1}^{r-1}].
\]

\[
e_{-\alpha_1}^{r+1} e_{-\beta}e_{-\alpha_2}e_{-\alpha_1} = e_{-\beta}e_{-\alpha_2}^2 + (r + 1)e_{-\alpha_1-\alpha_2}e_{-\alpha_1} + \left( \frac{r + 1}{2} \right) e_{-2\alpha_1-\alpha_2}e_{-\alpha_1}^{r-1}.
\]

\[
+ (r + 1)e_{-\beta}e_{-\alpha_2}e_{-\alpha_1} + re_{-\alpha_1-\alpha_2}e_{-\alpha_1} + (r - 1) \left( \frac{r + 1}{2} \right) e_{-\beta}e_{-\alpha_1-\alpha_2}e_{-\alpha_1}^{r-1}.
\]

The above equations allow us to write \( e_{-\alpha_1}^{r+1}\theta_2 \) in terms of elements \( e_{-\pi} \) with \( \pi \) a partition of \( \beta + (r + 2)\alpha_1 + \alpha_2 \). We see that the term \( e_{-\beta-\alpha_1}e_{-2\alpha_1-\alpha_2}e_{-\alpha_1}^{r-1} \) occurs in \( e_{-\alpha_1}^{r+1}\theta_2 \) with coefficient

\[
\left( \frac{r + 1}{2} \right) [(p + 1) - 2q + (r - 1)] = 0.
\]

This is predicted by the cancelation step in the proof of Lemma 4.4. In the remaining terms, \( e_{-\alpha_1}^{r} \) can be factored on the right, and this yields

\[
\theta_3 = (p + 1)(q + 1)(r + 1)e_{-\beta-2\alpha_1-\alpha_2} + (p + 1)(q + 1)e_{-\beta-\alpha_1-\alpha_2}e_{-\alpha_1} \quad (6.1)
\]

\[
+ (q + 1)(r + 1)e_{-\beta-\alpha_1}e_{-\alpha_1-\alpha_2} - (p/2)(r + 1)e_{-\beta}e_{-\alpha_1-\alpha_2}
\]

\[
+ 2(q + 1)e_{-\beta}e_{-\alpha_2}e_{-\alpha_1} + (r - q + 1)e_{-\beta}e_{-\alpha_1-\alpha_2}e_{-\alpha_1} + e_{-\beta}e_{-\alpha_2}e_{-\alpha_1}^2.
\]

Using the opposite orders on positive roots to those used above to define the \( e_{-\pi} \) we obtain

\[
\theta_2 = p[qe_{-\beta-\alpha_1-\alpha_2} + e_{-\alpha_2}e_{-\beta-\alpha_1}] + [e_{-\alpha_2}e_{-\alpha_1} - qe_{-\alpha_1-\alpha_2}]e_{-\beta},
\]

and

\[
\theta_3 = pqr e_{-\beta-2\alpha_1-\alpha_2} + pq e_{-\alpha_1}e_{-\beta-\alpha_1-\alpha_2} \quad (6.2)
\]

\[
+ qre_{-\alpha_1-\alpha_2}e_{-\beta-\alpha_1} - (r/2)(p + 1)e_{-\alpha_1-\alpha_2}e_{-\beta}
\]

\[
+ 2qe_{-\alpha_1}e_{-\alpha_2}e_{-\beta-\alpha_1} + (r - q - 1)e_{-\alpha_1}e_{-\alpha_1-\alpha_2}e_{-\beta} + e_{-\alpha_1}e_{-\alpha_2}e_{-\beta}.
\]

Remark 6.2. It seems remarkable that all the coefficients of \( \theta_3 \) in (6.1) and (6.2) are products of linear factors. This is also true in the Type A case, see Equations (7.1) and (7.2). A partial explanation of this phenomenon is given by specializing these coefficients to zero. Vanishing of these coefficients gives rise to factorizations of \( \theta_3 \) as in the examples below. Factorizations of Šapovalov elements will be discussed in detail elsewhere.
(a) If \( p = (\lambda + \rho, \alpha_1^\vee) = 0 \), then \( r = 2q \) and we have
\[
\theta_3 = 2q^2 e_{-\alpha_1 - \alpha_2} e_{-\beta - \alpha_1} - q e_{-2\alpha_1 - \alpha_2} e_{-\beta} + 2 e_{-\alpha_1} e_{-\alpha_2} e_{-\beta} + e_{-\alpha_2}^2 e_{-\alpha_2} e_{-\beta} = \theta_{\alpha_1 + \alpha_2} \theta_{\beta + \alpha_1}.
\]

(b) If \( q = (\lambda + \rho, (2\alpha_1 + \alpha_2)^\vee) = 0 \) then \( p = -r, \theta_2 = \theta_{\beta + \alpha_1 + \alpha_2} = \theta_{\alpha_2} \theta_{\beta + \alpha_1} \), and we have
\[
\theta_3 = \frac{(p/2)(p + 1)e_{-2\alpha_1 - \alpha_2} - (p + 1)e_{-\alpha_1 - \alpha_2 - \alpha_2} + e_{-\alpha_1}^2 e_{-\alpha_2} e_{-\beta}}{\theta_{\alpha_1 + \alpha_2} \theta_{\beta}}.
\]

(c) If \( r = (\lambda + \rho, (\alpha_1 + \alpha_2)^\vee) = 0 \), then \( p = 2q \), and we have
\[
\theta_3 = 2q^2 e_{-\alpha_1} e_{-\beta - \alpha_1 - \alpha_2} + 2q e_{-\alpha_1} e_{-\beta - \alpha_1} - (q + 1)e_{-\alpha_1} e_{-\alpha_1 - \alpha_2} e_{-\beta} + e_{-\alpha_1}^2 e_{-\alpha_2} e_{-\beta} = \theta_{\alpha_1} [2q^2 e_{-\beta - \alpha_1 - \alpha_2} 2q e_{-\alpha_2} e_{-\beta - \alpha_1} - (q + 1)e_{-\alpha_1 - \alpha_2} e_{-\beta} + e_{-\alpha_1} e_{-\alpha_2} e_{-\beta}] = \theta_{\alpha_1} \theta_{\beta + \alpha_1 + \alpha_2}.
\]

Similarly if \( p = -1 \), (resp. \( q = -1, r = -1 \)) then (6.2) yields the factorizations \( \theta_3 = \theta_{\beta + \alpha_1} \theta_{\alpha_1 + \alpha_2} \) (resp. \( \theta_2 = \theta_{\beta + \alpha_1} \theta_{\alpha_2}, \theta_3 = \theta_{\beta} \theta_{2\alpha_1 + \alpha_2} \), and \( \theta_3 = \theta_{\beta + \alpha_1 + \alpha_2} \theta_{\alpha_1} \)). On the other hand we see that \( p \) divides the coefficients of \( e_{-\beta - 2\alpha_1 - \alpha_2} \) and \( e_{-\alpha_1} e_{-\beta - \alpha_1 - \alpha_2} \) in (6.2) since when \( p = 0, \theta_3 = \theta_{\alpha_1 + \alpha_2} \theta_{\beta + \alpha_1} \) can be written as a linear combination of different \( e_{-\pi} \). In this way we obtain explanations for all the linear factors in (6.1) and (6.2) with the exception of the coefficients \( r - q \pm 1 \) of \( e_{-\beta} e_{-\alpha_1 - \alpha_2} e_{-\alpha_1} \) and \( e_{-\alpha_1} e_{-\alpha_1 - \alpha_2} e_{-\beta} \). At this point it may be worthwhile mentioning that \( r - q = (\lambda + \rho, \alpha_2^\vee) \). In addition equality holds in the upper bounds given in Theorem 1.2 for the degrees of all the coefficients in (6.1) and (6.2).

7 The Type A Case.

7.1 Lie Superalgebras.

We construct the elements \( \theta_\gamma \) in Theorem 1.2 explicitly when \( g = \mathfrak{gl}(m, n) \). Suppose that \( \gamma = \epsilon_\tau - \delta_\kappa \). For \( 1 \leq i < j \leq m \) and \( 1 \leq k < \ell \leq n \) define roots \( \sigma_{i,j}, \tau_{k,\ell} \) by
\[
\sigma_{i,j} = \epsilon_i - \epsilon_j, \quad \tau_{k,\ell} = \delta_k - \delta_\ell.
\]

Suppose \( B = (b_{i,j}) \) is a \( k \times \ell \) matrix with entries in \( U(n^-) \), \( I \subseteq \{1, \ldots, k\} \) and \( J \subseteq \{1, \ldots, \ell\} \). We denote the submatrix of \( B \) obtained by deleting the \( i \)th row for \( i \in I \), and the \( j \)th column for \( j \in J \) by \( iB_j \). If either set is empty, we omit the corresponding subscript. When \( I = \{i\} \), we write \( iB \) in place of \( iB \) and likewise when \( |J| = 1 \).
If \( k = \ell \) we define two noncommutative determinants of \( B \), the first working from left to right, and the second working from right to left.

\[
\overleftarrow{\det}(B) = \sum_{w \in S_k} \text{sign}(w)b_{1,w(1)} \cdots b_{k,w(k)},
\]

\[
\overrightarrow{\det}(B) = \sum_{w \in S_k} \text{sign}(w)b_{k,w(k)} \cdots b_{1,w(1)}.
\]

If \( k = 0 \) we make the convention that \( \overleftarrow{\det}(B) = \overrightarrow{\det}(B) = 1 \). Consider the following matrices with entries in \( U(n^-) \).

\[
A^+(\lambda, r) = \begin{bmatrix}
\varepsilon_{r+1,r} & (\lambda + \rho, \sigma_{r,r+1}) & 0 & \cdots & 0 \\
\varepsilon_{r+2,r} & \varepsilon_{r+2,r+1} & (\lambda + \rho, \sigma_{r,r+2}) & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\varepsilon_{m,r} & \varepsilon_{m,r+1} & \cdots & \cdots & \varepsilon_{m,m-1}
\end{bmatrix},
\]

\[
A^-(\lambda, s) = \begin{bmatrix}
(\lambda + \rho, \tau_{1,s}^-) & 0 & \cdots & 0 \\
\varepsilon_{m+2,m+1} & (\lambda + \rho, \tau_{2,s}^-) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\varepsilon_{m+s-1,m+1} & \varepsilon_{m+s-1,m+2} & \cdots & (\lambda + \rho, \tau_{s-1,s}^-)
\end{bmatrix},
\]

and let \( B^+(\lambda, r) \) (resp. \( B^-(\lambda, s) \)) be the matrices obtained from \( A^+(\lambda, r) \) (resp. \( A^-(\lambda, s) \)) by increasing all entries on the superdiagonals by one. Observe that in \( A^+(\lambda, r) \) (resp. \( A^-(\lambda, s) \)) the number of columns exceeds the number of rows by one (resp. the number of rows exceeds the number of columns by one). We also consider two degenerate cases: if \( r = m \) then \( A^+(\lambda, r) \) and \( B^+(\lambda, r) \) are “matrices with zero rows”. In this case we ignore the summation over \( j \) in the following formulas, replacing \( \overleftarrow{\det}(A^+(\lambda, r)_j) \) by 1 and \( i + j + r + m \) by \( i + 1 + r + m \). Similar remarks apply to the case where \( s = 1 \).

**Theorem 7.1.**

\[
\theta_\gamma(\lambda) = \sum_{j=1}^{m-r+1} \sum_{i=1}^s (-1)^{i+j+r+m} \overleftarrow{\det}(A^+(\lambda, r)_j) \overrightarrow{\det}(iA^-(\lambda, s))e_{m+i,j+r-1}. \tag{7.1}
\]

\[
= \sum_{j=1}^{m-r+1} \sum_{i=1}^s e_{m+i,j+r-1} (-1)^{i+j+r+m} \overleftarrow{\det}(B^+(\lambda, r)_j) \overrightarrow{\det}(iB^-(\lambda, s)). \tag{7.2}
\]

**Proof.** We prove (7.1) only. The proof of (7.2) is similar. For the isotropic simple root \( \beta = \varepsilon_m - \delta_1 \), (7.1) reduces to \( \theta_\beta(\lambda) = e_{m+1,m} \). Suppose that \( \alpha = \delta_s - \delta_{s+1}, \gamma = \varepsilon_r - \delta_s \) and \( \gamma' = \varepsilon_r - \delta_{s+1} = s_\alpha \gamma \). Assuming the result for \( \gamma \) we prove it for \( \gamma' \). The result for \( \varepsilon_r - \delta_s \) can be deduced in a similar way. Set \( e_{-\alpha} = e_{m+s+1,m+s} \). For \( 1 \leq i \leq s - 1 \) we have

\[
t_\gamma s_{i+1} = \delta_i - \delta_{s+1} = s_\alpha t_i s.
\]
Consider the matrix

\[
A^{-}(\lambda', s + 1) = \begin{bmatrix}
(\lambda' + \rho, \tau_{i,s+1}^{\lambda'}) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
(\lambda' + \rho, \tau_{2,s+1}) & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
e_{m+2,m+1} & e_{m+s-1,m+1} & \cdots & e_{m+s+1,m+1} \\
e_{m+s,m+1} & e_{m+s,m+2} & \cdots & e_{m+s+1,m+2} \\
e_{m+s+1,m+1} & e_{m+s+1,m+2} & \cdots & e_{m+s+1,m+s}
\end{bmatrix}.
\]

The matrix \(A^{-}(\lambda', s + 1)\) replaces the matrix \(A^{-}(\lambda, s)\) in Equation (7.1) in the analogous expression for \(\theta_{\gamma'}(\lambda')\). Suppose that \((\lambda + \rho, \alpha^\gamma) = p\) and let \(\lambda' = s_{\alpha} \cdot \lambda\). Then

\[
(\lambda' + \rho, \tau_{i,s+1}^{\lambda'}) = (\lambda + \rho, \tau_{i,s}^{\lambda}),
\]

for \(1 \leq i \leq s - 1\), and this means that the first \(s - 1\) diagonal entries of \(A^{-}(\lambda', s + 1)\) and \(A^{-}(\lambda, s)\) are equal. Also the entry in row \(s\) and column \(s\) of \(A^{-}(\lambda', s + 1)\) is equal to \(-p\). If we remove the last row (row \(s + 1\)) from \(A^{-}(\lambda', s + 1)\) the last column of the resulting matrix will have only one non-zero entry \(-p\). If in addition we remove this column, we obtain the matrix \(_{s}A^{-}(\lambda, s)\). Therefore

\[
-p \det(_{s}A^{-}(\lambda, s)) = \det(_{s+1}A^{-}(\lambda', s + 1)).
\]  

(7.3)

Similarly by removing row \(s\) from \(A^{-}(\lambda', s + 1)\) we see that

\[
\det(_{s}A^{-}(\lambda, s))e_{m+s+1,m+s} = \det(_{s}A^{-}(\lambda', s + 1)).
\]  

(7.4)

Equation (3.7) in this situation takes the form

\[
e^{p+1}_{-\alpha} \theta_{\gamma}(\lambda) = \theta_{\gamma'}(\lambda') e^{p}_{-\alpha}.
\]  

(7.5)

If \(r \leq k \leq m + s - 1\) we have

\[
e^{p+1}_{-\alpha} e_{m+s,k} = (p e_{m+s+1,k} + e_{m+s+1,m+s} e_{m+s,k}) e^{p}_{-\alpha}.
\]  

(7.6)

We now consider two cases: in the first entries in \(\det(_{i}A^{-}(\lambda, s))\) are replaced by entries in \(\det(_{i}A^{-}(\lambda', s + 1))\). Suppose \(1 \leq i \leq s - 1\), and \(m + 1 \leq k \leq m + s - 1\). Then \(e_{-\alpha}\) commutes with \(e_{m+i,k}\) and all entries in the matrix \(_{i}A^{-}(\lambda, s)\) except for those in the last row. Replacing \(e_{m+s,k}\) in the matrix \(_{i}A^{-}(\lambda, s)\) by \(e_{m+s+1,k}\) yields the matrix \(_{i,s}A^{-}(\lambda', s + 1)\) by Equation (7.6) gives the first equality below. For the second we use a cofactor expansion,

\[
e^{p+1}_{-\alpha} \det(_{i}A^{-}(\lambda, s))e_{m+i,k} = [p \det(_{i,s}A^{-}(\lambda', s + 1)) e_{m+i,k} + \det(_{i}A^{-}(\lambda, s)) e_{m+s+1,m+s} e^{p}_{-\alpha} = \det(_{i}A^{-}(\lambda', s + 1)) e_{m+i,k} e^{p}_{-\alpha}.
\]  

(7.7)
Now consider the case $i = s$. Here entries in $\det(iA^-(\lambda,s))$ are unchanged but the factor $e_{m+i,k}$ is replaced. If $r \leq k \leq m$, then all entries in the matrix $sA^-(\lambda,s)$, commute with $e_{m+s,k}$ and $e_{-\alpha}$, so by Equation (7.6) we get the first equality below, and the second equality comes from Equations (7.3) and (7.4)

$$
\begin{align*}
\theta_{p+1} & \det(sA^-(\lambda,s))e_{m+s,k}\theta_p = \det(sA^-(\lambda,s))[pe_{m+s+1,k} + e_{m+s+1,m+s}e_{m+s,k}] \\
& = -\det(s+1A^-)e_{m+s+1,k} + \det(sA^-(\lambda',s+1))e_{m+s,k}.
\end{align*}
\tag{7.8}
$$

Since $\det(A^+(\lambda,r_j))$ commutes with $e_{-\alpha}$ and $e_{m+j,i}$ for all $i,j$, it follows from Equations (7.5), (7.7) and (7.8) that

$$\theta_p(\lambda') = \sum_{j=1}^{m-r+1} \sum_{i=1}^{s+1} (-1)^{i+j+r+m} \det(A^+(\lambda,r_j)) \det(iA^-(\lambda',s+1)) e_{m+i,j+r-1}.
$$

as desired. \qed

### 7.2 Lie Algebras.

Let $\mathfrak{g} = \mathfrak{gl}(m)$, and $\alpha = \epsilon_r - \epsilon_i$. We give a determinantal formula for the Šapovalov element $\theta_{\alpha,1}$. Consider the following matrix with entries in $U(n^-)$.

$$C = \begin{bmatrix}
e_{r+1,r} & -(\lambda + \rho, \sigma_{r,r+1}^\vee) & 0 & \ldots & 0 \\
e_{r+2,r} & e_{r+2,r+1} & -(\lambda + \rho, \sigma_{r,r+2}^\vee) & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
e_{t-1,r} & e_{t-1,r+1} & \ldots & e_{t-1,t-2} & -(\lambda + \rho, \sigma_{t,t-1}^\vee) \\
e_{t,r} & e_{t,r+1} & \ldots & e_{t,t-2} & e_{t,t-1}
\end{bmatrix}.
$$

**Theorem 7.2.** The Šapovalov element $\theta_{\alpha,1}$ is given by

$$\theta_{\alpha,1} = \det(C).
$$

**Proof.** Similar to the proof of Theorem 7.1. \qed

**Corollary 7.3.** If $(\lambda + \rho, \sigma_{r,s}^\vee) = 0$ then $\theta_{\epsilon_r,1-\epsilon_i,1} = \theta_{\epsilon_S,1-\epsilon_i,1} \theta_{\epsilon_r,1-\epsilon_s,1}$.

**Proof.** Under the given hypothesis the matrix $C$ in the Theorem is block lower triangular. \qed

Determinants similar to those in Theorem 7.2 are introduced in [CL74] (see Proposition 2.5 and Theorem 2.7), where they are used to construct homomorphisms between Weyl modules. See also [Bru88], [Car87].

### 8 Survival of Šapovalov elements in factor modules.

Let $v_\lambda$ be a highest weight vector in a Verma module $M(\lambda)$ with highest weight $\lambda$, and suppose $\gamma$ is an odd root with $(\lambda + \rho, \gamma) = 0$. We are interested in the condition that the image of $\theta_\gamma v_\lambda$ is non-zero in various factor modules of $M(\lambda)$.
8.1 Independence of Šapovalov elements.

Given \( \lambda \in \mathfrak{h}^* \) recall the set \( B(\lambda) \) defined in Section 2, and define a “Bruhat order” \( \leq \) on \( B(\lambda) \) by \( \gamma' \leq \gamma \) if \( \gamma - \gamma' \) is a sum of positive even roots. Then introduce a relation \( \downarrow \) on \( B(\lambda) \) by \( \gamma' \downarrow \gamma \) if \( \gamma' \leq \gamma \) and \( (\gamma, \gamma') \neq 0 \). If \( \gamma \in B(\lambda) \), we say that \( \gamma \) is \( \lambda \)-minimal if \( \gamma' \downarrow \gamma \) and \( \gamma' \in B(\lambda) \) implies that \( \gamma' = \gamma \). For \( \gamma \in B(\lambda) \) set \( B(\lambda) - \gamma = B(\lambda) \setminus \{\gamma\} \).

We say \( \gamma \) is independent at \( \lambda \) if

\[
\theta_\gamma v_\lambda \notin \sum_{\gamma' \in B(\lambda) - \gamma} U(g) \theta_{\gamma'} v_\lambda.
\]

**Proposition 8.1.** If \( \gamma' \downarrow \gamma \) with \( \gamma' \in B(\lambda) \) and \( \gamma' < \gamma \), then \( \theta_\gamma v_\lambda \in U(g) \theta_{\gamma'} v_\lambda \).

**Proof.** The hypothesis implies that \( (\gamma, \alpha^\vee) > 0 \) and \( \gamma = s_\alpha \gamma' \). Thus the result follows from [Mus]. \( \square \)

By the Proposition, if we are interested in the independence of the Šapovalov elements \( \theta_\gamma \) for distinct isotropic roots, it suffices to study only \( \lambda \)-minimal roots \( \gamma \).

For the rest of this section we assume that \( g = gl(m, n) \). We use Equation (7.1), and order the positive roots of \( g \) so that each summand in this equation is a constant multiple of \( e^{-\pi} \) for some \( \pi \in \mathfrak{p}(\gamma) \). For such \( \pi \) the odd root vector is the rightmost factor of \( e^{-\pi} \), that is we have \( e^{-\pi} \in U(n^-_{n_0})n_1^- \).

**Lemma 8.2.** If \( \gamma \) is \( \lambda \)-minimal, then \( e^{-\gamma} v_\lambda \) occurs with non-zero coefficient in \( \theta_\gamma v_\lambda \).

**Proof.** Assume \( \gamma = \epsilon_r - \delta_s \). Then if \( \alpha = \epsilon_r - \epsilon_i \) with \( r < i \), or \( \alpha = \delta_j - \delta_s \) with \( j < s \) we have \( (\lambda + \rho, \alpha^\vee) \neq 0 \), since \( \gamma \) is \( \lambda \)-minimal. In other words the entries on the superdiagonals of \( A^+(\lambda, r) \) and \( A^-(\lambda, s) \) are non-zero. Thus the result follows from Theorem 7.1. \( \square \)

**Theorem 8.3.** The isotropic root \( \gamma \) is independent at \( \lambda \) if and only if \( \gamma \) is \( \lambda \)-minimal.

**Proof.** Set \( B = B(\lambda) - \gamma \). If \( \gamma \) is not \( \lambda \)-minimal then \( \gamma \) is not independent at \( \lambda \) by Proposition 8.1. Suppose that \( \gamma \) is \( \lambda \)-minimal and

\[
\theta_\gamma v_\lambda = \sum_{\gamma' \in B} U(g) \theta_{\gamma'} v_\lambda = \sum_{\gamma' \in B} U(n^-_0) \theta_{\gamma'} v_\lambda,
\]

then by comparing weights

\[
\theta_\gamma v_\lambda \in \sum_{\gamma' \in B} U(n^-_0) e^{-\gamma'} v_\lambda, \tag{8.1}
\]

But Lemma 8.2 implies that

\[
\theta_\gamma v_\lambda \equiv c e^{-\gamma} v_\lambda \mod \sum_{\gamma' \in B} U(n^-_0) e^{-\gamma'} v_\lambda
\]

for some non-zero constant \( c \). Combined with (8.1) we obtain a contradiction to the PBW Theorem. \( \square \)
8.2 Survival of Šapovalov elements in Kac modules.

For $g = \mathfrak{gl}(m, n)$ we have $g_1 = g_1^+ \oplus g_1^-$, where $g_1^+$ (resp. $g_1^-$) is the set of block upper (resp. lower) triangular matrices. Let $\mathfrak{h}$ be the Cartan subalgebra of $g$ consisting of diagonal matrices, and set $\mathfrak{p} = g_0 \oplus g_1^+$. Next let

$$P^+ = \{ \lambda \in \mathfrak{h}^* | (\lambda, \alpha^\vee) \in \mathbb{Z}, (\lambda, \alpha^\vee) \geq 0 \text{ for all } \alpha \in \Delta_0^+ \}$$

For $\lambda \in P^+$, let $L^0(\lambda)$ be the (finite dimensional) simple $g_0$-module with highest weight $\lambda$. Then $L^0(\lambda)$ is naturally a $\mathfrak{p}$-module and we define the Kac module $K(\lambda)$ by

$$K(\lambda) = U(g) \otimes_{U(\mathfrak{p})} L^0(\lambda).$$

Note that as a $g_0$-module

$$K(\lambda) = \Lambda(g_1^-) \otimes L^0(\lambda).$$

The next result is well-known. Indeed two methods of proof are given in Theorem 4.37 of [Bru03]. The second of these is based on Theorem 5.5 in [Ser96]. We give a short proof using Theorem 7.1. We now assume that the roots are ordered as in Equation (7.2), that is with the odd root vector first.

**Theorem 8.4.** If $\lambda$ and $\lambda - \epsilon_r + \delta_s$ belong to $P^+$ and $(\lambda + \rho, \epsilon_r - \delta_s) = 0$, then

$$[K(\lambda) : L(\lambda - \epsilon_r + \delta_s)] \neq 0.$$  

**Proof.** Set $\gamma = \epsilon_r - \delta_s$. Let $\theta_\gamma(\lambda)$ be as in Theorem 7.1. Then $w = \theta_\gamma(\lambda)v_\lambda$ is a highest weight vector in the Verma module $M(\lambda)$ with weight $\lambda - \gamma$. It suffices to show that the image of $w$ in the Kac module $K(\lambda)$ is nonzero. We have an embedding of $g_0$-modules

$$g_1^- \otimes L^0(\lambda) \subseteq U(g_1^-) \otimes L^0(\lambda).$$

The elements $e_{m+j,i+r-1}$ in Equation (7.2) form part of a basis for $g_1^-$. Furthermore the coefficient of $e_{m+j,i+r-1}$ belong to $U(n_0^-)$. Therefore it suffices to show that the coefficient of $e_{m+s,r}$ in this equation is nonzero. This coefficient is found by deleting the first column of the matrix $B^+(\lambda,r)$ and the last row of $B^-(\lambda,s)$ and taking determinants of the resulting matrices, which have only zero entries above the main diagonal. We find that the coefficient of $e_{m+s,r}$ is

$$\pm \prod_{k=1}^{m-r} (1 - (\lambda + \rho, \sigma_{r,r+k}^\vee)) \prod_{k=1}^{s-1} (1 - (\lambda + \rho, \tau_{k,s}^\vee)).$$

Since $\lambda \in P^+$, $(\lambda + \rho, \sigma_{r,r+k}^\vee) \geq 1$ with equality if and only if $k = 1$ and $(\lambda, \epsilon_r - \epsilon_{r+1}) = 0$. This cannot happen if $\lambda - \gamma \in P^+$, so the first product above is nonzero, and similarly so is the second. \qed

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9 Changing the Borel subalgebra.

9.1 Adjacent Borel subalgebras.

We consider the behavior of Šapovalov elements when the Borel subalgebra is changed. Let \( b', b'' \) be arbitrary adjacent Borel subalgebras, and suppose

\[
g^\alpha \subset b', \quad g^{-\alpha} \subset b''
\]  

(9.1)

for some isotropic root \( \alpha \). Let \( S \) be the intersection of the sets of roots of \( b' \) and \( b'' \), \( p = b' + b'' \) and \( r = \bigoplus_{\beta \in S} ke_{-\beta} \). Then \( r, p \) are subalgebras of \( g \) with \( g = p \oplus r \). Furthermore \( r \) is stable under \( e_{\pm \alpha} \), and consequently, so is \( U(r) \). Note that

\[
\rho(b'') = \rho(b') + \alpha. 
\]

(9.2)

Also if \( v_\mu \) is a highest weight vector with weight \( \mu \), then

\[
e_\alpha e_{-\alpha} v_\mu = h_\alpha v_\mu = (\mu, \alpha)v_\mu. 
\]

(9.3)

As is well known (see for example Corollary 8.6.3 in [Mus12]), if \((\mu + \rho(b'), \alpha) \neq 0\), then

\[
\mu(b'') + \rho(b'') = \mu(b') + \rho(b'). 
\]

(9.4)

In this situation we call the change of Borel subalgebras from \( b' \) to \( b'' \) (or vice-versa) a typical change of Borels. Consider the Zariski dense subset \( \Lambda_{\gamma,m} \) of \( H_{\gamma,m} \) given by

\[
\Lambda_{\gamma,m} = \{\mu \in H_{\gamma,m}|(\mu + \rho, \alpha) \notin \mathbb{Z} \text{ for all positive roots } \alpha \neq \gamma\}. 
\]

When \( \gamma \) is isotropic, we write \( \Lambda_{\gamma} \) in place of \( \Lambda_{\gamma,1} \). Since the coefficients of \( \theta_{\gamma,m} \) are polynomials, \( \theta_{\gamma,m} \) is determined (as usual modulo a left ideal) by the values of \( \theta_{\gamma,m} v_\mu \) for \( \mu \in \Lambda_{\gamma,m} \).

Suppose that \( \gamma \) is a positive root of both \( b' \) and \( b'' \), and that \( \theta_{\gamma,m} \) is a Šapovalov element corresponding to the pair \((\gamma, m)\) using the negatives of the roots of \( b'' \), and for brevity set \( \theta = \theta_{\gamma,m}(\mu) \). Assume that \( v_\mu \) is a highest weight vector in a Verma module \( M_{b'}(\mu') \) for \( b' \) with highest weight \( \mu' \in \Lambda_{\gamma,m} \). Then \( v_\mu = e_{-\alpha} v_{\mu'} \) is a highest weight vector for \( b'' \) which also generates \( M_{b''}(\mu') \). Thus we can write

\[
M_{b''}(\mu') = M_{b'}(\mu). 
\]

Next note that \( u = \theta e_{-\alpha} v_{\mu'} \) is a highest weight vector for \( b'' \), and \( e_\alpha \theta e_{-\alpha} v_{\mu'} \) is a highest weight vector for \( b' \) of weight \( \mu' - m\gamma \) that generates the same submodule of \( M_{b''}(\mu') \) as \( u \). We can write \( \theta \) in a unique way as \( \theta = e_\alpha \theta_1 + \theta_2 \) with \( \theta_1 \in U(r) \). Then

\[
e_\alpha \theta e_{-\alpha} v_{\mu'} = e_\alpha \theta_2 e_{-\alpha} v_{\mu'} = \theta_1' e_{-\alpha} v_{\mu'} + \theta_2' v_{\mu'}
\]

where \( \theta_1' = [e_\alpha, \theta_2] \), \( \theta_2' = (\mu', \alpha) \theta_2 \in U(r) \). Note that the term \( e_{-m\pi, \gamma} \) cannot occur in \( e_\alpha \theta_1 \) or \( \theta_1' e_{-\alpha} \). Allowing for possible re-ordering of positive roots used to define the \( e_{-\pi} \) (compare Lemma 4.3) we conclude that modulo terms of lower degree, the coefficient of \( e_{-m\pi, \gamma} \) in \( e_\alpha \theta e_{-\alpha} v_{\mu'} \) is equal to \( e_{-m\pi, \gamma} \) in \( \theta_2 e_\alpha v_{\mu'} \). Since \( \mu' \in \Lambda_{\gamma,m} \), each change of Borels in (1.12) is typical, and thus the foregoing applies to each link in the chain.
9.2 Chains of Borel subalgebras.

Using adjacent Borel subalgebras it is possible to give an alternative construction of Šapovalov elements corresponding to an isotropic root \( \gamma \) which is a simple root for some Borel subalgebra. This condition always holds in type A, but for other types, it is quite restrictive: if \( \mathfrak{g} = \mathfrak{osp}(2m, 2n + 1) \) the assumption only holds for roots of the form \( \pm(e_i - \delta_j) \), while if \( \mathfrak{g} = \mathfrak{osp}(2m, 2n) \) it holds only for these roots and the root \( \epsilon_m + \delta_n \). (Theorem 1.3 on the other hand applies to any positive isotropic root, provided we choose the appropriate Borel subalgebra satisfying Hypothesis (1.6).)

Suppose that \( \mathfrak{b} \) is the distinguished or anti-distinguished Borel subalgebra, and let \( \mathfrak{b}' \) be another Borel subalgebra with the same even part as \( \mathfrak{b} \). This condition always holds in type A, but for other types, it is quite restrictive: if \( \mathfrak{g} = \mathfrak{osp}(2m, 2n + 1) \) the assumption only holds for roots of the form \( \pm(e_i - \delta_j) \), while if \( \mathfrak{g} = \mathfrak{osp}(2m, 2n) \) it holds only for these roots and the root \( \epsilon_m + \delta_n \). (Theorem 1.3 on the other hand applies to any positive isotropic root, provided we choose the appropriate Borel subalgebra satisfying Hypothesis (1.6).)

Lemma 9.1. The leading term of the coefficient of \( e_{-\gamma}v_\lambda \) in

\[
e_{\alpha_1} \cdots e_{\alpha_r} e_{-\gamma} e_{-\alpha_r} \cdots e_{-\alpha_1} v_\lambda
\]

is, up to a constant multiple, equal to \( \prod_{i=1}^r (\lambda, \alpha_i) \).

Lemma 9.2. Suppose \( \gamma = w\beta \) with \( w \in W_{\text{nonisotropic}} \) and \( \beta \) a simple isotropic root, and that \( \gamma \) is a simple root of some Borel. Then \( q(w, \alpha) = 1 \) for all \( \alpha \in N(w^{-1}) \), and

\[
\{ \gamma - \alpha_i | i = 1, \ldots, r, (\gamma, \alpha_i^\vee) \neq 0 \} = N(w^{-1}).
\]

Proof. The first statement can be checked on a case-by-case basis. The second is clearly true for \( \gamma \) simple. Otherwise we can find a simple root \( \alpha \) such that \( (\gamma, \alpha^\vee) > 0 \). Then write \( w = s_{\alpha} u \) with \( \ell(w) = \ell(u) + 1 \) and \( u \in W_{\text{nonisotropic}} \), and set \( \gamma' = s_{\alpha} \gamma \). The result then follows by induction on \( \ell(w) \) and Equation (1.3). \( \Box \)

There exists a unique (modulo a suitable left ideal in \( U(\mathfrak{g}) \)) Šapovalov element \( \theta_{\gamma}^{(i)} \) for the Borel subalgebra \( \mathfrak{b}^{(i)} \) and polynomials \( g_i(\lambda), h_i(\lambda) \) such that

\[
e_{-\alpha_i} \theta_{\gamma}^{(i-1)} e_{\alpha_i} v_i = g_i(\lambda) \theta_{\gamma}^{(i)} v_i \tag{9.5}
\]

and

\[
e_{\alpha_i} \theta_{\gamma}^{(i)} e_{-\alpha_i} v_{i-1} = h_i(\lambda) \theta_{\gamma}^{(i-1)} v_{i-1} \tag{9.6}
\]

Lemma 9.3. We have

\[
g_i(\lambda) h_i(\lambda) = (\lambda + \rho, \alpha_i)(\lambda + \rho - \gamma, \alpha_i). \tag{9.7}
\]

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Proof. Since \( \lambda_i = \lambda_{i-1} - \alpha_i \),
\[
(\alpha_i, \lambda_{i-1})(\alpha_i, \lambda_{i-1} - \gamma)\theta_\gamma^{(i-1)} v_{i-1} = e_{-\alpha_i}e_{\alpha_i}\theta_\gamma^{(i)} e_{-\alpha_i} v_{i-1} = h_i(\lambda) e_{-\alpha_i} \theta_\gamma^{(i-1)} e_{\alpha_i} v_{i-1} = g_i(\lambda) h_i(\lambda) \theta_\gamma^{(i-1)} v_{i-1}.
\]
Now each change of Borels is typical and \( \alpha_i \) is simple isotropic for \( b^{(i-1)} \), so we have \( (\alpha_{i-1}, \lambda_i) = (\lambda + \rho, \alpha_i) \). The result follows.

**Theorem 9.4.** Set \( F(\gamma) = \{ i \mid 1 \leq i \leq r \text{ and } (\gamma, \alpha_i) = 0 \} \). Then
\[
e_{\alpha_1} \cdots e_{\alpha_r} e_{-\gamma} e_{-\alpha_r} \cdots e_{-\alpha_1} v_\lambda = \prod_{i \in F(\gamma)} (\lambda + \rho, \alpha_i) \theta_\gamma v_\lambda
\]
Proof. Since \( v_i = e_{-\alpha_i} v_{i-1} \), and \( u_{i-1} = e_{\alpha_i} u_i \), Equation (9.6) and induction yield
\[
u_{i-1} = \prod_{j=i}^r h_j(\lambda) \theta_\gamma^{(i-1)} v_{i-1}.
\]
Hence
\[
u_0 = \prod_{j=1}^r h_j(\lambda) \theta_\gamma v_0.
\]
On the other hand, by Theorem 1.3 and Lemma 9.2 the leading term of the coefficient of \( e_{-\gamma} \) in \( \theta_\gamma v_\lambda \) is given by \( \prod_{i=1}^r (\gamma, \alpha_i) \neq 0 (\lambda, \alpha_i) \) up to a scalar multiple. Therefore by comparing the coefficient of \( e_{-\gamma} v_\lambda \) on both sides of (9.9), and using Lemma 9.1, we have modulo terms of lower degree, that
\[
\prod_{j=1}^r h_j(\lambda) = \prod_{i \in F(\gamma)} (\lambda + \rho, \alpha_i).
\]
We note that the functions \( \lambda \rightarrow (\lambda + \rho, \alpha_i) \) for \( 1 \leq i \leq r \) are linearly independent on \( H_\gamma \). It follows that \( h_j(\lambda) \) is constant if \( j \notin F(\gamma) \), and \( h_j(\lambda) = (\lambda + \rho, \alpha_j) \) if \( j \in F(\gamma) \). However we know from Lemma 9.3 that if \( (\lambda, \alpha_j) = 0 \) then \( h_j(\lambda) \) divides \( (\alpha_j, \lambda + \rho) \). Thus the result follows.

Next suppose that \( b'', b' \) are adjacent Borel subalgebras as in Equation (9.1), and that \( d(b, b'') = d(b, b') + 1 \). We can find a sequence of Borel subalgebras as in (1.12) such that \( b' = b^{(i-1)} \) and \( b'' = b^{(i)} \). Adopting the notation of Equations (9.5) and (9.6), we can now clarify the relationship between the Šapovalov elements \( \theta_\gamma^{(i-1)} \) and \( \theta_\gamma^{(i)} \).

**Corollary 9.5.** With the above notation, we have up to constant multiples,

(a) If \( (\gamma, \alpha_i) = 0 \), then \( h_i(\lambda) = g_i(\lambda) = (\lambda + \rho, \alpha_i) \).

(b) If \( (\gamma, \alpha_i) \neq 0 \), then \( h_i(\lambda) = 1 \) and \( g_i(\lambda) = (\lambda + \rho, \alpha_i)(\lambda + \rho - \gamma, \alpha_i) \).

Proof. This was shown in the course of the proof.
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