GENERALIZED PÓLYA URN SCHEMES WITH NEGATIVE BUT LINEAR REINFORCEMENTS

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Abstract. In this paper, we consider a new type of urn scheme, where the selection probabilities are proportional to a weight function, which is linear but decreasing in the proportion of existing colours. We refer to it as the negatively reinforced urn scheme. We establish almost sure limit of the random configuration for any balanced replacement matrix \( R \). In particular, we show that the limiting configuration is uniform on the set of colours, if and only if, \( R \) is a doubly stochastic matrix. We further establish almost sure limit of the vector of colour counts and prove central limit theorems for the random configuration, as well as, for the colour counts.

1. Introduction

1.1. Background and Motivation. Various kinds of random reinforcement models have been of much interest in recent years \([23, 35, 31, 7, 25, 36, 41, 19, 34, 18, 17]\). Urn schemes, which were first studied by Pólya \([42]\), are perhaps the simplest reinforcement models. They have many applications and generalizations \([27, 26, 5, 6, 28, 30, 31, 7, 13, 16, 22, 21, 33, 17, 12, 10, 11]\). In general, reinforcement models typically adhere to the structure of “rich get richer”, which can also be termed as positive reinforcement. However, there have been some studies on negative reinforcements models in the context of percolation, such as the forest fire-type models from the point-of-view of self-destruction \([46, 43, 20, 11]\) and frozen percolation-type models from the point-of-view of stagnation \([38, 48, 47, 49]\). For urn schemes, a type of “negative reinforcement” have been studied when balls can be thrown away from the urn, as well as, added \([24, 50, 32, 33, 21]\). In such models, it is usually assumed that the model is tenable, that is, regardless of the stochastic path taken by the process, it is never required to remove a ball of a colour not currently present in the urn. Perhaps the most famous of such scheme is the Ehrenfest urn \([24, 39]\), which models the diffusion of a gas between two chambers of a box. There are some models without tenability, such as the OK Corral Model \([50, 32, 33]\) or Simple Date: September 26, 2018.

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Harmonic Urn \cite{21} in two colors. Typically these are used for modeling destructive competition.

In recent days, there has been some work on negative reinforcements, random graphs \cite{14, 15, 9} from a different point-of-view, where attachment probabilities of a new vertex are decreasing functions of the degree of the existing vertices. Such models have also been referred as “de-preferential attachment” \cite{9} as opposed to usual “preferential” attachment models \cite{13}. Motivated by this later set of works, in this paper, we present a specific model of negatively reinforced urn scheme, where the selection probabilities are linear but decreasing function of the proportion of colors. Negatively reinforced urn schemes are natural models for modeling problems with resource constrains. In particular, multi-server queuing systems with capacity constrains \cite{37, 38}. For such cases, it is desirable that at the steady state limit, all agents are having equal loads. In this work, we show that for a negative but linearly reinforced urn scheme such a limit is obtained under fairly general conditions on the replacement mechanism.

1.2. Model Description. In this work, we will only consider balanced urn schemes with \( k \)-colours, index by \( S := \{0, 1, \ldots, k-1\} \). More precisely, if \( R := ((R_{i,j}))_{0 \leq i, j \leq k-1} \) denotes the replacement matrix, that is, \( R_{i,j} \geq 0 \) is the number of balls of colour \( j \) to be placed in the urn when the colour of the selected ball is \( i \), then for a balanced urn, all row sums of \( R \) are constant. In this case, dividing all entries by the common row total, we may assume \( R \) is a stochastic matrix. We will also assume that the starting configuration \( U_0 := (U_{0,j})_{0 \leq j \leq k-1} \) is a probability distribution on the set of colours \( S \). As we will see from the proofs of our main results, this apparent loss of generality can easily be removed.

Denote by \( U_n := (U_{n,j})_{0 \leq j \leq k-1} \in [0, \infty)^k \) the random configuration of the urn at time \( n \). Also let \( F_n := \sigma(U_0, U_1, \cdots, U_n) \) be the natural filtration. We define a random variable \( Z_n \) by

\[
\mathbb{P}(Z_n = j \mid F_n) \propto w_\theta \left( \frac{U_{n,j}}{n+1} \right), \quad 0 \leq j \leq k-1.
\]

(1)

where \( w_\theta : [0,1] \to \mathbb{R}_+ \) is given by

\[
w_\theta(x) = \theta - x.
\]

(2)

\( \theta \geq 1 \) will be considered as a parameter for the model. Note that, \( Z_n \) represents the colour chosen at the \((n+1)\)-th draw. Starting with \( U_0 \) we define \( (U_n)_{n \geq 0} \) recursively as follows:

\[
U_{n+1} = U_n + \chi_{n+1} R.
\]

(3)

where \( \chi_{n+1} := (1(Z_n = j))_{0 \leq j \leq k-1} \).

We call the process \( (U_n)_{n \geq 0} \), a negative but linearly reinforced urn scheme with initial configuration \( U_0 \) and replacement matrix \( R \). In this work, we study the asymptotic properties of the following two processes:
**Random configuration of the urn:** Observe that for all \( n \geq 0 \),

\[
\sum_{j=0}^{k-1} U_{n,j} = n + 1. \tag{4}
\]

This holds because \( R \) is a stochastic matrix and \( U_0 \) is a probability vector. Thus the random configuration of the urn, namely, \( \frac{U_n}{n+1} \) is a probability mass function. Further,

\[
P(Z_n = j \mid \mathcal{F}_n) = \frac{\theta}{k\theta - 1} - \frac{1}{k\theta - 1} \frac{U_{n,j}}{n+1}, \quad 0 \leq j \leq k - 1. \tag{5}
\]

Thus, \( \frac{U_n A}{n+1} \) is the conditional distribution of the \((n + 1)\)-th selected colour, namely \( Z_n \), given \( U_0, U_1, \ldots, U_n \), where

\[
A_{k \times k} = \frac{\theta}{k\theta - 1} J_k - \frac{1}{k\theta - 1} I_k, \tag{6}
\]

and \( J_k := 1^T 1 \) is the \( k \times k \) matrix with all entries equal to 1 and \( I_k \) is the \( k \times k \)-identity matrix.

**Color count statistics:** Let \( N_n := (N_{n,0}, \ldots, N_{n,k-1}) \) be the vector of length \( k \), whose \( j \)-th element is the number of times colour \( j \) was selected in the first \( n \) trials, that is

\[
N_{n,j} = \sum_{m=0}^{n-1} 1(Z_m = j), \quad 0 \leq j \leq k - 1. \tag{7}
\]

It is easy to note that from (3) it follows

\[
U_{n+1} = U_0 + N_{n+1} R. \tag{8}
\]

1.3. **Outline.** In Section 2 we present the main results of the paper and the proofs are given in Section 3 and Section 4.

**2. The Main Results**

We define a new \( k \times k \) stochastic matrix, namely

\[
\hat{R} := RA = \frac{1}{k\theta - 1} (\theta J_k - R), \tag{9}
\]

where \( A \) is as defined in (6). As we state in the sequel, the asymptotic properties of \((U_n)_{n \geq 0}\) and \((N_n)_{n \geq 0}\) depends on whether the stochastic matrix \( \hat{R} \), is irreducible or reducible. We first state a necessary and sufficient condition for that.
2.1. A Necessary and Sufficient Condition for \( \hat{R} \) to be Irreducible.

We start with the following definitions, which are needed for stating our main results.

**Definition 1.** A directed graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) is called the graph associated with a \( k \times k \) stochastic matrix \( R = ((R_{i,j}))_{0 \leq i,j \leq k-1} \), if

\[ \mathcal{V} = \{0, 1, \ldots, k-1\} \text{ and } \mathcal{E} = \{(i,j) | R_{i,j} > 0; i, j \in \mathcal{V}\}. \]

**Definition 2.** A stochastic matrix \( R \) is called a *star*, if there exists a \( j \in \{0, 1, \ldots, k-1\} \), such that,

\[ R_{i,j} = 1 \text{ for all } i \neq j, \]

and in that case, we say \( j \) is the central vertex.

By definition, for the graph associated with a star replacement matrix, there is a central vertex such that each vertex other than the central vertex has only one outgoing edge and that is towards the central vertex. We note that in the definition of a star we allow the central vertex to have a self loop.

As we will see in the sequel, the asymptotic properties will depend on the irreducibility of the (new) stochastic matrix \( \hat{R} \), as defined in (9). Following lemma provides a necessary and sufficient condition for \( \hat{R} \) to be irreducible.

**Proposition 1.** Let \( R \) be a \( k \times k \) stochastic matrix with \( k \geq 2 \), then \( \hat{R} \) is irreducible, if and only if either \( \theta > 1 \) or \( \theta = 1 \) but \( R \) is not a star.

2.2. Asymptotics of the Random Configuration of the Urn.

2.2.1. Case when \( \hat{R} \) is Irreducible. Our first result is the almost sure asymptotic of the colour proportions.

**Theorem 1.** Let \( \hat{R} \) be irreducible. Then, for every starting configuration \( U_0 \),

\[ \frac{U_{n,j}}{n+1} \to \mu_j, \text{ a.s. } \forall 0 \leq j \leq k-1, \]  

(10)

where \( \mu = (\mu_0, \mu_1, \ldots, \mu_{k-1}) \) is the unique solution of the following matrix equation

\[ (\theta \mathbf{1} - \mu) \hat{R} = (k\theta - 1) \mu. \]  

(11)

**Remark 1.** Notice that if we define \( \nu = \mu A \), then from the equations (6) and (11), it follows that \( \nu \) is the unique solution of the matrix equation \( \nu \hat{R} = \nu \). Further, from equation (11) we have \( \mu = \nu \hat{R} \).

**Remark 2.** Since \( \frac{U_{n,j}}{n+1} \) is a bounded random variable, thus we get

\[ \frac{E[U_{n,j}]}{n+1} \to \mu_j, \text{ a.s., } \forall 0 \leq j \leq k-1, \]  

(12)

where \( \mu \) satisfies equation (11).
Remark 3. It is worth to note here that, the stochastic matrices $R$ and $\hat{R}$ both have uniform distribution as their unique stationary distribution, if and only if, $R$ is doubly stochastic, that is when $1R = 1$.

Our next result is a central limit theorem for the colour proportions.

**Theorem 2.** Suppose $\hat{R}$ is irreducible then there exists a $k \times k$ variance-covariance matrix $\Sigma = \Sigma(\theta, k)$, such that,

$$\frac{U_n - n\mu}{\sigma_n} \Rightarrow \mathcal{N}_k(0, \Sigma),$$

(13)

where for $k \geq 3$,

$$\sigma_n = \begin{cases} \sqrt{n \log n} & \text{if } k = 3, \theta = 1 \text{ and one of the eigenvalue of } R \text{ is } -1, \\ \sqrt{n} & \text{otherwise.} \end{cases}$$

(14)

and for $k = 2$ and $\theta \in [1, \frac{3}{2}]$,

$$\sigma_n = \begin{cases} \sqrt{n \log n} & \text{if the eigenvalues of } R \text{ are } 1 \text{ and } \lambda = \frac{1 - 2\theta}{2}; \\ \sqrt{n} & \text{if the eigenvalues of } R \text{ are } 1 \text{ and } \lambda < \frac{1 - 2\theta}{2}. \end{cases}$$

(15)

**Remark 4.** Note that $\Sigma$ is necessarily a positive semi-definite matrix because of (1).

**Remark 5.** It is worth noting here that the scaling is always by $\sqrt{n}$ for any parameter value $\theta \geq 1$ when $k \geq 4$. However, for small number of colors, namely, $k \in \{2, 3\}$, and certain specific parameter values, as given in equation (14) and (15) above has an extra factor of $\sqrt{\log n}$.

2.2.2. Case when $\hat{R}$ is Reducible. By Proposition 1, we know that $\hat{R}$ can be reducible, if and only if, $R$ is star and $\theta = 1$. Suppose $R$ is a star with $k \geq 2$ colours, then without any loss of generality we can write

$$R = \begin{pmatrix} \alpha_0 & \alpha_1 & \ldots & \alpha_{k-1} \\ 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \ldots & 0 \end{pmatrix} \text{ with } \sum_{j=0}^{k-1} \alpha_j = 1, \text{ and } \alpha_j \geq 0, \forall j,$$

(16)

by taking 0 as the central vertex. Taking $\theta = 1$, the matrix $\hat{R}$ is

$$\hat{R} = \frac{1}{k-1} \begin{pmatrix} 1 - \alpha_0 & 1 - \alpha_1 & \ldots & 1 - \alpha_{k-1} \\ 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \ldots & 1 \end{pmatrix},$$

(17)

which is clearly reducible. In the next theorem, we describe the limit of the urn configuration.
Theorem 3. Let $\theta = 1$ and replacement matrix $R$ be a star matrix as given in equation (10) and $R \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then,

$$
\frac{U_{n,0}}{n+1} \rightarrow 1, \text{ a.s.}
$$

Further, there exists a random variable $W \geq 0$, with $\mathbb{E}[W] > 0$, such that,

$$
\frac{U_{n,j}}{n^\gamma} \rightarrow \frac{\alpha_j}{k-1} W, \text{ a.s. } \forall j = 1, 2, \ldots, k-1,
$$

where $\gamma = \frac{1-\alpha}{k-1} < 1$.

Remark 6. In a trivial case, when $\gamma = 0$ or $(\alpha_0 = 1)$ we have

$$
U_{n,0} = U_{0,0} + n
$$

and

$$
U_{n,j} = U_{0,j} \quad \text{for all } j = 1, 2, \ldots, k-1.
$$

That is, at every time $n$, only colour 1 is reinforced into the urn.

Remark 7. When $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we get $\hat{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Notice that then $\hat{R}$ is the reinforcement rule for the classical Pólya urn scheme. Now using (3) we have

$$
\mathbb{E}\left[U_{n+1} \mid \mathcal{F}_n\right] = U_n + \frac{U_n}{n+1} = (n+2) \frac{U_n}{n+1},
$$

which implies that each coordinate of the vector $\frac{U_n}{n+1}$, is a positive martingale and hence converges. Moreover, by exchangeability and arguments similar to the classical Pólya urn, we can easily show that,

$$
\frac{U_{n,0}}{n+1} \rightarrow Z \text{ a.s.,}
$$

where $Z \sim \text{Beta}(U_{0,0}, U_{0,1})$.

2.3. Asymptotics of the Colour Count Statistics.

2.3.1. Case when $\hat{R}$ is Irreducible.

Theorem 4. Suppose $\hat{R}$ is irreducible then,

$$
\frac{N_{n,j}}{n} \rightarrow \frac{1}{k\theta - 1} \left[ \theta - \mu_j \right], \text{ a.s. } \forall 0 \leq j \leq k-1,
$$

where $\mu = (\mu_0, \mu_1, \ldots, \mu_{k-1})$ satisfies equation (11).

Theorem 5. Suppose $\hat{R}$ is irreducible, then there exists a variance-covariance matrix $\Sigma \equiv \Sigma(\theta, k)$, such that

$$
\frac{N_n - \frac{n}{k\theta - 1}(\theta \mathbf{1} - \mu)}{\sigma_n} \rightarrow \mathcal{N}_k \left(0, \Sigma\right),
$$

where $\sigma_n$ is given in the equations (14) and (15). Moreover,

$$
\Sigma = R^T \hat{\Sigma} R,
$$

(20)
where $\Sigma$ is as in Theorem 2.

**Remark 8.** Note that from definition (7), it follows that $\sum_{j=0}^{k-1} N_{n,j} = n$, thus $\tilde{\Sigma}$ is a positive semi-definite matrix. Further, from equation (20) it follows that $\text{rank}(\Sigma) \leq \text{rank}(\tilde{\Sigma})$ and equality holds, if and only if, the replacement matrix $R$ is non-singular.

2.3.2. Case when $\hat{R}$ is Reducible. Recall that $\hat{R}$ has the form given in (16) when it is reducible.

**Theorem 6.** Let $R$ be a star matrix with 0 as a central vertex and $\theta = 1$, such that $R \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then

$$\frac{N_{n,j}}{n} \rightarrow 0, \quad \text{a.s.}$$

and,

$$\frac{N_{n,j}}{n} \rightarrow \frac{1}{k-1}, \quad \text{a.s.} \forall 1 \leq j \leq k - 1.$$

**Remark 9.** For $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ using equation (8) and Remark (7) we get

$$\frac{N_{n,0}}{n+1} \rightarrow 1 - Z \quad \text{a.s.,}$$

where as before, $Z \sim \text{Beta}(U_0,0,U_0,1)$.

**Theorem 7.** Let $R$ be a star matrix with 0 as a central vertex and $\theta = 1$, such that $R \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then

1. if $\gamma = \frac{1-\alpha_0}{k-1} < 1/2$, then

$$\frac{1}{\sqrt{n}} \left( N_{n,j} - \frac{n}{k-1} \right) \Rightarrow N_k \left( 0, \frac{1}{k-1} I - \frac{1}{(k-1)^2} J \right),$$

where $N_{n,-} = (N_{n,1}, \ldots, N_{n,k-1})$, and

$$\frac{N_{n,0}}{\sqrt{n}} \rightarrow 0.$$

2. if $\gamma = \frac{1-\alpha_0}{k-1} \geq 1/2$, then

$$\frac{1}{n^\gamma} \left( N_{n,j} - \frac{n}{k-1} \right) \rightarrow P \frac{\alpha_j}{k-1} W, \quad \forall j \neq 0$$

and

$$\frac{N_{n,0}}{n^\gamma} \rightarrow \frac{1-\alpha_0}{k-1} W.$$ 

where $W$ is as given in Theorem 2.

**Remark 10.** Note that $\gamma < 1/2$, if and only if, $k \geq 4$ or $k = 3$ and $\alpha_0 > 0$ or $k = 2$ and $\alpha_0 > 1/2$. 

3. Proof of the Necessary and Sufficient Condition for \( \hat{R} \) to be Irreducible

Suppose \( G \) and \( \hat{G} \) are the directed graphs associated with the matrices \( R \) and \( \hat{R} \) respectively, as defined earlier. Observe that, \( \hat{R} \) is the product of two stochastic matrices, \( R \) and \( A \). The underlying Markov chain of \( \hat{R} \) can be seen as a two step Markov chain where the first step is taken according to \( R \) and the second step is taken according to \( A \). Recall from equation (\[9\]) that

\[
\hat{R} = \frac{1}{k\theta - 1} (\theta J - R).
\]

Now, to show that the Markov chain associated with \( \hat{R} \) is irreducible, it is enough to show that there exist a directed path between any two fixed vertices say \( u \) and \( v \), in \( \hat{G} \).

Clearly for \( \theta > 1 \), \( \hat{R}_{uv} > 0 \) for all \( u, v \), and thus \( \hat{R} \) is irreducible. Therefore, we only have to verify irreducibility for \( \theta = 1 \) case. For this we first fix two vertices, say \( u \) and \( v \). From equation (\[9\]) we get

\[
\hat{R}_{uv} = \frac{1 - R_{u,v}}{k - 1}.
\] (21)

To complete the proof, we will show that there is a path from \( u \) to \( v \) of length at most 2. We consider the following two cases:

**Case 1** \( R_{u,v} < 1 \): In this case, from equation (21) we get, \( \hat{R}_{uv} > 0 \). Therefore \((u, v)\) is an edge in \( \hat{G} \) and trivially there is a path of length 1 from \( u \) to \( v \) in \( \hat{G} \).

**Case 2** \( R_{u,v} = 1 \): In this case, \( u \) has no \( R \)-neighbor other than \( v \), that is \((u, v)\) is the only incoming edge to \( v \) in \( G \) and from equation (21), we have

\[
\hat{R}_{uv} = 0.
\]

As mentioned earlier for \( \theta = 1 \) and \( k = 2 \), \( \hat{R} \) is reducible only when \( R \) is the Friedman urn scheme, which is a star with two vertices. Thus in the rest of the proof we take \( k > 2 \), and show that \( \hat{R}_{uv}^2 > 0 \), that is there is a path of length 2.

Now, if \( R \) is not a star then there must exists a vertex \( l \) such that it leads to a vertex other than the central vertex, say \( m \) that is \( R_{l,m} > 0 \) \( (m \neq v) \). Now, according to \( \hat{R} \) chain, there is a positive probability of going from \( u \) to \( l \) in one step (first take a \( R \)-step from \( u \) to \( v \) which happens with probability 1 in this case, as \( R_{u,v} = 1 \), and then take a \( A \)-step to \( l \) with probability \( 1/(k-1) \)) and a positive probability of going from \( l \) to \( v \) in one step (first take a \( R \)-step from \( l \) to \( m \) with probability \( \hat{R}_{l,m} \), and then take a \( A \)-step to \( v \) with probability \( 1/(k - 1) \)). Therefore, there is path of length two in \( \hat{G} \) from \( u \) to \( v \) and thus the chain is irreducible.

**Remark 11.** Note that from the proof it follows that for a replacement matrix \( R \) with \( k > 2 \) such that, \( \hat{R} \) is irreducible, then \( \hat{R} \) is also aperiodic.
4. Proofs of the Main Results

We begin by observing the following fact. From equations (3), (5), (6) we get,
\[ E \left[ U_{n+1} \mid F_n \right] = U_n + E \left[ \chi_{n+1} \mid F_n \right] R = U_n + \frac{U_n}{n+1}AR. \] (22)
Thus,
\[ E \left[ U_{n+1} A \mid F_n \right] = U_n A + \frac{U_n}{n+1}ARA. \] (23)
Let \( \hat{U}_n := U_n A, n \geq 0 \), then
\[ \hat{U}_{n+1} = \hat{U}_n + \chi_{n+1}RA. \] (24)
and from equation (24) we get
\[ E \left[ \hat{U}_{n+1} \mid F_n \right] = \hat{U}_n + \frac{\hat{U}_n}{n+1}RA. \]

Therefore \( \left( \hat{U}_n \right)_{n \geq 0} \) is a classical urn scheme (uniform selection), with replacement matrix \( RA \). The construction \( \left( \hat{U}_n \right)_{n \geq 0} \) is essentially a coupling of a negative but linearly reinforced urn \( \left( U_n \right)_{n \geq 0} \) with replacement matrix \( R \), to a classical (positively reinforced) urn \( \left( \hat{U}_n \right)_{n \geq 0} \) with replacement matrix \( \hat{R} \). Note that, we get a one to one correspondence, as \( A \) is always invertible.

Proof of Theorem 1. Recall that, \( \hat{U}_n = U_n A \) is the configuration of a classical urn model with replacement matrix \( \hat{R} \). Since by our assumption, \( \hat{R} \) is irreducible therefore by Theorem 2.2. of [7], the limit of \( \frac{1}{n+1} \hat{U}_n \) is the normalized left eigenvector of \( \hat{R} \) associated with the maximal eigenvalue 1. That is
\[ \frac{\hat{U}_n}{n+1} \to \nu, \ a.s. \]
where \( \nu \) satisfies
\[ \nu \hat{R} = \nu. \]
Since \( U_n = \hat{U}_n A^{-1} \), we have
\[ \frac{U_n}{n+1} \to \mu, \ a.s., \]
where \( \mu = \nu A^{-1} \), and it satisfies the following matrix equation:
\[ (\theta 1 - \mu) R = (k \theta - 1) \mu. \]
This completes the proof. □

Proof of Theorem 2. Let \( 1, \lambda_1, \ldots, \lambda_s \) be the distinct eigenvalues of \( R \), such that, \( 1 \geq \Re(\lambda_1) \geq \cdots \geq \Re(\lambda_s) \geq -1 \), where \( \Re(\lambda) \) denotes the real part of the eigenvalue \( \lambda \). Recall from equation (9) that \( \hat{R} = \frac{1}{k \theta - 1} (\theta J_k - R) \).
So the eigenvalues of \( \hat{R} \) are \( 1, b \lambda_1, \ldots, b \lambda_s \), where \( b = \frac{-1}{k \theta - 1} \). Let \( \tau = \frac{\theta}{k \theta - 1} \).
max\{0, b \Re(\lambda_s)\}. Since \( \hat{U}_n = U_nA \), is a classical urn scheme with replacement matrix \( \hat{R} \), using Theorem 3.2 of [7], if
\[
b \Re(\lambda_s) \leq \frac{1}{2}
\] (25)
then there exists a variance-co-variance matrix \( \Sigma' \), such that
\[
\frac{\hat{U}_n - n\mu}{\sigma_n} \Rightarrow N_k (0, \Sigma')
\]
where
\[
\sigma_n = \begin{cases} \sqrt{n \log n} & \text{if } b \Re(\lambda_s) = \frac{1}{2}, \\ \sqrt{n} & \text{if } b \Re(\lambda_s) < \frac{1}{2}. \end{cases}
\] (26)
Notice that,
\[
b \Re(\lambda_s) \leq \frac{1}{2} \iff \Re(\lambda_s) \geq -\frac{1}{2}(k\theta - 1).
\] (27)
Now since \( \theta \geq 1 \) and \( \Re(\lambda_s) \geq -1 \) the above equation (27) holds whenever \( k \geq 3 \). Further, for \( k \geq 3 \), equality in (27) holds if and only if, \( \theta = 1 \), and \( k = 3 \). Moreover, for \( k = 2 \), the condition is equivalent to \( \Re(\lambda_s) \leq \frac{1-2\theta}{2} \). Thus, \( \sigma_n \) is given in (14) and (15) Therefore,
\[
\frac{U_n - n\mu}{\sigma_n} \Rightarrow N_k (0, \Sigma)
\]
where \( \Sigma = A^T\Sigma' A \). \( \square \)

**Proof of Theorem 3.** Without lose of any generality, we will assume \( \gamma > 0 \) (equivalently \( \alpha_0 < 1 \)), as otherwise the result is trivial as described in Remark 6. Since the matrix \( \hat{R} \), as given in (17) is reducible without isolated blocks. Using Proposition 4.3 of [28] we get,
\[
\frac{\hat{U}_{n,0}}{n+1} \to 0 \quad \text{and} \quad \frac{\hat{U}_{n,j}}{n+1} \to \frac{1}{k-1}, \quad \forall j \neq 0.
\]
which implies
\[
\frac{U_{n,0}}{n+1} \to 1 \quad \text{and} \quad \frac{U_{n,j}}{n+1} \to 0, \quad \forall j \neq 0.
\]
Now, note that the matrix \( \hat{R} \) given in (17) has eigenvalues 1, \( \gamma \) and 0, 0, \ldots, 0 (\( k-2 \) times), where \( \gamma = (1 - \alpha_0)/(k - 1) \). The eigenvector corresponding to the non-principal eigenvalue \( \gamma \) is
\[
\xi = \frac{1}{\gamma} (0, 1, 1, \ldots, 1)'.
\]
Therefore,
\[
\mathbb{E} \left[ U_{n+1}\xi \mid \mathcal{F}_n \right] = U_n \left[ I + \frac{\hat{R}}{n+1} \right] \xi = U_n \xi \left[ 1 + \frac{\gamma}{(n+1)} \right].
\]
Let $\Pi_n(\gamma) = \prod_{i=1}^{n} \left[ 1 + \frac{\gamma}{i} \right]$ then, $W_n := U_n\xi/\Pi_n(\gamma)$ is a non-negative martingale and using Euler’s product, for large $n$

$$\Pi_n(\gamma) \sim \frac{n^\gamma}{\Gamma(\gamma + 1)}.$$ 

We now show that this martingale is $L^2$ bounded, which will then imply that

$$\frac{U_n\xi}{n^\gamma} \to W$$

where $W$ is a non-degenerate random variable. More precisely, $W$ is nonzero with positive probability. We can write,

$$E\left[ W_n^2 \mid F_n \right] = W_n^2 + E\left[ (W_{n+1} - W_n)^2 \mid F_n \right]$$

and

$$W_{n+1} - W_n = \frac{1}{\Pi_{n+1}(\gamma)} \left[ U_{n+1}\xi - U_n\xi \left( 1 + \frac{\gamma}{n+1} \right) \right]$$

$$= \frac{1}{\Pi_{n+1}(\gamma)} \left[ \chi_n R\xi - \frac{\gamma}{n+1} U_n\xi \right]$$

$$= \frac{1}{\Pi_{n+1}(\gamma)} \left[ (k-1)\chi_{n,0} - \frac{(n+1) - U_{n,0}}{n+1} \right]$$

$$= \frac{k-1}{\Pi_{n+1}(\gamma)} \left[ \chi_{n,0} - E\left[ \chi_{n,0} \mid F_n \right] \right]$$

Therefore,

$$E\left[ W_{n+1}^2 \mid F_n \right] W_n^2 + \frac{(k-1)^2}{\Pi_{n+1}(\gamma)} E\left[ \chi_{n,0} \mid F_n \right] - E\left[ \chi_{n,0} \mid F_n \right]^2$$

$$\leq W_n^2 + \frac{1 - \alpha_0}{(n+1)\Pi_{n+1}(\gamma)} \frac{U_n\xi}{\Pi_{n+1}(\gamma)}$$

$$\leq W_n^2 + \frac{1 - \alpha_0}{(n+1)\Pi_{n+1}(\gamma)} W_n$$

$$\leq W_n^2 + \frac{1 - \alpha_0\Gamma(\gamma + 1)}{2(n+1)^{\gamma+1}} (1 + W_n^2)$$

The last inequality holds because $2W_n \leq 1 + W_n^2$. Let $c := \frac{1}{2}(1 - \alpha_0)\Gamma(\gamma + 1)$, then
\[ E \left[ W_{n+1}^2 + 1 \mid F_n \right] \leq \left( 1 + \frac{c}{(n+1)^{\gamma+1}} \right) (1 + W_n^2) \]

\[ \leq (1 + W_0^2) \prod_{j=1}^{n} \left( 1 + \frac{c}{(j+1)^{\gamma+1}} \right) \]

\[ \leq (1 + W_0^2) \exp \left( \sum_{j=1}^{n} \frac{c}{(j+1)^{\gamma+1}} \right) < \infty \quad (\text{since } \gamma > 0). \]

Thus \( W_n \) is \( \mathcal{L}^2 \)-bounded and hence converges to a non-degenerate random variable say \( W \). Now for a star matrix \( R \) (as given in equation (16)), the recursion (3) reduces to

\[ U_{n+1,0} = U_{n,0} + \alpha_0 \chi_{n+1,0} + (1 - \chi_{n+1,0}). \]

and

\[ U_{n+1,h} = U_{n,h} + \alpha_h \chi_{n+1,0} \quad \forall h \neq 0 \]  

(29)

Recall that for \( h \neq 0 \), \( \alpha_h > 0 \), dividing both sides by \( \alpha_h \), we get

\[ \frac{U_{n+1,h}}{\alpha_h} = \frac{U_{0,h}}{\alpha_h} + \sum_{j=1}^{n+1} \chi_j. \]

Since the above relation holds for every choice of \( h > 0 \), we get

\[ \frac{U_{n+1,h}}{\alpha_h} - \frac{U_{n+1,l}}{\alpha_l} = \frac{U_{0,h}}{\alpha_h} - \frac{U_{0,l}}{\alpha_l} \]

(30)

for any \( h, l \in \{1, 2, \ldots, k-1\} \). Multiplying the above equation by \( \frac{\alpha_h}{1-\alpha_0} \) and taking sum over \( l \neq 0 \), we get

\[ \frac{U_{n,h}}{\alpha_h} - \frac{1}{1-\alpha_0} \sum_{l \neq 0} U_{n,l} = \frac{U_{0,h}}{\alpha_h} - \frac{1}{1-\alpha_0} \sum_{l \neq 0} U_{0,l}, \]

which can be written as,

\[ \frac{U_{n,h}}{\alpha_h} - \frac{1}{k-1} U_n \xi = \frac{U_{0,h}}{\alpha_h} - \frac{1}{k-1} U_0 \xi. \]

Now dividing both sides by \( n^\gamma \),

\[ \frac{1}{n^\gamma} \frac{U_{n,h}}{\alpha_h} - \frac{1}{k-1} \frac{U_n \xi}{n^\gamma} = \frac{1}{n^\gamma} \left[ \frac{U_{0,h}}{\alpha_h} - \frac{1}{k-1} U_0 \xi \right]. \]

Note that the right hand side of the above expression goes to 0 as \( n \) tends to infinity. Therefore

\[ \lim_{n \to \infty} \frac{1}{n^\gamma} \frac{U_{n,h}}{\alpha_h} - \frac{1}{k-1} \frac{U_n \xi}{n^\gamma} = 0 \]
Using the limit from (28) we get,
\[ \frac{U_{n,h}}{n^\gamma} \to \frac{\alpha_h}{k-1} W. \]
\[ \square \]

**Proof of Theorem 4.** Note that from equation (7) and (8), we can write
\[ N_n = \sum_{i=1}^{n} \left( \chi_i - E \left[ \chi_i \left| F_{i-1} \right. \right] \right) + \sum_{i=1}^{n} E \left[ \chi_i \left| F_{i-1} \right. \right], \]
\[ = \sum_{i=1}^{n} \left( \chi_i - E \left[ \chi_i \left| F_{i-1} \right. \right] \right) + \frac{1}{k\theta - 1} \left[ \theta 1 - \frac{U_{i-1}}{i} \right], \quad (31) \]

Since \( \left( \chi_i - E \left[ \chi_i \left| F_{i-1} \right. \right] \right)_{i \geq 1} \) is a bounded martingale difference sequence, using Azuma’s inequality (see [14]) we get
\[ \frac{1}{n} \sum_{i=1}^{n} \left( \chi_i - E \left[ \chi_i \left| F_{i-1} \right. \right] \right) \to 0, \text{ a.s..} \]
\[ (32) \]
Using Theorem [11] and Cesaro Lemma (see [4]), we get
\[ \frac{N_{n,j}}{n} \to \frac{1}{k\theta - 1} \left[ \theta - \mu_j \right], \quad \text{a.s. } \forall 0 \leq j \leq k - 1. \]
\[ \square \]

**Proof of Theorem 5.** Notice that under our coupling \( N_n \) remains same for the two processes, namely, \( (U_n)_{n \geq 0} \) and \( \left( \hat{U}_n \right)_{n \geq 0} \). Thus applying Theorem 4.1 of [7] on the urn process \( \left( \hat{U}_n \right)_{n \geq 0} \) we conclude that there exists a matrix \( \hat{\Sigma} \) such that,
\[ \frac{N_n - n\mu A}{\sigma_n} \Rightarrow \mathcal{N} \left( 0, \hat{\Sigma} \right) \]
Finally the equation (20) follows from (8). This completes the proof. \[ \square \]

**Proof of Theorem 6.** The proof follows from equation (31) and (32). \[ \square \]

**Proof of Theorem 7.** Let \( M_n = (M_{n,0}, M_{n,1}, \ldots, M_{n,k-1}) \) be a martingale, where \( M_{n,j} := \sum_{i=1}^{n} \chi_{i,j} - E \left[ \chi_{i,j} \left| F_{i-1} \right. \right] \). Note that \( (M_n) \) is a a bounded increment martingale. and \( X_n := \frac{1}{\sqrt{n}} M_n \). That is, for a fixed colour \( j \), \( X_{n,j} = \frac{1}{\sqrt{n}} \left( \chi_{i,j} - E \left[ \chi_{i,j} \left| F_{i-1} \right. \right] \right) \). Let \( M_{n,-} := (M_{n,1}, \ldots, M_{n,k-1}) \) and \( X_{n,-} := (X_{n,1}, \ldots, X_{n,k-1}) \).
In this proof, we first provide a central limit theorem for $M_n$, and then for $N_n$. Observe that the $(l, m)$-th entry of the matrix $E \left[ X_i^T X_i \mid \mathcal{F}_{i-1} \right]$ is

$$\begin{align*}
\frac{1}{n} E \left[ \chi_{i,l} \chi_{i,m} \mid \mathcal{F}_{i-1} \right] &= E \left[ \chi_{i,l} \mid \mathcal{F}_{i-1} \right] E \left[ \chi_{i,m} \mid \mathcal{F}_{i-1} \right] \\
&= \begin{cases} 
\frac{1}{n} E \left[ \chi_{i,l} \mid \mathcal{F}_{i-1} \right] \left( 1 - E \left[ \chi_{i,l} \mid \mathcal{F}_{i-1} \right] \right) & \text{if } l = m, \\
\frac{1}{n} E \left[ \chi_{i,l} \mid \mathcal{F}_{i-1} \right] E \left[ \chi_{i,m} \mid \mathcal{F}_{i-1} \right] & \text{if } l \neq m
\end{cases}
\end{align*}$$

$$\begin{align*}
\frac{1}{n} E \left[ \chi_{i,l} \mid \mathcal{F}_{i-1} \right] &= \begin{cases} 
\frac{1}{n} \left( 1 - \frac{U_{i-1,l}}{1} \right) \left( 1 - \frac{1}{k-1} \left( 1 - \frac{U_{i-1,l}}{1} \right) \right) & \text{if } l = m, \\
\frac{1}{n(k-1)^2} \left( 1 - \frac{U_{i-1,l}}{1} \right) \left( 1 - \frac{U_{i-1,m}}{1} \right) & \text{if } l \neq m
\end{cases}
\end{align*}$$

So, as $n \to \infty$, (using Theorem 3) we have

$$\sum_{i=1}^{n} E \left[ X_i^T X_i \mid \mathcal{F}_{i-1} \right]_{(l,m)} \to \begin{cases} 
\frac{(k-2)}{(k-1)^2} & \text{if } l = m, \\
\frac{-1}{(k-1)^2} & \text{if } l \neq m
\end{cases}$$

Therefore,

$$\sum_{i=1}^{n} E \left[ X_i^T X_i \mid \mathcal{F}_{i-1} \right] \to \frac{1}{k-1} I - \frac{1}{(k-1)^2} J,$$

and by the martingale central limit theorem [29], we get

$$\frac{1}{\sqrt{n}} M_{n,-} \Longrightarrow N_k \left( 0, \frac{1}{k-1} I - \frac{1}{(k-1)^2} J \right).$$

(33)

Now for colour 0, we have

$$\frac{1}{\sqrt{n}} M_{n,0} = \frac{1}{\sqrt{n}} \sum_{j=1}^{k-1} M_{n,-}$$

which implies

$$\frac{1}{\sqrt{n}} M_{n,0} \xrightarrow{P} 0.$$ 

We now prove the central limit theorem for $N_n$. By equation (61), we have

$$N_n = M_n + \sum_{i=1}^{n} E \left[ \chi_{i} \mid \mathcal{F}_{i-1} \right]$$

Therefore,

$$N_{n,-} - \frac{n}{k-1} = M_{n,-} - \frac{1}{k-1} \sum_{i=1}^{n} U_{i-1,-}$$

(34)

Form Theorem 3 we know that for each $j \neq 0$

$$\frac{U_{i-1,j}}{i^2} \to \frac{\alpha_j}{k-1} W, \ a.s..$$
\[
\sum_{i=1}^{n} \frac{U_{i-1,j}}{i} \sim \frac{\alpha_j}{k-1} W \sum_{i=1}^{n} i^{\gamma-1} \sim \frac{\alpha_j}{k-1} W n^{\gamma}.
\]

Therefore,
\[
\frac{1}{n^\gamma} \sum_{i=1}^{n} \frac{U_{i-1,j}}{i} \rightarrow \frac{\alpha_j}{k-1} W \ a.s. \quad (35)
\]

Therefore for \( \gamma < 1/2 \), using equation (33), (34) and (35) we get
\[
\frac{1}{\sqrt{n}} \left( N_{n,-} - \frac{n}{k-1} \right) \implies N_k \left( 0, \frac{1}{k-1} I - \frac{1}{(k-1)^2} J \right),
\]
and for \( \gamma \geq 1/2 \),
\[
\frac{1}{n^\gamma} \left( \frac{n}{k-1} - N_{n,j} \right) \overset{P}{\rightarrow} \frac{\alpha_j}{k-1} W \forall j,
\]
since then \( M_{n,j}/n^\gamma \rightarrow 0 \). For \( j = 0 \), we have
\[
N_{n,0} = n - \sum_{j=1}^{k-1} N_{n,j} = \sum_{j=1}^{k-1} \left( \frac{n}{k-1} - N_{n,j} \right)
\]

Therefore for \( \gamma \geq 1/2 \), we have
\[
\frac{1}{n^\gamma} N_{n,0} = \sum_{j=1}^{k-1} \frac{1}{n^\gamma} \left( \frac{n}{k-1} - N_{n,j} \right) \overset{P}{\rightarrow} \frac{1}{k-1} W \sum_{j=1}^{k-1} \alpha_j = \frac{1 - \alpha_0}{k-1} W.
\]
and for \( \gamma > 1/2 \) we have
\[
\frac{N_{n,0}}{\sqrt{n}} \overset{P}{\rightarrow} 0.
\]

\[\Box\]

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