A STOCHASTIC APPROACH TO MODEL HOUSING MARKETS: THE US HOUSING MARKET CASE

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Abstract. This study aims to estimate the price changes in housing markets using a stochastic process, which is defined in the form of stochastic differential equations (SDEs). It proposes a general SDEs system on the price structure in terms of house price index and mortgage rate to establish an effective process. As an empirical analysis, it applies a calibration procedure to an SDE on monthly S&P/Case-Shiller US National Home Price Index (HPI) and 30-year fixed mortgage rate to estimate parameters of differentiable functions defined in SDEs. The prediction power of the proposed stochastic model is justified through a Monte Carlo algorithm for one-year ahead monthly forecasts of the HPI returns. The results of the study show that the stochastic processes are flexible in terms of the choice of structure, compact with respect to the number of exogenous variables involved, and it is a literal method. Furthermore, this approach has a relatively high estimation power in forecasting the national house prices.

1. Introduction. The pattern of house prices, directions in the price changes, or volatility of prices can be used as an indicator to understand housing market dynamics. Given the implications of such changes on the welfare, there has been a considerable interest in identifying the driving factors of the house price volatility and its further implications [11]. The volatility of house prices can be managed by understanding its influencing factors. For this reason, it is vital to reveal the determinants of house prices or the reasons of experiencing volatility in housing markets. Although many fundamental factors are well-known, especially after the recent global financial crisis, the studies on the influence of macroeconomic indicators on housing market have received a considerable attention in recent years (see for instance [1, 16, 21, 22, 36, 38]). Along with the impact of many macroeconomic indicators on financial market behavior, mainly the interest rate takes the first place as an external financing factor to purchase a property. The demand for real estate is known to be highly related to the cost of borrowing funds to purchase the property. Therefore, an increase or decrease in interest rate influences the house price movements.

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In real estate markets, external funding is mostly done through mortgages, which have a high association with the interest rate movements. Taking into account some important financial indicators such as 30-year fixed mortgage rate (FRM), 3-Month Treasury Bill, 6-Month Treasury Bill, 3-Month Eurodollar deposit rate (London), and housing market indicator, S&P/Case-Shiller National Home Price Index (HPI), the United States (US) series between 1975 and 2016, we see that interest rates and mortgage rate follow a similar pattern at which mortgage rate yields higher rate than the interest rates (Figure 1). It is an expected result as the treasury notes are the safest investment instruments since the US government issues guarantees on them where as mortgage rate is not. Furthermore, the duration of mortgage rate has to be longer due to the nature of house financing business. Figure 1 illustrates that both mortgage and interest rates show a reverse pattern and negative association to HPI. In Figure 1, we also observe two striking dynamic structures. First, the periodic patterns and reverse trend components are consistent features of the housing market. Second, a strong rise is consistent, even during the 2001 recession, reaching to a remarkable rise in 2006 compared to the position of mortgage rate. This case is commonly perceived as a bubble, indicated by a circle quoting to the financial crises [9, 10]. It is important to note that an increase in the prices, even resembling a bubble, is triggered by preceding prices in time. In addition, the mortgage rate appears to be less vulnerable to financial crises compared to interest rates and house prices.

![Figure 1](image_url)

**Figure 1.** Development of US National Home Price Index (1975–2016) with respect to the selected financial market indicators.

In housing markets, traditional price prediction methods are based on the comparison of the cost and selling prices without any internationally pronounced standards and certification process. A vast number of studies describing housing market dynamics utilizes many approaches, mostly in the frame of econometric models. These models usually reproduce and capture a sub-sample of characteristics in housing markets. Common valuation methods such as hedonic and multiple regression enable researchers to display the importance and impact of the significant housing characteristics and recommend a wide range of variables such as market regulation, physical conditions, spatial and national economic indicators, on the evolvement of house prices [12, 14, 28]. However, especially, the rigid assumptions underlying in
conventional models such as normality, independence among explanatory variables and linearity become an obstacle for assuring the prediction accuracy. Therefore, a model having a high forecasting power, and requiring a minimum number of contributing variables will enable market players to portray housing market tendency conveniently. Without dealing with collecting all related variables in the time frame defined, one can explain housing market price behavior and capture the stochastic pattern in terms of a major indicator. Besides, the multicollinearity among contributing variables may distort the accuracy of the prediction power in conventional methods. Nevertheless, selecting mostly linked contributing variable as a major indicator of house price movement reduces the cumbersome search for all relevant information, simplifies the housing market modeling and may increase the price prediction accuracy.

The strong dependence on housing markets and mortgage rate leads us to define a stochastic process which captures the behavior of house prices as well as its underlying factors using SDE approach. Hence, unlike many other studies, this study is based on one indicator affecting the house price structure and therefore, it reduces the complexity of dealing with many other contributing variables. In the proposed process SDE is utilized, as its rigorous mathematical descriptions include the influence of the historical prices as well as the impact of other variables. From the modeling point of view, to emphasize on the stochastic pattern of house price developments in time, we consider US housing price index and mortgage rate whose dynamics are observable over long years. The mortgage rate is taken as a basic indicator due to its major role in increasing (decreasing) the demand in housing markets and its high association with the interest rate. Based on the proposed SDEs, we analytically derive theoretical house prices associated with mortgage rate whose pattern also follow a certain SDE. In order to predict future housing market behavior, a calibration procedure is employed to the SDE and the required parameter sets are estimated based on the historical data. To achieve this, an optimization algorithm is adapted to the estimation process which takes into account the underlying probabilistic assumptions of the random terms in the SDEs. The critical and cumbersome part of this pricing approach is the precise description of the stochastic process governing the behavior of the house price and the mortgage rate. The main characteristics of this process are to capture the exact nature of both variables. This study contributes to the existing literature by deriving HPI returns in terms of mortgage rate using SDEs as an alternative method to the econometric methods.

The organization of the paper is as follows: Section 2 introduces the literature on traditional housing market forecasting. Section 3 is dedicated to present the system of SDE equations in modeling house price index in terms of mortgage rate and its theoretical derivation. Model calibration and forecasting the house prices based on real-world data are illustrated in Section 4. The last section of the study concludes the paper.

2. Literature Review. In recent years, many studies forecasting the house prices published due to its growing impact on consumption and financial markets [26, 29, 32, 34]. Many of the authors employ econometric models to determine the effect of particular housing characteristics with the selling price [5, 13, 24, 25, 35]. However, these models require numerous variables, which increase the model cost and due to characteristics of the explanatory variables, have high potential in creating multicollinearity. Moreover, the performance of house price prediction based on the
suggested models may lose their accuracy over time. As suggested by [40], the outcomes of such models differ not only in terms of the size of the parameters and their statistical significance but also in some cases, the sign of the coefficients. Therefore, they may reflect only the outcomes for a specific time and a specific location. Moreover, [32] emphasize that there is no single variable that can be considered as the most contributing one and therefore, it is difficult to identify apriori and particular variable or small set of variables. In this regard, the utilization of econometric models in forecasting house prices may not always be precise as expected. However, the existing literature indicates that relatively few studies use alternative modeling techniques for house price forecasting such as multivariate time series, which requires a strong underlying theoretical relationship [2, 17, 39]. There exist studies employing uni-variate time series approach with a special focus on nonlinear price dynamics [8, 26, 31]. These studies explore a variety of predictors of the housing market, which are beyond simple autoregressive models. For instance, [32] focuses on the forecastability of house prices in the states of US, and it shows that autoregressive models perform relatively well for interior states whereas they do not perform well for coastal states. It interprets the result as an evidence of a disconnection among prices and fundamentals.

As summarized above housing market modeling is mostly done using linear and nonlinear time series techniques and some other econometric techniques. However, to our knowledge, the implementation of a fully stochastic model in forecasting prices in housing markets does not exist in the literature. In a similar direction to [23] as a guiding literature, we implement an SDE structure to understand housing markets in terms of HPI and mortgage rate. This paper differs from [23] in the following aspects: i) SDE structure is composed of different contributing functions and their analytical solutions under these assumptions are derived. ii) We employ the model on US housing index and long-term mortgage rate whereas the guiding literature considers only the prices and interest rate. iii) The Ornstein–Uhlenbeck process is included to model the mortgage rate and the model is calibrated using observed data, whereas the guiding literature generally illustrates the findings with simulations.

3. The Economic Implications and the Model. Stochastic models, especially in the form of SDEs are commonly used in financial derivatives pricing and in real options analysis under the assumption of a fully competitive market [4, 15, 19, 33]. Among those Gaussian or Geometric Brownian motion structures are the most commonly implemented ones due to their nice properties [3, 7, 30]. The proposed house pricing model is an arbitrage model in which the stochastic behavior of the house prices and a co-factoring variable, mortgage rate are exogenously given. The SDE approach in developing a price structure in housing markets is defined according to the assumptions on its certain characteristics for its analytical simplicity. The price of any house on the market can then be derived as a function of these primitives, imposing the condition that no arbitrage profits exist as in the perfect markets.

In an efficient market, where prices instantly change to reflect a new information, the price would be an accurate reflection of all currently known supply and demand factors and would include an adjustment based on expected future changes in the supply-demand relations. Although the efficient market hypothesis in housing markets was first conducted in mid-eighties by [27] and some of other studies
following this study support the weak and semi-strong forms of the efficient market hypothesis [18], there is still lack of unanimous conclusion in the efficiency of this market. Under the assumption that the actions of the uninformed and the irrational investors are uncorrelated and random, their actions would cancel out and the market would agree on the same prices. For this reason, a continuous-time model relying on stationary diffusion processes is aimed to represent the price behavior of housing markets.

Most of the buyers finance their housing with primarily through a debt which causes a high correlation between house price and the mortgage rate as illustrated earlier (Figure 1). Such close association and high statistical dependence between house prices and mortgage rate is expected to define a stochastically interacting system of equations. Therefore, the model of house price is assumed to follow an evaluation process similar to the Black-Scholes-Merton (BSM) defined as follows.

**Definition 3.1.** Let \((\Omega, \mathcal{F}, (\mathcal{F})_{t \in [0,T]}, \mathbb{P})\) be a filtered probability space with required conditions. Consider an economy with two correlated state variables which are two non-negative adapted processes, the house price, \(h_t\), and mortgage rate, \(r_t\), defining a joint stochastic process,

\[
\begin{align*}
\frac{dh_t}{h_t} &= f_1(r_t, h_t) \, dt + f_2(h_t, \sigma_h) \, dZ_t, \\
\frac{dr_t}{r_t} &= f_3(r_t) \, dt + f_4(r_t, \sigma_r) \, dW_t, \\
\frac{dZ_t}{dW_t} &= \rho \, dt,
\end{align*}
\]

where, \(Z_t\) and \(W_t\) are two correlated Gaussian random variables, with corresponding means zero and variances \(t\), defined under the same natural probability measure \(\mathbb{P}\) and the correlation is \(\rho\). Here, at least twice differentiable functions, \(f_i, i = 1, 2, 3, 4\), determine the random components at which \(f_i, i = 1, 3\) correspond to the drift rates, whereas \(f_i, i = 2, 4\) describe the instantaneous part of the unanticipated HPI price and mortgage rate due to the volatiles related to each variable.

It should be rephrased in Equation (2) that the mortgage rate is expected to change at any time at a rate \(f_3(r_t)\). However, the actual change stems from the unbiased component \(f_4(r_t, \sigma_r)\) which is serially uncorrelated and follows normally distributed disturbances in the economy. It should be noted that the mortgage rate diffusion processes can be either identified with observed historical data or in particular with any of the well-known short rate interest rate models [6, 20, 37], which may increase the complexity of the model. For this reasons, the SDE system given in Definition 3.1 sets up a generalization of this feature to construct a more robust pricing structure. Another point to be mentioned is that the model can be easily tractable and flexible with respect to the choice of the functions \(f_i, i = 1, 2, 3, 4\). In this frame, the selection of the most appropriate model is an important issue and may not always be trivial. Based on the guiding literature [3, 7, 30], a selection of \(f_i\’s\) may be done. Specifically, selecting a linear relation between the house prices based on [33] resembles the econometric models commonly utilized in the literature for estimating the house price. Additionally, it should be emphasized that Equation (1) is a standard stochastic process structure for the underlying asset price allowing for a convenience yield, which evolves with a stochastic process structure introduced in Equation (2). In this setting, if one selects a fixed rate, \(r\), the model will become a one-factor model and the effect of the mortgage rate on the house price will not be observed properly.
3.1. Model Selection. The SDE approach is assumed to explain well the characteristics of prices in terms of the underlying distribution of housing returns (log-prices) interpreted intuitively from observed house prices. One-factor models such as Geometric Brownian Motion for the log return of the index price process and an Ornstein-Uhlenbeck (OU) process to represent the mortgage rate are proposed with prespecified parameters. Analogously, the selection of \( f_i, i = 1, 4 \), in Proposition 1, yields the proposed model for HPI. In index return modeling (Equation (4)), the parameter \( \mu_h \) represents the total expected return on HPI returns, \( \sigma_h \) is the constant volatility and \( \lambda \) is the rate at which \( r_t \) reverts to \( \mu_h \). Similarly, the mortgage rate model contains the parameters \( \mu_r \) representing the mean level at which the short rate will evolve around, in the long run, \( \kappa \) denoting the rate of reversion that characterizes the speed at which future trajectories will revert back to \( \mu_r \), and \( \sigma_r \) standing for the volatility of mortgage rate (Equation (5)).

This system of modeling in equity markets is firstly introduced by [15], which is taken in this paper as the guiding model to adopt the process of house price index log-returns \( h_t \), as a geometric Brownian motion where the growth rate is adjusted by mortgage rate \( r_t \). Since HPI is represented as a function of mortgage rate, this modest and flexible model embeds the properties in both dimensions.

**Proposition 1.** Given the parameters \( \lambda > 0, \mu_h \in \mathbb{R}, \sigma_h > 0, \kappa > 0, \mu_r \in \mathbb{R} \) and \( \sigma_r > 0 \), we define

\[
\begin{align*}
\frac{dh_t}{h_t} &= \lambda (\mu_h - r_t) \, dt + \sigma_h dZ_t, \\
\frac{dr_t}{r_t} &= \kappa (\mu_r - r_t) \, dt + \sigma_r dW_t, \\
dZ_t dW_t &= d\rho.
\end{align*}
\]

whose solutions are derived as

\[
\begin{align*}
h_T &= h_t e^{\left(\lambda \mu_h - \frac{\sigma_h^2}{2}\right)(T-t) + \sigma_h (Z_T - Z_t) - \lambda \int_t^T r_s \, ds}, \\
r_T &= r_t e^{-\kappa(T-t)} + \kappa \mu_r \left(1 - e^{-\kappa(T-t)}\right) + \sigma_r \int_t^T e^{-\kappa(T-s)} \, dW_s.
\end{align*}
\]

**Proof.** Using Itô’s Theorem and taking the logarithm we find

\[
\ln(h_T) = \ln(h_t) + \int_t^T \frac{1}{h_s} \frac{dh_s}{h_s} - \frac{1}{2} \int_t^T \frac{1}{h_s^2} d[h_h]_s
\]

\[
= \ln(h_t) + \int_t^T \lambda (\mu_h - r_s) \, ds + \int_t^T \sigma_h dZ_s - \frac{1}{2} \int_t^T \sigma_h^2 (\rho^2 + 1 - \rho^2) \, ds.
\]

By rearranging Equation (9) we obtain,

\[
\ln \left( \frac{h_T}{h_t} \right) = \int_t^T \left( \lambda (\mu_h - r_s) - \frac{\sigma_h^2}{2} \right) ds + \int_t^T \sigma_h dZ_s,
\]

which yields the house price index as

\[
h_T = h_t e^{\left(\lambda \mu_h - \frac{\sigma_h^2}{2}\right)(T-t) + \sigma_h (Z_T - Z_t) - \lambda \int_t^T r_s \, ds}.
\]

To derive the expression for, \( r_t \), we solve the O.U. process by defining a process

\[
X_t = r_t e^{\kappa t},
\]
which results in
\[ \begin{align*}
    dX_t &= e^{\kappa t} (\kappa r_t dt + dr_t) \\
    &= e^{\kappa t} (\kappa \mu_r dt + \sigma_r dW_t). 
\end{align*} \]  

(13)

By integrating Equation (13), we obtain
\[ X_T = x_t + \kappa \mu_r \left(e^{\kappa T} - e^{\kappa t}\right) + \sigma_r \int_t^T e^{\kappa s} dW_s. \]  

(14)

Rearrangement of Equation (14) ends up with
\[ r_T = r_t e^{-\kappa (T-t)} + \kappa \mu_r \left(1 - e^{-\kappa (T-t)}\right) + \sigma_r \int_t^T e^{-\kappa (T-s)} dW_s. \]

Note that the solution of \( r_t \) is composed the sum of a deterministic term and an integral of a deterministic function with respect to a Wiener process. Having this property in \( r_t \), we assume, under the normality assumption, the mean and the variance of mortgage rate can be derived using the martingale and Itô isometry of Brownian motion.

**Corollary 1.** Given \( r_t \) is normally distributed, the expected value, \( E[r_T] \) and the variance, \( \text{Var}(r_T) \), of mortgage rate are derived as follows:
\[ \begin{align*}
    E[r_T] &= r_t e^{-\kappa (T-t)} + \kappa \mu_r \left(1 - e^{-\kappa (T-t)}\right) \\
    \text{Var}(r_T) &= \frac{\sigma_r^2}{2\kappa} \left(1 - e^{-2\kappa (T-t)}\right). 
\end{align*} \]

4. **Model Justification and Forecasting: S&P Case-Shiller Case.** The main obstacle in the empirical implementation of house pricing with SDEs arises from the determination of the parameters associated to the proposed models. The reason is that the underlying assets in the housing markets are physically produced immobile products and the market liquidation is not as easy as in the fully competitive markets. Moreover, since housing markets are characterized by experimentation and bargaining among the potential buyer and seller, both parties naturally use their experience which may manipulate the prices in this market [12]. More importantly, houses are not traded frequently and the selling prices are not directly observable by the market.

The validity of our model is performed in a market whose historical data in a long-time frame is available and is known to be robust against to the extreme shocks in financial markets. We implement our model to monthly collected S&P Case-Shiller Home Price index, \( h \), and the monthly mortgage rate, \( r \), collected from Federal Reserve Bank of St. Louis for the period 1975 – 2016.

The descriptive statistics of the variables (Table 1) illustrate that HPI (log-returns) exposes a heavy tail and high kurtosis which can be a sign of the frequent occurrence of extreme events. On the other hand, the mortgage rate follows approximately normal distribution (kurtosis \( \approx 3.0 \)) with a mean rate of 8.38\% and a standard deviation of 3.24\%.

The model introduced in Proposition 1 holds the potential to represent the behavior of the real-world market. However, before any practical application, the coefficients in the model-specific values need to be observed and estimated using
Table 1. Descriptives of house price index, $h$, and mortgage rate, $r$, (1975-2015).

|        | Min   | Max   | Mean  | Std  | Skewness | Kurtosis |
|--------|-------|-------|-------|------|----------|----------|
| $h$    | 25.2  | 184.62| 96.26 | 47.55| 0.36     | 1.84     |
| log-$h$| -0.023| 0.02  | 0.004 | 0.006| -0.76    | 4.84     |
| $r$(%) | 3.32  | 18.44 | 8.38  | 3.24 | 0.79     | 3.33     |

the real market data. Therefore, the calibration of the model to determine the market reflecting the parameters is vital. Indeed, calibration is a procedure that minimizes the sum of the differences between the market data and the estimated values based on the prespecified model. In other words, we are minimizing the error to find the model parameters values.

In order to implement the prescribed model, we first discretize the time into a finite set of intervals $\{t_i\}_{i=1}^N$, where $t_i < t_{i+1}$ for all $i \in [0,N]$, with $t_0 = 0$ and $t_N = T$. Using a sufficiently large $N$ and an evenly spaced time-lattice $t_i = \frac{iT}{N}$, we approximate the HPI returns and mortgage rate. Starting from initial values observed from the real data, the HPI returns and mortgage rate are determined as follows:

$$h_{t+\Delta t} = h_t + \lambda (\mu_h - r_t) h_t \Delta t + \sigma_h Z_{t+\Delta t},$$

$$r_{t+\Delta t} = r_t + \kappa (\mu_r - r_t) \Delta t + \sigma_r W_{t+\Delta t},$$

where $\Delta t = t_{i+1} - t_i$.

Now, by rephrasing Equations (15) and (16) in terms of their parameters, we obtain

$$\hat{(\lambda, \mu_h)} = \arg\min_{\lambda, \mu_h} \sum_{i=1}^{N-1} \left( \frac{h_{i+1} - h_i}{h_i} - (\lambda \mu_h \Delta t + \lambda r_i \Delta t) \right)^2,$$

$$\hat{(\kappa, \mu_r)} = \arg\min_{\kappa, \mu_r} \sum_{i=1}^{N-1} \left( r_{i+1} - r_i - \kappa \mu_r \Delta t + \lambda r_i \Delta t \right)^2.$$

Using least square method (LSM) the parameter pairs $\kappa, \mu_r$ and $\lambda, \mu_h$, which minimizes Equations (17) and (18) are estimated. Afterwards, to determine the dispersion in fitted model based on these calibrated estimates, the standard deviations of the residuals between actual and estimated values in both SDEs are calculated. Table 2 shows that log-return of HPI reverts to the mean value 5.23 with a rate of 16.30 with a volatility rate 6.26% which are much higher than the ones in the log-transformed index returns. Similarly, the mortgage rate yields a high $\kappa$ referring to a fast rate of reversion that future observations reverting to the mean value of $-0.01\%$ and much higher volatility, $0.31\%$ compared to the original mortgage rate values which are also justified by higher standard deviation value in the mortgage rate. This illustrates that the proposed model captures also the hidden volatility in the series.

As next, we demonstrate the accuracy of the joint calibration (parameter estimation) by performing a simulation of monthly HPI returns based on the estimated SDE model and observed value returns collected for the period 1975-2015. The simulated HPI returns are illustrated in Figure 2. Our goal here is to visualize the observed HPI returns versus model predicted returns as circles on figure, and the
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Table 2. Estimates of the parameters using calibration.

| $\hat{\lambda}$ | $\hat{\mu}_h$ (%) | $\hat{\sigma}_h$ (%) | $\hat{\kappa}$ | $\hat{\mu}_r$ (%) | $\hat{\sigma}_r$ (%) | $\rho$  |
|-----------------|-------------------|---------------------|--------------|-------------------|---------------------|------|
| 16.30           | 5.23              | 6.23                | -0.01        | 7.74              | -0.01               | 0.31 |
|                 |                   |                     |              |                   |                     | -0.77|

The figure shows that the model generates plausible variability in index returns with the calibrated parameters. The accuracy of the fit is justified by quantifying the root mean square error (RMS) (0.45%) and mean absolute error (MAE) (0.31%). The errors are sufficiently small, which show us our algorithm and model works well with the real world data.

Figure 2. Simulated SDEs compared with observed S&P Case-Shiller Home Price Indices (1975-2015).

Here, it is worth to emphasize that there are two sources of errors appearing systematically in our numerical computations. First, the error related to the model structure, for instance discretization of the SDEs, and second, the error related to the estimation of the model parameters. The error from the model structure cannot be avoided since it arises from our limited capacity to describe mathematically the complexity of the housing market. It’s effect on the predictions is not random, but systematic. Therefore, the structural error does not have any probabilistic properties that can be exploited in the construction of the model performance criterion. It may also cause a misleading evaluation of prediction uncertainty associated with the parameter error since the model sensitivity to model parameters may be quite different than that of the correct model. Hence, an improper model selection increases the structural error and it may significantly decrease the usefulness of the model calibration, and in many cases, it may decrease the reliability of predictions since the parameter estimates are forced to compensate for the structural errors. The error associated with parameter estimation is random and it is related to the calibration method. It can be enhanced by changing the method or initial parameters to start the calibration. However, changing initial parameters may lead to restarting optimization routine and would have resulted in different parameter estimates. Such changes in estimates might influence the calibration results. Furthermore, most of the calibration methods have a potential of the risk to end up in a local minimum. As it is seen from the price prediction power of the model (Figure 2)
and from relatively small error values the model selection and its parameter estimations are relatively good even there exist errors associated with model structure and parameter estimation. With the calibrated model, we obtain a satisfactory low pricing error for the HPI returns.

The estimated HPI model is empowered to forecast the monthly future realizations with a crude Monte Carlo algorithm for a duration of a year which are shown in Figure 3. Here, the gray lines in the graphs represent the path space trajectory forecasts of the future 12 months with 100 simulation and the red line represents the Monte Carlo result of these forecasts. Needless to say, the forecasting values using SDE as future realizations capture the pattern in the index properly. These results verify the grasping ability of proposed SDE’s.

Figure 3. Observed and predicted S&P Case-Shiller Home Price Indices (2015-2016).

5. Conclusion. The past two decades have seen a proliferation of financial instruments that are linked to the price of house prices, such as futures, options and mortgage pool linked bonds. In addition, we witnessed a strong rise in the house prices in early 2000s followed by a severe global financial crisis related to the real estate market in between late 2006 and early 2009. These breakdowns urge more sophisticated, and at the same time flexible models to explain the behavior of housing markets and their related assets. Although there have been many econometric models studied in the literature which aim to resemble the behavior of housing markets, in this paper we propose a fully stochastic, flexible model whose outcomes are expected to allow better accuracy in prediction with a reduced number of exogenous variables. In particular, we offer a two-factor stochastic differential equation system for HPI returns. Unlike other models, our approach incorporates the uncertainties of the market via a major significative factor, mortgage rate.

The superiority of our model compared to conventional ones is that unlike many other pricing approaches it only needs the most directly contributing factors to the pricing: mortgage rate and HPI. Therefore, it reduces the cumbersome search for all relevant information and it reduces the complexity of dealing with many variables. In this respect, it simplifies the housing market modeling and may increase the price prediction accuracy. Furthermore, the model has a relatively high estimation power in forecasting HPI returns.
The outcomes of this paper enable researchers to understand the house price behavior in terms of the random pattern in the mortgage rate. This approach implemented through a real-life case yields a good prediction of future HPI returns as well as it captures the real volatility which is not foreseen accurately in original series. Measuring the price fluctuations and imitating the housing market evolution together with the mortgage rate using proposed approach gains importance, certainly for the markets whose historical observations in terms of all contributing factors are either scarce or not fully available.

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