Concentration inequalities via zero bias couplings

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Abstract

The tails of the distribution of a mean zero, variance $\sigma^2$ random variable $Y$ satisfy concentration of measure inequalities of the form $\Pr(Y \geq t) \leq \exp(-B(t))$ for

\[ B(t) = \begin{cases} \frac{t^2}{2(\sigma^2 + ct)} & \text{for } t \geq 0, \\ \frac{t}{c} \left( \log t - \log \log t - \frac{\sigma^2}{c} \right) & \text{for } t > e \end{cases} \]

whenever there exists a zero biased coupling of $Y$ bounded by $c$, under suitable conditions on the existence of the moment generating function of $Y$. These inequalities apply in cases where $Y$ is not a function of independent variables, such as for the Hoeffding statistic $Y = \sum_{i=1}^n a_i\pi(i)$ where $A = (a_{ij})_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n}$ and the permutation $\pi$ has the uniform distribution over the symmetric group, and when its distribution is constant on cycle type.

1 Introduction

Since the seminal work of [Talagrand (1995)], the concentration of measure phenomenon has attracted a great deal of attention of many researchers working in very diverse fields, see the extensive treatments of [Ledoux (2001)] and the recent text of [Boucheron et al. (2013)]. The work of [Chatterjee (2007)] uncovered connections between concentration phenomenon and Stein’s method, which produces non-asymptotic error bounds for distributional approximation, see Stein (1972, 1986), and also [Chen et al. (2011)] and [Ross (2011)] for overviews. Though the application of the majority of concentration results requires the quantity of interest to be a function of independent random variables, [Chatterjee (2007)] demonstrated that tail bounds for functions of dependent random variables, including Hoeffding’s statistic and the net magnetization in the Curie-Weiss model, can be derived using Stein’s exchangeable pair coupling. Use of the size bias coupling, another important technique from Stein’s method, was shown in [Ghosh and Goldstein (2011ab)] to produce concentration bounds for

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the number of relatively ordered subsequences of a random permutation, sliding window statistics, the number of local maxima of a random function on a graph, degrees of random graphs, multinomial occupation models and coverage problems in stochastic geometry.

Here we focus on the zero bias coupling, also borrowed from Stein’s method and first introduced in [Goldstein and Reinert (1997)], and show how it too may be used to yield concentration bounds. Recall from Goldstein and Reinert (1997) that for any mean zero random variable $Y$ with positive, finite variance $\sigma^2$, there exists a distribution for a random variable $Y^*$ satisfying

$$\mathbb{E}[Yf(Y)] = \sigma^2\mathbb{E}[f'(Y^*)]$$

for all absolutely continuous functions $f$ for which the expectation of either side exists; the variable $Y^*$ is said to have the $Y$-zero biased distribution. A restatement of a result of Stein (1972) shows that a mean zero random variable $Y$ is normal if and only if $Y = \mathcal{d} Y^*$. Thus, if $Y$ and $Y^*$ can be coupled closely it is natural to expect that the behavior of $Y$, including the decay of its tail probabilities, may have behavior similar to that of the normal. Our main results in the next section justify this heuristic. For the use of zero bias couplings to produce bounds in normal approximations see, for instance, Chen et al. (2011) and the references therein.

Applications of our results will be to the Hoeffding (1951) statistic given by

$$Y = \sum_{i=1}^{n} a_{i\pi(i)},$$

depending on an array $(a_{ij})_{1 \leq i,j \leq n}$ of real numbers and a random permutation $\pi$. The quantity $Y$ arises in many applications, permutation testing foremost among them, see Wald and Wolfowitz (1944) for a seminal reference. Our results provide concentration bounds for $Y$ when the distribution of $\pi$ is uniformly distributed over the symmetric group, and when its distribution is constant on the cycle type of $\pi$; for the latter, see Goldstein and Rinott (2003) for a statistical application where $\pi$ is chosen uniformly from the class of fixed point free involutions.

We present our main result, Theorem 2.1 in Section 2, applications in Section 3 and the proofs of Theorem 2.1 in Section 4.

## 2 Main Result

**Theorem 2.1** Let $Y$ be a mean zero random variable with variance $\sigma^2 \in (0, \infty)$ and moment generating function $m(s) = \mathbb{E}[e^{sY}]$, and let $Y^*$ have the $Y$-zero bias distribution and be defined on the same space as $Y$.

(a). If $Y^* - Y \leq c$ for some $c > 0$ and $m(s)$ exists for all $s \in [0, 1/c)$, then for all $t \geq 0$

$$\mathbb{P}(Y \geq t) \leq \exp \left(-\frac{t^2}{2(\sigma^2 + ct)}\right).$$

(2)

The same upper bound holds for $\mathbb{P}(Y \leq -t)$ if $Y - Y^* \leq c$ when $m(s)$ exists for all $s \in (-1/c, 0]$. If $|Y^* - Y| \leq c$ for some $c > 0$ and $m(s)$ exists for all $s \in [0, 2/c)$ then for all $t \geq 0$

$$\mathbb{P}(Y \geq t) \leq \exp \left(-\frac{t^2}{10\sigma^2/3 + ct}\right),$$

(3)
with the same upper bound holding for $\mathbb{P}(Y \leq -t)$ if $m(s)$ exists in $(-2/c, 0]$.

(b). If $Y^* - Y \leq c$ for some constant $c > 0$ and $m(s)$ exists at $\theta = (\log t - \log \log t)/c$ then for $t > e$

$$\mathbb{P}(Y \geq t) \leq \exp \left( -\frac{t}{c} \left( \log t - \log \log t - \frac{\sigma^2}{c} \right) \right) \leq \exp \left( -\frac{t}{2c} \left( \log t - \frac{2\sigma^2}{c} \right) \right). \quad (4)$$

If $Y - Y^* \leq c$ then the same bound holds for the left tail $\mathbb{P}(Y \leq -t)$ when $m(-\theta)$ is finite.

As regards part (a) and behavior in $n$, we remark that if $|Y^* - Y| \leq c$ and $m(s)$ exists in $[0, 2/c)$, the bound (2) is preferred over (3) for $|t| < 4\sigma^2/3c$, a set increasing to $\mathbb{R}$ asymptotically in typical applications where the variance of $Y$ increases to infinity in $n$ while $c$ remains constant. Regarding behavior in $t$, part (b) of Theorem 2.1 shows that the respective asymptotic orders as $t \to \infty$ of $\exp(-t/(2c))$ and $\exp(-t/c)$ of bounds (2) and (3), can be improved to $\exp(-t \log t/(2c))$. As the right tail bound (4) applies only when $t > e$ it should be considered as a complementary result to the bounds in (a) that hold for all $t \geq 0$.

Remark 2.1 Theorem 5.1 of Chen et al. (2011) states that when $Y$ is a mean zero random variable with variance one for which there exists a coupling to $Y^*$ such that $|Y^* - Y| \leq c$ for some $c$ then the Kolmogorov distance between $Y$ and the standard normal distribution is bounded by $2.03c$. Hence for small $c$ the distribution of $Y^*$ is close to the normal, and in the limiting case where $c$ takes the value zero inequality (2) is a valid bound when $Y$ has the standard normal distribution. For such $Y$ the inequality of Chu (1955) yields

$$\sup_{t \geq 0} \exp(t^2/2)\mathbb{P}(Y \geq t) = a,$$

for $a = 1/2$, while (2) yields the same bound with $a = 1$. The constant $a = 1$ also results when bounding $\mathbb{P}(Y \geq t)$ by $\inf_{s \geq 0} e^{-st}\mathbb{E}[e^{sY}]$.

On the other hand, again by Theorem 5.1 of Chen et al. (2011), when the distribution of $Y$ is not close to normal there cannot exist a coupling of $Y$ to $Y^*$ with a small value of $c$, and the bounds may perform poorly in that the tail decay of $Y$ may in fact be faster than what is indicated by (2).

We state some properties of the zero bias distribution. First, from (1), it is easy to see that whenever $a \neq 0$, we have

$$(aY)^* =_d aY^*. \quad (5)$$

Next, from Goldstein and Reinert (1997), if $Y$ is bounded by some constant, then $Y^*$ is also bounded by the same constant, that is,

$$|Y| \leq c \implies |Y^*| \leq c. \quad (6)$$

Though Theorem 2.1 may be invoked in the presence of dependence, and for variables not expressed as sums, we compare the performance of our bound to comparable results in the literature whose application is limited to the case where $Y$ is the sum of independent variables $X_1, \ldots, X_n$ with mean zero and variances $\sigma^2 = \text{Var}(X_i) \in (0, \infty), i = 1, \ldots, n$. Letting $\sigma^2 = \text{Var}(Y)$, following Goldstein and Reinert (1997), one can form a zero biased coupling
of $Y$ to $Y^*$ by replacing the $I^{th}$ summand $X_I$ of $Y$ by a random variable $X^*_I$, independent of the remaining summands, which has the $I^{th}$ summand’s zero bias distribution, where the index $I$ has distribution $\mathbb{P}(I = i) = \sigma^2_i / \sigma$ and is chosen independently of all else. When $|X_i| \leq c$ for all $i = 1, \ldots, n$, by (6) this construction satisfies

$$|Y^* - Y| = |X^*_I - X_I| \leq 2c,$$

and as the moment generating function of $Y$ exists everywhere in this case, using the bound, say, (2) we obtain

$$\mathbb{P}(Y \geq t) \leq \exp \left( -\frac{t^2}{2\sigma^2 + act} \right).$$

with $a = 4$. Perhaps the closest classical inequality to (7) that holds under the conditions above is the one of Bernstein, see Corollary 2.11 of Boucheron et al. (2013), which yields (7) with $a = 2/3$. Though the constant of $2/3$ is superior to 4, our results are more general as they provide concentration inequalities in the presence of dependence and for variables that need not be sums. Further, we also note that the rate for large $t$ of the bound (4) is superior to the rate in (7) for any $a > 0$.

The tail bounds in (4) can also be considered as a version of Bennett’s inequality for sums of independent random variables. In the same setting as for (7) where $Y$ is a sum of independent variables satisfying $|X_i| \leq c$, Bennett’s inequality, see Theorem 2.9 of Boucheron et al. (2013), provides the tail bound

$$\mathbb{P}(Y \geq t) \leq c^{t/c} \exp \left( -\frac{\sigma^2}{c^2} \left( 1 + \frac{ct}{\sigma^2} \right) \log \left( 1 + \frac{ct}{\sigma^2} \right) \right), \quad t \geq 0. \quad (8)$$

We note that in the case of independent summands, Bennett’s inequality will in general give better bounds than (4), but is again restricted to a sum of independent variables.

As the mean and variance pair $(\mu, \sigma^2)$ of a random variable $Y$ may in general take on any value in $\mathbb{R} \times (0, \infty)$, bounds for $Y$ expressed in terms of $\mu$, such as the method of self bounding functions, see McDiarmid (2006), and the use of size bias couplings, see Ghosh and Goldstein (2011ab), are not in general comparable to those of Theorem 2.1. In particular, in Remark 3.1 while handling an example involving dependent variables, we show how the bounds of Theorem 2.1 expressed in terms of the variance, may be superior to bounds expressed in terms of the mean.

3 Hoeffding’s permutation statistic

As discussed in the introduction, with $\pi$ a random permutation in the symmetric group $S_n$ and $A = (a_{ij})_{1 \leq i, j \leq n}$ an $n \times n$ matrix with real entries, Hoeffding’s statistic takes the form

$$Y = \sum_{i=1}^{n} a_{i\pi(i)}. \quad (9)$$

Hoeffding’s combinatorial central limit theorem Hoeffding (1951) gives conditions under which $Y$, properly centered and scaled, has an asymptotic normal distribution. The rate
of convergence of $Y$ to its normal limit is well studied, see for instance Chen et al. (2011) and references therein. Here we apply our main results from Section 2 to obtain concentration inequalities for $Y$ using zero bias couplings when $\pi$ is uniformly distributed over the symmetric group, and when its distribution is constant on conjugacy classes.

In the case where $\pi$ is uniform, when the rows of $A$ are monotone, or more generally, when they have the same relative order, the summand variables $\{a_{\pi(i)}\}_{1 \leq i \leq n}$ are negatively associated and the Bernstein and Bennett inequalities hold, (7) with $a = 2/3$ and (8), respectively, thus improving on the bound of Theorem 3.1 in this special case. However, for both the uniform and constant conjugacy class distributions considered below it is easy to show that negative association does not hold in general.

3.1 Uniform Distribution on Permutations

Let $\pi$ be chosen uniformly over $S_n$. Letting

$$a_i = \frac{1}{n} \sum_{j=1}^{n} a_{ij}, \quad a_j = \frac{1}{n} \sum_{i=1}^{n} a_{ij} \quad \text{and} \quad a_* = \frac{1}{n^2} \sum_{i,j=1}^{n} a_{ij},$$

straightforward calculations show that the mean of $Y$ is given by $\mu_A = na_*$, and its variance

$$\sigma_A^2 = \frac{1}{n-1} \sum_{1 \leq i,j \leq n} (a_{ij}^2 - a_i^2 - a_j^2 + a_*^2) = \frac{1}{n-1} \sum_{1 \leq i,j \leq n} (a_{ij} - a_i - a_j + a_*)^2.$$ (10)

Further, let $||a|| = \max_{1 \leq i,j \leq n} |a_{ij} - a_i|$. Avoiding trivialities, we assume $\sigma_A^2$ is non zero.

**Theorem 3.1** For $n \geq 3$ the bounds of Theorem 2.1 hold with $Y$ replaced by $Y - \mu_A$, $\sigma^2$ by $\sigma_A^2$ and $c = 8||a||$.

**Proof:** When $\pi$ has the uniform distribution over $S_n$, use of the exchangeable pair approach of Goldstein and Reinert (1997) for constructing zero bias couplings, as applied in Theorem 2.1 of Goldstein (2005), see also Theorem 6.1 of Chen et al. (2011), yields a coupling of $(Y - \mu_A)^*$ to $Y - \mu_A$ that satisfies

$$|(Y - \mu_A)^* - (Y - \mu_A)| \leq 8||a||.$$

An application of Theorem 2.1 now yields the claim. \qed

**Remark 3.1** (Chatterjee, 2007) obtained the concentration bound

$$\mathbb{P}(|Y - \mu_A| \geq t) \leq 2 \exp \left( - \frac{t^2}{4\mu_A + 2t} \right)$$ (11)

for the Hoeffding statistic $Y$ under the additional condition that $0 \leq a_{i,j} \leq 1$ for all $i, j$. In this case we may take $||a|| = 1$ and $c = 8$ in Theorem 3.1, yielding

$$\mathbb{P}(|Y - \mu_A| \geq t) \leq 2 \exp \left( - \frac{t^2}{2\sigma_A^2 + 16t} \right).$$ (12)

A simple computation shows that the bound (12) is smaller than (11) when $t \leq (2\mu_A - \sigma_A^2)/7$. 5
When $a_{ij}, 1 \leq i, j \leq n$ are themselves independent random variables with law $\mathcal{L}(U)$ having support in $[0, 1]$, then

$$E[\sigma_1^2] = (n - 1) \text{Var}(U) \leq (n - 1)E[U^2] < nE[U] = E[\mu_A],$$

where the first equality follows by a calculation using the first expression for the variance in (10), then applying $0 \leq U \leq 1$ to yield $E[U^2] \leq E[U]$ for use in the strict inequality. Hence if the array entries behave as independent and identically distributed random variables on $[0, 1]$, the bound (12) will be asymptotically preferred to (11) everywhere. Finally we note that the bound (13) further improves on (12), as regards its asymptotic order in $t$.

### 3.2 Permutation distribution constant on cycle type

We now consider Hoeffding’s statistic (9) when the distribution of $\pi$ is constant over cycle type. This framework includes two special cases of note, one where $\pi$ is a uniformly chosen fixed point free involution, considered by Goldstein and Rinott (2003) and Ghosh (2009), having applications to permutation testing in certain matched pair experiments, and the other where $\pi$ has the uniform distribution over permutations with a single cycle, considered by Kolchin and Chistyakov (1973), under the additional restriction that $a_{ij}$ factors into a product $b_i d_j$. Bounds on the error of the normal approximation to $Y$ when the distribution of $\pi$ is constant over cycle type were derived in Goldstein (2005).

We start by recalling some relevant definitions. For $q = 1, \ldots, n$ letting $f_q(\pi)$ be the number of $q$ cycles of $\pi$, the vector

$$f(\pi) = (f_1(\pi), \ldots, f_n(\pi))$$

is the cycle type of $\pi$. For instance, the permutation $\pi = ((1, 3, 7, 5), (2, 6, 4))$ in $S_7$ consists of one 4 cycle in which $1 \to 3 \to 7 \to 5 \to 1$, and one 3 cycle where $2 \to 6 \to 4 \to 2$, and hence has cycle type $(0, 0, 1, 1, 0, 0, 0)$. We say the permutations $\pi$ and $\sigma$ are of the same cycle type if $f(\pi) = f(\sigma)$, and that a distribution $\mathbb{P}$ on $S_n$ is constant on cycle type if $\mathbb{P}(\pi)$ depends only on $f(\pi)$, that is

$$\mathbb{P}(\pi) = \mathbb{P}(\sigma) \quad \text{whenever} \quad f(\pi) = f(\sigma).$$

With $\mathbb{N}_0$ the set of non-negative integers, clearly a vector $f = (f_1, \ldots, f_n)$ is a cycle type of a permutation in $S_n$ if and only if $f \in \mathcal{F}_n$ where

$$\mathcal{F}_n = \{(f_1, \ldots, f_n) \in \mathbb{N}_0^n : \sum_{i=1}^n if_i = n\}.$$ 

A special case of a distribution constant on cycle type is one uniformly distributed over all permutations having cycle type $f \in \mathcal{F}_n$, denoted $\mathcal{U}(f)$. The situations where $\pi$ is uniformly chosen from the set of all fixed point free involutions, and chosen uniformly from all permutations having a single cycle, are both distributions of type $\mathcal{U}(f)$, the first with $f = (0, n/2, 0, \ldots, 0)$ for even $n$ and the second with $f = (0, 0, \ldots, 0, 1)$.

We consider distributions over $S_n$ having no fixed points with probability one, as is true for the two special cases of most interest. Noting that under this condition no expression of the form $a_{ii}$ appears in the sum (9), let

$$a_{io} = \frac{1}{n-2} \sum_{j: j \neq i}^n a_{ij} \quad \text{and} \quad a_{oo} = \frac{1}{(n-1)(n-2)} \sum_{i \neq j}^n a_{ij}.$$
Under the symmetry condition $a_{ij} = a_{ji}$ for all $i \neq j$, see Chen et al. (2011), when $\pi_f$ has distribution $\mathcal{U}(f)$ with $f_1 = 0$, the mean $\mu$ and variance $\sigma_f^2 = \text{Var}(Y_f)$ of the corresponding variable $Y_f$ for $n \geq 4$ are given by

$$
\mu = (n - 2)a_{oo} \quad \text{and} \quad \sigma_f^2 = \left(\frac{1}{n - 1} + \frac{2f_2}{n(n - 3)}\right) \sum_{i \neq j} (a_{ij} - 2a_{io} + a_{oo})^2. \quad (13)
$$

For $n \geq 4$, when $n$ is even and $\iota$ is the cycle type of a fixed point free involution, then $\iota_k = (n/2)1(k = 2)$, and when $f$ is the cycle type of a permutation without any fixed points or two cycles, such as is the case for one long cycle, then $f_2 = 0$, and the variance in (13) specializes, respectively, to

$$
\sigma_\iota^2 = \frac{2(n - 2)}{(n - 1)(n - 3)} \sum_{i \neq j} (a_{ij} - 2a_{io} + a_{oo})^2 \quad \text{and} \quad \sigma_f^2 = \frac{1}{n - 1} \sum_{i \neq j} (a_{ij} - 2a_{io} + a_{oo})^2. \quad (14)
$$

Let also

$$
a_o = \max_{i \neq j} |a_{ij} - 2a_{io} + a_{oo}|. \quad (15)
$$

Lemma 6.5 of Chen et al. (2011) shows that when a distribution $\mathbb{P}$ on $S_n$ is constant on cycle type then it can be represented as the mixture of the uniform distributions $\mathcal{U}(f)$ for $f \in \mathcal{F}_n$,

$$
\mathbb{P} = \sum_{f \in \mathcal{F}_n} \rho_f \mathcal{U}(f) \quad \text{where} \quad \rho_f = \mathbb{P}(f(\pi) = f).
$$

In particular, when the distribution of $\pi$ is constant on cycle type and $Y$ is given by (9) then

$$
\mathbb{E}[Y] = \mu, \quad \text{and} \quad \mathbb{P}(Y - \mu \geq t) = \sum_{f \in \mathcal{F}_n} \rho_f \mathbb{P}(Y_f - \mu \geq t). \quad (16)
$$

Hence, bounds for $Y$ are implied by those for $Y_f, f \in \mathcal{F}_n$.

**Theorem 3.2** Let $n \geq 5$ and $(a_{ij})_{i,j=1}^n$ be an array of real numbers satisfying $a_{ij} = a_{ji}$. When $\pi$ is a uniformly chosen fixed point free involution, the bounds of Theorem 2.1 hold with $Y$ replaced by $Y - \mu$, $\sigma^2$ replaced by $\sigma_\iota^2$ of (14), and $c$ replaced by $24a_o$ with $a_o$ of (15). When the distribution of $\pi$ is constant on cycle type and has no fixed points or two cycles with probability one, the bounds of Theorem 2.1 hold with $Y$ replaced by $Y_f - \mu$, $\sigma^2$ replaced by $\sigma_f^2$ of (14), and $c$ replaced by $40a_o$.

**Proof**: When $\pi_\iota$ has the $\mathcal{U}(\iota)$ distribution, using the exchangeable pair approach of Goldstein and Reinert (1997), the construction in Lemma 6.10 of Chen et al. (2011) provides a zero biased coupling for $Y_{\iota} - \mu$ that satisfies

$$
|(Y_{\iota} - \mu)^* - (Y_{\iota} - \mu)| \leq 24a_o. \quad (17)
$$

Theorem 2.1 now obtains to yield the first claim. Similarly, Theorem 2.2 of Goldstein (2005) shows that the constant 24 in (17) can be replaced by 40 when $Y_\iota$ is replaced by $Y_f$ for any $f$ satisfying $f_1 = f_2 = 0$. The final claim of the theorem is now obtained by applying the
last equality of (16), noting that the variances and coupling constants for all such $Y_f$, and therefore the upper bounds on $\mathbb{P}(Y_f - \mu \geq t)$ produced by Theorem 2.1 are identical, and that $\rho_f$ sums to one.

We remark that bounds for the general situation where the distribution of $\pi$ is constant on cycle type, without fixed points, can be obtained using (16) in the same fashion, yielding a weighted sum of bounds of the two types appearing in Theorem 2.1.

4 Proof of Main Result

Proof of Theorem 2.1: Let $m(s) = \mathbb{E}[e^{sY}]$ and $m^*(s) = \mathbb{E}[e^{sY^*}]$. When $Y^* - Y \leq c$ for all $s \geq 0$ then

$$m^*(s) = \mathbb{E}[e^{sY^*}] = \mathbb{E}[e^{s(Y^* - Y)}e^{sY}] \leq \mathbb{E}[e^{cs}e^{sY}] = e^{cs}m(s). \quad (18)$$

In particular when $m(s)$ is finite then so is $m^*(s)$.

Part (a). If $m(s)$ exists in an open interval containing $s$ we may interchange expectation and differentiation at $s$ to obtain

$$m'(s) = \mathbb{E}[Ye^{sY}] = \sigma^2 \mathbb{E}[se^{sY^*}] = \sigma^2 sm^*(s), \quad (19)$$

where we have applied the zero bias relation (1) to yield the second equality.

We first prove (2). Starting with the well known inequality $1 - x \leq e^{-x}$, holding for all $x \geq 0$, we obtain

$$e^x \leq \frac{1}{1 - x} \quad \text{for} \quad x \in [0, 1).$$

Hence, for $\theta \in (0, 1/c)$ and $0 \leq s \leq \theta$ we have

$$m^*(s) = \mathbb{E}[e^{sY^*}] = \mathbb{E}[e^{s(Y^* - Y)}e^{sY}] \leq e^{sc}m(s) \leq \frac{1}{1 - sc}m(s).$$

Using the identity (19) to express $m^*(s)$ in terms of $m'(s)$ we obtain

$$m'(s) \leq \frac{\sigma^2 s}{1 - sc} m(s).$$

Dividing both sides by $m(s)$, integrating over $[0, \theta]$ and using that $m(0) = 1$ we obtain

$$\log m(\theta) = \int_0^\theta \frac{m'(s)}{m(s)} ds \leq \frac{\sigma^2}{1 - \theta c} \int_0^\theta s ds = \frac{\sigma^2 \theta^2}{2(1 - \theta c)},$$

and exponentiation yields

$$m(\theta) \leq \exp \left( \frac{\sigma^2 \theta^2}{2(1 - \theta c)} \right).$$

As (2) holds trivially for $t = 0$ consider $t > 0$ and apply Markov’s inequality to obtain

$$\mathbb{P}(Y \geq t) = \mathbb{P}(e^{\theta Y} \geq e^{\theta t}) \leq e^{-\theta t}m(\theta) \leq \exp \left( -\theta t + \frac{\sigma^2 \theta^2}{2(1 - \theta c)} \right).$$
Setting $\theta = t/(\sigma^2 + ct)$, and noting this value lies in the interval $(0, 1/c)$, (2) follows.

If now $Y - Y^* \leq c$ then letting $X = -Y$ we see from (5) with $a = -1$ that $X^* = -Y^*$ has the $X$-zero bias distribution, and applying (2) to $X$ yields the claimed left tail inequality.

Turning to (3), by the convexity of the exponential function we have

$$ e^y - e^x = \int_0^1 e^{ty + (1-t)x} dt \leq \int_0^1 (te^y + (1-t)e^x) dt = \frac{e^y + e^x}{2} \quad \text{for all } x \neq y, $$

and hence

$$ e^y - e^x \leq \frac{|y - x|(e^y + e^x)}{2} \quad \text{for all } x \text{ and } y. $$

Hence, when $|Y^* - Y| \leq c$, for all $\theta \in (0, 2/c)$ and $0 \leq s \leq \theta$,

$$ e^{sY^*} - e^{sY} \leq \frac{|s(Y^* - Y)|(e^{sY^*} + e^{sY})}{2} \leq \frac{cs}{2}(e^{sY^*} + e^{sY}). $$

Taking expectation yields

$$ m^*(s) - m(s) \leq \frac{cs}{2}(m^*(s) + m(s)), \quad \text{hence} \quad m^*(s) \leq \left(\frac{1 + cs/2}{1 - cs/2}\right) m(s), $$

and now relation (19) gives that

$$ m'(s) \leq \sigma^2 s \left(\frac{1 + cs/2}{1 - cs/2}\right) m(s). $$

Following steps similar to the ones above, we obtain

$$ m(\theta) \leq \exp \left(\frac{\alpha \sigma^2 \theta^2}{1 - c\theta/2}\right) \quad \text{for } \alpha = 5/6 \text{ and all } \theta \in (0, 2/c). $$

As the result holds trivially for $t = 0$, fix $t > 0$ and argue as before using Markov’s inequality to obtain

$$ \Pr(Y \geq t) \leq \exp \left(-\theta t + \frac{\alpha \sigma^2 \theta^2}{1 - c\theta/2}\right) \quad \text{for all } \theta \in (0, 2/c). $$

Letting $\theta = 2t/(4\alpha \sigma^2 + ct)$, and noting that this value lies in $(0, 2/c)$, we obtain the asserted right tail inequality (3). Replacing $Y$ by $-Y$ as before now demonstrates the remaining claim.

Proof of (b). For any $s \in [0, \theta)$ such that $m(\theta)$ exists, by (18) we have

$$ m'(s) = \mathbb{E}[Ye^{sY}] = \sigma^2 s \mathbb{E}[e^{sY^*}] = \sigma^2 s m^*(s) \leq \sigma^2 se^{cs} m(s), \quad \text{so that} \quad (\log m(s))' \leq \sigma^2 se^{cs}. $$

Integrating over $[0, \theta]$ and using that $m(0) = 1$ we obtain

$$ \log(m(\theta)) \leq \frac{\sigma^2}{c^2} (e^{c\theta} (c\theta - 1) + 1) $$
and exponentiation yields

\[ m(\theta) \leq \exp \left( \frac{\sigma^2}{c^2} \left( e^{\theta c} (c\theta - 1) + 1 \right) \right). \]

Applying Markov’s inequality as before,

\[ P(Y \geq t) \leq \exp \left( -\theta t + \frac{\sigma^2}{c^2} \left( e^{\theta c} (c\theta - 1) + 1 \right) \right). \]

For \( t > e \) letting \( \theta = (\log t - \log \log t)/c \) we obtain the first claim of (4) by

\[
P(Y \geq t) \leq \exp \left( -\frac{t}{c} \left( \log t - \log \log t \right) + \frac{\sigma^2}{c^2} \left( \frac{t}{\log t} \left( \log t - \log \log t - 1 \right) + 1 \right) \right)
\]

\[
\leq \exp \left( -\frac{t}{c} \left( \log t - \log \log t - \frac{\sigma^2}{c} \right) \right).
\]

The second claim follows by the inequality \((\log t)/2 \geq \log \log t\) for all \( t > 1 \). The left tail bound follows as in (a). □

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