PHASE TRANSITION OF AN ANISOTROPIC
GINZBURG–LANDAU EQUATION

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Abstract. We study the effective geometric motions of an anisotropic
Ginzburg–Landau equation with a small parameter $\varepsilon > 0$ which charac-
terizes the width of the transition layer. For well-prepared initial datum,
we show that as $\varepsilon$ tends to zero the solutions will develop a sharp inter-
face limit which evolves under mean curvature flow. The bulk limits of
the solutions correspond to a vector field $u(x, t)$ which is of unit length
on one side of the interface, and is zero on the other side. The proof
combines the modulated energy method and weak convergence methods.
In particular, by a (boundary) blow-up argument we show that $u$ must be
tangent to the sharp interface. Moreover, it solves a geometric evolution
equation for the Oseen–Frank model in liquid crystals.

Keywords: modulated energy methods, weak convergence methods,
blow-up analysis, mean curvature flow, phase-transition.

Mathematical Subject Classification: 53E10, 35R35, 35K58 35K57.

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1. Introduction

In the study of liquid crystals one often encounters elastic energies with anisotropy,
i.e. energies with distinct coefficients multiplying the square of the divergence and the
curl of the order parameters. Typical examples involve the Oseen–Frank model [30],
Ericksen’s model [11, 40] and the Landau–De Gennes model [5]. From a microscopic
point of view, the anisotropy of these models can be interpreted as excluded volume
potential of molecular interaction, cf. [29]. Anisotropic models also arise in the theory
of superconductivity, cf. [10]. The anisotropy brings various new challenges to the
studies of both variational problems and their gradient flows of the aforementioned models. In contrast to the convergence analysis of isotropic models, i.e. the (scalar) Allen–Cahn equations (cf. [19, 34, 6, 53, 54, 48, 50, 47]), the powerful analytic tools such as maximum principle and monotonicity formula are not readily established for anisotropic ones.

The attempt of this work is to study an anisotropic system modeling the isotropic-nematic phase transition of a liquid crystal droplet. Let $d \in \{2, 3\}$ be the dimension of the physical domain $\Omega$ with $C^3$ boundary $\partial \Omega$. We consider the anisotropic Ginzburg–Landau type energy

$$A_\varepsilon(u) = \int_\Omega \left( \frac{\varepsilon}{2} \mu |\nabla u|^2 + \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} F(u) \right) dx. \quad (1.1)$$

Here $u = (u_1, u_2, u_3) : \Omega \subset \mathbb{R}^d \mapsto \mathbb{R}^3$ is the order parameter describing the state of the system. The function $F(u)$ is a double equal-well potential which permits the isotropic-nematic phase transition. More precisely, it attains its global minimum value 0 at $\{0\} \cup S^2$. An example of $F$ is the Chern–Simons-Higgs model $F(u) = |u|^2(1 - |u|^2)^2$. See for instance [31, 36] for the physics and [9, 27, 28] for the mathematical analysis of related variational problems. The parameter $\varepsilon > 0$ denotes the relative intensity of elastic and bulk energy, which is usually quite small. The parameter $\mu > 0$ is material dependent which measures the degree of anisotropy.

The energy $\mathcal{L}$ is a simplified case of the full Landau–De Gennes energy (cf. [35, 45]). The variational investigations of the isotropic-nematic phase transition involving (1.1) were first done by Golovaty, Novack, Sternberg and Venkatraman [27, 28] in the static case in 2D. The present paper is concerned with the $L^2$-gradient flow of (1.1), i.e. the following system.

$$\partial_t u_\varepsilon - \mu \nabla(\nabla u_\varepsilon) = \Delta u_\varepsilon - \frac{1}{\varepsilon^2} DF(u_\varepsilon) \quad \text{in } \Omega \times (0, T), \quad (1.2a)$$

$$u_\varepsilon(x, 0) = u_\varepsilon^0(x) \quad \text{in } \Omega, \quad (1.2b)$$

$$u_\varepsilon(x, t) = 0 \quad \text{on } \partial \Omega \times (0, T), \quad (1.2c)$$

where $DF(u)$ is the gradient of $F(u)$ with respect to $u$. We shall study the small $\varepsilon$-asymptotics of this system with well-prepared initial datum $u_\varepsilon^0$ that undergoes a sharp transition across a co-dimensional one interface $I_0 \subset \mathbb{R}^d$. We shall show that the energy density $\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon)$ will be concentrated on a mean curvature flow $I := \bigcup_{t \geq 0} I_t \times \{t\}$ starting from $I_0$, namely

$$\lim_{\varepsilon \to 0} \int_\Omega \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) \right) dx = \sigma \mathcal{H}^{d-1}(I_t), \quad (1.3)$$

where $\mathcal{H}^{d-1}$ is the $(d - 1)$ dimensional Hausdorff measure, and $\sigma$ is a positive constant depending on $F$. Moreover, we shall derive bulk limit $u := \lim_{\varepsilon \to 0} u_\varepsilon$ away from $I_t$ and its boundary condition on $I_t$.

System (1.2a) is a vectorial and anisotropic generalization of the scalar parabolic Allen–Cahn equation. In the scalar case, there have been many developments on its co-dimensional one limit to the (two-phase) mean curvature flow during the last two decades. Here we mention two classes of results and postpone the discussions of some others in the sequel. One is the convergence to a Brakke’s flow by Ilmanen [34] using a version of Huisken’s monotonicity formula [32] and tools from geometric measure theory. See also [11, 33, 54, 48, 47, 50] and the references therein for further renovations. Despite of its energetic nature, a major difficulty of such an approach is the
control of the so-called discrepancy measure, and almost all existing literatures using this approach rely crucially on a version of Modica’s maximum principle [46]. However, it is not clear whether Modica’s maximum principle holds for elliptic/parabolic systems. Another approach, which relies more on parabolic comparison principle, is the global in time convergence towards the viscosity solution built by Chen–Giga-Goto [13] and independently by Evans–Spruck [20]. Such an approach has been implemented by Evans–Soner–Souganidis [19]. One can also refer to [6, 53] and the references therein for further discussions. These two approaches both give global in time (weak) convergences to weakly defined solutions of the mean curvature flow (up to their extinction times). However, as their technics involve parabolic maximum principle in one way or another, it is not clear how to use them to attack vectorial models in general. It is worth mentioning that for radially symmetric initial datum, Bronsard–Stoth [8] have obtained global in time convergence to the mean curvature flow of planar circles.

To the best of our knowledge, there are mainly two approaches to rigorously justify the convergence of the vectorial Allen–Cahn equations, both assuming that the limiting interface motion has a (local in time) classical solution. Compared with the aforementioned methods, which lead to global in time (weak) convergence, they have quite different natures. The first approach is the asymptotic expansion technics developed by De Mottoni–Schatzman [15] and by Alikakos–Bates–Chen [1]. It has been used recently by Fei–Wang–Zhang–Zhang [22] to study the isotropic-nematic phase transition in liquid crystals, and by Fei–Lin–Wang–Zhang [21] to study matrix-valued Allen–Cahn equations. The second approach, which also assumes a classical solution of the limiting interface motion (but not the limiting flows in the bulk regions), is the modulated energy method developed by Fischer–Laux–Simon [24]. Such a method is motivated by Jerrard–Smets [37] and Fischer–Hensel [23], and has been generalized to a matrix–valued model by Laux–Liu [40].

In the present work, we shall use the methods employed in [24, 40] to derive the energy convergence (1.3) and the bulk limit \( u = \lim_{\varepsilon_k \to 0} u_{\varepsilon_k} \) by establishing two modulated energy inequalities. Moreover, the derivation of the anchoring boundary condition of \( u \) (see (1.18c) below) uses a blow-up argument, which is inspired by a recent work of Lin–Wang [43]. There the authors have studied isotropic-nematic phase transitions in the static case based on an anisotropic Ericksen’s model.
To state the main result, we assume that

\[ I = \bigcup_{t \in [0, T]} I_t \times \{t\} \] is a smoothly evolving

\[(d - 1)\)-dimensional submanifold in \( \Omega \),

(1.4)

starting from a \((d - 1)\)-dimensional submanifold \( I_0 \subset \Omega \). Here a \((d - 1)\)-submanifold refers to an embedded closed smooth surface when \( d = 3 \) and curve when \( d = 2 \).

Let \( \Omega^+_t \) be the domain enclosed by \( I_t \), and \( d_I(x, t) \) be the signed-distance from \( x \) to \( I_t \) which takes negative values in \( \Omega^-_t \), and positive values in \( \Omega^+_t = \Omega \setminus \Omega^-_t \). Equivalently,

\[ \Omega^\pm_t := \{ x \in \Omega \mid d_I(x, t) \gtrless 0 \} \] (1.5)

For \( \delta > 0 \), the \((\text{open})\ \delta\)-neighborhood of \( I_t \) is denoted by

\[ B_\delta(I_t) := \{ x \in \Omega \mid |d_I(x, t)| < \delta \} \] (1.6)

Let \( \delta_0 \in (0, 1) \) be a sufficiently small number so that the nearest point projection

\[ P_I(\cdot, t) : B_{4\delta_0}(I_t) \to I_t \]

is smooth for any \( t \in [0, T] \), and that the interface (1.4) stays at least \( 4\delta_0 \) distant away from the physical boundary \( \partial \Omega \). A further description of the geometry can be found in Subsection 2.2 or in [12].

The first step to study the singular limit of (1.2) is to construct a modulated energy which encodes a distance between the energy in (1.1) and an energy corresponding to the moving interface \( I_t \) in (1.4). Following [37, 23, 24], we define an extension of the inward normal vector \( n(\cdot, t) \) of \( I_t \) by

\[ \xi(x, t) := \phi \left( \frac{d_I(x, t)}{\delta_0} \right) \nabla d_I(x, t) \quad \text{for } x \in \Omega, \]

where \( \phi \in C^\infty_c(\mathbb{R}; [0, 1]) \) is an appropriate cut-off function (see (2.11) below for its precise definition). Now we introduce

\[ E_\varepsilon[u_\varepsilon,I](t) := \int_\Omega \varepsilon \mu |\text{div} u_\varepsilon(\cdot, t)|^2 \, dx \]

\[ + \int_\Omega \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon(\cdot, t)|^2 + \frac{1}{\varepsilon} F(u_\varepsilon(\cdot, t)) - \xi \cdot \nabla \psi_\varepsilon(\cdot, t) \right) \, dx, \]

(1.7)

where \( \psi_\varepsilon \) is defined by

\[ \psi_\varepsilon(x, t) := \int_0^{||u_\varepsilon(x, t)||} g(s) \, ds. \]

(1.8)

We shall work with a class of potentials \( F(u) \) under standard assumptions (see e.g. [34, 11]). That is,

\[ F(u) = f(|u|) = g^2(|u|)/2, \]

(1.9)

where \( f \) is a double equal-well potential, namely,

\[ f \in C^\infty(\mathbb{R}_{\geq 0}), \quad f(s) > 0 \text{ for } s \in \mathbb{R}_{\geq 0} \setminus \{0, 1\}, \]

\[ g \geq 0 \text{ and is locally Lipschitz continuous, } g(0) = g(1) = 0. \]

(1.10a)
Moreover, the following structural assumptions on $f$ are made:
\begin{align}
\exists s_0 \in (0, 1) \text{ s.t. } f'(s) &> 0 \text{ on } (0, s_0) \text{ and } f'(s) < 0 \text{ on } (s_0, 1); \\
\exists c_0 \in (0, 1) \text{ s.t. } 2c_0^2s^2 &\leq f(s) \leq 2c_0^{-2}s^2 \text{ for any } s \geq 100.
\end{align}
After an appropriate modification for large $|s|$, the function $g(s) = |s||s^2 - 1|$, which corresponds to the Chern–Simons–Higgs potential, satisfies (1.11).

To control the bulk errors, we need another modulated energy:
\begin{equation}
B[u, I](t) := \int_{\Omega} \left( \sigma \chi - \sigma + 2(\psi - \sigma)^- \right) \eta \circ d_I \, dx + \int_{\Omega} (\psi - \sigma)^+ |\eta \circ d_I| \, dx.
\end{equation}
Here $\chi(\cdot, t) := 1_{\Omega^+} - 1_{\Omega^-}, h^\pm$ denote the positive/negative parts of a function $h$ respectively, and $\eta$ is a truncation of the identity function defined by
\begin{equation}
\eta(z) := \begin{cases} 
  z & \text{when } z \in [-\delta_0, 0], \\
  \delta_0 & \text{when } z \geq \delta_0, \\
  -\delta_0 & \text{when } z \leq -\delta_0.
\end{cases}
\end{equation}
Note that $\eta \circ d_I \chi \geq 0$ in $\Omega$ due to our convention on the signed-distance function, and thus the two integrands in (1.12) are both non-negative. We refer the readers to the proof of Theorem 4.1 below for more details on the positivity of (1.12).

Now we state the main result of this work:
\begin{Theorem}
Let $d \in \{2, 3\}$, and the assumptions (1.10) and (1.11) be in place. Assume that the moving interface $I$ in (1.3) evolves under mean curvature flow, and the initial datum of (1.2) satisfies the following conditions:
\begin{align}
\mathbf{u}_\varepsilon^{in} &\in W^{1,2}_0(\Omega), \\
A_\varepsilon(\mathbf{u}_\varepsilon^{in}) &\leq c_1, \\
E_\varepsilon[\mathbf{u}_\varepsilon^{in} I_0] + B[\mathbf{u}_\varepsilon^{in} |I_0] &\leq c_1\varepsilon,
\end{align}
where $c_1 > 0$ is independent of $\varepsilon$. Then there exists $C_1 > 0$ independent of $\varepsilon$ such that
\begin{equation}
\sup_{t \in [0,T]} E_\varepsilon[\mathbf{u}_\varepsilon I](t) + \sup_{t \in [0,T]} B[\mathbf{u}_\varepsilon I](t) \leq C_1\varepsilon,
\end{equation}
\begin{equation}
\sup_{t \in [0,T]} \int_{\Omega} |\psi - \sigma 1_{\Omega^+} | \, dx \leq C_1\varepsilon^{1/4}.
\end{equation}
Moreover, up to extraction of a subsequence $\varepsilon_k \downarrow 0$,
\begin{equation}
\mathbf{u}_\varepsilon \xrightarrow{k \to \infty} 1_{\Omega^+} \mathbf{u} \text{ in } C([0,T]; L^2_{loc}(\Omega \setminus I_t)),
\end{equation}
where $\mathbf{u}$ satisfies the following properties:
\begin{align}
\mathbf{u} &\in L^\infty(0, T; W^{1,6/5}(\Omega^+; S^2)), \quad \partial_t \mathbf{u} \in L^2(0, T; L^{6/5}(\Omega^+)), \\
\mathbf{u}(x, t) = 0 &\text{ for every } t \in [0, T] \text{ and for a.e. } x \in \Omega^-,
\end{align}
\begin{align}
(u \cdot \mathbf{n})(x, t) = 0 &\text{ for a.e. } t \in [0, T] \text{ and for } H^{d-1} \text{-a.e. } x \in I_t.
\end{align}

Among the conditions in (1.14), the crucial one is (1.14c), which is used to obtain the inequalities in Theorem 3.1 and in Theorem 4.1 below. To construct an initial datum satisfying (1.14), we need the following result.

Proposition 1.1. Let \( I_0 \subset \Omega \) be a \((d - 1)\)-dimensional submanifold. For any vector field
\[
\mathbf{u}_{\text{in}}^n \in W^{1,2}(\Omega; \mathbb{S}^2) \quad \text{with} \quad \mathbf{u}_{\text{in}}^n \big|_{I_0} \cdot \mathbf{n}_{I_0} = 0 \quad \text{a.e. on} \ I_0,
\]
there exists \( \mathbf{u}_\varepsilon^n \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega) \) such that
\[
\left\{ \begin{array}{ll}
\mathbf{u}_\varepsilon^n = \mathbf{u}_{\text{in}}^n & \text{in} \ \Omega_0^+ \setminus B_{2\delta_0}(I_0), \\
\mathbf{u}_\varepsilon^n = 0 & \text{in} \ \Omega_0^- \setminus B_{2\delta_0}(I_0),
\end{array} \right.
\]
and (1.13) holds for a constant \( c_1 > 0 \) which only depends on \( I_0 \) and \( \| \mathbf{u}_{\text{in}}^n \|_{W^{1,2}(\Omega)} \).

We comment on the conditions in (1.19). When \( d = 3 \), \( I_0 \) is a smooth closed surface in \( \Omega \). Due to topological obstructions, a vector field satisfying (1.19) is usually not smooth. For instance, when \( I_0 \) is diffeomorphic to a 2-sphere, due to the hairy ball theorem, \( \mathbf{u}_{\text{in}}^n \big|_{I_0} \) must have (at least) one pole. One example of such a pole, which is often encountered in the theory of liquid crystal, is given by the hedgehog profile. Locally, the tangent vector field near such a pole is \( C^1 \)-equivalent to the mapping \( \mathbf{h}(x) = x/|x| : B_1 \cap \mathbb{R}^2 \to \mathbb{S}^1 \).

Note that \( \mathbf{h} \in W^{\frac{1}{2},2}(B_1 \cap \mathbb{R}^2) \) but \( \mathbf{h} \notin W^{1,2}(B_1 \cap \mathbb{R}^2) \). When \( d = 2 \), there are fewer constraints to arrange a vector field \( \mathbf{f} : I_0 \mapsto \mathbb{S}^2 \subset \mathbb{R}^3 \) that is orthogonal to the planar curve \( I_0 \subset \mathbb{R}^2 \times \{0\} \). In general, using the extension lemma of Hardt–Lin (cf. [42 Lemma 2.2.10]), any tangent vector field \( \mathbf{f} \in W^{\frac{1}{2},2}(I_0; \mathbb{S}^2) \) has an extension \( \mathbf{u}_{\text{in}}^n \) satisfying (1.19).

An immediate consequence of Theorem 1.1 is the convergence in (1.3). Indeed, it follows from (1.15) and (2.26b) below that \( \int_\Omega \varepsilon \mu |\operatorname{div} \mathbf{u}|^2 \, dx \xrightarrow{\varepsilon \to 0} 0 \), and thus such an energy does not contribute to the surface energy in the limit. However, it forces \( \mathbf{u} \) to satisfy the boundary condition (1.18c). Now applying integration by parts to the last term of (1.7), and then using (1.16) and \( \xi |_{\partial \Omega} = 0 \), we find
\[
\lim_{\varepsilon \to 0} \int_\Omega \left( \frac{\varepsilon}{2} |\nabla \mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon} F(\mathbf{u}_\varepsilon) \right) \, dx = \lim_{\varepsilon \to 0} - \int_\Omega (\operatorname{div} \xi) \psi_\varepsilon \, dx = -\sigma \int_{\partial \Omega}^t (\operatorname{div} \xi) \, dx = \sigma \mathcal{H}^{d-1}(I_t).
\]

Note that the last step is due to the Green’s formula.

Under additional assumptions, we can show that the limit \( \mathbf{u} \) in (1.17) solves a geometric evolution equation in the bulk region \( \Omega^+ := \bigcup_{t \in [0,T]} \Omega^+_t \times \{t\} \).

Theorem 1.2. Let \( d = 2 \) and the assumptions of Theorem 1.1 be in place. Assume further that
\[
\begin{align*}
f(s) &= s^2 \text{ for } s \leq 1/4; \quad f(s) = (s - 1)^2 \text{ for } s \geq 3/4; \\
f(s) &\geq 1/16 \text{ for } s \in [1/4,3/4]; \\
\sup_{s \in [1/4,3/4]} |f'(s)| &\leq 4.
\end{align*}
\]
Then there exists a sufficiently small \( \mu > 0 \) (independent of \( \varepsilon \)) such that the vector field \( \mathbf{u} \) in (1.17) satisfies
\[
\int_\Omega \partial_t \mathbf{u} \wedge \mathbf{u} \cdot \mathbf{\Psi} \, dx + \int_\Omega (\nabla \mathbf{u} \wedge \mathbf{u}) \cdot \nabla \mathbf{\Psi} \, dx \\
= \mu \int_\Omega (\operatorname{div} \mathbf{u}) \left( (\operatorname{rot} \mathbf{\Psi}) \cdot \mathbf{u} - (\operatorname{rot} \mathbf{u}) \cdot \mathbf{\Psi} \right) \, dx
\]
for almost every \( t \in (0,T) \) and for every \( \mathbf{\Psi} \in C^1_c(\Omega^+_t; \mathbb{R}^3) \).
In the above equation $\wedge$ is the wedge product in $\mathbb{R}^3$ and rot is the curl operator. The equation (1.23) is the weak formulation of an Oseen–Frank flow, written as

$$\partial_t u = \Delta u + \mu(\mathbb{I}_3 - u \otimes u)\nabla(\text{div} u) + |\nabla u|^2 u, \quad \text{for } t \in (0, T], \, x \in \Omega_t^+. \quad (1.24)$$

It can be verified that when $u$ is sufficiently regular, then (1.23) implies (1.24). It is worth mentioning that equation of the form (1.24) is the $L^2$-gradient flow of the variational problem

$$\inf \int_U (\mu |\text{div} u|^2 + |\nabla u|^2) \, dx, \quad (1.25)$$

where the infimum is taken among mappings $u \in W^{1,2}(U; \mathbb{S}^2)$ fulfilling certain boundary conditions on $\partial U$. Note that (1.25) is a special case of the full Oseen–Frank model (cf. [30]).

This work will be organized as follows: In Section 2 we shall adapt the modulated energy method of [24] to the vectorial and anisotropic system (1.2), and then derive a differential inequality, i.e. Proposition 2.1. Such an inequality was previously derived in [40] for a matrix-valued equation. When applied to (1.2), it includes a term which does not have an obvious sign due to the additional $\text{div}$ term. This problem will be solved in Section 3 during the proof of the inequality in Theorem 3.1. This theorem, which precedes the first part of Theorem 1.1, is a major novelty of the present work, and will be employed in Section 4 (see Theorem 4.1) to derive the $L^1$-estimate of $\psi_\varepsilon$ in (1.16). Such an estimate will be used in Lemma 4.3 to identify appropriate level sets of $\psi_\varepsilon$ which converge to $I_t$ in certain sense. With this key lemma, we derive in Section 5 the anchoring boundary condition (1.18c), and thus finish the proof of Theorem 1.1. Section 6 is devoted to the proof of Theorem 1.2. The proof of Proposition 1.1 is quite similar to the construction given in [40]. We present a proof in Appendix A for the convenience of the readers.

2. Preliminaries

2.1. Notation and Conventions. We shall adopt the following conventions throughout the paper. Unless specified otherwise, $C > 0$ is a generic constant whose value might change from line to line, and will depend on the geometry of the interface (1.4) but not on $\varepsilon$ or $t \in [0, T]$. For two square matrices $A$ and $B$, their Frobenius inner product is defined by $A : B := \text{tr} A^T B$, which induces the norm $|A| := \sqrt{\text{tr} A^T A}$. We shall also use the following notation for a vector-valued function $u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$

$$x = \begin{cases} (x_1, x_2, x_3) & \text{when } d = 3, \\ (x_1, x_2, 0) & \text{when } d = 2. \end{cases} \quad (2.1)$$

$$\partial_0 = \partial_t, \quad \partial_i = \partial_{x_i}, \quad 1 \leq i \leq 3,$$

$$\nabla u_1 = (\partial_1 u_1, \partial_2 u_1, \partial_3 u_1), \quad \text{div } u = \sum_{i=1}^3 \partial_i u_i, \quad (2.2)$$

$$\text{rot } u = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1).$$
To ease computations when $d = 2$, $\nabla \mathbf{u}$ will be understood as the matrix
\[
\begin{pmatrix}
\partial_1 u_1 & \partial_2 u_1 & 0 \\
\partial_1 u_2 & \partial_2 u_2 & 0 \\
\partial_1 u_3 & \partial_2 u_3 & 0
\end{pmatrix},
\]
and any planar vector field is understood as a 3D vector field with vanishing 3rd component. (2.3)

In particular, the latter applies to the normal and the mean curvature vector fields (cf. (2.9) and (2.13) respectively below). For a function of $\mathbf{u}$, like $F(\mathbf{u})$, its gradient will be denoted by
\[
DF = (\partial_1 u_1 F, \partial_2 u_2 F, \partial_3 u_3 F).
\]

We end this section by the following assumptions regarding various constants. Theorem 1.1 will be proved for any fixed constant $\mu > 0$, while Theorem 1.2 is valid for a sufficiently small (fixed) $\mu$. To simplify the presentation, we shall assume without loss of generality that
\[
\mu \in (0, 1) \text{ is a fixed constant. (2.4)}
\]

Finally we can normalize $g$ (cf. (1.9)) to have
\[
\sigma := \int_0^1 g(s) \, ds = 1. \quad (2.5)
\]

As the $L^2$-gradient flow of (1.1), the system (1.2) enjoys the following energy dissipation law:
\[
A_\varepsilon(u_\varepsilon(\cdot, \hat{T})) + \int_0^{\hat{T}} \int_\Omega \varepsilon |\partial_t u_\varepsilon|^2 \, dx \, dt = A_\varepsilon(u_\varepsilon^m(\cdot)) \quad (2.6)
\]
for arbitrarily large time $\hat{T}$. Combining this with the theory of gradient flow and the regularity theory for elliptic system (cf. [4, 45]), one can construct a unique solution to system (1.2) that satisfies
\[
\mathbf{u}_\varepsilon \in L^2(0, \hat{T}; W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)) \text{ and } \partial_t \mathbf{u}_\varepsilon \in L^2(\Omega \times (0, \hat{T})).
\]

So for almost every $\hat{t} \in (0, \hat{T})$, we have
\[
\mathbf{u}_\varepsilon(\cdot, \hat{t}) \in W^{2,2}(\Omega) \hookrightarrow W^{1,6}(\Omega) \hookrightarrow C^{0,1/2}(\overline{\Omega}).
\]

Under the assumption (1.1a), the nonlinearity of (1.2a) has a linear growth. So considering the system with initial datum $\mathbf{u}_\varepsilon(\cdot, \hat{t})$, and using the Hölder estimates for parabolic system (cf. [21]), we deduce that
\[
\mathbf{u}_\varepsilon \text{ is a classical solution of (1.2a) in } \Omega \times (0, \hat{T}). \quad (2.7)
\]

For initial datum undergoing phase transitions near the initial interface $I_0$, formal asymptotic analysis suggests that $\nabla \mathbf{u}$ will be singular near $I_t$. However, the global dissipation law (2.6) is not sufficient to yield the (strong) convergence of $u_\varepsilon$, not even in the domain away from $I_t$. Following a recent work of Fisher et al. [24], we shall establish in this section a differential inequality which modulates the concentration and leads to the compactness of solutions in Sobolev spaces.
2.2. The modulated energy. We first set up the geometry of the moving interface $I$ defined in (1.4). Under a local parametrization $\varphi_t(s) : U \subset \mathbb{R}^{d-1} \rightarrow I_t$, the mean curvature flow reads

$$\partial_t \varphi_t(s) = \kappa n$$

(2.8)

where $\kappa = \kappa(\varphi_t(s), t)$ is the mean curvature and $n = n(\cdot, t) : I_t \mapsto \mathbb{S}^{d-1}$ is the inward normal vector. For any $t \in [0, T]$ we assume that the nearest-point projection $P_t(x, t) : B_{4\delta_0}(I_t) \mapsto I_t$ is smooth for some sufficiently small $\delta_0 \in (0, 1)$ which only depends on the geometry of $I$. Analytically we have $P_t(x, t) = x - \nabla d_I(x, t) d_I(x, t)$. So for each fixed $t \in [0, T]$, any point $x \in B_{4\delta_0}(I_t)$ corresponds to a unique pair $(r, s)$ with $r = d_I(x, t)$ and $s \in U$, and the identity

$$d_I(\varphi_t(s) + r n(\varphi_t(s), t), t) = r$$

holds with independent variables $(r, s, t)$. Differentiating this identity with respect to $r$ and $t$ leads to the following identities:

$$\nabla d_I(x, t) = n(P_t(x, t), t),$$

$$-\partial_t d_I(x, t) = \partial_t \varphi_t(s) \cdot n(\varphi_t(s), t) =: V(s, t).$$

(2.9)

The significance of these equations is that they extend the normal vector and the normal velocity from $I_t$ to a neighborhood of it. So we shall also use $n$ to denote $\nabla d_I$ when the latter is smooth. We shall extend $n$ to the whole computational domain $\Omega$ by defining

$$\xi(x, t) := \phi \left( \frac{d_I(x, t)}{\delta_0} \right) \nabla d_I(x, t)$$

(2.10)

where $\phi : \mathbb{R} \mapsto \mathbb{R}_+$ is an even, smooth function that decreases on $[0, 1]$, and satisfies

$$\begin{cases}
\phi(z) > 0 & \text{for } |z| < 1, \\
\phi(z) = 0 & \text{for } |z| \geq 1, \\
1 - 4z^2 \leq \phi(z) \leq 1 - \frac{1}{2}z^2 & \text{for } |z| \leq 1/2.
\end{cases}$$

(2.11)

To fulfill these requirements, we can simply choose

$$\phi(z) = \begin{cases}
\frac{1}{e^{2z^2 - 1} + 1} & \text{for } |z| < 1, \\
0 & \text{for } |z| \geq 1.
\end{cases}$$

We proceed with the extension of the mean curvature. Choosing a cut-off function $\eta_0(x, t)$ such that

$$\eta_0(\cdot, t) \in C_\infty_c(\mathbb{B}_{2\delta_0}(I_t); [0, 1]) \quad \text{and} \quad \eta_0 \equiv 1 \text{ in } \mathbb{B}_{\delta_0}(I_t),$$

(2.12)
we constantly extend the inward mean curvature vector by defining
\[ H(x, t) := \kappa \nabla d_I(x, t) \quad \text{with} \quad \kappa(x, t) = -\Delta d_I(P_I(x, t))\eta_0(x, t). \tag{2.13} \]

These combined with (2.10) imply that
\[ (n \cdot \nabla)H = 0 \text{ in } B_{\delta_0}(I_t), \tag{2.14a} \]
\[ (\xi \cdot \nabla)H = 0 \text{ in } \Omega, \tag{2.14b} \]
\[ \xi = 0 \text{ and } H = 0 \text{ on } \partial \Omega. \tag{2.14c} \]

**Lemma 2.1.** There exists a constant \( C > 0 \) depending only on the geometry of the interface (1.4) such that the following properties hold for every \( t \in [0, T] \):
\[
|\nabla \cdot \xi + H \cdot \xi| \leq C |d_I| \text{ in } B_{\delta_0}(I_t), \tag{2.15a}
\]
\[
\partial_t d_I + (H \cdot \nabla)d_I = 0 \text{ in } B_{\delta_0}(I_t), \tag{2.15b}
\]
\[
\partial_t \xi + (H \cdot \nabla)\xi + (\nabla H)^T \xi = 0 \text{ in } B_{\delta_0}(I_t), \tag{2.15c}
\]
where \( \nabla H := \{\partial_j H_i\}_{1 \leq i,j \leq 3} \) is a matrix with \( i \) being the row index.

**Proof.** By introducing \( \phi_0(\tau) := \phi(\frac{\tau}{\delta_0}) \), we can rewrite (2.10) as \( \xi = \phi_0(d_I) \nabla d_I \). Since \( \phi \) is even, we have \( \phi_0(0) = 0 \). This combined with Taylor’s expansion in \( d_I \) implies that
\[
\nabla \cdot \xi = |\nabla d_I|^2 \phi_0'(d_I) + \phi_0(d_I)\Delta d_I(x, t) = O(d_I) + \phi_0(d_I)\Delta d_I(P_I(x, t), t).
\]

This and (2.13) lead to (2.15a). Using (2.9) and (2.13), we can write (2.8) as the transport equation (2.15b), which leads to the following identities in \( B_{\delta_0}(I_t) \):
\[
\partial_t \nabla d_I + (H \cdot \nabla)\nabla d_I + (\nabla H)^T \nabla d_I = 0,
\]
\[
\partial_t \phi_0(d_I) + (H \cdot \nabla)\phi_0(d_I) = 0.
\]

These two equations together imply (2.15c). \( \Box \)

It will be convenient to introduce
\[
\psi_\varepsilon = d^F \circ u_\varepsilon \quad \text{where} \quad d^F(v) := \int_0^{|v|} g(s) \, ds. \tag{2.16}
\]

It can be verified using (1.10b) that
\[
d^F(v) \in C^1(\mathbb{R}^3), \quad \text{and} \quad Dd^F(v) = 0 \quad \text{iff} \quad v \in \{0, S^2\}. \tag{2.17}
\]

By (1.9) we have
\[
|Dd^F(v)| = \sqrt{2F(v)}, \quad \forall v \in \mathbb{R}^3. \tag{2.18}
\]

Recalling (2.7), we have
\[
\partial_t \psi_\varepsilon(x, t) = \partial_t u_\varepsilon(x, t) \cdot Dd^F(u_\varepsilon(x, t)) \quad \text{for any} \quad (x, t) \in \Omega \times (0, T), \tag{2.19a}
\]
\[
\nabla \psi_\varepsilon(x, t) = \nabla |u_\varepsilon(x, t)| \cdot g(|u_\varepsilon(x, t)|) \quad \text{if} \quad u_\varepsilon(x, t) \neq 0. \tag{2.19b}
\]

Now we define the phase-field analogues of the normal vector and the mean curvature vector respectively by
\[
\mathbf{n}_\varepsilon(x, t) := \begin{cases}
\nabla \psi_\varepsilon(x, t) / |\nabla \psi_\varepsilon| & \text{if } \nabla \psi_\varepsilon(x, t) \neq 0, \\
0 & \text{otherwise}.
\end{cases} \tag{2.20a}
\]
\[
\mathbf{H}_\varepsilon(x, t) := \begin{cases}
-\left(\varepsilon \Delta u_\varepsilon - \frac{1}{\varepsilon} D^2 F(u_\varepsilon) \right) \cdot \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} & \text{if } \nabla u_\varepsilon \neq 0, \\
0 & \text{otherwise}.
\end{cases} \tag{2.20b}
\]
Note that in (2.20b), the inner product is made with the column vectors of \( \nabla u_\varepsilon = (\partial_1 u_\varepsilon, \partial_2 u_\varepsilon, \partial_3 u_\varepsilon) \). We deduce from (2.20a) that
\[
\nabla \psi_\varepsilon = |\nabla \psi_\varepsilon| n_\varepsilon \quad \text{for any } (x, t).
\] (2.21)
Define also the orthogonal projection \( \Pi_{u_\varepsilon} \) by
\[
\Pi_{u_\varepsilon} \partial_i u_\varepsilon := \begin{cases} \left( \partial_i u_\varepsilon \cdot \frac{u_\varepsilon}{|u_\varepsilon|} \right) \frac{u_\varepsilon}{|u_\varepsilon|} & \text{if } u_\varepsilon \neq 0, \\ 0 & \text{otherwise.} \end{cases}
\] (2.22)

**Lemma 2.2.** The following equations hold:
\[
|\nabla \psi_\varepsilon| = |\Pi_{u_\varepsilon} \nabla u_\varepsilon||Dd^F(u_\varepsilon)| \quad \text{for any } (x, t),
\] (2.23a)
\[
\Pi_{u_\varepsilon} \nabla u_\varepsilon = \frac{|\nabla \psi_\varepsilon|}{|Dd^F(u_\varepsilon)|^2} Dd^F(u_\varepsilon) \otimes n_\varepsilon \quad \text{on } \{ x \mid |u_\varepsilon| \notin \{0, 1\} \}. \tag{2.23b}
\]

**Proof.** Concerning (2.23a), it suffices to work with the set \( \{ x \mid |u_\varepsilon| \notin \{0, 1\} \} \) where \( g(|u_\varepsilon|) > 0 \) (cf. (1.10)), for otherwise the equation will follow from (2.17) and (2.19a).

On this set we deduce from (2.17) that
\[
Dd^F(u_\varepsilon) = \frac{u_\varepsilon}{|u_\varepsilon|} g(|u_\varepsilon|) \neq 0,
\]
and we can rewrite (2.19a) as
\[
\partial_3 \psi_\varepsilon = \partial_i u_\varepsilon \cdot \frac{Dd^F(u_\varepsilon)}{|Dd^F(u_\varepsilon)|} |Dd^F(u_\varepsilon)| = \partial_i u_\varepsilon \cdot \frac{u_\varepsilon}{|u_\varepsilon|} |Dd^F(u_\varepsilon)|.
\] (2.24)
This combined with (2.22) implies (2.23a).

Now we turn to the proof of (2.23b). On the set \( \{ x \mid |u_\varepsilon| \notin \{0, 1\} \} \), we have
\[
\frac{|\nabla \psi_\varepsilon|}{|Dd^F(u_\varepsilon)|^2} Dd^F(u_\varepsilon) \otimes n_\varepsilon = \frac{Dd^F(u_\varepsilon)}{|Dd^F(u_\varepsilon)|^2} \otimes \nabla \psi_\varepsilon = \frac{u_\varepsilon}{|u_\varepsilon|} \otimes \nabla |u_\varepsilon|,
\]
and this implies (2.23b) in view of (2.22). \( \square \)

The following lemma establishes coercivity properties of the modulated energy (1.7).

**Lemma 2.3.** The following estimates hold for every \( t \in [0, T] \):
\[
\int_\Omega \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) - |\nabla \psi_\varepsilon| \right) \, dx \leq E_\varepsilon[u_\varepsilon|I|], \tag{2.26a}
\]
\[
\varepsilon \int_\Omega (\mu |\text{div} u_\varepsilon|^2 + |\nabla u_\varepsilon - \Pi_{u_\varepsilon} \nabla u_\varepsilon|^2) \, dx \leq 2E_\varepsilon[u_\varepsilon|I|], \tag{2.26b}
\]
\[
\int_\Omega \left( \varepsilon |\Pi_{u_\varepsilon} \nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}} |Dd^F(u_\varepsilon)| \right)^2 \, dx \leq 2E_\varepsilon[u_\varepsilon|I|], \tag{2.26c}
\]
\[
\int_\Omega \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) + |\nabla \psi_\varepsilon| \right) (1 - \xi \cdot n_\varepsilon) \, dx \leq 4E_\varepsilon[u_\varepsilon|I|], \tag{2.26d}
\]
\[
\int_\Omega \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) + |\nabla \psi_\varepsilon| \right) \min \left( d_I^2, 1 \right) \, dx \leq CE_\varepsilon[u_\varepsilon|I|] \tag{2.26e}
\]
where \( C = C(\delta_0, \phi) \).

**Proof.** The case when \( \mu = 0 \) has been done in [40], and the proof carries over to the present case. First, it follows from (2.22) that
\[
|\nabla u_\varepsilon - \Pi_{u_\varepsilon} \nabla u_\varepsilon|^2 + |\Pi_{u_\varepsilon} \nabla u_\varepsilon|^2 = |\nabla u_\varepsilon|^2.
\] (2.27)
Combining this with (2.21), we can write
\[
\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) - \xi \cdot \nabla \psi_\varepsilon
\]
\[= \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) - |\nabla \psi_\varepsilon| + |\nabla \psi_\varepsilon|(1 - \xi \cdot n_\varepsilon)
\]
\[= \frac{\varepsilon}{2} |\nabla u_\varepsilon - \Pi_{u_\varepsilon} \nabla u_\varepsilon|^2 + \left(\frac{\varepsilon}{2} |\Pi_{u_\varepsilon} \nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) - |\nabla \psi_\varepsilon|\right)
\]
\[+ |\nabla \psi_\varepsilon|(1 - \xi \cdot n_\varepsilon).
\] (2.28)

By (2.18) and (2.23a), the second term in the last display is non-negative. Since |\xi| \leq 1, we also have (2.26a), (2.26b), (2.26c) and
\[
E_\varepsilon[u_\varepsilon] \geq \int (1 - \xi \cdot n_\varepsilon) |\nabla \psi_\varepsilon| \, dx.
\] (2.29)

Combining (2.29) with (2.26a) and the inequality 1 - \xi \cdot n_\varepsilon \leq 2, we obtain (2.26d). Finally, by (2.11) and \(\delta_0 \in (0, 1)\) we have
\[1 - \xi \cdot n_\varepsilon \geq 1 - \phi\left(\frac{d_I}{\delta_0}\right) \geq \min\left(\frac{d_I^2}{2\delta_0^2}, 1 - \phi\left(\frac{1}{T}\right)\right) \geq C \min(d_I^2, 1).
\] (2.30)

This together with (2.26d) implies (2.26c).

The following result was first proved in [24] for the scalar Allen-Cahn equation, and was generalized to the vectorial case in [40].

**Proposition 2.1.** There exists a generic constant \(C > 0\) depending only on the geometry of the interface (1.4) such that
\[
\frac{d}{dt} E_\varepsilon[u_\varepsilon] + \int_\Omega \left(\varepsilon^2 |\partial_t u_\varepsilon|^2 - |H_\varepsilon|^2\right) \, dx + \frac{1}{2\varepsilon} \int_\Omega \varepsilon \partial_t u_\varepsilon - (\nabla \cdot \xi) DdF(u_\varepsilon)\right|^2 \, dx
\]
\[+ \frac{1}{2\varepsilon} \int_\Omega \left|H_\varepsilon - \varepsilon |\nabla u_\varepsilon| H\right|^2 \, dx \leq CE_\varepsilon[u_\varepsilon] \quad \text{for} \ t \in (0, T).
\] (2.31)

We present a proof of (2.31) in Appendix B for the convenience of the readers.

3. **Uniform estimates of solutions**

Observe that the second term on the left-hand side of (2.31) does not have an obvious sign. However, we have the following theorem.

**Theorem 3.1.** Under the assumptions of Theorem 1.7, there exists a constant \(C_0 > 0\), which depends only on the geometry of the interface (1.4) and \(c_1\) (cf. (1.14c)), such that
\[
\sup_{t \in [0,T]} \frac{1}{\varepsilon} E_\varepsilon[u_\varepsilon] + \int_0^T \int_\Omega \left( |\partial_t u_\varepsilon + (\nabla \cdot \xi) u_\varepsilon|^2 + |\partial_t u_\varepsilon - \Pi_{u_\varepsilon} \partial_t u_\varepsilon|^2 \right) \, dx dt \leq C_0.
\] (3.1)

It is worth mentioning that \(C_0\) is independent of \(\mu\). The proof of (3.1) relies on the following lemma.

**Lemma 3.2.** For any function \(\eta_1\) with \(\eta_1(\cdot, t) \in C_c(B_{4\theta_0}(I_t); \mathbb{R}_{\geq 0})\), there exists a universal constant \(C > 0\) which is independent of \(t\) and \(\varepsilon\) such that
\[
\int_\Omega \eta_1 \left|\nabla u_\varepsilon (\xi^3 - n \otimes n)\right|^2 \, dx \leq C \varepsilon^{-1} E_\varepsilon[u_\varepsilon](t) \quad \forall t \in [0, T].
\] (3.2)
Proof. On the set \( \{ x \mid g(\|u_\varepsilon\|) > 0 \} = \{ x \mid \|u_\varepsilon\| \notin \{ 0, 1 \} \} \) we can use (2.23a) and (2.23b) to estimate
\[
\begin{align*}
\left| \Pi_{u_\varepsilon} \nabla u_\varepsilon (I_3 - n_\varepsilon \otimes \xi) \right|^2 \\
= \frac{|\nabla \psi_\varepsilon|}{|Dd^F(u_\varepsilon)|^2} Dd^F(u_\varepsilon) \otimes (n_\varepsilon - \xi)^2 \\
\leq |n_\varepsilon - \xi|^2 |\Pi_{u_\varepsilon} \nabla u_\varepsilon|^2 \\
\leq 2(1 - \xi \cdot n_\varepsilon) |\nabla u_\varepsilon|^2.
\end{align*}
\]

On the set \( \{ x \mid |u_\varepsilon| = 0 \} \) we have \( \Pi_{u_\varepsilon} \nabla u_\varepsilon = 0 \) by the second case in (2.22). On the open set \( \{ x \mid |u_\varepsilon| > 0 \} \supset \{ x \mid |u_\varepsilon| = 1 \} \) we can write \( \Pi_{u_\varepsilon} \nabla u_\varepsilon = \nabla |u_\varepsilon| \otimes \frac{u_\varepsilon}{|u_\varepsilon|} \) by the first case in (2.22). This combined with [18, Theorem 4.4] implies that \( \Pi_{u_\varepsilon} \nabla u_\varepsilon = 0 \) for a.e. \( x \in \{ x \mid |u_\varepsilon| = 1 \} \). Altogether we have shown that
\[
\left| \Pi_{u_\varepsilon} \nabla u_\varepsilon (I_3 - n_\varepsilon \otimes \xi) \right|^2 \leq 2(1 - \xi \cdot n_\varepsilon) |\nabla u_\varepsilon|^2 \quad \text{a.e. in } \Omega. \tag{3.3}
\]
This together with (2.26d) implies
\[
\int_\Omega \left| \Pi_{u_\varepsilon} \nabla u_\varepsilon (I_3 - n_\varepsilon \otimes \xi) \right|^2 dx \leq C \varepsilon^{-1} E_\varepsilon |u_\varepsilon| I. \tag{3.4}
\]

In \( B_{4\delta_0}(I_\varepsilon) \) where \( n = \nabla d_\varepsilon \), we have the decomposition
\[
I_3 - n_\varepsilon \otimes n = I_3 - n_\varepsilon \otimes \xi + n_\varepsilon \otimes (\xi - n). \tag{3.5}
\]
Using (2.10) and (2.11), we can estimate the last term by
\[
|n_\varepsilon - \xi|^2 = |n_\varepsilon \otimes (\xi - n)|^2 \\
\leq 2|\xi - n| = 2 \left( 1 - \phi \left( \frac{d_\varepsilon}{\delta_0} \right) \right) \leq C \min \left( d_\varepsilon^2, 1 \right). \tag{3.6}
\]
These inequalities and (2.26e) lead to
\[
\int_\Omega \eta |\Pi_{u_\varepsilon} \nabla u_\varepsilon (I_3 - n_\varepsilon \otimes n)|^2 dx \leq C \varepsilon^{-1} E_\varepsilon |u_\varepsilon| I. \tag{3.7}
\]

Now using (3.6), (2.26d) and (2.26e) we find
\[
\int_\Omega \eta |\nabla u_\varepsilon|^2 \left( |n_\varepsilon - \xi|^2 + |\xi - n|^2 \right) dx \leq C \varepsilon^{-1} E_\varepsilon |u_\varepsilon| I.
\]

The above two estimates together with the formula
\[
(I_3 - n \otimes n) - (I_3 - n_\varepsilon \otimes n) = (n_\varepsilon - \xi) \otimes n + (\xi - n) \otimes n
\]
yield (3.2).

To proceed we need an \( L^4 \)-estimate of \( u_\varepsilon \).

Lemma 3.3. Under the assumption (1.14b), there exists a constant \( C = C(c_1) > 0 \) such that
\[
\begin{align*}
\sup_{t \in [0,T]} A_\varepsilon(u_\varepsilon(\cdot, t)) + \sup_{t \in [0,T]} \| \nabla \psi_\varepsilon(\cdot, t) \|_{L^1(\Omega)} \leq C, \tag{3.8a}
\sup_{t \in [0,T]} \| u_\varepsilon(\cdot, t) \|_{L^4(\Omega)} \leq C. \tag{3.8b}
\end{align*}
\]
Proof. It follows from (2.18), (2.23a) and the Cauchy–Schwarz inequality that
\[ A_\varepsilon(u_\varepsilon) \geq \int_\Omega \left( \frac{\varepsilon}{2} |\Pi u_\varepsilon \nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon} |DdF(u_\varepsilon)|^2 \right) \, dx \geq \int_\Omega |\nabla \psi_\varepsilon| \, dx. \]
This and (2.6) lead to (3.8a). To prove (3.8b), we first note that if $|u_\varepsilon| > 2$, then
\[ \psi_\varepsilon = \int_0^2 g(z) \, dz + \int_2^{u_\varepsilon} g(z) \, dz \geq c_0(|u_\varepsilon|^2 - 4). \]
This combined with Sobolev's embedding and $\psi_\varepsilon|_{\partial \Omega} = 0$ (cf. (1.2c)) leads to
\[ \int_\Omega |u_\varepsilon|^3 \, dx \leq C + \int_{\{x \in \Omega | |u_\varepsilon| > 2\}} |u_\varepsilon|^3 \, dx \leq C \left( 1 + \|\psi_\varepsilon\|_{L^{3/2}(\Omega)} \right) \leq C \left( 1 + \|\nabla \psi_\varepsilon\|_{L^1(\Omega)}^{3/2} \right). \]
□

Proof of Theorem 3.1. We shall only present the proof in 3D because the 2D case is analogous under the conventions made in Subsection 2.1. We shall employ Einstein summation notation by summing over repeated Latin indices.

We first use (2.31) to get
\[ \frac{2}{\varepsilon} \frac{d}{dt} E_\varepsilon[u_\varepsilon|I] + \frac{1}{\varepsilon^2} \int_\Omega \left[ (\varepsilon^2 |\partial_t u_\varepsilon|^2 - |H_\varepsilon|^2) + |H_\varepsilon - \varepsilon |\nabla u_\varepsilon|H| |^2 \right] \, dx \]
\[ + \frac{1}{\varepsilon^2} \int_\Omega |\varepsilon \partial_t u_\varepsilon - DdF(u_\varepsilon)(\nabla : \xi)|^2 \, dx \leq \frac{C}{\varepsilon} E_\varepsilon[u_\varepsilon|I]. \] (3.9)
Observe that the orthogonal projection (2.22) is parallel to $DdF(u_\varepsilon)$ when it does not vanish. So we can write
\[ |\varepsilon \partial_t u_\varepsilon - DdF(u_\varepsilon)(\nabla : \xi)|^2 \]
\[ = |\varepsilon \partial_t u_\varepsilon - \varepsilon \Pi u_\varepsilon \partial_t u_\varepsilon|^2 + |\varepsilon \Pi u_\varepsilon \partial_t u_\varepsilon - DdF(u_\varepsilon)(\nabla : \xi)|^2. \]
Substituting this identity into (3.9) we find
\[ \frac{2}{\varepsilon} \frac{d}{dt} E_\varepsilon[u_\varepsilon|I] + \frac{1}{\varepsilon^2} \int_\Omega \left[ (\varepsilon^2 |\partial_t u_\varepsilon|^2 - |H_\varepsilon|^2) + |H_\varepsilon - \varepsilon |\nabla u_\varepsilon|H| |^2 \right] \, dx \]
\[ + \int_\Omega |\partial_t u_\varepsilon - \Pi u_\varepsilon \partial_t u_\varepsilon|^2 \, dx \leq \frac{C}{\varepsilon} E_\varepsilon[u_\varepsilon|I]. \] (3.10)
To estimate the second term on the left-hand side, we use (1.2a) and (2.20b) to write
\[ H_\varepsilon = -\varepsilon \left( \partial_t u_\varepsilon - \mu \nabla \text{div} u_\varepsilon \right) \cdot \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} \quad \text{if} \quad \nabla u_\varepsilon \neq 0. \] (3.11)
Note that the inner product is made with the column vectors of $\nabla u_\varepsilon = (\partial_1 u_\varepsilon, \partial_2 u_\varepsilon, \partial_3 u_\varepsilon)$. Using the above formula, we expand the integrands of (3.10) and find

$$
\varepsilon^2 |\partial_t u_\varepsilon|^2 - |H_\varepsilon|^2 + |H_\varepsilon - \varepsilon \nabla u_\varepsilon|^2
$$

$$
= \varepsilon^2 |\partial_t u_\varepsilon|^2 + \varepsilon^2 |H_\varepsilon|^2 |\nabla u_\varepsilon|^2 + 2\varepsilon^2 \partial_t u_\varepsilon \cdot (H \cdot \nabla) u_\varepsilon
- 2\varepsilon^2 \mu \nabla (\text{div} u_\varepsilon) \cdot (H \cdot \nabla) u_\varepsilon
= \varepsilon^2 |\partial_t u_\varepsilon + (H \cdot \nabla) u_\varepsilon|^2 + \varepsilon^2 (|H_\varepsilon|^2 |\nabla u_\varepsilon|^2 - |(H \cdot \nabla) u_\varepsilon|^2)
- 2\varepsilon^2 \mu \nabla (\text{div} u_\varepsilon) \cdot (H \cdot \nabla) u_\varepsilon.
$$

Note that the second term in the last display is non-negative due to Cauchy-Schwarz’s inequality, and this implies that

$$
\int_\Omega |\partial_t u_\varepsilon + (H \cdot \nabla) u_\varepsilon|^2 dx
\leq \frac{1}{\varepsilon^2} \int_\Omega \left[ (\varepsilon^2 |\partial_t u_\varepsilon|^2 - |H_\varepsilon|^2) + |H_\varepsilon - \varepsilon \nabla u_\varepsilon|^2 \right] dx
+ 2\mu \int_\Omega \nabla (\text{div} u_\varepsilon) \cdot (H \cdot \nabla) u_\varepsilon dx.
$$

Adding the above inequality to (3.10) leads to

$$
2\varepsilon^{-1} \frac{d}{dt} E_\varepsilon[u_\varepsilon] + \int_\Omega |\partial_t u_\varepsilon + (H \cdot \nabla) u_\varepsilon|^2 dx + \int_\Omega |\partial_t u_\varepsilon - \Pi u_\varepsilon \partial_t u_\varepsilon|^2 dx
\leq C \varepsilon^{-1} E_\varepsilon[u_\varepsilon] + 2\mu \int_\Omega \nabla (\text{div} u_\varepsilon) \cdot (H \cdot \nabla) u_\varepsilon dx. 
$$

To estimate the last term, we write $u_\varepsilon = (u^i_\varepsilon)_{1 \leq i \leq 3}$ and $H = (H_i)_{1 \leq i \leq 3}$. Using integration by parts and (2.14c), we obtain

$$
\int_\Omega \nabla (\text{div} u_\varepsilon) \cdot (H \cdot \nabla) u_\varepsilon dx
= -\int_\Omega (\text{div} u_\varepsilon)(H \cdot \nabla) \text{div} u_\varepsilon dx - \int_\Omega (\text{div} u_\varepsilon)(\partial_j H \cdot \nabla) u^j_\varepsilon dx
\leq \frac{1}{2} \int_\Omega (\text{div} H)(\text{div} u_\varepsilon)^2 dx - \int_\Omega (\text{div} u_\varepsilon) \partial_k H_j \partial_k u^j_\varepsilon dx
- \int_\Omega (\text{div} u_\varepsilon)(\partial_j H_k - \partial_k H_j) \partial_k u^j_\varepsilon dx. 
$$

(3.13)

In view of (2.26b), the first integral in the last display of (3.13) is bounded by

$$
\mu^{-1} \varepsilon^{-1} \|\text{div} H\|_{L^\infty_\Omega} E_\varepsilon[u_\varepsilon] [I].
$$
The second integral can be estimated by decomposing $\nabla u_j^\epsilon$ and by using (2.14a):

\[
- \int_{\Omega} (\text{div} \, u_\epsilon) \nabla H_j \cdot \nabla u_j^\epsilon \, dx \\
= - \int_{\Omega} (\text{div} \, u_\epsilon) \nabla H_j \cdot \left( (I_3 - n \otimes n) \nabla u_j^\epsilon \right) \, dx - \int_{\Omega} (\text{div} \, u_\epsilon) (n \cdot \nabla H_j) (n \cdot \nabla u_j^\epsilon) \, dx \\
\leq \int_{\Omega} \left| \text{div} \, u_\epsilon \right|^2 \, dx + \int_{\Omega} \left| \nabla H \right|^2 \left| (I_3 - n \otimes n) \nabla u_\epsilon \right|^2 \, dx \\
+ C \int_{\Omega} \left| \nabla u_\epsilon \right|^2 \min \left( d_I^2, 1 \right) \, dx.
\] (3.14)

By (2.13) and (2.12), the second integral in the last display can be estimated using (3.2) with $\eta_1 := \left| \nabla H \right|^2$. The other two terms can be controlled by $(\mu^{-1} + 1)C \varepsilon^{-1} E_\varepsilon[u_\epsilon[I]$ using (2.26b) and (2.26c) respectively. To summarize we deduce from (3.13) and (3.14) that

\[
\int_{\Omega} \nabla (\text{div} \, u_\epsilon) \cdot (\nabla \cdot \nabla) u_\epsilon \, dx \\
\leq \mu^{-1} \varepsilon^{-1} \left\| \text{div} \, H \right\|_{L_{\infty}^2} E_\varepsilon[u_\epsilon[I] + (\mu^{-1} + 1)C \varepsilon^{-1} E_\varepsilon[u_\epsilon[I] \\
- \int_{\Omega} (\text{div} \, u_\epsilon) (\partial_j H_k - \partial_k H_j) \partial_k u_j^\epsilon \, dx.
\]

Combining this with (3.12), we find

\[
2 \varepsilon^{-1} \frac{d}{dt} E_\varepsilon[u_\epsilon[I] + \int_{\Omega} \left| \partial_t u_\epsilon + (\nabla \cdot \nabla) u_\epsilon \right|^2 \, dx + \int_{\Omega} \left| \partial_t u_\epsilon - \Pi u_\epsilon \partial_t u_\epsilon \right|^2 \, dx \\
\leq C \varepsilon^{-1} E_\varepsilon[u_\epsilon[I] - 2\mu \int_{\Omega} (\text{div} \, u_\epsilon) (\partial_j H_k - \partial_k H_j) \partial_k u_j^\epsilon \, dx.
\] (3.15)

Note that due to (2.4) the constant $C$ above can be made independent of $\mu$. It remains to estimate the last integral in (3.15). By orthogonal decompositions\(^{1}\),

\[
(\partial_j H_k - \partial_k H_j) \partial_k u_j^\epsilon = -(\text{rot} \, u_\epsilon) \cdot (\text{rot} \, H).
\]

We also need the following identity which follows by taking the wedge product of (1.2a) with $u_\epsilon$.

\[
\mu (\nabla \text{div} \, u_\epsilon) \wedge u_\epsilon = (\partial_t u_\epsilon - \Delta u_\epsilon) \wedge u_\epsilon.
\]

\(^{1}\text{For a square matrix } A, \text{ the decomposition } A = \frac{A + A^T}{2} + \frac{A - A^T}{2} \text{ is orthogonal under the Frobenius inner product } A : B \triangleq \text{tr}(A^T B).\)
Using the above two identities, we integrate by parts to obtain
\[-\mu \int_{\Omega} (\text{div} \, u_\varepsilon)(\partial_j H_k - \partial_k H_j) \partial_k u_\varepsilon \, dx\]
\[= \mu \int_{\Omega} (\text{div} \, u_\varepsilon)(\text{rot} \, \text{rot} \, H) \, dx\]
\[= \mu \int_{\Omega} (\text{div} \, u_\varepsilon) u_\varepsilon \cdot (\text{rot} \, \text{rot} \, H) \, dx - \int_{\Omega} \mu (\nabla \text{div} \, u_\varepsilon) \wedge u_\varepsilon \cdot (\text{rot} \, H) \, dx\]
\[= \mu \int_{\Omega} (\text{div} \, u_\varepsilon) u_\varepsilon \cdot (\text{rot} \, \text{rot} \, H) \, dx - \int_{\Omega} (\partial_t u_\varepsilon - \Delta u_\varepsilon) \wedge u_\varepsilon \cdot (\text{rot} \, H) \, dx\]
\[= \mu \int_{\Omega} (\text{div} \, u_\varepsilon) u_\varepsilon \cdot (\text{rot} \, \text{rot} \, H) \, dx - \int_{\Omega} (\partial_t u_\varepsilon + (H \cdot \nabla) u_\varepsilon) \wedge u_\varepsilon \cdot (\text{rot} \, H) \, dx\]
\[+ \int_{\Omega} (H \cdot \nabla) u_\varepsilon \wedge u_\varepsilon \cdot (\text{rot} \, H) \, dx + \int_{\Omega} \Delta u_\varepsilon \wedge u_\varepsilon \cdot (\text{rot} \, H) \, dx.\]

Inserting this identity into (3.15), and using the Cauchy–Schwarz inequality, (3.8b) and (2.26b), we find
\[2\varepsilon^{-1} \frac{d}{dt} E_\varepsilon[u_\varepsilon|I] + \frac{1}{2} \int_{\Omega} \left| \partial_t u_\varepsilon + (H \cdot \nabla) u_\varepsilon \right|^2 \, dx + \int_{\Omega} \left| \partial_t u_\varepsilon - \Pi u_\varepsilon \partial_k u_\varepsilon \right|^2 \, dx\]
\[\leq C \left( 1 + \varepsilon^{-1} E_\varepsilon[u_\varepsilon|I] \right) + 2 \int_{\Omega} (H \cdot \nabla) u_\varepsilon \wedge u_\varepsilon \cdot (\text{rot} \, H) \, dx + 2 \int_{\Omega} \Delta u_\varepsilon \wedge u_\varepsilon \cdot (\text{rot} \, H) \, dx\]
\[= C \left( 1 + \varepsilon^{-1} E_\varepsilon[u_\varepsilon|I] \right) + 2 \int_{\Omega} H_k \left( \partial_k u_\varepsilon - \Pi u_\varepsilon \partial_k u_\varepsilon \right) \wedge u_\varepsilon \cdot (\text{rot} \, H) \, dx\]
\[-2 \int_{\Omega} \left( \partial_k u_\varepsilon - \Pi u_\varepsilon \partial_k u_\varepsilon \right) \wedge u_\varepsilon \cdot (\text{rot} \, H) \, dx.\]

Note that in the last step we used integration by parts, the identity
\[(\Pi u_\varepsilon \partial_k u_\varepsilon) \wedge u_\varepsilon = 0\]
which follows from (2.22), and the identities \((\partial_k u_\varepsilon) \wedge (\partial_k u_\varepsilon) = 0\) for each fixed \(k \in \{1, 2, 3\}\). Finally, applying the Cauchy–Schwarz inequality and then (2.26b) and (3.8b) in the last two integrals of (3.16), we find
\[2\varepsilon^{-1} \frac{d}{dt} E_\varepsilon[u_\varepsilon|I] + \frac{1}{2} \int_{\Omega} \left| \partial_t u_\varepsilon + (H \cdot \nabla) u_\varepsilon \right|^2 \, dx + \int_{\Omega} \left| \partial_t u_\varepsilon - \Pi u_\varepsilon \partial_k u_\varepsilon \right|^2 \, dx\]
\[\leq C \left( 1 + \varepsilon^{-1} E_\varepsilon[u_\varepsilon|I] \right).\]

This combined with (1.14c) and Grönwall’s inequality leads to (3.1). \qed

Using (2.26c) and (3.1), we readily obtain the following corollary.

**Corollary 3.4.** Under the assumptions of Theorem 1.1, there exists a constant \(C > 0\), which depends only on the geometry of the interface (1.4) and \(c_1\), such that
\[\sup_{t \in [0,T]} \int_{\Omega_t^0 \setminus B_\delta(I_t)} \left( |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2} F(u_\varepsilon) + \frac{1}{\varepsilon} |\nabla \psi_\varepsilon| \right) \, dx \leq C \delta^{-2},\]
(3.19a)
\[\int_0^T \int_{\Omega_t^0 \setminus B_\delta(I_t)} |\partial_t u_\varepsilon|^2 \, dx \, dt \leq C \delta^{-2};\]
(3.19b)
hold for each fixed \(\delta \in (0, \delta_0)\).
Indeed, (3.19b) follows from (3.19a) and the inequality
\[ \int_0^T \int_\Omega \left| \partial_t \mathbf{u}_\varepsilon + (\mathbf{H} \cdot \nabla) \mathbf{u}_\varepsilon \right|^2 \, dx \, dt \leq C, \]
which is a consequence of (3.1). Another consequence of (3.1) is the following lemma concerning
\[ \hat{\mathbf{u}}_\varepsilon := \left\{ \begin{array}{ll} \frac{\mathbf{u}_\varepsilon}{|\mathbf{u}_\varepsilon|} & \text{if } \mathbf{u}_\varepsilon \neq 0, \\ 0 & \text{otherwise}. \end{array} \right. \] (3.21)

**Lemma 3.5.** Under the assumptions of Theorem 1.1, there exists a constant \( C > 0 \), which depends only on the geometry of the interface (1.4) and \( c_1 \), such that
\[ \sup_{t \in [0,T]} \int \Omega |\mathbf{u}_\varepsilon|^2 |\nabla \hat{\mathbf{u}}_\varepsilon|^2 \, dx + \sup_{t \in [0,T]} \int \Omega |\hat{\mathbf{u}}_\varepsilon \cdot \nabla |\mathbf{u}_\varepsilon||^2 \, dx \leq (1 + \mu^{-1})C, \] (3.22a)
\[ \sup_{t \in [0,T]} \int \Omega (\hat{\mathbf{u}}_\varepsilon \cdot \mathbf{n}_\varepsilon)^2 \left| \nabla \psi_\varepsilon \right| \, dx \leq (1 + \mu^{-1})(1 + \sqrt{\mu + 1})C\varepsilon. \] (3.22b)

**Proof.** We first deduce from (3.1) and (2.26b) that
\[ \sup_{t \in [0,T]} \int \Omega \left( \mu \left| \nabla \mathbf{u}_\varepsilon \right|^2 + |\nabla \mathbf{u}_\varepsilon - \Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon|^2 \right) \, dx \leq C. \] (3.23)

By (3.21) we have the identity \( \mathbf{u}_\varepsilon = |\mathbf{u}_\varepsilon| \hat{\mathbf{u}}_\varepsilon \). Using this and (2.22), we can write
\[ \nabla \mathbf{u}_\varepsilon - \Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon = |\mathbf{u}_\varepsilon| \nabla \hat{\mathbf{u}}_\varepsilon \ \text{if } \mathbf{u}_\varepsilon \neq 0. \] (3.24)

Substituting this formula into (3.23), we obtain the estimate of the first integral on the left-hand side of (3.22a). To control the second one, we use the following formula which follows from (2.22):
\[ \text{tr} \nabla \mathbf{u}_\varepsilon - \text{tr} (\Pi_{\mathbf{u}_\varepsilon} \nabla \mathbf{u}_\varepsilon) = \text{div} \mathbf{u}_\varepsilon - \hat{\mathbf{u}}_\varepsilon \cdot \nabla |\mathbf{u}_\varepsilon| \ \text{if } \mathbf{u}_\varepsilon \neq 0. \] (3.25)

Note that on the set \( \{ x \mid |\mathbf{u}_\varepsilon| = 0 \} \), we have \( \nabla |\mathbf{u}_\varepsilon| = 0 \) a.e., and thus the above formula is still valid. This and (3.23) yield the estimate of \( \hat{\mathbf{u}}_\varepsilon \cdot \nabla |\mathbf{u}_\varepsilon| \) and (3.22a) is proved.

Regarding (3.22b), it suffices to estimate over the set
\[ \{ x \mid \nabla \psi_\varepsilon \neq 0 \} \subset U_\varepsilon \]
because the integral over its complement vanishes. By (2.17) and (2.19a), we have \( U_\varepsilon \subset \{ x \mid |\mathbf{u}_\varepsilon| \neq 0, 1 \} \) where \( g(|\mathbf{u}_\varepsilon|) = |Dd^F|(\mathbf{u}_\varepsilon) > 0 \). This combined with (2.19b) and (2.20a) implies that
\[ \mathbf{n}_\varepsilon = \left( \frac{\nabla \psi_\varepsilon}{|\nabla \psi_\varepsilon|} \right) = \left( \frac{\nabla |\mathbf{u}_\varepsilon|}{|\nabla |\mathbf{u}_\varepsilon||} \right) \ \text{on } U_\varepsilon. \]

On the other hand, by the polar decomposition \( \mathbf{u}_\varepsilon = |\mathbf{u}_\varepsilon| \hat{\mathbf{u}}_\varepsilon \) and orthogonality \( \hat{\mathbf{u}}_\varepsilon \perp \partial_x \hat{\mathbf{u}}_\varepsilon \), we have
\[ |\nabla \mathbf{u}_\varepsilon|^2 = |\nabla |\mathbf{u}_\varepsilon||^2 + |\mathbf{u}_\varepsilon|^2 |\nabla \hat{\mathbf{u}}_\varepsilon|^2 \geq |\nabla |\mathbf{u}_\varepsilon||^2 \ \text{on } U_\varepsilon. \] (3.26)

Setting \( \hat{\mathbf{u}}_\varepsilon \cdot \mathbf{n}_\varepsilon := \cos \theta_\varepsilon \), we have
\[ \mu \int_{U_\varepsilon} \cos^2 \theta_\varepsilon |\nabla |\mathbf{u}_\varepsilon||^2 \, dx = \mu \int_{U_\varepsilon} |\hat{\mathbf{u}}_\varepsilon \cdot \mathbf{n}_\varepsilon|^2 |\nabla |\mathbf{u}_\varepsilon||^2 \, dx \leq (1 + \mu)C. \] (3.27)
This inequality, (2.26a) and (3.26), together imply that
\[
(1 + \mu)C \geq \int_{U_\varepsilon} \frac{\mu}{2} \cos^2 \theta \varepsilon |\nabla u_\varepsilon|^2 \, dx + \int_{U_\varepsilon} \left( \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2} F(u_\varepsilon) - \frac{1}{\varepsilon} |\nabla \psi_\varepsilon| \right) \, dx
\]
\[
\geq \frac{1}{\varepsilon} \int_{U_\varepsilon} \left( \sqrt{\mu \cos^2 \theta + 1} |\nabla u_\varepsilon| \sqrt{2F(u_\varepsilon) - |\nabla \psi_\varepsilon|} \right) \, dx
\]
\[
= \frac{1}{\varepsilon} \int_{U_\varepsilon} \left( \sqrt{\mu \cos^2 \theta + 1 - 1} \right) |\nabla \psi_\varepsilon| \, dx.
\]
Note that in the last step we have used the identity $|\nabla u_\varepsilon| \sqrt{2F(u_\varepsilon) - |\nabla \psi_\varepsilon|}$ which holds on $U_\varepsilon$. So (3.22b) follows from conjugation. \qed

4. Estimates of level sets

Recalling (2.5), the main result of this section is the following $L^1$-estimate of $\psi_\varepsilon$.

**Theorem 4.1.** Under the assumptions of Theorem 1.1, there exists $C > 0$ independent of $\varepsilon$ such that
\[
\sup_{t \in [0,T]} B[u_\varepsilon|I](t) \leq C\varepsilon, \quad (4.1)
\]
\[
\sup_{t \in [0,T]} \int_{\Omega} |\psi_\varepsilon - 1_{\Omega_t^\pm}| \, dx \leq C\varepsilon^{1/4}. \quad (4.2)
\]

**Proof.** We shall denote the positive and negative parts of a function $h$ by $h^+$ and $h^-$ respectively. For simplicity we shall suppress $dx$ in a volume integral. By [18, pp. 153], for any $h \in W^{1,1}(\Omega)$, we have
\[
\partial_i(h(x))^+ = (\partial_i h(x)) 1_{\{h(x) > 0\}}(x) \quad \text{for a.e. } x \in \Omega. \quad (4.3)
\]

Our goal is to estimate $2\psi_\varepsilon - 1 - \chi$ where $\chi(x, t) = \pm 1$ in $\Omega_t^\pm$. Using the formula $h = h^+ - h^-$, we can write
\[
2\psi_\varepsilon - 1 = 2(\psi_\varepsilon - 1)^+ + (1 - 2(\psi_\varepsilon - 1)^-) , \quad (4.4)
\]
and we shall estimate its difference with $\chi$. This will be done by establishing differential inequalities for the following energies which add up to (1.12):
\[
g_\varepsilon(t) := \int_{\Omega} (\psi_\varepsilon - 1)^+ \zeta \circ dI , \quad (4.5a)
\]
\[
h_\varepsilon(t) := \int_{\Omega} (\chi - [1 - 2(\psi_\varepsilon - 1)^-]) \eta \circ dI , \quad (4.5b)
\]
where $\eta(z)$ is defined by (1.13) and $|\eta|(z) := \zeta(z)$. It is obvious that the integrand of (4.5a) is non-negative. Since $\psi_\varepsilon \geq 0$, we have $(\psi_\varepsilon - 1)^- \in [0, 1]$ and thus $[1 - 2(\psi_\varepsilon - 1)^-]$ ranges in $[-1, 1]$. Using the identity $(\eta \circ dI) \chi = |\eta \circ dI|$, we deduce that the integrand of (4.5b) is also non-negative and
\[
h_\varepsilon(t) = \int_{\Omega} \left( 1 - 2(\psi_\varepsilon - 1)^- - \chi \right) \zeta \circ dI. \quad (4.6)
\]
Finally, we deduce from (1.14c) that
\[
g_\varepsilon(0) + h_\varepsilon(0) \leq c_1\varepsilon. \quad (4.7)
\]
Step 1: estimates of weighted errors. Using (1.8) and (1.9), we have

$$\partial_t \psi \varepsilon = \left( \partial_t u + (H \cdot \nabla) u \right) \cdot \frac{u}{\| u \|} \sqrt{2F(u)} - \nabla \psi \varepsilon. \quad (4.8)$$

Using this and (4.3) we can calculate

$$g' \varepsilon (t) = \int_{\{ \psi > 1 \}} \left( \partial_t u + (H \cdot \nabla) u \right) \cdot \frac{u}{\| u \|} \sqrt{2F(u)} \psi \varepsilon \circ dI$$

$$- \int_{\{ \psi > 1 \}} H \cdot \nabla \psi \varepsilon \psi \varepsilon \circ dI + \int_{\Omega} (\psi - 1)^+ \partial_t (\psi \varepsilon \circ dI)$$

$$= \int_{\{ \psi > 1 \}} \left( \partial_t u + (H \cdot \nabla) u \right) \cdot \frac{u}{\| u \|} \sqrt{2F(u)} \psi \varepsilon \circ dI$$

$$- \int_{\Omega} H \cdot \nabla (\psi - 1)^+ \psi \varepsilon \circ dI + \int_{\Omega} (\psi - 1)^+ H \cdot \nabla (\psi \varepsilon \circ dI)$$

$$+ \int_{\Omega} \left( \partial_t (\psi \varepsilon \circ dI) + H \cdot \nabla (\psi \varepsilon \circ dI) \right) (\psi - 1)^+.$$

By (2.15b), the integrand of the last integral vanishes on $B_{\delta \varepsilon} (I)$. Moreover, we can combine the second and the third integrals in the last display using integration by parts. Using also that $\| \text{div} \ H \|_{L^2_{\varepsilon,t}} \leq C$ and (2.26c), we find

$$g' \varepsilon (t) \leq \int_{\{ \psi > 1 \}} \left( \partial_t u + (H \cdot \nabla) u \right) \cdot \frac{u}{\| u \|} \sqrt{2F(u)} \psi \varepsilon \circ dI$$

$$+ \int_{\Omega} (\text{div} \ H)(\psi - 1)^+ \psi \varepsilon \circ dI + C \int_{\Omega \setminus B_{\delta \varepsilon} (I)} (\psi - 1)^+$$

$$\leq \int_{\Omega} \varepsilon \left| \partial_t u + (H \cdot \nabla) u \right|^2 + \left( \int_{\Omega} \frac{1}{\varepsilon} F(u) \zeta^2 \circ dI \right) + C g \varepsilon$$

$$\geq C \varepsilon [u] + C g \varepsilon + \int_{\Omega} \varepsilon \left| \partial_t u + (H \cdot \nabla) u \right|^2. \quad (4.9)$$

Now using (4.7), (3.20) and (3.1), we can apply the Grönwall lemma and obtain

$$\sup_{\varepsilon \in [0,T]} g \varepsilon (t) \leq C \varepsilon$$

for some $C$ which is independent of $\varepsilon$. Concerning $h \varepsilon$, for simplicity we introduce $w \varepsilon := \chi - [1 - (\psi - 1)^-].$ Using the identity $(\partial_t \chi) \psi \varepsilon \circ dI \equiv 0$ (in the sense of distribution), we find

$$(\partial_t w \varepsilon) \psi \varepsilon \circ dI = (2 \partial_t \psi \varepsilon) \chi \psi \varepsilon \circ dI. \quad (4.10)$$

So by the same calculation for $g \varepsilon$ we obtain

$$h' \varepsilon (t) = \int_{\{ \psi < 1 \}} 2(\partial_t u + (H \cdot \nabla) u) \cdot \frac{u}{\| u \|} \sqrt{2F(u)} \psi \varepsilon \circ dI$$

$$+ \int_{\Omega} (\text{div} \ H) w \varepsilon \psi \varepsilon \circ dI + \int_{\Omega} \left( \partial_t (\psi \varepsilon \circ dI) + (H \cdot \nabla) \psi \varepsilon \circ dI \right) w \varepsilon$$

$$\leq C \varepsilon [u] + C h \varepsilon (t) + \int_{\Omega} \varepsilon \left| \partial_t u + (H \cdot \nabla) u \right|^2.$$
Using (4.7) and (3.20), we can apply the Grönwall lemma and obtain $\sup_{t \in [0, T]} h_\varepsilon(t) \leq C\varepsilon$. Finally, by (4.4) and (4.6), we find
\[
\int_{\Omega} |2\psi_\varepsilon - 1 - \chi| \zeta \circ dI \leq \int_{\Omega} 2(\psi_\varepsilon - 1)^+ \zeta \circ dI + \int_{\Omega} |1 - 2(\psi_\varepsilon - 1)^- - \chi| \zeta \circ dI = 2g_\varepsilon(t) + h_\varepsilon(t) \leq C\varepsilon \quad \text{for all } t \in [0, T],
\]
and this proves (4.1).

**Step 2: remove the weight.** First note that (4.11) implies (4.2) with $\Omega$ replaced by $\Omega \setminus B_{\delta_0}(I_t)$. So we shall focus on the estimate on $B_{\delta_0}(I_t)$. We set $\chi_\varepsilon := 2\psi_\varepsilon - 1$ and abbreviate $\delta_0$ by $\delta$. For fixed $t \in [0, T]$ and $p \in I_t$ with normal vector $n = n(p)$, applying Hölder’s inequality and Lemma 4.2 below with $f(r, p, t) = |\chi(p + r n, t) - \chi_\varepsilon(p + r n, t)|$, we find
\[
\left( \int_{B_{\delta}(I_t)} |\chi(x, t) - \chi_\varepsilon(x, t)| \, dx \right)^{4/3} \leq C \left( \int_{I_t} \left( \int_{-\delta}^{\delta} f(r, p, t) \, dr \right)^{4/3} \, d\mathcal{H}^{d-1}(p) \right),
\]
(4.12)
\[
\leq C \int_{I_t} \|f(\cdot, p, t)\|_{L^{3/2}(-\delta, \delta)} \left( \int_{-\delta}^{\delta} f(r, p, t) \, dr \right)^{1/3} \, d\mathcal{H}^{d-1}(p) = C\|f(\cdot, t)\|_{L^{3/2}(B_{\delta_0}(I_t))} \left( \int_{I_t} \int_{-\delta}^{\delta} f(r, p, t) \, dr \, d\mathcal{H}^{d-1}(p) \right)^{1/3}.
\]
In view of (1.8) and (1.2c), we have $\psi_\varepsilon = 0$ on $\partial\Omega$. So by Sobolev’s embedding $W^{1,1} \hookrightarrow L^{3/2}$ we obtain
\[
\left( \int_{B_{\delta}(I_t)} |\chi(x, t) - \chi_\varepsilon(x, t)| \, dx \right)^4 \leq C \left( \|\chi\|_{L^{3/2}(\Omega)}^3 + \|\chi_\varepsilon\|_{L^{3/2}(\Omega)}^3 \right) \int_{\Omega} \zeta \circ dI |\chi_\varepsilon - \chi| \, dx \leq C(1 + \|\nabla \psi_\varepsilon\|_{L^1(\Omega)}^3) \int_{\Omega} \zeta \circ dI |\chi_\varepsilon - \chi| \, dx \leq C\varepsilon.
\]
Note that in the last step we employed (3.8a) and (4.11). This gives the desired estimate in $B_{\delta_0}(I_t)$ and thus the proof of (4.2) is finished. \(\square\)

**Lemma 4.2.** For any integrable function $f : [-\delta, \delta] \to \mathbb{R}_{\geq 0}$, we have
\[
\left( \int_{-\delta}^{\delta} f(r) \, dr \right)^4 \leq 6\|f\|_{L^{3/2}(-\delta, \delta)}^3 \int_{-\delta}^{\delta} |r| f(r) \, dr.
\]
(4.12)
Proof. We write \(x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)\) and \(F(x) = f(x_1)f(x_2)f(x_3)\). By symmetry and the Hölder inequality, we find

\[
\|f\|_{L^1([0,\delta])}^2 = \int_{[0,\delta]^3} F(x)F(y) \, dx \, dy \\
= 2 \int_{[0,\delta]^3} \int_{(x,y) \mid x_1+x_2+x_3 \le y_1+y_2+y_3} F(x)F(y) \, dx \, dy \\
= 2 \int_{[0,\delta]^3} \left( \int_{[0,\delta]^3} 1 \cdot F(x) \, dx \right) F(y) \, dy \\
\le 2 \int_{[0,\delta]^3} (y_1 + y_2 + y_3) \left( \int_{[0,\delta]^3} F^{3/2}(x) \, dx \right)^{2/3} F(y) \, dy \\
= 6\|f\|_{L^{3/2}([0,\delta])}^3 \|f\|_{L^1([0,\delta])}^2 \int_0^\delta r f(r) \, dr.
\]

\(\square\)

Now we turn to the study of the level sets of \(\psi_\varepsilon\). The main tool is the following estimate, which is a consequence of (2.20a), (2.26d) and (3.1).

\[
\sup_{t \in [0,T]} \int_U \left( |\nabla \psi_\varepsilon| - \xi \cdot \nabla \psi_\varepsilon \right) \, dx = \sup_{t \in [0,T]} \int_U \left( |\nabla \psi_\varepsilon| - \xi \cdot \nu_\varepsilon \nabla \psi_\varepsilon \right) \, dx \le C\varepsilon, \quad \forall U \text{ measurable in } \Omega.
\] (4.13)

Lemma 4.3. For each \(t \in [0,T]\) there exists a null set \(N_t^\varepsilon \subset (0,1/8)\) such that the following holds: for every \(\alpha \in (0,1/8)\setminus N_t^\varepsilon\), there exist

\[b_\varepsilon,\alpha(t) \in [1/2 - \alpha, 1/2 + \alpha] \quad \text{and} \quad q_\varepsilon,\alpha(t) \in [2 - \alpha, 2 + \alpha]\] (4.14)

such that the sets

\[\{x \mid \psi_\varepsilon(x,t) > b_\varepsilon,\alpha(t)\} \quad \text{and} \quad \{x \mid \psi_\varepsilon(x,t) < q_\varepsilon,\alpha(t)\}\] (4.15)

are of finite perimeter and

\[
\mathcal{H}^d-1(\{x \mid \psi_\varepsilon(x,t) = b_\varepsilon,\alpha(t)\}) - \mathcal{H}^d-1(I_t) \le C\varepsilon^{1/4} \alpha^{-1},
\] (4.16a)

\[
\mathcal{H}^d-1(\{x \mid \psi_\varepsilon(x,t) = q_\varepsilon,\alpha(t)\}) \le C\varepsilon^{1/4} \alpha^{-1},
\] (4.16b)

where \(C > 0\) is independent of \(t, \varepsilon\) and \(\alpha\).

Proof. To prove (4.16a), we consider the set

\[S_t^{\varepsilon,\alpha} = \{x \in \Omega \mid 2\psi_\varepsilon(x,t) - 1 \le 2\alpha\}, \quad \forall \alpha \in (0,1/8).
\] (4.17)

It follows from the co-area formula of BV function [18, section 5.5] that \(S_t^{\varepsilon,\alpha}\) has finite perimeter for every \(\alpha \in (0,1/8)\setminus \widetilde{N}_t^\varepsilon\) for some null set \(\widetilde{N}_t^\varepsilon \subset (0,1/8)\). Moreover, by (4.13), we have for every \(\alpha \in (0,1/8)\setminus \widetilde{N}_t^\varepsilon\) that

\[
C\varepsilon \ge \int_{S_t^{\varepsilon,\alpha}} (|\nabla \psi_\varepsilon| - \xi \cdot \nabla \psi_\varepsilon) \, dx \\
= \int_{\frac{1}{4} + \alpha} \mathcal{H}^d-1(\{x \mid \psi_\varepsilon = s\}) \, ds - \int_{\partial S_t^{\varepsilon,\alpha}} \xi \cdot \nu_\varepsilon \, d\mathcal{H}^d-1 + \int_{S_t^{\varepsilon,\alpha}} (\text{div} \xi) \psi_\varepsilon \, dx,
\] (4.18)
where $\nu$ is the outward normal of the set $S_t^{\varepsilon, \alpha}$, defined on its (measure-theoretic) boundary. Since $|\xi| \leq 1$ on $\Omega$ and $\psi_\varepsilon \leq 1$ on $S_t^{\varepsilon, \alpha}$, we have

$$\left| \int_{S_t^{\varepsilon, \alpha}} (\text{div } \xi) \psi_\varepsilon \, dx \right| \leq C |S_t^{\varepsilon, \alpha}|,$$

where $|A| = \mathcal{L}^d(A)$ is the $d$-Lebesgue measure of a set $A$. Combining this with (4.18), we find

$$\left| \int_{\frac{1}{2} - \alpha}^{\frac{1}{2} + \alpha} \mathcal{H}^{d-1} \left( \{ x \mid \psi_\varepsilon = s \} \right) \, ds - \int_{\partial S_t^{\varepsilon, \alpha}} \xi \cdot \nu \psi_\varepsilon \, d\mathcal{H}^{d-1} \right| \leq C (\varepsilon + |S_t^{\varepsilon, \alpha}|). \tag{4.19}$$

By the divergence theorem, we have

$$\int_{\partial S_t^{\varepsilon, \alpha}} \xi \cdot \nu \psi_\varepsilon \, d\mathcal{H}^{d-1} = - \left( \frac{1}{2} - \alpha \right) \int_{\{ x \mid \psi_\varepsilon < \frac{1}{2} - \alpha \}} \text{div } \xi \, dx - \left( \frac{1}{2} + \alpha \right) \int_{\{ x \mid \psi_\varepsilon > \frac{1}{2} + \alpha \}} \text{div } \xi \, dx,$$

$$-2\alpha \mathcal{H}^{d-1} (I_t) \overset{\text{(4.10)}}{=} \left( \frac{1}{2} - \alpha \right) \int_{\Omega_t^-} \text{div } \xi \, dx + \left( \frac{1}{2} + \alpha \right) \int_{\Omega_t^+} \text{div } \xi \, dx.$$

Inserting these two equations into (4.19), we find

$$\left| \int_{\frac{1}{2} - \alpha}^{\frac{1}{2} + \alpha} \mathcal{H}^{d-1} \left( \{ x \mid \psi_\varepsilon = s \} \right) \, ds - 2\alpha \mathcal{H}^{d-1} (I_t) \right| \leq C \left( \varepsilon + |S_t^{\varepsilon, \alpha}| + \left| \Omega_t^- \Delta \{ x \mid \psi_\varepsilon < \frac{1}{2} - \alpha \} \right| + \left| \Omega_t^+ \Delta \{ x \mid \psi_\varepsilon > \frac{1}{2} + \alpha \} \right| \right), \tag{4.20}$$

where $A \Delta B := (A - B) \cup (B - A)$ is the symmetric difference of two sets $A$ and $B$.

We first estimate $r_\varepsilon^+ := \left| \Omega_t^+ \Delta \{ x \mid \psi_\varepsilon > \frac{1}{2} + \alpha \} \right|$.

$$r_\varepsilon^+ = \left| \Omega_t^+ - \{ x \mid \psi_\varepsilon > \frac{1}{2} + \alpha \} \right| + \left| \{ x \mid \psi_\varepsilon > \frac{1}{2} + \alpha \} \right| - \left| \Omega_t^+ \right|$$

$$= \left| \left( \Omega_t^+ - \{ x \in \Omega_t^+ \mid \psi_\varepsilon > \frac{1}{2} + \alpha \} \right) - \{ x \in \Omega_t^- \mid \psi_\varepsilon > \frac{1}{2} + \alpha \} \right| + \left| \{ x \in \Omega_t^- \mid \psi_\varepsilon > \frac{1}{2} + \alpha \} \right|$$

$$\leq \left| \{ x \in \Omega_t^+ \mid \psi_\varepsilon \leq \frac{1}{2} + \alpha \} \right| + \left| \{ x \in \Omega_t^- \mid \psi_\varepsilon > \frac{1}{2} + \alpha \} \right|.$$

Now using Chebyshev’s inequality and (4.2), we find $r_\varepsilon^+ \leq C \varepsilon^{1/4}$. Similar estimates apply to $|S_t^{\varepsilon, \alpha}|$ and $r_\varepsilon^- := \left| \Omega_t^- \Delta \{ x \mid \psi_\varepsilon < \frac{1}{2} - \alpha \} \right|$. Substituting these estimates into (4.20), we find

$$\left| \frac{1}{2\alpha} \int_{\frac{1}{2} - \alpha}^{\frac{1}{2} + \alpha} \left( \mathcal{H}^{d-1} \left( \{ x \mid \psi_\varepsilon = s \} \right) - \mathcal{H}^{d-1} (I_t) \right) \, ds \right| \leq C \varepsilon^{1/4} \alpha^{-1}. \tag{4.21}$$

So (4.16a) follows from Fubini’s theorem.

To prove (4.16b), we consider the set

$$Q_t^{\varepsilon, \alpha} = \{ x \in \Omega \mid |\psi_\varepsilon(x, t) - 2| \leq \alpha \}, \quad \forall \alpha \in (0, 1/8), \tag{4.22}$$
Using (4.13) and the co-area formula, we have for every $\alpha \in (0, 1/8) \setminus \mathcal{N}_t^\varepsilon$ that

$$C\varepsilon \geq \int_{Q_t^{\varepsilon,\alpha}} \left( |\nabla \psi_s| - \xi \cdot \nabla \psi_s \right) \, dx$$

$$= \int_{\partial Q_t^{\varepsilon,\alpha}} \mathcal{H}^{d-1}(\{x \mid \psi_s = s\}) \, ds - \int_{\partial Q_t^{\varepsilon,\alpha}} \xi \cdot \nu \psi_s \, d\mathcal{H}^{d-1} + \int_{\mathcal{Q}_t^{\varepsilon,\alpha}} (\text{div} \, \xi) \psi_s \, dx,$$

where $\mathcal{N}_t^\varepsilon \supset \mathcal{N}_t^\varepsilon$ is a null set in $(0, 1/8)$ and $\nu$ is the outward normal of $\partial Q_t^{\varepsilon,\alpha}$. Since $\psi_s \leq 3$ on $Q_t^{\varepsilon,\alpha}$, we have $\int_{Q_t^{\varepsilon,\alpha}} (\text{div} \, \xi) \psi_s \, dx \leq C |Q_t^{\varepsilon,\alpha}|$, and thus

$$\int_{2-\alpha}^{2+\varepsilon} \mathcal{H}^{d-1}(\{x \mid \psi_s = s\}) \, ds \leq \left| \int_{\partial Q_t^{\varepsilon,\alpha}} \xi \cdot \nu \psi_s \, d\mathcal{H}^{d-1} \right| + C\varepsilon + C |Q_t^{\varepsilon,\alpha}|. \quad (4.23)$$

Using (2.14c), we have $\int_{\Omega} (\text{div} \, \xi) \, dx = 0$, and thus

$$\int_{\partial Q_t^{\varepsilon,\alpha}} \xi \cdot \nu \psi_s \, d\mathcal{H}^{d-1}$$

$$= (2 - \alpha) \int_{\{x \mid \psi_s \geq 2+\alpha\}} (\text{div} \, \xi) \, dx + (2 + \alpha) \int_{\{x \mid \psi_s \leq 2+\alpha\}} (\text{div} \, \xi) \, dx$$

$$= (2 - \alpha) \int_{\{x \mid \psi_s \geq 2+\alpha\}} (\text{div} \, \xi) \, dx - (2 + \alpha) \int_{\{x \mid \psi_s > 2+\alpha\}} (\text{div} \, \xi) \, dx.$$

This combined with Chebyshev’s inequality and (4.2) implies that

$$|Q_t^{\varepsilon,\alpha}| + \left| \int_{\partial Q_t^{\varepsilon,\alpha}} \xi \cdot \nu \psi_s \, d\mathcal{H}^{d-1} \right| \leq C\varepsilon^{1/4}.$$  

Substituting this in (4.23) leads to

$$\frac{1}{2\alpha} \int_{2-\alpha}^{2+\varepsilon} \mathcal{H}^{d-1}(\{x \mid \psi_s = s\}) \, ds \leq C\varepsilon^{1/4}\alpha^{-1}. \quad (4.24)$$

So (4.16b) follows from Fubini’s theorem. \qed

We end this section with the following result concerning the convergence of $u_s$.

**Proposition 4.1.** For every sequence $\varepsilon_k \downarrow 0$ there exists a subsequence, which we will not relabel, such that $u_k := u_{\varepsilon_k}$ satisfies

$$\partial_t u_k \wedge u_k \xrightarrow{k \to \infty} \partial_t u \wedge u \quad \text{weakly in } L^2(0, T; L^{6/5}(\Omega)), \quad (4.25a)$$

$$\partial_t u_k \wedge u_k \xrightarrow{k \to \infty} \partial_t u \wedge u \quad \text{weakly-star in } L^\infty(0, T; L^{6/5}(\Omega)), \quad 1 \leq i \leq 3, \quad (4.25b)$$

where $u = u(x, t)$ satisfies

$$u \in L^\infty(0, T; W^{1,2}_{\text{loc}} \cap W^{1,6/5}(\Omega^+_t; \mathbb{S}^2)), \quad (4.26a)$$

$$\partial_t u \in L^2(0, T; L^2_{\text{loc}} \cap L^{6/5}(\Omega^+_t)), \quad (4.26b)$$

$$u(x, t) = 0 \quad \text{for every } t \in [0, T] \text{ and for a.e. } x \in \Omega^+_t. \quad (4.26c)$$
Furthermore,

\[ \partial_t u_k \xrightarrow{k \to \infty} \partial_t u \text{ weakly in } L^2(0, T; L^2_{\text{loc}}(\Omega_t^\pm)), \]  

\[ \nabla u_k \xrightarrow{k \to \infty} \nabla u \text{ weakly-star in } L^\infty(0, T; L^2_{\text{loc}}(\Omega_t^\pm)), \]  

\[ u_k \xrightarrow{k \to \infty} u \text{ strongly in } C([0, T]; L^2_{\text{loc}}(\Omega_t^\pm)). \]  

Before proving this result, we state the Aubin–Lions–Simon lemma. See [41, Theorem 8.62, Exercise 8.63] or [52, Corollary 8] for the proof.

**Lemma 4.4.** Let \( I \subset \mathbb{R} \) be an open bounded interval, let \((Y_0, \| \cdot \|_{Y_0}), (Y_1, \| \cdot \|_{Y_1})\), and \((Y_2, \| \cdot \|_{Y_2})\) be Banach spaces with \( Y_0 \hookrightarrow Y_1 \hookrightarrow Y_2 \). Assume that the embedding \( Y_0 \hookrightarrow Y_1 \) is compact. Let \( \mathcal{V} \) be the Banach space of all functions \( u \in L^\infty(I; Y_0) \) whose distributional derivative \( \partial_t u \) belongs to \( L^2(I; Y_2) \) endowed with the norm

\[ \| u \|_\mathcal{V} := \| u \|_{L^\infty(I; Y_0)} + \| \partial_t u \|_{L^2(I; Y_2)}. \]

Then the embedding \( \mathcal{V} \hookrightarrow C(\bar{I}; Y_1) \) is compact.

**Proof of Proposition 4.1.** Define \( \Omega^\pm := \bigcup_{t \in [0, T]} \Omega^\pm_t \times \{t\} \). We first deduce from (3.1) and (2.26b) that

\[ \| \partial_t u_\varepsilon - \Pi_{u_\varepsilon} \partial_t u_\varepsilon \|_{L^2(0, T; L^2(\Omega))} + \| \nabla u_\varepsilon - \Pi_{u_\varepsilon} \nabla u_\varepsilon \|_{L^\infty(0, T; L^2(\Omega))} \leq C \]  

for some \( C \) independent of \( \varepsilon \). On the other hand, by (2.22) we find

\[ \Pi_{u_\varepsilon} \partial_t u_\varepsilon(x, t) \land u_\varepsilon(x, t) = 0 \quad \forall (x, t) \in \Omega \times (0, T) \]  

for \( 0 \leq i \leq 3 \) where \( \partial_0 := \partial_t \). Combining (4.29) and (4.28) with (3.8b), we deduce that

\[ \begin{align*}
\| \partial_t u_\varepsilon \land u_\varepsilon \|_{L^2(0, T; L^6(\Omega))} + \| \nabla u_\varepsilon \land u_\varepsilon \|_{L^\infty(0, T; L^6(\Omega))} \\
= \| (\partial_t u_\varepsilon - \Pi_{u_\varepsilon} \partial_t u_\varepsilon) \land u_\varepsilon \|_{L^2(0, T; L^6(\Omega))} \\
+ \| (\nabla u_\varepsilon - \Pi_{u_\varepsilon} \nabla u_\varepsilon) \land u_\varepsilon \|_{L^\infty(0, T; L^6(\Omega))} \leq C.
\end{align*} \]

So it follows from the Banach–Alaoglu theorem (cf. [41, A.5.]) that

\[ \begin{align*}
\partial_t u_k \land u_k & \xrightarrow{k \to \infty} g_0 \text{ weakly in } L^2(0, T; L^{6/5}(\Omega)), \\
\partial_t u_k \land u_k & \xrightarrow{k \to \infty} g_i \text{ weakly-star in } L^\infty(0, T; L^{6/5}(\Omega))
\end{align*} \]

where

\[ g_0 \in L^2(0, T; L^{6/5}(\Omega)) \text{ and } \{g_i\}_{1 \leq i \leq 3} \subset L^\infty(0, T; L^{6/5}(\Omega)). \]

It follows from (3.8b), (3.19a) and (3.19b) that, for any fixed \( \delta \in (0, \delta_0) \), up to extraction of subsequences there exists \( \varepsilon_k = \varepsilon_k(\delta) \xrightarrow{k \to \infty} 0 \) such that

\[ u_k \xrightarrow{k \to \infty} u \text{ weakly-star in } L^\infty(0, T; L^3(\Omega)), \]

\[ \tilde{u}_k \text{ weakly-star in } L^\infty(0, T; L^3(\Omega^\pm_{I_t}) \setminus B_{\delta}(I_t)), \]

\[ \partial_t \tilde{u}_k \text{ weakly in } L^2\left(0, T; L^2(\Omega^\pm_{I_t}) \setminus B_{\delta}(I_t)\right), \]

\[ \nabla \tilde{u}_k \text{ weakly-star in } L^\infty\left(0, T; L^2(\Omega^\pm_{I_t}) \setminus B_{\delta}(I_t)\right). \]

By (4.33a) and (4.33b), we have \( u = \tilde{u}_k \) a.e. in \( U^\pm(\delta) := \bigcup_{t \in [0, T]} (\Omega^\pm_{t} \setminus B_{\delta}(I_t)) \times \{t\} \).

This combined with (4.33c) and (4.33d) leads to

\[ u \in L^\infty(0, T; W^{1,2}_{\text{loc}}(\Omega_t^\pm)) \text{ with } \partial_t u \in L^2(0, T; L^2_{\text{loc}}(\Omega_t^\pm)). \]
Furthermore, employing \((4.33b)-(4.33d)\) and Lemma 4.4, we obtain
\[
\mathbf{u}_{\varepsilon_k} \xrightarrow{k \to \infty} \mathbf{u}_*=\mathbf{u} \text{ strongly in } C([0,T]; L^2(\Omega_t^+ \setminus B_\delta(I_t))).
\]
(4.35)
By passing to a sequential limit \(\delta = \delta_\ell \to 0\) and by a diagonal argument we obtain (4.27) up to extraction of subsequences.
Now we turn to the proof of (4.26). Using (3.19a), (4.35) and Fatou’s lemma, we deduce that
\[
f(|u|) = F(u) = F(\mathbf{u}_*) = 0 \text{ a.e. in } U^\pm(\delta)
\]
for any fixed \(\delta \in (0, \delta_0)\). This together with (1.10) implies that \(u\) ranges in \(\{0\} \cup S^2\) a.e. in \(\Omega \times (0,T)\). This combined with (4.2) and \((4.31)\) yields (4.26c) and
\[
u \in L^\infty(0,T; W^{1,2}_{loc}(\Omega_t^+; S^2)) \text{ with } \partial_t \mathbf{u} \in L^2(0,T; L^2_{loc}(\Omega_t^+)).
\]
(4.36)
Now we show the integrability of \(\nabla_x x\mathbf{u}\) up to the boundary. To this aim, we choose a sequence of functions
\[
\{\eta_k(\cdot, t)\}_{k \geq 1} \subset C^\infty_c(\Omega^+_t) \text{ with } \eta_k(\cdot, t) \xrightarrow{k \to \infty} 1_{\Omega^+_t} \text{ in } L^\infty(\Omega).
\]
(4.37)
By (4.31) and (4.27), we deduce that for \(0 \leq i \leq 3\),
\[
\eta_k \mathbf{g}_i = \eta_k \partial_t \mathbf{u} \wedge \mathbf{u} \text{ a.e. in } \Omega \times (0,T).
\]
(4.38)
By (4.32) and the dominated convergence theorem, we can take \(k \to \infty\) and get
\[
\mathbf{g}_i = \partial_t \mathbf{u} \wedge \mathbf{u} \text{ a.e. in } \Omega \times (0,T), \ 0 \leq i \leq 3.
\]
(4.39)
This and (4.31) lead to (4.25a) and (4.25b). Since \(\mathbf{u}\) maps \(\Omega^+\) into \(S^2\), we have
\[
|\partial_t \mathbf{u}|^2 = |\partial_t \mathbf{u} \wedge \mathbf{u}|^2 = |\mathbf{g}_i|^2 \text{ a.e. in } \Omega^+, \ 0 \leq i \leq 3.
\]
(4.40)
This and (4.32) improve (4.36) and yield (4.26a) and (4.26b).

\section{5. Proof of Theorem 1.1: Anchoring boundary condition}

The inequalities (1.15) and (1.16) have been proved in Theorem 3.1 and in Theorem 4.1. The assertions (1.17), (1.18a) and (1.18b) have been proved in Proposition 4.1 (cf. (4.27c) and (4.26)). It remains to verify (1.18c), and this will be done by applying Lemma 4.3 for every \(t \in [0,T]\) and by choosing an appropriate \(\alpha\) outside the null set \(\mathcal{N}_{t\varepsilon_k}^c \subset (0, 1/8)\). For simplicity we shall abbreviate \(\psi_{\varepsilon_k}\) and \(\mathbf{u}_{\varepsilon_k}\) by \(\psi_k\) and \(\mathbf{u}_k\) respectively.
For any \(k \geq 1\) we can choose \(\beta_k \in [1/2,1]\) such that \(\alpha = \alpha_k := \beta_k \varepsilon_k^{1/8} \notin \mathcal{N}_{t\varepsilon_k}^c\). Then by Lemma 4.3 there exist
\[
b_{\varepsilon_k,\alpha_k}(t) := b_k \in [\frac{1}{2} - \alpha_k, \frac{1}{2} + \alpha_k], \quad q_{\varepsilon_k,\alpha_k}(t) := q_k \in [2 - \alpha_k, 2 + \alpha_k]
\]
(5.1)
such that
\[
(b_k, q_k) \xrightarrow{k \to \infty} (\frac{1}{2}, 2),
\]
(5.2)
and such that the set
\[
\Omega_k^t := \{x \in \Omega | b_k < \psi_k(x,t) < q_k\} \text{ has finite perimeter}.
\]
(5.3)
Moreover, there exists \(C > 0\) which is independent of \(t\) and the particular choice of the subsequence \(\varepsilon_k\) such that
\[
|\mathcal{H}^{d-1}(\{x | \psi_k(x,t) = q_k\})| \leq C\varepsilon_k^{1/8},
\]
(5.4a)
\[
|\mathcal{H}^{d-1}(\partial\Omega_k^t) - \mathcal{H}^{d-1}(I_t)| \leq 2C\varepsilon_k^{1/8}.
\]
(5.4b)
Using these level sets, we can prove the following proposition which improves (1.27) to the convergence of $u_k$ up to the boundary $I_t$.

**Proposition 5.1.** Let $u$ be the limit vector field in Proposition 4.1. For a.e. $t \in [0, T]$, up to extraction of subsequences which we will not relabel, we have

\[
\begin{align*}
1_{\Omega^+_t} \hat{u}_k & \xrightarrow{k \to \infty} 1_{\Omega^+_t} u \quad \text{weakly-star in } BV(\Omega), \quad (5.5a) \\
1_{\Omega^+_t} \nabla \hat{u}_k & \xrightarrow{k \to \infty} 1_{\Omega^+_t} \nabla u \quad \text{weakly in } L^1(\Omega), \quad (5.5b) \\
1_{\Omega^+_t} \hat{u}_k & \xrightarrow{k \to \infty} 1_{\Omega^+_t} u \quad \text{strongly in } L^p(\Omega), \text{ for any fixed } p \in [1, \infty), \quad (5.5c)
\end{align*}
\]

where $\hat{u}_k = \hat{u}_{\epsilon_k}$ is defined in (3.21).

**Proof.** We first claim that there exists a positive constant $C_3$ depending only on $f$ (cf. (1.11)) such that the following statement holds for any $\delta \in (0, 1/8)$:

\[
|u_\varepsilon(x, t)| \geq C_3 \delta \quad \forall x \in \{ \psi_\varepsilon \geq \delta \}.
\]

Indeed, by (1.11a), $f$ (and also $g$) is increasing on $(0, s_0)$. If $|u_\varepsilon| \geq s_0$, we are done. Otherwise,

\[
\delta \leq \psi_\varepsilon = \int_0^{|u_\varepsilon|} g(s) \, ds \leq |u_\varepsilon| g(s_0),
\]

which implies (5.6). This combined with (3.22a) and (5.3) implies

\[
\sup_{t \in [0, T]} \int_{\Omega^+_t} |\nabla \hat{u}_k|^2 \, dx \leq C
\]

for $k$ sufficiently large. This and (5.3) imply that the distributional derivatives of $v_k(\cdot, t) := 1_{\Omega^+_t} \hat{u}_k(\cdot, t)$ have no Cantor parts, and the absolute continuous parts $\{1_{\Omega^+_t} \nabla \hat{u}_k\}_{k \geq 1}$ is bounded in $L^2(\Omega)$. Moreover, their jump parts enjoy the estimate

\[
\int_{\partial \Omega^+_t} |v_k(\cdot, t) - 0|^2 \, d\mathcal{H}^{d-1} \leq C,
\]

and $\{v_k(\cdot, t)\}_{k \geq 1}$ is bounded in $L^\infty(\Omega)$. With these properties, it follows from [2] (or [3] Section 4.1) that $\{v_k(\cdot, t)\}_{k \geq 1}$ is compact in SBV$(\Omega)$, the class of special functions of bounded variation on $\Omega$. More precisely, there exists $v(\cdot, t) \in SBV(\Omega)$ s.t. $v_k \to v$ weakly-star in $BV(\Omega)$ as $k \to \infty$, and the absolute continuous part of the gradient $\nabla^a v_k = 1_{\Omega^+_t} \nabla \hat{u}_k$ converges weakly in $L^1(\Omega)$ to $\nabla^a v$. To identify $v$, we use (4.2) to deduce that $1_{\Omega^+_t} \to 1_{\Omega^+_t}$ in $L^1(\Omega)$ as $k \to \infty$. This and (1.27a) yield $v(\cdot, t) = 1_{\Omega^+_t} u(\cdot, t)$ a.e. in $\Omega$, and thus (5.5a) and (5.5b) are proved. Finally by (5.5a), the compact embedding of $BV$ functions and the $L^\infty$ bound we get (5.5c).

To proceed we define the following measures for Borel sets $A \subset \Omega$:

\[
\theta(A) = \mathcal{H}^{d-1}(A \cap I_t), \quad (5.8a)
\]

\[
\theta_k(A) = \int_{A \cap \Omega^+_t} |\nabla \psi_k| \, dx. \quad (5.8b)
\]

**Lemma 5.1.** For a.e. $t \in [0, T]$, $\theta_k \xrightarrow{k \to \infty} \frac{1}{2} \theta$ weakly-star as Radon measures. (5.9)
Proof. We define truncation functions
\[
T_k(s) = \begin{cases} 
0 & \text{when } s \leq b_k, \\
 s - b_k & \text{when } b_k \leq s \leq q_k, \\
 q_k - b_k & \text{when } s \geq q_k,
\end{cases} \quad (5.10)
\]
\[
T(s) = \begin{cases} 
0 & \text{when } s \leq 1/2, \\
 s - 1/2 & \text{when } 1/2 \leq s \leq 2, \\
 3/2 & \text{when } s \geq 2.
\end{cases} \quad (5.11)
\]
By (5.2), we have \(T_k \xrightarrow{k \to \infty} T\) uniformly on \(\mathbb{R}\). Moreover,
\[
\nabla(T_k \circ \psi_k) = \nabla \psi_k \mathbf{1}_{\Omega^k_t} \quad \text{a.e. in } \Omega, \quad (5.12a)
\]
\[
T_k \circ \psi_k \xrightarrow{k \to \infty} \frac{1}{2} \mathbf{1}_{\Omega^+_t} \quad \text{strongly in } L^p(\Omega) \quad \text{for any fixed } p \in [1, \infty). \quad (5.12b)
\]
Indeed, by (2.7) and (2.17) we know that \(\psi_k(\cdot, t) \in C^1(\Omega)\). Also by (5.3) we have \(T' \circ \psi_k = \mathbf{1}_{\Omega^k_t}\) for a.e. \(x \in \Omega\). Therefore, (5.12a) follows from the chain rule (cf. [26, Proposition 3.24]), while (5.12b) follows from (4.2) and the dominated convergence theorem. By (4.13) we have for any \(g \in C^1_c(\Omega)\) that
\[
\int_{\Omega} g \, d\theta_k \xrightarrow{k \to \infty} \int_{\Omega^k_t} g |\nabla \psi_k| \, dx \xrightarrow{5.12b} O(\varepsilon_k) + \int_{\Omega^k_t} g \, \xi \cdot \nabla \psi_k \, dx \xrightarrow{5.12a} O(\varepsilon_k) + \int_{\Omega} g \, \xi \cdot \nabla(T_k \circ \psi_k) \, dx \xrightarrow{5.13} O(\varepsilon_k) - \int_{\Omega} \div (g \xi) \, T_k \circ \psi_k \, dx.
\]
Recalling that \(\xi\) is the inward normal of \(I_t\), we use (5.12b) to pass to the limit in the above equations and obtain
\[
\lim_{k \to \infty} \int_{\Omega} g \, d\theta_k \xrightarrow{5.12b} -\frac{1}{2} \int_{\Omega^+_t} \div (g \xi) \, dx = \frac{1}{2} \int_{I_t} g \, dH^{d-1} \xrightarrow{5.8b} \frac{1}{2} \int_{\Omega} g \, d\theta,
\]
for any \(g \in C^1_c(\Omega)\). By approximation, one can pass from \(C^1_c(\Omega)\) to \(C^0_c(\Omega)\), and this proves (5.9).

Now we finish the proof of Theorem 1.1 by verifying (1.18c). The proof here is inspired by the blow-up argument in [43]. See also [25] for the applications of such a method in the study of quasi-convex functionals.

Proof of (1.18c). For any \(x_0 \in I_t\) and any \(R > 0\), it follows from (5.5c), (5.12b) and the dominated convergence theorem that
\[
\lim_{k \to \infty} \int_{B_R(x_0)} \mathbf{1}_{\Omega^k_t} \hat{u}_k \cdot \frac{x - x_0}{|x - x_0|} T_k \circ \psi_k \, dx = \frac{1}{2} \int_{B_R(x_0)} \mathbf{1}_{\Omega^+_t} \mathbf{u} \cdot \frac{x - x_0}{|x - x_0|} \, dx.
\]
We can use spherical coordinate to rewrite the above two integrals in the form of \(\int_0^R \int_{\partial B_r(x_0)} (\cdot) \, dH^{d-1} \, dr\), and then apply Fubini’s theorem. Therefore, there exists \(r_j \downarrow 0\) such that for each \(j\) we have
\[
\lim_{k \to \infty} \int_{\partial B_{r_j}(x_0) \cap \Omega^k_t} \hat{u}_k \cdot \nu \, T_k \circ \psi_k \, dH^{d-1} = \frac{1}{2} \int_{\partial B_{r_j}(x_0) \cap \Omega^+_t} \mathbf{u} \cdot \nu \, dH^{d-1}. \quad (5.13)
\]
where $\nu$ is the outward normal of $\partial B_{r_j}(x_0)$. Moreover, we can arrange $r_j$ such that $\theta(\partial B_{r_j}(x_0)) = 0$. This combined with (5.12a) implies that

$$\lim_{k \to \infty} \theta_k(B_{r_j}(x_0)) = \frac{1}{2} \theta(B_{r_j}(x_0)).$$  \hspace{1cm} (5.14)$$

To proceed, we use convexity to write, for some $a_m, c_m \in \mathbb{R}$, that

$$s^2 = \sup_{m \in \mathbb{N}^+} (a_m s + c_m), \quad \forall s \in \mathbb{R}. \hspace{1cm} (5.15)$$

(cf. [3, Proposition 2.31]). For $\theta - a.e. x_0 \in \text{supp}(\theta) = I_t$, we have for each $m \geq 1$ that

$$0 \geq \lim_{k \to \infty} \int_{B_{r_j}(x_0)} (\vec{u}_k \cdot n_k)^2 \, d\theta_k \hspace{1cm} (5.16)$$

Note that in the last step we also used (5.14). It remains to compute the integral in the last display of (5.16) under the limit $k \to \infty$ for fixed $j, m$. To this aim, we use (5.12a) and integration by parts to find

$$\int_{B_{r_j}(x_0) \cap \Omega^k_t} \vec{u}_k \cdot \nabla (T_k \circ \psi_k) \, dx \hspace{1cm} (5.17)$$

$$= \int_{\partial (B_{r_j}(x_0) \cap \Omega^k_t)} (\vec{u}_k \cdot \nu) T_k \circ \psi_k \, dH^{d-1} - \int_{B_{r_j}(x_0)} 1_{\Omega^k_t} (\text{div} \, \vec{u}_k) T_k \circ \psi_k \, dx$$

$$= : A_k - B_k.$$  \hspace{1cm} (5.18)

Note that the integrand of $A_k$ is uniformly bounded in $L^\infty$. To compute the limit of $A_k$, we first deduce from (5.10) that $T_k \circ \psi_k = 0$ on the set $\{ x \in \Omega \mid \psi_k(x, t) = b_k \}$ which has finite perimeter (cf. (5.4a)). So we employ (5.3) to find

$$A_k = \int_{\partial B_{r_j}(x_0) \cap \Omega^k_t} (\vec{u}_k \cdot \nu) T_k \circ \psi_k \, dH^{d-1} + \int_{B_{r_j}(x_0) \cap \{ x \mid \psi_k = b_k \}} (\vec{u}_k \cdot \nu) T_k \circ \psi_k \, dH^{d-1}. \hspace{1cm} (5.19)$$

The limit of the first integral is given in (5.13), and that of the second vanishes in the limit $k \to \infty$ by (5.4a). So we conclude that

$$\lim_{k \to \infty} A_k = \frac{1}{2} \int_{\partial B_{r_j}(x_0) \cap \Omega^k_t} \vec{u} \cdot \nu \, dH^{d-1}. \hspace{1cm} (5.19)$$

Concerning the integral $B_k$, by (5.5b) the sequence $\{ 1_{\Omega^k_t} \text{div} \vec{u}_k \}_{k \geq 1}$ converges weakly in $L^1(\Omega)$. Moreover, $\{ T_k \circ \psi_k \}_{k \geq 1}$ is uniformly bounded in $L^\infty$, and converges a.e. in $\Omega$ to $\frac{1}{2} 1_{\Omega^k_t}$, due to (5.12b). Therefore, applying the Product Limit Theorem (cf. [16] or [49, pp. 169]), we obtain

$$\lim_{k \to \infty} B_k = \frac{1}{2} \int_{B_{r_j}(x_0) \cap \Omega^k_t} (\text{div} \, \vec{u}) \, dx. \hspace{1cm} (5.20)$$
Using (5.19) and (5.20), we can compute the limit in (5.17) and find

\[
\lim_{k \to \infty} \int_{B_r(x_0) \cap \Omega^+} \hat{u}_k \cdot \nabla (T_k \circ \psi_k) \, dx = \frac{1}{2} \int_{\partial B_r(x_0) \cap \Omega^+} u \cdot \nu \, d\mathcal{H}^{d-1} - \frac{1}{2} \int_{\partial(B_r(x_0) \cap \Omega^+)} u \cdot \nu \, d\mathcal{H}^{d-1}
\]

where in the last step we used \( \xi = -\nu \) on \( \partial \Omega^+ \). Note that \( \xi \) is the inward normal of \( I_t \) according to (2.10), and \( \Omega^+_t \) is the region enclosed by \( I_t \) with outward normal \( \nu \). Substituting (5.21) into (5.16) and then dividing the resulting inequality by \( \theta \left( B_{r_j}(x_0) \right) \) and taking \( j \to \infty \), we find

\[
0 \geq \lim_{j \to \infty} \frac{a_m}{\theta \left( B_{r_j}(x_0) \right)} \frac{1}{2} \int_{B_{r_j}(x_0) \cap I_t} u \cdot \xi \, d\mathcal{H}^{d-1} + \frac{c_m}{2} \quad \forall m \in \mathbb{N}^+.
\]

This together with (5.15) implies that

\[
(u \cdot \xi)^2(x_0) = 0 \quad \text{for} \quad \mathcal{H}^{d-1}\text{-a.e.} \quad x_0 \in I_t. \quad \square
\]

6. Proof of Theorem 1.2 Oseen–Frank limit in the bulk

The method here is inspired by [17, 38], which has a 2D nature. We set \( \tau_\varepsilon := \partial_t u_\varepsilon \) and write (1.2a) as

\[
\tau_\varepsilon = \mu \nabla (\text{div} \, u_\varepsilon) + \Delta u_\varepsilon - \varepsilon^{-2} DF(u_\varepsilon) \quad \text{in} \quad \Omega \times (0, T).
\]

By Corollary 3.4 and Proposition 4.1 (cf. (4.27c)), for a.e. \( t_0 \in (0, T) \) and for any compact set \( K \subset \subset \Omega^+_{t_0} \), we have

\[
\int_K |\tau_\varepsilon|^2 \, dx + \int_K \left( \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2} F(u_\varepsilon) \right) \, dx \leq \hat{c}^2 \quad \text{at} \quad t = t_0,
\]

\[
u u_{\varepsilon_k} (\cdot, t_0) \xrightarrow{k \to \infty} u(\cdot, t_0) \quad \text{strongly in} \quad L^2(K),
\]

where \( \hat{c} = \hat{c}(K, t_0) > 1 \) is independent of \( \mu \) and \( \varepsilon \).

**Proposition 6.1.** Let \( K \) be a compact set of \( \Omega^{+}_{t_0} \) and assume that (6.2a) and (6.2b) hold. There exists an absolute constant \( \Lambda \in (0, 1) \) with the following property: under the assumptions

\[
\hat{c} < \Lambda / \varepsilon^2 \quad \text{and} \quad \mu < \Lambda,
\]

\[
B_{2r}(x_0) \subset K \quad \text{with} \quad r < 1 \quad \text{and}
\]

\[
\int_{B_{2r}(x_0)} \left( \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2} F(u_\varepsilon) \right) \, dx \leq \hat{c}^2 \quad \text{at} \quad t = t_0,
\]

there exists a subsequence \( \varepsilon_k \downarrow 0 \), which we will not relabel, such that

\[
\nabla u_{\varepsilon_k} (\cdot, t_0) \xrightarrow{k \to \infty} \nabla u(\cdot, t_0) \quad \text{strongly in} \quad L^2(B_{r/2}(x_0)).
\]

We shall need the following inequality due to the special choice of \( f \) in (1.22):

\[
|f'(s)|^2 \leq C_4 f(s), \quad \forall s \geq 0,
\]

(6.5)
for some $C_4 > 1$. In the sequel $C_4 > 1$ will also be used as a generic constant that might change from line to line due to the use of the Sobolev embeddings and elliptic estimates. Note that $C_4$ is independent of $\mu$ and $r$.

**Lemma 6.1.** Under the assumptions (6.2a) and (6.3c) with a sufficiently small $\hat{\epsilon}$ (defined in (6.11) below), we have

$$3/4 \leq |u_\epsilon(\cdot, t_0)| \leq 5/4 \text{ on } B_r(x_0) \quad \text{for } \varepsilon \leq r/4.$$  

**Proof.** Without loss of generality, we assume $x_0 = 0$. For brevity we write $B_r(0)$ as $B_r$. Since all arguments are made at $t = t_0$, we shall suppress the time dependence of $u_\epsilon$.

**Step 1.** There exists $\hat{C} > 1$ depending on $\hat{\epsilon}$ such that for any $x_1 \in B_r$ we have

$$|u_\epsilon(x) - u_\epsilon(y)| \leq \hat{C} \sqrt{|x - y|} / \varepsilon, \quad \forall x, y \in B_\varepsilon(x_1). \quad (6.7)$$

To prove (6.7), let $u_\epsilon(z) = u_\epsilon(x_1 + \varepsilon z) : B_2 \to \mathbb{R}^3$. Then we can write (6.11) as

$$\mu \nabla \div \hat{u}_\epsilon(z) + \Delta \hat{u}_\epsilon(z) = \varepsilon^2 \tau_\epsilon(x_1 + \varepsilon z) + DF(\hat{u}_\epsilon(z)), \quad z \in B_2. \quad (6.8)$$

It follows from (6.2a) and a change of variable that $\{\varepsilon^2 \tau_\epsilon(x_1 + \varepsilon z)\}_{\varepsilon > 0}$ is uniformly bounded in $L^2(B_2)$. Using (6.3), we can estimate

$$\left\|DF(\hat{u}_\epsilon)\right\|_{L^2(B_2)}^2 \leq \varepsilon^{-2} \left\|f'(|u_\epsilon|)\right\|_{L^2(B_2)}^2 \leq \varepsilon^{-2} C_4 \int_{B_2(x_1)} F(u_\epsilon) \, dx \leq \varepsilon^2 C_4. \quad (6.9)$$

Altogether, we prove that the terms on the right-hand side of (6.8) is bounded in $L^2(B_2)$. Invoking the interior estimate for elliptic system (cf. [26, Theorem 4.9]), we obtain

$$\left\|\hat{u}_\epsilon\right\|_{W^{2,2}(B_1)} \leq C_4 (\hat{\epsilon} + \left\|\hat{u}_\epsilon\right\|_{L^2(B_2)}). \quad (6.10)$$

Note that $C_4$ is independent of $\mu$. Now we estimate the last term by

$$\left\|\hat{u}_\epsilon\right\|_{L^2(B_2)}^2 \leq C_4 \left(1 + \varepsilon^{-2} \int_{B_2(x_1) \cap \{|x| \geq 2\}} (|u_\epsilon| - 1)^2 \right) \leq C_4 \left(1 + \varepsilon^{-2} \int_{B_{2\varepsilon}(x_1)} f(|u_\epsilon|)\right) \leq (1 + \varepsilon^2) C_4. \quad (6.12)$$

Substituting this estimate in (6.10) and using Morrey’s embedding $W^{2,2} \hookrightarrow C^{1/2}$, we obtain $\left\|\hat{u}_\epsilon\right\|_{C^{1/2}(B_1)} \leq C_4 \hat{\epsilon}$. Rescaling back, we find (6.7) with

$$\hat{C} := C_4 \hat{\epsilon}.$$

**Step 2:** We claim that with the choice

$$\hat{\epsilon} < 16^{-8} C_4^{-2} \hat{\epsilon}^{-2} = 16^{-8} \hat{C}^{-2}, \quad (6.11)$$

we have either (6.6) or

$$|u_\epsilon| \leq 1/4 \text{ on } B_r \quad \text{for } \varepsilon \leq r/4. \quad (6.12)$$

Indeed, if neither of them were valid, then

$$\exists \varepsilon \in (0, r/4) \text{ and } x_1 \in B_r \text{ s.t. } |u_\epsilon(x_1)| \in (1/4, 3/4) \cup (5/4, +\infty). \quad (6.13)$$

Since $16\hat{\epsilon} < 1$, it follows from (6.7) that

$$|u_\epsilon(x_1) - u_\epsilon(x)| < 4^{-7} \quad \text{for } x \in B_{16\hat{\epsilon}}(x_1). \quad (6.14)$$

Using this and (1.22), we deduce one of the following two cases for $x \in B_{16\hat{\epsilon}}(x_1)$:

a) If $|u_\epsilon(x_1)| > 3$, then $|u_\epsilon(x)| > 2$ and thus $f(|u_\epsilon(x)|) \geq 1$. 

b) If \(|u_\varepsilon(x_1)| \in (1/4, 3/4) \cup (5/4, 3]|, then \(f(|u_\varepsilon(x_1)|) \geq 1/16\). By the third condition in (1.22) and (6.14), we have \(f(|u_\varepsilon(x)|) > 1/32\).

To summarize, we have the following inequality:

\[
F(u_\varepsilon(x)) = f(|u_\varepsilon(x)|) > 1/32 \quad \forall x \in B_{16\varepsilon}(x_1).
\]

(6.15)

Integrating this inequality over \(B_{16\varepsilon}(x_1)\) and using the assumption \(\varepsilon < r/4\), we find

\[
\varepsilon^{-2} \int_{B_{16\varepsilon}(x_1)} F(u_\varepsilon(x)) \, dx > 8\pi \varepsilon^2.
\]

However, this would contradict (6.3c) since \(B_{16\varepsilon}(x_1) \subset B_{2r}(x_0)\). So (6.13) is not valid and the claim is proved.

**Step 3:** We shall rule out (6.12).

Assuming (6.12), we deduce from (1.22) that \(F(u_\varepsilon) = |u_\varepsilon|^2\). By (6.1) we have

\[
\mu \nabla (\text{div } u_\varepsilon) + \Delta u_\varepsilon - 2\varepsilon^{-2} u_\varepsilon = \tau_\varepsilon \text{ in } B_r.
\]

(6.16)

For \(z \in B_1\), we introduce \(\tilde{u}_\varepsilon(z) := u_\varepsilon(rz)\) and \(\tilde{\tau}_\varepsilon(z) := \tau_\varepsilon(rz)\). Then

\[
\mu \nabla (\text{div } \tilde{u}_\varepsilon) + \Delta \tilde{u}_\varepsilon - 2r^2\varepsilon^{-2} \tilde{u}_\varepsilon = r^2 \tilde{\tau}_\varepsilon \text{ in } B_1.
\]

(6.17)

By the interior estimate for elliptic system, we have

\[
\|\tilde{u}_\varepsilon\|_{W^{2,2}(B_{1/2})} + r^2\varepsilon^{-2}\|\tilde{u}_\varepsilon\|_{L^2(B_{1/2})} \leq C_4 \left( \|\tilde{\tau}_\varepsilon\|_{L^2(B_1)} + \|\tilde{u}_\varepsilon\|_{L^2(B_1)} \right).
\]

(6.18)

Indeed, one can adapt the proof of [26, Theorem 4.9] to gain the term \(r^2\varepsilon^{-2}\|\tilde{u}_\varepsilon\|_{L^2(B_{1/2})}\).

By (6.18), (6.2a) and the conclusion in step 2, we find

\[
r^2\varepsilon^{-2}\|u_\varepsilon\|_{L^2(B_{1/2})} \leq C_4 \left( \|\tau_\varepsilon\|_{L^2(B_r)} + \|u_\varepsilon\|_{L^2(B_r)} \right) \leq C_4(\hat{c} + 1).
\]

This implies that \(u_\varepsilon \to 0\) strongly in \(L^2(B_{r/2})\), which contradicts (4.2) since \(B_{r/2} \subset K \subset K^+\). Therefore, we rule out (6.12) and obtain (6.6).

By (6.6), we have polar decomposition \(u_\varepsilon = \rho_\varepsilon v_\varepsilon\) where

\[
\rho_\varepsilon = |u_\varepsilon|, \quad v_\varepsilon = u_\varepsilon/|u_\varepsilon| \text{ in } B_r(x_0).
\]

(6.19)

We set

\[
w_\varepsilon := (v_\varepsilon, \rho_\varepsilon),
\]

(6.20)

and define the projection

\[
a_\parallel := (\mathbb{I} - v_\varepsilon \otimes v_\varepsilon)a
\]

(6.21)

for a vector field \(a\).

**Lemma 6.2.** Under the assumptions \(\varepsilon \leq r/4\), (6.2a) and (6.3c) for \(\varepsilon\) defined in (6.11), \(\rho_\varepsilon\) satisfies the following equation in \(B_r(x_0)\):

\[
\Delta \rho_\varepsilon - \varepsilon^{-2} f'(\rho_\varepsilon) + \mu \nabla^2 \rho_\varepsilon : (v_\varepsilon \otimes v_\varepsilon) + \mu \rho_\varepsilon (v_\varepsilon \cdot \nabla) \nabla v_\varepsilon = \tau_\varepsilon \cdot v_\varepsilon + \mathcal{N}_{1,\varepsilon}(\nabla w_\varepsilon, \nabla w_\varepsilon),
\]

(6.22)

where \(\mathcal{N}_{1,\varepsilon}(\cdot, \cdot) : \mathbb{R}^{4 \times 3} \times \mathbb{R}^{4 \times 3} \mapsto \mathbb{R}\) is bilinear with uniformly bounded coefficients. Also, \(v_\varepsilon\) satisfies the following equation in \(B_r(x_0)\):

\[
\rho_\varepsilon \Delta v_\varepsilon + \mu (|\nabla^2 \rho_\varepsilon| v_\varepsilon) + \mu \rho_\varepsilon (\nabla (\nabla v_\varepsilon))_\parallel = (\tau_\varepsilon)_\parallel + \mathcal{N}_{2,\varepsilon}(\nabla w_\varepsilon, \nabla w_\varepsilon),
\]

(6.23)

where \(\mathcal{N}_{2,\varepsilon}(\cdot, \cdot) : \mathbb{R}^{4 \times 3} \times \mathbb{R}^{4 \times 3} \mapsto \mathbb{R}^3\) is bilinear with uniformly bounded coefficients.
Proof. To simplify the presentation we will suppress the subscript $\varepsilon$. By (6.19) we have $|v|^2 \equiv 1$ and thus

$$\Delta v \cdot v = -|\nabla v|^2. \quad (6.24)$$

Substituting (6.19) into (6.1), we find

$$\tau = (\Delta \rho)v + 2(\nabla \rho \cdot \nabla)v + \rho \Delta v - \varepsilon^{-2} f'(\rho)v$$
$$+ \mu (\nabla^2 \rho)v + \mu \rho \nabla(\nabla v) + \mu (\nabla \rho \cdot \partial_i v)_{1 \leq i \leq 3} + \mu \nabla \rho(\nabla v). \quad (6.25)$$

Testing (6.25) with $v$ and using (6.24), we obtain

$$- \Delta \rho + \varepsilon^{-2} f'(\rho)$$
$$= - \tau \cdot v + \mu \nabla^2 \rho : (v \otimes v) + \mu \rho(\nabla \nabla v)$$
$$+ \mu(\nabla \rho \cdot \partial_i v)v_i + \mu(v \cdot \nabla \rho) \nabla \nabla v - \rho |\nabla v|^2. \quad (6.26)$$

The terms in the last line are bilinear with respect to $\nabla w = (\nabla v, \nabla \rho)$, and we denote their sum by $-N_{1,\varepsilon}(\nabla w, \nabla w)$. By (6.6), it has bounded coefficients and thus (6.22) is proved.

To derive (6.23), we shall use the following identities.

$$v || = 0 \quad \text{and} \quad (\partial_i v) || = \partial_i v. \quad (6.27)$$

These combined with (6.24) lead to

$$(\Delta v) || = \Delta v + |\nabla v|^2 v. \quad (6.28)$$

Now applying $(\cdot ||)$ to the equation in (6.25), and using (6.27) and (6.28), we obtain

$$\tau || = \rho \Delta v + \mu \left((\nabla^2 \rho)v|| + \mu \rho(\nabla(\nabla v))||\right)$$
$$+ 2((\nabla \rho \cdot \nabla)v) || + \rho v |\nabla v|^2 + \mu \left((\nabla \rho \cdot \partial_i v)_{1 \leq i \leq 3}\right) || + \mu(\nabla \rho)||((\nabla v). \quad (6.29)$$

The terms in the second line of the above equation are bilinear with respect to $\nabla w$, and we denote their sum by $-N_{2,\varepsilon}(\nabla w, \nabla w)$. By (6.6), it has bounded coefficients, and thus (6.23) is proved.

Proof of Proposition 6.1. We first show that, by choosing $\hat{\varepsilon}$ and $\mu$ sufficiently small, we have

$$\|\nabla^2(v_\varepsilon, \rho_\varepsilon)\|_{L^{4/3}(B_{r/2}(x_0))} \leq 2C_4 r^{-2}. \quad (6.30)$$

Recalling (6.20), we deduce from (6.3c) and (6.6) that

$$\|\nabla w_\varepsilon\|_{L^2(B_r(x_0))} \leq 4\hat{\varepsilon} \quad \text{on} \quad B_r(x_0) \quad \text{for} \quad \varepsilon \leq r/4. \quad (6.31)$$

Recalling that $r < 1$, let $\chi$ be a $C^2$ cut-off function such that

$$\chi \equiv \begin{cases} 1 \quad \text{in} \quad B_{r/2}(x_0) \\ 0 \quad \text{in} \quad B_1(x_0) \setminus B_r(x_0) \end{cases} \quad \text{and} \quad |\nabla^\ell \chi| \leq 8r^{-\ell} \quad \text{in} \quad B_1(x_0) \quad \text{for} \quad \ell \in \{1, 2\}. \quad (6.32)$$

and let $\bar{w}_\varepsilon := (\bar{v}_\varepsilon, \bar{\rho}_\varepsilon)$ with

$$\bar{\rho}_\varepsilon = \chi(\rho_\varepsilon - 1) \quad \text{and} \quad \bar{v}_\varepsilon = \chi v_\varepsilon. \quad (6.33)$$
Multiplying (6.23) by $\chi$ and using the linearity of $a_\parallel$ about $a$ (cf. (6.21)), we find
\[
\rho_\varepsilon \Delta \tilde{v}_\varepsilon + \rho_\varepsilon [\chi, \Delta]v_\varepsilon + \mu \left( [\chi, \nabla^2] (\rho_\varepsilon - 1) v_\varepsilon \right) + \mu \left( \nabla^2 \rho_\varepsilon v_\varepsilon \right)
+ \mu \rho_\varepsilon \left( [\chi, \nabla \text{div}] v_\varepsilon \right) + \mu \rho_\varepsilon \left( \nabla (\text{div} \tilde{v}_\varepsilon) \right)
= (T_\varepsilon) + \chi + \mathcal{N}_{2,\varepsilon} (\chi \nabla w_\varepsilon, \nabla \tilde{w}_\varepsilon)
\text{ in } B_1(x_0),
\] (6.34)
and $\tilde{v}_\varepsilon|_{\partial B_1(x_0)} = 0$. For brevity we denote $L^p(B_1(x_0))$ by $L^p$. Note that the commutators in (6.34) involve at most first order derivatives of $w_\varepsilon = (v_\varepsilon, \rho_\varepsilon)$, which satisfies (6.31). Now applying the $L^p$-estimate for elliptic equation [39] pp. 109 (componentwise) in (6.34), and invoking (6.31) and (6.3), we have
\[
\|\nabla^2 \tilde{v}_\varepsilon\|_{L^{4/3}} \leq C_4 \left( r^{-2} + r^{-1} + \mu \|\nabla^2 \tilde{w}_\varepsilon\|_{L^{4/3}} + \|\mathcal{N}_{2,\varepsilon} (\chi \nabla w_\varepsilon, \nabla \tilde{w}_\varepsilon)\|_{L^{4/3}} \right).
\] (6.35)
Note that the prefactors $r^{-1}$ and $r^{-2}$ are due to the differentiation of $\chi$ (cf. (6.32)), and that $C_4$ is independent of $r$. To estimate the last term, we employ the bi-linearity of $\mathcal{N}_{2,\varepsilon}$, (6.31) and (6.3):
\[
\|\nabla^2 \tilde{v}_\varepsilon\|_{L^{4/3}} \leq \|\mathcal{N}_{2,\varepsilon} (\chi \nabla w_\varepsilon, \nabla \tilde{w}_\varepsilon)\|_{L^{4/3}}
\leq \|\mathcal{N}_{2,\varepsilon} (\chi \nabla w_\varepsilon, \nabla \tilde{w}_\varepsilon)\|_{L^{4/3}} + \|\mathcal{N}_{2,\varepsilon} (\chi \nabla \tilde{w}_\varepsilon, \nabla w_\varepsilon)\|_{L^{4/3}} + C_4 r^{-1}
\leq C_4 \left( \|\nabla \tilde{w}_\varepsilon\|_{L^4} \|\nabla w_\varepsilon\|_{L^2} + r^{-1} \right)
\leq C_4 \left( \|\nabla^2 \tilde{w}_\varepsilon\|_{L^{4/3}} \|\nabla w_\varepsilon\|_{L^{2}} + r^{-1} \right).
\] (6.36)
Note that in the last step we used the Sobolev’s embedding $W^{1,4/3}(B_1) \subset L^4(B_1)$. Combining (6.36) with (6.35), we obtain
\[
\|\nabla^2 \tilde{v}_\varepsilon\|_{L^{4/3}} \leq C_4 \left( r^{-2} + \mu \|\nabla^2 (\tilde{v}_\varepsilon, \tilde{\rho}_\varepsilon)\|_{L^{4/3}} + \|\nabla^2 \tilde{w}_\varepsilon\|_{L^{4/3}} \|\nabla \tilde{w}_\varepsilon\|_{L^2} \right).
\] (6.37)
Now we turn to the estimate of $\rho_\varepsilon$. Using (6.36) and (1.22), we have $f'(\rho_\varepsilon) = 2(\rho_\varepsilon - 1)$ in $B_r(x_0)$. Now multiplying (6.22) by $\chi$ and using the linearity of (6.21), we find
\[
-2\varepsilon^{-2} \Delta \rho_\varepsilon + [\chi, \Delta] (\rho_\varepsilon - 1) + \mu (v_\varepsilon \otimes v_\varepsilon) \cdot \nabla^2 \rho_\varepsilon
+ \mu (v_\varepsilon \otimes v_\varepsilon) : [\chi, \nabla^2] (\rho_\varepsilon - 1) + \mu \rho_\varepsilon v_\varepsilon \cdot (\nabla \text{div} \tilde{v}_\varepsilon) + \mu \rho_\varepsilon v_\varepsilon \cdot ([\chi, \nabla \text{div}] v_\varepsilon)
= \chi T_\varepsilon \cdot \tilde{v}_\varepsilon + \mathcal{N}_{1,\varepsilon} (\chi \nabla \tilde{w}_\varepsilon, \nabla \tilde{w}_\varepsilon).
\]
In the same way as we did for (6.37), we find
\[
\|\nabla^2 \rho_\varepsilon\|_{L^{4/3}} \leq C_4 \left( r^{-2} + \mu \|\nabla^2 (\tilde{v}_\varepsilon, \tilde{\rho}_\varepsilon)\|_{L^{4/3}} + \|\mathcal{N}_{1,\varepsilon} (\chi \nabla \tilde{w}_\varepsilon, \nabla \tilde{w}_\varepsilon)\|_{L^{4/3}} \right)
\leq C_4 \left( r^{-2} + \mu \|\nabla^2 (\tilde{v}_\varepsilon, \tilde{\rho}_\varepsilon)\|_{L^{4/3}} + \|\nabla^2 \tilde{w}_\varepsilon\|_{L^{4/3}} \|\nabla \tilde{w}_\varepsilon\|_{L^2} \right).
\]
Combining this with (6.37) and (6.31) we discover
\[
\|\nabla^2 (\tilde{v}_\varepsilon, \tilde{\rho}_\varepsilon)\|_{L^{4/3}(B_r(x_0))} \leq C_4 \left( r^{-2} + \max\{\varepsilon, \mu\} \|\nabla^2 (\tilde{v}_\varepsilon, \tilde{\rho}_\varepsilon)\|_{L^{4/3}(B_r(x_0))} \right).
\] (6.38)
Note that before Lemma 6.1, we have assumed that $C_4 > 1$ and $\hat{c} > 1$. Recall also the choice of $\hat{c}$ in (6.11). By choosing
\[
\Lambda = 16^{-8} C_4^{-2} \text{ in (6.3)},
\]
we find $C_4 \max\{\varepsilon, \mu\} < 1/2$. This combined with (6.38) yields
\[
\|\nabla^2 (\tilde{v}_\varepsilon, \tilde{\rho}_\varepsilon)\|_{L^{4/3}(B_r(x_0))} \leq 2 C_4 r^{-2}.
\]
In view of (6.32) and (6.33), this implies (6.30).
Now using (6.2b), we have \( \rho_{\varepsilon_k}(\cdot, t_0) \xrightarrow{k \to \infty} |u|(\cdot, t_0) = 1 \) strongly in \( L^2(B_r(x_0)) \). Thus, using (6.6) we find
\[
\| v_{\varepsilon_k} - u \|_{L^2(B_r(x_0))}^2 \lesssim 2 \| u_{\varepsilon_k} - u \rho_{\varepsilon_k} \|_{L^2(B_r(x_0))}^2 \xrightarrow{k \to \infty} 0.
\]
These together with (6.30) and the Gagliardo–Nirenberg interpolation inequality yield
\[
(\varepsilon_k, \rho_{\varepsilon_k}) \xrightarrow{k \to \infty} (u, 1) \text{ strongly in } W^{1, 2}(B_{r/2}(x_0)). \tag{6.39}
\]
Finally, using (6.6) and (6.39) we find
\[
\nabla u_{\varepsilon_k} = \rho_{\varepsilon_k} \nabla v_{\varepsilon_k} + v_{\varepsilon_k} \nabla \rho_{\varepsilon_k} \xrightarrow{k \to \infty} \nabla u \text{ strongly in } L^2(B_{r/2}(x_0)),
\]
and finish the proof of (6.4).

**Proof of Theorem 1.2.** We employ the covering argument in [14]. For any test function \( \Psi \in C^1_c(\Omega_t^+; \mathbb{R}^3) \), we choose \( K = \text{supp}(\Psi) \subset \subset \Omega_t^+ \), and we define the singular set at time \( t \in (0, T] \) by
\[
\Sigma_t := \bigcap_{0 < r < 1} \left\{ x \in K \mid B_{2r}(x) \subset K, \lim_{k \to \infty} \int_{B_{2r}(x)} \left( \frac{1}{2} |\nabla u_{\varepsilon_k}|^2 + \frac{F(u_{\varepsilon_k})}{\varepsilon_k^2} \right) \, dx > \frac{\epsilon^2}{4} \right\}. \tag{6.40}
\]
We claim that \( \Sigma_t \) is discrete. Indeed, choose an arbitrary finite set \( \{y_j\}_{j=1}^J \subset \Sigma_t \) with mutually disjoint balls \( \{B_{2r_j}(y_j)\}_{j=1}^J \) inside \( K \) with \( r_j < 1/2 \). Since \( J \) is finite, there exist \( k_j > 0 \) such that for any \( k \geq k_j \) we have
\[
\int_{B_{2r_j}(y_j)} \left( \frac{1}{2} |\nabla u_{\varepsilon_k}|^2 + \frac{F(u_{\varepsilon_k})}{\varepsilon_k^2} \right) \, dx > \frac{\epsilon^2}{4} \quad \text{for all } j \in \{1, \ldots, J\}. \tag{6.41}
\]
Combined with (6.2a), this implies
\[
c^2 \geq \int_{\bigcup_{j=1}^J B_{2r_j}(y_j)} \left( \frac{1}{2} |\nabla u_{\varepsilon_k}|^2 + \frac{F(u_{\varepsilon_k})}{\varepsilon_k^2} \right) \, dx > \frac{\epsilon^2}{4} J. \tag{6.42}
\]
As a result, \( J \leq 4c^2\epsilon^{-2} \) and thus \( \Sigma_t \) is discrete. Therefore w.l.o.g. we can assume that \( \Sigma_t = \{x_0\} \) and \( B_{2r}(x_0) \subset K \). Let \( \eta \in C^1_c(B_2(0)) \) be a cut-off function which \( \equiv 1 \) in \( B_1(0) \). Then
\[
\Psi_\delta(x) := \Psi(x) \left( 1 - \eta\left( \frac{x - x_0}{\delta} \right) \right) \xrightarrow{\delta \to 0} \Psi(x) \text{ for any } x \neq x_0. \tag{6.43}
\]
It is obvious that \( \Psi_\delta = 0 \) in \( B_\delta(x_0) \). By (6.40) and Proposition 6.1 we have
\[
\nabla u_{\varepsilon_k} \xrightarrow{k \to \infty} \nabla u \text{ strongly in } L^2(K \setminus B_\delta(x_0)). \tag{6.44}
\]
Using these properties, we can apply \( \nabla u_{\varepsilon_k} \cdot \Psi_\delta \) to both sides of (1.2a), integrate by parts and then send \( k \to \infty \):
\[
\int_{\Omega} \partial_t u \land \Psi_\delta \, dx + \mu \int_{\Omega} (\text{div } u) \cdot (\text{rot } u) \cdot \Psi_\delta \, dx
\]
\[
+ \int_{\Omega} (\nabla u \land \Psi_\delta) \cdot \nabla \Psi_\delta \, dx - \mu \int_{\Omega} (\text{div } u) \cdot (\text{rot } \Psi_\delta) \cdot u \, dx = 0. \tag{6.45}
\]
Note that we have also used \( \partial_t u_{\varepsilon_k} \land u_{\varepsilon_k} \xrightarrow{k \to \infty} \partial_t u \land u \) weakly in \( L^2(0, T; L^{6/5}(\Omega)) \), which is due to Proposition 4.1. By (6.43) and the regularity of \( u \) (cf. (4.26a) and (4.26b)),
we can send $\delta \to 0$ in the first and the second integrals in (6.45) using the dominated convergence theorem. Concerning the third one, we have
\[
\int_{\Omega} (\nabla u \wedge u) \cdot \nabla \Psi_{\delta} \, dx = \int_{\Omega} (1 - \eta(\frac{x-x_0}{\delta})) (\nabla u \wedge u) \cdot \nabla \Psi \, dx
- \sum_{i=1}^{3} \int_{B_{2\delta}(x_0)} \frac{1}{\delta} (\partial_i \eta)(\frac{x-x_0}{\delta}) \partial_i u \wedge u \cdot \Psi \, dx.
\] (6.46)
We claim that the second integral on the right-hand side vanishes as $\delta \to 0$. Indeed, by the Cauchy–Schwarz inequality we have
\[
\left| \sum_{i=1}^{3} \int_{B_{2\delta}(x_0)} \frac{1}{\delta} (\partial_i \eta)(\frac{x-x_0}{\delta}) \partial_i u \wedge u \cdot \Psi \, dx \right|
\leq C \| \Psi \|_{L^\infty} \| \nabla \eta \|_{L^2(B_2)} \| \nabla u \|_{L^2(B_{2\delta}(x_0))} \delta \to 0 \to 0.
\] (6.47)
Now using $\lim_{\delta \to 0} \eta(\frac{x-x_0}{\delta}) = 0$ for any $x \neq x_0$, we can send $\delta \to 0$ in (6.46) and obtain
\[
\int_{\Omega} (\nabla u \wedge u) \cdot \nabla \Psi_{\delta} \, dx \xrightarrow{\delta \to 0} \int_{\Omega} (\nabla u \wedge u) \cdot \nabla \Psi \, dx.
\]
By the same argument we can compute the fourth integral in (6.45) and find
\[
\int_{\Omega} (\operatorname{div} u) (\operatorname{rot} \Psi_{\delta}) \cdot u \, dx \xrightarrow{\delta \to 0} \int_{\Omega} (\operatorname{div} u) (\operatorname{rot} \Psi) \cdot u \, dx.
\] (6.48)
Using the above two formulas, we can send $\delta \to 0$ in (6.45) and obtain (1.23). □

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Appendix A. Proof of Proposition 1.1

Proof of Proposition 1.1. We first recall that $\sigma = 1$ (cf. (2.5)), $I_0 \subset \Omega$ is the initial interface and $\eta_0$ is the cut-off function in (2.12). Then we define
\[
s_\varepsilon(x) := \eta_0(x) \theta \left( \frac{d_{I_0}(x)}{\varepsilon} \right) + \left( 1 - \eta_0(x) \right) 1_{\Omega_0^+},
\] (A.1)
where $\theta(z)$ is the solution of the ODE
\[-\theta''(z) + f'(\theta) = 0, \quad \theta(-\infty) = 0, \quad \theta(+\infty) = 1.
\] (A.2)
We note that $d_{I_0}$ is Lipschitz continuous in $\Omega$, and thus by Rademacher’s theorem we have $|\nabla d_{I_0}| \leq 1$ a.e. in $\Omega$. Recalling (1.19), we define
\[
u_{\varepsilon}^{in}(x) := s_\varepsilon(x) u^{in}(x).
\] (A.3)
One can verify that $u^{in}_{\varepsilon} \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega)$, $\|u^{in}_{\varepsilon}\|_{L^\infty(\Omega)} \leq 1$, and
\[
u_{\varepsilon}^{in} = \begin{cases} u^{in} & \text{if } x \in \Omega_0^+ \setminus B_{2\delta_0}(I_0), \\
\theta \left( \frac{d_{I_0}}{\varepsilon} \right) u^{in} & \text{if } x \in B_{\delta_0}(I_0), \\
0 & \text{if } x \in \Omega_0^- \setminus B_{2\delta_0}(I_0).
\end{cases}
\] (A.4)
So the condition (1.14a) is verified. To verify the others, we first compute the modulated energy in (1.7) for the initial datum $u^{in}_{\varepsilon}$. We write (A.1) as
\[
s_\varepsilon(x) = \theta \left( \frac{d_{I_0}(x)}{\varepsilon} \right) + \tilde{s}_\varepsilon(x),
\] (A.5)
where \( \hat{s}_\varepsilon(x) := (1 - \eta_0(x))(1_{Q^\varepsilon} - \theta \left( \frac{d_{\eta_0}(x)}{\varepsilon} \right)) \). Invoking (2.12) and the exponential convergence of \( \theta(z) \) as \( z \to \pm \infty \) (cf. (A.2)), we deduce that

\[
\| \hat{s}_\varepsilon \|_{L^\infty} + \| \nabla \hat{s}_\varepsilon \|_{L^\infty} \leq C e^{-\frac{C}{\varepsilon}}, \tag{A.6}
\]

for some constant \( C > 0 \) that only depends on \( I_0 \). By a Taylor’s expansion, we find

\[
F(u^{in}_\varepsilon) = f(\theta + \hat{s}_\varepsilon) = f(\theta) + O(e^{-C/\varepsilon}).
\]

Combining (A.3), (A.5) with (A.6), we obtain

\[
|\nabla u^{in}_\varepsilon|^2 = e^{-2\theta^2} + \theta^2|\nabla u^{in}|^2 + O(e^{-C/\varepsilon})(|\nabla u^{in}|^2 + 1).
\]

Note that we have also employed the identities \( \partial_x u^{in} \cdot u^{in} = 0 \) a.e. in \( \Omega \). Recalling (1.8), we have

\[
\psi_\varepsilon \big|_{t=0} = \int_0^{\theta + \hat{s}_\varepsilon} \sqrt{2f(s)} \, ds : \Omega \mapsto [0, 1]. \tag{A.7}
\]

So we can compute

\[
\frac{\varepsilon}{2} |\nabla u^{in}_\varepsilon|^2 + \frac{1}{\varepsilon} F(u^{in}_\varepsilon) - \xi \cdot \nabla \psi_\varepsilon \big|_{t=0}
= \frac{1}{2\varepsilon} \theta^2 + \frac{1}{\varepsilon} f(\theta) - \varepsilon^{-1} \xi \cdot n_{I_0} \theta' \sqrt{2f(\theta)} + \frac{\varepsilon}{2} \theta^2 |\nabla u^{in}|^2 + O(e^{-C/\varepsilon})(|\nabla u^{in}|^2 + 1). \tag{A.8}
\]

It follows from (2.10) that \( 1 - \xi \cdot n_{I_0} = O(d_{I_0}^2) \). So we have

\[
\varepsilon^{-1} \xi \cdot n_{I_0} \theta' \sqrt{2f(\theta)} = \varepsilon^{-1} \theta' \sqrt{2f(\theta)} + O(e^{-C/\varepsilon}) + \varepsilon^{-1} O(d_{I_0}^2) \theta' \sqrt{2f(\theta)}.
\]

Note that the last term can be written as

\[
\varepsilon^{-1} O(d_{I_0}^2) \theta' \sqrt{2f(\theta)} = O(\varepsilon) z^2 \theta'(z) \sqrt{2f(\theta(z))} \big|_{z= \frac{d_{I_0}(x)}{\varepsilon}}.
\]

Substituting the above two equations into (A.8), we find

\[
\int_\Omega \left( \frac{\varepsilon}{2} |\nabla u^{in}_\varepsilon|^2 + \frac{1}{\varepsilon} F(u^{in}_\varepsilon) - \xi \cdot \nabla \psi_\varepsilon \right) \, dx
= \int_\Omega \left( \frac{1}{2\varepsilon} \theta^2 + \frac{1}{\varepsilon} f(\theta) \right) \, dx
+ \int_\Omega \frac{\varepsilon}{2} \theta^2 |\nabla u^{in}|^2 \, dx + O(e^{-C/\varepsilon}) \int_\Omega (|\nabla u^{in}|^2 + 1) \, dx + O(\varepsilon) \quad \text{at } t = 0. \tag{A.9}
\]

Note that the integrand of the first integral on the right-hand side of (A.9) vanishes due to the identity \( \theta^2(z) = 2f(\theta(z)) \), which follows from (A.2). Now we turn to the first term in (1.7). Using (A.6) we can estimate

\[
|\text{div } u^{in}_\varepsilon|^2 \leq 2|\nabla \theta \cdot u^{in} + 2\theta^2|\text{div } u^{in}|^2 + O(e^{-C/\varepsilon})(1 + |\text{div } u^{in}|^2). \tag{A.10}
\]

By the exponential decay of \( \theta'(z) \) as \( z \to \pm \infty \), we deduce that

\[
|\nabla \theta \cdot u^{in}| = \begin{cases} 
\left| \frac{d_{I_0}}{\varepsilon} \theta' \left( \frac{d_{I_0}}{\varepsilon} \right) u^{in} \cdot n_{I_0} \right| \leq C \left| \frac{u^{in} \cdot n_{I_0}}{d_{I_0}} \right| \quad \text{in } B_{\delta_0}(I_0) \setminus I_0, \\
\left| \frac{1}{\varepsilon} \theta' \left( \frac{d_{I_0}}{\varepsilon} \right) u^{in} \cdot n_{I_0} \right| \leq e^{-\frac{C}{\varepsilon}} \quad \text{in } \Omega \setminus B_{\delta_0}(I_0). \tag{A.11}
\end{cases}
\]
Using this, (1.19) and Hardy’s inequality (cf. [7]), we find
\[
\int_{\Omega} |\nabla \theta \cdot u|^2 \, dx \leq C \int_{I_0} \int_{-\delta_0}^{\delta_0} \left| \frac{u_{in} \cdot n_{I_0}}{d_{I_0}} \right|^2 \, dr \, d\mathcal{H}^{d-1} + C
\]
\[
\leq C \left( \int_{\Omega} |\nabla u|^2 \, dx + 1 \right). \tag{A.12}
\]
Combining this with (A.10) and (A.9) we derive \( E_\varepsilon[u_{in}|I_0] \leq C \varepsilon \). Recalling (1.21), we have also obtained (1.14b). To verify (1.14c), we shall compute (1.12) at \( t = 0 \). By (A.7), we see that
\[
B[u_{in}|I_0] = 2 \int_{\Omega} \left( \frac{x+1}{2} - \psi_\varepsilon \right) \eta \circ d I \, dx.
\]
We shall only give the estimate in \( B_{\delta_0}(I_0) \cap \Omega_0^+ \) because the one in \( B_{\delta_0}(I_0) \cap \Omega_0^- \) follows in the same way, and the one in \( \Omega \setminus B_{\delta_0}(I_0) \) is due to (A.6) and the exponential convergence of \( \theta(z) \) at \( \pm \infty \).

\[
\int_{B_{\delta_0}(I_0) \cap \Omega_0^+} \left| \frac{\psi_\varepsilon - 1}{d I(x)} \right| \, dx \bigg|_{t=0} = \int_{B_{\delta_0}(I_0) \cap \Omega_0^+} \left( \int_{s_{\varepsilon}(x)}^{1} \sqrt{2f(s)} \, ds \right) d I(x) \, dx \bigg|_{t=0}
\]
\[
= \varepsilon \int_{B_{\delta_0}(I_0) \cap \Omega_0^+} \left( \int_{\theta_\varepsilon^{(d I(x))}}^{1} \sqrt{2f(s)} \, ds \right) \frac{d I(x)}{\varepsilon} \, dx \bigg|_{t=0} + O(e^{-C/\varepsilon}) \leq C \varepsilon^2, \tag{A.13}
\]
where the last step is due to the exponential decay of \( Q(z) := z \int_{\theta_\varepsilon^{(z)}}^{1} \sqrt{2f(s)} \, ds \) as \( z \uparrow \infty \).

\[\square\]

**Appendix B. Proof of Proposition 2.1**

**Lemma B.1.** The following identity holds:
\[
\int \nabla H : (\xi \otimes n_\varepsilon) |\nabla \psi_\varepsilon| \, dx - \int (\nabla \cdot H) \xi \cdot \nabla \psi_\varepsilon \, dx
\]
\[
= \int \nabla H : (\xi - n_\varepsilon) \otimes n_\varepsilon |\nabla \psi_\varepsilon| \, dx + \int \nabla \cdot H |\nabla u_\varepsilon| \, dx
\]
\[
+ \int \nabla \cdot H \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) \right) \, dx + \int \nabla \cdot H (|\nabla \psi_\varepsilon| - \xi \cdot \nabla \psi_\varepsilon) \, dx
\]
\[
- \int (\nabla H)_{ij} \varepsilon (\partial_i u_\varepsilon \cdot \partial_j u_\varepsilon) \, dx + \int \nabla H : (n_\varepsilon \otimes n_\varepsilon) |\nabla \psi_\varepsilon| \, dx. \tag{B.1}
\]

**Proof.** We introduce the stress tensor \((T_\varepsilon)_{ij} := (\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon)) \delta_{ij} - \varepsilon \partial_i u_\varepsilon \cdot \partial_j u_\varepsilon\). By (2.20b), we have the identity \( \nabla \cdot T_\varepsilon = H_\varepsilon |\nabla u_\varepsilon| \). Testing this identity with \( H \), integrating by parts and using (2.14c), we obtain
\[
\int H_\varepsilon \cdot H |\nabla u_\varepsilon| \, dx = - \int \nabla H : T_\varepsilon \, dx
\]
\[
= - \int \nabla \cdot H \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) \right) \, dx + \int (\nabla H)_{ij} \varepsilon (\partial_i u_\varepsilon \cdot \partial_j u_\varepsilon) \, dx.
\]
So adding zero leads to

\[
\int \nabla H : n_\varepsilon \otimes n_\varepsilon |\nabla \psi_\varepsilon| \, dx \\
= \int H_\varepsilon \cdot H |\nabla u_\varepsilon| \, dx + \int \nabla \cdot H \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) - |\nabla \psi_\varepsilon| \right) \, dx + \int \nabla \cdot H |\nabla \psi_\varepsilon| \, dx \\
- \int (\nabla H)_{ij} \varepsilon (\partial_i u_\varepsilon \cdot \partial_j u_\varepsilon) \, dx + \int (\nabla H) : (n_\varepsilon \otimes n_\varepsilon) |\nabla \psi_\varepsilon| \, dx,
\]

which yields \(B.1\).

Lemma B.2. Under the assumptions of Theorem 1.1, the following identity holds:

\[
\frac{d}{dt} E[u_\varepsilon |I] + \frac{1}{2\varepsilon} \int (\varepsilon^2 |\partial_t u_\varepsilon|^2 - |H_\varepsilon|^2) \, dx \\
+ \frac{1}{2\varepsilon} \int |\varepsilon \partial_t u_\varepsilon - (\nabla \cdot \xi) Dd^F(u_\varepsilon)|^2 \, dx + \frac{1}{2\varepsilon} \int |H_\varepsilon - \varepsilon |\nabla u_\varepsilon|H| \, dx \\
= \frac{1}{2\varepsilon} \int |(\nabla \cdot \xi)|Dd^F(u_\varepsilon)|n_\varepsilon + \varepsilon |\Pi u_\varepsilon \nabla u_\varepsilon|H|^2 \, dx \tag{B.2a}
\]

\[
+ \frac{\varepsilon}{2} \int |H|^2 \left( |\nabla u_\varepsilon|^2 - |\Pi u_\varepsilon \nabla u_\varepsilon|^2 \right) \, dx - \int \nabla H \cdot (\xi - n_\varepsilon)^{\otimes 2} |\nabla \psi_\varepsilon| \, dx \tag{B.2b}
\]

\[
+ \int (\nabla \cdot H) \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) - |\nabla \psi_\varepsilon| \right) \, dx \tag{B.2c}
\]

\[
+ \int (\nabla \cdot H) (1 - \xi \cdot n_\varepsilon) |\nabla \psi_\varepsilon| \, dx + \int (J^1_\varepsilon + J^2_\varepsilon) \, dx, \tag{B.2d}
\]

where

\[
J^1_\varepsilon := \nabla H : n_\varepsilon \otimes n_\varepsilon \left( |\nabla \psi_\varepsilon| - \varepsilon |\nabla u_\varepsilon|^2 \right) \\
+ \varepsilon \nabla H : (n_\varepsilon \otimes n_\varepsilon) \left( |\nabla u_\varepsilon|^2 - |\Pi u_\varepsilon \nabla u_\varepsilon|^2 \right) \\
- \sum_{ij} \varepsilon (\nabla H)_{ij} \left( \partial_i u_\varepsilon - \Pi u_\varepsilon \partial_i \varepsilon \right) \cdot (\partial_j u_\varepsilon - \Pi u_\varepsilon \partial_j \varepsilon), \tag{B.3}
\]

\[
J^2_\varepsilon := - \left( \partial_t \xi + (H \cdot \nabla) \xi + (\nabla H)^T \xi \right) \cdot |\nabla \psi_\varepsilon|. \tag{B.4}
\]

Proof. We shall employ the Einstein summation convention by summing over repeated indices. Using the energy dissipation law in \(2.6\) and adding zero, we find

\[
\frac{d}{dt} E_\varepsilon[u_\varepsilon |I] + \varepsilon \int |\partial_t u_\varepsilon|^2 \, dx - \int (\nabla \cdot \xi) Dd^F(u_\varepsilon) \cdot \partial_t u_\varepsilon \, dx \\
= \int (H \cdot \nabla) \xi \cdot \nabla \psi_\varepsilon \, dx + \int (\nabla H)^T \xi \cdot \nabla \psi_\varepsilon \, dx + \int J^2_\varepsilon \, dx. \tag{B.5}
\]

By the symmetry of \(\nabla^2 \psi_\varepsilon\) and the boundary conditions in \(2.14c\), we have

\[
\int \nabla \cdot (\xi \otimes H) \cdot \nabla \psi_\varepsilon \, dx = \int \nabla \cdot (H \otimes \xi) \cdot \nabla \psi_\varepsilon \, dx.
\]
Hence, the first integral on the right-hand side of (B.5) can be rewritten as

\[
\int (H \cdot \nabla) \xi \cdot \nabla \psi_\epsilon \, dx = \int \nabla \cdot (\xi \otimes H) \cdot \nabla \psi_\epsilon \, dx - \int (\nabla \cdot H) \xi \cdot \nabla \psi_\epsilon \, dx = \int (\nabla \cdot \xi) H \cdot \nabla \psi_\epsilon \, dx + \int (\xi \cdot \nabla) H \cdot \nabla \psi_\epsilon \, dx - \int (\nabla \cdot H) \xi \cdot \nabla \psi_\epsilon \, dx.
\]

Therefore,

\[
\frac{d}{dt} E_\epsilon[u_\epsilon | I] + \epsilon \int |\partial_t u_\epsilon|^2 \, dx - \int (\nabla \cdot \xi) Dd^F(u_\epsilon) \cdot \partial_t u_\epsilon \, dx = \int (\nabla \cdot \xi) H \cdot \nabla \psi_\epsilon \, dx + \int (\xi \cdot \nabla) H \cdot \nabla \psi_\epsilon \, dx - \int (\nabla \cdot H) \xi \cdot \nabla \psi_\epsilon \, dx + \int \nabla H : (\xi \otimes n_\epsilon) |\nabla \psi_\epsilon| \, dx + \int J^2_\epsilon \, dx.
\]

Now using (B.1) to replace the third and the fourth integrals on the right-hand side of the above equation, we find

\[
\frac{d}{dt} E_\epsilon[u_\epsilon | I] + \epsilon \int |\partial_t u_\epsilon|^2 \, dx - \int (\nabla \cdot \xi) Dd^F(u_\epsilon) \cdot \partial_t u_\epsilon \, dx = \int (\nabla \cdot \xi) H \cdot \nabla \psi_\epsilon \, dx + \int (\xi \cdot \nabla) H \cdot \nabla \psi_\epsilon \, dx + \int \nabla H : (\xi \otimes n_\epsilon) |\nabla \psi_\epsilon| \, dx + \int J^2_\epsilon \, dx.
\]

We shall show that \( J^1_\epsilon \) arises from the second and the third to last integrals by proving the following identity:

\[
\Pi_{u_\epsilon} \partial_i u_\epsilon \cdot \Pi_{u_\epsilon} \partial_j u_\epsilon = n_\epsilon \epsilon^i n_\epsilon^j |\Pi_{u_\epsilon} \nabla u_\epsilon|^2 \quad \text{a.e. in } \Omega, \tag{B.7}
\]

where \( (n_\epsilon^i)_{1 \leq i \leq 3} = n_\epsilon \). Such an identity holds obviously on the set \( \{x \mid u_\epsilon = 0\} \) by (2.22). It also holds on \( \{x \mid g(|u_\epsilon|) > 0\} \) due to the following identity which follows from (2.22) and (2.23a):

\[
\Pi_{u_\epsilon} \partial_i u_\epsilon \cdot \Pi_{u_\epsilon} \partial_j u_\epsilon |Dd^F(u_\epsilon)|^2 = \partial_i \psi_\epsilon \partial_j \psi_\epsilon = n_\epsilon n_\epsilon^i n_\epsilon^j |\Pi_{u_\epsilon} \nabla u_\epsilon|^2 |Dd^F(u_\epsilon)|^2.
\]

On the open set \( \{x \mid |u_\epsilon| > 0\} \) which includes \( \{x \mid |u_\epsilon| = 1\} \), we deduce from (2.22) and (2.19a) that \( \Pi_{u_\epsilon} \partial_j u_\epsilon = (\partial_j |u_\epsilon|) u_\epsilon \). By [18, Theorem 4.4] we have \( \partial_j |u_\epsilon| = 0 \) a.e. on \( \{x \mid |u_\epsilon| = 1\} \). We thus complete the proof of (B.7).

Now by (B.7) and adding zero, we find

\[
\nabla H : n_\epsilon \otimes n_\epsilon |\nabla \psi_\epsilon| - (\nabla H)_{ij} \epsilon (\partial_i u_\epsilon \cdot \partial_j u_\epsilon)
\]

\[
\overset{(2.22)}{=} \nabla H : n_\epsilon \otimes n_\epsilon |\nabla \psi_\epsilon| - \epsilon (\nabla H)_{ij} (\Pi_{u_\epsilon} \partial_i u_\epsilon \cdot \Pi_{u_\epsilon} \partial_j u_\epsilon)
\]

\[
- (\nabla H)_{ij} \epsilon ((\partial_i u_\epsilon - \Pi_{u_\epsilon} \partial_i u_\epsilon) \cdot (\partial_j u_\epsilon - \Pi_{u_\epsilon} \partial_j u_\epsilon)) \overset{(B.3)}{=} J^1_\epsilon \quad \text{a.e. in } \Omega.
\]
Using the identities $\nabla \psi = n_\varepsilon |\nabla \psi| \xi$ and $\nabla H : (\xi \otimes \xi) = 0$ (due to (2.14b)), we merge the second and third integrals on the right-hand side of (B.6):

$$\frac{d}{dt} E_\varepsilon[u_\varepsilon][I] = -\varepsilon \int |\partial_t u_\varepsilon|^2 dx + \int (\nabla \cdot \xi) DdF(u_\varepsilon) \cdot \partial_t u_\varepsilon dx$$

$$+ \int (\nabla \cdot \xi) H \cdot \nabla \psi dx + \int H : \nabla u_\varepsilon dx - \int \nabla H : (\xi - n_\varepsilon) \otimes \nabla \psi dx$$

$$+ \int (\nabla \cdot H) \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} F(u_\varepsilon) - |\nabla \psi| \right) dx$$

$$+ \int (\nabla \cdot H) (1 - \xi \cdot n_\varepsilon) |\nabla \psi| dx + \int (J_\varepsilon^1 + J_\varepsilon^2) dx.$$  \hspace{1cm} (B.8)

Now we complete squares for the first four terms on the right-hand side of (B.8). Reordering terms, we have

$$-\varepsilon |\partial_t u_\varepsilon|^2 + (\nabla \cdot \xi) DdF(u_\varepsilon) \cdot \partial_t u_\varepsilon + (\nabla \cdot \xi) H \cdot \nabla \psi + H \cdot H |\nabla u_\varepsilon|$$

$$= -\frac{1}{2\varepsilon} \left( |\partial_t u_\varepsilon|^2 - 2(\nabla \cdot \xi) DdF(u_\varepsilon) \cdot \varepsilon \partial_t u_\varepsilon + (\nabla \cdot \xi)^2 |DdF(u_\varepsilon)|^2 \right)$$

$$- \frac{1}{2\varepsilon} |\varepsilon \partial_t u_\varepsilon|^2 + \frac{1}{2\varepsilon} (\nabla \cdot \xi)^2 |DdF(u_\varepsilon)|^2 + (\nabla \cdot \xi) H \cdot \nabla \psi$$

$$- \frac{1}{2\varepsilon} |H_\varepsilon|^2 - 2\varepsilon |\nabla u_\varepsilon| H \cdot \nabla \psi$$

$$+ \frac{1}{2\varepsilon} \left( |H_\varepsilon|^2 + \varepsilon^2 |\nabla u_\varepsilon|^2 |H|^2 \right)$$

$$+ \frac{1}{2\varepsilon} \left( (\nabla \cdot \xi)^2 |DdF(u_\varepsilon)|^2 + 2\varepsilon (\nabla \cdot \xi) \nabla \psi \cdot H + \varepsilon \Pi u_\varepsilon |\nabla u_\varepsilon|^2 |H|^2 \right)$$

$$+ \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 - |\Pi u_\varepsilon \nabla u_\varepsilon|^2 |H|^2.$$  \hspace{1cm} (B.8)

Substituting this identity into (B.8), we arrive at (B.2). \hspace{1cm} □

*Proof of Proposition 2.1* The proof here is the same as the case $\mu = 0$, done in [40, Lemma 4.4]. This is because the form of the energy dissipation law (2.6) remains unchanged in the presence of the divergence term in (1.2a).

We first estimate the right-hand side of (B.2) by $E_\varepsilon[u_\varepsilon][I]$ up to a constant that only depends on $I_t$. Concerning (B.2a), it follows from the triangle inequality that

$$\int \left| \frac{1}{\sqrt{\varepsilon}} (\nabla \cdot \xi) DdF(u_\varepsilon) |n_\varepsilon + \sqrt{\varepsilon} |\Pi u_\varepsilon \nabla u_\varepsilon| H \right|^2 dx$$

$$\leq \int \left| (\nabla \cdot \xi) \left( \frac{1}{\sqrt{\varepsilon}} DdF(u_\varepsilon) - \sqrt{\varepsilon} |\Pi u_\varepsilon \nabla u_\varepsilon| \right) n_\varepsilon \right|^2 dx$$

$$+ \int \left| (\nabla \cdot \xi) \sqrt{\varepsilon} |\Pi u_\varepsilon \nabla u_\varepsilon| (n_\varepsilon - \xi) \right|^2 dx$$

$$+ \int \left| ((\nabla \cdot \xi) \xi + H) \sqrt{\varepsilon} |\Pi u_\varepsilon \nabla u_\varepsilon| \right|^2 dx.$$  \hspace{1cm} (2.26c)

The first integral on the right-hand side of the above inequality is controlled using (2.26a). Due to the elementary inequality $|\xi - n_\varepsilon|^2 \leq 2(1 - n_\varepsilon \cdot \xi)$, the second integral is controlled by (2.26a). The third integral can be treated using the relation $H = (H \cdot \xi) \xi + O(d_I(x,t))$ and (2.15a). So it can be controlled by (2.26c).
The integrals in (B.2b) can be controlled using (2.26c) and (2.26d). The one in (B.2c) is controlled by (2.26a). The first term in (B.2d) can be controlled using (2.26d). It remains to estimate (B.3) and (B.4). The integrals of the last two terms defining \( J^1_\varepsilon \) can be controlled by (2.26b). Therefore,

\[
\int J^1_\varepsilon \, dx \leq C \left( \int |n_\varepsilon - \xi| \left( \varepsilon |\nabla u_\varepsilon|^2 - \varepsilon |\Pi u_\varepsilon \nabla u_\varepsilon|^2 \right) \, dx + E_\varepsilon[u_\varepsilon|I] \right)
\]

The first and the third integrals in the last display can be estimated using (2.26b) and (2.26c) respectively. Then we employ (2.23a) to find

\[
\int J^1_\varepsilon \, dx \leq C \left( \int |n_\varepsilon - \xi| \left( \varepsilon |\Pi u_\varepsilon \nabla u_\varepsilon|^2 - |\nabla \psi_\varepsilon| \right) \, dx + E_\varepsilon[u_\varepsilon|I] \right)
\]

Finally applying the Cauchy-Schwarz inequality and then (2.26c) and (2.26d), we obtain \( \int J^1_\varepsilon \, dx \leq CE_\varepsilon[u_\varepsilon|I] \). As for \( J^2_\varepsilon \) (B.4), we employ (2.15c) and (2.26e) to obtain \( \int J^2_\varepsilon \, dx \leq CE_\varepsilon[u_\varepsilon|I] \). All in all, we have proved that the right-hand side of (B.2) is bounded by \( E_\varepsilon[u_\varepsilon|I] \) up to a multiplicative constant which only depends on \( I \).

\[ \square \]

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