Andreev-Bashkin effect and knot solitons in interacting mixture of a charged and a neutral superfluids with possible relevanece for neutron stars

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We discuss a mixture of interacting a neutral and a charged Bose condensates, which is supposed being realized in interior of neutron stars in form of coexistent neutron superfluid and protonic superconductor. We show that in this system, besides ordinary vortices of $S^1 \rightarrow S^1$ map, the neutron condensate also allows for (meta)stable finite-length knotted solitons, which are characterized by a nontrivial Hopf invariant and in some circumstancses are stabilized by Faddeev-Skyrme term induced by drag effect. We also consider a helical protonic fluxtube in this system and show that, in contrast, it does not induce a Faddeev-Skyrme term.

INTRODUCTION

In a standard model for a neutron star its interior features superfluidity of neutron Cooper pairs and superconductivity of proton Cooper pairs (see e.g. [1, 2]). Both these condensates allow vortices of $S^1 \rightarrow S^1$ map. Earlier it was suggested that the phenomenon of glitches in Crab and Vela pulsars is connected with vortex matter in these stars [3]. This remains a topic of intensive studies and discussions (for recent developments and citations see [4]). Besides that a standard model for a neutron star is a special system being a mixture of interacting a charged and a neutral condensates which makes it also being a topic of abstract academic interest [5] since such a system allows for interesting phenomena with no direct counterparts in e.g. superconducting metals. Studies of topological defects in a mixture of a charged and a neutral condensates, so far, concerned only ordinary Abrikosov-like columnar vortices (see e.g. [6, 7] and references therein). In this paper we argue that, possibly, this is not the only one type of stable topological defects allowed in neutron stars. We show that due to the drag effect in a mixture of a neutral and a charged superfluid (Andreev-Bashkin effect) the system also allows under certain conditions stable finite-length topological defects characterized by a nontrivial Hopf invariant, more precisely a special version of knot solitons.

Finite-length topological defects characterized by a nontrivial Hopf invariant were attracting interest for a long time in condensed matter physics: earlier it was discussed in spin-1 neutral superfluids [8, 9], in magnets [10], in charged and neutral two-component Bose condensates [11, 12, 13], in spin-triplet superconductors [14] and in other systems. In neutral systems finite-length closed vortices are not stable against shrinkage unless their size is stabilized by a conservation of some dynamic quantity, like in case of a propagating vortex loop. A special case is a neutral two-component system with a phase separation, where a vortex loop made up of one condensate with confined in its core circulating second condensate is stable against shrinkage [13]. Intrinsically stable topological defects characterized by a nontrivial Hopf invariant (the knot solitons) have been discussed in the Faddeev nonlinear $O(3)$ sigma model [15] where its stability is ensured by a special fourth-order derivative term:

$$ F_F = \left( \partial \vec{n} \right)^2 + \alpha \left( \vec{n} \cdot \partial_i \vec{n} \times \partial_j \vec{n} \right)^2 + \kappa \left( 1 - \vec{n} \cdot \vec{n}_0 \right)^2, \quad (1) $$

where $\vec{n} = (n_1, n_2, n_3)$ is a three-component unit vector. A knot soliton (being in the simplest case a toroidal vortex loop) is a configuration where the vector $\vec{n}$ resides in the core on e.g. the south pole of the unit sphere, at infinity it reaches the north pole, while in between the core and the vortex boundary it performs $n$ rotations if one goes once around the core and $m$ rotations if one goes once along a closed curve in toroidal direction. The stability of knots in this model was extensively studied in numerical simulations [16]. Recently it was realized that this model is relevant for wide class of physical systems. First, it was suggested that this model may be relevant in the infrared limit of QCD with the knots solitons being a candidate for glueballs [17]. Besides that an extended version of Faddeev model has been derived for two-band superconductors [11] and for triplet superconductors [14].

Below we discuss a possibility of formation of finite length stable topological defects in a mixture of interacting charged and neutral Bose condensates.

A MIXTURE OF INTERACTING CONDENSATES

A mixture of a charged (made up of protonic Cooper pairs) and neutral (made up of neutronic Cooper pairs) Bose condensates in the interior of neutron stars can be described in the hydrodynamic limit by the follow-
ing Ginzburg-Landau functional

$$F = \frac{1}{2} \rho^{pp} \nabla_p^2 v_p^2 + \frac{1}{2} \rho^{nn} v_n^2 + \rho^{pn} v_p \cdot v_n + V + \frac{B^2}{8\pi}$$  \tag{2}

where $B$ is magnetic field, and

$$V = a_p |\Psi_p|^2 + \frac{b_p}{2} |\Psi_p|^4 + a_n |\Psi_n|^2 + \frac{b_n}{2} |\Psi_n|^4 + c |\Psi_p|^2 |\Psi_n|^2$$  \tag{3}

is the potential term. We begin with a discussion of the simplest case of two $s$-wave condensates (so $\Psi_p = |\Psi_p| e^{i\phi_p}$ and $\Psi_n = |\Psi_n| e^{i\phi_n}$ are complex scalar fields which discribe proton and neutron condensates correspondingly). In the above expression

$$v_n = (\hbar/2m_n) \nabla \phi_n$$  \tag{4}

and

$$v_p = (\hbar/2m_p) \nabla \phi_p - (2e/m_pc) A$$  \tag{5}

are superfluid velocities of neutron and proton condensates. The key feature of this system is the Andreev-Bashkin effect: due to interaction between two superfluids the particle current of one of the condensates is carried by the superfluid velocity of another so the superfluid mass current of protons and neutrons in such a system is

$$w_p = \rho^{pp} v_p + \rho^{pn} v_n, \quad w_n = \rho^{nn} v_n + \rho^{np} v_p,$$  \tag{6}

where $\rho^{mn} = \rho^{np}$ is the superfluid density of one of the condensates which is carried by superfluid velocity of another. Because of the Andreev-Bashkin effect the charged supercurrent in this system depends on gradients of neutron condensate (as it follows from \[5\]):

$$J = \frac{e \hbar \rho^{pp}}{m_p^2} \left( \rho^{mn} m_p \nabla \phi_n + \nabla \phi_p - \frac{4e}{c \hbar} A \right)$$  \tag{7}

Let us discuss topological defects, allowed in \[2\], other than Abrikosov vortices.

**HELICAL NEUTRON VORTEX LOOP**

Let us consider a vortex loop made up of neutron condensate with zero density of neutron Cooper pairs in its core. Let us introduce a new variable $\theta$ as follows: $\rho^{nn} m_n = \sin^2(\theta/2)$. We will consider a defect where if we go from the core center to the boundary of the fluxtube in a cross section to the vortex, the variable $\theta$ grows from 0 to $\pi$. Since at the center of the vortex we have chosen that the density of the neutron condensate vanishes then indeed there is no drag effect in the center of the fluxtube and correspondingly $\rho^{nn}$ is zero in the core. This allows one to choose the boundary condition $\sin^2(\theta/2) = 0$ in the center of the vortex. Let us now impose the following configuration of $\phi_n$: if we go once around the vortex core the $\phi_n$ changes $2\pi n$, while if we cover the vortex loop once in toroidal direction (a closed curve along the core) $\phi_n$ changes $2\pi l$ with $n, l$ being integer. Such a situation naturally occurs if a loop is formed around rotation-induced vortex line or in case of two interlinked loops. This configuration corresponds to a spiral superflow of the neutron Cooper pairs in such a vortex ring. Topologically such a vortex is equivalent to knot solitons considered in \[1\] and can also be characterized by a unit vector $\mathbf{e} = (\cos \phi_n \sin \theta, \sin \phi_n \sin \theta, \cos \theta)$ with a nontrivial winding. We stress that we do not impose a nontrivial winding on $\phi_p$ (compare with discussion of knot solitons in the two-gap model \[1\]) where, in contrast, in a knot soliton the phases of both condensates must have a nontrivial winding number, however, as discussed below, neutral-charged mixture is principally different from the system in Ref. \[1\].

Indeed the nontrivial superflow of neutron Cooper pairs induces a drag current of proton Cooper pairs which in turn induces a magnetic field which can be calculated from \[7\]:

$$B = \text{curl} \left[ -J \frac{cm_p^2}{4e^2 \rho^{pp}} + c \hbar \frac{m_n \rho^{mn} \rho^{pp} \nabla \phi_n}{4e \rho^{pp} m_p} \right]$$  \tag{8}

which can also be written as

$$B_k = - \frac{cm_p^2}{4e \rho^{pp}} \left[ \nabla_i J_j - \nabla_j J_i \right] + \frac{c \hbar}{8e} \sin \theta \left[ \nabla_i \theta \nabla_j \phi_n - \nabla_j \theta \nabla_i \phi_n \right]$$  \tag{9}

This self-induced magnetic field gives the following contribution to the free energy \[2\]:

$$F_m = \frac{B^2}{8\pi} = \frac{c^2 \hbar^2}{512 \pi e^2} \left[ \frac{2m_p^2}{\hbar^2 e^2 \rho^{pp}} \left| \nabla_i J_j - \nabla_j J_i \right| \right]^2$$  \tag{10}

Which is a version of the Faddeev fourth-order derivative term analogous to the fourth-order derivative term in \[1\] closely related to the fourth-order derivative term in \[1\]. The fourth order derivative terms of this type provide stability to finite length topological defects \[12, 16\]. Physically, in a mixture of a charged and a neutral condensates this effect corresponds to the following situation: as mentioned above, the nontrivial configuration of phase and density of neutron condensate induces a charged drag current of proton Cooper pairs which results in the configuration of magnetic field \[9\]. This configuration has the special feature that if the vortex shrinks then the magnitude of the self-induced magnetic field grows.
We also remark that $\rho^{pp}$ is a measure of background density of proton condensate which is not required to vary to produce a knot soliton.

In the two-gap model in [11] there is a competition of the fourth-order derivative term (which corresponds to self-induced magnetic field) versus a second-order derivative term and a mass term for the third component of the $O(3)$-symmetric order parameter $\mathbf{n}$ (the third component of $\mathbf{n}$ is related to condensate densities in [11] and thus it is massive). In contrast, in the present model in the competition also participates kinetic energy of superflow of neutron Cooper pairs (which is minimized if the vortex shrinks). A necessary condition for (meta)stability of such a vortex loop is that the competition of kinetic energy of superflows, gradients of condensate density versus the self-induced magnetic field would stabilize the vortex loop at a length scale which corresponds to magnitude of magnetic field $|\mathbf{B}(x)|$ smaller than the field which could break proton Cooper pairs. We also emphasize that one of the differences with the system of two charged scalar fields in [11] is that in the present case the self induced magnetic field comes from drag current in the vicinity of the core while the superflow of neutron Cooper pairs is extended (not localized on length scale shorter or equal to penetration length like the field inducing currents in [11]). We also remark that indeed the effective action [2] is assumed being derived from a microscopic theory in the approximation of small gradients. Indeed one can derive higher-order derivative terms from a microscopic theory but this sort of terms, in contrast to the term [11] is irrelevant for discussion of the stability of finite-length topological defects in this system. Indeed a competition between second- and fourth-order derivative terms obtained in a derivative expansion would stabilize a topological defect at a characteristic length scale where all the higher-order derivative terms become of the same order of magnitude. So, at such length scales the derivative expansion fails. We also would like to stress that in the present system the knot soliton is prevented against a collapse by a finite energy barrier, in contrast to an infinite energy barrier in the case of the Faddeev’s nonlinear $\sigma$-model considered in mathematical physics [15]. That is, a zero in proton condensate density, outside core, may lead to unwinding of a knot soliton since in such a point the unit vector $\mathbf{n}$ is ill-defined and thus the Hopf map is ill-defined as well. However the modulus of proton condensate order parameter is massive so producing such a singularity is energetically expensive. Thus, it is a finite energy barrier which prevents a knot soliton in a superfluid/superconductor from collapsing.

### AN EXAMPLE OF GENERALIZATION TO OTHER PAIRING SYMMETRIES

Let us generalize the discussion to the case of a mixture of a spin-triplet neutron condensate and a singlet proton condensate in order to show that the picture does not depend significantly on pairing symmetry. The order parameter of the spin-1 neutral condensate is $|\Psi_n(x)|^2\zeta_n(x)$ where $(q = 1, 0, -1)$ and $\zeta$ is a normalized spinor $\zeta^\dagger \zeta = 1$. Free energy of a neutral spin-1 system is (see e.g. [13]):

$$F_t = \frac{\hbar^2}{2m_n} (\nabla |\Psi_n|)^2 + \frac{\hbar^2}{2m_n} |\Psi_n|^2 (\nabla \zeta)^2 - \mu |\Psi_n|^2 + \frac{|\Psi_n|^4}{2} [c_0 + c_2 < S >^2],$$

where $< S > = \zeta^\dagger S_{qj} \zeta$ is spin. Degenerate spinors are related to each other by gauge transformation $e^{i\phi_s}$ and spin rotations $U(\alpha, \beta, \tau) = e^{-i\tau} \alpha e^{-i\phi_s} \beta e^{-i\tau}$, where $(\alpha, \beta, \tau)$ are the Euler angles. The topological defects in the neutral system like this have been intensively studied (see e.g. [3, 4]). A charged counterpart of this system in ferromagnetic state allows stable knot solitons as it was shown in [14].

Let us consider first the ferromagnetic state (which emerges when $c_2 < 0$) in context of a mixture of superfluids. The energy in this case is minimized by $< S >^2 = 1$ and the ground state spinor and density are [15]:

\[ \zeta = e^{i(\phi_n - \tau)}(e^{i\phi_s} \cos^2 \frac{\beta}{2} \sqrt{2} \cos \frac{\beta}{2} \sin \frac{\beta}{2} e^{i\phi_s} \sin^2 \frac{\beta}{2}); \quad |\Psi_n|^2 = \frac{\hbar^2}{m_n} \mu. \]

The superfluid velocity in ferromagnetic case is [15]:

\[ \mathbf{v}_n = \frac{\hbar n}{m_n} [\nabla (\phi_n - \tau) - \cos \beta \nabla \alpha]. \]

So in a mixture of a neutral ferromagnetic triplet condensate and a charged singlet condensate the equation for charged current is:

\[ J = \frac{e\hbar}{m_p m_p \rho^{pp}} \left( \rho^{em} m_p |\nabla (\phi_n - \tau) - \cos \beta \nabla \alpha| \right) \]

\[ + \nabla \phi_p - \frac{4e}{\hbar} A \]  

(12)

From this expression we can see that assuming e.g. that there is no nontrivial windings in the variables $\alpha$ and $\beta$, the system reduces to (7) and thus allows for the topological defects in the form described in the first part of the paper. We emphasize that there are no knots of this type in a charged ferromagnetic triplet system considered in [14] because in the current equation of a charged triplet superconductor, the ratio of the coefficients in front of the vector potential term and the gradient term analogous to $\nabla (\phi_n - \tau)$ does not depend on the carrier density and thus one can not obtain a contribution analogous to Faddeev term to the free energy by imposing a nontrivial configuration of the first gradient term in the current equation similar to (12) in the system [14]. In a charged triplet case the knot soliton may form only as a spin texture [14]. So a neutral-charged mixture with drag effect in its magnetic properties is principally different
from a genuine charged system. Spin-texture knots can be formed in the present system too, as a configuration of the order parameter \( \mathbf{s} = (\cos \beta \sin \alpha, \sin \beta \sin \alpha, \cos \beta) \) characterized by a nontrivial Hopf invariant. Such a texture generates magnetic field due drag current induced by the superflow of the neutron Cooper pairs, which is produced by the spin texture. So, in general, there is the following nontrivial magnetic energy contribution to the free energy functional:

\[
F_m^t = \frac{e^2 \hbar^2}{32 \pi e^2} \left[ \frac{2n^2}{c^3 e^2 \rho} \left( \nabla_i J_j - \nabla_j J_i \right) - \sin \theta \left[ \nabla_i \nabla_j \phi_n \right. \right.
\]

\[
\left. - \nabla_j \nabla_i \phi_n \right] - \sin \beta \left[ \nabla_i \beta \nabla_i \alpha - \nabla_j \beta \nabla_j \alpha \right]^2 \tag{13}
\]

It must be observed that the spin-texture knot soliton is structurally different from the topologically equivalent knot of the type considered in the first part of the paper. The spin-texture knot is coreless (there are no zeroes of the condensate density in it). The third component of the order parameter \( \mathbf{s} = (\cos \beta \sin \alpha, \sin \beta \sin \alpha, \cos \beta) \) is massless in this case, thus the spin-texture knot solitons in this system are energetically less expensive and have larger characteristic size than the topologically equivalent knots in the variable \( \mathbf{e} = (\cos \phi_n, \sin \theta, \sin \phi_n, \sin \theta, \cos \theta) \).

Let us now consider the “polar” phase of triplet superconductors which is the case when \( c_2 > 0 \) in (11). The energy is minimized then by \( \mathbf{S} = 0 \). The spinor \( \zeta \) and the condensate density in the ground state are (10):

\[ \zeta = e^{i \phi_n}(-i e^{-i \alpha} \sin \beta, \cos \beta, \frac{1}{\sqrt{2}} e^{i \alpha} \sin \beta); \quad |\Psi_n|^2 = \mu / e_0. \]

The superfluid velocity in this case is (see e.g. (11)):

\[ \mathbf{v}_n = \frac{2n_{m_n}}{m_n} \nabla \phi_n \]

which is analogous to singlet case. Thus in the antiferromagnetic case the allowed knot solitons are equivalent to knots in a mixture of two singlet condensates considered in the first part of the paper.

We also remark that it is generally assumed that there is no proton-neutron pairing in a neutron star because of large differences in their Fermi energies.

While we can not make at this stage any definite predictions (which would require large-scale numerical simulations), let us however discuss possible mechanisms of formation of knot solitons of the discussed above types in neutron stars. As it is known, ordinary vortices in superconductors form e.g. as an energetically preferred state in external magnetic field. Indeed existence of such a term could result in a range of parameters where knots would have a finite negative energy if spins of neutron Cooper pairs in the knot soliton are aligned along the self-induced magnetic field. A definite answer to this question may however be only obtained in a large scale numerical simulation. Thus, if an ensemble of knot solitons is formed in a neutron star then one of the apparent consequences would be its interaction with ordinary columnar neutron vortices, then apparently in such a case knot solitons would disturb a regular lattice of neutron vortices.

**HEXICAL PROTONIC FLUXTUBE FORMED AROUND A NEUTRON VORTEX**

Above we considered knot solitons which appear due to notrivial helical windings of neutron condensate phase. In principle there is a theoretical mechanism which would allow formation of helical vortex loops of proton condensate. Let us show however that a helical protonic vortex loop is not a knot soliton and it is not stable. Here we stress the most recent studies indicate type-I behaviour of proton condensates. Let us now however consider the model (21). In that model a neutron star possesses a lattice of uniform neutron vortices and a complicated structure of sparse entangled proton flux tubes (see Fig. 3 in (21)). In the dynamical processes discussed in (21) one should expect that entangled proton fluxtubes may dynamically form rings around columnar neutron vortices as shown on Fig 1. Let us now consider such a ring. The charge current in such a ring is given by eq. (9).

When we go around such a flux tube, the protonic phase \( \phi_p \) changes \( 2\pi \), however there is also a current along such a vortex due to drag effect by superfluid neutron Cooper...
pairs which is characterized by a nontrivial phase winding of $\phi_n$, which changes $2\pi$ when we cover flux tube once in toroidal direction. This results in a spiral net charge current in such a vortex loop resembling that of a knot soliton considered in the first part of the paper. Let us show however that such a vortex loop is not stable: The magnetic field in such a helical fluxtube is given by

$$B = \frac{\hbar}{4e_c} \nabla \cdot \left[ -J \frac{m_p^2}{e \hbar \rho p n} - \nabla \phi_p - \rho \rho p n \nabla \phi_n \right]$$  \hspace{1cm} (15)

In such a configuration, in spite of helical net charge current, the individual phase configurations of $\phi_p$ and $\phi_n$ are not helical, besides that, the ratio of the vector potential term to gradient term for $\phi_p$ is constant. Thus such a helical superflow does not result into a self-induced Faddeev-Skyrme-like term, which, if it would be present, would significantly affect the considerations in [4, 21].

CONCLUSION

In conclusion we studied topological defects other than Abrikosov vortices in an interacting mixture of a neutral and a charged condensates. Such a system is believed being realized in the interior of neutron stars. We have shown that due to Andreev-Bashkin effect the system possesses a large variety of knot solitons of different nature than the knot solitons in the systems studied before. We also suggested that due to Zeeman coupling term, there could be a theoretical possibility of an exotic inhomogeneous ground state in this system: a spontaneous formation of a dense ensemble of knot solitons.

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