The Necessary Condition of Hypercyclicity of Truncated Toeplitz Operator

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Abstract. We study hypercyclicity of truncated Toeplitz operators in the model space \( H^2(D) \ominus \theta H^2(D) \) where \( \theta \) is inner function and \( H^2(D) \) is Hardy space. In this paper, the necessary condition of hypercyclicity of truncated Toeplitz operator is given.

1. Introduction
Since Robert Devaney provided chaos conditions in (1986, [1]), the chaotic systems became imaginable. Nowadays, the Devaney's definition of chaos is one of the most studied objects in chaos theory which says that the topological dynamical system \((X, f)\), i.e. topological space with continuous map \( f \) is chaotic if satisfied the following conditions: 1) if \( f \) has sensitivity to initial conditions; 2) if \( f \) is topologically transitive; 3) the set of periodic orbits of \( f \) are dense in \( X \).

Actually, chaos is familiar to non-linear systems (as well non-linear operators), but the examples found by G.D. Birkhoff in (1929, [3]), G.R. MacLane in (1952, [4]) and S. Rolewicz in (1969, [5]) showed that some linear operators have dense orbit in the corresponded space. Later was observed that this property means those operators satisfied the second condition of Devaney's definition in infinite dimensional spaces and the operators mentioned were called hypercyclic operators. In recent years, mathematicians studied the hypercyclicity of operator as the most important characterization of linear chaotic operators. For more details, we refer to [6].

An operator \( T \) in Banach space \( X \) is called hypercyclic if there is a vector \( x \in X \) such that the orbit
\[
\text{Orb}(T,x) = \{ T^n x , n = 0,1,2, \ldots \}
\]
is dense in \( X \).

Here, we remind that by \( \sigma(T) \) we mean the spectrum of \( T \). By \( D \) and \( T \) we denote the unite disc and the unit circle, respectively. In the next theorem, Kitai observed that the spectrum of an operator involved to its necessary condition to be hypercyclic:

**Theorem 1.1.** [6] Let \( T \) be a hypercyclic operator. Then \( \sigma(T) \) meets the unit circle, i.e. \( \sigma(T) \cap T \neq \emptyset \).

Moreover, the Rolewicz's example showed that the Toeplitz operator \( T_{\eta \bar{z}} \) on Hardy space \( H^2 \) is hypercyclic if \( |\eta| > 1 \). Godefory and Shapiro in [7] showed that the Toeplitz operator with antianalytic function \( \Phi \in H^\infty \) i.e. \( T_{\Phi} \) is hypercyclic if \( \Phi(D) \cap T \neq \emptyset \). By Shkarin in [8] was described hypercyclic Toeplitz operators with symbols of the form...
\[ \Phi(z) = az + b + cz \]

and by Baranov and Lishanski in [9] was described the hypercyclicity of \( T_\theta \) with symbols of the form

\[ \Phi(z) = p(\frac{z^\gamma}{z^\delta} + \varphi(z)) \]

where \( p \) is a polynomial and \( \varphi \in H^\infty \). Now, let \( H^2(D) \) be the classical Hardy space and \( \theta \) is an inner function with \( |\theta(z)| = 1 \) almost everywhere. It’s well known that \( \theta H^2(D) \) is closed subspace of \( H^2(D) \). The \emph{model space} is \( K_\theta = H^2(D) \ominus \theta H^2(D) \). Therefore, the truncated Toeplitz operator is

\[ A_\theta = A_\theta f = P_\theta (\varphi f) \quad (1.1) \]

Where \( f \in K_\theta \cap H^2(D) \) and \( P_\theta \) is the orthogonal projection of \( H^2(D) \) onto \( K_\theta \).

In this note, we obtain the necessary condition of hypercyclicity of truncated Toeplitz operator in term of Kitai’s Theorem 1.1.

2. Main Result

In order to obtain the result, we will review the Livshitz-Moller Theorem from [10]:

**Theorem 2.1.** If \( \theta \) be inner function, \( K = (\theta H^2)^{-1} \) and \( T = P_\theta S \) over \( K_\theta \), where \( S \) be backward shift operator and \( P_\theta \) is the orthogonal Projection of \( L^2 \) onto \( K_\theta \). Then

\[ \sigma(T) = \sigma(\theta). \]

**Theorem 2.2.** Let \( \varphi \) be a Riemann conformal mapping. Then

\[ \sigma(\varphi(A_\theta)) \cap T \neq \emptyset. \]

**Proof.** Without lose the generality we consider truncated Toeplitz operator with the following singular inner function:

\[ \theta(z) = \exp\left[ \frac{z^{r+1}}{z-1} \right]. \quad (1.2) \]

It’s well known that the function in (1.2) is analytic in \( H^2(D) \) such that for \( z \in \partial D, z \neq 1 \) satisfies \( |\theta| = 1 \) a.e. and for \( z \in D \) satisfies \( |\theta(z)| < 1 \).

Assume \( \eta : [0,1] \to [0, +\infty) \) be a \( C^2[0,1] \) function such that \( \eta(0) = \eta(1) = 0 \) and \( \eta > 0 \) on \([0,1]\).

Now, we will define a (closed) domain \( \Omega \) in the complex plane by

\[ \Omega = \{ x \in C : \Re(z) \in [0,1] - \eta(x) \leq y \leq \eta(x) \}. \]

Next, we fix a Riemann conformal mapping \( \varphi : D \to \Omega \) such that \( \varphi(1) = 1 \). In term of the \( H^\infty \) functional calculus (see [10]) one can set the following bounded operator

\[ T = \varphi(A_\theta). \]

It is obvious that the spectrum of \( A_\theta \) coincides with the support of the singular measure, defining the singular inner function \( \theta(z) \) such that \( \sigma(A_\theta) = \{1\}. \)

Since \( \varphi \) can be extended to a continuous function from \( \partial D \) to \( \partial \Omega \). In other words, \( \varphi \) is in the disc algebra, the spectral mapping theorem (Livshitz-Moller theorem) yields

\[ \sigma(T) = \varphi(\sigma(A_\theta)) = \{1\}. \]

In other words,

\[ \sigma(T) \cap T \neq \emptyset. \]

**References**

[1] Hirsch M. W., Smale S. and Devany R., Differential equations, dynamical systems, and an introduction to chaos. Elsevier, 2013.

[2] G. D. Birkhoff, Demonstration dun theoreme elementaire sur les fonctions entieres, C.R. Acad. Sci. Paris, 198, (1929), P. 473-475

[3] G. R. Maclane, Sequences of derivatives and normal families, J. Anal. Math., 2, (1952), P. 72-87.

[4] S. Rolewicz, On orbits of elements, Studia Math., 32, (1969), P. 17-22.

[5] Karl-G. Gross- Erdman and Alfred Manguillo, Linear Chaos, Springer, 2011.

[6] G. Godefroy, J. H. Shapiro, Operators with dense, invariant cyclic vector manifolds, J. Funct.
Anal., 98, (1991), P. 229-269.

[7] S. Shkarin, Orbits of coanalytic Toeplitz operators and weak hypercyclicity, arXiv:2012.

[8] A. Baranov and A. Lishanshkiii, Hypercyclic Toeplitz operators, Results Math., 70,(2016), P. 337-347.

[9] N.K. Nikolski, Lectures on the shift operator, Nauka:Moskova, 1980  (In Russian).

[10] B. Sz. Nagy, C. Foias, Harmonic analysis operators on Hilbert space, Northholand, Amesterdam and Akad. Kiado. Budapest, 1970.