A note on the Harris-Kesten Theorem

Béla Bollobás\footnote{Department of Mathematical Sciences, University of Memphis, Memphis TN 38152, USA} \footnote{Trinity College, Cambridge CB2 1TQ, UK} \footnote{Research supported in part by NSF grant ITR 0225610} \hfill Oliver Riordan\footnote{Royal Society Research Fellow, Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, UK} \\

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Abstract

A short proof of the Harris-Kesten result that the critical probability for bond percolation in the planar square lattice is $1/2$ was given in \cite{1}, using a sharp threshold result of Friedgut and Kalai. Here we point out that a key part of this proof may be replaced by an argument of Russo \cite{6} from 1982, using his approximate zero-one law in place of the Friedgut-Kalai result. Russo’s paper gave a new proof of the Harris-Kesten Theorem that seems to have received little attention.

Let $\mathbb{Z}^2$ be the planar square lattice, i.e., the graph with vertex set $\mathbb{Z}^2$ in which each pair of nearest neighbours is joined by an edge. Let $X = E(\mathbb{Z}^2)$ be the edge-set of $\mathbb{Z}^2$, and let $\Omega = \{-1, +1\}^X$. We write $\omega = (\omega_e)_{e \in X}$ for an element of $\Omega$, and say that the edge $e$ is open (in the state $\omega$) if $\omega_e = +1$, and closed if $\omega_e = -1$. An event $A \subset \Omega$ is local if it depends on only finitely many coordinates. As usual, let $\Sigma$ be the sigma-field generated by local events, and let $\mathbb{P}_p$ be the probability measure on $(\Omega, \Sigma)$ in which each edge is open with probability $p$, and these events are independent. Let $\theta(p)$ be the $\mathbb{P}_p$-probability that the origin is in an infinite open cluster, i.e., an infinite connected subgraph $C$ of $\mathbb{Z}^2$ with every edge of $C$ open. In 1960, Harris \cite{3} proved that $\theta(1/2) = 0$; in 1980, Kesten \cite{5} showed that $\theta(p) > 0$ for $p > 1/2$, establishing that $p_c = 1/2$ is the ‘critical probability’ for this model. A short proof of these results was given in \cite{1}, using a sharp-threshold result of Friedgut and Kalai \cite{2}, itself based on a result of Kahn, Kalai and Linial \cite{4}.

In 1982, Russo \cite{6} proved a general sharp-threshold result (weaker than the more recent results described above) and applied it to percolation, to give a new proof of the ‘equality of critical probabilities’ for site percolation in $\mathbb{Z}^2$. Although Russo does not explicitly say this, his application applies equally well to bond percolation, giving a new proof of the Harris-Kesten Theorem that seems not to be well known. Here we shall present Russo’s general sharp-threshold result, and then give a complete version of his application, to bond percolation in $\mathbb{Z}^2$. 
Replacing the appropriate section of [1] with this argument gives an even simpler proof of the Harris-Kesten Theorem; we are grateful to Professor Ronald Meester for bringing this to our attention.

An event \( A \subset \Omega \) is increasing if \( \omega \in A \) and \( \omega_e \leq \omega_e' \) for every \( e \) imply \( \omega' \in A \), i.e., if \( A \) is preserved when the state of one or more edges is changed from closed to open. An edge \( e \) is pivotal for an event \( A \) if changing the state of \( e \) affects whether or not \( A \) holds. Let \( \delta_e A \) be the event that \( e \) is pivotal for \( A \), so \( \omega \in \delta_e A \) if and only if exactly one of \( \omega_+ \), \( \omega_- \) is in \( A \), where \( \omega_\pm \) are the states that agree with \( \omega \) on all edges other than \( e \), with \( \omega_+ = 1 \) and \( \omega_- = -1 \). In [6], Russo proved the following result about the product measure \( \mathbb{P}_p \); in this result the structure of \( \mathbb{Z}^2 \) is irrelevant, i.e., the groundset \( X \) can be any countable set.

**Theorem 1.** For every \( \varepsilon > 0 \) there is an \( \eta > 0 \) such that if \( A \) is an increasing local event with
\[
\mathbb{P}_p(\delta_e A) < \eta
\]
for every \( e \in X \) and every \( p \in [0, 1] \), then there is a \( p_0 \in [0, 1] \) with
\[
\mathbb{P}_{p_0 - \varepsilon}(A) \leq \varepsilon \quad \text{and} \quad \mathbb{P}_{p_0 + \varepsilon}(A) \geq 1 - \varepsilon.
\]

As in [1], by a \( k \) by \( \ell \) rectangle we mean a rectangle \([a, b] \times [c, d]\) with \( a, b, c, d \in \mathbb{Z} \) and \( b - a = k \), \( d - c = \ell \). We identify a rectangle with the corresponding subgraph of \( \mathbb{Z}^2 \), including the boundary. A rectangle \( R \) has a horizontal open crossing if there is a path in \( R \) consisting of open edges, joining a vertex on the left-hand side of \( R \) to one on the right; we write \( H(R) \) for this event. Our starting point will be the following consequence of the Russo-Seymour-Welsh Lemma (see [1] and the references therein): there is a constant \( c > 0 \) such that
\[
\mathbb{P}_{1/2}(H(R)) \geq c,
\]
for any \( 3n \) by \( n \) rectangle \( R \). This is essentially the case \( \rho = 3 \) of Corollary 7 in [1]. (The latter result has an irrelevant restriction to \( n \) even; the present statement is immediate from the case \( \rho = 4 \) of this result.)

Our aim is to deduce Lemma 11 of [1], restated below.

**Lemma 2.** Let \( p > 1/2 \) be fixed. If \( R_n \) is a \( 3n \) by \( n \) rectangle, then \( \mathbb{P}_p(H(R_n)) \to 1 \) as \( n \to \infty \).

It is well known that Lemma 2 implies Kesten’s Theorem; see [1]. We shall deduce Lemma 2 from [1] using Theorem 1 and Harris’ result, that \( \theta(1/2) = 0 \). We shall need the concept of the dual lattice \( (\mathbb{Z}^2)^* \): this is the planar dual of the graph \( \mathbb{Z}^2 \), having a vertex for each face of \( \mathbb{Z}^2 \), and an edge \( e^* \) for each edge \( e \) of \( \mathbb{Z}^2 \), joining the two vertices corresponding to the faces of \( \mathbb{Z}^2 \) in whose boundary \( e \) lies. We take \( e^* \) to be open if and only if \( e \) is closed. The following argument is based on that of Russo [6].

**Proof of Lemma 2.** Let \( p_1 > 1/2 \) be fixed. Let \( D \) be a constant to be chosen below, and let \( R \) be a \( 3n \) by \( n \) rectangle with \( n \geq 2D + 1 \). Suppose that \( \omega \in \delta_e H(R) \), and define \( \omega_\pm \) as above. Note that \( e \) must be an edge of \( R \), as
$H(R)$ depends only on such edges. Then, in $\omega^+$ there is an open path in $R$ from the left-hand side to the right using the edge $e$. Hence, in $\omega$, the endpoints of $e$ are joined by open paths to the left- and right-hand sides of $R$. One of these paths must have length at least $(3n - 1)/2 \geq D$. Thus, for any $p$,

$$\mathbb{P}_p(\delta_e H(R)) \leq 2\mathbb{P}_p(0 \to D),$$

where $0 \to D$ is the event that there is an open path of length $D$ starting at the origin. Our assumption that $e$ is pivotal also implies that $H(R)$ does not hold in $\omega^-$. It follows (by Lemma 3 of [1]) that in $\omega^-$ there is an open path in the dual lattice joining the top of $R$ to the bottom, using the edge $e^*$. Hence, in the dual lattice, one of the endpoints of $e^*$ is in an open path of length at least $D$. As edges of the dual lattice are open independently with probability $1 - p$,

$$\mathbb{P}_p(\delta_e H(R)) \leq 2\mathbb{P}_{1-p}(0 \to D).$$

Let $0 < \varepsilon < \min\{(p_1 - 1/2)/2, c\}$ be arbitrary, where $c > 0$ is a constant for which (1) holds. Let $\eta = \eta(\varepsilon)$ be as in Theorem 1. For any $p$ we have $\mathbb{P}_p(0 \to D) \leq \eta$ as $D \to \infty$. Hence, by Harris’ Theorem (Theorem 8 in [1]), $\mathbb{P}_{1/2}(0 \to D) \to 0$, so we may choose $D$ such that $\mathbb{P}_{1/2}(0 \to D) \leq \eta/3$. As the event $0 \to D$ is increasing, for $p \leq 1/2$ we have

$$\mathbb{P}_p(0 \to D) \leq \mathbb{P}_{1/2}(0 \to D) \leq \eta/3.$$

Using (2) for $p \leq 1/2$ and (3) for $p \geq 1/2$, it follows that for any $p \in [0,1]$ and any edge $e$ in $R$ we have

$$\mathbb{P}_p(\delta_e H(R)) \leq 2\eta/3 < \eta.$$

As $H(R)$ is an increasing local event, and $\delta_e H(R)$ is empty for edges outside $R$, the conditions of Theorem 1 are satisfied. Hence, $\mathbb{P}_p(H(R))$ increases from at most $\varepsilon < c$ to at least $1 - \varepsilon$ in some interval of width at most $2\varepsilon < p_1 - 1/2$. As $\mathbb{P}_{1/2}(H(R)) \geq c$ by (1), it follows that $\mathbb{P}_{p_1}(H(R)) \geq 1 - \varepsilon$. In other words, we have shown that for $p_1 > 1/2$ and $\varepsilon > 0$ fixed and $R_n$ a $3n$ by $n$ rectangle, we have $\mathbb{P}_{p_1}(H(R_n)) \geq 1 - \varepsilon$ if $n$ is large enough. As $\varepsilon > 0$ is arbitrary, this completes the proof.

In Section 5 of [1], the Friedgut-Kalai sharp threshold result is used to deduce from (1) a result (Lemma 9 in [1]) that is somewhat stronger than Lemma 2. This stronger form was used in the first proof of Kesten’s Theorem given in [1]; however, in [1] two more very simple proofs are given, both of which need only Lemma 3.

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References

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