A Canonical Model Proof of Strong Completeness for BQL

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Abstract

I prove using a canonical model construction that a simple extension of Visser’s natural deduction system for Basic Propositional Logic is both sound and strongly complete with respect to Basic First-Order Logic (BQL). I utilize the canonical model construction to show that BQL satisfies both the Disjunction Property and the Existence Property.

1 Introduction

Basic Propositional Logic (first studied by Visser [4]) is obtained by dropping the requirement on the Kripke models for Intuitionistic Propositional Logic that the accessibility relation is reflexive. The most obvious way to add quantifiers to Basic Propositional Logic is to drop the corresponding reflexivity requirement on the Kripke models for Intuitionistic First-Order Logic. Unfortunately, the resulting logic invalidates the following extremely natural inferences:

\[ \forall v \phi(v) \quad (\forall-\text{Elim}) \]
\[ \forall v (\phi(v) \rightarrow \psi(v)) \rightarrow \forall v \phi(v) \rightarrow \forall v \psi(v) \quad (\forall-\text{Distribution}) \]
\[ \forall v (\phi(v) \rightarrow \psi(v)) \rightarrow \exists v \phi(v) \rightarrow \exists v \psi(v) \quad (\exists-\text{Conversion}) \]
\[ \forall \forall v (\phi(v)) \rightarrow \forall v \phi(v) \quad (\forall-\text{Permutation}) \]
where $\rho(\overline{v})$ is an arbitrary permutation of $\overline{v}$. We therefore need to find a better way of adding quantifiers to Basic Propositional Logic. A natural suggestion, tracing back to Restall [3], is to further modify the Kripke semantics for Intuitionistic First-Order Logic by, in addition to dropping reflexivity, restricting the models to those with constant domains and giving $\forall v \phi$ its Tarskian satisfaction condition. This results in Basic First-Order Logic (BQL), which successfully validates $\forall$-Elim, $\forall$-Distribution, $\exists$-Conversion and $\forall$-Permutation. In [2], Ishigaki and Kikuchi prove using a tree-sequent calculus that a sentence is valid in BQL iff it is provable in a simple Hilbert system. In this paper I go further by proving that a simple extension of Visser’s natural deduction system for Basic Proposition Logic [4] is both sound and strongly complete with respect to BQL. Furthermore, unlike Ishigaki and Kikuchi, I use a canonical model construction. I utilize the canonical model construction to show that BQL satisfies both the Disjunction Property and the Existence Property.

2 Basic First-Order Logic

Let $\mathcal{L}$ be a first-order language with primitive operators $\{\land, \lor, \to, \bot, \forall, \exists\}$ and no function symbols. We do not treat $=$ as a logical constant and consequently do not require that $\mathcal{L}$ contains $=$. Note, however, that $\mathcal{L}$ must contain at least one relation symbol.

2.1 Model Theory

In this section I give a semantic definition of BQL in terms of Kripke models and prove compactness using an ultraproduct construction. A transitive frame is a pair $\langle W, \prec \rangle$ such that $W$ is a non-empty set (the set of worlds) and $\prec$ is a transitive binary relation on $W$ (the accessibility relation). Let $V_\mathcal{L}$ be the set of constant symbols and relation symbols in $\mathcal{L}$. An $\mathcal{L}$-model (for BQL) is a 4-tuple $\mathcal{M} = \langle W, \prec, M, |\cdot| \rangle$ such that $\langle W, \prec \rangle$ is a transitive frame, $M$ is a non-empty set (the domain of $\mathcal{M}$) and $|\cdot|$ is a function with domain $V_\mathcal{L}$ such that $|c| \in M$ for every constant symbol $c$ and $|R^n| : W \to \mathcal{P}(M^n)$ for every $n$-ary relation symbol $R^n$, subject to the constraint that $w \prec u$ only if $|R^n|(w) \subseteq |R^n|(u)$. For a term $t(\overline{v}) \in \mathcal{L}$ and $\overline{a} \in M^n$, we let

$$|t|(\overline{a}) = \begin{cases} |c| & \text{if } t = c \\ a_i & \text{if } t = v_i. \end{cases}$$
For a formula $\phi(\overline{a}) \in \mathcal{L}$, $\overline{a} \in M^n$ and $w \in W$, we recursively define the satisfaction relation $\mathcal{M}, w \models \phi(\overline{a})$ as follows:

$$
\mathcal{M}, w \models \bot(\overline{a}) \\
\mathcal{M}, w \models R^n(t_1, \ldots, t_n)(\overline{a}) \iff \langle |t_1|(|\overline{a}|), \ldots, |t_n|(|\overline{a}|) \rangle \in |R^n|(w) \\
\mathcal{M}, w \models (\phi \land \psi)(\overline{a}) \iff \mathcal{M}, w \models \phi(\overline{a}) \text{ and } \mathcal{M}, w \models \psi(\overline{a}) \\
\mathcal{M}, w \models (\phi \lor \psi)(\overline{a}) \iff \mathcal{M}, w \models \phi(\overline{a}) \text{ or } \mathcal{M}, w \models \psi(\overline{a}) \\
\mathcal{M}, w \models (\phi \rightarrow \psi)(\overline{a}) \iff \text{ for all } u > w : \text{ if } \mathcal{M}, u \models \phi(\overline{a}) \text{ then } \mathcal{M}, u \models \psi(\overline{a}) \\
\mathcal{M}, w \models \exists \psi(\overline{a}) \iff \text{ for some } b \in M : \mathcal{M}, w \models \phi(b, \overline{a}) \\
\mathcal{M}, w \models \forall \psi(\overline{a}) \iff \text{ for all } b \in M : \mathcal{M}, w \models \phi(b, \overline{a}).
$$

**Theorem 1.** (Persistence Theorem) If $\mathcal{M}, w \models \phi(\overline{a})$ and $w \prec u$ then $\mathcal{M}, u \models \phi(\overline{a})$.

**Proof.** A straightforward induction on the construction of $\mathcal{L}$-formulas. \qed

For sentences $\Gamma \cup \{\phi\} \subseteq \mathcal{L}$, we say $\Gamma \models_{\text{BQL}} \phi$ iff for every $\mathcal{L}$-model $\mathcal{M}$ and every world $w \in \mathcal{M}$, if $\mathcal{M}, w \models \Gamma$ then $\mathcal{M}, w \models \phi$. Let $\{\mathcal{M}_i\}_{i \in I}$ be a non-empty family of $\mathcal{L}$-models. For an ultrafilter $U$ on $I$, the ultraproduct of $\{\mathcal{M}_i\}_{i \in I}$ over $U$ is the $\mathcal{L}$-model $\mathcal{M}_U = \langle \prod_i W_i, \prec, \prod_i M_i, |\cdot| \rangle$ such that $w \prec u$ iff $\{i : w_i \prec_i u_i\} \in U$, $|c| = \{|c_i\}_{i \in I}$ and $\overline{a} \in |R^n|(w)$ iff $\{i : \overline{a}_i \in |R^n_i|(w_i)\} \in U$. Note that

$$
w \prec u \prec z \implies \{i : w_i \prec_i u_i\} \in U \text{ and } \{i : u_i \prec_i z_i\} \in U \\
\implies \{i : w_i \prec_i u_i \prec_i z_i\} \in U \\
\implies \{i : w_i \prec_i z_i\} \in U \\
\implies w \prec z.
$$

So $\langle \prod_i W_i, \prec \rangle$ is in fact a transitive frame. To verify that $\mathcal{M}_U$ is in fact an $\mathcal{L}$-model, suppose $w \prec u$ and $\overline{a} \in |R^n|(w)$. Then $\{i : w_i \prec_i u_i\} \in U$ and $\{i : \overline{a}_i \in |R^n_i|(w_i)\} \in U$. So $\{i : w_i \prec_i u_i \text{ and } \overline{a}_i \in |R^n_i|(w_i)\} \in U$. But then $\{i : \overline{a}_i \in |R^n_i|(u_i)\} \in U$. So $\overline{a} \in |R^n|(u)$. 

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Lemma 1. (Union Lemma) If $X \cup Y \in U$ then $X \in U$ or $Y \in U$.

Proof. Suppose $X \cup Y \in U$. Suppose for a reductio that $X \notin U$ and $Y \notin U$. Then $\neg X \in U$ and $\neg Y \in U$. So $\neg (X \cup Y) = \neg X \cap \neg Y \in U$. But then $\emptyset = (X \cup Y) \cap (X \cup Y) \in U$, which is a contradiction. \hfill \Box

Lemma 2. (Ultraproduct Lemma) $M_\pi, w \models \phi(\overline{a})$ iff $\{i : M_i, w_i \models \phi(\overline{a}_i)\} \in U$.

Proof. By induction on the construction of $L$-formulas.

Base Cases $M_\pi, w \not\models \bot(\overline{a})$ and $\{i : M_i, w_i \models \bot(\overline{a}_i)\} = \emptyset \notin U$. Also, we have

$M_\pi, w \models R^n(t_1, ..., t_n)(\overline{a}) \iff \langle t_1|\overline{a}, ..., t_n|\overline{a} \rangle \in |R^n| (w)$

$\iff \{i : \langle t_1|\overline{a}_i, ..., t_n|\overline{a}_i \rangle \in |R^n| (w_i)\} \in U$

$\iff \{i : \langle t_1|\overline{a}_i, ..., t_n|\overline{a}_i \rangle \in |R^n| (w_i)\} \in U$

$\iff \{i : M_i, w_i \models R^n(t_1, ..., t_n)(\overline{a}_i)\} \in U$.

Induction Steps The induction steps for conjunction and disjunction are straightforward (we use the Union Lemma to get the disjunction step through).

Conditional Suppose $M_\pi, w \not\models (\phi \rightarrow \psi)(\overline{a})$. Then there exists $u \succ w$ such that $M_\pi, u \models \phi(\overline{a})$ and $M_\pi, u \not\models \psi(\overline{a})$. Hence $\{i : w_i \prec_i u_i\} \in U$. Also, by the induction hypothesis, $\{i : M_i, u_i \models \phi(\overline{a}_i)\} \in U$ and $\{i : M_i, u_i \not\models \psi(\overline{a}_i)\} \notin U$. So $\{i : M_i, u_i \not\models \phi(\overline{a}_i)\} \in U$ and $\{i : M_i, u_i \not\models \psi(\overline{a}_i)\} \in U$. Hence $\{i : M_i, w_i \not\models (\phi \rightarrow \psi)(\overline{a}_i)\} \notin U$.

For the converse, suppose $\{i : M_i, w_i \models (\phi \rightarrow \psi)(\overline{a}_i)\} \notin U$. Then $\{i : M_i, w_i \not\models (\phi \rightarrow \psi)(\overline{a}_i)\} \notin U$. So $\{i : M_i, u \models \phi(\overline{a}_i)\}$ and $\{i : M_i, u \not\models \psi(\overline{a}_i)\}$ for some $u \succ_i w_i \in U$. Choose $z_i \in \prod_i W_i$ such that

\[
z_i = \begin{cases} u \text{ such that } w_i \prec_i u \text{ and } M_i, u \models \phi(\overline{a}_i) \text{ and } M_i, u \not\models \psi(\overline{a}_i) \text{ if such a } u \text{ exists} \\ 
\text{anything otherwise.}
\end{cases}
\]

Then $\{i : M_i, z_i \models \phi(\overline{a}_i)\} \in U$ and $\{i : M_i, z_i \not\models \psi(\overline{a}_i)\} \notin U$. Hence, by the induction hypothesis, $M_\pi, z \models \phi(\overline{a})$ and $M_\pi, z \not\models \psi(\overline{a})$. Also, $\{i : w_i \prec_i z_i\} \in U$. So $w \prec z$. But then $M_\pi, w \not\models (\phi \rightarrow \psi)(\overline{a})$. \hfill \Box
Existential Quantifier

\[
\mathcal{M}_\pi, w \models \exists v \phi(\overline{a}) \implies \mathcal{M}_\pi, w \models \phi(\overline{b}, \overline{a}) \text{ for some } b \in \prod_i M_i \\
\implies \{ i : \mathcal{M}_i, w_i \models \phi(b_i, \overline{a_i}) \} \in \mathcal{U} \text{ for some } b \in \prod_i M_i \\
\implies \{ i : \mathcal{M}_i, w_i \models \exists v \phi(\overline{a_i}) \} \in \mathcal{U}.
\]

For the converse, suppose \( \{ i : \mathcal{M}_i, w_i \models \exists v \phi(\overline{a_i}) \} \in \mathcal{U} \). Then \( \{ i : \mathcal{M}_i, w_i \models \phi(b, \overline{a}) \text{ for some } b \in M_i \} \in \mathcal{U} \). Choose \( d \in \prod_i M_i \) such that

\[
d_i = \begin{cases} 
  b \text{ such that } \mathcal{M}_i, w_i \models \phi(b, \overline{a_i}) \text{ if such a } b \text{ exists} \\
  \text{anything otherwise.}
\end{cases}
\]

Then \( \{ i : \mathcal{M}_i, w_i \models \phi(d_i, \overline{a_i}) \} \in \mathcal{U} \). So, by the induction hypothesis, \( \mathcal{M}_\pi, w \models \phi(\overline{d}, \overline{a}) \).

But then \( \mathcal{M}_\pi, w \models \exists v \phi(\overline{a}) \).

Universal Quantifier

\[
\{ i : \mathcal{M}_i, w_i \models \forall v \phi(\overline{a_i}) \} \in \mathcal{U} \implies \{ i : \mathcal{M}_i, w_i \models \phi(b, \overline{a}) \text{ for all } b \in M_i \} \in \mathcal{U} \\
\implies \{ i : \mathcal{M}_i, w_i \models \phi(b, \overline{a_i}) \} \in \mathcal{U} \text{ for all } b \in \prod_i M_i \\
\implies \mathcal{M}_\pi, w \models \phi(b, \overline{a}) \text{ for all } b \in \prod_i M_i \\
\implies \mathcal{M}_\pi, w \models \forall v \phi(\overline{a}).
\]

For the converse, suppose \( \{ i : \mathcal{M}_i, w_i \models \forall v \phi(\overline{a_i}) \} \not\in \mathcal{U} \). Then \( \{ i : \mathcal{M}_i, w_i \not\models \forall v \phi(\overline{a_i}) \} \in \mathcal{U} \). So \( \{ i : \mathcal{M}_i, w_i \not\models \phi(b, \overline{a_i}) \text{ for some } b \in M_i \} \in \mathcal{U} \). Choose \( d \in \prod_i M_i \) such that

\[
d_i = \begin{cases} 
  b \text{ such that } \mathcal{M}_i, w_i \not\models \phi(b, \overline{a_i}) \text{ if such a } b \text{ exists} \\
  \text{anything otherwise.}
\end{cases}
\]

Then \( \{ i : \mathcal{M}_i, w_i \not\models \phi(d_i, \overline{a_i}) \} \in \mathcal{U} \). So \( \{ i : \mathcal{M}_i, w_i \not\models \phi(d_i, \overline{a_i}) \} \not\in \mathcal{U} \). Hence, by the induction hypothesis, \( \mathcal{M}_\pi, w \not\models \phi(\overline{d}, \overline{a}) \). But then \( \mathcal{M}_\pi, w \not\models \forall v \phi(\overline{a}) \).

X \subseteq P(I) \text{ is said to have the Finite Intersection Property (FIP) just in case for all } S_1, \ldots, S_n \in X, S_1 \cap \ldots \cap S_n \neq \emptyset.$$
**Lemma 3.** (FIP Lemma) Every non-empty $X \subseteq \mathcal{P}(I)$ with the FIP can be extended to an ultrafilter on $I$.

**Proof.** Let $X^* = \{ Y \subseteq I : S_1 \cap \ldots \cap S_n \subseteq Y \text{ for some } S_1, \ldots, S_n \in X \}$. Then $X^* \supseteq X$ is a filter on $I$. So, by Zorn’s Lemma, $X^*$ can be extended to an ultrafilter on $I$. \qed

**Theorem 2.** (Compactness) If $\Gamma \models_{\text{BQL}} \phi$ then $\Delta \models_{\text{BQL}} \phi$ for some finite $\Delta \subseteq \Gamma$.

**Proof.** For finite $\Gamma$ the claim is trivial. So let $\Gamma$ be infinite and suppose $\Delta \not\models_{\text{BQL}} \phi$ for every finite $\Delta \subseteq \Gamma$. Let $I$ denote the set of finite subsets of $\Gamma$. Then we can find a family $\{ (\mathcal{M}_\Delta, w_\Delta) \}_{\Delta \in I}$ such that $\mathcal{M}_\Delta, w_\Delta \models \Delta$ and $\mathcal{M}_\Delta, w_\Delta \not\models \phi$. Clearly, $X = \{ \{ \Delta \in I : \psi \in \Delta \} : \psi \in \Gamma \}$ is non-empty and has the FIP. So, by the FIP Lemma, there exists an ultrafilter $U \supseteq X$ on $I$. Let $\mathcal{M}_\pi$ be the ultraproduct of $\{ \mathcal{M}_\Delta \}_{\Delta \in I}$ over $U$ and let $w = \{ w_\Delta \}_{\Delta \in I}$. For $\psi \in \Gamma$, we have $\{ \Delta \in I : \mathcal{M}_\Delta, w_\Delta \models \psi \} \supseteq \{ \Delta \in I : \psi \in \Delta \} \in U$. So, by the Ultraproduct Lemma, $\mathcal{M}_\pi, w \models \Gamma$. On the other hand, $\{ \Delta \in I : \mathcal{M}_\Delta, w_\Delta \models \phi \} = \emptyset \notin U$. Hence, by the Ultraproduct Lemma, $\mathcal{M}_\pi, w \not\models \phi$. So $\Gamma \not\models_{\text{BQL}} \phi$. \qed

### 2.2 Proof Theory

From now on I suppose that $\mathcal{L}$ is countable and contains $\omega$-many constant symbols. The natural deduction system $\mathcal{N}_{\text{BQL}}$ consists of all proofs in $\mathcal{L}$ constructed using the following inference rules (note that we do not allow open formulas to occur in proofs):

\[
\begin{align*}
\vdash \phi & \quad (\bot-\text{Elim}) \\
\phi, \psi & \quad (\land-\text{Int}) \\
\phi \land \psi & \quad (\land-\text{Elim}_1) \\
\phi \land \psi & \quad (\land-\text{Elim}_2) \\
\psi & \quad (\lor-\text{Int}_1) \\
\phi \lor \psi & \quad (\lor-\text{Int}_2) \\
\phi \lor \psi & \quad (\lor-\text{Elim}) \\
\end{align*}
\]
\[
\begin{array}{ll}
\phi \rightarrow \psi & (\rightarrow\text{-Int}) \\
\phi \rightarrow \psi \rightarrow \chi & (\text{Internal Transitivity})
\end{array}
\]

\[
\begin{array}{ll}
\phi \rightarrow \psi \rightarrow \chi & (\text{Internal } \land\text{-Int}) \\
\phi \rightarrow \chi \rightarrow \psi & (\text{Internal } \lor\text{-Elim})
\end{array}
\]

\[
\begin{array}{ll}
\forall v(\phi \rightarrow \psi(v)) & (\text{Internal } \forall\text{-Int}) \\
\phi \rightarrow \forall v\psi(v) & (\exists\text{-Elim})
\end{array}
\]

\[
\begin{array}{ll}
\forall v(\phi \lor \psi(v)) & (\text{CD})
\end{array}
\]

\[
\begin{array}{ll}
\forall v(\phi \lor \psi(v)) & (\exists\text{-Int}) \\
\exists v\phi(v) & (\exists\text{-Elim})
\end{array}
\]

For sentences \( \Gamma \cup \{ \phi \} \subseteq \mathcal{L} \), we say \( \Gamma \vdash_{\text{BQL}} \phi \) iff there exists a proof of \( \phi \) from \( \Gamma \) in \( \mathcal{N}_{\text{BQL}} \).

**Proposition 1.** (Distribution) \( \phi \land (\psi \lor \chi) \vdash_{\text{BQL}} (\phi \land \psi) \lor (\phi \land \chi) \)
Proof.

\[
\begin{align*}
\frac{\phi \land (\psi \lor \chi)}{
\phi} \\
\frac{\phi \land \psi}{\phi} \\
\frac{\phi \land (\psi \lor \chi)}{\phi} \\
\frac{\phi \land \chi}{\phi}
\end{align*}
\]

\[
\begin{align*}
\frac{\psi \lor \chi}{\phi \land (\psi \lor \chi)} \\
\frac{(\phi \land \psi) \lor (\phi \land \chi)}{(\phi \land \psi) \lor (\phi \land \chi)}
\end{align*}
\]

\[
\frac{(\phi \land \psi) \lor (\phi \land \chi)}{\phi \land (\psi \lor \chi)}
\]

\[
\frac{\phi \land (\psi \lor \chi)}{\phi \land \psi}
\]

Proposition 2. (Infinite Distribution) \(\phi \land \exists v \psi(v) \vdash_{\text{BQL}} \exists v (\phi \land \psi(v))\)

Proof.

\[
\begin{align*}
\frac{\phi \land \exists v \psi(v)}{\exists v \psi(v)} \\
\frac{\phi \land \exists v \psi(v)}{\exists v (\phi \land \psi(v))}
\end{align*}
\]

For sentences \(\Sigma \subseteq L\), we let \(\text{N}BQL(\Sigma)\) denote the natural deduction system obtained by adding

\[
\frac{\Sigma}{\phi \to \psi}
\]

to \(\text{N}BQL\). We say \(\Gamma \vdash_{\text{BQL}(\Sigma)} \phi\) iff there exists a proof of \(\phi\) from \(\Gamma\) in \(\text{N}BQL(\Sigma)\). The next theorem is the key to proving completeness. In this paper I always let \(\bot \lor \bot = \bot \to \bot\) and abbreviate \(\bot \to \bot\) as \(\top\).

Theorem 3. (Relative Deduction Theorem) For finite \(\Gamma\): \(\Sigma \cup \Gamma \vdash_{\text{BQL}(\Sigma)} \phi\) iff \(\Sigma \vdash_{\text{BQL}} \Gamma \land \Gamma \to \phi\).
Proof. \(\iff\) Suppose \(\Sigma \vdash_{\text{BQL}} \Gamma \rightarrow \phi\). There are two cases.

Case 1 \(\Gamma \neq \emptyset\). Then we can find a proof of the form

\[
\frac{\Gamma \land \text{-Ints}}{\land \Gamma \rightarrow \phi} \quad \frac{\Sigma}{\land \Gamma \rightarrow \phi}
\]

in \(\text{N\text{-BQL}}(\Sigma)\).

Case 2 \(\Gamma = \emptyset\). Then we can find a proof of the form

\[
\frac{\bot}{\bot \rightarrow \bot} \quad \frac{\Sigma}{\Gamma \rightarrow \phi}
\]

in \(\text{N\text{-BQL}}(\Sigma)\).

\(\implies\) By induction on the construction of proofs in \(\text{N\text{-BQL}}(\Sigma)\).

Base Case Suppose we have a one-line proof in \(\text{N\text{-BQL}}(\Sigma)\) of \(\phi\) from \(\Sigma \cup \Gamma\). Then \(\phi \in \Sigma\) or \(\phi \in \Gamma\). If \(\phi \in \Sigma\) then

\[
\frac{\phi}{\land \Gamma \rightarrow \phi}
\]

is a proof of \(\land \Gamma \rightarrow \phi\) from \(\Sigma\) in \(\text{N\text{-BQL}}\). If \(\phi \in \Gamma\) then

\[
\frac{\land \Gamma}{\land \text{-Elims}} \frac{\phi}{\land \Gamma \rightarrow \phi}
\]

is a proof of \(\land \Gamma \rightarrow \phi\) from \(\Sigma\) in \(\text{N\text{-BQL}}\).

Induction Steps

Case 1 Suppose we have a proof of the form
in NBQL(Σ), where the final inference is ⊥-Elim, ∧-Elim₁, ∧-Elim₂, ∨-Int₁, ∨-Int₂, Internal ∀-Int, Internal ∃-Elim, ∀-Elim, CD or ∃-Int. Then, by the induction hypothesis, we can find a proof of the form

\[
\begin{array}{c}
\Sigma, \Gamma \\
\vdots \\
\overset{\phi}{\alpha}
\end{array}
\]

\begin{array}{c}
\Lambda \Gamma \rightarrow \alpha \\
\phi
\end{array}

\[
\frac{\alpha \rightarrow \phi}{\Lambda \Gamma \rightarrow \phi}
\]

in NBQL.

**Case 2** Suppose we have a proof of the form

\[
\begin{array}{c}
\Sigma, \Gamma \\
\vdots \\
\overset{\phi}{\alpha}
\end{array}
\]

\[
\begin{array}{c}
\Sigma, \Gamma \\
\vdots \\
\overset{\phi}{\beta}
\end{array}
\]

in NBQL(Σ), where the final inference is ∧-Int, Internal Transitivity, Internal ∧-Int or Internal ∨-Elim. Then, by the induction hypothesis, we can find a proof of the form

\[
\begin{array}{c}
\Sigma, \Gamma \\
\vdots \\
\overset{\phi}{\alpha \wedge \beta}
\end{array}
\]

\[
\begin{array}{c}
\Sigma, \Gamma \\
\vdots \\
\overset{\phi}{\alpha \wedge \beta}
\end{array}
\]

\[
\frac{\alpha \wedge \beta}{\alpha \rightarrow \phi}
\]

\[
\frac{\alpha \wedge \beta}{\beta}
\]

\[
\frac{\alpha \wedge \beta}{\phi}
\]

\[
\frac{\alpha \wedge \beta}{\alpha \wedge \beta \rightarrow \phi}
\]

in NBQL.

**Case 3** Suppose we have a proof of the form
in $\mathcal{AL}BQL(\Sigma)$. There are two subcases.

**Subcase 1** $\Gamma = \emptyset$. Then, by the induction hypothesis, we can find a proof of the form

$\Sigma, \Gamma \vdash \phi \lor \psi$

$\Gamma \vdash \chi$

in $\mathcal{AL}BQL$.

**Subcase 2** $\Gamma \neq \emptyset$. Then, by Distribution and the induction hypothesis, we can find a proof of the form

$\Gamma \vdash \phi \lor \psi$

$\Gamma \vdash \chi$

in $\mathcal{AL}BQL$. So, by the induction hypothesis, we can find a proof of the form

$\Gamma \vdash \chi$

in $\mathcal{AL}BQL$.

**Case 4** Suppose we have a proof of the form
$\Sigma, \Gamma, \overline{\varphi}$

\[
\begin{array}{c}
\vdots \\
\psi \\
\varphi \rightarrow \psi \\
\Gamma \rightarrow (\varphi \rightarrow \psi)
\end{array}
\]

in $\mathcal{N}BQL(\Sigma)$. There are two subcases.

**Subcase 1** $\Gamma = \emptyset$. Then, by the induction hypothesis, we can find a proof of the form

\[
\begin{array}{c}
\Sigma \\
\vdots \\
\varphi \rightarrow \psi \\
\Gamma \rightarrow (\varphi \rightarrow \psi)
\end{array}
\]

in $\mathcal{N}BQL$.

**Subcase 2** $\Gamma \neq \emptyset$. Then, by the induction hypothesis, we can find a proof of the form

\[
\begin{array}{c}
\Lambda \Gamma \\
\Lambda \Gamma \land \varphi \\
\varphi \rightarrow \Lambda \Gamma \land \varphi \\
\Lambda \Gamma \land \varphi \rightarrow \psi \\
\Lambda \Gamma \rightarrow (\varphi \rightarrow \psi)
\end{array}
\]

in $\mathcal{N}BQL$.

**Case 5** Suppose we have a proof of the form

\[
\begin{array}{c}
\Sigma, \Gamma \\
\vdots \\
\varphi \\
\varphi \rightarrow \psi \\
\psi
\end{array}
\]

in $\mathcal{N}BQL(\Sigma)$. Then, by the induction hypothesis, we can find a proof of the form

\[
\begin{array}{c}
\Sigma \\
\vdots \\
\Lambda \Gamma \rightarrow \varphi \\
\Lambda \Gamma \rightarrow \psi
\end{array}
\]

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in \( \mathcal{N} \text{BQL} \).

**Case 6** Suppose we have a proof of the form

\[
\frac{\Sigma, \Gamma \quad \phi(c)}{\forall v \phi(v)}
\]

in \( \mathcal{N} \text{BQL}(\Sigma) \). Then, by the induction hypothesis, we can find a proof of the form

\[
\frac{\Sigma \quad \bigwedge \Gamma \rightarrow \phi(c)}{\forall v (\bigwedge \Gamma \rightarrow \phi(v))}
\]

\[
\frac{\forall v (\bigwedge \Gamma \rightarrow \phi(v)) \quad \bigwedge \Gamma \rightarrow \forall v \phi(v)}{\bigwedge \Gamma \rightarrow \forall v \phi(v)}
\]

in \( \mathcal{N} \text{BQL} \).

**Case 7** Suppose we have a proof of the form

\[
\begin{array}{c}
\Sigma, \Gamma \\
\vdots
\end{array}
\begin{array}{c}
\Sigma, \Gamma, \phi(c) \\
\vdots
\end{array}
\begin{array}{c}
\exists v \phi(v) \\
\vdots
\end{array}
\begin{array}{c}
\psi
\end{array}
\]

\[
\frac{\phi(c) \rightarrow \psi}{\forall v (\phi(c) \rightarrow \psi)}
\]

\[
\frac{\forall v (\phi(c) \rightarrow \psi) \quad \exists v \phi(v) \rightarrow \psi}{\exists v \phi(v) \rightarrow \psi}
\]

\[
\frac{\exists v \phi(v) \rightarrow \psi \quad \top \rightarrow \exists v \phi(v)}{\top \rightarrow \psi}
\]

in \( \mathcal{N} \text{BQL} \). There are two subcases.

**Subcase 1** \( \Gamma = \emptyset \). Then, by the induction hypothesis, we can find a proof of the form

\[
\begin{array}{c}
\Sigma
\end{array}
\begin{array}{c}
\vdots
\end{array}
\begin{array}{c}
\phi(c) \rightarrow \psi
\end{array}
\]

\[
\frac{\forall v (\phi(c) \rightarrow \psi) \quad \exists v \phi(v) \rightarrow \psi}{\exists v \phi(v) \rightarrow \psi}
\]

\[
\frac{\exists v \phi(v) \rightarrow \psi \quad \top \rightarrow \exists v \phi(v)}{\top \rightarrow \psi}
\]

in \( \mathcal{N} \text{BQL} \).
Subcase 2 $\Gamma \neq \emptyset$. Then, by Infinite Distribution and the induction hypothesis, we can find a proof of the form

\[
\frac{\bigwedge \Gamma \land \exists \forall \phi(v)}{\exists v(\bigwedge \Gamma \land \phi(v))} \quad \frac{\bigwedge \Gamma \land \phi(c) \rightarrow \psi}{\bigwedge \Gamma \land \exists \forall \phi(v) \rightarrow \psi}
\]

\[
\frac{\bigwedge \Gamma \land \exists \forall \phi(v) \rightarrow \exists v(\bigwedge \Gamma \land \phi(v))}{\bigwedge \Gamma \land \exists \forall \phi(v) \rightarrow \psi}
\]

in $\mathcal{L}'BQL$. So, by the induction hypothesis, we can find a proof of the form

\[
\frac{\bigwedge \Gamma}{\bigwedge \Gamma \rightarrow \bigwedge \Gamma} \quad \frac{\bigwedge \Gamma \rightarrow \exists v \phi(v)}{\bigwedge \Gamma \rightarrow \exists v \phi(v) \rightarrow \psi} \quad \frac{\bigwedge \Gamma \rightarrow \exists v \phi(v) \rightarrow \psi}{\bigwedge \Gamma \rightarrow \psi}
\]

in $\mathcal{L}'BQL$. \qed

**Proposition 3.** (Relative Cut) For finite $\Gamma, \Pi$: if $\Sigma \cup \Gamma \vdash_{BQL(\Sigma)} \phi$ and $\Sigma \cup \{\phi\} \cup \Pi \vdash_{BQL(\Sigma)} \psi$ then $\Sigma \cup \Gamma \cup \Pi \vdash_{BQL(\Sigma)} \psi$.

**Proof.** Suppose $\Sigma \cup \Gamma \vdash_{BQL(\Sigma)} \phi$ and $\Sigma \cup \{\phi\} \cup \Pi \vdash_{BQL(\Sigma)} \psi$. There are two cases.

**Case 1** $\Pi = \emptyset$. Then, by the Relative Deduction Theorem, $\Sigma \vdash_{BQL} \bigwedge \Gamma \rightarrow \phi$ and $\Sigma \vdash_{BQL} \phi \rightarrow \psi$. So, by Internal Transitivity, $\Sigma \vdash_{BQL} \bigwedge \Gamma \rightarrow \psi$. But then, by the Relative Deduction Theorem, $\Sigma \cup \Gamma \vdash_{BQL(\Sigma)} \psi$.

**Case 2** $\Pi \neq \emptyset$. Then, by the Relative Deduction Theorem, $\Sigma \vdash_{BQL} \bigwedge \Gamma \rightarrow \phi$ and $\Sigma \vdash_{BQL} \phi \land \bigwedge \Pi \rightarrow \psi$. But we can construct the following proof in $\mathcal{L}'BQL$:  

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So $\Sigma \vdash_{\text{BQL}} \Gamma \land \Pi \rightarrow \psi$. There are two subcases.

Subcase 1 $\Gamma \neq \emptyset$. Then, by the Relative Deduction Theorem, $\Sigma \cup \Gamma \cup \Pi \vdash_{\text{BQL}(\Sigma)} \psi$.

Subcase 2 $\Gamma = \emptyset$. Then we can construct the following proof in $\mathcal{N}$BQL:

So $\Sigma \vdash_{\text{BQL}} \Pi \rightarrow \psi$. Hence, by the Relative Deduction Theorem, $\Sigma \cup \Pi \vdash_{\text{BQL}(\Sigma)} \psi$. □

3 Soundness and Completeness

Theorem 4. (Soundness) If $\Gamma \vdash_{\text{BQL}} \phi$ then $\Gamma \models_{\text{BQL}} \phi$.

Proof. By induction on the construction of proofs in $\mathcal{N}$BQL. The base case is straightforward. The induction steps are also straightforward except for the step for $\rightarrow$-Int, where we need to appeal to the Persistence Theorem.

$\rightarrow$-Int Suppose we have a proof of the form

$$
\Sigma, \phi \\
\vdots \\
\psi \\
\phi \rightarrow \psi
$$
in $\mathcal{N}^\text{BQL}$. Then, by the induction hypothesis, we have that $\Sigma, \phi \vdash_{\text{BQL}} \psi$. Suppose for a reductio that $\Sigma \not\vdash_{\text{BQL}} \phi \rightarrow \psi$. Then there exists an $\mathcal{L}$-model $\mathcal{M}$ such that for some world $w \in \mathcal{M}$, $\mathcal{M}, w \models \Sigma$ and $\mathcal{M}, w \not\models \phi \rightarrow \psi$. Hence there exists $u \succ w$ such that $\mathcal{M}, u \models \phi$ and $\mathcal{M}, u \not\models \psi$. So, by the Persistence Theorem, $\mathcal{M}, u \models \Sigma$. But then $\mathcal{M}, u \models \psi$, which is a contradiction. \qed

The proof of completeness is more involved. We adapt the canonical model argument of Visser [4] to handle quantification, following the method used by Gabbay, Shehtman and Skvortsov [1] to prove weak completeness for Constant Domain Intuitionistic First-Order Logic. Let $S \in \{\text{BQL}\} \cup \{\text{BQL}(\Sigma) : \Sigma \subseteq \mathcal{L}\}$. A set of sentences $\Gamma \subseteq \mathcal{L}$ is called a prime saturated $S$-theory just in case $\Gamma$ satisfies the following properties:

(Consistency) $\bot \not\in \Gamma$,
(S-Closure) $\Gamma \vdash_S \phi \implies \phi \in \Gamma$,
(Disjunction Property) $\phi \lor \psi \in \Gamma \implies \phi \in \Gamma$ or $\psi \in \Gamma$,
(Existence Property) $\exists v \phi(v) \in \Gamma \implies \phi(c) \in \Gamma$ for some $c \in \mathcal{L}$.
(Totality Property) $\phi(c) \in \Gamma$ for every $c \in \mathcal{L} \implies \forall v \phi(v) \in \Gamma$.

Let $\text{Sat}(S)$ denote the set of prime saturated $S$-theories.

**Lemma 4.** (Finite Extension Lemma) For finite $\Gamma$: if $\Gamma \not\vdash_{\text{BQL}} \phi$ there exists $\Gamma^* \supseteq \Gamma$ such that $\Gamma^* \in \text{Sat}(\text{BQL})$ and $\phi \not\in \Gamma^*$.

*Proof.* Suppose $\Gamma \not\vdash_{\text{BQL}} \phi$. Let $\{\psi_n\}_{n \in \omega}$ be an enumeration of $\mathcal{L}$-sentences ($\mathcal{L}$ is countable by assumption). We inductively define a pair $\{\Gamma_n\}_{n \in \omega}, \{\Delta_n\}_{n \in \omega}$ of increasing sequences of sets of sentences $\Gamma_n, \Delta_n \subseteq \mathcal{L}$ as follows, where $\Pi_n = \Gamma_n \cup \Delta_n \cup \{\psi_n\}$:
\[\Gamma_0 = \Gamma\]

\[
\Gamma_{n+1} = \begin{cases} 
\Gamma_n \text{ if } \Gamma_n \cup \{\psi_n\} \vdash_{\text{BQL}} \bigvee \Delta_n, \\
\Gamma_n \cup \{\psi_n\} \text{ if } \Gamma_n \cup \{\psi_n\} \not\vdash_{\text{BQL}} \bigvee \Delta_n \text{ and } \psi_n \neq \exists v \chi(v), \\
\Gamma_n \cup \{\psi_n, \chi(c)\} \text{ for } c \in \mathcal{L} \setminus \Pi_n \text{ if } \Gamma_n \cup \{\psi_n\} \vdash_{\text{BQL}} \bigvee \Delta_n \text{ and } \psi_n = \exists v \chi(v) 
\end{cases}
\]

\[\Delta_0 = \{\phi\}\]

\[
\Delta_{n+1} = \begin{cases} 
\Delta_n \text{ if } \Gamma_n \cup \{\psi_n\} \not\vdash_{\text{BQL}} \bigvee \Delta_n, \\
\Delta_n \cup \{\psi_n\} \text{ if } \Gamma_n \cup \{\psi_n\} \vdash_{\text{BQL}} \bigvee \Delta_n \text{ and } \psi_n \neq \forall v \chi(v), \\
\Delta_n \cup \{\psi_n, \chi(c)\} \text{ for } c \in \mathcal{L} \setminus \Pi_n \text{ if } \Gamma_n \cup \{\psi_n\} \not\vdash_{\text{BQL}} \bigvee \Delta_n \text{ and } \psi_n = \forall v \chi(v). 
\end{cases}
\]

This construction is well-defined because for every \(n\), \(\Pi_n\) is finite and so \(\bigvee \Delta_n\) exists and there are \(\omega\)-many constant symbols in \(\mathcal{L} \setminus \Pi_n\).

**Lemma 5.** (Finite Separation Lemma) For all \(n\): \(\Gamma_n \not\vdash_{\text{BQL}} \bigvee \Delta_n\).

**Proof.** By induction on \(n\). The base case is immediate. For the induction step, suppose \(\Gamma_n \not\vdash_{\text{BQL}} \bigvee \Delta_n\). There are two cases.

**Case 1** \(\Gamma_n \cup \{\psi_n\} \not\vdash_{\text{BQL}} \bigvee \Delta_n\). Then \(\Delta_{n+1} = \Delta_n\). So if \(\psi_n \neq \exists v \chi(v)\) then \(\Gamma_{n+1} = \Gamma_n \cup \{\psi_n\}\) and we're done. Suppose, then, that \(\psi_n = \exists v \chi(v)\). Then \(\Gamma_{n+1} = \Gamma_n \cup \{\exists v \chi(v), \chi(c)\}\) for some \(c \not\in \Gamma_n \cup \Delta_n \cup \{\chi(v)\}\). Suppose for a reductio that \(\Gamma_{n+1} \vdash_{\text{BQL}} \bigvee \Delta_{n+1}\). Then we can construct the following proof in \(\mathcal{N}\text{BQL}\):

\[
\begin{array}{c}
\Gamma_n, \exists v \chi(v), \chi(c) \\
\vdots \ \\
\exists v \chi(v) \\
\bigvee \Delta_n \\
\end{array}
\]

So \(\Gamma_n \cup \{\psi_n\} \not\vdash_{\text{BQL}} \bigvee \Delta_n\), which is a contradiction.

**Case 2** \(\Gamma_n \cup \{\psi_n\} \vdash_{\text{BQL}} \bigvee \Delta_n\). Then \(\Gamma_{n+1} = \Gamma_n\). There are two subcases.

**Subcase 1** \(\psi_n \neq \forall v \chi(v)\). Then \(\Delta_{n+1} = \Delta_n \cup \{\psi_n\}\). Suppose for a reductio that \(\Gamma_{n+1} \vdash_{\text{BQL}} \bigvee \Delta_{n+1}\). Then we can construct the following proof in \(\mathcal{N}\text{BQL}\):
So $\Gamma_n \vdash_{\text{BQL}} \bigvee \Delta_n$, which contradicts the induction hypothesis.

Subcase 2 $\psi_n = \forall v \chi(v)$. Then $\Delta_{n+1} = \Delta_n \cup \{\forall v \chi(v), \chi(c)\}$ for some $c \not\in \Gamma_n \cup \Delta_n \cup \{\chi(v)\}$. Suppose for a reductio that $\Gamma_{n+1} \vdash_{\text{BQL}} \bigvee \Delta_{n+1}$. Then we can construct the following proof in $\mathcal{N}\text{BQL}$:

\[
\begin{array}{c}
\Gamma_n \\
\vdots \\
\bigvee \Delta_n \lor \psi_n \\
\vdots \\
\bigvee \Delta_n \\
\bigvee \Delta_n
\end{array}
\]

So $\Gamma_n \vdash_{\text{BQL}} \bigvee \Delta_n$, which contradicts the induction hypothesis. \(\square\)

Let $\Gamma^* = \bigcup_{n \in \omega} \Gamma_n$. Clearly, $\Gamma \subseteq \Gamma^*$. Furthermore, $\phi \not\in \Gamma^*$, for otherwise, by $\lor$-Int, $\Gamma_k \vdash_{\text{BQL}} \bigvee \Delta_k$ for the least $k$ such that $\phi \in \Gamma_k$, which contradicts the Finite Separation Lemma. It remains to verify that $\Gamma^*$ satisfies consistency, BQL-closure, the Disjunction Property, the Existence Property and the Totality Property.

Consistency Suppose for a reductio that $\bot \in \Gamma^*$. Then, by $\bot$-Elim, $\Gamma_k \vdash_{\text{BQL}} \bigvee \Delta_k$ for the least $k$ such that $\bot \in \Gamma_k$, which contradicts the Finite Separation Lemma.

BQL-Closure Suppose for a reductio that $\Gamma^* \vdash_{\text{BQL}} \psi_n$ and $\psi_n \not\in \Gamma^*$. Then, in particular, $\psi_n \not\in \Gamma_{n+1}$. So $\psi_n \in \Delta_{n+1}$. On the other hand, since proofs in $\mathcal{N}\text{BQL}$ are finite, $\Gamma_k \vdash_{\text{BQL}} \psi_n$ for some $k$. But then, by $\lor$-Int, $\Gamma_{\max\{k,n+1\}} \vdash \bigvee \Delta_{\max\{k,n+1\}}$, which contradicts the Finite Separation Lemma.

Disjunction Property Suppose for a reductio that $\psi_n \lor \psi_m \in \Gamma^*$ and $\psi_n, \psi_m \not\in \Gamma^*$. Then, in particular, $\psi_n \not\in \Gamma_{n+1}$ and $\psi_m \not\in \Gamma_{m+1}$. So $\psi_n \in \Delta_{n+1}$ and $\psi_m \in \Delta_{m+1}$.
Suppose \( \Gamma = \emptyset \).

But then, by \( \lor\)-Int, \( \Gamma_k \vdash_{\text{BQL}} \lor \Delta_k \) for the least \( k \geq \max\{n + 1, m + 1\} \) such that \( \psi_n \lor \psi_m \in \Gamma_k \), which contradicts the Finite Separation Lemma.

Existence Property Suppose \( \exists v \chi(v) \in \Gamma^* \). Suppose for a reductio that \( \exists v \chi(v) \not\in \Gamma_{n+1} \) for the unique \( n \) such that \( \psi_n = \exists v \chi(v) \). Then \( \exists v \chi(v) \in \Delta_{n+1} \). So, by \( \lor\)-Int, \( \Gamma_k \vdash_{\text{BQL}} \lor \Delta_k \) for the least \( k \geq n + 1 \) such that \( \exists v \chi(v) \in \Gamma_k \), which contradicts the Finite Separation Lemma. Therefore \( \exists v \chi(v) \in \Gamma_{n+1} \). But then \( \chi(c) \in \Gamma_{n+1} \subseteq \Gamma^* \) for some \( c \in \mathcal{L} \).

Totality Property Suppose \( \chi(c) \in \Gamma^* \) for every \( c \in \mathcal{L} \). Suppose for a reductio that \( \forall v \chi(v) \not\in \Gamma^* \). Then, in particular, \( \forall v \chi(v) \not\in \Gamma_{n+1} \) for the unique \( n \) such that \( \psi_n = \forall v \chi(v) \). So \( \forall v \chi(v) \in \Delta_{n+1} \) and hence \( \chi(c) \in \Delta_{n+1} \) for some \( c \in \mathcal{L} \). But then, by \( \lor\)-Int, \( \Gamma_k \vdash_{\text{BQL}} \lor \Delta_k \) for the least \( k \geq n + 1 \) such that \( \chi(c) \in \Gamma_k \), which contradicts the Finite Separation Lemma. \( \square \)

**Lemma 6.** (Existential Witness Lemma) For \( \Sigma \in \text{Sat(BQL)} \) and finite \( \Gamma \): if \( \Sigma \cup \Gamma \cup \{\exists v \phi(v)\} \not\vdash_{\text{BQL}(\Sigma)} \psi \) then there exists \( c \in \mathcal{L} \) such that \( \Sigma \cup \Gamma \cup \{\phi(c)\} \not\vdash_{\text{BQL}(\Sigma)} \psi \).

**Proof.** Suppose \( \Sigma \cup \Gamma \cup \{\exists v \phi(v)\} \not\vdash_{\text{BQL}(\Sigma)} \psi \). There are two cases.

**Case 1** \( \Gamma = \emptyset \). Then, by the Relative Deduction Theorem, \( \Sigma \not\vdash_{\text{BQL}} \exists v \phi(v) \rightarrow \psi \). So, by Internal \( \exists\)-Int, \( \Sigma \not\vdash_{\text{BQL}} \forall v(\phi(v) \rightarrow \psi) \). Hence, since \( \Sigma \in \text{Sat(BQL)} \), there exists \( c \in \mathcal{L} \) such that \( \Sigma \not\vdash_{\text{BQL}} \phi(c) \rightarrow \psi \). But then, by the Relative Deduction Theorem, \( \Sigma \cup \{\phi(c)\} \not\vdash_{\text{BQL}(\Sigma)} \psi \).

**Case 2** \( \Gamma \neq \emptyset \). Then, by the Relative Deduction Theorem, \( \Sigma \not\vdash_{\text{BQL}} \forall v(\Gamma \land \exists v \phi(v) \rightarrow \psi) \). But, by Infinite Distributivity, we can construct the following proof in \( \land_{\text{BQL}} \):

\[
\begin{array}{c}
\forall v(\exists \Gamma \land \phi(v)) \\
\forall v(\exists \Gamma \land \phi(v) \rightarrow \psi) \\
\Gamma \land \exists v \phi(v) \rightarrow \exists v(\exists \Gamma \land \phi(v)) \\
\end{array}
\]

So \( \Sigma \not\vdash_{\text{BQL}} \forall v(\Gamma \land \phi(v) \rightarrow \psi) \). Hence, since \( \Sigma \in \text{Sat(BQL)} \), there exists \( c \in \mathcal{L} \) such that \( \Sigma \not\vdash_{\text{BQL}} \Gamma \land \phi(c) \rightarrow \psi \). But then, by the Relative Deduction Theorem, \( \Sigma \cup \Gamma \cup \{\phi(c)\} \not\vdash_{\text{BQL}(\Sigma)} \psi \). \( \square \)
Lemma 7. (Universal Witness Lemma) For $\Sigma \in \text{Sat}(\text{BQL})$ and finite $\Gamma$: if $\Sigma \cup \Gamma \not\models_{\text{BQL}(\Sigma)} \psi \lor \forall v \phi(v)$ then there exists $c \in \mathcal{L}$ such that $\Sigma \cup \Gamma \not\models_{\text{BQL}(\Sigma)} \psi \lor \phi(c)$.

Proof. Suppose $\Sigma \cup \Gamma \not\models_{\text{BQL}(\Sigma)} \psi \lor \forall v \phi(v)$. Then, by CD, $\Sigma \cup \Gamma \not\models_{\text{BQL}} \forall v(\psi \lor \phi(v))$. Hence, by the Relative Deduction Theorem, $\Sigma \not\models_{\text{BQL}} \land \Gamma \rightarrow \forall v(\psi \lor \phi(v))$. So, by Internal $\forall$-Int, $\Sigma \not\models_{\text{BQL}} \forall v((\land \Gamma \rightarrow \psi \lor \phi(v))$. Hence, since $\Sigma \in \text{Sat}(\text{BQL})$, there exists $c \in \mathcal{L}$ such that $\Sigma \not\models_{\text{BQL}} \land \Gamma \rightarrow \psi \lor \phi(c)$. But then, by the Relative Deduction Theorem, $\Sigma \cup \Gamma \not\models_{\text{BQL}(\Sigma)} \psi \lor \phi(c)$.

Lemma 8. (Relative Extension Lemma) For $\Sigma \in \text{Sat}(\text{BQL})$ and finite $\Gamma$: if $\Sigma \cup \Gamma \not\models_{\text{BQL}(\Sigma)} \phi$ then there exists $\Sigma^* \supseteq \Sigma \cup \Gamma$ such that $\Sigma^* \in \text{Sat}(\text{BQL}(\Sigma))$ and $\phi \notin \Sigma^*$.

Proof. Suppose $\Sigma \cup \Gamma \not\models_{\text{BQL}(\Sigma)} \phi$. Let $\{\psi_n\}_{n \in \omega}$ be an enumeration of $\mathcal{L}$-sentences. We inductively define a pair $\{n\}_{n \in \omega}$, $\{\Delta_n\}_{n \in \omega}$ of increasing sequences of sets of sentences $\Gamma_n, \Delta_n \subseteq \mathcal{L}$ as follows, where $\Pi^L_n(\chi(v)) = \{c \in \mathcal{L} : \Sigma \cup \Gamma_n \cup \{\chi(c)\} \not\models_{\text{BQL}(\Sigma)} \lor \Delta_n\}$ and $\Pi^R_n(\chi(v)) = \{c \in \mathcal{L} : \Sigma \cup \Gamma_n \not\models_{\text{BQL}(\Sigma)} \lor \Delta_n \lor \chi(c)\}$:

$$
\begin{align*}
\Gamma_0 &= \Gamma \\
\Gamma_{n+1} &= \begin{cases} 
\Gamma_n & \text{if } \Sigma \cup \Gamma_n \cup \{\psi_n\} \not\models_{\text{BQL}(\Sigma)} \lor \Delta_n, \\
\Gamma_n \cup \{\psi_n\} & \text{if } \Sigma \cup \Gamma_n \cup \{\psi_n\} \not\models_{\text{BQL}(\Sigma)} \lor \Delta_n \text{ and } \psi_n \neq \exists v \chi(v), \\
\Gamma_n \cup \{\psi_n, \chi(c)\} & \text{for } c \in \Pi^L_n(\chi(v)) \text{ if } \Sigma \cup \Gamma_n \cup \{\psi_n\} \not\models_{\text{BQL}(\Sigma)} \lor \Delta_n \text{ and } \\
\psi_n &= \exists v \chi(v) 
\end{cases} \\
\Delta_0 &= \{\phi\} \\
\Delta_{n+1} &= \begin{cases} 
\Delta_n & \text{if } \Sigma \cup \Gamma_n \cup \{\psi_n\} \not\models_{\text{BQL}(\Sigma)} \lor \Delta_n, \\
\Delta_n \cup \{\psi_n\} & \text{if } \Sigma \cup \Gamma_n \cup \{\psi_n\} \not\models_{\text{BQL}(\Sigma)} \lor \Delta_n \text{ and } \psi_n \neq \forall v \chi(v), \\
\Delta_n \cup \{\psi_n, \chi(c)\} & \text{for } c \in \Pi^R_n(\chi(v)) \text{ if } \Sigma \cup \Gamma_n \cup \{\psi_n\} \not\models_{\text{BQL}(\Sigma)} \lor \Delta_n \text{ and } \\
\psi_n &= \forall v \chi(v). 
\end{cases}
\end{align*}
$$

Lemma 9. (Relative Separation Lemma) For all $n$: $\Gamma_n, \Delta_n$ exist, are finite and $\Sigma \cup \Gamma_n \not\models_{\text{BQL}(\Sigma)} \lor \Delta_n$.

Proof. By induction on $n$. The base case is immediate. For the induction step, suppose $\Gamma_n, \Delta_n$ exist, are finite and $\Sigma \cup \Gamma_n \not\models_{\text{BQL}(\Sigma)} \lor \Delta_n$. There are two cases.
Case 1 \( \Sigma \cup \Gamma_n \cup \{ \psi_n \} \not\vdash_{BQL(\Sigma)} \bigvee \Delta_n \). Then \( \Delta_{n+1} = \Delta_n \). So if \( \psi_n \neq \exists v \chi(v) \) then \( \Gamma_{n+1} = \Gamma_n \cup \{ \psi_n \} \) and we’re done. Suppose, then, that \( \psi_n = \exists v \chi(v) \). Then, by the Existential Witness Lemma, \( \Pi^L_n(\chi(v)) \neq \emptyset \) and so \( \Gamma_{n+1} = \Gamma_n \cup \{ \exists v \chi(v), \chi(c) \} \) for some \( c \in \Pi^L_n(\chi(v)) \) exists. Suppose for a reductio that \( \Sigma \cup \Gamma_{n+1} \not\vdash_{BQL(\Sigma)} \bigvee \Delta_{n+1} \). By \( \exists \text{-Int} \), \( \Sigma \cup \{ \chi(c) \} \not\vdash_{BQL(\Sigma)} \exists v \chi(v) \). But then, by Relative Cut, \( \Sigma \cup \Gamma_n \cup \{ \chi(c) \} \not\vdash_{BQL(\Sigma)} \bigvee \Delta_n \), which contradicts the fact that \( c \in \Pi^L_n(\chi(v)) \).

Case 2 \( \Sigma \cup \Gamma_n \cup \{ \psi_n \} \not\vdash_{BQL(\Sigma)} \bigvee \Delta_n \). Then \( \Gamma_{n+1} = \Gamma_n \). There are two subcases.

Subcase 1 \( \psi_n \neq \forall v \chi(v) \). Then \( \Delta_{n+1} = \Delta_n \cup \{ \psi_n \} \). So, by a similar argument to Subcase 1 of the proof of the Finite Separation Lemma, \( \Sigma \cup \Gamma_{n+1} \not\vdash_{BQL(\Sigma)} \bigvee \Delta_{n+1} \).

Subcase 2 \( \psi_n = \forall v \chi(v) \). By a similar argument to Subcase 1 of the proof of the Finite Separation Lemma, \( \Sigma \cup \Gamma_n \not\vdash_{BQL(\Sigma)} \bigvee \Delta_n \lor \forall v \chi(v) \). So, by the Universal Witness Lemma, \( \Pi^R_n(\chi(v)) \neq \emptyset \) and hence \( \Delta_{n+1} = \Delta_n \cup \{ \forall v \chi(v), \chi(c) \} \) for some \( c \in \Pi^R_n(\chi(v)) \) exists. Suppose for a reductio that \( \Sigma \cup \Gamma_{n+1} \not\vdash_{BQL(\Sigma)} \bigvee \Delta_{n+1} \). Then we can construct the following proof in \( \mathcal{N}BQL(\Sigma) \):

\[
\begin{array}{c}
\Sigma, \Gamma_n \\
\vdots \\
(\bigvee \Delta_n \lor \chi(c)) \lor \forall v \chi(v) \\
\bigvee \Delta_n \lor \chi(c) \\
\end{array}
\]

So \( \Sigma \cup \Gamma_n \not\vdash_{BQL(\Sigma)} \bigvee \Delta_n \lor \chi(c) \), which contradicts the fact that \( c \in \Pi^R_n(\chi(v)) \). \( \square \)

Let \( \Sigma^* = \Sigma \cup \bigcup_{n \in \omega} \Gamma_n \). Clearly, \( \Sigma \cup \Gamma \subseteq \Sigma^* \). Furthermore, by similar arguments to those given in the proof of the Finite Extension Lemma, we can use the Relative Separation Lemma to show that \( \phi \not\in \Sigma^* \) and \( \Sigma^* \in \text{Sat}(BQL(\Sigma)) \). \( \square \)

The canonical frame for \( BQL \) is the structure \( \langle \text{Sat}(BQL), \prec \rangle \) such that \( \Sigma \prec \Gamma \) iff \( \Sigma \subseteq \Gamma \) and \( \Gamma \in \text{Sat}(BQL(\Sigma)) \). It is clear that \( \prec \) is transitive. By soundness, \( R^m(c_1, \ldots, c_n) \not\vdash_{BQL} \bot \). Hence, by the Finite Extension Lemma, \( \text{Sat}(BQL) \) is non-empty. So the canonical frame is in fact a transitive frame. The canonical model for \( BQL \) is the structure \( \mathcal{M}_\sigma = \langle \text{Sat}(BQL), \prec, C, |\cdot| \rangle \) such that \( C \) is the collection of constant symbols in \( \mathcal{L} \), \( |c| = c \) and \( |R^m| = \{ (c_1, \ldots, c_n) : R^m(c_1, \ldots, c_n) \in \Sigma \} \). Note
that
\[
\Sigma \prec \Gamma \implies \Sigma \subseteq \Gamma \\
\implies |R^n(\Sigma)| \subseteq |R^n(\Gamma)|.
\]

So $\mathcal{M}_\sigma$ is in fact an $\mathcal{L}$-model.

**Lemma 10.** (Truth Lemma) For all $\Sigma \in \text{Sat}(\mathcal{BQL}) : \phi \in \Sigma$ iff $\mathcal{M}_\sigma, \Sigma \vDash \phi$.

**Proof.** By induction on the complexity of $\mathcal{L}$-sentences. The base case is straightforward. The induction step is also straightforward except for the case of the conditional.

**Conditional**
\[
\phi \rightarrow \psi \in \Sigma \implies \text{for all } \Gamma \succ \Sigma : \text{if } \phi \in \Gamma \text{ then } \psi \in \Gamma \text{ (\Sigma-Restricted Modus Ponens)} \\
\implies \text{for all } \Gamma \succ \Sigma : \text{if } \mathcal{M}_\sigma, \Gamma \vDash \phi \text{ then } \mathcal{M}_\sigma, \Gamma \vDash \psi \\
\implies \mathcal{M}_\sigma, \Sigma \not\vDash \phi \rightarrow \psi
\]

For the converse, suppose $\phi \rightarrow \psi \not\in \Sigma$. Then $\Sigma \not\vDash_{\mathcal{BQL}} \phi \rightarrow \psi$. So, by the Relative Deduction Theorem, $\Sigma \cup \{\phi\} \not\vDash_{\mathcal{BQL}(\Sigma)} \psi$. Hence, by the Relative Extension Lemma, we can find $\Sigma^* \supseteq \Sigma \cup \{\phi\}$ such that $\Sigma^* \in \text{Sat}(\mathcal{BQL}(\Sigma))$ and $\psi \not\in \Sigma^*$. It follows that $\Sigma^* \in \text{Sat}(\mathcal{BQL})$. So, by the induction hypothesis, $\mathcal{M}_\sigma, \Sigma^* \vDash \phi$ and $\mathcal{M}_\sigma, \Sigma^* \not\vDash \psi$. But $\Sigma \prec \Sigma^*$. So $\mathcal{M}_\sigma, \Sigma \not\vDash \phi \rightarrow \psi$. \qed

**Lemma 11.** (Weak Completeness) For finite $\Gamma$: if $\Gamma \models_{\mathcal{BQL}} \phi$ then $\Gamma \vdash_{\mathcal{BQL}} \phi$.

**Proof.** Suppose $\Gamma \not\vdash_{\mathcal{BQL}} \phi$. Then, by the Finite Extension Lemma, there exists $\Gamma^* \supseteq \Gamma$ such that $\Gamma^* \in \text{Sat}(\mathcal{BQL})$ and $\phi \not\in \Gamma^*$. So, by the Truth Lemma, $\mathcal{M}_\sigma, \Gamma^* \vDash \Gamma$ and $\mathcal{M}_\sigma, \Gamma^* \not\vDash \phi$. But then $\Gamma \not\models_{\mathcal{BQL}} \phi$. \qed

**Theorem 5.** (Completeness) If $\Gamma \models_{\mathcal{BQL}} \phi$ then $\Gamma \vdash_{\mathcal{BQL}} \phi$.

**Proof.** Suppose $\Gamma \not\vdash_{\mathcal{BQL}} \phi$. Then, by compactness, there exists a finite $\Delta \subseteq \Gamma$ such that $\Delta \models_{\mathcal{BQL}} \phi$. Hence, by weak completeness, $\Delta \vdash_{\mathcal{BQL}} \phi$. But then $\Gamma \not\vdash_{\mathcal{BQL}} \phi$. \qed
4 Further Results

In this section I prove that BQL satisfies the Disjunction Property, the Existence Property and the Converse Deduction Theorem. The arguments are similar to the standard Kripke model proof of the Disjunction Property for Intuitionistic Propositional Logic. Let \( \{M_i\}_{i \in I} \) be a non-empty family of \( L \)-models such that \( W_i \cap W_j = \emptyset \) for all \( i \neq j \), \( M_i = M \) for all \( i \) and \( |c|_i = |c|_j \) for all \( i, j \). Let \( \{w_i\}_{i \in I} \) be such that \( w_i \in W_i \) and let \( W_i[w_i] = \{w_i\} \cup \{u \in W_i : w_i \prec_i u\} \). For \( z \notin \bigcup_i W_i \), we let \( \bigcup_i M_i[w_i] + z \) denote the \( L \)-model \( \langle \bigcup_i W_i[w_i] \cup \{z\}, \prec, M, |\cdot| \rangle \), where \( \prec = \bigcup_i (\prec_i \cap W_i[w_i]^2) \cup \{\langle z, u \rangle : u \in \bigcup_i W_i[w_i]\} \), \( |c| = |c|_i \) and \( |R^n| = \bigcup_i (|R^n|_i \upharpoonright W_i[w_i]) \cup \{\langle z, \emptyset \rangle\} \). Intuitively, \( \bigcup_i M_i[w_i] + z \) is the result of deleting all the worlds from each \( M_i \) except \( w_i \) and the worlds which \( w_i \) sees, gluing the resulting \( L \)-models together and then adding a new world \( z \) where all relation symbols have empty extension underneath the \( w_i \).

Lemma 12. (Cut and Splice Lemma) For all \( u \in W_i[w_i] : \bigcup_i M_i[w_i] + z, u \vDash \phi(\overline{a}) \) iff \( M_i, u \vDash \phi(\overline{a}) \).

Proof. A straightforward induction on the construction of \( L \)-formulas. \( \square \)

Theorem 6. (Converse Deduction Theorem) If \( \vdash_{\text{BQL}} \phi \rightarrow \psi \) then \( \phi \vdash_{\text{BQL}} \psi \).

Proof. Suppose \( \phi \not\vdash_{\text{BQL}} \psi \). Then, by completeness, there exists an \( L \)-model \( M \) with worlds \( W \) such that for some \( w \in W \), \( M, w \vDash \phi \) and \( M, w \not\vDash \psi \). Let \( z \notin W \). Then, by the Cut and Splice Lemma, \( M[w] + z, w \vDash \phi \) and \( M[w] + z, w \not\vDash \psi \). But \( z \) sees \( w \) in \( M[w] + z \). So \( M[w] + z, z \not\vDash \phi \rightarrow \psi \). Hence, by soundness, \( \not\vdash_{\text{BQL}} \phi \rightarrow \psi \). \( \square \)

Corollary 1. If \( \vdash_{\text{BQL}} \phi \) and \( \vdash_{\text{BQL}} \phi \rightarrow \psi \) then \( \vdash_{\text{BQL}} \psi \).

Let \( M_1 \) and \( M_2 \) be \( L \)-models. An isomorphism of \( M_1 \) onto \( M_2 \) is a pair \( \langle f, g \rangle \) of bijections \( f : W_1 \rightarrow W_2 \), \( g : M_1 \rightarrow M_2 \) such that \( w \prec_1 u \) iff \( f(w) \prec_2 f(u) \), \( g(|c|_1) = |c|_2 \) and \( \overline{a} \in |R^n|_1(w) \) iff \( g(\overline{a}) \in |R^n|_2(f(w)) \).
Lemma 13. (Isomorphism Lemma) Let \( \langle f, g \rangle \) be an isomorphism of \( \mathcal{M}_1 \) onto \( \mathcal{M}_2 \). Then \( \mathcal{M}_1, w \vdash \phi(\overline{a}) \) iff \( \mathcal{M}_2, f(w) \vdash \phi(g(\overline{a})) \).

Proof. A straightforward induction on the construction of \( \mathcal{L} \)-formulas.

We say that \( \mathcal{M}_2 \) is a copy of \( \mathcal{M}_1 \) iff \( M_1 = M_2 \) and there exists an isomorphism \( \langle f, g \rangle \) of \( \mathcal{M}_1 \) onto \( \mathcal{M}_2 \) such that \( g \) is the identity map. In this case we write \( \mathcal{M}_2 = f(\mathcal{M}_1) \).

Theorem 7. (Disjunction Property) If \( \vdash_{\text{BQL}} \phi \lor \psi \) then \( \vdash_{\text{BQL}} \phi \) or \( \vdash_{\text{BQL}} \psi \).

Proof. Suppose \( \models_{\text{BQL}} \phi \lor \psi \). Then, by the Finite Extension Lemma, there exist \( \Sigma_\phi, \Sigma_\psi \in \text{Sat}(\text{BQL}) \) such that \( \phi \not\in \Sigma_\phi \) and \( \psi \not\in \Sigma_\psi \). Hence, by the Truth Lemma, \( \mathcal{M}_\phi, \Sigma_\phi \not\models \phi \) and \( \mathcal{M}_\psi, \Sigma_\psi \not\models \psi \). Take a copy \( f(\mathcal{M}_\phi) \) of \( \mathcal{M}_\phi \) such that \( \text{Sat}(\text{BQL}) \cap f(\text{Sat}(\text{BQL})) = \emptyset \). Then, by the Isomorphism Lemma, \( f(\mathcal{M}_\phi), f(\Sigma_\phi) \not\models \psi \). Pick an arbitrary \( z \not\in \text{Sat}(\text{BQL}) \cup f(\text{Sat}(\text{BQL})) \) and let \( \mathcal{M} = \mathcal{M}_\phi[\Sigma_\phi] \cup f(\mathcal{M}_\phi)[f(\Sigma_\phi)] + z \).

Then, by the Cut and Splice Lemma, \( \mathcal{M}, \Sigma_\phi \models \phi \) and \( \mathcal{M}, f(\Sigma_\phi) \not\models \psi \). But \( z \) sees both \( \Sigma_\phi \) and \( f(\Sigma_\phi) \) in \( \mathcal{M} \). Hence, by the Persistence Theorem, \( \mathcal{M}, z \not\models \phi \) and \( \mathcal{M}, z \not\models \psi \).

So \( \mathcal{M}, z \not\models \phi \lor \psi \). But then, by soundness, \( \not\models_{\text{BQL}} \phi \lor \psi \).

Theorem 8. (Existence Property) If \( \vdash_{\text{BQL}} \exists v \phi(v) \) then \( \vdash_{\text{BQL}} \phi(c) \) for some \( c \in \mathcal{L} \).

Proof. Suppose \( \models_{\text{BQL}} \phi(c) \) for every \( c \in \mathcal{L} \). Then, by the Finite Extension Lemma, there exists a family \( \{ \Sigma_c \}_{c \in \mathcal{L}} \subseteq \text{Sat}(\text{BQL}) \) such that \( \phi(c) \not\in \Sigma_c \). Hence, by the Truth Lemma, \( \mathcal{M}_c, \Sigma_c \not\models \phi(c) \). Let \( \{ f_c(\mathcal{M}_c) \}_{c \in \mathcal{L}} \) be a family of copies of \( \mathcal{M}_c \) such that \( f_c(\text{Sat}(\text{BQL})) \cap f_d(\text{Sat}(\text{BQL})) = \emptyset \) for all \( c \neq d \). Then, by the Isomorphism Lemma, \( f_c(\mathcal{M}_c), f_c(\Sigma_c) \not\models \phi(c) \). Pick an arbitrary \( z \not\in \bigcup_c f_c(\text{Sat}(\text{BQL})) \) and let \( \mathcal{M} = \bigcup_c f_c(\mathcal{M}_c)[f_c(\Sigma_c)] + z \). Then, by the Cut and Splice Lemma, \( \mathcal{M}, f_c(\Sigma_c) \not\models \phi(c) \). But \( z \) sees every \( f_c(\Sigma_c) \) in \( \mathcal{M} \). So, by the Persistence Theorem, \( \mathcal{M}, z \not\models \phi(c) \) for every \( c \in \mathcal{L} \). Hence, since every element in the domain of \( \mathcal{M} \) is named by some \( c \in \mathcal{L} \), \( \mathcal{M}, z \not\models \exists v \phi(v) \). But then, by soundness, \( \not\models_{\text{BQL}} \exists v \phi(v) \).

Corollary 2. If \( \vdash_{\text{BQL}} \exists v \phi(v) \) and \( \phi(v) \) contains no constant symbols then \( \vdash_{\text{BQL}} \forall v \phi(v) \).
5 References

[1] Gabbay D. M., Shehtman V. B., and Skvortsov D. P., Quantification in Non-Classical Logic: Volume 1, Elsevier, 2009.

[2] Ishigaki R., and Kikuchi K., Tree-Sequent Methods for Subintuitionistic Predicate Logics, in Olivetti N. (ed.), Automated Reasoning with Analytic Tableaux and Related Methods: 16th International Conference Proceedings, Springer, 2007, pp. 149 – 164.

[3] Restall G., Sub intuitionistic Logics, Notre Dame Journal of Formal Logic 35(1): 116 – 129, 1994.

[4] Visser A., A Propositional Logic with Explicit Fixed Points, Studia Logica 40(2): 155 – 175, 1981.