Stochastic quantization and holographic Wilsonian renormalization group of scalar theories with arbitrary mass

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Abstract

We have studied a mathematical relationship between holographic Wilsonian renormalization group (HWRG) and stochastic quantization (SQ) of scalar field with arbitrary mass in AdS spacetime. In the stochastic theory, the field is described by an equation with a form of harmonic oscillator with time dependent frequency and its Euclidean action also shows explicit time dependent kernel in it. We have obtained the stochastic 2-point correlation function and demonstrate that it reproduces the radial evolution of the double trace operator correctly via the suggested relation given in arXiv:1209.2242. Moreover, we justify our stochastic procedure with time dependent kernel by showing that it can map to a new stochastic theory with a standard kernel without time dependence.
1 Introduction

AdS/CFT correspondence has shed light on the various strongly coupled field theories by providing very useful insights on them. Recently, fluid/gravity duality and AdS/CMT have been widely studied and much useful information has been obtained. Among such studies, especially holographic Wilsonian renormalization group (HWRG)\(^4\)\(^5\) provides the renormalization group flows of interesting multi-trace operators in the dual (conformal) field theories defined on a hypersurface with a certain radial cut-off in AdS space since it turns out that the radial cut-off in AdS space where the gravity theories are defined correspond to the energy scale in the dual field theories.

Another interesting challenge is to understand HWRG in the frame of stochastic quantization (SQ)\(^1\)\(^2\)\(^3\), i.e. to figure out a mathematical relationship between them. The mathematical relation between stochastic quantization (SQ) and holographic Wilsonian renormalization group (HWRG) has been addressed in the paper by J. Oh and D. P. Jatkar \(^6\)\(^7\)\(^8\) and they developed the relation to various theories in AdS space. The dictionary that the authors in \(^6\) have found is that once we identify the boundary on-shell action, \(I_{os}\) (without holographic renormalization, i.e. by keeping divergent pieces in it) with Euclidean action, \(S_E\) in stochastic quantization (i.e. \(S_E = -2I_{os}\)) and request \(t = r\), where \(t\) is stochastic time and \(r\) is radial variable in AdS space, the stochastic procedure precisely reproduce the radial evolution of the double trace operators in the dual field theory defined on the \(r = \epsilon\) hypersurface in AdS space.

In \(^6\), the authors provide two explicit examples to support their claim. One is massless scalar field in AdS\(_2\) and another is one-form field in AdS\(_4\). In these examples, they reproduced the radial evolution of the double trace operators in the dual field theories from the stochastic 2-point correlation functions precisely via the following relationship:

\[
\langle \Psi_p(r)\Psi_{-p}(r) \rangle^{-1}_H = \langle \Phi_p(t)\Phi_{-p}(t) \rangle^{-1}_S - \frac{1}{2}\frac{\delta^2 S_E}{\delta \Phi_p(t)\delta \Phi_{-p}(t)},
\]

(1)

where \((2\pi)^d\langle \Psi_p(r)\Psi_{-p}(r) \rangle^{-1}_H\) is the double trace operator, \(\langle \Phi_p(t)\Phi_{-p}(t) \rangle_S\) is the stochastic 2-point correlation function, and \(S_E\) is the Euclidean action. \(\Psi\) is the field defined in AdS space and \(\Phi\) is the stochastic field.

However, these examples are rather restrictive in a sense that their bulk action defined in AdS space effectively becomes theories of them defined on half of the flat space, \(\mathbb{R}_+\) once

\(^2\)The reason that it is not entire flat space is that the radial variable \(r\) runs from 0 to \(\infty\).
we substitute the explicit form of the AdS metric into the action. In fact, these theories are the ones which are optimized to recover the relation since in the usual stochastic quantization there is no notion of metric which explicitly depends on the stochastic time. The space of the stochastic quantization is a product space as \( M^d \times \mathbb{R} \), where \( M^d \) is \( d \)-dimensional manifold where the Euclidean action is defined and \( \mathbb{R} \) is the real line for the stochastic time \( t \).

In [7], the relation is extended to conformally coupled scalar in AdS\(_{d+1}\) space, which does not enjoy the property that the previous examples present. One sickness in this example is that the Euclidean action shows explicit stochastic time dependence via the identification \( S_E = -2I_{os} \). The formal form of the Euclidean action obtained via the identification is given by

\[
S_E = \int g(t)\Phi^2(x)d^dx,
\]

where \( \Phi \) is the stochastic field and \( g(t) \) carries the time dependence in \( S_E \). The form of Langevin equation is

\[
\frac{\partial \Phi(x,t)}{\partial t} = -\frac{1}{2} \frac{\delta S_E}{\delta \Phi(x,t)} + \eta(x,t),
\]

where \( \eta \) is the white Gaussian noise.

To evaluate the explicit form of the Langevin equation, we plug the Euclidean action into the Langevin equation and promote the field \( \Phi(x) \rightarrow \Phi(x,t) \). The Euclidean action before such promotion has no notion of stochastic time \( t \) in it. The purpose of the stochastic quantization is that one gets correlation functions in the very late time of \( t \) as the consequence of the quantization of \( S_E \). The correlation function is that of \( d \)-dimensional theory. The information about the stochastic time \( t \) is completely washed out by taking \( t \rightarrow \infty \) in the correlators. Therefore, it may be nonsense if there is \( t \) dependence in the Euclidean action, \( S_E \).

However, one can justify this by using a field redefinition. This scalar theory is mapped to a theory of massless scalar field theory in \( \mathbb{R}_+ \) by an appropriate field redefinition. In this field frame, the Langevin equation and the Euclidean action show no explicit time dependence in them and the radial evolution of the double trace operator is precisely reproduced from the stochastic 2-point function via the relation.

In this paper, we have extended this relation to scalar field theory in AdS space with arbitrary mass. This theory shares the similar problem, whose Euclidean action obtained via the identification \( S_E = -2I_{os} \) contains explicit time dependence. One can apply the same field redefinition used in [7] but he/she obtains an action with a form of harmonic oscillator with time dependent frequency \( f \), implying that its boundary on-shell action has explicit cut-off dependence in it too.

A resolution for this problem comes from [9]. In [9], the author argues the following. Langevin equation of harmonic oscillator with time dependent frequency depends on stochastic

\[ ds^2 = \frac{1}{r^2}(dr^2 + \sum_{i=1}^{d} dx^i dx^i). \]  

\[ \text{In conformally coupled scalar case, this time dependence is completely canceled out in the transformed field frame.} \]
time $t$ explicitly through a function $\rho(t)$, which satisfies a certain differential equation providing a relation between $\rho(t)$ and the time dependent frequency. The correlation functions of white Gaussian noise fields satisfy the standard form of them, e.g. $\langle \eta_\rho(t)\eta_\rho(t') \rangle = \delta(t - t')\delta^d(p - p')$. One can construct a mapping from this frame to another of stochastic quantization, where there is no explicit time dependence on the Langevin equation and a new Gaussian noise yields the standard correlation functions.

Such map is achieved by a rescaling of the stochastic time as well as appropriate field redefinition. This idea applies to the scalar field theory with arbitrary mass and we have shown an example of the explicit mapping. The method justifies that one can construct Langevin equation out of the Euclidean action with time dependent kernel in it. By using such time dependent Euclidean action from the relation $S_E = -2I_{os}$, we have reproduced the radial evolution of the double trace operator from stochastic process precisely.

One special property of the scalar theory with arbitrary mass in AdS is that the solution of the Langevin equation is no more exponential form of the function. Usually, the Langevin equation is a type of diffusion equation and its solution is a form of $e^{-t}$. However, in this case, the on-shell action provides an Euclidean action where its kernel contains a combination of Bessel function and its derivative. Such nontrivial Euclidean action gives the correct behavior of the stochastic correlation functions.

In Sec.2 we review HWRG for the scalar field with arbitrary mass in AdS$_{d+1}$ and the radial evolution of its double trace operator. In Sec.3 we develop stochastic quantization for the theory and show that the stochastic 2-point correlation function reproduces the radial evolution of the double trace operator correctly. In Sec.4 we discuss a justification for the stochastic process performing in Sec.3 with time dependent kernel by showing that the primitive Langevin equation is mapped to a new type of that without explicit time dependence and the standard form of correlations with the transformed white Gaussian noise.

### 2 Holographic Wilsonian renormalization group

We start with a free massive scalar field action defined in AdS$_{d+1}$ as

$$S = \int_{r > \epsilon} d^d x \sqrt{g} \mathcal{L}(\phi, \partial \phi) + S_B,$$

(5)

where $r$ is AdS radial coordinate, $\epsilon$ is radial cut-off and $S_B$ is the boundary effective action at $r = \epsilon$. $\mathcal{L}$ is the Lagrangian density of the massive scalar field defined in the AdS space, which is given by

$$\mathcal{L} = \frac{1}{2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} m^2 \phi^2,$$

(6)

where $g_{\mu \nu}$ is the AdS metric:

$$ds^2 = g_{\mu \nu} dx^\mu dx^\nu = \frac{dr^2 + \sum_{i=1}^{d} dx^i dx^i}{r^2}. $$

(7)

The bulk spacetime indices $\mu, \nu$ run from 1 to $d + 1$, we define that $x^{d+1} \equiv r$ and $x^1 \ldots x^d$ are boundary coordinates.
Flow of double trace operator in dual CFT  The boundary effective action is sum of boundary multi trace operators multiplied by the boundary value of the bulk field $\phi$ with the boundary momentum integration at $r = \epsilon$ hypersurface. Therefore,

$$S_B = \text{single trace part} - \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} D(p, \epsilon) \phi_p(\epsilon) \phi_{-p}(\epsilon) + \text{multi trace part},$$  

(8)

where $D(p, \epsilon)$ is the double trace operator. For the free field in the bulk, we have at most double trace operators and we are interested in the double trace part of the boundary effective action. Equation of the double trace part of the boundary effective action, $S_B$ [4, 5] is given by

$$\partial_\tau D(p, \epsilon) = \frac{1}{\sqrt{g} g^{rr}} \frac{1}{(2\pi)^d} D(\epsilon, p) D(\epsilon, -p) - (2\pi)^d \sqrt{g} (r^2 p^2 + m^2)$$  

(9)

and its solution is

$$D(\epsilon, p) = -(2\pi)^d \frac{\Pi_\phi}{\phi},$$  

(10)

where $\Pi_\phi$ is the canonical momentum of the field $\phi$. The canonical momentum satisfies following equations:

$$\Pi_\phi = \sqrt{g} g^{rr} \partial_r \phi \quad \text{and} \quad \partial_r \Pi_\phi = \sqrt{g} (r^2 p^2 + m^2) \phi,$$

(11)

where the first equation is nothing but definition of the canonical momentum. By combining these two equations, one can write down the bulk equation of motion as

$$\partial_r (r^{1-d} \partial_r \phi) = \frac{1}{r^{d+1}} (r^2 p^2 + m^2) \phi,$$

(12)

as well.

The solution of this equation in the zero boundary momentum $p = 0$ case is

$$\phi(r) = b_1 r^{\frac{d}{2} - \nu} + b_2 r^{\frac{d}{2} + \nu},$$

(13)

where $b_1$ and $b_2$ are arbitrary constants and

$$\nu \equiv \frac{1}{2} \sqrt{d^2 + 4m^2}.$$  

(14)

Once we turn on the boundary momentum $p$ of the field, then the solution is a linear combination of the Bessel $K$ and $I$ as

$$\phi_p(r) = c_1 r^{d/2} K_\nu(|p|r) + c_2 r^{d/2} I_\nu(|p|r),$$

(15)

where $c_1$ and $c_2$ are $p$ dependent constants.
Field redefinition we discuss HWRG in different field frame. In this field frame, the massive scalar field in AdS space becomes harmonic oscillator with \( r \) dependent frequency. By using the explicit form of the metric(7), the bulk action is written as

\[
S = \frac{1}{2} \int d^dr \phi (r^1 - d) \partial_r \phi \partial_r - p^2 r^1 - d \phi \partial_r \phi + m^2 r^1 - d \phi \phi) .
\]

(16)

We define a new field \( f_p \) as \( \phi_p = r^\frac{d-1}{2} f_p \), and substitute it into the action. Then the action is transformed to

\[
S = \frac{1}{2} \int d^dr d_p \left[ \partial_r f_p \partial_r f - p^2 \phi f - p^2 \phi f + \frac{1}{r^2} (m^2 + \frac{d^2 - 1}{4}) \phi f \right]
\]

(17)

Its equation of motion is given by

\[
0 = -\partial_r \partial_r f_p + p^2 f_p + r^{-2} \left( m^2 + \frac{d^2 - 1}{4} \right) f_p,
\]

(18)

which is a form of that of harmonic oscillator with \( r \)-dependent frequency \( \omega(r) \), where \( \omega^2(r) = p^2 + r^{-2}(m^2 + \frac{d^2 - 1}{4}) \). The solutions of this equation are

\[
f(r) = b_1 r^{\frac{1}{2} - \nu} + b_2 r^{\frac{1}{2} + \nu} \text{ for zero momentum case},
\]

(19)

\[
f_p(r) = c_1 r^{1/2} K_\nu(|p|r) + c_2 r^{1/2} I_\nu(|p|r) \text{ for } p \neq 0,
\]

(20)

where \( b_1 \) and \( b_2 \) are arbitrary constants and \( c_1 \) and \( c_2 \) are arbitrary boundary momentum \( p \) dependent functions.

HWRG in this field frame is given as follows. Equation of motion of the double trace operator, \( D_f \) satisfies Hamilton-Jacobi equation which is given by

\[
\partial_t D_f(p, \epsilon) = \frac{1}{(2\pi)^d} D_f(p, \epsilon) D_f(-p, \epsilon) - (2\pi)^d \left[ p^2 + \frac{1}{r^2} \left( m^2 + \frac{d^2 - 1}{4} \right) \right],
\]

(21)

where \( \Pi_f \) is canonical momentum of the field \( f_p \), which satisfies the following equations:

\[
\Pi_f = \partial_r f_p \text{ and } \partial_r \Pi_f = p^2 f_p + \frac{1}{r^2} \left( m^2 + \frac{d^2 - 1}{4} \right).
\]

(22)

In the zero boundary momentum case, the solution of the double trace operator is

\[
D_f(r) = -(2\pi)^d \frac{\partial_r f(r)}{f(r)} = -(2\pi)^d \frac{d_1 \left( \frac{1}{2} - \nu \right) + d_2 \left( \frac{1}{2} + \nu \right)r^{2\nu}}{d_1 r + d_2 r^{2\nu + 1}},
\]

(23)

where \( d_1 \) and \( d_2 \) are arbitrary constants. Then, the double trace part of the boundary effective action is given by

\[
S_B = \frac{1}{2} \frac{d_1 \left( \frac{1}{2} - \nu \right) + d_2 \left( \frac{1}{2} + \nu \right)r^{2\nu}}{d_1 r + d_2 r^{2\nu + 1}} f^2.
\]

(24)
Once we turn on the boundary directional momentum $p$, the solution of the double trace operator is given by

$$D_f(p, \epsilon) = -(2\pi)^d \frac{\partial}{\partial p} = -(2\pi)^d \partial_x \ln[\epsilon^{1/2}(c_1 K_\nu(|p|\epsilon) + c_2 I_\nu(|p|\epsilon))].$$

(25)

and so the double trace part of the boundary effective action becomes

$$S_B = -\frac{1}{2(2\pi)^d} \int d^d p D_f(p, \epsilon) f_p^{(0)}(\epsilon) f_{-p}^{(0)}(\epsilon).$$

(26)

**On-shell action** We discuss zero momentum case first. To evaluate the on-shell action, we need to choose regular solution (which should not be divergent in AdS interior). Therefore, $b_2 = 0$ is chosen in (19). Then, the form of the regular solution is given by

$$f(r) = f^{(0)} \frac{r^{1/2 - \nu}}{\epsilon^{1/2 - \nu}},$$

(27)

where $f^{(0)}$ is boundary value of the bulk field $f(r)$ at $r = \epsilon$. The on-shell action is, by definition, the bulk action up to equation of motion evaluated on $r = \epsilon$ hypersurface, which is given by

$$I_{os} = \frac{1}{2} \int_{r=\epsilon} d^d p f_p \partial_r f_{-p}. $$

(28)

We plug the regular bulk solution into this and we get

$$I_{os} = \frac{1}{2\epsilon} \left( \frac{1}{2} - \nu \right) \left( f^{(0)} \right)^2.$$

(29)

For nonzero momentum ($p \neq 0$) case, we set $c_2 = 0$ for the bulk solution to be regular. We rewrite this regular solution in terms of boundary value of the field $\phi$ as

$$f_p^{(0)} = f_p^{(0)} \frac{r^{1/2} K_\nu(|p| \epsilon)}{\epsilon^{1/2} K_\nu(|p| \epsilon)},$$

(30)

where $f_p^{(0)}$ is the boundary value of the bulk field $f_p(r)$ at $r = \epsilon$. By substituting this regular solution into the formal form of the on-shell action (28), the on-shell action is given by

$$I_{os} = \frac{1}{2} \int_{r=\epsilon} d^d p f_p^{(0)} \partial_r f_{-p} \ln[r^{1/2} K_\nu(|p| \epsilon)].$$

(31)

### 3 Stochastic quantization

In this section, we develop stochastic quantization by identifying the Euclidean action, $S_E$ with the on-shell actions (29) and (31) through the suggested relation $S_E = -2I_{os}$. It will be shown
that the radial evolution of the double trace operator can be reproduced by stochastic 2-point correlation function via the relation suggested in [6]. It is given by

\[
\langle f_p(0)f_{-p}(r) \rangle_H^{-1} = \langle \Phi_p(0)\Phi_{-p}(t) \rangle_S^{-1} - \frac{1}{2} \frac{\delta^2 S_E}{\delta f_p^{(0)} \delta f_{-p}^{(0)}},
\]

(32)

provided by \( r = t \) identification. \( \langle \Phi_p(0)\Phi_{-p}(t) \rangle_S \) is the stochastic 2-point correlation function and \( \langle f_p(0)f_{-p}(r) \rangle_H \) = \( \delta^2 S_B \delta f_p(0) \delta f_{-p}(0) - \delta^2 S_E \delta f_p(t) \delta f_{-p}(t) \).

Stochastic partition function We start with the Langevin equation, which is given by

\[
\frac{\partial \Phi(x,t)}{\partial t} = -\frac{1}{2} \delta S_E + \eta(x,t),
\]

(33)

where \( t \) is stochastic time, \( \Phi \) is stochastic field, \( \eta \) is the white Gaussian noise and \( S_E \) is the Euclidean action. We take this Euclidean action to be the following form:

\[
S_E = \frac{1}{2} \int d^d x g(t) \Phi^2(x,t),
\]

(34)

which contains stochastic time dependent kernel \( g(t) \) in it. By using such a form of the Euclidean action, the Langevin equation becomes

\[
\frac{\partial \Phi(x,t)}{\partial t} = -g(t)\Phi(x,t) + \eta(x,t).
\]

(35)

To evaluate the stochastic correlation functions, we consider the stochastic partition function. The stochastic partition function is

\[
Z[\eta] = \int \mathcal{D}[\eta] \exp (-\mathcal{S}),
\]

(36)

where

\[
\mathcal{S} = \frac{1}{2} \int_{t_0}^t dt' dx \eta^2(x,t'),
\]

(37)

and \( t \) and \( t_0 \) are final and initial stochastic time respectively. This suggests that

\[
\langle \eta_p(t)\eta_{p'}(t') \rangle = \frac{1}{Z} \int D[\eta] \eta_p(t)\eta_{p'}(t') e^{-\mathcal{S}} = \delta(t-t')\delta^d(p-p').
\]

(38)

By using the Langevin equation, we replace the stochastic noise, \( \eta \) with the stochastic field, \( \Phi \) and then we have

\[
\mathcal{S} = \int_{t_0}^t dt' dx \left( \frac{1}{2} \delta \Phi^2 + \frac{1}{2} (g^2(t) - \partial_t g(t)) \Phi^2 + \frac{1}{2} \partial_t (g(t)\Phi^2) \right),
\]

(39)

\[
\equiv S_{FP} + S_E|_{t=t_0},
\]

5The relation is given by the equation (2.38) in [6].
where $S_{FP}$ is called the Fokker-Planck action which is given by

$$S_{FP} = \int dt d^d x \mathcal{L}_{FP}$$

(40)

and $\mathcal{L}_{FP}$ is the Fokker-Planck Lagrangian density:

$$\mathcal{L}_{FP} = \frac{1}{2} (\partial_t \Phi)^2 + \frac{1}{2} (g^2(t) - \partial_t g(t)) \Phi^2.$$  

(41)

One can define this Fokker-Planck action in the momentum space too, by using the Fourier transform as

$$\Phi(x, t) = \frac{1}{(2\pi)^{d/2}} \int d^d p e^{-ipx} \Phi_p(t),$$  

(42)

then the Fokker-Planck action becomes

$$S_{FP} = \frac{1}{2} \int dt d^d x [\partial_t \Phi_p \partial_t \Phi_{-p} + (g^2(t) - \partial_t g(t)) \Phi_p \Phi_{-p}]$$  

(43)

As followed by [6], this Fokker-Planck action is identified with the action (17). For this, we demand that the Ricatti term appearing in (43) is equal to the radial coordinate dependent mass term in (17) along with the identification of the AdS radial coordinate $r$ with the stochastic time coordinate $t$. This is a part of the Stochastic Quantization and AdS/CFT dictionary. In other words, we replace the radial coordinate $r$ by the stochastic time $t$. This gives us the equation,

$$g^2(t) - \partial_t g(t) = p^2 + \frac{1}{t^2} \left( m^2 + \frac{d^2 - 1}{4} \right).$$  

(44)

For the zero boundary momentum case, $p = 0$ the solution of $g(t)$ is either

$$g_1(t) = \frac{-1/2 + \nu}{t}, \quad \text{or} \quad g_2(t) = \frac{-1/2 - \nu}{t}.$$  

(45)

Once we choose the solution $g_1(t)$, then the Euclidean action (34) is consistent with the prescription for the choice of the Euclidean action from the on-shell action given in [6]. The boundary value of the bulk scalar field $f(r)$ is to be the stochastic field. For the nonzero momentum case, the solution of this equation is

$$g(t) = -\partial_t \log[\sqrt{t} K_\nu(|p|t) + a_0 \sqrt{t} \ I_\nu(|p|t)].$$  

(46)

To match this with the on-shell action (31), we take $a_0$ to vanish. In fact, the term being proportional to $a_0$ comes from the irregular part of the bulk solution in the on-shell action. Therefore, a choice of $a_0 = 0$ is consistent with the Euclidean action derived from the on-shell action.
**Langevin approach**  We discuss the zero boundary momentum case first. The Langevin equation \(^{(33)}\) with the Euclidean action \(^{(29)}\) and \(r = t\) identification is given by

\[
\frac{\partial \Phi(t)}{\partial t} = \frac{1}{t} (1 - 2\nu) \Phi(t) + \eta(t),
\]

and its solution is

\[
\Phi(t) = \int_{t_0}^{t} \frac{t_{1}^{\frac{1}{2} - \nu}}{t_{0}^{\frac{1}{2} - \nu}} \eta(t') dt',
\]

where \(t_0\) is initial time to be specified soon. Now, we compute stochastic 2-point correlation function as

\[
\langle \Phi(t) \Phi(t') \rangle_S = \int_{t_0}^{t} \int_{t_0}^{t'} \frac{t_{1}^{\frac{2}{2} - \nu} \eta(t_0) \eta(t'_0)}{t_{0}^{\frac{2}{2} - \nu}} dt_0 dt'_0
\]

\[
= \int_{t_0}^{t} \frac{t_{1}^{\frac{1}{2} - \nu} \eta(t') \eta(t_0)}{t_{0}^{\frac{1}{2} - \nu}} dt_0
\]

\[
= \frac{t_{1}^{\frac{1}{2} - \nu} \eta(t') \eta(t_0)}{2\nu} (t^{2\nu} - t_{0}^{2\nu}).
\]

We have used the relation of 2-point function of the white Gaussian noise \(\eta\) for the second equality in the above computation:

\[
\langle \eta(t) \eta(t') \rangle = \delta(t - t').
\]

The equal time commutator is

\[
\langle \Phi(t) \Phi(t) \rangle_S = \frac{t^{1 - 2\nu}}{2\nu} (t^{2\nu} - t_{0}^{2\nu}).
\]

It is manifest that this stochastic 2-point correlation function can reproduce the solution of double trace operator \(^{(23)}\) via the suggested relation in \[^6\] as

\[
\langle f^{(0)}(r) f^{(0)}(r) \rangle_H^{-1} = \langle \Phi(t) \Phi(t) \rangle_S^{-1} - \frac{1}{2} \frac{\delta^2 S_E}{\delta \Phi(t) \delta \Phi(t)},
\]

provided by the identification \(r = t\) and requesting that a condition for \(t_0\) as \(t_0 = \left( -\frac{d_1}{d_2} \right)^{\frac{1}{2\nu}}\).

Next, we discuss nonzero momentum case. We start with the Euclidean action obtained from the on-shell action \(^{(31)}\) via the identification \(S_E = -2I_{os}\). From this form of the Euclidean action, we derive the Langevin equation as

\[
\frac{\partial \Phi_p(t)}{\partial t} = \partial_t \ln(t^{1/2} K_\nu(|p|)) \Phi_p(t) + \eta_p(t).
\]

Solution of this equation is

\[
\phi_p(t) = \int_{t_0}^{t} \frac{t^{1/2} K_\nu(|p|)}{t^{1/2} K_\nu(|p|)} \eta_p(t') dt'.
\]
Let us compute stochastic 2-point correlation function, which is given by
\[
\langle \phi_p(t)\phi_{p'}(t') \rangle = \int_{t_0}^{t} \int_{t_0}^{t'} \frac{t^{1/2} t'^{1/2} K_\nu(|p|t) K_\nu(|p'|t')} {t^{1/2} t'^{1/2} K_\nu(|p|t) K_\nu(|p'|t')} \langle \eta_p(t) \eta_{p'}(t') \rangle d\bar{t} d\bar{t}' \tag{55}
\]
\[
= \int_{t_0}^{t} \frac{t^{1/2} t'^{1/2} K_\nu(|p|t) K_\nu(|p'|t')} {t^{1/2} K_\nu(|p|t)^2} d\bar{t} \delta^d(p - p') \tag{56}
\]
\[
= t^{1/2} t'^{1/2} K_\nu(|p|t) K_\nu(|p'|t') \left. \frac{I_\nu(|p|t)} {K_\nu(|p|t)} \right|_{t'=t} \, ,
\]
where for the second equality we have used
\[
\langle \eta_p(t) \eta_{p'}(t') \rangle = \delta(t - t') \delta^d(p - p') \tag{57}
\]
and for the third equality the following integral formula has been used:
\[
\int \frac{dx} {x[K_\alpha(x)]^2} = \frac{I_\alpha(x)} {K_\alpha(x)} \, .
\tag{58}
\]
We are interested in equal time commutator, so we take \( t' = t \). The equal time commutator is given by
\[
\langle \phi_p(t)\phi_{p'}(t) \rangle = \delta^d(p - p') t[K_\nu(|p|t) I_\nu(|p|t) - \beta(t_0) K_\nu(|p|t)^2] \, ,
\tag{59}
\]
where
\[
\beta(t_0) = \frac{I_\alpha(|p|t_0)} {K_\alpha(|p|t_0)} 
\tag{60}
\]
To check if the stochastic correlation function(58) reproduces the radial evolution of the double trace operator(25) correctly via the relation(32), we compute
\[
\langle \Phi_p(t)\Phi_{-p}(t) \rangle - \frac{1} {2} \frac{\delta^2 S_E} {\delta \Phi_p(t) \delta \Phi_{-p}(t)} = \partial_t \ln[t^{1/2} (-\beta(t_0) K_\nu(|p|t) + I_\nu(|p|t))] \, .
\tag{61}
\]
Once we identify the radial variable \( r \) with the stochastic time \( t \) and request \( \beta(t_0) = -\frac{\alpha_1}{\epsilon_2} \), then (61) perfectly match with (25).

**Fokker-Planck approach** For the zero boundary momentum case, the Fokker-Planck action is given by
\[
S_{FP} = \frac{1} {2} \int dt d^4p \left[ \partial_t \Phi_p \partial_t \Phi_{-p} + \left( m^2 + \frac{d^2 - 1} {4} \right) \Phi_p \Phi_{-p} \right] .
\tag{62}
\]
By using its equation of motion, we get the on-shell evaluation of this action, which is
\[
S_{FP} = \frac{1} {2} \int \Phi(t') \partial_t \Phi(t') \bigg|_{t'=t_0} \, ,
\tag{63}
\]
where \( \Phi(t) \) becomes the most general solution of the equation of motion as
\[
\Phi(t') = \Phi(t) \frac{h_1 t_{1-\nu} + h_2 t_{1+\nu}^2} {h_1 t_{2-\nu} + h_2 t_{2+\nu}^2} \, .
\tag{64}
\]
where $h_1$ and $h_2$ are arbitrary constants. This is designed to satisfy the stochastic boundary condition at $t' = t$ as $\Phi(t' = t) = \Phi(t)$. To reproduce the double trace part of the boundary effective action in HWRG, we have chosen the initial boundary condition too as

$$t_0 = \left(-\frac{h_1}{h_2}\right)^{\frac{1}{2\nu}}. \tag{64}$$

Once the solution (63) is plugged into the Fokker-Planck action, we get

$$S_{FP} = \frac{1}{2} \frac{h_1}{h_1 t + h_2 t^{2\nu+1}} \left[\frac{1}{2} - \nu\right] t + h_2 \left(\frac{1}{2} + \nu\right) t^{2\nu+1}. \tag{65}$$

Once we identify $d_1 = h_1$ and $d_2 = h_2$, the this Fokker-Planck action correctly reproduce the boundary effective action, $S_B$ given in (24) provided by $r = t$.

Next we discuss nonzero momentum case ($p \neq 0$). From the action (43) together with (44) (or with (46)), one can derive equation of motion as

$$0 = -\partial_t^2 \Phi_p(t) + \left[\frac{1}{2} - \nu\right] \left(m^2 + \frac{d^2 - 1}{4}\right), \tag{66}$$

and its solution is

$$\Phi_p(t) = d_1 t^{1/2} K_\nu(|p|t) + d_2 t^{1/2} I_\nu(|p|t), \tag{67}$$

where $d_1$ and $d_2$ are $p$-dependent arbitrary constants. For this solution, we impose stochastic boundary condition as $\Phi(\tilde{t} = t) = \Phi(t)$, then we have

$$\Phi_p(\tilde{t}) = \Phi(t) \frac{t^{1/2}[d_1 K_\nu(|p|t) + d_2 I_\nu(|p|t)]}{t^{1/2}[d_1 K_\nu(|p|t) + d_2 I_\nu(|p|t)]}. \tag{68}$$

By substituting this solution into the Fokker-Planck action, we obtain

$$S_{FP} = \frac{1}{2} \int d^d p \partial_t \left[\frac{1}{2} t^{1/2}(d_1 K_\nu(|p|t) + d_2 I_\nu(|p|t))\right] \tag{69}$$

provided by a initial boundary condition as

$$\frac{d_1}{d_2} = -\frac{K_\nu(t_0)}{I_\nu(t_0)}. \tag{70}$$

In sum, once we take $d_1 = c_1$, $d_2 = c_2$ and $r = t$ then (69) reproduces (26) precisely.

### 4 Stochastic quantization with time dependent kernel

In this section, we justify our computation in the previous sections by providing resolution for the subtle issue that we have mentioned in introduction.
The Euclidean action necessarily shows explicit time dependence. Via the identification $S_E = -2I_{os}$, the Euclidean action has a form of (34) and this action includes time dependent kernel $g(t)$ in it. In the usual sense of stochastic quantization, the Euclidean action is defined on the $d$-dimensional Euclidean space and stochastic process is defined in $d+1$-dimensional space since it contains stochastic time $t$ as well as the $d$-dimensional Euclidean space.

The kernel in the Euclidean action does not have the stochastic time dependence. The purpose of the stochastic quantization is that one gets $n$-point correlation functions (for our case, 2-point correlation function only) as the consequence of the quantization of $S_E$, in the very late time of $t$. The correlation function has no notion of stochastic time $t$ dependence since it is the correlator of the $d$-dimensional theory. The information about the stochastic time $t$ is completely washed out by taking $t \to \infty$ in the correlators. Therefore, it may be nonsense if there is $t$ dependence in the Euclidean action, $S_E$.

By the way, the on-shell action, $I_{os}$ is in fact divergent since we did not perform holographic renormalization to obtain it. According to the prescription in [7], the way of getting $I_{os}$ is by computing boundary contribution of the bulk action upto the bulk equation of motion without holographic renormalization. Therefore, the on-shell action $I_{os}$ is ill-defined on the AdS boundary.

One can resolve this issue if one defines the on-shell action on the $r = \epsilon$ hyper surface near AdS boundary i.e. $\epsilon \ll 1$. Once the on-shell action is defined on the hyper surface away from AdS boundary, it contains explicit $\epsilon$ dependence, which is promoted to stochastic time by the identification ‘$r = t$’ as addressed in [6]. Due to this rule, it is impossible to avoid for $S_E$ not to have time dependent kernel in it.

Resolution of the time dependent kernel issue. One way to resolve this issue is that one develop a mapping from the Langevin equation with explicit time dependence to that with a standard kernel without time dependence. Our Langevin equation is (35) and the Fokker-Planck action becomes a form of harmonic oscillator with time dependence frequency $\omega(t)$, which is given by

$$\omega^2(t) = g^2(t) - \dot{g}(t).$$

(71)

The 2-point correlation of white Gaussian noise is given by (56).

Now, we develop a mapping from this frame to a new stochastic frame with standard kernel without time dependence. Let us consider the following map:

$$\Phi_k(t) \equiv u(T)\Psi_k(T), \quad \eta_k(t) = \frac{\zeta_k(T)}{u(T)} \quad \text{and} \quad t = \int u^2(T)dT,$$

(72)

where $u(T)$ satisfies

$$g(T) = \frac{1}{u^2(T)} - \frac{\dot{u}(T)}{u^3(T)} \quad \text{or equivalently,} \quad g(t) = \frac{1}{u^2(t)} - \frac{\dot{u}(t)}{u(t)}.$$

(73)

The field $\Psi_k(T)$ is a new stochastic field, $\zeta_k(T)$ is a new white Gaussian noise and $T$ is a new stochastic time. The ‘·’ denotes derivative with respect to its argument. Under such mapping
the Langevin equation and the 2-point correlation of white Gaussian noise transform as
\[ \frac{\partial \Psi_k(T)}{\partial T} = -\Psi_k(T) + \zeta_k(T) \text{ and } \langle \zeta_k(T) \zeta_{k'}(T') \rangle = \delta(T - T') \delta(k - k'), \tag{74} \]
respectively. This means that in this new frame, the Langevin equation is derived from a new Euclidean action with a kernel of identity as
\[ S_E = \int \Psi_k \Psi_{-k} d^d k. \tag{75} \]
As the simplest example, one choose
\[ u(t) = \frac{1}{\sqrt{\nu}} t^{1/2}, \text{ or equivalently } u(T) = \frac{1}{\sqrt{\nu}} e^{T/\nu}, \tag{76} \]
for the zero boundary momentum case discussed in the previous section. Then, one can get the same answer with (74) via the mapping (72) from (50) and (47).

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6We have omitted \( \delta^d(k - k') \) in the result since it is the case that the boundary momentum is turned off.