CONSTANT SCALAR CURVATURE METRICS ON BOUNDARY
COMPLEXES OF CYCLIC POLYTOPES

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ABSTRACT. In [7], a notion of constant scalar curvature metrics on piecewise flat manifolds is defined. Such metrics are candidates for canonical metrics on discrete manifolds. In this paper, we define a class of vertex transitive metrics on certain triangulations of $S^3$; namely, the boundary complexes of cyclic polytopes. We use combinatorial properties of cyclic polytopes to show that, for any number of vertices, these metrics have constant scalar curvature.

1. Introduction

A classical question in differential geometry is that of finding canonical metrics on manifolds. For example, the well-known Yamabe problem asks whether a given smooth compact Riemannian manifold $(M^n, g)$, for $n \geq 3$, has a constant scalar curvature metric $\tilde{g}$ which is conformal to $g$. This problem was posed by Yamabe in 1960, and a complete solution to the problem is due to the combined work of Yamabe, Trudinger, Aubin, and Schoen. See [11] for background on the problem and its solution.

Constant scalar curvature metrics arise as critical points in a conformal class of the volume normalized Einstein-Hilbert functional. To see this, recall that the Einstein-Hilbert functional is given by

$$\mathcal{E}H(g) = \int_{M^n} R_g dV_g,$$

where $R_g$ is the scalar curvature of $g$ and $dV_g$ is the volume form of $g$. Under a conformal variation $\delta g = f g$, the scalar curvature satisfies

$$\delta R[f g] = (1 - n) \Delta f - R f,$$

and the volume form satisfies

$$\delta dV[f g] = \frac{n}{2} f dV.$$

Thus the variation of $\mathcal{E}H$ is

$$\delta \mathcal{E}H[f g] = \left( \frac{n}{2} - 1 \right) \int_{M^n} R f dV.$$

Critical points of this equation have $R = 0$. To obtain constant scalar curvature metrics as critical points, one can take the variation of the volume normalized $\mathcal{E}H$ functional:

$$\mathcal{N} \mathcal{E}H(g) = \frac{\int_{M^n} R_g dV_g}{(\int_{M^n} dV_g)^{\frac{n}{2}}}.$$
One might like to pose an analogue to the Yamabe problem in a non-smooth setting; for example, on piecewise flat manifolds. To do so, one would need a discrete notion of curvature and of conformal variations. In [7], such concepts are defined on piecewise flat manifolds. The constant scalar curvature metrics in the piecewise flat setting should be somehow analogous to those in the smooth setting. Therefore, these metrics are defined to be critical points of a discrete analogue of the Einstein-Hilbert functional defined by T. Regge in 1961 [14]. We will call this functional the Einstein-Hilbert-Regge functional, and we will denote it $\mathcal{EHR}$. See [9] for a survey of the study of this functional as an action for general relativity and of its use in the Regge calculus and lattice gravity. In [4], it was shown that the $\mathcal{EHR}$ functional converges to the $\mathcal{EH}$ functional in a certain sense. The idea of discretizing smooth geometric notions forms the basis for fields such as discrete differential geometry and discrete exterior calculus ([1], [5], [6], [13]).

In [3], the behavior of the $\mathcal{EHR}$ functional was analyzed on the double tetrahedron. The double tetrahedron is the simplest triangulation of $S^3$, and it is not simplicial. However, the triangulation is neighborly and vertex transitive. This note can be thought of as an extension of that work, as the piecewise flat manifolds we will consider are homeomorphic to $S^3$ and are both neighborly and vertex transitive.

This paper is organized as follows. In §2, we will provide background about piecewise flat manifolds, and we will define discrete constant scalar curvature metrics. In §3, we will define cyclic polytopes. The boundary complexes of the cyclic polytopes will be our main object of study, and we will recall some facts about these triangulations of $S^3$. We will define a certain class of metrics on the boundary complexes of cyclic polytopes known as cyclic length metrics in §4, and we will collect some combinatorial results. In §5, we will show that the metrics do indeed have constant scalar curvature. Finally, in §6, we mention some open questions and directions for future research.

**Acknowledgement.** We would like to thank the participants and organizers of the 2010 Arizona Summer Program, during which this work took place. We would especially like to thank David Glickenstein for helpful discussions.

2. **Background and Notation**

In this section, we will provide the necessary background on piecewise flat manifolds. We will also describe a notion of conformal structure in this setting and will define constant scalar curvature metrics. We will follow closely the definitions in [3] and [7].

2.1. **Piecewise flat manifolds.** We begin with a *triangulated piecewise flat manifold*. The dimension of a triangulation is that of its highest dimensional simplex. A three-dimensional triangulation $\mathcal{T} = (V, E, F, T)$ has a collection of vertices (denoted $V$), edges (denoted $E$), faces (denoted $F$), and tetrahedra (denoted $T$). In this paper, the triangulations we consider will be three-dimensional simplicial complexes. Thus we will require that every face borders exactly two tetrahedra and that the link of every vertex is homeomorphic to $S^2$.

**Definition 2.1.** A triangulation is said to be neighborly if, for any pair of vertices, the edge connecting them belongs to the simplicial complex.
Definition 2.2. An automorphism is a function \( f : V \rightarrow V \) such that, for every edge \( e = (u, v) \in E \), \( f(e) = (f(u), f(v)) \) is also an edge. A triangulation is said to be vertex transitive if the automorphism group acts transitively on the vertices.

A triangulated piecewise flat manifold is denoted as \( (M, \mathcal{T}, \ell) \), where \( M \) is a manifold, \( \mathcal{T} \) is a triangulation of \( M \), and \( \ell \) is a metric according to the following definition.

Definition 2.3. A vector \( \ell \in \mathbb{R}^{|E|} \) such that each simplex can be realized as a Euclidean simplex with edge lengths determined by \( \ell \) is called a metric for the triangulated manifold \( (M, \mathcal{T}) \), and \( (M, \mathcal{T}, \ell) \) is called a triangulated piecewise flat manifold.

The condition for a metric can be described using Cayley-Menger determinants of the type described in [3]. Notice that a choice of metric (i.e., a choice of edge lengths) completely determines the geometry of the triangulation in the following sense. The angles of a face are determined by the edge lengths via the cosine law. The dihedral angles of a tetrahedron can then be calculated from the angles at the faces using the spherical cosine law. We use \( \beta_e \) to refer to the dihedral angle of a tetrahedron at edge \( e \). If we wish to emphasize that it is in tetrahedron \( t \), we denote it as \( \beta_{e<t} \). In what follows, \( \tau < \sigma \) or \( \sigma > \tau \) will mean that \( \tau \) is a sub-simplex of \( \sigma \). For computational purposes, we will sometimes use the notation \( \beta_{ij,kl} \) to refer to the dihedral angle at edge \( e_{ij} \) in tetrahedron \( t_{ijkl} \).

We will also call a metric \( \ell \) on \( (M, \mathcal{T}) \) vertex transitive if the automorphism group acts transitively on the vertices. In this case, every vertex will “look the same” both combinatorially and geometrically.

2.2. Discrete conformal structures and constant scalar curvature metrics.

In this section, we will consider a certain conformal structure that has been studied in [3], [7], [12], [15]. This choice of conformal structure will allow us to define a notion of vertex curvature and ultimately of constant scalar curvature metrics.

Definition 2.4. Let \( \{L_e\}_{e \in E} \) be such that \( (M, \mathcal{T}, L) \) is a piecewise flat manifold. Let \( V^* \) denote the real-valued functions on the vertices, and let \( U \subset V^* \) be an open set. A conformal structure is a map \( U \rightarrow \text{met} (M, \mathcal{T}) \) determined by

\[
\ell_e (f) = \exp \left[ \frac{1}{2} (f_v + f_{v'}) \right] L_e,
\]

where \( e \) is the edge between \( v \) and \( v' \). The conformal class is the image of \( U \) in \( \text{met} (M, \mathcal{T}) \), and it is entirely determined by \( L \). A conformal variation \( f(t) \) is a smooth curve \( (-\varepsilon, \varepsilon) \rightarrow U \) for small \( \varepsilon > 0 \), and it induces a conformal variation of metrics \( \ell (f(t)) \).

Remark 2.5. There is a more general notion of conformal structure on piecewise flat manifolds that is described in [7]. The conformal structure described here is called the perpendicular bisector conformal structure in that paper.

We define both the edge and vertex curvatures. The edge curvature is independent of a conformal structure. However, as described in [7], the choice of conformal structure allows us to define the vertex curvature. This will lead to a definition of constant scalar curvature metrics.
Definition 2.6. The edge curvature of an edge $e$ is
\begin{equation}
K_e = (2\pi - \sum_{t \in T} \beta_{e < t}) \ell_e,
\end{equation}
where $\ell_e$ is the edge length of $e$.

The vertex curvature of a vertex $v$ is
\begin{equation}
K_v = \frac{1}{2} \sum_{e > v} K_e,
\end{equation}
where the sum is over all edges containing vertex $v$.

We now define a discrete analogue of the Einstein-Hilbert functional.

Definition 2.7. The Einstein-Hilbert-Regge functional is
\begin{equation}
\mathcal{EH}(M, T, \ell) = \sum_{v \in V} K_v.
\end{equation}

Our goal is to define constant scalar curvature metrics in a way that is analogous to the smooth setting; namely, we would like to let these metrics be critical points of the normalized $\mathcal{EH}(\ell)$ functional. In the smooth case, one usually considers a volume normalization. However, in the discrete case, since the formula for volume of a simplex is quite complicated, one may also consider a normalization which is linear in the edge lengths. Thus, we will consider the two normalizations of the $\mathcal{EH}(\ell)$ functional given in [3].

To define these normalizations, we need the definitions of the total length and the total volume of the triangulation.

Definition 2.8. The total length of $(M, T, \ell)$ is
\begin{equation}
\mathcal{L}(M, T, \ell) = \sum_{e \in E} \ell_e.
\end{equation}

Let $V_t$ be the volume of tetrahedron $t$. Then the volume of $(M, T, \ell)$ is
\begin{equation}
\mathcal{V}(M, T, \ell) = \sum_{t \in T} V_t.
\end{equation}

Definition 2.9. The length normalized Einstein-Hilbert-Regge functional is
\begin{equation}
\mathcal{LEHR}(M, T, \ell) = \frac{\mathcal{EH}(M, T, \ell)}{\mathcal{L}(M, T, \ell)}.
\end{equation}
The volume normalized Einstein-Hilbert-Regge functional is
\begin{equation}
\mathcal{VEHR}(M, T, \ell) = \frac{\mathcal{EH}(M, T, \ell)}{(\mathcal{V}(M, T, \ell))^{1/3}}.
\end{equation}

The normalizations are defined so that the functionals take the same value if all lengths are scaled by the same positive constant.

Given these definitions of normalized $\mathcal{EH}(\ell)$ functionals, we can compute their derivatives with respect to a conformal variation. We define constant scalar curvature metrics to be critical points in a conformal class of the normalized $\mathcal{EH}(\ell)$ functionals. The subsequent definitions follow immediately from the variation formulas of $\mathcal{LEHR}$ and $\mathcal{VEHR}$, which can be found in [3], Lemma 4.9.
Definition 2.10. A three-dimensional piecewise flat manifold \((M, T, \ell)\) has constant scalar curvature if it is a critical point of one of the normalized \(\mathcal{EHR}\) functionals with respect to a conformal variation.

We say \((M, T, \ell)\) is LCSC if

\[ (2.9) \quad K_v = \lambda_L L_v, \]

for all \(v \in V\). Here \(\lambda_L = \mathcal{EHR}\), and \(L_v = \frac{1}{2} \sum_{e>v} \ell_e\).

We say \((M, T, \ell)\) is VCSC if

\[ (2.10) \quad K_v = \lambda_V V_v, \]

for all \(v \in V\). Here \(\lambda_V = \frac{\mathcal{EHR}}{3V}, V_v = \frac{1}{3} \sum_{t>f>v} h_{f<t} A_f, h_{f<t}\) is the signed distance between the circumcenters of the face \(f\) and tetrahedron \(t\), and \(A_f\) is the area of face \(f\).

Remark 2.11. Note that \(\sum_{v \in V} L_v = L, \) and \(\sum_{v \in V} V_v = 3V.\)

3. Cyclic Polytopes

Our primary objects of interest in the remainder of the paper will be certain triangulations of \(S^3\). Specifically, we will study the piecewise flat 3-manifolds that are the boundary complexes of 4-dimensional cyclic polytopes. These classical combinatorial manifolds are well-known and provide examples of neighborly triangulations of \(S^3\). For the convenience of the reader, we define cyclic polytopes as in [8], and we recall some important facts.

Definition 3.1. The moment curve is a map \(M_4 : \mathbb{R} \to \mathbb{R}^4\) defined parametrically as

\[ (3.1) \quad x(t) = (t, t^2, t^3, t^4). \]

Definition 3.2. A 4-dimensional cyclic polytope, \(C(n, 4)\), is the convex hull of \(n \geq 5\) points \(x(t_i)\) on \(M_4\), where \(t_1 < t_2 < \cdots < t_n\).

We will define the set \(V\) as follows:

\[ (3.2) \quad V = \{x(t_i) \in \mathbb{R}^4 : 1 \leq i \leq n\}. \]

Here we will denote elements of \(V\) as \(v_i\) for \(1 \leq i \leq n\), where \(v_i = x(t_i)\).

The following condition on the tetrahedra in 4-dimensional cyclic polytopes is due to Gale (see [8]).

Theorem 3.3 (Gale’s Evenness Condition). Consider a cyclic 4-polytope \(C(n, 4)\). A set of four points \(\tilde{V} \subseteq V\) determines a tetrahedron in \(C(n, 4)\) if and only if every two points in \(V \setminus \tilde{V}\) are separated on \(M_4\) by an even number of points of \(\tilde{V}\).

It is well-known that for any \(n \geq 5\), the boundary complex of \(C(n, 4)\) forms a neighborly triangulation of \(S^3\). This triangulation is invariant under the vertex transitive action of the dihedral group \(D_n\) [10]. The triangulation has \(n\) vertices (given by \(V\) in Eq. 3.2), \(\frac{n(n-1)}{2}\) edges, \(n^2 - 3n\) faces, and \(\frac{n^2 - 3n}{2}\) tetrahedra.

Definition 3.4. We will let \(\partial C(n, 4)\) denote the triangulation of \(S^3\) given by the boundary complex of \(C(n, 4)\).
We define a distinguished cycle, $C$, to be the following set of edges:

$$C = \{ e \in E : e = v_i v_{i+1} \text{ for } 1 \leq i < n \text{ or } e = v_n v_1 \}. \tag{3.3}$$

A straightforward application of Theorem 3.3 allows us to ascertain when four points in $V$ determine a tetrahedron in $\partial C(n, 4)$.

**Corollary 3.5.** A set of four points in $\partial C(n, 4)$ forms a tetrahedron if and only if those four points yield two distinct non-local edges in $C$.

The simplest example of a cyclic 4-polytope is $C(5, 4)$, which is the 4-simplex. In this case, $\partial C(5, 4)$ is known as the pentachoron, which is the smallest simplicial triangulation of $S^3$. See [2] for many geometric results about this manifold.

4. Cyclic length metrics on $\partial C(n, 4)$

In this section, we define certain metrics on $\partial C(n, 4)$, known as cyclic length metrics. We will collect some useful facts about these metrics. In the next section, we will use these facts to show that these metrics have constant scalar curvature.

**Definition 4.1.** Let $\ell$ be a metric on $\partial C(n, 4)$. Then $\ell$ is called a cyclic length metric if the length of any edge $e_{ij}$ is a function of the minimum number of edges between $v_i$ and $v_j$ on $C$, where $C$ is the distinguished cycle defined by Eq. (3.3).

See Figure 1 for a representation of a cyclic length metric on $\partial C(11, 4)$.

We will denote the minimal number of edges between $v_i$ and $v_j$ on $C$ as $D_{ij}$. Note also that although the lengths of each edge $e_{ij}$ are a function of $D_{ij}$, these lengths must still satisfy the criteria required of a metric; i.e. the Cayley-Menger determinant must be positive for all tetrahedra.

**Remark 4.2.** By construction, a cyclic length metric $\ell$ on $\partial C(n, 4)$ is vertex transitive.

On the pentachoron, $\partial C(5, 4)$, any vertex transitive metric is a cyclic length metric. However, in general there may be vertex transitive metrics on $\partial C(n, 4)$ that are not cyclic length metrics.

When considering cyclic length metrics on $\partial C(n, 4)$, we require additional information about the combinatorics and symmetries of these manifolds. The analysis will vary slightly depending on whether $n$ is odd or even; for reasons to be made clear below, we let odd $n = 2m + 3$ and even $n = 2m + 2$. We summarize the results of this section in the following table:

| Table 1. Properties of cyclic length metrics on $\partial C(n, 4)$ |
|-----------------------------|-----------------------------|
| Quantity of interest | $n = 2m + 3$ | $n = 2m + 2$ |
| Number of distinct edge lengths | $m + 1$ | $m + 1$ |
| Number of tetrahedra | $2m^2 + 3m$ | $2m^2 + m - 1$ |
| Types of tetrahedra | $m$ | $m$ |
| Number of each type | $2m + 3$ | $2m + 2$ ( $m + 1$ of type $T_m^e$ ) |
Lemma 4.3. Let $\ell$ be a cyclic length metric on $\partial C(n, 4)$.

1. Let $m \in \mathbb{N}$ be such that $n = 2m + 3$. The triangulation has the following properties:
   (a) There can be up to a total of $m + 1$ distinct edge lengths.
   (b) There are $2m^2 + 3m$ total tetrahedra in the triangulation.

2. Let $m \in \mathbb{N}$ be such that $n = 2m + 2$. The triangulation has the following properties:
   (a) There can be up to a total of $m + 1$ distinct edge lengths.
   (b) There are $2m^2 + m - 1$ total tetrahedra in the triangulation.

Proof. We begin with the case of $n = 2m + 3$. To prove the first statement, we choose two arbitrary distinct $v_i, v_j \in \mathcal{V}$. Since there are $2m + 3$ vertices, there are $2m + 3$ edges in the distinguished cycle $\mathcal{C}$ (see Eq. 3.3). Recall that the edge lengths in a cyclic length metric are determined by the minimum number of edges between $v_i$ and $v_j$ on $\mathcal{C}$. Notice that the minimum number of edges in $\mathcal{C}$ separating $v_i$ and $v_j$ is at most $\lfloor \frac{2m + 3}{2} \rfloor$. This is $m + 1$, as claimed.

For the second statement, choose an arbitrary edge in $\mathcal{C}$ and a second edge in $\mathcal{C}$ not local to the first edge. By Corollary 3.3, all such combinations will generate the tetrahedra in the triangulation. The number of these combinations, and thus the number of tetrahedra, is exactly

$$\frac{(2m + 3)(2m + 3) - 3}{2} = 2m^2 + 3m.$$

In the case that $n = 2m + 2$, the analysis follows in the same fashion when one recalls there are $2m + 2$ edges in the distinguished cycle $\mathcal{C}$.

Corollary 4.4. Let $\ell$ be a cyclic length metric on $\partial C(n, 4)$. Recall that the minimum number of edges between $v_i$ and $v_j$ on $\mathcal{C}$ is denoted $D_{ij}$.
Lemma 4.5. Let \( m \in \mathbb{N} \) be such that \( n = 2m + 3 \). If there exists a path of \( p \) distinct edges between \( v_i \) and \( v_j \) on \( C \) and \( p \leq m + 1 \), then \( D_{ij} = p \). If \( m + 1 < p \), then \( D_{ij} = (2m + 3) - p \).

(2) Let \( m \in \mathbb{N} \) be such that \( n = 2m + 2 \). If there exists a path of \( p \) distinct edges between \( v_i \) and \( v_j \) on \( C \) and \( p \leq m + 1 \), then \( D_{ij} = p \). If \( m + 1 < p \), then \( D_{ij} = (2m + 2) - p \).

**Proof.** These facts follow directly from Lemma 4.3.

Recall that Euclidean tetrahedra are completely determined by their edge lengths. The following lemma gives an upper bound for the different length structures that appear on tetrahedra in \( (\partial C(n, 4), \ell) \).

**Lemma 4.5.** Let \( \ell \) a cyclic length metric on \( \partial C(n, 4) \).

(1) Let \( n = 2m + 3 \). There are at most \( m \) distinct types of tetrahedra in the triangulation, denoted type \( T_k \) for \( 1 \leq k < m \) and \( T_m^o \) for \( k = m \). For each \( k \), there are \( 2m+3 \) tetrahedra of type \( T_k \) in the triangulation.

(2) Let \( n = 2m + 2 \). There are \( m \) distinct types of tetrahedra in the triangulation, denoted type \( T_k \) for \( 1 \leq k < m \) and \( T_m^e \) for \( k = m \). For \( 1 \leq k < m \), there are \( 2m + 2 \) type \( T_k \) tetrahedra in the triangulation, whereas there are \( m + 1 \) type \( T_m^e \) tetrahedra.

**Proof.** To start, let \( n = 2m + 3 \). By the proof of Lemma 4.3, there are at most \( m \) edge lengths in the triangulation. Let the set of these edge lengths be given by \( \{L_1, L_2, \ldots, L_{m+1}\} \), where \( L_k \) corresponds to the case that \( D_{ij} = k \). Clearly the maximum number of tetrahedra is generated when all of the \( L_i \) are distinct, so we will consider that case. Choose two non-local edges in \( \mathcal{C} \) that have vertices \( v_i, v_{i+1}, v_j, v_{j+1} \). By Corollary 3.5, these four vertices generate a tetrahedron in \( \partial C(2m + 3, 4) \). We claim that this tetrahedron is determined by the distance between \( v_{i+1} \) and \( v_j \) on \( \mathcal{C} \), i.e., by \( D_{(i+1)j} \).

We first consider the case that \( D_{(i+1)j} < m \); say, \( D_{(i+1)j} = p \) for \( p < m \). Notice that the tetrahedron generated by the vertices \( v_i, v_{i+1}, v_j, v_{j+1} \) will have the following edge lengths: \( \ell(e_{i(i+1)}) = \ell(e_{j(j+1)}) = L_1 \), \( \ell(e_{(i+1)j}) = L_p \), \( \ell(e_{ij}) = \ell(e_{(i+1)(j+1)}) = L_{p+1} \), \( \ell(e_{ij}) = L_{p+2} \). See Figure 2. Since the \( L_p \) are distinct, this gives \( m - 1 \) distinct tetrahedra in \( \partial C(2m + 3, 4) \). We will call each of these types of tetrahedra type \( T_p \).

The second case is that \( D_{(i+1)j} = m \). In this case, the tetrahedron generated by the vertices \( v_i, v_{i+1}, v_j, v_{j+1} \) will have the following edge lengths: \( \ell(e_{i(i+1)}) = \ell(e_{j(j+1)}) = L_1 \), \( \ell(e_{(i+1)j}) = L_m \), \( \ell(e_{ij}) = \ell(e_{(i+1)(j+1)}) = L_{m+1} \). Since \( L_m \) is distinct, this gives \( m - 1 \) distinct tetrahedra of this form in \( \partial C(2m + 3, 4) \). We will call this type of tetrahedron type \( T_m^o \).

It is also possible that \( D_{(i+1)j} > m \); however, by Corollary 4.4 we must then have \( D_{(j+1)} < m \), so we do not generate any additional types of tetrahedra.

Finally, we show how many of each type of tetrahedra exist in our triangulation when \( n = 2m + 3 \). There are \( 2m + 3 \) edges on \( \mathcal{C} \). Notice that exactly two tetrahedra of each type will be formed with each of these edges. However, in this process we
double count the tetrahedra, so the total number of tetrahedra of a given type is $2m + 3$.

Now consider the case that $n = 2m + 2$. By Lemma 4.3.2, there are at most $m + 1$ edge lengths in the triangulation. As above, let the set of all these edge lengths be given by $\{L_1, L_2, ..., L_{m+1}\}$, where $L_k$ corresponds to the case that $D_{ij} = k$, and all of the $L_k$ are distinct. Choose two non-local edges in $C$ having vertices $v_i, v_{i+1}, v_j, v_{j+1}$. These four vertices generate a tetrahedron in $\partial C(2m + 2, 4)$, and we claim that this tetrahedron is determined by $D_{(i+1)j}$.

The case that $D_{(i+1)j} < m$ follows in exactly the same fashion as the corresponding case in the proof for $n = 2m + 3$. We obtain $m - 1$ distinct types of tetrahedra.
with edge lengths as in Figure 2 and we call each of these types of tetrahedra type $T_p$.

In the case that $D^{(i+1)}(j) = m$, we have a difference in the edge lengths as compared to the odd vertex case. The key fact is that, in this setting, $D^{(j+1)}(i) = m$. Thus, the tetrahedron generated by $v_i, v_{i+1}, v_j, v_{j+1}$ will have the following edge lengths: $\ell(e_{i(i+1)}) = \ell(e_{j(j+1)}) = L_1$, $\ell(e_{i(i+1)}) = \ell(e_{j(j+1)}) = L_m$, $\ell(e_{ij}) = \ell(e_{(i+1)(j+1)}) = L_{m+1}$. See Figure 4. We call this a type $T'_m$ tetrahedron.

As we saw previously, if $D^{(i+1)}(j) > m$, no new tetrahedra are generated.

Finally, we show how many of each type of tetrahedra are in our triangulation when $n = 2m + 2$. There are $2m + 2$ edges on $C$. Exactly two tetrahedra of each type $T_k$ for $1 \leq k < m$ will be formed with each of these edges. Further, each edge will form exactly one type $T'_m$ tetrahedron. In this process, we double count the tetrahedra. So for $1 \leq k < m$, the total number of type $T_k$ is $\frac{2(2m+2)}{2} = 2m + 2$. Similarly, the total number of type $T'_m$ is $\frac{2m+2}{2} = m + 1$.

Notice that the tetrahedra of type $T'_m$ have all pairs of opposite edges equal. These tetrahedra are known as “equihedral”, “isosceles”, or “equifacial” tetrahedra in the literature, and they are very well-studied. Indeed, there are over 100 equivalent conditions that characterize the equihedral tetrahedra. They also give rise to constant scalar curvature metrics on the double tetrahedron (see [3]). One of the equivalent conditions that determines an equihedral tetrahedron is that opposite dihedral angles are equal. The following proposition shows an analogue of that condition for tetrahedra of type $T_k$.

**Proposition 4.6.** Let $\ell$ be a cyclic length metric on $\partial C(n,4)$. In a tetrahedron of type $T_k$ for $1 \leq k \leq m$, edges with equal lengths have the same dihedral angles.

**Proof.** Consider a type $T_k$ tetrahedron for some $k \in [1, m]$. By Lemma 4.5 there are two pairs of opposite edges that have equal lengths and one pair that may or may not. (If all of the edge lengths are distinct, then this pair will not have equal
Thus we see that $\beta_{ij} = \beta_{jk}$. The argument that $\beta_{ij} = \beta_{jk}$ follows in the same way.

One can use the spherical law of cosines to compute $\beta_{il}$ to be

$$\beta_{il} = \cos^{-1} \left( \frac{a^2+b^2-c^2}{2ab} - \frac{(-a^2+b^2+c^2)(a^2+c^2-d^2)}{4abc^2} \sqrt{1 - \frac{(-a^2+b^2+c^2)^2}{4b^2c^2}} \right).$$

Similarly,

$$\beta_{jk} = \cos^{-1} \left( \frac{a^2-c^2+d^2}{2ad} - \frac{(a^2-b^2+c^2)(-a^2+c^2+d^2)}{4ac^2d} \sqrt{1 - \frac{(a^2-b^2+c^2)^2}{4c^2d^2}} \right).$$

Algebraic manipulation shows that Eq. (4.1) becomes

$$\beta_{il} = \cos^{-1} \left( \frac{a^4 - (b^2 - 2c^2 + d^2) a^2 - 3c^4 + c^2 d^2 + b^2 (c^2 + d^2)}{4abc^2 \sqrt{(a^4-2(b^2+c^2)a^2+(b^2-c^2)^2)(a^4-2(c^2+d^2)a^2+(c^2-d^2)^2)}} \right),$$

while Eq. (4.2) becomes

$$\beta_{jk} = \cos^{-1} \left( \frac{a^4 - (b^2 - 2c^2 + d^2) a^2 - 3c^4 + c^2 d^2 + b^2 (c^2 + d^2)}{4ac^2d \sqrt{(a^4-2(b^2+c^2)a^2+(b^2-c^2)^2)(a^4-2(c^2+d^2)a^2+(c^2-d^2)^2)}} \right).$$

Thus we see that $\beta_{il} = \beta_{jk}$. \hfill $\square$

**Lemma 4.7.** Let $\ell$ be a cyclic length metric on $\partial C(n,4)$.

1. Let $n = 2m + 3$. Any vertex $v_i \in V$ is local to four tetrahedra of the types $T_k$, with $1 \leq k < m$, and to four tetrahedra of the type $T_m$.

2. Let $n = 2m + 2$. Any vertex $v \in V$ is local to four tetrahedra of the type $T_k$, with $1 \leq k < m$, and to two tetrahedra of the type $T_m$.

**Proof.** We begin with the case that $n = 2m + 3$, and we choose a vertex $v_0 \in V$.

It is now possible to construct a tetrahedron of type $T_k$ with vertices $v_0$, $v_1$, $v_{k+1}$, $v_{k+2}$. (Notice that this construction is valid for all $1 \leq k < m$; in particular, for type $T_m$.) Let this specific tetrahedron be labeled $t_{k1}$.

Define a map $f : V \to V$ by

$$f(v_i) = v_{i+h(\text{mod } 2m+3)},$$

where $h$ solves the equation $0 \equiv k + 1 + h(\text{mod } 2m+3)$. Note that under this map, $v_0 \to v_h$, $v_1 \to v_{h+1}$, $v_{k+1} \to v_0$, and $v_{k+2} \to v_1$. Since this map preserves $D_{ij}$ for any two vertices $v_i$ and $v_j$, this is still a type $T_k$ tetrahedron. Also note that this tetrahedron is distinct from $t_{k1}$; i.e. that the new set of vertices $v_h, v_{h+1}, v_0, v_1$ is distinct from our original set of vertices. If not, then we would have $h \equiv k + 1(\text{mod } 2m+3)$, which would imply that $h \equiv 0(\text{mod } 2m+3)$, which contradicts our choice of $h$. Let this tetrahedron be labeled $t_{k2}$.

Define the tetrahedron $t_{k3}$ by vertices $v_{2m+2}, v_0, v_k, v_{k+1}$. This is distinct from $t_{k1}$ and $t_{k2}$, and is clearly a tetrahedron of type $T_k$. Now apply $f$ to $t_{k3}$. Under this map, $v_{2m+2} \to v_{h-1}$, $v_0 \to v_h$, $v_{k+1} \to v_0$, and $v_k \to v_{2m+2}$. Label this tetrahedron $t_{k4}$. 

...
Again, we note that \( t_{k4} \) is distinct from \( t_{k3} \). If not, then we would have \( h - 1 \equiv k \pmod{2m + 3} \), which gives a similar contradiction as above. Also note that \( t_{k4} \) is distinct from \( t_{k3} \) since \( k + 1 \neq k + 2 \), and it is distinct from \( t_{k2} \) since \( h \neq h + 1 \). As our choice of \( v_0 \) was arbitrary, every vertex is part of at least four distinct tetrahedra of type \( T_k \), with \( 1 \leq k \leq m \).

We claim that in fact each vertex is contained in exactly four distinct tetrahedra of type \( T_k \). Let \( v_{\text{val}} \) denote the number of tetrahedra incident to vertex \( v \). From above, we know that \( v_{\text{val}} \geq 4m \). In fact, since our triangulation is vertex transitive, \( v_{\text{val}} = \frac{4T}{V} \), where \( T \) is the total number of tetrahedra in the triangulation and \( V \) is the total number of vertices. Thus \( v_{\text{val}} = \frac{4(2m^3 + 3m)}{2m + 3} = 4m \), so each vertex is contained in exactly four distinct tetrahedra of type \( T_k \) for \( 1 \leq k \leq m \) (including type \( T^e_k \)) as required.

The proof in the case that \( n = 2m + 2 \) follows in much the same way for type \( T_k \) tetrahedra when \( 1 \leq k < m \). One can simply change the definition of the function \( f \) in Eq. 4.5 to be \( f(v_i) = v_{i+h} \pmod{2m+2} \). However, to construct the two tetrahedra of type \( T_{\ell, m} \), we choose the vertex sets \( v_0, v_1, v_{m+1}, v_{m+2} \) and \( v_2m+2, v_0, v_m, v_{m+1} \). Using a similar counting argument as above, we see that each vertex lies in four type \( T_k \) tetrahedra for \( 1 \leq k < m \) and in two type \( T^e_m \) tetrahedra.

\[ \Box \]

5. Cyclic Length Metrics have Constant Scalar Curvature

In this section, we will use the results from the previous section to show that cyclic length metrics on \( \partial C(n, 4) \) have constant scalar curvature; namely, that they solve both Eq. 2.9 and Eq. 2.10.

We first show that both the edge and vertex curvatures (Eqs. 2.2 and 2.3) are constant for cyclic length metrics on \( \partial C(n, 4) \).

**Lemma 5.1.** Let \( \ell \) be a cyclic length metric on \( \partial C(n, 4) \). For all \( v \in V \), \( K_v \) is constant.

**Proof.** We will provide the proof for \( n = 2m + 3 \). The case of \( n = 2m + 2 \) follows similarly.

First we will show that all edges with equal length have equal edge curvature, \( K_v \). We will denote the \( m + 1 \) possible edge lengths as \( \{L_1, L_2, \ldots, L_{m+1}\} \) as in the proof of Lemma 4.5. Suppose \( i < j \). Choose an arbitrary \( e_{ij} \in E \) with length \( L_p \). Recall this corresponds to the case that \( D_{ij} = p \). We will show that every edge with length \( L_p \) is in the same number of certain types of tetrahedra. By Proposition 4.6 this will imply that the dihedral angles, and thus the edge curvatures, are the same.

First, let \( p \neq 1, 2, m + 1 \). The tetrahedra containing edge \( e_{ij} \) clearly contain the vertices \( v_i \) and \( v_j \). By Corollary 3.5, the other two vertices can be combinations of \( v_{i+1} \) or \( v_{i-1} \) and \( v_{j+1} \) or \( v_{j-1} \). Simply by observation, one can determine what type of tetrahedra corresponds to each vertex choice. For example, both the vertex sets \( v_{i-1}, v_i, v_j \) and \( v_{i+1}, v_i, v_{j+1} \) yield a type \( T_{p-1} \) tetrahedron. The vertex set \( v_{i-1}, v_i, v_{j+1}, v_j \) yields a type \( T_p \) tetrahedron, while the set \( v_{i+1}, v_i, v_{j-1}, v_j \) gives a type \( T_{p-2} \) tetrahedron.

Secondly, let \( p = m + 1 \). Again, we simply list the tetrahedra containing \( e_{ij} \) and note their types. The vertex set \( v_{i+1}, v_i, v_{j-1}, v_j \) generates a type \( T_{m-1} \) tetrahedron. However, all of the sets \( v_{i-1}, v_i, v_{j-1}, v_j \); \( v_{i+1}, v_i, v_{j+1}, v_j \); and \( v_{i-1}, v_i, v_{j+1}, v_j \) yield type \( T^e_m \) tetrahedra.
For the third case, let \( p = 2 \). In this case, notice that only one of \( v_{i+1} \) and \( v_{j-1} \) can be vertices of a tetrahedron, unless \( i = 1 \) and \( j = 2m + 2 \). For the former case, the vertex sets \( v_{i-1}, v_i, v_{j-1}, v_j \) and \( v_{i+1}, v_i, v_{j+1}, v_j \) yield tetrahedra of type \( T_1 \), whereas \( v_{i-1}, v_i, v_{j+1}, v_j \) gives a type \( T_2 \) tetrahedron. In the latter case, the analysis is the same, but notice that only one of \( v_{i-1} \) and \( v_{j+1} \) can be vertices.

Finally, let \( p = 1 \). Since there are \( 2m + 3 \) edges in \( \mathcal{C} \), Corollary 3.3 implies that there are \( 2m \) tetrahedra containing both \( e_{ij} \) and an edge that is not local to \( e_{ij} \) in \( \mathcal{C} \). Notice also that the tetrahedron with vertices \( v_{i-1}, v_i, v_j, v_{j+1} \) contains \( e_{ij} \) and is not one of the \( 2m \) tetrahedra previously mentioned. Thus, in this case, \( e_{ij} \) is contained in \( 2m + 1 \) tetrahedra. The vertex set \( v_{i-1}, v_i, v_j, v_{j+1} \) yields a tetrahedron of type \( T_1 \). Additionally, one can see that \( e_{ij} \) is in two of each type of \( T_k \) for \( 1 \leq k < m \) and two of type \( T_m^o \).

Therefore the edge curvature of any given edge depends only on the length of that edge, so edges with equal length have equal edge curvature. By Remark 4.2, the metric is vertex transitive, so the sets of edge lengths of the edges incident to each vertex are the same. Together, these facts imply that for all \( v \in V \), 
\[
K_v = \frac{1}{3} \sum_{e \ni v} K_e
\]
is constant. \( \square \)

**Corollary 5.2.** Let \( \ell \) be a cyclic length metric on \( \partial C(n, 4) \). Then \( \sum_e K_e = nK_v \).

*Proof. This result follows directly from Lemma 5.1. \( \square \)*

**Lemma 5.3.** Let \( \ell \) be a cyclic length metric on \( \partial C(n, 4) \). For all \( v \in V \), \( L_v = \frac{1}{n} L \).

*Proof. Recall \( L_v = \frac{1}{3} \sum_{e \ni v} \ell_e \), as in Definition 2.10. Since a cyclic length metric is vertex transitive, \( L_v \) is clearly constant for all \( v \in V \). Additionally, since \( \sum_{v \in V} L_v = L \), we see that \( L_v = \frac{1}{n} L \) as required. \( \square \)*

**Lemma 5.4.** Let \( \ell \) be a cyclic length metric on \( \partial C(n, 4) \). For all \( v \in V \), \( V_v = \frac{2}{n} V \).

*Proof. We begin the proof by letting \( n = 2m + 3 \). Recall the definition of \( V_v \) as given in Definition 2.10. For any vertex \( v \in V \), \( V_v = \frac{1}{3} \sum_{t \geq v} \sum_{(j,k,l) \leq t} h_{ijk,l} A_{ijk} \). Note that this sum is over all tetrahedra containing \( v \). By Lemma 4.1, each vertex is contained in exactly \( 4m \) tetrahedra. More precisely, each vertex is contained in four of each type \( T_p \), tetrahedra for \( 1 \leq p < m \) and four of tetrahedra of type \( T_m^o \). Thus we can split up \( V_v \) into \( m \) partial sums, one for each type of tetrahedron. Note that every tetrahedron of type \( T_p \) is geometrically identical, so the sum \( \sum_{(j,k,l) \leq t} h_{ijk,l} A_{ijk} \) only varies in regards to the choice of vertices.

In the tetrahedra of type \( T_p \), a given vertex \( v \) has each of the sets of possible incident edges appear one time. In other words, the vertex \( v \) “plays the role” of each of the vertices \( i, j, k, l \) in Figure 2 or Figure 3. There are four possible sets of incident edges. Thus for all \( v \in V \), the partial sums of \( V_v \) with respect to the tetrahedra of type \( T_k \) are equal. Since this holds for every type \( T_k \) for \( 1 \leq k \leq m \), the total sums \( V_v \) are equal for all \( v \in V \). Thus, \( V_v \) is constant, so \( \sum_{v \in V} V_v = n V_v \) for all \( v \in V \). Recall that \( \sum_{v \in V} V_v = 3V \), so \( V_v = \frac{2}{3} V \) as required.

The proof for \( n = 2m + 2 \) follows in the same way. The only difference is that a vertex \( v \) is contained in only two type \( T_m^o \) tetrahedra. However, in this case, notice that all of the vertices have the same sets of incident edge lengths (see Figure 4), so again the sum \( V_v \) equal to the same constant for all \( v \). \( \square \)

We now prove the main theorem of this section; namely that cyclic length metrics on \( \partial C(n, 4) \) have constant scalar curvature.
Theorem 5.5. Let $\ell$ be a cyclic length metric on $\partial C(n, 4)$. This metric is both $LCSC$ and $VCSC$; i.e. it satisfies both Eq. 2.9 and Eq. 2.10.

Proof. We will begin by showing this metric is $LCSC$. By Corollary 5.2 and Lemma 5.3,

$$\lambda_L = \sum_e K_e = \frac{nK_v}{L_v} = n \frac{v}{L_v}$$

for all $v \in V$. Thus, for all $v \in V$, $K_v = \lambda_L L_v$. Therefore $\ell$ is $LCSC$.

We will now show that this metric is $VCSC$. Applying Corollary 5.2 and Lemma 5.4 yields

$$\lambda_V = \sum_e K_e = \frac{nK_v}{3V} = \frac{K_v}{V_v}$$

for all $v \in V$. Thus, for all $v \in V$, $K_v = \lambda_V V_v$. Therefore, $\ell$ is $VCSC$. $\square$

6. Open Questions

As mentioned in Definition 2.10, $LCSC$ and $VCSC$ metrics occur at critical points of the $LEHR$ and $VEHR$ functionals within a conformal class. We have shown that cyclic length metrics on $\partial C(n, 4)$ are both $LCSC$ and $VCSC$, and we would like to know with what frequency these metrics occur; i.e. in which conformal classes these cyclic length metrics appear. One may hope that they occur in every conformal class. However the next proposition shows that this is not the case.

Proposition 6.1. On $\partial C(n, 4)$, there exists a conformal class that does not admit a cyclic length metric.

Proof. Recall from Definition 2.4 that a conformal class is equivalent to a choice of $L_{ij}$ for all $e_{ij} \in E$. Then the edge lengths within this conformal class are given by $\ell_{ij} = \exp \left[ \frac{1}{2}(f_i + f_j) \right] L_{ij}$ for $e_{ij}$.

We begin by supposing that all conformal classes on $\partial C(n, 4)$ admit a cyclic length metric, $\ell$. Choose a vertex $v_0$ and label the remaining vertices $v_1, v_2, \ldots, v_{n-1}$ proceeding clockwise around $C$. Recall that in a cyclic length metric (6.1)

$$\ell_{01} = \ell_{12} = \ell_{23} = \ell_{34}, \text{ and}$$

(6.2)

$$\ell_{02} = \ell_{13} = \ell_{24}.$$

Now consider the conformal class such that $L_{ij}=1$ for all $e_{ij} \in E$, except for $L_{12} = a \neq 1$.

Equation 6.1 implies both that $e^{\frac{1}{2}f_0} = ae^{\frac{1}{2}f_2}$ and $e^{\frac{1}{2}f_3} = ae^{\frac{1}{2}f_1}$. Substituting this into the first equality in Equation 6.2 shows that $f_1 = f_2$. The equality $\ell_{12} = \ell_{34}$ then yields

$$ae^{\frac{1}{2}f_1}e^{\frac{1}{2}f_2} = e^{\frac{1}{2}f_3}e^{\frac{1}{2}f_4},$$

$$a(e^{\frac{1}{2}f_1})^2 = ae^{\frac{1}{2}f_1}e^{\frac{1}{2}f_4}$$

$$f_1 = f_4.$$

But this implies that $l_{24} = e^{f_2}$ and $l_{13} = ae^{f_1}$. Equation 6.2 then gives $ae^{f_1} = e^{f_1}$, which contradicts our choice of $a$. Therefore this conformal class does not admit a cyclic length metric. $\square$
One would like to know if there is some sort of generalization of a cyclic length metric that has constant scalar curvature and that occurs in all conformal classes. Additionally, one might do an analysis similar to that in [3] of the Hessians of the $\mathcal{LEHR}$ and $\mathcal{VEHR}$ at a cyclic length metric to determine the local behavior at these critical points.

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