A \textit{q}-ANALOG OF EULER’S DECOMPOSITION FORMULA FOR THE DOUBLE ZETA FUNCTION

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Abstract. The double zeta function was first studied by Euler in response to a letter from Goldbach in 1742. One of Euler’s results for this function is a decomposition formula, which expresses the product of two values of the Riemann zeta function as a finite sum of double zeta values involving binomial coefficients. In this note, we establish a \textit{q}-analog of Euler’s decomposition formula. More specifically, we show that Euler’s decomposition formula can be extended to what might be referred to as a “double \textit{q}-zeta function” in such a way that Euler’s formula is recovered in the limit as \textit{q} tends to 1.

1. Introduction

The Riemann zeta function is defined for \( \Re(s) > 1 \) by

\[
\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1}
\]

Accordingly,

\[
\zeta(s, t) := \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k=1}^{n-1} \frac{1}{k^t}, \quad \Re(s) > 1, \quad \Re(s+t) > 2, \tag{2}
\]

is known as the double zeta function. The sums (2), and more generally those of the form

\[
\zeta(s_1, s_2, \ldots, s_m) := \sum_{k_1 > k_2 > \cdots > k_m > 0} \prod_{j=1}^{m} \frac{1}{k_j^{s_j}}, \quad \sum_{j=1}^{m} \Re(s_j) > n, \quad n = 1, 2, \ldots, m, \tag{3}
\]

have attracted increasing attention in recent years; see eg. [2, 4, 5, 6, 8, 9, 10, 12, 13, 18]. The survey articles [7, 14, 22, 23, 25] provide an extensive list of references. In (3) the sum is over all positive integers \( k_1, \ldots, k_m \) satisfying the indicated inequalities. Note that with positive integer arguments, \( s_1 > 1 \) is necessary and sufficient for convergence.

The problem of evaluating sums of the form (2) for integers \( s > 1, t > 0 \) seems to have been first proposed in a letter from Goldbach to Euler [16] in 1742. (See also [15, 17] [16] [18] [22] [23] [25]...
Among other results for (2), Euler proved that if \( s - 1 \) and \( t - 1 \) are positive integers, then the decomposition formula

\[
\zeta(s)\zeta(t) = \sum_{a=0}^{s-1} \left( \frac{a + t - 1}{t - 1} \right) \zeta(t, a, s - a) + \sum_{a=0}^{t-1} \left( \frac{a + s - 1}{s - 1} \right) \zeta(s, a, t - a)
\]

(4) holds. A combinatorial proof of Euler’s decomposition formula (4) based on the Drinfel’d integral representations \([1, \text{eq. (10)}]\). It is of course well-known that (4) can also be proved algebraically by summing the partial fraction decomposition \([20, \text{p. 48}] \) \([19, \text{Lemma 3.1}]\)

\[
\frac{1}{x^s(c - x)^t} = \sum_{a=0}^{s-1} \left( \frac{a + t - 1}{t - 1} \right) \frac{1}{x^{s-a}} c^{t+a} + \sum_{a=0}^{t-1} \left( \frac{a + s - 1}{s - 1} \right) c^{s+a} (c - x)^{t-a}
\]

(6) over appropriately chosen integers \( x \) and \( c \). (See eg. \([3]\).)

A \( q \)-analog of (3) was independently introduced in \([11, 21, 24]\) as

\[
\zeta[s_1, s_2, \ldots, s_m] := \sum_{k_1 > k_2 > \cdots > k_m > 0} \prod_{j=1}^{m} q^{(s_j - 1)k_j} [k_j]_q^{s_j},
\]

(7) where

\[
[k]_q := \sum_{j=0}^{k-1} q^j = \frac{1 - q^k}{1 - q}, \quad 0 < q < 1.
\]

Observe that we now have

\[
\zeta(s_1, \ldots, s_m) = \lim_{q \to 1^-} \zeta[s_1, \ldots, s_m],
\]

so that (7) represents a generalization of (3). In this note, we establish a \( q \)-analog of Euler’s decomposition formula (4).
2. Main Result

Our \( q \)-analog of Euler’s decomposition formula naturally requires only the \( m = 1 \) and \( m = 2 \) cases of (7); specifically the \( q \)-analogs of (1) and (2) given by

\[
\zeta[s] = \sum_{n>0} \frac{q^{(s-1)n}}{[n]^q} \quad \text{and} \quad \zeta[s,t] = \sum_{n,k>0} \frac{q^{(s-1)n}q^{(k-1)t}}{[n]^q[k]^q}.
\]  

We also define, for convenience, the sum

\[
\varphi[s] := \sum_{n=1}^\infty (n-1)q^{(s-1)n} \frac{nq^{(s-1)n}}{[n]^q} - \zeta[s].
\]

We can now state our main result.

**Theorem 1.** If \( s - 1 \) and \( t - 1 \) are positive integers, then

\[
\zeta[s]\zeta[t] = \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+t-1}{t-1} \binom{t-1}{b} (1-q)^b \zeta[t+a,s-a-b] \\
+ \sum_{a=0}^{t-1} \sum_{b=0}^{t-1-a} \binom{a+s-1}{s-1} \binom{s-1}{b} (1-q)^b \zeta[s+a,t-a-b] \\
- \sum_{j=1}^{\min(s,t)} \frac{(s+t-j-1)!}{(s-j)!(t-j)!} \frac{(1-q)^j}{(j-1)!} \varphi[s+t-j].
\]

Observe that the limiting case \( q = 1 \) of Theorem 1 reduces to Euler’s decomposition formula (1).

3. A Differential Identity

Our proof of Theorem 1 relies on the following identity.

**Lemma 1.** Let \( s \) and \( t \) be positive integers, and let \( x \) and \( y \) be non-zero real numbers. Then for all real \( q \),

\[
\frac{1}{x^sy^t} = \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+t-1}{t-1} \binom{t-1}{b} (1-q)^b(1+(q-1)y)^a(1+(q-1)x)^{t-1-b} \\
\times x^{s-a-b}(x+y+(q-1)xy)^{t+a} \\
+ \sum_{a=0}^{t-1} \sum_{b=0}^{t-1-a} \binom{a+s-1}{s-1} \binom{s-1}{b} (1-q)^b(1+(q-1)x)^a(1+(q-1)y)^{s-1-b} \\
\times y^{t-a-b}(x+y+(q-1)xy)^{s+a} \\
- \sum_{j=1}^{\min(s,t)} \frac{(s+t-j-1)!}{(s-j)!(t-j)!} \frac{(1-q)^j}{(j-1)!} \frac{(1+(q-1)y)^{s-j}(1+(q-1)x)^{t-j}}{(x+y+(q-1)xy)^{s+t-j}}.
\]
Proof. Apply the partial differential operator
\[
\frac{1}{(s-1)!} \left( -\frac{\partial}{\partial x} \right)^{s-1} \left( -\frac{\partial}{\partial y} \right)^{t-1}
\]
to both sides of the identity
\[
\frac{1}{xy} = \frac{1}{x+y+(q-1)xy} \left( \frac{1}{x} + \frac{1}{y} + q - 1 \right).
\]

Observe that when \( q = 1 \), Lemma 1 reduces to the identity
\[
\zeta[s] \zeta[t] = \sum_{a=0}^{s-1} \sum_{b=0}^{t-1-a} \frac{1}{x^{s-a}(x+y)^{t+a}} + \sum_{a=0}^{t-1} \sum_{b=0}^{s-1-a} \frac{1}{x^s(y)^{t-1}} (1-\frac{q}{x+y})^{s+a} y^{t-a},
\]
from which the partial fraction identity (6) (proved by induction in [19]) trivially follows.

4. Proof of Theorem 1

First, observe that if \( s > 1 \) and \( t > 1 \), then from (8),
\[
\zeta[s] \zeta[t] = \sum_{n=1}^{\infty} \sum_{u+v=n} \frac{q^{(s-1)u}}{[u]_q^s} \cdot \frac{q^{(t-1)v}}{[v]_q^t},
\]
where the inner sum is over all positive integers \( u \) and \( v \) such that \( u + v = n \). Next, apply Lemma 1 with \( x = [u]_q, y = [v]_q \), noting that then
\[
1 + (q - 1)x = q^u, \quad 1 + (q - 1)y = q^v, \quad x + y + (q - 1)xy = [u + v]_q.
\]
After interchanging the order of summation, there comes
\[
\zeta[s] \zeta[t] = \sum_{a=0}^{s-1} \sum_{b=0}^{t-1-a} \frac{1}{x^{s-a}(x+y)^{t+a}} \left( \begin{array}{c} t-1 \\ b \end{array} \right) \left( \begin{array}{c} t-1 \\ b \end{array} \right) (1-\frac{q}{x+y})^{s+a} y^{t-a} S[s+t-1-j, s, t, a, b]
\]
\[
+ \sum_{a=0}^{t-1} \sum_{b=0}^{s-1-a} \frac{1}{x^s(y)^{t-1}} \left( \begin{array}{c} s-1 \\ b \end{array} \right) \left( \begin{array}{c} s-1 \\ b \end{array} \right) (1-\frac{q}{x+y})^{s+a} y^{t-a} S[t, s, a, b]
\]
\[
- \sum_{j=1}^{\min(s,t)} \frac{(s+t-j-1)!}{(s-j)! (t-j)!} \frac{(1-q)^j}{(j-1)!} T[s+t, j],
\]
where

\[ S[s, t, a, b] = \sum_{n=1}^{\infty} \sum_{u+v=n} q^{(s-1)u} q^{(t-1)v} q^{(t-1-b)u} q^{av} = \sum_{n=1}^{\infty} \sum_{u+v=n} q^{(t+a-1)(u+v)} q^{(s-a-b-1)u} \]

\[ = \sum_{n=1}^{\infty} q^{(t+a-1)n} \sum_{u=n}^{n-1} q^{(s-a-b-1)u} \]

\[ = \zeta[t + a, s - a - b] \]

and

\[ T[s, t, j] = \sum_{n=1}^{\infty} \sum_{u+v=n} q^{(s-1)u} q^{(t-1)v} q^{(t-j)u} q^{(s-j)v} = \sum_{n=1}^{\infty} \sum_{u+v=n} q^{(s+t-j-1)(u+v)} \]

\[ = \varphi[s + t - j]. \]

5. Final Remarks

In [24], Zhao also gives a formula for the product \( \zeta[s] \zeta[t] \). However, Zhao’s formula is considerably more complicated than ours, as it is derived based on the \( q \)-shuffle rule [7, 11] satisfied by the Jackson \( q \)-integral analogs of the representations [5]. Of course, we also have the very simple \( q \)-stuffle [11] formula \( \zeta[s] \zeta[t] = \zeta[s, t] + \zeta[t, s] + \zeta[s+t] + (1-q) \zeta[s+t-1] \).

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