Unitary Representations of Lattices of Free Nilpotent Lie Groups of Step-Two

Vignon Oussa
Dept. of Mathematics
Bridgewater State University
Bridgewater, MA 02325 U.S.A.

June 2013

Abstract

Using a theorem proved by Bekka and Driutti, we show that if \( \mathfrak{g} \) is a freely generated nilpotent Lie algebra of step-two, then almost every irreducible representation of the corresponding Lie group restricted to some lattice \( \Gamma \) is an irreducible representation of \( \Gamma \) if the dimension of the Lie algebra is odd. However, if the dimension of the Lie algebra is even, then almost every unitary irreducible representation of the Lie group restricted to \( \Gamma \) is reducible.

1 Introduction

As Gabor theory continues to surge in popularity, it is increasingly drawing a lot of attention to the representation theory of nilpotent groups. This unusual connection is easily discovered through Fourier analysis. In fact, the conjugation of a translation operator by the Plancherel transform defined on \( L^2(\mathbb{R}) \) is a modulation operator. The group generated by the continuous family of translation and modulation operators is called the reduced Heisenberg group. Its universal covering group is a simply connected, connected nilpotent Lie group with Lie algebra spanned by \( X_1, X_2, X_3 \) with non trivial Lie brackets \( [X_3, X_2] = X_1 \). In fact the Heisenberg Lie algebra is a free nilpotent Lie algebra of step-two and two generators. The set of infinite dimensional
unitary irreducible representations of the Heisenberg group up to equivalence is parametrized by the punctured line \( \mathbb{R}^* \) (Chapter 7, [3]). For each element in the punctured line, the corresponding unitary irreducible representation is a Schrödinger representation which plays a central role in Gabor theory. Let \( \pi_\lambda \) be a Schrödinger representation. This representation acts on \( L^2(\mathbb{R}) \) as follows

\[
\begin{align*}
\pi_\lambda (\exp x_3 X_3) F(t) &= F(t - x_3) \\
\pi_\lambda (\exp x_2 X_2) F(t) &= e^{-2\pi i \lambda x_2} F(t) \\
\pi_\lambda (\exp x_1 X_1) F(t) &= e^{2\pi i \lambda x_1} F(t).
\end{align*}
\]

Thus, the family of functions \( \pi_\lambda (\exp (Z X_2) \exp (Z X_3)) F \) is a Gabor system (see [4]). Now, if \(|\lambda| \leq 1\), then there exists a function \( F \) such that the countable family of vectors \( \pi_\lambda (\exp (Z X_2) \exp (Z X_3)) F \) is a complete set in \( L^2(\mathbb{R}) \) and forms what is called a Gabor wavelet system (see [4]). In fact, the construction of Gabor wavelets is a very active area of research (see [4] [5]). We remark that although the restriction of \( \pi_\lambda \) to the discrete group \( \exp (Z X_1) \exp (Z X_2) \exp (Z X_2) \exp (Z X_1) \) is also irreducible for \( \mu \)-a.e. \( \lambda \) in the dual of the Lie algebra. This surprising fact is easily explained. In fact, using the orbit method (see [2]), and some formal calculations, it is not hard to see that almost every infinite dimensional irreducible representation of the group is parametrized by the manifold

\[
\Lambda = \{ (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^4 : \lambda_3 \neq 0 \}.
\]
For each \((\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \Lambda\), the corresponding irreducible representation is realized as acting in \(L^2(\mathbb{R})\) as follows

\[
\begin{align*}
\pi(\lambda_1, \lambda_2, \lambda_3, \lambda_4) (\exp x_3X_3) F(t) &= e^{2\pi itx_3\lambda_3} F(t) \\
\pi(\lambda_1, \lambda_2, \lambda_3, \lambda_4) (\exp x_2X_2) F(t) &= F(t - x_2) \\
\pi(\lambda_1, \lambda_2, \lambda_3, \lambda_4) (\exp x_1X_1) F(t) &= e^{2\pi ix_1\lambda_4} e^{2\pi itx_1\lambda_1} e^{-2\pi ix_1^2\lambda_4\lambda_3} e^{-\frac{2\pi ix_1^2\lambda_2\lambda_1}{\lambda_3}} F\left(t - \frac{\lambda_2}{\lambda_3} x_1\right) \\
\pi(\lambda_1, \lambda_2, \lambda_3, \lambda_4) (\exp z_kZ_k) F(t) &= e^{2\pi iz_k\lambda_k} F(t) .
\end{align*}
\]

Based on the action of the irreducible representation described above, it is clear that in fact \(\pi_\lambda|_\Gamma\) is irreducible a.e. Before we introduce the general case considered in this paper, we recall a theorem which is due to Bekka and Driutti \([1]\).

**Theorem 1** Let \(N\) be a nilpotent Lie group with rational structure. Let \(\Gamma\) be a lattice subgroup of \(N\). Let \(\lambda \in n^*\) and \(\pi_\lambda\) its corresponding representation. Then the representation \(\pi_\lambda|_\Gamma\) which is the restriction of \(\pi_\lambda\) to \(\Gamma\) is irreducible if and only if the null-space of the matrix \([\lambda [X_i, X_j]]_{1 \leq i, j, n}\) with respect to a fixed Jordan H"{o}lder basis of the Lie algebra is not contained in a proper rational ideal of \(n\).

Bekka and Driutti have also provided a rather simple algorithm to determine whether a sub-algebra of \(n\) is contained in a rational ideal of \(n\) or not (Proposition 1.1. \([1]\)).

1. Let \(\mathfrak{h}\) be a sub-algebra of \(n\). Fix a Jordan H"{o}lder basis \(\{X_1, \ldots, X_n\}\) for \(n\) passing through \([n, n]\)

2. If there exists \(X \in \mathfrak{h}\) such that \(X = \sum_{i=1}^{n} x_iX_i\) and if

\[
\dim_{\mathbb{Q}} (x_{\dim[n, n]+1}, \ldots, x_n) = n - \dim [n, n]
\]

then \(\mathfrak{h}\) is **not** contained in a proper rational ideal of \(n\)

3. Otherwise, \(\mathfrak{h}\) is contained in a proper rational ideal of \(n\)
Coming back to the example of the Heisenberg group which was discussed earlier, we observe that for each linear functional \( \lambda \in \mathbb{R}^* \) satisfying \( \lambda(X_1) \neq 0 \), the nullspace of the matrix

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \lambda(X_1) \\
0 & -\lambda(X_1) & 0
\end{pmatrix}
\]

with respect to the ordered basis \( \{X_1, X_2, X_3\} \) is the center of the Lie algebra which is a rational ideal itself. Thus according to Theorem [1], the restriction of every Schrödinger representation to the lattice \( \exp(ZX_1) \exp(ZX_2) \exp(ZX_2) \) is always reducible.

Using the theorem and algorithm above, we are able to show the following. Let \( f_{m,2} \) be the free nilpotent Lie algebra of step-two on \( m \) generators \( (m > 1) \). Let \( \Gamma \) be a lattice subgroup of \( f_{m,2} \) and let \( \mu \) be the canonical Lebesgue measure on \( f_{m,2}^* \). Then

**Theorem 2** If \( m \) is odd then for \( \lambda \in \mathbb{R}^* \), \( \pi_{\lambda}|\Gamma \) is irreducible \( \mu \)-a.e.

**Theorem 3** If \( m \) is even then for \( \lambda \in \mathbb{R}^* \), \( \pi_{\lambda}|\Gamma \) is reducible \( \mu \)-a.e.

This paper is organized as follows. In the second section of the paper, we introduce some basic facts about free nilpotent Lie algebras of step-two. In the third section of the paper, we prove Theorem 2 and in the last section, we prove Theorem 3.

### 1.1 Preliminaries

Let \( n \) be a nilpotent Lie algebra of dimension \( n \) over \( \mathbb{R} \) with corresponding Lie group \( N = \exp \mathfrak{n} \). We assume that \( N \) is simply connected and connected. It is well-known that if we write

\[
\mathfrak{n}_1 = [\mathfrak{n}, \mathfrak{n}]
\]

\[
: : : \]

\[
\mathfrak{n}_k = [\mathfrak{n}, \mathfrak{n}_{k-1}]
\]

\[
: : : \]

4
then there exists \( s \in \mathbb{N} \) such that \( n_s = \{0\} \). Let \( \mathfrak{v} \) be a subset of \( n \) and let \( \lambda \) be a linear functional in \( n^* \). We define the corresponding sets \( \mathfrak{v}^\lambda \) and \( \mathfrak{v}(\lambda) \) such that

\[
\mathfrak{v}^\lambda = \{ Z \in n : \lambda[Z,X] = 0 \text{ for every } X \in \mathfrak{v}\}
\]

and \( \mathfrak{v}(\lambda) = \mathfrak{v}^\lambda \cap \mathfrak{v} \). \( \mathfrak{z} \) denotes the center of the Lie algebra of \( n \), and the coadjoint action on the dual of \( n \) is simply the dual of the adjoint action of \( N \) on \( n \). Given \( X \in n, \lambda \in n^* \), the coadjoint action is defined multiplicatively as follows:

\[
\exp X \cdot \lambda (Y) = \lambda (\text{Ad}_{\exp -X} Y).
\]

We fix a Jordan Hölder basis \( \{X_i\}_{i=1}^n \) for \( n \). Given any linear functional \( \lambda \in n^* \), we construct the following skew-symmetric matrix:

\[
\textbf{M}(\lambda) = [\lambda [X_i,X_j]]_{1 \leq i,j \leq n}.
\]  \hspace{1cm} (1)

It is easy to see that \( n(\lambda) = \text{nullspace (M(\lambda))} \) with respect to the fixed Jordan Hölder basis, and that the center of the Lie algebra is always contained inside the vector space \( n(\lambda) \). It is also well-known that all coadjoint orbits have a natural symplectic smooth structure, and therefore are even-dimensional manifolds. Also, thanks to Kirillov’s theory, it is well-known that for each \( \lambda \in n^* \), there is a corresponding analytic subgroup \( P_\lambda = \exp (p(\lambda)) \) such that \( p(\lambda) \) is a maximal sub-algebra of \( n \) which is self orthogonal with respect to the bilinear form

\[
(X,Y) \mapsto \lambda [X,Y].
\]

There is also a character \( \chi_\lambda \) of \( P_\lambda \) such that the pair \( (P_\lambda, \chi_\lambda) \) determines up to unitary equivalence a unique irreducible representation of \( N \). More precisely, if \( \chi_\lambda (\exp X) = e^{2\pi i \lambda(X)} \) defines a character on \( P_\lambda \) then the unitary representation of \( N \)

\[
\pi_\lambda = \text{Ind}_{P_\lambda}^N (\chi_\lambda)
\]

is irreducible. This construction exhausts the set of all unitary irreducible representations of \( N \). We refer the interested reader to [2] which is a standard reference for this class of Lie groups. Now, we will recall some basic facts about nilpotent Lie groups which are also found in [2]

1. \( n \) has a **rational structure** if and only if there is an \( \mathbb{R} \)-basis \( \mathfrak{B} \) for \( n \) having rational structure constants and if \( n_\mathbb{Q} = \mathbb{Q} \text{-span (}\mathfrak{B}) \) then

\[
n \cong n_\mathbb{Q} \otimes \mathbb{Q}.
\]
2. If $N$ has a uniform subgroup $\Gamma$ then $n$ has a rational structure such that $n\mathbb{Q} = \mathbb{Q}$-span $(\log(\Gamma))$

3. If $n$ has a rational structure then $N$ has a uniform subgroup $\Gamma$ such that $\log(\Gamma) \subseteq n\mathbb{Q}$

### 2 Free Nilpotent Lie Algebras of Step Two and Examples

Let $f_{m,2}$ be the free nilpotent Lie algebra of step two on $m$ generators ($m > 1$). Let $Z_1, \ldots, Z_m$ be the generators of $f_{m,2}$. Then

$$f_{m,2} = \mathfrak{z} \oplus \mathbb{R}\text{-span} \{Z_1, \ldots, Z_m\}$$

such that $\mathfrak{z} = \mathbb{R}\text{-span} \{Z_{ik} : 1 \leq i \leq m, i < k \leq m\}$. The Lie brackets of this Lie algebra are described as follows.

$$[Z_i, Z_j] = Z_{ij} \text{ for } (1 \leq i \leq m \text{ and } i < j \leq m)$$

It is then easy too see that $\dim(\mathfrak{z}) = \frac{m(m-1)}{2}$. Also, it is convenient to relabel the basis elements of the Lie algebra as follows:

$$\begin{cases} 
X_1 = Z_{12} & X_{\frac{m(m-1)}{2}+1} = Z_1 \\
X_2 = Z_{13} & \\
\vdots & \\
X_{\frac{m(m-1)}{2}} = Z_{m-1} & X_{n-1} = Z_{m-1} \\
X_n = Z_m & 
\end{cases}$$

Clearly, $\{X_1, X_2, \ldots, X_{n-1}, X_n\}$ is a Jordan H"older basis through $\mathfrak{z} = [f_{m,2}, f_{m,2}]$ which is fixed from now on. We remark that since $f_{m,2}$ has a rational structure, then its admits a lattice subgroup which we will denote throughout this paper by $\Gamma$.

**Example 4** Let us suppose that $m = 5$. We write $\lambda_{ij} = \lambda [Z_i, Z_j]$ for $i < j$ and $\lambda \in f_{5,2}^*$. Put

$$\Omega = \{f \in f_{5,2}^* : f_{14}f_{23} - f_{13}f_{24} + f_{12}f_{34} \neq 0\}.$$
With some formal calculations, we obtain for \(\lambda \in \Omega\), that

\[
f_{m,2}(\lambda) = \mathfrak{z} \oplus \mathbb{R}(\alpha_1(\lambda) Z_1 + \alpha_2(\lambda) Z_2 + \alpha_3(\lambda) Z_3 + \alpha_4(\lambda) Z_4 + Z_5)
\]

where

\[
\alpha_k(\lambda) = \begin{cases} 
\lambda_{25} \lambda_{34} - \lambda_{24} \lambda_{35} + \lambda_{23} \lambda_{45} & \text{if } k = 1 \\
\lambda_{14} \lambda_{23} - \lambda_{13} \lambda_{24} + \lambda_{12} \lambda_{43} \\
\lambda_{15} \lambda_{34} - \lambda_{14} \lambda_{35} + \lambda_{13} \lambda_{45} & \text{if } k = 2 \\
\lambda_{14} \lambda_{23} - \lambda_{13} \lambda_{24} + \lambda_{12} \lambda_{34} \\
\lambda_{15} \lambda_{24} - \lambda_{14} \lambda_{25} + \lambda_{12} \lambda_{35} & \text{if } k = 3 \\
\lambda_{14} \lambda_{23} - \lambda_{13} \lambda_{24} + \lambda_{12} \lambda_{34} \\
\lambda_{15} \lambda_{25} - \lambda_{14} \lambda_{26} + \lambda_{12} \lambda_{35} & \text{if } k = 4
\end{cases}
\]

Clearly \(\dim_{\mathbb{Q}} (\alpha_1(\lambda), \alpha_2(\lambda), \alpha_3(\lambda), \alpha_4(\lambda), 1) = 5\) almost everywhere. Thus, \(\pi_{\lambda|\Gamma}\) is almost everywhere irreducible.

## 3 Proof of Theorem 2

Suppose that \(m\) is odd. For any linear functional \(\lambda \in f^*_{m,2}\), we consider the corresponding skew-symmetric matrix

\[
M(\lambda) = \begin{bmatrix}
0_{m(m-1) \times m(m-1)} & 0_{m(m-1) \times m} \\
0_{m \times m(m-1)} & [\lambda[Z_i, Z_j]]_{1 \leq i,j,m}
\end{bmatrix}.
\]  \hspace{1cm} (2)

Since \(m\) is odd then \(m = 2k + 1\) for some positive integer \(k\) greater than or equal to one. If \([\lambda[Z_i, Z_j]]_{1 \leq i,j,m}\) denotes the transpose of \([\lambda[Z_i, Z_j]]_{1 \leq i,j,m}\), it is easy to see that the null-space of \(M(\lambda)\) is equal to

\[
\mathfrak{z} \oplus \text{nullspace} \left([\lambda[Z_i, Z_j]]_{1 \leq i,j,m}\right) = \mathfrak{z} \oplus \text{nullspace} \left([\lambda[Z_i, Z_j]]_{1 \leq i,j,m}^T\right)
\]

since \([\lambda[Z_i, Z_j]]_{1 \leq i,j,m}\) is a skew-symmetric matrix. Now, let

\[
\alpha = \begin{bmatrix}
\alpha_1 \\
\vdots \\
\alpha_{2k}
\end{bmatrix}, \text{ and } \beta = \begin{bmatrix}
\beta_1 \\
\vdots \\
\beta_{2k}
\end{bmatrix} \in \mathbb{R}^{2k}.
\]

Also, let \(\langle \cdot, \cdot \rangle\) be the standard inner product defined on \(\mathbb{R}^{2k}\).
Lemma 5 If $M$ is a skew-symmetric matrix in $GL(2k, \mathbb{R})$ and if $M\alpha = \beta$ then $\langle \alpha, \beta \rangle = 0$

Proof. Suppose that $M\alpha = \beta$. Then

\[
M\alpha = \beta \Rightarrow \langle \alpha, \beta \rangle = \langle \alpha, M M^{-1} \beta \rangle \\
= \langle M^T \alpha, M \beta \rangle \\
= \langle -M\alpha, M^{-1} \beta \rangle \\
= \langle -\beta, \alpha \rangle \\
= -\langle \alpha, \beta \rangle.
\]

We conclude that $\langle \alpha, \beta \rangle = 0$. ■

Let us define the following vector-valued functions on $f^*_{m,2}$:

\[
\lambda \mapsto \alpha (\lambda) = \begin{bmatrix} \alpha_1 (\lambda) \\
\vdots \\
\alpha_{2k} (\lambda) \end{bmatrix} \text{ and } \gamma \mapsto \gamma (\lambda) = \begin{bmatrix} \alpha (\lambda) \\
1 \end{bmatrix}.
\]

Lemma 6 For almost every linear functional $\lambda$, there exists some $\gamma (\lambda) \in \mathbb{R}^{2k+1}$ such that

\[
\text{nullspace } (M(\lambda)) = \mathbb{R} \oplus \mathbb{R} (\gamma^T (\lambda) Z)
\]

where $Z^T = \begin{bmatrix} Z_1, \cdots, Z_{2k+1} \end{bmatrix}$.

Proof. We consider the equation

\[
[\lambda [Z_i, Z_j]]^T_{1 \leq i,j,m} (\gamma) = 0 \tag{3}
\]

where

\[
\gamma = \begin{bmatrix} \alpha \\
1 \end{bmatrix} \in \mathbb{R}^{2k+1} \text{ is unknown.}
\]

Equation (3) is equivalent to the following system of $m$ equations and $m - 1$ unknowns

\[
\begin{cases}
\alpha_1 \lambda [Z_1, Z_1] + \alpha_2 \lambda [Z_2, Z_1] + \cdots + \alpha_{2k} \lambda [Z_{2k}, Z_1] + \lambda [Z_{2k+1}, Z_1] = 0 \\
\alpha_1 \lambda [Z_1, Z_2] + \alpha_2 \lambda [Z_2, Z_2] + \cdots + \alpha_{2k} \lambda [Z_{2k}, Z_2] + \lambda [Z_{2k+1}, Z_2] = 0 \\
\vdots \\
\alpha_1 \lambda [Z_1, Z_{2k}] + \alpha_2 \lambda [Z_2, Z_{2k}] + \cdots + \alpha_{2k} \lambda [Z_{2k}, Z_{2k}] + \lambda [Z_{2k+1}, Z_{2k}] = 0 \\
\alpha_1 \lambda [Z_1, Z_{2k+1}] + \alpha_2 \lambda [Z_2, Z_{2k+1}] + \cdots + \alpha_{2k} \lambda [Z_{2k}, Z_{2k+1}] + \lambda [Z_{2k+1}, Z_{2k+1}] = 0
\end{cases} \tag{4}
\]
We define a matrix-valued function $\lambda \mapsto M(\lambda)$ on $\mathfrak{f}_{m,2}^*$ such that

$$M(\lambda) = [\lambda[Z_i, Z_j]]_{1 \leq i,j,m}^T = 
\begin{bmatrix}
\lambda[Z_1, Z_1] & \lambda[Z_2, Z_1] & \cdots & \lambda[Z_{2k}, Z_1] \\
\lambda[Z_1, Z_2] & \lambda[Z_2, Z_2] & \cdots & \lambda[Z_{2k}, Z_2] \\
\vdots & \vdots & \ddots & \vdots \\
\lambda[Z_1, Z_{2k}] & \lambda[Z_2, Z_{2k}] & \cdots & \lambda[Z_{2k}, Z_{2k}]
\end{bmatrix}$$

and

$$\beta(\lambda) = 
\begin{bmatrix}
-\lambda[Z_{2k+1}, Z_1] \\
-\lambda[Z_{2k+1}, Z_2] \\
\vdots \\
-\lambda[Z_{2k+1}, Z_{2k}]
\end{bmatrix}.$$ 

Let $\Omega = \{\lambda \in \mathfrak{f}_{m,2}^* : \det M(\lambda) \neq 0\}$. $\Omega$ is a Zariski open subset of $\mathfrak{f}_{m,2}^*$ since $\det M(\lambda)$ is a non-trivial homogeneous polynomial defined over $\mathfrak{f}_{m,2}^*$. Therefore $\Omega$ is a dense and open subset of $\mathfrak{f}_{m,2}^*$. Now, let $\lambda \in \Omega$. Although Equation (4) is equivalent to solving the system of equations

$$\begin{cases}
M(\lambda) \alpha = \beta(\lambda) \\
\langle \alpha, \beta(\lambda) \rangle = 0
\end{cases},$$

according to Lemma 5, if $M(\lambda) \alpha = \beta(\lambda)$ then

$$\langle \alpha, \beta(\lambda) \rangle = 0.$$ (5)

The point here is that, in order to find a complete solution to Equation (4), we only need to solve $M(\lambda) \alpha = \beta(\lambda)$ for $\alpha$. Let $\alpha$ be a solution to the equation. Since $M(\lambda)$ in non-singular, then $\alpha = \alpha(\lambda) = M(\lambda)^{-1} \beta(\lambda)$ and it follows that nullspace $(M(\lambda)) = Z \oplus \mathbb{R} \left( \gamma^T(\lambda) Z \right)$. This completes the proof. ■

Now, let

$$\Omega_1 = \bigcap_{j=1}^{2k} \{ \lambda \in \mathfrak{f}_{m,2}^* : \lambda[Z_{2k+1}, Z_j] \neq 0 \}.$$ 

We observe that since $\Omega_1$ is Zariski open in $\mathfrak{f}_{m,2}^*$, it is a dense subset of $\mathfrak{f}_{m,2}^*$.

Lemma 7 For every linear functional $\lambda \in \Omega \cap \Omega_1$,

$$\mathfrak{f}_{m,2}(\lambda) = Z \oplus \mathbb{R} (\alpha_1(\lambda) Z_1 + \alpha_2(\lambda) Z_2 + \cdots + \alpha_{2k}(\lambda) Z_{2k} + Z_{2k+1})$$

and each function $\lambda \mapsto \alpha_j(\lambda)$ is a non-vanishing rational function for all $1 \leq j \leq 2k$. 
Proof. First, we notice that \( \lambda \mapsto \beta(\lambda) \) is a non-zero vector-valued function on \( \Omega \cap \Omega_1 \). Let \( \mathcal{B} = \{b_1, \ldots, b_{2k}\} \) be the canonical column basis for \( \mathbb{R}^{2k} \). Then the coordinates of \( \beta(\lambda) : \langle \beta(\lambda), b_j \rangle \) are non-zero monomials in \( \mathbb{R} \left[ \lambda \left( Z_{(2k+1)1} \right), \cdots, \lambda \left( Z_{(2k+1)2k} \right) \right] \). Since \( \beta(\lambda), \alpha(\lambda) \) are orthogonal vectors (see (5)), then there exists some rotation matrix \( \theta(\lambda) \) in the orthogonal Lie group \( \text{O}(2k,\mathbb{R}) \) such that

\[
\alpha(\lambda) = \frac{\|M(\lambda)^{-1} \beta(\lambda)\| \theta(\lambda) \beta(\lambda)}{\|\beta(\lambda)\|}.
\]

If \( \theta(\lambda)^{-1} b_k = b_l \in \mathcal{B} \) then

\[
\langle \alpha(\lambda), b_k \rangle = \frac{\|M(\lambda)^{-1} \beta(\lambda)\| \theta(\lambda) \beta(\lambda), b_k}{\|\beta(\lambda)\|} = \frac{\|M(\lambda)^{-1} \beta(\lambda)\| \langle \beta(\lambda), b_l \rangle}{\|\beta(\lambda)\|} \neq 0 \quad \text{for all } \lambda \in \Omega \cap \Omega_1.
\]

We conclude that for any \( 1 \leq j \leq m \), \( \alpha_j(\lambda) \) is a non-zero rational function of \( \lambda \). Moreover, for any \( \lambda \in \Omega \cap \Omega_1 \), \( f_{m,2}(\lambda) = \text{nullspace} \left( M(\lambda) \right) \) which is equal to \( \mathfrak{j} \oplus \mathbb{R}(\alpha_1(\lambda) Z_1 + \alpha_2(\lambda) Z_2 + \cdots + \alpha_{2k}(\lambda) Z_{2k} + Z_{2k+1}) \).

Proof of Theorem 2. Let \( \lambda \in \Omega \cap \Omega_1 \) (a Zariski open and dense subset) and define \( X \in f_{2k+1,2}(\lambda) \) such that

\[
X = X_1 + \cdots + X_{m(m-1)/2} + \alpha_1(\lambda) Z_1 + \alpha_2(\lambda) Z_2 + \cdots + \alpha_{2k}(\lambda) Z_{2k} + Z_{2k+1}.
\]

Clearly for every \( \lambda \in \Omega \cap \Omega_1 \),

\[
\dim_{\mathbb{Q}}(\alpha_1(\lambda), \alpha_2(\lambda), \cdots, \alpha_{2k}(\lambda), 1) = 2k + 1.
\]

So, the vector space \( f_{m,2}(\lambda) \) is not contained in a proper rational ideal of \( f_{m,2} \) for all \( \lambda \in \Omega \cap \Omega_1 \). Appealing to Theorem III the corresponding irreducible representation \( \pi_\lambda \) restricted to \( \Gamma \) is an \textbf{irreducible representation} of \( \Gamma \) for all \( \lambda \in \Omega \cap \Omega_1 \).

Remark 8 If \( \lambda \in \Omega \cap \Lambda \) such that

\[
\Lambda = \bigcap_{j=1}^{2k} \left\{ \lambda \in f_{m,2}^* : [Z_{2k+1}, Z_j] \in \ker \lambda \right\}
\]
then \( \lambda \mapsto \beta(\lambda) \) is the zero function defined on \( \Omega \cap \Lambda \) and clearly,

\[
\frak{f}_{m,2}(\lambda) = \text{nulspan}(M(\lambda)) = \mathfrak{z} \oplus \mathbb{R}(Z_{2k+1})
\]

which is contained in a rational ideal of \( \frak{f}_{m,2} \). Therefore, each unitary irreducible representation \( \pi_\lambda \) of \( \exp(\frak{f}_{m,2}) \) restricted to \( \Gamma \) is a reducible representation of \( \Gamma \) for all \( \lambda \in \Omega \cap \Lambda \) (meager subset of \( \frak{f}_{m,2}^* \)).

4 Proof of Theorem 3

Suppose that \( m \) is even. This case is much easier than the odd case. The proof will be very short. Since \( m \) is even, then \( m = 2k \) for some positive integer \( k \) greater than or equal to one. Thus the null-space of \( M(\lambda) \) is equal to

\[
\mathfrak{z} \oplus \text{nulspan} \left( \left[ \lambda [Z_i, Z_j] \right]_{1 \leq i,j,m}^T \right).
\]

Let \( \lambda \) be an element of \( \frak{f}_{m,2}^* \). We consider the matrix equation

\[
M(\lambda) = \left[ \lambda [Z_i, Z_j] \right]_{1 \leq i,j,m}^T \alpha \in \mathbb{R}^{2k}
\]

\[
\begin{pmatrix}
\lambda [Z_1, Z_1] & \lambda [Z_2, Z_1] & \cdots & \lambda [Z_{2k}, Z_1] \\
\lambda [Z_1, Z_2] & \lambda [Z_2, Z_2] & \cdots & \lambda [Z_{2k}, Z_2] \\
\vdots & \vdots & & \vdots \\
\lambda [Z_1, Z_{2k}] & \lambda [Z_2, Z_{2k}] & \cdots & \lambda [Z_{2k}, Z_{2k}]
\end{pmatrix}
\]

which we want to solve for \( \alpha \). Put \( \Omega = \{ \lambda \in \frak{f}_{m,2}^* : \det M(\lambda) \neq 0 \} \). For every \( \lambda \in M(\lambda) \), we observe that \( M(\lambda) \) is a skew-symmetric matrix of even rank. Therefore \( \Omega \) is a Zariski open subset of \( \frak{f}_{m,2}^* \) and is dense and open in \( \frak{f}_{m,2}^* \). Moreover, solving the given matrix equation above, we obtain that \( \alpha = 0 \). As a result, for any \( \lambda \in \Omega, \frak{f}_{m,2}(\lambda) = \mathfrak{z} \). So, in summary, if the dimension of the Lie algebra \( \frak{f}_{m,2} \) is even then \( \frak{f}_{m,2}(\lambda) \) is contained inside a rational ideal of \( \frak{f}_{m,2} \) almost everywhere. As a result, almost every irreducible representation \( \pi_\lambda \) of \( \frak{f}_{m,2} \) restricted to \( \Gamma \) is a reducible representation of \( \Gamma \).
References

[1] M. Bekka, P. Driutti, Restrictions of irreducible unitary representations of nilpotent Lie groups to lattices. J. Funct. Anal. 168 (1999), no. 2, 514–528

[2] L. Corwin, F.P. Greenleaf, Representations of Nilpotent Lie Groups and Their Applications, Cambridge Univ. Press, Cambridge (1990)

[3] G. Folland, A Course in Abstract Harmonic Analysis, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[4] D. Han, Y. Wang, Yang, The existence of Gabor bases and frames, Wavelets, frames and operator theory. 183–192, Contemp. Math., 345, Amer. Math. Soc., Providence, RI, 2004

[5] C. Heil, History and Evolution of the Density Theorem for Gabor frames, J. Fourier Anal. Appl., 13 (2007), 113-166.