Physical symmetries and gauge choices in the Landau problem

Masashi Wakamatsu$^{1,a}$, Akihisa Hayashi$^{2,b}$

1 KEK Theory Center, Institute of Particle and Nuclear Studies, High Energy Accelerator Research Organization (KEK), Oho 1–1, Tsukuba, Ibaraki 305-0801, Japan
2 Department of Applied Physics, University of Fukui, Bunkyo 3-9-1, Fukui, Fukui 910-8507, Japan

Received: 14 April 2022 / Accepted: 16 June 2022 / Published online: 6 July 2022
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Communicated by Reinhard Alkofer

Abstract Due to a special nature of the Landau problem, in which the magnetic field is uniformly spreading over the whole two-dimensional plane, there necessarily exist three conserved quantities, i.e. two conserved momenta and one conserved orbital angular momentum for the electron, independently of the choice of the gauge potential. Accordingly, the quantum eigen-functions of the Landau problem can be obtained by diagonalizing the Landau Hamiltonian together with one of the above three conserved operators with the result that the quantum mechanical eigen-functions of the Landau problem can be written down for arbitrary gauge potential. The purpose of the present paper is to clarify the meaning of gauge choice in the Landau problem based on this gauge-potential-independent formulation, with a particular intention of unraveling the physical significance of the concept of gauge-invariant-extension of the canonical orbital angular momentum advocated in recent literature on the nucleon spin decomposition problem. At the end, our analysis is shown to disclose a physically vacuous side face of the gauge symmetry.

1 Introduction

For a quantum mechanical formulation of the Landau problem, one must necessarily introduce gauge potential instead of magnetic field itself [1,2]. The problem here is that the gauge potential, which reproduces the given magnetic field, is not unique at all. Practically useful choices of gauge are known to be the two Landau gauges and the symmetric gauge. It is widely believed that the choice of a particular gauge is inseparably connected with a specific symmetry of the corresponding electron eigen-functions [3–7]. In fact, the choice of one of the two Landau gauges naturally leads to the Hermite-type eigen-functions, which respects the translational symmetry along the $x$-direction or $y$-direction. On the other hand, the choice of symmetric gauge leads to the Laguerre-type eigen-functions, which respects the rotational symmetry around the coordinate origin. However, although it is not so familiar, the connection between the choice of gauge potential and the symmetry of the electron eigen-states is not an absolute demand of the quantum theory. The reason is because, in the Landau problem, there exist two conserved momenta and one conserved orbital angular momentum (OAM) of the electron irrespectively of the choice of the gauge potential. (In some previous literature, these conserved quantities were called the pseudo momenta and the pseudo OAM [8–10].) In pursuit of the meaning of gauge choice in the Landau problem, the authors of the paper [11] solved the Landau problem based on the gauge-invariant (but path-dependent) formulation of the quantum electrodynamics proposed by DeWitt many years ago [12]. In this theoretical framework, though the solutions of the Landau problem can be written down without specifying the gauge potential, they instead depend on the choice of path of the nonlocal gauge link also called the Wilson line. Choosing rectangular paths connecting the coordinate origin and the position of the relevant fields in the rectangular coordinate representation, it leads to a class of solutions, which may be regarded as a gauge-invariant extension based on either of the two Landau gauges. On the other hand, choosing a straight-line path connecting the coordinate origin and the position of the relevant fields in the 2-dimensional spherical coordinate system, it leads to a class of solutions, which can be regarded as a gauge-invariant extension based on the symmetric gauge. Using these two classes of solutions, they analysed the expectation values of the six operators [11]. Those are the
canonical momentum, pseudo momentum, and mechanical momentum operators of the electron as well as the canonical OAM, pseudo OAM, and the mechanical OAM operators. This analysis clarified the nature of the pseudo momentum as a gauge-covariant extension of the canonical momentum based on the Landau gauge, and also the nature of the pseudo OAM as a gauge-covariant extension of the canonical OAM based on the symmetric gauge. Unfortunately, the achievement of their analysis is still half way, in the sense that they did not fully elucidate the relations between the two classes of solutions, i.e. the gauge-potential-independent extension based on the two Landau gauge eigen-states and that based on the symmetric gauge eigen-states. Particularly interesting questions to be asked here would be the following. First, do the expectation values of the mechanical momentum and that of the mechanical OAM in fact coincide in these two different classes of eigen-functions? Since it is generally believed that both of those are manifestly gauge-invariant quantities, it seems natural to expect that the answer is “Yes”. If it is really so, what would be an answer to the same question asked for the pseudo momenta and the pseudo OAM? The second question is especially interesting, because the pseudo momentum and pseudo OAM operators are known to transform covariantly under an arbitrary gauge transformation just like the mechanical momentum and the mechanical OAM do.

Aiming at answering the above nontrivial questions as transparently as possible, we propose in the present paper a gauge-potential-independent formulation of the Landau problem without using nonlocal gauge link. Instead, the treatment is based on more familiar unitary transformation method in the standard non-relativistic quantum mechanics. In this formulation, the eigen-functions of the Landau problem will be obtained by diagonalizing the Landau Hamiltonian together with one of the above three conserved quantities for arbitrary choice of the gauge potential, which causes that the quantum mechanical eigen-functions of the Landau problem can be expressed for arbitrary choice of the gauge potential. As a nontrivial byproduct, we will be able to show that this elementary formulation of the Landau problem helps us to unravel the physically insubstantial nature of the concept of gauge-invariant-extension of the canonical OAM advocated in recent literature on the nucleon spin decomposition problem [13–17]. (For reviews on the nucleon spin decomposition problem, see [17, 18].)

The paper is organized as follows. In Sect. 2, we explain the reason why there exist three conserved quantities in the Landau problem, i.e. two conserved momenta and one conserved OAM, independently of the choice of the gauge potential. Utilizing this fact, we give a gauge-potential-independent formulation of the Landau problem, in which the eigen-functions of the Landau problem are written down for arbitrary gauge potentials. We show that, in this formulation, the eigen-functions of the Landau problem are devided into three classes, which we would call the gauge-potential-independent extension based on the 1st Landau gauge eigen-states, that based on the 2nd Landau gauge eigen-states, and that based on the symmetric gauge eigen-states. Next, in Sect. 3, we evaluate the matrix elements of three kinds of momentum and OAM operators, i.e. the canonical ones, the mechanical ones, and the conserved ones, between the Landau eigen-states belonging to different classes mentioned above, with a particular intention of verifying gauge-independence or gauge-dependence of the matrix elements of the two types of quantities, i.e. the mechanical momentum and the mechanical OAM, and the conserved momentum and the conserved OAM. In Sect. 4, we try to make clear the physical meaning of several findings reported in the previous sections. In particular, we argue that the pseudo momenta as well as the pseudo OAM are not gauge-invariant physical quantities in the standard or physical sense. Finally, in Sect. 5, we summarize what we can learn from the present investigation, especially on about highly delicate nature of the gauge symmetry concept in physics.
The existence of these three conserved quantities is inseparably connected with a special nature of the Landau problem [8–10,19]. That is, in the setting of the Landau problem, the magnetic field is uniformly spreading over the whole 2-dimensional plane, which means that there is no special point in this plane. One can choose any point as a coordinate origin for describing the motion of the electron. In other words, the Landau system has a translational invariance along an arbitrary direction in the 2-dimensional plane, which dictates the conservation of the two momenta \( p_{\text{cons}}^x \) and \( p_{\text{cons}}^y \) along the two orthogonal directions. Next, even after the coordinate origin is suitably fixed, there still remains a freedom to choose the direction of two perpendicular axes. This is equivalent to saying that the Landau system has a rotational symmetry (or the axial symmetry) around the coordinate origin. Undoubtedly, the Noether current corresponding this symmetry is the conserved OAM \( L_{z}^{\text{cons}} \). Since the classical equations of motion can be written down without introducing gauge-variant potential, the conservations of the above three quantities has little to do with the problem of gauge choice in the Landau problem. We recall that, in some of the previous literature, these conserved quantities were called the pseudo momenta and pseudo OAM [8–11]. In the present paper, we shall call them the conserved momenta and conserved OAM to emphasize that they are conserved quantities irrespectively of the choice of gauge potential in the Landau problem. Complexity occurs, since the quantum mechanical formulation of the Landau problem needs to introduce the vector potential, which is necessarily gauge-dependent, and there is a wide-spread misbelief that the conservation of the above-mentioned quantities are inseparably connected with the choice of gauge or gauge potential. To understand this delicate issue of gauge choice in the Landau problem in quantum mechanics, we think it instructive to first reformulate the classical mechanics of the Landau electron within the Lagrangian formulation.

### 2.1 Lagrangian formulation of the Landau problem

Note that, in order to write down the Lagrangian of the Landau system, we must introduce (gauge-variant) vector potential \( A \). As is widely-known, there are three popular choices of gauge or gauge potential in the Landau problem. They are the 1st Landau gauge \( A^{(L_1)} \), the 2nd Landau gauge \( A^{(L_2)} \), and the symmetric gauge \( A^{(S)} \), specified by

\[
A^{(L_1)}(r) = B (− y, 0),
\]

\[
A^{(L_2)}(r) = B (0, x),
\]

\[
A^{(S)}(r) = \frac{1}{2} B (− y, x).
\]

First, we consider the symmetric gauge. In this gauge, the Lagrangian of the Landau system is given by

\[
L(A^{(S)}) = \frac{1}{2} m_e v^2 - e v \cdot A^{(S)}
\]

\[
= \frac{1}{2} m_e v^2 + \frac{1}{2} e B (\dot{x} y - \dot{y} x)
\]

\[
= \frac{1}{2} m_e \left( \ddot{r}^2 + r^2 \dot{\phi}^2 \right) - \frac{1}{2} e B r^2 \dot{\phi}.
\]

Here \( r \) and \( \phi \) are the radial and angular coordinates in the 2-dimensional polar coordinate system. Since \( L(A^{(S)}) \) does not depend on the cyclic coordinate \( \phi \), the corresponding canonical conjugate momentum defined by

\[
p_{\phi}(A^{(S)}) = \frac{\partial L(A^{(S)})}{\partial \dot{\phi}} = m_e r^2 \dot{\phi} - \frac{1}{2} e B r^2
\]

\[
= L_{\text{mech}}^{(S)} - \frac{1}{2} e B r^2.
\]

should be conserved, i.e. \( \frac{d}{dt} p_{\phi}(A^{(S)}) = 0 \). Note that this just coincides with \( L_{z}^{\text{cons}}(A^{(S)}) \) given by (8), i.e. one confirms that

\[
p_{\phi}(A^{(S)}) = L_{z}^{\text{cons}}(A^{(S)}).
\]

Next, in the rectangular coordinate system, \( L(A^{(S)}) \) does depend on \( x \), but its partial derivative with respect to \( x \) turns out to be a total derivative on time as

\[
\frac{\partial L(A^{(S)})}{\partial x} = - \frac{1}{2} e B \dot{y}.
\]

This means that there must be a corresponding conserved quantity given by

\[
p_x(A^{(S)}) + \frac{1}{2} e B y,
\]

where \( p_x(A^{(S)}) \) is the canonical conjugate momentum with respect to \( x \) defined by

\[
p_x(A^{(S)}) = \frac{\partial L(A^{(S)})}{\partial \dot{x}}.
\]

Then, the conserved quantity indicated by (16) becomes

\[
p_x(A^{(S)}) + \frac{1}{2} e B y = m_e \dot{x} + \frac{1}{2} e B y + \frac{1}{2} e B y = m_e \dot{x} + e B y.
\]

This just coincides with the conserved moment \( p_x^{\text{cons}} \) defined by (6), i.e. we find that

\[
p_x(A^{(S)}) + e B y = p_x^{\text{cons}}.
\]

A lesson learned from the above analysis is therefore the following. Although the conservation of \( p_x^{\text{cons}} \) in the symmetric gauge is somewhat difficult to see as compared with \( L_{z}^{\text{cons}} \), the former is nevertheless a conserved quantity even in the symmetric gauge.

Since the two Landau gauges are of similar character, we consider below only the 1st Landau gauge specified by
In contrast, in the polar coordinate, $L(A^{(L_1)})$ does not depend on the cyclic coordinate $x$, the corresponding canonical momentum defined by:

$$p_x(A^{(L_1)}) = \frac{\partial L(A^{(L_1)})}{\partial \dot{\phi}} = m_e \dot{\phi} + e B y,$$

must be conserved, i.e. $\frac{d}{dt} p_x(A^{(L_1)}) = 0$. In fact, this just coincides with the conserved momentum $p_x^\text{cons}$ given by (6), i.e. we see that

$$p_x(A^{(L_1)}) = p_x^\text{cons}(A^{(L_1)}).$$

(22)

In contrast, in the polar coordinate, $L(A^{(L_1)})$ does depend on $\phi$. In fact, we have

$$L(A^{(L_1)}) = \frac{1}{2} m_e (\dot{r}^2 + r^2 \dot{\phi}^2)
+ e B (r \cos \phi - r \sin \phi) r \sin \phi
= \frac{1}{2} m_e (\dot{r}^2 + r^2 \dot{\phi}^2)
+ e B (r \dot{r} \cos \phi \sin \phi - r^2 \dot{\phi} \sin^2 \phi).$$

(23)

Nonetheless, its derivative on $\phi$ turns out to be a total derivative with respect to time as

$$\frac{\partial L(A^{(L_1)})}{\partial \phi} = e B r \dot{r} (-\sin^2 \phi + \cos^2 \phi)
- e B r^2 \dot{\phi} 2 \sin \phi \cos \phi
= \frac{d}{dt} \frac{1}{2} e B r^2 (\cos^2 \phi - \sin^2 \phi).$$

(24)

This indicates that the following quantity should be conserved

$$p_\phi(A^{(L_1)}) - \frac{1}{2} e B r^2 (\cos^2 \phi - \sin^2 \phi),$$

(25)

where $p_\phi(A^{(L_1)})$ is the canonical conjugate momentum with respect to $\phi$ in the $L_1$ gauge, defined by

$$p_\phi(A^{(L_1)}) = \frac{\partial L(A^{(L_1)})}{\partial \dot{\phi}}.$$

(26)

Note that the conserved quantity indicated by (25) can be rewritten as

$$p_\phi(A^{(L_1)}) - \frac{1}{2} e B r^2 (\cos^2 \phi - \sin^2 \phi)
= m_e r^2 \dot{\phi} - e B r^2 \sin^2 \phi - \frac{1}{2} e B r^2 (\cos^2 \phi - \sin^2 \phi)
= m_e r^2 \dot{\phi} - \frac{1}{2} e B r^2.$$

(27)

This precisely coincides with the conserved angular momentum $L_z^\text{cons}$ in the $L_1$-gauge, i.e. we find that

$$p_\phi(A^{(L_1)}) - \frac{1}{2} e B r^2 (\cos^2 \phi - \sin^2 \phi)
= L_z^\text{cons}(A^{(L_1)}).$$

(28)

Thus, although the conservation of $L_z^\text{cons}$ is somewhat difficult to see in the $L_1$ gauge, it nevertheless is a conserved quantity just like $p_x^\text{cons}$.

From the above considerations, it is clear that the following two quantities are conserved in both of the symmetric gauge and the 1st Landau gauge and that this fact can be extended to arbitrary gauge field configuration $A$:

$$p_x^\text{cons} = p_x^\text{mech}(A) + e B y,$$

(29)

$$L_z^\text{cons} = L_z^\text{mech}(A) - \frac{1}{2} e B r^2.$$

(30)

(Although we do not repeat a similar analysis, it is evident that the quantity $p_y^\text{cons} = p_y^\text{mech}(A) - e B x$ is also a conserved quantity independently of the choice of $A$.)

At this point, we recommend readers to keep in mind the fact that, since the second terms in the r.h.s. of (29) and (30) are intact under a gauge transformations, the transformation property of $p_x^\text{cons}$ and $L_z^\text{cons}$ under an arbitrary gauge transformation is exactly the same as $p_x^\text{mech}$ and $L_z^\text{mech}$. Namely, $p_x^\text{cons}$ and $L_z^\text{cons}$ transform covariantly under gauge transformation just like $p_x^\text{mech}$ and $L_z^\text{mech}$ do. As we shall see in the rest of the paper, what is meant by this fact provides with us a highly nontrivial question, the pursuit of whose answer leads us to unexpectedly deep insight into delicate nature of the gauge symmetry concept.

2.2 Hamilton formulation and quantum mechanics

When going to quantum mechanics in coordinate representation, the momentum operator is replaced by a derivative operator as $\hat{\mathbf{p}} \rightarrow -i \nabla$. (Throughout the paper, we use the natural unit, $\hbar = c = 1$.) The Hamiltonian of the Landau problem is then represented as

$$\hat{H}(A) = \frac{1}{2m_e} (-i \nabla + e A)^2.$$ 

(31)

The conserved momentum and the conserved OAM given by (29) and (30) also become quantum operators as

$$\hat{p}_x^\text{cons} = -i \frac{\partial}{\partial x} + e A_x + e B y,$$

(32)

$$\hat{L}_z^\text{cons} = -i \frac{\partial}{\partial \phi} + e r A_\phi - \frac{1}{2} e B r^2,$$

(33)

where $A_x$ represents the $x$-component of the vector potential $A$, while $A_\phi$ stands for its azimuthal component. The classical conservations of these quantities can be translated into
the commutation relations between the corresponding quantum operators and the Landau Hamiltonian as

\[
[\hat{p}_x^{\text{cons}}, \hat{H}(A)] = 0, \quad (34)
\]

\[
[\hat{L}_z^{\text{cons}}, \hat{H}(A)] = 0. \quad (35)
\]

In quantum mechanics, however, these two conserved quantities \(\hat{p}_x^{\text{cons}}\) and \(\hat{L}_z^{\text{cons}}\) do not commute with each other, as

\[
[\hat{p}_x^{\text{cons}}, \hat{L}_z^{\text{cons}}] \neq 0. \quad (36)
\]

This means that one is forced to choose either of these two operators as an operator which will be simultaneously diagonalized with the Landau Hamiltonian.

If one selects the symmetric gauge potential, it is conventional to choose \(\hat{L}_z^{\text{cons}}(A^{(S)})\) as an operator to be diagonalized with the Landau Hamiltonian, so that the eigen-states \(|\Psi_{n,m}\rangle\) in the symmetric gauge is customarily taken as the simultaneous eigen-functions of \(\hat{L}_z^{\text{cons}}(A^{(S)})\) and \(\hat{H}(A^{(S)})\) as \([11, 19]\)

\[
\hat{L}_z^{\text{cons}}(A^{(S)}) | \Psi_{n,m}^{(S)} \rangle = m | \Psi_{n,m}^{(S)} \rangle, \quad (37)
\]

\[
\hat{H}(A^{(S)}) | \Psi_{n,m}^{(S)} \rangle = E_n | \Psi_{n,m}^{(S)} \rangle, \quad (38)
\]

where \(E_n = (2n + 1) \omega_L\) with \(\omega_L = eB/(2m_e)\) being the so-called Larmor frequency. In fact, in the symmetric gauge, \(\hat{L}_z^{\text{cons}}(A)\) reduces to

\[
\hat{L}_z^{\text{cons}}(A^{(S)}) = -i \frac{\partial}{\partial \phi} + e r A^{(S)} - \frac{1}{2} eB r^2 = -i \frac{\partial}{\partial \phi}, \quad (39)
\]

which is nothing but the polar-coordinate representation of the ordinary canonical OAM operator \(\hat{L}_z^{\text{can}} = -i (r \times \nabla)\). Accordingly, the eigen-functions of (37) and (38) are given as

\[
\Psi_{n,m}^{(S)}(x, y) = \frac{e^{im\phi}}{\sqrt{2\pi}} N_{n,m} e^{-\frac{i}{2} \xi |m|/2} L^{|m|/2}_{n-|m|}(\xi), \quad (40)
\]

where \(\xi = r^2 / (2l_B^2)\) with \(l_B^2 = 1 / (eB)\), while

\[
N_{n,m} = \frac{1}{l_B} \left[ \frac{n - |m| + m}{n + |m| - m} \right]! \quad (41)
\]

is the normalization constant, and \(L^m_n(\xi)\) is the familiar associated Laguerre polynomials.

Although somewhat unconventional, however, even with the choice of the symmetric gauge potential \(A^{(S)}\), one can make a choice in which \(\hat{p}_x^{\text{cons}}(A^{(S)})\) and \(\hat{H}(A^{(S)})\) are simultaneously diagonalized. (This is so because \(\hat{p}_x^{\text{cons}}\) is also a conserved quantity irrespectively of a choice of the gauge potential.) This amounts to choosing the basis vectors \(|\Psi_{n,k_x}^{(S)}\rangle\) defined by the simultaneous eigen-equations:

\[
\hat{p}_x^{\text{cons}}(A^{(S)}) | \Psi_{n,k_x}^{(S)} \rangle = k_x | \Psi_{n,k_x}^{(S)} \rangle, \quad (42)
\]

\[
\hat{H}(A^{(S)}) | \Psi_{n,k_x}^{(S)} \rangle = E_n | \Psi_{n,k_x}^{(S)} \rangle. \quad (43)
\]

Here, we have assumed that the eigen-energies of the Landau Hamiltonian depends only on the Landau quantum number \(n\), which is in fact the case as \(E_n = (2n + 1) \omega_L\). Next, if one chooses to work by selecting the 1st Landau gauge potential \(A^{(L_1)}\), the relevant conserved quantities take the form:

\[
\hat{p}_x^{\text{cons}}(A^{(L_1)}) = -i \frac{\partial}{\partial x}, \quad (44)
\]

\[
\hat{L}_z^{\text{cons}}(A^{(L_1)}) = -i \frac{\partial}{\partial \phi} - \frac{1}{2} eB (x^2 - y^2). \quad (45)
\]

Note that \(\hat{p}_x^{\text{cons}}(A^{(L_1)})\) takes a simpler form in accordance with the fact that \(\hat{H}(A^{(L_1)})\) takes a separable form in the rectangular coordinate representation. Then, with the choice of the \(L_1\)-gauge potential, one usually adopts the basis, in which \(\hat{p}_x^{\text{cons}}(A^{(L_1)})\) and \(\hat{H}(A^{(L_1)})\) are simultaneously diagonalized as

\[
\hat{p}_x^{\text{cons}}(A^{(L_1)}) | \Psi_{n,k_x}^{(L_1)} \rangle = k_x | \Psi_{n,k_x}^{(L_1)} \rangle, \quad (46)
\]

\[
\hat{H}(A^{(L_1)}) | \Psi_{n,k_x}^{(L_1)} \rangle = E_n | \Psi_{n,k_x}^{(L_1)} \rangle. \quad (47)
\]

The explicit form of \(\Psi_{n,k_x}^{(L_1)}(x, y)\) is known to be given in the form

\[
\Psi_{n,k_x}^{(L_1)}(x, y) = \frac{e^{i k_x x}}{\sqrt{2\pi}} Y_n(y), \quad (48)
\]

where

\[
Y_n(y) = N_n H_n \left( \frac{y - y_0}{l_B} \right) e^{-\frac{(y-y_0)^2}{2l_B^2}}, \quad (49)
\]

with

\[
N_n = \left( \frac{1}{\sqrt{\pi} 2^n n! l_B} \right)^{1/2}, \quad y_0 = \frac{k_x}{eB} = l_B^2 k_x. \quad (50)
\]

These are nothing but the familiar Landau eigen-functions in the 1st Landau gauge. Although somewhat unconventional, however, even with the choice of \(A^{(L_1)}\), one can choose the basis, in which \(\hat{L}_z^{\text{cons}}(A^{(L_1)})\) and \(\hat{H}(A^{(L_1)})\) are simultaneously diagonalized as

\[
\hat{L}_z^{\text{cons}}(A^{(L_1)}) | \Psi_{n,k_x}^{(L_1)} \rangle = m | \Psi_{n,k_x}^{(L_1)} \rangle, \quad (51)
\]

\[
\hat{H}(A^{(L_1)}) | \Psi_{n,k_x}^{(L_1)} \rangle = E_n | \Psi_{n,k_x}^{(L_1)} \rangle. \quad (52)
\]

In this way, we now have totally four types of eigen-vectors of the Landau Hamiltonian,

\[
| \Psi_{n,m}^{(S)} \rangle, \quad | \Psi_{n,k_x}^{(S)} \rangle, \quad | \Psi_{n,k_x}^{(L_1)} \rangle, \quad | \Psi_{n,k_x}^{(L_1)} \rangle. \quad (53)
\]

A natural question is mutual relations between these four eigen-functions of the Landau problem. To answer this question, we recall that the gauge potential in the symmetric gauge
and the 1st Landau gauge are connected by the following gauge transformation:

\[ A^{(L_1)} = A^{(S)} + \nabla \chi, \quad \text{with} \quad \chi = -\frac{1}{2} B x y. \tag{54} \]

In quantum mechanics, the above gauge transformation can be realized as a unitary transformation represented as

\[ -i \nabla + e A^{(L_1)} = U \left( -i \nabla + e A^{(S)} \right) U^\dagger, \tag{55} \]

with

\[ U = e^{-i e \chi} = e^{i \frac{1}{2} e B x y}. \tag{56} \]

It can easily be shown that

\[ U \hat{H}(A^{(S)}) U^\dagger = \hat{H}(A^{(L_1)}), \tag{57} \]

\[ U \hat{p}_x^{\text{cons}}(A^{(S)}) U^\dagger = \hat{p}_x^{\text{cons}}(A^{(L_1)}), \tag{58} \]

\[ U \hat{L}_z^{\text{cons}}(A^{(S)}) U^\dagger = \hat{L}_z^{\text{cons}}(A^{(L_1)}). \tag{59} \]

Multiplying \( U \) on both sides of (37) from the left and inserting the identity \( U U^\dagger = \text{identity} \), we obtain

\[ \hat{L}_z^{\text{cons}}(A^{(S)}) U^\dagger U \left| \Psi_{n,m}^{(S)} \right> = m U \left| \Psi_{n,m}^{(S)} \right>, \tag{60} \]

Then, by using \( U \hat{L}_z^{\text{cons}}(A^{(S)}) U^\dagger = \hat{L}_z^{\text{cons}}(A^{(L_1)}) \), this equation becomes

\[ \hat{L}_z^{\text{cons}}(A^{(L_1)}) U \left| \Psi_{n,m}^{(S)} \right> = m U \left| \Psi_{n,m}^{(S)} \right>. \tag{61} \]

Comparing it with (51), we therefore conclude that (up to phase)

\[ \left| \Psi_{n,m}^{(L_1)} \right> = U \left| \Psi_{n,m}^{(S)} \right>. \tag{62} \]

Similarly, we can readily confirm the relation

\[ \left| \Psi_{n,k_x}^{(L_1)} \right> = U \left| \Psi_{n,k_x}^{(S)} \right>. \tag{63} \]

What is meant by these two relations is nothing surprising. First, the relation (62) dictates that the two Landau eigenstates \( \left| \Psi_{n,m}^{(L_1)} \right> \) and \( \left| \Psi_{n,m}^{(S)} \right> \) are related through the gauge transformation matrix \( U \). This indicates that these two eigenstates belong to the same family. Similarly, the relation (63) shows that \( \left| \Psi_{n,k_x}^{(L_1)} \right> \) and \( \left| \Psi_{n,k_x}^{(S)} \right> \) are related through the gauge transformation matrix \( U \), which implies that these two eigenstates \( \left| \Psi_{n,k_x}^{(L_1)} \right> \) and \( \left| \Psi_{n,k_x}^{(S)} \right> \) belong to the same family. More generally, any eigen-states belonging to the 1st family and any eigen-states belonging to the 2nd family can be constructed by using the gauge-transformation matrix as follows:

\[ \left| \Psi_{n,m}^{(S)} \right> \equiv U(x) \left| \Psi_{n,m}^{(S)} \right>, \tag{64} \]

and

\[ \left| \Psi_{n,k_x}^{(S)} \right> \equiv U(x') \left| \Psi_{n,k_x}^{(S)} \right>, \tag{65} \]

with \( x \) and \( x' \) being arbitrary harmonic functions. What is nontrivial is the relation between the 1st class of eigenstates \( \left| \Psi_{n,m}^{(S)} \right> \) and the 2nd class of eigen-states \( \left| \Psi_{n,k_x}^{(S)} \right> \). As we shall confirm through the following analysis, it turns out that they are not simply related by a single gauge transformation matrix, and that they actually belong to different family. (The meaning of the terminology family here will become clear in the next section.)

### 3 Matrix elements of 3 kinds of momentum and OAM operators between 4 types of Landau eigen-states

Our objective here is to evaluate the matrix elements of the 3 kinds of momentum operators

\[ \hat{p}_x^{\text{can}} = -i \frac{\partial}{\partial x}, \tag{66} \]

\[ \hat{p}_x^{\text{cons}} = -i \frac{\partial}{\partial x} + e A_x + e B y, \tag{67} \]

\[ \hat{p}_x^{\text{mech}} = -i \frac{\partial}{\partial x} + e A_x, \tag{68} \]

and the 3 kinds of OAM operators

\[ \hat{L}_z^{\text{can}} = -i \frac{\partial}{\partial \phi}, \tag{69} \]

\[ \hat{L}_z^{\text{cons}} = -i \frac{\partial}{\partial \phi} + e r A_y - \frac{1}{2} e B r^2, \tag{70} \]

\[ \hat{L}_z^{\text{mech}} = -i \frac{\partial}{\partial \phi} + e r A_y, \tag{71} \]

between the 4 different types of Landau eigen-states shown by (53). The reason of such studies will become clear at the end of calculations.

#### 3.1 Matrix elements in the \( |n, k_x\rangle \)-basis

We first consider the matrix elements between the eigenstates \( \left| \Psi_{n,k_x}^{(L_1)} \right> \) and \( \left| \Psi_{n,k_x}^{(S)} \right> \), which simultaneously diagonalize \( \hat{p}_x^{\text{cons}} \) and the Landau Hamiltonian \( \hat{H} \). Hereafter, we call them the \( |n, k_x\rangle \)-basis. Remember that these two eigenfunctions are related by the gauge transformation matrix \( U^\dagger \) as \( \left| \Psi_{n,k_x}^{(S)} \right> = U^\dagger \left| \Psi_{n,k_x}^{(L_1)} \right> \).

The calculations of these matrix elements are straightforward but very tedious. We therefore describe the detailed derivation in Appendices A and B, and show here only the final answers. Summarized in Table 1 are the matrix elements of the three momentum operators as well as those of the three OAM operators in the \( |n, k_x\rangle \)-basis.

This table reveals several important facts. First, we notice that

\[ \left< \Psi_{n,k_x}^{(L_1)} | \hat{p}_x^{\text{can}} | \Psi_{n,k_x}^{(L_1)} \right> \neq \left< \Psi_{n,k_x}^{(S)} | \hat{p}_x^{\text{can}} | \Psi_{n,k_x}^{(S)} \right>, \tag{72} \]

\[ \left< \Psi_{n,k_x}^{(L_1)} | \hat{L}_z^{\text{can}} | \Psi_{n,k_x}^{(L_1)} \right> \neq \left< \Psi_{n,k_x}^{(S)} | \hat{L}_z^{\text{can}} | \Psi_{n,k_x}^{(S)} \right>. \tag{73} \]

These results are nothing surprising, since the canonical quantities, \( \hat{p}_x \) and \( \hat{L}_z^{\text{can}} \), are widely recognized to be gauge-
variant operators. On the other hand, we see that
\[
\langle \psi^{(L_1)}_{n,k_x} | \hat{p}^{\text{cons}}_x (A^{(L_1)}) | \psi^{(L_1)}_{n,k_x} \rangle = \langle \psi^{(S)}_{n,k_x} | \hat{p}^{\text{cons}}_x (A^{(S)}) | \psi^{(S)}_{n,k_x} \rangle,
\]
\[
\langle \psi^{(L_1)}_{n,k_x} | \hat{p}^{\text{mech}}_x (A^{(L_1)}) | \psi^{(L_1)}_{n,k_x} \rangle = \langle \psi^{(S)}_{n,k_x} | \hat{p}^{\text{mech}}_x (A^{(S)}) | \psi^{(S)}_{n,k_x} \rangle,
\]
and
\[
\langle \psi^{(L_1)}_{n,k_x} | \hat{L}^{\text{cons}}_z (A^{(L_1)}) | \psi^{(L_1)}_{n,k_x} \rangle = \langle \psi^{(S)}_{n,k_x} | \hat{L}^{\text{cons}}_z (A^{(S)}) | \psi^{(S)}_{n,k_x} \rangle,
\]
\[
\langle \psi^{(L_1)}_{n,k_x} | \hat{L}^{\text{mech}}_z (A^{(L_1)}) | \psi^{(L_1)}_{n,k_x} \rangle = \langle \psi^{(S)}_{n,k_x} | \hat{L}^{\text{mech}}_z (A^{(S)}) | \psi^{(S)}_{n,k_x} \rangle.
\]

These are also expected relations, because all these four operators \(\hat{p}^{\text{cons}}_x (A), \hat{p}^{\text{mech}}_x (A), \hat{L}^{\text{cons}}_z (A), \hat{L}^{\text{mech}}_z (A)\) transform covariantly under a gauge transformation, and because the two bases \(\{| \psi^{(S)}_{n,k_x} \rangle\}\) and \(\{| \psi^{(L_1)}_{n,k_x} \rangle\}\) belong to the same family as related by the gauge transformation matrix matrix \(U^{(S)}\) as \(\hat{U}^{(S)} | \psi^{(S)}_{n,k_x} \rangle = U^{(S)} | \psi^{(L_1)}_{n,k_x} \rangle\).

Not so obvious is the following observation. Namely, although the mathematical properties of the conserved momentum operator and the mechanical momentum operator, i.e. their covariant nature under a gauge transformation, are entirely the same, Table 1 apparently reveals the existence of some critical differences between those. Note that the matrix element of the mechanical momentum operator \(\hat{p}^{\text{mech}}_x (A)\) is zero in both of the eigen-states \(| \psi^{(L_1)}_{n,k_x} \rangle\) and \(| \psi^{(S)}_{n,k_x} \rangle\). This appears to indicate physical nature of the mechanical momentum. In fact, in classical mechanics, the Landau electron makes a cyclotron (or circular) motion around some center. The position of the center of this cyclotron motion is arbitrary because of the special nature of the Landau problem, in which the magnetic field is uniformly spreading over the whole 2-dimensional plane. Nevertheless, the center of this cyclotron motion is a constant in time in both of classical mechanics and quantum mechanics, so that the time-average or the expectation value of the physical electron momentum along the \(x\)-direction must evidently be zero.

In contrast, the physical meaning of the conserved momentum is not necessarily obvious. From Table 1, we see that
\[
\langle \psi^{(L_1)}_{n,k_x} | \hat{p}^{\text{can}}_x (A^{(L_1)}) | \psi^{(L_1)}_{n,k_x} \rangle = \langle \psi^{(S)}_{n,k_x} | \hat{p}^{\text{can}}_x (A^{(S)}) | \psi^{(S)}_{n,k_x} \rangle,
\]
but
\[
\langle \psi^{(S)}_{n,k_x} | \hat{p}^{\text{can}}_x | \psi^{(S)}_{n,k_x} \rangle \neq \langle \psi^{(S)}_{n,k_x} | \hat{p}^{\text{cons}}_x (A^{(S)}) | \psi^{(S)}_{n,k_x} \rangle.
\]

The first equality is understandable because \(\hat{p}^{\text{cons}}_x\) just reduces to \(\hat{p}^{\text{can}}_x\) with the choice of the \(L_1\)-gauge potential. The second inequality however shows that, in general, the matrix elements of \(\hat{p}^{\text{cons}}_x\) and \(\hat{p}^{\text{can}}_x\) do not have any simple relation. One may also notice that the matrix elements of \(\hat{p}^{\text{cons}}_x\) and \(\hat{p}^{\text{can}}_x\) in the \(|n,k_x\rangle\)-basis are proportional to the quantum number \(k_x\). In the framework of quantum mechanics, \(k_x\) is an important quantum number, which characterizes the Landau eigen-states in the \(|n,k_x\rangle\)-basis. Nonetheless, one must recognize the fact that this quantum number is not such a quantity, which has a direct connection with observables of the Landau electron. (This is also indicated by the fact that this quantum number never appears in the eigen-functions in the symmetric gauge.)

Also puzzling is fairly clumsy expression for the matrix element of the conserved OAM operator \(\hat{L}^{\text{cons}}_z (A)\), which should be contrasted with the very simple form for the matrix element of the mechanical OAM operator \(\hat{L}^{\text{mech}}_z (A)\). In fact, the following equalities
\[
\langle \psi^{(L_1)}_{n,k_x} | \hat{L}^{\text{mech}}_z (A^{(L_1)}) | \psi^{(L_1)}_{n,k_x} \rangle = \langle \psi^{(S)}_{n,k_x} | \hat{L}^{\text{mech}}_z (A^{(S)}) | \psi^{(S)}_{n,k_x} \rangle = (2n + 1) \delta(k_x' - k_x),
\]
for the matrix elements of the mechanical OAM is already indicating close resemblance with the familiar answer for the expectation value of the same operator between the Landau
eigen-states $| \Psi_{n,m}^{(S)} \rangle$ in the symmetric gauge given as

$$\langle \Psi_{n,m}^{(S)} | \mathbf{L}_{z}^{\text{mech}} (A^{(S)}) | \Psi_{n,m}^{(S)} \rangle = 2n + 1.$$  \hspace{1cm} (81)

The difference is the appearance of the Dirac’s delta function $\delta(k_x' - k_x)$ in the matrix elements given by (80), which comes from the non-normalizable nature of the $| n, k_x \rangle$ basis functions. However, corresponding normalizable basis functions can easily be constructed, if we consider wave-packet states. To confirm it, we first recall that the Landau eigenfunctions in the $L_1$-gauge can be expressed as a product of non-normalizable plane-wave function $f_{k_x}(x)$ and normalizable functions $Y_n(y)$ as

$$\Psi_{n,k_x}^{(L_1)}(x, y) = f_{k_x}(x) Y_n(y).$$  \hspace{1cm} (82)

with $f_{k_x}(x) = (1/\sqrt{2\pi}) e^{i k_x x}$. We now replace the above plane-wave functions by wave-packets as

$$f_{k_x}(x) \rightarrow F_{k_x}(x) \equiv \int_{-\infty}^{\infty} g(k - k_x) f_{k_x}(x) \, dk = \int_{-\infty}^{\infty} dk \frac{d k_x}{\sqrt{2\pi}} g(k - k_x) e^{i k_x x}.$$  \hspace{1cm} (83)

Here, $g(k)$ is an appropriate weight function of superposition, which has a peak at $k = 0$, and is normalized as,

$$\int_{-\infty}^{\infty} dk \, |g(k)|^2 = 1. \hspace{1cm} (84)$$

In correspondence with the above replacement, the eigenfunctions in the $L_1$-gauge are also replaced as follows:

$$\Psi_{n,k_x}^{(L_1)}(x, y) \rightarrow \tilde{\Psi}_{n,k_x}^{(L_1)}(x, y) \equiv F_{k_x}(x) Y_n(y).$$  \hspace{1cm} (85)

On account of the definition of $F_{k_x}(x)$, the normalizable functions above can also be expressed as

$$\tilde{\Psi}_{n,k_x}^{(L_1)}(x, y) = \int_{-\infty}^{\infty} dk \, g(k - k_x) \Psi_{n,k_x}^{(L_1)}(x, y).$$  \hspace{1cm} (86)

Now, it is an easy exercise to show the equality

$$\langle \tilde{\Psi}_{n,k_x}^{(L_1)} | \mathbf{L}_{z}^{\text{mech}} (A^{(L_1)}) | \tilde{\Psi}_{n,k_x}^{(L_1)} \rangle = 2n + 1.$$  \hspace{1cm} (87)

It is a little surprising that this simple relation has never been written down before. It must be a highly nontrivial finding, since it shows that the expectation value of the mechanical OAM operator precisely coincides between both of the symmetric gauge eigen-states and the 1st Landau gauge eigen-states. (It is also obvious that this property can be extended to any eigen-states with arbitrary-gauge potentials.) Surprisingly and importantly, this gauge-class-independence of the matrix elements does not hold for the conserved OAM operator, even though the mechanical OAM operator and the conserved OAM operator have exactly the same covariant gauge transformation property under an arbitrary gauge transformation. (See the next section for further discussion.)

To sum up, the consideration above already indicates the existence of critical physical difference between the conserved quantities and the mechanical ones, despite the fact that both transform covariantly under a gauge transformation. We shall pursue this nontrivial observation further through the investigation of the matrix elements of the same operators in the $| n, m \rangle$-basis in the next subsection.

### 3.2 Matrix elements in the $| n, m \rangle$-basis

In this subsection, we evaluate the matrix elements of the three momentum operators and the three OAM operators between the two Landau eigen-states $| \Psi_{n,m}^{(S)} \rangle$ and $| \Psi_{n,m}^{(L_1)} \rangle$ belonging to the $| n, m \rangle$-basis class. Since these two eigenstates are related as $| \Psi_{n,m}^{(L_1)} \rangle = U | \Psi_{n,m}^{(S)} \rangle$, our main task below is to evaluate the matrix elements of the above operators between the symmetric-gauge eigen-states $| \Psi_{n,m}^{(S)} \rangle$. Since the diagonal matrix elements $\langle \Psi_{n,m}^{(S)} | O(A^{(S)}) | \Psi_{n,m}^{(S)} \rangle$ were already investigated in many previous papers, we extend these analyses to the non-diagonal matrix elements $\langle \Psi_{n,m}^{(S)} | O(A^{(L_1)}) | \Psi_{n,m}^{(L_1)} \rangle$ along the $x$-direction. To evaluate the non-diagonal matrix elements $\langle \Psi_{n,m}^{(S)} | O(A^{(L_1)}) | \Psi_{n,m}^{(L_1)} \rangle$, we find it much easier to use the algebraic method rather than analytical method. In the following, we therefore prepare necessary algebraic formulation to handle the problem.

First, besides the familiar mechanical momentum $\mathbf{p} + e A(r)$, it is convenient to introduce the following quantity, which was called in Ref. [5] the pseudo-momentum :

$$\tilde{\mathbf{p}} \equiv \mathbf{p} - e A(r).$$  \hspace{1cm} (88)

(Note that the pseudo-momentum above has nothing to do with the quantity with the same name appearing in Refs. [8–10].) Clearly, this quantity does not transform covariantly under a gauge transformation, and, as we shall see shortly, it is useful only in the symmetric gauge. Here and hereafter, we omit the hat symbol for the quantum operator including the momentum operator $\hat{\mathbf{p}}$ to avoid notational complexity. Note that, with use of the quantities $\mathbf{p}$ and $\tilde{\mathbf{p}}$, the canonical momentum and the vector potential can be expressed as [5]

$$\mathbf{p} = \frac{1}{2} \left( \mathbf{\Pi} + \tilde{\mathbf{p}} \right), \quad A(r) = \frac{1}{2e} \left( \mathbf{\Pi} - \tilde{\mathbf{p}} \right).$$  \hspace{1cm} (89)

It is easy to verify the following commutation relations (C.R.s) :

$$[\mathbf{\Pi}_x, \mathbf{\Pi}_y] = -i \frac{1}{l_B^2}, \quad [\tilde{\mathbf{p}}_x, \tilde{\mathbf{p}}_y] = +i \frac{1}{l_B^2}. \hspace{1cm} (90)$$

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We can also show that
\begin{equation}
\begin{aligned}
[\Pi_x, \tilde{\Pi}_x] &= 2i e \frac{\partial A_y}{\partial x}, \\
[\Pi_y, \tilde{\Pi}_y] &= 2i e \frac{\partial A_x}{\partial y}, \\
[\Pi_x, \tilde{\Pi}_y] &= -[\tilde{\Pi}_x, \Pi_y] = ie \left( \frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x} \right).
\end{aligned}
\tag{91}
\end{equation}

These mixed C.R.s also mean that \( \tilde{\Pi}_x \) and \( \tilde{\Pi}_y \) do not commute with the Landau Hamiltonian:
\begin{equation}
[\tilde{\Pi}_x, H] \neq 0, \quad [\tilde{\Pi}_y, H] \neq 0.
\tag{94}
\end{equation}

However, the above unwanted mixed commutators can be avoided with a particular choice of gauge, i.e. with the choice of the symmetric gauge potential, \( A(r) \rightarrow A^{(S)}(r) = \frac{i}{2} ( -y, x ) \). In fact, in this special choice of gauge potential, it holds that
\begin{equation}
[\Pi_x, \tilde{\Pi}_x] = [\Pi_y, \tilde{\Pi}_y] = [\Pi_x, \tilde{\Pi}_y] = [\tilde{\Pi}_x, \Pi_y] = [\tilde{\Pi}_x, \Pi_y] = 0,
\tag{95}
\end{equation}

which also means that \( [\tilde{\Pi}_x, H] = [\tilde{\Pi}_y, H] = 0 \).

Now, it is convenient to introduce two kinds of ladder operator by
\begin{equation}
\begin{aligned}
a &= i \frac{1}{\sqrt{2}} ( \Pi_x - i \Pi_y ), \\
a^\dagger &= -i \frac{1}{\sqrt{2}} ( \Pi_x + i \Pi_y ), \\
b &= i \frac{1}{\sqrt{2}} ( \tilde{\Pi}_x + i \tilde{\Pi}_y ), \\
b^\dagger &= -i \frac{1}{\sqrt{2}} ( \tilde{\Pi}_x - i \tilde{\Pi}_y ).
\end{aligned}
\tag{96}
\end{equation}

They satisfy the following C.R.s:
\begin{equation}
[a, a^\dagger] = 1, \quad [b, b^\dagger] = 1, \quad [a, b] = [a^\dagger, b] = [a, b^\dagger] = 0.
\tag{98}
\end{equation}

Since we have
\begin{equation}
\begin{aligned}
\Pi_x &= p_x - \frac{1}{2} e B y, \quad \Pi_y = p_y + \frac{1}{2} e B x, \\
\tilde{\Pi}_x &= p_x + \frac{1}{2} e B y, \quad \tilde{\Pi}_y = p_y - \frac{1}{2} e B x,
\end{aligned}
\tag{100}
\end{equation}

in the symmetric gauge, we can express as
\begin{equation}
\begin{aligned}
p_x &= \frac{1}{2} ( \Pi_x + \tilde{\Pi}_x ), \\
p_y &= \frac{1}{2} ( \Pi_y + \tilde{\Pi}_y ), \\
x &= \frac{1}{l_B^2} ( \Pi_y - \tilde{\Pi}_y ), \\
y &= -\frac{1}{l_B^2} ( \Pi_x - \tilde{\Pi}_x ).
\end{aligned}
\tag{102}
\end{equation}

Thus, \( \Pi \) and \( \tilde{\Pi} \) can eventually be expressed with the ladder operators as
\begin{equation}
\begin{aligned}
\Pi_x &= -i \frac{1}{\sqrt{2} l_B} \left( a - a^\dagger \right), \\
\Pi_y &= + i \frac{1}{\sqrt{2} l_B} \left( a + a^\dagger \right), \\
\tilde{\Pi}_x &= -i \frac{1}{\sqrt{2} l_B} \left( b - b^\dagger \right), \\
\tilde{\Pi}_y &= - \frac{1}{\sqrt{2} l_B} \left( b + b^\dagger \right). \\
\end{aligned}
\tag{104}
\end{equation}

The three momenta of our interest are then expressed with the ladder operators as
\begin{equation}
\begin{aligned}
\rho_x^{\text{can}} &= \frac{1}{2} ( \Pi_x + \tilde{\Pi}_x ) = -i \frac{1}{2 \sqrt{2} l_B} \left( a - a^\dagger + b - b^\dagger \right), \\
\rho_x^{\text{cons}}(A^{(S)}) &= \Pi_x = -i \frac{1}{\sqrt{2} l_B} \left( b - b^\dagger \right), \\
\rho_x^{\text{mech}}(A^{(S)}) &= \Pi_x = -i \frac{1}{\sqrt{2} l_B} \left( a - a^\dagger \right).
\end{aligned}
\tag{106}
\end{equation}

Similarly, the three OAMs of our interest can be expressed as
\begin{equation}
\begin{aligned}
\rho_x^{\text{can}}(\rho_y) &= x p_y - y p_x = a^\dagger a - b^\dagger b, \\
\rho_x^{\text{cons}}(A^{(S)}) &= L_z^{\text{can}} = a^\dagger a - b^\dagger b, \\
\rho_x^{\text{mech}}(A^{(S)}) &= x \Pi_y - y \Pi_x = 2 a^\dagger a + 1 + (a b + a^\dagger b^\dagger). \\
\end{aligned}
\tag{111}
\end{equation}

Finally, the Landau eigen-state \( |\Psi_{n,m}^{(S)}\rangle \) in the symmetric gauge are represented as
\begin{equation}
|\Psi_{n,m}^{(S)}\rangle = |n\rangle^A |n - m\rangle^B 
\tag{112}
\end{equation}

with \( n \) and \( m \) being integers satisfying the constraint \( m \leq n \). Here, \( |n\rangle^A \) and \( |n\rangle^B \) are the eigen-states of the Harmonic oscillator, respectively corresponding to the creation operators \( a^\dagger \) and \( b^\dagger \), represented as
\begin{equation}
|n\rangle^A = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle, \quad |n\rangle^B = \frac{(b^\dagger)^n}{\sqrt{n!}} |0\rangle.
\tag{113}
\end{equation}

In the following, we can also use the following familiar identities:
\begin{equation}
\begin{aligned}
a^\dagger |n\rangle^A &= \sqrt{n + 1} |n + 1\rangle^A, \\
a |n\rangle^A &= \sqrt{n} |n - 1\rangle^A, \\
b^\dagger |n\rangle^B &= \sqrt{n' + 1} |n' + 1\rangle^B, \\
b |n\rangle^B &= \sqrt{n'} |n' - 1\rangle^B.
\end{aligned}
\tag{114}
\end{equation}

We are now ready to evaluate the required matrix elements in the \( |n, m\rangle \)-basis. Although the calculations are straightforward, we describe them in Appendix C for the sake of
Table 2  Matrix elements of three momenta and OAMs in $| n, m \rangle$-basis

| $\hat{O}$ | $\langle \Psi_{n,m}^{(L)} | \hat{O} (A^{(L)}) | \Psi_{n,m}^{(L)} \rangle$ | $\langle \Psi_{n,m}^{(S)} | \hat{O} (A^{(S)}) | \Psi_{n,m}^{(S)} \rangle$ |
|-----------|-------------------------------------------------|-------------------------------------------------|
| $\hat{p}_x^{\text{can}}$ | $-i \sqrt{\frac{e B}{2}} \left( \sqrt{n} - m \delta_{m',m+1} \right)$ | $-i \sqrt{\frac{e B}{2}} \left( \sqrt{n} - m \delta_{m',m+1} \right)$ |
| $\hat{p}_x^{\text{mech}(A)}$ | 0 | 0 |
| $\hat{p}_x^{\text{cons}(A)}$ | $-i \sqrt{\frac{e B}{2}} \left( \sqrt{n} - m \delta_{m',m+1} \right)$ | $-i \sqrt{\frac{e B}{2}} \left( \sqrt{n} - m \delta_{m',m+1} \right)$ |
| $\hat{L}_z^{\text{can}}$ | $m \delta_{0,m',m}$ | $m \delta_{0,m',m}$ |
| $\hat{L}_z^{\text{mech}(A)}$ | $(2n+1)m \delta_{0,m',m}$ | $(2n+1)m \delta_{0,m',m}$ |
| $\hat{L}_z^{\text{cons}(A)}$ | $m \delta_{0,m',m}$ | $m \delta_{0,m',m}$ |

completeness. Summarized in Table 2 are the matrix elements of the three momentum operators and the three OAM operators in the $| n, m \rangle$-basis. Tables 1 and 2 together with their interpretations given below are the central achievement of the present paper.

First, from Table 2, we confirm the following relations.

\begin{align}
\langle \Psi_{n,m}^{(L)} | \hat{p}_x^{\text{can}} | \Psi_{n,m}^{(L)} \rangle &\neq \langle \Psi_{n,m}^{(S)} | \hat{p}_x^{\text{can}} | \Psi_{n,m}^{(S)} \rangle, \\
\langle \Psi_{n,m}^{(L)} | \hat{L}_z^{\text{can}} | \Psi_{n,m}^{(L)} \rangle &\neq \langle \Psi_{n,m}^{(S)} | \hat{L}_z^{\text{can}} | \Psi_{n,m}^{(S)} \rangle,
\end{align}

while

\begin{align}
\langle \Psi_{n,m'}^{(L)} | \hat{p}_x^{\text{can}} (A^{(L)}) | \Psi_{n,m}^{(L)} \rangle &\neq \langle \Psi_{n,m'}^{(S)} | \hat{p}_x^{\text{can}} (A^{(S)}) | \Psi_{n,m}^{(S)} \rangle, \\
\langle \Psi_{n,m'}^{(L)} | \hat{p}_x^{\text{mech}} (A^{(L)}) | \Psi_{n,m}^{(L)} \rangle &\neq \langle \Psi_{n,m'}^{(S)} | \hat{p}_x^{\text{mech}} (A^{(S)}) | \Psi_{n,m}^{(S)} \rangle,
\end{align}

and

\begin{align}
\langle \Psi_{n,m'}^{(L)} | \hat{L}_z (A^{(L)}) | \Psi_{n,m}^{(L)} \rangle &\neq \langle \Psi_{n,m'}^{(S)} | \hat{L}_z (A^{(S)}) | \Psi_{n,m}^{(S)} \rangle, \\
\langle \Psi_{n,m'}^{(L)} | \hat{L}_z^{\text{mech}} (A^{(L)}) | \Psi_{n,m}^{(L)} \rangle &\neq \langle \Psi_{n,m'}^{(S)} | \hat{L}_z^{\text{mech}} (A^{(S)}) | \Psi_{n,m}^{(S)} \rangle.
\end{align}

This is in some sense nothing surprising, in view of the fact that the $| n, k_x \rangle$-basis and $| n, m \rangle$-basis are characterized by totally different quantum numbers $k_x$ and $m$. However, highly nontrivial observation here is that the matrix elements of the mechanical momentum operator are exactly zero in both of the $| n, k_x \rangle$- and $| n, m \rangle$-basis eigen-states:

\begin{align}
\langle \Psi_{n,k_x}^{(L)} | \hat{p}_x^{\text{mech}} (A^{(L)}) | \Psi_{n,k_x}^{(L)} \rangle &= 0, \\
\langle \Psi_{n,k_x}^{(S)} | \hat{p}_x^{\text{mech}} (A^{(S)}) | \Psi_{n,k_x}^{(S)} \rangle &= 0.
\end{align}

The reasonable nature of these inequalities and equalities was already explained for the similar relations observed for the matrix elements in the $| n, k_x \rangle$-basis. (See the explanation given at the end of the previous subsection.)

Truly new insight can be obtained by comparing Tables 1 and 2. Let us first compare the matrix elements of the momentum operators. We find that, for the matrix elements of the canonical and conserved momentum operator in the $| n, k_x \rangle$-basis and the $| n, m \rangle$-basis, there is no simple correspondence between them:

\begin{align}
\langle \Psi_{n,m}^{(L)} | \hat{p}_x^{\text{can}} | \Psi_{n,m}^{(L)} \rangle &\neq \langle \Psi_{n,m}^{(S)} | \hat{p}_x^{\text{can}} | \Psi_{n,m}^{(S)} \rangle, \\
\langle \Psi_{n,m}^{(L)} | \hat{p}_x^{\text{mech}} (A^{(L)}) | \Psi_{n,m}^{(L)} \rangle &\neq \langle \Psi_{n,m}^{(S)} | \hat{p}_x^{\text{mech}} (A^{(S)}) | \Psi_{n,m}^{(S)} \rangle, \\
\langle \Psi_{n,m}^{(L)} | \hat{p}_x^{\text{cons}} (A^{(L)}) | \Psi_{n,m}^{(L)} \rangle &\neq \langle \Psi_{n,m}^{(S)} | \hat{p}_x^{\text{cons}} (A^{(S)}) | \Psi_{n,m}^{(S)} \rangle.
\end{align}

Undoubtedly, these relations imply physical nature of the mechanical (or kinetic) momentum as compared with the canonical momentum as well as the conserved momentum. (The terminology physical here means the observable nature of the quantity in question.) In fact, we have already explained the physical or dynamical reason in the previous subsection why the expectation value of the mechanical momentum along the $x$-direction (and also the $y$-direction) vanishes exactly.

Next, let us compare the matrix elements of the OAM operators in the $| n, k_x \rangle$-basis and the $| n, m \rangle$-basis. Just like the momentum operators, we confirm that, for the matrix
elements of the canonical and conserved OAM operators in the \(|n, k_z\)-basis and the \(|n, m\)-basis, there is no physically meaningful correspondence between them,

\[
\langle \Psi^{(L_1)}_{n,k_z} | \hat{L}^{\text{mech}}_{z} | \Psi^{(L_1)}_{n,k_z} \rangle = \langle \Psi^{(S)}_{n,k_z} | \hat{L}^{\text{mech}}_{z} | \Psi^{(S)}_{n,k_z} \rangle = (2n + 1) \delta(k_z^I - k_z),
\]

and

\[
\langle \Psi^{(L_1)}_{n,m} | \hat{L}^{\text{mech}}_{z} | \Psi^{(L_1)}_{n,m} \rangle = \langle \Psi^{(S)}_{n,m} | \hat{L}^{\text{mech}}_{z} | \Psi^{(S)}_{n,m} \rangle = (2n + 1) \delta(m - m').
\]

As already explained at the end of Sect. 3.1, the main difference comes from the non-normalizable plane-wave nature of the basis functions \(|n, k_z\). If we introduce the normalizable wave-packet states \(|\tilde{\Psi}^{(L_1)}_{n,k_z}\rangle\) and \(|\tilde{\Psi}^{(S)}_{n,k_z}\rangle\), respectively corresponding to the states \(|\Psi^{(L_1)}_{n,k_z}\rangle\) and \(|\Psi^{(S)}_{n,k_z}\rangle\), we are led to a beautiful relationship for the expectation values (diagonal matrix elements) of the mechanical OAM operator given as

\[
\langle \tilde{\Psi}^{(L_1)}_{n,k_z} | \hat{L}^{\text{mech}}_{z} | \tilde{\Psi}^{(L_1)}_{n,k_z} \rangle = \langle \tilde{\Psi}^{(S)}_{n,k_z} | \hat{L}^{\text{mech}}_{z} | \tilde{\Psi}^{(S)}_{n,k_z} \rangle = (2n + 1).
\]

In our opinion, this is interpreted as showing genuinely gauge-invariant nature of the expectation value of the mechanical OAM operator, or physical nature of the mechanical OAM.

### 4 Interpretation of the analysis results

The issue of gauge choice in the Landau problem is an unexpectedly perplex problem, which sometimes causes confusion. Remember first the wide-spread traditional viewpoint as follows. There are three typical choices of gauge in the Landau problem, i.e. the symmetric gauge and two Landau gauges. The choice of the symmetric gauge naturally leads to the Laguerre type solutions, which respect the rotational (or axial) symmetry around the origin. On the other hand, if one chooses either of the two Landau gauges, one is naturally led to the Hermite type solutions, which respects the translational symmetry along the \(x\)-axis or the \(y\)-axis. We think that this wide-spread understanding is nothing wrong and it offers an established point of view on the issue of gauge choice in the Landau problem. In another viewpoint, however, the connection between the choice of the gauge potential and the choice of the types of the Landau wave functions is not necessarily mandatory [19,20]. In fact, as we have seen, independently of the choice of the gauge potential, there exist three conserved quantities in the Landau problem. They are the conserved OAM \(L^{\text{cons}}_{z}\) and the two conserved momenta \(p^{\text{cons}}_{x}\) and \(p^{\text{cons}}_{y}\). The eigen-functions of the Landau problem can be obtained by simultaneously diagonalizing the Landau Hamiltonian and one of the above three conserved operators. If we choose \(L^{\text{cons}}_{z}\) there, the eigen-functions are characterized by two quantum numbers, \(n\) and \(m\), where \(n\) is the familiar Landau quantum number and \(m\) is the eigen-value of the OAM operator \(L^{\text{cons}}_{z}\). We emphasize once again that this is true for any choice of the gauge potential configuration. Naturally, the simplest candidate of this type of solutions is the familiar eigen-states \(|\Psi^{(S)}_{n,m}\rangle\) obtained with the choice of the symmetric gauge potential \(A^{(S)}\). However, let us consider such states that are obtained from the symmetric gauge eigen-states \(|\Psi^{(S)}_{n,m}\rangle\) by operating a gauge transformation matrix \(U(\chi)\) as

\[
|\Psi^{(\chi)}\rangle = U^{(\chi)} |\Psi^{(S)}_{n,m}\rangle.
\]

where \(\chi(r)\) is assumed to be an arbitrary harmonic function. Then, it seems obvious that the new states \(|\Psi^{(\chi)}\rangle\) are also characterized by the two quantum numbers \(n\) and \(m\). Since the dependence on the Landau quantum number needs little explanation, here we check the dependence on the second quantum number \(m\), just to be sure. Operating \(L^{\text{cons}}_{z} A^{(\chi)}\) on \(|\Psi^{(\chi)}\rangle\), we obtain

\[
L^{\text{cons}}_{z} A^{(\chi)} |\Psi^{(\chi)}\rangle = L^{\text{cons}}_{z} U^{(\chi)} |\Psi^{(S)}\rangle = U^{(\chi)} U^{(\chi)} L^{\text{cons}}_{z} A^{(\chi)} U^{(\chi)} |\Psi^{(S)}\rangle = U^{(\chi)} U^{(\chi)} L^{\text{cons}}_{z} A^{(\chi)} |\Psi^{(S)}\rangle = m U^{(\chi)} |\Psi^{(S)}_{n,m}\rangle.
\]

Here, use has been made of the covariant gauge-transformation property of \(L^{\text{cons}}_{z}\), i.e. the relation \(U^{(\chi)} U^{(\chi)} L^{\text{cons}}_{z} A^{(\chi)} U^{(\chi)} = L^{\text{cons}}_{z} A^{(\chi)}\). The above equation shows that \(|\Psi^{(\chi)}\rangle\) are the eigen-functions of \(L^{\text{cons}}_{z}\) with the eigen-value \(m\). We thus confirm that the states \(|\Psi^{(\chi)}\rangle\) are in fact characterized by the quantum numbers \(n\) and \(m\), so that it is legitimate to write \((134)\) as \(|\Psi^{(\chi)}_{n,m}\rangle = U^{(\chi)} |\Psi^{(S)}_{n,m}\rangle\). Note that the eigenstates \(|\Psi^{(\chi)}_{n,m}\rangle\) obtained with arbitrary (regular) gauge func-
tion $\chi$ are different from the symmetric-gauge eigen-states only by the phase factor $e^{-i e \chi}$, so that the electron probability densities corresponding to $|\Psi_{n,m}^{(S)}\rangle$ and $|\Psi_{n,m}^{(L)}\rangle$ must be exactly the same and they show the axial symmetry around the coordinate origin. Thus, the set of eigen-functions $|\Psi_{n,m}^{(S)}\rangle$ obtained in this way may be called the gauge-potential-independent extensions of the symmetric gauge eigen-states $|\Psi_{n,m}^{(S)}\rangle$). (Alternatively, mimicking the terminology advocated in the recent literature [16,17], they might simply be called the gauge-invariant extension based on the symmetric-gauge eigen-states $|\Psi_{n,m}^{(S)}\rangle$.)

Exactly by the same logic, we can define the states, which may be called the gauge-potential-independent extension of the 1st Landau-gauge eigen-states $|\Psi_{n,k_x}^{(L1)}\rangle$ by

$$|\Psi_{n,k_x}^{(L1)}\rangle = U(\chi) |\Psi_{n,k_x}^{(L1)}\rangle.$$  \hspace{1cm} (136)

Here, $U(\chi) = e^{-i e \chi}$ with $\chi$ being an arbitrary harmonic function. Naturally, any of the eigen-states $|\Psi_{n,k_x}^{(L1)}\rangle$ have exactly the same electron probability densities as the 1st Landau-gauge eigen-states $|\Psi_{n,k_x}^{(L1)}\rangle$. However, it is also obvious that these densities are absolutely different from the probability densities corresponding to the symmetric-gauge eigen-states $|\Psi_{n,m}^{(S)}\rangle$. Undoubtedly, it is related to the fact that there is no gauge transformation which directly connects the eigen-states of the symmetric gauge and those of the 1st Landau gauge. The reason may further be traced back to the (infinitely-many) degeneracy of the Landau levels, which happens in both of the symmetric-gauge eigenstates and of the Landau-gauge eigenstates. To understand this state of affairs in a more concrete manner, suppose that we operate the gauge transformation matrix $U = e^{i \frac{1}{2} e B x y}$ on the symmetric-gauge eigen-states $|\Psi_{n,m}^{(S)}\rangle$. Then, using the completeness relation for the 1st Landau-gauge eigen-states within the Hilbert space of fixed number of the Landau quantum number $n$ given as

$$\int dk_x |\Psi_{n,k_x}^{(L1)}\rangle \langle\Psi_{n,k_x}^{(L1)}| = 1,$$  \hspace{1cm} (137)

we obtain

$$U |\Psi_{n,m}^{(S)}\rangle = \int dk_x |\Psi_{n,k_x}^{(L1)}\rangle \langle\Psi_{n,k_x}^{(L1)}| U |\Psi_{n,m}^{(S)}\rangle = \int dk_x U_{n,k_x ; n,m} \langle\Psi_{n,k_x}^{(L1)}|,$$  \hspace{1cm} (138)

with the definition

$$U_{n,k_x ; n,m} \equiv \langle\Psi_{n,k_x}^{(L1)}| U |\Psi_{n,m}^{(S)}\rangle.$$  \hspace{1cm} (139)

This means that the gauge-transformed states $U |\Psi_{n,m}^{(S)}\rangle$ are superpositions of the 1st Landau-gauge eigen-states $|\Psi_{n,k_x}^{(L1)}\rangle$ with the weight function $U_{n,k_x ; n,m}$. The explicit form of this weight function is already written down in some previous literature [11,19]. It is given as

$$U_{n,k_x ; n,m} = C_{n,m} H_{n-m} \left( \frac{y_0}{l_B} \right) e^{-\frac{y_0^2}{2 l_B^2}},$$  \hspace{1cm} (140)

with $y_0 = k_x / (e B)$ and

$$C_{n,m} = l_B \left( \frac{1}{\sqrt{\pi} 2^{n-m} (n-m)!} \right)^{1/2}.$$  \hspace{1cm} (141)

Here, one may notice that the states obtained by operating $U = e^{i \frac{1}{2} e B x y}$ on $|\Psi_{n,m}^{(S)}\rangle$ are nothing but the states $|\Psi_{n,m}^{(L1)}\rangle$ defined before by Eqs.(51) and (52). From this fact, we now realize that the states $|\Psi_{n,m}^{(L1)}\rangle$ introduced there are actually the following superposition of the 1st Landau-gauge eigenstates $|\Psi_{n,k_x}^{(L1)}\rangle$,

$$|\Psi_{n,m}^{(L1)}\rangle = \int dk_x U_{n,k_x ; n,m} |\Psi_{n,k_x}^{(L1)}\rangle.$$  \hspace{1cm} (142)

Obviously, this relation can be generalized to arbitrary states defined as $|\Psi_{n,m}^{(L1)}\rangle = U(\chi) |\Psi_{n,m}^{(S)}\rangle = e^{-i e \chi} |\Psi_{n,m}^{(S)}\rangle$ with $\chi$ being an arbitrary harmonic function, and they are also the members of the gauge-potential-independent extension based on the symmetric gauge eigen-states $|\Psi_{n,m}^{(S)}\rangle$. In this way, we are led to the conclusion that there are totally three such extensions, i.e. the gauge-potential-independent extension based on the symmetric gauge eigen-states, that based on the 1st Landau-gauge eigen-states and that based on the 2nd Landau-gauge eigen-states. What is important to recognize here is that these three types of eigen-states belong to totally different (or inequivalent) gauge classes.

What is meant by the above statement would be understood by checking the following two properties:

- The expectation values of a genuinely gauge-invariant physical quantity should be the same irrespective of the choice of three types of eigen-functions.
- The expectation values of the gauge-variant physical quantity can be different for the eigen-states belonging to different gauge classes.

Through the analyses in the previous section, we have verified that the expectation value of the canonical momentum operator $p_x^{can}$ and the canonical OAM operator $L_z^{can}$ are in fact different between the two states belonging to different gauge classes, i.e. between the gauge-potential-independent extension based on the symmetric gauge eigen-states and that based on the 1st Landau gauge eigen-states. This is not surprising at all, since these canonical quantities are widely believed to be gauge-variant ones. In sharp contrast, we found that the expectation value of the mechanical momentum operator $p_x^{mech}$ and the mechanical OAM operator $L_z^{mech}$ perfectly coincide between the two states belonging to different gauge classes. Somewhat perplexing are the preserved
momentum operator $P_z^{\text{cons}}$ and the conserved OAM operator $L_z^{\text{cons}}$. Despite the fact that these two operators transform in a gauge-covariant manner just like the mechanical operators, we found that the expectation values of these conserved operators do not coincide between the eigen-states belonging to two gauge-inequivalent classes. What is the cause of this remarkable difference?

It can be understood as follows. Notice first that the following equality holds:

$$
\sum_m U_{n,k';n,m} U^*_{n,k;n,m} = \sum_m \langle \Psi_{n,k'}^{(L_1)} | U | \Psi_{n,m}^{(S)} \rangle \times \langle \Psi_{n,m}^{(S)} | U^\dagger | \Psi_{n,k}^{(L_1)} \rangle
$$

$$
= \langle \Psi_{n,k'}^{(L_1)} | U U^\dagger | \Psi_{n,k}^{(L_1)} \rangle
$$

$$
= \delta(k'_z - k_z).
$$

(143)

By using this relation, (142) can be inverted as

$$
| \Psi_{n,k}^{(L_1)} \rangle = \sum_m U^*_{n,k;n,m} | \Psi_{n,m}^{(L_1)} \rangle.
$$

(144)

Now we compare the matrix elements of the mechanical OAM operator and the conserved OAM operator between the states $| \Psi_{n,k}^{(L_1)} \rangle$ and $| \Psi_{h,n}^{(L_1)} \rangle$. For the former quantity, we find that

$$
\langle \Psi_{n,k}^{(L_1)} | \hat{L}_z^{\text{mech}} (A^{(L_1)}) | \Psi_{n,k}^{(L_1)} \rangle
$$

$$
= \sum_{m,m'} U_{n,k';n,m} U^*_{n,k;nm} \langle \Psi_{n,m}^{(S)} | \hat{L}_z^{\text{mech}} (A^{(L_1)}) | \Psi_{n,k}^{(S)} \rangle
$$

$$
= \sum_{m,m'} U_{n,k';n,m} U^*_{n,k;nm} \times (2n+1) \delta_{m',m}
$$

$$
= (2n+1) \sum_{m,m'} U_{n,k';n,m} U^*_{n,k;nm}
$$

$$
= (2n+1) \delta(k'_z - k_z),
$$

(145)

which should be compared with the matrix elements between the symmetric-gauge eigen-states given as $\langle \Psi_{n,m}^{(S)} | \hat{L}_z^{\text{mech}} (A^{(S)}) | \Psi_{n,m}^{(S)} \rangle = (2n+1) \delta_{m',m}$. A noteworthy fact here is that the Landau quantum number $n$ appears as a common quantum number in the 1st Landau-gauge eigen-states and the symmetric gauge ones and that, except for the appearance of the delta function and the Kronecker delta related to the different normalization conditions, the matrix elements of the mechanical OAM operators in both types of eigen-states depend only on $n$ and they are just the same, which may be interpreted as showing the physical nature of the mechanical OAM.

In contrast, for the matrix elements of the conserved OAM operator, we have

$$
\langle \Psi_{n,k}^{(L_1)} | \hat{L}_z^{\text{cons}} (A^{(L_1)}) | \Psi_{n,k}^{(L_1)} \rangle
$$

$$
= \sum_{m,m'} U_{n,k';n,m} U^*_{n,k;nm} \langle \Psi_{n,m}^{(S)} | \hat{L}_z^{\text{cons}} (A^{(L_1)}) | \Psi_{n,m}^{(S)} \rangle
$$

$$
= \sum_{m,m'} U_{n,k';n,m} U^*_{n,k;nm} \times m \delta_{m',m}
$$

$$
= \sum_{m} U_{n,k';n,m} U^*_{n,k;nm} \times m
$$

$$
= \left\{ n + 1 - \frac{k_z^2}{2eB} \right\} \delta(k'_z - k_z) + \frac{eB}{2} \delta''(k'_z - k_z),
$$

(146)

which should be compared with the matrix element between the symmetric-gauge eigen-states given as $\langle \Psi_{n,m}^{(S)} | \hat{L}_z^{\text{cons}} (A^{(S)}) | \Psi_{n,m}^{(S)} \rangle = m \delta_{m',m}$.

One should not overlook the fact that the matrix element of the conserved OAM operator between the 1st Landau-gauge eigen-states depend on both of $n$ and $k_z$, while the matrix element of the conserved OAM operator between the symmetric-gauge eigen-states depends only on $m$. Undoubtedly, this comparison indicates unphysical (or nonobservable) nature of the quantum number $k_z$ and $m$ appearing in the two different types of Landau eigen-functions Hamiltonian. In fact, we recall that nonobservability of the quantum number $m$ or the canonical OAM in the Landau problem was an object of extensive discussion in recent papers [21,22]. (Clearly, the same can be said also for the quantum-number $k_z$ or the canonical momentum.)

In any case, the above comprehensive analysis, combined with the comparison of Tables 1 and 2, confirms the fact that the gauge-independence of the expectation value of the conserved OAM operator is restricted within the same gauge class and it cannot be extended to different or inequivalent gauge classes. In other words, the gauge-covariance of a certain operator does not necessarily mean the gauge-independence of the corresponding quantity. We emphasize that this is a highly nontrivial finding which has never been explicitly stated before at least in an analytically solvable model like the Landau problem.

Before ending this section, we think it enlightening as well as important to point out an interesting relationship between the conserved OAM discussed in this paper and the idea of gauge-invariant (or more precisely gauge-covariant) extension of the canonical OAM advocated in the literature on the nucleon spin decomposition problem. The latter quantity was introduced based on the idea of Chen et al. [13,14], who proposed the decomposition of the gauge field into the physical component and the pure-gauge component as

$$
A = A^{\text{phys}} + A^{\text{pure}}.
$$

(147)

The basic postulate of theirs is that, under a gauge transformation specified by $U = e^{i\chi}$, the above two components transform as

$$
A^{\text{phys}} \rightarrow A'^{\text{phys}} = A^{\text{phys}},
$$

(148)

$$
A^{\text{pure}} \rightarrow A'^{\text{pure}} = A^{\text{pure}} + \nabla \chi.
$$

(149)
That is, the gauge degrees of freedom are totally carried by the pure-gauge part $A^{\text{pure}}$, while the physical component $A^{\text{phys}}$ is intact under a gauge transformation.

Suppose for the moment that such a decomposition in fact exists. Then, one may introduce the following quantities:

$$p_x^{\text{g.c.c.}[A^{\text{phys}}]} = p_x^{\text{can}} + e (A_x - A^{\text{phys}}_x),$$

$$L_z^{\text{g.c.c.}[A^{\text{phys}}]} = L_z^{\text{can}} + e (\mathbf{r} \times (A - A^{\text{phys}}))_z.$$

which may be called the gauge-covariant-canonical (g.c.c.) momentum and the g.c.c OAM. (In the previous literature [17], they were called the gauge-invariant-canonical (g.i.c.) momentum and g.i.c OAM. However, in the context of quantum mechanics or quantum field theory, it would be more legitimate to use the word covariant rather than the word invariant.) In fact, it is obvious that these operators transform covariantly under an arbitrary gauge transformation, because the mechanical operators transform covariantly, while $A^{\text{phys}}$ is intact. A pitfall of the above argument is that the decomposition of the vector potential into the physical and pure-gauge potential is not always unique. Remember that, when Chen et al. proposed the above decomposition of the gauge field, what was in their mind was the familiar transverse-longitudinal decomposition of the vector potential [23]. Once the Lorentz frame of reference is fixed, the transverse-longitudinal decomposition is known to be unique as long as the condition for the Helmholtz theorem is satisfied [24]. Unfortunately, in our 2-dimensional Landau problem, in which the condition for the Helmholtz theorem is not satisfied, there is no way to uniquely fix the physical component of the vector potential. This is reflected by the fact that the three practical gauge choices in the Landau problem all satisfy the transverse condition $\nabla \cdot A^{(S)} = \nabla \cdot A^{(L_1)} = \nabla \cdot A^{(L_2)} = 0$. Thus, although a unique identification of the physical component is not possible, suppose that we assign by hand the symmetric gauge potential as the physical component, i.e. $A^{(S)} = A^{\text{phys}}$. Then, it follows that

$$p_x^{\text{g.c.c.}[A^{(S)}]} = p_x^{\text{can}} + e A_x + \frac{1}{2} e B y,$$

$$L_z^{\text{g.c.c.}[A^{(S)}]} = L_z^{\text{can}} + e (\mathbf{r} \times A)_z - \frac{1}{2} e B r^2.$$

Here, one may notice that $L_z^{\text{g.c.c.}[A^{(S)}]}$ above precisely coincides with our conserved OAM, i.e.

$$L_z^{\text{g.c.c.}[A^{(S)}]} = L_z^{\text{cons}}.$$

On the other hand, if the 1st Landau gauge potential is identified with the physical component as $A^{(L_1)} = A^{\text{phys}}$, we have

$$p_x^{\text{g.c.c.}[A^{(L_1)}]} = p_x^{\text{can}} + e A_x + e B y,$$

$$L_z^{\text{g.c.c.}[A^{(L_1)}]} = L_z^{\text{can}} + e (\mathbf{r} \times A)_z - e B y^2.$$

Here, one finds that $p_x^{\text{g.c.c.}[A^{(L_1)}]}$ precisely coincides with our conserved momentum, i.e.

$$p_x^{\text{g.c.c.}[A^{(L_1)}]} = p_x^{\text{cons}}.$$

What can we learn from the above consideration? We have already shown that the conserved momentum as well as the conserved OAM are not truly gauge-invariant quantities despite their gauge-covariant transformation property. Undoubtedly, the same can be said for the gauge-covariant extension of the canonical OAM advocated in the recent literature [15–17].

The argument above also reminds us of the debates on the gauge-invariant or gauge-variant nature of the gluon spin as well as the canonical OAM of quarks in the nucleon spin decomposition problem. Originally, the canonical quark OAM inside the nucleon appearing in the famous Jaffe-Manohar decomposition was believed to be a gauge-variant quantity [25]. However, after Chen et al.'s paper appeared [13,14], several authors proposed a concept of gauge-covariant extension of the canonical OAM operator and the belief, that this extended canonical OAM can be thought of as a gauge-invariant quantity, became popular. (See, for example, the review [17].) However, this extension of the canonical OAM needs the concept of physical component of the gauge field. As we have clearly shown in the present paper, when there exist plural possibilities for the choice of the physical component of the gauge field, such extensions of the canonical OAM actually depends on the basis gauge of extensions and they are not gauge-invariant quantities in the rigorous sense. Note that the situation is totally different for the mechanical quark OAM inside the nucleon, which is defined without need of the physical component of the gauge field. One can then say that the mechanical OAM is a genuinely gauge-invariant quantity [26–28]. Unfortunately, to make the gluon spin inside the nucleon gauge-invariant, we also need the concept of physical component of the gluon field and the gauge-invariant extension based on it. Although the gauge-invariant extension based on the light-cone gauge is practically the most useful choice in the deep-inelastic-scattering physics, still one should keep in mind the fact that the gauge-invariance of the gluon spin attained in that way is not a gauge-invariance in the true sense. Undoubtedly, this obstruction must have a deep connection with the long-known fact in the field of perturbative QCD that there is no local and gauge-invariant gluon spin operator.
5 Conclusion

In the present paper, we proposed a simple quantum mechanical formulation of the famous Landau problem, which enables us to avoid a specific choice of gauge potential when writing down the eigen-functions of the Landau problem. The formalism is based on the existence of three conserved quantities in the Landau problem, i.e. the two conserved momenta and one conserved OAM, absolutely independently of the choice of gauge potential. A prominent feature of the above conserved momenta and the conserved OAM, which are also called the pseudo momenta and the pseudo OAM in some literature, is that they have covariant transformation properties under an arbitrary gauge transformation, just like the familiar mechanical momentum and mechanical OAM operators. (The latters are widely believed to be manifestly gauge-invariant quantities.) In this gauge-potential-independent formulation, although the quantum mechanical eigen-functions of the Landau Hamiltonian can be written down without fixing the gauge potential, it turns out that these solutions are divided into three classes, which we may call the gauge-potential-independent extensions based on three different basis gauges, i.e. the 1st Landau gauge, the 2nd Landau gauge, and the symmetric gauge. We have carried out a comparative analysis of the matrix elements of the mechanical operators and the conserved operators of the mechanical operators and the conserved operators between the three different classes of eigenstates. We then found that the matrix elements of the mechanical momentum and mechanical OAM operators between the three different eigen-states perfectly coincide with each other. This is interpreted to verify the genuinely gauge-invariant nature of the mechanical quantities. On the other hand, it turned out that the matrix elements of the conserved momentum and conserved OAM operators between the three different eigen-states do not coincide with each other. This means that three gauge-potential-independent extensions do actually belong to different gauge classes, and that the conserved momentum and conserved OAM are not truly gauge-invariant physical quantities despite their invariant gauge-transformation property. This also dictates that little physical meaning can be given to the idea of the gauge-invariant extension of the canonical OAM advocated in the recent literature on the nucleon spin decomposition problem. We can also say that the present analysis provides us with one concrete and clear example in which the gauge symmetry is just a redundancy of the description with no substantial physical contents [29,30].

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: All the outcomes of the investigation are given in the main text containing two tables as well as in three appendices.]

Appendix A Calculation of the matrix elements in the $|n, k_x\rangle$-basis

By using the explicit form of $\Psi^{(L_1)}_{n,k_x}(x, y)$ given by (48), it can readily be shown that

$$\langle \Psi^{(L_1)}_{n,k_x} | \hat{p}_x^{\text{can}} | \Psi^{(L_1)}_{n,k_x} \rangle = \langle \Psi^{(L_1)}_{n,k_x} | - i \frac{\partial}{\partial x} | \Psi^{(L_1)}_{n,k_x} \rangle = k_x \delta(k_x' - k_x). \quad (A1)$$

Furthermore, since $\hat{p}_x^{\text{cons}}(A)$ reduces to the canonical momentum $\hat{p}_x^{\text{can}}$ in the $L_1$-gauge, we naturally have

$$\langle \Psi^{(L_1)}_{n,k_x} | \hat{p}_x^{\text{cons}}(A^{(L_1)}) | \Psi^{(L_1)}_{n,k_x} \rangle = \langle \Psi^{(L_1)}_{n,k_x} | \hat{p}_x^{\text{can}} | \Psi^{(L_1)}_{n,k_x} \rangle = k_x \delta(k_x' - k_x). \quad (A2)$$

Next, the matrix element of the mechanical momentum operator becomes

$$\langle \Psi^{(L_1)}_{n,k_x} | \hat{p}_x^{\text{mech}}(A^{(L_1)}) | \Psi^{(L_1)}_{n,k_x} \rangle = \langle \Psi^{(L_1)}_{n,k_x} | \hat{p}_x^{\text{can}} + e (-B y) | \Psi^{(L_1)}_{n,k_x} \rangle = k_x \delta(k_x' - k_x) - eB \delta(k_x' - k_x) \langle Y_n | y | Y_n \rangle. \quad (A3)$$

Here, the matrix element $\langle Y_n | y | Y_n \rangle$ can be evaluated as follows:

$$\langle Y_n | y | Y_n \rangle = N_n^2 \int_{-\infty}^{\infty} dy e^{-\frac{(y-y_0)^2}{l_n^2}} \times H_n \left( \frac{y - y_0}{l_B} \right) = y_0 = \frac{k_x}{eB}. \quad (A4)$$

We therefore find a remarkable relation as

$$\langle \Psi^{(L_1)}_{n,k_x} | \hat{p}_x^{\text{mech}}(A^{(L_1)}) | \Psi^{(L_1)}_{n,k_x} \rangle = k_x \delta(k_x' - k_x) - eB \frac{k_x}{eB} \delta(k_x' - k_x) = 0, \quad (A5)$$

the physical significance of which is explained in the main text.

Next, we turn to the matrix elements between the eigenstates $|\Psi^{(S)}_{n,k_x}\rangle$. With the use of the relation $|\Psi^{(S)}_{n,k_x}\rangle = U^\dagger |\Psi^{(L_1)}_{n,k_x}\rangle$, we obtain

$$\langle \Psi^{(S)}_{n,k_x} | \hat{p}_x^{\text{can}} | \Psi^{(S)}_{n,k_x} \rangle = \langle \Psi^{(L_1)}_{n,k_x} | U \hat{p}_x^{\text{can}} U^\dagger | \Psi^{(L_1)}_{n,k_x} \rangle. \quad (A6)$$

Noting that

$$U \hat{p}_x^{\text{can}} U^\dagger = -\hat{p}_x^{\text{can}} - \frac{1}{2} eB y, \quad (A7)$$

we therefore find that

$$\langle \Psi^{(S)}_{n,k_x} | \hat{p}_x^{\text{can}} | \Psi^{(S)}_{n,k_x} \rangle = \langle \Psi^{(L_1)}_{n,k_x} | \hat{p}_x^{\text{can}} - \frac{1}{2} eB y | \Psi^{(L_1)}_{n,k_x} \rangle = k_x \delta(k_x' - k_x) - \frac{1}{2} eB \frac{k_x}{eB} \delta(k_x' - k_x)$$
Similarly, we have
\[
\langle \Psi^{(S)}_{n,k_x} | \hat{p}_{x}^{\text{cons}} (A^{(S)}) | \Psi^{(S)}_{n,k_x} \rangle = \langle \Psi^{(L_1)}_{n,k_x} | U \hat{p}_{x}^{\text{cons}} (A^{(L_1)}) U^{\dagger} | \Psi^{(L_1)}_{n,k_x} \rangle.
\] 
(A9)

Taking care of the covariant gauge-transformation property of \( \hat{p}_{x}^{\text{cons}} \), which means the relation \( U \hat{p}_{x}^{\text{cons}} (A^{(S)}) U^{\dagger} = \hat{p}_{x}^{\text{cons}} (A^{(L_1)}) \), we therefore find that
\[
\langle \Psi^{(S)}_{n,k_x} | \hat{p}_{x}^{\text{cons}} (A^{(S)}) | \Psi^{(S)}_{n,k_x} \rangle = \langle \Psi^{(L_1)}_{n,k_x} | \hat{p}_{x}^{\text{cons}} (A^{(L_1)}) | \Psi^{(L_1)}_{n,k_x} \rangle = k_x \delta (k_x' - k_x).
\] 
(A10)

For the matrix element of the mechanical momentum operator, we obtain
\[
\langle \Psi^{(S)}_{n,k_x} | \hat{p}_{x}^{\text{mech}} (A^{(S)}) | \Psi^{(S)}_{n,k_x} \rangle = \langle \Psi^{(L_1)}_{n,k_x} | \hat{p}_{x}^{\text{mech}} (A^{(L_1)}) | \Psi^{(L_1)}_{n,k_x} \rangle = k_x \delta (k_x' - k_x).
\] 
(A11)

Here, we have used the fact that \( \hat{p}_{x}^{\text{mech}} \) also transforms covariantly under a gauge transformation.

Next, we evaluate the matrix elements of the three OAM operators in the two \( |n, k_x\rangle \)-basis functions, \( \Psi^{(L_1)}_{n,k_x} (x,y) \) and \( \Psi^{(S)}_{n,k_x} (x,y) \). Although it may sound strange, this has never been done before. The reason is probably because the symmetric gauge eigen-states \( |\Psi^{(S)}_{n,k_x}\rangle \) are the most natural and convenient basis to deal with the OAM operators, and one seldom paid attention to calculating the expectation values of the OAM operators between the eigen-functions \( |\Psi^{(L_1)}_{n,k_x}\rangle \) in the 1st Landau gauge.

With the use of the definitions of \( \hat{L}_{z}^{\text{cons}} (A) \) and \( \hat{L}_{z}^{\text{mech}} (A) \), we obtain
\[
\langle \Psi^{(L_1)}_{n,k_x} | \hat{L}_{z}^{\text{cons}} (A^{(L_1)}) | \Psi^{(L_1)}_{n,k_x} \rangle = \langle \Psi^{(L_1)}_{n,k_x} | \hat{L}_{z}^{\text{mech}} (A^{(L_1)}) | \Psi^{(L_1)}_{n,k_x} \rangle = \langle \Psi^{(L_1)}_{n,k_x} | \hat{L}_{z}^{\text{mech}} (A^{(L_1)}) | \Psi^{(L_1)}_{n,k_x} \rangle
\] 
(A12)

Similarly, using the relation \( |\Psi^{(S)}_{n,k_x}\rangle = U^{\dagger} |\Psi^{(L_1)}_{n,k_x}\rangle \) together with the gauge-covariant transformation properties of \( \hat{L}_{z}^{\text{cons}} (A) \) and \( \hat{L}_{z}^{\text{mech}} (A) \), we obtain
\[
\langle \Psi^{(S)}_{n,k_x} | \hat{L}_{z}^{\text{cons}} (A^{(S)}) | \Psi^{(S)}_{n,k_x} \rangle = \langle \Psi^{(L_1)}_{n,k_x} | \hat{L}_{z}^{\text{can}} (A^{(L_1)}) | \Psi^{(L_1)}_{n,k_x} \rangle
\] 
(A13)

Thus, for evaluating the matrix elements of the three OAM operators in the \( |n, k_x\rangle \)-basis, we have only to know the following three matrix elements:
\[
\langle \Psi^{(L_1)}_{n,k_x} | \hat{L}_{z}^{\text{can}} | \Psi^{(L_1)}_{n,k_x} \rangle, \quad \langle \Psi^{(L_1)}_{n,k_x} | x^2 | \Psi^{(L_1)}_{n,k_x} \rangle,
\] 
(A14)
\[
\langle \Psi^{(L_1)}_{n,k_x} | y^2 | \Psi^{(L_1)}_{n,k_x} \rangle.
\] 
(A15)

As one can easily verify, the two of these matrix elements, i.e. \( \langle \Psi^{(L_1)}_{n,k_x} | \hat{L}_{z}^{\text{can}} | \Psi^{(L_1)}_{n,k_x} \rangle \) and \( \langle \Psi^{(L_1)}_{n,k_x} | x^2 | \Psi^{(L_1)}_{n,k_x} \rangle \), can be calculated without much difficulty. On the other hand, the calculation of the second matrix element \( \langle \Psi^{(L_1)}_{n,k_x} | y^2 | \Psi^{(L_1)}_{n,k_x} \rangle \) needs some care owing to the plane-wave nature of the eigen-states \( |\Psi^{(L_1)}_{n,k_x}\rangle \) along the x-direction. (Note that this cumbersome term appears in the matrix elements of \( \hat{L}_{z}^{\text{cons}} \) and \( \hat{L}_{z}^{\text{mech}} (A) \), but it does not in those of \( \hat{L}_{z}^{\text{mech}} (A) \).)

First, note that the eigen-functions \( \Psi^{(L_1)}_{n,k_x} (x,y) \) are normalized as
\[
\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \Psi^{(L_1)}_{n,k_x} (x,y) \Psi^{(L_1)}_{n,k_x} (x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \Psi^{(L_1)}_{n,k_x} (x,y) \Psi^{(L_1)}_{n,k_x} (x,y) = 1.
\] 
(A16)

Next, we have used the fact that \( Y_{m}(\xi) \) is normalized as
\[
\int_{-\infty}^{\infty} d\xi [Y_{m}(\xi)]^2 = 1.
\] 
(A17)

We begin with the calculation of the matrix element of the canonical OAM operator:
\[
\langle \Psi^{(L_1)}_{n,k_x} | \hat{L}_{z}^{\text{can}} | \Psi^{(L_1)}_{n,k_x} \rangle = \frac{1}{\sqrt{L_B}} \int_{-\infty}^{\infty} d\xi [\psi_{n}(\xi)]^2 = 1.
\] 
(A18)

We begin with the calculation of the matrix element of the canonical OAM operator:
\[
\langle \Psi^{(L_1)}_{n,k_x} | \hat{L}_{z}^{\text{can}} | \Psi^{(L_1)}_{n,k_x} \rangle = 1 / \sqrt{L_B} \int_{-\infty}^{\infty} d\xi [\psi_{n}(\xi)]^2 = 1.
\] 
(A19)
Let us first consider the first part in the r.h.s. Using the variables \( \xi = (y - i \frac{l_B}{l_B} k_x) / l_B \) and \( \xi' = (y - i \frac{l_B^2}{l_B} k'_x) / l_B \) together with the relation \( \frac{d}{dy} = \frac{i}{l_B} \frac{d}{d\xi} \), we obtain

\[
- i \langle \Psi_{n,k_x}^{(L_1)} | x \frac{\partial}{\partial y} | \Psi_{n,k_x}^{(L_1)} \rangle \\
= - i \left\{ \frac{1}{2 \pi} \int_{-\infty}^{\infty} dx x e^{-i (k'_x - k_x) x} \right\} \\
\times \frac{1}{l_B} \int_{-\infty}^{\infty} d\xi \psi_n(\xi') \frac{\partial}{\partial \xi} \psi_n(\xi) \\
= \left\{ - \frac{\partial}{\partial k_x} \delta(k'_x - k_x) \right\} \frac{1}{l_B} \int_{-\infty}^{\infty} d\xi \psi_n(\xi') \frac{\partial}{\partial \xi} \psi_n(\xi). \tag{A21}
\]

Here, we can make use of the identity of Dirac’s delta function:

\[
f(k) \delta'(k) = - f'(0) \delta(k), \tag{A22}
\]

where \( f(k) \) is any functions of \( k \), which satisfies the condition \( f(0) = 0 \). This gives

\[
- i \langle \Psi_{n,k'_x}^{(L_1)} | x \frac{\partial}{\partial y} | \Psi_{n,k_x}^{(L_1)} \rangle \\
= \left\{ - \frac{\partial}{\partial k_x} \delta(k'_x - k_x) \right\} \frac{1}{l_B} \int_{-\infty}^{\infty} d\xi \psi_n(\xi') \frac{\partial}{\partial \xi} \psi_n(\xi) \bigg|_{\xi' = \xi} \times \delta(k'_x - k_x). \tag{A23}
\]

if the following equality holds

\[
\int_{-\infty}^{\infty} d\xi \psi_n(\xi') \frac{\partial}{\partial \xi} \psi_n(\xi) \bigg|_{\xi' = \xi} = 0, \tag{A24}
\]

which can be easily verified to hold. We therefore find that

\[
- i \langle \Psi_{n,k'_x}^{(L_1)} | x \frac{\partial}{\partial y} | \Psi_{n,k_x}^{(L_1)} \rangle \\
= \delta(k'_x - k_x) \left( - \frac{\partial}{\partial \xi} \right) \frac{1}{l_B} \int_{-\infty}^{\infty} d\xi \psi_n(\xi') \frac{\partial}{\partial \xi} \psi_n(\xi) \bigg|_{\xi' = \xi} \\
= \delta(k'_x - k_x) \int_{-\infty}^{\infty} \psi_n(\xi) \frac{\partial^2}{\partial \xi^2} \psi_n(\xi) \\
= \frac{1}{2} (2n + 1) \delta(k'_x - k_x). \tag{A25}
\]

The second half part of (A20) can be evaluated as

\[
i \langle \Psi_{n,k'_x}^{(L_1)} | y \frac{\partial}{\partial x} | \Psi_{n,k_x}^{(L_1)} \rangle = - k_x \delta(k'_x - k_x) \\
\times \int_{-\infty}^{\infty} dy \frac{\partial}{\partial x} Y_n(y) \\
= - k_x \delta(k'_x - k_x) y_0 = - \frac{k_x^2}{eB} \delta(k'_x - k_x). \tag{A26}
\]

Combining the two terms, we therefore find that

\[
\langle \Psi_{n,k'_x}^{(L_1)} | \hat{L}_{can}^{\dagger} | \Psi_{n,k_x}^{(L_1)} \rangle = \left\{ n + \frac{1}{2} - \frac{k_x^2}{eB} \right\} \delta(k'_x - k_x). \tag{A27}
\]

Since the calculation of the matrix element \( \langle \Psi_{n,k'_x}^{(L_1)} | y^2 | \Psi_{n,k_x}^{(L_1)} \rangle \) is straightforward, here we show only the answer given as

\[
\langle \Psi_{n,k'_x}^{(L_1)} | y^2 | \Psi_{n,k_x}^{(L_1)} \rangle = \frac{1}{eB} \left( n + \frac{1}{2} + \frac{k_x^2}{eB} \right) \delta(k'_x - k_x). \tag{A28}
\]

Finally, as already pointed out, the calculation of the matrix element \( \langle \Psi_{n,k'_x}^{(L_1)} | x^2 | \Psi_{n,k_x}^{(L_1)} \rangle \) is a little technical, so that here we show only the final answer by leaving the explicit derivation to another Appendix B. The answer reads as

\[
\langle \Psi_{n,k'_x}^{(L_1)} | x^2 | \Psi_{n,k_x}^{(L_1)} \rangle = \frac{1}{eB} \left( n + \frac{1}{2} \right) \delta(k'_x - k_x) \\
- \delta''(k'_x - k_x), \tag{A29}
\]

where \( \delta''(k) \) represents the second derivative of \( \delta(k) \). In this way, we now have all the necessary matrix elements as follows:

\[
\langle \Psi_{n,k'_x}^{(L_1)} | \hat{L}_z^{\text{can}} | \Psi_{n,k_x}^{(L_1)} \rangle = \left\{ n + \frac{1}{2} - \frac{k_x^2}{eB} \right\} \delta(k'_x - k_x), \tag{A30}
\]

\[
eB \langle \Psi_{n,k'_x}^{(L_1)} | y^2 | \Psi_{n,k_x}^{(L_1)} \rangle = \left\{ n + \frac{1}{2} + \frac{k_x^2}{eB} \right\} \delta(k'_x - k_x), \tag{A31}
\]

\[
eB \langle \Psi_{n,k'_x}^{(L_1)} | x^2 | \Psi_{n,k_x}^{(L_1)} \rangle = \left( n + \frac{1}{2} \right) \delta(k'_x - k_x) \\
- eB \delta''(k'_x - k_x). \tag{A32}
\]

Using these answers, we can now readily write down the final answers for the matrix elements of the three OAM operators in the \( |n, k_x\rangle \)-basis. The answers are summarized in Table 1 in the main text together with the matrix elements of the three momentum operators.

### Appendix B Calculation of the matrix elements \( \langle \Psi_{n,k'_x}^{(L_1)} | x^2 | \Psi_{n,k_x}^{(L_1)} \rangle \)

The calculation starts with

\[
\langle \Psi_{n,k'_x}^{(L_1)} | x^2 | \Psi_{n,k_x}^{(L_1)} \rangle \\
= \left\{ \frac{1}{2 \pi} \int_{-\infty}^{\infty} dx e^{-i (k'_x - k_x) x} x^2 \right\} \\
\times \int_{-\infty}^{\infty} dy \frac{\partial}{\partial y} Y_n(y - y_0) \\
= \left\{ - \frac{\partial^2}{\partial k'_x^2} \delta(k'_x - k_x) \right\} \\
\times \int_{-\infty}^{\infty} dy \frac{\partial}{\partial y} Y_n(y - y_0). \tag{B33}
\]
With use of the following identity of Dirac's delta function,
\[ \delta''(k) f(k) = f''(0) \delta(k) - 2 f'(0) \delta'(k) + f(0) \delta''(k), \]
we obtain
\[
\langle \Psi_{n,k'_i}^{(L_1)} | x^2 | \Psi_{n,k_i}^{(L_1)} \rangle = - \left\{ \frac{\partial^2}{\partial k_x^2} \int_{-\infty}^{\infty} dy Y_n(y-y_0) Y_{n}(y-y_0) \right\}_{k'_i=k_i} \times \delta(k'_i - k_i) \\
+ \left\{ \frac{\partial}{\partial k_x} \int_{-\infty}^{\infty} dy Y_n(y-y_0) \right\}_{k'_i=k_i} \times \delta''(k'_i - k_i) \\
- \left\{ \int_{-\infty}^{\infty} dy Y_n(y-y_0) \right\}_{k'_i=k_i} \times \delta''(k'_i - k_i) \tag{B34}
\]
It is not so difficult to verify the equalities
\[
\left\{ \int_{-\infty}^{\infty} dy Y_n(y-y_0) Y_n(y-y_0) \right\}_{k'_i=k_i} = 1, \tag{B36}
\]
and
\[
\left\{ \frac{\partial}{\partial k_x} \int_{-\infty}^{\infty} dy Y_n(y-y_0) \right\}_{k'_i=k_i} = 0, \tag{B37}
\]
so that we show below how to evaluate
\[
\left\{ \frac{\partial^2}{\partial k_x^2} \int_{-\infty}^{\infty} dy Y_n(y-y_0) \right\}_{k'_i=k_i} = - i^2 \frac{e}{B} \int_{-\infty}^{\infty} d\xi \psi_n(\xi) \frac{\partial^2}{\partial \xi^2} \psi_n(\xi). \tag{B38}
\]
Using the familiar recursion formulas for the harmonic oscillator wave functions
\[
\frac{\partial}{\partial \xi} \psi_n(\xi) = \sqrt{\frac{n}{2}} \psi_{n-1}(\xi) - \sqrt{\frac{n+1}{2}} \psi_{n+1}(\xi), \tag{B39}
\]
\[
\frac{\partial^2}{\partial \xi^2} \psi_n(\xi) = - \frac{2n+1}{2} \psi_n(\xi) + \sqrt{\frac{n(n-1)}{2}} \psi_{n-2}(\xi) + \sqrt{\frac{(n+1)(n+2)}{2}} \psi_{n+2}(\xi), \tag{B40}
\]
we find that
\[
\int_{-\infty}^{\infty} d\xi \psi_n(\xi) \frac{\partial^2}{\partial \xi^2} \psi_n(\xi) = - \frac{1}{2} (2n + 1), \tag{B41}
\]
which in turn gives
\[
\left\{ \frac{\partial^2}{\partial k_x^2} \int_{-\infty}^{\infty} dy Y_n(y-y_0) \right\}_{k'_i=k_i} = - i^2 \frac{e}{B} (2n + 1). \tag{B42}
\]
Collecting the above formulas, we finally get
\[
\langle \Psi_{n,k'_i}^{(L_1)} | x^2 | \Psi_{n,k_i}^{(L_1)} \rangle = \frac{1}{eB} (n + \frac{1}{2}) \delta(k'_i - k_i) \\
- \delta''(k'_i - k_i). \tag{B43}
\]
which reproduces (A29) in the Appendix A.

**Appendix C Calculation of the matrix elements in the \([n,m]\)-basis**

Let us start with the calculation of the momentum operators. Using (107), we obtain
\[
\langle \psi_{n,m}^{(S)} | p_m \psi_{n,m}^{(S)} \rangle = - i \sqrt{\frac{eB}{2}} A(n | B | n-m) \frac{1}{| a\rangle^A n-m} B = 0. \tag{C45}
\]
On the other hand, we get
\[
\langle \psi_{n,m}^{(S)} | p_m^{mech} \psi_{n,m}^{(S)} \rangle = - i \sqrt{\frac{eB}{2}} A(n | B | n-m) \frac{1}{| a\rangle^A n-m} B = 0. \tag{C45}
\]
The matrix element of the canonical momentum operator can be calculated by using the relation \( p_m^{can} = \frac{1}{2} \left( p_m^{mech} + p_m^{cons} \right) \), which gives
\[
\langle \psi_{n,m}^{(S)} | p_m^{can} | \psi_{n,m}^{(S)} \rangle = - i \sqrt{\frac{eB}{2}} \left( \sqrt{n-m} \delta_{m',m+1} - \sqrt{n-m+1} \delta_{m',m-1} \right). \tag{C46}
\]
The matrix elements of the momentum operators between the eigen-states \( | \psi_{n,m}^{(L_1)} \rangle = U | \psi_{n,m}^{(S)} \rangle \). For the canonical momentum operator, we get
\[
\langle \psi_{n,m}^{(L_1)} | p_m^{can} | \psi_{n,m}^{(L_1)} \rangle = \langle \psi_{n,m}^{(S)} | U^\dagger p_m^{can} U | \psi_{n,m}^{(S)} \rangle = \langle \psi_{n,m}^{(S)} | p_m^{can} + \frac{1}{2} eB | \psi_{n,m}^{(S)} \rangle. \tag{C47}
\]
The second part can be evaluated by using
\[
\frac{1}{2} eB y = i \frac{1}{2} \sqrt{\frac{eB}{2}} (a - b - a^\dagger + b^\dagger). \tag{C48}
\]
which gives

\[ \frac{1}{2} e B \langle \Psi_{n,m}^{(S)} | y | \Psi_{n,m}^{(S)} \rangle = -i \sqrt{\frac{e B}{2}} \left\{ \sqrt{n-m + 1} \delta_{m',m+1} - \sqrt{n-m + 1} \delta_{m,m-1} \right\}. \]  

(C49)

Combining the two terms, we obtain

\[ \langle \Psi_{n,m'}^{(L)} | x_{\text{mech}}^{\text{can}} | \Psi_{n,m}^{(L)} \rangle = -i \sqrt{\frac{e B}{2}} \left\{ \sqrt{n-m + 1} \delta_{m',m+1} - \sqrt{n-m + 1} \delta_{m,m-1} \right\}. \]  

(C50)

Similarly, we can readily show that

\[ \langle \Psi_{n,m}^{(S)} | x_{\text{mech}}^{\text{can}} | \Psi_{n,m}^{(S)} \rangle = m \delta_{m',m}. \]  

(C52)

Next, we evaluate the matrix elements of the three OAM operators. First, for the canonical OAM operator, we obtain

\[ \langle \Psi_{n,m}^{(S)} | L_{z}^{\text{can}} | \Psi_{n,m}^{(S)} \rangle = A^{n} B^{n - m} |a_{n}^{\dagger} a_{n} - b_{n}^{\dagger} b_{n}^{\dagger} U | n^{\dagger} U^{n} | \Psi_{n,m}^{(S)} \rangle = n \delta_{m,m} - (n - m) \delta_{m',m} = m \delta_{m',m}. \]  

(C53)

Since \( L_{z}^{\text{cons}} (A) \) reduces to \( L_{z}^{\text{can}} \) in the symmetric gauge, we naturally get

\[ \langle \Psi_{n,m}^{(S)} | L_{z}^{\text{cons}} (A) | \Psi_{n,m}^{(S)} \rangle = m \delta_{m',m}. \]  

(C54)

Finally, for the matrix element of the mechanical OAM operator, we find that

\[ \langle \Psi_{n,m}^{(S)} | L_{z}^{\text{mech}} (A^{(S)}) | \Psi_{n,m}^{(S)} \rangle = (2n + 1) \delta_{m,m}. \]  

(C55)

and

\[ y^{2} = l_{B}^{2} (\Pi_{y} - \tilde{\Pi}_{y})^{2} = \frac{1}{2} l_{B}^{2} (a - a^{\dagger} + b + b^{\dagger} b). \]  

(C58)

which gives

\[ x^{2} - y^{2} = l_{B}^{2} \left\{ a^{2} + b^{2} + (a^{\dagger})^{2} + (b^{\dagger})^{2} + a b^{\dagger} + b a^{\dagger} + a + b \right\}. \]  

(C59)

After some tedious but straightforward algebra, we find that

\[ \frac{1}{2} e B \langle \Psi_{n,m}^{(S)} | x^{2} - y^{2} | \Psi_{n,m}^{(S)} \rangle = \frac{1}{2} \left\{ \sqrt{(n-m)(n-m+1)} \delta_{m',m+2} + \sqrt{(n-m+1)(n-m+2)} \delta_{m,m-2} \right\}. \]  

(C60)

which in turn gives

\[ \langle \Psi_{n,m}^{(S)} | L_{z}^{\text{cons}} (A^{(S)}) | \Psi_{n,m}^{(S)} \rangle = m \delta_{m',m}. \]  

(C62)

and

\[ \langle \Psi_{n,m}^{(S)} | L_{z}^{\text{mech}} (A^{(S)}) | \Psi_{n,m}^{(S)} \rangle = (2n + 1) \delta_{m',m}. \]  

(C63)

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