The invertibility of $2 \times 2$ operator matrices

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Abstract

In this paper the properties of right invertible row operators, i.e., of $1 \times 2$ surjective operator matrices are studied. This investigation is based on a specific space decomposition. Using this decomposition, we characterize the invertibility of a $2 \times 2$ operator matrix. As an application, the invertibility of Hamiltonian operator matrices is investigated.

Keywords: $2 \times 2$ operator matrix, Hamiltonian operator matrix, invertibility, row operator

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1 Introduction

The invertibility of a linear operator is one of the most basic problems in operator theory, and, obviously, appears in the study of the linear equation $Tx = y$ with a linear operator $T$.

This problem becomes even more involved if one considers the invertibility of $2 \times 2$ operator matrices. For this let $A, B, C$ and $D$ be bounded linear operators on a Hilbert space. If, e.g., they are pairwise commutative, then the operator matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

(1.1)
is invertible if and only if $AD - BC$ is invertible (cf. Problem 70). If only $C$ and $D$ are commutative, and if, in addition, $D$ is invertible, then the operator matrix $M$ is invertible if and only if $AD - BC$ is invertible (cf. Problem 71). In fact, the commutativity is essential in the above characterization, see Problem 71. The situation is even more involved if $A$ and $D$ are not defined on the same space and, hence, the formal expression $AD - BC$ has no meaning.

In general, there is no complete description of the invertibility of operator matrices in the non-commutative case. But if at least one of the entries $A$ or $D$ of
the operator matrix $M$ is invertible, one can describe the invertibility of $M$ in terms of the Schur complement. A similar statement holds also in the case of invertible entries $B$ or $C$. Moreover, the Schur complement method can be effectively used also in the case where the entries of $M$ are unbounded operators under additionally assumptions on the domain of the entries, such as the diagonally (or off-diagonally) dominant or upper (lower) dominant cases, see, e.g., the monograph \[^7\]. We also refer to \[^2,3\] for sufficient conditions for nonnegative Hamiltonian operators to have bounded inverses.

However, it is easy to see that there are many invertible $2 \times 2$ operator matrices with non invertible entries $A, B, C$ and $D$ (see, e.g., Theorem 2.11 below). Obviously, in such cases, the Schur complement method is not applicable.

It is the aim of the present article to give a full characterization for the invertibility of bounded $2 \times 2$ operator matrices. We do this in the following manner: A necessary condition for the invertibility of a $2 \times 2$ operator matrix $M$ in (1.1) is the fact that the row operator $(A \ B)$ is right invertible (that is, the range $\mathcal{R}((A \ B))$ of the operator $(A \ B)$ covers all of the spaces). A further necessary condition is $\mathcal{N}((A \ B)) = \{0\}$, where $\mathcal{N}((A \ B))$ denotes the kernel of $(A \ B)$ (see Corollary 3.3 below). This non-zero kernel $\mathcal{N}((A \ B))$ plays a crucial role. Its projection $P_X(\mathcal{N}((A \ B)))$ onto the first component is a subset of the kernel of $P_{R(B)}A$, where $P_{R(B)}$ denotes the orthogonal projection onto $R(B)^\perp$. Similarly, the projection of $\mathcal{N}((A \ B))$ onto the second component is a subset of $\mathcal{N}(P_{R(A)^\perp B})$.

Therefore we investigate a right invertible row operator $(A \ B)$ and choose a decomposition of the space into six parts which is built out of the subspaces $\mathcal{N}(A), \mathcal{N}(B), \mathcal{N}(P_{R(B)^\perp}A)$ and $\mathcal{N}(P_{R(A)^\perp B})$. As a result, we show that the operator $B_2^{-1}\tilde{A}_2$ considered as an operator from $P_X(\mathcal{N}((A \ B)))$ to $\mathcal{N}(B)^\perp \oplus \mathcal{N}(P_{R(A)^\perp B})^\perp$ is correctly defined. Here $\tilde{A}_2$ ($B_2$) denote the restriction of $A$ ($B$, respectively) to $\mathcal{N}(P_{R(B)^\perp}A)$ ($\mathcal{N}(B)^\perp \oplus \mathcal{N}(P_{R(A)^\perp B})^\perp$, respectively).

The main result of the present article is a full characterization of the invertibility of a $2 \times 2$ matrix operator $M$ in terms of its entries $A, B, C, D$, or to be more precise, in terms of the restrictions $\tilde{A}_2, B_2, C_2$ and $D_2$ which are, in some sense, all related to $\mathcal{N}((A \ B))$: A $2 \times 2$ operator matrix $M$ is invertible if and only if the following two statements are satisfied

(i) The restriction $D|_{\mathcal{N}(B)}$ is left invertible and

(ii) the operator

$$C_2 - D_2B_2^{-1}\tilde{A}_2 : P_X(\mathcal{N}((A \ B))) \to (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp$$

is one-to-one and surjective.

Here $C_2$ ($D_2$) is the restriction of $C$ ($D$, respectively) to $\mathcal{N}(P_{R(B)^\perp}A)$ ($\mathcal{N}(B)^\perp \oplus \mathcal{N}(P_{R(A)^\perp B})^\perp$, respectively) projected onto $(\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp$. This characterization is especially helpful if the spaces $\mathcal{N}((A \ B)), \mathcal{N}(P_{R(B)^\perp}A)$ or $\mathcal{N}(P_{R(A)^\perp B})$ are known explicitly, see, e.g., Theorem 2.11 in Section 2. Moreover, we use it to derive a characterization for isomorphic row operators in Section 3. Finally, in Section 4 we give an application to Hamiltonian operators.
2 Main result

We always assume that $\mathcal{X}$ and $\mathcal{Y}$ are complex separable Hilbert spaces. Let $T$ be a bounded operator between $\mathcal{X}$ and $\mathcal{Y}$. We write $T \in B(\mathcal{X}, \mathcal{Y})$ and, if $\mathcal{X} = \mathcal{Y}$, $T \in B(\mathcal{X})$. The range of $T$ is denoted by $R(T)$, the kernel by $N(T)$. The term isomorphism is reserved for linear bijections $T : \mathcal{X} \to \mathcal{Y}$ that are homeomorphisms, i.e., $T \in B(\mathcal{X}, \mathcal{Y})$ and $T^{-1} \in B(\mathcal{Y}, \mathcal{X})$.

A subspace in $\mathcal{Y}$ is an operator range if it coincides with the range of some bounded operator $T \in B(\mathcal{X}, \mathcal{Y})$. The following lemma is from [2, Theorem 2.4].

**Lemma 2.1** Let $R_1$ and $R_2$ be operator ranges in $\mathcal{Y}$ such that $R_1 + R_2$ is closed.

(i) If $R_1 \cap R_2$ is closed, then $R_1$ and $R_2$ are closed.

(ii) If $R_1$ and $R_2$ are dense in $\mathcal{Y}$, then $R_1 \cap R_2$ is dense in $\mathcal{Y}$.

From [1, Proposition 2.14, Theorem 2.16], we have the following basic facts, which are important in the proofs of our main results.

**Lemma 2.2** Let $\Omega_1$ and $\Omega_2$ be two closed subspaces in $\mathcal{X}$. Then

$$\Omega_1 \cap \Omega_2 = (\Omega_1^\perp + \Omega_2^\perp)^\perp, \quad \Omega_1^\perp \cap \Omega_2^\perp = (\Omega_1 + \Omega_2)^\perp,$$

and we further have the following equivalent descriptions:

(i) $\Omega_1 + \Omega_2$ is closed;

(ii) $\Omega_1^\perp + \Omega_2^\perp$ is closed;

(iii) $\Omega_1 + \Omega_2 = (\Omega_1^\perp \cap \Omega_2^\perp)^\perp$;

(iv) $(\Omega_1 \cap \Omega_2)^\perp = \Omega_1^\perp + \Omega_2^\perp$.

As usual, the symbol $\oplus$ denotes the orthogonal sum of two closed subspaces in a Hilbert space whereas the symbol $\dot{+}$ denotes the direct sum of two (not necessarily closed) subspaces in a Hilbert space. If $\Omega, \Omega_1$ are closed subspaces, $\Omega_1 \subset \Omega$, we denote by $\Omega \ominus \Omega_1$ the uniquely determined closed subspace $\Omega_2$ in $\Omega$ with $\Omega = \Omega_1 \oplus \Omega_2$.

The next lemma is well known, see, e.g., [3, Proposition 1.6.2] or [4, 6].

**Lemma 2.3** Let $A \in B(\mathcal{X}), B \in B(\mathcal{Y}, \mathcal{X}), C \in B(\mathcal{X}, \mathcal{Y}),$ and $D \in B(\mathcal{Y})$. Let $A$ (resp. $B$) be an isomorphism. Then the $2 \times 2$ operator matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in B(\mathcal{X} \oplus \mathcal{Y})$$

is an isomorphism if and only if $D - CA^{-1}B$ (resp. $C - DB^{-1}A$) is an isomorphism.
Recall that an operator \( T \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) \) is called right invertible if there exists an operator \( S \in \mathcal{B}(\mathcal{Y}, \mathcal{X}) \) with \( TS = I_\mathcal{Y} \), where \( I_\mathcal{Y} \) stands for the identity mapping in \( \mathcal{Y} \). Hence, if \( T \) is right invertible then it is surjective. Conversely, if \( T \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) \) then the restriction \( T|_{\mathcal{N}(T)\perp} \) maps \( \mathcal{N}(T)\perp \) onto \( \mathcal{R}(T) \) and, if \( \mathcal{R}(T) = \mathcal{Y} \), then \( T|_{\mathcal{N}(T)\perp} : \mathcal{N}(T)\perp \to \mathcal{Y} \) is an isomorphism. Then with

\[
S := \begin{pmatrix} 0 & (T|_{\mathcal{N}(T)\perp})^{-1} \end{pmatrix} : \mathcal{Y} \to \mathcal{N}(T)\oplus \mathcal{N}(T)\perp
\]

(2.1)

considered as an operator in \( \mathcal{B}(\mathcal{Y}, \mathcal{X}) \) we see that \( T \) is right invertible. This shows the equivalence of (i)-(iii) in the following (well-known) lemma.

**Lemma 2.4** For \( T \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) \) the following assertions are equivalent.

(i) The operator \( T \) is right invertible.

(ii) \( \mathcal{R}(T) = \mathcal{Y} \).

(iii) The operator \( T|_{\mathcal{N}(T)\perp} \) considered as an operator from \( \mathcal{N}(T)\perp \) into \( \mathcal{Y} \) is an isomorphism.

(iv) There exists an isomorphism \( U \in \mathcal{B}(\mathcal{Y}) \) such that \( UT \) is a right invertible operator.

**Proof.** It remains to show the equivalence of (iv) with (i)-(iii). Choose \( U = I_\mathcal{Y} \) and we see that (i) implies (iv). Conversely, let \( U \in \mathcal{B}(\mathcal{Y}) \) be an isomorphism. If \( UT \) is right invertible, then by (ii) \( \mathcal{R}(UT) = \mathcal{Y} \). As \( \mathcal{R}(T) = \mathcal{R}(UT) \), again (ii) shows that \( T \) is right invertible.

\( \square \)

Similarly, \( T \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) \) is called left invertible if there exists an operator \( S \in \mathcal{B}(\mathcal{Y}, \mathcal{X}) \) with \( ST = I_\mathcal{X} \). Hence, if \( T \) is left invertible then it is injective and for a sequence \( (y_n) \) in \( \mathcal{R}(T) \) with \( y_n \to y \) as \( n \to \infty \) we find \( (x_n) \) with \( Tx_n = y_n \) and

\[
x_n = STx_n = Sy_n \to Sy \quad \text{and} \quad y_n = Tx_n \to TSy,
\]

which shows the closedness of \( \mathcal{R}(T) \).

Conversely, if \( \mathcal{N}(T) = \{0\} \) and \( \mathcal{R}(T) \) is closed, then \( T \) considered as an operator from \( \mathcal{X} \) into \( \mathcal{R}(T) \) is an isomorphism and its inverse \( T^{-1} \) acts from \( \mathcal{R}(T) \) into \( \mathcal{X} \). Then with

\[
S := \begin{pmatrix} 0 & T^{-1} \end{pmatrix} : \mathcal{R}(T)\perp \oplus \mathcal{R}(T) \to \mathcal{X},
\]

(2.2)

considered as an operator in \( \mathcal{B}(\mathcal{Y}, \mathcal{X}) \), we see that \( T \) is left invertible. We collect these statements in the following lemma, where the equivalence of (i)-(iii) follows from the above considerations and the equivalence of (i)-(iii) with (iv) is obvious.

**Lemma 2.5** For \( T \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) \) the following assertions are equivalent.
The operator $T$ is left invertible.

(ii) $\mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T)$ is closed.

(iii) The operator $T$ considered as an operator from $X$ into $\mathcal{R}(T)$ is an isomorphism.

(iv) There exists an isomorphism $V \in \mathcal{B}(X)$ such that $TV$ is a left invertible operator.

Remark 2.6 The following observation for $T \in \mathcal{B}(X, Y)$ follows immediately from the Lemmas 2.4 and 2.5. If $T$ is right invertible, then there exists a left invertible operator $S \in \mathcal{B}(Y, X)$ (cf. (2.1)) with $TS = I_Y$ and $\mathcal{R}(S) = N(T)^\perp$. If $T$ is left invertible, then there exists a right invertible operator $S \in \mathcal{B}(Y, X)$ (cf. (2.2)) with $ST = I_X$.

For the orthogonal projection onto a closed subspace $\Omega$ in some Hilbert space we shortly write $P_\Omega$.

Theorem 2.7 Let $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y, X)$ and assume that the row operator $(AB) \in \mathcal{B}(X \oplus Y, X)$ is right invertible. Then $X$ admits the decomposition

$$X = (\mathcal{R}(A)^\perp + \mathcal{R}(B)^\perp) \oplus \mathcal{R}(A) \cap \mathcal{R}(B)$$

(2.3)

and the space $X \oplus Y$ admits the decomposition

$$X \oplus Y = X_1 \oplus X_2 \oplus X_3 \oplus Y_3 \oplus Y_2 \oplus Y_1,$$

(2.4)

where

$$X_1 := \mathcal{N}(A), \quad X_2 := \mathcal{N}(A)^\perp \ominus \mathcal{N}(P_{\mathcal{R}(B)^\perp} A)^\perp, \quad X_3 := \mathcal{N}(P_{\mathcal{R}(B)^\perp} A)^\perp,$$

$$Y_1 := \mathcal{N}(B), \quad Y_2 := \mathcal{N}(B)^\perp \ominus \mathcal{N}(P_{\mathcal{R}(A)^\perp} B)^\perp, \quad Y_3 := \mathcal{N}(P_{\mathcal{R}(A)^\perp} B)^\perp.$$

(2.5)

The row operator $(AB)$ from $X \oplus Y$ into $X$ admits the following representation with respect to the decompositions (2.3) and (2.4)

$$
\begin{pmatrix}
0 & 0 & 0 & B_3 & 0 & 0 \\
0 & 0 & A_3 & 0 & 0 & 0 \\
0 & A_2 & A_0 & B_0 & B_2 & 0
\end{pmatrix},
$$

(2.6)

where

$$A_0 \in \mathcal{B}\left(X_3, \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}\right), \quad A_2 \in \mathcal{B}\left(X_2, \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}\right), \quad A_3 \in \mathcal{B}\left(X_3, \mathcal{R}(B)^\perp\right);$$

$$B_0 \in \mathcal{B}\left(Y_3, \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}\right), \quad B_2 \in \mathcal{B}\left(Y_2, \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}\right), \quad B_3 \in \mathcal{B}\left(Y_3, \mathcal{R}(A)^\perp\right).$$

Then the operators $A_3$ and $B_3$ are isomorphisms and the row operator $(A_2, B_2) : X_2 \oplus Y_2 \to \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}$ is right invertible and

$$\overline{\mathcal{R}(A_2)} = \overline{\mathcal{R}(A) \cap \mathcal{R}(B)} = \overline{\mathcal{R}(B_2)}.$$  

(2.7)
Proof. Step 1. We prove (2.3)–(2.10).

The row operator \((A \ B) : X \oplus Y \rightarrow X\) is right invertible and we have with Lemma 2.4
\[ \mathcal{R}(A) + \mathcal{R}(B) = X. \] (2.8)

We claim
\[ P_{\mathcal{R}(A)\perp}(\mathcal{R}(B)) = \mathcal{R}(A)\perp. \] (2.9)

To see this, it suffices to show the inclusion \(P_{\mathcal{R}(A)\perp}(\mathcal{R}(B)) \subseteq \mathcal{R}(A)\perp\). Let \(x \in \mathcal{R}(A)\perp\). Then there exist \(x_1 \in \mathcal{R}(A)\) and \(x_2 \in \mathcal{R}(B)\) such that \(x = x_1 + x_2\), so \(x = P_{\mathcal{R}(A)\perp} x_2 \in P_{\mathcal{R}(A)\perp}(\mathcal{R}(B))\). This proves the claim. Similarly, we obtain
\[ P_{\mathcal{R}(B)\perp}(\mathcal{R}(A)) = \mathcal{R}(B)\perp. \] (2.10)

Moreover, by 2.8, we have
\[ \{0\} = X\perp = (\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)})\perp = \overline{\mathcal{R}(A)\cap \mathcal{R}(B)}\perp \]
and also the sum \(\overline{\mathcal{R}(A)} + \overline{\mathcal{R}(B)}\) is closed. By Lemma 2.2 (iv) it follows that
\[ \overline{\mathcal{R}(A)\cap \mathcal{R}(B))\perp} = \overline{\mathcal{R}(A)\perp} + \overline{\mathcal{R}(B)\perp}. \]

To sum up, we have the space decomposition (2.3). As \(\mathcal{N}(A) \subseteq \mathcal{N}(P_{\mathcal{R}(B)\perp} A)\), we have \(\mathcal{N}(P_{\mathcal{R}(B)\perp} A)\perp \subseteq \mathcal{N}(A)\perp\). Analogously we see \(\mathcal{N}(P_{\mathcal{R}(A)\perp} B)\perp \subseteq \mathcal{N}(B)\perp\) and, hence, decomposition (2.4) follows.

For \(x \in X_3\perp = \mathcal{N}(P_{\mathcal{R}(B)\perp} A)\) we have
\[ Ax = \left( I - P_{\mathcal{R}(B)\perp} \right) Ax = P_{\overline{\mathcal{R}(B)}} Ax. \]
Hence, \(x \in \mathcal{N}(P_{\mathcal{R}(B)\perp} A)\) if and only if
\[ Ax \in \overline{\mathcal{R}(B)}. \] (2.11)

Similarly, \(y \in \mathcal{N}(P_{\mathcal{R}(A)\perp} B)\) if and only if \(By \in \overline{\mathcal{R}(A)}\). Therefore, if \(x_2 \in X_2\) \((y_2 \in Y_2)\), then it follows that \(x_2 \in \mathcal{N}(P_{\mathcal{R}(B)\perp} A)\) (resp. \(y_2 \in \mathcal{N}(P_{\mathcal{R}(A)\perp} B)\)) and, by (2.11)
\[ Ax_2 \in \overline{\mathcal{R}(B)} \quad \text{(resp. } By_2 \in \overline{\mathcal{R}(A)}\text{)}). \] (2.12)

Then the zero entries in (2.10) follow from the fact that \(Ax = 0\) for \(x \in \mathcal{N}(A)\), \(By = 0\) for \(y \in \mathcal{N}(B)\), \(Ax \in \mathcal{R}(A)\), \(By \in \mathcal{R}(B)\), and (2.12).

Step 2. We show that \((A_2 \ B_3)\) is right invertible.

We have \(\mathcal{N}(A) \subseteq \mathcal{N}(P_{\mathcal{R}(B)\perp} A)\), \(\mathcal{N}(B) \subseteq \mathcal{N}(P_{\mathcal{R}(A)\perp} B)\) and by 2.8 and 2.4 we see that \(A_3\) and \(B_3\) are isomorphisms. Thus, there exists an isomorphism
\[ U \in \mathcal{B}(\overline{\mathcal{R}(A)\perp} \oplus \overline{\mathcal{R}(B)\perp}) \]
\[ U := \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-B_0B_3^{-1} & -A_0A_3^{-1} & 1
\end{pmatrix}. \]
such that
\[
U \begin{pmatrix} 0 & 0 & 0 & B_3 & 0 & 0 \\ 0 & 0 & A_3 & 0 & 0 & 0 \\ 0 & A_2 & A_0 & B_0 & B_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & B_3 & 0 & 0 \\ 0 & 0 & A_3 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & B_2 & 0 \end{pmatrix}.
\]

As \((A \ B)\) is right invertible, Lemma 2.4 shows that \((A_2 \ B_2) : X_2 \oplus Y_2 \to \mathcal{R}(A) \cap \mathcal{R}(B)\) is right invertible.

Step 3. We show (2.7).

By definition, we have \(\mathcal{R}(A_2) \subset \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}\) and \(\mathcal{R}(B_2) \subset \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}\). We will only show \(\overline{\mathcal{R}(A) \cap \mathcal{R}(B)} \subset \overline{\mathcal{R}(A_2)}\). The proof for \(\overline{\mathcal{R}(A) \cap \mathcal{R}(B)} \subset \overline{\mathcal{R}(A_2)}\) is the same and, hence, we omit this proof.

Let \(z \in \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}\). Then there exists a sequence \((z_n)\) in \(\mathcal{R}(B)\) which converges to \(z\). By the block representation (2.6) for \(B\) we find \(z_{1,n} \in \mathcal{R}(A)^\perp\) and \(z_{3,n} \in \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}\) with
\[
z_n = z_{1,n} + z_{3,n}, \quad n \in \mathbb{N},
\]
where we have
\[
z_{1,n} = B_3 y_{3,n} \quad \text{and} \quad z_{3,n} = B_0 y_{3,n} + B_2 y_{2,n} \quad \text{for } n \in \mathbb{N}
\]
for some \(y_{2,n} \in Y_2\) and \(y_{3,n} \in Y_3\). The convergence of \((z_n)\) implies the convergence of \((z_{1,n})\) to some \(z_1 \in \mathcal{R}(A)^\perp\) and of \((z_{3,n})\) to some \(z_3 \in \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}\),
\[
z = z_1 + z_3.
\]
The vectors \(z\) and \(z_3\) belong to \(\overline{\mathcal{R}(A)}\), thus \(z_1 \in \mathcal{R}(A)\) and \(z_1 = 0\) follows. Therefore \((B_3 y_{3,n})\) in (2.14) converges to zero. The fact that \(B_3\) is an isomorphism implies \(y_{3,n} \to 0\) as \(n \to \infty\). We conclude
\[
z = z_3 = \lim_{n \to \infty} z_{3,n} = \lim_{n \to \infty} B_2 y_{2,n}
\]
and \(z \in \overline{\mathcal{R}(B_2)}\) follows. Relation (2.7) is proved. \(\square\)

The following proposition will be used in the proof of the main result.

**Proposition 2.8** Let \(A \in B(\mathcal{X})\) and \(B \in B(\mathcal{Y}, X)\) and let the row operator \((A \ B) \in B(\mathcal{X} \oplus Y, \mathcal{X})\) be right invertible. The following assertions are equivalent.

(i) \(\mathcal{R}(B)\) is closed.

(ii) \(P_X(\mathcal{N}((A \ B)))\) is a closed subspace in \(\mathcal{X}\).

(iii) \(\mathcal{R}(B_2)\) is closed.
Thus, with (2.6), and as isomorphisms, we have

\[ P_\mathcal{X}(\mathcal{N}(A B)) = \{ x \in \mathcal{X} : Ax \in \mathcal{R}(A) \cap \mathcal{R}(B) \} \]

and \( P_\mathcal{X}(\mathcal{N}(A B)) \) is the pre-image of \( \mathcal{R}(B) \) under \( A \), and, hence, it is a closed subspace and (ii) holds.

If \( P_\mathcal{X}(\mathcal{N}(A B)) \) is closed, then also

\[ \Omega := P_\mathcal{X}(\mathcal{N}(A B)) \cap \mathcal{N}(A) = \{ x \in \mathcal{X} : x \in \mathcal{N}(A) \}, \quad Ax \in \mathcal{R}(A) \cap \mathcal{R}(B) \]

is closed. Decompose \( x \in \Omega \) with respect to the decomposition, cf. Theorem 2.7, \( \mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \) as \( x = x_1 + x_2 + x_3 \) with \( x_j \in \mathcal{X}_j \) for \( j = 1, 2, 3 \). Then \( x_1 = 0 \) and for some \( y \in \mathcal{Y} \) we have \( Ax = By \). Decompose \( y \) with respect to \( \mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2 \oplus \mathcal{Y}_3 \) (cf. Theorem 2.7) as \( y = y_1 + y_2 + y_3 \) with \( y_j \in \mathcal{Y}_j \) for \( j = 1, 2, 3 \). Relation (2.6) shows

\[ Ax = A \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ A_3 x_3 \\ A_2 x_2 + A_0 x_3 \end{pmatrix} = \begin{pmatrix} B_3 y_3 \\ 0 \\ B_0 y_3 + B_2 y_2 \end{pmatrix} = B \begin{pmatrix} y_3 \\ y_2 \\ y_1 \end{pmatrix} = By \]

and, as \( A_3 \) is an isomorphism, we obtain \( x_3 = 0 \). Therefore \( \Omega \subset \mathcal{X}_2 \) and we write

\[ \mathcal{X}_2 = \Omega \oplus (\mathcal{X}_2 \oplus \Omega). \]

By Theorem 2.7 \((A_2 \ B_2)\) is right invertible and we obtain with Lemma 2.4

\[ A_2(\mathcal{X}_2 \oplus \Omega) + B_2(\mathcal{Y}_2) = \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}, \quad A_2(\mathcal{X}_2 \oplus \Omega) \cap B_2(\mathcal{Y}_2) = \{0\}. \]

Thus, using Lemma 2.4, we deduce that \( A_2(\mathcal{X}_2 \oplus \Omega) \) and \( \mathcal{R}(B_2) \) are closed.

Assume that (iii) holds. Then, by 2.7, the operator \( B_2 \) is an isomorphism. Let \( z \in \overline{\mathcal{R}(B)} \). Then there exists a sequence \((z_n)\) in \( \mathcal{R}(B) \) which converges to \( z \). By the block representation (2.6) for \( B \) we find \( z_{1,n} \in \mathcal{R}(A) \) and \( z_{3,n} \in \overline{\mathcal{R}(A) \cap \mathcal{R}(B)} \) such that (2.13) and (2.14) hold for some \( y_{2,n} \in \mathcal{Y}_2 \) and \( y_{3,n} \in \mathcal{Y}_3 \). The convergence of \((z_n)\) implies the convergence of \((z_{1,n})\) to some \( z_1 \in \mathcal{R}(A) \) and of \((z_{3,n})\) to some \( z_3 \in \overline{\mathcal{R}(A) \cap \mathcal{R}(B)} \), \( z = z_1 + z_3 \). As the operators \( B_2 \) and \( B_3 \) (cf. Theorem 2.7) are isomorphisms, we have

\[ y_{3,n} \rightarrow B_3^{-1} z_1 \quad y_{2,n} \rightarrow -B_2^{-1} B_0 B_3^{-1} z_1 + B_2^{-1} z_3 \quad \text{as } n \rightarrow \infty. \]

Thus, with (2.6),

\[ B \begin{pmatrix} B_3^{-1} z_1 \\ -B_2^{-1} B_0 B_3^{-1} z_1 + B_2^{-1} z_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ 0 \\ z_3 \end{pmatrix} = z, \]

and \( z \in \mathcal{R}(B) \).
Lemma 2.9 Let $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y, X)$ and assume that the row operator $(A \ B) \in \mathcal{B}(X \oplus Y, X)$ is right invertible. Let $A_2$ and $B_2$ be as in Theorem 2.7. Then $B_2$ considered as an operator from $Y_2$ to $\mathcal{R}(B_2)$ is one-to-one and has an inverse $B_2^{-1} : \mathcal{R}(B_2) \to Y_2$. Define

$$\tilde{A}_2 := (0 \ A_2) : X_1 \oplus X_2 \to \overline{\mathcal{R}(A)} \cap \overline{\mathcal{R}(B)}.$$  

Then $\tilde{A}_2|_{P_X(\mathcal{N}((A \ B)))}$ maps to $\mathcal{R}(B_2)$ and the operator

$$B_2^{-1} \tilde{A}_2|_{P_X(\mathcal{N}((A \ B)))} : P_X(\mathcal{N}((A \ B))) \to Y_2$$

is correctly defined.

If $\mathcal{R}(B)$ is closed, then $B_2$ is an isomorphism and we have

$$X_1 \oplus X_2 = \mathcal{N}(P_{\mathcal{R}(B)^\perp} A) = P_X(\mathcal{N}((A \ B)))$$

and the operator

$$B_2^{-1} \tilde{A}_2 : \mathcal{N}(P_{\mathcal{R}(B)^\perp} A) \to Y_2$$

(2.15)

is correctly defined.

Proof. As $Y_2 \subset \mathcal{N}(B)^\perp$ the operator $B_2$ is one-to-one, hence its inverse $B_2^{-1} : \mathcal{R}(B_2) \to Y_2$ exists. From

$$P_X(\mathcal{N}((A \ B))) = \{x \in X : Ax \in \mathcal{R}(A) \cap \mathcal{R}(B)\} \subset \{x \in X : Ax \in \overline{\mathcal{R}(B)}\} \quad (2.16)$$

we conclude

$$P_X(\mathcal{N}((A \ B))) \subset \mathcal{N}(P_{\mathcal{R}(B)^\perp} A) = X_1 \oplus X_2.$$  

Moreover, we decompose $x \in P_X(\mathcal{N}((A \ B)))$ with respect to the decomposition $X = X_1 \oplus X_2 \oplus X_3$ (cf. Theorem 2.7) as $x = x_1 + x_2 + x_3$ with $x_j \in X_j$ for $j = 1, 2, 3$. Then $x_3 = 0$ and for some $y \in Y$ we have $Ax = By$. Decompose $y$ with respect to $Y = Y_1 \oplus Y_2 \oplus Y_3$ (cf. Theorem 2.7) as $y = y_1 + y_2 + y_3$ with $y_j \in Y_j$ for $j = 1, 2, 3$. Relation (2.6) shows

$$Ax = A \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ A_2 x_2 \end{pmatrix} = \begin{pmatrix} B_3 y_3 \\ 0 \\ B_0 y_3 + B_2 y_2 \end{pmatrix} = B \begin{pmatrix} y_3 \\ y_2 \\ y_1 \end{pmatrix} = By$$

and, as $B_3$ is an isomorphism, we obtain $y_3 = 0$ and $A_2 x_2 = B_2 y_2$. Thus $\tilde{A}_2 x \in \mathcal{R}(B_2)$ for $x \in P_X(\mathcal{N}((A \ B)))$ and $B_2^{-1} \tilde{A}_2|_{P_X(\mathcal{N}((A \ B)))}$ is correctly defined. If $\mathcal{R}(B)$ is closed, then by Proposition 2.3 also $\mathcal{R}(B_2)$ is closed and by (2.7) we see that $B_2$ is an isomorphism. Moreover, from (2.16) we see in this case $X_1 \oplus X_2 = \mathcal{N}(P_{\mathcal{R}(B)^\perp} A) = P_X(\mathcal{N}((A \ B)))$ and (2.15) follows. \qed

The following theorem is the main result. It provides a full characterization of isomorphic $2 \times 2$ operator matrices in terms of their entries.
Theorem 2.10  Let $A \in \mathcal{B}(\mathcal{X})$, $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Assume that the row operator $(A\ B) \in \mathcal{B}(\mathcal{X} \oplus \mathcal{Y}, \mathcal{X})$ is right invertible and, hence, adopt the notions $A_2$, $B_2$, and $X_j$, $Y_j$, $j = 1, 2, 3$, as in Theorem 2.7 and $\tilde{A}_2$ as in Lemma 2.9. Let $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $D \in \mathcal{B}(\mathcal{Y})$. Define the operator matrix $M$ by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$ 

Define the operator $B_2^{-1}\tilde{A}_2|_{P_{\mathcal{X}}(\mathcal{N}((A\ B))))}$ as in Lemma 2.4 and define

$$C_2 := P_{(\mathcal{R}(D|_{\mathcal{N}(B)}))} C|_{X_1 \oplus X_2} : X_1 \oplus X_2 \to (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp$$

and

$$D_2 := P_{(\mathcal{R}(D|_{\mathcal{N}(B)}))} D|_{Y_2} : \mathcal{Y}_2 \to (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp.$$ 

Then $M$ is an isomorphism if and only if the following two statements are satisfied:

(i) The restriction $D|_{\mathcal{N}(B)} : \mathcal{N}(B) \to \mathcal{Y}$ is left invertible.

(ii) The operator

$$\left( C_2 - D_2 B_2^{-1} \tilde{A}_2 \right)|_{P_{\mathcal{X}}(\mathcal{N}((A\ B))))} : P_{\mathcal{X}}(\mathcal{N}((A\ B))) \to (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp$$

is one-to-one and surjective.

Proof. Let $M$ be an isomorphism. Then the row operator $(A\ B) : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ is right invertible, see Lemma 2.4, and the column operator $(\begin{pmatrix} B \\ D \end{pmatrix}) : \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$ is injective. Moreover, if the range of $(\begin{pmatrix} B \\ D \end{pmatrix})$ is not closed then there exists a sequence $(y_n)$ in $\mathcal{Y}$ with $\|y_n\| = 1$, $n \in \mathbb{N}$, and $(\begin{pmatrix} B \\ D \end{pmatrix})y_n \to 0$ as $n \to \infty$. But this implies $A_2 y_n \to 0$, a contradiction as $M$ is assumed to be an isomorphism. Therefore the column operator $(\begin{pmatrix} B \\ D \end{pmatrix})$ is left invertible, cf. Lemma 2.5.

Now let $z \in \mathcal{R}(D|_{\mathcal{N}(B)})$. Then, there exists $z_n \in \mathcal{N}(B)$ such that $Dz_n \to z$ as $n \to \infty$, and we further have

$$\begin{pmatrix} B \\ D \end{pmatrix} z_n = \begin{pmatrix} 0 \\ Dz_n \end{pmatrix} \to \begin{pmatrix} 0 \\ z \end{pmatrix},$$

which together with Lemma 2.4 implies

$$\begin{pmatrix} B \\ D \end{pmatrix} x = \begin{pmatrix} 0 \\ z \end{pmatrix}$$

for some $x \in \mathcal{N}(B)$, and hence $D|_{\mathcal{N}(B)}x = z$. This proves that $\mathcal{R}(D|_{\mathcal{N}(B)})$ is closed, hence, $D|_{\mathcal{N}(B)}$ is left invertible by Lemma 2.5 and (i) is proved.

As $\mathcal{R}(D|_{\mathcal{N}(B)})$ is a closed subspace in $\mathcal{Y}$, we decompose $\mathcal{Y}$,

$$\mathcal{Y} = (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp \oplus \mathcal{R}(D|_{\mathcal{N}(B)}).$$ (2.17)
Similar to the proof of Theorem 2.7, $M$ as an operator from $\mathcal{N}(P_{R(B)} \downarrow A) \oplus X_3 \oplus Y_3 \oplus Y_2 \oplus Y_1$ into

$$(R(A) \downarrow + R(B) \downarrow) \oplus \overline{R(A)} \cap \overline{R(B)} \oplus (R(D|N(B))) \downarrow \oplus R(D|N(B))$$

has the following block representation

$$M = \begin{pmatrix} 0 & 0 & B_3 & 0 & 0 \\ 0 & A_3 & 0 & 0 & 0 \\ \tilde{A}_2 & A_0 & B_0 & B_2 & 0 \\ C_2 & C_3 & D_1 & D_2 & 0 \\ C_4 & C_5 & D_3 & D_4 & D_5 \end{pmatrix}. \quad (2.18)$$

By Theorem 2.7, $A_3$ and $B_3$ are isomorphisms. Additionally, as $M$ is an isomorphism, $D_5$ is also an isomorphism. Then there exist isomorphisms

$$U \in B \left( (R(A) \downarrow + R(B) \downarrow) \oplus \overline{R(A)} \cap \overline{R(B)} \oplus (R(D|N(B))) \downarrow \oplus R(D|N(B)) \right),$$

$$V \in B \left( \mathcal{N}(P_{R(B)} \downarrow A) \oplus X_3 \oplus Y_3 \oplus Y_2 \oplus Y_1 \right)$$

with

$$U := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -B_0 B_3^{-1} & -A_0 A_3^{-1} & 1 & 0 & 0 \\ -D_1 B_5^{-1} & -C_3 A_3^{-1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$V := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -D_5^{-1} C_4 & -D_5^{-1} C_5 & -D_5^{-1} D_3 & -D_5^{-1} D_4 & 1 \end{pmatrix}$$

such that

$$UMV = \begin{pmatrix} 0 & 0 & B_3 & 0 & 0 \\ 0 & A_3 & 0 & 0 & 0 \\ \tilde{A}_2 & 0 & 0 & B_2 & 0 \\ C_2 & 0 & 0 & D_2 & 0 \\ 0 & 0 & 0 & 0 & D_5 \end{pmatrix}. \quad (2.19)$$

Thus, $M$ is an isomorphism if and only if

$$\Delta := \left( \begin{array}{cc} \tilde{A}_2 & B_2 \\ C_2 & D_2 \end{array} \right) : \mathcal{N}(P_{R(B)} \downarrow A) \oplus Y_2 \rightarrow (\overline{R(A)} \cap \overline{R(B)}) \oplus (R(D|N(B))) \downarrow \quad (2.20)$$

is an isomorphism.
Case 1: $\mathcal{R}(B)$ is closed. In this case, from Lemma 2.9, $B_2 : \mathcal{V}_2 \rightarrow \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}$ is an isomorphism and $B_2^{-1} \tilde{A}_2 : \mathcal{N}(P_{\mathcal{R}(B)\perp A}) \rightarrow \mathcal{V}_2$ is correctly defined, see Lemma 2.9. According to Lemma 2.8, $\Delta$ is an isomorphism if and only if

$$C_2 - D_2 B_2^{-1} \tilde{A}_2 : \mathcal{N}(P_{\mathcal{R}(B)\perp A}) \rightarrow (\mathcal{R}(D|_{\mathcal{N}(B)}))^\perp$$

is an isomorphism. By Lemma 2.9, $\mathcal{N}(P_{\mathcal{R}(B)\perp A}) = P_{\mathcal{X}}(\mathcal{N}((A \perp B)))$ and (ii) is satisfied.

Case 2: $\mathcal{R}(B)$ is not closed. By Proposition 2.8, also $\mathcal{R}(B_2)$ is not closed which implies $\dim \mathcal{R}(B_2) = \infty$ and $\dim \mathcal{V}_2 = \infty$. The dimension does not change when we close a subspace, therefore we conclude from (2.7)

$$\dim \overline{\mathcal{R}(A) \cap \mathcal{R}(B)} = \dim \mathcal{R}(B_2) = \infty.$$  (2.21)

By Theorem 2.7, $(A \perp B_2)$ is right invertible, (2.7) and Lemma 2.1 imply

$$\mathcal{R}(A_2) \cap \mathcal{R}(B_2) = \mathcal{R}(A) \cap \mathcal{R}(B).$$

Obviously, $\mathcal{R}(A_2) \cap \mathcal{R}(B_2) \subset \mathcal{R}(A) \cap \mathcal{R}(B)$ and we obtain $\overline{\mathcal{R}(A) \cap \mathcal{R}(B)} \subset \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}$. Thus

$$\mathcal{R}(A) \cap \mathcal{R}(B) = \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}.$$  (2.21)

From this and from $\mathcal{R}(A) \cap \mathcal{R}(B) \subset \mathcal{R}(A) \cap \overline{\mathcal{R}(B)} \subset \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}$ we conclude with (2.21)

$$\infty = \dim \overline{\mathcal{R}(A) \cap \mathcal{R}(B)} = \dim \mathcal{R}(A) \cap \mathcal{R}(B) = \dim \mathcal{R}(A) \cap \overline{\mathcal{R}(B)}.$$  (2.22)

We will use (2.22) to show

$$\dim \mathcal{N}((A \perp B_2)) = \dim \mathcal{N}(P_{\mathcal{R}(B)\perp A}).$$  (2.23)

For this we consider

$$\mathcal{N}((A \perp B)) = \{(\tilde{x}) : x \in \mathcal{N}(A)\} \oplus \left\{(\tilde{y}) : y \in \mathcal{N}(A)^\perp, Ay = -Bz\right\}$$  (2.24)

and

$$\mathcal{N}(P_{\mathcal{R}(B)\perp A}) = \mathcal{N}(A) \oplus \left\{x : x \in \mathcal{N}(A)^\perp, Ax \in \overline{\mathcal{R}(B)}\right\}.$$  (2.22)

As $A$ restricted to $\mathcal{N}(A)^\perp$ is injective, we obtain with (2.22)

$$\dim \left\{(\tilde{y}) : y \in \mathcal{N}(A)^\perp, Ay = -Bz\right\} = \dim \mathcal{R}(A) \cap \mathcal{R}(B) = \dim \overline{\mathcal{R}(A) \cap \mathcal{R}(B)}$$

$$= \dim \left\{x : x \in \mathcal{N}(A)^\perp, Ax \in \overline{\mathcal{R}(B)}\right\}.$$  (2.22)

Therefore

$$\dim \mathcal{N}((A \perp B)) = \dim \mathcal{N}(P_{\mathcal{R}(B)\perp A})$$
and with \( (2.19) \) we obtain \( \dim N((\tilde{A}_2B_2)) = \dim N(P_{R(B)\perp}A) \), hence \( (2.23) \) is proved. Two separable Hilbert spaces of the same dimension are unitarily equivalent, therefore there exists a left invertible operator
\[
\begin{pmatrix} G \\ H \end{pmatrix} : \mathcal{Y}_2 \to N(P_{R(B)\perp}A) \oplus \mathcal{Y}_2 \text{ with range } N((\tilde{A}_2B_2)).
\] (2.25)

Since \( \mathcal{X}_1 \oplus \mathcal{X}_2 = N(P_{R(B)\perp}A) \) and by Theorem \( 2.7 \) and Lemma \( 2.9 \) \( (\tilde{A}_2B_2) : N(P_{R(B)\perp}A) \oplus \mathcal{Y}_2 \to R(A) \cap R(B) \) is a right invertible operator. Then, see Remark \( 2.6 \) there exists a left invertible operator
\[
\begin{pmatrix} E \\ F \end{pmatrix} : R(A) \cap R(B) \to N(P_{R(B)\perp}A) \oplus \mathcal{Y}_2.
\] (2.26)

such that
\[
\tilde{A}_2E + B_2F = I_{R(A) \cap R(B)} \quad \text{with } R\left(\begin{pmatrix} E \\ F \end{pmatrix}\right) = (N((\tilde{A}_2B_2)))^\perp
\] (2.27)

Define
\[
W = \begin{pmatrix} E & G \\ F & H \end{pmatrix} : \overline{R(A)} \cap \overline{R(B)} \to N(P_{R(B)\perp}A) \oplus \mathcal{Y}_2.
\] (2.28)

As \( \begin{pmatrix} G \\ H \end{pmatrix} \) and \( \begin{pmatrix} E \\ F \end{pmatrix} \) are left invertible and from \( (2.25) \) and \( (2.27) \) we obtain easily that \( W \) is an isomorphism. We have
\[
\Delta W = \begin{pmatrix} I_{\overline{R(A)} \cap \overline{R(B)}} & 0 \\ C_2E + D_2F & C_2G + D_2H \end{pmatrix}.
\] (2.29)

As \( M \) is an isomorphism, \( \Delta \) is an isomorphism (see \( (2.20) \)) and the operator \( C_2G + D_2H : \mathcal{Y}_2 \to (R(D|_{\mathcal{N}(B)}))^\perp \) is an isomorphism. Moreover, the operator \( B_2 \) considered as an operator from \( \mathcal{Y}_2 \) to \( R(B_2) \) is one-to-one and has an inverse, see Lemma \( 2.4 \) From \( \tilde{A}_2G + B_2H = 0 \) we conclude \( -B_2^{-1}\tilde{A}_2G = H \) and
\[
C_2G + D_2H = (C_2 - D_2B_2^{-1}\tilde{A}_2)G.
\] (2.30)

Therefore, \( C_2 - D_2B_2^{-1}\tilde{A}_2 : R(G) \to (R(D|_{\mathcal{N}(B)}))^\perp \) is one-to-one with range equal to \( (R(D|_{\mathcal{N}(B)}))^\perp \). From
\[
R\left(\begin{pmatrix} G \\ H \end{pmatrix}\right) = N((\tilde{A}_2B_2)) = \left( N(A) \oplus \begin{pmatrix} y \\ -x \end{pmatrix} : x \in N(A)^\perp, y \in N(B)^\perp, Ax = -By \right) = N((A \mathcal{B}),
\] (2.31)

see \( (2.24) \), it follows that \( R(G) = P_{X}(N((A \mathcal{B})) \) and (ii) is shown.
Now let us assume that (i) and (ii) hold. Then $R(D|_{\mathcal{N}(B)})$ is a closed subspace and $\mathcal{Y}$ admits a decomposition as in (2.17) and we obtain the representation of $M$ as in (2.18), where $A_3, B_3$ and $D_5$ are isomorphisms. Then, taking the same $U$ and $V$ as above, we obtain the relation (2.19). Moreover, if $\Delta$ in (2.20) is an isomorphism, then $M$ is an isomorphism.

If $R(B)$ is closed, then from Lemma 2.9, $B_2 : \mathcal{Y}_2 \to R(A) \cap R(B)$ is an isomorphism and $B_2^{-1} \tilde{A}_2 : N(P_{R(B)} A) \to \mathcal{Y}_2$ is correctly defined. Moreover, Lemma 2.9 $N(P_{R(B)} A) = P_X N((A B))$. Then, by (ii),

$$C_2 - D_2 B_2^{-1} \tilde{A}_2 : N(P_{R(B)} A) \to (R(D|_{\mathcal{N}(B)}))^\perp$$

is an isomorphism and according to Lemma 2.3 $\Delta$ is an isomorphism and, hence, $M$ is an isomorphism.

If $R(B)$ is not closed, then as above, we define the operators $G, H, E, F$, and $W$ as in (2.25), (2.26), (2.27), and (2.28). Moreover, the operator $W$ in (2.28) is an isomorphism and also (2.30) and (2.31) hold. By (2.31) $G = P_X N((A B))$ and as $B_2$ is one-to-one, we see that the operator $G$ in (2.25) is one-to-one. Hence, together with (ii), the operator $(C_2 - D_2 B_2^{-1} \tilde{A}_2)G : \mathcal{Y}_2 \to (R(D|_{\mathcal{N}(B)}))^\perp$ is one-to-one with range equal to $(R(D|_{\mathcal{N}(B)}))^\perp$. Therefore, by (2.30), $C_2 G + D_2 H$ is an isomorphism and, by (2.29) and as $W$ is an isomorphism, also $\Delta$ is an isomorphism. Therefore, see (2.20), $M$ is an isomorphism.

Finally, we consider the following special case.

**Theorem 2.11** Let $A, B, C, D \in \mathcal{B}(X)$ and let $\mathcal{X}', \mathcal{X}''$ be closed subspaces of $\mathcal{X}$ with

$$\mathcal{X} = \mathcal{X}' \oplus \mathcal{X}''$$

such that

$$R(A) = \mathcal{X}', \quad N(A) = \mathcal{X}'', \quad R(B) = \mathcal{X}'', \quad N(B) = \mathcal{X}'$$

Moreover assume that the restriction $D|_{\mathcal{X}'} : \mathcal{X}' \to \mathcal{X}$ is left invertible. Then the $2 \times 2$ operator matrix $M$,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

is an isomorphism if and only if

$$C_2 := P_{(R(D|_{\mathcal{X}'})^\perp)^\perp} C|_{\mathcal{X}''} : \mathcal{X}'' \to (R(D|_{\mathcal{X}'})^\perp)^\perp$$

is an isomorphism.

In particular, if, in addition, $R(B) \neq \{0\}$ and the operator $D|_{\mathcal{X}'} : \mathcal{X}' \to \mathcal{X}$ is an isomorphism, then for every operator $C \in \mathcal{B}(X)$ the $2 \times 2$ operator matrix $M$ is not an isomorphism.
Proof. Denote by \( P_X \) the orthogonal projection in \( X \oplus X \) onto the first component. Then
\[
P_X(\mathcal{N}((A \ B))) = \mathcal{N}(A) = \mathcal{X}''.
\]
Moreover, we have \( \mathcal{N}(P_{R(B)\perp A}) = \mathcal{N}(P_{X'}) = \mathcal{N}(A) = \mathcal{X}' \cap \mathcal{X}'' = \{0\} \). Then the space \( X_2 \) in Theorem 2.7 equals zero and the operators \( A_2 \) and \( \tilde{A}_2 \) in Theorem 2.10 are zero. Then the statements of Theorem 2.11 follow from Theorem 2.10. \(Q.E.D.\)

3 A characterization of isomorphic row operators

In this section let \( A, B, C, D \) and \( M \) be as in Theorem 2.10. In the following we use Theorems 2.7 and 2.10 to characterize the case of an isomorphic row operator \((A \ B)\) and to derive a necessary condition for \( M \) to be an isomorphism.

**Proposition 3.1** Let \( A \in B(X) \) and \( B \in B(Y, X) \). The row operator \((A \ B)\) is an isomorphism (i.e. \((A \ B)\) is left and right invertible) if and only if the following two statements are satisfied:

(i) \( \mathcal{N}(A) = \mathcal{N}(B) = \{0\} \).

(ii) \( \mathcal{R}(A) = \mathcal{R}(B) \cap \mathcal{R}(B) = \mathcal{R}(A) \cap \mathcal{R}(B) \).

**Proof.** If (i) and (ii) hold, then \( Ax + By = 0 \) for some \( x \in X \), \( y \in Y \) implies \( Ax = -By \in \mathcal{R}(B) \). By (ii), \( Ax = 0 \) and, hence, \( By = 0 \) follows. Then (i) implies \( x = y = 0 \) and \( \mathcal{N}((A \ B)) = \{0\} \). Moreover, we have with (ii)
\[
\mathcal{R}((A \ B)) \subset \mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(A) + \mathcal{R}(A) \cap \mathcal{R}(B) = \mathcal{X}
\]
and the row operator \((A \ B)\) is an isomorphism.

For the contrary let the row operator \((A \ B)\) be an isomorphism. If for some \( x \in X \) we have \( Ax = 0 \) then \((A \ B)(x) = 0 \) and, as \( \mathcal{N}(A \ B) = \{0\} \), \( x = 0 \) follows. That is, \( \mathcal{N}(A) = \{0\} \) and, similarly, we see \( \mathcal{N}(B) = \{0\} \). This shows (i). In order to show (ii) let \( x \in \overline{\mathcal{R}(A) \cap \mathcal{R}(B)} \) and assume \( x \neq 0 \). Then there exists sequences \((x_n)\) in \( X \) and \((y_n)\) in \( Y \) such that \((Ax_n)\) and \((By_n)\) converge both to \( x \) with \( \lim \inf_{n \to \infty} ||x_n|| > 0 \) and \( \lim \inf_{n \to \infty} ||y_n|| > 0 \). But then \((A \ B)(\frac{x_n}{y_n}) = Ax_n - By_n\) tends to zero and \( \overline{\mathcal{R}((A \ B))} \) is not closed, a contradiction. This shows
\[
\overline{\mathcal{R}(A) \cap \mathcal{R}(B)} = \{0\} \quad (3.1)
\]
As \( x \in \mathcal{N}(P_{R(B)\perp A}) \) if and only if \( Ax \in \overline{\mathcal{R}(B)} \) (see also (2.11)), we conclude with \( \mathcal{N}(A) = \{0\} \) and (3.1)
\[
\mathcal{N}(P_{R(B)\perp A}) = \{0\}.
\]
In the same way we obtain from (3.1) and $\mathcal{N}(B) = \{0\}$ that $\mathcal{N}(P_{\mathcal{R}(A)}B) = \{0\}$. Then for the spaces $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3$ from Theorem 2.7 we conclude

$$\mathcal{X}_1 = \{0\}, \quad \mathcal{X}_2 = \{0\}, \quad \mathcal{X}_3 = \mathcal{X}, \quad \mathcal{Y}_1 = \{0\}, \quad \mathcal{Y}_2 = \{0\}, \quad \text{and} \quad \mathcal{Y}_3 = \mathcal{Y}$$

and the row operator $(A \ B)$ admits a representation according to Theorem 2.7 with respect to the decompositions $\mathcal{X} \oplus \mathcal{Y}$ and $\mathcal{X} = \mathcal{R}(A) \perp \mathcal{R}(B)$ of the form

$$\begin{pmatrix} 0 & B_3 \\ A_3 & 0 \end{pmatrix},$$

where $A_3 \in \mathcal{B}(\mathcal{X}, \mathcal{R}(B))$ and $B_3 \in \mathcal{B}(\mathcal{Y}, \mathcal{R}(A))$ are isomorphisms. This shows (ii).

**Example 3.2** Let $\mathcal{X} = \mathcal{Y} = \ell^2(\mathbb{N})$ and consider the following operators $A$ and $B$ in $\mathcal{X}$:

$$A(x_n)_{n \in \mathbb{N}} := (x_1, 0, x_2, 0 \ldots) \quad \text{and} \quad B(x_n)_{n \in \mathbb{N}} := (0, x_1, 0, x_2 \ldots).$$

Then the row operator $(A \ B)$ satisfies (i) and (ii) of Proposition 3.1 and, hence, $(A \ B)$ is an isomorphism.

As a consequence, we derive the following condition for $M$ to be an isomorphism.

**Corollary 3.3** Let $A \in \mathcal{B}(\mathcal{X}), \ B \in \mathcal{B}(\mathcal{Y}, \mathcal{X}), \ C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $D \in \mathcal{B}(\mathcal{Y})$. If $\mathcal{Y} \neq \{0\}$ and $\mathcal{N}((A \ B)) = \{0\}$ then the operator matrix $M$

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is not a isomorphism.

**Proof.** If $M$ is an isomorphism, then as noted in the proof of Theorem 2.10, the row operator $(A \ B)$ is right invertible. Assume $\mathcal{N}((A \ B)) = \{0\}$. Then $(A \ B)$ is an isomorphism, and, by Proposition 3.1, $\mathcal{N}(B) = \{0\}$. Hence, we obtain $(\mathcal{R}(D)_{\mathcal{N}(B)})^\perp = \mathcal{Y}$ and (ii) in Theorem 2.10 cannot be true unless $\mathcal{Y} = \{0\}$. Therefore, either $\mathcal{Y} = \{0\}$ or $\mathcal{N}((A \ B)) \neq \{0\}$ holds. □

### 4 Application to Hamiltonian operators

In this section we consider the special case of Hamiltonian operators, i.e., in the situation of Theorem 2.10, $\mathcal{X} = \mathcal{Y}$, the operators $B, C$ are self-adjoint and $D = -A^*$. Under these assumptions, Theorem 2.10 takes the following simple form.
Theorem 4.1 Let $A, B, C \in \mathcal{B}(\mathcal{X})$. Assume that the row operator $(A \ B) \in \mathcal{B}(\mathcal{X} \oplus \mathcal{X}, \mathcal{X})$ is right invertible and that $B$ and $C$ are self-adjoint operators in $\mathcal{X}$, i.e. $B = B^*$ and $C = C^*$. Adopt the notions $A_2$, $B_2$, and $X_j$, $j = 1, 2, 3$, as in Theorem 2.7 and $\tilde{A}_2$ as in Lemma 2.9. Define the operator $B_2^{-1}\tilde{A}_2 \mid_{P_X(\mathcal{N}((A \ B)))}$ as in Lemma 2.9 and define $C_2 := P_{\mathcal{N}(P_{\mathcal{R}(B)} \perp A)}C \mid_{X_1 \oplus X_2} : X_1 \oplus X_2 \to \mathcal{N}(P_{\mathcal{R}(B)} \perp A)$ and $(-A^*)_2 := -P_{\mathcal{N}(P_{\mathcal{R}(B)} \perp A)}A^* \mid_{Y_2} : Y_2 \to \mathcal{N}(P_{\mathcal{R}(B)} \perp A)$. Then the Hamiltonian operator $H = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}$ is an isomorphism if and only if

(i) the operator

$\begin{pmatrix} C_2 - (-A^*)_2 B_2^{-1} \tilde{A}_2 \end{pmatrix} \mid_{P_X(\mathcal{N}((A \ B)))} : P_X(\mathcal{N}((A \ B))) \to \mathcal{N}(P_{\mathcal{R}(B)} \perp A)$

is one-to-one and surjective.

If in this case we have, in addition, that $\mathcal{R}(B)$ is closed, then $C_2 - (-A^*)_2 B_2^{-1} \tilde{A}_2 \in \mathcal{B}(\mathcal{N}(P_{\mathcal{R}(B)} \perp A))$ is an isomorphism.

Proof. By assumption, the row operator $(A \ B)$ is right invertible, hence (see Lemma 2.4) its range is closed and $\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{X}$. The same applies to $(B - A)$ and thus its adjoint,

$$(B - A)^* = \begin{pmatrix} B \\ -A^* \end{pmatrix},$$

has a closed range and is one-to-one. Let $z \in \mathcal{R}(-A^* \mid_{\mathcal{N}(B)})$. Then, there exists $z_n \in \mathcal{N}(B)$ such that $-A^* z_n \to z$ as $n \to \infty$, and we further have

$$(B - A^*) z_n = \begin{pmatrix} 0 \\ -A^* z_n \end{pmatrix} \to \begin{pmatrix} 0 \\ z \end{pmatrix},$$

which together with the closedness of the range of $(B - A)^*$ implies

$$(B - A^*) x = \begin{pmatrix} 0 \\ z \end{pmatrix}$$

for some $x \in \mathcal{N}(B)$, and hence $-A^* \mid_{\mathcal{N}(B)} x = z$. This proves that $\mathcal{R}(-A^* \mid_{\mathcal{N}(B)})$ is closed and (i) in Theorem 2.10 is satisfied for $D = -A^*$. 

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Next, we verify
\[
(\mathcal{R}(-A^*|_{\mathcal{N}(B)}))^\perp = \mathcal{N}(P_{\mathcal{R}(B)^\perp}A). \tag{4.1}
\]
Indeed, if \( x \in (\mathcal{R}(-A^*|_{\mathcal{N}(B)}))^\perp \), we have \((-Ax, y) = (x, -A^*y) = 0\) for every \( y \in \mathcal{N}(B) \), hence \(-Ax \in \mathcal{N}(B)^\perp\), which together with the self-adjointness of \( B \) deduces \( Ax \in \mathcal{R}(B) \), and hence \( x \in \mathcal{N}(P_{\mathcal{R}(B)^\perp}A) \); while if \( x \in \mathcal{N}(P_{\mathcal{R}(B)^\perp}A) \), then \( Ax \in \mathcal{R}(B) \), and hence we have for \( y \in \mathcal{N}(B) \) that \((-Ax, y) = (Ax, y) = 0\), i.e., \( x \in (\mathcal{R}(-A^*|_{\mathcal{N}(B)}))^\perp \).

Now the equivalence of (i) and the fact that \( H \) is an isomorphism follows from (4.1) and Theorem 2.10. The additional statement in the case of a closed range of \( B \) follows from Lemma 2.9.

\[ \square \]

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