Derivation of String Field Theory from the Large $N$ BMN Limit

Robert de Mello Koch†, Aristomenis Donos∗, Antal Jevicki∗ and João P. Rodrigues†

Department of Physics and Center for Theoretical Physics†,

University of the Witwatersrand,

Wits, 2050, South Africa

robert,joao@neo.phys.wits.ac.za

Department of Physics∗,

Brown University,

Providence, RI 02912, USA

donos,antal@het.brown.edu

We continue the development of a systematic procedure for deriving closed string pp wave string field theory from the large $N$ Berenstein-Maldacena-Nastase limit. In the present paper the effects of the Yang-Mills interaction are considered in detail for general BMN states. The SFT interaction with the appropriate operator insertion at the interaction point is demonstrated.
1. Introduction

The Berenstein-Maldacena-Nastase correspondence outlines a precise relationship between large $N$ $\mathcal{N} = 4$ super Yang-Mills theory and closed string theory in the ppwave background\[1\]. This limit simplifies and extends the AdS/CFT correspondence\[2\],\[3\],\[4\] and is a subject of detailed studies. Of particular interest is the derivation of pp wave string field theory from the Yang-Mills/matrix theory.

In the present paper we continue the work begun in \[5\] (here after referred to as (I)) on developing a direct and systematic approach for mapping large $N$ gauge theory to closed string theory. We have applied in I the methods of collective field theory and adapted them to the large $N$ BMN limit. The approach was described in the context of a quantum mechanical model. The essential feature of the collective field approach is the mechanism of joining and splitting (of loops) represented in an effective hamiltonian. Consequently the simplest example of obtaining string type interactions was seen to appear already from a free matrix theory. It was shown in particular that the SUGRA type amplitudes are correctly reproduced through the above mechanism. Likewise the 3-string overlap vertex was seen to naturally arise. In this approach the effect of Yang-Mills interactions is expected to renormalize the zeroth order expressions and we discuss this in the present work.

The matrix model language was used essentially for notational simplicity. The matrix model contains the effects, which in our framework, are identical in the full gauge theory. It may also be that there is an effective matrix model description of the full pp theory and our result have shed light on this possibility. (For a most recent discussion of holography in the pp limit see\[6\].)

Our plan is as follows: In Sect.2 we summarize the basics of the approach reviewing and extending the methods of I. In Sect.3 we give a simple example of string type amplitudes and exhibit the manner in which Yang-Mills type interactions modify the basic couplings. We make use of a coherent state picture and describe a map between this and the physical picture of I. In Sect.4 we then take these interactions into account concentrating on general string-type states. In the continuum BMN limit we derive the emerging 3-string interaction with the appropriate operator prefactor. We reproduce the spectrum of BMN loops in an appendix.
2. Collective field theory

In this section, we give a general overview of collective field theory as applied to the large N limit of Berenstein, Maldacena and Nastase. This section defines the formalism and gives a short summary of the results reached in I. The collective method in general provides a systematic formalism for describing the dynamics of observable, physical degrees of freedom in a theory with large \( N \) symmetry. In gauge or matrix theory the physical observables are given by loops or traces of matrix products (words). The method then provides a direct change of variables to the invariant observables. The resulting effective or collective Hamiltonian describes the full dynamics of these invariants. The essential two terms that define the effective hamiltonian are the interaction terms containing joining and splitting of the loops. In \( \mathcal{N}=4 \) SUSY Yang-Mills gauge theory we follow the degrees of freedom consisting of the Higgs fields: \( \phi_1, \phi_2, \ldots, \phi_5, \phi_6 \). In the proposal of Berenstein, Maldacena and Nastase two of the Higgs matrices \( (\phi_5, \phi_6) \) are chosen to play a special role, they are selected to define the light cone coordinates \( Z = \phi_5 + i\phi_6 \). The above matrix variables define a sector of the full theory that we are interested in. In addition we consider these as functions of time only, so that the dynamics reduces to matrix quantum mechanics. This gives a minimal set of fields that are capable of capturing the essential features of the BMN correspondence.

We therefore concentrate on a complex multi-matrix system with a Hamiltonian that is invariant under

\[
Z_i \rightarrow U^\dagger Z_i U \quad \bar{Z}_i \rightarrow U^\dagger \bar{Z}_i U.
\]

The basic equal time variables are given by single trace operators

\[
\text{Tr} \left( \ldots \prod_{i=1}^{M} Z_i^{n_i} \bar{Z}_i^{\bar{n}_i} \prod_{j=1}^{M} Z_j^{m_j} \bar{Z}_j^{\bar{m}_j} \ldots \right).
\] (2.1)

In the large \( N \) limit, one then has a change of variables from the original Yang-Mills fields to the invariant loop variables denoted collectively by \( \phi_C \), where \( C \) stands for a loop or word index. This index also includes complex conjugate loops \( \bar{\phi}_C \). One then considers a change of variables to this new set in the canonical operator formalism. Concentrating on the kinetic term, there follows
\[ T = - \text{Tr} \left( \sum_{i=1}^{M} \frac{\partial}{\partial Z_i} \frac{\partial}{\partial \bar{Z}_i} \right) = - \sum_{C,C'} \Omega(C,C') \frac{\partial}{\partial \phi_C} \frac{\partial}{\partial \phi_{C'}} + \sum_{C} \omega(C) \frac{\partial}{\partial \phi_C} \]  

with

\[ \Omega(C,C') = \text{Tr} \left( \sum_{i=1}^{M} \frac{\partial \bar{\phi}_C}{\partial Z_i} \frac{\partial \phi_{C'}}{\partial Z_i} \right) = \bar{\Omega}(C',C) \]

and

\[ \omega(C) = - \text{Tr} \left( \sum_{i=1}^{M} \frac{\partial^2 \phi_C}{\partial Z_i \partial \bar{Z}_i} \right). \]

These two operations define the processes of joining and splitting. In particular, \( \Omega(C,C') \) “joins” loops, or words. As an example, if \( \phi_C = \text{Tr}(Z_{i1}) \) and \( \phi_{C'} = \text{Tr}(Z_{i1}') \) then \( \Omega = JJ' \text{Tr}(Z_{i1}^{-1} Z_{i1}') \). In general, one has schematically

\[ \Omega(C,C') = \sum \phi_{C+C'} \]

where \( C + C' \) is obtained by adding the two words \( C \) and \( C' \). Similarly, \( \omega \) “splits” loops. Again,

\[ \omega(C) = \sum \phi_{C'C''} \]  

represents the processes of splitting the word \( C \) into \( C' \) and \( C'' \). The hermiticity of the new collective description is assured through a field transformation and this completes the Hamiltonian\[\text{7}\].

The non-triviality in applying collective field theory to multimatrix models comes from the enormous set of loops (words) that can be generated. For the present problem Berenstein, Maldacena and Nastase have identified a set of observables (traces) which have a mapping into the pp wave string. If we denote the BMN set of loops by \( C \) and their conjugates by \( C' \), the gauge theory process of “joining” (contained in \( \Omega(C,C') \)) generates new loops not in the original set. It the question of extra degrees of freedom that was first addressed and clarified in I. Consider the collective variables that have a direct relation with (lattice) string fields

\[ \Phi_J(l) = \text{Tr} \left( T_l \prod_{i=1}^{n} \phi(l_i) Z^J \right), \quad \phi(l_i) = Z^{l_i} \phi Z^{-l_i}, \]

\[ (2.4) \]
with $T_l$ the $l$ ordering operator - it orders the $\phi(l)$ factors so that $l_i$ increases from left to right. We also have

$$\bar{\Phi}_J(\{l\}) = \text{Tr}\left( \bar{Z}^J \tilde{T}_l \prod_{i=1}^n \bar{\phi}(l_i) \right), \quad \bar{\phi}(l_i) = \bar{Z}^{-l_i} \bar{\phi} \bar{Z}^{l_i}. \quad (2.5)$$

$\tilde{T}_l$ is a second $l$ ordering operator - it orders the $\bar{\phi}(l)$ factors so that $l_i$ decreases from left to right.

The joining and splitting processes applied to these lattice string fields proceed as follows. One first has the open loop defined by

$$P_J(\{l\})_{ij} = \sum_{a=1}^J T_l \left( \prod_{i=1}^n \phi(l_i - a) Z^{J-1} \right)_{ij} = \frac{\partial \Phi_J(\{l\})}{\partial Z_{ji}}$$

where we have $l_i - a \mod J$ so that $0 \leq l_i - a \leq J - 1$. Similarly

$$Q_J(\{l\})_{ij} = \sum_{j=1}^J T_l \left( \prod_{i=1, i \neq j}^n \phi(l_i - l_j) Z^J \right)_{ij} = \frac{\partial \Phi_J(\{l\})}{\partial \phi_{ji}}$$

In terms of these split (open) loops we generate through the joining operation the composite, joined loop. In particular these are contained in the interaction of the collective hamiltonian through the composite trace $\Omega$

$$\frac{\partial \Phi_{J_1}(\{l\})}{\partial Z_{ij}} \frac{\partial \Phi_{J_2}(\{\bar{l}\})}{\partial \phi_{ji}} + \frac{\partial \Phi_{J_1}(\{l\})}{\partial \phi_{ij}} \frac{\partial \Phi_{J_2}(\{\bar{l}\})}{\partial \bar{\phi}_{ji}} = \text{Tr}(P_{J_1}(\{l\}) \bar{P}_{J_2}(\{\bar{l}\})) + \text{Tr}(Q_{J_1}(\{l\}) \bar{Q}_{J_2}(\{\bar{l}\}))$$

$$= \sum_{a=1}^{J_1} \sum_{b=1}^{J_2} \text{Tr} \left[ T_l \left( \prod_{i=1}^{n_1} \phi(l_i - a) Z^{J_1-1} \bar{Z}^{J_2-1} \tilde{T}_l \left( \prod_{j=1}^{n_2} \bar{\phi}(\bar{l}_j - b) \right) \right) \right]$$

$$+ \sum_{k=1}^{n_1} \sum_{m=1}^{n_2} \text{Tr} \left[ T_l \left( \prod_{i=1, i \neq k}^{n_1} \phi(l_i - l_k) Z^{J_1} \bar{Z}^{J_2} \tilde{T}_l \left( \prod_{j=1, j \neq m}^{n_2} \bar{\phi}(\bar{l}_j - \bar{l}_m) \right) \right) \right]$$

This composite trace involves mixed combinations of $Z$ and $\bar{Z}$ which do not survive as excitations in the BMN correspondence. In I we have introduced a mechanism of factor-ization based on which we replace such new, unwanted loops by a sum of physical, BMN approved loops. Concretely,

$$\text{Tr} \left( P_{J_1}(\{l\}) \bar{P}_{J_2}(\{\bar{l}\}) \right) = \sum_{J_1, J_2} \sum_{\{m\}} C_{J_1 J_2}^{\{m\}} \bar{\Phi}_J(\{m\})$$
The coefficients (structure constants) in this relation can be generated by Schwinger-Dyson
equations (as we have discussed in (I)). A more complete procedure for finding these
coefficients is through consistency conditions obtained by taking expectation values of
both sides of the equation. In this way they can be read of from the correlator

$$\langle \Phi \ J \ (\{m\}) \ Tr \ (P_{J_1}(\{l\}) \ P_{J_2}(\{\bar{l}\})) \rangle$$

to equal

$$C_{J_1 J_2}^{\{m\}\{l\}\{\bar{l}\}} = J_1 \sum_{\kappa=1}^{n} \sum_{l=1}^{n_1} \sum_{q=1}^{n_2} \prod_{i=1}^{n_1-1} \prod_{j=1}^{n_1-1} \delta(m_{\kappa+i \mod n} - m_{\kappa \mod J_1}, \bar{l}_{q+i \mod n_2} - \bar{l}_{q \mod J_2}) \times \delta(l_{l+j \mod n_1} - l_{l \mod J_1} + \bar{l}_{q+n \mod n_2} - \bar{l}_{q \mod J_2}, \bar{l}_{q+n+j \mod n_2} - \bar{l}_{q \mod J_2}) \times \min\{m_{\kappa} - m_{\kappa \mod n}, l_l - l_l \mod n_1\}.$$ 

Similarly

$$Tr \ (Q_{J_1}(\{l\}) \bar{Q}_{J_2}(\{\bar{l}\})) = \sum_{J, \{m\}} G_{J_1 J_2}^{\{m\}\{l\}\{\bar{l}\}} \bar{\Phi}_J (\{m\})$$

with

$$G_{J_1 J_2}^{\{m\}\{l\}\{\bar{l}\}} = n_1 \sum_{\kappa=1}^{n} \sum_{l=1}^{n_1} \sum_{q=1}^{n_2} \prod_{i=1}^{n_1-1} \prod_{j=1}^{n_1-1} \delta(m_{\kappa+i \mod n} - m_{\kappa \mod J_1}, \bar{l}_{q+i \mod n_2} - \bar{l}_{q \mod J_2}) \times \delta(l_{l+j \mod n_1} - l_{l \mod J_1} + \bar{l}_{q+n \mod n_2} - \bar{l}_{q \mod J_2}, \bar{l}_{q+n+j \mod n_2} - \bar{l}_{q \mod J_2}) \times \min\{m_{\kappa} - m_{\kappa \mod n}, l_l - l_l \mod n_1\},$$

is again most simply deduced from the correlator

$$\langle \Phi \ J \ (\{m\}) \ Tr \ (Q_{J_1}(\{l\}) Q_{J_2}(\{\bar{l}\})) \rangle$$

The above sums appearing in $C$ and $G$ are equal and represent a form of a (lattice) three-
string vertex $|V_3^{0}\rangle$. In order to demonstrate that, let us consider the string states

$$|\psi_1\rangle = \sum_{p=0}^{J_1-1} \prod_{i=1}^{n_1} b_{p+l_i \mod J_1}^{(1)} |0\rangle_1$$
\[ |\psi_2\rangle = \sum_{q=0}^{J-1} \prod_{j=1}^{n} b_{q+m_j \text{ mod } J}^{(2)\dagger} |0\rangle_2 \]
\[ |\psi_3\rangle = \sum_{r=0}^{J_2-1} \prod_{k=1}^{n_1} b_{r+l_k \text{ mod } J_1}^{(3)\dagger} |0\rangle_3 \]

these are in direct correspondence with our matrix theory fields (states) \( \Phi_J (\{m\}) \), \( \Phi_{J_1} (\{l\}) \) and \( \Phi_{J_2} (\{\bar{l}\}) \).

The sums appearing in the matrix theory result correspond to reparametrizations that occur in the calculation of \( \langle \psi_1 | \langle \psi_2 | \langle \psi_3 | V_{30}^0 \rangle \). The sum in \( C \) would appear in the above calculation if we fix one of the delta functions between string (2) and (3) and one of the delta functions between (1) and (3). This allowed us to write the part of our collective Hamiltonian coming from \( H_0 \) as

\[
H_{0}^{\text{col}} = \sum_{J,\{l\}} (J + n) \Phi_J (\{l\}) \frac{\partial}{\partial \Phi_J (\{l\})} + \\
\sum_{J_1, J_2, J_3} \sum_{\{l^{(1)}\}, \{l^{(2)}\}, \{l^{(3)}\}} (\Delta J + \Delta n) \langle \psi_1 | \langle \psi_2 | \langle \psi_3 | V_{30}^0 \rangle \Phi_{J_1} (\{l^{(1)}\}) \Phi_{J_2} (\{l^{(2)}\}) \Phi_{J_3} (\{l^{(3)}\}) \frac{\partial}{\partial \Phi_{J_1} (\{l^{(1)}\})} \frac{\partial}{\partial \Phi_{J_2} (\{l^{(2)}\})} \frac{\partial}{\partial \Phi_{J_3} (\{l^{(3)}\})} \\
+ \sum_{J_1, J_2, J_3} \sum_{\{l^{(1)}\}, \{l^{(2)}\}, \{l^{(3)}\}} (\Delta J + \Delta n) \langle \psi_1 | \langle \psi_2 | \langle \psi_3 | V_{30}^0 \rangle \Phi_{J_1} (\{l^{(1)}\}) \Phi_{J_2} (\{l^{(2)}\}) \Phi_{J_3} (\{l^{(3)}\}) \frac{\partial}{\partial \Phi_{J_1} (\{l^{(1)}\})} \frac{\partial}{\partial \Phi_{J_2} (\{l^{(2)}\})} \frac{\partial}{\partial \Phi_{J_3} (\{l^{(3)}\})} 
\]

where

\[
\Delta J + \Delta n = J_1 + n_1 + J_2 + n_2 - J_3 - n_3
\]

and \( |\psi_i\rangle \) are the lattice string states associated with \( \Phi_J (\{l\}) \). The characteristic feature of this interaction is that it appears proportional to \( 1/N \) and exhibits the prefactor \( (E_3^0 - E_1^0 - E_2^0) \). The occurrence of the energy prefactor was an early conjecture of \( S \). It was explained in I in a specific example as resulting from a projection to light cone fields. It is expected that this form will be modified once Yang-Mills interactions are taken into account.

We note that this interaction can be transformed away through a (nonlinear) field redefinition. The Hamiltonian \( H_2 + H_3 \) can be reduced (in leading order of \( 1/N \)) to \( H_2 \). Another way to understand this fact is to realize that in the creation-annihilation (coherent state) basis the free oscillator hamiltonian is first order in the derivatives and its collective representation in this basis is still a quadratic Hamiltonian

\[
H_2 = \sum E_i A_i^{\dagger}(\{l_i\}) A(\{l_i\}).
\]
The nonlinear transformation

\[ A^\dagger(l_i) = \Phi^\dagger(l_i) + \frac{1}{4N} C(l_i, l_j l_k) \Phi^\dagger(l_j) \Phi^\dagger(l_k) \]
\[ + \frac{1}{2N} C(l_j, l_i l_k) \Phi^\dagger(l_k) \Phi^\dagger(l_j) \]

from coherent state fields \( A(l_i) \) to the physical collective fields relates the two representations. The coherent state picture will in the next section provide the simplest framework for incorporating the effect of Yang-Mills interactions.

3. Examples

We will start by considering the \( g_{YM}^2 \) effects of the dimensionally reduced Yang-Mills system and discuss simple, illustrative examples of corrections that this term gives. Consider the Hamiltonian

\[ H = \sum_{i=1}^{6} \text{Tr} \left( -\frac{\partial^2}{\partial \phi_i^2} + \phi_i^2 \right) - g_{YM}^2 \sum_{i<j} \text{Tr} \left( [\phi_i, \phi_j]^2 \right), \quad i, j = 1, ..., 6. \]

With

\[ \phi = \phi_1 + i\phi_2, \quad \psi = \phi_3 + i\phi_4, \quad Z = \phi_5 + i\phi_6, \]

the interaction term in the above Hamiltonian is equivalently written as

\[ H_1 = g_{YM}^2 \left( \frac{1}{4} [\phi, \bar{\phi}] [\phi, \bar{\phi}] + \frac{1}{4} [\psi, \bar{\psi}] [\psi, \bar{\psi}] + \frac{1}{4} [Z, \bar{Z}] [Z, \bar{Z}] \right. \]
\[ + \frac{1}{2} [\phi, \bar{\phi}] [\psi, \bar{\psi}] + \frac{1}{2} [\phi, \bar{\phi}] [Z, \bar{Z}] + \frac{1}{2} [\psi, \bar{\psi}] [Z, \bar{Z}] \]
\[ \left. - [\bar{\phi}, \bar{Z}] [\phi, Z] - [\bar{\psi}, \bar{Z}] [\psi, Z] - [\phi, \bar{\psi}] [\phi, \psi] \right), \quad (3.1) \]

showing the usual split into \( D \) and \( F \) terms respectively. Having in mind the passage to the infinite momentum frame, we work in a coherent state basis and project

\[ Z \to A^\dagger + B \to A^\dagger, \quad Z = A + B^\dagger \to \frac{\partial}{\partial A^\dagger}. \quad (3.2) \]

We don’t consider \( B^\dagger \) quanta. These quanta correspond, in the pp-wave string field theory, to modes with \( p^+ < 0 \). In the infinite momentum frame these modes decouple.

For the complex impurity fields \( \psi \) and \( \phi \) we project
\[ \bar{\phi} \rightarrow b^\dagger + d \rightarrow b^\dagger, \quad \phi \rightarrow b + d^\dagger \rightarrow \frac{\partial}{\partial b^\dagger}, \]
\[ \bar{\psi} \rightarrow c^\dagger + e \rightarrow c^\dagger, \quad \psi \rightarrow c + e^\dagger \rightarrow \frac{\partial}{\partial c^\dagger}. \]

This last projection ensures that our Hamiltonian keeps us within the subspace of loops that are near to chiral primary operators. This truncation could be justified by appealing to supersymmetry. Here we follow a more pedestrian approach and justify the truncation by showing that the resulting Hamiltonian reproduces the impurity number conserving amplitudes of the light cone string field theory vertex. We have also checked that the correct string mass spectrum (i.e. anomalous dimensions) is obtained to order \(1/N^2\).

An immediate consequence of the above reduction of degrees of freedom is that the \(D\) terms in (3.1) are trivial as a result of the commutator

\[ \left[ \left( \frac{\partial}{\partial A^\dagger} \right)_{ij}, A^\dagger_{kl} \right] = \left[ \frac{\partial}{\partial A^\dagger_{ji}}, A^\dagger_{kl} \right] = \delta_{jk}\delta_{il}. \]

We therefore arrive at the Hamiltonian

\[ \hat{H} = -g^2_{YM} \left( [b^\dagger, A^\dagger] \left[ \frac{\partial}{\partial b^\dagger}, \frac{\partial}{\partial A^\dagger} \right] + [c^\dagger, A^\dagger] \left[ \frac{\partial}{\partial c^\dagger}, \frac{\partial}{\partial A^\dagger} \right] + [b^\dagger, c^\dagger] \left[ \frac{\partial}{\partial b^\dagger}, \frac{\partial}{\partial c^\dagger} \right] \right) \]

which can be recognized as the (dimensionally reduced) operator \(\hat{\Delta} - \hat{J}\) in a coherent state basis. In obtaining this expression, we have subtracted the free terms in the Hamiltonian, which in the large \(J\) limit simply contribute an additive term equal to \(\hat{J}\).

In [I], we have already demonstrated agreement for a class of sugra states given by the loop variables

\[ O^J_{n,m} = \sum \text{Tr}(\phi^n \psi^m Z^J). \]

The sum is over all possible permutations of the \(\phi\) and \(\psi\) fields, that is, the above loop is a chiral primary operator. Based on the free part of the above Hamiltonian the following interacting cubic Hamiltonian was shown to arise

\[ H = 2\mu \delta_{J_1, J_2 + J_3} \delta_{n_1, n_2 + n_3} \delta_{m_1, m_2 + m_3} \sqrt{J_1 J_2 J_3} \sqrt{n_1!} \sqrt{n_2! n_3!} \sqrt{m_1!} \sqrt{m_2! m_3!} \]
\[ \times \left( \frac{J_2}{J_1} \right)^{n_2 + m_2} \left( \frac{J_3}{J_1} \right)^{n_3 + m_3} \prod'_{n_1, m_1} \prod'_{n_2, m_2} O^J_{n_3, m_3}. \]
The sugra amplitudes are special in that they do not seem to receive corrections from the Yang-Mills interaction. This is clear for their energies (i.e. anomalous dimensions) where non-renormalization theorems have been obtained. For the 3-point couplings one can also demonstrate an absence of corrections (in the leading order). Here we consider the next set of operators of interest, which contain stringy excitations and consequently do receive corrections from the Yang-Mills interaction. Let us concentrate on the subset of the full loop space consisting of the gauge theory operators

\[ \tilde{O}^J = \sqrt{JN^3}O^J = \text{Tr}(A^J), \]

\[ \tilde{O}^J_n = \sqrt{JN^3}O^J_n = \sum_{l=0}^{J} q^l \text{Tr}(b^l(A^J)^l c^l(A^J)^{-1}) = \sum_{l=0}^{J} q^l O_l. \]

where \( q = e^{\frac{2\pi i n}{N+1}} \). Begin by considering the action of \( \hat{H} \) on the two impurities state

\[ \hat{H}\tilde{O}^J_n = -g_{YM}^2 \left( 2N \left[ \sum_{l=1}^{J} q^l (O^J_{l-1} - O^J_l) + \sum_{l=0}^{J-1} q^l (O^J_{l+1} - O^J_l) \right] \right. \]

\[ + \sum_{l=2}^{J} q^l \sum_{l'=1}^{l-1} \tilde{O}^l (O^J_{l-l'-1} - O^J_{l'-1}) + \sum_{l=0}^{J-2} q^l \sum_{l'=0}^{J-1} \tilde{O}^l (O^J_{1+l'-1} - O^J_{l+l'+1}) \]

\[ + \sum_{l=2}^{J} q^l \sum_{l'=0}^{l-2} \tilde{O}^l (O^J_{l-l'+1} - O^J_{l'-1}) + \sum_{l=0}^{J-2} q^l \sum_{l'=1}^{J-1} \tilde{O}^l (O^J_{l+l'-1} - O^J_{l'-1}) \]

\[ + \sum_{l=0}^{J} q^l \tilde{O}^l (O^J_0 - O^J_l) + \sum_{l=0}^{J} q^l \tilde{O}^l (O^J_{J-l} - O^J_{J-1}). \]

By changing the order of the double summations, expressing \( O_l = \sum_n e^{-\frac{2\pi i n l}{J+1}} \tilde{O}^J_n/(J+1) \), performing the intermediate sums, taking into account the normalization of the states and taking the large \( J \) limit, we obtain (\( y = \frac{J+l}{J+1} \to \frac{J}{J+1} \))

\[ \hat{H}|O^J_n\rangle = \lambda' 8\pi^2 n^2 |O^J_n\rangle - g_2 \lambda' \sum_{J_1+J_2=J} \sum_{J_1,J_2,J_3,J_4} \frac{1}{\sqrt{J}} \sqrt{\frac{1-y}{y}} \left( \frac{8m}{ny-m} \right) \sin^2(\pi ny) |O^J_{m}O^J_{n}\rangle \]

(3.8)

\[ ^1 \text{In the large } J \text{ limit, the difference between } J \text{ and } J+1 \text{ is inconsequential. We use } J+1 \text{ to simplify the transform between } O_l \text{ and } \tilde{O}^J_n. \]
where $\lambda'$ and $g_2$ have their usual meanings

$$\lambda' = \frac{g_Y^2 N}{J^2}, \quad g_2 = \frac{J^2}{N}.$$  

As expected, apart from the diagonal term which gives the first string tension correction to the anomalous dimension, $\hat{H}$ provides a splitting of the impurity loop into two loops.

Consider next the action of $\hat{H}$ on two loops $O_{1m}^J O_{2n}^J$. Apart from the diagonal term and the splitting into a 3 trace state, this will exhibit the effect of joining loops $O_{1m}^J$ and $O_{2n}^J$ into a single BMN state $O_{m+n}^{J_1+J_2}$. This process is obtained when a derivative acts on $O_{1m}^J$ and the other on $O_{2n}^J$.

Explicitly ($J = J_1 + J_2$)

$$\hat{H}(\tilde{O}_{1}^{J_1} \tilde{O}_{2}^{J_2}) = -g_Y^2 \left[ 2N \left( \sum_{l=0}^{J_1-1} q_m^J (O_{l+1}^{J_1} - O_{l}^{J_1}) + \sum_{l=1}^{J_2} q_m^J (O_{l-1}^{J_2} - O_{l}^{J_2}) \right) \right] + 3 \text{ trace states}.$$  

Again by re-expressing $O_{1}^{J_1}$ in terms of $\tilde{O}_{n}^{J_1}$, performing the intermediate sums, taking into account the normalization of the states and taking the large $J$ limit we obtain

$$\hat{H}|O_{1m}^J O_{2n}^J\rangle = \lambda' \left( \frac{8\pi^2 m^2}{y^2} \right) |O_{1m}^J O_{2n}^J\rangle + 3 \text{ trace states} \quad (3.9)$$  

$$- \lambda' g_2 \sqrt{\frac{1-y}{y}} \left( \frac{8ny}{m-ny} \right) \sin^2(\pi ny) |O_{n}^{J_2}\rangle.$$  

There is another 2 impurity 2 trace state comprising of two loops of one impurity each

$$\tilde{O}_{\phi} = \sqrt{N^{J_1+1}} O_{\phi}^J = \text{Tr}(b^\dagger A^{iJ}), \quad \tilde{O}_{\psi} = \sqrt{N^{J_2+1}} O_{\psi}^J = \text{Tr}(c^\dagger A^{iJ}).$$

Inspection of (3.8) shows that our Hamiltonian apparently does not split the single 2 impurity trace into 2 single impurity traces. However, it can join the two single impurity loops

$$\hat{H}(\tilde{O}_{\phi}^J \tilde{O}_{\psi}^J) = -g_Y^2 \left( \sum_{l=0}^{J_2-1} \left[ (O_{J_1}^{l-J_1} - O_{J_2-l}^{J_2-J_1}) + (O_{J_2-l}^{J_2} - O_{J_2-l+J_2}^{J_2}) \right] \right)$$

$$+ \sum_{l=0}^{J_2-1} \left[ (O_{l}^{J_1} - O_{l+J_2}^{J_1}) + (O_{l+J_2}^{J_1} - O_{l+J_2+l}^{J_2}) \right] \right).$$
Carrying out steps similar to those leading to equations (3.8) and (3.9), we obtain

\[ \hat{H}|O_\phi^{J_1} O_\psi^{J_2}\rangle = -g_2\lambda' \sum_{n=-J/2}^{J/2} \frac{8}{\sqrt{J}} \sin^2(\pi ny)|O_n^{J_1}\rangle + \ldots \]  

(3.10)

Combining the results (3.8), (3.9) and (3.10) we obtain the following Hamiltonian acting on loop space

\[ \hat{H} = \lambda'(8\pi^2 n^2)O_n^J \frac{\partial}{\partial O_n^J} + \sum_{n,m,y} \lambda' g_2 D_{n,m,y} O_m^J O_n^J \frac{\partial}{\partial O_n^J} \] 

\[ + \sum_n \lambda' g_2 D_{m,n,y} O_n^J \frac{\partial}{\partial O_m^J} \frac{\partial}{\partial O_n^J} \] 

\[ + \lambda' g_2 D_{y,n} O_n^J \frac{\partial}{\partial O_\phi} \frac{\partial}{\partial O_\psi}. \]  

(3.11)

The coefficients can be read by inspection from equations (3.8), (3.9) and (3.10). In this coherent representation, we in general reach a Hamiltonian of the form

\[ H_{coh} = \sum_i E_i A_i^\dagger A_i + \sum_{ijl} D_{i,jl} A_j^\dagger A_l^\dagger A_i + \sum_{ijl} F_{jli} A_i^\dagger A_j A_l \]  

which is not hermitean. The transformation from coherent to physical fields given in Section 2 reads

\[ A_i^\dagger = \psi_i^\dagger + \frac{1}{4N} \bar{C}_{i,pq} \psi_p^\dagger \psi_q^\dagger + \frac{1}{2N} C_{p,iq} \psi_q \psi_p^\dagger. \]  

(3.13)

It leads to an interacting Hamiltonian of the form

\[ H = \sum_i E_i \psi_i^\dagger \psi_i + \frac{1}{N} \sum_{ijl} \left((E_i - E_j - E_l) \cdot \frac{1}{4} C_{i,jl} + D_{i,jl}\right) \psi_j^\dagger \psi_l^\dagger \psi_i + \frac{1}{N} \sum_{ijl} \left(-(E_i - E_j - E_l) \cdot \frac{1}{4} C_{i,jl} + F_{jli}\right) \psi_i^\dagger \psi_j \psi_l. \]  

(3.14)

For the example of three states denoted as (in what follows \( y \equiv \frac{J}{J} \))

\[ O_{J_1} \rightarrow \psi_0^y, \] 

\[ O_{n} \rightarrow \psi_n^y, \] 

\[ O_\phi \rightarrow \psi_1^y, \] 

\[ O_{J_1} \rightarrow \psi_2^y, \] 

\[ O_\psi \rightarrow \psi_3^y. \]
our transformation (3.13) makes use of the following results:

\[
\langle O^J_m O^J_n \rangle = g_2 C_{m,n} = g_2 C_{n,m} = \langle O^J_n O^J_m \rangle = g_2 \frac{y^{3/2} \sqrt{1 - y} \sin^2(\pi ny)}{\sqrt{J \pi^2} (m - ny)^2},
\]

\[
\langle O^J_{\phi \psi} O^J_{\psi} \rangle = g_2 C_{\psi \phi} = g_2 C_{\phi \psi} = \langle O^J_{\psi} O^J_{\phi \psi} \rangle = -g_2 \frac{\sin^2(\pi ny)}{\pi^2 \sqrt{J n^2}},
\]

\[
\bar{C}_{m,py} = C_{py,m} \quad \bar{C}_{m,y} = C_{y,m}.
\]

Some care must be exercised when computing these correlators, since the use of coherent states implies that one has a non-trivial inner product. These overlaps can equivalently be computed as correlators in the free matrix model in accord with the state operator map. As discussed in section 2, our transformation makes use of these correlators. The cubic contribution from (3.14) becomes:

\[
H_3 = \left[ -\frac{1}{2} C_{n,qz} \left( 8\pi^2 n^2 - 8\pi^2 \frac{q^2}{z^2} - 0 \right) + D_{qz,n} \right] \psi_1 \psi_2 \psi_0^{1-z}
\]

\[
+ \left[ -\frac{1}{2} C_{n,z} \left( 8\pi^2 n^2 - 0 - 0 \right) + D_{z,n} \right] \psi_1 \psi_2 \psi_3^{1-z}
\]

\[
+ \left[ \frac{1}{2} C_{m,qy} \left( 0 + 8\pi^2 m^2 - 8\pi^2 \frac{q^2}{y^2} \right) + D_{m,qy} \right] \psi_0^{1-y} \psi_1^{y \psi_1 \psi_1^{m1}}
\]

\[
+ \left[ \frac{1}{2} C_{m,y} \left( 8\pi m^2 - 0 - 0 \right) \right] \psi_2 \psi_3 (1 - y) \psi_1^{m1}
\]

We then evaluate the coefficient in the respective couplings:

\[
-\frac{1}{2} C_{n,qz} \left( 8\pi^2 n^2 - 8\pi^2 \frac{q^2}{z^2} \right) + D_{qz,n} = -4\pi^2 \left( n^2 - \frac{q^2}{z^2} \right) \frac{z^{3/2} \sqrt{1 - z} \sin^2(\pi nz)}{\sqrt{J \pi^2} (q - nz)^2}
\]

\[
- \frac{1}{\sqrt{J}} \sqrt{\frac{1 - z}{z}} \frac{8nz}{(q - nz)} \sin^2(\pi nz)
\]

\[
= 8\pi^2 \left( \sqrt{\frac{1 - z}{z}} \frac{1}{2 \sqrt{J \pi^2}} \sin^2(\pi nz) \right)
\]
Collecting these couplings, we obtain

\[ H_3 = \lambda' g_2 8\pi^2 \left( \tilde{\Gamma}^{(1)}_{n,my} \psi_1^{n1\dagger} \psi_1^{my} \psi_1^{1-y} + \tilde{\Gamma}^{(1)}_{my,n} \psi_0^{(1-y)\dagger} \psi_0^{my} \psi_0^{n1} \\
+ \tilde{\Gamma}^{(1)}_{n,y} \psi_2^{n1\dagger} \psi_2^{y} \psi_2^{(1-y)} + \tilde{\Gamma}^{(1)}_{y,n} \psi_3^{n1\dagger} \psi_3^{y} \psi_3^{(1-y)} \right), \]

with

\[ \tilde{\Gamma}^{(1)}_{n,my} = \tilde{\Gamma}^{(1)}_{my,n} = \sqrt{\frac{1 - y \sin^2(\pi ny)}{y} \frac{2}{\sqrt{J}}} \sin^2(\pi ny), \]

\[ \tilde{\Gamma}^{(1)}_{n,y} = \tilde{\Gamma}^{(1)}_{y,n} = -\frac{1}{\sqrt{J}} \sin^2(\pi ny). \]

These are indeed the cubic couplings generated from the pp wave SFT \cite{9, 10, 11, 12}.

4. Lattice Strings and The Vertex

We have in sect.2 and in (I) given the collective Hamiltonian corresponding to \( H_0 \) and we now concentrate on \( H_1 \). We start with the contributions to the cubic interaction.

In what follows we will consider loops of the form

\[ \Phi_J(\{l_i\}) = \text{Tr} \left( T_l \prod_{i=1}^n b(l_i) A^{l_i} \right), \]

where
\[ b_l = A^l b^\dagger A^{-l\dagger}. \]

The subscript \( l \) on \( b \) is understood as \( l \) mod \( J \), so that we can assume this index is in the range \( 0 \leq l \leq J - 1 \). The symbol \( T_l \) orders the \( b_l \) so that \( l \) increases from left to right. These loops correspond to the string states

\[ \Phi_J (\{l_i\}) \leftrightarrow \sum_{p=0}^{J-1} \prod_{i=1}^{n_1} b_{p+l_i \text{mod} J}^{(1)\dagger} |0; J\rangle. \]

In the continuum limit the sparse occupation of lattice sites becomes trivial.

For simplicity, we focus on a single complex impurity. The cubic interaction is represented as

\[ H_{3}^{\text{col}} = \sum_{J_1, J_2, \{l^{(1)}\}, \{l^{(2)}\}} (H_1 \Phi_{J_1}^{n_1}(\{l^{(1)}\})\Phi_{J_2}^{n_2}(\{l^{(2)}\})) \frac{\partial}{\partial \Phi_{J_1}^{n_1}(\{l^{(1)}\})} \frac{\partial}{\partial \Phi_{J_2}^{n_2}(\{l^{(2)}\})} \]

where, because we have a single impurity, joining is generated by

\[ \hat{H}_1 = -g_Y^2 \text{Tr} \left( [b^\dagger, A^\dagger] \left[ \frac{\partial}{\partial b^\dagger}, \frac{\partial}{\partial A^\dagger} \right] \right). \]

Towards this end, consider the following derivatives

\[ \frac{\partial \Phi_J (\{l_i\})}{\partial A^\dagger_{ij}} = (P \{l_i\})_{ji} \]

\[ = \sum_{a=0}^{J-1} \left[ T_l \prod_{l=1}^{n} b(l_i - a \text{ mod } J) A^{J-1\dagger} \right]_{ji}, \]

and

\[ \frac{\partial \Phi_J (\{l_i\})}{\partial b^\dagger_{ij}} = (Q \{l_i\})_{ji} \]

\[ = \sum_{k=1}^{n} \left[ T_l \prod_{i=1, i \neq k}^{n} b(l_i - l_k \text{ mod } J) A^{J\dagger} \right]_{ji}. \]

which produce
\[ (H_1 \Phi_J \{\{I_1^{(1)}\}\}) \Phi_{J_2} \{\{I_2^{(2)}\}\}) = g_{YM}^2 \left( \text{Tr}(b^\dagger A^\dagger P_1 Q_2) - \text{Tr}(A^\dagger b^\dagger P_1 Q_2) + \text{Tr}(b^\dagger A^\dagger P_2 Q_1) - \text{Tr}(A^\dagger b^\dagger P_2 Q_1) - \text{Tr}(P_2 b^\dagger A^\dagger Q_1) + \text{Tr}(P_2 A^\dagger b^\dagger Q_1) - \text{Tr}(P_1 b^\dagger A^\dagger Q_2) + \text{Tr}(P_1 A^\dagger b^\dagger Q_2) \right), \]

\[ (P_a)_{ji} = \frac{\partial \Phi_{J_a} \{\{I_a^{(1)}\}\}}{\partial A_{ij}^\dagger}, \quad (Q_a)_{ji} = \frac{\partial \Phi_{J_a} \{\{I_a^{(1)}\}\}}{\partial b_{ij}}, \quad a = 1, 2 \]

for the loop joining terms.

In I the free matrix model was seen to generate loop joinings that were reproducing the string field theory interaction vertex with the trivial energy prefactor (zeroth order in \(\mu\)). These free matrix model loop joining operations were associated with

\[ \text{Tr} \left( P_1 P_2 \right) = \sum_{J, \{I\}} C_{J_{J_1 J_2}}^{\{I_1^{(1)}\} \{I_2^{(2)}\}} \Phi_J \{\{I\}\}, \]

and

\[ \text{Tr} \left( Q_1 Q_2 \right) = \sum_{J, \{I\}} G_{J_{J_1 J_2}}^{\{I_1^{(1)}\} \{I_2^{(2)}\}} \Phi_J \{\{I\}\}. \]

Now from the action of (4.1) we will find another sequence of joining contributions to the full cubic interaction. As before we may again use our correspondence but this time with a different state from \( |V_3^0 \rangle \). However, in the continuum limit (large \(J\)) the new state will reduce to be \( |V_3^0 \rangle \) multiplied by an insertion which will be shown to be, in combination to the effect of the field redefinition, precisely the large \(\mu\) limit of the string field theory prefactor. Consider the term

\[ \text{Tr}(P_1 b^\dagger A^\dagger Q_2) = \]

\[ \sum_{a=0}^{J_1-1} \sum_{k=1}^{n_2-1} \left( \text{Tr} \left( T_i^{\sum_{i=1}^{n_1} b(I_i^{(1)} - a)A^{J_1-1}} \right) b^\dagger A^\dagger \left( T_j^{\prod_{j=1,j\neq k}^{n_2} b(I_j^{(2)} - I_k^{(2)})} \right) A^{J_2} \right) = \]

\[ \sum_{a=0}^{J_1-1} \sum_{k=1}^{n_2-1} \left( \text{Tr} \left( T_i^{\sum_{i=1}^{n_1} b(I_i^{(1)} - a)} \right) b(J_1 - 1) \left( T_j^{\prod_{j=1,j\neq k}^{n_2} b(I_j^{(2)} - I_k^{(2)} + J_1)} \right) A^{J_1 + J_2} \right). \]

(4.2)

The summation in \(P_1\) reflects the averaging in order to have the physical state \( |\psi_1 \rangle \), a part of the non-triviality of the above comes from \(Q_2\). We would expect the same to happen
with the second string variables after seen in the third string in the case of $|V_3^0\rangle$. In this case we observe the necessity of the insertion $b_{J_1}^{(2)\dagger}$. Another factor comes from $b^{\dagger}$ which gives an insertion $b_{J_1-1}^{(3)\dagger}$. From (4.2) we can easily read off the appropriate state for this term to be,

$$b_{J_1}^{(2)\dagger} b_{J_1-1}^{(3)\dagger} |V_0^3\rangle.$$

In exactly the same way we can argue for the following identifications

$$\text{Tr}(P_1 A^{\dagger} b^{\dagger} Q_2) \rightarrow b_{J_1}^{(2)\dagger} b_{J_1}^{(3)\dagger} |V_0^3\rangle.$$

The remaining terms follow

$$\text{Tr}(P_2 b^{\dagger} A^{\dagger} Q_1) = \text{Tr}(Q_1 P_2 b^{\dagger} A^{\dagger}) \rightarrow b_0^{(1)\dagger} b_{J_1+J_2-1}^{(3)\dagger} |V_0^3\rangle,$n

$$\text{Tr}(P_2 A^{\dagger} b^{\dagger} Q_1) = \text{Tr}(b^{\dagger} Q_1 P_2 A^{\dagger}) \rightarrow b_0^{(1)\dagger} b_0^{(3)\dagger} |V_0^3\rangle.$$

Next we have the four remaining contributions. Consider

$$\text{Tr}(b^{\dagger} A^{\dagger} P_1 Q_2) = \sum_{a=0}^{J_1-1} \sum_{k=1}^{n_2-1} \text{Tr} \left( b^{\dagger} \left( T_l \prod_{i=1}^{n_1} b(l_i^{(1)} - a + 1) \right) \right)$$

$$\left( T_j \prod_{j=1, j \neq k}^{n_2} b(l_j^{(2)} - l_k^{(2)} + J_1) \right) A^{J_1+J_2}\dagger).$$

After employing once more our correspondence we have

$$b_0^{(3)\dagger} b_{J_1}^{(2)\dagger} |V_{3,1}^0\rangle$$

where

$$|V_{3,1}^0\rangle = e^{\sum_{i=0}^{J_1-1} b_i^{(3)\dagger} b_i^{(1)\dagger} + \sum_{j=J_1}^{J_1+J_2-1} b_j^{(3)\dagger} b_j^{(2)\dagger}} |0\rangle_{123}$$

Similarly

$$\text{Tr}(A^{\dagger} b^{\dagger} P_1 Q_2) \rightarrow b_1^{(3)\dagger} b_{J_1}^{(2)\dagger} |V_{3,1}^0\rangle$$

Analogously

$$\text{Tr}(b^{\dagger} A^{\dagger} P_2 Q_1) = \text{Tr}(Q_1 b^{\dagger} A^{\dagger} P_2) \rightarrow b_{J_1}^{(3)\dagger} b_0^{(1)\dagger} V_{3,2}^0$$

$$\text{Tr}(A^{\dagger} b^{\dagger} P_2 Q_1) = \text{Tr}(Q_1 A^{\dagger} b^{\dagger} P_2) \rightarrow b_{J_1+1}^{(3)\dagger} b_0^{(1)\dagger} V_{3,2}^0.$$
where

\[ |V_{3,2}^0\rangle = e^{\sum_{i=0}^{J_1-1} b_{i}^{(3)\dagger} b_{i}^{(1)\dagger} + \sum_{j=J_1}^{J_1+J_2-1} b_{j}^{(3)\dagger} b_{j}^{(2)\dagger}} |0\rangle_{123} \]

We are now ready to collect the contributions to the three string vertex and take the continuum limit. Let us recall the contribution from the free part of the Hamiltonian

\[ (E_3 - E_2 - E_1) |V_{3}^0\rangle \]

This is essentially the result of I, but now we have the improved \( O(\lambda) \) corrected energies \( E_i \) in the prefactor multiplying \( V_{3}^0 \). This form is deduced from our free coherent Hamiltonian after performing the coherent physical space map. The relevant transformation given generally in sect.3 by eqs. (3.11)-(3.14). From the action of \( H_1 \) we have generated eight terms. In the continuum limit, the difference between \( V_{3,1}^0 \) and \( V_{3}^0 \) is negligible so the eight terms appear with the same 3-vertex:

\[
\left[ \left( b_0^{(3)\dagger} - b_{J_1+J_2-1}^{(3)\dagger} \right) b_0^{(1)\dagger} + \left( b_{J_1}^{(3)\dagger} - b_{J_1-1}^{(3)\dagger} \right) b_{J_1}^{(2)\dagger} \\
+ \left( b_0^{(3)\dagger} - b_1^{(3)\dagger} \right) b_{J_1}^{(2)\dagger} + \left( b_{J_1}^{(3)\dagger} - b_{J_1+1}^{(3)\dagger} \right) b_0^{(1)\dagger} \right] |V_{3}^0\rangle
\]

Altogether we therefore find the total matrix theory vertex comes with the prefactor

\[ \hat{V}_{Matrix} = \left[ (E_3 - E_1 - E_2) + 2 \left( \Delta b_0^{(3)\dagger} - \Delta b_{J_1}^{(3)\dagger} \right) \left( b_{J_1}^{(2)\dagger} - b_0^{(1)\dagger} \right) \right] |V_{3}^0\rangle \]

where \( \Delta b \) stands for a finite difference. We now clearly see the leading effect of the YM interaction: first it induces a correction to the energies(dimensions) in the prefactor renormalizing the free matrix theory result, in addition there are novel contributions to the prefactor.

In the continuum limit these become operator insertions at the interaction point, in particular we get the derivative \( b^+(0)^1 \) of the string creation coordinate at the interaction point. In order to demonstrate agreement with SFT now show that the SFT prefactor can be written in an identical form and \( \hat{P}_{Matrix} \to \hat{P}_{SFT} \).

To perform the comparison consider the SFT prefactor

\[ \hat{P} = \sum_{r=1}^{3} \sum_{n>0} \frac{\omega_n^{(r)}}{\mu \alpha^{(r)}} \left( a_n^{(r)\dagger} a_n^{(r)} - a_{-n}^{(r)\dagger} a_{-n}^{(r)} \right) \]

We can “separate” out an energy contribution to write it as
\[
\hat{P} = 2 \sum_{n>0} \left\{ \frac{\omega_n^{(1)}}{\mu \alpha^{(1)}} a_n^{(1)\dagger} a_n^{(1)} + \frac{\omega_n^{(2)}}{\mu \alpha^{(2)}} a_n^{(2)\dagger} a_n^{(2)} - \frac{\omega_n^{(3)}}{\mu \alpha^{(3)}} a_n^{(3)\dagger} a_n^{(3)} \right\} - (E_1 + E_2 - E_3)
\]

where

\[E_r = \sum_n \frac{\omega_n^{(r)}}{\mu \alpha^{(r)}} a_n^{(r)\dagger} a_n^{(r)}\]

Denoting the first term containing positive modes only by \(P_+\) we have the 3-vertex

\[P_+ |V_3^0\rangle = 2 \sum_{n>0} \left\{ \left( \frac{\omega_n^{(1)}}{\mu \alpha^{(1)}} - \frac{\omega_m^{(3)}}{\mu \alpha^{(3)}} \right) a_m^{(3)\dagger} a_n^{(1)\dagger} N_{nm}^{(13)} \right. \]

\[+ \left. \left( \frac{\omega_n^{(2)}}{\mu \alpha^{(2)}} - \frac{\omega_m^{(3)}}{\mu \alpha^{(3)}} \right) a_m^{(3)\dagger} a_n^{(2)\dagger} N_{nm}^{(23)} \right\} |V_3^0\rangle \]

using

\[N_{nm}^{(13)} \to \frac{2}{\pi} (-)^{m+n+1} \beta^{3/2} \frac{m \sin(\beta m \pi)}{m^2 \beta^2 - n^2} \]

\[N_{nm}^{(23)} \to \frac{2}{\pi} (-)^{m(1-\beta)}^{3/2} \frac{m \sin(\beta m \pi)}{(1-\beta)^2 m^2 - n^2} \]

one gets

\[P_+ |V_3^0\rangle = 2 \frac{\beta^{3/2} \alpha^{(3)}}{\mu^2 \alpha^{(1)2} \pi} \left( \sum_{m>0} (-)^m m \sin(\beta m \pi) a_{m}^{(3)\dagger} \right) \left( \sum_{n>0} (-)^n a_{n}^{(1)\dagger} \right) - \]

\[2 \frac{(1-\beta)^{3/2} \alpha^{(3)}}{\mu^2 \alpha^{(2)2} \pi} \left( \sum_{m>0} (-)^m m \sin(\beta m \pi) a_{m}^{(3)\dagger} \right) \left( \sum_{n>0} a_{n}^{(2)\dagger} \right) |V_3^0\rangle \]

since we may use the expansion

\[b^{(3)\dagger}(\sigma) = \frac{1}{\mu \sqrt{2\pi \alpha^{(3)}}} \sum_{m>0} (-)^m \left( a_{m}^{(3)\dagger} \cos \left( \frac{m \sigma}{\alpha^{(3)}} \right) + a_{-m}^{(3)\dagger} \sin \left( \frac{m \sigma}{\alpha^{(3)}} \right) \right)\]

and

\[b^{(r)\dagger}(\sigma) = \frac{1}{\mu \sqrt{2\pi \alpha^{(r)}}} \sum_{m>0} \left( a_{m}^{(r)\dagger} \cos \left( \frac{m \sigma}{\alpha^{(r)}} \right) + a_{-m}^{(r)\dagger} \sin \left( \frac{m \sigma}{\alpha^{(r)}} \right) \right), \text{ where } r = \{1, 2\}\]

For all three strings we have:
\[-b^{(1)\dagger}(\pi \alpha_{(1)}) = \frac{1}{\mu \sqrt{2\pi \alpha_{(1)}}} \sum_{n} (-)^{n+1} a_{n}^{(1)\dagger}\]

\[b^{(2)\dagger}(0) = \frac{1}{\mu \sqrt{2\pi \alpha_{(2)}}} \sum_{n} a_{n}^{(2)\dagger}\]

\[\frac{1}{2} \left( b^{(3)\dagger}(-\pi \alpha_{1}) - b^{(3)\dagger}(\pi \alpha_{1}) \right) = \frac{1}{\mu \sqrt{2\pi \alpha_{(3)}}} \sum_{m} \frac{m}{\alpha_{3}} \sin(\beta m \pi) a_{m}^{(3)\dagger}\]

and

\[P_{+} = 2 \left( b^{(3)\dagger'}(\pi \alpha_{1}) - b^{(3)\dagger'}(-\pi \alpha_{1}) \right) \left( b^{(2)\dagger}(0) - b^{(1)\dagger}(\pi \alpha_{1}) \right)\]

a form predicted in our matrix model calculation.

5. Conclusions

We have in the present paper constructed the pp wave cubic SFT interaction from large N matrix theory using the Berenstein-Maldacena-Nastase limit. Even though this construction was performed at leading order in the Yang-Mills theory coupling constant, the method that we employ is generally not limited to weak coupling. We have concentrated on the sector of the theory generated by the Higgs (scalar field) degrees of freedom. This and the use of a matrix model language was for the purpose of notational simplicity. The construction can be extended to include fermionic fields of SUSY Yang-Mills theory and also states generated by the (covariant) derivatives of the Higgs fields. In the basic scheme that we presented the closed string theory cubic interactions are seen to be correctly generated. Most importantly we have seen how the correct prefactor or operator insertion at the interaction point is obtained from the large N matrix theory construction. There has been some debate on the form of the prefactor, our direct calculations provide a unique and well defined form. Apart from the extension of the present approach to include the fermionic and derivative degrees of freedom of the full Yang-Mills theory the most important future goal is that of presenting a derivation of the full nonperturbative SFT interaction. This implies an extension of the calculations done presently without the use of weak coupling methods. The collective field approach that we employ is in general not limited to weak Yang-Mills coupling, its application to various phases of large N has been demonstrated in past studies. For that reason its further study offers a possibility for the nonperturbative understanding of the gauge theory/string theory correspondence.
Acknowledgements: The work of RdMK and JPR is supported by NRF grant number Gun 2053791. The work of AD and AJ is supported by DOE grant DE FGO2/19ER40688(Task A).

APPENDIX: Lattice Strings and The Spectrum

In this section we consider in detail the quadratic piece of the collective field theory hamiltonian. This will be done fully with a goal of deriving the first quantized lattice string hamiltonian of BMN. We consider the case of a real impurity matrix coordinates and work in the coherent state basis. Consider the nontrivial contribution from the interacting piece of the matrix theory. It is given by

\[ H_{2}^{\text{col}} = \sum_{J,\{\ell\}} \left( H_{1}(\Phi_{J}(\{\ell\})) \right) \frac{\partial}{\partial \Phi_{J}(\{\ell\})} \]

where in the action of \( H_{1}(\Phi_{J}(\{\ell\})) \) we concentrate only on the contribution linear in the collective field \( \Phi \). In the creation annihilation operator basis, we consider the action of

\[ -g_{YM}^{2} Tr \left( [b^\dagger, A^\dagger] \left[ \frac{\partial}{\partial b^\dagger}, \frac{\partial}{\partial A^\dagger} \right] + \left[ \frac{\partial}{\partial b^\dagger}, A^\dagger \right] \left[ b^\dagger, \frac{\partial}{\partial A^\dagger} \right] + \left[ b^\dagger, A^\dagger \right] \left[ b^\dagger, \frac{\partial}{\partial A^\dagger} \right] + \left[ \frac{\partial}{\partial b^\dagger}, A^\dagger \right] \left[ \frac{\partial}{\partial b^\dagger}, \frac{\partial}{\partial A^\dagger} \right] \right) \].

We begin by considering the most general trace

\[ \Phi_{J}^{n}(\{\ell\}) = \frac{1}{\sqrt{N^{J+n}}} Tr \left( T_{l} \prod_{i=1}^{n} b^{n_{i}}(l_{i}) A_{j}^{\dagger} \right) \]

where we have \( l_{i} \neq l_{j} \) for \( i \neq j \). The first term in our matrix hamiltonian produces at first order in \( N \),

\[ - Tr \left( [b^\dagger, A^\dagger] \left[ \frac{\partial}{\partial b^\dagger}, \frac{\partial}{\partial A^\dagger} \right] \right) \Phi_{J}^{n}(\{\ell\}) = \frac{2N}{\sqrt{N^{J+n}}} \sum_{i=1}^{n} Tr \left( T_{l} \prod_{i=1}^{n} b^{n_{i}}(l_{i}) A_{j}^{\dagger} \right) \]

\[ - \frac{N}{\sqrt{N^{J+n}}} \sum_{j=1}^{n} Tr \left( T_{l} \prod_{i=1}^{n} b^{\delta(i,j)}(l_{i} - \delta(i,j) \text{ mod } J)b^{n_{j}-\delta(i,j)}(l_{i}) A_{j}^{\dagger} \right) \]

\[ - \frac{N}{\sqrt{N^{J+n}}} \sum_{j=1}^{n} Tr \left( T_{l} \prod_{i=1}^{n} b^{\delta(i,j)}(l_{i} + \delta(i,j) \text{ mod } J)b^{n_{j}-\delta(i,j)}(l_{i}) A_{j}^{\dagger} \right) . \]
This implies a contribution of the form

\[ H_2^{\text{col}} = \sum_{J, \{l\}} (\hat{h}_{\text{col}} \Phi_J(\{l\})) \frac{\partial}{\partial \Phi_J(\{l\})} \]

with \( \hat{h}_{\text{col}} \) being a first quantized lattice string operator. We see from the above that we have a contribution

\[ N \sum_{i=0}^{J-1} : \left( b_{i+1}^\dagger b_{i+1} + b_i^\dagger b_i - b_{i+1}^\dagger b_i - b_i^\dagger b_{i+1} \right) :. \]

Next from the second and third term in \( H_1 \) we find

\[ -Tr(\left( \frac{\partial}{\partial b^\dagger}, A^\dagger \right) [b^\dagger, \frac{\partial}{\partial A^\dagger}]) \Phi^n_J(\{l_i\}) = \]

\[ -\frac{N}{\sqrt{NJ+n}} \sum_{j=1}^n Tr \left( T_j \prod_{i=1}^n b^{\delta(i,j)}(l_i - \delta(i,j) \mod J) b^{n_j - \delta(i,j)}(l_i) A^\dagger_j \right) \]

\[ -\frac{N}{\sqrt{NJ+n}} \sum_{j=1}^n Tr \left( T_j \prod_{i=1}^n b^{\delta(i,j)}(l_i + \delta(i,j) \mod J) b^{n_j - \delta(i,j)}(l_i) A^\dagger_j \right), \]

and

\[ -Tr([b^\dagger, A^\dagger] [b^\dagger, \frac{\partial}{\partial A^\dagger}]) \Phi^n_J(\{l_i\}) = \frac{N}{\sqrt{NJ+n}} \sum_{j=1}^n Tr \left( T_j b^2(j + 1 \mod J) \prod_{i=1}^n b^{n_i}(l_i) A^\dagger_j \right) \]

\[ + \frac{N}{\sqrt{NJ+n}} \sum_{j=1}^n Tr \left( T_j b^2(j) \prod_{i=1}^n b^{n_i}(l_i) A^\dagger_j \right) \]

\[ -2 \frac{N}{\sqrt{NJ+n}} \sum_{j=1}^n Tr \left( T_j b(j + 1 \mod J) b(j) \prod_{i=1}^n b^{n_i}(l_i) A^\dagger_j \right) \]

with the last term giving
$$-\text{Tr}(\frac{\partial}{\partial b_i^\dagger}, A) \frac{\partial}{\partial A_j}) \Phi_n(\{l_i\}) =$$

$$\frac{N}{\sqrt{NJ+n}} \sum_{j=1}^n \text{Tr} \left( T_l \theta(n_j - 1) \prod_{i=1}^n b_i^{n_i - 2\delta(i,j)} (l_i) A_i^\dagger J \right)$$

$$+ \frac{N}{\sqrt{NJ+n}} \sum_{j=1}^n \text{Tr} \left( T_l \theta(n_{j+1} - 1) \prod_{i=1}^n b_i^{n_{i+1} - 2\delta(i,j)} (l_{i+1}) A_i^\dagger J \right)$$

$$- 2 \frac{N}{\sqrt{NJ+n}} \sum_{j=1}^n \text{Tr} \left( T_l \theta(n_j) \theta(n_{j+1}) \prod_{i=1}^n b_i^{n_i - \delta(i,j) - \delta(i,j+1)} (l_i) A_i^\dagger J \right)$$

Collecting all contribution we obtain the first quantized lattice hamiltonian of the form

$$h_{\text{col}} = g^2_{YM} N \sum_{i=0}^{J-1} \left( b_{i+1}^{\dagger 2} + b_i^{\dagger 2} - 2b_{i+1} b_i^{\dagger} + b_i^2 + b_{i+1}^2 - 2b_{i+1} b_{i+1}\right)$$

which is precisely what someone would get from $h$ by neglecting the constant coming from $b_{i+1} b_{i+1}$ and $b_i b_i^\dagger$. This is recognized as the lattice BMN string hamiltonian

$$h_{BMN} = g^2_{YM} N \sum_{i=0}^{J-1} \left( b_{i+1}^{\dagger} + b_{i+1} - b_i^{\dagger} - b_i \right)^2 = \frac{1}{\epsilon^2} \sum_{i=0}^{J-1} \left( b_{i+1}^{\dagger} + b_{i+1} - b_i^{\dagger} - b_i \right)^2$$

with the understanding that as pointed out in\cite{1} the creation-annihilation operators are to be of Cuntz type. The physical basis behind these oscillators is the fact that the lattice sites should be sparsely occupied (i.e. not more than one oscillator at a lattice site).
References

[1] “Strings in Flat Space and pp Waves from N=4 Super Yang-Mills”, D. Berenstein, J. Maldacena and H. Nastase, JHEP 0204:013, (2002), hep-th/0202024.
[2] “The Large N Limit of Superconformal Field Theories and Supergravity”, J.M. Maldacena, Adv. Theor. Math. Phys. 2 231, (1998), hep-th/9711200.
[3] “Gauge Theory Correlators from Noncritical String Theory”, S.S. Gubser, I.R. Klebanov and A.M. Polyakov, Phys. Lett. B428 105 (1998).
[4] “Anti-de Sitter Space and Holography,” E. Witten, Adv. Theor. Math. Phys. 2 253 (1998), hep-th/9802150.
[5] “Collective String Field Theory of Matrix Models in the BMN Limit”, R. de Mello Koch, A. Jevicki and J. P. Rodrigues, hep-th/0209155.
[6] “What is Holography in the plane-wave limit of AdS/CFT Correspondence”, T. Yoneya, hep-th/0304183.
[7] “Quantum Theory of Many Variable Systems and Fields,” B. Sakita, World Sci. Lect. Notes Phys.1:1-217 (1985).
[8] “pp Wave String Interactions from Perturbative Yang-Mills Theory,” N. R. Constable, D. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnikov, W. Skiba JHEP 0207:017, (2002), hep-th/0205089.
[9] Three Point Functions in N=4 Yang-Mills Theory and pp Waves,” Chong-Sun Chu, V. V. Khoze and G. Travaglini, JHEP 0206:011 (2002) hep-th/0206005.
[10] “BMN Correlators and Operator Mixing in N=4 SYM Theory,” N. Beisert, C. Kristjansen, J. Plefka, G.W. Semenoff and M. Staudacher, Nucl.Phys. B650 125-161 (2003), hep-th/0208178.
[11] “Operator Mixing and the BMN Correspondence,” N. Constable, D.Z. Freedman, M. Headrick and S.Minwalla, hep-th/0209002.
[12] “Tracing the String: BMN Correspondence at Finite J^2/N,” J. Pearson, M. Spradlin, D. Vaman, H. Verlinde and A. Volovich, hep-th/0210102.
[13] “SYM Description of pp Wave String Interactions: Singlet Sector and Arbitrary Impurities,” J. Gomis, S. Moriyama and Jong-won Park, hep-th/0301250.
[14] See for example “Explicit formulas for Neumann coefficients in the plane wave geometry,” Y.H. He, J. H. Schwarz, M. Spradlin and A. Volovich, Phys. Rev. D67:086005, 2003, hep-th/0211198.
[15] “An Alternative Formulation of Light Cone String Field Theory on the Plane wave,” A.Pankiewicz. hep-th/0304232.