SPREADING WITH TWO SPEEDS AND MASS SEGREGATION IN A DIFFUSIVE COMPETITION SYSTEM WITH FREE BOUNDARIES

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ABSTRACT. We investigate the spreading behavior of two invasive species modeled by a Lotka-Volterra diffusive competition system with two free boundaries in a spherically symmetric setting. We show that, for the weak-strong competition case, under suitable assumptions, both species in the system can successfully spread into the available environment, but their spreading speeds are different, and their population masses tend to segregate, with the slower spreading competitor having its population concentrating on an expanding ball, say $B_t$, and the faster spreading competitor concentrating on a spherical shell outside $B_t$ that disappears to infinity as time goes to infinity.

1. Introduction

In this paper, we investigate the spreading behavior of two competing species described by the following free boundary problem in $\mathbb{R}^N$ ($N \geq 1$) with spherical symmetry:

\[
\begin{aligned}
(P) \quad u_t &= d \Delta u + ru(1 - u - kv) \quad \text{for } 0 < r < s_1(t), \quad t > 0, \\
v_t &= \Delta v + v(1 - v - hu) \quad \text{for } 0 < r < s_2(t), \quad t > 0, \\
u_r(0, t) &= v_r(0, t) = 0 \quad \text{for } t > 0, \\
u \equiv 0 &\quad \text{for all } r \geq s_1(t) \text{ and } t > 0, \quad v \equiv 0 \quad \text{for all } r \geq s_2(t) \text{ and } t > 0, \\
 s_1'(t) &= -\mu_1 u_r(s_1(t), t) \quad \text{for } t > 0, \quad s_2'(t) = -\mu_2 v_r(s_2(t), t) \quad \text{for } t > 0, \\
 s_1(0) &= s_1^0, \quad s_2(0) = s_2^0, \quad u(r, 0) = u_0(r), \quad v(r, 0) = v_0(r) \quad \text{for } r \in [0, \infty),
\end{aligned}
\]

where $u(r, t)$ and $v(r, t)$ represent the population densities of the two competing species at spatial location $r (= |x|)$ and time $t$; $\Delta \varphi := \varphi_{rr} + \frac{(N-1)}{r} \varphi_r$ is the usual Laplace operator acting on spherically symmetric functions. All the parameters are assumed to be positive, and without loss of generality, we have used a simplified version of the Lotka-Volterra competition model, which can be obtained from the general model by a standard change of variables procedure (see, for example, [13]). The initial data $(u_0, v_0, s_1^0, s_2^0)$ satisfies

\[
\begin{aligned}
\begin{cases}
 s_1^0 > 0, \quad s_2^0 > 0, \quad u_0 \in C^2([0, s_1^0]), \quad v_0 \in C^2([0, s_2^0]), \quad u_0'(0) = v_0'(0) = 0, \\
 u_0(r) > 0 \quad \text{for } r \in [0, s_1^0), \quad u_0(r) = 0 \quad \text{for } r \geq s_1^0, \\
 v_0(r) > 0 \quad \text{for } r \in [0, s_2^0), \quad v_0(r) = 0 \quad \text{for } r \geq s_2^0.
\end{cases}
\end{aligned}
\]

(1.1)

In this model, both species invade the environment through their own free boundaries: the species $u$ has a spreading front at $r = s_1(t)$, while $v$’s spreading front is at $r = s_2(t)$. For the mathematical treatment, we have extended $u(r, t)$ from its population range $r \in [0, s_1(t)]$ to $r > s_1(t)$ by 0, and extended $v(r, t)$ from $r \in [0, s_2(t)]$ to $r > s_2(t)$ by 0.
The global existence and uniqueness of the solution to problem \((P)\) under (1.1) can be established by the approach in [15] with suitable changes. In fact, the local existence and uniqueness proof can cover a rather general class of such free boundary systems. The assumption in \((P)\) that \(u\) and \(v\) have independent free boundaries causes some difficulties but this can be handled by following the approach in [15] with suitable modifications and corrections. The details are given in the Appendix at the end of the paper.

We say \((u, v, s_1, s_2)\) is a (global classical) solution of \((P)\) if
\[
(u, v, s_1, s_2) \in C^{2,1}(D^1) \times C^{2,1}(D^2) \times C^1([0, +\infty)) \times C^1([0, +\infty)),
\]
where
\[
D^1 := \{(r, t) : r \in [0, s_1(t)], t > 0\}, \quad D^2 := \{(r, t) : r \in [0, s_2(t)], t > 0\},
\]
and all the equations in \((P)\) are satisfied pointwisely. By the Hopf boundary lemma, it is easily seen that, for \(i = 1, 2\) and \(t > 0\), \(s'_i(t) > 0\). Hence
\[
s_{i,\infty} := \lim_{t \to \infty} s_i(t)
\]
is well-defined.

We are interested in the long-time behavior of \((P)\). In order to gain a good understanding, we focus on some interesting special cases. Our first assumption is that
\[
(1.2) \quad 0 < k < 1 < h.
\]
It is well known that under this assumption, when restricted over a fixed bounded domain \(\Omega\) with no-flux boundary conditions, the unique solution \((\tilde{u}(x, t), \tilde{v}(x, t))\) of the corresponding problem of \((P)\) converges to \((1, 0)\) as \(t \to \infty\) uniformly for \(x \in \overline{\Omega}\). So in the long run, the species \(u\) drives \(v\) to extinction and wins the competition. For this reason, condition (1.2) is often referred to as the case that \(u\) is superior and \(v\) is inferior in the competition. This is often referred to as a weak-strong competition case. A symmetric situation is \(0 < h < 1 < k\).

The case \(h, k \in (0, 1)\) is called the weak competition case (see [33]), while the case \(h, k \in (1, +\infty)\) is known as the strong competition case. In these cases, rather different long-time dynamical behaviors are expected.

In this paper, we will focus on problem \((P)\) for the weak-strong competition case (1.2), and demonstrate a rather interesting phenomenon, where \(u\) and \(v\) both survive in the competition, but they spread into the new territory with different speeds, and their population masses tend to segregate, with the population mass of \(v\) shifting to infinity as \(t \to \infty\).

For \((P)\) with space dimension \(N = 1\), such a phenomenon was discussed in [15], though less precisely than here. It is shown in Theorem 5 of [15] that under (1.2) and some additional conditions, both species can spread successfully, in the sense that
\[
(i) \quad s_{1,\infty} = s_{2,\infty} = \infty,
\]
Sharp criteria for spreading and vanishing of $u$ is an interesting question is whether there exists an asymptotic spreading speed, namely whether $r$ the environment, while the superior competitor $u$ where $0 < k < h$... 

At the end of the paper [15], the question of determining the spreading speeds for both species was raised as an open problem for future investigation.

In this paper, we will determine, for such a case, $\lim_{t \to \infty} \frac{s_i(t)}{t}$, $i = 1, 2$; so in particular, the open problem of [15] on the spreading speeds is resolved here. Moreover, we also obtain a much better understanding of the long-time behavior of $u(\cdot, t)$ and $v(\cdot, t)$, for all dimensions $N \geq 1$. See Theorem 1 and Corollary 1 below for details.

A crucial new ingredient (namely $c_{\mu_1}^*$ below) in our approach here comes from recent research on another closely related problem (proposed in [7]):

\[
(Q) \begin{cases}
u_t = d \Delta u + ru(1 - u - kv), & 0 < r < h(t), \ t > 0, \\
u_t = \Delta v + v(1 - v - hu), & 0 < r < \infty, \ t > 0, \\
u_r(0, t) = v_r(0, t) = 0, & u(r, t) = 0, \ h(t) \leq r < \infty, \ t > 0, \\
\mu_1 u_r(h(t), t), & t > 0, \\
h(0) = h_0, u(r, 0) = \hat{u}_0(r), & 0 \leq r \leq h_0, \\
v(r, 0) = \hat{v}_0(r), & 0 \leq r < \infty,
\end{cases}
\]

where $0 < k < 1 < h$ and

\[
(u_0 \in C^3([0, h_0]), \ u_0'(0) = \hat{u}_0(h_0) = 0, \ \hat{u}_0 > 0 \text{ in } [0, h_0), \\
\hat{v}_0 \in C^3([0, \infty)) \cap L^\infty((0, \infty)), \ \hat{u}_0'(0) = 0, \ \hat{v}_0 \geq (\neq) 0 \text{ in } [0, \infty).
\]

In problem $(Q)$ the inferior competitor $v$ is assumed to be a native species already established in the environment, while the superior competitor $u$ is invading the environment via the free boundary $r = h(t)$. Theorem 4.3 in [7] gives a spreading-vanishing dichotomy for (Q): Either

- (Spreading of $u$) $\lim_{t \to \infty} h(t) = \infty$ and

\[
\lim_{t \to \infty} (u(r, t), v(r, t)) = (1, 0) \text{ locally uniformly for } r \in [0, \infty), \text{ or}
\]

- (Vanishing of $u$) $\lim_{t \to \infty} h(t) < \infty$ and

\[
\lim_{t \to \infty} (u(r, t), v(r, t)) = (0, 1) \text{ locally uniformly for } r \in [0, \infty).
\]

Sharp criteria for spreading and vanishing of $u$ are also given in [7]. When spreading of $u$ happens, an interesting question is whether there exists an asymptotic spreading speed, namely whether $\lim_{t \to \infty} \frac{h(t)}{t}$ exists. This kind of questions, similar to the one being asked in [15] mentioned above, turns out to be rather difficult to answer for systems of equations with free boundaries. Recently, Du, Wang and Zhou [13] successfully established the spreading speed for (Q), by making use of
the following so-called semi-wave system:

\[
\begin{cases}
    cU'' + dU'' + rU(1 - U - kV) = 0, & -\infty < \xi < 0, \\
    cV'' + V'' + v(1 - V - hU) = 0, & -\infty < \xi < \infty, \\
    U(-\infty) = 1, & U(0) = 0, \\
    U'(\xi) < 0 = U(-\xi), & \xi < 0, \\
    V(-\infty) = 0, & V(+\infty) = 1, \\
    V'(\xi) > 0, & \xi \in \mathbb{R}.
\end{cases}
\]

(1.4)

It was shown that (1.4) has a unique solution if \(c \in [0,c_0)\), and it has no solution if \(c \geq c_0\), where \(c_0 \in [2\sqrt{rd(1-k)},2\sqrt{rd}]\) is the minimal speed for the traveling wave solution studied in [20]. More precisely, the following result holds:

**Theorem A.** (Theorem 1.3 of [13]) Assume that \(0 < k < 1 < h\). Then for each \(c \in [0,c_0)\), (1.4) has a unique solution \((U_c,V_c) \in [C(\mathbb{R}) \cap C^2([0,\infty))] \times C^2(\mathbb{R})\), and it has no solution for \(c \geq c_0\). Moreover,

(i) if \(0 \leq c_1 < c_2 < c_0\), then

\[
U'_{c_1}(0) < U'_{c_2}(0), \quad U_{c_1}(\xi) > U_{c_2}(\xi) \text{ for } \xi < 0, \quad V_{c_2}(\xi) > V_{c_1}(\xi) \text{ for } \xi \in \mathbb{R};
\]

(ii) the mapping \(c \mapsto (U_c,V_c)\) is continuous from \([0,c_0)\) to \(C^2_{\text{loc}}((-\infty,0]) \times C^2_{\text{loc}}(\mathbb{R})\) with

\[
\lim_{c \to c_0} (U_c,V_c) = (0,1) \quad \text{in } C^2_{\text{loc}}((-\infty,0]) \times C^2_{\text{loc}}(\mathbb{R});
\]

(iii) for each \(\mu_1 > 0\), there exists a unique \(c = c^*_{\mu_1} \in (0,c_0)\) such that

\[
\mu_1 U'_{c^*_{\mu_1}}(0) = c^*_{\mu_1} \quad \text{and } c^*_{\mu_1} \to c_0 \text{ as } \mu_1 \to \infty.
\]

The spreading speed for (Q) is established as follows.

**Theorem B.** (Theorem 1.1 of [13]) Assume that \(0 < k < 1 < h\). Let \((u,v,h)\) be the solution of (Q) with (1.3) and

\[
(1.5) \quad \liminf_{t \to \infty} \hat{v}_0(r) > 0.
\]

If \(h_\infty := \lim_{t \to \infty} h(t) = \infty\), then

\[
\lim_{t \to \infty} \frac{h(t)}{t} = c^*_{\mu_1},
\]

where \(c^*_{\mu_1}\) is given in Theorem A.

It turns out that \(c^*_{\mu_1}\) also plays an important role in determining the long-time dynamics of (P). In order to describe the second crucial number for the dynamics of (P) (namely \(s^*_{\mu_2}\) below), let us recall that, in the absence of the species \(u\), problem (P) reduces to a single species model studied by Du and Guo [2], who generalized the model proposed by Du and Lin [6] from one dimensional space to high dimensional space with spherical symmetry. In such a case, a spreading-vanishing
dichotomy holds for $v$, and when spreading happens, the spreading speed of $v$ is related to the following problem
\begin{equation}
\left\{
\begin{array}{l}
dq'' + sq' + q(a - bq) = 0 \quad \text{in } (-\infty, 0), \\
q(0) = 0, \quad q(-\infty) = a/b, \quad q(\xi) > 0 \quad \text{in } (-\infty, 0).
\end{array}
\right.
\end{equation}

More precisely, by Proposition 2.1 in [1] (see also Proposition 1.8 and Theorem 6.2 of [8]), the following result holds:

**Theorem C.** For fixed $a, b, d, \mu_2 > 0$, there exists a unique $s = s^*(a, b, d, \mu_2) \in (0, 2\sqrt{ad})$ and a unique solution $q^*$ to (1.6) with $s = s^*(a, b, d, \mu_2)$ such that $(q^*)'(0) = -s^*(a, b, d, \mu_2)/\mu_2$. Moreover, $(q^*)'(\xi) < 0$ for all $\xi \leq 0$.

Hereafter, we shall denote $s^*_{\mu_2} := s^*(1,1,1,\mu_2)$. It turns out that the long-time behavior of (P) depends crucially on whether $c^*_{\mu_1} < s^*_{\mu_2}$ or $c^*_{\mu_1} > s^*_{\mu_2}$. As demonstrated in Theorems 1 and 2 below, in the former case, it is possible for both species to spread successfully, while in the latter case, at least one species has to vanish eventually.

Let us note that while the existence and uniqueness of $s^*_{\mu_2}$ is relatively easy to establish (and has been used in [15] and other papers to estimate the spreading speeds for various systems), this is not the case for $c^*_{\mu_1}$, which takes more than half of the length of [13] to establish. The main advance of this research from [15] is achieved by making use of $c^*_{\mu_1}$.

**Theorem 1.** Suppose (1.2) holds and
\begin{equation}
c^*_{\mu_1} < s^*_{\mu_2}.
\end{equation}

Then one can choose initial functions $u_0$ and $v_0$ properly such that the unique solution $(u, v, s_1, s_2)$ of (P) satisfies
\begin{align*}
&\lim_{t \to \infty} \frac{s_1(t)}{t} = c^*_{\mu_1}, \quad \lim_{t \to \infty} \frac{s_2(t)}{t} = s^*_{\mu_2}, \\
&\text{and for every small } \epsilon > 0,
\end{align*}
\begin{equation}
\lim_{t \to \infty} (u(r, t), v(r, t)) = (1, 0) \text{ uniformly for } r \in [0, (c^*_{\mu_1} - \epsilon)t],
\end{equation}
\begin{equation}
\lim_{t \to \infty} v(r, t) = 1 \text{ uniformly for } r \in [(c^*_{\mu_1} + \epsilon)t, (s^*_{\mu_2} - \epsilon)t].
\end{equation}

Before giving some explanations regarding the condition (1.7) and the choices of $u_0$ and $v_0$ in the above theorem, let us first note that the above conclusions indicate that the $u$ species spread at the asymptotic speed $c^*_{\mu_1}$, while $v$ spreads at the faster asymptotic speed $s^*_{\mu_2}$. Moreover, (1.8) and (1.9) imply that the population mass of $u$ roughly concentrates on the expanding ball $\{r < c^*_{\mu_1}t\}$, while that of $v$ concentrates on the expanding spherical shell $\{c^*_{\mu_1}t < r < s^*_{\mu_2}t\}$ which shifts to infinity as $t \to \infty$. We also note that, apart from a relatively thin coexistence shell around $r = c^*_{\mu_1}t$, the population masses of $u$ and $v$ are largely segregated for all large time. Clearly this gives a more precise description for the spreadings of $u$ and $v$ than that in Theorem 5 of [15] (for $N = 1$) mentioned above.
We now look at some simple sufficient conditions for (1.7). We note that $c^*_\mu_1$ is independent of $\mu_2$ and the initial functions. From the proof of Lemma 2.9 in [13], we see that $c^*_\mu_1 \to 0$ as $\mu_1 \to 0$. Therefore when all the other parameters are fixed, (1.7) holds for all small $\mu_1 > 0$.

A second sufficient condition can be found by using Theorem A (iii), which implies $c^*_\mu_1 < c_0 \leq 2\sqrt{rd}$ for all $\mu_1 > 0$. It follows that (1.7) holds for all $\mu_1 > 0$ provided that $2\sqrt{rd} \leq s^*_\mu_2$.

Note that $2\sqrt{rd} \leq s^*_\mu_2$ holds if $\sqrt{rd} < 1$ and $\mu_2 \gg 1$ since $s^*_\mu_2 \to 2$ as $\mu_2 \to \infty$.

For the conditions in Theorem 1 on the initial functions $u_0$ and $v_0$, the simplest ones are given in the corollary below.

**Corollary 1.** Assume (1.2) and (1.7). Then there exists a large positive constant $C_0$ depending on $s^0_1$ such that the conclusions of Theorem 1 hold if

(i) $\|u_0\|_{L^\infty([0,s^0_1])} \leq 1$ with $s^0_1 \geq R^* \sqrt{d/[r(1-k)]}$,

(ii) for some $x_0 \geq C_0$ and $L \geq C_0$, $v_0(r) \geq 1$ for $r \in [x_0,x_0+L]$.

Here $R^*$ is uniquely determined by $\lambda_1(R^*) = 1$, where $\lambda_1(R)$ is the principal eigenvalue of

$$-\Delta \phi = \lambda \phi \quad \text{in } B_R, \quad \phi = 0 \quad \text{on } \partial B_R.$$ 

Roughly speaking, conditions (i) and (ii) above (together with (1.2) and (1.7)) guarantee that $u$ does not vanish yet it cannot spread too fast initially, and the initial population of $v$ is relatively well-established in some part of the environment where $u$ is absent, so with its fast spreading speed $v$ can outrun the superior but slower competitor $u$. In Section 2, weaker sufficient conditions on $u_0$ and $v_0$ will be given (see (B1) and (B2) there).

Next we describe the long-time behavior of (P) for the case

(1.10) $c^*_\mu_1 > s^*_\mu_2$.

We will show that, in this case, no matter how the initial functions $u_0$ and $v_0$ are chosen, at least one of $u$ and $v$ will vanish eventually. As in [15], we say $u$ (respectively $v$) vanishes eventually if

$s_{1,\infty} < +\infty$ and \( \lim_{t \to +\infty} \|u(\cdot,t)\|_{L^\infty([0,s_1(t)])} = 0 \)

(respectively, $s_{2,\infty} < +\infty$ and \( \lim_{t \to +\infty} \|v(\cdot,t)\|_{L^\infty([0,s_1(t)])} = 0 \));

and we say $u$ (respectively $v$) spreads successfully if

$s_{1,\infty} = \infty$ and there exists $\delta > 0$ such that,

$u(x,t) \geq \delta$ for $x \in I_u(t)$ and $t > 0$, 

where $I_u(t)$ is an interval of length at least $\delta$ that varies continuously in $t$ (respectively, $s_{2,\infty} = \infty$ and there exists $\delta > 0$ such that
\[ v(x,t) \geq \delta \text{ for } x \in I_u(t) \text{ and } t > 0, \]
where $I_v(t)$ is an interval of length at least $\delta$ that varies continuously in $t$).

For fixed $d, r, h, k > 0$ satisfying (1.2), we define
\[ B = B(d, r, h, k) := \left\{ (\mu_1, \mu_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : c_{\mu_1}^* > s_{\mu_2}^* \right\}. \]
Note that $B \neq \emptyset$ since $s_{\mu_2}^* \to 0$ as $\mu_2 \to 0$ and $c_{\mu_1}^* > 0$ is independent of $\mu_2$.

We have the following result.

**Theorem 2.** Assume that (1.2) holds. If $(\mu_1, \mu_2) \in B$, then at least one of the species $u$ and $v$ vanishes eventually. More precisely, depending on the choice of $u_0$ and $v_0$, exactly one of the following occurs for the unique solution $(u,v,s_1,s_2)$ of (P):

(i) Both species $u$ and $v$ vanish eventually.

(ii) The species $u$ vanishes eventually and $v$ spreads successfully.

(iii) The species $u$ spreads successfully and $v$ vanishes eventually.

Note that $(\mu_1, \mu_2) \in B$ if and only if (1.10) holds. Theorem 2 can be proved along the lines of the proof of [15, Corollary 1] with some suitable changes. When $N = 1$, Theorem 2 slightly improves the conclusion of Corollary 1 in [15], since it is easily seen that $A \subset B$ (due to $s^*(r(1-k), r, d, \mu_1) \leq c_{\mu_1}^*$), where $A := \left\{ (\mu_1, \mu_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : s^*(r(1-k), r, d, \mu_1) > s_{\mu_2}^* \right\}$ is given in [15].

**Remark 1.1.** We note that by suitably choosing the initial functions $u_0$ and $v_0$ and the parameters $\mu_1$ and $\mu_2$, all the three possibilities in Theorem 2 can occur. For example, for given $u_0$ and $v_0$ with $s_1^0 < R^* \sqrt{\frac{d}{r}}$ and $s_2^0 < R^*$, then scenario (i) occurs as long as both $\mu_1$ and $\mu_2$ are small enough and $(\mu_1, \mu_2) \in B$ (which can be proved by using the argument in [6, Lemma 3.8]). If next we modify $v_0$ such that $s_2^0 \geq R^*$, then $u$ still vanishes eventually but $v$ will spread successfully, which leads to scenario (ii). For scenario (iii) to occur, we can take $s_1^0 \geq R^* \sqrt{\frac{d}{r(1-k)}}$ and $\mu_2$ small enough.

Our results here suggest that in the weak-strong competition case, co-existence of the two species over a common (either moving or stationary) spatial region can hardly happen. This contrasts sharply to the weak competition case ($h, k \in (0, 1)$), where coexistence often occurs; see, for example [33, 31].

Before ending this section, we mention some further references that form part of the background of this research. Since the work [6], there have been tremendous efforts towards developing analytical tools to deal with more general single species models with free boundaries; see [1, 3, 4, 5, 8, 9, 11, 12, 17, 19, 21, 22, 24, 25, 27, 35] and references therein. Related works for two species models can be found in, for example, [7, 13, 14, 26, 28, 29, 30, 32, 33, 34]. The issue of
the spreading speed for single species models in homogeneous environment has been well studied, and we refer to [11, 12] for some sharp estimates. Some of the theory on single species models can be used to estimate the spreading speed for two species models; however, generally speaking, only rough upper and lower bounds can be obtained via this approach.

The rest of this paper is organized as follows. In Section 2, we shall prove our main result, Theorem [1] based on the comparison principle and on the construct of various auxiliary functions as comparison solutions to (P). Section 3 is an appendix, where we prove the local and global existence and uniqueness of solutions to a wide class of problems including (P) as a special case, and we also sketch the proof of Theorem 2.

2. PROOF OF THEOREM [1]

We start by establishing several technical lemmas.

**Lemma 2.1.** Let $\mu_2 > 0$ and $s_{\mu_2}^*$ be given in Theorem C. Then for each $s \in (0, s_{\mu_2}^*)$, there exists a unique $z = z(s) > 0$ such that the solution $q_s$ of the initial value problem

\[
\begin{aligned}
q'' + sq' + q(1 - q) &= 0 \quad \text{in } (-\infty, 0), \\
q(0) &= 0, \quad q'(0) = -s_{\mu_2}^*/\mu_2
\end{aligned}
\]

satisfies $q'_s(-z(s)) = 0$ and $q'_s(z) < 0$ for $z \in (-z(s), 0)$. Moreover, $q_s(-z(s))$ is continuous in $s$ and

\[
z(s) \nearrow \infty, \quad q_s(-z(s)) \nearrow 1 \quad \text{as } s \nearrow s_{\mu_2}^*.
\]

**Proof.** The conclusions follow directly from Proposition 2.4 in [18].

**Lemma 2.2.** Let $(u, v, s_1, s_2)$ be a solution of (P) with $s_{1,\infty} = s_{2,\infty} = \infty$. Suppose that

\[
\limsup_{t \to \infty} \frac{s_1(t)}{t} < c_1 < c_2 < \liminf_{t \to \infty} \frac{s_2(t)}{t}
\]

for some positive constants $c_1$ and $c_2$. Then for any $\varepsilon > 0$, there exists $T > 0$ such that

\[
v(r, t) < 1 + \varepsilon \quad \text{for all } t \geq T \text{ and } r \in [0, \infty), \tag{2.1}
\]

\[
v(r, t) > 1 - \varepsilon \quad \text{for all } t \geq T \text{ and } r \in [c_1 t, c_2 t]. \tag{2.2}
\]

**Proof.** Let $\bar{w}$ be the solution of $w'(t) = w(1 - w)$ with initial data $w(0) = \|v_0\|_{L^\infty}$. By the standard comparison principle, $v(x, t) \leq \bar{w}(t)$ for all $t \geq 0$. Since $\bar{w} \to 1$ as $t \to \infty$, there exists $T > 0$ such that (2.1) holds.

Before proving (2.2), we first show $\limsup_{t \to \infty} s_2(t)/t \leq s_{\mu_2}^*$ by simple comparison. Indeed, it is easy to check that $(v, s_2)$ forms a subsolution of

\[
\begin{aligned}
\bar{w}_t &= \Delta \bar{w} + \bar{w}(1 - \bar{w}), \quad 0 < r < \bar{\eta}(t), \quad t > 0, \\
\bar{w}_r(0, t) &= 0, \quad \bar{w}(\bar{\eta}(t), t) = 0, \quad t > 0, \\
\bar{\eta}'(t) &= -\mu_2 \bar{w}_r(\bar{\eta}(t), t), \quad t > 0, \\
\bar{\eta}(0) &= s_2^0, \quad \bar{w}(r, 0) = v_0(r), \quad r \in [0, s_2^0],
\end{aligned}
\]
By the comparison principle (Lemma 2.6 of [2]), $\bar{\eta}(t) \geq s_2(t)$ for all $t$, which implies that $\bar{\eta}(\infty) = \infty$. It then follows from Corollary 3.7 of [2] that $\bar{\eta}(t)/s_{\mu_2}^\ast$ as $t \to \infty$. Consequently, we have

$$\limsup_{t \to \infty} \frac{s_2(t)}{t} \leq \lim_{t \to \infty} \frac{\bar{\eta}(t)}{t} = s_{\mu_2}^\ast.$$ 

It follows that $c_2 < s_{\mu_2}^\ast$.

We now prove (2.2) by using a contradiction argument. Assume that the conclusion does not hold. Then there exist small $\epsilon_0 > 0$, $t_k \uparrow \infty$ and $x_k \in [c_1 t_k, c_2 t_k]$ such that

$$v(x_k, t_k) \leq 1 - \epsilon_0 \text{ for all } k \in \mathbb{N}. \tag{2.3}$$

Up to passing to a subsequence we may assume that $p_k := x_k/t_k \to p_0$ for some $p_0 \in [c_1, c_2]$ as $k \to \infty$.

We want to show that

$$\limsup_{k \to \infty} v(x_k, t_k) > 1 - \epsilon_0, \tag{2.4}$$

which would give the desired contradiction (with (2.3)). To do so, we define

$$w_k(R, t) = v(R + p_k t, t).$$

Then $w_k$ satisfies

$$w_t = w_{RR} + \left[ \frac{N - 1}{R + p_k t} + p_k \right] w_R + w(1 - w) \text{ for } -p_k t < R < s_2(t) - p_k t, \quad t \geq t_1.$$ 

Recall that $0 < c_1 < c_2 < s_{\mu_2}^\ast < 2$, $x_k = p_k t_k$ and $p_k \to p_0 \in [c_1, c_2]$ such that $p_0 \in (0, 2)$. Hence there exists large positive $L$ such that for all $L_1, L_2 \in [L, \infty)$, the problem

$$z_{RR} + p_0 z_R + z(1 - z) = 0 \text{ in } (-L_2, L_1), \quad z(-L_2) = z(L_1) = 0 \tag{2.5}$$

has a unique positive solution $z(R)$ and $z(0) > 1 - \epsilon_0$.

Fix $L_1 \geq L$, $p \in (p_0, 2)$ and define

$$\phi(R) = e^{-\frac{p}{2} R} \cos \frac{\pi R}{2}.$$ 

It is easily checked that

$$-\phi'' - p \phi' = \left[ \frac{p^2}{4} + \frac{\pi^2}{4 L_1^2} \right] \phi \text{ for } R \in [-L_1, L_1], \quad \phi(\pm L_1) = 0.$$ 

Moreover, there exists a unique $R_0 \in (-L_1, 0)$ such that

$$\phi'(R) < 0 \text{ for } R \in (R_0, L_1), \quad \phi'(R_0) = 0.$$ 

We may assume that $L_1$ is large enough such that

$$\tilde{p} := \frac{p^2}{4} + \frac{\pi^2}{4 L_1^2} < 1.$$ 

We then choose $L_2 > L$ such that

$$\tilde{L} := L_2 + R_0 > 0 \text{ and } \frac{\pi^2}{4 L_2^2} < \tilde{p}.$$
Set
\[
\tilde{\phi}(R) := \begin{cases} 
\phi(R), & R \in [R_0, L_1], \\
\phi(R_0) \cos \frac{z(R-R_0)}{2L}, & R \in [-L_2, R_0).
\end{cases}
\]

Then clearly
\[-\tilde{\phi}'' = \frac{\pi^2}{4L^2} \tilde{\phi}, \quad \tilde{\phi}' > 0 \quad \text{in} \quad (-L_2, R_0), \quad \tilde{\phi}(-L_2) = 0 = \tilde{\phi}'(R_0).
\]

Since
\[
\frac{N-1}{R+p_k t} + p_k < p
\]
for all large \(k\) and large \(t\), we further obtain, for such \(k\) and \(t\), say \(k \geq k_0\) and \(t \geq T_1\),
\[(2.6) \quad -\tilde{\phi}'' - \left[ \frac{N-1}{R+p_k t} + p_k \right] \tilde{\phi}' \leq -\tilde{\phi}'' - p \chi_{[R_0, L_1]} \tilde{\phi}' \leq -\delta \tilde{\phi} \quad \text{for} \quad R \in (-L_2, L_1).
\]

The above differential inequality should be understood in the weak sense since \(\tilde{\phi}''\) may have a jump at \(R = R_0\).

We now fix \(T_0 \geq T_1\) and observe that
\[v(R, T_0) > 0 \quad \text{for} \quad R \in [0, s_2(T_0)], \quad c_2 < \liminf_{t \to \infty} \frac{s_2(t)}{t}.
\]

Hence if \(T_0\) is large enough then for \(R \in [-L_2, L_1]\) and \(t \geq T_0\) we have
\[0 < -L_2 + c_1 t \leq R + p_k t \leq L_1 + c_2 t < s_2(t) \quad \text{for all} \quad k \geq 1.
\]

It follows that
\[w_k(R, T_0) = v(R+p_k T_0, T_0) \geq \sigma_0 := \min_{R \in [0, L_1+e_2 T_0]} v(R, T_0) > 0 \quad \text{for} \quad R \in [-L_2, L_1], \quad k \geq 1.
\]

Let \(z_k(R, t)\) be the unique solution of
\[z_t = z_{RR} + \left[ \frac{N-1}{R+p_k t} + p_k \right] z_R + z(1 - z), \quad z(L_1, t) = z(-L_2, t) = 0.
\]

with initial condition
\[z_k(R, T_0) = w_k(R, T_0), \quad R \in [-L_2, L_1].
\]

The comparison principle yields
\[w_k(R, t) \geq z_k(R, t) \quad \text{for} \quad R \in [-L_2, L_1], \quad t \geq T_0, \quad k \geq 1
\]
since \(w_k(R, t) > 0 = z_k(R, t)\) for \(R \in \{-L_2, L_1\}\) and \(t > T_0, \quad k \geq 1\).

On the other hand, if we choose \(\delta > 0\) sufficiently small, then \(\zeta(R) := \delta \tilde{\phi}(R) \leq \sigma_0\) for \(R \in [-L_2, L_1]\) and due to \((2.6)\), \(\zeta(R)\) satisfies
\[-\zeta'' - \left[ \frac{N-1}{R+p_k t} + p_k \right] \zeta' \leq \zeta(1 - \zeta) \quad \text{for} \quad R \in (-L_2, L_1), \quad t \geq T_0, \quad k \geq k_0.
\]

We thus obtain
\[z_k(R, t) \geq \zeta(R) \quad \text{for} \quad R \in [-L_2, L_1], \quad t \geq T_0, \quad k \geq k_0.
\]

We claim that
\[(2.7) \quad \lim_{k \to \infty} z_k(0, t_k) = z(0) > 1 - \epsilon_0,
\]
where \( z(R) \) is the unique positive solution of (2.5), which then gives
\[
\limsup_{k \to \infty} v(x_k, t_k) = \limsup_{k \to \infty} w_k(0, t_k) \geq \limsup_{k \to \infty} z_k(0, t_k) > 1 - \epsilon_0,
\]
and so (2.4) holds.

It remains to prove (2.7). Set
\[
Z_k(R, t) := z_k(R, t_k + t).
\]
Then \( Z_k \) satisfies
\[
(Z_k)_t = (Z_k)_{RR} + \left[ \frac{N - 1}{R + p_k(t_k + t)} + p_k \right] (Z_k)_R + Z_k(1 - Z_k) \text{ for } R \in (-L_2, L_1), \ t \geq T_0 - t_k,
\]
and
\[
Z_k(-L_2, t) = Z_k(L_1, t) = 0, \ Z_k(R, t) \geq z(R) \text{ for } R \in [-L_2, L_1], \ t \geq T_0 - t_k, \ k \geq k_0.
\]

By a simple comparison argument involving a suitable ODE problem we easily obtain
\[
Z_k(R, t) \leq M := \max\{||v(\cdot, T_0)||_{L^\infty} \} \text{ for } R \in [-L_2, L_1], \ t \geq T_0 - t_k, \ k \geq 1.
\]

Since \( \frac{N - 1}{R + p_k(t_k + t)} + p_k \to p_0 \) uniformly as \( k \to \infty \), we may apply the parabolic \( L^p \) estimate to the equations satisfied by \( Z_k \) to conclude that, for any \( p > 1 \) and \( T > 0 \), there exists \( C_1 > 0 \) such that, for all large \( k \geq k_0 \), say \( k \geq k_1 \),
\[
||Z_k||_{W^{2, 1}_p([-L_2, L_1] \times [-T, T])} \leq C_1.
\]

It then follows from the Sobolev embedding theorem that, for every \( \alpha (0, 1) \) and all \( k \geq k_1 \),
\[
||Z_k||_{C^{1+\tilde{\alpha}, (1+\tilde{\alpha})/2}([-L_2, L_1] \times [-T, T])} \leq C_2
\]
for some constant \( C_2 \) depending on \( C_1 \) and \( \alpha \). Let \( \tilde{\alpha} \in (0, \alpha) \). Then by compact embedding and a well known diagonal process, we can find a subsequence of \( \{Z_k\} \), still denoted by itself for the seek of convenience, such that
\[
Z_k(R, t) \to Z(R, t) \text{ as } k \to \infty \text{ in } C^{1+\tilde{\alpha}, (1+\tilde{\alpha})/2}_{loc}([-L_2, L_1] \times \mathbb{R}).
\]

From the equations satisfied by \( Z_k \) we obtain
\[
Z_t = Z_{RR} + p_0 Z_R + Z(1 - Z) \text{ for } R \in (-L_2, L_1), \ t \in \mathbb{R},
\]
and
\[
Z(-L_2, t) = Z(L_1, t) = 0, \ M \geq Z(R, t) \geq z(R) \text{ for } R \in [-L_2, L_1], \ t \in \mathbb{R}.
\]

We show that \( Z(R, t) \equiv z(R) \). Indeed, if we denote by \( Z \) the unique solution of
\[
z_t = z_{RR} + p_0 z_R + z(1 - z) \text{ for } R \in (-L_2, L_1), \ t > 0
\]
with boundary conditions \( z(-L_2, t) = z(L_1, t) = 0 \) and initial condition \( z(R, 0) = z(R) \), while let \( \bar{Z} \) be the unique solution to this problem but with initial condition replaced by \( z(R, 0) = M \), then clearly
\[
\lim_{t \to \infty} Z(R, t) = \lim_{t \to \infty} \bar{Z}(R, t) = z(R).
\]
On the other hand, for any $s > 0$, by the comparison principle we have

$$Z(R, t + s) \leq Z(R, t) \leq Z(R, t + s) \text{ for } R \in [-L_2, L_1], \; t \geq -s.$$  

Letting $s \to \infty$ we obtain $z(R) \leq Z(R, t) \leq z(R)$. We have thus proved $Z(R, t) \equiv z(R)$ and hence

$$z_k(0, t_k) = Z_k(0, 0) \to Z(0, 0) = z(0) \text{ as } k \to \infty.$$  

This proves (2.7) and the proof of Lemma 2.2 is complete. \hfill \square

We now start to construct some auxiliary functions by modifying the unique solution $(U, V)$ of (1.4) with $c = c_{\mu_1}^*$. Firstly, for any given small $\varepsilon \in (0, 1)$ we consider the following perturbed problem

$$(2.8) \quad \begin{cases} cU'' + dU''' + rU(1 + \varepsilon - U - hV) = 0 \text{ for } -\infty < \xi < 0, \\ cV'' + V'' + V(1 - \varepsilon - V - hU) = 0 \text{ for } -\infty < \xi < \infty, \\ U(-\infty) = 1 + \varepsilon, \; U(0) = 0, \; U'_\varepsilon(0) = -\mu_1/c, \; U'\varepsilon(\xi) < 0 = U(-\xi) \text{ for } \xi < 0 \\ V(-\infty) = 0, \; V(+\infty) = 1 - \varepsilon, \; V'\varepsilon(\xi) > 0 \text{ for } \xi \in \mathbb{R}. \end{cases}$$

Taking $U = (1 + \varepsilon)\hat{U}$ and $V = (1 - \varepsilon)\hat{V}$, then $(\hat{U}, \hat{V})$ satisfies (1.4) for $k$ and $h$ replaced by some $\hat{k}_\varepsilon$ and $\hat{h}_\varepsilon$ with $0 < \hat{k}_\varepsilon < 1 < \hat{h}_\varepsilon$, and $\hat{k}_\varepsilon \to k$, $\hat{h}_\varepsilon \to h$ as $\varepsilon \to 0$. Hence by Theorem A, there exists a unique $c = c_{\mu_1}^\varepsilon > 0$ such that (2.8) with $c = c_{\mu_1}^\varepsilon$ admits a unique solution $(U_\varepsilon, V_\varepsilon)$. As in [13], $(U_\varepsilon, V_\varepsilon)$ and $c_{\mu_1}^\varepsilon$ depends continuously on $\varepsilon$, and in particular, $c_{\mu_1}^\varepsilon \to c_{\mu_1}^*$ as $\varepsilon \to 0$. Moreover, as in the proof of Lemma 2.5 in [13], we have the asymptotic expansion

$$(2.9) \quad V_\varepsilon(\xi) = Ce^{\mu\varepsilon}(1 + o(1)), \quad V'_\varepsilon(\xi) = C\mu e^{\mu\varepsilon}(1 + o(1)) \text{ as } \xi \to -\infty$$

for some $C > 0$, where

$$\mu = \mu(\varepsilon) := \frac{-c_{\mu_1}^\varepsilon + \sqrt{(c_{\mu_1}^\varepsilon)^2 + 4[h(1 + \varepsilon) - 1 + \varepsilon]}}{2} > 0.$$  

Next we modify $(U_\varepsilon(\xi), V_\varepsilon(\xi))$ to obtain the required auxiliary functions. The modification of $V_\varepsilon$ is rather involved, and for simplicity, we do that for $\xi \geq 0$ and $\xi \leq 0$ separately.

We first consider the case $\xi \geq 0$. For fixed $\varepsilon \in (0, 1)$ sufficiently small, we define

$$(2.10) \quad Q'_+(\xi) = \begin{cases} V_\varepsilon(\xi) & \text{for } 0 \leq \xi \leq \xi_0, \\ V_\varepsilon(\xi) - \delta(\xi - \xi_0)^2V_\varepsilon(\xi_0) & \text{for } \xi_0 \leq \xi \leq \xi_0 + 1, \end{cases}$$

where $\xi_0 = \xi_0(\varepsilon) > 0$ is determined later and

$$(2.11) \quad \delta = \delta(\varepsilon) := \frac{\varepsilon}{4 + 2c_{\mu_1}^\varepsilon} \in (0, 1).$$

It is straightforward to see that $Q'_+ \in C^1([0, \xi_0 + 1])$. The following result will be useful later.

**Lemma 2.3.** For any small $\varepsilon > 0$, there exist $\xi_0 = \xi_0(\varepsilon) > 0$ and $\xi_1 = \xi_1(\varepsilon) \in (\xi_0, \xi_0 + 1)$ such that $\lim_{\varepsilon \to 0} \xi_0(\varepsilon) = \infty$ and

$$(Q'_+)'(\xi_1) = 0, \ (Q'_+)'(\xi) > 0 \text{ for } \xi \in [0, \xi_1),$$

and
Hence equation of (2.8), it is straightforward to see that the inequality in (2.12) holds for \( s, \xi \) and (2.11), we deduce
\[
(2.14)
\]
that
\[
Q^\varepsilon_+(\xi_1) = q_{s^\varepsilon}(-z(s^\varepsilon)),
\]
where \( z(s^\varepsilon) \) and \( q_{s^\varepsilon} \) are defined in Lemma 2.1 with \( s = s^\varepsilon \).

Proof. For convenience of notation we will write \( Q^\varepsilon_+ = Q \). Since \( V^\varepsilon_-(\infty) = 1 - \varepsilon, V^\varepsilon_+'(\infty) = 0 \) and \( V^\varepsilon_+'' > 0 \), we can choose \( \xi_0 = \xi_0(\varepsilon) \gg 1 \) such that \( \lim_{\varepsilon \to 0} \xi_0(\varepsilon) = \infty \) and
\[
1 - 2\varepsilon \leq V^\varepsilon_-(\xi) \leq 1 - \varepsilon, \quad V^\varepsilon_+''(\xi + 1) - 2\delta V^\varepsilon_-(\xi) < 0 \quad \text{for all} \quad \xi \geq \xi_0.
\]
In particular, we have
\[
Q'(\xi_0 + 1) = V^\varepsilon_-(\xi_0 + 1) - 2\delta V^\varepsilon_-(\xi_0) < 0.
\]
Note that \( Q'(\xi_0) = V^\varepsilon_-(\xi_0) > 0 \). By the continuity of \( Q' \), we can find \( \xi_1 = \xi_1(\xi_0) \in (\xi_0, \xi_0 + 1) \) such that
\[
(2.14)
\]
Hence we have \( Q' > 0 \) in \([0, \xi_1]\) since \( Q' = V^\varepsilon_+'' > 0 \) in \([0, \xi_0]\).

We now prove (2.12). For \( \xi \in [0, \xi_0) \), we have \( Q = V^\varepsilon_-(\xi) \). Using \( U^\varepsilon_-(\xi) \equiv 0 \) for \( \xi \geq 0 \) and the second equation of (2.8), it is straightforward to see that the inequality in (2.12) holds for \( \xi \in [0, \xi_0) \). For \( \xi \in (\xi_0, \xi_1] \), direct computation gives us
\[
Q'(\xi) = V^\varepsilon_-(\xi) - 2\delta(\xi - \xi_0)V^\varepsilon_-(\xi_0), \quad Q''(\xi) = V^\varepsilon_+''(\xi) - 2\delta V^\varepsilon_-(\xi_0).
\]
Hence
\[
c^\varepsilon_{\mu_1} Q' + Q'' + Q(1 - Q) = c^\varepsilon_{\mu_1} V^\varepsilon_-(\xi) - 2c^\varepsilon_{\mu_1}(\xi - \xi_0)V^\varepsilon_-(\xi_0) + V^\varepsilon_+''(\xi) - 2\delta V^\varepsilon_-(\xi_0) + \big[ V^\varepsilon_-(\xi) - 2\delta V^\varepsilon_-(\xi_0) \big] \big[ 1 - V^\varepsilon_-(\xi) + \delta(\xi - \xi_0)^2 V^\varepsilon_-(\xi_0) \big]
\]
Using \( 0 \leq \xi - \xi_0 \leq \xi_1 - \xi_0 < 1, \quad 1 > V^\varepsilon_-(\xi) > V^\varepsilon_-(\xi_0) \) for \( \xi \in [\xi_0, \xi_1] \), the identity
\[
c^\varepsilon_{\mu_1} V^\varepsilon_+'' + V^\varepsilon_+'' = -(1 - \varepsilon - V^\varepsilon_+) V^\varepsilon_-, \quad V^\varepsilon_-(\xi) - 2c^\varepsilon_{\mu_1}(\xi - \xi_0)V^\varepsilon_-(\xi_0) - 2\delta V^\varepsilon_-(\xi_0) - \delta^2 V^\varepsilon_+(\xi_0)
\]
and (2.11), we deduce
\[
c^\varepsilon_{\mu_1} Q' + Q'' + Q(1 - Q) \geq \varepsilon V^\varepsilon_-(\xi) - 2c^\varepsilon_{\mu_1}(\xi - \xi_0) - 2\delta V^\varepsilon_-(\xi_0) - \delta^2 V^\varepsilon_+(\xi_0)
\]
\[
\geq V^\varepsilon_-(\xi_0) - 2c^\varepsilon_{\mu_1}(\xi - \xi_0) - 4\delta
\]
\[
= 0 \quad \text{for} \quad \xi \in [\xi_0, \xi_1].
\]
To complete the proof, it remains to show the existence of \( s^\varepsilon \). Note that
\[
Q(\xi_0) = V^\varepsilon_-(\xi_0) \in [1 - 2\varepsilon, 1 - \varepsilon].
\]
By \[(2.13)\], we have
\[(2.15)\]
\[1 - 2\varepsilon \leq Q(\xi_0) \leq Q(\xi_1) \leq V_\varepsilon(\xi_1) \leq 1 - \varepsilon.
\]

By Lemma 2.4, \(q_s(-z(s))\) is a continuous and increasing function of \(s\) for \(s \in (0, s^*_{\mu_2})\), and \(q_s(-z(s)) \to 1\) as \(s \to s^*_{\mu_2}\). Therefore, in view of \[(2.15)\], for each small \(\varepsilon > 0\) there exists \(s^\varepsilon \in (0, s^*_{\mu_2})\) such that
\[Q(\xi_1) = q_{s^\varepsilon}(-z(s^\varepsilon)).\]
Moreover, \(s^\varepsilon \to s^*_{\mu_2}\) as \(\varepsilon \to 0\). Thus \[(2.13)\] holds. The proof of Lemma 2.4 is now complete. \(\square\)

We now consider the case \(\xi \leq 0\). We define
\[(2.16)\]
\[Q^\varepsilon(\xi) := \begin{cases} V_\varepsilon(\xi) & \text{for } \xi_2 \leq \xi \leq 0, \\ V_\varepsilon(\xi) + \gamma(\xi - \xi_2)V_\varepsilon(\xi_2) & \text{for } -\infty < \xi \leq \xi_2, \end{cases}\]
where
\[\gamma(\xi) = \gamma(\xi; \lambda) := -(e^{\lambda\xi} + e^{-\lambda\xi} - 2),\]
with \(\lambda > 0\) and \(\xi_2 < 0\) to be determined below.

**Lemma 2.4.** Let \(\varepsilon > 0\) be sufficiently small and \((U_\varepsilon, V_\varepsilon)\) be the solution of \[(2.8)\] with \(c = c^\varepsilon_{\mu_1}\). Then there exist \(\lambda = \lambda(\varepsilon) > 0\) sufficiently small and \(\xi_2 = \xi_2(\varepsilon) < 0\) such that \(V_\varepsilon(\xi_2) = Q^\varepsilon_-(\xi_2) < \varepsilon\) and
\[(2.17)\]
\[Q^\varepsilon_-(\xi) \in C^1((-\infty, 0]) \cap C^2((-\infty, 0]\{\xi_2\}), \quad (Q^\varepsilon_-)'(\xi) > 0 \text{ for all } \xi < 0.
\]
Moreover, there exists a unique \(\xi_3 \in (-\infty, \xi_2)\) depending on \(\xi_2\) and \(\lambda\) such that \(Q^\varepsilon_-(\xi_3) = 0\) and the following inequality holds:
\[(2.18)\]
\[c^\varepsilon_{\mu_1}(Q^\varepsilon_-)' + (Q^\varepsilon_-)'' + Q^\varepsilon_-(1 - Q^\varepsilon_- - hU_\varepsilon) \geq 0 \text{ for } \xi \in (\xi_3, 0)\{\xi_2\}.
\]

**Proof.** We write \(Q^\varepsilon_- = Q\) for convenience of notation. Using \(\gamma'(0) = 0\), it is straightforward to see that
\[Q \in C^1((-\infty, 0]) \cap C^2((-\infty, 0]\{\xi_2\})\]
for any choice of \(\xi_2 < 0\). Since \(V_\varepsilon' > 0\) in \(\mathbb{R}\) and \(\gamma'(\xi) > 0\) for \(\xi < 0\), we have
\[Q'(\xi) = \begin{cases} V_\varepsilon'(\xi) > 0 & \text{for } \xi_2 \leq \xi \leq 0, \\ V_\varepsilon'(\xi) + \gamma'(\xi - \xi_2)V_\varepsilon(\xi_2) > 0 & \text{for } \xi \leq \xi_2. \end{cases}\]
Hence \[(2.17)\] holds for any choice of \(\xi_2 < 0\).

For any given \(\lambda > 0\), we take \(K_\lambda > 0\) such that
\[(2.19)\]
\[e^{K_\lambda\xi} + e^{-K_\lambda\xi} - 2 > e^{-K_\lambda\mu},\]
where \(\mu > 0\) is given in \[(2.9)\]. By \[(2.9)\], we have
\[\frac{V_\varepsilon(\xi_2 - K_\lambda)}{V_\varepsilon(\xi_2)} \to e^{-K_\lambda\mu} \text{ as } \xi_2 \to -\infty.
\]
Together with (2.19), and \((U_\varepsilon, V_\varepsilon)(-\infty) = (1+\varepsilon, 0)\), we can take \(\xi_2 = \xi_2(\lambda)\) close to \(-\infty\) such that

\[
Q(\xi_2 - K_\lambda) = V_\varepsilon(\xi_2) \left[ \frac{V_\varepsilon(\xi_2 - K_\lambda)}{V_\varepsilon(\xi_2)} - (e^{K_\lambda \lambda} + e^{-K_\lambda \lambda} - 2) \right] < 0,
\]

\( (2.20) \)

\[
V_\varepsilon(\xi_2) < \min \left\{ \varepsilon, \frac{-\varepsilon}{-\gamma(-K_\lambda)} \right\}, \quad U_\varepsilon(\xi_2) > 1.
\]

On the other hand, since \(Q(\xi_2) = V_\varepsilon(\xi_2) > 0\), we can apply the intermediate value theorem to obtain \(\xi_3 \in (\xi_2 - K_\lambda, \xi_2)\) such that \(Q(\xi_3) = 0\). Such \(\xi_3\) is unique because of the monotonicity of \(Q\).

Next we show that, if \(\lambda > 0\) has been chosen small enough, with the above determined \(\xi_2\) and \(\xi_3\), (2.18) holds. To do this, we consider (2.18) for \(\xi \in (\xi_3, \xi_2)\) and \(\xi \in [\xi_2, 0)\) separately.

For \(\xi \in (\xi_3, \xi_2)\), we write \(V_\varepsilon = V_\varepsilon(\xi)\), \(\gamma = \gamma(\xi - \xi_2)\) and obtain

\[
c_{\mu_1}^{\varepsilon} Q' + Q'' + Q(1 - Q - hU_\varepsilon)
= c_{\mu_1}^{\varepsilon} V_\varepsilon' + V_\varepsilon'' + V_\varepsilon(\xi_2) \left[ c_{\mu_1}^{\varepsilon} \gamma' + \gamma'' \right] + V_\varepsilon(\xi_2) \left[ 1 - V_\varepsilon - \gamma V_\varepsilon(\xi_2) - hU_\varepsilon \right]
= -V_\varepsilon(1 - \varepsilon - V_\varepsilon - hU_\varepsilon) + V_\varepsilon(\xi_2) \left[ c_{\mu_1}^{\varepsilon} \gamma' + \gamma'' \right] + V_\varepsilon(\xi_2) \left[ 1 - V_\varepsilon - \gamma V_\varepsilon(\xi_2) - hU_\varepsilon \right]
\geq \varepsilon V_\varepsilon + V_\varepsilon(\xi_2) \left[ c_{\mu_1}^{\varepsilon} \gamma' + \gamma'' \right] - \gamma V_\varepsilon(\xi_2) \left[ hU_\varepsilon + \gamma V_\varepsilon(\xi_2) - 1 \right].
\]

By (2.20), for \(\xi \in (\xi_3, \xi_2)\),

\[
U_\varepsilon \geq 1, \quad 0 > \gamma(\xi - \xi_2)V_\varepsilon(\xi_2) \geq (1 - \varepsilon)V_\varepsilon(\xi_2) > -\varepsilon.
\]

It follows that

\[
(2.21)
\geq \varepsilon V_\varepsilon + V_\varepsilon(\xi_2) \left[ c_{\mu_1}^{\varepsilon} \gamma' + \gamma'' \right] - \gamma V_\varepsilon(\xi_2) \left[ h - \varepsilon - 1 \right] \quad \text{for} \quad \xi_3 < \xi < \xi_2.
\]

Using (2.9), we see that the right side of (2.21) is nonnegative if the following inequality holds:

\[
(2.22)
\varepsilon e^{\mu(\xi - \xi_3)} + c_{\mu_1}^{\varepsilon} \gamma' + \gamma'' - [h - \varepsilon - 1] \gamma > 0 \quad \text{for} \quad \xi_3 < \xi < \xi_2.
\]

We shall show that (2.22) indeed holds provided that \(\lambda > 0\) has been chosen small enough. To check this, for \(t = \xi_2 - \xi \geq 0\) we define

\[
F(t) := \varepsilon e^{-\mu t} - c_{\mu_1}^{\varepsilon} \lambda(e^{-\lambda t} - e^\lambda t) - \lambda^2(e^{-\lambda t} + e^\lambda t) + [h - \varepsilon - 1](e^{-\lambda t} + e^\lambda t - 2).
\]

By Lemma 2.5 below, we can take small \(\lambda\) depending only on \(\varepsilon\) such that \(F(t) > 0\) for all \(t \geq 0\). This implies (2.22), and so (2.18) holds for \(\xi \in (\xi_3, \xi_2)\).

For \(\xi \in (\xi_2, 0)\), we have \(Q'(\xi) = V_\varepsilon(\xi)\). From (2.8), it is straightforward to see that (2.18) holds for \(\xi \in (\xi_2, 0)\). This completes the proof.

\[
\square
\]

**Lemma 2.5.** Let \(\varepsilon > 0\) and \(F : [0, \infty) \to \mathbb{R}\) be defined by (2.23). Then \(F(t) > 0\) for all \(t \geq 0\) as long as \(\lambda > 0\) is small enough.
Proof. The argument is similar to [13, Lemma 3.3]. Let \( \kappa := h - \varepsilon - 1 \). Note that \( \kappa > 0 \) since \( h > 1 \). By direct computations,

\[
F'(t) = -\varepsilon \mu e^{-\mu t} + \lambda^2 c_{\mu_1}^\varepsilon (e^{\lambda t} + e^{-\lambda t}) + (\kappa \lambda - \lambda^3)(e^{\lambda t} - e^{-\lambda t}),
\]

\[
F''(t) = \varepsilon \mu^2 e^{-\mu t} + \lambda^3 c_{\mu_1}^\varepsilon (e^{\lambda t} - e^{-\lambda t}) + (\kappa \lambda^2 - \lambda^4)(e^{\lambda t} + e^{-\lambda t}),
\]

where \( \mu > 0 \) is given in (2.9). By taking \( \lambda \in \left(0, \min\left\{ \sqrt{\frac{\varepsilon \mu}{2}}, \sqrt{\frac{\varepsilon \mu}{2c_{\mu_1}^\varepsilon}}, \sqrt{\kappa} \right\} \right) \), we have \( F(0) > 0, F'(0) < 0, F'(\infty) = \infty \) and \( F''(t) > 0 \) for \( t \geq 0 \). If follows that \( F \) has a unique minimum point \( t = t_\lambda \). Consequently, to finish the proof of Lemma 2.5, it suffices to show the following:

(2.24) \[ F(t_\lambda) \geq 0 \] as long as \( \lambda > 0 \) is small.

By direct calculation, \( F'(t_\lambda) = 0 \) implies that

(2.25) \[ \varepsilon \mu e^{-\mu t_\lambda} = \lambda^2 c_{\mu_1}^\varepsilon (e^{\lambda t_\lambda} + e^{-\lambda t_\lambda}) + (\kappa \lambda - \lambda^3)(e^{\lambda t_\lambda} - e^{-\lambda t_\lambda}). \]

From (2.25), we easily deduce \( t_\lambda \to \infty \) as \( \lambda \to 0 \), for otherwise, the left hand side of (2.25) is bounded below by a positive constant while the right hand side converges to 0 as \( \lambda \to 0 \) along some sequence. Multiplying \( t_\lambda \) to both sides of (2.25) and we obtain, by a similar consideration, that \( \lambda t_\lambda \) is bounded from above by a positive constant as \( \lambda \to 0 \). It then follows that

\[ \kappa \lambda t_\lambda e^{\lambda t_\lambda} \to 0 \] as \( \lambda \to 0 \),

which implies \( \lambda t_\lambda \to 0 \) as \( \lambda \to 0 \). We thus obtain

(2.26) \[ t_\lambda \to \infty, \quad \lambda t_\lambda \to 0 \quad \text{as} \quad \lambda \to 0^+. \]

It follows that

(2.27) \[ \lim_{\lambda \to 0^+} \frac{e^{\lambda t_\lambda} - e^{-\lambda t_\lambda}}{2 \lambda t_\lambda} = \lim_{\lambda \to 0^+} \frac{e^{\lambda t_\lambda} + e^{-\lambda t_\lambda}}{2} = 1. \]

We now prove (2.24). Substituting (2.25) into \( F \), we have

\[
F(t_\lambda) = (e^{\lambda t_\lambda} - e^{-\lambda t_\lambda}) \left( \frac{c_{\mu_1}^\varepsilon \lambda}{\mu} - \frac{\lambda^3}{\mu} + \frac{\kappa}{\mu} \right) + \lambda^2 (e^{\lambda t_\lambda} + e^{-\lambda t_\lambda}) \left( \frac{c_{\mu_1}^\varepsilon}{\mu} - 1 \right) + \kappa (e^{\lambda t_\lambda} + e^{-\lambda t_\lambda} - 2)
\]

Using (2.26) and (2.27), for small \( \lambda > 0 \),

\[
F(t_\lambda) \geq 2 \lambda t_\lambda \left[ 1 + o(1) \right] \left( \frac{c_{\mu_1}^\varepsilon \lambda}{\mu} - \frac{\lambda^3}{\mu} + \frac{\kappa}{\mu} \lambda \right) + \lambda^2 \left[ 2 + o(1) \right] \left( \frac{c_{\mu_1}^\varepsilon}{\mu} - 1 \right)
\]

\[
= 2 \lambda^2 t_\lambda \left[ c_{\mu_1}^\varepsilon + \frac{\kappa}{\mu} + o(1) \right]
\]

\[
> 0.
\]

This completes the proof.
Combining (2.10) and (2.16) we now define
\[
\hat{Q}_\varepsilon(\xi) := \begin{cases} 
Q_\varepsilon^+(\xi) & \text{for } \xi \in [0, \xi_1], \\
Q_\varepsilon^-(\xi) & \text{for } \xi \in [\xi_3, 0], \\
0 & \text{for } \xi \in (-\infty, \xi_3], 
\end{cases}
\]
where \(\xi_1 > 0\) and \(\xi_3 < 0\) are given in Lemmas 2.3 and 2.4, respectively. We then define
\[
W_\varepsilon(r) := \begin{cases} 
0 & \text{for } r \in [\xi_1 + z(s_\varepsilon), \infty), \\
q_\varepsilon(r - \xi_1 - z(s_\varepsilon)) & \text{for } r \in [\xi_1, \xi_1 + z(s_\varepsilon)], \\
\hat{Q}_\varepsilon(r) & \text{for } r \in (-\infty, \xi_1], 
\end{cases}
\]
with \(s_\varepsilon \in (0, s_{\mu_2}^*)\) given in Lemma 2.3. Then clearly \(W_\varepsilon \in C(\mathbb{R})\) has compact support \([\xi_3, \xi_1 + z(s_\varepsilon)]\).

We are now ready to describe the conditions in Theorem 1 on the initial functions \(u_0\) and \(v_0\).

Since \(s_\varepsilon \to s_{\mu_2}^* > c_{\mu_1}^*\) and \(\xi_0(\varepsilon) \to \infty\) as \(\varepsilon \to 0\), where \(\xi_0(\varepsilon)\) is defined in Lemma 2.3, we can fix \(\varepsilon_0 > 0\) small so that
\[
s_\varepsilon - c_{\mu_1}^* > \frac{N - 1}{\xi_0(\varepsilon)} > 0 \quad \text{for all } \varepsilon \in (0, \varepsilon_0],
\]
where \(N\) is the space dimension. Our first condition is

\((B1)\): For some \(z_0 > 0\) and small \(\varepsilon_0 > 0\) as above,
\[
u_0(r) \leq U_{\varepsilon_0}(r - z_0), \quad \nu_0(r) \geq W_{\varepsilon_0}(r - z_0) \quad \text{for } r \geq 0.
\]
We note that \((B1)\) implies
\[
s_1^0 \leq z_0\quad \text{and} \quad s_2^0 \geq \xi_1(\varepsilon_0) + z(s_\varepsilon) + z_0.
\]

Our second condition is

\((B2)\): \(s_1^0 \geq R^* \sqrt{\frac{d}{r(1-k)}}\), where \(R^* > 0\) is defined in Corollary 1.

Since \(\limsup_{t \to \infty} v(r, t) \leq 1\) uniformly in \(r \in [0, s_2(t)]\), it is easy to see that \((B2)\) guarantees \(s_1, s_\infty = \lim_{t \to \infty} s_1(t) = \infty\) (see also the proof of Theorem 2).

We are now ready to prove Theorem 1, which we restate as

**Theorem 3.** Suppose that \((1.2), (1.7), (B1)\) and \((B2)\) hold. Then the solution \((u, v, s_1, s_2)\) of \((P)\) satisfies
\[
\lim_{t \to \infty} \frac{s_1(t)}{t} = c_{\mu_1}^*, \quad \lim_{t \to \infty} \frac{s_2(t)}{t} = s_{\mu_2}^*,
\]
and for every small \(\varepsilon > 0\), \((1.8), (1.9)\) hold.

Before giving the proof of Theorem 3, let us first observe how Corollary 1 follows easily from Theorem 3. It suffices to show that assumptions (i) and (ii) in Corollary 1 imply \((B1)\) and \((B2)\). Recall that
\[
U_{\varepsilon_0}(-\infty) = 1 + \varepsilon_0 \quad \text{and} \quad U_{\varepsilon_0}(\xi) = 0 \quad \text{for } \xi \geq 0.
\]
Therefore, for fixed $s_1^0 \geq R^* \sqrt{\frac{d}{r(1-k)}}$, there exists $C_1 > 0$ large such that
\[
U_{\xi_0}(r - z_0) \geq 1 \quad \text{for} \quad r \in [0, s_1^0], \quad z_0 \geq C_1.
\]
Hence for any given $u_0$ satisfying (i) of Corollary 1, (B2) and the first inequality in (B1) are satisfied if we take $z_0 \geq C_1$.

From the definition of $W_{\xi_0}$ we see that
\[
W_{\xi_0}(\xi) \leq 1 \quad \text{for} \quad \xi \in \mathbb{R}^1, \quad \text{and} \quad W_{\xi_0}(\xi) = 0 \quad \text{for} \quad \xi \notin [\xi_3, \xi_1 + z(s_{\xi_0})].
\]
If we take
\[
C_0 := \max \{C_1, \xi_1 + z(s_{\xi_0}) - \xi_3\},
\]
and for $x_0 \geq C_0$ and $L \geq C_0$, we let $z_0 := x_0 - \xi_3$, then
\[z_0 \geq x_0 \geq C_1, \quad [\xi_3, \xi_1 + z(s_{\xi_0})] \subset [x_0 - z_0, x_0 + L - z_0],
\]
and hence $W_{\xi_0}(r - z_0) = 0$ for $r \notin [x_0, x_0 + L]$. Thus when (ii) in Corollary 1 holds, we have
\[v_0(r) \geq W_{\xi_0}(r - z_0) \quad \text{for} \quad r \geq 0,
\]
which is the second inequality in (B1). This proves what we wanted.

**Proof of Theorem 3.** We break the rather long proof into 4 steps.

**Step 1:** We show
\[
\limsup_{t \to \infty} \frac{s_1(t)}{t} \leq c_{\mu_1} \varepsilon_0, \quad \liminf_{t \to \infty} \frac{s_2(t)}{t} \geq s_{\varepsilon_0} - \sigma_{\varepsilon_0},
\]
where
\[\sigma_\varepsilon := \frac{N - 1}{\xi_0(\varepsilon)} \quad \text{for} \quad \varepsilon \in (0, \varepsilon_0].
\]

By (2.29), we have
\[
c_{\mu_1}^\varepsilon < s_\varepsilon - \sigma_\varepsilon \quad \text{for} \quad \varepsilon \in (0, \varepsilon_0].
\]
We prove (2.30) by constructing suitable functions $(\overline{U}(r, t), \underline{V}(r, t), l(t), g(t))$ which satisfy certain differential inequalities that enable us to use a comparison argument to relate them to $(u(r, t), v(r, t), s_1(t), s_2(t))$. Set
\[l(t) := c_{\mu_1}^\varepsilon t + z_0, \quad g(t) := (s_{\varepsilon_0} - \sigma_{\varepsilon_0}) t + \xi_1 + z(s_{\varepsilon_0}) + z_0,
\]
\[\overline{U}(r, t) := U_{\xi_0}(r - l(t)) \quad \text{for} \quad r \in [0, l(t)], \quad t \geq 0,
\]
\[\underline{V}(r, t) := \begin{cases} 
\hat{Q}_{\varepsilon_0}(r - l(t)) & \text{for} \quad r \in [0, l(t) + \xi_1], \quad t \geq 0, \\
\hat{Q}_{\varepsilon_0}(\xi_1) & \text{for} \quad r \in [l(t) + \xi_1, g(t) - z(s_{\varepsilon_0})], \quad t \geq 0, \\
q_{s_{\varepsilon_0}}(r - g(t)) & \text{for} \quad r \in [g(t) - z(s_{\varepsilon_0}), g(t)], \quad t \geq 0,
\end{cases}
\]
where $\hat{Q}_{\varepsilon_0}$ is defined in (2.28) and $\xi_1 = \xi_1(\varepsilon_0)$ is given in Lemma 2.3. We note that
\[
\xi_1(\varepsilon_0) > \xi_0(\varepsilon_0) = \frac{N - 1}{\sigma_{\varepsilon_0}}.
\]
By the assumption (B1), we have
\[ (2.33) \quad u(r, 0) \leq U(r, 0) \text{ for } r \in [0, s_0^1]; \quad v(r, 0) \geq V(r, 0) \text{ for } r \in [0, s_0^2]. \]

We now show the wanted differential inequality for \( U \):
\[ (2.34) \quad U_t - dU_{rr} - \frac{N-1}{r} U_r - rU(1 - U - kV) \geq 0 \text{ for } 0 \leq r \leq l(t), \ t > 0. \]

Using \( U_r = U_{r0} < 0 \), direct computation gives us
\[ (2.35) \]
\[ \begin{align*}
J(r, t) := & \quad U_t - dU_{rr} - \frac{N-1}{r} U_r - rU(1 - U - kV) \\
\geq & \quad -c_{\mu_0} U_{r0}' - dU_{r0}'' - rU_{r0}(1 - U_{r0} - kV) \\
= & \quad rU_{r0}(1 + \varepsilon_0 - U_{r0} - kV_{r0}) - rU_{r0}(1 - U_{r0} - kV) \\
= & \quad rU_{r0}(\varepsilon_0 - kV_{r0} + kV) \\
\end{align*} \]

When \( U > 0 \), we have \( r \leq l(t) \) and so we can divide into two cases: when \( r - l(t) \in (\xi_2, 0) \), we have
\[ V(r, t) = \hat{Q}_{r0}(r - l(t)) = V_{r0}(r - l(t)). \]

Hence from (2.35) we see that \( J(r, t) > 0 \). When \( r - l(t) \in (-l(t), \xi_2) \), by Lemma 2.4 with \( \varepsilon = \varepsilon_0 \), we have
\[ kV_{r0}(r - l(t)) < V_{r0}(r - l(t)) < V_{r0}(\xi_2) < \varepsilon_0. \]

Again we obtain from (2.35) that \( J(r, t) > 0 \). Hence (2.34) holds.

We next show the wanted differential inequality for \( V_r \):
\[ (2.36) \quad V_r - V_{rr} - \frac{N-1}{r} V_r - V(1 - V - hU) \leq 0 \text{ for } 0 \leq r \leq g(t), \ t > 0. \]

We divide the proof into three parts.

(i) For \( r \in [0, l(t) + \xi_1] \), using \( V_{r}(r, t) = (\hat{Q}_{r0})'(r - l(t)) \geq 0 \), Lemma 2.3 and Lemma 2.4 with \( \varepsilon = \varepsilon_0 \),
\[ \begin{align*}
V_r - V_{rr} - \frac{N-1}{r} V_r - V(1 - V - hU) & \leq -\frac{N-1}{r} V_r \leq 0. \\
\end{align*} \]

(ii) For \( r \in [l(t) + \xi_1, g(t) - z(s_{r0})] \), we have \( U \equiv 0 \) and \( V \equiv \hat{Q}_{r0}(\xi_1) < 1 \). So clearly (2.36) holds.

(iii) For \( r \in (g(t) - z(s_{r0}), g(t)) \), we observe that \( r \geq g(t) - z(s_{r0}) \geq \xi_1 \). Also, by (2.32), we have
\[ \sigma_{r0} - \frac{N-1}{r} \geq \sigma_{r0} - \frac{N-1}{\xi_1} > 0. \]

Together with the fact that \( (q_{s0})'(r - g(t)) < 0 \) for \( r \in (g(t) - z(s_{r0}), g(t)) \) and \( t > 0 \), we have
\[ \begin{align*}
V_r - V_{rr} - \frac{N-1}{r} V_r - V(1 - V - hU) & = -g'(t)q_{s0}' - q_{s0}'' - \frac{N-1}{r} q_{s0}' - q_{s0}(1 - q_{s0}) \\
\leq & \quad \left( \sigma_{r0} - \frac{N-1}{r} \right) q_{s0}'(r - g(t)) \\
& \leq 0. \\
\end{align*} \]

We have thus proved (2.36).
In order to use the comparison principle to compare \((u, v, s_1, s_2)\) with \((\overline{U}, \overline{V}, l, g)\), we note that on the boundary \(r = 0\),
\begin{equation}
(2.37) \quad \overline{U}_r(0, t) < 0, \quad \overline{V}_r(0, t) > 0 \text{ for } t > 0.
\end{equation}
Regarding the free boundary conditions, we have
\begin{equation}
(2.38) \quad l'(t) = \frac{c_1^0}{\mu} = -\mu_1 U_r'(0) = -\mu_1 \overline{U}(l(t), t) \text{ for } t > 0,
\end{equation}
\begin{equation}
(2.39) \quad g'(t) = \frac{v^0}{\nu} - \sigma_0 = s^0_{\mu_2} = -\mu_2 \overline{V}(g(t), t) \text{ for } t > 0.
\end{equation}
By (2.33), (2.34), (2.36)-(2.39), we can apply the comparison principle ([33, Lemma 3.1] with minor modifications) to deduce that
\[
\text{s}_1(t) \leq l(t), \quad u(r, t) \leq \overline{U}(r, t) \text{ for } r \in [0, s_1(t)], \quad t > 0;
\]
\[
\text{s}_2(t) \geq g(t), \quad v(r, t) \geq \overline{V}(r, t) \text{ for } r \in [0, g(t)], \quad t > 0.
\]
In particular,
\[
\limsup_{t \to \infty} \frac{s_1(t)}{t} \leq \lim_{t \to \infty} \frac{l(t)}{t} = c_1^0, \quad \liminf_{t \to \infty} \frac{s_2(t)}{t} \geq \lim_{t \to \infty} \frac{g(t)}{t} = s^0_\nu - \sigma_0.
\]
We have thus proved (2.30).

**Step 2:** We refine the definitions of \((\overline{U}(r, t), \overline{V}(r, t), l(t), g(t))\) in Step 1 to obtain the improved estimates
\begin{equation}
(2.40) \quad \limsup_{t \to \infty} \frac{s_1(t)}{t} \leq c_1^*, \quad \liminf_{t \to \infty} \frac{s_2(t)}{t} \geq s_\mu^*.
\end{equation}
For any given \(\varepsilon \in (0, \varepsilon_0)\), we redefine \((l, g, \overline{U}, \overline{V})\) as
\[
l(t) = c_1^0 \mu_1 (t - T_\varepsilon) + z_1, \quad g(t) = (s^0 - \sigma_0)(t - T_\varepsilon) + \xi_1 + z(s^0) + z_1,
\]
\[
\overline{U}(r, t) = U_\varepsilon(r - l(t)) \text{ for } r \in [0, l(t)], \quad t \geq 0,
\]
\[
\overline{V}(r, t) = \begin{cases} 
\widehat{Q}^\varepsilon(r - l(t)) & \text{for } r \in [0, l(t) + \xi_1], \quad t \geq 0, \\
\widehat{Q}^\varepsilon(\xi_1) & \text{for } r \in [l(t) + \xi_1, g(t) - z(s^0)], \quad t \geq 0, \\
q_s^\varepsilon(r - g(t)) & \text{for } r \in [g(t) - z(s^0), g(t)], \quad t \geq 0, \\
0 & \text{for } r \in [g(t), \infty), \quad t \geq 0,
\end{cases}
\]
where \(\xi_1 = \xi_1(\varepsilon)\) is given in Lemma 2.3 and \(z_1, T_\varepsilon \gg 1\) are to be determined later.

We want to show that there exist \(z_1 \gg 1\) and \(T_\varepsilon \gg 1\) such that
\begin{equation}
(2.41) \quad u(r, T_\varepsilon) \leq \overline{U}(r, T_\varepsilon) \text{ for } r \in [0, s_1(T_\varepsilon)]; \quad v(r, T_\varepsilon) \geq \overline{V}(r, T_\varepsilon) \text{ for } r \in [0, s_2(T_\varepsilon)].
\end{equation}
Since \(\limsup_{t \to \infty} u(r, t) \leq 1\) uniformly in \(r\), there exists \(T_{1, \varepsilon}\) such that
\begin{equation}
(2.42) \quad u(r, t) \leq 1 + \varepsilon/2 \text{ for } r \in [0, s_1(t)] \text{ and } t \geq T_{1, \varepsilon}.
\end{equation}
By (2.31) and Lemma 2.2, we can find \(0 < \nu \ll 1\) and then \(T_{2, \varepsilon} \gg 1\) such that
\begin{equation}
(2.43) \quad c_1 := c_1^0 + \nu < s_\mu^0 - \sigma_\nu =: c_2,
\end{equation}
\begin{equation}
(2.44) \quad v(r, t) \geq 1 - \varepsilon \text{ for } r \in [c_1t, c_2t] \text{ and } t \geq T_{2, \varepsilon}.
\end{equation}
We now prove \((2.41)\) by making use of \((2.42)\) and \((2.44)\). By the definition of \(V(r, t)\), we see that \(\|V(\cdot, t)\|_{L^\infty} < 1 - \varepsilon\) for all \(t > 0\). Also, note that \(V(\cdot, T_\varepsilon) = W_\varepsilon(\cdot - z_1)\) has compact support \([\xi_3 + z_1, \xi_1 + z(s^\varepsilon) + z_1]\), whose length equals to \(\xi_1 - \xi_3 + z(s^\varepsilon)\) which is independent of the choice of \(T_\varepsilon\).

Next, we show the following claim: there exist \(z_1 \gg 1\) and \(T_\varepsilon \gg 1\) such that
\[
[\xi_3 + z_1, \xi_1 + z(s^\varepsilon) + z_1] \subset [c_1 T_\varepsilon, c_2 T_\varepsilon].
\]
Since \(U_\varepsilon(-\infty) = 1 + \varepsilon\), we can find \(T_{3,\varepsilon} \gg 1\) such that
\[
U_\varepsilon(r) > 1 + \frac{\varepsilon}{2} \text{ for } r \leq -T_{3,\varepsilon}.
\]
By \((2.39)\), we can find \(T_{4,\varepsilon} \gg 1\) so that
\[
s_1(t) \leq (c_{\varepsilon_1}^0 + \frac{\nu}{2})t = (c_1 - \frac{\nu}{2})t \text{ for } t \geq T_{4,\varepsilon}.
\]
We now take \(z_1 := c_1 T_\varepsilon - \xi_3\) with \(T_\varepsilon > \max\{T_{1,\varepsilon}, T_{2,\varepsilon}, T_{4,\varepsilon}\}\) chosen such that
\[
s_1(T_\varepsilon) - c_1 T_\varepsilon + \xi_3 < -\frac{\nu}{2} T_\varepsilon + \xi_3(\varepsilon) < -T_{3,\varepsilon},
\]
\[
\xi_1 + z(s^\varepsilon) + c_1 T_\varepsilon - \xi_3 < c_2 T_\varepsilon.
\]
It follows that
\[
[\xi_3 + z_1, \xi_1 + z(s^\varepsilon) + z_1] = [c_1 T_\varepsilon, \xi_1 + z(s^\varepsilon) + c_1 T_\varepsilon - \xi_3] \subset [c_1 T_\varepsilon, c_2 T_\varepsilon],
\]
and
\[
\mathcal{U}(r, T_\varepsilon) = U_\varepsilon(r - z_1) = U_\varepsilon(r - c_1 T_\varepsilon + \xi_3) \geq 1 + \frac{\varepsilon}{2} \text{ for } r \leq s_1(T_\varepsilon).
\]
Thus we may use \((2.42)\) and \((2.44)\) to obtain
\[
u(r, T_\varepsilon) \leq 1 + \frac{\varepsilon}{2} \leq \mathcal{U}(r, T_\varepsilon) \text{ for } r \in [0, s_1(T_\varepsilon)],
\]
\[
\mathbf{V}(\cdot, T_\varepsilon) < 1 - \varepsilon \leq v(\cdot, T_\varepsilon) \text{ for } r \in [0, s_2(T_\varepsilon)].
\]
We have thus proved \((2.41)\).

It is also easily seen that, with \(t > 0\) replaced by \(t > T_\varepsilon\) and \(\varepsilon_0\) replaced by \(\varepsilon\), the inequalities \((2.34)\) and \((2.36) - (2.39)\) still hold. Thus we are able to use the comparison principle as before to deduce
\[
s_1(t) \leq l(t), \quad u(r, t) \leq \mathcal{U}(r, t) \text{ for } r \in [0, s_1(t)), \quad t > T_\varepsilon;
\]
\[
s_2(t) \geq g(t), \quad v(r, t) \geq \mathbf{V}(r, t) \text{ for } r \in [0, g(t)), \quad t > T_\varepsilon.
\]
In particular,
\[
\limsup_{t \to \infty} \frac{s_1(t)}{t} \leq \lim_{t \to \infty} \frac{l(t)}{t} = c_{\mu_1}^0, \quad \liminf_{t \to \infty} \frac{s_2(t)}{t} \geq \lim_{t \to \infty} \frac{g(t)}{t} = s^\varepsilon - \sigma_\varepsilon.
\]
Since \(\varepsilon \in (0, \varepsilon_0)\) is arbitrary, taking \(\varepsilon \to 0\) we obtain \((2.40)\).
Step 3: We prove the following conclusions:

\[
\begin{align*}
\lim_{t \to \infty} \frac{s_1(t)}{t} &= c_{\mu_1}^*, \quad \lim_{t \to \infty} \left[ \max_{r \in [0, (c_{\mu_1}^* - \epsilon)t]} |u(r, t) - 1| \right] = 0. \\
\lim_{t \to \infty} \frac{s_2(t)}{t} &= s_{\mu_2}^*, \quad \lim_{t \to \infty} \left[ \max_{r \in [(c_{\mu_1}^* + \epsilon)t, (s_{\mu_2}^* - \epsilon)t]} |v(r, t) - 1| \right] = 0.
\end{align*}
\]

We note that for \( r \in [l(t) + \xi_1, g(t) - z(s^\epsilon)] \) and \( t > 0 \),

\[
V(r, t) = \hat{Q}^\epsilon(\xi_1) = V_\epsilon(\xi_1) - \delta(\xi_1 - \xi_0)^2 V_\epsilon(\xi_0) \\
\geq V_\epsilon(\xi_0) - \delta V_\epsilon(\xi_0) = (1 - \delta) V_\epsilon(\xi_0) \\
\geq (1 - \frac{\epsilon}{4})(1 - 2\epsilon).
\]

Thus for any given \( \epsilon > 0 \) we can choose \( \epsilon^* > 0 \) small enough so that for all \( \epsilon \in (0, \epsilon^*) \),

\[
V(r, t) \geq 1 - \epsilon \quad \text{for} \quad r \in [l(t) + \xi_1, g(t) - z(s^\epsilon)], \quad t > 0.
\]

In view of

\[
\lim_{\epsilon \to 0} c_{\mu_1}^\epsilon = c_{\mu_1}^*, \quad \lim_{\epsilon \to 0} s^\epsilon = s_{\mu_2}^*,
\]

and the inequality (2.31), by further shrinking \( \epsilon^* \) we may also assume that for all \( \epsilon \in (0, \epsilon^*) \),

\[
c_{\mu_1}^\epsilon < c_{\mu_1}^*, \quad s^\epsilon - \sigma_\epsilon > s_{\mu_2}^* - \frac{\epsilon}{2}.
\]

Hence for every \( \epsilon \in (0, \epsilon^*) \) we can find \( \tilde{T}_\epsilon \geq T_\epsilon \) such that

\[
[(c_{\mu_1}^* + \epsilon)t, (s_{\mu_2}^* - \epsilon)t] \subset [l(t) + \xi_1, g(t) - z(s^\epsilon)] \quad \text{for} \quad t \geq \tilde{T}_\epsilon.
\]

It follows that

\[
v(r, t) \geq V(r, t) \geq 1 - \epsilon \quad \text{for} \quad r \in [(c_{\mu_1}^* + \epsilon)t, (s_{\mu_2}^* - \epsilon)t], \quad t \geq \tilde{T}_\epsilon, \quad \epsilon \in (0, \epsilon^*],
\]

which implies

\[
\lim_{t \to \infty} \inf_{r \in [(c_{\mu_1}^* + \epsilon)t, (s_{\mu_2}^* - \epsilon)t]} \min_{r \in [(c_{\mu_1}^* + \epsilon)t, (s_{\mu_2}^* - \epsilon)t]} v(r, t) \geq 1.
\]

Next we obtain bounds for \((u, v, s_1, s_2)\) from the other side.

By comparison with an ODE upper solution,

\[
\lim_{t \to \infty} \sup_{r \to \infty} v(r, t) \leq 1 \quad \text{uniformly for} \quad r \in [0, \infty),
\]

which, combined with (2.47), yields

\[
\lim_{t \to \infty} \left[ \max_{r \in [(c_{\mu_1}^* + \epsilon)t, (s_{\mu_2}^* - \epsilon)t]} v(r, t) - 1 \right] = 0.
\]

This proves the second identity in (2.46).

As seen in the proof of Lemma 2.2, we have

\[
\lim_{t \to \infty} \sup_{r \to \infty} \frac{s_2(t)}{t} \leq s_{\mu_2}^*.
\]
Combining this with (2.40), we obtain the first identity in (2.46), namely
\[
\lim_{t \to \infty} \frac{s_2(t)}{t} = s_{\mu^2}^*.
\]

We next prove (2.45). Consider the problem (Q) with initial data in (1.3) chosen the following way: \( \hat{u}_0 = u_0, \ h_0 = s_0^1 \) and \( \hat{v}_0 \in C^2([0, \infty)) \cap L^\infty((0, \infty)) \) satisfies (1.5) and
\[
(2.48) \quad \hat{v}_0(r) \geq v_0(r) \quad \text{for} \ r \in [0, s_0^1].
\]
We denote its unique solution by \((\hat{u}, \hat{v}, h)\). Then by (B2) and Theorem 4.4 in [7], we have
\[
\lim_{t \to \infty} h(t) = \infty.
\]
Moreover, it follows from Theorem B that
\[
\lim_{t \to \infty} \frac{h(t)}{t} = c_{\mu_1}^*.
\]
Due to (2.48), we can apply the comparison principle ([33, Lemma 3.1] with minor modifications) to derive
\[
s_1(t) \geq h(t), \quad u(r, t) \geq \hat{u}(r, t) \quad \text{for} \ r \in [0, h(t)], \ t \geq 0,
\]
which in particular implies
\[
(2.49) \quad \liminf_{t \to \infty} \frac{s_1(t)}{t} \geq c_{\mu_1}^*.
\]
Moreover, by the definition of \( \psi_\delta \) and \( h(t) \) in [13], and the estimate
\[
\hat{u}(r, t) \geq \psi_\delta(h(t - T) - r) \quad \text{for} \ t > T, \ r \in [0, h(t - T)],
\]
we easily obtain the following conclusion:

For any given small \( \epsilon > 0 \), there exists \( \delta^* > 0 \) small such that for every \( \delta \in (0, \delta^*) \), there exists \( T_\delta^* > 0 \) large so that
\[
\hat{u}(r, t) \geq \psi_\delta(h(t - T) - r) \geq 1 - \epsilon \quad \text{for} \ r \in [0, (c_{\mu_1}^* - \epsilon)t], \ t \geq T_\delta^*.
\]
It follows that
\[
\liminf_{t \to \infty} \left[ \min_{r \in [0, (c_{\mu_1}^* - \epsilon)t]} \hat{u}(r, t) \right] \geq 1.
\]
Hence
\[
\liminf_{t \to \infty} \left[ \min_{r \in [0, (c_{\mu_1}^* - \epsilon)t]} u(r, t) \right] \geq 1.
\]
By comparison with an ODE upper solution, it is easily seen that
\[
\limsup_{t \to \infty} u(r, t) \leq 1 \quad \text{uniformly for} \ r \in [0, \infty).
\]
We thus obtain, for any small \( \epsilon > 0 \),
\[
\lim_{t \to \infty} \left[ \max_{r \in [0, (c_{\mu_1}^* - \epsilon)t]} |u(r, t) - 1| \right] = 0.
\]
This proves the second identity in (2.45).

Combining (2.49) and (2.40), we obtain the first identity in (2.45):
\[
\lim_{t \to \infty} \frac{s_1(t)}{t} = c_{\mu_1}^*.\]
Step 4: We complete the proof of Theorem 3 by finally showing that, for any small \( \epsilon > 0 \),
\[
(2.50) \quad \lim_{t \to \infty} \max_{r \in [0,(c_{p_1}^{-1} - \epsilon)t]} v(r, t) = 0.
\]

We prove this by making use of (2.45). Suppose by way of contradiction that (2.50) does not hold. Then for some \( \epsilon_0 > 0 \) small there exist \( \delta_0 > 0 \) and a sequence \( \{(r_k,t_k)\}_{k=1}^\infty \) such that
\[
\lim_{k \to \infty} t_k = \infty, \quad r_k \in [0,(c_{p_1}^{-1} - \epsilon_0)t_k], \quad v(r_k,t_k) \geq \delta_0 \quad \text{for all} \ k \geq 1.
\]

By passing to a subsequence, we have either (i) \( r_k \to r^* \in [0,\infty) \) or (ii) \( r_k \to \infty \) as \( k \to \infty \).

In case (i) we define
\[
v_k(r,t) := v(r,t + t_k), \quad u_k(r,t) := u(r,t + t_k) \quad \text{for} \ k \geq 1.
\]

Then
\[
\partial_t v_k = \Delta v_k + v_k(1 - v_k - h u_k) \quad \text{for} \ r \in [0,s_2(t + t_k)], \ t \geq -t_k.
\]

By (2.45), we have \( u_k \to 1 \) in \( L^\infty_{loc}([0,\infty) \times \mathbb{R}^1) \). Since \( v_k(1 - v_k - h u_k) \) has an \( L^\infty \) bound that is independent of \( k \), by standard parabolic regularity and a compactness consideration, we may assume, by passing to a subsequence involving a diagonal process, that
\[
v_k(r,t) \to v^*(r,t) \quad \text{in} \ C^{\frac{1 + \alpha}{2},\frac{1}{2}}_{loc}([0,\infty) \times \mathbb{R}^1), \ \alpha \in (0,1),
\]
and \( v^* \in W^{2,1}_{p,loc}([0,\infty) \times \mathbb{R}^1) \) \( (p > 1) \) is a solution of
\[
\begin{cases}
  v^*_t = \Delta v^* + v^*(1 - h - v^*) & \text{for} \ r \in [0,\infty), \ t \in \mathbb{R}^1, \\
  v^*_r(0,t) = 0 & \text{for} \ t \in \mathbb{R}^1.
\end{cases}
\]

Moreover, \( v^*(r^*,0) \geq \delta_0 \) and due to \( \limsup_{t \to \infty} v(r,t) \leq 1 \) we have \( v^*(r,t) \leq 1 \).

Fix \( R > 0 \) and let \( \hat{v}(r,t) \) be the unique solution of
\[
\begin{cases}
  \hat{v}_t = \Delta \hat{v} + \hat{v}(1 - h - \hat{v}) & \text{for} \ r \in [0,R), \ t > 0, \\
  \hat{v}_r(0,t) = 0, \ \hat{v}(R,t) = 1 & \text{for} \ t > 0, \\
  \hat{v}(r,0) = 1 & \text{for} \ r \in [0,R].
\end{cases}
\]

By the comparison principle we have, for any \( s > 0 \),
\[
0 \leq v^*(r,t) \leq \hat{v}(r,t + s) \quad \text{for} \ r \in [0,R], \ t \geq -s.
\]

On the other hand, by the well known properties of logistic type equations, we have
\[
\hat{v}(r,t) \to V_R(r) \quad \text{as} \ t \to \infty \quad \text{uniformly for} \ r \in [0,R],
\]

where \( V_R(r) \) is the unique solution to
\[
\Delta V_R + V_R(1 - h - V_R) \quad \text{in} \ [0,R]; \ V'_R(0) = 0, \ V_R(R) = 1.
\]

It follows that
\[
(2.51) \quad \delta_0 \leq v^*(r^*,0) \leq \lim_{s \to \infty} \hat{v}(r^*,s) = V_R(r^*).
\]
By Lemma 2.1 in [10], we have $V_R \leq V_{R'}$ in $[0, R']$ if $0 < R' < R$. Hence $V_\infty(r) := \lim_{R \to \infty} V_R(r)$ exists, and it is easily seen that $V_\infty$ is a nonnegative solution of
\[ \Delta V_\infty + V_\infty(1 - h - V_\infty) = 0 \text{ in } \mathbb{R}^1. \]
Since $1 - h < 0$, by Theorem 2.1 in [10], we have $V_\infty \equiv 0$. Hence $\lim_{R \to \infty} V_R(r) = 0$ for every $r \geq 0$.

We may now let $R \to \infty$ in (2.51) to obtain $\delta_0 \leq 0$. Thus we reach a contradiction in case (i).

In case (ii), $r_k \to \infty$ as $k \to \infty$, and we define
\[ v_k(r,t) := v(r + r_k, t + t_k), \quad u_k(r,t) := u(r + r_k, t + t_k) \text{ for } k \geq 1. \]

Since $r_k \leq (c_{\mu_1} - \epsilon_0) t_k$, by (2.45) we see that $u_k(r,t) \to 1$ in $L_{loc}^\infty(\mathbb{R}^1 \times \mathbb{R}^1)$. Then similarly, by passing to a subsequence, $v_k(r,t) \to \tilde{v}^*(r,t)$ in $C_{loc}^{1+\alpha, \frac{1+\alpha}{1+2\alpha}}(\mathbb{R}^1 \times \mathbb{R}^1)$, $\alpha \in (0, 1)$, and $\tilde{v}^* \in W^{2,1}_{p,loc}(\mathbb{R}^1 \times \mathbb{R}^1)$ ($p > 1$) is a solution of
\[ \tilde{v}^*_t = \tilde{v}^*_{rr} + \tilde{v}^*(1 - h - \tilde{v}^*) \text{ for } (r,t) \in \mathbb{R}^2. \]
Moreover, $\tilde{v}^*(0,0) \geq \delta_0$ and $\tilde{v}^*(r,t) \leq 1$. We may now compare $\tilde{v}^*$ with the one-dimensional version of $\hat{v}(r,t)$ used in case (i) to obtain a contradiction. We omit the details as they are just obvious modifications of the arguments in case (i).

As we arrive at a contradiction in both cases (i) and (ii), (2.50) must hold. The proof is now complete. \qed

3. Appendix

This section is divided into three subsections. In subsection 3.1, we establish the local existence and uniqueness of solutions for a rather general system including (P) as a special case. In subsection 3.2, we prove the global existence with some additional assumptions on the general system considered in subsection 3.1, but the resulting system is still much more general than (P). In the final subsection, we give the proof of Theorem 2.

3.1. Local existence and uniqueness. In this subsection, for possible future applications, we show the local existence and uniqueness of the solution to a more general system than (P). Our approach follows that in [15] with suitable changes, and in particular, we will fill in a gap in the argument of [15].

More precisely, we consider the following problem:
\[
\begin{aligned}
&u_t = d_1 \Delta u + f(r,t,u,v) \text{ for } 0 < r < s_1(t), \ t > 0, \\
v_t = d_2 \Delta v + g(r,t,u,v) \text{ for } 0 < r < s_2(t), \ t > 0, \\
u_r(0,t) = v_r(0,t) = 0 \text{ for } t > 0, \\
u \equiv 0 \text{ for } r \geq s_1(t) \text{ and } t > 0; \ v \equiv 0 \text{ for } r \geq s_2(t) \text{ and } t > 0, \\
s_1'(t) = -\mu_1 u_r(s_1(t),t), \ s_1^2(t) = -\mu_2 v_r(s_2(t),t) \text{ for } t > 0, \\
(s_1(0), s_2(0)) = (s_0^1, s_0^2), \ (u,v)(r,0) = (u_0, v_0)(r) \text{ for } r \in [0, \infty),
\end{aligned}
\]

\[ (3.1) \]
where \( r = |x| \), \( \Delta \varphi := \varphi_{rr} + \frac{(N-1)}{r} \varphi_r \), and the initial data satisfies (1.1). We assume that the nonlinear terms \( f \) and \( g \) satisfy

\[
(H1): \quad \begin{align*}
(\text{i}) & \quad f \text{ and } g \text{ are continuous in } r, t, u, v \in [0, \infty), \\
(\text{ii}) & \quad f(r, t, 0, v) = 0 = g(r, t, u, 0) \text{ for } r, t, u, v \geq 0, \\
(\text{iii}) & \quad \text{and } f \text{ and } g \text{ are locally Lipschitz continuous in } r, u, v \in [0, \infty), \\
& \quad \text{uniformly for } t \text{ in bounded subsets of } [0, \infty). 
\end{align*}
\]

We have the following local existence and uniqueness result for (3.1).

**Theorem 4.** Assume (H1) holds and \( \alpha \in (0, 1) \). Suppose for some \( M > 0 \),

\[
\|u_0\|_{C^2([0, s_0^1])} + \|v_0\|_{C^2([0, s_0^2])} + s_1^0 + s_2^0 \leq M.
\]

Then there exist \( T \in (0, 1) \) and \( \tilde{M} > 0 \) depending only on \( \alpha \), \( M \) and the local Lipschitz constants of \( f \) and \( g \) such that problem (3.1) has a unique solution

\[
(u, v, s_1, s_2) \in C^{1+\alpha,(1+\alpha)/2}(D_T^1) \times C^{1+\alpha,(1+\alpha)/2}(D_T^2) \times C^{1+\alpha/2}([0, T]) \times C^{1+\alpha/2}([0, T])
\]

satisfying

\[
(3.2) \quad \|u\|_{C^{1+\alpha,(1+\alpha)/2}(D_T^1)} + \|v\|_{C^{1+\alpha,(1+\alpha)/2}(D_T^2)} + \sum_{i=1}^2 \|s_i\|_{C^{1+\alpha/2}([0, T])} \leq \tilde{M},
\]

where \( D_T^i := \{(x, t) : 0 \leq x \leq s_i(t), \ 0 \leq t \leq T\} \) for \( i = 1, 2 \).

**Proof.** Firstly, for given \( T \in (0, 1) \), we introduce the function spaces

\[
\Sigma_T^i := \{s \in C^1([0, T]) : s(0) = s_0^i, \ s'(0) = s_1^i, \ 0 \leq s'(t) \leq s_1^i \leq s_2^i + 1, \ t \in [0, T]\}, \quad i = 1, 2,
\]

where

\[
s_1^* := -\mu_1 u_0'(s_0^1), \quad s_2^* := -\mu_2 v_0'(s_0^2).
\]

Clearly \( s(t) \geq s_1^* \) for \( t \in [0, T] \) if \( s \in \Sigma_T^i \).

For given \((\hat{s}_1, \hat{s}_2) \in \Sigma_T^1 \times \Sigma_T^2\), we introduce two corresponding function spaces

\[
X_T^1 = X_T^1(\hat{s}_1, \hat{s}_2) := \{u \in C([0, \infty) \times [0, T]) : u \equiv 0 \text{ for } r \geq \hat{s}_1(t), \ t \in [0, T], \ u(r, 0) \equiv u_0(r), \ \|u - u_0\|_{L^\infty([0, \infty) \times [0, T])} \leq 1\};
\]

\[
X_T^2 = X_T^2(\hat{s}_1, \hat{s}_2) := \{v \in C([0, \infty) \times [0, T]) : v \equiv 0 \text{ for } r \geq \hat{s}_2(t), \ t \in [0, T], \ v(r, 0) \equiv v_0(r), \ \|v - v_0\|_{L^\infty([0, \infty) \times [0, T])} \leq 1\}.
\]

We note that \( X_T^1 \) and \( X_T^2 \) are closed subsets of \( C([0, \infty) \times [0, T]) \) under the \( L^\infty([0, \infty) \times [0, T]) \) norm.
Given \((\hat{s}_1, \hat{s}_2) \in \Sigma^1_T \times \Sigma^2_T\) and \((\hat{u}, \hat{v}) \in X^1_T \times X^2_T\), we consider the following problem

\[
\begin{aligned}
& u_t = d_1 \Delta u + f(r, t, \hat{u}, \hat{v}) \text{ for } 0 < x < \hat{s}_1(t), \ 0 < t < T, \\
& v_t = d_2 \Delta v + g(r, t, \hat{u}, \hat{v}) \text{ for } 0 < x < \hat{s}_2(t), \ 0 < t < T, \\
& u_r(0, t) = v_r(0, t) = 0 \text{ for } 0 < t < T, \\
& u \equiv 0 \text{ for } r \geq \hat{s}_1(t) \text{ and } t > 0; \ v \equiv 0 \text{ for } r \geq \hat{s}_2(t) \text{ and } t > 0,
\end{aligned}
\]

(3.3)

Then and define

\[
\begin{aligned}
& (\hat{s}_1, \hat{s}_2)(0) = (s^0_1, s^0_2), \ (u, v)(r, 0) = (u_0, v_0)(r) \text{ for } r \in [0, \infty).
\end{aligned}
\]

To solve (3.3) for \(u\), we straighten the boundary \(r = \hat{s}_1(t)\) by the transformation \(R := r/\hat{s}_1(t)\) and define

\[
U(R, t) := u(r, t), \quad V(R, t) := v(r, t), \quad \hat{U}(R, t) := \hat{u}(r, t), \quad \hat{V}(R, t) := \hat{v}(r, t).
\]

Then \(U\) satisfies

\[
\begin{aligned}
& U_t = \frac{d_1 \Delta U}{(\hat{s}_1(t))^2} + \frac{\hat{s}_1'(t)R}{\hat{s}_1(t)} U_R + \hat{f}(R, t) \quad \text{for } R \in (0, 1), \ t \in (0, T), \\
& U_R(0, t) = U(1, t) = 0 \quad \text{for } t \in (0, T), \\
& U(R, 0) = U^0(R) := u_0(s^0_1R) \quad \text{for } R \in [0, 1],
\end{aligned}
\]

(3.4)

where

\[
\Delta U := U_{RR} + \frac{N - 1}{R} U_R, \quad \hat{f}(R, t) := f(\hat{s}_1(t)R, t, \hat{U}, \hat{V}).
\]

Since

\[
\begin{aligned}
& s^0_1 \leq \hat{s}_1(t) \leq s^*_1 + 1 \text{ for } t \in [0, T], \quad \text{and} \\
& \|\hat{s}_1'/\hat{s}_1\|_{L^\infty([0,T])} + \|\hat{f}\|_{L^\infty([0,\infty) \times [0,T])} < \infty,
\end{aligned}
\]

one can apply the standard parabolic \(L^p\) theory and the Sobolev embedding theorem (see [16, 23]) to deduce that (3.4) has a unique solution \(U \in C^{1+\alpha,(1+\alpha)/2}([0,1] \times [0, T])\) with

\[
\|U\|_{C^{1+\alpha,(1+\alpha)/2}([0,1] \times [0, T])} \leq C_1(\|\hat{f}\|_{\infty} + \|u_0\|_{C^2})
\]

for some \(C_1\) depending only on \(\alpha \in (0,1)\) and \(M\). It follows that \(u(r, t) = U(r/\hat{s}_1(t), t)\) satisfies

\[
\|u\|_{C^{1+\alpha,(1+\alpha)/2}(D^1_T)} \leq \tilde{C}_1(\|\hat{f}\|_{\infty} + \|u_0\|_{C^2})
\]

(3.5)

where \(\tilde{C}_1\) depends only on \(\alpha\) and \(M\), and

\[
D^1_T := \{(r, t) : r \in [0, \hat{s}_1(t)], \ t \in [0, T]\}.
\]

Similarly we can solve (3.3) to find a unique \(v \in C^{1+\alpha,(1+\alpha)/2}(D^2_T)\) satisfying

\[
\|v\|_{C^{1+\alpha,(1+\alpha)/2}(D^2_T)} \leq \tilde{C}_2(\|\hat{g}\|_{\infty} + \|v_0\|_{C^2}),
\]

(3.6)

where \(\tilde{C}_2\) depends only on \(\alpha\) and \(M\), and

\[
\begin{aligned}
& \hat{g}(R, t) = g(\hat{s}_2(t)R, t, \hat{U}, \hat{V}), \\
& D^2_T := \{(r, t) : r \in [0, \hat{s}_2(t)], \ t \in [0, T]\}.
\end{aligned}
\]
We now define a mapping $\mathcal{G}$ over $X^1_T \times X^2_T$ by

$$
\mathcal{G}(\hat{u}, \hat{v}) := (u, v),
$$

and show that $\mathcal{G}$ has a unique fixed point in $X^1_T \times X^2_T$ as long as $T \in (0, 1)$ is sufficiently small, by using the contraction mapping theorem.

For $R \in [0, 1]$ and $t \in [0, T]$,

$$
|U(R, t) - U(R, 0)| \leq T^{(1+\alpha)/2} \|U\|_{C^{1+\alpha, (1+\alpha)/2}([0,1] \times [0,T])} \leq C_1 T^{(1+\alpha)/2}(\|\hat{f}\|_{\infty} + \|u_0\|_{C^2}).
$$

It follows that

$$
\|u - u_0\|_{L^\infty([0,\infty) \times [0,T])} = \|U - U^0\|_{C([0,1] \times [0,T])} \leq C_1 T^{(1+\alpha)/2}(\|\hat{f}\|_{\infty} + \|u_0\|_{C^2}).
$$

Similarly,

$$
\|v - v_0\|_{L^\infty([0,\infty) \times [0,T])} \leq C_2 T^{(1+\alpha)/2}(\|\hat{g}\|_{\infty} + \|v_0\|_{C^2}).
$$

This implies that $\mathcal{G}$ maps $X^1_T \times X^2_T$ into itself for small $T \in (0, 1)$.

To see that $\mathcal{G}$ is a contraction mapping, we choose any $(\hat{u}_i, \hat{v}_i) \in X^1_T \times X^2_T$, $i = 1, 2$, and set

$$
\tilde{u} := \hat{u}_1 - \hat{u}_2, \quad \tilde{v} := \hat{v}_1 - \hat{v}_2.
$$

Then $(\tilde{u}, \tilde{v})$ satisfies

$$
\begin{cases}
\tilde{u}_t = d_1 \Delta \tilde{u} + f(r, t, \hat{u}_1, \hat{v}_1) - f(r, t, \hat{u}_2, \hat{v}_2) & \text{for } 0 < r < \hat{s}_1(t), \ 0 < t < T, \\
\tilde{v}_t = d_2 \Delta \tilde{v} + g(r, t, \hat{u}_1, \hat{v}_1) - g(r, t, \hat{u}_2, \hat{v}_2) & \text{for } 0 < r < \hat{s}_2(t), \ 0 < t < T, \\
\tilde{u}(0, t) = \tilde{v}(0, t) = 0 & \text{for } 0 < t < T, \\
\tilde{u} \equiv 0 & \text{for } r \geq \hat{s}_1(t), \ 0 < t < T; \quad \tilde{v} \equiv 0 & \text{for } r \geq \hat{s}_2(t), \ 0 < t < T, \\
(\tilde{u}, \tilde{v})(R, 0) = (0, 0), \ r \in [0, \infty).
\end{cases}
$$

By the Lipschitz continuity of $f$ and $g$, there exists $C_0 > 0$ such that for $r \in [0, \max\{\hat{s}_1(T), \hat{s}_1(T)\}]$ and $t \in [0, T]$,

$$
|f(r, t, \hat{u}_1, \hat{v}_1) - f(r, t, \hat{u}_2, \hat{v}_2)| \leq C_0 (|\hat{u}_1 - \hat{u}_2| + |\hat{v}_1 - \hat{v}_2|),
$$

$$
|g(r, t, \hat{u}_1, \hat{v}_1) - g(r, t, \hat{u}_2, \hat{v}_2)| \leq C_0 (|\hat{u}_1 - \hat{u}_2| + |\hat{v}_1 - \hat{v}_2|).
$$

We may then repeat the arguments leading to (3.5) and (3.6) to obtain

$$
\|\tilde{u}\|_{C^{1+\alpha, (1+\alpha)/2}(D^1_T)} + \|\tilde{v}\|_{C^{1+\alpha, (1+\alpha)/2}(D^2_T)} \leq C_2 (\|\hat{u}_1 - \hat{u}_2\|_{L^\infty([0,\infty) \times [0,T])} + \|\hat{v}_1 - \hat{v}_2\|_{L^\infty([0,\infty) \times [0,T])}),
$$

for some $C_2 = C_2(\alpha, M, C_0)$.

If we define

$$
\hat{U}(R, t) := \tilde{u}(\hat{s}_1(t)R, t), \quad \hat{V}(R, t) := \tilde{v}(\hat{s}_2(t)R, t),
$$

then $\hat{U}$ and $\hat{V}$ satisfy

$$
\begin{cases}
\alpha \hat{U}_t = \Delta \hat{U} + \hat{f}(\hat{U}, \hat{V}) & \text{in } D^1_T, \\
\alpha \hat{V}_t = \Delta \hat{V} + \hat{g}(\hat{U}, \hat{V}) & \text{in } D^2_T.
\end{cases}
$$

Here $\hat{f}$ and $\hat{g}$ are given by

$$
\hat{f}(u, v) := f(u, v) - f(\hat{u}, \hat{v}), \quad \hat{g}(u, v) := g(u, v) - g(\hat{u}, \hat{v}).
$$

Moreover, $(\hat{u}(R, t), \hat{v}(R, t))$ solves

$$
\begin{cases}
\alpha \hat{u}_t = \Delta \hat{u} + \hat{f}(\hat{u}, \hat{v}) & \text{in } D^1_T, \\
\alpha \hat{v}_t = \Delta \hat{v} + \hat{g}(\hat{u}, \hat{v}) & \text{in } D^2_T, \\
\hat{u} - \hat{V} & \text{on } \partial D^1_T, \\
\hat{v} - \hat{U} & \text{on } \partial D^2_T.
\end{cases}
$$

This completes the proof.
This implies that $G$.

Furthermore, from (3.5) and (3.6), we have

$$C \left( \parallel \tilde{u} \parallel_{C^{1+\alpha, (1+\alpha)/2}([0,1] \times [0,T])} + \parallel \tilde{v} \parallel_{C^{1+\alpha, (1+\alpha)/2}([0,1] \times [0,T])} \right)$$

for some $C = C(M)$. Hence from the above estimate for $\tilde{u}$ and $\tilde{v}$ we obtain

$$\parallel \tilde{U} \parallel_{C^{1+\alpha, (1+\alpha)/2}([0,1] \times [0,T])} + \parallel \tilde{V} \parallel_{C^{1+\alpha, (1+\alpha)/2}([0,1] \times [0,T])} \leq C_1 \left( \parallel \hat{u}_1 - \hat{u}_2 \parallel_{L^\infty([0,\infty) \times [0,T])} + \parallel \hat{v}_1 - \hat{v}_2 \parallel_{L^\infty([0,\infty) \times [0,T])} \right),$$

for some $C_1' = C_1'(\alpha, M, C_0)$. Since $\tilde{U}(R, 0) = \tilde{V}(R, 0) \equiv 0$, it follows that

$$\parallel \tilde{U} \parallel_{C([0,1] \times [0,T])} + \parallel \tilde{V} \parallel_{C([0,1] \times [0,T])} \leq C_2(T^{(1+\alpha)/2}) \left( \parallel \hat{u}_1 - \hat{u}_2 \parallel_{L^\infty([0,\infty) \times [0,T])} + \parallel \hat{v}_1 - \hat{v}_2 \parallel_{L^\infty([0,\infty) \times [0,T])} \right),$$

and hence

$$\parallel \tilde{u} \parallel_{L^\infty([0,\infty) \times [0,T])} + \parallel \tilde{v} \parallel_{L^\infty([0,\infty) \times [0,T])} \leq C_2'(T^{(1+\alpha)/2}) \left( \parallel \hat{u}_1 - \hat{u}_2 \parallel_{L^\infty([0,\infty) \times [0,T])} + \parallel \hat{v}_1 - \hat{v}_2 \parallel_{L^\infty([0,\infty) \times [0,T])} \right).$$

This implies that $G$ is a contraction mapping as long as $T \in (0, 1)$ is sufficiently small. By the contraction mapping theorem, $G$ has a unique fixed point in $X_T^1 \times X_T^2$, which we denote by $(\hat{u}, \hat{v})$.

Furthermore, from (3.5) and (3.6), we have

$$C \left( \parallel \tilde{u} \parallel_{C^{1+\alpha, (1+\alpha)/2}(D_T^1)} + \parallel \tilde{v} \parallel_{C^{1+\alpha, (1+\alpha)/2}(D_T^2)} \right) \leq \hat{C}_1,$$

for some $\hat{C}_1 = \hat{C}_1(\alpha, M, C_0)$.

For such $(\hat{u}, \hat{v})$, we introduce the mapping

$$\mathcal{F}(\tilde{s}_1, \tilde{s}_2) = \mathcal{F}(\tilde{s}_1, \tilde{s}_2; \hat{u}, \hat{v}) := (\tilde{s}_1, \tilde{s}_2)$$

with

$$\tilde{s}_1(t) = s_1^0 - \mu_1 \int_0^t \hat{u}_r(\tilde{s}_1(\tau), \tau) d\tau, \quad t \in [0, T];$$

$$\tilde{s}_2(t) = s_2^0 - \mu_2 \int_0^t \hat{v}_r(\tilde{s}_2(\tau), \tau) d\tau, \quad t \in [0, T].$$

Clearly

$$\tilde{s}'_1(t) = -\mu_1 \hat{u}_r(\tilde{s}_1(t), t) \geq 0, \quad \tilde{s}'_2(t) = -\mu_2 \hat{v}_r(\tilde{s}_2(t), t) \geq 0 \text{ for } t \in [0, T].$$

We shall again apply the contraction mapping theorem to deduce that $\mathcal{F}$ defined on $\Sigma_T^1 \times \Sigma_T^2$ has a unique fixed point. By (3.7) and (3.8), we see that $\tilde{s}'_i \in C^{\alpha/2}([0, T])$ with

$$\sum_{i=1}^{2} \parallel \tilde{s}'_i \parallel_{C^{\alpha/2}([0, T])} \leq (\mu_1 + \mu_2) \hat{C}_1.$$
It follows that
\[
\sum_{i=1}^{2} \| \hat{s}_i' - s_i' \|_{C([0,T])} \leq (\mu_1 + \mu_2) \hat{C}_1 T^{\alpha/2}.
\]
Hence \( \mathcal{F} \) maps \( \Sigma_T^1 \times \Sigma_T^2 \) into itself as long as \( T \in (0,1) \) is sufficiently small.

To show that \( \mathcal{F} \) is a contraction mapping, we let \((\hat{u}^s, \hat{v}^s)\) and \((\hat{u}^\sigma, \hat{v}^\sigma)\) be two fixed points of \( \mathcal{G} \) associated with \((\hat{s}_1, \hat{s}_2)\) and \((\hat{\sigma}_1, \hat{\sigma}_2)\) \( \in \Sigma_T^1 \times \Sigma_T^2 \), respectively; and for \( i = 1, 2 \), we denote \( D_{I_T}^i \) associated to \((\hat{s}_1, \hat{s}_2)\) and \((\hat{\sigma}_1, \hat{\sigma}_2)\) by, respectively
\[
D_{I_T}^{i,s} \text{ and } D_{I_T}^{i,\sigma}.
\]

Let us straighten \( r = \hat{s}_1(t) \) and \( r = \hat{\sigma}_1(t) \), respectively. To do so for \( r = \hat{s}_1(t) \), we define
\[
U^s(R, t) := \hat{u}^s(r, t), \quad V^s(R, t) := \hat{v}^s(r, t), \quad R = \frac{r}{\hat{s}_1(t)};
\]
then \( U^s \) satisfies
\[
\begin{align*}
U_t^s &= \frac{d_1 \Delta U^s}{(\hat{s}_1(t))^2} + \frac{\hat{s}_1'(t) R}{\hat{s}_1(t)} U_R^s + \hat{f}^s(R, t) \quad \text{for } R \in (0, 1), \ t \in (0, T), \\
U_R^s(0, t) &= U^s(1, t) = 0 \quad \text{for } t \in (0, T), \\
U^s(R, 0) &= u_0(s_1^0 R) \quad \text{for } R \in [0, 1],
\end{align*}
\]
where
\[
\hat{f}^s(R, t) := f(\hat{s}_1(t) R, t, U^s, V^s).
\]

Similarly we set
\[
U^\sigma(R, t) := \hat{u}^\sigma(r, t), \quad V^\sigma(R, t) := \hat{v}^\sigma(r, t), \quad R = \frac{r}{\hat{\sigma}_1(t)},
\]
and find that (3.10) holds with \((U^s, V^s, \hat{s}_1(t))\) replaced by \((U^\sigma, V^\sigma, \hat{\sigma}_1(t))\) everywhere.

Next we introduce
\[
\eta(t) := \hat{s}_1(t)/\hat{s}_2(t), \quad \xi(t) := \hat{\sigma}_1(t)/\hat{\sigma}_2(t), \\
P(R, t) := U^s(R, t) - U^\sigma(R, t), \quad Q(R, t) := V^s(R, t) - V^\sigma(R, t).
\]

By some simple computations, \( P \) satisfies
\[
\begin{align*}
P_t &= \frac{d_1 \Delta P}{(\hat{s}_1(t))^2} + \frac{\hat{s}_1'(t) R P_R}{\hat{s}_1(t)} + d_1 B_1(t) U_{RR}^s + R B_2(t) U_R^\sigma + F(R, t) \\
&\quad \text{for } R \in [0, 1], \ t \in [0, T], \\
P_R(0, t) &= P(1, t) = 0 \quad \text{for } t \in [0, T], \\
P(R, 0) &= 0 \quad \text{for } R \in [0, 1],
\end{align*}
\]
where
\[
B_1(t) := \frac{1}{(\hat{s}_1(t))^2} - \frac{1}{(\hat{\sigma}_1(t))^2}, \quad B_2(t) := \frac{\hat{s}_1'(t)}{\hat{s}_1(t)} - \frac{\hat{\sigma}_1'(t)}{\hat{\sigma}_1(t)};
\]
\[
F(R, t) := f(R \hat{s}_1(t), t, U^s, V^s) - f(R \hat{\sigma}_1(t), t, U^\sigma, V^\sigma).
\]
In view of (3.3),
\[
s_1'(t) = -\mu_1 \frac{U^\sigma_R(1,t)}{s_1(t)}, \quad \tilde{\sigma}_1'(t) = -\mu_1 \frac{U^\sigma_R(1,t)}{\tilde{\sigma}_1(t)},
\]
and hence
\[
\hat{s}_1'(t) - \hat{\sigma}_1'(t) = \frac{\mu_1}{s_1(t)} \hat{s}_1(t) - \frac{\mu_1}{\tilde{\sigma}_1(t)} \hat{\sigma}_1(t) + \frac{\mu_1 U^\sigma_R(1,t)}{s_1(t) \tilde{\sigma}_1(t)} \left[ \hat{s}_1(t) - \hat{\sigma}_1(t) \right].
\]

From now on, we will depart from the approach of [15] and fill in a gap which occurs in the argument there towards the proof that \( F \) is a contraction mapping.

It follows from the above identity that
\[
\| \hat{s}_1 - \hat{\sigma}_1 \|_{C^{1+\alpha/2}([0,T])} \leq C \left( \| U^\sigma_R(1,\cdot) - U^\sigma_R(1,\cdot) \|_{C^{1+\alpha}([0,T])} + \| \hat{s}_1 - \hat{\sigma}_1 \|_{C^{1+\alpha}([0,T])} \right),
\]
where \( C \) depends on \( \mu_1 \) and the upper bounds of \( \| \hat{s}_1 \|_{C^{1+\alpha/2}([0,T])} \), \( \| \hat{\sigma}_1 \|_{C^{1+\alpha/2}([0,T])} \) and \( \| U^\sigma_R(1,\cdot) \|_{C^{1+\alpha/2}([0,T])} \).

Hence \( C = C(\alpha, M, C_0) \).

Since \( T \leq 1 \), clearly
\[
\| \hat{s}_1 - \hat{\sigma}_1 \|_{C^{1+\alpha}([0,T])} \leq \| \hat{s}_1' - \hat{\sigma}_1' \|_{C([0,T])}.
\]
We also have
\[
\| U^\sigma_R(1,\cdot) - U^\sigma_R(1,\cdot) \|_{C^{1+\alpha}([0,T])} \leq \| P \|_{C^{1+\alpha,1+\alpha/2}([0,1] \times [0,T])}.
\]
We thus obtain
\[
\| \hat{s}_1' - \hat{\sigma}_1' \|_{C^{1+\alpha}([0,T])} \leq C \left( \| P \|_{C^{1+\alpha,1+\alpha/2}([0,1] \times [0,T])} + \| \hat{s}_1' - \hat{\sigma}_1' \|_{C([0,T])} \right).
\]

Applying the \( L^p \) estimate and the Sobolev embedding theorem to the problem (3.11), we obtain, for some \( p > 1 \),
\[
\| P \|_{C^{1+\alpha,1+\alpha/2}([0,1] \times [0,T])} \leq M_4 \left( \| B_1 \|_{C([0,T])} \| U^\sigma_R \|_{L^p([0,1] \times [0,T])} \right.
\]
\[
+ \| B_2 \|_{C([0,T])} \| U^\sigma_R \|_{L^p([0,1] \times [0,T])} + \| F \|_{L^p([0,1] \times [0,T])} \right)
\]
for some \( M_4 > 0 \) depending only on \( \alpha \) and \( M \). Due to the \( W^{1,1}_p([0,1] \times [0,T]) \) bound for \( U^\sigma \), we hence obtain
\[
\| P \|_{C^{1+\alpha,1+\alpha/2}([0,1] \times [0,T])} \leq M_5 \left( \| B_1 \|_{C([0,T])} + \| B_2 \|_{C([0,T])} + \| F \|_{L^p([0,1] \times [0,T])} \right)
\]
for some \( M_5 > 0 \) depending only on \( \alpha \), \( M \) and the Lipschitz constant of \( f \).

By the definitions of \( B_1(t) \), \( B_2(t) \) and \( F(R, t) \), we have
\[
\| B_1 \|_{C([0,T])} \leq C \| \hat{s}_1 - \hat{\sigma}_1 \|_{C([0,T])},
\]
\[
\| B_2 \|_{C([0,T])} \leq C \left( \| \hat{s}_1 - \hat{\sigma}_1 \|_{C([0,T])} + \| \hat{s}_1' - \hat{\sigma}_1' \|_{C([0,T])} \right),
\]
and
\[
\| F \|_{L^p([0,1] \times [0,T])} \leq C \left( \| P \|_{C([0,1] \times [0,T])} + \| Q \|_{C([0,1] \times [0,T])} + \| \hat{s}_1 - \hat{\sigma}_1 \|_{C([0,T])} \right).
\]
for some $C > 0$ depending only on $M$ and the Lipschitz constants of $f$.

We next estimate $\|P\|_{C([0,1] \times [0,T])}$ and $\|Q\|_{C([0,1] \times [0,T])}$ by using the estimate in Lemma 2.2 of [15], namely

$$\|\hat{u}^s - \hat{u}^\sigma\|_{C(\Gamma^1_T)} + \|\hat{v}^s - \hat{v}^\sigma\|_{C(\Gamma^2_T)} \leq C \sum_{i=1}^2 \|\hat{s}_i - \hat{\sigma}_i\|_{C([0,T])},$$

where

$$\Gamma^i_T := \{(r,t) : 0 \leq r \leq \min\{\hat{s}_i(t), \hat{\sigma}_i(t)\}\}, \ i = 1, 2.$$  

Without loss of generality, we may assume $\hat{s}_i(t) \leq \hat{\sigma}_1(t)$. Then for any $R \in [0,1]$ and $t \in [0,T]$, we have

$$|P(R,t)| = |\hat{u}^s(R\hat{s}_1(t),t) - \hat{u}^\sigma(R\hat{\sigma}_1(t),t)|$$

$$\leq |\hat{u}^s(R\hat{s}_1(t),t) - \hat{u}^\sigma(R\hat{\sigma}_1(t),t)| + |\hat{u}^\sigma(R\hat{s}_1(t),t) - \hat{u}^\sigma(R\hat{\sigma}_1(t),t)|$$

$$\leq C \sum_{i=1}^2 \|\hat{s}_i - \hat{\sigma}_i\|_{C([0,T])} + \|\hat{u}^\sigma\|_{C^{1+\alpha,1+\alpha/2}(D^1_T)} \|\hat{s}_1 - \hat{\sigma}_1\|_{C([0,T])}$$

$$\leq \tilde{C} \sum_{i=1}^2 \|\hat{s}_i - \hat{\sigma}_i\|_{C([0,T])}.$$  

It follows that

$$\|P\|_{C([0,1] \times [0,T])} \leq \tilde{C} \sum_{i=1}^2 \|\hat{s}_i - \hat{\sigma}_i\|_{C([0,T])}.$$  

For any $R \in [0,1]$ and $t \in [0,T]$, we have

$$|Q(R,t)| = |\hat{v}^s(R\hat{s}_1(t),t) - \hat{v}^\sigma(R\hat{\sigma}_1(t),t)|$$

$$= |\hat{v}^s(R\hat{s}_2(t),t) - \hat{v}^\sigma(R\hat{\sigma}_2(t),t)|.$$  

We now consider all the possible cases:

(i) If $R\eta(t) \geq 1$ and $R\xi(t) \geq 1$, then we immediately obtain

$$|Q(R,t)| = 0.$$  

(ii) If $R\eta(t) < 1$ and $R\xi(t) < 1$, assuming without loss of generality $\hat{s}_2(t) \leq \hat{\sigma}_2(t)$, then

$$|Q(R,t)| = |\hat{v}^s(R\eta(t)\hat{s}_2(t),t) - \hat{v}^\sigma(R\xi(t)\hat{\sigma}_2(t),t)|$$

$$\leq |\hat{v}^s(R\eta(t)\hat{s}_2(t),t) - \hat{v}^\sigma(R\eta(t)\hat{s}_2(t),t)|$$

$$+ |\hat{v}^\sigma(R\eta(t)\hat{s}_2(t),t) - \hat{v}^\sigma(R\xi(t)\hat{\sigma}_2(t),t)|$$

$$\leq C \sum_{i=1}^2 \|\hat{s}_i - \hat{\sigma}_i\|_{C([0,T])}$$

$$+ \|\hat{v}^\sigma\|_{C^{1+\alpha,1+\alpha/2}(D^2_T)} \|\hat{s}_1 - \hat{\sigma}_1\|_{C([0,T])}$$

$$\leq \tilde{C} \sum_{i=1}^2 \|\hat{s}_i - \hat{\sigma}_i\|_{C([0,T])}.$$
(iii) If $R\eta(t) < 1 \leq R\xi(t)$ and $R\eta(t)\dot{s}_2(t) \leq \dot{\sigma}_2(t)$, then

$$|Q(R, t)| = |\dot{s}^\alpha(R\eta(t)\dot{s}_2(t), t) - \dot{s}^\alpha(R\xi(t)\dot{\sigma}_2(t), t)|$$

$$\leq |\dot{s}^\alpha(R\eta(t)\dot{s}_2(t), t) - \dot{s}^\alpha(R\eta(t)\dot{s}_2(t), t)|$$

$$+ |\dot{s}^\alpha(R\eta(t)\dot{s}_2(t), t) - \dot{s}^\alpha(\dot{\sigma}_2(t), t)|$$

$$\leq C \sum_{i=1}^{2} \|\dot{s}_i - \dot{\sigma}_i\|_{C([0, T])}$$

$$+ \|\dot{s}^\sigma\|_{C^1([0, T])} |\eta(t)\dot{s}_2(t) - \dot{\sigma}_2(t)|.$$

From $R\eta(t) < 1 \leq R\xi(t)$ and $R\eta(t)\dot{s}_2(t) \leq \dot{\sigma}_2(t)$ we obtain

$$|R\eta(t)\dot{s}_2(t) - \dot{\sigma}_2(t)| \leq \left| \eta(t) \right| \dot{s}_2(t) - \dot{\sigma}_2(t) |$$

$$\leq \|1/\xi\|_{C([0, T])} |\eta(t)\dot{s}_2(t) - \dot{\sigma}_2(t)|$$

$$= \|1/\xi\|_{C([0, T])} |\dot{s}_1(t) - \dot{\sigma}_1(t)|.$$

Thus in this case we also have

$$|Q(R, t)| \leq \tilde{C} \sum_{i=1}^{2} \|\dot{s}_i - \dot{\sigma}_i\|_{C([0, T])}.$$

(iv) If $R\eta(t) < 1 \leq R\xi(t)$ and $R\eta(t)\dot{s}_2(t) > \dot{\sigma}_2(t)$, then

$$|R\eta(t)\dot{s}_2(t) - \dot{\sigma}_2(t)| \leq |\dot{\sigma}_2(t) - \dot{s}_2(t)|$$

and

$$|Q(R, t)| = |\dot{s}^\alpha(R\eta(t)\dot{s}_2(t), t)|$$

$$\leq \|\dot{s}^\sigma\|_{C^1([0, T])} |R\eta(t)\dot{s}_2(t) - \dot{\sigma}_2(t)|$$

$$\leq \|\dot{s}^\sigma\|_{C^1([0, T])} \|\dot{s}_2(t) - \dot{\sigma}_2(t)|$$

$$\leq 3 \sum_{i=1}^{2} \|\dot{s}_i - \dot{\sigma}_i\|_{C([0, T])}.$$

(v) If $R\eta(t) \geq 1 > R\xi(t)$, we are in a symmetric situation to cases (iii) and (iv) above, so we similarly obtain

$$|Q(R, t)| \leq \tilde{C} \sum_{i=1}^{2} \|\dot{s}_i - \dot{\sigma}_i\|_{C([0, T])}.$$

Thus in all the possible cases (3.17) always holds. It follows that

$$\|Q\|_{C([0,1] \times [0, T])} \leq \tilde{C} \sum_{i=1}^{2} \|\dot{s}_i - \dot{\sigma}_i\|_{C([0, T])}.$$
We thus obtain from (3.16) that

$$\|F\|_{L^p([0,1] \times [0,T])} \leq \tilde{C} \sum_{i=1}^{2} \|\hat{s}_i - \hat{\sigma}_i\|_{C([0,T])}. $$

(3.18)

We may now substitute (3.14), (3.15) and (3.18) into (3.13) to obtain

$$\|P\|_{C^{1+\alpha,(1+\alpha)/2}([0,1] \times [0,T])} \leq \tilde{C}' \left( \|\hat{s}_1' - \hat{\sigma}_1'\|_{C([0,T])} + \sum_{i=1}^{2} \|\hat{s}_i - \hat{\sigma}_i\|_{C([0,T])} \right).$$

It thus follows from (3.12) that

$$\|\hat{s}_1' - \hat{\sigma}_1'\|_{C^{1+\alpha,(1+\alpha)/2}([0,1] \times [0,T])} \leq \tilde{C}' \left( \|\hat{s}_1' - \hat{\sigma}_1'\|_{C([0,T])} + \sum_{i=1}^{2} \|\hat{s}_i - \hat{\sigma}_i\|_{C([0,T])} \right).$$

Since \(\hat{s}_1'(0) - \hat{\sigma}_1'(0) = 0\), this implies

$$\|\hat{s}_1' - \hat{\sigma}_1'\|_{C([0,T])} \leq T^{1+\alpha} \tilde{C}' \left( \|\hat{s}_1' - \hat{\sigma}_1'\|_{C([0,T])} + \sum_{i=1}^{2} \|\hat{s}_i - \hat{\sigma}_i\|_{C([0,T])} \right).$$

Hence for \(T > 0\) sufficiently small we have

$$\|\hat{s}_1' - \hat{\sigma}_1'\|_{C([0,T])} \leq T^{1+\alpha} \tilde{C}_1 \sum_{i=1}^{2} \|\hat{s}_i - \hat{\sigma}_i\|_{C([0,T])} $$

(3.19)

with \(\tilde{C}_1 > 0\) depending only on \(\alpha, M\) and the Lipschitz constant of \(f\).

In a similar manner, we can straighten \(r = \hat{s}_2(t)\) and \(r = \hat{\sigma}_2(t)\) to obtain

$$\|\hat{s}_2' - \hat{\sigma}_2'\|_{C([0,T])} \leq T^{1+\alpha} \tilde{C}_2 \sum_{i=1}^{2} \|\hat{s}_i - \hat{\sigma}_i\|_{C([0,T])} $$

(3.20)

with \(\tilde{C}_2 > 0\) depending only on \(\alpha, M\) and the Lipschitz constant of \(g\).

Finally, using \(\hat{s}_i(0) = \hat{\sigma}_i(0) = s_i^0, i = 1, 2\), we see that

$$\|\hat{s}_i - \hat{\sigma}_i\|_{C([0,T])} \leq T \|\hat{s}_i' - \hat{\sigma}_i'\|_{C([0,T])}, \ i = 1, 2.$$ 

(3.21)

Combining (3.19), (3.20) and (3.21), we see that \(\mathcal{F}\) is a contraction mapping as long as \(T > 0\) is sufficiently small. Hence \(\mathcal{F}\) has a unique fixed point \((s_1, s_2) \in \Sigma^1_T \times \Sigma^2_T\) for such \(T\).

Let \((u, v)\) be the unique fixed point of \(\mathcal{G}\) in \(X^1_T(s_1, s_2) \times X^2_T(s_1, s_2)\); then it is easily seen that \((u, v, s_1, s_2)\) is the unique solution of (3.1). Furthermore, (3.2) holds because of (3.7) and (3.9). We have now completed the proof of Theorem 4. \(\square\)

By Theorem 4 and the Schauder estimate, we see that the solution of (P) defined for \(t \in [0, T]\) is actually a classical solution.

3.2. Global existence. In this subsection, we show that the unique local solution of (3.1) can be extended to all positive time if the following extra assumption is imposed:

(H2) There exists a positive constant \(K\) such that \(f(r, t, u, v) \leq K(u + v)\) and \(g(r, t, u, v) \leq K(u + v)\) for \(r, t, u, v \geq 0\).
Theorem 5. Under the assumptions of Theorem 4 and (H2), problem (3.1) has a unique globally in time solution.

Proof. The proof is similar to that of [7, Theorem 2.4]. For the reader’s convenience, we present a brief proof. Let \([0, T^*)\) be the largest time interval for which the unique solution of (3.1) exists. By Theorem 4, \(T^* > 0\). By the strong maximum principle, we see that \(u(r, t) > 0\) in \([0, s_1(t)) \times [0, T^*)\) and \(v(r, t) > 0\) in \([0, s_2(t)) \times [0, T^*)\). We will show that \(T^* = \infty\). Aiming for a contradiction, we assume that \(T^* < \infty\). Consider the following ODEs

\[
\begin{align*}
dU/dt &= K(U + V), \quad t > 0, \quad U(0) = \|u_0\|_{L^\infty([0, s_0])}, \\
dV/dt &= K(U + V), \quad t > 0, \quad V(0) = \|v_0\|_{L^\infty([0, s_0])}.
\end{align*}
\]

Take \(M^* > T^*\). Clearly,

\[
0 < U(t) + V(t) < \left(\|u_0\|_{L^\infty([0, s_0])} + \|v_0\|_{L^\infty([0, s_0])}\right)e^{2KM^*} := C_1, \quad t \in [0, T^*)
\]

By (H2), we can compare \((u, v)\) with \((U, V)\) to obtain

\[
\|u\|_{L^\infty([0, s_1(t)] \times [0, T^*)]} + \|v\|_{L^\infty([0, s_2(t)] \times [0, T^*)]} \leq C_1.
\]

Next, we can use a similar argument as in [6, Lemma 2.2] to derive

\[
0 < s_i'(t) \leq C_2, \quad t \in (0, T^*), \quad i = 1, 2
\]

for some \(C_2\) independent of \(T^*\). Furthermore, we have

\[
s_i^0 \leq s_i(t) \leq s_i^0 + C_2 t \leq s_i^0 + C_2 M^*, \quad t \in [0, T^*), \quad i = 1, 2.
\]

Taking \(\epsilon \in (0, T^*)\), by standard parabolic regularity, there exists \(C_3 > 0\) depending only on \(K, M^*, C_1\) and \(C_2\) such that

\[
\|u(\cdot, t)\|_{C^2([0, s_1(t)])} + \|v(\cdot, t)\|_{C^2([0, s_2(t)])} \leq C_3, \quad t \in [\epsilon, T^*).
\]

By Theorem 4 there exists \(\tau > 0\) depending only on \(K, M^*\) and \(C_i (i = 1, 2, 3)\) such that the solution of problem (3.1) with initial time \(T^* - \tau/2\) can be extended uniquely to the time \(T^* + \tau/2\), which contradicts the definition of \(T^*\). This completes the proof of Theorem 5. \(\square\)

3.3. Proof of Theorem 2.

Proof of Theorem 2. Define

\[
s_* = R^* \sqrt{d/r}, \quad s^* = R^* \sqrt{\frac{d}{r} \frac{1}{\sqrt{1 - k}}}, \quad s^{**} = R^*.
\]

First, following the same lines in [15, Theorem 2] with some minor changes, we can prove the following three results:

(i) If \(s_{1, \infty} \leq s_*\), then \(u\) vanishes eventually. In this case, \(v\) spreads successfully (resp. vanishes eventually) if \(s_{2, \infty} > s^{**}\) (resp. \(s_{2, \infty} \leq s^{**}\)),

(ii) If \(s_* < s_{1, \infty} \leq s^*\), then \(u\) vanishes eventually, and \(v\) spreads successfully.
(iii) If $s_{1,\infty} > s^*$, then $u$ spreads successfully.

Next, we shall show

(iv) If $s_{1,\infty} > s^*$ and $(\mu_1, \mu_2) \in \mathcal{B}$, then $u$ spreads successfully and $v$ vanishes eventually.

By a simple comparison consideration we see that

$$
\limsup_{t \to \infty} \frac{s_2(t)}{t} \leq s^*_{\mu_2}.
$$

By (iii), we see that $u$ spreads successfully and so $s_{1,\infty} = \infty$. It follows that there exists $T \gg 1$ such that

$$
s_1(T) \geq R^* \sqrt{\frac{d}{r} \frac{1}{\sqrt{1-k}}}.
$$

This allows us to use a similar argument to that leading to (2.49) but taking $T$ as the initial time to obtain

$$
\liminf_{t \to \infty} \frac{s_1(t)}{t} \geq c^*_{\mu_1}.
$$

To show that $s_{2,\infty} < \infty$ we argue by contradiction and assume $s_{2,\infty} = \infty$. Since $(\mu_1, \mu_2) \in \mathcal{B}$, from (3.23) and (3.22) we can find $\tau \gg 1$ and $\hat{c}$ such that

$$
s^*_{\mu_2} < \hat{c} < c^*_{\mu_1}, \quad s_2(t) < \hat{c} t < s_1(t) \quad \text{for all } t \geq \tau.
$$

Then by the same process used in deriving the second identity in (2.45), we have

$$
\lim_{t \to \infty} \left[ \max_{r \in [0, \hat{c} t]} |u(r, t) - 1| \right] = 0.
$$

Also, noting $h > 1$ and $s_2(t) < \hat{c} t$, there exist $\tilde{\tau} > \tau$ such that

$$
v_t = \Delta v + v(1 - v - hu) \leq \Delta v, \quad 0 < r < s_2(t), \quad t \geq \tilde{\tau},
$$

which leads to $s_{2,\infty} < \infty$ by simple comparison (cf. [15, Theorem 3]). This reaches a contradiction. Hence we have proved $s_{2,\infty} < \infty$. Finally, using $s_{2,\infty} < \infty$ we can show $\lim_{t \to \infty} \|v(\cdot, t)\|_{C([0, s_2(t)])} = 0$ (cf. [15, Lemma 3.4]). Hence $v$ vanishes eventually and then (iv) follows.

The conclusions of Theorem 2 follow easily from (i)-(iv). □

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References

[1] G. Bunting, Y. Du, K. Krakowski, Spreading speed revisited: Analysis of a free boundary model, Netw. Heterog. Media, 7 (2012), 583–603.

[2] Y. Du, Z. M. Guo, Spreading-vanishing dichotomy in a diffusive logistic model with a free boundary II, J. Diff. Equns., 250 (2011), 4336–4366.

[3] Y. Du, Z.M. Guo, The Stefan problem for the Fisher-KPP equation, J. Diff. Equns., 253 (2012), 996–1035.

[4] Y. Du, Z. M. Guo, R. Peng, A diffusive logistic model with a free boundary in time-periodic environment, J. Funct. Anal., 265 (2013), 2089–2142.

[5] Y. Du, X. Liang, Pulsating semi-waves in periodic media and spreading speed determined by a free boundary model, Ann. Inst. H. Poincar’e Anal. Non Lin’eaire., 32 (2015), 279–305.

[6] Y. Du, Z.G. Lin, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, SIAM J. Math. Anal., 42 (2010), 377–405.

[7] Y. Du, Z.G. Lin, The diffusive competition model with a free boundary: Invasion of a superior or inferior competitor, Discrete Cont. Dyn. Syst. (Ser. B), 19 (2014), 3105–3132.

[8] Y. Du, B. Lou, Spreading and vanishing in nonlinear diffusion problems with free boundaries, J. Eur. Math. Soc., 17 (2015), 2673–2724.

[9] Y. Du, B. Lou, M. Zhou, Nonlinear diffusion problems with free boundaries: Convergence, transition speed and zero number arguments, SIAM J. Math. Anal., 47 (2015), 3555–3584.

[10] Y. Du and L. Ma, Logistic type equations on $\mathbb{R}^N$ by a squeezing method involving boundary blow-up solutions, J. London Math. Soc., 64 (2001), 107-124.

[11] Y. Du, H. Matsuzawa, M. Zhou, Sharp estimate of the spreading speed determined by nonlinear free boundary problems, SIAM J. Math. Anal., 46 (2014), 375–396.

[12] Y. Du, H. Matsuzawa, M. Zhou, Spreading speed and profile for nonlinear Stefan problems in high space dimensions, J. Math. Pures Appl., 103 (2015), 741–787.

[13] Y. Du, M.X. Wang, M. Zhou, Semi-wave and spreading speed for the diffusive competition model with a free boundary, J. Math. Pures Appl., 107 (2017), 253–287.

[14] J.-S. Guo, C.-H. Wu, On a free boundary problem for a two-species weak competition system, J. Dyn. Differ. Equ., 24 (2012), 873–895.

[15] J.-S. Guo, C.-H. Wu, Dynamics for a two-species competition-diffusion model with two free boundaries, Nonlinearity, 28 (2015), 1–27.

[16] B. Hu, Blow-up Theories for Semilinear Parabolic Equations, Lecture Notes in Math. 2018, Springer, Heidelberg, New York, 2011.

[17] Y. Kaneko, Spreading and vanishing behaviors for radially symmetric solutions of free boundary problems for reaction diffusion equations, Nonlinear Analysis: Real World Applications, 18 (2014), 121–140.

[18] Y. Kaneko, H. Matsuzawa, Spreading speed and sharp asymptotic profiles of solutions in free boundary problems for nonlinear advection-diffusion equations, J. Math. Anal. Appl., 428 (2015), 43-76.

[19] Y. Kaneko, Y. Yamada, A free boundary problem for a reaction-diffusion equation appearing in ecology, Adv. Math. Sci. Appl., 21 (2011), 467–492.

[20] Y. Kan-on, Fisher wave fronts for the Lotka-Volterra competition model with diffusion, Nonlinear Anal., 28 (1997), 145–164.

[21] Y. Kawai, Y. Yamada, Multiple spreading phenomena for a free boundary problem of a reaction-diffusion equation with a certain class of bistable nonlinearity, J. Diff. Equns., 261 (2016), 538–572.

[22] C.X. Lei, Z.G. Lin, Q.Y. Zhang, The spreading front of invasive species in favorable habitat or unfavorable habitat, J. Diff. Equns., 257 (2014), 145–166.

[23] D.A. Ladyzenskaja, V.A. Solonnikov, N.N. Uralceva Linear and Quasilinear Equations of Parabolic Type American Mathematical Society., Providence, R.I. 1968.

[24] H. Monobe, C.-H. Wu, On a free boundary problem for a reaction-diffusion-advection logistic model in heterogeneous environment, J. Diff. Equns., 261 (2016), 6144–6177.

[25] R. Peng, X.-Q. Zhao, The diffusive logistic model with a free boundary and seasonal succession, Discrete Contin. Dyn. Syst. (Ser. A), 33 (2013), 2007–2031.

[26] M.X. Wang, On some free boundary problems of the prey-predator model, J. Diff. Equns., 256 (2014), 3365–3394.

[27] M.X. Wang, The diffusive logistic equation with a free boundary and sign-changing coefficient, J. Diff. Equns., 258 (2015), 1252–1266.

[28] M.X. Wang, Y. Zhang, Note on a two-species competition-diffusion model with two free boundaries, Nonlinear Anal., 159 (2017), 458–467.

[29] M.X. Wang, J.F. Zhao, A free boundary problem for the predator-prey model with double free boundaries, J. Dyn. Diff. Equat., DOI: 10.1007/s10884-015-9503-5.
[30] M.X. Wang, J.F. Zhao, *Free boundary problems for a Lotka-Volterra competition system*, J. Dyn. Differ. Equ., 26 (2014), 655–672.

[31] Z.G. Wang, H. Nie and Y. Du, *Asymptotic spreading speed for the weak competition system with a free boundary*, preprint, 2017.

[32] C.-H. Wu, *Spreading speed and traveling waves for a two-species weak competition system with free boundary*, Discrete Cont. Dyn. Syst. (Ser. B), 18 (2013), 2441–2455.

[33] C.-H. Wu, *The minimal habitat size for spreading in a weak competition system with two free boundaries*, J. Diff. Eqns., 259 (2015), 873–897.

[34] J.F. Zhao, M.X. Wang, *A free boundary problem of a predator-prey model with higher dimension and heterogeneous environment*, Nonlinear Anal. Real World Appl., 16 (2014), 250–263.

[35] P. Zhou, D.M. Xiao, *The diffusive logistic model with a free boundary in heterogeneous environment*, J. Diff. Eqns., 256 (2014), 1927–1954.

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