Decay of superfluid turbulence via Kelvin wave radiation and vortex reconnections.

M. Leadbeater\textsuperscript{1}, D. C. Samuels\textsuperscript{2}, C. F. Barenghi\textsuperscript{2}, and C. S. Adams\textsuperscript{1}

\textsuperscript{1}Department of Physics, University of Durham, Durham DH1 3LE.
\textsuperscript{2}Department of Mathematics, University of Newcastle, Newcastle NE1 7RU.
(May 14, 2019)

The elementary processes involved in the decay of superfluid turbulence in the limit of low temperature are studied by numerical simulations of vortex ring collisions. We find that small vortex rings produced by reconnections eventually annihilate in a collision where all their energy is converted into Kelvin waves and sound. We show that sound emission due to Kelvin waves also leads to a loss of vortex line length but this dissipation mechanism alone is too small to account for the experimentally observed decay.

03.75.Fi, 67.40.Vs, 67.57.De

One of the most challenging problems in fluid mechanics is the development of a complete hydrodynamical description of turbulence. Progress is often made by studying fluids where the theoretical description can be simplified. For example, low temperature superfluids such as liquid helium\textsuperscript{1} or the recently discovered dilute Bose-Einstein condensates\textsuperscript{2} are particularly attractive due to the dramatic reduction in viscosity. In a superfluid, the turbulent state consists of a network of interacting vortices known as a ‘vortex tangle’\textsuperscript{3}. In superfluid helium-4 the vortex tangle is strongly coupled to an interpenetrating normal fluid and the system shares many of the features of classical turbulence\textsuperscript{4}. However, in the limit of low temperature, the normal fluid is negligible, but experiments still indicate a temperature independent decay of the turbulence state\textsuperscript{5}. In this case, the emission of sound either by vortex reconnections or vortex motion is the only active dissipation mechanism. Similarly, the time scale for the formation of vortex arrays in dilute Bose-Einstein condensates is also temperature independent suggesting that vortex-sound interactions may also be important in this relaxation process\textsuperscript{6}. In an earlier work we showed that significant sound energy is released during vortex reconnections\textsuperscript{7}. In the typical vortex line densities of superfluid helium-4 experiments\textsuperscript{8}, or in the regular structures found in a vortex lattice\textsuperscript{9}, one might expect that vortex reconnections are relatively infrequent, and that the continuous emission of sound by accelerating vortices may be the dominant dissipation mechanism. Unfortunately there is a scarcity of quantitative predictions of the significance of vortex motion as a dissipation mechanism in superfluid turbulence. Analytical results for the sound radiated by moving vortices exist only for simple cases such as a co-rotating pair\textsuperscript{10}. Furthermore, conventional numerical simulations of superfluid turbulence based on vortex filaments governed by incompressible Euler dynamics (the Biot-Savart law)\textsuperscript{11} are unable to describe sound emission.

An elegant model of quantum fluid mechanics is provided by the Gross-Pitaevskii (GP) equation. The GP model represents an extension of Euler fluid mechanics to include the quantisation of circulation, vortex core structure and vortex-sound interactions. Although the GP model does not accurately represent the physics of superfluid helium-4, it has been shown to provide qualitative and sometimes quantitative insight into the critical velocity for vortex nucleation\textsuperscript{12}, vortex line reconnections\textsuperscript{13}, vortex ring collisions\textsuperscript{14}, the decay of superfluid turbulence\textsuperscript{15}, and sound emission due to vortex reconnections\textsuperscript{16}.

To study sound emission due to vortex motion we excite Kelvin waves on a vortex ring and measure the length of the ring as a function of time. The Kelvin waves are produced by colliding two or more vortex rings. The initial state is constructed from stationary vortex ring solutions of the uniform flow equation found by Newton’s method\textsuperscript{17}. The desired configuration is obtained by multiplying the individual vortex ring states. This initial state is then evolved according to the dimensionless GP equation,

\[ i\partial_t \psi = -\frac{1}{2} \nabla^2 \psi + (|\psi|^2 - 1)\psi , \]  

using a semi-implicit Crank-Nicholson algorithm. In dimensionless units, distance and velocity are measured in terms of the healing length, $\xi$, and the sound speed, $c$, respectively. In addition, the asymptotic number density, $n_0$, is rescaled to unity. The computation box with volume, $V = (50)^3$, is divided into $10^6$ grid points with a spacing of 0.5. A grid spacing of 0.25 was also used to test the accuracy of the numerical methods. The time step is 0.02 and a typical simulation is run for $1.5 \times 10^5$ steps. Simulations have been performed with two, three and four vortex rings with radii ranging from 2.86 to 18.1 healing lengths. To convert the dimensionless units into values applicable to superfluid helium-4, we take the number density as $n_0 = 2.18 \times 10^{28}$ m$^{-3}$, the quantum of circulation as $\kappa = \hbar/m = 9.92 \times 10^{-8}$ m$^2$s$^{-1}$, and the healing length as $\xi/\sqrt{2} = 0.128$ nm\textsuperscript{18}. This gives a time unit $2\pi\xi^2/\kappa = 2$ ps, and therefore for superfluid helium-4 our simulations would correspond to a real time of 6 ns. Whereas for a sodium vapour condensate with $\xi \sim 0.2 \mu$m\textsuperscript{19}, the time unit is $\sim 15$ ns giving a simulation time of approximately 45 $\mu$s.

In the first example we consider a collision between a large and a small vortex ring. Such collisions are impor-
tant in vortex tangle dynamics because small vortex rings are often produced in collisions between larger structures [12]. A sequence of density isosurface plots illustrating the collision are shown in Fig. 1. Both vortex rings propagate in the positive $x$ direction with their propagation axes offset by the radius of the larger ring, $R_1 = 18.1$. The collision occurs at $t = 110$, and leads to the destruction of the small vortex ring. The vortex energy is converted into a sound pulse and Kelvin waves. The sound pulse propagates away from the large vortex ring and appears as a density minimum at $t = 120$ in Fig. 1. The Kelvin waves appear as two kinks which propagate around the vortex ring in opposite directions.

![Sequence of density isosurfaces](image)

**FIG. 1.** Sequence of density isosurfaces ($|\psi|^2 = 0.35$) illustrating a vortex ring collision with ring radii $R_1 = 18.1$ and $R_2 = 2.86$. A top view is shown below each frame. The rings collide at $t = 110$. The reconnection produces a sound pulse which appears as a small dot in the $t = 120$ frame.

In Fig. 2 we plot the vortex line length, $\ell$, as a function of time. The vortex line length is evaluated numerically by searching for phase singularities, estimating the length of vortex line within a grid cube and summing over all cubes. We have checked the accuracy of the method by performing the same calculation with a grid spacing of $0.25$ and $0.5$. The two calculations agree to within a fraction of the healing length. During the reconnection at $t = 110$, the vortex line length is stretched from its initial value of $2\pi(R_1 + R_2) \sim 130$, and then decreases dramatically due to the conversion of energy into sound [3]. After the collision the small vortex ring disappears completely while the mean radius of the large vortex ring is slightly larger, $R_1' = 18.8$.

The vortex line length oscillates due to the motion of the Kelvin waves around the vortex ring. By fitting the oscillations to a sine wave (shown bold in Fig. 2) we find that the period is $134 \pm 1$. This corresponds to a Kelvin wave velocity of $0.44$ times the speed of sound. The dispersion relationship for Kelvin waves is [10]

$$\omega = \frac{\kappa}{4\pi} k^2 \ln \left(\frac{2}{ka}\right),$$

where in dimensionless units, $\kappa = 2\pi$ and the core size $a = 1/\sqrt{2}$, therefore the group velocity of a Kelvin wave with center wavelength $\lambda$ is

$$v_g = \frac{\pi}{\lambda} \left[2 \ln \left(\frac{\sqrt{2} \lambda}{\pi}\right) - 1\right].$$

A velocity of $0.44$ corresponding to a Kelvin wavelength of $5.3$ healing lengths, which agrees with a measurement made from Fig. 1. In this intermediate wavelength region, the Kelvin wave group velocity is only weakly dependent on wavelength ($0.4 \leq v_g \leq 0.63$ for $5 \leq \lambda \leq 35$). Consequently, the dispersion of the Kelvin wave packet is relatively slow.

During the Kelvin wave oscillations the mean vortex line length gradually decreases. It is interesting to compare the observed length decrease with that expected for a Kelvin wave cascade, where energy is transferred to shorter and shorter wavelengths with a cut-off below a critical wavelength. In this case the vortex line density $L = \ell/V$, can be described by an equation of the form [17],

$$\frac{dL}{dt} = -\frac{\kappa}{2\pi} \chi_2 L^2,$$

where $\kappa = 2\pi$ in dimensionless units, and $\chi_2$ is a dimensionless coefficient. A linear fit to $1/L$ using the data from Fig. 2 for times between $t = 500$ and $t = 2900$ gives a coefficient, $\chi_2 \sim 0.006 \pm 0.001$. Both superfluid turbulence experiments [17] and numerical vortex dynamics simulations where an energy cut-off is assumed [3] suggest a coefficient, $\chi_2 \sim 0.3$, in the limit of low temperature. This suggests that Kelvin wave radiation alone is not sufficient to account for the decay of superfluid turbulence.

---

1 The original animated movies for figures 1 and 4 can be found at [http://massey.dur.ac.uk/ml].
The length of vortex line destroyed during a reconnection is strongly dependent on the reconnection angle \(\phi\). Consequently, to model a completely disordered vortex tangle we need to simulate a system which samples a complete range of reconnection angles. The simplest case where this is realised is four initial vortex rings, propagating inwards, as in the first frame of Fig. 4. The propagation axes are offset by one healing length to break the symmetry (the symmetric case completely annihilates in the first few hundred time units). A similar system, although with a very different length scale, has been studied using a classical vortex filament calculation [18]. The decay of the vortex line length is shown in Fig. 5. The initial vortex line density is much larger than in the previous examples, therefore there are a larger number of reconnections with a broad distribution of reconnection angles. A sequence of reconnections before \(t \sim 800\) results in the formation of two large vortex rings which avoid each other until \(t \sim 2600\). Again the decay coefficient for Kelvin wave radiation is consistent with Eq. (4). The collision at \(t \sim 2600\) sets up another phase of repeated reconnections producing smaller vortex rings which annihilate in collisions with larger structures. This scenario of vortex tangle decay involving the production of small vortex rings is also observed in the classical vortex filament calculations of Tsubota et al. [8]. Our main conclusion that Kelvin wave radiation alone cannot account for the observed decay of superfluid turbulence is illustrated by plotting a decay characterised by \(\chi_2 = 0.3\) in Fig. 5.
but that Kelvin waves plus vortex reconnections could be.

FIG. 4. Sequence of density isosurfaces ($|\psi|^2 = 0.35$) illustrating a collision involving four converging vortex rings each with radius $R = 18.1$. A large number of reconnections result in the generation of small structures which annihilate into sound energy producing only two vortex rings by $t = 740$. Note that because of the periodic boundary conditions the loop structures connect on opposite faces of the numerical box.

FIG. 5. The decay of vortex line length (thin line) for the collision involving four converging vortex rings shown in Fig. 4. For $t < 800$ there are a large number of reconnections resulting in a dramatic decrease in length. After $t \sim 800$ only two rings remain and there are no more reconnections until $t = 2400$. In this region the loss of vortex line length follows the prediction for Kelvin waves with a decay coefficient consistent with the data from Figs. 2 and 3. A decay characterised by $\chi^2 = 0.3$ is plotted (thick line) to illustrate the difference between this case and the much slower decay rate due to Kelvin wave radiation only.

ACKNOWLEDGMENTS

Financial support was provided by the EPSRC.

[1] C. F. Barenghi, R. J. Donnelly, and W. F. Vinen, Eds., Quantized vortex dynamics and superfluid turbulence, (Springer-Verlag, Heidelberg, 2001).
[2] M. Inguscio, S. Stringari and C. Wieman, Eds. Bose-Einstein condensation in atomic gases, Proc. Int. School of Physics Enrico Fermi, (IOS Press, Amsterdam, 1999).
[3] W. F. Vinen, Phys. Rev. B 61, 1410 (2000).
[4] S. L. Davies, P. C. Hendry, and P. V. E. McClintock, Physica B 280, 43 (2000).
[5] J. R. Abo-Shaeer, C. Raman, and W. Ketterle, Phys. Rev. Lett. 86, 1410 (2002).
[6] M. Leadbetter, T. Winiecki, D. C. Samuels, C. F. Barenghi, and C. S. Adams, Phys. Rev. Lett. 86, 1410 (2001).
[7] L. M. Pismen, Vortices in nonlinear fields (OUP, Oxford, 1999).
[8] M. Tsubota, T. Araki, and K. Nemirovskii, Phys. Rev. B 62, 11751 (2000).
[9] T. Frisch, Y. Pomeau, and S. Rica, Phys. Rev. Lett. 69, 1644 (1992).
[10] T. Winiecki and C. S. Adams, Europhys. Lett. 52, 257 (2000).
[11] J. Koplik and H. Levine, Phys. Rev. Lett. 71, 1375 (1993).
[12] J. Koplik and H. Levine, Phys. Rev. Lett. 76, 4745 (1996).
[13] C. Nore, M. Abid, and M. E. Brachet, Phys. Rev. Lett. 78, 3896 (1997).
[14] T. Winiecki, J. F. McCann, and C. S. Adams, Europhys. Lett. 48, 475 (1999).
[15] G. W. Rayfield and F. Reif, Phys. Rev. 136, 1194 (1964).
[16] R. J. Donnelly, Quantized vortices in Helium II, (CUP, Cambridge, 1991).
[17] W. F. Vinen, Proc. R. Soc. London, Ser. A 242, 493 (1957).
[18] The power radiated is proportional to a high power of the Kelvin wave frequency $\frac{3}{2}$. From equation (2) one finds that the power spectrum is sharply peaked around a wavelength of order 5 healing lengths. As the spectrum of Kelvin waves produced in a typical vortex ring collision is also peaked around 5 healing lengths, see Fig. 1, the decay is dominated by this wavelength region.
[19] D. Kivotides, J. C. Vassilicos, D. C. Samuels and C. F. Barenghi, Phys. Rev. Lett. 86, 3080 (2001).