Privacy-Preserving Multi-Party Contextual Bandits

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Abstract

Contextual bandits are online learners that, given an input, select an arm and receive a reward for that arm. They use the reward as a learning signal and aim to maximize the total reward over the inputs. Contextual bandits are commonly used to solve recommendation or ranking problems. This paper considers a learning setting in which multiple parties aim to train a contextual bandit together in a private way: the parties aim to maximize the total reward but do not want to share any of the relevant information they possess with the other parties. Specifically, multiple parties have access to (different) features that may benefit the learner but that cannot be shared with other parties. One of the parties pulls the arm but other parties may not learn which arm was pulled. One party receives the reward but the other parties may not learn the reward value. This paper develops a privacy-preserving multi-party contextual bandit for this learning setting by combining secure multi-party computation with a differentially private mechanism based on epsilon-greedy exploration.

1. Introduction

Contextual bandits are an important learning paradigm used by many recommendation systems (Langford & Zhang, 2008). The paradigm considers a series of interactions between the learner and the environment: in each interaction, the learner receives a context feature and selects an arm based on that context. The environment provides the learner with a reward after the arm is pulled (i.e., an action is executed). In traditional contextual bandit scenarios, the learner is a single party: that is, the party that pulls the arm is also the party that has access to all context features and that receives the reward. In many practical scenarios, however, contextual bandit learning involves multiple parties: for example, recommendation systems may involve content producers, content consumers, and the party that operates the recommendation service itself. These parties may not be willing or allowed to share all the information with each other that is needed to produce high-quality recommendations. For instance, a travel-recommendation service could recommend better itineraries by taking prior airline bookings, hotel reservations, and airline and hotel reviews into account as context. To do so, the travel-recommendation service requires data from booking, reservation, and review systems that may be operated by other parties. Similarly, a restaurant-recommendation service may be improved by considering a user’s prior reservations made via a restaurant-reservation service operated by another party.

In this paper, we develop a privacy-preserving contextual bandit that learns models in such multi-party settings. We study a multi-party contextual bandit setting in which: (1) all parties may provide some of the context features but none of the parties may learn each other’s features, (2) the party that pulls the arm is the only one that may know which arm was pulled, and (3) the party that receives the reward is the only one that may observe the reward value. We develop a learning algorithm that combines techniques from secure multi-party computation (Ben-Or et al., 1988) and differential privacy (Dwork et al., 2006). The algorithm achieves a high degree of privacy with limited losses in prediction accuracy compared to a non-private learner by using exploration mechanisms that naturally provide differential privacy. We provide theoretical guarantees on the privacy of our algorithm and empirically demonstrate its efficacy.

2. Problem Statement

Learning setting. We consider a multi-party contextual bandit setting (cf. Figure 1) with a set of parties \( \mathcal{P} \), a finite and fixed set of arms \( \mathcal{A} \), and \( T \) iterations. Each iteration, \( t \), in our learning setting consists of five main stages:

1. Each party \( p \in \mathcal{P} \) provides context features \( \mathbf{x}_{t,p} \in \mathbb{R}^{D_p} \) to the learner in a privacy-preserving way. We assume features are samples from a context distribution, \( p(\mathbf{x}) \).

2. The parties select an arm \( a_t \) by jointly evaluating policy \( \pi(\mathbf{x}_{t,1}, \ldots, \mathbf{x}_{t,|\mathcal{P}|}) \). Action \( a_t \) is not revealed to any of the parties, and \( \mathbf{x}_{t,q} \) is not revealed to parties \( p \neq q \).
3. The parties in $\mathcal{P}$ reveal the selected action, $a_t$, to party $p'$ but not to any of the other parties. Party $p'$ pulls the corresponding arm.

4. Party $p'' \not\in \mathcal{P}$ receives reward $r_t \in \mathbb{R}$, which is a sample from reward distribution $p(r | a_t, x_t)$, from the environment. Party $p''$ does not reveal $r_t$ to any other party.

5. The parties in $\mathcal{P}$ update policy $\pi$ without having access to the reward, $r_t$, or each other’s contexts, $x_{t,p}$.

Together, the parties learn a policy $\pi(x_{t,1}, \ldots, x_{t,|\mathcal{P}|}) \in \Pi$ that maximizes the average reward over all $T$ iterations:

$$V(\pi) = \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\left[r_t | \pi(x_{t,1}, \ldots, x_{t,|\mathcal{P}|}) \right],$$

where the expectation is over the context distribution, $p(x)$, and reward distribution, $p(r | a, x)$, that are given by the environment. We assume these distributions are fixed. We also assume the reward, $r$, has a fixed range, $r \in [0, 1]$.

Our policy set, $\Pi$, is the set of all epsilon-greedy policies that use models with a linear relation between context features $x_{t,p}$ and the corresponding scores for arm $a_t$ (Li et al., 2010). In particular, let $x_t \in \mathbb{R}^D$ be the concatenated context features of all $|\mathcal{P}|$ parties. To compute the score for arm $a$ at iteration $t$ we use a linear model $s_a = w_a^\top x_a$. Let $X_a$ be the $N_t \times D$ design matrix at iteration $t$ which consists of the $N_t$ context vectors observed before pulling arm $a$. Similarly, $r_a \in \mathbb{R}^{N_t}$ is the vector of observed rewards after pulling that same arm. The learner aims to find the model parameters at iteration $t$ by minimizing the least-squares error $\|X_a w_a - r_a\|^2_2$ using the linear least-squares solution: $w_a = (X_a^\top X_a)^{-1} X_a^\top r_a$.

Security model. In line with the cooperative nature of our learning setting, we assume a honest-but-curious security model (Goldreich, 2009): we assume parties do not collude and follow the specified learning algorithm, but parties may try and learn as much as possible from the information they observe when executing the algorithm.

Our learning algorithm comes with a differential-privacy guarantee on the information that parties $p'$ and $p''$ can obtain on context features $\forall p \in \mathcal{P}$: $x_{t,p}$. The action must be revealed to party $p'$ so that it can pull the corresponding arm. Similarly, party $p''$ must receive the reward. Hence, some information is ultimately revealed to parties $p'$ and $p''$. We provide a differential-privacy guarantee on the information leaked about $x_t$ by exploiting the randomness introduced by the epsilon-greedy exploration of the bandit algorithm.

Our algorithm guarantees that the other parties, $p \in \mathcal{P}$, do not gain any additional knowledge about context features $x_{t,q}$ with $q \neq p$. To obtain this guarantee, the algorithm does assume that all parties have access to privately shared random numbers generated during an off-line phase.  

3. Privacy-Preserving Contextual Bandits

Our privacy-preserving, multi-party contextual bandit employs an epsilon-greedy policy that assumes a linear relation between the context features and the score for an arm. To obtain privacy guarantees, we use arithmetic secret sharing techniques commonly used in secure multi-party computation to implement our learner (Damgård et al., 2011). We rely on the differentially private properties of epsilon-greedy policies when performing actions (see Section 4).

In arithmetic secret sharing, a scalar value $x \in \mathbb{Z}/Q\mathbb{Z}$ (where $\mathbb{Z}/Q\mathbb{Z}$ denotes a ring with $Q$ elements, and $Q$ is large) is shared across $|\mathcal{P}|$ parties such that the sum of the shares reconstructs the original value $x$. We denote the secret sharing of $x$ by $[x] = \{\forall p \in \mathcal{P} : [x]_p\}$, where $[x]_p \in \mathbb{Z}/Q\mathbb{Z}$ indicates party $p$’s share of $x$. The representation has the property that $\sum_{p \in \mathcal{P}} [x]_p \text{ mod } Q = x$. We use a simple encoding to represent the real-valued context features and model weights in $\mathbb{Z}/Q\mathbb{Z}$: to obtain $x \in \mathbb{Z}/Q\mathbb{Z}$, we multiply $x_R \in \mathbb{R}$ with a large scaling factor $B$ and round to the nearest integer: $x = \lfloor B x_R \rfloor$, where $B = 2^L$ for some precision parameter, $L$. We decode an encoded value, $x$, by computing $x_R \approx x/B$. Encoding real values this way incurs a precision loss that is inversely proportional to $L$.

Party $p \in \mathcal{P}$ shares a context feature, $x$, by drawing $|\mathcal{P}| - 1$ numbers uniformly at random from $\mathbb{Z}/Q\mathbb{Z}$ and distributing them among the other parties. The random numbers are the shares, $\forall q \neq p : [x]_q$, for those parties. Subsequently,
party \( p \) computes its own share as \( [x]_p = x - \sum_{q \neq p} [x]_q \).
None of the other parties can infer any information about \( x \) from their share. The policy is evaluated using exclusively arithmetically shared data and parameters. To allow party \( p' \) to pull an arm \( a_t \in A \), all parties communicate their shares, \([a_t]_p\), to party \( p'\), which computes \( a_t = \sum_{p \in \mathcal{P}} [a_t]_p \) and performs the corresponding action, \( a_t \). Subsequently, party \( p'' \) receives the reward, \( r_t \), which it secret shares with parties \( p \in \mathcal{P} \). The parties use \([r_t]_p\) to update the policy, again, by performing computations exclusively on arithmetic shares.

Algorithm 1 gives an overview of our privacy-preserving contextual bandit learner, which follows Li et al. (2010) but uses epsilon-greedy rather than UCB exploration. Unless otherwise stated, all computations in the algorithm are performed by all parties in \( \mathcal{P} \) on the secret-shared values without leaking any information to the other parties.

The algorithm relies on the homomorphic properties of arithmetic secret sharing in order to perform computations directly on the encrypted data. Below, we give an overview of how these computations are implemented. The primary cost in executing Algorithm 1 is the number of communication rounds between parties for certain operations, such as the evaluation of the argmax function. This communication can sometimes be overlapped with other computations.

Addition. The addition of two encrypted values, \([z] = [x] + [y]\), can be trivially implemented by having each party \( p \) sum their shares of \([x]\) and \([y]\). That is, each party \( p \in \mathcal{P} \) computes \([z]_p \leftarrow [x]_p + [y]_p\).

Multiplication. To facilitate multiplication of two secret shared values, the parties use random Beaver triples (Beaver, 1991) that were generated in an off-line preprocessing phase. A Beaver triple of secret shared values \([\alpha], [b], [c]\) satisfies the property \( c = ab \). The parties use the Beaver triple to compute \([\alpha] = [x] - [a]\) and \([\beta] = [y] - [b]\) and decrypt \( \alpha \) and \( \beta \). This does not leak information if \( a \) and \( b \) were drawn uniformly at random from the ring \( \mathbb{Z}/Q\mathbb{Z} \). The product \([x][y]\) can now be evaluated by computing \([c] + [\alpha][b] + [\beta][a] + [\alpha][\beta] \). It is straightforward to confirm that the result of the private multiplication is correct:

\[
[c] + [\alpha][b] + [\beta][a] + [\alpha][\beta] = \\
[a][b] + [x][b] - [a][b] + [y][a] - [b][a] + ([x] - [a])([y] - [b]) = [x][y].
\]

To decrypt \( \alpha \) and \( \beta \), all parties to communicate their shares of \([\alpha]\) and \([\beta]\) to each other: a communication round. The required correction for the additional scaling term from the encoding, \( B \), incurs a second communication round.

Square. To compute the square \([x^2]\), the parties use a Beaver pair \(([a], [b])\) such that \( b = a^2 \). Akin to before, the parties use the Beaver pair compute \([\alpha] = [x] - [a]\), decrypt \( \alpha \), and obtain the result via \([x^2] = [b] + 2\alpha[a] + \alpha^2\).

Dot product, matrix-vector, and matrix-matrix multiplication. The operations on scalars we described above can readily be used to perform operations on vectors and matrices that are secret-shared in element-wise fashion. Specifically, dot products combine multiple element-wise multiplications and additions. Matrix-vector and matrix-matrix multiplication is implemented by repeated computation of dot products of two arithmetically secret-shared vectors.

Matrix inverse. At each round, the algorithm computes \([\mathcal{A}]^{-1}\) matrix inverses of the \( D \times D \) matrices \( W_a \), which is computationally costly. Because we only perform rank-one updates of each \( W_a \), we can maintain a representation of \( W_a^{-1} \) instead, rendering the matrix inversion in the algorithm superfluous, and use the Sherman-Morrison formula (Bartlett, 1951) to perform the parameter update:

\[
W_a^{-1} \leftarrow W_a^{-1} - a_x x^\top W_a^{-1} \left(1 / (1 + x^\top W_a^{-1} x)\right). \tag{1}
\]

This expression comprises only multiplications, additions, and a reciprocal (see below).

Reciprocal. We compute the reciprocal \([1/x]\) using a Newton-Rhapson approximation with iterates \([x_{t+1}] \leftarrow [2x_t - xx_t^2]\). The Newton-Rhapson approximation converges rapidly when \( x_0 \) is initialized well. The reciprocal is only used in the Sherman-Morrison formula, so we choose the initialization with this in mind. Because \( W_a^{-1} \) in Equation 1 is the inverse of a positive-definite matrix, it is itself positive definite; the denominator in Equation 1, therefore, lies in the range \([1, c]\). Empirically, we found that \( c < 10 \) and that an initial value of \( x_0 = 3e^{-0.5} + 0.003 \) leads to good approximations in this range.

Argmax. We compute the index of the maximum value of a vector \([x]\) where \( x \in (\mathbb{Z}/Q\mathbb{Z})^{|\mathcal{A}|} \) as a one-hot vector of the same size as \( x \). Our algorithm for evaluating the argmax of \([x]\), called \([z]\), has three main stages:

1. Use Algorithm 2 to construct a vector \([y]\) of the same length as \([x]\) that contains ones at the indices of all maximum values of \([x]\) and zeros elsewhere.

2. Break ties in \([y]\) by multiplying it element-wise with a random permutation of \( \{1, \ldots, |\mathcal{A}|\} \). This permutation, \([\gamma]\), is generated and securely shared off-line.

3. Use Algorithm 2 to construct a one-hot vector \([z]\) that indicates the maximum value of \([y \circ \gamma]\).

The permutation in step 2 randomly breaks ties for the index of the maximum value. We opt for random tie-breaking because breaking ties deterministically may leak information. For example, if ties were broken by always selecting the last maximum value from \([x]\), the adversary would learn that \([x]\) did not have multiple maximum values if it observed a \([z]\) that has a 1 as its first element.
Algorithm 1 Privacy-preserving contextual bandit. (Follows Li et al. (2010) but uses epsilon-greedy rather than UCB exploration.)

**Input:**
- Exploration parameter, $\epsilon$.
- Party $p'$sel $\not\in P$ that pulls arms; party $p'' \not\in P$ that receives rewards.
- Set of parties, $P$; set of arms, $A$; number of context features produced by all parties, $D$.
- A store of Beaver triples and private (Bernoulli and uniform) samples generated off-line.

**Output:** Secretly shared weights, $[W]$, and biases, $[B]$

Party $p'$ initializes weights $W = \{W_1, \ldots, W_{|A|}\}$ with $W_a \leftarrow I_{D \times D}$.

Party $p'$ initializes biases $B = \{b_1, \ldots, b_{|A|}\}$ with $b_a \leftarrow 0_{D \times 1}$.

For $t \in \{1, \ldots, T\}$ do

- Parties observe context features $\forall p \in P : x_{t,p} \in \mathbb{R}^D$ (with $\sum_{p \in P} D_p = D$).
- Parties secretly share context features $\forall p \in P : [x_{t,p}]$.
- Parties concatenate shares $\forall p \in P : [x_{t,p}]$ into single share $[x_t]$ with $x_t \in \mathbb{R}^D$.
- Select next private sample $[y_t]$ with $y_t \sim Bernoulli(\epsilon)$.
  
  For $a \in A$ do
  
  - Compute $[w_a] \leftarrow [W^{-1}b_a]$ (weights for arm $a$).
  - Compute $[s_{t,a}] \leftarrow [w_a x_t]$ (score for arm $a$).
  - Select next random private sample $[v_a]$ with $v_a \sim Uniform(0,1)$.
  - Compute $[s_{t,a}] \leftarrow [y_t v_a + (1 - y_t) s_{t,a}]$ (differentially private score for arm $a$).
  
  End for

  Compute $[s_t] \leftarrow [\text{argmax}_a s_{t,a}]$ (arm to be pulled).

  Party $p'$ opens up $a_t$ and pulls arm.

  Party $p'$ constructs $\forall a \in A : o_a \leftarrow I(a = a_t)$ (binary values indicating selected action).

  Party $p'$ secretly shares all indicator variables, $\forall a \in A : [o_a]$.

  Party $p''$ receives reward $r_t$ and secretly shares $[r_t]$.

  For $a \in A$ do

  - Compute $[w_a] \leftarrow [W_a + o_a x_t x_t^\top]$.
  - Compute $[b_a] \leftarrow [b_a + o_a r_t x_t]$.

  End for

End for

Algorithm 2 Privacy-preserving identification of all maximum elements in a secret-shared vector.

**Input:** An arithmetically secret-shared vector $[x]$.

**Output:** A secret shared vector $[y] \in \{0, 1\}^{1, |A|}$ with ones indicating the maximum values in $[x]$.

For $i, j \in A \times A$ do

  - Compute $[f_{i,j}] \leftarrow [x_i \geq x_j]$.

End for

Set $[y] \leftarrow [1]$.

For $i \in A$ do

  For $j \in A$ do

    - Compute $[y_i] \leftarrow [y_i][f_{i,j}]$.

  End for

End for

In Algorithm 2, the evaluation of all $[x_i \geq x_j]$ terms is performed on a binary secret share of $x_i$ and $x_j$. A binary secret share is a a special type of arithmetic secret sharing for binary data in which the ring size $Q = 2$ (Goldreich et al., 1987). To convert an arithmetic share $[x]$ into a binary share $\langle x \rangle$, each party first secretly shares its arithmetic share with the other parties and then performs addition of the resulting shares. To construct the binary share $\langle [x]_p \rangle$ of its arithmetic share $[x]_p$, party $p \in P$ (1) draws $|P| - 1$ random bit strings $\langle [x]_p \rangle_q$ and shares those with the other parties and (2) computes its own binary share $\langle [x]_p \rangle_p = \bigoplus_{q \neq p} \langle [x]_p \rangle_q$. The parties now each obtained a binary share of $[x]_p$ without having to decrypt $x$. This process is repeated for each party $p \in P$ to create binary shares of all $|P|$ arithmetic shares $[x]_p$. Subsequently, the parties compute $\langle x \rangle = \sum_{p \in P} \langle [x]_p \rangle$. The summation is implemented by a Ripple-carry adder in $\log_2(|P|\log_2 Q)$ rounds (Catrina & De Hoogh, 2010). Subsequently, the $[x_i \geq x_j]$ operation is performed by computing $\langle [y] \rangle \leftarrow [x_i] - [x_j]$, constructing the binary secret sharing $\langle y \rangle$ per the procedure outlined above, obtaining the most significant bits, $\langle y \rangle^{(MSB)}$, and converting those bits back to an arithmetic share. To convert from a binary share $\langle y \rangle$ to an arithmetic share $\langle y \rangle$, the parties compute $\langle y \rangle = \sum_{b=1}^{B'} 2^b \langle (y)^{(b)} \rangle$, where $\langle y \rangle^{(b)}$ contains the $b$-th bits of the binary share $\langle y \rangle$ and $B'$ is the total number of bits.
in the shared secret. To create the arithmetic share of a bit, \([\{y\}]_b\), each party \(p \in P\) draws bits uniformly at random, and shares the difference between their bits and the random bits with the other parties. The parties sum all resulting shares to obtain \([\{y\} ]_s = [x_i - x_j]\).

Overall, the evaluation of \([x_i \geq x_j]\) requires seven communication rounds. We parallelize the reduction over \(i\), and perform the reduction over \(j\) using a binary reduction tree in \([\log_2(|A| - 1)]\) communication rounds.

### 4. Privacy Guarantee

The privacy guarantees for our algorithm rely on: (1) well-known guarantees on the security of arithmetic and binary secret sharing mechanisms and (2) the differentially private opening of actions \(a_t \in A\) by party \(p'\). For security guarantees of secret sharing, we refer the reader to Damgård et al. (2011). We focus on the differentially private opening of actions. Our primary observation is a natural link between epsilon-greedy policies and differential privacy.

A mechanism \(\mathcal{M}\) is \(\eta\)-differentially private if for all datasets \(D\) and \(\mathcal{D}'\) that differ by a single example and for all output sets \(S \subseteq \text{Range}(\mathcal{M})\) the following holds (Dwork, 2011):

\[
P(\mathcal{M}(D) \in S) \leq e^\eta P(\mathcal{M}(D') \in S).
\]

**Theorem 1.** If only the selected action \(a_t \in A\) at round \(t\) is revealed, then a policy that uses \(\epsilon\)-greedy exploration is \(\log(|A|/\epsilon)\)-differentially private.

**Proof.** The probability of selecting the action that corresponds to the maximum score is given by \(P(\pi(s) = \arg\max_{a \in A} s_a) = (1 - \epsilon) + \frac{\epsilon}{|A|}\). We use this probability to bound the privacy loss, \(\eta\):

\[
\log P(\pi(s) = j \mid j = \arg\max_{a \in A} s_a)
= \log \frac{1 - \epsilon + \epsilon/|A|}{\epsilon/|A|}
= \log \left(\frac{|A|}{\epsilon} - |A| + 1\right) \leq \log \left(\frac{|A|}{\epsilon}\right).
\]

Using the fact that the exploration parameter \(\epsilon \in [0, 1]\), we also observe that:

\[
\log P(\pi(s) = j \mid j \neq \arg\max_{a \in A} s_a)
\leq \log (1 - \epsilon) = 0.
\]

To complete the proof, we observe that:

\[
\log \frac{P(\pi(s) = k \mid k \neq \arg\max_{a \in A} s_a)}{P(\pi(s) = k \mid k = \arg\max_{a \in A} s_a)} = \log \frac{\epsilon/|A|}{1 - \epsilon + \epsilon/|A|}
\leq 0.
\]

The above result is a generalization of the randomized response protocol (Warner, 1965) to \(|A| > 2\) arms and arbitrary \(\epsilon \in [0, 1]\). Because the privacy loss grows logarithmically with the number of actions, we obtain high differential privacy for reasonable settings of exploration parameter \(\epsilon\).
5. Experiments

We perform experiments on the MNIST dataset to evaluate the efficiency and reward-privacy trade-off of our algorithm. We reduce the data dimensionality by projecting each digit image to the first 20 principal components of the dataset, and normalize each resulting vector to have unit length. We perform a single sweep through the 60,000 images in the MNIST training set. At each iteration, the parties receive a new (secret-shared) image and need to select one of 10 arms (digit classes). The reward is 1 if the selected arm corresponds to the correct digit class, and 0 otherwise. We implement Algorithm 1 on a ring \( \mathbb{Z}/Q\mathbb{Z} \) with \( Q = 2^{64} \). We rely on the property of 64-bit integer operations, where a result that is too large to fit in 64 bits is automatically mapped to \( \mathbb{Z}/Q\mathbb{Z} \) with \( Q = 2^{64} \). We use \( L = 20 \) bits of precision to encode floating-point values into \( \mathbb{Z}/Q\mathbb{Z} \) and 7 Newton-Rhapson iterations for computing the reciprocal function.

Most of our experiments are performed using \(|\mathcal{P}| = 2\) parties. Code reproducing the results of our experiments is publicly available at http://www.anonymized.com.

Reward and privacy. Figure 2 shows the average reward that our private contextual bandits obtain for four different values of \( \epsilon \). We compared the results obtained by our algorithm to that of a non-private implementation of the epsilon-greedy contextual bandit learner, and confirmed that the observed rewards are the same for a range of \( \epsilon \) values.

Figure 3 shows the average reward (averaged over 5 runs) observed as a function of the differential privacy parameter, \( \eta \) (higher values represent less privacy). The results were obtained by varying \( \epsilon \) and are shown for experiments with three different dataset sizes, \( T \). The results show that at certain levels of differential privacy, the reward obtained by the private algorithm is higher than that of its non-private counterpart (\( \eta = \infty \)). Indeed, some amount of exploration benefits the learner whilst also providing differential privacy. For higher levels of privacy, however, the reward obtained starts to decrease because too much exploration is needed to obtain the required level of privacy.

Efficiency and scale. Table 1 reports the run-time of the key operations in Algorithm 1, compared to an implementation of the same operations in PyTorch. In our experiments, the private contextual-bandit implementation with \(|\mathcal{P}| = 2\) parties and \(|\mathcal{A}| = 10\) actions is nearly \( 500 \times \) slower than a regular implementation. In real-world settings, the slowdown would likely be even higher because of network latency: our experiments were performed on a single machine where each party is implemented as a separate process. There are two key sources of inefficiency in Algorithm 1:

1. The weight update is \( \mathcal{O}(D) \) in a regular contextual-bandit implementation (only the weights for the selected arm are updated) but \( \mathcal{O}(|\mathcal{A}|D) \) in Algorithm 1: the private implementation cannot reveal the selected arm and, therefore, has to update all the weights. We note that these weight updates are parallelizable over arms.

| Operation  | Rounds | Slowdown |
|------------|--------|----------|
| Addition   | 0      | 11x      |
| Multiplication | 2  | 380x     |
| Reciprocal | \(30\) | \(6,000\times\) |
| Argmax     | \(\mathcal{O}(|\mathcal{P}| + \log_2(|\mathcal{A}|))\) | \(34,000\times\) |
We also perform experiments in which we vary the number of arms, $|A|$. To increase the number of arms, we construct a $K$-means clustering of the dataset and set $|A| = K$. We define the rewards to be Bernoulli-distributed with $P(r_a = 1) = \nu_a$. The probabilities $\nu_a$ are Gaussian-kernel values based on the distances from a data point to the inferred cluster centers: $\nu_a \propto e^{-\frac{1}{2}\|c_a - x\|^2}$, where $c_a$ is the $a$-th cluster and $\sigma$ is used to control the difficulty of the problem. We set $\sigma = 1/2$ in our experiments. Figure 4 demonstrates how the contextual-bandit algorithm scales with the number of arms. For small numbers of arms ($|A| \leq 20$), the implementation overhead dominates the computation time. For larger numbers of arms ($|A| \geq 40$), we observe quadratic scaling. Figure 5 shows how the algorithm scales as a function of the number of parties, $|P|$. The results illustrate that the run-time of our algorithm is $\mathcal{O}(|P|^2)$: all parties communicate with each other in every communication round, which leads to the quadratic scaling observed.

Figure 6 demonstrates how the reward changes as a function of the privacy loss $\eta$ when the number of arms, $|A|$, is varied. The privacy loss increases logarithmically in the number of arms, but the amount of exploration needed also increases. As a result, the optimal privacy loss in terms of reward only tends to increase slightly as the number of arms in the bandit increases. Indeed, this increase may be prohibitively large for web-scale recommendation applications in which the bandit has to select one arm out of millions of arms.

Membership inference attacks. To empirically measure the privacy of our contextual-bandit algorithm, we also performed experiments in which an adversary tries to infer whether or not a sample was part of the training dataset by applying the membership inference attack of Yeom et al. (2018) on model checkpoints saved at various points during training. The membership inference attack computes an empirical estimate of the joint action-reward distribution on the data that was used to train the model and on a held-out test set, respectively (we use the MNIST test set as held-out set). We use the resulting empirical distributions, $p_{\text{train}}(r|a)$ and $p_{\text{test}}(r|a)$, to infer training-data membership for an example $x_{T+1}$. Specifically, we: (1) evaluate the model on $x_{T+1}$, (2) observe the selected arm $a_{T+1}$, (3) receive corresponding reward $r_{T+1}$, and (4) predict training data membership if $p_{\text{train}}(r_{T+1}|a_{T+1}) > p_{\text{test}}(r_{T+1}|a_{T+1})$.

Following Yeom et al. (2018), we measure the advantage of the resulting adversary: the difference in the true positive rate and the false positive rate in predicting training set membership. The adversary advantage during training is shown in Figure 7 for models trained with different values of $\epsilon$. The results show that in the early stages of learning, the adversary has a slight advantage of 1–2%, this advantage rapidly decreases below 0.75% after the learner has observed more training examples\(^2\). The advantage slightly increases in the later stages of training: interestingly, this happens because the model slightly underfits on the MNIST dataset. Overall, the results of our experiments suggest that our contextual bandit learner is, indeed, maintaining privacy of the context features, $x$, well in practice.

\(^2\)For higher values of $\epsilon$, there is more variance in the model parameters during training, which is reflected in higher variance in the advantage values.
6. Related Work

This study fits into a larger body of work on privacy-preserving machine learning. Prior work has used similar techniques from secure multi-party computation (and homomorphic encryption) for secure evaluation and/or training of deep networks (Dowlin et al., 2016; Hynes et al., 2018; Juvekar et al., 2018; Mohassel & Zhang, 2017; Riazi et al., 2017; Shokri & Shmatikov, 2015; Wagh et al., 2018) and other classification models (Cock et al., 2019; Reich et al., 2019). Other related work has developed secure data-aggregation techniques (Bonawitz et al., 2017) for use in federated-learning scenarios (Bonawitz et al., 2019). To the best of our knowledge, our study is the first to use this family of techniques in the online-learning setting of contextual bandits, using the randomness introduced by exploration mechanisms to obtain a differential-privacy guarantee on the output produced by the learner.

Most closely related to our work are studies on differential private online learning (Dwork et al., 2010; Jain et al., 2012; Thakurta & Smith, 2013). In particular, Mishra & Thakurta (2015) develops UCB and Thompson samplers for (non-contextual) bandits with differential-privacy guarantees based on tree-based aggregation (Chan et al., 2010; Dwork et al., 2009). Follow-up work improved differentially private UCB to have better regret bounds (Tossou & Dimitrakakis, 2016). Recent work (Shariff & Sheffet, 2018) also developed a joint differentially private version of LinUCB (Li et al., 2010). In contrast to those prior studies, we study a more challenging setting in which the parties that implement the learner may not leak information about their observed contexts, actions, and rewards to each other. Having said that, our algorithm may be improved using the differentially private mechanisms of Mishra & Thakurta (2015); Tossou & Dimitrakakis (2016). In this study, we opted for the simpler epsilon-greedy mechanism because it can be implemented efficiently on arithmetically shared data. We leave the implementation of differentially private UCB and Thompson samplers in our secure multi-party computation framework to future work.

7. Summary and Future Work

We presented a privacy-preserving, multi-party contextual bandit algorithm that works correctly in practice and that comes with theoretical guarantees on (differential) privacy of the context features. Although our experimental evaluation of this algorithm demonstrates its effectiveness, several avenues for improvement remain:

- **Increase numerical stability.** Repeated use of the Sherman-Morrison formula is known to produce cancellation errors that may lead to numerical errors. A numerically stable algorithm would regularly compute the actual matrix inverse to eliminate such errors, and add a diagonal-regularizer to prevent ill-conditioning issues.

- **Robustness to disappearing parties.** In many practical settings, parties may temporarily disappear because of system failures (Bonawitz et al., 2019). To allow the algorithm to operate in such scenarios, different types of secret sharing (e.g., Shamir sharing (Shamir, 1979)) may be needed. The contextual bandit itself could learn to be robust to failing parties by employing a kind of “party dropout” at training time (Srivastava et al., 2014).

- **Security under stricter security models.** The current algorithm assumes parties are honest-but-curious, which means that parties do not deviate from the protocol in Algorithm 1. It is important to note that our privacy guarantees do not hold in stricter security models in which one or more parties operate adversarially or in settings in which the parties collude. Our current algorithm can be extended to provide guarantees under stricter security models: for instance, extending the algorithm to use message authentication codes (Goldreich, 2009) would allow the parties to detect attacks in which a minority of the parties behaves adversarially. Unfortunately, such extensions generally increase the computational and communication requirements of the learner.

- **Robustness to side-channel attacks.** In practical scenarios, it may be possible to break the privacy of our learner via side-channels attacks. For example, there is a delay between taking the action and receiving the reward that may make the learner susceptible to timing attacks (Kocher, 1996): if the distribution of reward delays depends on the action being selected, parties $p \in \mathcal{P}$ may be able to infer the selected action from the observed time delay, counter to our guarantees. A real-world implementation of our algorithm should, therefore, introduce random time delays in the operations performed by parties $p'$ and $p''$ to prevent information leakage.

- **Stronger membership-inference attacks.** The membership-inference attacks we considered in this study (Yeom et al., 2018) are not designed to use the full action-reward sequence as side information in the attack. It may thus be possible to strengthen these membership-inference attacks by using the full action-reward sequence, which may be observed by an external observer of the algorithm. The development of such stronger attacks may help to obtain better empirical insights into the level of privacy provided by our privacy-preserving contextual bandits.

- **Scaling to larger problems.** Our algorithm was tested on the task of digit recognition where the action set is small and the feature space was reduced. We leave as follow-on work scaling the algorithm to more realistic problems with larger actions sets, for example, by developing efficient approximations to the argmax function.
Acknowledgements

The authors thank Mark Tygert, Ilya Mironov and Xing Zhou for helpful discussions and comments on early drafts of this paper.

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A. Secret Sharing

Our privacy-preserving contextual bandits use two different types of secret sharing: (1) arithmetic secret sharing (Damgård et al., 2011); and (2) binary secret sharing (Goldreich et al., 1987). Below, we describe the secret sharing methods for single values \( x \) but they can trivially be extended to real-valued vectors \( \mathbf{x} \).

Arithmetic secret sharing. Arithmetic secret sharing is a type of secret sharing in which the sum of the shares reconstruct the original data \( x \). We refer to the shared representation of \( x \) as \([x]_P\). The shared representation across parties, \( P \), is given by \([x] = \{\forall p \in P : [x]_p\}\), where \([x]_p\) indicates the share of \( x \) that party \( p \in P \) has. The representation has the property that \(\sum_{p \in P} [x]_p \mod Q = x\). To make sure that none of the parties can learn any information about \( x \) from their share \([x]_p\), shares need to be sampled uniformly from a ring of size \( Q, \mathbb{Z}/Q\mathbb{Z} \), and all computations on shares must be performed modulus \( Q \). If \( x \) is real-valued, it is encoded to lie in \( \mathbb{Z}/Q\mathbb{Z} \) using the mechanism described in Appendix B before it is encrypted.

To encrypt the unencrypted data \( x \), party \( p \) that possesses \( x \) draws \(|P|−1\) numbers uniformly at random from \( \mathbb{Z}/Q\mathbb{Z} \) and distributes them among the other parties. Subsequently, party \( p \) computes its own share as \([x]_p = x - \sum_{q \neq p} [x]_q\). Thus all the parties (including party \( p \)) obtain a random number that is uniformly distributed over the ring, from which they cannot infer any information about \( x \). To decrypt \([x]\), the parties communicate their shares and compute \( x = \sum_{p \in P} [x]_p \mod Q \).

Binary secret sharing. Binary secret sharing is a special type of arithmetic secret sharing for binary data in which the ring size \( Q = 2 \) (Goldreich et al., 1987). Because addition modulo two is equivalent to taking an exclusive OR (XOR) of the bits, this type of sharing is often referred to as XOR secret sharing. To distinguish binary shares from arithmetic shares, we denote a binary share of variable \( x \) across \( P \) parties by \( \langle x \rangle = \{\forall p \in P : \langle x \rangle_p\} \). Just as with arithmetic sharing, binary secret shares allow for “linear” operations on bits without decryption. For example, binary sharing allows for the evaluation of any circuit expressed as XOR and AND gates. While it is much more efficient to do addition and multiplication of integers with arithmetic shares, logical expressions such as \(\text{max}(0, x)\) are more efficient to compute with binary shares. In equations, we denote AND by \(\odot\) and XOR by \(\oplus\).

To encrypt the unencrypted bit \( x \), party \( p \in P \) that possesses \( x \) draws \(|P|−1\) random bits and distributes those among the other parties. These form the shares \(\forall q \neq p : \langle x \rangle_q\) for \(|P|−1\) parties. Subsequently, party \( p \in P \) computes its own share as \(\langle x \rangle_p = x \oplus \bigoplus_{q \neq p} \langle x \rangle_q\). Thus all the parties...
We rely on binary secret sharing to implement logical operations such as XOR and AND are addition and multiplication modulo 2 of the numbers belong to the set \( \{0, 1\} \). As a result, the techniques we use for addition and multiplication of arithmetically shared values (see paper) can be used to implement XOR and AND as well. Evaluating \( \langle x \rangle \oplus a \) function amounts to one party \( p \in \mathcal{P} \) computing \( \langle x \rangle_p \oplus a \), and evaluating \( \langle x \rangle \otimes (y) \) amounts to each party \( p \in \mathcal{P} \) computing \( \langle x \rangle_p \otimes (y)_p \). Similarly, \( \langle x \rangle \otimes a \) is evaluated by having each party compute \( \langle x \rangle_p \otimes a \). The AND operation between two private values, \( \langle x \rangle \langle y \rangle \), is implemented akin to the private multiplication protocol using Beaver triples.

### A.1. Converting Between Secret-Sharing Types

Contextual bandit algorithms involve both functions that are easier to compute on arithmetic secret shares (e.g., matrix multiplication) and functions that are easier to implement via on binary secret shares (e.g., argmax) using binary circuits. Therefore, we use both types of secret sharing and convert between the two types using the techniques proposed in (Demmler et al., 2015).

**From \([x] \) to \( \langle x \rangle \):** To convert from an arithmetic share \( [x] \) to a binary share \( \langle x \rangle \), each party first secretly shares its arithmetic share with the other parties and then performs addition of the resulting shares. To construct the binary share \( \langle [x]_p \rangle \) of its arithmetic share \( [x]_p \), party \( p \in \mathcal{P} \): (1) draws \( \mathcal{P} \) a random bit string \( \langle [x]_p \rangle_q \) and shares those with the other parties and (2) computes its own binary share \( \langle [x]_p \rangle_p = \bigoplus_{q \neq p} \langle [x]_p \rangle_q \). The parties now each obtain a binary share \( [x]_p \) without having to decrypt \( x \). This process is repeated for each party \( p \in \mathcal{P} \) to create binary secret shares of all \( |\mathcal{P}| \) arithmetic shares \( [x]_p \). Subsequently, the parties compute \( \langle x \rangle = \sum_{p \in \mathcal{P}} \langle [x]_p \rangle \). The summation is implemented by Ripple-carry adder that can be evaluated in \( \log_2(|\mathcal{P}| \log_2 2) \) rounds (Catrina & De Hoogh, 2010; Damgård et al., 2005).

**From \( \langle x \rangle \) to \( [x] \):** To convert from a binary share \( \langle x \rangle \) to an arithmetic share \( [x] \), the parties compute \( [x] = \sum_{b=1}^{2^b} \langle \langle x \rangle \rangle_b \), where \( \langle \langle x \rangle \rangle_b \) denotes the b-th bit of the binary share \( \langle x \rangle \) and \( B \) is the total number of bits in the shared secret. To create the arithmetic share of a bit, \( \langle \langle x \rangle \rangle_b \), each party \( p \in \mathcal{P} \) draws a number uniformly at random from \( \{0, 1\} \) and shares the difference between their bit and the random number with the other parties. The parties sum all resulting shares to obtain \( \langle \langle x \rangle \rangle_b \).

### A.2. Logical Operations and the Sign Function

We rely on binary secret sharing to implement logical operations and the sign function.

**XOR and AND.** XOR and AND are addition and multiplication modulo 2 where the numbers belong to the set \( \{0, 1\} \). As a result, the techniques we use for addition and multiplication of arithmetically shared values (see paper) can be used to implement XOR and AND as well. Evaluating \( \langle x \rangle \oplus a \) function amounts to one party \( p \in \mathcal{P} \) computing \( \langle x \rangle_a \oplus a \), and evaluating \( \langle x \rangle \otimes (y) \) amounts to each party \( p \in \mathcal{P} \) computing \( \langle x \rangle_p \otimes (y)_p \). Similarly, \( \langle x \rangle \otimes a \) is evaluated by having each party compute \( \langle x \rangle_p \otimes a \). The AND operation between two private values, \( \langle x \rangle \langle y \rangle \), is implemented akin to the private multiplication protocol using Beaver triples.

### Algorithm 3 Private computation of the number of wraps in an arithmetically shared value.

**Input:**
- Arithmetically secret-shared value, \( [x] \).
- A store of random numbers and the number of wraps in those numbers generated off-line.

**Output:** Arithmetic sharing of the number of wraps in \([x] \), denoted by \([\theta_x] \).

Select next random number, \([r] \), and the number of wraps \([\theta_r] \) in \([r] \).

for \( p \in \mathcal{P} \) do

Party \( p \) computes \( \beta_{x_p r_p} \) from \([x]_p \) and \([r]_p \) such that

\[ [z]_p = [x]_p [r]_p - \beta_{x_p r_p} Q \]

end for

Parties decrypt \([z] \) to obtain \( z \) (note that \( z \) contains no information about \( x \)).

Parties compute number of wraps in \( z, \theta_z \).

Parties compute \( \eta_{z r} \) ← \( z < [r] \).

Parties compute \([\theta_x] \) ← \( \theta_x + [\beta_{x r}] - [\theta_r] - [\eta_{z r}] \) (number of wraps in \([x] \)).

**Sign function.** We express the sign function on an arithmetically shared value as \([x \geq 0] \). Using this expression, the sign function can be implemented by first converting the arithmetic share, \([x] \), to a binary share, \( \langle x \rangle \), using the conversion procedure described above. Subsequently, we obtain the most significant bit, \( \langle x \rangle^{(MSB)} \), and convert it back to an arithmetic share to obtain \([x \geq 0] \).

### B. Fixed-Precision Encoding

Contextual bandit algorithms generally use real-valued parameters and data. Therefore, we need to encode the real-valued numbers as integers before we can arithmetically share them. We do so by multiplying \( x \in \mathbb{R} \) with a large scaling factor \( B \) and rounding to the nearest integer: \( \hat{x} = \lfloor Bx \rfloor \), where \( B = 2^L \) for some precision parameter, \( L \). We decode an encoded value, \( \hat{x} \), by computing \( x = \hat{x}/B \). Encoding real-valued numbers this way incurs a precision loss that is inversely proportional to \( L \).

Since we scale by a factor \( B = 2^L \) to encode floating-point numbers, we must scale down by a factor \( 2^L \) after every multiplication. We do this using the public division protocol described in Appendix C.

### C. Public Division

A simple method to divide an arithmetically shared value, \([x] \), by a public value, \( \ell \), would simply divide the share of
each party by $\ell$). However, such a method can produce incorrect results when the sum of shares “wraps around” the ring size, $Q$. Defining $\theta_x$ to be the number of wraps such that $x = \sum_{p \in P} [x]_p - \theta_x Q$, indeed, we observe that:

$$
\frac{x}{\ell} = \sum_{p \in P} \frac{[x]_p}{\ell} - \frac{\theta_x}{\ell} Q \neq \sum_{p \in P} \frac{[x]_p}{\ell} - \theta_x Q.
$$

Therefore, the simple division method fails when $\theta_x \neq 0$, which happens with probability $P(\theta_x \neq 0) = \frac{\theta}{Q}$ in the two-party case. Many prior MPC implementations specialize to the $|P| = 2$ case and rely on this probability being negligible (Mohassel & Zhang, 2017; Riazi et al., 2017; Wagh et al., 2018). However, when $|P| > 2$ the probability of failure grows rapidly and we must account for the number of wraps, $\theta_x$.

We do so by privately computing a secret share of the number of wraps in $x$, $[\theta_x]$, using Algorithm 3. We use $[\theta_x]$ to compute the correct value of the division by $\ell$:

$$
\frac{x}{\ell} = [z] - [\theta_x]Q \ell
$$

where $[z] = \{\forall p \in P: [x]_p/\ell\}$.

In practice, it can be difficult to compute $[\eta_{xr}]$ in Algorithm 3 (line 8). We note that $\eta_{xr}$ has a fixed probability of being non-zero, irrespective of whether the number of parties is two or larger, i.e., regardless of the number of parties $P(\eta_{xr} \neq 0) = \frac{\theta}{Q}$. In practice, we therefore skip the computation of $[\eta_{xr}]$ and simply set $\eta_{xr} = 0$. This implies that incorrect results can be produced by our algorithm with small probability. For example, when we multiply two real-values, $\hat{x}$ and $\hat{y}$, the result will be encoded as $B^2 \hat{x} \hat{y}$ which has probability $\frac{B^2 \hat{x} \hat{y}}{Q}$ of producing an error. This probability can be reduced by increasing $Q$ or reducing the precision parameter, $B$.

### D. Numerical Precision

In addition to the experiments presented in the paper, we also performed experiments to measure the impact of varying the precision in the fixed-point encoding and the numerical approximations. Figure 8 shows the average reward as a function of the bits of precision used in encoding of floating-point values. The optimal precision is 20 bits with a sharp drop in reward obtained below 18 and above 22 bits. The drop below 18 bits is due to precision loss causing numerical instability. Algorithm 1 in the main paper is susceptible to three forms of numerical instability: (1) ill-conditioning due to relying on the normal equations to solve the least-squares problem, (2) degeneracies in the matrix $W^{-1}$ which can become singular or non-positive-definite, and (3) cancellation errors due to use of the Sherman-Morrison update. The drop in observed reward when using more than 22 bits of precision is due to wrap-around errors that arise because we do not correctly compute $\eta_{xr}$ (see Section B). This causes public divisions to fail catastrophically with higher probability, impeding the accuracy of the learning algorithm.

Figure 9 shows how the average reward changes as a function of the number of Newton-Rhapson iterations used to privately computing the reciprocal function (see Section 3 in the main paper for details).