Delta shocks and vacuum states for the isentropic magnetogasdynamics equations for Chaplygin gas as pressure and magnetic field vanish

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Abstract
This paper is concerned with the Riemann problem for the isentropic Chaplygin gas magnetogasdynamics equations and the formation of δ-shocks and vacuum states as pressure and magnetic field vanish. Firstly, the Riemann problem of the isentropic magnetogasdynamics equations for Chaplygin gas is solved analytically. Secondly, it is rigorously proved that, as both the pressure and the magnetic field vanish, the Riemann solution containing two shocks tends to a δ-shock solution to the transport equations, and the intermediate density between the two shocks tends to a weighted δ-measure which form the δ-shock; while the Riemann solution containing two rarefaction waves tends to a two-contact-discontinuity solution to the transport equations if we do not consider the virtual velocity in the vacuum region, the intermediate state between the two contact discontinuities is a vacuum state. Moreover, we also give some numerical simulations to confirm the theoretical analysis.

Keywords Isentropic magnetogasdynamics · Chaplygin gas · Riemann problem · Transport equations · δ-shock · Vacuum state · Numerical simulations

Mathematics Subject Classification 35L65 · 35L67

1 Introduction

In this paper, we are concerned with the system of conservation law governing the one-dimensional unsteady simple flow of an isentropic, inviscid and perfectly conducting
compressible fluid subjected to a transverse magnetic field (see [1, 2]):

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (p + \rho u^2 + B^2/2\mu)_x &= 0,
\end{align*}
\]

(1.1)

where \(\rho > 0, u, p : \mathbb{R} \rightarrow \mathbb{R}, B \) and \(\mu > 0\) represent the density, velocity, pressure, transverse magnetic field and magnetic permeability, respectively; \(p\) and \(B\) are known functions defined as

\[ p = -\frac{k_1}{\rho} \]

(1.2)

and \(B = k_2\rho\), where \(k_1\) and \(k_2\) are positive constants. The independent variables \(t\) and \(x\) denote time and space, respectively. The adiabatic exponent in (1.2) can be viewed as \(\gamma = -1\) by comparing with the state equation \(p = k_1 \rho^\gamma\) with \(\gamma \geq 1\) for the polytropic gas. Equation (1.2) was introduced by Chaplygin [3], Tsien [4] and von Karman [5] as a suitable mathematical approximation for calculating the lifting force on a wing of an airplane in aerodynamics. A gas is called the Chaplygin gas if it satisfies the equation of state (1.2). The Chaplygin gas possesses a negative pressure and occurs in certain theories of cosmology. It is one of the most important candidates for the dark energy in describing the present accelerated expansion of the universe. It is also a unified model of dark matter and dark energy since it behaves like pure dark matter at early times (high density) and like dark energy at late times (small density) [6–8].

For the isentropic Euler equations, Brenier [9] firstly studied the 1-D Riemann problem and obtained solutions with concentration when initial data belongs to a certain domain in the phase plane. Furthermore, Guo et al. [10] abandoned this constrain and constructively obtained the global solutions to the 1-D Riemann problem, in which the \(\delta\)-shock is developed. Moreover, they also systematically studied the 2-D Riemann problem for isentropic Chaplygin gas equations. For the 2-D case, we can also refer to [11] in which D. Serre studied the interaction of the pressure waves for the 2-D isentropic irrotational Chaplygin gas and constructively proved the existence of transonic solutions for two cases, saddle and vortex of 2-D Riemann problem. Recently, Sheng, Wang and Yin [12] and Wang [13] studied the Riemann problem for the generalized Chaplygin gas, obtained the solutions to the Riemann problem and the interactions of elementary waves. The Riemann solutions to the transport equations in zero-pressure flow in gas dynamics were presented by Sheng and Zhang in [14], in which \(\delta\)-shocks and vacuum states appeared. Concerning the pressureless gas dynamics, we also refer the readers to the work by Bouchut [15–17] or by Boudin [18].

In related researchs of the \(\delta\)-shocks, one very important and interesting topic is to study the phenomena of concentration and cavitation and the formation of \(\delta\)-shocks and vacuum states in solutions. In earlier paper [19], Chen and Liu [19] studied the formation of \(\delta\)-shocks and vacuum states of the Riemann solutions to the isentropic Euler equations for polytropic gas as \(\varepsilon \to 0\), in which they took the equation of state as \(P = \varepsilon p\) for \(p = \rho^{\gamma}/\gamma (\gamma > 1)\). Further, they also obtained the same results for the Euler equations for nonisentropic fluids in [20]. The same problem for the isentropic Euler equations for isothermal case was studied by Li [21], in which he proved that
when temperature drops to zero, the solution containing two shocks converges to the \( \delta \)-shock solution to the transport equations and the solution containing two rarefaction waves converges to the solution involving vacuum to the transport equations. Then, the results were extended to the relativistic Euler equations for polytropic gas by Yin and Sheng [22] and for Chaplygin gas by Yin and Song [23], the isentropic Euler equations for the generalized Chaplygin gas by Sheng, Wang and Yin [12] and for modified Chaplygin gas by Yang and Wang [24], the isentropic magnetogasdynamics equations for generalized Chaplygin gas by Chen and Sheng [25], the generalized pressureless gas dynamics model with a scaled pressure term by Mitrovic and Nedeljkov [26], the perturbed Aw-Rascle model by Shen and Sun [27], etc.

When the pressure law is given by a standard pressure law \( p = k_1 \rho^\gamma \) \((1 \leq \gamma \leq 2)\), the Riemann problem and interactions of elementary waves for (1.1) are well investigated in [1]. Recently, the limiting behavior of Riemann solutions to this system as both the pressure and the magnetic field vanish was also studied by Shen [2]. However, the adiabatic constant \( \gamma = -1 \) in (1.2) makes the problem different from the polytropic gas (for which the adiabatic constant \( \gamma \) is greater or equal than 1). Thus, in the present paper, the author restricts to a pressure law given by an equation of state (1.2) which was introduced by Chaplygin. Both pressure law and transverse magnetic field are governed by two parameters. The author studies for the behavior of the Riemann solution in the limit of the two parameters to zero. In these limits, the model coincides, in a sense to be prescribed, with the pressureless gas dynamics well-known to involve solutions containing \( \delta \)-shocks and/or vacuum regions. Exhibiting the Riemann solution of the system under consideration, the author recovers the expected \( \delta \)-shock solutions and vacuum solutions. The paper is also supplemented with numerical experiments to illustrate the presented solution behaviors.

The rest of this paper is organized as follows. In Sects. 2 and 3, the Riemann problems for the isentropic Chaplygin gas magnetogasdynamics equations and the transport equations are analyzed by characteristic analysis. In Sects. 4 and 5, we investigate the formation of \( \delta \)-shocks and vacuum states of the Riemann solutions to the isentropic magnetogasdynamics equations for Chaplygin gas as pressure and magnetic field vanish. In Sect. 6, we present some representative numerical simulations to demonstrate the validity of the theoretical analysis.

2 Riemann problem for the isentropic Chaplygin gas magnetogasdynamics equations

In this section, we discuss the Riemann solutions of (1.1) and (1.2) with initial data

\[
(\rho, u)(x, 0) = (\rho_{\pm}, u_{\pm}), \quad \pm x > 0, \quad (2.1)
\]

where \( \rho_{\pm} > 0 \) and \( u_{\pm} \) are arbitrary constants.

For smooth solution, system (1.1) is equivalent to

\[
\left( \begin{array}{c}
\rho \\
u \\
\end{array} \right)_t + \left( \begin{array}{c}
u \\
\frac{w^2}{\rho} \\
u \\
\end{array} \right)_x = 0, \quad (2.2)
\]
where \( w = (c^2 + b^2)^{1/2} \) is the magneto-acoustic speed with \( c = (p'(\rho))^{1/2} \) as the local sound speed and \( b = (B^2(\rho)/\mu\rho)^{1/2} \) the Alfven speed. In page 138 of Francis F. Chen [28], the Alfven speed is defined (also see Chen [29], Pages 319–324). The eigenvalues of system (1.1) and (1.2) are

\[
\lambda_1 = u - \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}}, \quad \lambda_2 = u + \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}}.
\]

Therefore, system (1.1) and (1.2) is strictly hyperbolic for \( \rho > 0 \).

The corresponding right eigenvectors are

\[
\vec{r}_1 = \left( -\rho, \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}} \right)^T, \quad \vec{r}_2 = \left( \rho, \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}} \right)^T.
\]

By simple calculation, we get

\[
\nabla \lambda_i \cdot \vec{r}_i = \frac{3k_2^2\rho}{2\mu \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}}} \neq 0, \quad i = 1, 2.
\]

Therefore, both the characteristic fields are genuinely nonlinear.

Since system (1.1), (1.2) and the Riemann data (2.1) are invariant under stretching of coordinates: \((x, t) \rightarrow (\alpha x, \alpha t)\) (\(\alpha\) is constant), we seek the self-similar solution

\[(\rho, u)(x, t) = (\rho, u)(\xi), \quad \xi = \frac{x}{t}.
\]

Then Riemann problem (1.1), (1.2) and (2.1) is reduced to the following boundary value problem of ordinary differential equations:

\[
-\xi \rho \xi + (\rho u)\xi = 0,
\]

\[
-\xi (\rho u)\xi + \left( -\frac{k_1}{\rho} + \rho u^2 + \frac{(k_2\rho)^2}{2\mu} \right)\xi = 0,
\]

(2.3)

with \((\rho, u)(\pm \infty) = (\rho_\pm, u_\pm)\).

For any smooth solution, system (2.3) can be written as

\[
\begin{pmatrix}
  u - \xi \\
  -\xi u + \frac{k_1}{\rho^2} + u^2 + \frac{k_2^2\rho}{\mu} - \xi \rho + 2\rho u
\end{pmatrix}
\begin{pmatrix}
  \rho \xi \\
  u_\xi
\end{pmatrix} = 0.
\]

(2.4)

It provides either general solutions (constant states)

\[(\rho, u)(\xi) = \text{constant} \quad (\rho > 0)\]
or singular solutions called the rarefaction waves \( R_1 \) and \( R_2 \) which denote, respectively, 1-rarefaction waves and 2-rarefaction waves,

\[
R_1: \begin{cases} 
  \xi = \lambda_1 = u - \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2 \rho}{\mu}}, \\
  u - u_- = -\int_{\rho_-}^{\rho} \frac{\sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2 \rho}{\mu}}}{s} ds, \quad \rho < \rho_-,
\end{cases}
\]

and

\[
R_2: \begin{cases} 
  \xi = \lambda_2 = u + \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2 \rho}{\mu}}, \\
  u - u_- = \int_{\rho_-}^{\rho} \frac{\sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2 \rho}{\mu}}}{s} ds, \quad \rho > \rho_-.
\end{cases}
\]

Differentiating the second equation of (2.5) with respect to \( \rho \) yields

\[
u_\rho = -\frac{\sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2 \rho}{\mu}}}{\rho} < 0,
\]

and subsequently,

\[
u_{\rho\rho} = \frac{4k_1 + \frac{k_2^2 \rho}{\mu}}{2\rho^2 \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2 \rho}{\mu}}} > 0,
\]

which mean that the 1-rarefaction wave curve \( R_1 \) is monotonic decreasing and convex in the \((\rho, u)\) plane \((\rho > 0)\). Similarly, from the second equation of (2.6), we have \( u_\rho > 0 \) and \( u_{\rho\rho} < 0 \), which mean that the 2-rarefaction wave curve \( R_2 \) is monotonic increasing and concave in the \((\rho, u)\) plane \((\rho > 0)\). In addition, it can be verified that \( \lim_{\rho \to 0^+} u = +\infty \) for the 1-rarefaction wave curve \( R_1 \), which implies that \( R_1 \) has the \( u \)-axis as its asymptotic line. It can also be proved that \( \lim_{\rho \to +\infty} u = +\infty \) for the 2-rarefaction wave curve \( R_2 \).

For a bounded discontinuity at \( \xi = \sigma \), the Rankine-Hugoniot conditions hold:

\[
\begin{align*}
  -\sigma [\rho] + [\rho u] &= 0, \\
  -\sigma [\rho u] + [-\frac{k_1}{\rho} + \rho u^2 + \frac{(k_2 \rho)^2}{2\mu}] &= 0,
\end{align*}
\]

where \([\rho] = \rho - \rho_-\), etc. Solving (2.7), we obtain two shocks \( S_1 \) and \( S_2 \)

\[
S_1: \begin{cases} 
  \sigma = u_- - \rho \sqrt{\frac{1}{\rho_-} \left( \frac{k_1}{\rho_-} + \frac{k_2^2 (\rho + \rho_-)}{2\mu} \right)}, \\
  u = u_- - \frac{1}{\rho_-} \left( \frac{k_1}{\rho_-} + \frac{k_2^2 (\rho + \rho_-)}{2\mu} \right) (\rho - \rho_-), \quad \rho > \rho_-, \quad \rho > \rho_-.
\end{cases}
\]
Fig. 1 Curves of elementary waves

\[
S_2 : \begin{cases}
\sigma = u_- + \rho \sqrt{\frac{1}{\rho \rho_-} \left( \frac{k_1}{\rho \rho_-} + \frac{k_2^2(\rho + \rho_-)}{2\mu} \right)}, \\
u = u_- + \frac{1}{\rho \rho_-} \left( \frac{k_1}{\rho \rho_-} + \frac{k_2^2(\rho + \rho_-)}{2\mu} \right) (\rho - \rho_-), \ \rho < \rho_-.
\end{cases}
\] (2.9)

Differentiating the second equation of (2.8) with respect to \( \rho \), for \( \rho > \rho_- \) we have

\[
u_\rho = -\frac{1}{2} \sqrt{\frac{1}{\rho \rho_-} \left( \frac{k_1}{\rho \rho_-} + \frac{k_2^2(\rho + \rho_-)}{2\mu} \right)} \left( \frac{2k_1}{\rho - \rho_-} + \frac{k_2^2}{\rho - \mu} + \frac{k_2^2}{2\rho \mu} + \frac{k_2^2}{2\rho^2 \mu} \right) < 0,
\]

which mean that the 1-shock curve \( S_1 \) is monotonic decreasing in the \((\rho, u)\) plane \((\rho > 0)\). Similarly, from the second equation of (2.9), for \( \rho < \rho_- \) we have \( u_\rho > 0 \), which mean that the 2-shock curve \( S_2 \) is monotonic increasing in the \((\rho, u)\) plane \((\rho > 0)\). In addition, it is easily derived from (2.9) that \( \lim_{\rho \to 0^+} u = -\infty \) for the 2-shock curve \( S_2 \), which implies that \( S_2 \) has the \( u \)-axis as its asymptotic line. It can also be derived from (2.8) that \( \lim_{\rho \to +\infty} u = -\infty \) for the 1-shock curve \( S_1 \).

In the phase plane \((\rho > 0, u \in \mathbb{R})\), through point \((\rho_-, u_-)\), we draw the elementary wave curves \( R_1, R_2, S_1 \) and \( S_2 \), respectively. Then the phase plane is divided into four regions I, II, III and IV \((\rho_-, u_-)\) (see Fig. 1).

By the analysis method in phase plane, for any given state \((\rho_+, u_+)\), one can construct the Riemann solutions as follows:

1. \((\rho_+, u_+) \in I(\rho_-, u_-) : R_1 + R_2;\)
2. \((\rho_+, u_+) \in II(\rho_-, u_-) : R_1 + S_2;\)
3. \((\rho_+, u_+) \in III(\rho_-, u_-) : S_1 + R_2;\)
4. \((\rho_+, u_+) \in IV(\rho_-, u_-) : S_1 + S_2.\)

By Lax [30], we have the following result.

**Theorem 1** (This result is due to Lax [30]). For Riemann problem (1.1), (1.2) and (2.1), there exists a unique entropy solution, which consists of shocks, rarefaction waves, and constant states.
3 Riemann problem for the transport equations

In this section, we briefly review the Riemann solutions to the transport equations in zero-pressure flow and \( B = 0 \)

\[
\begin{aligned}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2)_x &= 0,
\end{aligned}
\]  

(3.1)

with the following Riemann initial data:

\[
(\rho, u)(x, 0) = (\rho_{\pm}, u_{\pm}), \quad \pm x > 0.
\]  

(3.2)

See Sheng and Zhang [14] for more details. Concerning the pressureless gas dynamics, we also refer the readers to the work by Bouchut [15–17] or by Boudin [18].

The system (3.1) has a double eigenvalue \( \lambda = u \) and only one right eigenvector \( \mathbf{r} = (r, 0)^T \). Furthermore, we have \( \nabla \lambda \cdot \mathbf{r} = 0 \), which means that (3.1) is nonstrictly hyperbolic and \( \lambda \) is linearly degenerate.

As usual, we seek the self-similar solution

\[
(\rho, u)(x, t) = (\rho, u)(\xi), \quad \xi = \frac{x}{t}.
\]

Then Riemann problem (3.1) and (3.2) is reduced to the following boundary value problem of ordinary differential equations:

\[
\begin{aligned}
-\xi \rho_\xi + (\rho u)_\xi &= 0, \\
-\xi (\rho u)_\xi + (\rho u^2)_\xi &= 0,
\end{aligned}
\]  

(3.3)

with \((\rho, u)(\pm \infty) = (\rho_{\pm}, u_{\pm})\).

For any smooth solution, system (3.3) can be written as

\[
\begin{pmatrix}
  u - \xi & \rho \\
  0 & \rho (u - \xi)
\end{pmatrix}
\begin{pmatrix}
  \rho_\xi \\
  u_\xi
\end{pmatrix} = 0.
\]  

(3.4)

It provides either the general solution (constant state)

\[
(\rho, u)(\xi) = \text{constant} \quad (\rho \neq 0)
\]

or the singular solution

\[
\begin{aligned}
\rho &= 0, \\
u &= \xi,
\end{aligned}
\]  

(3.5)

which is called the vacuum state (see [14]), where \( u(\xi) \) is an arbitrary smooth function.

For a bounded discontinuity at \( \xi = \sigma \), the Rankine–Hugoniot condition holds:

\[
\begin{aligned}
-\sigma [\rho] + [\rho u] &= 0, \\
-\sigma [\rho u] + [\rho u^2] &= 0,
\end{aligned}
\]  

(3.6)
where \( \lfloor q \rfloor = q_+ - q_- \) denotes the jump of \( q \) across the discontinuity. By solving (3.6), we obtain

\[
J : \xi = \sigma = u_- (= \lambda_-) = u_+ (= \lambda_+),
\]

which is a contact discontinuity. It is a slip line and just the characteristic of solutions on both sides in \((x, t)\)-plane.

The Riemann problem (3.1) and (3.2) can be solved by contact discontinuities, vacuum or \( \delta \)-shock connecting two constant states \((\rho_\pm, u_\pm)\).

For the case \( u_- < u_+ \), there is no characteristic passing through the region \( u_- t < x < u_+ t \) and the vacuum appears in this region. The solution can be expressed as

\[
(\rho, u)(\xi) = \begin{cases} 
(\rho_-, u_-), & -\infty < x < u_- , \\
(0, \xi), & u_- < \xi < u_+ , \\
(\rho_+, u_+), & u_+ < \xi < +\infty .
\end{cases}
\]

(3.8)

For the case \( u_- = u_+ \), it is easy to see that the constant states \((\rho_\pm, u_\pm)\) can be connected by a contact discontinuity.

For the case \( u_- > u_+ \), the characteristic lines originating from the origin will overlap in a domain \( \Omega \), as shown in Fig. 2. So, singularity must happen in \( \Omega \). It is easy to know that the singularity is impossible to be a jump with finite amplitude because the Rankine–Hugoniot condition is not satisfied on the bounded jump. In other words, there is no solution which is piecewise smooth and bounded. Motivated by [14], we seek solutions with \( \delta \)-distribution at the jump.

To do so, a two-dimensional weighted \( \delta \)-function \( w(s)\delta_L \) supported on a smooth curve \( L = \{(t(s), x(s)) : a < s < b\} \) is defined by

\[
\langle w(s)\delta_L, \varphi \rangle = \int_a^b w(s)\varphi(t(s), x(s))\, ds
\]

(3.9)

for any \( \varphi \in C_0^\infty (R \times R_+) \).
Let us consider a solution of (3.1) and (3.2) of the form

\[
(\rho, u)(x, t) = \begin{cases} 
(\rho_-, u_-), & x < \sigma t, \\
(w(t)\delta(x - \sigma t), \sigma), & x = \sigma t, \\
(\rho_+, u_+), & x > \sigma t,
\end{cases}
\] (3.10)

where \(\sigma\) is a constant, \(w(t) \in C^1[0, +\infty)\), and \(\delta(\cdot)\) is the standard Dirac measure. \(x(t), w(t)\) and \(\sigma\) are the location, weight and velocity of the \(\delta\)-shock, respectively. Then the following generalized Rankine–Hugoniot conditions hold:

\[
\begin{align*}
\frac{dx(t)}{dt} &= \sigma, \\
\frac{dw(t)}{dt} &= \sigma [\rho] - [\rho u], \\
\frac{d(w(t)\sigma)}{dt} &= \sigma [\rho u] - [\rho u^2].
\end{align*}
\] (3.11)

where \([\rho] = \rho_+ - \rho_-\), with initial data

\[(x, w)(0) = (0, 0).\] (3.12)

In addition, to guarantee uniqueness, the \(\delta\)-shock should satisfy the entropy condition:

\[u_+ < \sigma < u_- .\]

Solving the system of simple ordinary differential equations (3.11) with initial data (3.12), we have, when \(\rho_- = \rho_+\),

\[x(t) = \frac{1}{2}(u_- + u_+)t, \quad w(t) = (\rho_-u_- - \rho_+u_+)t, \quad \sigma = \frac{1}{2}(u_- + u_+);\]

when \(\rho_- \neq \rho_+\),

\[x(t) = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}t, \quad w(t) = \sqrt{\rho_-}\rho_+(u_- - u_+)t, \quad \sigma = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}.\]

4 Formation of \(\delta\)-shocks

In this section, we study the formation of \(\delta\)-shocks in the Riemann solutions of system (1.1) and (1.2) in the case \((\rho_+, u_+) \in IV(\rho_-, u_-)\) with \(u_- > u_+\) as both the pressure and the magnetic field vanish.
4.1 Limit behavior of Riemann solutions as $k_1, k_2 \to 0$

When $(\rho_+, u_+) \in IV(\rho_-, u_-)$, for each pair of fixed $k_1 > 0$ and $k_2 > 0$, suppose that $(\rho_*, u_*)$ is the intermediate state connected with $(\rho_-, u_-)$ by a 1-shock $S_1$ with speed $\sigma_1$ and $(\rho_+, u_+)$ by a 2-shock $S_2$ with speed $\sigma_2$. Then it follows

\begin{align*}
S_1 : & \quad \sigma_1 = u_- - \rho_* \sqrt{\frac{1}{\rho_+ \rho_-} \left( \frac{k_1}{\rho_+ \rho_-} + \frac{k_2^2 (\rho_+ + \rho_-)}{2\mu} \right)}, \\
& \quad u_* = u_- - \sqrt{\frac{1}{\rho_+ \rho_-} \left( \frac{k_1}{\rho_+ \rho_-} + \frac{k_2^2 (\rho_+ + \rho_-)}{2\mu} \right)} (\rho_* - \rho_-), \quad \rho_* > \rho_-,
\end{align*}

(4.1)

\begin{align*}
S_2 : & \quad \sigma_2 = u_* + \rho_+ \sqrt{\frac{1}{\rho_+ \rho_-} \left( \frac{k_1}{\rho_+ \rho_-} + \frac{k_2^2 (\rho_+ + \rho_-)}{2\mu} \right)}, \\
& \quad u_+ = u_* + \sqrt{\frac{1}{\rho_+ \rho_-} \left( \frac{k_1}{\rho_+ \rho_-} + \frac{k_2^2 (\rho_+ + \rho_-)}{2\mu} \right)} (\rho_+ - \rho_*), \quad \rho_+ < \rho_*.
\end{align*}

(4.2)

The addition of (4.1) and (4.2) gives

\begin{align*}
u_- - u_+ &= \sqrt{\frac{1}{\rho_-} - \frac{1}{\rho_*}} \left[ k_1 \left( \frac{1}{\rho_-} - \frac{1}{\rho_*} \right) + \frac{k_2^2 (\rho_*^2 - \rho_+^2)}{2\mu} \right] \\
&\quad + \sqrt{\frac{1}{\rho_+} - \frac{1}{\rho_*}} \left[ k_1 \left( \frac{1}{\rho_+} - \frac{1}{\rho_*} \right) + \frac{k_2^2 (\rho_*^2 - \rho_-^2)}{2\mu} \right], \quad \rho_* > \rho_\pm. \tag{4.3}
\end{align*}

For any given $\rho_\pm > 0$, if \( \lim_{k_1,k_2 \to 0} \rho_* = M \in [\max(\rho_-, \rho_+), +\infty) \), then by taking the limit $k_1, k_2 \to 0$ in (4.3), we have $u_- - u_+ = 0$, which contradicts with $u_- > u_+$. Therefore, $\lim_{k_1,k_2 \to 0} \rho_* = +\infty$. Letting $k_1, k_2 \to 0$ in (4.3), we obtain the following result.

Lemma 1 It holds that

\begin{align*}
\lim_{k_1,k_2 \to 0} k_2^2 \rho_*^2 &= \frac{2\mu \rho_- \rho_+ (u_- - u_+)^2}{(\sqrt{\rho_-} + \sqrt{\rho_+})^2}.
\end{align*}

(4.4)

Lemma 2 Setting $\sigma = \frac{\sqrt{\rho_- u_-} + \sqrt{\rho_+ u_+}}{\sqrt{\rho_-} + \sqrt{\rho_+}}$, we have

\begin{align*}
\lim_{k_1,k_2 \to 0} u_* = \lim_{k_1,k_2 \to 0} \sigma_1 = \lim_{k_1,k_2 \to 0} \sigma_2 = \sigma,
\end{align*}

(4.5)

and

\begin{align*}
\lim_{k_1,k_2 \to 0} \int_{\sigma_1 t}^{\sigma_2 t} \rho_* dx = (\sigma [\rho] - [\rho u]) t = \sqrt{\rho_- \rho_+ (u_- - u_+)} t = w(t).
\end{align*}

(4.6)
**Proof** Letting $k_1, k_2 \to 0$ in (4.1) and noting Lemma 1, we have

$$
\lim_{k_1,k_2 \to 0} u_* = u_- - \lim_{k_1,k_2 \to 0} \frac{k_1}{\rho_-} \sqrt{\frac{1}{\rho_-} - \frac{1}{\rho_*}} \sqrt{k_1 \left( \frac{1}{\rho_-} - \frac{1}{\rho_*} \right) + \frac{k_2^2 (\rho_*^2 - \rho_-^2)}{2\mu}}
$$

$$
= u_- - \sqrt{\frac{\rho_- \rho_+(u_- - u_+)^2}{(\sqrt{\rho_-} + \sqrt{\rho_+})^2}} = \frac{\sqrt{\rho_- u_-} + \sqrt{\rho_+ u_+}}{\sqrt{\rho_-} + \sqrt{\rho_+}} = \sigma. \quad (4.7)
$$

From the first equation of (4.1), by Lemma 1, we obtain

$$
\lim_{k_1,k_2 \to 0} \sigma_1 = u_- - \lim_{k_1,k_2 \to 0} \frac{k_1}{\rho_-^2} + \frac{k_2^2 \rho_+^2 \left( \frac{1}{\rho_-} + \frac{1}{\rho_*} \right)}{2\mu}
$$

$$
= u_- - \sqrt{\frac{\rho_+(u_- - u_+)^2}{(\sqrt{\rho_-} + \sqrt{\rho_+})^2}} = \frac{\sqrt{\rho_- u_-} + \sqrt{\rho_+ u_+}}{\sqrt{\rho_-} + \sqrt{\rho_+}} = \sigma. \quad (4.8)
$$

From (4.2) and (4.7), we can easily get

$$
\lim_{k_1,k_2 \to 0} \sigma_2 = \lim_{k_1,k_2 \to 0} u_* + \lim_{k_1,k_2 \to 0} \frac{k_1}{\rho_*^2} + \frac{k_2^2 \rho_+^2 \left( \frac{1}{\rho_*} + \frac{1}{\rho_+} \right)}{2\mu} = \sigma. \quad (4.9)
$$

Thus it can be seen from (4.8) and (4.9) that when $k_1, k_2 \to 0$, the two shocks $S_1$ and $S_2$ will coincide whose velocities are identical with that of the $\delta$-shock of the transport equations with the same Riemann initial data $(\rho_\pm, u_\pm)$.

Using the Rankine–Hugoniot conditions (2.7) for $S_1$ and $S_2$, we have

$$
\begin{align*}
\sigma_1(\rho_* - \rho_-) &= \rho_* u_* - \rho_- u_-, \\
\sigma_2(\rho_+ - \rho_*) &= \rho_+ u_+ - \rho_* u_.
\end{align*} \quad (4.10)
$$

Then from (4.8) and (4.9) it follows that

$$
\lim_{k_1,k_2 \to 0} (\sigma_1 - \sigma_2) \rho_* = \lim_{k_1,k_2 \to 0} (\rho_+ u_+ - \rho_- u_- + \sigma_1 \rho_- - \sigma_2 \rho_+) = [\rho u] - \sigma [\rho]. \quad (4.11)
$$

This implies that

$$
\lim_{k_1,k_2 \to 0} \int_{\sigma_1 t}^{\sigma_2 t} \rho_* \, dx = (\sigma [\rho] - [\rho u]) t = \sqrt{\rho_- \rho_+(u_- - u_+)} t = w(t). \quad (4.12)
$$

The proof is completed. \(\square\)

**Remark 1** From the above results, it can be seen that the limit of the Riemann solution of system (1.1) and (1.2) as $k_1, k_2 \to 0$ in the case $(\rho_+, u_+) \in IV(\rho_-, u_-)$ is just the $\delta$-shock solution of (3.1)–(3.2) when $u_- > u_+$.
4.2 δ-shocks and concentration

Now, we give the following results which give a very nice depiction of the limit in the case \((\rho_+, u_+) \in IV(\rho_-, u_-)\).

**Theorem 2** Let \(u_- > u_+\) and \((\rho_+, u_+) \in IV(\rho_-, u_-)\). For any fixed \(k_1, k_2 > 0\), assuming that \((\rho, u)\) is a solution containing two shocks \(S_1\) and \(S_2\) of (1.1)–(1.2) with Riemann initial data (2.1), constructed in Sect. 2, it is obtained that as \(k_1, k_2 \to 0\), \((\rho, u)\) converges in the sense of distributions, and the limit functions \(\rho\) and \(\rho u\) are the sums of a step function and a \(\delta\)–measure with weights

\[
(\sigma[\rho] - [\rho u])t \quad \text{and} \quad (\sigma[\rho u] - [\rho u^2])t,
\]

respectively, which form a δ-shock solution of (3.1) with the same Riemann initial data \((\rho_\pm, u_\pm)\).

**Proof** Let \(\xi = x/t\). Then for any fixed \(k_1 > 0\) and \(k_2 > 0\), the Riemann solution to the isentropic magnetogasdynamics equations for Chaplygin gas (1.1)–(1.2) can be written as

\[
(\rho, u)(\xi) = \begin{cases}
(\rho_-, u_-), & \xi < \sigma_1, \\
(\rho_*, u_*), & \sigma_1 < \xi < \sigma_2, \\
(\rho_+, u_+), & \xi > \sigma_2,
\end{cases}
\]

which satisfies the following weak formulations:

\[
\int_{-\infty}^{+\infty} (\xi - u(\xi))\rho(\xi)\psi'(\xi)d\xi + \int_{-\infty}^{+\infty} \rho(\xi)\psi(\xi)d\xi = 0 \quad (4.14)
\]

and

\[
\int_{-\infty}^{+\infty} (\xi - u(\xi))\rho(\xi)u(\xi)\psi'(\xi)d\xi - \int_{-\infty}^{+\infty} \left(-\frac{k_1}{\rho(\xi)} + \frac{k_2(\rho(\xi))^2}{2\mu}\right)\psi'(\xi)d\xi + \int_{-\infty}^{+\infty} \rho(\xi)u(\xi)\psi(\xi)d\xi = 0 \quad (4.15)
\]

for any test function \(\psi \in C_0^\infty(-\infty, +\infty)\).

The first integral on the left-hand side of (4.15) can be decomposed into

\[
\left\{ \int_{-\infty}^{\sigma_1} + \int_{\sigma_1}^{\sigma_2} + \int_{\sigma_2}^{+\infty} \right\} (\xi - u(\xi))\rho(\xi)u(\xi)\psi'(\xi)d\xi. \quad (4.16)
\]

The sum of the first and the last terms in (4.16) is

\[
\int_{-\infty}^{\sigma_1} (\xi - u(\xi))\rho(\xi)u(\xi)\psi'(\xi)d\xi + \int_{\sigma_2}^{+\infty} (\xi - u(\xi))\rho(\xi)u(\xi)\psi'(\xi)d\xi
\]
\[
\begin{align*}
&= \rho_- u_- \sigma_1 \psi(\sigma_1) - \rho_- u_-^2 \psi(\sigma_1) - \rho_- u_- \int_{-\infty}^{\sigma_1} \psi(\xi) d\xi \\
&\quad - \rho_+ u_+ \sigma_2 \psi(\sigma_2) + \rho_+ u_+^2 \psi(\sigma_2) - \rho_+ u_+ \int_{\sigma_2}^{+\infty} \psi(\xi) d\xi.
\end{align*}
\] (4.17)

Taking the limit \( k_1, k_2 \to 0 \) in (4.17) leads to

\[
\lim_{k_1, k_2 \to 0} \left( \int_{-\infty}^{\sigma_1} + \int_{\sigma_2}^{+\infty} \right) (\xi - u(\xi)) \rho(\xi) u(\xi) \psi'(\xi) d\xi
\]

\[
= ([\rho u^2] - \sigma [\rho u]) \psi(\sigma) - \int_{-\infty}^{+\infty} (\rho_0 u_0)(\xi - \sigma) \cdot \psi(\xi) d\xi,
\] (4.18)

where \( (\rho_0 u_0)(\xi) = \rho_- u_- + [\rho u]H(\xi) \) and \( H \) is the Heaviside function.

For the second term in (4.16), integrating by parts again, we obtain

\[
\int_{\sigma_1}^{\sigma_2} (\xi - u(\xi)) \rho(\xi) u(\xi) \psi'(\xi) d\xi = \int_{\sigma_1}^{\sigma_2} (\xi - u(\xi)) \rho^*_u u^*_\sigma \psi'(\xi) d\xi
\]

\[
= -\rho_+ u^*_\sigma^2 (\psi(\sigma_2) - \psi(\sigma_1)) + \rho_+ u^*_\sigma (\sigma_2 \psi(\sigma_2) - \sigma_1 \psi(\sigma_1)) - \rho_+ u^*_\sigma \int_{\sigma_1}^{\sigma_2} \psi(\xi) d\xi
\]

\[
= -u^*_\sigma \rho^*_u (\sigma_2 - \sigma_1) \left( \frac{\psi(\sigma_2) - \psi(\sigma_1)}{\sigma_2 - \sigma_1} \right) + \frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} \psi(\xi) d\xi
\]

(4.19)

Taking the limit \( k_1, k_2 \to 0 \) in (4.19), noting (4.11) and the fact that both \( \psi \in C^\infty_0(-\infty, +\infty) \) and \( \lim_{k_1, k_2 \to 0} u_* = \lim_{k_1, k_2 \to 0} \sigma_1 = \lim_{k_1, k_2 \to 0} \sigma_2 = \sigma \), we deduce that

\[
\lim_{k_1, k_2 \to 0} \int_{\sigma_1}^{\sigma_2} (\xi - u(\xi)) \rho(\xi) u(\xi) \psi'(\xi) d\xi
\]

\[
= \sigma ([\rho u] - \sigma [\rho]) (\sigma \psi'(\sigma) - \sigma \psi'(\sigma) - \psi(\sigma) + \psi(\sigma)) = 0.
\] (4.20)

Similarly, the first integral on the left-hand side of (4.15) can be decomposed into three parts as

\[
- \left\{ \int_{-\infty}^{\sigma_1} + \int_{\sigma_1}^{\sigma_2} + \int_{\sigma_2}^{+\infty} \right\} \left( -\frac{k_1}{\rho(\xi)} + \frac{k_2^2 (\rho(\xi))^2}{2\mu} \right) \psi'(\xi) d\xi,
\] (4.21)

which equals to

\[
\int_{-\infty}^{\sigma_1} \left( \frac{k_1}{\rho_-} - \frac{k_2^2 \rho_-^2}{2\mu} \right) \psi'(\xi) d\xi + \int_{\sigma_1}^{\sigma_2} \left( \frac{k_1}{\rho_*} \right)
\]
\[
- \frac{k_2^2 \rho_s^2}{2\mu} \psi'(\xi) d\xi + \int_{\sigma_2}^{+\infty} \left( \frac{k_1}{\rho_+} - \frac{k_2^2 \rho_s^2}{2\mu} \right) \psi'(\xi) d\xi = \left( \frac{k_1}{\rho_-} - \frac{k_2^2 \rho_s^2}{2\mu} \right) \psi(\sigma_1) + \frac{k_1}{\rho_s} (\psi(\sigma_2) - \psi(\sigma_1))
\]

\[
- \frac{k_2^2 \rho_s^2}{2\mu} (\psi(\sigma_2) - \psi(\sigma_1)) - \left( \frac{k_1}{\rho_+} - \frac{k_2^2 \rho_s^2}{2\mu} \right) \psi(\sigma_2).
\]

(4.22)

Taking the limit \( k_1, k_2 \to 0 \) in (4.22), by Lemmas 1–2, we have

\[
\lim_{k_1, k_2 \to 0} \int_{-\infty}^{+\infty} \left( \frac{k_1}{\rho(\xi)} - \frac{k_2^2 (\rho(\xi))^2}{2\mu} \right) \psi'(\xi) d\xi = 0.
\]

(4.23)

Summarizing (4.18), (4.20) and (4.23) leads to

\[
\lim_{k_1, k_2 \to 0} \int_{-\infty}^{+\infty} ((\rho u)(\xi) - (\rho_0 u_0)(\xi - \sigma)) \psi(\xi) d\xi = (\sigma [\rho u] - [\rho u^2]) \psi(\sigma),
\]

(4.24)

which is true for any \( \psi \in C_0^\infty(-\infty, +\infty) \).

As done previously, we can obtain the limit for the first integral on the left-hand side of (4.14) as

\[
\lim_{k_1, k_2 \to 0} \int_{-\infty}^{+\infty} (\xi - u(\xi)) \rho(\xi) \psi'(\xi) d\xi
\]

\[
= ([\rho u] - \sigma [\rho]) \psi(\sigma) - \int_{-\infty}^{\sigma} \rho_+ \psi(\xi) d\xi - \int_{\sigma}^{+\infty} \rho_+ \psi(\xi) d\xi
\]

\[
= ([\rho u] - \sigma [\rho]) \psi(\sigma) - \int_{-\infty}^{+\infty} \rho_0 (\xi - \sigma) \psi(\xi) d\xi,
\]

(4.25)

where \( \rho_0(\xi) = \rho_- + [\rho] H(\xi) \). Then returning to the formulation (4.14), we have

\[
\lim_{k_1, k_2 \to 0} \int_{-\infty}^{+\infty} (\rho(\xi) - \rho_0(\xi - \sigma)) \psi(\xi) d\xi = (\sigma [\rho] - [\rho u]) \psi(\sigma).
\]

(4.26)

which is true for any \( \psi \in C_0^\infty(-\infty, +\infty) \).

Finally, we study the limits of \( \rho \) and \( \rho u \) as \( k_1, k_2 \to 0 \), by tracing the time-dependence of weights of the \( \delta \)-measure. Let \( \phi(x, t) \in C_0^\infty((-\infty, +\infty) \times [0, +\infty)) \), then we have
\[
\lim_{k_1,k_2 \to 0} \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho(x/t) \phi(x,t) dx dt = \lim_{k_1,k_2 \to 0} \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho(\xi) \phi(\xi,t) d\xi dt.
\]

(4.27)

Regarding \(t\) as a parameter and applying (4.26), one can easily see that

\[
\lim_{k_1,k_2 \to 0} \int_{-\infty}^{+\infty} \rho(\xi) \phi(\xi,t) d\xi = \int_{-\infty}^{+\infty} \rho_0(\xi - \sigma) \phi(x,t) dx + (\sigma[\rho] - [\rho u]) \phi(\sigma t, t),
\]

(4.28)

Substituting (4.28) into (4.27) and noting \(\rho_0(\frac{x}{t} - \sigma) = \rho_0(x - \sigma t)\), we have

\[
\lim_{k_1,k_2 \to 0} \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho(x/t) \phi(x,t) dx dt = \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho_0(x - \sigma t) \phi(x,t) dx dt + \int_0^{+\infty} t(\sigma[\rho] - [\rho u]) \phi(\sigma t, t) dt.
\]

(4.29)

By definition (3.9), the last term on the right-hand side of (4.29) equals to \(\langle w_1(t) \delta_S, \phi(\cdot, \cdot) \rangle\), where

\[
w_1(t) = (\sigma[\rho] - [\rho u]) t.
\]

With the same reason as before, we arrive at

\[
\lim_{k_1,k_2 \to 0} \int_0^{+\infty} \int_{-\infty}^{+\infty} \rho(x/t) u(x/t) \phi(x,t) dx dt = \int_0^{+\infty} \int_{-\infty}^{+\infty} (\rho_0 u_0)(x - \sigma t) \phi(x,t) dx dt + \int_0^{+\infty} t(\sigma[\rho u] - [\rho u^2]) \phi(\sigma t, t) dt.
\]

(4.30)

The last term on the right-hand side of (4.30) equals to \(\langle w_2(t) \delta_S, \phi(\cdot, \cdot) \rangle\), where

\[
w_2(t) = (\sigma[\rho u] - [\rho u^2]) t.
\]

The proof is completed. \(\square\)

5 Formation of vacuum states

In this section, we study the formation of vacuum states in the Riemann solutions of system (1.1) and (1.2) in the case \((\rho_+, u_+) \in I(\rho_-, u_-)\) with \(u_- < u_+\) and \(\rho_\pm > 0\) as
both the pressure and the magnetic field vanish. In this case, we know that the Riemann solution consists of a backward rarefaction wave \( R_1 \), a forward rarefaction wave \( R_2 \) and an intermediate state \((\rho_*, u_*)\) besides two constant states \((\rho_\pm, u_\pm)\), which are as follows

\[
\begin{align*}
R_1: \quad & \xi = \lambda_1 = u - \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}}, \\
& u = u_- - \int_{\rho_-}^{\rho} \frac{k_1 + k_2^2 s}{s^2 + \frac{k_2^2 s}{s}} \mu d\rho, \quad \rho_* \leq \rho \leq \rho_-, \\
\end{align*}
\]

(5.1)

and

\[
\begin{align*}
R_2: \quad & \xi = \lambda_2 = u + \sqrt{\frac{k_1}{\rho^2} + \frac{k_2^2\rho}{\mu}}, \\
& u = u_+ + \int_{\rho_+}^{\rho} \frac{k_1 + k_2^2 s}{s^2 + \frac{k_2^2 s}{s}} \mu d\rho, \quad \rho_* \leq \rho \leq \rho_+. \\
\end{align*}
\]

(5.2)

From (5.1) and (5.2), we can derive

\[
\begin{align*}
u_+ - u_- = \int_{\rho_*}^{\rho_-} \frac{\sqrt{\frac{k_1}{s^2} + \frac{k_2^2 s}{s}}}{s} d\rho \\
+ \int_{\rho_*}^{\rho_+} \frac{\sqrt{\frac{k_1}{s^2} + \frac{k_2^2 s}{s}}}{s} d\rho, \quad \rho_* \leq \rho \leq \rho_+. \\
\end{align*}
\]

(5.3)

For any given \( \rho_\pm > 0 \), if \( \lim_{k_1, k_2 \to 0} \rho_* = K \in (0, \min(\rho_-, \rho_+)) \), then by

\[
\begin{align*}
\int_{\rho_*}^{\rho} \frac{\sqrt{A + \frac{B}{s^2}}}{s} d\rho &= -\sqrt{A + \frac{B}{\rho_*^2}} + \sqrt{A} \ln \left( \sqrt{A + \frac{B}{\rho_*^2}} + \sqrt{A} \right) + \sqrt{A} \ln \rho \\
+ \sqrt{A + \frac{B}{\rho_*^2}} - \sqrt{A} \ln \left( \sqrt{A + \frac{B}{\rho_*^2}} + \sqrt{A} \right) - \sqrt{A} \ln \rho_*, \quad A > 0, \\
\end{align*}
\]

(5.4)

it follows that

\[
\begin{align*}
0 \leq \int_{\rho_*}^{\rho_-} \frac{\sqrt{\frac{k_1}{s^2} + \frac{k_2^2 s}{s}}}{s} d\rho \leq \int_{\rho_*}^{\rho_-} \sqrt{\frac{k_1}{s^2} + \frac{k_2^2 \rho_-}{s}} \mu d\rho \\
= \sqrt{\frac{k_1}{\rho_*^2} + \frac{k_2^2 \rho_-}{\mu}} - \sqrt{\frac{k_2^2 \rho_-}{\mu}} \ln \left( \sqrt{\frac{k_1}{\rho_*^2} + \frac{k_2^2 \rho_-}{\mu}} + \sqrt{\frac{k_2^2 \rho_-}{\mu}} \right) - \sqrt{\frac{k_2^2 \rho_-}{\mu}} \ln \rho_*
\end{align*}
\]
\[
- \sqrt{\frac{k_1}{\rho_-^2} + \frac{k_2 \rho_-}{\mu}} + \sqrt{\frac{k_2 \rho_-}{\mu}} \ln \left( \sqrt{\frac{k_1}{\rho_-^2} + \frac{k_2 \rho_-}{\mu}} + \sqrt{\frac{k_2 \rho_-}{\mu}} \right) + \frac{k_2 \rho_-}{\mu} \ln \rho_- \rightarrow 0, \quad \text{as} \ k_1, k_2 \rightarrow 0. \tag{5.5}
\]

Therefore, by the squeeze theorem in multivariable calculus, we arrive at
\[
\lim_{k_1, k_2 \to 0} \int_{\rho_*}^{\rho_-} \frac{\sqrt{\frac{k_1}{\rho^2} + \frac{k_2 \rho}{\mu}}}{s} ds = 0. \tag{5.6}
\]

Similarly, we can obtain that
\[
\lim_{k_1, k_2 \to 0} \int_{\rho_*}^{\rho_+} \frac{\sqrt{\frac{k_1}{\rho^2} + \frac{k_2 \rho}{\mu}}}{s} ds = 0. \tag{5.7}
\]

Combining (5.3), (5.6) and (5.7), we have \( u_- - u_+ = 0 \), which contradicts with \( u_- < u_+ \). Therefore, \( \lim_{k_1, k_2 \to 0} \rho_* = 0 \), which implies that a vacuum occurs.

The Riemann solution of (1.1) and (1.2) can be expressed explicitly as
\[
(\rho, u)(\xi) = \begin{cases} 
(\rho_-, u_-), & -\infty < \xi < \lambda_1(\rho_-, u_-), \\
R_1, & \lambda_1(\rho_-, u_-) \leq \xi < \lambda_1(\rho_*, u_*), \\
(\rho_*, u_*), & \lambda_1(\rho_*, u_*) < \xi < \lambda_2(\rho_*, u_*), \\
R_2, & \lambda_2(\rho_*, u_*) \leq \xi < \lambda_2(\rho_+, u_+), \\
(\rho_+, u_+), & \lambda_2(\rho_+, u_+) < \xi < +\infty,
\end{cases}
\]

where the state \((\rho, u)\) in the rarefaction wave \( R_1 \) is determined uniquely by (5.1), and the state \((\rho, u)\) in the rarefaction wave \( R_2 \) is determined uniquely by (5.2). By (5.1) and (5.2), it is easy to see that
\[
R_1:\begin{cases} 
\xi = \lambda_1(\rho, u) = u - \sqrt{\frac{k_1}{\rho^2} + \frac{k_2 \rho}{\mu}},\\
u_- - u_- = \int_{\rho_*}^{\rho_-} \frac{\sqrt{\frac{k_1}{s^2} + \frac{k_2 s}{\mu}}}{s} ds,\\
u_- \leq u \leq u_*, \rho_* \leq \rho \leq \rho_-,
\end{cases} \tag{5.8}
\]

and
\[
R_2:\begin{cases} 
\xi = \lambda_2(\rho, u) = u + \sqrt{\frac{k_1}{\rho^2} + \frac{k_2 \rho}{\mu}},\\
u_+ - u_* = \int_{\rho_*}^{\rho_+} \frac{\sqrt{\frac{k_1}{s^2} + \frac{k_2 s}{\mu}}}{s} ds,\\
u_* \leq u \leq u_+, \rho_* \leq \rho \leq \rho_+.
\end{cases} \tag{5.9}
\]
It follows from (5.8) and (5.9) that

\[2u_* - u_- - u_+ = \int_{\rho_*}^{\rho} \frac{\sqrt{\frac{k_1 s}{\rho^2} + \frac{k_2^2 s}{\mu}}}{s} ds. \quad (5.10)\]

This implies that

\[\lim_{k_1, k_2 \to 0} u_* = \frac{u_- + u_+}{2}. \quad (5.11)\]

On the other hand, taking the limit \(k_2 \to 0\) in (5.3) leads to

\[u_+ - u_- = \int_{\rho_*(k_1,0)}^{\rho_-} \frac{\sqrt{k_1}}{s^2} ds + \int_{\rho_*(k_1,0)}^{\rho_+} \frac{\sqrt{k_1}}{s^2} ds = -\frac{\sqrt{k_1}}{\rho_-} + \frac{\sqrt{k_1}}{\rho_*(k_1, 0)} - \frac{\sqrt{k_1}}{\rho_+} + \frac{\sqrt{k_1}}{\rho_*(k_1, 0)}. \quad (5.12)\]

Thus, from (5.12) we have

\[\lim_{k_1 \to 0} \frac{\sqrt{k_1}}{\rho_*(k_1, 0)} = \frac{u_+ - u_-}{2}. \quad (5.13)\]

Then, noting the fact that \(\rho_*(k_1, k_2) \leq \rho(k_1, k_2) \leq \rho_\pm\), it follows from (5.11) and (5.13) that

\[\lim_{k_1, k_2 \to 0} \lambda_1(\rho_*, u_*) = \lim_{k_1, k_2 \to 0} \left( u_* - \frac{k_1}{\sqrt{\rho_*^2 + \frac{k_2^2 \rho_*}{\mu}}} \right) = \lim_{k_1 \to 0} u_* - \lim_{k_1 \to 0} \frac{\sqrt{k_1}}{\rho_*(k_1, 0)} = \frac{u_- + u_+}{2} - \frac{u_+ - u_-}{2} = u_- . \quad (5.14)\]

By (5.8), it is obvious that

\[\lim_{k_1, k_2 \to 0} \lambda_1(\rho_-, u_-) = \lim_{k_1, k_2 \to 0} \left( u_- - \frac{k_1}{\sqrt{\rho_-^2 + \frac{k_2^2 \rho_-}{\mu}}} \right) = u_-. \quad (5.15)\]

Similarly, we can obtain that

\[\lim_{k_1, k_2 \to 0} \lambda_2(\rho_*, u_*) = u_+ \quad \text{and} \quad \lim_{k_1, k_2 \to 0} \lambda_2(\rho_+, u_+) = u_+. \quad (5.16)\]

With the above discussions, in the case \((\rho_+, u_+) \in I(\rho_-, u_-)\) with \(u_- < u_+\), as \(k_1, k_2 \to 0\), the limit of the Riemann solution of (1.1) and (1.2) can be written in the form
\[(\rho, u)(\xi) = \begin{cases} 
(\rho_-, u_-), & -\infty < \xi < u_-, \\
(0, \frac{u_- + u_+}{2}), & u_- < \xi < u_+, \\
(\rho_+, u_+), & u_+ < \xi < +\infty. 
\end{cases}\]

Then from above we have proved the following results.

**Theorem 3** In the case \((\rho_+, u_+) \in I(\rho_-, u_-)\) with \(u_- < u_+\), as \(k_1, k_2 \to 0\), the vacuum state occurs and two rarefaction waves \(R_1\) and \(R_2\) become two contact discontinuities \(u = u_-\) and \(u = u_+,\) respectively, connecting the constant states \((\rho_\pm, u_\pm)\) with the vacuum \((\rho = 0)\).

**Theorem 4** Here if we do not consider the virtual velocity in the vacuum region, then we can see that, in the case \((\rho_+, u_+) \in I(\rho_-, u_-)\) with \(u_- < u_+\), as \(k_1, k_2 \to 0\), the limit of the Riemann solution of (1.1) and (1.2) with initial data (2.1) is just the Riemann solution of the transport Eq. (3.1) for zero pressure flow with the same initial data, which contains two contact discontinuities \(\xi = x/t = u_\pm\) and a vacuum state besides two constant states.

**6 Numerical simulations**

In order to verify the validity of the formation of \(\delta\)-shocks and vacuum states mentioned in Sects. 4 and 5, we presents two selected groups of representative numerical simulations. A number of iterative numerical trials are executed to guarantee what we demonstrate are not numerical objects. To discretize the system, we use the fifth-order weighted essentially non-oscillatory scheme and third-order Runge–Kutta method [31, 32] with the mesh 400 cells. The numerical simulations are consistent with the theoretical analysis.

**6.1 Formation of \(\delta\)-shocks**

The numerical simulations are corresponding to the theoretical analysis in Sect. 4. When \((\rho_+, u_+) \in IV(\rho_-, u_-)\) with \(u_- > u_+\), we compute the solution of the Riemann problem (1.1)–(1.2) with \(\mu = 1\) and take the initial data as follows:

\[(\rho, u)(0, x) = \begin{cases} 
(2, 2), & x < 0, \\
(1, -1), & x > 0. 
\end{cases}\]

The numerical simulations for different choices of \(k_1\) and \(k_2\) are presented in Figs. 3, 4 and 5 for \(t = 0.35\). We start with \(k_1 = 1, k_2 = 1\), then \(k_1 = 0.25, k_2 = 0.15\) and finally \(k_1 = 0.001, k_2 = 0.001\).

We can observe from these numerical results that, when \(k_1, k_2\) decrease, the locations of the two shocks become closer and closer, and the density of the intermediate state increases dramatically, while the velocity is closer to a step function. As \(k_1, k_2 \to 0\), along with the intermediate state, the two shocks coincide to form a \(\delta\)-shock of the transport Eq. (3.1), while the velocity keeps a step function.
Fig. 3 Density (left) and velocity (right) for $k_1 = 1, k_2 = 1$

Fig. 4 Density (left) and velocity (right) for $k_1 = 0.25, k_2 = 0.15$

Fig. 5 Density (left) and velocity (right) for $k_1 = 0.001, k_2 = 0.001$

What’s more, by Theorem 2 and (4.1)–(4.2), if $(\rho_+, u_+) \in IV(\rho_-, u_-)$ with $u_- > u_+$, then the solution of the Riemann problem (1.1)–(1.2) with $\mu = 1$ with the Riemann initial data $(\rho_\pm, u_\pm)$ can be expressed as
\((\rho, u)(t, x) = \begin{cases} 
(\rho_-, u_-), & x < x_1(t), \\
(\rho_*, u_*), & x_1(t) < x < x_2(t), \\
(\rho_+, u_+), & x > x_2(t), \end{cases} \quad (6.2)\)

where

\[
x_1(t) = \left( u_- - \rho_* \sqrt{\frac{1}{\rho_* \rho_-} \left( \frac{k_1}{\rho_* \rho_-} + \frac{k_2^2 (\rho_* + \rho_-)}{2} \right)} \right) t, \quad (6.3)
\]

\[
x_2(t) = \left( u_* + \rho_+ \sqrt{\frac{1}{\rho_+ \rho_*} \left( \frac{k_1}{\rho_+ \rho_*} + \frac{k_2^2 (\rho_+ + \rho_*)}{2} \right)} \right) t, \quad (6.4)
\]

and the state \((\rho_*, u_*)\) is determined uniquely by

\[
\begin{cases}
  u_* = u_- - \sqrt{\frac{1}{\rho_* \rho_-} \left( \frac{k_1}{\rho_* \rho_-} + \frac{k_2^2 (\rho_* + \rho_-)}{2} \right)} (\rho_* - \rho_-), \\
  u_+ = u_* + \sqrt{\frac{1}{\rho_+ \rho_*} \left( \frac{k_1}{\rho_+ \rho_*} + \frac{k_2^2 (\rho_+ + \rho_*)}{2} \right)} (\rho_+ - \rho_*). \end{cases} \quad (6.5)
\]

Because these Eq. (6.5) are too complex to be solved exactly. From (6.5), we have

\[
\sqrt{\frac{1}{\rho_* \rho_-} \left( \frac{k_1}{\rho_* \rho_-} + \frac{k_2^2 (\rho_* + \rho_-)}{2} \right)} (\rho_* - \rho_-) - \sqrt{\frac{1}{\rho_+ \rho_*} \left( \frac{k_1}{\rho_+ \rho_*} + \frac{k_2^2 (\rho_+ + \rho_*)}{2} \right)} (\rho_+ - \rho_*)) = u_- - u_+ = 3.
\]

To obtain approximate solution \(\rho_*\), we compute

\[
\sqrt{\frac{1}{\rho_* \rho_-} \left( \frac{k_1}{\rho_* \rho_-} + \frac{k_2^2 (\rho_* + \rho_-)}{2} \right)} (\rho_* - \rho_-) - \sqrt{\frac{1}{\rho_+ \rho_*} \left( \frac{k_1}{\rho_+ \rho_*} + \frac{k_2^2 (\rho_+ + \rho_*)}{2} \right)} (\rho_+ - \rho_*)) = 3,
\]

by using numerical methods on MATLAB. Then we compute \(u_*\) by using the first equation in (6.5) and compute \(x_i(t) (i = 1, 2)\) by using (6.3)–(6.4) and the time \(t = 0.35\). Thus, for \(k_1 = 1, k_2 = 1\), we can calculate that

\[
(\rho, u)(0.35, x) = \begin{cases} 
(2, 2), & x < -1.575, \\
(3.43, 1.08), & -1.575 < x < -0.824, \\
(1, -1), & x > -0.824. \end{cases}
\]
Similarly, for \( k_1 = 0.25, \ k_2 = 0.15 \),

\[
(\rho, u)(0.35, x) = \begin{cases} 
(2, 2), & x < -1.277, \\
(16.9, 0.80), & -1.277 < x < -1.181, \\
(1, -1), & x > -1.181.
\end{cases}
\]

And for \( k_1 = 0.001, \ k_2 = 0.001 \),

\[
(\rho, u)(0.35, x) = \begin{cases} 
(2, 2), & x < -1.235, \\
(+\infty, -1), & x = -1.235, \\
(1, -1), & x > -1.235.
\end{cases}
\]

The numerical results are in complete agreement with the theoretical analysis in Sect. 4.

### 6.2 Formation of vacuum states

The numerical simulations are corresponding to the theoretical analysis in Sect. 5. In the case \((\rho_+, u_+) \in I(\rho_-, u_-)\) with \( u_- < u_+ \), we compute the solution of the Riemann problem (1.1)–(1.2) with \( \mu = 1 \) and take the initial data as follows:

\[
(\rho, u)(0, x) = \begin{cases} 
(1, 1), & x < 0, \\
(3, 2.7), & x > 0.
\end{cases}
\]  

(6.6)

The numerical simulations for different choices of \( k_1 \) and \( k_2 \) are displayed in Figs. 6, 7 and 8 for \( t = 0.2 \). We start with \( k_1 = 2, \ k_2 = 0.01 \), then \( k_1 = 0.5, \ k_2 = 0.01 \), finally \( k_1 = 0.00001, \ k_2 = 0.0001 \).

From these numerical results, we can clearly see that, when \( k_1, \ k_2 \) decrease, the boundaries of two rarefaction waves become closer and closer, the intermediate state tends to a vacuum state, while the velocity tends to a linear function. As \( k_1, \ k_2 \to 0 \), a two-rarefaction-wave solution tends to a two-contact-discontinuity solution with a vacuum state of the transport Eq. (3.1).

![Fig. 6 Density (left) and velocity (right) for \( k_1 = 2, \ k_2 = 0.01 \)]
What’s more, by (5.1)–(5.2), if \((\rho_+, u_+) \in I(\rho_-, u_-)\) with \(u_- < u_+\), then the Riemann solution of (1.1)–(1.2) with \(\mu = 1\) with the Riemann initial data \((\rho_\pm, u_\pm)\) can be expressed as

\[
(\rho, u)(t, x) = \begin{cases} 
(\rho_-, u_-), & x < x_1^-(t), \\
(\rho_1, u_1), & x_1^- (t) \leq x \leq x_1^+(t), \\
(\rho_*, u_*), & x_1^+ (t) < x < x_2^-(t), \\
(\rho_2, u_2), & x_2^- (t) \leq x \leq x_2^+(t), \\
(\rho_+, u_+), & x > x_2^+(t),
\end{cases}
\]

where

\[
x_1^-(t) = u_- t - \sqrt{\frac{k_1}{\rho_-^2} + k_2 \rho_- t}, \quad x_1^+(t) = u_* t - \sqrt{\frac{k_1}{\rho_*^2} + k_2 \rho_* t},
\]

\[
x_2^- (t) = u_* t + \sqrt{\frac{k_1}{\rho_*^2} + k_2 \rho_* t}, \quad x_2^+ (t) = u_+ t + \sqrt{\frac{k_1}{\rho_+^2} + k_2 \rho_+ t},
\]
and the states \((\rho_1, u_1)\) can be calculated by
\[
\begin{align*}
  u_1 &= \frac{x}{t} + \sqrt{\frac{k_1}{\rho_1^2} + k_2^2 \rho_1}, \\
  u_1 &= u_- - \int_{\rho_-}^{\rho_1} \frac{\sqrt{\frac{k_1}{s^2} + k_2^2}}{s} ds,
\end{align*}
\]
and the states \((\rho_2, u_2)\) can be calculated by
\[
\begin{align*}
  u_2 &= \frac{x}{t} - \sqrt{\frac{k_1}{\rho_2^2} + k_2^2 \rho_2}, \\
  u_2 &= u_+ + \int_{\rho_-}^{\rho_2} \frac{\sqrt{\frac{k_1}{s^2} + k_2^2}}{s} ds,
\end{align*}
\]
and the states \((\rho_*, u_*)\) is determined uniquely by
\[
\begin{align*}
  u_* &= u_- - \int_{\rho_-}^{\rho_*} \frac{\sqrt{\frac{k_1}{s^2} + k_2^2}}{s} ds, \\
  u_* &= u_+ + \int_{\rho_2}^{\rho_*} \frac{\sqrt{\frac{k_1}{s^2} + k_2^2}}{s} ds.
\end{align*}
\]
Then, we can calculate that for \(k_1 = 2, k_2 = 0.01\),
\[
(\rho, u)(0.2, x) =
\begin{array}{ll}
  (1, 1), & x < -0.0828, \\
  (-2110x - 173.708, 3790x + 32.382), & -0.0828 \leq x \leq -0.0827, \\
  (0.789, 1.379), & -0.0827 < x < 0.6343, \\
  (5527.5x - 3505.30425, 33025x - 20946.3785), & 0.6343 \leq x \leq 0.63434, \\
  (3, 2.7), & x > 0.63434.
\end{array}
\]
Similarly, for \(k_1 = 0.5, k_2 = 0.01\),
\[
(\rho, u)(0.2, x) =
\begin{array}{ll}
  (1, 1), & x < 0.0586, \\
  (31.2115 - 515.55x, 685.55x - 38.06244), & 0.0586 \leq x \leq 0.0595, \\
  (0.536, 1.617), & 0.0595 < x < 0.5781, \\
  (273.77x - 157.7349, 120.33x - 67.147), & 0.5781 \leq x \leq 0.5871, \\
  (3, 2.7), & x > 0.5871.
\end{array}
\]
And for \(k_1 = 0.00001, k_2 = 0.0001\),
\[
(\rho, u)(0.2, x)
\]
\[
\begin{aligned}
(1, 1), & \quad x < 0.1994, \\
((1.49825 - 2.49875x)^{1000}, 1), & \quad 0.1994 \leq x \leq 0.205, \\
(0, 5.0716x - 0.039678), & \quad 0.205 \leq x \leq 0.389959999, \\
(0, 1.84980), & \quad 0.389959999 < x < 0.389960000, \\
(0, 5.0716x - 0.039678), & \quad 0.389960000 \leq x \leq 0.45, \\
((−0.29875 + 2.49875x)^{1000}, 5.0716x - 0.039678), & \quad 0.45 \leq x \leq 0.5402, \\
(3, 2.7), & \quad x > 0.5402,
\end{aligned}
\]

which shows that there is a mass loss in the local region around \( x = 0.3899 \).

The numerical methods are in complete agreement with the theoretical analysis in Sect. 5.

7 Conclusions

In this paper, the solutions of the Riemann problem of the isentropic magnetogasdynamics equations for Chaplygin gas is constructed globally. In particular, the \( \delta \)-shock is not discovered in the Riemann solutions. Both pressure law and transverse magnetic field are governed by two parameters. We study for the behavior of the Riemann solution in the limit of the two parameters to zero. In these limits, the model coincides, in a sense to be prescribed, with the pressureless gas dynamics well-known to involve solutions containing \( \delta \)-shocks and/or vacuum regions. Exhibiting the Riemann solution of the system under consideration, we recover the expected \( \delta \)-shock solutions and vacuum solutions. The paper is also supplemented with numerical experiments to illustrate the presented solution behaviors. The current result reveals from the mathematical point of view that a sufficiently strong transverse magnetic field has a stabilizing effect and can prevent the \( \delta \)-shock from occurring. From the viewpoint of the physical meaning of the result, it may be used to illustrate a sufficiently strong transverse magnetic field can prevent the concentration phenomenon for the dark matter and dark energy in the evolution of the universe.

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Declarations

Conflict of interest The author states that there is no conflict of interest.

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