A NOTE ON GROTHENDIECK GROUPS OF PERIODIC DERIVED CATEGORIES

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ABSTRACT. We determine Grothendieck groups of periodic derived categories. In particular, we prove that the Grothendieck group of the \( m \)-periodic derived category of finitely generated modules over an Artin algebra is a free \( \mathbb{Z} \)-module if \( m \) is even but an \( \mathbb{F}_2 \)-vector space if \( m \) is odd. Its rank is equal to the number of isomorphism classes of simple modules in both cases. As an application, we prove that the number of non-isomorphic summands of a strict periodic tilting object \( T \), which was introduced in [S21] as a periodic analogue of tilting objects, is independent of the choice of \( T \).

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1. Introduction

1.1. Background. The representation theory of Artin algebras studies properties of Artin algebras which is preserved under Morita equivalence. Two Artin algebras are Morita equivalent if the categories \( \text{mod } \Lambda \) of finitely generated modules over them are equivalent. Grothendieck groups control many invariants of Morita equivalence. They are defined for each abelian category and invariant under equivalence of them, and thus Grothendieck groups themselves are a Morita invariant. For example, the Grothendieck group \( K_0(\text{mod } \Lambda) \) of finitely generated modules over an Artin algebra \( \Lambda \) is a free \( \mathbb{Z} \)-module whose rank is equal to the number of isomorphism classes of simple \( \Lambda \)-modules. Thus the number of isomorphism classes of simple modules is a Morita invariant.

Derived equivalence is a more flexible framework to study Artin algebras than Morita equivalence. Two Artin algebras are derived equivalent if the bounded derived categories \( D^b(\text{mod } \Lambda) \) of finitely generated modules over them are triangulated equivalent. Of course, Morita equivalence implies derived equivalence. We can define Grothendieck groups of triangulated categories and have an isomorphism \( K_0(\text{mod } \Lambda) \cong K_0(D^b(\text{mod } \Lambda)) \). It means that Grothendieck groups are not only a Morita invariant but also a derived invariant. In particular, the number of isomorphism classes of simple modules is still a derived invariant. The purpose of this paper is to give an analogy of the above results for \( m \)-periodic derived categories (Theorem 1.1 and Corollary 1.2) and to apply it to periodic tilting theory (Corollary 1.6).

Let \( m > 0 \) be a positive integer. \( m \)-periodic complexes are a \( \mathbb{Z}/m\mathbb{Z} \)-graded version of usual complexes. For example, a 2-periodic complex \( V \) consists of objects and morphisms \( V_0 \xrightarrow{d^0} V_1 \) with \( d^0d^0 = d^0d^1 = 0 \). For an abelian category \( \mathcal{A} \), the \( m \)-periodic derived category \( D_m(\mathcal{A}) \) of \( \mathcal{A} \) is the localization of the category of \( m \)-periodic complexes with respect to quasi-isomorphisms. See §2.2 for the precise definition. These were introduced by Peng and Xiao in [PX97] to construct a categorification of full semisimple Lie algebras via Ringel-Hall algebras. Inspired by this work, Bridgeland used 2-periodic derived categories of hereditary algebras to construct full quantum groups of symmetric Kac-Moody Lie algebras in [Br13]. Motivated by these studies, several authors analyzed the structure of \( m \)-periodic derived categories. See [Fu12, Go13, Zhao14, St18].
1.2. **Main results.** Let \( m > 0 \) be a positive integer. The following theorem is the main result of this paper, which will be shown in §4. It indicates \( m \)-periodic derived categories behave like usual derived categories if \( m \) is even. In contrast, they behave strangely if \( m \) is odd.

**Theorem 1.1.** For an essentially small enough projective abelian category \( \mathcal{A} \) of finite global dimension, we have an isomorphism

\[
K_0(D_m(\mathcal{A})) \simeq \begin{cases} 
K_0(\mathcal{A}) & \text{if } m \text{ is even}, \\
K_0(\mathcal{A}) \otimes \mathbb{F}_2 := K_0(\mathcal{A}) \otimes \mathbb{Z} \mathbb{F}_2 & \text{if } m \text{ is odd},
\end{cases}
\]

induced by the natural functor \( \mathcal{A} \rightarrow D_m(\mathcal{A}) \), where \( \mathbb{F}_2 \) is the finite field of two elements.

The result of the even periodic case is a direct generalization of [Fu12, Proposition 2.11], but the odd periodic case is a new one and includes a new insight of odd periodic triangulated categories.

As an immediate corollary, we have the following.

**Corollary 1.2.** Let \( \Lambda \) be an Artin algebra of finite global dimension, and let \( n \) be the number of isomorphism classes of simple modules. Then we have

\[
K_0(D_m(\text{mod } \Lambda)) \simeq \begin{cases} 
\mathbb{Z}^n & \text{if } m \text{ is even}, \\
\mathbb{F}_2^n & \text{if } m \text{ is odd},
\end{cases}
\]

We explain a motivation of Theorem 1.1 and Corollary 1.2. In [S21], the author proves the periodic tilting theorem, which gives a sufficient condition for a given triangulated category to be equivalent to the periodic derived category of an algebra. We review some definitions and results in [S21].

For a positive integer \( m > 0 \), a triangulated category \( \mathcal{T} \) is called \( m \)-periodic if its suspension functor \( \Sigma : \mathcal{T} \rightarrow \mathcal{T} \) satisfies \( \Sigma^m \simeq \text{Id}_T \) as additive functors.

**Definition 1.3.** Let \( \mathcal{T} \) be an \( m \)-periodic triangulated category.

1. An object \( T \in \mathcal{T} \) is \( m \)-periodic tilting if it satisfies \( \text{Hom}_\mathcal{T}(T, \Sigma^i T) = 0 \) for any \( i \not\in m \mathbb{Z} \) and the smallest thick triangulated category containing \( T \) coincides with \( \mathcal{T} \).
2. An \( m \)-periodic tilting object \( T \in \mathcal{T} \) is called strict if the global dimension of the endomorphism algebra \( \text{End}_\mathcal{T}(T) \) is less than \( m \).

We will not inform the algebraic and idempotent complete conditions in the following theorem. See [S21] for the detail. These mild assumptions are satisfied by almost all concrete triangulated categories appearing in the study of representations of algebras.

**Theorem 1.4** (The periodic tilting theorem [S21, Corollary 5.4]). Let \( \mathcal{T} \) be an idempotent complete algebraic \( m \)-periodic triangulated category over a perfect field \( k \) (e.g., if \( k \) is an algebraically closed field). Suppose that \( \text{Hom}_\mathcal{T}(X, Y) \) is finite dimensional over \( k \) for all objects \( X, Y \in \mathcal{T} \). If \( \mathcal{T} \) has a strict \( m \)-periodic tilting object \( T \), then there exists a triangulated equivalence \( \mathcal{T} \rightarrow D_m(\text{mod } \Lambda) \), where \( \Lambda := \text{End}_\mathcal{T}(T) \).

The periodic tilting theorem and periodic tilting objects are periodic analogue of usual tilting theorem and tilting objects (cf. [Tilt07]). Hence we expect that periodic tilting objects have properties similar to the usual one. However, there is the following example which is taught by Professor Osamu Iyama in the conference Algebraic Lie Theory and Representation Theory, 2021.

**Example 1.5.** Let \( k \) be a perfect field, and \( kA_3 \) the path algebra of the quiver \( 1 \leftarrow 2 \leftarrow 3 \) of type \( A_3 \). The Auslander-Reiten quiver of \( D_2(\text{mod } kA_3) \) is the following. (See Example 2.13)

```
          X_3 ---- Y_2 ------ o ---- Y_4 ------
             \  \    \        \      \    \\
             X_2 ------ Y_1 ------ o ------ Y_3 \\
                \  \    \        \      \    \\
                X_1  Y_0        Y_0  Y_3
```

Then \( X := \bigoplus_{i=1}^3 X_i \) and \( Y := \bigoplus_{i=1}^4 Y_i \) are both 2-periodic tilting objects in \( D_2(\text{mod } kA_3) \). Thus the number of non-isomorphic summands of a periodic tilting object is not constant, while the number for the usual one is constant.
In this example, we observe that End(X) ∼= kA₁ and End(Y) is isomorphic to a self-injective Nakayama algebra, and hence X is strict but Y is not. We expect that the number of non-isomorphic summands of a strict periodic tilting object is constant. This is true by Corollary 1.2 and Theorem 1.4.

**Corollary 1.6.** Fix a positive integer m > 0. Let T be an idempotent complete algebraic m-periodic triangulated category over a perfect field k. Suppose that Hom_T(X, Y) is finite dimensional over k for all objects X, Y ∈ T. Then the number of non-isomorphic summands of a strict periodic tilting object is constant.

**Proof.** Suppose Tᵢ ∈ T (i = 1, 2) are strict m-periodic tilting objects and set Λᵢ := End_T(Tᵢ). Then we have two triangulated equivalences T ∼ T̂ ∼ D₀(mod Λᵢ) by Theorem 1.2. A triangulated equivalence D₀(mod Λ₁) ∼ T ∼ D₀(mod Λ₂) induces an isomorphism K₀(D₀(mod Λ₁)) ∼ K₀(D₀(mod Λ₂)) on the Grothendieck groups. Hence Λ₁ and Λ₂ have the same number of isomorphism classes of simple modules by Corollary 1.2. Because the number of non-isomorphic summands of Tᵢ is equal to the number of isomorphism classes of simple modules over Λᵢ, the corollary follows. □

**Organization.** This paper is organized as follows. In Section 2 we collect basic properties of triangulated categories and periodic derived categories which we use throughout this paper. In Section 3 we investigate general properties of the Grothendieck group of a periodic triangulated category. In particular, we deal with the relationship between cohomological functors on periodic triangulated categories and homomorphisms between the Grothendieck groups of them. In Section 4 we give a proof of Theorem 1.4.

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2. Preliminaries

2.1. Triangulated categories. In this subsection, we gather basic notions on triangulated categories, which we will use. Throughout this paper, we assume that all categories and functors are additive. We denote by Σ the suspension functor of a triangulated category.

Let T be an essentially small triangulated category, that is, the isomorphism classes [T] of objects and Hom_T(X, Y) (X, Y ∈ T) form sets. For an object X ∈ T, the isomorphism class of X is denoted by [X] ∈ [T]. The Grothendieck group K₀(T) of T is defined as a quotient of a free abelian group:

\[ \bigoplus_{[X] \in [T]} \mathbb{Z}[X] / \langle [X] - [Y] + [Z] | X \to Y \to Z \to \Sigma X \text{ is an exact triangle} \rangle. \]

The residue class of [X] ∈ [T] in K₀(T) is also denoted by [X]. It is obvious that a triangle equivalence T ∼ T′ induces an isomorphism K₀(T) ∼ K₀(T′).

For an essentially small abelian category A, we can similarly define the Grothendieck group K₀(A) of A. Triangulated categories often relate to abelian categories and it gives a homomorphism between Grothendieck groups of them.

**Definition 2.1.** Let A be an abelian category and let T be a triangulated category.

1. A δ-functor \( A \to T \) from A to T is a pair \((F, \delta)\) of a functor \( F : A \to T \) and functorial morphisms \( \delta_{W,U} : \text{Ext}_A^1(W, U) \to \text{Hom}_T(FW, \Sigma FU) \) for all \( U, W \in A \) such that for any exact sequence \( 0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0 \) in A,

\[
FU \xrightarrow{Ff} FV \xrightarrow{Fg} FW \xrightarrow{\delta(c)} \Sigma FU
\]

is an exact triangle in T. If no ambiguity can arise, we will often say that \( F : A \to T \) is a δ-functor.

2. A functor \( F : T \to A \) is said to be cohomological if for any exact triangle \( X \to Y \to Z \to \Sigma X \) in T, \( FX \to FY \to FZ \) is an exact sequence in A. We set \( F^\circ := F \circ \Sigma^m : T \to A \).

Typical examples of δ-functors and cohomological functors are the natural inclusion \( A \to D(A) \) from an abelian category to its derived category and the 0th cohomology functor \( H^0 : D(A) \to A \), respectively. In §2.2 we will see the counterparts of these examples in periodic derived categories \( D_m(A) \). A δ-functor \( A \to T \) naturally induces a homomorphism \( K₀(A) \to K₀(T) \) but a cohomological functor \( T \to A \) does not induce a homomorphism \( K₀(T) \to K₀(A) \) in general. It is a difficulty we deal with in this paper.
Definition 2.2. Let $\mathcal{T}$ be a triangulated category.

(1) For a positive integer $m > 0$, $\mathcal{T}$ is $m$-periodic if $\Sigma^m \simeq \text{Id}_\mathcal{T}$ as additive functors.

(2) The period of $\mathcal{T}$ is the smallest positive integer $m$ such that $\mathcal{T}$ is $m$-periodic.

We study Grothendieck groups of periodic triangulated categories in §3.

2.2. Periodic derived categories. In this subsection, we give a review of periodic derived categories. See [S21, §3] for a detailed account. Fix a positive integer $m > 0$. $\mathbb{Z}_m$ denotes the cyclic group of order $m$. Roughly speaking, an $m$-periodic complex is a $\mathbb{Z}_m$-graded complex. In the following definition, replacing $\mathbb{Z}_m$ to $\mathbb{Z}$, we get the usual notion of complexes.

Definition 2.3. Let $\mathcal{C}$ be an additive category.

(1) An $m$-periodic complex $V$ is a family $(V_i, d_i)_{i \in \mathbb{Z}_m}$ of objects $V_i \in \mathcal{C}$ and morphisms $d_i : V_i \to V_{i+1}$ of $\mathcal{C}$ satisfying $d_{i+1} d_i = 0$ for all $i \in \mathbb{Z}_m$.

(2) A chain map $f : V \to W$ between $m$-periodic complexes $V$ and $W$ is a family $(f_i)_{i \in \mathbb{Z}_m}$ of morphisms $f_i : V_i \to W_i$ in $\mathcal{C}$ satisfying $f_{i+1} d_i = d_i f_i$ for all $i \in \mathbb{Z}_m$.

(3) $C_m(\mathcal{C})$ denotes the category of $m$-periodic complexes and chain maps.

Example 2.4. Let $\mathcal{C}$ be an additive category.

(1) A 1-periodic complex is a morphism $d : V \to V$ in $\mathcal{C}$ with $d^2 = 0$.

(2) A 2-periodic complex is a diagram $V^0 \to V^1$ in $\mathcal{C}$ with $d^1 d^0 = d^0 d^1 = 0$.

Two chain maps $f, g : V \to W$ of $m$-periodic complexes is homotopic if there exist $s : V^i \to W^{i-1}$ ($i \in \mathbb{Z}_m$) with $f^i - g^i = d^i_{W} s^i + s^{i+1} d^i_{V}$ for all $i \in \mathbb{Z}_m$. This gives rise to the homotopy category $K_m(\mathcal{C})$ of $m$-periodic complexes. The shift functor $[1] : C_m(\mathcal{C}) \to C_m(\mathcal{C})$ is defined by

$$V \mapsto V[1] := (V^{i+1}, -d^{i+1}_V)_{i \in \mathbb{Z}_m}.$$ 

The homotopy category $K_m(\mathcal{C})$ with the shift functor $[1] : K_m(\mathcal{C}) \to K_m(\mathcal{C})$ as the suspension functor is a triangulated category.

For an abelian category $\mathcal{A}$, the category $C_m(\mathcal{A})$ is also an abelian category. A sequence $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ in $C_m(\mathcal{A})$ is exact if and only if $0 \to U^i \xrightarrow{f^i} V^i \xrightarrow{g^i} W^i \to 0$ is exact in $\mathcal{A}$ for all $i \in \mathbb{Z}_m$.

Define the $i$th cohomology of $V \in C_m(\mathcal{A})$ by $H^i(V) := \text{Ker} d_i/\text{Im} d_{i+1}$ for $i \in \mathbb{Z}_m$. It gives rise to a functor $H^i : K_m(\mathcal{A}) \to \mathcal{A}$ for all $i \in \mathbb{Z}_m$. A chain map $f : V \to W$ of $m$-periodic complexes is a quasi-isomorphism if $H^i(f) : H^i(V) \to H^i(W)$ is an isomorphism for all $i \in \mathbb{Z}_m$.

Definition 2.5. For an abelian category $\mathcal{A}$, the $m$-periodic derived category $D_m(\mathcal{A})$ is the localization of $K_m(\mathcal{A})$ with respect to quasi-isomorphisms.

The category $D_m(\mathcal{A})$ is a triangulated category and the canonical functor $K_m(\mathcal{A}) \to D_m(\mathcal{A})$ is a triangulated functor.

Remark 2.6. The $m$-periodic derived category $D_m(\mathcal{A})$ of an abelian category $\mathcal{A}$ is a fundamental example of periodic triangulated category (See Definition 22), but its period is not necessarily $m$. The period depends on the parity of $m$. This phenomenon is caused by the change of signs of differential by the shift functor. For example, the shift of a 1-periodic complex is $(M, d)[1] = (M, -d)$. Hence $(M, d)$ and $(M, d)[1]$ is not isomorphic in general. There are three cases for the period $p$ of $D_m(\mathcal{A})$ (See [S21, Proposition 5.1]):

(i) $m$ is even and $p = m$,

(ii) $m$ is odd and $p = m$, and

(iii) $m$ is odd and $p = 2m$.

The $i$th cohomology functor $H^i : K_m(\mathcal{A}) \to \mathcal{A}$ induces a functor $D_m(\mathcal{A}) \to \mathcal{A}$. We also denote it by $H^i$. It is an advantage of localizing the category $K_m(\mathcal{A})$ that an exact sequence in $\mathcal{A}$ gives an exact triangle in $D_m(\mathcal{A})$.

Fact 2.7 ([S21, Proposition 3.12, 3.19]). Let $\mathcal{A}$ be an abelian category.

(1) The natural functor $C_m(\mathcal{A}) \to D_m(\mathcal{A})$ is a $\delta$-functor (See Definition 21(1)).

(2) The $i$th cohomology functor $H^i : D_m(\mathcal{A}) \to \mathcal{A}$ is a cohomological functor (See Definition 21 (2)).
The purpose of this paper is to study the Grothendieck groups of periodic derived categories. When do we define the Grothendieck groups of periodic derived categories? In other words when are periodic derived categories essentially small? The following theorem answers this question. See Proposition 2.10 below.

Fact 2.8 ([Go13] Lemma 9.5, cf. [S21] Corollary 3.28)). Let \( \mathcal{A} \) be an enough projective abelian category of finite global dimension, and \( \text{Proj}\mathcal{A} \) is the full subcategory of projective objects in \( \mathcal{A} \). Then the natural functor \( K_m(\text{Proj}\mathcal{A}) \to D_m(\mathcal{A}) \) is a triangulated equivalence.

The following fact implies surjectivity of a homomorphism \( K_0(\mathcal{A}) \to K_0(D_m(\mathcal{A})) \) induced by the natural inclusion \( \mathcal{A} \to D_m(\mathcal{A}) \). See Proposition 2.10 below.

Fact 2.9 ([Go13] Proposition 9.7, cf. [S21] Lemma 3.26)). Let \( \mathcal{A} \) be an enough projective abelian category of finite global dimension. Then the smallest triangulated subcategory of \( D_m(\mathcal{A}) \) containing \( \mathcal{A} \) coincides with \( D_m(\mathcal{A}) \).

We summarize the facts about periodic derived categories and rephrase them as statements about its Grothendieck groups.

Proposition 2.10. Let \( \mathcal{A} \) be an essentially small enough projective abelian category of finite global dimension.

1. \( D_m(\mathcal{A}) \) is essentially small. In particular, we can define the Grothendieck group of \( D_m(\mathcal{A}) \).
2. The natural functor \( \mathcal{A} \to D_m(\mathcal{A}) \) is a \( \delta \)-functor. In particular, we have an induced homomorphism
   \[ \psi: K_0(\mathcal{A}) \to K_0(D_m(\mathcal{A})). \]
3. The smallest triangulated subcategory of \( D_m(\mathcal{A}) \) containing \( \mathcal{A} \) coincides with \( D_m(\mathcal{A}) \). In particular, the homomorphism \( \psi: K_0(\mathcal{A}) \to K_0(D_m(\mathcal{A})) \) is surjective.

Proof. (1) It is not obvious that \( \text{Hom}_{D^b_m(\mathcal{A})}(V,W) \) forms a set in general since \( D_m(\mathcal{A}) \) is the localization of the category \( K_m(\mathcal{A}) \). However, by Fact 2.8 the natural functor \( K_m(\text{Proj}\mathcal{A}) \to D_m(\mathcal{A}) \) is an equivalence. Note that \( \text{Mor}_\mathcal{A} := \bigcup_{M,N \in \mathcal{A}} \text{Hom}_\mathcal{A}(M,N) \) forms a set because \( \mathcal{A} \) is essentially small. The category \( K_m(\text{Proj}\mathcal{A}) \) is essentially small since \( \text{Hom}_{K_m(\mathcal{A})}(V,W) \) is a quotient of a set \( \text{Hom}_{C_m(\mathcal{A})}(V,W) \in \prod_{i=0}^{m-1} \text{Hom}_\mathcal{A}(V^i,W^i) \) and \( [K_m(\text{Proj}\mathcal{A})] \subset \prod_{i=0}^{m-1} \text{Mor}_\mathcal{A}. \) Thus \( D_m(\mathcal{A}) \) is also essentially small.

2. Since the natural inclusion \( \mathcal{A} \hookrightarrow C_m(\mathcal{A}) \) is exact and the natural functor \( C_m(\mathcal{A}) \to D_m(\mathcal{A}) \) is a \( \delta \)-functor, their composition \( \mathcal{A} \to D_m(\mathcal{A}) \) is also a \( \delta \)-functor. Thus an exact sequence \( 0 \to L \to M \to N \to 0 \) in \( \mathcal{A} \) gives an exact triangle \( L \to M \to N \to L[1] \) in \( D_m(\mathcal{A}) \). It implies \( \psi: K_0(\mathcal{A}) \ni [M] \to K_0(D_m(\mathcal{A})) \) is a well-defined homomorphism.

3. For a class \( S \) of objects in a triangulated category, it is well-known that an object of the smallest triangulated category containing \( S \) is a (finite) iterated extension of shifts of objects of \( S \). Hence Fact 2.9 implies an object of \( D_m(\mathcal{A}) \) is an iterated extension of shifts of objects of \( \mathcal{A} \), and thus \( \psi \) is surjective.

Finally, we explain the relationship between periodic complexes and usual complexes. We do not use the following results and explanations in this paper but it gives a good picture of periodic derived categories. Let \( \mathcal{A} \) be an abelian category. \( C(\mathcal{A}) \) (resp. \( C^b(\mathcal{A}) \)) denotes the category of usual (resp. bounded) complexes over \( \mathcal{A} \). Define functors

\[ \iota: C_m(\mathcal{A}) \to C(\mathcal{A}), \quad V \mapsto \left( V^{(i \mod m)}, d_V^{(i \mod m)} \right)_{i \in \mathbb{Z}} \]

and

\[ \pi: C^b(\mathcal{A}) \to C_m(\mathcal{A}), \quad V \mapsto \left( \bigoplus_{j \equiv i \mod m} V^j, \bigoplus_{j \equiv i \mod m} d_V^j \right)_{i \in \mathbb{Z}}. \]

The functors \( \iota \) and \( \pi \) preserve quasi-isomorphisms, and induce triangulated functors \( \iota: D_m(\mathcal{A}) \to D(\mathcal{A}) \) and \( \pi: D^b(\mathcal{A}) \to D_m(\mathcal{A}), \) respectively. The functor \( \pi: D^b(\mathcal{A}) \to D_m(\mathcal{A}) \) is called the covering functor. This name comes from the following fact.

Fact 2.11 ([S21] Corollary 3.29)). Let \( \mathcal{A} \) be an enough projective abelian category of finite global dimension. For any \( V, W \in D^b(\mathcal{A}) \), we have \( \text{Hom}_{D^b(\mathcal{A})}(\pi V, \pi W) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_\mathcal{A}(V[i],W[i]). \)

For an additive category \( \mathcal{C} \) and an auto equivalence \( F: \mathcal{C} \to \mathcal{C} \), the orbit category \( \mathcal{C}/F \) of \( \mathcal{C} \) by \( F \) is defined by

\[ \text{Ob}(\mathcal{C}/F) := \text{Ob}\mathcal{C}, \quad \text{Hom}_{\mathcal{C}/F}(X,Y) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_\mathcal{C}(X, F^i Y). \]
We get \( \text{Ker} (\phi) \impliedby \text{equality (3.1)} \) sequences in \( A \). Proof \( \rightarrow \) \( X \). Let \( \pi \) be an essentially small category. Then for any \( m \)-periodic complex \( V \in D_m(\mathcal{A}) \), there exists an isomorphism \( V \simeq \bigoplus H^i(V)[-i] \) in \( D_m(\mathcal{A}) \).

In particular, the covering functor \( \pi : D^b(\mathcal{A}) \to D_m(\mathcal{A}) \) is essentially surjective, and thus the \( m \)-periodic derived category \( D_m(\mathcal{A}) \) coincide with the orbit category \( D^b(\mathcal{A})/[m] \).

Example 2.13. Let \( kQ \) be a path algebra over a field \( k \). By Fact 2.12, an indecomposable objects of \( D_m(\text{mod } kQ) \) is of the form \( M[i] \) for some \( M \in \text{mod } kQ \) and some \( i \in \mathbb{Z}_m \). If \( m \geq 2 \), then we have

\[
\text{Hom}_{D_m(\text{mod } kQ)}(M, N[i]) = \begin{cases} 
\text{Hom}_{kQ}(M, N) & \text{if } i \equiv 0 \mod m \\
\text{Ext}^1_{kQ}(M, N) & \text{if } i \equiv 1 \mod m \\
0 & \text{if } i \not\equiv 0, 1 \mod m
\end{cases}
\]

for any \( M, N \in \text{mod } kQ \) by Fact 2.12.

The category \( D_m(\text{mod } kQ) \) admits Auslander-Reiten sequences [Fu12 Theorem 2.10] and the covering functor \( \pi : D^b(\text{mod } kQ) \to D_m(\text{mod } kQ) \) preserves Auslander-Reiten sequences [Fu12 Theorem 3.1]. Thus if \( Q = 1 \leftarrow 2 \leftarrow 3 \), then the Auslander-Reiten quiver of \( D_2(\text{mod } kQ) \) is the following as explained in Example 2.3.

3. Grothendieck groups of periodic triangulated categories

In this section, we investigate properties of the Grothendieck group of a periodic triangulated category.

3.1. Even periodic case. Let \( m > 0 \) be an even integer, and \( T \) be an essentially small \( m \)-periodic triangulated category.

Lemma 3.1. A cohomological functor \( F : T \to A \) induces a homomorphism \( K_0(T) \to K_0(\mathcal{A}) \).

Proof. Set \( \phi(X) := \sum_{i=0}^{m-1} (-1)^i [F^i(X)] \in K_0(\mathcal{A}) \). We have to show that for any exact triangle \( X \to Y \to Z \to \Sigma X \) in \( T \), the equality \( \phi(X) - \phi(Y) + \phi(Z) = 0 \) holds in \( K_0(\mathcal{A}) \). We have two exact sequences in \( \mathcal{A} \):

\[
(3.1) \quad F^{m-1}(Z) \xrightarrow{f} F^m(X) \simeq F^0(X) \xrightarrow{g} F^0(Y),
\]

\[
(3.2) \quad 0 \to \ker g \to F^0(X) \xrightarrow{g} F^0(Y) \to \cdots \to F^{m-1}(Y) \to F^{m-1}(Z) \xrightarrow{f} \text{Im } f \to 0.
\]

We get \( [\ker g] = \phi(X) - \phi(Y) + \phi(Z) + (-1)^m [\text{Im } f] \) by (3.2). The assumption that \( m \) is even and (3.1) imply the equality \( \phi(X) - \phi(Y) + \phi(Z) = 0 \). \qed
3.2. Odd periodic case. Let $m > 0$ be an odd integer, and $\mathcal{T}$ be an essentially small $m$-periodic triangulated category.

**Lemma 3.2.** $K_0(\mathcal{T})$ is an $\mathbb{F}_2$-vector space, that is, for any element $\alpha \in K_0(\mathcal{T})$, we have $2\alpha = 0$.

**Proof.** By the axiom of triangulated categories, $X \to 0 \to \Sigma X \to \Sigma X$ is an exact triangle for any $X \in \mathcal{T}$. It implies $[\Sigma X] = -[X]$ in $K_0(\mathcal{T})$. Hence we have $[X] = [\Sigma^m X] = (-1)^m[X] = -[X]$. Thus we conclude that $2[X] = 0$ for any $X \in \mathcal{T}$. □

**Lemma 3.3.** A cohomological functor $\mathcal{T} \to \mathcal{A}$ induces a homomorphism $K_0(\mathcal{T}) \to K_0(\mathcal{A})$, where $K_0(\mathcal{A})_{\mathbb{F}_2} := K_0(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{F}_2 \cong K_0(\mathcal{A})/2K_0(\mathcal{A})$.

**Proof.** Let $X \to Y \to Z \to \Sigma X$ be an exact triangle in $\mathcal{T}$. Set $\phi(X) := \sum_{i=0}^{m-1} (-1)^i [F^i(X)] \in K_0(\mathcal{A})$. Then we have, for some object $K \in \mathcal{A}$,

$$[K] = \phi(X) - \phi(Y) + \phi(Z) + (-1)^m[K]$$

in $K_0(\mathcal{A})$ by the same calculation as in Lemma 3.1. Because $m$ is odd, We get $\phi(X) - \phi(Y) + \phi(Z) = 2[K] \equiv 0 \mod 2K_0(\mathcal{A})$. Hence the assignment $[T] \ni [X] \mapsto \phi(X) \mod 2K_0(\mathcal{A}) \in K_0(\mathcal{A})_{\mathbb{F}_2}$ extends to a homomorphism $K_0(\mathcal{T}) \to K_0(\mathcal{A})_{\mathbb{F}_2}$. □

4. Proof of Theorem 1.3

Let $\mathcal{A}$ be an enough projective abelian category of finite global dimension, $m > 0$ a positive integer, and $p$ the period of $D_m(\mathcal{A})$. There are three cases by Remark 2.6

(i) $m$ is even and $p = m$;
(ii) $m$ is odd and $p = m$, and
(iii) $m$ is odd and $p = 2m$.

We prove Theorem 1.1 separately in the three cases above. Note that the proof of the case (iii) also works the case (ii), but we give separate proofs since the proof of the case (ii) is simple and motivates the proof of (iii).

In any cases, we have a surjective homomorphism $\psi : K_0(\mathcal{A}) \to K_0(D_m(\mathcal{A}))$ induced by the natural $\delta$-functor $\mathcal{A} \to D_m(\mathcal{A})$ by Proposition 2.10.

(i) If $m$ is even, the cohomology functor $H : D_m(\mathcal{A}) \to \mathcal{A}$ induces a homomorphism

$$\phi : K_0(D_m(\mathcal{A})) \to K_0(\mathcal{A}), \quad [V] \mapsto \sum_{i=1}^{m} (-1)^i[H^i(V)]$$

by Lemma 3.1. The homomorphism $\phi$ is a retraction of $\psi$, and hence $\psi$ is injective. Thus $\psi$ is an isomorphism.

(ii) If $m$ is odd and $p = m$, then $D_m(\mathcal{A})$ is an odd periodic triangulated category. Thus $K_0(D_m(\mathcal{A}))$ is an $\mathbb{F}_2$-vector space by Lemma 3.2, and $\psi$ induces a surjective homomorphism $\psi_{\mathbb{F}_2} : K_0(D_m(\mathcal{A}))_{\mathbb{F}_2} \to K_0(D_m(\mathcal{A}))$. The cohomology functor $H : D_m(\mathcal{A}) \to \mathcal{A}$ also induces a homomorphism

$$\phi_{\mathbb{F}_2} : K_0(D_m(\mathcal{A})) \to K_0(D_m(\mathcal{A}))_{\mathbb{F}_2}, \quad [V] \mapsto \sum_{i=1}^{m} [H^i(V)] \mod 2K_0(\mathcal{A})$$

by Lemma 3.3. The homomorphism $\phi_{\mathbb{F}_2}$ is clearly a retraction of $\psi_{\mathbb{F}_2}$, and hence $\psi_{\mathbb{F}_2}$ is an isomorphism.

(iii) If $m$ is odd and $p = 2m$, then $D_m(\mathcal{A})$ is an even periodic triangulated category. Applying Lemma 3.4 to the cohomology functor $H : D_m(\mathcal{A}) \to \mathcal{A}$, we have an induced homomorphism $\phi : K_0(D_m(\mathcal{A})) \to K_0(\mathcal{A})$, but it is a zero map. Indeed, we have

$$\psi(V) = \sum_{i=1}^{2m} (-1)^i[H^i(V)] = \sum_{i=1}^{m} (-1)^i[H^i(V)] + \sum_{i=m+1}^{2m} (-1)^i[H^i(V)]$$

$$= \sum_{i=1}^{m} (-1)^i[H^i(V)] - \sum_{i=1}^{m} (-1)^i[H^i(V)] = 0.$$

Thus we cannot prove the theorem by the same way as (i).

Although $D_m(\mathcal{A})$ is even periodic, we can prove the similar results as Lemma 3.2 and 3.3 that is, $K_0(D_m(\mathcal{A}))$ is an $\mathbb{F}_2$-vector space and the assignment $\phi_{\mathbb{F}_2}(V) = \sum_{i=1}^{m} [H^i(V)] \mod 2K_0(\mathcal{A})$ defines a homomorphism $\phi_{\mathbb{F}_2} : K_0(D_m(\mathcal{A})) \to K_0(D_m(\mathcal{A}))_{\mathbb{F}_2}$. We first prove that $K_0(D_m(\mathcal{A}))$ is an $\mathbb{F}_2$-vector space. For an $m$-periodic complex $V = (V^i, d^i)_{i \in \mathbb{Z}_m}$, $V$ and $\Sigma^m V = (V^i, -d^i)_{i \in \mathbb{Z}_m}$ is not necessary isomorphic in $D_m(\mathcal{A})$ in general, but they define the same class $[V] = [\Sigma^m V]$ in the Grothendieck group $K_0(D_m(\mathcal{A}))$.
We prove this by induction on the number \( n_v \) of \( i \in \mathbb{Z}_m \) with \( V^i \neq 0 \). It is clear if \( n_v = 0, 1 \). Suppose \( n_v \geq 2 \). Then there exists \( i \in \mathbb{Z}_m \) such that \( V^i \neq 0 \). We may assume that \( i = 0 \). There exists the following exact sequences in \( C_m(\mathcal{A}) \).

\[
\begin{array}{ccccccc}
0 & \to & 0 & \to & 0 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
U & \to & \cdots & \to & V^{m-1} & \to & Z^0(V) & \to & 0 & \to & V^1 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
V & \to & \cdots & \to & V^{m-1} & \to & V^0 & \to & 0 & \to & V^1 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
V^0/Z^0(V) & \to & \cdots & \to & 0 & \to & V^0/Z^0(V) & \to & 0 & \to & \cdots, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\]
and

\[
\begin{array}{ccccccc}
0 & \to & 0 & \to & 0 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Z^0(V) & \to & \cdots & \to & 0 & \to & Z^0(V) & \to & 0 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
U & \to & \cdots & \to & V^{m-1} & \to & Z^0(V) & \to & 0 & \to & V^1 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
W & \to & \cdots & \to & V^{m-1} & \to & 0 & \to & V^1 & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\]

Noting that \( V^i = (\Sigma^m V)^i \) and \( Z^i(V) = Z^i(\Sigma^m V) \), we also have exact sequences \( 0 \to \Sigma^m U \to \Sigma^m V \to V^0/Z^0(V) \to 0 \) and \( 0 \to Z^0(V) \to \Sigma^m U \to \Sigma^m W \to 0 \). Since \( n_W < n_V \), we have \( [W] = [\Sigma^m W] \) in \( K_0(D_m(\mathcal{A})) \) by the induction hypothesis. The canonical \( \delta \)-functor \( C_m(\mathcal{A}) \to D_m(\mathcal{A}) \) carries the exact sequences above to exact triangles in \( D_m(\mathcal{A}) \), and thus we have

\[
[V] = [W] + [Z^0(V)] + [V^0/Z^0(V)] = [\Sigma^m W] + [Z^0(V)] + [V^0/Z^0(V)] = [\Sigma^m V] = -[V].
\]

Hence \( K_0(D_m(\mathcal{A})) \) is an \( \mathbb{F}_2 \)-vector space. The method above is Gorsky’s induction technique for periodic complexes, which appears in the proof of Fact 2.9. See [Go13, Proposition 9.7].

Next, we prove that the assignment \( \phi_{\mathcal{F}_2}(V) = \sum_{i=1}^m (H^i(V)) \mod 2K_0(\mathcal{A}) \) defines a homomorphism \( \phi_{\mathcal{F}_2} : K_0(D_m(\mathcal{A})) \to K_0(\mathcal{A})_{\mathcal{F}_2} \). Let \( U \to V \to W \to \Sigma U \) be an exact triangle in \( D_m(\mathcal{A}) \). \( m \)-periodic complexes \( U \) and \( \Sigma^m U \) are not isomorphic but \( H^m(U) = H^0(U) \) holds. Thus we have two exact sequences in \( \mathcal{A} \):

\[
H^{m-1}(W) \xrightarrow{f} H^m(U) \simeq H^0(U) \xrightarrow{g} H^0(V),
\]

\[
0 \to \ker g \to H^0(U) \xrightarrow{g} H^0(V) \to \cdots \to H^{m-1}(V) \xrightarrow{f} \text{im } f \to 0.
\]

A similar discussion as in Lemma 3.3 implies \( \phi_{\mathcal{F}_2}(U) - \phi_{\mathcal{F}_2}(V) + \phi_{\mathcal{F}_2}(W) \equiv 0 \mod 2K_0(\mathcal{A}) \).

The rest of the proof is similar to (ii). \( \square \)

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