Anti-self-dual Maxwell solutions on hyperkähler manifold and $N = 2$ supersymmetric Ashtekar gravity

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Abstract

Anti-self-dual (ASD) Maxwell solutions on 4-dimensional hyperkähler manifolds are constructed. The $N = 2$ supersymmetric half-flat equations are derived in the context of the Ashtekar formulation of $N = 2$ supergravity. These equations show that the ASD Maxwell solutions have a direct connection with the solutions of the reduced $N=2$ supersymmetric ASD Yang-Mills equations with a special choice of gauge group. Two examples of the Maxwell solutions are presented.

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1 Introduction

The Ashtekar formulation of Einstein gravity gives a new insight to the search for anti-self-dual (ASD) solutions without cosmological constant. These are constructed from the solutions of certain differential equations for volume-preserving vector fields on a 4-dimensional manifold. This characterization of the ASD solutions has been originally given by Ashtekar, Jacobson and Smolin [1], and further elaborated by Mason and Newman [2]. In the following, we call their differential equations the half-flat equations. These equations clarify the relationship between the ASD solutions of the Einstein and the Yang-Mills equations. Indeed, if we specialize the gauge group to be a volume-preserving diffeomorphism group, the reduced ASD Yang-Mills equations on the Euclidean space are identical to the half-flat equations [3].

Looking at geometrically, the Ashtekar formulation emphasizes the hyperkähler structures that naturally exist on ASD Einstein solutions. A hyperkähler manifold is a 4n-dimensional Riemannian manifold $(M, g)$ such that (1) $M$ admits three complex structures $J^a(a = 1, 2, 3)$ which obey the quaternionic relations $J^aJ^b = -\delta_{ab} - \epsilon_{abc}J^c$; (2) the metric...
is preserved by $J^a$; (3) the 2-forms $B^a$ defined by $B^a(X, Y) = g(J^a X, Y)$ for all vector fields $X, Y$ are three kähler forms, i.e. $dB^a = 0$ $(a = 1, 2, 3)$. The solutions of the half-flat equations ensure the conditions above and hence 4-dimensional hyperkähler metrics are ASD Einstein solutions.

Recently, making use of the half-flat equations we have explicitly constructed several hyperkähler metrics [3]. Subsequently here we extend the half-flat equations to the case of $N = 2$ supergravity. Our formulation has the advantage that the setting of $N = 2$ supersymmetric Yang-Mills theory is automatically provided. In particular ASD Maxwell solutions on hyperkähler manifolds are elucidated through the relationship to the reduced $N = 2$ ASD Yang-Mills equations. In the literature [4, 5] the $N = 2$ ASD supergravity has been investigated by using the superfield formulation, but our approach is very different and the results in the present work are more concrete.

In Section 2 we review the half-flat equations. In Section 3 we present a new construction of ASD Maxwell solutions on hyperkähler manifolds and derive the $N = 2$ supersymmetric half-flat equations. Finally, in Section 4 two examples of ASD Maxwell solutions are given.

The following is a summary of the notation used in this paper. The $so(3)$ generators and the Killing form are denoted by $E_a (a = 1, 2, 3)$ and $\langle , \rangle$, respectively. The symbols $\eta^a_{\mu\nu}$ and $\eta^a_{\mu\nu}$ $(a = 1, 2, 3; \mu, \nu = 0, 1, 2, 3)$ represent the 't Hooft matrices satisfying the relations:

$$\eta^a_{\mu\nu} = -\frac{1}{2} e_{\mu\nu\lambda\sigma} \eta^a_{\lambda\sigma}, \quad \eta^a_{\mu\nu} = \frac{1}{2} e_{\mu\nu\lambda\sigma} \eta^a_{\lambda\sigma}$$

(1.1)

and

$$\eta^a_{\mu\nu} \eta^b_{\lambda\sigma} = \delta_{ab} \delta_{\nu\sigma} + \epsilon_{abc} \eta^c_{\nu\sigma}, \quad (\eta^a_{\mu\nu} \text{ satisfy the same relations}).$$

(1.2)

In Section 3 we consider a space-time with the signature $(++--)$. Then the metrics $\hat{g}_{\mu\nu} = \text{diag}(1, 1, -1, -1)$ and $\kappa_{ab} = \text{diag}(1, -1, -1)$ are used to lower and rise the indices of $\eta^a_{\mu\nu}$ ($\tilde{\eta}^a_{\mu\nu}$).

## 2 Half-flat equations

In this section, we briefly describe the 4-dimensionial hyperkähler geometry from the point of view of Ashtekar gravity [1, 2]. We use the metric of the Euclidean signature for avoiding complex variables. The Ashtekar gravity consists of an $so(3)$ connection 1-form $A = A^a \otimes E_a$ and an $so(3)$-valued 2-form $B = B^a \otimes E_a$ on a 4-dimensional manifold $M$. The action is given by

$$S_{Ash} = \int_M \langle B \wedge F \rangle - \frac{1}{2} \langle C(B) \wedge B \rangle,$$

(2.1)

where $F = dA + \frac{1}{2}[A \wedge A]$, $C(B) = C^a_b B^b \otimes E_a$ and $C = C^a_b E_a \otimes E_b$ is a Lagrange multiplier field which obeys the conditions, $C^a_b = C^b_a$ and $C^a_a = 0$. The equations of

\footnote{\textit{1} $N = 1$ half-flat equations are obtained from our equations (3.20) \sim (3.23) by putting $T = 0.$}
motion are

\[ F - C(B) = 0 \]  
(2.2)
\[ DB = 0 \]  
(2.3)
\[ B^1 \wedge B^2 = B^2 \wedge B^3 = B^3 \wedge B^1 = 0 \]  
(2.4)
\[ B^1 \wedge B^1 = B^2 \wedge B^2 = B^3 \wedge B^3, \]  
(2.5)

where \( D \) is the covariant derivative with respect to \( A \). The algebraic equations (2.4) and (2.5) represent the constraints of this system.

To solve the constraints we introduce linearly independent four vector fields \( V^\mu (\mu = 0,1,2,3) \) and a volume form \( \omega \) on \( M \). Then the solutions become the self-dual \( 2 \)-forms

\[ B^a = \frac{1}{2} \eta^a_{\mu \nu} \iota_{V^\mu} \iota_{V^\nu} \omega, \]  
(2.6)

where \( \iota_{V^\mu} \) denotes the inner derivation with respect to \( V^\mu \). We proceed to solve the remaining equations (2.2) and (2.3). For the hyperkähler geometry, which we will focus on in this paper, \( C \) must be taken to be zero because \( C^a_b \) are the coefficients of self-dual Weyl curvature; this is equivalent to the requirement that the holonomy group is contained in subgroup \( \text{Sp}(1) \) of \( \text{SO}(4) \). With this choice, (2.2) becomes \( F = 0 \) and if we take the gauge fixing \( A = 0 \), (2.3) reduces to

\[ dB^a = 0 \quad (a = 1,2,3). \]  
(2.7)

Thus (2.6) implies the half-flat equations [1, 2],

\[ \frac{1}{2} \eta^a_{\mu \nu} [V^\mu, V^\nu] = 0, \]  
(2.8)
\[ L_{V^\mu} \omega = 0. \]  
(2.9)

This can be seen by applying the formula :

\[ d(\iota_X \iota_Y \alpha) = \iota_{[X,Y]} \alpha + \iota_Y L_X \alpha - \iota_X L_Y \alpha + \iota_X \iota_Y d \alpha \]  
(2.10)

for vector fields \( X, Y \) and a form \( \alpha \). Given a solution of (2.8) and (2.9), we have a metric

\[ g(V^\mu, V^\nu) = \phi \delta^\mu_\nu, \]  
(2.11)

where \( \phi = \omega(V_0, V_1, V_2, V_3) \). This metric is invariant by the three complex structures

\[ J^a(V^\mu) = \eta^a_{\mu \nu} V^\nu \quad (a = 1,2,3), \]  
(2.12)

which obey the relations \( J^a J^b = -\delta^a_b - \epsilon^{abc} J^c \) and give the three Kähler forms \( B^a(V^\mu, V^\nu) = g(J^a(V^\mu), V^\nu) \). Thus the triplet \( (M, g, J^a) \) is a hyperkähler manifold. Conversely, it is known that every 4-dimensional hyperkähler manifold locally arises by this construction [2, 7].

This formulation yields that the vector fields \( V^\mu \) may be identified with the components of a space-time independent ASD Yang-Mills connection on \( \mathbb{R}^4 \). Indeed, (2.7) is the
assertion that the gauge group is the diffeomorphism group \( \text{SDiff}_\omega(M) \) preserving the volume form \( \omega \), and (2.8) are explicitly written as

\[
[V_0, V_1] + [V_2, V_3] = 0 \quad (2.13)
\]
\[
[V_0, V_2] + [V_3, V_1] = 0 \quad (2.14)
\]
\[
[V_0, V_3] + [V_1, V_2] = 0, \quad (2.15)
\]

which are equivalent to the reduced ASD Yang-Mills equations \([2]\).

3 N=2 supersymmetric Ashtekar gravity

We start with the chiral action for \( \mathbb{N}=2 \) supergravity without cosmological constant \([8, 9]\). The bosonic part, which is the chiral action of Einstein-Maxwell theory, contains a \( U(1) \) connection 1-form \( a \) and a 2-form \( b \) in addition to \( A, B \) in (2.1) \([8]\). The fermionic fields (two gravitino fields) are expressed by Weyl spinor 1-forms \( \psi^i \) and Weyl spinor 2-forms \( \chi^i \), where \( i (= 1, 2) \) is a \( \text{Sp}(1) \) index representing the two supersymmetric charges.

By using the 2-component spinor notation, the chiral action is written as

\[
S_{\text{Ash}}^{\mathbb{N}=2} = \int B^{AB} \wedge F_{AB} + b \wedge f + \chi^i_A \wedge D\psi^A_i - \frac{1}{2} b \wedge b - \frac{1}{8} b \wedge \psi^A_i \wedge \psi^j_A - \frac{1}{2} C_{ABCD} B^{AB} \wedge B^{CD} - \frac{1}{2} \kappa^{iABC} B^{AB} \wedge \chi^C - \frac{1}{2} H_{AB} (B^{AB} \wedge b - \chi^A_i \wedge \chi^B_i) \quad (3.1)
\]

where \( f = da \), and \( C_{ABCD}, \kappa^{iABC} \) and \( H_{AB} \) are totally symmetric Lagrange multiplier fields.

Let us focus on ASD solutions. Then we can put \( A_{AB} = C_{ABCD} = 0 \) as stated in Sect.2, and further impose the conditions \( H_{AB} = \kappa^{iABC} = \psi^A_i = 0 \). It should be noticed that these restrictions preserve the \( \mathbb{N}=2 \) supersymmetry; as we will see in Sect.3.2 this symmetry is properly realized in the \( \mathbb{N}=2 \) supersymmetric ASD Yang-Mills equations with the gauge group \( \text{SDiff}_\omega(M) \). Now the equations of motion derived from \( S_{\text{Ash}}^{\mathbb{N}=2} \) reduce to

\[
f = b \quad (3.2)
\]
\[
dB^{AB} = db = d\chi^A_i = 0 \quad (3.3)
\]
\[
B^{(AB} \wedge B^{CD)} = 0 \quad (3.4)
\]
\[
B^{(AB} \wedge \chi^C_i) = 0 \quad (3.5)
\]
\[
B^{AB} \wedge b - \chi^A_i \wedge \chi^B_i = 0. \quad (3.6)
\]

3.1 Maxwell solutions on hyperkähler manifolds

We first consider the bosonic sector \( (b = f, B) \) in a space-time with the Euclidean signature. The relevant equations are obtained from (3.2) \( \sim (3.6) \) by putting \( \chi^A_i = 0 \). In the

\footnote{We have re-named the variables in \([10]\) as \((A_{AB}, A, B, \psi^A, B, \chi^A_i, \psi_{ABCD}, \kappa^{iABC}, \phi_{AB}) \mapsto (A_{AB}, a, \sqrt{2} \psi^A_i, -A^{AB}, -\frac{1}{2} b, -\sqrt{2} \chi^A_i, -C_{ABCD}, -\frac{1}{\sqrt{2}} \kappa^{iABC}, -\frac{1}{2} H_{AB}) \).}
previous section we have seen that the solutions $B^a(a = 1, 2, 3)$ are self-dual Kähler forms on a hyperkähler manifold $M$. Thus the equations (3.3) and (3.6) imply that $b$ is an ASD closed 2-form (ASD Maxwell solution) on $M$. The following proposition holds.

**Proposition.** Let $M$ be a hyperkähler manifold expressed by linear independent vector fields $V_\mu(\mu = 0, 1, 2, 3)$ and a volume form $\omega$ as mentioned in (2.8) and (2.9). If the vector field $T = T_\mu V_\mu$ satisfies

\begin{align*}
L_T \omega &= 0, \\
[V_\mu, [V_\mu, T]] &= 0,
\end{align*}

then $b$ defined by

\[ b = \frac{1}{2} b^a \eta^a_{\mu \nu} \iota_{V_\mu} \iota_{V_\nu} \omega \quad \text{for} \quad b^a = \eta^a_{\mu \nu} V_\mu T_\nu, \tag{3.9} \]

is an ASD closed 2-form on $M$.

**Proof.** The ASD condition of $b$ immediately follows from (3.9). Therefore it suffices to prove that $b$ is a closed 2-form. Using the identity of the 't Hooft matrices

\[ \eta^a_{\mu \nu} \eta^a_{\lambda \sigma} = \delta_{\mu \lambda} \delta_{\nu \sigma} - \delta_{\mu \sigma} \delta_{\nu \lambda} - \epsilon_{\mu \nu \lambda \sigma}, \tag{3.10} \]

we rewrite (3.3) in the form,

\[ b = \iota_{V_\mu} \iota_{V_\nu} L_{V_\mu} (T_\nu \omega) - \frac{1}{2} \epsilon_{\mu \nu \lambda \sigma} \iota_{V_\mu} \iota_{V_\sigma} L_{V_\lambda} (T_\nu \omega). \tag{3.11} \]

Let us define the vector fields

\[ W_{\mu \nu} = [V_\mu, V_\nu] + \frac{1}{2} \epsilon_{\mu \nu \lambda \sigma} [V_\lambda, V_\sigma]. \tag{3.12} \]

Then,

\[ b + \iota_{V_\mu} \iota_{W_{\mu \nu}} T_\nu \omega = \iota_{V_\mu} L_{V_\nu} (T_\nu \omega) + \frac{1}{2} \epsilon_{\mu \nu \lambda \sigma} \iota_{V_\mu} \iota_{V_\sigma} L_{V_\lambda} (T_\nu \omega). \tag{3.13} \]

The exterior derivative of (3.13) is evaluated as follows: Since both the vector fields $V_\mu$ and $T$ preserve the volume form $\omega$, we have

\[ d(\iota_{V_\mu} L_{V_\nu} \iota_{T_\nu} \omega) = L_{V_\mu} L_{V_\nu} \iota_{T_\nu} \omega = \iota_{[V_\mu, [V_\nu, T]]} \omega \tag{3.14} \]

and

\[ d(\epsilon_{\mu \nu \lambda \sigma} \iota_{V_\mu} L_{V_\lambda} \iota_{V_\sigma} (T_\nu \omega)) = \frac{1}{2} \epsilon_{\mu \nu \lambda \sigma} (L_{V_\mu} L_{V_\lambda} \iota_{V_\sigma} - \iota_{V_\mu} L_{V_\lambda} L_{V_\sigma})(T_\nu \omega) \]

\[ = \frac{1}{4} \epsilon_{\mu \nu \lambda \sigma} \iota_{[V_\mu, [V_\lambda, V_\sigma]]} T_\nu \omega = 0. \tag{3.15} \]

We thus find

\[ d(b + \iota_{V_\mu} \iota_{W_{\mu \nu}} T_\nu \omega) = \iota_{[V_\mu, [V_\nu, T]]} \omega. \tag{3.16} \]

Finally, making use of (2.8), i.e. $W_{\mu \nu} = 0$, combined with the condition (3.8), we obtain the required formula $db = 0$. \[ \square \]
Remark. Using the hyperkähler metric \((2.11)\), we can rewrite \((3.9)\) as
\[
b = dg(T) + \iota_{[V, T]}\omega.
\]
(3.17)

This expression is convenient to the explicit calculation in Sect.4.

3.2 \(N = 2\) supersymmetric half-flat equations

Let us return to the equations \((3.2)\)∼\((3.6)\) and assume a space-time with the signature \((++--)\). It is known that the hyperkähler manifolds with this signature provide the consistent backgrounds for closed \(N=2\) strings \([11, 12]\). We follow the paper for the spinor notation of \([4]\); the spinor indices \(A,B,C\) · · · in \((3.1)\) are replaced by the dotted indices \(\dot{A}, \dot{B}, \dot{C}\) · · · . To solve the constraints we introduce spinor valued vector fields \(V^A\) in addition to the vector fields \(V_\mu\) (or \(V_{\dot{A}}\)) and \(T\). Referring to \((3.9)\), we put
\[
\chi_{i\dot{A}} = \iota_{V_{\dot{B}}} V^{i\dot{A}} \omega,
\]
(3.18)
and
\[
\frac{1}{2} \bar{\eta}^{\mu\nu} [V_\mu, V_\nu] = 0
\]
(3.20)

\[
[V_\mu, [V_\mu, T]] + [V_i^A, V_i^A] = 0
\]
(3.21)

and
\[
[V_{\dot{B}}, V_i^A] = 0
\]
(3.22)

\[
L_{V_\mu} \omega = L_{V_i^A} \omega = L_T \omega = 0.
\]
(3.23)

This result is satisfactory in that it gives the direct correspondence between the ASD solutions of the \(N = 2\) supergravity and the \(N = 2\) supersymmetric Yang-Mills theory; the equations \((3.20)\)∼\((3.23)\) can be regarded as \(N = 2\) supersymmetric extension of the half-flat equations. To say more precisely, let us recall the \(N = 2\) ASD Yang-Mills equation in a flat space-time with the signature \((++--)\) \([4, 5]\). The \(N = 2\) Yang-Mills theory has the field content \((A_\mu, \lambda_i^A, \tilde{\lambda}_{i\dot{A}}, S, \tilde{S})\), where \(\lambda_i^A\) and \(\tilde{\lambda}_{i\dot{A}}\) are chiral and anti-chiral Majorana-Weyl spinors, while the fields \(S\) and \(\tilde{S}\) are real scalars. All the fields are in the adjoint representation of gauge group. By the supersymmetric ASD condition, i.e. \(\tilde{S} = 0\), the equations of motion reduce to
\[
\frac{1}{2} \bar{\eta}^{\mu\nu} [D_\mu, D_\nu] = 0
\]
(3.24)

\[
D^\mu D_\mu S + [\lambda_i^A, \lambda_i^A] = 0
\]
(3.25)

\[
(\sigma^\mu D_\mu)_{BA} \lambda_i^B = 0,
\]
(3.26)

where \(D_\mu = \partial_\mu + [A_\mu, \cdot]\). If we require that the fields are all constant on the space-time, and further choose the gauge group as \(SDiff_\omega(M)\), then the equations \((3.24)\)∼\((3.26)\) just become the \(N = 2\) supersymmetric half-flat equations \((3.20)\)∼\((3.23)\) with the identification \(A_\mu = V_\mu, \lambda_i^A = V_i^A\) and \(S = T\).
4 Examples of ASD Maxwell solutions

As an application of the proposition, we present two examples of ASD Maxwell solutions on 4-dimensional hyperkähler manifolds with one isometry generated by a Killing vector field $K = \frac{\partial}{\partial \tau}$. The first example gives the well-known Maxwell solution and the second one leads to a new solution as far as the authors know. We use local coordinates $(\tau, x^1, x^2, x^3)$ and a volume form $\omega = d\tau \wedge dx^1 \wedge dx^2 \wedge dx^3$ for the background manifold.

4.1 Gibbons-Hawking background

In this case we choose the vector fields $V_\mu$ as \[ V_0 = \phi \frac{\partial}{\partial \tau}, \]
\[ V_i = \frac{\partial}{\partial x^i} + \psi_i \frac{\partial}{\partial \tau} \quad (i = 1, 2, 3), \]
where the functions $\phi, \psi_i$ are all independent of $\tau$. Then these vector fields preserve the volume form $\omega$ and (2.8) implies
\[ *d\phi = d\psi, \]
where $\psi = \psi_i dx^i$ and $*$ denotes the Hodge operator on $\mathbb{R}^3 = \{ (x^1, x^2, x^3) \}$. The resultant metric is the Gibbons-Hawking multi-center metric \[ ds^2 = \phi^{-1}(d\tau + \psi)^2 + \phi dx^i dx^i. \]
The Killing vector field $T = K$ clearly satisfies (3.7) and (3.8). Applying the proposition, we have an ASD Maxwell solution \[ b = da \quad \text{with} \quad a = \phi^{-1}(d\tau - \psi). \]

4.2 Real heaven background

We choose the vector fields $V_\mu$ as \[ V_0 = e^\psi \left( \partial_3 \psi \cos \left( \frac{\tau}{2} \right) \frac{\partial}{\partial \tau} + \sin \left( \frac{\tau}{2} \right) \frac{\partial}{\partial x^3} \right), \]
\[ V_1 = e^\psi \left( -\partial_3 \psi \sin \left( \frac{\tau}{2} \right) \frac{\partial}{\partial \tau} + \cos \left( \frac{\tau}{2} \right) \frac{\partial}{\partial x^3} \right), \]
\[ V_2 = \frac{\partial}{\partial x^1} + \partial_2 \psi \frac{\partial}{\partial \tau}, \]
\[ V_3 = \frac{\partial}{\partial x^2} - \partial_1 \psi \frac{\partial}{\partial \tau}, \]
If the function $\psi$ is independent of $\tau$ and satisfies the 3-dimensional continual Toda equation:
\[ \partial_1^2 \psi + \partial_2^2 \psi + \partial_3^2 e^\psi = 0, \]
these vector fields are solutions of the half-flat equations (2.8) and (2.9). Then, the hyperkähler metric (the real heaven solution) is given by [16]

\[ ds^2 = (\partial_3 \psi)^{-1}(d\tau + \beta)^2 + (\partial_3 \psi)\gamma_{ij}dx^i dx^j, \] (4.11)

where

\[ \beta = -\partial_2 \psi dx^1 + \partial_1 \psi dx^2, \] (4.12)

and \( \gamma_{ij} \) is the diagonal metric \( \gamma_{11} = \gamma_{22} = e^{\psi}, \gamma_{33} = 1. \)

In this case we find solution of (3.7) and (3.8):

\[ T = c_1(\partial_1 \psi) \frac{\partial}{\partial \tau} + c_2(\partial_2 \psi) \frac{\partial}{\partial \tau} \] for constants \( c_i \) (i = 1, 2). (4.13)

The corresponding ASD Maxwell solution is given by

\[ b = c_1 da^{(1)} + c_2 da^{(2)}, \] (4.14)

where

\[ a^{(1)} = \partial_1 \psi(\partial_3 \psi)^{-1}(d\tau + \beta) + \partial_3 e^{\psi} dx^2 - \partial_2 \psi dx^3, \] (4.15)

\[ a^{(2)} = \partial_2 \psi(\partial_3 \psi)^{-1}(d\tau + \beta) - \partial_3 e^{\psi} dx^1 + \partial_1 \psi dx^3. \] (4.16)

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