PATHS, CYCLES AND SPRINKLING IN RANDOM HYPERGRAPHS

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Abstract. We prove a lower bound on the length of the longest $j$-tight cycle in a $k$-uniform binomial random hypergraph for any $2 \leq j \leq k - 1$. We first prove the existence of a $j$-tight path of the required length. The standard “sprinkling” argument is not enough to show that this path can be closed to a $j$-tight cycle - we therefore show that the path has many extensions, which is sufficient to allow the sprinkling to close the cycle.

1. Introduction

1.1. Paths and cycles in random graphs. Over the years there has been a considerable amount of research into the length of the longest paths and cycles in random graphs. This goes back to the work of Ajtai, Komlós and Szemerédi [1], who showed that in the Erdős-Rényi binomial random graph $G(n, p)$, the threshold $p = 1/n$ for the existence of a giant component is also the threshold for a path of linear length. In the supercritical regime, a standard sprinkling argument shows that whp the lengths of the longest path and the longest cycle are asymptotically the same, and therefore whp $G(n, p)$ also contains a cycle of linear length. This has been strengthened by various researchers, including Łuczak [11], and Kemkes and Wormald [10].

We note, however, that when $p = (1 + \varepsilon)/n$ for some small $\varepsilon > 0$, the asymptotic length $L_C$ of the longest cycle is still not known precisely: the best known lower and upper bounds are approximately $4n/3$ (see [11]) and $1.7395n$ (see [10]) respectively. On the other hand, Anastos and Frieze [2] determined the asymptotic length of the longest cycle precisely when $p = c/n$ for some sufficiently large constant $c$.

A similar problem, although one requiring very different techniques, is to determine the length of the longest induced path, which was achieved very recently by Glock [9] in the regime when $p = c/n$.

1.2. Paths and cycles in random hypergraphs. Given an integer $k \geq 2$, a $k$-uniform hypergraph consists of a set $V$ of vertices and a set $E \subset \binom{V}{k}$ of edges. (A 2-uniform hypergraph is simply a graph.) Among the many possible definitions of paths and cycles in hypergraphs, perhaps the most natural and well-studied is that of $j$-tight paths and cycles, which is in fact a family of definitions for $1 \leq j \leq k - 1$.

Definition 1. Given integers $1 \leq j \leq k - 1$ and a natural number $\ell$, a $j$-tight path of length $\ell$ in a $k$-uniform hypergraph consists of a sequence of distinct vertices $x_1, \ldots, x_{j+(k-j)\ell}$ and a sequence of edges $e_1, \ldots, e_\ell$ such that $e_i = \{x_{(k-j)(i-1)+1}, \ldots, x_{(k-j)(i-1)+k}\}$.

A $j$-tight cycle of length $\ell$ is similar except that $x_i = x_{(k-j)\ell+i}$ for $1 \leq i \leq j$ (and otherwise all vertices are distinct).

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1 with high probability, meaning with probability tending to 1 as $n$ tends to infinity.
In the literature, 1-tight paths/cycles are often called loose paths/cycles, while \((k - 1)\)-tight is often abbreviated simply to tight.

Let \(H^k(n, p)\) denote the \(k\)-uniform binomial random hypergraph, in which each \(k\)-set of vertices forms an edge with probability \(p\) independently. The analogue of the result of Ajtai, Komlós and Szemerédi showing a threshold for the existence of a \(j\)-tight path of linear length in \(H^k(n, p)\) was proved by the author together with Garbe, Hng, Kang, Sanhueza-Matamala and Zalla [3] for all \(k\) and \(j\). In contrast to the graph case, in general the threshold is not the same as the threshold for a giant \(j\)-tuple component (which was determined in [7]).

Let \(a\) be the unique integer satisfying \(1 \leq a \leq k - j\) and \(a \equiv k \mod k - j\), and let \(p_0 = p_0(n, k, j) := \frac{1}{\binom{k - j}{a} \binom{n}{k - j}}\). The results of [3] show that \(p_0\) is a threshold for the existence of a \(j\)-tight path of linear length in \(H^k(n, p)\). Furthermore, in the case when \(p = (1 + \varepsilon)p_0\) for some constant \(\varepsilon > 0\), upper and lower bounds on the length of the longest \(j\)-tight path were proved. In the case when \(j \geq 2\), these bounds are \(\Theta(\varepsilon n)\) and differ by a factor of 8. In the case when \(j = 1\), the lower bound was \(\Theta(\varepsilon^2 n)\) while the upper bound was \(\Theta(\varepsilon n)\).

This upper bound in the case when \(j = 1\) was subsequently improved by the author, Kang and Zalla [8] and shown to be \(\Theta(\varepsilon^2 n)\) in the range when \(p = (1 + \varepsilon)p_0\) (although the results of that paper also cover the range \(p = c/n\) for any constant \(c > 1\)). The strategy used was to prove an upper bound on the length of the longest loose cycle which transfers to an upper bound for loose paths using a standard sprinkling argument, just as has been often observed for graphs. Similarly, sprinkling can also be used to extend the lower bound on loose paths from [3] to an asymptotically identical lower bound for loose cycles.

1.3. Sprinkling in hypergraphs. This raises an obvious question: can we also use the sprinkling technique for \(j \geq 2\), and obtain a \(j\)-tight cycle from a \(j\)-tight path without significantly decreasing the length? Unfortunately, the naive approach does not work.

To see why first consider the case \(j \leq k/2\), when we have \(p = \Theta(n^{-(k-j)})\) and a path of length \(\Theta(n)\). Now for some \(\omega \to \infty\), sprinkle an extra probability of \(p/\omega\). We can identify \(n/\omega\) many \(j\)-sets from the start and from the end of the path with which we attempt to close to a cycle, and we need a further \(k - 2j\) vertices from outside the cycle to complete an edge. Thus the number of potential edges which would close the cycle is \(\Theta((n/\omega)^2 n^{k-2j})\), and the expected number of suitable edges we find is

\[
\frac{p}{\omega} \cdot \Theta\left(\frac{n^{k-2j+2}}{\omega^2}\right) = \Theta\left(\frac{n^{2-j}}{\omega^2}\right).
\]

This will be clearly enough if \(j = 1\) and if \(\omega\) tends to infinity sufficiently slowly, but for \(j \geq 2\) the argument fails. Indeed, for \(j > k/2\), the situation becomes even worse: here we even need more than one edge in order to be able to close the path to a cycle.

The essential reason why the sprinkling no longer works stems from the interplay between the \(j\)-sets and the vertices: a \(j\)-tight path “lives” on vertices, but is extended (or closed to a cycle) via \(j\)-sets. The number of \(j\)-sets within the path is naturally bounded by \(\Theta(n)\), but this is tiny compared to the number of \(j\)-sets in the world (namely \(\binom{\binom{n}{j}}{j}\)).

1.4. Main result. The main contribution of this paper is to provide a variant of the sprinkling argument which does work for \(j \geq 2\). In particular, we provide a search algorithm which whp will construct a long \(j\)-tight cycle in \(H^k(n, p)\). We thus provide a lower bound on the length of the longest \(j\)-tight cycle. Along the
way, we generalise the lower bound for $j$-tight paths given in [3] to be applicable for a larger range of $p$.

Let $L_C = L_C(n, k, j, p)$ be the random variable denoting the length of the longest $j$-tight path in $H^k(n, p)$.

**Theorem 2.** Let $k, j \in \mathbb{N}$ satisfy $2 \leq j \leq k - 1$ and let $a$ be the unique integer satisfying $1 \leq a \leq k - j$ and $a \equiv k \mod k - j$. Let $p_0 = p_0(n, k, j) := \left(\frac{1}{n\cdot\binom{n}{a}}\right)$.

For any $\delta > 0$, for any constant $c > 1$ and for any sequence $(c_n)_{n \in \mathbb{N}}$ satisfying $c_n \to c$ the following is true. Suppose that $p = c_n p_0$. Then whp

$$L_C \geq (1 - \delta) \cdot \frac{1 - c^{-1/(k-j)}}{k-j} \cdot n.$$ 

Note that it is trivially true that $L_P \geq L_C - O(1)$, where $L_P$ denotes the length of the longest $j$-tight path in $H^k(n, p)$. Therefore as a corollary we also obtain a lower bound on $L_P$ which generalises the one in [3].

2. Preliminaries

2.1. Notation and terminology. In this section we introduce some notation and terminology, and fix various parameters for the rest of the paper.

Throughout the paper, let $k, j$ be fixed natural numbers satisfying $2 \leq j \leq k - 1$. In particular, for the rest of the paper we will usually simply refer to paths and cycles rather than $j$-tight paths and $j$-tight cycles, since $j$ is understood.

All asymptotics in the paper are as $n \to \infty$, and in particular we will use the standard Landau notation $o(\cdot), O(\cdot), \Theta(\cdot)$ with respect to these asymptotics. We consider $k, j$ to be constants, so for example a bound of $O(n)$ may have a constant that is implicitly dependent on $k$ and $j$.

Let us further define the following parameters. Let $a = a(k, j)$ be the unique integer satisfying $1 \leq a \leq k - j$ and

$$a \equiv k \mod k - j.$$ 

The motivation for this parameter will become clear in Section 2.2. Given a natural number $\ell$, let $v_\ell = v_\ell(j, k) := j + \ell(k - j)$ denote the number of vertices in a path of length $\ell$. When $\ell = \Theta(n)$, we will often approximate $v_\ell$ simply by $\ell(k - j) = (1 + O(1/n))v_\ell$.

Let $p_0 = p_0(n, k, j) := \left(\frac{1}{n\cdot\binom{n}{a}}\right)$ denote the threshold for a long tight path. Given $p = p(n) = c_n p_0$ for some sequence $c_n$ of positive real numbers, let

$$L_1 = L_1(p) := \frac{1 - c^{-1/(k-j)}}{k-j} \cdot n.$$ 

Note that the parameters $n, c, k, j$ are implicit in $p$ and will be clear from the context. Further, let $L_C = L_C(n, p, k, j)$ denote the length of the longest $j$-tight cycle in $H^k(n, p)$.

For an integer $m$, we denote $[m] := \{1, \ldots, m\}$ and $[m]_0 := [m] \cup \{0\}$. We omit floors and ceilings when this does not significantly affect calculations.

2.2. The structure of paths. In graphs, there are only two paths with the same edge set (the second is obtained by reversing orientation), but depending on the values of $k$ and $j$, there may be many ways of reordering the vertices of a $j$-tight path within the edges which give a different path with the same edges. For example, in Figure 1, we may re-order $x_1, x_2, x_3$ arbitrarily. Even in the middle of the path, we may switch the order of $x_8$ and $x_9$ to give a new path. Nevertheless, we will identify paths which have the same set of edges, and indeed often identify a path with its edge set. We similarly identify cycles with their edge sets.
A further important point to note is which $j$-sets we can continue from: For example, in Figure 1, it seems natural to continue from the 4-set $\{x_{13}, \ldots, x_{16}\}$, but since the vertices $x_{11}, x_{12}, x_{13}$ may be rearranged arbitrarily, we could just as well replace $x_{13}$ by either of $x_{11}, x_{12}$ in this 4-set.

To account for this, we will borrow the following terminology from [3].

**Definition 3.** An extendable partition of a $j$-set $J$ is an ordered partition $(C_0, C_1, \ldots, C_r)$ of $J$, where $r = \lfloor \frac{k}{j} \rfloor$, with $|C_0| = a$ and $|C_i| = k - j$ for all $i \in [r]$.

In the example above, the 4-set $\{x_{10}, \ldots, x_{13}\}$ would have extendable partition $(C_0, C_1)$, where $C_0 = \{x_{10}\}$ and $C_1 = \{x_{11}, x_{12}, x_{13}\}$. In a search process, the final edge $\{x_{10}, \ldots, x_{16}\}$ added to this path would give rise to three new 4-sets from which we can continue, namely $J_i := \{x_i, x_{14}, x_{15}, x_{16}\}$, where $i = 11, 12, 13$. The extendable partition of $J_i$ would be $(C_0(i), C_1(i))$, where $C_0(i) = \{x_i\}$ and $C_1(i) = \{x_{14}, x_{15}, x_{16}\}$.

For general $k$ and $j$, when we discover an edge $K$ from a $j$-set $J$ with extendable partition $(C_0, \ldots, C_r)$, the new $j$-sets from which we can continue will be those consisting of $a$ vertices from $C_1$, all vertices of $C_2, \ldots, C_r$ and all vertices of $K \setminus J$, and these sets (in this order) will naturally form an extendable partition of the new $j$-set.

We refer the reader to [3] for a more detailed discussion of the structure of paths.

### 3. Proof outline

The initial, naive proof idea is to construct a long path using a search process, and then apply a sprinkling argument to close this path into a cycle. However, in its most basic form this argument fails for the reasons outlined in the introduction: we have too few potential attachment $j$-sets and too many required edges for the sprinkling to work.

Nevertheless, this will still be our overarching strategy, it just needs to be modified slightly. More precisely, we will aim to construct a family of long paths, all of which are identical along most of their length, but which diverge towards the two ends. This will give us many more potential attachment $j$-sets, and allow us to push the sprinkling argument through.

As such, we have two main lemmas in the proof. Let $L_P$ denote the length of the longest path in $H^k(n, p)$.

**Lemma 4.** Under the conditions of Theorem 2, whp $L_P \geq \left(1 - \frac{3}{\delta}\right)L_1(p)$.

The proof of this lemma is essentially the same as the proof for the special case of $p = (1 + \varepsilon)p_0$ in [3]. We first define an appropriate depth-first search process for constructing $j$-tight paths. Heuristically, this DFS is supercritical for as long as the path constructed has length significantly smaller than $L_1$. However, the algorithm will avoid re-using $j$-sets that have already been tried, if they led to dead-ends, and we need to know that this will not slow down the
growth too much. For this, we need a bounded degree lemma which shows that, in an appropriate sense, these j-sets are evenly distributed in the hypergraph, rather than clustered together. The proof of Lemma 4 is given in Section 4.

The main original contribution of this work is the second lemma, which guarantees the existence of a family of j-tight paths with many different endpoints. The DFS algorithm is well-suited to creating long paths quickly, but in order to fan out towards the ends, we will switch to a breadth-first search algorithm. The result of this algorithm will be the following structure.

**Definition 5.** Given integers \( \ell_1, \ell_2 \), a j-set \( J \) and a path \( P \) of length \( \ell_1 \) with end \( J' \), we say that \( J \) \( \ell_2 \)-augments the pair \( (P, J') \) if there exists a path \( P_{J,J'} \) starting at \( J \) and ending at \( J' \) such that \( P_{J,J'} \cup P \) is again a path, and has length at most \( \ell_1 + \ell_2 \).

In other words, we can extend \( P \) by length at most \( \ell_2 \) to end at \( J \) instead of \( J' \).

**Lemma 6.** Under the conditions of Theorem 2, there exists some constant \( \varepsilon \in (0, 1) \) such that the following holds. Whp \( H^k(n, p) \) contains a j-tight path \( P_0 \) of length \( (1 - \delta/2)L_1(p) \) with ends \( J_s, J_e \), and collections \( \mathcal{A}, \mathcal{B} \) of j-sets such that:

- \( |\mathcal{A}|, |\mathcal{B}| = \varepsilon^2 n^2 \);
- Every j-set \( A \in \mathcal{A} \) 2(\log n)^2\-augments \( (P_0, J_s) \);
- Every j-set \( B \in \mathcal{B} \) 2(\log n)^2\-augments \( (P_0, J_e) \);
- For at least \( (1 - \varepsilon)\varepsilon^4 n^{2j} \) pairs \( (A, B) \in \mathcal{A} \times \mathcal{B} \), the augmenting paths \( P_{A,J_s}, P_{B,J_e} \) are vertex-disjoint.

We will show how Lemma 6 follows from Lemma 4 in Section 5. Before continuing with the proofs of these two lemmas, let us first show how Lemma 6 implies our main theorem.

**Proof of Theorem 2.** Let \( \omega \) be some function of \( n \) tending to infinity arbitrarily slowly, and let \( p' := (1 - 1/\omega)p \). We apply Lemma 6 with \( p' \) in place of \( \omega \). Let us observe that \( p' = c'_n p_0 \), where \( c'_n := (1 - 1/\omega)c_n \rightarrow c \), and therefore we have \( L_1(p') = L_1(p) \). It follows that the path \( P_0 \) provided by Lemma 6 has length at least \( (1 - \delta/2)L_1(p) \geq (1 - \delta)L_1(p) \).

Now for each pair \( (A, B) \in \mathcal{A} \times \mathcal{B} \) satisfying the last condition of Lemma 6, concatenating the paths \( P_{A,J_s}, P_{0, B,J_e} \) gives a path \( P_{A,B} \) with ends \( A \) and \( B \) and containing \( P_0 \), which therefore has length at least \( (1 - \delta)L_1(p) \) (the length of \( P_0 \)), but also of length at most \( (1 - \delta/2)L_1(p') + 2(\log n)^2 = (1 - \Theta(1))n \). In other words, \( P_{A,B} \) leaves a set \( V_{A,B} \) of \( \Theta(n) \) vertices uncovered.

Let us now sprinkle an additional probability of \( p'' := p - p' \) onto the hypergraph. In order to close \( P_{A,B} \) to a cycle, we need to find a configuration containing \( s = \lfloor k - j \rfloor \) edges and \( b = k - j - a \) vertices of \( V_{A,B} \). For a fixed choice of \( A, B \) and \( b \) vertices of \( V_{A,B} \), the probability that the \( s \) required edges exist is simply \( (p'')^s \). For fixed \( A \) and \( B \), but for different choices of the \( b \) vertices of \( V_{A,B} \), these edges are all distinct. However, given two choices \( A_1, B_1, R_1 \) and \( A_2, B_2, R_2 \) of \( A, B \) and \( b \) vertices from \( V_{A,B} \), it is possible that the configurations require the same \( k \)-set to be an edge, and thus we no longer have independence. We therefore show that there are sufficiently many choices for which the \( k \)-sets are all distinct.

To see this, observe that there are \( \Theta \left( \varepsilon^4 n^{2j+b+k} \right) \) choices for the triple \( (A, B, R) \), and any particular \( k \)-set is required to be an edge by at most \( O(n^{2j+b-k}) \) triples. Therefore any choice of triple shares a \( k \)-set with at most \( O(n^{2j+b-k}) \) other triples, and we may greedily choose \( \Theta \left( \varepsilon^4 n^{2j+b-(2j+b-k)} \right) = \Theta(\varepsilon^4 n^k) \) without conflicts.
By choosing this many triples, we observe that the probability that none of them closes a cycle is

\[
(1 - (p')^s)\Theta(\varepsilon^4n^k) \leq \exp \left( -\Theta \left( \frac{\varepsilon^4n^k}{\omega^s n^{s(k-j)}} \right) \right).
\]

Now recall that \( s = \lceil \frac{j}{k-j} \rceil = \lceil \frac{k}{k-j} \rceil - 1 \leq \frac{k-1}{k-j} \). Thus the probability that none of the choices of \( A, B, R \) admits the edges necessary to close a cycle is at most

\[
\exp \left( -\Theta \left( \frac{\varepsilon^4n}{\omega^{(k-1)/(k-j)}} \right) \right) = o(1),
\]

where the last estimate follows since \( \omega \) tends to infinity arbitrarily slowly, so in particular we have \( \omega^{(k-1)/(k-j)} = o(n) \). \( \square \)

4. Depth-first search: proof of Lemma 4

Since the proof of Lemma 4 is essentially the same as that of the special case when \( p = (1+\varepsilon)p_0 \) from [3], we will not go into full detail here. However, we will outline the argument, partly to make this paper self-contained and partly because some of the ideas will reappear in the more complicated proof of Lemma 6 in Section 5.

In order to prove the existence of a long path, we borrow the Pathfinder algorithm from [3]. This is in essence a depth-first search algorithm; however, there are a few complications in comparison to the graph case.

Recall from Section 2.2 that, depending on the values of \( k \) and \( j \), when we add an edge to the current path, we may have multiple new \( j \)-sets from which we could extend the path. For this reason, each time we increase the length of the path, we produce a batch of \( j \)-sets with which the path could potentially end. In the example in Figure 1, the batch would consist of the three 4-sets containing the three new vertices and one of the previous three vertices; more generally, a batch will contain any \( j \)-set from which the path can be extended if we discover a further edge containing that \( j \)-set (and no other vertices from the current path).

During the algorithm, at each time step we will query a \( k \)-set to determine whether it forms an edge or not. This may be thought of as revealing the outcome of a \( \text{Ber}(p) \) random variable corresponding to this \( k \)-set (with these variables being mutually independent).

We will describe \( j \)-sets as being neutral, active or explored; initially all \( j \)-sets are neutral; a \( j \)-set \( J \) becomes active if we have discovered a path which can end in \( J \) (in which case a whole batch becomes active); \( J \) becomes explored once we have queried all possible \( k \)-sets from \( J \).

Of course, in order to produce a path we will not query any \( k \)-sets from \( J \) that contain any further vertices (apart from \( J \)) of the current path. But more than this, in order to allow analysis of the algorithm, we place an additional restriction: specifically, we do not query any \( k \)-set that contains any other active or explored \( j \)-set. This ensures that we never query the same \( k \)-set twice from different \( j \)-sets, and therefore the outcome of each query is independent of all other queries.

Whenever a new \( j \)-set becomes active, it is added to the end of the current path. Since we are considering a depth-first search, we will always query \( k \)-sets from the last active \( j \)-set in the queue. Whenever the queue of active \( j \)-sets is empty (so also the current path is empty), we choose a new neutral \( j \)-set from which to continue uniformly at random, and this \( j \)-set becomes active.

A formal description of the Pathfinder algorithm can be found in [3].
Let us observe that in the algorithm, whenever we find an edge from a \( j \)-set with extendable partition \((C_0, \ldots, C_r)\), \({\binom{\ell_j}{a}} = \binom{k-j}{a}\) new \( j \)-sets become active. Heuristically, towards the start of the process we will query approximately \(\binom{n-v_j}{k-j}^{\ell_j}\) many \( k \)-sets from a \( j \)-set, where \( \ell \) is the current length of the path (and recall that \( v_j = j + \ell(k - j) \) denotes the number of vertices in a path of length \( \ell \)). This gives a clear intuition for why we should find a path of length \( L_1(p) \): the expected number of \( j \)-sets that become active from any current \( j \)-set is approximately

\[
\binom{k-j}{a} \binom{n-v_j}{k-j} p = \left(1 + o(1)\right) \left(1 - \frac{\ell(k-j)}{n}\right)^{k-j} c.
\]

When \( \ell = L_1 = \frac{1 - e^{-1/(n-j)}}{k-j} \cdot n \), up to the \( 1 + o(1) \) error term this gives precisely \( 1 \)—in other words, \( L_1 \) is the length at which this process changes from being supercritical to subcritical.

The main difficulty in the proof comes in the approximation of the number of \( k \)-sets that we query from each \( j \)-set, which above we estimated by \( \binom{n-v_j}{k-j}^{\ell_j} \). In fact, this is an obvious upper bound, whereas we need a lower bound. The upper bound takes account of \( k \)-sets that may not be queried because they contain a vertex from the current path, but \( k \)-sets may also be forbidden because they contain another active or explored \( j \)-set (apart from the one we are currently querying from).

We call a \( j \)-set discovered if it is either active or explored. The set \( G_{\text{disc}} = G_{\text{disc}}(t) \) of discovered \( j \)-sets at time \( t \) may be thought of as the edge set of a \( j \)-uniform hypergraph. It is intuitive that at the start of the search process (i.e. for small \( t \)), this hypergraph is sparse, but we need to quantify this more precisely. Given \( 0 \leq i \leq j - 1 \), let \( \Delta_i(t) = \Delta_i(G_{\text{disc}}(t)) \) denote the maximum \( i \)-degree of \( G_{\text{disc}}(t) \), that is the maximum over all \( i \)-sets \( I \) of the number of \( j \)-sets of \( G_{\text{disc}}(t) \) that contain \( I \). (Note in particular that \( \Delta_0(t) = |G_{\text{disc}}(t)| \).) The purpose of this parameter is highlighted in the following proposition.

**Proposition 7.** Suppose that a \( j \)-set \( J \) becomes active when the length of the path is \( \ell = \ell_J \) and that \( n - v_{\ell_J} = \Theta(n) \). Then the number of \( k \)-sets that are eligible to be queried from \( J \) at time \( t \) is at least

\[
\left(1 - \sum_{i=0}^{j-1} O\left(\frac{\Delta_i(t)}{n^{j-i}}\right)\right) \left(\frac{n - v_{\ell_J}}{k-j}\right).
\]

**Proof.** Let us consider how many \( k \)-sets may not be queried from a \( j \)-set \( J \) because they contain a second, already discovered \( j \)-set \( J' \). We will make a case distinction based on the possible intersection size \( i = |J \cap J'| \in [j - 1]_0 \), and note that for each \( i \in [j - 1]_0 \), the number of discovered \( j \)-sets \( J' \) which intersect \( J \) in \( i \) vertices is at most \( \binom{j}{i} \Delta_i(t) \), and the number of \( k \)-sets that are forbidden because they contain both \( J \) and \( J' \) is (crudely) at most \( n^{k-2j+i} \). Therefore the number of forbidden \( k \)-sets is certainly at most

\[
\sum_{i=0}^{j-1} \binom{j}{i} \Delta_i(t)n^{k-2j+i} = \sum_{i=0}^{j-1} O\left(\frac{\Delta_i(t)}{n^{j-i}}\right) \left(\frac{n - v_{\ell_J}}{k-j}\right),
\]

where the approximation follows because \( \binom{n-v_{\ell_J}}{k-j} = \Theta(n^{k-j}) \). \( \square \)

It follows from this proposition that if \( \Delta_i(t) \ll n^{j-i} \) for each \( i \), the number of forbidden \( k \)-sets is insignificant compared to the number of \( k \)-sets that may be queried, and the calculation above will go through with the addition of some smaller order error terms.
We will therefore run the **Pathfinder** algorithm until one of the three stopping conditions is satisfied. Let us fix a constant $0 < \varepsilon \ll \delta$ and further constants $1 \ll c_0 \ll c_1 \ll \ldots \ll c_{j-1} \ll 1/\sqrt{\varepsilon}$.

- (DFS1) $\ell = (1 - \delta/3)L_1$;
- (DFS2) $t = \varepsilon^2 n^k =: t_0$;
- (DFS3) $\Delta_i(t) \geq \varepsilon c_i n^{j-i}$ for some $0 \leq i \leq j - 1$.

Now our goal is simply to show that whp the algorithm terminates when (DFS1) is invoked. As such, we have two main auxiliary results.

**Proposition 8.** Whp (DFS2) is not invoked.

**Lemma 9.** Whp (DFS3) is not invoked.

We note that Lemma 9 is a form of *bounded degree lemma* similar to the one first proved in [7] and subsequently used in one form or another in [3, 4, 6]. A far stronger form also appeared in [5]. In its original form, the bounded degree lemma roughly states that no $i$-degree is larger than the average $i$-degree by more than a bounded factor. The stronger form in [5] even provides a lower bound, showing that whp all $i$-degrees are approximately equal, a phenomenon we call *smoothness*.

For our purposes we need only the upper bound, and allow a deviation from the average of $O(c_i/\varepsilon)$. This could certainly be improved, and it seems likely that even smoothness is satisfied, but since we do not require an especially strong result for this paper, for simplicity we make no effort to optimise the parameters.

**Proof of Proposition 8.** Let us suppose (for a contradiction) that at time $t_0 = \varepsilon^2 n^k$, neither (DFS1) nor (DFS3) has been invoked. Since (DFS3) has not been invoked, we have $\Delta_i \leq \varepsilon c_i n^{j-i} \leq \sqrt{\varepsilon} n^{j-i}$ for each $i \in [j - 1]_0$, and so by Proposition 7, from each explored $j$-set we certainly made at least

$$\left(1 - O(\sqrt{\varepsilon})\right) \left(\frac{n - v_{(1-\delta/3)L_1}}{k - j}\right) \geq (1 + \delta^2) \left(\frac{n - v_{L_1}}{k - j}\right)$$

queries. We also observe that at time $t_0$ the number of edges we have discovered is distributed as Bi$(t_0, p)$, which has expectation $t_0 p = \Theta(\varepsilon^2 n^j)$. By a Chernoff bound, whp we have discovered at least $(1 - \delta^3)t_0p$ edges, and therefore at least $(1 - \delta^3)t_0p\binom{k-j}{a}$ many $j$-sets have become active. At any time, the number of currently active $j$-sets is $O(L_1) = O(n)$, and therefore the number of fully explored $j$-sets is at least

$$\left(1 - \delta^3\right)t_0p\binom{k-j}{a} - O(n) \geq (1 - \delta^2/2)t_0p\binom{k-j}{a},$$

since $t_0 p = \Theta(\varepsilon^2 n^j) \gg n$.

Thus the total number of queries made by time $t_0$ is at least

$$\left(1 - \delta^2/2\right)t_0p\binom{k-j}{a}(1 + \delta^2) \left(\frac{n - v_{L_1}}{k - j}\right) \geq (1 + \delta^2/3)t_0,$$

which is clearly a contradiction since by definition we have made precisely $t_0$ queries. \hfill $\square$

**Proof outline of Lemma 9.** We give only an outline of the proof here to introduce the main ideas. An essentially identical argument was used to prove [3, Lemma 34].

First consider the case $i = 0$, when the desired bound follows from the fact that, by a Chernoff bound we have found at most $2pt_0 = O(\varepsilon^2 n^j)$ edges, each of which leads to $O(1)$ many $j$-sets becoming active. Some further $j$-sets may also become active without finding an edge each time the queue of active $j$-sets
is empty and we pick a new \( j \)-set from which to start. It is easy to bound the number of times this happens by \( O(t_0n^{j-k}) = \Theta(\varepsilon^2n^j) \) (see the argument for “new starts” below).

Now given \( i \in [j-1] \) and an \( i \)-set \( I \), there are three ways in which the degree of \( i \) in \( G_{\text{disc}} \) could increase.

- A new start at \( I \) occurs when the current path is fully explored and we pick a new (ordered) \( j \)-set from which to start a new exploration process. If this \( j \)-set contains \( I \), then the degree of \( I \) increases by 1.
- A jump to \( I \) occurs when a \( k \)-set containing \( I \) is queried from a \( j \)-set not containing \( I \) and this \( k \)-set is indeed an edge. Then the degree of \( I \) increases by at most \( \binom{k-a}{j} \).
- A pivot at \( I \) occurs when an edge is discovered from a \( j \)-set already containing \( I \). Then the degree of \( I \) increases by at most \( \binom{k-a}{j} \).

We bound the contributions to the degree of \( I \) made by these three possibilities separately.

**New starts.** We can crudely bound the number of new starts by observing that for each starting \( j \)-set we must certainly have made at least \( \Theta(n^{k-j}) \) queries to fully explore it, and therefore at time \( t \) we can have made at most \( \Theta(tn^{j-k}) \) many new starts in total (when \( t \geq n^{k-j} \)). Since we chose the \( j \)-set for a new start uniformly at random, the probability that such a new start contains \( I \) is \( \Theta(n^{-i}) \), and the probability that the number of new starts at \( I \) by time \( t_0 = \varepsilon^2n^k \) is larger than twice its expectation (which itself is \( \Theta(t_0n^{j-k-i}) = \Theta(\varepsilon n^{j-i}) \geq \sqrt{n} \)) is exponentially small.

**Jumps.** We further subclassify jumps according to the size \( z \) of the intersection \( I \cap J \) between \( I \) and the \( j \)-set \( J \) from which the jump to \( I \) occurs. Observe that since for a jump we cannot have \( I \subset J \), we must have \( 0 \leq z \leq i - 1 \).

The number of \( j \)-sets of \( G_{\text{disc}}(t) \) with intersection of size \( z \) with \( I \) is at most \( \binom{n}{z} \Delta_z(t) = O(\Delta_z(t)) = O(\varepsilon czn^{j-z}) \), where for the last approximation we used condition (DFS3) with \( z \) in place of \( i \). For each such \( j \)-set \( J \), the number of \( k \)-sets which contain both \( J \) and \( I \), and which could therefore result in a jump to \( I \), is at most \( \binom{n}{k-j-i+z} = O(n^{k-j-i+z}) \).

Thus the total number of queries made which could result in jumps to \( z \) is at most \( \sum_{z=0}^{n} O(\varepsilon czn^{j-z}) \cdot O(n^{k-j-i+z}) = O(\varepsilon c_{i-1}n^{k-i}) \). Each such query gives a jump with probability \( p = O(n^{k-j-i}) \), and a Chernoff bound implies that the number of jumps is \( O(\varepsilon c_{i-1}n^{j-i}) \). Since each jump contributes at most \( \binom{k-a}{j} = O(1) \) to the degree of \( I \), the total contribution made by jumps is \( O(\varepsilon c_{i-1}n^{j-i}) \).

**Pivots.** We observe that from any \( j \)-set containing \( I \), the expected number of pivots at \( I \) is at most \( \binom{n}{k-j}p = O(1) \). Furthermore since we are studying a DFS process creating a path, the number of consecutive pivots at \( I \) can be at most \( \frac{\sum_{z=0}^{n} O(\varepsilon czn^{j-z}) \cdot O(n^{k-j-i+z})}{\binom{n}{k-j}} = k \) before the path has left \( I \). Since the number of new starts and jumps to \( I \) is \( O(\varepsilon c_{i-1}n^{j-i}) \), it follows that also whp the number of pivots at \( I \) is \( O(\varepsilon c_{i-1}n^{j-i}) \).

Now we have bounded the contribution to the degree of \( I \) made by each of the three possibilities as \( O(\varepsilon c_{i-1}n^{j-i}) \), and summing these three terms, together with the fact that \( c_{i-1} \ll c_i \), gives the desired result. \( \square \)

5. **Breadth-first search: Proof of Lemma 6**

In this section we aim to show how we can use the single long path guaranteed whp by Lemma 4 and extend it using a breadth-first search process to a family of paths with many ends, as required by Lemma 6.
5.1. The BFS algorithm: Pathbranch.

5.1.1. Motivation and setup. We will use Lemma 4 as a black box, and let $P_0'$ be some path of length $(1 - \delta/2)L_1 + 2(\log n)^2 \leq (1 - \delta/3)L_1$, which is guaranteed to exist whp. Let $P_0$ be the subpath of length $(1 - \delta/2)L_1$ obtained by removing $(\log n)^2$ edges from each end of $P_0'$. Furthermore, let $J_1, J_2$ be the collections of $(\log n)^2$ many $j$-sets which are ends of a subpath of $P_0'$, but not of $P_0$, divided naturally into two collections according to which end of $P_0'$ they are closest to.

Our aim is to start two breadth-first processes starting at $J_1, J_2$ to extend $P_0$, and to show that these processes quickly grow large. This fact in itself would be easy to prove by adapting the proof strategy from Section 4, since the length of $P_0$ is such that the processes are (just) supercritical, and intuitively we only need a logarithmic number of steps to grow to polynomial size.

More delicate, however, is to show that the search process produces path ends that are compatible with each other, in the sense that there are many choices of pairs of ends between which we have a path. In order to construct compatible sets of ends, having run the algorithm once to find augmenting paths at one end, we will have a set $F$ of forbidden vertices; roughly speaking, these are vertices which lie in too many of the augmenting paths from the first application of the algorithm, and therefore we would like to avoid them when constructing augmenting paths at the other end.

5.1.2. Informal description. Let us first describe the algorithm informally. We will start with paths $P_0 \subset P_0'$ and a set of ends $J$ (which will be either $J_1$ or $J_2$). These ends come with the natural extendable partition induced by $P_0'$. As in the DFS algorithm, we will label $j$-sets as neutral, active or explored. Initially the $j$-sets of $J$ are active and all others are neutral.

At each time $t$ we will query a $k$-set containing an active $j$-set $J$ to determine whether it is an edge. If it is, then a new $j$-sets are potential ends with which we can extend the path from $J$, and these become active, also inheriting an appropriate extendable partition. In order to ensure that we are always creating a path, we will forbid queries of $k$-sets which contain vertices of the path ending in $J$ (except the vertices of $J$ itself). We will also forbid $k$-sets with vertices from the forbidden set $F$. Finally, to ensure independence of the queries we will forbid $k$-sets which contain some explored $j$-set. (Note that because we are using a breadth-first search, we do not need to exclude other active $j$-sets $J'$ because $J'$ will be dealt with later once $J$ is explored, and such $k$-sets will be forbidden from $J'$ because they contain $J$.)

We will proceed in a standard BFS manner, i.e. from the first active $j$-set in the queue we will query all permissible $k$-sets, and any new $j$-sets we discover are added to the end of the queue.

We note that during the BFS process, the $j$-sets which become explored including those in $J_1 \cup J_2$, are certainly ends of a path containing $P_0$, and therefore candidates in our later sprinkling step. Since we have used Lemma 4 as a black box, and consider the BFS algorithm as a fresh start, we initially have a blank slate of explored $j$-sets, and therefore any $j$-set which is explored or active during the new process is an appropriate end.

5.1.3. Formal description. Given a path $P$ and a $j$-set $J$ which is an end of some subpath $P'$ of $P$, let us denote by $P_j[P]$ the extendable partition of $J$ which is naturally induced by the path $P'$. We will also denote by $P \mid J$ the longer of the two maximal subpaths of $P$ ending in $J$.

The formal description of the Pathbranch algorithm appears below.

We will run this algorithm twice, once with $J = J_1$ and once with $J = J_2$. Of course, the two instances of the algorithm will not be independent of each other.
Algorithm: Pathbranch

Input: Integers $k, j$ such that $1 \leq j \leq k - 1$.
Input: $H$, a $k$-uniform hypergraph.
Input: Paths $P_0 \subset P_0'$, set of $j$-sets $\mathcal{J} \in P_0' \setminus P_0$
Input: $F$, a set of forbidden vertices of size at most $\delta^2 n$

1. Let $a \in [k - j]$ be such that $a \equiv k \mod (k - j)$
2. Let $r = \lfloor \frac{n}{2j} \rfloor - 1$
3. $A \leftarrow \mathcal{J}$ ordered lexicographically  
   // active $j$-sets
4. $N \leftarrow \left( V(H) \setminus (\mathcal{J}_1 \cup \mathcal{J}_2) \right)$  
   // neutral $j$-sets
5. $E \leftarrow \emptyset$  
   // explored $j$-sets

for all $J \in \mathcal{J}$ do
7. $P_J \leftarrow P_0'_J$  
   // current $j$-tight path to $J$
8. $t_J \leftarrow |P_J|$  
   // length of $P_J$
9. $\mathcal{P}_J \leftarrow \mathcal{P}_J[P_0'_J]$  
   // extendable partition of $J$
10. $t \leftarrow 0$  
   // "time", number of queries made so far

while $A \neq \emptyset$ do
11. Let $J$ be the first $j$-set in $A$
12. Let $K$ be the set of $k$-sets $K \subset V(H)$ such that $K \supset J$, such that $K \setminus J$ is 
   vertex-disjoint from $P_J$, from $P_0'$ and from $F$, and such that $K$ does not contain 
   any $J' \in E$
while $K \neq \emptyset$ do
14. Let $K$ be the first $k$-set in $K$ according to the lexicographic order
15. $t \leftarrow t + 1$  
   // a new query is made
16. if $K \in H$ then  
   // "query $K$"
17. for each $Z \in \binom{C_i}{a}$ do
18.   $J_Z \leftarrow Z \cup C_2 \cup \ldots \cup C_r \cup (K \setminus J)$  
   // $j$-set to be added
19.   $P_{J_Z} \leftarrow P_J + K$  
   // Path ending at $J_Z$
20.   $\mathcal{P}_{J_Z} \leftarrow (Z, C_2, \ldots, C_r, K \setminus J)$  
   // extendable partition
21.   $\ell_{J_Z} \leftarrow \ell_J + 1$
22.   $A \leftarrow A + J_Z$  
   // $j$-set becomes active
23.   $N \leftarrow N - J_Z$  
   // $j$-set is no longer neutral
24.   $(A_i, E_i) \leftarrow (A, E)$  
   // update "snapshot" at time $t$
25.   $K \leftarrow K - K$  
   // update $K$
26.   $E \leftarrow E + J$  
   // $J$ becomes explored
27.   $A \leftarrow A - J$  
   // $J$ is no longer active

in general. However, if we can show that each instance satisfies some desired 
properties w.r.t. a union bound shows that also w.h.p both instances satisfy these 
properties. We will subsequently show that the desired properties will be enough 
to combine the two outputs of the algorithm in an appropriate way.

5.2. Analysing the algorithm. Let us define $\ell_t := \max_{J \in \mathcal{A}_t} \ell_J$. Let us also 
fix a constant $0 < \varepsilon < \delta$ and further constants $1 \ll c_0 \ll c_1 \ll \ldots \ll c_{j-1} \ll 
1/\varepsilon$. As in the DFS algorithm, let $G_{\text{disc}}(t) := \mathcal{A}_t \cup \mathcal{E}_t$, and we denote $\Delta_j(t) := 
\Delta_j(G_{\text{disc}}(t))$ for any $0 \leq i \leq j - 1$ and $t \in \mathbb{N}$. We will run the Pathbranch 
algorithm until time $T_{\text{stop}}$, the first time at which one of the following stopping 
conditions is satisfied.

- (S1) The algorithm has terminated.
- (S2) $\ell_t = \left(1 - \delta/2\right) L_1 + 2(\log n)^2 =: L_0$.
- (S3) $\Delta_j(t) \geq \varepsilon c_j n^{j-i}$ for some $i$.
- (S4) $|G_{\text{disc}}(t)| = \varepsilon^2 n^j$.

Our principal aim is to show that w.h.p it is (S4) that is invoked. The following 
proposition will be critical.
Proposition 10. For any \( t \leq T_{\text{stop}} \), the number of \( k \)-sets which are eligible to be queried from an active \( j \)-set is at least \( \binom{n-v(1-\delta/3)L_1}{k-j} = (1 + \Theta(\delta))(\binom{n-vL_0}{k-j}) \).

Proof. We first observe that an essentially identical proof to that of Proposition 7 shows the analogous result in this case: here we have that \( \ell_j \leq L_0 \) and \( \Delta_i(E_i) \leq \epsilon n^{d-1} \) because the stopping conditions (S2) and (S3) have not been invoked, and we obtain that the number eligible \( k \)-sets is at least \( (1 - O(\epsilon))(\binom{n-|F|-vL_0}{k-j}) \). It remains only to observe that

\[
(1 - O(\epsilon)) \binom{n - |F| - vL_0}{k-j} \geq \binom{n - v(1-\delta/3)L_1}{k-j},
\]

which holds since \( |F| \leq \delta^2 n \), since \( L_1 = \Theta(n) \) and since \( \epsilon \ll \delta \ll 1 \). \( \square \)

Claim 11. Whp (S1) is not invoked.

Proof. In order for (S1) to be invoked, all \( j \)-sets which became active would need to be fully explored at time \( T_{\text{stop}} \). Let \( m := G_{\text{disc}}(T_{\text{stop}}) = E_{\text{stop}} \) denote the number of \( j \)-sets which became active (or were active initially).

By Proposition 10, the algorithm has made at least \( M := m \cdot (1 + \Theta(\delta))(\binom{n-vL_0}{k-j}) \) queries, from which we certainly discovered at most \( m' := m(k-j)^{-1} \) new active \( j \)-sets. Thus we have

\[
M/m' = (1 + \Theta(\delta)) \binom{n-vL_1}{k-j} \binom{k-j}{a} \geq \frac{1}{(1 - \delta^2)p},
\]

or in other words \( m' \leq (1 - \delta^2)pM \). Thus we have made at least \( M \) queries during which we discovered at most \( m' \leq (1 - \delta^2)pM \) edges. A Chernoff bound will show that this is very unlikely provided \( pM \) is large enough.

More precisely, note that since the \( j \)-sets of \( J \) were initially active, we certainly have \( m \geq |J| \geq (\log n)^2 \), and therefore \( M \geq \Theta((\log n)^2 n^{k-j}) \). For any \( t \geq \Theta((\log n)^2 n^{k-j}) \), the probability that in the first \( t \) queries we find at most \( (1 - \delta^2)p \) edges is at most \( \exp(-\Theta(pt)) \leq \exp(-\Theta((\log n)^2)) = o(n^{-k}) \). Therefore we may take a union bound over all times \( t \) between 0 and \( n^k \) (which is a trivial upper bound on the total number of queries that can be made) and deduce that whp (S1) was not invoked during this time. \( \square \)

Proposition 12. Whp (S2) is not invoked.

Proof. We consider the generation of the BFS process, where the \( j \)-sets of \( J \) form generation 0 and a \( j \)-set lies in generation \( i \) if it was discovered from a \( j \)-set in generation \( i - 1 \). Observe that since for each \( J \in J \) we have \( \ell_j \leq (1 - \delta/2)L_1 + (\log n)^2 \), (S2) can only be invoked if we have reached generation at least \( (\log n)^2 \).

Let us define \( X_i \) to be the number of \( j \)-sets in generation \( i \), so in particular \( X_0 = |J| = (\log n)^2 \) deterministically. By Proposition 10, we make at least \( X_i(1 + \Theta(\delta))(\binom{k-j}{a}p)^{-1} \) queries to obtain generation \( i + 1 \) from generation \( i \), and therefore \( \mathbb{E}(X_{i+1} | X_i = x_i) \geq (1 + \Theta(\delta))x_i \). A repeated application of the Chernoff bound and a union bound over all at most \( n^j \) generations shows that whp \( X_i \geq (1 - \delta^2)^i X_0 \) for every \( i \) until time \( T_{\text{stop}} \) (after which Proposition 10 no longer applies).

In order to reach generation \( (\log n)^2 \), this would involve discovering a generation of size at least \( (1 + \delta^2)((\log n)^2 X_0 > n^j \), which is clearly impossible since this is larger than the total number of \( j \)-sets available (and indeed (S4) would already have been applied long before this point).

Lemma 13. Whp (S3) is not invoked.
Proof. The proof is broadly similar to the proof of Lemma 9. The case when \( i = 0 \) is trivial, since \( \Delta_0(t) = |G_{\text{disc}}(t)| \leq \varepsilon^2 n^j \) because of \( (S4) \). Let us therefore suppose that \( 1 \leq i \leq j - 1 \) and \( I \) is an \( i \)-set. We observe that there are two ways in which the degree of \( I \) can increase in \( G_{\text{disc}} \).

- A new start at \( I \) consists of a \( j \)-set of \( J \) which contains \( I \). This contributes one to the degree of \( I \).
- A jump to \( I \) occurs when a \( k \)-set containing \( I \) is queried from a \( j \)-set not containing \( I \) and this \( k \)-set is indeed an edge. Then the degree of \( I \) increases by at most \( \binom{k-1}{a} \).
- A pivot at \( I \) occurs when an edge is discovered from a \( j \)-set already containing \( I \). Then the degree of \( I \) increases by at most \( \binom{k-1}{a} \).

We bound the contributions made by new starts, jumps and pivots separately.

New starts. Note that in contrast to the DFS algorithm, new starts are already determined by the input, since we start only once with this input and terminate the algorithm if we have no more active \( j \)-sets. It is also clear that since the \( j \)-sets of \( J \) lie within a path, the degree of \( I \) in \( J \) is \( O(1) \) (deterministically).

Jumps. The number of \( j \)-sets of \( G_{\text{disc}} \) which intersect \( I \) in \( z \leq i - 1 \) vertices is at most
\[
\binom{j}{i} \Delta_z(G_{\text{disc}}) = O(\varepsilon c_n n^{j-z}),
\]
where we have used the condition \( (S3) \) with \( z \) in place of \( i \).

Furthermore, each such \( j \)-set gives rise to \( O(n^{k-j-i+z}) \) queries which would result in a jump to \( I \) if the corresponding \( k \)-set is an edge, and thus the total number of queries which would result in a jump to \( I \) is
\[
\sum_{z=0}^{i-1} O(\varepsilon c_n n^{j-z}) \cdot O(n^{k-j-i+z}) = O(\varepsilon c_{i-1} n^{k-i}).
\]
Each such query gives a jump with probability \( p = O(n^{k-j}) \), and a Chernoff bound implies that with probability at least \( 1 - \exp(-\sqrt{n}) \) the number of jumps is \( O(\varepsilon c_{i-1} n^{j-i}) \). Since each jump gives a contribution of at most \( \varepsilon c_{i-1} n^{j-i} \), the total contribution is \( O(\varepsilon c_{i-1} n^{j-i}) \).

Pivots. For each \( j \)-set \( J \) arising from either a new start at \( I \) or a jump to \( I \), we start a new pivot process consisting of all of the \( k \)-sets we discover from \( J \) and its descendants which contain \( I \). It is important that, while we now have a BFS rather than a DFS process, we are still constructing paths (albeit many simultaneously) and therefore the number of consecutive pivots at \( I \) is at most \( \frac{k-1}{a} \). Therefore each pivot process runs for at most \( k \) generations.

Furthermore, the number of queries made from each \( j \)-set in the pivot process is at most \( n^{k-j} \), and therefore the expected number of pivots discovered from each \( j \)-set is at most \( \varepsilon n \). Since each pivot gives rise to at most \( \binom{k}{a} \) \( j \)-sets, the expected total size of each pivot process is \( O(1) \).

Now since we start at most \( O(\varepsilon c_{i-1} n^{j-i}) \) pivot processes, and the expected size of each of these is bounded, it is easy to see that with very high probability, the total size of all of these pivot processes is \( O(\varepsilon c_{i-1} n^{j-i}) \). Indeed, this can be shown using a Chernoff bound on the total number of edges we discover in all the pivot processes combined, and using the fact that \( \varepsilon c_{i-1} n^{j-i} \geq c n \geq \sqrt{n} \).

Furthermore the failure probability provided by the Chernoff bound is at most \( \exp(-\sqrt{n}) \).

Summing the contributions from new starts, jumps and pivots, and applying a union bound on the failure probabilities, we deduce that with probability at least \( 1 - 2 \exp(-\sqrt{n}) \) the degree of \( I \) in \( G_{\text{disc}} \) is at most \( 1 + O(\varepsilon c_{i-1} n^{j-i}) \leq \varepsilon c n^{j-i} \). A union bound over all \( O(n^i) = o(\exp(\sqrt{n})) \) choices of \( I \) completes the argument.

Now combining Claim 11, Proposition 12 and Lemma 13 immediately gives the following corollary.
Corollary 14. Whp stopping condition (S4) is invoked first. □

5.3. Proof of Lemma 6. We can now use this corollary to prove Lemma 6.

As described above, we let $P'_0$ be a path of length $(1 - \delta/2)L_1 + 2(\log n)^2$ in $H^k(n, p)$, which exists whp by Lemma 4, and let $P_0$ be the path obtained by removing $(\log n)^2$ edges from the start and from the end of $P'_0$. Let $J_1, J_2$ be the sets of $j$-sets which can form the end of a path which lies within $P'_0$ but not within $P_0$, divided into two classes in the natural way.

We first run the Pathbranch algorithm with input $k, j, H = H^k(n, p), P_0, P'_0, J = J_1$ and with forbidden vertex set $F = \emptyset$. Let $A$ be the resulting outcome of $G_{\text{disc}}(t) = A_t \cup E_t$ at the stopping time $t = T_{\text{stop}}$.

We now aim to run the algorithm again, with $J_2$ in place of $J_1$. However, it is in theory possible that all of the augmenting paths ending in a $j$-set of $A$ share some common vertex $x$, and that the same happens when we construct augmenting paths at the other end, meaning that all pairs of paths will be incompatible. Of course, intuitively this is very unlikely. To formalise this intuition, we will make use of our ability to forbid a set $F$ of vertices, which we did not need to do in the first iteration.

Let us define a heavy vertex to be a vertex which does not lie in $P'_0$, but lies in at least $\varepsilon^2(\log n)^3n^{j-1}$ many augmented paths $P_J$, where $J \in A$ (i.e. which lies in at least a $(\log n)^3/n$ proportion of the augmented paths).

Claim 15. Whp there are at most $\delta^2n$ heavy vertices.

Proof. Let $q$ be the number of pairs $(v, P)$ consisting of a heavy vertex $v$ and an augmenting path containing $v$. We will estimate $q$ in two different ways.

By (S4), there are at most $\varepsilon^2n^j$ choices for $P$, each of which contains at most $j + 2(\log n)^2(k - j) \leq 2k(\log n)^2$ vertices due to (S2), and therefore certainly at most $2k(\log n)^2$ heavy vertices $v$, which implies that $q \leq \varepsilon^2n^j \cdot 2k(\log n)^2$.

On the other hand, letting $h$ denote the number of heavy vertices, we have $q \geq h \cdot \varepsilon^2(\log n)^3n^{j-1}$ by the definition of a heavy vertex. Combining these two estimates, we obtain $h \leq 2kn/(\log n) \leq \delta^2n$. □

We now run the algorithm again, this time with input $J = J_2$ and with $F$ being precisely the set of heavy vertices (and all other inputs as before). Let $B$ be the resulting outcome of $G_{\text{disc}} = A_t \cup E_t$ at the stopping time. We claim that whp $P_0, A$ and $B$ satisfy the conditions of Lemma 6.

First, recall that $P_0$ has length $(1 - \delta/2)L_1$ by definition. Next, observe that by Corollary 14 we have $|A|, |B| = \varepsilon^2n^j$ whp.

Let $J_s, J_e$ be the two end $j$-sets of $P_0$. It is clearly true by construction that every $j$-set $J \in A$ augments $(P_0, J_s)$ while every $j$-set $J \in B$ augments $(P_0, J_e)$ (without loss of generality this way round). Furthermore, the length of the augmenting paths is at most $2(\log n)^2$ due to (S2).

Finally, we show the trickiest of the properties, that for most pairs $(A, B) \in A \times B$ the augmenting paths are disjoint. Given $A \in A$, let $P_{A, J_s}$ denote the corresponding augmenting path (i.e. $P_A$ without $P_0 - J_s$), and let $\mathcal{P}_A = \{P_{A, J_s} : A \in A\}$ denote the set of these augmenting paths. For $B \in B$, we define $P_{B, J_e}$ and $\mathcal{P}_B$ similarly. Observe that for each $B \in B$ the path $P_{B, J_e}$ has length at most $2(\log n)^2$, and therefore contains $O((\log n)^2)$ vertices. Since when constructing $B$ we excluded heavy vertices, each of these vertices lies in at most $\varepsilon^2(\log n)^3n^{j-1}$ of the paths $\mathcal{P}_A$, and therefore each path of $\mathcal{P}_B$ intersects with at most $O(\varepsilon^2(\log n)^5n^{j-1}) = O\left(\frac{(\log n)^5}{n}|A|\right)$ of the paths in $\mathcal{P}_A$. Therefore the
number of pairs \((A, B) \in A \times B\) such that the paths \(P_{A, J}, P_{B, J}\) are vertex-disjoint is at least
\[
|B| \left(1 - O\left(\frac{(\log n)^5}{n}\right)\right) |A| \geq (1 - \varepsilon)^4 n^2 j,
\]
as required.

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