We consider a new subclass of quadratic stochastic (evolutionary) operators on the simplex indexed by a finite Abelian group $G$ with heredity law $\mu$. With the help of the notion of $s(\mu)$-invariant subgroups, where $s(\mu)$ denotes the support of $\mu$ in $G$, we prove that almost all (w.r.t. Lebesgue measure) trajectories of such operators converge to a unique fixed point which is the center of the simplex. We also identify and describe the periodic trajectories of the operator and give conditions for regularity and periodicity.

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Key words. Quadratic stochastic operator, Volterra and non-Volterra operators, evolutionary operator.

1. Introduction

The notion of quadratic stochastic operators (QSOs) was introduced by S.N. Bernstein in [1], and since then the theory of quadratic stochastic operators has been developed for more than 85 years (see e.g. [4] – [9], [10], [12], [13], [17], [18] for some classic as well as recent results). While QSOs were originally introduced as “evolutionary operators” describing the dynamics of gene frequencies for given laws of heredity in mathematical population genetics (see [14] for a comprehensive account), they are also interesting from a purely mathematical point of view. Their full classification remains a challenging open problem.

A quadratic stochastic (evolutionary) operator arises in population genetics as follows. Consider a (large) population with $m \in \mathbb{N}$ different genetic types. Let $[m] := \{1, 2, \ldots, m\}$ and $x^0 = (x_1^0, \ldots, x_m^0)$ be the relative frequencies of the genetic types within the whole population in the present generation, which is a probability distribution and hence an element of the simplex indexed by $[m]$ which we denote by $S^{m-1}$. To determine the (expected) gene frequencies in the next generation, let $p_{ij,k}$ be the probability that two individuals of type $i$ resp. $j$ interbreed to produce an offspring with genetic type $k$. Then, the probability distribution $x' = (x_1', \ldots, x_m') \in S^{m-1}$ describing the (expected)
gene frequencies in the next generation is given by

\[ x'_k = \sum_{i,j=1}^{m} p_{ij,k} x_i^0 x_j^0, \quad k = 1, ..., m. \] (1.1)

The association \( x^0 \mapsto x' \) defines a map \( V : S^{m-1} \to S^{m-1} \) called evolutionary operator. The population evolves by starting from an arbitrary frequency distribution \( x^0 \), then passing to the state \( x' = V(x^0) \) (the next “generation”), then to the state \( x'' = V(V(x^0)) \), and so on. Thus the evolution of gene frequencies of the population can be considered as a dynamical system

\[ x^0, \quad x' = V(x^0), \quad x'' = V^2(x^0), \quad x''' = V^3(x^0), \quad \ldots \]

Note that \( V \) (defined by (1.1)) is a non-linear (quadratic) operator, and its dimension increases with \( m \). Higher dimensional dynamical systems are important but there are relatively few dynamical phenomena that are currently understood ([2], [3], [16]).

One of the main objects of study for QSOs is the asymptotic behavior of their trajectories depending on the initial value. This has been determined so far only for certain special subclasses of QSOs. Indeed, a natural choice for the \( p_{ij,k} \), also called “coefficients of heredity”, with an obvious biological interpretation, is given by

\[ p_{ij,k} = 0, \quad \text{if} \quad k \notin \{i, j\}, \quad i, j, k = 1, ..., m. \] (1.2)

In this case, we speak of a Volterra QSO, and the corresponding asymptotic behaviour of their trajectories has been analysed in [4], [5] and [6] using the theory of Lyapunov functions and tournaments. In [15], infinite dimensional Volterra operators and their dynamics have been studied.

However, in the non-Volterra case (i.e., where condition (1.2) is violated), many questions remain open and there seems to be no general theory available. See [7] for a recent review of QSOs.

In the present article, we investigate a certain class of non-Volterra QSOs which exhibit an additional group structure in the definition of the \( p_{ij,k} \) that allows us to obtain rather complete asymptotic results. More precisely, instead of considering the simplex \( S^{m-1} \) over \([m]\), we regard a finite Abelian group \( G = (G, +) \), say of order \( m \), and the corresponding simplex \( S^G \) indexed by \( G \) (which can also be regarded as the space of all probability measures on \( G \)). Then, for any measure \( \mu \in S^G \) we define the “coefficients of heredity” by

\[ p_{ij,k} = \mu_{k-i-j}, \quad \forall i, j, k \in G. \]

If \( s(\mu) \) denotes the support of \( \mu \) in \( G \), we introduce the notion of an “\( s(\mu) \)-invariant subgroup”, with the help of which we prove that the trajectory of such operators always converges either to a periodic trajectory or to a fixed point. In particular, they are all ergodic. We also show that the speed of convergence to the limit resp. to the periodic orbit is rather fast (in fact, double-exponential). Finally, we give criteria for regularity.
and periodicity.

Note that in [8], the authors also consider a class of quadratic stochastic operators corresponding to a finite Abelian group, however with a different choice of the $p_{ij,k}$. We will discuss their model and result below.

The paper is organized as follows. The next chapter provides some preliminaries and previously known results from the theory of QSOs. In Chapter 3 we introduce our new class of nonlinear operators and state and prove our results.

2. Preliminaries and known results

A quadratic stochastic operator (QSO) is a mapping $V$ of the simplex

\[ S^{m-1} = \left\{ x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_i \geq 0, \sum_{i=1}^{m} x_i = 1 \right\} \]  

(2.1)

into itself, of the form $V(x) = x' \in S^{m-1}$ where

\[ x'_k = \sum_{i,j=1}^{m} p_{ij,k} x_i x_j, \quad (k = 1, \ldots, m), \]  

(2.2)

and the $p_{ij,k}$ satisfy

\[ p_{ij,k} = p_{ji,k} \geq 0, \quad \sum_{k=1}^{m} p_{ij,k} = 1, \quad (i, j, k = 1, \ldots, m). \]  

(2.3)

The trajectory (orbit) \( \{x^{(n)}\} \) for an initial value \( x^{(0)} \in S^{m-1} \) is defined by

\[ x^{(n+1)} = V(x^{(n)}) = V^{n+1}(x^{(0)}), \quad n = 0, 1, 2, \ldots \]

We now recall some definitions and results from the theory of QSOs.

**Definition 2.1.** A point \( x \in S^{m-1} \) is called a fixed point of a QSO \( V \) if \( V(x) = x \).

**Definition 2.2.** A QSO \( V \) is called regular if for any initial point \( x \in S^{m-1} \) the limit

\[ \lim_{n \to \infty} V^n(x) \]

exists. We call the \( V \) almost regular, if the above condition holds for almost all (w.r.t. Lebesgue-measure) initial points \( x \).

Note that our QSOs are continuous operators and that the simplex over a finite set is compact and convex, so that by the Brouwer Fixed-Point Theorem there is always at least one fixed point. Further, by continuity, any limit point of a QSO is also a fixed point. Limit behavior of trajectories and fixed points of QSOs play an important role in many applied problems, see e.g. [4], [5], [11], [12], [14]. The intuitive meaning of the regularity of a QSO in terms of mathematical genetics is obvious: In the long run the distribution of gene frequencies tends to an equilibrium, no matter what the initial condition was. Further, if a given limit point is strictly inside the simplex, then this
means that there is long-term coexistence of genetic types under the respective initial distribution.

Of course, it is not necessarily clear whether a given set of “coefficients of heredity” has a direct biological interpretation at all, but this is not the main point of this paper. Instead, as mentioned above, we focus on a certain part of the classification problem for QSOs that is inspired a priori purely by mathematical curiosity.

**Definition 2.3.** \( V \) is said to be ergodic if the limit

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} V^k(x) \tag{2.4}
\]

exists for any \( x \in S^{m-1} \).

Evidently, any regular QSO and – more generally – any QSO for which every trajectory converges to a (not necessarily strict) periodic orbit is ergodic, but the converse is not necessarily true.

On the basis of numerical calculations Ulam conjectured [19] that any QSO is ergodic. In 1977, Zakharevich [20] proved that this conjecture is false in general. Later in [9] necessary and sufficient conditions for ergodicity of a QSO defined on \( S^2 \) were established.

Now we introduce some notation and results from [8]. Recall that the simplex \( S^{m-1} \) is the set of all probability measures on \( [m] = \{1, \ldots, m\} \). We now consider instead of \( [m] \) a finite Abelian group \( G \) of order \( m \) and the corresponding simplex \( S^G \) over \( G \). Let \( U \subset G \) be a subgroup of \( G \) and \( \{g+U : g \in G\} \) be the cosets of \( U \) in \( G \). Suppose \( \lambda \in S^G \) is a fixed positive measure, that is \( \lambda_i = \lambda(i) > 0 \) for any \( i \in G \). Then we define the coefficients of heredity as in [8] by

\[
p_{ij,k} = \begin{cases} \frac{\lambda_k}{\lambda(i+j+U)}, & \text{if } k \in \{i+j+U\}; \\ 0, & \text{otherwise, } i,j,k \in G. \end{cases} \tag{2.5}
\]

Note that if \( U = \{0\} \), where 0 is the identity element of the group \( G \), then the corresponding QSO has the form

\[
x'_k = \sum_{i,j \in G, \ i+j=k} x_i x_j \tag{2.6}
\]

For convenience, we will freely use the obvious analogs of Definitions (2.1), (2.2) and (2.3) for QSO on the simplex \( S^G \) instead of \( S^{m-1} \).

**Theorem 2.4.** [8] Almost all (w.r.t Lebesgue measure) orbits of the QSO defined by (2.6) converge to the center of the simplex.

**Corollary 2.5.** The QSO (2.6) is almost regular.
3. Asymptotic behaviour of \((G, \mu)-\text{quadratic stochastic operators}\)

Let \((G, +)\) be a finite Abelian group, \(|G| = m\) and \(S^G\) be the set of all probability measures on \(G\), where \(|.|\) denotes the cardinality of a set. We denote the identity element of \(G\) by \(0\). Let \(\mu \in S^G\) be a fixed measure. Then, we define coefficients of heredity by

\[
p_{ij,k} := \mu_{k-i-j}, \quad \forall i, j, k \in G. \tag{3.1}
\]

It is easy to check that for arbitrary \(i, j, k \in G\) the conditions (2.3) are satisfied.

**Definition 3.1.** The QSO satisfying (2.2), (2.3) and (3.1) is called \((G, \mu)-\text{quadratic stochastic operator}\).

Any \((G, \mu)-\text{QSO}\) has the form

\[
V : x_k' = \sum_{i,j \in G} p_{ij,k} x_i x_j = \sum_{i \in G} \sum_{l \in G} \mu_l x_i x_{k-l-i}. \tag{3.2}
\]

**Remark 3.2.** If \(\mu_0 = 1\) then the corresponding \((G, \mu)-\text{QSO}\) coincides with the QSO obtained from (2.0).

**Definition 3.3.** Let \(U\) be a subgroup of \(G\) and \(A \subseteq G\) a nonempty set. Then, \(U\) is called \(A\)-invariant if \(|U + A| = |U|\).

Recall that a coset of a subgroup \(U\) in an Abelian group \(G\) is of the form \(\{g + u, u \in U\}\) for some \(g \in G\). It is easy to derive the following basic properties of \(A\)-invariant sets.

**Proposition 3.4.** Let \(U\) be any subgroup of \(G\) and \(A \subseteq G\).

- \(U\) is \(A\)-invariant iff \(A\) is contained in a coset of \(U\).
- If \(U\) is \(A\)-invariant and \(\emptyset \neq \tilde{A} \subseteq A\), then \(U\) is \(\tilde{A}\)-invariant.
- \(|A| = 1\) implies that \(U\) is \(A\)-invariant.
- If \(|A| > m/2\), then the only \(A\)-invariant subgroup is \(U = G\).

**Proposition 3.5.** Let \(B\) be a non-empty subset of \(G\). Then, \(|B + B| = |B|\) iff \(B\) is a coset of some subgroup of \(G\).

**Proof.** If \(B\) is coset of \(G\), then it can be written as \(g + U\) for some subgroup \(U\), and the first part of the result follows from

\[
|B + B| = |g + g + U + U| = |g + g + U| = |B|.
\]

Conversely let \(B = \{b_1, b_2, \ldots, b_k\}\). Then \(B = b_1 + \tilde{B}\), where \(\tilde{B} = \{0, b_2 - b_1, \ldots, b_k - b_1\}\). With this notation, we obtain

\[
B + B = (b_1 + b_1) + \tilde{B} + \tilde{B} \supseteq (b_1 + b_1) + \tilde{B}
\]

since \(0 \in \tilde{B}\). If \(|B + B| = |B| = k\), then equality holds and we have \(\tilde{B} + \tilde{B} = \tilde{B}\), so \(\tilde{B}\) is a subgroup and \(B\) is a coset of \(\tilde{B}\). \(\square\)

For \(x \in S^G\) denote by \(s(x) = \{i \in G : x_i > 0\}\) the support of \(x\). We now investigate whether the cardinality of support \(s(x)\) of a state \(x\) grows after an application of \(V\), depending on the support \(s(\mu)\) of \(\mu\) (where, by slight abuse of notation, we do not distinguish between the support of a “point” \(x \in S^G\) and a “measure” \(\mu \in S^G\)).
Proposition 3.6. Let $V$ be a $(G,\mu)-$QSO. Then:

a) For any $x \in S^G$, we have $|s(Vx)| \geq |s(x)|$.

b) For $x \in S^G$ we have $|s(Vx)| = |s(x)|$ iff $s(x)$ is equal to a coset of an $s(\mu)$-invariant subgroup.

Proof. a) From (3.2) we have
$$s(Vx) = s(x) + s(x) + s(\mu),$$
so a) holds.

b) First assume that $s(x) = g + U$ and $U$ is an $s(\mu)$-invariant subgroup. Then
$$s(Vx) = g + U + g + U + s(\mu) = (g + g) + U + s(\mu),$$
so that
$$|s(Vx)| = |(g + g) + U + s(\mu)| = |U| = |s(x)|.$$  
Conversely, assume that $|s(Vx)| = |s(x)|$, i.e. $|s(x) + s(x) + s(\mu)| = |s(x)|$. Since
$$|s(x) + s(x) + s(\mu)| \geq |s(x) + s(x)| \geq |s(x)|,$$
we have $|s(x) + s(x)| = |s(x)|$, so $s(x)$ is a coset by Proposition 3.5. say $s(x) = g + U$. Then
$$|U| = |s(x)| = |s(Vx)| = |g + U + g + U + s(\mu)| = |U + s(\mu)|,$$
so $U$ is $s(\mu)$-invariant. \hfill \square

Remark 3.7. The previous proposition implies that for every $x \in S^G$ there exists some $n_0 \in \mathbb{N}_0$ such that $|s(V^{n+1}x)| = |s(V^n x)|$ for all $n \geq n_0$ and $|s(V^{n-1}x)| < |s(V^n x)|$ for all $n \leq n_0$. Further, $s(V^n x)$ is the coset of an $s(\mu)$-invariant subgroup $U$ iff $n \geq n_0$ (note that $U$ does not depend on $n$). Note that $s(V^n x)$ depends on $x$ only via $s(x)$ and on $\mu$ only via its support $s(\mu)$.

Definition 3.8. For a nonempty subset $B$ of $G$ the uniform distribution $u(B) \in S^G$ on $B$ is defined as
$$u(B)_k = \begin{cases} 
1/|B|, & k \in B, \\
0, & k \notin B.
\end{cases}$$

The next theorem is our key result.

Theorem 3.9. For each $x \in S^G$ we have
$$\lim_{n \to \infty} (V^n x - u(s(V^n x))) = 0.$$

Proof. For given $x \in S^G$ and $n_0$ as in Remark 3.7 let $k = |s(V^{n_0} x)|$. The case $k = 1$ is clear, so we assume that $k > 1$. Assume that the support of $y \in S^G$ equals a coset of an $s(\mu)$-invariant subgroup $U$ of cardinality $k$ (as is the case for $V^n x$ whenever $n \geq n_0$) and let $v_1 \geq v_2 \geq \cdots \geq v_k$ be the numbers $y_j \in s(y)$ in decreasing order. Then, the well known rearrangement inequality gives
$$\sum_{r=1}^{k} v_r v_{k+1-r} \leq \sum_{i \in s(y)} y_i y_{\sigma(i)} \leq \sum_{r=1}^{k} v_r^2,$$
which holds for any permutation \( \sigma \) of \( s(y) \). Combined with (3.2), this gives for every \( j \in s(Vy) \)

\[
(Vy)_j = \sum_{l \in j - i - s(y)} \mu_l \sum_{i \in s(y)} y_i y_{j - i} - i \\
\geq \sum_{r=1}^{k} v_r v_{k+1-r} \\
= v_k \sum_{r=1}^{k} v_r + \sum_{r=1}^{k} (v_{k+1-r} - v_k) v_r \\
\geq v_k \sum_{r=1}^{k} v_r + (v_1 - v_k) v_k \\
= v_k + (v_1 - v_k) v_k.
\]

(3.5)

In particular, the smallest element of \((Vy)\) is strictly larger than the smallest element of \(y\) unless \( y = u(s(y)) \). Hence, by (3.5),

\[
(0,\frac{1}{k}] \ni \alpha = \min_{j \in s(Vy)} (Vy)_j \geq v_k + (v_1 - v_k) v_k.
\]

(3.6)

Note further that \( 1 = \sum_{j=1}^{k} v_j \leq (k-1)v_1 + v_k \), so that

\[
(v_1 - v_k) v_k = \frac{(k-1)v_1 v_k}{k-1} - \frac{(k-1)v_k^2}{k-1} \\
\geq \frac{(1-v_k) v_k}{k-1} - \frac{(k-1)v_k^2}{k-1} \\
= \frac{v_k}{k-1} - \frac{kv_k^2}{k-1} = \left(\frac{1}{k} - v_k\right) \frac{kv_k}{k-1}.
\]

Putting this and (3.6) together gives

\[
\frac{1}{k} - \alpha \leq \frac{1}{k} - v_k - (v_1 - v_k) v_k \leq \left(\frac{1}{k} - v_k\right) \left(1 - \frac{kv_k}{k-1}\right)
\]

which immediately implies the statement in the theorem. \( \square \)

**Remark 3.10.** If \( k := |s(x)| > m/2 \), then the only subgroup of \( G \) with order at least \( k \) is \( G \) itself, hence \( V^n(x) \) converges to the center \( u(G) \) of \( S^G \) thus showing that \( V \) is almost regular. Further, Theorem 3.9 shows that \( V \) is ergodic.

The following theorem shows that the speed of convergence in the previous theorem is double-exponential, i.e. rather fast. Let \( \|\cdot\| \) denote an arbitrary norm on \( \mathbb{R}^G \).

**Theorem 3.11.** For each \( x \in S^G \),

\[
\limsup_{n \to \infty} \frac{1}{n} \log \log \|V^n x - u(s(V^n x))\| \leq \log 2.
\]
Proof. For given $x \in S^G$ and $n_0$ as in Remark 3.7, let $k = |s(V^{n_0}x)|$. If the support of $y \in S$ equals a coset of an $s(\mu)$-invariant subgroup $U$ of cardinality $k$ (as is the case for $V^n x$ whenever $n \geq n_0$), then, denoting

$$\varepsilon = \max_{i \in s(y)} \left\{ \left| y_i - \frac{1}{k} \right| \right\},$$

we have for $j \in s(Vy)$

$$(Vy)_j - \frac{1}{k} = \sum_{i \in s(y)} \sum_{l \in j-i-s(y)} \mu_l \left( y_i - \frac{1}{k} \right) \left( y_{j-l-i} - \frac{1}{k} \right),$$

and therefore

$$\left| (Vy)_j - \frac{1}{k} \right| \leq \varepsilon k.$$

Using Theorem 3.9 the claim follows.

We also have the following kind of converse to Theorem 3.9.

**Theorem 3.12.** For any $s(\mu)$-invariant subgroup $U$ and any $a \in s(\mu)$ and $g \in G$ for which $n \mapsto 2^n g, n \in \mathbb{N}$ is periodic, the sequence of uniform distributions $u(2^n g - a + U), n \in \mathbb{N}$, is a periodic orbit of the map $V$.

**Proof.** Let $U$ be an any $s(\mu)$-invariant subgroup and $a \in s(\mu)$. Assume that the mapping $n \mapsto 2^n g$ is periodic. Then, for $x = u(g - a + U)$ we obtain from (3.4) that

$$V^n(x) = u(2^n g - a + U)$$

which is periodic in $n$.

One can ask under which conditions on $G$ and $s(\mu)$ the QSO (3.2) is regular.

**Proposition 3.13.** A $(G, \mu)-$QSO is regular iff for each $s(\mu)$-invariant subgroup $U$ and every $g \in G$ there exists some $n \in \mathbb{N}$ such that $2^n g \in U$. In this case, all trajectories converge to the uniform distribution on some coset of some $s(\mu)$-invariant subgroup $U$.

**Proof.** Theorem 3.9 and Theorem 3.12 show that all periodic (possibly constant) trajectories are of the form $u(2^n g - a + U)$ for some $a \in s(\mu), g \in G$ and $U$ an $s(\mu)$-invariant subgroup of $G$, and that, conversely, all such trajectories are periodic. It remains to investigate which of these trajectories have a minimal period at least 2.

Suppose we are given a periodic trajectory corresponding to $g, a, U$ as above. If - on the one hand - all $2^n g, n \in \mathbb{N}$ eventually end up in $U$, then there is no strict periodicity, i.e. the trajectory becomes constant.

If - on the other hand - $g, a, U$ are such that $2^n g \notin U$, then

$$2^{n+1} g - a + U \neq 2^n g - a + U.$$ 

Consequently, if $2^n g \notin U$ holds for all $n \in \mathbb{N}$, then $V$ is not regular. The last statement follows from Theorem 3.9.

Finally, we state some more explicit conditions for regularity or non-regularity.
Corollary 3.14. If \( m = |G| = 2^k \) for some \( k \in \mathbb{N}_0 \), then the \((G, \mu)\)-QSO is regular for any choice of \( \mu \).

Proof. In this case, the order of any \( g \in G \) is of the form \( 2^\nu \) for some \( \nu \leq k \), so the claim follows from Proposition 3.13. \( \square \)

Corollary 3.15. If \( m = |G| \) is not a power of 2, then there exists some \( \mu \in S^G \) for which the \((G, \mu)\)-QSO is not regular, e.g. \( \mu \in S^G \) defined by \( \mu_0 = 1 \).

Proof. Since \( m \) is not a power of 2 there exists some prime number \( p \neq 2 \) and some \( g \in G \) of order \( p \) (this follows from the fact that every finite Abelian group is the sum of cyclic groups whose orders are powers of primes). Then \( 2^n g \) is different from 0 for any \( n \in \mathbb{N}_0 \). Therefore, \( x \in S^G \) defined as \( x_h = \delta_{gh} \) is the initial point of a trajectory of \( V \) of minimal period \( p \). In particular, \( V \) is not regular. \( \square \)

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References

[1] Bernstein S.N. The solution of a mathematical problem related to the theory of heredity. Uchn. Zapiski. NI Kaf. Ukr. Otd. Mat. 1924. no. 1., 83-115 (Russian).
[2] Devaney R.L. An Introduction to Chaotic Dynamical Systems. Westview Press, 2003.
[3] Elaydi S.N. Discrete Chaos. Chapman Hall/CRC, 2000.
[4] Ganikhodzhaev R.N. Quadratic stochastic operators, Lyapunov functions, and tournaments, Sb. Math. 76 No.2 (1993), 489–506.
[5] Ganikhodzhaev R.N. Map of fixed points and Lyapunov functions for a class of discrete dynamical systems, Math. Notes 56 No. 5, (1994) 1125–1131.
[6] Ganikhodzhaev R.N. and Eshmamatova D.B. Quadratic automorphisms of a simplex and the asymptotic behavior of their trajectories, Vladikavkas. Mat. Zh. 8 No. 2, (2006) 1228, (in Russian).
[7] Ganikhodzhaev R.N., Mukhamedov F.M., Rozikov U.A. Quadratic stochastic operators: Results and open problems. Inf. Dim. Anal., Quantum Prob. and Rel. Top.. Vol. 14. no. 2, 279-335 (2011).
[8] Ganikhodjaev N.N., Wahiddin M.R.B., Zanin D.V. Regularity of some class of nonlinear transformations, arXiv:math.DS/07080697.
[9] Ganikhodjaev N.N., Zanin D.V. On a necessary condition for the ergodicity of quadratic operators defined on the two-dimensional simplex, Russian Math. Surveys 59:3, 571–572, (2004).
[10] Hofbauer J. and Sigmund K. The Theory of Evolution and Dynamical Systems. Mathematical aspects of selection, London Math. Soc. Stud. Texts, vol. 7, Cambridge University Press, Cambridge 1988.
[11] Jenks, R.D. Quadratic Differential Systems for Interactive Population Models. J. Diff. Eqs. Vol.5, (1969), 497–514.
[12] Kesten H. Quadratic transformations: a model for population growth. I, Adv. Appl. Prob. 2 no. 1 (1970), 182.
[13] Kesten H. Quadratic transformations: a model for population growth. II, Adv. Appl. Prob. 2 no. 2 (1970), 179–228.
[14] Lyubich Yu.I. Mathematical Structures in Population Genetics. Biomathematics, 22, Springer-Verlag, 1992.
[15] Mukhamedov F., Akin H. and Temir S. On infinite dimensional quadratic Volterra operators, *J. Math. Anal. Appl.* **310**, (2005), 533–556.

[16] Robinson R.C., *An Introduction to Dynamical Systems: Continuous and Discrete*, Pearson Education, 2004.

[17] Rozikov U.A. and Jamilov U.U. F-quadratic stochastic operators, *Math. Notes* **83** No. 4, (2008) 554–559.

[18] Rozikov U.A. and Jamilov U.U. The dynamics of strictly non-Volterra quadratic stochastic operators on the two dimensional simplex, *Sb. Math.* **200** No.9 (2009), 1339–1351.

[19] Ulam S. *A Collection of Mathematical Problems*, Interscience Publishers, New-York-London 1960.

[20] Zakharevich M.I., On behavior of trajectories and the ergodic hypothesis for quadratic transformations of the simplex, *Russian Math. Surveys* **33**:6, 265–266 (1978).

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