Fidelity of recovery, geometric squashed entanglement, and measurement recoverability

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Abstract

This paper defines the fidelity of recovery of a tripartite quantum state on systems $A$, $B$, and $C$ as a measure of how well one can recover the full state on all three systems if system $A$ is lost and a recovery operation is performed on system $C$ alone. The surprisal of the fidelity of recovery (its negative logarithm) is an information quantity which obeys nearly all of the properties of the conditional quantum mutual information $I(A; B | C)$, including non-negativity, monotonicity under local operations, duality, and a dimension bound. We then define an entanglement measure based on this quantity, which we call the geometric squashed entanglement. We prove that the geometric squashed entanglement is an entanglement monotone, that it vanishes if and only if the state on which it is evaluated is unentangled, and that it reduces to the geometric measure of entanglement if the state is pure. We also show that it is subadditive, continuous, and normalized on maximally entangled states. We next define the surprisal of measurement recoverability, which is an information quantity in the spirit of quantum discord, characterizing how well one can recover a share of a bipartite state if it is measured. We prove that this discord-like quantity satisfies several properties, including non-negativity, faithfulness on classical-quantum states, invariance under local isometries, dimension bounds, and normalization on maximally entangled states. This quantity combined with a recent breakthrough of Fawzi and Renner allows to characterize states with discord nearly equal to zero as being approximate fixed points of entanglement breaking channels (equivalently, they are recoverable from the state of a measuring apparatus). Finally, we discuss a multipartite fidelity of recovery and several of its properties.

1 Introduction

The conditional quantum mutual information (CQMI) is a central information quantity that finds numerous applications in quantum information theory [DY08, YD09], the theory of quantum correlations [OZ01, CW04], and quantum many-body physics [Kim13b, Bas12]. For a quantum state $\rho_{ABC}$ shared between three parties, say, Alice, Bob, and Charlie, the CQMI is defined as

$$I(A; B | C)_\rho \equiv H(AC)_\rho + H(BC)_\rho - H(C)_\rho - H(ABC)_\rho,$$

(1.1)

where $H(F)_\sigma \equiv -\text{Tr}\{\sigma_F \log \sigma_F\}$ is the von Neumann entropy of a state $\sigma_F$ on system $F$ and we unambiguously let $\rho_C \equiv \text{Tr}_{AB}\{\rho_{ABC}\}$ denote the reduced density operator on system $C$, for

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example. The CQMI captures the correlations present between Alice and Bob from the perspective of Charlie in the independent and identically distributed (i.i.d.) resource limit, where an asymptotically large number of copies of the state $\rho_{ABC}$ are shared between the three parties. It is non-negative \cite{LR73a,LR73b}, non-increasing under the action of local quantum operations on systems $A$ or $B$, and obeys a duality relation for a four-party pure state $\psi_{ABCD}$, given by $I(A;B|C)_{\psi} = I(B;A|D)_{\psi}$. It finds operational meaning as twice the optimal quantum communication cost in the i.i.d. state redistribution protocol \cite{DY08,YD09}. It underlies the squashed entanglement \cite{CW04}, which is a measure of entanglement that satisfies all of the axioms desired for such a measure \cite{AP04,KW04,BCY11}, and furthermore underlies the quantum discord \cite{OZ01}, which is a measure of quantum correlations different from those due to entanglement.

In an attempt to develop a version of the CQMI, which would be relevant for the “one-shot” or finite resource regimes, we along with Berta \cite{BSW14} recently proposed Rényi generalizations of the CQMI. We proved that these Rényi generalizations of the CQMI retain many of the properties of the original CQMI in (1.1). While the application of these particular Rényi CQMIs in one-shot state redistribution remains to be studied, we have used them to define a Rényi squashed entanglement and a Rényi quantum discord \cite{SBW14}, which retain several properties of the respective, original, von Neumann entropy based quantities.

One contribution of \cite{BSW14} was the conjecture that the proposed Rényi CQMIs are monotone increasing in the Rényi parameter, as is known to be the case for other Rényi entropic quantities. That is, for a tripartite state $\rho_{ABC}$, and for a Rényi conditional mutual information $\tilde{I}_\alpha(A;B|C)_{\rho}$ defined as \cite{BSW14,Section 6}

$$\tilde{I}_\alpha(A;B|C)_{\rho} \equiv \frac{1}{\alpha - 1} \log \left\| \rho_{1/2}^{1/2} \rho_{AC}^{(1-\alpha)/2\alpha} \rho_{C}^{(\alpha-1)/2\alpha} \rho_{BC}^{(1-\alpha)/2\alpha} \right\|^{2\alpha}_{2\alpha},$$

\cite{BSW14,Section 8} conjectured that the following inequality holds for $0 \leq \alpha \leq \beta$:

$$\tilde{I}_\alpha(A;B|C)_{\rho} \leq \tilde{I}_\beta(A;B|C)_{\rho}.$$  \hspace{1cm} (1.3)

Proofs were given for this conjectured inequality when the Rényi parameter $\alpha$ is in a neighborhood of one and when $1/\alpha + 1/\beta = 2$ \cite{BSW14,Section 8}.

We also pointed out implications of the conjectured inequality for understanding states with small conditional quantum mutual information \cite{BSW14,Section 8} (later stressed in \cite{Ber14}). In particular, we pointed out that the following lower bound on the conditional quantum mutual information holds as a consequence of the conjectured inequality in (1.3) by choosing $\alpha = 1/2$ and $\beta = 1$:

$$I(A;B|C)_{\rho} \geq - \log \mathcal{F}(\rho_{ABC}; \mathcal{R}^P_{C\rightarrow AC}(\rho_{BC})) \geq \frac{1}{4} \left\| \rho_{ABC} - \mathcal{R}^P_{C\rightarrow AC}(\rho_{BC}) \right\|_1^2,$$

where $\mathcal{R}^P_{C\rightarrow AC}$ is a quantum channel known as the Petz recovery map \cite{HJPW04}, defined as

$$\mathcal{R}^P_{C\rightarrow AC}(\cdot) \equiv \rho_{AC}^{1/2} \rho_C^{-1/2}(\cdot) \rho_C^{-1/2} \rho_{AC}^{1/2}.$$  \hspace{1cm} (1.6)

The fidelity is a measure of how close two quantum states are and is defined for positive semidefinite operators $P$ and $Q$ as

$$\mathcal{F}(P,Q) \equiv \left\| \sqrt{P} \sqrt{Q} \right\|_1^2.$$  \hspace{1cm} (1.7)

\footnote{However, see the recent progress on one-shot state redistribution in \cite{BCT14,DHO14}.}
The trace distance bound in (1.4) was conjectured previously in [Kim13a] and a related conjecture (with a different lower bound) was considered in [WL12].

The conjectured inequality in (1.4) revealed that (if it is true) it would be possible to understand tripartite states with small conditional mutual information in the following sense: If one loses system $A$ of a tripartite state $\rho_{ABC}$ and is allowed to perform the Petz recovery map on system $C$ alone, then the fidelity of recovery in doing so will be high. The converse statement was already established in [BSW14, Proposition 35] and independently in [FR14, Eq. (8)]. Indeed, suppose now that a tripartite state $\rho_{ABC}$ has large conditional mutual information. Then if one loses system $A$ and attempts to recover it by acting on system $C$ alone, then the fidelity of recovery will not be high no matter what scheme is employed (see [BSW14, Proposition 35] for specific parameters). These statements are already known to be true for a classical system $C$, but the main question is whether the inequality in (1.4) holds for a quantum system $C$.

2 Summary of results

When studying the conjectured inequality in (1.4), we can observe that a simple lower bound on the RHS is in terms of a quantity that we call the surprisal of the fidelity of recovery:

$$-\log F(\rho_{ABC}, R_{C\rightarrow AC} (\rho_{BC})) \geq I_F (A; B|C)_{\rho},$$

where the fidelity of recovery is defined as

$$F (A; B|C)_{\rho} \equiv \sup_{R} F (\rho_{ABC}, R_{C\rightarrow AC} (\rho_{BC})).$$

That is, rather than considering the particular Petz recovery map, one could consider optimizing the fidelity with respect to all such recovery maps. One of the main objectives of the present paper is to study the fidelity of recovery in more detail.$^2$

2.1 Properties of the surprisal of the fidelity of recovery

Our conclusions for $I_F (A; B|C)_{\rho}$ are that it obeys many of the same properties as the conditional mutual information $I(A; B|C)_{\rho}$:

1. (Non-negativity) $I_F (A; B|C)_{\rho} \geq 0$ for any tripartite quantum state. (This one is obvious because the fidelity between two quantum states is bounded from above by one.)

2. (Monotonicity) $I_F (A; B|C)_{\rho}$ is monotone under quantum operations on systems $A$ or $B$, in the sense that

$$I_F (A; B|C)_{\rho} \geq I_F (A'; B'|C)_{\omega},$$

where $\omega_{ABC} \equiv (N_{A\rightarrow A'} \otimes M_{B\rightarrow B'}) (\rho_{ABC})$ and $N_{A\rightarrow A'}$ and $M_{B\rightarrow B'}$ are quantum channels acting on systems $A$ and $B$, respectively.

3. (Duality) For a four-party pure state $\psi_{ABCD}$, the following duality relation holds

$$I_F (A; B|C)_{\psi} = I_F (A; B|D)_{\psi}.$$  

$^2$Note: After the completion of this work, we learned of the recent breakthrough result of [FR14], in which the inequality $I(A; B|C)_{\rho} \geq -\log F(A; B|C)_{\rho}$ was established for any tripartite state $\rho_{ABC} \in S(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$. Thus, for states with small conditional mutual information (near to zero), the fidelity of recovery is high (near to one).
4. (Dimension bound) The following dimension bound holds
\[ I_F (A; B|C)_\psi \leq 2 \log |A|, \] (2.5)
where $|A|$ is the dimension of the system $A$.

5. (Weak chain rule) The chain rule for conditional mutual information of a four-party state $\rho_{ABCD}$ is as follows:
\[ I (AC; B|D)_\rho = I (A; B|CD)_\rho + I (C; B|D)_\rho. \] (2.6)

We find something weaker than this for $I_F$, which we call the weak chain rule for $I_F$:
\[ I_F (AC; B|D)_\rho \geq I_F (A; B|CD)_\rho. \] (2.7)

2.2 Geometric squashed entanglement

Our next contribution is to define an entanglement measure of a bipartite state that we call the geometric squashed entanglement. To motivate this quantity, recall that the squashed entanglement of a bipartite state $\rho_{AB}$ is defined as
\[ E^{\text{sq}} (A; B)_{\rho} \equiv \frac{1}{2} \inf_{\omega_{\mathcal{A}B\mathcal{E}}} \{ I (A; B|E)_\omega : \rho_{AB} = \text{Tr}_E \{ \omega_{\mathcal{A}B\mathcal{E}} \} \}, \] (2.8)
where the infimum is over all extensions $\omega_{\mathcal{A}B\mathcal{E}}$ of the state $\rho_{AB}$ [CW04]. The interpretation of $E^{\text{sq}} (A; B)_{\rho}$ is that it quantifies the correlations present between Alice and Bob after a third party (often associated to an environment or eavesdropper) attempts to “squash down” their correlations. In light of the above discussion, we define the geometric squashed entanglement simply by replacing the conditional mutual information with $I_F$:
\[ E_F^{\text{sq}} (A; B)_{\rho} \equiv \frac{1}{2} \inf_{\omega_{\mathcal{A}B\mathcal{E}}} \{ I_F (A; B|E)_\omega : \rho_{AB} = \text{Tr}_E \{ \omega_{\mathcal{A}B\mathcal{E}} \} \}. \] (2.9)

We also employ the related quantity throughout the paper:
\[ F^{\text{sq}} (A; B)_{\rho} \equiv \sup_{\omega_{\mathcal{A}B\mathcal{E}}} \left\{ F (A; B|E)_{\rho} : \rho_{AB} = \text{Tr}_E \{ \omega_{\mathcal{A}B\mathcal{E}} \} \right\}, \] (2.10)
with the two of them being related by
\[ E_F^{\text{sq}} (A; B)_{\rho} = -\frac{1}{2} \log F^{\text{sq}} (A; B)_{\rho}. \] (2.11)

We prove the following results for the geometric squashed entanglement, justifying it as an entanglement measure in its own right:

1. (Entanglement Monotone) The geometric squashed entanglement of $\rho_{AB}$ does not increase under local operations and classical communication. That is, the following inequality holds
\[ E_F^{\text{sq}} (A; B)_{\rho} \geq E_F^{\text{sq}} (A'; B')_{\omega}, \] (2.12)
where \( \omega_{AB} \equiv \Lambda_{AB \rightarrow A'B'}(\rho_{AB}) \) and \( \Lambda_{AB \rightarrow A'B'} \) is a quantum channel realized by local operations and classical communication. The geometric squashed entanglement is also convex, i.e.,

\[
\sum_x p_X(x) E^\text{sq}_F(A;B)_{\rho^x} \geq E^\text{sq}_F(A;B)_{\bar{\rho}},
\]

where

\[
\bar{\rho}_{AB} \equiv \sum_x p_X(x) \rho^x_{AB}.
\]

2. **(Faithfulness)** The geometric squashed entanglement of \( \rho_{AB} \) is equal to zero if and only if \( \rho_{AB} \) is a separable (unentangled) state. In particular, we prove the following bound by appealing directly to the argument in [WL12]:

\[
E^\text{sq}_F(A;B)_{\rho} \geq \frac{1}{512} \left\| \rho_{AB} - \text{SEP}(A : B) \right\|_1^4,
\]

where the trace distance to separable states is defined by

\[
\left\| \rho_{AB} - \text{SEP}(A : B) \right\|_1 \equiv \inf_{\sigma_{AB} \in \text{SEP}(A : B)} \left\| \rho_{AB} - \sigma_{AB} \right\|_1.
\]

3. **(Reduction to geometric measure)** The geometric squashed entanglement of a pure state \( |\phi\rangle_{AB} \) reduces to a variant of the well known geometric measure of entanglement [WG03] (see also [CAH14] and references therein):

\[
E^\text{sq}_F(A;B)_{\phi} = -\frac{1}{2} \log \sup_{|\varphi\rangle_A} \langle \phi|_{AB} (\varphi_A \otimes \phi_B) |\phi\rangle_{AB}
\]

4. **(Normalization)** The geometric squashed entanglement of a maximally entangled state \( \Phi_{AB} \) is equal to \( \log d \), where \( d \) is the Schmidt rank of \( \Phi_{AB} \). It is larger than \( \log d \) when evaluated for a private state [HHHO05, HHHO09] of \( \log d \) private bits.

5. **(Subadditivity)** The geometric squashed entanglement is subadditive for tensor-product states, i.e.,

\[
E^\text{sq}_F(A_1A_2;B_1B_2)_{\omega} \leq E^\text{sq}_F(A_1;B_1)_{\rho} + E^\text{sq}_F(A_2;B_2)_{\sigma},
\]

where \( \omega_{A_1B_1A_2B_2} \equiv \rho_{A_1B_1} \otimes \sigma_{A_2B_2} \).

6. **(Continuity)** If two quantum states \( \rho_{AB} \) and \( \sigma_{AB} \) are close in trace distance, then their respective geometric squashed entanglements are close as well.

### 2.3 Surprisal of measurement recoverability

The quantum discord \( D(\overline{A};B)_{\rho} \) is an information quantity which characterizes quantum correlations of a bipartite state \( \rho_{AB} \), by quantifying how much correlation is lost through the act of a quantum measurement [OZ01] (we give a full definition later on). By a chain of reasoning detailed in Section 6 which begins with the original definition of quantum discord, we define the surprisal of measurement recoverability of a bipartite state as follows:

\[
D_F(\overline{A};B)_{\rho} \equiv -\log \sup_{\mathcal{E}_A} F(\rho_{AB}, \mathcal{E}_A(\rho_{AB})),
\]
where the supremum is over the convex set of entanglement breaking channels \cite{HSR03}. Since every entanglement breaking channel can be written as a concatenation of a measurement map followed by a preparation map, $D_F (\overline{A}; B)_{\rho}$ characterizes how well one can recover a bipartite state after performing a quantum measurement on one share of it. Equivalently, the quantity captures how close $\rho_{AB}$ is to being a fixed point of an entanglement breaking channel.

We establish several properties of $D_F (\overline{A}; B)_{\rho}$, which are analogous to properties known to hold for the quantum discord \cite{MBC+12}:

1. **(Non-negativity)** This follows trivially because the fidelity between two quantum states is always a real number between zero and one.

2. **(Invariance under local isometries)** $D_F (\overline{A}; B)_{\rho}$ is invariant under local isometries, in the sense that
   \[
   D_F (\overline{A}; B)_{\rho} = D_F (\overline{A'}; B')_{\sigma},
   \]  
   where
   \[
   \sigma_{A'B'} \equiv (\mathcal{U}_{A \to A'} \otimes \mathcal{V}_{B \to B'}) (\rho_{AB})
   \]
   and $\mathcal{U}_{A \to A'}$ and $\mathcal{V}_{B \to B'}$ are isometric quantum channels.

3. **(Faithfulness)** $D_F (\overline{A}; B)_{\rho}$ is equal to zero if and only if $\rho_{AB}$ is a classical-quantum state (classical on system $A$).

4. **(Dimension bound)** $D_F (\overline{A}; B)_{\rho} \leq 2 \log |A|$.

5. **(Normalization)** $D_F (\overline{A}; B)_{\Phi}$ for a maximally entangled state $\Phi_{AB}$ is equal to $\log d$, where $d$ is the Schmidt rank of $\Phi_{AB}$.

6. **(Monotonicity)** The suprisingal of measurement recoverability is monotone with respect to quantum operations on the unmeasured system, i.e.,
   \[
   D_F (\overline{A}; B)_{\rho} \geq D_F (\overline{A'}; B')_{\sigma},
   \]
   where $\sigma_{A'B'} \equiv \mathcal{N}_{B \to B'} (\rho_{AB})$.

7. **(Continuity)** If two quantum states $\rho_{AB}$ and $\sigma_{AB}$ are close in trace distance, then the respective $D_F (\overline{A}; B)$ quantities are close as well.

Finally, we use $D_F (\overline{A}; B)_{\rho}$ and a recent result of Fawzi and Renner \cite{FR14} to establish that the quantum discord of $\rho_{AB}$ is nearly equal to zero if and only if $\rho_{AB}$ is an approximate fixed point of entanglement breaking channel (i.e., if it is possible to nearly recover $\rho_{AB}$ after performing a measurement on the system $A$). We then argue that several discord-like measures appearing throughout the literature \cite{MBC+12} have a more natural physical grounding if they are based on how far a given bipartite state is from being a fixed point of an entanglement breaking channel.
3 Preliminaries

**Norms, states, extensions, channels, and measurements.** Let \( \mathcal{B}(\mathcal{H}) \) denote the algebra of bounded linear operators acting on a Hilbert space \( \mathcal{H} \). We restrict ourselves to finite-dimensional Hilbert spaces throughout this paper. For \( \alpha \geq 1 \), we define the \( \alpha \)-norm of an operator \( X \) as

\[
\|X\|_{\alpha} \equiv \text{Tr}((\sqrt{X^*X})^\alpha)^{1/\alpha}.
\]

Let \( \mathcal{B}(\mathcal{H})_+ \) denote the subset of positive semi-definite operators. We also write \( X \geq 0 \) if \( X \in \mathcal{B}(\mathcal{H})_+ \). An operator \( \rho \) is in the set \( \mathcal{S}(\mathcal{H}) \) of density operators (or states) if \( \rho \in \mathcal{B}(\mathcal{H})_+ \) and \( \text{Tr}\{\rho\} = 1 \). The tensor product of two Hilbert spaces \( \mathcal{H}_A \) and \( \mathcal{H}_B \) is denoted by \( \mathcal{H}_A \otimes \mathcal{H}_B \) or \( \mathcal{H}_{AB} \).

Let \( \rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B) \), we unambiguously write \( \rho_A = \text{Tr}_B \{\rho_{AB}\} \) for the reduced density operator on system \( A \). We use \( \rho_{AB} \), \( \sigma_{AB} \), \( \tau_{AB} \), \( \omega_{AB} \), etc. to denote general density operators in \( \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B) \). We use \( \psi_{AB} \), \( \varphi_{AB} \), \( \phi_{AB} \), etc. denote rank-one density operators (pure states) in \( \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B) \) (with it implicit, clear from the context, and the above convention implying that \( \psi_A \), \( \varphi_A \), \( \phi_A \) are mixed if \( \psi_{AB} \), \( \varphi_{AB} \), \( \phi_{AB} \) are pure and entangled).

We also say that pure-state vectors \( |\psi\rangle \) in \( \mathcal{H} \) are states. Any bipartite pure state \( |\psi\rangle_{AB} \) in \( \mathcal{H}_{AB} \) is written in Schmidt form as

\[
|\psi\rangle_{AB} \equiv \sum_{i=0}^{d-1} \sqrt{\lambda_i} |i\rangle_A |i\rangle_B,
\]

where \( \{|i\rangle_A\} \) and \( \{|i\rangle_B\} \) form orthonormal bases in \( \mathcal{H}_A \) and \( \mathcal{H}_B \), respectively, \( d \) is the Schmidt rank of the state, and \( \sum_{i=0}^{d-1} \lambda_i = 1 \). By a maximally entangled state, we mean a bipartite pure state of the form

\[
|\Phi\rangle_{AB} \equiv \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_A |i\rangle_B.
\]

A state \( \gamma_{ABA'B'} \) is a private state [HHHO05] if Alice and Bob can extract a secret key from it by performing local von Neumann measurements on the \( A \) and \( B \) systems of \( \gamma_{ABA'B'} \), such that the resulting secret key is product with any purifying system of \( \gamma_{ABA'B'} \). The systems \( A' \) and \( B' \) are known as “shield systems” because they aid in keeping the key secure from any eavesdropper possessing the purifying system. Interestingly, a private state of log \( d \) private bits can be written in the following form [HHHO05] [HHHO09]:

\[
\gamma_{ABA'B'} = U_{AB'A'B'} \left( \Omega_{AB} \otimes \rho_{A'B'} \right) U_{AB'A'B'}^\dagger,
\]

where

\[
U_{AB'A'B'} = \sum_{i,j} |i\rangle_A \langle j|_B \otimes U_{AB}'^{ij}.
\]

The unitaries can be chosen such that \( U_{AB}'^{ij} = V_{AB}'^{ij} \) or \( U_{AB}'^{ij} = V_{AB}'^{ij} \). This implies that the unitary \( U_{AB'A'B'} \) can be implemented either as

\[
U_{AB'A'B'} = \sum_{i} |i\rangle_A \otimes I_B \otimes V_{AB}'^{i},
\]

or

\[
U_{AB'A'B'} = I_A \otimes \sum_{i} |i\rangle_B \otimes V_{AB}'^{i}.
\]
The trace distance between two quantum states \( \rho, \sigma \in \mathcal{S}(\mathcal{H}) \) is equal to \( \|\rho - \sigma\|_1 \). It has a direct operational interpretation in terms of the distinguishability of these states. That is, if \( \rho \) or \( \sigma \) are prepared with equal probability and the task is to distinguish them via some quantum measurement, then the optimal success probability in doing so is equal to \( (1 + \|\rho - \sigma\|_1) / 2 \).

A linear map \( \mathcal{N}_{A \to B} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B) \) is positive if \( \mathcal{N}_{A \to B}(\sigma_A) \in \mathcal{B}(\mathcal{H}_B)_+ \) whenever \( \sigma_A \in \mathcal{B}(\mathcal{H}_A)_+ \). Let id\(_A\) denote the identity map acting on a system \( A \). A linear map \( \mathcal{N}_{A \to B} \) is completely positive if the map id\(_R\) \(\otimes\) \( \mathcal{N}_{A \to B} \) is positive for a reference system \( R \) of arbitrary size. A linear map \( \mathcal{N}_{A \to B} \) is trace-preserving if \( \text{Tr}\{\mathcal{N}_{A \to B}(\tau_A)\} = \text{Tr}\{\tau_A\} \) for all input operators \( \tau_A \in \mathcal{B}(\mathcal{H}_A) \). If a linear map is completely positive and trace-preserving (CPTP), we say that it is a quantum channel or quantum operation. An extension of a state \( \rho_A \in \mathcal{S}(\mathcal{H}_A) \) is some state \( \Omega_{RA} \in \mathcal{S}(\mathcal{H}_R \otimes \mathcal{H}_A) \) such that \( \text{Tr}_R\{\Omega_{RA}\} = \rho_A \). An isometric extension \( U^N_{A \to BE} \) of a channel \( \mathcal{N}_{A \to B} \) acting on a state \( \rho_A \in \mathcal{S}(\mathcal{H}_A) \) is a linear map that satisfies the following:

\[
\text{Tr}_E \left\{ U^N_{A \to BE} \rho_A (U^N_{A \to BE})^\dagger \right\} = \mathcal{N}_{A \to B}(\rho_A), \quad U^N_A U_A^\dagger = I_A, \quad U^N_A U^\dagger_A = \Pi_{BE},
\]

where \( \Pi_{BE} \) is a projection onto a subspace of the Hilbert space \( \mathcal{H}_B \otimes \mathcal{H}_E \).

\section{Fidelity of recovery}

In this section, we formally define the fidelity of recovery for a tripartite state \( \rho_{ABC} \), and we prove that it possesses various properties, demonstrating that the quantity \( I_F(A;B|C)_\rho \) defined in (2.1) is similar to the conditional mutual information.

**Definition 1 (Fidelity of recovery)** Let \( \rho_{ABC} \) be a tripartite state. The fidelity of recovery for \( \rho_{ABC} \) with respect to system \( A \) is defined as follows:

\[
F(A;B|C)_\rho \equiv \sup_{\mathcal{R}_{C \to AC}} F(\rho_{ABC};\mathcal{R}_{C \to AC}(\rho_{BC})).
\]

This quantity characterizes how well one can recover the full state on systems \( ABC \) from system \( C \) alone if system \( A \) is lost.

**Proposition 2 (Duality)** Let \( \phi_{ABCD} \) denote a four-party pure state. Then

\[
F(A;B|C)_{\phi} = F(A;B|D)_{\phi},
\]

which is equivalent to

\[
I_F(A;B|C)_{\phi} = I_F(A;B|D)_{\phi}.
\]

**Proof.** By definition,

\[
F(A;B|C)_{\phi} = \sup_{\mathcal{R}^1_{C \to AC}} F(\phi_{ABC};\mathcal{R}^1_{C \to AC}(\phi_{BC})).
\]

Let \( U_{C \to ACE}^{\mathcal{R}^1} \) be an isometric extension of \( \mathcal{R}^1_{C \to AC} \). Since \( \phi_{ABCD} \) is a purification of \( \phi_{ABC} \) and \( U_{C \to ACE}^{\mathcal{R}^1}(\phi_{BCA'D}) \) is a purification of \( \mathcal{R}^1_{C \to AC}(\phi_{BC}) \), we can apply Uhlmann’s theorem for fidelity to conclude that

\[
\sup_{\mathcal{R}^1_{C \to AC}} F(\phi_{ABC};\mathcal{R}^1_{C \to AC}(\phi_{BC})) = \sup_{U_{D \to A'DE}^1} \sup_{U_{C \to ACE}^{\mathcal{R}^1}} F(U_{D \to A'DE}(\phi_{ABCD}), U_{C \to ACE}^{\mathcal{R}^1}(\phi_{BCA'D})).
\]
Figure 1: This figure helps to illustrate the main idea behind the proof of Proposition 2 and furthermore highlights the dual role played by an isometric extension of the recovery map on system D and an Uhlmann isometry acting on system A (and vice versa).

Now consider that

\[ F(A; B|D) = \sup_{R^2_{D\rightarrow AD}} F(\phi_{ABD}, R^2_{D\rightarrow AD}(\phi_{BD})). \]  

(4.6)

Let \( U^R_{D\rightarrow ADE} \) be an isometric extension of \( R^2_{D\rightarrow AD} \). Since \( \phi_{ABCD} \) is a purification of \( \phi_{ABD} \) and \( U^R_{D\rightarrow ADE}(\phi_{BDA'C}) \) is a purification of \( R^2_{D\rightarrow AD}(\phi_{BD}) \), we can apply Uhlmann’s theorem for fidelity to conclude that

\[ \sup_{R^2_{D\rightarrow AD}} F(\phi_{ABD}, R^2_{D\rightarrow AD}(\phi_{BD})) = \sup_{U^R_{C\rightarrow A'E}} \sup_{U^R_{D\rightarrow ADE}} F(U_{C\rightarrow A'E}(\phi_{ABCD}), U^R_{D\rightarrow ADE}(\phi_{BDA'C})). \]  

(4.7)

By inspecting the RHS of (4.5) and the RHS of (4.7), we see that the two expressions are equivalent so that the statement of the proposition holds. Figure 1 gives a graphical depiction of this proof which should help in determining which systems are “connected together” and furthermore highlights how the duality between the recovery map and the map from Uhlmann’s theorem is reflected in the duality for the fidelity of recovery.

Remark 3 The physical interpretation of the above duality is as follows: beginning with a four-party pure state \( \phi_{ABCD} \), suppose that system A is lost. Then one can recover the state on systems ABC from system C alone just as well as one can recover the state on systems ABD from system D alone.

Proposition 4 (Monotonicity) The fidelity of recovery is monotone under local operations on systems A and B, in the sense that

\[ F(A; B|C) \leq F(A'; B'|C), \]  

(4.8)

where \( \tau_{A'B'C} \equiv (N_{A\rightarrow A'} \otimes M_{B\rightarrow B'})(\rho_{ABC}) \). The above inequality is equivalent to

\[ I_F(A; B|C) \geq I_F(A'; B'|C). \]  

(4.9)
Proposition 7 (Dimension bound) The fidelity of recovery obeys the following dimension bound:

\[
F(\rho_{ABC}, R_{C\to AC} (\rho_{BC})) \leq F((N_{A\to A'} \otimes M_{B\to B'}) (\rho_{ABC}), (N_{A\to A'} \otimes M_{B\to B'}) (R_{C\to AC} (\rho_{BC}))) \leq F((N_{A\to A'} \otimes M_{B\to B'}) (\rho_{ABC}), (N_{A\to A'} \circ R_{C\to AC}) (M_{B\to B'} (\rho_{BC}))) \leq \sup_{R_{C\to A'C}} F((N_{A\to A'} \otimes M_{B\to B'}) (\rho_{ABC}), R_{C\to A'C} (M_{B\to B'} (\rho_{BC}))) \leq F(A'\otimes B'|C)_{(N\otimes M)(\rho)},
\]

where the first inequality is due to monotonicity of the fidelity under quantum operations. Since the chain of inequalities holds for all $R_{C\to AC}$, it follows that

\[
F(A; B|C)_\rho = \sup_{R_{C\to AC}} F(\rho_{ABC}, R_{C\to AC} (\rho_{BC})) \leq F(A'; B'|C)_{(N\otimes M)(\rho)}.
\]

\[\begin{align*}
\text{Remark 5} & \quad \text{The only property of the fidelity used to prove the above proposition is that it is monotone under quantum operations. This suggests that we can construct a fidelity-of-recovery-like measure from any “generalized divergence” (a function that is monotone under quantum operations).} \\
\text{Remark 6} & \quad \text{The physical interpretation of the above monotonicity under local operations is as follows: for a tripartite state } \rho_{ABC}, \text{ suppose that system } A \text{ is lost. Then it is easier to recover the state on systems } ABC \text{ from } C \text{ alone if there is local noise applied to systems } A \text{ or } B \text{ or both, before system } A \text{ is lost (and thus before attempting the recovery).}
\end{align*}\]

Proposition 7 (Dimension bound) The fidelity of recovery obeys the following dimension bound:

\[
F(A; B|C)_\rho \geq \frac{1}{|A|^2},
\]

which is equivalent to

\[
I_F(A; B|C)_\rho \leq 2 \log |A|.
\]

Proof. For any recovery map $R_{C\to AC}$, we have that

\[
F(\rho_{ABC}, R_{C\to AC} (\rho_{BC})) \leq F((N_{A\to A'} \otimes M_{B\to B'}) (\rho_{ABC}), (N_{A\to A'} \otimes M_{B\to B'}) (R_{C\to AC} (\rho_{BC}))) \leq F((N_{A\to A'} \otimes M_{B\to B'}) (\rho_{ABC}), (N_{A\to A'} \circ R_{C\to AC}) (M_{B\to B'} (\rho_{BC}))) \leq \sup_{R_{C\to A'C}} F((N_{A\to A'} \otimes M_{B\to B'}) (\rho_{ABC}), R_{C\to A'C} (M_{B\to B'} (\rho_{BC}))) \leq F(A'\otimes B'|C)_{(N\otimes M)(\rho)},
\]

which is a consequence of the monotonicity of the fidelity under quantum operations. This suggests that we can construct a fidelity-of-recovery-like measure from any “generalized divergence” (a function that is monotone under quantum operations).
The first equality follows by recognizing that the second term is a conditional Rényi entropy of order 1/2 [TCR09, Definition 3]. The second equality follows from a duality relation for this conditional Rényi entropy [TCR09, Lemma 6]. The second inequality is a consequence of the quantum data processing inequality for conditional Rényi entropies [TCR09, Lemma 5] (with the map taken to be a partial trace over system $D$). The last inequality follows from a dimension bound which holds for any Rényi entropy. ■

**Proposition 8 (Weak chain rule)** Given a four-party state $\rho_{ABCD}$, the following inequality holds

$$I_F(A; B|D)_\rho \geq I_F(A; B|CD)_\rho.$$ (4.26)

**Proof.** The inequality is equivalent to

$$F(AC; B|D)_\rho \leq F(A; B|CD)_\rho,$$ (4.27)

which follows from the fact that it is easier to recover $A$ from $CD$ than it is to recover both $A$ and $C$ from $D$ alone. Indeed, let $R_{D\rightarrow ACD}$ be any recovery map. Then

$$F(\rho_{ABCD}, R_{D\rightarrow ACD}(\rho_{BD})) = F(\rho_{ABCD}, (R_{D\rightarrow ACD} \circ \text{Tr}_C)(\rho_{BCD})) \leq \sup_{R_{CD\rightarrow ACD}} F(\rho_{ABCD}, (R_{CD\rightarrow ACD})(\rho_{BCD})) = F(A; B|CD)_\rho.$$ (4.29)

Since the chain of inequalities holds for any recovery map $R_{D\rightarrow ACD}$, we can conclude (4.27) from the definition of $F(AC; B|D)_\rho$. ■

**Proposition 9 (Conditioning on classical information)** Let $\omega_{ABCX}$ be a state for which system $X$ is classical:

$$\omega_{ABCX} = \sum_x p_X(x) \omega_{ABC}^x \otimes |x\rangle \langle x|_X,$$ (4.31)

where $\{|x\rangle_X\}$ is an orthonormal basis, $p_X(x)$ is a probability distribution, and each $\omega_{ABC}^x$ is a state. Then the following equalities hold

$$F(A; B|CX)_\omega \left[\sum_x p_X(x) \sqrt{F(A; B|C)_{\omega^x}} \right]^2,$$ (4.32)

$$I_F(A; B|CX)_\omega = -2 \log \left[\sum_x p_X(x) \exp \left\{-\frac{1}{2} I_F(A; B|C)_{\omega^x}\right\}\right].$$ (4.33)

**Proof.** We first prove the inequality

$$\sqrt{F(A; B|CX)_\omega} \leq \sum_x p_X(x) \sqrt{F(A; B|C)_{\omega^x}}.$$ (4.34)

Let $R_{CX\rightarrow CXA}$ be any recovery map taking systems $CX$ to systems $CXA$, and let $R_{C\rightarrow CXA}^x$ be defined by

$$R_{C\rightarrow CXA}^x(\sigma_C) \equiv R_{CX\rightarrow CXA}(\sigma_C \otimes |x\rangle \langle x|_X).$$ (4.35)
Then
\[
\sqrt{F\left(\sum_x p_X(x) \omega_{ABC}^x \otimes |x\rangle \langle x| \cdot R_{CX \rightarrow CXA} \left(\sum_x p_X(x) \omega_{BC}^x \otimes |x\rangle \langle x| \right)\right)}
\]
\[
= \sqrt{F\left(\sum_x p_X(x) \omega_{ABC}^x \otimes |x\rangle \langle x| \cdot \sum_x p_X(x) \mathcal{R}_{C \rightarrow CXA}^x \left(\omega_{BC}^x \right)\right)}
\]
\[
\leq \sum_x p_X(x) \sqrt{F\left(\omega_{ABC}^x \otimes |x\rangle \langle x| \cdot \mathcal{R}_{C \rightarrow CXA}^x \left(\omega_{BC}^x \right)\right)}
\]
\[
\leq \sum_x p_X(x) \sqrt{F\left(\omega_{ABC}^x, \mathcal{R}_{C \rightarrow CA}^x \left(\omega_{BC}^x \right)\right)}
\]
\[
\leq \sum_x p_X(x) \sup_{\mathcal{R}_{C \rightarrow CA}} \sqrt{F\left(\omega_{ABC}^x, \mathcal{R}_{C \rightarrow CA}^x \left(\omega_{BC}^x \right)\right)}
\]
\[
= \sum_x p_X(x) \sqrt{F(A; B|C), \omega^x}.
\]
\[\text{(4.36)}\]

The first inequality follows from joint concavity of the root fidelity \[\text{NC10, Theorem 9.7}\], the second from monotonicity of the fidelity under the discarding of system \(X\), and the third by taking a supremum over each of the individual recovery maps \(\mathcal{R}_{C \rightarrow CA}\). Since the inequality holds universally for any recovery map \(\mathcal{R}_{CX \rightarrow CXA}\), we can conclude \[\text{(4.34)}\].

We now prove the other inequality:
\[
\sqrt{F(A; B|CX), \omega} \geq \sum_x p_X(x) \sqrt{F(A; B|C), \omega^x}.
\]
\[\text{(4.41)}\]

For any set of recovery maps \(\mathcal{R}_{C \rightarrow CA}^x\), define \(\mathcal{R}_{CX \rightarrow CXA}\) as follows:
\[
\mathcal{R}_{CX \rightarrow CXA}^x (\tau_{CX}) \equiv \sum_{\mathcal{R}_{C \rightarrow CA}} \mathcal{R}_{C \rightarrow CA}^x (\langle x| \langle x (\tau_{CX}) | x \rangle \rangle \langle x \rangle \langle x|)
\]
\[\text{(4.42)}\]
so that it first measures the system \(X\) in the basis \(\{|x\rangle \langle x|\}\), places the outcome in the same classical register, and then acts with the particular recovery map \(\mathcal{R}_{C \rightarrow CA}^x\). Then
\[
\left[\sum_x p_X(x) \sqrt{F(\omega_{ABC}^x, \mathcal{R}_{C \rightarrow CA}^x \left(\omega_{BC}^x \right))}\right]^2
\]
\[
= F\left(\sum_x p_X(x) \omega_{ABC}^x \otimes |x\rangle \langle x| \cdot \sum_x p_X(x) \mathcal{R}_{C \rightarrow CA}^x \left(\omega_{BC}^x \right) \otimes |x\rangle \langle x| \right)
\]
\[
= F\left(\sum_x p_X(x) \omega_{ABC}^x \otimes |x\rangle \langle x| \cdot \mathcal{R}_{CX \rightarrow CXA} \left(\sum_x p_X(x) \omega_{BC}^x \otimes |x\rangle \langle x| \right)\right)
\]
\[
\leq F(A; B|CX), \omega.
\]
\[\text{(4.43)}\]

Since the inequality holds for any set of individual recovery maps \(\{\mathcal{R}_{C \rightarrow CA}^x\}\), we obtain \[\text{(4.41)}\]. Combining \[\text{(4.34)}\] with \[\text{(4.41)}\] gives \[\text{(4.32)}\].

Finally, we recover \[\text{(4.33)}\] from the equality in \[\text{(4.32)}\] and definitions. 

Applying convexity of \(x^2\) and convexity of \(-\log\) to Proposition \[\text{9}\] gives the following corollary:
Corollary 10  Let $\omega_{ABCX}$ be a state for which system $X$ is classical. Then

$$F(A; B|CX)_\omega \leq \sum_x p_x(x) F(A; B|C)_{\omega^x},$$

(4.46)

$$I_F(A; B|CX)_\omega \leq \sum_x p_x(x) I_F(A; B|C)_{\omega^x}.$$

(4.47)

Proposition 11  (Conditioning on a product system) Let $\rho_{ABC} = \sigma_{AB} \otimes \omega_C$. Then

$$F(A; B|C)_\rho = F(A; B|\sigma) \equiv \sup_{\tau_A} F(\sigma_{AB}; \tau_A \otimes \sigma_B),$$

(4.48)

$$I_F(A; B|C)_\rho = I_F(A; B|\sigma) \equiv -\log F(A; B|\sigma).$$

(4.49)

**Proof.** Consider that, for any recovery map $R_{C \rightarrow AC}$

$$F(\sigma_{AB} \otimes \omega_C, R_{C \rightarrow AC}; (\sigma_B \otimes \omega_C)) = F(\sigma_{AB} \otimes \omega_C, \sigma_B \otimes R_{C \rightarrow AC}(\omega_C))$$

(4.50)

$$\leq F(\sigma_{AB}, \sigma_B \otimes R_{C \rightarrow A}(\omega_C))$$

(4.51)

$$\leq \sup_{\tau_A} F(\sigma_{AB}, \sigma_B \otimes \tau_A).$$

(4.52)

The first inequality follows because fidelity is monotone under partial trace over the $C$ system. The second inequality follows by optimizing the second argument to the fidelity over all states on the $A$ system. Since the inequality holds independent of the recovery map $R_{C \rightarrow AC}$, this proves that

$$F(A; B|C)_\rho \leq F(A; B|\sigma).$$

(4.53)

To prove the other inequality $F(A; B|\sigma) \leq F(A; B|C)_\rho$, consider for any state $\tau_A$ that

$$F(\sigma_{AB}, \sigma_B \otimes \tau_A) = F(\sigma_{AB} \otimes \omega_C, \sigma_B \otimes \tau_A \otimes \omega_C)$$

(4.54)

$$= F(\sigma_{AB} \otimes \omega_C, (\mathrm{id}_C \otimes P_\lambda^x)(\sigma_B \otimes \omega_C))$$

(4.55)

$$\leq \sup_{R_{C \rightarrow AC}} F(\sigma_{AB} \otimes \omega_C, R_{C \rightarrow AC}(\sigma_B \otimes \omega_C)).$$

(4.56)

The first equality follows because fidelity is multiplicative under tensor-product states. The second equality follows by taking $(\mathrm{id}_C \otimes P_\lambda^x)$ to be the recovery map that does nothing to system $C$ and prepares $\tau_A$ on system $A$. The inequality follows by optimizing over all recovery maps. Since the inequality is independent of the prepared state, we obtain the other inequality

$$F(A; B|\sigma) \leq F(A; B|C)_\rho.$$

(4.57)

The equality $I_F(A; B|C)_\rho = I_F(A; B|\sigma)$ follows by applying a negative logarithm to $F(A; B|C)_\rho = F(A; B|\sigma)$. We note in passing that the quantity on the RHS in (4.49) is closely related to the sandwiched Rényi mutual information of order 1/2 [MLDS+13, WWY14, Bei13, GW13].

The following proposition gives a simple proof of the main result of [FR14] when the tripartite state of interest is pure:

**Proposition 12  (Approximate quantum Markov chain)** The conditional mutual information $I(A; B|C)_\psi$ of a pure tripartite state $\psi_{ABC}$ has the following lower bound:

$$I(A; B|C)_\psi \geq -\log F(A; B|C)_\psi.$$

(4.58)
Proof. Let $\varphi_D$ be a pure state on an auxiliary system $D$, so that $|\psi\rangle_{ABC} \otimes |\varphi\rangle_D$ is a purification of $|\psi\rangle_{ABC}$. Consider the following chain of inequalities:

\begin{align*}
I(A; B|C)_\psi &= I(A; B|D)_{\psi \otimes \varphi} \quad (4.59) \\
&= I(A; B)_\psi \quad (4.60) \\
&\geq - \log F(\psi_{AB}, \psi_A \otimes \psi_B) \quad (4.61) \\
&\geq - \log F(A; B)_\psi \quad (4.62) \\
&= - \log F(A; B|D)_{\psi \otimes \varphi} \quad (4.63) \\
&= - \log F(A; B|C)_\psi. \quad (4.64)
\end{align*}

The first equality follows from duality of conditional mutual information. The second follows because system $D$ is product with systems $A$ and $B$. The first inequality follows from monotonicity of the sandwiched Rényi relative entropies [MLDS+13, Theorem 7]:

\begin{equation}
\tilde{D}_\alpha(\rho \| \sigma) \leq \tilde{D}_\beta(\rho \| \sigma),
\end{equation}

for states $\rho$ and $\sigma$ and Rényi parameters $\alpha$ and $\beta$ such that $0 \leq \alpha \leq \beta$. We apply this with the choices $\alpha = 1/2$, $\beta = 1$, $\rho = \psi_{AB}$, and $\sigma = \psi_A \otimes \psi_B$. The second inequality follows by optimizing over states on system $A$ and applying the definition in (4.49). The second-to-last equality follows from Proposition 11 and the last from Proposition 2. ■

5 Geometric squashed entanglement

In this section, we formally define the geometric squashed entanglement of a bipartite state $\rho_{AB}$, and we prove that it obeys the properties claimed in Section 2.

**Definition 13 (Geometric squashed entanglement)** The geometric squashed entanglement of a bipartite state $\rho_{AB}$ is defined as follows:

\begin{equation}
E_{sq}^{F}(A; B)_\rho \equiv - \frac{1}{2} \log F_{sq}^{eq}(A; B)_\rho, \quad (5.1)
\end{equation}

where

\begin{align*}
F_{sq}^{eq}(A; B)_\rho &\equiv \sup_{\omega_{ABE}} \left\{ F(A; B|E)_\rho : \rho_{AB} = \text{Tr}_E \{\omega_{ABE}\} \right\} \\
&= \sup_{\omega_{ABE}} \sup_{\mathcal{R}_{E\to AE}} \left\{ F(\omega_{ABE}, \mathcal{R}_{E\to AE}(\omega_{BE})) : \rho_{AB} = \text{Tr}_E \{\omega_{ABE}\} \right\}. \quad (5.2)
\end{align*}

The geometric squashed entanglement can equivalently be written in terms of an optimization over “squashing channels” acting on a purifying system of the original state (cf. [CW04, Eq. (3)]):

**Proposition 14** Let $\rho_{AB}$ be a bipartite state and let $|\psi\rangle_{ABE'}$ be a fixed purification of it. Then

\begin{equation}
F_{sq}^{eq}(A; B)_\rho = \sup_{S_{E'\to E}} F(A; B|E)_{S(\psi)}, \quad (5.4)
\end{equation}

where the optimization is over squashing channels $S_{E'\to E}$. 

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Proof. We first prove the inequality $F_{\text{sq}}(A;B)_{\rho} \geq \sup_{S_{E'\rightarrow E}} F(A;B|E)_{S(\psi)}$. Indeed, for a given squashing channel $S_{E'\rightarrow E}$ and purification $\psi_{ABE'}$, the state $S_{E'\rightarrow E}(\psi_{ABE'})$ is an extension of $\rho_{AB}$. So it follows by definition that

$$F(A;B|E)_{S(\psi)} \leq F_{\text{sq}}(A;B)_{\rho}. \quad (5.5)$$

Since the choice of squashing channel was arbitrary, the first inequality follows.

We now prove the other inequality

$$F_{\text{sq}}(A;B)_{\rho} \leq \sup_{S_{E'\rightarrow E}} F(A;B|E)_{S(\psi)}. \quad (5.6)$$

Let $\omega_{ABE}$ be an extension of $\rho_{AB}$. Let $\varphi_{ABEE_1}$ be a purification of $\omega_{ABE}$, which is in turn also a purification of $\rho_{AB}$. Since all purifications are related by isometries acting on the purifying system, we know that there exists an isometry $U_{E'\rightarrow EE_1}$ (depending on $\omega$) such that

$$|\varphi\rangle_{ABEE_1} = U_{E'\rightarrow EE_1}|\psi\rangle_{ABE'}. \quad (5.7)$$

Furthermore, we know that

$$\omega_{ABE} = \text{Tr}_{E_1}\left\{U_{E'\rightarrow EE_1}^{\dagger}\varphi_{ABE}(U_{E'\rightarrow EE_1})^{\dagger}\right\} \quad (5.8)$$

$$\equiv S_{E'\rightarrow E}(\varphi_{ABE'}), \quad (5.9)$$

where we define the squashing channel $S_{E'\rightarrow E}$ from the isometry $U_{E'\rightarrow EE_1}$. So this implies that

$$F(A;B|E)_{\omega} = F(A;B|E)_{S_{-}(\psi)} \leq \sup_{S_{E'\rightarrow E}} F(A;B|E)_{S(\psi)}. \quad (5.10)$$

Since the inequality above holds for all extensions, the inequality in (5.6) follows. 

The following statement is a direct consequence of Proposition 4.

**Corollary 15** The geometric squashed entanglement is monotone under local operations on both systems $A$ and $B$:

$$E_{F}^{\text{sq}}(A;B)_{\rho} \geq E_{F}^{\text{sq}}(A';B')_{\tau}, \quad (5.11)$$

where $\tau_{A'B'} \equiv (N_{A\rightarrow A'}\otimes M_{B\rightarrow B'})(\rho_{AB})$. This is equivalent to

$$F_{\text{sq}}(A;B)_{\rho} \leq F_{\text{sq}}(A';B')_{\tau}. \quad (5.12)$$

**Proposition 16** The geometric squashed entanglement is invariant under classical communication, in the sense that

$$E_{F}^{\text{sq}}(AX_A;B)_{\rho} = E_{F}^{\text{sq}}(AX_A;BX_B)_{\rho} = E_{F}^{\text{sq}}(A;BX_B)_{\rho}, \quad (5.13)$$

for a state $\rho_{XAXBAB}$ defined as

$$\rho_{XAXBAB} \equiv \sum_{x} p_{X}(x)|x\rangle_{X_A} \otimes |x\rangle_{X_B} \otimes \rho_{\tilde{A}B}. \quad (5.14)$$

This is equivalent to

$$F_{\text{sq}}(AX_A;B)_{\rho} = F_{\text{sq}}(AX_A;BX_B)_{\rho} = F_{\text{sq}}(A;BX_B)_{\rho}. \quad (5.15)$$
Proof. From monotonicity under local operations, we have that

$$F^{\text{eq}}(AX_A; BX_B) \leq F^{\text{eq}}(AX_A; B) ,$$  \hspace{1cm} (5.16) \\
$$F^{\text{eq}}(AX_A; BX_B) \leq F^{\text{eq}}(A; BX_B) ,$$  \hspace{1cm} (5.17) \\

So we need to show the opposite inequalities. Let \( \rho_{X_A ABE} \) be any extension of \( \rho_{X_A AB} \). Let \( |\varphi^0\rangle_{X_A X_R ABE} \) be a purification of \( \rho_{X_A ABE} \), taken as

$$|\varphi^0\rangle_{X_A X_R ABE} \equiv \sum_x \sqrt{p_X(x)} |x\rangle_{X_A} |x\rangle_{X_R} |\varphi^0_x\rangle_{ABER} ,$$  \hspace{1cm} (5.18) \\

where each \( |\varphi^0_x\rangle_{ABER} \) purifies \( \rho^0_{ABE} \). Let \( \varphi^0_{X_A X_B X_R ABE} \) be defined as

$$\varphi^0_{X_A X_B X_R ABE} \equiv \sum_x p_X(x) |x\rangle_{X_A} \otimes |x\rangle_{X_R} \langle x\rangle_{X_B} \otimes |\varphi^0_x\rangle_{ABER} .$$  \hspace{1cm} (5.19) \\

Let \( \theta_{X_A X_B X_R ABEF} \) be a purification of \( \varphi^0_{X_A X_B X_R ABE} \). We then have that

$$F(AX_A; B|E)_{\rho} = F(AX_A; B|X_R)_{\varphi^0} \leq F(AX_A; B|X_R)_{\varphi^0} ,$$  \hspace{1cm} (5.20) \\

where the inequality follows from processing system \( X_A \) with a completely dephasing map. Let \( R_{X_R \rightarrow X_R RAX_A} \) be any CPTP map. When such a map acts on a classical system \( X_R \), we can think of it as a collection of maps, conditioned on the value in the register \( X_R \), i.e., \( \{ R^x_{R \rightarrow X_R RAX_A} \} \). So we then have that

$$F\left( \varphi^0_{AX_A X_B X_R RAX_A}; R_{X_R \rightarrow X_R RAX_A} \left( \varphi^0_{BX_B X_R R} \right) \right)$$  \\
$$= \left[ \sum_x p_X(x) \sqrt{F\left( |\varphi^0_x\rangle_{ABER} \otimes |x\rangle_{X_R} \langle x\rangle_{X_B} \otimes |\varphi^0_x\rangle_{ABER} \right)} \right]^2$$  \hspace{1cm} (5.22) \\
$$= \left[ \sum_x p_X(x) \sqrt{F\left( |\varphi^0_{ABER} \otimes |x\rangle_{X_B} \langle x\rangle_{X_B} \otimes |\varphi^0_x\rangle_{ABER} \right)} \right]^2$$  \hspace{1cm} (5.23) \\
$$= F\left( \varphi^0_{AX_A X_B X_R X_R RAX_A}; R_{X_R \rightarrow X_R RAX_A} \left( \varphi^0_{BX_B X_R R} \right) \right)$$  \hspace{1cm} (5.24) \\
$$\leq F(AX_A; BX_B|RX_R)_{\varphi^0}$$  \hspace{1cm} (5.25) \\
$$\leq F^{\text{eq}}(AX_A; BX_B)_{\rho}$$  \hspace{1cm} (5.26) \\

The first equality is a property of the quantum fidelity. The second equality results because the fidelity is multiplicative under tensor-product states. The third equality follows for the same reason that the first one does. The first inequality follows from Definition 11 and the last inequality follows from Definition 13. Since the chain of inequalities above holds for all extensions of \( \rho_{AXAB} \), we can conclude the following inequality:

$$F^{\text{eq}}(AX_A; B)_{\rho} \leq F^{\text{eq}}(AX_A; BX_B)_{\rho} .$$  \hspace{1cm} (5.27) \\

By the same line of reasoning, we can conclude

$$F^{\text{eq}}(A; BX_B)_{\rho} \leq F^{\text{eq}}(AX_A; BX_B)_{\rho} ,$$  \hspace{1cm} (5.28) \\

and thus the statement of the proposition. \( \blacksquare \)

The following theorem is a direct consequence of Corollary 15 and Proposition 16.
**Theorem 17 (LOCC monotone)** The geometric squashed entanglement is an LOCC monotone.

**Theorem 18 (Convexity)** The geometric squashed entanglement is convex, i.e.,
\[
\sum_x p_X(x) E^\text{sq}_F(A;B)_\rho x \geq E^\text{sq}_F(A;B)_\rho ,
\]
where
\[
\bar{\rho}_{AB} \equiv \sum_x p_X(x) \rho^x_{AB}.
\]

**Proof.** Let \( \rho^x_{ABE} \) be an extension of each \( \rho^x_{AB} \), so that
\[
\omega_{XABE} \equiv \sum_x p_X(x) |x\rangle \langle x|_X \otimes \rho^x_{ABE}
\]
is some extension of \( \bar{\rho}_{AB} \). Then the definition of \( E^\text{sq}_F(A;B)_\rho \) and Corollary 10 give that
\[
2E^\text{sq}_F(A;B)_\rho \leq I_F(A;B|EX)_\omega \leq \sum_x p_X(x) I_F(A;B|E)_{\rho x}.
\]

Since the inequality holds independent of each particular extension of \( \rho^x_{AB} \), we can conclude (5.29).

Theorems 17 and 18 immediately lead to the following corollary:

**Corollary 19 (Entanglement monotone)** The geometric squashed entanglement is an entanglement monotone.

**Theorem 20 (Faithfulness)** The geometric squashed entanglement is faithful, in the sense that
\[
E^\text{sq}_F(A;B)_\rho = 0 \text{ if and only if } \rho_{AB} \text{ is separable.}
\]

This is equivalent to
\[
F^\text{sq}(A;B)_{\rho} = 1 \text{ if and only if } \rho_{AB} \text{ is separable.}
\]
Furthermore, we have the following bound holding for states with small geometric squashed entanglement:
\[
E^\text{sq}_F(A;B)_\rho \geq \frac{1}{512 |A|^4} \| \rho_{AB} - \text{SEP}(A:B) \|_1^4.
\]

**Proof.** We first prove the if-part of the theorem. So, given by assumption that \( \rho_{AB} \) is separable, it has a decomposition of the following form:
\[
\rho_{AB} = \sum_x p_X(x) |\psi_x\rangle \langle \psi_x|_A \otimes |\phi_x\rangle \langle \phi_x|_B.
\]

Then an extension of the state is of the form
\[
\rho_{ABE} = \sum_x p_X(x) |\psi_x\rangle \langle \psi_x|_A \otimes |\phi_x\rangle \langle \phi_x|_B \otimes |x\rangle \langle x|_E.
\]
Clearly, if the system $A$ becomes lost, someone who possesses system $E$ could measure it and prepare the state $|\psi_x\rangle_A$ conditioned on the measurement outcome. That is, the recover map $R_{E \rightarrow AE}$ is as follows:

$$R_{E \rightarrow AE}(\sigma_E) = \sum_x \langle x|\sigma_E|x\rangle |\psi_x\rangle \langle \psi_x|_A \otimes |x\rangle \langle x|_E.$$  

(5.38)

So this implies that

$$F(\rho_{ABE}, R_{E \rightarrow AE}(\rho_{BE})) = 1,$$  

(5.39)

and thus $F^{\text{sq}}(A; B)_{\rho} = 1$.

The only-if-part of the theorem is a direct consequence of the reasoning in [WL12]. We repeat the argument from [WL12] here for the convenience of the reader. The reasoning from [WL12] establishes that the trace distance between $\rho_{AB}$ and the set SEP($A : B$) of separable states on systems $A$ and $B$ is bounded from above by a function of $-1/2 \log F^{\text{sq}}(A; B)_{\rho}$ and $|A|$. This will then allow us to conclude the only-if-part of the theorem.

Let

$$\varepsilon \equiv -1/2 \log F^{\text{sq}}(A; B)_{\rho}$$  

(5.40)

for some bipartite state $\rho_{AB}$ and let

$$\varepsilon_{\omega, R} \equiv -1/2 \log F(\omega_{ABE}, R_{E \rightarrow AE}(\omega_{BE})),$$  

(5.41)

for some extension $\omega_{ABE}$ and a recovery map $R_{E \rightarrow AE}$. By definition, we have that

$$\varepsilon = \inf_{\omega, R_{E \rightarrow AE}} \varepsilon_{\omega, R}.$$  

(5.42)

Then consider that

$$\varepsilon_{\omega, R} \geq \frac{1}{8}\|\omega_{ABE} - R_{E \rightarrow AE}(\omega_{BE})\|_1^2,$$  

(5.43)

where the inequality follows from a well known relation between the fidelity and trace distance [FvdG98]. Therefore, by defining $\delta_{\omega, R} = \sqrt{8\varepsilon_{\omega, R}}$ we have that

$$\delta_{\omega, R} \geq \|\omega_{ABE} - R_{E \rightarrow AE}(\omega_{BE})\|_1$$  

(5.44)

$$= \|\omega_{ABE} - (R_{E \rightarrow A_2E} \circ \text{Tr}_{A_2})(\omega_{A_1BE})\|_1,$$  

(5.45)

where the systems $A_1$ and $A_2$ are defined to be isomorphic to system $A$. Now consider applying the same recovery map again. We then have that

$$\delta_{\omega, R} \geq \|(R_{E \rightarrow A_3E} \circ \text{Tr}_{A_3})(\omega_{A_2BE}) - \bigcirc_{i=2}^3 (R_{E \rightarrow A_iE} \circ \text{Tr}_{A_{i-1}})(\omega_{A_{i+1}BE})\|_1,$$  

(5.46)

which follows from the inequality above and monotonicity of the trace distance under the quantum operation $R_{E \rightarrow A_3E} \circ \text{Tr}_{A_3}$. Combining via the triangle inequality, we find for $k \geq 2$ that

$$\|\omega_{ABE} - \bigcirc_{i=2}^3 (R_{E \rightarrow A_iE} \circ \text{Tr}_{A_{i-1}})(\omega_{A_{i+1}BE})\|_1 \leq 2\delta_{\omega, R}$$  

(5.47)

$$\leq k\delta_{\omega, R}.$$  

(5.48)

We can iterate this reasoning in the following way: For $j \in \{4, \ldots, k\}$ (assuming now $k \geq 4$), apply the maps $R_{E \rightarrow A_jE} \circ \text{Tr}_{A_{j-1}}$ along with monotonicity of trace distance to establish the following inequalities:

$$\|\bigcirc_{i=3}^j (R_{E \rightarrow A_iE} \circ \text{Tr}_{A_{i-1}})(\omega_{A_2BE}) - \bigcirc_{i=2}^3 (R_{E \rightarrow A_iE} \circ \text{Tr}_{A_{i-1}})(\omega_{A_{i+1}BE})\|_1 \leq \delta_{\omega, R}$$  

(5.49)
Figure 2: This figure illustrates the global state after performing a recovery map $k$ times on system $E$.

Apply the triangle inequality to all of these to establish the following inequalities for $j \in \{1, \ldots, k\}$:

$$
\| \omega_{ABE} - O_{i=2}^j (R_{E\rightarrow A_iE} \circ \text{Tr}_{A_{i-1}}) (\omega_{A_1BE}) \|_1 \leq k\delta_{\omega,R},
$$

(5.50)

with the interpretation for $j = 1$ that there is no map applied. From monotonicity of trace distance under quantum operations, we can then conclude the following inequalities for $j \in \{1, \ldots, k\}$:

$$
\| \rho_{AB} - \text{Tr}_E \left\{ O_{i=2}^j (R_{E\rightarrow A_iE} \circ \text{Tr}_{A_{i-1}}) (\omega_{A_1BE}) \right\} \|_1 \leq k\delta_{\omega,R}.
$$

(5.51)

Let $\gamma_{A_1A_2\ldots A_kBE}$ denote the following state:

$$
\gamma_{A_1A_2\ldots A_kBE} \equiv R_{E\rightarrow A_kE} (\cdots (R_{E\rightarrow A_2E} (\omega_{A_1BE}))).
$$

(5.52)

(See Figure 2 for a graphical depiction of this state.) Then the inequalities in (5.51) are equivalent to the following inequalities for $j \in \{1, \ldots, k\}$:

$$
\| \rho_{AB} - \gamma_{A_jB} \|_1 \leq k\delta_{\omega,R},
$$

(5.53)

which are in turn equivalent to the following ones for any permutation $\pi \in S_k$:

$$
\| \rho_{AB} - \text{Tr}_{A_2\ldots A_k} \left\{ W_{A_1A_2\ldots A_k}^\pi \gamma_{A_1A_2\ldots A_kB} (W_{A_1A_2\ldots A_k}^\pi)^\dagger \right\} \|_1 \leq k\delta_{\omega,R},
$$

(5.54)

with $W_{A_1A_2\ldots A_k}^\pi$ a unitary representation of the permutation $\pi$. We can then define $\overline{\gamma}_{A_1\ldots A_kB}$ as a symmetrized version of $\gamma_{A_1\ldots A_kB}$:

$$
\overline{\gamma}_{A_1\ldots A_kB} \equiv \frac{1}{k!} \sum_{\pi \in S_k} W_{A_1A_2\ldots A_k}^\pi \gamma_{A_1\ldots A_kB} (W_{A_1A_2\ldots A_k}^\pi)^\dagger.
$$

(5.55)
The inequalities in (5.54) allow us to conclude that
\[
k \delta_{\omega, \mathcal{R}} \geq \frac{1}{k!} \sum_{\pi \in S_k} \left\| \rho_{AB} - \text{Tr}_{A_2 \cdots A_k} \left\{ W_{\pi}^{\gamma} A_2 \cdots A_k \gamma A_1 A_2 \cdots A_k B \left( W_{\pi}^{\gamma} A_1 A_2 \cdots A_k \right) \right\} \right\|_1
\]
(5.56)
where the second inequality is a consequence of the convexity of trace distance. So what the reasoning in [WL12] accomplishes is to construct a \( k \)-extendible state \( \gamma_{A_1 B} \) that is \( k \delta_{\omega, \mathcal{R}} \)-close to \( \rho_{AB} \) in trace distance.

Following [WL12], we now recall a particular quantum de Finetti result in [CKMR07, Theorem II.7']. Consider a state \( \omega_{A_1 \cdots A_k B} \) which is permutation invariant with respect to systems \( A_1 \cdots A_k \). Let \( \omega_{A_1 \cdots A_n B} \) denote the reduced state on \( n \) of the \( k \) \( A \) systems when \( n \leq k \). Then, for large \( k \), \( \omega_{A_1 \cdots A_n B} \) is close in trace distance to a convex combination of product states of the form \( \int \sigma_A^{\otimes n} \otimes \tau (\sigma)_B \, d\mu(\sigma) \), where \( \mu \) is a probability measure on the set of mixed states on a single \( A \) system.

Let \( \mu \) be a probability measure on the set of mixed states on a single \( A \) system and \( \{ \tau (\sigma) \}_\sigma \) a family of states parametrized by \( \sigma \), with the approximation given by
\[
\frac{2 |A|^2}{k} \geq \left\| \omega_{A_1 \cdots A_n B} - \int \sigma_A^{\otimes n} \otimes \tau (\sigma)_B \, d\mu(\sigma) \right\|_1.
\]
(5.59)
Applying this theorem in our context (choosing \( n = 1 \)) leads to the following conclusion:
\[
\frac{2 |A|^2}{k} \geq \left\| \gamma_{A_1 B} - \int \sigma_A \otimes \tau (\sigma) B \, d\mu(\sigma) \right\|_1
\]
(5.60)

\[
\geq \left\| \gamma_{A_1 B} - \text{SEP}(A : B) \right\|_1,
\]
(5.61)
because the state \( \int \sigma_A \otimes \tau (\sigma) B \, d\mu(\sigma) \) is a particular separable state.

We can now combine (5.58) and (5.61) with the triangle inequality to conclude the following bound
\[
\left\| \rho_{AB} - \text{SEP}(A : B) \right\|_1 \leq \frac{2 |A|^2}{k} + k \delta_{\omega, \mathcal{R}}.
\]
(5.62)
By choosing \( k \) to diverge slower than \( \delta_{\omega, \mathcal{R}}^{-1} \), say as \( k = |A| \sqrt{2/\delta_{\omega, \mathcal{R}}} \), we obtain the following bound:
\[
\left\| \rho_{AB} - \text{SEP}(A : B) \right\|_1 \leq |A| \sqrt{8 \delta_{\omega, \mathcal{R}}}
\]
(5.63)
\[
= (512)^{1/4} |A| \varepsilon_{\omega, \mathcal{R}}^{1/4}.
\]
(5.64)
Since the above bound holds for all extensions and recovery maps, we can obtain the tightest bound by taking an infimum over all of them. By substituting with (5.40) and (5.41), we find that
\[
\left\| \rho_{AB} - \text{SEP}(A : B) \right\|_1 \leq (512)^{1/4} |A| \left( -1/2 \log F^{\text{sq}} (A ; B) \right)^{1/4},
\]
(5.65)
or equivalently
\[
E_F^{\text{sq}} (A ; B)_{\rho} = -1/2 \log F^{\text{sq}} (A ; B)_{\rho} \geq \frac{1}{512 |A|^4} \left\| \rho_{AB} - \text{SEP}(A : B) \right\|_1^4.
\]
(5.66)
This proves the converse part of the faithfulness of the geometric squashed entanglement. ■
Proposition 21 (Reduction to geometric measure) Let $\phi_{AB}$ be a bipartite pure state. Then

$$E^{sq}_F(A;B)_{\phi} = \frac{1}{2} \log \sup_{|\varphi\rangle} \langle \varphi | (\varphi_A \otimes \phi_B) | \phi \rangle_{AB}$$  \hspace{1cm} (5.67)

Proof. Any extension of a pure bipartite state is of the form $\phi_{AB} \otimes \omega_E$. Applying Proposition 11, we find that

$$F(A;B|E)_{\phi \otimes \omega} = F(A;B)_\phi$$  \hspace{1cm} (5.68)

$$= \sup_{\sigma_A} F(\phi_{AB};\phi_B \otimes \sigma_A)$$  \hspace{1cm} (5.69)

$$= \sup_{|\varphi\rangle} \langle \varphi | (\varphi_A \otimes \phi_B) | \phi \rangle_{AB}.$$  \hspace{1cm} (5.70)

The last equality follows due to a convexity argument applied to

$$F(\phi_{AB};\phi_B \otimes \sigma_A) = \langle \phi |_{AB} \phi_B \otimes \sigma_A | \phi \rangle_{AB}.$$  \hspace{1cm} (5.71)

Since the equality holds independent of any particular extension of $\phi_{AB}$, we obtain the statement of the proposition upon applying a negative logarithm and dividing by two. \hfill \qed

Proposition 22 (Normalization) For a maximally entangled state $\Phi_{AB}$ of Schmidt rank $d$,

$$E^{sq}_F(A;B)_{\Phi} = \log d.$$  \hspace{1cm} (5.72)

Proof. For any pure-state vector $|\varphi\rangle_A$, it follows from the so-called “transpose trick” for the maximally entangled state that

$$\langle \Phi |_{AB} (|\varphi\rangle_A \otimes \Phi_B) | \Phi \rangle_{AB} = \frac{1}{d} \langle \varphi^* |_B \Phi_B | \varphi^* \rangle_B$$  \hspace{1cm} (5.73)

$$= \frac{1}{d^2},$$  \hspace{1cm} (5.74)

from which the statement of the proposition follows by combining with the result of Proposition 21. \hfill \qed

Proposition 23 For a private state $\gamma_{ABA'B'}$ of log $d$ private bits, the geometric squashed entanglement obeys the following bound:

$$E^{sq}_F(A'A';BB')_{\gamma} \geq \log d.$$  \hspace{1cm} (5.75)

Proof. The proof is in a similar spirit to the proof of [Chr06 Proposition 4.19], but tailored to the fidelity of recovery quantity. Recall (3.4)-(3.7). Any extension $\gamma_{ABA'B'} E$ of a private state $\gamma_{ABA'B'}$ takes the form:

$$\gamma_{ABA'B'} E = U_{ABA'B'} (\Phi_{AB} \otimes \rho_{A'B'E}) U_{ABA'B'}^\dagger,$$  \hspace{1cm} (5.76)

where $\rho_{A'B'E}$ is an extension of $\rho_{A'B'}$. This is because the state $\Phi_{AB}$ is not extendible. Then consider that

$$F(A'A';BB'|E)_{\gamma} = \sup_{R} F(\gamma_{ABA'B'} E;R_{E \rightarrow AA'E} (\gamma_{BB'E} E)),$$  \hspace{1cm} (5.77)
where \( \mathcal{R}_{E \rightarrow A'A'E} \) is a recovery map. From (3.4)-(3.7), we can write

\[
\gamma_{ABA'B'E} = \frac{1}{d} \sum_{i,j} |i\rangle_A \langle i| \otimes |j\rangle_B \otimes V^i_{A'B'} \rho_{A'B'E} \left( V^j_{A'B'} \right)^\dagger, \tag{5.78}
\]

which implies that

\[
\gamma_{BB'E} = \frac{1}{d} \sum_i |i\rangle_B \otimes \text{Tr}_{A'} \left\{ V^i_{A'B'} \rho_{A'B'E} \left( V^i_{A'B'} \right)^\dagger \right\}. \tag{5.79}
\]

So then consider the fidelity of recovery for a particular recovery map \( \mathcal{R}_{E \rightarrow A'A'E} \):

\[
F(\gamma_{ABA'B'E}, \mathcal{R}_{E \rightarrow A'A'E} (\gamma_{BB'E})) = F(U_{ABA'B'} (\Phi_{AB} \otimes \rho_{A'B'E}) U^\dagger_{ABA'B'}, \frac{1}{d} \sum_i |i\rangle_B \otimes \mathcal{R}_{E \rightarrow A'A'E} \left( \text{Tr}_{A'} \left\{ V^i_{A'B'} \rho_{A'B'E} \left( V^i_{A'B'} \right)^\dagger \right\} \right)) \tag{5.80}
\]

\[
= F \left( (\Phi_{AB} \otimes \rho_{A'B'E}), U^\dagger_{ABA'B'} \left[ \frac{1}{d} \sum_i |i\rangle_B \otimes \mathcal{R}_{E \rightarrow A'A'E} \left( \text{Tr}_{A'} \left\{ V^i_{A'B'} \rho_{A'B'E} \left( V^i_{A'B'} \right)^\dagger \right\} \right) \right] U_{ABA'B'} \right), \tag{5.81}
\]

where the second equality follows from invariance of the fidelity under unitaries. Then consider that

\[
U^\dagger_{ABA'B'} \left[ \frac{1}{d} \sum_i |i\rangle_B \otimes \mathcal{R}_{E \rightarrow A'A'E} \left( \text{Tr}_{A'} \left\{ V^i_{A'B'} \rho_{A'B'E} \left( V^i_{A'B'} \right)^\dagger \right\} \right) \right] U_{ABA'B'}
\]

\[
= \left( I_A \otimes \sum_j |j\rangle_B \otimes \left( V^j_{A'B'} \right)^\dagger \right) \left[ \frac{1}{d} \sum_i |i\rangle_B \otimes \mathcal{R}_{E \rightarrow A'A'E} \left( \text{Tr}_{A'} \left\{ V^i_{A'B'} \rho_{A'B'E} \left( V^i_{A'B'} \right)^\dagger \right\} \right) \right] \times
\]

\[
\left( I_A \otimes \sum_{j'} |j'\rangle_B \otimes V^{j'}_{A'B'} \right) \tag{5.82}
\]

\[
= \frac{1}{d} \sum_i |i\rangle_B \otimes \left( V^i_{A'B'} \right)^\dagger \mathcal{R}_{E \rightarrow A'A'E} \left( \text{Tr}_{A'} \left\{ V^i_{A'B'} \rho_{A'B'E} \left( V^i_{A'B'} \right)^\dagger \right\} \right) V^i_{A'B'}. \tag{5.83}
\]
If we trace over systems $A'B'$, the fidelity only goes up, so consider that the state above becomes as follows under this partial trace:

$$\frac{1}{d} \sum_i |i\rangle \langle i|_B \otimes \text{Tr}_{A'B'} \left\{ (V_{A'B'}^i)^\dagger \mathcal{R}_{E \rightarrow AA'E} \left( \text{Tr}_{A'} \left\{ V_{A'B'}^i \rho_{A'B'E} \left( V_{A'B'}^i \right)^\dagger \right\} \right) V_{A'B'}^i \right\}$$

$$= \frac{1}{d} \sum_i |i\rangle \langle i|_B \otimes \text{Tr}_{A'} \left\{ \mathcal{R}_{E \rightarrow AA'E} \left( \text{Tr}_{A'} \left\{ V_{A'B'}^i \rho_{A'B'E} \left( V_{A'B'}^i \right)^\dagger \right\} \right) \right\}$$

(5.84)

$$= \frac{1}{d} \sum_i |i\rangle \langle i|_B \otimes \text{Tr}_{A'} \left\{ \mathcal{R}_{E \rightarrow AA'E} \left( \text{Tr}_{A'} \left\{ \rho_{A'B'E} \right\} \right) \right\}$$

(5.85)

$$= \frac{1}{d} \sum_i |i\rangle \langle i|_B \otimes \text{Tr}_{A'} \left\{ \mathcal{R}_{E \rightarrow AA'E} (\rho_E) \right\}$$

(5.86)

$$= \pi_B \otimes \mathcal{R}_{E \rightarrow AE} (\rho_E),$$

(5.87)

where $\pi_B$ is a maximally mixed state on system $B$. So an upper bound on (5.81) is given by

$$F (\Phi_{AB} \otimes \rho_E, \pi_B \otimes \mathcal{R}_{E \rightarrow AE} (\rho_E)) \leq F (\Phi_{AB}, \pi_B \otimes \mathcal{R}_{E \rightarrow A} (\rho_E))$$

(5.89)

$$= 1/d^2.$$  
(5.90)

Since this upper bound is universal for any recovery map and any extension of the original state, we obtain the following inequality:

$$\sup_{\gamma'_{AA'B'B'}: \gamma_{AA'B'B'} \equiv \mathcal{R}_{E \rightarrow AA'E}} F (AA'; BB'|E) \leq 1/d^2.$$  
(5.91)

After taking a negative logarithm, we recover the statement of the proposition. 

Proof. Let $\omega_{A_1B_1A_2B_2}$ be an extension of $\rho_{A_1B_1}$ and let $\tau_{A_2B_2E_2}$ be an extension of $\tau_{A_2B_2}$. Let $\mathcal{R}_{E_1 \rightarrow A_1E_1}$ and $\mathcal{R}_{E_2 \rightarrow A_2E_2}$ be recovery maps. Then

$$F (\rho_{A_1B_1E_1}, \mathcal{R}_{E_1 \rightarrow A_1E_1} (\rho_{B_1E_1})) \cdot F (\tau_{A_2B_2E_2}, \mathcal{R}_{E_2 \rightarrow A_2E_2} (\tau_{B_2E_2}))$$

(5.92)

$$= F (\rho_{A_1B_1E_1} \otimes \tau_{A_2B_2E_2}, \mathcal{R}_{E_1 \rightarrow A_1E_1} (\rho_{B_1E_1}) \otimes \mathcal{R}_{E_2 \rightarrow A_2E_2} (\tau_{B_2E_2}))$$

(5.93)

$$\leq \sup_{\omega_{A_1A_2B_1B_2}} \sup_{\mathcal{R}_{E \rightarrow A_1A_2E}} \left\{ F (\omega_{A_1A_2B_1B_2}, \mathcal{R}_{E \rightarrow A_1A_2E} (\omega_{B_1B_2E})): \rho_{A_1B_1} \otimes \tau_{A_2B_2} = \text{Tr}_E \left\{ \omega_{A_1A_2B_1B_2E} \right\} \right\}$$

(5.94)

$$= F_{\text{sq}} (A_1A_2; B_1B_2)_{\rho \otimes \tau}.$$  
(5.95)

Proposition 24 (Subadditivity / Supermultiplicativity) Let $\omega_{A_1B_1A_2B_2} \equiv \rho_{A_1B_1} \otimes \sigma_{A_2B_2}$. Then

$$E_{F}^{\text{sq}} (A_1A_2; B_1B_2)_{\omega} \leq E_{F}^{\text{sq}} (A_1; B_1)_{\rho} + E_{F}^{\text{sq}} (A_2; B_2)_{\sigma},$$

(5.96)
Since the inequality holds for all extensions \( \rho_{A_1B_1E_1} \) and \( \tau_{A_2B_2E_2} \) and recovery maps \( R_{E_1 \to A_1E_1} \) and \( R_{E_2 \to A_2E_2} \), we can conclude that

\[
F^\text{sq}(A_1; B_1)_{\rho} \cdot F^\text{sq}(A_2; B_2)_{\tau} \leq F^\text{sq}(A_1A_2; B_1B_2)_{\rho \otimes \tau}
\]  

(5.97)

By taking negative logarithms and dividing by 1/2, we arrive at the subadditivity statement for \( E_F^\text{sq} \).

**Proposition 25 (Continuity)** The geometric squashed entanglement is a continuous function of its input. That is, given two bipartite states \( \rho_{AB} \) and \( \sigma_{AB} \) such that \( F(\rho_{AB}, \sigma_{AB}) \geq 1 - \varepsilon \) where \( \varepsilon \in [0,1] \), then the following inequalities hold

\[
\left| F^\text{sq}(A; B)_{\rho} - F^\text{sq}(A; B)_{\sigma} \right| \leq 8\sqrt{\varepsilon},
\]  

(5.98)

\[
\left| E_F^\text{sq}(A; B)_{\rho} - E_F^\text{sq}(A; B)_{\sigma} \right| \leq 4|A|^2 \sqrt{\varepsilon}.
\]  

(5.99)

**Proof.** One of the main tools for our proof is the purified distance \[TCR10\, Definition 4\], defined for two quantum states as

\[
P(\rho, \sigma) \equiv \sqrt{1 - F(\rho, \sigma)},
\]  

(5.100)

and which for our case implies that

\[
P(\rho_{AB}, \sigma_{AB}) \leq \sqrt{\varepsilon}.
\]  

(5.101)

Letting \( \sigma_{ABE} \) be an arbitrary extension of \( \sigma_{AB} \), \[TCR10\, Corollary 9\] implies that there exists an extension \( \rho_{ABE} \) of \( \rho_{AB} \) such that

\[
P(\rho_{ABE}, \sigma_{ABE}) = P(\rho_{AB}, \sigma_{AB}) \leq \sqrt{\varepsilon}.
\]  

(5.102)

Let \( R_{E \to AE} \) be an arbitrary recovery map. Then the above and monotonicity of the purified distance under quantum operations \[TCR10\, Lemma 7\] imply that

\[
P(R_{E \to AE}(\rho_{BE}), R_{E \to AE}(\sigma_{BE})) \leq P(\rho_{ABE}, \sigma_{ABE}) \leq \sqrt{\varepsilon}.
\]  

(5.103)

So consider that the triangle inequality for purified distance \[TCR10\, Lemma 5\] implies that

\[
P(\rho_{ABE}, R_{E \to AE}(\rho_{BE})) \leq P(\rho_{ABE}, \sigma_{ABE}) + P(\sigma_{ABE}, R_{E \to AE}(\sigma_{BE})) + P(R_{E \to AE}(\sigma_{BE}), R_{E \to AE}(\rho_{BE}))
\]  

\[
\leq P(\sigma_{ABE}, R_{E \to AE}(\sigma_{BE})) + \sqrt{\varepsilon}
\]  

(5.104)

\[
= P(\sigma_{ABE}, R_{E \to AE}(\sigma_{BE})) + 2\sqrt{\varepsilon}.
\]  

(5.105)

This is equivalent to

\[
\sqrt{1 - F(\rho_{ABE}, R_{E \to AE}(\rho_{BE}))} \leq \sqrt{1 - F(\sigma_{ABE}, R_{E \to AE}(\sigma_{BE}))} + 2\sqrt{\varepsilon}
\]  

(5.107)

which upon squaring gives

\[
1 - F(\rho_{ABE}, R_{E \to AE}(\rho_{BE})) \leq 1 - F(\sigma_{ABE}, R_{E \to AE}(\sigma_{BE})) + 8\sqrt{\varepsilon},
\]  

(5.108)
where we used that $F(\rho, \sigma) \in [0, 1]$ and $\varepsilon \leq \sqrt{\varepsilon}$ for $\varepsilon \in [0, 1]$. This in turn implies the following inequality
\[ F(\rho_{ABE}, \mathcal{R}_{E\to AE}(\rho_{BE})) + 8\sqrt{\varepsilon} \geq F(\sigma_{ABE}, \mathcal{R}_{E\to AE}(\sigma_{BE})). \] (5.109)

By taking a supremum, we find that
\[ F^{sq}(A; B)_\rho + 8\sqrt{\varepsilon} \geq F^{sq}(A; B)_\sigma. \] (5.110)

Since the extension of $\sigma_{AB}$ and the recovery map $\mathcal{R}_{E\to AE}$ were arbitrary, it follows that
\[ F^{sq}(A; B)_\sigma + 8\sqrt{\varepsilon} \geq F^{sq}(A; B)_\rho, \] (5.112)

which gives us (5.98).

By dividing (5.111) by $F^{sq}(A; B)_\rho$ and taking a logarithm, we find that
\[
\log\left(\frac{F^{sq}(A; B)_\sigma}{F^{sq}(A; B)_\rho}\right) \leq \log\left(1 + \frac{8\sqrt{\varepsilon}}{F^{sq}(A; B)_\rho}\right)
\leq \frac{8\sqrt{\varepsilon}}{F^{sq}(A; B)_\rho},
\] (5.113)
\[
\leq |A|^2 8\sqrt{\varepsilon}. \] (5.115)

where we used that $\log (x + 1) \leq x$ and the dimension bound from Proposition [7]. Applying this to the other inequality in (5.112) gives that
\[
\log\left(\frac{F^{sq}(A; B)_\rho}{F^{sq}(A; B)_\sigma}\right) \leq |A|^2 8\sqrt{\varepsilon},
\] (5.116)

from which we can conclude (5.99) upon dividing both sides by 1/2. ■

6 Fidelity of measurement recovery

In this section, we propose an alternative measure of quantum correlations, the surprisal of measurement recoverability, which follows the original motivation behind the quantum discord [OZ01]. However, our measure has a clear operational meaning in the “one-shot” setting, being based on how well one can recover a bipartite quantum state if one system is measured. We begin by recalling the definition of the quantum discord and proceed from there with the motivation behind the newly proposed measure.

**Definition 26 (Quantum discord)** The quantum discord of a bipartite state $\rho_{AB}$ is defined as the difference between the quantum mutual information of $\rho_{AB}$ and the classical correlation [HV01] of $\rho_{AB}$:
\[
D(A; B)_\rho \equiv I(A; B)_\rho - \sup_{\{\Lambda^x\}} I(X; B)_{\sigma^x}
= \inf_{\{\Lambda^x\}} \left[ I(A; B)_\rho - I(X; B)_{\sigma^x} \right],
\] (6.1)
where \( \{ \Lambda^x \} \) is a POVM with \( \Lambda^x \geq 0 \) for all \( x \) and \( \sum_x \Lambda^x = I \) and \( \sigma_{XB} \) is defined as

\[
\sigma_{XB} \equiv \sum_x |x\rangle \langle x|_X \otimes \text{Tr}_A \{ \Lambda^x_A \rho_{AB} \}.
\] (6.3)

We now recall some developments from [BSW14] and [SBW14]. Let \( \mathcal{M}_{A \rightarrow X} \) denote the following measurement map:

\[
\mathcal{M}_{A \rightarrow X} (\omega_A) \equiv \sum_x \text{Tr} \{ \Lambda^x_A \omega_A \} |x\rangle \langle x|_X.
\] (6.4)

Using this, we can write (6.3) as \( \sigma_{XB} = \mathcal{M}_{A \rightarrow X} (\rho_{AB}) \). Now, to every measurement map \( \mathcal{M}_{A \rightarrow X} \), we can find an isometric extension of it with the following form:

\[
U^M_{A \rightarrow X E} |\psi\rangle_A \equiv \sum_x |x\rangle_X |x, y\rangle_E \langle \varphi_{x,y} | \psi\rangle_A,
\] (6.5)

where the vectors \( \{ |\varphi_{x,y}\rangle_A \} \) are part of a rank-one refinement of the POVM \( \{ \Lambda^x_A \} \):

\[
\Lambda^x_A = \sum_y \langle \varphi_{x,y} | \varphi_{x,y} \rangle.
\] (6.6)

Thus,

\[
\mathcal{M}_{A \rightarrow X} (\omega_A) = \text{Tr}_E \{ U^M_{A \rightarrow X E} (\omega_A) \},
\] (6.7)

where

\[
U^M_{A \rightarrow X E} (\omega_A) \equiv U^M_{A \rightarrow X E} (\omega_A) (U^M_{A \rightarrow X E})^\dagger.
\] (6.8)

Let \( \sigma_{XEB} \) denote the following state:

\[
\sigma_{XEB} = U^M_{A \rightarrow X E} (\rho_{AB}).
\] (6.9)

We can use the above development to rewrite the objective function of the quantum discord in (6.2) as follows:

\[
I (A; B) \rho - I (X; B) \sigma = I (XE; B) \sigma - I (X; B) \sigma
= I (E; B|X) \sigma.
\] (6.10)

So this means that we can rewrite the discord in terms of the conditional mutual information as

\[
D (\overline{A}; B) = \inf_{\{\Lambda^x\}} I (E; B|X) \sigma,
\] (6.12)

with the state \( \sigma_{XEB} \) understood as described above, as arising from an isometric extension of a measurement map applied to the state \( \rho_{AB} \). We are now in a position to define the surprisal of measurement recoverability:

**Definition 27 (Surprisal of measurement recoverability)** We define the following information quantity:

\[
D_F (\overline{A}; B) \rho \equiv \inf_{\{\Lambda^x\}} I_F (E; B|X) \sigma,
\] (6.13)

where we have simply substituted the conditional mutual information in (6.12) with \( I_F \). Writing out the right-hand side of (6.13) carefully, we find that

\[
D_F (\overline{A}; B) = -\log \sup_{U^M_{A \rightarrow X E}, \mathcal{R}_{X \rightarrow X E}} F (U^M_{A \rightarrow X E} (\rho_{AB}) , \mathcal{R}_{X \rightarrow X E} (\mathcal{M}_{A \rightarrow X} (\rho_{AB}))),
\] (6.14)

where \( \mathcal{M}_{A \rightarrow X} \) is defined in (6.4), \( U^M_{A \rightarrow X E} \) is defined in (6.5), and \( U^M_{A \rightarrow X E} \) is defined in (6.8).
This quantity has a similar interpretation as the original discord, as summarized in the following quote from [OZ01]:

“A vanishing discord can be considered as an indicator of the superselection rule, or — in the case of interest — its value is a measure of the efficiency of einselection. When [the discord] is large for any measurement, a lot of information is missed and destroyed by any measurement on the apparatus alone, but when [the discord] is small almost all the information about [the system] that exists in the [system–apparatus] correlations is locally recoverable from the state of the apparatus.”

Indeed, exploiting the invariance of fidelity under isometries gives a simple rewriting of (6.14):

\[
D_F(\rho_{AB}) = \log \sup_{U_{A\to XE}, R_{X\to XE}} F(\rho_{AB}, (U_{A\to XE})^\dagger (R_{X\to XE} (M_{A\to X} (\rho_{AB}))))
\]

(6.15)

\[
= \log \sup_{M_{A\to X}, P_{X\to A}} F(\rho_{AB}, (P_{X\to A} (M_{A\to X} (\rho_{AB}))))
\]

(6.16)

The last line follows the interpretation given in the quote above: the measurement map \(M_{A\to X}\) is performed on the \(A\) system of the state \(\rho_{AB}\), which is followed by a recovery map \(P_{X\to A}\) that attempts to recover the \(A\) system from the state of the measuring apparatus. Since the measurement map has a classical output, any recovery map acting on such a classical system is equivalent to a preparation map. So the quantity \(D_F(\rho_{AB})\) captures how difficult it is to recover the full bipartite state after some measurement is performed on it, following the original spirit of the quantum discord. However, the quantity \(D_F(\rho_{AB})\) defined above has the advantage of being a “one-shot” measure, given that the fidelity has a clear operational meaning in a “one-shot” setting. If \(D_F(\rho_{AB})\) is near to zero, then \(F(\rho_{AB}, (P_{X\to A} (M_{A\to X} (\rho_{AB}))))\) is close to one, so that it is possible to recover the system \(A\) by performing a recovery map on the state of the apparatus. Conversely, if \(D_F(\rho_{AB})\) is far from zero, then the measurement recoverability is far from one, so that it is not possible to recover system \(A\) from the state of the measuring apparatus.

Exploiting the observation that any entanglement breaking channel can be written as a concatenation of a measurement map with a preparation map [HSR03], we can rewrite \(D_F(\rho_{AB})\) once again, this time seeing that it captures how close \(\rho_{AB}\) is to being a fixed point of an entanglement-breaking channel:

\[
D_F(\rho_{AB}) = \log \sup_{\mathcal{E}_A} F(\rho_{AB}, \mathcal{E}_A (\rho_{AB}))
\]

(6.17)

where the optimization is over the convex set of entanglement breaking channels. This observation leads to the following proposition, which characterizes quantum states with discord nearly equal to zero.

**Proposition 28 (Approximate faithfulness)** A bipartite quantum state \(\rho_{AB}\) has quantum discord nearly equal to zero if and only if it is an approximate fixed point of an entanglement breaking channel \(\mathcal{E}_A\) and \(\varepsilon \in [0, 1]\) such that

\[
\|\rho_{AB} - \mathcal{E}_A (\rho_{AB})\|_1 \leq \varepsilon,
\]

(6.18)

then the quantum discord \(D(\rho_{AB})\) obeys the following bound

\[
D(\rho_{AB}) \leq 4h_2(\varepsilon) + 8\varepsilon \log |A|,
\]

(6.19)
where $h_2(\varepsilon)$ is the binary entropy with the property that $\lim_{\varepsilon \downarrow 0} h_2(\varepsilon) = 0$. Conversely, if the quantum discord $D(\overline{A};B)_\rho$ obeys the following bound for $\varepsilon \in [0,1]$:

$$D(\overline{A};B)_\rho \leq \varepsilon,$$

(6.20)

then there exists an entanglement breaking channel $E_A$ such that

$$\|\rho_{AB} - E_A(\rho_{AB})\|_1 \leq 2\sqrt{\varepsilon}.$$  

(6.21)

**Proof.** We begin by proving (6.18)-(6.19). Since any entanglement breaking channel $E_A$ consists of a measurement map $M_{A \rightarrow X}$ followed by a preparation map $P_{X \rightarrow A}$, we can write $E_A = P_{X \rightarrow A} \circ M_{A \rightarrow X}$. Then consider that

$$D(\overline{A};B)_\rho = I(A;B)_\rho - \sup_{\{\Lambda_X\}} I(X;B)_{\Lambda_X}$$

(6.22)

$$\leq I(A;B)_\rho - I(X;B)_{\Lambda_X(\rho)}$$

(6.23)

$$\leq I(A;B)_\rho - I(A;B)_{P\circ M(\rho)}$$

(6.24)

$$= I(A;B)_\rho - I(A;B)_{E(\rho)}$$

(6.25)

$$\leq 4h_2(\varepsilon) + 8\varepsilon \log |A|.$$  

(6.26)

The first inequality follows because the measurement given by $M_{A \rightarrow X}$ is not necessarily optimal. The second inequality is a consequence of the quantum data processing inequality, in which quantum mutual information is non-increasing under the local operation $P_{X \rightarrow A}$. The last equality follows because $E_A = P_{X \rightarrow A} \circ M_{A \rightarrow X}$. The last inequality is a consequence of the Alicki-Fannes inequality [AF04].

We now prove (6.20)-(6.21). The Fawzi-Renner inequality $I(A;B|C)_\rho \geq -\log F(A;B|C)_\rho$ which holds for any tripartite state $\rho_{ABC}$ [FR14], combined with other observations recalled in this section connecting discord with conditional mutual information, gives us that there exists an entanglement breaking channel $E_A$ such that

$$D(\overline{A};B)_\rho \geq -\log F(\rho_{AB},E_A(\rho_{AB}))$$

(6.27)

$$\geq -\log \left(1 - \frac{1}{4} \|\rho_{AB} - E_A(\rho_{AB})\|_2^2\right)$$

(6.28)

$$\geq \frac{1}{4} \|\rho_{AB} - E_A(\rho_{AB})\|_1^2,$$

(6.29)

where the second inequality follows from well known relations between trace distance and fidelity [VvdG98] and the last from $-\log (1 - x) \geq x$, valid for $x \leq 1$. This is sufficient to conclude (6.20)-(6.21).

**Remark 29** The main conclusion we can take from Proposition 28 is that quantum states with discord nearly equal to zero are such that they are recoverable after performing some measurement on one share of them, making precise the quote from [OZ01] given above. In prior work [Hay06, Lemma 8.12], quantum states with discord exactly equal to zero were characterized as being entirely classical on the system being measured, but this condition is perhaps too restrictive for characterizing states with discord approximately equal to zero.
Remark 30 In prior work, discord-like measures of the following form have been widely considered throughout the literature [MBC+12]:

\[
\inf_{\chi_{AB} \in \mathcal{C} \mathcal{Q}} \Delta (\rho_{AB}, \chi_{AB}),
\]

(6.30)

\[
\inf_{\chi_{AB} \in \mathcal{C} \mathcal{C}} \Delta (\rho_{AB}, \chi_{AB}),
\]

(6.31)

where \( \mathcal{C} \mathcal{Q} \) and \( \mathcal{C} \mathcal{C} \) are the respective sets of classical-quantum and classical-classical states and \( \Delta \) is some suitable (pseudo-)distance measure such as relative entropy, trace distance, or Hilbert-Schmidt distance. The larger message of Proposition 28 is that it seems more reasonable from the physical perspective argued in this section and in the original discord paper [OZ01] to consider discord-like measures of the following form:

\[
\inf_{\mathcal{E}} \Delta (\rho_{AB}, \mathcal{E}_A (\rho_{AB})),
\]

(6.32)

\[
\inf_{\mathcal{E}_A, \mathcal{E}_B} \Delta (\rho_{AB}, (\mathcal{E}_A \otimes \mathcal{E}_B) (\rho_{AB})),
\]

(6.33)

where the optimization is over the convex set of entanglement breaking channels and \( \Delta \) is again some suitable (pseudo-)distance measure as mentioned above.

We now establish some properties of the surprisal of measurement recoverability:

Proposition 31 (Invariance under local isometries) \( D_F (A; B)_\rho \) is invariant under local isometries, in the sense that

\[
D_F (A; B)_\rho = D_F (A'; B')_\sigma,
\]

(6.34)

where

\[
\sigma_{A'B'} \equiv (U_{A \rightarrow A'} \otimes V_{B \rightarrow B'}) (\rho_{AB})
\]

(6.35)

and \( U_{A \rightarrow A'} \) and \( V_{B \rightarrow B'} \) are isometric CPTP maps.

Proof. Let \( \mathcal{E}_A \) be some entanglement-breaking channel and let \( U_A \) and \( V_B \) denote the local isometries (for simplicity, we suppress the fact that they can map to systems of size different from their inputs). Then from invariance of fidelity under isometries, we find that

\[
F (\rho_{AB}, \mathcal{E}_A (\rho_{AB}))
= F \left( (U_A \otimes V_B) \rho_{AB} (U_A \otimes V_B)^\dagger, (U_A \otimes V_B) (\mathcal{E}_A (\rho_{AB})) (U_A \otimes V_B)^\dagger \right)
\]

(6.36)

\[
= F \left( (U_A \otimes V_B) \rho_{AB} (U_A \otimes V_B)^\dagger, U_A (\mathcal{E}_A \left( U_A^\dagger \left( (U_A \otimes V_B) \rho_{AB} (U_A \otimes V_B)^\dagger \right) U_A \right) U_A^\dagger \right)
\]

(6.37)

\[
\leq \sup_{\mathcal{E}_A} F \left( (U_A \otimes V_B) \rho_{AB} (U_A \otimes V_B)^\dagger, \mathcal{E}_A \left( (U_A \otimes V_B) \rho_{AB} (U_A \otimes V_B)^\dagger \right) \right)
\]

(6.38)

Since the inequality is true for any map \( \mathcal{E}_A \), we find after applying a negative logarithm that

\[
D_F (A; B)_\rho \geq D_F (A; B)_{\mathcal{U}_{A \otimes V} (\rho)}
\]

(6.39)

With essentially the same proof (by redefining \( \rho_{AB} \) to be \( (U_A \otimes V_B) \rho_{AB} (U_A \otimes V_B)^\dagger \) and taking \( U_A \otimes V_B \) to be \( (U_A \otimes V_B)^\dagger \)), we find that

\[
D_F (A; B)_{(U \otimes V) \rho (U \otimes V)^\dagger} \geq D_F (A; B)_\rho,
\]

(6.40)

which gives the statement of the proposition.
Proposition 32 (Exact faithfulness) The surprisal of measurement recoverability $D_F(\rho)$ is equal to zero if and only if $\rho_{AB}$ is a classical-quantum state, having the form

$$\rho_{AB} = \sum_x p_X(x) |x\rangle_A \langle x| \otimes \rho^x_B,$$

for some orthonormal basis \{\{|x\rangle\}\}, probability distribution $p_X(x)$, and states \{\rho^x_B\}.

Proof. Suppose that the state is classical-quantum. Then it is a fixed point of the entanglement breaking map $\sum_x |x\rangle_A \langle x| \otimes \rho^x_B$, so that the fidelity of measurement recovery is equal to one and its surprisal is equal to zero. On the other hand, suppose that $D_F(\rho_{AB}) = 0$. Then this means that there exists an entanglement breaking channel $E_A$ of which $\rho_{AB}$ is a fixed point (since $F(\rho_{AB}, E_A(\rho_{AB})) = 1$ is equivalent to $\rho_{AB} = E_A(\rho_{AB})$), and furthermore, applying the fixed point projection

$$\overline{E}_A \equiv \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^K E_A^k,$$

leaves $\rho_{AB}$ invariant. The map $\overline{E}_A$ has been characterized in [FNW14, Theorem 5.3] to be an entanglement breaking channel of the following form:

$$\overline{E}_A(\cdot) = \sum_i \text{Tr}\{M_i(\cdot)\} \sigma_i,$$

where the states $\sigma_i$ have orthogonal support. Applying this channel to $\rho_{AB}$ then gives a classical-quantum state, and since $\rho_{AB}$ is invariant under the action of this channel to begin with, it must have been classical-quantum from the start. \[\blacksquare\]

Proposition 33 (Dimension bound) The surprisal of measurement recoverability obeys the following dimension bound:

$$D_F(\rho_{AB}) \leq 2 \log |A|,$$

or equivalently,

$$\sup_{E_A} F(\rho_{AB}, E_A(\rho_{AB})) \geq \frac{1}{|A|^2}.$$

Proof. The idea behind the proof is to consider an entanglement breaking channel $E_A$ that traces out the input and replaces with the maximally mixed state. Furthermore, let $\psi_{ABC}$ be a purification of $\rho_{AB}$. We then find that

$$D_F(\rho_{AB}) \leq -\log F(\rho_{AB}, \pi_A \otimes \rho_B)$$

$$\leq -\log \left[\text{Tr}\left\{\sqrt{\rho_{AB}} \sqrt{\pi_A \otimes \rho_B}\right\}\right]^2$$

$$= \log |A| - \log \left[\text{Tr}\left\{\sqrt{\rho_{AB}} \sqrt{I_A \otimes \rho_B}\right\}\right]^2$$

$$= \log |A| - H_{1/2}(A|B)_{\rho}$$

$$= \log |A| + H_{3/2}(A|C)_{\psi}$$

$$\leq \log |A| + H_{3/2}(A)_{\rho}$$

$$\leq 2 \log |A|.$$
The reasons behind these steps are quite similar to those in the proof of Proposition 7, so we omit them.

By making use of the special form of the entanglement fidelity for a quantum channel (see, e.g., [Wil13, Theorem 9.5.1]), we arrive at the following form for $D_F(\overline{A};B)$ when evaluated for a pure state:

**Proposition 34 (Pure states)** Let $\psi_{AB}$ be a pure state. Then

$$D_F(\overline{A};B)_\psi = -\log \sup_{|\phi_x\rangle,|\varphi_x\rangle: \sum_x |\varphi_x\rangle \langle \varphi_x|=I} \sum_x |\varphi_x\rangle |\psi_A\varphi_x\rangle^2,$$  

(6.53)

where the optimization is over pure-state vectors $|\phi_x\rangle$ and corresponding measurement vectors $|\varphi_x\rangle$ satisfying $\sum_x |\varphi_x\rangle \langle \varphi_x|=I$.

**Proposition 35 (Normalization)** The surprisal of measurement recoverability $D_F(\overline{A};B)_\Phi$ is equal to $\log d$ for a maximally entangled state with Schmidt rank $d$.

**Proof.** The following bound is a consequence of [Rai99, Lemma 2]

$$F(\Phi_{AB}, \mathcal{E}_A(\Phi_{AB})) \leq \frac{1}{d}$$  

(6.54)

because $\mathcal{E}_A(\Phi_{AB})$ is a separable state. Since the bound holds for any entanglement breaking channel, we get

$$D_F(\overline{A};B)_\Phi \geq \log d.$$  

(6.55)

On the other hand, an entanglement breaking channel $\mathcal{E}_A$ achieving this bound is one that dephases system $A$ in the Schmidt basis of $\Phi_{AB}$, so that

$$F(\Phi_{AB}, \mathcal{E}_A(\Phi_{AB})) = \langle \Phi_{AB} | \left(\frac{1}{d} \sum_i |i\rangle \langle i| \otimes |i\rangle \langle i|_B\right) |\Phi_{AB}\rangle$$  

(6.56)

$$= \frac{1}{d}.$$  

(6.57)

So this implies that $D_F(\overline{A};B)_\Phi \leq \log d$, which concludes the proof.

**Proposition 36 (Monotonicity)** The surprisal of measurement recoverability is monotone with respect to quantum operations on the unmeasured system, i.e.,

$$D_F(\overline{A};B)_\rho \geq D_F(\overline{A};B')_{\sigma},$$  

(6.58)

where $\sigma_{AB'} \equiv \mathcal{N}_{B\rightarrow B'}(\rho_{AB})$.

**Proof.** Intuitively, this follows because it is easier to recover from a measurement when the state is noisier to begin with. Indeed, let $\mathcal{E}_A$ be an entanglement breaking channel. Then

$$F(\rho_{AB}, \mathcal{E}_A(\rho_{AB})) \leq F(\sigma_{AB'}, \mathcal{E}_A(\sigma_{AB'}))$$  

(6.59)

$$\leq \sup_{\mathcal{E}_A} F(\sigma_{AB'}, \mathcal{E}_A(\sigma_{AB'})),$$  

(6.60)
where the first inequality is due to the fact that $\mathcal{E}_A$ commutes with $N_{B \to B'}$ and monotonicity of the fidelity under quantum operations. Since the inequality holds for all entanglement breaking channels, we can conclude that

$$\sup_{\mathcal{E}_A} F (\rho_{AB}, \mathcal{E}_A (\rho_{AB})) \leq \sup_{\mathcal{E}_A} F (\sigma_{AB'}, \mathcal{E}_A (\sigma_{AB'})). \quad (6.61)$$

Taking a negative logarithm gives the statement of the proposition.

With a proof nearly identical to that for Proposition 25, we find that $D_F (\mathcal{F} ; \mathcal{B})$ is continuous:

**Proposition 37 (Continuity)** $D_F (\mathcal{F} ; \mathcal{B})$ is a continuous function of its input. That is, given two bipartite states $\rho_{AB}$ and $\sigma_{AB}$ such that $F (\rho_{AB}, \sigma_{AB}) \geq 1 - \varepsilon$ where $\varepsilon \in [0, 1]$, then the following inequalities hold

$$\left| \sup_{\mathcal{E}_A} F (\rho_{AB}, \mathcal{E}_A (\rho_{AB})) - \sup_{\mathcal{E}_A} F (\sigma_{AB}, \mathcal{E}_A (\sigma_{AB})) \right| \leq 8 \sqrt{\varepsilon}, \quad (6.62)$$

$$\left| D_F (\mathcal{F} ; \mathcal{B})_\rho - D_F (\mathcal{F} ; \mathcal{B})_\sigma \right| \leq |A|^2 8 \sqrt{\varepsilon}. \quad (6.63)$$

7 Multipartite fidelity of recovery

We state here that it is certainly possible to generalize the fidelity of recovery to the multipartite setting. Indeed, by following the same line of reasoning mentioned in the introduction (starting from the Rényi conditional multipartite information [BSW14 Section 10.1] and understanding the $\alpha = 1/2$ quantity in terms of several Petz recovery maps), we can define the multipartite fidelity of recovery for a multipartite state $\rho_{A_1 \cdots A_l C}$ as follows:

$$F (A_1; A_2; \cdots; A_l|C)_\rho = \sup_{\mathcal{R}_C \to A_l C, \mathcal{R}_{C_1} \to A_1 C, \cdots, \mathcal{R}_{C_l} \to A_{l-1} C} F \left( \rho_{A_1 \cdots A_l C}, \mathcal{R}_{C_1}^{1} \circ \cdots \circ \mathcal{R}_{C_l}^{l-1} \mathcal{R}_{C} \to A_{l-1} C (\rho_{A_l C}) \right). \quad (7.1)$$

The interpretation of this quantity is as written: systems $A_1$ through $A_{l-1}$ of the state $\rho_{A_1 \cdots A_l C}$ are lost, and one attempts to recover them one at a time by performing a sequence of recovery maps on system $C$ alone. We can then define a quantity analogous to the multipartite conditional mutual information as follows:

$$I_F (A_1; A_2; \cdots; A_l|C)_\rho \equiv - \log F (A_1; A_2; \cdots; A_l|C)_\rho, \quad (7.2)$$

and one can easily show along the lines given here for the bipartite case that the resulting quantity is non-negative, monotone under local operations, and obeys a dimension bound.

We leave it as an open question to develop fully a multipartite geometric squashed entanglement, defined by replacing the conditional multipartite mutual information in the usual definition [YHH+09] with $I_F$ given above. One could also explore multipartite versions of the surprisal of measurement recoverability.

8 Conclusion

We have defined the fidelity of recovery $F (A; B|C)_\rho$ of a tripartite state $\rho_{ABC}$ to quantify how well one can recover the full state on all three systems if system $A$ is lost and the recovery map
can act only on system $C$. By taking the negative logarithm of the fidelity of recovery, we obtain an entropic quantity $I_F(A;B|C)_{\rho}$ which obeys nearly all of the entropic relations that the conditional mutual information does (non-negativity, monotonicity under local operations, duality, and dimension bounds). The quantities $F(A;B|C)_{\rho}$ and $I_F(A;B|C)_{\rho}$ are rooted in our earlier work on seeking out R"enyi generationalsiations of the conditional mutual information [BSW14]. Whereas we have not been able to prove that all of the aforementioned properties hold for the R"enyi conditional mutual informations from [BSW14], it is pleasing to us that it is relatively straightforward to show that these properties hold for $I_F(A;B|C)_{\rho}$.

Another contribution was to define a geometric squashed entanglement measure $E_{\text{sq}}^F(A;B)_{\rho}$, inspired by the original squashed entanglement measure from [CW04]. We proved that $E_{\text{sq}}^F(A;B)_{\rho}$ is an entanglement monotone, is faithful, reduces to a variant of the well known geometric measure of entanglement [WG03] [CAH14], normalized on maximally entangled states, subadditive, and continuous. The new entanglement measure could find applications in “one-shot” scenarios of quantum information theory, since it is fundamentally a one-shot measure based on the fidelity.

Our final contribution was to define the surprisal of measurement recoverability $D_F(A;B)_{\rho}$, a quantum correlation measure having physical roots in the same vein as those used to justify the definition of the quantum discord. We showed that it is non-negative, invariant under local isometries, faithful on classical-quantum states, obeys a dimension bound, and is continuous. Furthermore, we used this quantity to characterize quantum states with discord nearly equal to zero, finding that such states are approximate fixed points of an entanglement breaking channel.

From here, there are several interesting lines of inquiry to pursue. Can we prove a stronger chain rule for the fidelity of recovery? If something along these lines holds, it might be helpful in establishing that the geometric squashed entanglement is monogamous. Can we use geometric squashed entanglement to characterize the one-shot distillable entanglement or secret key of a bipartite state? Is it possible to improve our continuity bounds? Can one show that geometric squashed entanglement is nonlockable [Chr06]? Preliminary evidence from considering the strongest known locking schemes from [FHS11] suggests that it might not be lockable. We are also interested in a multipartite geometric squashed entanglement, but we face similar challenges as those discussed in [LW14] for establishing its faithfulness.

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