We study the dynamics of quantum fluctuations which take place at the earliest stage of high-energy processes and the conditions under which the data from \( e-p \) deep-inelastic scattering may serve as an input for computing the initial data for heavy-ion collisions at high energies. Our method is essentially based on the space-time picture of these seemingly different phenomena. We prove that the ultra-violet renormalization of the virtual loops does not bring any scale into the problem. The scale appears only in connection with the collinear cut-off in the evolution equations and is defined by the physical properties of the final state. In heavy-ion collisions the basic screening effect is due to the mass of the collective modes (plasmons) in the dense non-equilibrium quark-gluon system, which is estimated. We avoid the standard parton phenomenology and suggest a dedicated class of evolution equations which describe the dynamics of quantum fluctuations in heavy ion collisions.

1. INTRODUCTION

In this study, we wish to approach the theory of high energy collisions of heavy ions based on a minimal number of first principles instead of using the elaborate technique of factorization-based scattering theory \(^1\). Our ultimate goal is to provide a description of the transition process which converts two initial-state composite systems (the stable nuclei in the “normal” nonperturbative vacuum) into the quark-gluon plasma (QGP), which is a dense system of quarks and gluons with a perturbative vacuum as its ground state. This scenario was suggested by Shuryak 20 years ago \(^2\) and it is our basic assumption that this phenomenon does indeed take place. Our main result is that this dense system can be formed only \textit{in a single quantum transition}. This transition is similar to the process described by the evolution equations of deep-inelastic scattering (DIS) and differs from it only by the scale parameter inherent to the final state. The limit of resolution for the emission process in DIS is connected with the minimal mass of a jet, while in the case of the QGP final-state, the scale is much smaller and defined by the screening properties of the QGP itself.

To pose this essentially quantum–mechanical problem properly, one (ideally) needs an exact definition of two main elements: the initial state of the system and the observables in the expected final state.

We know very little about the initial state, and thus must rely on the following phenomenological input only: The nuclei are stable bound states of QCD and therefore, their configuration is dominated by the stationary quark and gluon fields. The nuclei are well shaped objects; the uncertainty of their boundaries does not exceed the typical Yukawa interaction range. In the laboratory frame, both nuclei are Lorentz contracted to a longitudinal size \( R_0/\gamma \sim 0.1\,fm \).

The tail of the Yukawa potential is contracted in the same proportion. The world lines of the nuclei are the two opposite generatrices of the light-cone that has its vertex at the interaction point. Therefore, no interaction between the nuclei is possible before they overlap geometrically.

The final state is defined more accurately. We believe that single-particle distributions of quarks and gluons at some early moment after the nuclei have intersected, describe it sufficiently. Thus, we may rely on a reasonably
well-defined quantum observable. The corresponding operator should count the number of final-state particles defined as the excitations above the perturbative vacuum. To develop the theory for this transition process we have to cope with a binding feature that the “final” state has to be defined at a finite time. This may look disturbing for readers well versed in scattering theory.

Is it possible to compute the quantum-mechanical average of the operator that counts the final-state particles without precise knowledge of the wave function of the initial state? We prove that this is indeed possible, provided we have the data from a much simpler process, like $e^{-}p$ DIS. Moreover, we argue that this is possible even without constructing any model for the proton or colliding nuclei like, $e.g.$, the parton model. Such a possibility is provided solely by the inclusive character of the measurements in both cases. Indeed, the measurement of one-particle distributions is as inclusive as the measurement of the distribution of the final-state electron in DIS.

The key to our proof and to the suggested algorithm is the principle of causality in the quantum-mechanical measurement which has two major aspects. First, any statement concerning the time ordering must be in manifest agreement with the light cone boundaries. This is always guaranteed in the theory based on the relativistic wave equations. Second, the process of measurement physically interrupts the evolution of a quantum system, and any dynamical information about the quantum-mechanical evolution reveals itself only after the wave function is collapsed.

In order to implement these principles in a practical design of a theory, one has to start with a space-time description of the measurement, $i.e.$, the Heisenberg picture of quantum mechanics.

To give a flavor of how the method works practically, let us start with a qualitative description of the inclusive $e^{-}p$ DIS measurement (for now, at the tree level without discussion of the effects of interference). In this experiment, the only observable is the number of electrons with a given momentum in the final state. Something in the past has to create the electromagnetic field that deflects the electron. Before this field is created, the electromagnetic current, which is the source of this field, has to be formed. Since the momentum transfer in the process is very high, the current has to be sufficiently localized. This localization requires, in its turn, that the electric charges which carry this current must be dynamically decoupled from the bulk of the proton before the scattering field is created (to prevent a recoil to the other parts of the proton which could spread the emission domain). Such a dynamical decoupling of a quark requires a proper rearrangement of the gluonic component of the proton with the creation of short-wave components of a gluon field. By causality, corresponding gluonic fluctuation must happen before the current has decoupled, etc. Thus we arrive at the picture of the sequential-dynamical fluctuations which create an electromagnetic field probed by the electron. It is very important that the picture of the fluctuation is dynamical. The short-wave Fourier components of the quark and gluon fields are more typical of free propagation with high momentum, than for a smooth static design of a stable proton. The lifetimes of these fluctuations are very short and they all coherently add up to form a stable proton, unless the interaction of measurement breaks the proper balance of phases. This intervention freezes some instantaneous picture of the fluctuations, but with wrong “initial velocities” which results in a new wave function, and collapse of the old one. An example of such a fluctuation is depicted in Fig. 1, where the arrows point to the later moments of time.

This qualitative picture has been described many times and with many variations in the literature, starting with the pioneering lecture by Gribov [3], and including a recent textbook [4]; however, the sequential temporal ordering of the fluctuations has never been a key issue. We derive this ordering as a consequence of the Heisenberg equations of motion for the observables. The ordering appears to be universal only at the Born level. In general, the interference
effects break the sequential temporal ordering, unless the measurement prohibits the interference itself. We eventually arrive at a time-ordered picture which resembles the picture used by Lepage and Brodsky. The difference is that they applied time ordering exclusively in connection with the light-cone Hamiltonian dynamics, while our picture holds in any dynamics as long as the expansion in terms of two-point correlators makes sense. Qualitative arguments supporting the strong time ordering in fast hadrons were put forward earlier by Gribov. For our practical goals, such an ordering is significant for several reasons.

First, it makes the whole picture of the evolution causal and clearly indicates that the inclusively measured fluctuations are independent of the nature of the inclusive probe.

Second, the causality allows one to prove important dispersion relations. These relations provide a tool for the renormalization which is also subjected to causality, viz., there cannot be radiative corrections to a stable proton (or nucleus) configuration. This condition is physical and fully equivalent to the on-mass-shell renormalization of the asymptotic states in a standard scattering theory. Thus, we do not treat the intermediate quark and gluon fields as fundamental fields that have certain states. Only hadrons (in DIS) or collective modes (in dense matter) can be the states of emission.

Third, the causality in the inclusive measurement requires that the quark and gluon fields, which might exist after the measurement, are fully developed as the virtual fluctuations before the measurement. This is true both for freely-propagating hard quarks and gluons and for their final-state hadronic equivalents (jets). Therefore, the fluctuations have to satisfy the condition of emission. This condition requires that (i) the field of emission exists as a detectable state, and (ii) that two different states can be resolved. Unlike QED, QCD has an intrinsic criterion of resolution which requires that the states of emission in the physical vacuum must be hadrons. Thus, e.g., there cannot be a quark or gluon emission which carries away an invariant mass $m_0$ less than several hadronic masses. In other words, the modes with wave-length $\lambda > m_0^{-1} \sim 1 \text{Gev}^{-1}$ cannot propagate. The process of formation of the final-state jets has to be fully accomplished in a few $\text{fm}$. This general rule, however, must be changed if a quark or gluon becomes a part of a dense system, like the QGP, where the boundary of the soft (and hopefully still perturbative) dynamics
is determined by a parameter similar to the plasmon mass (or the Debye screening length) and not a hadronic scale \( \Lambda^2_{QCD} \). This condition selects fluctuations that are materialized as the QGP after the coherence of the nuclei is broken.

Fourth, our analysis, based on the causal picture of the evolution, indicates that the only scale that may appear in the theory is connected to the physical properties of the final states. This scale penetrates into the theory only in connection with the infra-red (collinear) problem but \textit{not} in connection with ultra-violet renormalization. This observation allows one to outline the scenario of formation of the dense matter at the first 1 fm of a nuclear collision as follows: The nuclei probe each other locally at all scales which are consistent with the spectrum of the allowed final states. The most localized interactions create the final-state fields with the highest \( p_t \) which are the least sensitive to the collective properties of the final state. (A similar process takes place in DIS at sufficiently high \( p^2 \gg m^2_{hadr} > \Lambda^2_{QCD} \).) These states are formed as (quasi-) particles by the time \( \tau \sim 1/p_t \). In this domain, the structure functions and their rates of evolution must be very similar for both nuclear collisions and DIS. Therefore, the DIS data obtained with precise control of the momentum transfer may serve as an input for the calculation of the nuclei collision. The states with lower \( p_t \) are formed at later times. This does not much affect the QCD evolution in DIS, where the phase-space of the final states is almost unpopulated and only the shape of the jets may lead to the power corrections. In the case of nuclei collisions this is not true, as the low–\( p_t \) states are formed in a space which is already populated with the earlier formed (virtually decoupled) fast particles. These fast particles produce screening effects. The process of the formation of the new states saturates at \( p_t \sim m_D \), where \( m_D \) is an effective mass of partons in the (highly non-equilibrium) collective system formed by the time \( \tau_D \sim 1/m_D \). To quantify this scenario one has to derive evolution equations that respect specific properties of the final states in the QGP. A draft of such a derivation is given in the last two chapters of the paper.

To conclude, the standard inclusive e-p DIS delivers information about quantum fluctuations which may dynamically develop in the proton. We do not need to know the “proton’s wave function” (since the proton is a true fundamental mode of QCD, this notion is not even well defined). The theory can be built on two premises: causality, and the condition of emission. The latter is also known as the principle of cluster decomposition which must hold in any reasonable field theory. What the “resolved clusters” are is a very delicate question. These states should be defined with an explicit reference as to how they are detected. Conventional detectors deal with hadrons and allow one to hypothesize about jets. QGP turns out to be a kind of collective detector for quarks and gluons.

A vast search for a scale in ultra-relativistic nuclear collisions has been undertaken by McLerran \textit{et al.} This study started with the formulation of the \textit{model} for a large nucleus as a system of valence quarks moving at the speed of light (McLerran–Venugopalan model \cite{McLerran}). The original idea of this approach was that the scale is associated with the density of valence quarks, that is, with the parameters of the initial state of the nuclear collision. The model was gradually improved \cite{McLerran, Venugopalan} by the accounting for the radiative corrections with emphasis on the domain of small \( x_F \) and a derivation of the BFKL equation, which is known to have no scale parameter.

The idea that screening effects should be taken into account, even at the earliest moments of a collision of two nuclei has been articulated earlier and with different motivations by Shuryak and Xiong \cite{Shuryak} and by Eskola, Muller and X.-N. Wang \cite{Eskola}. In Ref. \cite{Shuryak}, the cut-off mass (expressed via the temperature) was introduced to screen the singularities of the cross-section of the purely gluonic process, \( gg \leftrightarrow ng \), in the chemically non-equilibrated plasma, \( m^2 \sim g^2 n_{gluon}/T \). In Ref. \cite{Eskola}, the non-equilibrium Debye screening was introduced in order to prevent the singular infrared behavior of the hard part of the factorized cross-section of the mini-jet production. The expression for the
screening mass in \[ \text{[8]} \] is very similar to our equation \( \text{[10.16]} \). However, in all of these papers, the origin of the structure functions of the colliding nuclei is not discussed, and there still remains an uncertainty connected with the choice of the factorization scale. We go one step further and prove that the screening parameters of the collective system determine the collinear cut-off self-consistently; these screening parameters are \textit{already in the evolution equations} for the structure functions of individual nucleons (and therefore, in the nuclear structure functions).

\section*{II. OUTLINE OF CALCULATIONS, MAIN RESULTS, AND CONCLUSION}

The QCD evolution equations are the main focus of this study. Historically, their development started along two qualitatively different lines which eventually merged into what is currently known as the DGLAP equations. Despite their common name, there are still certain concerns whether the marriage is happy. The one-to-one correspondence between the DGL \[ \text{[10]} \] and the AP \[ \text{[11]} \] equations is evident only at the Born level (without the virtual radiative corrections). A discussion of what happens in the higher orders and particularly, what the argument of the running coupling is, still continues (see, e.g. Refs. \[ \text{[12,13]} \]). The variations appear due to \textit{ab initio} different approaches. While the AP equations re-interpret the equations of the operator product expansion (OPE) based on the factorization and the renormalization group (RG) for \( e-p \) DIS, the DGL equations are derived as a probabilistic picture of the multiple process. The AP equations just \textit{accept} the running coupling of the RG for the exclusive Feynman amplitudes, while in the DGL scheme one has to calculate the radiative corrections as constituents of a real process explicitly and assemble them into an effective quantity (the running coupling).

The first claim about a closer correspondence between the DGL and the AP equations, at least in one loop beyond the Born approximation, was made in Ref. \[ \text{[14]} \] (widely known as DDT) almost 20 years ago. The ultimate goal was to recover the running coupling in the natural environment of the QCD evolution ladder by collecting various radiative corrections into an effective form-factor instead of relying (as does the OPE method) on the RG for the Feynman amplitudes of the S-matrix scattering theory. This program was a success only for one subgroup of ladder diagrams, \textit{vis.}, when the gluon is an offspring of the fermion line and the dynamics is driven solely by the splitting kernel \( P_{qq} \).

This was a smart choice, and success was almost predetermined by the well known fact that in physical gauges the renormalization of the coupling is fully defined by the gluon propagator. Nevertheless, the DDT study did mighty battle with the double-logarithmic terms which naturally appear in radiative corrections. It was a formidable task to show that the running coupling depends on the highest transverse momentum in the splitting point. Neither DDT, nor anybody else since that time, attempted to do the same for other types of splittings, though it is not obvious if the same result can be obtained for the coupling constant which accompanies the kernel \( P_{qg} \) (when a single gluon line carries the lowest of three transverse momenta). We intend to bridge this gap and assemble the running coupling constant of all other kernels. The kernel \( P_{gg} \) (when all three lines are gluons) is of extreme interest since one may expect competition between three gluon self-energies with all different momenta. The case of pure glue-dynamics provides an ultimate test for the understanding of the dynamics of QCD evolution. We address it first, and use it later as a touchstone for all the tools we design.

For the sake of completeness, we begin in Sec. \[ \text{[12]} \] with the basics of the so-called Keldysh-Schwinger technique \[ \text{[15]} \], in the form adjusted for the calculations of the inclusive observables \[ \text{[16]} \]. The net yield of this section is summarized in Eq. \[ \text{[3.36]} \] for the fixed-order two-point field correlators. This equation shows that the correlator \( D(x, y) \) has two
contributions: from the initial data, \(D(\xi, \eta)\), and the source, \(\Pi(\xi, \eta)\), acting at an earlier time. In Sec. III C we discuss the connection between Eq. (3.36) and the spectral representation of field correlators and explain why the initial data can be dropped. In Sec. IV these equations are used to derive the equations that lead to the picture in Fig. 1 and prove that the photon of e-p DIS must be treated as part of the proton.

We also examine (in Sec. IV) what types of fluctuation are active in the e-p DIS which is inclusive with respect to a quark jet (without observation of the final-state electron). In this problem, the photon “belongs” to the initial-state electron and the measurement is affected both by QED and QCD fluctuations which may develop in the initial-state electron.

In Sec. V we derive the evolution equations of glue-dynamics yet without the radiative corrections. Even at this level, we discover the second independent element of the evolution, \(\Pi\), the longitudinal fields, which are absent both in the DGL and the AP equations. The reason is that, in our derivation, the evolution equation naturally appear in a tensor form \(V_0\). Projecting this equation onto the normal modes we obtain two interconnected evolution equations, (5.4) and (5.5), for the transverse and longitudinal (static, non-propagating) gluon modes. This should not be considered a surprise. Indeed, it is well known that the photon on the top of the evolution ladder (just proved to be a part of the proton) has transverse and longitudinal polarizations which are probed by the structure functions \(F_2\) and \(F_L\), respectively. It is not clear, \(a\ priori\), why the gluon field should be only transverse. If the longitudinal field is eliminated from Eqs. (5.4) and (5.5) by a fiat, the remaining single equation is exactly the DGLAP evolution equation of glue-dynamics. Its kernel is singular, and it is common to regulate it with the aid of the (+)-prescription. In Ref. [18] this prescription which incorporated self-energy corrections (in the fixed coupling approximation) was derived. For a similar singularity in Eq. (5.5) this method does not work. Moreover, it is not clear what kind of coupling accompanies the transition between the longitudinal and transverse fields in the evolution equations. Potential consequence of this new-type of infrared divergence might be a disaster since the standard remedies of scattering theory are not designed to handle any problems related to the longitudinal fields. In scattering theory the latter are never treated as observables. The specifics of heavy-ion collisions are different because the observables must be defined at a finite time of evolution. Unlike scattering theory, the longitudinal fields cannot be absorbed into the definition of the final states. Before looking for a cure, we must make sure that this is not a false alarm, \(e.g.\) perhaps the infrared divergences do not cancel between the different rungs of the emission ladder. Thus, we need to study the full set of radiative corrections in the order \(g^4\).

A full set of radiative corrections includes different types of diagrams. Two diagrams in Fig. 5 are virtual vertex corrections. Their vertex loops are formed by three \(T\)-ordered propagators, \(D_{00}\), in \(V_{000}\) (diagram (a)) and three \(T^\dagger\)-ordered propagators, \(D_{11}\), in \(V_{111}\) (diagram (b)). Even though all propagators which are connected to these vertices are retarded, the vertex loops have \(a\ priori\) arbitrary ordering of their space-time arguments. This may hurt the nice causal picture of the lowest order approximation. Our main observation is that the net yield of the sum of these two diagrams does not change if the vertex \(V_{000}\) is replaced by the retarded vertex \(V_{ret}\) (diagram (c)) and the vertex \(V_{111}\) is replaced by the advanced vertex \(V_{adv}\) (diagram (d)). In this formulation, the picture of the evolution remains

\[1\] It was Jianwei Qiu [17], who first paid attention to the special role of the static part of the quark and gluon propagators (he called them \textit{special} or \textit{contact} propagators) in computing the twist-4 corrections to the DIS process. However, the main emphasis of this study was on reproducing the high-twist effects in the perturbative part of the OPE-expansion.
totally causal even in the order $g^4$.

The retarded properties of the new vertex functions in coordinate space can be translated into the analytic properties in momentum space. To comply with the light-cone dynamics and the light-cone gauge $A^+ = 0$, we must treat the coordinate $x^+$ as the temporal variable. The conjugated variable is the light-cone energy $p^-$. We prove that the vertex function $V_{ret}(p,k-p,-k)$ is analytic in the upper half-plane of complex $p^-$. The light-cone energy $p^-$ corresponds to the latest momentum $p^\mu$ resolved in the course of the evolution (which has the lowest component $p^+$ of the light-cone momentum and is, in fact, probed in the inclusive measurement). These results are obtained in Sec. VI B and Appendix 1. Knowing the analytical properties of the vertex function we are able to prove the dispersion relations which allow one to find the real part of the vertex via its imaginary part. It occurs that all singularities which may affect the final answer are collinear and already appear in the calculation of the imaginary part which is connected with real processes. Thus, these singularities are physical and do not change in the course of a calculation of the dispersion integrals. Besides, they are always connected with emission into the final states and never with the absorption from the initial states.

The dispersive method provides a tool for the UV-renormalization of the vertex function, viz., the subtractions in dispersion relations. These subtractions must be made at a value $p^- = -\Omega$ of the light-cone energy in the domain where the imaginary part is zero. This domain occurs to be limited to a finite segment of the real axis in the complex plane of $p^-$. The position $p^- = -\Omega$ of the subtraction point can be transformed into the value of $p^2$, viz., $p^2 = -\mu^2 = -p^+\Omega - p_t^2$. Under the assumption of strong ordering of the transverse momenta, $p_t >> k_t$, the allowed values of $\mu^2$ are $0 < \mu^2 < p_t^2$. Even though this limitation is a direct consequence of causality, it may seem abnormal from the RG point of view. Indeed, we cannot take an asymptotically high renormalization point in $p^2$. However, in a causal picture of gradually increasing resolution along the line of evolution, this limitation is absolutely natural. It is illegal to renormalize what is not yet resolved!

If we proceed with renormalization at an arbitrary scale $\mu^2$, then the calculation of the vertex function leads to the terms $\log^2(p_t^2/\mu^2)$ along with $\log(p_t^2/\mu^2)$. The double-logarithms find no counterparts among either virtual loops or real emission processes. Therefore, we immediately undermine the leading logarithmic approximation (LLA) in which the DGLAP equations are derived (see Eq. (7.6)). Besides, in order to have $\log(p_t^2/\mu^2) >> 1$ one should choose a very low renormalization scale $\mu^2$ which would contradict all beliefs about perturbative QCD. These problems compel us to look for a physically-motivated condition for the renormalization. Surprisingly, this condition is readily found at the opposite end of the allowed interval. If we take $\mu^2 = p_t^2$ which corresponds to the subtraction point at $p^- = 0$, then all of the above problems miraculously disappear. The vertex function, as well as all retarded self-energies (also calculated via dispersion relations), become free of logarithms, $\log(p_t^2/\mu^2)$, which originate from ultraviolet renormalization (where they are most expected to come from). Technically, the toll is that we have to look for the large logarithms in real processes. However, from the physical point of view, this is a gain rather than a loss, since we get rid of the logarithms which are just an artifact of the renormalization prescription. In fact, we can promote the status of the running coupling; instead of being a formal attribute of the renormalization group it may become a dynamical form-factor due to the real processes. Before we search for these large logarithms, let us try to understand why the subtraction point $p^- = 0$ is a natural physical point for the renormalization.

According to the principle of causality, the full picture of the QCD evolution develops before the measurement. Thus, the system of fields must be coherent. It represents, at most, virtual decomposition of a hadron in terms of an
unnatural set of modes which emerge as propagating fields only after the interaction happens. Therefore, the radiative corrections to the propagation of fields should not cause any phase shifts along the line of hadron propagation and hence, the real part of virtual loops should vanish for the state of free propagation. This is exactly what is achieved by the on-mass-shell renormalization of the asymptotic states in scattering theory. If the hadron is assumed to move along the light cone, the real parts of the vertices and self-energies should vanish at $p^- = 0$.

Contributions $(7.9)$ and $(7.10)$ of the retarded gluon self-energy and vertex (both renormalized at the point $p^- = 0$) to the evolution equation are double-logarithmic with respect to the collinear cut-off $\epsilon$. This cut-off already appears in the imaginary parts of the self-energy and vertex. Therefore, it is connected with real processes. Even if the divergent terms do cancel, one still has to understand why. The physical mechanism of this cancelation can only be due to interference, and it has to work even at the classical level. The collinear divergence is both a classical effect and an artifact of theoretical model with massless charged particles. Therefore, it should not be treated as an artificial phenomenon caused by spurious poles of a special gauge. Indeed, the physical mechanism of the collinear divergence can be easily translated into the language of space-time.

Consider a charged particle that instantaneously changes its velocity from $\mathbf{v}_1$ to $\mathbf{v}_2$ at a time $t = 0$. At some later time $t$, the picture of the field is as follows. Outside the sphere of radius $R = t$, there is an unchanged static proper field of the initial-state particle. Inside this sphere, there is the newly created proper field of the final-state particle. The field of radiation (which has the old and the new static fields as the boundary conditions) is located on the sphere itself. If $v_2 < 1$ then the charge is inside the sphere and decoupled from the radiation field. At $v_2 = 1$, the charge never decouples from the radiation field and this leads to the divergence in the integration over the time of the interaction. Thus, assigning a finite mass $m$ to the field is a test for true collinear divergence. This is also a cure, since then $v_2 < 1$ and the problem disappears. Alternatively, one may provide at least a single interaction with a third body which interrupts the emission process after some time $\Delta t$. In both cases, we encounter a dimensional physical parameter which regulates the time-length of interaction between the charge and the radiation field. If $m \neq 0$, then the smallest of two quantities, $\Delta t$ and $m^{-1}$, is the physical cut-off for the emission process. Let us assume that some mass $m_0$ can be introduced as a collinear cut-off. If this cut-off cancels in the full assembly of diagrams in any given order, then the theory has no single dimensional parameter. This is exactly the case of pQCD where the parameter $\Lambda_{QCD}$ establishes, at most, an absolute boundary of its validity. All practical calculations are possible only for momenta higher than some $Q_0 \gg \Lambda_{QCD}$. All the divergences, including soft and collinear, are usually removed by means of the dimensional regularization with reference to the KLN theorem; however, this is not the case we address.

In Sec. VIII, we calculate contributions of real emission processes to the causal QCD evolution. The diagrams in Figs. 8a and 8b, with the unitary cut through the vertex, and the interference diagram in Fig. 8c, do not lead to terms with $\log(p_1^2)$. They, however, include many terms with the collinear cut-off which eliminates a part of phase-space which does not satisfy the emission criterion. In Sec. IX we study the radiative corrections in the LT-transition mode. They confirm the general trend.

Thus, we have proved that in the time-ordered picture of evolution the UV-renormalization should be performed in such a way that radiative corrections do not affect the initial state and bring no scale into the problem. The scale is brought in by the emission process and is defined by parameters of the final state that emerge after the collision. The most important conclusion that follows from this observation is that the evolution equations must be modified, \textit{viz.}, the universal running coupling that accompanies splitting kernels in the DGLAP equations must be exchanged.
for the form-factors which depend on the type and scale of the emission process. This is required by causality of the inclusive measurement.

It is high time to understand why we need a theory with a physical cut-off of the collinear emission and why we cannot rely on the standard approach of pQCD. The latter was constructed as a theory of the S-matrix at asymptotic energies. The concept of asymptotic freedom is formulated precisely in this context. The running coupling is always introduced as a certain assembly of radiative corrections regardless of whether the perturbation series is divergent. In pQCD, the way to assemble vertices and self-energies is defined by the structure of Feynman amplitudes of exclusive processes, and it is very important that the collinear (mass) singularities are, at any price, removed from this assembly. The most popular technical tool to do this is based on dimensional regularization of the Feynman amplitudes. This method does not distinguish between UV-, IR-, and mass-singularities, despite the different physical nature of these divergences. Sometimes, one may trace that cancelations of certain divergences, are due to interference of various partial amplitudes. In other cases, one may observe that the soft amplitudes exponentiate and die out in the quantities like total cross-sections. The underlying calculations always implicitly rely on the structure of the space of states, which includes *arbitrarily soft* states of emission (otherwise, the exponentiation makes no sense). Such a space of states is completely physical in QED but not in realistic QCD. Usually, the collinear singularities are forced to cancel with the aid of the KLN theorem which redefines the states of scattering by admixing a probabilistic distribution of soft (collinear) states. The proof of the KLN theorem is intimately connected with the definition of the states, *vis.*, it is sensitive to the parameter of resolution, which is hidden in the definition of the degenerate states. Once again, the theorem was derived under the influence of QED where even very soft photons can be resolved in the asymptotic states, provided a sufficiently long time is allowed for the measurement. To approach this idealized regime in QCD, one has to limit the theoretical analysis to exclusive processes with very energetic final-state jets (insensitive to several additional soft hadrons which might simulate the emission of extra soft gluons). However, even in QED there are cases when the recipe of KLN is not supposed to work. For instance, in the QED plasma, the space of states is populated, and we are not free to assign arbitrary statistical weights to the collinear states.

In nuclear collisions, the space of states has a quite different structure. Unlike a scattering problem, the states must be defined at a finite time of evolution. Furthermore, the longitudinal fields cannot be absorbed into the definition of stable charged particles (by means of on-mass-shell renormalization) and are not geometrically separated from the fields of emission. Nevertheless, the technical problems are not frightening, since the normal modes in dense and exited (hot) nuclear matter (to the first approximation) can be computed perturbatively. In Sec. X, we compute the mass of a transverse plasmon (a collective mode), which is now the final state for the emission in the QCD evolution process. The distribution of gluons that induce a plasmon mass is defined by the process of multiple-gluon production, and the plasmon mass is the function of the time of the distribution measurement. The result of this calculation clarifies the correspondence between the two approaches. It is summarized by Eq. (10.14): The larger the transverse momentum $p_t$ of the emission is, the earlier it is formed as the radiation field, the less it is affected by the interaction with other partons, and the smaller its effective mass is. This is exactly the limit when the OPE-based calculations of DIS and the here advocated self-screened evolution should merge. The waves with a very large $p_t$ just do not “see” the medium and have to hadronize into jets as if there was no medium. In order to quantify this observation, one has to estimate the corrections to the LLA on both sides. These corrections lead to the broadening of jets in DIS and radiation losses of fast partons in AA-collisions.
Formula (10.14) was derived in three steps. We started (in Sec. X A) with an attempt to compute the mass of the transverse plasmon in the null-plane dynamics, which we essentially relied on in the derivation of the evolution equations. It turns out that the dispersive part of the retarded self-energy is always proportional to $p^2$ and may contribute only to the renormalization of the propagator. At first glance, the non-dispersive part seems to be capable of generating the effective mass of the plasmon. However, even in the case of a finite density of emitted gluons, the non-dispersive part remains quasi-local because of the singular nature of longitudinal fields in the null-plane dynamics, and it vanishes in the course of renormalization. We conclude that it is impossible to generate an effective mass of a quasi-particle in the null-plane dynamics.

In order to smooth out the geometry of the longitudinal fields, we had to choose another Hamiltonian dynamics which is described in Sec. X B. This new dynamics uses the proper time $\tau$, $\tau^2 = t^2 - z^2$, as the time variable. For each slice in rapidity, $\tau$ is just the local time of the co-moving reference frame. An advantage of this dynamics is that it naturally incorporates the longitudinal velocity of a particle as one of its quantum numbers and visualizes the process of the particle formation by the time $\sim 1/pt$. In its global formulation, the proper-time dynamics uses the gauge $A^\tau = 0$. The physical mechanisms of the screening effects (like the generation of the plasmon mass) are known to be localized in a finite space-time domain. This understanding allowed us to use the gauge $A^\tau = 0$ in a local fashion. In order to estimate the plasmon mass we approximated this gauge by the temporal-axial gauge (TAG) $A^0 = 0$ of the local (in rapidity) reference frame.

An estimate of the plasmon mass is accomplished in Sec. X C. Once again, in local TAG only the non-dispersive part of the retarded self-energy (in which the propagation is mediated by the longitudinal field) and the tadpole term (with the inertia-less contact interaction) lead to the plasmon mass, which is given by Eq. (10.14). The low-$pt$ mode of the radiation field acquires a finite effective mass as a result of its forward scattering on the strongly localized (and formed earlier) particles with $qt \gg pt$. The smaller $pt$ is, the larger the plasmon mass is. Thus, the whole process, that converts the nuclei into the dense system of (yet non-equilibrium) quarks and gluons, must saturate. In this way, the scenario approaches the end of the "earliest stage", i.e., the dense system is already created and the collisions between the partons-plasmons start to take over the dynamics of the quark-gluon system.

To derive the evolution equations with screening, we had to find a space-time domain where the proper-time dynamics is close to the null-plane dynamics. The limit is found at large negative rapidity, viz., in the photon fragmentation region. In the last section of the paper, we show that when the effective mass of the plasmon is used as the pole mass of the radiation field in the evolution equations, the evolution equations become regular. The collinear singularity of the splitting kernel $P_{gg}$ as well as a similar singularity in the LT-transition mode are screened by the plasmon mass. Thus, we obtain a simple version of the evolution equations with screening, which can be applied to the fluctuations in the photon fragmentation region. They become the DGLAP equations when $s \to \infty$. In order to obtain the evolution equations valid in the central-rapidity region, we have to consider the whole problem in the scope of the “wedge dynamics” [20,21]. The last thing we estimate within our “local approximation” is that the plasmon mass leads to strong suppression of all radiative corrections in the order $\alpha_s^2$.

The net yield of this paper is as follows. The interaction between the two ultrarelativistic nuclei switches on almost instantaneously. This interaction explores all possible quantum fluctuations which could have developed by the moment of the collision and freezes (as the final states) only the fluctuations compatible with the measured observable. These snapshots cannot have an arbitrary structure, since the emerging configurations must be consistent
with all the interactions which are effective on the time-scale of the emission process. In other words, the modes of the radiation field which are excited in the course of the nuclear collision should be the collective excitations of the dense quark-gluon system. This conclusion is the result of an intensive search of the scale inherent in the process of a heavy-ion collision. We proved that the scale is determined only by the physical properties of the final state. Eventually, the scenario for the ultra-relativistic nuclear collision promises to be more perturbative than the standard pQCD. It will be free from the ambiguities of the standard factorization scheme inherent in the cascade models.

This approach is novel for the field of ultrarelativistic nuclear collisions, but the underlying physics is not new. Quantum transitions in solids always excite collective modes, like photons in the medium, phonons, or electrons with effective masses and charges (polarons). Similar phenomena are known in particle and atomic physics. For example:

1. Let an electron-positron pair be created by two photons. If the energy of the collision is large, then the electron and positron are created in the states of freely propagating particles and the cross-section of this process accurately agrees with the tree-level perturbative calculation. However, if the energy of the collision is near the threshold of the process, then the relative velocity of the electron and positron is small, and they are likely to form positronium. It would be incredibly difficult to compute this case using scattering theory. Indeed, one has to account for the multiple emission of soft photons which gradually builds up the Coulomb field between the electron and positron and binds them into the positronium. However, the problem is easily solved if we realize that the bound state is the final state for the process. We can still use low-order perturbation theory to study the transition between the two photons and the bound state of a pair [19].

2. Let an excited atom be in a cavity with ideally conducting walls. The system is characterized by three parameters: the size $L$ of the cavity, the wave-length $\lambda \ll L$ of the emission, and the life-time $\Delta t = 1/\Gamma$ of the excited state. The questions are, in what case will the emitted photon bounce between the cavity walls, and when will the emission field be one of the normal cavity modes. The answer is very simple. If $c\Delta t \ll L$, the photon will behave like a bouncing ball. When the line of emission is very narrow, $c\Delta t \gg L$, the cavity mode will be excited. It is perfectly clear that in the first case, the transition current that emits the photon is localized in the atom. In the second case it is not. By the time of emission, the currents in the conducting walls have to rearrange charges in such a way that the emission field immediately satisfies the proper boundary conditions. We thus have a collective transition in an extended system.

From a practical point of view, these two different problems are united by the method of obtaining their solutions. A part of the interaction (Coulomb interaction in the first case, and the interaction of radiation with the cavity walls in the second case) is attributed to the new “bare” Hamiltonian which is diagonalized by the wave functions of the final state modes. This allows one to reach the solution in a most economical way. The modes of the proper-time dynamics are introduced with the same goal of optimizing the solution of the nuclear collision problem by an explicit account of the collision geometry (adequate choice of the quantum numbers) already at the level of normal modes.

A list of known examples can continue. The example of nuclear collisions is new in one respect only. Traditionally, the collective modes are constructed against the existing “heavy matter background”. In nuclear collisions, the collective modes are created simultaneously with the matter which supports them.
III. THE EQUATIONS OF RELATIVISTIC QUANTUM FIELD KINETICS

The equations of quantum field kinetics were derived in Ref. [16] with the goal of calculating the observables which cannot be reduced to the form of the matrix elements of the S-matrix, viz., the composite operators that cannot be written as the T-ordered products of the fundamental fields. The inclusive rates (or inclusive cross-sections) which are really the number-of-particles operators, are quantities of this sort. In [16], we concentrated on the phenomena associated with the dynamics of fermions. In this study, the main emphasis is on the detail calculations in the gluon sector, and thus, we use equations of QFK for vector gauge field.

A. Basic definitions

The calculation of observables such as inclusive cross sections is based on work by Keldysh [15]. It incorporates a specific set of exact (dressed) field correlators. These correlators are products of Heisenberg operators, averaged with the density matrix of the initial state. For the gluon field they read

\[
D_{10}^{ab\alpha\beta}(x,y) = -i\langle A^{a\alpha}(x)A^{b\beta}(y) \rangle,
D_{01}^{ab\alpha\beta}(x,y) = -i\langle A^{b\beta}(y)A^{a\alpha}(x) \rangle,
D_{00}^{ab\alpha\beta}(x,y) = -i\langle T(A^{a\alpha}(x))A^{b\beta}(y) \rangle,
D_{11}^{ab\alpha\beta}(x,y) = -i\langle T^\dagger(A^{a\alpha}(x))A^{b\beta}(y) \rangle,
\]

(3.1)

where \(T\) and \(T^\dagger\) are the symbols of the time and anti-time ordering. They may be rewritten in a unified form,

\[
D_{AB}^{ab\alpha\beta}(x,y) = -i\langle T_c(A^{a\alpha}(x))A^{b\beta}(y) \rangle,
\]

(3.2)

in terms of a special ordering \(T_c\) along a contour \(C = C_0 + C_1\) (the doubled time axis) with \(T\)-ordering on \(C_0\) and \(T^\dagger\)-ordering on \(C_1\). The operators labeled by ‘1’ are \(T^\dagger\)-ordered, and stand before the \(T\)-ordered operators labeled by ‘0’. Recalling that

\[
A(x) = S^\dagger T(A(x))S \equiv T^\dagger(A(x))S,
\]

(3.3)

we may introduce the formal operator \(S_c = S^\dagger S\), and rewrite (3.3) using the operators of the \(in\)-interaction picture,

\[
D_{AB}^{ab\alpha\beta}(x,y) = -i\langle T_c(A^{a\alpha}(x))A^{b\beta}(y)S_c \rangle.
\]

(3.4)

Except for the matrix form, the Schwinger-Dyson equations for the Heisenberg correlators remain the same as in any other technique. An elegant and universal way to derive them (that does not rely on the initial diagram expansion) can be found in Ref. [22]. For the gluon field these equations are of the form

\[
D_{AB} = D_{AB} + \sum_R D_{AR} \circ (V\mathcal{A}_R) \circ D_{RB} + \sum_{RS} D_{AR} \circ \Pi_{RS} \circ D_{SB},
\]

(3.5)

where the dot stands for both convolution in coordinate space, and the usual product in momentum space (providing the system can be treated as homogeneous in space and time). Indeed, the only tool used to derive these equations was the Wick theorem for the ordered products of the operators. (The type of ordering is not essential for the proof of the Wick theorem [22]). The second term on the right corresponds to the interaction with the “external” field \(\mathcal{A}\).

The formal solution to this matrix equation (3.5) can be cast in the following symbolic form,
Explicit expressions for the self-energies $\Pi_{AB}$ emerge automatically in the course of the derivation of the Schwinger-Dyson equations; we find,

$$\Pi_{AB}^{\alpha \beta}(x, y) = -\frac{i}{2}(-1)^{A+B} \sum_{R,S=0}^1 (-1)^{R+S} \int d\xi d\eta V_{cai f}(\xi, x, \eta) D_{AR}^{f f', \lambda \nu}(\eta, \eta') V_{RBS; f'bc}(\eta', y, \xi') D_{c c', \sigma \mu}(\xi', \xi)$$

$$-\frac{i}{2}(-1)^{A+B} \sum_{R,S=0}^1 (-1)^{R+S} \int d\xi d\eta V_{ci f}(\xi, x, \eta) D_{RBS; cabf}(\eta, y, \eta') D_{f'bc c', \lambda \mu}(\eta', \xi) .$$

where the first and the second term correspond to the loop and tadpole diagrams, respectively. The 3-gluon vertex (in coordinate representation) is defined as

$$V_{bcf, RSP}(x, y, z) = (-1)^{R+S+\rho} \frac{\delta[D^{-1}(x, y)]^{bc; \alpha \beta}}{g \delta A_\rho(z_P)} .$$

The coordinate expression for the bare 3-gluon vertex is

$$V_{abc}^{\alpha \beta \gamma}(x) \equiv V_{abc}^{\alpha \beta \gamma}(x_1, x_2, x_3) = -ig f_{abc} [g^{\alpha \beta} (\partial^\gamma_1 - \partial^\gamma_2) + g^{\beta \gamma} (\partial^\alpha_2 - \partial^\alpha_3) + g^{\gamma \alpha} (\partial^\beta_3 - \partial^\beta_1)] ,$$

where one has to put $x_1 = x_2 = x_3 = x$ at the end of the calculation. In the momentum representation, it may be written as

$$V_{ABC; abc}^{\alpha \beta \gamma}(p_1, p_2, p_3) = -ig \delta_{AB} \delta_{AC} f^{abc} [g^{\alpha \beta} (p_1 - p_2)^\gamma + g^{\beta \gamma} (p_2 - p_3)^\alpha + g^{\gamma \alpha} (p_3 - p_1)^\beta] .$$

The next (third) order correction to the three-gluon vertex is obtained by means of the Eq. (3.7), after differentiating the second-order self-energy with respect to the field $A$,

$$(^{(3)}V_{bcf, RSP}^{\alpha \beta \gamma}(x, y, z) = (-1)^{1+R+S+\rho} \frac{\delta[D^{-1}(x, y)]^{bc; \alpha \beta}}{g \delta A_\rho(x_P)} =$$

$$-ig V(x_R) D_{RS}(x, y) V(y_S) D_{SB}(y, z) V(z_B) D_{BR}(z, x) ,$$

where all Lorentz and color indices in the second equation are dropped for brevity.

In this paper, the tadpole term in the gluon self-energy will not be considered beyond the one-loop approximation. Hence, we need only the bare four-gluon vertex,

$$V_{ABRS; abrs}^{\alpha \beta \sigma} = -g^2 \delta_{AB} \delta_{AR} \delta_{AS} \left[ f_{nar} f_{nbs} (g^{\alpha \beta} g^{\rho \sigma} - g^{\alpha \sigma} g^{\beta \rho}) + f_{nas} f_{nbr} (g^{\alpha \beta} g^{\rho \sigma} - g^{\rho \alpha} g^{\beta \sigma}) + f_{nab} f_{nrs} (g^{\alpha \beta} g^{\rho \sigma} - g^{\alpha \sigma} g^{\beta \rho}) \right] .$$

The four types of operator ordering which enter Eqs. (3.1) are not linearly independent, i.e., there exists a set of relations between the field correlators and the self-energies:

$$D_{00} + D_{11} = D_{10} + D_{01}, \quad \Pi_{00} + \Pi_{11} = -\Pi_{10} - \Pi_{01} .$$

These indicate that only three elements of the $2 \times 2$ matrices $D, \Pi, \ etc.$ are independent. To remove the over-determination let us introduce new functions

$$D_{ret} = D_{00} - D_{01}, \quad D_{adv} = D_{00} - D_{10}, \quad D_1 = D_{00} + D_{11};$$

$$\Pi_{ret} = \Pi_{00} + \Pi_{01}, \quad \Pi_{adv} = \Pi_{00} + \Pi_{10}, \quad \Pi_1 = \Pi_{00} + \Pi_{11} .$$

(3.14)
One possible way to exclude the extraneous quantities is to use the following unitary transformation \([15]\):

\[
\tilde{D} = R^{-1} DR, \quad \tilde{\Pi} = R^{-1} \Pi R, \quad R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.
\]

(3.15)

In this new representation, the matrices of the field correlators and self-energies have a triangle form,

\[
\tilde{D} = \begin{pmatrix} 0 & D_{\text{adv}} \\ D_{\text{ret}} & D_1 \end{pmatrix}, \quad \tilde{\Pi} = \begin{pmatrix} \Pi_{\text{ret}} & \Pi_{\text{adv}} \\ \Pi_{\text{adv}} & 0 \end{pmatrix}.
\]

(3.16)

Applying the transformation \((3.16)\) to the matrix Schwinger-Dyson equations \((3.5)\) we may rewrite them in the following form:

\[
D_{\text{ret}} = D_{\text{ret}} + D_{\text{ret}} \circ \Pi_{\text{ret}} \circ D_{\text{ret}},
\]

(3.17)

\[
D_{\text{adv}} = D_{\text{adv}} + D_{\text{adv}} \circ \Pi_{\text{adv}} \circ D_{\text{adv}},
\]

(3.18)

\[
D_1 = D_1 + D_{\text{ret}} \circ \Pi_{\text{ret}} \circ D_1 + D_1 \circ \Pi_{\text{adv}} \circ D_{\text{adv}} + D_{\text{ret}} \circ \Pi_{\text{ret}} \circ D_{\text{adv}}.
\]

(3.19)

Using the definitions \((3.7), (3.14)\) and the identities,

\[
D_{00}(x_1, x_2)D_{00}(y_1, y_2) - D_{01}(x_1, x_2)D_{10}(y_1, y_2) = D_{10}(x_1, x_2)D_{01}(y_1, y_2) - D_{11}(x_1, x_2)D_{11}(y_1, y_2) = \frac{1}{2} \left[ D_{\text{ret}}(x_1, x_2)D_1(y_1, y_2) + D_1(x_1, x_2)D_{\text{adv}}(y_1, y_2) \right],
\]

(3.20)

we may write the explicit expressions for the retarded and advanced gluon self-energies in the approximation of the bare vertex,

\[
\Pi^{\alpha\beta,ab}_{\text{ret},\text{adv}}(x,y) = -\frac{i}{4} \gamma_{\mu}^{\alpha\lambda}(x)D^{\epsilon\sigma}_{\text{adv},\text{ret}}(x,y)V^{\nu\beta}_{\epsilon\beta}(y)D^{f,s\lambda}_{1}(y,x)
\]

\[
+V_{\mu\lambda}^{\epsilon\sigma}(x)D_{1}^{e,\mu\nu}(x,y)V_{\nu\beta}^{\epsilon\sigma}(y)D_{\text{adv}}^{f,s\lambda}(y,x) + \delta(x-y)\Pi_{\text{adv}}^{\alpha\beta,ab}(x,x).
\]

(3.21)

It is easy to see that one of the propagators in the loop keeps track on the temporal ordering of the arguments of these correlators. The second propagator, \(D_1\), is nothing but the density of physical states with the on-mass-shell momentum which mediates the propagation of the gluon.

FIG. 2. The retarded gluon self-energy. The arrows point to the latest temporal argument. The crossed lines correspond to correlator \(D_1\) which represents the density of states of free propagation.
In order to find the tadpole part of the retarded gluon self-energy, $\Pi_{tadpl}(x, x)$, we have to go back to the Schwinger-Dyson equation which contains the following fragment,

$$D_{\text{ret}} = \ldots - \frac{i}{2} \sum_{R} (-1)^{R} \int dz [D_{0R}^{\sigma}(x, z) V^{\sigma \lambda \mu \nu} D_{RR}^{\lambda}(z, y) D_{R1}^{\mu}(z, y) - D_{0R}^{\sigma}(x, z) V_{\sigma \lambda \mu \nu} D_{RR}^{\lambda}(z, y) D_{R1}^{\mu}(z, y)] .$$

The sum over the contour index $R$ involves two Green functions, $D_{00}(z, z)$ and $D_{11}(z, z)$, with coinciding arguments which are ill-defined since the arguments are assumed to be ordered. Only the sum $D_{00}(z, z) + D_{11}(z, z) = D_{11}(z, z)$ is unambiguous. Using the following chain of transformations (which is very similar to (3.20)), we obtain

$$D_{\text{ret}} = \ldots + D_{0R} V D_{RR} D_{R0} - D_{0R} V D_{RR} D_{R1} = D_{1R} V D_{RR} D_{R0} - D_{1R} V D_{RR} D_{R1} = -\frac{1}{2} [D_{\text{ret}} V D_{1} D_{\text{ret}} + D_{1} V (D_{00} - D_{11}) D_{\text{ret}}] .$$

(3.22)

Since the correlator $D_{1}$ in the last equation is on mass shell, and $D_{\text{ret}}$ on the left is not, the second term on the right drops out and the tadpole term of the retarded self-energy becomes

$$\Pi^{\gamma \beta, ab}_{tadpl}(z, z) = -\frac{i}{2} V_{\alpha \lambda \mu \beta}^{\lambda \mu \nu} D_{1}^{\alpha \lambda \mu \nu}(z, z) .$$

(3.23)

This term is entirely due to the instantaneous contact interaction of the propagating field with on-mass-shell modes.

**B. The formal solution of the integral equations**

The solution we shall look for now is equivalent to the rearrangement of the perturbation series for observables. This rearrangement is useful when specific features of the non-equilibrium system must be taken into account. First, let us introduce two new correlators:

$$D_{0} = D_{\text{ret}} - D_{\text{adv}} = D_{10} - D_{01} ,$$

(3.24)

which coincides with a commutator of the gluon fields and thus disappears outside the light cone, and

$$\Pi_{0} = \Pi_{\text{ret}} - \Pi_{\text{adv}} = -\Pi_{10} - \Pi_{01} ,$$

(3.25)

which is the commutator of two charged currents and has the same causal properties as (3.24). The integral equation for $D_{0}$ may be derived by subtracting Eq. (3.17) from Eq. (3.18),

$$D_{0} = D_{0} + D_{\text{ret}} \circ \Pi_{\text{ret}} \circ D_{0} + D_{0} \circ \Pi_{\text{adv}} \circ D_{\text{adv}} + D_{\text{ret}} \circ \Pi_{0} \circ D_{\text{adv}} .$$

(3.26)

The sum and the difference of Eqs. (3.19) and (3.26) give corresponding equations for the off-diagonal correlators $D_{10}$ and $D_{01}$,

$$D_{01} = D_{01} + D_{\text{ret}} \circ \Pi_{\text{ret}} \circ D_{01,10} + D_{01} \circ \Pi_{\text{adv}} \circ D_{\text{adv}} - D_{\text{ret}} \circ \Pi_{0} \circ D_{\text{adv}} .$$

(3.27)

Since Eq. (3.17) for the retarded propagator may be identically rewritten in the same form,

$$D_{\text{ret}} = D_{\text{ret}} + D_{\text{ret}} \circ \Pi_{\text{adv}} \circ D_{\text{adv}} + D_{\text{ret}} \circ \Pi_{\text{ret}} \circ D_{\text{ret}} - D_{\text{ret}} \circ \Pi_{\text{adv}} \circ D_{\text{adv}} ,$$

(3.28)

we may use Eq. (3.27) and derive the corresponding equations for the $T$- and $T^{\dagger}$-ordered propagators,
\[ D_{00}^{11} = D_{00} + D_{\text{ret}} \circ \Pi_{\text{ret}} \circ D_{00} + D_{00} \circ \Pi_{\text{adv}} \circ D_{\text{adv}} + D_{\text{ret}} \circ \Pi_{11} \circ D_{\text{adv}}. \]  

(3.29)

This chain of routine transformations reduces the equations that make up all elements of the matrix correlator \( D_{AB} \) to a unified form. On the one hand, this representation shows that linear relations between the correlators (or, explicitly, different types of orderings) hold even for the equations that the correlators obey. On the other hand, this representation of the equations singles out the role of retarded and advanced propagators over all other correlators.

In order to understand why their role is special, let us transform them further, and begin by rewriting of Eq. (3.19) for the density of states \( D_1 \) identically as

\[ (1 - D_{\text{ret}} \circ \Pi_{\text{ret}}) \circ D_1 = D_1 \circ (1 + \Pi_{\text{adv}} \circ D_{\text{adv}}) + D_{\text{ret}} \circ \Pi_1 \circ D_{\text{adv}}. \]  

(3.30)

Since \( D_{\text{ret}} \circ \Pi_{\text{ret}} \circ D_{\text{ret}} = D_{\text{ret}} - D_{\text{ret}} \), it is easy to show that

\[ (1 + D_{\text{ret}} \circ \Pi_{\text{ret}})(1 - D_{\text{ret}} \circ \Pi_{\text{ret}}) = 1. \]  

(3.31)

Furthermore, we have the following two relations:

\[ (1 + \Pi_{\text{adv}} \circ D_{\text{adv}}) = \overleftarrow{D}^{-1}_{(0)} \circ D_{\text{adv}}, \quad (1 + D_{\text{ret}} \circ \Pi_{\text{ret}}) = D_{\text{ret}} \circ \overleftarrow{D}^{-1}_{(0)}, \]  

(3.32)

where \( \overleftarrow{D}^{-1}_{(0)}(x) \) and \( \overrightarrow{D}^{-1}_{(0)}(x) \) are the left and the right differential operators of the wave equation, respectively. They explicitly depend on the type of the Hamiltonian dynamics, including the gauge condition imposed on the field \( A(x) \).

Multiplying Eq. (3.30) by \( (1 + D_{\text{ret}} \circ \Pi_{\text{ret}}) \) from the left, we find the final form of the equation,

\[ D_1 = D_{\text{ret}} \circ D_1 \circ \Pi_1 \circ D_{\text{adv}} \]  

(3.33)

Repeating these transformations for the other equations, we obtain the corresponding forms that are most convenient for the subsequent analysis:

\[ D_{10}^{11} = D_{\text{ret}} \circ D_{10} \circ \Pi_{01} \circ D_{\text{adv}} - D_{\text{ret}} \circ \Pi_{11} \circ D_{\text{adv}}, \]  

(3.34)

\[ D_{00}^{11} = D_{\text{ret}} \circ D_{00} \circ \Pi_{11} \circ D_{\text{adv}} + D_{\text{ret}} \circ \Pi_{11} \circ D_{\text{adv}}. \]  

(3.35)

While equations (3.17) and (3.18) are the standard integral equations which have an unknown function on both sides; however, after transformation the unknown function appears only on the l.h.s., and we may consider Eqs. (3.33)–(3.35) to be the formal representation of the required solution.

At this point, the first and the most naive idea is to ignore the arrows indicating the direction the differential operators act in, and to rewrite Eqs. (3.34)–(3.35) in the momentum representation. Then the first term in each of Eqs. (3.34) will contain the expression like \( p^2 \delta(p^2) \), which equals zero. This feature reflects the simple fact that the off-diagonal correlators \( D_{10} \) and \( D_{01} \) are solutions of the linearized homogeneous Yang-Mills equations. However, this approach does not appear to be sufficiently consistent: we lose the identity between Eqs. (3.34) and (3.26), and make it impossible to generate the standard perturbation expansion in powers of the coupling constant. Eqs. (3.35) will be corrupted as well.

A more careful examination of Eqs. (3.34)-(3.35) shows that all four dressed correlators \( D_{AB} \) can be found as the formal solution of the retarded Cauchy problem, with bare field correlators being the initial data and the self-energies
being the sources. Indeed, integrating the first term of each these equations twice by parts, we find for all four elements of $D_{AB}$ to be

$$D_{01}(x, y) = \int d^4\Sigma_\mu^{(g)} D_{ret}(x, \xi) \hat{\partial}_\mu D_{01}(\xi, \eta) \hat{\partial}_\mu D_{adv}(\eta, y) - \int d^4\xi d^4\eta D_{ret}(x, \xi) [\Pi_{00}(\xi, \eta) D_{adv}(\eta, y)$$

$$D_{00}(x, y) = \int d^4\Sigma_\mu^{(g)} D_{ret}(x, \xi) \hat{\partial}_\mu D_{00}(\xi, \eta) \hat{\partial}_\mu D_{adv}(\eta, y) - \int d^4\xi d^4\eta D_{ret}(x, \xi) [\pm D_{01}^{-1} + \Pi_{00}(\xi, \eta)] D_{adv}(\eta, y) .$$

The equations (3.36) and (3.37) are the basic equations of relativistic quantum field kinetics. They are identical to the initial set of Schwinger-Dyson equations, but have the advantage that the time direction is stated explicitly. There are two terms of different origin that contribute to any correlator (and, consequently, to any observable). The first term retains some memory of the initial data. The length of time for which this memory is kept depends on the retarded and advanced propagators. The second term describes the current dynamics of the system. A comparison of these two contributions allows one to judge if the system has various time scales.

C. Initial data and spectral densities.

We perform all calculations in the gauge $A^+ = 0$ which is complementary to the null-plane dynamics with the the time variable $x^+$. In this gauge, the on-shell correlators and the propagators of the perturbation theory are as follows,

$$D^{\#ab,\mu\nu}_{01}(p) = -2\pi i \delta_{\mu\nu} (p^0 \pm i0) \delta(p^2) ,$$

$$D^{\#ab,\mu\nu}_{ret}(p) = \frac{d^{\mu\nu}(p)}{p^+(p^- \pm i0) - p^2} ,$$

$$d^{\mu\nu}(p) = -g^{\mu\nu} + n^\mu p^\nu + n^\nu p^\mu .$$

(3.38)

The polarization tensor $\Pi^{\mu\nu}$ appears only between the retarded and advanced propagators, i.e., in the combination $[D_{ret}(p)\Pi(p)D_{adv}(p)]^{\mu\nu}$. Both propagators contain the same projector, $d^{\mu\nu}(p)$ which (by the gauge condition) is orthogonal to the 4-vector $n^\mu$. So, of the general tensor form, only two terms survive,

$$\Pi^{\mu\nu}(p) = g^{\mu\nu} p^2 w_1(p) + p^\mu p^\nu w_2(p) + ... .$$

(3.39)

The others, like $p^\mu n^\nu + n^\mu p^\nu$ or $n^\mu n^\nu$, cancel out. The invariants $w_1$ and $w_2$ can be found from the two contractions,

$$-\vec{d}_{\mu\nu}(p)\Pi^{\mu\nu}(p) = 2p^2 w_1(p) ,$$

$$n_\mu n_\nu\Pi^{\mu\nu}(p) = (p^+)^2 w_2(p) .$$

(3.40)

independently of the other invariants accompanying the missing tensor structures. (The projector $\vec{d}_{\mu\nu}(p)$ is defined below in Eq. (3.42).) The solutions of the Schwinger-Dyson equations (3.17) and (3.18) for the retarded and advanced gluon propagators,

$$D_{ret} = D_{ret} + D_{ret} \Pi_{ret} + D_{adv} \Pi_{adv} + D_{adv} ,$$

can be cast in the form,

$$D^{\mu\nu}_{(ret)}(p) = \frac{\vec{d}^{\mu\nu}(p)}{p^2 + p^2 w_1^{(0)}(p)} + \frac{1}{(p^+)^2} \frac{n^\mu n^\nu}{1 - w_2^{(0)}(p)} .$$

(3.41)

The first term is the propagator of the transverse field and has a normal pole corresponding to the causal propagation. It contains the polarization sum for the two transverse modes,
\[\bar{\mathcal{D}}^{\mu\nu}(p) \equiv -d^{\mu\rho}(p) d^\rho_\nu(p) = -g^{\mu\nu} + \frac{p^\mu n^\nu + n^\mu p^\nu}{p^+} - \frac{p^2 n^\mu n^\nu}{(p^+)^2},\]  
(3.42)

which is orthogonal to both vectors \(n^\nu\) and \(p^\nu\). The second term in Eq. (3.41) is the response function of the longitudinal field which does not lead to causal propagation. When radiative corrections \(w_1\) and \(w_2\) are dropped, the bare propagators are recovered with a clear separation of the transverse and longitudinal parts. This separation tells us that the longitudinal field in the null-plane dynamics has only the component \(A^-\) which is generated by the charge density \(j^+\) of the source current which enters Gauss’ law but not the equations of motion. If all other components of the current vanish, then (by the current conservation) we have \(\partial_+ j^+ = 0\) and, consequently, \(j^+ = j^+(x^-) = j^+(x^0 - x^3)\).

Therefore, the purely static field of the null-plane dynamics is produced by the charge traveling at the speed of light in the positive \(x^3\)-direction. The radiation field is everything else minus this static field. Physically, the field is called the radiation field if it geometrically decouples from the source. Therefore, we can formulate a criterion for radiation in the null-plane dynamics. The radiation field has to be slowed down with respect to the static source. There is no problem with real massive sources, they are never static in the afore-mentioned sense. However, if the source is put into the infinite momentum frame, a paradox arises and the problem re-appears through the poles \(1/p^+\) in the propagators. Fortunately, in the case of QCD the radiation is always massive, which resolves the paradox and screens the light-cone singularity in the emission process. However, since the screening is a non-local effect associated with the longitudinal fields, it cannot be derived in the singular geometry of the null-plane dynamics (see Sec. X).

With the shorthand notation, \(W_1^{(2)}(p) = p^2 + p^2 w_1^{(2)}(p)\), and \(W_2^{(2)}(p) = 1 - w_2^{(2)}(p)\), we readily obtain,

\[
\left[ D_{ret}(k)\Pi^{(2)}_{(2)}(k)D_{adv}(k) \right]^{\mu\nu} = -\frac{\bar{\mathcal{D}}^{\mu\nu}(p)p^2 w_1^{(2)}(p)}{W_1^{(2)}(p)W_1^{(2)}(p)} + \frac{w_2^{(2)}(p)n^\mu n^\nu}{(p^+)^2 W_2^{(2)}(p)W_2^{(2)}(p)}, \tag{3.43}
\]

which also exhibits a clear separation between the transverse and static fields. The first term on the right side is extracted by contraction with the metric tensor \(g^{\mu\nu}\); the second one is extracted with the aid of the projector \(p^\mu p^\nu/(p^+)^2\). The combination (3.43) enters the right side of Eq. (3.34) and represents the density of states into which the gluon field may decay.

The first term in Eq. (3.34) corresponds to the initial data for the field correlator. The role of this term is two-fold. According to the picture qualitatively described in the Introduction, the final states in the inclusive measurement are fully developed being the virtual fluctuations before the measurement. Therefore, we encounter two types of initial fields. The fields of the first type (with the correlators labeled as \(D^*\)) are real quark and gluon fields of the proton or nuclei before the collision. They always have sources and no initial data which can be expressed in terms of the free fields. Hence, the first term in Eq. (3.34) for these fields has to be dropped. To understand further consequences of this step, let us assume that the full retarded and advanced propagators in Eq. (3.34) are renormalized according to some condition (to be specified later). At the renormalization point, the retarded and advanced self-energies vanish and, therefore, \(D_{ret}^{-1}D_{(0)}^{-1}D_{adv} = 1\). This leads to

\[
D^{*}_{10} = Z D^{*}_{10} - D^{*}_{ret} \Pi^{(2)}_{(2)}D^{*}_{adv}, \tag{3.44}
\]

where \(Z\) is the residue of the pole of the renormalized propagator. Thus, the absence of the initial data for the free gluon field is equivalent to the requirement that its renormalization factor \(Z\) equals zero. Hence, this is not a fundamental field and its spectral density is formed solely by multiparticle states into which it decays. (This option to make a distinction between the elementary and composite fields was intensively discussed by Weinberg [26]).
The fields of the second type (with the correlators labeled as $D^\#$) are used to decompose the colliding composite objects in terms of the true normal modes of the final state which are excited only after the collision. For example, in the case of the two nuclei colliding, the normal modes are plasmons which have effective masses $m$. To some approximation (connected to the possible damping of excitations in the collective systems) we have, $D^\#_{10} \equiv D^\#_+ \sim \delta(p^2 - m^2)\theta(p^0)$, and we may transform Eq. (3.34) into the known expression for the spectral density,

$$D^\#_{+\mu\nu}(p^2) = \int_0^\infty dm^2 \rho_{\mu\nu}(m^2)\delta(p^2 - m^2) = Z d^{\mu\nu}(p)\delta(p^2 - m^2)\theta(p^0) + \int_0^\infty dm^2 \sigma_{\mu\nu}(m^2)\delta(p^2 - m^2)\theta(p^0).$$  (3.45)

Masses that enter this spectral representation shield the abundant collinear singularities that may appear in the evolution equations.

**IV. TEMporal ORDER IN INClusive PROCESSES**

The program for computing the quark and gluon distributions in heavy-ion collisions relies on the data obtained in seemingly more simple processes, e.g., $ep$-DIS. It turns out that differently triggered sets of data may carry significantly different information. In this section, we discuss two examples and show that even minor change in the way the data are taken may strongly affect what is actually observed.

Let us consider a collision process where the parameters of only one final state particle explicitly measured. Let this particle be an electron with momentum $k'$ and spin $\sigma'$. Deep inelastic electron-proton scattering is an example of such an experiment. All vectors of final states which are accepted into the data ensemble are of the form $a_{\sigma'}(k')|X\rangle$ where the vectors $|X\rangle$ form a complete set. The initial state consists of the electron with momentum $k$ and spin $\sigma$ and the proton carrying quantum numbers $P$. Thus, the initial state vector is $a_\sigma(k)|P\rangle$. The inclusive transition amplitude reads as $\langle X|a_{\sigma'}(k') S a_\sigma^\dagger(k)|P\rangle$ and the inclusive momentum distribution of the final-state electron is the sum of the squared moduli of these amplitudes over the full set of the non-controlled states $|X\rangle$. This yields the following formula,

$$\frac{dN_e}{dk'\sigma} = \langle P|a_{\sigma'}(k)S^\dagger a_\sigma^\dagger(k')a_\sigma(k')Sa_\sigma^\dagger(k)|P\rangle,$$  (4.1)

which is just an average of the Heisenberg operator of the number of the final-state electrons over the initial state. Since the state $|P\rangle$ contains no electrons, one may commute electron creation and annihilation operators with the $S$-matrix and its conjugate $S^\dagger$ pulling the Fock operators $a$ and $a^\dagger$ to the right and to the left, respectively. Let $\psi_{k\sigma}^{(+)}(x)$ be the one-particle wave function of the electron. This procedure results in

$$\frac{dN_e}{dk'\sigma} = \frac{1}{2} \sum_{\sigma'} \int dx'dy \overline{\psi_{k\sigma}^{(+)}(x')\psi_{k\sigma}^{(+)}(y)} \langle P| \frac{\delta^2}{\delta \Psi(x)\delta \Psi(y)} \left( \frac{\delta S^\dagger}{\delta \Psi(y')} \frac{\delta S}{\delta \Psi(x')} \right) |P\rangle \psi_{k\sigma}^{(+)}(y)\psi_{k'\sigma'}^{(+)}(y').$$  (4.2)

Since the electron couples only to the electromagnetic field, then, to the lowest order,

$$\frac{dN_e}{dk'\sigma} = \frac{1}{2} \sum_{\sigma'} \int dx dy \overline{\psi_{k'\sigma'}^{(+)}(y)\psi_{k\sigma}^{(+)}(x)} \langle P| A^{(\gamma)}(x) A^{(\gamma)}(y)|P\rangle \psi_{k\sigma}^{(+)}(y)\psi_{k'\sigma'}^{(+)}(x),$$  (4.3)

where $A^{(\gamma)}(x)$ is the Heisenberg operator of the electromagnetic field. Already at this very early stage of calculations, the answer has a very clear physical interpretation. Since only the final-state electron is measured, the probability of the electron scattering is entirely defined by the electromagnetic field produced by the rest of the system evolved from its initial state until the moment of interaction with the electron.
The correlator of the two electromagnetic fields in Eq. (4.3) is the function \( iD^{(\gamma)}_{10}(x, y) \). Therefore, we can use Eq. (3.36) to express it via the electromagnetic polarization tensor of the proton. Summation over the spins of the electrons brings in the leptonic tensor, \( L_{\mu\nu}(k, k') \). If \( q = k - k' \) is the space-like momentum transfer, then the DIS cross-section is given by

\[
\frac{dN_e}{dk'} = \frac{e^2 L_{\mu\nu}(k, k')}{4(2\pi)^6 EE'} D_{\text{rel}}^{(\gamma)}(q) \pi^{\mu\nu}_{10}(q) D^{(\gamma)}_{\text{adv}}(q),
\]

where \( \pi^{\mu\nu}_{10}(q) \) is the notation for the correlator of the two electromagnetic currents,

\[
\pi^{\mu\nu}_{10}(x, y) = \langle P| j^\mu(x) j^\nu(y) |P \rangle.
\]

(Alternatively, we may obtain the answer by means of the Yang-Feldman equation [23],

\[
A(x) = \int d^4y D^{(\gamma)}_{\text{rel}}(x, y) j(y),
\]

where \( j(y) \) is the Heisenberg operator of the electromagnetic current and \( D^{(\gamma)}_{\text{rel}}(x, y) \) is the retarded propagator of the photon.) The correlator of the currents is the source of the field which has scattered the electron. Here, both photon propagators are retarded and indicate the causal order of the process. The measurement analyzes all those fluctuations in the hadron, which have developed before the scattering of the electron and created the virtual photon probed by the electron.

The tensor \( \pi^{\mu\nu}_{10}(x, y) \) is naturally expressed via the two quark correlators,

\[
\pi^{\mu\nu}_{10}(x, y) = \text{Tr} \gamma^\mu G_{10}(x, y) \gamma^\nu G_{01}(y, x).
\]

The correlators \( G_{10} \) and \( G_{01} \) are evolved from the earlier times via equations (3.26) and (3.27) of Ref [27]. The field correlator generated via the source of the previous step of evolution is

\[
G_{01} \rightarrow G^*_0 G_{01} = -G_{\text{rel}} \Sigma^*_0 G_{\text{adv}}.
\]

The newly created correlation \( G_{10} \) in the state of free propagation is connected with the density of the final states,

\[
G_{10} \rightarrow G_{10}^\# = G_{10}^\# + G_{\text{rel}}^\# \Sigma_{10}^\# G_{\text{adv}}^\#.
\]

Similarly, the self-energies \( \Sigma_{01} \) are defined via the correlators \( G_{01} \) and \( D_{10} \) which can also be causally evolved from previous stages of the evolution. In this way, the causal ladder of the fluctuations is built even without the concept of partons.

Early discussions of the role of the light cone distances in high energy collisions [24] resulted in Gribov’s idea of the two-step treatment of inelastic processes [25]: the gamma-quantum first decays into virtual hadrons and later these hadrons interact with the nuclear target. This idea looks very attractive since it explains the origin of the two leading jets in electro-production events, corresponding to the target and the projectile (photon) fragmentation. It also provides a reasonable explanation of the plateau in the rapidity distribution of the hadrons. However, this elegant qualitative picture contains a disturbing element, i.e., the way the words “first” and “later” are used.

In order to understand how Gribov’s process may be observed practically, let us change the observable in the same deep inelastic process initiated by the \( ep \)-interaction. Let us trigger events on the high-\( p_t \) quark or gluon jet in the
final state regardless of the momentum of the electron. This means that all states that are included into the data set are of the form, \(\alpha_\lambda(p)|X\rangle\), where \(\alpha_\lambda(p)\) is the Fock operator for the final state quark. The corresponding observable is the number of the final-state quarks. To the lowest order of perturbation theory, this is a process of the type \(2 \rightarrow 1\).

Its inclusive probability is

\[
\frac{dN_q}{dp} = \langle P|\sigma_2(k) S^\dagger \alpha_\lambda(p) \alpha_\lambda(p) S \sigma_1(k)|P\rangle.
\]

(4.9)

After commutation of the Fock operators with \(S\) and \(S^\dagger\) we arrive at the expression,

\[
\frac{dN_q}{dp} = \frac{1}{2} \sum_{\sigma\lambda} \int dx dx' dy dy' \psi^{(+)}(x) \pi^{(+)}(x') \langle P| \frac{\delta^2}{\delta \psi(x) \delta \psi(y')} \left( \frac{\delta S^\dagger}{\delta q(y')} \frac{\delta S}{\delta q(x')} \right) |P\rangle \psi^{(+)}(y) q^{(+)}(y') \rangle.
\]

(4.10)

With reference to the Bogolyubov’s form of the micro-causality principle [22], which (in its simplest form) reads as

\[
\frac{\delta}{\delta \phi(x)} \left( S^\dagger \frac{\delta S}{\delta \phi(y)} \right) = 0, \quad \text{unless} \quad (x - y)^2 > 0 \quad \text{and} \quad x^0 > y^0,
\]

(4.11)

we may argue that the space-time points \(x'\) and \(y'\) (where the quark jet is created) are inside the forward light cone of the possible points \(x\) and \(y\) of the electron scattering. To the lowest order, the graph for this process is depicted at Fig. 3.

![FIG. 3. Fluctuation which is “active” in the inclusive measurement of jet distribution in DIS.](image)

The photon propagator is retarded. To produce the quark, the hadronic target has to be hit either by the photon coming from the first step of the virtual fragmentation of the electron or by one of the partons formed by the further virtual fragmentation. By causality, both the electron and the hadronic target must develop appropriate fluctuations before the moment of the creation of the final-state quark. Thus, the change of the trigger drastically affects the information read out of the data. Triggering on the high-\(p_t\) jet in the same process allows one to filter out fluctuations corresponding to Gribov’s picture.

V. THE EVOLUTION LADDER TO THE LOWEST ORDER

The way to derive evolution equations in the causal picture of QFK has been shown in Refs. [16,27]. In the case of pure glue-dynamics, we start with the Eq. (3.7) for the tensor \(\Pi_{(10)}\), and iterate this equation with the aid of the
Eqs. (3.34). To the lowest order in $g^2$, the vertices are considered to be bare (without the loop corrections). The iteration process explicitly accounts for the two types of fields corresponding either to the core of the proton (these are not yet touched by the evolution) or to the fields of anomalous fluctuations (which eventually become the field of radiation.) Therefore, every off-diagonal correlator, $D_{01}$ or $D_{10}$, corresponds either to the initial field of colliding composite particles, or to the final state of the gluon field. In the first case, we use equations (3.34) and express the field correlators via their sources formed during the preceding evolution,

$$
D_{01} \rightarrow D_{01}^\# = -D_{ret}^{\Pi^\#_{01}}D_{adv}. \tag{5.1}
$$

The first term in Eq. (3.34) is dropped, which corresponds to the main physical assumption that the proton or nucleus is a fundamental mode of QCD and that there is no free gluon fields defined by the initial data. In the second case, we replace $D_{01}$ by the densities of the final states,

$$
D_{01} \rightarrow D_{01}^\# = D_{10}^\# - D_{ret}^{\Pi^\#_{10}}D_{adv}. \tag{5.2}
$$

Here, the first term corresponds to the immediate creation of the final state on-mass shell gluons (therefore, their propagators have no radiative corrections). By keeping this term intact, we implicitly recognize that the states of freely propagating color fields do exist. This is both the main assumption and the corner-stone of perturbative QCD. Though this step allows one to approach the exclusive high-energy processes from the "safe" side of the QED-like perturbation theory, it completely disregards the existence of the hadronic scale and creates artificial problems for collinear processes. Further analysis indicates that in the picture with realistic final states, this term has either to be modified by the redefinition of the normal modes (e.g., in dense matter), or be abandoned (if the final states are hadronic jets). The second term corresponds to multiparticle emission (via the intermediate off-mass-shell gluon field) and is more physical.

For now, we limit ourselves by the condition that only one two-point correlator of this expansion can originate from the QCD evolution of the initial state and that the emitted gluon is on-mass-shell,

$$
\Pi^{\mu\nu}_{01}(p) = -\frac{i}{2} \int \frac{d^4k}{(2\pi)^4} V^{\mu\nu}_{acf}(p,k-p,-k) [D_{ret}(k)\Pi_{01}(k)D_{adv}(k)]_{\alpha\beta}^{\gamma\delta} V^{\gamma\delta}_{hc.f}(-p,k-k)D_{ret}^{\Pi^\#_{10}}D_{adv}. \tag{5.3}
$$

This is the full tensor form of the evolution equations of pure glue-dynamics to order $g^2$. They have to be projected onto the normal modes of the gluon field and be rewritten in the form of scalar equations for the invariant functions.

Our next goal is to find the evolution equations for the invariants, $w_1$ and $w_2$ of the gluon field. Using the Eqs. (3.40) and (3.43) we extract evolution equations for the various invariants from the tensor evolution equations (5.3). We obtain two equations for the invariants of the gluon source:

$$
\frac{p^2 w_1^{(01)}(p)}{W_1^R(p)W_1^A(p)} = \frac{N_c \alpha_s}{\pi^2} \int d^4k \delta_+\left[(k-p)^2\right] \left[\frac{-p^2/z + k^2}{W_1^R(p)W_1^A(p)}P_{gg}(z)\frac{k^2 w_1^{(01)}(k)}{W_1^R(k)W_1^A(k)} - \frac{(z-1/2)^2}{W_1^R(p)W_1^A(p)} \frac{w_2^{(01)}(k)}{W_2^R(k)W_2^A(k)} \right], \tag{5.4}
$$

$$
\frac{w_2^{(01)}(p)}{W_2^R(p)W_2^A(p)} = \frac{2N_c \alpha_s}{\pi^2} \int d^4k \delta_+\left[(k-p)^2\right] \left[\frac{1}{W_2^R(p)W_2^A(p)} \frac{k^2 w_2^{(01)}(k)}{W_2^R(k)W_2^A(k)} \right], \tag{5.5}
$$

where $P_{gg}(z)$ is a well-known splitting kernel,
\[ P_{gg}(z) = \frac{1 - z}{z} + \frac{z}{1 - z} + z(1 - z). \]  

(5.6)

Within the accuracy of this approximation, viz, when \(^{(2)}\Pi_{01} \sim g^2\) is used as the basis for the iteration procedure, the propagators \(W_R(p)\) and \(W_A(p)\) on the right-hand side do not include loop corrections.

The first term on the right side of Eq. (5.4) has a transverse source in it, which describes a step of evolution when the initially transverse field pattern remains transverse after the emission. In what follows, we shall refer to this type of transition as the TT-mode. The second term in this equation contains a longitudinal (static) source and after one act of real emission the new field pattern becomes transverse. This field can be, e.g., the proper field of the static source, which became a propagating transverse field after the static source has been accelerated (TL-transition mode).

The second equation describes the process of creation of the new static field of the source after the acceleration has been terminated (LT-transition mode). There is no LL-transition mode since any rearrangement of a charged system between two different static configurations requires at least two emissions. The first emission extinguishes the old static field, the second emission creates a new one.

Connections between the invariants \(w^{01}_j(p)\) and the structure functions follow from the description of measurement,

\[ c \int dp^- \frac{ip^+p^2w_1(p)}{W_R^1(p)W_A^1(p)} = \frac{dG(x,p_t^2)}{dp_t^2}, \quad c \int dp^- \frac{ip^+w_2(p)}{W_R^2(p)W_A^2(p)} = G(x,p_t^2), \]  

(5.7)

where \(c\) is some common normalization constant. In terms of these functions the evolution equations acquire a habitual form,

\[
\frac{dG(p^+,p_t^2)}{dp_t^2} = -\frac{N_c\alpha_s}{\pi^2} \int \frac{dk^+_+dk}{2} \left[ \int dp^- \frac{\delta_+[(k-p)^2]}{p^2} P_{gg}(z) \frac{dG(k^+,k_t^2)}{dk_t^2} + \int dp^- \frac{\delta_+[(k-p)^2]}{[p^2]^2} \left(z - \frac{1}{2}\right)^2 G(k^+,k_t^2) \right], \]  

(5.8)

\[
G(p^+,p_t^2) = -\frac{N_c\alpha_s}{\pi^2} \frac{dk^+_+dk}{2} \int dp^- \frac{\delta_+[(k-p)^2]}{2} \left(z - \frac{1}{2}\right)^2 \frac{dG(k^+,k_t^2)}{dk_t^2}. \]  

(5.9)

We must account for the static components in course of the QCD evolution as long as we wish to describe this process as developing in time, and as long as the fields are considered not only in the far zone, but also in the near zone, where they continuously interact with the sources.

None of the integrations \(dp^-\) in Eq. (5.8) is singular at the point \(k^+ = p^+\). Indeed, assuming the strong ordering of the emission, \(p_k \gg k_t\), we obtain,

\[
\frac{dG(p^+,p_t^2)}{dp_t^2} = \frac{N_c\alpha_s}{\pi^2} \left[ \int_{p^+} d^p^+ \frac{k^+_+}{2k^+} \frac{1}{p_t^2} P_{gg}(z) \int dk_t^+ \frac{dG(k^+,k_t^2)}{dk_t^2} + \frac{1 - z}{p_t^4} \left(z - \frac{1}{2}\right)^2 \int dk_t^+ G(k^+,k_t^2) \right]. \]  

(5.10)

It is common to regulate the singularity of the splitting kernel \(P_{gg}(z)\) at \(z = 1\) with the aid of (+)-prescription. In Ref. [18], this prescription to order \(g^2\) was obtained using two diagrams depicted in Fig. 4. The singularity in the one-loop correction to the free propagation compensates for the divergence due to the collinear emission.
In the case of emission (that converts transverse field into the longitudinal one), we encounter the same type of end-point singularity. Integration $dp^−$ in Eq. (5.9) is singular and leads to the pole $(1−z)^{−1}$ at $k^+=p^+$, i.e.,

$$\Pi(x,p_1^2) = -\frac{N_c\alpha_s}{2\pi} N_c \int_{p^+}^{P^+} \frac{dk^+}{k^+-p^+} \frac{(2p^+-k^+)^2}{4p^+k^+} G(k^+) ,$$

(5.11)

where

$$G(p^+) = \int_{Q^2_0}^{\infty} dp_1^2 \frac{dG(p_1^+,p_1^2)}{dp_1^2} ,$$

(5.12)

is the $x$-fraction of the glue converted into radiation, integrated over all transverse momenta starting from some low (confinement?) scale $Q_0^2$.

Though the pole in Eq. (5.11) is similar to the one in $P_{gg}(z)$ no self-energy correction to this process exists, since the two-point function cannot convert a T-mode into an L-mode. The only hope for the possible removal of this divergence is connected to the radiative corrections of the emission process. Besides, unlike (transverse) partons the longitudinal fields are beyond the factorization scheme and the RG for the S-matrix. It is not at all obvious what kind of running coupling accompanies the splitting kernel. Therefore, we have to consider the evolution with the elementary cell at least to the order $g^4$.

**VI. THE EVOLUTION EQUATIONS TO THE ORDER $\alpha_s^2$**

There are two kinds of problems which cannot be resolved within the approach of a simple ladder. First, we encounter collinear singularities which may be (at least partially) compensated by loop corrections like $W_i^{R,A}(p)$. Second, the loop corrections and the additional real emission processes are required to assemble the running coupling.

Let us start with the full expression for $(4)\Pi_{01}(p)$, which emerges after the vertex of the $g^3$-order (3.11) is substituted into the Eq.(3.1),

$$(4)\Pi_{01}^{\alpha\beta,ab}(x,y) = -\frac{1}{2} \int dz_1dz_2 \sum_{R,S=0}^1 (-1)^{R+S} V_{\alpha_1\alpha_2}(x) D_{0R}^{x\sigma_1\sigma_2}(x,z_1) V_{\rho_1\rho_2}(z_1) D_{R1}^{\rho_1\rho_2}(z_1) \times$$

$$\times V_{b_1b_2}(y) D_{1S}^{y\sigma_1\sigma_2}(y,z_2) V_{\sigma_1\sigma_2}(z_2) D_{S0}^{\sigma_1\sigma_2}(z_2,x) D_{RS}^{\sigma_1\sigma_2}(z_2,z_2) .$$

(6.1)

In this equation, we still keep all the gluon correlators dressed. We decompose it using the following guidelines. First, we divide terms into two major groups corresponding to $R = S$ and $R \neq S$. The former gives rise to the virtual
vertex corrections, while the latter describes all possible real emission processes. The loops of virtual corrections are formed by the diagonal propagators. They are ultraviolet-divergent and require renormalization. Inside each group, every off-diagonal correlator, $D_{10}$ or $D_{11}$ corresponds either to the initial field of the colliding composite particles, or to the final state of the gluon field. Thus, we again use Eqs. (5.1) and (5.2) along with the previous agreement that only one two-point correlator of this expansion can originate from the QCD evolution of the initial state.

In this approximation, two types of radiative corrections appear in the evolution equations. First, we have to calculate the basic polarization tensor $\Pi_{01}$ to order $g^4$ by accounting for the vertex corrections. Second, we should account for the loop corrections in all propagators. The formal counting of the order comes from the following decomposition,

$$D_{01} = -D_{ret}\Pi_{01}D_{adv} = -D_{ret}[1 + \Pi_{ret}D_{ret}]\Pi_{01}[1 + D_{adv}\Pi_{adv}]D_{adv}$$

$$\approx -D_{ret}[^{(4)}\Pi_{01} + (^{(2)}\Pi_{ret}D_{ret}^{(2)}\Pi_{01} + (^{(2)}\Pi_{01}D_{adv}^{(2)}\Pi_{adv})D_{adv}],$$

where $^{(n)}\Pi \sim g^n$. After separation of the transverse and longitudinal modes, we obtain,

$$\frac{p^2}{W(p^2)}w_{11}^{(p)} = \frac{(4)^2w_{11}^{(p)}}{[p^2]^2} + 2p^2\frac{(2)^2w_{1}^{(p)}}{[p^2]^2},$$

$$\frac{p^2}{W(p^2)}w_{22}^{(p)} = (4)w_{1}^{(p)} + 2^{(2)}w_{2}^{(p)}(2)u_{1}^{(p)}.$$  (6.3)

### A. The gluon self-energy corrections

Various gluon self-energies $^{(2)}\Pi_{AB}(p)$ of the order $g^2$ are needed to compute the elementary cell of the QCD ladder to order $g^4$. Though the results are well-known, we consider it instructive to go through their derivation in some details.

The simplest of $^{(2)}\Pi_{AB}$, the self-energy $^{(2)}\Pi_{10}^{(p - k)}$, is the correction to the emission process due to the “mass” of the decoupled gluon which splits further into two massless gluons. These two gluons belong to the space of the final freely-propagating states. Therefore,

$$\Pi_{10}^{\alpha\beta}(p) = i\frac{g^2}{2(2\pi)^2} \int d^4 q \theta(q^+)\theta(p^+ - q^+)\delta(q^+ - q^-)\delta(q^+ - q^-)(p^+ - q^-)$$

$$\times V_{\mu\nu}(q - p, p, -q)d_{\rho\sigma}(q)V^\rho\sigma(q, -p, p - q)d_{\nu\mu}(q - p).$$  (6.5)

After integration over $q^-$ with the aid of the first $\delta$-function we obtain the following expressions for the transverse and longitudinal invariants $w_{1}^{10}$ and $w_{2}^{10}$:

$$[w_{1}^{10}(p), w_{2}^{10}(p)] = -i\frac{g^2N_c}{(2\pi)^2} \int_0^{p^+} dq^+\frac{dq^+}{q^+} \int d^2q^+\delta[(p^+ - q^+)(p^- - q^-) - (p_\perp - q_\perp)^2].$$  (6.6)

The divergence of this integral can be viewed in different ways. First, let us notice that the integration $d^2q_\perp$ involves only a $\delta$-function and the result of this integration is $\pi p^+/q^+$. Thus, the divergence becomes connected with the $dq^+$ integration, and requires both lower, and upper cut-offs in the remaining integral for $w_{2}^{10}(p)$.
\[ w_j^{10}(p) = -\frac{i g^2 N_c}{8\pi} \theta(p^2) \theta(p^+ - \epsilon) \beta_j(p^+, \epsilon), \]  

where we have introduced the functions

\[ \beta_1(p^+, \epsilon) = -2N_c \int_{\epsilon}^{p^+ - \epsilon} \frac{dq^+}{p^+} P_{gg}(q^+ / p^+) \rightarrow N_c \left( \frac{11}{3} - 4 \ln \frac{p^+}{\epsilon} \right), \text{ when } \epsilon \rightarrow 0, \]  

\[ \beta_2(p^+, \epsilon) = N_c \int_{\epsilon}^{p^+ - \epsilon} \frac{dq^+}{p^+} \left( 1 - \frac{q^+}{p^+} \right)^2 \rightarrow \frac{N_c}{3}, \text{ when } \epsilon \rightarrow 0. \]  

Being introduced in this way, the cut-off may be thought of as a regulator for the spurious poles in the gluon correlators \( D_{10}(q) \) and \( D_{01}(q - p) \) which form the loop. However, such an interpretation is excluded by the fact that these correlators are on-mass-shell and include only transverse fields. Therefore, the poles of the polarization sums \( \tilde{d}(q) \) and \( \tilde{d}(q - p) \) are due to the transverse momenta of the on-mass-shell loop gluons and the singularity of the remaining integration is a physical collinear (mass) singularity. To remove this singularity, one has to eliminate the collinear region by fiat, \( \text{viz.} \) by introducing a physical parameter of resolution for the emission process. The value of this parameter has to be extracted from the data, \( \text{e.g.} \) from the shape of jets in DIS. In the case of heavy-ion collisions, the cut-off will naturally come from the finite density of the final-state particles. Indeed, in the coordinate space, a collinear singularity is caused by the infinite time-of-interaction between the radiation and its massless source. A single interaction with the “third body” suffices to interrupt the emission process and to allow the field of radiation to decouple from its source.

If the cut-off parameter is sufficiently small, than the integration results in the well known numbers associated with the Gell-Mann-Low beta-function. However, one should keep in mind that if the ln \( \epsilon \)-term is not miraculously canceled due to some interference process and has to be kept finite, then the magic \( "(11 N_c / 3)" \) becomes an approximate number. In fact, the cut-off \( \epsilon \) can be re-expressed via the lower limit of the invariant mass of the two final-state gluons and it has to be even kept in the finite quantity \( \beta_2 \).

To the lowest order, the retarded self-energy of a gluon can be computed via its imaginary part. This statement is not trivial. Indeed, the retarded self-energy is as follows,

\[
\Pi_{\text{ret}}^{\alpha\beta,ab}(p) = -\frac{i}{4} \int \frac{d^4q}{(2\pi)^4} \left[ V_{f \alpha c}(q - p, p, -q) D_{10}^{bc,\alpha \sigma}(q) V_{c \beta f}(q, -p, p - q) D_{10}^{f,\nu \mu}(q - p) + V_{f \alpha c}(q - p, p, -q) D_{10}^{bc,\alpha \sigma}(q) V_{c \beta f}(q, -p, p - q) D_{10}^{f,\nu \mu}(q - p) \right].
\]  

(6.10)

Though in both terms we encounter the retarded propagators, only their transverse parts are truly causal. Thus, if we wish to use the dispersion relations to calculate the real part of \( \Pi_{\text{ret}} \) (which is responsible for the phase shifts in the propagation of the gluon field) via the imaginary part (which is directly connected with the real processes), we have to separate causal and non-causal parts in Eq. (6.11),

\[ D_{\text{ret}}(q) = D_{\text{ret}}^{(T)}(q) + D_{\text{ret}}^{(L)}(q), \quad D_{\text{adv}}(q) = D_{\text{adv}}^{(T)}(q) + D_{\text{adv}}^{(L)}(q). \]  

(6.11)

Thus, we obtain the dispersive part of the retarded (advanced) self-energy,

\[
\Pi_{\text{ret}}^{(D)}(p) = -\frac{i}{4} \int \frac{d^4q}{(2\pi)^4} \left[ V(q - p, p, -q) D_{\text{ret}}^{(T)}(q) V(q, -p, p - q) D_{\text{ret}}^{(T)}(q - p) + V(q - p, p, -q) D_{\text{ret}}^{(T)}(q) V(q, -p, p - q) D_{\text{ret}}^{(T)}(q - p) \right],
\]  

(6.12)
and the non-dispersive part,

$$\Pi^{(ND)}_{\text{ret}}(p) = -\frac{i}{4} \int \frac{d^4 q}{(2\pi)^4} \left[ V(q-p,p,-q)D^{(L)}(q)V(q,-p,q)D_1(q-p) + V(q-p,p,-q)D_1(q)V(q,-p,q)D^{(L)}(q) \right]. \tag{6.13}$$

The difference,

$$\Pi_0 = \Pi_{\text{ret}} - \Pi_{\text{adv}} = \Pi^{(D)}_{\text{ret}} - \Pi^{(D)}_{\text{adv}} = \Pi_{01} - \Pi_{10}, \tag{6.14}$$

is the imaginary part of the retarded self-energy, which has been already computed. Projecting onto the two normal modes, we obtain the real part of the invariants as the dispersion integral of the imaginary part with one subtraction at some value $p^- = -\Omega$,

$$\text{Re} w_2^\text{ret}(p) \equiv w_2^\text{ret}(p) = \frac{1}{\pi} \int d\omega^- \left( \frac{1}{\omega^- - p^-} - \frac{1}{\omega^+ + \Omega} \right) \text{Im} w_2^\text{ret}(\omega), \tag{6.15}$$

where

$$\text{Im} w_2^\text{ret}(\omega) = \frac{g^2 N_c}{16\pi} \theta(p^+\omega^- - p_i^2) \text{sign}(p^+ \beta_j(p^+, \epsilon). \tag{6.16}$$

An explicit calculation of the dispersion integral results in the renormalized retarded self-energy,

$$\text{Re} w_2^\text{ret}(p) = -\frac{g^2 N_c}{16\pi^2} \beta_j(p^+, \epsilon) \ln \frac{\mu^2}{-p^2}, \tag{6.17}$$

where the subtraction point $p^- = -\Omega$ is translated into $p^2 = -\mu^2 = -p^+\Omega - p_i^2$.

The non-dispersive part identically vanishes for the real part of $w_2^\text{ret}$ (because of the polarization properties of the three-gluon vertex). For the non-renormalized non-dispersive function we obtain a divergent integral,

$$\text{Re} \left[ p^2 w_1^{(ND)}(p) \right] = -\frac{g^2 N_c}{2(2\pi)^3} \int dq^+ dq^+ \left[ \frac{1}{|q^+|} \left( \frac{q^+ + p^+}{q^+ - p^+} \right)^2 + \frac{1}{|q^+ - p^+|} \left( \frac{q^+ - 2p^+}{q^+} \right)^2 \right], \tag{6.18}$$

which is independent of $p^-$ and is, therefore, a quasi-local function proportional to $\delta(x^+)$. It identically vanishes after one subtraction at any value of $p^-$. Thus, after the renormalization, the retarded self-energy is totally causal both for the transverse and the longitudinal modes. An explicit choice of the renormalization point will be done later (together with the virtual vertex corrections).

**B. General properties of the virtual vertex corrections**

Computing this type of correction is most important for our goals because the vertex strongly depends on the renormalization condition and is affected by infrared singularities. It is a very sensitive tool for the physical analysis of various renormalization prescriptions and from this calculation we draw our major conclusions. The part of the gluon polarization density of the order $g^4$ with the virtual vertex corrections has the following form,

$$\left[ \Pi_{\text{ret}}^{\beta,ab}(p) \right]_{\text{V}} = \frac{g^2 N_c}{2} \int \frac{d^4 k}{(2\pi)^4} V^{a_1,\alpha_2}(q-p,p,-q)V^{\rho_2,\rho_1}(q-k,q,-k) \times \left[ -D_{\text{ret}}^{\beta,\rho_1}(k)D_{\text{adv}}^{\rho_1,\rho_2}(k-p) + D_{\text{ret}}^{\rho_1,\rho_2}(k)D_{\text{adv}}^{\rho_1,\rho_2}(k-p) \right]. \tag{6.19}$$
By virtue of the relations, \(D_{00} = D_s + D_1/2\) and \(D_{11} = -D_s + D_1/2\), the string, \(D_{00}(q)D_{00}(q - p)D_{10}(q - k) + D_{11}(q)D_{11}(q - p)D_{11}(q - k)\), which consists of propagators that form the loop of the virtual vertex, can be rearranged into

\[
D_1(q)D_s(k - p)D_s(q - k) + D_s(q)D_1(k - p)D_s(q - k) + D_s(q)D_s(k - p)D_1(q - k). \tag{6.20}
\]

**FIG. 5.** Virtual vertex corrections. The original sum \(V_{000} + V_{111}\) can be transformed into \(V_{\text{ret}} + V_{\text{adv}}\), thus providing the causal behavior of the ladder with the radiative corrections in the order \(\alpha_s^2\).

In Appendix 1, we show that the vertex function inside (7.14), rewritten in the form (6.20) is, in fact, the real part of the retarded vertex, viz. a triangle graph in which the space-time point \(x\) (corresponding to the final momentum \(p\)) is the latest point on a real time scale. Therefore, one can safely replace the original \(V_{000}\) by \(V_{\text{ret}}\) and \(V_{111}\) (of the conjugated graph) by \(V_{\text{adv}}\), since the non-causal remainders do not contribute to the virtual vertex corrections. Thus, we proved that the virtual loop corrections do not break the causal picture of the QCD evolution, which has been established above, at the tree level. Two retarded virtual vertex loops (depicted at Figs. 5c and 5d) lead to the following analytic expression

\[
\left[\Pi_{01}^{\alpha\beta, ab}(p)\right]_{VV} = \delta^{ab}[\gamma^{\alpha\beta} p^2 \omega_1^{(01)}(p) + p^\alpha p^\beta \omega_2^{(01)}(p)] = \delta^{ab} \frac{-iN_c^2}{2} \int \frac{d^4k}{(2\pi)^4} 2\text{Re}^{(3)}V_{\text{ret}}^{\sigma_1\rho_2}(k - p, p, -k) \times
\]

\[
\times \left[\bar{d}_{\rho_2} \delta_{\beta_2} k^2 \omega_1^{(01)}(k) W_2^{(\sigma_1)}(k) W_1^{(\rho_2)}(k) + n_{\rho_2} n_{\sigma_1} w_2^{(01)}(k) W_2^{(\sigma_1)}(k) W_1^{(\rho_2)}(k)\right] V_{\beta_2\beta_1}(k, -p, p - k) D_{10}^{\rho_2, \sigma_1}(k - p). \tag{6.21}
\]

Since the QCD evolution fully develops before the measurement, the physical system of the emitted fields must be coherent. These fields represent, at most, a virtual decomposition of the hadron in terms of a set of modes which should emerge as propagating fields only after the interaction happens. This understanding of the QCD evolution provides us with the natural condition for the renormalization of the fields in the QCD evolution process. Namely, radiative corrections to the propagation of the fields should not cause any phase shifts along the line of propagation of the hadron. If the hadron is assumed to move along the light cone, the real parts of the vertices and the self-energies of the retarded propagators should vanish at \(p^- = 0\). Different subtractions are also explored below.

The retarded vertex function \(V_{\text{ret}}(p, k - p, -k)\) is proved to be analytic in the lower half-plane of the complex light-cone energy, \(p^-\). Equivalently, the advanced function \(V_{\text{adv}}(p, k - p, -k)\) is analytic in the upper half-plane of the complex light-cone energy, \(p^-\). According to the Schwartz symmetry principle, they form a single analytic function in
the entire complex plane with the cuts along the segments of the real axis where the imaginary part of this function does not vanish. These segments are $p^t/p^+ < p^- < \infty$ and $-\infty < p^- < -(\vec{p}_t - \vec{k}_t)^2/(k^+ - p^+)$. Applying the Cauchy integral formula to the contour depicted at Fig. 6 one can prove the dispersion relation (with the subtraction at the point $p^- = -\Omega$),

$$\text{Re}^{(3)} V^{(D)}_{ret}(p, -k, k - p) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1}{\omega - p^-} \left( \frac{1}{\omega - p^+} - \frac{1}{\omega + \Omega} \right) \text{Im}^{(3)} V^{(D)}_{ret}(\omega, -k, k - \omega), \quad (6.22)$$

where a four-vector $\omega^\mu = (p^+, \omega^-, \vec{p}_t)$ was introduced for brevity. The subtraction in the dispersion relation $(6.22)$ performs an ultraviolet renormalization of the vertex function. The subtracted term is quasi-local and does not affect the causal properties of the vertex. The subtraction can only be real and can be performed only at a point where the imaginary part is zero. This limits the position of the subtraction point to the segment

$$\frac{-(\vec{p}_t - \vec{k}_t)^2}{k^+ - p^+} < p^- < \frac{p^t_2}{p^+}, \quad (6.23)$$

of the real axis, where $\text{Im}^{(3)} V_{ret} = \text{Im}^{(3)} V_{adv} = 0$. Position $p^- = -\Omega$ of the subtraction point can be translated into the value of $p^2$, $p^2 = -\mu^2 = -p^+\Omega - p^t_2$. (Since we discuss the vertex function in its natural environment (which includes the mass-sell delta-function of the emitted gluon, $\delta[(k - p)^2]$) $p^2$ should be replaced by $-k^+(\vec{p}_t - \vec{k}_t)^2/(k^+ - p^+)$. Under an assumption of strong ordering of the transverse momenta, $p_t >> k_t$, the allowed values of $\mu^2$ are

$$0 < \mu^2 < p^t_2. \quad (6.24)$$

FIG. 6. Contour in the plane of complex $p^-$ used in the proof of the dispersion relations for the vertex function. It is closed by two arcs at the infinity.

This inequality may look abnormal since it does not allow one to choose the renormalization point for the vertex function (corresponding to the currently resolved momentum, $p$), e.g., at $p^2 = Q^2$ (the much higher momentum which is measured at the end point of the QCD evolution). Such a choice would be almost obvious and even preferable in the S-matrix approach to the QCD-evolution which does not require any space-time ordering in the evolution process. Nevertheless, in our approach, the upper boundary in the inequality $(6.24)$ is a strict consequence of causality. Indeed, the momentum transfer $Q$ will be detected later on, and it cannot affect the earlier loops which correspond to the currently resolved momentum, $p$. Causality forbids one to renormalize what is not yet resolved. However, we are free
to choose for the renormalization any of the already resolved lower scales. The limits prescribed by the inequalities \[ (3.39) \] and \[ (1.24) \] are obtained from the dispersive approach which expresses the real parts of the loops (regulating the phase shifts) via the imaginary parts (which are related to the real emission processes and are causally ordered in space and time). The renormalized real parts of the propagators and vertices will eventually be combined into the effective coupling. Thus, the running coupling itself becomes a dynamically-defined quantity that experiences evolution not only in terms of the momenta, but also in terms of the space-time variables. This is manifestly in contrast with the S-matrix theory which achieves a running regime of the coupling constant by means of the renormalization group method and assigns the same value of coupling to the entire scattering process.

**VII. RETARDED VERTEX IN THE TT-MODE**

The dispersive part of the \( V_{\text{ret}} \) yields the following term in the evolution,

\[
\left[ \frac{dG(p^+, p^2)}{dp_i^2} \right]^{V,(D)}_{TT} = \frac{g^4 N_c^2}{4(2\pi)^6} \int \frac{dk^+dk_1 dk^+_2}{2k^+_2} \Phi_1(p,k) + p^+ \Phi_2(p,k) \]

(7.1)

where the kernels \( \Phi_1(p,k) \) and \( \Phi_2(p,k) \) come from the dispersion integrals along the right and left cuts, respectively.

\[
\Phi_1(p,k) = \int dp^+ \frac{\delta \left[ \frac{(k-p)^2}{p^2} \right]}{k^2} \int \frac{d(\omega^+ + \Omega) d\omega^-}{(\omega^- - p^-)(\omega^- + \Omega)} \int dq^+ dq^- \delta_+ \left[ \frac{(q^2 + ((q - \omega)^2)}{q^2} \right] \times 8 \left\{ 2C_1 p^2(\omega^- - p^-) - 2C_2(k - q)^2(\omega^- - p^-) - C_3(\omega^- - p^-)^2 + C_4((k - q)^2)^2 + C_5 [p^2]^2 - 2C_6 p^2(k - q)^2 \right\},
\]

(7.2)

\[
\Phi_2(p,k) = -\int dp^+ \frac{\delta \left[ \frac{(k-p)^2}{p^2} \right]}{k^2} \int \frac{d(\omega^+ + \Omega) d\omega^-}{(\omega^- - p^-)(\omega^- + \Omega)} \left\{ \int dq^+ dq^- \delta_+ \left[ \frac{(q^2 + ((q - \omega)^2)}{q^2} \right] \times 8 \left\{ 16H_1 p^2 q^2 + 16H_2 q^2(\omega^- - p^-) - 16H_3(\omega^- - p^-)^2 - 8H_4(\omega^- - p^-)^2 + 8H_5 [p^2]^2 + 8H_6 [q^2]^2 \right\} \right\},
\]

(7.3)

where the expressions in curly brackets are the result of computing the trace of the product of many Lorentz matrices. The functions \( C_j = C_j(k^+, p^+, q^+) \) and \( H_j = H_j(k^+, p^+, q^+) \) are lengthy rational functions. They were obtained and further handled with the aid of Mathematica and FeynCalc. The terms proportional to \( k^2 \) are omitted because of the assumption of strong ordering of the transverse momenta along the ladder, \( k^2 << p_i^2 \). Here we give only an explanation of the main steps. As is clearly seen from (7.1) and (7.2), (7.3), the real part of the retarded vertex is not calculated and renormalized separately. It is put (in its tensor form) into the environment of the ladder in the form of the dispersive part of the vertex \( 8 \left\{ \cdots \right\} \). The terms proportional to \( k^2 \) are omitted because of the assumption of strong ordering of the transverse momenta along the ladder, \( k^2 << p_i^2 \). Here we give only an explanation of the main steps. As is clearly seen from (7.1) and (7.2), (7.3), the real part of the retarded vertex is not calculated and renormalized separately. It is put (in its tensor form) into the environment of the ladder in the form of the dispersion integral with one subtraction, and only after that is the trace computed. We had to take this approach, because it turned out to be very difficult to find the tensor decomposition of the third rank vertex tensor (similar to Eq. (3.39) for the self-energies). Technically, the calculations are conducted as follows. We start with integrating out the variables \( q^- \), \( \omega^- \), and \( p^- \), with the aid of three delta functions. Then, we integrate over the angle in the plane of the vector \( q_i \). All these integrations are finite. The last step is integration over \( q_i = |q_i| \). Some of the integrals happen to be divergent at lower or upper limits of integration, or even at both of them. However, due to miraculous relations between different coefficients \( C_j \) and \( H_j \), the sums of the integrals are finite. They are as follows:

\[
p^+ \Phi_1(p,k) = \frac{8\pi}{p_i^2} \int_0^{p^+} dq^+ k^+ \left( \frac{k^+ - q^+}{p^+} \right)^2 C_4 \ln \left( 1 + \frac{(k^+ - q^+)^2}{p^+(k^+ - q^+)} \right) - \left( \frac{(k^+ - p^+)^2}{p^+} \right) - \left( \frac{(q^+)^2}{q^+} \right) \ln \left( 1 + \frac{(q^+)^2}{(p^+ - q^+) \mu^2} \right) \right),
\]

(7.4)
\[ p^+ \Phi_2(p, k) = \frac{8\pi}{p_t^2} \int_{p^+}^{k^+} \frac{dq^+}{k^+} \left( \frac{k^+ - q^+}{p^+} C_4 \ln \left( 1 + \frac{k^+(p^+ - q^+)}{p^+(k^+ - q^+)} \right) - \frac{C_3}{k^+(k^+ - q^+)} \left( \frac{1 + \frac{k^+(p^+ - q^+)}{k^+ p_t^2}}{1 + \frac{k^+(p^+ - q^+)}{k^+ p_t^2}} \right) \right) \]  

(7.5)

Recall that the parameter \( \mu^2 = p^+ \Omega + p_t^2 \) is an equivalent of the original subtraction point at \( p^- = -\Omega \). The remaining integration over the light-cone component \( q^+ \) (of the momentum which flows around the loop) is singular at the end points. We have carefully checked that no spurious poles connected with the incomplete fixing of the gauge for the longitudinal part of the gluon field is involved in the formation of these physical singularities. They all are collinear (mass) singularities associated with the end points of the allowed phase space. Moreover, these singularities already appear in the calculation of the imaginary part of the vertex function thus being connected to real processes in the complementary domain of the external momenta. Hence, we are allowed to introduce a cut-off parameter \( \epsilon \) (to shield the singular behavior), which defines an actual resolution of the process and serves as a definition of a part of the newly created gluon field, that should not be treated as real emission.

Before the integration in Eqs. (7.4) and (7.5), it is expedient to select a large parameter that defines major contribution to the final answer. The logarithms \( \ln (p_t^2 / \mu^2) \) are good candidates since they can be large if the ratio \( p_t^2 / \mu^2 \) is either very large or very small. Traditionally, one would wish to associate \( \mu^2 \) with the highest momentum transfer \( Q^2 \) in the process. Unfortunately, inequality (6.23) deprives us this kind of luxury. Moreover, equations (7.4) and (7.5) contain \( \mu^2 \) in the first power, which also makes \( \mu^2 = Q^2 \) very unlikely. Thus, if we wish to hunt for large logarithms of \( p_t^2 / \mu^2 \) formally, we are forced to choose the renormalization point somewhere near the left tip of the right cut in the plane of complex \( p^- \). This corresponds to the on-mass-shell (with respect to the momentum \( p^\mu \) ) renormalization of the vertex function, that is, in the domain which is most unphysical in the context of the QCD evolution. We even start the perturbative expansion with coupling typical for the non-perturbative domain. Regardless to these reservations, we formally obtain \( p_t^2 / \mu^2 \gg 1 \) as a large parameter and can continue expecting to arrive at the leading logarithmic approximation (LLA) with the largest term \( \sim \ln (p_t^2 / \mu^2) \). The explicit calculation yields

\[
\left[ \frac{dG(p^+, p_t^2)}{dp_t^2} \right]_{TT}^{VV,(D)} = \frac{4 N_c^2}{(2\pi)^6} \frac{2 \pi}{p_t^2} \int \frac{dk^+ dk_t^+ dk_G(k^+, k_t^+)}{2k^+} \left\{ \frac{1 - 2z}{6} \right. + \frac{2}{3} \ln \frac{p_t^2}{\mu^2} - 2 \ln \frac{p_t^2}{\mu^2} \ln \frac{p_t^2}{(1-z)\mu^2} \\
+ \ln^2 \frac{p_t^2}{\epsilon} - \frac{11}{3} \ln \frac{1-z}{z} + 2 \ln \frac{p_t^2}{\epsilon} \ln \frac{1-z}{z} - 4 \ln \frac{1-z}{z} \right\},
\]

(7.6)

where \( z = p^+/k^+ \). The answer occurs to be doubly logarithmic which, may invalidate the perturbative expansion, because no terms \( \sim \ln^2 p_t^2 \) appear in other diagrams which could compensate it in the vertex.

There is no other way to proceed except for changing the renormalization point. Let us make the new choice relying on the causal picture of the process of the QCD evolution, which has been discussed previously, and make a subtraction at the point \( p^- = -\Omega = 0 \) which corresponds to \( \mu^2 = p_t^2 \). Then the parameter \( \mu^2 / p_t^2 \) becomes unity, and equations (7.4) and (7.5) acquire the following form:

\[
p^+ \Phi_1(p, k) = \frac{8\pi}{p_t^2} \int_{p^+}^{k^+} \frac{dq^+}{k^+} \left( \frac{C_3}{(k^+)^2(p^+ - q^+)} \left\{ \ln \left( 1 + \frac{p^+}{k^+} \right) - \frac{p^+}{k^+} \ln \left( 1 + \frac{q^+}{p^+} \right) \right\} - \frac{k^+ - q^+}{k^+ p_t^2} C_4 \ln \frac{k^+ - q^+}{k^+ - p^+} \right),
\]

(7.7)

\[
p^+ \Phi_2(p, k) = \frac{8\pi}{p_t^2} \int_{p^+}^{k^+} \frac{dq^+}{k^+} \left( \frac{-p^+ C_3}{(k^+)^3(p^+ - q^+)} \ln \frac{k^+ - p^+}{q^+ - p^+} - \frac{k^+ - q^+}{k^+ p_t^2} C_4 \ln \frac{k^+ - q^+}{k^+ - p^+} \right).
\]

(7.8)

The remaining integration is straightforward and results in the expression
It is interesting to trace the origin of the $\log^2(p^+/\epsilon)$ in this formula. Originally, the calculation leads to three doubly-logarithmic terms,

$$z \ln^2 \frac{p^+}{\epsilon} - z \ln^2 \frac{k^+}{\epsilon} + \ln^2 \frac{h^+}{\epsilon},$$

of which the first term comes from the vertex cut through the momentum $p$, while the second and the third terms come from the cut through the momentum $k - p$. Even though there is partial cancelation between the first and the second terms (which removes two of the double logs), this is not a systematic effect. Indeed, since the kinematic regions of two cuts do not intersect, this cancelation cannot be an effect of destructive interference.

Finally, we have a correction due to the retarded self-energies $D_{\text{(ret)}}(p)$. They are depicted in Fig. 7. The result with the arbitrary subtraction point is as follows,

$$\left[ \frac{dG(p^+, p_t^2)}{dp_t^2} \right]^{V,(D)}_{TT} = \frac{g^4 N_c^2}{(2\pi)^6} \int \frac{dk^+ d\vec{k}_t}{2k^+} \frac{dG(k^+, k_t^2)}{dk_t^2} \left\{ -2\frac{2}{5} + 11z^2 - 6z^3 \right\} +$$

$$+ \mathcal{P}_{gg}(z) \left[ \pi^2 \frac{2 + z}{6} - \frac{11}{3} + (1 - 2z) \ln^2 z + \frac{3}{2} - z \right] \ln^2 (1 - z) -$$

$$- \frac{11}{3} \ln (1 - z) + \left( 6 \ln (1 - z) - 2 \ln z - 2 \ln \frac{1 - z}{z} \right) \ln \frac{p^+}{\epsilon} + \ln^2 \frac{p^+}{\epsilon} \right\}.$$

(7.9)

Since the renormalization of the self-energy of the propagator is subject to the same condition as the vertex function, we have to take $\mu^2 = p_t^2$ and thus gain no large $\ln p_t^2$ coming from the ultraviolet renormalization.

Several remarks are in order:

1. The dispersive part of the retarded vertex function, renormalized in accordance with the requirement of causality, does not contain any $p_t^2$-dependent large numbers coming from the UV cut-off. The non-dispersive part of the retarded vertex $\text{Re}^{(3)}(V_{\text{ref}}^{(N)}(D))$ may undermine the causal picture of the evolution, but it naturally combines with the residue $\Delta \text{Re}^{(3)}(V_{\text{ref}}^{(D)})$, so that in the TT-mode of the evolution equations they together produce the $p^-$-independent quasilocal function, which identically vanishes after renormalization. This is demonstrated by an explicit calculation in Appendix 2. The same is true for the terms originating from the four-gluon interaction. They prove to be quasilocal as well, and vanish after subtraction at any value of $p^-$. The proof is also given in Appendix 2.
2. The retarded self-energy with momentum $p^\mu$ also must be renormalized by the subtraction at the point $p^- = 0$, and thus carries no logarithmic dependence on $p^2$. Therefore, we come to the conclusion, that in the causal picture of QCD evolution, the renormalization of virtual loops does not produce large logarithms of $p^2$. Hence, the only source of these logarithms may be the real processes.

3. Numerous collinear cut-offs in Eqs. (7.9) and (7.10) appear already at the level of the calculation of the imaginary parts of the retarded vertex and retarded self-energy (unitary cuts) and thus are exclusively due to real processes. These singularities are intimately connected with the definition of emission as a physical process and require the parameter of resolution.

4. The expected large logarithms of $p^2$ which are known to drive the QCD evolution in the DGLAP equations cannot be an artifact of the renormalization prescription. Hence, they have to be outside of the jurisdiction of the renormalization-dependent virtual loops and, therefore, must originate from the real-emission processes. The renormalization prescription itself has to be motivated physically.

VIII. REAL EMISSION IN THE TT-MODE

There are four diagrams which contribute to real emission. Three of them originate from the vertex correction (3.7) with $R \neq S$ (Fig. 8a,b,c). They differ by the place where the initial state density $D^\ast_{01}$ is built into the diagram. If $D^\ast_{01}$ is put on one of the lines which enters the external vertex of $^4\Pi_{01}$ we obtain the diagrams which include the cut vertex (Fig. 8a,b). If $D^\ast_{01}$ replaces the internal line $D_{RS}$, then we obtain a fragment of the ladder with the crossed rungs (Fig. 8c, the interference term for the emission of two off-spring gluons). The fourth diagram (Fig. 8d) is a direct descendent of the simple ladder diagram and its order is enhanced due to the further splitting of the decoupled gluon.

![FIG. 8. Real processes in the order $\alpha_s$.](image)

The two cut vertex diagrams, added together, have a fragment

$$D_{00}(q)D_{11}(k-p) + D_{11}(q)D_{00}(k-p) = \frac{1}{2}D_1(q)D_1(k-p) - 2D_s(q)D_s(k-p).$$

(8.1)

The first singular term on the right side contributes nothing, while both propagators of the second term acquire a principal value prescription. The analytic expression for the two cut vertex diagrams is as follows,
where $R_j = R_j(k^+, p^+, q^+)$ are lengthy rational functions. The general line of calculations is approximately the same as in the case of the virtual vertex. First, we integrate out the variables $q^-$ and $p^-$ using the two delta functions. Then, we integrate over the angle in the plane of the vector $\vec{q}_t$. As previously, these integrations are finite. The last step is the integration over $q_t = |\vec{q}_t|$. Some of the integrals are divergent but, due to relations between different coefficients $R_j$ the sum of the integrals is finite. It already contains $\ln(k^+ − q^+)$; therefore, the final integration $dq^+$ between $p^+$ and $k^+$ is double-logarithmic. The full answer is,

$$
\left[\frac{dG(p^+, p_t^2)}{dp_t^2}\right]^{RV}_{TT} = \frac{g^4 N_c^2}{(2\pi)^6} \int \frac{dk^+ d\vec{k}_t}{2k^+} \frac{dG(k^+, k_t^2)}{dk_t^2} \int \frac{dp^-}{|p_t^2|_{P.V.}} \int d^4q^\pm \frac{\delta_+(\vec{q} - q^\pm)\delta_+(q - p^\pm)}{|q^2|_{P.V.}} \times 
\{ R_1(\vec{q} - k^+)^2 + R_2(q^2) + R_3(q^2) + R_4(p^2)^2 + R_5(p^2) + R_6p^2(\vec{q} - k)^2 + R_6p^2q^2 \},
$$

(8.2)

Here, the term $\ln^2(\frac{p^+}{\epsilon})$ is a partner of the similar term in the virtual vertex (7.9) which originates from the dispersion cut due to the same real process. If a cancelation between these terms had taken place, one could have considered it as a result of the interference (which is possible at least owing to the same geometry of the emission.) However, these terms not only differ by a factor of two, but also have the same sign.

The interference diagram (Fig. 8c) is calculated in a similar way to the cut-vertex diagram. The initial analytic expression is

$$
\left[\frac{dG(p^+, p_t^2)}{dp_t^2}\right]^{INT}_{TT} = \frac{2g^4 N_c^2}{(2\pi)^6} \int \frac{dk^+ d\vec{k}_t}{2k^+} \frac{dG(k^+, k_t^2)}{dk_t^2} \int \frac{dp^-}{|p_t^2|_{P.V.}} \int d^4q^\pm \frac{\delta_+(\vec{q} - q^\pm)\delta_+(q - p^\pm)}{|q^2|_{P.V.}} \times 
\{ U_1[\kappa^2]^2 + U_2q^2\kappa^2 + U_3q^2 + U_4[p^2]^2 + U_5p^2\kappa^2 + U_6p^2q^2 \},
$$

(8.4)

where $\kappa = p + k - q$. The final answer can be cast in a form

$$
\left[\frac{dG(p^+, p_t^2)}{dp_t^2}\right]^{INT}_{TT} = \frac{g^4 N_c^2}{(2\pi)^6} \int \frac{dk^+ d\vec{k}_t}{2k^+} \frac{dG(k^+, k_t^2)}{dk_t^2} \left\{ 2P_{gg}(-z) \left( \frac{\pi^2}{6} + 2\text{Li}_2(-z) + \ln(1 + z) \ln z \right) - \frac{(1 - z)(1088 + 143z + 1088z^2)}{36z} + \frac{(1 + z)(22 + 29z + 22z^2)}{6z} \ln z - P_{gg}(z) \left( \ln^2 z + 4 + \frac{k^+ - p^+}{\epsilon} \right) \right\}.
$$

(8.5)

Neither the “cut vertex”, nor the “interference term” has logarithms $\ln p_t^2$, but both include logarithms $\ln(1/x)$, which are not small at low $x$.

The last of the three diagrams related to real processes is due to the decoupled heavy cluster as an emission field in the final state. To the lowest possible order, one can mimic it by the diagram in Fig. 8d, where the “heavy off-spring gluon” decays into two massless gluons. In the lowest order, we have to keep the propagator of the heavy gluon bare, which results in the following expression,

$$
\left[\frac{dG(p^+, p_t^2)}{dp_t^2}\right]^{HG}_{TT} = \frac{ig^2 N_c}{(2\pi)^4} \int \frac{dk^+ d\vec{k}_t}{2k^+} \frac{dG(k^+, k_t^2)}{dk_t^2} \int_{-\infty}^{(k^+ - \kappa)^2 + m^2} dp^- \times 
\left\{ 8P_{gg}(z) \left[ \frac{1}{p^2} + \frac{z}{1 - z} \frac{(k - p)^2}{m^2} \right] \frac{w^{#01}_{k}(k - p)}{(k - p)^2} + 2z \left[ \frac{1 + z}{1 - z} \right] \frac{w^{#01}_{k}(k - p)}{(k - p)^2} \right\},
$$

(8.6)

where the cut-off mass $m^2$, is defined by the properties of the real emission processes, of which the most important is the existence of the minimal mass $m$ of the emitted jet, which is of the same order of magnitude as the hadronic mass. The two integrations $dp^-$ are straightforward:
\[
\int \frac{(\xi - p)^2 + m^2}{p^2 - (k - p)^2} \frac{dp^-}{p^2} = \frac{1}{k^+ p_t^1} \ln \frac{m^2 p^+}{p_t^1 k^+} \quad \int \frac{(\xi - p)^2}{|p^2|^2} \frac{dp^-}{p^2} = 1 - \frac{z}{p^+ p_t^1}. \tag{8.7}
\]

Thus, we have obtained the first and the only logarithm of \( p_t^2 \) which is scaled by a hadronic mass. This scale does not coincide with an artificial renormalization scale \( \mu^2 \). This is a real emission process, which produces large logarithms, not virtual loops! The final answer becomes

\[
\times \left\{ \mathcal{P}_{gg}(z) \left[ -\frac{11}{6} + 2 \ln \frac{k^- + p^+}{\epsilon} \right] \left[ 1 + \ln \frac{m^2 p^+}{p_t^1 k^+} \right] - \frac{1}{24} \frac{(k^+ + p^+)^2}{k^+ (k^- + p^-)} \right\}.
\tag{8.8}
\]

Since we have isolated the mechanism responsible for the large log's, it is natural to try to elevate its status. This does not need special effort, since we have already started from the dressed form of the correlator, which corresponds to the emission rung, \( D^\#_{ret}(k - p)\Pi^\#_{10}(k - p)D^\#_{adv}(k - p) \). This string is nothing but the imaginary part of the full propagator \( D^\#_{ret}(k - p) \) in the kinematic region \( k^+ - p^+ > 0 \). Indeed, since the polarization loop \( \Pi^\#_{10}(q) \) vanishes at positive \( q^+ \), we can write a chain of transformations,

\[
- [D_{ret}(p)\Pi_{10}(p)D_{adv}(p)]^\mu_\nu = -D_{ret}(\Pi_{10} - \Pi_{01})D_{adv} = D_{ret}(\Pi_{ret} - \Pi_{adv})D_{adv} = (D_{ret}D^{-1}_{ret})D_{adv} - D_{ret}(D^{-1}_{ret}D_{adv} - 1) = D_{ret} - D_{adv} + D_{ret}(D^{-1}_{ret} - D^{-1}_{adv})D_{adv} = 2i \text{ Im } D_{ret} = 2i \text{ Im } \left[ \frac{\tilde{d}^{\mu\nu}(p)}{p^2(1 - w_{ret}(p))} + \frac{n^\mu n^\nu}{(p^+)^2(1 - w_{adv}(p))} \right]. \tag{8.9}
\]

Substituting the last string of this form for the final-state density of the radiated glue, we arrive at

\[
\times \left\{ 8\mathcal{P}_{gg}(z) \left[ \frac{1}{p^2} + \frac{z}{1 - z} \frac{(p - k)^2}{|p^2|^2} \right] \frac{1}{(k - p)^2(1 - w_{ret}(k - p))} + 2z \left( \frac{1 + z}{1 - z} \right)^2 \frac{1}{|p^2|^2} \frac{1}{(1 - w_{adv}(k - p))} \right\}.
\tag{8.10}
\]

To compute the integrals \( dp^- \) in this equation we may use the Schwartz symmetry principle. Indeed, basic analytic properties of the retarded and advanced gluon self-energies in the plane of complex \( p^- \) stimulate similar analytic properties of the gluon propagator. Therefore, the real integration can be replaced by the complex integration along the path \( \mathcal{C} \), which envelopes the cut in the complex plane of the function

\[
\tilde{w}_j(k - p) = \begin{cases} 
 w_{j,ret}(k - p), & \text{Im } p^- < 0, \\
 w_{j,adv}(k - p), & \text{Im } p^- > 0
\end{cases} \tag{8.11}
\]

and is closed by a big circle at the infinity. Then the integral is calculated as the residue in the pole of the integrand. For example,

\[
\int_{-\infty}^{(\xi - p)_+} \frac{dp^-}{p^+ p^- - p_t^2} \text{ Im } \frac{1}{(k - p)^2(1 - w_{ret}(k - p))} = \int_{C} \frac{dp^-}{p^+ p^- - p_t^2} \frac{1}{(k - p)^2(1 - \tilde{w}_1(k - p))} = -\frac{\pi}{k^+ p_t^2} \frac{1}{1 - \text{Re} w_{ret}(k - p; p^- = p_t^2/p^+)}.
\tag{8.12}
\]

Two properties of the integrand should be noted. First, the pole at the point \( (k - p)^2 = 0 \) (corresponding to the free propagation) is obviously located at the second sheet and does not contribute to the integral (this pole has even to be read with the principal value assignment). Second, at the pole \( p^- = p_t^2/p^+ \), the function \( \tilde{w}(k - p) \) is real. The
other two integrals are computed in the same way, the only difference being, the pole at $p^2 = 0$ is of the second order. In the simplest case, when the final-state jet is simulated via a two-gluon system, the functions $\text{Re} w^{\mu\nu}(k - p)$, which enter the final answer are,

$$\text{Re} w^{\mu\nu}(k - p) = \frac{g^2 N_c}{16\pi^2} \beta_j(k^+ - p^+, \epsilon) \ln \frac{\mu^2}{-(k-p)^2}.$$  \hspace{1cm} (8.13)

We have to substitute the value of the argument in the pole, $-(k-p)^2 = p_t^2/z$, (in the approximation of strong ordering, $p_t \gg k_t$) into Eq. (8.13). Eventually, we express the result in terms of two effective couplings: the transverse coupling $\alpha_T(p_t^2/z)$ due to the transverse mode of the emitted heavy gluon, and the longitudinal coupling $\alpha_L(p_t^2/z)$, due to the longitudinal mode of the emitted gluon, i.e.

$$\left[ \frac{dG(p^+, p^2_t)}{dp^2_t} \right]_{TT}^{\mu\nu} = \frac{N_c}{2(2\pi)^3} \frac{8\pi}{p_t^2} \int \frac{d^4k^+}{2k^+} \frac{dG(k^+, k^2_t)}{dk^2_t} \{ P_{bb}(z) (\alpha_T(p_t^2/z) - \beta_T \alpha_T^2(p_t^2/z) + \frac{(1+z)^2}{1-z} \beta_L \alpha_L^2(p_t^2/z) \}, \hspace{1cm} (8.14)$$

where,

$$\beta_T = \frac{g^2}{16\pi^2} \beta_1(k^+ - p^+, \epsilon), \hspace{1cm} \beta_L = \frac{g^2}{16\pi^2} \beta_2(k^+ - p^+, \epsilon),$$

$$\alpha_T = \frac{g^2/4\pi}{1 + \beta_T \ln \frac{\mu^2}{\mu^2 p_t^2}}, \hspace{1cm} \alpha_L = \frac{g^2/4\pi}{1 + \beta_L \ln \frac{\mu^2}{\mu^2 p_t^2}}.$$  \hspace{1cm} (8.15)

The running coupling with the same argument, $\alpha(p_t^2/z)$, was obtained by Dokshitzer et al. \textsuperscript{[12]} by means of the “dispersive method” based on an ad hoc dispersion formula for the effective running coupling. Our construction (Eqs. (8.14) – (8.15)) provides a formal proof to this approach. In general, we should not rely on a simple two-body model for the density of final states used in these equations. Instead, as is conceived in Ref. [12], we can consider real density of the final state jets which are not yet calculable from a theory, but can be extracted from the data. In our picture, with the proton (or nuclei) kept intact before the collision we have to renormalize the gluon self-energy $\Pi^{\mu}(p)$ at the point $p^- = 0$. Thus, we have to set $\mu^2 = p_t^2$ in Eqs. (8.14) and (8.15). Then the $p_t$-dependence of the effective coupling may only come from the the collinear cut-off, $\epsilon$, which accounts for the properties of the states of emission.

**IX. RADIATIVE CORRECTIONS IN THE LT-MODE**

There are no major differences in the calculations for this mode compared to the TT-mode, since we only change a projector. Therefore, we shall omit details in the analysis below. The dispersive part of the virtual vertex before the final $q^+$-integration is

$$[G(p^+, p^2_t)]^{V, (D)}_{LT} = \frac{g^2 N_c^2}{(2\pi)^3} \int \frac{dk^+ dk_t}{2k^+} \frac{dG(k^+, k^2_t)}{dk^2_t} \left\{ \int_0^{p^+} \psi(q^+, k^+, p^+) dq^+ \right. \right.$$  \hspace{1cm} \left. \times \left[ \ln \left( 1 + \frac{k^+(p^+ - q^+)}{p^+(k^+ - q^+)} \right) - \left( 1 + \frac{p^+ - q^+}{k^+ p_t^2} \right) \ln \left( \frac{k^+(p^+ - q^+)}{p^+(k^+ - q^+)} \right) \right] \right. + \left. \int_{p^+}^{k^+} \psi(q^+, k^+, p^+) dq^+ \ln \left( 1 + \frac{k^+(q^+ - p^+)}{p^+(q^+ - k^+)} \right) \right\},  \hspace{1cm} (9.1)$$

The UV-renormalization is performed by means of one subtraction at the point $p^- = -\Omega$. The first and the second integrals are obtained via the dispersion relation from the imaginary parts corresponding to the unitary cuts of the
Finally, we have a correction due to the retarded self-energies $\mu$. Since the renormalization of the self-energy of the propagator is subjected to the same condition as the vertex function, with the same choice of the renormalization point at $k_0$, the non-dispersive part of the vertex correction is not quasi-local. After renormalization, it looks like

$$\left[\mathcal{G}(p^+, p_I^2)\right]_{LT}^{VV,(D)} = \frac{g^4 N_c^2}{(2\pi)^5} \int \frac{dk^+ d\tilde{k}_i}{2k^+} \frac{dG(k^+, k_I^2)}{dk_i^2} \{ (2-z)(132 - 132z + 35z^2) - \frac{7(2-z)^2}{3z(1-z)} \ln(1-z) - \\
\frac{(2-z)(16 - 14z + z^2)}{2z^2(1-z)} \ln^2(1-z) - \frac{(2-z)^2(2 - 2z + z^2)}{z(1-z)^2} \text{Li}_2(z) \} \right. \quad (9.2)$$

Unlike the TT-mode, the non-dispersive part of the vertex correction is not quasi-local. After renormalization, it looks like

$$\left[\mathcal{G}(p^+, p_I^2)\right]_{LT}^{VV,(ND)} = \frac{g^4 N_c^2}{(2\pi)^5} \int \frac{dk^+ d\tilde{k}_i}{2k^+} \frac{dG(k^+, k_I^2)}{dk_i^2} \int_0^{p^+} dq^+ \psi_L(q^+, k^+, p^+) \ln \frac{(1-z)\mu^2}{p_I^2} \left. \right. \quad (9.3)$$

With the same choice of the renormalization point at $\mu^2 = p_I^2$, we obtain

$$\left[\mathcal{G}(p^+, p_I^2)\right]_{LT}^{VV,(ND)} = \frac{g^4 N_c^2}{(2\pi)^5} \int \frac{dk^+ d\tilde{k}_i}{2k^+} \frac{dG(k^+, k_I^2)}{dk_i^2} \left\{ \frac{(2-z)(8 - 8z + z^2)}{z^2(1-z)} \ln^2(1-z) - 4 \frac{(2-z)^2}{2z(1-z)} \ln(1-z) \right\} \right. \quad (9.4)$$

Finally, we have a correction due to the retarded self-energies $D_{(\text{ret})}^{(\mu)}(p)$. Before specifying the subtraction point, it is given by,

$$\left[\mathcal{G}(p^+, p_I^2)\right]_{LT}^{SE} = \frac{g^4 N_c^2}{2(2\pi)^5} \int \frac{dk^+ d\tilde{k}_i}{2k^+} \frac{dG(k^+, k_I^2)}{dk_i^2} (2-z)^2 \frac{\beta_2(p^+, \epsilon) \ln \left[ \frac{\mu^2(1-z)}{p_I^2} \right]}{z(1-z)} \right. \quad (9.5)$$

Since the renormalization of the self-energy of the propagator is subjected to the same condition as the vertex function, we have to take $\mu^2 = p_I^2$ and thus obtain no large $\ln p_I^2$. Thus, as in the case of the TT-mode we can perform the UV-renormalization in such a way that no large $\ln p_I^2$ arises.

Real processes have log's of $p_I^2$. One of them, the cut vertex, is gone. The two remaining diagrams correspond to the emission of a heavy gluon and the interference between two sequential gluon emissions.

Let us begin with the heavy gluon. As previously, we start with the lowest order contribution (with the bare propagator in the intermediate state),

$$\left[\mathcal{G}(p^+, p_I^2)\right]_{LT}^{HG} = \frac{ig^2 N_c}{(2\pi)^4} \int \frac{dk^+ d\tilde{k}_i}{2k^+} \frac{dG(k^+, k_I^2)}{dk_i^2} \frac{2(2-z)^2}{z} \int_{-\infty}^{-(k_i^2 - p_I^2)^2} dp^- \frac{w_i^{#01}(k-p)}{(k-p)^2} \right. \quad (9.6)$$

Since the density of states $w_i^{#01}(k-p)$ only establishes the limit of the integration $dp^-$ (being constant above the threshold), the integral $dp^-$ requires two cut-offs, the minimal mass $m$ of a gluon jet, and the maximal mass, $M$, resulting in

$$\left[\mathcal{G}(p^+, p_I^2)\right]_{LT}^{HG} = -\frac{g^2 N_c^2}{2(2\pi)^5} \int \frac{dk^+ d\tilde{k}_i}{2k^+} \frac{dG(k^+, k_I^2)}{dk_i^2} \frac{(2-z)^2}{z(1-z)} \beta_1(k_i^2 - p^2, \epsilon) \ln \left[ \frac{M^2}{m^2} \right] \right. \quad (9.7)$$

No dependence on $p_I^2$ appears here, just a big number connected with the end-points of the mass spectrum of the final-state jets. This has to be compared with the case of the TT-mode (Eqs. (8.6) and (8.7)). The difference is easily
understood, since the propagator of the longitudinal field has no causal pole. The next logical step is to consider the same process with the dressed propagator of the final-state gluon field, as has been done for the case of the TT-transition mode. This makes the integral converge; the cut-off \( M \) is unnecessary, and we find,

\[
G(p^+, p_2^T)_{LT} = i g^2 N_c \left( \frac{2}{2\pi} \right)^4 \int \frac{dk^+ dk_T^2}{2k^+} \frac{dG(k^+, k_T^2)}{dk_T^2} \frac{2(2-z)^2}{z} \times
\]

\[
\times \int_{-\infty}^{-\frac{(k_T-p_T)^2+m^2}{k_T+p_T}} dp^+ \frac{w_1^01(k-p)}{(k-p)^2(1-w_1^01(k-p))(1-w_1^{\text{adv}}(k-p))}.
\]

In our model, the propagator of the final-state time-like gluon has no resonance poles and the integrand of the equation (9.8) lacks the propagating pole at \( p^2 = 0 \). Thus we cannot continue with the contour integration and have to proceed in a straightforward way with the simple model of the two-body final state. Then, the integral in the previous equation is calculated as follows,

\[
G(p^+, p_2^T)_{LT} = -g^2 N_c \left( \frac{2}{2\pi} \right)^3 \int \frac{dk^+ dk_T}{2k^+} \frac{dG(k^+, k_T^2)}{dk_T^2} \frac{2(2-z)^2}{z(1-z)} \frac{\arctan \beta_T}{\pi \beta_T}.
\]

The pole at \( z = 1 \), which was a subject of concern in the order \( g^2 \), is persistently reproduced in all radiative corrections in the order \( g^4 \), and it requires the physical screening.

X. SCREENING EFFECTS IN THE QCD EVOLUTION

We have shown above that the UV-renormalization of the evolution equations can be performed in such a way that it does not bring any scale into the problem. Therefore, the only scale which is unambiguously present in the equations of the QCD evolution is the one associated with the properties of the final states that may regulate mass divergences. In the evolution picture based on null-plane dynamics, this scale systematically enters via the collinear cut-off \( \epsilon \). Now, we want to find an explicit expression for this cut-off in terms of the parameters of the collective modes of the final state. In the simplest case this will be the mass of the transverse plasmon.

It turns out that the null-plane dynamics, which employs the light-like direction \( x^+ \) as the Hamiltonian time, is ill-suited for computing the screening effects. In null-plane dynamics, the screening is just absent because the geometry of fields of these dynamics is singular and cannot provide a mass to a plasmon. In what follows, we prove this statement and suggest an alternative type of dynamics, which allows one to derive the evolution equations with screening.

To compute the screening effects, one has to rely on some distribution of the excited modes. One can consistently introduce the distribution only in connection with the procedure of its measurement. Throughout this paper, we concentrat on the inclusively measured one-particle distribution and argue that the full final state of the collision is always prepared as a fluctuation before the moment of the inclusive measurement (and that the measurement only excites the system of the allowed final states.) Below, supporting this idea, we construct the basis of states that naturally complies with the Lorentz contraction of the incoming objects and includes both strongly and weakly localized field modes. In terms of this basis, a snapshot of a virtual fluctuation before a collision (with as of, yet, coherently balanced phases) already looks like a distribution of hard particles (with high \( p_T \)) moving through the slowly-varying fields (with low \( p_T \)). The interaction just freezes this fluctuation with the wrong phases and converts it into the true distribution, in which the slow fields are immediately screened by the interaction with the hard particles.
Thus, even at a very early time after a collision of two nuclei, the emerging final state is rather a system of plasmons, than an ensemble of free partons.

A. The absence of screening in null-plane dynamics

The QCD evolution in high-energy processes is always described in the null–plane dynamics, which is the only one where the standard variable of evolution equations, the Feynman $x$, is unambiguously defined. It is tempting to try to compute the screening effects staying within framework of these dynamics. The procedure seems to be straightforward. One has to compute the retarded propagator [3.41] for the field of emission,

$$D^{\mu\nu}_\text{ret}(p) = \frac{\mathcal{J}^{\mu\nu}(p)}{p^2 + p^2 w_1^{R}(p)} + \frac{1}{(p^+)^2} \frac{n^\mu n^\nu}{1 - w_2^{R}(p)},$$

accounting for the effects of the final-state interaction (collective nature of the final states) in the invariants $p^2 w_1^{R}(p)$ and $w_2^{R}(p)$ of the gluon self-energy. In this formula, the first and the second terms are the full propagators of the transverse and longitudinal fields of the null-plane dynamics, respectively. The self-energy $p^2 w_1^{R}(p)$ of the transverse field has a dispersive part and non-dispersive part. The former is given by the Eq. (6.12), and it always is zero at $p^2 = 0$. Mathematically, this zero appears in the imaginary part of $\Pi^{(D)}_\text{ret}(p)$, and survives through the calculation of the dispersion integral (6.15). Physically, this happens because the propagation of the field in the dispersive term of the self-energy is mediated by the transverse propagator $D^{T}_\text{ret}(p)$, which maintains the causality of $\Pi^{(D)}_\text{ret}(p)$. This is true for all physical gauges, which explicitly separate the transverse and longitudinal fields.

In the non-dispersive part (6.13) of $\Pi^{(D)}_\text{ret}(p)$, the field propagation is mediated by the longitudinal field, which results in Eq. (6.18) for $p^2 w_1^{(N,D)}(p)$. In the case, when the states are populated with density $n(k) = n(k^+, k_\perp)$ this equation is modified to,

$$\text{Re} \left[ p^2 w_1^{(N,D)}(p) \right] = -\frac{g^2 N_c}{2(2\pi)^3} \int dq^+ d^2q_\perp \left\{ \frac{1 + 2n(q)}{|q^+|} \left( \frac{q^+ + p^+}{q^+ - p^+} \right)^2 + \frac{1 + 2n(q - p)}{|q^+ - p^+|} \left( \frac{q^+ - 2p^+}{q^+} \right)^2 \right\}. \quad (10.1)$$

This expression has no zero at $p^2 = 0$ and seems to be a good candidate to form the mass term in the dispersion equation for the transverse field. However, as before, this term is quasi-local in $x^+$ (it is independent of the light-cone energy $p^-$) and vanishes after one subtraction at any value of $p^-$.

The contact interaction in the tadpole part of the gluon self-energy behaves in the same way, and results in the $p^-$-independent quasi-local term.

Thus, we can draw two conclusions. (i) The mass of a plasmon with momentum $p$ is generated by the amplitude of the forward scattering of a gluon on some distribution of states only, if the gluon field between the two successive interactions is longitudinal, or if the interaction is contact. (ii) In the null-plane dynamics, these interactions are local in $x^+$. Therefore, they are incapable of producing any screening effects, which are genuinely non-local. In order to incorporate screening effects into the evolution equations, we have to exchange the null-plane Hamiltonian dynamics for another dynamics in which the fields would behave in a less singular way.

B. Field states in the proper-time dynamics

It was explicitly demonstrated that it is impossible to adequately describe the basic process of forward scattering, which is responsible for the generation of the plasmon mass, in the null-plane dynamics. The problem arises due
to the singular behavior of the field pattern which is defined as the static field with respect to the time \( x^+ \). This singular behavior shows that the choice of the Hamiltonian dynamics and proper definition of the field states is highly nontrivial and important. The problem we encounter in connection with screening is even more general. Indeed, a search for the QGP in heavy-ion collisions is, in the first place, a search for evidence of entropy production. Before the collision, the quark and gluon fields are assembled into two coherent wave packets (the nuclei), and therefore, the initial entropy equals zero. The coherence is lost, and entropy is created due to the interaction. Though one may wish to rely on the invariant formula \( S = \text{Tr} \rho \ln \rho \), which expresses the entropy \( S \) via the density matrix \( \rho \), at least one basis of states should be found explicitly. Thus, it is imperative to find a way to describe quarks and gluons of both nuclei as well as the products of their interaction using the same Hamiltonian dynamics. An appropriate choice for the gluons is always difficult because the gauge is a global object (as are the Hamiltonian dynamics) and both nuclei should be described using the same gauge condition.

Quantum field theory has a strict definition of dynamics. This notion was introduced by Dirac \([28]\) at the end of the 1940’s in connection with his attempt to build a quantum theory of the gravitational field. Every (Hamiltonian) dynamics includes its specific definition of the quantum mechanical observables on the (arbitrary) space-like surfaces, as well as the means to describe the evolution of the observables from the “earlier” space-like surface to the “later” one.

The primary choice of the degrees of freedom is effective if, even without any interaction, the dynamics of the normal modes adequately reflects the main physical features of the phenomenon. The intuitive physical picture clearly indicates that the normal modes of the fields participating in the collision of the two nuclei should be compatible with their Lorentz contraction. Unlike the incoming plane waves of the standard scattering theory, the nuclei have a well-defined shape and the space-time domain of their intersection is also well-defined. Of the ten symmetries of the Poincaré group, only rotation around the collision \( z \)-axis, boost along it, and the translations and boosts in the transverse \( x \) and \( y \)-directions survive. The idea of the collision of two plane sheets immediately leads us to the wedge form; the states of quark and gluon fields before and after the collision must be confined within the past and future light cones (wedges) with the \( xy \)-collision plane as the edge. Therefore, it is profitable to choose, in advance, the set of normal modes which have the symmetry of the localized interaction and carry quantum numbers adequate to this symmetry. These quantum numbers are the transverse components of momentum and the rapidity of the particle (which replaces the component \( p^z \) of its momentum). In this \textit{ad hoc} approach, all the spectral components of the nuclear wave functions ought to collapse in the two-dimensional plane of the interaction, even if all the confining interactions of the quarks and gluons in the hadrons and the coherence of the hadronic wave functions are neglected.

In the wedge form of dynamics, the states of free quark and gluon fields are defined (normalized) on the space-like hyper-surfaces of the constant proper time \( \tau \), \( \tau^2 = t^2 - z^2 \). The main idea of this approach is to study the dynamical evolution of the interacting fields along the Hamiltonian time \( \tau \). The gauge of the gluon field is fixed by the condition \( A^\tau = 0 \). This simple idea solves several problems. On the one hand, it becomes possible to treat the two different light-front dynamics which describe each nucleus of the initial state separately, as two limits of this single dynamics. On the other hand, after the collision, this gauge simulates a local (in rapidity) temporal-axial gauge. This feature provides a smooth transition to the boost-invariant regime of the created matter expansion (as a first approximation). Particularly, addressing the problem of screening, we will be able to compute the plasmon mass in a uniform fashion, considering each rapidity interval separately and using the (local) temporal axial gauge.
The feature of the states to collapse at the interaction vertex is crucial for understanding the dynamics of the collision. A simple optical prototype of the wedge dynamics is the camera obscura (a dark chamber with the pin-hole in the wall). Amongst the many possible a priori ways to decompose the incoming light, the camera selects only one. Only the spherical harmonics centered at the pin-hole can penetrate inside the camera. The spherical waves reveal their angular dependence at some distance from the center and build up the image on the opposite wall. Here, we suggest to view the collision of two nuclei as a kind of diffraction of the initial wave functions through the “pin-hole” $t = 0$, $z = 0$ in $tz$-plane.

The states of the wedge dynamics appear to be almost ideally suited for the analysis of the processes that are localized at different times $\tau$ and intervals of rapidity $\eta$ and are characterized by a different transverse momentum transfer. With respect to any particular process, these states are easily divided into slowly varying fields and localized particles. In this way, one may introduce the distribution of particles and study their effect on the dynamics of the fields. Thus, we now can calculate the plasmon mass as a local (at some scale) effect which agrees with our understanding of its physical origin.

For the simplest qualitative estimates, it is enough to consider the one-particle wave functions of the scalar field. Let us rewrite the wave function $\psi_{\theta, p_\perp} (x)$ in the following form,

$$\psi_{\theta, p_\perp} (x) = \frac{1}{4\pi^{3/2}} e^{-ip^0 t + ip^z z + ip_\perp r_\perp} \equiv \begin{cases} 4^{-1} \pi^{-3/2} e^{-im_\perp \tau \cosh(\eta - \theta)_{\perp} \pmb{r}_\perp}, & \tau^2 > 0, \\ 4^{-1} \pi^{-3/2} e^{-im_\perp \tau \sinh(\eta - \theta)_{\perp} \pmb{r}_\perp}, & \tau^2 < 0. \end{cases} \quad (10.2)$$

where $p^0 = m_\perp \cosh \theta$, $p^z = m_\perp \sinh \theta$ ($\theta$ being the rapidity of the particle), and, as usual, $m_\perp^2 = p_\perp^2 + m^2$. The above form implies that $\tau$ is positive in the future of the wedge vertex and negative in its past. Even though this wave function is obviously a plane wave which occupies the whole space, it carries the quantum number $\theta$ (rapidity of the particle) instead of the momentum $p_\perp$. A peculiar property of this wave function is that it may be normalized in two different ways, either on the hypersurface where $t = \text{const}$,

$$\int_{t=\text{const}} \psi^*_{\theta', p'_\perp} (x) \mathbf{i} \frac{\partial}{\partial t} \psi_{\theta, p_\perp} (x) = \delta(\theta - \theta') \delta(p_\perp - p'_\perp), \quad (10.3)$$

or, equivalently, on the hypersurfaces $\tau = \text{const}$ in the future- and the past–light wedges of the collision plane, where $\tau^2 > 0$,

$$\int_{\tau=\text{const}} \psi^*_{\theta', p'_\perp} (x) \mathbf{i} \frac{\partial}{\partial \tau} \psi_{\theta, p_\perp} (x) = \delta(\theta - \theta') \delta(p_\perp - p'_\perp). \quad (10.4)$$

The norm of a particle’s wave function always corresponds to the conservation of its charge or probability to find it. Since the norm given by Eq. (10.4) does not depend on $\tau$, the particle with a given rapidity $\theta$ (or velocity $v = \tanh \theta = p^z / p^0$), which is “prepared” on the surface $\tau = \text{const}$ in the past light wedge, cannot flow through the light-like wedge boundaries; the particle is predetermined to penetrate in the future light wedge through its vertex. The dynamics of the penetration process can be understood in the following way.

At large $m_\perp |\tau|$, the phase of the wave function $\psi_{\theta, p_\perp}$ is stationary in a very narrow interval around $\eta = \theta$ (outside this interval, the function oscillates with exponentially increasing frequency); the wave function describes a particle with rapidity $\theta$ moving along the classical trajectory. However, for $m_\perp |\tau| \ll 1$, the phase is almost constant along the surface $\tau = \text{const}$. The smaller $\tau$ is, the more uniformly the domain of stationary phase is stretched out along the light cone. A single particle with the wave function $\psi_{\theta, p_\perp}$ begins its life as the wave with the given rapidity $\theta$ at large
negative $\tau$. Later, it becomes spread out over the boundary of the past light wedge as $\tau \to -0$. Still being spread, it appears on the boundary of the future light wedge. Eventually, it again becomes a wave with rapidity $\theta$ at large positive $\tau$. The size and location of the interval where the phase of the wave function is stationary plays a central role in all subsequent discussions, since it is equivalent to the localization of a particle. Indeed, the overlapping of the domains of stationary phases in space and time provides the most effective interaction of the fields.

The size $\Delta \eta$ of the $\eta$-interval around the particle rapidity $\theta$, where the wave function is stationary, is easily evaluated. Extracting the trivial factor $e^{-im_\perp \tau}$ which defines the evolution of the wave function in the $\tau$-direction, we obtain an estimate from the exponential of Eq. (10.2),

$$2m_\perp \tau \sinh^2(\Delta \eta/2) \sim 1.$$  (10.5)

The two limiting cases are as follows,

$$\delta \eta \sim \sqrt{\frac{2}{m_\perp \tau}}, \quad \text{when } m_\perp \tau \gg 1, \quad \text{and} \quad \Delta \eta \sim \ln \frac{2}{m_\perp \tau}, \quad \text{when } m_\perp \tau \ll 1.$$  (10.6)

In the first case, one may boost this interval into the laboratory reference frame and see that the interval of stationary phase is Lorentz contracted (according to the rapidity $\theta$) in $z$-direction.

The modes normalized according to (10.4) play the same role as the wave functions of bound states in the examples described at the end of Sec. [Il]. Indeed, the packets of the waves with the same rapidity $\theta$ do not experience dispersion in the longitudinal direction. Therefore, an expansion in terms of such waves can be employed to form moving objects. The amplitudes and phases of the coefficients in this expansion are balanced in such a way that, before the collision, the wave packets represent the finite-sized nuclei with given rapidities. These wave packets must at least partially diagonalize the Hamiltonian that includes interactions which maintain the shapes of the nuclei.

If two localized objects simultaneously pass through the vertex, then the partial waves that form their wave functions effectively overlap in the vicinity of the light wedge. The interval of time when the partial wave with transverse momentum $p_\perp$ is spread out, is of the order $\tau \sim 1/m_\perp$. The high-$p_\perp$ components of the wave function localize around the world line of the initial rapidity $\eta = \theta$ earlier, than the low-$p_\perp$ components and have less time to interact. If no interaction occurs at sufficiently small $\tau$, then the amplitudes and the phases of these waves remain unchanged and the packet assembles into the initial finite-sized object.

If the interaction takes place and the balance of phases becomes broken, then the estimates of Eqs. (10.5) and (10.6) show how to construct the picture of screening at sufficiently small $\tau$, even before the secondary collisions come into the game. The states with $p_t \ll \tau^{-1}$ vary only slowly at the rapidity interval $\Delta \eta \sim \ln(2/p_t \tau) \gg 1$, and cannot be considered to be particles. At the same time, the states with $k_t \gg \tau^{-1}$ are well localized in the rapidity direction. For these states, $\delta \eta \sim \sqrt{2/k_t \tau} \ll 1$ and we may safely view them as the particles with the distribution which can be measured inclusively at the time $\tau$. The forward scattering of the particles with transverse momentum $k_t$ on the field with the momentum $p_t \ll k_t$ results in an adjoint mass of the soft field which screens the possible collinear singularity in the evolution equations. The process of evolution naturally saturates at the proper time $\tau$ when the adjoint mass becomes comparable to the transverse momentum $p_t \sim 1/\tau$. This is the end of the “earliest stage”. At later times, collisions between the partons-plasmons take over.

Another important question is how early the “earliest stage” begins. It is easy to understand that the shortest time scale that may be seriously considered in the theory is defined by the full energy of collision, $\tau_{\min} \sim 1/\sqrt{s}$. (For nuclei
collisions, \( \tau_{\text{min}} \sim 1/A^{1/3} \sqrt{s} \) where \( s \) is the invariant mass per participating nucleon. At this time, the stationary phase of a partial wave is stretched over the widest rapidity interval \( \delta \eta \sim \ln(\sqrt{s}/m_\perp) \). It is not surprising that this estimate coincides with the well known kinematically allowed width \( 2Y \) of the rapidity plateau, \( 2Y \approx \log(s/m_{\text{char}}^2) \).

We just have two complementary ways to obtain the same quantity.

The entire design of pQCD is aimed at getting rid of logarithms like \( \log(s/m^2) \); they lead to mass singularities. Their presence invalidates the power counting in the calculation of the scattering amplitudes and makes the dimensional analysis (including the RG methods) impossible. On the contrary, in AA–collisions, this logarithms play a major role defining the energy density in the emerging dense system and, consequently, the screening effects. However, even in this case, we may find subprocesses (with the largest \( p_t \)–transfer), which are insensitive to the collective effects, and where the two strategies (screening and removing) must be not too far away from each other.

A full analysis of the QCD evolution and the collective effects in the scope of the wedge dynamics is a subject of separate study which is now underway. To treat these problems in a global fashion, one has to derive propagators and the gluon self-energy \( \Pi^{\mu\nu} \) where the two strategies (screening and removing) must be not too far away from each other. This case, we may find subprocesses (with the largest \( p_t \)–transfer), which are insensitive to the collective effects, and where the two strategies (screening and removing) must be not too far away from each other.

C. Local screening and the mass of a plasmon

The local parameterization of the dynamics based on the system of the space-like surfaces \( \tau = \text{const} \) is easily implemented with the aid of the vector \( u^\mu(\eta) = (\cosh \eta, 0, \sinh \eta) \), which is normal to the hyper-surface and indicates the local “time” direction. As long as we are interested in the process, which takes place in a limited slice in rapidity, we may consider \( \eta \) as a label of this slice and perform all calculations in the Lorentz frame which moves with the rapidity \( \eta \). In this frame, \( u^\mu = (1, 0, 0, 0) \), and the global gauge condition, \( u(\eta)A = A^\tau = 0 \), becomes just \( A^0 = 0 \). A dynamical formation of the plasmon mass is exactly the case when we can proceed in this simplified manner.

The bare propagator in this (local) temporal-axial gauge is of the following form,

\[
D^{\mu\nu}(u, k) = \frac{1}{k^2} \left( -g^{\mu\nu} + \frac{k^\mu u^\nu + k^\nu u^\mu}{(ku)^2} - \frac{k^\mu k^\nu}{(ku)^2} \right) = \frac{d^{\mu\nu}(T)(k)}{k^2} + \frac{d^{\mu\nu}(L)(k)}{(ku)^2}, \tag{10.7}
\]

where the first and the second terms in this equation are the propagators of the transverse and longitudinal fields, respectively. The polarization matrices are,

\[
d^{\mu\nu}(T)(k) = -g^{\mu\nu} + u^\mu u^\nu - d^{\mu\nu}(L)(k), \quad d^{\mu\nu}(L)(k) = \frac{[k^\mu - u^\mu(ku)][k^\nu - u^\nu(ku)]}{(ku)^2 - k^2}, \tag{10.8}
\]

and the gluon self-energy \( \Pi^{\mu\nu} \) is of the following form,

\[
\Pi^{\mu\nu}(p) = d^{\mu\nu}(T)(p) w_T(p) + \frac{[p^\mu u^\nu - (pu)p^\nu][p^\mu u^\nu - (pu)p^\nu]}{p^2 (pu)^2 - p^2} w_L(p). \tag{10.9}
\]

Since \( \Pi^{\mu\nu} \) always appears in an assembly \( D^{\mu\nu} D^{\nu\alpha} \) the terms, like \( p^\mu u^\nu + u^\mu p^\nu \) or \( u^\mu u^\nu \), which are necessary to provide transversality of \( \Pi^{\mu\nu} \), cancel out. The invariants \( w_T \) and \( w_L \) can be found from the two contractions,

\[
- g_{\mu\nu} \Pi^{\mu\nu}(p) = 2 w_T(p) + w_L(p), \quad u_\mu u_\nu(p) \Pi^{\mu\nu}(p) = \frac{(pu)^2 - p^2}{p^2} w_L(p). \tag{10.10}
\]
The solution of the Schwinger-Dyson equation for the retarded propagator can be cast in the form

$$D^{\mu \nu}_{ret}(p) = \frac{d^{\mu \nu}_{(T)}(p)}{p^2 - w^R_T(p)} + \frac{p^2 d^{\mu \nu}_{(L)}(p)}{(pu)^2[p^2 - w^R_T(p)]}, \tag{10.11}$$

which exhibits an explicit separation of the transverse and longitudinal propagators.

Our minimal goal is to estimate the mass of the transverse plasmon (the collective mode) which is expected to replace the final state of free propagation in the QCD evolution equations. Therefore, we need to compute the self-energy \(w^B_T(p)\) of the soft transverse mode accounting for the effect of the hard part of the (already measured) final-state inclusive gluon distribution, \(dN_g/dk_t dy\). To pick up the dominant effect, we have to compute \(w^B_T(p)\) at the point \(p^2 = 0\) \[2\]. Therefore, we can parameterize the momentum \(p\) of the soft mode as \(p^\mu = (p_t \cosh \theta, \vec{p}_t, p_t \sinh \theta)\). The self-energy \(w^B_T(p)\) has dispersive and non-dispersive parts, which are defined by equations (6.12) and (6.13), respectively. Even before the loop integral is computed, it is straightforward to check that the dispersive part is proportional to \(p^2\) and can be disregarded in our simple estimate. The non-dispersive part of \(w^B_T(p)\) is represented by the integral,

\[
w^{(ND)}_T(p) = \frac{g^2 N_c}{2\pi R^2} \int \frac{d^4 q}{V^{(3)}(\tau)} \left\{ \frac{\delta(q^2)}{q^2_0} \left[ 1 + 2n(q) \right] \left[ \frac{(p - q)^2}{4} \right] + \frac{(q_0 + p_0)^2}{4(p - q)^2} - \frac{q_0^2 + p_0^2}{2} \right\} + \mathcal{O}(p^2), \tag{10.12} \]

where the “normalization volume”, \(V^{(3)}(\tau)\), for the hard modes with transverse momenta \(q_t \gg p_t\) is defined as the volume of the domain on the hyper-surface \(\tau = const\), where the wave function of the soft plasmon mode is stationary, and is given by

\[V^{(3)}(\tau) = \pi R^2 \tau \Delta \eta, \quad \Delta \eta \sim \ln \frac{2}{p_t \tau},\]

where \(R\) is the radius of the colliding nuclei, \(\pi R^2 \approx \pi r_0^2 A^{2/3}\), and \(r_0\) is the proton radius. The expression (10.12) includes a vacuum part which is UV-divergent and requires renormalization (which we neglect for now). Integrating \(q_0\) out and setting \(p^2 = 0\) we arrive at

\[
w^{(ND)}_T(p) = \frac{g^2 N_c}{2\pi R^2 \Delta \eta} \int \frac{d^3 q}{|q|^3} \frac{d N_g}{d^3 q} \left[ \frac{(p - q)^2}{2} + \frac{(5|q|^4 + 2|q|^2 p_t^2 + |p|^4)}{2(p - q)^2} - q^2 - p^2 \right]. \tag{10.13} \]

As for the distribution \(d N_g/d^3 q = d N_g / q_t d^2 \vec{q} dy\), we shall assume that it is homogeneous in the rapidity interval \(\Delta \eta\) centered at the rapidity \(\theta\) of the soft mode, \(n = n(q_t, \theta - y) \approx n(q_t, \theta)\). Furthermore, we can take \(\theta = 0\) and consequently, \(p_z \approx 0, p_0 \approx p_t\). With the same accuracy, we may take \(q_z = q_t \sinh y \approx 0\), \(dq_z = q_t \cosh y dy \approx q_t dy\) and integrate over \(q_t\) with the condition \(q_t \gg p_t\). In this approximation, we obtain

\[w^{(ND)}_T(p_t^2, \theta; p^2 = 0) = \frac{g^2 N_c}{\pi R^2} \int_{p_t^2}^{\infty} \frac{dq_t^2}{\tau q_t} \frac{d N_g(q_t^2, \theta)}{d q_t^2}, \tag{10.14} \]

where the function \(dN_g/dq_t^2\) is the density of the final-state gluons in the unit interval near rapidity \(\theta\). In the same way we compute the contribution of the tadpole term,

\[w^{(tadpl)}(p_t^2, \theta; p^2 = 0) = \frac{3g^2 N_c}{\pi R^2 \Delta \eta} \int d^3 q \left( \frac{1}{2} + n(q) \right) = \frac{3g^2 N_c}{\pi R^2} \int_{p_t^2}^{\infty} \frac{dq_t^2}{\tau q_t} \frac{d N_g(q_t^2, \theta)}{d q_t^2}, \tag{10.15} \]

where the divergent vacuum term vanishes after UV-renormalization.

\[44\]
Now, the renormalization can be performed in the usual way by subtracting the quasilocal terms after the integration with a weight function $n(\theta)$. Indeed, the point of renormalization can be chosen in the domain $\theta \to \mp \infty$ ($p^- \to 0$ or $p^+ \to 0$) where it does not affect the finite part of $w(p)_T$ produced by the physical distribution $n(\theta, p_t)$ which vanishes for modes with infinite rapidity. Assembling these two contributions, we obtain,

$$m^2_{pl}(p^2_t, \theta) = w^{(1D)}_{T}(p^2_t, \theta; p^2 = 0) + w^{(2D)}_{T}(p^2_t, \theta; p^2 = 0) = (3 + 1) \frac{g^2 N_c}{\pi R^2} \int_{p^2_t}^{\infty} dq^2_t d q^2_t \int_{\tau q_t}^{\infty} d q^2_t d N_g(q^2_t, \theta).$$

This quantity is defined locally in rapidity and it works as a feedback that limits the possible energy of the emission field from below and thus screens the mass singularity in the evolution equations.

Because the product $p_t \tau$ in Eq.(10.16) is close to unity, the proper time $\tau$ and momentum $p_t$ are the complementary parameters, this equation can be read in two ways. On the one hand, the smaller $p_t$ is, the more particles with higher momentum interacting with the soft mode are emitted, and the heavier this soft mode is. On the other hand, we know that the mode with momentum $p_t$ acquires the status of the emission field only by a time $\tau \sim 1/p_t$. The later this happens, the more that particles which participate in the formation of the plasmon mass are created, and the larger the mass of plasmon is.

**XI. EVOLUTION EQUATIONS WITH FINAL STATE SCREENING**

**A. Screening in the order $\alpha_s$.**

Our last problem is to show that the replacement of the free radiation field by the collective modes of the true final state screens the collinear singularities that we encountered in the null-plane dynamics. Previously, these singularities were coming either from the poles of the polarization sum $d^\mu\nu$, or from the integration $dp^-$ of the $\delta_+[(k - p)^2]$. In order to smooth out this behavior, we consider the evolution equations in the same dynamics which allows for the effect of local mass generation. While the calculation of the plasmon mass could be performed in the co-moving reference frame, the evolution must be treated in a more global fashion. In order to find the space-time domain where the QCD evolution described by the DGLAP equation develops, let us go back to the general four-vector $u^\mu = (\cosh \eta, \vec{0}, \sinh \eta)$, that defines the local time direction at the rapidity $\eta$ and keep the latter as a parameter. The gauge of the gluon field is fixed by the condition $uA = 0$ which may be conveniently rewritten as

$$u(\eta)A = \frac{1}{2} e^{-\eta} A^+ + \frac{1}{2} e^{\eta} A^- = 0.$$  

From this, we infer that the gauge condition $A^+ = 0$ holds as $\eta \to -\infty$, that is, in the nearest vicinity of the hyper-plane $x^+ = 0$. In the process of $e^-p$ DIS, the Lorentz frame moving with this rapidity is the rest frame for the electron, and the infinite-momentum frame for the proton. The momentum of the virtual photon of the DIS has components $q^+ = 0$ and $q^- = 2\nu/P^+$. Thus, the DGLAP equations describe the process which is localized in the space-time domain of the electron fragmentation.

In what follows, we attempt to answer only two questions:

(i) Does the plasmon mass shield the singularity in the evolution equations?

(ii) Does the equation with screening allow for a smooth transition to the DGLAP equation when the screening mass becomes small?
For these limited goals, we can drop all regular terms. To derive the evolution equation, we start with Eq. (5.3) and extract the scalar equations by means of Eqs. (10.10). In this way we obtain an analog of the Eqs. (5.4), (5.8) for the evolution of the transverse fields,

$$\frac{dG(x, p_t^2)}{dp_t^2} = -\frac{N_c \alpha_s}{\pi^2} \int dp_0 \int \frac{dk_t^2 k_t^2 \delta(k_0 - p_0 - \omega) dG(k^2, k_0^2)}{2 \omega} \frac{p_{t0}^2}{p^2} P_{gg}(p^0) \left[ \frac{1}{p^2} - \frac{p^2}{k_0^2} \right]^2 + O\left(\frac{p_t^2}{s}, \frac{k_t^2}{s}\right), \quad (11.2)$$

where only the terms which do not vanish at $s \to \infty$ ($\eta \to -\infty$) are retained. In this equation,

$$\omega^2 = (k_x - p_x)^2 + m^2((k_z^2 - p_z^2)^2) = (k_x^2 - p_x^2) + (k_x^2 - p_x^2) + m^2((k_z^2 - p_z^2)^2),$$

and the temporal and longitudinal components of momenta are those of the infinite momentum frame,

$$k_0^+ \equiv (k_0 - \frac{1}{2} \epsilon_0^+ k^+ + \frac{1}{2} \epsilon_0^- k^- \approx k_0^+ / 2, \quad k_z^+ = \frac{1}{2} \epsilon_0^+ k^+ - \frac{1}{2} \epsilon_0^- k^- \approx k_0^+ / 2, \quad \eta \to -\infty. \quad (11.3)$$

Both are large and scale with the momentum of the proton. (In Eq. (11.2), and in what follows the label $\ast$ is dropped.)

The operational definition of the structure function in the new variables changes slightly with respect to the previous definition [11.7],

$$\int dp_0 \frac{p_{t0}}{p^2} w_{t0}^{(01)}(p) = \frac{dG(x, p_t^2)}{dp_t^2}, \quad (11.4)$$

where by virtue of (11.3), the domain of the integration $dp_0$ is strongly localized near the point $p_0 = p_t$. The mass $m^2((k_z^2 - p_z^2)^2)$ was computed in Sec. [X.C] and is used as the pole mass of the emitted gluon-plasmon. It has the property to increase with decreasing transverse momentum. Integrating out $p_0$ with the aid of the delta-function in the Eq. (11.2), we arrive at

$$\frac{dG(p_t^2, p_t^2)}{p_t^2} = -\frac{N_c \alpha_s}{\pi^2} \int dk_t^2 k_t^2 \frac{dG(k^2, k_0^2)}{2 \omega} \frac{(k_z - \omega)^2}{k_z^2} \left[ \frac{k_z}{\omega} + \frac{k_z}{k_z - \omega} - 2 + \frac{\omega(k_z - \omega)}{k_z^2} \right], \quad (11.5)$$

where the expression in the square brackets is just the splitting kernel $P_{gg}$ and the strong ordering ($p_t^2 \gg k_t^2$) is assumed from now on. Once again, this equation is written in the first approximation with respect to the small parameter $p_t^2 / k_t^2 \sim p_t^2 / s$, and the former pole at $p_0 = k_0$ is translated into the non-singular factor $1/\omega$.

By examination, Eq. (11.5) has no singularity in the remaining integration $dk_z$. It is reliably screened by the finite $p_t^2$ regardless of the presence of the plasmon mass. Such a screening happens because now the proton has finite rapidity; and the dynamics are free from the artificial singularity of the null-plane. However, if we consider a non-ordered emission, or just take $p_t$ to be small, then only the mass, $m((k_z^2 - p_z^2))$, would provide the desired protection. Only in the formal limit $m^2 / s \to 0$, the singularity at $k_z = p_z$ becomes a reality, and we recover the usual DGLAP equation. It is easy to demonstrate, that the same mechanism of shielding of the mass singularity works for Eq. (5.9), which describes the transition between the longitudinal and transverse modes of the gluon field.

With the mass computed according to (10.14), Eq. (11.3) can be considered as the simplest prototype of the evolution equation with screening for the fluctuations in the AA-collisions. One should keep in mind, that (by virtue of (11.3)) the set of limits, $k_z - p_z \to 0, k^+ - p^+ \to 0$, and $\eta \to -\infty$, which has been employed for its derivation is intrinsically ambiguous. In order to obtain the evolution equations that can handle the emission in the central rapidity region we cannot approach the limit of $\eta \to -\infty$. These equations can be derived only in the framework of the “wedge dynamics” [20,21].
B. Suppression of the radiative corrections in the order $\alpha_s^2$.

The last thing that we are going to demonstrate with our limited “local” approximation is that the radiative corrections of the order $\alpha_s^2$ are doubly-suppressed by the plasmon masses. Indeed, a review of our previous calculations in this order shows that all of them are connected with processes with at least two real emissions. An example of such a process with two consecutive emissions is depicted at Fig. 9.

![Fragment of the ladder with two rungs of emission.](image)

FIG. 9. Fragment of the ladder with two rungs of emission.

In any of these cases, we have to integrate the expressions with two mass-shell delta functions,

$$\int d^4k \delta_+[(k-p)^2-m_1^2] \delta_+[(l-k)^2-m_2^2] P_1(p^+/k^+) P_2(k^+/l^+) \ ,$$

where $P_{1,2}(z)$ may have a pole at $z=1$, $m_1 = m_1((k_t-p_t)^2)$, $m_2 = m_2((l_t-k_t)^2)$, and the total emitted momentum is $l-p=s$. For example, let the collinear singularity emerge when the integration $dk^+$ reaches one or both ends of the interval $p^+ < k^+ < l^+$. Introducing the new variables, $f = k-p$, we can integrate out $f^-$ with the aid of one of the delta-functions and remove the second one by integrating over the angle between the transverse vectors $f_t$ and $s_t$.

The latter integration defines the limits for the magnitude of the transverse component $f_t$, which has to be confined between the two positive roots of the equation,

$$[(s_t+f_t)^2-(s^++f^+)(s^- - \frac{f_2^2-m_2^2}{f^+})-m_2^2] \ [(s^+-f^+)(s^- - \frac{f_1^2-m_1^2}{f^+})-m_1^2-(s_t-f_t)^2] = 0 \ .$$

(11.7)

These roots exist (the measure of integration $df_t$ does not vanish) only if the variable $f^+$ is within the following limits,

$$1 + \frac{m_2^2-m_1^2}{s^2} - \sqrt{1-2\frac{m_1^2+m_2^2}{s^2} + \left(\frac{m_1^2-m_2^2}{s^2}\right)^2} < 2 \frac{f^+}{s^+} <
$$

$$< 1 + \frac{m_2^2-m_1^2}{s^2} + \sqrt{1-2\frac{m_1^2+m_2^2}{s^2} + \left(\frac{m_1^2-m_2^2}{s^2}\right)^2} \ .$$

(11.8)

This inequality shows that the domain of integration over $k^+$ gets smaller with the growth of the masses, and it just vanishes when, e.g., either $m_1^2 \geq s^2$ or $m_2^2 \geq s^2$, and the emission of two gluons becomes impossible. Hence, the finite plasmon masses not only screen the mass singularities, but also strongly reduce the phase-space for the terms of the order $\alpha_s^2$. 

47
ACKNOWLEDGMENTS

We are grateful to Yu. Dokshitzer, A. Kovner, L. McLerran, B. Muller, and H. Weigert for many stimulating discussions. We wish to thank Edward Shuryak for helpful conversations at various stages in the development of this work. We are grateful to Jianwei Qiu for attracting our attention to his paper [17]. We appreciate the help of Scott Payson who critically read the manuscript.

This work was supported by the U.S. Department of Energy under Contract No. DE–FG02–94ER40831.

XII. APPENDIX 1. CAUSAL PROPERTIES OF THE EVOLUTION LADDER

In the course of computing the $g^4$-elementary ladder cell we met various types of radiative corrections with retarded order of their temporal arguments. This circumstance allowed us to make a firm conclusion (which was verified at the tree-level) about the time ordering in process the QCD evolution process. It is also fortunate in one more respect. The retarded (advanced) functions have the unique analytic properties that they are regular in the upper (lower) half-plane of the complex energy. Since we use Hamiltonian dynamics with the coordinate $x^+$ chosen as the “time” direction, the corresponding conjugate variable is the light-cone energy, $p^-$, and we can rely on the analytic properties in the complex plane of $p^-$. Beyond the tree-level calculations, we include radiative corrections with loops, self-energies and vertex functions. We must check whether they maintain the aforementioned causal behavior. Self-energy corrections to the propagators are of an explicitly retarded type, and they do not cause any problems. The loops of the vertices have yet to be examined. The two vertex functions, $V_{000}$ and $V_{111}$, which come from the terms with $R = S$ in Eq.(6.19), have the same surroundings and appear in the combination $V_{000} + V_{111}$. This sum forms the following string of propagators which constitute the loop,

$$D_{00}(q)D_{00}(k-p)D_{00}(q-k) + D_{11}(q)D_{11}(k-p)D_{11}(q-k).$$

(A1.1)

Since $D_{00} = D_s + D_1/2$ and $D_{11} = -D_s + D_1/2$, this string can be identically rewritten as,

$$D_{1}(q)D_{s}(k-p)D_{s}(q-k) + D_{s}(q)D_{1}(k-p)D_{s}(q-k) + D_{s}(q)D_{s}(k-p)D_{1}(q-k),$$

(A1.2)

where we omitted the term $D_{1}D_{1}D_{1}/2$, which identically vanishes in the surroundings of the evolution ladder. We want to show that expression (A1.2) is nothing but the real part of the retarded vertex, which is an analytic function in the lower half-plane of energy $p^-$. To prove this statement, we should derive the formula for the retarded vertex to order $g^3$. This is more easily done in the coordinate form.

Let us start with the definition of the retarded vertex in the form of the functional derivative of the retarded self-energy with respect to the field $\mathcal{A}(z)$,

$$\frac{\delta}{\delta \mathcal{A}(z)} \Pi_{\text{ret}}^{\nu\beta\sigma}_{\text{bcf};\text{ret}}(x, y, z) = -\frac{\delta^{(2)} \Pi_{\text{ret}}^{\nu\beta\sigma}}{g\delta_{\nu\beta}}(x, y).$$

(A1.3)

The self-energy $\Pi_{\text{ret}}$ is built from the propagators $D_{\text{ret}}$, $D_{\text{adv}}$, and $D_1$. The simplest way to differentiate them is to use the equations [3,5] for $D_{\text{ret}}$, $D_{\text{adv}}$, and $D_1$ in the presence of the classical external field. These equations can be written as follows:

$$D_{(\text{ret})}(x, y) = D_{(\text{adv})}(x, y) + \int dz D_{(\text{adv})}(x, z)V(z)\mathcal{A}(z)D_{(\text{ret})}(z, y),$$

(A1.4)
\[ D_1(x, y) = D_1(x, y) + \int dz [D_{\text{ret}}(x, z)V(z)A(z)D_1(z, y) + D_1(x, z)V(z)A(z)D_{\text{adv}}(z, y)]. \]  

(A1.5)

It is straightforward to obtain,

\[
(3) V_{\text{rel}}^{\rho\sigma}(x, y, z) = -\frac{i}{4} \left[ V^{\alpha_1\alpha_2}(x)D_{\text{ret}}^{\rho_2}(x, z)V^{\sigma_2\sigma_1}(z)D_{\text{ret}}^{\rho_1}(y)D^{ho_1\alpha_1}(y, x) 
+ V^{\alpha_1\alpha_2}(x)D_{\text{ret}}^{\rho_2}(x, z)V^{\sigma_2\sigma_1}(z)D_{\text{adv}}^{\rho_1}(y, x)D^{ho_1\alpha_1}(y, x) 
+ V^{\alpha_1\alpha_2}(x)D_{\text{adv}}^{\rho_2}(y, z)V^{\sigma_2\sigma_1}(z)D_{\text{adv}}^{\rho_1}(y, x) \right] 
+ V^{\alpha_1\alpha_2}(x)D_{\text{ret}}^{\rho_2}(x, y)V^{\sigma_2\sigma_1}(z)D_{\text{adv}}^{\rho_1}(y, z)V^{\sigma_2\sigma_1}(z)D_{\text{adv}}^{\rho_1\alpha_1}(y, x) 
+ V^{\alpha_1\alpha_2}(x)D_{\text{adv}}^{\rho_2}(x, y)V^{\sigma_2\sigma_1}(z)D_{\text{adv}}^{\rho_1\alpha_1}(y, x) \right]. \]  

(A1.6)

Here, the groups of the first three, and of the last three terms differ only by the direction in which the arguments are going around the loop. Therefore, these two groups are identical. In the momentum representation, we obtain

\[
(3) V_{\text{rel}}(p, -k, k - p) = -\frac{i}{2} \int dxdydz e^{ip(x-z)+ik(y-z)} \left[ V(x)D_{\text{ret}}(x-y)V(y)D_{\text{ret}}(y-z)V(z)D_1(z-x) 
+ V(x)D_{\text{ret}}(x-y)V(y)D_{\text{adv}}(z-x) + V(x)D_1(x-y)V(y)D_{\text{adv}}(y-z)V(z)D_{\text{adv}}(z-x) \right]. \]  

(A1.7)

FIG. 10. Three diagrams contributing to the retarded vertex. Lines with arrows correspond to the propagators \( D_{\text{ret}} \) and point \( x \) is the latest. The crossed lines correspond to the correlators \( D_1 \), the densities of real states.

The elements which are responsible for the analytic properties of the retarded vertex with respect to momentum \( p \) are underlined in the formula and marked by the time-directed arrows in Fig. 10. In the same way, we may introduce an advanced vertex function and put it in a form similar to |A1.7|.

\[
(3) V_{\text{adv}}(p, -k, k - p) = -\frac{i}{2} \int dxdydz e^{ip(x-z)+ik(y-z)} \left[ V(x)D_{\text{adv}}(x-y)V(y)D_{\text{adv}}(y-z)V(z)D_1(z-x) 
+ V(x)D_{\text{adv}}(x-y)V(y)D_{\text{ret}}(z-x) + V(x)D_1(x-y)V(y)D_{\text{ret}}(y-z)V(z)D_{\text{ret}}(z-x) \right]. \]  

(A1.8)

which shows that at the real light-cone energy \( p^- \), \( V_{\text{adv}} \) is the complex conjugate of \( V_{\text{ret}} \). This is exactly the sum \( V_{\text{ret}} + V_{\text{adv}} = V_{000} + V_{111} \) (twice the real part of \( V_{\text{ret}} \) that defines the vertex loop correction to the elementary cell of the ladder. Indeed, since

\[ D_{\text{ret}} = D_s + \frac{1}{2} D_0, \quad D_{\text{adv}} = D_s - \frac{1}{2} D_0, \]  

(A1.9)

we find that
Thus, it is natural to use the dispersion relation (6.22) to compute the real part of $V_{ret}$. However, one should remember that each of the retarded propagators of the gauge field includes a part that corresponds to longitudinal fields, which are not driven by the Hamiltonian equations of motion. Only the transverse fields propagate and obey causality. Therefore, those retarded and advanced functions, which maintain the causal properties of the vertex (and are underlined in Eq. (A1.7)) have to be split into longitudinal and transverse parts,

$$D_{ret}(q) = D^{(T)}_{ret}(q) + D^{(L)}(q), \quad D_{adv}(q) = D^{(T)}_{adv}(q) + D^{(L)}(q),$$

and only the causal transverse part has the proper analytic behavior, which results in dispersion relations linking the real and imaginary parts of the vertex function. Thus we encounter a necessity of separating the dispersive part of the vertex, $V^{(D)}$, from the non-dispersive part, $V^{(ND)}$.

For our practical goals, we need only the real part of the dispersive term of the vertex function, which is

$$\text{Re} \ (3) V_{ret}^{(D)}(p, -k, k - p) = -\frac{i}{4} \int \frac{d^4q}{(2\pi)^4} \left[ V_x D_1(q) V_y D_s(q - k) V_x D_s(q - p) + V_z D_z(q) V_y D_1(q - k) V_z D_1(q - p) \right].$$

(A1.10)

It can be computed by means of dispersion relation via the imaginary part, which consists of two terms. The main term, is symmetric with respect to the singular functions (commutators $D_0$ and densities of states $D_1$), and is given by,

$$\text{Im} \ (3) V_{ret}^{(D)}(\omega, -k, k - \omega) = -\frac{i}{4} \int \frac{d^4q}{(2\pi)^4} \times$$

\[ \times \left( \left[ V_x D_0(q) V_y D_s^{(T)}(q - k) V_x D_1(q - \omega) - V_x D_1(q) V_y D_s^{(T)}(q - k) V_x D_0(q - \omega) \right] + \\
+ \left[ V_x D_s^{(T)}(q) V_y D_0(q - k) V_x D_1(q - \omega) - V_x D_s^{(T)}(q) V_y D_1(q - k) V_x D_0(q - \omega) \right] + \\
+ \left[ V_x D_0(q) V_y D_1(q - k) V_x D_s^{(T)}(q - \omega) - V_x D_1(q) V_y D_0(q - k) V_x D_s^{(T)}(q - \omega) \right] \right).$$

(A1.13)

Symmetrization has been achieved by an extra splitting of the two retarded propagators which do not depend on the momentum $p$ and do not participate in maintaining the analytic properties with respect to the (light-cone) energy $p^-$. Therefore, we have an additional residue of the splitting procedure which contains the longitudinal propagators $D^{(L)}$, i.e.,

$$\Delta \text{Im} \ (3) V_{ret}^{(D)}(\omega, -k, k - \omega) = -\frac{i}{4} \int \frac{d^4q}{(2\pi)^4} \times$$

\[ \times \left[ -V_x D_s^{(L)}(q) V_y D_1(q - k) V_x D_0(q - \omega) - V_x D_1(q) V_y D_s^{(L)}(q - k) V_x D_0(q - \omega) \right].$$

(A1.14)

In the last two equations, we have introduced the four-vector $\omega^\mu = (p^+, \omega^-, p^-)$. The first two terms in Eq. (A1.13) correspond to the unitary cut of the diagram for $V_{ret}$ near the external line with momentum $p$. The next two terms correspond to the unitary cut near the external line with momentum $p - k$. At real $\omega^-$, the imaginary part due to these terms differs from zero for $\omega^- > p_-^2/p^+$ and for $\omega^- < -(p_-^2 - k_-^2)/(k^+ - p^+)$, respectively. Therefore, the subtraction point in the dispersion relations with respect to the light-cone energy $p^-$ is confined to the finite interval.
of $p^-$ between the tips of the two cuts. The last two terms in Eq. (A1.13) would have corresponded to the unitary cut near the external line with momentum $k$, if the momentum $k$ were time-like. As long as dispersion relations do not involve the momentum $k$, these terms vanish in the ladder kinematics, since we always have $k^2 < 0$.

The non-dispersive part of the vertex is not causal and can only be calculated in a straightforward manner,

$$
(3) \mathcal{V}_{\text{ret}}^{(ND)}(p, -k, k - p) = -\frac{i}{2} \int \frac{d^4 q}{(2\pi)^4} \left[ \mathcal{V}_x D_s^{(T)}(q) \mathcal{V}_y D_s^{(L)}(q - k) \mathcal{V}_z D_1(q - p) + \mathcal{V}_x D_s^{(L)}(q) \mathcal{V}_y D_s^{(T)}(q - k) \mathcal{V}_z D_1(q - p) + \mathcal{V}_x D_1(q) \mathcal{V}_y D_s(q - k) \mathcal{V}_z D^{(L)}(q - p) \right].
$$

We use the vertex function only when it is embedded in the ladder diagram, and therefore, we do not need to compute it separately.

**XIII. APPENDIX 2. THE VANISHING CONTRIBUTIONS**

There are several types of terms that do not contribute to the evolution equations for various reasons.

**A. The terms with the four-gluon vertices**

In the order $g^4$, we have eleven diagrams with four-gluon vertices. They can appear only in the TT-transition mode. Ten of them turn out to be zero. These are diagrams (a–e) in Fig. 11 and their “mirror reflections”.

![Diagrams with four-gluon vertices](image)

**FIG. 11. Diagrams with four-gluon vertices.**

Diagrams a–c include the virtual loops that are the real parts of the retarded functions. Hence, they can be computed via the dispersion relations. Even though the dispersion integral is found to be UV-finite, it still requires renormalization. Therefore, the dispersion integral has to be rewritten with the subtraction at some point $p^- = -\Omega$.

For the diagram (a) we get,

$$
\left[ \frac{dG(p^+, p_t^+)}{dp_t^+} \right]_{(a)} = -\frac{ig^2 N_c}{(2\pi)^2} \int \frac{dk^+ dk^-}{2k^+} \left( \mathcal{V}_x D_s^{(T)}(q) \mathcal{V}_y D_s^{(L)}(q - k) \mathcal{V}_z D_1(q - p) + \mathcal{V}_x D_s^{(L)}(q) \mathcal{V}_y D_s^{(T)}(q - k) \mathcal{V}_z D_1(q - p) + \mathcal{V}_x D_1(q) \mathcal{V}_y D_s(q - k) \mathcal{V}_z D^{(L)}(q - p) \right) \times
$$

$$
\int dq^+ d^2 q_\perp \delta_+(\omega - q^+ q_\perp^2) \left( \frac{\omega^- - p^-}{p^+} + \frac{(k - q)^2}{k^+(p^+ - q^+)} - \frac{p_\perp^2 (2k^+ q^- - k^+ p^+ - q^+ p^-)}{k^+ p^+ (p^+ - q^+)} \right) = 0.
$$

This virtual correction identically vanishes after subtraction at any value of the light-cone energy. Therefore, in the coordinate space this term is quasi-local ($\sim \delta(x^+)$) and does not contribute to the evolution.
Diagram (b) behaves similarly: it is finite and we calculate it using the dispersion relation over $p^-$ making a subtraction at some point $p^- = -\Omega$ to renormalize it,

$$
\left[ \frac{dG(p^+, p_1^2)}{dp_1^2} \right]_{(b)} = \frac{ig^2 N_c}{(2\pi)^2} \int \frac{dk^+ dk_1}{2k^+} \frac{dG(k^+, k_1^2)}{dk_1^2} \frac{\delta[(k - p)^2]}{(k^+ - q^+)(k^+ - p^+)} \int \left( \frac{d\omega^-}{\omega^- - p^-} - \frac{d\omega^-}{\omega^- + \Omega} \right) \frac{dq^+ dq^- \delta[(q - \omega)^2] \delta[q^2]}{[p^2]^2} \times \left\{ \frac{(\omega^- - p^-)}{k^+(q^- - p^+)} + \frac{q^2}{k^+(q^- - p^+)} \right\} = 0. \tag{A2.2}
$$

Without renormalization, these two terms would produce finite corrections comparable to those coming from the three-gluon vertex (they include both the Coulomb logarithm, and the collinear cut-off).

Diagram (c) is of the form,

$$
\left[ \frac{dG(p^+, p_1^2)}{dp_1^2} \right]_{(c)} = \frac{ig^2 N_c}{(2\pi)^2} \int \frac{dk^+ dk_1}{2k^+} \frac{dG(k^+, k_1^2)}{dk_1^2} \frac{\delta[(k - p)^2]}{[p^2]^2} \int \frac{dq^+ dq^- \delta[(q - \omega)^2] \delta[q^2]}{[p^2]^2} \times \left\{ \frac{(k - q)^2(\omega - p^-)}{p^+} + \frac{[k(q^- - p^+)]^2}{k^+(p^- - q^+)} \right\} = 0. \tag{A2.3}
$$

It proves to be zero after delta-functions are used to integrate out $p^-$ and $q^-$. (The expression in the curly brackets is zero.)

The diagrams (d) and (e) correspond to real processes. They vanish identically, which is seen only after the integrations $dp^-$, $dq^-$, and $dq_1^-$ are carried out.

The only term which has a four-gluon vertex that survives, is given by diagram (f). It corresponds to the real emission

$$
\left[ \frac{dG(p^+, p_1^2)}{dp_1^2} \right]_{(f)} = -\frac{24ig^4 N_c^2}{(2\pi)^2} \int \frac{dk^+ dk_1}{2k^+} \frac{dG(k^+, k_1^2)}{dk_1^2} \frac{\delta[(k - p)^2]}{[p^2]^2} \int \frac{dp^- dq^+ dq^- \delta[(q - p)^2] \delta[(q - \omega)^2]}{[p^2]^2} \times \frac{q^2}{k^+(p^- - q^+)} \frac{k^+ p^+}{k^+ p^+ p_1^2}. \tag{A2.4}
$$

B. The non-dispersive and relative terms in three gluon vertex.

The two terms in the three-gluon vertex in the TT-transition mode, which were not discussed in the body of the paper, are given by Eqs. (A1.14) and (A1.15). To compute the contribution of the longitudinal (non-causal) fields in the dispersive part we have to substitute (A1.14) into Eq. (3.22) and then proceed with Eq. (3.21). These steps result in

$$
\left[ \frac{dG(p^+, p_1^2)}{dp_1^2} \right]^{VV, (\Delta L)}_{TT} = \frac{g^4 N_c^2}{4(2\pi)^6} \int \frac{dk^+ dk_1}{2k^+} \frac{dG(k^+, k_1^2)}{dk_1^2} \frac{\delta[(k - p)^2]}{[p^2]^2} \int dp^- \frac{\delta[(k - p)^2]}{[p^2]^2} \left[ p^+ \Phi_{\Delta L1}(p, k) + p^+ \Phi_{\Delta L2}(p, k) \right], \tag{A2.5}
$$

where $\Phi_{\Delta L1}$ and $\Phi_{\Delta L2}$ are given by

$$
\Phi_{\Delta L1} = -\int \left( \frac{d\omega^-}{\omega^- - p^-} - \frac{d\omega^-}{\omega^- + \Omega} \right) \frac{dq^+ dq^- \text{sign}(q^+ - p^+)}{2} \frac{\delta[(q - \omega)^2] \delta[q^2]}{[p^2]^2} \times \left\{ \frac{(\omega^- - p^-)}{p^+} + \frac{(k - q)^2}{k^+(p^- - q^+)} \right\} := \Psi_1(q^+) \tag{A2.6}
$$
\[ \Phi_{\Delta L_2} = \int \left( \frac{d\omega^-}{\omega^- - p^-} - \frac{d\omega^-}{\omega^- + \Omega} \right) \int \frac{dq^+ dq^-}{2} \text{sign}(q^+ - p^+) \int d^2 \vec{q}_t \frac{\delta[(q - \omega)^2] \delta[(k - q)^2]}{[p^2]^2} \times \]
\[ \times \left\{ \frac{(\omega - p^-)}{(k^+ - q^+)(k^+ - p^+)} - \frac{q^2}{k^+(q^+ - p^+)(k^+ - q^+)} \right\} \Psi_2(q^+) \],  
(A2.7)

where \( \Psi_{1,2}(q^+) \) are rational functions. After the integration over \( q^- \), \( \omega^- \), and \( \vec{q}_t \), both \( \Phi_{\Delta L_1} \) and \( \Phi_{\Delta L_2} \) vanish for an arbitrary subtraction point \( \Omega \). Therefore, the longitudinal fields contribute only a quasi–local term to the vertex function. The non-dispersive part, given by Eq. (A1.17), being substituted into Eq. (6.21), results in

\[ \left[ \frac{dG(p^+, p_t^2)}{dp_t^2} \right]_{TT}^{VV,(N_D)} = \frac{g^4 N_c^2}{4(2\pi)^6} \int \frac{dk^+ dk_t}{2k^+} \frac{dG(k^+, k_t^2)}{dk_t^2} \int dp^- [p^+ \Phi_{N D 1}(p, k) + p^+ \Phi_{N D 2}(p, k)] \delta[(k - p)^2] \],  
(A2.8)

with \( \Phi_{N D 1} \) and \( \Phi_{N D 2} \) given by

\[ \Phi_{N D 1} = \int q^4 \frac{\delta[(q - p)^2]}{[p^2]^2} \left\{ p^2(2k^+ + k^+ p^- - q^+ p^+) - (k - q)^2(p^+)^2 \right\} \tilde{\Psi}_1(q^+) \]  
(A2.9)

and

\[ \Phi_{N D 2} = \int q^4 \frac{\delta[(q - p)^2]}{[p^2]^2} \left\{ p^2(2k^+ - q^-) - q^2(k^+ - p^+) \right\} \tilde{\Psi}_2(q^+) \].  
(A2.10)

As with all other constituents of the vertex function, the non-dispersive part, even being finite, requires renormalization. We make a subtraction at \( p^- = -\Omega \). This results in both functions, \( \Phi_{N D 2} \) and \( \Phi_{N D 2} \), separately being zero. Thus, all terms associated with the longitudinal fields in the vertex function in the TT-transition mode are quasi–local.

[1] G. Sterman et al., *Handbook of perturbative QCD*, Rev.Mod.Phys. 67(1995)157.
[2] E.V. Shuryak, Sov. Phys. JETP 47 (1978) 212.
[3] V.N. Gribov, *Space-time description of hadron interactions at high energies*, in Proceedings of the 8-th Leningrad Nuclear Physics Winter School, February 16-27, 1973.
[4] Yu.L. Dokshitzer et al., *Basics of perturbative QCD*, Editions Frontieres, 1991.
[5] G.P. Lepage and S.J. Brodsky, Phys.Rev. D22, (1979)2157.
[6] L. McLerran, R. Venugopalan, Phys.Rev. D49 (1994) 2233; D49 (1994)3352.
[7] J. Jalilian-Marian, A. Kovner, L.McLerran, H.Weigert, Phys.Rev. D55 (1997) 5414; J. Jalilian-Marian, A. Kovner, H.Weigert, The wilson renormalization group for low x physics: gluon evolution at finite parton density, Preprint TPI-MINN-97-26,1997 ( hep-ph/9709492).
[8] L. Xiong, E.V. Shuryak, Nucl. Phys A590 (1995) 589.
[9] Kari J. Eskola, Berndt Muller, and Xin-Nian Wang, Selfscreened parton cascades, Preprint DUKE-TH-96-120 ( nucl-th/9608013).
[10] L.N. Lipatov, Sov.J.Nucl.Phys. 20 (1975) 94; V.N. Gribov, L.N. Lipatov, Sov.J.Nucl.Phys. 15(1975)438 and 675; Yu.L. Dokshitzer, Sov.Phys. JETP 46 (1977)641.
[11] G. Altarelli, G. Parisi, Nucl.Phys. B126 (1977)298;
[12] Yu.L. Dokshitzer, G. Marchesini, B.R. Webber, Nucl.Phys. B469 (1996) 93.
[13] Yu.L. Dokshitzer, D.V. Shirkov Z.Phys.C67(1995) 449.
[14] Yu.L. Dokshitzer, D.I. Dyakonov and S.I. Troyan, Phys.Rep. 58(1980)269.
[15] L.V. Keldysh, Sov. Phys. JETP 20 (1964) 1018; E.M. Lifshits, L.P. Pitaevsky, *Physical kinetics*, Pergamon Press, Oxford, 1981.
[16] A. Makhlin, Phys.Rev. C 51 (1995) 3454.
[17] Jianwei Qui, Phys.Rev. D42, 30(1990).
[18] J.C. Collins and Jianwei Qui, Phys.Rev. D39, 1398(1989).
[19] A.D. Sakharov, JETP 18, 631 (1948).
[20] A. Makhlin, The wedge form of relativistic dynamics, Preprint WSU-NP-96-11, 1996, hep-ph/9608251.
[21] A. Makhlin, The wedge form of dynamics. II. The gluons. Preprint WSU-NP-13, 1996, hep-ph/9608261.
[22] N.N. Bogolyubov, D.V. Shirkov, Introduction to the theory of quantized fields, Interscience, NY, 1959.
[23] C.N. Yang, D. Feldman, Phys.Rev. 79 (1950) 972.
[24] V.N. Gribov, B.L. Ioffe, and I.Ya. Pomeranchuk, Sov. J. Nucl. Phys. 2, 549 (1966).
[25] V.N. Gribov, Sov. Phys. JETP, 30, 709 (1970).
[26] S. Weinberg, Phys.Rev. 130 (1962) 776; Phys.Rev. 137B (1964) 672; The Quantum theory of fields, Ch.10, Cambridge Univ. Press, 1995.
[27] A. Makhlin, Phys.Rev. C 52 (1995) 995.
[28] P.A.M. Dirac, Rev.Mod. Phys, 21, 392 (1949).