Finite-Size Scaling in the transverse Ising Model on a Square Lattice

C. J. Hamer
School of Physics,
The University of New South Wales,
Sydney, NSW 2052, Australia.

Abstract

Energy eigenvalues and order parameters are calculated by exact diagonalization for the transverse Ising model on square lattices of up to 6x6 sites. Finite-size scaling is used to estimate the critical parameters of the model, confirming universality with the three-dimensional classical Ising model. Critical amplitudes are also estimated for both the energy gap and the ground-state energy.

PACS Indices: 02.70.+d, 05.30.-d, 05.50.+q, 64.40.Cn

(Submitted to J. Phys. A)
I. INTRODUCTION

Recent advances in computer technology have allowed the exact diagonalization of Ising-type quantum spin systems up to 36 sites in size. Schulz, Ziman and Poilblanc (1996), for example, studied the J1-J2 XXZ Heisenberg spin model on square lattices up to 6x6 sites. Our aim in this paper is to carry out an exact diagonalization study of the transverse Ising model on the square lattice, in order to estimate its critical parameters and study its finite-size scaling behaviour.

The transverse Ising model in (2+1)D is well-known to be the quantum Hamiltonian corresponding to the classical 3D Ising model (Suzuki 1976, Fradkin and Susskind 1978), and exhibits a quantum phase transition in the same universality class as the classical 3D Ising thermal transition. It was first studied by series expansion methods by Pfeuty and Elliott (1971), and there have been several further series expansion calculations since then, both ‘low-temperature’ (Marland 1981, Yanase et al 1976, Oitmaa et al 1991) and ‘high-temperature’ (Hamer and Irving 1984, Hamer and Guttmann 1989, He et al 1990). Exact finite-lattice calculations have also been carried out previously (Roomany and Wyld 1980, Hamer 1983, Henkel 1984, 1987) for square lattices of up to 5x5 sites, and similar calculations have also been done for the triangular lattice (Hamer and Johnson 1986, Henkel 1990, Price et al 1993). Here we extend these calculations for the first time to the 6x6 lattice, and use finite-size scaling theory to obtain improved estimates of the critical point and critical index \( \nu \).

The finite-size scaling amplitudes at the critical point are also of interest. In (1+1)D, it is well-known that the theory of conformal invariance relates the scaling amplitudes to fundamental parameters of the underlying effective field theory at the critical point, such as the conformal anomaly and scaling indices. In higher dimensions, a similar scenario is known to hold at ‘first-order’ transitions, where a continuous symmetry is spontaneously broken, giving rise to Goldstone bosons (Hasenfratz and Leutwyler 1990): the finite-size scaling amplitudes are related to parameters of the Goldstone bosons such as the spin-wave stiffness and spin-wave velocity. Does something similar apply at second-order transitions in higher dimensions? Apart from a discussion by Cardy (1985), little has been done in this area. One peculiar result was obtained by Henkel (1986, 1987) and Weston (1990), and confirmed recently by Weigel and Janke (1999): the scaling amplitudes of the spin-spin and energy-energy correlation lengths on antiperiodic lattices have a universal ratio:

\[
\frac{A_\sigma}{A_\epsilon} = \frac{x_\sigma}{x_\epsilon}
\]

where the \( x_i \) are the scaling indices in the respective sectors. This phenomenon appears to have no good theoretical explanation at the present time.

Our exact diagonalization methods are outlined briefly in Section 2, and the numerical results are presented in Section 3. The critical parameters so obtained are compared with other estimates in Section 4, and the critical amplitudes are discussed.

II. METHOD

The transverse Ising model on the square lattice has the Hamiltonian
\[ H = \sum_i (1 - \sigma_3(i)) - x \sum_{<ij>} \sigma_1(i)\sigma_1(j) - h \sum_i \sigma_1(i) \]  

(2.1)

where the sum \(<ij>\) runs over nearest neighbour pairs on the lattice, and the \(\sigma\) matrices are the usual Pauli spin operators acting on a 2-state spin-variable at each site. The coupling \(x\) is analogous to an inverse ‘temperature’, and \(h\) represents an external ‘magnetic field’. We shall employ a representation in which the \(\sigma_3(i)\) are diagonal. Periodic boundary conditions are assumed.

The unperturbed ground-state of the model at \(x = 0\) has all spins ‘up’, i.e. \(\sigma_3(i) = +1\), all \(i\). The interaction term will induce an admixture of states with ‘flipped’ spins. The Hilbert space of the model consists of two sectors, containing an odd and even number of flipped spins respectively.

Exact diagonalizations have been carried out for \(L \times L\) lattices, \(L = 1,..,6\). The methods employed are fairly standard, for the most part, and will not be described in detail here. First, a list of allowed basis states in the given sector was prepared, using the ‘sub-lattice coding’ technique of Lin (1990). This efficient technique produces a sorted list of states, requiring only one integer word of storage per state. Since only the zero-momentum states are considered here, the states were ‘symmetrized’: that is, all copies of a given state under translations, reflections and rotations were represented by a single state. Thus for the 6x6 lattice in the even sector, the total number of ‘unsymmetrized’ states is approximately \(2^{35}\), whereas under symmetrization this is reduced by a factor of approximately 288, down to 119,539,680.

Next, the Hamiltonian matrix elements are generated, by applying the interaction operators of equation (2.1) to each initial state, symmetrizing the resulting final state, and looking it up in the master file. The elements were grouped into blocks, each of which acts between small sub-sets of the initial and final state vectors, to avoid ‘thrashing’ during the matrix multiplications. Within each sub-set, the initial and final addresses can each be fitted into a half-integer, so that the matrix elements occupied 35 Gbyte of storage over all.

Finally, the lowest eigenvalue and eigenvector of the Hamiltonian were found in each sector, using the conjugate gradient method. Nightingale et al (1993) showed that the conjugate gradient method converges faster than the Lanczos method for large problems such as this. We found that the eigenvalue converged to an accuracy of 1 part in \(10^{10}\) in 20-25 iterations for the 6x6 lattice in the neighbourhood of the critical point.

Having determined the quantities of interest for each finite lattice, it is then necessary to make an extrapolation to \(L \to \infty\), to estimate the bulk behaviour of the system. In the vicinity of the critical point, the finite-lattice sequence will typically behave as

\[ f_L = f_{\infty} + a_1 L^{-\omega_1} + a_2 L^{-\omega_2} + \cdots \]  

(2.2)

where the \(\omega_i\) are non-integer exponents, in general (Barber 1983). The problem of extrapolating such a sequence has been discussed in several reviews (Smith and Ford 1982, Barber and Hamer 1982, Guttmann 1989). We have employed a number of different algorithms, including:

i) the Neville table (Guttmann 1989), which is best suited to a simple polynomial sequence, with integer exponents \(\omega_i\);
ii) the alternating VBS algorithm (van den Broeck and Schwartz 1979, Barber and Hamer 1982), which can give good convergence for sequences of type $(2,2)$, but needs at least two iterations to work well;

iii) the Lubkin algorithm (1952), which is more suitable for short sequences;

iv) the Bulirsch-Stoer algorithm (1964), which has been applied in this context by Henkel and Patkos (1987), and Henkel and Schütz (1988). This algorithm involves an explicit parameter $\omega$ which can be optimized to match the leading power-law correction. It has been claimed by Henkel and Schütz that the algorithm is more robust and more accurate than the VBS algorithm, especially for short sequences.

III. RESULTS

A. Finite-lattice data

The pseudo-critical point at lattice size $L$ can be defined according to finite-size scaling theory (Barber 1983) as the coupling $x_L$ such that

$$R_L(x_L) = 1$$

(3.1)

where $R_L(x)$ is the scaled energy-gap ratio

$$R_L(x) = \frac{LF_L(x)}{(L-1)F_{L-1}(x)}$$

(3.2)

and $F_L(x)$ is the energy gap for lattice size $L$. This point is found by calculating the energy eigenvalues at a cluster of 5 equally spaced points in the neighbourhood of $x_L$, and then finding $x_L$ by interpolation between them. The spacing between the points was chosen as $\Delta x = 0.001$, estimated to balance the truncation and round-off errors in the calculation. The values of all other observables can then be estimated at $x_L$ by the same finite-difference interpolation procedures. Tables 1 and 2 list the pseudo-critical points $x_L$, and the values of the calculated observables at coupling $x_L$ for each pair of lattice sizes $L$ and $(L-1)$, for $L = 2, 3, 4, 5$ and 6. The values of $x_L$ for $L = 2$ to 5 listed in Table 1 agree through six figures with those calculated previously (Hamer 1983).

Table 1 lists values for the ground-state energy per site $\epsilon_{0,L}$ for lattice size $L$, and its derivatives $\epsilon'_0, L$ and $\epsilon''_0, L$, where the prime denotes differentiation with respect to $x$. The values are expected to be accurate to the figures quoted (or better) as regards round-off error. The truncation error in the 5-point interpolation process is harder to estimate, since it involves unknown higher derivatives of $\epsilon_0$, but we estimate it should be no more than about 1 part in $10^{12}$ for $\epsilon_0$ and 1 part in $10^6$ for $\epsilon''_0$.

We have also listed values in Table 1 for the magnetic susceptibility, defined by

$$\chi_L = -\frac{1}{L^2} \frac{\partial^2 E_{0,L}(x,h)}{\partial h^2} |_{h=0}$$

(3.3)

This derivative was also estimated by a finite difference method, using a cluster of 5 data points around $h = 0$, with a spacing $\Delta h = 0.0003$, giving an estimated truncation error of
no more than 1 part in $10^6$ in the susceptibility. This calculation was a little too large to carry through for \( L = 6 \), with the facilities available.

Table 2 lists the energy gap \( F_L \) between the odd and even sectors, and its derivatives \( F'_L \) and \( F''_L \), at each \( x_L \). Values are also listed here for the quantity \( M_L \) defined by

\[
M_L = \langle 0 | \sigma_1(1) | 1 \rangle \tag{3.4}
\]

where \( |0\rangle, |1\rangle \) are the lowest-lying energy eigenvectors in the even and odd sectors, respectively. It can be shown (Yang 1952, Uzelac 1980, Hamer 1982) that this quantity converges towards the spontaneous magnetization in the bulk limit. Unfortunately, for technical reasons we were again unable to calculate this quantity for \( L = 6 \). Since the accuracy of the wavefunction is only the square root of that of the eigenvalue, the round-off error in these values is expected to be about 1 part in $10^6$.

**B. Critical Point**

The sequence of pseudo-critical points \( x_L \) converges rapidly, as can be seen in Fig. 1, where \( x_L \) is plotted against \( 1/L^4 \). To estimate the bulk limit \( (L \to \infty) \) of this sequence, we have employed various algorithms discussed above, as well as a simple polynomial fit in \( 1/L^4 \) and higher powers.

Our final estimate of the critical point is

\[
x_c = 0.32841(2) \tag{3.5}
\]

This is consistent with our earlier finite-size estimate of \( x_c = 0.3289(10) \) (Hamer 1983), but nearly two orders of magnitude more accurate. Henkel (1987) obtained an improved estimate \( x_c = 0.3282(1) \) from lattices up to 5x5 sites.

**C. Critical Indices**

Finite-size scaling theory (Barber 1983) also tells us how to estimate the critical indices for the model. The finite-lattice susceptibility \( \chi_L \), for instance, is predicted to scale at the critical point like

\[
\chi_L(x_c) \sim L^{\gamma/\nu}, \quad L \to \infty \tag{3.6}
\]

and hence one finds that

\[
L(1 - \frac{\chi_L(x_L)}{\chi_{L-1}(x_L)}) \sim -\frac{\gamma}{\nu}, \quad L \to \infty \tag{3.7}
\]

Similarly, ratios of the finite-lattice ‘magnetizations’ (equation 3.4) give estimates of \( \beta/\nu \). Finally, estimates of the index \( 1/\nu \) can be obtained from the Callan-Symanzik ‘beta function’ (Barber 1983),

\[
\frac{\beta_L(x)}{g} = \frac{F_L(x)}{(F_L(x) - 2xF'_L(x))} \tag{3.8}
\]
via

\[ L(1 - \frac{\beta_L(x_L)}{\beta_{L-1}(x_L)}) \sim \frac{1}{\nu}, \quad L \to \infty \tag{3.9} \]

One would expect to obtain estimates of the ratio \( \alpha/\nu \) in a similar fashion from the ‘specific heat’,

\[ C_L(x) = -\frac{x^2 \partial^2 \epsilon_0}{L^2 \partial x^2}, \tag{3.10} \]

but it is known (Hamer 1983) that these estimates are very poor, too high by a factor of nearly 2. The reason is easily found: the ground-state energy or specific heat contains a ‘regular’ or analytic piece as well as the singular term (Privman and Fisher 1984). Henkel (1987) has cleverly sidestepped this problem, using a transition amplitude to find \( \alpha/\nu \), in analogy to equation (3.4). Here, we eliminate the regular term by subtracting:

\[ \epsilon''_{0,L} - \epsilon''_{0,L-1} \sim L^{\alpha/\nu - 1}, \quad L \to \infty, \tag{3.11} \]

and using successive ratios of these differences to estimate \( 1 - \alpha/\nu \) The estimates so obtained for the critical index ratios are listed in Table 3.

Alternatively, ‘logarithmic’ estimates of the critical indices may be obtained as follows:

\[ \ln \left[ \frac{\chi_L(x_L)/\chi_{L-1}(x_L)}{\ln[L/(L - 1)]} \right] \sim \frac{\gamma}{\nu}, \quad L \to \infty \tag{3.12} \]

These alternative estimates are listed in Table 4. The finite-size corrections are generally smaller for the logarithmic estimates.

These finite-size estimates of the critical indices agree closely with the previous calculation of Hamer (1983) up to \( L = 5 \); and remarkably enough, most of them agree to within 4 significant figures with the equivalent results obtained for the triangular lattice (Hamer and Johnson 1986, Price et al 1993).

The same algorithms mentioned above have been employed to extrapolate these sequences to their bulk limit. The sequences are very short, and may have slight irregularities, so that the tabular algorithms are generally no more accurate than simple graphical methods or polynomial fits in the extrapolation. The resulting estimates are listed at the foot of Tables 3 and 4. The errors in these estimates are inevitably rather subjective, but the variation between different algorithms gives some indication of the likely error.

Figure 2 graphs the estimates of \( 1/\nu \) from Tables 3 and 4 as a function of \( 1/L \): it can be seen that the behaviour is almost precisely linear for the estimates from Table 3. Correspondingly, the Neville tables and polynomial fits give stable results, while the Lubkin and Bulirsch-Stoer algorithms give less stable results, possibly a little higher. We conclude that

\[ \frac{1}{\nu} = 1.591(2) \tag{3.13} \]

The estimates for \( \alpha/\nu \) are not quite so well-behaved, but our final estimate is
\[
\frac{\alpha}{\nu} = 0.16(1) \tag{3.14}
\]

This is a much better result than can be obtained directly from the specific heat, equation (3.10).

The estimates for the other indices $\beta/\nu$ and $\gamma/\nu$ are rapidly convergent, but we only have data up to $L = 5$, which were known previously. We find

\[
\frac{\beta}{\nu} = 0.522(2) \tag{3.15}
\]

\[
\frac{\gamma}{\nu} = 1.96(1) \tag{3.16}
\]

### D. Energy Amplitudes

The finite-size behaviour of the energy gap at the critical point is

\[
F_L(x_L) \sim \frac{A_1}{L}, \quad L \to \infty \tag{3.17}
\]

so the amplitude $A_1$ can be estimated by

\[
LF_L(x_L) \sim A_1, \quad L \to \infty \tag{3.18}
\]

The sequence of estimates for $A_1$ is shown in Figure 3. It extrapolates to a value

\[
A_1 = 1.39(1) \tag{3.19}
\]

A value of 1.42 was previously estimated by Henkel (1987).

The finite-size scaling behaviour of the ground-state energy per site $\epsilon_0$ at the pseudo-critical point is shown in Figure 4. The finite-size scaling corrections appear to decrease like $1/L^3$, in accordance with the Privman-Fisher scaling hypothesis (1984), which states that the singular part of the free energy density of a system of finite size $L$ should scale as $L^{-d}$ (here $d = 3$). A polynomial fit on this assumption gives

\[
\epsilon_{0,L}(x_L) \sim \epsilon_0^* - \frac{A_0}{L^3}, \quad L \to \infty \tag{3.20}
\]

with

\[
\epsilon_0^* = -0.624(1) \tag{3.21}
\]

and

\[
A_0 = 0.38(5) \tag{3.22}
\]

Further evidence for this power-law behaviour can be obtained as follows. Suppose

\[
\epsilon_{0,L}(x_L) \sim \epsilon_0^* - A_0/L^p, \quad L \to \infty \tag{3.23}
\]
then
\[
L[1 - \frac{(\epsilon_{0,L}(x_L) - \epsilon_{0,L-1}(x_L))}{(\epsilon_{0,L-1}(x_{L-1}) - \epsilon_{0,L-2}(x_{L-1}))}] \equiv p_L \sim p \quad \text{as } L \to \infty \tag{3.24}
\]
and
\[
\ln\left[\frac{\epsilon_{0,L}(x_L) - \epsilon_{0,L-1}(x_L)}{\epsilon_{0,L-1}(x_{L-1}) - \epsilon_{0,L-2}(x_{L-1})}\right] / \ln[L/L - 1] \sim -p \quad \text{as } L \to \infty \tag{3.25}
\]
The sequences of finite lattice estimates for \( p \) are shown in Figure 5. It can be seen that the ‘linear’ sequence comes down towards 3 from above, whilst the ‘logarithmic’ sequence comes up towards 3 from below. The sequences are a little irregular, however, and the best estimate we can obtain for the bulk limit is
\[
p = 2.8(2) \tag{3.26}
\]
a little lower than, but still consistent with 3.

Assuming that \( p = 3 \), the scaling amplitude \( A_0 \) for the ground-state energy can be found by
\[
\frac{L^4}{3} (\epsilon_{0,L}(x_L) - \epsilon_{0,L-1}(x_L)) \sim A_0, \quad L \to \infty \tag{3.27}
\]
The sequence of estimates for \( A_0 \) is graphed in Figure 6, and extrapolates to a value
\[
A_0 = 0.35(2) \tag{3.28}
\]
which is in reasonable agreement with equation (3.22). Henkel (1987) previously obtained an estimate of 0.39 for this quantity (allowing for the different normalization of his Hamiltonian). Mon (1985) obtained a Monte Carlo estimate of the corresponding free energy amplitude in the 3D classical model.

In order to ‘calibrate’ this result, we need to know the “speed of light” \( v \), or in other words the scale factor needed in this model to make the long-range correlations isotropic in space and time at the critical point. We have attempted to estimate this using the dispersion relation for the lowest excited state at the critical point, expected to be of the form
\[
E(k) = vk \tag{3.29}
\]
in the bulk system. We have calculated the finite-lattice eigenvalues for low-lying excited states with non-zero momentum for lattice sizes \( L = 2 \) to 5, and set
\[
v_L = \frac{L}{2\pi} (F_L(x_L, \frac{2\pi}{L}) - F_L(x_L, 0)) \sim v, \quad L \to \infty \tag{3.30}
\]
where \( F_L(x, k) \) is the energy at coupling \( x \) for momentum \( k \). Figure 7 shows the sequence of finite-lattice estimates for \( v \) as a function of \( 1/L \). They extrapolate to a bulk value
\[
v = 0.99(3) \tag{3.31}
\]
The ratio $A_0/v$ should be a universal number, independent of the normalization of the Hamiltonian. From the results above, we find

$$\frac{A_0}{v} = 0.35(2), \quad (3.32)$$

to be compared with values of 0.719 expected according to effective field theory for a single free boson (Hasenfratz and Niedermayer 1993), or 0.211 for a single free fermion degree of freedom (Appendix). The result (3.31) matches neither of these values. This is not surprising, since the effective field theory at the critical point is expected to be a non-trivial interacting theory. It might be possible to estimate this quantity via the $\epsilon$-expansion, using a Landau-Ginzburg effective field theory. This has not yet been done, as far as we are aware.

**IV. CONCLUSIONS**

We have calculated the lowest-lying energy eigenvalues of the transverse Ising model on the square lattice with periodic boundary conditions for lattice sizes up to 6x6 sites, using the conjugate gradient method. Finite-size scaling theory has been employed to estimate the critical parameters, which are compared with previous estimates in Table 5.

It can be seen that our present estimates agree well with earlier finite-size scaling results. We have achieved a substantial increase in accuracy for the critical point, but only a more modest increase for the critical index $\nu$. The results appear very compatible with previous series analyses, and also with recent estimates for the classical 3D Ising model, and field theory. This provides further confirmation of the universality between these transitions. Finally, it can be seen that the accuracy of the exponents for the quantum model is now not very far behind that for the classical model.

We have also estimated the finite-size scaling amplitudes for the energy eigenvalues at the critical point. For the spin gap we find

$$A_1 = 1.39(1) \quad (4.1)$$

and estimated

$$A_0 = 0.35(2), \quad (4.3)$$

to be compared with a previous estimate of 0.39 by Henkel (1987). It should be possible to predict this amplitude from Landau-Ginzburg effective field theory.

An extension to 7x7 sites of these exact diagonalization calculations is hardly feasible at the present time, but there are some very precise approximate methods now available, such as the density matrix renormalization group (White 1992) and path integral Monte Carlo techniques (Sandvik 1992). These might well be able to extend the results to larger lattice sizes, and allow much improved finite-size scaling estimates of the critical parameters. They
could also confirm whether or not the Casimir energy scales as in equation (4.2). Our exact
diagonalization results should provide a useful calibration for such studies. We look forward
to seeing such calculations in the future.

We have chosen here to work on the square lattice rather than the triangular one, because
the Hamiltonian matrix is somewhat smaller, and the leading finite-size corrections are
expected to be much the same for both lattices. There are, however, some hints of irregularity
or alternating behaviour in some of the square lattice sequences. It might well be that the
triangular lattice results are smoother.

ACKNOWLEDGMENTS

I would like to thank Dr. P.F. Price and Prof. I. Affleck for useful discussions. Part
of this work was carried out while on study leave at the Institute for Theoretical Physics,
University of California at Santa Barbara, and at the Centre for Nonlinear Studies, Los
Alamos National Laboratory. I would like to thank Prof. R. Singh and the organizers of the
Workshop on ‘Magnetic Materials in Novel Materials and Geometries’ for their hospitality
in Santa Barbara, and Dr. J. Gubernatis for his kind hospitality in Los Alamos. The
calculations were performed using facilities at the New South Wales Centre for Parallel
Computing, and at the Centre for Nonlinear Studies, Los Alamos. I am very grateful to
Prof. R. Standish and Mr. D. Neal for their assistance in this regard. This research was
supported in part by the National Science Foundation under grant no. PHY94-07194, and
also by a grant from the Australian Research Council.

Appendix

The finite-size scaling amplitude for the ground-state energy (“Casimir amplitude”) can
be calculated for free fields as follows.

1. Free boson case

The zero-point energy of a free boson field is given in $d$ space dimensions by

$$E_0 = \frac{1}{2} \sum_k \omega_k$$

i.e. $\omega_k/2$ for each momentum mode. On a lattice, the free particle Hamiltonian can be
written in a finite-difference form

$$H = \frac{1}{2} \sum_n \left[ \dot{\phi}^2(n) + \sum_{i=1}^d \left( \frac{\phi(n+i)-\phi(n-i)}{2} \right)^2 \right]$$

where the lattice spacing has been set to 1. The eigenmodes are plane waves

$$\phi(n, t) = \frac{1}{N^{1/2}} \sum_k [a_k e^{i(k.n-\omega_k t)} + a_k^\dagger e^{-i(k.n-\omega_k t)}]$$

whence
\[ \omega_k = [2 \sum_{i=1}^{d} (1 - \cos k_i)]^{1/2} \]  

and for periodic boundary conditions the allowed momenta are

\[ k_i = \frac{2\pi}{L} l_i, \quad l_i = 0, 1, 2, \ldots \]

for a lattice of \(N = L^d\) sites. Hence

\[ E_0 = \frac{1}{2} \sum_{k} [2 \sum_{i=1}^{d} (1 - \cos k_i)]^{1/2} \]

i.e.

\[ \epsilon_0 = \frac{1}{L^d} \sum_{\{l_i\} = 0}^{L-1} \left[ \sum_{i=1}^{d} \left( \frac{\pi l_i}{L} \right)^2 \right]^{1/2} \]

Now the leading finite-size correction to this sum arises from the infrared (small momentum) behaviour of the lattice sum (Hasenfratz and Leutwyler 1990), and does not depend on the cutoff or regularization at large momentum. Thus we may approximate for our purposes

\[ \epsilon_0 \approx \frac{1}{L^d} \sum_{\{l_i\} = -\infty}^{\infty} \left[ \sum_{i=1}^{d} \left( \frac{\pi l_i}{L} \right)^2 \right]^{1/2} \]

Now use the Poisson resummation formula

\[ \sum_{m=-\infty}^{+\infty} f(mL) = \frac{1}{L^d} \sum_{n=-\infty}^{+\infty} \left( \frac{2\pi n}{L} \right) \]

with

\[ g(k) = \int_{-\infty}^{+\infty} e^{ik \cdot x} f(x) d^d x \]

and

\[ \epsilon_0' = \frac{1}{4\pi} \sum_{\{m_i\}'} \int_0^{+\infty} k^d dk \frac{J_{d/2-1}(kx)}{(2\pi kx)^{d/2-1}} \quad (x = L|m|) \]

\[ = -\frac{1}{2\pi^{d+1} L^{d+1}} \sum_{\{m_i\}'}^{+\infty} \frac{1}{|m|^{d+1}} \]

The dash here implies removal of the term \(m = 0\), which corresponds to the (infinite, non-universal) bulk ground-state energy per site, which we simply drop.

The sum involved here is a generalization of the Riemann zeta function. It gives

\[ \epsilon_0' = -\frac{A_0}{L^{d+1}} \]
where for $d = 1$, $A_0$ is easily evaluated

$$A_0 = \frac{\pi}{6} = 0.5236 \tag{17}$$

the result familiar from conformal field theory. For higher dimensions, we have evaluated the sum numerically

$$d = 2 : A_0 = 0.7189 \tag{18}$$

$$d = 3 : A_0 = 0.8375 \tag{19}$$

The result for $d = 2$ was given previously by Hasenfratz and Niedermayer (1993).

### 2. Free fermion case

A similar naive argument can be given for the case of a single species of free Weyl (spinless) fermions. The filled Dirac sea has energy

$$E_0 = - \sum_k \omega_k \tag{20}$$

where again

$$\omega_k = [2 \sum_{i=1}^d (1 - \cos k_i)]^{1/2} \tag{21}$$

and we assume antiperiodic boundary conditions for the fermions

$$k_i = \frac{\pi}{L} (2l_i + 1), \quad l_i = 0, 1, 2, \cdots \tag{22}$$

Hence

$$\epsilon_0 \simeq - \frac{1}{L^d} \sum_{\{l_i\} = -\infty}^{\infty} \left[ \sum_{i=1}^d \left( \frac{\pi(2l_i + 1)}{2L} \right)^2 \right]^{1/2} \tag{23}$$

Use the Poisson resummation formula again to find

$$\epsilon'_0 = - \frac{1}{4\pi} \sum_{\{m_i\}'}^{\infty} (-1)^{\sum_i m_i} \int_0^{\infty} k^d dk \frac{J_{d-1}(kx)}{(2\pi kx)^{d/2 - 1}} \quad (x = L|m|) \tag{24}$$

$$= \frac{\Gamma\left(\frac{d+1}{2}\right)}{2\pi^{d+2} L^{d+1}} \sum_{\{m_i\}'}^{\infty} (-1)^{\sum_i m_i} \frac{1}{|m|^{d+1}} \tag{25}$$

For $d = 1$ $A_0$ is easily evaluated to give

$$A_0 = \frac{\pi}{12} = 0.2618 \tag{26}$$

also familiar from conformal field theory; while for higher dimensions, we find numerically

$$d = 2 : A_0 = 0.2106 \tag{27}$$

$$d = 3 : A_0 = 0.1957 \tag{28}$$

These numbers have not been obtained previously, as far as we are aware.
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* Email address: c.hamer@unsw.edu.au

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FIGURES

**Figure 1.** Graph of the finite-lattice pseudo-critical points $x_L$ as a function of $1/L^4$. The line is merely to guide the eye.

**Figure 2.** Finite-lattice estimates of the index $1/\nu$ graphed against $1/L$. The full circles are ‘linear’ estimates, the open circles are ‘logarithmic’ estimates. The lines are merely to guide the eye.

**Figure 3.** The energy gap amplitude $A_{1,L}$ graphed against $1/L^2$.

**Figure 4.** The ground-state energy per site at lattice size $L$ graphed against $1/L^3$.

**Figure 5.** The effective exponent $p_L$ graphed against $1/L$.

**Figure 6.** The Casimir amplitude $A_{0,L}$ graphed against $1/L$.

**Figure 7.** Finite-lattice estimates of the ‘speed of light’ $v$ graphed against $1/L$. 

### TABLE I.
Finite-lattice data at the pseudo-critical points $x_L$, calculated for the pair of lattice sites $L$ and $(L-1)$ in each case. Given are the ground-state energy per site $\epsilon_0$, its first two derivatives $\epsilon'_0$ and $\epsilon''_0$ with respect to $x$, and the susceptibility $\chi$.

| $x_L$       | $L$ | $\epsilon_0$   | $\epsilon'_0$  | $\epsilon''_0$ | $\chi$  |
|------------|-----|----------------|----------------|----------------|---------|
| 0.26034238222 | 1   | -0.520684764436 | -2.00000000   | 0.00000       | -0.09000 |
|             | 2   | -0.074060535180 | -0.6163273    | -2.90881      | -0.32656 |
| 0.31600008772 | 2   | -0.112709188696 | -0.77866508   | -3.06919      | -0.45374 |
|             | 3   | -0.070818161984 | -0.59879739   | -4.15119      | -1.00929 |
| 0.32424925229 | 3   | -0.075901188836 | -0.63384113   | -4.34370      | -1.13177 |
|             | 4   | -0.066430308096 | -0.57114518   | -5.09931      | -1.99844 |
| 0.32669593806 | 4   | -0.067843120636 | -0.58378935   | -5.23641      | -2.11883 |
|             | 5   | -0.064637823298 | -0.55295527   | -5.83324      | -3.29209 |
| 0.32758326752 | 5   | -0.065130784765 | -0.55817035   | -5.92135      | -       |
|             | 6   | -0.063752757694 | -0.54012509   | -6.42027      | -       |

### TABLE II.
Finite-lattice data as in Table 1, consisting of the mass gap $F$, its first two derivatives $F'$ and $F''$ with respect to $x$, and the ‘magnetization’ $M$.

| $x_L$       | $L$ | $F$             | $F'$            | $F''$           | $M$    |
|------------|-----|----------------|----------------|----------------|--------|
| 0.26034238222 | 1   | 2.000000000000 | 0.000000      | 0.00000        | 1.0000  |
|             | 2   | 1.000000000000 | -3.407921     | 6.05294        | 0.67337 |
| 0.31600008772 | 2   | 0.820891162135 | -3.0223297    | 7.44433        | 0.71908 |
|             | 3   | 0.547260774756 | -4.4695972    | 13.9564        | 0.58357 |
| 0.32424925229 | 3   | 0.510885529302 | -4.3470817    | 15.7416        | 0.59705 |
|             | 4   | 0.383164146983 | -5.4088281    | 25.0603        | 0.51520 |
| 0.32669593806 | 4   | 0.370007329425 | -5.3452110    | 26.9450        | 0.52130 |
|             | 5   | 0.296005863592 | -6.2337554    | 39.3896        | 0.46488 |
| 0.32758326752 | 5   | 0.290490201648 | -6.1980387    | 41.1167        | -      |
|             | 6   | 0.2420751678   | -6.9850090    | 56.7636        | -      |
TABLE III. Finite-size scaling estimates of the critical indices, as defined by equation (3.7) and following, where L is the larger of the two lattice sizes used in the estimate.

| L   | 1/ν   | 1 - α/ν | β/ν     | γ/ν     | ρ     |
|-----|-------|---------|---------|---------|-------|
| 2   | 1.27817 | -       | 0.653264 | 1.44880 | -     |
| 3   | 1.38021 | 1.88408 | 0.565352 | 1.65132 | 1.71862 |
| 4   | 1.43242 | 1.20661 | 0.548326 | 1.73469 | 2.09567 |
| 5   | 1.46377 | 1.05069 | 0.541083 | 1.78195 | 2.30781 |
| 6   | 1.48479 | 0.98436 | -        | -       | 2.42047 |
| ∞   | 1.591(2) | 0.84(1) | 0.523(2) | 1.95(1) | 2.8(2) |

TABLE IV. ‘Logarithmic’ finite-size scaling estimates of the critical indices, as defined by equation (3.12) and following, where L is the larger of the two lattice sizes used in the estimate.

| L   | 1/ν   | 1 - α/ν | β/ν     | γ/ν     | ρ     |
|-----|-------|---------|---------|---------|-------|
| 2   | -     | -       | -       | -       | -     |
| 3   | 1.52002 | 2.43901 | 0.514989 | 1.97178 | 4.83686709 |
| 4   | 1.54105 | 1.24804 | 0.512493 | 1.97642 | 4.16837815 |
| 5   | 1.55226 | 1.05715 | 0.513266 | 1.97479 | 3.85524695 |
| 6   | 1.55937 | 0.98287 | -        | -       | 3.63001774 |
| ∞   | 1.593(3) | 0.84(1) | 0.521(3) | 1.96(1) | 2.5(2) |

TABLE V. A comparison of critical parameters obtained in the present work with some others obtained elsewhere. Key: HT = high-temperature series; LT = low-temperature series; FS = finite-size scaling; MC = Monte Carlo; TR = triangular lattice; SQ = square lattice

| (2+1)-dimensional Ising model | ν     | α     | β     | γ     | x_c   |
|-------------------------------|-------|-------|-------|-------|-------|
| FS SQ Hamer (1983)            | 0.635(5) | 0.3289(10) |
| FS SQ Henkel (1984, 1987)     | 0.629(2) | 0.11(1) | 0.324(9) | -     | 0.3282(1) |
| FS SQ This work               | 0.629(1) | 0.10(1) | 0.328(2) | 1.23(1) | 0.32841(2) |
| FS TR Hamer & Johnson (1986)  | 0.627(4) | -      | 0.332(6) | 1.236(8) |
| FS TR Price et al (1993)      | 0.627(2) | 0.12(2) | 0.324(3) | 1.23(1) |
| HT SQ He et al (1990)         | 0.637(4) | 0.11(2) | -      | 1.244(4) | 0.32851(8) |
| LT SQ Oitmaa et al (1991)     | 0.64(3)  | 0.096(6) | 0.318(4) | 1.25(2) |

| 3-dimensional Ising model     | ν     | α     | β     | γ     | x_c   |
|-------------------------------|-------|-------|-------|-------|-------|
| HT BCC Butera and Comi (1997) | 0.6308(5) | -      | 1.2384(6) |
| MC Hasenbusch (1999)          | 0.6296(3)(4) | 1.2367(11) |

Field Theory

| Guida and Zinn-Justin (1997) | ν     | α     | β     | γ     | x_c   |
|-------------------------------|-------|-------|-------|-------|-------|
|                               | 0.6304(13) | 0.109(4) | 0.3258(14) | 1.2396(13) |
