Quasi-modular forms attached to Hodge structures

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Abstract The space $D$ of Hodge structures on a fixed polarized lattice is known as Griffiths period domain and its quotient by the isometry group of the lattice is the moduli of polarized Hodge structures of a fixed type. When $D$ is a Hermition symmetric domain then we have automorphic forms on $D$, which according to Baily-Borel theorem, they give an algebraic structure to the mentioned moduli space. In this article we slightly modify this picture by considering the space $U$ of polarized lattices in a fixed complex vector space with a fixed Hodge filtration and polarization. It turns out that the isometry group of the filtration and polarization, which is an algebraic group, acts on $U$ and the quotient is again the moduli of polarized Hodge structures. This formulation leads us to a notion of quasi-automorphic forms which generalizes quasi-modular forms attached to elliptic curves.

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In 1970 Griffiths in his article [6] introduced the period domain $D$ and described a project to enlarge $D$ to a moduli space of degenerating polarized Hodge structures. He also asked for the existence of a certain automorphic form theory for $D$, generalizing the usual notion of automorphic forms on Hermitian symmetric domains. Since then there have been much effort made on the first part of Griffiths’s project (see [8, 15] and the references there). For the second part Griffiths himself introduced the theory of automorphic cohomology; however, the generating function role of automorphic forms is somewhat lacking in this theory.

Some years ago, I was looking for some analytic spaces over $D$ for which one may state the Baily-Borel theorem on the unique algebraic structure of quotients of Hermitian symmetric domains by discrete arithmetic groups. I realized that even in the simplest case of Hodge structures, namely $h^{01} = h^{10} = 1$, such spaces are not well studied. This led me to the definition of a class of holomorphic functions on the Poincaré upper-half-plane which generalize the classical modular forms (see [16]). Since a differential operator acts on them I called them differential modular forms. Soon after I realized that such functions play a central role in mathematical physics and, in particular, in mirror symmetry (see [11] and the references therein). Inspired by this special case of Hodge structures with its fruitful applications, I felt the necessity to develop as much as possible similar theories for an arbitrary type of Hodge structure.

In this note we construct an analytic variety $U$ and an action of an algebraic group $G_0$ on $U$ from the right such that $U/G_0$ is the moduli space of polarized Hodge structures of a fixed type. We may pose the following algebraization problem for $U$, in parallel to
the Baily-Borel theorem in (1): construct functions on $U$ which have some automorphic properties with respect to the action of $G_0$ and have some finite growth when a Hodge structure degenerates. There must be enough of them in order to enhance $U$ with a canonical structure of an algebraic variety such that the action of $G_0$ is algebraic. In the case for which the Griffiths period domain is Hermitian symmetric, for instance for the Siegel upper half-plane, this problem seems to be promising but needs a reasonable amount of work if one wants to construct such functions through the inverse of the generalized period maps (see (4,1)). Among them are calculating explicit affine coordinates in certain moduli spaces and calculating Gauss-Manin connections. Some main ingredients of such a study for K3 surfaces endowed with polarizations is already done by many authors, see for instance (2) and the references therein. For the case in which the Griffiths period domain is not Hermitian symmetric, we reformulate the algebraization problem further (see (4,2) and § 4.1). Among them are calculating explicit affine coordinates in certain moduli spaces $T$ is the moduli space of pairs $(\delta_1, \delta_2)$ is a basis of the $\mathbb{Z}$-module $H_1(E_1, \mathbb{Z})$ with $\langle \delta_1, \delta_2 \rangle = -1$. The algebraic group

$$G_0 = \{ \begin{pmatrix} k & k' \\ 0 & k^{-1} \end{pmatrix} \mid k, k' \in \mathbb{C}, k \neq 0 \}$$

acts from the right on $U$ by the usual multiplication of matrices. Under $pm$ the action of $G_0$ is given by

$$t \cdot g = (t_1k^{-2} + k'k^{-1}, t_2k^{-4}, t_3k^{-6}),$$

$$t = (t_1, t_2, t_3) \in \mathbb{C}^3, \quad g = \begin{pmatrix} k & k' \\ 0 & k^{-1} \end{pmatrix} \in G_0.$$ 

In fact, $T$ is the moduli space of pairs $(E, \{\omega_1, \omega_2\})$, where $E$ is an elliptic curve and $\{\omega_1, \omega_2\}$ is a basis of $H^1_{\text{DR}}(E)$ such that $\omega_1$ is represented by a differential form of the first kind and $\frac{1}{2\pi i} \int_E \omega_1 \cup \omega_2 = 1$.

The algebra of quasi-modular forms arises in the following way: We consider the composition of maps

$$\mathbb{H} \overset{i}{\rightarrow} P \overset{\text{pm}^{-1}}{\rightarrow} U \overset{\text{pm}}{\rightarrow} T \overset{\tau}{\rightarrow} T,$$

where $\mathbb{H} = \{ \tau \in \mathbb{C} \mid \text{Im}(\tau) > 0 \}$ is the upper half-plane.
In this section we define the generalized period domain $U$ is the quotient map and $\tilde{T} = C^3$ is the underlying complex manifold of the affine variety $\text{Spec}(C[t_1, t_2, t_3])$. The pullback of the function ring $C[t_1, t_2, t_3]$ of $T$ by the composition $\mathbb{H} \to \tilde{T}$ is a $C$-algebra which we call the $C$-algebra of quasi-modular forms for $\text{SL}(2, \mathbb{Z})$. Three Eisenstein series

$$g_i(\tau) = a_k \left( 1 + b_k \sum_{d=1}^{\infty} d^{2k-1} \frac{e^{2\pi i d \tau}}{1 - e^{2\pi i d \tau}} \right), \quad k = 1, 2, 3,$$

where

$$(b_1, b_2, b_3) = (-24, 240, -504), \quad (a_1, a_2, a_3) = \left( \frac{2\pi i}{12}, \frac{2\pi i}{12}, \frac{2\pi i}{12}, \frac{2\pi i}{12}, \frac{2\pi i}{12} \right)$$

are obtained by taking the pullback of the $t_i$'s. Our reformulation of the algebraization problem is based on §3.3 and the pullback argument, see [3.3]

We fix some notations from linear algebra. For a basis $\omega_1, \omega_2, \ldots, \omega_h$ of a vector space we denote by $\omega$ an $h \times 1$ matrix whose entries are the $\omega_i$'s. In this way we also say that $\omega$ is a basis of the vector space. If there is no danger of confusion we also use $\omega$ to denote an element of the vector space. We use $A^t$ to denote the transpose of the matrix $A$. Recall that if $\delta$ and $\omega$ are two bases of a vector space, $\delta = \rho \omega$ for some $\rho \in \text{GL}(h, C)$ and a bilinear form on $V_0$ in the basis $\delta$ (resp. $\omega$) has the matrix form $A$ (resp. $B$) then $\rho BP^t = A$. By $a_{ij}$ we mean an $h \times h$ matrix whose $(i, j)$ entry is $a_{ij}$.

## 1 Moduli of polarized Hodge structures

In this section we define the generalized period domain $U$ and we explain its comparison with the classical Griffiths period domain.

### 1.1 The space of polarized lattices

We fix a $C$-vector space $V_0$ of dimension $h$, a natural number $m \in \mathbb{N}$ and a $h \times h$ integer-valued matrix $\Psi_0$ such that the associated bilinear form

$$\mathbb{Z}^h \times \mathbb{Z}^h \to \mathbb{Z}, \quad (a, b) \mapsto a^t \Psi_0 b$$

is non-degenerate, symmetric if $m$ is even and skew if $m$ is odd. Note that, in the case of $\mathbb{Z}$-modules, by non-degenerate we mean that the associated morphism

$$\mathbb{Z}^h \to (\mathbb{Z}^h)^\vee, \quad a \mapsto (b \mapsto a^t \Psi_0 b)$$

is an isomorphism, where $\vee$ means the dual of a $\mathbb{Z}$-module.

A lattice $V_\mathbb{Z}$ in $V_0$ is a $\mathbb{Z}$-module generated by a basis of $V_0$. A polarized lattice $(V_\mathbb{Z}, \Psi_\mathbb{Z})$ of type $\Psi_0$ is a lattice $V_\mathbb{Z}$ together with a bilinear map $\Psi_\mathbb{Z} : V_\mathbb{Z} \times V_\mathbb{Z} \to \mathbb{Z}$ such that in a $\mathbb{Z}$-basis of $V_\mathbb{Z}$, $\Psi_\mathbb{Z}$ has the form $\Psi_0$.

Let $\mathcal{L}$ be the set of polarized lattices of type $\Psi_0$ in $V_0$. It has a canonical structure of a complex manifold of dimension $\dim_C(V_0)^2$. One can take a local chart around $(V_\mathbb{Z}, \Psi_\mathbb{Z})$ by fixing a basis of the $\mathbb{Z}$-module $V_\mathbb{Z}$. Usually, we denote an element of $\mathcal{L}$ by $(x, y, \ldots, \text{resp. } \Psi(x), \Psi(y), \ldots)$. Let $R$
be any subring of \( \mathbb{C} \). For instance, \( R \) can be \( \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z} \). We define
\[
V_R(x) := V_\mathbb{Z}(x) \otimes \mathbb{Z} R \quad \text{and} \quad \psi_R(x) : V_R(x) \times V_R(x) \to R \quad \text{the induced map.}
\]

Conjugation with respect to \( x \in \mathcal{L} \) of an element \( \omega = \sum_{i=1}^{\delta} a_i \delta_i \in V_0 \), where \( V_\mathbb{Z}(x) = \sum_{i=1}^{\delta} \mathbb{Z} \delta_i \), is defined by
\[
\overline{\omega} := \sum_{i=1}^{\delta} \overline{a_i} \delta_i,
\]
where \( \overline{s}, s \in \mathbb{C} \) is the usual conjugation of complex numbers.

### 1.2 Hodge filtration

We fix Hodge numbers
\[
h^{i,m-i} \in \mathbb{N} \cup \{0\}, \quad h^i := \sum_{j=i}^{m} h^{i,m-j}, \quad i = 0, 1, \ldots, m, \quad h^0 = h
\]
a filtration
\[
F_0^* : \{0\} = F_0^{m+1} \subset F_0^m \subset \cdots \subset F_0^0 = V_0, \quad \dim(F_0^i) = h^i
\]
on \( V_0 \) and a bilinear form
\[
\psi_0 : V_0 \times V_0 \to \mathbb{C}
\]
such that in a basis of \( V_0 \) its matrix is \( \Psi_0 \) and it satisfies
\[
\psi_0(F^i_0, F^j_0) = 0, \quad \forall i, j, i + j > m.
\]
A basis \( \omega_i, i = 1, 2, \ldots, h \) of \( V_0 \) is compatible with the filtration \( F_0^* \) if \( \omega_i, i = 1, 2, \ldots, h \) is a basis of \( F_0^i \) for all \( i \). It is sometimes convenient to fix a basis \( \omega_i, i = 1, 2, \ldots, h \) of \( V_0 \) which is compatible with the filtration \( F_0^* \) and such that the polarization matrix \( [\psi_0(\omega_i, \omega_j)] \) is a fixed matrix \( \Phi_0 \):
\[
[\psi_0(\omega_i, \omega_j)] = \Phi_0.
\]
The matrices \( \Psi_0 \) and \( \Phi_0 \) are not necessarily the same. For any \( x \in \mathcal{L} \) we define
\[
H^{i,m-i}(x) := F_0^i \cap \overline{F_0^{m-i}}
\]
and the following properties for \( x \in \mathcal{L} \):
1. \( \psi_\mathbb{C}(x) = \psi_0 \);
2. \( V_0 = \bigoplus_{m=0}^{m-1} H^{i,m-i}(x) \);
3. \( (-1)^{i+m-i}(\sqrt{-1}^{-m}) \psi_\mathbb{C}(x)(\omega, \overline{\omega}) > 0, \quad \forall \omega \in H^{i,m-i}(x), \quad \omega \neq 0. \)

Throughout the text we call these properties P1, P2 and P3. Fix a polarized lattice \( x \in \mathcal{L} \). P1 implies that
\[
\psi_0(H^{i,m-i}(x), H^{j,m-j}(x)) = 0 \quad \text{except for} \quad i + j = m.
\]
This is because if \( i + j > m \) then \( \psi_0(F^i_0, F^j_0) = 0 \) and if \( i + j < m \) then \( \psi_0(F^i_0, F^j_0) = 0 \). We have also \( \sum_i H^{i,m-i}(x) = \bigoplus_i H^{i,m-i}(x) \) if and only if
\[
F^i_0 \cap \overline{F^j_0} = 0, \quad \forall i + j > m.
\]
If $a_{m-k} + \cdots + a_{0,m} = 0$, $a_{i,m-i} \in H^{i,m-i}(x)$ for some $0 \leq k \leq m$ with $a_{m-k} \neq 0$, then

$$-a_{m-k} = a_{m-k-1,k+1} + \cdots + a_{0,m} \in F_0^{m-k} \cap F_0^{k+1} \Rightarrow a_{k,m-k} = 0$$

which is a contradiction. The proof in the other direction is a consequence of $F_i^0 \cap F_j^0 = H_i^{m-i}(x) \cap H_j^{m-j}(x)$, $i + j > m$.

### 1.3 Period domain $U$

Define

$$X := \{ x \in \mathcal{L} | x \text{ satisfies P1} \},$$

$$U := \{ x \in \mathcal{L} | x \text{ satisfies P1, P2, P3} \}.$$

**Proposition 1.** The set $X$ is an analytic subset of $\mathcal{L}$ and $U$ is an open subset of $X$.

**Proof.** Take a basis $\omega_i$, $i = 1, 2, \ldots, h$ of $V_0$ compatible with the Hodge filtration. The property (5) is given by

$$\psi_\mathcal{C}(x)(\omega_r, \omega_s) = 0, \ r \leq h^i, \ s \leq h^i, \ i + j > m$$

and so $X$ is an analytic subset of $\mathcal{L}$.

Now choose a basis $\delta$ of $V_Z(x)$ and write $\delta = p \omega$. Using $\omega$ we may assume that $V_0 = \mathbb{C}^h$ and $\delta$ is constituted by the rows of $p$. We have

$$\omega = p^{-1} \delta \quad \Rightarrow \quad \overline{\omega^*} = \overline{\delta^*} = \overline{\delta} = \overline{\delta} = \overline{p \omega}$$

Therefore, the rows of $\overline{p \omega}$ are complex conjugates of the entries of $\omega$. Now it is easy to verify that if the property (6), dim$(H^{i,m-i}(x)) = h^i$ and P3 are valid for one $x$ then they are valid for all points in a small neighborhood of $x$ (for P3 we may first restrict $\psi_0$ to the product of sphere of radius 1 and center $0 \in \mathbb{C}^h$).

### 1.4 An algebraic group

Let $G_0$ be the algebraic group

$$G_0 := \text{Aut}(F_0^*, \psi_0) := \{ g : V_0 \rightarrow V_0 \text{ linear} | g(F_0^i) = F_0^i, \ \psi_0(g(\omega_1), g(\omega_2)) = \psi_0(\omega_1, \omega_2), \ \omega_1, \omega_2 \in V_0 \}.$$

It acts from the right on $\mathcal{L}$ in a canonical way:

$$xg := g^{-1}(x), \ \psi_\mathcal{L}(xg)(\cdot, \cdot) := \psi_\mathcal{L}(g(\cdot), g(\cdot)), \ g \in G_0, \ x \in \mathcal{L}.$$
Proposition 2. The properties P1, P2 and P3 are invariant under the action of $G_0$.

Proof. The property P1 for $xg$ follows from the definition. Let $x \in \mathcal{L}$, $g \in G_0$ and $\omega \in V_0$. We have
\[
H^{m-i}(xg) = F_0^i \cap F_0^{m-i} = F_0^i \cap g^{-1}(F_0^{m-i}) = F_0^i \cap g^{-1}(F_0^{m-i}) = g^{-1}(F_0^i \cap F_0^{m-i}) = g^{-1}(H^{m-i}(x)) \]
and
\[
\psi_C(xg)(\omega, \overline{\omega}) = \psi_C(x)(g(\omega), g^{-1}(g(\overline{\omega}))) = \psi_C(x)(g(\omega), g(\overline{\omega})).
\]
These equalities prove the proposition.

The above proposition implies that $G_0$ acts from the right on $U$. We fix a basis $\omega_i, i = 1, 2, \ldots, h$, of $V_0$ compatible with the Hodge filtration $F_0^\bullet$ and, if there is no danger of confusion, we identify each $g \in G_0$ with the $h \times h$ matrix $\tilde{g}$ given by
\[
[g^{-1}(\omega_1), g^{-1}(\omega_2), \ldots, g^{-1}(\omega_h)] = [\omega_1, \omega_2, \ldots, \omega_h] \tilde{g}. \tag{7}
\]

1.5 Griffiths period domain

In this section we give the classical approach to the moduli of polarized Hodge structures due to P. Griffiths. The reader is referred to [9, 8] for more developments in this direction.

Let us fix the $\mathbb{C}$-vector space $V_0$ and the Hodge numbers as in [12]. Let also $F$ be the space of filtrations $F_0^\bullet$ in $V_0$. In fact, $F$ has a natural structure of a compact smooth projective variety. We fix the polarized lattice $x_0 \in \mathcal{L}$ and define the Griffiths domain
\[
D := \{ F^\bullet \in F \mid (V_Z(x_0), \psi_Z(x_0), F^\bullet) \text{ is a polarized Hodge structure} \}.
\]
The group
\[
\Gamma_Z := \text{Aut}(V_Z(x_0), \psi_Z(x_0))
\]
acts on $V_0$ from the right in the usual way and this gives us an action of $\Gamma_Z$ on $D$. The space $\Gamma_Z \setminus D$ is the moduli space of polarized Hodge structures.

Proposition 3. There is a canonical isomorphism
\[
\beta : U / G_0 \to \Gamma_Z \setminus D.
\]

Proof. We take $x \in U$ and an isomorphism
\[
\gamma : (V_Z(x), \psi_Z(x)) \to (V_Z(x_0), \psi_Z(x_0)).
\]
The pushforward of the Hodge filtration $F_0^\bullet$ under this isomorphism gives us a Hodge filtration on $V_0$ with respect to the lattice $V_Z(x_0)$ and so it gives us a point $\beta(x) \in D$. Different choices of $\gamma$ lead us to the action of $\Gamma_Z$ on $\beta(x)$. Therefore, we have a well-defined map
\[
\beta : U \to \Gamma_Z \setminus D.
\]
Since $G_0 = \text{Aut}(V_0, F_0^\bullet, \psi_0)$, $\beta$ induces the desired isomorphism (it is surjective because for any polarized Hodge structure $(V_Z(x_0), \psi_Z(x_0), F^\bullet)$ we have $V_Z(x_0) = V_0$, $\psi_Z(x_0) = \psi_0$ and $F^\bullet = g(F_0^\bullet)$ for some $g \in G_0$).

The Griffiths domain is the moduli space of polarized Hodge structures of a fixed type and with a $\mathbb{Z}$-basis in which the polarization has a fixed matrix form. Our domain $U$ is the
moduli space of polarized Hodge structures of a fixed type and with a $\mathbb{C}$-basis compatible with the Hodge filtration and for which the polarization has a fixed matrix form.

2 Period map

In this section we introduce Poincaré duals, period matrices and Gauss-Manin connections in the framework of polarized Hodge structures.

2.1 Poincaré dual

In this section we explain the notion of Poincaré dual. Let $(V_Z(x), \psi_Z(x))$ be a polarized lattice and $\delta \in V_Z(x)^\vee$, where $\vee$ means the dual of a $\mathbb{Z}$-module. We will use the symbolic integral notation

$$\int_\delta \omega := \delta(\omega), \forall \omega \in V_0.$$ 

The equality

$$\int_\delta \omega \omega^\vee = \int_\delta \omega, \forall \omega \in V_0, \delta \in V_Z(x)^\vee$$

follows directly from the definition. The Poincaré dual of $\delta \in V_Z(x)^\vee$ is an element $\delta_{pd} \in V_Z(x)$ with the property

$$\int_\delta \omega = \psi_Z(x)(\omega, \delta_{pd}), \forall \omega \in V_Z(x).$$

It exists and is unique because $\psi_Z$ is non-degenerate. Using the Poincaré duality one defines the dual polarization

$$\psi_Z(x)^\vee(\delta_i, \delta_j) := \psi_Z(x)(\delta_i^{pd}, \delta_j^{pd}), \delta_i, \delta_j \in V_Z(x)^\vee.$$ 

We have

$$(A^\vee \delta)^{pd} = A^{-1} \delta^{pd}, \forall A \in \Gamma_Z, \delta \in V_Z(x_0)^\vee,$$

where $A^\vee : V_Z(x_0)^\vee \to V_Z(x_0)^\vee$ is the induced dual map. This follows from:

$$\int_{A^\vee \delta} \omega = \int_\delta A \omega = \psi_Z(x_0)(A \omega, \delta^{pd}) = \psi_Z(x_0)(\omega, A^{-1} \delta^{pd}), \forall \omega \in V_0.$$ 

We define

$$\Gamma_Z^\vee := \text{Aut}(V_Z(x_0)^\vee, \psi_Z(x_0)^\vee).$$

It follows that $\Gamma_Z \to \Gamma_Z^\vee$, $A \mapsto A^\vee$ is an isomorphism of groups.

2.2 Period matrix

Let $\omega_i, i = 1, 2, \ldots, h$ be a $\mathbb{C}$-basis of $V_0$ compatible with $F_0^\bullet$. Recall that $\omega$ means the $h \times 1$ matrix with entries $\omega_i$. For $x \in U$, we take a $\mathbb{Z}$-basis $\delta_i, i = 1, 2, \ldots, h$ of $V_Z(x)^\vee$ such that the matrix of $\psi_Z(x)^\vee$ in the basis $\delta$ is $\Psi_0$. We define the abstract period matrix/period map in the following way:
Instead of the period matrix it is useful to use the matrix
\[ q = q(x), \quad \text{where} \quad \delta^\text{nd} = q\omega. \]

Then we have
\[ \Psi_0^t = p m \cdot q^t. \]

If we identify \( V_0 \) with \( \mathbb{C}^h \) through the basis \( \omega \) then \( q \) is a matrix whose rows are the entries of \( \delta \). We define \( P \) to be the set of period matrices \( pm \). We write an element \( A \) of \( \Gamma_{\mathbb{Z}} \) in a basis of \( V_{\mathbb{Z}}(x_0) \), and redefine \( \Gamma_{\mathbb{Z}} \):
\[ \Gamma_{\mathbb{Z}} := \{ A \in \text{GL}(h, \mathbb{Z}) \mid A \Psi_0 A^t = \Psi_0 \}. \]

The group \( \Gamma_{\mathbb{Z}} \) acts on \( P \) from the left by the usual multiplication of matrices and
\[ U = \Gamma_{\mathbb{Z}} \setminus P. \]

In a similar way, if we identity each element \( g \) of \( G_0 \) with the matrix \( \tilde{g} \) in (7) then \( G_0 \) acts from the right on \( P \) by the usual multiplication of matrices.

### 2.3 A canonical connection on \( \mathcal{L} \)

We consider the trivial bundle \( \mathcal{H} = \mathcal{L} \times V_0 \) on \( \mathcal{L} \). On \( \mathcal{H} \) we have a well-defined integrable connection
\[ \nabla : \mathcal{H} \to \Omega^1_{\mathcal{L}} \otimes \Theta_{\mathcal{H}} \]

such that a section \( s \) of \( \mathcal{H} \) in a small open set \( V \subset \mathcal{L} \) with the property
\[ s(x) \in \{ x \} \times V_{\mathbb{Z}}(x), \quad x \in V. \]

is flat. Let \( \omega_1, \omega_2, \ldots, \omega_h \) be a basis of \( V_0 \) compatible with the Hodge filtration \( F^*_0 \). We can consider \( \omega_i \) as a global section of \( \mathcal{H} \) and so we have
\[ \nabla \omega = A \otimes \omega, \quad A = \begin{pmatrix} \omega_{i1} & \omega_{i2} & \cdots & \omega_{i1} \\
 \omega_{21} & \omega_{22} & \cdots & \omega_{2h} \\
 \vdots & \vdots & \ddots & \vdots \\
 \omega_{h1} & \omega_{h2} & \cdots & \omega_{hh} \end{pmatrix}, \quad \omega_{ij} \in H^0(\mathcal{L}, \Omega^1_{\mathcal{L}}). \quad (9) \]

\( A \) is called the connection matrix of \( \nabla \) in the basis \( \omega \). The connection \( \nabla \) is integrable and so \( dA = A \wedge A \):
\[ d\omega_j = \sum_{k=1}^h \omega_k \wedge \omega_{kj}, \quad i, j = 1, 2, \ldots, h. \quad (10) \]

Let \( \delta \) be a basis of flat sections. Write \( \delta = q\omega \). We have
\[ \omega = q^{-1} \delta \Rightarrow \nabla(\omega) = d(q^{-1})q\omega \Rightarrow \]
\[ A = dq^{-1} \cdot q = dq \cdot (pm^t \cdot \Psi_0^{-1}) \cdot (\Psi_0^t \cdot pm^{-t}) = dq \cdot pm^{-t}. \]
and so
\[ A = d(pm^t) \cdot pm^{-t}. \tag{11} \]
where \( pm \) is the abstract period map. We have used the equality \( \Psi_0 = pm \cdot q^t \). Note that the entries of \( A \) are holomorphic 1-forms on \( L \) and a fundamental system for the linear differential equation \( dY = A \cdot Y \) in \( L \) is given by \( Y = pm^t \):
\[ d(pm^t) = A \cdot pm^t. \]

We define the Griffiths transversality distribution by:
\[ F_{gr} : \omega_{ij} = 0, \; i \leq h^{m-x}, \; j > h^{m-x} - 1, \; x = 0, 1, \ldots, m - 2. \tag{12} \]
A holomorphic map \( f : V \rightarrow U \), where \( V \) is an analytic variety, is called a period map if it is tangent to the Griffiths transversality distribution, that is, for all \( \omega_{ij} \) as in (12) we have
\[ f^{-1} \omega_{ij} = 0. \]

### 2.4 Some functions on \( L \)

For two vectors \( \omega_1, \omega_2 \in V_0 \), we have the following holomorphic function on \( L \):
\[ L \rightarrow \mathbb{C}, \; x \mapsto \psi_{\mathbb{C}}(x)(\omega_1, \omega_2). \]
We choose a basis \( \omega \) of \( V_0 \) and \( \delta \) of \( V_\mathbb{Z}^\vee \) for \( x \in L \) and write \( \delta^{pd} = q \cdot \omega \). Then
\[ F := [\psi_{\mathbb{C}}(x)(\omega_i, \bar{\omega}_j)] = (q^{-1})\Psi_0 q^{-t} = pm^t \Psi_0^{-1} pm \tag{13} \]
we have used the identity \( \Psi_0 = pm \cdot q^t \). The matrix \( F \) satisfies the differential equation
\[ dF = A \cdot F + F \cdot A^t, \tag{14} \]
where \( A \) is the connection matrix. The proof is a straightforward consequence of (13) and (11):
\[ dF = d(pm^t \Psi_0^{-1} pm) \\
= (d(pm^t) \Psi_0^{-1} pm + pm^t \Psi_0^{-1} (d(pm^t)) \\
= A \cdot F + F \cdot A^t. \]

It is easy to check that every solution of the differential equation (14) is of the form \( pm^t \cdot C \cdot pm \) for some constant \( h \times h \) matrix \( C \) with entries in \( \mathbb{C} \) (if \( F \) is a solution of (14) then \( F \cdot pm^{-1} \) is a solution of \( dY = A \cdot Y \)). We restrict \( F, A \) and \( pm \) to \( U \) and we conclude that
\[ \Phi_0 = pm^t \Psi_0^{-1} pm \tag{15} \]
\[ A \cdot \Phi_0 = -\Phi_0 \cdot A^t, \]
where by definition \( F|_U \) is the constant matrix \( \Phi_0 \).
We have a plenty of non-holomorphic functions on \( L \). For two elements \( \omega_1, \omega_2 \in V_0 \) we define
\[ L \rightarrow \mathbb{C}, \; x \mapsto \psi_{\mathbb{C}}(x)(\omega_1, \overline{\omega_2}). \]
Let \( \omega \) and \( \delta \) be as before. We write \( \delta^{pd} = \overline{\sigma} \cdot \sigma^t \) and we have
\[ G := [\psi_{\mathbb{C}}(x)(\omega_i, \bar{\omega}_j)] = pm^t \Psi_0^{-1} \overline{pm} = (q^{-1})\Psi_0 \overline{q}^{-t} \tag{16} \]
The matrix $G$ satisfies the differential equation
\[ dG = A \cdot G + G \cdot \overline{\alpha}, \quad (17) \]
where $A$ is the connection matrix.

### 3 Quasi-modular forms attached to Hodge structures

In this section we explain what is a quasi-modular form attached to a given fixed data of Hodge structures and a full family of enhanced projective varieties.

#### 3.1 Enhanced projective varieties

Let $X$ be a complex smooth projective variety of a fixed topological type. This means that we fix a $C^\infty$ manifold $X_0$ and assume that $X$ as a $C^\infty$-manifold is isomorphic to $X_0$ (we do not fix the isomorphism). Let $n$ be the complex dimension of $X$ and let $m$ be an integer with $1 \leq m \leq n$. We fix an element $\theta \in H^{2n-2m}(X, \mathbb{Z}) \cap H^{m-n-m}(X)$. By $H^i(X, \mathbb{Z})$ we mean its image in $H^i(X, \mathbb{C}) = H^i_{\text{dR}}(X)$; therefore, we have killed the torsion. We consider the bilinear map
\[ \langle \cdot, \cdot \rangle_{\mathbb{C}} : H^m_{\text{dR}}(X) \times H^m_{\text{dR}}(X) \to \mathbb{C}, \quad \langle \omega, \alpha \rangle = \frac{1}{(2\pi i)^m} \int_X \omega \cup \alpha \cup \theta. \]

The $(2\pi i)^{-m}$ factor in the above definition ensures us that the bilinear map $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ is defined for the algebraic de Rham cohomology (see for instance Deligne’s lecture in [3]). We assume that it is non-degenerate. The cohomology $H^m_{\text{dR}}(X)$ is equipped with the so-called Hodge filtration $F^\ast$. We assume that the Hodge numbers $h^{i,m-i}$, $i = 0, 1, 2, \ldots, m$ coincide with those fixed in this article. We consider Hodge structures with an isomorphism
\[ (H^m_{\text{dR}}(X), F^\ast, \langle \cdot, \cdot \rangle_{\mathbb{C}}) \cong (V_0, F^m_{0\ast}, \psi_0). \]

From now on, by an enhanced projective variety we mean all the data described in the previous paragraph.

We also need to introduce families of enhanced projective varieties. Let $V$ be an irreducible affine variety and $\mathcal{O}_V$ be the ring of regular functions on $V$. By definition $V$ is the underlying complex space of $\text{Spec}(\mathcal{O}_V)$ and $\mathcal{O}_V$ is a finitely generated reduced $\mathbb{C}$-algebra without zero divisors. Also, let $X \to V$ be a family of smooth projective varieties as in the previous paragraph. We will also use the notations $\{X_t\}_{t \in V}$ or $X/V$ to denote $X \to V$. The de Rham cohomology $H^m_{\text{dR}}(X/V)$ and its Hodge filtration $F^\ast H^m_{\text{dR}}(X/V)$ are $\mathcal{O}_V$-modules (see for instance [7]) and in a similar way we have $\langle \cdot, \cdot \rangle_{\mathcal{O}_V} : H^m_{\text{dR}}(X/V) \times H^m_{\text{dR}}(X/V) \to \mathcal{O}_V$. Note that we fix an element $\theta \in F^{n-m}H^{(n-2m)}_{\text{dR}}(X/V)$ and assume that it induces in each fiber $X_t$ an element in $H^{2n-2m}(X_t, \mathbb{Z})$. We say that the family is enhanced if we have an isomorphism
\[ (H^m_{\text{dR}}(X/V), F^\ast H^m_{\text{dR}}(X/V), \langle \cdot, \cdot \rangle_{\mathcal{O}_V}) \cong (V_0 \otimes_{\mathbb{C}} \mathcal{O}_V, F^m_{0\ast} \otimes_{\mathbb{C}} \mathcal{O}_V, \psi_0 \otimes_{\mathbb{C}} \mathcal{O}_V). \quad (18) \]

We fix a basis $\omega_0$, $i = 1, 2, \ldots, h$ of $V_0$ compatible with the filtration $F^\ast$. Under the above isomorphism we get a basis $\partial_\alpha$, $i = 1, 2, \ldots, h$ of the $\mathcal{O}_V$-module $H^m_{\text{dR}}(X/V)$ which is compatible with the Hodge filtration and the bilinear map $\langle \cdot, \cdot \rangle_{\mathcal{O}_V}$ written in this basis is a constant matrix. This gives us another formulation of an enhanced family of projective varieties. An enhanced family of projective varieties $\{X_t\}_{t \in V}$ is full if we have an algebraic
action of $G_0$ (defined in §1.4) from the right on $V$ (and hence on $\mathcal{O}_V$) such that it is compatible with the isomorphism (13). This is equivalent to saying that for $X_t$ and $\tilde{\omega}_i, i = 1, 2, \ldots, h$ as above, we have an isomorphism

$$(X_t, [\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_h]) \cong (X_t, [\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_h]g), \ t \in V, \ g \in G_0,$$

(recall the matrix form of $g \in G_0$ in (7)). A morphism $Y/W \to X/V$ of two families of enhanced projective varieties is a commutative diagram

$$\begin{array}{ccc}
Y & \to & X \\
\downarrow & & \downarrow \\
W & \to & V
\end{array}$$

such that

$$H^m(X/V) \to H^m(Y/W)$$

$$\downarrow \quad \downarrow$$

$$V_0 \otimes_C \mathcal{O}_V \to V_0 \otimes_C \mathcal{O}_W$$

is also commutative.

### 3.2 Period map

For an enhanced projective variety $X$, we consider the image of $H^m(X, \mathbb{Z})$ in $H^m(X, \mathbb{C}) \cong H^m_{\text{DR}}(X) \cong V_0$ and hence we obtain a unique point in $U$. Note that by this process we kill torsion elements in $H^m(X, \mathbb{Z})$. We fix bases $\omega_i$ and $\tilde{\omega}_i$ as in §1.1 and a basis $\delta_i, i = 1, 2, \ldots, h$ of $H^0(X, \mathbb{Z}) = H^0(X, \mathbb{Z})^\vee$ with $\langle [\delta_i, \delta_j] \rangle = \Psi_0$ and we see that the corresponding point in $U := \Gamma_{\mathbf{Z}} \setminus P$ is given by the equivalence class of the geometric period matrix $[\int_{\tilde{\omega}_i} \delta_j]$.

For any family of enhanced projective varieties $\{X_t\}_{t \in V}$ we get

$$\text{pm} : V \to U$$

which is holomorphic. It satisfies the so-called Griffiths transversality, that is, it is tangent to the Griffiths transversality distribution. It is called a geometric period map. The pullback of the connection $\nabla$ constructed in §2.3 by the period map $\text{pm}$ is the Gauss-Manin connection of the family $\{X_t\}_{t \in V}$. If the family is full then the geometric period map commutes with the action of $G_0$:

$$\text{pm}(tg) = \text{pm}(t)g, \ g \in G_0, \ t \in V.$$
topological data, but not necessarily enhanced and smooth, and with the discriminant variety \( \Delta \subset S \), the map \( Y \setminus f^{-1}(\Delta) \to S \setminus \Delta \) is an underlying morphism of an enhanced family, and hence, we have the map \( S \setminus \Delta \to T \) which extends to \( S \to \tilde{T} \). The conjecture is about the existence of \( \tilde{T} \) with such an extension property.

Similar to Shimura varieties, we expect that \( T \) and \( \tilde{T} \) are affine varieties defined over \( \bar{\mathbb{Q}} \).

Both conjectures are true in the case of elliptic curves (see the discussion in the Introduction). In this case, the function ring of \( T \) (resp. \( \tilde{T} \)) is \( \mathbb{C}[t_1, t_2, t_3, 1/127t^3 - t^2] \) (resp. \( \mathbb{C}[t_1, t_2, t_3] \)).

We have also verified the conjectures for a particular class of Calabi-Yau varieties (see §4.2 and [13]).

Now, consider the case in which both conjectures are true. We are going to explain the rough idea of the algebra of quasi-modular forms attached to all fixed data that we had. It is the pullback of the \( \mathbb{C} \)-algebra of regular functions in \( \tilde{T} \) by the composition

\[
\mathbb{H} \to P|_{\text{Im}(\text{pm})} \to U|_{\text{Im}(\text{pm})} \to T \to \tilde{T}.
\]

Here \( \text{pm} \) is the geometric period map. We need that the period map is locally injective (local Torelli problem) and hence \( \text{pm}^{-1} \) is a local inverse map. The set \( \mathbb{H} \) is a subset of the set of period matrices \( P \) and it will play the role of the Poincaré upper half-plane. If the Griffiths period domain \( D \) is Hermitian symmetric then it is biholomorphic to \( D \) (see §4.1); however, in other cases it depends on the universal period map \( T \to U \) and its dimension is the dimension of the deformation space of the projective variety. In this case we do not need to define \( \mathbb{H} \) explicitly (see §4.2). More details of this discussion will be explained by two examples of the next section.

4 Examples

In this section we discuss two examples of Hodge structures and the corresponding quasi-modular form algebras: those attached to mirror quintic Calabi-Yau varieties and principally polarized Abelian varieties. The details of the first case are done in [13, 14] and we will sketch the results which are related to the main thread of the present text. For the second case there is much work that has been done and I only sketch some ideas. Much of the work for K3 surfaces endowed with polarizations has been already done by many authors, see [2] and the references therein. The generalization of such results to Siegel quasi-modular forms is work for the future.

4.1 Siegel quasi-modular forms

We consider the case in which the weight \( m \) is equal to 1 and the polarization matrix is:

\[
\Psi_0 = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix},
\]

where \( I_g \) is the \( g \times g \) identity matrix. In this case \( g := h^{10} = h^{01} \) and \( h = 2g \). We take a basis \( \omega_i, i = 1, 2, \ldots, 2g \), of \( V_0 \) compatible with \( F_0^* \), that is, the first \( g \) elements form a basis of \( F_1^0 \). We further assume that the polarization \( \psi_0 : V_0 \times V_0 \to \mathbb{C} \) in the basis \( \omega \) has the form \( \Phi_0 := \Psi_0 \). Because of the particular format of \( \Psi_0 \), both these assumptions do not contradict each other. We take a basis \( \delta \) of \( V_2(x)^* \) such that the intersection form in this basis is of the form \( \Psi_0 \) and we write the associated period matrix in the form.
The matrix
\[ \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} \begin{pmatrix} x_1^t & x_2^t \\ x_3^t & x_4^t \end{pmatrix} = \begin{pmatrix} x_1^t & x_2^t \\ x_3^t & x_4^t \end{pmatrix} \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \]
where \( x_i, i = 1, \ldots, 4, \) are \( g \times g \) matrices. Since \( \Psi_0^t = \Psi_0 \), we have
\[ \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} \begin{pmatrix} x_1^t & x_2^t \\ x_3^t & x_4^t \end{pmatrix} = \begin{pmatrix} x_1^t & x_2^t \\ x_3^t & x_4^t \end{pmatrix} \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \]
and
\[ \begin{pmatrix} \omega \end{pmatrix} \begin{pmatrix} \omega_j \end{pmatrix} = \begin{pmatrix} x_1^t & x_2^t \\ x_3^t & x_4^t \end{pmatrix} \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}. \]

The properties P1, P2 and P3 are summarized in the properties
\[ x_3^t x_1 = x_1^t x_3, \ x_3^t x_2 - x_1^t x_4 = -I_g, \]
\[ x_1, x_2 \in \text{GL}(g, \mathbb{C}), \]
\[ \sqrt{-1}(x_3^t x_1 - x_1^t x_3) \text{ is a positive matrix.} \]

By definition \( P \) is the set of all \( 2g \times 2g \) matrices \( \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \) satisfying the above properties:
The matrix \( x := x_1 x_3^{-1} \) is well-defined and invertible and satisfies the well-known Riemann relations:
\[ x^t = x, \ \text{Im}(x) \text{ is a positive matrix.} \]

The set of matrices \( x \in \text{Mat}^{2g \times 2g}(\mathbb{C}) \) with the above properties is called the Siegel upper half-space and is denoted by \( \mathbb{H} \). We have \( U = \Gamma_2 \setminus P, \) where
\[ \Gamma_2 = \text{Sp}(2g, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2g, \mathbb{Z}) \mid ab^t = ba^t, \ cd^t = dc^t, \ ad^t - bc^t = I_g \right\}. \]

We have also
\[ G_0 = \left\{ \begin{pmatrix} k & k' \\ 0 & k^{-1} \end{pmatrix} \in \text{GL}(2g, \mathbb{C}) \mid kk'^t = k'k \right\} \]
which acts on \( P \) from the right. The group \( \text{Sp}(2g, \mathbb{Z}) \) acts on \( \mathbb{H} \) by
\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : x = (ax + b)(cx + d)^{-1}, \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z}), \ x \in \mathbb{H} \]
and we have the isomorphism
\[ U/G_0 \rightarrow \text{Sp}(2g, \mathbb{Z}) \setminus \mathbb{H}, \]
given by
\[ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \rightarrow x_1 x_3^{-1}. \]
To each point \( x \) of \( P \) we associate a triple \( (A_x, \theta_x, \alpha_x) \) as follows: We have \( A_x := \mathbb{C}^g / \Lambda_x, \)
where \( \Lambda_x \) is the \( \mathbb{Z} \)-submodule of \( \mathbb{C}^g \) generated by the rows of \( x_1 \) and \( x_3. \) We have cycles
\[ \delta_i \in H_1(A_x, \mathbb{Z}), \ i = 1, 2, \ldots, 2g, \] which are defined by the property
\[ \int_{\delta_i} dz_j = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}, \] where
the Hodge filtration

The polarization in triples

1

H

polarized abelian variety with a polarization

which satisfies

framework of algebraic geometry. We have to construct an algebraic variety to introduce Siegel quasi-modular forms, we have to study the same moduli space in the

impose a functional property for

and the second is the canonical map. The period map in this case is a biholomorphism. If we



\[ \theta \]

\[ z \]

\[ 14 \]

Hossein Movasati

\[ x_1 \]

\[ x_2 \]

\[ x_3 \]

\[ x_4 \]

\[ \theta \rightarrow \left( \begin{array}{c} z \\ I_g \\ 0 \end{array} \right) \]

and the second is the canonical map. The period map in this case is a biholomorphism. If we impose a functional property for \( f \) regarding the action of \( G_0 \) then this will be translated into a functional property of a Siegel quasi-modular form with respect to the action of \( Sp(2g, \mathbb{Z}) \). In this way we can even define a Siegel quasi-modular form defined over \( \mathbb{Q} \) (recall that we expect \( \hat{T} \) to be defined over \( \mathbb{Q} \)). It is left to the reader to verify that the \( \mathbb{C} \)-algebra of Siegel quasi-modular forms is closed under derivations with respect to \( z_{ij} \) with \( z = [z_{ij}] \in \mathbb{H} \). For the realization of all these in the case of elliptic curves, \( g = 1 \), see the Introduction and [10]. See the books [10][4][12] for more information on Siegel modular forms.

4.2 Hodge numbers, 1,1,1,1

In this section we consider the case \( m = 3 \) and the Hodge numbers \( h^{30} = h^{21} = h^{12} = h^{03} = 1 \), \( h = 4 \). The polarization matrix written in an integral basis is given by

\[ \Psi_0 = \left( \begin{array}{ccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right) \]
Let us fix a basis \( \omega_1, \omega_2, \omega_3, \omega_4 \) of \( V_0 \) compatible with the Hodge filtration \( F^*_0 \), a basis \( \delta_1, \delta_2, \delta_3, \delta_4 \in V_2(x) \) with the intersection matrix \( I_0 \) and let us write the period matrix in the form \( \text{pm}(x) = [x_{ij}]_{i,j=1,2,\ldots,4} \). We assume that the polarization \( \psi_0 \) in the basis \( \omega_i \) is given by the matrix

\[
\Phi_0 := \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}.
\]

The algebraic group \( G_0 \) is defined to be

\[
G_0 := \left\{ g = \begin{pmatrix}
  g_{11} & g_{12} & g_{13} & g_{14} \\
  g_{22} & g_{23} & g_{24} & 0 \\
  0 & g_{33} & g_{34} & 0 \\
  0 & 0 & 0 & g_{44}
\end{pmatrix}, \ g^t \Phi_0 g = \Phi_0, \ g_{ij} \in \mathbb{C} \right\}.
\]

One can verify that it is generated by six one-dimensional subgroups, two of them isomorphic to the multiplicative group \( \mathbb{G}_m \) and four of them isomorphic to the additive group \( \mathbb{G}_a \). Therefore, \( G_0 \) is of dimension 6. We consider the subset \( \mathbb{H} \) of \( P \) consisting of matrices

\[
\tau = \begin{pmatrix}
  \tau_0 & 1 & 0 & 0 \\
  1 & 0 & 0 & 0 \\
  \tau_1 & \tau_2 & 1 & 0 \\
  \tau_2 - \tau_0 \tau_3 + \tau_1 - \tau_0 & 1
\end{pmatrix}, \quad \tau = (20)
\]

where \( \tau_i, \ i = 0, 1, 2, 3, \) are some variables in \( \mathbb{C} \) (they are coordinates of the corresponding moduli space of polarized Hodge structures and so this moduli space is of dimension four). The particular expressions for the \( (4,2) \) and \( (4,3) \) entries of the above matrix follow from the polynomial relations \( (15) \) between periods. The connection matrix \( A \) restricted to \( \mathbb{H} \) is

\[
d\tau^* \cdot \tau^{-t} = \begin{pmatrix}
  0 & d\tau_0 - \tau_2 d\tau_1 + d\tau_1 - \tau_1 d\tau_0 + \tau_0 d\tau_1 + d\tau_2 \\
  0 & 0 & d\tau_1 \\
  0 & 0 & 0 & -d\tau_0 \\
  0 & 0 & 0 & 0
\end{pmatrix}.
\]

The Griffiths transversality distribution is given by

\[
-\tau_3 d\tau_0 + d\tau_1 = 0, \quad -\tau_1 d\tau_0 + \tau_0 d\tau_1 + d\tau_2 = 0.
\]

and so, if we consider \( \tau_0 \) as an independent parameter defined in a neighborhood of \( +\sqrt{-1} \mathbb{H} \), and all other quantities \( \tau_i \) depending on \( \tau_0 \), then we have

\[
\tau_3 = \frac{\partial \tau_1}{\partial \tau_0}, \quad \frac{\partial \tau_2}{\partial \tau_0} = \tau_1 - \tau_0 \frac{\partial \tau_1}{\partial \tau_0}.
\]

In \( (13) \) we have checked the conjectures in \( (3) \) for the Calabi-Yau threefolds of mirror quintic type. In this case \( \dim(T) = 7 = 1 + 6 \), where 1 is the dimension of the moduli space of mirror quintic Calabi-Yau varieties and 6 is the dimension of the algebraic group \( G_0 \). Hence, we have constructed an algebra generated by seven functions in \( \tau_0 \), which we call it the algebra of quasi-modular forms attached to mirror quintic Calabi-Yau varieties. The image of the geometric period map lies in \( \mathbb{H} \) with

\[
\tau_1 = -\frac{25}{12} + \frac{5}{2} \tau_0 (\tau_0 + 1) + \frac{1}{(2\pi i)^2} \sum_{n=1}^{\infty} \left( \sum_{d|n} d^2 \right) e^{2\pi i \frac{n\tau_0}{n^2}}. \quad (22)
\]
Here, \( n_d \)'s are instanton numbers and the second derivative of \( \tau_1 \) with respect to \( \tau_0 \) is the Yukawa coupling. The Yukawa coupling itself turns out to be a quasi-modular form in our context but not its double primitive \( \tau_1 \). The set \( \mathbb{H} \) is a subset of \( \tilde{\mathbb{H}} \) defined by (21) and (22). As far as I know this is the first case in which the Griffiths period domain is not Hermitian symmetric and we have an attached algebra of quasi-modular forms and even the Global Torelli problem is true; that is, the period map is globally injective (see [5]). However, note that in [13] we have only used the local injectivity of the period map. In this case we can prove that the pullback map from the algebra of regular functions on \( \tilde{T} \) to the algebra of holomorphic functions on \( \mathbb{H} \) is injective. Our quasi-modular form theory in this example is attached to mirror quintic Calabi-Yau varieties and not the corresponding period domain. There are other functions \( \tau_1 \) attached to one-dimensional families of varieties and the corresponding period maps. They may have their own quasi-modular forms algebra different from the one explained in this section.

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**References**

1. Walter L. Baily Jr. and Armand Borel. Compactification of arithmetic quotients of bounded symmetric domains. *Ann. of Math. (2)*, 84:442–528, 1966.
2. Adrian Clingher and Charles F. Doran. Lattice polarized K3 surfaces and Siegel modular forms. To appear in Advances in Mathematics, arXiv:1004.3503, 2012.
3. Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih. *Hodge cycles, motives, and Shimura varieties*, volume 900 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1982. Philosophical Studies Series in Philosophy, 20.
4. Eberhard Freitag. *Siegelsche Modulfunktionen*, volume 254 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1983.
5. Philip A. Griffiths. Logarithmic Hodge structures (report on the work of Kato-Usui). 2009.
6. Philip A. Griffiths. Periods of integrals on algebraic manifolds: Summary of main results and discussion of open problems. *Bull. Amer. Math. Soc.*, 76:228–296, 1970.
7. Alexander Grothendieck. On the de Rham cohomology of algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (29):95–103, 1966.
8. Kazuya Kato and Sampei Usui. Borel-Serre spaces and spaces of \( SL(2) \)-orbits. In *Algebraic geometry 2000, Azumino (Hotaka)*, volume 36 of *Adv. Stud. Pure Math.*, pages 321–382. Math. Soc. Japan, Tokyo, 2002.
9. Kazuya Kato and Sampei Usui. Classifying spaces of degenerating polarized Hodge structures. *Annals of Mathematics Studies, Princeton University Press*, 169:xii+336, 2009.
10. Helmut Klingen. *Introductory lectures on Siegel modular forms*, volume 20 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.
11. Maxim Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 120–139, Basel, 1995. Birkhäuser.
12. Hans Maass. *Siegel’s modular forms and Dirichlet series*. Springer-Verlag, Berlin, 1971. Lecture Notes in Mathematics, Vol. 216.
13. Hossein Movasati. Eisenstein type series for Calabi-Yau varieties. *Nuclear Physics B*, 847:460–484, 2011.
14. Hossein Movasati. Modular-type functions attached to mirror quintic Calabi-Yau varieties. *Preprint*, arXiv:1111.0357, 2011.
15. Hossein Movasati. Moduli of polarized Hodge structures. *Bull. Braz. Math. Soc. (N.S.)*, 39(1):81–107, 2008.
16. Hossein Movasati. On differential modular forms and some analytic relations between Eisenstein series. *Ramanujan J.*, 17(1):53–76, 2008.