We study the effective $D=4, N=1$ supergravity description of five-dimensional heterotic M-theory in the presence of an M5 brane, and derive the Killing vectors and isometry group for the Kähler moduli-space metric. The group is found to be a non-semisimple maximal parabolic subgroup of $Sp(4, \mathbb{R})$, containing a non-trivial $SL(2, \mathbb{R})$ factor. The underlying moduli-space is then naturally realised as the group space $Sp(4, \mathbb{R})/U(2)$, but equipped with a nonhomogeneous metric that is invariant only under that maximal parabolic group. This nonhomogeneous metric space can also be derived via field truncations and identifications performed on $Sp(8, \mathbb{R})/U(4)$ with its standard homogeneous metric. In a companion paper we use these symmetries to derive new cosmological solutions from known ones.

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I. INTRODUCTION

It has been known for some time that the classical supergravity actions often possess scalar fields with highly constrained interactions. In particular, the $D=4, N=1$ theories have scalar field kinetic terms that must collectively arise from the Kähler potential of a complex Kähler manifold [1]. This means that the scalars can be reinterpreted as real-valued coordinates on a complex space, and so their equations of motion possess an underlying geometric significance. Therefore, when investigating the scalar-field cosmology of supergravity models, it is often useful to identify the structure of these scalar-field manifolds and their symmetries [2]. Indeed, this knowledge sometimes obviates the need to solve the full equations of motion of the system, since repeated symmetry transformations on special solutions often “builds” the general solution for us.

In this paper we will be considering a scalar-field cosmology that descends from Horava-Witten (HW) theory coupled to an M5 brane [3, 4]. In particular, we will consider a compactification of HW theory on a Calabi-Yau three-fold $CY_3$, leading to the known form of five-dimensional heterotic M-theory [5, 6]. Within the context of this theory, it was shown in Ref. [7] how a further compactification to four-dimensions leads to an $N=1$ supergravity theory with an unusual braneworld cosmology. For example, it transpires that a new scalar corresponding to the M5 brane position must be included in the set of cosmologically significant fields, and this leads to a forcing effect whereby the ambient dimensions change size as the brane moves. Moreover, the frictional forces acting back on the brane are such that it accelerates and then decelerates back to rest, mimicking a time-dependent force of finite duration. The internal dimensions of the model are thus subjected to a brief period of brane-induced changes, but eventually act as free scalars that are contracting or expanding with fixed rates.

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However, not all of the scalar fields were considered in Ref. [7], since the axionic fields were consistently truncated away. This means that only a portion of the full scalar manifold was explored, and that the M5 brane behaviour is liable to be only an approximation once the axions are restored. Therefore, following on from the work of Ref. [7], we wish to analyse the structure and symmetries of the full Kähler metric, and also discuss the form of the underlying manifold on which the metric is placed. In a companion paper [8] we will then use this information to find new solutions to the equations of motion.

We now summarise our results. We find that the isometry group of the Kähler metric is a maximal parabolic subgroup of \( Sp(4, \mathbb{R}) \), which possesses an \( SL(2, \mathbb{R}) \) group of T-duality transformations. The underlying Kähler manifold is the Siegel plane \( SH_2 \cong Sp(4, \mathbb{R})/U(2) \) equipped with a nonhomogeneous metric. We show this by assembling the metric from the right-invariant one forms of the group \( Sp(4, \mathbb{R})/U(2) \). We also demonstrate how to derive the nonhomogeneous metric from a homogeneous metric on the higher-dimensional Siegel plane \( SH_4 \cong Sp(8, \mathbb{R})/U(4) \), by making certain field truncations and identifications.

II. THE FOUR-DIMENSIONAL ACTION

We now review the \( D = 4, N = 1 \) supergravity action presented in Ref. [7]. Recall that this was derived via a compactification of 11D supergravity on \( S^1/Z_2 \times CY_3 \), leading to two four-dimensional boundary planes separated along a fifth dimension. A single M5 brane was also included in the space, by wrapping it on a holomorphic 2-cycle of the \( CY_3 \). The brane then appears as a three-brane of charge \( q \) that lies parallel to the boundaries, and which can move along the interval. Importantly, the interaction between the boundaries and brane leads to the existence of a static, triple-domain wall BPS solution. One can then consider further reducing on this solution, so as to find a supergravity theory describing slowly varying fluctuations about the static BPS vacuum. This contains the six scalar fields \( \beta, \chi, \phi, \sigma, z, \nu \) with the following non-standard kinetic terms

\[
S_4 = -\frac{1}{2\kappa_4^2} \int_{M_4} d^4 x \sqrt{-g} \left[ \frac{1}{2} R + \frac{3}{4} (\partial \beta)^2 + 3 e^{-2\beta} (\partial \chi)^2 + \frac{1}{4} (\partial \phi)^2 + \frac{1}{4} e^{-2\phi} (\partial \sigma + 4 q z \partial \nu)^2 + \frac{1}{2} q e^{\beta-\phi} (\partial z)^2 + 2 q e^{-\beta-\phi} (\partial \nu - \chi \partial z)^2 \right]
\]

Each of these scalars has an underlying significance in terms of the \( D = 5 \) parent theory from which it descends. The scalar \( \beta \) is the zero-mode of the \( g_{55} \) component in the \( D = 5 \) metric, and measures the separation between the boundaries. Specifically, the separation is given by \( \pi \rho e^\beta \) in terms of some dimensionful reference size \( \pi \rho \). The field \( \phi \) represents the orbifold-averaged Calabi-Yau volume, such that the physical size is given by \( v e^\phi \) in terms of a dimensionful reference volume \( v \). The scalars \( \sigma, \chi \) originate from the bulk three-form and graviphoton field respectively. The field \( z \) measures the position of the bulk brane between the boundaries, and has been normalised such that \( z \in (0, 1) \). The points \( z = 0, 1 \) then correspond to the boundaries themselves. Lastly, the field \( \nu \) arises from the self-dual two-form on the brane worldvolume.

This reduction on a BPS solution guarantees that the scalars must group into supersymmetric multiplets described by a supersymmetric action. One can verify that they naturally fall into the pairs \( (\phi, \sigma), (\beta, \chi), (z, \nu) \), which are the bosonic components of chiral superfields \( S, T, Z \) as follows

\[
S = e^\phi + q z^2 e^\beta + i (\sigma + 2 q z^2 \chi) \quad , \quad T = e^\beta + 2 i \chi \quad , \quad Z = e^\beta z + 2 i (\nu - z \chi)
\]

This naturally leads to a Kähler manifold expression for the scalar part of the action

\[
S_4 = -\frac{1}{2\kappa_4^2} \int d^4 x \sqrt{-g} \left( \frac{1}{2} R + K_{ij} \partial_i \Phi^j \partial_i \bar{\Phi}^j \right)
\]

where the superfields are grouped into a coordinate vector \( \Phi = (S, T, Z) \), with the complex conjugate coordinates denoted by \( \bar{\Phi} \). The Kähler metric \( K_{ij} \) is given by

\[
K_{ij} = \frac{\partial^2 K}{\partial \Phi^i \partial \bar{\Phi}^j}
\]

in terms of the Kähler potential

\[
K = -\ln \left[ S + S - q (Z + Z)^2 \right] - 3 \ln (T + T)
\]
III. KILLING VECTORS AND ISOMETRY GROUP

To understand the scalar-field dynamics of the action Eq. (1), it is important to study the isometries of the associated Kähler metric. The rationale for this is as follows. Usually the underlying scalar-field manifold is a group $G/H$ where $G$ is a Lie group of diffeomorphisms that acts transitively on $G/H$, and $H$ is a normal subgroup of $G$. Moreover, the Kähler metric on $G/H$ is often invariant under some isometry group $G' \subseteq G$. The natural consequence of this is as follows. If we are given any solution to the scalar-field equations of motion, then this can be transformed into a new and generally more complicated solution by systematically applying an isometry transformation in $G'$. Although only $G$ is strictly transitive on the underlying group space $G/H$, so that the general solution cannot be built unless $G' = G$, we can nonetheless make significant progress by applying isometries. In particular, we can find complicated new solutions that might otherwise be inaccessible using conventional solving techniques.

Based on this, we consider the isometries of Eq. (1), which have not yet been discussed in the braneworld literature. To this end, we recognise that the action Eq. (1) is written in the Einstein frame, such that the gravitational contribution is cleanly decoupled from the scalar fields. Consequently, the spacetime manifold will transform as a singlet under the isometry group of the scalar manifold, and so can be considered some fixed background that does not affect the permissible scalar field symmetries. Therefore, from now on we ignore the Ricci scalar term in the action. For simplicity, we will present our results in terms of the scalar fields $\beta, \chi, \phi, \sigma, z, \nu$, so that the scalar-field manifold is treated as real and six-dimensional. Where necessary, we will comment upon the analogous complex results in terms of the natural complex fields $S, T, Z$.

The infinitesimal isometries of the metric Eq. (1) are generated by left-invariant Killing vector fields $L^i$. By solving the Killing equations,

$$\nabla_i L_k + \nabla_k L_i = 0$$

one can verify that there are seven real Killing vector fields as follows

$$L^1 = \partial_\beta + \chi \partial_\chi - \frac{1}{2} z \partial_z + \frac{1}{2} \nu \partial_\nu$$
$$L^2 = -2\chi \partial_\beta + \left(\frac{1}{4} e^{2\beta} - \chi^2\right) \partial_\chi + \nu \partial_z - 2q \nu^2 \partial_\sigma$$
$$L^3 = \partial_\chi + z \partial_\nu - 2q z^2 \partial_\sigma$$
$$L^4 = \partial_\phi + \frac{1}{2} z \partial_z + \sigma \partial_\sigma + \frac{1}{2} \nu \partial_\nu$$
$$L^5 = \partial_z - 4q \nu \partial_\sigma$$
$$L^6 = 4q \partial_\sigma$$
$$L^7 = \partial_\nu$$

We note that each of these real vector fields also corresponds to a holomorphic Killing vector field of the complex manifold with coordinates $S, T, Z$. That is, every such field can be decomposed as the cleanly-separated sum

$$L^a = Y^a(\Phi) + \overline{Y}^a(\bar{\Phi})$$

where the $Y^a$ satisfy the complex Killing equations

$$K_{ij} \nabla_i Y^j + K_{ji} \nabla_j Y^i = 0$$

This means that as we drag the geometry along the flowlines of the $L^i$, we consistently preserve the complex-structure identifications Eq. (2) in each of the tangent spaces that we pass through. Consequently, every real Killing vector is also holomorphic, and it does not matter whether we compute the vectors using the real or complex Killing equations. In Appendix A we present the $L^i$ rewritten in terms of $S, T, Z$.

---

1 If $H$ is not a normal subgroup, then the coset $G/H$ is a manifold with no group structure. Thus, not every coset $G/H$ is a group, and care should be taken to distinguish between “cosets” and “cosets that are also groups”. In the following we will always consider spaces of the form $G/H$ where $H$ is normal in $G$, and so these spaces are automatically Lie groups in their own right.
These vector fields $L^i$ now define seven symmetry directions of the Kähler manifold, such that infinitesimal transformations along these directions leave the scalar-field geometry (and hence interactions) invariant. Therefore, these vectors constitute a basis for the Lie algebra of the isometry group, and their exponentiations will lead to the finite symmetry transformations of the isometry group itself. The algebra can be determined by investigating the commutation relations of the $L^i$, which are presented in Table 1 below.

$$
\begin{array}{ccccccc}
L^1 & L^2 & L^3 & L^4 & L^5 & L^6 & L^7 \\
L^1 & 0 & L^2 & -L^3 & 0 & \frac{1}{2}L^5 & 0 & -\frac{1}{2}L^7 \\
L^2 & -L^2 & 0 & 2L^1 & 0 & 0 & 0 & -L^5 \\
L^3 & L^3 & -2L^1 & 0 & 0 & -L^5 & 0 & 0 \\
L^4 & 0 & 0 & 0 & 0 & -\frac{4}{3}L^5 & -L^7 & -\frac{1}{3}L^5 \\
L^5 & -\frac{1}{2}L^5 & 0 & L^7 & \frac{1}{2}L^5 & 0 & 0 & L^6 \\
L^6 & 0 & 0 & 0 & L^6 & 0 & 0 & 0 \\
L^7 & \frac{4}{3}L^5 & L^5 & 0 & \frac{1}{2}L^7 & -L^6 & 0 & 0 \\
\end{array}
$$

\textit{TABLE I: Commutation relations of the Killing vector fields $L^i$}

By inspection of the table we can see that $\{L^1, L^2, L^3\}$ form an $sl(2, \mathbb{R})$ semisimple subalgebra, leaving over a solvable subalgebra $s \equiv \{L^4, L^5, L^6, L^7\}$. Hence, using the standard Levi-Malcev decomposition, the total Lie algebra $g$ must read

$$
g = sl(2, \mathbb{R}) \overset{\oplus}{\longrightarrow} s
$$

where $\overset{\oplus}{\longrightarrow}$ indicates a semidirect sum. To identify $s$ explicitly, we first note that it possesses a three-dimensional, non-Abelian subalgebra $h_3 = \{L^5, L^6, L^7\}$ that can be uniquely identified as a Heisenberg algebra. To see this, recall that the Heisenberg algebras $h_p$ are $p = 2n + 1$ dimensional generalisations of the basic ‘uncertainty’ relation, and take the form

$$
[P_i, P_j] = [Q_i, Q_j] = [P_i, C] = [Q_i, C] = [C, C] = 0, \quad [P_i, Q_j] = C\delta_{ij}
$$

They can be viewed as central extensions of the commutative algebra $\mathbb{R}^{2n}$ by $\mathbb{R}$, with central term $C$. If we now relabel the generators as $L^5 = P, L^6 = C, L^7 = Q$ then the only non-trivial commutator for these three generators becomes

$$
[P, Q] = C
$$

as it should be for the Heisenberg algebra $h_3$. Using this information, we define $D = -L^4$ and note that the total set of commutators for $s$ becomes

$$
[P, Q] = C, \quad [D, C] = C, \quad [D, P] = \frac{1}{2}P, \quad [D, Q] = \frac{1}{2}Q
$$

so that all the elements of the Heisenberg subalgebra $h_3$ are eigenvectors of the additional generator $D$. The unique four-dimensional, solvable Lie algebra $s$ that satisfies these requirements is the Lie algebra $su(1,2)/u(2)$ of the group $SU(1,2)/U(2)$, i.e. the Borel (or Iwasawa) subalgebra of the Lie algebra $su(1,2)$ (see for example Ref. [10])\textsuperscript{2}.

Therefore, we can now identify the total Lie algebra $g$ as

$$
g = sl(2, \mathbb{R}) \overset{\oplus}{\longrightarrow} su(1,2)/u(2) \quad (6)
$$

\textsuperscript{2} Note that $SU(1,2)/U(2)$ is indeed a Lie group, since $U(2)$ is a normal subgroup of $SU(1,2)$.\textsuperscript{2}
This happens to be isomorphic to one of the maximal parabolic subalgebras of $sp(4,\mathbb{R})$, and has a unique lift via the exponential map to

$$G = SL(2,\mathbb{R}) \ltimes SU(1,2)/U(2)$$

where $\ltimes$ defines the semidirect product (see for example Ref. [11]). This group $G$ now defines the corresponding maximal parabolic subgroup of $Sp(4,\mathbb{R})$. To find a suitable matrix description of this group, we note that a $4 \times 4$ matrix basis for $g$ can obviously be found by considering a subset of the basis matrices for $sp(4,\mathbb{R})$. Specifically, if $N_i, i = 1 \ldots 10,$ denotes the ten basis matrices of $sp(4,\mathbb{R})$ (see Appendix B), then $N_i, i = 1 \ldots 7,$ is a suitable basis for $g$ with commutation relations identical to the Killing vector fields $L_i$. Hence, by simple Lie exponentiation, one can show that the corresponding maximal parabolic subgroup $G$ of $Sp(4,\mathbb{R})$ can be represented by real matrices of the form

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with

$$A = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}, B = \begin{pmatrix} * & * \\ * & * \end{pmatrix}, C = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$$A^T D - C^T B = 1_2, \quad A^T C, B^T D \quad \text{symmetric}$$

Here the $*$ indicates an entry that is not forced to be zero. This is a closed, symplectic Lie group, and defines an arbitrary element of the isometry group $G$.

IV. THE GROUP ACTION

It is now useful to double-check this result, by showing that the structure of $G$ precisely reproduces the finite group transformations represented by integrating the Killing vectors. As an important corollary, we will also uncover the natural structure of the scalar-field moduli space upon which the symmetry group $G$ acts.

To integrate the Killing vector fields, we first make the formal identification

$$L^i = \sum_j \frac{d\zeta_j}{dc_i} \frac{\partial}{\partial \zeta_j}$$

for each $i$, where $c_i$ is a set of parameters that measures how far one has moved along the flowlines of the associated $L^i$, and $\zeta = (\beta, \chi, \phi, \sigma, z, \nu)$ is a row vector of moduli fields. By comparing coefficients on each side one finds that a set of coupled differential equations must be satisfied, whose solution as a function of the $c_i$ determines the finite, integrated transformations. These transformations are shown below, with the equivalent complex results presented in Appendix C.

$$L^1 : \quad \beta \rightarrow \beta + c_1$$

$$L^2 : \quad e^\beta \rightarrow \frac{e^\beta}{(1 + c_2 \chi)^2 + \frac{1}{4}c_2^2 e^{2\beta}}$$

$$L^3 : \quad \chi \rightarrow \chi + c_3$$

$$L^4 : \quad \phi \rightarrow \phi + c_4$$

$$L^5 : \quad \sigma \rightarrow \sigma - 4q\nu \cdot c_5$$

$$L^6 : \quad \sigma \rightarrow \sigma + 4q\nu$$

$$L^7 : \quad \nu \rightarrow \nu + c_7$$

$$\chi \rightarrow \chi e^{c_1}$$

$$\chi \rightarrow \chi (1 + c_2 \chi) + \frac{1}{4}c_2 c_2 e^{2\beta}$$

$$\sigma \rightarrow \sigma - c_3 \cdot 2q\nu^2$$

$$\sigma \rightarrow \sigma e^{c_4}$$

$$z \rightarrow z + c_5$$

$$z \rightarrow z e^{c_5/2}$$

$$\nu \rightarrow \nu e^{c_1/2}$$

$$\nu \rightarrow \nu e^{c_4/2}$$

$$\nu \rightarrow \nu e^{c_1/2}$$

$$\nu \rightarrow \nu e^{c_4/2}$$

$$\nu \rightarrow \nu e^{c_1/2}$$

$$\nu \rightarrow \nu e^{c_4/2}$$

(10)
These transformations can be implemented as a matrix multiplication on the scalar fields, where the latter have themselves been embedded as the entries of a symplectic $4 \times 4$ matrix. To understand why this is possible, consider the following. It is well known that the group $Sp(4, \mathbb{R})$ acts as a transitive group of diffeomorphisms on its own solvable subgroup, which is the group manifold $\mathcal{M} = Sp(4, \mathbb{R})/U(2)$. These diffeomorphisms are simply implemented by the multiplications

$$ V \rightarrow UVS $$

(11)

where $V \in \mathcal{M}$, $S \in Sp(4, \mathbb{R})$ and $U \in Sp(4, \mathbb{R}) \cap O(4, \mathbb{R}) \cong U(2)$. That is, $Sp(4, \mathbb{R})$ acts by right multiplications on $\mathcal{M}$, with a compensating local (field-dependent) left multiplication by a member of the maximal compact subgroup $U(2)$.

We can then equip $\mathcal{M}$ with a metric invariant under $Sp(4, \mathbb{R})$, by taking a unique (up to scaling) linear combination of the right-invariant one-forms on $\mathcal{M}$. This forms $\mathcal{M}$ into a Riemannian homogeneous space.

However, it is perfectly reasonable to consider a non-transitive set of diffeomorphisms corresponding to right-multiplication on $\mathcal{M}$ only by $G \subset Sp(4, \mathbb{R})$, and define a metric on $\mathcal{M}$ that is only $G$-invariant. To see that this is consistent, consider that the unique Iwasawa decomposition of $S \in Sp(4, \mathbb{R})$ takes the form $S = U\mathcal{N}$, where $\mathcal{N} \equiv SU(4)$, by taking a unique (up to scaling) linear combination of the right-invariant one-forms on $\mathcal{M}$. This transforms $\mathcal{M}$ into a Riemannian homogeneous space.

Thus, it must be possible to represent the transformations Eq. 11 in the form Eq. 12, thus verifying that the isometry group is $G$ and the underlying target space $\mathcal{M}$. To do this, consider gathering the scalar fields $\beta, \chi, \phi, \sigma, z, \nu$ into the $4 \times 4$ matrix $V$, defined by

$$ V = e^{-\frac{1}{2}N^1}e^{2\chi N^2}e^{\frac{1}{2}\phi N^3}e^{\sqrt{2}N^4}e^{-(\sigma+4qz\nu)N^5}e^{\sqrt{2}N^7} $$

$$ = \left( \begin{array}{cccc} e^{-\beta/2} & 0 & 2\chi e^{-\beta/2} & 2\sqrt{(\nu - \chi z)e^{-\beta/2}} \\ \sqrt{\gamma}e^{-\phi/2} & e^{-\phi/2} & 2\sqrt{(\nu + \chi z)e^{-\phi/2}} & (\sigma + 2qz\nu)e^{-\phi/2} \\ 0 & 0 & e^{\beta/2} & -\sqrt{\gamma}e^{\beta/2} \\ 0 & 0 & 0 & e^{\phi/2} \end{array} \right) $$

(13)

This is nothing but an arbitrary element of $\mathcal{M} = Sp(4, \mathbb{R})/U(2)$, which in turn is simply an arbitrary element of the solvable subgroup of $Sp(4, \mathbb{R})$ defined by exponentiation of the solvable subalgebra generators $N^i, i = 1, 2, 4, 5, 6, 7$. Consequently, the six scalars are naturally related to the six group parameters of $Sp(4, \mathbb{R})/U(2)$. We also take a matrix $\mathcal{G} \in G$ with constant entries as follows

$$ \mathcal{G} = \left( \begin{array}{cccc} e^{-c_1/2} & 0 & 2c_3 & 2\sqrt{q}c_7 \\ \sqrt{q}c_5 & e^{-c_2/2} & 2\sqrt{(c_3c_5 + c_7 e^{-c_2/2})} e^{c_1/2} & 4qc_6 \\ \frac{1}{2}e^2 & 0 & (1 + e_2c_3) e^{c_1/2} & \sqrt{q}(c_2c_7 - c_5 e^{-c_2/2}) e^{c_1/2} \\ 0 & 0 & 0 & e^{c_2/2} \end{array} \right)$$
Finally, if we define the function \( u \equiv 2 \left( e^{-c_1/2} + c_2 \chi \right) e^{-\beta} \), then the \( U(1) \) “compensator” matrix \( U \) can be written:

\[
U = \begin{pmatrix}
\frac{u}{\sqrt{c_3 + u^2}} & 0 & \frac{c_2}{\sqrt{c_3 + u^2}} & 0 \\
0 & 1 & 0 & 0 \\
-\frac{c_2}{\sqrt{c_3 + u^2}} & 0 & \frac{u}{\sqrt{c_3 + u^2}} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Overall, we then find that the total set of isometry transformations as defined in Eq. (10) can indeed be derived from the multiplications

\[ V \rightarrow U V G \]

Consequently, the isometry group is \( G \) and the underlying manifold is \( \mathcal{M} = Sp(4, \mathbb{R})/U(2) \cong G/U(1) \). Note that the parabolic \( G \) action is not transitive, and so not every point in \( \mathcal{M} \) can be connected to every other by a \( G \) isometry (for a description of the fixed points see Ref. \cite{12}). This stands in stark contrast to the transitive isometry groups of homogeneous Riemannian spaces, and suggests that there will be some obstruction to generating the general solution to the equations of motion using the isometries alone.

If, instead, we wish to visualise these transformations in terms of the natural complex fields, then there is an equivalent and elegant alternative to the above procedure. One can verify that there is a bijective map from \( \mathcal{M} \) into the space of complex, symmetric \( 2 \times 2 \) matrices \( \Psi \) with positive definite imaginary part. This space is called the upper Siegel plane \( SH_2 = \{ \Psi \in \text{Sym}_2 \mathbb{C} : \text{Im}(\Psi) > 0 \} \) (see for example \cite{13, 14, 15}). Moreover, there is a natural action of \( Sp(4, \mathbb{R}) \) (and so by restriction \( G \)) as a set of fractional linear transformations over \( SH_2 \). Concretely, we first define the matrix

\[ \Psi = 2i \begin{pmatrix} S & \sqrt{qZ} \\ \sqrt{qZ} & T \end{pmatrix} \]

We then consider the transformation action

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \Psi \rightarrow (A\Psi + B)(C\Psi + D)^{-1}
\]

where \( A, B, C, D \) are \( 2 \times 2 \) real matrices satisfying the symplectic conditions

\[
A^T D - C^T B = I_2, \quad A^T C, \quad B^T D \quad \text{symmetric}
\]

If we now restrict the action to those matrices of the form

\[
G_2 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

with

\[
A = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \quad B = \begin{pmatrix} * & * \\ * & * \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}, \quad D = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}
\]

then these produce the correct isometry transformations at the level of the complex fields. One can verify explicitly that the matrices \( G_2 \) do indeed define the maximal parabolic subgroup \( G \) again, but with a slightly different matrix representation to the one specified by \( G \) in Eq. (8) (defined by a different embedding in \( Sp(4, \mathbb{R}) \)).

---

3 To clarify: \( U \) defines a subgroup of \( Sp(4, \mathbb{R}) \cap O(4, \mathbb{R}) \) that is isomorphic to \( U(1) \), and is represented by \( 4 \times 4 \) matrices.
V. BUILDING THE METRIC

To come full circle, it should now be possible to build the $G$-invariant metric from the invariant one-forms of the underlying manifold $M$. To see this, recall that $M$ is a coset of the form $Sp(4, \mathbb{R})/U(2)$ with $U(2)$ a normal subgroup of $Sp(4, \mathbb{R})$, and so $M$ is a Lie group as well as a manifold. Consequently, there is a set of canonically discriminated left-invariant vector fields on $M$, and an associated “dual” set of right-invariant one-forms. These dual one-forms can be readily constructed from the matrix $\mathcal{V}$ in Eq. (13), by forming the combination

$$
\begin{pmatrix}
-\frac{1}{2} d\beta \\
\sqrt{q} e^{\frac{1}{2}(\beta - \phi)} d\nu \\
0 \\
-rac{1}{2} d\phi \\
2 \sqrt{q} e^{\frac{1}{2}(\beta - \phi)} (d\nu - \chi dz) \\
\frac{1}{2} d\beta \\
0 \\
0 \\
2 \sqrt{q} e^{\frac{1}{2}(\beta - \phi)} (d\nu - \chi dz) \\
\frac{1}{2} d\beta \\
e^{-\beta} (d\sigma + 4qz d\nu) \\
e^{-\phi} (d\sigma + 4qz d\nu) \\
e^{-\phi} (d\sigma + 4qz d\nu) \\
\sqrt{q} e^{\frac{1}{2}(\beta - \phi)} dz \\
\sqrt{2} q e^{\frac{1}{2}(\beta - \phi)} (d\nu - \chi dz)
\end{pmatrix}
$$

(15)

Each entry now defines a right-invariant one-form of the group $M$, and these forms are extremely useful for the following reason. The Killing vectors are left-invariant vector fields, and one can verify that these fields correspond to transformations enacted by infinitesimal right multiplications on the Lie algebra of the group. Hence, if we wish to assemble an invariant metric, this metric should be right-invariant and built from the entries above. So let us define the following right-invariant one-forms

$$
\theta_1 = \frac{1}{2} d\beta \\
\theta_2 = e^{-\beta} d\chi \\
\theta_3 = \frac{1}{2} d\phi \\
\theta_4 = \frac{1}{2} e^{-\phi} (d\sigma + 4qz d\nu) \\
\theta_5 = \sqrt{q} e^{\frac{1}{2}(\beta - \phi)} dz \\
\theta_6 = \sqrt{2} q e^{\frac{1}{2}(\beta - \phi)} (d\nu - \chi dz)
$$

We now note that a choice of right-invariant metric corresponds to a choice of how to combine these forms together. This is, implicitly, a choice of inner product on the Killing vectors in the Lie algebra. For example, one simple choice is to take

$$
ds^2 = \sum_{i=1}^{7} (\theta_i)^2
$$

This defines the natural homogeneous, symmetric metric on $Sp(4, \mathbb{R})/U(2)$ and can be derived from the Kähler potential

$$
K = -\ln \left[ S + \overline{S} - q \frac{(Z + \overline{Z})^2}{T + \overline{T}} \right] - \ln (T + \overline{T}) = -\ln [\det \text{Im} (\Psi)]
$$

(16)

However, another equally valid choice – which has an isometry group that is smaller than $Sp(4, \mathbb{R})$ – is given by

$$
ds^2 = k \sum_{i=1}^{2} (\theta_i)^2 + \sum_{i=3}^{7} (\theta_i)^2
$$

where $k \neq 1$ and is a positive real constant. This derives from the Kähler potential

$$
K = -\ln \left[ S + \overline{S} - q \frac{(Z + \overline{Z})^2}{T + \overline{T}} \right] - k \ln (T + \overline{T})
$$

(17)

and cannot be written in a neat way in terms of the Siegel plane coordinate $\Psi$ because the metric is nonhomogeneous. However, it is still a perfectly valid way of measuring distances on the manifold, albeit with a reduced isometry group $G$. If we fix $k = 3$ then this is obviously the Kähler potential implied by the action Eq. (1), which in turn is fixed by the structure of the higher-dimensional supergravity from which it descends.
VI. ALTERNATIVE DERIVATION OF THE METRIC

There is an alternative way to derive the metric, which allows one to rewrite it in a more compact and elegant form. We noted previously that the inhomogeneous Kähler potential Eq. (5) cannot be rewritten in terms of the natural complex coordinate \( \Psi \) on \( SH_2 \). This is fundamentally because the obstructing factor of 3 (which makes the metric inhomogeneous) does not allow the terms to be gathered as the determinant of \( \text{Im}(\Psi) \). However, one can bypass this obstruction by jumping up to the higher-dimensional Siegel plane \( SH_4 \cong Sp(8,\mathbb{R})/U(4) \), using a homogeneous metric, and then killing off some fields. Consider, for example, defining the complex coordinate

\[
\Phi = 2i \begin{pmatrix} S & Z \\ Z^T & T \end{pmatrix}
\]

where \( S, T, Z \) are complex \( 2 \times 2 \) matrices, with \( S, T \) symmetric. If \( \text{Im}(\Phi) > 0 \) then \( \Phi \) belongs to the upper Siegel plane \( SH_4 \), defined as usual by

\[
SH_4 = \{ \Phi \in \text{Sym}_4 \mathbb{C} : \text{Im}(\Phi) > 0 \}.
\]

Moreover, there is a natural homogeneous metric on \( SH_4 \) with a transitive group of \( Sp(8,\mathbb{R}) \) isometries, which derives from the Kähler potential

\[
K = -\ln[\det \text{Im}(\Phi)]
\]

Now consider restricting the 10 independent complex entries in \( \Phi \), by demanding that it reduce to the form

\[
\tilde{\Phi} = 2i \begin{pmatrix} \sqrt{q}Z & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{pmatrix}
\]

This now defines a Kähler submanifold of \( SH_4 \), with an induced Kähler potential given by the restriction of Eq. (18) to this submanifold. This induced potential takes the simple form

\[
K = -\ln[\det \text{Im}(\tilde{\Phi})]
\]

This is precisely the Kähler potential (and associated metric) that we have been analysing in Eq. (1), now written in a more compact form. One can now readily determine the isometry group of the metric, by looking for the subgroup \( G \subset Sp(8,\mathbb{R}) \) that preserves the form of \( \tilde{\Phi} \) under fractional linear transformations. One can verify that \( G \) is represented by the following \( Sp(8,\mathbb{R}) \) matrices:

\[
G_3 = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}
\]

where

\[
\mathcal{A} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}
\]

with the additional constraints

\[
\mathcal{A}^T \mathcal{D} - \mathcal{C}^T \mathcal{B} = 1_4, \quad \mathcal{A}^T \mathcal{C}, \mathcal{B}^T \mathcal{D} \quad \text{symmetric}
\]

Due to the diagonal nature of the \( 2 \times 2 \) submatrices in the lower right blocks of \( \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \) above, the non-trivial group structure is all captured by the \( 2 \times 2 \) submatrices in the upper left blocks. However, this set of \( 2 \times 2 \) matrices defines the matrix representation \( G_3 \) as given in Eq. (14). Consequently, \( G_2 \) and \( G_3 \) must be two different representations of the same abstract group, which is none other than the maximal parabolic group \( G \). Thus, we can derive the nonhomogeneous metric from an ordinary homogeneous one, by making appropriate field identifications and truncations. That is, the nonhomogeneous character arises by imposing restrictions on the fields in an otherwise homogeneous Riemannian space.
VII. CONCLUSION

We have investigated the effective $D = 4$ description of heterotic M-theory, paying particular attention to the isometries of the Kähler metric defined by the cosmological fields. In doing so, we have presented our results in a variety of ways, and have had occasion to switch between the real scalar field $s$ and the complex superfields that they group into.

We now state the main results of this paper in the most convenient and transparent form, in terms of the superfields $S, T, Z$. We discovered that the underlying complex manifold is a submanifold of the upper Siegel plane $SH_4$, with each point of this submanifold labelled by the matrix

$$\tilde{\Phi} = 2i \begin{pmatrix} S & \sqrt{q}Z & 0 & 0 \\ \sqrt{q}Z & T & 0 & 0 \\ 0 & 0 & T & 0 \\ 0 & 0 & 0 & T \end{pmatrix}$$

The Kähler potential can then be written in terms of this matrix as

$$K = -\ln[\det \text{Im}(\tilde{\Phi})]$$

The isometries of the associated Kähler metric can be readily determined, by asking for those transformations that preserve the form of $\tilde{\Phi}$ and act only as Kähler transformations on the Kähler potential. The isometry group was found to be a group $G \subset \text{Sp}(8, \mathbb{R})$, where $G$ is in fact a maximal parabolic subgroup of $\text{Sp}(4, \mathbb{R})$ containing a set of $\text{SL}(2, \mathbb{R})$ $T$-duality transformations. Specifically, the matrix $\tilde{\Phi}$ transforms as

$$\tilde{\Phi} \to (A\tilde{\Phi} + B)(C\tilde{\Phi} + D)^{-1}$$

where the matrices $A, B, C, D$ are given by

$$A = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}, \quad B = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}, \quad D = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$$

subject to the symplectic conditions

$$A^T D - C^T B = 1_4, \quad A^T C, B^T D \text{ symmetric}$$

One can now use these $G$ isometries to make significant progress toward the general solution to the equations of motion. In particular, one can build complicated new solutions by applying these $G$ isometries to simpler, specific solutions (say where certain fields are set to zero). In a companion paper we will derive several new classes of previously unknown cosmological solutions to Eq. (1), including a solution where the M5 brane can reverse direction despite the absence of any explicit potentials.

It would also be interesting to clarify the origin of the exact $\text{SL}(2, \mathbb{R})$ symmetry that we have found. For example, we know that in the absence of bulk M5 branes the Kähler potential contribution for the $T$ modulus is just $-3\ln(T + \overline{T})$, and that this leads to an exact set of $T$-duality transformations. However, it is a non-trivial feature that a modified version of this $T$-duality persists in the presence of an M5 brane.
Appendix A

The holomorphic Killing vector fields are given by

\[
L^1 = (T \partial_T + \frac{1}{2} Z \partial_Z) + (\overline{T} \partial_{\overline{T}} + \frac{1}{2} \overline{Z} \partial_{\overline{Z}})
\]

\[
L^2 = \frac{i}{2} (qZ^2 \partial_S + T^2 \partial_T + ZT \partial_Z) - \frac{i}{2} (qZ^2 \partial_{\overline{S}} + \overline{T}^2 \partial_{\overline{T}} + \overline{Z} \overline{T} \partial_{\overline{Z}})
\]

\[
L^3 = 2i (\partial_T - \partial_{\overline{T}})
\]

\[
L^4 = (S \partial_S + \frac{1}{2} Z \partial_Z) + (\overline{S} \partial_{\overline{S}} + \frac{1}{2} \overline{Z} \partial_{\overline{Z}})
\]

\[
L^5 = (2qZ \partial_S + T \partial_Z) + (2q \overline{Z} \partial_{\overline{S}} + \overline{T} \partial_{\overline{T}})
\]

\[
L^6 = 4qi (\partial_S - \partial_{\overline{S}})
\]

\[
L^7 = -2i (\partial_Z - \partial_{\overline{Z}})
\]
Appendix B

A $4 \times 4$ matrix basis for the ten-dimensional Lie algebra $sp(4, \mathbb{R})$ is given by

\[
N^1 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \\
N^2 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \\
N^3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \\
N^4 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
N^5 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \\
N^6 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix} \\
N^7 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} \\
N^8 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
N^9 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix} \\
N^{10} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

Appendix C

The finite superfield transformations corresponding to $L^1 \ldots L^7$ are

\[
L^1 : \quad S = S_0 \\
T = T_0 e^{c_1} \\
Z = Z_0 e^{c_1/2}
\]

\[
L^2 : \quad S = S_0 + \frac{qZ_0^2}{T_0} \left( \frac{ic_2T_0}{2 - ic_2T_0} \right) \\
T = \frac{2T_0}{2 - ic_2T_0} \\
Z = \frac{2Z_0}{2 - ic_2T_0}
\]

\[
L^3 : \quad S = S_0 \\
T = T_0 + 2ic_3 \\
Z = Z_0
\]

\[
L^4 : \quad S = S_0 e^{c_4} \\
T = T_0 \\
Z = Z_0 e^{c_4/2}
\]

\[
L^5 : \quad S = S_0 + 2qZ_0c_5 + qT_0c_5^2 \\
T = T_0 \\
Z = Z_0 + c_5T_0
\]

\[
L^6 : \quad S = S_0 + 4qic_6 \\
T = T_0 \\
Z = Z_0
\]

\[
L^7 : \quad S = S_0 \\
T = T_0 \\
Z = Z_0 - 2ic_7
\]