THE RIEMANN PROBLEM AND THE LIMIT SOLUTIONS AS MAGNETIC FIELD VANISHES TO MAGNETOGASDYNAMICS FOR GENERALIZED CHAPLYGIN GAS

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Abstract. This paper is concerned with the Euler equations in the magnetogasdynamics for generalized Chaplygin gas. The global solutions to the Riemann problems of the Euler equations in the magnetogasdynamics for generalized Chaplygin gas are obtained constructively by using phase plane analysis method. The formation of delta shock wave is studied as magnetic field vanishes. The limit behaviors of the Riemann solutions as magnetic field vanishes are also obtained.

1. Introduction. Magnetogasdynamics plays a very important role in engineering physics, astrophysics, nuclear science, plasma physics and many other areas [9]. It can be described mathematically by the Euler equations for the ideal magnetofluid. Since the full governing system for magnetogasdynamics is highly nonlinear and more complicated, it is necessary to study the various simplified models. One of the simplified model is the assumption that the flow wherein magnetic and velocity fields are orthogonal everywhere, such as in [5, 10, 11, 12]. In the present paper, we consider the system of equations which governs the one dimensional unsteady simple flow of an isentropic, inviscid and perfectly conducting compressible fluid, subject to a transverse magnetic field, can be written as the following conservation laws

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + \left(\rho u^2 + p + \frac{B^2}{2\mu}\right)_x &= 0.
\end{align*}
\]

(1)

where \(\rho \geq 0, u, p, B = \kappa \rho\) and \(\mu > 0\) denote the density, the velocity, the pressure, the transverse magnetic field and the magnetic permeability, respectively. \(\kappa\) is a positive constant.

The system (1) with the equation of state

\[p = -s\rho^{-\alpha}, \quad 0 < \alpha \leq 1, s > 0.\]

(2)

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is called the generalized Chaplygin gas magnetogasdynamic system, where \( s \) and \( \alpha \) are constants. When \( \alpha = 1, s = 1 \), it is called a Chaplygin gas, which was introduced by Chaplygin [3], Tsien [19] and von Karman [6] as a suitable mathematical approximation for calculating the lifting force on a wing of an airplane in aerodynamics.

The vanishing pressure limit method was investigated first by Li [8] in 2001, in which he obtained the limit of Riemann solutions of the isentropic Euler equations as pressure vanishes. For the magnetogasdynamic system, in [14], Shen studied the limits of Riemann solutions to the isentropic magnetogasdynamics for polytropic gas. In [15], Shao proved that the Riemann solutions of the isentropic Chaplygin gas magnetogasdynamics equations tends to a delta shock solution to transport equations as pressure and magnetic field vanish. For results on the vanishing pressure limit of Riemann solutions, the reader is referred to the papers ([1, 2, 7, 16, 17, 20, 22]) and the references cited therein for more details.

In this paper, we are concerned with the Riemann problem for (1) and (2) with the following initial data

\[
(u, \rho) (x, 0) = \begin{cases} 
(u_+, \rho_+), & \text{if } x > 0, \\
(u_-, \rho_-), & \text{if } x < 0,
\end{cases}
\]

where \((u_+, \rho_+)\) are two constant states.

The organization of this paper is as follows. In Section 2, we solve the Riemann problem of (1) and (3) with (2) by characteristic analysis in phase plane. In Section 3, we recall the Riemann problem to the generalized Chaplygin gas dynamics system. In Section 4, we discuss the limit of Riemann solutions to (1), (2) and (3) as the magnetic field vanish, i.e. \( \kappa \to 0 \).

2. Solutions to the Riemann problem (1) and (3) with (2). For any smooth solution, the system (1) can be written in the form

\[
\begin{pmatrix}
u \\
\rho
\end{pmatrix}_t + \begin{pmatrix}
u w^2/\rho \\
\rho u
\end{pmatrix}_x = 0,
\]

where \( w = \sqrt{c^2 + b^2} \) is the magneto-acoustic speed, \( c = \sqrt{p'(\rho)} \) is the sound speed, and \( b^2 = B^2(\rho)/(\mu \rho) \) is the Alfvén speed.

The eigenvalues of system (1) are

\[
\lambda_1 = u - \sqrt{s\alpha/\rho^{\alpha+1} + \kappa^2\rho/\mu}, \quad \lambda_2 = u + \sqrt{s\alpha/\rho^{\alpha+1} + \kappa^2\rho/\mu}.
\]

It follows that the system is strictly hyperbolic.

The corresponding right-eigenvectors are

\[
\vec{\eta}_1 = \left( \rho, -\sqrt{s\alpha/\rho^{\alpha+1} + \kappa^2\rho/\mu} \right)^T, \quad \vec{\eta}_2 = \left( \rho, \sqrt{s\alpha/\rho^{\alpha+1} + \kappa^2\rho/\mu} \right)^T.
\]

By same computations we have

\[
\nabla \lambda_i \cdot \vec{\eta}_i = \mp \frac{(1 - \alpha)s\alpha/\rho^{\alpha+1} + 3\kappa^2\rho/\mu}{2\sqrt{s\alpha/\rho^{\alpha+1} + \kappa^2\rho/\mu}} \neq 0 \quad i = 1, 2,
\]

where \( \nabla \) denotes the gradient with respect to \((\rho, u)\). So, the first and second characteristic fields are genuinely nonlinear.
Since equations (1) and the Riemann data (3) are invariant under uniform stretching of coordinates
\[(x, t) \rightarrow (kx, kt),\]
where \(k\) is a constant. We consider self-similar solutions of (1) and (2)
\[(u, \rho)(x, t) = (u, \rho)(\xi), \quad \xi = \frac{x}{t}.\]
Then the Riemann problem is reduced to a boundary value problem of ordinary differential equations at infinity
\[
\begin{align*}
\xi \rho \rho_s + (\rho u)_\xi &= 0, \\
-\xi (\rho u)_\xi + \left(\rho u^2 + p + \frac{B^2}{2\mu}\right)_\xi &= 0,
\end{align*}
\]
with
\[(u, \rho)(\pm \infty) = (u_\pm, \rho_\pm).\]
For smooth solutions, the equations (8) can be written as
\[
\begin{pmatrix}
\rho(u - \xi) \\
\rho^\alpha + 1 \\
\rho u - \xi
\end{pmatrix}

\begin{pmatrix}
\frac{s\alpha}{\rho^\alpha + 1} + \frac{\kappa^2 \rho}{\mu} \\
\frac{\rho^2 + p + \frac{B^2}{2\mu}}{\rho}
\end{pmatrix}

\begin{pmatrix}
u \\
\rho
\end{pmatrix}_\xi = 0.
\]
(9)
The general solution of (9) is constant states
\[(u, \rho)(\xi) = \text{Constant, \quad} (\rho > 0),\]
and the singular solution consists of forward or backward rarefaction waves
\[
\begin{align*}
\xi &= \lambda_{1,2} = u \mp \sqrt{\frac{s\alpha}{\rho^\alpha + 1} + \frac{\kappa^2 \rho}{\mu}}, \\
\xi_{R_\kappa}(u_-, \rho_-) : u &= u_- \mp \int_{\rho_-}^{\rho} \sqrt{\frac{s\alpha/\tau^\alpha + 1 + \kappa^2 \tau/\mu}{\tau}} \, d\tau, \quad \rho < (>) \rho_-.
\end{align*}
\]
(10)
For a bounded discontinuous solution, the following Rankine–Hugoniot conditions hold
\[
\begin{align*}
-\sigma [\rho] + [\rho u] &= 0, \\
-\sigma [\rho u] + \left[\rho u^2 + p + \frac{B^2}{2\mu}\right] &= 0,
\end{align*}
\]
(11)
where \([\rho] = \rho - \rho_-\), etc., and \(\sigma\) is the velocity of the discontinuity.
For (11), we have two shock waves \(\xi_{S_\kappa}\) and \(\xi_{S_\kappa}\) as follows
\[
\begin{align*}
\sigma_\pm &= u_- \mp \sqrt{\frac{\rho_- s(\rho_-^\alpha - \rho_-^{-\alpha}) + \frac{s^2 \alpha}{\mu} (\rho_-^2 - \rho_-^{-2})}{\rho - \rho_-}} (\rho - \rho_-), \\
\xi_{S_\kappa}(u_-, \rho_-) : u &= u_- \mp \sqrt{\frac{1}{\rho \rho_-} - s(\rho_-^\alpha - \rho_-^{-\alpha}) + \frac{s^2 \alpha}{\mu} (\rho_-^2 - \rho_-^{-2})} (\rho - \rho_-), \\
\rho > (>) \rho_-.
\end{align*}
\]
(12)
The rarefaction waves \(\xi_{R_\kappa}\) and shock waves \(\xi_{S_\kappa}\) are called the elementary waves of the Euler equations for magnetogasdynamics.
**Lemma 2.1.** The curve \( \overrightarrow{\mathcal{R}}_\kappa(u_-, \rho_-) \) is monotonic decreasing and convex in the \((u, \rho)\) plane. It has a straight line \( \rho = 0 \) as its asymptote. The curve \( \overrightarrow{\mathcal{S}}_\kappa(u_-, \rho_-) \) is monotonic increasing and concave in the \((u, \rho)\) plane, and \( \lim_{\rho \to +\infty} u = +\infty \). The curve \( \overrightarrow{\mathcal{S}}_\kappa(u_-, \rho_-) \) is monotonic decreasing in the \((u, \rho)\) plane, and \( \lim_{\rho \to +\infty} u = -\infty \).

The curve \( \overrightarrow{\mathcal{S}}_\kappa(u_-, \rho_-) \) is monotonic increasing in the \((u, \rho)\) plane. It has a straight line \( \rho = 0 \) as its asymptote. (See Fig.1)

**Proof.** Differentiating the second equation of \( \overrightarrow{\mathcal{R}}_\kappa(u_-, \rho_-) \) with respect to \( \rho \), we get
\[
 u_\rho = -\frac{1}{\rho} \sqrt{\alpha/\rho^{\alpha+1} + \kappa^2 \rho/\mu} < 0,
 u_{\rho \rho} = \frac{-s(\rho^{-\alpha} - \rho_-^{-\alpha}) + \frac{\kappa^2}{2 \mu} (\rho^2 - \rho_-^2)}{2 \sqrt{-s(\rho^{-\alpha} - \rho_-^{-\alpha}) + \frac{\kappa^2}{2 \mu} (\rho^2 - \rho_-^2)}} < 0.
\]

It is easy to see that \( u \to +\infty \) when \( \rho \to 0 \). Similarly, for the curve \( \overrightarrow{\mathcal{S}}_\kappa(u_-, \rho_-) \), we can obtain \( u_\rho > 0, u_{\rho \rho} < 0 \) and it has \( u \to +\infty \) when \( \rho \to +\infty \).

Differentiating the second equation of \( \overrightarrow{\mathcal{S}}_\kappa(u_-, \rho_-) \) with respect to \( \rho \), for \( \rho > \rho_- \), we get
\[
 u_\rho = -\frac{(-s(\rho^{-\alpha} - \rho_-^{-\alpha}) + \frac{\kappa^2}{2 \mu} (\rho^2 - \rho_-^2)) \rho^2 + (s \alpha \rho^{-\alpha+1}) \rho_\rho + \frac{\kappa^2 \rho}{2 \mu} (\rho^2 - \rho_-^2)}{2 \rho_\rho}
\]
In addition, we obtain that \( u \to -\infty \) when \( \rho \to +\infty \). Similarly, for the curve \( \overrightarrow{\mathcal{S}}_\kappa(u_-, \rho_-) \), we can obtain \( u_\rho > 0 \) and it has \( u \to -\infty \) when \( \rho \to 0^+ \).

For any given \((u_-, \rho_-)\), the phase plane \((\rho > 0)\) is divided into five regions. Then, by the analysis method in phase plane, according to the right state \((u_+, \rho_+)\) in the different region, we can construct the unique global Riemann solution connecting two constant states \((u_-, \rho_-)\) and \((u_+, \rho_+)\). As shown in Fig.1.

**Theorem 2.2.** For Riemann problem (1), (2) and (3), there exists a unique piecewise smooth solution, which consists of constant states, shocks and rarefaction waves. For any given \((u_+, \rho_+)\), the Riemann solutions are shown as follows:

1. \( (u_+, \rho_+) \in I_\kappa(u_-, \rho_-) : \overrightarrow{\mathcal{R}}_\kappa + \overrightarrow{\mathcal{S}}_\kappa \);
2. \( (u_+, \rho_+) \in \Pi_\kappa(u_-, \rho_-) : \overrightarrow{\mathcal{R}}_\kappa + \overrightarrow{\mathcal{R}}_\kappa \);
3. \( (u_+, \rho_+) \in \mathcal{W}_\kappa(u_-, \rho_-) : \overrightarrow{\mathcal{R}}_\kappa + \overrightarrow{\mathcal{S}}_\kappa \);
4. \( (u_+, \rho_+) \in \mathcal{V}_\kappa(u_-, \rho_-) : \overrightarrow{\mathcal{S}}_\kappa + \overrightarrow{\mathcal{S}}_\kappa \).

**Fig.1.** The elementary wave curves in the phase plane.
3. Riemann problems of the Euler equations for generalized Chaplygin gas. The Riemann problems of the Euler equations modeling isentropic compressible fluids for Chaplygin gas were studied by D. Serre ([13]), Guo, Sheng and Zhang [4], Wang [21], etc. The Riemann problem to the Euler equations are

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
\left(\rho u\right)_t + \left(\rho u^2 + p\right)_x &= 0.
\end{align*}
\]  

(13)

with initial data (3) and the equation of state (2).

For smooth solutions in the self-similar plane, we have the backward (forward) rarefaction wave curves

\[
\begin{align*}
\xi &= \lambda_{1,2} = u \mp \sqrt{s} \rho \frac{\alpha + 1}{2} \\
\frac{k}{\rho} R(u_-, \rho_-) : u &= u_- \pm \frac{2\sqrt{s} \alpha}{1 + \alpha} \left(\rho_--\rho_-^{\frac{\alpha + 1}{2}}\right), \quad \rho \leq (\geq) \rho_-.
\end{align*}
\]  

(14)

For a bounded discontinuous solution, we get the backward (forward) shock wave curves

\[
\begin{align*}
\sigma_\pm &= u_- \mp \sqrt{s} \rho \frac{\rho_--\rho_-^{\alpha + 1}}{\rho_-^{\alpha + 1} - \rho_--\rho_-}, \\
\frac{k}{\rho} S(u_-, \rho_-) : u &= u_- \pm \sqrt{s} \rho_--\rho_-^{\alpha + 1} \left(\rho_--\rho_-\right), \quad \rho \geq (\leq) \rho_-.
\end{align*}
\]  

(15)

From the above results, we see that the phase plane is divided into five regions (See Fig.2). In addition, we draw an \(L_\delta\) curve, which is determined as follows:

\[u = u_- - \sqrt{s} \left(\rho_--\rho_-^{\frac{\alpha + 1}{2}}\right).\]

For any given state \((\rho_+, u_+)\), the Riemann solutions are shown as follows

1. \((u_+, \rho_+) \in I(u_-, \rho_-) : R + R\); 2. \((u_+, \rho_+) \in II(u_-, \rho_-) : S + R\); 3. \((u_+, \rho_+) \in III(u_-, \rho_-) : R + S\); 4. \((u_+, \rho_+) \in IV(u_-, \rho_-) : S + S\); 5. \((u_+, \rho_+) \in V(u_-, \rho_-) :\) Delta shock wave.

When \((u_+, \rho_+) \in V(u_-, \rho_-)\), we consider the delta shock wave solution.

Fig.2. The curves of elementary waves.
Definition 3.1. A two-dimensional weighted delta function $w(t)\delta_L$ supported on a smooth curve $L = \{(x(s), t(s)) : a < s < b\}$ is defined by

$$< w(\cdot)\delta_L, \phi(\cdot, \cdot) > = \int_a^b w(t(s))\phi(x(s), t(s))ds.$$  
for any $\phi \in C_0^\infty(R \times R_+)$.

With this definition, the delta shock wave solutions can be written as

$$(u, \rho)(t, x) = \begin{cases} 
(u_-, \rho_-), & x < x(t), \\
(u_\delta, w(t)\delta(x - x(t))), & x = x(t), \\
(u_+, \rho_+), & x > x(t).
\end{cases}$$  
(16)

For the delta shock wave solutions, we have the following generalized Rankine-Hugoniot conditions:

$$\begin{align*}
\frac{dx(t)}{dt} &= u_\delta, \\
\frac{dw(t)}{dt} &= u_\delta[\rho] - [\rho u], \\
\frac{d}{dt}(w(t)u_\delta) &= u_\delta[\rho u] - [\rho u^2 + p],
\end{align*}$$  
(17)

with initial data: $x(0) = 0, w(0) = 0$. To guarantee uniqueness, the discontinuity must satisfy the $\delta$-entropy condition $u_+ + \sqrt{s\rho_+} < u_\delta < u_- - \sqrt{s\rho_-}$. 

Solving (17), we get, when $\rho_- = \rho_+$,

$$w(t) = \rho_- (u_- - u_+) t, \quad u_\delta = \frac{u_- + u_+}{2},$$

and when $\rho_- \neq \rho_+$,

$$w(t) = \sqrt{[\rho u]^2 - [\rho][\rho u^2 - s\rho_-] t}, \quad u_\delta = \frac{\rho_+ u_+ - \rho_- u_- + \sqrt{[\rho u]^2 - [\rho][\rho u^2 - s\rho_-]}}{\rho_+ - \rho_-}.$$  

Fig.3. The limiting behaviors of the curves of elementary waves.
4. The limiting behaviors of the Riemann solutions as $\kappa \to 0$.

4.1. Case 1. $(u_+, \rho_+) \in I(u_-, \rho_-)$.

**Lemma 4.1.** For any $\kappa_1 > \kappa_2 \geq 0$, $\overrightarrow{R}_{\kappa_1}$ is located at the right side of $\overrightarrow{R}_{\kappa_2}$ and $\overrightarrow{R}_{\kappa_1}$ is located above $\overrightarrow{R}_{\kappa_2}$. Furthermore, for any $\kappa > 0$, $\overrightarrow{R}_\kappa$ is located at the right side of $\overrightarrow{R}$ and $\overrightarrow{R}_\kappa$ is located above $\overrightarrow{R}$ in the $(u, \rho)$ plane, i.e., $\overrightarrow{R}_\kappa \cup \overrightarrow{R}_\kappa \subseteq I(u_-, \rho_-)$ (see Fig.3).

**Proof.** For any $\kappa_1 > \kappa_2 \geq 0$ and $\rho > \rho_-$, take $(u_{\kappa_1}, \rho) \in \overrightarrow{R}_{\kappa_1}(u_-, \rho_-)$ and $(u_{\kappa_2}, \rho) \in \overrightarrow{R}_{\kappa_2}(u_-, \rho_-)$ (when $\kappa_2 = 0$, $\overrightarrow{R}_0(u_-, \rho_-) = \overrightarrow{R}(u_-, \rho_-)$). By (10), we have

$$u_{\kappa_1} - u_{\kappa_2} = \int_{\rho_-}^{\rho} \frac{\sqrt{s\alpha/\tau + 1 + \kappa_1 \tau/\mu} - \sqrt{s\alpha/\tau + 1 + \kappa_2 \tau/\mu}}{\tau} d\tau > 0, \quad \rho_- < \rho.$$

(18)

For any $u \geq u_-$, take $(u, \rho_{\kappa_1}) \in \overrightarrow{R}_{\kappa_1}(u_-, \rho_-)$ and $(u, \rho_{\kappa_2}) \in \overrightarrow{R}_{\kappa_2}(u_-, \rho_-)$, we have

$$\int_{\rho_-}^{\rho_{\kappa_1}} \frac{\sqrt{s\alpha/\tau + 1 + \kappa_1^2 \tau/\mu}}{\tau} d\tau = \int_{\rho_-}^{\rho_{\kappa_2}} \frac{\sqrt{s\alpha/\tau + 1 + \kappa_2^2 \tau/\mu}}{\tau} d\tau.$$  

(19)

Thus, we get $\rho_{\kappa_1} > \rho_{\kappa_2}$. By (18) and (19), we have $\overrightarrow{R}_{\kappa_1}$ is located at the right side of $\overrightarrow{R}_{\kappa_2}$ and $\overrightarrow{R}_{\kappa_1}$ is located above $\overrightarrow{R}_{\kappa_2}$. Furthermore, for any $\kappa > 0$, $\overrightarrow{R}_\kappa \cup \overrightarrow{R}_\kappa \subseteq I(u_-, \rho_-)$.

**Lemma 4.2.** For $(u_+, \rho_+) \in I(u_-, \rho_-)$, there exists a constant $\kappa_0 > 0$, such that $(u_+, \rho_+) \in I_\kappa(u_-, \rho_-)$ as $0 < \kappa < \kappa_0$.

**Proof.** For $(u_+, \rho_+) \in I(u_-, \rho_-)$, if $\rho_+ > \rho_-$, there exists a $\kappa_1 > 0$, such that $(u_+, \rho_+) \in \overrightarrow{R}_{\kappa_1}(u_-, \rho_-)$. In fact, from $(u_+, \rho_+) \in \overrightarrow{R}_\kappa(u_-, \rho_-)$, we have

$$u_+ = u_- + \int_{\rho_-}^{\rho_+} \frac{\sqrt{s\alpha/\tau + 1 + \kappa_1^2 \tau/\mu}}{\tau} d\tau, \quad \rho_- < \rho_+.$$

(20)

By the mean value theorem, we take

$$\kappa_1 = \mu \left( \rho_0 \left( \frac{u_+ - u_-}{\rho_+ - \rho_-} \right)^2 - \frac{s\alpha}{\rho_0^{\alpha+2}} \right), \quad \rho_- < \rho_0 < \rho_+.$$

(21)

By Lemma 4.1, when $0 < \kappa < \kappa_1$, there exists a $(u_*, \rho_*) \in \overrightarrow{R}_\kappa(u_-, \rho_-) \supseteq \overrightarrow{R}_\kappa(u_*, \rho_*)$, and the Riemann solution consists of a backward rarefaction $\overrightarrow{R}_\kappa$ and a forward rarefaction $\overrightarrow{R}_\kappa$:

$$\begin{cases}
\frac{\overrightarrow{R}_\kappa(u_-, \rho_-)}{\overrightarrow{R}_\kappa(u_*, \rho_*)} : u_* = u_- - \int_{\rho_-}^{\rho_+} \frac{\sqrt{s\alpha/\tau + 1 + \kappa_1^2 \tau/\mu}}{\tau} d\tau, \quad \rho_* < \rho_-, \\
\frac{\overrightarrow{R}_\kappa(u_*, \rho_*)}{\overrightarrow{R}_\kappa(u_+, \rho_+)} : u_+ = u_* + \int_{\rho_-}^{\rho_+} \frac{\sqrt{s\alpha/\tau + 1 + \kappa_1^2 \tau/\mu}}{\tau} d\tau, \quad \rho_* < \rho_+.
\end{cases}$$

(22)

For $(u_+, \rho_+) \in I(u_-, \rho_-)$, if $\rho_+ < \rho_-$, there exists a $\kappa_2 > 0$, such that $(u_+, \rho_+) \in \overrightarrow{R}_{\kappa_2}(u_-, \rho_-)$. In fact, from $(u_+, \rho_+) \in \overrightarrow{R}_\kappa(u_-, \rho_-)$, we have

$$u_+ = u_- - \int_{\rho_-}^{\rho_+} \frac{\sqrt{s\alpha/\tau + 1 + \kappa_2^2 \tau/\mu}}{\tau} d\tau, \quad \rho_+ < \rho_-.$$

(23)
By the mean value theorem and (23), we take
\[ κ_2 = \frac{1}{\mu} \left( \frac{μ_0 (u_+ - u_-)}{ρ_+ - ρ_-} \right)^2 - \frac{sα}{ρ_0^2}, \quad ρ_+ < ρ_0 < ρ_- \]  
(24)

Then, by Lemma 4.1, when $0 < κ < κ_2$, the Riemann solution consists of a backward rarefaction wave $\overrightarrow{R}_κ$ and a forward rarefaction wave $\overleftarrow{R}_κ$.

Denote $κ_0 = \min\{κ_1, κ_2\} > 0$. We get the result. □

In view of (22), we have
\[ \int_{ρ_+}^{ρ_0} \frac{\sqrt{sα/τ^{α+1}} + κ^2τ/μ}{τ} dτ = \frac{u_+ - u_-}{2} - \frac{1}{2} \int_{ρ_-}^{ρ_0} \frac{\sqrt{sα/τ^{α+1}} + κ^2τ/μ}{τ} dτ. \]
(25)

Letting $κ → 0$ in (25), we get
\[ ρ_κ = \left( \frac{u_+ - u_-}{2} + \frac{1 + α}{4\sqrt{sα}} (u_- - u_+) \right)^{-\frac{1}{1+α}}. \]
(26)

Thus when $κ → 0$, we get
\[ \begin{cases} \overrightarrow{R}(u_-, ρ_-) : u_κ - \frac{2\sqrt{sα}}{1+α} ρ_κ^{-\frac{1+α}{2}} = u_- - \frac{2\sqrt{sα}}{1+α} ρ_-^{-\frac{1+α}{2}}, \quad ρ_- < ρ_κ, \\ \overleftarrow{R}(u_κ, ρ_κ) : u_+ + \frac{2\sqrt{sα}}{1+α} ρ_κ^{-\frac{1+α}{2}} = u_κ + \frac{1+α}{4\sqrt{sα}} (u_- - u_+), \quad ρ_κ < ρ_+. \end{cases} \]
(27)

**Theorem 4.3.** When $(u_+, ρ_+) ∈ Π(u_-, ρ_-)$ and the magnetic field vanishes, i.e. $κ → 0$, the limit of the solution of (1), (2) and (3) which may consist of $(\overrightarrow{S}_κ$ and $\overleftarrow{R}_κ$) or $(\overrightarrow{R}_κ$ and $\overrightarrow{S}_κ$) or $(\overrightarrow{R}_κ$ and $\overleftarrow{S}_κ$) are the corresponding Riemann solution to the Euler equations (13) with initial data (3) which consist of $\overrightarrow{R}$ and $\overleftarrow{R}$.

**4.2. Case 2.** $(u_+, ρ_+) ∈ Π(u_-, ρ_-) ∪ Π(u_-, ρ_-)$.

**Lemma 4.4.** When $κ > 0$, $\overrightarrow{S}_κ(u_-, ρ_-)$ is above $\overrightarrow{S}(u_-, ρ_-)$ and $\overrightarrow{S}_κ(u_-, ρ_-)$ is at the left side of $\overrightarrow{S}(u_-, ρ_-)$. It follows that $Π(u_-, ρ_-) ⊆ Π_κ(u_-, ρ_-)$ and $Π_κ(u_-, ρ_-) ⊆ Π_κ(u_-, ρ_-)$.

**Proof.** Let $\overrightarrow{S}(u_-, ρ_-)$ and $\overrightarrow{S}_κ(u_-, ρ_-)$ intersect with the line $ρ = ρ_0$ at the points $(u, ρ_0)$ and $(u_κ, ρ_κ)$, respectively, by (12), we have
\[ u_κ - u = \sqrt{1 - \frac{1}{ρ_- - ρ_0}} \left( \sqrt{s \left( \frac{1}{ρ_κ^2} - \frac{1}{ρ_0^2} \right)} + \frac{κ^2}{2μ} (ρ_κ^2 - ρ_0^2) \right) < 0. \]
(28)

Let $\overrightarrow{S}(u_-, ρ_-)$ and $\overrightarrow{S}_κ(u_-, ρ_-)$ intersect with the line $u = u_0$ at the points $(u_0, ρ_0)$ and $(u_0, ρ_κ)$, by (15), we have
\[ \sqrt{\left( \frac{1}{ρ_κ} - \frac{1}{ρ_-} \right) \left( \frac{1}{ρ_κ^2} - \frac{1}{ρ_-^2} \right) + \frac{κ^2}{2μ} (ρ_κ^2 - ρ_-^2)} = \sqrt{s \left( \frac{1}{ρ_κ} - \frac{1}{ρ_-} \right) \left( \frac{1}{ρ_κ^2} - \frac{1}{ρ_-^2} \right)}. \]
(29)

Then we have $ρ < ρ_κ$. Thus, we get $Π(u_-, ρ_-) ⊆ Π_κ(u_-, ρ_-)$ and $Π_κ(u_-, ρ_-) ⊆ Π_κ(u_-, ρ_-)$. □
When \((u_+, \rho_+) \in \mathcal{I}(u_-, \rho_-) \subseteq \mathcal{I}(u_-, \rho_-)\), the solution of (1) and (3) with (2) consists of a backward shock wave \(\overline{S}_\kappa\) and a forward rarefaction wave \(\overline{R}_\kappa\). Let \((u_*, \rho_*)\) is the intermediate state, \((u_*, \rho_*) \in \overline{S}_\kappa(u_-, \rho_-) \cap \overline{R}_\kappa(u_+, \rho_+)\), which are as follows

\[
\begin{cases}
\overline{S}_\kappa(u_-, \rho_-) : u_* = u_- - \sqrt{\frac{1}{\rho_* \rho_-} \cdot \frac{s(\rho_* - \rho_-) + \frac{\kappa^2}{\mu}(\rho_*^2 - \rho_-^2)}{\rho_- - \rho_*} (\rho_* - \rho_-)}, \\
\quad \rho_* > \rho_-, \\
\overline{R}_\kappa(u_+, \rho_+) : u_+ = u_* + \int_{\rho_*}^{\rho} \frac{\sqrt{s\alpha/\tau + 1 + \kappa^2 \rho/\mu}}{\tau} d\tau, \quad \rho_+ > \rho_*.
\end{cases}
\]

We have \(u_- - \frac{\sqrt{s}}{\rho_* - \rho_-} < u_* < u_+ < u_- + \frac{2\sqrt{s\alpha}}{1 + \alpha} \rho_* - \frac{\alpha + 1}{2} \). Thus, letting \(\kappa \to 0\), we get

\[
\begin{cases}
\overline{S}_\kappa(u_-, \rho_-) \to \overline{S}(u_-, \rho_-) : u_* = u_- - \sqrt{s \left( \frac{1}{\rho_-} - \frac{1}{\rho_*} \right) \left( \frac{1}{\rho_*^\alpha} - \frac{1}{\rho_*^\kappa} \right)}, \\
\quad \rho_* > \rho_-, \\
\overline{R}_\kappa(u_+, \rho_+) \to \overline{R}(u_+, \rho_+) : u_+ = u_* + \frac{2\sqrt{s\alpha}}{1 + \alpha} \rho_* - \frac{\alpha + 1}{2} = u_* + \frac{2\sqrt{s\alpha}}{1 + \alpha} \rho_* - \frac{\alpha + 1}{2}, \quad \rho_+ > \rho_*.
\end{cases}
\]

When \((u_+, \rho_+) \in \mathcal{I}(u_-, \rho_-) \subseteq \mathcal{I}(u_-, \rho_-)\), the solution of (1) and (3) with (2) consists of a backward rarefaction wave \(\overline{R}_\kappa\) and a forward shock wave \(\overline{S}_\kappa\). Then we have

\[
\begin{cases}
\overline{R}_\kappa(u_-, \rho_-) : u_* = u_- - \int_{\rho_*}^{\rho} \frac{\sqrt{s\alpha/\tau + 1 + \kappa^2 \rho/\mu}}{\tau} d\tau, \quad \rho_* < \rho_-, \\
\overline{S}_\kappa(u_+, \rho_+) : u_+ = u_* + \frac{1}{\rho_* \rho_+} \frac{s(\rho_* - \rho_-) + \frac{\kappa^2}{\mu}(\rho_*^2 - \rho_-^2)}{\rho_+ - \rho_*} (\rho_* - \rho_-), \quad \rho_* < \rho_+.
\end{cases}
\]

By \(\rho_* < \rho_* < \rho_-\), letting \(\kappa \to 0\), we get

\[
\begin{cases}
\overline{R}_\kappa(u_-, \rho_-) \to \overline{R}(u_-, \rho_-) : u_* = u_- - \frac{2\sqrt{s\alpha}}{1 + \alpha} \rho_* - \frac{\alpha + 1}{2} = u_* - \frac{2\sqrt{s\alpha}}{1 + \alpha} \rho_* - \frac{\alpha + 1}{2}, \quad \rho_- > \rho_*, \\
\overline{S}_\kappa(u_+, \rho_+) \to \overline{S}(u_+, \rho_+) : u_+ = u_* - \sqrt{s \left( \frac{1}{\rho_*} - \frac{1}{\rho_*} \right) \left( \frac{1}{\rho_*^\alpha} - \frac{1}{\rho_*^\kappa} \right)}), \quad \rho_+ < \rho_*.
\end{cases}
\]

**Theorem 4.5.** When \((u_+, \rho_+) \in \mathcal{I}(u_-, \rho_-) \cup \mathcal{I}(u_-, \rho_-)\) and the magnetic field vanishes, the limit of the solution of (1) and (3) with (2) is the Riemann solution to the Euler equations (13) with the same initial data.

4.3. **Case 3.** \((u_+, \rho_+) \in \mathcal{I}(u_-, \rho_-)\).

**Lemma 4.6.** For any \(\kappa_1 > \kappa_2 \geq 0\), \(\overline{S}_{\kappa_1}\) is located at the left side of \(\overline{S}_{\kappa_2}\) and \(\overline{S}_{\kappa_1}\) is located above \(\overline{S}_{\kappa_2}\). Furthermore, for any \(\kappa > 0\), \(\overline{S}_\kappa\) is located at the left side of \(\overline{S}\) and \(\overline{S}_\kappa\) is located above \(\overline{S}\) in the \((u, \rho)\) plane (see Fig.3).
Proof. For any \( \kappa_1 > \kappa_2 \geq 0 \) and \( \rho > \rho_- (\rho < \rho_-) \), take \((u_{\kappa_1}, \rho) \in \overline{S}_{\kappa_1}(S_{\kappa_1})\) and \((u_{\kappa_2}, \rho) \in \overline{S}_{\kappa_2}(S_{\kappa_2})\) (when \( \kappa_2 = 0 \), \( S_0 = \overline{S}_0 \)). By \((12)\), we have

\[
-\sqrt{\frac{\rho - \rho_-}{\rho \rho_-}} \left( -s(\rho_-^{-\alpha} - \rho_-^{-\alpha}) + \frac{\kappa_2^2}{2\mu}(\rho^2 - \rho_-^2) \right) < (>) 0.
\]

Therefore, we have \( \overline{S}_{\kappa_1} \) is located at the right side of \( \overline{S}_{\kappa_2} \) and \( \overline{S}_{\kappa_1} \) is located above \( \overline{S}_{\kappa_2} \).

**Lemma 4.7.** For \((u_+, \rho_+) \in IV(u_-, \rho_-)\), there exists a constant \( \kappa_0 > 0 \) such that the Riemann solution of \((1)\) and \((3)\) with \((2)\) consists of a backward shock wave \( \overline{S}_\kappa \) and a forward shock wave \( \overline{S}_\kappa \) as \( 0 < \kappa < \kappa_0 \).

**Proof.** For \((u_+, \rho_+) \in IV(u_-, \rho_-)\), if \( \rho_+ > \rho_- \), there exists a \( \kappa_3 > 0 \), such that \((u_+, \rho_+) \in \overline{S}_{\kappa_3}(u_-, \rho_-)\). In fact, from \((u_+, \rho_+) \in \overline{S}_{\kappa}(u_-, \rho_-)\), we have

\[
u_+ = u_- - \sqrt{\left( \frac{1}{\rho_-} - \frac{1}{\rho_+} \right) \left( \frac{s}{\rho_-} - \frac{1}{\rho_+} \right) + \frac{\kappa_2^2}{2\mu}(\rho_+^2 - \rho_-^2)}}.
\]

Then we take

\[
\kappa_3 = \frac{1}{\rho_+ - \rho_-} \sqrt{2\mu \left( (u_- - u_+) \frac{\rho_+ \rho_-}{\rho_- - \rho_+} - s \left( \frac{1}{\rho_-} - \frac{1}{\rho_+} \right) \right)}.
\]

Thus, by Lemma 4.6, when \( 0 < \kappa < \kappa_3 \), the Riemann solution consists of a backward shock wave \( \overline{S}_\kappa \) and a forward shock wave \( \overline{S}_\kappa \)

\[
\begin{cases}
\overline{S}_\kappa(u_-, \rho_-) : u_* = u_- - \sqrt{\left( \frac{1}{\rho_-} - \frac{1}{\rho_*} \right) \left( -s(\rho_-^{-\alpha} - \rho_-^{-\alpha}) + \frac{\kappa_3^2}{2\mu}(\rho_*^2 - \rho_-^2) \right)}, \\
\rho_* > \rho_-,
\end{cases}
\]

\[
\begin{cases}
\overline{S}_\kappa(u_+, \rho_+) : u_+ = u_+ + \sqrt{\left( \frac{1}{\rho_+} - \frac{1}{\rho_*} \right) \left( -s(\rho_*^{-\alpha} - \rho_+^{-\alpha}) + \frac{\kappa_3^2}{2\mu}(\rho_+^2 - \rho_*^2) \right)}, \\
\rho_+ < \rho_*.
\end{cases}
\]

For \((u_+, \rho_+) \in IV(u_-, \rho_-)\), if \( \rho_+ < \rho_- \), there exists a \( \kappa_4 > 0 \), such that \((u_+, \rho_+) \in \overline{S}_{\kappa_4}(u_-, \rho_-)\). In fact, from \((u_+, \rho_+) \in \overline{S}_{\kappa}(u_-, \rho_-)\), we have

\[
u_+ = u_- - \sqrt{\left( \frac{1}{\rho_+} - \frac{1}{\rho_-} \right) \left( s \left( \frac{1}{\rho_+} - \frac{1}{\rho_-} \right) + \frac{\kappa_4^2}{2\mu}(\rho_+^2 - \rho_-^2) \right)}.\]

Then we take

\[
\kappa_4 = \frac{1}{\rho_- - \rho_+} \sqrt{2\mu \left( (u_- - u_+) \frac{\rho_+ \rho_-}{\rho_- - \rho_+} - s \left( \frac{1}{\rho_+} - \frac{1}{\rho_-} \right) \right)}.
\]

Thus, when \( 0 < \kappa < \kappa_4 \), the Riemann solution consists of a backward shock wave \( \overline{S}_\kappa \) and a forward shock wave \( \overline{S}_\kappa \). Denote \( \kappa_0 = \min\{\kappa_3, \kappa_4\} > 0 \), we get the result. \(\square\)
Because the curve $\overline{S}_\kappa(u_-, \rho_-)$ is monotonic decreasing and $L_\delta$ is monotonic increasing with $\frac{da}{dp} = \frac{(1+\kappa)\sqrt{\rho}}{2}\rho^{\frac{3+2\kappa}{2}} > 0$, thus the curve $\overline{S}_\kappa(u_-, \rho_-)$ and $L_\delta$ must intersect at a point $(\tilde{u}, \tilde{\rho})$ when $\kappa \neq 0$. Thus, we have

$$\begin{cases}
\dot{u} + \sqrt{s}\dot{\rho}^{\frac{\alpha+1}{2}} = u_- - \sqrt{s}\dot{\rho}^{\frac{\alpha-1}{2}}, \\
\dot{u} = u_- - \sqrt{1 + \frac{1}{\dot{\rho}\rho_-} \cdot \frac{-s(\dot{\rho}^-\alpha - \dot{\rho}^-\alpha) + \frac{\alpha^2}{2\mu}(\dot{\rho}^2 - \rho_-^2)}{\dot{\rho} - \rho_-}}, \quad \rho > \rho_-.
\end{cases}$$

(40)

Then, we get

$$\frac{\kappa^2}{2s\mu} = \frac{(\dot{\rho}^\alpha + \rho^-\alpha^2/2)^2}{(\dot{\rho} - \rho_-)^2(\dot{\rho} + \rho_-)} + \hat{\rho}^{-\alpha} - \rho^{-\alpha}.$$  

(41)

Therefore, we have $\lim_{\kappa \to 0} \dot{\rho} = +\infty$, and $\lim_{\kappa \to 0} \dot{u} = u_- - \sqrt{s}\rho_-^{\frac{\alpha+1}{2}}$. Thus when $(u_+, \rho_+) \in N(u_-, \rho_-)$ and $\kappa \to 0$, we get

$$\begin{cases}
\overline{S}_\kappa(u_-, \rho_-) \to \overline{S}(u_-, \rho_-) : u_* = u_- - \sqrt{1 + \frac{1}{\rho_-\rho_\ast} \cdot \frac{-s(\rho_\ast^{-\alpha} - \rho_\ast^{-\alpha}) + \frac{\alpha^2}{2\mu}(\rho_\ast^2 - \rho_-^2)}{\rho_\ast - \rho_-}}, \\
\rho_\ast > \rho_-,
\end{cases}$$

\begin{align}
\overline{S}_\kappa(u_-, \rho_*) &\to \overline{S}(u_*, \rho_*) : u_+ = u_* - \sqrt{1 + \frac{1}{\rho_+\rho_\ast} \cdot \frac{-s(\rho_\ast^{-\alpha} - \rho_\ast^{-\alpha}) + \frac{\alpha^2}{2\mu}(\rho_\ast^2 - \rho_+^2)}{\rho_\ast - \rho_+}}, \\
\rho_+ < \rho_\ast.
\end{align}

(42)

**Theorem 4.8.** When $(u_+, \rho_+) \in N(u_-, \rho_-)$ and the magnetic field vanishes, i.e. $\kappa \to 0$, the limit of the solution of (1) and (3) with (2) is the corresponding Riemann solution to the Euler equations (13) with initial data (3) which consists of $\overline{S}$ and $\overline{S}$.

4.4. Case 4. $(u_+, \rho_+) \in V(u_-, \rho_-)$. In this section, we will study the phenomenon of concentration and the formation of delta shock waves in the Riemann solutions of system (1) and (3) with (2) as $(u_+, \rho_+) \in V(u_-, \rho_-)$ and the magnetic field vanishes, i.e. $\kappa \to 0$. In this case, similar to the last subsection, there exists a $\tilde{\kappa}_0 > 0$, such that $(u_+, \rho_+) \in N_\kappa(u_-, \rho_-)$ as $0 < \kappa < \tilde{\kappa}_0$. The solution of (1) and (3) with (2) consists of $\overline{S}_\kappa(u_-, \rho_-)$ and $\overline{S}_\kappa(u_*, \rho_*)$, where $(u_*, \rho_*) \in \overline{S}_\kappa(u_-, \rho_-)$.

**Lemma 4.9.** In the case $(u_+, \rho_+) \in V(u_-, \rho_-)$, we have

1. $\lim_{\kappa \to 0} \rho_\ast = +\infty$.
2. $\lim_{\kappa \to 0} \kappa^2 \rho_\ast^2 = 2\mu A$, where

$$A = \rho_+ - \rho_+ \left(s[\frac{\rho^{-\alpha-1}}{\rho}] + \rho_+ + \rho_+ \left(\frac{|u|}{\rho}\right)^2 + 2 \frac{|u|}{\rho} \sqrt{s[\frac{\rho^{-\alpha}}{\rho}] + \rho_+ + \rho_+ \left(\frac{|u|}{\rho}\right)^2}\right).$$

Proof. According to (37), we obtain

$$u_- - u_+ = \sqrt{\frac{1}{\rho_+\rho_-} \cdot \frac{-s(\rho_\ast^{-\alpha} - \rho_\ast^{-\alpha}) + \frac{\alpha^2}{2\mu}(\rho_\ast^2 - \rho_-^2)}{\rho_\ast - \rho_-}} (\rho_\ast - \rho_-) + \sqrt{\frac{1}{\rho_+\rho_\ast} \cdot \frac{-s(\rho_\ast^{-\alpha} - \rho_\ast^{-\alpha}) + \frac{\alpha^2}{2\mu}(\rho_\ast^2 - \rho_+^2)}{\rho_\ast - \rho_+}} (\rho_\ast - \rho_+).$$  

(43)
Assuming that \( \lim_{\kappa \to 0} \rho_* = M \neq +\infty \), then we can get

\[
\nu_- - \nu_+ = \sqrt{s \left( \frac{1}{\rho_* - \frac{1}{M}} \right) \left( \frac{1}{\rho_*^2} - \frac{1}{M^2} \right)} + \sqrt{s \left( \frac{1}{\rho_* + \frac{1}{M}} \right) \left( \frac{1}{\rho_*^2} - \frac{1}{M^2} \right)} < \sqrt{s\rho_-^{-\frac{\alpha + 1}{2}}} + \sqrt{s\rho_+^{-\frac{\alpha + 1}{2}}}.
\]

That is to say, we have

\[
\nu_+ + \sqrt{s\rho_+^{-\frac{\alpha + 1}{2}}} > \nu_- - \sqrt{s\rho_-^{-\frac{\alpha + 1}{2}}},
\]

which contradicts with \( \nu_+ + \sqrt{s\rho_+^{-\frac{\alpha + 1}{2}}} < \nu_- - \sqrt{s\rho_-^{-\frac{\alpha + 1}{2}}} \). Therefore, \( \lim_{\kappa \to 0} \rho_* = +\infty \).

Let \( \lim_{\kappa \to 0} \kappa^2 \rho_*^2 = 2\mu A \), by (43), we have

\[
\nu_- - \sqrt{s\rho_-^{-\alpha - 1} + \rho_-^{-1} A} = \nu_+ + \sqrt{s\rho_+^{-\alpha - 1} + \rho_+^{-1} A},
\]
denoted by \( u_\delta \). By the direct calculations and simplification, we get (2).

\( \square \)

**Lemma 4.10.** In the case \( (\nu_+ + \rho_+) \in \mathcal{V}(\nu_-, \rho_-, \nu_-) \), we have

1. \( \lim_{\kappa \to 0} \nu_+ = \nu_\delta \in \left[ \nu_+ + \sqrt{s\rho_+^{-\frac{\alpha + 1}{2}}} - \nu_- - \sqrt{s\rho_-^{-\frac{\alpha + 1}{2}}} \right] \).
2. \( \lim_{\kappa \to 0} \sigma_- = \lim_{\kappa \to 0} \sigma_+ = \lim_{\kappa \to 0} \nu_+ = \nu_\delta \).

**Proof.** In view of (37) and Lemma 4.9, we get

\[
u_\delta = \lim_{\kappa \to 0} \nu_+ = \nu_- \lim_{\kappa \to 0} \frac{1}{\rho_* \rho_-} \cdot \frac{-s(\rho_*^{-\alpha} - \rho_-^{-\alpha}) + \frac{s^2}{20}(\rho_*^2 - \rho_-^2)}{(\rho_* - \rho_-)} (\rho_* - \rho_-) = \]

\[
u_- - \sqrt{s\rho_-^{-\alpha - 1} + \rho_-^{-1} A} = \nu_+ + \sqrt{s\rho_+^{-\alpha - 1} + \rho_+^{-1} A},
\]

Similarly, we get

\[
u_\delta = \lim_{\kappa \to 0} \nu_+ = \nu_+ \lim_{\kappa \to 0} \frac{1}{\rho_+ \rho_*} \cdot \frac{-s(\rho_+^{-\alpha} - \rho_*^{-\alpha}) + \frac{s^2}{20}(\rho_+^2 - \rho_*^2)}{(\rho_* - \rho_+)} (\rho_* - \rho_+) = \]

\[
u_+ + \sqrt{s\rho_+^{-\alpha - 1} + \rho_+^{-1} A} \geq \nu_+ + \sqrt{s\rho_+^{-\alpha - 1} + \rho_+^{-1} A},
\]

So we obtain (1).

Using the Rankine-Hugoniot conditions (11) for \( \overline{S}_\kappa \) and \( \overline{S}_\kappa \) and Lemma 3.1, we have

\[egin{align*}
-\sigma_- (\rho_- - \rho_*) + (\rho_- \nu_+ - \rho_\nu_*) = 0, \\
-\sigma_+ (\rho_+ - \rho_-) + (\rho_\nu_+ - \rho_\nu_-) = 0.
\end{align*}
\]

It follows that

\[
\lim_{\kappa \to 0} \frac{\rho_- \nu_- - \rho_\nu_+}{\rho_- - \rho_*} = \lim_{\kappa \to 0} \frac{\rho_- \nu_- - \rho_\nu_-}{\rho_- - \rho_*} = \lim_{\kappa \to 0} \nu_+ = \nu_\delta,
\]
In addition, we have
\[
\lim_{\kappa \to 0} \sigma_+ = \lim_{\kappa \to 0} \frac{\rho_* u_* - \rho_+ u_+}{\rho_* - \rho_+} = \lim_{\kappa \to 0} \frac{u_* - \frac{\rho_- u_-}{\rho_*}}{1 - \frac{\rho_-}{\rho_*}} = \lim_{\kappa \to 0} u_* = u_\delta.
\]

Lemma 4.11.
\[
\lim_{\kappa \to 0} \int_{\sigma_-}^{\sigma_+} \rho_* d\rho = u_\delta [\rho] - [\rho u], \quad \lim_{\kappa \to 0} \int_{\sigma_-}^{\sigma_+} \rho_* u_* d\rho = u_\delta [\rho u] - [\rho u^2 + s \rho^{-\alpha}].
\]
In addition
\[
u = \frac{u_- + u_+}{2},
\]
as \rho_- = \rho_+ and
\[
u = \frac{\rho_+ u_+ - \rho_- u_- + \sqrt{[\rho u]^2 - [\rho] [\rho u^2 - s \rho^{-\alpha}]} \rho_+ - \rho_-}{\rho_+ - \rho_-}.
\]
as \rho_\neq \rho_+.

Proof. Using the Rankine-Hugoniot conditions (11), we have
\[
\begin{cases}
-\sigma_-(\rho_- - \rho_*) + (\rho_- u_- - \rho_* u_*) = 0, \\
-\sigma_-(\rho_- u_- - \rho_* u_*) + \left(\rho_- u_-^2 - s \rho_-^{-\alpha} + \frac{\kappa^2 \rho^2_-}{2\mu} - \rho_* u_*^2 + s \rho_*^{-\alpha} - \frac{\kappa^2 \rho^2_*}{2\mu}\right) = 0.
\end{cases}
\]
and
\[
\begin{cases}
-\sigma_+(\rho_+ - \rho_*) + (\rho_+ u_+ - \rho_* u_*) = 0, \\
-\sigma_+(\rho_+ u_+ - \rho_* u_*) + \left(\rho_+ u_+^2 - s \rho_+^{-\alpha} + \frac{\kappa^2 \rho^2_+}{2\mu} - \rho_* u_*^2 + s \rho_*^{-\alpha} - \frac{\kappa^2 \rho^2_*}{2\mu}\right) = 0.
\end{cases}
\]
According to the first equation in (44), (45) and Lemma 4.10, we obtain
\[
\lim_{\kappa \to 0} \rho_* (\sigma_+ - \sigma_-) = \lim_{\kappa \to 0} (-\sigma_- \rho_- + \sigma_+ \rho_+ + \rho_- u_- - \rho_+ u_+) = u_\delta [\rho] - [\rho u],
\]
According to the second equations in (44) and (45) and Lemma 4.10, we obtain
\[
\lim_{\kappa \to 0} \rho_* u_* (\sigma_+ - \sigma_-) = \lim_{\kappa \to 0} \left(-\sigma_- \rho_- u_- + \sigma_+ \rho_+ u_+ - [\rho u^2 + s \rho^{-\alpha}] + \frac{\kappa^2}{2\mu} [\rho^2]\right)
\]
\[
= u_\delta [\rho u] - [\rho u^2 + s \rho^{-\alpha}].
\]
Due to
\[
\lim_{\kappa \to 0} \rho_* u_*(\sigma_+ - \sigma_-) = \lim_{\kappa \to 0} \rho_* (\sigma_+ - \sigma_-) \cdot \lim_{\kappa \to 0} u_*,
\]
we have
\[
u [\rho u] - [\rho u^2 + s \rho^{-\alpha}] = (u_\delta [\rho] - [\rho u]) u_\delta.
\]
Therefore, we get
\[
u = \frac{u_- + u_+}{2}, \quad \rho_\neq \rho_+;
\]
\[
u = \frac{\rho_+ u_+ - \rho_- u_- + \sqrt{[\rho u]^2 - [\rho] [\rho u^2 - s \rho^{-\alpha}]} \rho_+ - \rho_-}{\rho_+ - \rho_-}, \quad \rho_\neq \rho_+.
\]
The proof is completed. \qed
Theorem 4.12. Let $0 < s < s_0$ and $(u_+, \rho_+) \in V(u_-, \rho_-)$. For a fixed $\kappa > 0$, the Riemann solution of (1) and (3) with (2) is that $\overline{S}_\kappa$ with $\overline{S}_\kappa$ or $\overline{S}_\kappa$ with $\overline{R}_\kappa$, constructed in Section 2. Then $\rho$ and $\rho u$ converge to the Riemann solution of (13) and (3) in the sense of distributions as $\kappa \to 0$. The limit functions $\rho$ and $\rho u$ are the sums of a step function and a $\delta$-measure with weights

$$w_1(t) = \big(\sigma[\rho] - [\rho u]\big)t,$$

and

$$w_2(t) = \big(\sigma[\rho u] - [\rho u^2 + s\rho^{-\alpha}]\big)t,$$

respectively, which form a delta shock wave solution.

Proof. Let $\xi = \frac{\zeta}{t}$. Due to $(u_+, \rho_+) \in IV$, the Riemann solution can be written as,

$$(u, \rho)(\xi) = \begin{cases} (u_+, \rho_+), & \xi < \sigma_-, \\ (u_+, \rho_+), & \sigma_- < \xi < \sigma_+, \\ (u_+, \rho_+), & \xi > \sigma_+. \end{cases} \quad (46)$$

From (8), the Riemann solution (46) satisfies the following weak formulations

$$\int_{-\infty}^{+\infty} (\xi - u(\xi))\rho(\xi)\phi'(\xi)d\xi + \int_{-\infty}^{+\infty} \rho(\xi)\phi(\xi)d\xi = 0, \quad (47)$$

$$\int_{-\infty}^{+\infty} (\xi - u(\xi))\rho(\xi)\phi'(\xi)d\xi - \int_{-\infty}^{+\infty} \left(p(\xi) + \frac{B^2(\xi)}{2\mu}\right)\phi'(\xi)d\xi + \int_{-\infty}^{+\infty} \rho(\xi)\phi(\xi)d\xi = 0. \quad (48)$$

for any test function $\phi \in C_0^1(-\infty, +\infty)$.

For the first integral in (48), we have

$$\int_{-\infty}^{+\infty} (\xi - u(\xi))\rho(\xi)\phi'(\xi)d\xi = \left(\int_{-\infty}^{\sigma_-} + \int_{\sigma_-}^{\sigma_+} + \int_{\sigma_+}^{+\infty}\right) (\xi - u(\xi))\rho(\xi)\phi'(\xi)d\xi = I_1 + I_2 + I_3.$$

Due to

$$I_1 + I_3 = \left(\int_{-\infty}^{\sigma_-} + \int_{\sigma_-}^{+\infty}\right) (\xi - u(\xi))\rho(\xi)\phi'(\xi)d\xi - \rho_-\int_{\sigma_-}^{+\infty} \phi(\xi)d\xi - \rho_+ u_+\sigma_+\phi(\sigma_+) + \rho_+ u_+\int_{\sigma_+}^{+\infty} \phi(\xi)d\xi,$$

we have

$$\lim_{\kappa \to 0} (I_1 + I_3) = \left([\rho u^2] - \sigma_\delta[\rho u]\right)\phi(\sigma_\delta) - \int_{-\infty}^{+\infty} \rho_0 u_0(\xi - \sigma_\delta)\phi(\xi)d\xi,$$

where $\rho_0 u_0(\xi - \sigma) = \frac{1}{2}(\rho_- u_- + \rho_+ u_+ + [\rho u]H(\xi - \sigma))$ and $H$ is the Heaviside function.

$$\int_{\sigma_-}^{+\infty} (\xi - u(\xi))\rho(\xi)\phi'(\xi)d\xi = \int_{\sigma_-}^{\sigma_+} (\xi - u_\sigma)\rho_\sigma u_\sigma\phi'(\xi)d\xi$$

$$= -\rho_+ u_\sigma (\sigma_+ - \sigma_-) \left(\frac{\phi(\sigma_+ - \sigma_-)u_\sigma}{\sigma_+ - \sigma_-} - \frac{\sigma_+\phi(\sigma_+) - \sigma_-\phi(\sigma_-)}{\sigma_+ - \sigma_-} + \frac{1}{\sigma_+ - \sigma_-} \int_{\sigma_-}^{\sigma_+} \phi(\xi)d\xi\right). \quad (49)$$
By lemma (4.10), we obtain
\[
\lim_{\kappa \to 0} \int_{\sigma_-}^{\sigma_+} \left( \xi - u(\xi) \right) \rho(\xi) u(\xi) \phi'(\xi) d\xi = 0.
\]
Similarly, we have
\[
- \int_{-\infty}^{+\infty} \left( -s \rho^{-\alpha} + \frac{\kappa^2 \rho^2(\xi)}{2\mu} \right) \phi'(\xi) d\xi = - \left( \int_{-\infty}^{\sigma_1} + \int_{\sigma_1}^{\sigma_2} + \int_{\sigma_2}^{+\infty} \right) \left( -s \rho^{-\alpha} + \frac{\kappa^2 \rho^2(\xi)}{2\mu} \right) \phi'(\xi) d\xi.
\]
Then we get
\[
\int_{-\infty}^{\sigma_1} \left( s \rho^{-\alpha} - \frac{\kappa^2 \rho^2(\xi)}{2\mu} \right) \phi'(\xi) d\xi + \int_{\sigma_1}^{\sigma_2} \left( s \rho^{-\alpha} - \frac{\kappa^2 \rho^2(\xi)}{2\mu} \right) \phi'(\xi) d\xi + \int_{\sigma_2}^{+\infty} \left( s \rho^{-\alpha} - \frac{\kappa^2 \rho^2(\xi)}{2\mu} \right) \phi'(\xi) d\xi
\]
\[
= \left( s \rho^{-\alpha} - \frac{\kappa^2 \rho^2(\xi)}{2\mu} \right) \phi(\sigma_1) + \left( s \rho^{-\alpha} - \frac{\kappa^2 \rho^2(\xi)}{2\mu} \right) \phi(\sigma_2) - \phi(\sigma_1) - \phi(\sigma_2).
\]
Thus,
\[
\lim_{k \to 0} \int_{-\infty}^{+\infty} \left( -s \rho^{-\alpha} + \frac{\kappa^2 \rho^2(\xi)}{2\mu} \right) \phi'(\xi) d\xi = - \left[ s \rho^{-\alpha} \right] \phi(\sigma).
\]
Then, we obtain that
\[
\lim_{k \to 0} \int_{-\infty}^{+\infty} \left( \rho(\xi) u(\xi) - \rho_0 u(\xi - \sigma) \right) \phi(\xi) d\xi = \left( \sigma \left[ \rho u - [\rho u^2] - [s \rho^{-\alpha}] \right] \phi(\sigma) \right) \phi(\sigma), \quad (50)
\]
for any test function \( \phi \in C^1_0(-\infty, +\infty) \).

As done previously for (47), we obtain that
\[
\lim_{k \to 0} \int_{-\infty}^{+\infty} \left( \rho(\xi) - \rho_0(\xi - \sigma) \right) \phi(\xi) d\xi = \left( \sigma \left[ \rho - [\rho u] \right] \phi(\sigma) \right) \phi(\sigma). \quad (51)
\]
Finally, we study the limits of \( \rho \) and \( \rho u \) as \( \kappa \to 0 \). By tracing the time dependence of weights of the \( \delta \)-measure and let \( \phi(x,t) \in C^1_0((-\infty, +\infty) \times [0, \infty)) \) and \( \hat{\phi}(\xi,t) := \phi(\xi,t) \), then we have
\[
\lim_{k \to 0} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho \left( \frac{x}{t} \right) \phi(x,t) d\xi dt = \lim_{k \to 0} \int_{-\infty}^{+\infty} t \int_{-\infty}^{+\infty} \rho(\xi) \phi(\xi,t) d\xi dt.
\]
Applying (51), we have
\[
\lim_{k \to 0} \int_{-\infty}^{+\infty} t \int_{-\infty}^{+\infty} \rho(\xi) \phi(\xi,t) d\xi dt
\]
\[
= \int_{-\infty}^{+\infty} \rho_0(x - \sigma t) \phi(x,t) dx dt + \int_{0}^{+\infty} \left( \sigma \left[ \rho - [\rho u] \right] \right) t \phi(\sigma,t) dt. \quad (52)
\]
By definition, the last term on the right-hand side of (52) equals to
\[
< w_1(t) \delta_\sigma, \phi(\cdot, \cdot) >, \quad \text{where} \quad w_1(t) = \left( \sigma \left[ \rho - [\rho u] \right] \right) t.
\]
Similarly, from (50), we arrive at
\[
\lim_{k \to 0} \int_{-\infty}^{+\infty} \rho \left( \frac{x}{t} \right) u_0 \left( \frac{x}{t} \right) \phi(x,t) d\xi dt
\]
\[
= \int_{-\infty}^{+\infty} \rho_0 u_0(x - \sigma t) \phi(x,t) dx dt + \int_{0}^{+\infty} \left( \sigma [\rho u] - [\rho u^2 + s \rho^{-\alpha}] \right) t \phi(\sigma,t) dt, \quad (53)
\]
with
\[
\begin{aligned}
  w_2(t) &= \left(\sigma[\rho] - [\rho u + s\rho^{-\alpha}]\right)t.
\end{aligned}
\]

Then we get the result. \hfill \square

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