Geometrically nonlinear free vibration of Euler-Bernoulli shallow arch

O Outassafte\textsuperscript{1}, A Adri\textsuperscript{1}, S Rifai\textsuperscript{1} and R Benamar\textsuperscript{2}

\textsuperscript{1}Laboratory of Mechanics, Production, and Industrial Engineering, LMPGI, Higher School of Technology of Casablanca, ESTC, Hassan II University of Casablanca, B.P 8112 Oasis, Casablanca, Morocco

\textsuperscript{2}Mohammed V University in Rabat, EMI-Rabat, LERSIM, B.P.765 Agdal, Rabat, Morocco

omar.outassafte@ensem.ac.ma

Abstract. The purpose of this work is to investigate the geometrical non-linearity in free vibrations of the Euler-Bernoulli shallow arch with clamped ends. The nonlinear governing equilibrium equation of the shallow arch is obtained after the Euler Bernoulli theory and Won Karman geometrical nonlinearity assumptions. The initial curvature of the arch is not due to the axial displacement of the beam but is due to the geometric of the beam itself. Taking into account a harmonic motion, the kinetic and total strain energy are discretized into a series of finite functions, which are a combination of the linear modes calculated before and the coefficients of contribution. The discretized expressions are derived by applying a Hamilton principle energy and spectral analysis. A cubic nonlinear algebraic system is obtained and solved numerically using an approximation method (the so-called second formulation) is applied to resolve various nonlinear vibration problems. To illustrate the effect of the curvature on the fundamental nonlinear mode and nonlinear frequency, the corresponding backbone curves, nonlinear amplitude vibration, and curvature of the arch are presented for the first modes shapes.

1. Introduction
Curved beams are structural elements widely used in civil, mechanical, and aerospace engineering. These structural elements often exhibit significant amplitude vibrations, leading to structural collapse due to low damping or light loading\cite{1}. The height of the initial curvature can be used as a control parameter to adjust the frequency conveniently.
Many research is studied the static and dynamic of a shallow arch. Raymond H. Plaut and Niels Olhoff [2] determine the arch’s form, which maximizes the fundamental vibration frequency.

Nayfeh et al. [3] obtained an exact solution of the eigenvalue problems governing linear undamped free vibrations of a buckled beam. W. Lacarbona and G. Rega [4] obtained the general conditions for orthogonality of the nonlinear modes in two-to-one, two-to-three, and one-to-one resonance. Zhuang Peng Yi et al. [1] used the multiple scales method to construct the NNMs considering quadratic and cubic nonlinearities for a shallow arch with one end elastically restrained in vertical and rotational directions in the case of two-to-one internal resonances. In reference [5], the displacement method has been applied to studying the transverse displacement of the shallow curved beam. After George C. Tsiatas and Nick G. Babouskos [6] investigated the linear and geometrically nonlinear response of non-uniform shallow arches under a concentrated central force, they assume that the shallow arches have increased overall stiffness contrary to straight beams. The harmonic balance method and pseudo-arc-length method are applied by Si-Qin Ye et al. [7] to study the nonlinear vibration of curved structures with nonlinear boundary conditions. All these authors confirmed that linear and nonlinear frequencies and modes are sensitive to the initial linear curvature.

For this present paper, we will extend Benamar’s method for a shallow sinusoidal arch to investigate the effect of initial curvature in the geometrical nonlinearity of the shallow arch in a free vibration case.

2. Theoretical formulation
Consider a shallow arch of length \( L \), with initial rise \( q \), a cross-section \( A \), a mass per unit length \( m \), a moment of inertia \( I \), and modulus of elasticity \( E \). It is possible to neglect damping and axial force of displacement.

\[
\delta \int_{t_1}^{t_2} (V - T) dt = 0
\]

(2)

Where, \( \delta \) is the variation taken within the specified time interval.

With the material density of the beam \( \rho \) and the cross-sectional area \( A \), the kinetic energy of the shallow arch can be expressed as:
\[ T = \rho A \int_0^L \left( \frac{\partial w(x,t)}{\partial t} \right)^2 dx \]  

(3)

The non-linear strain–displacement relationships of a uniform beam undergoing large deflections with clamped ends \cite{7} \( u(0) = u(L) = 0 \) are:

\[ \varepsilon_x = \frac{1}{2} \left( \frac{\partial w(x,t)}{\partial x} \right)^2 + \frac{d w_0(x)}{dx} \frac{\partial w(x,t)}{\partial x} \]  

(4)

Using Hooke’s law, one can write the following relationships for the axial resultant force \( N_x \) and the bending moment \( M \).

\[ N_x = EA\varepsilon_x \quad \quad M = EI\varepsilon_x \]  

(5)

Where the potential energy along the beam is composed of the axial tension potential energy and the bending potential energy:

\[ V = \frac{1}{2} \int_0^L \left( N_x \varepsilon_x + Mw_{xx} \right) dx \]  

(6)

Substituting Eq. (3) and (6) into Eq. (2) and using the variation principle, the governing equation of motion can be expressed as \cite{1}:

\[ \rho A \frac{\partial^2 w(x,t)}{\partial t^2} + EI \frac{\partial^4 w(x,t)}{\partial x^4} = \frac{EI}{2L} \left( \frac{\partial^2 w(x,t)}{\partial x^2} + d^2 w_0(x) \frac{\partial w(x,t)}{\partial x} \right) \int_0^L \left( \frac{\partial w(x,t)}{\partial x} \right)^2 + 2 \frac{dw_0(x)}{dx} \frac{\partial w(x,t)}{\partial x} dx = 0 \]  

(7)

And the corresponding boundary conditions for clamped-clamped ends:

\[ w(x = 0) = \frac{\partial w}{\partial x} (x = 0, t) = 0 \quad ; \quad w(x = L) = \frac{\partial w}{\partial x} (x = L, t) = 0 \]

3. **Free vibration of a shallow arch (Linear case)**

For convenience, the following non-dimensional variables are used:

\[ x^* = \frac{x}{L}, \quad w^* = \frac{w}{r}, \quad w_0^* = \frac{w_0}{r}, \quad r = \sqrt{\frac{I}{A t}}, \quad t' = \sqrt{\frac{EI}{\rho A L^4}} \]  

(8)

Where, \( r \) is the radius of gyration of the cross-section, \( x^* \) is the dimensionless length, \( w^* \) is the dimensionless deflection from the initial configuration, and \( w_0^* \) is the dimensionless initial curvature of the arch.

Substituting equation (10) into equation (7) gives the dimensionless nonlinear equilibrium equation:

\[ \frac{\partial^2 w^*(x,t)}{\partial t^2} + \frac{\partial^4 w^*(x,t)}{\partial x^4} = \left( \frac{\partial^2 w^*(x,t)}{\partial x^2} + d^2 w_0^*(x) \frac{\partial w^*(x,t)}{\partial x} \right) \int_0^L \left( \frac{\partial w^*(x,t)}{\partial x} \right)^2 + 2 \frac{dw_0^*(x)}{dx} \frac{\partial w^*(x,t)}{\partial x} dx^* = 0 \]  

(9)
Figure 2. Variation of the natural frequencies of the five modes of the clamped-clamped shallow arch with non-dimensional rise, dash line for symmetrical modes, continuous line is for antisymmetric mode.

To find the linearized frequencies and modes shapes of the arch, we will linearize equation (11) and employ the separation of variables by assuming a time-harmonic solution equation (12). Thus, we let:

\[ w(x,t) = w(x)e^{i\omega t} \]  

we will have five homogeneous algebraic equations that define an eigenvalue problem for the mode shapes \( w_i(x) \) and the natural frequencies \( \omega_i \).

In figure 2 we represented a variation of the five first fundamentals of a clamped-clamped shallow arch with initial curvature. As we see in this figure, the frequencies of antisymmetric modes shape varying in the function of the initial curvature contrary to the frequencies of symmetric modes shapes which stayed constant along the axis of initial curvature, and this phenomenon is mentioned in [3].

4. Non-linear formulation

To develop the non-linear theory, the transverse displacement function is expanded as a series of \( N \) basic spatial functions:

\[ w(x,t) = q_j(t)w_j(x) = a_jw_j(x) \]

Using a generalized parameterization and the usual summation convention defined in [8] and [9], the kinetic energy \( T \) and potential axial \( V \) of the shallow arch can be expressed as:

\[ T = \frac{1}{2} a_j a_j m_{ij} \omega^2 \sin^2(\omega t) \]

\[ V = \frac{1}{2} a_i a_j k_{ij} \cos^2(\omega t) + \frac{1}{2} a_i a_j a_k b_{ijkl} \cos^3(\omega t) + \frac{1}{2} a_i a_j a_k b_{ijkl} \cos^4(\omega t) \]

With:

\[ m_{ij} = \rho A \int_0^L w_j w_i dx \]
\[ k_{i,j} = EA \int_0^L w_i' w_j' dx + \frac{EA}{L} \int_0^L w_0 w_i' dx \int_0^L w_j' dx \]  
\[ b_{i,j,k} = \frac{EA}{L} \int_0^L w_i w_j' dx \int_0^L w_k' dx \]  
\[ b_{i,j,k,l} = \frac{EA}{L} \int_0^L w_i w_j' dx \int_0^L w_k w_l' dx \]  

Where, \( m_{ij}, k_{ij}, b_{ijk}, b_{ijkl} \) are respectively, the mass tensor, linear, cubic nonlinear, and nonlinear quadratic rigidity of the shallow arch.

Now we put this following non-dimensional parameter to obtain these expressions as written below:

\[ \frac{m_{ij}}{m_{ij}^*} = \frac{\rho A r^2}{L}, \quad \frac{k_{ij}}{k_{ij}^*} = \frac{EA}{L^3} r^4, \quad \frac{k_{ijk}}{k_{ijk}^*} = \frac{EA}{L^3} r^4, \quad \frac{k_{ijkl}}{k_{ijkl}^*} = \frac{EA}{L^3} r^4 \]  

In which \( m_{ij}^*, k_{ij}^*, b_{ijk}^* \) and \( b_{ijkl}^* \) are non-dimensional tensors given by:

\[ m_{ij}^* = \int_0^1 w_i^* w_j^* dx^* \]  
\[ k_{ij}^* = \int_0^1 w_i^* w_j^* dx^* + \int_0^1 w_i^* w_j^* dx^* \int_0^1 w_0^* w_j^* dx^* \]  
\[ b_{ijk}^* = \int_0^1 w_i^* w_j^* dx^* \int_0^1 w_k^* w_j^* dx^* \]  
\[ b_{ijkl}^* = \int_0^1 w_i^* w_j^* dx^* \int_0^1 w_k^* w_l^* dx^* \]

The dynamic behavior of the structure is governed by Hamilton's principle, which is symbolically written in equation (5). After replacing \( V \) and \( T \) by their discretized expressions in Equations, the time functions have to be integrated. The range of integration was chosen equal to \( 0, \frac{2\pi}{\omega} \) which corresponds to a quarter of a period of motion.

After calculations, we obtained the following non-linear system:

\[ 2[K] \{ A \} + 3[B(\{ A \})] \{ A \} - 2\omega^2 [M] \{ A \} = 0 \]  

Where \( [K], [M] \) and \( B(A) \) are the classical linear rigidity, mass matrices, and nonlinear rigidity, respectively.

5. Numerical results and discussion

In the problem herein, we consider a shallow arch clamped at both ends, with initial curvature \( q \).
Figure 3. The comparison between nonlinear frequency of shallow arch when the non-dimensional rise equal $q=4$ and 8 according to the first mode.

In figure 3 the nonlinear frequency ratio versus the non-dimensional vibration amplitude curves is plotted for the first mode shape with different values of initial curvature $q$. The normalized non-linear amplitude vibration and curvature are presented in Figures 4 and 5.

Figure 4. The first normalized non-linear mode shape for $q=4$. 
The linear and non-linear curvatures were corresponding to the first mode shape for \( q=4 \).

6. Conclusion

Foremost, the geometrical nonlinearity in transverse free vibration of the shallow arch with clamped ends based on the Euler-Bernoulli theory and Von Karman assumptions has been studied. The initial curvature of the shallow arch has been considered as a parameter of control to see its influence on nonlinear frequencies. The corresponding backbones curves in the first and second mode shape have been illustrated and analyzed. The results show that the frequency ratio of antisymmetric modes shapes decreases when the initial value of initial curvature increases; these results are since when the initial curvature is increased, the flexural rigidity of the arch increases.

References

[1] Zhuangpeng Yi · Ilinca Stanciulescu 2015 Nonlinear normal modes of a shallow arch with elastic constraints for two-to-one internal resonances Journal Nonlinear Dynamics
[2] Raymond H. Flaut and Niels Olhoff 2020 Optimal Forms of Shallow Arches with Respect to Vibration and Stability Journal of Structural Mechanics vol 11, No 1.
[3] A.H. Nayfeh, W. Kreider and T.J. Anderson 1995 Investigation of natural frequencies and mode shapes of buckled beams AIAA Journal Vol. 33, pp. 1121-1126.
[4] W. Lacarbonara and G. Rega 2003 Resonant non-linear normal modes. Part II: activation/orthogonality conditions for shallow structural systems Int. J. Non-Linear Mech., vol. 38, no. 6, pp. 873–887.
[5] D. J. Dawe The transverse vibration of shallow arches using the displacement method 1971 Int. J. Mech. Sci., vol. 13, no. 8, pp. 713–720.
[6] G. C. Tsiatas and N. G. Babouskos 2017 Linear and geometrically nonlinear analysis of non-uniform shallow arches under a central concentrated force Int. J. Non-Linear Mech., vol. 92, pp. 92–101
[7] S.-Q. Ye, X.-Y. Mao, H. Ding, J.-C. Ji, and L.-Q. Chen 2020 Nonlinear vibrations of a slightly curved beam with nonlinear boundary conditions Int. J. Mech. Sci., vol. 168, p. 105294.
[8] M. El kadiri, R. Benamar, and R. G. White 2002 IMPROVEMENT OF THE SEMI-ANALYTICAL METHOD, FOR DETERMINING THE GEOMETRICALLY NON-LINEAR RESPONSE OF THIN STRAIGHT STRUCTURES. PART I: APPLICATION TO CLAMPED–CLAMPED AND SIMPLY SUPPORTED–CLAMPED BEAMS J. Sound Vib, vol. 249, no. 2, pp. 263–305,
[9] R. Benamar, M. M. K. Bennouna, and R. G. White 1991 The effects of large vibration amplitudes on the mode shapes and natural frequencies of thin elastic structures part I. Simply supported and clamped-clamped beams J. Sound Vib, vol. 149, no. 2, pp. 179–195,