Ergodic properties of Fractional Stochastic Burgers Equation

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Abstract

We prove the existence and uniqueness of invariant measures for the fractional stochastic Burgers equation (FSBE) driven by fractional power of the Laplacian and space-time white noise. We show also that the transition measures of the solution converge to the invariant measure in the norm of total variation. To this end we show first two results which are of independent interest: that the semigroup corresponding to the solution of the FSBE is strong Feller and irreducible.

Key words: stochastic fractional Burgers equation, ergodic properties, invariant measure, strong feller property, strongly mixing property, cylindrical Wiener noise.

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1 Introduction

Dynamics of many complex phenomena is driven by nonlocal interactions. Such problems are usually modeled by means of evolution equations including fractional powers of differential operators or more general pseudo-differential operators. For example, such equations arise in the theory of the quasi-geostrophic flows, the fast rotating fluids, the dynamic of the frontogenesis in meteorology, the diffusions in fractal or disordered medium, the pollution problems, the mathematical finance and the transport problems, see for a short

The frontogenesis is the terminology used by atmosphere scientists for describing the formation in finite time of a discontinuous temperature front.
Kakutani and Sugimoto studied the wave propagation in complex solids, especially, viscoelastic materials (for example Polymers). They proved that the viscoelasticity affects the behavior of the wave. In particular, they showed that the relaxation function has the form $k(t) = ct^{-\nu}$, $0 < \nu < 1$, $c \in \mathbb{R}$, instead of the exponential form known in the standard models (Maxwell-Voigt and Voigt). This polynomial relaxation, called slow relaxation, is due to the non uniformity of the material. The main reason of this non uniformity is the accumulation of several relaxations in different scales. The far field is then described by a Burgers equation with the leading operator $(-\Delta)^{\frac{1+\nu}{2}}$ instead of the Laplacian:

$$\partial_t u = -(-\Delta)^{\frac{1+\nu}{2}} u + \partial_x u^2.$$  

The above equation also describes the far-field evolution of acoustic waves propagating in a gas-filled tube with a boundary layer and has been used to study the acoustic waves in tunnels during the passage of the trains. Indeed, in the last case the geometrical configurations can yield a memory effect and other types of resonance phenomena.

Frequently, lack of information about properties of the system makes it natural to introduce stochastic models. Moreover, the stochastic models are also powerful tools in the study of stability of deterministic systems under small perturbations. The stochastic character is visible when the initial data is random and/or when the coefficients are random.

Stochastic Partial Differential Equations play an essential role in the mathematical modeling of many physical phenomena. These equations are not only generalizations of the deterministic cases, but they lead to new and important phenomena as well. E.g. Crauel and Flandoli in [13] showed that the deterministic pitchfork bifurcation disappears as soon as an additive white noise of arbitrarily small intensity is incorporated the model. In [26], Hairer and Mattingly characterized the class of noises for which the 2 dimensional stochastic Navier-Stokes equation is ergodic. In recent series of papers and lectures, Flandoli and his Co-authors proved that for several examples of deterministic partial differential equations which are illposedness a suitable random noise can restore the illposedness see e.g. [3, 14, 22, 23].

In particular, the stochastic Burgers equation (briefly SBE) emerges in the modeling of many phenomena, such as in hydrodynamics, in cosmology and in turbulence see for short list e.g. [2, 16, 32] and the references therein. While the physical models leading to Burgers equation are simple, the mathematical study is quite difficult and complex. The nonlinear term in this equation comes from kinematical considerations, hence it cannot be replaced by some simplifications or modifications. Let us also denote that recently some other generalizations of stochastic Burgers equation have also been investigated see e.g. [37, 44].
The aim of this paper is to study the ergodic properties of the solution of the fractional stochastic Burgers equation. The ergodic properties of several stochastic partial differential equations have been extensively studied, e.g. [15, 17, 18, 20, 24, 25, 26, 36, 39]. These equations do not recover the FSBE. More precisely, we are interested in the fractional stochastic Burgers equation (FSBE) given by:

$$\frac{\partial u}{\partial t} = -(\Delta)^{\frac{\alpha}{2}} u + \partial_x u^2 + g(u) \partial_{tx}^2 W_{tx}, \quad t > 0, \ x \in (0,1),$$  \hspace{1cm} (1.1)

with the boundary conditions,

$$u(t,0) = u(t,1) = 0, \quad t > 0,$$ \hspace{1cm} (1.2)

and the initial condition

$$u(0,x) = u_0(x), \quad x \in [0,1]$$ \hspace{1cm} (1.3)

where $u_0 \in L^2(0,1)$. We will denote by $A$ the negative Dirichlet Laplacian in the space $H = L^2(0,1)$, that is

$$A = -\Delta, \quad D(A) = H^{2,2}(0,1) \cap H^1_0(0,1).$$

Then the fractional power $(-\Delta)^{\frac{\alpha}{2}}$ is defined as a fractional power of the operator $A$ (see [35, pp. 72-73]):

$$(-\Delta)^{\frac{\alpha}{2}} := A_{\alpha} u := \frac{\sin \frac{\alpha \pi}{2}}{\pi} \int_0^{\infty} t^{\frac{\alpha}{2}-1} A(tI + A)^{-1} dt.$$ \hspace{1cm} (1.4)

The diffusion coefficient $g$ is a bounded and Lipschitz continuous map from $\mathbb{R}$ to $\mathbb{R}$ and $\partial_{tx}^2 W_{tx}$ stands for the space-time white noise i.e. $\partial_{tx}^2 W_{tx}$ is the distributional derivative of a mean zero Gaussian field $\{W(t,x), t \geq 0, x \in [0,1]\}$ defined on a filtered probability space $(\Omega, F, \mathcal{F}, \mathbb{P})$, where $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$, such that the covariance function is given by:

$$\mathbb{E}[W(t,x)W(s,y)] = (t \wedge s)x \wedge y, \quad t, s \geq 0, \ x, y \in [0,1].$$ \hspace{1cm} (1.5)

It is now customary, see for instance [19] (and [6]), to rewrite the problem (1.1-1.4) as a stochastic evolution problem in the Hilbert space $H = L^2(0,1)$ in the following way

$$\begin{cases} 
du(t) = (-A_{\alpha} u(t) + Bu^2(t))dt + g(u(t)) \ dW_t, \ t > 0, \\
u(0) = u_0 \in L^2(0,1).
\end{cases}$$ \hspace{1cm} (1.6)

Here $B := \frac{\partial}{\partial x}$ and $W := \{W(t), t \geq 0\}$ is an $H$-cylindrical Wiener process given by (with $(e_j)_{j=1}^{\infty}$ is an ONB of $H$),

$$W(t) := \sum_{j=1}^{\infty} \beta_j(t)e_j,$$ \hspace{1cm} (1.7)
where \((\beta_j)_{j=1}^{\infty}\) is a sequence of independent real Brownian motions.

Let us recall the definition of a solution we will use throughout the whole paper.

**Definition 1.** Suppose that \(1 < \alpha \leq 2\). An \(\mathbb{F}\)-adapted \(L^2(0,1)\)-valued continuous process \(u = (u(t), t \geq 0)\) is said to be a mild solution of equation (1.6) if for some \(p > \frac{2\alpha}{\alpha-1}\)

\[
\mathbb{E} \sup_{t \in [0,T]} |u(t)|_{L^2}^p < \infty, \quad T > 0,
\]

(1.8)

the function \((0, t) \ni s \rightarrow S_\alpha(t-s)Bu^2(s) \in L^2(0,1)\) is Bochner integrable and for every \(t \geq 0\), the following identity holds \(\mathbb{P}\)-a.s. in \(L^2(0,1)\)

\[
u(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(t-s)Bu^2(s) \, ds + \int_0^t S_\alpha(t-s)g(u(s)) \, dW(s).
\]

(1.9)

For the reader’s convenience we recall the existence and uniqueness result for the case \(u_0 \in L^2\). For the general case and proof see [6].

**Theorem 1.** Assume that \(g : \mathbb{R} \to \mathbb{R}\) is a bounded and Lipschitz continuous function and that \(\frac{3}{2} < \alpha < 2\). Then for every \(u_0 \in L^2(0,1)\) there exists a unique mild solution \(u\) of equation (1.6).

In what follows the initial data \(u_0 \in L^2(0,1)\) will usually be denoted by \(x\) and the unique mild solution \(u\) of equation (1.6) whose existence is guaranteed by Theorem 1 will be denote by \(u(t, x), t \geq 0\).

Using standard arguments, see for example [19], we can prove that the solution of equation (1.6) is an \(L^2\)-valued strong-Markov and Feller process. Let us define, the transition semigroup corresponding to the FSBE (1.6), i.e. a family \(U_t = (U_t)_{t \geq 0}\) of bounded linear operators on the space of bounded Borel functions \(B_b(L^2)\) by

\[
(U_t\varphi)(x) = \mathbb{E}[\varphi(u(t, x))], \quad x \in L^2(0,1), \quad t \geq 0.
\]

(1.10)

Note that \(U\) is a semigroup (although not strongly continuous) on \(B_b(L^2(0,1))\).

Let

\[
\mu_t(x, B) = \mathbb{P}(u(t, x) \in B), \quad t \geq 0, x \in L^2(0,1), B \subset L^2(0,1), \text{ Borel},
\]

be the family of transition measures of the process \(u\). Then, clearly

\[
U_t\phi(x) = \int_{L^2} \phi(y)\mu_t(x, dy), \quad \phi \in \mathcal{B}(L^2(0,1)).
\]

The main aim of our article is to study the ergodic properties of the FSBE (1.6), or equivalently, the ergodic properties of the semigroup \(U\). Our main result is the following one.
Theorem 2. (a) Assume that $g : \mathbb{R} \to \mathbb{R}$ is a bounded and Lipschitz continuous function. Suppose that $\frac{3}{2} < \alpha < 2$. Then there exists an invariant measure $\mu$ for the transition semigroup $U$ corresponding to the fractional stochastic Burgers equation (1.6).

(b) Assume additionally that

$$\inf_{x \in \mathbb{R}} |g(x)| > 0.$$ 

Then the invariant measure $\mu$ is unique and for every $x \in L^2(0,1)$

$$\lim_{t \to \infty} \|\mu_t(x, \cdot) - \mu(\cdot)\|_{\text{TV}} = 0,$$  \hspace{1cm} (1.11)

where $\| \cdot \|_{\text{TV}}$ denotes the total variation norm.

We note that (1.11) trivially yields the strong mixing property, that is,

$$\lim_{t \to \infty} \int_{L^2} \left| U_t\phi - \int_{L^2} \phi d\mu \right|^2 d\mu = 0,$$  \hspace{1cm} (1.12)

for every $\phi : L^2(0,1) \to \mathbb{R}$ such that

$$\int_{L^2} |\phi|^2 d\mu < \infty.$$

The paper is organized as follows. Section 2 contains some auxiliary facts and basic estimates that will be needed later. In the third and fourth sections, we will study the regularity of the semigroup corresponding to the solution of the FSBE. In particular, we will prove the strong Feller property and the irreducibility in sections three and four respectively. In the last section we will prove the existence of an invariant measure and, under stronger assumptions, its uniqueness and the strong mixing property (1.12).

2 Preliminaries and a priori estimates

Let $u = (u(t), t \geq 0)$ be a given process and let $\gamma \in \mathbb{R}_+$. We introduce the following stochastic process $z_\gamma = (z_\gamma(t), t \geq 0)$ defined by

$$z_\gamma(t) := \int_0^t e^{-(t-s)(A_\alpha + \gamma)} g(u(s)) \, dW(s), \hspace{1cm} t \geq 0.$$  \hspace{1cm} (2.13)

The process $z_\gamma$ satisfies the following stochastic differential equation

$$\begin{cases}
  d z_\gamma(t) = -(A_\alpha + \gamma) z_\gamma(t) dt + g(u(t)) \, dW(t), & t \geq 0, \\
  z_\gamma(0) = 0.
\end{cases}$$
Let us remark that if a process $u$ is a solution of (1.6) then the stochastic process $v_\gamma := u - z_\gamma$ satisfies pathwise the following equation

$$
\begin{align*}
\left\{ \begin{array}{l}
\frac{d}{dt}v_\gamma(t) = -A_\alpha v_\gamma(t) + B(v_\gamma(t) + z_\gamma(t))^2 + \gamma z_\gamma(t), \quad t > 0, \\
v(0) = u_0.
\end{array} \right.
\end{align*}
$$

(2.14)

The following two results are the main tools allowing us to study the long time behaviour of the norm $|v_\gamma(t)|_{L^2}^2$. For $s \in \mathbb{R}_+ \setminus \mathbb{N}$ we will denote by $H^{s,p}$ the fractional order Sobolev space (called also Bessel potential spaces) defined by the complex interpolation method, i.e.

$$
H^{s,p}(0, 1) = [H^{k,p}(0, 1), H^{m,p}(0, 1)]_\vartheta,
$$

(2.15)

where $k, m \in \mathbb{N}$, $\vartheta \in (0, 1)$, $k < m$, are chosen to satisfy

$$
s = (1 - \vartheta)k + \vartheta m.
$$

(2.16)

These spaces coincide with the Sobolev spaces $W^{s,p}(0, 1)$ for integer values of $s$ if $1 < p < \infty$ and for all $s$ when $p = 2$ see e.g. [1, p 219], [38] and [43, p 310]. Moreover, under condition (2.16), we have, see e.g. [43, p 103, p 196 & p 336]

$$
H^{s,p}(0, 1) = D((-\Delta)^{\frac{s}{2}}) = [L^p(0, 1), H^{m,p}(0, 1)]_\vartheta.
$$

(2.17)

In what follows by $H^{s,p}_0(0, 1)$, $s \geq 0$, $p \in (1, \infty)$, we will denote the closure of $C_0^\infty(0, 1)$ in the Banach space $H^{s,p}(0, 1)$. It is well known, see e.g. [30, Theorem 11.1] and [43, Theorem 1.4.3.2, p.317] that $H^{s,p}_0(0, 1) = H^{s,p}(0, 1)$ iff $s \leq \frac{1}{p}$. The norms in various Sobolev spaces $H^{s,p}(0, 1)$ will be denoted by $|\cdot|_{H^{s,p}}$. Similarly, the norm in the $L^p(0, 1)$ space will be denoted by $|\cdot|_{L^p}$.

**Lemma 3.** [6, Lemma 3.4] There exist $\nu_1 > 0$, $q > 2$, $s \in \left(\frac{1}{q}, \frac{1}{2}\right)$ and $C > 0$, such that

$$
\left[\frac{d}{dt}|v_\gamma(t)|_{L^2}^2\right]^{\frac{1}{2}} \leq \frac{\nu_1}{2}|v_\gamma(t)|_{H_0^{\frac{1}{2}}-2}^2 + C|z_\gamma(t)|_{H^{\frac{1}{2}}-q}^{\frac{\nu}{2}}|v_\gamma(t)|_{L^2}^{\frac{2}{\nu}} + C|z_\gamma(t)|_{H^{1-\frac{1}{2}-q}}^4 + C|z_\gamma(t)|_{L^2}^2, \quad t \geq 0.
$$

(2.18)

**Proof.** The proof is similar to the proof of [6, Proposition 3.3].

**Lemma 4.** Let $(v_\gamma(t))_{t \geq 0}$ be the solution of the equation (2.14). Then there exist $\nu \geq 0$, $C > 0$, $s \in \left(\frac{1}{q}, \frac{1}{2}\right)$, $q \in (2, \infty)$ and $s_0 \in \left(\frac{1}{2}, \frac{s}{2}\right)$, such that

$$
\left[\frac{d}{dt}|v_\gamma(t)|_{L^2}^2\right]^{\frac{1}{2}} \geq \nu|v_\gamma(t)|_{H_0^{\frac{1}{2}}-2}^2 + C|z_\gamma(t)|_{H^{1-\frac{1}{2}-q}}^4 + C|z_\gamma(t)|_{L^2}^2, \quad t \geq 0.
$$

(2.19)
Lemma 4 is an improved version of Lemma 3. The proofs are only slightly different from those in [6]. They are based on the Sobolev embedding theorems [38, Theorem 1, Tr 6, 3.3.1 section 2.4.4, p 82 & Proposition Tr 6, 2.3.5 section 2.1.2, p 14] as well as the following one which we recall for the readers convenience.

**Theorem 5.** [38, Theorem 2, section 4.4.4, p.177 & Proposition Tr 6, 2.3.5 section 2.1.2, p.14] Assume that

\[
0 < s_1 \leq s_2, \quad \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}, \quad s_1 + s_2 > \frac{n}{p_1} + \frac{n}{p_2} - n
\]

and if \(s_1 < s_2\), \(p \geq p_1\). Assume also that

\[
\frac{n}{p} - s_1 = \begin{cases} 
\left( \frac{n}{p_1} - s_1 \right)_+ + \left( \frac{n}{p_2} - s_2 \right)_+, & \text{if } \max_i \left( \frac{n}{p_i} - s_i \right) > 0, \\
\max_i \left( \frac{n}{p_i} - s_i \right) > 0, & \text{otherwise}
\end{cases}
\]  

(2.20)

Then, provided that \(\{i \in \{1, 2\} : s_i = \frac{n}{p_i} \text{ and } p_i > 1\} = \emptyset\),

\[
H^{s_1,p_1}(0,1) \cdot H^{s_2,p_2}(0,1) \hookrightarrow H^{s_1,p}(0,1).
\]

**Proposition 1.** [6, Proposition 3.3] Assume that $$\beta \in \left(\frac{1}{2}, 1\right)$$. Then there exists a constant \(C > 0\) such that for each \(u \in H_0^{\beta,2}(0,1)\) and each \(v \in H^{1-\beta,2}(0,1)\) the following inequality is satisfied

\[
\left| \int_0^1 u(x) Dv(x) \, dx \right| \leq C |u|_{H_0^{\beta,2}} |v|_{H^{1-\beta,2}}. 
\]  

(2.21)

The following result is a generalization of [18, Lemma 5.1]. In order to formulate and prove it we need to recall some basic facts about the Itô integration in martingale type-2 Banach spaces. For more information, see e.g. [7, 5, 9, 45]. For brevity reason and to focus on the stochastic integral required in this paper, we restrict ourselves to the case when the separable Banach space \(E\) is \(L^q(O, \mathcal{O}, \nu) = L^q(O)\), where \((O, \mathcal{O}, \nu)\) is a \(\sigma\)-finite measure space. In this case and for \(2 \leq q < \infty\) the space \(E = L^q(O)\) is both a UMD and type-2 (and hence, see [5] a martingale type 2). The construction of an Itô integral for such spaces is described in the paper [5] by the first named author.

**Definition 2.** Let \(H\) be a separable Hilbert space and let \(E\) be a separable Banach space. A bounded linear operator \(L : H \to E\), is called \(\gamma\)-radonifying operator if and only if for some (or equivalently for every) orthonormal basis \((ONB) (h_j)_{j=1}^\infty\) of \(H\) and some (equivalently every) sequence \((\gamma_j)_{j=1}^\infty\) of iid \(N(0,1)\) random variables defined on a probability space \((\Omega', \mathcal{F}', \mathbb{P}')\). we have\(^2\)

\[
\|L\|_{R_\gamma(H,E)}^2 := \mathbb{E}' \left\{ \sum_{j=1}^\infty \gamma_j L h_j \right\}_E^2 < \infty. 
\]  

(2.22)

\(^2\)Actually, in view of the Itô-Nisio Theorem, the class \(R_\gamma(H,E)\), the exponent 2 below can be replaced by any \(p \in (1, \infty)\).
The space of all $\gamma$-radonifying operators from $H$ to $E$ will be denoted by $R_\gamma(H, E)$.

Let us mention here that for certain classes of the target Banach spaces, a complete characterization of the $\gamma$-radonifying operators can be given in a purely non-probabilistic terms. For example, if $E$ is a Hilbert space, then a bounded linear operator $L \in R_\gamma(H, E)$ if and only if it is a Hilbert-Schmidt operator. In this case $\|L\|_{R_\gamma(H, E)} = \|L\|_{HS}$. If, see [2], $E = L_q(O, \mathcal{O}, \nu)$, as described above, then $L \in R_\gamma(H, E)$ if and only if there exists a function $\kappa \in L^q(O, \mathcal{O}, \nu; H) = L^q(O, H)$ such that for every $h \in H$, $(Lh)(x) = \langle \kappa(x), h \rangle$ for $\nu$-a.a. $x \in O$. In particular, see also [15, Proposition 13.7], if $1 < q < \infty$ and $(h_j)_{j \in J}$ is a fixed ONB of $H$ then $L \in R_\gamma(H, E)$ if and only if $(\sum_{j \in J} |Lh_j|^2)^{\frac{1}{2}}$ is summable in $L^q(O)$. In the former, respectively the latter case, $\|L\|_{R_\gamma(H, L^q)}$ is equivalent to $|\kappa|_{L^q(O, \mathcal{O}, \nu; H)}$, resp. $(\sum_{j \in J} |Lh_j|^2)^{\frac{1}{2}}|_{L^q}$.

Let us now recall the fundamental property of such an integral resembling the Burkholder inequality for local martingales, see [5, 21, 33]. There exists a constant $C = C_{p, q} > 0$ such that for every progressively measurable process $\Phi \in L^p(\Omega; L^2(0, T; R_\gamma(L^2, L^q)))$, where $2 \leq q < \infty$ and $1 < p < \infty$, we have

$$\mathbb{E}\left|\sup_{t \in [0, T]} \int_0^t \Phi(s) dW(s)\right|_{L^q}^p \leq C_{p, q} \mathbb{E}\left( \int_0^T \|\Phi(t)\|_{R_\gamma(L^2, L^q)}^2 dt \right)^{\frac{p}{2}}. \tag{2.23}$$

Finally, let us recall that $A$ is a selfadjoint operator with compact inverse $A^{-1}$ and that for every $\alpha \in \mathbb{R}$

$$A^{\frac{\alpha}{2}} e_j = \lambda_j^{\frac{\alpha}{2}} e_j, \quad j \geq 1,$$

where

$$e_j(\xi) = \sqrt{2} \sin \pi j \xi, \quad \lambda_j = \pi^2 j^2.$$

**Lemma 6.** For all numbers $\varepsilon > 0$, $\sigma < \frac{\alpha - 1}{2}$, $p \geq 1$ and $q > 1$ we can find a positive number $\gamma_0$ such that for any $\gamma > \gamma_0$,

$$\mathbb{E}|z_\gamma(t)|_{R_{p,q}}^p \leq \varepsilon, \quad t \geq 0. \tag{2.24}$$

**Proof.** By inequality (2.23) there exists a constant $C := C_{p,q} > 0$, such that for every $t \geq 0$,

$$\mathbb{E}\left|(-A)^{\sigma/2} z_\gamma(t)\right|_{L^q}^p \tag{2.25}$$

$$= \mathbb{E}\left|(-A)^{\sigma/2} \int_0^t e^{-(t-s)(A_\alpha + \gamma)} g(u(s)) dW(s)\right|_{L^q}^p$$

$$\leq C_{p,q} \mathbb{E}\left( \int_0^t \|(-A)^{\sigma/2} e^{-(t-s)(A_\alpha + \gamma)} g(u(s))\|_{R_\gamma(L^2, L^q)}^2 ds \right)^{\frac{p}{2}}.$$
Thanks to [9], see also [45, Proposition 13.7], there exists a positive constant $C := C_q > 0$, (we keep the same notation for all constants), such that for all $r \geq 0$

$$\|(-A)^{\frac{q}{2}} e^{-r(A_{\alpha}+\gamma)} g(u(s))\|_{R_q(L^2_{eq})} \leq C_q \left( \sum_{j=1}^{\infty} \|(-A)^{\frac{q}{2}} e^{-r(A_{\alpha}+\gamma)} g(u(s)) e_j \|^2 \right)^{\frac{1}{2}}_{L^q}. \quad (2.26)$$

By arguing as in [6] around inequality (C.15), in particular using Proposition C.1.6 and Lemma 2.4, we infer that

$$\left(\sum_{j=1}^{\infty} \|(-A)^{\frac{q}{2}} e^{-r(A_{\alpha}+\gamma)} g(u(s)) e_j \|^2 \right)^{\frac{1}{2}}_{L^q} \leq b_0 \left( \sum_{k=1}^{\infty} \lambda_k^{\frac{q}{2}} e^{-r(\lambda_k^{\frac{q}{2}} + \gamma)} \right), \quad r > 0, \quad (2.27)$$

Inserting the above inequality in (2.26) and using (2.25) and the Hölder inequality for series, we get

$$\mathbb{E} \left| (-A)^{\sigma/2} z_\gamma(t) \right|^p_{L^q} \leq C_{p,q,b_0} \left( \int_0^t \left( \sum_{k=1}^{\infty} \lambda_k^{\frac{q}{2}} e^{-s(\lambda_k^{\frac{q}{2}} + \gamma)} \right)^2 ds \right)^{\frac{p}{2}} \leq C_{p,q,b_0} \left( \int_0^t \left( \sum_{k=1}^{\infty} \lambda_k^{\sigma} e^{-s(\lambda_k^{\frac{q}{2}} + \gamma)} \right) \left( \sum_{k=1}^{\infty} e^{-s(\lambda_k^{\frac{q}{2}} + \gamma)} \right) ds \right)^{\frac{p}{2}}. \quad (2.28)$$

It is easy to see that there exist $\rho \in \left( \frac{1}{\alpha}, 1 \right)$, (which we will let later tend to $\frac{1}{\alpha}$), and $C_{\rho}$, such that

$$\left( \sum_{k=1}^{\infty} e^{-s(\lambda_k^{\frac{q}{2}} + \gamma)} \right) \leq C_{\rho} s^{-\rho}, \quad \gamma > 0.$$

By the change of variables $\xi := s(\lambda_k^{\frac{q}{2}} + \gamma)$ and letting $t$ go to infinity in the RHS of the inequality above, we infer that for every $t > 0$,

$$\mathbb{E} \left| (-A)^{\sigma/2} z_\gamma(t) \right|^p_{L^q} \leq C_{p,q,b_0,\rho} \left( \sum_{k=1}^{+\infty} \frac{\lambda_k^{\sigma}}{(\lambda_k^{\frac{q}{2}} + \gamma)^{1-\rho}} \int_0^{+\infty} \xi^{-\rho} e^{-\xi} d\xi \right)^{\frac{p}{2}}. \quad (2.29)$$

The series on the RHS of (2.29) converges provided $\sigma \in [0, \frac{\alpha-1}{2}]$. In order to get the inequality (2.24), it is sufficient to observe that in view of the Lebesgue dominated convergence theorem

$$\lim_{\gamma \to \infty} \sum_{k=1}^{+\infty} \frac{(k\pi)^{2\sigma}}{(k\pi)^{\alpha} + \gamma)^{1-\rho}} = 0.$$

This completes the proof of the Lemma.
**Proof of Lemma 4**

We note first that for every $T > 0$ and each $z_\gamma \in L^\infty([0, T], H^{s,q}(0, 1))$, $q > 2$ and where $s \in \left(\frac{\alpha}{q}, \frac{1}{2}\right)$, the deterministic problem (2.14) has a unique solution $v_\gamma \in C(0, T; L^2(0, 1)) \cap L^2(0, T; H_0^{s,q}(0, 1))$. Multiplying the both sides of (2.14) by the solution $v_\gamma(t)$ and applying Lemma III.1.2 from [42], then we get

$$
2 \frac{d}{dt} |v_\gamma(t)|^2_{L^2} = -\langle A_\alpha v_\gamma(t), v_\gamma(t) \rangle + \langle B(v_\gamma(t)^2), v_\gamma(t) \rangle
$$

(2.30)

$$
+ 2\langle B(z_\gamma(t)v_\gamma(t)), v_\gamma(t) \rangle + \langle Bz_\gamma(t)^2), v_\gamma(t) \rangle + \gamma\langle z_\gamma(t), v_\gamma(t) \rangle.
$$

In what follows, the time argument is, for simplicity of notations, omitted. It is easy to see that $\langle B(v_\gamma^2), v_\gamma \rangle = 0$ and that

$$
\gamma |\langle z_\gamma, v_\gamma \rangle | \leq \gamma^2 |z_\gamma|^2_{L^2} + \nu_1 |v_\gamma|^2_{H_0^{s,q,2}}.
$$

(2.31)

Moreover, since $\frac{\alpha}{q} \in \left(\frac{3}{8}, \frac{1}{2}\right)$ and $\lambda_1 = \pi^2$, we obtain

$$
\langle A_\alpha v_\gamma, v_\gamma \rangle = \langle A_{\beta} v_\gamma, v_\gamma \rangle = |A_{\beta} v_\gamma|^2_{L^2} \geq 2^\alpha |v_\gamma|^2_{H_0^{s,q,2}}, \ v_\gamma \in D(A_\alpha).
$$

(2.32)

Applying Proposition 11 with $\beta = \alpha/2$, we infer that

$$
|\langle B(z_\gamma v_\gamma), v_\gamma \rangle | \leq C |v_\gamma|^2_{H_0^{s,q,2}} |z_\gamma v_\gamma|_{H^{1-\frac{s}{2},2}}.
$$

By Theorem 3 then there exist $C > 0$, $q' > 1$ and $s' > \max \left\{ \frac{1}{q'}, 1 - \frac{\alpha}{2} \right\}$, such that

$$
|z_\gamma v_\gamma|_{H^{1-\frac{s}{2},2}} \leq C |z_\gamma|_{H^{1-\frac{s}{2},2}} |v_\gamma|_{H^{s',q'}}.
$$

Furthermore, using [38], Theorem 1, Tr 6, 3.3.1 section 2.4.4, p 82 & Proposition Tr 6, 2.3.5 section 2.1.2, p 14], the following embedding

$$
H^{s_0,2}(0, 1) \hookrightarrow H^{s',q'}(0, 1)
$$

holds for all $s_0 \geq s' + \frac{1}{2} - \frac{1}{q'}$ and $s_0 > s'$. Hence, there exists a constant $C > 0$, such that

$$
|v_\gamma|_{H^{s',q'}} \leq C |v_\gamma|_{H^{s_0,2}}.
$$

Now we choose $s_0 \leq \frac{\alpha}{2}$. This choice is possible thanks to the conditions $\alpha > \frac{3}{2}$ and $q' > 1$. In fact, thanks to the above conditions, the following inequalities hold

$$
\max \left\{ \frac{1}{q'}, 1 - \frac{\alpha}{2} \right\} < s' < s_0 \leq \frac{\alpha}{2} - \frac{1}{2} + \frac{1}{q'}
$$

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and
\[ \max\{ \frac{1}{q'}, 1 - \frac{\alpha}{2} \} < s' < \frac{\alpha}{2} - \frac{1}{2} + \frac{1}{q'}. \]
By interpolation, there exists a constant \( C > 0 \), such that
\[ |v_{\gamma}|_{H^{s',2}} \leq C |v_{\gamma}|_{L^2}^{1-\frac{2\alpha}{2}} |v_{\gamma}|_{H^{s',2}}^{\frac{2\alpha}{2}}, \]
and thereby
\[ |\langle B(z_{\gamma}, v_{\gamma}) \rangle| \leq \nu_2 |v_{\gamma}|_{H^{s',2}}^{2} + C |z_{\gamma}|_{H^{1-\frac{\alpha}{2}}}^\alpha |v_{\gamma}|_{L^2}^{2}. \tag{2.33} \]
Invoking the classical Young inequality we find that for some generic constant \( C > 0 \)
\[ |\langle B(z_{\gamma}^2, v_{\gamma}) \rangle| \leq v_2 |v_{\gamma}|_{H^{s',2}}^{2} + C |z_{\gamma}|_{H^{1-\frac{\alpha}{2}}}^\alpha |v_{\gamma}|_{L^2}^{2}. \tag{2.33} \]
To estimate the term \( |\langle B(z_{\gamma}^2, v_{\gamma}) \rangle| \), we follow the same calculation from [6]. Applying again Proposition 1 with \( \beta = \alpha/2 \) we infer that
\[ |\langle B(z_{\gamma}^2, v_{\gamma}) \rangle| \leq C |v_{\gamma}|_{H^{s',2}}^{2} |z_{\gamma}|_{H^{1-\frac{\alpha}{2}}}^2. \] Arguing as before with the choice \( \frac{1}{q} < s < \frac{1}{2}, \ q > 2 \), we get
\[ |\langle B(z_{\gamma}^2, v_{\gamma}) \rangle| \leq v_3 |v_{\gamma}|_{H^{s',2}}^{2} + C |z_{\gamma}|_{H^{s,q}}^4 + C |z_{\gamma}|_{H^{1-\frac{\alpha}{2}}}^4. \tag{2.34} \]
Combining equality (2.30), inequalities (2.31) and (2.32) with inequalities (2.33) and (2.34) and choose \( \nu_1, \nu_2 \) and \( \nu_3 \), such that \( -\frac{\pi^{2\alpha}}{2} + \frac{\nu_1}{2} + \nu_2 + \frac{\nu_3}{2} = -\nu \), where \( \nu > 0 \), we get (2.19).

3 Strong Feller Property

We recall first the Feller property of the transition semigroup associated to the solution of equation (1.6).

Proposition 2. The transition semigroup \( U \) corresponding to the fractional stochastic Burgers equation (1.6) is a Markov Feller semigroup. i.e.
\[ U_t \phi \in C_b \left( L^2(0, 1) \right), \quad \phi \in C_b \left( L^2(0, 1) \right). \tag{3.35} \]
Proof. Omitted. \( \square \)
In order to show the uniqueness of the invariant measure we will need stronger properties of the transition semigroup. Below, for the reader’s convenience we recall the concept of the strong Feller property and topological irreducibility.

**Definition 3.** A Markov semigroup $U = (U_t)_{t \geq 0}$ on a Polish space $X$ is said to be (topologically) irreducible if for any $t > 0$,

$$U_t 1_\Gamma(x) > 0, \; x \in X,$$

for arbitrary non empty open set $\Gamma \subset X$. It is said to have the strong Feller property if for any $t > 0$

$$U_t \phi \in C_b(X), \; \phi \in B_b(X),$$

where $B_b(X)$ denotes the space of bounded Borel functions on $X$.

In order to prove the strong Feller property of the Markov semigroup $U$ corresponding to the fractional stochastic Burgers equation (1.6) we introduce a family of truncated systems. As in [6], for each natural number $n$ we define a map $\pi_n : L^2(0,1) \to L^2(0,1)$ by

$$\pi_n(v) = \begin{cases} v & \text{if } |v|_{L^2} \leq n, \\ \frac{v}{|v|_{L^2}} & \text{if } |v|_{L^2} > n. \end{cases} \quad v \in L^2(0,1)$$

Then for each $n \geq 1$ we can consider an equation

$$\begin{cases} du_n(t) = (-A_\alpha u_n(t) + B (\pi_n (u_n(t)))^2) \, dt + g (u_n(t)) \, dW(t), \\ u_n(0) = u_0, \end{cases} \quad (3.38)$$

By [6], for each $u_0 = x \in L^2(0,1)$, equation (3.38) admits a unique mild solution $(u_n(t,x), t \geq 0)$. Let $U^n = (U^n_t)_{t \geq 0}$ be transition semigroup associated to the solution of equation (3.38), i.e. defined by

$$(U^n_t \varphi)(x) = \mathbb{E} [\varphi(u_n(t,x))], \; \varphi \in B_b(L^2(0,1)), \; x \in L^2(0,1).$$

**Lemma 7.** for every $n \geq 1$ the semigroup $U^n$ is strongly Feller on $H = L^2(0,1)$. More precisely, for every $R > 0$, $t > 0$ there exists $C(R, t) > 0$ such that for any $\phi \in B_b(H)$

$$|(U^n_t \phi)(x) - (U^n_t \phi)(y)| \leq C(R, t)|x - y|_{L^2}, \quad |x|_{L^2} \leq R, \; |y|_{L^2} \leq R.$$

**Proof.** The proof is a straightforward modification of the proof of analogous statement in [18], hence omitted. \qed
Theorem 8. Assume that $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that there exist $a_0, b_0 > 0$ such that

$$|g(x)| \in [a_0, b_0], \ x \in \mathbb{R}. \quad (3.40)$$

Then the Markov semigroup $U$ corresponding to the fractional stochastic Burgers equation (1.6) is strong Feller.

Proof. For $x \in L^2(0, 1)$, let $(u(t, x), t \geq 0)$, respectively $(u_n(t, x), t \geq 0)$, be the unique mild solution of equation (1.6), respectively (3.38), with the initial condition $u(0, x) = x$. For $n \in \mathbb{N}$ and $x \in L^2(0, 1)$ let $\tau_n(x)$ be an $\mathbb{F}$-stopping time defined by

$$\tau_n(x) := \inf\{t \geq 0 : |u(t, x)|_{L^2} \geq n\}.$$ 

Let us fix $t > 0$. It follows from [6, inequality (3.18)] that one can find a constant $C(t) > 0$ and a sequence $(q_n)_{n=1}^{\infty}$, independent of $t$, such that

$$q_k \not\to \infty$$

and

$$\mathbb{P}(\tau_n(x) < t) \leq \frac{C(t)}{q_n}(1 + \ln^+ |x|_{L^2}), \ n \in \mathbb{N}. \quad (3.41)$$

On the other hand, due to the uniqueness property of the solution $u$ proved in [6, section 3], $u(t, x) = u_n(t, x)$ for $t < \tau_n(x)$. Therefore, for all $x \in L^2(0, 1)$ and $\phi \in \mathcal{B}_b(L^2(0, 1))$

$$|(U_t\phi)(x) - (U^n_t\phi)(x)| = |\mathbb{E}\phi(u(t, x)) - \mathbb{E}\phi(u_n(t, x))|$$

$$= |\mathbb{E}[\phi(u(t, x)) - \phi(u_n(t, x))]1_{\tau_n(x) < t}|$$

$$\leq 2|\phi|_{L^\infty}\mathbb{P}(\tau_n(x) < t). \quad (3.42)$$

Hence, in view of inequality (3.41), we infer that $U^n_t\phi$ converges to $U_t\phi$ uniformly on balls in $L^2(0, 1)$. Hence, by applying Lemma 7, the proof of Theorem 8 is complete. \hfill \square

4 Irreducibility

The main result of this section is as follows.

Theorem 9. Assume that $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that condition (3.40) holds. Then the Markov semigroup $U$ corresponding to the fractional stochastic Burgers equation (1.6) is irreducible.

Proof. For $x \in L^2(0, 1)$, let $(u(t, x), t \geq 0)$, respectively $(u_n(t, x), t \geq 0)$, be the unique mild solution of equation (1.9), respectively (3.38), with the initial condition $u(0, x) = x$. Let us fix $\varepsilon > 0$, $T > 0$, $t \in [0, T]$ and $x, y \in L^2(0, 1)$. To prove the topological irreducibility, it is enough to show that

$$\mathbb{P}(u(t, x) \in B(y, \varepsilon)) > 0. \quad (4.43)$$
Let us first observe that because $D(A_\alpha)$ is a dense subspace of $L^2(0,1)$, it is enough to prove (4.43) for $y \in D(A_\alpha)$. In what follows we make this additional assumption.

For $n \geq 1$ and $0 < \tau < t$, let us define an $\mathbb{F}$-progressively measurable process

$$f^{\tau,n}(s, y) := \begin{cases} \frac{1}{t-\tau}e^{-(s-\tau)A_\alpha}(y - u(\tau, x)) - A_\alpha y1_{B(0,n)}(u(\tau, x)), & \text{if } s > \tau, \\ 0, & \text{if } s \leq \tau. \end{cases}$$

(4.44)

Later on we will choose $\tau$ to be sufficiently close to $t$. Let us consider the following stochastic evolution equation:

$$\begin{cases} du^{\tau,n}(s) = (-A_\alpha u^{\tau,n}(s) + \frac{\partial}{\partial x}(u^{\tau,n}(s))^2 + f^{\tau,n}(s, y))ds + g(u^{\tau,n}(s))dW(s), \\ u^{\tau,n}(0) = x, \end{cases}$$

(4.45)

Repeating the argument from [6] we can prove that the problem (4.45) has a unique mild solution $(u^{\tau,n}(s, x)), s \geq 0$, such that

$$M := \mathbb{E}\sup_{s \in [0,T]} |u^{\tau,n}(s)|^2_{L^2} < \infty.$$  

(4.46)

Next we define an $\mathbb{F}$-progressively measurable process $(\beta^{\tau,n}(s))_{s \geq 0}$ by

$$\beta^{\tau,n}(s) := (g(u^{\tau,n}(s, x)))^{-1}f^{\tau,n}(s, y),$$

where for $u \in L^2(0,1)$, $g(u)$ is the multiplication operator on $L^2(0,1)$. Since the range of $|g|$ is a subset of $[a_0, b_0]$, $g(u)$ is a linear isomorphism of $L^2(0,1)$. By employing the Girsanov Theorem we will prove that the laws of the processes $u(\cdot, x)$ and $u^{\tau,n}(\cdot, x)$ are equivalent on $C([0,T], L^2(0,1))$. Indeed, since

$$\mathbb{E}e^{\int_0^T |\beta^{\tau,n}(s)|^2_{L^2} ds} < \infty,$$  

(4.47)

[19] Lemmata 10.14 and 10.15, the process $\hat{W}$ defined by

$$\hat{W}(t) := W(t) + \int_0^t \beta^{\tau,n}(s)ds, \quad t \in [0,T]$$

(4.48)

is an $L^2$-cylindrical Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}_T, \mathbb{P}_T)$, where $\mathbb{F}_T := \{\mathcal{F}_t\}_{t \in [0,T]}$ and the probability measure $\mathbb{P}_T$ is equivalent to $\mathbb{P}$ with the Radon-Nikodym derivative $\frac{d\mathbb{P}_T}{d\mathbb{P}}$ equal to

$$Z_T := \exp\left(\int_0^T \langle \beta^{\tau,n}(s), dW(s) \rangle - \frac{1}{2} \int_0^T |\beta^{\tau,n}(s)|^2_{L^2} ds \right).$$  

(4.49)
Therefore, using standard arguments we find that the process $u^\tau,n(\cdot, x)$ solves the equation on $[0, T]$,

$$
\begin{align*}
du^\tau,n(s, x) &= (-A_\alpha u^\tau,n(s, x) + \frac{\partial}{\partial x}(u^\tau,n(s, x))^2 + f^\tau,n(s, y)) ds \\
&\quad + g(u^\tau,n(s, x))d\dot{W}(s) - g(u^\tau,n(s, x))(g(u^\tau,n(s, x)))^{-1}[f^\tau,n(s, y)] ds \\
&\quad + \left(-A_\alpha u^\tau,n(s, x) + \frac{\partial}{\partial x}(u^\tau,n(s, x))^2 ds + g(u^\tau,n(s, x))dW(s) \right) \\
&\quad - \left[ 1 - \left( -A_\alpha u^\tau,n(s, x) + \frac{\partial}{\partial x}(u^\tau,n(s, x))^2 ds + g(u^\tau,n(s, x))dW(s) \right) \right].
\end{align*}
$$

It follows from the uniqueness in law of the solution of the (1.6) that the laws on $C([0, T], L^2(0, 1))$ of $u(\cdot, x)$ under $P$ and of $u^\tau,n(\cdot, x)$ under $P^*_T$ are the same. Hence,

$$
\mathbb{P}(u(t, x) \in B(y, \varepsilon)) = \mathbb{P}^*_T(u^\tau,n(t, x) \in B(y, \varepsilon)). \quad (4.50)
$$

Therefore, in order to prove that (4.43) holds, it is enough to show that we can choose $n$ and $\tau$ such that $\mathbb{P}^*_T(u^\tau,n(t, x) \in B(y, \varepsilon)) > 0$.

Since $u^\tau,n(\cdot, x)$ is a mild solution to (4.45), we infer that

$$
\begin{align*}
u^\tau,n(t, x) &= e^{-(t-\tau)A_\alpha}u(\tau, x) + \int_\tau^t e^{-(t-s)A_\alpha} \left( \frac{\partial}{\partial x}(u^\tau,n(s, x))^2 + f^\tau,n(s, y) \right) ds \\
&\quad + \int_\tau^t e^{-(t-s)A_\alpha}g(u^\tau,n(s, x)) dW(s), \quad t > \tau.
\end{align*}
$$

Direct calculations based on the semigroup property of $(e^{-tA_\alpha})$ and equality (4.44), yield, see also (36),

$$
e^{-(t-\tau)A_\alpha}u(\tau) + \int_\tau^t e^{-(t-s)A_\alpha}f^\tau,n(s, y) ds = y, \quad t > \tau.
$$

Consequently,

$$
u^\tau,n(t, x) = y + \int_\tau^t e^{-(t-s)A_\alpha} \frac{\partial}{\partial x}(u^\tau,n(s, x))^2 ds + \int_\tau^t e^{-(t-s)A_\alpha}g(u^\tau,n(s, x)) dW(s).
$$

Hence

$$
\begin{align*}
\mathbb{P}\left( u^\tau,n(t, x) \in B(y, \varepsilon) \right) &\geq 1 - \mathbb{P}\left( \left| \int_\tau^t e^{-(t-s)A_\alpha} \frac{\partial}{\partial x}(u^\tau,n(s, x))^2 ds \right|_{L^2} > \frac{\varepsilon}{2} \right) \\
&\quad - \mathbb{P}\left( \left| \int_\tau^t e^{-(t-s)A_\alpha}g(u^\tau,n(s, x)) dW(s) \right|_{L^2} > \frac{\varepsilon}{2} \right) \quad (4.51)
\end{align*}
$$

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Invoking the Chebyshev inequality, inequality (2.23), the boundedness of the operator \( g \) and the inequality \( ||AB||_{HS} \leq ||A||_{HS}||B||_{L(H)} \), we infer that

\[
P(\left| \int_{\tau}^{t} e^{-(t-s)A_{\alpha}} g(u^{\tau,n}(s)) \, dW(s) \right|_{L^2} > \frac{\varepsilon}{2}) 
\leq \frac{2p}{\varepsilon^p} E \left( \int_{\tau}^{t} e^{-(t-s)A_{\alpha}} g(u^{\tau,n}(s)) \, dW(s) \right)^p 
\leq \frac{2p}{\varepsilon^p} \left( \int_{0}^{t-\tau} ||e^{-rA_{\alpha}}||_{HS}^2 ds \right)^{\frac{p}{2}} 
\leq \frac{4p}{\varepsilon^p} \left( \int_{0}^{t-\tau} ||e^{-rA_{\alpha}}||_{HS}^2 ds \right)^{\frac{p}{2}}.
\]

Since \( \int_{0}^{T} ||e^{-rA_{\alpha}}||_{HS}^2 ds < \infty \), we can choose \( \tau < t \) such that

\[
P(\left| \int_{\tau}^{t} e^{-(t-s)A_{\alpha}} g(u^{\tau,n}(s)) \, dW(s) \right|_{L^2} > \frac{\varepsilon}{2}) 
\leq \frac{1}{4}.
\]

Furthermore, by the Chebyshev inequality, Lemma 2.11 and (4.46), we obtain

\[
P \left( \left| \int_{\tau}^{t} e^{-(t-s)A_{\alpha}} \frac{\partial}{\partial x} (u^{\tau,n}(s))^2 \, ds \right|_{L^2} > \frac{\varepsilon}{2} \right) 
\leq \frac{4}{\varepsilon^2} E \left( \int_{\tau}^{t} e^{-(t-s)A_{\alpha}} \frac{\partial}{\partial x} (u^{\tau,n}(s))^2 \, ds \right)^2 
\leq \frac{4C_{\alpha}}{\varepsilon^2} (t-\tau)^{1-\frac{3}{2\alpha}} E \sup_{s \leq t} |u^{\tau,n}(s)|_{L^2}^2 
\leq \frac{4MC_{\alpha}}{\varepsilon^2} (t-\tau)^{1-\frac{3}{2\alpha}}.
\]

Therefore, there exists \( \tau < t \) such that

\[
P \left( \left| \int_{\tau}^{t} e^{-(t-s)A_{\alpha}} \frac{\partial}{\partial x} (u^{\tau,n}(s))^2 \, ds \right|_{L^2} > \frac{\varepsilon}{2} \right) 
\leq \frac{1}{4}
\]

and combining this result with (4.52) we find that

\[
P(\{ u^{\tau,n}(t, x) \in B(y, \varepsilon) \}) \geq \frac{1}{2}.
\]

Thanks to the equivalence of \( P \) and \( P^\ast_T \), we obtain

\[
P_T^\ast(\{ u^{\tau,n}(t, x) \in B(y, \varepsilon) \}) > 0.
\]

and invoking identity (4.50), we find that

\[
P(\{ u(t, x) \in B(y, \varepsilon) \}) = P_T^\ast(\{ u^{\tau,n}(t, x) \in B(y, \varepsilon) \}) > 0.
\]

This proves (4.43) for \( y \in D(A_{\alpha}) \). As we observed at the beginning of the proof this implies the general case and therefore the proof of Theorem 9 is complete. \[\Box\]
5 Existence and properties of the invariant measure

The main task in this section is to prove the existence of an invariant measure. Our main tool is the Krylov-Bogoliubov Theorem. Our result here is

**Theorem 10.** There exists at least one invariant measure for the transition semigroup $U$, corresponding to the fractional stochastic Burgers equation (1.6).

To prove this theorem we start with some auxiliary lemmas.

**Lemma 11.** There exists $\gamma_1$ such that for every $\gamma > \gamma_1$ and every $\varepsilon > 0$ we can find $M > 0$ such that

\[
\frac{1}{t} \int_0^t \mathbb{P} \left( |v_\gamma(s)|_{L^2}^2 \geq M \right) ds < \varepsilon, \quad t > 0. \tag{5.54}
\]

**Proof.** Let us fix $\varepsilon > 0$. Let $C$ denote a constant greater than the third power of the maximum of the embedding constant of $H^{1-\frac{\alpha}{4},2}(0,1) \hookrightarrow L^2(0,1)$, the embedding constant of $H^{1-\frac{\alpha}{4},2}(0,1) \hookrightarrow H^{1-\frac{\alpha}{4},2}(0,1)$ and the constant $C$ from (2.19). By Lemma 6 we can find $\gamma_0 > 0$ such that, for all $\gamma \geq \gamma_0$

\[
\max \left[ \sup_{[0,\infty)} \mathbb{E} |z_\gamma(t)|_{H^{1-\frac{\alpha}{4},2}}^{\frac{2\alpha}{\alpha-\alpha}}, \sup_{[0,\infty)} \mathbb{E} |z_\gamma(t)|_{H^{1-\frac{\alpha}{4},2}}^{4}, \sup_{[0,\infty)} \mathbb{E} |z_\gamma(t)|_{H^{1-\frac{\alpha}{4},2}}^{2} \right] \leq \frac{\varepsilon^\nu}{4C} \tag{5.55}
\]

Let $M > 1$. We define a process $\zeta_\gamma$ by $\zeta_\gamma(t) = \log \left( |v_\gamma(t)|_{L^2}^2 \vee M \right)$, $t \geq 0$. Using (2.19) and the weak derivative of $\zeta_\gamma(t)$, we get

\[
\zeta'_\gamma(t) = \frac{1}{\mathbb{E}|v_\gamma(t)|_{L^2}^2-1}|v_\gamma(t)|_{L^2}^2 \frac{d}{dt} |v_\gamma(t)|_{L^2}^2 \\
\leq \frac{1}{\mathbb{E}|v_\gamma(t)|_{L^2}^2-1}|v_\gamma(t)|_{L^2}^2 \left( -\nu |v_\gamma(t)|_{L^2}^2 + C |z_\gamma(t)|_{H^{1-\frac{\alpha}{4},2}}^{\frac{2\alpha}{\alpha-\alpha}} |v_\gamma(t)|_{L^2}^2 \right. \\
\left. + C |z_\gamma(t)|_{H^{1-\frac{\alpha}{4},2}}^{4} + C |z_\gamma(t)|_{H^{1-\frac{\alpha}{4},2}}^{4} + \gamma_1^2 |z_\gamma(t)|_{L^2}^2 \right). \tag{5.56}
\]

Taking expectation of the both sides, we get

\[
\mathbb{E}\zeta_\gamma(t) - \mathbb{E}\zeta_\gamma(0) + \nu \int_0^t \mathbb{P} \left( |v_\gamma(s)|_{L^2}^2 \geq M \right) ds \tag{5.57}
\]

\[
\leq t \left( C \sup_{(0,\infty)} \mathbb{E}|z_\gamma(s)|_{H^{1-\frac{\alpha}{4},2}}^{\frac{2\alpha}{\alpha-\alpha}} + C \frac{\gamma_1^2}{M} \sup_{(0,\infty)} \mathbb{E}|z_\gamma(s)|_{L^2}^2 \right) \tag{5.58}
\]

\[
\leq t \left( C \sup_{(0,\infty)} \mathbb{E}|z_\gamma(s)|_{H^{1-\frac{\alpha}{4},2}}^{\frac{2\alpha}{\alpha-\alpha}} + C \frac{\gamma_1^2}{M} \sup_{(0,\infty)} \mathbb{E}|z_\gamma(s)|_{L^2}^2 \right). \tag{5.59}
\]
It is easy to see that if $M \geq |v(0)|^2_{L^2}$, then
\[ \mathbb{E} \zeta_{t_1}(t) - \mathbb{E} \zeta_{t_1}(0) \geq 0, \quad t \geq 0. \]

Hence
\[
\frac{1}{t} \int_0^t \mathbb{P} \left( |v_{t_1}(s)|^2_{L^2} \geq M \right) ds \leq \frac{C}{M^\nu} \left[ M \sup_{(0, \infty)} \mathbb{E} |z_{t_1}(s)|_{H^{1-\frac{3}{2}}} \right]^{\frac{2}{\alpha}} \\
+ \sup_{(0, \infty)} \mathbb{E} |z_{t_1}(s)|_{H^{1-\frac{3}{2}}}^{\frac{4}{3}} + \gamma_1^2 \sup_{(0, \infty)} \mathbb{E} |z_{t_1}(s)|^2_{L^2}.
\]

(5.58)

Let us remark that thanks to the condition $\alpha > \frac{3}{2}$, we have $H^{1-\frac{3}{2}}(0, 1) \hookrightarrow H^{1-\frac{3}{2}}(0, 1)$, hence $|z_{t_1}(s)|_{H^{1-\frac{3}{2}}} \leq C|z_{t_1}(s)|_{H^{1-\frac{3}{2}}}$. Consequently, from the formulae $\text{(5.55)}$ and for the choice $\frac{1}{q} < s < \frac{\alpha}{2} - \frac{1}{2} < \frac{1}{2}$, $q > \frac{2}{\alpha-1}$, Lemma [6] and thanks to the condition $M > \gamma^2$, then we get
\[
\frac{1}{t} \int_0^t \mathbb{P} \left( |v_{t_1}(s)|^2_{L^2} \geq M \right) ds < \varepsilon.
\]

The following Lemma is a slight generalization of [6, Lemma 2.11].

**Lemma 12.** For every $T > 0$ there exist $C > 0$ and $\theta > 0$ such that for all $v \in C([0, T]; L^1(0, 1))$,
\[
\int_0^t \left| (-A)^{\frac{2\alpha-3}{4}} e^{-(t-s)A} Bv(s) \right|_{L^2} ds \leq Ct^\theta \left| v \right|_{L^\infty([0, t]; L^1(0, 1))}^2, \quad t \geq 0.
\]

(5.60)

**Proof.** Let us choose (and fix) $\beta > 0$ and $v \in C([0, T]; L^1(0, 1))$. Then we have
\[
\left| (-A)^{\beta} e^{-(t-s)A} Bv(s) \right|^2_{L^2} = \sum_{k=1}^{+\infty} \langle (-A)^{\beta} e^{-(t-s)A} Bv(s), e_k \rangle^2 \\
= \sum_{k=1}^{+\infty} \langle v(s), B e^{-(t-s)A} (-A)^{\beta} e_k \rangle^2 \leq \sum_{k=1}^{+\infty} \left( k\pi \right)^2 \lambda_k^{2\beta} e^{-\lambda_k^{\frac{2}{3}} (t-s)} \langle v(s), f_k \rangle^2,
\]
where $f_k(\cdot) := \cos(k \pi \cdot)$. Thanks to the inequality $|\langle v, f_k \rangle| \leq |v|_{L^1} |f_k|_{L^\infty}$, we get
\[
\int_0^t \left| (-A)^{\beta} e^{-(t-s)A} Bv(s) \right|_{L^2} ds \leq \int_0^t \left( \sum_{k=1}^{+\infty} \left( k\pi \right)^{4\beta+2} e^{-(k \pi)^{\alpha} (t-s)} \right)^{\frac{1}{2}} |v(s)|_{L^1(0, 1)} ds \\
\leq \sup_{[0, t]} |v(s)|_{L^1(0, 1)} \int_0^t \left( \sum_{k=1}^{+\infty} \left( k\pi \right)^{4\beta+2} e^{-(k \pi)^{\alpha} s} \right)^{\frac{1}{2}} ds.
\]

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Since $e^{-(k\pi)^\sigma s} \leq C_\sigma s^{-\sigma} (k\pi)^{-\sigma \alpha}$ for all $s \geq 0$ and a certain constant $C_\sigma > 0$, we get

\[
\int_0^t \left( \sum_{k=1}^{+\infty} (k\pi)^{4\beta + 2} e^{-(k\pi)^\alpha s} \right)^{\frac{1}{2}} ds \leq C_N \left( \sum_{k=1}^{+\infty} (k\pi)^{4\beta + 2 - \sigma \alpha} \right)^{\frac{1}{2}} \int_0^t s^{-\frac{\alpha}{2}} ds. \tag{5.61}
\]

The series and the integral in the RHS of (5.61) converge provided $\frac{4\beta + 3}{4} < \sigma < 2$. Such a value of $\sigma$ exists provided $\beta < \frac{2\alpha - 3}{4}$. Moreover, if $\alpha > \frac{3}{2}$, then we can choose $0 \leq \beta < \frac{2\alpha - 3}{4}$, such that

\[
\int_0^t \left| (-A)^\beta e^{-(t-s)A} B v(s) \right|_{L^2} ds \leq C t^{1 - \frac{\alpha}{2}} \sup_{[0,t]} |v(s)|_{L^1}. \tag{5.62}
\]

To get the inequality (5.60), we take $\theta := 1 - \frac{\sigma}{2}$ and $\beta = \frac{2\alpha - 3}{4}$. □

**Proof of Theorem 10**

We use the Krylov-Bogoliubov method, in particular, we follow the proof in [16, 18]. Let us remark that for all $\theta > 0$ the embedding $H^{\theta,2}(0,1) \hookrightarrow L^2(0,1)$ is compact, see e.g. [16] and [20, p273]. Hence the ball

\[
B_{H^{\theta,2}}(0, M) := \{ v \in H^{\theta,2}(0,1), |v|_{H^{\theta,2}} \leq M \}
\]

is a compact subset of $L^2(0,1)$.

In the first step, let us prove that for all $\beta \in (0, \frac{2\alpha - 3}{4})$ and $\varepsilon > 0$ there exists $M > 0$, large enough such that $\frac{1}{t} \int_0^t P \left( \left| (-A)^\beta v_\gamma(s) \right|_{L^2}^2 \geq M \right) ds \leq \varepsilon$, where $v_\gamma$ is the solution of (2.14). We have,

\[
v_\gamma(t+1) = e^{-A_\alpha} v_\gamma(t) + \int_t^{t+1} e^{-A_\alpha (t+1-s)} B(v_\gamma(s) + z_\gamma(s))^2 ds \\
+ \gamma \int_t^{t+1} e^{-A_\alpha (t+1-s)} z_\gamma(s) ds. \ a.s. \tag{5.63}
\]

Let $0 < \beta < \frac{2\alpha - 3}{4}$. Then

\[
\left| (-A)^\beta v_\gamma(t+1) \right|_{L^2} \leq \left| A^{2\beta}_{\alpha} e^{-A_\alpha} v_\gamma(t) \right|_{L^2} \\
+ \int_t^{t+1} \left| (-A)^\beta e^{-A_\alpha (t+1-s)} B(v_\gamma(s) + z_\gamma(s))^2 \right|_{L^2} ds \\
+ \gamma \int_t^{t+1} \left| (-A)^\beta e^{-A_\alpha (t+1-s)} z_\gamma(s) \right|_{L^2} ds \ a.s. \tag{5.64}
\]

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We denote the three terms in the RHS of (5.64) by $I_1, I_2, I_3$ respectively. It is easy to see, using the properties of the heat semigroup that

$$I_1 := |A_\alpha^{2\beta} e^{-A_\alpha t} v_\gamma(t)|_{L^2} \leq C |v_\gamma(t)|_{L^2} \quad \text{and} \quad I_3 \leq \gamma \int_t^{t+1} |z_\gamma(s)|_{H^{2\alpha,2}} ds.$$

(5.65)

For the second term, arguing as in the proof of Lemma 12, we get for some $0 < \eta < 1$,

$$I_2 \leq \int_t^{t+1} \left| (-A)^\beta e^{-A_\alpha (t+1-s)} B(v_\gamma(s)) \right|^2_{L^2} ds$$

$$+ 2 \int_t^{t+1} \left| (-A)^\beta e^{-A_\alpha (t+1-s)} B(v_\gamma(s) z_\gamma(s)) \right|^2_{L^2} ds$$

$$+ \int_t^{t+1} \left| (-A)^\beta e^{-A_\alpha (t+1-s)} B(z_\gamma(s)) \right|^2_{L^2} ds$$

$$\leq \sup_{0 \leq s \leq 1} |v_\gamma(t+r)|_{L^2}^2 + 2 \int_t^{t+1} (t+1-s)^{-\eta} |v_\gamma(s)|_{L^2} |z_\gamma(s)|_{L^2} ds$$

$$+ \int_t^{t+1} (t+1-s)^{-\eta} |z_\gamma(s)|_{L^2}^2 ds$$

$$\leq c_\eta \sup_{0 \leq r \leq 1} |v_\gamma(t+r)|_{L^2}^2 + 2 \int_t^{t+1} (t+1-s)^{-\eta} |z_\gamma(s)|_{L^2}^2 ds.$$

(5.66)

By subsisting (5.65) - (5.66) on (5.64), we get

$$|(-A)^\beta v_\gamma(t+1)|_{L^2} \leq C |v_\gamma(t)|_{L^2} + c_\eta \sup_{0 \leq r \leq 1} |v_\gamma(t+r)|_{L^2}^2$$

$$+ 2 \int_t^{t+1} (t+1-s)^{-\eta} |z_\gamma(s)|_{L^2}^2 ds + \gamma \int_t^{t+1} |z_\gamma(s)|_{H^{2\alpha,2}} ds. \ a.s.$$

(5.67)

Hence

$$\frac{1}{T} \int_0^T \mathbb{P} \left( \left| (-A)^\beta v_\gamma(t+1) \right|^2_{L^2} > M \right) dt \leq \frac{1}{T} \int_0^T \mathbb{P} \left( |v_\gamma(t)|_{L^2} > \sqrt{\frac{M}{4C}} \right) dt$$

$$+ \frac{1}{T} \int_0^T \mathbb{P} \left( \sup_{0 \leq r \leq 1} |v_\gamma(t+r)|_{L^2}^4 > \frac{M}{4c_\eta} \right) dt$$

$$+ \frac{1}{T} \int_0^T \mathbb{P} \left( \int_t^{t+1} (t+1-s)^{-\eta} |z_\gamma(s)|_{L^2}^2 ds > \frac{M}{8C} \right) dt$$

$$+ \frac{1}{T} \int_0^T \mathbb{P} \left( \int_t^{t+1} |z_\gamma(s)|_{H^{2\alpha,2}} ds > \frac{M}{4\gamma} \right) dt.$$

(5.68)
From Lemma 11 there exists $M_1 > 0$, such that $\forall M > M_1$, we have
\[
\frac{1}{T} \int_0^T \mathbb{P}\left( |v_\gamma(t)|^2_{L^2} > \sqrt{\frac{M}{4C}} \right) dt < \frac{\varepsilon}{4}. \tag{5.69}
\]
By Chebyshev inequality and Lemma 6 and considering the choice $M > \max\{M_1, 16\gamma, 32C\}$ and the condition $\alpha < 2$ which guaranty that $\beta < \frac{2\alpha - 3}{4}$, we infer
\[
\frac{1}{T} \int_0^T \mathbb{P}\left( \int_t^{t+1} (t + 1 - s)^{-\eta} |z_\gamma(s)|^2_{L^2} ds > \frac{M}{8C} \right) dt 
\leq \frac{8C}{M} \frac{1}{T} \int_0^T \left( \int_t^{t+1} (t + 1 - s)^{-\eta} \mathbb{E}|z_\gamma(s)|^2_{L^2} ds \right) dt < \frac{\varepsilon}{4}, \tag{5.70}
\]
and
\[
\frac{1}{T} \int_0^T \mathbb{P}\left( \int_t^{t+1} |z_\gamma(s)|_{H^{2,2}} ds > \frac{M}{4\gamma} \right) dt \leq \frac{4\gamma}{M} \frac{1}{T} \int_0^T \left( \int_t^{t+1} \mathbb{E}|z_\gamma(s)|_{H^{2,2}} ds \right) dt < \frac{\varepsilon}{4}. \tag{5.71}
\]
To estimate the second term in RHS of the inequality (5.68), we apply Gronwall Lemma to the formulae (2.18), then we get
\[
|v_\gamma(t + s)|^2_{L^2} \leq |v_\gamma(t)|^2_{L^2} e^{C \int_0^s \frac{|z_\gamma(t + r)|^4_{H^{1,2}}}{H^{1,2}} dr} + \int_0^s \left[ C |z_\gamma(t + r)|^4_{H^{1,2}} + \gamma^2 C |z_\gamma(t + r)|^2_{L^2} \right] e^{C \int_r^s \frac{|z_\gamma(t + \xi)|^4_{H^{1,2}}}{H^{1,2}} d\xi} dr 
\leq e^{C \int_0^s \frac{|z_\gamma(t + r)|^4_{H^{1,2}}}{H^{1,2}} dr} \left[ |v_\gamma(t)|^2_{L^2} + \int_0^1 C |z_\gamma(t + r)|^4_{H^{1,2}} dr \right] 
+ \int_0^1 C |z_\gamma(t + r)|^4_{H^{1,2}} dr + \gamma^2 C \int_0^1 |z_\gamma(t + r)|^2_{L^2} dr. \tag{5.72}
\]
Hence,
\[
\frac{1}{T} \int_0^T \mathbb{P}\left( \sup_{0 \leq r \leq 1} |v_\gamma(t + r)|^4_{L^2} > \frac{M}{4c_\eta} \right) dt 
\leq \frac{1}{T} \int_0^T \mathbb{P}\left( e^{C \int_0^r \frac{|z_\gamma(t + r)|^4_{H^{1,2}}}{H^{1,2}} dr} > \sqrt{\frac{M}{4c_\eta}} \right) dt 
+ \frac{1}{T} \int_0^T \mathbb{P}\left( |v_\gamma(t)|^2_{L^2} > \sqrt{\frac{M}{4^3c_\eta}} \right) dt 
+ \frac{1}{T} \int_0^T \mathbb{P}\left( \int_0^1 |z_\gamma(t + r)|^4_{H^{1,2}} dr > \sqrt{\frac{M}{4^3c_\etaC^2}} \right) dt.
\]
Arguing as above and use Lemmata 6, 11 and Chebychev inequality, then for enough large $M$, we get

$$\frac{1}{T} \int_0^T \mathbb{P} \left( \sup_{0 \leq r \leq 1} \left| v_\gamma(t + r) \right|_{L^2}^4 > \frac{M}{4c_\eta} \right) dt < \frac{\varepsilon}{4}. \quad (5.74)$$

Finally, from (5.69)–(5.71) and (5.74), we get

$$\frac{1}{T} \int_0^T \mathbb{P} \left( \left| (-A)^\beta v_\gamma(t + 1) \right|_{L^2}^2 > M \right) dt < \varepsilon. \quad (5.75)$$

Let us recall that $u = v_\gamma + z_\gamma$, hence using (5.74) and Lemma 6 we prove that

$$\frac{1}{T} \int_0^T \mathbb{P} \left( \left| (-A)^\beta u(t + 1) \right|_{L^2}^2 > M \right) dt < \varepsilon. \quad (5.76)$$

Hence the family of probability measures $\mu_T(\cdot) := \frac{1}{T} \int_0^T (\mathcal{L}(X_t+))(\cdot)dt$ is tight. By Porokhorov’s Theorem, we can subtract a weak convergent sequence $\mu_{T_{n_k}}(\cdot), T_{n_k} \uparrow +\infty$. Using Krylov-Bogoliubov existence Theorem, see e.g. [20], we confirm the existence of at least one invariant measure.

**Proof of the Theorem 2**

The proof of the main Theorem 2 follows, in a standard way, as a conclusion of the above results. In fact, thanks to Theorem 10 there exists an invariant measure. The semigroup $U$ is strong Feller and irreducible for all $t > 0$, hence by the Khas’minskii Theorem the semigroup is regular for all $t > 0$, see e.g. [20, Proposition 4.1.1]. As a consequence of the regularity of the semigroup and using Doob’s Theorem, we conclude that the invariant measure is unique. The convergence in (1.11) follows immediately from [40, Theorem 1] and [20, Proposition 4.2.1], see also [31]. Finally, the ergodicity of the invariant measure follows from the uniqueness, see e.g. [20, Theorem 3.2.6].

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References

[1] A. R. Adams, “Sobolev Spaces”, Academic Press New York, London 1975.

[2] S. Albeverio, F. Flandoli and Y. G. Sinai, “SPDE in hydrodynamics: recent progress and prospects”, Lect. Not. in Math. 1942. Springer-Verlag C.I.M.E. Florence 2008.

[3] D. Barbato, F. Flandoli, and F. Morandin, “Uniqueness for a stochastic inviscid dyadic model”, Proc. Amer. Math. Soc. 138 no. 7, 2010.

[4] P. Biler, T. Funaki, and W. A. Woyczynski, “Fractal Burgers equations”, J. Differential Equations 148 no. 1, pp. 9-46, 1998.

[5] Z. Brzeźniak, “On stochastic convolution in Banach spaces and applications”, Stochastics Stochastics Rep. 61 no. 3-4, pp. 245-295, 1997.

[6] Z. Brzeźniak, and L. Debby, “On stochastic Burgers equation driven by a fractional Laplacian and space-time white noise”, Stochastic differential equations: theory and applications, pp. 135-167, Interdiscip. Math. Sci., 2, World Sci. Publ., Hackensack, NJ, 2007.

[7] Z. Brzeźniak, and D. Gatarek, “Martingale solutions and invariant measures for stochastic evolution equations in Banach spaces”, Stochastic Process. Appl. 84 no. 2, pp. 187-225, 1999.

[8] Z. Brzeźniak, and Y. Li, “Asymptotic compactness and absorbing sets for 2D stochastic Navier-Stokes equations on some unbounded domains”, Trans. Am. Math. Soc., N 358, pp. 5587-5629, 2006.

[9] Z. Brzeźniak and J.M.A.M. van Neerven, “Space-time regularity for linear stochastic evolution equations driven by spatially homogeneous noise”, J. Math. Kyoto Univ. 43, no. 2, pp. 261-303, 2003.

[10] L. A. Caffarelli, “Some nonlinear problems involving non-local diffusions”, ICIAM 07-6th Intern. Congress on Industrial and Applied Math. pp. 43-56 Eur. Math. Soc. Zurich 2009.

[11] L. A. Caffarelli, and A. Vasseur, “Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation”, Ann. of Math. 2, 171 no. 3, pp. 1903-1930, 2010.

[12] D. Chae, P. Constantin, and J. Wu “Dissipative models generalizing the 2D Navier-Stokes and the surface quasi-geostrophic equations”, arXiv:1011.0171 31 Oct 2010.
[13] H. Crauel, and F. Flandoli, “Hausdorff dimension of invariant sets for random dynamical systems”, J. Dynam. Differential Equations 10 no. 3, pp. 449-474, 1998.

[14] G. Da Prato, and F. Flandoli, “Pathwise uniqueness for a class of SDE in Hilbert spaces and applications”, J. Funct. Anal. 259 no. 1, pp. 243-267, 2010.

[15] G. Da Prato, and A. Debussche, “Differentiability of the transition semigroup of the stochastic Burgers equation, and application to the corresponding Hamilton-Jacobi equation”, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei 9 Mat. Appl. 9 no. 4, pp. 267-277, 1998.

[16] G. Da Prato, A. Debussche, and R. Temam, “Stochastic Burgers’ equation”, NoDEA Nonlinear Differential Equations Appl. 1 no. 4, pp. 389-402, 1994.

[17] G. Da Prato, K. D. Elworthy, and J. Zabczyk, “Strong Feller property for stochastic semilinear equations”, Stochastic Anal. Appl. 13 no. 1, pp. 35-45, 1995.

[18] G. Da Prato, and D. Gatarek, “Stochastic Burgers equation with correlated noise”, Stochastics Stochastics Rep. 52 no. 1-2, pp. 29-41, 1995.

[19] G. Da Prato, and J. Zabczyk, “Stochastic equations in infinite dimensions”, Encyclopedia of Mathematics and its Applications 44. Cambridge University Press 1992.

[20] G. Da Prato, and J. Zabczyk, “Ergodicity for infinite-dimensional systems”, London Mathematical Society Lecture Note Series 229. Cambridge University Press, Cambridge 1996.

[21] E. Dettweiler, “Stochastic integration of Banach space valued functions”, in Arnold L., Kotelenez P. (Eds.) STOCHASTIC SPACE-TIME MODELS AND LIMIT THEORIES, D. Reidel, Dordrecht, pp. 53-79, 1985.

[22] F. Flandoli, “Random perturbation of PDEs and fluid dynamic models”, Lectures Notes in Mathematics 2015, Ecole d’Été de Probabilités de Saint-Flour XL 2010.

[23] F. Flandoli, and M. Gubinelli, and E. Priola, “Well-posedness of the transport equation by stochastic perturbation”, Invent. Math. 180 no. 1, pp. 1-53, 2010.

[24] D. Gatarek, and B. Goldys, “On invariant measures for diffusions on Banach spaces”, Potential Anal. 7 no. 2, pp. 539-553, 1997.
[25] B. Goldys, and B. Maslowski, “Exponential ergodicity for stochastic Burgers and 2D Navier-Stokes equations”, J. Funct. Anal. 226 no. 1, pp. 230-255, 2005.

[26] M. Hairer, and J. C. Mattingly, “Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing”, Ann. of Math. (2) 164 no. 3, pp. 993-1032, 2006.

[27] A. Kiselev, “Regularity and blow up for active scalars”, arXiv:1009.0540v1, 6 Sep 2010.

[28] A. Kiselev, and F. Nazarov, and R. Shterenberg,“ Blow up and regularity for fractal Burgers equation”, Dyn. Partial Differ. Equ. 5 no. 3, pp. 211-240, 2008.

[29] T. Kukatani, and N. Sugimoto, “Generalized Burgers Equations for Nonlinear Viscoelastic Waves”, Wave Motion 7, pp. 447-458, 1985.

[30] J. L. Lions, and E. Magenes, “Non-Homogeneous Boundary Value Problems and Applications”, Vol. I English translation. Springer-Verlag Berlin, New York 1972.

[31] B. Maslowski, and J. Seidler, “Probabilistic approach to the strong Feller property”, Probab. Theory Related Fields 118, No. 2, pp. 187-210, 2000.

[32] A. D. Neate, and A. Truman, “On the stochastic Burgers equation and some applications to turbulence and astrophysics”, Analysis and stochastics of growth processes and interface models, pp. 281-305 Oxford Univ. Press, Oxford 2008.

[33] M. Ondreját, “Uniqueness for stochastic evolution equations in Banach spaces”, Dissertationes Math. (Rozprawy Mat.) No 426, 2004.

[34] A. D. Pablo, F. Quiros, A. Rodriguez, and J. L. VazquezA, “Fractional porous medium equation”, arXiv:1001.2383v1 [math.AP] 14 Jan 2010.

[35] A. Pazy, “Semigroups of linear operators and applications to partial differential equations”, Applied Mathematical Sciences 44. Springer-Verlag, New York 1983.

[36] S. Peszat, and J. Zabczyk, “Strong Feller property and irreducibility for diffusions on Hilbert spaces”, Ann. Probab. 23 No. 1, pp. 157-172, 1995.

[37] M. Röckner, and Z. Sobol, “Kolmogorov equations in infinite dimensions: well-posedness and regularity of solutions, with applications to stochastic generalized Burgers equations”, Ann. Probab. 34 No. 2, pp. 663-727, 2006.
[38] T. Runst, and W. Sickel, “Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations”, de Gruyter Series in Nonlinear Analysis and Applications 3. Walter de Gruyter & Co. Berlin 1996.

[39] A. Shirikyan, “Ergodicity for a class of Markov processes and applications to randomly forced PDE’s”, I. Russ. J. Math. Phys. 12 No. 1, pp. 81-96, 2005.

[40] L. Stettner, “Remarks on ergodic conditions for Markov processes on Polish spaces”, Bull. Polish Acad. Sci. Math. 42, No. 2, pp. 103-114, 1994.

[41] N. Sugimoto, “Generalized Burgers equations and Fractional Calculus”, Nonlinear Wave Motion. (A. Jeffery, Ed) pp. 162-179, 1991.

[42] R. Temam, “Navier-Stokes equations. Theory and numerical analysis”, Reprint of the 1984 edition. AMS Chelsea Publishing, Providence, RI, 2001.

[43] H. Triebel, “Interpolation theory, function spaces, differential operators”, North-Holland Mathematical Library, 18. North-Holland Publishing Co., Amsterdam-New York 1978.

[44] A. Truman, and J. L. Wu, “Fractal Burgers' equation driven by Lévy noise”, Lect. Notes Pure Appl. Math. 245 Chapman and Hall/CRC, Boca Raton, FL 2006.

[45] J. Van Neerven, “γ-radonifying operators-a survey”, The AMSI-ANU Workshop on Spectral Theory and Harmonic Analysis, pp. 1-61, Proc. Centre Math. Appl. Austral. Nat. Univ. 44 Austral. Nat. Univ. Canberra 2010.