Parity odd equilibrium partition function in $2 + 1$ dimensions

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Abstract:

We use Schwinger’s proper time method to compute the parity odd contributions to the $U(1)$ current and energy-momentum tensor of an ideal gas of fermions in $2 + 1$ dimensions in the presence of static gauge and gravitational backgrounds. From these results the equilibrium partition function at first order in the derivative expansion is explicitly obtained by integration. The form of the computed partition function is consistent with general arguments based on Kaluza-Klein and gauge invariance.
1 Introduction

Apart from the direct thermodynamic description, the equilibrium partition function as a functional of stationary gauge and gravitational fields plays an important role in the realm of fluid dynamics. In particular, the structure of the first few terms of an expansion in derivatives of these fields has an interplay with the non-dissipative part of the constitutive relations of hydrodynamics, which connect expectation values of microscopic currents with macroscopic fields such as temperature, chemical potential and fluid velocities. This has led to significant progress in understanding some parity-odd susceptibilities related to anomalous conservation equations of field theories with anomalies [1–11], without the need to resort to entropy production arguments [12–15].

The general form of the thermal equilibrium partition function that includes first order terms in the derivative expansion of the background fields has been discussed previously (see e.g. [14]). For an arbitrary time-independent background given by the line element and $U(1)$ gauge connection

$$ds^2 = -e^{2\sigma(x)}(dt + a_i(x)dx^i)^2 + g_{ij}(x)dx^i dx^j, \quad i, j = 1, \ldots p,$$

$$A_\mu = (A_0(x), A(x)), \quad (1.1)$$

the partition function in one and three spatial dimensions depends only on a few constant coefficients, which appear in the anomalous conservation equations. In two spatial dimensions the form of the corresponding partition function is less restricted. In this case there are not anomalous conservation laws, but the mass of the Dirac field acts as a source of
parity violation, and it turns out that the partition function may depend on arbitrary functions of the mass and certain combinations of the background fields. Arguments based on Kaluza-Klein and gauge invariance \cite{14} constrain the form of the most general parity-odd partition function at first derivative order in $2+1$ dimensions to be

$$
\mathcal{W} = \frac{1}{2} \int d^2x \left( \alpha(\sigma, A_0) \epsilon^{ij} \partial_i \tilde{A}_j + T_0 \beta(\sigma, A_0) \epsilon^{ij} \partial_i a_j \right),
$$

where $\tilde{A}_i = A_i - A_0 a_i$, $T_0$ is the equilibrium temperature and the functions $\alpha(\sigma, A_0)$ and $\beta(\sigma, A_0)$ are specific to the system considered.

The main purpose of this paper is precisely the computation of the parity violating partition function for a noninteracting massive Dirac fermion in $2+1$ dimensions at first order in the derivatives of the background fields. In order to do this we first obtain the parity-odd contributions to the $U(1)$ currents and energy-momentum tensor in the nonuniform static background defined by the gauge field and the metric (1.1) and identify our results with the variational formulas from $\mathcal{W}$. Integrating the corresponding equations we are able to completely determine the functions $\alpha(\sigma, A_0)$ and $\beta(\sigma, A_0)$. As far as we know, the piece in (1.2) proportional to the curl of the vector field $a$ had never been computed before and this is one of our main results.

We find that the functions $\alpha$ and $\beta$ thus determined vanish for massless fermions. This, however, does not imply the absence of parity violating effects in this limit. The theory has to be regularized, and a convenient gauge-invariant regularization procedure in two spatial dimensions is the Pauli-Villars method, which uses massive fermions as regulator fields. From a study of the asymptotic behavior of the partition function for large fermion masses we conclude that, at first order in the derivative expansion, there are parity-odd contributions to the currents of massless fermions —which produce the nonzero Hall currents from the parity anomaly \cite{16–19}— but not to their energy-momentum tensor. This is consistent with the fact that the purely gravitational contribution at zero temperature in $2+1$ dimensions is described by the gravitational Chern-Simons action, which gives rise to the third (derivative) order Cotton tensor \cite{19, 20}.

The contents of this paper is organized as follows. In section 2 we express the components of the currents and energy-momentum tensor in terms of the coincidence limit of the thermal Green function $\mathcal{G}(x, x', \omega_n)$ and its first derivatives. The derivative expansion for the thermal Green function is computed in section 3 using Schwinger’s proper time method, and from the Green function the parity-odd contributions to the currents and energy-momentum tensor are derived in section 4. These results are used to determine the partition function in section 5, where we also discuss the small and large fermion mass limits and the effects of a Pauli-Villars regulator. Our conclusions and possible extensions of this work are presented in section 6.

2 General expressions for the $U(1)$ current and energy-momentum tensor

In this section we obtain general formulas for the current and energy-momentum tensor in a static background and express their components in terms of the coincidence limit of the
thermal Green function and its first derivatives. The action for a charged spin-1/2 fermion in a curved background is given by $S = \int d^3x \sqrt{-g}\mathcal{L}$, where

$$\mathcal{L} = -i\bar{\psi}\gamma^\mu \nabla_\mu \psi + im\bar{\psi}\psi$$

(2.1)

and the space-time dependent matrices $\gamma^\mu(x)$ are related to the constant Minkowski matrices by $\gamma^\mu(x) = e^\mu_a(x)\gamma^a$, where $e^\mu_a$ is the vielbein. They satisfy $\{\gamma^\mu(x),\gamma^\nu(x)\} = 2g^{\mu\nu}(x)$ and $\{\gamma^a,\gamma^b\} = 2\eta^{ab}$, with $\eta^{ab} = \text{diag}(-1,1,1)$ in 2 + 1 dimensions. We choose a representation where the constant Minkowski matrices are given in terms of the Pauli matrices by

$$\gamma^0 = -i\sigma_3 \quad , \quad \gamma^1 = \sigma_2 \quad , \quad \gamma^2 = -\sigma_1 .$$

(2.2)

Note that $\bar{\psi}$ is related to $\psi^\dagger$ by $\bar{\psi} = \psi^\dagger \gamma^0 = -i\psi^\dagger \sigma_3$.

The covariant derivative of the Dirac field $\psi$ is given by

$$\nabla_\mu \psi = (\partial_\mu + \frac{1}{4}\omega^a_{\mu\nu}\gamma^{ab} - iA_\mu)\psi, \quad \gamma^{ab} = \frac{1}{2}[\gamma^a,\gamma^b] ,$$

(2.3)

where, in the absence of torsion, the spin connection is related to the vielbein $e^\mu_a$ by

$$\omega^a_{\mu\nu} = -e^\nu_b(\partial_\mu e^a_b - \Gamma^a_{\mu\nu}e^b_\sigma) .$$

(2.4)

Here $\Gamma^a_{\mu\nu}$ are the Christoffel symbols obtained from the metric. For the rest of the paper, we will consider a static metric of the form

$$ds^2 = -e^{2\sigma(x)}(dt + a_i(x)dx^i)^2 + \delta_{ij}dx^i dx^j , \quad i,j = 1,2$$

(2.5)

and assume that the gauge fields are time-independent as well. Although the most general static metric would have $g_{ij} \neq \delta_{ij}$, the one considered here can be used to obtain all the parity-odd contributions to the $U(1)$ current and energy-momentum tensor. This form of the metric is preserved by redefinitions of time of the form

$$t' = t + \phi(x)$$

(2.6)

with the Kaluza-Klein (KK) field $a_i$ transforming like a connection

$$a'_i = a_i - \partial_i \phi .$$

(2.7)

As pointed out in [14], one can easily check that tensors with upper spatial indices and lower temporal indices are automatically invariant under the KK transformation (2.6). Combinations of the form $V_i - a_iV_0$ are also KK-invariant.

Using the variational definition of the $U(1)$ current

$$J_\mu = \frac{g_{\mu\nu}}{\sqrt{-g}}\frac{\delta S}{\delta A_\mu}$$

(2.8)

yields the following formulas for the KK-invariant components

$$J_0 = -e^\sigma \psi^\dagger \psi \quad , \quad J^i = \psi^\dagger \sigma_i \psi \quad , \quad i = 1,2 .$$

(2.9)
Similarly, using the standard formula for the energy-momentum tensor \([21, 22]\)

\[
T_{\mu\nu} = \frac{i}{4} \bar{\psi} \left[ \gamma_{\mu} \nabla_{\nu} - \nabla_{\mu} \gamma_{\nu} + (\mu \leftrightarrow \nu) \right] \psi
\]  

(2.10)

with

\[
\bar{\psi} \nabla_{\mu} = \bar{\psi} \left( \frac{\nabla_{\mu}}{2} - \frac{1}{4} \omega_{\mu} \gamma_{\nu} + i A_{\mu} \right),
\]  

(2.11)

the following expressions are obtained

\[
T_{00} = \frac{i}{2} e^\sigma (\psi^\dagger \partial \psi - \partial \psi^\dagger \psi) + e^\sigma A_0 \psi^\dagger \psi - \frac{1}{4} e^{3\sigma} \epsilon^{ij} \partial_i a_j \psi^\dagger \sigma_3 \psi,
\]

\[
T^i_0 = \frac{i}{4} e^\sigma (\psi^\dagger \partial_i \psi - \partial_i \psi^\dagger \psi) + \frac{1}{2} e^\sigma A_i \psi^\dagger \psi + \frac{1}{4} e^{3\sigma} \epsilon^{ij} \partial_j \sigma \psi^\dagger \sigma_3 \psi
\]  

(2.12)

On general grounds \([14]\) the KK-invariant components \(T^{ij}\) can have no parity-odd contributions at first derivative order and will not be considered here. See, however, our comments at the end of section 6.

In the next section we will compute a derivative expansion for the thermal Green function \(G(x, x', \omega_n)\) defined by \([23]\)

\[
\langle T \psi(-i\tau, x) \psi^\dagger(0, x') \rangle_\beta = T_0 \sum_n e^{-i\omega_n \tau} G(x, x', \omega_n),
\]  

(2.13)

where the sum runs over fermionic Matsubara frequencies \(\omega_n = \frac{2\pi}{\beta} (n + \frac{1}{2})\) and \(\beta = 1/T_0\).

In terms of this Green functions the currents and energy-momentum tensor are given by

\[
J^i = T_0 \sum_n \left[ \text{tr} \sigma_i G(x, x, \omega_n) \right],
\]

\[
J_0 = T_0 \sum_n \left[ -e^\sigma \text{tr} G(x, x, \omega_n) \right]
\]  

(2.14)

and

\[
T_{00} = T_0 \sum_n \left[ e^{3\sigma} (i\omega_n + A_0) \text{tr} G(x, x, \omega_n) - \frac{1}{4} e^{3\sigma} \epsilon^{ij} \partial_i a_j \text{tr} \sigma_3 G(x, x, \omega_n) \right],
\]

\[
T^i_0 = T_0 \sum_n \left[ \frac{i}{4} e^\sigma \text{tr} \left( \frac{\partial}{\partial x^i} G(x, x', \omega_n) - \frac{\partial}{\partial x'^i} G(x, x', \omega_n) \right) \right]_x'=x
\]

\[
+ T_0 \sum_n \left[ \frac{1}{2} e^\sigma A_i \text{tr} G(x, x, \omega_n) + \frac{1}{4} e^{3\sigma} \epsilon^{ij} \partial_j \sigma \text{tr} \sigma_3 G(x, x, \omega_n) \right]
\]

\[
+ T_0 \sum_n \left[ -\frac{1}{2} (i\omega_n + A_0) \text{tr} \sigma_i G(x, x, \omega_n) \right]
\]

\[
+ T_0 \sum_n \left[ -\frac{1}{2} e^\sigma a_i (i\omega_n + A_0) \text{tr} G(x, x, \omega_n) \right].
\]  

(2.15)
3 Derivative expansion of the Green function

In this section we compute the two-point Green function for an ideal gas of fermions at finite density and temperature in the static $U(1)$ and gravitational background considered in the previous section. As a first step, we rewrite the action as

$$S = -\int d^3x \psi^\dagger \sqrt{-g} \gamma^0 \gamma^0 [i\partial_t - H] \psi,$$

(3.1)

where the hamiltonian density operator $H$ is given by [24]

$$H = -i(\frac{1}{4} \omega_0^{ab} - iA_0) - \frac{im}{g^{00}} \gamma^0 - \frac{i}{g^{00}} \gamma^0 k \nabla_k, \quad k = 1, 2.$$

(3.2)

Next we rotate to euclidean time $t \rightarrow -i\tau$ and consider the thermal Green function $G(x, x', \omega_n)$ defined in Eq. (2.13). This Green function satisfies the following differential equation

$$-\sqrt{-g} \gamma^0 \gamma^0 (i\omega_n - H)G(x, x', \omega_n) = \delta(x - x').$$

(3.3)

For the background metric (2.5), the prefactor takes the simple form

$$-\sqrt{-g} \gamma^0 \gamma^0 = 1 - e^\sigma a \cdot \sigma,$$

(3.4)

where 1 is the $2 \times 2$ unit matrix. By construction, the hamiltonian (3.2) depends only on the background fields and their first derivatives. After some lengthy algebra we obtain

$$-\sqrt{-g} \gamma^0 \gamma^0 H = H_0 + H_1,$$

(3.5)

where

$$H_0 = -A_0 1 + e^\sigma m \sigma_3 - e^\sigma A_0 \cdot (i\partial + A - A_0 a)$$

(3.6)

$$H_1 = -\frac{e^{2\sigma}}{4} \epsilon^{ij} a_j \sigma_3 - \frac{i}{2} e^\sigma \sigma \cdot \partial \sigma.$$  

(3.7)

Then Eq. (3.3) can be written

$$[(1 - e^\sigma a \cdot \sigma) i\omega_n - H(x)] G(x, x', \omega_n) = \delta(x - x'),$$

(3.8)

where $H(x) = H_0(x) + H_1(x)$. For arbitrary non-uniform background fields this equation can not be solved in closed form, but we may consider a derivative expansion

$$G \sim G_0 + G_1 + \ldots,$$

(3.9)

where the subscript indicates the order in derivatives of the background fields. In particular, $G_0$ is the Green function for a constant background specified by the values of $A_0, \sigma, a$ and $A$ at some fixed reference point.
3.1 The Green function at leading order

The Green function at leading order is obtained by neglecting the derivatives of the background fields. Then Eq. (3.8) simplifies to

$$\left[1 - e^{\sigma(z)} a(z) \cdot \sigma \right] i \omega_n - H_0(z) \right] G_0(x, x', \omega_n) = \delta(x - x'),$$  \hspace{1cm} (3.10)

where the background fields are evaluated at the reference point \( z \) but the derivatives in \( H(z) \) act on \( x \)

$$H_0(z) = -A_0(z)1 + e^{\sigma(z)} m \sigma_3 - e^{\sigma(z)} \sigma_i \left( i \frac{\partial}{\partial x^i} + A_i(z) - A_0(z) a_i(z) \right).$$  \hspace{1cm} (3.11)

Taking the space Fourier transform gives

$$G_0(p, \omega_n) = \frac{(A_0 + i \omega_n)1 + me^\sigma \sigma_3 + e^\sigma \sigma \cdot \tilde{p}}{(A_0 + i \omega_n)^2 - e^{2\sigma}(m^2 + \tilde{p}^2)},$$  \hspace{1cm} (3.12)

where

$$\tilde{p}_i \equiv p_i - A_i + (A_0 + i \omega_n) a_i.$$  \hspace{1cm} (3.13)

Note the occurrence of the KK-invariant combination \( A_i - A_0 a_i \). As Eq. (3.13) implies the following relation in position space

$$G_0(x, x', \omega_n) = e^{i(A - (A_0 + i \omega_n) a) \cdot (x - x')} \left|_{A=a=0} \right.$$  \hspace{1cm} (3.14)

we may restrict ourselves to the case \( A = a = 0 \), adding the phase at the end of the computation. Given the structure of (3.12), it is convenient to consider first the scalar Green function with Fourier transform

$$\Delta_0(p, \omega_n) = \frac{1}{(A_0 + i \omega_n)^2 - e^{2\sigma}(m^2 + \tilde{p}^2)},$$  \hspace{1cm} (3.15)

which in position space is given by

$$\Delta_0(x, x', \omega_n) = -\frac{1}{2\pi} e^{-2\sigma} K_0 \left( |x - x'| \sqrt{m^2 - e^{-2\sigma}(A_0 + i \omega_n)^2} \right),$$  \hspace{1cm} (3.16)

where \( K_0 \) is the modified Bessel function of the second kind. Then the fermionic Green function at leading order is given by

$$G_0(x, x', \omega_n) = e^{i(A - (A_0 + i \omega_n) a) \cdot (x - x')} D_L(x) \Delta_0(x, x', \omega_n),$$  \hspace{1cm} (3.17)

where \( D_L(x) \) is the Fourier transform of the numerator in (3.12)

$$D_L(x) = (A_0 + i \omega_n)1 + m e^\sigma \sigma_3 - ie^\sigma \sigma \cdot \partial.$$  \hspace{1cm} (3.18)

This finally yields the following explicit expression for the Green function at leading order

$$G_0(x, x', \omega_n) = -\frac{e^{i(A - (A_0 + i \omega_n) a) \cdot (x - x') - 2\sigma}}{2\pi} \left[ \left( (A_0 + i \omega_n)1 + m e^\sigma \sigma_3 \right) K_0(b|x - x'|) + i(\hat{x} \cdot \sigma) bK_1(b|x - x'|) \right],$$  \hspace{1cm} (3.19)
where \[ b^2 \equiv m^2 - e^{-2\sigma}(A_0 + i\omega_n)^2 \] (3.20) and we have used
\[ \partial_i K_0(b|x|) = -\frac{x_i}{|x|}bK_1(b|x|). \] (3.21)

Remember that in (3.19) all the fields are evaluated at the reference point \( z \), which in general will be different from \( x \) and \( x' \).

### 3.2 The Green function at first derivative order

Here we will use perturbation theory to compute \( \mathcal{G}_1(i\omega_n, x, x') \), i.e., the contribution to the Green function at first order in the derivatives of the background fields. In order to simplify the calculations we will assume that, by a combination of gauge and KK transformations, the fields \( A \) and \( a \) have been set to zero at a reference point \( z \). As explained at the end of this section, the complete dependence on \( A \) and \( a \) of physical observables computed from \( \mathcal{G}_1(i\omega_n, x, x') \) can be restored at the end of the calculation by invoking gauge and KK invariance. We expand the fields around the reference point
\[
A_0(x) \simeq A_0(z) + (x^i - z^i)\partial_i A_0(z) \\
\sigma(x) \simeq \sigma(z) + (x^i - z^i)\partial_i \sigma(z) \\
A_j(x) \simeq (x^i - z^i)\partial_i A_j(z) \\
a_j(x) \simeq (x^i - z^i)\partial_i a_j(z)
\]
and, correspondingly, the hamiltonian
\[ H(x) = H_0(z) + \delta H(x) + \ldots, \] (3.23)
where \( \delta H(x) \) is linear in the derivatives of the fields and is given by
\[ \delta H(x) = (x^i - z^i) \partial_i H_0|_z + H_1(z). \] (3.24)

We must also expand the prefactor
\[ -\sqrt{-g} \gamma^0 \Gamma^0(x) \simeq 1 - e^{\sigma(z)}\delta a(x) \cdot \sigma \equiv 1 - (x^i - z^i)e^{\sigma(z)}\sigma^i \partial_j a_j(z). \] (3.25)

Then, substituting \( \mathcal{G} \sim \mathcal{G}_0 + \mathcal{G}_1 + \ldots \) into (3.8) and using these expansions yields the following differential equation for \( \mathcal{G}_1 \)
\[
(i\omega_n - H_0(z))\mathcal{G}_1(x, x', \omega_n) = (\delta H(x) + e^{\sigma(z)}\delta a(x) \cdot \sigma)\mathcal{G}_0(x, x', \omega_n). \] (3.26)

This equation is solved by the first term in a Schwinger-Dyson expansion
\[
\mathcal{G}_1(x, x', \omega_n) = \int d^2 x'' \mathcal{G}_0(x, x'', \omega_n)(\delta H(x'') + i\omega_n e^{\sigma(z)}\delta a(x'') \cdot \sigma)\mathcal{G}_0(x'', x', \omega_n). \] (3.27)

Noting that \( A = a = 0 \) at the reference point \( z \), we can use (3.17) to write
\[
\mathcal{G}_0(x, x', \omega_n)|_{A=a=0} = D_L(x)\Delta_0(x, x', \omega_n). \] (3.28)
We may also use the analogous relation
\[ G_0(x, x', \omega_n) \bigg|_{A=a=0} = \Delta_0(x, x', \omega_n) \mathcal{D}_R(x'), \]
where
\[ \mathcal{D}_R(x') = (A_0(z) + i\omega_n)1 + m e^{\sigma(z)}\sigma_i + i e^{\sigma(z)}\sigma_i \frac{\partial}{\partial x'_i}, \]
to rewrite Eq. (3.27) in terms of the scalar Green function
\[ G_1(x, x', \omega_n) = \mathcal{D}_L(x) \left[ \int d^2x'' \Delta_0(x, x'', \omega_n) \left( \delta H(x'') + i\omega_ne^{\sigma(z)}\delta a(x'') \cdot \sigma \right) \Delta_0(x'', x', \omega_n) \right] \mathcal{D}_R(x'), \]
with \( \delta a_i(x) = (x^j - z^j)\partial_j a_i(z). \) The integral over \( x'' \) in (3.31) can be turned into a Gaussian by using the proper time representation of the scalar Green function
\[ \Delta_0(x, x', \omega_n) = -\frac{e^{-2\sigma}}{2\pi} K_0 \left( b|z - x'| \right) = -\frac{e^{-2\sigma}}{2\pi} \int_0^\infty \frac{ds}{2s} e^{-\frac{|x-x'|^2}{4s} - b^2s}, \]
where \( b \) has been defined in Eq. (3.20). The evaluation of \( G_1(x, x', \omega_n) \) is straightforward in principle but rather lengthy and is best carried out by computer.

According to Eq. (3.22) all the fields and their derivatives, including those present in \( \mathcal{D}_L(x) \) and \( \mathcal{D}_R(x), \) are evaluated at the reference point \( z, \) but at the end of the calculation we may set \( z = x. \) We may also restore the whole dependence on \( A_i \) and \( a_i, \) which have been set to zero at the reference point, by invoking gauge and KK invariance of the components of the currents and energy-momentum tensor. In practice, this amounts to the substitution
\[ \epsilon^{ij} \partial_i A_j \rightarrow \epsilon^{ij} (\partial_i A_j + a_i \partial_j A_0), \]
which is justified by noting that the right hand side can be written as
\[ \epsilon^{ij} (\partial_i A_j + a_i \partial_j A_0) = \epsilon^{ij} \partial_i \tilde{A}_j + \epsilon^{ij} A_0 \partial_i a_j, \]
where \( \tilde{A}_i = A_i - A_0 a_i. \) This is obviously KK and gauge invariant and reduces to the left hand side for \( a_i = 0. \) The complete expression for \( G_1(x, x', \omega_n) \) is rather cumbersome and will not be given here. Instead, in the next section we will extract the relevant parity odd pieces from \( G_1(x, x', \omega_n) \) and \( G_0(x, x', \omega_n). \)

4 Computation of the \( U(1) \) current and energy-momentum tensor

In this section we will use the results of the previous section to derive the parity odd contributions to the currents and energy-momentum tensor. A look at Eq. (3.19) shows that there are no parity odd contributions to the Green function at leading order. As a consequence, in order to extract the parity-odd contributions to Eqs. (2.14) and (2.15) we must use the leading approximation \( G_0 \) to the Green function for terms that contain
explicit derivatives of the background fields and $G_1$ otherwise. For instance, the formula for $T_{00}$ becomes

$$T_{00} = T_0 \sum_n \left[ e^\sigma (i\omega_n + A_0) \text{tr} G_1(x, x, \omega_n) - \frac{1}{4} e^{3\sigma} \epsilon^{ij} \partial_i a_j \text{tr} \sigma_3 G_0(x, x, \omega_n) \right].$$

(4.1)

Then simple inspection of Eqs. (2.14) and (2.15) shows that only four traces of the Green functions are needed. Using the results in the previous section for $G_0$ (Eq. (3.19)) and $G_1$ (Eq. (3.31)), one finds that the following combination vanishes

$$\text{tr} \left( \frac{\partial}{\partial x^i} G_1(x, x', \omega_n) - \frac{\partial}{\partial x^0} G_1(x, x', \omega_n) \right) \bigg|_{x' = x} = 0,$$

(4.2)

while the remaining traces are given by

$$\text{tr} G_1(x, x, \omega_n) = -\frac{me^{-\sigma}}{4\pi} T_0 \sum_n \int_0^\infty ds e^{-b^2 s} \left[ 2\epsilon^{ij} (\partial_i A_j + a_i \partial_j A_0) - (A_0 + i\omega_n) \epsilon^{ij} \partial_i a_j \right],$$

$$\text{tr} \sigma_1 G_1(x, x, \omega_n) = -\frac{me^{-2\sigma}}{2\pi} T_0 \sum_n \int_0^\infty ds e^{-b^2 s} \left[ \epsilon^{ij} \partial_j A_0 - (A_0 + i\omega_n) \epsilon^{ij} \partial_j \sigma \right],$$

$$\text{tr} \sigma_3 G_0(x, x, \omega_n) = -\frac{me^{-\sigma}}{2\pi} T_0 \sum_n \int_0^\infty \frac{ds}{s} e^{-b^2 s}. $$

(4.3)

As mentioned above, all the sum run over fermionic Matsubara frequencies $\omega_n = \frac{2\pi n}{\beta} (n + \frac{1}{2})$ with $\beta = 1/T_0$. Note that these are the complete expressions for the traces and, as a consequence, all the contributions at first derivative order are parity violating. This is consistent with the fact that one can not construct a parity invariant contribution to the partition function at this order.

Substitution of these expressions into Eqs. (2.14) and (2.15) gives the following formulas

$$J^i = -\frac{me^{-2\sigma}}{2\pi} \epsilon^{ij} [I_0 \partial_j A_0 - I_1 \partial_j \sigma],$$

$$J_0 = \frac{m}{4\pi} \epsilon^{ij} [2I_0 (\partial_i A_j + a_i \partial_j A_0) - I_1 \partial_i a_j],$$

$$T^i_0 = \frac{me^{-2\sigma}}{4\pi} \epsilon^{ij} [I_1 \partial_j A_0 - I_2 \partial_j \sigma],$$

$$T_{00} = -\frac{m}{4\pi} \epsilon^{ij} [2I_1 (\partial_i A_j + a_i \partial_j A_0) - I_2 \partial_i a_j],$$

(4.4)

where the first order contributions to the currents and energy-momentum tensor have been expressed in terms of just three Matsubara sums

$$I_0 = T_0 \sum_n \int_0^\infty ds e^{-b^2 s}$$

$$I_1 = T_0 \sum_n \int_0^\infty ds e^{-b^2 s} (A_0 + i\omega_n)$$

$$I_2 = T_0 \sum_n \int_0^\infty ds e^{-b^2 s} \left( A_0 + i\omega_n \right)^2.$$ 

(4.5)
with $b^2 = m^2 - e^{-2\sigma}(A_0 + i\omega_n)^2$ according to Eq. (3.20). Doing the sum over Matsubara frequencies as described in the Appendix finally yields the following results

\[ J^i = \frac{e^{-\sigma}}{8\pi} f_- (\sigma, A_0) e^{ij} \partial_j A_0 - \frac{m}{8\pi} f_+ (\sigma, A_0) e^{ij} \partial_j \sigma, \]

\[ J_0 = -\frac{e^\sigma}{8\pi} f_- (\sigma, A_0) e^{ij} (\partial_i A_j + a_i \partial_j A_0) + \frac{me^{2\sigma}}{16\pi} f_+ (\sigma, A_0) e^{ij} \partial_j a_j, \]

\[ T_0^i = -\frac{m}{16\pi} f_+ (\sigma, A_0) e^{ij} \partial_j A_0 + \frac{m^2 e^{2\sigma}}{16\pi} f_+ (\sigma, A_0) e^{ij} \partial_j \sigma + \frac{m}{8\pi \beta} f_0 (\sigma, A_0) e^{ij} \partial_j a_j, \]

\[ T_{00} = \frac{me^{2\sigma}}{8\pi} f_+ (\sigma, A_0) e^{ij} (\partial_i A_j + a_i \partial_j A_0) - \frac{m^2 e^{3\sigma}}{16\pi} f_- (\sigma, A_0) e^{ij} \partial_j a_j \]

\[ - \frac{me^{2\sigma}}{8\pi \beta} f_0 (\sigma, A_0) e^{ij} \partial_j a_j, \]

where we have defined the functions

\[ f_\pm (\sigma, A_0) = \tanh \left[ \frac{\beta}{2} (A_0 - e^\sigma m) \right] \pm \tanh \left[ \frac{\beta}{2} (A_0 + e^\sigma m) \right] , \]

\[ f_0 (\sigma, A_0) = \log \left[ 2 \cosh (A_0 \beta) + 2 \cosh (e^\sigma \beta m) \right]. \]

Equations (4.6) to (4.11) are the main results in this section. It is worth mentioning that, as observed in the Appendix, the sums over Matsubara frequencies (4.5) are finite without subtractions, and so are all the contributions to the currents and energy-momentum tensor at first derivative order.

5 The parity-odd equilibrium partition function

In this section our results for the parity odd contributions to the currents and energy-momentum tensor are used to obtain a completely explicit expression for the equilibrium partition function. The general form of the parity odd partition function at first order in the derivative expansion has been given in [14]

\[ \mathcal{W} = \frac{1}{2} \int d^2 x \left( \alpha (\sigma, A_0) e^{ij} \partial_i \tilde{A}_j + T_0 \beta (\sigma, A_0) e^{ij} \partial_i A_j \right), \]

where $\tilde{A}_i = A_i - A_0 a_i$. Then using the variational formulae [14]

\[ J^i = \frac{T_0}{\sqrt{-g}} \frac{\delta \mathcal{W}}{\delta A_i} , \quad J_0 = -\frac{e^{2\sigma} T_0}{\sqrt{-g}} \frac{\delta \mathcal{W}}{\delta A_0} \]

\[ T_0^i = \frac{T_0}{\sqrt{-g}} \left( \frac{\delta \mathcal{W}}{\delta a_i} - A_0 \frac{\delta \mathcal{W}}{\delta A_i} \right) , \quad T_{00} = -\frac{e^{2\sigma} T_0}{\sqrt{-g}} \frac{\delta \mathcal{W}}{\delta \sigma} \]

with Eq. (5.1) gives

\[ J^i = T_0 e^{-\sigma} e^{ij} \partial_j \alpha \]

\[ J_0 = -T_0 e^{\sigma} \left( \frac{\partial \alpha}{\partial A_0} e^{ij} \partial_i \tilde{A}_j + T_0 \frac{\partial \beta}{\partial A_0} e^{ij} \partial_i a_j \right) \]

\[ T_0^i = T_0 e^{-\sigma} (T_0 e^{ij} \partial_j \beta - A_0 e^{ij} \partial_j \alpha) \]

\[ T_{00} = -T_0 e^{\sigma} \left( \frac{\partial \alpha}{\partial \sigma} e^{ij} \partial_i \tilde{A}_j + T_0 \frac{\partial \beta}{\partial \sigma} e^{ij} \partial_i a_j \right). \]
Using (4.6) for $J_i$, Eq. (5.3) can be readily integrated, yielding
\[ \alpha(\sigma, A_0) = \frac{1}{4\pi} \log \left[ \cosh \left( \frac{\beta}{2} (A_0 - e^\sigma m) \right) \right] - \frac{1}{4\pi} \log \left[ \cosh \left( \frac{\beta}{2} (A_0 + e^\sigma m) \right) \right]. \] (5.7)

One can then use, for instance, Eqs. (4.8) and (5.5) for $T^i_0$ to obtain
\[ \beta(\sigma, A_0) = -\frac{\beta^2}{4\pi} A_0 e^\sigma m + \frac{\beta}{8\pi} e^\sigma m \log \left[ 2 \cosh (A_0 \beta) + 2 \cosh (e^\sigma \beta m) \right] - \frac{\beta}{4\pi} (A_0 + e^\sigma m) \log \left[ 1 + e^{-\beta(A_0 + e^\sigma m)} \right] + \frac{\beta}{4\pi} (A_0 - e^\sigma m) \log \left[ 1 + e^{-\beta(A_0 - e^\sigma m)} \right] + \frac{1}{4\pi} \text{Li}_2 \left[ -e^{-\beta(A_0 + e^\sigma m)} \right] - \frac{1}{4\pi} \text{Li}_2 \left[ -e^{-\beta(A_0 - e^\sigma m)} \right] \right], \] (5.8)

where $\text{Li}_2$ is the polylogarithm function. It can be easily checked that the expressions for $J_0$ and $T_{00}$ derived from the partition function $\mathcal{W}$ agree with the results obtained in the last section. Note that the existence of a (unique) solution implies non-trivial integrability conditions for the currents and energy-momentum tensor, thus providing a stringent check of the correctness of our results. Eqs. (5.7) and (5.8) are the main results in this paper.

In the limit of small fermion mass the functions $\alpha$ and $\beta$ take the following form
\[ \alpha(\sigma, A_0) \to -\frac{me^\sigma \beta}{4\pi} \tanh \left( \frac{A_0 \beta}{2} \right) + \mathcal{O}(m^2) \]
\[ \beta(\sigma, A_0) \to -\frac{me^\sigma \beta}{8\pi} \left[ -\log \left[ 2 (1 + \cosh (A_0 \beta)) \right] + 2 A_0 \beta \tanh \left( \frac{A_0 \beta}{2} \right) \right] + \mathcal{O}(m^2) \] (5.9)

and vanish for massless fermions. However, this does not imply the absence of parity violating effects in this limit [17]. The reason is that, even though the contributions to the currents and energy-momentum tensor at first derivative order are finite, there are divergent contributions at leading order. Specifically, at zero derivative order $T_{00}$ is given by
\[ T_{00} = T_0 \sum_n \sigma (i \omega_n + A_0) \text{ tr } \mathcal{G}_0(\sigma, x, \omega_n) = -\frac{e^{-\sigma}}{2\pi} \sum_n \int_0^\infty \frac{ds}{s} e^{-bs} (A_0 + i \omega_n)^2. \] (5.10)

The divergence can be extracted in the $T_0 \to 0$ limit, where the sum over Matsubara frequencies reduces to an integral. In this limit we find
\[ T_{00}^{\text{div}} = \lim_{\epsilon \to 0} \frac{e^{2\pi}}{8\pi^{3/2}} \int_{-\infty}^{\infty} \frac{ds}{s^{5/2}} e^{-m^2 s} = \lim_{\epsilon \to 0} \frac{e^{2\pi}}{12\pi^{3/2}} \left[ e^{-3/2} - 3m^2 \epsilon^{-1/2} + \ldots \right], \] (5.11)

where the dots stand for contributions that are finite in the $\epsilon \to 0$ limit.

Thus the theory has to be regularized, and this can be done in a gauge-invariant way by means of a Pauli-Villars regulator, which amounts to the introduction of auxiliary fermions with large masses $\{M_i\}$ and weights $\{C_i\}$. In order to cancel the divergences in Eq. (5.11) we must impose
\[ \sum_{i=0}^n C_i = 0 \quad \sum_{i=0}^n C_i M_i^2 = 0, \] (5.12)
where $i = 0$ refers to the physical fermion, i.e., $C_0 = 1$ and $M_0 = m$. A minimal system satisfying these constraints includes three auxiliary fermions with weights $C_1 = C_2 = -1$, $C_3 = 1$, and large masses $M_1 = M_2 = M, M_3 = \sqrt{2}M$ of the same sign as the mass of the physical fermion\footnote{Strictly speaking, Eq. (5.12) leaves the signs of the large masses undetermined. However, Pauli-Villars gives the correct coefficient for the radiatively induced Chern-Simons term in the effective action computed by other methods \cite{18} only if the masses of the auxiliary fermions have the same sign as the physical fermion.}. Thus, in the regularized theory instead of Eq. (5.1) we must consider
\[
W_{\text{reg}} = \lim_{M \to \infty} \left( W(m) - 2W(M) + W(\sqrt{2}M) \right). \tag{5.13}
\]
Now, the infinite mass limits
\[
\lim_{M \to \infty} \alpha(\sigma, A_0) = -\frac{A_0 \beta}{4\pi} \text{sgn}(M), \quad \lim_{M \to \infty} \beta(\sigma, A_0) = -\frac{A_0^2 \beta^2}{8\pi} \text{sgn}(M), \tag{5.14}
\]
where sgn is the sign function, and Eq. (5.13) with sgn($M$) = sgn($m$), imply the following contributions from the auxiliary fields
\[
\delta \alpha(\sigma, A_0) = \frac{A_0 \beta}{4\pi} \text{sgn}(m), \quad \delta \beta(\sigma, A_0) = \frac{A_0^2 \beta^2}{8\pi} \text{sgn}(m). \tag{5.15}
\]
Then, Eqs. (5.3)-(5.6) yield
\[
\delta J^i = -\frac{e_\sigma}{4\pi} \text{sgn}(m) e^{ij} \partial_j A_0, \quad \delta J_0 = \frac{e_\sigma}{4\pi} \text{sgn}(m) e^{ij} (\partial_i A_j + a_i \partial_j A_0), \tag{5.16}
\]
together with $\delta T_{00} = \delta T_0 = 0$. Thus, at first order in the derivative expansion we find parity-odd contributions to the currents of massless fermions, but not to their energy-momentum tensor. Eq. (5.16) differs from the well known result for the currents in the absence of gravitational fields \cite{18} by a term proportional to $e^{ij} a_i \partial_j A_0$. This term represents a mixed gauge-gravitational contribution to the parity anomaly. As mentioned in the introduction, in 2 + 1 dimensions the purely gravitational contribution at zero temperature takes the form of the Cotton tensor \cite{19} which, being of third derivative order, can not be seen here. Note also that, for massive fermions, the contribution from the auxiliary fields (5.16) must be added to the finite results in Eqs. (4.6)-(4.7).

6 Discussion

In this paper we have used perturbation theory combined with Schwinger’s proper time method to obtain a derivative expansion for the thermal two-point Green function of an ideal gas of massive fermions in non-trivial static gauge and gravitational backgrounds in 2 + 1 dimensions. After relating the currents and energy-momentum tensor to traces of the Green function (2.14)-(2.15), we have extracted all the parity violating contributions at first derivative order in the background fields. These contributions are finite without subtractions and are explicitly given by Eqs. (4.6)-(4.10). We have also shown that there are no parity preserving contributions at first order in the derivative expansion.
These results have been used to obtain a completely explicit expression for the equilibrium partition function, determining the two unknown functions $\alpha(\sigma, A_0)$ and $\beta(\sigma, A_0)$ defined in [14]. The role of a Pauli-Villars regulator has also been analyzed showing that, at first derivative order, massless fermions have parity-odd contributions to their currents but not to their energy-momentum tensor (5.16).

As discussed in detail in [14], several adiabatic transport coefficients such as Hall electric and thermal conductivities can be derived from the knowledge of the functions $\alpha(\sigma, A_0)$ and $\beta(\sigma, A_0)$ in the equilibrium partition function (5.1). There are, however, additional adiabatic transport coefficients which can not be derived in this formulation. Among them is the Hall viscosity, which involves the spatial components $T^{ij}$ of the energy-momentum tensor and has been connected with the intrinsic angular momentum density [25] and with the gravitational response of the system in the presence of geometric torsion [26, 27]. For the torsionless backgrounds considered in this paper and in [14], the spatial components $T^{ij}$ vanish at first derivative order.

Although the possible interplay between torsion and Hall viscosity has been recently questioned [28], it seems to us that a more detailed analysis including non-zero torsion might be of interest. Indeed, in $2 + 1$ dimensions the angular momentum density is proportional to $\bar{\psi}\psi$ and acquires a non-zero equilibrium expectation value, in contrast to the situation in $3 + 1$ dimensions. Since this quantity is the zero component of a conserved current, it should be possible to include it in the thermodynamic description by switching on the appropriate conjugate source, which must be a component of the contortion tensor (see, for instance, ref. [29]). This component would play the role of additional background data in the partition function. It is not clear to us that the spatial components of the energy-momentum tensor $T^{ij}$ would continue to vanish after these modifications. We hope to pursue these issues in future work.

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A Matsubara sums

In this Appendix we evaluate the three sums over the Matsubara frequencies $\omega_n = \frac{2\pi}{\beta}(n + \frac{1}{2})$ which appeared in section 4:

$$I_0 = T_0 \sum_n \int_0^\infty ds e^{-b^2s}$$

$$I_1 = T_0 \sum_n \int_0^\infty ds e^{-b^2s}(A_0 + i\omega_n)$$

$$I_2 = T_0 \sum_n \int_0^\infty ds e^{-b^2s} \left( A_0 + i\omega_n \right)^2 + \frac{e^{2\sigma}}{2s}$$ (A.1)
where \( b^2 = m^2 - e^{-2\sigma}(A_0 + i\omega_n)^2 \). The evaluation of \( I_0 \) and \( I_1 \) is straightforward if one does the integrals first. For instance
\[
I_0 = T_0 \sum_n \int_0^\infty ds e^{-b^2 s} = T_0 \sum_n \frac{1}{b^2} = T_0 \sum_n \frac{1}{m^2 - e^{-2\sigma}(A_0 + i\omega_n)^2} = -e^{\sigma} \left[ \tanh \frac{\beta}{2}(A_0 - e^\sigma m) - \tanh \frac{\beta}{2}(A_0 + e^\sigma m) \right].
\] (A.2)

In the same way one obtains
\[
I_1 = -e^{2\sigma} \left[ \tanh \frac{\beta}{2}(A_0 - e^\sigma m) + \tanh \frac{\beta}{2}(A_0 + e^\sigma m) \right].
\] (A.3)

Obviously, we can not apply this method to \( I_2 \), as the integral of the second term diverges at the lower limit. Instead, we use the identity
\[
(A_0 + i\omega_n)^2 e^{-b^2 s} = e^{2\sigma} \left( \frac{d}{ds} + m^2 \right) e^{-b^2 s}
\] (A.4)
to write
\[
I_2 = T_0 \sum_n \int_0^\infty ds \left( \frac{d}{ds} + 1 \frac{1}{2s} + m^2 \right) e^{-b^2 s + 2\sigma}.
\] (A.5)

The required sum evaluates to
\[
T_0 \sum_n e^{-b^2 s + 2\sigma} = e^{-m^2 s + 3\sigma} \frac{1}{2\sqrt{\pi s}} \vartheta_3 \left( \frac{1}{2}(\pi - iA_0 \beta), e^{-\frac{2\sigma s^2}{4}} \right),
\] (A.6)

where \( \vartheta_3 \) is a Jacobi \( \Theta \) function, which admits the expansion
\[
\vartheta_3(u, q) = 1 + 2 \sum_{n=1}^\infty q^n \cos(2nu).
\] (A.7)

This gives
\[
T_0 \sum_n \left( \frac{d}{ds} + \frac{1}{2s} + m^2 \right) e^{-b^2 s + 2\sigma} = \frac{\beta^2 e^{5\sigma}}{8\sqrt{\pi s^{5/2}}} \exp[-\frac{e^{2\sigma} \beta^2}{4s}] \vartheta_3' \left( \frac{1}{2}(\pi - iA_0 \beta), e^{-\frac{2\sigma s^2}{4s}} \right),
\] (A.8)

where
\[
\vartheta_3'(u, q) = \frac{\partial \vartheta_3(u, q)}{\partial q} = 2 \sum_{n=1}^\infty n^2 q^{n-1} \cos(2nu).
\] (A.9)

Now the integral over \( s \) can be done term by term without encountering any divergence, giving
\[
I_2 = \sum_{n=1}^\infty e^{-\sigma|n|n\beta + 3\sigma} \left( m + \frac{e^{-\sigma}}{n\beta} \right) (-)^n \cosh(A_0 n \beta).
\] (A.10)

Finally, summing the trigonometric series yields
\[
I_2 = -\frac{me^{3\sigma}}{4} \left[ \tanh \frac{\beta}{2}(A_0 - e^\sigma m) - \tanh \frac{\beta}{2}(A_0 + e^\sigma m) \right]
- \frac{e^{2\sigma}}{2\beta} \log \left[ 2 \cosh(A_0 \beta) + 2 \cosh(e^\sigma \beta m) \right].
\] (A.11)
Note that the three sums computed in this Appendix are finite from the outset, without the need for infinite subtractions: the potential divergence in the integral of the second term of $I_2$ cancels against another divergence from the first term, and the net result is finite.

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