Mean-square Stabilizability via Output Feedback for Non-minimum Phase Networked Feedback Systems

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Abstract

This work studies mean-square stabilizability via output feedback for a networked linear time invariant (LTI) feedback system with a non-minimum phase plant. In the feedback system, the control signals are transmitted to the plant over a set of parallel communication channels with possible packet dropout. Our goal is to analytically describe intrinsic constraints among channel packet dropout probabilities and the plant’s characteristics, such as unstable poles, non-minimum phase zeros in the mean-square stabilizability of the system. It turns out that this is a very hard problem. Here, we focus on the case in which the plant has relative degree one and each non-minimum zero of the plant is only associated with one of control input channels. Then, the admissible region of packet dropout probabilities in the mean-square stabilizability of the system is obtained. Moreover, a set of hyper-rectangles in this region is presented in terms of the plant’s non-minimum phase zeros, unstable poles and Wonham decomposition forms which is related to the structure of controllable subspace of the plant. When the non-minimum phase zeros are void, it is found that the supremum of packet dropout probabilities’ product in the admissible region is determined by the product of plant’s unstable poles only. A numerical example is presented to illustrate the fundamental constraints in the mean-square stabilizability of the networked system.

Key words: Networked control system, output feedback, mean-square stabilization, non-minimum phase zero

1 Introduction

In the last two decades, stabilization problems for networked feedback systems have attracted a great amount of research interests (for example, see [4], [6], [9], [10], [13] and the references therein). These works mainly focus on coping with new challenges caused by limited resources, uncertainties/unreliability in communication channels. Indeed, great success has been achieved in this research area, in particular, for stabilization via state feedback. In [1], networked multi-input multi-output (MIMO) LTI feedback systems are studied where control signals are transmitted to actuators over fading channels. Uncertainties in the channels are modeled as multiplicative noises and then a design scheme is presented for mean-square stabilization via state feedback. Moreover, fundamental constraints in mean-square stabilizability caused by channel uncertainties are studied for the networked systems in [1]. It is shown for a networked single-input feedback system that the minimum capacity required for mean-square stabilization via state feedback is determined by the product of all the unstable poles of the plant. In [16], this problem is studied for a networked MIMO system where the total capacity of the feedback control channels is given and can be allocated to each channels associated with individual control inputs. It is found that the minimum total channel capacity for the mean-square stabilization problem is also determined by the product of all the unstable poles of the plant. Some new developments in stabilization and state estimation for networked systems over packet dropping channels are presented in [2] for systems with both actuators and sensors connected to controllers over communication channels.

In this work, we study the mean-square stabilizability via output feedback for a networked MIMO LTI system where the control signals are transmitted over packet dropping channels. The channel uncertainties are also modeled as multiplicative noises. The difficulties for mean-square stabilization with multiplicative noises are well recognized (see e.g.
For any complex number $z$, we denote its complex conjugate by $\overline{z}$. For any vector $u$, we denote its transpose by $u^T$ and conjugate transpose by $u^*$. For any matrix $A$, the transpose, conjugate transpose, spectral radius and trace are denoted by $A^T$, $A^*$, $\rho(A)$ and $\text{Tr}(A)$, respectively. Denote a state-space model of an LTI system by $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. For any real rational function matrix $G(z)$, $z \in \mathbb{C}$, define $G^r(z) = G^T(1/z)$. Denote the expectation operator by $E\{\cdot\}$. Let the open unit disc be denoted by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, the closed unit disc by $\bar{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$, the unit circle by $\partial \mathbb{D}$, and the complements of $\mathbb{D}$ and $\bar{\mathbb{D}}$ by $\mathbb{D}^c$ and $\bar{\mathbb{D}}^c$, respectively. The space $\mathcal{L}_2$ is a Hilbert space and consists of all complex matrix functions $G(z)$ which are measurable in $\partial \mathbb{D}$ and

$$\frac{1}{2\pi} \int_{-\pi}^\pi \text{Tr} \left[ G^*(e^{j\theta}) G(e^{j\theta}) \right] d\theta < \infty.$$

For $F, G \in \mathcal{L}_2$, the inner product is defined as

$$\langle F, G \rangle = \frac{1}{2\pi} \int_{-\pi}^\pi \text{Tr} \left[ F^*(e^{j\theta}) G(e^{j\theta}) \right] d\theta$$

and the induced norm is defined by

$$\|G\|_2 = \sqrt{\langle G, G \rangle}.$$  

It is well-known that $\mathcal{L}_2$ admits an orthogonal decomposition into the subspaces

$$\mathcal{H}_2 := \left\{ G : G(z) \text{ analytic in } \mathbb{D}^c, \right\}$$

and

$$\mathcal{H}_2^\perp := \left\{ G : G(z) \text{ analytic in } \mathbb{D}, G(0) = 0, \right\}$$

Note that for any $F \in \mathcal{H}_2^\perp$ and $G \in \mathcal{H}_2$,  

$$\langle F, G \rangle = 0.$$  

Define the Hardy space

$$\mathcal{H}_\infty := \left\{ G : G(z) \text{ bounded and analytic in } \mathbb{D}^c \right\}.$$

A subset of $\mathcal{H}_\infty$, denoted by $\mathcal{H}_2^r$, is the set of all proper stable rational transfer function matrices in the discrete-time sense. Note that we have used the same notation $\| \cdot \|_2$ to denote the corresponding norm for spaces $\mathcal{L}_2$, $\mathcal{H}_2$ and $\mathcal{H}_2^r$.

### 2 Problem Formulation

The networked feedback system under study is depicted in Fig. 1. The plant $G$ in the system is a MIMO LTI system and the signal $y(k)$ is the measurement. The control signal $u(k)$ for the plant is generated by the feedback controller $K$. It includes $r$ entries $u_1(k), \ldots, u_r(k)$ which are sent to the plant $G$ over $r$ parallel packet dropping channels, respectively. The signal $v(k) = [v_1(k), \ldots, v_r(k)]^T$ is the received control signal at the plant side.

Let $\{\alpha_j(k), k = 0, 1, 2, \ldots, \infty\}, j = 1, \ldots, r$ be random processes with independent identical Bernoulli probability distributions, respectively. It indicates the receipt of the control signal $u(k)$, i.e., $\alpha_j(k) = 1$ if $u_j(k)$ is received, otherwise
\( \alpha_j(0) = 0 \). Let the probability of \( \alpha_j(k) = 0 \) be \( p_j \). The averaged receiving rate of data packets is \( \mathbf{E}(\alpha_j(k)) = 1 - p_j \) in the \( j \)-th channel. Let \( \omega_j(k) = \alpha_j(k) - (1 - p_j) \). Subsequently, the received control signal \( v_j(k) \) is written as:

\[
v_j(k) = \alpha_j(k)u_j(k) = (1 - p_j)u_j(k) + \omega_j(k)u_j(k). \tag{4}\]

It is clear that \( \{ \omega_j(k), k = 0, 1, 2, \ldots, \infty \}, j = 1, \ldots, r \) have independent identical probability distributions, referred to as \( i.i.d \) random processes, respectively. The \( i.i.d \) random process \( \{ \omega_j(k), k = 0, 1, 2, \ldots, \infty \} \) has zero mean and variance \((1 - p_j)p_j \). Now, it is assumed that \( \{ \alpha_j(k) \}, j = 1, \ldots, r \) are mutually independent. And then, it holds for any \( i, j \in \{ 1, \ldots, r \}, i \neq j \) that \( \mathbf{E}(\omega_i(k_1)\omega_j(k_2)) = 0, \forall k_1, k_2 > 0 \).

Denote the averaged channel gain by

\[
\mu = \text{diag}\{1 - p_1, \ldots, 1 - p_r\}
\]

and the multiplicative noise in the channels by

\[
\omega(k) = \text{diag}\{\omega_1(k), \ldots, \omega_r(k)\}. \tag{5}\]

It follows from the discussion above that \( \mathbf{E}(\omega(k)) = 0 \) and

\[
\mathbf{E}(\omega(k)\omega^T(k)) = \text{diag}\{p_1(1 - p_1), \ldots, p_r(1 - p_r)\}.
\]

Let \( \hat{\omega}(k) = \mu^{-1}\omega(k) \). From (4), the packet dropout channels in the system shown in Fig. 1 are modeled as follows (also see [11]):

\[
v(k) = \mu u(k) + \mu \hat{\omega}(k)u(k). \tag{6}\]

It is verified from the mean and covariance of \( \omega(k), k = 0, 1, 2, \ldots \) that

\[
\mathbf{E}(\hat{\omega}(k)) = 0 \quad \text{and} \quad \mathbf{E}(\hat{\omega}(k)\hat{\omega}^T(k)) = \Sigma
\]

where

\[
\Sigma = \text{diag}\left\{ \frac{p_1}{1 - p_1}, \ldots, \frac{p_r}{1 - p_r} \right\}.
\]

**Definition 1 (see [14])** For any initial state, if it holds for the control signal and the output that

\[
\lim_{k \to \infty} \mathbf{E} \{u(k)u^T(k)\} = 0, \quad \lim_{k \to \infty} \mathbf{E} \{v(k)v^T(k)\} = 0,
\]

then the feedback system in Fig. 1 is said to be mean-square stable.

To study the mean-square stability for the networked feedback system in Fig. 1, it is re-diagrammed as an LTI system with a multiplicative noise as shown in Fig. 2. Let \( \Delta(k) = \hat{\omega}(k)u(k) \). The channel model (6) is rewritten as

\[
v(k) = \mu u(k) + \mu \Delta(k).
\]

Thus, the transfer function \( T \) from \( \Delta(k) \) to \( u(k) \) in the nominal system is given by

\[
T = (I - KG\mu)^{-1}KG\mu \tag{7}
\]

where \( G\mu \) is considered as a new plant involved with the averaged gain of the channel. Let \( \hat{T}_{ij}, i, j = 1, \ldots, r \) be the \( \{i, j\} \)-th entry of the transfer function matrix \( T \) and

\[
\hat{T} = \begin{bmatrix}
||T_{11}||_2^2 & \cdots & ||T_{1r}||_2^2 \\
\vdots & \ddots & \vdots \\
||T_{r1}||_2^2 & \cdots & ||T_{rr}||_2^2
\end{bmatrix}.
\]

**Lemma 1 (see [8])** The LTI system with a multiplicative noise in Fig. 2 is mean-square stable if and only if it holds that

\[
\rho(\hat{T}\Sigma) < 1. \tag{9}
\]

To design an output feedback controller \( K \) which stabilizes the system in Fig. 2 in the mean-square sense is referred to as mean-square stabilization via output feedback. If this problem is solvable, the system is referred to as mean-square stabilizable. Intuitively, the mean-square stabilizability of the system is related to the packet dropout probabilities \( p_1, \ldots, p_r \) and the transfer function \( T \) of the nominal closed-loop system. In this work, fundamental constraints in mean-square stabilizability via output feedback are studied for the networked system in terms of the packet dropout probabilities and characteristics of the plant \( G \).

### 3 Upper Triangular Coprime Factorization

To study the mean-square stabilizability of the networked system, we consider the set of all possible stabilizing controllers for the plant \( G\mu \), which is described by Youla parametrization in terms of its coprime factorizations. A useful tool for the mean-square stabilization design, referred
to as upper triangular coprime factorization, is introduced in this section.

Suppose that the state-space model of the plant \( G \mu \) is given by \( G \mu = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \). and \( \{A, B\} \) is controllable, \( \{A, C\} \) is detectable. Let the right coprime factorization of the plant \( G \mu \) be \( NM^{-1} \), where the factors \( N \) and \( M \) are from \( \mathbb{R} \mathcal{H}_\infty \). Moreover, \( N \) and \( M \) are given by

\[
M = I - F(zI - A + BF)^{-1}B, \quad N = C(zI - A + BF)^{-1}B, \tag{10}
\]

\[
\text{where } F \text{ is any stabilizing state feedback gain (for details, see e.g. [17]).}
\]

It is shown in [15] that, with certain state transformation, the state-space model of \( G \mu \) can be transformed into so-called Wonham decomposition form \( \begin{bmatrix} A_w & B_w \\ C_w & 0 \end{bmatrix} \) with

\[
A_w = \begin{bmatrix} A_1 & \cdots & \ast \\ 0 & A_2 & \cdots & \ast \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_r \end{bmatrix}, \quad B_w = \begin{bmatrix} b_1 & \cdots & \ast \\ 0 & b_2 & \cdots & \ast \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_r \end{bmatrix},
\]

where

\[
A_j = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_{j1} & -a_{j(j-1)} & -a_{j(j-2)} & \cdots & -a_{j1} \end{bmatrix}, \quad b_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{12}
\]

Since the pairs \( \{A_j, b_j\}, j = 1, \ldots, r \), are all controllable, it is always possible to find row vectors \( f_j \) such that \( A_j + b_j f_j \) is stable for all \( j = 1, \ldots, r \). Now, we select a block diagonal state feedback gain \( F = \text{diag} \{ f_1, f_2, \ldots, f_r \} \). Applying Wonham decomposition forms and the state feedback gain \( F \) into (10) and (11) yields a right coprime factorization \( G \mu = NM^{-1} \) in which the factor \( M \) is an upper triangular matrix. In this work, this coprime factorization is referred to as upper triangular coprime factorization. It is summarized in the following result.

**Lemma 2** For a given plant \( G \mu \), there exist coprime matrices \( N \) and \( M \in \mathbb{R} \mathcal{H}_\infty \) such that \( G \mu = NM^{-1} \) and the matrix \( M \) is an upper triangular matrix. Furthermore, the diagonal elements \( m_{jj}, j = 1, \ldots, r \) of \( M \) are given by

\[
m_{jj} = 1 - f_j(zI - A_j + b_j f_j)^{-1}b_j, \quad j = 1, \ldots, r.
\]

Taking account of the structures of \( A_j \) and \( b_j \), we can see that the numerator polynomial of \( m_{jj} \) is the characteristic polynomial of \( A_j \). Denote the unstable poles of \( A_j \) by \( \lambda_{j1}, \ldots, \lambda_{jl} \). Note the fact that \( \{A_j, b_j\} \) is controllable. By selecting a proper \( f_j \), the poles of \( m_{jj} \) are assigned as \( 1/\lambda_{j1}, \ldots, 1/\lambda_{jl} \) and all stable poles of \( A_j \). This yields that the diagonal elements \( m_{jj} \) is given by

\[
m_{jj} = \frac{(z - \lambda_{j1}) \times \cdots \times (z - \lambda_{jl})}{(\lambda_{j1}^* z - 1) \times \cdots \times (\lambda_{jl}^* z - 1)}.
\]

It is clear that \( m_{jj} \) is an inner, i.e., \( m_{jj}(z) \big|_{m_{jj}(z) = 1} \) (for definition of an inner, see e.g. [17]). Denote it by \( m_{j,\text{in}} \). For this particular upper triangular coprime factorization, let \( M_{\text{in}} = \text{diag} \{ m_{1,\text{in}}, \ldots, m_{r,\text{in}} \} \) referred to as diagonal inner. Moreover, a balanced realization of \( m_{j,\text{in}} \), which is used in remainder of this work, is denoted by

\[
m_{j,\text{in}} = \begin{bmatrix} A_{j,\text{in}} & B_{j,\text{in}} \\ C_{j,\text{in}} & D_{j,\text{in}} \end{bmatrix}.
\]

In general, for a given plant \( G \), there is a finite number of Wonham decomposition forms to \( G \mu \) in which poles of the plant could be assigned to different diagonal sub-matrices in the state matrix \( A_w \), respectively. This comes out a set of upper triangular coprime factorizations and associated diagonal inners \( M_{\text{in}} \) for the plant, which are dependent to unstable poles in diagonal sub-matrices in Wonham decomposition forms. It will be shown in next section that the interaction between this feature and non-minimum phase zeros of the plant leads to the non-convexity in analyzing the mean-square stabilizability for the non-minimum phase system.

### 4 Mean-square stabilizability

In this section, fundamental constraints in mean-square stabilizability via output feedback, caused by the uncertainties in network channels, are studied for the system in Fig. 1. This is a very hard problem in general since non-minimum phase zeros make the mean-square stabilization via output feedback to be a non-convex problem (see for example [11]). Our study focuses on a non-minimum phase plant under Assumption 1.

**Assumption 1** The plant \( G \) has non-minimum phase zeros \( z_1, \ldots, z_r \). Each of them is associated with a column of \( G \), i.e.,

\[
G = G_0 \text{diag} \{ 1 - z_1 z^{-1}, \ldots, 1 - z_r z^{-1} \}
\]

where \( G_0 \) is a minimum phase system and with relative degree one, i.e., \( \lim_{|z| \to \infty} z G_0(z) \) is invertible.
At first glance, this assumption is quite artificial. However, due to multi-path transmission in wireless communication, multiple paths with different propagation lengths yield a channel with finite impulse response (FIR) which may include a nonminimum phase zero. In general, there is as called "common sub-channel zero" induced by multi-path transmission which is a difficult issue in channel identification and estimation (for example see [7] and [12]). This is a case which fits Assumption 1. On the other hand, we attempt to analytically investigate inherent constraints on the mean-square stabilizability imposed by interaction between Wonham decomposition forms and non-minimum phase zeros of the plant for the networked system. To seek a simplicity, the plants under this assumption are studied, which would be an interesting case as shown in Example 1. It should be noted that the results in this work can be extended to the case:

\[ G = G_0 \text{diag} \{ z^{-\tau_1}g_1, \ldots, z^{-\tau_r}g_r \} \]

where scale transfer functions \( g_j, j = 1, \ldots, r \) have more than one non-minimum phase zeros and relative degree zero, \( \tau_j, j = 1, \ldots, r \) are positive integers, \( G_0 \) is a minimum phase system and with relative degree one.

Now, we consider all stabilizing controllers for the nominal closed-loop system \( T \). Let \( NM^{-1} \) be an upper triangular right coprime factorization of the plant \( G \mu \), which is discussed in the preceding section. And let \( \tilde{M}^{-1}N, \tilde{N}, \hat{M}, N \in \mathcal{BH}_\infty \), be the left coprime factorization of the plant \( G \mu \) associated with \( N M^{-1} \). It is well known (see [17] for details) that the factors \( N, \tilde{M}, \tilde{N}, \hat{M} \) with some \( X, \tilde{X}, \bar{X}, \tilde{Y} \in \mathcal{BH}_\infty \) satisfy the Bezout Identity below:

\[
\begin{bmatrix}
M & Y \\
N & X
\end{bmatrix}
\begin{bmatrix}
\tilde{X} & -\tilde{Y} \\
-\tilde{N} & \hat{M}
\end{bmatrix} = I.
\tag{14}
\]

All stabilizing controllers for the nominal system are given

\[
K = (\tilde{X} - Q\tilde{N})^{-1}(\tilde{Y} - Q\tilde{M}),
\tag{15}
\]

where \( Q \in \mathcal{BH}_\infty \) is a parameter to be designed. Applying the controller (15) to the system, we obtain the nominal closed-loop system \( T \) in (7) as follows:

\[
T = (Y - MQ)\tilde{N}.
\tag{16}
\]

According to Lemma 1, the system is mean-square stabilizable if and only if there exists a \( Q \) satisfying the inequality

\[
\rho(\tilde{T}\Sigma) < 1.
\]

To this end, we need the following result (see [5] for details),

**Lemma 3** Suppose \( W \) is an \( r \times r \) positive matrix and \( w_{ij} \) is the \( \{i, j\} \)-th entry of \( W \). Then, it holds that

\[
\rho(W) = \inf_{\Gamma} \max_{j} \sum_{i=1}^{r} \frac{\gamma_i^2}{\Gamma_j} w_{ij}
\]

where \( \Gamma = \text{diag} \{ \gamma_1^2, \ldots, \gamma_r^2 \} \), with \( \gamma_i > 0, i = 1, \ldots, r \).

Denote the \( j \)-th column of \( T \) by \( T_j \). Applying Lemma 3, we have

\[
\rho(\tilde{T}\Sigma) = \inf_{\Gamma} \max_{j} \left\| \Gamma_{j}^{1/2} T_{j} \right\| F \cdot \frac{p_j}{1 - p_j}. \tag{17}
\]

From Lemma 1 and the spectral radius given in (17), we have the next result.

**Lemma 4** The closed-loop system in Fig. 2 is mean-square stabilizable if and only if it holds for some \( \Gamma \) and \( Q \) that

\[
\left\| \Gamma_{j}^{1/2} T_{j} \right\| F \cdot \frac{p_j}{1 - p_j} < 1, \quad j = 1, \ldots, r.
\tag{18}
\]

Now, it is studied to minimize \( \rho(\tilde{T}\Sigma) \). From (16), it holds for the system that

\[
\Gamma_{j}^{1/2} T_{j} \tilde{X}_{j}^{-1} = \Gamma_{j}^{1/2} (Y - MQ)\tilde{N}_{j}^{-1/2} e_{j} \tag{19}
\]

where \( e_j \) is the \( j \)-th column of the \( r \times r \) identity matrix \( I \).

Applying Bezout identity (14) into (19) leads to

\[
\Gamma_{j}^{1/2} T_{j} \tilde{Y}_{j}^{-1} = \Gamma_{j}^{1/2} M(\tilde{X} - Q\tilde{N}) - I \Gamma_{j}^{-1/2} e_{j}.
\tag{20}
\]

Let \( M_{\Gamma} = \Gamma_{j}^{1/2} M_{j}^{-1/2}, \tilde{N}_{\Gamma} = \Gamma_{j}^{1/2} \tilde{N}_{j}^{-1/2}, \tilde{X}_{\Gamma} = \Gamma_{j}^{1/2} \tilde{X}_{j}^{-1/2}, \) and \( Q_{\Gamma} = \Gamma_{j}^{1/2} Q_{j}^{-1/2} \). We rewrite (20) as

\[
\Gamma_{j}^{1/2} T_{j} \tilde{Y}_{j}^{-1} = [M_{\Gamma}(\tilde{X}_{\Gamma} - Q_{\Gamma}\tilde{N}_{\Gamma}) - I]e_{j}.
\]

Let the inner-outer factorization of \( M_{\Gamma} \) given by \( M_{\Gamma} = M_{\Gamma_{\text{in}}} M_{\Gamma_{\text{out}}} \) where \( M_{\Gamma_{\text{in}}}, M_{\Gamma_{\text{out}}} \) are inner and outer, respectively (see e.g. [17]). Noting the identity \( M_{\Gamma_{\text{in}}} M_{\Gamma_{\text{in}}} = I \) and the definition of \( \mathcal{L}_2 \) norm, we have that

\[
\left\| \Gamma_{j}^{1/2} T_{j} \tilde{Y}_{j}^{-1} \right\| F_{2} = \left\| \left[ M_{\Gamma_{\text{out}}}(\tilde{X}_{\Gamma} - Q_{\Gamma}\tilde{N}_{\Gamma}) - M_{\Gamma_{\text{in}}}^{-1} \right] e_{j} \right\| F_{2}.
\tag{21}
\]

Due to the facts that \( M_{\Gamma_{\text{in}}}^{-1} - M_{\Gamma_{\text{in}}}^{-1}(\infty) \in \mathcal{H}_\infty \) and \( M_{\Gamma_{\text{out}}}(\tilde{X}_{\Gamma} - Q_{\Gamma}\tilde{N}_{\Gamma}) - M_{\Gamma_{\text{in}}}^{-1}(\infty) \in \mathcal{H}_\infty \), it holds

\[
\langle M_{\Gamma_{\text{in}}}^{-1} - M_{\Gamma_{\text{in}}}^{-1}(\infty), M_{\Gamma_{\text{out}}}(\tilde{X}_{\Gamma} - Q_{\Gamma}\tilde{N}_{\Gamma}) - M_{\Gamma_{\text{in}}}^{-1}(\infty) \rangle = 0.
\]

Hence, (21) is written as follows:

\[
\left\| \Gamma_{j}^{1/2} T_{j} \tilde{Y}_{j}^{-1} \right\| F_{2} = \left\| \left[ M_{\Gamma_{\text{out}}}(\tilde{X}_{\Gamma} - Q_{\Gamma}\tilde{N}_{\Gamma}) - M_{\Gamma_{\text{in}}}^{-1}(\infty) \right] e_{j} \right\| F_{2} + \left\| M_{\Gamma_{\text{in}}}^{-1} - M_{\Gamma_{\text{in}}}^{-1}(\infty) \right\| F_{2}. \tag{22}
\]

Let

\[
J_{\Gamma}(Q_{\Gamma}) = \max_{1 \leq j \leq r} \left\{ \left\| \left[ M_{\Gamma_{\text{out}}}(\tilde{X}_{\Gamma} - Q_{\Gamma}\tilde{N}_{\Gamma}) - M_{\Gamma_{\text{in}}}^{-1}(\infty) \right] e_{j} \right\| F_{2} \right. + \left. \left\| M_{\Gamma_{\text{in}}}^{-1} - M_{\Gamma_{\text{in}}}^{-1}(\infty) \right\| F_{2} \right\}. \tag{23}
\]
Thus, from analysis optimal solution in minimizing \( J_f(Q_f) \), in general. To explain the intrinsic constraints caused by interaction between packet dropout probabilities, unstable poles and non-minimum phase zeros in this mean-square stabilization problem, the analytic optimal solution in minimizing \( J_f(Q_f) \) is studied for the system under Assumption 1.

From Assumption 1, an inner-outer factorization of \( \tilde{N}_{\Gamma} \) is given by \( \tilde{N}_{\Gamma} = \tilde{N}_{\text{out}} \text{diag} \{ n_{1, in}, \ldots, n_{r, in} \} \) where \( \tilde{N}_{\text{out}} \) is an outer of \( \tilde{N}_{\Gamma} \) and \( n_{j, in} = \frac{z - z_j}{z_j - 1}, j = 1, \ldots, r \) are inner factors.

Thus, from \( n_{j, in} = 1 \), we obtain that
\[
\left\| \left[ \tilde{M}_{\text{out}}(\tilde{X}_f - Q_f \tilde{N}_{\text{out}}) - M_{\Gamma_{in}}^{-1}(\infty) \right] e_j n_{j, in}^{-1} \right\|^2 = \left\| \tilde{M}_{\text{out}} Q_f \tilde{N}_{\text{out}} e_j - \left[ \tilde{M}_{\text{out}}(\tilde{X}_f - M_{\Gamma_{in}}^{-1}(\infty)) e_j n_{j, in}^{-1} \right] \right\|^2. \tag{25}
\]

Subsequently, it follows from fraction decomposition that
\[
\left[ \tilde{M}_{\text{out}}(\tilde{X}_f - M_{\Gamma_{in}}^{-1}(\infty)) e_j n_{j, in}^{-1} \right] = - \left[ \tilde{M}_{\text{out}}(z_j) \tilde{X}_f(z_j) - M_{\Gamma_{in}}^{-1}(\infty) \right] e_j \frac{1 - z_j^2}{z - z_j} L_j, \tag{26}
\]

where \( L_j \) is the remainder part of this fraction decomposition which belongs to \( \mathcal{H}_2 \). Note the fact that
\[
\left[ \tilde{M}_{\text{out}}(z_j) \tilde{X}_f(z_j) - M_{\Gamma_{in}}^{-1}(\infty) \right] e_j \frac{1 - z_j^2}{z - z_j} \in \mathcal{H}_2^-. \]

Then, substituting (26) into (25) leads to
\[
\left\| \left[ \tilde{M}_{\text{out}}(\tilde{X}_f - Q_f \tilde{N}_{\text{out}}) - M_{\Gamma_{in}}^{-1}(\infty) \right] e_j \right\|^2 = \left\| \tilde{M}_{\text{out}} Q_f \tilde{N}_{\text{out}} e_j - L_j \right\|^2 + \left\| \left[ \tilde{M}_{\text{out}}(z_j) \tilde{X}_f(z_j) - M_{\Gamma_{in}}^{-1}(\infty) \right] e_j \frac{1 - z_j^2}{z - z_j} \right\|^2. \tag{27}
\]

Let \( L = [L_1 \ldots L_r] \). Select \( Q_f = M_{\text{out}}^{-1} L \tilde{N}_{\text{out}}^{-1} \). It is clear from (22) and (27) that this \( Q_f \) minimizes \( \| \Gamma^{1/2} T_j Y_j^{-1} \|_2 \), \( j = 1, \ldots, r \) simultaneously. Moreover, it holds that
\[
\min_{Q_f \in \mathcal{HR}_{\infty}} J_f(Q_f) = \max_{1 \leq j \leq r} \left\{ \gamma_j, \frac{p_j}{1 - p_j}, j = 1, \ldots, r \right\} \tag{28}
\]

where
\[
J_f = \min_{Q_f} \left\| \Gamma^{1/2} T_j Y_j^{-1} \right\|_2 = \left\| M_{\Gamma_{in}}^{-1} - M_{\Gamma_{in}}^{-1}(\infty) \right\|_2 \quad + \quad \left\| M_{\text{out}}(z_j) \tilde{X}_f(z_j) - M_{\Gamma_{in}}^{-1}(\infty) \right\|_2 \frac{1 - z_j^2}{z - z_j}. \tag{29}
\]

Denote the packet dropout probability vector by \( p = (p_1, \ldots, p_r) \) and the mean-square stabilizable region of \( p \) to the closed-loop system by \( \mathcal{P} \). This mean-square stabilizable region \( \mathcal{P} \) is studied in terms of non-minimum phase zeros and a balanced realization of \( M_{\Gamma_{in}} \) given by
\[
M_{\Gamma_{in}} = \left[ \begin{array}{c|c}
A_{\Gamma_{in}} & B_{\Gamma_{in}} \\
C_{\Gamma_{in}} & D_{\Gamma_{in}} \\
\end{array} \right].
\]

**Theorem 1** Suppose that the plant \( G \) satisfies Assumption 1. The system in Fig. 1 is mean-square stabilizable if and only if the packet dropout probability vector \( p = (p_1, \ldots, p_r) \in \mathcal{P} \) is given by
\[
\mathcal{P} = \left\{ p = (p_1, \ldots, p_r) \mid p_j < (e_j^T \Phi_{\Gamma,j} e_j + 1)^{-1}, j = 1, \ldots, r, \Gamma > 0 \right\}, \tag{30}
\]

where
\[
\Phi_{\Gamma,j} = D_{\Gamma_{in}}^{-1} B_{\Gamma_{in}} N_{j, in} (A_{\Gamma_{in}}^{-1}) B_{\Gamma_{in}} D_{\Gamma_{in}}^{-1}
\]

and
\[
N_{j, in} (A_{\Gamma_{in}}^{-1}) = (z_j^2 A_{\Gamma_{in}}^{-1} - I) (z_j I - A_{\Gamma_{in}}^{-1})^{-1}.
\]

The proof of this theorem is given in Appendix A.

In general, the mean-square stabilizable region \( \mathcal{P} \) given by this theorem is non-convex. Now, a set of convex subregions of \( \mathcal{P} \) is studied in terms of diagonal inners \( M_{\Gamma_{in}} \) associated with Wonham decomposition forms of the plant \( G_{\mu} \).

**Theorem 2** Suppose that the plant \( G \) satisfies Assumption 1. Then, the system in Fig. 1 is mean-square stabilizable if, for all \( j = 1, \ldots, r \), the packet dropout probability \( p_j \) in \( j \)-th channel satisfies:
\[
p_j \leq \hat{\rho}_j \tag{31}
\]

where
\[
\hat{\rho}_j = D_{j, in}^{-1} B_{j, in} N_{j, in} (A_{j, in}^{-1}) B_{j, in} D_{j, in}^{-1} + 1.
\]

The proof of this theorem is presented in Appendix B.
It is worth to be mentioned that, in terms of the signal-to-noise ratio, \( \log \frac{1}{p_j} \) is the counterpart of the channel capacity which is studied in [1] and [16]. Instead of considering the minimum channel capacities for the mean-square stabilizability, Theorems 1 and 2 describe the mean-square stabilizable region and one of its subregions for the packet dropout probability vector \( p = (p_1, \ldots, p_r) \), respectively. In particular, the subregion presented in Theorem 2 is a hyper-rectangle determined by a diagonal inner \( M_{in} \) which is associated with a given Wonham decomposition form of the plant \( GM \). Moreover, it has two diagonal vertices which are the origin and \( V = (\hat{\rho}_1, \ldots, \hat{\rho}_r) \), respectively. The latter is in the boundary of the mean-square stabilizable region \( \mathcal{P} \).

Since Wonham decomposition form of the plant \( GM \) may not be unique, if this is a case, there is a set of such mean-square stabilizable hyper-rectangles for the packet dropout probability vector \( p \). Denote the diagonal inner associated with the \( s \)-th Wonham decomposition form by \( M_{s,in} \) and denote its diagonal entries by \( m_{s,1,n}, \ldots, m_{s,r,n} \). Let

\[
\begin{bmatrix}
\Lambda_{s,j,n} & B_{s,j,n} \\
C_{s,j,n} & D_{s,j,n}
\end{bmatrix}
\]

be a balance realization of \( m_{s,j,n}, j = 1, \ldots, r \). Denote the mean-square stabilizable hyper-rectangle associated with the \( s \)-th Wonham decomposition form by \( \mathcal{P}_s \). Its vertex \( V_s = (\hat{\rho}_{s1}, \ldots, \hat{\rho}_{sr}) \) is obtained by using Theorem 2.

**Corollary 1** If the packet dropout probability vector \((p_1, \ldots, p_r)\) is in the union of all \( \mathcal{P}_s \), i.e.,

\[
(p_1, \ldots, p_r) \in \bigcup_s \mathcal{P}_s,
\]

then the networked feedback system in Fig. 1 is mean-square stabilizable.

If the plant \( G \) has only one Wonham decomposition form, the mean-square stabilizable hyper-rectangles merge to one hyper-rectangle. The equation (32) becomes the necessary and sufficient condition for the mean-square stabilizability of the system. For a SIMO plant \( G \), there is only one Wonham decomposition form, the mean-square stabilizable region and hyper-rectangle studied in Theorems 1 and 2, respectively, degrade to a common interval in one dimension space. In this case, Theorem 2 presents a necessary and sufficient condition for the mean-square stabilizability of the system, i.e., \( p_1 \) given by the theorem is the supremum of the packet dropout probability which is allowed for the mean-square stabilizability of the network feedback system. In particular, for a SISO plant with one unstable pole \( \lambda_1 \) and one non-minimum phase zero \( z_1 \), this supremum is given by

\[
\hat{p}_1 = \left( \frac{\lambda_1^2 - 1}{(z_1 \lambda_1 - 1)^2/(z_1 - \lambda_1)^2} + 1 \right)^{-1}.
\]

Notice the fact that the product \( \prod_{j=1}^r p_j \) is the probability with which data packets over all channels are dropped simultaneously. In this work, it is referred to as blocking packet dropout probability. The volume of a hyper-rectangle \( \mathcal{P}_s \) is the maximum of the blocking packet dropout probability for all \((p_1, \ldots, p_r) \in \mathcal{P}_s \). Thus, it leads to:

**Corollary 2** If the blocking packet dropout probability \( \prod_{j=1}^r p_j \) of the channels satisfies the inequality

\[
\prod_{j=1}^r p_j < \max_s \left\{ \prod_{j=1}^r \hat{p}_{sj} \right\},
\]

then, there exists a set of data dropout probabilities \( p_1, \ldots, p_r \) with which the networked feedback system in Fig. 1 is mean-square stabilizable.

Now, we study the case in which the non-minimum phase zeros are void, i.e., the plant is a minimum phase system with relative degree one.

**Lemma 5** Suppose that the plant \( G \) is minimum phase with relative degree one. Then, for any given \( \Gamma > 0 \), there exists a \( Q \) to jointly minimize \( \| \Gamma^{1/2} T_j Y_j^{-1} \|_2^2, j = 1, \ldots, r \). It holds that

\[
\min_Q \| \Gamma^{1/2} T_j Y_j^{-1} \|_2^2 = e_j^T D_{\text{in}}^{-1} D_{\text{in}}^{-1} e_j - 1.
\]

The proof of this lemma is given in Appendix C.

**Theorem 3** If the plant \( G \) is minimum phase with relative degree one and \( \lambda_1, \ldots, \lambda_l \) are unstable poles of the plant, then the closed-loop system is mean-square stabilizable for some packet dropout probabilities \( p_1, \ldots, p_r \) if and only if the blocking packet dropout probability of the channels satisfy:

\[
\prod_{j=1}^r p_j < \left( \prod_{i=1}^l |\lambda_i| \right)^{-1}.
\]

The proof of this theorem is given in Appendix D.

Theorem 3 presents the supremum of the blocking packet dropout probability in the mean-square stabilizability of the networked feedback system. Once the blocking packet dropout rate is less than this supremum, the system is mean-square stabilizable by allocating packet dropout probabilities among channels.

**Example 1** Suppose that the plant in the networked feedback system shown in Fig 1. is a two-input two-output system. The transfer function of the plant is given as below:

\[
G = \begin{bmatrix}
\frac{(z - 0.25)z + 2}{z(z - 2)(z + 1.5)} & \frac{z - 1.5}{z(z + 1.5)} \\
\frac{z - 2}{z(z + 1.5)} & \frac{(2z - 2.75)(z - 1.5)}{z(z - 2)}
\end{bmatrix}.
\]
Let \( p_1 \) and \( p_2 \) be packet dropout probabilities of two channels, respectively. Applying Theorem 1, we can obtain the mean-square stabilizable region \( O V_{11}V_1V_{22} \) in Fig. 3 for the packet dropout probabilities, numerically.

Note the fact that there are two Wonham decomposition forms for the plant. Two diagonal inner matrices with these forms are

\[
M_{1,in} = \text{diag} \left\{ \frac{z-2}{2z-1}, \frac{(z+1.5)(z-2.5)}{(-1.5z-1)(2.5z-1)} \right\}
\]

and

\[
M_{2,in} = \text{diag} \left\{ \frac{(z-2)(z+1.5)}{2z-1}, \frac{z-2.5}{(-1.5z-1)(2.5z-1)} \right\}.
\]

The balance realizations of \( M_{1,in} \) and \( M_{2,in} \) are given by

\[
M_{1,in} = \begin{bmatrix}
0.5 & 0 & 0 & -0.866 & 0 \\
0 & 0.4 & -0.683 & 0 & 0.611 \\
0 & 0 & -0.667 & 0 & -0.745 \\
0.866 & 0 & 0 & 0 & 0 \\
0 & 0.917 & 0.298 & 0 & -0.267
\end{bmatrix}
\]

and

\[
M_{2,in} = \begin{bmatrix}
0.5 & 0.646 & 0 & 0.578 & 0 \\
0 & -0.667 & 0 & 0.745 & 0 \\
0 & 0 & 0.4 & 0 & -0.917 \\
0.866 & -0.373 & 0 & -0.333 & 0 \\
0 & 0 & 0.817 & 0 & 0.4
\end{bmatrix}
\]

(35)

respectively. According to Theorem 2, the mean-square stabilizable regions \( p_1 < \hat{p}_{11} = 0.1758 \) and \( p_2 < \hat{p}_{12} = 0.0142 \) are obtained from the balance realization (35) of \( M_{1,in} \) for \( p_1 \) and \( p_2 \), respectively. Subsequently, the mean-square stabilizable rectangle \( O V_{11}V_1V_{12} \) shown in Fig. 3 is obtained for the packet dropout probability vector \( (p_1, p_2) \). Similarly, the mean-square stabilizable regions \( p_1 < \hat{p}_{21} = 0.0476 \) and \( p_2 < \hat{p}_{22} = 0.0246 \) are obtained from the balance realization (36) of \( M_{2,in} \) for \( p_1 \) and \( p_2 \), respectively. The mean-square stabilizable rectangle \( O V_{21}V_2V_{22} \) shown in Fig. 3 is obtained for the packet dropout probability vector. Two vertices of these rectangles are \( V_1 = (0.1758, 0.0142) \) and \( V_2 = (0.0476, 0.0246) \), the areas of the rectangles are \( 2.50 \times 10^{-3}, 1.17 \times 10^{-3} \), respectively. In other words, the mean-square stabilizable rectangle obtained from \( M_{1,in} \) is bounded by the green curve \( p_1 p_2 = 2.50 \times 10^{-3} \). While, the mean-square stabilizable rectangle obtained from \( M_{2,in} \) is bounded by the green curve \( p_1 p_2 = 1.17 \times 10^{-3} \). The upper bound of the blocking packet dropout probability for mean-square stabilizability of the system is \( 2.50 \times 10^{-3} \). If the plant had only one Wonham decomposition form, these two green curves would merge to one curve and the two rectangles would merge to one rectangle as well.

5 Conclusion

This work studies the mean-square stabilizability via output feedback for a networked MIMO feedback system over several parallel packet dropping communication channels. The admissible region of packet dropout probabilities is discussed in the mean-square stabilizability of a non-minimum phase networked system. The trade-off among these packet dropout probabilities, plant’s characteristics and structure in the mean-square stabilizability of the system is presented by an upper bound of blocking packet dropout probability in the region. And then, it is found that, for a minimum phase plant with relative degree one, the supremum of blocking packet dropout probability which is allowed for the mean-square stabilizability is only determined by the product of the plant’s unstable poles.

Appendix A Proof of Theorem 1

We first review three basic results in LTI systems.

Lemma 6 (see [3]) For a balanced realization

\[
\begin{bmatrix}
A_{j,in} & B_{j,in} \\
C_{j,in} & D_{j,in}
\end{bmatrix}
\]

of the inner \( m_{j,in} \), it holds that

\[
\begin{bmatrix}
A_{j,in}^* & C_{j,in}^* \\
B_{j,in}^* & D_{j,in}^*
\end{bmatrix}
\begin{bmatrix}
A_{j,in} & B_{j,in} \\
C_{j,in} & D_{j,in}
\end{bmatrix}
= \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}.
\]  

(A.1)
Hence, we rewrite (29) as follows:

$$\|I - M_{\Gamma m} M_{\Gamma m}^{-1}(\infty)\|_2^2 = e_j^T D_{\Gamma m}^{-1} B_{\Gamma m} B_{\Gamma m} D_{\Gamma m}^{-1} e_j. \quad (A.5)$$

On the other hand, it holds that

$$\| [M_{\Gamma m}^{-1}(z_j) - M_{\Gamma m}^{-1}(\infty)] e_j \|_2^2 = (z_j^2 - 1) e_j^T \left[ M_{\Gamma m}^{-1}(z_j) - M_{\Gamma m}^{-1}(\infty) \right]^2 \times [M_{\Gamma m}^{-1}(z_j) - M_{\Gamma m}^{-1}(\infty)] e_j \quad (A.6)$$

where $M_{\Gamma m}^{-1}$ is given by Lemma 8 that

$$M_{\Gamma m}^{-1} = \begin{bmatrix} A_{\Gamma m} - B_{\Gamma m} D_{\Gamma m}^{-1} C_{\Gamma m} & -B_{\Gamma m} D_{\Gamma m}^{-1} C_{\Gamma m} \\ D_{\Gamma m}^{-1} C_{\Gamma m} & D_{\Gamma m}^{-1} C_{\Gamma m} \end{bmatrix}.$$ 

It is verified by Lemma 6 that $A_{\Gamma m}^{-1} = A_{\Gamma m} - B_{\Gamma m} D_{\Gamma m}^{-1} C_{\Gamma m}$.

Substituting (A.5), (A.6), (A.7) into (A.3) leads to

$$J_{\Gamma,j} = e_j^T D_{\Gamma m}^{-1} B_{\Gamma m} (z_j I - A_{\Gamma m}^{-1})^{-1} [z_j^2 - 1] C_{\Gamma m} D_{\Gamma m}^{-1} D_{\Gamma m}^{-1} e_j \quad (A.8)$$

It follows from Lemma 6 that

$$C_{\Gamma m} D_{\Gamma m}^{-1} D_{\Gamma m}^{-1} C_{\Gamma m} + I = A_{\Gamma m}^{-1} A_{\Gamma m}^{-1}.$$ 

Then it turns out that

$$J_{\Gamma,j} = e_j^T D_{\Gamma m}^{-1} B_{\Gamma m} (z_j I - A_{\Gamma m}^{-1})^{-1} [z_j^2 - 1] C_{\Gamma m} D_{\Gamma m}^{-1} D_{\Gamma m}^{-1} e_j. \quad (A.8)$$

Consequently, from (A.4) and (A.8), we obtain that the system is mean-square stabilizable if and only if $p = (p_1, \ldots, p_r) \in \mathcal{P}$.

### Appendix B Proof of Theorem 2

Since $NM^{-1}$ is an upper triangular coprime factorization (see Section 5), matrices $M$ and $M_\Gamma$ are written as follows:

$$M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1r} \\ 0 & m_{22} & \cdots & m_{2r} \\ & & \ddots & \vdots \\ 0 & 0 & \cdots & m_{rr} \end{bmatrix}, \quad M_\Gamma = \begin{bmatrix} m_{11} \hat{m}_{12} & \cdots & \hat{m}_{1r} \\ 0 & m_{22} \cdots & \hat{m}_{2r} \\ & & \ddots & \vdots \\ 0 & 0 & \cdots & m_{rr} \end{bmatrix}.$$
where \( \tilde{m}_{ij} = m_{ij} \frac{2}{f_j} \), \( 1 \leq i < j \leq r \).

Let \( \Gamma_{e}^{1/2} = \text{diag} \{ 1, \epsilon^{-1}, \ldots, \epsilon^{-r+1} \} \).

\[
M_{\Gamma_e} = \begin{bmatrix}
m_{11} & m_{12} \epsilon & \cdots & m_{1r} \epsilon^{-2} \\
0 & m_{22} & \cdots & m_{2r} \epsilon^{-3} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & m_{rr}
\end{bmatrix}.
\]

As studied in Section 3, selecting proper \( f_j, j = 1, \ldots, r \) in Lemma 2 yields that \( m_{11}, \ldots, m_{rr} \) are inner factors \( m_{1,in}, \ldots, m_{r,in} \). This leads to \( \lim_{\epsilon \to 0} M_{\Gamma_e} = \text{diag} \{ m_{1,in}, \ldots, m_{r,in} \} \), i.e., \( \lim_{\epsilon \to 0} M_{\Gamma_e} \) is an inner. Let \( M_{\Gamma_e,in} = M_{\Gamma_e} - \text{inner-out} \) factorization. It holds that

\[
\lim_{\epsilon \to 0} M_{\Gamma_e,in} = \text{diag} \{ m_{1,in}, \ldots, m_{r,in} \}. \tag{B.1}
\]

It is clear that

\[
\inf_Q \inf_\Gamma \left\| \Gamma^{1/2} T_j \Phi_j^{-1} \right\|_2^2 \leq \liminf_{\epsilon \to 0} \left\| \Gamma_e^{1/2} T_j \epsilon^{-1} \right\|_2^2.
\]

From (A.3) and (B.1), it holds that

\[
\liminf_{\epsilon \to 0} \left\| \Gamma_e^{1/2} T_j \epsilon^{-1} \right\|_2^2 = \left\{ \left\| m_{j,in} \right\|_2^2 - \left\| m_{j,in}^{-1} \right\|_2^2 + \left\| m_{j,in}^{-1}(z_j) - m_{j,in}^{-1}(\infty) \right\|_2 \frac{1 - z_j^2}{z_j} \right\}.
\]

In the light of the proof for Theorem 1, we can see that

\[
\left\| m_{j,in}^{-1}(\infty) \right\|_2^2 + \left\| m_{j,in}^{-1}(z_j) - m_{j,in}^{-1}(\infty) \right\|_2 \frac{1 - z_j^2}{z_j} \leq \left\| D_{j,in}^{-1} B_{j,in}^* N_{j,in} (A_{j,in}^{-1}) N_{j,in} (A_{j,in}^{-1}) B_{j,in} D_{j,in}^{-1} \right\|_2.
\]

If it holds for \( j = 1, \ldots, r \) that

\[
\frac{p_j}{1 - p_j} D_{j,in}^{-1} B_{j,in}^* N_{j,in} (A_{j,in}) N_{j,in} (A_{j,in}) B_{j,in} D_{j,in}^{-1} < 1,
\]

i.e., the inequality (31) holds for \( r = 1, \ldots, r \). Then, we can design \( Q \) such that the inequalities hold

\[
\inf_Q \inf_\Gamma \left\| \Gamma^{1/2} T_j \Phi_j^{-1} \right\|_2^2 < 1, \quad j = 1, \ldots, r.
\]

Therefore, the system is mean-square stabilizable.

**Appendix C Proof of Lemma 5**

It is shown in the proof of Theorem 1 that

\[
\left\| \Gamma^{1/2} T_j \Phi_j^{-1} \right\|_2^2 = \left\| [M_{\Gamma_{out}}(\tilde{X}_j - Q_j \tilde{N}_j) - M_{\Gamma_{in}}^{-1}(\infty)] e_j \right\|_2^2
\]

Since the plant \( G \) is minimum phase with relative degree one, we can design \( Q_j \) such that

\[
M_{\Gamma_{out}}(\tilde{X}_j - Q_j \tilde{N}_j) - M_{\Gamma_{in}}^{-1}(\infty) = 0.
\]

Thus, it holds for \( j = 1, \ldots, r \) that

\[
\min_Q \left\| \Gamma^{1/2} T_j \Phi_j^{-1} \right\|_2^2 = \left\| [M_{\Gamma_{in}}^{-1}] e_j \right\|_2^2. \tag{C.1}
\]

Applying (A.5), we write (C.1) as follows:

\[
\min_Q \left\| \Gamma^{1/2} T_j \Phi_j^{-1} \right\|_2^2 = e_j^T D_{\Gamma_{in}}^{-1} B_{\Gamma_{in}}^* B_{\Gamma_{in}} D_{\Gamma_{in}}^{-1} e_j, \quad j = 1, \ldots, r. \tag{C.2}
\]

Taking into account of the fact given by Lemma 6

\[
B_{\Gamma_{in}}^* B_{\Gamma_{in}} + D_{\Gamma_{in}}^* D_{\Gamma_{in}} = I,
\]

we rewrite (C.2) as follows:

\[
\min_Q \left\| \Gamma^{1/2} T_j \Phi_j^{-1} \right\|_2^2 = e_j^T D_{\Gamma_{in}}^{-1} D_{\Gamma_{in}}^{-1} e_j - 1, \quad j = 1, \ldots, r. \tag{C.2}
\]

**Appendix D Proof of Theorem 3**

According to Lemmas 4 and 5, the system is mean-square stabilizable if and only if it holds for some \( \Gamma > 0 \) that

\[
e_j^T D_{\Gamma_{in}}^{-1} D_{\Gamma_{in}}^{-1} e_j - 1 < \frac{1 - p_j}{p_j}, \quad j = 1, \ldots, r. \tag{D.1}
\]

Thus, the \( j \)-th channel’s packet dropout probability \( p_j \) satisfies the inequality for the \( \Gamma > 0 \) as below:

\[
e_j^T D_{\Gamma_{in}}^{-1} D_{\Gamma_{in}}^{-1} e_j < \frac{1}{p_j}. \tag{D.2}
\]

Subsequently, it holds for the blocking packet dropout probability of the channel that

\[
\prod_{j=1}^r \frac{1}{p_j} \geq \prod_{j=1}^r e_j^T D_{\Gamma_{in}}^{-1} D_{\Gamma_{in}}^{-1} e_j \geq \det (D_{\Gamma_{in}}^{-1} D_{\Gamma_{in}}^{-1}) \tag{D.3}
\]

where the second inequality follows from a property of a positive definite matrix (see [5]).
Since the plant $G$ has $l$ unstable poles, the inner $M_{in}$ has $l$ factors $M_{in,1}, \cdots, M_{in,l}$, i.e., $M_{in} = M_{in,1} \times \cdots \times M_{in,l}$. Each of these factors is associated with an unstable pole $\lambda_i$ and is given by

$$M_{in,i} = U_i \eta_i^* \frac{z - \lambda_i}{\lambda_i^* z - 1}$$

where $\eta_i$ is direction vector of $\lambda_i$ and $[U_i \quad \eta_i] \begin{bmatrix} U_i^* \\ \eta_i^* \end{bmatrix} = I$.

Hence, it holds for the balanced realization of $M_{in}$ that

$$D_{in} = \prod_{i=1}^{n} [U_i \quad \eta_i] \begin{bmatrix} I & 0 \\ 0 & 1/\lambda_i \end{bmatrix} \begin{bmatrix} U_i^* \\ \eta_i^* \end{bmatrix}.$$ (D.4)

Substituting (D.4) into (D.3) leads to

$$\prod_{j=1}^{r} p_j < \prod_{i=1}^{l} |\lambda_i|^{-1}. \quad (D.5)$$

On the other hand, if the blocking packet dropout probability satisfies (D.5), it is always possible to find a set of $p_1, \cdots, p_r$ so that the inequalities hold:

$$p_j < \prod_{i=1}^{l_j} |\lambda_{ji}|^{-1}, \quad j = 1, \cdots, r \quad (D.6)$$

where $\lambda_{ji}, i = 1, \cdots, l_j$ are unstable poles of $A_j$.

Let $\Gamma^{1/2} = \{1, \epsilon, \cdots, \epsilon^{r-1}\}$. In the light of the proof for Theorem 2, we can design a mean-square stabilizing controller for the system when some $p_1, \cdots, p_r$ satisfy the inequalities (D.6), respectively, and $\epsilon \to 0$.

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