SOME REMARKS ABOUT THE ZARISKI TOPOLOGY OF THE CREMONA GROUP

IVAN PAN AND ALVARO RITTATORE

Abstract. For an algebraic variety $X$ we study the behavior of algebraic morphisms from an algebraic variety to the group $\text{Bir}(X)$ of birational maps of $X$ and obtain, as application, some insight about the relationship between the so-called Zariski topology of $\text{Bir}(X)$ and the algebraic structure of this group, in the case where $X$ is rational.

1. Introduction

Let $k$ be an algebraically closed field and denote by $\mathbb{P}^n$ the projective space of dimension $n$ over $k$. The set $\text{Bir}(\mathbb{P}^n)$ of birational maps $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is the so-called Cremona group of $\mathbb{P}^n$. For an element $f \in \text{Bir}(\mathbb{P}^n)$ there exist homogeneous polynomials of the same degree $f_0, \ldots, f_n \in k[x_0, \ldots, x_n]$, without nontrivial common factors, such that if $x = (x_0 : \cdots : x_n)$ is not a common zero of the $f_i$’s, then $f(x) = (f_0(x) : \cdots : f_n(x))$. The (algebraic) degree of $f$ is the common degree of the $f_i$’s, and is denoted by $\text{deg}(f)$.

A natural way to produce an “algebraic family” of birational maps is to consider a birational map $f = (f_0 : \cdots : f_n) \in \text{Bir}(\mathbb{P}^n)$ and to allow the coefficients of the $f_i$’s to vary in an affine (irreducible) $k$-variety $T$. That is, we consider polynomials $f_0, \ldots, f_n \in k[T] \otimes k[x_0, \ldots, x_n]$, homogeneous and of the same degree in $x$ and we define $\varphi : T \to \text{Bir}(\mathbb{P}^n)$ by

$$\varphi(t, x) = (f_0(t, x) : \cdots : f_n(t, x));$$

in particular we assume that for all $t \in T$ the map $\varphi_t := \varphi(t, \cdot) : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is birational.

As pointed out by Serre in [Ser08, §1.6] the family of the topologies on $\text{Bir}(\mathbb{P}^n)$ which make any such algebraic family a continuous function, has a finest element, designed in loc. cit as the Zariski Topology of $\text{Bir}(\mathbb{P}^n)$. Moreover, we can replace $\mathbb{P}^n$ with an irreducible algebraic variety $X$ of dimension $n$ and the same holds for $\text{Bir}(X)$.

The aim of this work is to study the behavior of these “morphisms” $T \to \text{Bir}(X)$ and obtain, as application, some insight about the relationship between the topology and the algebraic structure of the group $\text{Bir}(X)$, where $X$ is a rational variety.

More precisely, in Section 2 we present some basic results about $\text{Bir}(X)$ that show the relationship between the algebraic structure and the Zariski topology.

Both authors are partially supported by the ANII, MathAmSud and CSIC-Udelar (Uruguay).
In Section 3, the main one, we deal with the case $X = \mathbb{P}^n$, or more generally the case where $X$ is a rational variety (see Lemma [2]). We begin by stating two deep results about the connectedness and simplicity of $\text{Bir}(\mathbb{P}^n)$ proved in [Bl2011] and [CaLa2011] (Proposition [16]) and extract as an easy consequence that a nontrivial normal subgroup of $\text{Bir}(\mathbb{P}^2)$ has trivial centralizer. Next we prove that for a morphism $\varphi : T \to \text{Bir}(\mathbb{P}^n)$, the function $t \mapsto \deg(\varphi_t)$ is lower semicontinuous ($\S$ 3.2). This result has some nice consequences:

(a) every Cremona transformation of degree $d$ is a specialization of Cremona transformations of degree $> d$ (Corollary [29]);
(b) the degree map $\deg : \text{Bir}(\mathbb{P}^n) \to \mathbb{Z}$ is lower semicontinuous ($\S$ 3.3);
(c) a morphism $T \to \text{Bir}(\mathbb{P}^n)$ maps constructible sets into constructible sets ($\S$3.5);
(d) the Zariski topology of $\text{Bir}(\mathbb{P}^n)$ is not Noetherian ($\S$3.6);
(e) there exist (explicit, non canonical) closed immersions of $\text{Bir}(\mathbb{P}^{n-1}) \hookrightarrow \text{Bir}(\mathbb{P}^n)$ ($\S$3.7);
(f) the subgroup consisting of the elements $f \in \text{Bir}(\mathbb{P}^n)$ which stabilize the set of lines passing through a fixed point is closed ($\S$3.7).

2. Generalities

Following [De1970, §2] we have:

**Definition 1.** A birational map $\varphi : T \times X \dashrightarrow T \times X$, where $T$ and $X$ are $k$-varieties and $X$ is irreducible, is said to be a pseudo-automorphism of $T \times X$, over $T$, if there exists a dense open subset $U \subset T \times X$ such that:

(a) $\varphi$ is defined on $U$;
(b) $U_t := U \cap \{t\} \times X$ is dense in $\{t\} \times X$ for all $t \in T$, and
(c) there exists a morphism $f : U \to X$ such that $\varphi|_U(t,x) = (t,f(t,x))$, and $\varphi|_{U_t} : U_t \to \{t\} \times X$ is a birational morphism.

In particular, a pseudo-automorphism $\varphi$ as above induces a family $T \to \text{Bir}(X)$ of birational maps $\varphi_t : X \dashrightarrow X$. Following [Bl2011] we call this family an algebraic family in $\text{Bir}(X)$ or a morphism from $T$ to $\text{Bir}(X)$.

We will identify a morphism $\varphi : T \to \text{Bir}(X)$ with its corresponding pseudo-automorphism and denote $\varphi_t = \varphi(t)$.

Note that if $\varphi : T \to \text{Bir}(X)$ is a morphism, the map $\psi : T \to \text{Bir}(X)$ defined by $\psi_t = \varphi_t^{-1}$ is also a morphism where $\varphi_t^{-1}$ denotes the inverse map of $\varphi_t$.

We say $F \subset \text{Bir}(X)$ is closed if its pullback under every morphism $T \to \text{Bir}(X)$ is closed in $T$, for all $T$. This defines the so-called Zariski topology on $\text{Bir}(X)$ ([Mu1974], [Ser08, §1.6], [Bl2011]).

In order to define the Zariski topology, as above, it suffices to consider morphisms from an affine variety $T$. Indeed, notice that a subset $F \subset T$ is closed if and only if there exists
a cover by open sets \( T = \bigcup V_i \), with \( V_i \) affine, such that \( F \cap V_i \) is closed in \( V_i \), for all \( i \). Then we may restrict a pseudo-automorphism \( \varphi : T \times X \rightarrow T \times X \) to each \( V_i \times X \) and obtain a pseudo-automorphism \( \varphi_i : V_i \times X \rightarrow V_i \times X \), for every \( i \). The assertion follows easily from the previous remark. Clearly, we may also suppose \( T \) is irreducible.

Unless otherwise explicitly stated, in the sequel we always suppose \( T \) is affine and irreducible.

**Lemma 2.** Let \( F : X \rightarrow Y \) be a birational map between two algebraic varieties. Then the map \( F^* : \text{Bir}(Y) \rightarrow \text{Bir}(X) \) defined by \( F^*(f) = F^{-1} \circ f \circ F \) is a homeomorphism, with inverse \( (F^{-1})^* \).

**Proof.** The result follows once we observe that \( \varphi : T \times Y \rightarrow T \times Y \) is a pseudo-automorphism if and only if \((\text{id} \times F^{-1}) \circ \varphi \circ (\text{id} \times F) : T \times X \rightarrow T \times X \) is a pseudo-automorphism.

We consider \( \text{Bir}(X) \times \text{Bir}(Y) \subset \text{Bir}(X \times Y) \) by taking \((f, g) \in \text{Bir}(X) \times \text{Bir}(Y) \) into the rational map \( F : X \times Y \rightarrow X \times Y \) defined as \( F(x, y) = (f(x), g(y)) \).

**Lemma 3.** Let \( X, Y \) be algebraic varieties and \( F \in \text{Bir}(X \times Y) \) a birational map; write \( F(x, y) = (F_1(x, y), F_2(x, y)) \) for \((x, y) \in X \times Y \) in the domain of \( F \). Then \( F \in \text{Bir}(X) \times \text{Bir}(Y) \subset \text{Bir}(X \times Y) \) if and only if there exist dense open subsets \( U \subset X, V \subset Y \) such that \( F \) is defined on \( U \times V \) and \( F_1(x, y) = F_1(x, y') \), \( F_2(x, y) = F_1(x', y) \) for \( x, x' \in U \), \( y, y' \in V \).

**Proof.** First suppose there exist \( f \in \text{Bir}(X) \) and \( g \in \text{Bir}(Y) \) such that \( F(x, y) = (f(x), g(y)) \). Consider nonempty open sets \( U \subset X \) and \( V \subset Y \) such that \( f \) and \( g \) are defined on \( U \) and \( V \) respectively. Hence, \( F_1 \) and \( F_2 \) are defined on \( U \times V \) and we have that \( F_1(x, y) = f(x) \) and \( F_2(x, y) = g(y) \), from which the “only if part” follows.

Conversely, suppose there exist nonempty open sets \( U \) and \( V \) as stated. Then \( F_1 \) and \( F_2 \) induce morphisms \( f : U \rightarrow X \) and \( g : V \rightarrow Y \) such that \( F(x, y) = (f(x), g(y)) \) for \((x, y) \in U \times V \). Since \( U \times V \) is dense in \( X \times Y \), this completes the proof.

**Proposition 4.** If \( X, Y \) are algebraic (irreducible) varieties, then \( \text{Bir}(X) \times \text{Bir}(Y) \subset \text{Bir}(X \times Y) \) is a closed subgroup.

**Proof.** In view of Lemma 2 we can assume that \( X \subset \mathbb{A}^n \), \( Y \subset \mathbb{A}^m \) are affine varieties. Let \( \varphi : T \times X \times Y \rightarrow T \times X \times Y \) be a pseudo-automorphism (over \( T \)). Then

\[
\varphi(t, x, y) = (f_1(t, x, y), \ldots, f_n(t, x, y), g_1(t, x, y), \ldots, g_m(t, x, y)),
\]

where \( f_i, g_j \in \mathbb{k}(T \times X \times Y) \) are rational functions on \( T \times X \times Y \) (of course, \( f_i, g_j \) verify additional conditions).

Let \( A := \varphi^{-1}(\text{Bir}(X) \times \text{Bir}(Y)) \) and denote by \( \overline{A} \) the closure of \( A \) in \( T \). Following Lemma 3 it suffices to prove that the restrictions of the \( f_i \)’s (resp. the \( g_j \)’s) to \( \overline{A} \times X \times Y \) do not depend on \( y \) (resp. on \( x \)), which implies \( A = \overline{A} \).
Up to restrict ϕ to each irreducible component of A we may suppose that A is dense in T. By symmetry we only consider the case relative to the f_i’s and write f = f_i for such a rational function.

Since the poles of f are contained in a proper subvariety of T × X × Y, we deduce that there exists y_0 ∈ Y such that the restriction of f to T × X × {y_0} induces a rational function on this subvariety. If p : T × X × Y → T × X × {y_0} denotes the morphism (t, x, y) ↦→ (t, x, y_0) we conclude f ◦ p is a rational function on T × X × Y.

Our assumption implies f coincides with f ◦ p along A × X × Y, which is dense in T × X × Y, so f = f ◦ p and the result follows.

Remark 5. Two pseudo-automorphisms ϕ : T × X → T × X and ψ : T × Y → T × Y induce a morphism (ϕ, ψ) : T → Bir(X) × Bir(Y), that is, an algebraic family in Bir(X) × Bir(Y). As in the proof of Proposition 4 it follows from Lemma 3 that F ⊂ Bir(X) × Bir(Y) is closed if and only if (ϕ, ψ)^{-1}(F) is closed for every pair ϕ, ψ. Moreover, it is easy to prove that the topology on Bir(X) × Bir(Y) induced by the Zariski topology of Bir(X × Y) is the finest topology for which all the morphisms (ϕ, ψ) are continuous.

Observe that the Zariski topology of Bir(X) × Bir(Y) is finer than the product topology of the Zariski topologies of its factors, as it is the case for algebraic varieties.

Proposition 6. If ϕ, ψ : T → Bir(X) are morphisms, then t ↦→ ϕ_t ◦ ψ_t defines an algebraic family in Bir(X). Moreover, the product homomorphism Bir(X) × Bir(X) → Bir(X) and the inversion map Bir(X) → Bir(X) are continuous.

Proof. To prove the first assertion it suffices to note that the family t ↦→ ϕ_t ◦ ψ_t corresponds to the pseudo-automorphism ϕ ◦ ψ : T × X → T × X. Applying Remark 5 the first part of the second assertion follows. Indeed, if F ⊂ Bir(X) is a closed subset, then (ϕ, ψ)^{-1}(m^{-1}(F)) = (ϕ ◦ ψ)^{-1}(F). For the rest of the proof it suffices to note that for a family ψ as above the map t ↦→ ψ_t^{-1} defines an algebraic family.

Lemma 7. The Zariski topology on Bir(X) is T1. In particular, if ϕ, ψ : T → Bir(X) are two morphisms, then the subset \{t ∈ T; ϕ(t) = ψ(t)\} is closed.

Proof. It suffices to show that id ∈ Bir(X) is a closed point. Without loss of generality we may suppose X ⊂ P^m is a projective variety. Then a morphism ϕ : T → Bir(X) may be represented as

ϕ_t = (f_0(t, x) : ⋯ : f_m(t, x)), (t, x) ∈ U,
where $U$ is as in Definition 1 and $f_i \in k[T][x_0, \ldots, x_m]$, $i = 0, \ldots, m$, are homogeneous of same degree in the variables $x_0, \ldots, x_m$. Therefore

$$\{ t \in T; \varphi(t) = id \} = \bigcap_{i,j=0}^m \{ t \in T : x_j f_i(t, x) - x_i f_j(t, x) = 0, \forall(t, x) \in U_t \}$$

$$= \bigcap_{i,j=0}^m \{ t \in T : x_j f_i(t, x) - x_i f_j(t, x) = 0, \forall x \in X \}$$

$$= \bigcap_{i,j=0,x \in X} \{ t \in T : x_j f_i(t, x) - x_i f_j(t, x) = 0 \}.$$  

Since for all $i,j$ the equations

$$x_j f_i(t, x) - x_i f_j(t, x) = h_1(x) = \cdots = h_\ell(x) = 0$$

define a closed set in $T \times X$, and $X$ is projective we deduce $\{ t \in T; \varphi(t) = id \}$ is closed in $T$.

**Corollary 8.** Let $\psi : Y \to Bir(X)$ be a morphism, where $Y$ is a projective variety. Then $\psi(Y)$ is closed.

**Proof.** A morphism $\varphi : T \to Bir(X)$ induces a morphism $\phi : T \times Y \to Bir(X)$ defined by $(t, y) \mapsto \varphi(t) \circ \psi(y)^{-1}$. Then $\phi^{-1}(\{id\}) = \{(t, y); \varphi(t) = \psi(y)\}$ is closed in $T \times Y$. The projection of this set onto the first factor is exactly $\phi^{-1}(\psi(Y))$ which is closed. \hfill \Box

**Corollary 9.** The centralizer of an element $f \in Bir(X)$ is closed. In particular, the centralizer $C_{Bir(X)}(G)$ of a subgroup $G \subset Bir(X)$ is closed.

**Proof.** Since the commutator map $c_f : Bir(X) \to Bir(X)$, $c_f(h) = hfh^{-1}f^{-1}$, is continuous, $c_f^{-1}(\{id\})$ is closed. \hfill \Box

Another consequence of Lemma 4 (and Remark 5) is that for an arbitrary topological subspace $A \subset Bir(X)$ and a point $f \in Bir(X)$, the natural identification map $\{f\} \times A \to A$ is an homeomorphism. As in [Sha, Chap.I, Thm. 3] we obtain:

**Corollary 10.** If $A, B \subset Bir(X)$ are irreducible subspaces, then $A \times B$ is an irreducible subspace of $Bir(X) \times Bir(X)$.

**Proposition 11.** The irreducible components of $Bir(X)$ do not intersect. Moreover, $Bir(X)^0$, the unique irreducible component of $Bir(X)$ which contains $id$, is a normal (closed) subgroup.

**Proof.** Let $A, B$ be irreducible components containing $id$. Corollary 10 implies $A \cdot B$ is irreducible. Since $id \in A \cap B$ then $A \cup B \subset A \cdot B$ from which it follows $A = A \cdot B = B$. This proves the uniqueness of $Bir(X)^0$.

The rest of the proof works as in [FSRi, Chapter 3, Thm. 3.8]. \hfill \Box
We have also the following easy result:

**Proposition 12.** Let $H \subset \text{Bir}(X)$ be a subgroup.

(a) The closure $\overline{H}$ of $H$ is a subgroup. Moreover, if $H$ is normal, then $\overline{H}$ is normal.

(b) If $H$ contains a dense open set, then $H = \overline{H}$.

**Proof.** The proof of this result follows the same arguments that the analogous case for algebraic groups (see [FSRi, Chapter 3, Section 3]). For example, in order to prove the second part of (a) it suffices to note that since $g \mapsto fgf^{-1}$ is an homeomorphism, then $fHf^{-1} = Hf^{-1}$.

\[\square\]

### 3. The Cremona group

In this section we consider the case $X = \mathbb{P}^n$; we fix homogeneous coordinates $x_0, \ldots, x_n$ in $\mathbb{P}^n$. As in the introduction, if $f : \mathbb{P}^n \to \mathbb{P}^n$ is a birational map, the degree of $f$ is the minimal degree $\deg(f)$ of homogeneous polynomials in $\mathbb{k}[x_0, \ldots, x_n]$ defining $f$.

#### 3.1. Connectedness and simplicity.

In [Bl2011, Thms. 4.2 and 5.1] Jérémie Blanc proves the following two results:

**Theorem 13** (J. Blanc). $\text{Bir}(\mathbb{P}^2)$ does not admit nontrivial normal closed subgroups.

**Theorem 14** (J. Blanc). If $f, g \in \text{Bir}(\mathbb{P}^n)$, then there exists a morphism $\theta : U \to \text{Bir}(\mathbb{P}^n)$, where $U$ is an open subset of $\mathbb{A}^1$ containing 0, 1, such that $\theta(0) = f, \theta(1) = g$. In particular $\text{Bir}(\mathbb{P}^n)$ is connected.

In Theorem 14 the open set $U$ is irreducible and the morphism $\theta$ is continuous. Hence we deduce that $\text{Bir}(\mathbb{P}^n)$ is irreducible.

On the other hand, in [CaLa2011] Serge Cantat and Stéphane Lamy prove the following result:

**Theorem 15** (S. Cantat-S. Lamy). $\text{Bir}(\mathbb{P}^2)$ is not a simple (abstract) group, i.e., it contains a non trivial normal subgroup.

In fact they prove that for a “very general” birational map $f \in \text{Bir}(\mathbb{P}^2)$ of degree $d$, with $d \gg 0$, the minimal normal subgroup containing $f$ is nontrivial. From Theorems 13 and 14 it follows that all non trivial normal subgroup in $\text{Bir}(\mathbb{P}^2)$ are dense.

Putting all together we obtain:

**Proposition 16.** Let $G \subset \text{Bir}(\mathbb{P}^2)$ be a nontrivial normal subgroup. Then $C_{\text{Bir}(\mathbb{P}^2)}(G) = \{id\}$.

**Proof.** Suppose $C_{\text{Bir}(\mathbb{P}^2)}(G) \neq \{id\}$. The closure $\overline{G}$ of $G$ is a normal subgroup, then it coincides with the entire Cremona group. If $f \in C_{\text{Bir}(\mathbb{P}^2)}(G)$, then $G$ is contained in the centralizer of $f$, which is closed. We deduce that $f$ commute with all the elements.
of Bir($\mathbb{P}^2$), that is $C_{\text{Bir($\mathbb{P}^2$)}}(G)$ coincides with the center $Z(\text{Bir($\mathbb{P}^2$)})$ of Bir($\mathbb{P}^2$). Since $Z(\text{Bir($\mathbb{P}^2$)}) = \{id\}$, the result follows. For the convenience of the reader we give a proof of the well known fact that $Z(\text{Bir($\mathbb{P}^2$)}) = \{id\}$.

Recall that Bir($\mathbb{P}^2$) is generated by quadratic transformations of the form $g_1\sigma g_2$ where $g_1, g_2 \in \text{PGL}(3, \mathbb{k})$ and $\sigma = (x_1x_2 : x_0x_2 : x_0x_1)$ is the standard quadratic transformation. Take $f \in Z(\text{Bir($\mathbb{P}^2$)})$. If $L \subset \mathbb{P}^2$ is a general line, then we may construct a quadratic transformation $\sigma_L$ which contracts $L$ to a point and such that $f$ is well defined in this point. Since $f\sigma_L = \sigma_L f$ and we may suppose $f$ is well defined and injective on an open set of $L$ we deduce $f$ transforms $L$ into a curve contracted by $\sigma_L$, that is, the strict transform of $L$ under $f$ is a line, and then $f \in \text{PGL}(3, \mathbb{k})$, so $f \in Z(\text{PGL}(3, \mathbb{k})) = \{id\}$. \hfill $\square$

3.2. Writings and degree of a pseudo-morphism.

Let $\varphi : T \to \text{Bir($\mathbb{P}^n$)}$ be a morphism, where $T$ is an irreducible variety. Denote by $\pi : T \times \mathbb{P}^n \to T$ the projection onto the first factor. Then the pseudo-automorphism $\varphi$ (Definition 1) verifies the following commutative diagram

$$
\begin{array}{ccc}
T \times \mathbb{P}^n & \xrightarrow{\varphi} & T \times \mathbb{P}^n \\
\pi \downarrow & & \downarrow \pi \\
T & & T
\end{array}
$$

In other words, $\varphi$ induces a commutative diagram

$$
\begin{array}{ccc}
\mathbb{k}(T \times \mathbb{P}^n) & \xrightarrow{\varphi^*} & \mathbb{k}(T \times \mathbb{P}^n) \\
\pi^* \downarrow & & \downarrow \pi^* \\
\mathbb{k}(T) & & \mathbb{k}(T)
\end{array}
$$

We deduce that there exist rational functions $\varphi_0, \ldots, \varphi_n \in \mathbb{k}(T \times \mathbb{P}^n)$ such that

$$
\varphi(t, x) = (\varphi_0(t, x) : \cdots : \varphi_n(t, x)),
$$

where the formula above holds for $(t, x)$ in an open set $U \subset T \times \mathbb{P}^n$. Moreover, we may suppose $U \cap (\{t\} \times \mathbb{P}^n) \neq \emptyset$ for all $t \in T$. Observe that we are assuming that $U$ is contained in the domain of definition of $\varphi_i$, for all $i$. Hence, for all $t \in T$, there exists an open set $U_t \subset \{t\} \times \mathbb{P}^n$ where all $\varphi_i|_{U_t}$ are well defined. We can also assume that there exists $i_t$ such that $\varphi_{i_t}$ does not vanish in $U_t$.

Let $V \subset T$ be an affine nonempty open subset. From the remarks above, we deduce that there exists a (non necessarily unique) representation of $\varphi$ of the form

$$
\varphi(t, x) = (f_0(t, x) : \cdots : f_n(t, x)), (t, x) \in U' \subset U \cap (V \times \mathbb{P}^n),
$$

where $U' \subset U \cap (V \times \mathbb{P}^n)$ is an open subset and $f_0, \ldots, f_n \in \mathbb{k}[V \times \mathbb{A}^{n+1}] = \mathbb{k}[V] \otimes \mathbb{k}[x_0, \ldots, x_n]$ are homogeneous polynomials in $x_0, \ldots, x_n$, of the same degree. In particular,
if $U' \cap \{ (t_0) \times \mathbb{P}^n \} \neq \emptyset$, then

$$\varphi_{t_0}(x) = (f_0(t_0, x), \ldots, f_n(t_0, x))$$

for $x$ in an open set $U'_{t_0} \subset \mathbb{P}^n$; that is, there exist $x_0 \in U'_{t_0}$ and $i_0$ such that $f_{i_0}(t_0, x_0) \neq 0$. Observe that $\{ t_0 \} \times U'_{t_0} \subset U_{t_0}$.

**Definition 17.** With the notations above, consider the $(n + 1)$-uple $(f_0, \ldots, f_n)$ satisfying (i) and let $\ell = \deg(f_i)$. We say that $w_V^\varphi = (f_0, \ldots, f_n)$ is a writing of $\varphi$ on $V$. The positive integer $\deg(w_V^\varphi) := \ell$ is said to be the degree of $w_V^\varphi$.

**Remark 18.** Let $w = w_V^\varphi = (f_0, \ldots, f_n)$ be a writing of $\varphi$ on an affine open subset $V \subset T$. We introduce the ideal $I(w) \subset \mathbb{k}[V] \otimes \mathbb{k}[x]$ generated by $f_0, \ldots, f_n$. Then $I(w)$ defines a subvariety $X^w \subset V \times \mathbb{A}^{n+1}$. Notice that $X^w$ is stable under the action of $\mathbb{k}^*$ on $V \times \mathbb{A}^{n+1}$ defined by $\lambda \cdot (t, x) \mapsto (t, \lambda x)$. Moreover, the projection $\pi : X^w \to V$ onto the first factor is equivariant and, by definition, surjective. The function $t \mapsto \dim \pi^{-1}(t)$ is upper-semicontinuous, from which we deduce $V_i := \{ t ; \dim \pi^{-1}(t) \geq i \}$ is closed in $V$ for all $i = 1, \ldots, n + 1$.

Since $\pi^{-1}(t) = X^w \cap (\{ t \} \times \mathbb{A}^{n+1})$, it follows that $\dim \pi^{-1}(t) > n$ if and only if $\pi^{-1}(t) = \{ t \} \times \mathbb{A}^{n+1}$. In other words, an element $t \in V$ belongs to $V_{n+1}$ if and only if $\{ t \} \times \mathbb{P}^n \cap U' = \emptyset$, where $U' \subset V \times \mathbb{P}^n$ is the domain of definition of the rational map $(t, x) \mapsto (t, (f_0(t, x), \ldots, f_n(t, x)))$. Observe that $V_{n+1} \subset V$.

The preceding remark motivates the following

**Definition 19.** Let $\varphi : T \to \text{Bir}(\mathbb{P}^n)$ be a morphism and $t \in T$. A writing passing through $t$ is a writing $w_V^\varphi$ of $\varphi$ such that $t \in V \setminus V_{n+1}$.

**Lemma 20.** Let $\varphi : T \to \text{Bir}(\mathbb{P}^n)$ be a morphism and $t_0 \in T$. Then there exists a writing $w_V^\varphi$ passing through $t_0$.

**Proof.** By definition, there exists $x_0 \in \mathbb{P}^n$ such that $\varphi$ is defined in $(t_0, x_0)$. Hence, there exist $f_0, g_0, \ldots, f_n, g_n \in \mathbb{k}[T] \otimes \mathbb{k}[x]$ such that $(g_0 \cdots g_n)(t_0, x_0) \neq 0$ and $\varphi(t, x) = \left( f_0/g_0(t, x), \ldots, f_n/g_n(t, x) \right)$, where the equality holds in an open neighborhood $A$ of $(t_0, x_0)$ in $T \times \mathbb{P}^n$. Eliminating denominators, we deduce that

$$\varphi(t, x) = \left( h_0(t, x), \ldots, h_n(t, x) \right),$$

where $h_i \in \mathbb{k}[T] \otimes \mathbb{k}[x]$ and the above formula holds in an open subset $A' \subset A$, containing $(t_0, x_0)$. If $V \subset T$ is an affine open subset such that for all $t \in V$ there exists $x \in \mathbb{P}^n$ with $(t, x) \in A'$, it is clear that $w_V^\varphi = (h_0, \ldots, h_n)$ is a writing of $\varphi$ through $t_0$.

**Definition 21.** Let $\varphi : T \to \text{Bir}(\mathbb{P}^n)$ be a morphism, where $T$ is an irreducible variety. Denote by $\mathcal{V}$ the family of nonempty affine open sets in $T$ on which there exists, at most, a writing of $\varphi$. The degree of $\varphi$ is the positive integer

$$\text{Deg}(\varphi) := \min\{ \deg(w_V^\varphi) : V \in \mathcal{V} \}.$$
Note that two \( n \)-uples \((f_0, \ldots, f_n)\) and \((f'_0, \ldots, f'_n)\), with \(\deg(f_i) = \deg(f'_i) = \text{Deg}(\varphi)\) define the same writing on an open set \(V\) if and only if they coincide up to multiplication by a nonzero element in \(k(V) = k(T)\).

For \(t \in T\) we denote by \(\deg(\varphi_t)\) the usual algebraic degree of the map \(\varphi_t : \mathbb{P}^n \rightarrow \mathbb{P}^n\); it is the minimal degree of components among the \((n+1)\)-uples of homogeneous polynomials defining \(\varphi_t\).

By applying (1) we obtain that if \(t \in T\), then \(\deg(\varphi_t) \leq \deg(w^\varphi_t)\) for every writing \(w^\varphi_t\) passing through \(t\). Moreover, we have the following

**Lemma 22.** Let \(w = w^\varphi_t\) be a writing for the morphism \(\varphi : T \rightarrow \text{Bir}(\mathbb{P}^n)\) and \(t \in V \setminus V_{n+1}\). Then the following assertions are equivalent:

(a) \(t \in V_n\).

(b) There is a codimension 1 subvariety \(X^w_t \subset \mathbb{A}^{n+1}\) such that \(\pi^{-1}(t) = \{t\} \times X^w_t\).

(c) \(\deg(\varphi_t) < \deg(w)\).

*Proof.* The equivalence of assertions (a) and (b) is obvious. In order to prove that (b) is equivalent to (c) let \(w = (f_0, \ldots, f_n)\); then for every \(t \in V \setminus V_{n+1}\) the rational map

\[
x \mapsto (f_0(t, x) : \ldots : f_n(t, x))
\]

coincides with \(\varphi_t\). Therefore \(\deg(\varphi_t) < \deg(f_i)\) if and only if the polynomials \(g_0, \ldots, g_n \in \mathbb{k}[x]\) defined by \(g_i(x) = f_i(t, x)\), where \(t\) is fixed and \(i = 0, \ldots, n\), admit a nontrivial factor. \(\square\)

The following example is taken from [BlFu2013 Lemma 2.13]

**Example 23.** Let \(T \subset \mathbb{P}^2\) be the projective nodal cubic curve of equation \(a^3 + b^3 - abc = 0\), with singular point \(o = (0 : 0 : 1)\), and consider the morphism \(\varphi : T \rightarrow \text{Bir}(\mathbb{P}^n)\) defined by

\[
\varphi(a : b : c) = (x_0f : x_1g : x_2f : \ldots : x_nf),
\]

where

\[
f = bx_0^2 + cx_0x_2 + ax_2^2, \quad g = (a + b)x_0^2 + (b + c)x_0x_2 + ax_2^2;
\]

note that \(\varphi_o = (x_0^2x_2 : x_0x_1x_2 : x_0^2x_2 : \ldots : x_0^2x_n)\) is the identity map.

Set \(f' = abf\) and \(g' = abg\), that is

\[
f' = ab^2x_0^2 + (a^3 + b^3)x_0x_2 + a^2bx_2^2, \quad g' = ab(a + b)x_0^2 + (ab^2 + a^3 + b^3)x_0x_2 + a^2bx_2^2.
\]

If \(V \subset T\) is the affine open set defined by \(c = 1\), then \(w^\varphi_V = (x_0f', x_1g', x_2f', \ldots, x_nf')\) is a writing of \(\varphi\) on \(V\) with degree 3. Clearly \(o \in V_{n+1}\) and \(w^\varphi_V\) is through all non-singular point in \(T\). As it follows from loc. cit the polynomial \(ax_0 + bx_2\) defines (locally) a higher common divisor for \(f'\) and \(g'\) in \(\mathbb{k}[V'] \otimes \mathbb{k}[x]\) where \(V' = V \setminus \{o\}\). Hence \(V = V_n\). Dividing all components in \(w^\varphi_V\) by \(ax_0 + bx_2\) we obtain a new writing on \(V\) of degree 2. One deduces \(\text{Deg}(\varphi) = 2\).
Remark 24. Consider a morphism \( \varphi: T \to \text{Bir}(\mathbb{P}^n) \) and let \( U \) and \( f: U \to \mathbb{P}^n \) be as in Definition 1. If \( \sigma: S \to T \) is a birational morphism it follows that the morphism

\[ (\sigma \times \text{id})^{-1}(U) \to S \times \mathbb{P}^n, (s, x) \mapsto (s, f(\sigma(s), x)) \]

induces a morphism \( \varphi \circ \sigma: S \to \text{Bir}(\mathbb{P}^n) \).

If \( s \in S \), then \( ((\sigma \times \text{id})^{-1}(U))_s \simeq U_{\sigma(s)} \) and up to this isomorphism the birational map \( \varphi_{\sigma(s)} \) coincides with \( (\varphi \circ \sigma)_s \).

Lemma 25. Let \( \varphi: T \to \text{Bir}(\mathbb{P}^n) \) be a morphism and consider a birational morphism \( \sigma: S \to T \). Then \( \text{Deg}(\varphi) = \text{Deg}(\varphi \circ \sigma) \).

Proof. Notice that if \( \varphi: T \to \text{Bir}(\mathbb{P}^n) \) is a morphism and \( U \subset T \) is an open subset, then \( \varphi|_U: U \to \text{Bir}(\mathbb{P}^n) \) is also a morphism, and clearly \( \text{Deg}(\varphi) = \text{Deg}(\varphi|_U) \). Then it suffices to prove the result when \( \sigma \) is an isomorphism, in which case the result is trivial.

3.3. Degree and semicontinuity.

Proposition 26. Let \( \varphi: T \to \text{Bir}(\mathbb{P}^n) \) be a morphism. Consider the set \( U_\varphi := \{ t \in T : \text{deg}(\varphi_t) = \text{Deg}(\varphi) \} \). Then

(a) \( U_\varphi \) is a nonempty open subset of \( T \).

(b) \( \text{deg}(\varphi_t) \leq \text{Deg}(\varphi) \) for all \( t \in T \).

Proof. We may reduce the proof to the case where \( T \) is smooth. Indeed, if \( T \) is singular we consider a proper birational surjective morphism \( \sigma: S \to T \), where \( S \) is smooth, and set \( \psi := \varphi \circ \sigma \); assume that assertions (a) and (b) hold on \( S \). Then Remark 24 implies (b) holds on \( T \) and that remark together with Lemma 25 imply \( U_\psi = \sigma^{-1}(U_\varphi) \). Since \( \sigma \) is proper and surjective it follows that \( \sigma \) is an open morphism; hence \( \sigma(U_\psi) = U_\varphi \) is a nonempty open subset of \( T \) which proves that (a) also holds on \( T \).

Now assume \( T \) is smooth. In order to prove that \( U_\varphi \) is not empty we consider a writing \( w^\varphi_t \) such that \( \text{deg}(w^\varphi_t) = \text{Deg}(\varphi) \). By Lemma 22 it suffices to prove that \( V \setminus V_n \neq \emptyset \). Assume that \( V_n = V \) and consider the variety \( X^w \subset V \times \mathbb{A}^{n+1} \) defined by the ideal \( I(w) \) generated by the components of \( w \). Since \( V_{n+1} \subset V \), it follows that \( X^w \) has codimension 1; denote by \( Z \) the union of codimension 1 irreducible components of \( X^w \) which project onto \( V \). If \( t_0 \in V \setminus V_{n+1} \), then the ideal \( I(Z)_{t_0} \subset \mathcal{O}_{V,t_0}[x] = \mathcal{O}_{T,t_0}[x] \) of elements in \( \mathcal{O}_{V,t_0}[x] \) vanishing in a neighborhood of \( \{(t_0) \times \mathbb{A}^{n+1} \} \cap Z \) is principal; let \( g \in \mathcal{O}_{V,t_0}[x] \) be a polynomial, homogeneous in \( x_0, \ldots, x_n \), which generates \( I(Z)_{t_0} \). Hence there exist a positive integer \( \ell \), an index \( 0 \leq j \leq n \) and homogeneous polynomials \( h_0, \ldots, h_n \in \mathcal{O}_{V,t_0}[x] \) such that \( f_i = g^j h_i \), for all \( i = 0, \ldots, n \), and \( h_j \notin I(Z)_{t_0} \).

There exists an affine open neighborhood \( V' \) of \( t_0 \) in \( V \setminus V_{n+1} \) such that \( f_i, g, h_i \in k[V'] \otimes k[x] \). Then \( w^\varphi_{t_0} := (h_0, \ldots, h_n) \) defines a writing of \( \varphi \) on \( V' \) through \( t_0 \), with \( \text{deg}(w^\varphi_{t_0}) < \text{deg}(w^\varphi_t) = \text{Deg}(\varphi) \), and we obtain a contradiction.

In order to prove that \( U_\varphi \) is open, let \( t_0 \in U_\varphi \) and consider a writing \( w' = w_U^\varphi = (f'_0, \ldots, f'_n) \) passing through \( t_0 \).
If \( U \setminus U_n \neq \emptyset \) then \( A = (V \setminus V_n) \cap (U \setminus U_n) \neq \emptyset \) and it follows from Lemma 22 that for all \( t \in A \)
\[
\deg(w') = \deg(\varphi_t) = \deg(w) = \deg(\varphi) = \deg(\varphi_{t_0}).
\]
Hence \( t_0 \in U \setminus U_n \subset U \varphi \).

If \( U = U_n \), by arguing as in the preceding part of the proof we deduce the existence of an affine open neighborhood \( U' \subset U \setminus U_{n+1} \) of \( t_0 \) and a writing \( w_{U'}^e = (h'_0, \ldots, h'_n) \), with \( f_i = g_i' h'_i \) for some \( g', h'_i \in k[U'] \otimes k[x] \). Since \( h'_j \) does not belong to \( I(Z')_t_0 \) (obvious notations), Lemma 22(c) implies \( \deg(w_{U'}^e) \leq \deg(\varphi_{t_0}) \), and thus \( \deg(w_{U'}^e) = \deg(\varphi) \). Hence \( t_0 \in U' \setminus U_{n+1} \subset U \varphi \) which completes the proof of (a).

In order to prove that \( U \varphi \) is open, let \( t_0 \in U \varphi \) and consider a writing \( w_{\tilde{U}}^e = (f'_0, \ldots, f'_n) \) passing through \( t_0 \). If \( t_0 \notin U \setminus U_n \) there is nothing to prove. Otherwise \( \deg(w_{\tilde{U}}^e) > \deg(\varphi_{t_0}) = \deg(\varphi) \), hence \( U = U_n \). By arguing as in the preceding part of the proof we deduce the existence of an affine open neighborhood \( U' \subset U \setminus U_{n+1} \) of \( t_0 \) and a writing \( w_{U'}^e = (h'_0, \ldots, h'_n) \), with \( f_i = g_i' h'_i \) for some \( g', h'_i \in k[U'] \otimes k[x] \). Since \( h'_j \) does not belong to \( I(Z')_t_0 \) (obvious notations), Lemma 22(c) implies \( \deg(w_{U'}^e) \leq \deg(\varphi_{t_0}) \), and thus \( \deg(w_{U'}^e) = \deg(\varphi) \). Hence \( t_0 \in U' \setminus U_{n+1} \subset U \varphi \) which completes the proof of (a).

To prove (b) we consider a writing \( w = w_{U}^e = (g_0, \ldots, g_n) \) such that \( \deg(w) = \deg(\varphi) \). Since \( g_i \in k[V] \otimes k[x] \subset k(T)[x] \) for all \( i \), there exists \( a \in k[T] \) such that \( a g_i \in k[T] \otimes k[x] \) for \( i = 1, \ldots, n \). Write
\[
ag_i = \sum_{I \in J} a_I^i x^I, \quad J = \{ I = (i_0, \ldots, i_n); i_0 + \cdots + i_n = \deg(\varphi) \}, a_I^i \in k[T],
\]
for \( i = 0, \ldots, n \).

If \( t \in T \) we take an irreducible smooth curve \( C \subset T \) passing through \( t \) such that \( C \cap U \varphi \neq \emptyset \). If \( \alpha \) is a local parameter for the local ring \( O_{C,t} \) of \( C \) at \( t \), there exists \( m \) such that \( \alpha^m \) does divide the restriction of \( a_I^i \) to \( C \), for all \( I \) and all \( i \), but \( \alpha^{m+1} \) does not; set
\[
g'_i := \sum_{I \in J} b_I^i x^I,
\]
where \( b_I^i := (a_I^i)|_C/\alpha^m \in O_{C,t}, i = 0, \ldots, n \). By construction \((g'_0, \ldots, g'_n) \) defines a writing of the morphism \( \varphi|_C : C \to Bir(\mathbb{P}^n) \) on an open neighborhood of \( t \) in \( C \). It follows \( \deg(\varphi) \leq \deg(g'_i) = \deg(\varphi_t) = \deg(\varphi) \).

As a consequence of (the proof of) Proposition 26 we have the following:

**Corollary 27.** Let \( \varphi : T \to Bir(\mathbb{P}^n) \) be a morphism, then:

(a) \( \deg(\varphi) = \max \{ \deg(\varphi_t) : t \in T \} \). Moreover, a writing \( w_{U}^e \) is of minimum degree, that is \( \deg(w_{U}^e) = \deg(\varphi) \), if and only if \( V \setminus V_n \neq \emptyset \).

(b) If \( t \in T \) is such that \( \deg(\varphi_t) = \deg(\varphi) \), then there exists a writing \( w = w_{U}^e \) through \( t \), with \( \deg(w) = \deg(\varphi) \).
Clearly the function \( t \mapsto \deg(\varphi_t) \) takes finitely many values, say \( d_1 = \text{Deg}(\varphi) > d_2 > \cdots > d_\ell \geq 1 \). Consider the decomposition \( T \setminus U_\varphi = X_1 \cup \cdots \cup X_r \) in irreducible components. We may restrict \( \varphi \) to each \( X_i \) and apply Proposition \[26\] to conclude \( \deg(\varphi_t) = d_2 \) for \( t \) in an open set (possibly empty for some \( i \)) \( U_i \subset X_i \) and \( \deg(\varphi_t) < d_2 \) on \( X_i \setminus U_i, i = 1, \ldots, r \). Repeating the argument with \( d_3 \), and so on, we deduce:

**Theorem 28.** Let \( \varphi : T \to \text{Bir}(\mathbb{P}^n) \) be a morphism. Then

(a) There exists a stratification by locally closed sets \( T = \bigcup_{j=1}^\ell V_j \) such that \( \deg(\varphi_t) \) is constant on \( V_j \), for all \( j = 1, \ldots, \ell \).

(b) The function \( \deg \circ \varphi : T \to \mathbb{N}, t \mapsto \deg(\varphi_t) \), is lower-semicontinuous. \( \square \)

**Corollary 29.** If \( d, e \in \mathbb{Z} \) are positive integers numbers with \( d \leq e \), then every Cremona transformation of degree \( d \) is specialization of Cremona transformations of degrees \( \geq e \).

**Proof.** Let \( f \) be a Cremona transformation of degree \( d \). Consider a morphism \( \theta : T \to \text{Bir}(\mathbb{P}^n) \), where \( T \) is a dense open set in \( \mathbb{A}^1 \) containing \( 0, 1 \) such that \( \theta(0) = f \) and \( \theta(1) \) is a Cremona transformation of degree \( e \) (Theorem \[13\]). The proof follows from Proposition \[26\] applied to the morphism \( \theta \). \( \square \)

**Corollary 30.** The degree function \( \deg : \text{Bir}(\mathbb{P}^n) \to \mathbb{N} \) is lower-semicontinuous, i.e. for all \( d \) the subset \( \text{Bir}_{\leq d}(\mathbb{P}^n) \) of birational maps of degree \( \leq d \) is closed. In particular, a subset \( \mathcal{F} \subset \text{Bir}(\mathbb{P}^n) \) is closed if and only if \( \mathcal{F} \cap \text{Bir}_{\leq d}(\mathbb{P}^n) \) is closed for all \( d > 0 \).

**Proof.** The assertion relative to semicontinuity is a direct consequence of Theorem \[28\](b). For the last assertion we note that if \( \varphi : T \to \text{Bir}(\mathbb{P}^n) \) is a morphism and \( e = \text{Deg}(\varphi) \), then \( \varphi^{-1}(\mathcal{F}) = \varphi^{-1}(\mathcal{F} \cap \text{Bir}(\mathbb{P}^n)_{\leq e}) \). \( \square \)

**Remark 31.** Note that \( \text{Bir}(\mathbb{P}^n) = \bigcup_{d \geq 1} \text{Bir}_{\leq d}(\mathbb{P}^n) \), with \( \text{Bir}_{\leq d}(\mathbb{P}^n) \subsetneq \text{Bir}_{\leq d+1}(\mathbb{P}^n) \) and \( \text{Bir}(\mathbb{P}^n)_1 = \text{PGL}(n+1, \mathbb{k}) \).

### 3.4. Algebraization of morphisms.

In this paragraph we deal with the morphisms \( \varphi : T \to \text{Bir}(\mathbb{P}^n) \) and their relationship with the stratification described in Theorem \[28\]. We consider the locally closed sets \( \text{Bir}(\mathbb{P}^n)_d := \text{Bir}(\mathbb{P}^n)_{\leq d} \setminus \text{Bir}(\mathbb{P}^n)_{\leq d-1} \), where \( d \geq 2 \). If \( \text{Deg}(\varphi) = d \), then \( U_\varphi = \varphi^{-1}(\text{Bir}(\mathbb{P}^n)_d) \).

Nguyen has shown in his doctoral thesis (\[Ngu2009\]) that \( \text{Bir}(\mathbb{P}^n)_d \) (with the induced Zariski topology) supports a structure of algebraic variety (see also \[BlFu2013\] Prop.2.15]). We give here some details on this construction, as a preliminary result for Theorem \[33\].

For integers \( d, n, r \), with \( d, n > 0 \) and \( r \geq 0 \), we consider the vector space \( V = k[x_0, \ldots, x_n]_{d+1}^r \) of \((r+1)\)-uples of \( d \)-forms. Notice that the projective space \( \mathbb{P}(d, n, r) = \mathbb{P}(V) \) consisting of dimension 1 subspaces in \( V \) has dimension \( N(d, n, r) = \binom{n+d}{d}(r + 1) - 1 \).

The following lemma shows how to identify \( \text{Bir}(\mathbb{P}^n)_d \) with a locally closed subset of \( \mathbb{P}(d, n, n) \). In particular, \( \text{Bir}(\mathbb{P}^n)_d \) is a quasi-projective variety and \( \text{Bir}(\mathbb{P}^n)_{\leq d} \) is a finite
union of quasi-projective varieties. The reader should be aware that the topology induced by Bir($\mathbb{P}^n$) on Bir($\mathbb{P}^n)_{\leq d}$ is not the one given by this union.

**Lemma 32.** There exists a canonical bijection between Bir($\mathbb{P}^n)_d$ and a locally closed subset of $\mathbb{P}_{(d,n,n)}$. In particular, Bir($\mathbb{P}^n)_d$ is a quasi-projective variety.

**Proof.** Let $e < d$ be a non-negative integer number. Consider the projective spaces $\mathbb{P}_{(d,n,n)}$, $\mathbb{P}_{(d-e,n,n)}$ and $\mathbb{P}_{(e,n,0)}$. Then there exists a “Segre type” morphism $s : \mathbb{P}_{(d-e,n,n)} \times \mathbb{P}_{(e,n,0)} \to \mathbb{P}_{(d,n,n)}$ which to a pair of elements $(g_0 : \cdots : g_n) \in \mathbb{P}_{(d-e,n,n)}$, $(f) \in \mathbb{P}_{(e,n,0)}$ it associates $(g_0 f : \cdots : g_n f)$. We denote by $W_e \subset \mathbb{P}_{(d,n,n)}$ the image of $s$, which is a projective subvariety.

Now consider the open set $U \subset \mathbb{P}_{(d,n,n)}$ consisting of points $(f_0 : f_1 : \cdots : f_n)$ where the Jacobian determinant $\partial(f_0,f_1,\ldots,f_n)/\partial(x_0,\ldots,x_n)$ is not identically zero. Clearly, an element $(f_0 : f_1 : \cdots : f_n) \in \mathbb{P}_{(d,n,n)} \cap U$ can be identified with a dominant rational map $\mathbb{P}^n \to \mathbb{P}^n$ defined by homogeneous polynomials (without common factors) of degree $\leq d$, and any such dominant rational map can be described in this way. Under this identification, points in $U_d := \left[\mathbb{P}_{(d,n,n)} \setminus \left(\cup_{e<n} W_e\right)\right] \cap U$ are in one-to-one correspondence with dominant rational maps defined by polynomials of degree exactly $d$.

As it follows readily from [RPV2001, Annexe B, Pro. B], the (bijective) image of Bir($\mathbb{P}^n)_d$ under the correspondence above is closed in $U_d$. Hence it is a quasi-projective variety. □

The topology given by the preceding construction coincides with the Zariski topology, inducing a structure of algebraic variety on Bir($\mathbb{P}^n)_d$:

**Theorem 33** (Blanc and Furter). Let $\varphi : T \to \text{Bir}(\mathbb{P}^n)$ be a morphism with $d = \text{Deg}(\varphi)$, and let $U_\varphi$ be as in Proposition 24. Then we have:

(a) the induced map $U_\varphi \to \text{Bir}(\mathbb{P}^n)_d$ is a morphism of algebraic varieties.

(b) the topology on Bir($\mathbb{P}^n)_d$ induced by Bir($\mathbb{P}^n)$ coincides with the topology of Bir($\mathbb{P}^n)_d$ induced by $\mathbb{P}_{(d,n,n)}$ as in Lemma 32. □

### 3.5. Chevalley type Theorem.

**Theorem 34.** Let $X$ be a rational variety. If $\varphi : T \to \text{Bir}(X)$ is a morphism and $C \subset T$ is a constructible set, then $\varphi(C)$ is constructible and contains a dense open subset of $\varphi(C)$.

**Proof.** By Lemma 2 we may suppose $X = \mathbb{P}^n$ and $\varphi$ with degree $d = \text{Deg}(\varphi)$. Hence $\varphi(T) \subset \text{Bir}(\mathbb{P}^n)_{\leq d}$; we consider the morphism $\varphi_0 : U_0 = U_\varphi \to \text{Bir}(\mathbb{P}^n)_d$ induced by $\varphi$.

On the other hand, Theorem 28 gives a stratification $T \setminus U_0 = \cup V_j^f$ by locally closed sets such that $d_j := \text{deg}(\varphi(t))$ is constant on each $V_j$; set $\varphi_j : V_j \to \text{Bir}(\mathbb{P}^n)_{d_j}$ the morphism induced by $\varphi$ on $V_j$. 


We deduce that \( \varphi(C) \) is constructible by using Theorem 33 and applying the standard Chevalley Theorem to the morphisms \( \varphi_0, \varphi_1, \ldots, \varphi_\ell \). The last assertion of the theorem is a general topology result: since \( \varphi(C) \) is constructible, then \( \varphi(C) = \bigcup_{i=1}^\ell Z_i \), where \( Z_i \) is a locally closed subset for all \( i = 1, \ldots, \ell \). Then

\[
\varphi(C) \setminus \bigcup_i (Z_i \setminus Z_i) = \overline{\varphi(C)} \setminus \bigcup_i (\overline{Z_i} \setminus \overline{Z_i})
\]
is a dense open subset of \( \overline{\varphi(C)} \). \( \square \)

3.6. Cyclic closed subgroups.

**Corollary 35.** Let \( \{f_m\} \subset \text{Bir}(\mathbb{P}^n) \) be an infinite sequence of birational maps. Then \( \{f_m\} \) is closed if and only if \( \lim_{m \to \infty} \deg(f_m) = \infty \). In particular, the Zariski topology on \( \text{Bir}(\mathbb{P}^n) \) is not Noetherian.

**Proof.** Let \( \varphi : T \to \text{Bir}(\mathbb{P}^n) \) be a morphism, with \( \text{Deg}(\varphi) = d \). Then there exists \( m_0 \) such that \( \deg(f_m) \geq d \) for all \( m \geq m_0 \), and thus \( \varphi^{-1}(\{f_m\}) \) is finite. Hence, the if follows from Corollary 30 and Theorem 33.

Conversely, suppose that \( \liminf_{m \to \infty} \deg(f_m) = d < \infty \). Then there exist infinitely many \( f_i \) whose degree is \( d \). Hence, \( \{f_m\} \cap \text{Bir}(\mathbb{P}^n)_d \) is an infinite countable subset of the algebraic variety and thus it is not closed. \( \square \)

**Corollary 36.** Let \( f \in \text{Bir}(\mathbb{P}^n) \) be a birational map of degree \( d \). The cyclic subgroup \( \langle f \rangle \) generated by \( f \) is closed if and only if either \( f \) is of finite order or \( \lim_{m \to \infty} \deg(f^m) = \infty \).

**Proof.** Indeed, following [DiFa2001], if \( \langle f \rangle \) is infinite, then the sequence \( \deg(f^m) \) either is bounded or grows with order at least \( m \). Hence, the infinite cyclic group \( \langle f \rangle \) is not closed only when the sequence \( \deg(f^m) \) is bounded. The remaining equivalence follows from [BlDe2013, Thm. A]. \( \square \)

3.7. Some big closed subgroups.

Let \( o \in \mathbb{P}^n \) be a point. Consider the subgroup \( \text{St}_o(\mathbb{P}^n) \subset \text{Bir}(\mathbb{P}^n) \) of birational transformations which stabilize (birationality) the set of lines passing through \( o \). If \( o' \) is another point \( \text{St}_o(\mathbb{P}^n) \) and \( \text{St}_{o'}(\mathbb{P}^n) \) may be conjugated by mean of a linear automorphism; in the sequel we fix \( o = (1 : 0 : \cdots : 0) \). In [Do2011] the group \( \text{St}_o(\mathbb{P}^n) \) is introduced in a different form and is called the de Jonquières subgroup of level \( n - 1 \) (see also [Pa2000]).
Let $\pi : \mathbb{P}^n \rightarrow \mathbb{P}^{n-1}$ be the projection of center $o$ defined by

$$x_0 : x_1 : \cdots : x_n \mapsto (x_1 : \cdots : x_n).$$

Then $\text{St}_o(\mathbb{P}^n) = \{ f \in \text{Bir}(\mathbb{P}^n) : \exists \tau \in \text{Bir}(\mathbb{P}^{n-1}), \pi f = \tau \pi \}$. Moreover, note that $\text{St}_o(\mathbb{P}^n)$ is the semidirect product

$$1 \longrightarrow \text{Jon}_o(\mathbb{P}^n) \longrightarrow \text{St}_o(\mathbb{P}^n) \longrightarrow \text{Bir}(\mathbb{P}^{n-1}) \longrightarrow 1$$

where $\text{Jon}_o(\mathbb{P}^n) = \{ f \in \text{Bir}(\mathbb{P}^n) : \pi f = \pi \}$ and $\rho$ is the evident homomorphism, and $\tau = \rho(f)$. Indeed, the morphism $\sigma : \text{Bir}(\mathbb{P}^{n-1}) \rightarrow \text{Bir}(\mathbb{P}^n)$ given by

$$(h_1 : \cdots : h_n) \mapsto (x_0h_1 : x_1h_1 : \cdots : x_nh_n)$$

is injective and such that $\sigma(\text{Bir}(\mathbb{P}^{n-1})) \subset \text{St}_o(\mathbb{P}^n)$. Clearly, $\rho \circ \sigma = \text{id}$.

Moreover, we affirm that $\rho$ is continuous, and $\sigma$ is a continuous closed immersion. Indeed, if $\varphi : T \rightarrow \text{Bir}(\mathbb{P}^n)$ is a morphism then the composition $\rho \circ \varphi$ defines a morphism $T \rightarrow \text{Bir}(\mathbb{P}^{n-1})$; therefore $\rho$ is a continuous function. Clearly, $\sigma$ is continuous. In order to prove, among other things, that $\sigma$ is a closed immersion we need the following:

**Lemma 38.** Let $f \in \mathbb{k}[T] \otimes \mathbb{k}[x_0, \ldots, x_n]$ be a polynomial, homogeneous in $x$; denote by $\deg_{x_0}(f)$ its degree in $x_0$. Then for all integer $m \geq 0$ and $i = 0, \ldots, n$ the sets

$$R = \{ t \in T : x_i|f(t, x) \} , \ S_m = \{ t \in T : \deg_{x_0}(f) \leq m \}$$

are closed in $T$.

**Proof.** Let $a_1, \ldots, a_N \in \mathbb{k}[T]$ be the coefficients of $f$ as polynomial in $x_0, \ldots, x_n$. It is clear that $R$ and $S_m$ are defined as common zeroes of a subset of the polynomials $\{a_1, \ldots, a_N\} \subset \mathbb{k}[T]$. \(\square\)

**Theorem 39.** The subgroups $\text{Jon}_o(\mathbb{P}^n)$ and $\text{St}_o(\mathbb{P}^n)$ are closed and $\sigma(\text{Bir}(\mathbb{P}^{n-1}))$ is closed in $\text{Bir}(\mathbb{P}^n)$. In particular, $\sigma$ is a closed immersion.

**Proof.** Let $\varphi : T \rightarrow \text{Bir}(\mathbb{P}^n)$ be a morphism, say with $\text{Deg}(\varphi) = d$. In order to prove that $\varphi^{-1}(\text{Jon}_o(\mathbb{P}^n))$ is closed it suffices to consider a net $(t_\xi)$ in $\varphi^{-1}(\text{Jon}_o(\mathbb{P}^n))$, where $\xi$ varies in a directed set, and show that every limit point $t_\infty \in T$ of that net satisfies $\varphi(t_\infty) \in \text{Jon}_o(\mathbb{P}^n)$. Let $t_\infty$ be such a limit point and $T = \bigcup_{j=0}^d V_j$ be the stratification given by Theorem $\text{(28)}(a)$, where $V_0 = U_\varphi$ is the open set introduced in Proposition $\text{(26)}$; $\xi$ is the semidirect product

Then there exists $j$ such that the subnet $(t_\xi) \cap V_j$ has $t_\infty$ as limit point. Thus, we may assume $t_\xi \in U_\varphi$ for all $\xi$, that is that $\deg(\varphi(t_\xi)) = d$. By shrinking $T$, if necessary, we may assume

$$\varphi(t, x) = (f_0(t, x) : \cdots : f_n(t, x)),$$

where $f_i \in \mathbb{k}[T] \otimes \mathbb{k}[x]$ are homogeneous in $x = \{x_0, \ldots, x_n\}$ of degree $d$ (see Corollary $\text{(27)}(b)$). From the description given in $\text{(P2000)} \S 2$ it follows that for all $\xi$ there exists a homogeneous polynomial $q_\xi \in \mathbb{k}[x]$ such that:

(a) $f_i(t_\xi, x) = x_iq_\xi(x)$, for $i > 0$;
(b) \( f_0(t_ξ, x) \) and \( q_ξ(x) \) have degrees \( \leq 1 \) in \( x_0 \);
(c) \( f_0(t_ξ, x)q_ξ(x) \) has degree \( \geq 1 \) in \( x_0 \).

By Lemma 38, when \( t_ξ \) specializes to \( t_∞ \), then \( ϕ_ξ = ϕ(t_ξ) \) specializes to the birational map \( ϕ_{t∞} = (f : x_1q : \cdots : x_nq) : \mathbb{P}^n −→ \mathbb{P}^n \), where \( f(x) \) and \( x_1q(x) \), \( i > 0 \), are polynomials in \( x \) of degree \( d \) and with degree \( \leq 1 \) in \( x_0 \). Suppose that \( f \) and \( q \) admit a common factor \( h \in \mathbb{K}[x_0, \ldots, x_n] \), of degree \( \geq 0 \). Since the limit map \( ϕ_{t∞} \) is birational (of degree \( \leq d \)) we deduce that \( h \in \mathbb{K}[x_1, \ldots, x_n] \); otherwise \( h \) would have degree \( 1 \) in \( x_0 \) and the map \( ϕ_{t∞} \) would be defined by polynomials in \( x_1, \ldots, x_n \) contradicting birationality. Hence \( f' := f/h \) and \( q' := q/h \) satisfy the conditions (b) and (c) above. We conclude \( ϕ_{t∞} = (f' : x_1q' : \cdots : x_nq') \). Applying again the description of [Pa2000 §2], we deduce that \( πϕ_{t∞} = π \), that is \( ϕ_{t∞} ∈ \text{Jon}_o(\mathbb{P}^n) \), which proves \( \text{Jon}_o(\mathbb{P}^n) \) is closed.

In order to prove that \( σ(\text{Bir}(\mathbb{P}^{n−1})) \) is closed, consider a net \((t_ξ) ∈ ϕ^{-1}(σ(\mathbb{P}^{n−1})) \), with limit point \( t_∞ \). As before, we can assume that \( t_ξ ∈ U_ϕ \) for all \( ξ \). With the notation introduced above we have that

(a) \( f_i(t_ξ, x) = x_1h_i,ξ(x) \), for \( i > 0 \), and
(b) \( f_0(t_ξ, x) = x_0h_1,ξ(x) \),

where \( τ_ξ = (h_1,ξ : \cdots : h_n,ξ) : \mathbb{P}^{n−1} −→ \mathbb{P}^{n−1} \) is birational. From Lemma 38 we obtain that \( h_i,ξ \) specializes to a polynomial \( h_i \in \mathbb{K}[x_1, \ldots, x_n] \), \( i > 0 \) and that \( ϕ_{t∞} = (x_0h_1 : x_1h_1 : \cdots : x_nh_n) \). Since \( πϕ_{t∞} = τ_ξπ \) we conclude that \( ϕ_{t∞} ∈ \text{St}_o(\mathbb{P}^n) \) and thus \( (h_1 : \cdots : h_n) ∈ \text{Bir}(\mathbb{P}^{n−1}) \) ([Pa2000 Prop. 2.2]). Since \( σ((h_1 : \cdots : h_n)) = ϕ_{t∞} \), it follows that \( σ(\text{Bir}(\mathbb{P}^{n−1})) \) is closed.

Finally, since for elements \( f ∈ \text{Jon}_o(\mathbb{P}^n) \) and \( h ∈ \text{Bir}(\mathbb{P}^{n−1}) \) the product \( f × h \) is the composition \( f_0σ(h) \), then \( \text{St}_o(\mathbb{P}^n) = \text{Jon}_o(\mathbb{P}^n)/\text{Im}(σ) \) (product in \( \text{Bir}(\mathbb{P}^n) \)). The fact that \( \text{St}_o(\mathbb{P}^n) \) is closed follows then from the two assertions we have just proved together with the continuity of the functions \( ρ : \text{St}_o(\mathbb{P}^n) −→ \text{Bir}(\mathbb{P}^{n−1}) \), the group product and the group inversion. Indeed, let \((f_ξ × h_ξ) \) be a net in \( \text{St}_o(\mathbb{P}^n) \) which specializes to \( s ∈ \text{Bir}(\mathbb{P}^n) \). Then \( ρ(f_ξ × h_ξ) = ρ(1 × h_ξ) = h_ξ \) specializes to \( ρ(s) = h ∈ \text{Bir}(\mathbb{P}^{n−1}) \). Since \( (f_ξ × h_ξ)(1 × h_ξ) = f_ξ × 1 ∈ \text{Jon}_o(\mathbb{P}^n) \), the net \((f_ξ × 1) \) specializes to \( sσ(h) ∈ \text{Jon}_o(\mathbb{P}^n) \). Thus \( s ∈ \text{St}_o(\mathbb{P}^n) \).

Remark 40. More generally, for \( ℓ = 1, \ldots, n \), the map \( σ_ℓ : \text{Bir}(\mathbb{P}^{n−1}) −→ \text{Bir}(\mathbb{P}^n) \) defined by

\[
σ_ℓ((h_1 : \cdots : h_n)) = (x_0h_ℓ : x_1h_1 : \cdots : x_nh_n)
\]

is a continuous, closed, homomorphism whose image is contained in \( \text{St}_o(\mathbb{P}^n) \) and such that \( ρσ_ℓ = id \). In this notation, the map \( σ \) of Theorem 39 is \( σ_1 \). Moreover, one has

\[
\bigcap_{ℓ=1}^n σ_ℓ(\text{Bir}(\mathbb{P}^{n−1})) = \{id\}.
\]

If \( U_ℓ \) is the dense open set \( \text{Bir}(\mathbb{P}^n) \setminus σ_ℓ(\text{Bir}(\mathbb{P}^{n−1})) \), then \( \text{Bir}(\mathbb{P}^n) \setminus \{id\} = \bigcup_{ℓ=1}^n U_ℓ \).
References

[Bl2011] J. Blanc, *Groupes de Cremona, connexité et simplicité*, Ann. Sci. Éc. Norm. Supér. 43 (2010), no. 2, pp. 357-364.

[BlDe2013] J. Blanc and J. Dserti, *Degree growth of birational maps of the plane*, to appear in Ann. Sc. Norm. Super. Pisa Cl. Sci.

[BlFu2013] J. Blanc and J.-Ph. Furter, *Topologies and Structures of Cremona Groups*, Ann. of Math. 178 (2013), no. 3, 1173-1198.

[CaLa2011] S. Cantat and S. Lamy, *Normal subgroups of the Cremona group*, Acta Math, Vol. 210, 1 (2013), pp 31-94.

[De1970] M. Demazure, *Sous-groupes algébriques de rang maximum du groupe de Cremona*, Ann. Sci. Éc. Norm. Supér. 4e série, t. 3, no 4 (1970), 507-588.

[DiFa2001] J. Diller and C. Favre, *Dynamics of Bimeromorphic Maps of Surfaces*, American Journal of Mathematics Vol. 123, No. 6 (Dec., 2001), pp. 1135-1169.

[Do2011] I. Dolgachev, *Lectures on Cremona transformations*, Ann Arbor-Rome, preprint 2011.

[FaWu11] Ch. Favre and E. Welc, *Degree Growth of Monomial maps and McMullen’s Polytope Algebra*, Indiana Univ. Math. J. 61 (2012), 493-524.

[FSRI] W. Ferrer-Santos and A. Rittatore, *Actions and Invariants of Algebraic Groups*, CRC Press, 2005.

[Mu1974] D. Mumford, *Algebraic Geometry in Mathematical developments arising from Hilbert problems. Proceedings of the Symposium in Pure Mathematics of the American Mathematical Society held at Northern Illinois University*, De Kalb, Ill., May, 1974. 4445.

[Ngu2009] D. Nguyen *Groupe de Cremona*, PhD thesis, Université de Nice-Sophia Antipolis 2009.

[RPV2001] F. Ronga, I. Pan and T. Vust, *Transformation quadratiques de l’espace projective à trois dimensions*, Ann. Inst. Fourier, Grenoble 51, 5 (2001), 1153-1187.

[Pa2000] I. Pan, *Les transformations de Cremona stellaires*, Proc. American Math. Soc, V. 129, N. 5, 12571262.

[Sha] I. R. Shafarevich, *Basic Algebraic Geometry 1*, Springer-Verlag, 1988.

[Ser08] J. P. Serre, *Le groupe de Cremona et ses sous-groups finis*, Séminaire BOURBAKI, No 1000, 2008-2009. 2008-2009.

Ivan Pan, Centro de Matemática, Facultad de Ciencias, Universidad de la República, Iguá 4225, 11400 - Montevideo - URUGUAY

E-mail address: ivan@cmat.edu.uy

Alvaro Rittatore, Centro de Matemática, Facultad de Ciencias, Universidad de la República, Iguá 4225, 11400 - Montevideo - URUGUAY

E-mail address: alvaro@cmat.edu.uy