Abstract: Inspired by the soft theorem in gravity theory by Cachazo and Strominger, the soft theorem for color-ordered Yang-Mills amplitudes has also been identified by Casali. In this note, the soft theorem of $\mathcal{N} = 4$ SYM using the Grassmannian formulation is studied. Explicitly, in the holomorphic soft limit, we reduce the $n$-particle amplitude in terms of Grassmannian contour integrations into the deformed $(n-1)$-particle amplitude by localizing $k$ variables relevant to the $n$-th particle. Meanwhile, the universal soft factor emerges as a byproduct. In passing, the ‘particle interpretation’ of this formulation for all $k$’s is understood thoroughly.

Keywords: Amplitudes, Grassmannian, Soft Theorem.
1. Introduction

Recently a new soft theorem for tree level gravity amplitudes was studied in [1]. By using the BCFW construction and imposing the holomorphic soft limit, Cachazo and Strominger have proved that

\[ M_n (\lambda_n \rightarrow \varepsilon \lambda_n) = \varepsilon^{-3} \sum_{a=1}^{n-2} \frac{(n-1,a)a^n}{(n-1,a)^2} M_{n-1} (\tilde{\lambda}_{n-1} \rightarrow \tilde{\lambda}_{n-1} + \varepsilon \frac{\langle an \rangle}{a, n-1} \tilde{\lambda}_n, \tilde{\lambda}_1 \rightarrow \tilde{\lambda}_1 + \varepsilon \frac{\langle n-1, n \rangle}{n-1, a} \tilde{\lambda}_n) + O(\varepsilon^0), \]  

(1.1)

here for \( M_n \) and \( M_{n-1} \), the unmentioned external kinematic data are un-deformed and we prefer to suppress them for conciseness\(^1\). Taylor expansion in \( \varepsilon \) exhibits three singular terms in orders \( \varepsilon^{-3} \), \( \varepsilon^{-2} \) and \( \varepsilon^{-1} \), while higher order terms in \( \varepsilon \) will be mixed with the less interesting \( O(\varepsilon^0) \) parts.

A similar relation for tree level Yang-Mills amplitudes using the BCFW construction, proved by Casali [2], takes the form

\[ A_n (\lambda_n \rightarrow \varepsilon \lambda_n) = \varepsilon^{-2} \frac{(n-1,1)}{(n-1,n)\langle n1 \rangle} A_{n-1} (\tilde{\lambda}_{n-1} \rightarrow \tilde{\lambda}_{n-1} + \varepsilon \frac{\langle 1n \rangle}{1, n-1} \tilde{\lambda}_n, \tilde{\lambda}_1 \rightarrow \tilde{\lambda}_1 + \varepsilon \frac{\langle n-1, n \rangle}{n-1, 1} \tilde{\lambda}_n) + O(\varepsilon^0), \]

(1.2)

where two singular terms in orders \( \varepsilon^{-2} \) and \( \varepsilon^{-1} \) appear after Taylor expansion. Mixing between higher order terms from the deformed \( A_{n-1} \) and \( O(\varepsilon^0) \) parts also pertains to this case.

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\(^1\)Any amplitude mentioned in this note contains the delta function which imposes momentum conservation.
It is fruitful to study the same object from various viewpoints in physics because it will deepen our original understanding and reveal many hidden facts. Many related studies have been achieved including: the soft limit from Poincaré symmetry and gauge invariance \cite{5, 6}, Feynman diagram approach \cite{7}, conformal symmetry approach for the soft limit in Yang-Mills theory \cite{8}, loop corrections of the soft limit \cite{9, 10, 11, 12}, string-theory approach for the soft limit \cite{13, 14, 15, 17}, ambitwistor string approach \cite{18, 19}, and KLT approach \cite{20}.

In this note, a relatively novel way using the Grassmannian contour integral is shown to reproduce the soft theorem for amplitudes in $\mathcal{N} = 4$ SYM. Relevant background on Grassmannian can be found in \cite{3, 4}, and a brief review of key formulae related to $N^{k-2}$MHV amplitudes $A_{n}^{[k]}$ is given here. The first example is the NMHV (super) amplitude, written as

$$A_{n}^{[3]} = \int \frac{g_{n}^{[3]}}{(n-1)(1)(3)} \frac{1}{f_{6} \ldots f_{n}},$$

(1.3)

where $(l)$ is short for the consecutive 3-minor $(l + l + 2)$ in terms of $c_{Ii}$’s for the $k = 3$ case. The integral symbol above denotes

$$\int d^{3(n-3)} c_{Ii} \delta^{2(n-3)}(\lambda_{I} - \lambda_{I} c_{Ii}) \delta^{2}(\tilde{\lambda}_{I} + c_{Ii} \tilde{\lambda}_{I}) = \delta^{4} \left( \sum \lambda_{I} \tilde{\lambda}_{I} \right) \times J \int d^{n-5} \tau,$$

(1.4)

note that we have suppressed the supersymmetric content \(\delta^{4}(\tilde{\eta}_{I} + c_{Ii} \tilde{\eta}_{I})\), as it is not involved in solving $c_{Ii}$’s. In the latter $\tau$ parameterization, the $(n - 5)$ variables are localized by $(n - 5)$ $f_{i}$’s in (1.3). Although as shown in \cite{3}, the integrand in (1.3) is nothing but

$$\frac{1}{(1 2 3)(2 3 4) \ldots (n 1 2)},$$

(1.5)

after the cancelation of non-consecutive minors between the numerator and denominator, for practical calculations (1.3) is adopted and the reason is: It is known that residue (or contour) integrations demand non-consecutive minors for physical outputs, while some consecutive minors are redundant. The advantage of the integrand in (1.3) is that all unwanted minors are ‘lifted away’ from the sequence of minors mapped to zero (i.e., they are only evaluated as parts of the residues at zero minors).

Another convenience of this integrand is that in process of the inverse soft operation (or ‘add one particle at a time’), the soft factor is recovered in the soft limit. More explicitly, this factor is given via

$$I_{n}^{[3]} = S_{(n-1) \rightarrow n}^{[3]} I_{n-1}^{[3]}$$

where $I_{n}^{[3]}$ is short for the integrand in (1.3), with

$$S_{(n-1) \rightarrow n}^{[3]} = \frac{(n - 2)!}{(n - 1)! (n - 2) 2 3},$$

(1.6)

$\frac{1}{(n - 1)! f_{n}}$.

\footnote{For general $N^{k-2}$MHV amplitudes, the counting is the following. The $(k \times n)$ matrix $c_{Ii}$ has $\text{GL}(k)$ gauge invariance, hence there are $k(n - k)$ actual integration variables. Imposing $(2n - 4)$ delta functions, there are $(k - 2)(n - k - 2)$ variables left to be fixed by contour integrations.}
here the prime means that the corresponding consecutive minor is for the \((n - 1)\) case, namely \((n - 2)' = (n - 2, n - 11)\) and \((n - 1)' = (n - 11, 2)\). In the limit \(\lambda_n \rightarrow \varepsilon \lambda_n\),

\[
S_{(n-1) \rightarrow n}^{[3]} \rightarrow \frac{1}{\varepsilon^2} \frac{(n - 1, 1)}{(n - 1, n)(n1)},
\]

this is the desired soft factor after getting three \(c_{I_n}\)’s (in general \(k c_{I_n}\)’s) localized. In \([3]\) only the leading singular term is mentioned, while in fact the sub-leading term is automatically included as well, which will be demonstrated in this note.

Having explained the integrand for NMHV amplitudes, now let’s present the universal structure of general \(N^{k-2}\)MHV amplitudes derived by the conjugation construction. To see it, one can compare the integrand for NMHV amplitudes \((1.3)\), with the one for \(N^{2}\)MHV amplitudes given by

\[
A^{[4]}_n = \int_{\{F_7 = \ldots = F_n = 0\}} \frac{g^{[4]}_n}{(n - 1)(1)(3)} \frac{1}{F_7 \ldots F_n},
\]

where

\[
g^{[4]}_n = \prod_{j=7}^{n-1} (1 2 3 j)(23 j - 2 j - 1)(1 j - 2 j - 1 j) \prod_{j=4}^{n-3} (13 j j + 1)(12 j j + 3),
\]

and \(F_l = f_{l1} f_{l2}\) with

\[
f_{l1} = (l - 3 l - 2 l - 1 l)(l - 3 l 1 2)(l - 3 2 l 2 - 2),
\]

\[
f_{l2} = (1 l - 2 l - 1 l)(1 l 2 3)(13 l - 3 l - 2).
\]

The conjugation construction is contained in the steps of getting \(g^{[4]}_n\) and \(F_l\)’s from \(g^{[3]}_n\) and \(f_1\)’s, as well as transforming \((n - 1)(1)(3)\) of \(k = 3\) into the same product but of \(k = 4\). By using this construction, one can show that for general \(N^{k-2}\)MHV amplitudes,

\[
A^{[k]}_n = \int_{\{F_{k+3} = \ldots = F_n = 0\}} \frac{g^{[k]}_n}{(n - 1)(1)(3)} \frac{1}{F_{k+3} \ldots F_n}, \quad F_l = f_{l1} \ldots f_{l,k-2},
\]

here each \(F_l\) is a product of \((k - 2)\) \(f_{ij}\)’s which enforce \((k - 2)\) minors to be zero, hence offsetting the \((k - 2)\) variables brought by each newly added particle. Details of this approach can be found in appendix \([4]\).

Returning to the inverse soft operation, we need to emphasize a relation. Assume that the integrand \(I^{[k]}_{n-1}\) is known, to get \(I^{[k]}_n\), one simply needs to multiply \(I^{[k]}_{n-1}\) by the inverse soft factor

\[
S^{[k]}_{(n-1) \rightarrow n} = \frac{(n - k + 1)'(n - k + 2)' \ldots (n - 1)'}{(n - k + 1)(n - k + 2) \ldots (n - 1)(n)},
\]

which has been ‘over-simplified’ due to the cancelation of all non-consecutive minors between the numerator and denominator, same as what happens in \([1,3]\). This observation tells the simplicity in handling the soft limit: We need only focus on the consecutive minors.

After getting familiar with the Grassmannian formulation, in this note we will show how to use this approach to reproduce \((1.2)\) for all \(k\)’s, especially two key components in \((1.2)\): the overall soft factor and
the deformation of anti-holomorphic spinor pair \((\tilde{\lambda}_{n-1}, \tilde{\lambda}_1)\) in terms of \(\varepsilon\). Although it is quite difficult to get the explicit expression for \(N^2\text{MHV}\) amplitudes by performing all residue integrations, let alone general \(N^{k-2}\text{MHV}\) amplitudes, still in the soft limit, with sufficient tricks one is able to find this generic relation for all \(k\)'s while keeping the Grassmannian contour integrations of \(A^{[k]}_{n-1}\) unsolved.

This note is organized as follows. Section 2 briefly reviews the inverse soft operation for \(N_{\text{NMHV}}\) amplitudes and presents the corresponding soft theorem via a detailed proof. Section 3 explores the same aspects of \(N^2_{\text{MHV}}\) and \(N^3_{\text{MHV}}\) amplitudes and attempts to find the pattern for general \(N^{k-2}_{\text{MHV}}\) ones. Having gathered enough sense of this procedure, section 4 provides a general proof of the soft theorem for \(N^{k-2}_{\text{MHV}}\) amplitudes. Appendix A proves the general structure we have claimed for \(N^{k-2}_{\text{MHV}}\) amplitudes by using the conjugation construction. Appendix B explains why the unmentioned but possibly singular parts are actually regular in the soft limit.

2. NMHV Amplitudes Redux

In this section let’s consider the simplest case, i.e., the \(N_{\text{NMHV}}\) amplitude formulated by

\[
A^{[3]}_n = \int \frac{g^{[3]}_n}{(n-1)(1)(3)} \frac{1}{f_6 \ldots f_{n-1} f_n},
\]

(2.1)

where to simplify the notation, underlines are used to indicate the zero factors for the contour integrations. By applying the global residue theorem, this expression can be manipulated into

\[
\begin{align*}
\int \frac{g^{[3]}_n}{(n-1)(1)(3)} \frac{1}{f_6 \ldots f_{n-1} f_n} &= - \int \frac{g^{[3]}_n}{(n-1)(1)(3)} \frac{1}{f_6 \ldots f_{n-1} f_n} - \int \frac{g^{[3]}_n}{(n-1)(1)(3)} \frac{1}{f_6 \ldots f_{n-1} f_n} - \int \frac{g^{[3]}_n}{(n-1)(1)(3)} \frac{1}{f_6 \ldots f_{n-1} f_n},
\end{align*}
\]

(2.2)

as will be explained in appendix B, among three terms above, only the \((n-1)\) term has singular contribution in the holomorphic soft limit \(\lambda_n \to \varepsilon \lambda_n\).

To work out the calculation, let’s write the integral symbol explicitly as

\[
\int = \delta^4 \left( \sum \lambda_i \hat{\lambda}_i \right) \times J \int d^{n-5} \tau = \int d^{3(n-3)} c_{Ij} \delta^{2(n-3)} (\lambda_i - \lambda_I c_{Ij}) \delta^{2,3} (\hat{\lambda}_I + c_{Ii} \hat{\lambda}_i),
\]

(2.3)

where again \(\delta^{4,3}(\eta_I + c_{Ii} \eta_i)\) has been suppressed. We have found that the latter representation in terms of \(c_{Ii}\)'s is more convenient for our purpose. In the Grassmannian formulation of \(c_{Ii}\)'s, one needs to choose a gauge. Among many ones, the following gauge provides maximal simplicity:

\[
C = \begin{pmatrix}
\ldots & 1 & 0 & c_{n-2,n} & 0 & c_{n-2,2} & \ldots \\
\ldots & 0 & 1 & c_{n-1,n} & 0 & c_{n-1,2} & \ldots \\
\ldots & 0 & 0 & c_1 n & 1 & c_{12} & \ldots \\
\end{pmatrix},
\]

(2.4)
where three columns \((n-2,n-1,1)\) have been fixed to be a unit matrix\(^3\).

Now we attempt to write the integrand possessing \((n-1)\) in \((2.4)\) into the form of the deformed \(A_{n-1}^{[m]}\) multiplied by the soft factor, hence the integral can be split as

\[
- \int d^{3(n-4)}c_{I2} \delta^{2(n-4)}(\lambda_{\tilde{2}} - \lambda_I c_{I2}) \delta^{(3)}(\tilde{\lambda}_I + c_{I2} \tilde{\lambda}_1 + c_{In} \tilde{\lambda}_n) I_{n-1}^{[m]} I_{n-1}^{[m]} \left\{ \int d^3 c_{In} \delta^2(\lambda_n - \lambda_I c_{In}) \frac{(n-2)'(n-1)'}{(n-2)(n-1)} \right\},
\]

where \(\tilde{2} = 1, \ldots, n-1\), and integrations over all \(c_{In}\)’s are collected inside the curly bracket. Keep in mind that the integrand inside the curly bracket is the inverse soft factor \(S^{3_3}_{n-1\rightarrow n}\), which later turns into the soft factor when its residue is evaluated at \((n-1) = 0\). To show this, let’s compute the relevant minors as

\[
\begin{align*}
(n-2)' &= 1, \quad (n-1)' = c_{n-2,2}, \quad (2.6) \\
(n-2) &= c_{1n}, \quad (n-1) = -c_{n-2,n}, \quad (n) = -\begin{vmatrix} c_{n-2,n} & c_{n-2,2} \\ c_{n-1,n} & c_{n-1,2} \end{vmatrix}, \quad (2.7)
\end{align*}
\]

and the integration measure is defined as (be aware of the reversed cyclic order\(^4\))

\[
\int d^3 c_{In} \equiv \int dc_{In} dc_{n-1,n} dc_{n-2,n}, \quad (2.8)
\]

since \((n-1) = 0\) fixes \(c_{n-2,n} = 0\), hence the delta function \(\delta^2(\lambda_n - \lambda_I c_{In})\) solves

\[
c_{n-1,n} = \frac{\langle 1n \rangle}{\langle 1, n-1 \rangle}, \quad c_{1n} = \frac{\langle n-1,n \rangle}{\langle n-1,1 \rangle} \quad (2.9)
\]

Putting every piece together,

\[
- \int d^3 c_{In} \delta^2(\lambda_n - \lambda_I c_{In}) \frac{(n-2)'(n-1)'}{(n-2)(n-1)} = \frac{1}{\langle n-1,n \rangle c_{n-1,n} c_{1n}} = \frac{\langle n-1,1 \rangle}{\langle n-1,1 \rangle} = \frac{\langle n-1,1 \rangle}{\langle n-1,1 \rangle}.
\]

(2.10)

which matches the soft factor as promised.

Then what is going on in the rest parts of the Grassmannian integral? Recall \((2.3)\), first note that delta functions \(\delta^{2(n-4)}(\lambda_{\tilde{2}} - \lambda_I c_{I2})\) are unaffected, while delta functions \(\delta^{(3)}(\tilde{\lambda}_I + c_{I2} \tilde{\lambda}_1 + c_{In} \tilde{\lambda}_n)\) reduce to

\[
\begin{align*}
\delta^2(\tilde{\lambda}_{n-1} + c_{n-1,\tilde{2}} \tilde{\lambda}_1 + c_{n-1,n} \tilde{\lambda}_n) \delta^2(\tilde{\lambda}_1 + c_{I2} \tilde{\lambda}_1 + c_{I1} \tilde{\lambda}_n) \delta^2(\tilde{\lambda}_2 + c_{I2} \tilde{\lambda}_2 + c_{I2} \tilde{\lambda}_1) \\
= \delta^2(\tilde{\lambda}_{n-1} + \frac{\langle 1n \rangle}{\langle 1, n-1 \rangle} \tilde{\lambda}_n + c_{n-1,\tilde{2}} \tilde{\lambda}_1) \delta^2(\tilde{\lambda}_1 + \frac{\langle n-1,n \rangle}{\langle n-1,1 \rangle} \tilde{\lambda}_n + c_{I2} \tilde{\lambda}_1) \delta^2(\tilde{\lambda}_2 + c_{I2} \tilde{\lambda}_2).
\end{align*}
\]

(2.11)

\(^3\)Our gauge choice is a bit different from the standard one where columns of negative helicities are often fixed, since under such a choice the supersymmetric counterpart can be integrated over most conveniently \([3]\). Here the three columns chosen to be fixed are not necessarily associated with negative helicities. But for the soft particle \(n\), which has positive helicity, it is natural to have an unfixed column.

\(^4\)The reason to choose this order will be explained in section \([4]\).
where the third one is also unaffected. Plug (2.11) back into (2.5), we just recover the soft theorem for NMHV amplitudes in $\mathcal{N} = 4$ SYM at tree level, namely, (2.2) becomes

$$A^{[3]}_n (\lambda_n \to \varepsilon \lambda_n) = \frac{1}{\varepsilon^2} \frac{(n-1,1)}{(n-1,n)(n1)} A^{[3]}_{n-1} \left( \tilde{\lambda}_{n-1} \to \tilde{\lambda}_{n-1} + \varepsilon \frac{(1n)}{(1,n-1)} \tilde{\lambda}_n, \tilde{\lambda}_1 \to \tilde{\lambda}_1 + \varepsilon \frac{(n-1,n)}{(n-1,1)} \tilde{\lambda}_n \right) + \text{(pure regular parts)},$$

(2.12)

where $\lambda_n$ is replaced by $\varepsilon \lambda_n$ to manifest the soft divergence. Here let’s call the first item above ‘the singular parts’, but it in fact contains ‘mixed regular parts’ after Taylor expansion in $\varepsilon$. In contrast, the ‘pure regular parts’ do not involve the $1/\varepsilon^2$ prefactor. One more comment is that these pure regular parts correspond to terms whose $A_L$’s are not 3-particle amplitudes in the BCFW construction, as the reader can see more explanations in $[1, 2]$.

3. More Extensions: $N^2$MHV and $N^3$MHV Amplitudes

Having accomplished the simplest case, now we would like to explore the $N^2$MHV and $N^3$MHV amplitudes. From these two further examples, the pattern for general $N^{k-2}$MHV amplitudes, which will be revealed in the next section, starts to emerge.

To begin with the $N^2$MHV amplitude, recall (1.8) and (1.10), by applying the global residue theorem, the relevant integral becomes

$$\int g^{[4]}_n \frac{1}{f' F_7 \cdots F_{n-1}(f_{n1} f_{n2})} = - \int g^{[4]}_n \frac{1}{f' F_7 \cdots F_{n-1}(f_{n1} f_{n2})} = - \int g^{[4]}_n \frac{1}{(n-1)(1)(3)} F_7 \cdots F_{n-1} f_{n1} (n-2)(n) \cdots,$$

(3.1)

where $f' = (n-1)(1)(3)$ for conciseness, we also remind the reader that $f_{n1} = (n-3)(\ldots)(\ldots)$ and $f_{n2} = (n-2)(n)(\ldots)$ where dots in parentheses denote the less important non-consecutive minors. In the second line above, among $3^2$ choices of zero minors in $f'$ and $f_{n2}$, let’s single out the term with $(n-1) = (n-2) = 0$, since other terms do not give singular contributions in the soft limit.

After the key integral is identified in (3.1), following the similar recipe and using (1.12), we split the integrand into the part of remaining $(n-1)$ particles and the part of inverse soft factor, as treated in (2.5). Now let’s focus on the following integral

$$- \int d^4 c_{In} \delta^2 (\lambda_n - \lambda_I c_{In}) \frac{(n-3)'(n-2)'(n-1)'}{(n-3)(n-2)(n-1)(n)},$$

(3.2)

as mentioned before, the reason to assign $(n-2)$ to be the second zero minor is that this choice turns out to be the only singular contribution in the soft limit. Its proof (and the proof for general $k$’s) is given in appendix $[3]$. Note that there is another minus sign appeared due to swapping the positions of $(n-1)$ and

$$\mathbf{4.13}$$
(n - 2), since by default (n - 1) locates at the first place in the sequence of zero minors. The latter fact is also true for all k’s, as shown in appendix [4].

To proceed, with the previous experience, we choose the gauge

$$C = \begin{pmatrix} 
\ldots & c_{n-2,n-3} & 1 & 0 & c_{n-2,n} & 0 & 0 & c_{n-2,3} & \ldots \\
\ldots & c_{n-1,n-3} & 0 & 1 & c_{n-1,n} & 0 & 0 & c_{n-1,3} & \ldots \\
\ldots & c_{1,n-3} & 0 & 0 & c_{1n} & 1 & 0 & c_{13} & \ldots \\
\ldots & c_{2,n-3} & 0 & 0 & c_{2n} & 0 & 1 & c_{23} & \ldots 
\end{pmatrix},$$  

(3.3)

where four columns \((n - 2, n - 1, 1, 2)\) have been fixed to be a unit matrix. Hence the relevant minors are computed as

\[(n - 3)' = -c_{2, n-3}, \quad (n - 2)' = 1, \quad (n - 1)' = -c_{n-2,3},\]

(3.4)

\[(n - 3) = \begin{vmatrix} 
c_{1,n-3} & c_{1n} 
c_{2,n-3} & c_{2n} 
\end{vmatrix}, \quad (n - 2) = -c_{2n}, \quad (n - 1) = -c_{n-2,n}, \quad (n) = \begin{vmatrix} 
c_{n-2,n} & c_{n-2,3} 
c_{n-1,n} & c_{n-1,3} 
\end{vmatrix},\]

(3.5)

and the integration measure is

$$\int d^4c_{I\bar{n}} = \int dc_{2n}dc_{1n}dc_{n-1,n}dc_{n-2,n} = \int dc_{1n}dc_{n-1,n} \int dc_{2n} \int dc_{n-2,n}.\]

(3.6)

Pay attention to the order of \(dc_{I\bar{n}}\)’s as they in fact anticommute. A little subtlety here is that we have to match the orders of \(dc_{I\bar{n}}\)’s and zero minors, namely \(dc_{2n}dc_{n-2,n}\) must be associated with \((n - 2)(n - 1)\), otherwise a sign factor will arise due to altering the order of either \(dc_{I\bar{n}}\)’s or zero minors. But nicely, there is no such a worry in this case and that’s the reason to adopt the reversed cyclic order for \(dc_{I\bar{n}}\)’s. Also note that we always leave the integrations over \(c_{1n}\) and \(c_{n-1,n}\) to the last step, after performing the \((k - 2)\) residue integrations.

The two integrations easily fix \(c_{2n} = c_{n-2,n} = 0\), and using the remaining two delta functions in \(\{3, 2\}\) we find the solution \([3, 2]\). Hence the final result is

$$\frac{1}{(n - 1, 1)} \frac{1}{c_{n-1,n}c_{1n}} = \frac{\langle n - 1, 1 \rangle}{\langle n - 1, n \rangle \langle n 1 \rangle},$$

(3.7)

and the anti-holomorphic spinor pair \(\tilde{\lambda}_{n-1, \tilde{\lambda}_1}\) is deformed as indicated in \(\{2, 1\}\), regardless of the increase of \(k\). Not surprisingly, the soft theorem is again recovered for the \(N^2\)MHV case.

Next we move on to the case of \(N^3\)MHV amplitudes, because it contains a non-trivial feature that cannot be seen in the \(N^2\)MHV case. By applying the global residue theorem, the relevant integral becomes\(^5\)

$$\int g_n^\text{[5]} \frac{1}{F_{\tilde{\lambda}_8}\cdots F_{n-1}(\tilde{f}_{n1}\tilde{f}_{n2}\tilde{f}_{n3})} = - \int g_n^\text{[5]} \frac{1}{F_{\tilde{\lambda}_8}\cdots F_{n-1}(\tilde{f}_{n1}\tilde{f}_{n2}\tilde{f}_{n3})}.\]

(3.8)

\(^5\)This form for \(k = 5\) will be explained in appendix [4].
where \( f' = (n-1)(1)(3) \). In the right hand side above, among \( 3^3 \) choices of zero minors in \( f', f_{n2} \) and \( f_{n3} \), as you may guess, we single out the one picking \( (n-1), (n-2) \) and \( (n-3) \) respectively, where following expressions of \( f_{nj} \)'s are used,

\[
f_{n1} = (n-4)(\ldots)(\ldots), \quad f_{n2} = (n-3)(\ldots)(\ldots), \quad f_{n3} = (n-2)(n)(\ldots).
\]

(3.9)

Following the pattern of (3.2) together with (1.12), let’s calculate

\[
- \int d^5 c_{1n} \delta^2 (\lambda_n - \lambda_I c_{1n}) \frac{(n-4)'(n-3)'(n-2)'(n-1)'}{(-)^2(n-4)(n-3)(n-2)(n-1)(n)},
\]

(3.10)

the reason to assign \( (n-3) \) to be the third zero minor is the same as previous. Here a sign factor also arises when \( (n-1) \) is pulled through \( (n-3)(n-2) \), since only the zero minors care about their order, while others trivially commute.

To proceed as trickily as before, we choose the gauge

\[
C = \begin{pmatrix}
\ldots & c_{n-2,n-4} & c_{n-2,n-3} & 1 & 0 & c_{n-2,n} & 0 & 0 & 0 & c_{n-2,4} & \ldots \\
\ldots & c_{n-1,n-4} & c_{n-1,n-3} & 0 & 1 & c_{n-1,n} & 0 & 0 & 0 & c_{n-1,4} & \ldots \\
\ldots & c_{1,n-4} & c_{1,n-3} & 0 & 0 & c_{1n} & 1 & 0 & 0 & c_{14} & \ldots \\
\ldots & c_{2,n-4} & c_{2,n-3} & 0 & 0 & c_{2n} & 0 & 1 & 0 & c_{24} & \ldots \\
\ldots & c_{3,n-4} & c_{3,n-3} & 0 & 0 & c_{3n} & 0 & 0 & 1 & c_{34} & \ldots
\end{pmatrix},
\]

(3.11)

where five columns \( (n-2, n-1, 1, 2, 3) \) have been fixed to be a unit matrix. Compare this choice with those in NMHV and \( \text{N}^2\text{MHV} \) amplitudes, keen eyes will immediately see the pattern: We always fix \( k \) columns \( (n-2, n-1, 1, 2, \ldots, k-2) \) to be a unit matrix. Hence the relevant minors are computed as

\[
(n-4)' = \begin{vmatrix}
c_{2,n-4} & c_{2,n-3} \\
c_{3,n-4} & c_{3,n-3}
\end{vmatrix}, \quad (n-3)' = c_{3,n-3}, \quad (n-2)' = 1, \quad (n-1)' = c_{n-2,4},
\]

(3.12)

\[
(n-4) = \begin{vmatrix}
c_{1,n-4} & c_{1,n-3} & c_{1n} \\
c_{2,n-4} & c_{2,n-3} & c_{2n} \\
c_{3,n-4} & c_{3,n-3} & c_{3n}
\end{vmatrix}, \quad (n-3) = \begin{vmatrix}
c_{2,n-3} & c_{2n} \\
c_{3,n-3} & c_{3n}
\end{vmatrix},
\]

(3.13)

\[
(n-2) = c_{3n}, \quad (n-1) = -c_{n-2,n}, \quad (n) = -\begin{vmatrix}
c_{n-2,n} & c_{n-2,4} \\
c_{n-1,n} & c_{n-1,4}
\end{vmatrix}.
\]

and the integration measure is

\[
\int d^5 c_{1n} = \int dc_{3n} dc_{2n} dc_{1n} dc_{n-1,n} dc_{n-2,n} = - \int dc_{1n} dc_{n-1,n} \int dc_{2n} dc_{3n} \int dc_{n-2,n},
\]

(3.14)

where the order above is chosen to fit \( (n-3)(n-2)(n-1) \). One must start the integrations from the rightmost, so the residue at \( (n-3) = 0 \) must be evaluated after finishing \( (n-2) \) and \( (n-1) \). In this way the cancelation of all other un-localized \( c_{IJ} \)'s is guaranteed, as these minors factorize with particular zero
entries. This is a general pattern of $N^{k-2}$MHV amplitudes but it only starts to emerge from the $N^3$MHV case.

As expected, the final result of (3.10) is

$$\frac{1}{\langle n-1,1 \rangle} \frac{1}{c_{n-1,n}c_{1n}} = \langle n-1,1 \rangle \langle n-1,n \rangle \langle n1 \rangle,$$

with $c_{2n} = c_{3n} = c_{n-2,n} = 0$, the spinor pair $(\tilde{\lambda}_{n-1}, \tilde{\lambda}_1)$ is deformed exactly as (2.11). Once more, the soft theorem is recovered for the $N^3$MHV case.

4. General $N^{k-2}$MHV Amplitudes

In the previous $N^3$MHV case, we actually presume that $N^{k-2}$MHV amplitudes have a universal structure, where $(n-1)$ and $(n)$ play special roles, namely

$$A_{[k]}^n = \int g_n^{[k]} \frac{1}{f'} F_{k+3} \ldots F_n, \quad f' = (n-1)(1)(3), \quad F_i = f_{i1} \ldots f_{ik-2},$$

which is constructed by conjugation in appendix [A]. Applying the global residue theorem, yields

$$\int g_n^{[k]} \frac{1}{f'} F_{k+3} \ldots F_{n-1} (f_{n1} f_{n2} \ldots f_{nk-2}) = - \int g_n^{[k]} \frac{1}{f'} F_{k+3} \ldots F_{n-1} (f_{n1} f_{n2} \ldots f_{nk-2}),$$

where each underlined $F_i$ enforces all $(k-2)$ $f_{ij}$’s it contains to be zero, and each $f_{ij}$ contributes one zero minor at a time respectively. The form of $f_{nj}$’s is given by

$$f_{n1} = (n-k+1)(\ldots)(\ldots), \quad \ldots f_{nk-3} = (n-3)(\ldots)(\ldots), \quad f_{nk-2} = (n-2)(n)(\ldots).$$

Among $3^{k-2}$ choices of zero minors in the right hand side of (4.2), let’s single out the one picking $(n-1) = (n-k+2) = (n-k+3) = \ldots = (n-2) = 0$, selected from $f'$ and $f_{nj}$’s with $j = 2, \ldots, k-2$. This one is the only term that has singular contribution in the holomorphic soft limit.

To proceed, we choose the gauge

$$C = \begin{pmatrix}
\cdots & c_{n-2,n-1} & c_{n-2,n-k+1} & c_{n-2,n-k+2} & \cdots & c_{n-2,n-k} & 1 & 0 & c_{n-2,n} & 0 & \ldots & 0 & 0 & c_{n-2,k-1} & \ldots \\
\cdots & c_{n-1,n-1} & c_{n-1,n-k+1} & c_{n-1,n-k+2} & \cdots & c_{n-1,n-k} & 0 & 1 & c_{n-1,n} & 0 & \ldots & 0 & 0 & c_{n-1,k-1} & \ldots \\
\cdots & c_{1,n-1} & c_{1,n-k+1} & c_{1,n-k+2} & \cdots & c_{1,n-k} & 0 & 0 & c_{1,n} & 0 & \ldots & 0 & 0 & c_{1,k-1} & \ldots \\
\cdots & c_{2,n-1} & c_{2,n-k+1} & c_{2,n-k+2} & \cdots & c_{2,n-k} & 0 & 0 & c_{2,n} & 0 & \ldots & 0 & 0 & c_{2,k-1} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & c_{k,n-1} & c_{k,n-k+1} & c_{k,n-k+2} & \cdots & c_{k,n-k} & 0 & 0 & c_{k,n} & 0 & \ldots & 0 & 1 & c_{k,k-1} & \cdots \\
\cdots & c_{k-1,n-1} & c_{k-1,n-k+1} & c_{k-1,n-k+2} & \cdots & c_{k-1,n-k} & 0 & 0 & c_{k-1,n} & 0 & \ldots & 0 & 1 & c_{k-1,k-1} & \cdots \end{pmatrix},$$

where $k$ columns $(n-2, n-1, 1, 2, \ldots, k-3, k-2)$ have been fixed to be a unit matrix. Next let’s calculate the following integral

$$- \int d^k c_{In} \delta^2(\lambda_n - \lambda_I c_{In}) \frac{(n-k+1)'(n-k+2)'(n-k+3)' \ldots (n-1)'}{(-)^{k-3}(n-k+1)(n-k+2)(n-k+3) \ldots (n-2)(n-1)(n)},$$

(4.5)
which is obtained by extending (3.10) for a generic \( k \) with (1.12). The sign factor in the denominator arises when \( (n-1) \) is pulled through other \((k-3)\) zero minors. As previous, the only singular contribution is from the sequence of zero minors selected above.

In this gauge, the relevant minors are computed as

\[
(n-k+1)''' = (-)^{(k-3)\cdot 3} \begin{vmatrix} c_{2,n-k+1} & \cdots & c_{2,n-3} \\ \vdots & \ddots & \vdots \\ c_{k-2,n-k+1} & \cdots & c_{k-2,n-3} \end{vmatrix},
\]

\[
(n-k+2)''' = (-)^{(k-4)\cdot 4} \begin{vmatrix} c_{3,n-k+2} & \cdots & c_{3,n-3} \\ \vdots & \ddots & \vdots \\ c_{k-2,n-k+2} & \cdots & c_{k-2,n-3} \end{vmatrix},
\]

\[\vdots\]

\[
(n-4)''' = (-)^{2(k-2)} \begin{vmatrix} c_{k-3,n-4} & c_{k-3,n-3} \\ c_{k-2,n-4} & c_{k-2,n-3} \end{vmatrix},
\]

\[
(n-3)''' = (-)^{1(k-1)} c_{k-2,n-3},
\]

\[
(n-2)''' = 1,
\]

\[
(n-1)''' = (-)^{k-1} c_{n-2,k-1},
\]

and

\[
(n-k+1) = (-)^{(k-2)\cdot 2} \begin{vmatrix} c_{1,n-k+1} & \cdots & c_{1,n-3} & c_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ c_{k-2,n-k+1} & \cdots & c_{k-2,n-3} & c_{k-2,n} \end{vmatrix},
\]

\[
(n-k+2) = (-)^{(k-3)\cdot 3} \begin{vmatrix} c_{2,n-k+2} & \cdots & c_{2,n-3} & c_{2n} \\ \vdots & \ddots & \vdots & \vdots \\ c_{k-2,n-k+2} & \cdots & c_{k-2,n-3} & c_{k-2,n} \end{vmatrix},
\]

\[\vdots\]

\[
(n-4) = (-)^{3(k-3)} \begin{vmatrix} c_{k-4,n-4} & c_{k-4,n-3} & c_{k-4,n} \\ c_{k-3,n-4} & c_{k-3,n-3} & c_{k-3,n} \\ c_{k-2,n-4} & c_{k-2,n-3} & c_{k-2,n} \end{vmatrix},
\]

\[
(n-3) = (-)^{2(k-2)} \begin{vmatrix} c_{k-3,n-3} & c_{k-3,n} \\ c_{k-2,n-3} & c_{k-2,n} \end{vmatrix},
\]

\[
(n-2) = (-)^{1(k-1)} c_{k-2,n},
\]

\[
(n-1) = -c_{n-2,n},
\]

\[
(n) = (-)^{k-2} \begin{vmatrix} c_{n-2,n} & c_{n-2,k-1} \\ c_{n-1,n} & c_{n-1,k-1} \end{vmatrix}.
\]
And the integration measure is

\[
\int d^k c_{fn} = \int dc_{k-2,n} dc_{k-3,n} \ldots dc_{2n} dc_{1n} dc_{n-1,n} dc_{n-2,n} \\
= (-)^{1+2+\ldots+(k-4)} \int dc_{1n} dc_{n-1,n} \int dc_{2n} \ldots dc_{k-3,n} dc_{k-2,n} \int dc_{n-2,n},
\]  

(4.8)

note that we have reversed the order of \( dc_{k-2,n} dc_{k-3,n} \ldots dc_{2n} \) to fit \((n-k+2)(n-k+3)\ldots(n-2)\), hence a sign factor arises. The \((k-2)\) residue integrations at \((n-k+2) = (n-k+3) = \ldots = (n-2) = (n-1) = 0\) fix \((k-2)\) \(c_{fn}\)’s to be zero when proceeded from right to left, while \(c_{n-1,n}\) and \(c_{1n}\) are localized by the delta function \(\delta^2(\lambda_n - \lambda_I c_{fn})\) in the last step.

After that, (4.7) reduces to

\[
(n-k+1) = (-)^{(k-2)2} c_{1n-k+1} \ldots c_{1,n-3} c_{1n} \ldots c_{k-2,n-k+1} \ldots c_{k-2,n-3} c_{k-2,n} \\
(n-k+2) = (-)^{(k-3)3} c_{2,n-k+2} \ldots c_{2,n-3} c_{2n} \ldots c_{k-2,n-k+2} \ldots c_{k-2,n-3} c_{k-2,n} \\
\vdots \\
(n-4) = (-)^{(3(k-3)} c_{k-2,n-k+1} c_{k-2,n-3} c_{k-2,n} \\
(n-3) = (-)^{(2(k-2)} c_{k-3,n-k+1} c_{k-3,n-3} c_{k-3,n} \\
(n-2) = (-)^{(1(k-1)} c_{k-2,n-2,n} = -c_{n-2,n}, \\
(n) = (-)^{(k-2)} c_{n-2,n} \ldots c_{n-2,1} = (-)^{k-1} c_{n-1,n} c_{n-2,k-1}.
\]

(4.9)

Now the pattern is clear from (4.9): \((n-1) = 0\) fixes \(c_{n-2,n} = 0\), hence \((n)\) factorizes into \((-)^{k-1} c_{n-1,n} c_{n-2,k-1}\). \((n-2) = 0\) fixes \(c_{k-2,n} = 0\), hence \((n-3)\) factorizes into \((-)^{2(k-2)+1} c_{k-3,n} c_{k-2,n-3}\), and a further integration fixes \(c_{k-3,n} = 0\). Having \(c_{k-2,n} = c_{k-3,n} = 0\), \((n-4)\) again factorizes. This pattern will repeat to the ‘top’ minor \((n-k+1)\). Therefore the correct order to proceed residue integrations is crucial. Combine the result above with (4.4) and plug them back into (4.3), all un-localized \(c_{I_J}\)’s cancel, and we reach the
longing answer:
\[
\frac{n - 1}{n - 1, n} \times \text{Sign},
\]
where
\[
\text{Sign} = (-)^{1+2+\ldots+(k-4)}(-)^{k-1}(k-2)\ldots(-)^{k-3}3\ldots(-)^{k-2}1\ldots(-)^{k-3}3+(k-4)(-)^{(k-2)+1}1\ldots(-)(k-3)3+(k-4)\ldots(-)(k-2)+2(k-3) = 1,
\]
which is not a coincidence, but a consequence of cautiously chosen conventions. Since \(c_{2n} = c_{3n} = \ldots = c_{k-2,n} = c_{n-2,n} = 0\), the spinor pair \((\tilde{\lambda}_{n-1}, \tilde{\lambda}_1)\) is deformed identically as (2.11). The soft theorem for general \(N^{k-2}\)MHV amplitudes is now proved.

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A. Conjugation Construction of General \(N^{k-2}\)MHV Amplitudes

Given an amplitude \(A_n^{[k]}\), one can construct two other amplitudes: \(A_{n+m}^{[k]}\) with \(m\) extra particles of positive helicity, and \(A_{n+l}^{[k]}\) with \(l\) positive helicities flipped. It is easy to construct \(A_{n+m}^{[k]}\) by applying the inverse soft operation (‘add one particle at a time’) successively. However, for \(A_{n}^{[k]}\) it is not so straightforward and we need to use a trick, i.e., the conjugation.

Assume that \(k = 3\) and \(n = 6 + l\), the conjugation gives
\[
A_{6+l}^{[3]} = A_{6+l}^{[3]},
\]
then the general \(A_{n}^{[3+l]}\) can be obtained from \(A_{6+l}^{[3]}\), which we named as the ‘seed amplitude’, since the inverse soft operation will grow it into amplitudes for all \(n\)’s while fixing \(k’ = 3 + l\).

Based on the observation above, we now present an incomplete approach to construct amplitudes for all \(k\)’s. It is incomplete because we will leave the unnecessary part, which involves overwhelming products of non-consecutive minors, unspecified in this derivation.

Let’s start by rewriting (4.1) as
\[
A_n^{[k]} = \int \frac{g_n^{[k]} [F_{k+3} \ldots F_{n}]}{[(n-1)(1)(3)]_k [F_{k+3} \ldots F_{n}]} = \frac{[F_1]_k = [f_{i1} \ldots f_{i,k-2}]}{[F_{6+l}]} ,
\]
where \([\ldots]_k\) is a collective type label, for instance, \([abc]_k = a_k b_k c_k\), and it is introduced to distinguish ‘the same functions’ of different \(k\)’s. Setting \(k = 3\) and \(n = 6 + l\), then the seed amplitude is
\[
A_{6+l}^{[3]} = \int \frac{g_{6+l}^{[3]} [F_{6+l}]}{[(8+l)(4)(6)]^{3+l} [F_{6+l}]} = \frac{[F_{6+l}]}{[F_{6+l}]} ,
\]
after recalling that the conjugate of \((l)_k = (l \, l + 1 \ldots l + k - 1)\) is \((l + k)_{n-k} = (l + k \, l + \ldots \, l + 1)\). To return to the analogous form of \((A.2)\), we use the cyclic invariance to perform the following shift of labels: \(i \rightarrow i - 3\), now

\[
A_{6+l}^{[3+l]} = \int \frac{g_{6+l}^{[3]}}{[\hat{s} f_{6 \ldots \hat{s} f_{6+l}]}_{3+l}},
\]

where the \(\hat{s}\) operator denotes the cyclic shift. Define new variables via \(\hat{s} g_{6+l}^{[3]} = g_{6+l}^{[3+l]}\), and

\[
\hat{s} f_{6 \ldots \hat{s} f_{6+l} \equiv f_{6+l,1} \ldots f_{6+l,1+l} = F_{6+l},
\]

hence we achieve

\[
A_{6+l}^{[3+l]} = \int \frac{g_{6+l}^{[3+l]}}{[\hat{s} f_{6+l}]}_{3+l} \frac{1}{[F_{6+l}]_{3+l}},
\]

extending from \((6 + l)\) to generic \(n\) by using ‘add one particle a at time’ successively, yields

\[
A_n^{[3+l]} = \int \frac{g_{6+l}^{[3+l]}}{[\hat{s} F_{n,3+l}]}_{3+l} \frac{1}{[F_{1} \ldots \hat{F}_{n}]_{3+l}}, [F_{1} = [f_{1} \ldots f_{k+1+l}]}_{3+l},
\]

which has the same form as \((A.2)\). From the initial amplitude \(A_6^{[3]}\), we know that each \(f_{ij}\) contains three minors and this is true for all \(n\)’s and \(k\)’s, as implied by the construction above. It is also confirmed by the first non-trivial example of \(N^2\)MHV amplitudes, as mentioned in section 3.

We have left the explicit expressions of \(f_{ij}\)’s unspecified, but for the sake of proving the soft theorem, one key fact must be emphasized: The form of \(f_{nj}\)’s is given by

\[
f_{n1} = [(n-k+1)(\ldots)]_k, \ldots \, f_{n,k-3} = [(n-3)(\ldots)]_k, \, f_{n,k-2} = [(n-2)(n)]_k, \quad (A.8)
\]

Obviously the minor \((n)\) plays a special role above, in addition to the special minor \((n-1)\). Also note that all consecutive minors involving column \(n\), except \((n-1)\), must locate in \(F_{n} = f_{n1} \ldots f_{n,k-2}\). For given generic \(n\) and \(k\), the proof of this arrangement is the following.

By default, this arrangement trivially extends to the \((n+m)\) case while fixing \(k\), then let’s fix \(n = 6 + l\) and replace \(k = 3\) by \(k' = 3 + l\). Firstly \((A.8)\) is valid for all \(n\)’s in cases of \(k = 3\) and \(k = 4\), as confirmed in section 4. To extend it by induction, note that in process \((A.5)\) of constructing \(F_{6+l}\) from \(f_6 \ldots f_{6+l}\), the operation of conjugation followed by label shift exactly preserves the minor labels, while \(k = 3\) is replaced by \(k' = 3 + l\). Explicitly, we find the following form identical to \((A.8)\), which is

\[
f_{n,1} = [(n-k'+1)(\ldots)]_{k'}, \ldots \, f_{n,k'-3} = [(n-3)(\ldots)]_{k'}, \, f_{n,k'-2} = [(n-2)(n)]_{k'}, \quad (A.9)
\]

where \(n\) and \(k'\) are kept instead of \(l\). Therefore the proof is done.

The order of minors involving column \(n\) in \((A.8)\) is \((n-k+1)(n-k+2) \ldots (n-2)(n)\), which justifies the order in \((A.5)\).
B. Pure Regular Parts

In this part, we will show that after using the global residue theorem, only one term has singular contribution. For the reader’s convenience, let’s write (4.2) again here

$$\int \frac{g_n[k]}{f'} \frac{1}{F_{k+3} \ldots F_{n-1}(f_1f_2 \ldots f_{n,k-2})} = - \int g_n[k] \frac{1}{f' F_{k+3} \ldots F_{n-1}(f_1f_2 \ldots f_{n,k-2})},$$

(B.1)

with $f' = (n-1)(1)(3)$ and

$$f_{n1} = (n-k+1)(\ldots)(\ldots), \ldots f_{n,k-3} = (n-3)(\ldots)(\ldots), f_{n,k-2} = (n-2)(n)(\ldots).$$

(B.2)

It has been claimed that the only singular contribution in the soft limit is from the particular sequence of zero minors selected above, namely $(n-1)(n-k+2)(n-k+3)\ldots(n-2)$, while all other choices give regular terms.

To see why, let’s explore the origin of soft divergence in the Grassmannian formulation. From section [4], we see that the $(k-2)$ residue integrations enforce $c_{2n} = c_{3n} = \ldots = c_{k-2,n} = c_{n-2,n} = 0$, leaving only $c_{n-1,n}$ and $c_{1n}$ non-vanishing. Write (4.3) again here,

$$c_{n-1,n} = \frac{\langle 1n \rangle}{\langle 1, n-1 \rangle}, \quad c_{1n} = \frac{\langle n-1, n \rangle}{\langle n-1, 1 \rangle},$$

(B.3)

with

$$\lambda_n = \lambda_{n-1,c_{n-1,n}} + \lambda_{1,c_{1n}}.$$  

(B.4)

It is natural to conceive that if there are some extra pieces besides $\lambda_n$ on the left hand side of the equation above, $c_{n-1,n}$ and $c_{1n}$ would be ‘protected’ from vanishing in the soft limit $\lambda_n \to \varepsilon \lambda_n$, and hence the denominator involving $c_{n-1,n}$ and $c_{1n}$ would not cause divergence since it is non-zero.

Favorably this is the right hint to catch. Rewrite the $\lambda_n$ equation before localizing all $c_{I,n}$’s as

$$\lambda_n - \lambda_{I,c_{I,n}} = \lambda_{n-1,c_{n-1,n}} + \lambda_{1,c_{1n}},$$

(B.5)

where $I = n-2,n-3,\ldots,k-2$. Any selected sequence of zero minors other than $(n-k+2)(n-k+3)\ldots(n-2)(n-1)$ fails to localize all $(k-2)$ $c_{I,n}$’s to be zero, since these are the only minors involving column $n$ besides $(n-k+1)$ and $(n)$, while the latter two are localized by the delta function $\delta^2(\lambda_n - \lambda_I c_{I,n})$ as always. Consider the extreme case where all $(k-2)$ $c_{I,n}$’s are not localized by the $(k-2)$ zero minors mentioned before, then they must be localized by other constraints in the full Grassmannian integral. The relevant part in the integral involving $c_{I,n}$’s is

$$\int d^k c_{I,n} \delta^2(\lambda_n - \lambda_I c_{I,n}) \frac{\delta^{2-k}(\tilde{\lambda}_I + c_{I,n}\tilde{\lambda}_n)}{(n-k+1)(n-k+2)(n-k+3)\ldots(n-2)(n-1)(n)}$$

$$= \frac{1}{\langle n-1, 1 \rangle} \int d^{k-2} c_{I,n} \delta^{2-k}(\tilde{\lambda}_I + c_{I,n}\tilde{\lambda}_n) \delta^2(\tilde{\lambda}_{n-1} + c_{n-1,n}\tilde{\lambda}_n) \delta^2(\tilde{\lambda}_{n-1} + c_{1n}\tilde{\lambda}_n),$$

(B.6)
where
\[ c_{n-1,n} = \frac{\langle 1n \rangle - \langle 1I \rangle c_{1n}}{\langle 1, n-1 \rangle}, \quad c_{1n} = \frac{\langle n-1, n \rangle - \langle n-1, 1 \rangle c_{1n}}{\langle n-1, 1 \rangle}. \] (B.7)

Since \(2(k - 2) \geq k - 2\), all \(c_{I,n}\)'s can be localized by delta functions selected from the \((k - 2)\) constraints on spinors. And these values of \(c_{I,n}\)'s only depend on anti-holomorphic spinors, hence they are free from the holomorphic soft limit. When \(c_{I,n}\)'s are finite, \(c_{n-1,n}\) and \(c_{1n}\) also remain finite even though \(\lambda_n \to \varepsilon \lambda_n\), hence the possible divergence caused by these two variables safely dissolves in the integral (B.6).

When \((k - 2)\) \(c_{I,n}\)'s are partially localized, the same argument works analogously. This is why only one selected sequence of \((k - 2)\) zero minors can cause soft divergence, as this choice delicately removes all ‘protections’ against pushing \(c_{n-1,n}\) and \(c_{1n}\) to zero.

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