An Exact Solution of a Generalization of the Rabi Model

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There has been renewed theoretical interest recently in the Rabi model due to Braak’s analytical solution and introduction of a new criterion for integrability. We focus not on the integrability of the system but rather why it is solvable in the first place. We show that the Rabi model is the limiting case of a more general finite dimensional system by use of a contraction and suggest that this is the reason for it’s solvability, which still applies in the case of non-integrable but solvable variations.

I. INTRODUCTION

In classical mechanics the terms integrable and solvable are often used interchangeably, implying that a system that is integrable is also said to be solvable while a non-integrable system exhibits chaos. In this sense it became clear in the early twentieth century that most classical dynamical systems are not analytically solvable, for example the perturbation series for the three body problem is only convergent in certain regions of the phase space. Later work by Kolmogorov, Arnold and Moser [1] established that this region covers a large volume for small perturbations but has a complicated fractal structure. Thus, chaos and instability are still possible for small bodies in nearly Keplerian orbits and the solar system appears stable because such bodies were either kicked out or fell into the Sun or Jupiter.

An analogous understanding still does not exist in the case of quantum mechanics. The superficial observation that the problem to be solved is linear (the Schrodinger equation) misses the point that the Hilbert spaces of most systems of interest are infinite dimensional: a linear problem in infinite dimensions has many of the analytical subtleties of finite dimensional non-linear ordinary differential equations [2]. One way to approach the question of solvability of a quantum system is to ask if it has sensible solutions. Unlike Braak, who was interested in the integrability of the system, we ask another fundamental question “What makes a quantum mechanical system solvable?”

To view this problem from a different perspective we look at the Hamiltonian

$$H_L = \omega L_3 + \Delta R_3 + gL_1R_1,$$

(2)

where $L$ and $R$ are angular momentum matrices with magnitude $l$ and $r$ respectively and we have absorbed a factor of 2 into $g$. This looks eerily similar to the Rabi model and in section V we use a contraction of the algebra to show that in the limit as $l \to \infty$ with $r = \frac{1}{2}$, (2) becomes the Rabi Hamiltonian (1). The advantage of studying this new, more general Hamiltonian, is that unlike the Rabi model it has finite dimension.

To find the matrix elements of $H_L$ we use the basis states

$$|\psi_L\rangle = |l, m_l, r, m_r\rangle = |l, m_l\rangle \otimes |r, m_r\rangle$$

(3)

where $m_l = -l, -l+1, ..., l$ and $m_r = -r, -r+1, ..., r$ are the azimuthal components of angular momenta in the $L_3, R_3$ basis. This allows us to write the Hamiltonian as a block tridiagonal matrix

$$H_L = \begin{pmatrix}
A_{-l} & B_{-l-1} & 0 & \cdots & 0 \\
B_{-l} & A_{-l+1} & B_{-l+2} & \cdots & 0 \\
0 & B_{-l+2} & A_{-l+3} & \ddots & 0 \\
0 & 0 & \ddots & \ddots & B_l \\
0 & 0 & \cdots & B_{l-1} & A_l
\end{pmatrix}$$

(4)

with

$$A_k = k\omega I_r + \Delta R_3$$

$$B_k = \sqrt{l(l+1) - k(k-1)}gR_1,$$

(5)

where $I_r$ is an $r$ dimensional identity matrix.

II. CONTINUED FRACTIONS AND TRIDIAGONAL MATRICES

The general theory of analytical continued fractions was developed by Stieltjes in the late 19th century while
studying divergent power series[6]. These continued fractions are of the form
\[ f(z) = \frac{1}{z + a_1 - \frac{b_1^2}{z + a_2 - \frac{b_2^2}{z + a_3 - \ldots}}} \]  
and are intimately connected with ordinary tridiagonal matrices of the form
\[ A = \begin{pmatrix} a_0 & b_1 & 0 & 0 & \ldots \\ c_1 & a_1 & b_2 & 0 & \ldots \\ 0 & c_2 & a_2 & b_3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \]  
This connection can be seen by defining
\[ \Delta_{-1}(z) = 1, \]
\[ \Delta_0(z) = a_0 - z, \]
\[ \Delta_k(z) = (a_k - z) \Delta_{k-1}(z) - b_k c_k \Delta_{k-2}(z) \]
which gives us a continued fraction
\[ S_k(z) = \frac{\Delta_k(z)}{\Delta_{k-1}(z)} = \frac{(a_k - z) - \frac{b_k c_k}{S_{k-1}(z)}}{S_{k-1}(z)} \]  
whose roots are the eigenvalues of the matrix. If however, \( A \) is an infinite dimensional matrix, the roots of \( S_k \) represent the \( k^{th} \) approximation to the eigenvalues of \( A \). When \( a_k \) are real and \( b_k c_k \) &gt; 0 the function \( S_k \) is a Sturm sequence, meaning the zeros are real and the roots of \( S_{k-1} \) are the poles of \( S_k \). This gives us an easy way to calculate higher approximations to the eigenvalues: between any two pairs of poles of \( S_k \) is an eigenvalue.

This tells us that as long as \( S_k \) is a convergent continued fraction, even if the matrix it represents is infinite dimensional, it can still be solved to the desired level of precision. More importantly since the convergence of such continued fractions has been well established for over a hundred years this method gives a true check as to whether or not diagonalizing increasingly larger matrices will converge to the eigenvalues of an infinite dimensional matrix[6].

### III. EIGENVALUES OF BLOCK TRIDIAGONAL MATRICES

The usefulness of tridiagonal form to prove convergence and the block tridiagonal form of \( H_L \) prompts us to ask “is it also possible to develop relations similar to (8) and (9) for block tridiagonal matrices?” The answer is, in some cases, yes. Using the transfer matrix method of Molinari [7] we can find the eigenvalues of an \((n + 1) \times (n + 1)\) block tridiagonal matrix
\[ M = \begin{pmatrix} A_0 & B_1 & B_2 & \cdots \\ B_1 & A_1 & B_2 & \cdots \\ B_2 & A_2 & \cdots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \]  
where \( A_k, B_k \) are \( m \times m \) square matrices, and \( A_k \) is of the form \( A_k = -z I_m \), where \( z \) are the eigenvalues. In this case \( \det M = 0 \), allowing us to set
\[ M \Psi = \begin{pmatrix} A_0 & B_1 & B_2 & \cdots \\ C_1 & A_1 & B_2 & \cdots \\ B_2 & A_2 & \cdots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \vdots \end{pmatrix} = 0 \]  
where \( \Psi \) is a null vector with components \( \psi_k \in \mathbb{C}^m \), giving us a set of equations
\[ A_0 \psi_0 + B_1 \psi_1 = 0, \]
\[ B_{k+1} \psi_{k+1} + A_k \psi_k + C_k \psi_{k-1} = 0, \]
\[ A_n \psi_n + C_n \psi_{n+1} = 0. \]  
This can be written recursively as
\[ \begin{bmatrix} \psi_{k+1} \\ \psi_k \end{bmatrix} = \begin{bmatrix} -B_{k+1}^{-1} A_k & -B_{k+1}^{-1} C_k \\ \phantom{-}0 & \phantom{-}0 \end{bmatrix} \begin{bmatrix} \psi_k \\ \psi_{k-1} \end{bmatrix}, \]  
which defines the transfer matrix
\[ T_k = \begin{bmatrix} -B_{k+1}^{-1} A_k & -B_{k+1}^{-1} C_k \\ \phantom{-}0 & \phantom{-}0 \end{bmatrix} T_{k-1}, \]
\[ T_n = \begin{bmatrix} A_n & C_n \\ \phantom{-}0 & \phantom{-}0 \end{bmatrix} \begin{bmatrix} -B_n^{-1} A_{n-1} & -B_n^{-1} C_{n-1} \\ \phantom{-}0 & \phantom{-}0 \end{bmatrix} \ldots \]
\[ \times \begin{bmatrix} -B_1^{-1} A_0 & -B_1^{-1} C_0 \\ \phantom{-}0 & \phantom{-}0 \end{bmatrix}. \]  
If we define the top left element of \( T_k \) as
\[ T_{k,11} = -B_{k+1}^{-1} A_k T_{k-1,11} - B_{k+1}^{-1} C_k T_{k-2,11}, \]  
since \( \psi_n + 2 = 0 \) and \( \psi_{-1} = 0 \)
\[ \begin{bmatrix} \psi_{n+1} \\ \psi_1 \end{bmatrix} = T_n \begin{bmatrix} \psi_1 \\ \phantom{-}0 \end{bmatrix}, \]  
so \( \det T_{n,11} = 0 \) is the same as \( \det M = 0 \), defining the eigenvalue equation.

### IV. SPECTRUM OF \( H_L \)

For the most general form of \( H_L \), integer values of \( r \) correspond to singular matrices that have no inverse, so the above method can be modified to the form of Salkuyeh[8]. It is possible to solve \( H_L \) for any half integer spin but the most elegant case, which is also interesting
obtain two terminating continued fractions where

$$ R_i = \frac{1}{2} \sigma_i $$

$$ A_k = kI + \frac{\Delta}{2} \sigma_3 $$

$$ B_k = C_k = g \sqrt{l(l+1)-k(k-1)} \sigma_1 $$

$$ B_k^{-1} = b_k \sigma_1 $$

(17)

Due to the special property of Pauli matrices $\sigma$ which is of the form

This gives us the eigenvalue equation $\det (H_L - zI) = 0 = \det (T_{11}(z))$, where the transfer matrix can be simplified by multiplying each $2 \times 2$ matrix in $T_k$ by $b_k$ to give

$$ T_k = \left[ \begin{array}{cc} \sigma_1 (A_k - zI) & b_k I \\ -b_k & 0 \end{array} \right] T_{k-1} $$

$$ T_{k,11} = \sigma_1 A_k T_{k-1,11} - b_k^2 T_{k-2,11} $$

$$ = \left((k\omega - z) \sigma_1 - \frac{\Delta}{2} \sigma_2\right) T_{k-1,11} - B_k^2 T_{k-2,11}. $$

(18)

Due to the special property of Pauli matrices $\sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma_k$ we see that if we define

$$ S_m = T_{m,11} T_{1,11} \quad (19) $$

we have the recursive matrix

$$ S_m = (m\omega - z) \sigma_1 - i \Delta \sigma_2 - B_k^2 S_{m-1}, $$

(20)

which is of the form

$$ S_m = \begin{pmatrix} 0 & a(z) \\ b(z) & 0 \end{pmatrix}, $$

(21)

with inverse

$$ S_m^{-1} = \begin{pmatrix} 0 & \frac{1}{a(z)} \\ \frac{1}{b(z)} & 0 \end{pmatrix}. $$

(22)

This allows us to write $S_m$ in matrix form

$$ S_m = \begin{pmatrix} 0 & a_m^+ - \frac{b_m^2}{a_m^{-1} - \frac{b_m^4}{a_m^{-2} - \ldots}} \\ a_m^- - \frac{b_m^2}{a_m^{-1} - \frac{b_m^4}{a_m^{-2} - \ldots}} & 0 \end{pmatrix}, $$

(23)

where $a_m^+ = m\omega - z + \frac{\Delta}{2}$, and by setting $S_m = 0$ we obtain two terminating continued fractions

$$ S_{m,\pm}(z) = m\omega - z \mp (-1)^m \frac{\Delta}{2} - \frac{b_m^2}{S_{m-1,\pm}(z)} $$

(24)

where $m = -l, -l+1, \ldots, l$, whose zeros are the eigenvalues of the $H_L$. Because these are finite dimensional matrices, calculating the roots of $S_{l,+}$ gives the even parity spectrum while the roots of $S_{l,-}$ give the odd parity spectrum.

V. CONTRACTION OF $H_L$ TO $H_R$

We could of course perform the same procedure on the Rabi model $H_R$ to find it’s eigenvalues in a similar way but a deeper connection between the two systems can be seen by performing a singular change of basis. It was discovered by Inonu and Wigner [9] that a transformation of this type changes one Lie Algebra into another using a process called contraction. We briefly summarize Gilmore’s [10] description of a different class of contractions to show the relationship between (1) and (2).

A Lie algebra defined by the basis vectors $X_i$ is closed under commutation, so the commutators

$$ [X_i, X_j] = C_{ij}^k X_k $$

(25)

are contained in the algebra. The structure constants $C_{ij}^k$ completely determine the algebra, however it is possible to perform a change of basis transformation

$$ Y_i = M_i^j X_j $$

(26)

where the new structure constant $C_{ij}^k$ becomes

$$ C_{ij}^k = \left( M^{-1}\right)_i^l \left( M^{-1}\right)_j^m C_{lm}^n M^k_n $$

(27)

due to a non-singular transformation. If we allow the transformation to be parameter dependent, where

$$ Y_i = M_i^j (\epsilon) X_j $$

$$ C_{ij}^k = C_{ij}^k (\epsilon) $$

(28)

the structure constant often converges to a new Lie Algebra if $C_{ij}^k (\epsilon)$ becomes singular in the limit as $\epsilon \to \infty$.

One representation of angular momentum is the compact unitary group $U(2)$, which is spanned by the operators $J_3, J_{\pm}, J_0$, ($J_0$ is the identity) which correspond to the commutation relations

$$ [J_3, J_{\pm}] = \pm J_\pm $$

$$ [J_+, J_-] = 2J_3 $$

$$ [J_0, J_\pm] = 0. $$

(29)

We can now change basis to the Heisenberg group $H_4$ by using

$$ \begin{pmatrix} h_+ \\ h_- \\ h_3 \\ h_0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} c \\ c \end{pmatrix} \begin{pmatrix} J_+ \\ J_- \end{pmatrix} $$

(30)

which gives us

$$ [h_3, h_{\pm}] = \pm h_{\pm} $$

$$ [h_+, h_-] = 2c^2 h_3 - h_0 $$

$$ [h_0, h] = 0 $$

(31)

and in the limit as $c \to 0$

$$ [h_3, h_{\pm}] = \pm h_{\pm} $$

$$ [h_+, h_-] = -h_0 = -I $$

(32)
which satisfy the same commutations as the single mode photon operators
\[
[N = a^\dagger a, a] = -a \\
[N = a^\dagger a, a^\dagger] = a^\dagger \\
[a^\dagger, a] = -1.
\] (33)

We can now identify (in the limit as \( c \to 0 \))
\[
h_3 = N \\
h_+ = a^\dagger \\
h_- = a.
\] (34)

To see how the basis states change we first operate \( h_3 \) on the angular momentum state \(| j, m \rangle \) to get
\[
h_3 | j, m \rangle = \left( J_3 + \frac{1}{2c^2} J_0 \right) | j, m \rangle = \left( m + \frac{1}{2c^2} \right) | j, m \rangle.
\] (35)

The ground state of this system corresponds to \( m = -j \), so the \( n^{th} \) state is \( n = j + m \). In order for the limit to be well defined we require
\[
\lim_{c \to 0} \left( m + \frac{1}{2c^2} \right) = \lim_{c \to 0} \left( n - j + \frac{1}{2c^2} \right)
\] (36)
to also be well defined. We have already equated \( h_3 \) with the number operator, so the requirement becomes
\[
\lim_{c \to 0} \left( -j + \frac{1}{2c^2} \right) = 0
\] (37)
telling us that \( 2jc^2 = 1 \), or in other words as \( c \to 0 \), \( j \to \infty \), and
\[
\lim_{j \to \infty} h_3 | j, m \rangle = n | \infty, n \rangle.
\] (38)

Similarly, we see that
\[
a^\dagger | n \rangle = \lim_{c \to 0} cJ_+ | j, m \rangle = \lim_{c \to 0} \sqrt{j(j+1) - m(m+1)} | j, m + 1 \rangle
\]
\[
= \lim_{c \to 0} \sqrt{(1-c^2n)(n+1)} | j, m + 1 \rangle
\]
\[
= \sqrt{n+1} | n + 1 \rangle.
\] (39)

So a contraction on \( H_L \) changes (24) to
\[
S_{k,\pm}(z) = k\omega - z \mp (-1)^k \frac{\Delta}{2} - \frac{g^2k}{S_{k-1,\pm}(z)},
\] (40)
k = 0, 1, 2, 3, ... Its zeros give us the spectrum of the Rabi model \( H_R \). As discussed in section II, there is a rapidly convergent algorithm to find the zeros, which takes advantage of the fact that there is a zero of \( S_{k,\pm}(z) \) in between two of its poles, which are simply the zeroes of the previous approximation, \( S_{k-1,\pm}(z) \). Thus we can limit the search for each zero to these intervals, increasing the size \( k \) of the matrix in each step.

VI. CONCLUSION

We show that the Rabi model can be thought of as the limit of a sequence of finite dimensional block tridiagonal Hamiltonians, each of which can be solved by a continued fraction method. The solvability of the Rabi model can thus be understood as due to this finite dimensional truncation and the existence of approximate conservation laws (selection rules for transition matrix elements) that ensure block tridiagonality, explaining why even a Hamiltonian with a broken symmetry can be solved [3]. Conversely, we should expect that systems which do not allow convergent finite dimensional approximations exhibit quantum chaos. We hope to construct such an example in a later publication.

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