On a sharp volume estimate for gradient Ricci solitons with scalar curvature bounded below

Shijin Zhang*
Chern Institute of Mathematics, Nankai University,
Tianjin, 300071, P.R. China
Department of Mathematics, University of California at San Diego,
La Jolla, CA 92093, USA
shijin_zhang@yahoo.com

September 3, 2009

Abstract

In this note, we obtain a sharp volume estimate for complete gradient Ricci solitons with scalar curvature bounded below by a positive constant. Using Chen-Yokota’s argument we obtain a local lower bound estimate of the scalar curvature for the Ricci flow on complete manifolds. Consequently, one has a sharp estimate of the scalar curvature for expanding Ricci solitons; we also provide a direct (elliptic) proof of this sharp estimate. Moreover, if the scalar curvature attains its minimum value at some point, then the manifold is Einstein.

Introduction

The Ricci flow \( \frac{\partial}{\partial t} g(x, t) = -2Ric(x, t) \), was introduced by Hamilton in [6]. We say that a quadruple \((M^n, g, f, \varepsilon)\), where \((M^n, g)\) is a Riemannian manifold, \(f\) is a smooth function on \(M^n\) and \(\varepsilon \in \mathbb{R}\), is a gradient Ricci soliton if

\[
R_{ij} + \nabla_i \nabla_j f + \frac{\varepsilon}{2} g_{ij} = 0.
\] (0.1)

We call \(f\) the potential function. We say that \(g\) is shrinking, steady, or expanding if \(\varepsilon < 0\), \(\varepsilon = 0\), or \(\varepsilon > 0\), respectively.

The following volume growth estimate for complete shrinking gradient Ricci solitons was proved by O. Munteanu [8], with an important special case was proved by H.-D. Cao and D.-T. Zhou [7]. Let \((M^n, g, f, -1)\) be a complete shrinking gradient Ricci soliton. Given \(o \in M^n\), there exists a constant \(C < \infty\) such that

---

*Shijin Zhang is currently a visiting PhD student at the Department of Mathematics UCSD. He is partially supported by China Scholarship Council.
for all \( r \geq 0 \), where \( B(o, r) \) is the ball of radius \( r \) at center \( o \) and \( V(B(o, r)) \) denotes the volume of \( B(o, r) \) with respect to the metric \( g \). From the proof of Proposition 2.1 in [4], we can obtain the following property; see Lemma 1.1 below.

**Lemma 0.1** Let \((M^n, g, f, -1)\) be a complete gradient shrinking Ricci soliton with \( R \geq \delta > 0 \). Then for any \( \eta > 0 \), there exists a constant \( C_1 < \infty \) depending on \( \eta \) and the soliton such that

\[ V(B(o, r)) \leq C_1(r + 1)^{n-(2-\eta)\delta} \]

for all \( r > 0 \).

We sharpen the above result as follows, which is our main theorem.

**Theorem 0.2** Let \((M^n, g, f, -1)\) be a complete shrinking gradient Ricci soliton with \( R \geq \delta > 0 \). Then given \( o \in M^n \), there exists a constant \( C < \infty \) depending only on \( \delta \), \( o \) and the soliton such that

\[ V(B(o, r)) \leq C(r + 1)^{n-2\delta} \]

for all \( r \geq 0 \).

**Remark 0.3** Above result is sharp. For example, a product \( M^n = N_k \times \mathbb{R}^{n-k}(k = 2, 3, \ldots, n) \), where \( N_k \) is an Einstein manifold with constant scalar curvature \( \frac{k}{2} \) and take \( f = \frac{|x|^2}{4} \) on \( \mathbb{R}^{n-k} \), then the equality in Theorem 0.2 holds.

The following property for gradient Ricci solitons is the second part of Theorem 1.3 in Z.-H. Zhang [11]. In fact, part 1 is a consequence of Corollary 2.5 in B.-L. Chen [2].

**Theorem 0.4** Let \((M^n, g, f, \varepsilon)\) be a noncompact complete gradient Ricci soliton.

1. If the gradient soliton is shrinking or steady, then \( R \geq 0 \).

2. If the gradient soliton is expanding, then there exists a positive constant \( C(n) \) such that \( R \geq -C(n)\varepsilon \).

The following property is an improvement of part 2 of Theorem 0.4, which is the sharp estimate for noncompact expanding gradient Ricci solitons. The compact case follows from a direct application of the maximum principle; we know the manifold is Einstein (see Proposition 9.43 in [3]).

**Theorem 0.5** Let \((M^n, g, f, 1)\) be a complete expanding gradient Ricci soliton. Then \( R \geq -\frac{n}{2} \). Furthermore, if there exists a point \( x_0 \in M^n \) such that \( R(x_0) = -\frac{n}{2} \), then \((M^n, g)\) is Einstein, i.e., \( R_{ij} = -\frac{1}{2}g_{ij} \).

The first part of Theorem 0.5 is a consequence of Corollary 2.3 (i) in B.-L. Chen [2](see Corollary 2.2 below); we also provide a direct (elliptic) proof of this sharp estimate.
1 Volume growth of complete noncompact gradient Ricci solitons

We consider the complete shrinking gradient Ricci solitons in this section, i.e. \( \varepsilon = -1 \). Normalizing \( f \), from (0.1) we have
\[
R + |\nabla f|^2 - f \equiv 0. \tag{1.1}
\]
Define
\[
\mathcal{V}(c) \equiv \int_{\{f \leq c\}} d\mu = Vol\{f \leq c\},
\]
\[
\mathcal{R}(c) \equiv \int_{\{f \leq c\}} Rd\mu.
\]
By the co-area formula
\[
\mathcal{V}'(c) = \int_{\{f = c\}} \frac{1}{|\nabla f|} d\sigma,
\]
\[
\mathcal{R}'(c) = \int_{\{f = c\}} \frac{R}{|\nabla f|} d\sigma.
\]
Since \( R \geq 0 \) (see Corollary 2.3 below), we have
\[
\mathcal{R}(c) \geq 0 \text{ and } \mathcal{R}'(c) \geq 0.
\]
Integrating \( R + \Delta f = \frac{n}{2} \) over \( \{f \leq c\} \) yields
\[
\frac{n}{2} \mathcal{V}(c) - \mathcal{R}(c) = \int_{\{f \leq c\}} \Delta f d\mu
= \int_{\{f = c\}} \frac{\partial f}{\partial \nu} d\sigma
= \int_{\{f = c\}} |\nabla f| d\sigma. \tag{1.2}
\]
In particular,
\[
\frac{n}{2} \mathcal{V}(c) \geq \mathcal{R}(c). \tag{1.3}
\]
Since by (1.1),
\[
|\nabla f| = \frac{(\nabla f)^2}{|\nabla f|} = \frac{R}{|\nabla f|}.
\]
So we have
\[
\frac{n}{2} \mathcal{V}(c) - \mathcal{R}(c) = c\mathcal{V}'(c) - \mathcal{R}'(c). \tag{1.4}
\]
That is
\[
\frac{n}{2} \mathcal{V}(c) - c\mathcal{V}'(c) = \mathcal{R}(c) - \mathcal{R}'(c). \tag{1.5}
\]
Lemma 1.1 Let \((M^n, g, f, -1)\) be a complete gradient shrinking Ricci soliton with \(R \geq \delta > 0\), then for any \(\eta > 0\), there exist \(c_0, C_1\) depending on \(\eta\) and \(\delta\), when \(c \geq c_0\), we have

\[
V(c) \leq C_1 c^{\frac{n-(2-\eta)\delta}{2}}.
\]

Proof. For the sake of completeness, we provide the detailed proof. The proof is similar to the proof of Theorem 1 in [8] (or the proof of Proposition 2.1 in [4]). If \(\eta \geq 2\), this has done by Theorem 1 in [8]. So we only consider \(\eta < 2\). Now using the positive lower bound for \(R\) and \(\eta < 2\), we have

\[
\frac{n-(2-\eta)\delta}{2} V(c) - \frac{\eta}{2} R(c) \geq \frac{n}{2} V(c) - R(c)
\]

This implies

\[
\frac{d}{dc}(c^{-\frac{n-(2-\eta)\delta}{2}} V(c)) = c^{-\frac{n+2-(2-\eta)\delta}{2}}(c V'(c) - \frac{n-(2-\eta)\delta}{2} V(c))
\leq c^{-\frac{n+2-(2-\eta)\delta}{2}}(R'(c) - \frac{\eta}{2} R(c)).
\]

Integrating this by parts on \([c_0, \bar{c}]\) yields

\[
\bar{c}^{-\frac{n-(2-\eta)\delta}{2}} V(\bar{c}) - c_0^{-\frac{n-(2-\eta)\delta}{2}} V(c_0) \leq \bar{c}^{-\frac{n+2-(2-\eta)\delta}{2}} R(\bar{c}) - c_0^{-\frac{n+2-(2-\eta)\delta}{2}} R(c_0)
\leq c_0^{-\frac{n+2-(2-\eta)\delta}{2}} \int_{c_0}^{\bar{c}} (R'(c) - \frac{\eta}{2} R(c)) dc.
\]

Since \(R(c) \geq 0\), for \(c_0 \geq \frac{n+2-(2-\eta)\delta}{\eta}\) we have

\[
\bar{c}^{-\frac{n-(2-\eta)\delta}{2}} V(\bar{c}) - c_0^{-\frac{n-(2-\eta)\delta}{2}} V(c_0) \leq \bar{c}^{-\frac{n+2-(2-\eta)\delta}{2}} R(\bar{c})
\leq \frac{\eta}{2} c_0^{-\frac{n+2-(2-\eta)\delta}{2}} V(c_0)
\]

the last inequality has used (1.3). Thus if \(\bar{c} \geq \max\{n, c_0\}, c_0 \geq \frac{n+2-(2-\eta)\delta}{\eta}\), then

\[
V(\bar{c}) \leq 2c_0^{-\frac{n-(2-\eta)\delta}{2}} V(c_0) c^{\frac{n-(2-\eta)\delta}{2}}.
\]

So Lemma holds. ■

Theorem 1.2 Let \((M^n, g, f, -1)\) be a complete gradient shrinking Ricci soliton with \(R \geq \delta > 0\), then there exists a positive constant \(C\) depending only on \(\delta\), \(o\) and the soliton such that

\[
V(B(o, r)) \leq C(1 + r)^{n-2\delta}.
\]
Proof. Since \( R \geq \delta \), \( R(c) \geq 0 \), we have
\[
\frac{n - 2\delta}{2} V(c) \geq \frac{n}{2} V(c) - R(c) = cV'(c) - R'(c).
\]

Since \( R'(c) \geq 0 \), we have
\[
\frac{d}{dc} \left( c - \frac{n - 2\delta}{2} V(c) \right) = c - \frac{n + 2 - 2\delta}{2} \left( cV'(c) - \frac{n - 2\delta}{2} V(c) \right) \leq c - \frac{n + 2 - 2\delta}{2} R'(c).
\]

Integrating this by parts on \([c_0, \bar{c}]\) yields
\[
\bar{c} - \frac{n - 2\delta}{2} V(\bar{c}) - c_0 - \frac{n - 2\delta}{2} V(c_0) \leq \int_{c_0}^{\bar{c}} c - \frac{n + 2 - 2\delta}{2} R'(c) dc
\]

By (1.3), we have
\[
\int_{c_0}^{\bar{c}} c - \frac{n + 4 - 2\delta}{2} R(c) dc \leq \frac{n}{2} \int_{c_0}^{\bar{c}} c - \frac{n + 4 - 2\delta}{2} V(c) dc.
\]

Let \( \eta = \frac{1}{\delta} \) in Lemma 1.1, so when \( c \) is large enough, we have
\[
V(c) \leq C_1 c^{n + 1 - 2\delta}.
\]

So
\[
\int_{c_0}^{\bar{c}} c - \frac{n + 4 - 2\delta}{2} R(c) dc \leq C_1 \frac{n}{2} \int_{c_0}^{\bar{c}} c - \frac{n + 4 - 2\delta}{2} c^{n + 1 - 2\delta} dc
\]
\[
= \frac{nC_1}{2} \int_{c_0}^{\bar{c}} c^{n - \frac{1}{2}} dc
\]
\[
= nC_1 (c_0^{n - 2\delta} - \bar{c}^{n - 2\delta})
\]
\[
\leq nC_1 c_0^{n - 2\delta}.
\]

Since (1.3) and \( V(c) \geq 0 \), \( \delta \leq \frac{n}{2} \). So
\[
\bar{c} - \frac{n - 2\delta}{2} V(\bar{c}) - c_0 - \frac{n - 2\delta}{2} V(c_0) \leq \frac{n}{2} \bar{c} - \frac{n + 2 - 2\delta}{2} V(\bar{c}) + \frac{n + 2 - 2\delta}{2} nC_1 c_0^{n - 2\delta}.
\]

Then same argument in the proof of Lemma 1.1, when \( \bar{c} \) is large enough, there exists a constant \( C_2 \) depending only on \( \delta \) such that
\[
V(\bar{c}) \leq C_2 \bar{c}^{n - 2\delta}.
\]

By Theorem 1.1 of [5] (or see [11]), there exists a constant \( C \) depending only on \( g \) and \( o \) such that
\[
\frac{1}{4} (r(x) - C)^2 \leq f(x) \leq \frac{1}{4} (r(x) + C)^2
\]
where \( r(x) \) denotes the distance from \( x \) to \( o \). Hence we obtain the result. ■
2 Lower bound of scalar curvature for Ricci flow

In this section, we observe that by a modification of B.-L. Chen’s theorem (Corollary 2.3(i) in [2]), we obtain a local lower bound estimate of the scalar curvature for the Ricci flow on complete manifolds, we follow Yokota’s argument in Proposition A.3 in [10].

**Theorem 2.1** For any $0 < \varepsilon < \frac{2}{n}$. Suppose $(M^n, g(t))$, $t \in [\alpha, \beta]$ is a complete solution to Ricci flow, $p \in M$, then there exist constants $C(p)$ depending on $p$ and the metrics $g(t)(t \in [\alpha, \beta])$ and $C$ such that when $c \geq C(p)$, we have

$$R(x, t) \geq -B\frac{e^{2AB(t-\alpha)} + 1}{e^{2AB(t-\alpha)} - 1}$$

whenever $x \in B_{g(t)}(p, c), t \in (\alpha, \beta]$, where $A(\varepsilon) = \frac{2}{n} - \varepsilon, B(\varepsilon) = \frac{3C}{2A\varepsilon c^2}$.

**Proof.** First we use the cutoff function in the proof of Proposition A.3 in [10]. Let $\eta : \mathbb{R} \rightarrow [0, 1]$ be a nonincreasing $C^2$ function such that $|\eta'|$ and $|\eta''|$ are bounded and $\eta(u) = 1$ for any $u \in (-\infty, 1], \eta(u) = 0$ for any $u \in [2, \infty)$ and $\eta(u) = (2 - u)^4$ for any $u \in \left[\frac{3}{2}, 2\right]$. Then there exists a positive constant $C$ such that

$$\frac{(\eta'(u))^2}{\eta(u)} \leq C\eta(u)\frac{1}{2},$$

$$|\eta''(u)| \leq C\eta(u)\frac{1}{2}.$$  \hfill (2.2)

Clearly we can choose a number $r_0 \in (0, 1)$, such that

$$Rc(g(t)) \leq (n - 1)r_0^{-2}$$

in $B_{g(t)}(p, r_0)$ for $t \in [\alpha, \beta]$. Let $C(p) = r_0 + \frac{5}{3}(n - 1)r_0^{-1}(\beta - \alpha)$ and given any $c \geq C(p)$.

Given any time $t_0 \in (\alpha, \beta]$ and suppose that

$$R(p, t_0) < 0.$$ \hfill (2.4)

Define $Q : M \times [\alpha, t_0] \rightarrow \mathbb{R}$ by

$$Q(x, t) = \eta(\frac{\tilde{r}(x, t)}{c})R(x, t),$$ \hfill (2.5)

where

$$\tilde{r}(x, t) \doteq d_{g(t)}(x, p) + \frac{5}{3}(n - 1)r_0^{-1}(t_0 - t).$$ \hfill (2.6)

Then $Q(x, t)$ is a compactly support function.

By Lemma 8.3 (a) in [9], we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)\tilde{r}(x, t) \geq 0,$$ \hfill (2.7)
whenever $d_{g(t)}(x, p) > r_0$, $t \in [\alpha, t_0]$, in the barrier sense. We have

$$\left( \frac{\partial}{\partial t} - \Delta \right) Q = \eta \left( \frac{\partial}{\partial t} - \Delta \right) R + \frac{2}{c} R \left( \frac{\partial}{\partial t} - \Delta \right) \tilde{r} - \frac{2}{c} \eta' \nabla \tilde{r}, \nabla R > - \frac{\eta''}{c} R$$

where $\eta$ denotes $\eta(\tilde{r})$. In the case of $d_{g(t)}(x, p) \leq r_0$, then $\tilde{r}(x, t) \leq c$, so at point $(x, t)$, $\eta = 1$, $\eta' = \eta'' = 0$, so

$$\left( \frac{\partial}{\partial t} - \Delta \right) Q = \left( \frac{\partial}{\partial t} - \Delta \right) R.$$  \hspace{1cm} (2.8)

In the case of $d_{g(t)}(x, p) > r_0$, then we applying (2.7) and $\eta' \leq 0$, we have at a point where $R \leq 0$

$$\left( \frac{\partial}{\partial t} - \Delta \right) Q \geq \eta \frac{2}{n} R^2 - \frac{2 \eta'}{c} \nabla \tilde{r}, \nabla Q > + \frac{1}{c^2} (2 \frac{(\eta')^2}{\eta} - \eta'') R$$  \hspace{1cm} (2.9)

whenever $\eta \neq 0$. Hence by both of cases, we have at point $(x, t)$ where $R \leq 0$, (2.9) holds whenever $\eta \neq 0$. Applying (2.2) and (2.3) to (2.9), we have at any point where $Q < 0$

$$\left( \frac{\partial}{\partial t} - \Delta \right) Q \geq \eta \frac{2}{n} R^2 - \frac{2 \eta'}{c} \nabla \tilde{r}, \nabla Q > + \frac{3C}{c^2} \eta^2 R.$$  \hspace{1cm} (2.10)

Let $Q_m(t) = \min_{x \in M} Q(x, t)$. Then

$$Q_m(t_0) \leq R(p, t_0) < 0.$$  

By (2.10) we have for any $t \in [\alpha, t_0]$ where $Q_m(t) < 0$ and for any $x_t$ such that $Q(x_t, t) = Q_m(t)$, then for any $\varepsilon \in (0, \frac{2}{n})$

$$\frac{d}{dt} Q_m(t) \geq \frac{2}{n} \eta R(x_t, t)^2 + \frac{3C}{c^2} \eta^2 R(x_t, t)$$

$$\geq \left( \frac{2}{n} - \varepsilon \right) \eta R(x_t, t)^2 - \frac{9C^2}{4c^2}$$

$$\geq \left( \frac{2}{n} - \varepsilon \right) Q_m^2 - \frac{9C^2}{4c^2}$$

using $ab \geq -\varepsilon a^2 - \frac{1}{\varepsilon} b^2$ and $0 < \eta \leq 1$.

Recall that the solution of ODE

$$\begin{cases}
\frac{dq}{dt} & = A(q^2 - B^2), \\
q(t_0) & = q_0
\end{cases}$$

on $[\alpha, t_0]$, then

$$q(t) = \begin{cases}
-B\frac{e^{-2A(t_0-t)+1}}{D}\frac{1}{q_0} & \text{if } B \neq -q_0, \\
q_0 & \text{if } B = -q_0
\end{cases}$$

where $D = \frac{q_0-B}{q_0+B}$ provided $B \neq -q_0$.

Taking $A = \frac{2}{n} - \varepsilon$, $B = \frac{3C}{2\sqrt{A\varepsilon}}$, and $q_0 = Q_m(t_0) < 0$, then we have
\[ Q_m(t) \leq q(t). \]

\( q(t) > -\infty \) for \( t \in [\alpha, t_0] \), since \( Q_m(t) > -\infty \) for \( t \in [\alpha, t_0] \).

**Case (1).** If \( q_0 \geq -B \), then we have
\[
R(x, t_0) \geq Q_m(t_0) \geq -B
\]
whenever \( x \in B_{g(t_0)}(p, c) \), since \( \eta = 1 \).

**Case (2).** If \( q_0 < -B \), then \( D > 1 \) and since \( q(t) > -\infty \) for \( t \in [\alpha, t_0] \), we have
\[
De^{-2AB(t_0-t)} - 1 > 0
\]
for all \( t \in [\alpha, t_0] \), so
\[
De^{-2AB(t_0-\alpha)} - 1 > 0 \text{ i.e., } q_0 > -\frac{B^2 e^{2AB(t_0-\alpha)} + 1}{e^{2AB(t_0-\alpha)} - 1}
\]
so that
\[
R(x, t_0) \geq q_0 > -\frac{B^2 e^{2AB(t_0-\alpha)} + 1}{e^{2AB(t_0-\alpha)} - 1}
\]
whenever \( x \in B_{g(t_0)}(p, c) \).

Since \( B > 0 \), (2.12) is a better estimate, we conclude that (2.12) holds in either case.

Since \( c \) independent of \( t \), we complete the proof of this theorem. ■

**Corollary 2.2** Suppose \( (M^n, g(t)), \ t \in [\alpha, \beta] \), is a complete solution to Ricci flow, then
\[
R \geq -\frac{n}{2(t-\alpha)}
\]
on \( M \times (\alpha, \beta] \).

**Proof.** For any \( t_0 \in (\alpha, \beta] \). Now fix \( \varepsilon \in (0, \frac{2}{n}) \) and let \( c \to \infty \). Then \( B \to 0 \).

Since
\[
\lim_{B \to 0} B\frac{e^{2AB(t_0-\alpha)} + 1}{e^{2AB(t_0-\alpha)} - 1} = \frac{1}{A(t_0 - \alpha)}
\]
and (2.12) independent of \( c \), we obtain
\[
R(x, t_0) \geq -\frac{1}{(\varepsilon + 1)/(t_0 - \alpha)}
\]
on \( M \times \{t_0\} \). Finally, taking \( \varepsilon \to 0 \), we obtain
\[
R(x, t_0) \geq -\frac{n}{2(t_0 - \alpha)}
\]
on \( M \times \{t_0\} \). Since above argument holds for any \( t \in (\alpha, \beta] \), we obtain the corollary. ■

The following property is Corollary 2.5 in [2]( or Proposition A.3 in [10]).
Corollary 2.3 If \((M^n, g(t))\), \(t \in (-\infty, 0]\), is a complete ancient solution to the Ricci flow, then
\[
R \geq 0
\]
on \(M \times (-\infty, 0]\).

Corollary 2.4 Suppose \((M^n, g, f, \varepsilon)\) be a noncompact complete gradient Ricci soliton. Then

1. If the gradient soliton is shrinking or steady, then \(R \geq 0\).

2. If the gradient soliton is expanding, then \(R \geq -\frac{n\varepsilon^2}{2}\). Moreover, if the scalar curvature attain the minimum value \(-\frac{n\varepsilon^2}{2}\) at some point, then \((M^n, g(t))\) is Einstein.

Proof. Part (1) is a consequence of Corollary 2.3.

As shown in Theorem 4.1 of [3], associated to the metric and the potential function \(f\), there exists a family of metrics \(g(t) = -2Ric(g(t))\), with the property that \(g(0) = g\), and a family of diffeomorphisms \(\phi(t)\), which is generated by the vector field \(X = \frac{1}{\tau} \nabla f\), such that \(\phi(0) = id\), and \(g(t) = \tau(t)\phi^*(tg)\) with \(\tau(t) = 1 + \varepsilon t > 0\), as well as \(f(x, t) = \phi^*(t)f(x)\).

For expanding gradient Ricci soliton, i.e. \(\varepsilon > 0\). We know \(t \in (-\frac{1}{\varepsilon}, \infty)\), so by Corollary 2.2 we obtain \(R(x, t) \geq -\frac{n\varepsilon^2}{2(\tau + \frac{1}{\varepsilon})}\), i.e., \(R(x, t) \geq -\frac{n\varepsilon^2}{2}\).

Let \(\tilde{R} = R + \frac{n\varepsilon^2}{2\tau}\), so \(\tilde{R} \geq 0\) and
\[
\frac{\partial}{\partial t} \tilde{R} = \Delta R + 2|\nabla R|^2 - \frac{n\varepsilon^2}{2\tau^2}
\]
\[
= \Delta \tilde{R} + 2|\nabla R|^2 + \frac{2R^2}{n} - \frac{n\varepsilon^2}{2\tau^2}
\]
\[
= \Delta \tilde{R} + 2|\nabla R|^2 + \frac{2R}{n} \tilde{R}(\tilde{R} - \frac{n\varepsilon}{\tau})
\]
\[
\geq \Delta \tilde{R} - \frac{2\varepsilon}{\tau} \tilde{R}.
\]

So
\[
\frac{\partial}{\partial t} (\tau^2 \tilde{R}) \geq \Delta (\tau^2 \tilde{R}).
\]

By strong maximum principle (Theorem 6.54 in [3]), we know that if there exists a point \(x_0\) such that \(\tilde{R}(x_0, t_0) = 0\) for some \(t_0 > -\frac{1}{\varepsilon}\), then \(\tilde{R}(x, t_0) \equiv 0\) for all \(x \in M\). So
\[
R(x, t_0) \equiv -\frac{n\varepsilon}{2\tau(t_0)}.
\]

So when \(t_0 = 0\), i.e. \(\tau(0) = 1\), then \(R(x, 0) \equiv -\frac{n\varepsilon}{2}\). From Lemma 3.1 (1) below (or see section 4.1 in [3]), we have
\[
\Delta R + 2|\nabla R|^2 - \langle \nabla f, \nabla R \rangle + \varepsilon R = 0.
\]

So by (2.16) we know
\[ |Rc + \frac{\varepsilon}{2} g|^2 = 0. \]

So
\[ R_{ij} = -\frac{\varepsilon}{2} g_{ij}. \]

Hence \((M, g, f, \varepsilon)\) is Einstein. \(\blacksquare\)

In next section, we will provide a direct (elliptic) proof of the first part of Theorem 0.5.

### 3 Direct proof for expanding Ricci solitons

In this section, we provide a direct (elliptic) proof of lower bound for scalar curvature for complete expanding gradient Ricci solitons. We use a cutoff function argument to equation (0.1).

**Lemma 3.1** Let \((M^n, g, f, \varepsilon)\) be a complete gradient Ricci soliton. Fix \(o \in M^n\), and define \(r(x) = d(x, o)\), then the following hold

1. \(\Delta R + 2|Rc|^2 - \langle \nabla f, \nabla R \rangle + \varepsilon R = 0.\)
2. Suppose \(\text{Ric} \leq (n-1)K\) on \(B(o, r_0)\), for some positive numbers \(r_0\) and \(K\). Then for any point \(x\), outside \(B(o, r_0)\)
   \[ (\Delta r - \langle \nabla f, \nabla r \rangle)(x) \leq -\langle \nabla f, \nabla r \rangle(o) + \frac{n}{2}r(x) + (n-1)\{\frac{2}{3}Kr_0 + r_0^{-1}\}. \]

Part (1) is well known. Part (2) follows from an idea of Perelman; see Lemma 8.3 in [9] and its antecedent in §17 on 'Bounds on changing distances', in [7]. For the detailed proof of part 2, also see [1].

Now we prove the first part of Theorem 0.5.

**Proof.** For expanding gradient Ricci solitons, let \(\varepsilon = 1\), so that (1) in Lemma 1.1 is
\[ \Delta R + 2|Rc|^2 - \langle \nabla f, \nabla R \rangle + R = 0. \tag{3.1} \]

If \(M^n\) were closed, by Proposition 9.43 in [3], we know \((M^n, g, f, 1)\) is Einstein. So we only consider the noncompact case. Fix \(o \in M^n\) and fix a large number \(b\). Let
\[ \eta : [0, \infty) \to [0, 1] \]
be a \(C^\infty\) nonincreasing cutoff function with \(\eta(u) = 1\) for \(u \in [0, 1]\) and \(\eta(u) = 0\) for \(u \in [1 + b, \infty)\). Define \(\Phi : M \to \mathbb{R}\) by
\[ \Phi(x) = \eta(\frac{r(x)}{c})R(x) \]
for \(c \in (0, \infty)\). Later we shall take \(c \to \infty\).

We have
\[ \Delta \Phi = \eta \Delta R + \frac{2\eta'}{c} \langle \nabla r, \nabla R \rangle + (\frac{\eta'}{c} \Delta r + \frac{\eta''}{c^2})R. \]
We have dropped $' \circ \tilde{r}'$ in our notation. By (3.1), we have

$$
\Delta \Phi = \eta (-2|Rc|^2 + <\nabla f, \nabla R > - R) + \frac{2}{c} \eta' < \nabla r, \nabla R > + \left( \frac{\eta'}{c} \Delta r + \frac{\eta''}{c^2} \right) R
$$

$$
= <\nabla f, \nabla \Phi > + \frac{2}{c} \eta' < \nabla r, \nabla \Phi > - \left( \frac{\eta'}{c^2} \right)^2 + \frac{\eta'}{c} < \nabla f, \nabla > R
$$

$$
+ \eta (-2|Rc|^2 - R) + \left( \frac{\eta'}{c} \Delta r + \frac{\eta''}{c^2} \right) R
$$

(3.2)

at all points where $\eta \neq 0$.

Suppose $x_0 \in M$ is such that

$$
\Phi(x_0) = \min_M \Phi < 0.
$$

(3.3)

Since $R(x_0) < 0$, at $x_0$ we have

$$
0 \geq \eta (-\frac{2}{n}R - 1) + \frac{\eta'}{c} (\Delta r - <\nabla f, \nabla r>) + \frac{1}{c^2} (\eta'' - 2 \frac{(\eta')^2}{\eta}).
$$

(3.4)

We consider two cases, depending on the location of $x_0$.

**Case (i).** Suppose $r(x_0) < c$, so that $\eta(\frac{r(x_0)}{c}) = 1$ in a neighborhood of $x_0$. Then (3.4) and (3.3) imply

$$
0 \geq -\frac{2}{n} \eta' - 1
$$

$$
= -\frac{2}{n} \Phi(x_0) - 1
$$

$$
\geq -\frac{2}{n} \eta(\frac{r(x_0)}{c}) R(x) - 1
$$

for all $x \in M$. This implies the desired estimate

$$
R(x) \geq -\frac{n}{2}
$$

(3.5)

for all $x \in B(o, c)$ since $\eta(\frac{r}{c}) = 1$ in $B(o, c)$.

**Case (ii).** Now suppose $r(x_0) \geq c$ and again consider (3.4). Note that we may choose $\eta$ so that

$$
\eta'' - 2 \frac{(\eta')^2}{\eta} \geq -C_2
$$

(3.6)

for some universal constant $C_2 < \infty$. Since $\eta' \leq 0$, applying Lemma 3.1 (2) and (3.6) to (3.4) yields for all $x \in M$

$$
\frac{2}{n} \Phi(x) \geq \frac{2}{n} \Phi(x_0)
$$

$$
\geq \eta(\frac{r(x_0)}{c}) \left( \frac{n-1}{r_0} - <\nabla f, \nabla r > (o) + \frac{1}{2} r(x_0) + \frac{2}{3} r_0 \max_{B(o, r_0)} Rc \right)
$$

$$
- \eta(\frac{r(x_0)}{c}) \frac{C_2}{c^2}
$$

(3.7)
where $C_2$ independent of $c$. Taking $r_0 = 1$ and $c \geq 2$, we have for all $x \in B(o, c)$

$$\frac{2}{n}R(x) \geq \frac{\eta'(r(x_0))}{c}(n - 1 + |\nabla f|(a) + \frac{1}{2}r(x_0) + \frac{2}{3}\max_{B(o, 1)} Rc)$$

$$- \eta \left( \frac{r(x_0)}{c} \right) - \frac{C_2}{c^2}$$

Since $-C_2 \leq \eta' \leq 0$ imply that for all $x \in B(o, c)$

$$\frac{2}{n}R(x) \geq - \frac{C_2}{c}(n - 1 + |\nabla f|(a) + \frac{2}{3}\max_{B(o, 1)} Rc + \frac{1}{c})$$

$$+ \frac{1}{2}\eta'(\frac{r(x_0)}{c}) \frac{r(x_0)}{c} - \eta \left( \frac{r(x_0)}{c} \right).$$

When take $c \to \infty$, then the first term of right hand side of (3.9) tends to 0. So we only consider to estimate the term $\frac{1}{2}\eta'(\frac{r(x_0)}{c}) \frac{r(x_0)}{c} - \eta \left( \frac{r(x_0)}{c} \right)$. Since $x_0 \in B(o, (1 + b)c) - B(o, c)$, we have $1 \leq \frac{r(x_0)}{c} < 1 + b$. Define $h_\eta(u)$ by

$$h_\eta(u) = \frac{1}{2}\eta'(u)u - \eta(u).$$

So we only estimate $h_\eta(u)$ for $u \in [1, 1 + b]$.

If we replace $\eta$ with nonnegative piecewise linear function $\theta(u)$ such that

$$\theta(u) = \begin{cases} 
1 & \text{if } u \in [0, 1], \\
\frac{1+b-u}{b} & \text{if } u \in [1, 1+b], \\
0 & \text{if } u \in [1+b, \infty)
\end{cases}$$

then $h_\eta(u) = -\frac{2b+2-u}{b}$ for $u \in [1, 1+b]$. So $h_\eta(u) \geq -1$ for $u \in [2, b]$ and $h_\eta(u) \geq -1 - \frac{1+b}{b}$ for $u \in [1, 2]$. For any small positive number $\delta$, we can obtain a $C^\infty$ cutoff function $\beta$ after smooth the linear function $\theta$ such that $\beta(u) = \theta(u)$ for $u \in [0, 1] \cup [2, b] \cup [1+b, \infty)$ and $-\frac{1+b}{b} \leq \beta'(u) \leq 0$ for $u \in [1, 2] \cup [b, 1+b]$. So when $b$ is large and $\delta \leq \frac{b-1}{b}$, we have $h_\beta(u) \geq -1 - \frac{1+b}{b}$ for $u \in [1, 1+b]$. Let $\eta$ equal $\beta$, take $c \to \infty, \delta \to 0, b \to \infty$, by (3.9) we obtain

$$\frac{2}{n}R(x) \geq -1$$

for all $x \in M$. So

$$R(x) \geq -\frac{n}{2}$$

(3.10)

for all $x \in M$. ■

**Acknowledgements**

The author would like to thank Professor Fuquan Fang and Professor Lei Ni for their encouragement and constant help. He would also like to thank Professor Ben Chow for encouragement and many helpful discussions and suggestions.
References

[1] B. Chow, *Expository notes on gradient Ricci solitons*, unpublished.

[2] B.-L. Chen, *Strong uniqueness of the Ricci flow*, J. Differential Geom., 82(2009), 363-382.

[3] B. Chow, P. Lu and L. Ni, *Hamilton’s Ricci flow*, Amer. Math. Soc., Providence, RI, 2006.

[4] J.A. Carrillo and L. Ni, *Sharp logarithmic sobolev inequalities on gradient solitons and applications*, arxiv:math.DG/0806.2417.

[5] H.D. Cao and D. Zhou, *On complete gradient shrinking solitons*, arXiv:math.DG/0903.3932.

[6] R.S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. 17 (1982), no.2, 255-306.

[7] R.S. Hamilton, *The formulation of singularities in the Ricci flow*, Surveys in Differential Geometry, Vol. II (Cambridge, MA, 1993), 7-136, Internat. Press, Cambridge, MA, 1995.

[8] O. Munteanu, *The volume growth of complete gradient Shrinking Ricci solitons*, arxiv:math.DG/0904.0798.

[9] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arxiv:math.DG/0211159.

[10] T. Yokota, *Perelman’s reduced volume and gap theorem for the Ricci flow*, Communications in Analysis and Geometry 17(2009), no 2, 227-263.

[11] Z.-H. Zhang, *On the completeness of gradient Ricci solitons*, arxiv:math.DG/0807.1581.