Universality classes in two-component driven diffusive systems

V. Popkov(1,2), J. Schmidt(1)

(1) Institut für Theoretische Physik, Universität zu Köln,
Zülpicher Str. 77, 50937 Cologne, Germany, and

(2) CSDC Università di Firenze, via G.Sansone 1, 50019 Sesto Fiorentino, Italy

G.M. Schütz(3,4)

(3) Institute of Complex Systems II, Theoretical Soft Matter and Biophysics,
Forschungszentrum Jülich, 52425 Jülich, Germany and

(4) Interdisziplinäres Zentrum für Komplexe Systeme,
Universität Bonn, Brühler Str. 7, 53119 Bonn, Germany

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We study time-dependent density fluctuations in the stationary state of driven diffusive systems with two conserved densities $\rho_i$. Using Monte-Carlo simulations of two coupled single-lane asymmetric simple exclusion processes, we present numerical evidence for universality classes with dynamical exponents $z = (1+\sqrt{5})/2$ and $z = 3/2$ (but different from the Kardar-Parisi-Zhang (KPZ) universality class), which have not been reported yet for driven diffusive systems. The numerical asymmetry of the scaling functions converge slowly for some of the non-KPZ superdiffusive modes with maximally asymmetric $z$-stable Lévy functions predicted by mode coupling theory.

We compute for general strictly hyperbolic two-component systems the exact mode coupling matrices with the current-density relation and the compressibility matrix as input parameters. These stationary input data determine completely all permissible universality classes. It turns out that the minimal current density relation which produces all theoretically permissible universality classes does not require nonlinearities in the currents of order higher than $\rho_i^2 \rho_j$.

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I. INTRODUCTION

Anomalous transport is the hallmark of many one-dimensional non-equilibrium systems even when interactions are short-ranged [1]. A common way of characterizing 1-d systems that exhibit anomalous transport is through the dynamical structure function which describes the time-dependent fluctuations of the long-lived modes in the stationary state. In systems with short-range interactions and one global conservation law (giving rise to one long-lived mode) only two universality classes are known to exist, the Gaussian universality class with dynamical exponent $z = 2$ (also describing diffusive fluctuations in equilibrium stationary states), and the superdiffusive Kardar-Parisi-Zhang (KPZ) universality class with dynamical exponent $z = 3/2$ [2] for systems driven out of equilibrium. The scaling form of the KPZ structure function was found some 10 years ago by Prähofer and Spohn for the polynuclear growth model [3] and for a driven diffusive system, viz. the asymmetric simple exclusion process [4]. Since then the scaling function, which is expected to be universal, has also been observed in various experiments [5, 6].

Superdiffusive fluctuations in systems with more than one conservation law are less well-studied. Stochastic dynamics have been considered for driven diffusive systems with two conservation laws. Naively one might expect both modes to be in the KPZ universality class. This guess is indeed confirmed for the Arndt-Heinzel-Rittenberg model [7] by using exact results for the steady state combined by fluctuating hydrodynamics and mode coupling theory [8] and also for a general class of multi-component exclusion processes [9]. It was also known for some time that one mode can be KPZ, while the other is diffusive, see [10] where exact microscopic and hydrodynamic limit arguments are used, and numerical work [11, 12] for related results.

Recently van Beijeren [13] studied a system with Hamiltonian dynamics with three conservation laws. He predicted KPZ-universality for the two sound modes of the system and a novel superdiffusive universality class with dynamical exponent $z = 5/3$ for the heat mode. The occurrence of a $5/3$ mode was subsequently demonstrated for FPU-chains [14] with three conservation laws and generally for anharmonic chains [15] and a family of exclusion process with two conservation laws [16]. Also recent mathematically rigorous work indicates non-trivial anomalous behaviour fluctuations in systems with two conservation laws [17].

Stochastic interacting particle systems with two conservation laws exhibit extremely rich
behaviour in one dimension, including spontaneous symmetry breaking \[7, 18–22\] or phase separation \[7, 19, 23–26\] in nonequilibrium stationary states, see \[27\] for a review. Studying the coarse-grained time evolution of two component systems with an umbilic point one finds shocks with unusual properties \[28, 29\]. It is the purpose of this paper to go beyond stationary and time-dependent mean properties and consider time-dependent fluctuations. Specifically, we show that the complete list of dynamical universality classes that, according to mode coupling theory, can appear in the presence of two conservation laws can be realized in driven diffusive systems with two conserved densities. To this end we compute the exact mode coupling matrices for general strictly hyperbolic two-component systems with the stationary current-density relation and stationary compressibility matrix as the only input. With these input data the scaling form of the dynamical structure function is completely determined, except in the presence of a diffusive mode where the phenomenological diffusion coefficient enters the scale factors in the scaling functions. We perform these calculations to use mode coupling theory for computing the dynamical structure function for two superdiffusive modes which have been not reported yet in the literature on driven diffusive systems. We also present simulation data for a family of exclusion processes which confirm the theoretical predictions.

This paper is organized in the following way. We first introduce the lattice model that we are going to study numerically (Section II). This is an extended version of the two-lane exclusion process presented in our earlier work \[16\] that allows us to relax constraints on the physically accessible parameter manifold. In Section III we first review predictions from mode coupling theory and then use the theory to make predictions for our model. The numerical tests of these predictions and some mode coupling computations are presented in Section IV. We finish with some conclusions in Section V. In the appendix we perform the full computation of the mode coupling matrices for arbitrary strictly hyperbolic two-component systems.

II. TWO-LANE ASYMMETRIC SIMPLE EXCLUSION PROCESS

We consider a two-lane asymmetric simple exclusion process where particles hop randomly on two parallel chains with \(L\) sites each and periodic boundary conditions. Particles do not change lanes and they obey the hard core exclusion principle which forbids occupancy of a
A particle on lane 1 (2) hops to the neighbouring site (provided this target site is empty) with to the right or left with rates \( r_1 \) - \( r_1 \) that depend on the particle configuration on the adjacent sites of the other lane that are marked by a cross.

A hopping event from site \( k \) to site \( k + 1 \) on the same lane may happen if site \( k \) is occupied and site \( k + 1 \) on the same lane is empty. The rate of hopping depends on the particle configuration on the adjacent lane as follows: Particles on lane \( i \) hop from site \( k \) to site \( k + 1 \) with rate \( r_i(k, k + 1) \) and from site \( k + 1 \) to site \( k \) with rate \( \ell_i(k + 1, k) \) (Fig. 1). The rates are given by

\[
\begin{align*}
    r_1(k, k + 1) &= p_1 + b_1 n_k^{(2)} + c_1 n_{k+1}^{(2)} + d_1 n_k^{(2)} n_{k+1}^{(2)} \\
    \ell_1(k + 1, k) &= q_1 + e_1 n_k^{(2)} + f_1 n_{k+1}^{(2)} + g_1 n_k^{(2)} n_{k+1}^{(2)} \\
    r_2(k, k + 1) &= p_2 + b_2 n_k^{(1)} + c_2 n_{k+1}^{(1)} + d_2 n_k^{(1)} n_{k+1}^{(1)} \\
    \ell_2(k + 1, k) &= q_2 + e_2 n_k^{(1)} + f_2 n_{k+1}^{(1)} + g_2 n_k^{(1)} n_{k+1}^{(1)}.
\end{align*}
\]

The hopping attempts of particles from site \( k \) on lane \( i \) to neighbouring sites occur independently of each other, after an exponentially distributed random time with mean \( \tau_i(k) = [r_i(k, k + 1) + \ell_i(k, k - 1)]^{-1} \) for a jump from site \( k \) on lane \( i \). Hopping attempts on an already occupied site are rejected.

Using pairwise balance \[31\], it is easy to verify that for any pair of total particle numbers \( N_i \) the stationary distribution for this model is the uniform distribution, provided that the symmetry constraints \( b_1 - e_1 = c_2 - f_2, b_2 - e_2 = c_1 - f_1, d_1 = g_1 \) and \( d_2 = g_2 \) are met for the interaction constants between the two lanes. The “bare” hopping rates \( p_1, p_2, q_1, q_2 \) are arbitrary. From the canonical uniform measures one constructs stationary grandcanonical
product measures where each site of lane \(i\) is occupied independently of the other sites with probability \(\rho_i \in [0,1] = N_i/L\). Hence the \(\rho_i\) are the conserved densities of the grandcanonical stationary distribution.

From the hopping rates (1) - (1) one reads off the corresponding stationary current vector \(\vec{j}\) with components

\[
\begin{align*}
  j_1(\rho_1, \rho_2) &= \rho_1 (1 - \rho_1)(a + \gamma \rho_2), \\
  j_2(\rho_1, \rho_2) &= \rho_2 (1 - \rho_2)(b + \gamma \rho_1).
\end{align*}
\]

with

\[
a = p_1 - q_1, \quad b = p_2 - q_2, \quad \gamma = b_1 + c_1 - e_1 - f_1.
\]

Notice that this current-density relation depends on the microscopic details of the model only through the parameter combinations \(a, b, \gamma\) which can take arbitrary real values. For \(a = 1\) we recover the current-density relation of the totally asymmetric model of [16] where \(q_\alpha = e_\alpha = f_\alpha = g_\alpha = d_\alpha = 0\) and \(b_\alpha = c_\alpha = \gamma/2\). This version of the model is used for Monte-Carlo simulations of the mode-coupling predictions of the next section. For \(b = 1\) we recover the totally asymmetric two-lane model of [32] which is a special case of the multi-lane model of [33]. Throughout this work we set \(a = 1, \gamma \neq 0\). For simulations we choose the totally asymmetric model where \(\gamma > - \min (1, b)\).

The product measure corresponds to a grandcanonical ensemble with a fluctuating particle number. These fluctuations are described by the symmetric compressibility matrix \(K\) with matrix elements

\[
K_{ij} = \frac{1}{L} < (N_i - \rho_i L)(N_j - \rho_j L) >= \rho_i (1 - \rho_i) \delta_{i,j}.
\]

In the notation defined in the appendix this corresponds to

\[
\kappa_i := K_{ii} = \rho_i (1 - \rho_i), \quad \bar{\kappa} := K_{12} = 0.
\]

As discussed below the current density relation \(\vec{j}\) given in (2) and the compressibility matrix \(K\) given (4) are the input data which completely determine the large scale behaviour of the particle system, up to a scale factor if a diffusive mode is relevant.
III. DYNAMICAL UNIVERSALITY CLASSES

A. Fluctuating hydrodynamics and mode coupling theory

Following the ideas set out in [34, 35] the starting point for investigating the large-scale dynamics of a microscopic lattice model is the system of conservation laws

$$\frac{\partial}{\partial t} \vec{\rho} + \frac{\partial}{\partial x} \vec{j} = 0 \quad (6)$$

where component \( \rho_i(x, t) \) of \( \vec{\rho} \) is the coarse-grained local density of component \( i \), and the component \( j_i(x, t) \) of \( \vec{j} \) is the associated current. It is a function of \( x \) and \( t \) only through its dependence on the local conserved densities. Hence these equations can be rewritten as

$$\frac{\partial}{\partial t} \vec{\rho} + J \frac{\partial}{\partial x} \vec{\rho} = 0 \quad (7)$$

where \( J \) is the current Jacobian with matrix elements \( J_{ij} = \partial j_i / \partial \rho_j \). The product \( JK \) of the Jacobian with the compressibility matrix [4] is symmetric [36] which guarantees that the system (7) is hyperbolic [37]. The eigenvalues \( v_i \) of \( J \) are the characteristic velocities of the system. If \( v_1 \neq v_2 \) the system is called strictly hyperbolic. Notice that in our convention \( \vec{\rho} \) and \( \vec{j} \) are regarded as column vectors. Transposition is denoted by a superscript \( T \).

Eq. (7) describes the deterministic time evolution of the density under Eulerian scaling where the lattice spacing \( a \) is taken to zero, lattice distance \( k \) taken to infinity such that \( x = ka \) fixed and at the same time the microscopic time \( \tau \) is taken to infinity such that the macroscopic time \( t = \tau a \) is finite. The effect of fluctuations, which occur on finer space-time scales where \( t = \tau a^z \) with dynamical exponent \( z > 1 \), can be captured by adding phenomenological white noise terms \( \xi_i \) and taking the non-linear fluctuating hydrodynamics approach together with a mode-coupling analysis of the non-linear equation. Following [15] we summarize here the main ingredients of this well-established description.

One expands the local densities \( \rho_i(x, t) = \rho_i + u_i(x, t) \) around their long-time stationary values \( \rho_i \) and keeps terms to first non-linear order in the fluctuation fields \( u_i \). For quadratic nonlinearities [17] then yields

$$\partial_t \vec{u} = -\partial_x \left( J \vec{u} + \frac{1}{2} \langle u | \vec{H} | u \rangle - \partial_x D \vec{u} + B \xi \right) \quad (8)$$

where \( \vec{H} \) is a column vector whose entries \( H_i = H^{(i)} \) are the Hessians with matrix elements \( H^{(i)}_{jk} = \partial^2 j_i / (\partial \rho_j \partial \rho_k) \) and the bra-ket notation represents the inner product in component
space, i.e., \( \langle u \mid = \vec{u}^T \), \( \mid u \rangle = \vec{u} \) and therefore \( \langle u \mid H_i \mid u \rangle = \vec{u}^T H^{(i)} \vec{u} = \sum_{jk} u_j u_k H^{(i)}_{jk} \). The diffusion matrix \( D \) is a phenomenological quantity. The noise strength \( B \) does not appear explicitly below, but plays an indirect role in the mode-coupling analysis. One recognizes in (8) a system of coupled noisy Burgers equations. If the quadratic non-linearity is absent one has diffusive behaviour, up to possible logarithmic corrections that may arise from cubic non-linearities [38].

In order to analyze this nonlinear equation we transform to normal modes \( \vec{ψ} = R\vec{u} \) where \( RJR^{-1} = \text{diag}(v_i) \) and the transformation matrix \( R \) is normalized such that \( RKR^T = 1 \), see the appendix. From (8) one thus arrives at

\[
\partial_t \psi_i = -\partial_x \left( v_i \psi_i + \langle \psi \mid G^{(i)} \mid \phi \rangle - \partial_x (\tilde{D} \vec{ψ}), + (\tilde{B} \tilde{ξ}) \right)
\]

(9)

with \( \tilde{D} = RDR^{-1} \), \( \tilde{B} = RB \) and

\[
G^{(i)} = \frac{1}{2} \sum_j R_{ij}(R^{-1})^T H^{(j)} R^{-1}
\]

(10)

are the mode coupling matrices.

To make contact of this macroscopic description with the microscopic model we first note that the current-density relation in the components of the current vector \( \vec{j} \) arises from the microscopic model by computing the stationary current-density relations \( j_i(\rho_1, \rho_2) \) and then substituting the stationary conserved densities by the coarse-grained local densities \( \rho_i(x, t) \) which are regarded as slow variables. Similarly, the compressibility matrix \( K \) is computed from the stationary distribution. Hence the mode coupling matrices (and with them the dynamical universality classes as shown below) are completely determined by these two stationary properties of the system. We stress that the exact stationary current-density relations and the exact stationary compressibilities are required. Approximations obtained e.g. from stationary mean field theory will can, in general, only accidentally provide the information necessary for determining the dynamical universality classes of the system.

Second, consider the dynamical structure matrix \( S_k(t) \). Its matrix elements are the dynamical structure functions

\[
S_k^{ij}(t) := \langle (n_k^{(i)}(t) - \rho_i)(n_k^{(j)}(t) - \rho_j) \rangle.
\]

(11)

which measure density fluctuations in the stationary state. This quantity has two different physical interpretations. On the one hand, one can regard the random variable \( f_k^{(i)}(t) := \)
\( n_k^{(i)}(t) - \rho_i \) as a stochastic process and then the dynamical structure function describes the stationary two-time correlations of this process. The long-time behaviour of the dynamical structure function can thus be determined from the fluctuation fields \( u_i(x, t) \), i.e., \( S_k^{ij}(t) \xrightarrow{k,t \to \infty} \langle u_i(x, t)u_j(0, 0) \rangle \). In a different interpretation the dynamical structure function measures the time evolution of the expectation of \( f_k^{(i)}(t) \) at time \( t \), i.e., the unnormalized density profiles \( \tilde{\rho}_k^{(i)}(t) := \langle (n_k^{(i)}(t) - \rho_i) \rangle \) that at time \( t = 0 \) have a delta-peak at site 0. Since the two conserved quantities interact, even an initial perturbation of only one component will cause a non-trivial evolution of both profiles. In each component the initial peak will evolve into two separate peaks, which move and spread with time. The characteristic velocities \( v_i \) are the collective velocities, i.e., the center-of-mass velocities of the two local perturbations \( \delta \rho \). The variance of the evolving density profiles determines the collective diffusion coefficient. This interpretation is quite natural from the viewpoint of regarding (8) as a more detailed description of (6) in the sense of describing fluctuation effects on finer space-time scales due to the randomness of the stochastic process from which (6) arises under Eulerian scaling.

Analogously one can regard the modes \( \psi_i(x, t) \) as stationary processes and the transformed dynamical structure functions \( \tilde{S}_k^{ij}(t) \) as the stationary space-correlations of this process. In the second interpretation the modes are seen as local perturbations of a stationary distribution with a specific choice of initial amplitudes in each component. The transformation of the dynamical structure functions \( \tilde{S}_k^{ij}(t) = \langle \psi_k^{(i)}(t)\psi_0^{(j)}(0) \rangle \) to the eigenmodes on the lattice, which is important for the numerical simulation of lattice models, is discussed in Appendix A. Since for strictly hyperbolic systems the two characteristic velocities are different, one expects that the off-diagonal elements of \( \tilde{S} \) decay quickly. For long times and large distances one is thus left with the diagonal elements \( \tilde{S}_k^{ii}(t) \) which we denote by by the \( \phi_i(x, t) \) with initial value \( \phi_i(x, 0) = \delta(x) \).

The large scale behaviour of the diagonal elements is expected to have the scaling form

\[
\phi_i(x, t) \sim t^{-1/z_i} \Phi_i((x - v_i t) / t)
\]

with a dynamical exponent \( z_i \) that may be different for the two modes. The exponent in the power law prefactor follows from mass conservation. In momentum space one has

\[
\hat{\phi}_i(k, t) \sim e^{-iv_i k t} \hat{\Phi}_i(k^2 t)
\]
for the Fourier transform

\[ \hat{\phi}_i(k, t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-ikx} \phi_i(x, t). \]  

(14)

(Notice the difference of the definition of the Fourier transform to that of [15, 30]. In order to compare with our results one has to make in [15, 30] the substitutions \( k \to k/(2\pi) \) for the momentum variables \( k \), and \( \hat{\phi} \to \sqrt{2\pi}\hat{\phi} \) for the amplitudes of Fourier transforms in momentum space.)

Whether the difference of the characteristic speeds vanishes or not plays an important role. For the case where \( v_1 = v_2 \), i.e., when the system (7) has an umbilic point, it was found numerically in the framework of dynamic roughening of directed lines that the dynamical exponent is \( z = 3/2 \), but the scaling functions are not KPZ [39]. On the other hand, for strictly hyperbolic systems the normal modes have different speeds and hence their interaction becomes very weak for long times. By identifying \( \phi_i \) with the gradient of a height variable (9) then turns generically into two decoupled KPZ-equations with coefficients \( G_{ii}^{(i)} \) determining the strength of the nonlinearity. Therefore, if the leading diagonal terms \( G_{ii}^{(i)} \) are all non-zero one expects for all modes \( z = 3/2 \) with KPZ scaling functions \( \Phi_i \). The other diagonal elements \( G_{jj}^{(i)} \) with \( i \neq j \) provide the leading corrections to the KPZ modes. All offdiagonal elements \( G_{jk}^{(i)} \) with \( j \neq k \) result in subleading corrections.

In order to analyze the system of nonlinear stochastic PDE’s in more detail we employ mode coupling theory [15]. The basic idea is to capture the combined effect of non-linearity and noise by a memory kernel. Thus the starting point for computing the \( \phi_i \) are the mode coupling equations

\[ \partial_t \phi_i(x, t) = (-v_i \partial_x + D_i \partial_x^2)\phi_i(x, t) + \int_0^t ds \int_{-\infty}^{\infty} dy \phi_i(x - y, t - s) \partial_y^2 M_{ii}(y, s) \]  

(15)

with the diagonal element \( D_i := \tilde{D}_{ii} \) of the phenomenological diffusion matrix and the memory kernel

\[ M_{ii}(y, s) = 2 \sum_{j,k} (G_{jk}^{(i)})^2 \phi_j(y, s)\phi_k(y, s). \]  

(16)

In writing these equations we have already neglected subleading contributions. The strategy is to plug into this equation, or into its Fourier representation, the scaling ansatz (12) (or (13)). One gets equations for the dynamical exponents arising from requiring non-trivial scaling solutions and using the known results \( z = 3/2 \) for KPZ and \( z = 2 \) for diffusion. In
a next step one can then solve for the actual scaling functions, see below. Since for \( v_1 \neq v_2 \) one has \( \phi_j(y, s)\phi_k(y, s) \approx 0 \) for \( j \neq k \) it is clear that the scaling behaviour of the solutions of (15) will be determined largely by the diagonal terms \( G_{jj}^{(i)} \) of the mode coupling matrices \( G^{(i)} \). If a leading self coupling term \( G_{ii}^{(i)} \) vanishes, one finds non-KPZ behaviour for mode \( i \). In the appendix we compute the mode coupling matrices of a general two-component system with the current vector and compressibility matrix as an input.

Following this strategy one finds after some algebra the complete list of possible universal behaviour of strictly hyperbolic systems from the structure of the mode coupling matrices \( G^{(i)} \) as shown in Table 1 [30]. The shorthand KPZ represents the KPZ scaling function, while KPZ' refers to modified KPZ, both with dynamical exponent \( z = 3/2 \). \( D \) represents a Gaussian scaling function \( \Phi \) with dynamical exponent \( z = 2 \), \( \alpha L \) represents an \( \alpha \)-stable Lévy distribution as scaling function \( \Phi \) with dynamical exponent \( z = \alpha \), GM (for golden mean) represents \( \varphi L \) with \( \varphi = (1 + \sqrt{5})/2 \). In what follows we apply these general results to the two-lane model defined above. It will transpire that all theoretically possible scenarios can actually be realized in this family of models.

**B. Mode-coupling matrix for the two-lane model**

The input data are the current-density relation (2) and the compressibility matrix (4). From the current-density relation one computes the current Jacobian and the Hessian, which are used together with the compressibility matrix to compute the basis for normal modes and finally the mode coupling matrices, as shown in detail in the appendix in the general case.

For the present model we remark first that the currents (2) are at most quadratic in each density. Hence no logarithmic corrections to diffusive behaviour are expected in the two-lane model defined above. Second, as discussed in the appendix, in any coupled two-component system a vanishing cross compressibility \( \bar{\kappa} = 0 \) implies that the cross derivatives \( \partial_{\alpha j \beta} \) (where \( \alpha \neq \beta \)) of the currents have to be non-zero except when one of the two components is frozen.

For our system the explicit form of \( J \) is

\[
J = \begin{pmatrix}
(1 + \gamma \rho_2)(1 - 2\rho_1) & \gamma \rho_1(1 - \rho_1) \\
\gamma \rho_1(1 - \rho_2) & (b + \gamma \rho_1)(1 - 2\rho_2)
\end{pmatrix}.
\] (17)

| $G^{(1)}$ | $G^{(2)}$ | $(\star,\star)$ | $(\star,0)$ | $(0,\star)$ | $(0,0)$ |
| --- | --- | --- | --- | --- | --- |
| $(\star,\star)$ | (KPZ,KPZ) | (KPZ,KPZ) | $(\frac{5}{2}L,KPZ)$ | (D,KPZ') |
| $(0,\star)$ | (KPZ,KPZ) | (KPZ,KPZ) | $(\frac{5}{2}L,KPZ)$ | (D,KPZ) |
| $(\star,0)$ | (KPZ,$\frac{5}{2}L$) | (KPZ,$\frac{5}{2}L$) | (GM,GM) | (D,$\frac{3}{2}L$) |
| $(0,0)$ | (KPZ',D) | (KPZ,D) | $(\frac{3}{2}L,D)$ | (D,D) |

**TABLE I:** Classification of universal behaviour of the two modes by the structure of the mode coupling matrices $G^{(i)}$. The acronyms denote: KPZ: KPZ universality class (superdiffusive), KPZ’: modified KPZ universality class (superdiffusive), D = Gaussian universality class (normal diffusion), $\alpha$L: superdiffusive universality class with $\alpha$-stable Lévy scaling function and GM = $\varphi$L with the golden mean $\varphi = (1 + \sqrt{5})/2$. An asterisk or star in the $G^{(i)}$ denotes a non-zero entry, no entry represents an arbitrary value (zero or non-zero). The selfcoupling terms $G_{ii}^{(i)}$ are marked as star or boldface 0, resp.

and the Hessians $H^{(i)}$ are

$$H^{(1)} = \begin{pmatrix} -2(1 + \gamma \rho_2) & \gamma(1 - 2\rho_1) \\ \gamma(1 - 2\rho_1) & 0 \end{pmatrix}, \quad H^{(2)} = \begin{pmatrix} 0 & \gamma(1 - 2\rho_2) \\ \gamma(1 - 2\rho_2) & -2(b + \gamma \rho_1) \end{pmatrix}.$$  \quad (18)

The “good” parameters are not the matrix elements of the current Jacobian and the Hessians, but the parameters $u$, $\omega = \tan \phi$ \cite{A20} and the transformed Hessian parameters \cite{A42}, \cite{A43}. From \cite{17} we read off

$$\omega = \frac{1 - b - (2 + b\gamma)\rho_1 + (\gamma + 2b)\rho_2}{2\gamma \sqrt{\rho_1(1 - \rho_1)\rho_2(1 - \rho_2)}} \left(1 + \sqrt{1 + \frac{4\gamma^2(\rho_1(1 - \rho_1)\rho_2(1 - \rho_2))}{(1 - b - (2 + b\gamma)\rho_1 + (\gamma + 2b)\rho_2)^2}}\right) \quad (19)$$

and

$$u = \sqrt{\frac{\rho_1(1 - \rho_1)}{\rho_2(1 - \rho_2)}} \quad (20)$$

For $J_{11} = J_{22}$ one has $\omega = 1$. 


The collective velocities $v_{1,2}$ are given in (A4). Notice that $J_{12}J_{21} = \gamma^2 \rho_1 (1 - \rho_1) \rho_2 (1 - \rho_2) \geq 0$ in the whole physical parameter regime of the model. Therefore $\delta \neq 0$ which implies that the model is strictly hyperbolic for all physical parameter values. Moreover, unless one of the lanes is frozen (fully occupied or fully empty) we have the strict inequality $J_{12}J_{21} > 0$. The frozen case is of no interest since the dynamics in the non-frozen lane reduce to the dynamics of a single exclusion process. Hence we shall assume $J_{12}J_{21} > 0$ throughout this paper.

From the Hessians (18) one obtains, using (A42), (A43),

$$g_1^{(1)} = -2(1 + \gamma \rho_2), \quad g_2^{(1)} = 0, \quad \bar{g}^{(1)} = \gamma \sqrt{\frac{\rho_2(1 - \rho_2)}{\rho_1 (1 - \rho_1)}} (1 - 2 \rho_1),$$

(21)

and

$$g_1^{(2)} = 0, \quad g_2^{(2)} = -2 \sqrt{\frac{\rho_2(1 - \rho_2)}{\rho_1 (1 - \rho_1)}} (b + \gamma \rho_1), \quad \bar{g}^{(2)} = \gamma (1 - 2 \rho_2).$$

(22)

The compressibility matrix enters the mode coupling coefficients only through the normalization factors for which we obtain from (A25)

$$z_{\pm} = \frac{1}{\sqrt{\kappa_1}} \notin \{0, \pm \infty\}.$$  

(23)

This yields

$$G_{jj}^{(i)}(\omega) = A_0 D_j^{(i)}(\omega)$$

(24)

with

$$D_1^{(1)}(\omega) = g_1^{(1)} - 2 \bar{g}^{(1)} \omega + 2 \bar{g}^{(2)} \omega^2 - g_2^{(2)} \omega^3$$

(25)

$$D_2^{(1)}(\omega) = \left(2 \bar{g}^{(1)} - g_2^{(2)}\right) \omega + \left(g_1^{(1)} - 2 \bar{g}^{(2)}\right) \omega^2$$

(26)

$$D_1^{(2)}(\omega) = \left(g_1^{(1)} - 2 \bar{g}^{(2)}\right) \omega + \left(g_2^{(2)} - 2 \bar{g}^{(1)}\right) \omega^2$$

(27)

$$D_2^{(2)}(\omega) = g_2^{(2)} + 2 \bar{g}^{(2)} \omega + 2 \bar{g}^{(1)} \omega^2 + g_1^{(1)} \omega^3.$$  

(28)

and

$$A_0 = \frac{1}{2} \sqrt{\kappa_1} \cos^3(\phi) \neq 0.$$  

(29)

As discussed in the appendix a vanishing cross-compressibility $\tilde{\kappa} = 0$ implies that $A_0 \neq 0$. Therefore a diagonal element $G_{jj}^{(i)}$ of a mode coupling matrix vanishes if and only if the polynomial $D_j^{(i)}$ defined in (25) - (28) vanishes. In order to see whether all scenarios listed
in Table II can be realized we study these cases.

**Purely diffusive case (D,D):**

First consider the purely diffusive case (D,D) which requires $D_1^{(1)} = D_2^{(1)} = D_1^{(2)} = D_2^{(2)} = 0$. Demanding that $D_2^{(1)} = D_1^{(2)} = 0$ leads to the constraints $g_1^{(1)} = 2\bar{g}^{(2)}$ and $g_2^{(2)} = 2\bar{g}^{(1)}$. In terms of the parameters $b, \gamma, \rho$, this reads $-\gamma = 1/(1-\rho_2) = b/(1-\rho_1)$. This is outside the physical parameter range $\gamma > -\min(1,b)$ of the totally asymmetric model of [16], but can be realized in the general two-lane model defined in Section II. Plugging this condition into $D_1^{(1)} = D_2^{(2)} = 0$ yields the further conditions that $g_1^{(1)} = g_2^{(2)} = 0$, i.e., both Hessians must vanish. This requires

$$\rho_1 = \rho_2 = 1/2, \quad b = 1, \quad \gamma = -2.$$  \hspace{1cm} (30)

The characteristic velocities are then $v_{1,2} = \mp 1$. It is somewhat counterintuitive that for these values $j_1 = j_2 = 0$, i.e. the system appears to be macroscopically in equilibrium, but the Gaussian mass fluctuations travel with non-zero velocities. A simple parameter choice for this scenario is $p_1 = p_2 = 1$, $q_1 = d_1 = g_1 = q_2 = d_2 = g_2 = 0$, $b_1 = c_1 = b_2 = c_2 = -1/2$, $e_1 = f_1 = e_2 = f_2 = 1/2$.

**Superdiffusive mixed cases (D,KPZ'), (D,KPZ), (D,\textfrac{2}{3}L), (KPZ,\textfrac{5}{3}L):**

Consider $b = 1$ where the hopping rates are completely symmetric with respect to the lane interchange and take $\rho_1 = \rho_2 =: \rho$. Then $g_1^{(1)} = g_2^{(2)} = -2(1 + \gamma \rho)$, $g_1^{(2)} = g_2^{(1)} = 0$, $\bar{g}^{(1)} = \bar{g}^{(1)} = \gamma/(1-2\rho)$ and $u = 1$, $\omega = 1$. This yields $D_1^{(1)} = D_2^{(1)} = 0$ and $D_2^{(2)} = 2A_0(\bar{g}_1^{(1)} + 2\bar{g}^{(1)})$, $D_1^{(2)} = 2A_0(\bar{g}_1^{(1)} - 2\bar{g}^{(1)})$ with $A_0 = \sqrt{\rho(1-\rho)/32}$. Computing the off-diagonal elements from (A41), (A38) we find the full mode coupling matrices

$$G^{(1)} = -4A_0(1 + \gamma \rho) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad G^{(2)} = -4A_0 \begin{pmatrix} 1 + \gamma(1-\rho) & 0 \\ 0 & 1 - \gamma(1-3\rho) \end{pmatrix}$$ \hspace{1cm} (31)

Thus generically this line is in the (D,KPZ') universality class (Fig. 2).

Notice that at $\gamma = -1/(1-\rho)$ one has $D_1^{(2)} = 0$, corresponding to the (D,KPZ) universality class which can be realized in the generalized two-lane model defined above and that occurs also in the single-lane multi-component asymmetric simple exclusion process.
FIG. 2: (Colour online) Location of points where $G^{(2)}_{22} = 0, G^{(2)}_{11} \neq 0$ for $b = 1$ and different values of $\gamma$. In the upper right (lower left) corner the points grouped along curves of increasing length correspond to $\gamma = 1.5, 2.5, 5$ ($\gamma = -0.6, -0.7, -0.85$). On these curves one generically has the $(\frac{5}{3}L,KPZ)$ universality class. On the diagonal line $\rho_1 = \rho_2$ one has $G^{(1)}_{11} = G^{(1)}_{22} = 0$, generically corresponding to the (D,KPZ$'$) universality class. On the intersection of this line with a curve parametrized by $\gamma$ one has the (D,$\frac{3}{2}L$) universality class.

with stationary product measure $\Phi_0$. For $\gamma = 1/(1 - 3\rho)$ one has $D^{(2)}_2 = 0$, corresponding to the (D,$\frac{3}{2}L$) scenario, see next section. If one moves away from the line $\rho_1 = \rho_2$, but stays on the curves indicated in Fig. 2 for special values of $\gamma$ the self-coupling coefficient $G^{(1)}_{11}$ is non-zero, but $G^{(2)}_{22} = 0$. Hence one has the (KPZ,$\frac{5}{3}L$) scenario. The three cases (D,KPZ$'$), (D,$\frac{3}{2}L$) and (KPZ,$\frac{5}{3}L$) can be realized in the totally asymmetric two-lane model.

Golden mean universality class (GM,GM):

Next consider $b \neq 1$. The formulas for the mode coupling matrices become cumbersome and we do not present them here in explicit form in full generality. It turns out that one can have that both self-coupling coefficients $G^{(i)}_{ii}$ vanish and both subleading diagonal elements $G^{(i)}_{jj}$ with $j \neq i$ are non-zero, corresponding to the ($\varphi L$,$\varphi L$) scenario where both dynamical exponents are the golden mean $\varphi = (1 + \sqrt{5})/2$, see Fig. 3. This can be realized by choosing
FIG. 3: Location of points where $G^{(1)}_{11} = 0, G^{(1)}_{22} \neq 0$ (crosses in the upper right corner) or $G^{(2)}_{22} = 0$ and $G^{(2)}_{11} \neq 0$ (thin bullets), for fixed $\gamma = -3/4$ and $b = 1.5$ (black), $b = 1.2$ green), $b = 0.9$ (red), $b = 0.8$ (blue), corresponding to the order from left to right in the lower half of the figure and opposite order in the upper part of the figure. Along the curves indicated by the dots (crosses) one has generically the ($\varphi L$,KPZ) or (KPZ,$\varphi L$) universality class. At the intersections of curves with the same colour one has the golden mean universality class ($\varphi L$,$\varphi L$).

unequal densities such that

$$(1 + \gamma \rho_2)(1 - 2\rho_1) = (b + \gamma \rho_1)(1 - 2\rho_2)$$

which corresponds to $J_{11} = J_{22}$ and hence $\omega = 1$. Then the requirement $D^{(1)}_1 = D^{(2)}_2 = 0$ yields

$$\rho_1 = \frac{1 - b}{3\gamma}, \quad \rho_2 = \frac{\gamma - 1}{3\gamma}$$

which implies $\gamma \in (-\infty, -1/2] \cup [1, \infty)$ and $b$ is in the range between $\gamma$ and $-2\gamma$. For general values of $\omega$ the analytical formulas for the lines $G^{(1)}_{11} = G^{(2)}_{22} = 0$ in the $\rho_1 - \rho_2$ plane are complicated. In order to demonstrate the existence of solutions we show numerical plots for fixed $\gamma = -3/4$ and various values $b$ in Fig. 3. Notice also that there are parameter ranges of $b$ without solutions in the physical range of densities $(\rho_1, \rho_2) \in [0, 1] \times [0, 1]$.

In what follows we investigate the two novel scenarios ($D,\varphi L$) and ($GM,GM$) which have not been reported yet in the literature on driven diffusive systems. We also comment on the shape of the structure function for the (KPZ,$\varphi L$) mode discussed in [16].
IV. SUPERDIFFUSIVE NON-KPZ UNIVERSALITY CLASSES

A. Diffusive mode and 3/2 - Lévy mode

We consider the case where mode 1 is Gaussian, and mode 2 has non-vanishing cross-coupling,

\[ G_{11}^{(1)} = G_{22}^{(1)} = G_{22}^{(2)} = 0, \quad G_{11}^{(2)} \neq 0 \]  

(34)

The mode coupling equation (15) for mode 2 reads in Fourier space

\[
\partial_t \hat{\phi}_2(k,t) = -ikv_2 \hat{\phi}_2(k,t) - k^2 D_2 \hat{\phi}_2 \\
-2(G_{11}^{(2)})^2 k^2 \int_0^t ds \hat{\phi}_2(k,t-s) \int_{-\infty}^{\infty} dq \hat{\phi}_1((k-q,s) \hat{\phi}_1(q,s)).
\]  

(35)

with \( D_2 = \tilde{D}_{22} \). For the Gaussian mode 1 the mode coupling equation is obtained by the exchange 1 \( \leftrightarrow \) 2 in (35) and dropping the term containing the integral. Note that we are interested in the large \( x \) behaviour of the scaling function, meaning \( k \to 0 \) in Fourier space.

We start with the observation that the Gaussian mode has the usual scaling form

\[
\phi_1(x,t) = \frac{1}{\sqrt{4\pi D_1 t}} e^{-\frac{(x-v_1 t)^2}{4D_1 t}}
\]  

(36)

with Fourier transform \( \hat{\phi}_1(k,t) = 1/\sqrt{2\pi} \exp(-iv_1 kt - D_1 k^2 t) \). Inserting this into (35) and performing the integration over \( q \), we obtain

\[
\partial_t \hat{\phi}_2(k,t) = -(iv_2 + k^2 D_2) \hat{\phi}_2(k,t) - k^2 \left(G_{11}^{(2)}\right)^2 \int_0^t ds \hat{\phi}_2(k,t-s) \frac{e^{-iv_1 k s - D_2 k^2 s/2}}{2\pi D_2 s}.
\]  

(37)

This equation can be solved in terms of the Laplace transform \( \tilde{\psi}_2(k,\omega) := \int_0^\infty dt e^{-\omega t} \hat{\phi}_2(k,t) \) which yields

\[
\tilde{\psi}_2(k,\omega) = \frac{\hat{\phi}_2(k,0)}{\omega + iv_2 + k^2 \left(D_2 + \left(G_{11}^{(2)}\right)^2 \left(2D_2(\omega + iv_1 + D_2 k^2/2)\right)^{-1}\right)}.
\]  

(38)

For large times we assume the real-space scaling form \( \phi_2(x,t) = t^{-1/z} h \left( \frac{(x-v_2 t)^z}{t} \right) \) with dynamical exponent \( z > 1 \). This is equivalent to the scaling forms

\[
\hat{\phi}_2(k,t) = e^{-iv_2 k t} f(|k|^z t), \quad \tilde{\psi}_2(k,\omega) = |k|^{-z} g \left( \frac{\omega + iv_2}{|k|^z} \right)
\]  

(39)

for the Fourier- and Laplace transforms respectively. Introducing the shifted Laplace parameter \( \tilde{\omega} := \omega + iv_2 \) and finds that the leading small-\( k \) behaviour of the Laplace transform
comes from the term proportional to $v_1 - v_2$ under the square root. This yields $z = 3/2$ and we obtain in the limit $\tilde{\omega} \to 0$ (with scaling variable $\tilde{\omega}/|k|^2$ kept fixed) after performing the inverse Laplace transformation

$$\hat{\phi}_2(k, t) = \frac{1}{\sqrt{2\pi}} \exp \left(-iv_2kt - C_0|k|^{3/2}t \left[1 - i \operatorname{sgn}(k(v_1 - v_2))\right]\right)$$

with

$$C_0 = \frac{(G^{(2)}_{11})^2}{2\sqrt{D_1|v_2 - v_1|}}$$

We recognize here the characteristic function of an $\alpha$-stable Lévy distribution

$$\phi(k; \mu, c, \alpha, \beta) := \exp \left(ik\mu - |ck|^\alpha(1 - i\beta \tan \left(\frac{\pi\alpha}{2}\right)\operatorname{sgn}(k))\right)$$

with $\mu = -v_2t$, $\alpha = 3/2$, maximal asymmetry $\beta = \operatorname{sgn}(v_1 - v_2)$ and

$$c = C_0^{2/3}/(2\pi)^{3/2}t^{2/3}.$$

The scaling function (40) is similar to the one found to describe the hydrodynamics of the anharmonic chain in the case of an "even potential", see [15].

Monte-Carlo simulation data for the 3/2-Lévy mode are shown in Fig. 4 for small times up to $t \approx 100$. The mode moves with a velocity that is very close to the theoretical prediction $v_2 \approx 1.2$. Indeed, one expects the error in the velocity to be small, since the velocity is not an approximate asymptotic property. It comes from mass conservation and is an exact constant for all times even on the lattice [32].

The scaling exponent and asymmetry predicted by Mode Coupling theory are in a good agreement with the Monte Carlo simulations, see Figs. 5, 6. In Fig. 5 we show the measured growth of the variance of the 3/2-Lévy mode. Since the $\alpha$-stable Lévy distributions have infinite variance, one expects this empirical variance to grow in time. Mass conservation together with dynamical scaling predicts a growth proportional to $t^\nu$ with $\nu = 2/z$ [16]. The measured exponent $\nu_{exp} \approx 1.32$ is very close to the theoretical value $\nu = 4/3$ even for the early time regime shown in the figures.

The only parameter that has slow convergence to the asymptotic value is the asymmetry of the scaling function. A similar phenomenon is discussed in [15] in terms of corrections...
FIG. 4: (Colour online) Dynamical structure functions for particles on chain 1, for 3/2-Lévy mode with $c_2 = 1.3$ at different times from Monte Carlo simulations, averaged over $18 \cdot 10^7$ histories. Parameters: $N = 400, \gamma = 2.5, B = 1, \rho_1 = 0.2, \rho_2 = 0.2$. Statistical errors are smaller than symbol size.

FIG. 5: (Color online) Variance of the dynamical structure function shown in Fig. 4 versus time. To scaling of the memory kernel for the 5/3-Lévy mode. They are shown to vanish slowly with a power law decay in time. Here we measure the deviation of the asymmetry from its asymptotic value. The measured quantity $1 + \beta_{exp}$ decreases monotonically with time. The decay is approximately algebraic with exponent $\approx 1/6$, see Fig. 7.
FIG. 6: (Colour online) Fit of dynamical structure functions for time $t = 88$ with 3/2-stable Levi distribution with asymmetry $\beta = -0.692$. For parameters see Fig. 4.

FIG. 7: (Colour online) Asymmetry $1 + \beta$ versus time, obtained by fitting the numerically obtained dynamical structure function with the PDF of 3/2 Levi stable law. The line with the power law $\propto t^{-1/6}$ is a guide for the eye. For parameters see Fig. 4.
B. Two golden mean modes

We consider now the case where both self-coupling coefficients \( G^{(i)}_{ii} \) of the mode coupling matrix vanish and both subleading coefficients \( G^{(i)}_{jj} \) are non-zero. In this case one cannot use the Gaussian or the KPZ scaling function as an input into the mode coupling equations. However, the equations give a self-consistency relation which allows one to compute the scaling function for the two modes, see [30] for the symmetric case where \( G^{(1)}_{22} = G^{(2)}_{11} \).

For the non-symmetric case \( G^{(1)}_{22} \neq G^{(2)}_{11} \) the calculation of [30] is not directly applicable. However, one can adopt a similar philosophy with two scaling functions

\[
\hat{\phi}_1(k, t) = e^{-iv_1kt}g(b|k|^\beta), \quad \hat{\phi}_2(k, t) = e^{-iv_2kt}h(c|k|^\gamma t)
\]  

as input, which in addition to the \textit{a priori} unknown dynamical exponents \( \gamma \) and \( 1/\beta \), have different scale factors \( b, c \) as free variables. With this ansatz one obtains the consistency conditions \( \gamma = 1 + \beta \) and \( \gamma = 1/\beta \) for the dynamical exponent. After lengthy but straightforward computations one also finds the scale factors and with the relabelling \( \hat{\phi}_\pm(k, t) \equiv \hat{\phi}_1(k, t), \hat{\phi}_+(k, t) \equiv \hat{\phi}_2(k, t) \) arrives at

\[
\hat{\phi}_\pm(k, t) = \frac{1}{\sqrt{2\pi}} \exp \left( -iv_\pm kt - C_\pm |k|^\varphi t \left[ 1 \pm \text{isgn}(k(v_- - v_\pm)) \tan \left( \frac{\pi \varphi}{2} \right) \right] \right)
\]

with golden mean \( \gamma = \varphi \equiv (1 + \sqrt{5})/2 \) and the scale factors

\[
C_\pm = \frac{1}{2} |v_+ - v_-|^1 - \frac{1}{\varphi} \left( \frac{2G^{(1)}_{22} G^{(2)}_{11}}{\varphi \sin \left( \frac{\pi \varphi}{2} \right)} \right)^{\varphi^{-1}} \left( \frac{G^{(1)}_{22}}{G^{(2)}_{11}} \right)^{\pm(1+\varphi)}.
\]

Notice that \( \varphi - 1 = 1/\varphi \).

In order to test the scaling function for the two-lane model we choose the parameter manifold [32] where one has the characteristic velocities

\[
v_\pm = (1 + \gamma \rho_2)(1 - 2 \rho_1) \pm \gamma \sqrt{\rho_1(1 - \rho_1)\rho_2(1 - \rho_2)}
\]

In order to measure the dynamical exponent \( \varphi \approx 1.61803 \), which is rather close to \( z = 5/3 \) appearing in the \((5/3,\text{KPZ})\) scenario studied in [16], we focus on the large-time regime. Fig. 8 shows the dynamical structure function \( S_k^{11}(t) \) from Monte Carlo simulations. The peaks move with the predicted velocities and they are well separated at the earliest time \( t = 480 \) shown in the figure. We have chosen from [32] \( \gamma = 2.5, b = 0.625 \) and \( \rho_1 = 0.25, \rho_2 = 0.2 \).
corresponding to mode coupling matrices

\[
G^{(1)} = \begin{pmatrix}
0 & -0.406416 \\
-0.406416 & -0.105726
\end{pmatrix}, \quad G^{(2)} = \begin{pmatrix}
-0.812833 & -0.052863 \\
-0.052863 & 0
\end{pmatrix}
\tag{48}
\]

and transformation matrix

\[
R^{-1} = \begin{pmatrix}
-0.734553 & 0.734553 \\
0.678551 & 0.678551
\end{pmatrix}.
\tag{49}
\]

The columns of \( R \) are the eigenmodes with velocities \( v_1 \equiv v_- = 0.316987, v_2 \equiv v_+ = 1.18301 \) respectively.

FIG. 8: (Colour online) Dynamical structure function for particles on chain 1, for golden mean mode with \( v_2 = 1.183 \) at different times from Monte Carlo simulations, averaged over all lattice sites, and over 900 histories, on a periodic two-lane system with \( L = 10^6 \) sites. Parameters: \( \gamma = 2.5, b = 0.625, \rho_1 = 0.25, \rho_2 = 0.2 \). Statistical errors are smaller than symbol size.

Notice that by definition the scale factor \( c \) entering the Lévy function (42) scales as \( t^{1/\phi} \). We have determined this scale factor by fitting the measured dynamic structure function for mode 2 with a maximally asymmetric \( \phi \)-stable Lévy distribution (Fig. 9). A least square fit with 95% confidence bounds gives a measured dynamical exponent \( z = 1.646 \), with error bars \( 1.632 < z < 1.661 \). This is not far from the theoretical value \( z = \phi \approx 1.618 \), but does not allow for a distinction from the dynamical exponent \( z = 5/3 \) that would be expected if the self-coupling coefficient \( G^{(2)}_{22} \) was non-zero.

One can see a clear difference between the two cases by fitting the measured scaling function both with a \( \phi \)-stable distribution and a 5/3-stable distribution, see Fig. 10 time
FIG. 9: (Color online) Scale factor, obtained by fitting the measured dynamic structure function with maximally asymmetric $\phi$-stable Lévy distribution, versus time, in double logarithmic scale. The line with the golden mean exponent is a guide for the eye. The line with golden mean exponent is a guide for an eye. Parameters: $\gamma = 2.5, B = 0.625, \rho_1 = 0.25, \rho_2 = 0.2, N = 10^6$. Histories: 100. Data correspond to mode 2 with velocity $v_2 = 1.183$.

$t = 600$. The $\varphi$-stable distribution is practically not distinguishable from the numerical data points. In contrast, the $5/3$-stable distribution shows a marked deviation from the numerical data particularly near the peak region and on the left algebraically decaying side of the function. The discrepancy increases with time.

We remark that the left peak in Fig. 8 corresponding to mode 1 is considerably less asymmetric than the peak of mode 2 shown in more detail in Fig. 10. To get some intuition for this observation consider the numerical values $G^{(1)}_{11}$ and $G^{(2)}_{22}$ (48). Their square of their ratio is $\approx 1/64$, so the coupling strengths differ by almost two orders of magnitude. If $G^{(1)}_{22}$ was zero, we would be back to the $(D,3L)$ scenario discussed in the previous subsection and mode 1 would be a symmetric Gaussian peak. Therefore one indeed expects for mode 1 at finite times a much more symmetric function for small relative coupling strength than predicted for the asymptotic regime.
FIG. 10: (Color online) Measured dynamic structure function for mode 2 at time $t = 600$. A fit with the $5/3$-stable Lévy distribution is shown, which exhibits deviations especially near the peak region. On the other hand, the fit with with the $\varphi$-stable Lévy distribution provides a very good fit everywhere. The asymmetry $\beta$ is maximal for both fits. For parameters see Fig. 9.

C. KPZ mode and $5/3$-Lévy mode

In [16] we reported the occurrence of the ($5/3L$,$KPZ$) universality class for the totally asymmetric version of the two-lane exclusion process. The measured dynamical exponents were shown to agree well with the theoretical prediction. Here we expand on these result by briefly discussing the scaling function. In Fig. 11 one can see that a reasonable fit of the numerical data can be obtained with a $5/3$-stable Lévy distribution predicted by mode coupling theory [15].

The measured dynamical structure function, however, exhibits an asymmetry much less than the predicted maximal value. Indeed, for small times its amplitude is rather small. We attribute this discrepancy to finite-time effects, cf. the argument for the left GM mode of the previous subsection. In order to substantiate this claim we show in Table III numerically determined asymmetries. They grow in time, thus supporting the argument. We do not have a theoretical prediction of how they should grow.
FIG. 11: (Color online) Dynamic structure function for particles on lane 1 at time $t = 200$. The curve is a fit with the $5/3$-stable Lévy distribution with non-maximal asymmetry.

| $t$  | 20  | 40  | 60  | 80  | 100 | 120 | 140 | 160 | 200 |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\beta$ | -0.0229 | -0.0504 | -0.0685 | -0.0797 | -0.0825 | -0.0872 | -0.0918 | -0.0916 | -0.1000 |

TABLE II: Asymmetry $\beta$ of a $5/3$-Lévy distribution obtained from a fit to the numerical data for the $5/3$-Lévy mode of [16] for different times $t$.

V. CONCLUSIONS

We consider time-dependent density fluctuations in driven diffusive system with two conservation laws. For one conservation law it is well-established that the appropriate tool to describe the universal properties of these fluctuations is non-linear fluctuating hydrodynamics [9]. Recent work, reviewed in [15], shows that the approach can be extended to anharmonic chains with more than one conservation law and also to Hamiltonian dynamics with three conservation laws [13]. From the present work and a recent paper [16] we conclude that the predictions of the theory apply also to driven diffusive systems with stochastic lattice gas dynamics with two conservation laws. Specifically, for a two-lane asymmetric simple exclusion process we argue that all theoretically possible universality classes for two-component systems [30], which are determined by the zeroes of the diagonal elements of the mode coupling matrices $G^{(i)}$, can be realized (see Table II). Among these, our Monte-Carlo simulations
of a two-lane asymmetric exclusion process confirm two superdiffusive universality classes which have gone unnoticed so far in the literature on driven diffusive systems.

Mode coupling theory not only predicts the dynamical exponents $z$ for these universality classes, but also the scaling forms of the dynamical structure functions for these novel superdiffusive modes. In most cases these scaling functions are $z$-stable Lévy distributions with maximal asymmetry. The numerical simulation confirms these predictions with great accuracy both for the $3/2$-mode and a golden mean mode with $z = (1 + \sqrt{5})/2$ shown to occur in anharmonic chains [30]. For some modes the $z$-stable Lévy distributions provide excellent fits, but with an effective asymmetry that is not maximal. However, our data show that the numerically fitted asymmetry increases with time in the cases we considered, thus supporting the notion that asymptotically the maximal value will be reached.

Which universality classes actually occur in a system at given values of the physical parameters of the model is completely encoded in the stationary current-density relation $\vec{j}(\rho_1, \rho_2)$, no other knowledge about a given model is required. The stationary compressibility matrix $K(\rho_1, \rho_2)$, related to the current-density relation through a time-reversal symmetry proved in [36], allows for the prediction also of the scale factors that enter the scaling functions, unless diffusive modes are relevant. Thus generically the scaling functions are completely determined by two simple stationary properties: The current-density relation $\vec{j}(\rho_1, \rho_2)$ and the compressibility matrix $K(\rho_1, \rho_2)$. Going beyond specific lattice gas models, we have computed the mode coupling matrices in general form for arbitrary input data, i.e., arbitrary current-density relation and compressibility matrix. From the diagonal matrix elements of these one can then directly read off the scaling functions for arbitrary two-component systems, except in the presence of the diffusive universality class where the scale factors contain diffusion coefficient not predicted by the theory.

It is interesting to notice that all possible scenarios of universality classes (see Table I) can be realized with the current-density relation (2). This relation is minimal in the sense that the non-linearity of the conserved current $i$ is only quadratic and the coupling of this non-linearity to the other conserved quantity is only linear. Thus it is not necessary to have more complicated current-density relation in order to observe all allowed universality classes. Moreover, this minimal current-density relation has the nice property that one does not expect logarithmic corrections to diffusive modes [38]. Our two-lane exclusion process, which is an extension of the model studied by us previously [16], provides a simple
microscopic realization for this minimal current-density relation.

Throughout this discussion we have assumed that the current-density relation is strictly hyperbolic, i.e., the collective velocities $v_i$ of the two modes are different. This assumption is crucial for the decoupling argument for the modes that underlies the mode-coupling computations. Indeed, the nonequilibrium time reversal symmetry \[36\] rules out umbilic points (where $v_1 = v_2$) in any model with minimal current-density relation and at the same time diagonal compressibility matrix. Therefore in the model presented here the issue does not actually arise. However, umbilic points are a generic feature of more complicated models, either with the same minimal current-density relation, but non-diagonal compressibility matrix \[28\], or for non-minimal current-density relations \[39\]. From numerical observations \[39\] one expects dynamical exponent $z = 3/2$ as for KPZ, but non-KPZ scaling functions. How mode-coupling theory can predict the behaviour at umbilic points is an open problem. It would also be interesting to extend mode coupling theory to predict the convergence of the finite-time asymmetry in the Lévy distribution to the asymptotic maximal value.

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Appendix A: Mode coupling matrices for strictly hyperbolic two-component systems

1. Notation

We consider a general system with two conservation laws. For definiteness we choose the language of driven diffusive systems with currents $j_i(\rho_1, \rho_2), i = 1, 2$ for the conserved
densities $\rho_i$. We define the general flux Jacobian

$$ J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} $$  \hspace{1cm} (A1)$$

with matrix elements

$$ J_{\alpha\beta} = \frac{\partial j_\alpha}{\partial \rho_\beta} $$  \hspace{1cm} (A2)$$

The transposed matrix is denoted $J^T$.

We define

$$ \delta := (J_{11} - J_{22}) \sqrt{1 + \frac{4J_{12}J_{21}}{(J_{11} - J_{22})^2}} $$  \hspace{1cm} (A3)$$

which is the signed square root of the discriminant of the characteristic polynomial of $J$ with the sign given by $J_{11} - J_{22}$ The two eigenvalues are

$$ v_{\pm} = \frac{1}{2} (J_{11} + J_{22} \pm \delta). $$  \hspace{1cm} (A4)$$

We associate velocity $v_-$ with eigenmode 1 and $v_+$ with eigenmode 2, irrespective of the sign of $v_- - v_+$ which is equal to the sign of $J_{22} - J_{11}$.

The Hessians are denoted

$$ H^{(i)} = \begin{pmatrix} h_1^{(i)} & \bar{h}^{(i)} \\ \bar{h}^{(i)} & h_2^{(i)} \end{pmatrix} $$  \hspace{1cm} (A5)$$

with

$$ h_1^{(i)} = (\partial_1)^2 j_i, \quad h_2^{(i)} = (\partial_2)^2 j_i, \quad \bar{h}^{(i)} = \partial_1 \partial_2 j_i. $$  \hspace{1cm} (A6)$$

They are symmetric by definition.

The compressibility matrix is denoted

$$ K = \begin{pmatrix} \kappa_1 & \bar{\kappa} \\ \bar{\kappa} & \kappa_2 \end{pmatrix} $$  \hspace{1cm} (A7)$$

It is symmetric by definition. Without loss of generality we can assume $\kappa_1\kappa_2 \neq 0$ since a vanishing self-compressibility corresponds to a “frozen” lane without fluctuations which would reduce the dynamics of the two-lane system to a dynamics with a single conservation law. Time-reversal yields the Onsager-type symmetry [36]

$$ JK = KJ^T $$  \hspace{1cm} (A8)$$
which implies

\[ J_{21}\kappa_1 - J_{12}\kappa_2 = (J_{11} - J_{22})\bar{\kappa}. \]  

(A9)

This relation also guarantees that the eigenvalues \( \delta \) of a physical flux Jacobian are generally real.

We point out that for any model with \( \kappa = 0 \), i.e., whenever the stationary distribution factorizes in the conserved quantities, the compressibilities satisfy \( J_{21}\kappa_1 = J_{12}\kappa_2 \). Thus a vanishing cross derivative \( J_{ij} \) for one of the currents implies a vanishing cross derivative \( J_{ji} \) also of the other, without any \textit{a priori} assumption on the stochastic dynamics. The same is true also on parameter manifolds where \( J_{11} = J_{22} \).

2. Normal modes

We focus on the strictly hyperbolic case \( v_+ \neq v_- \) corresponding to \( \delta \neq 0 \). Since \( J \) is not assumed to be symmetric we have to distinguish right (column) and left (row) eigenvectors, denoted by \( \vec{c}^\pm \) and \( \vec{r}^\pm \), respectively. Here

\[
\vec{c}^\pm = \begin{pmatrix} c_1^\mp \\ c_2^\mp \end{pmatrix}, \quad \vec{r}^\pm = \begin{pmatrix} r_1^\pm \\ r_2^\pm \end{pmatrix}.
\]  

(A10)

We normalize them to obtain a biorthogonal basis with scalar product

\[
\vec{r}^\alpha \cdot \vec{c}^\beta := r_1^\alpha c_1^\beta + r_2^\alpha c_2^\beta = \delta_{\alpha,\beta}
\]  

(A11)

with \( \alpha, \beta \in \{\pm\} \). Using

\[
\frac{J_{22} - J_{11} - \delta}{2\sqrt{J_{12}J_{21}}} = \frac{2\sqrt{J_{12}J_{21}}}{J_{11} - J_{22} - \delta}
\]  

(A12)

this yields

\[
\vec{c}^\pm = \frac{1}{2\delta y_\pm} \begin{pmatrix} 2J_{12} \\ J_{22} - J_{11} \pm \delta \end{pmatrix},
\]  

(A13)

\[
\vec{r}^\pm = \frac{y_\pm}{\delta \pm (J_{22} - J_{11})} (2J_{21}, J_{22} - J_{11} \pm \delta)
\]  

(A14)

with arbitrary normalization constants \( y_\pm \).

Next we introduce (bearing in mind that \( \delta \neq 0 \))

\[
R = \begin{pmatrix} r_1^- & r_2^- \\ r_1^+ & r_2^+ \end{pmatrix}, \quad R^{-1} = \begin{pmatrix} c_1^- & c_1^+ \\ c_2^- & c_2^+ \end{pmatrix}.
\]  

(A15)
Biorthogonality and normalization give $RR^{-1} = 1$. The fact that $R$ contains the left eigenvectors as its rows implies $RJ = \Lambda R$ where $\Lambda = \text{diag}(v_-, v_+)$. Therefore

$$RJR^{-1} = \Lambda. \quad (A16)$$

Then the linearized Eulerian hydrodynamic equations read

$$\frac{\partial}{\partial t} \vec{\phi} + \Lambda \frac{\partial}{\partial x} \vec{\phi} = 0 \quad (A17)$$

with $\vec{\phi} = R \vec{u}$.

The diagonalizer $R$ is uniquely defined up to multiplication by an invertible diagonal matrix which is reflected in the arbitrariness of the normalization factors $y_\pm$. In order to fix these constants we first observe that from (A8) it follows that $R(JK)^T = \Lambda(RKR^T) = R(KJ)^T = (RKR^T)\Lambda$. Hence $RKR^T$ must be diagonal since $\Lambda$ is diagonal. This allows us to fix the normalization constants $y_\pm$ by demanding

$$RKR^T = 1. \quad (A18)$$

This normalization condition has its origin in the fact that the structure matrix $S(k, t)$ (whose components are the dynamical structure functions) is by definition normalized such that $\sum_k S(k, t) = K$, see next subsection. For computing the normalization factors we first consider $J_{12}J_{21} \neq 0$.

It is convenient to parametrize $R$ by diagonal matrices $Z = \text{diag}(z_-, z_+)$, $U = \text{diag}(1, u)$ and an orthogonal matrix $O$ such that

$$R = ZOU = \begin{pmatrix} z_- \cos \phi & -uz_- \sin \phi \\
 z_+ \sin \phi & uz_+ \cos \phi \end{pmatrix} \quad (A19)$$

with

$$\tan \phi = \frac{J_{11} - J_{22} + \delta}{2\sqrt{J_{12}J_{21}}}, \quad u = \sqrt{\frac{J_{12}}{J_{21}}}. \quad (A20)$$

Notice that $J_{12}J_{21} \neq 0$ implies $u \neq 0$, $\sin \phi \neq 0$ and $\cos \phi \neq 0$. There are several useful identities involving the rotation angle $\phi$, viz. $\tan(2\phi) = 2\sqrt{J_{12}J_{21}}/(J_{22} - J_{11})$, $\sqrt{J_{12}J_{21}}(\cos^2 \phi - \sin^2 \phi) = (J_{22} - J_{11}) \cos \phi \sin \phi$ and $\delta = (J_{22} - J_{11})(\cos^2 \phi - \sin^2 \phi) + 4\sqrt{J_{12}J_{21}} \cos \phi \sin \phi = (J_{22} - J_{11}) \cos(2\phi) + 2\sqrt{J_{12}J_{21}} \sin 2\phi = (J_{22} - J_{11})/\cos(2\phi) = 2\sqrt{J_{12}J_{21}}/\sin(2\phi)$. 

Now we use that for $\kappa \neq 0$ one can write
\[
UJU^{-1} = \mu UKU + \nu \mathbf{1}
\]  
(A21)
with
\[
\mu = \frac{J_{21}}{\kappa}, \quad \nu = \frac{1}{2} \left( J_{11} + J_{22} - \frac{J_{21}\kappa_1 + J_{12}\kappa_2}{\kappa} \right).
\]  
(A22)
Therefore
\[
RKR^T = ZOUKUO^T Z = \frac{1}{\mu} (ZOUJU^{-1}O^T Z - \nu Z^2)
\]
\[
= \frac{1}{\mu} \begin{pmatrix}
    v_- - z_-^2 \nu & 0 \\
    0 & v_+ - z_+^2 \nu
\end{pmatrix}
\]  
(A23)
which yields
\[
z_\pm^2 = \frac{v_\pm - \mu}{\nu}. 
\]  
(A24)
By comparing with (A15) one finds that the normalization factors for the eigenvectors are given by $y_- = uz_- \sin \phi$, $y_+ = uz_+ \cos \phi$. For $\bar{\kappa} = 0$ one obtains directly from (A9) and (A19) that
\[
y_-^2 = \frac{\sin^2 \phi}{\kappa_2}, \quad y_+^2 = \frac{\cos^2 \phi}{\kappa_2}. 
\]  
(A25)

Even though not relevant for the two-lane model of this paper we mention for completeness that some care with limits has to be taken when $J_{12}J_{21} = 0$. First notice that in this case the physical requirement $\kappa_1\kappa_2 \neq 0$ implies $\bar{\kappa} \neq 0$. Specifically for $J_{12} = 0$, $J_{21} \neq 0$ one has $\delta = J_{11} - J_{22}$, $v_- = J_{11}$, $v_+ = J_{22}$, $J_{21}\kappa_1 = (J_{11} - J_{22})\bar{\kappa}$ and
\[
R = \begin{pmatrix}
    \tilde{z}_- & 0 \\
    \frac{\tilde{z}_+ J_{21}}{J_{22} - J_{11}} & \tilde{z}_+
\end{pmatrix}
\]  
(A26)
with
\[
\tilde{z}_-^{-2} = \kappa_1, \quad \tilde{z}_+^{-2} = \kappa_2 - \frac{\bar{\kappa}^2}{\kappa_1}.
\]  
(A27)
Notice that here strict hyperbolicity implies $J_{11} \neq J_{22}$ so that $R$ is well-defined.

Similarly one obtains for $J_{21} = 0$, $J_{12} \neq 0$ with $v_- = J_{11} \neq v_+ = J_{22}$ the relation $J_{12}\kappa_2 = (J_{22} - J_{11})\bar{\kappa}$ and
\[
R = \begin{pmatrix}
    \tilde{z}_- & \frac{\tilde{z}_- J_{12}}{J_{11} - J_{22}} \\
    0 & \tilde{z}_+
\end{pmatrix}
\]  
(A28)
with
\[
\hat{z}_-^{-2} = \kappa_1 - \frac{\bar{\kappa}^2}{\kappa_2}, \quad \hat{z}_+^{-2} = \kappa_2.
\] (A29)

If \( J_{12} = J_{21} = 0 \) then \( J \) is diagonal. For the strictly hyperbolic case \( J_{11} \neq J_{22} \) one necessarily has \( \bar{\kappa} = 0 \) and the normalization condition \( \text{(A18)} \) yields \( R = \text{diag}(\kappa_1^{-1}, \kappa_2^{-1}) \).

3. Normal modes and the microscopic dynamical structure function

In order to explain the origin of the normalization condition and to apply it to the two-lane model we define \( f_k^{(i)}(t) := n_k^{(i)}(t) - \rho_i \) and \( f_0^{(i)} := f_0^{(i)}(0) \) where the random variable \( n_k^{(i)}(t) \) is the particle number on site \( k \) of lane \( i \) at time \( t \). We also define the two-component column vector \( \vec{f}_k(t) \) with components \( f_k^{(i)}(t) \) and the two-component row vector \( \vec{f}_0^T := (f_0^{(1)}, f_0^{(2)}) \).

Expectation w.r.t. the stationary distribution is denoted by \( \langle \cdot \rangle \). The expectation of a matrix is understood as the matrix of the expectations of its components. Then the dynamical structure matrix with components \( \text{(III)} \) can be written \( S_k(t) = \langle \hat{S}_k(t) \rangle \) with \( \hat{S}_k(t) = \vec{f}_k(t) \otimes \vec{f}_0^T \).

The normalization of the dynamical structure matrix, defined by the sum over the whole lattice, is given by
\[
\sum_k S_k(t) = K. \quad \text{(A30)}
\]
It is independent of time because of particle number conservation. Now we consider the lattice normal modes
\[
\vec{\psi}_k(t) = R\vec{f}_k(t)
\] (A31)
with components \( \psi_k^{(i)}(t) \) where \( R \) is the diagonalizer \( \text{(A15)} \). In components
\[
\psi_k^{(1)}(t) = r_{11}f_k^{(1)}(t) + r_{12}f_k^{(2)}(t), \quad \psi_k^{(2)}(t) = r_{21}f_k^{(1)}(t) + r_{22}f_k^{(2)}(t)
\] (A32)
and similarly \( \psi_0^{(i)} := \psi_0^{(i)}(0) \). In terms of the lattice normal modes the structure matrix has the form \( \tilde{S}_k(t) := \langle \hat{S}_k(t) \rangle \) with \( \hat{S}_k(t) := \vec{\psi}_k(t) \otimes \vec{\psi}_0^T \). This yields
\[
\tilde{S}_k(t) = RS_k(t)R^T \quad \text{(A33)}
\]
with matrix elements \( \tilde{S}_{ij}(k,t) = \langle \psi_k^{(i)}(t)\psi_0^{(j)} \rangle \). The normalization
\[
\sum_k \tilde{S}_k(t) = RK R^T \quad \text{(A34)}
\]
leads to the requirement \( \text{(A18)} \).
4. Computation of the mode-coupling matrices

The mode-coupling coefficients are given by

\[ G^{(\gamma)}_{\alpha\beta} := \frac{1}{2} \sum_{\lambda} R_{\gamma\lambda} [(R^{-1})^T H^{(\lambda)} R^{-1}]_{\alpha\beta}. \]  

(A35)

where \( G^{(\gamma)}_{\alpha\beta} G^{(\gamma)}_{\beta\alpha} \). Using the previous results one finds for \( G^{(1)} \) the matrix elements

\[ G^{(1)}_{11} = \frac{1}{2 z_-} \left[ \cos^2 \phi \left( h_1^{(1)} \cos \phi - u h_1^{(2)} \sin \phi \right) + u^{-2} \sin^2 \phi \left( h_2^{(1)} \cos \phi - u h_2^{(2)} \sin \phi \right) - 2 u^{-1} \cos \phi \sin \phi \left( \bar{h}^{(1)} \cos \phi - \bar{u} h^{(2)} \sin \phi \right) \right], \]

(A36)

\[ G^{(1)}_{22} = \frac{z_-}{2 z_+} \left[ \sin^2 \phi \left( h_1^{(1)} \cos \phi - u h_1^{(2)} \sin \phi \right) + u^{-2} \cos^2 \phi \left( h_2^{(1)} \cos \phi - u h_2^{(2)} \sin \phi \right) + 2 u^{-1} \cos \phi \sin \phi \left( \bar{h}^{(1)} \cos \phi - \bar{u} h^{(2)} \sin \phi \right) \right], \]

(A37)

\[ G^{(1)}_{12} = \frac{1}{2 z_+} \left[ \cos \phi \sin \phi \left( h_1^{(1)} \cos \phi - u h_1^{(2)} \sin \phi - u^{-2} h_1^{(1)} \cos \phi + u^{-1} h_2^{(2)} \sin \phi \right) - u^{-1} (\cos^2 \phi - \sin^2 \phi) \left( \bar{h}^{(1)} \cos \phi - \bar{u} h^{(2)} \sin \phi \right) \right], \]

(A38)

and for \( G^{(2)} \) one has

\[ G^{(2)}_{22} = \frac{1}{2 z_+} \left[ \sin^2 \phi \left( h_1^{(1)} \sin \phi + u h_1^{(2)} \cos \phi \right) + u^{-2} \cos^2 \phi (h_2^{(1)} \sin \phi + u h_2^{(2)} \cos \phi) + 2 u^{-1} \cos \phi \sin \phi \left( \bar{h}^{(1)} \sin \phi + \bar{u} h^{(2)} \cos \phi \right) \right], \]

(A39)

\[ G^{(2)}_{11} = \frac{z_+}{2 z_-} \left[ \cos^2 \phi \left( h_1^{(1)} \sin \phi + u h_1^{(2)} \cos \phi \right) + u^{-2} \sin^2 \phi \left( h_2^{(1)} \sin \phi + h_2^{(2)} u \cos \phi \right) - 2 u^{-1} \cos \phi \sin \phi \left( \bar{h}^{(1)} \sin \phi + \bar{u} h^{(2)} \cos \phi \right) \right], \]

(A40)

\[ G^{(2)}_{12} = \frac{1}{2 z_-} \left[ \cos \phi \sin \phi \left( h_1^{(1)} \sin \phi + u h_1^{(2)} \cos \phi - u^{-2} h_1^{(1)} \sin \phi + u^{-1} h_2^{(2)} \cos \phi \right) - u^{-1} (\cos^2 \phi - \sin^2 \phi) \left( \bar{h}^{(1)} \sin \phi + \bar{u} h^{(2)} \cos \phi \right) \right]. \]

(A41)

In terms of the model parameters \( a, b, c, d, \kappa_{1,2}, \bar{\kappa} \) the quantities \( \phi \) and \( u \) are given in [A20] and the quantities \( z_\pm \) are given in [A24]. The parameter \( \delta \) appearing in [A20] is given in [A3].

In order to analyze the manifolds where diagonal elements of the mode coupling matrices vanish it is convenient to introduce

\[ g^{(1)}_1 := h_1^{(1)}, \quad g^{(1)}_2 := u^{-2} h_2^{(1)}, \quad \bar{g}^{(1)} := u^{-1} \bar{h}^{(1)} \]

(A42)

\[ g^{(2)}_1 := u h_1^{(2)}, \quad g^{(2)}_2 := u^{-1} h_2^{(2)}, \quad \bar{g}^{(2)} := \bar{h}^{(2)}. \]

(A43)
and define the polynomials
\[
D^{(1)}_1(\omega) := g^{(1)}_1 - \left( g^{(2)}_1 + 2g^{(1)}_1 \right) \omega + \left( g^{(1)}_2 + 2g^{(2)}_1 \right) \omega^2 - g^{(2)}_2 \omega^3 \quad (A44)
\]
\[
D^{(1)}_2(\omega) := g^{(1)}_2 - \left( g^{(2)}_2 - 2g^{(1)}_1 \right) \omega + \left( g^{(1)}_1 - 2g^{(2)}_1 \right) \omega^2 - g^{(2)}_1 \omega^3 \quad (A45)
\]
\[
D^{(2)}_1(\omega) := g^{(2)}_1 + \left( g^{(1)}_1 - 2g^{(2)}_1 \right) \omega + \left( g^{(2)}_2 - 2g^{(1)}_1 \right) \omega^2 + g^{(1)}_2 \omega^3 \quad (A46)
\]
\[
D^{(2)}_2(\omega) := g^{(2)}_2 + \left( g^{(1)}_2 + 2g^{(2)}_1 \right) \omega + \left( g^{(2)}_1 + 2g^{(1)}_1 \right) \omega^2 + g^{(1)}_1 \omega^3. \quad (A47)
\]

with \( \omega := \tan \phi \). Only the Hessian and the parameters \( u \) and \( \tan \phi \) given in (A20) enter these functions. They do not depend on the compressibilities. Then one has

\[
G^{(1)}_{11} = \frac{\cos^3 \phi}{2z_-} D^{(1)}_1(\omega), \quad G^{(2)}_{22} = \frac{z_- \cos^3 \phi}{2z_+^2} D^{(1)}_2(\omega) \quad (A48)
\]
\[
G^{(1)}_{11} = \frac{z_+ \cos^3 \phi}{2z_-^2} D^{(2)}_1(\omega), \quad G^{(2)}_{22} = \frac{\cos^3 \phi}{2z_+} D^{(2)}_2(\omega). \quad (A49)
\]

Notice the symmetry properties \( D^{(1)}_1(\omega) = -\omega^3 D^{(2)}_2(-\omega^{-1}) \) and \( D^{(1)}_2(\omega) = -\omega^3 D^{(1)}_2(-\omega^{-1}) \).

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