On the Maximal Diversity Order of Spatial Multiplexing with Transmit Antenna Selection

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Abstract

Zhang et. al. recently derived upper and lower bounds on the achievable diversity of an \( N_R \times N_T \) i.i.d. Rayleigh fading multiple antenna system using transmit antenna selection, spatial multiplexing and a linear receiver structure. For the case of \( L = 2 \) transmitting (out of \( N_T \) available) antennas the bounds are tight and therefore specify the maximal diversity order. For the general case with \( L \leq \min(N_R, N_T) \) transmitting antennas it was conjectured that the maximal diversity is \( (N_T - L + 1)(N_R - L + 1) \) which coincides with the lower bound. Herein, we prove this conjecture for the zero forcing and zero forcing decision feedback (with optimal detection ordering) receiver structures.

Index Terms

Diversity, Antenna Selection, Spatial Multiplexing, Zero Forcing Receiver.

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I. INTRODUCTION

The multiple antennas in a multiple input-multiple output (MIMO) wireless system can be used either to increase the data rate or reliability (diversity) of the wireless link [1]. In order to capitalize on the benefits offered by the MIMO wireless link while maintaining manageable complexity and cost the use of antenna selection has been previously suggested [2]. In a system using antenna selection only a small subset of the available antennas would typically be used, thereby limiting the number of RF chains required.

In [3] Zhang et. al. rigorously analyzed the maximal achievable diversity for a system transmitting $L$ independent data-streams from $L$ out of $N_T$ possible transmit antennas in conjunction with linear (decision feedback) processing at the receiver. In particular, for the case of a block i.i.d. Rayleigh fading channel it was shown that the maximal diversity of such a system is bounded between $M_L \triangleq (N_T-L+1)(N_R-L+1)$ and $M_U \triangleq (N_T-L+1)(N_R-L)$. Since $M_L = M_U$ for $L = 2$ these bounds uniquely determine the maximal diversity in the case of two transmitting antennas and thereby analytically prove some previous observations made in the literature [4], [5]. Further, for the general case where $2 < L < \min(N_R, N_T)$ it was in [3] conjectured that the maximal diversity coincides with the lower bound, $M_L$. Herein, we extend the analysis of [3] by proving this conjecture for the case of the zero forcing (ZF) and ZF-decision-feedback (DF) receivers (with optimal detection ordering). It should however be noted that the cases of the minimum mean square error (MMSE) and MMSE-DF receivers (although with a fixed detection ordering) also follow from our result by applying the analysis in [3].

The structure of this correspondence is as follows. The system model considered is covered in Section II mainly in order to introduce notation. The reader is referred to [3] for details regarding the systems model and a proper motivation of the problem considered. Our main contribution is then given in Section III in the form of Theorem I.
to columns of $\mathbf{H}$) and transmits independently coded data streams from each antenna. As in [3], let $U_j$ denote the $j$th possible antenna subset where

$$U_1 = \{h_1, h_2, \ldots, h_L\}$$
$$U_2 = \{h_1, h_2, \ldots, h_{L-1}, h_{L+1}\}$$
$$\vdots$$
$$U_{N_U} = \{h_{N_T-L+1}, \ldots, h_{N_T}\}$$

and where $N_U = \binom{N_T}{L}$ is the total number of such subsets. The channel can then be modeled according to

$$y = \sqrt{\frac{\rho_0}{L}} \mathbf{H}_j \mathbf{s} + \mathbf{n}.$$  \hspace{1cm} (2)

where in the above; $\mathbf{H}_j \in \mathbb{C}^{N_R \times L}$ is the channel matrix containing the columns in the selected subset $U_j$; where $y \in \mathbb{C}^{N_R \times T}$ is the signal block received during $T$ channel uses; where $\mathbf{s} \in \mathbb{C}^{L \times T}$ is the transmitted signal block; and where $\mathbf{n} \in \mathbb{C}^{N_R \times T}$ is the circularly symmetric complex Gaussian noise which is assumed spatially and temporally white and of unit variance.

At the receiver, a ZF front-end is used to separate the transmitted data streams according to

$$\tilde{\mathbf{s}} = \mathbf{H}_j^\dagger y = \sqrt{\frac{\rho_0}{L}} \mathbf{s} + \tilde{\mathbf{n}}$$  \hspace{1cm} (3)

where $\mathbf{H}_j^\dagger = (\mathbf{H}_j^\dagger \mathbf{H}_j)^{-1} \mathbf{H}_j^\dagger$ is the pseudo-inverse of $\mathbf{H}_j$. Since $L \leq N_R$ by assumption it follows that $\mathbf{Q}_j = \mathbf{H}_j^\dagger \mathbf{H}_j$ is invertible with probability one. The effective noise, $\tilde{\mathbf{n}}$, is spatially colored with covariance $\mathbf{Q}_j^{-1}$ and the effective post-processing signal to noise ratio (SNR) of the $k$th data stream is given by

$$\rho_k^{(j)} = \left(\frac{\rho_0}{L}\right) / [\mathbf{Q}_j^{-1}]_{kk}$$  \hspace{1cm} (4)

where $1 \leq k \leq L$ [1], [3]. A given data stream, $k$, is said to be in outage if the post-processing SNR drops below a given threshold, $\gamma > 0$ and the diversity order, $d_k^{(j)}$, of this stream is defined according to

$$d_k^{(j)} = \lim_{\rho_0 \to \infty} \frac{\ln \mathbb{P}(\rho_k^{(j)} \leq \gamma)}{\ln \rho_0^{-1}}.$$  \hspace{1cm} (5)

Similary, let $\bar{\rho}^{(j)}$ and $\underline{\rho}^{(j)}$ denote the maximal and minimal post-processing SNRs defined according to

$$\bar{\rho}^{(j)} \triangleq \max_{1 \leq k \leq L} \rho_k^{(j)} \quad \text{and} \quad \underline{\rho}^{(j)} \triangleq \min_{1 \leq k \leq L} \rho_k^{(j)}.$$  \hspace{1cm} (6)

Note also that $\underline{\rho}^{(j)} \leq \bar{\rho}^{(j)}$ and that $\bar{\rho}^{(j)} \leq \gamma$ imply that all streams are simultaneously in outage. Thus,

$$d_k^{(j)} \leq d^{(j)} \triangleq \limsup_{\rho_0 \to \infty} \frac{\ln \mathbb{P}(\bar{\rho}^{(j)} \leq \gamma)}{\ln \rho_0^{-1}}.$$  \hspace{1cm} (6)
and
\[ d_k^{(j)} \geq \tilde{d}_k^{(j)} \triangleq \liminf_{\rho_0 \to \infty} \frac{\ln P(\rho^{(j)} \leq \gamma)}{\ln \rho_0^{-1}} \]  
provides upper and lower bounds on the diversity order of the ZF receiver. It also provides upper and lower bounds on the ZF-DF receiver with optimal ordering since if \( \tilde{\rho}^{(j)} \leq \gamma \) no data can be reliably decoded and the first data stream decoded is likely to be in error, regardless of the detection ordering policy applied. Similarly, if \( \rho^{(j)} \geq \gamma \) all streams can be reliably decoded and (7) therefore provides lower bounds on the diversity of the ZF and ZF-DF receivers. The reader is referred to [3] for additional details.

B. Problem statement

In general terms, an antenna selection policy is characterized by some (measurable) function \( \varphi \)
\[ \varphi : \mathbb{C}^{N_T \times N_R} \mapsto \{1, 2, \ldots, N_U\} \]
which selects an antenna subset, \( U_j \), based on the channel matrix realization, \( H \), according to \( j = \varphi(H) \).
In [3] it is shown that there exists an antenna selection policy, \( j = \varphi(H) \), for which
\[ \underline{d}^{(j)} = (N_T - L + 1)(N_R - L + 1) . \]
This bound is also shown to be tight in the case where \( L = 2 \) using a geometrical approach. Further, the bound is conjectured to be tight when \( L > 2 \). Herein, we confirm this conjecture in a positive sense by proving that
\[ \bar{d}^{(j)} \leq (N_T - L + 1)(N_R - L + 1) \]
for any antenna selection policy, \( \varphi \). The proof is given in the following section.

III. PROOF OF CONJECTURE

In the proof, we let \( \succeq \) denote the partial matrix ordering induced by the positive semi-definite (PSD) cone [6]. For hermitian matrices \( A \) and \( B \), \( A, B \in \mathbb{C}^{n \times n} \), we write \( A \succeq B \) to denote that \( A - B \) is PSD. In particular, we will use that \( [A]_{kk} \geq [B]_{kk} \) whenever \( A \succeq B \) and where \( [A]_{kk} \) and \( [B]_{kk} \) denotes the \( k \)th diagonal value of \( A \) and \( B \). Also, \( A \succeq B \) is equivalent to \( A^{-1} \preceq B^{-1} \) for strictly positive definite matrices \( A \) and \( B \) and \( A \succeq 0 \) if and only if all principal sub-matrices of \( A \) are PSD [7].

We are now ready to state and prove the contribution of this work which is given by Theorem 1 below. Note also that the theorem yields the recently proved [8], [9] statement that detection ordering can not improve the ZF-DF diversity order as a special case by selecting \( L = N_T \leq N_R \). It should also be noted
that the proof of Theorem 1 is similar to a recently submitted proof [10] of this statement but that the antenna selection case represents a non-trivial extension.

**Theorem 1:** Given an arbitrary antenna selection policy \( j = \varphi(H) \) let \( \bar{d}^{(j)} \) be defined as in (6). Then

\[
\bar{d}^{(j)} \leq (N_T - L + 1)(N_R - L + 1).
\]  

(8)

**Proof:** Let \( Q \Delta H^H H \), \( Q_j \Delta H_j^H H_j \) and note that \( Q_j \) is an \( L \times L \) principal sub-matrix of \( Q \). Further, let the eigenvalue decomposition of \( Q \) be given by

\[
Q = U \Lambda U^H
\]

where \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_{N_T}) \) are the ordered eigenvalues, \( \lambda_1 \geq \ldots \geq \lambda_{N_T} \), of \( Q \) and where \( U = [u_1 \ \ldots \ u_{N_T}] \) are the corresponding eigenvectors. Since \( Q \) is unitarily invariant it can be assumed that \( U \) is a Haar matrix and independent of \( \Lambda \) [11]. Let \( V \Delta [u_1 \ \ldots \ u_{L-1}] \) and note that

\[
Q = \sum_{i=1}^{N_T} \lambda_i u_i u_i^H \leq \sum_{i=1}^{L-1} \lambda_1 u_i u_i^H + \lambda_L I = \lambda_1 VV^H + \lambda_L I.
\]

Let

\[
S \Delta \frac{\lambda_1}{\lambda_L} VV^H + I.
\]

and let \( S_j \) be the \( L \times L \) principal submatrix of \( S \) obtained by selecting the rows and columns corresponding to antenna subset \( j \). Note that since \( Q \preceq \lambda_L S \) it follows that \( Q_j \preceq \lambda_L S_j \) and in particular \( Q_j^{-1} \succeq \lambda_L^{-1} S_j^{-1} \) which implies that \([Q_j^{-1}]_{kk} \geq \lambda_L^{-1} [S_j^{-1}]_{kk} \) for \( k = 1, \ldots, L \).

Let \( V_j \in \mathbb{C}^{L \times (L-1)} \) be the matrix consisting of the \( L \) rows of \( V \) corresponding to antenna subset \( j \). Note also that \( S_j = \frac{\lambda_1}{\lambda_L} V_j V_j^H + I \). By the matrix inversion lemma it follows that

\[
S_j^{-1} = (\frac{\lambda_1}{\lambda_L} V_j V_j^H + I)^{-1}
= I - V_j (\frac{\lambda_1}{\lambda_L} I + V_j^H V_j)^{-1} V_j^H.
\]  

(9)

As \( \lambda_1 \geq \lambda_L \geq 0 \) it follows that

\[
\frac{\lambda_1}{\lambda_L} I + V_j^H V_j \succeq V_j^H V_j
\]

and therefore

\[
(\frac{\lambda_1}{\lambda_L} I + V_j^H V_j)^{-1} \preceq (V_j^H V_j)^{-1}
\]  

(10)

which is equivalent to

\[
-(\frac{\lambda_1}{\lambda_L} I + V_j^H V_j)^{-1} \preceq -(V_j^H V_j)^{-1}.
\]  

(11)
Note also that the inverse on the right hand side of (10) exists with probability one due the unitary invariance of $V$ (the probability that any $L$ rows are linearly dependent is zero). Now, inserting (11) into (9) yields

$$S_j^{-1} \succeq I - V_jV_j^HV_j^{-1} \equiv P_j.$$ (12)

In the above, $P_j$ corresponds to a projection onto the null-space of $V_j^H$ (which has dimension one since $V_j \in \mathbb{C}^{L \times L}$). Note also that for a fixed $j$ (independent of $H$) the distribution of $V_j$ is invariant to multiplication from the right by $L \times L$ unitary matrices. This follows from the unitary invariance of $U$ (and $V$). Therefore, the null-space of $V_j^H$ is unitarily invariant and

$$P([P_j]_{kk} = 0) = 0$$

for fixed $j$ and $k$ since $[P_j]_{kk} = e_k^HP_je_k$ is the squared length of the projection of the $k$th natural basis vector, $e_k$, onto the null-space of $V_j$ (the probability that $e_k$ is completely orthogonal to the null-space is zero). Since there are a finite number of possible $k$ and $j$ it follows that

$$P(\exists k, j [P_j]_{kk} = 0) = 0.$$ (13)

From (13) it follows that there is some constant, $\kappa > 0$, for which

$$P(\exists k, j [P_j]_{kk} < \kappa) < 1$$
or equivalently for which

$$P(\forall k, j [P_j]_{kk} \geq \kappa) > 0.$$ (14)

In particular, for $j = \varphi(H)$, it follows that

$$P([P_j]_{kk} \geq \kappa, k = 1, \ldots, L) > 0$$

which states that the probability that all diagonal values of $P_j$ are simultaneously large (in the sense that they are bounded away from zero) is strictly positive.

For notational conveniens in the following, let

$$\tau \equiv \min_{k,j}[P_j]_{kk}.$$

Since

$$[Q_j^{-1}]_{kk} \geq \lambda_L^{-1}[S_j^{-1}]_{kk} \geq \lambda_L^{-1}[P_j]_{kk} \geq \lambda_L^{-1}\tau$$

it follows, by (14), that

$$\rho_k^{(j)} = \left(\frac{\rho_0}{\tau L}\right) / [Q_j^{-1}]_{kk} \leq \frac{\lambda_L\rho_0}{\tau L}.$$
for $k = 1, \ldots, L$. Thus, if $\gamma \geq \kappa$ and $\lambda L \leq \kappa \gamma L \rho_0^{-1}$ it follows that $\rho_k^{(j)} \leq \gamma$ for $k = 1, \ldots, L$. This implies

$$P(\bar{\rho}^{(j)} < \gamma) \leq P(\lambda L \leq \kappa \gamma L \rho_0^{-1} \cap \kappa \leq \tau)$$

$$= P(\lambda L \leq \kappa \gamma L \rho_0^{-1}) P(\kappa \leq \tau)$$

where the last equality follows by the independence of $\tau$ (which is a function of $U$) and $\lambda L$. This implies

$$\frac{\ln P(\bar{\rho}^{(j)} < \gamma)}{\ln \rho_0^{-1}} \leq \frac{\ln P(\lambda L \leq \kappa \gamma L \rho_0^{-1})}{\ln \rho_0^{-1}} + \frac{\ln P(\kappa \leq \tau)}{\ln \rho_0^{-1}}$$

where

$$\limsup_{\rho_0 \to \infty} \frac{\ln P(\kappa \leq \tau)}{\ln \rho_0^{-1}} = 0$$

due to (14) and since $P(\kappa \leq \tau) > 0$ does not depend on $\rho_0$. Thus,

$$\bar{d}^{(j)} \triangleq \limsup_{\rho_0 \to \infty} \frac{\ln P(\bar{\rho}^{(j)} < \gamma)}{\ln \rho_0^{-1}}$$

$$\leq \limsup_{\rho_0 \to \infty} \frac{\ln P(\lambda L \leq \kappa \gamma L \rho_0^{-1})}{\ln \rho_0^{-1}} + \limsup_{\rho_0 \to \infty} \frac{\ln P(\kappa \leq \tau)}{\ln \rho_0^{-1}}$$

$$= \limsup_{\rho_0 \to \infty} \frac{\ln P(\lambda L \leq \rho_0^{-1})}{\ln \rho_0^{-1}}$$

$$= (N_T - L + 1)(N_R - L + 1)$$

where the last equality follows from [12, Equation (17)] or as a special case of [13, Equation (15)]. This completes the proof and established the assertion made by the theorem.

IV. CONCLUSIONS

We have proved the conjecture of Zhang et. al. in [3] regarding the diversity order of spatial multiplexing systems with transmit antenna selection.

REFERENCES

[1] D. Tse and P. Viswanath, Fundamentals of Wireless Communication. Cambridge University Press, 2005.

[2] A. F. Molisch and M. Z. Win, “Mimo systems with antenna selection,” IEEE Microwave Magazine, vol. 5, no. 1, pp. 46–56, Mar. 2004.

[3] H. Zhang, H. Dai, Q. Zhou, and B. L. Hughes, “On the diversity order of spatial multiplexing systems with transmit antenna selection: A geometrical approach,” IEEE Transactions on Information Theory, vol. 52, no. 12, pp. 5297–5311, Dec. 2006.

[4] R. W. Heath Jr., S. Sandhu, and A. Paulraj, “Antenna selection for spatial multiplexing systems with linear receivers,” IEEE Communications Letters, vol. 5, no. 4, pp. 142–144, Apr. 2001.
[5] R. W. Heath Jr. and A. Paulraj, “Antenna selection for spatial multiplexing systems based on minimum error rate,” in Proc. IEEE International Conference on Communications, ICC, 2001.

[6] S. Boyd and L. Vandenberghe, Convex Optimization. Cambridge University Press, 2004.

[7] R. A. Horn and C. R. Johnson, Matrix Analysis. Cambridge University Press, 1985.

[8] Y. Jiang, X. Zheng, and J. Li, “Asymptotic performance analysis of V-BLAST,” in Proc. IEEE Global Telecommunications Conference, GLOBECOM, 2005.

[9] H. Zhang, H. Dai, and B. L. Hughes, “On the diversity-multiplexing tradeoff for ordered SIC receivers over MIMO channels,” in Proc. IEEE International Conference on Communications, ICC, 2006.

[10] Y. Jiang, M. K. Varanasi, X. Zheng, and J. Li, “Performance analysis of V-BLAST at high SNR regime,” IEEE Transactions on Wireless Communications, Dec. 2005, submitted.

[11] A. Tulino and S. Verdú, “Random matrix theory and wireless communications,” Foundations and Trends in Communications and Information Theory, vol. 1, no. 1, June 2004.

[12] E. Sengul, E. Akay, and E. Ayanoglu, “Diversity analysis of single and multiple beamforming,” in Proc. IEEE Vehicular Technology Conference, VTC, Spring, 2005.

[13] L. Zheng and D. N. C. Tse, “Diversity and multiplexing: A fundamental tradeoff in multiple-antenna channels,” IEEE Transactions on Information Theory, vol. 49, no. 5, pp. 1073–1096, May 2003.