Affine perturbation series of the deflection angle of a ray near the photon sphere of a Reissner-Nordström black hole

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We investigate the affine perturbation series of the deflection angle of a ray near the photon sphere of an extreme Reissner-Nordström black hole. We compare the 0th and 1st orders of the affine perturbation series with the deflection angle in a strong deflection limit. We conclude that the 0th order of the affine perturbation series is more accurate than the deflection angle in the strong deflection limit and that we should improve a strong-deflection-limit analysis by using the 0th order of the affine perturbation series.

I. INTRODUCTION

Recently, LIGO Scientific and Virgo Collaborations have reported the detection of gravitational waves from a black hole binary [1]; Event Horizon Telescope Collaboration [2, 3] have reported the detection of the shadows of black hole candidates in the center of galaxies; and Wilkins et al. [4] have reported the detection of gravitational waves from a black hole binary [1]; Event Horizon Telescope Collaboration [2, 3] have reported the detection of the shadows of black hole candidates in the center of galaxies; and Wilkins et al. [4] have reported the detection of x-ray echoes caused by a strong gravity near a black hole candidate. Theoretical investigations on the strong gravity due to black holes will be important as the details of the black hole systems are observed.

Compact objects such as black holes and wormholes have a photon sphere formed by unstable circular photon orbits due to the strong gravity. The image of rays reflected near the photon sphere in a Schwarzschild spacetime was studied by Hagiwara in 1931 [18] and by Darwin in 1959 [19], and the image by the photon sphere has been revived several times [20–42]. Bozza expressed the deflection angle of a ray near the photon sphere in a strong deflection limit

where \( \bar{a} \) and \( \bar{b} \) were obtained as \( \bar{a} = 1 \) and \( \bar{b} = \log\left(\frac{b}{b_m} - 1\right) + \bar{b} + O(b - b_m) \), (1.1)

where \( \bar{a} \) and \( \bar{b} \) were obtained as \( \bar{a} = 1 \) and \( \bar{b} = \log\left(\frac{b}{b_m} - 1\right) + \pi \) in the Schwarzschild spacetime [28]. Note that the same deflection angle was obtained by using a radial coordinate by Darwin [19] and by Bozza et al. [21].

Eiroa et al. investigated a numerical method to obtain \( \alpha_{\text{SDL}} \) in an asymptotically-flat, static, and spherically symmetric spacetime with a photon sphere [43]. In Ref. [28], Bozza investigated the deflection angle \( \alpha_{\text{SDL}} \) in the strong deflection limit \( b \to b_m + 0 \), in a general asymptotically-flat, static, and spherically symmetric spacetime with the photon sphere. In Bozza’s method, \( \bar{a} \) is obtained in an analytical expression, but \( \bar{b} \) is often not obtained in an analytical form because a term cannot be integrated analytically. The analytical form of \( \bar{b} \) has been obtained only in a few spacetimes: The Schwarzschild spacetime [24, 28], a braneworld black hole spacetime [44], and five-dimensional and seven-dimensional Tangherlini spacetimes [45] were obtained by using Bozza’s method.

The strong-deflection-limit analysis was extended by several authors and was applied to many spacetimes and lens configurations [35, 37, 41, 43, 46–66, 80]. Tsukamoto extended Bozza’s method [28] to apply directly to ultrastatic spacetimes with the constant norm of a time-translational Killing vector [35, 55] and claimed that the error term \( O(b - b_m) \) in Eq. (1.1) should be read as \( O((b/b_m - 1) \log(b/b_m - 1)) \). By using the alternative method, the analytic expressions of \( \bar{a} \) and \( \bar{b} \) for an Ellis-Bronnikov wormhole [35, 53] and a spatial Schwarzschild wormhole [35] were obtained. Tsukamoto generalized the alternative method to a general asymptotically-flat, static, and spherically symmetric spacetime with the photon sphere [58]. The analytic expressions of \( \bar{a} \) and \( \bar{b} \) in a Reissner-Nordström spacetime [57, 58] and in Kerr and Kerr-Newman spacetimes [61] were calculated by the method.

Iyer and Petters [53] considered the affine perturbation series of the deflection angle of the ray near the photon sphere of the Schwarzschild black hole in the following form:

where \( b_p \) is defined as

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\[ b_p = 1 - \frac{b_m}{b} \]
where \( \sigma_0, \sigma_1, \sigma_2, \sigma_3, \lambda_0, \rho_0, \rho_1, \rho_2, \) and \( \rho_3 \) are coefficients. The affine perturbation series of the deflection angle close the gap that developed between the exact deflection angle and the one in the strong deflection limit. They showed that the 0th order of the affine perturbation series \( \alpha_{0th} \) defined by
\[
\alpha_{0th} = \sigma_0 \log \left( \frac{\lambda_0}{b_0} \right) + \rho_0
\]
does not correspond with the deflection angle \( \alpha_{SDL} \) in the strong deflection limit by Darwin \[10\] and by Bozza \[28\] and that the 0th order of the affine perturbation series \( \alpha_{0th} \) is more precise than \( \alpha_{SDL} \).

The Reissner-Nordström spacetime, which is obtained as an electrovacuum solution of Einstein equations, is often used as the toy model of compact objects with a richer structure than the Schwarzschild spacetime since it can be tractable in analytical calculations. The shadow image \[2, 69–73\], gravitational lensing \[28, 43, 48, 57, 58\], \[74, 75\], and time delay \[76\] in the Reissner-Nordström image can be tractable in analytical calculations. The shadow often used as the toy model of compact objects with a charged Reissner-Nordström black hole with a mass \( M \) and an electrical charge \( Q \) \[3\] as an electrovacuum solution of Einstein equations, is
\[
A(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}.
\]

It has an event horizon at \( r = M + \sqrt{M^2 - Q^2} \) for \( |Q| \leq M \). There are time translational and axial Killing vectors \( t^\mu \partial_\mu = \partial_t \) and \( \phi^\mu \partial_\mu = \partial_\phi \) because of the stationarity and axisymmetry of the spacetime, respectively. We can assume \( \theta = \pi/2 \) and \( Q \geq 0 \) without loss of generality.

From \( k^\mu k_\mu = 0 \), where \( k^\mu \equiv \dot{x}^\mu \) is the wave-number vector of a ray and the dot denotes a differentiation with respect to an affine parameter on the orbit of the ray, the orbit is expressed by
\[
-A(r) \dot{t}^2 + \frac{\dot{r}^2}{A(r)} + r^2 \dot{\phi}^2 = 0.
\]

II. DEFINITION ANGLE IN A REISSNER-NORDSTRÖM SPACETIME

A line element in a Reissner-Nordström spacetime is given by
\[
ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),
\]
where \( A(r) \) is given by
\[
A(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}.
\]

The motion of the ray is permitted in a region satisfying the photon sphere \[\sigma_{II} \equiv \frac{r_{VS}}{2} \] were also investigated.

In 2002, Eiroa et al. applied their numerical method to a Reissner-Nordström black hole with a mass \( M \) and an electrical charge \( Q \) and obtained numerical values corresponding to \( \bar{a} \) and \( \bar{b} \) \[43\]. In the same year, Bozza obtained the numerical expression of \( \bar{a} \) and the numerical expression of \( \bar{b} \) \[28\]. The analytical form of \( \bar{b} \) by using a series on a small charge was obtained without the series on the small charge. The deflection angles of the rays in the Reissner-Nordström naked singularity spacetime with the photon sphere in strong deflection limits \( \bar{b} \rightarrow b_m \pm 0 \) \[43, 74, 75\], and with a marginally unstable photon sphere in the strong deflection limit \( \bar{b} \rightarrow b_m + 0 \) \[40\], and the deflection angle in the Reissner-Nordström naked singularity spacetime without the photon sphere \[74\] were also investigated.

In this paper, we investigate the affine perturbation expansion of the deflection angle of the ray in analytical calculations near the photon sphere of an extreme charged Reissner-Nordström black hole. We discuss the error of the deflection angles and we conclude that a strong-deflection-limit analysis for a gravitational lensing can be improved by using \( \alpha_{0th} \). We use the units in which the light speed and Newton’s constant are unity.
the ray forms an unstable circular light sphere, called a photon sphere, at \( r = r_m \) and its effective potential holds \( V_m = V''_m = 0 \) and \( V'''_m < 0 \) for \( Q < 3M/(2\sqrt{2}) \). Here and therefore, quantities with the subscript \( m \) denote the quantities at \( r = r_m \) or \( r_0 = r_m \). The ray with \( b < b_m \) (or \( r_0 < r_m \)) goes inside of the photon sphere while the ray with \( b > b_m \) (or \( r_0 > r_m \)) is scattered on the outside of the photon sphere. We assume \( b > b_m \) (or \( r_0 > r_m \)) and \( Q < 3M/(2\sqrt{2}) \).

From Eq. (2.23), the deflection angle of the ray is obtained as

\[
\alpha = 2 \int_{r_0}^{\infty} \frac{bdr}{\sqrt{r^4 - b^2r^2 + 2Mb^2 r - b^2Q^2}} - \pi \\
= 2 \int_0^{\hbar_0} \frac{dh}{\sqrt{G(h)}} - \pi, \tag{2.12}
\]

where \( h \), \( h_0 \), and \( G(h) \) are defined by \( h \equiv M/r \), \( h_0 \equiv M/r_0 \), and

\[
G(h) \equiv -q^2h^4 + 2h^3 - h^2 + h_0^2, \tag{2.13}
\]

respectively, and where \( q \) and \( h_b \) are defined by \( q \equiv Q/M \) and \( h_b \equiv M/b \). By using the relation

\[
h_0^2 = q^2h^4 - 2h^3 + h^2, \tag{2.14}
\]

we can rewrite \( G(h) \) as

\[
G(h) = (h - h_0)[-q^2h^3 + (q^2h_0^2 + 2)h^2 \]
\[+ (q^2h_0^2 + 2h_0 - 1)h + (q^2h_0^2 + 2h_0 - 1)h_0] \]
\[= -q^2(h - h_0)(h - h_1)(h - h_1)(h - h_2), \tag{2.15}
\]

where \( h_1 \), \( h_2 \), and \( h_3 \) are three of four real solutions of \( G(h) = 0 \) and the relation \( h_1 < h_0 < h_1 < h_2 \) holds. The deflection angle can be expressed as

\[
\alpha = 2 \int_0^{\hbar_0} \frac{dh}{\sqrt{-(h - h_0)(h - h_1)(h - h_1)(h - h_2)}} - \pi \\
= \frac{2}{\sqrt{h_2 - h_1}} \left[K(k) - F(\Psi, k)\right] - \pi. \tag{2.16}
\]

where \( F(\Psi, k) \) is the incomplete elliptic integral of the first kind defined by

\[
F(\Psi, k) = \int_0^\Psi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \tag{2.17}
\]

\( K(k) \) is the complete elliptic integral of the first kind defined by \( K(k) \equiv F(\pi/2, k) \), \( k \) is defined by

\[
k = \sqrt{(h_2 - h_1)(h_0 - h_1)(h_2 - h_0)(h_1 - h_1)}, \tag{2.18}
\]

and \( \Psi \) is defined by

\[
\Psi = \arcsin \sqrt{\frac{(h_0 - h_2)(h_1 - h_1)}{(h_0 - h_1)(h_0 - h_2)}}. \tag{2.19}
\]

We define variables \( k_p \) and \( h_p \) by

\[
k_p = \sqrt{1 - k^2} \tag{2.20}
\]

and

\[
h_p = 1 - \frac{h_0}{h_m} = 1 - \frac{r_m}{r_0} = 1 - \frac{3 + \sqrt{9 - 8q^2}}{2} h_0, \tag{2.21}
\]

respectively, and where \( h_m \) is defined by

\[
h_m \equiv \frac{M}{r_m} = \frac{2}{3 + \sqrt{9 - 8q^2}}. \tag{2.22}
\]

In the strong deflection limit \( r_0 \rightarrow r_m + 0 \) or \( b \rightarrow b_m + 0 \), we obtain \( k \rightarrow 1 - 0 \), \( k_p \rightarrow +0 \), \( h_p \rightarrow +0 \), and \( b_p \rightarrow +0 \). First, we expand the deflection angle \( \alpha \) in powers of \( k_p \), around \( 0 \); next we expand it in powers of \( b_p \), around \( 0 \); then we substitute \( h_p \), which is expanded in powers of \( b_p \), around \( 0 \), into the deflection angle \( \alpha \); and finally, we obtain the affine perturbation series of the deflection angle \( \alpha \) for \( 0 \leq Q < 3M/(2\sqrt{2}) \). We can obtain the affine perturbation series of the deflection angle \( \alpha \), but we do not show its general and explicit formula for \( 0 \leq Q < 3M/(2\sqrt{2}) \) on this paper because it would have an extremely complicated form and because it is generally not easy to simplify it.

### A. Extreme charged black hole with \( Q = M \)

We concentrate on the extreme charged case with \( Q = M \) to show the explicit and very simple form of the affine perturbation series of the deflection angle. In this case, we get \( h_m = 1/2 \), \( h_p = 1 - 2h_0 \), \( h_1 = (1 - \sqrt{1 - 4(h_0 - 1)h_0})/2 \), \( h_1 = 1 - h_0 \), and \( h_2 = \left[1 + \sqrt{1 - 4(h_0 - 1)h_0}\right]/2 \), which are plotted in Fig. [1](#). By using them and the relation \( h_p = \sqrt{b_p} \), we obtain the affine perturbation series of the deflection angle \( \alpha \) as

\[
\alpha = -\pi + \frac{2 - 3\sqrt{2}}{4} b_p + \frac{108 - 149\sqrt{2}}{256} b_p^2 \\
+ \frac{2060 - 2763\sqrt{2}}{6144} b_p^3 + O(b_p^4) \\
+ \frac{2048 + 768b_0 + 456b_0^2 + 315b_0^3 + O(b_p^5)}{1024\sqrt{2}} \\
\times \log \left[ \frac{2^3(3 - 2\sqrt{2})}{b_p} \right]. \tag{2.23}
\]

By comparing Eq. (2.23) with Eq. (1.2), we get

\[
\rho_0 = -\pi, \quad \rho_1 = \frac{2 - 3\sqrt{2}}{4}, \quad \rho_2 = \frac{108 - 149\sqrt{2}}{256}, \\
\rho_3 = \frac{2060 - 2763\sqrt{2}}{6144}, \quad \sigma_0 = \sqrt{2}, \quad \sigma_1 = \frac{3\sqrt{2}}{8}, \\
\sigma_2 = \frac{57\sqrt{2}}{256}, \quad \lambda_0 = 2^5(3 - 2\sqrt{2}). \tag{2.24}
\]
FIG. 1. $h_2$, $h_1$, $h_0$, and $h_{-1}$ are denoted by wide (red) solid, wide (green) dashed, narrow (blue) solid, and narrow (black) dashed curves, respectively, as a function of $h_0$ in the extreme charged case with $Q = M$.

B. Schwarzschild black hole with $Q = 0$

As a reference, we show the coefficients of the affine perturbation series of the deflection angle in Eq. (1.2) in the Schwarzschild black hole case with $Q = 0$ obtained by Iyer and Petters

$$\rho_0 = -\pi, \quad \rho_1 = \frac{-17 + 4\sqrt{3}}{18}, \quad \rho_2 = \frac{-879 + 236\sqrt{3}}{1296},$$

$$\rho_3 = \frac{-321590 + 90588\sqrt{3}}{629856}, \quad \sigma_0 = 1, \quad \sigma_1 = \frac{5}{18},$$

$$\sigma_2 = \frac{205}{1296}, \quad \sigma_3 = \frac{68145}{629856}, \quad \lambda_0 = 216(7 - 4\sqrt{3}).$$

III. DISCUSSION

If the relation

$$\lim_{b \to b_m+0} \left( \frac{b}{b_m} - 1 \right) = \lim_{b \to b_m+0} \left( 1 - \frac{b_m}{b} \right)$$

is used, the 0th order of the affine perturbation series $\alpha_{\text{aff}}$ in the Schwarzschild spacetime obtained by Iyer and Petters corresponds with the deflection angle $\alpha_{\text{SDL}}$ in the strong deflection limit $r_0 \to r_m + 0$ or $b \to b_m + 0$ obtained by Darwin and by Bozza. From the correspondence, we can express $\bar{a}$ and $\bar{b}$ as

$$\bar{a} = \sigma_0$$

and

$$\bar{b} = \sigma_0 \log \lambda_0 + \rho_0,$$

respectively.

In the Reissner-Nordström spacetime for $0 \leq Q < 3M/(2\sqrt{2})$, the analytic form of $\bar{a}$ in deflection angle $\alpha_{\text{SDL}}$ in the strong deflection limit $b \to b_m + 0$ was obtained in Refs. 28, 57, 58 as

$$\bar{a} = \frac{r_m}{\sqrt{3Mr_m - 4Q^2}},$$

and $\bar{b}$ was obtained in Refs. 57, 58 as

$$\bar{b} = \bar{a} \log \left[ \frac{8(3Mr_m - 4Q^2)^3}{M^2 r_m^2 (Mr_m - Q^2)^2} \times \left( 2\sqrt{Mr_m - Q^2} - \sqrt{3Mr_m - 4Q^2} \right)^2 \right] - \pi.$$

Therefore, we can obtain the affine perturbation series of the deflection angle in the 0th order $\alpha_{0\text{th}}$ for $0 \leq Q < 3M/(2\sqrt{2})$ with

$$\sigma_0 = \frac{r_m}{\sqrt{3Mr_m - 4Q^2}},$$

$$\lambda_0 = \frac{8(3Mr_m - 4Q^2)^3}{M^2 r_m^2 (Mr_m - Q^2)^2} \times \left( 2\sqrt{Mr_m - Q^2} - \sqrt{3Mr_m - 4Q^2} \right)^2,$$

and

$$\rho_0 = -\pi.$$

We define the percent errors of $\alpha_{0\text{th}}$ and $\alpha_{\text{SDL}}$ as

the percent error of $\alpha_{0\text{th}} \equiv \frac{\alpha - \alpha_{0\text{th}}}{\alpha} \times 100$, and

the percent error of $\alpha_{\text{SDL}} \equiv \frac{\alpha - \alpha_{\text{SDL}}}{\alpha} \times 100$, respectively, where $\alpha$ is calculated by using Eq. (1.2), and the errors are plotted in Fig. 2. We notice that $\alpha_{0\text{th}}$ has a smaller error than $\alpha_{\text{SDL}}$. The errors of $\alpha_{0\text{th}}$ and $\alpha_{\text{SDL}}$ exponentially decrease as $\alpha$ increases.

We also notice that the errors increase as the charge increases and they become very large in the near extreme charged case $Q \sim M$. This is because the photon sphere exists only for $0 \leq Q < 3M/(2\sqrt{2})$ and it becomes a marginally unstable photon sphere with $V_m = V'_m = V''_m = 0$ and $V'''_m < 0$ for $Q = 3M/(2\sqrt{2})$ and the deflection angle reflected by the marginally unstable photon sphere diverges nonlogarithmically.

We define the affine perturbation expansion of the deflection angle of the ray in the 1st order $\alpha_{1\text{st}}$, the 2nd order $\alpha_{2\text{nd}}$, and the 3rd order $\alpha_{3\text{rd}}$ as

$$\alpha_{1\text{st}} \equiv (\sigma_0 + \sigma_1 b_p) \log \left( \frac{\lambda_0}{b_p} \right) + \rho_0 + \rho_1 b_p,$$

$$\alpha_{2\text{nd}} \equiv (\sigma_0 + \sigma_1 b_p + \sigma_2 b_p^2) \log \left( \frac{\lambda_0}{b_p} \right) + \rho_0 + \rho_1 b_p + \rho_2 b_p^2,$$

$$\alpha_{3\text{rd}} \equiv (\sigma_0 + \sigma_1 b_p + \sigma_2 b_p^2 + \sigma_3 b_p^3) \log \left( \frac{\lambda_0}{b_p} \right) + \rho_0 + \rho_1 b_p + \rho_2 b_p^2 + \rho_3 b_p^3,$$
The percent errors increase if the charge increases. The percent errors of the deflection angle increase and that the percent error of the affine perturbation series of the deflection angle in every order decreases exponentially as the charge increases. The percent errors of $\alpha_{1st}$, $\alpha_{2nd}$, and $\alpha_{3rd}$ are defined by

$$\alpha_{1st} = \frac{\alpha - \alpha_{1st}}{\alpha} \times 100, \quad \alpha_{2nd} = \frac{\alpha - \alpha_{2nd}}{\alpha} \times 100,$$

and

$$\alpha_{3rd} = \frac{\alpha - \alpha_{3rd}}{\alpha} \times 100,$$

respectively, where $\alpha$ is calculated by Eq. (2.12), and they are plotted in Fig. 3. Figure 3 shows that the percent error of the affine perturbation series of the deflection angle in every order decreases exponentially as the deflection angle increases and that the percent errors increase if the charge increases. The percent errors of $\alpha_{SDL}$, $\alpha_{0th}$, $\alpha_{1st}$, $\alpha_{2nd}$, and $\alpha_{3rd}$ for $\alpha = \pi$, 2$\pi$, 3$\pi$ are shown in Tables I and II since they are important in gravitational lensing by the photon sphere of the black hole.

By comparing with the errors of $\alpha_{0th}$ and $\alpha_{SDL}$, we notice that $\alpha_{SDL}$ has a hidden error which is given by a difference

$$\alpha_{0th} - \alpha_{SDL} = \tilde{a} \log \frac{b}{b_m} = \sigma_0 \log \left( \frac{1}{1 - b_p} \right).$$

If the hidden error is much smaller than the 1st order terms $O(b_p \log b_p) = -\sigma_1 b_p \log b_p$ and $O(b_p) = b_p (\rho_1 + \sigma_1 \log \lambda_0)$, we do not have to care about the hidden error term. Note that the higher order terms have been obtained from

$$\alpha_{1st} - \alpha_{0th} = \sigma_1 b_p \log \frac{\lambda_0}{b_p} + \rho_1 b_p$$

$$= -\sigma_1 b_p \log b_p + b_p (\rho_1 + \sigma_1 \log \lambda_0). \quad (3.18)$$

The hidden error $\alpha_{0th} - \alpha_{SDL} = \sigma_0 \log [1/(1 - b_p)]$ and the 1st order terms $-\sigma_1 b_p \log b_p$ and $b_p (\rho_1 + \sigma_1 \log \lambda_0)$ are plotted in Fig. 4. We find that the hidden error of $\alpha_{SDL}$ is comparable with the 1st order terms in a retro lensing configuration with $\alpha \sim \pi$.

IV. CONCLUSION

The deflection angle in the 0th order of the affine perturbation series $\alpha_{0th}$ is more accurate than the deflection angle $\alpha_{SDL}$ in the strong deflection limit because $\alpha_{SDL}$ has a hidden error. The hidden error given by $\alpha_{0th} - \alpha_{SDL}$ is comparable with 1st order terms when the deflection angle is $\pi$. This implies that we should improve a strong-deflection-limit analysis for a gravitational lensing by using $\alpha_{0th}$. The improvement would not be difficult since we obtain $\alpha_{0th}$ from $\alpha_{SDL}$ by using relation (3.1).
FIG. 3. The percent errors of $\alpha_{0th}$, $\alpha_{1st}$, $\alpha_{2nd}$, and $\alpha_{3rd}$ for $Q = 0$ (left panel) and $Q = M$ (right panel). Wide (red) solid, wide (green) dashed, narrow (blue) solid, and narrow (black) dashed lines denote the percent errors of $\alpha_{0th}$, $\alpha_{1st}$, $\alpha_{2nd}$, and $\alpha_{3rd}$, respectively, against $\alpha$ calculated by using Eq. (2.12).

TABLE I. Percent errors of $\alpha_{SDL}$, $\alpha_{0th}$, $\alpha_{1th}$, $\alpha_{2nd}$, and $\alpha_{3rd}$ when $\alpha$ calculated by using Eq. (2.12) is $\pi$, $2\pi$, and $3\pi$ for the Schwarzschild black hole with $Q = 0$.

| $\alpha$ in numerical | Percent error of $\alpha_{SDL}$ | Percent error of $\alpha_{0th}$ | Percent error of $\alpha_{1th}$ | Percent error of $\alpha_{2nd}$ | Percent error of $\alpha_{3rd}$ |
|------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $\pi$                  | 2.11                            | 1.14                            | $1.83 \times 10^{-2}$           | $3.65 \times 10^{-4}$           | $8.09 \times 10^{-6}$           |
| $2\pi$                 | $6.11 \times 10^{-2}$          | $4.11 \times 10^{-2}$          | $2.83 \times 10^{-5}$           | $2.39 \times 10^{-8}$           | $< 10^{-10}$                    |
| $3\pi$                 | $2.26 \times 10^{-3}$          | $1.68 \times 10^{-3}$          | $5.05 \times 10^{-8}$           | $< 10^{-10}$                    | $< 10^{-10}$                    |

TABLE II. The percent errors in the extreme charged black hole case with $Q = M$.

| $\alpha$ in numerical | Percent error of $\alpha_{SDL}$ | Percent error of $\alpha_{0th}$ | Percent error of $\alpha_{1th}$ | Percent error of $\alpha_{2nd}$ | Percent error of $\alpha_{3rd}$ |
|------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $\pi$                  | 7.40                            | 4.10                            | $1.62 \times 10^{-1}$           | $7.68 \times 10^{-3}$           | $4.01 \times 10^{-4}$           |
| $2\pi$                 | $4.98 \times 10^{-1}$          | $3.37 \times 10^{-1}$          | $1.37 \times 10^{-3}$           | $6.59 \times 10^{-6}$           | $3.48 \times 10^{-8}$           |
| $3\pi$                 | $4.50 \times 10^{-2}$          | $3.35 \times 10^{-2}$          | $1.47 \times 10^{-5}$           | $7.64 \times 10^{-9}$           | $< 10^{-10}$                    |

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FIG. 4. The hidden error of $\alpha_{\text{SDL}}$ given by $\alpha_{\text{SDL}} = \sigma_0 \log [1/(1 - b_p)]$ and the 1st order terms $-\sigma_1 b_p \log b_p$ and $\delta_p \sigma_0 \log \lambda_0$ are denoted by wide (red) solid, wide (green) dashed, and narrow (blue) solid curves, respectively, against $\alpha$ calculated by using Eq. (2.12) for $Q = 0$ (left panel) and $Q = M$ (right panel).

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