GLOBAL RIGIDITY FOR ULTRA-DIFFERENTIABLE QUASIPERIODIC COCYCLES AND ITS SPECTRAL APPLICATIONS

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Abstract. For quasiperiodic Schrödinger operators with one-frequency analytic potentials, from dynamical systems side, it has been proved that the corresponding quasiperiodic Schrödinger cocycle is either rotations reducible or has positive Lyapunov exponent for all irrational frequency and almost every energy [AFK11]. From spectral theory side, the “Schrödinger conjecture” [AFK11] and the “Last’s intersection spectrum conjecture” have been verified [JM12a]. The proofs of above results crucially depend on the analyticity of the potentials. People are curious about if the analyticity is essential for those problems, see open problems by Fayad-Krikorian [FK09, Kri] and Jitomirskaya-Marx [JM12a, MJ17]. In this paper, we prove the above mentioned results for ultra-differentiable potentials.

1. Introduction and main results

We consider smooth quasiperiodic $SL(2, \mathbb{R})$ cocycles
\[
(\alpha, A) : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{T} \times \mathbb{R}^2, \quad (\theta, w) \mapsto (\theta + \alpha, A(\theta)w),
\]
where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $A \in C^\infty(\mathbb{T}, SL(2, \mathbb{R}))$ and $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. Typical examples are Schrödinger cocycles where
\[
A(\theta) = S^V_E(\theta) = \begin{pmatrix}
E - V(\theta) & -1 \\
1 & 0
\end{pmatrix},
\]
which is equivalent to the eigenvalue equations of the one-dimensional quasiperiodic Schrödinger operator $H_{V,\alpha,\theta}$ defined by
\[
(H_{V,\alpha,\theta}u)_n = u_{n-1} + u_{n+1} + V(\theta + n\alpha)u_n.
\]
Quasiperiodic Schrödinger operators describe the conductivity of electrons in a two-dimensional crystal layer subject to an external magnetic field of flux acting perpendicular to the lattice plane. Due to the rich backgrounds in quantum physics, quasiperiodic Schrödinger operators have been extensively studied [Las05].

It has been proved that the (almost) reducibility of the above Schrödinger cocycles is a powerful tool in the study of the spectral theory of quasiperiodic Schrödinger operators [You18]. Recall that $(\alpha, A)$ is $C^r(r$ could be $\infty$ or $\omega)$
reducible, if there exist $B \in C^r(T, PSL(2, \mathbb{R}))$ and $C \in SL(2, \mathbb{R})$ such that
$$B(\cdot + \alpha)A(\cdot)B(\cdot) = C.$$ 
We remark that the reducibility is too restrictive since even an $\mathbb{R}$-valued cocycle is in general not reducible if the frequency is very Liouvillean. The appropriate notion is $C^r$ rotations reducibility, which means, there exist $B \in C^r(T, PSL(2, \mathbb{R}))$ and $C \in SL(2, \mathbb{R})$ such that
$$B(\cdot + \alpha)A(\cdot)B(\cdot) = C.$$ 

By Kotani’s theory, for Lebesgue almost every $E \in \mathbb{R}$, the Schrödinger cocycle $(\alpha, S^E_V)$ is either $L^2$ rotations reducible or has positive Lyapunov exponent. In many circumstances, especially for its dynamical and spectral applications, what’s important is the rigidity, i.e., whether $L^2$ conjugacy for analytic (resp. smooth) cocycles implies analytic (resp. smooth) conjugacy under some additional assumptions. In this paper, we are interested in the global rigidity results for smooth quasiperiodic Schrödinger cocycles and its spectral applications.

1.1. Global rigidity results for smooth quasiperiodic cocycles. Based on the powerful method of renormalization, Avila-Krikorian [AK06] proved that if $\alpha$ is recurrent Diophantine, $V \in C^\infty(T, \mathbb{R})$, then for Lebesgue almost every $E$, the Schrödinger cocycle $(\alpha, S^E_V)$ is either nonuniformly hyperbolic or $C^\infty$ reducible. Later, Fayad-Krikorian [FK09] proved that for all Diophantine $\alpha$, and $V \in C^\infty(T, \mathbb{R})$, then for Lebesgue almost every $E$, the Schrödinger cocycle $(\alpha, S^E_V)$ is either nonuniformly hyperbolic or $C^\infty$-reducible. Indeed, it was pointed out by Fayad-Krikorian [FK09], to extend the results of [FK09] to any irrational number is an interesting and important problem. The problem was later settled by Avila-Fayad-Krikorian [AFK11] in the analytic case. More precisely, for all irrational $\alpha$ and any $V \in C^\infty(T, \mathbb{R})$, Avila-Fayad-Krikorian [AFK11] proved that the Schrödinger cocycle $(\alpha, S^E_V)$ is either $C^\omega$ rotations reducible or has positive Lyapunov exponent for Lebesgue almost every $E$. Around 2011’s, R. Krikorian [Kri] asked the fourth author whether the global dichotomy results of [AFK11] hold in the $C^\infty$ topology.

In the present paper, we address R. Krikorian’s question for a large class of $C^\infty$ cocycles. We first introduce the definition of $M$-ultra-differentiable functions. It is known that the derivatives of a $C^\infty$ function $f$ may grow as fast as you like and its regularity is characterized by the growth of $D^sf$. For a given sequence of positive real numbers $M = (M_s)_{s \in \mathbb{N}}$, we say $f \in C^\infty(T, \mathbb{R})$
is $M$-ultra-differentiable if there exists $r > 0$ such that
\[ \|D^s f\|_{C^0} \leq r^{-s} M_s, \]
here $r$ is also called the “width”. The real-analytic and $\nu$-Gevrey functions are two special cases corresponding to $M_s = s!$ and $M_s = (s!)^{\nu^{-1}}$, $0 < \nu < 1$ respectively. Then our main result is the following:

**Theorem 1.1.** Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $V : T \to \mathbb{R}$ be an $M$-ultra-differentiable function with $M = (M_s)_{s \in \mathbb{N}}$ satisfying

(H1): **Log-convex:** $M_{s-k}^s < M_k^{s-\ell} M_\ell^{s-k}$, $s > \ell > k$,

(H2): **Sub-exponential growth:** $\lim_{s \to \infty} s^{-1} \ln(M_{s+1}/M_s) = 0$.

Then for Lebesgue almost every $E \in \mathbb{R}$, either the Schrödinger cocycle $(\alpha, S^E)$ is $C^\infty$ rotations reducible or it has positive Lyapunov exponent.

**Remark 1.1.** Theorem 1.1 proved the $C^\infty$ rigidity for a large class of $C^\infty$ quasiperiodic cocycles. The almost rigidity in $C^\infty$ topology was already proved by Fayad-Krikorian [FK09]. More precisely, they proved the cocycle either has positive Lyapunov exponent, or the cocycle is $C^\infty$ almost rotations reducible, which means the cocycle can be approximated by rotations reducible cocycles in the $C^\infty$ topology.

We remark that the assumptions (H1) and (H2) are not restrictive. It is obvious that both analytic and Gevrey class functions satisfy (H1) and (H2). Indeed, the log-convexity condition (H1) is a very classical assumption in the literature, which guarantees the space of $M$-ultra-differentiable functions form a Banach algebra. The sub-exponential condition (H2) was first introduced by Bounemoura-Fejoz [BFar] to guarantee the ultra-differentiable functions have an analogue of the Cauchy estimates for analytic functions, which is one of the main ingredients in KAM theory. We remark that the commonly used condition in the literature is called moderate growth condition:

\[ \sup_{s, \ell \in \mathbb{N}} \left( \frac{M_{s+\ell}}{M_s M_\ell} \right)^{1/(s+\ell)} < \infty, \]

which is stronger than (H2), see [BFar] for details.

Attached to the sequence $(M_s)_{s \in \mathbb{N}}$, one can define $\Lambda : [0, \infty) \to [0, \infty)$ by

\[ \Lambda(y) := \ln \left( \sup_{s \in \mathbb{N}} y^s M_s^{-1} \right) = \sup_{s \in \mathbb{N}} (s \ln y - \ln M_s), \]

which in fact describes the decay rate of the Fourier coefficients for periodic functions. For $C^\infty$ smooth periodic functions, the growth of $\Lambda(y)$ is faster than $\ln(y^s)$ for any $s \in \mathbb{N}$ as $y$ goes to infinity. Consequently, $C^\infty$ means

\[ \lim_{y \to \infty} \frac{\Lambda(y)}{\ln(y)} = \infty. \]
On the other hand, one can easily check that $M_s = \exp\{s^\delta(\delta-1)^{-1}\}$ satisfies \((H1)\) and \((H2)\) if and only if $\delta > 2$. Attached to this $M_s$, $\Lambda(y) = (\ln y)^\delta$. Notice that $\Lambda(y) = y^\nu$, $0 < \nu < 1$ for Gevrey functions. Thus the space of $M$-ultra-differentiable functions with \((H1)\) and \((H2)\) is much bigger than the space of Gevrey functions, and quite close to the whole space of $C^\infty$ functions. However, those $C^\infty$ functions with $\Lambda(y) \leq (\ln y)^2$ are not included. We don’t know it is essential or due to the shortage of our method.

The proof of Theorem 1.1 is based on renormalization technique and local KAM result. For $M$-ultra-differentiable functions, to describe the smallness of perturbation, we define the $\| \cdot \|_{M,r}$-norm by
\[
\|f\|_{M,r} = c \sup_{s \in \mathbb{N}} ((1 + s)^2 r^s \|D_\theta^s f(\theta)\|_{C^0} M_s^{-1}) < \infty, \ c = 4\pi^2 / 3,
\]
and denote by $U^M_r(\mathbb{T}, \ast)$ the set of all these $\ast$-valued functions ($\ast$ will usually denote $\mathbb{R}$, $sl(2, \mathbb{R})$ $SL(2, \mathbb{R})$).

Then our precise KAM-type result is the following:

**Theorem 1.2.** Let $r > 0$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $A \in U^M_r(\mathbb{T}, SL(2, \mathbb{R}))$ with $M$ satisfying \((H1)\) and \((H2)\). Then for every $\tau > 1$ and $\gamma > 0$, there exists $\varepsilon_* = \varepsilon_*(\gamma, \tau, r, M) > 0$, such that if $\|A - R\|_{M,r} \leq \varepsilon_*$ for some $R \in SO(2, \mathbb{R})$, and $\rho(\alpha, A) =: \rho_f \in DC_\alpha(\gamma, \tau)$, i.e.,
\[
\|k\alpha \pm 2\rho_f\|_\mathbb{Z} \geq \gamma(k)^{-\tau}, \ \forall k \in \mathbb{Z}, \langle k \rangle = \max\{1, |k|\},
\]
then $(\alpha, A)$ is $C^\infty$ rotations reducible.

We point out that, Theorem 1.2 is a semi-local result in the terminology of [FK18], i.e., the smallness of the perturbation $\varepsilon_*$ does not depend on the frequency $\alpha$. One should not expect that $\varepsilon_*$ is independent of $\rho_f$ (in terms of $\gamma, \tau$) as this is not true in the $C^\infty$ topology (or even Gevrey class) [AK16]. To this end, we mention another open problem of Fayad-Krikorian [FK18]: Is the semi-local version of the almost reducibility conjecture true for cocycles in quasi-analytic classes? In the analytic topology, it has been established in [HY12, YZ13].

The technical reason why we introduce \((H1)\) and \((H2)\) is the following: The proof of Theorem 1.2 is based on a non-standard KAM scheme developed in [HY12, KWYZ18]. The key idea is to prove that the homological equations
\[
e^{2i(2\pi \rho_f + \tilde{g}(\cdot))} f(\cdot + \alpha) - f + h = 0,
\]
has a smooth approximating solution, consult section 4.1 for more discussions. Here $\tilde{g}(\cdot)$ comes from the perturbation, in order to ensure that \((1.2)\) has a smooth approximating solution, we do need some kind of control for all derivatives $\|D^s \tilde{g}(\cdot)\|_{C^0}, s \in \mathbb{N}$ which is guaranteed by \((H2)\).

Next we give a short review of local reducibility results. The pioneering result of local reducibility was due to Dinaburg-Sinai [DS75], who proved that if $\alpha \in DC$, and $V$ is analytically small, then $(\alpha, S^V_E)$ is reducible for majoritiy of $E$. Eliasson [Eli92] further proved for Lebesgue almost surely
If $\alpha$ is Liouvillean, based on “algebraic conjugacy trick” developed in \cite{FK09}, Avila-Fayad-Krikorian \cite{AFK11} proved that in the local regime, $(\alpha, S^V_E)$ is reducible for majority of $E$, thus gives a generalization of Dinaburg-Sinai’s Theorem \cite{DS75} to arbitrary one-frequency. The result was also proved for analytic quasiperiodic linear systems by Hou-You in \cite{HY12}. Later, Zhou-Wang \cite{ZW12} generalized $SL(2, \mathbb{R})$ cocycles result \cite{AFK11} to $GL(d, \mathbb{R})$ cocycles by different method. Theorem 1.1 and Theorem 1.2 can be seen as a generalization of \cite{AFK11} from analytic functions to ultra-differentiable functions.

1.2. The spectral applications. We point out that global rigidity results in the analytic topology \cite{AK06, AFK11} have many important applications in the spectral theory of quasiperiodic Schrödinger operators. To name a few, it was used to verify the Schrödinger Conjecture \cite{VPMG93} in the Liouvillean context \cite{AFK11}, it also plays an essential role in solving “Last’s intersection spectrum conjecture” \cite{JM12a}, Aubry-Andre-Jitomirskaya’s conjecture \cite{AYZ17}. With Theorem 1.2, one can prove the first two conjectures also hold for quasiperiodic operators with M-ultra-differentiable potentials satisfying \textbf{(H1)} and \textbf{(H2)}.

1.2.1. Schrödinger conjecture. The Schrödinger conjecture \cite{VPMG93} says, for general discrete Schrödinger operators over uniquely ergodic base dynamics, all eigenfunctions are bounded for almost every energy in the support of the absolutely continuous part of the spectral measure. This conjecture has recently been disproved by Avila \cite{Avi15b}. However, it is still interesting to know, to what extend the conjecture is true. For example, the KAM scheme of \cite{AFK11} implies that the Schrödinger conjecture is true in the quasiperiodic case with analytic potentials, and this was the first time it was verified in a Liouvillean context. Indeed, as pointed by Jitomirskaya and Marx in \cite{MJ17} (page 2363 of \cite{MJ17}): addressing the Schrödinger conjecture for quasiperiodic operators with lower regularities of the potentials still remains an open problem.

With Theorem 1.1, we can prove the Schrödinger conjecture with M-ultra-differentiable quasiperiodic potentials.

**Corollary 1.** Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $V: \mathbb{T} \to \mathbb{R}$ be a M-ultra-differentiable function satisfying \textbf{(H1)} and \textbf{(H2)}. Then the Schrödinger conjecture is true.

1.2.2. Last’s intersection spectrum conjecture. Denote

$$S_-(\beta) = \cap_{\theta \in \mathbb{T}} \Sigma_{ac}(\beta, \theta),$$

where $\Sigma_{ac}(\beta, \theta)$ is the absolutely continuous spectrum of (quasi)periodic Schrödinger operator $H_{V, \beta, \theta}$ defined by \textbf{(1.1)}. For any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, it can be
approximated by a sequence of rational numbers \((p_n/q_n)\). It is well known that the rational frequency approximation is indispensable for numeric analysis, thus the existence of the limits \(S_-(p_n/q_n)\) as \(p_n/q_n \to \alpha\) is crucial. A conjecture of Y. Last says, up to a set of zero Lebesgue measure, the absolutely continuous spectrum can be obtained asymptotically from \(S_-(p_n/q_n)\), the spectrum of periodic operators associated with the continued fraction expansion of \(\alpha\).

Jitomirskaya-Marx [JM12a] settled the “Last’s intersection spectrum conjecture” for analytic quasiperiodic Schrödinger operators. They also pointed out, in [JM12a], that the analyticity of the potential \(V\) is essential for the proof of their result, and, whether or not the analyticity can be relaxed without reducing the range of frequencies for which the statement holds is an interesting open problem (page 5 of [JM12a]). In this work, we will give a positive answer to this problem for \(\nu\)-Gevrey potentials with \(1/2 < \nu \leq 1\).

**Theorem 1.3.** Let \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\) and \(V: \mathbb{T} \to \mathbb{R}\) be a \(\nu\)-Gevrey function with \(1/2 < \nu \leq 1\), there is a sequence \(p_n/q_n \to \alpha\) such that
\[
\lim_{n \to \infty} S_-(p_n/q_n) = S_-(\alpha) = \Sigma_{ac}(\alpha).
\]

**Remark 1.2.** As we will see, in fact we will prove
\[
\Sigma_{ac}(\alpha) \subset \liminf_{n \to \infty} S_-(p_n/q_n)
\]
for all M-ultra-differentiable potential satisfying (H1) and (H2) (Theorem 6.1). Gevrey property only plays a role in proving
\[
\limsup_{n \to \infty} S_-(p_n/q_n) \subset \Sigma_{ac}(\alpha)
\]
for Diophantine frequency (Theorem 6.2).

We briefly explain why analyticity is crucial for the proof of [JM12a]. On the one hand, the key of (1.3) is to prove that \(E \in \Sigma_{ac}(\alpha)\) implies exponentially small variation (in \(q_n\)) of the approximating discriminants (“generalized Chambers’ formula”). For the analytic potential, Jitomirskaya-Marx [JM12a] got this estimate as a corollary of Avila’s quantization of acceleration [Avi15a], which can be defined only for analytic cocycles. On the other hand, the proof of (1.4) was first obtained by Shamis [Sha11] as a corollary of the continuity of Lyapunov exponent: i.e. the Lyapunov exponent \(L(\beta + \cdot, \cdot): \mathbb{T} \times C^\omega(\mathbb{T}, SL(2, \mathbb{C}))\) is jointly continuous for any irrational \(\beta\) [BJ02, JKS09, JM12b]. However, the Lyapunov exponent \(L(\beta + \cdot, \cdot): \mathbb{T} \times C^\infty(\mathbb{T}, SL(2, \mathbb{C}))\) is not continuous [WY13].

In fact, it was also pointed out by Jitomirskaya and Marx in [JM12a] that analyticity should not be essential for their results, while one needs new methods in the non-analytic case. To generalize the result in [JM12a] to ultra-differentiable potential, we have to overcome the difficulty caused
by the non-analyticity of potential. One key issue is to prove the “generalized Chambers’ formula” in ultra-differentiable case. Instead of using Avila’s quantization of acceleration [Avi15a], we will use perturbative argument which avoids the analyticity, showing that if the cocycle is smoothly rotations reducible, then \( q \)-step transfer matrices grows sub-exponentially in \( q \). To do this, we will use inverse renormalization and quantitative KAM result, to show if the cocycle is almost reducible in ultra-differentiable topology, then we have a good control of the growth of \( q \)-step transfer matrices.

As for the proof of second inclusion, the key is to prove that Lyapunov exponent can be still continuous with respect to the rational approximation of the frequency for \( \nu \)-Gevrey potential \( V \) with \( 1/2 < \nu < 1 \), if \( \alpha \) is Diophantine, which is a generalization of the results in [BJ02]. See Theorem 6.3 for details. Recently, Ge-Wang-You-Zhao [LWJZ20] further constructed counter-examples for \( \nu \)-Gevrey potential \( V \) with \( 0 < \nu < 1/2 \), which shows Theorem 6.3 is optimal.

Finally, we review some related results. For general ergodic discrete Schrödinger operators, the relation between the absolutely continuous spectrum and the spectrum of certain periodic approximates has been studied by Last in [Las92, Las93], more precisely, [Las93] essentially proved that for \( V \in C^1(T) \) and a.e. \( \alpha , \lim \sup_{n \to \infty} S \cdot (p_n/q_n) \subset \Sigma_{ac}(\alpha) \) up to sets of zero Lebesgue measure. The conjecture is known for the almost Mathieu operator where \( V(\theta) = 2\lambda \cos \theta \) ([AvMS90, Las94] for a.e. \( \alpha , \lambda \) and [AK06, JK02, Las93, JM12a] extending to all \( \alpha \)). More recently, the conjecture was settled for a.e. \( \alpha \) and sufficiently smooth potential by Zhao [Zha19].

1.3. The structure of this paper. The paper is arranged as follows. In Section 2 we give some definitions and preliminaries. Before giving the proof of Theorem 1.2, we first derive condition (A) on Fourier coefficients from assumptions (H1) and (H2) on Taylor coefficients (Lemma 3.3) in Section 3. Then we prove Theorem 1.2 in Section 4 and prove Theorem 1.1 in Section 5. The proof of Theorem 1.3 is given in Section 6, which was based on Theorem 6.1 and Theorem 6.2. In Section 7 we give the proof of Generalized Chambers’ formula (Proposition 3), and in Section 8 we give the proof of the joint continuity of Lyapunov exponent (Theorem 6.3), these two results are bases of the proof of Theorem 6.1 and Theorem 6.2 respectively.

2. Definitions and preliminaries

2.1. Quasiperiodic cocycles. Given \( A \in C^0(T, SL(2, \mathbb{R})) \) and \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \), the iterates of \( (\alpha, A) \) are of the form \( (\alpha, A)^n = (n\alpha, A_n) \), where

\[
A_n(\cdot) := \begin{cases} 
A(\cdot + (n-1)\alpha) \cdots A(\cdot + \alpha)A(\cdot), & n \geq 0 \\
A^{-1}(\cdot + n\alpha)A^{-1}(\cdot + (n+1)\alpha) \cdots A^{-1}(\cdot - \alpha), & n < 0
\end{cases}
\]

3As pointed out in footnote 5 of [JM12a], this ideas was first pointed out by the fourth author after first preprint of [JM12a].
Define the finite Lyapunov exponent as

\[ L_n(\alpha, A) = \frac{1}{n} \int_T \ln \| A_n(\theta) \| d\theta, \]

then by Kingman’s subadditive ergodic theorem, the Lyapunov exponent of \((\alpha, A)\) is defined as

\[ L(\alpha, A) = \lim_{n \to \infty} L_n(\alpha, A) = \inf_{n>0} L_n(\alpha, A) \geq 0. \]

The cocycle \((\alpha, A)\) is called uniformly hyperbolic if there exists a continuous splitting \(E_s(\theta) \oplus E_u(\theta) = \mathbb{R}^2\), and \(C > 0, 0 < \lambda < 1\), such that for every \(n \geq 1\) we have

\[ \| A_n(\theta) w \| \leq C \lambda^n \| w \|, \quad \forall w \in E_s(\theta), \]
\[ \| A_{-n}(\theta) w \| \leq C \lambda^n \| w \|, \quad \forall w \in E_u(\theta). \]

Assume now \(A \in C^0(\mathbb{T}, SL(2, \mathbb{R}))\) is homotopic to the identity, then there exist \(\psi : \mathbb{T} \times \mathbb{T} \to \mathbb{R}\) and \(u : \mathbb{T} \times \mathbb{T} \to \mathbb{R}^+\) such that

\[ A(x) \cdot \begin{pmatrix} \cos 2\pi y \\ \sin 2\pi y \end{pmatrix} = u(x, y) \begin{pmatrix} \cos 2\pi(y + \psi(x, y)) \\ \sin 2\pi(y + \psi(x, y)) \end{pmatrix}. \]

The function \(\psi\) is called a lift of \(A\). Let \(\mu\) be any probability measure on \(\mathbb{T} \times \mathbb{T}\) which is invariant by the continuous map \(T : (x, y) \mapsto (x + \alpha, y + \psi(x, y))\), projecting over Lebesgue measure on the first coordinate (for instance, take \(\nu\) as any accumulation point of \(\frac{1}{n} \sum_{k=0}^{n-1} T_k^* \nu\) where \(\nu\) is Lebesgue measure on \(\mathbb{T} \times \mathbb{T}\)). Then the number

\[ \rho(\alpha, A) = \int \psi d\mu \mod \mathbb{Z} \]

does not depend on the choices of \(\psi\) and \(\mu\) and is called the fibered rotation number of \((\alpha, A)\), see [JM82] and [Her83]. It is immediate from the definition that

\[ |\rho(\alpha, A) - \rho| \leq \| A - R_\rho \|_{C^0}. \]

2.2. Continued fraction expansion. Let \(\alpha \in (0, 1)\) be irrational. Define \(a_0 = 0, a_0 = \alpha\), and inductively for \(k \geq 1\),

\[ a_k = [\alpha_{k-1}], \quad \alpha_k = \alpha_{k-1} - a_k = G(\alpha_{k-1}) = \{\alpha_{k-1}\}, \]

where \(G(\cdot)\) is the Gauss map. Let \(p_0 = 0, p_1 = 1, q_0 = 1, q_1 = 1\), then we define inductively \(p_k = a_k q_{k-1} + p_{k-2}, q_k = q_k q_{k-1} + q_{k-2}\). The sequence \((q_n)\) is the denominators of best rational approximations of \(\alpha\) since we have

\[ \| k\alpha \|_2 \geq \| q_{n-1}\alpha \|_2, \quad \forall 1 \leq k < q_n, \]

and

\[ (q_n + q_{n+1})^{-1} < \| q_n\alpha \|_2 \leq q_{n+1}^{-1}. \]

For sequence \((q_n)\), we will fix a particular subsequence \((q_{n_k})\) of the denominators of the best rational approximations for \(\alpha\), which for simplicity will be denoted by \((Q_k)\). Denote the sequences \((q_{n_k} + 1)\) and \((p_{n_k})\) by \((\mathcal{Q}_k)\) and \((\mathcal{P}_k)\) respectively.
Lemma 2.1. [AFK11] For any $\mathbb{A} \geq 1$, there exists a subsequence $(Q_k)$ of $(q_n)$ such that $Q_0 = 1$ and for each $k \geq 0$, $Q_{k+1} \leq Q_k^A$, either $Q_k \geq Q_k^A$, or the pairs $(\overline{Q}_{k-1}, Q_k)$ and $(Q_k, Q_{k+1})$ are both $CD(\mathbb{A}, \mathbb{A}, \mathbb{A}^3)$ bridges.

Set $\tau > 1$ and $\mathbb{A} > \tau + 23 > 24$, then for $\{\overline{Q}_n\}_{n \geq 0}$, the selected subsequence in Lemma 2.1, we have the following lemma.

Lemma 2.2. For $\{\overline{Q}_n\}_{n \geq 0}$, we have

$$\overline{Q}_{n+1} \geq Q_n^A, \quad \forall n \geq 0.$$  

Proof. Case one : $\overline{Q}_{n+1} \geq Q_n^A$. Obviously $\overline{Q}_{n+1} \geq Q_{n+1}^A \geq Q^n_A$.

Case two : $\overline{Q}_{n+1} \leq Q_n^A$. In this case we know that $(\overline{Q}_n, Q_{n+1})$ forms a CD $(\mathbb{A}, \mathbb{A}, \mathbb{A}^3)$ bridge. Thus $Q_{n+1} \geq Q^A_n$, which implies $\overline{Q}_{n+1} \geq Q^A_{n+1}$.  

2.3. Renormalization. In this subsection we give the notations and definitions about the renormalization which are given in [AK06, FK09, AK15].

2.3.1. $\mathbb{Z}^2$–actions. Consider the cocycle $(\alpha, A) \in (0, 1) \setminus \mathbb{Q} \times U^M_r(\mathbb{R}, SL(2, \mathbb{R}))$ and set $
abla^n = \sum_{l=0}^n \alpha_l = (-1)^n(q_n \alpha - p_n) = (q_{n+1} + \alpha_{n+1} q_{n+1})^{-1}$, where $\alpha_n = G^n(\alpha)$. Let $\Omega^r = \mathbb{R} \times U^M_\mathbb{R}(\mathbb{R}, SL(2, \mathbb{R}))$ be the subgroup of Diff($\mathbb{R} \times U^M_\mathbb{R}(\mathbb{R}, SL(2, \mathbb{R}))$) made of skew-product diffeomorphisms $(\alpha, A) \in \mathbb{R} \times U^M_\mathbb{R}(\mathbb{R}, SL(2, \mathbb{R}))$.

A $U^M_r$ fibered $\mathbb{Z}^2$–action is a homomorphism $\Phi : \mathbb{Z}^2 \to \Omega^r$. We denote by $A^r$ the space of such actions, and denote $\Phi = (\Phi(0, 1), \Phi(0, 1))$ for short. Let $\Pi_1 : \mathbb{R} \times U^M_r(\mathbb{R}, SL(2, \mathbb{R})) \to \mathbb{R}$, $\Pi_2 : \mathbb{R} \times U^M_r(\mathbb{R}, SL(2, \mathbb{R})) \to U^M_r(\mathbb{R}, SL(2, \mathbb{R}))$ be the coordinate projections. Let also $\gamma_{n,m} \equiv \Pi_1 \circ \Phi(n, m)$ and $A^\Phi_{n,m} = \Pi_2 \circ \Phi(n, m)$.

Two fibered $\mathbb{Z}^2$ actions $\Phi$, $\Phi'$ are said to be conjugate if there exists a smooth map $B : \mathbb{R} \to SL(2, \mathbb{R})$ such that $$\Phi'(n, m) = (0, B) \circ \Phi(n, m) \circ (0, B)^{-1}, \quad \forall (n, m) \in \mathbb{Z}^2.$$  

That is $$A^\Phi_{n,m}(\cdot) = B(\cdot + \gamma_{n,m}) A^\Phi_{n,m}(\cdot) B(\cdot)^{-1}, \quad \gamma_{n,m}^\Phi = \gamma_{n,m}^\Phi.$$  

We denote $\Phi' = \text{Conj}_B(\Phi)$ for short. We say that an action is normalized if $\Phi(1, 0) = (1, Id)$, and in that case, if $\Phi(0, 1) = (\alpha, A)$, the map $A \in U^M_r(\mathbb{R}, SL(2, \mathbb{R}))$ is clearly $\mathbb{Z}$–periodic.
For any $M$-ultra-differentiable function $f: \mathbb{R} \to \mathbb{R}$ (not necessary periodic), one can also define
\[
\|f\|_{r,T} = c \sup_{s \in \mathbb{N}} \left((1 + s)^2 r^s \|D^s f(\theta)\|_{C^0([0,T])}M_s^{-1}\right),
\]
where $c = 4\pi^2/3$.

If $f: \mathbb{T} \to \mathbb{R}$ is periodic, we also denote $\|f\|_{r,1} = \|f\|_{M,r}$.

**Lemma 2.3.** (Lemma 2 of [FK09]) If $\Phi \in \Lambda^r$ with $\gamma_{1,0} = 1$, then there exists $B \in U^M_r(\mathbb{R}, SL(2, \mathbb{R}))$ and a normalized action $\tilde{\Phi}$ such that $\tilde{\Phi} = \text{Conj}_B(\Phi)$. Moreover, for any $T \in \mathbb{R}^+$, we have estimate
\[
\|B - Id\|_{rK^{-1},1} \leq \|\Phi(1,0) - Id\|_{r,1},
\]
\[
\|B\|_{r(K_sT)^{-1},T} \leq \|\Phi(1,0)\|_{T^{-1},T}, \quad \forall T \in \mathbb{R}^+,
\]
where $K_s$ is an absolute constant.

### 2.3.2. Renormalization of actions.
Following [AK06, FK09, AK15], we introduce the scheme of renormalization of $\mathbb{Z}^2$ actions.

Fixing $\lambda \neq 0$. Define $M_\lambda: \Lambda^r \to \Lambda^r$ by
\[
M_\lambda(\Phi)(n,m) := (\lambda^{-1}\gamma_{n,m}^\Phi, A_{n,m}^\Phi(\lambda \cdot \cdot)).
\]

Let $\theta_s \in \mathbb{R}$. Define $T_{\theta_s}: \Lambda^r \to \Lambda^r$ by
\[
T_{\theta_s}(\Phi)(n,m) := (\gamma_{n,m}^\Phi, A_{n,m}^\Phi(\cdot + \theta_s)).
\]

Let $U \in GL(2, \mathbb{R})$. Define $N_U: \Lambda^r \to \Lambda^r$ by
\[
N_U(\Phi)(n,m) := \Phi(n',m'), \quad \text{where } \left(\begin{array}{c} n' \\ m' \end{array}\right) = U^{-1}\left(\begin{array}{c} n \\ m \end{array}\right).
\]

Let $\tilde{Q}_n = \left(\begin{array}{cc} q_n & p_n \\ q_{n-1} & p_{n-1} \end{array}\right)$, and define for $n \in \mathbb{N}$ and $\theta_s \in \mathbb{R}$ the renormalized actions
\[
R^n(\Phi) := M_{\beta_{n-1}} \circ N_{\tilde{Q}_n}(\Phi), \quad R^n_{\theta_s}(\Phi) := T_{\theta_s}^{-1}\left[R^n(T_{\theta_s}(\Phi))\right].
\]

For any given cocycle $(\alpha, A)$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we set $\Phi = ((1, Id), (\alpha, A))$. Then by the definitions of the operators above, we get
\[
R^n_{\theta_s}(\Phi) = ((1, A^{(n,0)}), (\alpha_n, A^{(n,1)})),
\]
where
\[
A^{(n,0)}(\theta) = A_{\theta}^{(n,0)}(\theta_s + \beta_{n-1}(\theta - \theta_s)),
\]
\[
A^{(n,1)}(\theta) = A_{\theta}^{(n,1)}(\theta_s + \beta_{n-1}(\theta - \theta_s)).
\]

Thus $A^{(n,0)}$ and $A^{(n,1)}$ are $\beta_{n-1}^{-1}$-periodic and can be regarded as cocycles over the dynamics on $\mathbb{R}$ given by $\theta \mapsto \theta + 1$ and $\theta \mapsto \theta + \alpha_n$. It is easy to see that $A^{(n,1)}(\theta + 1)A^{(n,0)}(\theta) = A^{(n,0)}(\theta + \alpha_n)A^{(n,1)}(\theta)$, which expresses the
commutation of the cocycles. Based on this fact, there exists $D_n$ which is a normalizing map such that

$$D_n(\theta + 1)A^{(n,0)}(\theta)D_n(\theta)^{-1} = Id,$$  
$$D_n(\theta + \alpha_n)A^{(n,1)}(\theta)D_n(\theta)^{-1} = A^{(n)}(\theta),$$

which satisfies $A^{(n)}(\theta + 1) = A^{(n)}(\theta)$. Thus $A^{(n)}$ can be seen as an element of $C^0(\mathbb{T}, SL(2, \mathbb{R}))$, and $(\alpha_n, A^{(n)})$ is called a representative of the $n$-th renormalization of $(\alpha, A)$.

2.3.3. Convergence of the renormalized actions. The following result on convergence of renormalized actions was essentially contained in [AK06, FK09, AK15], which deal with cocycles in $C^r$ setting with $\ell \in \mathbb{N}$ and $\ell = \infty, \omega$. We will sketch the proof in the ultra-differentiable setting, just for completeness.

**Proposition 1 ([AK06, FK09, AK15]).** Suppose that $(\alpha, A) \in (0, 1) \setminus \mathbb{Q} \times U_r(\mathbb{T}, SL(2, \mathbb{R}))$. If $(\alpha, A)$ is $L^2$-conjugated to rotations and homotopic to the identity, then for almost every $\theta_0 \in \mathbb{R}$, there exists $D_n \in U_{r/K}^2(\mathbb{R}, SL(2, \mathbb{R}))$ with

$$\|D_n\|_{r/(K^2 r^*), T} \leq C^{q-1}(T+1),$$

such that

$$\text{Conj}_{D_n}(R^n_{\theta_n}(\Phi)) = ((1, Id), (\alpha_n, R_{\rho_n}e^{F_n})),$$

with $\|F_n\|_{r/K} \to 0$, where $K_*$ is an absolute constant defined in Lemma 2.3.

**Proof.** We first prove $\{A^{(n,i)}(\theta)\}_{n \geq 0}, i = 1, 2$, are precompact in $U_{r/K}^M$. Indeed, Theorem 5.1 of [AK06] shows that for any $(\alpha, A) \in (0, 1) \setminus \mathbb{Q} \times C^s(\mathbb{T}, SL(2, \mathbb{R}))$, if it is $L^2$-conjugated to rotation, then for almost every $\theta_0 \in \mathbb{R}$, there exists $K_0 > 0$ such that for every $d > 0$ and for every $n > n_0(d)$,

$$\|\vartheta^\ell A^{(n,i)}(\theta)\| \leq K_*^{\ell+1}\|A(\theta)\|_{C^s}, \ i = 1, 2, \ 0 \leq \ell \leq s, \ |\theta - \theta_0| < d/n,$$

which implies that

$$\|A^{(n,i)}(\theta)\|_{C^s} \leq 2K_*^{s+1}\|A(\theta)\|_{C^s}, \ i = 1, 2,$$

therefore, by the definition of norms of ultra-differentiable functions, we have

$$\|A^{(n,i)}(\theta)\|_{r/K, 1} \leq 2K_*\|A(\theta)\|_{r, 1} < \infty, \ i = 1, 2.$$

That is the sequences $\{A^{(n,i)}(\theta)\}_{n \geq 0}, i = 1, 2$, are uniformly bounded in $U_{r/K}$, which implies $\{A^{(n,i)}(\theta)\}_{n \geq 0}, i = 1, 2$, are precompact in $U_{r/K}^M$.

Assume $B \in L^2(\mathbb{T}, SL(2, \mathbb{R}))$ is the conjugation such that

$$B(\theta + \alpha)A(\theta)B(\theta)^{-1} \in SO(2, \mathbb{R}),$$

consequently by Theorems 4.3, Theorem 4.4 in [AK15] we have

$$R^n_{\theta_n}(\text{Conj}_{B(\theta_n)}(\Phi)) = ((1, \tilde{C}_n^{(1)}(\theta)), (\alpha_n, \tilde{C}_n^{(2)}(\theta)))$$
with \( C^{(i)}_n = R_{\theta_n} e^{U^{(i)}_n(\theta)} \), \( i = 1, 2 \), and
\[
\|U_n^{(i)}(\theta)\|_{r/K_s,1} \to 0, \ i = 1, 2, \text{if } n \to \infty, |\theta - \theta_s| \leq d/n, n \geq n_0(d).
\]

Using Lemma 3.2, there is a normalizing conjugation \( \widetilde{D}_n \), which is closed to identity in \( \| \cdot \|_{rK^{-2,1}} \)-topology such that
\[
\widetilde{D}_n(\theta + 1)C_n^{(1)}(\theta)\widetilde{D}_n(\theta)^{-1} = I d.
\]

Denote \( D_n = \widetilde{D}_n B \), the action \( \text{Conj}_{D_n}(R^n_{\theta_n}(\Phi)) \) is of form \((1, Id, (\alpha_n, R_{\theta_n} e^{F_n})) \) with \( \|F_n\|_{rK^{-2,1}} \to 0 \). Moreover, for any \( T \in \mathbb{R}^+ \), by Lemma 2.3 we get
\[
\|D_n\|_{r/(K_s^2T), T} \leq \|C_n^{(1)}\|_{r/K_s, T}^{T+1} \leq \|B(\theta_s)\|^{2(T+1)} \|A^{n,0}\|_{r, T}^{T+1}
\]
\[
\leq \|B(\theta_s)\|^{2(T+1)} \|A\|_{r,1}^{q_{n-1}(T+1)} \leq C^{q_{n-1}(T+1)},
\]
then (2.2) follows directly.

\[\square\]

3. Ultra-differentiable functions

As we introduced, one way to define the modulus of ultra-differentiable functions is by the growth of \( D^sf \). For periodic function \( f \in C^\infty(\mathbb{T}, \mathbb{R}) \), an alternative way is to define the modulus of ultra-differentiability by the decay rate of its Fourier coefficient. Attached to the sequence \((M_s)_{s \in \mathbb{N}} \), we can define \( \Lambda : [0, \infty) \to [0, \infty) \) by
\[
\Lambda(y) := \ln \left( \sup_{s \in \mathbb{N}} y^s M_s^{-1} \right) = \sup_{s \in \mathbb{N}} (s \ln y - \ln M_s).
\]

This defines a function \( \Lambda : [0, \infty) \to [0, \infty) \), which is continuous, constant equal to zero for \( y \leq 1 \) and strictly increasing for \( y \geq 1 \) (see [BFar, CC94] or [Thi03]).

For any \( f \in U^M_r(\mathbb{T}, \mathbb{R}) \), write it as \( f(\theta) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2\pi ik \theta} \), one easily checks that \( \Lambda \) controls the decay of the Fourier coefficients in the sense that
\[
|\hat{f}(k)| \leq \|f\|_{M,r} \exp \{-\Lambda(2\pi|k|)\}, \ \forall k \in \mathbb{Z}.
\]

For periodic function, using \( \Lambda \) is more natural and convenient. Now we derive some properties of \( \Lambda \) for \( f \in U^M_r(\mathbb{T}, \mathbb{R}) \) from (H1) and (H2), which will be the bases of our whole proof of the KAM scheme.

Lemma 3.1 (Proposition 10 of [BFar]). Let \( f, g \in U^M_r(\mathbb{T}, \mathbb{R}) \) with \( M \) satisfying (H1). Then \( f \cdot g \in U^M_r(\mathbb{T}, \mathbb{R}) \), and we have
\[
\|f \cdot g\|_{M,r} \leq \|f\|_{M,r} \|g\|_{M,r}.
\]

Remark 3.1. As explained in [BFar], the role of the normalizing constant \( c > 0 \) in the definition of \( \|f\|_{M,r} \) is to ensure that \( U^M_r(\mathbb{T}, \mathbb{R}) \) forms a standard Banach algebra with respect to multiplication.
Lemma 3.2 (Proposition 8 of [BFar]). Let \( f \in U_r^M(\mathbb{T}, \mathbb{R}) \) with \( M \) satisfying (H2). Then \( \partial f \in U_{r/2}^M(\mathbb{T}, \mathbb{R}) \) with
\[
\| \partial f \|_{r/2} \leq C_M r^{-1} \| f \|_r,
\]
where
\[
C_M := \sup_{s \in \mathbb{N}} \{ 2^{-s} M_{s+1} M_s^{-1} \} < \infty.
\]

Lemma 3.3. (H1) and (H2) imply that there exists \( \Gamma : [1, \infty) \to \mathbb{R}^+ \) such that the following hold:
\[
(A) : \left\{ \begin{array}{l}
(I) : \lim_{x \to \infty} \Gamma(x) = \infty, \\
(II) : \Gamma(x) \ln x \text{ is non-decreasing}, \\
(III) : \Lambda(y) - \Lambda(x) \geq (\ln y - \ln x) \Gamma(x) \ln x, \ \forall y > x \geq 1.
\end{array} \right.
\]

Proof. For any \( x \geq 1 \), we select \( s(x) \in \mathbb{N} \) as the one such that
\[
(3.3) \quad \Lambda(x) = \sup_{s \in \mathbb{N}} (x^s M_s^{-1}) = (x^{s(x)} M_{s(x)}^{-1}).
\]

Claim 1. The function \( s(x) \in \mathbb{N} \) is well-defined, non-decreasing with
\[
(3.4) \quad \lim_{x \to \infty} s(x)(\ln x)^{-1} = \infty.
\]

Proof. It is quite standard \( s(x) \in \mathbb{N} \) is well-defined, we first prove that \( s(x) \) is non-decreasing. By the definition of \( s(x) \), we have
\[
x^{s(x)} M_{s(x)}^{-1} \geq x^{s(x)+1} M_{s(x)+1}^{-1}, \quad x^{s(x)} M_{s(x)}^{-1} \geq x^{s(x)-1} M_{s(x)-1}^{-1},
\]
which implies
\[
(3.5) \quad M_{s(x)} / M_{s(x)-1} \leq x \leq M_{s(x)+1} / M_{s(x)}.
\]
Assume that there exist \( y > x \geq 1 \) such that \( s(y) < s(x) \). The fact that \( s(\cdot) \in \mathbb{N} \) implies \( s(y) + 1 \leq s(x) \). First, by (3.5) we get
\[
y \leq M_{s(y)+1} / M_{s(y)}, \quad x \geq M_{s(x)} / M_{s(x)-1}.
\]
However, by (H1) we know that \( \{ M_{\ell+1} / M_\ell \}_{\ell \in \mathbb{N}} \) is increasing, which together with \( s(y) + 1 \leq s(x) \), implies that
\[
y \leq M_{s(y)+1} / M_{s(y)} \leq M_{s(x)} / M_{s(x)-1} \leq x,
\]
this contradicts with the assumption \( y > x \). Thus \( s(x) \) is non-decreasing.
By (3.5), we have
\[
s^{-1}(x) \ln(M_{s(x)} / M_{s(x)-1}) \leq s^{-1}(x) \ln x \leq s^{-1}(x) \ln(M_{s(x)+1} / M_{s(x)}),
\]
then (3.4) follows from the assumption (H2).

Let \( \Gamma(x) = s(x)(\ln x)^{-1} \). Then (3.4) implies (I). Moreover, note \( \Gamma(x) \ln x = s(x) \), which together with the fact \( s(x) \in \mathbb{N} \) is non-decreasing, implies (II).
Now we prove (III). For any \( y > x \), by the fact \( s(x) \in \mathbb{N} \) is non-decreasing, we can distinguish the proof into two cases:
Case 1: $s(y) = s(x)$. By the definitions of $\Lambda(x)$ and $s(x)$, we get
\[ \Lambda(y) - \Lambda(x) = (\ln y - \ln x)s(x) = (\ln y - \ln x)\Gamma(x)\ln x. \]

Case 2: $s(y) \geq s(x) + 1$. The inequality on the left hand of (3.5) and the fact that $\{M_{s+1}/M_s\}_{s\in\mathbb{N}}$ is increasing imply
\[ y \geq M_{s(y)}/M_{s(y)-1} \geq M_{s(x)+1}/M_{s(x)}, \]
that is $\ln y \geq \ln M_{s(x)+1} - \ln M_{s(x)}$. Together with the definitions of $\Lambda(x)$ and $s(x)$, it yields
\[ \Lambda(y) - \Lambda(x) \geq \ln(y^{s(x)+1}M_{s(x)+1}^{-1}) - \ln(x^{s(x)}M_{s(x)}^{-1}) \]
\[ = (\ln y - \ln x)s(x) + \ln y - (\ln M_{s(x)+1} - \ln M_{s(x)}) \]
\[ \geq (\ln y - \ln x)s(x) = (\ln y - \ln x)\Gamma(x)\ln x. \]
We thus finish the whole proof. \qed

**Remark 3.2.** To give a heuristic understanding of the function $\Gamma$, we can assume that $\Lambda$ is differentiable, by (3.3), we can rewrite it as
\[ \Gamma(x) = x\Lambda'(x)(\ln x)^{-1}. \]
Now if we fix $\Lambda(x) = (\ln x)^\delta$, then $\Gamma(x) = x\Lambda'(x)(\ln x)^{-1} = \delta(\ln x)^{\delta-2}$ and (I) and (II) are equivalent to
\[ \delta(\ln x)^{\delta-2} \to +\infty \]
and $\delta(\ln x)^{\delta-1}$ is non-decreasing, which means $\delta > 2$.

For the function $f \in C^\infty(T, \mathbb{R})$ and any $K \geq 1$, we define the truncation operator $T_K$ and projection operator $R_K$ as
\[ T_K f(\theta) = \sum_{k \in \mathbb{Z}, |k| < K} \hat{f}(k)e^{2\pi ik\theta}, \quad R_K f(\theta) = \sum_{k \in \mathbb{Z}, |k| \geq K} \hat{f}(k)e^{2\pi ik\theta}. \]
We denote the average of $f(\theta)$ on $T$ by $[f(\theta)]_\theta = \int_T f(\theta)d\theta = \hat{f}(0)$. The norm of $R_K f(\theta)$ in a shrunken regime has the following estimate for $C^\infty$ functions satisfying (H1) and (H2).

**Lemma 3.4.** Under the assumptions (A), there exists $T_1 = T_1(M)$, such that for any $f \in U_\rho^M(T, \mathbb{R})$, if $Kr \geq T_1$, then
\[ \|R_K f\|_{M,r/2} \leq C(Kr^2)^{-1}\|f\|_{M,r} \exp\{-9^{-1}\Gamma(4Kr)\ln(4Kr)\}. \]
Particularly,
\[ \|R_K f\|_{C^0} \leq (Kr^{-2})^{-1}\|f\|_{M,r} \exp\{-\Lambda(\pi Kr)\}. \]
**Proof.** First by (II) and (III) in (A), for any $|k| \geq K$, we have
\[ \Lambda(2\pi k|r) - \Lambda(2\pi k|7/8)r \]
\[ \geq \{\ln(2\pi k|r) - \ln(2\pi k|7/8)r\}\Gamma(2\pi k|7/8)r \ln(2\pi k|7/8)r \]
\[ \geq \ln(8/7)\Gamma(4k|r) \ln(4k|r) > 9^{-1}\Gamma(4Kr)\ln(4Kr). \]
Moreover, by (I) in (A), we know there exists $T_1 = T_1(M)$, such that if $Kr \geq T_1$ then $\Gamma(4|k|r) > 18, \forall |k| \geq K$, which implies

$$\Lambda(|2\pi k|(7/8)r) - \Lambda(|2\pi k|(3/4)r) > 2 \ln(|4k|r).$$

Consequently, direct calculations show that

$$\sum_{|k| \geq K} \exp\{-\Lambda(|2\pi k|r)}|2\pi k(3r/4)|^s M^{-1}_s$$

$$\leq \sum_{|k| \geq K} \exp\{-\Lambda(|2\pi k|r)} \exp\{\Lambda(|2\pi k|(3/4)r)}$$

$$\leq \sup_{|k| \geq K} \exp\{-\Lambda(|2\pi k|r)} + \Lambda(|2\pi k|(7/8)r)}$$

$$\sum_{|k| \geq K} \exp\{-\Lambda(|2\pi k|(7/8)r)} + \Lambda(|2\pi k|(3/4)r)}$$

$$\leq \exp\{-9^{-1}\Gamma(4Kr) \ln(4Kr)} \sum_{|k| \geq K} |4kr|^{-2}$$

$$\leq (4Kr)^{-1} \exp\{-9^{-1}\Gamma(4Kr) \ln(4Kr)}.$$  

Finally, by (3.2), we have

$$\|D^s_\theta Rf\|_{C^s} \leq \|f\|_{M,r} \sum_{|k| \geq K} \exp\{-\Lambda(|2\pi k|r)}|2\pi k|^s,$$

then

$$\|Rf\|_{M,r/2} = 3^{-1}4\pi^2 \sup_{s \in \mathbb{N}} (r/2)^s (1 + s)^2 \|D^s_\theta Rf\|_{C^s} M^{-1}_s$$

$$\leq \|f\|_{M,r} \sum_{|k| \geq K} \exp\{-\Lambda(|2\pi k|r)}|2\pi k|(3r/4)|^s M^{-1}_s$$

$$\leq C(Kr)^{-1} \|f\|_{M,r} \exp\{-9^{-1}\Gamma(4Kr) \ln(4Kr)}.$$

where the last inequality follows from (3.8).

The conclusion (3.7) follows from similar computations, we thus omit the details. \[\Box\]

For the given function $f \in U_r^M(T, \mathbb{R})$, we define the $\| \cdot \|_{\Lambda,r}$ norm by

$$\|f\|_{\Lambda,r} = \sum_{k \in \mathbb{Z}} |\tilde{f}(k)| e^{\Lambda(2\pi kr)},$$

with $\Lambda$ being the one defined by (3.1). With the help of Lemma 3.3, we can discuss the relationship between the spaces $\| \cdot \|_{M,r}$ and $\| \cdot \|_{\Lambda,r}$.

**Lemma 3.5.** Under the assumptions (H1) and (H2), we have

$$\|f\|_{M,r} \leq C\|f\|_{\Lambda,2r},$$

$$\|f\|_{\Lambda,r/2} \leq (2\pi r)^{-1} (4 + c_M) \|f\|_{M,r}.$$
where \( c_M \) is a constant that only depends on the sequence \( M \).

**Proof.** First, (3.1) implies that for any \( y > 0 \), any \( s \in \mathbb{N} \),
\[
\exp\{-\Lambda(y)\}y^s \leq M_s,
\]
which yields
\[
\exp\{-\Lambda(yr)\}y^s = \exp\{-\Lambda(yr)\}(yr)^s r^{-s} \leq M_s(r)^{-s}, \forall yr > 0.
\]
Then
\[
\sup_{k \in \mathbb{Z}} \exp\{-\Lambda(\lVert 2\pi k | 2r \rVert)\} \leq \sup_{k \in \mathbb{Z}} \exp\{-\Lambda(\lVert 2\pi k | 2r \rVert)\} \leq M_s(2r)^{-s}.
\]
Thus for any \( f \in U_r^M(\mathbb{T}, \mathbb{R}) \), for any \( s \in \mathbb{N} \), we get
\[
\|D_0^s f\|_{C^0} \leq \sum_{k \in \mathbb{Z}} |\hat{f}(k)| \lVert 2\pi k \rVert^s
\leq \sup_{k \in \mathbb{Z}} \exp\{-\Lambda(\lVert 2\pi k | 2r \rVert)\} \|\hat{f}(k)\| \exp\{\Lambda(\lVert 2\pi k | 2r \rVert)\}
\leq \|f\|_{\Lambda,2r} M_s(2r)^{-s},
\]
which implies that
\[
\|f\|_{M,r} = 3^{-1} 4\pi^2 \sup_{s \in \mathbb{N}} \left( r^s (1 + s)^2 \|D_0^s f\|_{C^0} M_s^{-1} \right)
\leq \|f\|_{\Lambda,2r} \sup_{s \in \mathbb{N}} 3^{-1} 4\pi^2 2^{-s} (1 + s)^2 < C \|f\|_{\Lambda,2r}.
\]
Now we turn to the inequality in (3.11). Easily, for \( y \geq 1 \), we have
\[
\exp\{-\Lambda(2y) + \Lambda(y)\} = \inf_{s \in \mathbb{N}} \{(2y)^{-s} M_s y^{s(y)} M_s^{-1}\}
\leq (2y)^{-s(y)+2} M_s y^{s(y)} M_s^{-1} \leq c_M(2y)^{-2},
\]
where (by (H2))
\[
c_M := \sup_{s \in \mathbb{N}} \{2^{-s} M_{s+2} M_s^{-1}\} < \infty.
\]
Note \( \Lambda(y) = 0 \) if \( y \leq 1 \). Consequently, we have
\[
\|f\|_{\Lambda,r/2} = \sum_{k \in \mathbb{Z}} |\hat{f}(k)| \exp\{\Lambda(\lVert \pi k \rVert r)\}
\leq \|f\|_{M,r} \sum_{k \in \mathbb{Z}} \exp\{-\Lambda(\lVert 2\pi k | r \rVert) + \Lambda(\lVert \pi k \rVert)\}
\leq \|f\|_{M,r} \{ \sum_{|k| < (\pi r)^{-1}} + \sum_{|k| \geq (\pi r)^{-1}} \} \exp\{-\Lambda(\lVert 2\pi k | r \rVert) + \Lambda(\lVert \pi k \rVert)\}
\leq 2(\pi r)^{-1} \|f\|_{M,r} + c_M \|f\|_{M,r} \sum_{|k| \geq (\pi r)^{-1}} (2\pi k)^{-2}
\leq 2(\pi r)^{-1} \|f\|_{M,r} + (2\pi r)^{-1} c_M \|f\|_{M,r}.
\]
\[\square\]
We have stated the above lemma only for $\|f\|_{\Lambda,r/2}$ in (3.11) as this is the only case we shall need; but clearly one could obtain an estimate for any $\|\partial^s f\|_{\Lambda,r/2}, s \in \mathbb{N}$, by the similar discussions above.

4. The inductive step

4.1. Sketch of the proof. The proof of Theorem 1.2 is based on a non-standard KAM scheme which was first developed in [KWYZ18]. Now let us briefly introduce the main idea of the proof. We start from the cocycle $(\alpha, R_{\rho_f} e^{F_n})$ with $\|F_n\|$ of size $\varepsilon_n$, to conjugate it into $(\alpha, R_{\rho_f} e^{F_{n+1}})$ with a smaller perturbation, a crucial ingredient is to solve the homological equations

\begin{align}
f_1(\cdot + \alpha) - f_1 &= -(g_1 - [g_1]_\theta), \nonumber \\
e^{4\pi i \rho_f} f_2(\cdot + \alpha) - f_2 + g_2 &= 0.
\end{align}

However, if $\alpha$ is Liouvillean, (4.1) can’t be solved at all, even if in the analytic category. This is essentially different from the classical KAM scheme. Therefore, we have to leave $g_1(\theta)$ (at least the resonant terms of $g_1(\theta)$) into the normal form. As a result, from the second step of iteration we need to consider the modified cocycle $(\alpha, R_{\rho_f + (2\pi)^{-1} g(\theta)} e^{F_n(\theta)})$, thus the second equation in (4.1) is of the form

\[ e^{2i(2\pi \rho_f + g(\theta))} f(\cdot + \alpha) - f + g_2 = 0. \]

In order to get desired result, we distinguish the discussions into three steps. In the first step we eliminate the lower order terms of $g(\theta) \in U^M_r(\mathbb{T}, \mathbb{R})$ by solving the equation

\[ v(\theta + \alpha) - v(\theta) = -(\mathcal{T}_{\mathcal{Q}_n} g - [g]_\theta). \]

Although $\|g(\theta)\|$ is of size $\varepsilon_0$, $\|e^{iv}\|_r$ could be very large in Liouvillean frequency case. To control $\|e^{iv}\|_r$, the trick is to control $\|\text{Im}v(\theta)\|$ at the cost of reducing the analytic radius greatly, which was first developed in analytic case in [YZ14]. The key point here is that $v(\theta)$ is in fact a trigonometric polynomial, one can analytic continue $v(\theta)$ to become a real analytic function, and the “width” $r$ just plays the role of analytic radius. Therefore, one can shrink $r$ greatly in order to control $\|e^{iv}\|_r$ (Lemma 4.1). Consequently, the “width” will go to zero rapidly, and the convergence of the KAM iteration only works in the $C^\infty$ category.

The second step is to make the perturbation much smaller by solving the homological equation

\[ e^{2i(2\pi \rho_f + \tilde{g}(\cdot))} f(\cdot + \alpha) - f + h = 0, \]

where $\|\tilde{g}\| = O(\|F_n\|)$. By the method of diagonally dominant [KWYZ18], we can solve its approximation equation and then to make the perturbation as small as we desire (Lemma 4.3).

By these two steps, we can already get $C^\infty$ almost reducible result (Corollary 2). However, to get $C^\infty$ rotations reducible result, at the end of one
KAM step we need to inverse the first step, such that the conjugation is close to the identity (Lemma 4.6).

For simplicity, in the following parts we will shorten $U_r^M(\mathbb{T}, *)$ and $\| \cdot \|_{M,r}$ as $U_r(\mathbb{T}, *)$ and $\| \cdot \|_r$, also the letter $C$ denotes suitable (possibly different) large constant that do not depend on the iteration step.

4.2. **Main iteration lemma.** For the functions $\Lambda(x)$ and $\Gamma(x)$ in Lemma 3.3, by (I) in (A) we know that there exists $\bar{T} \geq T_1$, where $T_1$ is defined in Lemma 3.4, such that for any $x \geq \bar{T}$

\[(4.2) \quad \Gamma(x) \geq 64\Lambda^8 r^4, \]

\[(4.3) \quad \Lambda(x) \geq \ln x. \]

Denote

\[(4.4) \quad T = \max\{c_M^3, \bar{T}^3, (2^{-1}r)^{-12}, (4\gamma^{-1})^{2r} \}, \]

where $c_M$ is the one in (3.11). Then for the $T$ defined above, we claim that there exists $n_0$ such that $Q_{n_0+1} \leq T^4$ and $\overline{Q}_{n_0+1} \geq T$. Indeed, let $m_0$ be such that $Q_{m_0} \leq T \leq Q_{m_0+1}$. If $\overline{Q}_{m_0} \geq T$, then we set $n_0 = m_0 - 1$. Otherwise, if $\overline{Q}_{m_0} \leq T$, by the definition of $(Q_k)$, it then holds $Q_{m_0+1} \leq T^{4^r}$. By the selection, $\overline{Q}_{m_0+1} \geq T$, then $n_0 = m_0$ satisfy our needs. In the following we will shorten $n_0$ as 0, that is $\overline{Q}_r$ stands for $\overline{Q}_{r+n_0}$.

Without loss of generality we assume $0 < r_0 := 2^{-1}r \leq 1$. Set

\[(4.5) \quad \varrho_0 = 0, \quad \varrho_0 = T^{-8\Lambda^4 r^2}, \]

then $\varrho_0$ just depends on $\gamma, \tau, r, M$, but not on $\alpha$. Once we have this, we can define the iterative parameters as following, for $n \geq 1$

\[(4.6) \quad \tau_n = 2\overline{Q}_n^{-2} r_0, \quad r_n = \overline{Q}_{n-1}^{-2} r_0, \]

\[ \varepsilon_n = \varrho_{n-1} \overline{Q}_n^{-1} \varrho_n^{r_0}, \quad \varepsilon_n = C \sum_{l=0}^{n-1} \varepsilon_l. \]

To simplify the notations, for any $g \in C^0(\mathbb{T}, \mathbb{R})$, we denote

\[ R_g := \left( \begin{array}{cc} \cos 2\pi g & -\sin 2\pi g \\ \sin 2\pi g & \cos 2\pi g \end{array} \right) = e^{-2\pi gJ}, \quad J = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \]

and set

\[ F_r(\rho_f, \eta, \tilde{\eta}) := \left\{ (\alpha, R_{\rho_f + (2\pi)^{-1}g(\theta)}e^{F(\theta)}) : \| g \|_r \leq \eta, \| F \|_r \leq \tilde{\eta}, \right. \]

\[ \left. \rho_f = \rho(\alpha, R_{\rho_f + (2\pi)^{-1}g(\theta)}e^{F(\theta)}) \right\}. \]

Then the main inductive lemma is the following:

**Proposition 2.** Assume that $\rho_f \in DC_\alpha(\gamma, \tau)$, then for $n \geq 1$, the cocycle

\[(4.7) \quad (\alpha, R_{\rho_f + (2\pi)^{-1}g_n}e^{F_n(\theta)}) \in F_r(\rho_f, \varepsilon_n, \varepsilon_n), \]
with $R_{\mathbb{T}_n} g_n = 0$ can be conjugated to
\begin{equation}
(\alpha, R_{\rho f + (2\pi)^{-1} g_{n+1}} e^{F_{n+1}(\theta)}) \in \mathcal{F}_{r_{n+1}}(\rho_f, \varepsilon_{n+1}, \varepsilon_{n+1}),
\end{equation}
with $R_{\mathbb{T}_{n+1}} g_{n+1} = 0$ by the conjugation $\Phi_n$ with the estimate
\begin{equation}
\|\Phi_n - I\|_{r_{n+1}} \leq C \varepsilon_n^{\frac{1}{2}}.
\end{equation}

The construction of the conjugation in Proposition 2 is divided into three steps given in Lemma 4.1, Lemma 4.3 and Lemma 4.6 of the following.

**Lemma 4.1.** For $n \geq 1$, the cocycle
\begin{equation}
(\alpha, R_{\rho f + (2\pi)^{-1} g_n} e^{F_n(\theta)}) \in \mathcal{F}_n(\rho_f, \varepsilon_n),
\end{equation}
with $R_{\mathbb{T}_n} g_n = 0$ can be conjugated to the cocycle
\begin{equation}
(\alpha, R_{\rho f} e^{F_n(\theta)}) \in \mathcal{F}_n(\rho_f, 0, C \varepsilon_n),
\end{equation}
via the conjugation $(0, e^{-v_n J})$ with $\|e^{-v_n J}\|_{\mathfrak{r}_n} \leq C$.

Before giving the proof of Lemma 4.1 we give an auxiliary lemma. To this end, for $f(\theta) = \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{2\pi i k \theta} \in U_r(\mathbb{T}, \mathbb{R})$, we set $\hat{\theta} = \theta + i\tilde{\theta}$, $(\theta \in \mathbb{T}, |\tilde{\theta}| \leq r)$
\[ \tilde{f}(\hat{\theta}) = \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{2\pi i k(\theta + i\tilde{\theta})}. \]

Then we, formally, define the analytic norm
\[ \|\tilde{f}\|^*_r = \sum_{k \in \mathbb{Z}} |\hat{f}(k)| \sup_{|\tilde{\theta}| \leq r, \theta \in \mathbb{T}} |e^{2\pi i k(\theta + i\tilde{\theta})}| = \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^{2\pi k|r|.} \]

If $\text{Im} \tilde{\theta} = \bar{\theta} = 0$, then $\tilde{f}(\hat{\theta}) = f(\theta)$, and if $0 < |\text{Im} \tilde{\theta}| \leq r$, one has
\[ \|f\|_{\Lambda, r} = \sum_{k \in \mathbb{Z}} |\hat{f}(k)|e^{\Lambda(|2\pi k|r|)} \leq \|\tilde{f}\|^*_r. \]

In general, $\|\tilde{f}\|^*_r = \infty$, however, if $f$ is a trigonometric polynomial, then $\tilde{f}$ really defines a real analytic function in the strip $|\text{Im} \tilde{\theta}| \leq r$, motivated by this we have the following:

**Lemma 4.2.** Assume that $v$ is the solution of
\begin{equation}
v(\theta + \alpha) - v(\theta) = -(\mathcal{T}_{\mathbb{T}_n} g - [g]_\theta),
\end{equation}
where $g \in U_r(\mathbb{T}, \mathbb{R})$ with $\|g(\theta)\|_{\mathfrak{r}_n} \leq C \varepsilon_0$. Then
\[ \|e^{iv(\theta)}\|_{\mathfrak{r}_n} \leq C. \]

**Proof.** By comparing the Fourier coefficients of (4.12) we have
\[ v(\theta) = \sum_{0 < |k| < \mathcal{T}_n} \hat{v}(k)e^{2\pi i k \theta} = - \sum_{0 < |k| < \mathcal{T}_n} \hat{g}(k)(e^{2\pi i k \alpha} - 1)^{-1} e^{2\pi i k \theta} \]
with estimate
\begin{equation}
|\hat{v}(k)| \leq \mathcal{T}_n |\hat{g}(k)|, \quad 0 < |k| < \mathcal{T}_n.
\end{equation}
For $\theta \in \mathbb{T}$, by the fact $g(\theta) \in \mathbb{R}$, one has $v(\theta) \in \mathbb{R}$. Thus for the function

$$
\tilde{v}(\theta) - v(\theta) = \sum_{0 < |k| < n} \tilde{v}(k)e^{2\pi ik\theta}(e^{-2\pi k\theta} - 1),
$$

we have $\text{Im}\tilde{v}(\theta) = \text{Im}(\tilde{v}(\theta) - v(\theta))$. Consequently, by (4.13), we have:

\begin{align*}
\|\text{Im}\tilde{v}(\theta)\|_{\mathcal{Q}_n^{-2}}^* &\leq \|\tilde{v}(\theta) - v(\theta)\|_{\mathcal{Q}_n^{-2}}^*
= \sum_{0 < |k| < n} |\tilde{v}(k)| \sup_{|\theta| \leq \mathcal{Q}_n^2, \theta \in \mathbb{T}} |e^{2\pi ik\theta}(e^{-2\pi k\theta} - 1)|
\leq \sum_{0 < |k| < n} \mathcal{Q}_n^{-1} |\tilde{g}(k)| 16\pi |k|
\leq 16\pi \mathcal{Q}_n^{-1} |\tilde{T}(\pi r_n)^{-1} \sum_{|k| < \tilde{T}(\pi r_n)^{-1}} \tilde{g}(k)|
+ 16(\mathcal{Q}_n r_n)^{-1} \sum_{\tilde{T}(\pi r_n)^{-1} \leq |k| < \mathcal{Q}_n} |\tilde{g}(k)| \exp\{\Lambda(\pi |k| r_n)\}
\leq 32\tilde{T}(\mathcal{Q}_n r_n)^{-1} \|g\|_{\Lambda, r_n/2},
\end{align*}

where the third inequality follows by (4.3).

By (3.11) of Lemma 3.5, one can further compute

$$
\|\text{Im}\tilde{v}(\theta)\|_{\mathcal{Q}_n^{-2}}^* \leq 32\tilde{T}(2\pi \mathcal{Q}_n)^{-1} (4 + c_M)\mathcal{Q}_n^{-1} \|\tilde{g}\|_{r_n} \leq \|\tilde{g}\|_{r_n},
$$

the last inequality follows by $\mathcal{Q}_n \geq \max\{T, \mathcal{Q}_n^{-1}\}$, $n \geq 1$ (by Lemma 2.2 and choice of $\mathcal{Q}_n$). Therefore, by (3.10) of Lemma 3.5, we have

\begin{align*}
\|e^{i\bar{v}(\theta)}\|_{r_n} &\leq C\|e^{i\bar{v}(\theta)}\|_{\Lambda, 2r_n} \leq C\|e^{i\tilde{v}(\theta)}\|_{2\mathcal{Q}_n} \leq C\|e^{i\tilde{v}(\theta)}\|_{\mathcal{Q}_n^{-2}}^*
\leq C\exp\{\|\text{Im}\tilde{v}(\theta)\|_{\mathcal{Q}_n^{-2}}^*\} \leq C\exp\{\|\tilde{g}\|_{r_n}\} < C.
\end{align*}

\textbf{Proof of Lemma 4.1:} Assume that $v_n$ is the solution of

$$
v_n(\theta + \alpha) - v_n(\theta) = -(g_n(\theta) - \tilde{g}_n(0)).
$$

Note $R\mathcal{Q}_n g_n = 0$, then by Lemma 4.2 we have

$$
\|e^{v_n J}\|_{r_n} \leq C.
$$

Direct computation shows that $(0, e^{-v_n J})$ conjugates the cocycle (4.10) into $(\alpha, R_{\rho_f + (2\pi)^{-1}\tilde{\theta}(0)} e^{F_n})$, with $F_n = e^{-v_n J} F_n(\theta) e^{v_n J}$. Thus by Lemma 3.1, we have

\begin{align*}
\|F_n\|_{r_n} &\leq \|e^{-v_n J}\|_{r_n} \|F_n\|_{r_n} \|e^{v_n J}\|_{r_n} \leq C\varepsilon_n.
\end{align*}
we give one important lemma, which means

\[(4.15) \quad |\widehat{g}_n(0)| \leq C\varepsilon_n.\]

Also note if \(B,D\) are small \(sl(2,\mathbb{R})\) matrices, then there exists \(E \in sl(2,\mathbb{R})\) such that

\[e^B e^D = e^{B+D+E},\]

where \(E\) is a sum of terms at least 2 orders in \(B,D\). Consequently, by (4.14), (4.15) and Lemma 3.1, there exists \(F_n \in U_T(\mathbb{T}, sl(2,\mathbb{R}))\) such that \(R_{\rho_f+(2\pi)^{-1}\widehat{g}_n(0)}e^\widehat{F}_n = R_{\rho_f}e^\widehat{F}_n\) with estimate \(\|\widehat{F}_n\|_{\tau_n} \leq C\varepsilon_n.\)

Once we get (4.11), we will further conjugate it to another cocycle with much smaller perturbation. We will give a lemma which can be applied to more general cocycles rather than just (4.11).

**Lemma 4.3.** Consider the cocycle \(\alpha, R_{\rho_f}e^{\widehat{F}(\theta)}\) with \(\rho_f = \rho(\alpha, R_{\rho_f}e^{\widehat{F}(\theta)}) \in DC_\alpha(\gamma, \tau)\) and

\[(4.16) \quad \|\widehat{F}\|_{\tau_n} \leq 8^{-2}\gamma^2 Q_{n+1}^{-2} \gamma^2, \quad n \geq 0.\]

Then there is a conjugation map \(\Psi_n \in U_{\tau_{n+1}}(\mathbb{T}, SL(2,\mathbb{R}))\) with

\[\|\Psi_n - I\|_{\tau_{n+1}} \leq \|\widehat{F}\|_{\tau_n}^{\frac{1}{2}}, \quad n \geq 0,\]

such that \(\Psi_n\) conjugates the cocycle \(\alpha, R_{\rho_f}e^{\widehat{F}(\theta)}\) into

\[(4.17) \quad (\alpha, R_{\rho_f+(2\pi)^{-1}\widehat{g}_n}e^{G(\theta)}) \in \mathcal{F}_{\tau_{n+1}}(\rho_f, 2\|\widehat{F}\|_{\tau_n}, \varepsilon),\]

with \(R_{\tau_{n+1}}\widehat{g}_n = 0\) and \(n \geq 0\), where \(\varepsilon = C^{-2}\|\widehat{F}\|_{\tau_n}^{-\frac{1}{4}}(Q_{n+1}^{-1})^{\frac{1}{4}}\).

Before giving the proof of Lemma 4.3 we give one important lemma, which is about the estimate of small divisors and serves as the fundamental ingredients of the proof. Although the proof is quite simple, it is the key observation that to obtaining semi-local results.

**Lemma 4.4.** For any \(0 < \gamma < 1, \tau > 1\), assume that \(Q_{n+1} \geq T\) and \(\rho \in DC_\alpha(\gamma, \tau)\), then for any \(|k| \leq Q_{n+1}^{-\frac{1}{2}}\), we have

\[(4.18) \quad |e^{2\pi i(k\alpha+2\rho)} - 1| \geq \gamma Q_{n+1}^{-\tau^2}.\]

**Proof.** We just need to estimate \(|e^{2\pi i(k\alpha+2\rho)} - 1|\) since

\[|e^{2\pi i(k\alpha-2\rho)} - 1| = |e^{2\pi i(-k\alpha+2\rho)} - 1|.\]

**Case 1.** \(Q_{n+1} \geq Q_{n+1}^{2\tau}\). Then our assumptions imply

\[|e^{2\pi i(k\alpha+2\rho)} - 1| = 2|\sin \pi (k\alpha + 2\rho)| > \|k\alpha + 2\rho\|_2 \geq \gamma Q_{n+1}^{-\tau^2} \geq \gamma Q_{n+1}^{-\tau^2}.\]
Case 2. \( \Omega_{n+1} > Q_{n+1}^{2\tau} \). Write \( k = \tilde{k} + mQ_{n+1}, \ m \in \mathbb{Z} \) with \( |\tilde{k}| < Q_{n+1} \). Then we have

\[
|m| \leq |k|/Q_{n+1} < Q_{n+1}^{3/2}/Q_{n+1}.
\]

Consequently, by the assumption that \( \rho \in DC_\alpha(\gamma, \tau) \), one has

\[
|e^{2\pi i (k + 2\rho) \omega} - 1| > \|k + mQ_{n+1} + 2\rho\|_2 \geq \|k + 2\rho\|_2 - |m|Q_{n+1}\| \geq \gamma Q_{n+1}^{-\tau} - |m|/Q_{n+1} \geq \gamma Q_{n+1}^{-\tau} - Q_{n+1}^{3/2}Q_{n+1}^{-1} > 2^{-1}\gamma Q_{n+1}^{-\tau},
\]

where the last inequality is by

\[
\frac{1}{Q_{n+1}^{3/2}} = Q_{n+1}^{2\tau} \frac{1}{Q_{n+1}^{2\tau}} \geq 2^{\gamma^{-1}}Q_{n+1}^{-\tau},
\]

which is guaranteed by \( \Omega_{n+1} \geq (2\gamma^{-1})^{2\tau} \) and \( \Omega_{n+1} > Q_{n+1}^{2\tau} \). \( \square \)

Set \( su(1, 1) \) be the space consisting of matrices of the form \( \begin{pmatrix} t & v \\ \overline{v} & -t \end{pmatrix} \) \( (t \in \mathbb{R}, \ v \in \mathbb{C}) \), we simply denote such a matrix by \( \{t, v\} \). Recall that \( sl(2, \mathbb{R}) \) is isomorphic to \( su(1, 1) \) by the rule \( A \mapsto MAM^{-1} \), where \( M = \frac{1}{1+i} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \),

and a simple calculation yields

\[
M \begin{pmatrix} x & y + z \\ y - z & -x \end{pmatrix} M^{-1} = \begin{pmatrix} iz & x - iy \\ x + iy & -iz \end{pmatrix}, \ x, y, z \in \mathbb{R}.
\]

Motivated by Lemma 4.4, we can define the following non-resonant and resonant spaces.

\[
\mathcal{B}_r^{(nre)} = \{0, T_{\Omega_{n+1}^{3/2}} g(\theta) : g \in U_r(\mathbb{T}, \mathbb{C})\},
\]

\[
\mathcal{B}_r^{(re)} = \{f(\theta), T_{\Omega_{n+1}^{3/2}} g(\theta) : g \in U_r(\mathbb{T}, \mathbb{C}), \ f \in U_r(\mathbb{T}, \mathbb{R})\}.
\]

It follows that \( U_r(\mathbb{T}, su(1, 1)) = \mathcal{B}_r^{(nre)} \oplus \mathcal{B}_r^{(re)} \). In order to prove Lemma 4.3, we will need the following lemma:

**Lemma 4.5.** Assume that \( A = \text{diag}\{e^{-2\pi i \rho}, e^{2\pi i \rho}\} \) and \( g \in U_r(\mathbb{T}, su(1, 1)) \). If \( \rho \in DC_\alpha(\gamma, \tau) \), and

\[
\|g\|_r \leq 8^{-2\gamma^2} Q_{n+1}^{-2\tau^2}, \ n \geq 0,
\]

then there exist \( Y \in \mathcal{B}_r^{(nre)} \) and \( g^{(re)} \in \mathcal{B}_r^{(re)} \) such that

\[
e^{Y(\cdot + \alpha)} A e^{g^{(\cdot)}(\cdot)} = A e^{g^{(re)}(\cdot)}
\]

with \( \|Y\|_r \leq \|g\|_r^{1/2} \) and \( \|g^{(re)}\|_r \leq 2\|g\|_r \).
The proof of this lemma, which involves the homotopy method, is postponed to Appendix. Similar proofs appeared in [YZ14, Dia06].

**Proof of Lemma 4.3:** Since $SL(2, \mathbb{R})$ is isomorphic to $SU(1, 1)$, instead of $(\alpha, R_\rho, e^{\tilde{F}(\theta)})$, we just consider $(\alpha, A e^{W(\theta)})$, where $A = MR_{\rho_f} M^{-1} = \text{diag}\{e^{-2\pi i \rho_f}, e^{2\pi i \rho_f}\} \in SU(1, 1), W = M \tilde{F} M^{-1} \in su(1, 1)$.

Since $\rho_f \in D_\alpha(\gamma, \tau)$ and $\|\tilde{F}\|_{\tau_n} \leq 8^{-2} \gamma^2 Q_{n+1}^{-2d^2}$, by Lemma 4.5, there exist $Y \in \mathcal{B}_{\tau_n}^{(re)}$ and $W^{(re)} \in \mathcal{B}_{\tau_n}^{(re)}$ such that $e^Y$ conjugates $(\alpha, A e^W)$ to $(\alpha, A e^{W^{(re)}})$ with

$$
\|Y\|_{\tau_n} \leq \|\tilde{F}\|_{\tau_n}^{1/2}, \quad \|W^{(re)}\|_{\tau_n} \leq 2\|\tilde{F}\|_{\tau_n}.
$$

(4.19)

Denote $W^{(re)}(\theta) = \{\tilde{f}(\theta), R_\gamma^{-1} \tilde{g}(\theta)\} \in \mathcal{B}_{\tau_n}^{(re)}$. Thus by (4.19)

$$
\|\tilde{f}(\theta)\|_{\tau_n} \leq \|W^{(re)}(\theta)\|_{\tau_n} \leq 2\|\tilde{F}(\theta)\|_{\tau_n}.
$$

Note $\mathcal{Q}_{n+1} \geq \mathcal{Q}^{24} (\text{by Lemma 2.2})$ and $\mathcal{Q}_{n+1} \geq T \geq r_0^{-12}$ (by (4.4)) we get

$$
\frac{\mathcal{Q}^{1/2}}{\mathcal{Q}_{n+1}} > \frac{\mathcal{Q}^{1/2}}{\mathcal{Q}_{n+1}} > 4^{-1} \mathcal{Q}_n^{-2} \tau_n^{-2} \gg 1, \ n \geq 0,
$$

which implies

$$
\frac{\mathcal{Q}^{1/2}}{\mathcal{Q}_{n+1}} T \mathcal{T} > \frac{\mathcal{Q}^{1/2}}{\mathcal{Q}_{n+1}} + T^{1/2} \geq T_1.
$$

(4.20)

Set $P(\theta) = W^{(re)}(\theta) - \{T_{\mathcal{Q}^{1/2}} ^{-1} \tilde{f}(\theta), 0\}$, thus $T_{\mathcal{Q}^{1/2}} ^{-1} P(\theta) = 0$, then by (3.6) in Lemma 3.4, we get

$$
\|P(\theta)\|_{\tau_n/2} \leq C(\mathcal{Q}^{1/2} / \mathcal{Q}_{n+1}^{1/2}) \|P(\theta)\|_{\tau_n} \exp\{-9^{-1} \Gamma(4 \mathcal{Q}^{1/2}_{n+1} / \mathcal{Q}_{n+1}) \ln(4 \mathcal{Q}^{1/2}_{n+1} / \mathcal{Q}_{n+1})\}
$$

$$
\leq 12 C \|\tilde{F}(\theta)\|_{\tau_n} \exp\{-9^{-1} \Gamma(4 \mathcal{Q}^{1/2}_{n+1} / \mathcal{Q}_{n+1}) \ln(4 \mathcal{Q}^{1/2}_{n+1})\}
$$

$$
\leq (2C^2)^{-1} \|\tilde{F}(\theta)\|_{\tau_n} \frac{1}{2} \left(\frac{\mathcal{Q}^{1/2}}{\mathcal{Q}_{n+1}}\right),
$$

where the second inequality is by (4.20) and the fact that $\Gamma(x) \ln x$ is non-decreasing, i.e., $(\Pi)$ in (A), and the last inequality is by (4.2), that is $\Gamma(\mathcal{Q}^{1/2}_{n+1}) > 64 A^8 r^4$.

Note

$$
A e^{W^{(re)}} = A e^{\{\mathcal{Q}_{n+1}^{-1} \tilde{f}(\theta), 0\}} E, \ E = e^{-\{\mathcal{Q}_{n+1}^{-1} \tilde{f}(\theta), 0\}} e^{W^{(re)}}.
$$
Then by Lemma 3.1 we have
\[
\|E - I\|_{\tau_n^2} \leq e^{\|T_{\tau_{n+1}}^* f_{\tau\tau_n^2}\|} e^{W_{(\epsilon)}} - e^{\{T_{\tau_{n+1}}^* f_{\tau\tau_n^2}\}} \|_{\tau_n^2} \\
= e^{\|T_{\tau_{n+1}}^* f_{\tau\tau_n^2}\|} e^{\{T_{\tau_{n+1}}^* f_{\tau\tau_n^2}\} + P - e^{\{T_{\tau_{n+1}}^* f_{\tau\tau_n^2}\}}} \|_{\tau_n^2} \\
\leq e^{2\|T_{\tau_{n+1}}^* f_{\tau\tau_n^2}\|} e^{\|P\|_{\tau_n^2}} \|_{\tau_n^2} \\
\leq 2\|P\|_{\tau_n^2} \leq C^{-2} \|\tilde{F}\|_{\tau_n^2}^{-1/2} (\tau_{n+1})^2.
\]

Thus by implicit function theorem, there exists \( \tilde{G} \in U_{\tau_n^2}(\mathbb{T}, su(1, 1)) \) such that \( E = e^{\tilde{G}} \) with
\[
\|\tilde{G}\|_{\tau_n^2} \leq \|E - I\|_{\tau_n^2} < C^{-2} \|\tilde{F}\|_{\tau_n^2}^{-1/2} (\tau_{n+1}).
\]

Now we go back to \( SL(2, \mathbb{R}) \). Let \( \Psi_n = e^{M_{+1} Y \epsilon} \). Then \( \|\Psi_n - I\|_{\tau_n^2} \leq \|Y\|_{\tau_n} \). Moreover, \( \Psi_n \) conjugates the cocycle \((\alpha, \ R_{\rho_f} e^{\tilde{F} (\theta)})\) to (4.17) with \( G = M_{+1} \tilde{G} \), \( g_n(\theta) = -T_{\tau_{n+1}} \tilde{f}(\theta) \). Obviously, \( \mathcal{R}_{\tau_{n+1}} g_n = 0 \).

To ensure the composition of the conjugations is close to the identity, we do one more conjugation which is the inverse of transformation in Lemma 4.1:

**Lemma 4.6.** Assume that \( v_n \) is the one defined in Lemma 4.1. Then for any \( n \geq 1 \), \((0, e^{v_n(\theta)J})\) further conjugates the cocycle
\[
(\alpha, \ R_{\rho_f + (2\pi)^{-1} g_n} e^{G(\theta)}) \in \mathcal{F}_{\tau_{n+1}} (\rho_f, C\epsilon_n, C^{-1} \epsilon_{n+1}),
\]
with \( \mathcal{R}_{\tau_{n+1}}^* \tilde{g}_n = 0 \), to the cocycle
\[
(\alpha, R_{\rho_f + (2\pi)^{-1} g_n + e^{F_{n+1}}} e^{F_{n+1}}) \in \mathcal{F}_{\tau_{n+1}} (\rho_f, \tilde{\epsilon}_{n+1}, \epsilon_{n+1})
\]
with \( \mathcal{R}_{\tau_{n+1}} g_{n+1} = 0 \).

**Proof.** Since \( v_n \) is the solution of \( v_n(\theta + \alpha) - v_n(\theta) = -g_n(\theta) + \tilde{g}_n(0) \), then \((0, e^{v_n(\theta)J})\) conjugates the cocycle \((\alpha, \ R_{\rho_f + (2\pi)^{-1} \tilde{g}_n} e^{G(\theta)})\) to
\[
(\alpha, R_{\rho_f + (2\pi)^{-1} (g_n + g_n - \tilde{g}_n(0))} e^{F_{n+1}(\theta)}),
\]
where \( F_{n+1} = e^{v_n(\theta)J} G e^{-v_n(\theta)J} \). Let \( g_{n+1} = \tilde{g}_n + g_n - \tilde{g}_n(0) \), then by Lemma 3.2 and Lemma 4.1, we have estimates
\[
\|g_{n+1}\|_{\tau_{n+1}} \leq \|\tilde{g}_n\|_{\tau_{n+1}} + \|g_n - \tilde{g}_n(0)\|_{\tau_n} \leq C\epsilon_n + \tilde{\epsilon}_n = \tilde{\epsilon}_{n+1},
\]
\[
\|F_{n+1}\|_{\tau_{n+1}} \leq \|e^{v_n J}\|_{\tau_{n+1}}^2 \|G\|_{\tau_{n+1}} \leq \epsilon_{n+1}.
\]

Obviously, \( \mathcal{R}_{\tau_{n+1}} g_{n+1} = 0 \). Moreover, the fibered rotation number does not change since \( e^{v_n J} \) is homotopic to the identity. \( \square \)
Now we are in the position to prove Proposition 2. First by Lemma 4.1, 
\((0, e^{-v_nJ})\) conjugates the cocycle (4.7) to
\[
(\alpha, R_{\rho_f} e^{\bar{F}_n(\theta)}) \in \mathcal{F}_{r_n}(\rho_f, 0, C\varepsilon_n).
\]
Moreover, by our definition of \(\varepsilon_n\), one can easily check that
\[
C\varepsilon_n = C\varepsilon_{n-1}Q_n^{-1/2} (Q_{n+1}^\frac{1}{2}) \leq CQ_n^{-8\Delta^4 r^2} \leq 8^{-2\gamma^2} Q_{n+1}^{-2\tau}, n \geq 1,
\]
the last inequality holds since, by Lemma 2.1, \(Q_n^\Delta \geq Q_{n+1}\) and \(Q_n^\tau \geq T \geq (4\gamma^{-1})^{2\tau}, n \geq 1\). That is (4.16) holds with \(C\varepsilon_n\) in place of \(\|\bar{F}_n\|_{\tau_n}\). Then by the assumption \(\rho_f \in DC_\alpha(\gamma, \tau)\), one can apply Lemma 4.3, and there exists \(\Psi_n \in U_{r_{n+1}}(\mathbb{T}, SL(2, \mathbb{R}))\) with
\[
\|\Psi_n - I\|_{r_{n+1}} \leq C\varepsilon_n^\frac{1}{r},
\]
which further conjugates the obtained cocycle into
\[
(\alpha, R_{\rho_f + (2\pi)^{-1}\bar{g}_n} e^{G(\theta)}) \in \mathcal{F}_{r_{n+1}}(\rho_f, 2C\varepsilon_n, C^{-2}C\varepsilon_n Q_{n+1}^{-1/2} (Q_{n+1}^\frac{1}{2}))
\]
\[
= \mathcal{F}_{r_{n+1}}(\rho_f, C\varepsilon_n, C^{-1}e^{-1} \varepsilon_{n+1}).
\]
Finally, by Lemma 4.6, \((0, e^{v_n(\theta)J})\) further conjugates the cocycle above to (4.8) with desired estimates. Let \(\Phi_n = e^{v_n(\theta)J} \Psi_n e^{-v_n(\theta)J}\), then by Lemma 3.1 and Lemma 4.1, we have
\[
\|\Phi_n - I\|_{r_{n+1}} = \|e^{v_n(\theta)J(\Psi_n - I)e^{-v_n(\theta)J}}\|_{r_{n+1}}
\]
\[
\leq \|e^{v_n(\theta)J}\|_{\tau_n}^2 \|\Psi_n - I\|_{r_{n+1}} < C\varepsilon_n^\frac{1}{r},
\]
which finishes the whole proof. \(\square\)

5. Proof of Theorem 1.1 and Theorem 1.2

5.1. Proof of Theorem 1.2. Set \(A = R_{\rho} e^{F}\). By the assumption \(\rho(\alpha, R_{\rho} e^{F}) = \rho_f\) and (2.1) one has \(\rho_f - g \leq 2\|F\|_{c_0}\), thus one can rewrite \((\alpha, R_{\rho_f} e^{\bar{F}})\) with \(\|\bar{F}\|_r \leq C\|F\|_r\). Set \(\varepsilon_* := C^{-1} \varepsilon_0\), where \(\varepsilon_0\) is the one defined by (4.5).

Set \(\tau_0 = r\), by the selection of \(\varepsilon_0\) and \(Q_1 \leq T^{\Delta^4}, T \geq (4\gamma^{-1})^{2\tau}, \) we get
\[
\|\bar{F}\|_{\tau_0} \leq C\varepsilon_* = \varepsilon_0 = T^{-8\Delta^4 r^2} \leq 8^{-2\gamma^2} Q_{1}^{-2\tau^2}.
\]
Since we further assume \(\rho_f \in DC_\alpha(\gamma, \tau)\), one can apply Lemma 4.3, then there exists \(\Psi_0 \in U_{r_1}(\mathbb{T}, SL(2, \mathbb{R}))\) with
\[
\|\Psi_0 - I\|_{r_1} \leq C\varepsilon_0^\frac{1}{r},
\]
which conjugates the cocycle \((\alpha, R_{\rho_f} e^{\bar{F}})\) into
\[
(\alpha, R_{\rho_f + (2\pi)^{-1}\bar{g}_0} e^{G_0}) \in \mathcal{F}_{r_1}(\rho_f, \varepsilon_1, \varepsilon_1).
\]
We emphasize that in the first iteration step, we only apply Lemma 4.3, without applying Proposition 2, which is quite different from the rest steps.

Now we set $\tilde{g}_0 = g_1$, $G_0 = F_1$, and $\Phi_0 = \Psi_0$. Then one can apply Proposition 2 inductively, and get a sequence of transformations $\{\Phi_n\}_{n \geq 0}$ with estimate $\|\Phi_n - I\|_{r_{n+1}} \leq C\varepsilon_n^\frac{1}{n}$. Furthermore, let

$$\Phi^{(n)} = \Phi_{n-1} \circ \Phi_{n-2} \circ \cdots \circ \Phi_0, \quad \Phi = \lim_{n \to \infty} \Phi^{(n)},$$

then $\Phi^{(n)}$ conjugates the original cocycle $(\alpha, R_{\rho_j} e^{F})$ to $(\alpha, R_{\rho_j + (2\pi)^{-1} g_n} e^{F_n(\theta)})$.

Finally, let’s show the convergence of $\Phi^{(n)}$. Let $\Phi = \lim_{n \to \infty} \Phi^{(n)}$, we will show $\Phi \in C^\infty(\mathbb{T}, SL(2, \mathbb{R}))$. Indeed, by the definition of $\|\cdot\|_r$-norm we have

$$\|D^j_{g}f\|_{C^0} \leq \|f\|_r r^{-j} M_j, \quad \forall f \in U_r(\mathbb{T}, SL(2, \mathbb{R})),$$

and by (I) of (A), for any $j \in \mathbb{N}$, there exists $n_j \in \mathbb{N}$, such that for any $n \geq n_j$, we have $CM_j \leq \frac{1}{\epsilon_{n-1}}$, and $\Gamma^j \left(\frac{\partial}{\partial_n}\right) \geq 24j$. By (4.9) and standard computation, we get $\|\Phi^{(n+1)} - \Phi^{(n)}\|_{r_{n+1}} \leq C\varepsilon_n^\frac{1}{n}$, then by (5.1) we can further compute

$$\|D^j(\Phi^{(n+1)} - \Phi^{(n)})\|_{C^0} \leq \|\Phi^{(n+1)} - \Phi^{(n)}\|_{r_{n+1}} M_j r^{-j}_{n+1} \leq C\varepsilon_n^\frac{1}{n} M_j r^{-j}_{n+1}$$

$$= \frac{1}{Q_n^{\frac{1}{2}} + \frac{1}{2}(\frac{\partial}{\partial_n})^j} \frac{1}{\epsilon_{n-1}} C M_j Q_n^{2j} T_0^{-j}$$

$$< \frac{1}{Q_n^{\frac{1}{2}} + \frac{1}{2}(\frac{\partial}{\partial_n})^j} \frac{1}{\epsilon_{n-1}} Q_n^{\frac{1}{2}} < \frac{1}{Q_n^{\frac{1}{2}} + \frac{1}{2}(\frac{\partial}{\partial_n})^j} \frac{1}{\epsilon_{n-1}} = \frac{1}{\epsilon_n},$$

which means $\Phi \in C^\infty(\mathbb{T}, SL(2, \mathbb{R}))$. Let $g_\infty = \lim_{n \to \infty} g_n$, then $\Phi$ conjugates the cocycle $(\alpha, R_{\rho_j} e^{F})$ to $(\alpha, R_{\rho_j + (2\pi)^{-1} g_\infty})$, where $g_\infty \in C^\infty(\mathbb{T}, \mathbb{R})$. □

Note the proof of the proposition 2 is separated into three steps and if we just manipulate the first two steps, we get the local almost reducibility.

**Corollary 2.** Under the assumptions of Theorem 1.2, there exists a sequence of $B_\ell \in U_{r_\ell}(\mathbb{T}, SL(2, \mathbb{R}))$ transforming $(\alpha, A)$ into $(\alpha, R_{\rho_j} e^{F_\ell})$ with estimates

$$\|B_\ell\|_{r_\ell} \leq C, \quad \|F_\ell\|_{r_\ell} \leq \varepsilon_\ell.$$

**Proof.** Set $B_\ell = \Psi_{\ell-1} e^{\nu_{\ell-1} F}_{\Phi^{(\ell-1)}}$, where $\Psi_{\ell-1}, \nu_{\ell-1}$ are the ones in Section 4 and $\Phi^{(\ell-1)}$ is the one defined above with $\ell - 1$ in place of $n$. Obviously, $B_\ell$ transforming $(\alpha, A)$ into $(\alpha, R_{\rho_j} e^{F_\ell})$ and the estimates of $B_\ell$ and $F_\ell$ follow from the estimates of $\Psi_{\ell-1}, \nu_{\ell-1}$ and $\Phi^{(\ell-1)}$. □

**5.2. Proof of Theorem 1.1.** The proof of Theorem 1.1 relies on the renormalization theory of one-frequency quasiperiodic $SL(2, \mathbb{R})$ cocycles. Recall for any $0 < \gamma < 1$ and $\tau > 1$, $DC_\alpha(\gamma, \tau)$ denotes the set of all $\rho$ such that

$$\|k\alpha + 2\rho\|_\mathbb{Z} \geq \gamma\langle k \rangle^{-\tau}, \quad \langle k \rangle = \max\{1, |k|\}, \quad \forall k \in \mathbb{Z}.$$
Let $\mathcal{P} \subset [0, 1/2)$ be the set of all $\rho$ such that there exist $0 < \gamma < 1$ and $\tau > 1$ with $\rho \beta_{n-1}^{-1} \in DC_{\alpha_n}(\gamma, \tau)$ for infinitely many $n$. By Borel-Cantelli lemma, $\mathcal{P}$ is full measure in $[0, 1/2)$. We will fix the sequence $\{n_j\}_{j \in \mathbb{N}}$ such that $\beta_{n_j-1}^{-1} \rho_f \in DC_{\alpha_{n_j}}(\gamma, \tau)$.

We also recall the following well-known Kotani’s theory [Kot84].

**Theorem 5.1 ([Kot84]).** Let $\tilde{\mathcal{P}} \subset [0, 1/2)$ be any full measure subset. For every $V \in C^\infty(\mathbb{T}, \mathbb{R})$, for almost every $E \in \mathbb{R}$, we have

- either $(\alpha, S_E^V)$ has a positive Lyapunov exponent, or
- $(\alpha, S_E^V)$ is $L^2$-conjugated to an $SO(2, \mathbb{R})$-valued cocycle and the fibered rotation number of $(\alpha, S_E^V)$ belongs to $\tilde{\mathcal{P}}$.

We start from $(\alpha, S_E^V)$ which can be $L^2$-conjugated to an $SO(2, \mathbb{R})$-valued cocycle. By definition of $\mathcal{P}$, if $\rho(\alpha, S_E^V) = \rho_f$ belongs to $\mathcal{P}$, we can find $0 < \gamma < 1$ and $\tau > 1$, and arbitrary large $j > 0$, such that $\rho_f \beta_{n_j-1}^{-1} \in DC_{\alpha_{n_j}}(\gamma, \tau)$. Now Proposition 1 ensures that $\|F_{n_j}\|_{rK^{-2}} \to 0$, then we can further choose $j$ large enough, such that

$$\|F_{n_j}\|_{rK^{-2}} \leq \varepsilon_*(\gamma, \tau, rK^{-2}, M),$$

where $\varepsilon_* = \varepsilon_*(\gamma, \tau, r, M) > 0$ is the one in Theorem 1.2. Since $(-1)^{n_j} \rho_f \beta_{n_j-1}^{-1}$ is just the rotation number of $(\alpha_{n_j}, R_{\rho_{n_j}} e^{F_{n_j}})$, by Theorem 1.2 we know that $(\alpha_{n_j}, R_{\rho_{n_j}} e^{F_{n_j}})$ is $C^\infty$ rotations reducible. Note $(\alpha_{n_j}, R_{\rho_{n_j}} e^{F_{n_j}})$ is rotations reducible (or reducible) implies $(\alpha, S_E^V)$ is rotations reducible (or reducible) in the same regularity class (consult Proposition 4.2 of [AK15] for example), then Theorem 1.1 follows directly.

6. Last’s intersection spectrum conjecture

Consider the Schrödinger operator $H_{V, \beta, \theta}$ defined by (1.1) with ultra-differentiable potential $V \in U_r(\mathbb{T}, \mathbb{R})$, frequency $\beta \in \mathbb{T}$ and phase $\theta \in \mathbb{T}$. For fixed $\theta$, denote by $\sigma(\beta, \theta)$ and $\sigma_{ac}(\beta, \theta)$ the spectrum of $H_{V, \beta, \theta}$ and its absolutely continuous (ac)-component, respectively. It is well known that in the case $\beta = p/q$, $\sigma(p/q, \theta)$ is purely absolutely continuous and consists of $q$, possibly touching, bands. Moreover, in the case $\beta = \alpha$ is irrational, the spectrum and ac spectrum do not depend on $\theta$:

$$\sigma(\alpha, \theta) =: \Sigma(\alpha), \quad \sigma_{ac}(\alpha, \theta) =: \Sigma_{ac}(\alpha), \quad \forall \theta \in \mathbb{T}.$$

In order to treat rational and irrational frequencies on the same footing, similar to Avron et al. [AvMS90], given $\beta \in \mathbb{T}$, we introduce the sets

$$S_+ (\beta) := \bigcup_{\theta \in \mathbb{T}} \sigma(\beta, \theta) = \Sigma(\beta),$$

and

$$S_- (\beta) := \bigcap_{\theta \in \mathbb{T}} \sigma(\beta, \theta) = \Sigma_{ac}(\beta).$$
Note that it was proved in [JM12a] that
\[ S_+(\alpha) = \Sigma(\alpha) = \lim_{n \to \infty} S_+(p_n/q_n). \]

Theorem 1.3 follows immediately from the following Theorem 6.1 and Theorem 6.2, while the key arguments are “generalized Chambers’ formula” (Proposition 3) and continuity of Lyapunov exponent (Theorem 6.3).

6.1. Generalized Chambers’ formula.

**Theorem 6.1.** Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q}, \ V : \mathbb{T} \to \mathbb{R} \) be an \( M \)-ultra-differentiable function satisfying (H1) and (H2), then we have
\[ S_-(-\alpha) = \Sigma_{ac}(\alpha) \subset \lim \inf_{n \to \infty} S_-(p_n/q_n). \]

The proof of Theorem 6.1 depends on the following generalized Chambers’ formula. To state this, recall that for each \( \theta \in \mathbb{T} \), \( H_{V,p/q,\theta} \) is a periodic operator whose spectrum, \( \sigma(p/q, \theta) \), is given in terms of the discriminant by
\[ \sigma(p/q, \theta) = t_{p/q}(\cdot, \theta)^{-1}[-2, 2], \]
where
\[ t_{p/q}(E, \theta) = \text{tr}\{\Pi_{s=q-1}^{0}S^V_E(\theta + sp/q)\}, \]
which is called as the discriminant of \( H_{V,p/q,\theta} \), here “tr” stands for the trace. In general, this discriminant is a polynomial of degree \( q \) in \( E \) and \( q^{-1} \)-periodic in \( \theta \), whence one may write
\[ (6.1) \quad t_{p/q}(E, \theta) = \sum_{k \in \mathbb{Z}} a_{q,k}(E)e^{2\pi i q k \theta}. \]

For the almost Mathieu operator, the potential \( V = 2\lambda \cos 2\pi \theta \) is in fact a trigonometric polynomial of degree 1. Thus in the formula (6.2) only the Fourier coefficients with \( k = 0, \pm 1 \) survive, resulting the celebrated Chamber formula [Cha65, BS82, BHJ19]
\[ t_{p/q}(E, \theta) = a_{q,0}(E) + 2\lambda q \cos(2\pi q \theta). \]

Note the classical Chamber’s formula holds for any \( \lambda \). In particular, it shows that phase variations of the discriminant for the subcritical almost Mathieu operator (thus has absolutely continuous spectrum) are exponentially small in \( q \). Now for any \( C^\infty \) potential \( V : \mathbb{T} \to \mathbb{R} \) satisfying (H1) and (H2), \( E \in \Sigma_{ac}(\alpha) \), we will show that the difference between the determines \( t_{p/q}(E, \theta) \) of rational approximates of \( \alpha \), and its phase-average \( a_{q,0}(E) \), is in fact sub-exponentially small in \( q \):

**Proposition 3.** Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q}, \ V : \mathbb{T} \to \mathbb{R} \) be an \( M \)-ultra-differentiable function satisfying (H1) and (H2), then for almost every \( E \in \Sigma_{ac}(\alpha) \), there exist \( n_* = n(V, \alpha, E) \in \mathbb{N}, c = c(E) \) such that
\[ (6.3) \quad \|t_{p_n/q_n}(E, \theta) - a_{q_n,0}(E)\|_{C^0} \leq 4 \exp\{-\Lambda(cq_n)\} \]
whenever \( n \geq n_* \).
Remark 6.1. Indeed, one can select \( c(E) \), such that \( cq_n > q_n^{\frac{3}{4}} \).

If \( V \) is analytic, Jitomirskaya-Marx (Proposition 3.1 in \([JM12a]\)) proved that \( \|t_{p_n/q_n}(E, \theta) - a_{q_n,0}(E)\|_{c_0} \) is exponentially small in \( q \). Their proof depends on Avila’s quantization of acceleration \([Avi15a]\), key of his global theory. While our proof is a perturbation argument, completely different from theirs.

Proof of Theorem 6.1. We will first prove Theorem 6.1 assuming Proposition 3 and postpone the proof of Proposition 3 to Section 7. We point out the ideas of the proof was essentially given by Avila and sketched in \([JM12a]\). We give the full proof here for completeness. Let \( K \subset [0, 1/2) \) be the set of all \( \rho \) such that

\[
\inf_{p \in \mathbb{Z}} |q_n \rho - p| \geq n^{-2}, \quad \text{eventually.}
\]

A simple Borel Cantelli argument shows \( |K| = 1/2 \). For any \( \beta \in \mathbb{T} \), we denote by \( N(\beta, E) \) the integrated density of states (IDS). Note the set \( \mathcal{P} \subset [0, 1/2) \) we defined in Section 5.2 is also full of measure, i.e., \( |\mathcal{P}| = 1/2 \), thus \( \mathcal{P} \equiv \mathcal{K} \). Moreover, Theorem 1.1 actually implies that for almost every \( E \in \Sigma_{ac}(\alpha) \), \( \mathcal{P} \equiv \mathcal{K} \), \( N(\beta, E) \) is Lipschitz in \( \alpha \), i.e., there exists some \( \Gamma(E) \)

\[
|N(\alpha, E) - N(p_n/q_n, E)| < q_n^{-2} \Gamma(E).
\]

Since \( N(\alpha, E) \in \mathcal{K} \), then by (6.4), for \( n \) sufficiently large, we have

\[
p - 1 + \frac{1}{2q_n} < q_n N(p_n/q_n, E) < p - \frac{1}{2q_n},
\]

for some \( 1 \leq p \leq q_n \). On the other hand, it was calculated in \([AD08]\) that if \( E \) belongs to the \( k \)-th band of \( S_{+}(p_n/q_n) \), we have

\[
q_n N\left(p_n/q_n, E\right) = k - 1 + 2(-1)^{q_n+k-1} \int_{\mathbb{T}} \rho(p_n/q_n, E, \theta) d\theta + \frac{1 - (-1)^{q_n-k+1}}{2},
\]

where

\[
\rho\left(p_n/q_n, E, \theta\right) = \begin{cases} 
0 & \text{if} \ t_{p_n/q_n}(E, \theta) > 2, \\
(2\pi)^{-1} \arccos(2^{-1}t_{p_n/q_n}(E, \theta)) & \text{if} \ |t_{p_n/q_n}(E, \theta)| \leq 2, \\
1/2 & \text{if} \ t_{p_n/q_n}(E, \theta) < -2.
\end{cases}
\]

Then (6.5) and (6.6) imply that

\[
2 \left| \cos \left(2\pi \int_{\mathbb{T}} \rho(p_n/q_n, E, \theta) d\theta\right)\right| < 2 - \frac{1}{q_n^2}.
\]
Since \( \rho(p_n/q_n, E, \theta) \) is continuous in \( \theta \), (6.8) and (6.7) imply that there exists \( \tilde{\theta} \in \mathbb{T} \), such that
\[
|t_{p_n/q_n}(E, \tilde{\theta})| = 2|\cos(2\pi \rho(p_n/q_n, E, \tilde{\theta}))| < 2 - \frac{1}{q_n^2}.
\]
Then by (6.3) in Proposition 3 we have
\[
|a_{q_n,0}(E)| \leq 2 - \frac{1}{q_n^2} + 4 \exp\{-\Lambda(cq_n)\} \leq 2 - \frac{1}{2q_n^r}.
\]
By (6.3) again, for any \( \theta \in \mathbb{T} \), we have \( |t_{p_n/q_n}(E, \theta)| \leq 2 \), which means \( E \in S_-(p_n/q_n) \). \( \square \)

6.2. Continuity of the Lyapunov exponent. Theorem 6.1 proves \( \Sigma_{ac}(\alpha) \subset \lim_{n \to \infty} S_-(p_n/q_n) \) for any M-ultra-differentiable potentials satisfying (H1) and (H2), however, when we come to the inverse inclusion, we can only prove the result for \( \nu \)-Gevrey potentials with \( 1/2 < \nu < 1 \). It is interesting to extend the conclusion below to the cocycle with ultra-differential potentials, even with \( C^\infty \) potentials.

**Theorem 6.2.** Let \( V : \mathbb{T} \to \mathbb{R} \) be a \( \nu \)-Gevrey function with \( 1/2 < \nu < 1 \), and assume that \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \). Then there is a sequence \( p_n/q_n \to \alpha \), such that
\[
\limsup_{n \to \infty} S_-(p_n/q_n) \subset S_-(\alpha) = \Sigma_{ac}(\alpha).
\]

**Remark 6.2.** The sequence \( p_n/q_n \) will be the full sequence of continued fraction approximations in the case \( \alpha \) is Diophantine, and an appropriate subsequence of it otherwise. For practical purposes of making conclusions about \( S_-(\alpha) \) based on the information on \( S_-(p_n/q_n) \), it is sufficient to have convergence along a subsequence. However, in the latter case, the potential can be any stationary bounded ergodic one.

The proof of Theorem 6.2 depends on the continuity of the Lyapunov exponent for the more general Gevrey cocycles. For a Gevrey (possibly matrix valued) function \( f \), we let
\[
\|f\|_{\nu,r} = \sum_{k \in \mathbb{Z}} |f(k)|e^{2\pi k|\nu|_r}, \quad 0 < \nu < 1.
\]
We denote by \( G^\nu_r(\mathbb{T}, \ast) \) the set of all these \( \ast \)-valued functions (\( \ast \) will usually denote \( \mathbb{R} \), \( SL(2, \mathbb{R}) \)). If we set \( r = \tilde{\nu}^\ast \), we get \( \|f\|_{\nu,r} = \|f\|_{\Lambda_\nu,\tilde{\nu}} \), where \( \| \cdot \|_{\Lambda_\nu,\tilde{\nu}} \)-norm is the one defined by (3.9) with \( \Lambda_\nu(x) = x^{\tilde{\nu}} \). To simplify the notation, we introduce \( \| \cdot \|_{\nu,r} \). Note the function \( \Lambda_\nu(x) = x^{\nu}, 0 < \nu < 1 \), satisfies the subadditivity, thus \( G^\nu_r(\mathbb{T}, \ast), 0 < \nu < 1 \), is a Banach algebra.

**Theorem 6.3.** Let \( \rho > 0, 2^{-1} < \nu < 1 \). Consider the cocycle \( (\alpha, A) \in (0,1) \setminus \mathbb{Q} \times G^\nu_\rho(\mathbb{T}, SL(2, \mathbb{R})) \) with \( \alpha \in DC \). Then \( L(\alpha, A) \) is jointly continuous in the sense that
\[
\lim_{n \to \infty} L(p_n/q_n, A_n) = L(\alpha, A),
\]
where $p_n/q_n$ is the continued fraction expansion of $\alpha$, and $A_n \in G^\nu_\rho(T, SL(2, \mathbb{R}))$ with $A_n \to A$ under the topology derived by $\| \cdot \|_{\nu, \rho}$-norm.

The full joint continuity was first proved by Bourgain-Jitomirskaya [BJ02] for analytic cocycles, which also plays a fundamental role in establishing the global theory of Schrödinger operator [Avi15a]. However, due to lack of analyticity, it’s very difficult to generalize the above result to all irrational $\alpha$.

The main reason is that in our large deviation theorem estimates, there is an upper bound for $N$ (see the assumptions in Proposition 5), thus we cannot deal with the extremely Liouvillean frequency. It’s an interesting open question whether one can prove the continuity of the Lyapunov exponent for Liouvillean frequency and non-analytic potentials.

**Proof of Theorem 6.2.** We will first prove Theorem 6.2 and left the proof of Theorem 6.3 to Section 8. We separate the proof into two cases.

**Case I:** $\alpha \in DC(v, 10)$. We can assume that $L(\alpha, E) > 0$, then by Theorem 6.3, for any $p_n/q_n$ sufficiently close to $\alpha$, we have $L(p_n/q_n, E) > 0$. This implies that $E \notin \sigma(p_n/q_n, \theta)$ for some $\theta \in \mathbb{T}$, hence $E \notin S_-(p_n/q_n)$.

**Case II:** $\alpha \notin DC(v, 10)$, we define a sequence $\{V^\theta_m(n)\}_{m=1}^{\infty}$ periodic potentials by:

$$V^\theta_m(n) = V(\theta + n\alpha), \quad n = 1, 2, \cdots, m,$$

$$V^\theta_m(n + m) = V^\theta_m(n),$$

such that $V^\theta_m$ is obtained from $V_\omega$ by “cutting” a finite piece of length $m$, and then repeating it. We denote by $\sigma_m(\theta)$ the spectrum of the periodic Schrödinger operators

$$Hu = u_{n+1} + u_{n-1} + V^\theta_m(n)u_n.$$  

By Theorem 1 in [Las93], for a.e. $\theta \in \mathbb{T}$,

$$\lim_{m \to \infty} \sup_{\sigma(m(\theta) \subset \Sigma_{ac}(\alpha).}$$

We define

$$A^\theta_{q_n}(\xi) = \begin{pmatrix} V^\theta_{q_n}(1) & 1 & \cdots & e^{-i\xi m} \\ 1 & V^\theta_{q_n}(2) & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ e^{i\xi m} & \cdots & \cdots & 1 \\ \end{pmatrix},$$

$$\bar{A}^\theta_{q_n}(\xi) = \begin{pmatrix} V(\theta + p_n/q_n) & 1 & \cdots & e^{-i\xi m} \\ 1 & V(\theta + 2p_n/q_n) & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ e^{i\xi m} & \cdots & \cdots & 1 \\ \end{pmatrix}.$$
It is standard that
\[ \sigma_{q_n}(\theta) = \bigcup_\xi \text{Spec}(A_{q_n}^0(\xi)), \quad \sigma(p_n/q_n, \theta) = \bigcup_\xi \text{Spec}(A_{q_n}^0(\xi)), \]
where Spec(A) denotes the sets of all eigenvalues of A. We need the following perturbation theory of matrices.

**Proposition 4** (Corollary 12.2 of [Bha87]). Let A and B be normal with \( \|A - B\| = \varepsilon \). Then within a distance of \( \varepsilon \) of every eigenvalue of A there is at least one eigenvalue of B and vice versa.

Fix \( \theta_0 \) such that (6.9) holds with \( \theta_0 \) in place of \( \theta \). Notice for any \( E_0 \in \sigma_{q_n}(\theta_0) \), there exists \( \xi_n \) such that \( E_0 \in \text{Spec}(A_{q_n}^{\theta_0}(\xi_n)) \). Applying Proposition 4 to \( A_{q_n}^{\theta_0}(\xi_n) \) and \( A_{q_n}^{\theta_0}(\xi_n) \), there exists \( E_0' \in \text{Spec}(A_{q_n}^{\theta_0}(\xi_n)) \) such that
\[ |E_0 - E_0'| \leq \| A_{q_n}^{\theta_0}(\xi_n) - \overline{A_{q_n}^{\theta_0}(\xi_n)} \| \leq C(V) \sum_{j=1}^{q_n} |j(\alpha - p_n/q_n)| \leq C(V)q_n^2|\alpha - p_n/q_n|. \]

Since \( E_0' \in \sigma(p_n/q_n, \theta_0) \), it follows that
\[ \| \sigma_{q_n}(\theta_0) - \sigma(p_n/q_n, \theta_0) \|_H \leq C(V)q_n^2|\alpha - p_n/q_n|, \]
where \( \|A - B\|_H \) denotes the Hausdorff distance of two sets. We denote \( \sigma_{q_n}(\theta_0) = \bigcup_{i=1}^{q_n'} [a_{n,i}, b_{n,i}], q_n' \leq q_n \). (6.10) implies that
\[ \sigma(p_n/q_n, \theta_0) \subset \bigcup_{i=1}^{q_n'} \left[ a_{n,i} - C(V)q_n^2|\alpha - p_n/q_n|, b_{n,i} + C(V)q_n^2|\alpha - p_n/q_n| \right]. \]

It follows that
\[ |\sigma(p_n/q_n, \theta_0) \setminus \sigma_{q_n}(\theta_0)| \leq C(V)q_n^3|\alpha - p_n/q_n|. \]

Since \( \alpha \notin DC(v, 10) \), by (6.11), there exists a subsequence \( p_n/q_n \) such that
\[ \limsup_{n \to \infty} \sigma(p_n/q_n, \theta_0) \subset \limsup_{n \to \infty} \sigma_{q_n}(\theta_0) \subset \Sigma_{ac}(\alpha). \]

Moreover, notice that
\[ \limsup_{n \to \infty} S_-(p_n/q_n) \subset \limsup_{n \to \infty} \sigma(p_n/q_n, \theta_0), \]

hence
\[ \limsup_{n \to \infty} S_-(p_n/q_n) \subset \Sigma_{ac}(\alpha). \]

7. **Proof of Proposition 3**

Suppose that \( V : T \to \mathbb{R} \) is an M-ultra-differentiable function satisfying (\( H1 \)) and (\( H2 \)), then for almost every \( E \in \Sigma_{ac}(\alpha) \), by Theorem 1.1, \( (\alpha, S_E^V) \) is \( C^\infty \) rotations reducible. However, this is not enough for us to conclude
\[ \|t_{p_n/q_n}(E, \theta) - a_{q_n,0}(E)\|_{C^0} \leq 4 \exp\{-\Lambda(cq_n)\}. \]
Since even we assume \((\alpha, S_{\rho}^\psi) = (\alpha, R_{\psi(\theta)})\) with \(\psi(\theta) \in C^\infty(T, \mathbb{R})\), which only gives
\[\|t_{p_n/q_n}(E, \theta) - a_{q_n,0}(E)\|_{C^0} \leq c q_n^{-\infty}.\]

The idea is that our KAM scheme not only gives \(C^\infty\) rotations reducibility, but also almost reducibility in the ultra-differentiable topology (Corollary 2), however, this only works for cocycles which are close to constant. Coupled with the renormalization argument, we will show that if the cocycle is \(L^2\)-conjugated to rotations and \(\rho(\alpha, A) \in \mathcal{P}\), then \((\alpha, A)\) is also almost reducibility in the ultra-differentiable topology (Lemma 7.3). Consequently, Proposition 3 follows from the perturbation arguments.

7.1. **Global almost reducibility.** To get desired quantitative estimates, the main method is the inverse renormalization which was first developed in [FK09]. We introduce the notation \(\Psi := J\mathcal{R}_{\theta_n}^{\Phi}(\Psi')\) if \(\Psi' = \mathcal{R}_{\theta_n}^{\Phi}(\Psi)\). It is easy to check that
\[J\mathcal{R}_{\theta_n}^{\Phi}(\Phi) = T_{\beta_n}^{-1} \circ N_{\alpha} \circ M_{\alpha_n} \circ T_{\beta_n}(\Phi),\]
In our setting, one of the bases is the following:

\[\text{Lemma 7.1. Let } \Phi_1 = ((1, Id), (\alpha_n, R_{\rho_n} e^{F(\theta)})), \Phi_2 = ((1, Id), (\alpha_n, R_{\rho_n})) \text{ where } F \in U_r(\mathbb{T}, sl(2, \mathbb{R})) \text{ with estimate } \|F\|_{r,1} \leq q_n^{-1}. \text{ Then,}\]
\[\|J\mathcal{R}_{\theta_n}^{\Phi_1}(\Phi_1) - J\mathcal{R}_{\theta_n}^{\Phi_2}(\Phi_2)\|_{\beta_n-1, r, 1} < 2q_n^{-1}\|F(\theta)\|_{r,1}.\]

**Proof.** Direct computation shows that
\[M_{\beta_n}^{-1}(\Phi_1) = ((\beta_n^{-1}, Id), (\beta_n^{-1}\alpha_n, R_{\rho_n} e^{F(\beta_n^{-1}\theta)})),\]
\[M_{\beta_n}^{-1}(\Phi_2) = ((\beta_n^{-1}, Id), (\beta_n^{-1}\alpha_n, R_{\rho_n})),\]
and consequently
\[N_{\alpha} \circ M_{\beta_n}^{-1}(\Phi_1) = ((1, R_{\rho_n}q_n^{-1} e^{F_1(\theta)}), (\alpha, R_{\rho_n}p_n^{-1} e^{F_2(\theta)})),\]
\[N_{\alpha} \circ M_{\beta_n}^{-1}(\Phi_2) = ((1, R_{\rho_n}q_n^{-1}), (\alpha, R_{\rho_n}p_n^{-1})).\]

Note for any \(\lambda \neq 0, \|D_{\Phi}^\lambda M_{\lambda}(\Phi)\|^{C^0([0,T])} = \lambda^s \|D_{\Phi}^\lambda \Phi\|^{C^0([0,T])},\) which gives
\[\|M_\lambda(\Phi)\|_{\lambda^{-1} r, T} = \|\Phi\|_{r, \lambda T}.\]

Thus the main task is to estimate the norm under iteration of the cocycles. To do this, we need the following simple observation:

**Lemma 7.2.** (Lemma 4.3 of [ZW12]) Let \(F_i \in U_r(\mathbb{R}, sl(2, \mathbb{R})), i = 1, \cdots, j.\) Then it holds that
\[R_{\rho} e^{F_j(\theta)} R_{\rho} e^{F_{j-1}(\theta)} \cdots R_{\rho} e^{F_1(\theta)} = R_{\rho} e^{\bar{F}(\theta)},\]
with estimate
\[\|\bar{F}\|_{r,T} \leq \sum_{i=1}^{j} \|F_i\|_{r,T}.\]
By (7.2) and Lemma 7.2, we have
\[ \|F_1\|_{\beta_{n-1} r, 1} \leq q_{n-1} \|F\|_{r, 1}, \quad \|F_2\|_{\beta_{n-1} r, 1} \leq p_{n-1} \|F\|_{r, 1}. \]
Then (7.1) follows directly.

\[ \text{(7.3)} \quad B_{j, \ell}(\theta + \alpha)A(\theta)B_{j, \ell}(\theta)^{-1} = R_{\rho_{\ell}}e^{F_{j, \ell}(\theta)}, \quad \ell > \hat{n}, \]
where \( \bar{\tau}_\ell = r_{\beta_{n-1}/2K_s \bar{Q}_{\ell-1}} \), and \( \hat{n} \in \mathbb{N} \) is the smallest one such that \( \bar{Q}_{\hat{n}} \geq C^d_{\alpha, T_j}. \) Moreover, we have estimate
\[ \|F_{j, \ell}\|_{\bar{\tau}_\ell} \leq \varepsilon_{\ell}^j, \quad \|B_{j, \ell}\|_{\bar{\tau}_\ell} \leq 4C^3q_{n-1}q_{n_j}. \]

**Proof.** Since \((\alpha, A)\) is \(L^2\)-conjugated to rotations and homotopic to the identity, by Proposition 1 there exists \(D_n \in U_{rK_s^{-2}}(\mathbb{R}, SL(2, \mathbb{R}))\) such that
\[ \|D_n\|_{r(K_s^2T)^{-1}, T} \leq C^{q_{n-1}(T+1)}, \]
such that
\[ \text{Conj}_{D_n}(\mathcal{R}_{\theta_n}^n(\Phi)) = ((1, Id), (\alpha_n, R_{\rho_n}e^{F_n})), \]
with \(\|F_n\|_{rK_s^{-2}, 1} \to 0\). Since \(\rho_f = \rho(\alpha, A) \in \mathcal{P}\), we can further choose \(j\) large enough, such that
\[ \|F_{n_j}\|_{rK_s^{-2}, 1} \leq \varepsilon_*\gamma, rK_s^{-2}, M, \]
where \(\varepsilon_* = \varepsilon_*(\gamma, r, M) > 0\) is the one in Theorem 1.2. In the following, we will write \(D_j\) for \(D_n\) short, and denote \(T_n = \beta_{n-1}^{-1} = q_n + \alpha_n q_{n-1}, \ r_\ell = 2^{-1}rK_s^{-2}\bar{Q}_{\ell-1}^2, \) then \(\bar{\tau}_\ell = (K_sT_n)^{-1}r_\ell. \)

Now we apply Corollary 2 to the action \(\text{Conj}_{D_j}(\mathcal{R}_{\theta_n}^n(\Phi))\) and denote \(Z_{j, \ell} = B_{\ell}D_j\), then
\[ \text{Conj}_{Z_{j, \ell}}(\mathcal{R}_{\theta_n}^n(\Phi)) = \bar{\Phi}_{1}^{j, \ell}(\Phi) := ((1, Id), (\alpha_{n_j}, R_{\rho_{n_j}}e^{F_{j, \ell}})), \]
which, together with (5.2) and (7.4) and the fact \(r_{\ell} \ll rK_s^{-2}T_n^{-1}, \) implies
\[ \|F_{j, \ell}\|_{r_{\ell}, 1} \leq \varepsilon_{\ell}, \quad \|Z_{j, \ell}\|_{r_{\ell}, T_{n_j}} \leq \|D_j\|_{r_{\ell}, T_{n_j}} \|B_\ell\|_{r_{\ell}, 1} \leq C^{3q_{n_j-1}q_{n_j}}. \]

Once we have these, we can set \(\bar{\Phi}_2^{j, \ell}(\Phi) = ((1, Id), (\alpha_{n_j}, R_{\rho_{n_j}}))\) and define \(\Phi_{2, j, \ell}\) by
\[ \text{Conj}_{Z_{j, \ell}}(\mathcal{R}_{\theta_n}^n(\Phi_{2, j, \ell})) = \bar{\Phi}_2^{j, \ell}. \]

For given \( G \in SL(2, \mathbb{R}) \), \( T_{\theta} \), and \( N_U \) commute with \( \text{Conj}_G \) while \( M_\lambda \circ \text{Conj}_G = \text{Conj}_{G(\lambda)} \circ M_\lambda \), then by (7.5) and (7.7), we get

\[
\begin{align*}
\Phi &= \mathcal{J}R_{\theta,n}^{\alpha}(\text{Conj}_{Z_{j,\ell}}(\Phi_1^{(j,\ell)})) = \text{Conj}_{Z_{j,\ell}}^{-1}(\phi_{(j,\ell)}) \circ R_{\theta,n}^{\alpha}, \\
\Phi_{2,j,\ell} &= \mathcal{J}R_{\theta,n}^{\alpha}(\text{Conj}_{Z_{j,\ell}}(\Phi_2^{(j,\ell)})) = \text{Conj}_{Z_{j,\ell}}^{-1}(\phi_{(j,\ell)}) \circ R_{\theta,n}^{\alpha},
\end{align*}
\]

where

\[
\begin{align*}
\Phi_1^{(j,\ell)} &= \mathcal{J}R_{\theta,n}^{\alpha}(\Phi_1^{(j,\ell)}), \\
\Phi_2^{(j,\ell)} &= \mathcal{J}R_{\theta,n}^{\alpha}(\Phi_2^{(j,\ell)}) = ((1, R_{\rho_j q_{n,j}}), (\alpha, R_{\rho_j q_{n,j}})),
\end{align*}
\]

Set \( \bar{Z}_j(\theta) = R_{\rho_j q_{n,j}^{-1}, \theta} \), then

\[
\text{Conj}_{\bar{Z}_j}(\Phi_{2,j,\ell}) = ((1, Id), (\alpha, R_{\rho_j})) := \Phi_{**},
\]

which, together with (7.8) and (7.9), implies

\[
\begin{align*}
\Phi &= \text{Conj}_{Z_{j,\ell}}^{-1}(\phi_{(j,\ell)}), \\
\Phi_{2,j,\ell} &= \text{Conj}_{Z_{j,\ell}}^{-1}(\phi_{(j,\ell)}) \circ R_{\theta,n}^{\alpha},
\end{align*}
\]

where \( \Phi_\ast = \text{Conj}_{Z_{j,\ell}}(\Phi_1^{(j,\ell)}) \). Moreover, by our selection \( Q_\ell > Q_\ell n \geq C^2\).

In the following, we will give the estimate of the distance of \( \Phi \) and \( \Phi_{2,j,\ell} \).

First, we apply Lemma 7.1 with \( \Phi_\ast^{(j,\ell)} \) in place of \( \Phi_\ast \), and \( n_{j} \) in place of \( n \) respectively, then by (7.1)

\[
\| \Phi_{1,j,\ell}^{(j,\ell)} - \Phi_{2,j,\ell}^{(j,\ell)} \|_{T_{n_j}^{-1}, r_{\ell,1}} < \| F_{j,\ell} \|_{T_{r_{\ell,1}}} \leq \varepsilon \| \Phi \|_{T_{r_{\ell,1}}} \leq \frac{3}{\varepsilon}.
\]

Finally, by (7.6) and (7.8) and the inequality above we get

\[
\| \Phi - \Phi_{2,j,\ell}^{(j,\ell)} \|_{T_{n_j}^{-1}, r_{\ell,1}} \leq \| Z_{j,\ell} \|_{T_{\ell,1}} \| \Phi_{1,j,\ell}^{(j,\ell)} - \Phi_{2,j,\ell}^{(j,\ell)} \|_{T_{n_j}^{-1}, r_{\ell,1}} \leq C^6 q_{n_j}^{-1} q_{n_j}^{-1} \varepsilon \| \Phi \|_{T_{r_{\ell,1}}} \leq \frac{4}{\varepsilon}.
\]

Notice that \( \Phi_{2,j,\ell} \) may not be normalized, however, by (7.12) we know that

\[
\| \Phi_{2,j,\ell}(1,0) - Id \|_{T_{n_j}^{-1}, r_{\ell,1}} = \| \Phi_{2,j,\ell}(1,0) - \Phi(1,0) \|_{T_{n_j}^{-1}, r_{\ell,1}} \leq C^6 q_{n_j}^{-1} q_{n_j}^{-1} \varepsilon \| \Phi \|_{T_{r_{\ell,1}}} \leq \frac{4}{\varepsilon}.
\]

Thus by Lemma 2.3, there exists a conjugation \( \tilde{B}_{j,\ell} \in U_{r_{\ell}}(\mathbb{R}, SL(2, \mathbb{R})) \) such that \( \tilde{B}_{j,\ell} = \text{Conj}_{\tilde{B}_{j,\ell}}(\Phi_{2,j,\ell}) \) is a normalized action. Moreover, we have estimate

\[
\| B_{j,\ell} - Id \|_{T_{r_{\ell,1}}} \leq \| B_{j,\ell}(1,0) - Id \|_{T_{n_j}^{-1}, r_{\ell,1}} \leq C^6 q_{n_j}^{-1} q_{n_j}^{-1} \varepsilon \| \Phi \|_{T_{r_{\ell,1}}} \leq \frac{4}{\varepsilon}.
\]
Since \( \Phi_{j,\ell}(0,1) = \bar{B}_{j,\ell}(\theta + \alpha)\Phi_{2,j,\ell}(0,1)\bar{B}_{j,\ell}(\theta)^{-1} \), by (7.13), (7.14) we have

\[
\|\Phi_{j,\ell} - \Phi_{2,j,\ell}\|_{\tilde{r}_\varepsilon,1} \leq 2C^{6q_{n_j}q_{n_j}^{-1}}\varepsilon_{\ell}^{1/2}.
\]

The inequality above, together with (7.12), yields

\[
(7.15) \quad \|\Phi - \Phi_{j,\ell}\|_{\tilde{r}_\varepsilon,1} \leq 3C^{6q_{n_j}q_{n_j}^{-1}}\varepsilon_{\ell}^{4/3}.
\]

Set \( B_{j,\ell}(\cdot) = \tilde{Z}_j(\cdot)Z_{j,\ell}(\beta_{n_j}^{-1})B_{j,\ell}^{-1}(\cdot) \), then by (7.10) we get

\[
(7.16) \quad \text{Conj}_{B_{j,\ell}}(\Phi_{j,\ell}) = \text{Conj}_{\tilde{Z}_jZ_{j,\ell}(\beta_{n_j}^{-1})}(\Phi_{2,j,\ell}) = \Phi_{**}.
\]

Thus \( B_{j,\ell} \) is 1–periodic since both \( \Phi_{j,\ell} \) and \( \Phi_{**} \) are normalized. Moreover, (7.6), (7.11) and (7.14) imply

\[
(7.17) \quad \|B_{j,\ell}\|_{\tilde{r}_\varepsilon,1} \leq \|\tilde{Z}_j\|_{\tilde{r}_\varepsilon,1}\|Z_{j,\ell}\|_{\tilde{r}_\varepsilon,T_{n_j}}\|\bar{B}_{j,\ell}\|_{\tilde{r}_\varepsilon,1} \leq 4C^{3q_{n_j}^{-1}q_{n_j}}.
\]

By (7.15)–(7.17) we get,

\[
\|\Phi(0,B_{j,\ell}(\cdot + \alpha)) \circ (\alpha, A) \circ (0, B_{j,\ell})^{-1} - (\alpha, R_{q_{n_j}})\|_{\tilde{r}_\varepsilon,1} \\
\leq \|\text{Conj}_{B_{j,\ell}}(\Phi) - \text{Conj}_{B_{j,\ell}}(\Phi_{j,\ell})\|_{\tilde{r}_\varepsilon,1} \leq C^{13q_{n_j}^{-1}q_{n_j}}\varepsilon_{\ell}^{3/4} \leq \varepsilon_{\ell}^{3/2},
\]

where the last inequality follows from our selection \( \overline{Q}_{\ell} > \overline{Q}_{\hat{n}} \geq C^{q_{n_j}} \) and definition of \( \varepsilon_{\ell} \). Thus, by implicit function theorem, there exists a unique \( F_{j,\ell} \in U_{\tilde{r}_\varepsilon}(\mathbb{T}, sl(2, \mathbb{R})) \), such that

\[
B_{j,\ell}(\theta + \alpha)A(\theta)B_{j,\ell}(\theta)^{-1} = R_{q_{n_j}}e^{F_{j,\ell}(\theta)}
\]

with \( \|F_{j,\ell}\|_{\tilde{r}_\varepsilon} \leq \varepsilon_{\ell}^{1/2}. \)

### 7.2. Proof of Proposition 3

First we construct the desired sequence. For the sequence \( (q_{\ell})_{\ell \in \mathbb{N}} \) and subsequence \( (Q_{\ell})_{\ell \in \mathbb{N}} \) constructed in Lemma 2.1, first set \( n_1 \geq n_0 \) to be the smallest integer such that

\[
(7.18) \quad \max\{16CM^{6q_{n_j}}(2 + 2\|V\|_\tau), 16r^{-1}q_{n_j}K^3_x\} \leq \overline{Q}_{n_1},
\]

where \( n_0 \) is the one in section 4. Then we set \( n_* \) be the smallest integer number such that

\[
(7.19) \quad q_{n_*} > \overline{Q}_{n+1}^{2k^{4}r^2}.
\]

Then, for any fixed \( n \) with \( n \geq n_* \), we set \( \ell \in \mathbb{N} \) be the smallest integer number such that \( Q^3_{\ell} \geq q_n \). That is

\[
(7.20) \quad \overline{Q}_{\ell-1}^{2k^{4}r^2} < q_n \leq \overline{Q}_{\ell}^{2k^{4}r^2}.
\]

Thus by (7.19) and (7.20) we get \( \ell - 1 \geq n_1 \geq \tilde{n} \), where \( \tilde{n} \) is the one defined in Lemma 7.3.
By our construction, for almost every $E \in \Sigma_{\text{ac}}(\alpha)$, $(\alpha, S^V_E)$ is $L^2$-conjugated to rotations, and $\rho_f = \rho(\alpha, S^V_E) \in \mathcal{P}$. Then by (7.3) in Lemma 7.3, there exist $B_{j,\ell}, F_{j,\ell} \in U_{\tilde{r}_\ell}(\mathbb{T}, SL(2, \mathbb{R}))$ such that

$$B_{j,\ell}(\theta + \alpha)S^V_E(\theta)B_{j,\ell}(\theta)^{-1} = R_{\rho_f}e^{F_{j,\ell}},$$

with estimate

$$\|F_{j,\ell}\|_{\tilde{r}_\ell} \leq \varepsilon_{\ell}^{\frac{1}{2}}, \quad \|B_{j,\ell}\|_{\tilde{r}_\ell} < 4C^{3q_{n_j}-1q_{n_j}}.$$  

We shorten $B_{j,\ell}$ and $F_{j,\ell}$ as $B$ and $F$, respectively. By (7.21) we get

$$B(\theta + p_n/q_n)S^V_E(\theta)B(\theta)^{-1} = R_{\rho_f} + f(\theta),$$

where

$$f(\theta) = R_{\rho_f}(e^{F(\theta)} - I) + \{B(\theta + p_n/q_n) - B(\theta + \alpha)\}S^V_E(\theta)B(\theta)^{-1} = (I) + (II).$$

Note for any $E \in \Sigma_{\text{ac}}(\alpha) \subset \Sigma(\alpha)$, we have $|E| < 2 + \|V\|_r$, thus by Cauchy’s estimate (Lemma 3.2), we get

$$||I||_{\tilde{r}_\ell/2} \leq |\alpha - \frac{p_n}{q_n}| \|\partial B\|_{\tilde{r}_\ell/2} \|S^V_E\|_{\tilde{r}_\ell} \|B^{-1}\|_{\tilde{r}_\ell} \leq C_M\tilde{r}_\ell^{-1} q_n^{-2} \|B^{-1}\|_{\tilde{r}_\ell} \|B\|_{\tilde{r}_\ell} (2 + 2\|V\|_r) \leq 16C_M C^{6q_{n_j}-1q_{n_j}} (2 + 2\|V\|_r)\tilde{r}_\ell^{-1} q_n^{-2}.$$  

Since $B$ is 1-periodic, by (7.22), we have

$$B(\theta + q_np_n/q_n)\Pi^0_{s=q_n-1}S^V_E(\theta + sp_n/q_n)B(\theta)^{-1} = \Pi^0_{s=q_n-1}B(\theta + (s + 1)p_n/q_n)S^V_E(\theta + sp_n/q_n)B(\theta + sp_n/q_n)^{-1} = \Pi^0_{s=q_n-1}\{R_{\rho_f} + f(\theta + sp_n/q_n)\}.$$  

As a consequence,

$$\text{tr}\Pi^0_{s=q_n-1}S^V_E(\theta + sp_n/q_n) = \text{tr}\Pi^0_{s=q_n-1}\{R_{\rho_f} + f(\theta + sp_n/q_n)\},$$

which, together with (6.1) and (6.2), implies

$$t_{p_n/q_n}(E, \theta) = \text{tr}\Pi^0_{s=q_n-1}\{R_{\rho_f} + f(\theta + sp_n/q_n)\} = \sum_{k \in \mathbb{Z}} a_{q_n,k}(E)e^{2\pi ikq_n\theta}. $$

The first equality in (7.24) implies

$$\|t_{p_n/q_n}(E, \theta)\|_{\tilde{r}_\ell/2} \leq 2\{1 + \|f\|_{\tilde{r}_\ell/2}\}^{q_n}. $$

In the following we will give the estimate of $\|f\|_{\tilde{r}_\ell/2}$, indeed, by (7.18) and $\ell - 1 \geq n_1$, we have

$$\tilde{r}_\ell^{-1} = 2r^{-1} \beta_{n_j-1}^{-1} K^3_{\ell-1} \cdot Q_{\ell-1}^2 < 4^{-1} Q_{\ell-1}^2.$$  

Again, by (4.6), (7.18), (7.20) and (7.23), we have

$$\|f\|_{\tilde{r}_\ell/2} \leq \|I\|_{\tilde{r}_\ell/2} + \|\Pi\|_{\tilde{r}_\ell/2} \leq \frac{1}{2} Q_{\ell}^{-4A^4r^2} + \frac{1}{2} Q_{\ell-1}^2 q_n^{-2} \leq q_n^{-2(1-A^{-1}r^{-2})}. $$
Consequently, by (7.25)
\[ \|t_{p_n/q_n}(E, \theta)\|_{\ell_2} \leq 2\{1 + q_n^{-2(1 - A^{-4}r^{-2})}\}q_n < 4. \]

On the other hand, by (7.26), we have
\[ q_n^{-1}\tilde{r}_\ell > q_nQ_{\ell-1}^{-3} > q_n^{-1}2A^{-4}r^{-2} > T_1. \]
Moreover, the second equality in (7.24) implies
\[ t_{p_n/q_n}(E, \theta) - a_{q_n,0}(E) = R_{q_n}t_{p_n/q_n}(\theta, E). \]
Thus by Lemma 3.4 and (7.27), we have
\[ \|t_{p_n/q_n}(E, \theta) - a_{q_n,0}(E)\|_{C^0} \leq \|t_{p_n/q_n}(E, \theta)\|_{2-1\tilde{r}_\ell} \exp\{-\Lambda(q_n^{-1}2A^{-4}r^{-2})\} \leq 4 \exp\{-\Lambda(q_n^{-1}2A^{-4}r^{-2})\}, \]
the last inequality follows from the fact that \( \Lambda(\cdot) \) is non-decreasing on \( \mathbb{R}^+ \).

8. Proof of Theorem 6.3

In this section we give the proof of Theorem 6.3 which is based on the large deviation theorem and avalanche principle. Notice that the cocycle in Theorem 6.3 is \( \nu \)-Gevrey with \( 1/2 < \nu < 1 \), we will follow the method in [Kle05] to approximate the Gevrey cocycle by its truncated cocycle which is analytic in the certain strip. For the continuity argument, our scheme is in the spirit of [BJ02] with some modifications. Compared to the result in [Kle05] with \( 1/2 < \nu < 1 \), our large deviation theorem also works for more general cocycles (other than Schrödinger cocycle) and rational frequencies, which is an analogue of Bourgain-Jitomirskaya [BJ02]. More concretely, if we truncate the cocycle \( A(\alpha, \theta) \) to \( \tilde{A}(\alpha, \theta) \), and denote \( \tilde{A}_N(\alpha, \theta) \) to be the transfer matrix, then \( \det\tilde{A}_N(\alpha, \theta) \) depends on \( \theta \), is not constant anymore (of course not identical to 1), thus we have to prove that the subharmonic extension of \( N^{-1}\ln\|\tilde{A}_N(\alpha, \theta)\| \) is bounded. This boundedness will enable us to give an enhanced version of the large deviation bound shown in [BJ02]. For more results and methods to prove the continuity of the Lyapunov exponents, we refer readers to [AJS14, JKS09, JM12b, JM11].

8.1. Large deviation theorem. In this subsection we give a large deviation theorem for the \( \nu \)-Gevrey cocycle with \( 1/2 < \nu < 1 \). Let \( A_N(\alpha, \theta) \) and \( L_N(\alpha, A) \) be the associated transfer matrix and finite Lyapunov exponent of the cocycle \( (\alpha, A) \). Then we have the following:

**Proposition 5.** Let \( \rho > 0, \frac{1}{2} < \nu < 1, 0 < \kappa < 1 \). Assume that \( A \in C^\nu(\mathbb{T}, SL(2, \mathbb{R})) \), and
\[ |\alpha - a| < \frac{1}{q^2}, \quad (a, q) = 1. \]
Then there exist $c, C_i(\kappa) > 0, i = 1, 2, \sigma_1 > \sigma > 1 > \gamma > 0$ and $q_0(\kappa, \rho, \nu) \in \mathbb{N}^+$ such that for $q \geq q_0, C_1(\kappa)q^\sigma < N < C_2(\kappa)q^{\sigma_1}$,

$$\text{mes}\left\{ \theta : \frac{1}{N} \ln \| A_N(\alpha, \theta) \| - L_N(\alpha, A) > \kappa \right\} < e^{-cq^\gamma}.$$  

8.1.1. Averages of shifts of subharmonic functions. Let $u = u(\theta)$ be a function on $\mathbb{T}$ having a subharmonic extension on the strip $|\text{Im} \theta| \leq \rho$, and $\alpha \in \mathbb{T}$. We prove that the mean of $u$ is close to the Fejér average of $u(\theta)$ for $\theta$ outside a small set (here being ‘close’ or ‘small’ is expressed in terms of the number of shifts considered).

Consider the Fejér kernel of order $p$:

$$K^p_{R}(t) = \left( \frac{1}{R} \sum_{j=0}^{R-1} e^{2\pi i jt} \right)^p,$$

then we have

$$|K^p_{R}(t)| = \frac{1}{R^p} \left| \frac{1 - e^{2\pi R t}}{1 - e^{2\pi i t}} \right|^p \leq \frac{1}{R^p \| t \|_{\mathbb{Z}}^p}.$$  

Notice also $|K^p_{R}(t)| \leq 1$, we have

$$|K^p_{R}(t)| \leq \min \left\{ 1, \frac{1}{R^p \| t \|_{\mathbb{Z}}^p} \right\} \leq \frac{2}{1 + R^p \| t \|_{\mathbb{Z}}^p}.$$  

We can rewrite (8.1) as

$$K^p_{R}(t) = \frac{1}{R^p} \sum_{j=0}^{p(R-1)} c^p_{R}(j) e^{2\pi i jt},$$

where $c^p_{R}(j)$ are positive integers so that

$$\frac{1}{R^p} \sum_{j=0}^{p(R-1)} c^p_{R}(j) = 1.$$  

Notice that if $p = 1$ then $K^1_{R}(t) = \frac{1}{R} \sum_{j=0}^{R-1} e^{2\pi i jt}$, thus $c^1_{R}(j) = 1$ for all $j$.

**Proposition 6.** Let $\rho > 0$. Assume that $u : \mathbb{T} \to \mathbb{R}$ has a bounded subharmonic extension to the strip $|\text{Im} \theta| \leq \rho$ and $\| u \|_{C^0} \leq S$. If

$$|\alpha - \frac{a}{q}| < \frac{1}{q^2}, \quad (a, q) = 1,$$

and $\sigma > 1, 0 < \varsigma < \sigma^{-1}, \varsigma(1 - \sigma^{-1})^{-1} < p < (\sigma - 1)^{-1}$, then

$$\text{mes}\left\{ \theta : \left| \frac{1}{R^p} \sum_{j=0}^{p(R-1)} c^p_{R}(j) u(\theta + j\alpha) - [u(\theta)]_\theta \right| > \varsigma_2 R^{-\varsigma_1} \right\} < \frac{R^{2\varsigma_1}}{2^p \exp\{R^{3}\}},$$

provided $R = q^\sigma \geq q(\varsigma, p, \sigma)$ ($\varsigma_1 = p(1 - \sigma^{-1}), \varsigma_2 = 2^{p+5}S\rho^{-1}, \varsigma_3 = \frac{1+p}{\sigma} - p$).
Proof. The proof is divided into the following 3 steps.

1. **Shift of Fejér average.** Since $u$ is subharmonic in the strip $[|\text{Im}\theta| \leq \rho]$, then from Corollary 4.7 in [Bou05], we get

\begin{equation}
|\widehat{u}(k)| \leq \frac{S}{\rho|k|}.
\end{equation}

Consider the Fejér average of $u_N(\theta)$, and notice that

\[ u(\theta + j\alpha) = [u(\theta)]_\theta + \sum_{k \neq 0} \widehat{u}(k)e^{2\pi ik(\theta+j\alpha)}, \]

thus, by shortening $K^p_R(\cdot)$ as $K_R(\cdot)$, we get

\[
\frac{1}{Rp} \sum_{j=0}^{p(R-1)} e^p_R(j)u(\theta + j\alpha) - [u(\theta)]_\theta = \sum_{k \neq 0} \widehat{u}(k) \left( \frac{1}{Rp} \sum_{j=0}^{p(R-1)} e^p_R(j)e^{2\pi ij\alpha} \right) e^{2\pi ik\theta}
\]

\begin{equation}
= \sum_{k \neq 0} \widehat{u}(k) \cdot K_R(k\alpha)e^{2\pi ik\theta} := w(\theta) = T_Kw(\theta) + R_Kw(\theta),
\end{equation}

where $K > q$ is a large constant that will be determined later.

2. **Estimate of the $w(\theta)$.** In the following, we will give the estimate of $w(\theta)$. Let $I_\ell = \left[ \frac{2}{q}\ell, \frac{2}{q}(\ell + 1) \right)$, then we write $T_Kw(\theta)$ as

\[
T_Kw(\theta) = \sum_{\ell=0}^{[4Kq^{-1}]+1} \sum_{k \in I_\ell} \widehat{u}(k) \cdot K_R(k\alpha)e^{2\pi ik\theta}.
\]

Note \(|\alpha - \frac{q}{q}\theta| < \frac{1}{2q}\), it follows that for $|k| \leq \frac{q}{2}$ with $k \neq 0$, we have \(|k\alpha - \frac{ka}{q}| < \frac{1}{2q}\), hence $\|k\alpha\|_\mathbb{Z} > \frac{1}{2q}$. Let $\alpha_1, \ldots, \alpha_{q/4}$ be the decreasing rearrangement of $\{\|k\alpha\|_\mathbb{Z}^{-1}\}_{0 < |k| \leq \frac{q}{4}}$. Then we have $\alpha_i \leq \frac{2q}{q+1}$. Moreover, for any interval of length $q/4$, same is true for $\{\|k\alpha\|_\mathbb{Z}^{-1}\}_{|k| \in I}$ if we exclude at most one value of $k$. By (8.2) and (8.3), we have

\begin{equation}
\sum_{0 < |k| < \frac{q}{4}} |\widehat{u}(k)K_R(k\alpha)| \leq \sum_{0 < |k| < \frac{q}{4}} \frac{S\|k\alpha\|_\mathbb{Z}^{-p}}{|k|\rho R^p} \leq \sum_{1 \leq i < \frac{q}{4}} \frac{2S(2q/i)^p}{\rho R^p} \leq \frac{2p+3S}{\rho} \left( \frac{q}{R} \right)^p,
\end{equation}

and for each $\ell \geq 1$, we have

\[
\sum_{|k| \in I_\ell} |\widehat{u}(k)K_R(k\alpha)| \leq \frac{2S}{4\rho \ell} \left( 1 + \sum_{1 \leq i < \frac{q}{4}} \frac{(2q/i)^p}{R^p} \right) \leq \frac{8S}{\rho q \ell} (1 + c(q/R)^p).
\]
Thus we have
\[
\left| \sum_{\ell=1}^{[4Kq^{-1}]+1} \sum_{|k| \in I_\ell} \tilde{u}(k) K_R(k\alpha)e^{2\pi ik\theta} \right| \leq \sum_{\ell=1}^{[4Kq^{-1}]+1} \frac{8S}{\rho q^\ell} (1 + c(q/R)^p) \\
\leq \frac{8S}{\rho q} (1 + c(q/R)^p) \ln[4Kq^{-1} + 1] \\
\leq \frac{8S}{\rho q} (1 + c(q/R)^p) \ln K.
\]
(8.6)

On the other hand, again by (8.3), we have
\[
\|R_Kw(\theta)\|_{l^2}^2 \leq \sum_{|k| \geq K} \frac{S^2}{(\rho|k|)^2} \leq \frac{S^2}{\rho^2} K^{-1}.
\]
(8.7)

3. Choose approximate $K, p$.

Now we can finish the proof of Proposition 6. By (8.4)-(8.6), we have
\[
\left|\left[u(\theta)\right]_a - \frac{1}{R^p} \sum_{j=0}^{p^{(R-1)}} c_R^p(j) u(j\theta + j\alpha)\right| \\
\leq \frac{2^{p+3}}{\rho} \left(\frac{q}{R}\right)^p + \frac{8S}{\rho q} \left(1 + c \left(\frac{q}{R}\right)^p\right) \ln K + |R_Kw(\theta)|.
\]

Take $q = R^{\sigma^{-1}}, \sigma > 1, K = \exp\{R^{\sigma^{-1} - p(1-\sigma^{-1})}\}$ and $0 < \varsigma < \sigma^{-1}, \varsigma(1 - \sigma^{-1})^{-1} < p < (\sigma - 1)^{-1}$. Once we fixed the parameters above, we have,
\[
\left|\frac{2^{p+3}}{\rho} \left(\frac{q}{R}\right)^p\right| = 2^{p+3} S \rho^{-1} R^{-\varsigma_1}, \quad \left|\frac{8S}{\rho q} \left(1 + c \left(\frac{q}{R}\right)^p\right) \ln K\right| \leq 16 \rho^{-1} S q^{-1} \ln K = 16 S \rho^{-1} R^{-\varsigma_1},
\]
where $\varsigma_1 = p(1 - \sigma^{-1})$. By Chebyshev’s inequality and (8.7), one has
\[
mes\left\{\theta : |R_Kw(\theta)| > 2^4 S \rho^{-1} R^{-\varsigma_1}\right\} \leq (2^4 S \rho^{-1} R^{-\varsigma_1})^{-2} \|R_Kw\|_{l^2}^2 \\
\leq 2^{-8} R^{2\varsigma_1} \exp\{-R^{\varsigma_3}\},
\]
where $\varsigma_3 = (1 + p)\sigma^{-1} - p$. By the above argument, the desired result follows directly.

8.1.2. Trigonometric polynomial approximations. Since $A \in G^u_\rho(\mathbb{T}, SL(2, \mathbb{R}))$, then we can write $A(\theta) = \sum_{k \in \mathbb{Z}} \hat{A}(k)e^{2\pi ik\theta}$ with estimate
\[
|\hat{A}(k)| \leq \|A\|_{\nu, \rho} e^{-\rho|2\pi k|^{\nu}}, \quad \forall k \in \mathbb{Z}.
\]
(8.8)

For any $N > 0$, denote $\tilde{N} = N^{b\nu^{-1}},$ where $b = \delta(\nu^{-1} - 1)^{-1}$ and $\delta \in (0, 1)$ will be fixed later. Once we have this, we can consider the truncated cocycle $\tilde{A}(\theta) := T_{\tilde{N}} A(\theta)$, denote by $\tilde{A}_N(\alpha, \theta)$ and $\tilde{L}_N(\alpha, \tilde{A})$ the associated transfer
matrix and finite Lyapunov exponent by substituting $\tilde{A}(\theta)$ for $A(\theta)$. Then we have the following lemma.

**Lemma 8.1.** There exists $N(\rho, \nu, \|A\|_{\nu, \rho}) \in \mathbb{N}$, $c = c(\rho, \nu, \|A\|_{\nu, \rho})$ such that if $N \geq N(\rho, \nu, \|A\|_{\nu, \rho})$, then we have the following estimates:

\begin{equation}
\|A(\theta) - \tilde{A}(\theta)\| \leq e^{-cN}, \quad e^{-cN_b} = e^{-cN_b}, \\
N^{-1} \ln \|A_N(\alpha, \theta)\| - N^{-1} \ln \|\tilde{A}_N(\alpha, \theta)\| \leq e^{-\frac{c}{2}N_b}, \\
|L_N(\alpha, A) - \tilde{L}_N(\alpha, \tilde{A})| < e^{-\frac{c}{2}N_b}.
\end{equation}

**Proof.** The estimate (8.9) follows directly from (8.8). Moreover, by telescoping argument, for $N \geq N(\rho, \nu, \|A\|_{\nu, \rho})$ which is large enough, we have

\[ \|A_N(\alpha, \theta) - \tilde{A}_N(\alpha, \theta)\| \leq (\|A\|_{\nu, \rho} + 1)^N e^{-cN_b} \leq e^{-\frac{c}{2}N_b}. \]

It follows that

\[ \left| \frac{1}{N} \ln \|A_N(\alpha, \theta)\| - \frac{1}{N} \ln \|\tilde{A}_N(\alpha, \theta)\| \right| \leq \frac{1}{N} \|A_N(\alpha, \theta) - \tilde{A}_N(\alpha, \theta)\| < e^{-\frac{c}{2}N_b}. \]

By averaging, one thus has $|L_N(\alpha, A) - \tilde{L}_N(\alpha, \tilde{A})| < e^{-\frac{c}{2}N_b}$. \(\square\)

Since $\tilde{A}(\theta)$ is a trigonometric polynomial, then one can analytic continue $\tilde{A}(\theta)$ to become an analytic function. Indeed, let

\[ \rho_N = \frac{\rho}{4\pi} N^{-b(\nu^{-1} - 1)} := \frac{\rho}{4\pi} N^{-\delta}, \]

and set $\vartheta = \theta + i\tilde{\theta}$, then $\tilde{A}(\vartheta)$ is analytic in the strip $|\tilde{\theta}| \leq \rho_N$:

\begin{equation}
\|\tilde{A}\|_{\rho_N}^* = \sum_{|k| < \tilde{N}} |\tilde{A}(k)| e^{2\pi |\nu| k \rho_N} \leq \|A\|_{\nu, \rho} \sum_{|k| < \tilde{N}} e^{-\rho|2\pi k|} e^{2\pi k |\nu| \rho_N} := e^{C_1} < \infty.
\end{equation}

For $\vartheta = \theta + i\tilde{\theta}$ with $|\tilde{\theta}| \leq \rho_N$, set

\begin{equation}
\tilde{u}_N(\vartheta) := \frac{1}{N} \ln \|\tilde{A}_N(\vartheta)\|.
\end{equation}

In the following lemma, we will prove that $|\tilde{u}_N(\vartheta)|$ is indeed a bounded subharmonic function in the strip $[\ln \vartheta| < \rho_N]$.

**Lemma 8.2.** We have the estimate

\begin{equation}
\sup_{\tilde{\theta} \in \mathbb{T}} \sup_{|\tilde{\theta}| \leq \rho_N} |\tilde{u}_N(\vartheta)| \leq \max\{\ln 2, C_1\},
\end{equation}

where $C_1$ is the one in (8.10).

**Proof.** We will prove that the analytic continuation $\tilde{A}(\vartheta)$ in the strip $[\ln \vartheta| < \rho_N]$ is not singular, which implies that $|\tilde{u}_N(\vartheta)|$ is a bounded subharmonic function.
We first give a estimate about $\| \widetilde{A}(\theta) - \widetilde{A}(\theta) \|$ as follows:

$$
\| \widetilde{A}(\theta) - \widetilde{A}(\theta) \| \leq \sum_{0 < |k| < N} |\widetilde{A}(k)| \sup_{\theta \in \mathbb{T}, |\bar{\theta}| \leq \rho N} |e^{2\pi ik\theta} (e^{-2\pi k\bar{\theta}} - 1)|
$$

$$
= \sum_{0 < |k| \leq N^{\delta/2}} |\widetilde{A}(k)| e^{2\pi k|\rho N} \left( 1 - e^{-|2\pi k\rho N|} \right)
+ \sum_{N^{\delta/2} < |k| < N^{bu-1}} |\widetilde{A}(k)| e^{2\pi k|\nu|} \left( e^{2\pi k|\rho N|} - 1 \right) e^{-2\pi k|\nu|}.
$$

To estimate the first term, note if $0 < |k| \leq N^{\delta/2}$, then one has $|2\pi k|\rho N \leq \rho N^{-\delta/2} / 2 \ll 1$, which implies

$$
1 - e^{-|2\pi k\rho N|} \leq 2|2\pi k\rho N| \leq \rho N^{-\delta/2}.
$$

To estimate the second term, note for all $k$ with $N^{\delta/2} < |k| < N^{bu-1}$, one has

$$
|2\pi k|\nu|/2 - |2\pi k|\rho N = 2^{-1} \rho (|2\pi k|\nu - |k|N^{-\delta}) \geq 0,
$$

which implies

$$
(e^{2\pi k|\rho N|} - 1)e^{-|2\pi k|\nu|/2} < 2, \forall N^{\delta/2} < |k| < N^{bu-1}.
$$

Consequently, one has

$$
\| \widetilde{A}(\theta) - \widetilde{A}(\theta) \| \leq \rho N^{-\delta/2} \| \widetilde{A} \|\| \rho \| + 2\| A \|\nu, \rho \| e^{-2\pi N^{\delta/2}|\nu|\rho/2}
\leq 2\rho N^{-\delta/2} e^{C_1}.
$$

(8.13)

To see this, one only needs to check that $f(x) = x^{\nu} - (2\pi)^{-1}xN^{-\delta} > 0$ on the interval $[2\pi, 2\pi N^{bu-1}]$.

Now we give the estimate $| \det \widetilde{A}(\theta) |$. First, the inequality in (8.13) implies

(8.14)

$$
| \det \widetilde{A}(\theta) - \det \widetilde{A}(\theta) | \leq 4\| A(\theta) \|\| \widetilde{A}(\theta) - \widetilde{A}(\theta) \| \leq 8\rho e^{2C_1} N^{-\delta/2} \ll 1.
$$

Moreover, note $A \in SL(2, \mathbb{R})$, then by Lemma 8.1, we have

(8.15)

$$
|1 - \det \widetilde{A}(\theta)| \leq 8\| A(\theta) \|\| \widetilde{A}(\theta) - \widetilde{A}(\theta) \| \leq Ce^{-CN^b},
$$

that is $| \det \widetilde{A}(\theta) | \geq 1/2, \forall \theta \in \mathbb{T}$, which, together with (8.14), yields

(8.16)

$$
| \det \widetilde{A}(\theta) | \geq 1/4, \forall (\theta, \bar{\theta}) \in \mathbb{T} \times [-\rho N, \rho N].
$$

Once we get the inequality in (8.16), we are ready to estimate $|\bar{u}_N(\theta)|$. Indeed,

$$
\| \bar{A}_N(\theta) \| \geq | \det \bar{A}_N(\theta) | = |\Pi_{\ell=0}^{N-1} \det \bar{A}(\theta + \ell \alpha) |
= \Pi_{\ell=0}^{N-1} | \det \bar{A}(\theta + \ell \alpha) | \geq 4^{-N},
$$

which yields $2^{-N} \leq \| \bar{A}_N(\theta) \| \leq e^{NC_1}$, or

$$
\sup_{\theta \in \mathbb{T}} \sup_{|\bar{\theta}| \leq \rho N} |\bar{u}_N(\theta)| \leq \max\{ \ln 2, C_1 \}.
$$

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8.1.3. Proof of Proposition 5. In this section we will give the proof of Proposition 5 by applying Proposition 6. First by Lemma 8.1, we have
\[
\frac{1}{N} \ln \|A_N(\theta)\| - L_N(\alpha, A) \leq |N^{-1} \ln \|A_N(\theta)\| - \tilde{u}_N(\theta)| \\
+ |\tilde{u}_N(\theta) - [\tilde{u}_N(\theta)]_{\theta}| + |[\tilde{u}_N(\theta)]_{\theta} - L_N(\alpha, A)| \\
\leq |\tilde{u}_N(\theta) - [\tilde{u}_N(\theta)]_{\theta}| + 2e^{-\frac{\pi}{2}N^b}.
\]
(8.17)
Thus we only need to estimate \(|\tilde{u}_N(\theta) - [\tilde{u}_N(\theta)]_{\theta}|\), which is controlled by the sum
\[
|\tilde{u}_N(\theta) - F_{R,p}[\tilde{u}_N](\theta)| + |F_{R,p}[\tilde{u}_N](\theta) - [\tilde{u}_N(\theta)]_{\theta}|,
\]
where \(F_{R,p}[u](\theta) = \frac{1}{Mp^{(R-1)}} \sum_{j=0}^{p} c^p_R(j)\tilde{u}_N(\theta + j\alpha)\).

In the following, we will give the estimates of the two terms above. First we would like to bound \(|\tilde{u}_N(\theta) - [\tilde{u}_N(\theta + \alpha)]|\). For the function \(\tilde{u}_N(\theta)\) defined by (8.11), the inequality in (8.12) and \(C_1 > \ln 2\) imply \(\tilde{u}_N(\theta)\) is a bounded subharmonic function on \([|\text{Im} \theta| < \rho_N]\). It follows that
\[
||\tilde{u}_N(\theta)|| \leq C_1 = 2^{-1}C_2, \ C_2 = 2C_1.
\]
That is this function \(\tilde{u}_N(\theta)\) satisfies the hypotheses in Proposition 6. Consequently,
\[
|\tilde{u}_N(\theta) - \tilde{u}_N(\theta + \alpha)| = \frac{1}{N} \left| \ln \|\tilde{A}_N(\theta)\| - \ln \|\tilde{A}_N(\theta + \alpha)\| \right| \\
= \frac{1}{N} \left| \ln \frac{\|\tilde{A}_N(\theta)\|}{\|\tilde{A}_N(\theta + \alpha)\|} \right| \\
\leq \frac{1}{N} \left| \ln \left( \|\tilde{A}(\theta + N\alpha)^{-1}\| \cdot ||\tilde{A}(\theta)|| \right) \right|.
\]
Thus we only need to estimate \(||\tilde{A}(\theta + N\alpha)^{-1}||\). Indeed, (8.15) implies
\[
||\tilde{A}(\theta)^{-1}|| \leq \|1/ \det \tilde{A}(\theta)\||\tilde{A}(\theta)|| \\
\leq \{1 + 2\|I - \det \tilde{A}(\theta)\|\}||\tilde{A}(\theta)|| \leq 2e^{C_1} \leq e^{2C_1}.
\]
Once we have this we get
\[
(8.20) \quad |\tilde{u}_N(\theta) - \tilde{u}_N(\theta + \alpha)| \leq \frac{3C_2}{2N}.
\]
For the fixed \(\nu\) with \(1/2 < \nu < 1\), there exists \(\delta \in (0, 1)\) such that \(\nu^{-1} < 1 + \delta\). Once we fix \(\nu\) and \(\delta\) by this way, we set \(b = \delta(\nu^{-1} - 1)^{-1} > 1\).
Then we choose $\sigma$ and $\varsigma$ in Proposition 6 as $1 < \sigma < \min\{2, \delta^{-1}\}$, $\varsigma = \delta$. That is the parameters $\sigma$ and $p > 1$ in Proposition 6 satisfy

$$\delta = b(1/\nu - 1) < \frac{1}{\sigma} < 1, \quad \frac{\delta\sigma}{\sigma - 1} < p < \frac{1}{\sigma - 1}.$$ 

Take $\gamma = 1 + p(1 - \sigma), \sigma_1 = \frac{2}{\delta}(\sigma - 1)$. It is obvious that

$$1 > \gamma = 1 + p(1 - \sigma) > 0, \quad \sigma - \sigma_1 = \frac{\delta\sigma - p(\sigma - 1)}{\delta} < 0.$$ 

Notice that $1 < \sigma < \delta^{-1}$ and $\nu^{-1} - 1 < \delta < 1$, thus $\delta, \sigma \to 1$ as $\nu \to 1/2$, which imply $\sigma_1 \to \sigma, p \to \infty$ and $\gamma \to 0$ as $\nu \to 1/2$.

For $q, R$ with $R = q^{\sigma}$ (as Proposition 6), set $\frac{g_1 p C_2}{C} q^{\sigma} < N < \left(\frac{C_2}{2^{p+\nu p C_2}}\right)^{\frac{1}{3}} q^{\sigma_1}$ and $K$ as the one in Proposition 6. Now we give the estimate of the first term in (8.18). More concretely,

$$F_{R,p}[\tilde{u}_N](\theta) - \bar{u}_N(\theta) \leq \frac{1}{R^p} \sum_{j=0}^{p(R-1)} |\tilde{u}_N(\theta + j\alpha) - \bar{u}_N(\theta)|c_R^p(j)$$

$$\leq \frac{1}{R^p} \sum_{j=0}^{p(R-1)} c_R^p(j) \frac{j3C_2}{2N} < \frac{3p(R-1)C_2}{2N}$$

$$\leq 3p(R-1)C_2 \frac{\kappa}{18pC_2} q^{-\sigma} \leq \frac{\kappa}{6},$$

where the third inequality is by (8.20).

We will apply Proposition 6 to get the estimate of the second term in (8.18). More concretely, let $\varsigma_i, i = 1, 2, 3$ be the ones in Proposition 6 with $2^{-1}C_2$ and $\rho_N$ in place of $S$ and $\rho$, respectively and note $N < \left(\frac{C_2}{2^{p+\nu p C_2}}\right)^{\frac{1}{3}} q^{\sigma_1}$ we get

$$\varsigma_2 R^{-\varsigma_1} = 2^{p+5}q^{-1}C_2 4\pi \rho^{-1} N^\delta q^{-p(\sigma - 1)} < \kappa.$$ 

Thus by Proposition 6 we know that there is a set such that for all $\theta$ outside this set we have

$$\left|F_{R,p}[\tilde{u}_N](\theta) - [\bar{u}_N(\theta)]_\theta\right| \leq \varsigma_2 R^{-\varsigma_1} < \kappa.$$ 

Moreover, the measure of this set is less than

$$2^{8} R^{2\varsigma_1} \exp\{-R^{\varsigma_1}\} = 2^{-8} q^{2p(\sigma - 1)} \exp\{-q^{\gamma}\} < \exp\{-2^{-1} q^{\gamma}\}.$$ 

Set $C_1(\kappa) = \frac{g_1 p C_2}{C} \kappa, C_2(\kappa) = \left(\frac{C_2}{2^{p+\nu p C_2}}\right)^{\frac{1}{3}}, c = \frac{1}{\sigma}$ and $q \geq q_0$, with $q_0$ depending on $\kappa, \rho, \nu$ (by Lemma 8.1) and sufficiently large, then by (8.17)-(8.23) we finish the proof of Proposition 5.
8.2. Application of avalanche principle.

Proposition 7 ([GS01, Bou05]). Let $A_1, A_2, \cdots, A_n$ be a sequence in $SL(2, \mathbb{R})$ satisfying the conditions

$$
\min_{1 \leq j \leq n} \|A_j\| \geq \mu > n,
$$

$$
\max_{1 \leq j \leq n} [\ln \|A_j\| + \ln \|A_{j+1}\| - \ln \|A_{j+1}A_j\|] < \frac{1}{2} \ln \mu.
$$

Then there exists a constant $C_A < \infty$ such that

$$
\left| \ln \prod_{j=1}^{n} A_j + \sum_{j=2}^{n-1} \ln \|A_j\| - \sum_{j=1}^{n-1} \ln \|A_{j+1}A_j\| \right| < C_A \frac{n}{\mu}.
$$

Following the ideas in [BJ02], in case of positive Lyapunov exponent, the large deviation theorem provides us a possibility to apply avalanche principal (Proposition 7) to $A(\theta + jN\alpha)$ for $\theta$ in a set of large measure and therefore pass on to a larger scale.

Lemma 8.3. Assume that $|\alpha - \frac{2}{q}| < \frac{1}{q}$, $(a, q) = 1$. Let $C_1(\kappa)q^\alpha < N < C_2(\kappa)q^{\alpha+1}$ and $q \geq q_0(\kappa)$ be the same as Proposition 5. Assume that $L_N(\alpha, A) > 100\kappa > 0$ and $L_{2N}(\alpha, A) > \frac{19}{20}L_N(\alpha, A)$. Then for $N'$ such that $m = N'N^{-1}$ satisfies $e^{\frac{q}{2}q^{\gamma/4}} < m < e^{q^{\gamma}}$, we have

$$
\left| L_{N'}(\alpha, A) + L_N(\alpha, A) - 2L_{2N}(\alpha, A) \right| < C e^{-\frac{c}{2}q^{\gamma}},
$$

where $c$ is the one from the large deviation bound of Proposition 5.

Proof. We use the avalanche principal (Proposition 7) on $A_j^N(\theta) = A_N(\theta + jN\alpha)$ with $\theta$ being restricted to the set $\Omega \subset \mathbb{T}$, defined by $2m$ conditions:

$$
\left| \frac{1}{N} \ln \|A_j^N(\theta)\| - L_N(\alpha, A) \right| \leq \kappa, \quad 1 \leq j \leq m,
$$

$$
\left| \frac{1}{2N} \ln \|A_j^{2N}(\theta)\| - L_{2N}(\alpha, A) \right| \leq \kappa, \quad 1 \leq j \leq m.
$$

By Proposition 5, we have for any $j$

$$
\text{mes}\left\{ \theta : \left| \frac{1}{N} \ln \|A_j^N(\theta)\| - L_N(\alpha, A) \right| > \kappa \right\} < e^{-cq^\gamma},
$$

$$
\text{mes}\left\{ \theta : \left| \frac{1}{2N} \ln \|A_j^{2N}(\theta)\| - L_{2N}(\alpha, A) \right| > \kappa \right\} < e^{-cq^\gamma}.
$$

Thus we have

$$
\text{mes}(\mathbb{T} \setminus \Omega) < 2me^{-cq^\gamma}.
$$

For each $A_j^N(\theta)$ with $\theta \in \Omega$,

$$
e^{N(L_N(\alpha,A) - \kappa)} < \|A_j^N(\theta)\| < e^{N(L_N(\alpha,A) + \kappa)}.
$$

Note that since $L_N(\alpha, A) > 100\kappa$, then

$$
\|A_j^N(\theta)\| > e^{0.999NL_N(\alpha,A)} := \mu.
$$
For large enough $q$, and hence $N$ by hypothesis, we have $\mu > 2m$ (since $\sigma > 1 > \gamma$). Also for $j < m$, by the fact that $A_j^N(\theta)A_{j+1}^N(\theta) = A_j^{2N}(\theta)$, we have
\[
\left| \ln \| A_j^N(\theta) \| + \ln \| A_{j+1}^N(\theta) \| - \ln \| A_{j+1}^N(\theta) A_j^N(\theta) \| \right|
< 4N\kappa + 2N|L_N(\alpha, A) - L_{2N}(\alpha, A)|
< \frac{1}{25}NL_N(\alpha, A) + 2N\left(\frac{1}{20}L_N(\alpha, A)\right) < \frac{1}{2}\ln \mu,
\]
where the second inequality follows by $L_{2N}(\alpha, A) > \frac{10}{29}L_N(\alpha, A)$. Thus, we can apply the avalanche principal (Proposition 7) for $\theta \in \Omega$ to obtain
\[
\left| \ln \prod_{j=1}^m A_j^N(\theta) + \sum_{j=2}^{m-1} \ln \| A_j^N(\theta) \| - \sum_{j=1}^{m-1} \ln \| A_{j+1}^N(\theta) A_j^N(\theta) \| \right|
< CAm/\mu < me^{-\frac{1}{2}NL_N(\alpha, A)}.
\]
Integrating on $\Omega$, we get
\[
\left| \int_{\Omega} \ln \| A_j^N(\theta) \| d\theta + \sum_{j=2}^{m-1} \int_{\Omega} \ln \| A_j^N(\theta + jN\alpha) \| d\theta \right.
\left. - \sum_{j=1}^{m-1} \int_{\Omega} \ln \| A_{j+1}^N(\theta + jN\alpha) \| d\theta \right| < me^{-\frac{1}{2}NL_N(\alpha, A)},
\]
therefore, recalling (8.24) and the assumption $N > C_1(\kappa)q^\sigma$, we have
\[
\left| L_N(\alpha, A) + \frac{m-2}{m}L_N(\alpha, A) - \frac{2(m-1)}{m}L_{2N}(\alpha, A) \right|
< mN^{\sigma-1}e^{-\frac{1}{2}NL_N(\alpha, A)} + Ce^{-\frac{1}{2}q^\gamma} < Ce^{-\frac{1}{2}q^\gamma}.
\]
It follows that
\[
\left| L_N(\alpha, A) + L_N(\alpha, A) - 2L_{2N}(\alpha, A) \right|
< Ce^{-\frac{1}{2}q^\gamma} + 2m^{-1}|L_N(\alpha, A) - L_{2N}(\alpha, A)|
< Ce^{-\frac{1}{2}q^\gamma} + L_N(\alpha, A)(10m)^{-1} < Ce^{-\frac{1}{2}q^\gamma},
\]
where the second inequality is by $L_{2N}(\alpha, A) > \frac{10}{29}L_N(\alpha, A)$ and the last inequality is by $m > e^{\frac{1}{2}q^\gamma/4}$.

Actually, the condition "$L_{2N}(\alpha, A) > \frac{10}{29}L_N(\alpha, A)$" is not necessary if $q$ is sufficiently large and $L(\alpha, A) > 0$.

**Lemma 8.4.** Assume that $L(\alpha, A) > 100\kappa > 0$, then there exists $N_0 \in \mathbb{N}$ with $C_1(\kappa)q_0^\sigma < N_0 < C_2(\kappa)q_0^\sigma$, $q_0$ is the one defined in Proposition 5 such that
\[
L_{2N_0}(\alpha, A) > \frac{99}{100}L_{N_0}(\alpha, A).
\]
Proof. Note that for any \( n \), by subadditivity, we have
\[
100 \kappa < L(\alpha, A) = \inf L_n(\alpha, A) \leq L_{2n}(\alpha, A) \leq L_n(\alpha, A) \leq C_1,
\]
where \( C_1 \) is the one in (8.10).

Set \( j_0 = \lfloor (\ln(100/99))^{-1} \ln(C_1/100\kappa) \rfloor \), that is
\[
(8.26) \quad (99/100)^{j_0 + 1} C_1 < 100 \kappa \leq (99/100)^{j_0} C_1.
\]

Consider the sequence \( \{L_{2jN}(\alpha, A)\} \) where \( N = |C_1(q_0^\gamma)| + 1, j \in \mathbb{N} \). If
\[
L_{2j+1N}(\alpha, A) \leq (99/100)L_{2jN}(\alpha, A)
\]
hold for all \( 0 \leq j \leq j_0 \), then
\[
100 \kappa < L_{2j_0+1N}(\alpha, A) \leq (99/100)^{j_0 + 1} L_N(\alpha, A) \leq (99/100)^{j_0 + 1} C_1 < 100 \kappa,
\]
where the last inequality is by first inequality in (8.26). Thus there exists \( j_0 \)'s with \( 0 \leq j_0 \leq j_0 \) such that
\[
L_{2j_0+1N}(\alpha, A) > (99/100)L_{2j_0N}(\alpha, A).
\]

Moreover, since \( j_0 \) is fixed, we can set \( q_0 \) large enough such that
\[
2^{j_0} < 2^{-1}C_1(\kappa)^{-1}C_2(\kappa)q_0^{\sigma_1-\sigma}.
\]

Set \( N_0 = 2^{j_0}N \). Thus we have the estimates
\[
C_1(\kappa)q_0^\sigma \leq N \leq N_0 \leq 2^{j_0}N \leq C_2(\kappa)q_0^{\sigma_1}
\]
with
\[
L_{2N_0}(\alpha, A) > (99/100)L_{N_0}(\alpha, A).
\]

\( \square \)

8.3. Inductive argument. Once one has Lemma 8.3, one can follow the induction arguments developed in [BJ02]. However, in our case there is an upper bound of \( N \) in the large deviation theorem. Thus we can only deal with Diophantine frequencies and their continued fraction expansions. Moreover, we need to deal with the Diophantine frequencies and their continued fraction expansions separately. Let \( p_n/q_n \) be the continued fraction expansion of \( \alpha \). To apply Lemma 8.3 inductively, we first fix \( \alpha \in DC(v, \tau) \), and inductively choose the following sequences:
\[
q_0 = \tilde{q}_0 < N_0 < \tilde{q}_1 < N_1 < \cdots < N_s < \tilde{q}_{s+1} < N_{s+1} < \cdots,
\]
where \( \tilde{p}_i/\tilde{q}_i \) is a subsequence of the continued fraction expansion of \( \alpha \) with
\[
\tilde{q}_{s+1} \quad \text{is the smallest } q_j \text{ such that } \tilde{q}_{s+1} > e^{\tilde{q}_s^{\gamma/2}}, s \geq 0,
\]
\[
C_1(\kappa)\tilde{q}_s^{\sigma} < N_s < C_2(\kappa)\tilde{q}_s^{\sigma_1}, \quad \tilde{q}_s|N_s, s \geq 0,
\]
\[
N_{s+1} = m_{s+1}N_s, \quad e^{\tilde{q}_s^{\gamma/4}} < m_{s+1} < 2m_{s+1} < e^{\tilde{q}_s^{\gamma}}, s \geq 0.
\]

Actually, we can inductively select a sequence \( \{N_s\} \) such that (8.28)-(8.30) hold, indeed the starting case \( s = 0 \) follows from Lemma 8.4. First by the selection of \( \tilde{q}_s \) and the Diophantine condition of \( \alpha \), one can check that
It's easy to check that
\[ C_1(\kappa)\tilde{q}_{s+1}^\sigma < C_1(\kappa)\tilde{q}_{s}^\sigma \tilde{q}_{s+1}^\sigma < N_{s+1} < 2C_2(\kappa)\tilde{q}_{s}^\sigma \tilde{q}_{s+1}^\sigma < C_2(\kappa)\tilde{q}_{s}^\sigma, \]
\[ e^{e^{7/2}} < \tilde{q}_{s+1} < e^{2e^{7/2}}. \]
Thus such a choice of \( N_s \) satisfies all estimates in (8.28)-(8.30) if \( \tilde{q}_0 \) is sufficiently large.

**Lemma 8.5.** Assume that \( \alpha \in DC(v, \tau) \) and \( L(\alpha, A) > 100\kappa > 0 \). There exist \( c'' > 0 \) and \( C_1(\kappa){\tilde{q}_0^\sigma} < N_0 < C_2(\kappa){\tilde{q}_0^\sigma} \) such that
\[ |L(\alpha, A) + L_{N_0}(\alpha, A) - 2L_{2N_0}(\alpha, A)| < e^{-c''{\tilde{q}_0}^{\gamma/4}}. \]

**Proof.** Let \( c/100 < c_3 < c_2 < c_1 < c/2, \) \( 2C_1 < C < \infty, \) and \( \tilde{q}_{-1} = 0 \). We use induction to show that the sequences \( \{N_s\} \) and \( \{\tilde{q}_s\} \), defined by (8.27), additionally satisfy, for \( s \geq 0 \),
\[ |L_{N_{s+1}}(\alpha, A) + L_{N_s}(\alpha, A) - 2L_{2N_s}(\alpha, A)| < Ce^{-c_1{\tilde{q}_s}^{\gamma/4}}, \]
(8.31)
\[ |L_{2N_{s+1}}(\alpha, A) - L_{N_{s+1}}(\alpha, A)| < Ce^{-c_2{\tilde{q}_s}^{\gamma/4}}, \]
(8.32)
\[ |L_{N_{s+1}}(\alpha, A) - L_{N_s}(\alpha, A)| < Ce^{-c_3{\tilde{q}_s}^{\gamma/4}}. \]
(8.33)
We first check the case \( s = 0 \). Fix \( N_1 \) satisfying (8.30). We will show
\[ |L_{N_1}(\alpha, A) + L_{N_0}(\alpha, A) - 2L_{2N_0}(\alpha, A)| < Ce^{-c_1{\tilde{q}_0}^{\gamma/4}}, \]
\[ |L_{2N_1}(\alpha, A) - L_{N_1}(\alpha, A)| < Ce^{-c_2{\tilde{q}_0}^{\gamma/4}}, \]
\[ |L_{N_1}(\alpha, A) - L_{N_0}(\alpha, A)| < Ce^{-c_3{\tilde{q}_0}^{\gamma/4}} = C. \]
In this case, the last inequality holds automatically since \( \tilde{q}_{-1} = 0 \), one only needs to check the first two inequalities. By (8.25) and (8.29) with \( s = 0 \) we know that the conditions in Lemma 8.3 are all satisfied with \( N' = N_1, N = N_0 \) and \( q = \tilde{q}_0 \). Therefore, by Lemma 8.3, we have
\[ |L_{N_1}(\alpha, A) + L_{N_0}(\alpha, A) - 2L_{2N_0}(\alpha, A)| < Ce^{-c_2{\tilde{q}_0}^{\gamma/4}} < Ce^{-c_1{\tilde{q}_0}^{\gamma/4}}. \]
On the other hand, (8.30) ensures one can also apply Lemma 8.3 to \( N' = 2N_1 \), thus we have
\[ |L_{2N_1}(\alpha, A) + L_{N_0}(\alpha, A) - 2L_{2N_0}(\alpha, A)| < Ce^{-c_1{\tilde{q}_0}^{\gamma/4}}. \]
It follows that
\[ |L_{2N_1}(\alpha, A) - L_{N_1}(\alpha, A)| < 2Ce^{-c_1{\tilde{q}_0}^{\gamma/4}} < Ce^{-c_3{\tilde{q}_0}^{\gamma/4}}, \]
and we have completed the initial case \( s = 0 \).
For $j \geq 1$, assume that (8.31)-(8.33) hold for all $s$ with $s \leq j - 1$. Now we consider the case $s = j$. Fix $N_{j+1}$ satisfying (8.30). By induction we have

$$|L_{2N_j}(\alpha, A) - L_{N_j}(\alpha, A)| < e^{-c_2\tilde{q}^{\gamma/4}} \leq C e^{-c_2\tilde{q}^{\gamma/4}}.$$  

This implies $L_{2N_j}(\alpha, A) > (19/20) L_{N_j}(\alpha, A)$, which together with (8.29), implies $N_j$ satisfies the two conditions of $N$ in Lemma 8.3 with $\tilde{q}_j$ in place of $q$. Moreover, by (8.30), $m_{j+1} = N_{j+1}^{-1}$ satisfies the estimate of $m$ in Lemma 8.3 with $\tilde{q}_j$ in place of $q$. Thus by Lemma 8.3, with $N' = N_{j+1}, N = N_j$ and $q = \tilde{q}_j$ we get

$$|L_{N_{j+1}}(\alpha, A) + L_{N_j}(\alpha, A) - 2L_{2N_j}(\alpha, A)| < e^{-c_1\tilde{q}_j^{\gamma/4}} < C e^{-c_1\tilde{q}_j^{\gamma/4}}.$$  

Similarly, (8.30) ensures one can also apply Lemma 8.3 to $N' = 2N_{j+1}$, and we have

$$|L_{2N_{j+1}}(\alpha, A) + L_{N_j}(\alpha, A) - 2L_{2N_j}(\alpha, A)| < C e^{-c_1\tilde{q}_j^{\gamma/4}}.$$  

Thus

$$|L_{2N_{j+1}}(\alpha, A) - L_{N_{j+1}}(\alpha, A)| < 2C e^{-c_1\tilde{q}_j^{\gamma/4}} < C e^{-c_2\tilde{q}_j^{\gamma/4}},$$  

$$|L_{N_{j+1}}(\alpha, A) - L_{N_j}(\alpha, A)| \leq |L_{N_{j+1}}(\alpha, A) + L_{N_j}(\alpha, A) - 2L_{2N_j}(\alpha, A)| + |2L_{2N_j}(\alpha, A) - 2L_{N_j}(\alpha, A)|$$  

$$< C e^{-c_1\tilde{q}_j^{\gamma/4}} + 2C e^{-c_2\tilde{q}_j^{\gamma/4}} < C e^{-c_3\tilde{q}_j^{\gamma/4}}.$$  

That is the estimates in (8.31)-(8.33) hold for all $s \in \mathbb{N}$. As a consequence of (8.31) with $s = 0$ and (8.33)

$$|L(\alpha, A) + L_{N_0}(\alpha, A) - 2L_{2N_0}(\alpha, A)|$$  

$$\leq |L_{N_1}(\alpha, A) + L_{N_0}(\alpha, A) - 2L_{2N_0}(\alpha, A)|$$  

$$+ \sum_{s \geq 1} |L_{N_{s+1}}(\alpha, A) - L_{N_s}(\alpha, A)| < e^{-c_0\tilde{q}_0^{\gamma/4}}.$$  

\[\square\]

For the rational frequency case, we will first estimate the difference between $L(p_j/q_j, A_j)$ and $L_n(p_j/q_j, A_j)$ with $n$ much larger than $q_j$ (Lemma 8.6), and then use avalanche principle estimate to get estimate of $L_n(p_j/q_j, A_j)$ (Lemma 8.7).

**Lemma 8.6.** Consider the cocycle $(p/q, A) \in \mathbb{Q} \times C^0(\mathbb{T}, SL(2, \mathbb{R}))$ with $p, q \in \mathbb{N}$, $(p, q) = 1$, and $|A| \leq C$. Set $n = mq + r, m \in \mathbb{N}, 0 \leq r < q$, then

$$L_n(p/q, A) \leq L(p/q, A) + 2n^{-1}(\ln n + qC_1).$$  

**Proof.** Set $A_q := A q(\theta) = A(\theta + (q-1)p/q) A(\theta + (q-2)p/q) \cdots A(\theta)$. For the matrix $A_q$ there exists a unitary $U$ such that $A_q = U \begin{pmatrix} \lambda & \psi \\ 0 & \lambda^{-1} \end{pmatrix} U^{-1}.$
Then for \( m \in \mathbb{N} \), we have \( A_q^m = U \begin{pmatrix} \lambda^m & R_m(\lambda, \psi) \\ 0 & \lambda^{-m} \end{pmatrix} U^{-1} \), \( m \geq 2 \) with

\[
R_m(\lambda, \psi) = \begin{cases} 
\sum_{l=1}^{k} \{ \lambda^{2l-1} + \lambda^{-(2l-1)} \} \psi & m = 2k, k \geq 1, \\
\psi + \sum_{l=1}^{k} \{ \lambda^{2l} + \lambda^{-2l} \} \psi & m = 2k + 1, k \geq 1.
\end{cases}
\]

Thus

\[
\| A_q^m \| \leq \| \lambda^m \| + \| R(\lambda, \psi) \| \leq \| \lambda \|^m (1 + m \| \psi \|) \leq \rho(A_q)^m (1 + m \exp\{qC_1\}),
\]

where \( \rho(A) \) stands for the spectrum radius. Note, for \( n = mq + r \), \( A_n(\theta) = A_r(\theta) A_q^n(\theta) \), then by the inequality above we get

\[
L_n(p/q, A) = \frac{1}{m} \int_T \ln \| A_n(\theta) \| d\theta \leq \frac{1}{n} \int_T \ln \| A_q^n(\theta) \| d\theta + \frac{rC_1}{n} \\
\leq \frac{1}{n} \left\{ \int_T \ln \rho(A_q)^m d\theta + \int_T \ln (1 + m \exp\{qC_1\}) d\theta \right\} + \frac{qC_1}{n} \\
\leq L(p/q, A) + 2n^{-1}(\ln m + qC_1).
\]

\[
L_n(p/q, A) = \frac{1}{m} \int_T \ln \| A_n(\theta) \| d\theta \leq \frac{1}{n} \int_T \ln \| A_q^n(\theta) \| d\theta + \frac{rC_1}{n} \\
\leq \frac{1}{n} \left\{ \int_T \ln \rho(A_q)^m d\theta + \int_T \ln (1 + m \exp\{qC_1\}) d\theta \right\} + \frac{qC_1}{n} \\
\leq L(p/q, A) + 2n^{-1}(\ln m + qC_1).
\]

\[
\Box
\]

**Lemma 8.7.** Assume that \( \alpha \in DC(v, \tau) \) and \( \{p_j/q_j\} \) is the sequence of continued fraction expansion of \( \alpha \), and \( A_j \rightarrow A \) under the topology derived by \( \| \cdot \|_{\nu, \rho} \)-norm. Then there exists a \( j_1 \) such that for \( j \geq j_1 \), we have

\[
|L(p_j/q_j, A_j) + L_{N_0}(p_j/q_j, A_j) - 2L_{2N_0}(p_j/q_j, A_j)| < 2e^{-c''\tilde{q}_0^{\gamma/4}}.
\]

**Remark 8.1.** Here \( N_0 \) and \( c'' \) are the ones in Lemma 8.5.

**Proof.** For the fixed \( N_0 \), note \( L_{2N_0}(1, 2) \) and \( L_{N_0}(1, 2) \) are continuous in both variables and \( L_{2N_0}(\alpha, A) > (99/100)L_{N_0}(\alpha, A), L_{N_0}(\alpha, A) > 100 \kappa > 0 \), then there exists \( j_1 \in \mathbb{Z}^+ \), such that if \( j > j_1 \), we have

\[
L_{N_0}(p_j/q_j, A_j) > 99\kappa
\]

(34)

\[
L_{2N_0}(p_j/q_j, A_j) > (49/50)L_{N_0}(p_j/q_j, A_j).
\]

(35)

For the fixed \( p_j/q_j \) and the sequence \( \{\tilde{q}_\ell\} \) defined by (8.27), there exists \( s_j \in \mathbb{N} \) such that \( \tilde{q}_{s_j} \leq q_j < \tilde{q}_{s_j+1} \). Then we define the same sequences \( \{\tilde{q}_{\ell}\}_{\ell=0}^{s_j} \) and \( \{N_\ell\}_{\ell=0}^{s_j+1} \) for \( p_j/q_j \) as \( \alpha \) such that (8.28)-(30) hold. Following Lemma 8.5, we will inductively show that

\[
|L_{N_{\ell+1}}(p_j/q_j, A_j) + L_{N_\ell}(p_j/q_j, A_j) - 2L_{2N_\ell}(p_j/q_j, A_j)| < Ce^{-c\tilde{q}_\ell^{\gamma/4}},
\]

(36)

\[
|L_{2N_{\ell+1}}(p_j/q_j, A_j) - L_{N_{\ell+1}}(p_j/q_j, A_j)| < Ce^{-c\tilde{q}_\ell^{\gamma/4}},
\]

(37)

\[
|L_{N_{\ell+1}}(p_j/q_j, A_j) - L_{N_\ell}(p_j/q_j, A_j)| < Ce^{-c\tilde{q}_\ell^{\gamma/4}},
\]

(38)

\[
|L_{N_{\ell+1}}(p_j/q_j, A_j) - L_{N_\ell}(p_j/q_j, A_j)| \leq Ce^{-c\tilde{q}_\ell^{\gamma/4}}.
\]

(39)
Moreover, by Lemma 8.3, it is a continuous function in both variables, thus, \( L \) which implies that
\[
(8.40) \quad L_{2N_0}(p_j/q_j, A_j) > (19/20)L_{N_0}(p_j/q_j, A_j),
\]
Indeed, by the property of continued fraction expansion, (8.39) holds for any \( 0 \leq \ell \leq s_j \), and (8.40) follows from (8.35) and (8.37). On the other hand, if \( \ell = 0 \), by (8.34), (8.35) and (8.36), we have
\[
|L_{N_1}(p_j/q_j, A_j) - L_{N_0}(p_j/q_j, A_j)| \leq |L_{N_1}(p_j/q_j, A_j) - L_{N_0}(p_j/q_j, A_j)| + 2|L_{2N_0}(p_j/q_j, A_j) - L_{N_0}(p_j/q_j, A_j)| \\
\leq Ce^{-c_1\tilde{q}_0^{\gamma/4}} + 2L_{N_0}(p_j/q_j, A_j)/50,
\]
which implies that
\[
(8.42) \quad L_{N_1}(p_j/q_j, A_j) \geq 48L_{N_0}(p_j/q_j, A_j)/50 - Ce^{-c_1\tilde{q}_0^{\gamma/4}} > 95\kappa.
\]
As for (8.41), in case \( \ell \geq 1 \), by (8.42) and (8.38), one has
\[
L_{N_{\ell+1}}(p_j/q_j, A_j) \geq L_{N_1}(p_j/q_j, A_j) - \sum_{k=1}^{\ell} |L_{N_{k+1}}(p_j/q_j, A_j) - L_{N_k}(p_j/q_j, A_j)| \\
> \left( 95\kappa - \sum_{k=0}^{\ell-1} Ce^{-c_3\tilde{q}_0^{\gamma/4}} \right) > 90\kappa.
\]
Therefore, the iteration can be conducted \( s_j \) times, and we obtain
\[
(8.43) \quad |L_{N_{s_j+1}}(p_j/q_j, A_j) + L_{N_0}(p_j/q_j, A_j) - 2L_{2N_0}(p_j/q_j, A_j)| < e^{-c''\tilde{q}_0^{\gamma/4}}.
\]
Moreover, by Lemma 8.6, we get
\[
L_{N_{s_j+1}}(p_j/q_j, A_j) \leq L(p_j/q_j, A_j) + 5C_1C_1(\kappa)^{-1}\tilde{q}_j^{-\sigma}(\sigma-1).
\]
The inequality above, together with (8.43) yields
\[
|L(p_j/q_j, A_j) + L_{N_0}(p_j/q_j, A_j) - 2L_{2N_0}(p_j/q_j, A_j)| < 2e^{-c''\tilde{q}_0^{\gamma/4}}.
\]
\[
\square \]

8.4. Proof of Theorem 6.3. Assume \( \alpha \in DC(v, \tau) \) and \( p_n/q_n \) be the continued fraction expansion of \( \alpha \). Notice that since for each \( N \), \( L_N(\alpha, A) \) is a continuous function in both variables, thus, \( L(\alpha, A) = \inf L_N(\alpha, A) \) is upper semi-continuous, then in the case \( L(\alpha, A) = 0 \) we get
\[
0 \leq \liminf_{n \to \infty} L(p_n/q_n, A_n) \leq \limsup_{n \to \infty} L(p_n/q_n, A_n) \leq L(\alpha, A) = 0,
\]
that is \( \lim_{n \to \infty} L(p_n/q_n, A_n) = 0 \). Therefore we may assume \( L(\alpha, A) > 100\kappa > 0 \).
Take \( j > j_1 \) and \( C_1(\kappa)q_0^\delta < N_0 < C_2(\kappa)q_0^{\gamma_1} \), by Lemma 8.5 and Lemma 8.7, we have
\[
|L(\alpha, A) + L_{N_0}(\alpha, A) - 2L_{2N_0}(\alpha, A)| < e^{-c''_0\gamma_0^{\alpha/4}},
\]
and
\[
|L(p_j/q_j, A_j) + L_{N_0}(p_j/q_j, A_j) - 2L_{2N_0}(p_j/q_j, A_j)| < 2e^{-c''_0\gamma_0^{\alpha/4}}.
\]
Hence, one can estimate
\[
|L(\alpha, A) - L(p_j/q_j, A_j)| \leq |L_{N_0}(p_j/q_j, A_j) - L_{N_0}(\alpha, A)|
+ 2|L_{2N_0}(p_j/q_j, A_j) - L_{2N_0}(\alpha, A)| + 3e^{-c''_0\gamma_0^{\alpha/4}}
\]
\[
\leq C(\kappa)^{N_0} \{|p_j/q_j - \alpha| + \|A - A_j\|_{\nu, \rho}\} + 3e^{-c''_0\gamma_0^{\alpha/4}},
\]
it follows that
\[
\limsup_{j \to \infty} |L(\alpha, A) - L(p_j/q_j, A_j)| \leq 4e^{-c''_0\gamma_0^{\alpha/4}},
\]
let \( \gamma_0 \to \infty \), we get the result.

**APPENDIX: PROOF OF LEMMA 4.5**

Define \( B_r(\delta) = \{ Y \in B_r(\text{nre}) : \| Y \|_r \leq \delta \} \) and set \( \epsilon = 8^{-2} r^2 Q_{n+1}^2 \). Then we define the nonlinear functional
\[
\mathcal{F} : B_r(\epsilon^{1/2}) \to B_r(\text{nre})
\]
by
\[
(8.44) \quad \mathcal{F}(Y) = \mathbb{P}_{\text{nre}} \ln(e^{A^{-1}Y(\theta + \alpha)A} e^{\theta} e^{-Y}),
\]
where \( \mathbb{P}_{\text{nre}} \) denotes the standard projections from \( B_r \) to \( B_r(\text{nre}) \).

We will find a solution of functional equation
\[
(8.45) \quad \mathcal{F}(Y_t) = (1 - t)\mathcal{F}(Y_0), \quad Y_0 = 0.
\]

The derivative of \( \mathcal{F} \) at \( Y \in B_r(\epsilon^{1/2}) \) along \( Y' \in B_r(\text{nre}) \) is given by
\[
(8.46) \quad D\mathcal{F}(Y)Y' = \mathbb{P}_{\text{nre}} \{ A^{-1}Y'(\theta + \alpha)A - Y' + O(\|A\|^2 g)Y' + P[A, Y, Y', g](\theta) \},
\]
where
\[
P[A, Y, Y', g](\theta) = O(A^{-1}Y(\theta + \alpha)A)A^{-1}Y'(\theta + \alpha)A + 2^{-1}[Y''', F + H] + \cdots
\]
\[
- O(Y)Y' + 2^{-1}[F + H', -Y''] + \cdots - O(\|A\|^2 g)Y',
\]
\[
O(\|A\|^2 g)Y' = O(g)A^{-1}Y'(\theta + \alpha)A + O(g)Y',
\]
with \( P[A, Y, Y', 0](\theta) = 0 \),
\[
Y'''(\theta + \alpha) = A^{-1}Y'(\theta + \alpha)A + O(A^{-1}Y(\theta + \alpha)A)A^{-1}Y'(\theta + \alpha)A,
\]
\[
Y''(\theta) = Y'(\theta) + O(Y(\theta))Y'(\theta),
\]
\[
F(\theta) = A^{-1}Y(\theta + \alpha)A + g(\theta) - Y(\theta),
\]
and $H, H'$ being sums of terms at least 2 orders in $A^{-1}Y'(\theta + \alpha)A, g(\theta), -Y(\theta)$. Moreover, the first “…” denotes the sum of terms which are at least 2 orders in $F + H$ but only 1 order in $Y''$. The second “…” denotes the sum of terms which are at least 2 orders in $F + H'$ but only 1 order in $Y''$.

We give a estimate about the operator $DF(Y)^{-1}$.

**Proposition 8.** For the fixed $Y \in B_r(\varepsilon^{1/2})$, $DF(Y)$ (defined by (8.46)) is a linear map from $B_r^{(nre)}$ to $B_r^{(nre)}$ with estimate

$$
(8.47) \quad \|DF(Y)^{-1}\| \leq 2^{-1}\varepsilon^{-1/2}.
$$

**Proof.** For the fixed $Y \in B_r(\varepsilon^{1/2})$, obviously, the operator $DF(Y)$ defined by (8.46) is a linear map from $B_r^{(nre)}$ to $B_r^{(nre)}$. In the following we prove the estimate in (8.47). To this end, we consider the operator $DF(0)$ given by

$$
DF(0)Y' = A^{-1}Y'(\theta + \alpha)A - Y' + P_{nre}O(\|A\|^2 g)Y', Y' \in B_r^{(nre)}.
$$

Note the operator $DF(0)$ is a linear map mapping $B_r^{(nre)}$ to $B_r^{(nre)}$. Next we give the estimate about $DF(0)^{-1}$.

Note $\overline{Q}_{n+1} \geq T > (2\gamma^{-1})^{2r}, n \geq 0 ((4.4))$, then by (4.18) in Lemma 4.4 we get

$$
\|k\alpha \pm 2\rho f\|_2 \geq \gamma Q_{n+1}^{-\gamma^2} = 8\varepsilon^{\frac{1}{2}}, |k| < \overline{Q}_{n+1}^{\frac{1}{2}}.
$$

By the inequality above, one can check, for $Y' \in B_r^{(nre)}$,

$$
A^{-1}Y'(\cdot + \alpha)A - Y' \in B_r^{(nre)},
$$

$$
\|A^{-1}Y'(\cdot + \alpha)A - Y'\|_r \leq 8\|A\|^2\varepsilon^{\frac{1}{2}}\|Y'\|_r.
$$

Moreover, by Lemma 3.1 (Banach algebra property) we get $\|O(\|A\|^2 g)\|_r \leq 2\|A\|^2\varepsilon$. Note $8\varepsilon^{1/2} - 12\varepsilon < 1$, then

$$
(8.48) \quad \|DF(0)^{-1}\|_r \leq 2(8\varepsilon^{1/2})^{-1} - 1 = 4^{-1}\varepsilon^{-1/2}.
$$

Once we get (8.48), we will turn to $\|DF(Y)^{-1}\|$. The calculations below also depends on Lemma 3.1, we omit the reference about this lemma.

Note $\{DF(Y) - DF(0)\}Y' = P_{nre}\{P(A, Y, Y'g) - P(A, 0, Y'g)\}$, we get

$$
(8.49) \quad \sup_{\|Y\|_r \leq \varepsilon^{\frac{1}{2}}, \|g\|_r \leq \varepsilon} \|DF(Y) - DF(0)\| \leq 2\varepsilon^{\frac{1}{2}}.
$$

(8.48) and (8.49) yield

$$
\|DF(0)^{-1}(DF(Y) - DF(0))\| \leq 4^{-1}\varepsilon^{-1/2}2\varepsilon^{\frac{1}{2}} = 2^{-1}.
$$

Finally, note

$$
DF(Y)^{-1} = \{1 + DF(0)^{-1}(DF(Y) - DF(0))\}^{-1}DF(0)^{-1},
$$
then by the inequality above we know that $D F(Y)$ is invertible with
\[ \| D F(Y)^{-1} \| \leq 2 \| D F(0)^{-1} \| \leq 2^{-1} \epsilon^{-1/2}. \]
\[ \square \]

Now we turn to functional equation (8.45). Formally, we get
\[ (8.50) \ Y_t = - \int_0^t D F(Y_s)^{-1} F(Y_0) ds = - \int_0^t D F(Y_s)^{-1} \mathbb{P}_{nre} g ds, 0 \leq t \leq 1. \]
Moreover, by (8.47)
\[ \| Y_t \|_r \leq \sup_{Y \in B_r(\epsilon^{1/2}), \| g \|_r \leq \epsilon} \| D F(Y)^{-1} \| \| g \|_r \leq 2^{-1} \epsilon^{-1/2} \epsilon < \epsilon^{1/2}, 0 \leq t \leq 1. \]

Therefore, the solution of (8.45) exists in $B_r(\epsilon^{1/2})$ and is given by (8.50).

For $Y_t$, $0 \leq t \leq 1$, given above, we know that $F(Y_1) = 0$, that is (by (8.44))
\[ \mathbb{P}_{nre} \ln(e^{A^{-1} Y_1 (\theta + \alpha)} A e^{g} e^{-Y_1}) = 0, \]
which implies that there exists a matrix $g^{(re)} \in B_r^{(re)}$ such that
\[ \ln(e^{A^{-1} Y_1 (\theta + \alpha)} A e^{g} e^{-Y_1}) = g^{(re)}. \]
That is
\[ e^{Y_1 (\theta + \alpha)} A e^{g} e^{-Y_1} = A e^{g^{(re)}}. \]
By standard calculations we get the estimate $\| g^{(re)} \| \leq 2 \epsilon$. This $Y_1$, with the estimate $\| Y_1 \|_r < \epsilon^{1/2}$, is the one we want.

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