Continuum Deformation Coordination of Multi-Agent Systems Using Cooperative Localization

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Abstract—This paper studies the problem of decentralized continuum deformation coordination of multi-agent systems aided by cooperative localization. We treat agents as particles inside a triangular continuum (deformable body) in a 2-D motion space and let the continuum deformation coordination be defined by three leaders located at vertices of a triangle, called the leading triangle. The leaders’ desired trajectories are assigned as the solution of a constrained optimal control problem such that safety requirements are satisfied in the presence of disturbance and measurement noise. Followers distributed inside the leading triangle acquire continuum deformation in a decentralized fashion by integrating cooperative localization and local communication. Specifically, cooperative localization estimates the global positions of all agents using relative position measurements based primarily on proximity of agents. Simulation results are presented for a network of ten agents.

I. INTRODUCTION

Cooperative localization (CL) has shown great promise in reducing state estimation errors for multi-agent systems particularly when no GPS is available [1], [2]. Cooperative localization algorithms are well-suited in decentralized coordination since they rely only on relative pose measurements and self propagation. Centralized cooperative localization algorithms [1], [3] have been shown to be better pose estimators but require each agent keeping track of their cross covariances with all other agents in ad hoc networks. However in networks with predefined communication graphs and/or formation, these same centralized algorithms can be applied in a decentralized manner since each agent only needs to keep track of nearby agents; i.e. those agents that it is able to measure their relative pose using on-board sensors. Hence, we can reliably use these algorithms for continuum deformation coordination of multi-agent systems.

Containment Control and Continuum Deformation Coordination [4], [5] are two existing decentralized leader-follower methods in which a desired coordination is guided by a finite number of leaders and acquired by the remaining followers through local communication. Stability and convergence of the multi-agent containment control method are studied in [6]–[8]. Multi-agent containment under fixed [9] and switching [10], [11] communication protocols have been investigated. Researchers have also studied finite-time containment control [12]–[14] as well as multi-agent containment under partial communication [15]. Continuum deformation coordination treats agents as particles of an n-D deformable body where the desired coordination is defined by a homogeneous transformation, and \( n \in \{1,2,3\} \) is the dimension of the continuum in a 3-D motion space. A desired n-D homogeneous transformation can be defined by \( n+1 \) leaders agents representing the vertices an n-D virtual simplex called leading simplex, where the desired trajectories of the leader agents are inferred through local communication. Continuum deformation coordination can formally specify and verify safety by assigning lower limits of eigenvalues of the Jacobian matrix of the homogeneous transformation. As a result, a large number of agents can aggressively deform in an obstacle-laden environment while inter-agent collision avoidance is assured.

This paper develops a framework for decentralized continuum deformation coordination through simultaneous cooperative localization and local communication. Without loss of generality, this paper assumes that each individual agent is modeled by a double integrator dynamics coordinating in a 2-D motion space. The desired continuum deformation is planned by the desired trajectories of the three leaders, located at vertices of the leading triangle, and acquired by followers through communication and localization. Assuming that the initial and final configurations of the leading triangles are known, the leaders’ desired trajectories are assigned as the solution of a coupled optimization problem. More specifically, the leaders’ optimal trajectories are determined as the solution of fixed-time constrained optimal control problem while an optimization algorithm is employed to minimize travel time between the initial and final formations subject to all safety constraints.

This paper is organized as follows: Preliminary notions of graph theory and a review of homogeneous transformation coordination are presented in Section II Problem Statement is presented in Section III and followed by continuum deformation planning and Cooperative Localization in Sections IV and V respectively. Safety of continuum deformation coordination is specified in Section VI Simulation Results are presented in Section VII followed by Conclusion in Section VIII.

II. PRELIMINARIES

A. Graph Theory Notions

1) Cooperative Coordination Graph

Inter-agent communication within the multi-agent system is defined by digraph \( G_c = (V,E_c) \) with node set \( V \) and edge set...
In the infinite-dimensional case, the in-neighbor agents at any time.

In this paper we will make the following assumptions:

Assumption 1. Leaders form a triangle at any time $t$.

Assumption 2. The in-neighbors of every follower $i$ form a triangle at any time $t$.

Assumption 3. Every follower $i \in \mathcal{V}$ is inside the communication triangle made by its in-neighbor agents.

Assumption 4. The digraph $\mathcal{G}$ is defined such that there exists at least one directed path from every leader to every follower agent.

This paper assumes that inner-agent communications have weights and the inter-agent communication topology is time-invariant. Let $w_{i,j}$ denote communication weight between agent $i$ and $j \in \mathcal{N}_i$. Then, we can define weight matrix $W = [W_{ij}] \in \mathbb{R}^{(N-3) \times N}$ as follows:

$$W_{ij} = \begin{cases} w_{i+n+1,j} & j \in \mathcal{N}_{i+n+1} \land (i+n+1) \in \mathcal{V}_F \\ -1 & j = i+n+1 \\ 0 & \text{otherwise} \end{cases}. \quad (1)$$

By partitioning $W$, we have

$$W = \begin{bmatrix} B & A \end{bmatrix}, \quad (2)$$

it has been proven that $A \in \mathbb{R}^{(N-n-1) \times (N-n-1)}$ and $B \in \mathbb{R}^{(N-n-1) \times (n+1)}$ hold the following properties [4]:

1) Matrix $A$ is a nonsingular M-matrix and Hurwitz, if there exists at least one path from every leader to every follower.
2) Diagonal elements of $A$ are all $-1$.
3) Matrix $B$ and off-diagonal elements of $A$ are non-negative.

2) Cooperative Localization Graph

We assume that leaders are equipped with GPS and thus no leader agent needs to estimate its own position. Follower agents rely on cooperative localization to estimate their own positions at any time $t$. Cooperative localization is defined by directed graph $\mathcal{G}_L(\mathcal{V}, \mathcal{E}_L)$ with node set $\mathcal{V}$ and edge set $\mathcal{E}_L \subset \mathcal{V} \times \mathcal{V}$. Note that node sets of the localization and coordination graphs are the same but edge sets $\mathcal{E}_c$ and $\mathcal{E}_l$ are different.

B. Position Notations

For every agent $i \in \mathcal{V}$, we define actual position denoted by $r_i(t)$ at time $t \geq t_0$, global desired position denoted by $r_{i,HT}(t)$ at time $t \geq t_0$, and reference position denoted by $r_{i,0}$ at time $t_0$. Note that global actual, global desired, and global reference positions of agent $i \in \mathcal{V}$ are expressed with respect to an inertial coordinate system with base vectors $\hat{e}_x$ and $\hat{e}_y$. We define $\hat{e}_x = [1 \ 0]^T$ and $\hat{e}_y = [0 \ 1]^T$, $r_i(t) = [x_i \ y_i]^T$, $r_{i,HT}(t) = [x_{i,HT} \ y_{i,HT}]^T$, and $r_{i,0} = [x_{i,0} \ y_{i,0}]^T$.\]

Assumption 5. Global desired position $r_{i,HT}$ is identical to Global reference position $r_{i,0}$ at time $t = t_0$ for every agent $i \in \mathcal{V}$, i.e. $r_{i,HT}(t_0) = r_{i,0}$ for every agent $i \in \mathcal{V}$.

C. Homogeneous Deformation Coordination

Homogeneous transformation of the multi-agent system is given by

$$\forall i \in \mathcal{V}, \ t \geq t_0, \quad r_{i,HT} = Q(t)r_{i,0} + d(t) \quad (3)$$

where $Q(t) \in \mathbb{R}^{2 \times 2}$ is non-singular at any time $t \geq t_0$ and $Q(t_0) = I_2 \in \mathbb{R}^{2 \times 2}$, $d \in \mathbb{R}^{2 \times 1}$ is the rigid-body displacement vector. Per Assumption 5, $d(t_0) = 0$.

Proposition 1. Let $Q(t)$ be expressed as

$$t \geq t_0, \quad Q(t) = R_D(t)U_D(t) \quad (4)$$

using polar decomposition, where $R_D(t)$ is an orthogonal (rotation) matrix, and $U_D(t)$ is a symmetric (pure deformation) matrix. If $Q(t_0) = I_2$ and matrix $Q(t)$ is non-singular at any time $t$, then, eigenvalues of matrix $U_D(t)$, denoted by $\lambda_1(t)$ and $\lambda_2(t)$, are all positive at any time $t$ which in turn implies that matrix $U_D(t)$ is positive definite at any time $t \geq t_0$.

Proof. Because $U_D$ is symmetric, it can be expressed as

$$U_D(t) = S(t)\Lambda(t)S^T(t)$$

at any time $t \geq t_0$, where $S(t)$ is orthogonal and $\Lambda(t) = \text{diag}(\lambda_1(t), \lambda_2(t))$ is diagonal. Now, matrix $Q(t)$ can be expressed as

$$Q(t) = R_D(t)S(t)\Lambda(t)S^T(t).$$

Because $R_D$ and $S$ are orthogonal and matrix $Q$ is non-singular at any time $t$, eigenvalues $\lambda_1(t)$ and $\lambda_2(t)$ are non-zero at any time $t$ where $\lambda_1(t_0) = \lambda_2(t_0) = 1$. Because $\lambda_1(t_0)$ and $\lambda_2(t_0)$ are positive at time $t = t_0$ and they never become zero due to nonsingularity of matrix $Q(t)$ at any time $t$, it implies that $\lambda_1(t)$ and $\lambda_2(t)$ are positive at any time $t \geq t_0$. Hence, $U_D(t)$ is positive definite at any time $t \geq t_0$. \qed
1) Homogeneous Deformation Definition
Since Assumption 1 holds at time $t_0$, leaders form a triangle at time $t_0$, and elements of matrix $Q$ and $d$ are uniquely defined by the leaders’ global desired positions as $\mathbf{Q}$

\[
\begin{bmatrix}
Q_{11}(t) \\
Q_{12}(t) \\
Q_{21}(t) \\
Q_{22}(t) \\
d_1(t) \\
d_2(t)
\end{bmatrix} =
\begin{bmatrix}
x_{1,0} & y_{1,0} & 0 & 0 & 1 & 0 \\
x_{2,0} & y_{2,0} & 0 & 0 & 1 & 0 \\
x_{3,0} & y_{3,0} & 0 & 0 & 1 & 0 \\
x_{1,0} & y_{1,0} & 0 & 0 & 1 & 1 \\
x_{2,0} & y_{2,0} & 0 & 0 & 1 & 1 \\
x_{3,0} & y_{3,0} & 0 & 0 & 1 & 1 
\end{bmatrix}^{-1}
\begin{bmatrix}
x_{1,HT}(t) \\
x_{2,HT}(t) \\
x_{3,HT}(t) \\
x_{1,HT}(t) \\
x_{2,HT}(t) \\
x_{3,HT}(t)
\end{bmatrix}.
\]

(5)

Therefore, a desired homogeneous deformation can be planned either by elements of matrix $Q$, denoted by $Q_{11}$, $Q_{12}$, $Q_{21}$, and $Q_{22}$, and vector $d$, denoted by $d_1$ and $d_2$, or by planning the leaders’ global desired position components, denoted by $x_{1,HT}$, $x_{2,HT}$, $x_{3,HT}$, $y_{1,HT}$, $y_{2,HT}$, and $y_{3,HT}$.

Because homogeneous transformation is a linear transformation, global desired position of follower $i \in \mathcal{V}_F$ can be equivalently defined by $\mathbf{Q}$ or expressed as a linear combination of the leaders’ global desired position by

\[
\text{r}_{i,HT}(t) = \sum_{j \in \mathcal{V}_L} \alpha_{i,j} \text{r}_{j,HT}(t)
\]

(6)

where $\alpha_{i,1}$, $\alpha_{i,2}$, and $\alpha_{i,3}$ are constants that are uniquely assigned by solving

\[
\begin{bmatrix}
\alpha_{i,1} \\
\alpha_{i,2} \\
\alpha_{i,3}
\end{bmatrix} =
\begin{bmatrix}
x_{1,0} & x_{2,0} & x_{3,0} & x_{1,0} \\
y_{1,0} & y_{2,0} & y_{3,0} & y_{1,0} \\
1 & 1 & 1 & 1
\end{bmatrix}
\]

(7)

for every agent $i \in \mathcal{V}_F$.

III. PROBLEM STATEMENT
This paper considers coordination of a double integrator agent team moving in the $x-y$ plane. Dynamics of agent $i \in \mathcal{V}$ is given by

\[
x_i(k + 1) = \mathbf{A}_i x_i(k) + \mathbf{B}_i [u_i(k) + \eta_i(k)]
\]

(8)

where $x_i = [\mathbf{r}_i^T \ \mathbf{v}_i^T]^T$ and $\eta_i$ refer to the state, and process noise vectors, respectively. Matrices

\[
\mathbf{A}_i =
\begin{bmatrix}
1 & 0 & \Delta T & 0 \\
0 & 1 & 0 & \Delta T \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

(9a)

\[
\mathbf{B}_i =
\begin{bmatrix}
0 & 0 & \Delta T & 0 \\
0 & 0 & 0 & \Delta T
\end{bmatrix}
\]

(9b)

where $\Delta T$ is the sample time step. For agent $i \in \mathcal{V}$, we define control input $u_i$ as follows:

\[
u_i = \begin{cases}
g_1(\hat{\mathbf{r}}_i, HT - \hat{\mathbf{r}}_i) + g_2(\mathbf{r}_i, HT - \mathbf{r}_i) & i \in \mathcal{V}_L \\
g_1 \sum_{j \in \mathcal{N}_i} w_{i,j} (\hat{\mathbf{r}}_j - \hat{\mathbf{r}}_i) + g_2 \sum_{j \in \mathcal{N}_i} w_{i,j} (\hat{\mathbf{r}}_j - \hat{\mathbf{r}}_i) & i \in \mathcal{V}_F
\end{cases}
\]

(10)

where $w_{i,j} > 0$ is a constant communication weight between agent $i \in \mathcal{V}_F$ and in-neighbor $j \in \mathcal{N}_i$, $\hat{\mathbf{r}}_i$ and $\mathbf{r}_j$ denote actual positions of agents $i$ and $j$, $\hat{\mathbf{r}}_i$ and $\hat{\mathbf{r}}_j$ denote the estimations of actual positions of agents $i$ and $j$, and $\hat{\mathbf{r}}_{i,HT}$ is the global desired position of leader agent $i \in \mathcal{V}_L$. This paper assumes that followers’ communication weights are consistent with agents’ reference positions and obtained by

\[
\begin{bmatrix}
w_{i,1,i} \\
w_{i,2,i} \\
w_{i,3,i}
\end{bmatrix} =
\begin{bmatrix}
x_{1,i,0} & x_{2,i,0} & x_{3,i,0} & x_{1,i,0} \\
x_{2,i,0} & x_{2,i,0} & x_{3,i,0} & x_{1,i,0} \\
x_{3,i,0} & x_{2,i,0} & x_{3,i,0} & x_{1,i,0}
\end{bmatrix}
\]

(11)

where $g_1, g_2 > 0$ are constant; $i_1$, $i_2$, and $i_3$ are the index numbers of the in-neighbors of follower $i \in \mathcal{V}_F$, i.e. $\mathcal{N}_i = \{i_1, i_2, i_3\}$ defines the index numbers of the in-neighbors of agent $i \in \mathcal{V}_F$.

The above continuum deformation coordination problem is defined as a decentralized leader-follower coordination problem. In Section IV, we assume that initial and final configurations of leaders are given, and assign the desired trajectories as a solution of a constrained optimal control problem. We offer a cooperative localization method in Section VII to acquire a desired continuum deformation coordination in a decentralized fashion. Furthermore, we provide safety conditions in Section VII to check and ensure that collision is avoided in decentralized continuum coordination, inferred by cooperative localization.

IV. CONTINUUM DEFORMATION COORDINATION PLANNING

In this section, we discuss how the desired positions of the leaders are planned. Specifically, we consider leaders’ desired positions planned according to the minimum control effort with fixed initial and final positions and velocities and the fixed area of the triangle made up by three leaders. Indeed, let $\mathbf{r}_{i,HT}$ be updated by the double integrator dynamics

\[
\mathbf{r}_i(t) = \mathbf{r}_i(0) + \mathbf{v}_i(0)t + \frac{1}{2} \mathbf{a}_i(t)
\]

where $\mathbf{a}_i(t)$ is the leaders’ control input.

\[
\dot{\mathbf{r}}_i(t) = \mathbf{v}_i(t)
\]

\[
\dot{\mathbf{v}}_i(t) = \mathbf{a}_i(t)
\]

\[
\mathbf{r}_i(0) = \mathbf{r}_i(0)
\]

\[
\mathbf{v}_i(0) = \mathbf{v}_i(0)
\]

\[
\mathbf{a}_i(t) = \begin{cases}
-g_1(\hat{\mathbf{r}}_i, HT - \hat{\mathbf{r}}_i) - g_2(\mathbf{r}_i, HT - \mathbf{r}_i) & i \in \mathcal{V}_L \\
g_1 \sum_{j \in \mathcal{N}_i} w_{i,j} (\hat{\mathbf{r}}_j - \hat{\mathbf{r}}_i) + g_2 \sum_{j \in \mathcal{N}_i} w_{i,j} (\hat{\mathbf{r}}_j - \hat{\mathbf{r}}_i) & i \in \mathcal{V}_F
\end{cases}
\]

(12)

\[
\int_{t_0}^{t_f} \left( \sum_{i \in \mathcal{V}_L} (\mathbf{v}_{x,i}^2 + \mathbf{v}_{y,i}^2) \right) dt
\]

is minimized subject to boundary conditions

\[
q \in \{x, y\}, \ i \in \mathcal{V}_L, \ \dot{q}_i, HT (t_0) = q_i,0
\]

(13a)

\[
q \in \{x, y\}, \ i \in \mathcal{V}_L, \ \dot{q}_i, HT (t_f) = q_i, f
\]

(13b)

\[
q \in \{x, y\}, \ i \in \mathcal{V}_L, \ \dot{q}_i, HT (t_0) = q_i, 0
\]

(13c)

\[
q \in \{x, y\}, \ i \in \mathcal{V}_L, \ \dot{q}_i, HT (t_f) = q_i, 0
\]

(13d)

and equality constraint

\[
2a_0 - \begin{bmatrix}
x_{1,HT}(t) & x_{2,HT}(t) & x_{3,HT}(t) \\
y_{1,HT}(t) & y_{2,HT}(t) & y_{3,HT}(t)
\end{bmatrix} = 0
\]

(14)

where $a_0$ is the area of the leading triangle with vertices occupied by leaders 1, 2, and 3, respectively. Note that the area $a_0$ and travel time $T = t_f - t_0$ are both fixed. The solution of the above optimal control problem was presented in Ref. [16].
V. COOPERATIVE LOCALIZATION

In order to control a multi-agent system, at least a position feedback from all agents is needed. However, under practical circumstances such feedback data might not be available. Alternatively, what might be available is the limited information about relative pose of the agents with respect to each other. In this section, we employ cooperative localization for state estimation in order to ensure availability of accurate feedback in the multi-agent system. Specifically, we present an algorithm to reconstruct the full state of each agent based on the limited information about each agent’s relative pose.

A. Dynamics of Individual Agents

Utilizing Extended Kalman Filter (EKF), we denote the predicted and updated state estimates for agent $i$ as $\hat{x}_{i-}$ and $\hat{x}_{i+}$, respectively. Referring to $\hat{x}_{i-}$ and $\hat{x}_{i+}$, agents $i = 1, \ldots, n$ can propagate their states from sample time $(k)$ to $k+1$ given initial conditions $x_i(0)$ and starting with covariance $P_i(0)$. In this section, we employ cooperative localization for each agent $i$ with the measurement vector $y_i$. Next, assume agent $i$ has an on-board sensor such as Lidar, then the measurements consist of relative range $c_i$ and azimuth angle $\theta_i$:

$$y_i(k+1) = c_{ij}(x_i(k), x_j(k)) + v_i(k)$$

where $v_i(k)$ represents measurement noise. Assuming agent $i$ has an on-board sensor such as Lidar, then the measurement vectors consist of relative range $d_{ij}$ and azimuth angle $\theta_{ij}$:

$$\begin{bmatrix} d_{ij} \\ \theta_{ij} \end{bmatrix} = \begin{bmatrix} \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2} \\ \arctan \frac{y_j - y_i}{x_j - x_i} \end{bmatrix}$$

At the update step, the state and covariance matrices remain the same if there are no relative measurements. Let us now assume that a follower agent $f$ takes relative measurements of a leader agent $l$. Then, the innovation residual error between the measurement $y_{f,l}$ and the estimated output through propagated states is

$$e_{f,l}(k+1) = y_{f,l}(k+1) - c_{f,l}(\hat{x}_{f-}(k+1), \hat{x}_{l-}(k+1))
\approx y_{f,l}(k+1) - \begin{bmatrix} C_{f,l}(k+1) \\ -C_{f,l}(k+1) \end{bmatrix} \begin{bmatrix} \hat{x}_{f-}(k+1) \\ \hat{x}_{l-}(k+1) \end{bmatrix}$$

where

$$C_{f,l} = \begin{bmatrix} \frac{\dot{x}_l - \dot{x}_f}{d_{f,l}} & -\frac{\dot{y}_l - \dot{y}_f}{d_{f,l}} & 0 & 0 \\ \frac{\dot{y}_l - \dot{y}_f}{\dot{x}_l - \dot{x}_f} & -\frac{1}{\dot{x}_l - \dot{x}_f} & 0 & 0 \end{bmatrix}.$$
B. MAS Collective Dynamics

Define

\[ X_{\text{SYS}} = \text{vec} \left( \begin{bmatrix} r_4 & \cdots & r_N & \dot{r}_4 & \cdots & \dot{r}_N \end{bmatrix}^T \right) \]  \hspace{1cm} (25a)  
\[ \dot{X}_{\text{SYS}} = \text{vec} \left( \begin{bmatrix} \ddot{r}_4 & \cdots & \ddot{r}_N & \ddot{r}_4 & \cdots & \ddot{r}_N \end{bmatrix}^T \right) \]  \hspace{1cm} (25b)

as the state vectors of the MAS control system and the estimator, respectively, and matrix

\[ K = \begin{bmatrix} 0 & 0 & g_2A & 0 \end{bmatrix} \]  \hspace{1cm} (26)

where \( A \) was previously defined in (2). Note that vec(·) is the matrix vectorization operator. Given agent dynamics (8), the matrix vectorization operator, respectively, and matrix

\[
X_{\text{SYS}}(k+1) = AX_{\text{SYS}}(k) + BU_{\text{SYS}}(k) + \eta_{\text{SYS}}(k) + V_{\text{SYS}}(k)
\]

\[
Y_{\text{SYS}}(k) = C_{\text{SYS}}X_{\text{SYS}}(k) + \nu_{\text{SYS}}(k)
\]

\[
\dot{X}_{\text{SYS}}(k+1) = AX_{\text{SYS}}(k) + BU_{\text{SYS}}(k) + \eta_{\text{SYS}}(k) + V_{\text{SYS}}(k)
\]

\[
\dot{Y}_{\text{SYS}}(k) = C_{\text{SYS}}X_{\text{SYS}}(k) + \nu_{\text{SYS}}(k)
\]

where \( C_{\text{SYS}} \in \mathbb{R}^{(N-3) \times 4(N-3)} \) is the observation matrix, \( \eta_{\text{SYS}} \in \mathbb{R}^{4(N-3) \times 1} \) and \( \nu_{\text{SYS}} \in \mathbb{R}^{4(N-3) \times 1} \) are Gaussian process and measurement noise vectors, respectively, and

\[
A_{\text{SYS}} = \begin{bmatrix} I_{(N-3)} & 0 \end{bmatrix} \in \mathbb{R}^{4(N-3) \times 4(N-3)}
\]

\[
B_{\text{SYS}} = \begin{bmatrix} I_{(N-3)} & 0 \end{bmatrix} \in \mathbb{R}^{4(N-3) \times 6}
\]

\[
U_{\text{SYS}} = \text{vec} \left( \begin{bmatrix} r_1 & r_2 & r_3 & \dot{r}_1 & \dot{r}_2 & \dot{r}_3 \end{bmatrix}^T \right) \in \mathbb{R}^{6 \times 1}
\]

Note that control \( g_1 \) and \( g_2 \) are chosen such that the eigenvalues of matrix \( A_{\text{SYS}} \) are strictly located on the left side of the complex plane. The block diagram of the controllable form of MAS collective dynamics is shown in Fig. 1.

\[ A_{\text{CL}} = O^T A_{\text{SYS}} O \]  \hspace{1cm} (29b)

\[ B_{\text{CL}} = O^T B_{\text{SYS}} \]  \hspace{1cm} (29c)

for relating \( X_{\text{SYS}}, A_{\text{SYS}}, \) and \( B_{\text{SYS}} \) to \( X_{\text{CL}}, A_{\text{CL}}, \) and \( B_{\text{CL}} \) respectively, where \( O = [O_{ih}] \in \mathbb{R}^{4(N-3) \times 4(N-3)} \) is orthonormal and defined as follows:

\[
O_{ih} = \begin{cases} 
1 & \text{if } l = 4(i-1)+1, \ h = i, \ i = 4, \cdots, N \\
1 & \text{if } l = 4(i-1)+2, \ h = i+N-3, \ i = 4, \cdots, N \\
1 & \text{if } l = 4(i-1)+3, \ h = i+2(N-3), \ i = 4, \cdots, N \\
0 & \text{otherwise}
\end{cases}
\]

\[
(30)
\]

Remark 1. To implement the proposed cooperative localization, we define the similarity transformations

\[ X_{\text{CL}} = OX_{\text{SYS}} \]  \hspace{1cm} (29a)

A flowchart for assignment travel time \( T \) ensuring safety of the MAS continuum deformation coordination is shown in Fig. 2.

Fig. 2: The flowchart for assignment travel time \( T \) ensuring safety of the MAS continuum deformation coordination.

VI. SAFETY SPECIFICATION AND VERIFICATION

The following conditions provide safety requirements such as collision avoidance, boundedness, and follower containment in a continuum deformation coordination acquired by cooperative localization.

Collision Avoidance Condition: Let \( \epsilon > 0 \) be the radius of the radius of the smallest ball enclosing every individual agent. Then, inter-agent collision is avoided, if

\[
\forall k, \quad \int_{i \neq j \in V} ||r_i(k) - r_j(k)|| > 2\epsilon.
\]

Bounding Condition: Deviation of every agent \( i \) from the global desired trajectory \( r_{i,HT}(k) \) is bounded, if

\[
\forall k, \quad \int_{i \neq j \in V} ||r_i(k) - r_{i,HT}(k)|| \leq \delta.
\]

Follower Containment Condition: Let \( r_m(k) = [x_m(k) \ y_m(k)]^T \), \( r_j(k) = [x_j(k) \ y_j(k)]^T \), \( r_h(k) = [x_h(k) \ y_h(k)]^T \), \( r_t(k) = [x_t(k) \ y_t(k)]^T \) be the position of arbitrary agents \( m, j, h, \) and \( t \), respectively, where \( m, j, h \) form a triangle at discrete time \( k \), i.e. agents \( m, j, h \) are not aligned at discrete time \( k \). We define function

\[
\Omega(r_m, r_j, r_h, r_t) = \begin{bmatrix} x_m & x_j & x_h & x_t \end{bmatrix} \begin{bmatrix} y_m & y_j & y_h & y_t \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}
\]

at discrete every time \( k \). Agent \( t \) is inside the triangle defined by vertices \( m, j, \) and \( h \), if \( \Omega(r_m, r_j, r_h, r_t) \geq 0 \). We can ensured
that all followers remain inside the leading triangle, defined by leaders 1, 2, and 3, at every discrete time $k$, if
\begin{equation}
\forall k, \quad \bigwedge_{i \in \mathcal{V}} \Omega((r_i(k), r_i(k), r_i(k))) \geq 0,
\end{equation}
where “$\bigwedge$” is the “wedge symbol. For given initial and final configurations of the leaders, we choose a sufficiently-large travel time $T = t_f - t_0 \geq T^*$, where $T^* = t_f^* - t_0$ is assigned as the solution of the following constrained programming problem:

\begin{equation}
T^* = \min T
\end{equation}
subject to safety constraints (31), (32), (34), MAS collective dynamics, estimation dynamics, and the following inequality constraint:
\begin{equation}
i \in \mathcal{V}, q \in \{x, y\}, \quad q_{i,HT}^* = v_{q,i}^*.
\end{equation}
where $q_{i,HT}^*$ is component $q \in \{x, y\}$ of desired trajectory of leader $i \in \mathcal{V}_L$. Note that $q_{i,HT}^*(t)$ can be determined, if initial and final configurations of the leaders are known, and $t_0$ and $t_f$. On the other hand, we need to know $q_{i,HT}^*$ of every leader agent $i \in \mathcal{V}_L$ to solve the minimum time optimization problem presented in this section. Therefore, leaders’ desired trajectories and travel times must be solved interactively. The flowchart shown in Fig. 2 illustrates how $T^*$ and $q_{i,HT}^*$ can be interactively determined for every leader $i \in \mathcal{V}_L$ such that all presented constraints are satisfied.

It must be noted that in EKF-based cooperative localization, random noise is introduced in both process and measurements. This leads to some uncertainty exist in the optimization process and results. Therefore, Monte Carlo simulations may be required to verify that all safety requirements are satisfied regardless of uncertainties in state estimation.

Since mobile robots are represented as double integrators, we must include process noise in the model to account for unmodeled dynamics. Hence we assume the standard deviation of the process noise to be 0.5 m/s². We also assume that the measurements obtained by all agents are polluted by additive Gaussian noise with a standard deviation of 0.03 m. In addition, all measurements are updated at a rate of 0.1 s.

The minimum time $T^*$ determined from (35) which satisfies conditions (31), (32), and (34) for this scenario is determined to be only 10.1 s assuming continuous noise-free full-state feedback. However, the minimum time is increased to 20.5 s when states are estimated using cooperative localization, as determined by optimization and verified via Monte Carlo simulations. In these simulations, the gains for the control law in (10) were selected as $g_1 = 6$, $g_2 = 9$. The selected safety threshold values were $\epsilon = 0.5$ and $\delta = 0.5$.

Figure 5 shows the paths of the 10 mobile robots in the simulation. Figure 6 shows the global position estimation errors in $X$ and $Y$ directions for a typical simulation with cooperative localization. Figure 7 shows the coordination tracking errors in $X$ and $Y$ directions. It can be concluded that the both estimation and tracking errors never exceed 0.8% of the distance travelled by each agent.
where leaders’ desired trajectories and travel time between the initial and final configurations are obtained by solving coupled optimization problem. More specifically, leaders’ desired trajectories are determined by solving a constrained optimal control problem while the final time is minimized such that all safety requirements are satisfied. We also showed how follower agents can acquire the desired continuum deformation coordination through simultaneous communication and localization. As a result, a relatively-large number of agents of a multi-agent system can acquire a desired continuum deformation with low computation cost while all safety requirements are met and the MAS is capable of aggressive deformation in the geometrically-constrained environments.

ACKNOWLEDGMENT

This research was supported in part by the National Science Foundation under Award No. 1914581 and the Office of Naval Research under Award No. N00014-19-1-2255.

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