Two-Time Quantum Mechanics

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A two-time quantum theory of a system of two particles with the direct electromagnetic interaction based on a quantum version of the action principle is considered. An analog of Schrödinger equation for the system is obtained.

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I. INTRODUCTION

In the preprint\textsuperscript{1} a generalized canonical form of a multi-time dynamical theory was proposed. This form may be taken as a basis for a new approach in quantum theory. This approach is based on so called quantum action principle (QAP) proposed in\textsuperscript{2,3}. In paper\textsuperscript{1}, a two-time dynamical system of two charges with a direct electromagnetic interaction was considered as a simple example. A quantum action operator for this system in the first-order approximation of perturbation theory in a parameter of the direct electromagnetic interaction was defined. In a maximally simplified non-relativistic limit this action, a two-time integral, which corresponds to the Coulomb interaction described, with relativistic accuracy, by the Feynman retarded-advanced propagator\textsuperscript{4}, is conserved. It is this term that makes impossible the ordinary canonical quantization procedure.

In the present work we develop a two-time quantum theory of the relativistic Coulomb interaction of charges as a toy model for a more complicated multi-time quantum electrodynamics. A two-time wave equation, which is similar to Schrödinger equation, is obtained.

II. QUANTUM ACTION PRINCIPLE FOR TWO-TIME DYNAMICAL SYSTEM

The action operator of a two-time dynamical theory has the form\textsuperscript{1}:

\[
\mathcal{I} = \int_0^{T_1} dt_1 \dot{x}_{1k}(t_1) \frac{\hbar}{i} \frac{\delta}{\delta x_{1k}(t_1)} + \int_0^{T_2} dt_2 \dot{x}_{2k}(t_2) \frac{\hbar}{i} \frac{\delta}{\delta x_{2k}(t_2)} - \hat{H}.
\]

Here \(x_{1k}(t_1)\) and \(x_{2k}(t_2)\) are trajectories of particles with own parameters of time, the dot denotes the time derivative with respect to the time parameter corresponding to each particle. The constant \(\hbar\) is not equal to the ordinary Plank constant \(\hbar\). The physical dimensionality of \(\hbar\) is \([\hbar] = \text{Joule} \cdot s^2\). A connection between two constants will be introduced latter. A Hamiltonian part of the action operator (1) in the non-relativistic limit is:

\[
\hat{H} = - \int_0^{T_1} \frac{\hbar^2}{2m_1} \frac{\delta^2}{\delta x_{1k}^2(t_1)} - \int_0^{T_2} \frac{\hbar^2}{2m_2} \frac{\delta^2}{\delta x_{2k}^2(t_2)} + \frac{1}{2} e_1 e_2 \int_0^{T_1} \int_0^{T_2} dt_1 dt_2 \delta (s_{12}^2),
\]

where (velocity of light \(c = 1\))

\[
s_{12}^2 \equiv (t_1 - t_2)^2 - (x_1 - x_2)^2
\]

is the interval in the Minkowsky space. The action operator (1) is defined in a space of wave functionals \(\Psi[x_1(t_1), x_2(t_2)]\) where in turn the variational derivatives are defined as follows:

\[
\delta \Psi = \int_0^{T_1} dt_1 \frac{\delta \Psi}{\delta x_{1k}(t_1)} \delta x_{1k}(t_1) + \int_0^{T_2} dt_2 \frac{\delta \Psi}{\delta x_{2k}(t_2)} \delta x_{2k}(t_2)
\]

The Hamiltonian operator (2) is formally Hermitian with respect to the scalar product in the space of wave functionals:

\[
(\Psi_1, \Psi_2) \equiv \int dt_1 d^3x_1(t_1) d^3x_2(t_2) \Psi_1^* [x_1(t_1), x_2(t_2)] \Psi_2 [x_1(t_1), x_2(t_2)].
\]

The first two canonical terms of the action operator (1) are non-Hermitian, but we shall overcome this problem by throwing away of corresponding imaginary parts of eigenvalues of the action operator.
We formulate QAP as the eigenvalue problem for the action operator [4]:

$$\tilde{I}\Psi = \Lambda \Psi.$$  \hspace{1cm} (6)

A solution $\Psi[x_1(t_1), x_2(t_2)]$ of the equation [6] describes a quantum state of motion of the system. $[\Psi[x_1(t_1), x_2(t_2)]]^2$ is a probability density of particles movement along trajectories from a small neighbourhood of given trajectories $x_{1k}(t_1), x_{2k}(t_2)$. It is the equation [6] that is equivalent to Schrödinger equation in the case of a single-time dynamical theory [3]. In the case of two-time dynamical theory considered here it will be reduced to a two-time analog of the Schrödinger equation.

It is useful to re-formulate the eigenvalue problem, introducing for any wave functional $\Psi$ an auxiliary functional:

$$\Lambda[x_1(t_1), x_2(t_2)] \equiv \tilde{I}\Psi[x_1(t_1), x_2(t_2)]/\Psi[x_1(t_1), x_2(t_2)].$$  \hspace{1cm} (7)

Then QAP may be formulated in terms of a necessary condition that the functional (7) is equal to an eigenvalue of the action operator. The following exponential representation

$$\Psi[x_1(t_1), x_2(t_2)] \equiv \exp\left(\frac{\tilde{I}}{\hbar} S[x_1(t_1), x_2(t_2)] + R[x_1(t_1), x_2(t_2)]\right)$$  \hspace{1cm} (8)

with real functionals $S[x_1(t_1), x_2(t_2)], R[x_1(t_1), x_2(t_2)]$ is useful for a wave functional, in particular, for quasiclassical decomposition of a solution. At this stage, analytic properties of the Coulomb potential are important. In the work [3], QAP was formulated for the case of a real-analytic potential. In order to use this concrete formulation, let us regularize the Coulomb potential, replacing the $\delta$-function by the exponent as follows:

$$\delta(s_{12}^2) \to \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(s_{12}^2)^2}{2\sigma^2}\right).$$  \hspace{1cm} (9)

At the final stage of calculations, the limit $\sigma \to 0$ is supposed. With this regularization the functionals $S[x_1(t_1), x_2(t_2)], R[x_1(t_1), x_2(t_2)]$ may be looked for as real-analytic functionals of trajectories of particles:

$$S[x_1, x_2] = \int_0^{T_1} dt_1 \int_0^{T_2} dt_2 s(t_1, t_2, x_1(t_1), x_2(t_2)), \hspace{1cm} (10)$$

$$s \equiv s_{k_1}^{(10)}(t_1, t_2) x_{1k}(t_1) + s_{k_2}^{(01)}(t_1, t_2) x_{2k}(t_2) + \sum_{l=1}^{N_2} s_{kl}^{(02)}(t_1, t_2) x_{2k}(t_2) x_{2l}(t_2) + ...,$$  \hspace{1cm} (11)

Substituting (10) and (11) into (7) and taking into account (11) and (13), we obtain the auxiliary functional $\Lambda[x_1(t_1), x_2(t_2)]$ as a functional series of particles coordinates $x_{1k}(t_1), x_{2k}(t_2)$ degrees. Then the condition that the functional is equal to an eigenvalue of the action operator leads to an infinite set of differential equations for coefficients of the series (11) and (13). We do not present here these equations (in the case of a single-time theory, see [4]). In the next section we shall obtain an equivalent formulation of this necessary condition in terms of a two-time wave equation.

III. TWO-TIME WAVE EQUATION

In order to re-formulate QAP in terms of a wave equation, let us introduce a multiplicative representation of a wave functional $\Psi[x_1(t_1), x_2(t_2)]$ in terms of a two-time wave function $\psi(t_1, t_2, x_1(t_2), x_2(t_2))$ as follows. Let us divide both time intervals $[0, T_{12}]$ on $N_{12}$ small intervals of the same length $\varepsilon = T_{12}/N_{12}$ by points $t_{n_1} = n_1\varepsilon, t_{n_2} = n_2\varepsilon$, and approximate a trajectory of the system of two particles $(x_{1k}(t_1), x_{2k}(t_2))$ in a configuration space by a broken line with vertices $(x_{1k}(n_1), x_{2k}(n_2)), n_1 = 1, 2, ..., N_1, n_2 = 1, 2, ..., N_2$, and the end points $(x_{1k}(0), x_{2k}(0)) \equiv (x_{1k}^0, x_{2k}^0)$, $(x_{1k}(N_1), x_{2k}(N_2)) \equiv (x_{1k}^{N_1}, x_{2k}^{N_2})$. Then the functionals (10) and (11) may be approximated by corresponding integral sums:

$$S = \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \varepsilon^2 s(t_{n_1}, t_{n_2}, x_1(n_1), x_2(n_2)), \hspace{1cm} (14)$$

$$R = \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \varepsilon^2 r(t_{n_1}, t_{n_2}, x_1(n_1), x_2(n_2)). \hspace{1cm} (15)$$

The central point of passage from a wave functional to a wave function is the following equality [2, 4]:

$$\tilde{h} = \varepsilon\hbar.$$  \hspace{1cm} (16)

Taking into account (14) and (15), the exponential representation of a wave functional (8) can be transformed to a product of values of the wave function at discrete
moments of time in the power \( \varepsilon \):

\[
\Psi [x] = \prod_{n_1=1}^{N_1} \prod_{n_2=1}^{N_2} \psi^\varepsilon (t_{n_1}, t_{n_2}, x_1 (n_1), x_2 (n_2)),
\]

(17)

\[
\psi^\varepsilon (t_{n_1}, t_{n_2}, x_1 (n_1), x_2 (n_2)) \equiv \exp \varepsilon \chi (t_{n_1}, t_{n_2}, x_1 (n_1), x_2 (n_2)),
\]

(18)

\[
\chi (t_{n_1}, t_{n_2}, x_1 (n_1), x_2 (n_2)) \equiv \frac{i}{\hbar} \varepsilon (t_{n_1}, t_{n_2}, x_1 (n_1), x_2 (n_2)) + \varepsilon r (t_{n_1}, t_{n_2}, x_1 (n_1), x_2 (n_2)).
\]

(19)

Here and further the product \( \varepsilon r \) will be considered as a single symbol. In this approximation the wave functional \( \Psi \) is a function of coordinates of vertices \( (x_{1k} (n_1), x_{2k} (n_2)) \) of a broken line. According to the definition (3), the variational derivatives of this wave functional must be replaced by the partial derivatives as follows (2):

\[
\frac{\delta \Psi}{\delta x_{1k} (n_1)} \equiv \frac{1}{\varepsilon} \frac{\partial \Psi}{\partial x_{1k} (n_1)}
\]

(20)

\[
\frac{\delta \Psi}{\delta x_{2k} (n_2)} \equiv \frac{1}{\varepsilon} \frac{\partial \Psi}{\partial x_{2k} (n_2)}
\]

(21)

Then the action operator can be approximated by a differential operator. The first term in the canonical part of the auxiliary functional \( \Lambda [x_1 (t_1), x_2 (t_2)] \) is approximated as follows:

\[
\int_0^{T_1} dt_1 \frac{\hbar}{i} \frac{\delta \ln \Psi}{\delta x_{1k} (t_1)} \left[ \chi (t_{n_1}, t_{n_2}, x_1 (n_1), x_2 (n_2)) - \chi (t_{n_1-1}, t_{n_2}, x_1 (n_1 - 1), x_2 (n_2)) \right] - \frac{\hbar}{i} \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \left[ \chi (t_{n_1}, t_{n_2}, x_1 (n_1), x_2 (n_2)) - \chi (t_{n_1-1}, t_{n_2}, x_1 (n_1 - 1), x_2 (n_2)) \right] - \frac{\hbar}{i} \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \left[ \chi (t_{n_1}, t_{n_2}, x_1 (n_1), x_2 (n_2)) - \chi (0, t_{n_2}, x_1 (0), x_2 (n_2)) \right]
\]

(22)

The second term in (12) is approximated in the similar way. In the limit \( \varepsilon \to 0 \) the full canonical part of the auxiliary functional \( \Lambda [x_1 (t_1), x_2 (t_2)] \) can be written in an integral form:

\[
\frac{\hbar}{i} \int_0^{T_1} dt_1 \int_0^{T_2} dt_2 \left[ \chi (t_1, t_2, x_1 (t_1), x_2 (t_2)) \right] - \chi (0, t_2, x_1 (0), x_2 (t_2)) + \frac{1}{T_1 T_2} \left( \chi (t_1, T_2, x_1 (t_1), x_2 (T_2)) - \chi (t_1, 0, x_1 (t_1), x_2 (T_2)) \right)
\]

(23)

Let us consider the Hamiltonian part of the auxiliary functional \( \Lambda [x_1 (t_1), x_2 (t_2)] \). The first term, which corresponds to the first free particle, is approximated as follows:

\[
\frac{\hbar}{2m_1} \int_0^{T_1} dt_1 \frac{\delta^2 \Psi}{\delta x_{1k}^2 (t_1)}\Psi
\]

(24)

\[
\approx \frac{\hbar}{2m_1} \sum_{n_1=1}^{N_1} \varepsilon \left[ \left( \sum_{n_2=1}^{N_2} \frac{\varepsilon}{\partial x_{1k} (n_1)} \right)^2 + \sum_{n_2=1}^{N_2} \left( \sum_{n_1=1}^{N_1} \frac{\varepsilon}{\partial x_{1k} (n_1)} \right)^2 \right]
\]

(25)

The second term, which corresponds to the second free particle, is approximated in the similar way. In the limit \( \varepsilon \to 0 \) the Hamiltonian part of the auxiliary functional \( \Lambda [x_1 (t_1), x_2 (t_2)] \) equals to (notice that the third term, which corresponds to Coulomb interaction, remains unchanged in this limit):

\[
- \frac{\hbar}{2m_2} \int_0^{T_2} dt_2 \left[ \frac{\partial^2 \chi (t_1, t_2, x_1 (t_1), x_2 (t_2))}{\partial x_{2k}^2} \right]
\]

(26)
Collecting together all parts of the auxiliary functional \( \Lambda [x_1(t_1), x_2(t_2)] \), one can write it in the form:

\[
\Lambda [x_1(t_1), x_2(t_2)] = \lambda + \int_0^{T_1} \int_0^{T_2} dt_1 \int_0^{T_2} dt_2 \tilde{W} \chi, \tag{26}
\]

where

\[
\lambda = \int_0^{T_1} dt_1 \int_0^{T_2} dt_2 f(t_1, t_2) \tag{27}
\]

\[
f(t_1, t_2) = \left[ \frac{1}{T_1} \left( s(T_1, t_2, 0, 0) - s(0, t_2, 0, 0) \right) + \frac{1}{T_2} \right] 
\]

\[
+ \frac{1}{2m_1} \left( s_k^{(10)}(t_1, t_2) \int_0^{T_2} d\bar{t}_2 s_k^{(10)}(\bar{t}_1, \bar{t}_2) \right) \tag{28}
\]

\[-\hbar^2 \varepsilon r_k^{(10)}(t_1, t_2) \int_0^{T_2} d\bar{t}_2 \varepsilon r_k^{(10)}(\bar{t}_1, \bar{t}_2) \]

\[-\hbar^2 \varepsilon r_k^{(20)}(t_1, t_2) \right] + e_1 e_2 \delta(t_1 - t_2)^2 \]

is an eigenvalue of the action operator, and

\[
\tilde{W} \chi = \frac{\hbar}{i} \left[ \frac{1}{T_1} \left( \chi(T_1, t_2, x_1^T, x_2 (t_2)) - \chi(0, t_2, x_1^0, x_2(2)) \right) + \frac{1}{T_2} \chi(t_1, T_2, x_1(t_1), x_2^T(t_2)) \right] \tag{29}
\]

\[
- \chi(t_1, 0, x_1(t_1), x_2^T(t_2)) - \frac{\partial \chi(t_1, t_2, x_1(t_1), x_2(t_2))}{\partial t_1} \frac{\partial \chi(t_1, t_2, x_1(t_1), x_2(t_2))}{\partial t_2} - \frac{\hbar^2}{2m_1} \left( \frac{\partial \chi(t_1, t_2, x_1(t_1), x_2(t_2))}{\partial x_{1k}} \right) \tag{30}
\]

\[
\times \int_0^{T_2} d\bar{t}_2 \frac{\partial \chi(t_1, \bar{t}_2, x_1(t_1), x_2(\bar{t}_2))}{\partial x_{1k}} - \frac{\partial^2 \chi(t_1, t_2, x_1(t_1), x_2(t_2))}{\partial x_{1k}^2} \right]
\]

is a part of the auxiliary functional \( \Lambda [x_1(t_1), x_2(t_2)] \) which depends only on non-zero degrees of coordinates of particles. It is this part of the auxiliary functional \( \Lambda [x_1(t_1), x_2(t_2)] \) that gives a two-time wave equation, which corresponds QAP:

\[
\tilde{W} \chi = 0. \tag{31}
\]

This equation must to be solved for an arbitrary trajectory \((x_{1k}(t_1), x_{2k}(t_2))\) of the system with fixed end points.

Let us consider the limiting case, when the interaction of charges is absent. In this case the wave equation \((31)\) has a simple additive solution for the exponent \(\chi\) of a wave function:

\[
\chi(t_1, t_2, x_1, x_2) = \chi_1(t_1, x_1) + \chi_2(t_2, x_2) \tag{32}
\]

\[
\chi_1(t_1, x_1) = -\frac{i}{\hbar} \left( W_{11} - p_{1k} x_{1k} \right), \tag{33}
\]

\[
\chi_2(t_2, x_2) = -\frac{i}{\hbar} \left( W_{22} - p_{2k} x_{2k} \right), \tag{34}
\]

where the energy \(W\) and the momentum \(p_k\) of each particle obey the ordinary relation:

\[
W = \frac{p^2}{2m}. \tag{35}
\]

In this case, the exponent \((31)\) defines a wave function of the system as a product:

\[
\psi(t_1, t_2, x_1, x_2) = \psi_1(t_1, x_1) \psi_2(t_2, x_2). \tag{36}
\]

The energy of the two-time quantum dynamical system with the direct interaction can be defined in the limit \(T_1 = T_2 = T \to \infty\) as follows:

\[
W = \frac{p^2}{2m_1} + \frac{p^2}{2m_2}, \tag{37}
\]

at the condition that the limit does not depend on the end points \(x_{1}^T, x_{2}^T\). In the limiting case of free particles we have as usually:

\[
W = \frac{p^2}{2m_1} + \frac{p^2}{2m_2}. \tag{38}
\]
IV. CONCLUSIONS

In conclusion, the new formulation of quantum mechanics based on a quantum version of the action principle is equivalent to ordinary Schrödinger formulation of quantum mechanics in the case of a single-time dynamical theory [4]. We have showed that the proposed new approach leads to a two-time analog of the Schrödinger equation in the case of the two-time dynamical theory of a two-particle system with the direct electromagnetic interaction.

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