Coexistence of Grass, Saplings and Trees
in the Staver-Levin Forest Model

Richard Durrett and Yuan Zhang *
Department of Mathematics, Box 90320
Duke U., Durham, NC 27708-0320
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Abstract

In this paper we prove the existence of a nontrivial stationary distribution for a forest model with Grass, Saplings and Trees, by comparing with the two type contact process model of Krone and considering the long range limit. Our proof shows that if a particle systems has states \{0, 1, 2\} and is attractive, then coexistence occurs in the long-range model when the absorbing state (0,0) is an unstable fixed point of the mean-field ODE for \(u_1, u_2\). The result we obtain in this way is asymptotically sharp for Krone’s model, but the Staver-Levin forest model, like the quadratic contact process, may have a nontrivial stationary distribution when (0,0) is attracting.

1 Introduction

In a recent paper published in Science [13], Carla Staver, Sally Archibald and Simon Levin argued that tree cover does not increase continuously with rainfall but rather is constrained to low (< 50%, “savannah”) or high (> 75%, “forest”) levels. In follow-up works published in Ecology [14] and the American Naturalist [15], they studied the following ODE for the evolution of the fraction of land covered by grass \(G\), saplings \(S\), and trees \(T\):

\[
\begin{align*}
\frac{dG}{dt} &= \mu S + \nu T - \beta GT \\
\frac{dS}{dt} &= \beta GT - \omega(G)S - \mu S \quad (1) \\
\frac{dT}{dt} &= \omega S - \nu T
\end{align*}
\]

Here \(\omega(G)\) is a decreasing function of \(G\) which reflects death of saplings due to the spread of fires and \(\mu \geq \nu\) are the death rates for saplings and trees. Studies suggest (see [15] for

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references) that regions with tree cover below about 40% burn frequently but fire is rare above this threshold, so they used an $\omega$ that is close to a step function.

The ODE in (1) has very interesting behavior: it may have two stable fixed points, changing the values of parameters may lead to Hopf bifurcations, and if the system has an extra type of savannah trees, there can be period orbits. In this paper, we will begin the study of the corresponding spatial model. The state at time $t$ is $\chi_t : \mathbb{Z}^d \to \{0, 1, 2\}$, where 0 = Grass, 1 = Sapling, and 2 = Tree. Given the application, it would be natural to restrict our attention to $d = 2$, but since the techniques we develop will be applicable to other systems we consider the general case.

In the forest model, it is natural to assume that dispersal of seeds is long range. To simplify our calculations, we will not use a continuous dispersal distribution for tree seeds, but instead let $f_i(x, L)$ denote the fraction of sites of type $i$ in the box $x + [-L, L]^d$ and declare that site $x$ changes

- $0 \to 1$ at rate $\beta f_2(x, L)$
- $1 \to 2$ at rate $\omega(f_0(x, K))$
- $1 \to 0$ at rate $\nu$
- $2 \to 0$ at rate $\mu$

The configuration with all sites 0 is an absorbing state. This naturally raises the question of finding conditions that guarantee the existence of a nontrivial stationary distribution. Our model has three states but it is “attractive,” i.e., if $\chi_0(x) \leq \chi'_0(x)$ for all $x$ then we can construct the process so that this inequality holds for all time. From this, it follows from the usual argument that if we start from $\chi^2_0(x) \equiv 2$ then $\chi^2_t$ converges to a limit $\chi^2_\infty$ that is a translation invariant stationary distribution, and there will be a nontrivial stationary distribution if and only if $P(\chi^2_\infty(0) = 0) < 1$. Since 2’s give birth to 1’s and 1’s grow into 2’s, if $\chi^2_\infty$ is nontrivial then both species will be present with positive density in $\chi^2_\infty$.

If $\omega \equiv \gamma$ is constant, $\nu = 1 + \delta$, and $\mu = 1$ then our system reduces to one studied by Krone [10]. In his model, 1’s are juveniles who are not yet able to reproduce. Krone proved the existence of nontrivial stationary distributions in his model by using a simple comparison between the sites in state 2 and a discrete time finite-dependent oriented percolation. In the percolation process we have an edge from $(x, n) \to (x + 1, n + 1)$ if a 2 at $x$ at time $ne$ will give birth to a 1 at $x + 1$, which then grows to a 2 before time $(n + 1)e$, and there are no deaths at $x$ or $x + 1$ in $[ne, (n + 1)e]$. As the reader can imagine, this argument produces a very crude upper bound on the critical value of the system.

A simple comparison shows that if we replace $\omega(G)$ in the Staver-Levin model, $\chi_t$, by the constant $\omega = \omega(1)$, to obtain a special case $\eta_t$ of Krone’s model, then $\chi_t$ dominates $\eta_t$ in the sense that given $\chi_0 \geq \eta_0$ the two processes can be coupled so that $\chi_t \geq \eta_t$ for all $t$. Because of this, we can prove existence of nontrivial stationary distribution in the Staver-Levin model by studying Krone’s model. To do this under the assumption of long range interactions, we
begin with the mean field ODE:

\[
\begin{align*}
\frac{dG}{dt} &= \mu S + \nu T - \beta GT \\
\frac{dS}{dt} &= \beta GT - (\omega + \mu)S \\
\frac{dT}{dt} &= \omega S - \nu T
\end{align*}
\]

Since \( G + S + T = 1 \), we can set \( G = 1 - S - T \) and reduce the system to two equations for \( S \) and \( T \).

To guess the condition for coexistence in the long range limit we note that:

Lemma 1.1. For the mean-field ODE (2), \( S = 0, T = 0 \) is not a attracting fixed point if and only if

\[
\mu \nu < \omega (\beta - \nu)
\]

Proof. When \((S, T) \approx (0, 0)\) and hence \( G \approx 1 \), the mean-field ODE is approximately:

\[
\begin{pmatrix}
\frac{dS}{dt} \\
\frac{dT}{dt}
\end{pmatrix} \approx A \begin{pmatrix} S \\ T \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} -[\omega + \mu] & \beta \\ \omega & -\nu \end{pmatrix}
\]

The trace of \( A \), which is the sum of its eigenvalues is negative, so \((0, 0)\) is not attracting if and only if the determinant of \( A \), which is the product of the eigenvalues is negative. Since \((\omega + \mu)\nu - \beta \omega < 0\) if and only if \(\mu \nu < (\beta - \nu)\omega\), we have proved the desired result.

The main theorem of this paper is:

\[
\text{Theorem 1.} \quad \text{Let } \eta \text{ be Krone's model with parameters that satisfy (3). Then } \eta \text{ has a a nontrivial stationary distribution when } L \text{ is large enough.}
\]

When (3) fails in the Staver-Levin model for \( \omega = \omega(1) \), the mean-field ODE may have a nontrivial fixed point, and there may be a stationary distribution. However, in Krone's model we have the following converse, which does not require the assumption of long range.

\[
\text{Theorem 2.} \quad \text{Suppose (3) does not hold. Then for any } L > 0, \text{ Krone’s model } \eta \text{ dies out, i.e. for any initial configuration } \eta_0 \text{ with finitely many non-zero sites:}
\]

\[
\lim_{t \to \infty} P(\eta_t(x) \equiv 0) = 1
\]

1.1 Sketch of the Proof of Theorem 1

1. The key idea is due to Granman and Swindle [8]. They consider a model of a catalytic surface in which atoms of type \( i = 1, 2 \) land at vacant sites (0’s) at rate \( p_i \), while adjacent 1, 2 pairs turn into 0,0 at rate \( \infty \). If after a landing event, several 1,2 pairs are created, one is chosen at random to be removed. The first type of event is the absorption of an atom onto the surface of the catalyst, while the second is a chemical reaction, e.g., carbon monoxide CO and oxygen O reacting to produce CO\(_2\). The last reaction occurs in the catalytic converted in your car, but the appropriate model for that system is more complicated. An oxygen
molecule $O_2$ lands and dissociates to two $O$ bound to the surface when a pair of adjacent sites is vacant. See Durrett and Swindle [7] for more details about the phase transition in the system.

Suppose without loss of generality that $p_1 + p_2 = 1$. In this case Grannan and Swindle [8] showed that if $p_1 \neq p_2$ the only possible stationary distributions concentrate configurations that are $\equiv 1$ or $\equiv 2$. Mountford and Sudbury [12] later improved this result by showing that if $p_1 > 1/2$ and the initial configuration has infinitely many 1’s then the system converges to the all 1’s state.

The key to the Grannan-Swindle argument was to consider

$$Q(\eta_t) = \sum_x e^{-\lambda\|x\|} q[\eta_t(x)],$$

where $\|x\| = \sup_i |x_i|$ is the $L^\infty$ norm, $q(0) = 0$, $q(1) = 1$, and $q(2) = -1$. If $\lambda$ is small enough then $dEQ/dt \geq 0$ so $Q$ is a bounded submartingale and hence converges almost surely to a limit. Since an absorption or chemical reaction in $[-K, K]^d$ changes $Q$ by an amount $\geq \delta_K$, it follows that such events eventually do not occur.

2. Recovery from small density is the next step. We will pick $\epsilon_0 > 0$ small, let $\ell = [\epsilon_0 L]$ be the integer part of $\epsilon_0 L$ and divide space into small boxes $\hat{B}_x = 2\ell x + (-\ell, \ell]^d$. To make the number of 1’s and 2’s in the various small boxes sufficient to describe the state of the process, we declare two small boxes to be neighbors if all of their points are within an $L^\infty$ distance $L$. For the “truncated process”, which is stochastically bounded by $\eta_t$, and in which births of trees can only occur between sites in neighboring small boxes, we will show that if $\alpha \in (d/2, d)$ and we start with a configuration that has $L^\alpha$ non-zero sites in $\hat{B}_0$ and 0 elsewhere, then the system will recover and produce a small box $\hat{B}_x$ at time $\tau$ in which the density of nonzero sites is $a_0 > 0$ and $P(\tau > t_0 \log L) < L^{d/2-\alpha}$. See Lemma 3.1. To prove this, we use an analogue of Grannan and Swindle’s $Q$. The fact that $(0,0)$ is an unstable fixed point implies $dEQ/dt > 0$ as long as the density in all small boxes is $\leq a_0$.

3. Bounding the location of the positive density box is the next step. To do this we use a comparison with branching random walk to show that the small box $\hat{B}_x$ with density $a_0$ constructed in Step 2 is not too far from 0, Random walk estimates will later be used to control how far it will wander as we iterate the construction. For this step it is important that the truncated process is invariant under reflection, so the mean displacement is 0. If we try to work directly with the original interacting particle system $\eta_t$ then it is hard to show that the increments between box locations are independent and have mean 0.

4. Moving particles. The final ingredient in the block construction is to show that given a small block $\hat{B}_x$ with positive density and any $y$ with $\|y - x\|_1 \leq [c \log L]$ then if $c$ is small enough it is very likely that there will be $\geq L^\alpha$ particles in $\hat{B}_y$ at time $[c \log L]$. Choosing $y$ appropriately and then using the recovery lemma we can get lower bounds on the spread of the process.

5. Block construction. Once we have completed steps 2, 3, and 4, it is straightforward to show that our system dominates a one-dependent oriented percolation and prove the existence of a nontrivial stationary distribution.
The remainder of the paper is devoted to the proof of Theorem 1. The truncated process is defined in Section 2 and a graphical representation is used to couple it, Krone’s model, and the Staver-Levin model. In Section 3 we use the Grannan-Swindler argument to prove the recovery from small density. In Section 4, we bound the movement of the positive density boxes produced by our construction. In Section 5, we show how to move particles. Finally in Section 6 we put all of the pieces together and complete the proof.

2 Box Process and Graphical Representation

For some fixed \( \epsilon_0 > 0 \) which will be specified in (11), let \( l = [\epsilon_0 L] \) and divide space \( Z^d \) into small boxes:

\[
\hat{B}_x = 2lx + (-l, l]^d \quad x \in Z^d
\]

For any \( x \in Z^d \) there is a unique \( x' \) such that \( x \in \hat{B}_{x'} \). Define the new neighborhood of interaction as follows: For any \( y \in \hat{B}_{y'} \), \( y \in \mathcal{N}(x) \) if and only if

\[
\sup_{z_1 \in \hat{B}_{x'}, z_2 \in \hat{B}_{y'}} \|z_1 - z_2\| \leq L.
\]

It is easy to see that \( \mathcal{N}(x) \subset B_x(L) \). To show that \( \mathcal{N}(x) \supset B_x((1 - 4\epsilon_0)L) \) we note that if \( \|x - y\| \leq (1 - 4\epsilon_0)L \), \( z_1 \in \hat{B}_{x'} \), and \( z_2 \in \hat{B}_{y'} \), where \( \hat{B}_{x'} \) and \( \hat{B}_{y'} \) are the small boxes containing \( x \) and \( y \):

\[
\|z_1 - z_2\| \leq \|z_1 - x\| + \|x - y\| + \|y - z_2\| \leq 4(\epsilon_0 L) + (1 - 4\epsilon_0)L \leq L.
\]

Given the new neighborhood \( \mathcal{N}(x) \), we define the truncated version of Krone’s model \( \xi_t \) by its transition rates:

\[
\text{transition at rate}
\]

\[
1 \to 0 \quad \mu
\]

\[
2 \to 0 \quad \nu
\]

\[
1 \to 2 \quad \omega
\]

\[
0 \to 1 \quad \beta N_2[\mathcal{N}(x)]/(2L + 1)^d
\]

where \( N_i[S] \) stands for the number of \( i \)'s in the set \( S \). For any \( x \in Z^d \) and \( \xi \in \{0, 1, 2\}^Z \), define \( n_i(x, \xi) \) to be the number of type \( i \)'s in the small box \( \hat{B}_x \) in the configuration \( \xi \). The box process

\[
\zeta_t(x) = [n_1(x, \xi_t), n_2(x, \xi_t)], \forall x \in Z^d.
\]

Then \( \zeta_t \) is a Markov process on \( \{(n_1, n_2) : n_1, n_2 \geq 0, n_1 + n_2 \leq |\hat{B}_0| \} \) in which:

\[
\text{transition at rate}
\]

\[
\zeta_t(x) \to \zeta_t(x) - (1, 0) \quad \mu \zeta_t^1(x)
\]

\[
\zeta_t(x) \to \zeta_t(x) - (0, 1) \quad \nu \zeta_t^2(x)
\]

\[
\zeta_t(x) \to \zeta_t(x) + (-1, 1) \quad \omega \zeta_t^1(x)
\]

\[
\zeta_t(x) \to \zeta_t(x) + (1, 0) \quad \sum_{y: \hat{B}_y \subset \mathcal{N}(x)} \beta \zeta_t^2(y)
\]
Because $\zeta_t$ only records the number of particles in any small box, and the neighborhood is defined so that all sites in the same small box have the same neighbors, the distribution of $\zeta_t$ is symmetric under reflection in any axis. The main use for this observation is that the displacement of the location of the positive density box produced by the recovery lemma in Section 3 has mean 0.

### 2.1 Graphical Representation

We will use the graphical representation similar as in [10] to construct Krone’s model $\eta_t$ and the truncated version $\xi_t$ on the same probability space, so that

(∗) If $\eta_0 \geq \xi_0$ then we will have $\eta_t \geq \xi_t$ for all $t$.

We use independent families of Poisson processes for each $x \in \mathbb{Z}^d$, as follows:

- $\{V_n^x : n \geq 1\}$ with rate $\nu$. We put an $\times$ at space-time point $(x, V_n^x)$ and write a $\delta_{12}$ next to it to indicate a death will occur if $x$ is occupied by a 1 or a 2.
- $\{U_n^x : n \geq 1\}$ with rate $\mu - \nu$. We put an $\times$ at space-time point $(x, V_n^x)$ and write a $\delta_1$ next to it to indicate a death will occur if $x$ is occupied by 1.
- $\{W_n^x : n \geq 1\}$ with rate $\omega$. We put an $\bullet$ at space-time point $(x, V_n^x)$ and write a $\beta$ next to it to indicate that if $x$ is in state 1, it will become a 2.
- $\{T_n^{x,y} : n \geq 1\}$ with rate $\beta/|B_0|$ all $y \in \mathcal{N}(x)$. We draw a solid arrow from $(x, T_n^{x,y})$ to $(y, T_n^{x,y})$ to indicate that if $x$ is occupied by a 2 and $y$ is vacant, then a birth will occur at $x$ in either process.
- $\{T_n^{x,y} : n \geq 1\}$ with rate $\beta/|B_0|$ all $y \in B_x(L) - \mathcal{N}(x)$. We draw a dashed arrow from $(x, T_n^{x,y})$ to $(y, T_n^{x,y})$ to indicate that if $x$ is occupied by a 2 and $y$ is vacant then a birth will occur at $x$ in the process $\xi_t$.

Standard arguments that go back to Harris [9] over forty years ago, guarantee that we have constructed the desired processes and (∗) holds. We can construct the Staver-Levin model on the same space so that it has $\chi_t \geq \xi_t$ by adding another family of Poisson process $\{\hat{W}_n^x : n \geq 1\}$ with rate $1 - \omega$, and independent random variables $w_{x,n}$ uniform on $(0, 1)$. At any time $W_n^x$ if $x$ is in state 1, it will increase to state 2 if

$$w_{n,x} > \frac{\omega(f_0(x, K)) - \omega}{1 - \omega}$$

### 3 Recovery Lemma

Given (3), one can pick a $\theta$, which must be $> 1$, such that

$$\frac{\mu + \omega}{\omega} < \theta < \frac{\beta}{\nu} \quad \text{(5)}$$
so we have
\[ \theta \omega - (\omega + \mu) > 0, \beta - \theta \nu > 0 \]
and since the inequalities above are strict, we can pick some \( a_0 > 0 \) and \( \rho > 0 \) such that
\[ \theta \omega - (\omega + \mu) \geq \rho, \beta(1 - 4a_0) - \theta \nu \geq \theta \rho. \] (6)

Now we can let the undetermined \( \epsilon_0 \) in the definition of \( \xi_t \) in Section 2 be a positive constant such that \((1 - 4\epsilon_0)^d > 1 - 2a_0\).

We start with an initial configuration in \( \Xi_0 \), the \( \xi_0 \) that have \( \xi_0(x) = 0 \) for all \( x \not\in \hat{B}_0 \) and the number of nonzero sites in \( \hat{B}_0 \) is at least \( L^\alpha \) for some \( \alpha \in (d/2, d) \). We define a stopping time \( \tau \):
\[ \tau = \inf\{ t : \exists x \in Z^d \text{ s.t. } n_1(x, \xi_t) + n_2(x, \xi_t) \geq a_0|\hat{B}_0| \} \] (7)

**Lemma 3.1. Recovery Lemma.** Suppose we start the truncated version of Krone’s model from a \( \xi_0 \in \Xi_0 \). Let \( t_0 = 2d/\rho \). When \( L \) is large,
\[ P(\tau > t_0 \log L) < L^{d/2 - \alpha} \] (8)

**Proof.** As mentioned in the introduction, we consider
\[ Q(\xi_t) = \lambda^d \sum_{x \in Z^d} e^{-\lambda \|x\|} w[\xi_t(x)] \]
where \( \lambda = L^{-1} a_0/2 \) and
\[ w[\xi(x)] = \begin{cases} 0 & \text{if } \xi(x) = 0 \\ 1 & \text{if } \xi(x) = 1 \\ \theta & \text{if } \xi(x) = 2 \end{cases} \]

If we imagine \( \mathbb{R}^d \) divided into cubes with centers at \( \lambda Z^d \) and think about sums approximating an integral then we see that
\[ \lambda^d \sum_{x \in Z^d} e^{-\lambda \|x\|} \leq e^{\lambda/2} \int_{\mathbb{R}^d} e^{-\|z\|} dz \leq U_0 \] (9)

for all \( \lambda \in (0, 1] \). From this it follows that \( Q(\xi_t) \leq \theta U_0 \). Here and in what follows \( U \)’s are upper bounds that are independent of \( \lambda \in (0, 1] \).

Our next step towards Lemma 3.1 is to study the infinitesimal mean
\[ \mu(\xi) = \lim_{\delta t \downarrow 0} \frac{E[Q(\xi_{t+\delta t}) - Q(\xi_t)|\xi_t = \xi]}{\delta t}. \]

**Lemma 3.2.** For all \( \xi \) such that \( n_1(x, \xi) + n_2(x, \xi) \leq a_0|\hat{B}_0| \) for all \( x \in Z^d \) and any \( t \geq 0 \), \( \mu(\xi) \geq \rho Q(\xi) \) where \( \rho \) is defined in (6).

**Proof.** Straightforward calculation gives
\[ \frac{\mu(\xi)}{\lambda^d} = \sum_{\xi(x)=1} [(\theta - 1)\omega - \mu]e^{-(\omega + \mu)\|x\|} + \sum_{\xi(x)=0} \beta \frac{N_2[N(x)]}{(2L + 1)^d} e^{-\lambda \|x\|} - \sum_{\xi(x)=2} \theta \mu e^{-\lambda \|x\|}. \] (10)
For the second term in the equation above, we interchange the roles of $x$ and $y$ then rearrange the sum:

$$
\sum_{\xi(x)=0} \frac{\beta N_2[\mathcal{N}(x)]}{(2L + 1)^d} e^{-\lambda \|x\|} = \sum_{\xi(x)=2} (2L + 1)^{-d} \sum_{y \in \mathcal{N}(x), \xi(y)=0} \beta e^{-\lambda \|y\|}.
$$

Noting that $\lambda = L^{-1} a_0 / 2$, and that for any $x$ and $y \in \mathcal{N}(x) \subset B_x$, $-L \leq \|y\| - \|x\| \leq L$, we have

$$e^{-\lambda \|y\|} \geq e^{-a_0/2} e^{-\lambda \|x\|} \geq (1 - a_0) e^{-\lambda \|x\|}.
$$

Using this with $n_1(x, \xi) + n_2(x, \xi) < a_0 |\hat{B}_0|$, and $B_x [(1 - 4\epsilon_0)L] \subset \mathcal{N}(x)$ from (11),

$$
\sum_{\xi(x)=0} \frac{\beta N_2[\mathcal{N}(x)]}{(2L + 1)^d} e^{-\lambda \|x\|} \geq (1 - a_0) \sum_{\xi(x)=2} \frac{\beta N_0(B_x[(1 - 4\epsilon_0)L])}{(2L + 1)^d} e^{-\lambda \|x\|}
\geq (1 - a_0)[(1 - 4\epsilon_0)^d - a_0] \sum_{\xi(x)=2} \beta e^{-\lambda \|x\|}.
$$

So we can now specify $\epsilon_0$ be small enough so that

$$(1 - 4\epsilon_0)^d > 1 - 2a_0. \quad (11)$$

This choice implies

$$\sum_{\xi(x)=0} \frac{\beta N_2[\mathcal{N}(x)]}{(2L + 1)^d} e^{-\lambda \|x\|} > (1 - a_0)(1 - 3a_0) \sum_{\xi(x)=2} \beta e^{-\lambda \|x\|} > (1 - 4a_0) \sum_{\xi(x)=2} \beta e^{-\lambda \|x\|}.$$ 

Combining inequality above with (11) and (8) gives

$$
\mu(\eta) \geq \lambda^d \sum_{\eta(x)=1} [((\theta - 1)\omega - \mu]e^{-\lambda \|x\|} + \lambda^d \sum_{\eta(x)=2} [(1 - 4a_0)\beta - \theta \nu] e^{-\lambda \|x\|} \geq \rho \mathcal{Q}(\eta).
$$

which proves the desired result. \qed

Let $\sigma^2(\xi)$ be the infinitesimal variance of $\mathcal{Q}(\xi_t)$:

$$
\sigma^2(\xi) = \lim_{\delta t \downarrow 0} \frac{E([\mathcal{Q}(\xi_{t+\delta t}) - \mathcal{Q}(\xi_t)]^2|\mathcal{Q}(\xi_t) = \xi)}{\delta t}.
$$

**Lemma 3.3.** There is a constant $U_1$ independent of $\lambda \in (0, 1]$, so that $\sigma^2(\xi) \leq U_1 L^{-d}$.

**Proof.** By considering the rates at which jumps occur and the changes they cause we see that

$$
\frac{\sigma^2(\xi)}{\lambda^d} = \sum_{\xi(x)=1} [(\theta - 1)^2 \omega + \mu] e^{-\lambda \|x\|} + \sum_{\xi(x)=0} \frac{\beta N_2[\mathcal{N}(x)]}{(2L + 1)^d} e^{-\lambda \|x\|} + 4 \sum_{\xi(x)=2} \theta \nu e^{-\lambda \|x\|}. \quad (12)
$$

Using (9) and recalling $\lambda = L^{-1} a_0 / 2$ now gives the desired result. \qed
For any initial configuration $\xi_0$, define

$$M_t = Q(\xi_t) - Q(\xi_0) - \int_0^t \mu(\xi_s)ds$$  \tag{13}$$

According to Dynkin’s formula, $M_t$ is a martingale with $EM_t = 0$.

**Lemma 3.4.** $EM_t^2 = E \int_0^t \sigma^2(\xi_s)ds \leq U_1 L^{-d}$ and hence

$$E \left( \sup_{s \leq t} M_s^2 \right) \leq 8U_1 L^{-d} t.$$  \tag{14}$$

**Proof.** Using (13) and (9) we see that

$$|\mu(\xi_t)| \leq U_2 = (\omega + \beta + \nu + \mu)U_0.$$  \tag{15}$$

To calculate $EM_t^2$, let $t^n_i = it/n$.

$$EM_t^2 = \sum_{i=0}^{n-1} E(M_{t_{i+1}^n} - M_{t_i^n})^2 = \sum_{i=0}^{n-1} E \left[ Q(\xi_{t_{i+1}^n}) - Q(\xi_{t_i^n}) - \int_{t_i^n}^{t_{i+1}^n} \mu(\xi_s)ds \right]^2.$$  \tag{16}$$

Using (15) we have

$$\sum_{i=0}^{n} \left[ \int_{t_i^n}^{t_{i+1}^n} \mu(\xi_s)ds \right]^2 \leq U_2 \sum_{i=1}^{n} (t_{i+1}^n - t_i^n)^2 = U_2 \frac{t^2}{n} \to 0.$$

From the definition of the infinitesimal variance, and the fact that the semigroup associated with a Markov process is a contraction operator, we get

$$0 \leq -\left| \frac{n}{t} E[Q(\xi_{t/n}) - Q(\xi_0)]^2 - \sigma^2(\xi_0) \right| \geq E \left| \left( \frac{n}{t} E[Q(\xi(t_{i+1}^n)) - Q(\xi(t_i^n))]^2 - \sigma^2(\xi(t_{i+1}^n)) \right) \right|$$

which implies that

$$\sum_{i=0}^{n-1} E[Q(\xi_{t_{i+1}^n}) - Q(\xi_{t_i^n})]^2 \to E \int_0^t \sigma^2(\xi_s)ds.$$  \tag{17}$$

The last two results and the Cauchy-Schwartz inequality imply that the other term generated by the square in (16) tends to 0 and we have established the first equality. Using lemma 3.3 and the $L^2$ maximal inequality now, completes the proof.

At this point, we have all the tools needed in the proof of Lemma 3.1. If $\xi_0 \in \Xi_0$ there is a $u_4 > 0$ such that for all $\xi_0$ in Lemma 3.1

$$u_4 L^{-d+\alpha} \leq Q(\xi_0).$$

Using (14) now

$$E \left( \sup_{s \leq t_0 \log L} M_s^2 \right) \leq 4U_1 L^{-d} t_0 \log L.$$
so by Chebyshev’s inequality and the fact that \( \alpha > d/2 \):

\[
P\left( \left| \sup_{s \leq t_0 \log L} M_s \right| \geq u_4 L^{-d+\alpha}/2 \right) \leq \frac{16U_1 L^{-d} t_0 \log L}{u_4 L^{2(d-\alpha)}} = O(L^{-2\alpha+d} \log L) = o(L^{d/2-\alpha}) \to 0.
\]

Consider the event \( \{ \tau > t_0 \log L \} \). For any \( s \leq t_0 \log L \), \( n_1(x, \xi_s) + n_2(x, \xi_s) < a_0|B_0| \), for all \( x \in \mathbb{Z}^d \), so by Lemma 3.2, \( \mu(\xi_s) \geq \rho Q(\xi_s) \). Consider the set:

\[
A = \left\{ \left| \sup_{s \leq t_0 \log L} M_s \right| < u_4 L^{-d+\alpha}/2 \right\} \cap \{ \tau > t_0 \log L \}.
\]

On \( A \) we will have that for all \( t \in [0, t_0 \log L] \),

\[
Q(\xi_t^0) \geq u_4 L^{-d+\alpha} + \rho \int_0^t Q(\xi_s^0) ds.
\]

If we let \( f(t) = e^{\rho t} u_4 L^{-d+\alpha}/2 \), then

\[
f(t) = u_4 L^{-d+\alpha}/2 + \rho \int_0^t f(s) ds.
\]

Reasoning as in the proof of Gronwall’s inequality:

**Lemma 3.5.** \( Q(\xi_t^0) \geq f(t) \) for all \( t \in [0, t_0 \log L] \)

**Proof.** Suppose the lemma does not hold. Let \( t_1 = \inf\{ t \in [0, t_0 \log L] : Q(\xi_t^0) < f(t) \} \). By right-continuity of \( Q(\xi_t^0) \), \( Q(\xi_{t_1}^0) \leq f(t_1) \) and \( t > 0 \). However, by definition of \( t_1 \), we have \( Q(\xi_t^0) \geq f(t) \) on \( [0, t_1) \), and by right-continuity of \( Q(\xi_t^0) \) near \( t = 0 \), the inequality can be strict in a neighborhood of 0. Thus

\[
Q(\xi_{t_1}^0) \geq u_4 L^{-d+\alpha}/2 + \rho \int_0^{t_1} Q(\xi_s^0) ds > u_4 L^{-d+\alpha}/2 + \rho \int_0^{t_1} f(s) ds = f(t_1)
\]

contradiction. \( \square \)

Recalling that \( t_0 = 2d/\rho \)

\[
f(t_0 \log L) = e^{\rho t_0 \log L} u_4 L^{-d+\alpha}/2 = u_4 L^{d+a}/2.
\]

When \( L \) is large this will be \( \geq \theta U_0 \) the largest possible value of \( Q(\xi_t) \), a contradiction, and we have finished the proof of Lemma 3.1. \( \square \)

### 3.1 Proof of Theorem 2

Using the same technique as in Lemma 3.2, we are able to prove the extinction result in Theorem 2 which does not require the assumption of long range.
Proof. When (3) does not hold, if \( \beta \leq \nu \), the system dies out since \( \eta_t \) can be bounded by a subcritical contact process with birth rate \( \beta \) and death rate \( \nu \) (the special case of \( \eta_t \) when \( \omega = \infty \)). Otherwise, we can find a \( \theta' \) such that

\[
\frac{\mu + \omega}{\omega} > \theta' > \frac{\beta}{\nu} > 1.
\]

For \( \eta_t \) starting from \( \eta_0 \) with a finite number of non-zero sites, consider

\[
S(\eta_t) = \sum_{x \in Z^d} 1_{\eta_t(x)=1} + \theta' 1_{\eta_t(x)=2}.
\]

Similarly, let \( \mu(\eta_t) \) be the infinitesimal mean of \( S(\eta_t) \), according to same calculation as in the proof of Lemma 3.2, we have

\[
\mu(\eta_t) = \sum_{x \in Z^d} [\omega(\theta' - 1) - \mu] 1_{\eta_t(x)=1} + [-\theta' \nu + f_0(x, \eta_t) \beta] 1_{\eta_t(x)=2}.
\]

Note that

\[
\omega(\theta' - 1) - \mu < 0
\]

and that

\[-\theta' \nu + f_0(x, \eta_t) \beta \leq -\theta' \nu + \beta < 0\]

we showed that \( \mu(\eta_t) \leq 0 \forall t \geq 0 \), \( S(\eta_t) \) is a nonnegative supermartingale. By martingale convergence theorem, \( S(\eta_t) \) converge to some limit as \( t \to \infty \). Note that each jump in \( \eta_t \) will change \( S(\eta_t) \) by \( 1 \), \( \theta' \) or \( \theta' - 1 > 0 \). Thus to have \( \eta_t \) converges, with probability one there must be only finite jumps in each path of \( \eta_t \), which implies that with probability one \( \eta_t \) will end up at configuration of all 0’s, which is the absorbing state. \( \square \)

4 Spatial Location of the Positive Density Box

The argument in the previous section proves the existence of a small box \( \hat{B}_x \) with positive density, but this is not useful if we do not have control over its location. To do this, we note that the graphical representation in Section 2, shows that box process \( \xi_t \) can be stochastically bounded by Krone’s Model \( \eta_t \) starting from a same initial configuration. Krone’s process can in turn be bounded by a branching random walk \( \gamma_t \) in which there are no deaths, 2’s give birth to 2 at rate \( \beta \), and births are not suppressed if the site is occupied.

**Lemma 4.1.** Suppose we start from \( \gamma_0 \) such that \( \gamma_0(x) = 2 \) for all \( x \in \hat{B}_0 \), \( \gamma_0(x) = 0 \) otherwise. Let \( M_k(t) \) be the largest \( k \)th coordinate among the occupied sites at time \( t \). If \( L \) is large enough then for any \( m > 0 \) we have

\[
P(M_k(t) \geq (2\beta + m)Lt) \leq e^{-mt|\hat{B}_0|}.
\]

From this it follows that there is a \( C_{4.1} < \infty \) so,

\[
E(M_k^2(t_0 \log L)) \leq C_{4.1}(L \log L)^2
\]
Proof. First we will start from the case where $\gamma_0$ has only one particle at 0. Rescale space by dividing by $L$. In the limit as $L \to \infty$ we have a branching random walk $\bar{\gamma}_t$ with births displaced by an amount uniform on $[-1, 1]^d$. It suffices to show that the corresponding maximum has $EM_k^2(t_0 \log L) \leq C(\log L)^2$. To this we note that mean number of particles in $A$ at time $t$

$$E(M(t, A)) = e^{\beta t} P(S_t \in A)$$

where $S_t$ is a random walk that makes jumps at rate $\beta$.

$$E \exp(\theta S_t) = \exp(\beta [\phi(\theta) - 1]) \quad \text{with} \quad \phi(\theta) = (e^{\theta} + e^{-\theta})/2\theta.$$ 

Large deviations implies that for any $\theta > 0$

$$P(S_t \geq x) \leq e^{-\theta x} e^{\beta t [\phi(\theta) - 1]}.$$ 

When $\theta = 1$, $\phi(1) - 1 = 0.543$ so if $L$ is large $\phi(1) - 1 \leq 1$ and it follows that

$$P(M_k(t) \geq (2\beta + m)t) \leq e^{-mt}.$$ 

Thus if we start with $|\hat{B}_0|$ particles in $\hat{B}_0/L \subset [-1, 1]^d$ in $\gamma_0$ then

$$P(M_k(t) \geq 1 + (2\beta + m)t) \leq |\hat{B}_0| e^{-mt}.$$ 

Taking $t = t_0 \log L$ now gives the desired result. 

Lemma 4.2. For any $a > 0$, let $M_j(t_0 \log L) 1 \leq j \leq L^a$ be independent and identically distributed. There is a $C_{1.2} < \infty$ so that for large $L$

$$P \left( \max_{1 \leq j \leq L^a} \|M_j(t_0 \log L)\| \geq C_{1.2} L \log L \right) \leq 1/L$$

Proof. Taking $t = t_0 \log L$ in (17) the quantity in the theorem is bounded by $L^{d+a} \exp(-mt_0 \log L)$. Taking the constant $m$ to be large enough gives thee desired result.

5 Moving Particles in $\eta_t$

Let $H_{t,x}$ be the number of nonzero sites of $\eta_t$ in $\hat{B}_x$ at time $t$. In this section, we will use the graphical representation in Section 2 and the technique in Durrett and Lanchier [4] to show that

Lemma 5.1. There are constants $\delta_{5.1} > 0$ and an $L_{5.1} < \infty$ such that for all $L \geq L_{5.1}$ and any initial configuration $\eta_0$ with $H_{0,0} \geq L^{d/2}$

$$P(H_{1, \delta x} \geq \delta_{5.1} H_{0,0}) > 1 - e^{-L^{d/4}}$$

for any $v \in \{0, \pm e_1, \cdots, \pm e_d\}$. 

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Proof. We begin with the case \( v = 0 \) which is easy. Define \( G_0^v \) to be the set of of points \( x \in \hat{B}_0 \), with (a) \( \eta_0(x) \geq 1 \), and (b) no death marks \( \times \)'s occur in \( \{x\} \times [0, 1] \). We have \( \xi_t(x) \geq 1 \) on \( S_0 = H_{0,0} \cap G_0 \), and \( |S_0| \sim \text{Binomial}(H_{0,0}, e^{-\mu}) \), so the desired result follows from large deviations for the Binomial.

For \( v \neq 0 \), define \( G_0 \) to be the set of of points in \( G_0^0 \) for which (c) there exists a (\( \bullet \)) in \( \{x\} \times [0, 1/2] \). We define \( G_v \) to be the set of points \( y \in \hat{B}_v \) so that there are no \( \times \)'s in \( \{y\} \times [0, 1] \). For any \( x \in \hat{B}_0 \) and \( y \in \hat{B}_v \) we say that \( x \) and \( y \) are connected (and write \( x \rightarrow y \) if there is an arrow from \( x \) to \( y \) in \( (1/2, 1) \). By definition of our process \( \eta_1(y) \geq 1 \) for all \( y \) in

\[
S = \{y : y \in G_v, \text{ there exists an } x \in G_0 \text{ so that } x \rightarrow y \}.
\]

It is easy to see that

\[
|G_0| \sim \text{Binomial}[H_{0,0}, e^{-\mu}(1 - e^{-\omega/2})] \tag{19} \]  

and conditional on \( |G_0| \), we can determine the conditional distribution of \( S \):

\[
|S| \sim \text{Binomial}(|\hat{B}_v|, e^{-\mu}[1 - e^{-\beta|G_0|/2|B_0|}]). \tag{20}
\]

Since being the recipient of a birth from \( \hat{B}_0 \) are independent for different sites in \( G_v \) by Poisson thinning.

Since that binomial distribution decays exponentially fast away from the mean, there is some constant \( c > 0 \) such that

\[
P(|G_0| > H_{0,0})e^{-\mu}(1 - e^{-\omega/2})/2 \geq 1 - e^{-cL^{d/2}}. \tag{21}
\]

To simplify the next computation, we note that \( 1 - e^{-\beta r} \sim \beta r \) as \( r \to 0 \) so if the \( \epsilon_0 \) in the definition of the small box is small enough \( 1 - e^{-\beta|G_v|/2|B_0|} \geq \beta|G_v|/4|B_0| \). Let \( p = e^{-\mu} \beta|G_0|/4|B_0| \). A standard large deviations result, see e.g., Lemma 2.8.5 in [3] shows that if \( X = \text{Binomial}(N, p) \) then

\[
P(X \leq Np/2) \leq \exp(-Np/8)
\]

from which the desired result follows.

Iterating this result \( O(\log L) \) we can move a positive density box to one with at least \( L^\alpha \) particles. Let \( \| \cdot \|_1 \) be the \( L^1 \)-norm on \( Z^d \).

**Lemma 5.2.** For any \( \alpha \in (d/2, d) \), let \( C_{5.2} \) be a constant such that \( C_{5.2} \log \xi_{5.1} > \alpha - d \). There is a finite \( L_{5.2} > L_{5.1} \) such that for all \( L > L_{5.2} \), any initial configuration \( \eta_0 \) with \( H_{0,0} \geq a_0|\hat{B}_0| \), and any \( x \in Z^d \) such that \( \|x\|_1 \leq C_{5.2} \log L \), where

\[
P(H_{x, |5.2\log L|} \geq L^\alpha) \geq 1 - e^{-L^{d/4}/2}
\]

**Proof.** Let \( n = \lfloor C_{5.2} \log L \rfloor \). We can find a sequence \( x_0 = 0, x_1, \ldots, x_n = x \) such that for all \( i = 0, \ldots, n - 1, x_{i+1} - x_i \in \{0, \pm \epsilon_1, \ldots, \pm \epsilon_d\} \). For any \( i = 1, \ldots, n \) define the event

\[
A_i = \{H_{x_i, i} \geq \xi_{5.1}/H_{0,0} \}.
\]
By the definition of $C_{5.2}$, $H_x[A_{5.2}^1] \geq L^a$ on $A_n$. To estimate $P(A_n)$ note that by Lemma 6.1

$$P(A_n) \geq 1 - \sum_{i=1}^{n} P(A_i^c) \geq 1 - \sum_{i=1}^{n} P(A_i^c | A_{i-1}) \geq 1 - C_{5.2}(\log L)e^{-L^{d/4}} \geq 1 - e^{-L^{d/4}/2}$$

when $L$ is large.

6 Block Construction

At this point, we have all the tools to construct the block event and complete the proof of Theorem 1. Let $0 < a < \alpha/2 - d/4$, $K = L^{1+2a/3}$, $\Gamma_m = 2mKe_1 + [-K, K]^d$, and $\Gamma'_m = 2mKe_1 + [-K/2, K/2]^d$. If $m + n$ is even, we say that $(m, n)$ is wet if there is a positive density small box in $\Gamma'_m$ at some time in $[nL^a, nL^a+C_6 \log L]$, where $C_6 = C_{5.2} + t_0$. Our goal is to show

**Lemma 6.1.** If $(m, n)$ is wet then with high probability so is $(m + 1, n + 1)$, and the events which produce this are measurable with respect to the graphical representation in $(\Gamma_m \cup \Gamma_{m+1}) \times [nL^a, (n+1)L^a + C_6 \log L]$.

Once this is done, Theorem 1 follows. See [11] for more details.

**Proof.** To prove Lemma 6.1, we will alternate two steps, starting from the location of the initial positive density box $\hat{B}_{y_0}$ at time $T_0$. Let $A_0 = \{T_0 < \infty\}$ which is the whole space. Assume given a deterministic sequence $\delta_i$ with $\|\delta_i\| \leq C_{5.2} \log L$. If we do not ever meet a failure, the construction will terminate at the first time that $T_i > (n + 1)L^a$. The actual number steps will be random but the number is $\leq N = \lceil L^a/[C_{5.2} \log L] \rceil$. We will do our estimate of the probability of success supposing that $N$ steps are required, to lower bound the probability of success when we stop at the first time $T_i \geq (n + 1)L^a$. Suppose $i \geq 1$.

1. **Deterministic Moving.** If at the stopping time $T_{i-1} < \infty$ we have a positive density small box $\hat{B}_{y_{i-1}}$, then we use results in Section 5 to produce a small box $\hat{B}_{y_i + \delta_i}$ with at least $L^a$ nonzero sites at time $S_i = T_{i-1} + [C_{5.2} \log L]$. If we fail we let $S_i = \infty$ and the construction terminates. Let $A_i^+ = \{S_i < \infty\}$.

2. **Random Recovery.** If at the stopping time $S_i < \infty$ we have a small box $\hat{B}_{y_{i-1} + \delta_i}$ with at least $L^a$ nonzero sites then we set all of the sites outside the box to 0, and we use the recovery lemma to produce a positive density small box $\hat{B}_{y_i}$ at time $S_i \leq T_i \leq S_i + t_0 \log L$. Again if we fail we let $T_i = \infty$ and the construction terminates. Let $A_i = \{T_i < \infty\}$. Let $\Delta_i(\omega) = y_i - (y_{i-1} - \delta_i)$ on $A_i$, and $= 0$ on $A_i^c$.

If we define the partial sums $\bar{y}_i = y_0 + \sum_{j=1}^{i} \delta_j$ and $\Sigma_i = \sum_{j=1}^{i} \Delta_j$ then we have $y_i = \bar{y}_i + \Sigma_i$. We think of $\bar{y}_i$ as the mean of the location of the positive density box and $\Sigma_i$ as the random fluctuations in its location. We make no attempt to adjust the deterministic movements $\delta_i$ to compensate for the fluctuations. Let $y_{\text{end}} = (y_{\text{end}}^1, 0, \cdots, 0) \in \mathbb{Z}^d$ be such that $2K(m + 1)e_1 \in \hat{B}_{y_{\text{end}}}$. We define the $\delta_i$ to reduce the coordinates $y_k^1$, $k = 2, \ldots, d$ to 0 and then increase $y_0^1$ to $y_{\text{end}}^1$, in all cases using steps of size $\leq C_{5.2} \log L$. Note that

$$\|y_0 - y_{\text{end}}\|_1 = O(K)/\ell = O(L^{2a/3}) = o(N)$$
Lemma 6.2. For any initial configuration η(T₀) so that there is a small box ˆB_y₀ ⊂ Γ'_m, and any sequence δ_i, i ≤ N with ∥δ_i∥ ≤ C_{δ,1} log L and any ε > 0, there is a good event G_N with G_N → 1 as L → ∞ so that (a) G_N ⊂ A_N, (b) on G_N, ∥y_i − ˘y_i∥ < εL^{2α/3} for 1 ≤ i ≤ N. (c) G_N depends only on the gadgets of graphical representation in Γ_m ∪ Γ_{m+1}

Proof. The first step is to show that P(A_N) → 1 as L → ∞. For the ith deterministic moving step, using the strong Markov property and Lemma 5.2, we have

\[ P(A_i^+|A_{i-1}) > 1 - e^{-L^{d/4}/2}. \]

Then for the random recovery phase, according to Lemma 3.1, we have the conditional probability of success:

\[ P(A_i|A_i^+) = P(\tau_i < t_0 \log L) > 1 - L^{d/2-α}. \]

Combining the two observations, we have

\[ P(A_i|A_{i-1}) > (1 - e^{-L^{d/4}/2})(1 - L^{d/2-α}) > 1 - e^{-L^{d/4}/2} - L^{d/2-α} \]

which implies

\[ P(A_N) ≥ 1 - \sum_{i=1}^{N} P(A_i^c) ≥ 1 - \sum_{i=1}^{N} P(A_i^c|A_{i-1}) ≥ 1 - L^α(e^{-L^{d/4}/2} + L^{d/2-α}) ≥ 1 - 2L^{d/4-α/2} → 1. \]

The next step is to control the fluctuations in the movement of our box.

Lemma 6.3. Let F(T_i) be the filtration generated by events in the graphical representation up to stopping time T_i. For any 1 ≤ k ≤ d, \{Σ_i \}_{i=1}^N is a martingale with respect to F(T_i). E(Σ_i^k) = 0, and var(Σ_i^k) ≤ C_{k,1}L^α log L so for any ε we have

\[ P\left(\max_{1 ≤ i ≤ N} ||Σ_i|| > εL^{2α/3}\right) → 0 \]

Proof. Consider the conditional expectation of Δ^k_i under F(S_i). By remarks in Section 2, we have E(Δ^k_i|F(S_i)) = 0. Noting that 2∥Δ^k_i∥ can be bounded by the largest kth coordinate among the occupies sites of the corresponding branching random walk at time t_0 log L, Lemma 11 implies that E((Δ^k_i)^2|F(S_i)) ≤ C(\log L)^2. By orthogonality of martingale increments var(Σ_i^k) ≤ NC(\log L)^2. Since N ≤ L^α/\[C_{δ,1} \log L] + 1, we have the desired bound on the variances and the desired result follows from L^2 maximal inequality for martingales.
To check (c) now note that under $A_i$ the success of $A_{i+1}^+$ depends only on gadgets in:

$$(2\ell y_i + [-\ell C_{\frac{5}{2}} \log L, \ell C_{\frac{5}{2}} \log L]^d) \times [T_i, S_{i+1}]$$

and that when the $i$th copy of the truncated process never wander outside of

$$D_i = 2\ell (y_{i-1} + \delta_i) + [-C_{\frac{4}{2}} (L \log L), C_{\frac{4}{2}} (L \log L)]^d$$

the success of $A_{i+1}$ under $A_i^+$ depends only on gadgets in $D_i$. According to Lemma 4.2 and the fact that $N < L^a$, with probability $\geq 1 - L^{-1} = 1 - o(1)$, our construction only depends on gadgets in the box:

$$\bigcup_{i=0}^{N-1} \left((2\ell y_i + [-\ell C_{\frac{5}{2}} \log L, \ell C_{\frac{5}{2}} \log L]^d) \times [T_i, S_{i+1}] \cup (2\ell (y_i + \delta_{i+1}) + [-C_{\frac{4}{2}} (L \log L), C_{\frac{4}{2}} (L \log L)]^d) \times [S_{i+1}, T_{i+1}]\right).$$

(24)

The locations of the $y_i$ are controlled by Lemma 6.3 so that one can easily found the box defined in (24) a subset of $\Gamma_n \cup \Gamma_{n+1}$, and proof of Lemma 6.2 is complete.

Back to the proof of Lemma 6.1. On $G_N$ it follows from (22) and Lemma 6.2 when we stop at the first time $T_i \geq (n + 1)L^a$:

$$\|y_i - 2K(m + 1)e_1\| \leq \ell (1 + 2\|y_i - \bar{y}_i\|) \leq 4\epsilon K$$

which implies

$$\hat{B}_{y_i} = 2\ell y_i + (-\ell, \ell]^d \subset \Gamma_{n+1}'.$$

Noting that the success of $G_N$ only depends on gadgets in $\Gamma_n \cup \Gamma_{n+1}$, we have proved that $G_N$ is a subset of the block event in Lemma 6.1 which completes the proof of Lemma 6.1 and Theorem 1.
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