A NONSPECTRAL DENSE BANACH SUBALGEBRA OF THE IRRATIONAL ROTATION ALGEBRA

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(Communicated by Palle E. T. Jorgensen)

Abstract. We give an example of a dense, simple, unital Banach subalgebra $A$ of the irrational rotation $C^*$-algebra $B$, such that $A$ is not a spectral subalgebra of $B$. This answers a question posed by T. W. Palmer (Spectral algebras, Rocky Mountain J. Math. 22 (1992), 293–328).

If $A$ is a subalgebra of an algebra $B$ (both algebras over the complex numbers), we say that $A$ is a spectral subalgebra of $B$ if the quasi-invertible elements of $A$ are precisely the quasi-invertible elements of $B$ which lie in $A$. In the language of [3], this is equivalent to saying that $A$ is a spectral invariant subalgebra of $B$.

There are many known examples of dense unital Banach subalgebras of $C^*$-algebras which are not spectral. For example, see Example 3.1 of [3]. The example we give here is of interest because the Banach algebra is simple and, thus, answers Question 5.12 of [1] in the negative.

Recall that the irrational rotation algebra associated with an irrational real number $\theta$ is the $C^*$-crossed product of the integers $\mathbb{Z}$ with the commutative $C^*$-algebra of continuous functions on the circle $C(\mathbb{T})$, where $n \in \mathbb{Z}$ acts via $\alpha_n(\varphi)(z) = \varphi(z - n\theta)$, for $\varphi \in C(\mathbb{T})$ and $z \in \mathbb{T}$. Let $B = \mathbb{Z} \times C(\mathbb{T})$ denote this crossed product.

Let $A$ be the set of functions $F$ from $\mathbb{Z}$ to $C(\mathbb{T})$ which satisfy the integrability condition
$$\|F\|_A = \sum_{n \in \mathbb{Z}} e^{\|n\|} \|F(n)\|_{C(\mathbb{T})} < \infty,$$
where $\| \|_{\infty}$ denotes the sup norm on $C(\mathbb{T})$. Then $A$ is complete for the norm $\| \|_A$ and is a Banach algebra. The algebra $A$ is contained in $L^1(\mathbb{Z}, C(\mathbb{T}))$ with dense and continuous inclusion and, hence, is contained in $B$ with dense and continuous inclusion. Recall that the multiplication (in both $A$ and $B$) is given by
$$F \ast G(n, z) = \sum_{m \in \mathbb{Z}} F(n, z)G(n - m, z - m\theta), \quad F, G \in A, n \in \mathbb{Z}, z \in \mathbb{T}.$$
Let \( u_n = \delta_n \otimes 1 \in A \) denote the delta function at \( n \in \mathbb{Z} \) tensored with the identity in \( C(T) \). Then \( u_0 \) is the unit in both \( A \) and \( B \).

**Theorem 1.** The Banach algebra \( A \) is simple.

**Proof.** We imitate the argument of [2]. Define a continuous linear map \( P: A \to C(T) \subseteq A \) by \( P(F) = F(0) \). Note that \( \|P(F)\|_A \leq \|F\|_A \) for \( F \in A \). Let \( J \) be a closed two-sided ideal in \( A \), which is not equal to \( A \). Since \( \mathbb{Z} \) acts ergodically on \( T \), we know that \( C(T) \) has no nontrivial closed \( \mathbb{Z} \)-invariant ideals. Hence, \( J \cap C(T) = 0 \).

We show that \( P(J) = 0 \). It suffices to show that \( P(J) \subseteq J \). Let \( \epsilon > 0 \) and \( F \in A \). Let \( N \) be a sufficiently large integer for which

\[
\sum_{|n| > N} e^{|n|} \|F(n)\|_\infty < \epsilon.
\]

Define \( F_1 \in A \) by \( F_1(n) = 0 \) if \( |n| > N \), and \( F_1(n) = F(n) \) if \( |n| \leq N \). By the proof of Lemma 6 of [2], there exists unimodular functions \( \theta_1, \ldots, \theta_M \in C(T) \) such that

\[
P(F_1) = \frac{1}{M} \sum_{n=1}^{M} \theta_n^* F_1 \theta_n.
\]

(Here unimodular means that \( |\theta_i(z)| = 1 \) for each \( z \in T \) and \( i = 1, \ldots, M \).) Hence,

\[
(*) \quad \left\| P(F) - \frac{1}{M} \sum_{n=1}^{M} \theta_n^* F \theta_n \right\|_A \leq \|P(F - F_1)\|_A + \|F - F_1\|_A < 2\epsilon.
\]

Now if \( F \in J \), \( (*) \) shows that \( P(F) \) can be approximated arbitrarily closely by elements of \( J \). Since \( J \) is closed, this shows that \( P(F) \in J \). Hence, \( P(J) \subseteq J \) and \( P(J) = 0 \).

If \( P(F u_n) = 0 \) for all \( n \), then \( F(n) = 0 \) for all \( n \) and so \( F = 0 \). Since \( J \) is a two-sided ideal and \( P(J) = 0 \), we have \( P(J u_n) = 0 \) for all \( n \). Hence, \( J = 0 \) and \( A \) is simple. \( \square \)

**Theorem 2.** The Banach algebra \( A \) is not a spectral subalgebra of \( B \).

**Proof.** We construct an algebraically irreducible \( A \)-module which is not contained in any \( * \)-representation of \( B \) on a Hilbert space. By Corollary 1.5 of [3], it will follow that \( A \) is not a spectral subalgebra of \( B \).

Let \( E \) be the Banach \( A \)-module \( C(T) \) with sup norm and with (continuous) action of \( A \) given by

\[
(F \varphi)(z) = \sum_{n} F(n, z)e^n \varphi(z - n\theta), \quad \varphi \in E, F \in A, z \in T.
\]

We show that \( E \) is in fact algebraically irreducible. Let \( \varphi \in E \) be not identically equal to zero. Since the complex conjugate of \( \varphi \) is in \( A \), the algebraic span \( A\varphi \) contains \( |\varphi|^2 \), which we denote by \( \psi \). Note \( u_n \psi(z) = e^n \psi(z - n\theta) \). Since \( \theta \) is irrational and \( T \) is compact, there exists finitely many \( n_1, \ldots, n_k \in \mathbb{Z} \) such that the sum of \( u_n \psi \) from \( i = 1 \) to \( k \) never vanishes on \( T \). If \( \chi \) is this sum, then \( 1/\chi \) is in \( C(T) \subseteq A \), so \( 1 \in A\varphi \) and, hence, \( E = A\varphi \). This proves that \( E \) is algebraically irreducible.
It remains to show that no \( \ast \)-representation of \( B \) on a Hilbert space contains \( E \). But the action of \( Z \) on \( l \in E \) is given by \( u_n l = e^n l \). Clearly the Hilbert space could not have a unitary, or even isometric, action of \( Z \). □

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