IMPROVED LEMAITRE-TOLMAN MODEL AND THE MASS AND TURN-AROUND RADIUS IN GROUP OF GALAXIES II: THE ROLE OF DARK ENERGY

ANTONINO DEL POPolo
Dipartimento di Fisica e Astronomia, University of Catania, Viale Andrea Doria 6, 95125, Catania, Italy

and

Institute of Astronomy, Russian Academy of Sciences, Pyatnitskaya str. 48, 119017 Moscow, Russia

MAN HO CHAN
Department of Science and Environmental Studies, The Education University of Hong Kong, Tai Po, New Territories, Hong Kong

ABSTRACT

In this paper, we extend our previous study Del Popolo et al. (2021) on the Lemaitre-Tolman (LT) model showing how the prediction of the model changes when the equation of state parameter ($w$) of dark energy is modified. In the previous study, it was considered that dark energy was merely constituted by the cosmological constant. In this paper, as in the previous study, we also took into account the effect of angular momentum and dynamical friction ($J\eta$ LT model) that modifies the evolution of a perturbation, initially moving with the Hubble flow. As a first step, solving the equation of motion, we calculated the relationship between mass, $M$, and the turn-around radius, $R_0$. If one knows the value of the turn-around radius $R_0$, it is possible to obtain the mass of the studied objects.

As a second step, we build up, as in the previous paper, a relationship between the velocity, $v$, and radius, $R$. The relation was fitted to data of groups and clusters. Since the relationship $v - R$ depends on the Hubble constant and the mass of the object, we obtained optimized values of the two parameters of the objects studied. The mass decreases of a factor of maximum 25% comparing the $J\eta$ LT results (for which $w = -1$) and the case $w = -1/3$, while the Hubble constant increases going from $w = -1$ to $w = -1/3$.

Finally, the obtained values of the mass, $M$, and $R_0$ of the studied objects can put constraints to the dark energy equation of state parameter, $w$.

Keywords: Dwarf galaxies; galaxy clusters; modified gravity; mass-temperature relation

1. INTRODUCTION

Several independent observations, and data collected in the last two decades revealed that the universe expansion is accelerated. Observations of supernovae of type IA (SNIa) were the first to indicate the accelerated expansion trend (Perlmutter et al. 1998; Riess et al. 1998). This result has been confirmed by several subsequent analyses, like that of the small-scale anisotropies in temperature of the cosmic microwave background radiation (Planck Collaboration et al. 2016), and other data. The acceleration is interpreted as due to a medium with a negative equation of state (EoS), with EoS parameter $w = P/\rho$, often considered in the range $-1 < w < -1/3$. Nevertheless, the nature of dark energy is unknown, observations indicate that it constitutes now about 70% of matter-energy in the Universe. A plethora of models of dark energy have been introduced, starting from a cosmological constant (Carroll 2001), or a scalar field (Peebles & Ratra 2003).

In a previous paper (Del Popolo et al. 2021), we extended the so called Lemaitre-Tolman model (that we discuss in a while) to take account of the cosmological constant ($w = -1$), angular momentum, and dynamical friction. The model was dubbed $J\eta$LT model. In the present paper, we want to see how the results we discussed in the previous paper, that we will indicate with Paper I (Del Popolo et al. 2021) changes for different values of the EoS parameter $w$.

The mass and the mass-to-light ($M/L$) ratios of group of galaxies are sometimes obtained by means of the Virial theorem. This last is known to give reliable results only in the case the system studied are in dynamical equilibrium. This assumption is often not correct as shown by Niemi et al. (2007), who also showed that $\simeq 20\%$ of the studied groups were not gravitationally bound. Just to give an example, while in the past Huchra & Geller (1982), using the Virial theorem, found values of the mass-to-light ($M/L$) ratios of groups of the order of $\simeq 170M_\odot/L_{B,M_\odot}$. More recent measurements, using methods different from the Virial
theorem, have given much smaller results (e.g., $10 - 30 M_\odot/L_{B,M}$, Karachentsev (2005)).

Lynden-Bell (1981) and Sandage (1986) proposed an alternative approach to the virial theorem based on the Lemaitre-Tolman (LT) model (Lemaître 1933; Tolman 1934). The quoted method, is used to describe the evolution of a system, similarly to the case of the spherical collapse model (SCM). This model assumes that a system is divided in a series of shells of given radius containing a mass $M$. The spherical perturbation described by the spherical model, initially expands following the Hubble flow, until a maximum radius, dubbed turn-around radius, $R_0$, is reached. Later the system starts to collapse. The simplest LT model, taking into account only the gravitational potential energy, is referred to as SLT (standard LT model) to distinguish it from extension of the model, is characterized by a central region in equilibrium, surrounded by the region that expands to a maximum radius and then collapses. A group of galaxies dominated by a central one or a binary system is well described by the LT model.

In the simplest case, in which one considers only the effect of the gravitational potential (SLT model), as shown by Sandage (1986), Peirani & de Freitas Pacheco (2006, 2008) the mass is given by

$$M = \frac{\pi^2 R_0^3}{8G T_0^2} = 3.06 \times 10^{12} h^2 R_0^3 M_\odot,$$

where $T_0$ is the age of the universe, and $h$ is the Hubble constant in units of 100 km$^{-1}$/Mpc. This simple model was applied to the Local Group (Sandage 1986) and to the Virgo cluster (Hoffman et al. 1980; Tully & Shaya 1984; Teerikorpi et al. 1992). The first extension of the model, taking into account the effect of the cosmological constant was presented by Peirani & de Freitas Pacheco (2006, 2008). The model is dubbed in the following "modified LT model" (MLT). They applied the model to Virgo cluster, the pair M31-MW, M81, the Centaurus A-M83 group, the IC342/Maffei-I group, and the NGC 253 group. This model shows a different relation between the mass $M$, and $R_0$, namely

$$M = \frac{1.69945 R_0^3}{G T_0^2} = 4.22 \times 10^{12} h^2 R_0^3 M_\odot$$

Peirani & de Freitas Pacheco (2006, 2008), proposed also another method to calculate the mass. They built up a velocity-distance relationship, $v - R$, describing the kinematic status of the systems studied. If one can determine the velocities $v$, and distance $R$ for the members of the groups studied, one can obtain the mass, $M$, and the Hubble constant by means of a non-linear fit of the $v - R$ relation to the data. Del Popolo et al. (2021) extended the work of Peirani & de Freitas Pacheco (2006, 2008) by taking into account the effect of angular momentum (JLT model) and dynamical friction (JηLT model)\(^1\).

In the present paper, we are interested in how the change of the EoS parameter $w$ influences the result of the $J_\eta$ LT model.

As a first step, we will solve the equations of motion of the system to see how the $M - R_0$ relation changes with $w$. Then, we will build up the $v - R$ relation, as done in Peirani & de Freitas Pacheco (2006, 2008), Del Popolo et al. (2021), and fit it to the data of the Virgo cluster, the pair M31-MW, M81, the Centaurus A-M83 group, the IC342/Maffei-I group, and the NGC 253 group.

The paper is organized as follows. Section 2 introduces the model, and shows how to solve it. In Section 3, we find the velocity-radius relations for the case $w = -2/3$ ($J_\eta$LT$-$2/3 model), and the case $w = -1/3$ ($J_\eta$LT$-$1/3 model). In Section 4, the $v - R$ relation was applied to groups and clusters of galaxies. In Section 6, we put constraints on the dark energy equation of state parameter. Section 7 is devoted to conclusions.

2. MODEL

As previously reported, the SCM introduced by Gunn & Gott (1972) is a simple method to study analytically the non-linear evolution of perturbations of dark matter (DM) and dark energy (DE). It describes the evolution of a spherically symmetric perturbation which is initially following the Hubble flow, and then detaches from it and collapses forming a structure. In the Gunn & Gott (1972) model was taken into account only the gravitational potential of the mass, and matter moves in a radial fashion (Gunn & Gott 1972; Gunn 1977). SCM was improved in several papers adding the effect of the cosmological constant (Lahav et al. 1991), tidal angular momentum (Peebles 1969; White 1984), random angular momentum (Ryden & Gunn 1987; Gurevich & Zybvin 1988a,b; White & Zaritsky 1992; Sikivie et al. 1997; Nusser 2001; Hiotelis 2002; Le Delliou & Henriksen 2003; Ascasibar et al. 2004; Williams et al. 2004; Zukin & Bertschinger 2010), and dynamical friction (Antonuccio-Delogu & Colafrancesco 1994; Del Popolo 2009). The model has been used to study a series of issues. Del Popolo et al. (2013a,b) studied the effects of shear and rotation for smooth DE models, while Pace et al. (2014a) studied them in clustering DE cosmologies, and Del Popolo et al. (2013c) in Chaplygin cosmologies. Several authors (Bernaldeau 1994; Bardeen et al. 1986; Ohta et al. 2003, 2004; Basilakos 2009; Pace et al. 2010; Basilakos et al. 2010) studied the SCM with

\(^1\) The effect of the specific angular momentum $J$, and dynamical friction on the turn-around, the threshold of collapse, the clusters of galaxies structure and evolution, their mass function, and their mass-temperature relation, have been studied in several papers (Del Popolo & Gambera 1998, 1999, 2000; Del Popolo 2006a,b,c; Del Popolo et al. 2017, 2019, 2020).
negligible DE perturbations, and others (Mota & van de Bruck 2004; Nunes & Mota 2006; Abramo et al. 2007, 2008, 2009b, a; Creminelli et al. 2010; Basse et al. 2011; Batista & Pace 2013) took account of DE fluid perturbation. The main parameters of the SCM are mass independent, but if the role of shear and rotation are taken into account they become mass dependent (Pace et al. 2010; Del Popolo et al. 2013b).

The equation of motion of the system are given following Peebles (1993), Bartlett & Silk (1993), Lahav et al. (1991), Del Popolo & Gambera (1998), Del Popolo & Gambera (1999), Del Popolo (2000c) by the equation given in Del Popolo et al. (2021):

$$\frac{dv}{dt} = -\frac{GM}{R^2} + \frac{J^2}{R^3} - \frac{1 + 3w}{2} \Omega_\Lambda H_0^2 \left(\frac{a_0}{a}\right)^{3(1+w)} R - \eta \frac{dR}{dt},$$  \hspace{1cm} (3)

where \( J = \frac{L}{M} \) is the specific angular momentum, \( L \) is the angular momentum and taken into account ordered angular momentum generated by tidal forces and random angular momentum (see Appendix C.2 of Del Popolo (2009)), \( \Omega_\Lambda = \frac{\dot{\rho}}{\rho_\Lambda}, \) where \( \rho_c \) is the critical density, \( \rho_\Lambda \) is the density related to the cosmological constant \( \Lambda, \eta \) (explicitly given in Del Popolo (2009) (Appendix D, Eq. D5)) the dynamical friction coefficient per unit mass, where \( w \) is the DE equation of state (EoS) parameter. DE is modeled by a fluid with an EoS given by \( P = w \rho, \) where \( \rho \) is the energy density. \( a \) is the expansion parameter. Eq.(3) satisfies the equation

$$H = \frac{\dot{a}}{a} = H_0 \sqrt{\Omega_m \left(\frac{a_0}{a}\right) + \Omega_\Lambda \left(\frac{a_0}{a}\right)^{3(1+w)}}.$$  \hspace{1cm} (4)

Assuming that \( J = kR^\alpha, \) with \( \alpha = 1, \) in agreement with Bullock et al. (2001)\(^2\), and \( k \) constant, in terms of the variables \( y = R/R_0, t = x/H_0, \) Eq.(3) can be written as

$$\frac{d^2 y}{dx^2} = -\frac{A}{2y^2} \left(1 + 3w/2 \right) \Omega_\Lambda y \left(\frac{a_0}{a}\right)^{3(1+w)} + K_j \frac{y}{y} - \eta \frac{dy}{dx},$$  \hspace{1cm} (5)

where \( K_j = k \frac{1}{(H_0 R_0)^2}, A = \frac{2GM}{H_0^2 R_0^3}. \) In terms of \( \xi = a/a_0, \) and \( x, \) Eq. (4) can be written as

$$\frac{d\xi}{dx} = \sqrt{\frac{\Omega_m}{\xi} + \frac{\Omega_\Lambda}{\xi^{1+3w}}}.$$ \hspace{1cm} (6)

or separating the variables, as

$$x = \int_0^1 \frac{d\xi}{\sqrt{\frac{\Omega_m}{\xi} + \frac{\Omega_\Lambda}{\xi^{1+3w}}}}.$$ \hspace{1cm} (7)

Eq.(5) has a first integral, given by

$$u^2 = \left(\frac{dy}{dx}\right)^2 = \frac{A}{y} - \frac{1 + 3w}{2} \Omega_\Lambda y^2 \left(\frac{a_0}{a}\right)^{3(1+w)} + 2K_j \log y - \frac{2}{H_0} \int \left(\frac{dy}{dx}\right)^2 \ dx + K$$ \hspace{1cm} (8)

where \( K = \frac{2E}{(H_0 R_0)^2}, \) and \( E \) is the energy per unit mass of a shell.

Eq.(5), and Eq.(6) were solved as described in Peirani & de Freitas Pacheco (2006, 2008). This can be done in a couple of ways. One way is to get the value of scale parameter and the corresponding time for a given redshift, from Eq. (6). Moreover, the gravitational term dominates at high redshift. By means of a Taylor expansion the initial conditions can be obtained. The parameter \( A, \) which is the goal of the calculation, is varied until the condition \( \frac{dy}{dx} = 0, \) and \( y = 1 \) are satisfied.

The second way to get \( A, \) is based on the use of the equation for the velocity (Eq.(8)), as described in Del Popolo et al. (2021).

In Del Popolo et al. (2021), we indicated the general solution of Eq. (5), the one with \( w = -1, \) \( \Omega_\Lambda = 0.7, \) \( \Omega_m = 0.3, \) \( K_j = 0.78, \) \( \eta/H_0 = 0.5, \) with \( J_\eta LT. \) As shown in Del Popolo et al. (2021), we got \( A = 6.05, \) or \( M = \frac{28111 R_0^3}{GT_0^2} = 6.9843 \times 10^{12}h^2R_0^3M_\odot. \)

We solved Eq.(5), and Eq.(6), also for the case \( w = -2/3 (\text{that we dub} J_\eta LT_{-2/3}), \) and \( w = -1/3 (\text{that we dub} J_\eta LT_{-1/3}). \)

In the case \( J_\eta LT_{-2/3}, A = 5.83, \) corresponding to \( M = 6.73 \times 10^{12}h^2R_0^3M_\odot, \) and in the case \( J_\eta LT_{-1/3}, A = 5.32, \) corresponding to \( M = 6.14 \times 10^{12}h^2R_0^3M_\odot. \)

\(^2\) In that paper \( \alpha = 1.1 \pm 0.3 \)
Figure 1. Evolution of shells with different energies ($K$) in the $J\eta LT_{-2/3}$ model. The green, red, and blue lines correspond to $K = -4.8$, $K = -6.18$, and $K = -6.5$, respectively. The vertical black line is the time $x = 0.903$, at which $R = R_0$.

Figure 2. Velocity profile for different $w$. The left panel shows the $J\eta LT$ model ($w = -1$). The central panel, the $J\eta LT_{-2/3}$ case, and the right panel, the $J\eta LT_{-1/3}$ case. The data points are solutions of Eq. (5) for different values of $K$ (see Eq. (8)). In Paper I, and its Fig. 2, and in the text of the present paper, there is a deeper discussion. The lines are fit to the data, in the form $v = -b/y^n + by$, whose parameters can be found in Table 1.

In Fig. 1, we plotted three solutions of the case $J\eta LT_{-2/3}$ for different values of $K$. The red line represents the case $K = -6.18$, and represents the solution that just reached turn-around. The blue line has $K = -6.5$, and reached turn-around in the past. The value of $K = -5.327$ is a threshold value. For $K < -5.327$, turn-around and collapse are allowed, while are not allowed for $K > -5.327$, as the case of the green line, having $K = -4.8$. The vertical black line is the time $x = 0.903$, at which $R = R_0$.

3. DETERMINATION OF THE VELOCITY-RADIUS RELATION

As we already reported, Peirani & de Freitas Pacheco (2006, 2008), and Del Popolo et al. (2021) found a relation between the velocity and the radius, $v - R$, that was fitted to the data of some groups of galaxies. The relation depends from the Hubble constant, $H$, and the mass, $M$, of the system. After $H$, and $M$ are obtained it is also possible to obtain the turn-around radius $R_0$, by means of the relation, already written, $A = \frac{2GM}{bH}$. As discussed, the coefficient $A$ is obtained solving the equations of motions. We already discussed how to obtain the $v - R$ relation in Del Popolo et al. (2021), and it was also discussed in Peirani & de Freitas Pacheco (2006, 2008). For reader’s convenience, we recall how we get the quoted relation. Let’s consider Fig. 1. The plot represents three solutions that are obtained from Eq. (5), and Eq. (6), for three different values of $K$. The black vertical line in Fig. 1 corresponds to $x = 0.903$, as can be obtained from Eq. (7). The intersection of the black line with each curve $y(x)$, corresponding to a given $K$, gives a succession $y_K(0.903)$. Moreover, solving Eq. (5), and Eq. (6), one can obtain the velocity $v(x)$, and again a succession $v_k(0.903)$. We will get a couple of value ($y, v$) for each intersection of the vertical line with the curves, and considering all the intersections, we have $y_k(0.903)$, $v_k(0.903)$. In Fig. 2, we plotted in red the succession of points in the cases $J\eta LT$ (left panel), $J\eta LT_{-2/3}$ (central panel), and $J\eta LT_{-1/3}$ (right panel). The series of points can be fitted with a relation of the form $v = -b/y^n + by$. In Fig. 2, the fit is represented by the brown lines. The parameters of the fit, $b$, and $n$ are reported in Table 1, for the three cases considered ($J\eta LT$ ($w = -1$), $J\eta LT_{-2/3}$, and $J\eta LT_{-1/3}$). The fit $v = -b/y^n + by$ can be written in terms of the physical units as

$$v(R) = -bH_0R_0 \left( \frac{R_0}{R} \right)^n + bH_0R. \quad (9)$$
Recalling that \( R_0 = \left( \frac{2GM}{H_0^2} \right)^{\frac{1}{3}} \), substituting in the previous equation, we get

\[
v(R) = -bH_0 \left( \frac{2GM}{AH_0^2} \right)^{\frac{3}{4}} + bH_0 R.
\] (10)

Using the parameters in Table 1, in the case \( J\eta LT_{-2/3} \), we have

\[
v(R) = -\frac{0.69H_0}{R^{0.82}} \left( \frac{GM}{H_0^2} \right)^{0.61} + 1.32H_0 R,
\] (11)

In the case of the \( J\eta LT_{-1/3} \) case, we have:

\[
v(R) = -\frac{0.74H_0}{R^{0.75}} \left( \frac{GM}{H_0^2} \right)^{0.58} + 1.29H_0 R,
\] (12)

and in the case \( J\eta LT \) (\( w = -1 \)), which was already studied in Paper I, we have

\[
v(R) = -\frac{0.59H_0}{R^{0.91}} \left( \frac{GM}{H_0^2} \right)^{0.64} + 1.34H_0 R.
\] (13)

Notice that this last \( v - R \) relation (Eq. (13)) is slightly different from that of Paper I, where the first term was 0.66, and here 0.59. We checked again all the fits and noticed that in the case \( w = -1 \) there was a small discrepancy. This produces also a change of the mass values and \( R_0 \) in Table (2) for the case \( J\eta LT \).

As already reported, in Fig. 2, we plot, from left to right, the velocity profiles of the cases \( J\eta LT \) (left panel), \( J\eta LT_{-2/3} \) (central panel), and \( J\eta LT_{-1/3} \), using adimensional variables.

All the previous equations satisfy the condition \( v(R_0) = 0 \). In the following, we will apply Eq.(11), Eq.(12) to some groups of galaxies and clusters. The parameters of the different models that were described in this paper, plus the case \( J\eta LT \) studied in paper I are summarized in Table 1. The first line corresponds to the \( J\eta LT \) model. The second line to the \( J\eta LT_{-2/3} \) model, and, the last line to the \( J\eta LT_{-1/3} \) case.

| model   | \( \eta/H_0 \) | \( K \)  | \( b \)  | \( n \) | \( A \) |
|---------|----------------|--------|--------|-------|-------|
| \( J\eta LT \) | 0.5            | 0.78   | 1.34   | 0.91  | 6.05  |
| \( J\eta LT_{-2/3} \) | 0.5            | 0.78   | 1.32   | 0.82  | 5.83  |
| \( J\eta LT_{-1/3} \) | 0.5            | 0.78   | 1.29   | 0.75  | 5.32  |

Table 1. The constant \( A \), and the fitting parameters \( b \), and \( n \) of the velocity-distance \( (v - R) \) relations, for the \( J\eta LT \), the \( J\eta LT_{-2/3} \), and \( J\eta LT_{-1/3} \) models.

Table 1, as well as Fig.2 shows that with decreasing \( w \), from \( w = -1 \) of the case \( J\eta LT \) to \( w = -2/3 \) of the case \( J\eta LT_{-2/3} \), and \( w = -1/3 \) of the case \( J\eta LT_{-1/3} \) the values of the parameter \( A = \frac{2GM}{H_0^2} \) decreases, and the velocity profile flattens. For a fixed value of \( R_0 \), and \( H_0 \), the mass of the structures decreases.

4. APPLICATION TO DATA FROM NEAR GROUPS AND ClUSTERS OF GALAXIES

As we already discussed, we obtained a \( v - R \) relation for the case \( J\eta LT_{-2/3} \), (Eq. 11), \( J\eta LT_{-1/3} \) (Eq. (12)), and in paper I that for the case \( J\eta LT \) (Eq. (13)). These relations depend on mass, \( M \), and the Hubble constant, \( H \). We will now fit the quoted relations to near groups, and to the Virgo cluster. To perform the fit, we need for each galaxy its distance and velocity with respect to the center of mass of the system studied. The data were obtained, as described in Peirani & de Freitas Pacheco (2006, 2008), Del Popolo et al. (2021), by Peirani & de Freitas Pacheco (2006, 2008). As done in Paper I, we will fit these data with the \( J\eta LT_{-2/3} \), (Eq. 11), and \( J\eta LT_{-1/3} \) (Eq. (12)).

Fig.3 plots the \( v - R \) relationships for the groups studied. From top to bottom we have that the top left panel represents the M31-MW group, the top right panel the M81 group, the central left panel the NGC 253 group, the central right panel the IC 342 group, the bottom left panel the Cen/M83 group, and the bottom right panel the Virgo cluster. The red diamonds represent the data from Peirani & de Freitas Pacheco (2006, 2008). The black line is the fit to the data. We represented only the case \( w = -2/3 \), because the other cases (\( w = -1 \), and \( w = -1/3 \)), are almost indistinguishable from the case \( w = -2/3 \) with the

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3 For precision’s sake, in the present paper we recalculated also the \( v - R \) relation, and performed the fit to the quoted groups for the \( J\eta LT \) case.
Table 2. Values of the parameters of the examined groups. The rows 1-3 represent the value of the Hubble constant in units of $100 \text{ km s}^{-1} / \text{Mpc}$ for the cases $J_\eta JLT_{-2/3}$, $J_\eta JLT_{-1/3}$, and $J_\eta LT$. The masses in units of $10^{12} M_\odot$, and for the same cases, are presented in rows 4-6, while in the rows 7-9 the values of the turn-around radius, $R_0$, in Mpc. The rows 10-12 give the velocity dispersion obtained as the other parameters by fitting the data to the $v - R$ relation.

| M31/MW | M81 | NGC 253 | IC 342 | CenA/M83 | Virgo |
|-------|-----|--------|--------|----------|-------|
| $h(J_\eta JLT_{-1/3})$ | 0.75 ± 0.04 | 0.7 ± 0.04 | 0.67 ± 0.06 | 0.60 ± 0.10 | 0.59 ± 0.04 | 0.69 ± 0.09 |
| $h(J_\eta JLT_{-2/3})$ | 0.72 ± 0.04 | 0.67 ± 0.04 | 0.65 ± 0.06 | 0.57 ± 0.10 | 0.57 ± 0.04 | 0.64 ± 0.08 |
| $h(J_\eta LT)$ | 0.69 ± 0.04 | 0.65 ± 0.04 | 0.63 ± 0.05 | 0.55 ± 0.10 | 0.55 ± 0.04 | 0.59 ± 0.09 |
| $M(J_\eta JLT_{-1/3}) [10^{12} M_\odot]$ | 3.27 ± 0.50 | 1.40 ± 0.10 | 0.21 ± 0.10 | 0.28 ± 0.10 | 2.81 ± 0.50 | 1681 ± 200 |
| $M(J_\eta JLT_{-2/3}) [10^{12} M_\odot]$ | 3.72 ± 0.50 | 1.53 ± 0.10 | 0.24 ± 0.15 | 0.312 ± 0.12 | 3.17 ± 0.50 | 1760 ± 200 |
| $M(J_\eta LT) [10^{12} M_\odot]$ | 4.31 ± 0.40 | 1.68 ± 0.10 | 0.33 ± 0.10 | 0.35 ± 0.10 | 3.65 ± 0.40 | 1844 ± 200 |
| $R(J_\eta JLT_{-1/3}) [\text{Mpc}]$ | 0.99 ± 0.10 | 0.78 ± 0.05 | 0.43 ± 0.10 | 0.51 ± 0.09 | 1.10 ± 0.08 | 8.35 ± 0.80 |
| $R(J_\eta JLT_{-2/3}) [\text{Mpc}]$ | 1.02 ± 0.10 | 0.80 ± 0.05 | 0.44 ± 0.10 | 0.52 ± 0.09 | 1.14 ± 0.08 | 8.62 ± 0.80 |
| $R(J_\eta LT) [\text{Mpc}]$ | 1.08 ± 0.10 | 0.83 ± 0.05 | 0.48 ± 0.10 | 0.55 ± 0.09 | 1.20 ± 0.08 | 9.12 ± 0.80 |
| $\sigma(J_\eta JLT_{-1/3}) [\text{km/s}]$ | 38.29 | 53.19 | 44.75 | 33.41 | 45.96 | 352.84 |
| $\sigma(J_\eta JLT_{-2/3}) [\text{km/s}]$ | 38.52 | 53.90 | 44.17 | 34.17 | 45.91 | 353.69 |
| $\sigma(J_\eta LT) [\text{km/s}]$ | 38.8 | 54.77 | 45.9 | 34.48 | 44.81 | 355.1 |

The Karachentsev et al. (2002a) data were fitted with Eq. (11) (case $J_\eta LT_{-2/3}$), and Eq. (12) (case $J_\eta LT_{-1/3}$). The results are shown in Table 2. By means of the SLT model Karachentsev et al. (2002a) found a mass $1.5 \times 10^{12} M_\odot$, and a turn-around radius of $0.94 \pm 0.1$ Mpc. Peirani & de Freitas Pacheco (2006), by means of the MLT model found $(2.5 \pm 0.7) \times 10^{12} M_\odot$, $R_0 = 1.0 \pm 0.1$ Mpc, and $h = 0.74 \pm 0.04$. The value of the mass of Peirani & de Freitas Pacheco (2006) is larger than that of Karachentsev et al. (2002a), that used the SLT model. As already discussed in Paper I, the class of the LT models give a higher masses, and smaller $h$ if the effect of the cosmological constant, angular momentum, and other effects which contribute with positive terms in the equation of motion are taken into account. The values of $R_0$, in our cases ($J_\eta LT_{-2/3}$, $J_\eta LT_{-1/3}$) are in agreement, within the estimated uncertainties, with estimate reported in Peirani & de Freitas Pacheco (2006). Concerning the values of $h$, it is in agreement to that of Peirani & de Freitas Pacheco (2006) in all cases, while the mass $M$ obtained by Peirani & de Freitas Pacheco (2006) is slightly smaller. The errors reported, come from the fitting procedure.

4.1. M31-MW

The M31 group has been studied by many authors. For instance Karachentsev et al. (2002a), Karachentsev et al. (2002b, 2006) found $R_0 = 0.89 \pm 0.05$ Mpc, and $M = (1.03 \pm 0.17) \times 10^{12} M_\odot$. Their value of $R_0$ is in agreement to that of our models, while our mass is larger. Peirani & de Freitas Pacheco (2008) found $M = (0.92 \pm 0.24) \times 10^{12} M_\odot$, smaller than our cases and $h = 0.67 \pm 0.04$, in agreement with our cases.

4.2. The M81 group

The M81 group has been studied by many authors. For instance Karachentsev et al. (2002a), Karachentsev et al. (2002b, 2006) found $R_0 = 0.89 \pm 0.05$ Mpc, and $M = (1.03 \pm 0.17) \times 10^{12} M_\odot$. Their value of $R_0$ is in agreement to that of our models, while our mass is larger. Peirani & de Freitas Pacheco (2008) found $M = (1.3 \pm 1.8) \times 10^{11} M_\odot$, and $h = 0.63 \pm 0.06$, both in agreement with our estimates.

4.3. The NGC253 group

Karachentsev et al. (2003b) obtained $R_0 = 0.7 \pm 0.1$ Mpc, and $M = (5.5 \pm 2.2) \times 10^{11} M_\odot$, both larger than our estimates. Peirani & de Freitas Pacheco (2008) found $M = (1.3 \pm 1.8) \times 10^{11} M_\odot$, and $h = 0.63 \pm 0.06$, both in agreement with our estimates.

4.4. The IC342 group

Karachentsev et al. (2003a) found that $R_0 = 0.9 \pm 0.1$, and $M = (1.07 \pm 0.33) \times 10^{12} M_\odot$, both larger than our estimates. Peirani & de Freitas Pacheco (2008) found $M = (2.0 \pm 1.3) \times 10^{11} M_\odot$, $R_0 (\approx 0.53$ Mpc), and $h = 0.57 \pm 0.10$. Our values of mass, $R_0$, and $h$ agree with Peirani & de Freitas Pacheco (2008) estimates.

4.5. The CenA/M83 group

Karachentsev et al. (2002c, 2007) studied this group taking into account the effect of cosmological constant, and found $R_0 = 1.55 \pm 0.13$ Mpc, and $M = (6.4 \pm 1.8) \times 10^{12} M_\odot$, both larger than our estimates. Peirani & de Freitas Pacheco (2008), found values 3-4 times smaller ($M = (2.1 \pm 0.5) \times 10^{12} M_\odot$), and $h = 0.57 \pm 0.04$. Both the values of $M$, and $h$ that we found are in agreement with Peirani & de Freitas Pacheco (2008).
Figure 3. Velocity-distance plots for the groups of galaxies studied. The top left panel represents the M31-MW group, the top right panel the M81 group, the central left panel the NGC 253 group, the central right panel the IC 342 group, the bottom left panel the Cen/M83 group, and the bottom right panel the Virgo cluster. The red diamonds represent the data from Peirani & de Freitas Pacheco (2006, 2008). The black line is the fit to the data obtained with the model $J\eta LT_{-2/3}$.

4.6. The Virgo cluster

Several authors studied this well known cluster with different methods. Hoffman et al. (1980), Fouque et al. (2001) used the SLT model, Tully & Shaya (1984) used the Virial theorem, obtaining masses smaller than $10^{15}M_{\odot}$. By converse, Fouque et al. (2001) found a larger value ($1.3 \times 10^{15}M_{\odot}$). Peirani & de Freitas Pacheco (2006), by means of the MLT model, got $M = (1.10 \pm 0.12) \times 10^{15}M_{\odot}$, smaller than our estimates, $h = 0.65 \pm 0.09$, and $R_0 = 8.6 \pm 0.8$ Mpc, both in agreement with our estimates.

From the previous discussion, the estimates of mass, $M$, turn-around, $R_0$, and Hubble constant, are usually in agreement with those of Peirani & de Freitas Pacheco (2006, 2008). From $w = -1$ ($J\eta LT$ case) to $w = -1/3$ ($J\eta LT_{-1/3}$ case) the value of the Hubble constant, in units of $100 \text{ kms}^{-1}/\text{Mpc}$, $h$, increases, and the mass, $M$, decreases (see Table 2). As discussed in Paper I, in the $J\eta LT$ case ($w = -1$), some of the values of $h$ are smaller than the range of values in which the Hubble constant is constrained. From two decades ago to now the constraints on the Hubble constant has changed from $72^{+9}_{-8}$ km/Mpc s, to the
range $67 - 75$ km/Mpc s (Freedman et al. 2019). Other constraints from gravitational wave physics give $70.3_{-5.0}^{+5.3}$ km/Mpc s (Hotokezaka et al. 2019), and smaller range are obtained with DES + BAO + BBN, $67.4_{-1.2}^{+1.1}$ km/Mpc s, and $67.5 \pm 1.1$ km/Mpc s (Schöneberg et al. 2019). The smaller value of all the previous constraints is $64$ km/Mpc s, which disagrees with the prediction of the $J\eta LT$ for CenA/M83. The predictions of $J\eta LT_{-2/3}$ and $J\eta LT_{-1/3}$ disagree slightly with the ranges indicated, only in the case of CenA/M83. As discussed in Paper I, this issue could be due to non completeness of the data used in 2008 by Peirani & de Freitas Pacheco (2008)\footnote{This incompleteness is confirmed by the fact that a significant amount of faint dwarf galaxy candidates were discovered by means of a survey of the Centaurus group performed few years ago (Müller et al. 2017).}

![Figure 4](image-url)  
**Figure 4.** $v - R$ relationship for the $J\eta LT$ model (red curve), $J\eta LT_{-2/3}$ (green curve), and $J\eta LT_{-1/3}$ (black curve).

In Fig. (5), we plot the $v - R$ relations for the $J\eta LT$ case (red line), the $J\eta LT_{-2/3}$ case (green line), and the $J\eta LT_{-1/3}$ case (black line). The plot shows that for distance smaller than $R_0$, the $J\eta LT$ case has larger negative velocities with respect to $J\eta LT_{-2/3}$ case, and this last larger negative velocities with respect to the $J\eta LT_{-1/3}$ case. This means that turn-around happens before in the case $J\eta LT$, then in the $J\eta LT_{-2/3}$ case, and finally in the $J\eta LT_{-1/3}$ case.

5. COMPARISON OF THE MODEL WITH OTHER RESULTS IN LITERATURE

In this section, we want to compare the results of the model described in Section 2 with other results focusing on the effect of DE on structure formation. For example, Kuhlen et al. (2005) studied the effect of DE on the density profile of DM haloes, considering models with constant equation of state models. They found that larger values of $w$ give rise to higher concentrations and higher halo central densities. This is due to the fact that collapse happens earlier and because haloes have higher virial densities. Similarly, Klypin et al. (2003) studied the properties of dark matter haloes in a variety of DE models. They also found that DE halos are denser than those in $\Lambda CDM$ because the DE haloes collapse earlier. This conclusion is in agreement with our model. As shown in Fig. 4, the turn-around and collapse happens earlier in larger $w$ cases.

Here, in Fig. 5, we compare our results with the non-linear overdensity at collapse, $\Delta_{\text{vir}}$, and the linear overdensity at collapse, $\delta_c$, in quintessence cosmologies, obtained by Kuhlen et al. (2005).

In the top panel of Fig. 5, we plot the non-linear overdensity at collapse, $\Delta_{\text{vir}}$, in terms of redshift $z$. The solid, dashed and dotted lines represent respectively the case $w = -1$ in the Kuhlen et al. (2005) paper, $w = -0.75$ in the Kuhlen et al. (2005) paper and $w = -0.50$ in the Kuhlen et al. (2005) paper. In all cases, the dot dashed line represents that in our model. In the bottom panel, we plot the linear overdensity at collapse, $\delta_c$, as a function of redshift. The lines style is as that for the top panel.

As we told, the collapse happens earlier for larger values of $w$. The increase in $\Delta_{\text{vir}}$ for larger $w$ results primarily from later collapse redshifts. The growth of structure is affected by the DE, more rapid for high values of $w$ (as shown in Fig. 4).

Baushev (2020), studied the Hubble stream near a massive object, while Baushev (2019), gave an analytic solution for the relation between the mass of group and the turn-around radius $R_0$ taking into account of DE. He found that

$$M = 2.278 \times 10^{12} \times \left( \frac{R_0}{1 \text{Mpc}} \right)^3 \left( \frac{H_0}{73 \text{km/s/Mpc}} \right)^2$$

(14)

In his papers he did not take account of shear and vorticity. As we showed in Del Popolo & Chan (2020), the angular momentum can be expressed in terms of shear and vorticity (see Eq. 12). Then, in order to compare our result to that of Baushev (2019) we neglect angular momentum and dynamical friction in our Eq. (3). Similarly, to compare our result to that of Kuhlen et al. (2005), we used the parameters and the assumptions they used.
Figure 5. Plot of the linear and non-linear overdensities at collapse in quintessence cosmologies. Top panel: $\Delta_{\text{vir}}$, the non-linear overdensity at collapse in terms of redshift $z$. The solid, dashed and dotted lines represent respectively the case $w = -1$ in the Kuhlen et al. (2005) paper, $w = -0.75$ in the Kuhlen et al. (2005) paper and $w = -0.50$ in the Kuhlen et al. (2005) paper, while the dot dashed lines represent the corresponding cases in our model. Bottom panel, $\delta_c$, the linear overdensity at collapse, as a function of redshift. The lines style is as that for the top panel.

So, as discussed in Del Popolo et al. (2021), and Eq. (2), in the case dynamical friction, shear and vorticity are neglected, the relation between mass and $R_0$ in Eq. (2), that can be written in terms of an Hubble constant equal to $73\text{km/s/Mpc}$ as

$$M = 2.25 \times 10^{12} (\frac{R_0}{1\text{Mpc}})^3 (\frac{H_0}{73\text{km/s/Mpc}})^2$$

which is in agreement with Eq. (14) of Kuhlen et al. (2005).

6. CONSTRAINTS ON THE DM EOS PARAMETER

The turn-around, as already discussed, is the shell of the structure at which the velocity of expansion is zero. It is the region of space that delimits the region expanding with Hubble flow from that which will recollapse. Because of this particularity, several authors have proposed it as a promising way to test cosmological models (Lopes et al. 2018). It has been used to study dark energy models, to disentangle between those models, the $\Lambda$CDM model, and modified gravity models (Pavlidou & Tomaras 2014; Pavlidou et al. 2014; Faraoni et al. 2015; Bhattacharya et al. 2017; Lopes et al. 2018, 2019).

In the majority of studies, the turn-around has been treated in the framework of General Relativity, or modified theories of gravity, as a geometric quantity. In real structures, the structure, and dimension of $R_0$ depends from several quantities like angular momentum, dynamical friction, or dark energy. It is then necessary to use a more physical model than those published.

$R_0$ is also modified by the presence of vorticity, and shear in the equation of motion. In order to see how $R_0$ is modified, in Del Popolo et al. (2020), we used an extended spherical collapse model (ESCM) (Del Popolo 2013; Del Popolo et al. 2013a; Pace et al. 2014b; Mehrabi et al. 2017; Pace et al. 2019). In Del Popolo & Chan (2020), we showed how taking also account of dynamical friction changed $R_0$.

In Del Popolo et al. (2020), Del Popolo & Chan (2020), we also showed how is possible to put some constraints on the DE EoS parameter $w$, by means of the $M - R_0$ plane.
Figure 6. Mass-radius relation of stable structures for different $w$. The solid lines, obtained by Pavlidou et al. (2014), from bottom to top represent $w = -2.5$ (red solid line), -2 (pink solid line), -1.5 (blue solid line), -1 (black solid line), and -0.5 (solid green line). The dashed lines correspond to the solid lines, but they were obtained taking the effect of the cosmological constant, angular momentum, and dynamical friction (Del Popolo & Chan 2020). The dots with error bars come from the case $J_\eta LT^{-1/3}$, and are taken from Table 2.

| Stable structure | range of $w$ |
|------------------|--------------|
| M81              | $w \geq -2$  |
| IC342            | $w \geq -1$  |
| NGC253           | $w \geq -0.9$|
| CenA/M83         | $w \geq -1.6$|
| Local Group      | $w \geq -2.2$|
| Virgo            | $w \geq -2.5$|

The constraints on $w$ depends on the estimated values of the mass and $R_0$ of galaxies, groups, and clusters.

In Del Popolo et al. (2020), Del Popolo & Chan (2020), we calculated the mass, and turn-around for several groups, and also studied how the $M - R_0$ relation is modified by shear, rotation, and dynamical friction.

Here, we recalculate the constraints shown in Del Popolo et al. (2020), Del Popolo & Chan (2020) by means of the revised value of mass, $M$, and $R_0$ presented in this paper.

Fig.6 plots the mass-radius relation, $M - R_0$, of stable structures for different $w$. The solid lines from bottom to top represent $w = -2.5$ (red solid line), -2 (pink solid line), -1.5 (blue solid line), -1 (black solid line), and -0.5 (solid green line) (see Pavlidou et al. (2014)). The dashed lines are the $M - R_0$ lines for the same values of $w$ described obtained using the model in Del Popolo & Chan (2020) considering the effect of shear, vorticity, angular momentum, dynamical friction, and dark energy only in the case $w = -1$. The dots with error bars, are data obtained in the previous sections, and reported in Table 2 (case $J_\eta LT^{-1/3}$).

Table 3 reproduces the constraints to $w$. They differ from previous constraints as those obtained by Pavlidou & Tomaras (2014), Pavlidou et al. (2014).

In order to explain the difference between the results in Pavlidou & Tomaras (2014), Pavlidou et al. (2014), or Del Popolo & Chan (2020), we recall the models used in those papers. As already discussed, in the majority of studies found in literature the turn-around is obtained starting from a metric in the framework of General Relativity, or modified theories of gravity. Pavlidou et al. (2014) started from the metric in their Eq. 3.1. They derived the equation of evolution of the Universe. They wrote the equations in the case of a spherically symmetric configuration taking into account dark energy. They then determined the region in which matter can decouple from expansion and collapse. Pavlidou & Tomaras (2014) followed a similar path. Turn-around is obtained using geometry, excluding fundamental effects like tidal interaction, random angular momentum, dynamical friction, that are
related to the content in baryons of the system. Real structures, like clusters of galaxies, are much more complicated than what the methods of Pavlidou et al. (2014), and Pavlidou & Tomaras (2014) can describe. It is then necessary to use a more physical model than those published. This is the reason why in Del Popolo et al. (2020), Del Popolo & Chan (2020) we used an extended spherical collapse model. The model takes into account shear, vorticity, dynamical friction, and DE. The effect of shear, vorticity, and dynamical friction, not taken into account in Pavlidou et al. (2014), and Pavlidou & Tomaras (2014), and also the majority of models dealing with the turn-around, change the relation between \( M \) and \( R_0 \). Consequently the constraints obtained in Del Popolo et al. (2020), Del Popolo & Chan (2020) differ from those in Pavlidou et al. (2014), and Pavlidou & Tomaras (2014).

A last observation, is that there is no correlation between the groups masses, and \( w \). If one looks at NGC253, and IC342 which are the less massive, one finds values of \( w = -0.9 \), and -1, while in the case of Virgo, the most massive one gets -2.5, and one can think to a correlation. In reality a simple plot shows that there is not a strict correlation between mass and \( w \).

7. CONCLUSIONS

In this paper, we extended one of the models studied in Paper I, the \( J\eta LT \) model, to consider the effect of changing the parameter \( w \) of the EoS of dark energy. The \( J\eta LT \) model, by converse, had been an extension of the SLT model taking into account cosmological constant, angular momentum, and dynamical friction. We showed that the change of \( w \) modifies the evolution of perturbations. For example, turn-around happens before for more negative values of \( w \). By solving the equation of motion, we got a relation between the mass, \( M \), and the turn-around radius \( R_0 \), similarly to what done in Peirani & de Freitas Pacheco (2006, 2008), Del Popolo et al. (2021). For a given, \( R_0 \), the perturbation mass of the case with \( w = -1 \) (\( J\eta LT \)) is larger than those having \( w = -2/3 \) (\( J\eta LT _{-2/3} \) case), and \( w = -1/3 \) (\( J\eta LT _{-1/3} \) case). After finding a relation between mass, \( M \), and turn-around, \( R_0 \), we found a velocity, \( v \), radius, \( R \), relation, depending on mass and the Hubble constant. The data of the local group, M81, NGC 253, IC342, CenA/M83, and Virgo were fitted with this relation obtaining best fitting values for the mass, \( M \), and Hubble constant of the group considered. The mass decreases for less negative value of \( w \), while the Hubble constant has the opposite behavior. Finally, constraints to \( w \) were obtained from the mass, \( M \), and turn-around radius obtained by the studied groups of galaxies, comparing with the \( M - R_0 \) relation obtained by Del Popolo & Chan (2020). The constraints differ from previous ones (Pavlidou & Tomaras 2014; Pavlidou et al. 2014), because those predictions were based on the calculation of the mass, \( M \), and \( R_0 \) by means of the virial theorem, and because the \( M - R_0 \) relationship did not account the effect of shear, rotation, and dynamical friction.

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