A 5-quantifier $(\in, =)$-expression
ZF-equivalent to
the Axiom of Choice

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Abstract  In this paper I present an $(\in, =)$-sentence, $AC^{**}$, with only 5 quantifiers, that logically implies the axiom of choice, $AC$. Furthermore, using a weak fragment of ZF set theory, I prove that $AC$ implies $AC^{**}$.

Up to now 6 quantifiers were the minimum and 3 quantifiers don’t suffice since all 3-quantifier $(\in, =)$-sentences are decided in a weak fragment of ZF set theory. Thus the gap is reduced to the undecided case of a 4 quantifier sentence ZF-equivalent to $AC$.

1 Acknowledgment and Introduction

First and foremost I would like to thank Norman Megill for reviewing this paper and its drafts and for checking the proofs. Secondly I would like to thank Harvey Friedman for the challenges he poses to people like myself by posting bold enough conjectures on the FOM\(^1\), an E-mail listing for people interested in the foundations of mathematics.

In one of his FOM-postings\(^2\) Harvey Friedman sums up about what was known, up to then, about quantifier complexity in set theory. He also makes a number of conjectures. I quote the part of interest to this paper below.

As I said many times on the FOM, all 3 quantifier sentences are decided in a weak fragment of ZF. There is a 5 quantifier sentence that is not decided in ZFC, and is provably equivalent to the existence of a subtle cardinal over ZFC.

\[\ldots\]

I conjecture that the axiom of choice cannot be stated with 5 quantifiers, but this isn’t known even for 4 quantifiers. We know that the AxC can be stated with 6 quantifiers (posting #195).

In this quote a sentence is understood to be part of the primitive language of set theory, which is standard first order predicate calculus with equality and

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one binary relation, $\in$. Furthermore, in counting the number of quantifiers, one counts every individual quantifier, not just alternations of quantifiers.

This inspired me to investigate the 4- and 5-quantifier sentences, and in particular I set out to disprove the conjecture that there isn’t a 5-quantifier ($\in,=)$-sentence equivalent to the axiom of choice. I succeeded and this paper presents the proof. A draft version of this result was checked by Norman Megill with Metamath, a system for computer-aided formal proof checking developed by Norman Megill.

All this poses the next question: “Can $\text{AC}$ be stated by an ($\in,=)$-sentence with only 4 quantifiers?” I personally believe this isn’t possible. This belief comes from the fact that all known (to me) sentences ZF-equivalent to $\text{AC}$ have a form like $\forall x \ldots \exists y \varphi(x, y)$. The dots allow for some premises which must be satisfied by $x$ and $\varphi(x, y)$ must, in some way, express that $y$ represents a maximal decision relative to $x$. Usually this $\varphi$ can’t be stated with 2 or less quantifiers. The only exception I am aware of is a reworked version of Zorn’s Lemma. But in this case the premises on $x$ become quantifier-loaded.

In section 2, I introduce the notion of choice-sets and prove that “$y$ is a choice-set for $x$” can be stated with a well-formed-formula (wff) in the primitive language of set-theory containing only 3 quantifiers. In particular this wff belongs to the complexity class $\forall \exists \forall$.

In section 3, I state the axiom of choice, $\text{AC}$, in terms of choice-sets. Furthermore, I introduce a new “stronger” statement $\text{AC}^*$, which is stronger in the sense that only first order predicate calculus is required to prove that $\text{AC}^*$ implies $\text{AC}$. This is possible since $\text{AC}^*$ exhibits a weaker hypothesis and a stronger conclusion then $\text{AC}$. Following this, I show that this new statement is implied by $\text{AC}$ in a weak fragment of ZF. Namely the following:

**Extensionality:** $\forall x, y(\forall z(z \in x \iff z \in y) \rightarrow x = y)$

**Pairing:** $\forall x, y \exists z \forall a(a \in z \iff a = x \lor a = y)$.

In ($\in,=)$-notation this becomes $\forall x, y \exists z \forall a(a \in z \iff a = x \lor a = y)$.

**Bounded Separation:** If $\varphi$ is a wff in which all quantifiers are bounded, then we have $\forall x \exists y \forall z(z \in y \iff z \in x \land \varphi)$.

In ($\in,=)$-notation this becomes $\forall x \exists y \forall z(z \in y \iff z \in x \land \varphi)$.

**Bounded Replacement:** If $\varphi$ is a wff in which all quantifiers are bounded and $y$ is not free, then we have $\forall x(\forall z \exists z' \varphi \rightarrow \exists y \forall z \in x \exists z' \in y \varphi)$. In ($\in,=)$-notation this becomes $\forall x(\forall z \exists z' (\varphi \land \exists z'' (\varphi(z''/z') \rightarrow z'' = z'))) \rightarrow \exists y \forall z \exists z'(z \in x \rightarrow z' \in y \land \varphi)).$

The proof is done by describing a method for constructing a set $x^*$ from any set $x$, such that the following properties are satisfied:

1. If $x$ satisfies the hypotheses of $\text{AC}^*$, then $x^*$ satisfies the hypotheses of $\text{AC}$.
2. If $y$ is a choice-set for $x^*$, then we can construct a choice-set from this set for $x$ which is not contained in $x$.

In section 4, I rewrite $\text{AC}^*$ to obtain $\text{AC}^{**}$, which is a 5-quantifier ($\in,=)$-sentence. The proof of their equivalence only involves first order predicate calculus without any of the axioms of ZF set theory.
2 Choice-sets

Definition 2.1 We call a set $y$ a choice-set for a set $x$, and write $C(y, x)$, if for any non-empty element, $z \in x$, the intersection of $z$ and $y$, $z \cap y$, is a singleton. I.e. $y$ chooses exactly one element from every non-empty $z$ in $x$. Formally this could be written as

$$\forall z \in x (z \neq \emptyset \rightarrow \exists a \in y \ a \in z).$$

If we want to use $(\epsilon, =)$-notation, this can be translated into

$$\forall z \in x (\exists! b \in z \rightarrow \exists a (a \in z \land a \in y \land \forall b (b \in z \land b \in y \rightarrow b = a))).$$

Lemma 2.2 Suppose we have three wff's $X(t)$, $Y(t)$ and $Z(r, t)$, in which the variables $a$ and $b$ do not occur freely. Suppose furthermore, that $\forall t (Y(t) \rightarrow X(t))$ is valid. If we now define the wff's $A$ and $B$ as below, then $A$ and $B$ are equivalent.

- $A \equiv \exists b X(b) \rightarrow \exists a (Y(a) \land \forall b Z(b, a))$,
- $B \equiv \exists a \forall b [(X(b) \rightarrow Y(a)) \land (Y(a) \rightarrow Z(b, a))]$.

Proof

We need to prove $A \vdash B$ and $B \vdash A$

$A \vdash B$: Suppose that we have $A$. We have to consider two possibilities.

$\exists b X(b)$: In this case, $A$ gives us an $a$ that satisfies both $Y(a)$ and $\forall b Z(b, a)$. Now take any $b$. Since we have $Y(a)$, we definitely have $X(b) \rightarrow Y(a)$. Because of $\forall b Z(b, a)$, we also have $Z(b, a)$, which in turn implies $Y(a) \rightarrow Z(b, a)$. Hence for our chosen $a$ and any $b$, we find that we have $(X(b) \rightarrow Y(a)) \land (Y(a) \rightarrow Z(b, a))$.

$\neg \exists b X(b)$: In this case we have $\forall b \neg X(b)$. Now take any $a$ and $b$. By assumption, we have $\neg X(b)$, which gives us $X(b) \rightarrow Y(a)$. Also by assumption, we have $\neg X(a)$. Since we required $\forall t (Y(t) \rightarrow X(t))$ to be valid, we have $Y(a) \rightarrow X(a)$. These two together give us $\neg Y(a)$, which result in the validity of $Y(a) \rightarrow Z(b, a)$. Again $B$ follows. We even got $B$ with a universal quantifier instead of an existential quantifier.

$B \vdash A$: Suppose we now have $B$. To prove $A$, we assume $\exists b X(b)$. Hence we may assume a $b$ that satisfies $X(b)$. $B$ gives us an $a$, that satisfies $\forall b [(X(b) \rightarrow Y(a)) \land (Y(a) \rightarrow Z(b, a))]$. So in particular our chosen $a$ and $b$ from $B$ satisfy both $X(b)$ and $Y(b)$, from which we can conclude $Y(a)$. Now take any $b$. Again because of our choice for $a$, we get that $a$ and $b$ satisfy $Y(a) \rightarrow Z(b, a)$. But since we have $Y(a)$, this implies $Z(b, a)$. Since $b$ was chosen arbitrarily, our chosen $a$ also satisfies $\forall b Z(b, a)$. This concludes the proof.

Corollary 2.3 Notice that that the premises of the above lemma are fulfilled when we choose $X(t)$, $Y(t)$ and $Z(r, t)$ as below.

- $X(t) \equiv t \in z$,
- $Y(t) \equiv X(t) \land t \in y$,
- $Z(r, t) \equiv Y(r) \rightarrow r = t$.

This gives the following $A$ and $B$:
A clearly states: If $z$ is non-empty, then the intersection between $z$ and $y$ is a singleton. While $B$ shows that this can be said with only two quantifiers. More in particular we may conclude that $C(y, x)$ is equivalent to

$$\forall z (z \in x \rightarrow B(x, y, z)),$$

Hence “$y$ is a choice-set for $x$” can be stated in $(\in, =)$-notation with as little as 3 quantifiers.

### 3 Axiom of Choice

**Definition 3.1** One of the formulations of the axiom of choice, $AC$, states: For any set $x$ consisting of non-empty pairwise disjoint elements, there exists a choice-set $y$ for $x$. Formally this can be written as:

$$AC \equiv \forall x [AC_{h,1}(x) \land AC_{h,2}(x) \rightarrow \exists y C(y, x)],$$

where we have

$$AC_{h,1}(x) \equiv \forall z \in x z \neq \emptyset \text{ and }$$

$$AC_{h,2}(x) \equiv \forall z, z' \in x (z \neq z' \rightarrow z \cap z' = \emptyset).$$

In the literature, one might find an alternative formulation for the axiom of choice that does not require the elements of $x$ to be non-empty to guarantee the existence of a choice-set. In the presence of the Axiom of Bounded Separation, these two formulations are equivalent, since $y$ is a choice-set for $x$ iff $y$ is a choice-set for $\{ z \in x \mid z \neq \emptyset \} = \{ z \in x \mid \exists a \in z a = a \}$. This follows easily from the fact that $C(y, x)$ only gives information about the non-empty elements of $x$.

**Definition 3.2** In what follows, we will be interested in the following statement:

$$AC^* \equiv \forall x (AC_h^*(x) \rightarrow \exists y (y \notin x \land C(y, x))),$$

where we have

$$AC_h^*(x) \equiv \forall z \in x \exists a \in z \forall z' \in x (z \neq z' \rightarrow a \notin z').$$

I.e. $AC_h^*(x)$ states that all elements of $x$ contain an element not contained in any other element of $x$. The main purpose of this section is to prove the equivalence of this statement with $AC$.

**Lemma 3.3** $AC_{h,1}(x) \land AC_{h,2}(x) \rightarrow AC_h^*(x)$.

**Proof** Take any $z \in x$. Since we have $AC_{h,1}(x)$, we have an $a$ in $z$. Now, $AC_{h,2}(x)$ gives us that this $a$ is not an element of any other element of $x$. Hence $AC_h^*(x)$ follows. □

**Theorem 3.4** $AC^* \rightarrow AC$. 
Proof Suppose $AC^*$ is valid. Now suppose $x$ satisfies the hypotheses of $AC$. The previous lemma states that $x$ then also satisfies the hypotheses of $AC^*$. Hence by $AC^*$, we have a choice-set $y$ for $x$, which even is guaranteed not to be an element of $x$. □

Notice that the proof of $AC^* \rightarrow AC$ did not use any of the axioms of set-theory. The reverse implication, on the other hand, requires some of the other axioms of set-theory.

In what follows, I will indicate in parentheses which axioms and which previous results are required to prove the stated result.

Lemma 3.5 (Extensionality, Bounded Separation) Given any set $x$, then $\varphi(z, z_x)$ defined below is a function in the sense of the Axiom of Bounded Replacement.

$$\varphi(z, z_x) \equiv z_x = \{a \in z | \forall z^* \in x(z \neq z^* \rightarrow a \notin z^*)\}$$

Furthermore, for this function, we have the following properties:

1. $\forall z \in x \, z_x \subseteq z$,
2. $\forall z, z' \in x(z \neq z' \rightarrow z_x \cap z'_x = \emptyset)$,
3. $\forall z, z' \in x(z \neq z' \rightarrow z_x \cap z'_x = \emptyset)$,
4. $AC_b(x) \rightarrow \forall z \in x \, z_x \neq \emptyset$.

Here $z_x$ (resp. $z'_x$) denotes the image of $z$ (resp. $z'$) for the function $\varphi$ defined above. Formally this means that property 3 actually stands for $\forall z, z', z'_x \in x \forall z_x, z'_x [\varphi(z, z_x) \land \varphi(z', z'_x) \rightarrow (z \neq z' \rightarrow z_x \cap z'_x = \emptyset)]$

Proof Because of the Axiom of Bounded Separation we have for all $z$, a set $z_x$ such that $\varphi(z, z_x)$ is satisfied. The Axiom of Extensionality guarantees the uniqueness of this $z_x$. Hence $\varphi$ does indeed define a function. Since $\varphi(z, z_x)$ can be stated in $\in, =$-notation with only bounded quantifiers as follows:

$$\varphi(z, z_x) \equiv \forall a \in z(\forall z^* \in x(z \neq z^* \rightarrow a \notin z^*) \rightarrow a \in z_x)$$

$$\land \forall a \in z_x (a \in z \land \forall z^* \in x(z \neq z^* \rightarrow a \notin z^*))$$

$\varphi$ does indeed satisfy all the conditions of the Axiom of Bounded Replacement. We now prove the properties of this function.

1. This follows immediately from the definition of $z_x$.
2. Suppose this property didn’t hold. Hence we have two distinct elements, $z$ and $z'$, in $x$ and an $a$, such that $a \in z_x \cap z$ holds. By the previous property, we would have $a \in z \cap z'$. But by definition of $z_x$, $a$ would then not be an elements of $z_x$. Which is in contradiction with our chosen $a$.
3. Given any two distinct elements, $z$ and $z'$, in $x$, the two previous properties allow us to derive $z_x \cap z'_x \subseteq z_x \cap z' = \emptyset$.
4. Suppose $x$ satisfies $AC_b^*(x)$. Now take any $z \in x$. $AC_b(x)$ gives us an $a \in z$, not contained in any $z'$ in $x$ different from $z$. Hence by definition we have $a \in z_x$.

□

Corollary 3.6 (Bounded Replacement, Previous Lemma) The Axiom of Bounded Replacement and the previous lemma guarantee the existence of a
set \( x^* \) as the image of the function \( \varphi \) restricted to \( x \).

\[
x^* = \{ z_x \mid z \in x \}
\]

One easily verifies that such an \( x^* \) satisfies the following properties:

- \( AC_{h,2}(x^*) \) (corresponds to property 3 in the previous lemma),
- \( AC_{h}^*(x) \leftrightarrow AC_{h,1}(x^*) \) (“\( \rightarrow \)” corresponds to property 4 in the previous lemma, and the reverse follows from properties 1 and 2 in the previous lemma).

**Theorem 3.7** (Pairing, Bounded Separation, Previous corollary, Previous lemma) \( AC \rightarrow AC^* \).

**Proof** Suppose \( AC \) is valid and \( x \) is a set that satisfies \( AC_{h}^*(x) \). Now consider a set \( x^* \) given by the above corollary. This same corollary and \( AC_{h}^*(x) \) guarantees that \( x^* \) satisfies the hypotheses of \( AC \). Hence \( AC \) gives us a choice-set \( y \) for \( x^* \). The Axiom of Bounded Separation guarantees us the existence of a set

\[
y' = \{ a \in y \mid \exists z \in x \ a \in z \}.
\]

One easily verifies that we have

\[
y' = y \cap (\bigcup x^*).\]

On the other hand, the previous lemma allows one to verify that

\[
z_x = z \cap (\bigcup x^*)
\]

is valid for any \( z \in x \). This allows us to derive

\[
z \cap y' = z \cap (y \cap (\bigcup x^*)) = (z \cap (\bigcup x^*)) \cap y = z_x \cap y.
\]

Since \( y \) is a choice-set for \( x^* \), we find that \( z \cap y' \) is a singleton for all \( z \in x \). Hence \( y' \) is a choice-set for \( x \).

If \( y' \) is not an element of \( x \), then nothing remains to be proven. So suppose \( y' \) is an element of \( x \). Since \( y' \) is a choice-set for \( x \), we find that

\[
y' = y' \cap y' = \{ a \}
\]

for some \( a \). However, since we have \( AC_{h}^*(x) \), \( a \) cannot be contained in any other element of \( x \) and no element of \( x \) is empty. On the other hand, since \( y' \) is a choice-set for \( x \), \( a \) would have to be contained in all non-empty elements of \( x \). Hence we get \( x = \{ y' \} \). Now if we where to find a set \( b \) different from \( a \), then the Axiom of Pairing gives a set \( y'' = \{ a, b \} \). Such a set would then still be a choice-set for \( x \) and it would not be an element of \( x \).

The search for this set \( b \) can be done by different means (read: using different axioms of set-theory). Since we have already been using the Axiom of Bounded Separation, we will use this route. If \( a \) is empty, then \( y' \) which is non-empty will do. If on the other hand \( a \) is non-empty, then the Axiom of Bounded Subsets \( \{ u \in a \mid u \neq u \} \) guarantees that there exists an empty set \( b \) as a subset of \( a \). This \( b \) will do in this case. \( \square \)

**Corollary 3.8** If we summarize, then we find that to prove the equivalence of \( AC \) and \( AC^* \) we used the following axioms of set-theory.

- Axiom of Extensionality,
- Axiom of Pairing,
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- Axiom of Bounded Separation,
- Axiom of Bounded Replacement.

Since \(x^*\) is a subset of the powerset of \(x\), one could also replace the Axiom of Bounded Replacement by the Axiom of Powersets.

Note that while we did use \(\cap, \cup\) and \(\subseteq\) quite a bit, these were always used to construct classes and their relationships as classes. Hence their use can be eliminated from the above, be it at the cost of readability.

4 5-quantifier Axiom of Choice

The following result requires only first-order predicate calculus, i.e. none of the axioms of ZF were used.

**Theorem 4.1**  The sentence

\[ AC^{\ast\ast} \equiv \forall x \exists y \forall z \exists a \forall b \left[(y \in x \land A(x, y, z, a)) \lor (y \notin x \land B(x, y, z, a, b))\right] \]

where \(A(x, y, z, a)\) and \(B(x, y, z, a, b)\) are given by

\[ A(x, y, z, a) \equiv z \in y \rightarrow a \in x \land a \neq y \land z \in a \]

and

\[ B(x, y, z, a, b) \equiv z \in x \rightarrow (b \in z \rightarrow a \in z \land a \in y) \]

\[ \land (a \in z \land a \in y \rightarrow (b \in z \land b \in y \rightarrow b = a)) \]

is equivalent to \(AC^*\).

**Proof**

1. By corollary 2.3, we find that

\[ \exists y (y \notin x \land C(y, x)) \]

is equivalent to

\[ \exists y (y \notin x \land \forall z (z \in x \rightarrow \exists a \forall b \left[ (b \in z \rightarrow a \in z \land a \in y) \land (a \in z \land a \in y \rightarrow (b \in z \land b \in y \rightarrow b = a))\right])) \]

Since \(z \in x\) doesn’t mention \(a\) nor \(b\), this is equivalent to

\[ \exists y (y \notin x \land \forall z \exists a \forall b B(x, y, z, a, b)). \]

Since \(y \notin x\) doesn’t mention \(z\), \(a\) nor \(b\), this is equivalent to

\[ \exists y \forall z \exists a \forall b (y \notin x \land B(x, y, z, a, b)). \]

2. \(\neg AC_h^*(x)\) can be rewritten as

\[ \neg \forall z [z \in x \rightarrow \exists a (a \in z \land \forall z' [z' \in x \rightarrow (z \neq z' \rightarrow a \notin z'])]]. \]

Carrying the negation through the quantifiers and the logical connections on its way, we find this to be equivalent to

\[ \exists z [z \in x \land \forall a (a \in z \rightarrow \exists z' [z' \in x \land z \neq z' \land a \in z'])]. \]

Since \(a \in z\) does not mention \(z'\), this is equivalent to

\[ \exists z [z \in x \land \forall a \exists z' (a \in z \rightarrow [z' \in x \land z \neq z' \land a \in z'])]. \]
Since $z \in x$ does not mention $a$ nor $z'$, this is equivalent to
\[ \exists z \forall a \exists z'[z \in x \land (a \in z \to [z' \in x \land z \neq z' \land a \in z')]) \].
Replacing the variables $z$, $a$ and $z'$ simultaneously with the variables $y$, $z$ and $a$, this becomes
\[ \exists z \forall y \exists a (y \in x \land A(x, y, a)). \]
Since this wff does not mention $b$, this is equivalent to
\[ \exists y \forall z \exists a \forall b (y \in x \land A(x, y, z, a)). \]

3. One easily verifies that $-[(y \in x \land A(x, y, z, a)) \land (y \notin x \land B(x, y, z, a, b))]$ is always valid.

The above points allow us to verify our theorem. This goes like this:

\[
\begin{align*}
AC^* & \\
\downarrow & \\
\forall x (\neg AC^*_h(x) \lor \exists y (y \notin x \land C(y, x))) & \\
\downarrow & (1, 2) & \\
\forall x [\exists y \forall z \exists a \forall b (y \in x \land A(x, y, z, a)) \lor \exists y \forall z \exists a \forall b (y \notin x \land B(x, y, z, a, b))] & \\
\downarrow & (3) & \\
\forall x \exists y \forall z \exists a \forall b [(y \in x \land A(x, y, z, a)) \lor (y \notin x \land B(x, y, z, a, b))] & \\
\end{align*}
\]

**Corollary 4.2** Hence we have an $(\in, =)$-sentence with only 5 quantifiers, $AC^{**}$, equivalent to the axiom of choice.

**Remark 4.3** There is a way to obtain a shorter 5-quantifier $(\in, =)$-sentence equivalent to the axiom of choice. This follows from the following observations.

- If $x$ is a set without empty elements, then $y$ is a choice-set for $x$ iff the intersection of $y$ and any element of $x$ (not restricted to non-empty elements of $x$) is a singleton.
- If $x$ satisfies $AC^*_h(x)$, then it does not contain any empty elements.

Hence we find that $AC^*$ is equivalent to $\overline{AC}^*$ obtained from $AC^*$ by replacing $C(y, x)$ with $\overline{C}(y, x)$ given by:
\[ \overline{C}(y, x) \equiv \forall z \in x \exists a (a \in z \land a \in y \land \forall b (b \in z \land b \in y \to b = a)). \]

We then could translate $\overline{AC}^*$ into $\overline{AC}^{**}$, as we did translate $AC^*$ into $AC^{**}$, not needing the step using corollary 2.3. This would result in replacing $B(x, y, z, a, b)$ with
\[ \overline{B}(x, y, z, a, b) \equiv z \in x \to a \in z \land a \in y \land (b \in z \land b \in y \to b = a). \]

This shortens our sentence by 16 symbols (4 parentheses, 3 logical connectors and 3 atomic formulas).
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**Notes**

1. More information can be found at
   [http://www.cs.nyu.edu/mailman/listinfo/fom](http://www.cs.nyu.edu/mailman/listinfo/fom).

2. The posting of Harvey Friedman I’m referring to is entitled “196:Quantifier complexity in set theory” and can be found at
   [http://www.cs.nyu.edu/pipermail/fom/2003-November/007680.html](http://www.cs.nyu.edu/pipermail/fom/2003-November/007680.html)

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