Axial-Vector Duality as a Gauge Symmetry and Topology Change in String Theory

Amit Giveon

Racah Institute of Physics
The Hebrew University
Jerusalem, 91904, ISRAEL

and

Elias Kiritsis

Theory Division, CERN, CH-1211
Geneva 23, SWITZERLAND

ABSTRACT

Lines generated by marginal deformations of WZW models are considered. The Weyl symmetry at the WZW point implies the existence of a duality symmetry on such lines. The duality is interpreted as a broken gauge symmetry in string theory. It is shown that at the two end points the axial and vector cosets are obtained. This shows that the axial and vector cosets are equivalent CFTs both in the compact and the non-compact cases. Moreover, it is shown that there are $\sigma$-model deformations that interpolate smoothly between manifolds with different topologies.

CERN-TH.6816/93
February 1993
Revised: August 1993

*e-mail address: GIVEON@HUJIVMS.bitnet
†e-mail address: KIRITSIS@NXTH08.CERN.CH
1 Introduction

The moduli space of flat d-dimensional toroidal string compactifications is generated by a group $O(d, d, R)$ [1]. This generalizes to curved string backgrounds with $d$ toroidal isometries (chiral and anti-chiral) where the moduli space again can be generated by the action of a group isomorphic to $O(d, d, R)$ [2, 3]. The global structure of the moduli space is rather striking; points corresponding to different backgrounds, are related by the generalized duality group of discrete symmetries, isomorphic to $O(d, d, Z)$ [4, 5].

In this paper, we will argue that the discrete symmetries of the moduli of target space should be interpreted as global residual symmetries of an underlying broken gauge group of string theory. In particular, we will show that the axial/vector duality of abelian WZW cosets [5] is a gauge symmetry, and therefore, it is an exact symmetry in string theory, both for compact as well as non-compact groups.

Moreover, we will describe deformations that generate a family of conformal backgrounds that interpolate between manifolds with different topologies. This provides a simple example of classical topology change in string theory, similar to the phenomenon described recently for Calabi-Yau compactifications [6, 7]. The topology change that will be described here occurs for cosmological string backgrounds, and is therefore important.

The view of discrete symmetries in target space as gauge symmetries was already discussed in the flat case. In ref. [8] it was argued that from the point of view of an effective space-time theory, the discrete symmetry relating a compact dimension of radius $r$ to that of radius $1/r$ is indeed a part of a continuous $SU(2)$ rotation. The argument goes as follows. When the compactified circle is at radius $r = 1$, there is an extended affine symmetry $SU(2)_L \times SU(2)_R$. This theory contains three chiral currents $J^a$, and three anti-chiral currents $\bar{J}^a$ in the adjoint representation of $SU(2)_L$ and $SU(2)_R$, respectively. The space of truly marginal independent directions is embedded in the 9-dimensional space generated by $J^a\bar{J}^b$. The set of critical points that can be reached from the $SU(2)$ point by conformal deformations (of the type $(\sum_{a=1}^{3} \alpha_a J^a)(\sum_{b=1}^{3} \beta_b\bar{J}^b)$) span a 5-dimensional surface in the 9-dimensional euclidean space. However, because different truly marginal perturbations are equivalent under continuous transformations in the group $SU(2)_L \times SU(2)_R$, the dimension of the physical moduli space is 1. In particular, the duality transformation relating $r$ to $1/r$ corresponds to the Weyl transformation in $SU(2)_L$ that takes $J^3$ to $-J^3$. Infinitesimally near the $SU(2)$ point, this corresponds to the identification of the theory given by the deformation $\alpha J^3 \bar{J}^3$, with the theory given by the deformation $-\alpha J^3 \bar{J}^3$.

The generalization to $d$-dimensional toroidal backgrounds was given ref. [9]. It was shown that any element of the generalized duality group $O(d, d, Z)$, is a product of discrete symmetries corresponding to continuous rotations in groups (Weyl transformations) around points with an extended affine symmetry. Moreover, it was suggested [9] that in string theory the generalized dualities are residual discrete symmetries of a broken infinite dimensional gauge group. This was realized for the heterotic string compactified
on a torus in ref. [10], where a completely duality invariant effective action of the $N = 4$ heterotic string was constructed. It was then shown that the infinite dimensional gauge algebra is broken at any classical background to a finite dimensional group, and that the generalized duality group $O(6, 22, Z)$ elements are residual discrete symmetries of the broken gauge algebra. It is therefore expected that target space dualities are residual discrete symmetries of the underlying gauge algebra of string theory [9, 10, 11, 12, 13].

The identification of target space dualities with continuous rotations around points with an extended symmetry is rather important. In particular, it proves that such dualities are exact symmetries (nonperturbatively in $\alpha'$): theories associated to different flat backgrounds that are related by duality correspond to the same CFT (to all orders and interactions).

Generalized target space dualities are not limited only to flat backgrounds; in ref. [3] it was shown that the elements of $O(d, d, Z)$ are discrete symmetries of the space of curved string backgrounds that are independent of $d$ coordinates. It was moreover shown in [14] that (as in the flat case [3]), in the compact WZW case, these are exact symmetries of CFT and string theory by relating them to invariance under the affine Weyl group. It was also argued that they should persist in the non-compact case.

Are these symmetries residual discrete symmetries of an underlying gauge algebra in string theory? In this work we present the answer concerning particular elements of the $O(d, d, Z)$ group. Namely, we will show that any element relating an axially gauged $U(1)$ of a WZW model to the vector gauging is a residual discrete symmetry of the broken gauge algebra in the sense discussed above. In particular, this proves that the axial/vector coset duality is an exact symmetry in string theory. This is true for compact groups, as well as for non-compact groups, and therefore, has important implications for black-hole duality [15] and the study of singularities in string theory. The extension to the full $O(d, d, Z)$ might be done along the lines of [3], and will appear elsewhere.

The paper is organized as follows: In section 2 we start with the simplest non-trivial case, the $SU(2)$ or $SL(2, R)$ model and its marginal deformation. In section 3 we present the partition function, and discuss the geometrical interpretation of the modulus parameter. In section 4 we study the target space geometry and topology change along the line of deformations, and we describe a smooth topology change in the extended moduli space of the $SU(2)$ WZW model. In section 5 we deal with the general case. In section 6 we discuss the duality on the line of marginal deformations and its relation to a broken gauge symmetry transformation. Finally, section 7 contains our conclusions and further comments. In the appendix we describe why the $\sigma$-model action we give for the deformed WZW model is exact to all orders in $\alpha'$. 


2 \ J \bar{J} \text{ deformation of } SU(2) \text{ or } SL(2) \text{ WZW model and duality as a broken gauge symmetry}

In this section we will consider duality as a broken gauge symmetry for the simplest nontrivial case, namely, duality acting on the deformation line of \( SU(2) \) or \( SL(2) \) WZW models.

If we parametrize the \( SU(2) \) group element as
\[
g = e^{i\theta_1 \sigma_3} e^{ix \sigma_2} e^{i\theta_2 \sigma_3}
\]
then the action for \( SU(2)_k \) WZW model is given by
\[
S[x, \theta_1, \theta_2] = S_1 + S_a + S[x],
\]
\[
S_1 = \frac{k}{2\pi} \int d^2 z (\partial \theta_1 \bar{\partial} \theta_1 + \partial \theta_2 \bar{\partial} \theta_2 + 2\Sigma(x) \partial \theta_2 \bar{\partial} \theta_1),
\]
\[
S_a = \frac{k}{2\pi} \int d^2 z (\partial \theta_2 \bar{\partial} \theta_1 - \partial \theta_1 \bar{\partial} \theta_2),
\]
\[
S[x] = \frac{k}{2\pi} \int d^2 z \partial x \bar{\partial} x - \frac{1}{8\pi} \int d^2 z \phi_0 R^{(2)},
\]
where \( \Sigma(x) = \cos 2x \), and \( \phi_0 \) is a constant dilaton. The action for \( SL(2)_k \) WZW model is given from (2.2) by taking \( x \to ix \) and \( k \to -k \).

The antisymmetric term \( S_a \) in (2.2) is (locally) a total derivative, and therefore may give only topological contributions, depending on the periodicity of the coordinates \( \theta \). To specify the periodicity, we define
\[
\theta = \theta_2 - \theta_1, \quad \bar{\theta} = \theta_1 + \theta_2,
\]
such that
\[
\theta \equiv \theta + 2\pi, \quad \bar{\theta} \equiv \bar{\theta} + 2\pi.
\]
In these coordinates the action becomes
\[
S[x, \theta, \bar{\theta}] = \frac{1}{2\pi} \int d^2 z (\partial \theta, \bar{\partial} \bar{\theta}, \partial x) \begin{pmatrix} E & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} \bar{\partial} \theta \\ \bar{\partial} \bar{\theta} \end{pmatrix} - \frac{1}{8\pi} \int d^2 z \phi_0 R^{(2)},
\]
where \( E \) is the \( 2 \times 2 \) matrix
\[
E = \frac{k}{2} \begin{pmatrix} 1 - \Sigma & 1 + \Sigma \\ - (1 + \Sigma) & 1 + \Sigma \end{pmatrix},
\]
* We define the CFT corresponding to the \( SL(2) \) WZW model as the analytic continuation of the (Euclidean) 3-d hyperboloid \( (H^+_3) \) \( \sigma \)-model. This definition provides with a stable path integral prescription for the \( SL(2) \) theory as shown in [16]. All subsequent remarks concerning \( SL(2) \) will assume this definition.
The action $S$ (2.3) is manifestly invariant under the $U(1)_L \times U(1)_R$ affine symmetry generated by the currents
\[
J = \frac{k}{2} (- (1 - \Sigma) \partial \theta + (1 + \Sigma) \partial \bar{\theta}), \quad \bar{J} = \frac{k}{2} ((1 - \Sigma) \partial \bar{\theta} + (1 + \Sigma) \partial \theta).
\]
(2.7)

In addition, there are two extra chiral currents, and anti-chiral currents, completing the affine $SU(2)_L \times SU(2)_R$ or $SL(2)_L \times SL(2)_R$ symmetry of the the WZW model.

It is possible to deform the action $S$ to new conformal backgrounds by adding to it any marginal deformation. We will focus on marginal deformations that are obtained as a linear combination of chiral currents times a linear combination of anti-chiral currents. It is now important to note that all the deformations that are equivalent under the action of the symmetry group give rise to equivalent CFTs (although not necessarily to backgrounds that are related by coordinate transformations!). In the following we deform the WZW action with the $J\bar{J}$ marginal operator, as was done in ref. [17]. This deformation gives rise to a one parameter family of theories, parametrized by the radius of the Cartan torus.

Once deforming with $J\bar{J}$, the affine symmetry is broken to $U(1)_L \times U(1)_R$.

The $J\bar{J}$ deformation is equivalent to a particular one parameter family of $O(2,2)$ rotations acting on the background matrix $E$ and the dilaton [17]. To show this point we begin by establishing our notation following ref. [9].

The group $O(d,d,R)$ can be represented as a $2d \times 2d$-dimensional matrices $g$ preserving the bilinear form $j$
\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad j = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},
\]
(2.8)

where $a,b,c,d$, and $I$ are $d \times d$-dimensional constant matrices, and
\[
g^t j g = j.
\]
(2.9)

We define the action of $g$ on $E$ by fractional linear transformations:
\[
g(E) = E' = (aE + b)(cE + d)^{-1}.
\]
(2.10)

\footnote{It is easy to see that for generic level these are the only potential marginal deformations. Any other possible deformation has to be generated by affine primaries, and these do not generically have dimension $(1,1)$. This can happen though at special levels. For example, in the case of $SU(2)_k$ when $k = (m - 1)(m + 2)$, then the $j = m$ primary has dimension one. It is not known if this is exactly marginal, although it is for $m = 2$. However it is suggestive that the central charge of the associated parafermion system is $1 + [1 - 6/m(m + 1)]$ so that it may be that the theory is a semidirect product of a $U(1)$ and a minimal model.}
The group $O(d, d, R)$ is generated by $GL(d)$ transformations:

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} = \begin{pmatrix} A & 0 \\
0 & (A^t)^{-1}
\end{pmatrix} \quad \text{s.t. } A \in GL(d),
\]

(2.11)

constant $\Theta$ shifts

\[
\begin{pmatrix} a & b \\
c & d
\end{pmatrix} = \begin{pmatrix} I & \Theta \\
0 & I
\end{pmatrix} \quad \text{s.t. } \Theta = -\Theta^t,
\]

(2.12)

and factorized duality

\[
\begin{pmatrix} a & b \\
c & d
\end{pmatrix} = \begin{pmatrix} I - e_1 & e_1 \\
e_1 & I - e_1
\end{pmatrix} \quad \text{s.t. } e_1 = \text{diag}(1, 0, ..., 0).
\]

(2.13)

The discrete group $O(d, d, Z)$ is defined to be the elements $g$ in (2.8) with integer entries.

Rotations with $O(2, 2)$ elements in $GL(2)$ and $\Theta$ shifts preserve the current algebra (although in general they change the spectrum). Therefore, once choosing a $J\bar{J}$ deformation of the WZW point (2.2) for specific $J$ and $\bar{J}$, the deformed theories are described in terms of a one parameter family of rotations. Next we describe this rotation.

The backgrounds corresponding to the $J\bar{J}$ deformation of the WZW point, with $J$ and $\bar{J}$ given in (2.7), are constructed by an $O(2, 2)$ transformation (2.10) of the WZW background (2.6) with the element 

\[
g_\alpha = \begin{pmatrix} I & \cos^2 \alpha (k - \tan \alpha) \epsilon \\
0 & I
\end{pmatrix} \begin{pmatrix} A(\alpha) & 0 \\
0 & (A(\alpha)^t)^{-1}
\end{pmatrix} \begin{pmatrix} C(\alpha) & S(\alpha) \\
S(\alpha) & C(\alpha)
\end{pmatrix} \begin{pmatrix} I & -k \epsilon \\
0 & I
\end{pmatrix},
\]

(2.14)

where

\[
I = \begin{pmatrix} 1 & 0 \\
0 & 1
\end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & 1 \\
-1 & 0
\end{pmatrix}, \quad A(\alpha) = \begin{pmatrix} \cos \alpha & 0 \\
0 & \cos \alpha (1 + k \tan \alpha)
\end{pmatrix},
\]

\[
C(\alpha) = \cos \alpha \ I, \quad S(\alpha) = \sin \alpha \ \epsilon,
\]

(2.15)

namely,

\[
\frac{1}{k} g_\alpha(E) \equiv E_{R(\alpha)} = \frac{1}{1 + R^2 \frac{1 - \Sigma}{1 + \Sigma}} \begin{pmatrix} \frac{1 - \Sigma}{1 + \Sigma} & 1 \\
-1 & R^2
\end{pmatrix},
\]

(2.16)

where

\[
R(\alpha)^2 = (1 + k \tan \alpha)^2.
\]

(2.17)

The deformation line parametrized by $-\pi/2 \leq \alpha \leq \pi/2$ is, therefore, a double cover of the line parametrized by the radius $0 \leq R \leq \infty$. The original WZW point is given at $R = 1$.

The dilaton field $\phi$ also transforms under the $O(2, 2)$ rotation. At the WZW point the dilaton is the constant $\phi = \phi_0$ appearing in (2.2). Then at the point $R$, the value of the dilaton is equal to that implied by $O(2, 2, R)$ transformations (see also appendix A)

\[
\phi = \phi_0 + \frac{1}{2} \log \left( \frac{\det G(1)}{\det G(R)} \right).
\]

(2.18)
With this dilaton, $\sqrt{G(R)}e^{\phi(R)}$ is independent of $R$, and moreover, it has the appropriate asymptotic behaviour as $R \to \infty$.

We have obtained the $\sigma$-model background (2.16, 2.18) using $O(2,2)$ transformations, which are correct to 1-loop order (but correctable to all orders [14]). In appendix A we show that there is a scheme in which this background solves the $\beta$-function equations to all orders.

Two special backgrounds occur at the boundaries of the $R$ modulus space. At $R = 0$ ($\alpha = \tan^{-1}(-\frac{1}{k})$) the background matrix is

$$E_{R=0} = \begin{pmatrix} \frac{1-S}{1+\Sigma} & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1-S}{1+\Sigma} & 0 \\ 0 & 0 \end{pmatrix} \mod \Theta - \text{shift.} \quad (2.19)$$

This background corresponds to the direct product of the vectorially gauged coset $SU(2)/U(1)_v$ [18] or $SL(2)/U(1)_v$, and a free boson at a compactification radius $r = 0$ (that is equivalent to a non-compact free boson via the $r \to 1/r$ duality). The constant antisymmetric tensor in (2.19) can be safely dropped since one of the two coordinates is non-compact.

At $R = \infty$ ($\alpha = \pi/2$) the background matrix is

$$E_{R=\infty} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1+\Sigma}{1-S} \end{pmatrix}. \quad (2.20)$$

This background corresponds to the direct product of the axially gauged coset $SU(2)/U(1)_a$ or $SL(2)/U(1)_a$ and a free scalar field at a compactification radius $r = 0$.

As was already mentioned, the $R = 1$ ($\alpha = 0$) point, corresponds to the original WZW model. Around this point, and for an infinitesimal deformation parameter $\delta \alpha$, the deformed action is given by

$$S_{R=1+\delta R} = S_{R=1} + \frac{\delta R^2}{4\pi k} \int J \bar{J}, \quad (2.21)$$

where $J$ and $\bar{J}$ are given in (2.7). This extends along the full line. The $U(1)_L$ affine symmetry is generated by $\theta \to \theta - \epsilon$, $\tilde{\theta} \to \tilde{\theta} + \epsilon/R^2$ with a conserved current

$$J(R) = k \frac{-(1-S)\partial \theta + (1+\Sigma)\partial \tilde{\theta}}{1+\Sigma + R^2(1-S)}, \quad (2.22)$$

whereas the $U(1)_R$ affine symmetry is generated by $\theta \to \theta + \epsilon$, $\tilde{\theta} \to \tilde{\theta} + \epsilon/R^2$ with a conserved current

$$\bar{J}(R) = k \frac{(1-S)\partial \theta + (1+\Sigma)\partial \tilde{\theta}}{1+\Sigma + R^2(1-S)}. \quad (2.23)$$

Thus

$$S_{R+\delta R} = S_R + \frac{\delta R^2}{4\pi k} \int J(R) \bar{J}(R), \quad (2.24)$$

and the variation of the dilaton provides the proper measure for the $\sigma$-model above.
We now arrive to the important point of this section. The Weyl transformation $J \rightarrow -J$ is given by a group rotation at the WZW point, and thus is a symmetry of the WZW model. Therefore, the deformation of the WZW model by $\delta \alpha J \bar{J}$ is equivalent infinitesimally to the deformation by $-\delta \alpha J \bar{J}$. The points $\delta \alpha$ and $-\delta \alpha$ along the $\alpha$-modulus are thus the same CFT. In string theory, we say that they are related by a residual $Z_2$ symmetry of the broken gauge symmetry of the extended symmetry point.

The residual discrete symmetry is the target space duality. This symmetry can be integrated to finite $\alpha$, giving rise to a $Z_2 \in O(2,2,\mathbb{Z})$ duality matrix

$$g_D = \begin{pmatrix} 0 & I & I - \epsilon & e_2 & e_1 \\ I & 0 & 0 & 0 & e_1 \\ 0 & I & 0 & 0 & e_2 \\ 1 & -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & I & I - \epsilon & e_2 & e_1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ (2.25)

$I$ is the 2-dimensional identity matrix, and $\epsilon$ is given in (2.15). The element $g_D$ acts on $E_R$ by (2.14) and gives

$$g_D(E_R) = E_{1/R}. \quad \text{(2.27)}$$

Therefore, duality takes the modulus $R$ to its inverse $1/R$.

As a consistency check, we note that $R \rightarrow 1/R$ is equivalent to $\tan \alpha \rightarrow \frac{-\tan \alpha}{1 + k \tan \alpha}$, and therefore, a small $\alpha$ is transformed by duality to $-\alpha$ as it should be.

In particular, we learn that the $R = 0$ and $R = \infty$ points are the same CFT. Thus the vector/axial duality of $SU(2)/U(1)$ and $SL(2)/U(1)$ is exact, and corresponds in string theory to a residual discrete symmetry of the broken gauge symmetry.

We should add some more comments here concerning equivalent versions of the WZW deformations. The first remark is that the background (2.16,2.18) can be obtained as a gauged WZW model, $SU(2) \times U(1)/U(1)$ where the $U(1)$ has radius $e$ and the gauged $U(1)$ is the sum of the $U(1)$ of the free boson and the Cartan of $SU(2)$. Gauging the axial current and integrating out the gauge fields we obtain half of the line in (2.16,2.18) with $R^2 = 1 + k/e^2$. Gauging the vector current we obtain the other half of the line with $R^2 = (1 + k/e^2)^{-1}$. It is obvious that at $e \rightarrow \infty$ we are left with the WZW model. When $e \rightarrow 0$, before gauging, we know that the CFT is that of a non-compact boson times the WZW model. The only compact $U(1)$ subgroup then is one that lies solely in $SU(2)$ and thus we obtain the direct product of a non-compact boson times the $SU(2)/U(1)$ coset model.

The discussion in the Appendix shows that the background (2.16,2.18) is a conformally exact $\sigma$-model to all orders in a certain scheme. In view of the relation to the coset
above, this implies that there is a scheme in which the semiclassical result (obtained by integrating out the gauge fields) is exact to all orders.

Another dual version of our model can be obtained by performing a factorized duality transformation \([2,13]\), followed by an appropriate \(GL(2, \mathbb{Z})\) transformation. The action obtained thus is the sum of the parafermionic action and the action of a free scalar field with radius \(\sqrt{kR}\), up to a \(Z_k\) orbifoldization which couples the two. The orbifoldizing symmetry acts as a \(Z_k\) transformation in the parafermionic theory and a simultaneous translation of the free scalar by \(2\pi/k\). In this form of the action, the factorization of the theory (at the boundary) to that of a non-compact boson and the \(SU(2)/U(1)\) parafermion theory is manifest: When \(R \to \infty\), the \(Z_k\) translation acts trivially on the non-compact scalar, so the orbifolding symmetry acts uniquely in the parafermion theory. However, the parafermion theory is invariant under such an orbifolding \([19]\). This also supports the statement that the scalar with a vanishing radius at the boundary is a bona-fide non-compact scalar. All of the above will be explicitly verified in the next section by an analysis of the exact torus partition function.

3 The partition function along the \(R\)-line

In this section we present the partition function, and discuss the geometrical interpretation of the parameter \(R\).

It is tempting to relate \(R\) to the compactification radius of the deformed Cartan torus. This gets support by looking at the torus partition function for such a deformation. The partition function is known in the \(SU(2)_k\) deformed case, and is given by \([20]\)

\[
Z(R) = \sum_{l,\bar{l}=0}^{k} \sum_{m=-k+1}^{k} \sum_{r=0}^{k-1} N_{l,\bar{l}} c_m^l(q) \bar{c}_{m-2r}^\bar{l}(\bar{q}) \sum_{M,N \in \mathbb{Z}} q^{\Delta_{M,N}} \bar{q}^{\bar{\Delta}_{M,N}},
\]

with

\[
\Delta_{M,N} = \frac{1}{4k} \left( \frac{kM + m - r}{R} + R(kN + r) \right)^2,
\]

\[
\bar{\Delta}_{M,N} = \frac{1}{4k} \left( \frac{kM + m - r}{R} - R(kN + r) \right)^2.
\]

In (3.1) \(N_{l,\bar{l}} = \delta_{l,\bar{l}}\) (corresponding to the diagonal modular invariant WZW model), \(c_m^l(q)\) are the standard string functions \([21]\), and \(q = e^{2\pi i \tau}\), where \(\tau = \tau_1 + i\tau_2\) is the complex torus moduli parameter.

This partition function, indeed, can be obtained by performing a \(Z_k\) orbifold on the direct tensor product of the \(SU(2)/U(1)\) parafermion theory and a free scalar field compactified on a circle of radius \(\sqrt{kR}\) \([20]\). The \(Z_k\) symmetry we orbifoldize with, is a combination of the \(Z_k\) parafermionic symmetry and a translation of the free scalar by
$2\pi/k$. This is explicitly demonstrated by writing (3.4) as

$$Z(R) = \frac{1}{k} \sum_{r,s \in \mathbb{Z}_k} \zeta_k(r,s) Z(r,s,R) \tag{3.4}$$

where $\zeta_k(r,s)$ is the parafermion partition function twisted by the elements $r,s \in \mathbb{Z}_k$ around the two non-trivial cycles and $Z(r,s,R)$ is the respective twisted partition function of the free scalar with radius $\sqrt{kR}$. By doing the sum on $s$ in (3.4) we get the expression given in (3.1).

The partition function (3.1) is invariant under the duality transformation $R \to 1/R$. The fixed point $R = 1$ corresponds to the WZW model. Next we will check the two boundary points: $R = \infty$ and $R = 0$.

At $R \to \infty$ the only contribution to $Z$ comes when $r = 0$ and $N = 0$ in the sums, namely

$$Z(R \to \infty) = \left| \eta(q) \right|^2 \sum_{l,l=0}^{k} \sum_{m=-k+1}^{k} N_{l,t} c^l_m(q) \bar{c}^f_m(\bar{q}) \left( \sqrt{kR} \tau_2^{-1/2} |\eta(q)|^{-2} \right), \tag{3.5}$$

where $\eta$ is the Dedekind eta-function. The first parentheses in (3.5) corresponds to the vectorially gauged coset $[14]$

$$Z^v_{SU(2)/U(1)} = |\eta(q)|^2 \sum_{l,l=0}^{k} \sum_{m=-k+1}^{k} N_{l,t} c^l_m(q) \bar{c}^f_m(\bar{q}), \tag{3.6}$$

which is the correct parafermionic partition function $[14]$. The second parentheses in (3.3) corresponds to a free scalar field at the decompactification limit or zero radius limit.

At $R \to 0$ the only three contributions to $Z$ come when the following conditions are obeyed in the sums in (3.1): (1) $m - r = 0$ and $M = 0$. (2) $m - r = k$ and $M = -1$. (3) $m - r = -k$ and $M = 1$. Therefore, one finds

$$Z(R \to 0) = \left[ \left| \eta(q) \right|^2 \sum_{l,l=0}^{k} N_{l,t} \left( \sum_{m=-k}^{-1} c^l_m(q) \bar{c}^f_{m-2k}(\bar{q}) + \sum_{m=0}^{k-1} c^l_m(q) \bar{c}^f_{m-2k}(\bar{q}) + c^l_k(q) \bar{c}^f_k(\bar{q}) \right) \right] \times \left( \sqrt{kR}^{-1} \tau_2^{-1/2} |\eta(q)|^{-2} \right). \tag{3.7}$$

The first parentheses in (3.7) corresponds to the axially gauged coset $[14]$

$$Z^a_{SU(2)/U(1)} = |\eta(q)|^2 \sum_{l,l=0}^{k} N_{l,t} \left( \sum_{m=-k}^{-1} c^l_m(q) \bar{c}^f_{m-2k}(\bar{q}) + \sum_{m=0}^{k-1} c^l_m(q) \bar{c}^f_{m-2k}(\bar{q}) + c^l_k(q) \bar{c}^f_k(\bar{q}) \right). \tag{3.8}$$

Using the symmetry of the string functions $c^l_m$ under the affine Weyl group one obtains that $[14]$

$$Z^a = Z^v. \tag{3.9}$$

The second parentheses in (3.7) corresponds to a free scalar field at the decompactification limit or zero radius limit.
We conclude that the partition function (3.1) corresponds to the deformation line of the $SU(2)$ WZW background described in this section. The parameter $R$ in (2.16) is therefore related to the radius of the Cartan torus. Although we do not know the exact partition function of $SL(2, R)$ for arbitrary Cartan radius, our previous argument implies that similar things happen also there.

4 Geometry along the $R$-line and smooth topology change

In this section we describe the geometry along the $R$-line and a smooth topology change in the extended moduli space of $SU(2)$ or $SL(2)$.

We will first present the description of the target-space geometry along the $R$-line of marginal deformations. The $\sigma$-model metric in the case of deformed $SU(2)$ (in the coordinates $\theta, \tilde{\theta}, x$) is given by

$$ G \sim k \begin{pmatrix} \frac{\sin^2 x}{\cos^2 x + R^2 \sin^2 x} & 0 & 0 \\ 0 & \frac{R^2 \cos^2 x}{\cos^2 x + R^2 \sin^2 x} & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

(4.1)

The scalar curvature $\hat{R}$ is

$$ \hat{R} = -\frac{2 \, 2 - 5 R^2 + 2 (R^4 - 1) \sin^2 x}{k \left( 1 + (R^2 - 1) \sin^2 x \right)^2}. $$

(4.2)

The manifold is regular except at the end-points where

$$ \hat{R}(R = 0) = -\frac{4}{k \cos^2 x}, \quad \hat{R}(R = \infty) = -\frac{4}{k \sin^2 x}. $$

(4.3)

At $R = 1$ we get the constant curvature of $S^3$, $\hat{R} = 6/k$.

It should be noted that the geometric data (metric, curvature, etc.) are invariant under $R \to 1/R$ and $x \to \pi/2 - x$. Another interesting object is the volume of the manifold as a function of $R$ that can be computed to be

$$ V(R) \sim \frac{R \log R}{R^2 - 1}. $$

(4.4)

satisfying $V(R) = V(1/R)$. The volume becomes singular only at the boundaries of moduli space, $R = 0, \infty$.

For $SL(2, R)$, the trigonometric functions in (4.2) are replaced by the corresponding hyperbolic functions. Here the manifold has a curvature singularity for $0 \leq R < 1$. Similar remarks apply to the Euclidean version, the 3-d hyperboloid.

The $R$ marginal deformation generates a continuous family of CFTs that interpolate between two manifolds with different topology: the “cigar” shape ($R = \infty$) with the
topology of the disk and the “trumpet” shape \((R = 0)\) with the topology of a cylinder. In the \(SU(2)\) case, the \(S^3\) group manifold is deformed to \(D_2 \times p\), i.e., the direct product of a two disc and a point. These topology changes occur only at the boundary of moduli space; a smooth topology change will be described below.

In the \(SU(2)\) case, the complete moduli space of backgrounds also includes target spaces that are a direct product of \(SU(2)/U(1)\) like backgrounds (namely, \(ds^2 = kdx^2 + k' \tan^2 x d\theta^2\), where \(k'\) is an arbitrary constant) with a finite radius circle \(S^1\). These backgrounds are smooth deformations of the \(SU(2)\) group manifold \(S^3\). For example, we can rotate the \(SU(2)\) background matrix \(E\) in (2.5,2.6) with the \(O(2,2)\) element

\[
\tilde{g}_{\alpha} = \begin{pmatrix} I & k \epsilon \\ 0 & I \end{pmatrix} \begin{pmatrix} C(\alpha) & S(\alpha) \\ S(\alpha) & C(\alpha) \end{pmatrix} \begin{pmatrix} I & -k \epsilon \\ 0 & I \end{pmatrix},
\]

(4.5)

where \(\epsilon, C(\alpha), S(\alpha)\) are given in (2.13). One finds

\[
\tilde{g}_{\alpha}(E) = k \frac{\Delta}{\Sigma} \begin{pmatrix} 1 - \Sigma & B \\ -B & 1 + \Sigma \end{pmatrix},
\]

(4.6)

where

\[
\Sigma = \cos 2x, \quad \Delta = \cos^2 \alpha (1 + \Sigma) + (\cos \alpha + k \sin \alpha)^2 (1 - \Sigma),
\]

\[
B = \frac{1}{k} \sin(2\alpha) - (\cos(2\alpha) + \frac{k}{2} \sin(2\alpha))(1 - \Sigma) + \Delta.
\]

(4.7)

The dilaton transforms by eq. (2.18).

The metric along the \(\alpha\) line of deformations is then given by the line element

\[
ds^2(\alpha) = \frac{k}{\Delta(\alpha)} [(1 - \Sigma)d\theta^2 + (1 + \Sigma)d\tilde{\theta}^2] + kdx^2.
\]

(4.8)

At the point \(\alpha = 0\) the background (4.6,4.8) is the \(SU(2)_k\) group manifold \(S^3\) (with an antisymmetric background). Along the line \(0 < \alpha < \pi/2\) the background (4.6) includes the metric (4.8) with the topology of \(S^3\), as well as an antisymmetric background and a dilaton field. At the point \(\alpha = \pi/2\) the background metric is

\[
ds^2(\alpha = \pi/2) = \frac{1}{k} [d\theta^2 + \frac{1 + \Sigma}{1 - \Sigma} d\tilde{\theta}^2] + kdx^2.
\]

(4.9)

At this point the manifold has a topology of \(D_2 \times S^1_{1/k}\), where \(D_2\) is a two disc and \(S^1_{1/k}\) is a circle with radius \(r^2 = 1/k\). One may continue to deform this theory by, for example, changing the compactification radius \(r\) of the free boson \(\theta\).

Therefore, we find that the quantum theories based on \(\sigma\)-models with topologically distinct target spaces in the extended moduli space of the WZW model are smoothly connected, even though classically a physical singularity is encountered.

It is remarkable that (for integer \(k\)) the neighborhood of the point \(\alpha = \pi/2\) is mapped to the neighborhood of the point \(\alpha = 0\) by an element of \(O(2,2,Z)\), namely, a target
space generalized duality \cite{22, 3, 23}. Therefore, a region in the moduli space where a
topology change occurs is mapped to a region where there is no topology change at all.
This is very similar to the observation made in the Calabi-Yau case \cite{6, 7}, where mirror
symmetry plays the same role as $O(2, 2, Z)$ plays here.

5 $J\bar{J}$ deformations in curved backgrounds

In this section we describe $J\bar{J}$ deformations of general WZW models. The relation of
these deformations to $O(d, d)$ transformations was described infinitesimally in \cite{14}
and in finite form in \cite{24}. We will start with a particular parametrization of a WZW model
for a group $G$. The group $G$ can be semisimple, although we will explicitly indicate one
of the levels $k$, while the others are hidden in the action. The relevant level is the one
corresponding to the simple component of the group whose Cartan we are deforming. The
following parametrization can be easily obtained using the Polyakov-Wiegmann formula
(see for example \cite{5, 14})

$$S[x^a, \theta_1, \theta_2] = S_1 + S_a + S[x],$$  \hspace{1cm} (5.1)

$$S_1 = \frac{k}{2\pi} \int d^2 z \left( \partial \theta_1 \bar{\partial} \theta_1 + \partial \theta_2 \bar{\partial} \theta_2 + 2\Sigma(x) \partial \theta_2 \bar{\partial} \theta_1 + \Gamma^1_a(x) \partial x^a \bar{\partial} \theta_1 + \Gamma^2_a(x) \partial \theta_2 \bar{\partial} x^a \right),$$  \hspace{1cm} (5.2)

$$S_a = \frac{k}{2\pi} \int d^2 z \left( \partial \theta_2 \bar{\partial} \theta_1 - \partial \theta_1 \bar{\partial} \theta_2 \right),$$  \hspace{1cm} (5.3)

$$S[x] = \frac{k}{2\pi} \int d^2 z \left( \Gamma^1_{ab}(x) \partial x^a \bar{\partial} x^b - \frac{1}{8\pi} \int d^2 z \phi_0 R^{(2)} \right),$$  \hspace{1cm} (5.4)

where $\Sigma(x), \Gamma^1(x), \Gamma^2(x)$ and $\Gamma(x)$ are independent of the coordinates $\theta_1, \theta_2$. With the coordinates $\theta$ and $\bar{\theta}$ defined in \cite{2, 3, 24}, the action becomes

$$S[x^a, \theta, \bar{\theta}] = \frac{1}{2\pi} \int d^2 z (\partial \theta, \bar{\partial} \bar{\theta}, \partial x^a) \begin{pmatrix} E & F^2_b \\ F^1_a & F_{ab} \end{pmatrix} \begin{pmatrix} \bar{\partial} \theta \\ \bar{\partial} \bar{\theta} \\ \partial x^b \end{pmatrix} - \frac{1}{8\pi} \int d^2 z \phi_0 R^{(2)},$$  \hspace{1cm} (5.5)

where the $2 \times 2$ background matrix $E$, the $D \times 2$ matrix $F^2_b$, the $2 \times D$ matrix $F^1_a$, and the $D \times D$ matrix $F_{ab}$ are given by

$$\begin{pmatrix} E & F^2_b \\ F^1_a & F_{ab} \end{pmatrix} = \frac{k}{2} \begin{pmatrix} 1 - \Sigma & 1 + \Sigma \\ -(1 + \Sigma) & 1 + \Sigma \\ -\Gamma^1_a & \Gamma^1_a \\ 2\Gamma_{ab} \end{pmatrix}.$$  \hspace{1cm} (5.6)

The action (5.5) is manifestly invariant under the $U(1)_L \times U(1)_R$ affine symmetry
generated by the currents

$$J = \frac{k}{2} \left( -(1 - \Sigma) \partial \theta + (1 + \Sigma) \bar{\partial} \bar{\theta} + \Gamma^1_a \partial x^a \right),$$

$$\bar{J} = \frac{k}{2} \left( (1 - \Sigma) \bar{\partial} \theta + (1 + \Sigma) \partial \bar{\theta} + \Gamma^2_a \bar{\partial} x^a \right).$$  \hspace{1cm} (5.7)
In addition, there are extra \( \text{dim}G - 1 \) chiral currents and \( \text{dim}G - 1 \) anti-chiral currents, completing the affine \( G_L \times G_R \) symmetry of the WZW model.

We now deform the action (5.5) with \( J \bar{J} \), where \( J \) and \( \bar{J} \) are given in (5.7). The \( U(1) \) chiral and anti-chiral currents at the deformed theory will be found.

The deformation with \( J \bar{J} \) is equivalent to \( g \in O(2, 2) \) rotations acting on the background matrices \( E, F^1, F^2, F \). The action of \( g \) in (2.8) on \( E \) is given in (2.10). Here we need also the action of \( g \) on the background matrices \( F^1, F^2, F \):

\[
\begin{align*}
g(F^1) &= F^1(cE + d)^{-1}, \\
g(F^2) &= (a - E'c)F^2, \\
g(F) &= F - F^1(cE + d)^{-1}cF^2,
\end{align*}
\]

(5.8)

where \( a, c, d \) are defined in (2.8), and \( E' \) is given in (2.10).

The one parameter family of the \( J \bar{J} \) deformation is given by acting on the background (5.6) with \( g_\alpha \) given in (2.14). Using eqs. (2.10, 5.8) one finds

\[
\frac{1}{k} g_\alpha(E) \equiv E_{R(\alpha)},
\]

(5.9)

\[
\frac{1}{k} g_\alpha(F^1) \equiv F^1_{R(\alpha)} = \frac{(1 + \Sigma)^{-1} \Gamma^1}{1 + R^2 \frac{1 + \Sigma}{1 + \Sigma}} \begin{pmatrix} -1 & R^2 \end{pmatrix},
\]

(5.10)

\[
\frac{1}{k} g_\alpha(F^2) \equiv F^2_{R(\alpha)} = \frac{(1 + \Sigma)^{-1} \Gamma^2}{1 + R^2 \frac{1 + \Sigma}{1 + \Sigma}} \begin{pmatrix} 1 & R^2 \end{pmatrix},
\]

(5.11)

\[
\frac{1}{k} g_\alpha(F) \equiv F_{R(\alpha)} = \Gamma + \frac{(R^2 - 1)(1 + \Sigma)^{-1} \Gamma^1 \Gamma^2}{2(1 + R^2 \frac{1 + \Sigma}{1 + \Sigma})},
\]

(5.12)

where \( E_R \) and \( R(\alpha) \) are given in (2.16, 2.17).

The constant dilaton \( \phi_0 \) in the WZW background (5.4) transforms under \( g_\alpha \) by eq. (2.18). For more details see ref. [3].

From eqs. (2.16, 5.10, 5.11, 5.12), we see that up to an overall \( k \) factor, the background is parametrized by one radius \( R^2 = (1 + k \tan \alpha)^2 \).

As in the \( SU(2) \) and \( SL(2) \) cases, the whole Cartan subalgebra survives along the deformation. In the parametrization we are using we have only one pair of these explicit

\[
\begin{align*}
J(R) &= k \frac{-(1 - \Sigma) \partial \theta + (1 + \Sigma) \partial \bar{\theta} + \Gamma^a \partial x^a}{1 + \Sigma + R^2(1 - \Sigma)}, \\
\bar{J}(R) &= k \frac{(1 - \Sigma) \partial \bar{\theta} + (1 + \Sigma) \partial \theta + \Gamma^a \partial x^a}{1 + \Sigma + R^2(1 - \Sigma)},
\end{align*}
\]

(5.13, 5.14)

\[
\text{Actually, one can bring the background in (5.3)-(5.4) into the form appearing in ref. [3] eq. (2.1) with } i = 1, ..., \text{rank}G. \text{ In this form the rank}G \text{ chiral currents and rank}G \text{ anti-chiral currents corresponding to the left-handed and right-handed Cartan tori are manifest.}
\]

13
and we can verify that that eq. (2.24) is still valid. In fact it does not matter with which current in the Cartan of a simple component we are deforming. Different choices are related by target space reparametrizations.

It is instructive here to discuss the counting of parameters of \( O(d,d) \) transformations related to that of marginal deformations. A general WZW model for a semi-simple group \( G \) has \( 2r \) Killing symmetries associated with the currents of the Cartan subalgebra of \( G_L \times G_R \), where \( r \) is the rank of \( G \). Thus the relevant group is \( O(2r,2r) \). \( GL(2r) \) transformations and antisymmetric tensor shifts preserve the presence of the current algebra, although they might change its spectrum. Out of this \((6r^2 - r)\)-dimensional subgroup no transformation preserves the action. There is a \( r(2r - 1) \)-dimensional manifold of left-over transformations which break the \( G \)-current algebra; these are of the type \( J\bar{J} \). Therefore, the \( J\bar{J} \) deformations correspond to a \( r(2r - 1) \) parameter family of the full \( O(2r,2r) \) group. Out of these, the \( r(r - 1) \)-dimensional subgroup \( O(r) \times O(r) \) leaves the action invariant. The rest \( r^2 \) transformations correspond precisely to all possible \( J\bar{J} \) marginal perturbations with currents in the Cartan subalgebra.

The deformation described here is true for any background with chiral and anti-chiral currents, and it is only for the purpose of discussing gauge symmetries that we assume that some point on this line (in our notation \( R = 1 \) ) is a WZW model.

In the next section we discuss target space duality on the \( R \)-line of deformations.

### 6 Duality on the \( J\bar{J} \) line is a residual broken gauge symmetry

Remarkably, the action of \( g_D \in O(2,2,Z) \) given in eq. (2.25) on the backgrounds \( E, F^1, F^2, F \) in eq. (5.6), gives a rather simple duality transformation on the \( R \)-modulus. By straightforward calculations using the transformations given in (2.10, 5.8) one finds

\[
\begin{pmatrix}
E_R & F^2_R \\
F^1_R & F_R
\end{pmatrix}
\begin{pmatrix}
E_{1/R} & F^2_{1/R} \\
F^1_{1/R} & F_{1/R}
\end{pmatrix} =
\begin{pmatrix}
E_R & F^2_R \\
F^1_R & F_R
\end{pmatrix}
\begin{pmatrix}
E_{1/R} & F^2_{1/R} \\
F^1_{1/R} & F_{1/R}
\end{pmatrix}.
\]

Therefore, duality takes the modulus \( R \) to its inverse \( 1/R \).

The results described in section 2 for the \( SU(2) \) and \( SL(2) \) cases are extended to the general case. The fixed point \( R = 1 \) corresponds to the extended symmetry point \( G_L \times G_R \), namely, the original WZW model. Infinitesimally around the extended symmetry point, duality corresponds to the transformation \( \alpha \rightarrow -\alpha \). This transformation can be achieved by a Weyl transformation in \( G_L \) (or \( G_R \) that reflects \( J \rightarrow -J \) (or \( \bar{J} \rightarrow -\bar{J} \)). Duality is related to a Weyl reflection and is, therefore, a residual discrete symmetry of the broken gauge algebra of the associated string theory. This provides a map between the two half lines, and in particular identify the two boundaries at \( R = 0 \) and \( R = \infty \). In the following we show that these boundaries correspond, respectively, to the direct product of the cosets \( G/U(1)_a \) and \( G/U(1)_a \) with a free scalar field.
At $R = 0$ ($\alpha = \tan^{-1}(-\frac{1}{k})$) the background matrix is

$$E_{R=0} = \begin{pmatrix}
\frac{1 - \Sigma}{1 + \Sigma} & 1 \\
-1 & 0
\end{pmatrix} = \begin{pmatrix}
\frac{1 - \Sigma}{1 + \Sigma} & 0 \\
0 & 0
\end{pmatrix} \mod \Theta \text{ shift},$$

$$F^1_{R=0} = (1 + \Sigma)^{-1} \Gamma^1(-1,0), \quad F^2_{R=0} = (1 + \Sigma)^{-1} \Gamma^2 \begin{pmatrix}
1 \\
0
\end{pmatrix},$$

$$F_{R=0} = \Gamma - \frac{1}{2} (1 + \Sigma)^{-1} \Gamma^1 \Gamma^2. \quad (6.2)$$

This background corresponds \cite{22,3} to the direct product of the vectorially gauged coset $G/U(1)_v$ and a free boson at a compactification radius $r = 0$ (that is equivalent to a non-compact free boson via the $r \to 1/r$ duality).

At $R = \infty$ ($\alpha = \pi/2$) the background matrix is

$$E_{R=\infty} = \begin{pmatrix}
0 & 0 \\
0 & \frac{1 - \Sigma}{1 + \Sigma}
\end{pmatrix}, \quad F_{R=\infty} = \Gamma + \frac{1}{2} (1 - \Sigma)^{-1} \Gamma^1 \Gamma^2,$$

$$F^1_{R=\infty} = (1 - \Sigma)^{-1} \Gamma^1(0,1), \quad F^2_{R=\infty} = (1 - \Sigma)^{-1} \Gamma^2 \begin{pmatrix}
0 \\
1
\end{pmatrix}. \quad (6.3)$$

This background corresponds \cite{22,3} to the direct product of the axially gauged coset $G/U(1)_a$ and a free boson at a compactification radius $r = 0$.

Therefore, like the $SU(2)$ and $SL(2,R)$ cases, because the end-points are equivalent theories we obtain that axial-vector duality is exact in general.

### 7 Comments and open problems

In this work we have studied some aspects of the global structure of the moduli space of curved string backgrounds with $d$ toroidal isometries. The moduli space of such CFTs (at the $\sigma$ model level) contains the moduli space that is generated by $O(d,d,R)$ transformations. The latter contains all marginal deformations of the $JJ$ kind, with $J, \bar{J}$ in the Cartan. The symmetries acting on the $O(d,d,R)$ space are generated by the $O(d,d,Z)$ group. What we have shown in this work is that some elements of this group, corresponding to axial-vector duality of abelian cosets of WZW models, can be viewed as residual gauge symmetries since around a WZW point they are related by a continuous group transformation. Do all the $O(d,d,Z)$ transformations correspond to residual gauge symmetries? This question was addressed in the flat case \cite{3} where it was shown that any $O(d,d,Z)$ transformation corresponds to a product of discrete residual gauge symmetries around points with extended symmetry.

At the $\sigma$-model level, the deformations given by $O(d,d)$ transformations are not the full story; one can generate more (equivalent) backgrounds by deforming, for example, with $JJ$ combinations different from the ones considered here (namely, of the type
\[
\left( \sum_{a=1}^{\dim G} \alpha_a J^a \right) \left( \sum_{b=1}^{\dim G} \beta_b \bar{J}^b \right), \quad \text{where } D = \dim G.
\]
Although these are related by transformations in the \( G_L \times G_R \) symmetry group, they give \( \sigma \)-models which are not related to the previous one by a coordinate transformation (unless the transformation is in the diagonal \( G \)). A similar effect appears when gauging \( U(1) \) sub-groups, that are embedded differently in the group, giving rise to a family of different actions for the same coset.

The deformation line of the \( SL(2) \) model described in section 2 interpolate between the \textit{euclidean} abelian cosets. Therefore, we proved duality for the euclidean \( SL(2)/U(1) \) coset. A proof for the lorentzian case proceeds along the same lines by deforming with the \( J\bar{J} \) corresponding to the non-compact \( U(1) \) in \( SL(2) \).

A final remark concerns “topology change”. The marginal deformations discussed in this paper provide some concrete examples of continuous change of topology by relating the axial to vector cosets, and to the original \( WZW \) points. We found that a smooth topology change occurs in the extended moduli space of \( SU(2)_k \) model. This can be extended to the moduli space of \( SL(2)_k \), and other curved string backgrounds with one time-like coordinate; it therefore provides an evidence to a \textit{classical topology change in cosmological string backgrounds} (and other curved space-time). Moreover, in these cases the different topologies might give rise to equivalent theories, a phenomenon that is not unheard of in string theory.

\section*{Acknowledgments}

We thank S. Elitzur, C. Kounnas, A. Polychronakos, E. Rabinovici, M. Roček and A. Shapere for discussions. We also thank C. Kounnas and M. Roček for useful comments on the paper. AG would like to thank the Institute for theoretical Physics at Santa Barbara where this work was completed. EK would like to thank the Tel Aviv University and especially the Hebrew University in Jerusalem for their warm hospitality while this work was done. This work was supported in part by NSF grant No. PHY89-04035. We would also like to thank the referee for prompting us to make our arguments clearer.

\section*{Note Added}

In \cite{25}, Moore gave a precise prescription of Conformal Perturbation Theory which preserves the duality symmetry \( R \rightarrow 1/R \) in the theory of compact scalar fields. It is straightforward to show that precisely the same prescription does the job here, both in the compact and non-compact theories, since an arbitrary correlation function factorizes into that of a free boson and a parafermionic one, which does not feel the perturbation. The interpretation of this result is that the duality symmetry is non-anomalous in perturbation theory. We thank the referee for bringing this to our attention.
In [26], Tseytlin showed that there is a scheme in $\sigma$-model perturbation theory where the semiclassical background for the (compact or non-compact) parafermion theory is exact to all orders. Our arguments show that such a scheme exists for the whole line of theories and not only for the boundary or WZW points. The above make the following conjecture highly plausible: For all coset models (compact or non-compact) there is a scheme where the semiclassical background (obtained from the gauged WZW model) is exact.
Appendix A.

In this Appendix we will give the detailed argument concerning the exactness of the \( \sigma \)-model picture. Conformal perturbation theory (CPT) indicates that there is a line of theories obtained by perturbing around the \( SU(2) \) or \( SL(2) \) WZW model by \( \int J^3\bar{J}^3 \). The theories along the line have a \( U(1)_L \times U(1)_R \) chiral symmetry. Thus the \( \sigma \)-model action of these theories must satisfy at least the following three properties

1) It should have \( U(1)_L \times U(1)_R \) chiral symmetry along the line.

2) It should have the group property: \( \delta S \sim \int J^3\bar{J}^3 \) at any point of the line (this is a property obvious in CPT).

3) At a specific point it should reduce to the known action of the \( SU(2) \) or \( SL(2) \) WZW model.

The most general 3-d action which satisfies property (1) is \[ S[x,\psi_1,\psi_2] = S_1 + S_2 + S_3, \] \( (A.1) \)

\[ S_1 = \frac{1}{2\pi} \int d^2 z \left( \partial \theta_1 \bar{\partial} \theta_1 + \partial \theta_2 \bar{\partial} \theta_2 + 2\Sigma(x,\lambda)\partial \theta_1 \bar{\partial} \theta_2 + \Gamma_1(x,\lambda)\partial x \bar{\partial} x + \Gamma_2(x,\lambda)\partial \theta_{1} \bar{\partial} x \right), \] \( (A.2) \)

\[ S_2 = \frac{1}{2\pi} \int d^2 z B(\lambda) (\partial \theta_1 \bar{\partial} \theta_2 - \partial \theta_2 \bar{\partial} \theta_1) + \frac{1}{2\pi} \int d^2 z \Gamma(x,\lambda)\partial x \bar{\partial} x, \] \( (A.3) \)

\[ S_3 = -\frac{1}{8\pi} \int d^2 z \phi(x,\lambda) R^{(2)}, \] \( (A.4) \)

where

\[ \theta_i = \sum_{j=1}^{2} \alpha_{ij}(\lambda)\psi_j, \] \( (A.5) \)

and we have explicitly indicated the dependence on the continuous parameter \( \lambda \). Here the \( \psi_i \) are \( \lambda \)-independent angular coordinates.

The chiral currents can be calculated to be

\[ J(\lambda) = \partial \theta_2 + \Sigma(x,\lambda)\partial \theta_1 + \frac{1}{2} \Gamma_1(x,\lambda)\partial x \] \( (A.6) \)

\[ \bar{J}(\lambda) = \bar{\partial} \theta_1 + \Sigma(x,\lambda)\bar{\partial} \theta_2 + \frac{1}{2} \Gamma_2(x,\lambda)\bar{\partial} x. \] \( (A.7) \)

We now impose property (2) to \( S_1 + S_2 \) \[ \frac{\partial}{\partial \lambda}S_{1+2}(\lambda) = \frac{g(\lambda)}{2\pi} \int J(\lambda)\bar{J}(\lambda), \] \( (A.8) \)

where \( g(\lambda) \) reflects the freedom of independent normalization of the currents along the line. However, it can always be set to any fixed number by a reparametrization in \( \lambda \).

\( \bar{J} \) The dilaton should be considered as part of the measure of the path integral since its contribution is visible at the one loop level. It should of course be determined and we will do so in two independent ways, either from the requirement that the measure is correct along the line, as it was explained in the main text, or just from the requirement of conformal invariance that we will use here.
Thus, from now on, we will assume without loss of generality that \( g = 2 \). Then (A.8) gives a system of first order non-linear differential equations which can be integrated explicitly. Imposing also the boundary conditions:

\[
\alpha_{ij}(0) = \begin{pmatrix} \sqrt{k} & 0 \\ 0 & \sqrt{k} \end{pmatrix}, \quad (A.9)
\]

\[B(0) = k, \quad \Sigma(x,0) = \Sigma(x), \quad (A.10)\]

\[
\Gamma_1(x,0) = \Gamma_2(x,0) = 0, \quad \Gamma(x,0) = k; \quad (A.11)
\]

we obtain

\[
\alpha_{ij}(\lambda) = \sqrt{k} \begin{pmatrix} \cosh(\lambda) & \sinh(\lambda) \\ \sinh(\lambda) & \cosh(\lambda) \end{pmatrix}, \quad (A.12)
\]

\[B = k, \quad \Sigma(x,\lambda) = \frac{1 - \frac{1 - \Sigma(x)}{1 + \Sigma(x)} e^{-2\lambda}}{1 + \frac{1 - \Sigma(x)}{1 + \Sigma(x)} e^{-2\lambda}}, \quad (A.13)\]

\[
\Gamma_1(x,\lambda) = \Gamma_2(x,\lambda) = 0, \quad \Gamma(x,\lambda) = k. \quad (A.14)
\]

This is precisely the background (2.16) with the identification \( R = e^{-\lambda} \).

So far we have not imposed conformal invariance. At one-loop the \( \beta \)-function equations amount to (prime indicates differentiation with respect to \( x \))

\[\phi = -\log[\Sigma'/\sqrt{1 - \Sigma^2}] + \text{constant} \quad (A.15)\]

and

\[
\left( \frac{\Sigma''}{\Sigma'} \right)' + \frac{2\Sigma\Sigma'' + \Sigma'^2}{1 - \Sigma^2} + 3 \frac{\Sigma^2\Sigma'^2}{(1 - \Sigma^2)^2} = 0 \quad (A.16)
\]

It is important to note that with \( \Sigma(x) = \cos(2x) \) (\( SU(2) \)) or \( \cosh(2x) \) (\( SL(2) \)) (A.13) satisfies the one-loop equation (A.16). The only way (A.13) can change consistent with our requirements (1-3) is by a redefinition of \( R \), which implies that there is a scheme in \( \sigma \)-model perturbation theory where the metric and the antisymmetric tensor receive no higher order corrections in \( \alpha' \). In such a case also the dilaton receives no higher order corrections \[^{27}\].

Consequently, the dilaton is given by

\[\phi(x, R) = \log[1 + \frac{1 - R^2}{1 + R^2} \Sigma(x)] + f(R). \quad (A.17)\]

The \( R \)-dependent constant can be fixed by the requirements that \( \sqrt{G(R)} e^{\phi(R)} \) (which represents the physical string coupling) is invariant along the line. This gives the formula for the dilaton presented in section 2.

[^{27}]: The dilaton \( \beta \)-function (central charge) does get corrections. This is what is happening also in the WZW model. However, like that case, one can replace \( k \) with \( k + 2 \) in front of the action. Then the central charge is given by the classical and 1-loop piece only, without spoiling the vanishing of the other \( \beta \)-functions.
The arguments above show that (2.16,2.18) describe the correct (to all orders in \( \alpha' \)) \( \sigma \)-model background associated with the deformation of the WZW model. One further comment is in order to ensure that our conclusions hold non-perturbatively in \( \alpha' \).

The correlators of a generic theory along the \( R \)-line (compact or not) are products of a parafermionic correlator coupled to a block of a boson at radius \( R/\sqrt{k} \). For the boson blocks we know their explicit structure. What remains to analyse is the non-perturbative structure of the parafermionic blocks (which is the same as that of the respective WZW).

In the compact case there are non-perturbative contributions which can be seen to come from the non-trivial affine null vectors. A look at the exact torus partition function (3.1) suffices to note that the non-pertrubative terms are precisely those coming from the subtraction of affine null vectors (with dimensions of \( O(k) \)). This persists for partition functions at any genus. An independent look at the sphere four-point amplitudes confirms again that the non-perturbative terms come from the part of the overall coefficient which enforces the affine cutoff (which we know to be the consequence of affine null vectors). Such non-perturbative corrections do not spoil the \( R \rightarrow 1/R \) duality. In the non-compact case no non-perturbative corrections are expected since there are no non-trivial affine null vectors. A computation of the partition function in this case at \( R = 1 \) corroborates this statement [16].
References

[1] K. Narain, M. Sarmadi and E. Witten, Nucl. Phys. B279 (1987) 369.

[2] K.A. Meissner and G. Veneziano, Phys. Lett. B267 (1991) 33;
M. Gasperini, J. Maharana and G. Veneziano, Phys. Lett. B272 (1991) 277;
A. Sen, Phys. Lett. B271 (1991) 295.

[3] A. Giveon and M. Roček, Nucl. Phys. B380 (1992) 128.

[4] A. Giveon, E. Rabinovici and G. Veneziano, Nucl. Phys. B322 (1989) 167;
A. Shapere and F. Wilczek, Nucl. Phys. B320 (1989) 669;
A. Giveon, N. Malkin and E. Rabinovici, Phys. Lett. B220 (1989) 551.

[5] E.B. Kiritsis, Mod. Phys. Lett. A6 (1991) 2871.

[6] P.S. Aspinwall, B.R. Greene and D. Morrison, Phys.Lett. B303 (1993) 249.

[7] E. Witten, Nucl. Phys. B403 (1993) 159.

[8] M. Dine, P. Huet and N. Seiberg, Nucl. Phys. B322 (1989) 301.

[9] A. Giveon, N. Malkin and E. Rabinovici, Phys. Lett. B238 (1990) 57.

[10] A. Giveon and M. Porrati, Phys. Lett. B246 (1990) 54;
A. Giveon and M. Porrati, Nucl. Phys. B355 (1991) 422.

[11] T.Kugo and B. Zwiebach, Prog. Theor. Phys. 87 (1992) 801.

[12] A. Giveon, Nucl. Phys. B391 (1993) 229.

[13] L. Ibanez and D. Lüst, Phys.Lett. B302 (1993) 38.

[14] E. Kiritsis, Nucl. Phys. B405 (1993) 109.

[15] A. Giveon, Mod. Phys. Lett. A6 (1991) 2843;
R. Dijkgraaf, E. Verlinde, and H. Verlinde, Nucl. Phys. B371 (1992) 269.

[16] K. Gawedski, Lectures given at the Cargese Summer Institute “ New Symmetry
Principles in QFT”, July 1991.

[17] S.F. Hassan and A. Sen, Nucl. Phys. B405 (1993) 143.

[18] K. Bardakci, M. Crescimanno and E. Rabinovici, Nucl. Phys. B344 (1990) 344.

[19] D. Gepner and Z. Qiu, Nucl. Phys. B285 (1987) 423.

[20] S.K. Yang, Phys. Lett. B209 (1988) 242.

[21] V. Kač and D. Peterson, Adv. Math. 53 (1984) 125.
[22] M. Roček and E. Verlinde, Nucl. Phys. B373 (1992) 630.

[23] A. Kumar, Phys. Lett. B293 (1992) 49.

[24] M. Henningson and C. Nappi, Phys.Rev. D48 (1993) 861.

[25] G. Moore, “Finite in All Directions”, hepth/9305139.

[26] A. Tseytlin, “On Field Redefinitions and Exact Solutions in String Theory”, hepth/9308042.

[27] R. Metsaev and A. Tseytlin, Nucl. Phys. B293 (1987) 385.

[28] I. Bakas and E. Kiritsis, Nucl. Phys. B343 (1990) 343.