OPTIMAL LENGTH ESTIMATES FOR STABLE CMC SURFACES IN 3-SPACE FORMS

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Abstract. In this paper, we study stable constant mean curvature $H$ surfaces in $\mathbb{R}^3$. We prove that, in such a surface, the distance from a point to the boundary is less than or equal to $\pi/(2H)$. This upper bound is optimal and is extended to stable constant mean curvature surfaces in space forms.

1. Introduction

A constant mean curvature (cmc) surface $\Sigma$ in a Riemannian 3-manifold $\mathbb{M}^3$ is stable if its stability operator, $L = -\Delta - \text{Ric}(n, n) - |A|^2$, is nonnegative, where $\Delta$ is the Laplace operator on $\Sigma$, Ric is the Ricci tensor on $\mathbb{M}^3$, $n$ is the normal along $\Sigma$, and $A$ is the second fundamental form on $\Sigma$. For minimal surfaces ($H = 0$), this characterization is only valid for two-sided surfaces, so in the following we restrict ourselves to such surfaces. The nonnegativity of the stability operator means that $\Sigma$ is a local minimizer of the area functional on surfaces with regard to the infinitesimal deformations fixing its boundary.

The stability hypothesis was studied by several authors and has many consequences (see [6] for an overview). For example, D. Fischer-Colbrie and R. Schoen [4] studied the case of complete stable minimal surfaces when $\mathbb{M}^3$ has nonnegative scalar curvature. They obtain that the universal cover of $\Sigma$ is not conformally equivalent to the disk and, as a consequence, prove that the plane is the only complete stable minimal surface in $\mathbb{R}^3$. From this, R. Schoen [9] has derived a curvature estimate for stable cmc surfaces.

In [2], T. H. Colding and W. P. Minicozzi introduced new techniques and obtained area and curvature estimates for stable cmc surfaces. Afterward, these techniques were used by P. Castillon [1] to answer a question asked in [4] about the consequences of the positivity of certain elliptic operators. Recently, the same ideas have been used by J. Espinar and H. Rosenberg [3] to obtain similar results.

In [7], A. Ros and H. Rosenberg study constant mean curvature $H$ surfaces in $\mathbb{R}^3$ with $H \neq 0$. They prove a maximum principle at infinity. One of their tools is a length estimate for stable cmc surfaces. In fact, they prove that the intrinsic distance from a point $p$ in a stable cmc surface $\Sigma$ to the boundary of $\Sigma$ is less than $\pi/H$. H. Rosenberg [8] has generalized this result to any ambient 3-manifolds and large mean curvature. The aim of this paper is to improve the result of Ros...
and Rosenberg. In fact, applying the ideas of [2], we prove that the distance is less than $\pi/(2H)$. This estimate is optimal since, for a hemisphere of radius $1/H$, the distance from the pole to the boundary is $\pi/(2H)$. Actually we prove that the hemisphere of radius $1/H$ is the only stable $\text{cmc} H$ surface where the distance $\pi/(2H)$ is reached. We can generalize this result to stable $\text{cmc} H$ surfaces in $\mathbb{M}^3(\kappa)$, where $\mathbb{M}^3(\kappa)$ is the 3-space form of sectional curvature $\kappa$. We prove that when $H^2 + \kappa > 0$ such an optimal estimate exists. In fact, it is already known that when $\kappa \leq 0$ and $H^2 + \kappa \leq 0$, there is no such estimate since there exist complete stable $\text{cmc} H$ surfaces. But, in some sense, our results are an extension of the fact that the planes (resp. the horospheres) are the only stable complete constant mean curvature $H$ surfaces in $\mathbb{R}^3$ (resp. $\mathbb{M}^3(\kappa), \kappa < 0$) when $H = 0$ (resp. $H^2 + \kappa = 0$).

2. Definitions

On a constant mean curvature surface $\Sigma$ in a Riemannian 3-manifold $M^3$, the stability operator is defined by $L = -\Delta - \text{Ric}(n, n) - |A|^2$, where $\Delta$ is the Laplace operator on $\Sigma$, $\text{Ric}$ is the Ricci tensor on $M^3$, $n$ is the normal along $\Sigma$ and $A$ is the second fundamental form on $\Sigma$. When it is necessary, we will denote the stability operator by $L_f$ to refer to the immersion $f$ of $\Sigma$ in $M^3$. The surface $\Sigma$ is called stable if the operator $L$ is nonnegative; i.e., for every compactly supported function $u$, we have

$$0 \leq \int_{\Sigma} uL(u)d\sigma = \int_{\Sigma} \|\nabla u\|^2 - (\text{Ric}(n, n) + |A|^2)u^2d\sigma.$$ 

We remark that this property is sometimes called strong stability since it means that the second derivatives of the area functional are nonnegative with respect to any compactly supported infinitesimal deformations $u$, whereas $\Sigma$ is critical for this functional only for compactly supported infinitesimal deformations with vanishing mean value, i.e. $\int_{\Sigma} ud\sigma = 0$.

In the following, on a $\text{cmc}$ surface, the normal $n$ is always chosen such that $H$ is nonnegative.

We will denote by $d_\Sigma$ the intrinsic distance on $\Sigma$ and by $K$ the sectional curvature of the surface.

3. Results

The main result of this paper is the following theorem.

**Theorem 3.1.** Let $H$ be positive. Let $\Sigma$ be a stable constant mean curvature $H$ surface in $\mathbb{R}^3$. Then, for $p \in \Sigma$, we have

$$d_\Sigma(p, \partial \Sigma) \leq \frac{\pi}{2H}.$$ \hspace{1cm} (3.1)

Moreover, if the equality is satisfied, $\Sigma$ is a hemisphere.

In $\mathbb{R}^3$, the stability operator can be written $L = -\Delta - 4H^2 + 2K$.

**Proof.** We denote by $R_0$ the distance $d_\Sigma(p, \partial \Sigma)$ and assume that $R_0 \geq \pi/(2H)$. If $R_0 < \pi/H$ we denote by $I$ the segment $[\pi/(2H), R_0]$; otherwise $I = [\pi/(2H), \pi/H)$. In fact, because of the work of Ros and Rosenberg [7], we already know that $R_0 \leq \pi/H$. Let $R$ be in $I$.

The surface $\Sigma$ has constant mean curvature $H$; thus its sectional curvature is less than $H^2$. So the exponential map $\exp_p$ is a local diffeomorphism on the disk
$D(0, R) \subset T_p \Sigma$ of center 0 and radius $R$. On this disk, we consider the induced metric and the operator \( L = -\Delta - 4H^2 + 2K \). The surface \( \Sigma \) is stable, so there exists a positive function \( g \) on \( \Sigma \) such that \( L(g) = 0 \) (see Theorem 1 in [4]). On \( D(0, R) \), the function \( \tilde{g} = g \circ \exp_p \) is then positive and satisfies \( L(\tilde{g}) = 0 \) since \( D(0, R) \) and \( \Sigma \) are locally isometric. The operator \( L \) is thus nonnegative on \( D(0, R) \) [4].

For \( r \in [0, R] \), we define \( l(r) \) as the length of the circle \( \{v, |v| = r\} \subset D(0, R) \) and \( K(r) = \int_{D(0, r)} K \sigma \). Since \( D(0, R) \) and \( \Sigma \) are locally isometric, the sectional curvature \( K \) of \( D(0, R) \) is less than \( H^2 \). Then

\[
l(r) \geq \frac{2\pi}{H} \sin Hr.
\]

By Gauss-Bonnet, we have

\[
K(r) = 2\pi - l'(r).
\]

Let us consider a function \( \eta : [0, R] \to [0, 1] \) with \( \eta(0) = 1 \) and \( \eta(R) = 0 \). Let us write the nonnegativity of \( L \) for the radial function \( u = \eta(r) \):

\[
0 \leq \int_0^R (\eta'(r))^2 l(r)dr - 4H^2 \int_0^R \eta^2(r)l(r)dr + 2 \int_0^R K'(r)\eta^2(r)dr.
\]

Hence, following the ideas in [2] and using (3.3) and the boundary values of \( \eta \), we have

\[
\int_0^R (4H^2\eta^2 - \eta'^2)ldr \leq 2 \left( [K(r)\eta^2(r)]_0^R - \int_0^R K(r)(\eta^2(r))'dr \right)
\]

\[
= -2 \int_0^R K(r)(\eta^2(r))'dr
\]

\[
= -2 \int_0^R (2\pi - l'(r))(\eta^2(r))'dr
\]

\[
= 4\pi + 2 \int_0^R (\eta^2(r))'l'(r)dr
\]

\[
= 4\pi + [2(\eta^2(r))'l(r)]_0^R - 2 \int_0^R (\eta^2(r))''l(r)dr
\]

\[
= 4\pi - 2 \int_0^R (\eta^2(r))''l(r)dr.
\]

Thus we obtain

\[
\int_0^R \left( 4H^2\eta^2 - \eta'^2 + 2(\eta^2)'' \right) ldr \leq 4\pi.
\]

We shall apply this equation to the function \( \eta(r) = \cos \frac{\pi r}{2R} \). In this case we have

\[
\eta^2 = \frac{\pi^2}{4R^2} \sin^2 \frac{\pi r}{2R},
\]

\[
(\eta^2)'' = \frac{\pi^2}{2R^2} \left( \cos^2 \frac{\pi r}{2R} - \sin^2 \frac{\pi r}{2R} \right).
\]

Thus

\[
4H^2\eta^2 - \eta'^2 + 2(\eta^2)'' = (4H^2 - \frac{\pi^2}{R^2}) \cos^2 \frac{\pi r}{2R} + \frac{3\pi^2}{4R^2} \sin^2 \frac{\pi r}{2R}.
\]
As $R \geq \pi / 2H$, $4H^2 \eta^2 - \eta'^2 + 2(\eta^2)''$ is nonnegative and, by (3.2),
\[
\left(4H^2 \eta^2 - \eta'^2 + 2(\eta^2)''\right) I \geq \left(4H^2 - \frac{\pi^2}{R^2}\right) \cos^2 \frac{\pi x}{2R} + \frac{3\pi^2}{4R^2} \sin^2 \frac{\pi x}{2R} \frac{2\pi}{H} \sin H r
\]

Thus integrating in (3.3), we obtain (we recall that $R < \pi / H$)
\[
4\pi \geq \frac{\pi}{H} \left(4H^2 - \frac{\pi^2}{4R^2}\right) \frac{1}{\pi}(1 - \cos HR)
\]
\[
+ \left(\frac{R}{\pi + HR}(1 - \cos(\pi + HR)) - \frac{R}{\pi - HR}(1 - \cos(\pi - HR))\right).
\]

After some simplifications in the above expression, we obtain
\[
4\pi \geq \frac{4\pi}{\pi^2 - H^2 R^2} \left(-32H^2 R^4 + 24\pi^2 H^2 R^2 - \pi^4 - 10\pi^2 H^2 R^2 - \pi^4 \cos HR\right).
\]

Now, passing $4\pi$ on the right-hand side of the above inequality and simplifying by $\pi$, we get
\[
F(R) := \frac{-4H^2 R^2 - \pi^2 - 10\pi^2 H^2 R^2 - \pi^4 \cos HR}{4H^2 R^2(\pi^2 - H^2 R^2)} \leq 0.
\]
If we write $R = \pi / (2H) + x$, we compute the Taylor expansion of $F$ and obtain
\[
F\left(\frac{\pi}{2H} + x\right) = 2H x + o(x),
\]
which is positive if $x > 0$. Thus, if $R_0 > \pi / (2H)$, we get a contradiction and the inequality (3.1) is proved.

Now if $R_0 = \pi / (2H)$, we have in fact equality all along the computation, so $l(r) = (2\pi / H) \sin H r$ and $K(r) = 2\pi - b'(r) = 2\pi(1 - \cos H r)$. But we also know that the sectional curvature is less than $H^2$; thus $K(r) \leq H^2$. Since this inequality is in fact an equality, the sectional curvature is in fact $H^2$ at every point. Thus the principal curvatures of a point in $\Sigma$ are $H$ and $H$; i.e. there are only umbilical points. Hence $\Sigma$ is a piece of a sphere of radius $1 / H$ and, since $d_S(p, \partial\Sigma) = \frac{\pi}{\pi R}$, it contains the hemisphere of pole $p$. A hemisphere cannot be strictly contained in a stable subdomain of the sphere, so $\Sigma$ is a hemisphere. □

With this result we have an important corollary.

**Corollary 3.2.** Let $H \geq 0$ and $\kappa \in \mathbb{R}$ such that $H^2 + \kappa > 0$. Let $\Sigma$ be a stable constant mean curvature $H$ surface in $\mathbb{M}^3(\kappa)$. Then for $p \in \Sigma$, we have
\[
d_S(p, \partial \Sigma) \leq \frac{\pi}{2\sqrt{H^2 + \kappa}}.
\]

Moreover, if the equality is satisfied, $\Sigma$ is a geodesical hemisphere of $\mathbb{M}^3(\kappa)$.

**Proof.** First, the case $\kappa = 0$ is Theorem 3.1.

Let $\Pi : \tilde{\Sigma} \to \Sigma$ be the universal cover of $\Sigma$. We then have a constant mean curvature immersion of $\Sigma$ in $\mathbb{M}^3(\kappa)$. Let $L = -\Delta - 2\kappa - |A|^2$ be the stability operator on $\Sigma$. $\Sigma$ is stable, so there exists a positive function $g$ on $\Sigma$ such that $L(g) = -\Delta g - (2\kappa + |A|^2) g = 0$. Thus the function $\tilde{g} = g \circ \Pi$ is a positive function.
on \( \tilde{\Sigma} \) satisfying \( L(\tilde{g}) = 0 \). Hence \( \tilde{\Sigma} \) is stable. Let \( I \) and \( S \) be respectively the first fundamental form and the shape operator on \( \tilde{\Sigma} \). They satisfy the Gauss and Codazzi equations for \( M^3(\kappa) \).

We define \( S' = S + (-H + \sqrt{H^2 + \kappa}) \text{id} \) on \( \tilde{\Sigma} \). Then \( I \) and \( S' \) satisfy the Gauss and Codazzi equations for \( M^3(0) = \mathbb{R}^3 \) (see [5]). Hence there exists an immersion \( f \) of \( \tilde{\Sigma} \) in \( \mathbb{R}^3 \) with first fundamental form \( I \) and shape operator \( S' \) (we notice that the induced metric is the same). Its mean curvature is then \( H + (-H + \sqrt{H^2 + \kappa}) = \sqrt{H^2 + \kappa} \); i.e., the immersion has constant mean curvature. The stability operator is

\[
L_f = -\Delta - \|S'\|^2
= -\Delta - (\|S\|^2 + 4H(-H + \sqrt{H^2 + \kappa}) + 2(-H + \sqrt{H^2 + \kappa})^2)
= -\Delta - (\|S\|^2 + 2\kappa)
= \mathcal{L}.
\]

Hence the surface \( f(\tilde{\Sigma}) \) is stable. So, from Theorem 3.1, we have

\[
d_{\Sigma}(p, \partial \Sigma) = d_{\tilde{\Sigma}}(\tilde{p}, \partial \tilde{\Sigma}) \leq \frac{\pi}{2\sqrt{H^2 + \kappa}},
\]

where \( \Pi(\tilde{p}) = p \).

The equality case comes from the equality case in Theorem 3.1 and since the Lawson correspondence sends spheres into spheres.

\[\square\]

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