The asymptotics of a generalised Beta function

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Abstract

We consider the generalised Beta function introduced by Chaudhry et al. [J. Comp. Appl. Math. 78 (1997) 19–32] defined by

\[ B(x, y; p) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left[-\frac{p}{4t(1-t)}\right] dt, \]

where \( \Re(p) > 0 \) and the parameters \( x \) and \( y \) are arbitrary complex numbers. The asymptotic behaviour of \( B(x, y; p) \) is obtained when (i) \( p \) large, with \( x \) and \( y \) fixed, (ii) \( x \) and \( p \) large, (iii) \( x, y \) and \( p \) large and (iv) either \( x \) or \( y \) large, with \( p \) finite. Numerical results are given to illustrate the accuracy of the formulas obtained.

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1. Introduction

In [1], Chaudhry et al. introduced a generalised beta function defined by the Euler-type integral

\[ B(x, y; p) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left[-\frac{p}{4t(1-t)}\right] dt, \quad (1.1) \]

where \( \Re(p) > 0 \) and the parameters \( x \) and \( y \) are arbitrary complex numbers. When \( p = 0 \), it is clear that when \( \Re(x) > 0 \) and \( \Re(y) > 0 \) the generalised function reduces to the well-known beta function \( B(x, y) \) of classical analysis. The justification for defining this extension of the beta function is given in [1] and an application of its use in defining extensions of the Gauss and confluent hypergeometric functions is discussed in [2]. It is evident from the definition in (1.1) that \( B(x, y; p) \) satisfies the symmetry property

\[ B(x, y; p) = B(y, x; p). \quad (1.2) \]

A list of useful properties of \( B(x, y; p) \) is detailed by Miller in [4], where it is established that \( B(x, y; p) \) may be expanded as an infinite series of Whittaker functions or Laguerre polynomials; see (A.1). He also obtained a Mellin-Barnes integral representation for

\footnote{The factor 4 is introduced in the exponential for presentational convenience.}
\[ B(x, y; p), \text{ which we exploit in Section 2, and expressed } B(x, x \pm n; p) \text{ and } B(1 \pm n, 1; p), \]

where \( n \) is an integer, as finite sums of Whittaker functions.

Our aim in this note is to derive asymptotic expansions for \( B(x, y; p) \) for large \( x, y \) and \( p \). We consider (i) \( |p| \to \infty \) in \( |\arg p| < \frac{1}{2}\pi \), with \( x \) and \( y \) fixed, (ii) \( x \) and \( p \) large, (iii) \( x, y \) and \( p \) large and (iv) either \( x \) or \( y \) large, with \( p \) finite. The expansion for large \( p \) is obtained using a Mellin-Barnes integral representation for \( p \) functions; see \([4, \text{Eq. (1.6)}]\).

The integral may be evaluated by the well-known Cahen-Mellin integral given by (see, for example, \([8, \text{p. 90}]\))

\[
\int_{\arg z \to \pi} \frac{\Gamma(s + \alpha)z^{-s}ds}{\Gamma(\alpha)} = z^\alpha e^{-z} \quad (|\arg z| < \frac{1}{2}\pi, \ c > -\Re(\alpha))
\]

2. The expansion of \( B(x, y; p) \) for large \( p \) with \( x, y \) finite

We start with the Mellin-Barnes integral representation given by Miller \([4]\)

\[
B(x, y; p) = 2^{1-x-y-\frac{1}{2}} \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{\Gamma(s) \Gamma(x + s) \Gamma(y + s)}{\Gamma(\frac{1}{2} x + \frac{1}{2} y + s) \Gamma(\frac{1}{2} x + \frac{1}{2} y + \frac{1}{2} + s)} p^{-s} ds \quad (2.1)
\]

valid in \( |\arg p| < \frac{1}{2}\pi \), where \( c > \max\{0, -\Re(x), -\Re(y)\} \) so that the integration path lies to the right of all the poles of the integrand situated at \( s = -k, s = -x - k \) and \( s = -y - k, k = 0, 1, 2, \ldots \). Displacement of the integration path to the left over the poles followed by evaluation of the residues (assuming that no two members of the set \( \{0, x, y\} \) differ by an integer – thereby avoiding the presence of higher-order poles) yields the result that \( B(x, y; p) \) can be expressed as the sum of three \( \hypergeom{2}{1}{\frac{1}{2} p} \) hypergeometric functions; see \([4, \text{Eq. (1.6)}]\).

Since there are no poles in the half-plane \( \Re(s) > c \) it follows that displacement of the integration path to the right can produce no algebraic-type asymptotic expansion; see \([8, \text{§5.4}]\). We can therefore displace the path as far to the right as we please; on such a displaced path, which we denote by \( L \), the variable \( |s| \) is everywhere large. The ratio of gamma functions in the integrand in (2.1) may then be expanded as an inverse factorial expansion given by \([8, \text{p. 39, Lemma 2.2}]\)

\[
\frac{\Gamma(s) \Gamma(x + s) \Gamma(y + s)}{\Gamma(\frac{1}{2} x + \frac{1}{2} y + s) \Gamma(\frac{1}{2} x + \frac{1}{2} y + \frac{1}{2} + s)} = \sum_{j=0}^{M-1} (-)^j c_j \Gamma(s - j - \frac{1}{2}) + \rho_M(s) \Gamma(s - M - \frac{1}{2}),
\]

where \( M \) is a positive integer and \( \rho_M(s) = O(1) \) as \( |s| \to \infty \) in \( |\arg s| < \pi \). The coefficients \( c_j \equiv c_j(x, y) \) are discussed below where the leading coefficient \( c_0 = 1 \).

Substitution of the above inverse factorial expansion into the integral (2.1) then produces

\[
B(x, y; p) = 2^{1-x-y-\frac{1}{2}} \frac{1}{2\pi i} \int_{L} \Gamma(s - j - \frac{1}{2}) p^{-s} ds + R_M,
\]

where

\[
R_M = \frac{1}{2\pi i} \int_{L} \rho_M(s) \Gamma(s - M - \frac{1}{2}) p^{-s} ds.
\]

The integral may be evaluated by the well-known Cahen-Mellin integral given by (see, for example, \([8, \text{p. 90}]\))

\[
\frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \Gamma(s + \alpha)z^{-s}ds = z^\alpha e^{-z} \quad (|\arg z| < \frac{1}{2}\pi, \ c > -\Re(\alpha))
\]
to yield

\[ B(x, y; p) = 2^{1-x-y} \pi^{\frac{1}{2}} \left\{ p^{-\frac{1}{2}} e^{-p} \sum_{j=0}^{M-1} (-)^j c_j p^{-j} + R_M \right\}. \]

A bound for the remainder \( R_M \) has been considered in [8, p. 71, Lemma 2.7], from which it follows that \( R_M = O(p^{-M-\frac{1}{2}} e^{-p}) \) as \( |p| \to \infty \) in \( |\arg p| < \frac{1}{2} \pi \).

Hence we obtain the asymptotic expansion

\[ B(x, y; p) = 2^{1-x-y} \pi^{\frac{1}{2}} p^{-\frac{x+y}{2}} e^{-p} \left\{ \sum_{j=0}^{M-1} (-)^j c_j p^{-j} + O(p^{-M}) \right\} \quad (2.2) \]

valid as \( |p| \to \infty \) in the sector \( |\arg p| < \frac{1}{2} \pi \). The expansion of \( B(x, y; p) \) for large \( p \) is seen to be exponentially small in \( |\arg p| < \frac{1}{2} \pi \); this is a standard result when there are no poles on the right of the path in (2.1) and routine path displacement does not produce any useful asymptotic information [8, §5.4].

The coefficients \( c_j \) for \( j \geq 1 \) can be generated by the algorithm described in [8, §2.2.4]. It is found that

\[ c_1 = \frac{1}{4} (1 + x + y + 2xy - x^2 - y^2), \]
\[ c_2 = \frac{1}{32} (9 + 6(2 + xy)(x + y + xy) - (7 + 4xy)(x^2 + y^2) - 6(x^3 + y^3) + x^4 + y^4 + 14xy), \]

which are symmetrical in \( x \) and \( y \) as required by (1.2). A closed-form representation for \( c_j \) is derived in the appendix, where it is shown that \( c_j \) can be expressed in terms of a terminating \( 3F_2(1) \) hypergeometric function given by

\[ c_j \equiv c_j(x, y) = \frac{(1/2)j}{j!} \gamma \left[ -j, \frac{1}{2} y - \frac{1}{2} x, \frac{1}{2} y - \frac{1}{2} x + \frac{1}{2}; 1 \right], \quad (2.3) \]

where \( (a)_j = \Gamma(a+j)/\Gamma(a) \) is the Pochhammer symbol. When \( x = y \), this reduces to the simpler expression

\[ c_j(x, x) = \frac{(1/2)j(x + 1/2)_j}{j!}. \quad (2.4) \]

We remark that the asymptotic expansion of \( B(x, y; p) \) for \( p \to \infty \) could also have been obtained by application of the method of steepest descents, which we shall employ in the subsequent sections. See also the appendix for a different approach.

3. The expansion of \( B(x, y; p) \) for large \( x \) and \( p \) with \( y \) finite

We consider the expansion of \( B(x, y; p) \) for large \( x \) and \( p \), with \( y \) finite, when it is supposed that \( p = ax \), where \( a > 0 \) and \( |\arg x| < \frac{1}{2} \pi \). By the symmetry property (1.2), the same result will also cover the case of large \( y \) and \( p \), with \( x \) finite. From (1.1), we have

\[ B(x, y; ax) = \int_0^1 f(t) e^{-x \psi(t)} dt \quad (|\arg x| < \frac{1}{2} \pi), \quad (3.1) \]

where

\[ \psi(t) = \frac{a}{4t(1-t)} - \log t, \quad f(t) = \frac{(1-t)^y}{t}. \]

Saddle points of the exponential factor are given by \( \psi'(t) = 0 \); that is, at the roots of the cubic

\[ t(1-t)^2 + \frac{1}{4} a(1-2t) = 0. \quad (3.2) \]
We label the three saddles \(t_0, t_1\) and \(t_2\). All three saddles lie on the real axis with \(t_0\) situated in the closed interval \([0,1]\), with \(t_1 > 1\) and \(t_2 < 0\). The \(t\)-plane is cut along \((-\infty,0]\). Paths of steepest descent through the saddles \(t_r\) \((r = 0,1)\) are given by

\[
\Im\{e^{i\theta}(\psi(t) - \psi(t_r))\} = 0, \quad \theta = \arg x;
\]

these paths terminate at \(t = 0\) and \(t = 1\) in the directions \(|\theta - \phi| < \frac{1}{2}\pi\) and \(\frac{1}{2}\pi < \theta - \phi < \frac{3}{2}\pi\), respectively, where \(\phi = \arg t\).

When \(x > 0\), the integration path coincides with the steepest descent path over the saddle \(t_0\); for complex \(x\) in the sector \(|\arg x| < \frac{1}{2}\pi\), the steepest descent path through \(t_0\) becomes deformed but still terminates at \(t = 0\) and \(t = 1\); see Fig. 1. Application of the saddle-point method then yields the leading behaviour

\[
B(x,y;ax) \sim \left[ \frac{2\pi}{x\psi''(t_0)} \right] f(t_0) e^{-x\psi(t_0)}
\]

\[
= \sqrt{\frac{2\pi}{x\psi''(t_0)}} t_0^{x-1}(1-t_0)^{y-1} \exp\left[ \frac{-ax}{4t_0(1-t_0)} \right]
\]

as \(|x| \to \infty\) in the sector \(|\arg x| < \frac{1}{2}\pi\), where some routine algebra combined with (3.2) shows that

\[
\psi''(t_0) = \frac{1-3t_0+4t_0^2}{t_0^2(1-t_0)(2t_0-1)}.
\]

We remark that the saddle \(t_0 \equiv t_0(a)\) has to be computed for a particular value of the parameter \(a\), either directly from (3.2) or as a cubic root.

The asymptotic expansion of \(B(x,y;ax)\) is given by [7, p. 47]

\[
B(x,y;ax) \sim 2e^{-x\psi(t_0)} \sum_{n=0}^{\infty} \frac{C_{2n}\Gamma(n+\frac{1}{2})}{x^{n+\frac{3}{2}}} \quad (|x| \to \infty, \ |\arg x| < \frac{1}{2}\pi).
\]
The coefficients $C_n$ can be obtained by an inversion process and are listed for $n \leq 8$ in [3, p. 119] and for $n \leq 4$ in [9, p. 13]. Alternatively, they can be obtained by an expansion process to yield Wojdylo’s formula [10] given by

$$C_n = \frac{1}{2a_0^{(n+1)/2}} \sum_{k=0}^{n} b_{n-k} \sum_{j=0}^{k} \frac{(-)^j(jn + \frac{j}{2})_j}{j! a_0^j} B_{kj}; \quad (3.5)$$

see also [5, 6]. Here $B_{kj} \equiv B_{kj}(a_1, a_2, \ldots, a_{k-j+1})$ are the partial ordinary Bell polynomials generated by the recursion

$$B_{kj} = \sum_{r=1}^{k-j+1} a_r B_{k-r,j-1}, \quad B_{k0} = \delta_{k0},$$

where $\delta_{mn}$ is the Kronecker symbol, and the coefficients $a_r$ and $b_r$ appear in the expansions

$$\psi(t) - \psi(t_0) = \sum_{r=0}^{\infty} a_r (t - t_0)^{r+2}, \quad f(t) = \sum_{r=0}^{\infty} b_r (t - t_0)^r \quad (3.6)$$

valid in a neighbourhood of the saddle $t = t_0$.

In numerical computations we choose a value of the parameter $a$ and compute the saddle $t_0$ from (3.2). With a value of $y$, Mathematica is used to determine the coefficients $a_r$ and $b_r$ for $0 \leq r \leq n_0$. The coefficients $C_{2n}$ can then be calculated for $0 \leq n \leq n_0$ from (3.5). We display the computed values of $C_{2n}$ for different values of $a$ and $y$ in Table 1. In Table 2, the values of the absolute relative error in the computation of $B(x, y; ax)$ from (3.4) are presented as a function of the truncation index $n$ when $x = 100$.

| $n$ | $a = 1, \ y = 1$ | $a = \frac{1}{2}, \ y = \frac{3}{2}$ | $a = \frac{3}{2}, \ y = \frac{5}{2}$ | $a = 2, \ y = \frac{1}{2}$ |
|-----|------------------|-----------------------------------|-----------------------------------|------------------|
| 0   | $+0.2668661228\,$ | $+0.1364219142\,$               | $+0.2036093538\,$               | $+0.3909054941\,$ |
| 1   | $+0.0982652355\,$ | $+0.2683883846\,$               | $+0.0762869817\,$               | $-0.0309094064\,$ |
| 2   | $-0.0636566665\,$ | $-0.1085963949\,$               | $-0.0456489054\,$               | $-0.0039290092\,$ |
| 3   | $+0.018602666\,$  | $+0.0151339630\,$               | $+0.0137423943\,$               | $+0.0024209801\,$ |
| 4   | $-0.0039253710\,$ | $-0.0003383888\,$               | $-0.0026770977\,$               | $-0.0005115807\,$ |
| 5   | $+0.0012059654\,$ | $+0.0004533741\,$               | $+0.0003423270\,$               | $+0.0009299402\,$ |

**4. The expansion of $B(x, y; p)$ for large $x$, $y$ and $p$**

We consider the expansion of $B(x, y; p)$ for large $x$, $y$ and $p$, when it is supposed that $p = ax$ and $y = bx$, where $a > 0$, $b > 0$ and $|\arg x| < \frac{1}{2}\pi$. From (1.1), we have

$$B(x, y; p) = \int_0^1 f(t)e^{-x\psi(t)} dt \quad (|\arg x| < \frac{1}{2}\pi), \quad (4.1)$$

*For example, this generates the values $B_{41} = a_4$, $B_{42} = a_4^2 + 2a_1a_3$, $B_{43} = 3a_2^2a_2$ and $B_{44} = a_4^2$. 

[2]
Saddle points of the exponential factor are given by the roots of the cubic
\[ t(1 - t)\{1 - (b + 1)t\} + \frac{1}{4}a(1 - 2t) = 0. \]  
Routine examination of this cubic shows that, when \( a > 0, b > 0 \), all roots are real, with one root greater than 1, one in the interval \([0, 1]\) and one negative root. The distribution of the saddles is thus similar to that in Section 3, where we continue to label the saddle situated in \([0, 1]\) by \( t_0 \). The topology of the path of steepest descent through the saddle \( t_0 \), given by \( \Im\{e^{i\theta}(\psi(t) - \psi(t_0))\} = 0 \) where \( \theta = \arg x \), is also similar to that depicted in Fig. 1.

Accordingly, the expansion of \( B(x, y; p) \) when \( p = ax \) and \( y = bx \), with \( a > 0, b > 0 \), is given by
\[ B(x, bx; ax) \sim 2e^{-x\psi(t_0)} \sum_{n=0}^{\infty} \frac{C_{2n}\Gamma(n + \frac{1}{2})}{x^{n+\frac{1}{2}}} \ \ (|x| \to \infty, |\arg x| < \frac{1}{2}\pi), \]  
where the coefficients \( C_{2n} \) can be determined from (3.5) when the coefficients \( a_r \) and \( b_r \) in (3.6) are evaluated from the definitions of \( \psi(t) \) and \( f(t) \) in (4.2).

The leading behaviour is
\[ B(x, bx; ax) \sim \sqrt{\frac{2\pi}{x^{b}(t_0)}} f(t_0)e^{-x\psi(t_0)} \]
\[ = \sqrt{\frac{2\pi}{x^{b}(t_0)}} t_0^{x-1}(1 - t_0)^{b-1} \exp \left[ \frac{-ax}{4t_0(1-t_0)} \right] \]  
(4.5)
as \( |x| \to \infty \) in the sector \( |\arg x| < \frac{1}{2}\pi \), where
\[ \psi''(t_0) = \frac{1}{t_0^2(1-t_0)(2t_0-1)} \left( 1 - \frac{bt_0}{1-t_0} \right) + \frac{b}{t_0(1-t_0)^2} \]  
and \( t_0 \equiv t_0(a, b) \) is the root of (4.3) situated in \( t \in [0, 1] \).

We note that when \( b = 1 \) we have the result [1, 4]
\[ B(x, x; p) = 2^{1-2x} \pi^{\frac{1}{2}} p^{(x-1)/2} e^{-\frac{x}{2p}} W_{-\frac{1}{2}, \frac{1}{2}, x} (p) \]
in terms of the Whittaker function \( W_{\kappa, \mu}(z) \); see (A.1).

### Table 2: Values of the absolute relative error in \( B(x, y; ax) \) when \( x = 100 \) for different truncation index.

| n  | \( a = 1, y = 1 \)   | \( a = \frac{1}{2}, y = \frac{3}{2} \) | \( a = \frac{3}{2}, y = \frac{5}{4} \) | \( a = 2, y = \frac{1}{2} \) |
|----|----------------------|-----------------------------|-----------------------------|-----------------------------|
| 0  | 1.838 × 10^{-3}      | 9.682 × 10^{-3}            | 1.853 × 10^{-3}            | 3.963 × 10^{-4}            |
| 1  | 1.770 × 10^{-5}      | 5.892 × 10^{-5}            | 1.666 × 10^{-5}            | 7.426 × 10^{-7}            |
| 2  | 1.295 × 10^{-7}      | 2.058 × 10^{-7}            | 1.255 × 10^{-7}            | 1.153 × 10^{-8}            |
| 3  | 9.506 × 10^{-10}     | 1.517 × 10^{-10}           | 8.562 × 10^{-10}           | 8.568 × 10^{-11}           |
| 4  | 1.295 × 10^{-11}     | 9.526 × 10^{-12}           | 5.011 × 10^{-12}           | 2.332 × 10^{-13}           |
| 5  | 3.688 × 10^{-12}     | 1.933 × 10^{-13}           | 5.472 × 10^{-14}           | 6.917 × 10^{-15}           |
5. The behaviour of $B(x, y; p)$ for large $x$ and finite $y$ and $p$

In this final section, we examine the behaviour of $B(x, y; p)$ for large complex $x = |x|e^{i\theta}$, with $0 \leq \theta \leq \pi$, when $y$ and $p > 0$ are finite. The situation when $-\pi \leq \theta \leq 0$ is analogous and, in the case of real $y$, $B(x, y; p)$ assumes conjugate values. This case has been discussed in [2, Appendix], but is repeated (with minor corrections) here for completeness. By the symmetry property (1.2), the same result will also cover the case of large $y$, with $x$ and $p$ finite.

From (1.1), we have upon interchanging $x$ and $y$ (by virtue of (1.2))

$$B(x, y; p) = \int_0^1 f(t)e^{-|x|\psi(t)}dt,$$

(5.1)

where

$$\psi(t) = \frac{\alpha}{t(1-t)} - e^{i\theta}\log(1-t), \quad f(t) = \frac{ty^{-1}}{1-t}, \quad \alpha := \frac{p}{4|x|}.$$

(5.2)

Because $p > 0$ is a fixed parameter, the integral (5.1) is valid for arbitrary complex values of $x$ and $y$. Saddle points of the exponential factor arise when $\psi'(t) = 0$; that is, when

$$t^2(t-1) + \alpha e^{-i\theta}(1-2t) = 0.$$  

(5.3)

We label the three saddles $t_0$, $t_1$ and $t_2$ as in Section 3. When $\theta = 0$, all three saddles are situated on the real axis with $t_0 \in [0, 1]$ and $t_1 > 1$, $t_2 < 1$. As $\theta$ increases, the saddles $t_0$ and $t_2$ rotate about the origin and $t_1$ rotates about the point $t = 1$. The result of this rotation is that, when $\theta = \pi$, $t_0$ and $t_2$ become a complex conjugate pair near the origin and $t_1$ is situated in the interval $[0, 1]$; see Fig. 2.

Figure 2: The steepest descent and ascent paths through the saddles $t_0$ and $t_1$ (heavy dots) when $\alpha = 1/3$ and (a) $\theta = 0.25\pi$, (b) $\theta = \theta_0 = 0.65595\pi$, (c) $\theta = 0.69\pi$, (d) $\theta = \theta_1 = 0.71782\pi$, (e) $\theta = 0.80\pi$ and (f) $\theta = \pi$. The arrows indicate the integration path. The steepest ascent paths spiral round $t = 1$ out to infinity passing onto adjacent Riemann surfaces. The saddle $t_2$ is not shown. The $t$-plane is cut along $[1, \infty)$. 

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When $\theta = 0$, the integration path coincides with the steepest descent path passing over the saddle $t_0$ given approximately by
\[ t_0 \simeq \alpha \frac{1}{2} - \frac{1}{2} \alpha \quad (x \to \infty). \]

Then, with the estimates
\[ x\psi(t_0) \simeq (px)^{1/2} + \frac{3}{8}p, \quad \psi''(t_0) \simeq 2\alpha^{-\frac{1}{2}}, \]
we find by application of the saddle-point method the leading behaviour
\[ B(x, y; p) \sim \sqrt{\frac{\pi}{x}} \left( \frac{p}{4x} \right)^{\frac{1}{2}} \exp \left[ -\left( px \right)^{1/2} - \frac{3}{8}p \right] \quad (\theta = 0, \ x \to +\infty). \quad (5.4) \]

When $\theta = \pi$, we find from (5.3) that the saddle $t_1$ close to the point $t = 1$ is given by
\[ t_1 \simeq 1 - \alpha + \alpha^3 \quad (|x| \to \infty) \]

and
\[ |x| \psi(t_1) \simeq |x| + \frac{1}{4}p - |x| \log \alpha, \quad \psi''(t_1) \simeq \alpha^{-2}. \]

The integration path again coincides with the steepest descent path through $t_1$, and so we obtain the behaviour
\[ B(x, y; p) \sim i \sqrt{\frac{\pi}{x}} \left( \frac{p}{4x} \right)^{x} e^{\pi x} \exp \left[ x - \frac{1}{4}p \right] \quad (\theta = \pi, \ x \to -\infty) \]
\[ = \sqrt{\frac{2\pi}{|x|}} \left( \frac{p}{4|x|} \right)^{|x|} \exp \left[ |x| - \frac{1}{4}p \right]. \quad (5.5) \]

The leading terms in (5.4) and (5.5) were given in [2, Appendix].

A detailed study of the topology of the steepest descent paths through the saddles $t_0$ and $t_1$ when $0 \leq \theta \leq \pi$ is summarised in Fig. 2 for the particular case $\alpha = \frac{1}{3}$. The $t$-plane is cut along $[1, \infty)$ and paths of steepest descent either terminate at $t = 0$ (with $|\arg t| < \frac{1}{2} \pi$), $t = 1$ (with $|\arg(1 - t)| < \frac{1}{2} \pi$) or at infinity. Paths that approach infinity spiral round the point $t = 1$ passing onto adjacent Riemann surfaces. The figures reveal that there are two critical values of the phase $\theta$, where the saddles $t_0$ and $t_1$ become connected (via a Stokes phenomenon). We denote these values by $\theta_0 \equiv \theta_0(\alpha)$ and $\theta_1 \equiv \theta_1(\alpha)$, where $\alpha$ is defined in (5.2). The values of these critical angles are tabulated in Table 3 for different $\alpha$.

When $0 \leq \theta < \theta_0(\alpha)$, the integration path can be deformed to coincide with the steepest descent path passing over $t_0$, so that the leading behaviour in (5.4) applies in this sector. When $\theta_0(\alpha) < \theta < \theta_1(\alpha)$, the integration path is deformed to pass over both saddles $t_0$ and $t_1$, where each steepest descent path spirals out to infinity. Finally, when $\theta_1(\alpha) < \theta \leq \pi$, the integration path is deformed to pass over only the saddle $t_1$.

Based on these considerations and on the approximation of the saddles $t_0 \simeq \alpha^{1/2} - \frac{1}{2} \alpha'$, $t_1 \simeq 1 + \alpha' - \alpha'^3$, where $\alpha' = p/(4x)$, the leading behaviour of $B(x, y; p)$ is found to be
\[ B(x, y; p) \sim \begin{cases} J_0 & 0 \leq \theta < \theta_1(\alpha) \\ J_0 - J_1 & \theta_1(\alpha) < \theta < \theta_2(\alpha) \\ J_1 & \theta_2(\alpha) < \theta \leq \pi \end{cases} \quad (5.6) \]

The saddle $t_2$ does not enter into our consideration as it plays no role in the asymptotic evaluation of $B(x, y; p)$ when $0 \leq \theta \leq \pi$. 

\[ \]
the expansion (2.2). Miller [4, Eq. (2.3a)] has shown that

\[ B \text{ convergent series of Whittaker functions in the form} \]

In this appendix we derive a closed-form expression for the coefficients... of (5.1). The parameter values chosen correspond to...\] of (5.1) the saddle \( t_0 \) is dominant, whereas when \( \theta > \theta^*(\alpha) \) the saddle \( t_1 \) is dominant in the large-\(|x|\) limit.

In Table 4 we present the results of numerical calculations using the asymptotic behaviour of \( B(x,y;p) \) in (5.6) compared to the values obtained by numerical integration of (5.1). The parameter values chosen correspond to \( \alpha = 0.01 \) and the saddles \( t_0 \) and \( t_1 \) are computed from (5.3), with the leading forms \( J_0 \) and \( J_1 \) computed from (5.7) and (5.8).

It is seen from Table 3 that the exchange of dominance between the two contributory... \( \theta \sim 0.60\pi \).

**Appendix:** A closed-form expression for the coefficients \( c_j \)

In this appendix we derive a closed-form expression for the coefficients \( c_j \) appearing in the expansion (2.2). Miller [4, Eq. (2.3a)] has shown that \( B(x,y;p) \) can be expressed as a convergent series of Whittaker functions in the form

\[ B(x,y;p) = 2^{1-x-y}π^{\frac{3}{4}} p^{(y-1)/2} e^{-\frac{p}{4} x} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2} y - \frac{1}{2} x\right) k \left(\frac{3}{2} + \frac{1}{2} y - \frac{1}{2} x\right) k}{k!} W_{-k-\frac{1}{2} y, \frac{1}{2} x}(p), \]
Table 4: Values of the asymptotic behaviour of $B(x, y; p)$ in (5.6) with the calculated value when $|x| = 50$, $p = 2$ ($\alpha = 0.01$) and $y = \frac{1}{4}$ for different $\theta = \arg x$.

| $\theta / \pi$ | Asymptotic value | Calculated value |
|---------------|------------------|------------------|
| 0             | $+5.175 \times 10^{-06}$ | $+5.187 \times 10^{-06}$ |
| 0.20          | $-8.210 \times 10^{-06} + 2.081 \times 10^{-06}i$ | $-8.223 \times 10^{-06} + 2.096 \times 10^{-06}i$ |
| 0.40          | $+3.468 \times 10^{-05} - 6.934 \times 10^{-06}i$ | $+3.470 \times 10^{-05} - 7.020 \times 10^{-06}i$ |
| 0.50          | $+2.647 \times 10^{-06} - 9.853 \times 10^{-05}i$ | $+2.402 \times 10^{-06} - 9.855 \times 10^{-05}i$ |
| 0.60          | $-8.387 \times 10^{-04} - 3.821 \times 10^{-03}i$ | $-8.781 \times 10^{-04} - 3.823 \times 10^{-03}i$ |
| 0.70          | $-5.944 \times 10^{+28} + 1.659 \times 10^{+28}i$ | $-5.952 \times 10^{+28} + 1.652 \times 10^{+28}i$ |
| 0.80          | $+2.786 \times 10^{+54} + 3.451 \times 10^{+54}i$ | $+2.786 \times 10^{+54} + 3.459 \times 10^{+54}i$ |
| 1.00          | $+4.146 \times 10^{+77}$ | $+4.154 \times 10^{+77}$ |

where $W_{\kappa, \mu}(x)$ is the Whittaker function. For $p \to \infty$ with bounded $k$, we have the expansion [7, Eq. (13.19.3)]

$$W_{-k-\frac{1}{2}, \frac{1}{2}}(p) = p^{-k-\frac{1}{2}} e^{-\frac{1}{2}p} \left\{ \sum_{n=0}^{N-1} (-)^n \frac{(\frac{1}{2} + k)_n (y + \frac{1}{2} + k)_n}{n! p^n} + O(p^{-N}) \right\},$$

where $N$ is a positive integer. Then we obtain from (A.1)

$$B(x, y; p) = 2^{1-x-y} \pi^{\frac{1}{2}} p^{-\frac{1}{2}} e^{-p} \left\{ S(x, y; p) + O(p^{-N}) \right\},$$

(A.2)

where

$$S(x, y; p) = \sum_{k=0}^{N-1} \frac{1}{k!} p^k \left\{ \sum_{n=0}^{N-1} (-)^n \frac{(\frac{1}{2} + k)_n (y + \frac{1}{2} + k)_n}{n! p^n} \right\},$$

and we have made the change of summation index $n \to j - k$. Use of the fact that $(-j)_k = (-)^k j! / (j - k)!$, the above double sum can be written as

$$\sum_{k=0}^{N-1} \frac{1}{k!} \left\{ \sum_{n=0}^{N-1} \frac{(-)^j j! (\frac{1}{2} + k)_j (y + \frac{1}{2} + k)_j}{j! (k + \frac{1}{2})_j (y + \frac{1}{2} + k)_j p^j} \right\}$$

$$\sum_{j=0}^{N-1} \frac{(-)^j}{j!} \left\{ \sum_{k=0}^{N-1} \frac{(\frac{1}{2} + k)_k (y + \frac{1}{2} + k)_k}{j! (k + \frac{1}{2})_j (y + \frac{1}{2} + k)_j p^j} \right\}$$

upon reversal of the order of summation and identification of the inner sum over $k$ as a terminating $3F_2$ series of unit argument.

Comparison of (A.2) and (A.3) with the expansion obtained in (2.2) then yields the final result

$$c_j = \frac{(\frac{1}{2} + k)_j (y + \frac{1}{2})_j}{j!} 3F_2 \left[ -j, \frac{1}{2} y - \frac{1}{2} x, \frac{1}{2} + \frac{1}{2} y - \frac{1}{2} x ; 1 \right].$$

(A.4)
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