Structure and stability of the Lukash plane-wave spacetime

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Abstract

We study the vacuum, plane-wave Bianchi $VII_h$ spacetimes described by the Lukash metric. Combining covariant with orthonormal frame techniques, we describe these models in terms of their irreducible kinematical and geometrical quantities. This covariant description is used to study analytically the response of the Lukash spacetime to linear perturbations. We find that the stability of the vacuum solution depends crucially on the background shear anisotropy. The stronger the deviation from the Hubble expansion, the more likely the overall linear instability of the model. Our analysis addresses rotational, shear and Weyl curvature perturbations and identifies conditions sufficient for the linear growth of these distortions.

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1 Introduction

There has long been an interest in the study of the spatially homogeneous Bianchi spacetimes and their cosmological applications to our understanding of singularities and of the observed level of isotropy in the universe. These studies analyse the problems within the manageable domain of ordinary differential equations and provide only a finite number of alternative cosmologies (see [1] and references therein). The most general Bianchi universes that contain the open Friedmann model as a special subcase are those of type $VII_h$. The late-time asymptotes for the non-tilted type $VII_h$ spacetimes, with $h \neq 0$ and a matter content that obeys the strong energy condition, evolve towards the vacuum plane-wave solution found by Doroshkevich et al and Lukash [2, 5] that is known as the Lukash metric. These spacetimes describe the most general effects of spatially homogeneous perturbations on open Friedmann universes; see for example [6]-[10]. The Lukash metric plays a guiding role in these investigations because of the subtle stability properties of isotropic expansion at late times in open universes. When the strong energy condition is obeyed, then isotropic expansion was found to be stable but not asymptotically stable at late times [7]-[10].

Traditionally, Bianchi spacetimes have been studied qualitatively, primarily by means of dynamical system methods [11]-[16]. The same techniques also facilitate the analysis of the less well understood tilted Bianchi models, namely those where the fluid 4-velocity is no longer orthogonal to the hypersurfaces of constant time [17]-[20]. In this paper we attempt an analytical

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approach. In particular, we combine covariant and orthonormal frame techniques to provide a description of the Lukash spacetime of Bianchi type $VII_h$ in terms of its irreducible kinematical and geometrical quantities. We then use the zero-order results to study inhomogeneous perturbations around the vacuum solution and discuss its linear stability. In so doing, we allow for the presence of a low-density matter component with a pressure-free equation of state. Our main interest is the evolution of perturbations in the kinematics and the geometry of the vacuum model and its linear response to these distortions. We find that stability depends primarily on the amount of the background shear anisotropy. In particular, our results show that the higher the background shear the more likely is the linear instability of the model. Another key factor is the relative orientations of the various kinematical and geometrical quantities. The positioning of the vorticity vector with respect to the principal axes of shear, for example, or the relative orientation between the shear and the spatial curvature eigenframes can also influence the linear stability of the Lukash spacetime.

We also consider rotational perturbations and study their evolution relative to the background volume expansion. Our analysis provides a condition for the linear growth of vortical distortions that depends primarily on the background shear anisotropy. We identify, in particular, the minimum amount of shear necessary for the linear instability of the Lukash universe against vorticity perturbations. The level of background shear can also determine whether kinematical anisotropies will remain bounded or dominate the linear expansion of the perturbed Lukash model. Additionally, by allowing for low-density dust matter, we find that its presence has no effect on the vorticity and shear of the perturbed spacetime. In contrast, the introduction of a material component can have an effect on the linear Weyl anisotropy of the model. More specifically, the presence of even a low-density dust fluid seems enough to ensure that the Lukash solution will diverge from its original plane-wave nature at late times.

2 The Lukash plane-wave attractor

The Bianchi $VII_h$ models belong to the non-exceptional family of the Behr class B spatially homogeneous spacetimes. The plane-wave Lukash solution is the late-time attractor of the Bianchi $VII_h$ models for a broad range of initial date and matter properties. These vacuum spacetimes correspond to equilibrium points of the associated autonomous dynamical system and are self-similar [21]-[24]. The line element of the Lukash metric takes the form

$$ds^2 = -dt^2 + t^2 dx^2 + t^2 e^{2rx} \left[ (Ady + Bdz)^2 + (Cdy + Adz)^2 \right], \quad (1)$$

where $r$ is an arbitrary constant parameter in the range $0 < r < 1$, $A = \cos v$, $B = f^{-1} \sin v$, $C = -f \sin v$ and $v = k(x + \ln t)$ [25, 26]. Note that $f$ and $k$ are constants related to $r$ by

$$\frac{k^2}{f^2} (1 - f^2)^2 = 4r(1 - r) \quad \text{and} \quad r^2 = hk^2, \quad (2)$$

where $h$ is the associated group parameter. As we shall see next, constraint (2) is the Lukash analogue of the Friedmann equation. We also point out that for $r = 1$ and $f^2 = 1$ the Lukash metric reduces to that of the empty Milne universe.

3 Covariant description

Consider a family of observers, with worldlines tangent to the timelike velocity field $u_a$ (normalised so that $u_a u^a = -1$). The latter, together with the associated projection tensor $h_{ab} = \ldots$
$g_{ab} + u_a u_b$, introduces a local 1+3 threading of the spacetime into time and space. One can then decompose the various kinematical, dynamical and geometrical quantities into their respective irreducible parts and obtain a completely covariant description of the spacetime [27, 28].

3.1 Covariant variables

The covariant formalism uses the irreducible kinematic quantities, the energy density and pressure of the matter fields and the gravito-electromagnetic tensors, instead of the metric which in itself does not provide a covariant description. The key equations are the Ricci and Bianchi identities, applied to the observers’ 4-velocity, while Einstein’s equations are incorporated via algebraic relations between the Ricci and the matter energy-momentum tensors. Thus, in the absence of matter and rotation, the plane-wave attractors of the Bianchi VII$\text{h}$ spacetimes are covariantly characterised by

$$
\mu = 0 = p = q_a = \pi_{ab} \quad \text{and} \quad \dot{u}_a = 0 = \omega_a,
$$

while

$$
\Theta, \sigma_{ab}, E_{ab}, H_{ab} \neq 0.
$$

Note that $\mu$, $p$, $q_a$ and $\pi_{ab}$ are respectively the energy density, the isotropic pressure, the heat flux and the anisotropic stresses of the matter, $\Theta$, $\sigma_{ab}$, $\omega_a$ and $u_a$ are the volume expansion, the shear, the vorticity and the acceleration, while $E_{ab}$ and $H_{ab}$ are the electric and magnetic parts of the Weyl tensor ($C_{abcd}$). The latter have equal magnitudes and are orthogonal to each other, in accord with the Petrov type N nature of the Lukash solution. In other words,

$$
E^2 = H^2 \quad \text{and} \quad E_{ab}H^{ab} = 0,
$$

where $E^2 = E_{ab}E^{ab}/2$ and $H^2 = H_{ab}H^{ab}/2$. The former of these constraints implies that $C_{abcd}C^{abcd} = 0$. The latter ensures that $C_{abcd} = \eta_{abpq}C^{pqcd}/2$ and $\eta_{abcd}$ is the 4-dimensional alternating tensor. Note that the Weyl curvature invariant $C_{abcd}C^{abcd}$ has been suggested and used as a measure of the gravitational entropy by several authors [29]-[33], but cannot on its own capture deviations from isotropic expansion in plane-wave spacetimes.

3.2 Covariant equations

The average volume expansion of the Lukash universe is described by the following version of the Raychaudhuri equation

$$
\dot{\Theta} = -\frac{1}{3} \Theta^2 - 2\sigma^2,
$$

where $\sigma^2 = \sigma_{ab}\sigma^{ab}/2$ is the magnitude of the shear tensor. As usual, the expansion scalar is used to define an average scale factor ($a$) via the standard relation $\Theta/3 = \dot{a}/a$.

The absence of matter means that the Lukash spacetime is Ricci flat. The curvature of the spatial sections, however, is not zero. In particular, zero rotation ensures that the 3-Ricci tensor ($\mathcal{R}_{ab}$) is completely determined by its scalar and its symmetric and trace-free parts. These are given respectively by

$$
\mathcal{R} = -\frac{2}{3} \Theta^2 + 2\sigma^2,
$$

$$
\mathcal{S}_{ab} = -\frac{1}{3} \Theta \sigma_{ab} + \sigma_{c(a}\sigma^{c}b) + E_{ab},
$$

1Throughout this article we employ a Lorentzian metric with signature $(-, +, +, +)$ and use geometrised units with $c = 1 = 8\pi G$. Consequently, all geometrical variables have physical dimensions that are integer powers of length. Also, Latin indices take the values 0,1,2,3 and Greek ones run from 1 to 3.
where $S_{ab} = R_{(ab)} = R_{(ab)} - Rh_{ab}/3$. As we will see below, the scalar $R$ is negative, which means that the model is spatially open. Note that expression (7) is the generalised Friedmann equation.

In covariant terms, gravitational waves are described by the electric and the magnetic parts of the Weyl tensor. The latter obey a set of three coupled propagation equations, which are accompanied by an equal number of constraints. In the case of the Lukash plane-wave spacetime the evolution equations take the form

\begin{align*}
\dot{E}_{ab} & = - \Theta E_{ab} + \text{curl} H_{ab} + 3 \sigma_{c(a} E_{b)}^c, \\
\dot{H}_{ab} & = - \Theta H_{ab} - \text{curl} E_{ab} + 3 \sigma_{c(a} H_{b)}^c, \\
\dot{\sigma}_{ab} & = - \frac{2}{3} \Theta \sigma_{ab} - E_{ab} - \sigma_{c(a} \sigma_{b)}^c.
\end{align*}

(9) (10) (11)

The constraints, on the other hand, are

\begin{align*}
D^b \sigma_{ab} & = \frac{2}{3} D_a \Theta, \\
D^b E_{ab} & = \epsilon_{abc} \sigma^b_{d} H^{cd}, \\
D^b H_{ab} & = - \epsilon_{abc} \sigma^b_{d} E^{cd},
\end{align*}

(12) (13) (14)

where $\epsilon_{abc} = \eta_{abcd} u^d$ is the spatial alternating tensor. In addition, the shear and the magnetic component of the Weyl tensor are directly related by

$$H_{ab} = \text{curl} \sigma_{ab},$$

(15)

with $\text{curl} H_{ab} = \epsilon_{cd(a} D^c H^{d)}_{b)}$ by definition. Clearly, exactly analogous expressions define $\text{curl} E_{ab}$ and $\text{curl} \sigma_{ab}$. Note that the presence of (standing) gravitational waves is guaranteed by the non-zero values of both $\text{curl} E_{ab}$ and $\text{curl} H_{ab}$. This is possible, despite the spatial homogeneity of the Lukash spacetime, because of the non-zero 3-curvature of the model. In other words, $D_c E_{ab}$, $D_c H_{ab} \neq 0$ due to non-zero Christoffel symbols.

4 Orthonormal-frame description

The orthonormal frame formalism is an 1+3 decomposition of the EFE into evolution and constraint equations relative to the timelike vector field $e_0$ of an orthonormal frame $\{e_a\}$ [34]-[37]. In cosmological studies $e_0$ is the fundamental 4-velocity field, usually identified with the motion of the cosmic medium. In models that accept an isometry group, $e_0$ can also be chosen as the normal to the spacelike group orbits.

4.1 Structure constants

The Lukash solution belongs to the non-exceptional family of the Bianchi class B spacetimes. For these models the structure constants $n_{\alpha \beta}$ and $a_\alpha$ (with $n_{\alpha \beta} a^\alpha = 0$) take the form

\begin{equation}
\begin{aligned}
n_{\alpha \beta} & = \text{diag} \left( 0, n_2, n_3 \right) \quad \text{and} \\
a_\alpha & = (a_1, 0, 0).
\end{aligned}
\end{equation}

(16)

\(^2\)Angled brackets denote the symmetric and trace-free part of orthogonally projected tensors and the orthogonally projected components of vectors.

\(^3\)In the orthonormal formalism the spacetime metric is $g_{ab} = \eta_{ab} = \text{diag} \left( -1, 1, 1, 1 \right)$ and the spatial frame vectors are $\{e_a\}$. Greek indices are raised and lowered by means of the spatial metric $g_{a \beta} = \delta_{a \beta}$. 

4
Moreover, the self-similarity of the Bianchi $VII_h$ plane-wave attractor guarantees that the three non-zero components of the structure constants are given by [25]

$$a_1 = -\frac{r}{t}, \quad n_2 = \frac{k}{ft} \quad \text{and} \quad n_3 = \frac{kf}{t},$$  \hspace{1cm} (17)

where $r$, $k$ and $f$ are constants related with each other and with the group parameter of the model (see Eq. (2)).

4.2 Kinematics

Non-exceptional, non-tilted Bianchi class B spacetimes, like the Bianchi $VII_h$ cosmologies, have $\sigma_{12} = 0 = \sigma_{13}$. Hence, the self similarity of the vacuum plane-wave attractor of these models means that the remaining components of the shear tensor are

$$\sigma_{11} = \frac{2(1-r)}{3t}, \quad \sigma_{22} = \sigma_{33} = -\frac{1-r}{3t} \quad \text{and} \quad \sigma_{23} = \frac{k(1-f^2)}{2ft}. \hspace{1cm} (18)$$

On using the above, one finds that the magnitude of the shear tensor associated with the Lukash solution is

$$\sigma^2 = \frac{1}{2} \sigma_{\alpha\beta} \sigma^{\alpha\beta} = \frac{(1-r)(1+2r)}{3t^2}, \hspace{1cm} (19)$$

where $\sigma_{\alpha\beta}$ are the non-zero orthonormal frame components of $\sigma_{ab}$. On the other hand, the mean Hubble volume expansion of the Lukash universe is determined by the scalar

$$\Theta = \frac{1+2r}{t}. \hspace{1cm} (20)$$

The above ensures that the average scale factor obeys the simple power law $a \propto t^{(1+2r)/3}$ with $0 < r < 1$. As expected, for $r \to 1$ the scale factor evolution reduces to that of the Milne universe (i.e. $a \propto t$). Note that the effect of the anisotropy is to reduce the expansion rate below that of the isotropic case.

When measuring the average anisotropy of the expansion, it helps to introduce the following dimensionless and expansion-normalised shear parameter

$$\Sigma \equiv \frac{3\sigma^2}{\Theta^2}. \hspace{1cm} (21)$$

In the Lukash spacetime the scalars $\sigma^2$ and $\Theta$ are given by (19) and (20) respectively. Using these expressions we obtain

$$\Sigma = \frac{1-r}{1+2r}, \hspace{1cm} (22)$$

This result reflects the model’s self similarity, which guarantees that all the expansion-normalised dimensionless variables remain constant in time. Given that $\Sigma > 0$ and $0 < r < 1$, we immediately deduce that $0 < \Sigma < 1$, in accord with $R < 0$ in (7). Thus, although the shear anisotropy is not asymptotically stable (in the Lyapunov sense), it is stable in the sense that any deviations from isotropy never diverge [8]-[10]. Note that for $r \to 1$ the $\Sigma$-parameter approaches zero and the expansion becomes isotropic. Maximum shear anisotropy (i.e. $\Sigma \to 1$), on the other hand, corresponds to the $r = 0$ limit.
Definition (21) also provides an alternative expression for the Raychaudhuri equation of the Lukash solution. In particular, combining (6) and (21) one arrives at
\[
\dot{\Theta} = -\frac{1}{3} \Theta^2 (1 + 2\Sigma),
\]
while the power-law evolution of the average scale factor now reads \(a \propto t^{1/(1+2\Sigma)}\). Thus, in the absence of any shear anisotropy we have \(a \propto t\), as in the Milne universe. For maximum shear anisotropy, on the other hand, we arrive at the familiar scale-factor evolution of the Kasner vacuum solutions (i.e. \(a \propto t^{1/3}\)). Note that the deceleration parameter of the Lukash model is \(q = 2\Sigma\), which means that \(q = 0\) when \(\Sigma = 0\) and takes value \(q = 2\) of the Kasner points as \(\Sigma \to 1\).

4.3 Spatial curvature

In a vacuum, plane-wave Bianchi VII\(_h\) spacetime, the trace of the 3-Ricci tensor \(\mathcal{R}_{\alpha\beta}\) associated with the surfaces of constant time is
\[
\mathcal{R} = -\frac{1}{2} (n_2 - n_3)^2 - 6a^2,
\]
where \(a_1, n_2\) and \(n_3\) are given in Eq. (17). According to the above \(\mathcal{R} < 0\) always, which guarantees the hyperbolic geometry of the spatial sections. Note that we may use (17) to recast Eq. (24) as
\[
\mathcal{R} = -\frac{k^2(1 - f^2)^2}{2f^2t^2} - \frac{6r^2}{t^2} = -\frac{2r(1 + 2r)}{t^2}.
\]

Then, combining results (19), (20) and (24), one can show that that expression (7) (i.e. the Lukash analogue of the Friedmann equation) reduces to the constraint (2a).

Spatial curvature anisotropies are described via the symmetric and trace-free tensor \(S_{\alpha\beta}\). In the Lukash model, the only non-zero components of \(S_{\alpha\beta}\) are
\[
S_{11} = -\frac{4r(1 - r)}{3f^2t^2}, \quad S_{22} = \frac{k^2(1 - f^2)(2 + f^2)}{3f^2t^2}, \quad S_{33} = -\frac{k^2(1 - f^2)(1 + 2f^2)}{3f^2t^2}
\]
and
\[
S_{23} = -\frac{kr(1 - f^2)}{ft^2}.
\]

According to (25)-(27), the spatial curvature of the model vanishes at the maximum shear limit, namely as \(r \to 0\). At the other end, as \(r\) approaches unity, only the isotropic part of \(\mathcal{R}_{\alpha\beta}\) survives. Recall that \(k^2(1 - f^2) = 0\) as \(r \to 0\) or 1 (see Eq. (2a)).

4.4 Weyl curvature

The only non-zero components of the Weyl curvature tensor associated with a vacuum, plane-wave Bianchi type VII\(_h\) spacetime are
\[
E_{22} = -E_{33} = -H_{23} = \frac{k^2(1 - f^4)}{2f^2t^2}
\]
and
\[
H_{22} = -H_{33} = E_{23} = \frac{k(1 - f^4)(1 - 2r)}{2ft^2}.
\]
This means that the Weyl curvature is minimised near the limits $r = 1$ and $r = 0$, where $k^2(1 - f^2) = 0$. Also, when $r = 1/2$, which corresponds to $\Sigma = 1/4$ (see Eq. (22)), we have $H_{22} = 0 = H_{33} = E_{23}$ and the Weyl tensor has only one independent component.

These relations between the Weyl tensor components also guarantee that the electric and magnetic Weyl tensors have equal magnitudes, although they are orthogonal to each other. More specifically, expressions (28) and (29) imply that

$$E^2 = H^2 = \frac{k^2(1 - f^2)^2}{4f^2t^4} \left[ \frac{k^2}{f^2} (1 + f^2)^2 + (1 + 2r)^2 \right], \quad (30)$$

and that

$$E_{\alpha\beta}H^{\alpha\beta} = 0. \quad (31)$$

As with the expansion anisotropy, it helps to measure the anisotropy of the Weyl field by means of the following expansion-normalised, dimensionless scalars

$$W_+ \equiv \frac{W_+}{\Theta^2} \quad \text{and} \quad W_- \equiv \frac{W_-}{\Theta^4}, \quad (32)$$

where $W_{\pm} = E^2 \pm H^2$ by definition. The self-similarity of the Lukash solution guarantees that $W_+$ is time independent, while the plane-wave nature of the model ensures that $W_- = 0$ (see constraint (30)).

5 The perturbed Lukash solution

The Lukash solution is the late-time attractor of the Bianchi $VII_h$ spacetimes [8, 9], which are known to contain the open FRW universe as a special subcase. In this respect, studying the behaviour of the perturbed Lukash model could provide useful clues to the final stages of ever-expanding FRW cosmologies with $\mu + 3\rho > 0$. If $\mu + 3\rho \leq 0$ then the expansion will approach the FRW (for power-law inflationary behaviour [38]) or de Sitter universe in accord with the cosmic no-hair theorems. Similar effects can arise from the effects of higher-order curvature corrections to the Einstein-Hilbert Lagrangian of general relativity [39].

5.1 Nonlinear equations

Consider a perturbed vacuum Bianchi $VII_h$ spacetime and allow for a low-density, pressure-free matter component. The nonlinear evolution of the latter is governed by the standard energy-density conservation law

$$\dot{\mu} = -\Theta \mu. \quad (33)$$

When the matter component is in the form of dust, there is no acceleration and the only additional kinematic contribution comes from possible rotational disturbances. This means that the cosmological velocity field, which is identified with the motion of the matter, remains geodesic although it is allowed to rotate. Rotation is monitored by the propagation equation of the vorticity vector

$$\dot{\omega}_a = -\frac{2}{3}\Theta \omega_a + \sigma_{ab}\omega^b, \quad (34)$$

which also satisfies the constraint

$$D^a\omega_a = 0. \quad (35)$$
According to (34), in addition to the expansion effect, which always reduces vorticity, there is a contribution due to the shear anisotropy. Note that for pressure-free matter there are no sources of rotation and vorticity remains zero if it was zero initially.

In the presence of a pressureless fluid and vorticity the nonlinear Friedmann and Raychaudhuri equations respectively give

\[ \mathcal{R} = -\frac{2}{3} \Theta^2 (1 - \Sigma - \Omega) - 2\omega^2 \] (36)

and

\[ \dot{\Theta} = -\frac{4}{3} \Theta^2 (1 + 2\Sigma + \frac{1}{2} \Omega) + 2\omega^2, \] (37)

where \( \Omega = 3\mu/\Theta^2 \) is the density parameter. Similarly, the introduction of matter and rotation modifies the rest of the propagation formulae given in section 1.2 as follows

\[ \dot{E}_{ab} = -\Theta E_{ab} + \text{curl} H_{ab} + 3\sigma_{c(a} E_{b)}^c - \frac{1}{3} \mu \sigma_{ab} - \omega^c \epsilon_{cd(a} E_{b)}^d, \] (38)

\[ H_{ab} = -\Theta H_{ab} - \text{curl} E_{ab} + 3\sigma_{c(a} H_{b)}^d - \omega^c \epsilon_{cd(a} H_{b)}^d, \] (39)

\[ \dot{\sigma}_{ab} = -\frac{2}{3} \Theta \sigma_{ab} - E_{ab} - \sigma_{c(a} \sigma^{c} b) - \omega_{(a} \omega_{b)}, \] (40)

while the associated constraints become

\[ H_{ab} = \text{curl} \sigma_{ab} + D_{(a} \omega_{b)}, \] (41)

\[ D^b \sigma_{ab} = \frac{2}{3} D_a \Theta + \text{curl} \omega_a, \] (42)

\[ D^b E_{ab} = \epsilon_{abc} \sigma^b_d H^{cd} - 3H_{ab} \omega^b + \frac{1}{3} D_a \mu, \] (43)

\[ D^b H_{ab} = -\epsilon_{abc} \sigma^b_d E^{cd} + 3E_{ab} \omega^b + \mu \omega_a. \] (44)

Note that vorticity affects the expansion and the shear evolution only at the nonlinear level, while it has a linear contribution in Eqs. (38), (39) and (41)-(44).

### 5.2 Linear vortices

The presence of matter means that one can identify the cosmological velocity field with that of the material component, which in turn gives physical substance to the idea of rotation. We measure the relative strength of rotational perturbations by means of the expansion-normalised dimensionless scalar

\[ \varpi = \frac{\omega}{\Theta}, \] (45)

with \( \omega = (\omega_a \omega^a)^{1/2} \). Taking the time derivative of \( \omega \) and using (23) and (34) we obtain the linear expression

\[ \dot{\varpi} = -\frac{1}{3} \tilde{\Theta} \left( 1 - 2\tilde{\Sigma} - 3\tilde{\Sigma}_{ab} n^a n^b \right) \varpi, \] (46)

where the tildas indicate background quantities. Here, \( \Sigma_{ab} = \sigma_{ab}/\Theta \) by definition and \( n_a \) is the unit vector along the rotation axis (i.e. \( \omega_a = \omega n_a \)). Written in an orthonormal frame the above reads

\[ \partial_t \varpi = -\frac{1}{3} \tilde{\Theta} \left( 1 - 2\tilde{\Sigma} \right) \varpi + \tilde{\Theta} \Sigma_{ab} n^a n^b \varpi, \] (47)

\(^4\)In a low density perturbed model with \( \Omega \ll 1 \), the linear Friedmann and Raychaudhuri equations retain their background functional form. The difference is that \( \Sigma \) is generally not constant, which means that the average scale factor of the perturbed Lukash model no longer obeys the simple power-law evolution given in section 3.2.
where $\tilde{\Sigma}_{\alpha\beta}$ and $n_\alpha$, $n_\beta$ are the non-zero orthonormal frame components of $\tilde{\Sigma}_{ab}$ and $n_a$, $n_b$ respectively. The first term on the right-hand side of this equation describes the average evolution of $\varpi$, while the second conveys the directional effects. The former increases (or decreases) $\varpi$ depending on whether $\tilde{\Sigma}$ is greater (or less) than $1/2$.  

Overall, linear vortices grow, relative to the average background expansion, when the following condition holds

$$2\tilde{\Sigma} + 3\tilde{\Sigma}_{\alpha\beta}n^\alpha n^\beta > 1.$$  

(48)

This condition implies that the growth of linear vortices also depends on the relative orientation between the background shear eigenframe and the rotation axis. Assuming that rotation takes place along the $e_1$ axis of the background orthonormal frame, we may use expressions (18a) and (22) to verify that condition (48) holds as long as $r < 1/2$.  In this case linear vortices grow relative to the background expansion as $\varpi \propto t^{1-2r}$. This means that the growth rate of $\varpi$ takes the maximum value $\varpi \propto t$ at the $r = 0$ limit, namely for maximum background shear. Alternatively, one may assume that the rotation axis lies along $e_2$ or $e_3$. Then, a similar calculation shows that linear vortices can never grow relative to the average background expansion. At best, $\varpi$ remains constant (when $r \to 0$).

Following expressions (46) and (47), the average shear distortion always increases the residual amount of rotation. Moreover, when condition (48) is fulfilled, the overall effect of the shear (including the direction dependent component $\Sigma_{\alpha\beta}$) will also boost vorticity perturbations. In other words, as far as rotation is concerned, shear distortions can mimic the effects of matter pressure. Recall that in the presence of pressure vorticity does not necessarily decay with time (e.g. see [27, 40, 41]). Instead, for matter with a stiff enough equation of state rotation will increase despite the universal expansion. In our case, non-zero pressure with $p = p(\mu)$ means that Eq. (46) takes the form

$$\dot{\varpi} = -\frac{1}{2} \tilde{\Theta} \left[ 1 - 2\tilde{\Sigma} - 3 \left( c_2^2 + \tilde{\Sigma}_{ab}n^a n^b \right) \right] \varpi,$$

(49)

where $c_2^2 = dp/d\mu$ is the square of the adiabatic sound speed. This demonstrates clearly the analogy between the shear and the pressure effects on rotation. For example, when $\tilde{\Sigma} = 1$ and the direction-dependent term on the right-hand side of the above is negligible, the shear effect on $\varpi$ is indistinguishable from that of a matter component with $p/\mu = 2/3$.

### 5.3 Linear shear anisotropies

In a perturbed Bianchi $VII_h$ model with low-density dust, Raychaudhuri’s formula (see Eq. (37)) ensures that the linear expansion proceeds unaffected by the presence of matter or by rotational distortions. Also, expression (40) guarantees that, to linear order, vortical perturbations do not affect the evolution of the expansion-normalised shear parameter. Thus, ignoring rotational and matter effects, we take the time derivative of (21) and then use Eqs. (6), (8) and (11) to arrive

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5 The value $\tilde{\Sigma} = 1/2$, which corresponds to $r = 1/4$ (see expression (22)), indicates a Lukash-type spacetime with half the allowed amount of shear anisotropy. In the case of rotational distortions, $\tilde{\Sigma} = 1/2$ also indicates the point where the background expansion rate drops faster than the vorticity. Recall the decreasing effect of the shear on the average expansion scalar (see Eqs. (6), (23)). Therefore, at $\tilde{\Sigma} = 1/2$ the scalar $\varpi$ starts to increase on average (i.e. excluding the direction dependent effect of the last term in (47)).

6 The condition $r < 1/2$ on the metric parameter translates into the constraint $\tilde{\Sigma} > 1/4$ on the background shear anisotropy. Accordingly, only three quarters of the allowed Lukash backgrounds are potentially unstable to linear rotational perturbations.
\[ \dot{\Sigma} = -\frac{4}{3} \Theta (1 - \Sigma) \Sigma - \frac{3}{\Theta^2} S_{ab} \sigma^{ab}. \] (50)

The first term in the right-hand side of this equation describes the average linear evolution of \( \Sigma \). Conversely, the last term of Eq. (50) describes directional effects and depends on the relative orientations of the \( S_{ab} \) and \( \sigma_{ab} \) eigenframes. Relative to an orthonormal coordinate system the above reads

\[ \partial_t \Sigma = -\frac{4}{3} \Theta (1 - \Sigma) \Sigma - \frac{3}{\Theta^2} S_{\alpha\beta} \sigma^{\alpha\beta}, \] (51)

with \( S_{\alpha\beta} \) and \( \sigma_{\alpha\beta} \) representing the non-zero orthonormal frame components of \( S_{ab} \) and \( \sigma_{ab} \) respectively.

Suppose that \( \Sigma \to 0 \) to zero order. This state corresponds to a background of minimum shear and 3-Ricci anisotropy (i.e. \( r \to 1 \)) with \( \tilde{\sigma}_{\alpha\beta} = 0 = \tilde{S}_{\alpha\beta} \), where tildas indicate the zero-order quantities. At this limit, which corresponds to a perturbed Milne universe, Eq. (50) takes the linear form

\[ \partial_t \Sigma = -\frac{4}{3} \Theta \Sigma. \] (52)

As a result, \( \Sigma \propto a^{-4} \propto t^{-4} \) to first order. In other words, when \( \tilde{\Sigma} \to 0 \) any linear expansion anisotropies that may occur will quickly disperse.

When \( \Sigma \to 1 \) we have \( r \to 0 \) and the background has maximum shear anisotropy and zero 3-Ricci curvature. This means that \( \tilde{S}_{\alpha\beta} = 0 \) but \( \tilde{\sigma}_{\alpha\beta} \neq 0 \) (see expressions (18), (26) and (27)). Here it helps to introduce the auxiliary linear variable \( S = 1 - \Sigma \) and rewrite Eq. (52) as follows

\[ \partial_t S = \frac{4}{3} \tilde{\Theta} S + \frac{3}{\Theta^2} \tilde{\sigma}_{\alpha\beta} S^{\alpha\beta}, \] (53)

given that \( \tilde{S} \to 0 \). Note that whenever \( S \) grows the \( \Sigma \)-parameter decreases and vice-versa. Also note that although the background \( S \) remains bounded within the open interval \((0, 1)\), this is not necessarily the case at the linear level. Indeed, using the zero-order relations (18) and the trace-free nature of \( S_{\alpha\beta} \), we find that the last term on the right-hand side of (53) equals \( 3S_{11}t \) in the \( r \to 0 \) limit. This in turn allows us to recast Eq. (53) as

\[ \partial_t S = \frac{4}{3t} S + 3S_{11}t, \] (54)

where \( S_{11} \) is a component of the perturbed spatial Ricci tensor. The above means that a decrease in \( S \) and therefore an increase in the linear shear anisotropy when \( \tilde{\Sigma} \to 1 \) is possible in principle. Suppose, for example, that the perturbed \( S_{11} \) component retains its background form, namely that \( S_{11} = -4r(1 - r)/3l^2 \) with \( 0 < r \ll 1 \). Then, Eq. (54) solves to give

\[ S = S_0 \left( \frac{t}{t_0} \right)^{4/3} + 3r(1 - r) \left[ 1 - \left( \frac{t}{t_0} \right)^{4/3} \right], \] (55)

where \( S_0 = S(t_0) \). Recalling that \( S = 1 - \Sigma \) by definition, this leads to the following expression for the perturbed shear parameter:

\[ \Sigma = 1 - 3r(1 - r) + \frac{3r^2(1 - 2r)}{1 + 2r} \left( \frac{t}{t_0} \right)^{4/3}, \] (56)
assuming that \( \Sigma_0 = (1 - r)/(1 + 2r) \). The latter means that the initial relation between the perturbed \( \Sigma \) and \( r \) has the background functional form. Given that \( r \ll 1 \), expression (56) implies that the perturbed \( \Sigma \)-parameter can break through the \( \Sigma = 1 \) barrier provided that

\[
\left( \frac{t}{t_0} \right)^{4/3} > \frac{(1 - r)(1 + 2r)}{r(1 - 2r)}.
\]  

(57)

For example, if \( r \sim 10^{-2} \), this will happen when \( t > 10^{3/2}t_0 \). Note, however, that the smaller the value of \( r \) the longer it takes for the perturbed \( \Sigma \) to cross through unity.

It should be emphasised that \( \Sigma > 1 \) to first order implies that the linear 3-Ricci curvature of the perturbed Lukash model becomes positive (see Eq. (36) with \( \Omega = 0 = \omega_0 \)). This is possible for maximum background shear (i.e. as \( r \to 0 \)), because the zero-order spatial Ricci tensor vanishes at that limit. So, in principle, the perturbed spacetime can have spatial sections with slightly positive curvature. Clearly, in this case the linear Lukash model perturbs away from the family of the Bianchi \( VII_h \) cosmologies, a fact demonstrated by the unbounded shear parameter. On the other hand, if we demand that the linear 3-curvature is never positive, then \( \Sigma \) will always remain bounded by unity.

### 5.4 Linear Weyl anisotropy

Consider the expansion-normalised dimensionless variable \( W_- \) defined in (32a). This scalar vanishes in the Lukash plane-wave background, which implies that a linear growth for \( W_- \) is a sign of instability at that perturbative order. In addition, \( W_- \) directly determines the Weyl curvature invariant \( C_{abcd}C^{abcd} \). In this respect, the linear evolution of \( W_- \) also monitors the gravitational entropy of the perturbed Lukash universe and, to a certain extent, that of low-density open FRW models.\(^7\)

For weakly rotating Lukash universes with a low-density dust component (i.e. when \( \pi, \Omega \ll 1 \)), the time derivative of definition (32b) leads to

\[
\dot{W}_- = -\frac{2}{3}(1 - 4\Sigma)\Theta W_- + \frac{1}{\Theta^4} \left( E^{ab}_{\text{curl}} H_{ab} + H^{ab}_{\text{curl}} E_{ab} \right) \\
+ \frac{3}{\Theta^4} \sigma_{ca} \left( E_c^b E^{ab} - H_c^b H^{ab} \right) - \frac{1}{\Theta^4} \omega^c \epsilon_{cda} \left( E_d^b E^{ab} - H_d^b H^{ab} \right) \\
- \frac{\mu}{2\Theta^4} \sigma_{ab} E^{ab},
\]  

(58)

on using Eqs. (37)-(39). The last two terms in the right-hand side of the above describe the effects of vorticity and matter respectively. Note that although matter does not directly contribute to the Weyl field, the later is not entirely arbitrary because of the contracted Bianchi identities. These are in a sense the field equations for the Weyl curvature and, among others, convey the matter effects on the propagation of the Weyl components (e.g. see [42]). On introducing an

\(^7\)A complete description of the Weyl anisotropy of the perturbed vacuum Bianchi \( VII_h \) spacetime, also requires to study the linear evolution of the expansion normalised scalar \( W_+ \) (see definition (32a)). Unlike \( W_- \), however, \( W_+ \) has nonzero background value. This complicates further the linear study of \( W_+ \) and allows only for relatively trivial analytic solutions of the associated propagation equation. It is conceivable that an improved version of the formalism presented here will be able to address the full Weyl anisotropy of the perturbed Lukash solution.
orthonormal frame, expression (58) reads

$$\partial_t \mathcal{W}_- = -\frac{2}{3} (1 - 4\Sigma) \Theta \mathcal{W}_- + \frac{1}{\Theta^4} \epsilon_{\mu\nu\alpha} \left( E^{\alpha\beta} \partial^{\mu} H^{\nu}_{\beta} + H^{\alpha\beta} \partial^{\mu} E^{\nu}_{\beta} \right)$$

$$- \frac{1}{\Theta^4} \epsilon_{\mu\nu\alpha} a^{\mu} \left( E^{\alpha\beta} H^{\nu}_{\beta} + H^{\alpha\beta} E^{\nu}_{\beta} \right) - \frac{3}{\Theta^4} \eta_{\mu\alpha} \left( E^{\alpha\beta} H^{\mu}_{\beta} + H^{\alpha\beta} E^{\mu}_{\beta} \right)$$

$$+ \frac{1}{\Theta^4} \eta_{\mu}^{\alpha\beta} H_{\alpha\beta} + \frac{3}{\Theta^4} \sigma_{\mu\alpha} \left( E^{\mu}_{\beta} E^{\alpha\beta} - H^{\mu}_{\beta} H^{\alpha\beta} \right)$$

$$- \frac{1}{\Theta^4} \omega^{\mu\nu\alpha\beta} \left( E^{\alpha\nu} E^{\beta\alpha} - H^{\alpha\nu} H^{\beta\alpha} \right) - \frac{\mu}{2\Theta^4} \sigma_{\alpha\beta} E^{\alpha\beta}. \quad (59)$$

Again, we notice that the first term on the right-hand side describes the average evolution of $\mathcal{W}_-$, while the rest describe direction-dependent effects. The former depends crucially on the background shear anisotropy and it is reversed in sign at $\Sigma = 1/4$. This threshold corresponds to $r = 1/2$ and indicates the point where the background expansion rate starts decreasing faster than the average $W_-$. Recall that at $\Sigma = 1/4$ the background Weyl field has only one independent component (see section 3.4).

To proceed further we consider homogeneous perturbations in the Weyl field, namely that $\partial_{\mu} E^{\alpha\beta} = 0 = \partial_{\mu} H_{\alpha\beta}$. This condition also monitors the large-scale behaviour of $\mathcal{W}_-$ in the presence of inhomogeneities. In addition, we assume that $E_{\alpha\beta} = E_{\alpha\beta} E_{\mu\nu}$ and $H_{\alpha\beta} H_{\mu\nu} = H_{\alpha\beta} H_{\mu\nu}$ to linear order.\(^8\) Then, the combined linear contribution of the second, third, fourth and fifth terms in the right-hand side of (59) is zero, while the sixth term reduces to $-2(1 - r)\mathcal{W}_- / t$. Also, employing the background relations between the Weyl tensor components given in (28) and (29), one can immediately show that the vorticity term in Eq. (59) vanishes to first order. Finally, on using the zero-order expressions (2), (18), (20), (28) and given that $\mu \propto t^{-(1+2r)}$ for dust (see Eq. (33)), we find that

$$\frac{\mu}{2\Theta^4} \sigma_{\alpha\beta} E^{\alpha\beta} = \frac{\mu}{\Theta^4} \bar{\sigma}_{23} \bar{E}_{23} \propto \frac{r(1 - r)(1 - 2r)}{(1 + 2r)^4} t^{-2r}, \quad (60)$$

to first order. Note that, according to this, the matter effect in Eq. (59) vanishes when either $\bar{\sigma}_{23}$ or $\bar{E}_{23}$ is zero. On these grounds the linear expression (59) reduces to

$$\partial_t \mathcal{W}_- = -2r t^{-1} \mathcal{W}_- + \frac{r(1 - r)(1 - 2r)}{(1 + 2r)^4} C t^{-2r}, \quad (61)$$

where the parameter $r$ varies in the open interval $(0, 1)$ and $C$ is a constant. This result implies that the expansion-normalised scalar $\mathcal{W}_-$ remains unchanged to linear order as $r \to 0$, that is for maximum background shear. The reason is that in the $r = 0$ limit the Weyl anisotropy of the Lukash solution disappears (see Eqs. (28), (29)). The matter effect also vanishes near the minimum background shear limit (i.e. as $r \to 1$) and at the $r = 1/2$ threshold. The former of these two results is not surprising, since the Lukash solution decays to the Milne universe at the $r = 1$ limit. When $r = 1/2$, however, the matter effect is zero because $\bar{E}_{23} = 0$ at that point (see Eq. (29)). Recall that at the $r = 1/2$ threshold the Weyl tensor has only one independent component.

When $r \to 0$ we find that $\partial_t \mathcal{W}_- = 0$, which implies that any deviations in $\mathcal{W}_-$ that may occur will remain constant. On the other hand, as $r \to 1$ or at the $r = 1/2$ threshold the

\(^8\)This assumption, which allows us to obtain analytic solutions for the linear evolution of $\mathcal{W}_-$, implies that the $E_{11}$ and $H_{11}$ components of the perturbed model vanish. This restriction already holds in the background.
expansion-normalised Weyl parameter decays as

\[ W_+ \propto t^{-2} \quad \text{and} \quad W_- \propto t^{-1}, \quad (62) \]

respectively. In general, ignoring the effect of matter leads to \( W_- \propto t^{-2r} \) and therefore ensures that \( W_- \) dies away at a rate inversely proportional to the background shear anisotropy. This in turn implies that \( W_- \propto t^{-2(2+r)} \), since \( \tilde{\Theta} \propto t^{-1} \). The linear decay of the \( W_- \) parameter is in agreement with the stability of the vacuum, plane-wave equilibrium points that consist the late time asymptotes of the Bianchi \( \text{VII}_h \) spacetimes (e.g. see [1, 9, 20]).

The situation changes in the presence of matter. The latter, even when it is in the form of a low-density dust component, introduces new degrees of freedom into the system and the decrease of \( W_- \) is not always guaranteed. Indeed, the general solution of Eq. (61) reads

\[ W_- = C_1 t^{1-2r} + C_2 t^{-2r}, \quad (63) \]

where \( C_1, C_2 \) are constants and \( C_1 = Cr((1 - r)(1 - 2r))/(1 + 2r)^4 \). According to the above, the expansion-normalised scalar \( W_- \) will start increasing as \( W_- \propto t^{1-2r} \) when \( 0 < r < 1/2 \). The latter corresponds to \( \tilde{\Sigma} > 1/4 \), which means that three quarters of the allowed Lukash models, those with the largest background shear anisotropy, are unstable against linear Weyl curvature distortions. Incidentally, the same family was also found vulnerable to linear rotational distortions (see section 5.2). Finally, we note that the aforementioned matter effects are sensitive to the precise evolution of the background model, namely to the properties of the Lukash vacuum solution. Given that, one should be careful before extrapolating these results to Bianchi \( \text{VII}_h \) models with dust.

6 Discussion

Bianchi models, particularly those that contain the FRW cosmologies as special subcases, are essential for our understanding of the large scale anisotropy of the universe. In the family of Bianchi spacetimes, those of type \( \text{VII}_h \) are the most general homogeneous models containing the spatially open FRW universe. In the absence of matter these spacetimes reduce to the plane-wave solution found by Doroshkevich and Lukash [2, 5]. These vacuum models also act as the future attractors for the non-tilted, perfect fluid Bianchi \( \text{VII}_h \) cosmologies [43]. For this reason the empty \( \text{VII}_h \) Lukash model has been used to study the late-time evolution of perturbed open FRW universes with a conventional matter content.

Here we have engaged a mixture of covariant and orthonormal frame methods to study analytically the linear response of the Lukash solution to a variety of perturbations. Our results show that the amount of the background shear anisotropy is crucial for the stability of the vacuum model. More specifically, by looking into rotational or shear perturbations, we found that the linear instability of these distortions is more likely in models with higher background shear anisotropy. When dealing with vorticity perturbations, in particular, the background shear can force linear rotational perturbations to grow, thus mimicking the effects of a fluid with a stiff equation of state. Also, when the unperturbed model has the maximum allowed kinematical anisotropy, linear shear distortions are no longer necessarily bounded. In the latter case the Lukash universe perturbs away from the family of the Bianchi \( \text{VII}_h \) spacetimes.

Our linear analysis also considered the effects of a non-zero matter component. When the latter was in the form of low-density dust, we found that matter had no effect on either the rotation or the kinematical anisotropy of the perturbed Lukash solution. However, the presence
of matter (even a non-relativistic pressureless fluid) plays a key role in the evolution of the Weyl anisotropy of the perturbed spacetime. In particular, our study showed that the introduction of a pressure-free component at the linear level is the catalyst that can force the vacuum $VII_h$ model to diverge from its original plane-wave nature.

In the present paper the study of the matter effects has been confined to the case of a pressure-free component. It is relatively straightforward to extend this formalism to include the effects of pressure. Generally speaking, nonzero pressure means that the matter term in Eq. (61) decays faster than in the case of dust. This in turn should make the matter effects on $W_{\text{an}}$ less pronounced. However, this rather intuitive picture will be probably complicated by the presence of pressure gradients, which for dust are identically zero. One could also consider the potential implications of large-scale magnetic fields or of a non-conventional, ‘dark’ matter component. The study of magnetic fields in Bianchi $VII_h$, for example, has lead to limits on the strength of a possible large-scale homogeneous field more stringent than those obtained from standard nucleosynthesis constraints [44]. This happens because anisotropic stresses play an important role in the evolution of simple anisotropic universes of Bianchi type I. The anisotropic trace-free stress mimics the part played by the anisotropic curvature in type $VII$ models and, in combination with a perfect fluid, slows the decay of the shear anisotropy in a subtle way [45]-[47]. The latter leads to more severe observational consequences for the CMB. A further consideration for future work is the role of dark energy in the universe. For a perfect fluid with $\mu + 3p < 0$, in violation of the strong-energy condition, the Lukash metric is unstable and approaches the flat Friedmann universe as $t \to \infty$, following with the course of power-law inflation. If $p = -\mu$, then the dynamics approach the de Sitter universe with exponential rapidity within the event horizon of any geodesically moving observer [48]-[50]. This case is less interesting from a mathematical point of view because all distortions are rapidly inflated away. However, it is likely to be of considerable astronomical interest because of the growing observational evidence that inflation has played some role in the very early evolution of the universe and that dark energy, with $\mu + 3p < 0$, is dominating the dynamics of the universe again today. Finally, we note that if the universe is negatively curved with compact topology there are severe constraints on the possibility of any homogeneous anisotropy existing in the expansion at all: all Bianchi $VII_h$ universes have to be isotropic [51]-[53].

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