SOME COMPACTNESS RESULTS RELATED TO
SCALAR CURVATURE DEFORMATION

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Abstract. Motivated by the prescribing scalar curvature problem, we study the equation $\Delta_g u + Ku^p = 0$ \((1 + \zeta \leq p \leq \frac{n + 2}{n - 2})\) on locally conformally flat manifolds \((M, g)\) with \(R(g) = 0\). We prove that when \(K\) satisfies certain conditions and the dimension of \(M\) is 3 or 4, any solution \(u\) of this equation with bounded energy has uniform upper and lower bounds. Similar techniques can also be applied to prove that on 4-dimensional scalar positive manifolds the solutions of $\Delta_g u - \frac{n - 2}{4(n - 1)} R(g)u + Ku^p = 0$, \(K > 0\), \(1 + \zeta \leq p \leq \frac{n + 2}{n - 2}\) can only have simple blow-up points.

1. Introduction

Let \((M^n, g)\) be an \(n\)-dimensional compact manifold with metric \(g\), and \(u > 0\) be a positive function defined on \(M\). The scalar curvature of the conformally deformed metric \(u^{\frac{4}{n-2}}g\) is given by

$$R(u^{\frac{4}{n-2}}g) = -c(n)^{-1} u^{-\frac{4}{n-2}} \left( \Delta_g u - c(n) R(g)u \right)$$

where \(c(n) = \frac{n - 2}{4(n - 1)}\).

The famous Yamabe conjecture says that given a compact Riemannian manifold \((M, g)\) of dimension \(n \geq 3\), \(g\) can be conformally deformed to a metric of constant scalar curvature. This conjecture was proved to be true by the work of Trudinger (1976), Aubin (1976) and Schoen (1984).

It is natural to ask for a prescribed smooth function \(K\) on \(M\) if it is possible to deform \(g\) to a metric with scalar curvature \(K\). J. Escobar and R. Schoen studied this question in (1991) and gave some conditions under which \(K\) can be a scalar curvature function.

Since the proofs of their results are variational, it is interesting to know if the general compactness theorems hold under the same conditions as well. On an \(n\)-dimensional compact manifold \((M, g)\), are all the positive solutions of the following equation compact in the \(C^2\) norm? This equation is the subcritical scalar curvature deformation equation

$$\Delta_g u - \frac{n - 2}{4(n - 1)} R(g)u + Ku^p = 0$$

where \(1 < p \leq \frac{n + 2}{n - 2}\)
If we can establish a uniform upper bound for the $C^0$ norm of $u$, then the compactness in the $C^2$ norm will follow easily from the bootstrap argument.

R. Schoen ([7]), and Y. Li and M. Zhu ([5]) gained compactness results on three dimensional manifolds not conformally diffeomorphic to $S^3$ when $K$ is a positive constant and positive function, respectively.

In this paper we investigate the zero scalar curvature case. This case is unknown and technically more difficult than the positive scalar curvature case. When the scalar curvature is zero, the equation becomes

$$\Delta_g u + Ku^p = 0$$

where $1 + \zeta \leq p \leq \frac{n + 2}{n - 2}$

The necessary conditions for (1) to have solutions are $K > 0$ somewhere on $M$ and $\int_M K dv_g < 0$, hence $K$ has to change signs on the manifold. The blow-up estimates used to prove the $K > 0$ case depends on the lower bound on $K$, so it cannot be used on the region where $K$ is small.

One way to overcome this problem is to assume an energy bound on $u$. Under this assumption, it can be proved that the maximum of $u$ is uniformly bounded where $K \leq \delta$ for $\delta > 0$ appropriately small. This implies that blow-up can not happen and the $C^0$ norms are bounded, which gives the compactness.

Furthermore, if the integrals $|\int_M K|$ have a uniform positive lower bound, it can be proved that any limit function of a convergence sequence of positive solutions of (1) is non-trivial, i.e. is strictly positive.

In summary, the main result in this paper is:

**Theorem 1.1.** Let $(M, g)$ be a three or four dimensional locally conformally flat compact manifold with $R(g) = 0$. Let $\mathcal{K} := \{K : K > 0$ somewhere on $M, \int_M K dv_g \leq -C_K^{-1} < 0, \text{ and } \|K\|_{C^3} \leq C_K\}$ for some constant $C_K$, and $S_\Lambda := \{u : u > 0 \text{ solves (1), } K \in \mathcal{K}, \text{ and } E(u) \leq \Lambda\}$. Then there exists $C = C(M, g, C_K, \Lambda, \zeta) > 0$ such that $u \in S_\Lambda$ satisfies $C^{-1} \leq \|u\|_{C^3(M)} \leq C$.

This theorem is consistent with the existence theorem of Escobar and Schoen.

Additionally, we can use some of the techniques in the proof of the above theorem to get a better understanding of the possible blow-up for 4 dimensional scalar positive manifolds.

**Theorem 1.2.** If $(M, g)$ is a four dimensional locally conformally flat compact manifold with positive scalar curvature, and $K > 0$ on $M$, then
then the possible blow-up of the solutions of equation

\[ \Delta_g u - c(n)R(g)u + Ku^p = 0 \quad (1 \leq p \leq \frac{n + 2}{n - 2}) \]

is always simple.

We will give the precise definition of simple blow-up in section 5. Roughly it means that blow-up points are isolated and consist of a simple “bubble”.

The rest of this paper is mostly devoted to the proof of Theorem 1.1 and it will also illustrate the proof of Theorem 1.2. In section 2 we prove the lower bound on \( u \) assuming the upper bound exists. In section 3 we show that \( u \) is uniformly bounded above on the region where \( K \) is sufficiently small. In section 4 we reduce the possible blow-up on the region where \( K \) is big to two cases. In section 5 we introduce the definition of simple blow-up and prove some important estimates. In sections 6 and 7 we show that neither case in section 4 can happen, hence prove the compactness theorem.

2. The Lower Bound on \( u \)

Suppose \( u \) with \( E(u) \leq \Lambda \) is a solution of equation (1) for some \( 1 + \zeta \leq p \leq \frac{n + 2}{n - 2} \), where \( K \) satisfies

\[ K > 0 \text{ somewhere on } M, \quad \|K\|_{C^3} \leq C_K, \quad \text{and} \quad \int_M Kdv_g \leq -\frac{1}{C_K} < 0. \]

**Lemma 2.1.** \( \int_M u^2dv_g \leq C \int_M |\nabla u|^2dv_g \) where \( C = C(M, g, n, C_K) \).

**Proof:** Let \( \bar{u} = \text{Vol}(M)^{-1} \int_M u dv_g \), then

\[ \int_M u^2dv_g \leq 2\int_M (u - \bar{u})^2dv_g + 2\int_M \bar{u}^2dv_g \quad (3) \]

By the Poincaré inequality

\[ \int_M (u - \bar{u})^2dv_g \leq C(M, g) \int_M |\nabla u|^2dv_g, \quad (4) \]

so we only need to find an upper bound for \( \bar{u} \).
\[
\bar{u}^{p+1} C_K^{-1} \\
\leq \bar{u}^{p+1} \left( \int_M -K \, dv_g \right) \\
= - \int_M \bar{u}^{p+1} \, dv_g + \int_M (\bar{u}^{p+1} - \bar{u}^{p+1}) \, dv_g \\
\leq \max_M |K| \int_M |u^{p+1} - \bar{u}^{p+1}| \, dv_g \\
\leq \max_M |K| \int_M (2^{p-1}(p+1)|u - \bar{u}|^{p+1} + 2^{2p-1}(p+1)\bar{u}^p|u - \bar{u}|) \, dv_g \\
\quad \text{(by calculus)} \\
\leq C(n, C_K) \int_M (|u - \bar{u}|^{p+1} + \bar{u}^p|u - \bar{u}|) \, dv_g.
\]
Therefore
\[
\bar{u}^{p+1} \leq C(n, C_K) \int_M (|u - \bar{u}|^{p+1} + \bar{u}^p|u - \bar{u}|) \, dv_g.
\]
The first term on the right hand side
\[
\int_M |u - \bar{u}|^{p+1} \, dv_g \leq C(M, g, n) \left( \int_M |\nabla u|^2 \, dv_g \right)^{\frac{p+1}{2}}
\]
by Hölder and Sobolev inequalities.
Similarly the second term on the right hand side
\[
\int_M |u - \bar{u}| \, dv_g \leq C(M, g, n) \left( \int_M |\nabla u|^2 \, dv_g \right)^{\frac{1}{2}}.
\]
Therefore
\[
\bar{u}^{p+1} \leq C(M, g, n, C_K) \left( \left( \int_M |\nabla u|^2 \, dv_g \right)^{\frac{p+1}{2}} + \bar{u}^p \left( \int_M |\nabla u|^2 \, dv_g \right)^{\frac{1}{2}} \right)
\]
Choose
\[C = \max\{1, C(M, g, n, C_K)\}, \quad \bar{u} \leq C \left( \int_M |\nabla u|^2 \, dv_g \right)^{\frac{1}{2}}.
\]
Thus by \([3]\) and \([4]\)
\[
\int_M u^2 \, dv_g \leq C(M, g, n, C_K) \int_M |\nabla u|^2 \, dv_g.
\]

Now assume \(u\) is bounded above, we claim that it is also bounded below away from 0. Suppose not, then \(\exists \ \{x_i\} \subset M, \{u_i\} \subset S_\lambda \) and \(\{K_i\} \subset K\) such that \(\Delta u_i + K_i u_i^p = 0\) and \(u_i(x_i) \to 0\). Since \(\|u_i\|_{C^3}

is bounded, then there is a subsequence also denoted as \( \{u_i\} \) which converges in \( C^2 \)-norm to some function \( u \geq 0 \). Similarly after passing to a subsequence \( \{K_i\} \) also converges to some function \( K \). Let \( p \) be the corresponding limit point of \( \{p_i\} \), then we have

\[
\Delta u + Ku^p = 0.
\]

The manifold \( M \) is compact, so \( \{x_i\} \) also has a limit point \( x \in M \). Since \( u_i(x_i) \to 0 \), \( u(x) = 0 \) and then by the strong maximum principle \( u \equiv 0 \).

On the other hand, by the Sobolev inequality and Lemma 2.1

\[
\left( \int_M u_i^{2n/(n-2)} \, dv_g \right)^{n-2/n} \leq C \int_M |\nabla u_i|^2 \, dv_g
\]

\[
= C \int_M K_i u_i^{p_i+1} \, dv_g
\]

\[
\leq C \left( \int_M u_i^{2n/(n-2)} \, dv_g \right)^{(n-2)/n} (p_i+1)
\]

This implies

\[
1 \leq C \left( \int_M u_i^{2n/(n-2)} \, dv_g \right)^{(n-2)/n} (p_i+1).
\]

Then since \( p_i - 1 \geq \zeta > 0 \), \( u_i \) cannot converge to 0, contradicting \( u \equiv 0 \).

Now it is only left to show that \( u \) has a uniform upper bound. Since \( u \) satisfies equation (1), the upper bound of \( \|u\|_{C^3} \) will follow easily by the standard elliptic theory and Sobolev embedding theorem once we establish a uniform upper bound on \( u \).

3. An Upper Bound on \( u \) on the set where \( K \) is Small

Let \( u > 0 \) be a function which satisfies

\[
\Delta u + Ku^p = 0, \quad 1 \leq p \leq \frac{n + 2}{n - 2}
\]

for some \( K \in K \).

Choose \( 1 < \beta < \frac{2n}{n-2} - 1 \). Let \( x \) be a point on \( M \) and \( \varphi \) be a cut-off function such that

\[
\varphi \equiv 1 \text{ on } B_{\frac{4\sigma}{n-2}}(x), \quad \varphi \equiv 0 \text{ on } M \setminus B_{\sigma}(x), \quad \text{and } |\nabla \varphi| \leq 2\sigma^{-2}.
\]

Multiplying (1) by \( \varphi^2 u^\beta \) and integrating by parts gives
\[ \int_M \varphi^2 Ku^{\beta+p} dv_g = -\int_M \varphi^2 u^\beta \Delta_g u \ dv_g \]
\[ = \frac{4\beta}{(\beta+1)^2} \int_M \varphi^2 |\nabla (u^{\frac{\beta+1}{2}})|^2 dv_g + \frac{2}{\beta + 1} \int_M u^{\frac{\beta+1}{2}} \nabla (u^{\frac{\beta+1}{2}}) \cdot \nabla (\varphi^2) dv_g \]

Let \( w = u^{\frac{\beta+1}{2}} \), then
\[ \frac{4\beta}{(\beta+1)^2} \int_M \varphi^2 |\nabla w|^2 \ dv_g \]
\[ = -\frac{4}{\beta + 1} \int_M \varphi w \nabla w \cdot \nabla \varphi \ dv_g + \int_M \varphi^2 Kw^2u^{p-1} \ dv_g \]
\[ \leq \frac{4}{\beta + 1} \epsilon \int_M \varphi^2 |\nabla w|^2 \ dv_g + \frac{4}{\beta + 1} \epsilon^{-1} \int_M w^2 |\nabla \varphi|^2 \ dv_g \]
\[ + \max_{\overline{B}_\sigma(x)} K \int_M \varphi^2 w^2u^{p-1} \ dv_g \]

Choosing \( \epsilon \) small enough (only depending on \( \beta \)) we can absorb the first integral into the left hand side and get
\[ \int_M \varphi^2 |\nabla w|^2 \ dv_g \leq C \sigma^{-2} \int_{\overline{B}_\sigma(x)} w^2 \ dv_g + \max_{\overline{B}_\sigma(x)} K \int_M \varphi^2 w^2u^{p-1} \ dv_g. \]

Therefore
\[ \int_M |\nabla (\varphi w)|^2 dv_g \leq 2 \int_M |\nabla \varphi|^2 w^2 \ dv_g + 2 \int_M \varphi^2 |\nabla w|^2 dv_g \]
\[ \leq C \sigma^{-2} \int_{\overline{B}_\sigma(x)} w^2 dv_g + 2 \max_{\overline{B}_\sigma(x)} K \int_M \varphi^2 w^2u^{p-1}dv_g \]

Since \( \beta + 1 < \frac{2n}{n-2} \),
(5)
\[ \int_{B_\sigma(x)} w^2 dv_g \leq \int_{B_\sigma(x)} u^{\frac{2n}{n-2}} dv_g + \text{Vol}(M, g) \leq C(M, g, n, C_K, \Lambda) \]
by the Sobolev inequality, Lemma 2.1 and the energy bound on \( u \).

Similarly \( (p-1) \frac{n}{2} \leq \frac{2n}{n-2} \) implies
\[ \int_M \varphi^2 w^2 u^{p-1} \, dv_g \]
\[ \leq \left( \int_{B_\sigma(x)} (\varphi^2 w^2)^{\frac{n-2}{n}} \, dv_g \right)^{\frac{n}{n-2}} \left( \int_{B_\sigma(x)} (u^{p-1})^{\frac{n}{p}} \, dv_g \right)^{\frac{p}{n}} \]
\[ \leq C(M, g, n, C_K, \Lambda) \left( \int_{B_\sigma(x)} (\varphi w)^{\frac{2n}{n-2}} \, dv_g \right)^{\frac{n-2}{n}}. \]

So we know
\[ \int_M |\nabla(\varphi w)|^2 dv_g \leq C(M, g, n, C_K, \Lambda) \left[ \sigma^{-2} + \max_{B_\sigma(x)} K \left( \int_{B_\sigma(x)} (\varphi w)^{\frac{2n}{n-2}} \, dv_g \right)^{\frac{n-2}{n}} \right] \] (6)

Then by the Sobolev inequality
\[ \left( \int_M (\varphi w)^{\frac{2n}{n-2}} \, dv_g \right)^{\frac{n-2}{n}} \leq C(M, g) \left( \int_M |\nabla(\varphi w)|^2 \, dv_g + \int_M (\varphi w)^2 \, dv_g \right) \]
\[ \leq C(M, g, n, C_K, \Lambda) \left[ \sigma^{-2} + \max_{B_\sigma(x)} K \left( \int_{B_\sigma(x)} (\varphi w)^{\frac{2n}{n-2}} \, dv_g \right)^{\frac{n-2}{n}} \right] \]

where the last inequality follows from (5) and (6).

Let \( \delta = \frac{1}{4} C(M, g, n, C_K, \Lambda)^{-1} \). If \( \max_{B_\sigma(x)} K < 2\delta \), then we can absorb the second term on the right hand side of the above inequality into the left hand side to get
\[ \left( \int_M (\varphi w)^{\frac{2n}{n-2}} \, dv_g \right)^{\frac{n-2}{n}} \leq C(M, g, n, C_K, \Lambda) \sigma^{-2}. \]

Therefore
\[ \int_{B_{\frac{3}{4}\sigma}(x)} u^{\frac{p+1}{2} \frac{2n}{n-2}} \, dv_g \leq C(M, g, n, C_K, \Lambda) \sigma^{-\frac{2n}{n-2}} \] (7)

Define \( K_\delta := \{ x \in M : K(x) < \delta \} \), and let \( x_1 \in \partial K_\delta \) and \( x_2 \in \partial K_{2\delta} \) be the points which realize the distance between \( \partial K_\delta \) and \( \partial K_{2\delta} \), i.e.,
\[ d_g(x_1, x_2) = d_g(\partial K\delta, \partial K_{2\delta}). \]

Then
\[ 2\delta = K(x_2) \leq K(x_1) + \max_M |\nabla K|d_g(x_1, x_2) \]
\[ \leq \delta + C_K d_g(x_1, x_2) \]

which implies
\[ d_g(x_1, x_2) > C(\delta, C_K) = C(M, g, n, \Lambda, C_K). \]

Let \( \sigma = \frac{1}{2}d_g(x_1, x_2) > \frac{1}{2}C(M, g, n, \Lambda, C_K). \) For any \( x \in K_\delta, \) we have
\[ B_{\frac{3}{4}\sigma}(x) \subset K_{2\delta}, \] therefore
\[ (8) \int_{B_{\frac{3}{4}\sigma}(x)} u^{\frac{\beta + 1}{2} \cdot \frac{2n}{n-2}} \ dv_g \leq C(M, g, n, \Lambda, C_K) \]

by (7) and the lower bound on \( \sigma. \)

This tells us that \( u \in L^{\frac{\beta + 1}{2} \cdot \frac{2n}{n-2}}(B_{\frac{3}{4}\sigma}(x)), \) hence \( K u^{p-1} \in L^r(B_{\frac{3}{4}\sigma}(x)) \)

where
\[ (9) \quad r = \frac{\beta + 1}{2} \cdot \frac{2n}{n-2} \geq \frac{\beta + 1}{2} \cdot \frac{n}{2} > n \] when \( p \neq 1 \)

and let \( r = \frac{\beta + 1}{2} \cdot \frac{n}{2} \) when \( p = 1. \)

By the elliptic theory
\[ (10) \quad \sup_{B_{\frac{1}{4}\sigma}(x)} u \leq C \left( M, g, r, \|K u^{p-1}\|_{L^r(B_{\frac{3}{4}\sigma}(x))} \right) \sigma^{-\frac{n}{2}}\|u\|_{L^2(B_{\frac{3}{4}\sigma}(x))} \]

The constant in the above inequality usually blows up when \( r \to \frac{n}{2} \) or when \( \|K u^{p-1}\|_{L^r(B_{\frac{3}{4}\sigma}(x))} \) is unbounded. But here we have a fixed lower bound on \( r \) from (9), and by (8)
\[ \|K u^{p-1}\|_{L^r(B_{\frac{3}{4}\sigma}(x))} \leq C(M, g, n, \zeta, \Lambda, C_K). \]

So the constant in (10) has an upper bound only depending on \( M, g, n, \zeta, \Lambda \) and \( C_K. \)

Since we also have a uniform lower bound on \( \sigma, \) and by Lemma 2.1 and the energy bound on \( u \) we know that
\[ \|u\|_{L^2(B_{\frac{3}{4}\sigma}(x))} \leq C(M, g, n, \zeta, C_K) \Lambda, \]

it can be concluded from (10) that
\[ \sup_{B_{\frac{1}{4}\sigma}(x)} u \leq C(M, g, n, \zeta, \Lambda, C_K). \]
Therefore
\[
\sup_{K_\delta} u \leq C(M, g, n, \zeta, \Lambda, C_K)
\]
since \(x\) is an arbitrary point in \(K_\delta\).

In the next few sections we are going to prove that \(u\) is also uniformly bounded on \(M \setminus K_\delta\).

4. REDUCTION TO THE ISOLATED BLOW-UP CASE

We first prove a lemma.

**Lemma 4.1.** Suppose \(W \supseteq K_\delta / 2\) (\(\delta\) is chosen as in section 3 and \(K_\delta / 2\) is also defined in the same way as in section 3) is a compact subset of \(M\) and \(u > 0\) is a solution of equation (1) with \(K \in K\) (as defined in Theorem 1.1). Given \(\epsilon, R > 0\), there exists \(C = C(\epsilon, R) > 0\) such that if
\[
\max_{M \setminus W} d(x)^{p-1} u(x) \geq C,
\]
where \(d(x) = \text{dist}_g(x, W)\) (let \(d(x) = 1\) in case \(W = \emptyset\)), then
- \(n + 2 - p < \epsilon\)
- there exists \(x_0 \in M \setminus W\) which is a local maximum point of \(u\), and the geodesic ball \(B_{Ru(x_0)}^{p-1}(x_0) \subset M \setminus W\)
- Choose the \(y\)-coordinates around \(x_0\) so that \(z \equiv y u(x_0)^{p-1} / v(x_0)\) is a geodesic normal coordinate system centered at \(x_0\), then
\[
\left\| u(0)^{-1} u \left( \frac{y}{u(0)^{p-1}} \right) - \bar{v}(y) \right\|_{C^2(B_{2R}(0))} < \epsilon
\]
where
\[
\bar{v}(y) = \left( 1 + \frac{K(x_0)}{n(n-2)} |y|^2 \right)^{-\frac{n+2}{2}}
\]
is the (unique) exact solution of
\[
\Delta v(y) + K(x_0)v(y)^{\frac{n+2}{n-2}} = 0, \quad y \in \mathbb{R}^n, \quad 0 < v \leq 1, \quad v(0) = 1
\]
where \(\Delta\) is the Euclidean Laplacian.

**Proof:** Suppose no such \(C\) exists, then there exists \(\{u_i\}, \{p_i\}, \{K_i\}\) and \(\{W_i\}\) such that
\[
\Delta_g u_i + K_i u_i^{p_i} = 0,
\]
\[
\max_{M \setminus W_i} d_i(x)^{p_i-1} u_i(x) \geq i \quad \text{where} \quad d_i(x) = \text{dist}_g(x, W_i),
\]
and
\[
W_i \supseteq K_\delta^{(i)} := \{ x \in M : K_i(x) < \delta / 2 \},
\]
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but for each \(i\) there doesn’t exist any local maximum point which satisfies the conditions in the lemma.

Define \(f_i(x) = d_i(x)\frac{2}{n-1}u_i(x)\), let \(x_i\) be a maximum point of \(f_i(x)\) in \(M \setminus \overline{W_i}\). Since on the boundary of \(M \setminus \overline{W_i}\), \(f_i \equiv 0\), we know \(x_i \in \overline{M \setminus \overline{W_i}}\). Let \(z\) be the geodesic normal coordinates with respect to the background metric \(g\) centered at \(x_i\), and let \(y = u_i(x_i)\frac{2}{n-1}z\). Define

\[
v_i(y) = u_i(x_i)^{-1}u_i\left(\frac{y}{u_i(x_i)^{\frac{2}{n-1}}}\right).
\]

Then \(v_i\) satisfies

\[
(11) \quad \Delta_{g^{(i)}} v_i + K_i \left(\frac{y}{u_i(x_i)^{\frac{2}{n-1}}}\right) v_i^{p_i} = 0
\]

where the metric \(g^{(i)}(y) = g_{\alpha\beta} \left(\frac{y}{u_i(x_i)^{\frac{2}{n-1}}}\right) dy^\alpha dy^\beta\). Since we have chosen coordinates for which \(g_{\alpha\beta}(0) = \delta_{\alpha\beta}\), for \(y\) on a bounded set, \(g^{(i)}(y)\) converges to the Euclidean metric.

Let \(r_i = \frac{d_i(x_i)}{2}\) and \(R_i = u_i(x_i)^{\frac{2}{n-1}}r_i\), then by our choice of \(x_i\), \(R_i \geq \frac{1}{2} r_i \to \infty\) as \(i \to \infty\). In the ball centered at \(x_i\) with radius \(r_i\),

\[
\frac{\max_{B_{R_i}} d_i(x)}{\min_{B_{R_i}} d_i(x)} = \frac{d_i(x_i) + \frac{d_i(x)}{2}}{d_i(x_i) - \frac{d_i(x)}{2}} = 3.
\]

Also from the choice of \(x_i\) we know

\[
d_i(x)\frac{2}{n-1}u_i(x) = f_i(x) \leq f_i(x_i) = d_i(x_i)\frac{2}{n-1}u_i(x_i),
\]

so

\[
u_i(x_i) \leq 3\frac{2}{n-1}u_i(x_i), \quad \text{i.e.,} \quad u_i\left(\frac{y}{u_i(0)^{\frac{2}{n-1}}}\right) \leq 3\frac{2}{n-1}u_i(0)
\]

in the \(y\) coordinates. Therefore

\[
(12) \quad v_i(y) \leq 3\frac{2}{n-1} \leq C \quad \text{on} \quad B_{R_i}(0) \subset \mathbb{R}^n(y)
\]

for some constant \(C\) independent of \(i\).

By our choice of \(x_i\), \(d_i(x_i)\frac{2}{n-1}u_i(x_i) \to \infty\) as \(i \to \infty\), but \(d_i(x_i)\) is bounded above on \(M\), so \(u_i(x_i) \to \infty\) as \(i \to \infty\). Hence \(\left|\frac{y}{u_i(x_i)^{\frac{2}{n-1}}}\right| \to 0\) on any compact subset on the \(y\)-plane.

After passing to a subsequence, we may assume \(x_i \to x_0 \in M\) and \(K_i\) converges to some function \(K\) in \(C^2\)-norm. By the choice of \(x_i\) and \(W_i\) we know \(K_i(x_i) \geq \delta\), so \(K(x_0) \geq \frac{\delta}{2}\).
Then with (11) and (12) by the elliptic estimates we know \( \{v_i\} \) converges to some function \( \hat{v} \) in \( C^2 \)-norm which satisfies \( \hat{v}(y) \geq 0 \) and

\[
\Delta \hat{v}(y) + K(x_0)\hat{v}(y) \frac{n+2}{n-2} = 0 \quad \text{on } \mathbb{R}^n(y)
\]

Here and throughout the rest of this chapter we use \( \Delta \) to denote the Laplacian with respect to the Euclidean metric. The reason for \( \lim_{i \to \infty} p_i = \frac{n+2}{n-2} \) is that \( K(x_0) > 0 \), \( v(0) = 1 \) (since \( v_i(0) \equiv 1 \)) and there is no non-trivial solution of \( \Delta v + cv^p = 0 \) on \( \mathbb{R}^n \) with \( c > 0 \) and \( 1 \leq p < \frac{n+2}{n-2} \) (proved in [2]). Another consequence of \( v(0) = 1 \) is that by the Harnack inequality \( v(y) > 0 \) for any \( y \).

As proved in [2], \( \hat{v}(y) \) has the expression

\[
\hat{v}(y) = \left( \frac{\lambda K(x_0)^{-\frac{1}{2}}}{1 + \frac{1}{n(n-2)} \lambda^2 |y - \bar{y}|^2} \right)^{\frac{n-2}{2}}
\]

for some \( \lambda > 0 \), where \( \bar{y} \) is the maximum point of \( \hat{v} \).

Since \( v_i \to \hat{v} \) in \( C^2 \)-norm on compact subsets on \( \mathbb{R}^n(y) \), there exists \( \{\bar{y}_i\} \) such that each \( \bar{y}_i \) is a local maximum point of \( v_i(y) \) and \( \bar{y}_i \to \bar{y} \). Let \( \bar{z}_i = u_i(x_i)^{-\frac{n-1}{2}} \bar{y}_i \) and \( \bar{x}_i = \exp_{x_i} \bar{z}_i \in M \). Note that \( R_i \to \infty \), so for large enough \( i \), \( \bar{y}_i \in B_{R_i}(0) \subset \mathbb{R}^n(y) \). Therefore \( \bar{z}_i \in B_{\rho_i}(0) \subset \mathbb{R}^n(z) \), hence \( \bar{x}_i \in B_{\rho_i}(x_i) \), consequently \( B_{\rho_i}(\bar{x}_i) \subset M \setminus W_i \) and \( K_i(\bar{x}_i) > \frac{\delta}{2} \).

Let \( l \) be any fixed large radius so that \( |y| < l \). Since \( R_i = r_i u_i(x_i)^{-\frac{n-1}{2}} \to \infty \), for large enough \( i \), \( 2l < R_i \). Hence

\[
B_{\frac{2l}{u_i(x_i)^{\frac{n-1}{2}}}}(x_i) \subset B_{\rho_i}(x_i) \subset M \setminus W_i.
\]

When \( i \) is large enough \( \bar{y}_i \) is also in \( B_l(0) \), so \( |\bar{z}_i| < \frac{l}{u_i(x_i)^{\frac{n-1}{2}}} \). This implies that

\[
B_{\frac{l}{u_i(x_i)^{\frac{n-1}{2}}}}(\bar{x}_i) \subset B_{\frac{2l}{u_i(x_i)^{\frac{n-1}{2}}}}(x_i) \subset M \setminus W_i.
\]

We chose \( \bar{y}_i \) to be a maximum point of \( v_i(y) \), and \( \bar{y}_i \to \bar{y} \) which is the only maximum point of \( \hat{v} \). Thus when \( i \) is sufficiently large \( \bar{y}_i \) is also the only maximum point of \( v_i \) for \( |y| \leq 2l \). Therefore \( \bar{x}_i \) is the maximum point of \( u_i \) on \( B_{\frac{2l}{u_i(x_i)^{\frac{n-1}{2}}}}(x_i) \). In particular \( u_i(x_i) \leq u_i(\bar{x}_i) \), so

\[
B_{\frac{l}{u_i(x_i)^{\frac{n-1}{2}}}}(\bar{x}_i) \supset B_{\frac{l}{u_i(\bar{x}_i)^{\frac{n-1}{2}}}}(\bar{x}_i).
\]

Hence

- \( B_{\frac{l}{u_i(\bar{x}_i)^{\frac{n-1}{2}}}}(\bar{x}_i) \subset M \setminus W_i \).
Now redefine \( z \) to be the geodesic normal coordinates centered at each \( \bar{x}_i \) and let \( y = u_i(\bar{x}_i) \frac{nu - 1}{2} z \). Define 

\[
\bar{v}_i(y) = u_i(\bar{x}_i) \frac{nu - 1}{2} y \frac{u_i(\bar{x}_i) \frac{nu - 1}{2} z}{u_i(\bar{x}_i) \frac{nu - 1}{2} z}.
\]

Since \( \bar{x}_i \) is the maximum point of \( u_i \) on 

\[
B \frac{nu - 1}{u_i(\bar{x}_i) \frac{nu - 1}{2}}(x_i) \supset B \frac{nu - 1}{u_i(\bar{x}_i) \frac{nu - 1}{2}}(\bar{x}_i) \supset B \frac{nu - 1}{u_i(\bar{x}_i) \frac{nu - 1}{2}}(\bar{x}_i),
\]

then for all \( |y| \leq l \), we have \( \bar{v}_i(y) \leq 1 \).

Then the same argument as that for \( v_i \) shows that \( \bar{v}_i \) converges in \( C^2 \)-norm to some function \( \bar{v} > 0 \) which satisfies

\[
\Delta \bar{v} + K(\bar{x}_0) \frac{nu - 2}{nu - 2} = 0
\]

where \( \bar{x}_0 = \lim_{i \to \infty} \bar{x}_i \) and \( K(\bar{x}_0) > 0 \) (because \( K_i(\bar{x}_i) > \delta \)).

Since \( y = 0 \) is a maximum point of \( \bar{v}_i \), the maximum of \( \bar{v} \) is attained at \( y = 0 \). So \( \bar{v} \) has the expression

\[
\bar{v}(y) = \left( 1 + \frac{K(\bar{x}_0)}{n(n - 2)} |y|^2 \right)^{-\frac{nu - 2}{2}}.
\]

Then since \( \bar{v}_i \to \bar{v} \) and \( \bar{x}_i \to \bar{x}_0 \), for large enough \( i \),

\[
\bullet \left\| u_i(\bar{x}_i)^{-1} u_i \left( \frac{nu - 1}{2} y \frac{u_i(\bar{x}_i) \frac{nu - 1}{2} z}{u_i(\bar{x}_i) \frac{nu - 1}{2} z} \right) - \left( 1 + \frac{K(\bar{x}_0)}{n(n - 2)} |y|^2 \right)^{-\frac{nu - 2}{2}} \right\| < \epsilon.
\]

This is a contradiction.

\[\Box\]

Now fix \( \epsilon > 0 \) and \( R >> 0 \). Suppose

\[
\max_{M \setminus K_{\frac{\epsilon}{2}}} d_g(x, M \setminus K_{\frac{\epsilon}{2}})^{\frac{2}{p - 1}} u(x) > C \quad (C \text{ is the constant in Lemma 4.1}).
\]

Applying Lemma 4.1 to the case \( W = K_{\frac{\epsilon}{2}} \), there exists \( x_1 \in M \) which is a local maximum point of \( u \) and satisfies the conditions in the lemma. Let \( r_1 = Ru(x_1) \frac{nu - 1}{2} \). We can stop the procedure if at any point \( x \in M, \ d_g(x, K_{\frac{\epsilon}{2}} \cup B_{r_1}(x_1))^\frac{2}{p - 1} u(x) \) is bounded by some constant only depending on \( \epsilon \) and \( R \). Otherwise let \( W = K_{\frac{\epsilon}{2}} \cup B_{r_1}(x_1) \) and apply Lemma 4.1 again to find another local maximum point \( x_2 \in M \) to satisfy the conditions in the lemma.

Repeating this procedure we will get a sequence of disjoint balls

\[
B_{r_1}(x_1), ..., B_{r_N}(x_N).
\]
The sequence must be finite because for each \( i \)
\[
r_i = Ru(x_i)^{-\frac{p-1}{2}} \geq R \left( \max_M u \right)^{-\frac{p-1}{2}},
\]
hence the volume of \( B_{r_i}(x_i) \) has a lower bound. Here we allow the number of balls \( N \) to depend on the function \( u \).
Thus we know
\[
u(x) \leq Cd_g \left( x, K_{\frac{\delta}{2}} \cup \bigcup_{i=1}^{N} B_{r_i}(x_i) \right)^{-\frac{2}{p-1}},
\]
for some constant \( C = C(\epsilon, R) \).

Consider an arbitrary point \( x \in M \setminus (K_{\frac{\delta}{2}} \cup \{x_1, ..., x_N\}) \).
If \( d_g(x, x_i) > 2r_i \) for all \( i = 1, ..., N \), then
\[
d_g \left( x, \bigcup_{i=1}^{N} B_{r_i}(x_i) \right) = d_g \left( x, B_{r_i}(x_i) \right) = d_g(x, x_i) - r_i > \frac{1}{2} d_g(x, x_i) \geq \frac{1}{2} d_g(x, \{x_1, ..., x_N\}).
\]
So
\[
u(x) \leq C \left( d_g \left( x, K_{\frac{\delta}{2}} \cup \{x_1, ..., x_N\} \right) \right)^{-\frac{2}{p-1}}.
\]
If \( x \in B_{2r_i}(x_i) \) for some \( i \in \{1, ..., N\} \), then \( 2r_i > d_g(x, x_i) \), i.e.,
\[
r_i > \frac{1}{2} d_g(x, x_i) \geq \frac{1}{2} d_g \left( x, K_{\frac{\delta}{2}} \cup \{x_1, ..., x_N\} \right).
\]
In the coordinate system \( y \) centered around \( x_i \) as in Lemma 4.1
\[
\left\| u(x_i)^{-1} u \left( \frac{y}{u(x_i)^{\frac{p-1}{2}}} \right) - \left( 1 + \frac{K(x_i)}{n(n-2)|y|^2} \right)^{-\frac{n-2}{2}} \right\| < \epsilon,
\]
therefore
\[
u(x) \leq (1 + \epsilon) u(x_i) = (1 + \epsilon) R^{\frac{2}{p-1}} r_i^{-\frac{2}{p-1}} \quad \text{since } r_i = \frac{R}{u(x_i)^{\frac{p-1}{2}}}
\]
\[
\leq C \left( d_g \left( x, K_{\frac{\delta}{2}} \cup \{x_1, ..., x_N\} \right) \right)^{-\frac{2}{p-1}}.
\]
We conclude that
Proposition 4.2. Given $\epsilon > 0$, $R >> 0$, there exists $C = C(\epsilon, R)$ such that if $u$ is a solution of equation (11) and
\[
\max_{x \in M} \left( \left( d_g(x, K_{\frac{\delta}{2}}) \right)^{\frac{2}{p-1}} u(x) \right) > C,
\]
then there exists $\{x_1, \ldots, x_N\} \subset M \setminus K_{\frac{\delta}{2}}$ with $N$ depending on $u$, and

- Each $x_i$ is a local maximum of $u$ and the geodesic balls $\{B_{r_i}(x_i)\}$ are disjoint.
- $\left| \frac{n+2}{n-2} - p \right| < \epsilon$ and in the coordinate system $y$ so chosen that $z = \frac{y}{u(x_i)^{\frac{p}{2}-1}}$ is the geodesic normal coordinate system centered at $x_i$, we have
\[
\left\| u(x_i)^{-1} u \left( \frac{y}{u(x_i)^{\frac{p}{2}-1}} \right) - \bar{v}(y) \right\|_{C^2(B_{2R}(0))} < \epsilon
\]
on the ball $B_{2R}(0) \subset \mathbb{R}^n(y)$, where
\[
\bar{v}(y) = \left( 1 + \frac{K(x_i)}{n(n-2)|y|^2} \right)^{-\frac{n-2}{2}}.
\]
- There exists $C = C(\epsilon, R)$ such that
\[
u(x) \leq C \left( d_g(x, K_{\frac{\delta}{2}} \cup \{x_1, \ldots, x_N\}) \right)^{-\frac{2}{p-1}}.
\]

Before we proceed with the proof, we need to give two definitions.

**Definition 4.3.** We call a point $\bar{x}$ on a manifold $M$ a **blow-up point** of the sequence $\{u_i\}$ if $\bar{x} = \lim_{i \to \infty} x_i$ for some $\{x_i\} \subset M$ and $u_i(x_i) \to \infty$.

**Definition 4.4.** Let $\{u_i\}$ be a sequence of functions satisfying $\Delta_g u_i + K_i u_i^p = 0$ on some manifold $M$ where the metrics $g_i$ converge to some metric $g_0$. A point $\bar{x} \in M$ is called an **isolated blow-up point** of $\{u_i\}$ if there exist local maximum points $x_i$ of $u_i$ and a fixed radius $r_0 > 0$ such that

1. $x_i \to \bar{x}$.
2. $u_i(x_i) \to \infty$.
3. $u_i(x) \leq C \left( d_g(x, x_i) \right)^{-\frac{2}{p-1}}$ \( \forall x \in B_{r_0}(x_i) \) for some constant $C$ independent of $i$.

Now we are going to prove that $u$ is uniformly bounded on $M \setminus K_{\frac{\delta}{2}}$. Suppose it is not, then there are sequences $\{u_i\}, \{p_i\}$ and $\{K_i\}$ such
that
\[ \Delta_g u_i + K_i u_i^{p_i} = 0 \quad \text{and} \quad \max_{M \setminus K_{i,\delta}} u_i \to \infty \quad \text{as} \quad i \to \infty \]
where \( K_{i,\delta} := \{ x \in M : K_i(x) < \delta \} \).

By an argument similar to that in section 3, \( d_g(\partial K_{i,\delta}, \partial K_{i,\delta}) \) is bounded below by some constant depending only on \( M, g, n, \zeta, \Lambda, C_K, \) so \( \max_{M \setminus K_{i,\delta}} \left( d_g(x, K_{i,\delta}) \right)^{\frac{n-2}{2}} u_i \to \infty \) as \( i \to \infty \). Thus for fixed \( \epsilon > 0 \) and \( R >> 0 \) we can apply Proposition 3.2 to each \( u_i \) and find \( x_{1,i}, \ldots, x_{N(i),i} \) such that

(15) each \( x_{j,i} \quad (1 \leq j \leq N(i)) \) is a local maximum of \( u_i \);
(16) the balls \( B_{\frac{R}{u(x_{j,i})^{p_i-1}}} (x_{j,i}) \) are disjoint;

and for coordinates \( y \) centered at \( x_{j,i} \) such that \( \frac{y}{u(x_{j,i})^{p_i-1}} \) is the geodesic normal coordinates,

(17) \[ \left\| u_i(x_{j,i})^{-1} u_i \left( \frac{y}{u(x_{j,i})^{p_i-1}} \right) - \left( 1 + \frac{K_i(x_{j,i})}{n(n-2)} |y|^2 \right)^{\frac{n-2}{2}} \right\|_{C^2(B_{2R}(0))} < \epsilon. \]

Let \( \sigma_i = \min \{ d_g(x_{\alpha,i}, x_{\beta,i}) : \alpha \neq \beta, 1 \leq \alpha, \beta \leq N(i) \} \). Without lost of generality we can assume \( \sigma_i = d_g(x_{1,i}, x_{2,i}) \). There are two possibilities which could happen.

**Case I:** \( \sigma_i \geq \sigma > 0 \).
Then the points \( x_{j,i} \) have isolated limiting points \( x_1, x_2, \ldots \), which are isolated blow-up points of \( \{ u_i \} \) as defined above.

**Case II:** \( \sigma_i \to 0 \).
Then we rescale the coordinates to make the minimal distance to be 1: let \( y = \sigma_{i}^{-1} z \) where \( z \) is the geodesic normal coordinate system centered at \( x_{1,i} \). We also rescale the function by defining

\[ v_i(y) = \sigma_{i}^{p_i-1} u_i(\sigma_i y). \]

\( v_i \) satisfies
\[ \Delta_{g(i)} v_i + K_i(y) v_i^{p_i} = 0 \]
where the metric \( g^{(i)}(y) = g_{\alpha\beta}(\sigma_i y) dy^\alpha dy^\beta. \)

Let \( y_{j,i} \) be the coordinate corresponding to \( x_{j,i} \) then the distance between any two points in \( \{ y_{1,i}, \ldots, y_{N(i),i} \} \) is at least 1. Let \( \{ y_1, y_2, \ldots \} \) be the limiting points of \( \{ y_{j,i} \} \).
Let $\Omega$ be any compact subset of $\mathbb{R}^n(y) \setminus \{y_1, y_2, \ldots\}$. Because we have proved that $u_i$ is uniformly bounded on $K_{i,\delta}$, we must have the maximum points $\{x_{j,i}\} \subset M \setminus K_{i,\delta}$ and therefore

$$d_g(x_{j,i}, K_{i,\frac{\delta}{4}}) \geq d_g(\partial K_{i,\delta}, \partial K_{i,\frac{\delta}{4}}) \geq C.$$ 

This means for $y \in \Omega$ and $x = \exp_{x_{1,i}}(\sigma_i y) \in M$, when $i$ is large enough,

$$d_g \left( x, K_{i,\frac{\delta}{2}} \cup \{x_{1,i}, \ldots, x_{N(i),i}\} \right) = d_g \left( x, \{x_{1,i}, \ldots, x_{N(i),i}\} \right).$$

Then by Proposition 4.2 and the fact that $g^{(i)}$ converges to the Euclidean metric on $\Omega$,

$$v_i(y) = \sigma_i^{\frac{2}{n+2}} u_i(\sigma_i y) \leq \sigma_i^{\frac{2}{n+2}} C(\epsilon, R) \left( d_g \left( x, \{x_{1,i}, \ldots, x_{N(i),i}\} \right) \right)^{-\frac{2}{n+2}} = C(\epsilon, R) \left( d_{g^{(i)}}(y, \{y_{1,i}, \ldots, y_{N(i),i}\}) \right)^{-\frac{2}{n+2}} \leq C(\epsilon, R, \Omega).$$

We also know that $K_i$ has uniform $C^3$ bound, so it converges in $C^2$ norm to some function $K$. Then by the standard elliptic estimates $\{v_i\}$ converge on $\Omega$ to some function $v$ in $C^2$-norm which satisfies $\Delta v + K(x_1)v^{\frac{n+2}{n+2}} = 0$ on $\Omega$, where $x_1$ is the limit point of $\{x_{1,i}\}$ and $K(x_1) \geq \delta > 0$. Since $\Omega$ is arbitrary, $\Delta v + K(x_1)v^{\frac{n+2}{n+2}} = 0$ on $\mathbb{R}^n(y) \setminus \{y_1, y_2, \ldots\}$.

If none of the points $0 = y_1, y_2, \ldots$ is a blow-up point for $\{v_i\}$, then $\{v_i\}$ is uniformly bounded near each of those points. Thus the convergence $v_i \to v$ holds near each of those points and therefore $\Delta v + K(x_1)v^{\frac{n+2}{n+2}} = 0$ on $\mathbb{R}^n(y)$. By the definition of $v_i$ and the choice of $\sigma_i$, $x_{1,i}$ and $x_{2,i}$,

$$v_i(0) = \sigma_i^{\frac{2}{n+2}} u_i(x_{1,i}) \leq \left( \frac{R}{u_i(x_{1,i})^{\frac{n+2}{2}}} \right)^{\frac{2}{n+2}} u_i(x_{1,i}) \leq R^{\frac{2}{n+2}}.$$ 

So $v(0)$ is bounded below away from 0 and hence $v \neq 0$. By similar argument as before we then know $v$ can be expressed as

$$v(y) = \left( \frac{\lambda K(x_1)^{-\frac{n}{2}}}{1 + \frac{1}{n(n+2)} \lambda^2 |y - \bar{y}|^2} \right)^{\frac{n+2}{n+2}}$$

for some $\lambda > 0$ and $\bar{y} \in \mathbb{R}^n(y)$. It implies that $v$ can only have one critical point. But we know each $v_i$ has at least two critical points $0$
and \( |y_{2,i}| = 1 \), and \( u_i \) is \( C^2 \)-close to \( v \), so \( v \) must also have at least two critical points. This is a contradiction. Thus \( \{v_i\} \) has at least one blow-up point, without lost of generality we can assume it to be 0.

If there are other blow-up points besides 0, they are at least distance 1 apart. For any \( \|y\| \leq \frac{1}{2} \), the corresponding \( x = \exp_{x_{1,i}}(\sigma_i y), x_{j,i} = \exp_{x_{1,i}}(\sigma_i y_{j,i}) \in M \) satisfy \( d_g(x, \{x_{1,i}, ..., x_{N(i)}\}) = d_g(x, x_{1,i}) \). So by Proposition 4.2

\[
v_i(y) = \frac{\sigma_i^{2/p_i - 1}}{2} u_i(\sigma_i y)
= \sigma_i^{2/p_i - 1} u_i(x)
\leq \sigma_i^{2/p_i - 1} C(\epsilon, R) d_g(x, x_{1,i})^{-\frac{2}{p_i - 1}}
= C(\epsilon, R) d_{g^{(i)}}(y, 0)^{-\frac{2}{p_i - 1}}
\]

Therefore we have reduced Case II to the following case:

There is a sequence of functions \( \{v_i\} \), each satisfies

\[
\Delta_{g^{(i)}} v_i + K_i(\sigma_i y) u_i^{p_i} = 0
\]

where \( g^{(i)}(y) = g_{\alpha\beta}(\sigma_i y) dy^\alpha dy^\beta \) converges to the Euclidean metric on compact subset of \( \mathbb{R}^n \). The sequence \( \{v_i\} \) has isolated blow-up point(s) \( \{0, ..., \} \).

In the following sections we are going to show neither Case I nor Case II can happen for \( n = 3, 4 \).

5. Simple blow-up and Related Estimates

In this section we are going to diverge from the proof of the compactness theorems temporarily. We will analyze the phenomenon of simple blow-up and obtain some estimates which are important in the rest of the proof.

**Definition 5.1.** \( x_0 \) is called a **simple blow-up point** of \( \{u_i\} \) if it is an isolated blow-up point and there exists \( r_0 > 0 \) independent of \( i \) such that \( \hat{w}_i(r) \) has only one critical point for \( r \in (0, r_0) \), where \( \hat{w}_i(r) = \text{Vol}(S_r)^{-1} \int_{S_r} |z|^{2/p_i - 1} u_i(z) d\Sigma_s \), and \( z \) is the local coordinate system centered at each \( x_i \).

We are going to derive some estimates of \( u_i \) near a simple blow-up point.

The first lemma actually only requires \( x_0 \) to be an isolated blow-up point.

**Lemma 5.2.** If \( x_0 = \lim_{i \to \infty} x_i \) is an isolated blow-up point of \( \{u_i\} \) which satisfies \( \Delta_{g^{(i)}} u_i + K_i u_i^{p_i} = 0 \), then there exists a constant \( C \)
There exist constants $\bar{\alpha}_i$ independent of $i$ and $r$ such that $\max_{\partial B_r(x_i)} u_i(x) \leq C \min_{\partial B_r(x_i)} u_i(x)$ for any $0 < r \leq r_0$ where $r_0$ is the fixed radius as in Definition \ref{def:fixed_radius}.

**Proof:** Let $z$ be the coordinates centered at each $x_i$ and $y = \frac{\hat{z}}{r}$. Define $\hat{u}_i(y) = r^{\frac{2}{n-1}} u_i(ry)$ and $\hat{y}_i(y) = g_{\alpha_i}^{(i)}(ry)dy^\alpha dy^\beta$. Then

$$\Delta_{\hat{y}(y)} \hat{u}_i + K_i(ry) \hat{u}_i^e = 0.$$ 

Since $u_i(z) \leq C |z|^{-\frac{n-2}{2}}$ for $|z| \leq r_0$, we know $|y|^{\frac{n-2}{2}} u_i(y) \leq C$ for $|y| \leq \frac{r_0}{r}$. In particular, when $|y| = 1$, $\hat{u}_i(y) \leq C$. Then we can apply the standard Harnack inequality to get $\max_{|y|=1} \hat{u}_i(y) \leq C \min_{|y|=1} \hat{u}_i(y)$ for some constant $C$ independent of $i$ and $r$, so

$$\max_{|z|=r} u_i(z) \leq C \min_{|z|=r} u_i(z).$$

\hfill \Box

**Proposition 5.3.** Let $x_0 = \lim_{i \to \infty} x_i$ be a simple blow-up point of $u_i$ with $p_i \to \frac{n+2}{2}$. Let $z$ be the geodesic coordinates centered at each $x_i$. There exist constants $\bar{r} \leq r_0$ and $C$ independent of $i$ such that

- if $0 \leq |z| \leq \bar{r}$, then
  $$u_i(z) \geq C u_i(x_i) \left( 1 + \frac{K_i(x_i)}{n(n-2)u_i(x_i)^{n-2}} |z|^2 \right)^{-\frac{n-2}{2}}$$

- if $0 \leq |z| \leq \frac{R}{u_i(x_i)^{\frac{n-2}{2}}}$, then
  $$u_i(z) \leq C u_i(x_i) \left( 1 + \frac{K_i(x_i)}{n(n-2)u_i(x_i)^{n-2}} |z|^2 \right)^{-\frac{n-2}{2}}$$

- if $\frac{R}{u_i(x_i)^{\frac{n-2}{2}}} \leq |z| \leq \bar{r}$, then
  $$u_i(z) \leq C u_i(x_i)^{l_i} |z|^{-l_i}$$

where $l_i, t_i$ are chosen such that $l_i \to \frac{6(n-2)}{7}$ when $i \to \infty$, and $t_i = 1 - \frac{(p_i-1)t_i}{2}$.

**Proof:** By Proposition \ref{prop:blow-up} when $0 \leq |z| \leq \frac{R}{u_i(x_i)^{\frac{n-2}{2}}}$,

$$\left( 1 - \epsilon \right) \frac{u_i(x_i)}{\left( 1 + \frac{K_i(x_i)}{n(n-2)u_i(x_i)^{n-2}} |z|^2 \right)^{\frac{n-2}{2}}} \leq u_i(z) \leq \left( 1 + \epsilon \right) \frac{u_i(x_i)}{\left( 1 + \frac{K_i(x_i)}{n(n-2)u_i(x_i)^{n-2}} |z|^2 \right)^{\frac{n-2}{2}}}.$$
Since $u_i(x_i) > 1$ and $p_i - 1 \leq \frac{4}{n-2}$,

$$u_i(x_i) \left(1 + \frac{K_i(x_i)}{n(n-2)} u_i(x_i)^{p_i-1} |z|^2 \right)^{-\frac{n-2}{2}}$$

$$\geq u_i(x_i) \left(1 + \frac{K_i(x_i)}{n(n-2)} u_i(x_i)^{\frac{4}{n-2}} |z|^2 \right)^{-\frac{n-2}{2}}.$$ 

So we only need to find the upper and lower bounds for $u_i(z)$ when $\frac{R}{R_i^{\frac{n-1}{2}}} \leq |z| \leq \bar{r}$.

First the lower bound.

Let $G_i$ be the Green's function of $\Delta g_i$ which is singular at 0 and $G_i = 0$ on $\partial B_r$. Then $G_i(z) = |z|^{2-n} + R_i(z)$ where $\lim_{|z| \to 0} |z|^{n-2} R_i(z) = 0$. Since $g_i$ converges uniformly to $g_0$, there exist constants $C_1$ and $C_2$ independent of $i$ such that

$$C_1 \leq 1 + |z|^{n-2} R_i(z) = |z|^{n-2} G_i(z) \leq C_2 \quad \text{for} \quad |z| \leq \bar{r},$$

i.e. $C_1 |z|^{2-n} \leq G_i(z) \leq C_2 |z|^{2-n}$.

When $|z| = Ru_i(x_i)^{-\frac{n-1}{2}}$,

$$u_i(z) \geq (1 - \epsilon) \frac{u_i(x_i)}{\left(1 + \frac{K_i(x_i)}{n(n-2)} u_i(x_i)^{p_i-1} |z|^2 \right)^{\frac{n-2}{2}}}$$

$$= (1 - \epsilon) \frac{u_i(x_i)}{\left(1 + \frac{K_i(x_i)}{n(n-2)} R^2 \right)^{\frac{n-2}{2}}}$$

and

$$u_i(x_i)^{-1} G_i(z) \leq C_2 |z|^{2-n} u_i(x_i)^{-1}$$

$$= C_2 R^{2-n} u_i(x_i)^{\frac{(n-2)(p_i-1)}{2} - 1}$$

$$\leq C_2 R^{2-n} u_i(x_i) \quad \text{(since} \quad \frac{n-2}{2}(p_i - 1) \leq 1 \text{)}. $$

For $R >> 0$,

$$R^{n-2} \left(1 + \frac{K_i(x_i)}{n(n-2)} R^2 \right)^{-\frac{n-2}{2}} = \left( R^{-2} + \frac{K_i(x_i)}{n(n-2)} \right)^{-\frac{n-2}{2}} \geq C(n, C_K).$$

Therefore $u_i(z) \geq Cu_i(x_i)^{-1} G_i(z)$ for some constant $C$ independent of $i$ when $|z| = Ru_i(x_i)^{-\frac{n-1}{2}}$.

For that constant $C$, $u_i(z) \geq Cu_i(x_i)^{-1} G_i(z) = 0$ when $|z| = \bar{r}$.
Then since
\[ \Delta \left( u_i(z) - C u_i(x_i)^{-1} G_i(z) \right) = \Delta u_i(z) = -K_i u_i(z)^{p_i} < 0 \]
on $B_r$, by the maximal principle
\[ u_i(z) - C u_i(x_i)^{-1} G_i(z) > 0 \]
i.e., \[ u_i(z) > C u_i(x_i)^{-1} G_i(z) \]
when \( \frac{R}{u_i(x_i)^{\frac{1}{n-2}}} \leq |z| \leq \bar{r} \).
Because \( u_i(x_i)^{-1} G_i(z) \geq C_1 |z|^{2-n} u_i(x_i)^{-1} \), we now need to compare \( |z|^{2-n} u_i(x_i)^{-1} \) with \( u_i(x_i) \cdot \left( 1 + \frac{K_i(x_i)}{n(n-2)} u_i(x_i)^{\frac{4}{n-2}} |z|^2 \right)^{-\frac{n-2}{2}} \) in order to get the desired lower bound.
\[
\begin{align*}
    u_i(x_i)^2 |z|^{n-2} & \left( 1 + \frac{K_i(x_i)}{n(n-2)} u_i(x_i)^{\frac{4}{n-2}} |z|^2 \right)^{-\frac{n-2}{2}} \\
    \leq u_i(x_i)^2 & \left( \frac{K_i(x_i)}{n(n-2)} u_i(x_i)^{\frac{4}{n-2}} \right)^{-\frac{n-2}{2}} \\
    \leq C 
\end{align*}
\]
where the constant $C$ is independent of $i$ because $K_i(x_i) \geq \delta$ which doesn’t depend on $i$. Therefore
\[ |z|^{2-n} u_i(x_i)^{-1} \geq C u_i(x_i) \left( 1 + \frac{K_i(x_i)}{n(n-2)} u_i(x_i)^{\frac{4}{n-2}} |z|^2 \right)^{-\frac{n-2}{2}}, \]
which then implies that
\[ u_i(z) \geq C u_i(x_i) \left( 1 + \frac{K_i(x_i)}{n(n-2)} u_i(x_i)^{\frac{4}{n-2}} |z|^2 \right)^{-\frac{n-2}{2}} \]
when \( \frac{R}{u_i(x_i)^{\frac{1}{n-2}}} \leq |z| \leq \bar{r} \).
Next the upper bound. We are going to apply the same strategy of constructing a comparison function and using the maximal principle.
Define $\mathcal{L}_i \varphi := \Delta_i \varphi + K_i u_i(z)^{p_i-1} \varphi$. By definition $\mathcal{L}_i u_i = 0$. Let $M_i = \max_{\partial B_r} u_i$ and $m_i = \min_{\partial B_r} u_i$. Let $C_i = (1 + \epsilon) \left( \frac{K_i(x_i)}{n(n-2)} \right)^{-\frac{n-2}{2}}$.
$C_i$ is bounded above and below by constants only depending on $\epsilon, n, C_K$ and $\delta$. Consider the function
\[ M_i(\bar{r}^{-1}|z|)^{-n+2+l_i} + C_i u_i(x_i)^{l_i} |z|^{-l_i}. \]
When $|z| = \frac{R}{u_i(x_i)^{\frac{1}{n-2}}}$, ...
\[ u_i(z) \leq (1 + \epsilon) \frac{u_i(x_i)}{\left(1 + \frac{K_i(x_i)}{n(n-2)} u_i(x_i)^{p_i-1}|z|^2\right)^{n-2}} \]

\[ = (1 + \epsilon) \frac{u_i(x_i)}{\left(1 + \frac{K_i(x_i)}{n(n-2)} R^2\right)^{n-2}} \]

\[ \leq C_i u_i(x_i) R^{-l_i} \quad \text{(because } l_i < n - 2) \]

\[ = C_i u_i(x_i)^{l_i} |z|^{-l_i} \quad \text{(by the choice of } t_i). \]

When \( |z| = \bar{r} \), by the definition of \( M_i \), \( u_i(z) \leq M_i = M_i(\bar{r}^{-1}|z|)^{-n+2+l_i} \).

So on \( \{|z| = \bar{r}\} \cup \{|z| = Ru_i(x_i)^{-\frac{p_i-1}{2}}\} \),

\[ u_i(z) \leq M_i(\bar{r}^{-1}|z|)^{-n+2+l_i} + C_i u_i(x_i)^{l_i} |z|^{-l_i}. \]

In the Euclidean coordinates, \( \Delta |z|^{-l_i} = -l_i(n - 2 - l_i)|z|^{-l_i-2} \) and \( \Delta |z|^{-n+2+l_i} = -l_i(n - 2 - l_i)|z|^{-n+l_i} \). Since \( \bar{z} \) is the geodesic normal coordinates, when \( \bar{r} \) is sufficiently small, \( g_0 \) and \( g_i \) are close to the Euclidean metric. Then when \( i \) is large enough

\[ \Delta_{\bar{z}} |z|^{-l_i} \leq -\frac{1}{2} l_i(n - 2 - l_i)|z|^{-l_i-2} \]

and

\[ \Delta_{g_i} |z|^{-n+2+l_i} \leq -\frac{1}{2} l_i(n - 2 - l_i)|z|^{-n+l_i}. \]

Thus

\[ \mathcal{L}_i (C_i u_i(x_i)^{l_i} |z|^{-l_i}) \]

\[ = C_i u_i(x_i)^{l_i} \Delta_{g_i} |z|^{-l_i} + C_i u_i(x_i)^{l_i} K_i u_i(z)^{p_i-1} |z|^{-l_i} \]

\[ \leq -C l_i(n - 2 - l_i) u_i(x_i)^{l_i} |z|^{-l_i-2} + C' u_i(x_i)^{l_i} u_i(z)^{p_i-1} |z|^{-l_i} \]

for some constants \( C, C' \) independent of \( i \).

The upper bound on \( u_i(z) \) when \( |z| = Ru_i(x_i)^{-\frac{p_i-1}{2}} \) and Lemma 5.2 implies that

\[ \tilde{u}_i \left( Ru_i(x_i)^{-\frac{p_i-1}{2}} \right) \leq \frac{(1 + \epsilon) u_i(x_i)}{\left[1 + \frac{K_i(x_i)}{n(n-2)} u_i(x_i)^{p_i-1} \left(Ru_i(x_i)^{-\frac{p_i-1}{2}} \right)^2\right]^{n-2}} \]

\[ \leq C u_i(x_i) R^{2-n}. \]
Since $x_0$ is a simple point of blow-up, $r^{\frac{-2}{n-1}}\bar{u}_i(r)$ is decreasing from $Ru_i(x_i)^{-\frac{n-1}{2}}$ to $\bar{r}$, which implies

$$|z|^{\frac{2}{n-1}}\bar{u}_i(|z|) \leq \left(Ru_i(x_i)^{-\frac{n-1}{2}}\right)^{\frac{2}{n-1}} \cdot \bar{u}_i \left(Ru_i(x_i)^{-\frac{n-1}{2}}\right) \leq CR^{\frac{2}{n-1}+2-n}.$$  

Thus by Lemma 5.2 again

$$u_i(z)^{p_i-1} \leq C|z|^{-2}R^{2-(n-2)(p_i-1)}$$

and hence

$$u_i(z)^{p_i-1}|z|^{-l_i} \leq C|z|^{-2-l_i}R^{2-(n-2)(p_i-1)}.$$  

So we know

$$\mathcal{L}_i(C_iu_i(x_i)|z|^{-l_i})$$

$$\leq (-C_l(n-2-l_i) + C'R^{2-(n-2)(p_i-1)}) u_i(x_i)^{l_i} |z|^{-l_i-2}$$

By our choice of $l_i$, $l_i(n-2-l_i)$ is always bounded below by some positive constant independent of $i$. When $i$ is sufficiently large, $2-(n-2)(p_i-1) < 0$, we can choose $R$ big enough such that $-C_l(n-2-l_i) + C'R^{2-(n-2)(p_i-1)} < 0$, hence $\mathcal{L}_i(C_iu_i(x_i)|z|^{-l_i}) < 0$.

Similarly,

$$\mathcal{L}_i(M_i(\bar{r}^{-1}|z|)^{-n+2+l_i})$$

$$= M_i\bar{r}^{n-2-l_i} \Delta_{y_i}|z|^{-n+2+l_i} + M_i\bar{r}^{n-2-l_i} K_i u_i(z)^{p_i-1} |z|^{-n+2+l_i}$$

$$\leq -\frac{1}{2}l_i(n-2-l_i)M_i\bar{r}^{n-2-l_i} |z|^{-n+l_i}$$

$$+ K_iM_i\bar{r}^{n-2-l_i} R^{2-(n-2)(p_i-1)} |z|^{-n+l_i}$$

by equations (20) and (21). We can choose $R$ large enough such that $-\frac{1}{2}l_i(n-2-l_i) + K_iR^{2-(n-2)(p_i-1)} < 0$ and hence

$$\mathcal{L}_i(M_i(\bar{r}^{-1}|z|)^{-n+2+l_i}) < 0.$$  

Therefore when $Ru_i(x_i)^{-\frac{n-1}{2}} \leq |z| \leq \bar{r}$,

$$\mathcal{L}_i\left(M_i(\bar{r}^{-1}|z|)^{-n+2+l_i} + C_iu_i(x_i)^{l_i}|z|^{-l_i}\right) < 0.$$  

Then by the maximal principle

$$u_i(z) \leq M_i(\bar{r}^{-1}|z|)^{-n+2+l_i} + C_iu_i(x_i)^{l_i}|z|^{-l_i}.$$
By Lemma 5.2 and because $x_0$ is a simple blow-up point, for \( R_{u_i(x_i)} \)
\[ \leq \theta \leq \bar{r}, \]
\[ \bar{r}^{\frac{n}{n-1}}M_i \leq \theta^{\frac{n}{n-1}}u_i(\theta) \]
\[ \leq \theta^{\frac{n}{n-1}}(M_i(\bar{r}^{-1}\theta)^{-n+2+l_i} + C_iu_i(x_i)^t_i\theta^{-l_i}) \]
\[ = \bar{r}^{n-2-l_i}\theta^{\frac{2}{n-1}n+2+l_i}M_i + \theta^{\frac{2}{n-1}}C_iu_i(x_i)^t_i\theta^{-l_i} \]
for some constant $C$ independent of $i$.

When $i \to \infty$, $\frac{2}{p_{i-1}} - n + 2 + l_i \to \frac{5}{14}(n - 2) > 0$.

Since $\frac{R_{u_i(x_i)}}{\bar{r}^{\frac{n}{n-1}} \to 0}$, we can choose $\theta$ small enough (fixed, independent of $i$) to absorb the first term on the right hand side of the above inequality into the left hand side to get $M_i \leq 2C_i\theta^{\frac{2}{p_{i-1}}-l_i}u_i(x_i)^t_i \leq Cu_i(x_i)^t_i$.

Therefore

\[ u_i(z) \leq M_i(\bar{r}^{-1}|z|)^{-n+2+l_i} + C_iu_i(x_i)^t_i|z|^{-l_i} \]
\[ \leq M_i(\bar{r}^{-1}|z|)^{-l_i} + C_iu_i(x_i)^t_i|z|^{-l_i} \]
\[ \leq Cu_i(x_i)^t_i|z|^{-l_i} \]

\[ \square \]

**Proposition 5.4.** If $x_0 = \lim_{i \to \infty} x_i$ is a simple blow-up point and $p_i \to \frac{n+2}{n-2}$. Let $\delta_i = \frac{n+2}{n-2} - p_i$, then $\lim_{i \to \infty} u_i(x_i)^{\delta_i} = 1$.

For the proofs of this proposition and theorems 1.1 and 1.2, we need to use the following **Pohozaev identity** as proved in [8].

**Proposition 5.5.** (Schoen, 1988) Let $(N, g)$ be an $n$-dimensional compact Riemannian manifold with smooth boundary $\partial N$. Let $R$ denote the scalar curvature function of $N$, and suppose $X$ is a conformal Killing vector field on $N$. We then have the identity

\[ \int_N (\mathcal{L}_X R)dv = \frac{2n}{n-2} \int_{\partial N} (\text{Ric} - n^{-1}Rg)(X, \nu)d\sigma, \]

where $\text{Ric}(\cdot, \cdot)$ denotes the Ricci tensor of $N$ thought of as a quadratic form on tangent vectors, $\mathcal{L}_X$ denotes the Lie derivative, $\nu$ denotes the outward unit normal vector to $\partial N$, $dv$ and $d\sigma$ are volume and surface measure (with respect to $g$), respectively.

We now prove Proposition 5.5.

**Proof:** Choose the conformal coordinate $z$ centered at $x_i$ such that on the small ball $|z| \leq \sigma$, $g$ can be written as $\lambda(z)^{\frac{4}{n-2}}g_0$ where $g_0$ is the Euclidean metric. Choose the conformal Killing field $X = \sum_{j=1}^{n} z^j \frac{\partial}{\partial z^j}$. 

\[ \square \]
we can apply the Pohozaev identity to get

\[(23) \quad \frac{n-2}{2n} \int_{B_\sigma} X(R_i) dv_{g_i} = \int_{\partial B_\sigma} T_i(X, \nu_i) d\Sigma_i\]

where the notations are

\[g_i = u_i^{\frac{4}{n-2}} g = (\lambda u_i)^{\frac{4}{n-2}} g_0,\]
\[R_i = R(g_i) = c(n)^{-1} K_i u_i^{-\delta_i},\]
\[dv_{g_i} = u_i^{\frac{2n}{n-2}} dv_g = (\lambda u_i)^{\frac{2n}{n-2}} dz,\]
\[\nu_i = (\lambda u_i)^{-\frac{2}{n-2}} \sigma^{-1} \sum_j z_j \partial_j\]

is the unit outer normal vector on \(\partial B_\sigma\) with respect to \(g_i\),
\[d\Sigma_i = (\lambda u_i)^{-\frac{2(n-1)}{n-2}} d\Sigma_\sigma\]
where \(d\Sigma_\sigma\) is the surface element of the standard \(S^{n-1}(\sigma)\),
\[T_i = (n-2)(\lambda u_i)^{\frac{2}{n-2}} \left( \text{Hess} \left((\lambda u_i)^{-\frac{2}{n-2}}\right) - \frac{1}{n} \Delta \left((\lambda u_i)^{-\frac{2}{n-2}}\right) g_0 \right)\]

where Hess and \(\Delta\) are taken with respect to the Euclidean metric \(g_0\).

We are going to study the decay of both sides of (23).

Up to a constant the left hand side is
\[c(n) \int_{B_\sigma} X(R_i) dv_{g_i}\]
\[= \int_{B_\sigma} X(K_i u_i^{-\delta_i})(\lambda u_i)^{\frac{2n}{n-2}} dz\]
\[= \int_{B_\sigma} X(K_i) u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} dz - \delta_i \int_{B_\sigma} K_i u_i^{p_i} X(u_i) \lambda^{\frac{2n}{n-2}} dz\]
\[= \int_{B_\sigma} |z| \frac{\partial K_i}{\partial r} u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} dz + \frac{\delta_i}{p_i+1} \int_{B_\sigma} r \frac{\partial K_i}{\partial r} \lambda^{\frac{2n}{n-2}} u_i^{p_i+1} dz\]
\[+ \frac{\delta_i}{p_i+1} \int_{B_\sigma} K_i u_i^{p_i+1} r \frac{\partial \lambda^{\frac{2n}{n-2}}}{\partial r} dz\]
\[- \frac{\delta_i}{p_i+1} \int_{B_\sigma} K_i u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} \text{div} X dz\]
\[+ \int_{\partial B_\sigma} K_i u_i^{p_i+1} \lambda^{\frac{2n}{n-2}} X \cdot \left( \sum_j z_j \frac{\partial}{\partial z_j} \sigma \right) d\Sigma_\sigma\]

which can be further written as
\[
= \left(1 + \frac{\delta_i}{p_i + 1}\right) \int_{B_{\sigma}} |z| \partial K_i u_i^{p_i + 1} \frac{\partial}{\partial r} \lambda^{\frac{2n}{n-2}} dz \\
+ \frac{\delta_i}{p_i + 1} \int_{B_{\sigma}} |z| K_i u_i^{p_i + 1} \lambda^{\frac{2n}{n-2}} dz \\
+ \frac{\delta_i}{p_i + 1} n \int_{B_{\sigma}} K_i u_i^{p_i + 1} \lambda^{\frac{2n}{n-2}} dz \cdot \frac{\delta_i}{p_i + 1} \int_{\partial B_{\sigma}} \sigma K_i u_i^{p_i + 1} \lambda^{\frac{2n}{n-2}} d\Sigma_{\sigma}.
\]

By Proposition 5.3
\[
\int_{|z| \leq \frac{R}{u_i(x_i) - l_i}} |z| u_i(z)^{p_i + 1} dz \leq C u_i(x_i)^{p_i + 1} \int_{|z| \leq \frac{R}{u_i(x_i) - l_i}} |z| dz \\
\leq C u_i(x_i)^{p_i + 1 - \frac{(n+1)(p_i-1)}{2}} \\
= C u_i(x_i)^{-\frac{2}{n-2} + \frac{n-1}{2} l_i}.
\]

Also since \(\lim_{i \to \infty} (n - l_i(p_i + 1) + 1) = \frac{-5}{7} n + 1 < 0\),
\[
\int_{|z| \leq \frac{R}{u_i(x_i) - l_i}} |z| u_i(z)^{p_i + 1} dz \\
\leq C \int_{|z| \leq \frac{R}{u_i(x_i) - l_i}} |z| (u_i(x_i)^{t_i} |z|^{-l_i})^{p_i + 1} \\
\leq C u_i(x_i)^{t_i(p_i + 1) - \frac{p_i-1}{2} (n-l_i(p_i+1)+1)} \\
= C u_i(x_i)^{p_i + 1 - \frac{(n+1)(p_i-1)}{2}} \\
= C u_i(x_i)^{-\frac{2}{n-2} + \frac{n-1}{2} l_i}.
\]

So
\[
\int_{|z| \leq \sigma} |z| u_i(z)^{p_i + 1} dz \leq C u_i(x_i)^{-\frac{2}{n-2} + \frac{n-1}{2} l_i}.
\]

and hence the first term in (24) decays in the order of \(u_i(x_i)^{-\frac{2}{n-2} + \frac{n-1}{2} l_i}\) and the second term decays even faster than that since \(\delta_i \to 0\).

By Proposition 5.3 on \(\partial B_{\sigma}\), \(u_i\) decays in the order of \(u_i(x_i)^{t_i}\), so the fourth term in (24) decays at least in the order of \(u_i(x_i)^{t_i(p_i+1)}\).

The third term
\[
\delta_i \int_{B_{\sigma}} K_i u_i^{p_i + 1} \lambda^{\frac{2n}{n-2}} dz \geq C \delta_i \int_{B_{\sigma}} u_i^{p_i + 1} dz.
\]
When $|z| \leq \frac{R}{u_i(x_i)^{p_i-1}}$, 

$$u_i(z) \geq (1 - \epsilon) \frac{u_i(x_i)}{\left(1 + \frac{K_i(x_i)}{n(n-2)} u_i(x_i)^{p_i-1} |z|^2 \right)^{\frac{n-2}{2}}}$$

$$\geq (1 - \epsilon) \frac{u_i(x_i)}{\left(1 + \frac{K_i(x_i)}{n(n-2)} R^2 \right)^{\frac{n-2}{2}}}$$

$$\geq C u_i(x_i),$$

thus

$$\int_{B_{\sigma}} u_i^{p_i+1} dz \geq \int_{|z| \leq \frac{R}{u_i(x_i)^{p_i-1}}} u_i^{p_i+1} dz$$

$$\geq C u_i(x_i)^{p_i+1 - \frac{n}{2}(p_i-1)}$$

$$= C u_i(x_i) \frac{n^{n-2}}{n} \delta_i$$

$$\geq C.$$  

(26)

So the third term is bounded below by $C \delta_i$.

Next we are going to study the decay of the right hand side of (23).

$$\int_{\partial B_{\sigma}} T_i(X, \nu_i) d\Sigma_i$$

$$= \int_{\partial B_{\sigma}} (n - 2)(\lambda u_i)^{-\frac{2}{n-2}} \left[ \text{Hess} \left( (\lambda u_i)^{-\frac{2}{n-2}} \right) \left(r \frac{\partial}{\partial r}, (\lambda u_i)^{-\frac{2}{n-2}} \sigma^{-1} r \frac{\partial}{\partial r} \right) \right.$$  

$$- \frac{1}{n} \Delta \left( (\lambda u_i)^{-\frac{2}{n-2}} \right) \left\langle r \frac{\partial}{\partial r}, (\lambda u_i)^{-\frac{2}{n-2}} \sigma^{-1} r \frac{\partial}{\partial r} \right\rangle \right] (\lambda u_i)^{\frac{2(n-1)}{n-2}} d\Sigma_\sigma$$

(27)

(where $\langle \cdot, \cdot \rangle$ is the Euclidean metric)

$$= (n - 2) \int_{\partial B_{\sigma}} \sigma^{-1} \text{Hess} \left( (\lambda u_i)^{-\frac{2}{n-2}} \right) \left(r \frac{\partial}{\partial r}, r \frac{\partial}{\partial r} \right)$$

$$- \frac{\sigma}{n} \Delta \left( (\lambda u_i)^{-\frac{2}{n-2}} \right) (\lambda u_i)^{\frac{2(n-1)}{n-2}} d\Sigma_\sigma$$

$$= (n - 2) \int_{\partial B_{\sigma}} \sigma^{-1} \left[ - \frac{2}{n-2} (\lambda u_i) \sum_{j,k} \frac{\partial}{\partial z^j} z^j \frac{\partial}{\partial z^k} \frac{\partial}{\partial z^k} (\lambda u_i) \right.$$

$$+ \frac{2n}{(n-2)^2} \sum_{j,k} z^j z^k \frac{\partial^2}{\partial z^j \partial z^k} (\lambda u_i) \left( \frac{\partial}{\partial z^j} \right)^2 - \sigma \cdot$$

$$\left[ - \frac{2}{n(n-2)} (\lambda u_i) \sum_j \frac{\partial^2}{\partial z^j} (\lambda u_i) \left( \frac{\partial}{\partial z^j} \right)^2 + \frac{2}{(n-2)^2} \sum_j \left( \frac{\partial}{\partial z^j} \right)^2 \right] d\Sigma_\sigma$$
On \( \partial B_2 \), by Proposition 5.3, \( u_i \leq C u_i(x_i)^{l_i} \), so by the elliptic regularity theory, \( \|u_i\|_{C^2(\partial B_2)} \leq C u_i(x_i)^{l_i} \). Thus we know (27) decays in the order of \( u_i(x_i)^{2t_i} \).

Then by comparing the decay rate of both sides of (28)

\[
\delta_i \leq C \left( u_i(x_i)^{-\frac{n-1}{2}+ \frac{n-1}{2} \delta_i} + u_i(x_i)^{2t_i} \right).
\]

Thus

\[
\delta_i \ln u_i(x_i) \leq C \left( u_i(x_i)^{-\frac{n-1}{2}+ \frac{n-1}{2} \delta_i} + u_i(x_i)^{2t_i} \right) \ln u_i(x_i).
\]

By our choice of \( l_i \),

\[
t_i = 1 - \frac{(p_i - 1)l_i}{2} \to -\frac{5}{7} < 0
\]

Since \( u_i(x_i) \to \infty \), \( \left( u_i(x_i)^{-\frac{n-1}{2}+ \frac{n-1}{2} \delta_i} + u_i(x_i)^{2t_i} \right) \ln u_i(x_i) \to 0 \). Consequently

\[
\lim_{i \to \infty} \delta_i \ln u_i(x_i) = 0
\]

which implies \( \lim_{i \to \infty} u_i(x_i)^{\delta_i} = 1 \).

\section{Ruling out Case I}

In section 4, we reduced the possible blow-up phenomenon of \( \{u_i\} \) which are solutions of equation (11) into two cases. In this section we are going to show that case I cannot happen, in the next section we will rule out case II and hence complete the proof of Theorem 1.1.

**Case I:** The sequence \( \{u_i\} \) has isolated blow-up points \( x_1, x_2, \ldots \in M \).

Suppose \( x_1, x_2, \ldots \) are all simple blow-up points. Choose \( P \in M \setminus \{x_1, x_2, \ldots \} \). On any compact subset \( \Omega \) of \( M \setminus \{x_1, x_2, \ldots \} \) containing \( P \), since \( x_1, x_2, \ldots \) are isolated blow-up points, \( u_i \) is bounded above by some constant independent of \( i \), so on \( \Omega \) the standard Harnack inequality holds for \( \{u_i\} \). Then by Proposition 5.3 and the Harnack inequality \( u_i(P) \to 0 \). In addition, the Harnack inequality also holds for \( \frac{u_i}{u_i(P)} \). In other words, for some constant \( C \) independent of \( i \),

\[
\max_{\Omega} \frac{u_i}{u_i(P)} \leq C \min_{\Omega} \frac{u_i}{u_i(P)} \leq C \frac{u_i(P)}{u_i(P)} = C.
\]

Since \( u_i \) satisfies (11),

\[
\Delta_g \left( \frac{u_i}{u_i(P)} \right) + u_i(P)^{p_i-1} K_i \left( \frac{u_i}{u_i(P)} \right)^{p_i} = 0.
\]

By the standard elliptic estimates, \( \frac{u_i}{u_i(P)} \) has uniform \( C^{2,\alpha} \)-norm on \( \Omega \). So on \( \Omega \), \( \frac{u_i}{u_i(P)} \to H \) in \( C^2 \)-norm where \( H \) satisfies \( \Delta_g H = 0 \). Since \( \Omega \) is arbitrary, \( H \) satisfies \( \Delta_g H = 0 \) on \( M \setminus \{x_1, x_2, \ldots \} \). By the fact
$u_i > 0$ we know that $H(x) \geq 0$, then the maximal principle gives $H > 0$ on $M \setminus \{x_1, x_2, \ldots\}$. Thus by the removable singularity theorems of harmonic functions $H$ is a constant.

Since we assume $x_1$ is a simple blow-up point, there exists a sequence of points $\{x_i^{(i)}\}$ approaching $x_1$ such that for the coordinates $z$ centered at each $x_i^{(i)}$, the function $|z|^{n/2} \bar{u}_i(|z|)$ is strictly decreasing in $|z|$ for $R_{u_i}(x_i) - \frac{\alpha_i}{2} \leq |z| \leq r_0$. In particular it is decreasing for $\frac{\alpha_i}{2} \leq |z| \leq r_0$ when $i$ is sufficiently large. This implies that $|z|^{n/2} \bar{u}_i(|z|)u_i(P)^{-1}$ is strictly decreasing in $|z|$ for $\frac{\alpha_i}{2} \leq |z| \leq r_0$. If $\frac{\alpha_i}{2} \leq |z| \leq r_0$, then the corresponding point $\exp_{x_i^{(i)}} z$ is at least distance $\frac{\alpha_i}{2}$ from $x_1$ because $\{x_i^{(i)}\}$ approaches $x_1$. Thus $\bar{u}_i(|z|)u_i(P)^{-1}$ converges in $C^2$-norm to $H$, which is a constant. Consequently $|z|^{n/2} \bar{u}_i(|z|)u_i(P)^{-1}$ converges in $C^2$-norm to $|z|^{n/2}H$ which is strictly increasing in $|z|$. This is a contradiction.

Therefore there must be a point in $\{x_1, x_2, \ldots\}$ which is not a simple blow-up point, without loss of generality we assume it to be $x_1$. To simplify the notations we are going to rename it to be $x_0$. Let $x_i$ be the local maximum points of $u_i$ such that $\lim_{i \to \infty} x_i = x_0$. Let $z$ be the local coordinate system centered at each $x_i$. Since $x_0$ is not a simple blow-up point, as a function of $|z|$, $|z|^{n/2} \bar{u}_i(|z|)$ has a second critical point at $|z| = r_i$ where $r_i \to 0$. Let $y = \frac{z}{r_i}$ and define $v_i(y) = r_i^{\alpha_i/2} u_i(r_i y)$. Then $v_i(y)$ satisfies

$$
\Delta g^{(i)} v_i(y) + \tilde{K}_i(y) v_i(y)^{p_i} = 0
$$

where $g^{(i)}(y) = g_{\alpha_\beta}(r_i y) dy^\alpha dy^\beta$ and $\tilde{K}_i(y) = K_i(r_i y)$.

By this definition $|y| = 1$ is the second critical point of $|y|^{n/2} \bar{v}_i(|y|)$. By Proposition 1.2 for $0 \neq |z| \leq \sigma$, $u_i(z) \leq C |z|^{-\frac{\alpha_i}{2} \sigma}$ where $\sigma$ is a positive constant. Then since $\frac{\alpha_i}{2} \sigma \to \infty$, $|v_i(y)| \leq C |y|^{-\frac{\alpha_i}{2} \sigma}$ for $|y| \neq 0$. Therefore by the same argument as before we know that $v_i(y)$ converges in $C^2$-norm on $\mathbb{R}^n \setminus \{0\}$ to some function $v$ which satisfies $\Delta v + K(x_0) v^{n+2 \alpha_i} = 0$ where here and in the rest of the proof $\Delta$ is the Euclidean Laplacian and $K$ is the limit function of $\{K_i\}$.

If $0$ is not a blow-up point of $\{v_i\}$, then $v$ satisfies $\Delta v + K(x_0) v^{n+2 \alpha_i} = 0$ on $\mathbb{R}^n$. Since $r_i > R_{u_i}(x_i)^{-\frac{\alpha_i}{2} \sigma}$, $v_i(0) > R_{\bar{u}_i}^{-\alpha_i/2}$ and hence $v > 0$. This implies that $v$ is the standard spherical solution and $|y|^{n/2} \bar{v}_i(|y|)$ only has one critical point. On the other hand, $|y|^{\frac{n-2}{2} \alpha_i} \bar{v}_i(|y|)$ has two critical
Thus we can conclude that $v$ is a blow-up point for $\lambda h$. Thus $0$ is a blow-up point for $|v_i|$. Since $|y|_p = 0$ on $\Omega$. Therefore $\Delta h = 0$ on $\mathbb{R}^n \setminus \{0\}$ since $\Omega$ is arbitrary.

Next we are going to apply the Pohozaev identity $\mathcal{P}_2$ to equation (22). Since $g$ is locally conformally flat, we can write $g(z) = \lambda^{\frac{4}{n-2}}(z)dz^2$. Hence we can write $g^{(i)}(y) = \lambda^{\frac{4}{n-2}}(r_i y)dy^2$. We are going to use $\lambda_i(y)$ to denote $\lambda(r_i y)$. Let $X = \sum_j y^j \frac{\partial}{\partial y^j}$, the Pohozaev identity
becomes

\[ n - 2 \left( \frac{1}{2n} \right) \int_{B_N} X(R_i)dv_{g_i} = \int_{\partial B_N} T_i(X, \nu_i)d\Sigma_i \]

where

\[ g_i(y) = v_i(y)\frac{1}{n-2}d^2y, \]
\[ R_i(y) = R(g_i) = c(n)^{-1} K_i v_i^{-\delta}, \]
\[ dv_{g_i} = v_i(y)\frac{2n}{n-2}dv_{g_i}, \]
\[ \nu_i = (\lambda_i v_i)^{\frac{1}{n-2}} \left( \sum_j y_j \frac{\partial}{\partial y_j} \right) \]

is the unit outer normal vector on \( \partial B_N \) with respect to \( g_i \),

\[ d\Sigma_i = (\lambda_i v_i)^{\frac{2(n-1)}{n-2}} d\Sigma_\sigma \]

where \( d\Sigma_\sigma \) is the surface element of the standard \( S^{n-1}(\sigma) \),

\[ T_i = \text{Ric}(g_i) - n^{-1}R(g_i)g_i. \]

We divide both sides of (30) by \( v_i^2(\bar{y}) \). The right hand side becomes

\[ \frac{1}{v_i^2(\bar{y})} \int_{\partial B_N} T_i(X, \nu_i)d\Sigma_i \]
\[ = \frac{1}{v_i^2(\bar{y})} \int_{\partial B_N} (\text{Ric}(g_i) - n^{-1}R(g_i)g_i)(X, \nu_i)d\Sigma_i \]
\[ = \frac{1}{v_i^2(\bar{y})} \int_{\partial B_N} \left( \text{Ric} \left( (\lambda_i v_i)^{\frac{4}{n-2}} g_0 \right) \right. \]
\[ - n^{-1}R \left( (\lambda_i v_i)^{\frac{4}{n-2}} g_0 \right) (X, \nu_0)(\lambda_i v_i)^2d\Sigma_\sigma \]

(31) \[ = \int_{\partial B_N} \left( \frac{\lambda_i v_i}{v_i(\bar{y})} \right)^2 \left[ \text{Ric} \left( \frac{\lambda_i v_i}{v_i(\bar{y})} \right)^{\frac{4}{n-2}} g_0 \right. \]
\[ - n^{-1}R \left( \frac{\lambda_i v_i}{v_i(\bar{y})} \right)^{\frac{4}{n-2}} g_0 \left( \frac{\lambda_i v_i}{v_i(\bar{y})} \right)^2 g_0 \right] (X, \nu_0)d\Sigma_\sigma \]

where \( g_0 \) denotes the Euclidean metric and \( \nu_0 = \sigma^{-1} \sum_j y_j \frac{\partial}{\partial y_j} \) is the unit outer normal on \( \partial B_N \) with respect to the Euclidean metric \( g_0 \).

When \( i \to \infty \), for \( |y| = \sigma, \lambda_i(y) = \lambda(r_iy) \to \lambda(x_0) \), without loss of generality we can assume it to be 1. Thus when \( i \) goes to \( \infty \), (31)
converges to
\[
\int_{\partial B} h^2 \left( \text{Ric} \left( h^{\frac{4}{n-2}} g_0 \right) - n^{-1} R \left( h^{\frac{4}{n-2}} g_0 \right) h^{\frac{4}{n-2}} g_0 \right) (X, \nu_0) d\Sigma_\sigma
\]
\[
= \int_{\partial B} h^2 \cdot (n-2) h^{\frac{2}{n-2}} \left[ \text{Hess} \left( h^{-\frac{2}{n-2}} \right) (X, \nu_0) - \frac{1}{n} \Delta \left( h^{-\frac{2}{n-2}} \right) g_0 (X, \nu_0) \right] d\Sigma_\sigma
\]
\[
= (n-2) \sigma^{-1} \int_{\partial B} \hat{h}^{\frac{2(n-1)}{n-2}}.
\]
\[
\left[ \text{Hess} \left( h^{-\frac{2}{n-2}} \right) (X, X) - \frac{1}{n} \Delta \left( h^{-\frac{2}{n-2}} \right) \sigma^2 \right] d\Sigma_\sigma
\]
By the expression of \( h \)
\[
h^{-\frac{2}{n-2}} = \left( \frac{1}{2} (1 + |y|^{2-n}) \right)^{-\frac{2}{n-2}} = 2\frac{n}{n-2} |y|^2 - \frac{2n}{n-2} |y|^n + O \left( |y|^{2(n-1)} \right)
\]
Then by direct computation
\[
\text{Hess} \left( \frac{2}{n-2} |y|^2 - \frac{2n}{n-2} |y|^n \right) (X, X) - \frac{1}{n} \Delta \left( \frac{2}{n-2} |y|^2 - \frac{2n}{n-2} |y|^n \right) \sigma^2
\]
\[
= -2\frac{n}{n-2} (n-1) \sigma^n
\]
Therefore
\[
\text{Hess} \left( h^{-\frac{2}{n-2}} \right) (X, X) - \frac{1}{n} \Delta \left( h^{-\frac{2}{n-2}} \right) \sigma^2 = -2\frac{n}{n-2} (n-1) \sigma^n + O \left( \sigma^{2(n-1)} \right).
\]
Also we know
\[
h^{\frac{2(n-1)}{n-2}} = \left( \frac{1}{2} \right)^{\frac{2(n-1)}{n-2}} |y|^{-2(n-1)} \left( 1 + O(|y|^{n-2}) \right).
\]
So we can conclude that
\[
= -\frac{1}{2} (n-1)(n-2) \sigma^{-1} \int_{\partial B_e} \left( |y|^{-2(n-1)} + O(|y|^{-n}) \right) \cdot
\]
\[
\left( |y|^n + O(|y|^{2(n-1)}) \right) \sigma^{n-1} d\Sigma_1
\]
\[
= -\frac{1}{2} (n-1)(n-2) + O(\sigma^{n-2})
\]
(33) \(< 0\)
when we choose \( \sigma \) to be sufficiently small.
On the other hand, after being divided by $v_i^2(y)$, the left hand side of (30) is
\[
\frac{n - 2}{2n} c(n)^{-1} \frac{1}{v_i^2(y)} \int_{B_\sigma} X(\tilde{K}_i v_i^{-\delta_i})(\lambda_i v_i)^{\frac{2n}{n - 2}} dy.
\]
We write
\[
\frac{1}{v_i^2(y)} \int_{B_\sigma} X(\tilde{K}_i v_i^{-\delta_i})(\lambda_i v_i)^{\frac{2n}{n - 2}} dy
\]
(34) = \frac{1}{v_i^2(y)} \int_{B_\sigma} X(\tilde{K}_i) v_i^{p_i + 1} \lambda_i^{\frac{2n}{n - 2}} dy - \frac{\delta_i}{v_i^2(y)} \int_{B_\sigma} \tilde{K}_i \lambda_i^{\frac{2n}{n - 2}} v_i^{p_i} X(v_i) dy.

The second term
\[
= - \frac{\delta_i}{p_i + 1} \frac{1}{v_i^2(y)} \int_{B_\sigma} \tilde{K}_i \lambda_i^{\frac{2n}{n - 2}} X(v_i^{p_i + 1}) dy
\]
\[
= - \frac{\delta_i}{p_i + 1} \frac{1}{v_i^2(y)} \int_{B_\sigma} \left( \text{div}(\tilde{K}_i \lambda_i^{\frac{2n}{n - 2}} v_i^{p_i + 1} X) - \tilde{K}_i \lambda_i^{\frac{2n}{n - 2}} v_i^{p_i + 1} \text{div} X \right) dy
\]
\[
+ \frac{\delta_i}{p_i + 1} \frac{1}{v_i^2(y)} \int_{B_\sigma} \lambda_i^{\frac{2n}{n - 2}} v_i^{p_i + 1} X(\tilde{K}_i) dy
\]
\[
+ \frac{\delta_i}{p_i + 1} \frac{1}{v_i^2(y)} \int_{B_\sigma} \tilde{K}_i v_i^{p_i + 1} X(\lambda_i^{\frac{2n}{n - 2}}) dy
\]
\[
= - \frac{\delta_i}{p_i + 1} \frac{\sigma}{v_i^2(y)} \int_{\partial B_\sigma} \tilde{K}_i \lambda_i^{\frac{2n}{n - 2}} v_i^{p_i + 1} d\Sigma_\sigma + \frac{\delta_i}{p_i + 1} \frac{1}{v_i^2(y)} \int_{\partial B_\sigma} \tilde{K}_i \lambda_i^{\frac{2n}{n - 2}} v_i^{p_i + 1} \left( n + X(\ln \tilde{K}_i) + \frac{2n}{n - 2} X(\ln \lambda_i) \right) dy.
\]

Since $X = r \frac{\partial}{\partial r}$ and $\frac{\partial}{\partial r}(\ln \tilde{K}_i)$, $\frac{\partial}{\partial r}(\ln \lambda_i)$ are uniformly bounded, we can choose $\sigma$ to be small (independent of $i$) to make $n + X(\ln \tilde{K}_i) + \frac{2n}{n - 2} X(\ln \lambda_i) > 0$. Because on $\partial B_\sigma$, $\frac{v_i}{v_i(y)} \to h(\sigma) > 0$ and $v_i \to 0$ uniformly,
\[
\frac{1}{v_i^2(y)} \int_{\partial B_\sigma} \tilde{K}_i \lambda_i^{\frac{2n}{n - 2}} v_i^{p_i + 1} d\Sigma_\sigma = \int_{\partial B_\sigma} \tilde{K}_i \lambda_i^{\frac{2n}{n - 2}} \left( \frac{v_i}{v_i(y)} \right)^2 v_i^{p_i - 1} d\Sigma_\sigma \to 0.
\]
Thus when $i \to \infty$, the limit of the second term of (34) is greater than or equal to 0.

As will be proved in Proposition 6.1 when the dimension $n = 3, 4$, the limit of the first term of (34)
\[
\lim_{i \to \infty} \frac{1}{v_i^2(y)} \int_{B_\sigma} X(\tilde{K}_i) v_i^{p_i + 1} \lambda_i^{\frac{2n}{n - 2}} dy = 0.
\]
This then implies that the limit of the left hand side of (30) is greater than or equal to 0, which contradicts (33). So we can rule out Case I.

**Proposition 6.1.** When $n = 3, 4$, 
\[
\lim_{i \to \infty} \frac{1}{v_i^2(y)} \int_{B_r} X(\tilde{K}_i) v_i^{p_i + 1} \lambda_i^{2n} dv = 0.
\]

Before we prove Proposition 6.1, we first need to carefully investigate the behaviour of $\tilde{K}_i$.

By Proposition 5.3 we have the following estimates:

- if $0 \leq |y| \leq 1$, $v_i(y) \geq C v_i(0) \left(1 + \frac{\tilde{K}_i(0)}{n(n-2)} v_i(0) \frac{4}{n-2} |y|^2\right)^{-\frac{n+2}{2}}$
- if $0 \leq |y| \leq R v_i(0)^{-\frac{p_i-1}{2}}$, then $v_i(y) \leq C v_i(0) \left(1 + \frac{\tilde{K}_i(0)}{n(n-2)} v_i(0) \frac{p_i-1}{2} |y|^2\right)^{-\frac{n+2}{2}}$
- if $\frac{R v_i(0)^{-\frac{p_i-1}{2}}}{v_i(0)} \leq |y| \leq 1$, then $v_i(y) \leq C v_i(0)^{t_i} |y|^{-l_i}$ where

\[l_i, t_i\] are chosen such that $l_i \to \frac{6(n-2)}{7}$, and $t_i = 1 - \frac{(p_i-1)t_i}{2}$.

**Lemma 6.2.** For any $j, 1, 2, ..., n$,
\[
\frac{\partial \tilde{K}_i}{\partial y^j}(0) \leq C \left( r_i v_i(0)^{-\frac{2}{n-2} + \frac{n+1}{2} t_i} + v_i(0)^{2 t_i}\right)
\]

**Proof:** Choose the conformal Killing vector field to be $X = \frac{\partial}{\partial y^j}$, we have the Pohozaev identity

\[
\frac{n-2}{2n} \int_{B_r} X(R_i) dv_{g_i} = \int_{\partial B_r} T_i(X, \nu_i) d\Sigma_i
\]

where

\[
g_i(y) = v_i(y)^{\frac{4}{n-2}} g^{(i)}(y) = (\lambda_i v_i)^{\frac{4}{n-2}} g_0
\]

where $g_0$ is the Euclidean metric,

\[
R_i(y) = R(g_i) = c(n)^{-1} \tilde{K}_i v_i^{-\delta_i},
\]

\[
dv_{g_i} = v_i(y)^{\frac{2n}{n-2}} dv_{g^{(i)}} = (\lambda_i v_i)^{\frac{2n}{n-2}} dy,
\]

\[
\nu_i = \left(\lambda_i v_i\right)^{-\frac{2}{n-2} \sigma^{-1}} \sum_j y^j \frac{\partial}{\partial y^j}
\]

is the unit outer normal vector on $\partial B_r$ with respect to $g_i$,

\[
d\Sigma_i = \left(\lambda_i v_i\right)^{\frac{2(n-1)}{n-2}} d\Sigma_{g_i}
\]

where $d\Sigma_{g_i}$ is the surface element of the standard $S^{n-1}(\sigma)$,

\[
T_i = (n - 2)(\lambda_i v_i)^{\frac{2}{n-2}} \left( \text{Hess} \left( \left(\lambda_i v_i\right)^{-\frac{2}{n-2}} \right) - \frac{1}{n} \Delta \left( \left(\lambda_i v_i\right)^{-\frac{2}{n-2}} \right) g_0 \right)
\]
Here Hess and $\Delta$ are taken with respect to the Euclidean metric $g_0$.

The left hand side of (35) is

\[
\frac{n-2}{2n} \int_{B_r} \frac{\partial}{\partial y^1} (R_i) dv_{g_i} = \frac{n-2}{2n} c(n)^{-1} \int_{B_r} \frac{\partial}{\partial y^1} (K_i v_i^{-\delta_i}) (\lambda_i v_i)^{2n \over n-2} dv \]

\[
= \frac{n-2}{2n} c(n)^{-1} \int_{B_r} \left(1 + \frac{\delta_i}{p_i + 1}\right) \lambda_i^{2n \over n-2} v_i^{p_i+1} \frac{\partial \tilde{K}_i}{\partial y^1} dv \\
+ \frac{n-2}{2n} c(n)^{-1} \int_{B_r} \frac{\delta_i}{p_i + 1} \frac{\partial \lambda_i^{2n \over n-2}}{\partial y^1} dv \\
- \frac{n-2}{2n} c(n)^{-1} \frac{\delta_i}{p_i + 1} \int_{\partial B_r} \lambda_i^{2n \over n-2} \tilde{K}_i v_i^{p_i+1} y^1 d\Sigma \\
(36)
\]

By Proposition 5.3, the third term in (36) is bounded above by

\[
C \delta_i \cdot v_i(0)^{t_i(p_i+1)} \leq C \delta_i v_i(0)^{2t_i}
\]

since $t_i < 0$ and $v_i(0) \to \infty$.

Same as in the proof of Proposition 5.4, the second term in (36) is bounded above by

\[
C \delta_i r_i \int_{|y| \leq \sigma} v_i(y)^{p_i+1} dy \\
\leq C \delta_i r_i \left( \int_{|z| \leq R v_i(0)^{\frac{n-1}{2}}} v_i(0)^{p_i+1} dv + \int_{R v_i(0)^{\frac{n-1}{2}} \leq |y| \leq \sigma} (v_i(0)^{t_i} |y|^{-t_i})^{p_i+1} dy \right) \\
\leq C \delta_i r_i \left( v_i(0)^{p_i+1 - 2(p_i-1)} + v_i(0)^{t_i(p_i+1)} \cdot v_i(0)^{-p_i-1}(n-1)(p_i+1)) \right) \\
= C \delta_i r_i v_i(0)^{p_i+1 - \frac{p_i-1}{2}} (\text{ since } t_i + \frac{(p_i-1)l_i}{2} = 1) \\
= C \delta_i r_i v_i(0)^{q-1} \delta_i \\
\leq C \delta_i r_i \\
\text{ (by Proposition 5.4)}
\]

By the estimates almost identical to those of the right hand side of (23), we know that the right hand of (35) decays in the rate of $v_i(0)^{2t_i}$. 


Therefore the first term in (36) which is
\[
\frac{n - 2}{2n} c(n)^{-1} \int_{B_x} \left( 1 + \frac{\delta_i}{p_i + 1} \right) \lambda_i^{\frac{2n}{n-2}} v_i^{p_i+1} \frac{\partial \tilde{K}_i}{\partial y^i} dy
\]
is bounded above by \(C(\delta_i v_i(0)^{2t_i} + \delta_i r_i + v_i(0)^{2t_i}) \leq C(\delta_i r_i + v_i(0)^{2t_i}).\)

By the Taylor expansion
\[
\frac{\partial \tilde{K}_i}{\partial y^i}(y) = \frac{\partial \tilde{K}_i}{\partial y^i}(0) + \nabla \left( \frac{\partial \tilde{K}_i}{\partial y^i} \right)(v) \cdot y
\]
for some \(|v| \leq |y|\).

As in the proof of Proposition 5.4 and also using the fact that \(\tilde{K}_i(y) = K_i(r_i y),\)
\[
\frac{n - 2}{2n} c(n)^{-1} \int_{B_x} \left( 1 + \frac{\delta_i}{p_i + 1} \right) \lambda_i^{\frac{2n}{n-2}} v_i^{p_i+1} \left| \nabla \left( \frac{\partial \tilde{K}_i}{\partial y^i} \right)(v) \cdot y \right| dy
\]
\[
\leq C r_i \int_{B_x} v_i^{p_i+1} |y| dy
\]
\[
\leq C r_i v_i(0) \frac{2n}{n-2} + \frac{n-1}{2} \delta_i
\]
where the last inequality is proved in the same way as (25).

Thus we know
\[
\left| \frac{\partial \tilde{K}_i}{\partial y^i}(0) \right| \int_{B_x} \lambda_i^{\frac{2n}{n-2}} v_i^{p_i+1} dy
\]
\[
\leq C \left( r_i v_i(0) - \frac{2n}{n-2} + \frac{n-1}{2} \delta_i + \delta_i r_i + v_i(0)^{2t_i} \right)
\]
\[
\leq C \left( r_i v_i(0) - \frac{2n}{n-2} + \frac{n-1}{2} \delta_i + r_i v_i(0)^{2t_i} + v_i(0)^{2t_i} \right)
\]
(by inequality (28))
\[
\leq C \left( r_i v_i(0) - \frac{2n}{n-2} + \frac{n-1}{2} \delta_i + v_i(0)^{2t_i} \right)
\]

Then by (28)
\[
\left| \frac{\partial \tilde{K}_i}{\partial y^i}(0) \right| \leq C \left( r_i v_i(0) - \frac{2n}{n-2} + \frac{n-1}{2} \delta_i + v_i(0)^{2t_i} \right).
\]

The same estimate holds for \(\left| \frac{\partial \tilde{K}_i}{\partial y^j}(0) \right|, j = 2, \ldots, n\) as well.

\(\Box\)

Now we can prove Proposition 6.1.
Proof: By the estimates of $v_i$ as stated between Proposition 6.1 and Lemma 6.2, it is equivalent to proving

$$
\lim_{i \to \infty} v_i^2(0) \int_{B_\sigma} X(\tilde{K}_i) v_i^{p_i+1} \lambda_i^{2n-2} dy = 0.
$$

When $n = 3, 4$,

$$
X(\tilde{K}_i)(y) = \left( \sum_j y^j \frac{\partial \tilde{K}_i}{\partial y^j} \right)(y)
$$

$$
= \left( \sum_j y^j \frac{\partial \tilde{K}_i}{\partial y^j} \right)(0) + \sum_k \frac{\partial}{\partial y^k} \left( \sum_j y^j \frac{\partial \tilde{K}_i}{\partial y^j} \right)(0) y^k + O(|y|^3)
$$

$$
= \sum_j \frac{\partial \tilde{K}_i}{\partial y^j}(0) y^j + \sum_{j,k} \frac{\partial^2 \tilde{K}_i}{\partial y^j \partial y^k}(0) y^j y^k + O(|y|^3)
$$

Since for $\kappa = 1$ or $\kappa = 2$,

$$
n - l_i(p_i + 1) + \kappa \to -\frac{5n}{7} + \kappa < 0,
$$

by similar calculation as in the proof of Proposition 5.4, We have

$$
\int_{B_\sigma} v_i^{p_i+1} |y|^\kappa dy
\leq C \left( \int_{|y| \leq R v_i(0)^{-\frac{2n-2}{2}}} v_i(0)^{p_i+1} |y|^\kappa dy 
+ \int_{R v_i(0)^{\frac{2n-2}{2}} \leq |y| \leq \sigma} (v_i(0)^{l_i} |y|^{-l_i})^{p_i+1} |y|^\kappa dy \right)
$$

(37)

$$
\leq C v_i(0)^{p_i+1 - \frac{(n+\kappa)(p_i+1)}{2}}
$$

Then

$$
v_i^2(0) \int_{B_\sigma} \frac{\partial^2 \tilde{K}_i}{\partial y^j \partial y^k}(0) y^j y^k v_i^{p_i+1} \lambda_i^{2n-2} dy
\leq C r_i^2 v_i^2(0) \int_{B_\sigma} v_i^{p_i+1} |y|^2 dy
\leq C r_i^2 v_i(0)^{\frac{2n-2}{n-2} + \frac{1}{2}}
\to 0 \quad \text{as } i \to 0.
$$

By Proposition 5.4 and Lemma 6.2.
\[ v^2_i(0) \left| \int_{B_\sigma} \frac{\partial \tilde{K}_i}{\partial y^j}(0) y^j v_i^{p_i+1} \lambda_i^{2n} dy \right| \]
\[ \leq C v^2_i(0) \left( r_i v_i(0)^2 \frac{\frac{4}{n-2} + (n-1) \delta_i}{\sigma} + v_i(0)^2 \frac{2}{n-2} + 2t_i + \frac{n-1}{2} \right) \int_{B_\sigma} |y| v_i^{p_i+1} dy \]
\[ \leq C \left( r_i v_i(0)^2 - \frac{4}{n-2} + v_i(0)^2 \frac{2}{n-2} + 2t_i + \frac{n-1}{2} \right) \]

Since \( t_i = 1 - \frac{(p_i-1) \delta_i}{2} \rightarrow 1 - \frac{2}{n-2} \frac{6(n-2)}{7} = -\frac{5}{7} \),
\[ \lim_{i \to \infty} \left( 2 - \frac{2}{n-2} + 2t_i \right) = 2 - \frac{2}{n-2} - \frac{10}{7} < 0. \]

We also have \( 2 - \frac{4}{n-2} \leq 0 \). Therefore
\[ v^2_i(0) \left| \int_{B_\sigma} \frac{\partial \tilde{K}_i}{\partial y^j}(0) y^j v_i^{p_i+1} \lambda_i^{2n} dy \right| \]
\[ \leq C \left( r_i v_i(0)^2 - \frac{4}{n-2} + v_i(0)^2 \frac{2}{n-2} + 2t_i \right) \]
\[ \rightarrow 0 \quad \text{as} \quad i \to 0. \]

Lastly,
\[ v^2_i(0) \int_{B_\sigma} v_i^{p_i+1} |y|^3 dy = v^2_i(0) \left( \int_{|y| \leq R_{v_i}(0)} \frac{p_i-1}{2} v_i(y)^{p_i+1} |y|^3 dy \right. \]
\[ + \int_{R_{v_i}(0)} \frac{p_i-1}{2} \leq |y| \leq \sigma v_i(y)^{p_i+1} |y|^3 dy \right) \]

The first term
\[ v^2_i(0) \int_{|y| \leq \frac{R}{v_i(0)}} v_i(y)^{p_i+1} |y|^3 dy \]
\[ \leq C v_i(0)^{2 + p_i + 1 - \frac{(n+3)(p_i-1)}{2}} \rightarrow 0 \]

since \( 2 + p_i + 1 - \frac{(n+3)(p_i-1)}{2} \rightarrow 2 - \frac{6}{n-2} < 0 \).

Because \( \lim_{i \to \infty} \left( - l_i(p_i+1) + 3 + n \right) = -\frac{5n}{7} + 3 > 0 \), the second term
\[ v_i^2(0) \int_{B_{\sigma \varepsilon \max}} \frac{R}{v_i(0)} \leq |y| \leq \sigma \ v_i(y)^{p_i+1} |y|^3 \, dy \]

\[ \leq C v_i^2(0) \int_{B_{\sigma \varepsilon \max}} \frac{R}{v_i(0)} \leq |y| \leq \sigma \ (v_i(0)^{t_i} |y|^{-l_i})^{p_i+1} |y|^3 \, dy \]

\[ \leq C v_i(0)^{2 + t_i(p_i + 1)} \to 0 \]

since \( 2 + t_i(p_i + 1) \to 2 - \frac{5}{n-2} < 0 \).

Therefore \( v_i^2(0) \int_{B_{\sigma \varepsilon \max}} v_i^{p_i+1} |y|^3 \, dy \) also converges to 0.

Thus we have proved that \( \lim_{i \to \infty} v_i^2(0) \int_{B_{\sigma \varepsilon \max}} X(\tilde{K}_i) v_i^{p_i+1} \lambda_i^{\frac{2n}{n-2}} \, dy = 0 \)
when \( n = 3, 4 \).  \( \square \)

7. Ruling Out Case II

In section 4, we have reduced Case II to the following:
There is a sequence of functions \( \{v_i\} \), each satisfies
\[ \Delta g^{(i)} v_i + K_i(\sigma_i y) v_i^{p_i} = 0 \]
where \( g^{(i)}(y) = g_{\alpha \beta}(\sigma_i y) dy^\alpha dy^\beta \) converges to the Euclidean metric on compact subset of \( \mathbb{R}^n(y) \). The sequence \( \{v_i\} \) has isolated blow-up point(s) \( \{0, \ldots\} \).

If 0 is not a simple blow-up point, then we can use the same argument as in the previous section to rescale the function and get a contradiction by examining both sides of the Pohozaev identity.

Thus 0 must be a simple blow-up point for \( \{v_i\} \), which satisfies
\[ \Delta g^{(i)} v_i + K_i(\sigma_i y) v_i^{p_i} = 0 \]
Then we can study this sequence of \( \{v_i\} \) in the same way as we studied the sequence of solutions \( \{v_i\} \) for equation (29). We can apply almost exactly the same argument and get a contradiction. The only difference is the expression of \( h = \lim_{i \to \infty} \frac{v_i(y)}{v_i(0)} \).

In Case I, we know \( h \) satisfies \( \Delta h = 0 \) on \( \mathbb{R}^n \setminus \{0\} \), \( h(1) = 1 \), and from the construction has a second critical point at \( |y| = 1 \), which implies that \( h = \frac{1}{2}(1 + |y|^{2-n}) \).

But here we only know \( h \) satisfies \( \Delta h = 0 \) where it is regular and \( h(1) = 1 \), but don't know whether it has a second critical point. What we do know though is that 0 is not the only blow-up point of \( \{v_i\} \). This is true because as defined in section 4, \( v_i(y) = \sigma_i^{p_i+1} u_i(\sigma_i y) \) and by the
choice of \( \sigma_i \), there exists \( \{ y_{2,i} \} \) such that \( |y_{2,i}| = 1 \) and

\[
v_i(y_{2,i}) = \sigma_i^{\frac{2}{n-1}} u_i(x_{2,i}) \geq \left( \frac{R}{u_i(x_{2,i})} \right)^{\frac{2}{n-1}} u_i(x_{2,i}) = R^{\frac{2}{n-1}}.
\]

Suppose \( 0 \) is the only blow-up point for \( \{ v_i \} \), then the Harnack inequality holds on any compact subset \( \Omega \) of \( \mathbb{R}^n(y) \) which contains \( \partial B_{\bar{r}} \) and a neighborhood of \( y_2 = \lim_{i \to \infty} y_{2,i} \), where \( \bar{r} \) is chosen as in Proposition 5.3. Therefore we know

\[
v_i(y_{2,i}) \leq \max_{\Omega} v_i \leq C \inf_{\Omega} v_i \leq C \inf_{\partial B_{\bar{r}}} v_i \to 0 \quad \text{as} \quad i \to \infty
\]

by Proposition 5.3. This is a contradiction.

Thus \( \{ v_i \} \) has two or more blow-up points \( \{ 0, y_2, \ldots \} \), and hence \( h \) also has blow-up points \( \{ 0, y_2, \ldots \} \). Since \( h \) is harmonic everywhere else, we can write

\[
h(y) = c_1 |y|^{2-n} + c_2 |y - y_2|^{2-n} + \bar{h}(y)
\]

where \( c_1, c_2 > 0 \) are constants and \( \bar{h}(y) \) is harmonic on \( \mathbb{R}^n \setminus \{ y_3, \ldots \} \) (if \( h(y) \) has blow-up points \( y_3, \ldots \) other than \( 0 \) and \( y_2 \)). By the harmonicity of \( \bar{h} \), the Harnack inequality and the maximal principle, the infimum of \( \bar{h} \) is approached when \( y \) goes off to \( \infty \). Now since we know \( h > 0 \) and

\[
\lim_{|y| \to \infty} c_1 |y|^{2-n} = \lim_{|y| \to \infty} c_2 |y - y_2|^{2-n} = 0,
\]

the infimum of \( \bar{h} \) must be non-negative. Thus when \( |y| \) is small, \( c_2 |y - y_2|^{2-n} + \bar{h}(y) > 0 \), i.e., near \( 0 \),

\[
h(y) = c_1 |y|^{2-n} + A + O(|y|), \quad \text{where} \quad A > 0.
\]

Then we can analyze \( \frac{1}{v_i^2(y)} \int_{\partial B_{\bar{r}}} T_i(X, \nu_i) d\Sigma_i \) in the same way as we did for (31). The positive “mass” term \( A > 0 \) guarantees that

\[
\lim_{i \to \infty} \frac{1}{v_i^2(y)} \int_{\partial B_{\bar{r}}} T_i(X, \nu_i) d\Sigma_i < 0.
\]

By exactly the same argument as in the previous section we can also show

\[
\lim_{i \to \infty} \frac{n - 2}{2n} c(n)^{-1} \frac{1}{v_i^2(y)} \int_{B_{\bar{r}}} X(R_i) d\nu_i \geq 0.
\]

This is a contradiction, so Case II is also ruled out.

Thus we have finished the proof of Theorem 1.1.

When \( R(g) > 0 \) and \( K > 0 \), we can similarly define isolated blow-up points and simple blow-up points for \( \{ u_i \} \) which satisfies \( \Delta_y u_i - c(n) R(g) u_i + K u_i^\theta = 0 \). After slight modification we have the same estimates as those in Proposition 5.3. If the blow-up is not simple,
then it is either not isolated blow-up or it is isolated but not simple blow-up.

If the blow-up is isolated but not simple, we can rescale the function and metric as in Case I of the scalar flat case to reduce it to the simple blow-up case. Then a contradiction follows from the Pohozaev identity as in the scalar flat case.

If the blow-up is not isolated, we can first rescale the function in the same way as in case II of the scalar flat case to reduce it to the isolated blow-up cases. Then we can use almost the identical argument as in the scalar flat case to rule it out.

Thus the possible blow-up could only be simple. This completes the proof of Theorem 1.2.

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