ESSENTIAL SELF-ADJOINTNESS OF A DISCRETE MAGNETIC
SCHÖDINGER OPERATOR

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Abstract. We prove essential self-adjointness for a semibounded from below
discrete magnetic Schrödinger operator in a space that represents a combina-
torial model of the two-dimensional Euclidean space. The Dezin discretization
scheme is used for constructing a discrete model.

1. Introduction

Let $\Lambda^p_k(\mathbb{R}^2)$ be the set of all $k$-smooth (i.e., of the class $C^k$) complex-valued
$p$-forms in $\mathbb{R}^2$ and let $\Lambda^p(\mathbb{R}^2) = \Lambda^p_\infty(\mathbb{R}^2)$. We define a magnetic potential as a
real-valued 1-form $A \in \Lambda^1_1(\mathbb{R}^2)$, i.e.,

$$A = A_1 dx^1 + A_2 dx^2,$$

where $A_1, A_2 \in C^1(\mathbb{R}^2)$ are real-valued functions. We introduce an invariant inner
product for $p$-forms with compact support in $\mathbb{R}^2$ in the following way

$$(\varphi, \psi) = \int_{\mathbb{R}^2} \varphi \wedge * \psi,$$

where $*$ is the operation of metric conjugation of forms (the Hodge star operator),
$\wedge$ is the operation of exterior multiplication and the bar over $\psi$ denotes complex
conjugation. Consider the completion of the linear spaces of smooth forms in norm
that is generated by the inner product $\langle \cdot, \cdot \rangle$. We denote the formed Hilbert spaces
by $L^2(\mathbb{R}^2)$ for 0-forms (functions) and by $L^2\Lambda^p(\mathbb{R}^2)$ for $p$-forms, $p = 1, 2$. Let $d$
be the operator of exterior differentiation. We introduce a deformed differential
according to the rule

$$d_A : C^\infty(\mathbb{R}^2) \to \Lambda^1_1(\mathbb{R}^2), \quad \varphi \to d\varphi + i\varphi A,$$

where $i^2 = -1$ and $A$ is the magnetic potential. The inner product $\langle \cdot, \cdot \rangle$ enables
us to define an operator formally adjoint to $d_A$ as follows

$$\delta_A : \Lambda^1_1(\mathbb{R}^2) \to C(\mathbb{R}^2).$$

Then we may define the magnetic Laplacian $\Delta_A$ (Laplacian $\Delta$ with potential $A$)
according to

$$- \Delta_A \equiv \delta_A d_A : C^\infty(\mathbb{R}^2) \to C(\mathbb{R}^2).$$

Identifying the magnetic potential $A$ with the multiplication operator

$$A : C^\infty(\mathbb{R}^2) \to \Lambda^1_1(\mathbb{R}^2), \quad \varphi \to \varphi A,$$

2000 Mathematics Subject Classification. 35J10, 39A12, 39A70.
Key words and phrases. Schrödinger operator, difference equations.
we may represent the operator $\delta_A$ in the form
\begin{equation}
(1.4) \\
\delta_A \omega = (\delta - iA^*)\omega,
\end{equation}
where $\delta$, $A^*$ are operators formally adjoint to $d$ and $A$, respectively. Using (1.2) and (1.4), we may rewrite the magnetic Laplacian $\Delta_A$ as
\begin{align*}
-\Delta_A \varphi &= (\delta - iA^*)(d\varphi + iA\varphi) = \\
&= -\Delta \varphi - iA^*d\varphi + iA^*A\varphi + \delta(A\varphi).
\end{align*}
Consider now the magnetic Schrödinger operator
\begin{equation}
(1.5) \\
H_{A,V} = -\Delta_A + V,
\end{equation}
where $V$ is a real-valued function, which is also called electric potential, and $V \in L^2_{\text{loc}}(\mathbb{R}^2)$. Suppose that the operator $H_{A,V}$ is semi-bounded from below on $C_0^\infty(\mathbb{R}^2)$, i.e., there exists a constant $c \in \mathbb{R}$ such that
\begin{equation}
(1.6) \\
(H_{A,V} \varphi, \varphi) \geq -c(\varphi, \varphi), \quad \varphi \in C_0^\infty(\mathbb{R}^2).
\end{equation}
Here $C_0^\infty(\mathbb{R}^2)$ is the set of all $C^\infty$ functions with compact support in $\mathbb{R}^2$. Then, as is well known (see [9]), operator (1.5) is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$.

The main aim of the present work is to study the essential self-adjointness of the discrete magnetic Schrödinger operator on a combinatorial object corresponding to $C_0^\infty(\mathbb{R}^2)$. In [11], we proposed a discrete model of the magnetic Laplacian (1.3) such that it preserves the geometrical structure of the initial continual object and also proved the self-adjointness of the operator of a discrete Dirichlet problem for the magnetic Laplacian. In bounded domains, which gives the finite dimensionality of the corresponding Hilbert spaces of a discrete problem, the results of [11] can easily be generalized for the case of the magnetic Schrödinger operator. In the present work we show that the semi-bounded from below discrete magnetic Schrödinger operator, as in the continual case, has a unique self-adjoint realization. It should be emphasized that, besides conditions (1.6), no other limitations are imposed on the behavior of discrete analogs of the potentials $A$ and $V$ at infinity. Our approach is based on the formalism proposed by Dezin in [4]. We shall also use the results described in [11].

Note that the discrete magnetic Laplacian and discrete magnetic Schrödinger operators are fairly popular subjects of inquiry among mathematicians and physicists. There exist numerous various approaches (different from that proposed in the present work) both to the construction of discrete models and to investigation of the corresponding difference operators (see, e.g., [1, 2, 5, 6, 8, 10] and references therein). In the overwhelming majority of these works particular attention is given to studying the spectral properties of discrete magnetic Schrödinger operators on infinite graphs. As to investigations of the essential self-adjointness of discrete operators a review of different aspects of this problem can be found in [3, 12].

2. Basic Combinatorial Constructions

In this section we briefly recall the definitions of the basic combinatorial operations, which will be used in constructing the discrete analogs of operators (1.3) and (1.5). Let $\mathcal{E}(2)$ be a two-dimensional complex, i.e., a combinatorial model of $\mathbb{R}^2$ (for more detail, see [3, 11]). The complex $\mathcal{E}(2)$ can be represented as $\mathcal{E}(2) = \mathcal{E}^0 \oplus \mathcal{E}^1 \oplus \mathcal{E}^2$, where $\mathcal{E}^p$ is a real linear space of $p$-dimensional chains, 

\begin{align*}
(1.3) \\
H_{A,V} &= -\Delta_A + V,
\end{align*}

where $V$ is a real-valued function, which is also called electric potential, and $V \in L^2_{\text{loc}}(\mathbb{R}^2)$. Suppose that the operator $H_{A,V}$ is semi-bounded from below on $C_0^\infty(\mathbb{R}^2)$, i.e., there exists a constant $c \in \mathbb{R}$ such that
\begin{align*}
(1.6) \\
(H_{A,V} \varphi, \varphi) \geq -c(\varphi, \varphi), \quad \varphi \in C_0^\infty(\mathbb{R}^2).
\end{align*}
Here $C_0^\infty(\mathbb{R}^2)$ is the set of all $C^\infty$ functions with compact support in $\mathbb{R}^2$. Then, as is well known (see [9]), operator (1.5) is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$.

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$p = 0, 1, 2$. We denote by $\{x_{k,s}\}, \{e_{k,s}^1, e_{k,s}^2\}, \{\Omega_{k,s}\}, k, s \in \mathbb{Z}$, the sets of basic elements of the spaces $\mathcal{C}^0$, $\mathcal{C}^1$ and $\mathcal{C}^2$, respectively. For convenience, we introduce the shift operators
\[ \tau k = k + 1, \quad \sigma k = k - 1 \]
on the set of indices. The boundary operator $\partial$ on the basic elements of $\mathcal{C}(2)$ is assigned as
\[ \partial x_{k,s} = 0, \quad \partial e_{k,s}^1 = x_{\tau k,s} - x_{k,s}, \quad \partial e_{k,s}^2 = x_{k,\tau s} - x_{k,s}, \]
(2.1)
\[ \partial \Omega_{k,s} = e_{k,s}^1 + e_{\tau k,s}^1 - e_{k,s}^1 - e_{k,s}^2. \]
We introduce a dual object, i.e., a complex conjugate to $\mathcal{C}(2)$. We denote it by $K(2)$, and let it be a linear space of complex-valued functions over $\mathcal{C}(2)$. Further, suppose that $\mathcal{C}^0$, $K^1$ and $K^2$ are linear spaces conjugate to $\mathcal{C}^0$, $\mathcal{C}^1$ and $\mathcal{C}^2$, i.e., they have bases of the form $\{x_{k,s}\}, \{e_{k,s}^1, e_{k,s}^2\}, \{\Omega_{k,s}\}$, respectively. Then we may consider $K(2) = \mathcal{C}^0 \oplus K^1 \oplus K^2$ as a complex of complex-valued cochains of the corresponding dimensionality. In what follows, these cochains are called forms, which emphasizes their proximity to the corresponding continual objects (differential forms). Then the 0-, 1-, and 2-forms $\varphi \in \mathcal{C}^0$, $\omega = (u, v) \in K^1$ and $\eta \in K^2$ look like
\[ \varphi = \sum_{k,s} \varphi_{k,s} x_{k,s}^1, \quad \omega = \sum_{k,s} (u_{k,s} e_{k,s}^1 + v_{k,s} e_{k,s}^2), \quad \eta = \sum_{k,s} \eta_{k,s} \Omega_{k,s}, \]
where $\varphi_{k,s}, u_{k,s}, v_{k,s}, \eta_{k,s} \in \mathbb{C}$ for all $k, s \in \mathbb{Z}$.

We define the operation of pairing for the basic elements of complexes $\mathcal{C}(2)$ and $K(2)$ according to the rule
\[ < x_{k,s}, x_{p,q}^r > = < e_{k,s}^1, e_{p,q}^r > = < e_{k,s}^2, e_{p,q}^r > = < \Omega_{k,s}, \Omega_{p,q}^r > = \delta_{k,s}^{p,q}, \]
(2.3)
where $\delta_{k,s}^{p,q}$ is the Kronecker delta. Pairing (2.3) is extended to arbitrary forms (2.2) by linearity. The boundary operator (2.1) induces in the conjugate complex $K(2)$ a dual operation, namely, a coboundary operator $d^c$:
\[ < \partial a, \alpha > = < a, d^c \alpha >, \]
(2.4)
where $a \in \mathcal{C}(2)$ and $\alpha \in K(2)$. We consider the coboundary operator
\[ d^c : K^p \rightarrow K^{p+1} \]
as a discrete analog of the operation of exterior differentiation $d$. In what follows, we use the following difference representations of the operator $d^c$:
\[ < e_{k,s}^1, d^c \varphi > = \varphi_{\tau k,s} - \varphi_{k,s} - \Delta_k \varphi_{k,s}, \]
\[ < e_{k,s}^2, d^c \varphi > = \varphi_{k,\tau s} - \varphi_{k,s} - \Delta_s \varphi_{k,s}, \]
\[ < \Omega_{k,s}, d^c \omega > = v_{k,s} - u_{k,s} - u_{k,\tau s} + u_{k,s} - \Delta_k v_{k,s} - \Delta_s u_{k,s}. \]
(2.5)
In the complex $K(2)$ we introduce the operation of multiplication, which is considered as an analog of the exterior multiplication of differential forms. We denote this operation by $\cup$ and define it as
\[ x_{k,s} \cup x_{k,s} = x_{k,s}, \quad e_{k,s}^1 \cup e_{k,\tau s}^1 = -\Omega_{k,s}, \]
\[ x_{k,s} \cup e_{k,s}^1 = e_{k,s}^1 \cup x_{k,s} = e_{k,s}, \quad x_{k,s} \cup e_{k,s}^2 = e_{k,s}^2 \cup x_{k,\tau s} = e_{k,s}, \]
\[ x_{k,s} \cup \Omega_{k,s} = \Omega_{k,s} \cup x_{k,\tau s} = e_{k,s}^1 \cup e_{k,\tau s}^2 = \Omega_{k,s}. \]
(2.6)
assuming that the product is equal to zero in all other cases. The $\cup$-multiplication is extended to forms (2.2) by linearity. We denote by $\varepsilon^{k,s}$ an arbitrary basic element of $K(2)$. Then we introduce an operation $\ast$ taking

$$(2.7) \quad \varepsilon^{k,s} \cup \ast \varepsilon^{k,s} = \Omega^{k,s}.$$  

Using (2.6), we obtain

$$\ast \varepsilon^{k,s} = \Omega^{k,s}, \quad \ast \varepsilon^{1,2} = \varepsilon^{1,2}, \quad \ast \varepsilon^{2,1} = - \varepsilon^{1,2}, \quad \ast \Omega^{k,s} = x^{k}.\tau.$$  

The operation $\ast$ is extended to arbitrary forms by linearity.

Let $\alpha \in K^p$ be an arbitrary $p$-form, i.e.,

$$(2.8) \quad \alpha = \sum_{k,s} \alpha_{k,s} \varepsilon^{k,s}.$$  

We denote by $K^p_0$ the set of all discrete $p$-forms with compact support, i.e., if $\alpha \in K^p_0$, then only a finite number of components $\alpha_{k,s}$ in (2.8) is different from zero. Let now

$$(2.9) \quad \Omega = \sum_{k,s} \Omega_{k,s}, \quad k, s \in \mathbb{Z},$$  

where $\Omega_{k,s}$ is a two-dimensional basic element of $\mathfrak{C}(2)$. Note that we shall also use the notation $\Omega = \Omega_N$, if sum (2.9) is finite and $-N \leq k, s \leq N$, $N \in \mathbb{N}$.

The relation

$$(2.10) \quad (\alpha, \beta) = \langle \Omega, \alpha \cup \ast \beta \rangle,$$  

where $\alpha, \beta \in K^p_0$, gives a correct definition of the inner product for discrete $p$-forms (see (1.1)). Using (2.3), (2.6) and (2.7), we may rewrite relation (2.10) as

$$(2.11) \quad (\alpha, \beta) = \sum_{k,s} \alpha_{k,s} \beta_{k,s}.$$  

The inner product (2.10) enables us to define an operator formally adjoint to $d^c$, namely, the operator $\delta^c : K^{p+1} \rightarrow K^p$ satisfying the following relation

$$(2.12) \quad (d^c \alpha, \beta) = (\alpha, \delta^c \beta), \quad \alpha \in K^p_0, \quad \beta \in K^{p+1}.$$  

It is easy to show that

$$(2.13) \quad \delta^c \beta = (-1)^{p+1} \ast^{-1} d^c \ast \beta,$$  

where $\ast^{-1}$ is the operation inverse to $\ast$, i.e., $\ast^{-1} \ast = 1$. Hence, we may consider the operator $\delta^c$ as a discrete analog of the codifferential $\delta$. Taking (2.5) into account, we have for $\omega \in K^1$

$$(2.14) \quad \delta^c \omega = \sum_{k,s} (-\Delta_k u_{k,s} - \Delta_s v_{k,s}) \varepsilon^{k,s}.$$  

Thus, the discrete analog of the Laplace operator looks like

$$(2.15) \quad -\Delta^c = \delta^c d^c + d^c \delta^c : K^p \rightarrow K^p.$$  

Obviously, since $\delta^c \varphi = 0$ for $\varphi \in K^0$, we have

$$(2.16) \quad -\Delta^c \varphi = \delta^c d^c \varphi.$$
3. Discrete Analog of the Magnetic Laplacian

Let a real-valued 1-form

\[ A = \sum_{k,s} (A^1_{k,s} e^{k,s}_1 + A^2_{k,s} e^{k,s}_2), \]

where \( A^1_{k,s}, A^2_{k,s} \in \mathbb{R} \), be a discrete analog of the magnetic potential. Then we define a discrete analog of the deformed differential \((1.2)\) in the following way:

\[(3.1) \quad d^c_A : K^0 \to K^1, \quad \varphi \to d^c \varphi + i \varphi \cup A.\]

In view of \((2.5)\) and \((2.6)\), we obtain

\[(3.2) \quad d^c_A \varphi = \sum_{k,s} \left((\Delta_k \varphi_{k,s} + i \varphi_{k,s} A^1_{k,s}) e^{k,s}_1 + (\Delta_s \varphi_{k,s} + i \varphi_{k,s} A^2_{k,s}) e^{k,s}_2\right).\]

Further, we identify the discrete magnetic potential \( A \) with the operator of multiplication as follows:

\[(3.3) \quad A : K^0 \to K^1, \quad \varphi \to \varphi \cup A.\]

Then it is easy to obtain

\[ A \varphi = \sum_{k,s} (\varphi_{k,s} A^1_{k,s} e^{k,s}_1 + \varphi_{k,s} A^2_{k,s} e^{k,s}_2). \]

Let \( A^* : K^1 \to K^0 \) be the operator formally conjugate to \( A \), i.e., it acts on an arbitrary 1-form \( \omega = (u, v) \) according to the rule

\[(3.4) \quad A^* \omega = \sum_{k,s} (A^1_{k,s} u_{k,s} + A^2_{k,s} v_{k,s}) x^{k,s}.\]

Hence (see \(11\) for more detail), the operator \( \delta^c_A : K^1 \to K^0 \), which is formally adjoint to the operator \( d^c_A \), has the form

\[(3.5) \quad \delta^c_A = \delta^c - i A^*.\]

Thus, we may define a discrete magnetic Laplacian as

\[ -\Delta^c_A = \delta^c_A d^c_A : K^0 \to K^0. \]

In view of \((3.1)\) and \((3.5)\), we obtain

\begin{align*}
-\Delta^c_A \varphi &= \delta^c_A (d^c \varphi + i \varphi \cup A) = \\
&= (\delta^c - i A^*) d^c \varphi + (\delta^c - i A^*) (i \varphi \cup A) = \\
&= -\Delta^c \varphi - i A^* d^c \varphi + i \delta^c (\varphi \cup A) + A^* (\varphi \cup A) = \\
&= -\Delta^c \varphi - i A^* d^c \varphi + i \delta^c A \varphi + A^* A \varphi.
\end{align*}

Using \((2.11)\) and \((2.14)\), it is easy to show that for forms \( \varphi \in K^0 \) and \( \omega \in K^1 \) (see \((2.2)\)) one may write

\[ \delta^c (\omega \cup \varphi) = \delta^c \omega \cup \varphi - \sum_{k,s} \left( u_{\sigma k,s} (\Delta_k \varphi_{k,s}) + v_{k,\sigma s} (\Delta_s \varphi_{k,s}) \right) x^{k,s}, \]

\[ \delta^c (\varphi \cup \omega) = \varphi \cup \delta^c \omega - \sum_{k,s} \left( (\Delta_k \varphi_{\sigma k,s}) u_{\sigma k,s} + (\Delta_s \varphi_{k,\sigma s}) v_{k,\sigma s} \right) x^{k,s}. \]
Then, in view of (3.4) and (3.5), the discrete analogs of the Leibniz rule for the operator $\delta_A$ will have the form

$$\delta_A^c(\omega \cup \varphi) = \delta^c(\omega \cup \varphi) - \sum_{k,s} (u_{\sigma k,s}(\Delta_k \varphi_{k,s}) + v_{k,\sigma s}(\Delta_s \varphi_{k,s})) x^{k,s}.$$  

(3.7)

$$-i \sum_{k,s} (A_{k,s}^1 u_{k,s} \varphi_{k,s} + A_{k,s}^2 v_{k,s} \varphi_{k,s}) x^{k,s},$$

(3.8)

$$\delta_A^c(\varphi \cup \omega) = \varphi \cup \delta_A^c \omega - \sum_{k,s} ((\Delta_k \varphi_{k,s}) u_{\sigma k,s} + (\Delta_s \varphi_{k,s}) v_{k,\sigma s}) x^{k,s}.$$  

In addition, we have for $\varphi, \psi \in K^0$

$$-\Delta_A^c(\varphi \cup \psi) = \delta_A^c(d^c(\varphi \cup \psi) + i\varphi \cup \psi \cup A) =$$

$$= \delta_A^c(d^c \varphi \cup \psi + \varphi \cup d^c \psi + i\varphi \cup (\psi \cup A)) =$$

$$= \delta_A^c(d^c \varphi \cup \psi) + \delta_A^c(\varphi \cup d^c \psi).$$

From here, replacing $\omega$ by the 1-form $d^c \varphi$ (see (2.5)) and $\varphi$ by $\psi$ in (3.7) as well as $\omega$ by the 1-form $d^c \psi$, looking like (3.9), in (3.10), we obtain

$$-\Delta_A^c(\varphi \cup \psi) = \varphi \cup \delta_A^c d^c \psi + d^c \varphi \cup \psi + \sum_{k,s} \Phi_{k,s} x^{k,s},$$

where

$$\Phi_{k,s} = (\Delta_k \varphi_{k,s})(\psi_{k,s} - \psi_{\sigma k,s} + i\psi_{\sigma k,s} A_{\sigma k,s}) + i(\Delta_k \varphi_{k,s}) \psi_{\sigma k,s} A_{\sigma k,s}^1 +$$

$$+ (\Delta_s \varphi_{k,s})(\psi_{k,s} - \psi_{k,s} + i\psi_{k,s} A_{k,s}^2) + i(\Delta_k \varphi_{k,s}) \psi_{k,s} A_{k,s}^2.$$  

4. DISCRETE MAGNETIC SCHRÖDINGER OPERATOR

Let a real-valued 0-form $V \in K^0$ be a discrete analog of the electric potential, i.e.,

$$V = \sum_{k,s} V_{k,s} x^{k,s}, \quad V_{k,s} \in \mathbb{R}.$$  

Then the discrete analog of the magnetic Schrödinger operator (1.5) has the form

$$H_{A,V}^c = -\Delta_A^c + V.$$  

Since we do not impose any restrictions on the behavior of components of the discrete forms $A$ and $V$ at infinity, operator (4.1), generally speaking, is unbounded.

For forms looking like (2.5), we introduce a linear space

$$\mathcal{H}^p = \{ \alpha \in K^p : \sum_{k,s} |\alpha_{k,s}|^2 < +\infty, \quad k, s \in \mathbb{Z} \}, \quad p = 0, 1, 2.$$  

(4.2)

Obviously, according to (2.11), the space $\mathcal{H}^p$ is a Hilbert space with the inner product (2.10) and a norm

$$\|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle} = \left( \sum_{k,s} |\alpha_{k,s}|^2 \right)^{\frac{1}{2}}.$$  

(4.3)

It should be noted that, if $\alpha \in \mathcal{H}^0$, then the sequence of components $(\alpha_{k,s})$ is an element of $\ell^2(\mathbb{Z}^2)$, i.e., the space of all square summable complex-valued sequences. Since the set of all finite sequences $\ell^n(\mathbb{Z}^2)$ is dense in $\ell^2(\mathbb{Z}^2)$, the space $K^n_0$ is dense in $\mathcal{H}^0$. Hence, the operator $H_{A,V}^c : K^0_0 \to \mathcal{H}^0$ is densely defined (i.e., $K^0_0 = \mathcal{H}^0$)
and symmetric. In what follows, we assume that operator $\tilde{\Omega}_0$ is semi-bounded from below on $K_0^0$, i.e., condition (1.6) is satisfied for $H_{\tilde{\Omega}_0}$. We define minimal and maximal operators associated with $H_{\tilde{\Omega}_0}$ in $\mathcal{H}^0$ as follows:

$$H_{min} : D(H_{min}) \to \mathcal{H}^0, \quad H_{max} : D(H_{max}) \to \mathcal{H}^0,$$

where

$$D(H_{min}) = K_0^0, \quad D(H_{max}) = \{ \varphi \in \mathcal{H}^0 | H_{\tilde{\Omega}_0}^c \varphi \in \mathcal{H}^0 \}.$$

The essential self-adjointness of the operator $H_{\tilde{\Omega}_0}^c$ means that $H_{min} = H_{max}$, i.e., the closure of the minimal operator in $\mathcal{H}^0$ coincides with the maximal operator.

Further, we introduce a cutting 0-form $\chi^N \in K^0$ by

$$\chi^N = \sum_{k,s} \chi_{k,s}^N x_{k,s}, \quad \chi_{k,s}^N = \begin{cases} 1, & |k|, |s| \leq N \\ 0, & |k|, |s| > N \end{cases}, \quad N \in \mathbb{N}.$$  

Hence, $\chi^N = \sum x_{k,s}$ and $k,s$ take values from $-N$ to $N$. We also denote the inner product (2.10) by $(\cdot, \cdot)_N$ if $\Omega = \Omega_N$ (see (2.9)).

**Lemma 4.1.** Let $\psi \in K^0$ and let $\psi^N \in K^0$ be a form looking like (4.4). Then

$$\left( H_{\tilde{\Omega}_0}^c (\chi^N \psi), \chi^N \psi \right)_N = \left( \psi^N, \psi \right)_N.$$  

**Proof.** Using relation (3.9), we obtain for arbitrary $\varphi, \psi \in K^0$

$$H_{\tilde{\Omega}_0}^c (\varphi \psi) = \delta_N^c d_{\tilde{\Omega}_0}^c (\varphi \psi) + V(\varphi \psi) + \sum_{k,s} \Phi_{k,s} x_{k,s}.$$

Since the components of form $\delta^c d^c \varphi$ look like the following difference operators

$$-\Delta_k (\Delta_k \varphi_{k,s}) - \Delta_s (\Delta_s \varphi_{k,s})$$

and all summands of the components of $\Phi_{k,s}$ have multipliers of the form $\Delta_k \varphi_{k,s}$ and $\Delta_s \varphi_{k,s}$ (see (3.10)), we find for $\varphi$ with constant components that $\delta^c d^c \varphi = 0$ and $\Phi_{k,s} = 0$. Let now $\varphi = \chi^N$. We denote by $\Phi^N$ the 0-form with components (3.10), where $\varphi_{k,s}$ are replaced by $\chi_{k,s}^N$. Substituting (1.6) in the inner product $\left( H_{\tilde{\Omega}_0}^c (\chi^N \psi), \chi^N \psi \right)_N$, we see that the components of the form $\chi^N$ are equal to 1 at points $x_{k,s}$ of the domain $\Omega_N$ and by a step beyond its boundary. This fact guarantees that the components of the forms $\delta^c d^c \chi^N$ and $\Phi^N$ are equal to zero at points of the boundary of the domain $\Omega_N$, i.e., for $k = \pm N$ or $s = \pm N$. From here we immediately obtain

$$\left( \delta^c d^c \chi^N, \chi^N \psi \right)_N = 0, \quad \left( \Phi^N, \chi^N \psi \right)_N = 0$$

and this means that equality (1.6) holds true. \hfill \Box

**Theorem 4.2.** Let the discrete magnetic Schrödinger operator $H_{\tilde{\Omega}_0}^c$ be semi-bounded from below on $K_0^0$. Then $H_{\tilde{\Omega}_0}^c$ is essentially self-adjoint.

**Proof.** Obviously, every semi-bounded operator becomes strictly positive if the corresponding constant is added to it. For example, adding $(\epsilon + 1)I d$ to $H_{\tilde{\Omega}_0}^c$, we get

$$H_{\tilde{\Omega}_0}^c (\psi) \geq ||\psi||^2, \quad \psi \in K_0^0,$$

where the norm $||\cdot||$ is given by expression (4.3). As is well known for such operators (see [7, Theorem X.26]) the essential self-adjointness of $H_{\tilde{\Omega}_0}^c$ is equivalent to the

1. \hfill \Box
condition that \( \text{Ker}(H^*_m) = \{0\} \). Here the kernel of this operator is denoted by \( \text{Ker} \). Then the essential self-adjointness of \( H^c_{A,V} \) means that the equation

\[
H^c_{A,V} \psi = 0
\]

has only a trivial solution in \( \mathcal{H}^0 \).

Let \( \psi \) be a solution of equation (4.7). We introduce notation \( \psi^N = \chi^N \cup \psi \) and suppose that \( H^c_{A,V} \psi^N = f^N \). Then

\[
(H^c_{A,V}(\psi^N, \psi^N))_N = \sum_{|k|, |s| \leq N} f^N_{k,s} \cdot \overline{\psi^N_{k,s}} \geq \sum_{|k|, |s| \leq N} |\psi^N_{k,s}|^2 = \sum_{|k|, |s| \leq N} |\psi_{k,s}|^2 = \|\psi^N\|^2.
\]

On the other hand, since \( H^c_{A,V} \psi = 0 \) according to our assumption, relation (4.6) yields

\[
(H^c_{A,V}(\psi^N, \psi^N))_N = 0.
\]

Hence,

\[
\|\psi^N\|^2 \leq 0.
\]

Passing to the limit as \( N \to +\infty \), we obtain \( \|\psi^N\|^2 \to \|\psi\|^2 \).

Thus, \( \psi = 0 \). \( \square \)

**Corollary 4.3.** Suppose that the discrete electric potential \( V \in K^0 \) is bounded from below, i.e., there exists \( c \) such that, for all \( k, s \in \mathbb{Z} \), the inequality \( V_{k,s} \geq c > -\infty \) is satisfied. Then the operator \( H^c_{A,V} \) is essentially self-adjoint.

**Proof.** Indeed, since the discrete magnetic Laplacian \( -\Delta^c \) is a positive operator on \( K^0 \) (see proof in [11]), the boundedness from below of the form \( V \in K^0 \) leads to the semiboundedness from below of the operator \( H^c_{A,V} \). \( \square \)

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