GEOMETRY OF WHIPS AND CHAINS

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1. Introduction

The purpose of this note is to explore the geometry of the inextensible string (or whip) which is fixed at one end and free at the other end. In the absence of gravity, the inextensible string is a geodesic motion in the space of curves in $\mathbb{R}^2$ parametrized by length and passing through a fixed point. This is a submanifold of the space of all curves in $\mathbb{R}^2$, and the Lagrange multiplier from the constraint is the tension in the string. Hence the string satisfies a wave equation, where the tension is generated by the velocity implicitly by solving a one-dimensional Laplace equation.

The situation seems closely analogous to the motion of an incompressible fluid, where the configuration space is the space of volume-preserving diffeomorphisms, a submanifold of the space of all diffeomorphisms, and the Lagrange multiplier is the pressure. This is also a geodesic motion, but the Riemannian geometry is still not well-understood. It is our hope that understanding the one-dimensional situation better will help in understanding the two- and three-dimensional versions.

Part of the problem with the inextensible string is that the geodesic equation is not an ordinary differential equation on an infinite-dimensional manifold (unlike the situation for incompressible fluids). The derivative loss cannot be avoided, and thus one must treat it as a hyperbolic partial differential equation. However, since the tension must be zero at the free end, the equation degenerates at the free end, and the standard approach to symmetric hyperbolic equations must be modified to deal with this. Although these issues are not discussed here, we will study them in a forthcoming paper.

The other serious issue is that since the tension is not given by a positive function of the speed, position, and time, but rather given implicitly by the solution of an ordinary differential equation, it is possible for tension to become negative. This corresponds to the differential equation changing type from hyperbolic to elliptic, and drastically complicates the problem. We show that in the absence of gravity, the
tension is always nonnegative, while with gravity, the tension is non-negative as long as the string hangs below the fixed point.

To better understand the necessary conditions on weak solutions (which has remained controversial in recent papers on falling strings with kinks), we consider the corresponding finite-dimensional model: a system of \( N \) particles linked by rigid massless rods. In the absence of gravity, this is also a geodesic motion on an \( N \)-torus with a certain Riemannian metric. The geodesic equation ends up being a sort of discretization of the wave equation for the inextensible string, which still conserves energy. Furthermore, the tension in the chain converges to the tension in the corresponding whip as \( N \to \infty \). As a result, the Riemannian curvature of \( T^N \) converges to the Riemannian curvature of the space of curves, and one can therefore understand the geometry of the infinite-dimensional space quite well by studying the finite-dimensional approximation.

This suggests that one may be able to use the same approach to study incompressible fluids. Namely, take a collection of particles on a torus \( T^2 \) in a grid, and constrain them so that the areas of quadrilaterals are constant. (One could consider constraining triangle areas, but this seems too rigid; it does not appear possible to approximate a general area-preserving diffeomorphisms by such maps.) The curvature of the finite-dimensional manifold should approach that of the full area-preserving diffeomorphism group, and hence one expects one could understand the geometry of fluid mechanics well by a finite-dimensional approximation.

2. Discrete string

Let us compute the geometry of a discrete string (a chain), which we model by \( n + 1 \) point masses joined in \( \mathbb{R}^2 \) by rigid rods of length \( \frac{1}{n} \) whose mass is negligible. We assume the 0th point is fixed at the origin while the \( n \)th point is free.

The unconstrained Lagrangian is

\[
\mathcal{L} = \frac{1}{2} \sum_{i=1}^{n} \dot{x}_i^2 - \sum_{i=1}^{n} g(x_i, e_2).
\]

In addition the constraints are given by

\[
|x_i - x_{i-1}|^2 = \frac{1}{n^2}, \quad 1 \leq i \leq n.
\]

Therefore the equations of motion are

\[
\ddot{x}_i = -g e_2 + \lambda_{i+1}(x_{i+1} - x_i) + \lambda_i(x_{i-1} - x_i)
\]
for $1 \leq i \leq n$, where we are fixing $x_0 = 0$ and $\lambda_{n+1} = 0$ to make the equations satisfied also at the endpoints.

The constraint equations determine the Lagrange multipliers $\lambda$, which are obviously just the discrete tensions. We get

$$|v_i - v_{i-1}|^2 = -\lambda_{i+1} \langle x_i - x_{i-1}, x_{i+1} - x_i \rangle + \frac{2}{n^2} \lambda_i - \lambda_{i-1} \langle x_i - x_{i-1}, x_{i-1} - x_{i-2} \rangle$$

for $1 \leq i < n$ (again using $\lambda_{n+1} = 0$), while for $i = 0$ we get

$$|v_1|^2 - g \langle x_1, e_2 \rangle = -\lambda_2 \langle x_1, x_2 - x_1 \rangle + \frac{1}{n^2} \lambda_1.$$

We can simplify these $\lambda$-equations somewhat if we change variables. For each $1 \leq i \leq n$, let $x_i = x_{i-1} + \frac{1}{n} (\cos \theta_i, \sin \theta_i)$, where $\theta_i \in S^1$. Then

$$\langle x_i - x_{i-1}, x_{i+1} - x_i \rangle = \frac{1}{n^2} \cos (\theta_{i+1} - \theta_i)$$

while

$$|v_i - v_{i-1}|^2 = \frac{1}{n^2} |\dot{\theta}_i|^2.$$

So the constraint equation for $\lambda$ becomes

$$-\cos (\theta_{i+1} - \theta_i) \lambda_{i+1} + 2\lambda_i - \cos (\theta_i - \theta_{i-1}) \lambda_{i-1} = |\dot{\theta}_i|^2$$

for $2 \leq i \leq n$ and

$$-\cos (\theta_2 - \theta_1) \lambda_2 + \lambda_1 = |\dot{\theta}_1|^2 - ng \sin \theta_1.$$

For the evolution equation, we have

$$\ddot{\theta}_i = \lambda_{i+1} \sin (\theta_{i+1} - \theta_i) - \lambda_{i-1} \sin (\theta_i - \theta_{i-1})$$

for $2 \leq i \leq n$ and

$$\ddot{\theta}_1 = \lambda_2 \sin (\theta_2 - \theta_1) - ng \cos \theta_1.$$

Our first question is whether the tension is positive. We can write equations (3) and (2) as

$$\sum_{j=1}^{n} M_{ij} \lambda_j = |\dot{\theta}_i|^2 - n g \delta_{i1} \sin \theta_1,$$

where the matrix $M$ is

$$M = \begin{pmatrix}
1 & -\cos (\theta_2 - \theta_1) & 0 & \cdots & 0 & 0 \\
-\cos (\theta_2 - \theta_1) & 2 & -\cos (\theta_3 - \theta_2) & \cdots & 0 & 0 \\
0 & -\cos (\theta_3 - \theta_2) & 2 & \cdots & 0 & 0 \\
& & & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & -\cos (\theta_{i-1} - \theta_{i-2}) & 2
\end{pmatrix}.$$
If we let \( a_i = \cos(\theta_{i+1} - \theta_i) \) for \( 1 \leq i \leq n-1 \), then standard Gaussian elimination yields

\[
\begin{pmatrix}
  b_1 - a_1 & 0 & \ldots & 0 \\
  0 & b_2 - a_2 & \ldots & 0 \\
  0 & 0 & b_3 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \ldots & 0 & b_n
\end{pmatrix}
\begin{pmatrix}
  1 \\
  0 \\
  \frac{a_1}{b_1} \\
  \frac{a_1 a_2}{b_1 b_2} \\
  \frac{a_1 a_2 a_3}{b_1 b_2 b_3} \\
  \vdots \\
  \frac{a_1 a_2 \ldots a_n}{b_1 b_2 \ldots b_n}
\end{pmatrix} =
\begin{pmatrix}
  1 \\
  0 \\
  0 \\
  \frac{a_i b_i}{\prod_{j=1}^{n} b_j} \\
  \frac{a_i a_{i+1} b_{i+1}}{\prod_{j=1}^{n} b_j} \\
  \vdots \\
  \frac{a_i a_{i+1} \ldots a_n b_n}{\prod_{j=1}^{n} b_j}
\end{pmatrix}
\]

where the sequence \( b_i \) is defined by \( b_1 = 1, b_{i+1} = 2 - \frac{a_i^2}{b_i} \). Since \( |a_i| \leq 1 \) for every \( i \), we see that every \( b_i \) satisfies \( 1 \leq b_i \leq 2 \).

Reducing further, we obtain the inverse matrix

\[
M^{ij} = \sum_{m=\max(i,j)}^{n} \frac{1}{b_m} \frac{a_k}{b_k} \frac{a_l}{b_l}
\]

Thus the tensions are

\[
\lambda_i = M^{i1}(\dot{\theta}_1^2 - ng \sin \theta_1) + \sum_{j=2}^{n} M^{ij} \dot{\theta}_j^2.
\]

If every \( a_i \geq 0 \), then every component of \( M^{-1} \) is nonnegative. So as long as \( |\theta_{i+1} - \theta_i| \leq \pi \) for every \( 1 \leq i \leq n-1 \) and \( \sin \theta_1 \leq 0 \), we will have all tensions \( \lambda_i \geq 0 \), regardless of \( |\dot{\theta}_i| \). If any \( a_i < 0 \), then some \( M^{ij} < 0 \), and thus if \( \dot{\theta}_k = \delta_{jk} \), then \( \lambda_j < 0 \). Additionally, we always have \( M^{11} > 0 \), so that if \( \sin \theta_1 > 0 \), then when all \( \dot{\theta}_k = 0 \), we have \( \lambda_1 = -ngM^{11} \sin \theta_1 \).

Let us now look at the curvature; we will assume for the moment that \( g = 0 \) to get a purely geometric problem. We consider \( M_n = (S^1)^n \) as a submanifold of \( R^{2n} \), with induced Riemannian metric given by the kinetic energy formula above. We know that the equation of a constrained geodesic is always

\[
\frac{d^2 x_k}{dt^2} \partial_k = B_k \left( \frac{dx_i}{dt} \partial_i, \frac{dx_j}{dt} \partial_j \right) \partial_k.
\]

For any vector \( u \in T_p M_n \), we have \( \langle u_k - u_{k-1}, x_k - x_{k-1} \rangle = 0 \), for every \( 1 \leq k \leq n \). So \( u_k - u_{k-1} = \frac{1}{n} \eta_k (-\sin \theta_k, \cos \theta_k) \) for some \( \eta_k \in \mathbb{R} \). In terms of the \( \eta \)’s, we can write (using (4))

\[
B_k(u, u) = \lambda(u, u)_{k+1}(x_{k+1} - x_k) + \lambda(u, u)_k(x_{k-1} - x_k),
\]

for \( 1 \leq k \leq n \), where

\[
\lambda(u, u)_i = \sum_{j=1}^{n} M^{ij}|\eta_j|^2.
\]
Since the ambient space is flat, the curvature is, by the Gauss-Codazzi formula,
\[ \langle R(u, v)v, u \rangle = \langle B(u, u), B(v, v) \rangle - \langle B(u, v), B(u, v) \rangle. \]

To simplify this, we first compute
\[
\langle B(u, u), B(v, v) \rangle = \frac{1}{n^2} \sum_{k=1}^{n} \lambda(u, u)_{k+1} \lambda(v, v)_{k+1} + \lambda(u, u)_k \lambda(v, v)_k
\]
\[ - \lambda(u, u)_{k+1} \lambda(v, v)_k \cos (\theta_{k+1} - \theta_k)
\]
\[ - \lambda(u, u)_k \lambda(v, v)_{k+1} \cos (\theta_{k+1} - \theta_k)
\]
\[ = \frac{1}{n^2} \lambda(v, v)_1 \left( \lambda(u, u)_1 - \lambda(u, u)_2 \cos (\theta_2 - \theta_1) \right)
\]
\[ + \frac{1}{n^2} \sum_{k=2}^{n} 2 \lambda(u, u)_k \lambda(v, v)_k
\]
\[ - \lambda(u, u)_{k-1} \lambda(v, v)_k \cos (\theta_k - \theta_{k-1})
\]
\[ - \lambda(u, u)_k \lambda(v, v)_{k-1} \cos (\theta_{k+1} - \theta_k)
\]
\[ = \frac{1}{n^2} \sum_{k=1}^{n} \sum_{l=1}^{n} M_{kl}(\theta) \lambda(u, u)_l \lambda(v, v)_k
\]
\[ = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} M^{ij}(\theta) |\eta_i|^2 |\xi_j|.
\]

Similarly we have
\[
\langle B(u, v), B(u, v) \rangle = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} M^{ij}(\theta) \eta_i \xi_i \eta_j \xi_j.
\]

Therefore the curvature is
\[
(9) \quad \langle R(u, v)v, u \rangle = \frac{1}{n^2} \sum_{i,j=1}^{n} M^{ij}(\theta) \left[ \eta_i^2 \xi_j^2 - \eta_i \xi_i \eta_j \xi_j \right]
\]
\[ = \frac{1}{2n^2} \sum_{i,j=1}^{n} M^{ij}(\theta) \left[ \eta_j \xi_j - \eta_i \xi_i \right]^2.
\]

Thus the curvature is nonnegative in every section if and only if every \( M^{ij}(\theta) \geq 0 \), which also happens if and only if every tension is nonnegative regardless of the initial velocity.

To get a bound on the curvature, we want a bound on the components \( M^{ij} \). We see from equation (6) that \( M^{ij} \leq \max(n + 1 - i, n + 1 - j) \).

The extreme case is when all \( \theta \)'s are the same (so that \( a_i = 1 \) for every \( i \)), which corresponds to a straight string. Then we have \( M^{ij} = \)
max\((n + 1 - i, n + 1 - j)\). Therefore the sectional curvature satisfies \(0 < K < n\).

3. Smooth string

Now let us consider the continuum model of a smooth inextensible string. The configuration space is the set \(\mathcal{C}_{L,0}\) of all \(C^\infty\) maps \(\eta: [0, 1] \to \mathbb{R}^2\) such that \(\eta(0) = 0\) and \(\langle \eta'(s), \eta'(s) \rangle \equiv 1\). This is a closed submanifold of \(\mathcal{C}_0 = \{\eta \in C^\infty([0, 1], \mathbb{R}^d) \mid \eta(0) = 0\}\). The tangent spaces are

\[
T_\eta \mathcal{C}_0 = \{X \in C^\infty([0, 1], \mathbb{R}^2) \mid X(s) \in T_{\eta(s)} \mathbb{R}^2 \quad \forall s, \quad X(0) = 0\}
\]

and

\[
T_\eta \mathcal{C}_{L,0} = \{X \in T_\eta \mathcal{C}_0 \mid \langle X'(s), \eta'(s) \rangle = 0 \quad \forall s\}.
\]

The Riemannian metric is defined by

\[
\langle\langle X, Y \rangle\rangle = \int_0^1 \langle X(s), Y(s) \rangle_{\eta(s)} ds.
\]

In this metric, the orthogonal space to \(T_\eta \mathcal{C}_{L,0}\) is

\[
\left( T_\eta \mathcal{C}_{L,0} \right) ^\perp = \left\{ Y \in C^\infty([0, 1], \mathbb{R}^2) \mid Y(s) = \frac{d}{ds}(\sigma(s)\eta'(s)) \right. \text{ for some } \sigma \in C^\infty([0, 1], \mathbb{R}) \text{ with } \sigma(1) = 0 \}.
\]

The orthogonal projection of a vector \(X \in T_\eta \mathcal{C}_0\) should be

\[
P_L(X) = X - \frac{d}{ds}(\sigma \eta'),
\]

where \(\sigma\) satisfies

\[
(10) \quad \sigma''(s) - \langle \eta''(s), \eta''(s) \rangle \sigma(s) = \langle X'(s), \eta'(s) \rangle, \quad \sigma(1) = 0,
\]

but we have a problem with the boundary condition at \(s = 0\). We want \(P_L\) to map into \(T_\eta \mathcal{C}_{L,0}\), but

\[
P_L(X)(0) = X(0) - \sigma'(0)\eta'(0) - \sigma(0)\eta''(0) = -\sigma'(0)\eta'(0) - \sigma(0)\eta''(0).
\]

Now \(\eta''(0)\) is orthogonal to \(\eta'(0)\), so if \(\eta''(0) \neq 0\), we need both \(\sigma(0) = 0\) and \(\sigma'(0) = 0\), which will generally be impossible. So we do not have a nice orthogonal projection. There are two ways to get around this.

(1) Consider only curves with \(\eta''(0) = 0\). This forces \(X''(0) = 0\) as well, and then we have to worry about \(P_L(X)\) also satisfying \((P_L(X))''(0) = 0\). This forces \(\eta''(0) = 0\) and \(X''(0) = 0\), and so on... We end up being forced to require that all derivatives of \(\eta\) exist at \(s = 0\) (which is why I assumed \(\eta \in C^\infty([0, 1], \mathbb{R}^2)\)) and that all even derivatives of \(\eta\) vanish at \(s = 0\). Alternatively,
\( \eta \) extends to a \( C^\infty \) curve on \([-1,1]\) such that \( \eta(-s) = -\eta(s) \). This observation leads to the second idea.

(2) We could assume from the start that \( \eta \) is the restriction of an odd curve on \([-1,1]\). Then the tangent space will automatically require \( X \) odd. We then have two boundary conditions for (10): \( \sigma(-1) = 0 \) and \( \sigma(1) = 0 \). Finally, the solution of (10) will automatically be even in \( s \) by symmetry, and then we can just restrict to \([0,1]\). The advantage of this approach is that we don’t need to assume any more smoothness than is absolutely necessary for (10) to make sense. This is of course the same kind of trick that’s used to get simple solutions to the constant-coefficient linear wave equation with one free and one fixed boundary.

Either of these two tricks will force \( \sigma'(0) = 0 \), so we may as well assume this as our other boundary condition.

These difficulties do not arise if both ends of the curve are free. If both ends of the curve are fixed, we need oddness through both endpoints, again as with the simplest wave equation.

Now, the first thing we do with the orthogonal projection is to compute the second fundamental form \( B : T^\perp \mathcal{C}_{L,0} \times T^\perp \mathcal{C}_{L,0} \to (T^\perp \mathcal{C}_{L,0})^\perp \) defined by \( B(X,Y) = \nabla_X Y - P_L(\nabla_X Y) \). We obtain

\[
B(X,Y) = \frac{d}{ds} \left( \sigma_{XY}(s) \eta'(s) \right)
\]

where

\[
\sigma_{XY}''(s) - \kappa^2(s) \sigma_{XY}(s) = -\langle X'(s), Y'(s) \rangle.
\]

This formula implies the geodesic equation

\[
\frac{d^2 \eta}{dt^2} = B\left( \frac{d\eta}{dt}, \frac{d\eta}{dt} \right),
\]

or more explicitly,

\[
\eta_{tt}(s,t) = \partial_s \left( \sigma_{XX}(s,t) \eta_s(s,t) \right),
\]

\[
\partial_s^2 \sigma_{XX}(s,t) - |\eta_{ss}(s,t)|^2 \sigma_{XX}(s,t) = -|\eta_{st}(s,t)|^2,
\]

with boundary conditions as discussed.

If we want to incorporate gravity, then we have

\[
\eta_{tt}(s,t) = \partial_s \left( \sigma(s,t) \eta_s(s,t) \right) - g \mathbf{e}_2
\]

\[
\sigma_{ss}(s,t) - |\eta_{ss}(s,t)|^2 \sigma(s,t) = -|\eta_{st}(s,t)|^2,
\]

with boundary condition \( \sigma(1,0) = 0 \). To determine the boundary condition at \( s = 1 \), we assume \( \eta_{ss}(0,t) = 0 \) as before. If we want
\[ \eta_{tt}(0, t) = 0, \text{ we must have } \sigma_s(0, t)\eta_s(0, t) = g e_2, \text{ which cannot be satisfied unless } e_2 \parallel \eta_s(0, t). \text{ Instead, motivated in part by the finite case above, we set } \sigma_s(0, t) = g(\mathbf{e}_2, \eta_s(0, t)), \text{ which is equivalent to forcing the acceleration parallel to the curve to be zero.} \]

If, as before, we use the fact that \(|\eta_s| \equiv 1\) to write \(\eta_s(s, t) = (\cos \theta(s, t), \sin \theta(s, t))\), then differentiating the first of equations (13) with respect to \(s\) yields

\[
\theta_{tt}(s, t) = 2\sigma_s(s, t)\theta_s(s, t) + \sigma(s, t)\theta_{ss}(s, t) = -\theta_t(s, t) \tag{15}
\]

\[
\sigma_{ss}(s, t) - \theta_s(s, t)\sigma(s, t) = -\theta_t(s, t) \tag{16}
\]

with boundary conditions \(\sigma_s(0, t) = g \cos \theta(0, t)\) and \(\sigma(1, t) = 0.\)

Now we want to compute the curvature; again we assume \(g = 0\) to get a purely geometric formula. Formula (11) gives, via the Gauss-Codazzi equations, the curvature formula:

\[
\langle\langle R(X, Y)Y, X \rangle \rangle = \langle\langle B(X, X), B(Y, Y) \rangle \rangle - \langle\langle B(X, Y), B(X, Y) \rangle \rangle
\]

\[
= \int_0^1 \left\langle \frac{d}{ds} (\sigma_{XX}(s)\eta'(s)), \frac{d}{ds} (\sigma_{YY}(s)\eta'(s)) \right\rangle - \left\langle \frac{d}{ds} (\sigma_{XY}(s)\eta'(s)), \frac{d}{ds} (\sigma_{XY}(s)\eta'(s)) \right\rangle \right\rangle ds
\]

\[
= \int_0^1 \sigma_{XX}(s)(-\sigma_{YY}(s) + \kappa^2(s)\sigma_{YY}(s)) ds
\]

\[
- \int_0^1 \sigma_{XY}(s)(-\sigma_{XY}(s) + \kappa^2(s)\sigma_{XY}(s)) ds.
\]

If we denote the Green function of equation (12) by \(G(s, q)\), so that

\[
\sigma_{XY}(s) = \int_0^1 G(s, q)\langle X'(q), Y'(q) \rangle dq,
\]

then the curvature formula becomes

\[
\langle\langle R(X, Y)Y, X \rangle \rangle = \int_0^1 \int_0^1 G(s, q) \left( |X'(q)|^2 |Y'(s)|^2 - \langle X'(q), Y'(q) \rangle \langle X'(s), Y'(s) \rangle \right) ds dq
\]

\[
= \frac{1}{2} \int_0^1 \int_0^1 G(s, q) \sum_{i,j=1}^d \left( X'_i(s)Y'_j(q) - X'_j(q)Y'_i(s) \right)^2 ds dq,
\]

using the symmetry of the Green function. Thus if \(G(s, q) \geq 0\) for all \(s\) and \(q\), we get nonnegative curvature.
To prove positivity of $G(s, q)$, we first compute:

$$G(q, q) = \int_0^1 G(s, q) \delta(s - q) \, ds$$

$$= - \int_0^1 G(s, q) \left( G_{ss}(s, q) - \kappa^2(s) G(s, q) \right) \, ds$$

$$= \int_0^1 G_s(s, q)^2 + \kappa^2(s) G(s, q)^2 \, ds > 0.$$ 

So $G(q, q) > 0$. We also know that $G_s(s = q^+, q) - G_s(s = q^-, q) = -1$. Since the boundary conditions are $G_s(0, q) = 0$ and $G(1, q) = 0$, we can say the following.

1. If $G(0, q) < 0$, then $G(s, q) < 0$ for $s$ near 0. As long as $G(s, q) < 0$, we have $G_{ss}(s, q) \leq 0$, which implies $G_s$ is nonincreasing. Since $G_s(0, q) = 0$, this means $G_s(s, q) \leq 0$ as long as $G(s, q) < 0$. Thus there is no way $G(s, q)$ can increase to 0 or any positive number on $0 < s < q$. This makes $G(q, q) > 0$ impossible, a contradiction. Therefore $G(0, q) > 0$, and furthermore $G(s, q) > 0$ for all $s \in (0, q)$.

2. If $G_s(1, q) > 0$ then $G(s, q) < 0$ for $s$ close to 1. As long as $G(s, q) < 0$, we have $G_{ss}(s, q) \leq 0$, and so $G_s(s, q)$ is nonincreasing. So $G_s(s, q) > 0$ as long as $G(s, q) < 0$, so $G$ cannot have a turning point on $q < s < 1$. This again makes $G(q, q) > 0$ impossible, implying a contradiction. So $G_s(1, q) < 0$, which implies $G(s, q) > 0$ for all $s \in (q, 1)$.

This proves that the curvature is nonnegative as long as $\eta$ is smooth. In fact the sectional curvature must be strictly positive in all nontrivial two-planes, since it can only vanish if there is a $c$ such that $X'(s) = cY'(s)$ for every $s$. Since $X(0) = Y(0)$, we must have $X(s) = cY(s)$ for all $s$, which means $X$ and $Y$ are linearly dependent in $T_s\mathcal{C}_{L,0}$.

Polarizing the formula for curvature, we obtain

$$\langle \langle R(Y, X)X, W \rangle \rangle = \int_0^1 \int_0^1 G(s, q) \left( |X'(q)|^2 \langle Y'(s), W'(s) \rangle - \langle X'(q), Y'(q) \rangle \langle X'(s), W'(s) \rangle \right) \, ds \, dq$$

$$= - \int_0^1 \int_0^1 \left( |X'(q)|^2 \left( \frac{\partial}{\partial s} G(s, q) Y'(s), W(s) \right) + \langle X'(q), Y'(q) \rangle \left( \frac{\partial}{\partial s} G(s, q) X'(s), W(s) \right) \right) \, ds \, dq.$$
Therefore the curvature operator is

\[ Y \mapsto P_L \left( \frac{\partial}{\partial s} \int_0^1 G(s, q) \left( -|X'(q)|^2 Y'(s) + \langle X'(q), Y'(q) \rangle X'(s) \right) dq \right). \]

Even for smooth \( X \), this operator is not bounded in \( L^2 \). (By contrast, the curvature operator in the volumorphism group for a fixed \( C^1 \) vector field \( X \) is bounded in \( L^2 \) and in any other Sobolev space.)

4. Convergence

We don’t have a good existence and uniqueness result for the continuous equation; the closest we have is Reeken [R2], who proved short-time existence for a chain in \( \mathbb{R}^3 \) in something like the Sobolev space \( H^{17} \) with initial conditions close to a chain hanging straight down (for technical reasons, Reeken assumes the chain is hanging from \((0, 0, \infty)\) with non-constant gravity). This result is already quite difficult. It is conceivable that we could get a simpler result using the two-dimensional equation derived above. Of course, the finite-dimensional approximation is the flow of a vector field on a compact manifold (a bounded subset of the tangent space \( T(S^1)^n \), since the kinetic energy is bounded). Thus the finite-dimensional approximation has a unique solution existing for all time and for any \( n \). We can thus ask whether solutions of the PDE can be constructed as limits of the ODE as \( n \to \infty \).

It is easy to see that if we knew a smooth solution of the PDE existed, then the discrete model would converge to it. Let us write

\[ \theta_n: \left\{ \frac{1}{n}, \frac{2}{n}, \ldots, 1 \right\} \times \mathbb{R} \to S^1 \text{ and } \sigma_n: \left\{ \frac{1}{n}, \frac{2}{n}, \ldots, 1 \right\} \times \mathbb{R} \to \mathbb{R}. \]

We will set \( \theta_n(\frac{k}{n}, t) = \theta_k(t) \) and \( \sigma_n(\frac{k}{n}, t) = \frac{1}{n} \lambda_k(t) \) for \( 1 \leq k \leq n \).

Then equations (2) and (4) become

\[
\frac{d^2}{dt^2} \theta_n \left( \frac{k}{n}, t \right) = n^2 \sigma_n \left( \frac{k+1}{n}, t \right) \sin \left[ \theta_n \left( \frac{k+1}{n}, t \right) - \theta_n \left( \frac{k}{n}, t \right) \right] \\
- n^2 \sigma_n \left( \frac{k-1}{n}, t \right) \sin \left[ \theta_n \left( \frac{k}{n}, t \right) - \theta_n \left( \frac{k-1}{n}, t \right) \right]
\]

and

\[
- \cos \left[ \theta_n \left( \frac{k+1}{n}, t \right) - \theta_n \left( \frac{k}{n}, t \right) \right] \sigma_n \left( \frac{k+1}{n}, t \right) + 2 \sigma_n \left( \frac{k}{n}, t \right) \\
- \cos \left[ \theta_n \left( \frac{k-1}{n}, t \right) - \theta_n \left( \frac{k}{n}, t \right) \right] \sigma_n \left( \frac{k-1}{n}, t \right) = \frac{1}{n^2} \left( \frac{d}{dt} \theta_n \left( \frac{k}{n}, t \right) \right)^2
\]

The operators appearing in equations (17) and (18) are symmetric discretizations of derivative operators; if \( \sigma \) and \( \theta \) are sufficiently
smooth, we can write

\[ n^2 \sigma(x + \frac{1}{n}, t) \sin[\theta(x + \frac{1}{n}, t) - \theta(x, t)] \]

\[ - n^2 \sigma(x - \frac{1}{n}, t) \sin[\theta(x, t) - \theta(x - \frac{1}{n}, t)] \]

\[ = \sigma(x)\theta''(x) + 2\sigma'(x)\theta'(x) + O(\frac{1}{n^2}) \]

and

\[ - n^2 \cos[\theta(x + \frac{1}{n}, t) - \theta(x, t)]\sigma(x + \frac{1}{n}, t) + 2n^2 \sigma(x, t) \]

\[ - n^2 \cos[\theta(x, t) - \theta(x - \frac{1}{n}, t)]\sigma(x - \frac{1}{n}, t) \]

\[ = \theta'(x)^2\sigma(x) - \sigma''(x) + O(\frac{1}{n^2}) \]

where the constant in \( O(\frac{1}{n^2}) \) depends on the \( C^4 \) norms of both \( \theta \) and \( \sigma \).

Therefore, we can say that if \( \sigma \) and \( \theta \) form a sufficiently solution of (13), then the motion of the discrete chain with \( n \) elements converges to the motion of the smooth chain as \( n \to \infty \). We have convergence not only of the position and velocity, but also of the acceleration and the tension, which is not necessarily typical (for example, a system with strong returning force to a submanifold has position and velocity converging to the geodesic motion on the submanifold, but the acceleration does not converge).

Because the convergence is so strong, we can speculate about convergence of the motion in the nonsmooth case. Suppose the continuous string has a kink in it. There has been some serious analysis of the possible jump conditions one should impose on the equations in this case. See for example Reeken [R1], who used energy conservation and momentum conservation to classify possible motions of a kink. Also D. Serre used an approach similar to yours, relaxing the constraint to \( |x'(s)| \leq 1 \) and requiring that the tension always be nonnegative, and derived jump conditions. But this only sets up the differential equations, and there’s no guarantee that they have solutions. (Serre found blowup phenomena in a couple of examples.)

Indeed, since we expect the motion of \( n \) constrained particles to converge to the motion of a chain, the fact that the particles can (ideally) support negative tension in certain motions (when there is an acute-angle kink or when the chain starts with the first link above the fixed point) seems to suggest that the motion of the continuous string should also support negative tension. Of course this destroys the actual equations, which suggests that the motion of \( n \) constrained particles has no limit in some cases.
We will look at the limit as \( n \to \infty \) by examining the inverse matrix \( M^{ij} \) given by (6). We expect the components of this matrix to converge to the Green function as \( \frac{1}{n} M^{ij} \approx G\left(\frac{i}{n}, \frac{j}{n}\right) \). To understand what happens to \( M^{ij} \), we first look at the sequence \( b_i \) appearing in its definition. We have

\[
b_{i+1} = 2 - \frac{a_i^2}{b_i}, \quad \text{where } a_i = \cos (\theta_{i+1} - \theta_i).
\]

Now if the discrete curve approaches something smooth, then we will have

\[
\theta_{i+1} - \theta_i \approx \frac{1}{n} \kappa_i,
\]

where \( \kappa_i = \dot{\theta}(\frac{i+1}{n}) \), the approximate curvature. Thus as long as the curve is smooth, we will have \( a_i \approx 1 \) and thus also \( b_i \approx 1 \).

Numerically experimenting, we find that the sequence \( b_i \) is approximated very well by

\[
b_i = 1 + \frac{1}{n} f\left(\frac{i}{n}\right),
\]

where \( f \) satisfies the differential equation

\[
f'(s) = \kappa^2(s) - f^2(s),
\]

with initial condition \( f(0) = 0 \). This equation of course comes from the Riccati trick \( f(s) = y'(s)/y(s) \), where \( y''(s) - \kappa^2(s)y(s) = 0 \).

Now suppose there is a kink in the curve, say of angle \( \alpha \) at position \( s_o \) with \( s_o \approx k/n \) for some integer \( k \). Then \( \theta_{k+1} - \theta_k = \alpha \), and so all \( a_i \)'s are close to 1 except for \( a_k = \cos \alpha \). We then have \( b_{k+1} \) being rather far from 1, which means for equation (19) to be valid, we must have \( \lim_{s \to s_o} f(s) = +\infty \). In fact this is precisely what happens numerically: \( f \) satisfies equation (20) to the left of \( s_o \) with initial condition \( f(0) = 0 \); \( f \) also satisfies (20) to the right of \( s_o \) with condition \( \lim_{s \to s_o} f(s) = +\infty \). In particular, \( f \) does not depend on the size of the kink, only the fact that there is one. If there are multiple kinks, we get the same condition: \( f \) approaches infinity from the right side of the kink.

Knowing \( a_i \) and \( b_i \), we know \( M^{ij} \). Recall from (6) the formula

\[
M^{ij} = \sum_{m=\max i,j}^n \frac{1}{b_m} \prod_{k=i}^{m-1} \frac{a_k}{b_k} \prod_{l=j}^{m-1} \frac{a_l}{b_l}.
\]
Pick any \( x \) and \( y \) in \([0, 1]\) and \( s \geq \max(x, y) \). For large \( n \), choose \( i = nx \), \( j = ny \), and \( m = ns \). Since \( a_k \approx 1 - \frac{s^2}{2n^2} \) except at the kink, we have

\[
\prod_{k=nx}^{ns} a_k \rightarrow \begin{cases} 
1 & \text{if } s_o \notin (x, s) \\
\cos \alpha & \text{if } s_o \in (x, s) 
\end{cases} \quad \text{as } n \to \infty
\]

for any \( x \) and \( s \). On the other hand, we have

\[
\prod_{k=nx}^{ns} b_k = \exp \left( \sum_{k=nx}^{ns} \ln \left[ 1 + \frac{1}{n} f \left( \frac{k}{n} \right) \right] \right)
\]

\[
= \exp \left( \frac{1}{n} \sum_{k=nx}^{ns} f \left( \frac{k}{n} \right) + O \left( \frac{1}{n} \right) \right) \to \exp \left( \int_x^s f(\sigma) \, d\sigma \right).
\]

Thus we have

\[
G(x, y) = \int_{\max(x,y)}^1 \phi(x, s)\phi(y, s)e^{-\int_x^s f(\sigma) \, d\sigma}e^{-\int_y^s f(\tau) \, d\tau} \, ds
\]

where

\[
\phi(x, s) = \begin{cases} 
\cos \alpha & x < s_o < s \\
1 & \text{otherwise}
\end{cases}
\]

Thus we see \( G(x, y) \geq 0 \) even if the curve has one or more kinks. So in the absence of gravity, the curvature is nonnegative in all sections.

If there is gravity, then \( G(x, y) \) may be negative, if the string is pointing upwards at the fixed point. If we use a generalized curvature that incorporates the Hessian of the potential energy, then we will also get negative generalized curvature in this case.

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