Lattice spin models for non-Abelian Chiral Spin Liquids

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(Dated: March 5, 2018)

We suggest a class of two-dimensional lattice spin Hamiltonians describing non-Abelian SU(2) chiral spin liquids - spin-analogues of fractional non-Abelian quantum Hall states with gapped bulk and gapless chiral edge excitations described by the SU(2)\(_n\) Wess-Zumino-Novikov-Witten conformal field theory. The models are constructed from an array of a generalized spin-\(n/2\) ladders with multi-spin exchange interaction which are coupled by isolated spins. Such models allow a controllable analytic treatement starting from the one-dimensional limit and are characterized by a bulk gap and non-Abelian SU(2)\(_n\) gapless edge excitations.

Historically, such a state was discovered by considering lattice versions of the bosonic \(\nu = 1/2\) Laughlin wave function as variational candidate ground-state wave functions for the triangular lattice spin-1/2 Heisenberg model \cite{11 2 7}. There is now a numerical evidence for its existence in the spin-1/2 Heisenberg model on kagome lattice with the first, second and third nearest neighbor interaction \cite{8 10}. Another piece of numerical evidence suggests that KL CSL exists for the spin-1/2 Heisenberg model on kagome and triangular lattices \cite{12 15} with an additional three-spin exchange interaction explicitly breaking the time-reversal symmetry. Both studies numerically demonstrate the required ground-state degeneracy and the full characterization of the underlying topological order for the KL CSL state.

Since numerics has its restrictions it is highly desirable to have microscopic models of CSLs which would allow a controlled analytical description. At the same time such controllable models should not be too unrealistic. While exact parent lattice Hamiltonians for KL CSL wave function have been constructed in Refs. \cite{16 13 14} they are very challenging to achieve experimentally since they contain complex variables as well as long-range interactions. A complementary approach is the so-called coupled-wires construction to FQHS which starts from an array of one-dimensional (1D) wires, coupled in such a way that a two-dimensional (2D) gapped phase with 1D gapless edge excitations emerges \cite{20}. It opens a possibility to find microscopic models which display Abelian and non-Abelian topologically ordered physics from arrays of quantum spin chains \cite{21 24}. For KL CSL such attempt was made by Gorohovsky et al. \cite{21} who suggested a model on anisotropic triangular lattice. Unfortunately the authors have found that in any realistic situation the CSL will be destroyed by competing magnetic orders. Since this puts them in a conflict with the numerical results of Refs. \cite{14 15} who presented a good evidence for CSL even for the isotropic lattice, things are not under control.

As was suggested in \cite{17 25}, the KL CLS is just one example of spin liquid among many and CSLs with non-Abelian statistics might be stabilized. We construct analytically tractable 2D microscopic models of non-Abelian SU(2) CSLs where the edge modes are described by the SU(2)\(_n\) WZNW theory. For \(n = 2\) this is an analogue of the bosonic Pfaffian state. For general \(n\) these CSLs correspond to the Read-Rezayi series at filling \(\nu = n/2\) with \(\mathbb{Z}_n\) parafermionic neutral states \cite{26}. For \(n \geq 2\), these quasiparticle excitations possess non-Abelian statistics like those of the Pfaffian. The lattice models consist of an array of coupled spin-\(n/2\) generalized zigzag ladders with three-spin exchange interactions which breaks the time-reversal and parity symmetries preserving their product. Although it looks unusual, such interaction may be generated either as a result of spontaneous symmetry breaking \cite{20} or, as was suggested in Ref. \cite{11} emerges in a Mott insulator as a result of a magnetic field.

Lattice spin models. The simplest lattice model for \(n = 1\) is depicted in Fig. 1 and the Hamiltonian for individual ladders in this case includes only two- and
three-spin interactions:

\[ \mathcal{H}_{\text{ladder}} = \mathcal{H}_J + \mathcal{H}_X, \]

\[ \mathcal{H}_J = \sum_n \left[ J S_n \cdot S_{n+2} + J' S_n \cdot S_{n+1} \right], \]

\[ \mathcal{H}_X = \frac{\chi}{2} \sum_{i,j,k \in \Delta} \left( S_i \cdot [S_j \times S_k] \right), \]

where the diagonal exchange is weaker than the exchange along the legs: \( J' < J \). For the three-spin interaction the sum goes on all triangular plaquettes and indices \( i,j,k \) appear in the clockwise order in each elementary triangle. The ladder model (1) allows analytic and controllable treatment in two cases. Either the interchain exchange is weak \( J' \ll J \) or one is close to the integrable point: \( \chi^2 = 2JJ' \). The low-energy spectrum of the \( n = 1 \) model (1) consists of gapless bosonic modes of opposite chirality located on different legs of the ladder [21,28]. This fulfills the necessary condition for the coupled-wires construction which requires that neighboring wires are coupled by chiral modes of opposite chirality [20]. The authors of Ref. [21] proceeded suggesting to couple the ladders in the fashion of triangular lattice with the conclusion that such coupling will destroy the CSL due to the competing magnetic order. This competition comes from the coupling of staggered magnetizations of the next-to-nearest neighbor chains generated by higher order virtual processes. On the triangular lattice this interaction is \( J_{nnn} \sim J''(J'/J) \), where \( J'' \) is the exchange between the ladders. On the other hand, the interaction between the gapless modes of the neighboring ladders is marginally relevant and the corresponding gap is exponentially small in \( 1/J'' \) and hence will be suppressed by the indirect exchange \( J_{nnn} \).

In model of Fig. 1 the ladders with three-spin interactions which couple spins around triangles are separated by localized spins. The interaction with the latter spins is strongly anisotropic. Such arrangement allows one to overcome the problems related to the unfrustrated interactions between next-to-nearest spins discussed above and to make the CSL more robust, still providing a possibility for analytical treatment of the model Hamiltonians. The fact that our model is formulated on a lattice allows numerical checks of our results.

In the case of generic \( n \), the integrable model (1) for spins \( S = n/2 \) was found by Zyryan [29]. The explicit form of the Hamiltonian is given in Refs. [29,30] and it represents coupled integrable critical spin-S = \( n/2 \) Heisenberg chains with extra multi-spin interactions around triangular plaquettes. The most important difference is that the gapless chiral modes now are described by the SU(2)_n WZNW model. A 2D lattice model of non-Abelian CSL with SU(2)_n edge modes is constructed then by coupling the generalized ladders as in Fig. 1.

The continuum limit of the individual ladder model. To derive the effective field theory describing the continuum limit of the ladder model one just follows the standard approach of weakly-coupled two-leg spin ladders [22,33]. Alternatively at the integrable point one may use the Bethe ansatz results of Refs. [25,29]. The continuum limit Hamiltonian is the sum of two perturbed SU(2)_n WZNW models with densities:

\[ \mathcal{H}_{\text{eff}} = \mathcal{H}_R + \mathcal{H}_L + \mathcal{V}, \]

\[ \mathcal{H}_R = \frac{2\pi v}{n+2} \left[ : \mathbf{J}_{R1}^2 : + : \mathbf{J}_{L2}^2 : \right] + 2\pi vg_R \mathbf{J}_{R1} \cdot \mathbf{J}_{R2}, \]

\[ \mathcal{H}_L = \frac{2\pi v}{n+2} \left[ : \mathbf{J}_{L1}^2 : + : \mathbf{J}_{R2}^2 : \right] + 2\pi vg_L \mathbf{J}_{L1} \cdot \mathbf{J}_{R2}, \]

\[ \mathcal{V} = \gamma_{tw} \mathbf{n}_1 \cdot \partial_x \mathbf{n}_2 + \gamma_{bs} (\mathbf{J}_{R1} \cdot \mathbf{J}_{L1} + \mathbf{J}_{R2} \cdot \mathbf{J}_{L2}), \]

where \( \mathbf{J}_{R,L} \) are SU(2)_n Kac-Moody chiral currents and \( v \sim J \) is the spin velocity. Operators \( \mathbf{n}_{1,2} \) are staggered magnetizations of the chains 1,2. They have scaling dimension \( d = 3/(n+2) \) and are related to the 2×2 matrix fields \( \mathbf{G}_{1,2} \) of the WZNW models describing individual chains [22,33]: \( \mathbf{n}_{1,2} \sim \text{Tr}[\hat{G}_{1,2}] (\hat{G} \text{ being a vector formed by the Pauli matrices}). For \( n = 1 \) all perturbations are marginal while for \( n > 1 \) the non-zero conformal spin perturbation with coupling constant \( \gamma_{tw} \), dubbed twist term in Ref. [34] is strongly relevant with a scaling dimension \( d_t = 1 + 3/(n+2) \). However, when the integrability condition is met, (i) \( \gamma_{tw} = 0, \gamma_{bs} < 0 \) and the continuum limit Hamiltonian (1) splits into two independent parts and (ii) \( g_R \) and \( g_L \) have different signs. As a consequence of (ii) at the integrable point one of the models \( \mathcal{H}_{R,L} \) be-
comes massive and the other one is massless depending on the sign of $\gamma$ with the marginally relevant dimensionless coupling $g = |g_R| \sim (J'/2J)^{1/2}$. For $g \ll 1$ the spectral gap for massive mode at the integrable point is exponentially small: $\Delta_{\text{ladder}} = Jg^{1/n}\exp(-1/g)$. However, the gapless mode remains robust independently of the value of $g$ and hence always admits a field-theory description even when the gap is of the order of $J$. The field theory for the gapless mode is the one of the SU(2)$_n$ WZNW models [30, 31] which has a negative coupling constant.

According to Ref. [24] who analyzed the one-loop renormalization group flows of model (4) with $n = 1$ outside of the integrable point, the picture of the spectrum given above remains qualitatively valid in some region outside of the integrable line. As a result one can be confident that the phase diagram of the generalized zigzag spin ladder contains a region where the right movers of (say) chain 1 are strongly coupled to the left movers of chain 2 and the left movers of chain 1 are weakly coupled to the right movers of chain 2. The part of the spectrum containing the strongly coupled modes is gapped. Such violation of parity is, of course, a consequence of the three-spin interaction. For $n > 1$ the situation is more delicate since the twist operator is relevant. Even if the gap is sufficiently large so that $\Delta_{\text{ladder}} > \epsilon_{\text{int}}^1/(2d_1)$, we have to make sure that the twist operator does not ruin the quantum criticality of the edge modes. In this respect, we need to consider its projection on the low-energy sector. Since the twist operator is a spin singlet and is a sum of the twist operators on different parities, as was described above.

(1) are such that the effective couplings for the current-current interactions on a given ladder have different signs for different parities, as was described above.

**Coupling the ladders.** When zigzag ladders are arranged in a 2D array, the critical nematic modes couple. Depending on the nature of this coupling the system may either order as a nematic or remain a CSL. The order prevails if the matrix order parameters $U_n$ from different ladders couple directly. As discussed above, one needs to isolate the ladders from each other more thoroughly to stabilize a CSL state. The arrangement shown on Fig. 1 makes it more difficult for the nematic order parameters to couple. In that case the chiral gapless modes from neighboring ladders $y,y + 1$ interact indirectly with each other through the intermediate spins interacting with a ferromagnetic Ising-like interaction. We will demonstrate that this arrangement produces a spin gap in the bulk. As will be shown the gap magnitude increases when the exchange with the intermediate spins is anisotropic. Meanwhile the exchange within the ladders must remain SU(2) invariant to preserve the Kac-Moody algebra of the currents. Taking into account that each leg of an individual ladder contains just one chiral gapless mode and that the in-chain staggered magnetizations are short-ranged fields at energies $\ll \Delta_{\text{ladder}}$, we arrive to the following low-energy model which describes the neighboring ladders $y,y + 1$ as a Kondo lattice model with a forward scattering:

\[
\mathcal{H} = \frac{2\pi v}{n + 2} \int dx \left[ J_{L,y}^I + J_{L,y}^R \right] S_{y+1/2}^a(x) S_{y+1}^a(x) + \sum_l \eta_a \left[ J_{L,y}^I(x_l) + J_{R,y}^R(x_l) \right] S_{y+1/2}^a(x_l) S_{y+1}^a(x_l) + \sum_l I_{ab}^r S_{y+1/2}^a(x_l) S_{y+1}^b(x_{l+1}).
\]

We will consider the case $\eta^z >> \eta^x = \eta^y$. According to Ref. [37] where somewhat similar system was studied, Hamiltonian (9) must be augmented by the RKKY interaction between the localized spins generated by the coupling to irrelevant operators. The strongest of those is $\eta_0 \xi_{z,x} (n_x^n + n_y^{n+1}) S_{y+1/2}^a$, in our anisotropic model. Since correlation functions of the staggered magnetization decay exponentially, the second-order perturbation theory in $\eta_0$ yields the short range interaction $\mathcal{J}_{r,l}^{ab} S_{n,l}^a S_{l+1}^b$, where the exchange $\mathcal{I}_{r,l}$ decays exponentially with distance. This interaction is antiferromagnetic and competes with the bare one. We will consider the case when the net interaction $\mathcal{I}_{r,l}^{ab} = -\mathcal{I}_{r,l}^{\delta_{r,1}} + \mathcal{J}_{r,l}^{ab}$ is small. For $n = 1$ model (9) with $I_{r} = 0$ was considered in Refs. [35, 39] and it was concluded that the spectrum is gapped and has no local order parameter. Using the Abelian bosonization approach, one can bosonize the chiral SU(2)$_1$ currents in terms of chiral bosons $\Phi^{R,L}_r$ and recast the Lagrangian density for model [9] as (from now on we set $v = 1$)

\[
\mathcal{L} = \frac{1}{2} \left\{ (\partial_x \Phi)^2 + (\partial_t \Phi)^2 \right\} + \rho \left\{ \frac{\eta_0}{2\pi} \cos(\sqrt{2\pi} \Phi) (S^+ e^{-i\sqrt{2\pi} \Phi} + S^- e^{i\sqrt{2\pi} \Phi}) \right\} + \frac{1}{2} \sum_{n,r} \mathcal{I}^{ab}_{r} S_{n}^{a} S_{n+r}^{b},
\]

where $\mathcal{I}^{ab}_{r}$ is the exchange interaction between the localized spins generated by the coupling to irrelevant operators. The strongest of those is $\eta_0 \xi_{z,x} (n_x^n + n_y^{n+1}) S_{y+1/2}^a$, in our anisotropic model. Since correlation functions of the staggered magnetization decay exponentially, the second-order perturbation theory in $\eta_0$ yields the short range interaction $\mathcal{J}_{r,l}^{ab} S_{n,l}^a S_{l+1}^b$, where the exchange $\mathcal{I}_{r,l}$ decays exponentially with distance. This interaction is antiferromagnetic and competes with the bare one. We will consider the case when the net interaction $\mathcal{I}_{r,l}^{ab} = -\mathcal{I}_{r,l}^{\delta_{r,1}} + \mathcal{J}_{r,l}^{ab}$ is small. For $n = 1$ model (9) with $I_{r} = 0$ was considered in Refs. [35, 39] and it was concluded that the spectrum is gapped and has no local order parameter. Using the Abelian bosonization approach, one can bosonize the chiral SU(2)$_1$ currents in terms of chiral bosons $\Phi^{R,L}_r$ and recast the Lagrangian density for model [9] as (from now on we set $v = 1$)
where we assume that the local spins can be located at arbitrary points along the chains with spin density $\rho_s$, and not necessarily in a regular fashion as depicted on Fig. 1. Here $\Theta = \Phi_L - \Phi_R$ is the field dual to $\Phi = \Phi_L + \Phi_R$ and $S^z = e^{\pm i\psi} \sin \theta$, $S^x = \cos \theta$. After the transformation $\psi \rightarrow \psi + \sqrt{2} \pi \Theta$ the coupling in the second term changes to $\eta_z \rightarrow \eta_z - 2\pi$ and the Hamiltonian becomes

$$H = \sum_{n,r} J^a_{nr} \tilde{S}^a_n \tilde{S}^a_{n+r} + \eta_z \sum_n \tilde{S}^z_n \cos(\sqrt{2} \pi \Phi_n) \tag{11}$$

$$+ \frac{\eta_z - 2\pi}{\sqrt{2} \pi} \sum_n \partial_x \Phi_n \tilde{S}^z_n + \frac{1}{2} \int dx [\left(\partial_x \Theta\right)^2 + \left(\partial_x \Phi\right)^2],$$

The situation simplifies when $\eta_z = 2\pi$. As we have said, we will always consider the case when $I_r$ is sufficiently small which can be always achieved by tuning the interactions. In the simplest case $I_r = 0$ we are left with the interaction term

$$V = \frac{\eta_z \rho_s}{2 \pi a_0} \cos(\sqrt{2} \pi \Phi) \tilde{S}^z. \tag{12}$$

where $\tilde{S}^a$ are transformed spin components which includes the staggered factor. Then since the operators $\tilde{S}^z(x)$ commute with the Hamiltonian, they can be replaced by constants. The ground state configuration corresponds to all $\tilde{S}^z(x)$ being equal. In this sector model is reduced to the integrable sine-Gordon model with a spectral gap $\Delta \sim (\rho_s \eta_{\perp})^{2/3}$. The dimensional analysis indicates that for general $\eta_z > \eta_{\perp}$ the gap is $\Delta \sim \eta_{\perp}^{4\pi/3\eta_z}$. So the bulk excitations are gapped and the boundary chiral modes are gapless as is expected for a QHS. The latter ones are described by the SU(2)$_1$ WZNW theory. A similar Toulouse approach can be performed in the general $n > 1$ case, as described in Ref. 30 with the emergence of a bulk gapped phase with gapless edge excitations described by the SU(2)$_n$ WZNW conformal field theory.

The gap $\Delta$ provides us with the window of tolerance for the magnitude of $I_r$: this exchange interaction should not exceed $\Delta$. We find it interesting to take a closer look at the bulk excitations in the $n = 1$ case. At the integrable point $\eta_z = 2\pi$, $I_r = 0$ model [11] has two kinds of excitations: the massive mobile sine-Gordon triplet modes and immobile defects corresponding to flips of $\tilde{S}^z$. Away from this point the latter defects become mobile solitons; each soliton carries spin 1/2 from the ladders and a zero Majorana mode from the interladder spins. To see this we need to consider a perturbation around $\eta_z = 2\pi$. Integrating out the massive $\Phi$ modes in the leading order in $\delta \eta = \eta_z - 2\pi$ changes $I_r$ by the amount $\delta I_r \sim (\delta \eta / \eta)^2 \delta (\Delta / \psi) \exp(-\Delta |r| / \psi)$. Treating the exponentially decaying Ising exchange as the nearest neighbor interaction we arrive to the Ising model in a transverse dynamical field. The Jordan-Wigner transformation brings us to the model of Majorana fermions which mass term changes sign when $\cos(\sqrt{2} \pi \Phi)$ does. Such model has been considered in Ref. 38 and in the limit when the Majoranas are slower than kinks possesses rich physics. It certainly has Majorana zero modes riding on kinks. There are also traces of the two-channel Kondo physics suggested for this case in Ref. 12 in the sense that the Majorana zero modes on kinks are similar to static Majorana zero modes on isolated spins in the two-channel Kondo effect. So we see that the bulk of our model is gapped and the excitations are nontrivial. A half of the spinon modes of the ladders are gapped inside of each ladder. These are spin-1/2 solitons. The remaining half are gapped due to the interladder coupling through the intermediate spins. These are non-Abelian kinks (visons?). Such fractionalized excitations cannot propagate in the transverse direction. However, there are probably their bound states with integer quantum numbers and these ones can. If the applied magnetic field exceeds the bulk gap we expect the bulk to become gapless with central charge $c = 3/2$ and the velocity $v \sim M$ (the magnetization). This may lead to interesting $T/M$ scaling in the thermodynamics.

In order to demonstrate the robustness of the described state we have to consider a possibility of interladder couplings other than the ones already described. In the low-energy theory such couplings are generated by high energy virtual processes and their magnitudes are proportional to powers of the couplings of the bare lattice Hamiltonian. In the given case one has to expect, for instance, the appearance of the effective interactions between the local (or Kondo) spins from different rows which may lead to their ordering. This process competes with the Kondo screening preventing the gap formation. Such unwanted interactions constitute a general problem for all quantum Hall constructions based on coupling of wires. Starting from the original idea by Teo and Kane all these scenario envisage that the original lattice model can at low energy be reduced to 1D critical models coupled by marginally relevant current-current interactions. The difficulty is that the gaps generated by the marginal interactions are exponentially small in the direct interwire exchange and the ones generated by high energy virtual processes have power law dependence on the same couplings of the lattice model. In the model with the intermediate spin we suggest this difficulty is avoided since the spin currents from neighboring ladders are not coupled directly which would generate exponentially small gap, but via the anisotropic exchange with intermediate spins.

We are grateful to C. Chamon, F. H. L. Essler, Y. Fuji, V. Gritsev, R. M. Konik, N. Robinson, O. A. Starykh and A. Wcislochbaum for inspirational discussions. A. M. T. thanks the Isaac Newton Institute in Cambridge for hospitality. The authors acknowledge the Yukawa Institute for Theoretical Physics for hospitality during the completion of this work. A.M.T. was supported by the US DOE under contract number DE-AC02-98 CH
P. L. would like to thank CNRS (France) for financial support (PICS grant).

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Supplemental Materials: Lattice spin models for non-Abelian Chiral Spin Liquids

THE EXPLICIT FORM OF THE INTEGRABLE SPIN-S ZIGZAG TWO-LEG LADDER

The integrable spin-$S$ two-leg zigzag ladder with a time-reversal breaking interaction is defined by the lattice Hamiltonian $[S1]$: \[ H = \sum_n \left[ \theta^2 (\mathcal{H}_{n_1,n_1+1} + \mathcal{H}_{n_2,n_2+1}) + 2(\mathcal{H}_{n_1,n_2} + \mathcal{H}_{n_2,n_1}) + 2i\theta[(\mathcal{H}_{n_1,n_1+1} + \mathcal{H}_{n_2,n_2+1})_1, (\mathcal{H}_{n_1,n_2} + \mathcal{H}_{n_2,n_1})_1]\right], \] (S1)

where \[ \mathcal{H}_{n,n+1}(x) = \sum_{j=1}^{2S} \sum_{k=1}^{j} \frac{2S}{k^2 + \theta^2} \times \prod_{l=0,l\neq j}^{2S} \frac{2x - l(l + 1) + 2S(S + 1)}{j(j + 1) - l(l + 1)}, \]

\[ x = S_n \cdot S_{n+1}. \] (S2)

When $\theta = 0$, the latter is the integrable spin-$S$ Heisenberg chain $[S2]$ which belongs to SU(2)$_{2S}$ universality class $[S3]$. An explicit form of model $[S1]$ for $S = 1$ can be found in Ref. $[S4]$. The Bethe ansatz equations for model $[S1]$ are $[S1] [S5]:$

\[ \left( \frac{\lambda_a - \theta + in/2}{\lambda_a - \theta - in/2} \right)^N \left( \frac{\lambda_a + \theta - in/2}{\lambda_a + \theta + in/2} \right)^N = \prod_{b=1}^{M} \frac{\lambda_a - \lambda_b + i}{\lambda_a - \lambda_b - i}, \] (S3)

\[ E = \sum_{a=1}^{\lambda} \sum_{\sigma = \pm 1} \frac{n}{(\lambda_a + \sigma \theta)^2 + n^2/4}. \] (S4)

NEMATIC ORDER PARAMETERS OF THE INDIVIDUAL LADDER MODEL

We consider a single generalized zigzag two-leg spin ladder where the sign of the T-breaking term is such that for instance $g_L > 0$ and $g_R < 0$. This means that the $2R$ and $1L$ modes hybridize and are gapful. In stark contrast, the other chiral modes $2L$ and $1R$ remain gapless and correspond to the edge modes with quantum critical behavior. Let us try to write the low-energy limit of various order parameters after averaging out the massive degrees of freedom as it is done in $[S6]$. We first consider the scalar product of staggered magnetizations:

\[ \mathbf{n}_1 \cdot \mathbf{n}_2 \sim -\langle \text{Tr}[\mathcal{G}_1^\dagger \mathcal{G}_2] \rangle \sim -2\langle \text{Tr}(\mathcal{G}_1 \mathcal{G}_2) + \text{Tr}(\mathcal{G}_1^\dagger \mathcal{G}_2) \rangle \sim -2\langle \mathcal{G}_{1La} \mathcal{G}_{2R_o} \mathcal{G}_{2L \beta} \mathcal{G}_{1R \beta} + \mathcal{G}_{1La} \mathcal{G}_{2R_\beta} \mathcal{G}_{2L \beta} \mathcal{G}_{1R_o} \rangle, \] (S5)

where $\langle A \rangle$ means the average of operator $A$ in the ground state of the massive theory of $\mathcal{H}_L$ and we have separated the chiral components of the SU(2)$_{2n}$ WZNW matrices $G_{1,2}^\alpha (\alpha, \beta = \uparrow, \downarrow)$. In the ground state of the current-current model $\mathcal{H}_L$, we have: $\langle \mathcal{G}_{1La} \mathcal{G}_{2R_\beta} \rangle = C \delta_{\alpha \beta}, C$ being a non-universal constant. By introducing the emerging SU(2)$_{2n}$ WZNW matrix $U_{\alpha \beta} = G_{2L \beta} G_{1R \beta}$, we find the low-energy description from Eq. $[S5]$: $\mathbf{n}_1 \cdot \mathbf{n}_2 \sim \text{Tr} U$. A similar approach can be performed for the vector chiral order parameter: $\mathbf{n}_1 \times \mathbf{n}_2 \sim i\text{Tr}(\partial U)$. Finally, the staggered magnetization of each individual chain $\mathbf{n}_{1,2}$ has a zero vacuum expectation value in the ground state of model $\mathcal{H}_L$ which means that their two-point correlators decay exponentially at large distance and at low-energy.

We now discuss the fate of the twist term $\mathbf{n}_1 \cdot \partial_\mathbf{x} \mathbf{n}_2$ after averaging out the massive degrees of freedom of the integrable model $\mathcal{H}_L$. By performing a similar approach as in Eq. $[S5]$, we get: $\mathbf{n}_1 \cdot \partial_\mathbf{z} \mathbf{n}_2 \sim \mathcal{G}_{2La} \partial_\mathbf{z} \mathcal{G}_{1R_o}$. Using $\partial_\mathbf{x} = i(\partial - \bar{\partial})$ and $\partial_\mathbf{z} = i(\partial + \bar{\partial})$, we find that the projection of the twist term in the low-energy limit reduces to a total derivative of the SU(2)$_{2n}$ WZNW field: $\mathbf{n}_1 \cdot \partial_\mathbf{z} \mathbf{n}_2 \sim \partial_\mathbf{y} \text{Tr} U$ and can be thus neglected.
THE EFFECTIVE INTERACTIONS

As is written in the main text, the exchange interactions between local spins and staggered magnetizations of the chains give rise to short range interactions between the latter spins. Below we will provide explicit calculations to support this statement. The interaction is generated by integration over massive modes encoded in staggered magnetization of the ladders. In the second order of perturbation theory in $\eta^a$ we get the following contribution to the action of the spins:

$$\delta S = -\frac{1}{2} \sum_{j,k} \int d\tau_1 d\tau_2 S^a_{j+1/2}(\tau_1) \langle \langle (n_j - n_{j+1})^a(\tau_1)(n_k - n_{k+1})^a(\tau_2) \rangle \rangle S^a_{k+1/2}(\tau_2)$$  \hspace{1cm} (S6)

We can calculate the correlation functions explicitly. Due to the SU(2) symmetry of the ladder Hamiltonian it is sufficient to do it for $n^z \sim \sin(\sqrt{2\pi}\Phi) = \sin(\sqrt{2\pi}(\varphi + \bar{\varphi}))$. We will take advantage of the fact that for a given chain $\varphi$ and $\bar{\varphi}$ do not couple to each other and one of these field is gapless and the other is gapped. Then we have

$$\langle \langle \sin(\sqrt{2\pi}\Phi(1))\sin(\sqrt{2\pi}\Phi(2)) \rangle \rangle = \frac{1}{2} \langle \langle e^{i\sqrt{2\pi}\varphi(1)}e^{-i\sqrt{2\pi}\varphi(2)} \rangle \rangle \sim \frac{1}{(v\tau_1 + ix_1)^{1/2}} D(1, 2).$$

The correlation function $D(1, 2)$ was calculated in [SS]:

$$D(\tau, x) \equiv \langle \langle e^{i\sqrt{2\pi}\varphi(\tau, x)} e^{-i\sqrt{2\pi}\varphi(0, 0)} \rangle \rangle \approx Z \langle \langle \frac{v\tau + ix}{v\tau - ix} \rangle \rangle^{1/4} r^{1/2} e^{-\Delta_{\text{ladder}} r}, \quad r^2 = r^2 + (x/v)^2,$$

where $Z \sim 1$, so that

$$\langle \langle n^a(1) n^a(2) \rangle \rangle \sim r_{12}^{-1} \exp(-\Delta_{\text{ladder}} r_{12}).$$

(S9)

Since $\eta^z \gg \eta^{x,y}$ the most important contribution is to the $S^z S^z$ interaction as is stated in the main text. The Fourier transform of the exchange integral is given by $\omega = 0$ transform of the correlation function:

$$\mathcal{J}_{\text{eff}}(q) \sim -\frac{[\eta^z]^2 q^2}{\sqrt{(qv)^2 + \Delta_{\text{ladder}}^2}} \to \frac{[\eta^z]^2}{q} \partial_x^2 K_0(\Delta_{\text{ladder}} x/v),$$

(S10)

This interaction is antiferromagnetic and decays fast in real space (as $1/x^2$) even at small distances.

The integration over the massive modes will also introduce interaction between localized spins of different chains. However, since the correlation function between $n_1$ and $n_2$ is zero, this interaction is generated only in the fourth order in $\eta$:

$$S_{j+1/2}^z(y) S_{k+1/2}^z(y + 1) S_{l+1/2}^z(y) S_{\ell+1/2}^z(y + 1) \langle \langle \partial_x n(y, x) \partial_x n(y + 1, x) \partial_x n(y, x) \partial_x n(y + 1, x) \rangle \rangle \sim x_1 \approx x_3, x_2 \approx x_4$$

so this will generate the interaction of energy densities of the Ising models.

TOULOUSE LIMIT SOLUTION OF THE KONDO-LATTICE MODEL WITH FORWARD SCATTERING

We extend the Toulouse limit solution of the Kondo-lattice model with forward scattering presented in the letter to the $n \geq 2$ case. The Hamiltonian is defined by:

$$\mathcal{H} = \frac{2\pi v}{n + 2} \int dx \left[ J^z_{R,y+1} : + : J^z_{L,y} : \right] + \sum_{\ell} \eta_{\ell} \left[ J^x_{L,y}(x_{\ell}) + J^x_{R,y+1}(x_{\ell}) \right] S^a_{y+1/2}(x_{\ell}),$$

(S11)

where $J_{R,y+1}$ and $J_{L,y}$ are chiral SU(2)$_n$ Kac-Moody currents. The starting point of the solution is to express these currents in terms of $Z_N$ parafermions currents $\psi_L, \psi_R$ with conformal weights $(1-1/n,0)$, $(0,1-1/n)$ and chiral bosonic fields $\Phi_{L,R}$ [SS]:

$$J^z_{L,y} = \sqrt{\frac{n}{2\pi}} \partial_x \Phi_L, \quad J^z_{R,y+1} = \sqrt{\frac{n}{2\pi}} \partial_x \Phi_R$$

$$J^+_L = \frac{\sqrt{n}}{2\pi} e^{i\sqrt{2\pi/\eta\Phi_L}} \psi^+_L, \quad J^+_R = \frac{\sqrt{n}}{2\pi} e^{-i\sqrt{2\pi/\eta\Phi_R}} \psi^+_R.$$
For the sake of simplicity, we assume that the chiral fields with different chiralities commute between themselves. The interacting part of model (9) reads then as follows:

\[ \mathcal{H}_{\text{int}} = \sum_{l} \eta_{z} \sqrt{\frac{n}{2\pi}} \partial_{x} \Phi S^{z} + \frac{\eta_{\perp} \sqrt{R}}{2\pi a_{0}} S^{+} e^{-i\sqrt{2\pi/n} \Theta} \left[ e^{-i\sqrt{2\pi/n} \Phi} \psi_{L} + e^{i\sqrt{2\pi/n} \Psi} \psi_{R} \right] + \text{H.c.,} \]  

(S14)

where \( \Phi = \Phi_{L} + \Phi_{R} \) and \( \Theta = \Phi_{L} - \Phi_{R} \) are respectively the total bosonic field and its dual field. Repeating the same procedure as for \( n = 1 \), we now absorb the phase factor \( e^{-i\sqrt{2\pi/n} \Theta} \) into the spin operator by a canonical transformation:

\[ U = e^{-i\sqrt{2\pi/n} \sum_{l} \Theta(x_{l}) S^{z}(x_{l})}. \]  

(S15)

Since \( [\Theta(x_{l}), \Phi(y_{l})] = -i\theta(x - y) \), we get the non-trivial transformation for the total bosonic field:

\[ U^{+} \Phi(x) U = \Phi(x) + \sqrt{\frac{2\pi}{n}} \sum_{l} S^{z}(x_{l}) \theta(x_{l} - x) \]  

(S16)

\[ \frac{1}{2} \int dx U^{+} (\partial_{x} \Phi)^{2} U = \frac{1}{2} \int dx (\partial_{x} \Phi)^{2} - \sqrt{\frac{2\pi}{n}} \sum_{l} S^{z}(x_{l}) \partial_{x} \Phi(x_{l}). \]  

(S17)

The Toulouse limit solution is then defined when \( \eta_{z} = 2\pi/n \):

\[ \mathcal{H}_{\text{int}}' = U^{+} \mathcal{H}_{\text{int}} U = \frac{\eta_{\perp} \sqrt{R}}{2\pi a_{0}} \sum_{l} S^{+}(x_{l}) \left[ e^{-i\sqrt{2\pi/n} \Phi} \psi_{L} + e^{i\sqrt{2\pi/n} \Phi} \psi_{R} \right] (x_{l}) + \text{H.c.,} \]  

(S18)

where, in the parafermion fields, we have absorbed phase factors which stem from the non-trivial transformation of the \( \Phi \) field (S16) under the canonical transformation. The resulting interaction is strongly relevant and opens spectral gaps for the bulk modes.