Non-adiabatic time-optimal edge mode transfer on mechanical topological chain

I. Brouzos, I. Kiorpelidis, F. K. Diakonos, and G. Theocharis

1LAUM, UMR-CNRS 6613, Le Mans Université, Av. O. Messiaen, 72085, Le Mans, France
2Department of Physics, University of Athens, 15771, Athens, Greece

We show that it is possible to successfully transfer topologically protected edge modes across a mechanical chain, with non-adiabatic optimal control schemes, in a time even shorter than their own period. The proposed protocols vastly outperform the adiabatic ones, both in time-scales and in robustness against disorder. Our control schemes possess non-adiabatic time intervals during which driving frequencies exceed characteristic frequencies of the system, shift the value of adiabatic invariant, and exchange a great amount of energy. As a bonus feature and in contrast with quantum chains, ultrafast pumping in classical chains is accompanied with amplification of the edge mode. Introducing non-adiabatic pumping of topologically protected states aims to challenge common approaches in this emerging field.

Introduction.— Thouless adiabatic quantized pumping [1] is a central concept of theoretical and experimental research in topological physics. It is based on an adiabatic cyclic modulation of the one-dimensional (1D) potential parameters, such that a flowing mode or state of the system transfers from one end to the other, carrying with it an amount of charge, spin or energy. The key feature of this transfer protocol is robustness against disorder attributed to the intrinsic topological protection of the 1D system [2, 3]. It has been realized in several experimental platforms such as semiconductor quantum dots [4, 5], cold atoms [6–9], photons [10–12], artificial spin systems [13, 14]. Very recently, the emerging field of mechanical metamaterials [15], where substantial progress has been achieved in realizing classical mechanical analogs of topological systems, see for example [16] and references in [17, 18], has also reported Thouless pumping both theoretically [19] and experimentally [20]. In the former case [19], elastic lattices simulated the Aubry-André-Haper model, and in the latter [20] magneto-mechanical resonators were arranged such that they map the benchmark 1D topological model of the Rice-Mele and Su-Schrieffer-Heeger (SSH) dimer chain [2]. Recent experiments on phononic spatiotemporal structures [21] pave the way for further experimental investigations of topological pumping in mechanics.

Both experimental and theoretical works consider adiabatic variation of the parameters, to apply the original Thouless concept [2, 20, 22, 23]. This is expressed as keeping a time scale for the parameter variation far from the relevant dynamical time scale of the system [20, 23]. Even in alternative protocols that have been proposed to speed up the transfer like coherent tunneling adiabatic process [24] or Landau-Zener type of transition [25], the parametric variation is quasi-adiabatic. For even-sized finite systems, the localized edge states are a superposition of symmetric and antisymmetric eigenstates with a finite overlap. Therefore one can consider Rabi tunneling mode [26] and wait for the relevant time or even carefully speed up this process [27]. Although succeeding in transferring the state quite robustly, the very slow time scales of those processes are subject to decay or decoherence rates of the mode as well as other lossy factors. This points at the need for time-optimal transfer protocols.

The scope of this work is to demonstrate that pumping of edge states in a mechanical SSH model [with an alternating spring constant, see Fig. 1(a),(b)] can be successfully and robustly done (with fidelity over 99%) with ultrafast optimal protocols possessing non adiabatic features. We employ optimal control schemes, which stem out from quantum control algorithms [28, 29], as shortcuts to adiabaticity [30]. The time-scale of the optimal driving -if we set the initial phases of the same order of magnitude, or, even faster than the characteristic period of the edge mode, indicating a strongly non-adiabatic process. Furthermore, as a consequence of large energy exchange, we illustrate that these control schemes result in magnifying the amplitudes of the mode. This feature could help overcome the inherent losses in classical systems or even be used to design phononic amplifiers. This amplification is a unique effect for the optimally driven mechanical system, since it is allowed due to its classical properties -in contrast to the quantum case where amplification is prohibited by unitarity. Finally, we show that, although our control schemes operate fast and with substantial modulation of the potential parameters and energy, they are even more robust against disorder compared to slow-variation and low energy exchange schemes.
SSH classical chain.— SSH chain was initially conceptualized as a diatomic linear quantum system, with next-neighbour interaction, exhibiting topological properties [2]. An equivalent classical system that we study here is shown in Fig. 1(a). It is a relevant model for various types of classical mechanical metamaterials, as for instance granular systems [31], phononic lattices [19] and magneto-mechanical structures [20]. It consists of an odd number of $N$ identical masses $m$, connected with springs with alternating stiffness $\kappa_1$ and $\kappa_2$, fixed at the two ends of the chain. With these boundary conditions, the system allows for the appearance of an edge state, localized on one side of the linear chain when $\kappa_1 \neq \kappa_2$. We set out from the edge state localized on the left part of the chain, when $\kappa_1 > \kappa_2$, as shown in Fig. 1(b). Our target for the optimal control is to transfer this state to the corresponding one on the right part, when $\kappa_1 < \kappa_2$ shown also in Fig. 1(b), by time-dependently modulating the spring constants which exchange values between initial and final time, namely $\kappa_1(0) = \kappa_2(T)$ and $\kappa_2(0) = \kappa_1(T)$. In Fig. 1(c) we show the eigenenergy of each eigenmode, where the single point in the middle of the spectrum, with eigenfrequency $\tilde{\omega} = \sqrt{\frac{\omega_1 + \omega_2}{m}}$, corresponds to the edge mode while the others compose the so-called bulk modes. Hereafter, every variable or parameter with a tilde (like $\tilde{\omega}$ above) corresponds to the edge mode.

The equations of motion for a spring-mass system with $N$ masses, as depicted in Fig. 1(a), read:

\[
\begin{align*}
\frac{d^2 q_{2n}}{dt^2} &= \kappa_1(t)(q_{2n+1} - q_{2n}) - \kappa_2(t)(q_{2n} - q_{2n-1}) \\
\frac{d^2 q_{2n+1}}{dt^2} &= \kappa_2(t)(q_{2n+2} - q_{2n+1}) - \kappa_1(t)(q_{2n+1} - q_{2n-2}),
\end{align*}
\]

with $n = 0, 1, ..., (N - 1)/2$ and fixed boundary conditions $q_0 = q_{N+1} = 0$. Stiffnesses $\kappa_1$ and $\kappa_2$ scale with $\kappa_0$ (with $\kappa_2(t = 0) = \kappa_0$), time $t$ scales as $\sqrt{m/\kappa_0}$, and amplitudes $q$ with normalization length $L$ of the edge state amplitude vector $L = ||\tilde{q}||$. The amplitude vector $\tilde{q}$ is derived from $-\tilde{\omega}^2 \tilde{q} = K \tilde{q}$ with $K = \begin{bmatrix} -\pi & \kappa_1 & 0 & \ldots \\ \kappa_1 & -\pi & \kappa_2 & 0 & \ldots \\ 0 & \kappa_2 & -\pi & \kappa_1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \kappa_1 & -\pi & \kappa_2 & \ldots & 0 \\ \ldots & 0 & \kappa_2 & -\pi \end{bmatrix}$, where $\pi = \kappa_1 + \kappa_2$.

Without loss of generality we consider here a system of $N = 21$ masses and $\kappa_1(0) = 3$, $\kappa_2(0) = 1$. These values result in a well localized edge mode. We checked that systems with different odd number of masses $N$ and initial stiffness values give qualitatively similar results. In addition, we note that by considering odd-sized systems we avoid the slow-mode Rabi oscillations.

Fidelity.— At every time instant $t$ of the process, one can project the instantaneous state of the system (see Supplemental material [32]), defined by the generalized coordinates $\mathbf{q}(t)$ and momenta $\mathbf{p}(t)$, to the instantaneous normal variables $\mathbf{Q}(t)$ and $\mathbf{P}(t)$, through the relations $\mathbf{Q}(t) = A(t)^{-1}\mathbf{q}(t)$ and $\mathbf{P}(t) = A(t)^{-1}\mathbf{p}(t)$. $A(t)$ is the instantaneous modal matrix composed of the instantaneous eigenmodes $\mathbf{q}_i(t)$ ($i = 1, ..., N$ numbering the eigenmode vectors) obtained by $-\omega_i^2(t)\mathbf{q}_i(t) = \mathbf{K} \mathbf{q}_i(t)$. Knowing this projection at every moment of time, the instantaneous energy distribution into the eigenmodes can be calculated by the relation $E_i(t) = \frac{1}{2} (\mathbf{P}_i^2(t) + \omega_i^2(t)\mathbf{Q}_i^2(t))$. As both, total instantaneous energy $E(t) = \sum_i E_i(t)$ as well as instantaneous energy of each mode $E_i(t)$ vary with time, and, most importantly, the initial and final energy need not to be the same [33] we define fidelity as $\mathcal{F} = \frac{\tilde{E}(T)}{\tilde{E}}$, where $\tilde{E}$ refers to the energy of the edge mode at the final time $T$. We additionally note that, since the initial edge state is an oscillating eigenmode, the fidelity in general depends on the initial phase. For all the cases we present (the one exception will be referred explicitly), we consider the initial phase as a free parameter ranging from $[0, 2\pi]$ and we present the minimum fidelity. Finally, we emphasize that in our case, there is an unavoidable crossing point of the two propagating bands of the dispersion relation when, at some time moment, $\kappa_1 = \kappa_2$. At the vicinity of this point, which may be referred as ‘closed gap’ configuration, (although in fact there is always a finite separation of the modes in our finite system) the edge mode becomes also delocalized and almost indistinguishable from the bulk modes. The crossing of the ‘closed gap’ results in general to unavoidable bulk excitation. Therefore, from here on, it is more reasonable to consider nearly perfect (and not perfect) state transfer with fidelity $\mathcal{F} > 99\%$.

From adiabatically slow to time optimal.— The first question we opt to answer here is what is the minimum total time $T$ needed to obtain this targeted fidelity for the successfully transfer of the edge mode, and what are the optimal control functions for the stiffness constants i.e. how should $\kappa_1(t)$, $\kappa_2(t)$ change in time to achieve this task. Let us first consider the case of keeping constant the frequency of the edge mode $\tilde{\omega} = 2$ throughout the transfer operation by applying the symmetry condition $\kappa_1(t) + \kappa_2(t) = 4$. First, we consider the protocol which is defined in [23] as a single frequency trigonometric control function $\kappa_1(t) = 2 + \cos(\pi t/T)$ [see Fig. 2(a)]. The resulting necessary time for the target fidelity is $T_{\text{trig}} = 297$, which is two orders of magnitudes longer than the period of oscillation of the edge mode $\left(T_{\text{trig}} \gg \tilde{T} = \pi\right)$. Then, we consider other analytic control functions, such as a linear ramp $[\kappa_1(t) = 3 - 2t/T]$ which results already to a shorter $T_{\text{lin}} = 192$. Our best guess shown in Fig. 2(a) is a tangential $\kappa_1(t) = c + \tan((t - a)/b)$ (see Supplemental Material [32]) which gives $T_{\text{tan}} = 144$. The advantage of the tangential control function over the other two (linear and trigonometric) is that it allows initially a rapid transition to the ‘closed gap’ configuration and at the same time a very slow crossing of this configuration during which $\kappa_1 \approx \kappa_2$. In contrary, far from the ‘closed gap’ configuration, faster parameter variation should be in principle possible since the edge modes are protected. To reduce further the transfer time, we search
for the optimal $\kappa_1(t)$ (see Supplemental Material [32]) which results to a shape shown in Fig. 2(a) and to $T_{sym} = 40$ almost an order of magnitude faster than the trigonometric scheme of [23]. This is the control protocol with the fastest transfer time we found under the constraint of the symmetric condition.

In order to shed more light into the underlying physics of the control schemes, we show three more quantities: (i) the total energy [Fig. 2(b)] (ii) the adiabatic invariant [34], which for our system is given by $I(t) = \sum \rho_i$ [Fig. 2(c)] and (iii) the ‘adiabatic ratio’ [Fig. 2(d)] $\alpha = \omega_s/\omega_r$ where $\omega_r = \max(\omega_1, \omega_2)$ with $\omega_{1,2} = \frac{1}{\kappa_{1,2}(t)} \frac{d\kappa_{1,2}(t)}{dt}$ and $\omega_s$ the difference of the edge mode frequency to the closest bulk mode frequency [35]. The adiabatic ratio compares the driving frequency $\omega_r$ with the characteristic transition frequency of the system $\omega_s$ therefore, the larger it is, the fastest is the variation.

When $\alpha > 1$, the process enters into the non-adiabatic regime.

For the symmetric non-optimal processes, for all the cases we obtain an adiabatic ratio $\alpha < 0.05$, which shows the adiabaticity of these processes. For the optimal symmetric case the driving frequency $\omega_r$ is of the same order with $\omega_s$ and this signals a step towards non-adiabaticity (still $\alpha < 0.25$). This is further confirmed by a careful investigation of $E(t)$ in Fig. 2(b), and $I(t)$ in Fig. 2(c), where one can see that the shorter the time transfer is, the more the energy oscillates to larger amplitudes and the more the adiabatic invariant oscillates around its initial value. Things become more interesting by looking Fig. 3(a-b) and the corresponding bulk mode excitation during the time process. For the ‘tan’ function, Fig. 3(a), almost only one eigenmode, the instantaneous edge mode, is excited during the process. However, for the optimal symmetric pulse, Fig. 3(b), a significant amount of bulk excitations is observed in the middle of the process. Eventually though, the energy is returned to the edge mode leading to an almost perfect state transfer. In other words, the symmetric optimal control does not avoid instantaneous bulk excitation and big exchange of energy, but uses genuinely the bulk states to transfer faster the edge mode to the other side. Also, if we interpret adiabaticity quantum mechanically, i.e. remaining throughout the process on the corresponding instantaneous eigenstate, then the optimal symmetric case is clearly the most non-adiabatic one. However, we will see that this comes with a drawback for the robustness against disorder.

Apparently, although optimal control speeds up the transfer by driving faster and less adiabatically to the ‘closed gap’, the adiabatic ratio $\alpha$ remains below unity during the process and also the final total energy and the invariant is almost equal to the initial one. The question is what happens if one relaxes the ‘symmetric condition’. The optimal control schemes presented in Fig. 2(e), with both $\kappa_1(t)$ and $\kappa_2(t)$ increasing and decreasing asymmetrically, results to an optimal time of $T_{asym} = 12 = 30\%T_{sym} < 3T$. Furthermore, setting the initial phase, say to maximum displacement-zero velocities [see Fig. 1(a)], the optimal time becomes subperiodic $T_{opt} = 1 < \tilde{T}$. Such ultrafast time-optimal schemes exchange a great amount of energy [Fig. 2(f)], shift the adiabatic invariant which largely deviates from the initial value [Fig. 2(g)] and break the adiabatic condition by having $\alpha > 1$ [Fig. 2(h)], especially in the beginning and end of the process. Although it is still unavoidable to cross the ‘closed gap’, this happens at larger values of $\kappa_1(t) = \kappa_2(t)$ where the modes differ significantly in frequency, due to the finite size of the chain, and thus the excitation of the bulk modes is significantly suppressed. Fig. 3(c) verifies this behaviour, showing that the instantaneous bulk excitation is very limited. Therefore, although the optimal asymmetric schemes are clearly non-adiabatic in terms of the adiabatic invariant and the adiabatic ratio $\alpha$, they
needs not to coincide with the energy of the initial state [see target state for the driven mechanical chain considered here, fact that, in contrary to the quantum case, the energy of the initial stiffness values to disorder. Indeed, if we apply a random disorder on the bulk states (optimal symmetric) will be more sensitive to disorder. Apart from confirming both desired results, the edge mode has been transferred but also if it remains at the other edge. Apart from confirming both desired results, we observe here one more feature for the (asymmetric) optimal control schemes in Fig. 3(e,f) namely the amplification appearing as darker spots after $t = T$. This is due to the fact that, in contrary to the quantum case, the energy of the target state for the driven mechanical chain considered here, needs not to coincide with the energy of the initial state [see Fig. 2(f)].

**Robustness against disorder.**— An important property of topological materials that has given such a strong boost to the related research recently, is the protection against disorder. One would naively expect that our proposed optimal schemes, being non-adiabatic (optimal asymmetric) or exciting bulk states (optimal symmetric) will be more sensitive to disorder. Indeed, if we apply a random disorder on the initial stiffness values $\kappa_{1,2} = \kappa_{1,2} (1 + \epsilon \ast w_{1,2})$ ($\epsilon = 0.2, w \in [-1, 1]$), and compare the distribution of fidelity for the tangential [Fig. 4(a)] and the symmetric optimal [Fig. 4(b)] driving, we see that the slower variation scheme is more robust. This is due to the fact that, within the symmetric condition, as we have mentioned, the instantaneous bulk excitation for shorter processes becomes larger [compare Fig. 3(a) and (b)]. Then, the addition of disorder breaks the genuine use of the bulk modes and a significant amount of energy get lost in the bulk excitations. The surprising result though is that the time-optimal asymmetric case in Fig. 4(c), is not only better than the symmetric, but also better from the slow varying tangential driving.

There are two reasons for this robust behavior of (asymmetric) time-optimal schemes. First, the fact that the main mechanism for the optimal pulses is the large exchange of energy keeping the excitation to the instantaneous bulk modes at the same level as in the slow-variation ones [compare Fig. 3(a) and (c)] and thus, more robust than the symmetric optimal case. In other words the non-adiabatic schemes decrease the necessary time by keeping restricted excitation to the instantaneous bulk modes like the adiabatics do, because they apply the non-adiabatic driving when it is safe (away from the ‘closed gap’ configuration), while when approaching the regime $\kappa_1 \approx \kappa_2$ they become very adiabatic [see again Fig. 2(b)]. The second reason is the fact that the stiffness values become much larger and the operation happens in shorter time so the effective (average in time) degree of disorder strength is lower.

**Conclusions and Perspectives.**— We have shown that in a classical SSH chain it is possible to pump the edge mode of the system non-adiabatically, in a time shorter than its period, by varying the two stiffness parameters of the chain independently. These ultrafast processes, obtained via optimal control schemes, are robust against disorder and amplify the edge mode at the same time. Further application of optimal control in edge state transfer may include other mechanical or quantum systems. A very interesting question concerns the speed limit for information transfer in driven mechanical chains, in analogy with Lieb-Robinson bound [27, 37] of the quantum systems. Preliminary results for our system indicate that the optimal times for sym-

![Figure 3.](https://example.com/figure3.png)

**Figure 3.** (Upper panel) Time evolution of the mode distribution $E_i$ for three cases: (a) tangential (b) symmetric optimal (c) asymmetric optimal. The symmetric optimal has the largest instantaneous bulk mode excitation. (Lower panel) Time evolution of the absolute value of the particle displacements of the chain for three cases: (d) symmetric optimal (e) asymmetric optimal independent of phase (f) sub-periodic asymmetric optimal (amplitude in log scale) for initial phase with maximum displacement [Fig.1 (b)]. The final state in (e) and (f) is an amplified edge mode.

![Figure 4.](https://example.com/figure4.png)

**Figure 4.** Robustness against disorder. Shown is the statistical distribution out of $10^4$ random realizations of the disorder on the initial stiffness values $\kappa_{1,2} = \kappa_{1,2} (1 + \epsilon \ast w_{1,2})$ for disorder strength $\epsilon = 0.2$ ($w \in [-1, 1]$) for 3 cases of control functions: (a) tangential (b) symmetric optimal (c) asymmetric optimal.

Transfer and Amplification.— In Fig. 3(d-f), we show the spatio-temporal evolution of the absolute value of the particle displacements for three cases: (d) symmetric optimal, (e) asymmetric optimal independent of phase, and (f) sub-periodic asymmetric optimal with fixed initial phase, zero velocities, maximum displacement [Fig. 1(b)]. Note that we evolve the system for a total time of $2T$, where $\kappa_1 (T < t < 2T) = \kappa_1 (T)$ and $\kappa_2 (T < t < 2T) = \kappa_2 (T)$, to check if the edge mode has been transferred but also if it remains at the other edge. Apart from confirming both desired results, we observe here one more feature for the (asymmetric) optimal control schemes in Fig. 3(e,f) namely the amplification appearing as darker spots after $t = T$. This is due to the fact that, in contrary to the quantum case, the energy of the target state for the driven mechanical chain considered here, needs not to coincide with the energy of the initial state [see Fig. 2(f)].
metric schemes have linear dependence with the system length (number of masses), as predicted by Lieb-Robinson while for non-optimal schemes the behaviour is quadratic.

Our work also shows that with optimal control schemes, it is possible to overcome adiabatic restrictions without loss in efficiency and this can be applied, among others, in the transfer of topological state in experiments, which flourish in the field [10–14, 20], or in adiabatic passage processes [24].

Acknowledgements.— This work has been funded by the project CS.MICRO funded under the program Etoiles Montantes of the Region Pays de la Loire. I. K. acknowledges financial support from Academy of Athens.

[1] D. J. Thouless, Phys. Rev. B 27, 6083 (1983).
[2] J. K. Asbóth, L. Oroszlány, and A. Pályi, A Short Course on Topological Insulators, Lecture Notes in Physics, Vol. 919 (Springer, Cham, 2016).
[3] N. R. Cooper, J. Dalibard, and R. B. Spielman, Rev. Mod. Phys. 91, 015005 (2019).
[4] M. Switkes, C. M. Marcus, K. Campman, and A. C. Gossard, Science 283, 1905 (1999).
[5] M. D. Blumenthal, B. Kaestner, L. Li, S. Giblin, T. J. B. M. Janssen, M. Pepper, D. Anderson, G. Jones, and D. A. Ritchie, Nat. Phys. 3, 343 (2007).
[6] H.-I. Lu, M. Schemmer, L. M. Aycock, D. Genkina, S. Sugawa, and I. B. Spielman, Phys. Rev. Lett. 116, 200402 (2016).
[7] S. Nakajima, T. Tomita, S. Tait, T. Ichinose, H. Ozawa, L. Wang, M. Troyer, and Y. Takahashi, Nat. Phys. 12, 296 (2016).
[8] M. Lohse, C. Schweizer, O. Zilberberg, M. Aidelevsburger, and I. Bloch, Nat. Phys. 12, 350 (2016).
[9] M. Lohse, C. Schweizer, H. M. Price, O. Zilberberg, and I. Bloch, Nature (London) 553, 55 (2018).
[10] Y. E. Kraus, Y. Lahini, Z. Ringel, M. Verbin, and O. Zilberberg, Phys. Rev. Lett. 109, 106402 (2012).
[11] M. Verbin, O. Zilberberg, Y. Lahini, Y. E. Kraus, and Y. Silberberg, Phys. Rev. B 91, 064201 (2015).
[12] O. Zilberberg, S. Huang, J. Guglielmon, M. Wang, K. P. Chen, Y. E. Kraus, and M. C. Rechtsman, Nature (London) 553, 59 (2018).
[13] M. D. Schroer, M. H. Kolodrubetz, W. F. Kindel, M. Sandberg, J. Gao, M. R. Vissers, D. P. Pappas, A. Polkovnikov, and K. W. Lehnert, Phys. Rev. Lett. 113, 050402 (2014).
[14] W. Ma, L. Zhou, Q. Zhang, M. Li, C. Cheng, J. Geng, X. Rong, F. Shi, J. Gong, and J. Du, Phys. Rev. Lett. 120, 120501 (2018).
[15] J. Christensen, M. Kadic, O. Kraft, and M. Wegener, MRS Commun. 5, 453 (2015).
[16] R. Susstrunk, and S. D. Huber, Science 349, 47 (2015); L. M. Nash, D. Kleckner, A. Read, V. Vitelli, A. M. Turner, and W. T. M. Irvine, Proc. Natl. Acad. Sci. USA 112, 14495 (2015).
[17] K. Bertoldi, V. Vitelli, J. Christensen, and M. Van Hecke, Nat. Rev. Mater. 2, 17066 (2017).
[18] G. Ma, M. Xiao, and C. T. Chan, Nat. Rev. Phys. 1, 281 (2019).
[19] M. I. N. Rosa, R. K. Pal, J. R. F. Arruda, and M. Ruzzene, Phys. Rev. Lett. 123, 034301 (2019).
[20] I. H. Grinberg, M. Lin, C. Harris, W. A. Benalcazar, C. W. Peterson, T. L. Hughes, and G. Bahl arXiv:1905.02778.
[21] Y. Wang, B. Yousefzadeh, H. Chen, H. Nassar, G. Huang, and C. Daraio, Phys. Rev. Lett. 121, 194301 (2018); Y. Chen, X. Li, H. Nassar, A. N. Norris, C. Daraio, and G. Huang, Phys. Rev. Applied, 11, 064052 (2019); J. Marconi, E. Riva, M. Di Ronco, G. Cazzulani, F. Braghin, and M. Ruzzene, arXiv:1909.13224.
[22] P. Boross, J. K. Ashbóth, G. Széchenyi, L. Oroszlány, and A. Pályi, Phys. Rev. B 100, 045414 (2019).
[23] F. Mei, G. Chen, L. Tian, S.-L. Zhu, and S. Jia, Phys. Rev. A 98, 012331 (2018).
[24] S. Longhi, Phys. Rev. B 99, 155150 (2019).
[25] S. Longhi, G. L. Giorgi, and R. Zambrini Landau-Zener topological quantum state transfer, Adv. Quant. Technol., 1800090 (2019).
[26] M. P. Estarellas, I. D’Amico, and T. P. Spiller, Sci. Rep. 7, 42904 (2017).
[27] N. Lang, and H. P. Büchler, npj Quantum Information 3, 47 (2017).
[28] J. Werschnik, and E. K. U. Gross, J. Phys: B: At. Mol. Opt. Phys. 40, R175 (2007); S. J. Glaser, U. Boscaín, T. Calarco, C. P. Koch, W. Köckenberger, R. Kosloff, I. Kuprov, B. Luy, S. Schirmer, T. S. Herbrüggen, D. Sugny, and F. K. Wilhelm, Eur. Phys. J. D 69, 279 (2015).
[29] T. Caneva, M. Murphy, T. Calarco, R. Fazio, S. Montangero, V. Giovannetti, and G. E. Santoro, Phys. Rev. Lett. 103, 240501 (2009); T. Caneva, T. Calarco, and S. Montangero, Phys. Rev. A 84, 022326 (2011).
[30] E. Torrontegui, S. Ibáñez, S. Martínez-Garaot, M. Modugno, A. del Campo, D. Guery-Odelin, A. Ruschhaupt, Xi Chen, and J. G. Maga, Adv. At. Mol. Opt. Phys. 62, 117 (2013).
[31] R. Chaunsali, E. Kim, A. Thakkar, P.G. Kevrekidis, and J. Yang, Phys. Rev. Lett. 119, 024301 (2017).
[32] See Supplemental Material at ... for the definition of fidelity, adiabatic invariant and details of the optimal control methods.
[33] We note that for a non-adiabatic process this leads to possible amplification.
[34] M. Devaud, V. Leny, J-C Bacri and T. Hocquet, Eur. J. Phys. 29, 831 (2008).
[35] We note that other choices of characteristic frequencies ωs of the system, eg. taking highest or lower frequency, or edge mode frequency do not change the characteristics analyzed for this quantity.
[36] J.-L. Wu and S. Zhang, Sci. Rep. 7, 46255 (2017).
[37] E. H. Lieb, and D. W. Robinson, Commun. Math. Phys. 28, 251 (1972).