Elliptic \((N,N')\)-Soliton Solutions of the lattice KP Equation

Sikarin Yoo-Kong\(^{1,\ast,1}\), Frank Nijhoff\(^{1,2}\)
\(^{1}\)Department of Physics, King Mongkut’s University of Technology Thonburi, Thailand, 10140.
\(^{\ast}\)King Mongkut’s University of Technology Thonburi Ratchaburi Campus, Thailand, 70150.
\(^{1}\)School of Mathematics, Department of Applied Mathematics, University of Leeds, United Kingdom, LS2 9JT.
\(^{1}\)syookong@gmail.com, \(^{2}\)nijhoff@maths.leeds.ac.uk

November 24, 2011

Abstract

Elliptic soliton solutions, i.e., a hierarchy of functions based on an elliptic seed solution, are constructed using an elliptic Cauchy kernel, for integrable lattice equations of Kadomtsev-Petviashvili (KP) type. This comprises the lattice KP, modified KP (mKP) and Schwarzian KP (SKP) equations as well as Hirota’s bilinear KP equation, and their successive continuum limits. The reduction to the elliptic soliton solutions of KdV type lattice equations is also discussed.

1 The lattice KP and Hirota equations

The study of the discrete versions of soliton systems, i.e., systems given by integrable partial difference equations, have become in recent years a focus of attention in the theory of integrable systems. Among those systems, the discrete analogue of Kadomtsev-Petviashvili (KP) equations which define in three dimensional lattice, seem to form a universal class of systems. The first equation of this type was found by Hirota in [12] and was referred to as DAGTE (Discrete analogue of generalised Toda equation) which is the bilinear equation

\[
(a_1 e^{D_1} + a_2 e^{D_2} + a_3 e^{D_3}) \tau \cdot \tau = 0,
\]

(1.1)

where the Hirota operators \(D_i\) produce finite forward-and backward shifts, when acting on a pair of functions, in the corresponding lattice direction, i.e.,

\[
e^{D_i} f \cdot g = f(n_1 + 1, \ldots, \ldots) g(n_1 - 1, \ldots) .
\]

Special reductions of this equation, are obtained when the coefficients \(a_1, a_2, a_3\) satisfy the condition \(a_1 + a_2 + a_3 = 0\), but the full equation is integrable in the sense of multidimensional consistency for arbitrary parameters, see [5]. Miwa [18] reparametrized the equation in that restricted case, and hence in that form it is often referred to as Hirota-Miwa equation. In this paper we will investigate a class of solutions of this and related three-dimensional lattice equations, comprising the following equations:

\(^{1}\)Sometimes the full equation (representing the bilinear discrete KP equation) is also (in our view erroneously) referred to as the Hirota-Miwa equation. In fact, in [18] only the restricted case was considered, and generalized to a four-term equation which is nowadays referred to as the Miwa equation.
The bilinear lattice KP equation
\[ \mathcal{J}f + \mathcal{J}\tilde{f} - \mathcal{J}\mathcal{J}f = 0. \] (1.2)

The lattice KP equation
\[ (\mathcal{w} - \mathcal{w})(\mathcal{w} - \tilde{\mathcal{w}}) = (\mathcal{w} - \tilde{\mathcal{w}})(\tilde{\mathcal{w}} - \mathcal{w}). \] (1.3)

The lattice modified KP equation
\[ \hat{v} - \tilde{v} + \mathcal{w} - \mathcal{w} + \mathcal{w} - \hat{\mathcal{w}} = 0. \] (1.4)

The asymmetric modified KP
\[ \hat{V}_0 - \tilde{V}_0 + \mathcal{V}_0 - \mathcal{V}_0 + \mathcal{V}_0 - \hat{\mathcal{V}}_0 = 0. \] (1.5)

The lattice Schwarzian KP
\[ \frac{(z - \tilde{z})(\tilde{z} - \hat{z})(\hat{z} - z)}{(z - \tilde{z})(\tilde{z} - \hat{z})(\hat{z} - z)} = 1. \] (1.6)

The notation in (1.2)-(1.6) is as follows: all dependent variables are functions defined on the multidimensional lattice with discrete coordinates \( (n, m, h) \in \mathbb{Z}^3 \), e.g., \( f = f(n, m, h) \), and the elementary shifts in the three discrete directions are denoted by \( \hat{f} = f(n + 1, m, h) \), \( \tilde{f} = f(n, m + 1, h) \) and \( \mathcal{f} = f(n, m, h + 1) \). The first equation (1.2) is Hirota’s DAGTE after a change of independent variables and a point transformation. The other KP type lattice equations (1.3)-(1.6) were established in [22] in a form containing parameters but which are equivalent to the normalised forms given above by point transformations. In particular, the Schwarzian KP (1.6) was first given in this normalised form in [10], and was subsequently recovered in [6], whilst its geometric significance was explored in [15, 14, 9]. All equations (1.3)-(1.6) appear as distinct parameter choices of a generalized lattice KP equation given in [22], whilst the (1.2) can be viewed as a potential equation for the asymmetric lattice MKP equation. With regard to terminology, we prefer to reserve the name lattice (potential) KP equation for (1.3) rather than for (1.2) or for (1.1) (as is commonly done in the literature), because it is actually the former equation that is more directly related to the actual continuous KP equation whilst the latter two equations are more related to the bilinear form of the KP equation. Similarly, the equations (1.4) and (1.6) can be shown to be related directly to the continuous (potential) modified and the Schwarzian KP respectively through continuum limits, cf. [22, 21, 31], hence their names. It was recently established in [5] that this list of five canonical equations exhausts all possible cases of octahedral type lattice equations, up to equivalence, which are multidimensionally consistent when embedded in the four dimensional lattice.

Some of these equations have been quite widely studied. In particular, eq. (1.1) has arisen in many contexts over the last few years, notably in connection with affine Weyl group description of discrete Painlevé equations, [25], and in connection with the representation theory of Kac-Moody algebras [13]. An interesting connection was found with the spectrum of Bethe Ansatz states of quantum solvable models, the eigenvalues of which were
shown to obey specific versions of Hirota’s bilinear equation, cf. [17, 32]. Furthermore, related to this is a connection found in [24] between, on the one hand, pole-type (in the sense of Airault, McKea and Moser [1]) and soliton solutions of the lattice KP equation and a class of discrete-time many-body systems which effectively are given rise to integrable correspondences (i.e., multi-valued dynamical maps) which are time-discretizations of the Ruijsenaar’s model [20, 27]. On the other hand the equations describing these same classical discrete dynamical systems coincide exactly with the Bethe-Ansatz equations for the excitations of quantum integrable models solvable by the quantum inverse scattering transform. It seems from all these intriguing connections that the lattice KP equations exhibit a certain universality in their connection with various types of integrable models involving partial difference equations, classical many-body systems and quantum solvable systems, a full picture of which is not yet understood at this juncture.

In this paper, we consider a novel class of solution of these lattice equations namely soliton type solutions based on an elliptic seed solution. The construction exploits elliptic $N \times N'$ Cauchy-matrices, and involve also $N' \times N$ coefficient matrices, which in principle allows for a classification of the solutions according to the Schubert decomposition of a corresponding Grassmannian. Thus, our main result, which provides a general expression for what we accordingly call the elliptic $(N,N')$-soliton solutions, could form a basis for a similar analysis as was performed in the case of the continuous KP equation by Kodama and Chakravarty, [16], to describe the soliton taxonomy of the KP equation. Furthermore, our approach yields the various Miura type relations between the different equations and allows to study the various intermediate continuum limits, as well as reductions to lower dimensional equations, in a systematic way.

## 2 Cauchy matrix scheme

We will develop now a scheme along the lines of the paper [19, 20] where Atkinson and one of the present authors developed the elliptic soliton solutions for the ABS (Adler/Bobenko/Suris, cf. [1]) list of two-dimensional lattice equations, based on elliptic Cauchy matrices. That construction provided solutions for all equations of the ABS list up to $Q_3$, whereas the case of $Q_4$ was treated separately using a different approach, cf. [3].

In this Section we derive the basic relations, and in the next Section we will use these relations to find a general elliptic $(N,N')$-soliton solution for KP.

### 2.1 Basic ingredients

At this point we find it useful to introduce the Lamé function

$$\Psi_\chi(\kappa) =: \Phi_\chi(\kappa) e^{-\zeta(\xi)\kappa}, \quad (2.1)$$

where

$$\Phi_\chi(\kappa) = \frac{\sigma(\kappa + \xi)}{\sigma(\xi)\sigma(\kappa)}, \quad (2.2)$$

with $\sigma$ is the sigma function and we have included an exponential factor together with the $\Phi$-function, breaking the symmetry between the argument of the function and the suffix. Although most of the results of this paper can be obtained in terms of the $\Phi$-function along i.e. without adopting the exponential factor, the exponential factor does provide a certain regularisation of the functions involved which may affect their analytic behaviour. The basic identities for the $\Psi$ function are the following:

$$\begin{align*}
\Psi_\xi(\kappa)\Psi_\delta(\lambda) &= e^{\eta_\kappa \xi}\Psi_{\xi+\delta}(\kappa)\Psi_{\xi}(\lambda-\kappa) + e^{\eta_\kappa \lambda}\Psi_{\xi}(\kappa-\lambda)\Psi_{\xi+\delta}(\lambda), \quad (2.3a) \\
\Psi_\xi(\kappa)\Psi_\delta(\kappa) &= e^{\eta_\kappa \xi}\Psi_{\xi+\delta}(\kappa) \left[\zeta(\xi) + \zeta(\delta) + \zeta(\kappa) - \zeta(\xi + \delta + \kappa)\right], \quad (2.3b) \\
\Psi_\xi(\kappa)\Psi_\xi(\lambda) &= \Psi_\xi(\kappa + \lambda) \left[\zeta(\xi) + \zeta(\kappa) + \zeta(\lambda) - \zeta(\xi + \kappa + \lambda)\right], \quad (2.3c)
\end{align*}$$

\footnote{The inclusion of the exponential factor amounts to a specific gauge transformation on the quantities defined later on in the construction, and hence could be removed without affecting the main results.}
in which we have introduced
\[ \eta_s = \eta_s(\xi) = \zeta(\xi + \delta) - \zeta(\xi) - \zeta(\delta) = \frac{1}{2} \frac{\psi'(\xi) - \psi'(\delta)}{\psi(\xi) - \psi(\delta)}. \] (2.4)

Furthermore, we have the symmetry: \( \Psi_{\delta}(-\kappa) = -\Psi_{-\delta}(\kappa). \)

The starting point for our construction is the “bare” non-autonomous Cauchy matrix
\[ M^0 = (M^0_{i,j})_{i,j=1,...,N}, \quad M^0_{i,j}(\xi) = \Psi_\xi(\kappa_i + \kappa_j'), \] (2.5)
depending on a variable \( \xi \) depending linearly on independent variables \( n,m \), namely \( \xi = \xi_0 + n\delta + m\epsilon + h\lambda \), with \( \delta, \epsilon \) and \( \lambda \) being the corresponding lattice parameters. We will assume that the set rapidity parameters \( \{\kappa_i, i = 1, \ldots, N\} \) is such that \( \kappa_i + \kappa_j' \neq 0 \) (modulo the period lattice of the \( \sigma \)-function).

Furthermore, defining
\[ p_\kappa = \Psi_\delta(\kappa), \quad q_\kappa = \Psi_\epsilon(\kappa), \quad l_\kappa = \Psi_\lambda(\kappa), \] (2.6)
and setting \( \kappa = \pm \kappa_i, \kappa' = \pm \kappa_j', \) we can derive from the basic addition formula \( 2.3a \) the following dynamical properties of the elliptic Cauchy matrix:
\[ M^0_{i,j}p_{k'+j} = \Psi_\xi(\kappa_i + \kappa_j')\Psi_\delta(\kappa_j'), \]
\[ = e^{\eta_i(\kappa_i + \kappa_j')}\Psi_{\delta+i}(\kappa_i + \kappa_j') + e^{\eta_i(\kappa_j')}\Psi_{\delta+\kappa_j'}(\kappa_i)\Psi(\kappa_i) \]
\[ = \tilde{M}^0_{i,j}(p_\kappa, \kappa_j') + e^{\eta_i(\kappa_j')}\Psi_{\delta+\kappa_j'}(\kappa_i)\Psi(\kappa_i) \].

We introduce now the plane-wave factors (i.e., discrete exponential functions)
\[ \rho(\kappa) = \left( e^{-\zeta(\delta)\kappa} p_{-\kappa} \right)^n \left( e^{-\zeta(\epsilon)\kappa} q_{-\kappa} \right)^m \left( e^{-\zeta(\lambda)\kappa} l_{-\kappa} \right)^h e^{\zeta(\xi)\kappa} \rho_{0,0,0}(\kappa), \]
\[ \nu(\kappa') = \left( e^{\zeta(\eta)\kappa'} p_{\kappa'} \right)^{-n} \left( e^{\zeta(\eta)\kappa'} q_{\kappa'} \right)^{-m} \left( e^{\zeta(\eta)\kappa'} l_{\kappa'} \right)^{-h} e^{\zeta(\xi)\kappa'} \rho_{0,0,0}(\kappa'), \]
and for the specific values \( \kappa = \kappa_i, \rho_i := \rho(\kappa_i), \kappa' = \kappa_j', \nu_j := \rho(\kappa_j') \) obeying the shift relations
\[ \frac{\rho_i}{\rho_j} = e^{\eta_i(\kappa_i) p_{-\kappa_i}}, \quad \frac{\rho_i}{\rho_j} = e^{\eta_i(\kappa_j) q_{-\kappa_j}}, \quad \frac{\rho_i}{\rho_j} = e^{\eta_i(\kappa_j) l_{-\kappa_j}}, \]
\[ \frac{\nu_i}{\nu_j} = e^{\eta_i(\kappa_j') (p_{\kappa_j'})^{-1}}, \quad \frac{\nu_i}{\nu_j} = e^{\eta_i(\kappa_j') (q_{\kappa_j'})^{-1}}, \quad \frac{\nu_i}{\nu_j} = e^{\eta_i(\kappa_j') (l_{\kappa_j'})^{-1}}, \]
where the superscripts “ \( \sim \)”, “ \( \sim \)” and “ \( \sim \)” denote the lattice shifts related to the shift by one unit in the variables \( n, m \) and \( h \) respectively, making use also of the relations \( \xi = \xi + \delta, \xi = \xi + \epsilon \) and \( \xi = \xi + \lambda \) which sit inside the coefficients \( \eta, \eta_\epsilon \) and \( \eta_\lambda \).

Now we can introduce the \( N \)- and \( N' \)-component vectors
\[ r = (\rho_i \Psi(\kappa_i))_{i=1,...,N}, \quad s = (\nu_j \Psi(\kappa_j'))_{j=1,...,N'}, \]
(2.11)
where \( \rho_i = \rho_{n,m,h}(\kappa_i) \) and \( \nu_j = \nu_{n,m,h}(\kappa_j') \) in terms of which we can define now the “dressed” Cauchy matrix:
\[ M = (M_{i,j})_{i,j=1,...,N}, \quad M_{i,j} = \rho_i M^0_{i,j} \nu_j, \]
(2.12)
As a consequence of the relations given earlier, and employing the definitions of the plane-wave factors, we can now describe the discrete dynamics as follows:

**Lemma 2.0.1.** The dressed Cauchy matrix \( M \), as defined in \( 2.12 \), obeys the following linear relations under elementary shifts of the independent variables \( n \)
\[ \tilde{M} = M - r s^T, \]
(2.13)
and under shifts of the variable in the similar relations:

\[ \widetilde{M} = M - r \tilde{s}^T, \]

and under shifts of the variable \( h \) the similar relations:

\[ \overline{M} = M - r \overline{r}^T, \]

where, as before, \( \widetilde{M}, \overline{M} \) and \( \overline{M} \) denote the shifted Cauchy matrices.

In what follows we will employ the relations (2.13), (2.14) and (2.15) to obtain nonlinear shift relations for specific objects defined in terms of the Cauchy matrix \( M \).

## 2.2 The \( \tau \)-function and related basic objects

Introduce now the \( \tau \)-function:

\[ \tau = \tau_{n,m,h} = \det_{N \times N} (1 + MC) = \det_{N' \times N'} (1 + CM), \]  

where, since \( M \) is in general not a square matrix, the \( N' \times N \) constant matrix \( C \) is introduced to compensate for the discrepancy. The matrix \( 1 \) is either the \( N \times N \) or \( N' \times N' \) unit matrices respectively in both determinants. The latter identity is a consequence of the general Weinstein-Aronszajn formula:

\[ \det_{N \times N} \left( 1 + \sum_{i=1}^{N'} m_i c_i \right) = \det_{N' \times N'} (1 + c'_i m_k), \]  

where \( m_i = (M_{i,j})_{i=1,...,N} \) and \( c'_i = (C_{i,j})_{j=1,...,N} \) are the \( N' \) column-resp. row vectors from the matrices \( M \) and \( C \).

From the definition of the \( \tau \)-function and the relations for the dressed Cauchy matrix \( M \) we obtain:

\[ \tilde{\tau} = \det_{N \times N} \left( 1 + \tilde{M} C \right) = \det_{N \times N} \left\{ 1 + (M - r \tilde{s}^T)C \right\} \]

\[ = \det_{N \times N} \left\{ (1 + MC) \left[ 1 - (1 + MC)^{-1} r \tilde{s}^T C \right] \right\} \]

\[ = \tau \det_{N \times N} \left\{ \left[ 1 - (1 + MC)^{-1} r \tilde{s}^T C \right] \right\}. \]

Introducing

\[ \chi_{\alpha,\beta} = \chi_{\alpha,\beta}(\xi) \equiv \zeta(\alpha) + \zeta(\beta) + \zeta(\xi) - \zeta(\xi + \alpha + \beta), \]  

and using the fact that

\[ \tilde{s} = e^{\mu K'} (p_{K'})^{-1} \frac{\Psi_k(K')}{\Psi_k(K)} s = [\chi_{\delta_{k},k'}]^{-1} s, \]

where \( p_{K'} \) is the diagonal matrix with entries \( p_{\kappa_j} \) and \( K' = \text{diag}(\kappa'_1, ..., \kappa'_N) \). Here and in what follows the notation \( \chi_{\delta_{k},k'} \) denotes the diagonal matrix with entries \( \chi_{\delta_{k},k'}, (j = 1, ..., N') \).

Now we have

\[ \tilde{\tau} = 1 - s^T [\chi_{\delta_{k},k'}]^{-1} C [1 + MC]^{-1} r = W_3. \]

The reverse fraction of Eq. (2.20) can be computed by processing the same computation

\[ \tau = \det_{N \times N} (1 + MC) = \det_{N \times N} \left\{ 1 + (\tilde{M} + r \tilde{s}^T)C \right\} \]

\[ = \det_{N \times N} \left\{ (1 + \tilde{M} C) \left[ 1 + (1 + \tilde{M})^{-1} r \tilde{s}^T C \right] \right\} \]

\[ = \tilde{\tau} \det_{N \times N} \left\{ \left[ 1 + (1 + \tilde{M})^{-1} r \tilde{s}^T C \right] \right\}.

5
Using also the fact that
\[ \tilde{r} = e^\gamma K p_{-K} \frac{\Psi(K)}{\Psi(K)} r = -[\tilde{\chi}_{-\delta, K}] r, \] (2.21)
where \( K = \text{diag}(\kappa_1, \ldots, \kappa_N) \) and then we have
\[ \tau \tilde{r} = 1 - \tilde{s}^T \left[ 1 + \tilde{MC} \right]^{-1} C [\tilde{\chi}_{-\delta, K}]^{-1} \tilde{r} = \tilde{V}_{-\delta}. \] (2.22)

For arbitrary parameter \( \alpha \) we introduce
\[ W_\alpha = 1 - s^T [\chi_\alpha, K]^{-1} C \left[ 1 + MC \right]^{-1} r, \] (2.23)
\[ V_\alpha = 1 - s^T [1 + CM]^{-1} C [\chi_\alpha, K]^{-1} r, \] (2.24)
where we conclude that
\[ \frac{\tau}{\tau} = W_\delta = \frac{1}{V_{-\delta}}. \] (2.25)

We will now proceed the derivation of the dynamical relations of \( W_\delta \) and \( V_\delta \).

### 2.3 Basic linear relations

In order to derive relations for the objects \( V_\alpha \) and \( W_\alpha \) we need to introduce the \( N \)- and \( N' \)-component column- resp. row vectors:
\[ \begin{align*}
    u_\alpha &= (1 + MC)^{-1} (\chi_\alpha, K)^{-1} r, \quad (2.26a) \\
    ^t u_\beta &= s^T (\chi_\beta, K)^{-1} (1 + CM)^{-1}. \quad (2.26b)
\end{align*} \]

Using these equations, we can write the functions \( V \) and \( W \) in the forms
\[ \begin{align*}
    V_\alpha &= 1 - s^T C u_\alpha, \quad (2.27a) \\
    W_\beta &= 1 - ^t u_\beta C r. \quad (2.27b)
\end{align*} \]

Performing the following calculation:
\[ \begin{align*}
\tilde{u}_\alpha &= (1 + \tilde{MC})^{-1} (\tilde{\chi}_\alpha, K)^{-1} \tilde{r} = (1 + \tilde{MC})^{-1} (\tilde{\chi}_\alpha, K)^{-1} e^\gamma K \frac{\Psi(K)}{\Psi(K)} \tilde{r} \\
\Rightarrow \quad (1 + \tilde{MC}) \tilde{u}_\alpha &= \frac{\Phi(K)}{\Psi(K)} \tilde{r}, \\
\Rightarrow \quad \left( 1 + MC \right) - r \tilde{s}^T C \tilde{u}_\alpha &= - \frac{\zeta(K)}{\zeta(K)} \left( \tilde{\zeta}(\xi + \alpha) - \zeta(\delta) - \zeta(K + \xi + \alpha) \right) r \\
&= \left( -1 + \frac{\zeta(\xi + \alpha) - \zeta(\xi) + \zeta(\xi) + \zeta(\xi + \alpha) - \zeta(K + \xi + \alpha)}{\zeta(K) + \zeta(\xi) + \zeta(\xi + \alpha) - \zeta(K + \xi + \alpha)} \right) r
\end{align*} \]
Multiplying both sides by \( (1 + MC)^{-1} \) and introducing the vector
\[ u_0 \equiv (1 + MC)^{-1} r, \] (2.28)
we get the relation
\[ \tilde{u}_\alpha = -u_0 \tilde{V}_\alpha + \chi_{\alpha, \delta} u_\alpha. \] (2.29)

A similar set of relations can be derived for the adjoint vectors \( \tilde{u}_\alpha \), which involves the adjoint vector to \( \tilde{u}_0 \), namely
\[ ^t u_0 \equiv s^T (1 + CM)^{-1}, \] (2.30)
and obviously these relations all have their counterparts involving the other lattice shift related to shifts in the discrete independent variables \( m \) and \( n \) instead of \( n \).

Summarising the results of these derivations, we have the following statement:
Lemma 2.0.2. The $N$- and $N'$-component vectors given in (2.26), together with the ones given in (2.28) and (2.30) obey the following set of linear shift relations

\[ \tilde{u}_\alpha = -u_0 V_\alpha + \chi_{\alpha,\delta} u_\alpha, \quad (2.31a) \]
\[ \tilde{u}_\beta = W_\beta \tilde{u}_0 - \chi_{\beta,\delta} \tilde{u}_\delta, \quad (2.31b) \]

and a similar set of relations involving the shifts in the variable $w$ is given

\[ \tilde{u}_\alpha = -u_0 \tilde{V}_\alpha + \chi_{\alpha,\epsilon} u_\alpha, \quad (2.32a) \]
\[ \tilde{u}_\beta = W_\beta \tilde{u}_0 - \chi_{\beta,\epsilon} \tilde{u}_\epsilon, \quad (2.32b) \]

and a similar set of relations involving the shifts in the variable $h$ is given

\[ \tilde{u}_\alpha = -u_0 \tilde{V}_\alpha + \chi_{\alpha,\lambda} u_\alpha, \quad (2.33a) \]
\[ \tilde{u}_\beta = W_\beta \tilde{u}_0 - \chi_{\beta,\lambda} \tilde{u}_\lambda. \quad (2.33b) \]

2.4 Basic nonlinear relations

The nonlinear relations can be obtained by multiply $s^T C$ on the left hand side of (2.31a). We have

\[ s^T C \tilde{u}_\alpha = -s^T C u_0 \tilde{V}_\alpha + \chi_{\alpha,\delta} s^T C u_\alpha \]
\[ \tilde{s}^T \chi_{\alpha, K} C \tilde{u}_\alpha = -w_0 \tilde{V}_\alpha + \chi_{\alpha,\delta}(1 - V_\alpha), \quad (2.34) \]

where $w_0 = s^T C u_0$. The same relations can be obtained in the same way for other shift directions

\[ \tilde{s}^T \chi_{\alpha, K} C \tilde{u}_\alpha = -w_0 \tilde{V}_\alpha + \chi_{\alpha,\epsilon}(1 - V_\alpha), \quad (2.35) \]
\[ \tilde{s}^T \chi_{\alpha, K} C \tilde{u}_\alpha = -w_0 \tilde{V}_\alpha + \chi_{\alpha,\lambda}(1 - V_\alpha). \quad (2.36) \]

Combining (2.34) and (2.35), we obtain

\[ \tilde{V}_\alpha(\tilde{w} - \tilde{w}) = \tilde{\chi}_{\alpha,\delta} \tilde{V}_\alpha - \tilde{\chi}_{\alpha,\epsilon} \tilde{V}_\epsilon \]

where $w = \zeta(\xi) - w_0 - \eta(\delta) - m(\epsilon) - h(\gamma)$.

Similarly, multiplying the right hand side of (2.31b) with $C \tilde{v}$, we obtain

\[ W_\beta(\tilde{w} - \tilde{w}) = \tilde{\chi}_{\beta,\delta} W_\delta - \tilde{\chi}_{\beta,\epsilon} W_\epsilon. \quad (2.38) \]

We now can write $\tilde{w} - \tilde{w}$ in the following

\[ (\tilde{w} - \tilde{w}) = \frac{1}{\tilde{V}_\alpha} \left( p_{\alpha} \tilde{V}_\alpha - q_{\alpha} \tilde{V}_\alpha \right) = \frac{1}{W_\beta} \left( p_{-\beta} \tilde{W}_\beta - q_{-\beta} \tilde{W}_\beta \right). \quad (2.41) \]

This equation has its counterpart involving the other lattice directions namely

\[ (\tilde{w} - \tilde{w}) = \frac{1}{\tilde{V}_\alpha} \left( l_{\alpha} \tilde{V}_\alpha - q_{\alpha} \tilde{V}_\alpha \right) = \frac{1}{W_\beta} \left( l_{-\beta} \tilde{W}_\beta - q_{-\beta} \tilde{W}_\beta \right). \quad (2.42a) \]
\[ (\tilde{w} - \tilde{w}) = \frac{1}{\tilde{V}_\alpha} \left( p_{\alpha} \tilde{V}_\alpha - l_{\alpha} \tilde{V}_\alpha \right) = \frac{1}{W_\beta} \left( p_{-\beta} \tilde{W}_\beta - l_{-\beta} \tilde{W}_\beta \right). \quad (2.42b) \]
Using (2.41), (2.42a) and (2.42b), we have
\[
\frac{p_\alpha \hat{V}_\alpha - q_\alpha \widetilde{V}_\alpha}{\hat{V}_\alpha} + \frac{q_\alpha \hat{V}_\alpha - l_\alpha \widetilde{V}_\alpha}{\hat{V}_\alpha} + \frac{l_\alpha \hat{V}_\alpha - p_\alpha \widetilde{V}_\alpha}{\hat{V}_\alpha} = 0 ,
\] (2.43)
or equivalently
\[
\frac{p_\beta \hat{W}_\beta - q_\beta \widetilde{W}_\beta}{\hat{W}_\beta} + \frac{q_\beta \hat{W}_\beta - l_\beta \widetilde{W}_\beta}{\hat{W}_\beta} + \frac{l_\beta \hat{W}_\beta - p_\beta \widetilde{W}_\beta}{\hat{W}_\beta} = 0 .
\] (2.44)
These two equations are the “modified lattice KP”.

Taking \( \alpha = -\delta \), (2.43) becomes
\[
\frac{q_\delta \hat{V}_{-\delta} - l_\delta \hat{V}_{-\delta}}{\hat{V}_{-\delta}} + \frac{l_\delta \hat{V}_{-\delta} - q_\delta \hat{V}_{-\delta}}{\hat{V}_{-\delta}} = 0 .
\] (2.45)
Taking \( \beta = \delta \), (2.44) becomes
\[
\frac{q_\delta \hat{W}_\delta - l_\delta \hat{W}_\delta}{\hat{W}_\delta} + \frac{l_\delta \hat{W}_\delta - q_\delta \hat{W}_\delta}{\hat{W}_\delta} = 0 .
\] (2.46)
(2.45) and (2.46) are the “asymmetric modified KP”.

Furthermore, for \( \alpha = -\delta \), we find that
\[
(\hat{w} - \hat{\tilde{w}}) = -q_{-\delta} \hat{V}_{-\delta}/\hat{V}_{-\delta} ,
\] (2.47)
\[
(\tilde{w} - \hat{w}) = -l_{-\delta} \hat{V}_{-\delta}/\hat{V}_{-\delta} .
\] (2.48)
The combination of these two equations yields
\[
(\hat{w} - \hat{\tilde{w}})(\tilde{w} - \hat{w}) = (\tilde{w} - \hat{\tilde{w}})(\hat{w} - \tilde{w}) ,
\] (2.49)
which is the “lattice KP equation”.

Using (2.25), we can write (2.47) and (2.48) in the forms
\[
(\hat{w} - \hat{\tilde{w}}) = -q_{-\delta} \hat{\tau} \hat{\tau}/\hat{\tau} \hat{\tau} ,
\] (2.50)
\[
(\tilde{w} - \hat{w}) = -l_{-\delta} \hat{\tau} \hat{\tau}/\hat{\tau} \hat{\tau} .
\] (2.51)
From (2.42a), if we take \( \alpha = -\varepsilon \) we also have
\[
(\hat{w} - \tilde{w}) = -q_{-\varepsilon} \hat{\tau} \hat{\tau}/\hat{\tau} \hat{\tau} ,
\] (2.52)
The combination of (2.50), (2.51) and (2.52) gives
\[
l_{-\delta} \hat{\tau} \hat{\tau} + q_{-\varepsilon} \hat{\tau} \hat{\tau} - q_{-\delta} \hat{\tau} \hat{\tau} = 0 ,
\] (2.53)
which is actually the “Hirota’s DAGTE”. In the rational case, the summation of coefficients would add up to be zero. In the elliptic case, this condition is no longer to be the case.
2.5 Schwarzian KP variables

From the previous relations for $u_\alpha$ we can now proceed as follows

$$s^T (\chi_{\beta,K})^{-1} C (\chi_{\alpha,\delta} u_\alpha - \tilde{V}_\alpha u_0) = s^T \Phi_\alpha (\tilde{\kappa}) \Phi_\beta (\tilde{\kappa}) \frac{\Phi_\alpha (\tilde{\kappa}) \Phi_\beta (\tilde{\kappa})}{\Phi_\beta (\tilde{\kappa})} e^{-\eta_\kappa \kappa} p_\kappa \tilde{u}_\alpha$$

$$= s^T \Phi_\alpha (\tilde{\kappa}) \Phi_\beta (\tilde{\kappa}) \tilde{u}_\alpha = s^T \frac{\zeta (\tilde{\kappa}) + \zeta (\kappa + \beta) - \zeta (\tilde{\kappa} + K' + \beta)}{\zeta (\tilde{\kappa}) + \zeta (\kappa + \beta) + \zeta (\kappa + \beta + \kappa)} \tilde{u}_\alpha$$

$$= \chi_{\alpha,\delta} S_{\beta,\alpha} - (1 - W_\beta) \tilde{V}_\alpha,$$

from which we get the following relation:

$$W_\beta \tilde{V}_\alpha = 1 - \tilde{\chi}_{\beta,-\alpha} \tilde{S}_{\beta,\alpha} - \chi_{\alpha,\delta} S_{\beta,\alpha}, \quad (2.54)$$

where

$$S_{\beta,\alpha} = s^T [\chi_{\beta,K}]^{-1} C [1 + MC]^{-1} [\chi_{\alpha,K}]^{-1} r, \quad (2.55)$$

$$= s^T u_\beta C [\chi_{\alpha,K}]^{-1} r = s^T [\chi_{\beta,K}]^{-1} C u_\alpha. \quad (2.56)$$

Similarly we have

$$W_\beta \tilde{V}_\alpha = 1 - \tilde{\chi}_{\beta,-\alpha} \tilde{S}_{\beta,\alpha} - \chi_{\alpha,\delta} S_{\beta,\alpha}, \quad (2.57)$$

$$W_\beta \tilde{V}_\alpha = 1 - \tilde{\chi}_{\beta,-\alpha} \tilde{S}_{\beta,\alpha} - \chi_{\alpha,\delta} S_{\beta,\alpha}, \quad (2.57b)$$

Note that if we multiply $C [\chi_{\alpha,K}]^{-1} \tilde{r}$ from the right hand side of Eq. (2.55) the computation leads to Eq. (2.54).

Multiplying $\Phi_\alpha (\tilde{\kappa}) \Phi_\beta (\tilde{\kappa})$ to the left hand side of (2.54), we have

$$(\Phi_\alpha (\tilde{\kappa}) \tilde{V}_\alpha) (\Phi_\beta (\tilde{\kappa}) W_\beta) = -\Phi_\beta (\tilde{\kappa}) \Phi_\alpha (\tilde{\kappa}) \Phi_\beta (\tilde{\kappa}) S_{\beta,\alpha} - \Phi_\beta (\tilde{\kappa}) \Phi_\beta (\tilde{\kappa}) \Phi_\beta (\tilde{\kappa}) S_{\beta,\alpha}$$

$$+ \Phi_\alpha (\tilde{\kappa}) \Phi_\beta (\tilde{\kappa}) \Phi_\beta (\tilde{\kappa}) \Phi_\beta (\tilde{\kappa}) \Phi_\beta (\tilde{\kappa}) + \Phi_\alpha (\tilde{\kappa}) \Phi_\beta (\tilde{\kappa}) \Phi_\beta (\tilde{\kappa}) \Phi_\beta (\tilde{\kappa}) \Phi_\beta (\tilde{\kappa}) \Phi_\beta (\tilde{\kappa})$$

$$= \Phi_\alpha (\tilde{\kappa}) (-p_{\beta} + p_{\beta} \tilde{\chi}_{\alpha,\beta} S_{\beta,\alpha})$$

$$+ \Phi_\alpha (\tilde{\kappa}) (p_{\alpha} - p_{\alpha} \chi_{\alpha,\delta} S_{\beta,\alpha}). \quad (2.58)$$

Introducing the new variable

$$Z_{\beta,\alpha} = \Phi_\beta (\tilde{\kappa}) (1 - \chi_{\alpha,\delta} S_{\beta,\alpha}), \quad (2.59a)$$

we can write (2.58) in the form

$$\tilde{V}_\alpha W_\beta = p_{\alpha} Z_{\beta,\alpha} - p_{\beta} \tilde{Z}_{\beta,\alpha}. \quad (2.60)$$

Similarly we have

$$\tilde{V}_\alpha W_\beta = q_{\alpha} Z_{\beta,\alpha} - q_{\beta} \tilde{Z}_{\beta,\alpha}, \quad (2.61)$$

$$\tilde{V}_\alpha W_\beta = l_{\alpha} Z_{\beta,\alpha} - l_{\beta} \tilde{Z}_{\beta,\alpha}. \quad (2.62)$$

Using the identity

$$(\tilde{V}_\alpha W_\beta, \tilde{V}_\alpha W_\beta, \tilde{V}_\alpha W_\beta) = (\tilde{V}_\alpha W_\beta) (\tilde{V}_\alpha W_\beta) \quad (2.63)$$

we can derive the equation

$$\frac{(p_{\alpha} Z_{\beta,\alpha} - p_{\beta} \tilde{Z}_{\beta,\alpha})}{(q_{\alpha} Z_{\beta,\alpha} - q_{\beta} \tilde{Z}_{\beta,\alpha})} = \frac{(l_{\alpha} Z_{\beta,\alpha} - l_{\beta} \tilde{Z}_{\beta,\alpha})}{(l_{\alpha} Z_{\beta,\alpha} - l_{\beta} \tilde{Z}_{\beta,\alpha})} \frac{(p_{\alpha} \tilde{Z}_{\beta,\alpha} - p_{\beta} \tilde{Z}_{\beta,\alpha})}{(q_{\alpha} \tilde{Z}_{\beta,\alpha} - q_{\beta} \tilde{Z}_{\beta,\alpha})}, \quad (2.64)$$

9
which is the "Schwarzian lattice KP equation", first given the explicit form in [10].

**Remark:** Let \( \partial \) denote the derivative w.r.t any independent variable on which solely the \( \rho_i \) and all the \( \nu_j \) depend, but not any of the other ingredients in the elliptic soliton solutions. Hence \( \partial \) only acts on the \( \rho_i \) and \( \nu_j \). Assuming that the latter can be written in the following form:

\[
r = \Re e, \quad s^T = e^T S \quad \text{where} \quad e^T = (1, \ldots, 1).
\]

we can use the following relation for the Cauchy matrix

\[
\partial M = (\partial R) R^{-1} M + M S^{-1} (\partial S).
\]

to derive the following expression for the action of \( \partial \) on the main variable \( S_{\alpha,\beta} \)

\[
\partial S_{\alpha,\beta} = u_{\beta} [S^{-1} \partial S C + C \partial R R^{-1}] u_{\alpha}.
\]

(2.65)

This expression, in fact, is the analogue of the square eigenfunction expansion (in the sense of the seminal paper by Deift, Lund and Trubowitz of 1980, cf. [8]) of the elliptic soliton solution of a continuous KP equation, e.g. by choosing \( \partial \) to represent the partial derivative w.r.t. one of the continuous independent variables of the KP equation.

In the discrete case one can derive a similar equation to (2.65). Let "~" denote here the shift w.r.t any discrete variable on which solely the \( \rho_i \) and \( \nu_j \) depend. Using the ingredients of the previous case, one can derive the formula:

\[
\tilde{S}_{\alpha,\beta} - S_{\alpha,\beta} = u_{\beta} \left[ C \partial R R^{-1} - \tilde{S} S^{-1} C \right] u_{\alpha}.
\]

(2.66)

which constitutes the discrete analogue to (2.65). Note that the shift "~" may also represent a composite shift, or a combined shift w.r.t multiple variables, the derivation of (2.66) only uses the fact that the shift distributes over products of functions of the discrete variable by \( \tilde{f} \tilde{g} = \tilde{f} \tilde{g} \) and that the discrete variable enters the solutions via the \( \rho_i \) and \( \nu_j \). There is, however, an important difference between the formulae (2.65) and (2.66), namely that the left hand side of the latter involves also shifts of the eigenfunctions. We expect that these formulae may prove useful in the derivation of conservation laws for the discrete equations considered in this paper.

### 2.6 Hirota form of the elliptic \( N \)-soliton solution

In this Section, we will derive some explicit formulae for the soliton solutions in terms of the \( \tau \)-function, which allow us to study their properties. Using the fact that the matrix \( M \) is actually a Cauchy matrix, the \( \tau \)-function can be explicitly computed by using the expansion

\[
\tau = \det (1 + MC) = 1 + \sum_{i=1}^{N} |B_{i,i}| + \sum_{i<j} |B_{i,i} B_{j,j}| + \cdots + \det(B),
\]

where \( B = MC \).

**Lemma 2.0.3. Cauchy-Binet formula:** For an arbitrary \( N \times M \) matrix \( A \) and \( M \times N \) matrix \( B \) we have the following formula for the \( N \times N \) determinant of the product \([A,B]\):

\[
\det_{N \times N} (AB) = \begin{cases} 
0 & \text{if } M < N \\
\det(A) \det(B) & \text{if } M = N \\
\sum_{1 \leq l_1 \leq \ldots \leq l_N \leq M} \det \left( A_{(1,\ldots,N)}(l_1,\ldots,l_N) \right) \det \left( B_{(l_1,\ldots,l_N)(1,\ldots,N)} \right) & \text{if } M > N
\end{cases}
\]

in which \( A_{(1,\ldots,N)(l_1,\ldots,l_N)} \) denotes the matrix obtained by selecting the \( l_1,\ldots,l_N \) columns from the matrix \( A \) and \( B_{(l_1,\ldots,l_N)(1,\ldots,N)} \) is the matrix obtained by selecting the \( l_1,\ldots,l_N \) rows from the matrix \( B \).
Using Cauchy-Binet formula, we may express the \( \tau \)-function in the form

\[
\tau = 1 + \sum_{i=1}^{N} \left( \sum_{l}^{
N'} M_{il} C_{l} \right) + \sum_{i<j}^{N} \left( \sum_{l_{1}<l_{2}}^{
N'} \begin{array}{c|c|c|c} M_{il_{1}} & M_{il_{2}} \\ \hline M_{jl_{1}} & M_{jl_{2}} \\ \end{array} \right) \times C_{l_{1}} C_{l_{2}} C_{l_{1}j} C_{l_{2}j} \\
+ \sum_{i<j<s}^{N} \left( \sum_{l_{1}<l_{2}<l_{3}}^{
N'} M_{il_{1}} M_{il_{2}} M_{il_{3}} M_{jl_{1}} M_{jl_{2}} M_{jl_{3}} M_{kl_{1}} M_{kl_{2}} M_{kl_{3}} \right) \times C_{l_{1}} C_{l_{2}} C_{l_{3}} C_{l_{1}j} C_{l_{2}j} C_{l_{3}j} C_{l_{1}s} C_{l_{2}s} C_{l_{3}s} \\
+ \ldots + \sum_{l_{1}<l_{2}<\ldots<l_{N'}} \det M_{(1,2,\ldots,N')(l_{1},l_{2},\ldots,l_{N'})} \det C_{(l_{1},l_{2},\ldots,l_{N})(1,2,\ldots,N)}. \tag{2.67}
\]

where \( C_{ij} \) are the entries of the matrix \( C \) and \( M_{(1,2,\ldots,N')(l_{1},l_{2},\ldots,l_{N'})} \) denotes the matrix obtained by selecting the \((l_{1},l_{2},\ldots,l_{N'})\) columns from the matrix \( M \) and \( C_{(l_{1},l_{2},\ldots,l_{N})(1,2,\ldots,N)} \) is the matrix obtained by selecting the \((l_{1},l_{2},\ldots,l_{N'})\) rows from \( C \).

Using the Frobenius formula for the relevant elliptic Cauchy determinants. Thus, from

\[
\det \left( \rho_{i} \Phi_{\kappa_{i}+\kappa'_{i}}(\xi) \eta_{j} \right) = \left( \prod_{i} \rho_{i} \nu_{i} \right) \frac{\sigma(\xi + \sum_{j}(\kappa_{i} + \kappa'_{i}))}{\sigma(\xi)} \times \frac{\prod_{i<j} \sigma(\kappa_{i} + \kappa_{j}) \sigma(\kappa'_{i} + \kappa'_{j})}{\prod_{i<j} \sigma(\kappa_{i} - \kappa_{j})}. \tag{2.68}
\]

Introducing the notations

\[
e^{A_{i,j}} \equiv \sigma(\kappa_{i} - \kappa_{j}), \quad e^{\theta_{i}} = \rho_{i} e^{-\xi(\kappa_{i})}, \quad e^{E_{i,j}} \equiv \sigma(\kappa_{i}' - \kappa_{j}'), \quad e^{\eta_{i}} = \nu_{i} e^{-\xi(\kappa_{i})'},
\]

the Hirota formula for the \( \tau \)-function thus takes the form:

\[
\tau = 1 + \sum_{i=1}^{N} e^{\theta_{i}} \sum_{j=1}^{N'} e^{\eta_{j}} \left| C_{li}^{(1 \times 1)} \right| \frac{\sigma(\xi + \kappa_{i} + \kappa_{i}')}{{\sigma(\xi)} \sigma(\kappa_{i} + \kappa_{i}')} \times \frac{\sigma(\xi + \kappa_{i} + \kappa_{j} + \kappa_{i}')}{{\sigma(\xi)} \prod_{f=1}^{3} \sigma(\kappa_{i} + \kappa_{j}) \sigma(\kappa_{j} + \kappa_{i})} \\
+ \sum_{i<j}^{N} e^{\theta_{i} + \theta_{j} + A_{i,j}} \left| C_{l_{1},l_{2}}^{(2 \times 2)} \right| \frac{\sigma(\xi + \kappa_{i} + \kappa_{j} + \kappa_{i}')}{{\sigma(\xi)} \prod_{f=1}^{3} \sigma(\kappa_{i} + \kappa_{j}) \sigma(\kappa_{j} + \kappa_{i})} \\
+ \sum_{i<j<s}^{N} e^{\theta_{i} + \theta_{j} + A_{i,j} + A_{i,s} + A_{j,s}} \left| C_{l_{1},l_{2},l_{3}}^{(3 \times 3)} \right| \frac{\sigma(\xi + \kappa_{i} + \kappa_{j} + \kappa_{s} + \kappa_{i}')}{{\sigma(\xi)} \prod_{f=1}^{3} \sigma(\kappa_{i} + \kappa_{j}) \sigma(\kappa_{j} + \kappa_{i})} + \ldots,
\tag{2.69}
\]

where \( \left| C_{l_{1},l_{2},\ldots,l_{N'}}^{(N \times N)} \right| \) is the determinant of the \( N \times N \) matrix of the selected entries of \( C \).

The dependence of \( \tau \) in (2.69) on the discrete dynamical variables \( n, m, h \) comes only through the \( \theta_{i} \) and \( \eta_{j} \), which in turn depend on \( n, m, h \) via the plane wave factors (2.7) and (2.8).

### 3 Reduction to the lattice KdV equations

Dimensional reductions of the lattice KP systems can be obtained by imposing certain symmetry conditions. This process could be done by imposing the condition that \( T_{-\delta} \circ T_{\delta} = 1 \) for all lattice directions. This implies that the interchange \( \delta \rightarrow -\delta \) leads to a reversal of the lattice shift.
We now consider the plane-wave factors shifted in the "\(\sim\)" direction

\[ \tilde{\rho} = \Phi_\delta(-\kappa)\rho, \]  

\[ \tilde{\nu} = \Phi_\delta^{-1}(\kappa')\nu. \]  

(3.1) (3.2)

We find that

\[ T_{-\delta} \circ T_\delta(\rho\nu) = \frac{\varphi(\kappa) - \varphi(\delta)}{\varphi(\kappa') - \varphi(\delta)}\rho\nu. \]  

(3.3)

If we take \(\kappa = \kappa'\) we have \(T_{-\delta} \circ T_\delta(\rho\nu) = \rho\nu\).

Let’s consider the lattice KP (2.49),

\[ \hat{w} - \tilde{w} \hat{w} - \tilde{w} = \tilde{w} - \tilde{w}, \]  

(3.4)

Setting \(\lambda = -\delta\), we have \(T_\lambda \circ T_\delta w = \hat{w} = w \Rightarrow \tilde{w} = \tilde{w}\), where \(w = w(n-1, m)\), yielding

\[ (\tilde{w} - \hat{w})(w - \hat{w}) = \left[(\hat{w} - \tilde{w})(w - \hat{w})\right]. \]  

(3.5)

Similarly, setting \(\lambda = -\epsilon\), we have \(T_\lambda \circ T_\epsilon w = \hat{w} = w \Rightarrow \tilde{w} = \tilde{w}\), where \(w = w(n, m-1)\), yielding

\[ (\tilde{w} - \hat{w})(w - \hat{w}) = \left[(\hat{w} - \tilde{w})(w - \hat{w})\right]. \]  

(3.6)

This implies that the product on the right-hand sides of these relations is conserved in all directions, and then constant:

\[ (\tilde{w} - \hat{w})(w - \hat{w}) = \text{constant}, \]  

(3.7)

which is actually the elliptic version of the "lattice KdV" equation.

From (2.42a), if we set \(\lambda = -\delta\) we find that

\[ (\hat{w} - w)(w - \tilde{w}) = \left[(\hat{w} - w)(w - \tilde{w})\right]. \]  

(3.8)

Combining this equation with (2.42b) and using the lattice KdV, we obtain

\[ p_\alpha(V_\alpha \hat{V}_\alpha - \hat{V}_\alpha \hat{V}_\alpha) = q_\alpha(V_\alpha \hat{V}_\alpha - \hat{V}_\alpha \hat{V}_\alpha), \]  

(3.9)

which is the “lattice modified KdV” equation.

From (2.53), if we set \(\lambda = -\delta\) we get

\[ q_\beta \hat{\tau} + p_\beta \hat{\tau} = p_\beta \hat{\tau}. \]  

(3.10)

and if we set \(\lambda = -\epsilon\), we get

\[ p_\beta \hat{\tau} + q_\beta \hat{\tau} = q_\beta \hat{\tau}. \]  

(3.11)

(3.10) and (3.11) are the “lattice Hirota” equations.

From (2.62), if we set \(\lambda = -\delta\) we get

\[ V_\alpha \hat{W}_\beta = -p_\alpha \hat{Z}_{\beta,\alpha} + p_{-\beta} \hat{Z}_{\beta,\alpha}, \]  

(3.12)

and if we set \(\lambda = -\epsilon\) we get

\[ V_\alpha \hat{W}_\beta = -q_\alpha \hat{Z}_{\beta,\alpha} + q_{-\beta} \hat{Z}_{\beta,\alpha}. \]  

(3.13)
Using the identity:
\[
\frac{\tilde{V}_\alpha W_\beta}{V_\alpha W_\beta} = \frac{\tilde{V}_\alpha W_\beta}{V_\alpha W_\beta},
\]  
we obtain the equation for \( Z_{\beta,\alpha} \)
\[
\frac{p_\alpha Z_{\beta,\alpha} - p_{-\beta} \tilde{Z}_{\beta,\alpha}}{q_\alpha Z_{\beta,\alpha} - q_{-\beta} \tilde{Z}_{\beta,\alpha}} = \frac{q_{-\beta} \tilde{Z}_{\beta,\alpha} - q_\alpha \tilde{Z}_{\beta,\alpha}}{p_{-\beta} \tilde{Z}_{\beta,\alpha} - p_\alpha \tilde{Z}_{\beta,\alpha}},
\]
which is the “lattice Schwarzian KdV” equation.

The dimensional reductions described here to lattice equations of the KdV class are obtained of the basis of the explicit form for the solutions described in the earlier sections. We would like to mention at this point that in the recent paper \[2\], it was shown how to embed all equations in the ABS list in the Schwarzian KP equation \[1.6\], through the canonical definition of a “Schwarzian variable” associated with each member of the list.

## 4 The continuum limit

In previous Sections, we derive the fully discrete KP equations. The derivation of intermediate and full continuum limits of the lattice KP systems were studied in \[22, 23\], as well as \[27, 30, 31\] where the derivation of hierarchy arising from the lattice KP equations was systematically studied. We will consider continuum limit on the level of the elliptic soliton solution. We will give the detail only for the lattice KP \[2.49\] and present the final results for the other lattice KP equations.

We find out that it is more convenient to express the lattice KP equation \[2.49\] in the following form
\[
\left( \tilde{\pi}_0 - \tilde{\pi}_0 + \zeta(\xi + \epsilon + \lambda) - \zeta(\xi + \delta + \lambda) - \zeta(\epsilon) + \zeta(\delta) \right) = \left( \tilde{\omega}_0 - \tilde{\omega}_0 + \zeta(\xi + \epsilon) - \zeta(\xi + \delta) - \zeta(\epsilon) + \zeta(\delta) \right),
\]
where \( \omega_0 = s^T C(1 + MC)^{-1} \).

### First continuum limit: The skew limit.
We start to perform the continuum limit by making change of discrete variables \((n, m) \to (N = n + m, m)\) and then taking the limit
\[
\epsilon - \delta = \eta \to 0, \ n \to \infty, \ n \to -\infty, \ s.t. \ mn \to \tau \text{ and } N \text{ fixed}.
\]

The plane wave functions \[2.7\] and \[2.8\] become
\[
\rho(\kappa) = \left(e^{-\zeta(\delta)\kappa} \rho_{-\kappa}\right)^N e^{\zeta(\xi)\kappa + \zeta(\delta - \kappa)\tau} \rho_{0,0}(\kappa),
\]
\[
\nu(\kappa') = \left(e^{\zeta(\delta)\kappa'} \nu_{\kappa'}\right)^{-N} e^{\zeta(\xi)\kappa' + \zeta(\delta + \kappa')\tau} \nu_{0,0}(\kappa'),
\]
and \(w_0[n, m, h] \to w_0[N, \tau, h]\). Then \(w_0[N, \tau, h]\) satisfies the differential-difference equation
\[
\frac{\partial \tilde{\omega}_0}{\partial \tau} + \varphi(\xi + \delta + \lambda) - \varphi(\delta) = \frac{\tilde{\omega}_0 - \tilde{\omega}_0 + \zeta(\xi + \delta + \lambda) - \zeta(\xi + 2\delta) - \zeta(\lambda) + \zeta(\delta)}{\tilde{\omega}_0 - \tilde{\omega}_0 + \zeta(\xi + \lambda) - \zeta(\xi + \delta) + \zeta(\lambda) + \zeta(\delta)}. \tag{4.4}
\]

### The second limit. Next, we perform the limit on the discrete variable \(h\):
\[
\lambda \to 0, \ h \to \infty, \ s.t. \ \gamma = \lambda h \text{ fixed}.
\]
The plane wave functions (4.2) and (4.3) become

\[
\rho(\kappa) = \left( e^{-\zeta(\delta)\kappa} p_{-\kappa} \right)^N e^{\zeta(\xi)\kappa + \zeta(\delta-\kappa)\tau - \zeta(\kappa)\gamma} w_{0,0,0} (\kappa),
\]

\[
\nu(\kappa) = \left( e^{\zeta(\delta)\kappa} p_{\kappa} \right)^N e^{\zeta(\xi)\kappa - \zeta(\delta+\kappa)\tau + \zeta(\kappa)\gamma} w_{0,0,0} (\kappa),
\]

and \(w_0[N, \tau, h] \to w_0[N, \tau, \gamma]\). Then \(w_0[N, \tau, \gamma]\) now satisfies the partial differential-difference equation

\[
-\left( \frac{\partial w_0}{\partial \gamma} + \varphi'(\xi) \right) = \left( \frac{\partial w_0}{\partial \tau} + \varphi(\xi) - \varphi(\delta) \right) \left( \tilde{w}_0 - 2w_0 + w_0 + 2\zeta(\xi) - \zeta(\xi + \delta) - \zeta(\xi - \delta) \right).
\]  

The third limit: The Full limit. This limit can be obtained by investigating the limiting behavior of the plane wave function. From (4.5), we have

\[
\left( e^{-\zeta(\delta)\kappa} p_{-\kappa} \right)^N e^{\zeta(\xi)\kappa + \zeta(\delta-\kappa)\tau - \zeta(\kappa)\gamma} \\
\to \exp \left( N \ln |\kappa| - \ln |\kappa(\kappa)| + \zeta(\delta - \kappa)\tau - \zeta(\kappa)\gamma + \zeta(\xi)\kappa \right) \\
\to \exp \left( N \left( -\delta(\kappa) - \frac{\delta^2}{2} \varphi(\kappa) + \frac{\delta^3}{6} \varphi'(\kappa) + ... \right) - \zeta(\kappa)\gamma \right) \\
+ \tau \left( -\zeta(\kappa) - \delta \varphi(\kappa) - \frac{\delta^2}{2} \varphi'(\kappa) + ... \right) + \zeta(\xi)\kappa \right) \\
\to \exp \left( (\zeta(\xi)\kappa + \zeta(\kappa)\tau + \varphi(\kappa)\gamma + \varphi'(\kappa)\gamma t + ....) \right),
\]

where we define \(b = N\delta\) and

\[
x = -\delta - \tau - \gamma, \quad y = -\frac{\delta b}{2} - \tau \delta, \quad \text{and} \quad t = \frac{\delta^2 b}{6} - \frac{\delta^2 \tau}{2}.
\]

Applying the Taylor expansions, we have

\[
\tilde{w}_0 = w_0 + \frac{\delta w_0}{\partial b} + \frac{\delta^2 w_0}{\partial b^2} + \frac{\delta^3 w_0}{\partial b^3} + \frac{\delta^4 w_0}{24 \partial b^4} + ....,
\]

where \(b = b_0 + N\delta\) and using the chain rule formulae, we have

\[
\frac{\partial w_0}{\partial \gamma} = - \frac{\partial w_0}{\partial x},
\]

\[
\frac{\partial w_0}{\partial \tau} = - \frac{\partial w_0}{\partial x} - \frac{\delta w_0}{\partial y},
\]

\[
\frac{\partial^2 w_0}{\partial \tau \partial \gamma} = - \frac{\partial^2 w_0}{\partial x^2} + 2 \frac{\partial^2 w_0}{\partial x \partial y} + \frac{\partial^2 w_0}{\partial x^2}.
\]

Using (4.10) and (4.11), we recover the “potential KP” equation in order \(O(\delta^2)\):

\[
[w_0]_{xt} = 6[w_0]_x[w_0]_{xx} + \frac{3}{2}[w_0]_{yy} + \frac{1}{2}[w_0]_{xxxx},
\]

where \(w_0[N, \tau, \gamma] \to w_0[x, y, t]\).

At the end of this Section, we would like to mention that the continuum limit can be performed on the rest of KP equations namely the modified and Schwarzian KP equations. We will give a list of the results in the following

The discrete modified KP equation:

\[
\frac{\bar{\chi}_{\alpha, \delta} \bar{V}_\alpha - \bar{\chi}_{\alpha, \epsilon} \bar{V}_\alpha}{\bar{V}_\alpha} + \frac{\bar{\chi}_{\alpha, \epsilon} \bar{V}_\alpha - \bar{\chi}_{\alpha, \lambda} \bar{V}_\alpha}{\bar{V}_\alpha} + \frac{\bar{\chi}_{\alpha, \lambda} \bar{V}_\alpha - \bar{\chi}_{\alpha, \delta} \bar{V}_\alpha}{\bar{V}_\alpha} = 0,
\]

(4.13)
where \( V_\alpha = 1 - s^T C (1 + MC)^{-1} (\chi_{-\alpha} K)^{-1} r \). Taking the skew limit, we have the differential-difference equation

\[
\frac{\partial \tilde{V}_\alpha}{\partial \tau} \tilde{V}_\alpha \left( \tilde{V}_\alpha \tilde{\tilde{V}}_\alpha - \tilde{\tilde{\tilde{V}}}_\alpha \right) + \frac{\partial \tilde{V}_\alpha}{\partial \tau} \tilde{V}_\alpha \left( \tilde{\tilde{V}}_\alpha \tilde{\tilde{\tilde{V}}}_\alpha - \tilde{\tilde{\tilde{\tilde{V}}}}_\alpha \right) = 0 ,
\]

(4.14)

where \( V_\alpha[n, m, h] \to V_\alpha[N, \tau, h] \).

Taking another limit on the discrete variable \( h \), we obtain the partial differential-difference equation

\[
\frac{\partial^2 \tilde{V}_\alpha}{\partial \tau \partial \tau} = \frac{\partial \tilde{V}_\alpha}{\partial \tau} \tilde{V}_\alpha \left( \tilde{\tilde{V}}_\alpha \tilde{\tilde{V}}_\alpha - \tilde{\tilde{\tilde{V}}}_\alpha \right) \left( \tilde{V}_\alpha \tilde{\tilde{V}}_\alpha - \tilde{\tilde{\tilde{V}}}_\alpha \right) + \frac{\partial \tilde{V}_\alpha}{\partial \tau} \tilde{V}_\alpha \left( \tilde{\tilde{V}}_\alpha \tilde{\tilde{V}}_\alpha - \tilde{\tilde{\tilde{V}}}_\alpha \right) + \frac{\partial \tilde{V}_\alpha}{\partial \tau} \tilde{V}_\alpha \left( \tilde{\tilde{V}}_\alpha \tilde{\tilde{V}}_\alpha - \tilde{\tilde{\tilde{V}}}_\alpha \right) = 0 ,
\]

(4.15)

where \( \chi_\alpha = \zeta(\alpha) + \zeta(\xi + \delta) - \zeta(\xi + \alpha + \delta) \) and \( V_\alpha[N, \tau, h] \to V_\alpha[N, \tau, \gamma] \).

Taking the full limit, we obtain the modified KP equation

\[
[V_\alpha]_{xx} = [V_\alpha]_{xxx} + 3[V_\alpha]_{yy} - 6[V_\alpha]^2[V_\alpha]_{xx} - 6[V_\alpha]_y[V_\alpha]_{xx} ,
\]

(4.16)

where \( V_\alpha[x, y, t] \to V_\alpha(x, y, t) \).

The lattice Schwarzian KP equation:

\[
\frac{1 - \chi_{\alpha, \delta} S_{\beta, \alpha} - \tilde{\tilde{\tilde{S}}}_{\beta, \alpha}}{1 - \chi_{\alpha, \lambda} S_{\beta, \alpha} - \tilde{\tilde{\tilde{S}}}_{\beta, \alpha}} \left[ 1 - \chi_{\alpha, \lambda} \tilde{\tilde{S}}_{\beta, \alpha} - \tilde{\tilde{\tilde{\tilde{S}}}}_{\beta, \alpha} \right] \left[ 1 - \chi_{\alpha, \lambda} \tilde{\tilde{S}}_{\beta, \alpha} - \tilde{\tilde{\tilde{\tilde{S}}}}_{\beta, \alpha} \right] = 1 ,
\]

(4.17)

where \( S_{\beta, \alpha} = s^T [\chi_{\beta, K}]^{-1} C [1 + MC]^{-1} [\chi_{\alpha, K}]^{-1} r \). Taking the skew limit, we derive the differential-difference equation

\[
(\varphi(\tilde{\xi} + \alpha) - \varphi(\tilde{\xi})) \tilde{S}_{\beta, \alpha} + (\varphi(\tilde{\xi} + \beta) - \varphi(\tilde{\xi})) \tilde{S}_{\beta, \alpha} + \chi_{\alpha, \lambda} \frac{\partial \tilde{S}_{\beta, \alpha}}{\partial \tau} + \chi_{\beta, \lambda} \frac{\partial \tilde{S}_{\beta, \alpha}}{\partial \tau} = 0 ,
\]

(4.18)

where \( S_{\beta, \alpha}[n, m, h] \to S_{\beta, \alpha}[N, \tau, h] \).

Taking another limit on the discrete variable \( h \), we have the partial differential-difference equation

\[
[\varphi(\xi + \beta) + \varphi(\xi + \alpha) - 2\varphi(\xi)] S_{\beta, \alpha} + [\chi_{\alpha, \delta} + \chi_{\beta, \delta}] \frac{\partial S_{\beta, \alpha}}{\partial \tau} + \varphi(\xi + \beta) S_{\beta, \alpha} + \chi_{\alpha, \delta} \frac{\partial S_{\beta, \alpha}}{\partial \tau} = 0 ,
\]

(4.19)
where \( S_{\beta,\alpha}[N, \tau, h] \rightarrow S_{\beta,\alpha}[N, \tau, \gamma] \).

Taking the full limit, we obtain the Schwarzian KP equation

\[
\frac{\partial}{\partial x} \left( \frac{[S_{\beta,\alpha}]_t - [S_{\beta,\alpha}]_{xxx}}{[S_{\beta,\alpha}]_x} + \frac{2[S_{\beta,\alpha}]_y^2 - 3[S_{\beta,\alpha}]_x^3}{2[S_{\beta,\alpha}]_x^2} \right) + \frac{\partial}{\partial y} \left( \frac{[S_{\beta,\alpha}]_y}{[S_{\beta,\alpha}]_x} \right) = 0, \quad (4.20)
\]

where \( S_{\beta,\alpha}[N, \tau, h] \rightarrow S_{\beta,\alpha}[x, y, t] \).

### 5 Summary

In this paper, we established the explicit form of a class of elliptic soliton solutions for all the lattice KP equations, based on a construction using elliptic Cauchy matrices denoted by \( M \). Furthermore, the construction exhibits numerous relations between the various lattice equations, as well as corresponding Lax pairs. The explicit form for the corresponding \( \tau \)-function depends crucially on the coefficient matrix \( C \), which opens the way to classify the various different lattice soliton behaviours according to the Schubert decompositions of the corresponding Grassmannians, following similar work in the continuous case by Kodama and Chalravarty, [16]. Several reductions were considered: i) dimensional reduction to KdV lattice systems, and ii) continuum limits to the semidiscrete and fully continuous KP equations. For all these equations the corresponding elliptic soliton solutions are derived in parallel. Furthermore, the result in (4.19) can be simplified to the cases of trigonometric/hyperbolic by taking: \( \sigma(x) \mapsto \sin(x) \) or \( \sigma(x) \mapsto \sinh(x) \) and for the rational case we have \( \sigma(x) \mapsto x \).

There are well-established connections between soliton solution of integrable PDE and the integrable many body systems [26, 27]. This can be made most explicit in he rational and trigonometric/hyperbolic cases. The elliptic case of this correspondence is more difficult to establish, but we expect that the elliptic solitons which we have studied here can be connected to elliptic case of the discrete-time Ruijsenaars model constructed in [24], but at this juncture this is still conjectural. Nevertheless, in [28, 29], an explicit connection between the rational discrete-time RS system and the KP lattice was established using the solution structure of the former model.

### Acknowledgements

SYK was supported by the Royal Thai Government and FWN is supported by a Royal Society/Leverhulme Trust senior research fellowship.

### References

[1] Airault H, McKean H and Moser J, Rational and Elliptic Solutions fo the Kortewegde Vries Equation and a Related Many-Body Problem, Commun Pure Appl. Math, 30, 95(1997).

[2] Atkinson J and Joshi N, The Schwarzian variable associated with discrete KdV-type equations, arXiv:1010.1916v1(2010).

[3] Atkinson J and Nijhoff F W, A constructive approach to the soliton solutions of integrable quadrilateral lattice equations, Commun Maths Phys, 299, 283(2010).

[4] Adler V E, Bobenko A I, Suris Y B, Classification of integrable equations on quad-graphs, the consistency approach, Commun. Math. Phys, 233, 513(2003).

[5] Adler V E, Bobenko A I, Suris Y B, Classification of integrable discrete equations of octahedron type, Int Math Res Notices, 17(2011).

[6] Bogdanov L V, Konopelchenko B G, Analytic-bilinear approach to integrable hierarchies. II. Multicomponent KP and 2D Toda lattice hierarchies, J. Math. Phys, 39, 4701(1998).

[7] Capel W H, Wiersma L G and Nijhoff W F, Linearizing integral transform for the multicomponent lattice KP, Physica, 138A, 76(1986).
[8] Deift P, Lund F and Trubowitz E, Nonlinear wave equations and constrained harmonic motion., Commun. Math. Phys, 74, 141(1980).
[9] Doliwa A, Desargues maps and the Hirota-Miwa equation, Proc. Royal Soc. A, 466, 1177(2010).
[10] Dorfman I Ya, Nijhoff W F, On a (2+1)-dimensional version of the Krichever-Novikov equation, Phys. Lett, 157A, 107(1991).
[11] Gragg W B, The Pade Table and Its Relation to Certain Algorithms of Numerical Analysis SIAM Review, 14, 1(1972).
[12] Hirota R, Discrete Analogue of a Generalised Toda Equation, J. Phys. Soc. Japan, 50, 3785(1981).
[13] Kac G V, Infinite dimensional Lie algebras, Cambridge University Press, 1994.
[14] King A D, Schief W K, Tetrahedra, octahedra and cubo-octahedra: integrable geometry of multi-ratios, J. Phys. A, 36, 785(2003).
[15] Konopelchenko B G, Schief W K, Menelaus’ theorem, Clifford configurations and inverse geometry of the Schwarzian KP hierarchy, J. Phys. A, 35, 6125(2002).
[16] Kodama Y, Young diagrams and N-soliton solutions of the KP equation, J. Phys. A, 37, 11169(2004).
[17] Lipan O, Wiegmann P and Zabrodin Z, Fusion Rules for Quantum transfer Matrices as Dynamical Systems on Grassmann Manifolds, Preprint UNiv. of Chicago(1997).
[18] Miwa T, On Hirota’s difference Equations, Proc. Japan Acad, 58, 9(1982).
[19] Nijhoff W F, Atkinson J and Hietarinta J, Soliton Solutions for ABS Lattice Equations: I Cauchy Matrix Approach, J. Phys. A: Special Issue dedicated to the Darboux Days, 42, 404005(2009).
[20] Nijhoff W F and Atkinson J, Elliptic N-soliton solutions of ABS lattice equations, Int. Math. Res. Notices, 20, 3837(2010).
[21] Nijhoff W F, Capel H W, and Wiersma G L, Integrable lattice systems in two and three dimensions, In: Geometric Aspects of the Einstein Equations and Integrable Systems, Ed. R. Martini, Lecture Notes in Physics, Berlin/New York, Springer Verlag, 263(1985).
[22] Nijhoff W F, Capel H W, Wiersma G L, and Quispel G R W, Backlund transformations and three-dimensional lattice equations, Phys. Lett, 105A, 267(1984).
[23] Nijhoff F W, Capel H W and Wiersma G L, Integrable Lattice Systems in Two and Three Dimensions, Ed. R. Martini, in: Geometric Aspects of the Einstein Equations and Integrable Systems, Lecture Notes in Physics, pp. 263–302, Berlin/New York, Springer Verlag(1985).
[24] Nijhoff W F, Ragnisco O and Kuznetsov V B, Integrable Time-Discretization of the Ruijsenaars-Schneider Model, Commun. Math. Phys, 176, 681(1996).
[25] Noumi M and Yamada Y, Affine Weyl groups, discrete dynamical systems and Painlevé equations, Commun. Math. Phys, 217 315(2001).
[26] Ruijsenaars S N M and Schneider H, A New Class of Integrable Systems and Its relation to Solitons, Ann. Phys, 170, 370(1986).
[27] Ruijsenaars S N M, Complete integrability of Relativistic Calogero-Moser systems and Elliptic Function Identities, Commun. Math. Phys, 110, 191(1987).
[28] Yoo-Kong S, Calogero-Moser type systems, associated KP systems, and Lagrangian structure, Thesis, University of Leeds(2011).
[29] Yoo-Kong S and Nijhoff W F, Discrete-time Ruijsenaars-Schneider system and Lagrangian 1-form structure, in preparation.
[30] Wiersma L G and Capel W H, Lattice equations, hierarchies and Hamiltonian structures: The Kadomtsev-Petviashvili equation, Phys. Lett. 124A, 124(1987).
[31] Wiersma L G and Capel W H, *Lattice equations, hierarchies and Hamiltonian structures: II. KP-type of hierarchies on 2D lattices*, Physica, 149A, 49(1988); ibid. *III. The 2D toda and KP hierarchies*, Physica, 149A, 75(1988).

[32] Zabrodin Z, *Discrete Hirota’s equation in Quantum Integrable Models*, Theor. Math. Phys, 113, 1347(1997).