How universal is the fractional-quantum-Hall edge Luttinger liquid?

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This article reports on our microscopic investigations of the edge of the fractional quantum Hall state at filling factor \( \nu = 1/3 \). We show that the interaction dependence of the wave function is well described in an approximation that includes mixing with higher composite-fermion Landau levels in the lowest order. We then proceed to calculate the equal time edge Green function, which provides evidence that the Luttinger exponent characterizing the decay of the Green function at long distances is interaction dependent. The relevance of this result to tunneling experiments is discussed.

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There has been interest in the physics of edge states in fractional quantum Hall effect (FQHE) \([1]\) for several reasons. On the quantum Hall plateaus, the Hall current flows along the edges (at least for sufficiently small currents) because that is the only place where gapless excitations are present. Presence of extended edge states is crucial for the QHE, as is clear in the explanation of the physics of the plateaus \([2]\). The physics of the edge states is also directly relevant to experiments studying tunneling into edge states \([3,4]\).

Theoretically, the electrons at the edge of a FQHE system constitute an example of a one dimensional liquid, the long-distance, low-energy physics of which is generically described by the Luttinger model. One of the most interesting aspects of the edge Luttinger liquid was the assertion, based on general arguments, that the quantization of the Hall conductance fixes the parameter characterizing its asymptotic behavior \([5]\). Consider, for example, the equal time edge Green function, defined as

\[
G_{\text{edge}}(x) = \frac{\langle \Psi \Psi_{\text{f}}(x) \Psi_{\text{e}}(0) \Psi \rangle}{\langle \Psi | \Psi \rangle} \tag{1}
\]

where \(\Psi\) is the ground state, \(\Psi_{\text{e}}\) and \(\Psi_{\text{f}}\) are annihilation and creation field operators, and \(x\) is the distance along the edge. Wen argued that the long-distance behavior of the Green function is given by \(|G_{\text{edge}}(x)| \sim x^{-\alpha}\) with \(\alpha = 3\) for \(\nu = 1/3\); such behavior was verified \([6]\) in an explicit calculation for Laughlin’s wave function for bosons at \(\nu = 1/2\).

However, the exponent measured in the experiment of Grayson et al. \([8]\) is a continuous function of the filling factor. More recent experiments \([9]\) show evidence for a plateau near \(\nu = 1/3\) but with \(\alpha \approx 2.7\). This raises the question that we wish to explore in this work: Does the actual form of the interaction affect the properties of the edge Luttinger liquid, and if so, how? There has been much work investigating this and related issues \([9,10]\), and our goal here is to compute the exponent in a microscopic approach.

The microscopic understanding of the FQHE is based on certain wave functions \([11,12]\), which are known to be quite accurate for the bulk, homogeneous FQHE states from extensive comparisons against exact results in the edge-less spherical geometry. However, the accuracy of these wave functions remains relatively untested at the edges. The validity of the wave functions in the bulk does not necessarily carry over to the edges. In the bulk, the ground state is separated from excitations by a gap, which renders it robust to the actual form of the interaction. On the other hand, at the edges, the system is compressible with gapless excitations, which may make it more susceptible to various perturbations and the actual form of the inter-electron interaction.

We will focus on the filling factor at which the edge properties have been most investigated, namely the lowest Landau level (LL) filling \(\nu = 1/3\). The bulk ground state here is well approximated by Laughlin’s wave function \([11]\):

\[
\Psi_{1/3}^0 = \prod_{j<k} (z_j - z_k)^3 \exp \left[ -\frac{1}{4l_0^2} \sum_i |z_i|^2 \right], \tag{2}
\]

where \(z_j = x_j - iy_j\) denotes the position of the \(j\)th particle as a complex number and \(l_0 = \sqrt{\hbar c/eB}\) is the magnetic length. In fact, \(\Psi_{1/3}^0\) is the exact ground state for a short range model potential \([13]\) denoted by \(V_1\). However, it is not exact for the Coulomb interaction, and significant deviations exist between this wave function and the
exact Coulomb ground state in the disk geometry [17].

FIG. 1. Schematic depiction of the composite-fermion exciton in which a composite fermion, shown as an electron bound to two flux quanta, is excited to a higher CF-Landau level. This state is described by the wave function $\Psi^{CF\text{-exciton}}_{m^*}$, explained in the text, with $m^* = 3$. The y-axis is the composite-fermion Landau level index ($n^*$), and the x-axis is the effective angular momentum $m^*$.

To gain an understanding of the physical origin of the corrections to $\Psi^0_{1/3}$ we consider the question of mixing of $\Psi^0_{1/3}$ with low energy excitations, as might be induced by the long range part of the Coulomb interaction. We construct excitations using the composite fermion theory of the FQHE [12]. In this approach, $\Psi^0_{1/3}$ is interpreted as one filled composite-fermion LL, rewritten as

$$\Psi^0_{1/3} = \Phi_1 \prod_{j<k} (z_j - z_k)^2 ,$$

where $\Phi_1$ is the wave function of one filled LL, given by

$$\Phi_1 = A[n_0(\vec{r}_1)\cdots n_{N-1}(\vec{r}_N)] \prod_{j<k} (z_j - z_k) \exp[-\frac{1}{4\delta^2} \sum_i |z_i|^2] ,$$

where $A$ is the antisymmetrization operator and $n_m(\vec{r})$ is the single particle wave function in the lowest LL with angular momentum $m$. The Jastrow factor $\prod_{j<k}(z_j - z_k)^2$ attaches to each electron in $\Phi_1$, to convert it into a composite fermion. The excitations of $\Psi^0_{1/3}$ are then images of excitations in $\Phi_1$. In particular, a particle hole pair excitation in $\Phi_1$ produces a particle-hole pair of composite fermions, namely a composite-fermion (CF) exciton, in the full wave function. We restrict our attention below to the states containing a single CF exciton [19], and furthermore, to those excitons for which the CF-LLL index changes by unity. (Remember: mixing with higher electronic LLs is not considered in this work.) The validity of this approximation will be checked below. These excitons are related to excitations in which one electron in $\Phi_1$ is promoted from the angular momentum $m^*$ state in the lowest electronic LL to the angular momentum $m^*$ state in the second electronic LL, producing an excited state denoted by

$$\Phi^{el\text{-exciton}}_{1,m^*} = A[n_0(\vec{r}_1)\cdots n_{m^*-1}(\vec{r}_{m^*-1})n_{m^*}(\vec{r}_{m^*+1})\cdots n_{N-1}(\vec{r}_N)]$$

where $\zeta_{m^*}$ is the wave function of an electron in the second LL in angular momentum $m^*$ state. The excitations that do not conserve $m^*$ are not relevant because they have a different total angular momentum than $\Psi^0_{1/3}$ and therefore do not couple to it. The corresponding CF excitation at $\nu = 1/3$ is shown schematically in Fig. 1, with the wave function given by:

$$\Psi^{CF\text{-exciton}}_{m^*} = \mathcal{P}_{LLL} \Phi^{el\text{-exciton}}_{1,m^*} \prod_{j<k} (z_j - z_k)^2 ,$$

where $\mathcal{P}_{LLL}$ denotes projection of the wave function into the lowest LL. The explicit form of the projected wave functions can be written in the standard manner [20]. The largest value of $m^*$, corresponding to the angular momentum of the outermost occupied state in $\Phi_1$, is $N - 1$, giving a total of $N$ CF-exciton states.

| $N$ | 10 | 11 | 12 | 20 | 30 |
|-----|-----|-----|-----|-----|-----|
| overlap | 0.982 | 0.980 | 0.979 | 0.971 | 0.965 |
| $E^0$ | 7.175 | 8.467 | 9.839 | 23.278 | 45.251 |
| $E^{mod}$ | 7.172 | 8.463 | 9.835 | 23.272 | 45.243 |

We next proceed to diagonalize the Coulomb Hamiltonian in the restricted Hilbert space of $N + 1$ states: $\Psi^0_{1/3}$ and $\Psi^{CF\text{-exciton}}_{m^*}$ (with $m^* = 0, \ldots, N - 1$). One difficulty is that these states are not mutually orthogonal. We obtain an orthogonal basis following the Gram-Schmidt procedure, compute the matrix elements of the Coulomb Hamiltonian, and then carry out diagonalization in the restricted Hilbert space, performing all integrals by the Monte Carlo method. While the procedure is rather cumbersome, it can be carried out satisfactorily, and we refer the reader to literature for further details and also for a trick used to evaluate off-diagonal matrix elements by Monte Carlo [22]. To obtain adequate accuracy, we need to perform up to 10 million Monte Carlo iterations for each point. The modified composite fermion ground state obtained in this manner, i.e., taking into account the mixing of the “unperturbed” CF ground state $\Psi^0_{1/3}$ with single CF excitons, will be denoted by $\Psi^{mod}_{1/3}$. We
note that like $\Psi^0_{1/3}$, $\Psi_{mod}^{1/3}$ also involves only single particle states with $m \leq 3(N-1)$; the states with non-zero occupation of $m > 3(N-1)$ would contain two or more CF-excitons, which we neglect in this work. We have studied several systems up to $N = 30$; the overlap between $\Psi_{mod}^{1/3}$ and $\Psi^0_{1/3}$, also evaluated by Monte Carlo, is shown in Table I. The density profile of $\Psi_{mod}^{1/3}$ is shown in Figs. (2) and (3) for several values of $N$.

Fig. 2. The density profile of $\Psi^0_{1/3}$ (dashed line) and the modified composite fermion state $\Psi_{mod}^{1/3}$ described in text (dots) for $N = 10$, 11, and 12 particles. The distance from the center of the disk, $r$, is quoted in units of the magnetic length, $l_0 = \sqrt{\hbar c/eB}$. The statistical uncertainty in Monte Carlo is smaller than the size of the dots. Also given is the density profile for the exact Coulomb ground state (solid line), taken from Tsisper and Goldman [17].

To test the validity of the approximations made above, we compare in Fig. (4) the modified density profile calculated above with the exact Coulomb density profile taken from Ref. [17]. The comparison demonstrates that $\Psi_{mod}^{1/3}$ is a significant improvement over $\Psi^0_{1/3}$, and is actually quite close to the exact wave function, thus establishing that much of the discrepancy between $\Psi_{mod}^{1/3}$ and the exact wave function can be accounted for by incorporating mixing with the lowest energy single CF-excitons. This, to our knowledge, is the first variational improvement of the FQHE wave function in the lowest LL space.

Fig. 3. The density profiles for $\Psi^0_{1/3}$ (dashed line) and $\Psi_{mod}^{1/3}$ (dots) for $N = 30$. The solid line is a fit according to $\rho(r) = 1/3 + \frac{r_0}{l_0} \cos \left( \frac{2\pi r - \alpha \gamma}{\gamma x_0} \right)$ with $x = r - r_0$, $r_0 = 11.407$, $\rho_1 = 0.113$, $x_0 = 4.02$, $\alpha = 0.029$, $\beta = 0.415$, and $\gamma = 0.086$. (All distances are quoted in units of $l_0$.)

Having demonstrated that $\Psi_{mod}^{1/3}$ correctly captures the interaction dependent physics at the edge, we proceed to calculate the edge Green function. For the disk geometry, the equal time edge Green function is defined as

$$G_{edge}(\theta) = \frac{<\Psi|\Psi^+_e(R_0)|\Psi_e(R)|\Psi>}{<\Psi|\Psi>} \quad (7)$$

where $\Psi$ is the ground state describing a disk of electrons centered at the origin, $z = R$ is the position at the edge along the x-axis and $z = R e^{i\theta}$ is the position of the edge at an angle $\theta$ relative to the x-axis. Following Lee and Wen [3], we take $R = \sqrt{6N} l_0$ at $\nu = 1/3$. The power law exponent $\alpha$ is defined by the equation appropriate for the geometry under consideration [3]:

$$|G_{edge}(\theta)| \sim \sin^{-\alpha}(\theta/2) \quad (8)$$

The Green function can be expressed as

$$G_{edge}(\theta) = N \int d^2 z_2 ... d^2 z_N \Psi^*(R_0, z_2, ..., z_N) \Psi(R, z_2, ..., z_N) \int d^2 z_1 ... d^2 z_N \Psi^*(z_1, ..., z_N) \Psi(z_1, ..., z_N) \quad (9)$$
We write the numerator as $\langle \Psi^\dagger \Psi \rangle$, where $\langle \rangle$ denotes integral over coordinates $z_2...z_N$. Using

$$2 \langle \Psi^\dagger \Psi \rangle = \langle |\Psi_\theta + i \Psi|^2 \rangle = \langle |\Psi|^2 \rangle = \langle |\Psi_\theta|^2 \rangle$$

we express the real and imaginary parts of the edge Green function in terms of integrals over positive definite, real integrands, which can be evaluated efficiently by Monte Carlo.

Fig. 3 shows $|G_{\text{edge}}(\theta)|$ for both $\Psi_{1/3}$ and $\Psi_{\text{mod}}$ for $N = 30$. For the former wave function, we recover $\alpha = 3$, as expected. The deviation of the Green function from the power-law behavior at short distances (small $\theta$) seen in Fig. 3 is quite similar to that found in Ref. 12 for bosons at $\nu = 1/2$. There is also a well defined exponent for $\Psi_{\text{mod}}$, indicating that the edge continues to be described as a Luttinger liquid. However, the Luttinger exponent now is $\alpha \approx 2.5$.

FIG. 4. The edge Green function as a function of the distance along the edge, parametrized by the angle $\theta$. The distance from the center is taken to be $R = \sqrt{6N} l_0$, $l_0$ being the magnetic length. The result is shown for $\Psi_{1/3}$, which is exact for the $V_1$ interaction, and the modified CF wave functions for two interactions: the Coulomb interaction ($r^{-1}$) and the Yukawa interaction ($r^{-1} e^{-r/l_0}$). In each case, the straight line is a fit with the exponent $\alpha$ shown near the line.

The change in the exponent is surprisingly large considering the smallness of the corrections to $\Psi_{1/3}$. Several points are in order here. It is sometimes stated that the arguments leading to $\alpha = 3$ require a sharp edge, i.e., when the density is approximately constant (at its bulk value) up to the edge and falls sufficiently sharply to zero at the edge. Independent of the question of whether and how a sharp edge may be realized, we note that both $\Psi_{1/3}$ and $\Psi_{\text{mod}}$ have rather similar density profiles at the edge, indicating that sharp-vs-smooth-edge type considerations are not likely to have any bearing on the difference in the exponents. We have also tested that the mixing with the CF excitons does not affect the compressibility of the state; the gap in the angular momentum $L = 3(N - 1)$ sector of the Hilbert space is only very weakly affected by it. In other words, the mixing with the second CF-LL merely renormalizes the states of the lowest CF-LL, while preserving the single channel character of the problem. Finally, while it is impossible to prove rigorously that the conclusion based on our finite system is valid in the thermodynamic limit, the system size ($N = 30$) is sufficiently large to reproduce the expected exponent $\alpha = 3$ for Laughlin’s wave function and also to give a well defined exponent for the modified wave function.

One might wonder if a universal exponent is obtained at least for all short-ranged interactions. To address this issue, we have also considered a short-ranged, Yukawa-type interaction $V(r) = r^{-1} \exp[-r/l_0]$. The edge Green function is computed as before, also shown in Fig. 3, from which the edge exponent $\alpha \approx 2.6$ is deduced. This suggests that the exponent $\alpha = 3$ is unique to $\Psi_{1/3}$, which has the very special property that all of the zeros of the wave function are bound to the electrons; any splitting of the zeros away from electrons seems to change the exponent.

We believe that our study provides compelling evidence that the edge exponent is sensitive to the detailed form of the wave function. Because the Hall quantization is not affected by small perturbations in the wave function, so long as the gap in the bulk is preserved, one must conclude that the Luttinger parameter $\alpha$ is not tied to the quantized Hall conductance, and is therefore not a fundamental property of the FQHE state. Investigations into the origin of the non-universality of the edge properties are bound to produce interesting physics.

Our results are generally consistent with the tunneling experiments of Chang and collaborators 13. The theoretical exponent obtained above is smaller than 3.0 for both the Coulomb and Yukawa interactions, consistent with experiment. (A more accurate estimation of the observed exponent will require a realistic treatment of the confining potential at the edge 14.) Also, unless the exponent is completely fixed by the Hall conductance, there is no reason to expect it to be precisely constant over the range of a Hall plateau.

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