A COMPARISON OF L-GROUPS FOR COVERS OF SPLIT REDUCTIVE GROUPS

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Abstract. In one article, the author has defined an L-group associated to a cover of a quasisplit reductive group over a local or global field. In another article, Wee Teck Gan and Fan Gao define (following an unpublished letter of the author) an L-group associated to a cover of a pinned split reductive group over a local or global field. In this short note, we give an isomorphism between these L-groups. In this way, the results and conjectures discussed by Gan and Gao are compatible with those of the author. Both support the same Langlands-type conjectures for covering groups.

Summary of two constructions

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Summary of two constructions

Let \( G \) be a split reductive group over a local or global field \( F \). Choose a Borel subgroup \( B = TU \) containing a split maximal torus \( T \) in \( G \). Let \( X = \text{Hom}(T, G_m) \) be the character lattice, and \( Y = \text{Hom}(G_m, T) \) be the cocharacter lattice of \( T \). Let \( \Phi \subset X \) be the set of roots and \( \Delta \subset \Phi \) the subset of simple roots. For each root \( \alpha \in \Phi \), let \( U_\alpha \) be the associated root subgroup. Let \( \Phi^\vee \) and \( \Delta^\vee \) be the associated coroots and simple coroots. The root datum of \( G \supset B \supset T \) is

\[
\Psi = (X, \Phi, \Delta, Y, \Phi^\vee, \Delta^\vee).
\]

Fix a pinning (épinglage) of \( G \) as well – a system of isomorphisms \( x_\alpha : G_\alpha \to U_\alpha \) for every root \( \alpha \).

The following notions of covering groups and their dual groups match those in [Wei15]. Let \( \tilde{G} = (G', n) \) be a degree \( n \) cover of \( G \) over \( F \); in particular, \( \#\mu_n(F) = n \). Here \( G' \) is a central extension of \( G \) by \( K_2 \) in the sense of [B-D], and write \( (Q, D, f) \) for the three Brylinski-Deligne invariants of \( G' \). Assume that if \( n \) is odd, then \( Q : Y \to \mathbb{Z} \) takes only even values (this is [Wei15, Assumption 3.1]).

Let \( \tilde{G}^\vee \supset \tilde{B}^\vee \supset \tilde{T}^\vee \) be the dual group of \( \tilde{G} \), and let \( \tilde{Z}^\vee \) be the center of \( \tilde{G}^\vee \). The group \( \tilde{G}^\vee \) is a pinned complex reductive group, associated to the root datum

\[
(Y_{Q, n}, \tilde{\Phi}^\vee, \tilde{\Delta}^\vee, X_{Q, n}, \tilde{\Phi}, \tilde{\Delta}).
\]

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Here \( Y_{Q,n} \subset \hat{Y} \) is a sublattice containing \( nY \). For each coroot \( \alpha^\vee \in \hat{\Phi}^\vee \), there is an associated positive integer \( n_\alpha \) dividing \( n \) and a “modified coroot” \( \tilde{\alpha}^\vee = n_\alpha \alpha^\vee \in \hat{\Phi}^\vee \). The set \( \hat{\Phi}^\vee \) consists of the modified coroots, and \( \Delta^\vee \) the modified simple coroots. Define \( Y_{Q,n}^{sc} \) to be the sublattice of \( Y_{Q,n} \) generated by the modified coroots. Then

\[
\tilde{T}^\vee = \text{Hom}(Y_{Q,n}/\mathbb{C}) \text{ and } \tilde{Z}^\vee = \text{Hom}(Y_{Q,n}/Y_{Q,n}^{sc}, \mathbb{C}).
\]

Let \( \bar{F}/F \) be a separable algebraic closure, and \( \text{Gal}_F = \text{Gal}(\bar{F}/F) \) the absolute Galois group. Fix an injective character \( \tilde{\epsilon}: \mu_n(F) \to \mathbb{C}^\times \). From this data, the constructions of [Wei13] and [GG14] both yield an \( L \)-group of \( \bar{G} \) via a Baer sum of two extensions. In both papers, an extension

(First twist)

\[
\tilde{Z}^\vee \hookrightarrow E_1 \to \text{Gal}_F,
\]

is described in essentially the same way. When \( F \) is local, this “first twist” \( E_1 \) is defined via a \( \tilde{Z}^\vee \)-valued 2-cocycle on \( \text{Gal}_F \). See [GG14 §5.2] and [Wei13 §5.4] (in the latter, \( E_1 \) is denoted \( (\tau_Q)_* \text{Gal}_F \)). Over global fields, the construction follows from the local construction and Hilbert reciprocity.

Both papers include a “second twist”. Gan and Gao [GG14 §5.2] describe an extension

(Second twist)

\[
\tilde{Z}^\vee \hookrightarrow E_2 \to \text{Gal}_F,
\]

following an unpublished letter (June, 2012) from the author to Deligne. In [Wei13], the second twist is the fundamental group of a gerbe, denoted \( \pi_1^{et}(E, (\bar{G}), \tilde{s}) \). In this article \( \tilde{s} = \text{Spec}(\bar{F}) \), and so we write \( \pi_1^{et}(E, (\bar{G}), \tilde{F}) \) instead.

Both papers proceed by taking the Baer sum of these two extensions, \( E = E_1 + E_2 \), to form an extension \( \tilde{Z}^\vee \hookrightarrow E \to \text{Gal}_F \). The extension \( E \) is denoted \( \tilde{L} \tilde{Z} \) in [Wei13 §5.4]. Then, one pushes out the extension \( E \) via \( \tilde{Z}^\vee \hookrightarrow \bar{G}^\vee \), to define the \( L \)-group

(L-group)

\[
\bar{G}^\vee \hookrightarrow \tilde{L} \bar{G} \to \text{Gal}_F.
\]

The two constructions of the \( L \)-group, from [GG14] and [Wei13] are the same, except for insignificant linguistic differences, and a significant difference between the “second twists”. In this short note, by giving an isomorphism,

\[
\pi_1^{et}(E, (\bar{G}), \tilde{F}) \text{ (described by the author) } \cong E_2 \text{ (described by Gan and Gao)}
\]

we will demonstrate that the second twists, and thus the \( L \)-groups, of both papers are isomorphic. Therefore, the work of Gan and Gao in [GG14] supports the broader conjectures of [Wei15].

Remark 0.1. Among the “insignificant linguistic differences,” we note that Gan and Gao use extensions of \( F^\times /F^\times n \) (for local fields) or the Weil group \( \mathcal{W}_F \) rather than \( \text{Gal}_F \). But pulling back via the reciprocity map of class field theory yields extensions of \( \text{Gal}_F \) by \( \tilde{Z}^\vee \) as above.

1. Computations in the gerbe

1.1. Convenient base points. Let \( E_0(\bar{G}) \) be the gerbe constructed in [Wei13 §3]. Rather than using the language of étale sheaves over \( F \), we work with \( \tilde{F} \)-points and trace through the \( \text{Gal}_F \)-action. Let \( \bar{T} = \text{Hom}(Y_{Q,n}/\tilde{F}^\times) \) and \( \bar{T}_{sc} = \text{Hom}(Y_{Q,n}^{sc}/\tilde{F}^\times) \).
Let $p: \hat{T} \to \hat{T}_{sc}$ be the surjective $Gal_F$-equivariant homomorphism dual to the inclusion $Y^\infty_{Q,n} \hookrightarrow Y_{Q,n}$. Define
\[
\hat{Z} = \text{Ker}(p) = \text{Hom}(Y_{Q,n}/Y^\infty_{Q,n}, \hat{F}^\times).
\]
The reader is warned not to confuse $\hat{T}, \hat{T}_{sc}, \hat{Z}$ with $\check{T}, \check{T}_{sc}, \check{Z};$ the former are nontrivial $Gal_F$-modules (Homs into $\hat{F}^\times$) and the latter are trivial $Gal_F$-modules (Homs into $\mathbb{C}^\times$ as a trivial $Gal_F$-module).

Write $\check{D} = \mathcal{D}(\check{F})$ and $D = \mathcal{D}(F)$, where we recall $\mathcal{D}$ is the second Brylinski-Deligne invariant of the cover $\check{G}$. We have a $Gal_F$-equivariant short exact sequence,
\[
\check{F}^\times \hookrightarrow \check{D} \to Y.
\]
By Hilbert’s Theorem 90, the $Gal_F$-fixed points give a short exact sequence,
\[
F^\times \to D \to Y.
\]
Let $\check{D}_{Q,n}$ and $\check{D}^\infty_{Q,n}$ denote the preimages of $Y_{Q,n}$ and $Y^\infty_{Q,n}$ in $\check{D}$. These are abelian groups, fitting into a commutative diagram with exact rows.
\[
\begin{array}{ccc}
\check{F}^\times & \longrightarrow & \check{D}^\infty_{Q,n} \\
\downarrow & & \downarrow \\
F^\times & \longrightarrow & D_{Q,n} \\
\end{array}
\]
Let $\text{Spl}(\check{D}_{Q,n})$ be the $\hat{T}$-torsor of splittings of $\check{D}_{Q,n}$, and similarly let $\text{Spl}(\check{D}^\infty_{Q,n})$ be the $\hat{T}_{sc}$-torsor of splittings of $\check{D}^\infty_{Q,n}$.

Let $\text{Whit}$ denote the $\hat{T}_{sc}$-torsor of nondegenerate characters of $U(\check{F})$. An element of $\text{Whit}$ is a homomorphism (defined over $\check{F}$) from $U$ to $G_a$ which is nontrivial on every simple root subgroup $U_\alpha$. $Gal_F$ acts on $\text{Whit}$, and the fixed points $\text{Whit} = \text{Whit}^{Gal_F}$ are those homomorphisms from $U$ to $G_a$ which are defined over $F$. The $\hat{T}_{sc}$-action on $\text{Whit}$ is described in [Wei13, §3.3].

The pinning $\{x_\alpha : \alpha \in \Phi\}$ of $G$ gives an element $\psi \in \text{Whit}$. Namely, let $\psi$ be the unique nondegenerate character of $U$ which satisfies
\[
\psi(x_\alpha(1)) = 1 \text{ for all } \alpha \in \Delta.
\]

In [Wei13, §3.3], we define an surjective homomorphism $\mu: \hat{T}_{sc} \to \check{T}_{sc}$, and a $Gal_F$-equivariant isomorphism of $\hat{T}_{sc}$-torsors,
\[
\check{\omega}: \mu_*\text{Whit} \to \text{Spl}(\check{D}^\infty_{Q,n}).
\]
The isomorphism $\check{\omega}$ sends $\psi$ to the unique splitting $s_\psi \in \text{Spl}(\check{D}^\infty_{Q,n})$ which satisfies
\[
s_\psi(\check{x}^\alpha) = r_\alpha \cdot [e_\alpha]^{n_\alpha}, \text{ with } r_\alpha = (-1)^{Q(\alpha^\vee)} \frac{2 - 2^{-1}}{\alpha(\alpha^\vee)}.
\]

We describe the element $[e_\alpha] \in D$ concisely here, based on [B-D, §11] and [GG14, §2.4]. Let $F((v))$ be the field of Laurent series with coefficients in $F$. The extension $K_2 \hookrightarrow G' \to G$ splits over any unipotent subgroup, and so the pinning homomorphisms $x_\alpha: F((v)) \to U_\alpha(F((v)))$ lift to homomorphisms
\[
\check{x}_\alpha: F((v)) \to U'_\alpha(F((v))).
\]
Define, for any $u \in F((v))^\times$,
\[
\check{n}_\alpha(u) = \check{x}_\alpha(u)\check{x}_{-\alpha}(-u^{-1})\check{x}_\alpha(u).
\]
This yields an element
\[ \tilde{t}_\alpha = \tilde{n}_\alpha(v) \cdot \tilde{n}_\alpha(-1) \in T'(F(\langle v \rangle)). \]
Then \( t_\alpha \) lies over \( \alpha^\vee(v) \in T(F(\langle v \rangle)) \). Its pushout via \( K_2(F(\langle v \rangle)) \xrightarrow{\partial} F^\times \) is the element we call \([e_\alpha] \in D\).

**Remark 1.1.** The element \( s_\psi(\tilde{\alpha}^\vee) = r_\alpha \cdot [e_\alpha]^{n_\alpha} \) coincides with what Gan and Gao call \( s_{Q^c}(\tilde{\alpha}^\vee) \) in [GG14, §5.2]; the sign \( r_\alpha \) arises from the formulae of [B-D, §11.1.4, 11.1.5].

Let \( j_0: \hat{T}_{sc} \twoheadrightarrow \mu_s \text{Whit} \) be the unique isomorphism of \( \hat{T}_{sc}\)-torsors which sends 1 to \( \psi \) (or rather the image of \( \psi \) via \( \text{Whit} \to \mu_s \text{Whit} \)). Since \( \psi \in \text{Whit} \) is \( \text{Gal}_F \)-invariant, this isomorphism \( j_0 \) is also \( \text{Gal}_F \)-invariant.

Finally, let \( s \in \text{Spl}((D_{Q,n}) \) be a splitting which restricts to \( s_\psi \) on \( Y_{Q,n}^\infty \). Such a splitting \( s \) exists, since the map \( \text{Spl}((D_{Q,n}) \to \text{Spl}(D_{Q,n}) \) is surjective (since the map \( \hat{T} \to \hat{T}_{sc} \) is surjective). Note that \( s \) is not necessarily \( \text{Gal}_F \)-invariant (and often cannot be).

Let \( h: \hat{T} \to \text{Spl}(\hat{D}_{Q,n}) \) be the function given by
\[ h(x) = x^n \cdot s \text{ for all } x \in \hat{T}. \]

The triple \( \hat{\varepsilon} = (\hat{T}, h, j_0) \) is an \( \hat{F} \)-object (i.e., a geometric base point) of the gerbe \( \mathbf{E}_s(\hat{G}) \). Note that the construction of \( \hat{\varepsilon} \) depends on two choices: a pinning of \( G \) (to obtain \( \psi \in \text{Whit} \)) and a splitting \( s \) of \( D_{Q,n} \) extending \( s_\psi \). We call such a triple \( \hat{\varepsilon} \) a convenient base point for the gerbe \( \mathbf{E}_s(\hat{G}) \).

### 1.2. The fundamental group.
For a convenient base point \( \hat{\varepsilon} \) associated to \( s \), we consider the fundamental group
\[ \pi_1^{et}(\mathbf{E}_s(\hat{G}), \hat{\varepsilon}) = \bigsqcup_{\gamma \in \text{Gal}_F} \text{Hom}(\hat{\varepsilon}, \gamma \hat{\varepsilon}). \]
This fundamental group fits into a short exact sequence
\[ \hat{Z}^\vee \rightarrow \pi_1^{et}(\mathbf{E}_s(\hat{G}), \hat{\varepsilon}) \rightarrow \text{Gal}_F, \]
where the fibre over \( \gamma \in \text{Gal}_F \) is \( \text{Hom}(\hat{\varepsilon}, \gamma \hat{\varepsilon}) \). Thus to describe the fundamental group, it suffices to describe each fibre (as a \( \hat{Z}^\vee \)-torsor), and the multiplication maps among fibres.

The base point \( \gamma \hat{\varepsilon} \) is the triple \( (\gamma \hat{T}, \gamma \circ h, \gamma \circ j_0) \), where \( \gamma \hat{T} \) is the \( \hat{T} \)-torsor with underlying set \( \hat{T} \) and twisted action
\[ u \ast_{\gamma} x = \gamma^{-1}(u) \cdot x. \]

To give an element \( f \in \text{Hom}(\hat{\varepsilon}, \gamma \hat{\varepsilon}) \) is the same as giving an element \( \zeta \in \hat{Z}^\vee \) and a map of \( \hat{T} \)-torsors \( f_0: \hat{T} \to \gamma \hat{T} \) satisfying
\[ (\gamma \circ h) \circ f_0 = h \text{ and } (\gamma \circ j_0) \circ p_* f_0 = j_0. \]
Any such map of \( \hat{T} \)-torsors is uniquely determined by the element \( \tau \in \hat{T} \) satisfying \( f_0(1) = \tau \). The two conditions above are equivalent to the two conditions
\[ (1.1) \]
\[ \tau^n = \gamma^{-1}s/s \text{ and } \tau \in \hat{Z}. \]
Thus, to give an element \( f \in \text{Hom}(\hat{\varepsilon}, \gamma \hat{\varepsilon}) \) is the same as giving a pair \( (\tau, \zeta) \in \hat{T} \times \hat{Z}^\vee \), where \( \tau \) satisfies the two conditions above. Therefore, in what follows, we
write \((τ, ζ) ∈ \text{Hom}(\bar{z}, \gamma \bar{z})\) to indicate that \(τ\) satisfies the two conditions above, and to refer to the corresponding morphism in the gerbe \(E_1(\tilde{G})\) in concrete terms.

We use \(ε: \mu_n(F) \xrightarrow{\sim} \mu_n(\mathbb{C})\) to identify \(\tilde{Z}_{[n]}\) with \(\tilde{Z}_{[n]}^\vee\). Two pairs \((τ, ζ)\) and \((τ', ζ')\) are identified in \(\text{Hom}(\bar{z}, \gamma \bar{z})\) if and only if there exists \(ξ \in \tilde{Z}_{[n]}\) such that

\[τ' = ξ · τ\text{ and }ζ' = ε(ξ)^{-1} · ζ.\]

The structure of \(\text{Hom}(\bar{z}, \gamma \bar{z})\) as a \(\tilde{Z}_\mathbb{F}\)-torsor is by scaling the second factor in \((τ, ζ) ∈ \tilde{T} × \tilde{Z}_\mathbb{F}\). To describe the fundamental group completely, it remains to describe the multiplication maps among fibres. If \(γ_1, γ_2 ∈ \text{Gal}_F\), and

\[(τ_1, ζ_1) ∈ \text{Hom}(\bar{z}, γ_1 \bar{z})\text{ and }τ_2, ζ_2) ∈ \text{Hom}(\bar{z}, γ_2 \bar{z}),\]

then their composition in \(\pi^d_1(\text{Hom}(\tilde{G}), \bar{z})\) is given by

\[(τ_1, ζ_1) ∘ (τ_2, ζ_2) = (γ_2^{-1}(τ_1) · τ_2, ζ_1 ζ_2).\]

Observe that

\[(γ_2^{-1}(τ_1) τ_2)^n = γ_2^{-1} (γ_1 s/s) · (γ_2^{-1} s/s) = (γ_1 γ_2)^{-1} s/s.\]

Therefore \((γ_2^{-1}(τ_1) · τ_2, ζ_1 ζ_2) ∈ \text{Hom}(\bar{z}, γ_1 γ_2 \bar{z})\) as required.

2. Comparison to the second twist

2.1. The second twist. The construction of the second twist in [GG14] does not rely on gerbes at all, at the expense of some generality; it seems difficult to extend the construction there to nonsplit groups. But for split groups, the construction of [GG14] offers significant simplifications over [Wei15]. The starting point in [GG14] is the same short exact sequence of abelian groups as in the previous section,

\[F^\times \xhookleftarrow{D_{Q,n}} \xrightarrow{Y_{Q,n}}.\]

And as before, we utilize the splitting \(s_ψ: Y_{Q,n}^\infty \xrightarrow{\sim} D^\infty_{Q,n}\). Taking the quotient by \(s_ψ(Y_{Q,n}^\infty)\), we obtain a short exact sequence

\[F^\times \xhookleftarrow{D_{Q,n}} \xrightarrow{s_ψ(Y_{Q,n})} Y_{Q,n}^\infty.\]

Apply \(\text{Hom}(\bullet, \mathbb{C}^\times)\) (and note \(\mathbb{C}^\times\) is divisible) to obtain a short exact sequence,

\[\tilde{Z}_\mathbb{F}^\vee \xrightarrow{\text{Hom}} \tilde{Z}_\mathbb{F} \xrightarrow{\text{Hom}(F^\times, \mathbb{C}^\times)} \text{Hom}(F^\times, \mathbb{C}^\times).\]

Define a homomorphism \(\text{Gal}_F \xrightarrow{\gamma} \text{Hom}(F^\times, \mathbb{C}^\times)\) by the Artin symbol,

\[γ \mapsto \left( u \mapsto ε(\frac{γ^{-1}(\sqrt[n]{u})}{\sqrt[n]{u}}) \right).\]

Pulling back the previous short exact sequence by this homomorphism yields a short exact sequence

\[\tilde{Z}_\mathbb{F}^\vee \xhookrightarrow{E_2} \xrightarrow{\text{Gal}_F}.\]

This \(E_2\) is the second twist described in [GG14].

Remark 2.1. There is an insignificant difference here – at the last step, over a local field \(F\), Gan and Gao pull back to \(F^\times/F^\times n\) via the Hilbert symbol whereas we pull further back to \(\text{Gal}_F\) via the Artin symbol.
Write \( E_{2, \gamma} \) for the fibre of \( E_2 \) over any \( \gamma \in \Gal_F \). Again, to understand the extension \( E_2 \), it suffices to understand these fibres (as \( \hat{Z}^\nu \)-torsors), and to understand the multiplication maps among them. The steps above yield the following (somewhat) concise description of \( E_{2, \gamma} \).

\( E_{2, \gamma} \) is the set of homomorphisms \( \chi : D_{Q, n} \to \mathbb{C}^\times \) such that

- \( \chi \) is trivial on the image of \( Y_{Q, n}^{sc} \) via the splitting \( s_\psi \).
- For every \( u \in F^\times \), \( \chi(u) = \epsilon (\gamma^{-1} \sqrt{u} / \sqrt{\psi u}) \).

Multiplication among fibres is given by usual multiplication, \( \chi_1, \chi_2 \mapsto \chi_1 \chi_2 \). The \( \hat{Z}^\nu \)-torsor structure on the fibres is given as follows: if \( \eta \in \hat{Z}^\nu \), then

\[
[\eta * \chi](d) = \eta(y) \cdot \chi(d) \quad \text{for all} \quad d \in D_{Q, n} \quad \text{lying over} \quad y \in Y_{Q, n}.
\]

### 2.2. Comparison

Now we describe a map from \( \pi_1^\nu(\mathbb{E}, \mathbb{G}) \) to \( E_2 \), fibrewise over \( \Gal_F \). From the splitting \( s \) (used to define \( \tilde{\varepsilon} \) and restricting to \( s_\psi \) on \( Y_{Q, n}^{sc} \)), every element of \( D_{Q, n} \) can be written uniquely as \( s(y) \cdot u \) for some \( y \in Y_{Q, n} \) and some \( u \in F^\times \). Such an element \( s(y) \cdot u \) is \( \Gal_F \)-invariant if and only if

\[
\gamma(s(y))\gamma(u) = s(y)u, \quad \text{or equivalently} \quad \gamma^{-1} \sqrt{u} \cdot \gamma^{-1} s \cdot y = 1, \quad \text{for all} \quad \gamma \in \Gal_F.
\]

Suppose that \( \gamma \in \Gal_F \) and \((\tau, 1) \in \Hom(\tilde{\varepsilon}, \gamma \tilde{\varepsilon}) \). Define \( \chi : D_{Q, n} \to \mu_n(\mathbb{C}) \) by

\[
\chi(s(y) \cdot u) = \epsilon \left( \gamma^{-1} \sqrt{u} / \sqrt{\psi u} \cdot \tau(y) \right).
\]

This makes sense, because \( \Gal_F \)-invariance of \( s(y) \cdot u \) implies

\[
\left( \frac{\gamma^{-1} \sqrt{u}}{\sqrt{\psi u}} \cdot \tau(y) \right)^n = \frac{\gamma^{-1} u}{u} \cdot \frac{\gamma^{-1} s \cdot y}{s} = 1.
\]

To see that \( \chi \in E_{2, \gamma} \), observe that

- \( \chi \) is a homomorphism (a straightforward computation).
- If \( y \in Y_{Q, n}^{sc} \), then \( \chi(s(y)) = \tau(y) = 1 \) since \( \tau \in \hat{Z} \).
- If \( u \in F^\times \), then \( \chi(u) = \epsilon (\gamma^{-1} \sqrt{u} / \sqrt{\psi u}) \) by definition.

#### Lemma 2.2

The map sending \((\tau, 1) \) to \( \chi \), described above, extends uniquely to an isomorphism of \( \hat{Z}^\nu \)-torsors from \( \Hom(\tilde{\varepsilon}, \gamma \tilde{\varepsilon}) \) to \( E_{2, \gamma} \).

**Proof:** If this map extends to an isomorphism of \( \hat{Z}^\nu \)-torsors as claimed, the map must send an element \((\tau, \zeta) \) in \( \Hom(\tilde{\varepsilon}, \gamma \tilde{\varepsilon}) \) to the element \( \zeta * \chi \in E_{2, \gamma} \). To demonstrate that the map extends to an isomorphism of \( \hat{Z}^\nu \)-torsors, it must only be checked that

\[
(\xi \cdot \tau, 1) \quad \text{and} \quad (\tau, \epsilon(\xi))
\]

map to the same element of \( E_{2, \gamma} \), for all \( \xi \in \hat{Z}_{[n]} \). For this, we observe that \((\xi \cdot \tau, 1) \) maps to the character \( \chi' \) given by

\[
\chi'(s(y) \cdot u) = \epsilon \left( \frac{\gamma^{-1} \sqrt{u}}{\sqrt{\psi u}} \xi(y) \tau(y) \right) = \epsilon(\xi(y)) \cdot \epsilon \left( \frac{\gamma^{-1} \sqrt{u}}{\sqrt{\psi u}} \tau(y) \right) = \epsilon(\xi(y)) \cdot \chi(s(y) \cdot u).
\]

Thus \( \chi' = \epsilon(\xi) * \chi \) and this demonstrates the lemma.

From this lemma, we have a well-defined “comparison” isomorphism of \( \hat{Z}^\nu \)-torsors,

\[
C_\gamma : \Hom(\tilde{\varepsilon}, \gamma \tilde{\varepsilon}) \to E_{2, \gamma},
\]

(Comparison)

\[
C_\gamma(\tau, \zeta)(s(y) \cdot u) = \epsilon \left( \frac{\gamma^{-1} \sqrt{u}}{\sqrt{\psi u}} \cdot \tau(y) \right) \cdot \zeta(y).
\]
Checking compatibility with multiplication yields the following.

**Lemma 2.3.** The isomorphisms $C_{\gamma}$ are compatible with the multiplication maps, yielding an isomorphism of extensions of $\text{Gal}_F$ by $\hat{Z}^\prime$,

$$C = C_2 : \pi_1^\text{ét}(E_\gamma(\hat{G}), \bar{z}) \to E_2.$$ 

**Proof.** Suppose that $(\tau_1, \zeta_1) \in \text{Hom}(\bar{z}, \gamma_1 \bar{z})$ and $(\tau_2, \zeta_2) \in \text{Hom}(\bar{z}, \gamma_2 \bar{z})$. Their product in $\pi_1^\text{ét}(E_\gamma(\hat{G}), \bar{z})$ is $(\gamma_2^{-1}(\tau_1) \tau_2, \zeta_1 \zeta_2)$. We compute

$$C_{\gamma_1 \gamma_2}(\tau_1 \gamma^{-1}(\tau_2), \zeta_1 \zeta_2)(s(y) \cdot u) = \epsilon \left( \frac{\gamma_1 \gamma_2^{-1}}{\sqrt[1]{u}} \cdot \frac{\gamma_2^{-1}(\tau_1(y)) \tau_2(y)}{\sqrt{u}} \cdot \frac{\zeta_1(y)}{\sqrt{u}} \cdot \frac{\zeta_2(y)}{\sqrt{u}} \right) \cdot \gamma_1(\bar{y}) \gamma_2(\bar{y})$$

$$= \epsilon \left( \frac{\gamma_2^{-1}}{\sqrt[1]{u}} \cdot \frac{\gamma_1^{-1}(\tau_1(y)) \tau_2(y)}{\sqrt{u}} \cdot \frac{\zeta_1(y)}{\sqrt{u}} \cdot \frac{\zeta_2(y)}{\sqrt{u}} \right) \cdot \gamma_1(\bar{y}) \gamma_2(\bar{y})$$

$$= \epsilon \left( \frac{\gamma_1^{-1}}{\sqrt[1]{u}} \cdot \frac{\gamma_1^{-1}(\tau_1(y)) \tau_2(y)}{\sqrt{u}} \cdot \frac{\zeta_1(y)}{\sqrt{u}} \cdot \frac{\zeta_2(y)}{\sqrt{u}} \right) \gamma_1(\bar{y}) \gamma_2(\bar{y})$$

In the middle step, we use the fact that $\left( \frac{\gamma_1^{-1}}{\sqrt[1]{u}} \cdot \frac{\gamma_1^{-1}(\tau_1(y)) \tau_2(y)}{\sqrt{u}} \right)$ is an element of $\mu_n(F)$, and hence is $\text{Gal}_F$-invariant. This computation demonstrates compatibility of the isomorphisms $C_{\gamma}$ with multiplication maps, and hence the lemma is proven. □

### 2.3. Independence of base point.

Lastly, we demonstrate that the comparison isomorphisms

$$C_{\bar{z}} : \pi_1^\text{ét}(E_\gamma(\hat{G}), \bar{z}) \to E_2$$

depend naturally on the choice of convenient base point. With the pinned split group $G$ fixed, choosing a convenient base point is the same as choosing a splitting of $D_{Q,n}$ which restricts to $s_{\psi}$.

So consider two convenient base points $\bar{z}_1$ and $\bar{z}_2$, arising from splittings $s_1, s_2$ of $D_{Q,n}$ which restrict to $s_{\psi}$ on $Y_{Q,n}^{sc}$. Any isomorphism $\iota$ from $\bar{z}_1$ to $\bar{z}_2$ in the gerbe $E_\gamma(\hat{G})$ defines an isomorphism

$$\iota : \pi_1^\text{ét}(E_\gamma(\hat{G}), \bar{z}_1) \to \pi_1^\text{ét}(E_\gamma(\hat{G}), \bar{z}_2).$$

See [Wei13, Theorem 19.6] for details. In fact, the isomorphism of fundamental groups above does not depend on the choice of isomorphism from $\bar{z}_1$ to $\bar{z}_2$; thus one may define a “Platonic” fundamental group

$$\pi_1^\text{ét}(E_\gamma(\hat{G}), F)$$

without reference to an object of the gerbe.

**Theorem 2.4.** For any two convenient base points $\bar{z}_1, \bar{z}_2$, and any isomorphism $\iota : \bar{z}_1 \to \bar{z}_2$, we have $C_{\bar{z}_2} \circ \iota = C_{\bar{z}_1}$. Thus $E_2$ is isomorphic to the fundamental group $\pi_1(E_\gamma(\hat{G}), F)$, as defined in [Wei13, Theorem 19.7, Remark 19.8].
Proof. Choose any isomorphism from $\tilde{z}_1 = (\hat{T}, h_1, j_0)$ to $\tilde{z}_2 = (\hat{T}, h_2, j_0)$ in the gerbe $E_\gamma(\tilde{G})$. Here $h_1(1) = s_1$ and $h_2(1) = s_2$, and $j_0(1) = s_0$. Such an isomorphism $\tilde{z}_1 \xrightarrow{\sim} \tilde{z}_2$ is given by an isomorphism $\iota: \hat{T} \to \hat{T}$ of $\hat{T}$-torsors satisfying the two conditions

$$h_2 \circ \iota = h_1 \quad \text{and} \quad j_0 \circ p_\ast \iota = j_0.$$  

Such an $\iota$ is determined by the element $b = \iota(1) \in \hat{T}$. The two conditions above are equivalent to the two conditions

$$b^n = s_1/s_2 \quad \text{and} \quad b \in \hat{Z}. $$

The isomorphism $\tilde{z}_1 \xrightarrow{\sim} \tilde{z}_2$ determined by such a $b \in \hat{T}$ yields an isomorphism $\gamma_{\iota}: \gamma \tilde{z}_1 \to \gamma \tilde{z}_2$, for any $\gamma \in \text{Gal}_F$. The isomorphism $\gamma_{\iota}$ is given by the isomorphism of $\hat{T}$-torsors from $\gamma \hat{T}$ to $\gamma \hat{T}$, which sends $1$ to $\gamma(b)$.

This allows us to describe the isomorphism

$$\iota: \pi^{	ext{Et}}_1(E_\gamma(\tilde{G}), \tilde{z}_1) \to \pi^{	ext{Et}}_1(E_\gamma(\tilde{G}), \tilde{z}_2)$$

fibrewise over $\text{Gal}_F$. Namely, for any $\gamma \in \text{Gal}_F$, and any $f \in \text{Hom}(\tilde{z}_1, \gamma \tilde{z}_1)$, we find a unique element $\iota(f) \in \text{Hom}(\tilde{z}_2, \gamma \tilde{z}_2)$ which makes the following diagram commute.

$$\begin{array}{ccc}
\tilde{z}_1 & \xrightarrow{f} & \gamma \tilde{z}_1 \\
\downarrow{\iota} & & \downarrow{\gamma_{\iota}} \\
\tilde{z}_2 & \xrightarrow{\iota(f)} & \gamma \tilde{z}_2
\end{array}$$

If $f = (\tau, 1)$, then $\iota(f) = (\tau b/\gamma^{-1}b, 1)$. Indeed, when $\tau^n = \gamma^{-1}s_1/s_1$, we have

$$\left(\frac{\tau b}{\gamma^{-1}b}\right)^n = \frac{\gamma^{-1}s_1}{s_1} \cdot \frac{b^n}{\gamma^{-1}b^n} = \frac{\gamma^{-1}s_1}{s_1} \cdot \frac{s_2}{\gamma^{-1}s_1} = \frac{\gamma^{-1}s_2}{s_2}.$$ 

Thus $\iota(f) \in \text{Hom}(\tilde{z}_2, \gamma \tilde{z}_2)$ as required. In this way,

$$\iota: \pi^{	ext{Et}}_1(E_\gamma(\tilde{G}), \tilde{z}_1) \to \pi^{	ext{Et}}_1(E_\gamma(\tilde{G}), \tilde{z}_2),$$

is given concretely on each fibre over $\gamma \in \text{Gal}_F$ by

$$\iota(\tau, \zeta) = \left(\tau \cdot \frac{b}{\gamma^{-1}b}, \zeta\right).$$

Note that the conditions $b^n = s_1/s_2$ and $b \in \hat{Z}$ uniquely determine $b$ up to multiplication by $\hat{Z}[n]$. Since $\hat{Z}[n]$ is a trivial $\text{Gal}_F$-module, the isomorphism $\iota$ of fundamental groups is independent of $b$. Finally, we compute, for any $y \in Y_{Q,n}$, $u \in \hat{F}^\times$ such that $s_1(y) \cdot u \in D_{Q,n}$, and any $(\tau, \zeta) \in \text{Hom}(\tilde{z}_1, \gamma \tilde{z}_1)$,

$$[C_{\tilde{z}_2} \circ \iota](\tau, \zeta)(s_1(y) \cdot u) = C_{\tilde{z}_2}(\tau \gamma(b)/\gamma^{-1}b, \zeta)(s_1(y) \cdot u)$$

$$= C_{\tilde{z}_2}(\tau \gamma(b)/b, \zeta)(s_2(y) \cdot b^n(y)u)$$

$$= \epsilon \left(\frac{\gamma^{-1}\sqrt{b^n(y)u}}{\sqrt{b^n(y)u}} \cdot \tau(y) \cdot \frac{b(y)}{\gamma^{-1}(b(y))}\right) \cdot \zeta(y)$$

$$= \epsilon \left(\frac{\gamma^{-1}\sqrt{u}}{\sqrt{u}} \cdot \tau(y)\right) \cdot \zeta(u)$$

$$= C_{\tilde{z}_1}(\tau, \zeta)(s_1(y) \cdot u).$$

□
As noted in the introduction, this demonstrates compatibility between two approaches to the L-group.

**Corollary 2.5.** The L-group defined in [Wei15] is isomorphic to the L-group defined in [GG14], for all pinned split reductive groups over local or global fields.

**References**

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