Predicting Multidimensional Data via Tensor Learning

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Abstract

The analysis of multidimensional data is becoming a more and more relevant topic in statistical and machine learning research. Given their complexity, such data objects are usually reshaped into matrices or vectors and then analysed. However, this methodology presents several drawbacks. First of all, it destroys the intrinsic interconnections among datapoints in the multidimensional space and, secondly, the number of parameters to be estimated in a model increases exponentially. We develop a model that overcomes such drawbacks. In particular, we proposed a parsimonious tensor regression based model that retains the intrinsic multidimensional structure of the dataset. Tucker structure is employed to achieve parsimony and a shrinkage penalization is introduced to deal with over-fitting and collinearity. An Alternating Least Squares (ALS) algorithm is developed to estimate the model parameters. A simulation exercise is produced to validate the model and its robustness. Finally, an empirical application to Foursquares spatio-temporal dataset and macroeconomic time series is also performed. Overall, the proposed model is able to outperform existing models present in forecasting literature.

Keywords: Tensor regression, Multiway data, ALS, Multilinear regression

1 Introduction

Big Data and multidimensional data are becoming a relevant topic in statistical and machine learning research [Anandkumar et al. (2014); Cichocki (2014); Romera-Paredes et al. (2013); Brandi (2018)]. Working with such huge and complex objects in order to extract information is difficult for numerous reasons [Acar et al. (2009); Tumminello et al. (2005)]. Among the possible difficulties we can encounter

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working with such objects, there are the dimension of the dataset and the complexity of model used for the analysis [Acar et al. (2009)]. In some cases, the procedure in order to analyse huge datasets is to reduce its dimensionality or to use approximate methods. The issue is even more pronounced when we deal with multidimensional data. Multidimensional data are generally referred to datasets characterized by more than two dimensions. Examples are 3D images, panel data (individuals × variables × time × location) or higher order (> 2) multivariate portfolio moments [Acar et al. (2009)]. Given their intrinsic complexity, such datasets are usually reshaped into matrices or vectors and then analysed using standard methods. However, this manipulation of the original datasets has some drawbacks. First of all, it destroys the intrinsic interconnections between the data points in the multidimensional space [Billio et al. (2018); Lock (2018); Kolda and Bader (2009); Anandkumar et al. (2014); Brandi (2018)]. Secondly, the number of parameters to be estimated increases exponentially [Bro (1998); Smilde, Bro, and Geladi (2005); Zhang (2017); Acar et al. (2009)]. In this paper we exploit the intrinsic multidimensional structure of the dataset via a the construction of a tensor variate regression model. In particular, we build a Tensor regression with Tucker structured coefficient. The use of the Tucker structure in the learning model has two main advantages. The first one is the reduction of the complexity of the model while the second one is, in the spirit of reduced rank regression, to exploit the common information shared by the modes of the tensors [Izenman (1975); Mukherjee and Zhu (2011); Acar et al. (2009)]. We also build a penalized version of the model in order to deal with data collinearity and we propose and Alternating Least Squares (ALS) algorithm to estimate the model’s parameters.

The paper is structured as follows. Section 2 is related to the notation and operation on tensors. Section 3 introduces the tensor regression and the proposed estimation procedure. Section 4 is devoted to the simulation study and Section 5 to the empirical application with the results. Section 6 concludes.

2 Notation and preliminaries

Tensors are generalization of scalars, vectors and matrices and their order defines them [Kroonenberg and De Leeuw (1980); Anandkumar et al. (2014); Kolda and Bader (2009)]. The order of a tensor, which is the number dimensions that characterises it, is also referred to as ways or modes. Scalars are 0th order tensors, vectors are 1st order tensors and matrices are 2nd order tensors. Whatever has more than two dimensions is referred to as higher order tensor or just as tensor. Throughout this work the notation will follow the standard convention introduced in Kolda and Bader (2009): \( x \) is a scalar, \( x \) is a vector, \( X \) is a matrix and \( \mathbf{X} \) is a tensor.

2.1 Operations on tensors

There are different operations which can be performed with tensors through the use of linear and multilinear algebra. The literature on the topic is vast and [Kolda (2006); Bader and Kolda (2006); Kolda and Bader (2009); De Lathauwer, De Moor, and Vandewalle (2000); Anandkumar et al. (2014); Liu and Trenkler (2008)] are only few of the papers focusing on tensor operations. They carefully present the various calculations involving tensors and matrices both formally and with examples. For the sake of conciseness, we present only the tensor operations inherent to this work.
Matricization

The matricization of a tensor (also called unfolding) is the process of reshaping a tensor into a matrix. Take a tensor \( \mathbf{X} \in \mathbb{R}^{I_1 \times I_2 \cdots \times I_N} \). Let the ordered sets \( \mathcal{R} = \{r_1, \ldots, r_L\} \) and \( \mathcal{C} = \{c_1, \ldots, c_M\} \) be a partitioning of the modes of the tensor \( \mathcal{N} = \{1, \ldots, N\} \). The matricized tensor is defined as:

\[
\mathbf{X}_{(\mathcal{R} \times \mathcal{C}, \mathcal{N})} \in \mathbb{R}^{J \times K}
\]

where \( I_N \) is the size of the original tensor, \( J = \prod_{n \in \mathcal{R}} I_n \) and \( K = \prod_{n \in \mathcal{C}} I_n \). The \( n \)-mode matricization is a special case in which the set \( \mathcal{R} \) is a singleton equal to \( n \) and \( \mathcal{C} = \{1, \ldots, n-1, n+1, \ldots, N\} \). It is defined as \( \mathbf{X}_{(\mathcal{R} \times \mathcal{C}, \mathcal{N})} \equiv \mathbf{X}_{(n)} \). In this case the fibers of mode \( n \) are aligned as the columns of the resulting matrix. When \( \mathcal{R} = \mathcal{N} \) and \( \mathcal{C} = \emptyset \), we have the vectorization of the tensor, i.e.:

\[
\mathbf{X}_{(\mathcal{R} \times \emptyset, \mathcal{N})} \equiv vec(\mathbf{X}).
\]

Tensor multiplication

As for the matrix case, it is possible to define different tensors multiplications through the use of linear and multilinear algebra \cite{Kolda2006b, Bader2006, Kolda2009}. In particular, we define the \( n \)-mode product and the contracted product.

\( n \)-mode product

The \( n \)-mode product of the tensor \( \mathbf{X} \in \mathbb{R}^{I_1 \times I_2 \cdots \times I_N} \) with a matrix \( \mathbf{V} \) of size \( J \times I_n \) is denoted as:

\[
\mathbf{y} = \mathbf{X} \times_n \mathbf{V}
\]

and the resulting tensor \( \mathbf{y} \) is of size \( I_1 \times \cdots \times I_{n-1} \times J \times I_{n+1} \times \cdots \times I_N \).

This can be expressed in terms of matricized tensors as:

\[
\mathbf{y} = \mathbf{X} \times_n \mathbf{V} \iff \mathbf{Y}_{(n)} = \mathbf{V} \mathbf{X}_{(n)}.
\]

where \( \mathbf{Y}_{(n)} \) and \( \mathbf{X}_{(n)} \) are the \( n \)-mode matricization of \( \mathbf{y} \) and \( \mathbf{X} \).

Contracted product

The contracted product, also known as tensor contraction, is the tensor multiplication between tensors which extend the matrix product to higher dimension. Let \( \mathbf{X} \in \mathbb{R}^{J_1 \times J_2 \cdots \times J_N} \) and \( \mathbf{V} \in \mathbb{R}^{J_1 \times J_2 \cdots \times J_M} \). The constructed product is conveniently written as:

\[
\mathbf{y} = \langle \mathbf{X}, \mathbf{V} \rangle_{(J_1 \times J_2 \cdots J_N)}
\]

Where the subscripts \( J_X \) and \( J_Y \) are the modes over which the product is carried out. The dimension of \( J_X \) and \( J_Y \) can be different but the size of the modes over which the contraction is performed must be identical. This can be computed in matricized form as:

\[
\mathbf{X}_{(J_X \times J_Y^{\emptyset}, \mathcal{N})} \mathbf{V}_{(J_Y \times J_Y^{\emptyset}, \mathcal{M})}
\]
where $\mathcal{J}_X$ and $\mathcal{J}_V$ are the set of modes not involved in the product. The case in which $\mathbf{X}$ and $\mathbf{V}$ have the same dimension and the contraction is performed over all the modes, it defines the inner product and results in a scalar, i.e.:

$$y = \langle \mathbf{X}, \mathbf{V} \rangle$$

This results definition can be used to compute the Frobenious norm of a tensor, i.e.:

$$\|\mathbf{X}\|_F^2 = \langle \mathbf{X}, \mathbf{X} \rangle.$$  

This can be written in matrix form as $\|\mathbf{X}\|_F^2 = \|\mathbf{X}_{(n)}\|_F$ as presented in Bader and Kolda (2006). This shows that the Frobenious norm of a tensor is equivalent to the Frobenious norm of its $n$-mode matricization.

**Tensor decomposition**

Tensors, like matrices, can be decomposed in smaller (in terms of rank) objects Kolda and Bader (2009); Acar et al. (2009); Anandkumar et al. (2014). One factorization method employed in multi-way analysis is the Tucker decomposition theorized by Tucker (1964); Tucker (1966) and it represents an extension of the bilinear factor analysis to the higher dimensional case. It is also referred to as N-mode PCA Kapteyn, Neudecker, and Wansbeek (1986) and Higher-order SVD De Lathauwer, De Moor, and Vandewalle (2000). Take an $n$-th order tensor $\mathbf{X} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$. The Tucker decomposition of $\mathbf{X}$ takes the form of a $n$-mode product, i.e.:

$$\mathbf{X} \approx \mathbf{G} \times_1 \mathbf{U}^{(1)} \times_2 \mathbf{U}^{(2)} \ldots \times_N \mathbf{U}^{(N)} = \mathbf{G} \times \{\mathbf{U}^{(n)}\}. \quad (4)$$

Where $\mathbf{G}$ is the core tensor and it is usually of dimension smaller than $\mathbf{X}$ and $\mathbf{U}^{(n)}$ are the factor matrices. The Tucker decomposition can be be employed for two main reason, i.e. factor analysis or dimensionality reduction. The former is performed by analysing the factor matrices $\mathbf{U}^{(n)}$ corresponding to each mode of the decomposed tensor. The latter is achieved when $\mathbf{G}$ is of lower dimension than $\mathbf{X}$. In fact, it is possible to rewrite (approximately) $\mathbf{X}$ with a much lower number of components. In this paper, we use the Tucker structure on the regression coefficient in order to achieve parsimony and to exploit common information among the datapoints in different dimensions of the tensors, in the same spirit of reduce rank regression.

### 3 Tensor regression

In this section we propose a Tensor regression with Tucker coefficient and develop an ALS type of algorithm to estimate the model parameters. Tensor regression can be formulated in different ways: the tensor structure is only in the response or the regression variable or it can be on both. The literature related to the first specification is ample Zhou, Li, and Zhu (2013); Sun and Li (2017); Li and Zhang (2017); Guhaniyogi, Qamar, and Dunson (2017); Li et al. (2018); He et al. (2018) whilst the fully tensor variate regression received attention only recently from the statistics and machine learning communities employing different approaches Hoff (2015); Yu, Cheng, and Liu (2015); Lock (2018); Brandi (2018); Billio et al. (2018). In this work we will extend the tensor regression model in Brandi (2018) to the arbitrary dimensional case and, in same spirit of Lock (2018), we introduce a penalty to take into account the possible collinearity between the variables in the dataset. The fully tensor variate regression models proposed by Lock (2018); Billio et al. (2018) have the common
feature of having a PARAFAC (Andersen and Bro (2003); Bro (1997, 1998); Kolda and Bader (2009)) structure in the tensor coefficient. This is a limitation since the PARAFAC decomposition imposes the factor matrices to have the same number of components. In fact, the PARAFAC decomposition can be seen as a constrained Tucker decomposition with super-diagonal core tensor and with the same number of factor components along each dimension. The limitation becomes more evident when we deal with dimensionally skewed tensors, for which imposing the same number of components is excessively restrictive. The proposed tensor regression is formulated making use of the contracted product and can be expressed as:

$$Y = A + \langle X, B \rangle_{(J_X; J_B)} + E$$  \hspace{1cm} (5)$$

Where $$X \in \mathbb{R}^{N \times I_1 \times \cdots \times I_N}$$ is the regressor tensor, $$Y \in \mathbb{R}^{N \times J_1 \times \cdots \times J_M}$$ is the response tensor, $$E \in \mathbb{R}^{N \times J_1 \times \cdots \times J_M}$$ is the error tensor, $$A \in \mathbb{R}^{1 \times J_1 \times \cdots \times J_M}$$ is the intercept tensor while the slope coefficient tensor we are interested to learn is $$B \in \mathbb{R}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_M}$$. The subscripts $$J_X$$ and $$J_B$$ are the modes over which the product is carried out. Unfolding $$Y$$, $$X$$, $$E$$ and $$A$$ over the first $$(n + 1)$$ modes and $$B$$ over the first $$(N - n - 1)$$ modes, we can rewrite the regression (omitting subscripts for the sake of simplicity) as:

$$Y = A + XB + E$$

The regression takes the form of a multivariate regression for which the Least Squares (LS) solution (given sufficient data) of $$B$$ is:

$$\hat{B} = (X^TX)^{-1}X^TY$$

and

$$\hat{A} = E[Y] - E[X] \hat{B}.$$  \hspace{1cm} (6)$$

This equivalent representation of eq. 5 shows the relationship between multivariate regression and tensor regression. In fact, eq. 5 reduces to multivariate regression if $$Y$$ and $$X$$ are 2-nd order tensors (matrices). However, the $$B$$ coefficient is high dimensional. In order to resolve the issue and to enhance parsimony, we impose a Tucker structure on $$B$$ such that it is possible to recover the original $$B$$ with smaller objects, i.e.:

$$B \approx G \times_1 U^{(1)} \cdots \times_N U^{(N)} \times_{N+1} V^{(1)} \times \cdots \times_{N+M} V^{(M)}$$

where $$G$$ is the core tensor which drives the complexity of the reconstruction, $$U^{(n)}$$ are the factor matrices related to the input modes while $$V^{(m)}$$ are the factor matrices related to the output modes. This specification has two advantages. The first one is the aforementioned parsimony that in some cases is a necessary condition to estimate a model. The second one is that the factor structure in the regressor coefficient permits to link the dependent and independent variables through a restricted number of multilinear relationship. In fact, this specification can be seen as a reduced rank regression with multilinear interactions. This is very important feature when we deal with rank deficient (or near rank deficient) tensors or when we want to make forecasting because the links between regressor and response are restricted only to the fewer strong relations. In addition, the Tucker structure, contrarily of the PARAFAC, can handle dimension asymmetric (modality skewness) tensor regression since each dimension do not need to have the same number of components.
3.1 Estimation

In this section we introduce the supervised learning procedure to estimate the parameters of model presented in eq. [5]. Following the standard literature in regression analysis, we select the model parameters which minimise the the sum of squared residuals [1]:

\[ \| Y - \langle X, B \rangle_{(j_X:j_B)} \|^2_F = \| E \|^2_F = \langle E, E \rangle \]  

(7)

Imposing the Tucker structure of eq. [3] in the coefficient the optimization takes the form

\[ \hat{B} = \arg \min_{\text{Trk}(B) \leq R} \| Y - \langle X, B \rangle_{(j_X:j_B)} \|^2_F \]  

(8)

where Trk(B) is the Tucker rank of B and R is the chosen/estimated dimension of the core tensor G. The learning of the model parameter is a nonlinear optimization problem which can be solved by iterative algorithms such as ALS. The ALS algorithm was introduced for the Tucker decomposition by Kroonenberg and De Leeuw [1980]; Kapteyn, Neudecker, and Wansbeek [1986]. This methodology solves the optimization problem by dividing it into small least squares problems. This allows to solve the optimization problem described in eq. [8] for each factor matrix (and the core tensor G) of the Tucker structure separately, keeping constant the other ones at the previous iteration value. Then, it iterates alternatively among all the Tucker components until the algorithm converges (a minimum error threshold is reached) or a maximum number of iterations is reached. The partial least square solutions over each component \( U^{(n)} \) and G for the general case of an N-th order coefficient tensor are retrieved by Algorithm 1.

Algorithm 1 Alternating least squares

1. Initialize the algorithm to some \( U^{(n)}_0 \) and \( G_0 \).
2. repeat
3. \( U^{(1)}_i = \phi(U^{(2)}_{i-1}, \ldots, U^{(N)}_{i-1}, V^{(1)}_{i-1}, \ldots, V^{(M)}_{i-1}, G_{i-1}, X, Y) \)
4. \( \vdots \)
5. \( U^{(N)}_i = \phi(U^{(1)}_{i}, \ldots, U^{(N-1)}_{i}, V^{(1)}_{i-1}, \ldots, V^{(M)}_{i-1}, G_{i-1}, X, Y) \)
6. \( V^{(1)}_{i} = \phi(U^{(1)}_{i}, \ldots, U^{(N)}_{i}, V^{(2)}_{i}, \ldots, V^{(M)}_{i}, G_{i-1}, X, Y) \)
7. \( \vdots \)
8. \( V^{(M)}_{i} = \phi(U^{(1)}_{i}, \ldots, U^{(N)}_{i}, V^{(1)}_{i}, \ldots, V^{(M-1)}_{i-1}, G_{i}, X, Y) \)
9. \( G_{i} = \phi(U^{(1)}_{i}, \ldots, U^{(N)}_{i}, V^{(1)}_{i}, \ldots, V^{(M-1)}_{i-1}, X, Y) \)
10. until Convergence or Maximum iterations reached.
11. Return \( \hat{B} \)

Iterating over each component taking the others fixed, permits to solve the “big” problem as “small” least squares problems. In Algorithm 1, \( \phi(\cdot) \) is the LS problem for each subcomponent and makes clear the dependence of each sub-problem to the remaining components. We now explicitly show the LS solution for each Tucker component, i.e. \( U^{(1)}, \ldots, U^{(N)}, V^{(1)}, \ldots, V^{(M)} \) and \( G \). We start by estimating the input modes matrices \( U^{(1)}, \ldots, U^{(N)} \). We show the solution of the general \( U^{(n)} \) taking

\[ 1 \] Without loss of generality we omit \( A \) from the derivation.
the other components fixed. Define $\mathcal{H}_n$ to be the contracted product between $X$ and the Tucker coefficient $B$ without the $n$-th component matrix, i.e.

\[
\mathcal{H}_n = \langle X, B_{-n} \rangle (I_{x_{n+1}}; I_{y_{n}})
\]

\[
B_{-n} = \mathcal{J} \times_1 U^{(1)} \cdots \times_{n-1} U^{(n-1)} \times_{n+1} U^{(n+1)} \cdots \times_N U^{(N)} \times_{N+1} V^{(1)} \times \cdots \times_{N+M} V^{(M)}
\]

where $\mathcal{J}_{x_{n+1}}$ and $\mathcal{J}_{y_{n}}$ are the set of indices without the mode corresponding to the $n$-th component matrix. The resulting tensor $\mathcal{H}_n$ is of dimension $N \times I_n \times F_n \times J_1 \times \cdots \times J_M$. Unfolding $\mathcal{H}_n$ over the 2-nd and 3-rd modes, we get $H_n \in \mathbb{R}^{F_n \times NM}$ where $\mathcal{M} = \prod_{i=1}^{M} J_i$. This is the design matrix used to predict vec$(Y)$ from all the columns of $U^{(n)}$. Consequently, the LS solution for the $n$-th input mode matrix is

\[
\text{vec}(U^{(n)}) = (H_n H_n^T)^{-1} H_n \text{vec}(Y).
\]

The procedure to find the LS solution of the outcome modes is similar. We show the result for the general $V^{(m)}$ matrix taking all the other Tucker components fixed. Define $O_m$ to be the contracted product between $X$ and the tucker coefficient $B$ without the $m$-th outcome component matrix, i.e.

\[
O_m = \langle X, B_{-m} \rangle (I_{x}; I_{y_{m}})
\]

\[
B_{-m} = \mathcal{J} \times_1 U^{(1)} \cdots \times_N U^{(N)} \times N+1 V^{(1)} \cdots \times m-1 V^{(N+m-1)} \cdots \times N+m+1 V^{(m+1)} \times_{N+M} V^{(M)}
\]

where $\mathcal{J}_{x}$ and $\mathcal{J}_{B}$ are the full set of indices already defined for eq. [5]. The resulting tensor $O_m$ is of dimension $N \times \times J_1 \times \cdots \times J_{m-1} \times F_m \times J_{m+1} \times J_M$. Unfolding $O_m$ over the $(m+1)$-th mode gives $O_{m} \in \mathbb{R}^{F_m \times NM_m}$ where

\[
\mathcal{M}_m = \prod_{i=1}^{M \neq m} J_i.
\]

This is the design matrix used to predict the unfolded response tensor $Y$ along the $(m+1)$-th outcome mode, i.e. $Y_{(M+1)} \in \mathbb{R}^{J_m \times NM_m}$. Therefore, the LS solution for the $m$-th outcome mode matrix is

\[
V^{(m)} = (O_m O_m^T)^{-1} O_m Y_{(M+1)}.
\]

Finally, the unfolded core tensor is updated as:

\[
G = (X^{*T}_{(1)} X^{*}_{(1)})^{-1} X^{*T}_{(1)} Y^{*}_{(1)}.
\]

Where $X^{*}_{(1)}$ and $Y^{*}_{(1)}$ are the 1-st mode matricization of

\[
X^{*} = X \times_2 U^{(1)} \cdots \times_{N+1} U^{(N)}
\]

\[
Y^{*} = Y \times_2 V^{(1)} \cdots \times_{M+1} V^{(M)}
\]

where $V^{\dagger} = V^{\dagger}$ and $\dagger$ is the Moore–Penrose pseudo-inverse. If the dataset is not centered, the intercept tensor $\mathcal{A}$ is computed as:

\[
\mathcal{A} = \mathbb{E}[Y] - \langle \mathbb{E}[X], \mathcal{B} \rangle (I_{x}; I_{y_{m}}).
\]

\[\text{One can alternatively compute the } H_n \text{ matrix related to the } f_n \text{-th column of } U^{(n)} \text{ and then concatenate all these matrices to get } H_n = [H_{n}^{1}, \ldots, H_{n}^{F_n}].\]
3.2 Penalized Tensor regression

Even if the reduce rank specification is able to reduce the redundant information across the tensor dimension, it is still prone to over-fitting when intra-mode collinearity is present. In this case, a shirkage estimator is necessary for a stable solution. In fact, the presence of collinearity between the variables of the dataset degrades the forecasting capabilities of the regression model. In this work we will use the Tikhonov regularization Tikhonov (1943); Tikhonov and Arsenin (n.d.); Kennedy (2003). Known also as Ridge regularization would modify the optimization problem in eq. 7 as

$$\hat{B} = \arg \min_{\text{Trk}(\mathcal{B}) \leq \mathcal{R}} \| Y - \langle \mathcal{X}, \mathcal{B} \rangle \|_F^2 + \lambda \| B \|_F^2.$$  (15)

Where $\lambda > 0$ is the regularization parameter. The greater the $\lambda$ the stronger is the shrinkage effect on the parameters. However, high values of $\lambda$ increase the bias of the tensor coefficient $B$. For this reason, the shrinkage parameter is usual set via data driven procedures rather then input by the user. The optimization problem presented in eq. 15 can be rewritten as an unregularised problem with modified regressor and response tensors. In particular, we can rewrite eq. 15 as

$$\hat{B} = \arg \min_{\text{Trk}(\mathcal{B}) \leq \mathcal{R}} \| \bar{Y} - \langle \bar{\mathcal{X}}, \bar{\mathcal{B}} \rangle \|_F^2.$$  (16)

where $\bar{\mathcal{X}}$ is the modified version of $\mathcal{X}$. Using the same argument in Lock (2018), we can easily rewrite the LS solution without the need of running the regression with the augmented predictor and response variables, which can be high dimensional. The solution for the general $U^{(n)}$ taking the other components fixed is given by

$$\text{vec}(U^{(n)}) = (H_{[n]}H_{[n]}^T + I_{I_n \times I_n} \otimes \lambda \langle B_n B_n \rangle_{(n;n)})^{-1}H_{[n]} \text{vec}(Y).$$  (17)

where $I_{I_n \times I_n}$ is the Identity matrix of dimension $I_n \times I_n$. For the outcome modes matrices the solution becomes

$$V^{(m)} = (O_{[m]}O_{[m]}^T + \lambda \langle B_m B_m \rangle_{(m;m)})^{-1}O_{[m]} Y_{(m+1)}^T.$$  (18)

The constructed products $\langle B_n B_n \rangle_{(n;n)}$ and $\langle B_m B_m \rangle_{(m;m)}$ result in matrices fo dimensions $F_n \times F_n$ and $F_m \times F_m$ respectively. In this work we leave the $G$ unregularized since we want that the core tensor, which regulates the interconnections between the tensor dimensions, is not biased. However, the Ridge solution of $tG$ takes the form

$$G = (X^{*T}_{(1)}X^{*}_{(1)} + \lambda I)^{-1}X^{*T}_{(1)}Y^{*}_{(1)}.$$  (19)

4 Simulation study

In this section, we test the reliability of the model and its estimation procedure via simulation study. For this purpose, we implement two experiments. The first one is the ability of the model to retrieve the real, structural $B$ coefficient with a compressed Tucker version. To do this, we use as structural $B$ the image presented in fig. 1 which is a 3-rd order tensor of dimension $149 \times 118 \times 3$. The first two dimensions relates to the activation of the image pixels while the third mode controls the colour of each pixel.

We then generate a regression dataset as follows:
- Generate $X \in \mathbb{R}^{250 \times 149}$ from $N(0, 1)$.
- Generate $E \in \mathbb{R}^{250 \times 118 \times 3}$ from $N(0, 1)$.
- $Y = \langle X, B \rangle_{(2;1)} + E$.

The resulting $Y$ is of dimension $250 \times 118 \times 3$. In fig. 2 we report the tensor regression coefficient $B$ corresponding to different core specifications and their respective Bayesian information criterion (BIC). The Bayesian information criterion is a data-driven method for model selection. It makes into relationship the goodness of fit and its complexity, penalizing non-parsimonious models. It is computed as follows:

$$BIC = u \ln \left( \frac{SSR}{u} \right) + w \ln(u)$$

where $u$ is the number of data-points, $w$ is the number of estimated elements in the Tucker coefficient $B$ and $SSR$ are the sum of squared residuals of the regression. The BIC is formed by two additive components. The first one tends to decrease as the model becomes more complex, since it will reduce the sum of squared residuals while the second component increases as the model complexity increases. Therefore, the best model is associated with the minimum BIC. As we can see from fig. 2, the BIC is minimised for the setting corresponding to a core tensor of size $33 \times 33 \times 3$. This is a considerably high compression with respect to the original dimension ($\approx 77\%$ compression). Also from a graphical point of view, the reconstruction seems pretty accurate.

For the second experiment, we build a tensor dataset with strong collinearity within the modes. Collinearity is a condition for which different regressors are strongly correlated, making the estimation unstable. To simulate the dataset we use the Array Normal model with separable covariance matrices introduced in [Hoff et al. (2011); ?]. The model use the Tucker product to generate a tensor for which
Figure 2: Results for the tensor regression with image coefficient tensor $\mathcal{B}$. Number between square brackets refer to the size of the core tensor. The data-driven BIC criterion of eq. 20 are also reported. Each dimension has its own covariance matrix. Take a tensor $X \in \mathbb{R}^{T \times I_1 \times \cdots \times I_N}$. $X$ is an Array Normal dataset with covariance matrices $\Sigma_T$, $\Sigma_{I_1}$, $\ldots$, $\Sigma_{I_N}$ if we can rewrite it as:

$$X = Z \times_1 \Sigma_{T}^{\frac{1}{2}} \times_2 \Sigma_{I_1}^{\frac{1}{2}} \cdots \times_{N+1} \Sigma_{I_N}^{\frac{1}{2}}$$

(21)

where $Z$ is a normal variate tensor and $\Sigma_i^{\frac{1}{2}}$ is the square root of the covariance matrix for the $i$-th dimension. For this simulation, we generate a regression dataset as follows:

- Generate a standard Gaussian tensor $Z \in \mathbb{R}^{100 \times 6 \times 19}$.
- Generate the Covariance matrix for mode $i$ as $\Sigma_i = \rho_i^{\vert p-q \vert}$, where $p = 1, \ldots, P$ and $q = 1, \ldots, Q$ are the indices of the matrix components.
- Compute the $\Sigma_i^{\frac{1}{2}}$ as the Cholesky decomposition of $\Sigma_i$. 
- Build $\mathbf{X}$ as in eq. [21].
- Generate $\mathbf{B}$ to be Tucker tensor of dimension $6 \times 19 \times 6 \times 19$ with core tensor of size $2 \times 3 \times 2 \times 3$.
- Generate $\mathbf{E} \in \mathbb{R}^{100 \times 6 \times 19}$ from $N(0, 1)$.

The response tensor $\mathbf{Y}$ is then formed as

$$\mathbf{Y} = \langle \mathbf{X}, \kappa \mathbf{B} \rangle_{(2, 3; 1, 2)} + \mathbf{E}$$

where $\kappa$ is a scalar such that the Signal to Noise Ratio (SNR) is equal to a specified value, i.e.

$$\text{SNR} = \frac{\|\langle \mathbf{X}, \kappa \mathbf{B} \rangle_{(2, 3; 1, 2)}\|_F^2}{\|\mathbf{E}\|_F^2}.$$ 

The resulting $\mathbf{Y}$ is of dimension $100 \times 19 \times 6$. For the covariance matrices we used $\rho_T = 0.1$, $\rho_1 = 0.95$ and $\rho_2 = 0.8$. In this way, we have a strong collinearity in the second mode and a moderate one in the third mode. To see the effect of the shrinkage on prediction, we generate a new dataset of the same dimension and with the same $\mathbf{B}$, i.e.

$$\mathbf{Y}_{\text{new}} = \langle \mathbf{X}_{\text{new}}, \kappa \mathbf{B} \rangle_{(2, 3; 1, 2)} + \mathbf{E}_{\text{new}}.$$ 

We then compute the BIC for different specification of the model presented in eq. [15], using $\mathbf{Y}_{\text{new}}$ and $\hat{\mathbf{Y}}_{\text{new}} = \langle \mathbf{X}_{\text{new}}, \hat{\mathbf{B}} \rangle_{(2, 3; 1, 2)}$ for the computation of the SSR. For this simulation study we used the following specifications:

- Tucker rank $\mathbf{R}_\ast = [f, g, f, g]$ with $f = 1, \ldots, 6$ and $g = 1, \ldots, 19$.
- Shrinkage parameter $\lambda = 0, 0.5, 1, 5, 50$ as in Lock (2018).
- SNR = 1, 5 as in Lock (2018).

Table 1 provides the results for the simulation experiment. Given the huge dimension of the result, we provide for each $\lambda$ and SNR, the minimum value of the BIC and the corresponding estimated $\mathbf{R}_\ast$.

|       | SNR=1       |       | SNR=5       |
|-------|-------------|-------|-------------|
|       | BIC         | $\hat{f}$ | $\hat{g}$ | BIC         | $\hat{f}$ | $\hat{g}$ |
| $\lambda = 0$ | $1.7924e+03$ | 2      | 3       | $1.9677e+04$ | 1      | 1       |
| $\lambda = 0.5$ | $\textbf{1.7792e+03}$ | 2      | 3       | $1.9630e+04$ | 1      | 1       |
| $\lambda = 1$  | $1.7847e+03$ | 2      | 3       | $1.9595e+04$ | 1      | 1       |
| $\lambda = 5$  | $1.9243e+03$ | 2      | 3       | $1.9438e+04$ | 1      | 1       |
| $\lambda = 50$ | $2.4547e+03$ | 2      | 3       | $\textbf{1.8134e+04}$ | 1      | 2       |

Table 1: Table for the BIC computed on predicted values.

As it is possible to notice, for the low SNR scenario, the model has its optimum in correspondence to a shrinkage parameter $\lambda = 0.5$ and $\mathbf{R}_\ast = [2, 3, 2, 3]$ which is the core tensor size we used to create the regression. In the case in which the SNR is stronger, the model strongly shrinks the coefficient tensor ($\lambda = 50$) and the low-rank approximation suggests a one factor model in most of the cases. This
could be due to the fact that the redundant information is very strong due to the importance the numerator of the SNR has with respect to the denominator and the model captures it. With these results we showed in two simulation experiments that the model is able to correctly estimate the true structural coefficient even with a strong compression and to predict the response tensor variable in presence of redundant information induced by the collinearity condition. In the next section, we apply the Tensor regression model on real data.

5 Empirical application

In this section, we apply the proposed tensor regression model to forecasting. The purpose of this analysis is to test the prediction performance of the model against existing ones in different contexts. To this aim, we run two applications with the use of two different datasets. The first one is a spatio-temporal dataset based on the Foursquare platform. This dataset has been already used in the literature to test the forecasting ability of multilinear multitasking learning models [Yu, Cheng, and Liu (2015)]. For this application, we compare the Accelerated Low-rank Tensor Online Learning (ALTO) model proposed in [Yu, Cheng, and Liu (2015)] with the Tensor regression. The second application consists in forecasting a panel composed by 6 macroeconomic time series for 19 countries. In this case, we use as benchmark model the Vector Autoregression (VAR), which have become the workhorse model for macroeconomic forecasting [Karlsson (2013)].

5.1 Datasets

The Foursquare dataset contains the users’ check-in records in the Pittsburgh area between 24/02/2012 and 24/05/2012. On Foursquare, a person can check-in at his current location, leave tips about the venue, explore discounts around his current location or add other people as their friends. The dataset is composed by the number of check-ins by 121 users belonging to the 15 different types of venues over 1198 time periods. The second dataset used is a subsample of the data presented in [Pesaran, Schuermann, and Smith (2009)] on Global VAR. The subsample contains quarterly data from 1979Q2 to 2016Q4 of real GDP growth, the rate of inflation, log-returns of the nominal equity price index, log-returns of the exchange rate, nominal short-term interest rate and nominal long-term interest rate. The countries in the subsample (the ones for which all the data is available) are: Australia, Austria, Belgium, Canada, France, Germany, India, Italy, Japan, Korea, Netherlands, Norway, New Zeland, South Africa, Spain, Sweden, Switzerland, UK, USA. The data is checked for stationarity via the KPSS test and the Augmented Dickey fuller test. All the series results to be trend stationary at 5% level. The dataset is a tensor composed by $150 \times 6 \times 19$ variables hence the $B$ tensor coefficient to be estimated is of dimension $6 \times 19 \times 6 \times 19$ that implies 12996 parameters (assuming no intercept).

5.2 Forecasting

We now consider the Tensor regression in which the predictor ($X$) is a lagged version of the response variable ($Y$). It is the case of Autoregression in which a set of variables are regressed on their past values that we name Tensor Autoregression (TAR). In the case of a Tensor Autoregressive model, the coefficient tensor can be seen as a sort of Multilayer causality network in which each coefficient determines the effect in time and space from one variable to itself and to the other variables. To test
the TAR forecasting ability and compare it with alternative models (ALTO and VAR), we compute the mean root square forecasted error for each time series across different models, i.e.:

$$\text{RMSFE}_t = \sqrt{\sum_{h=1}^{H} \left( \frac{Y_{t+h} - \hat{Y}_{t+h}}{h} \right)^2}$$

where $t$ is the forecasting horizon. The model with a lower RMSFE$_t$ is the one with a better performance in forecasting.

5.3 Empirical results

In this section, we show the empirical results of the Tensor Autoregression in the two applications.

5.3.1 Foursquare dataset

For the Foursquare dataset we compare the forecasting error of the Accelerated Low-Rank Tensor Learning (ALTO) model proposed in Yu, Cheng, and Liu [2015] with the TAR model. We use the authors’ code for the ALTO model and compare the RMSFE of the two models with up to 4 lags and Tucker rank up to 4. Table 2 summarizes the results obtained using a 80% of training set.

| Rank | ALTO | | |  |  |  |  |  |  |
|------|------|---|---|---|---|---|---|---|---|
| Lag  | 1    | 2  | 3  | 1  | 2  | 3  | 1  | 2  | 3  |
| 1    | 0.13737 | 0.16135 | 0.16556 | 0.12689 | 0.12398 | 0.12353 |  |
| 2    | 0.12890 | 0.14400 | 0.14641 | 0.12551 | 0.12400 | 0.12364 |  |
| 3    | 0.12997 | 0.13887 | 0.14001 | 0.12454 | 0.12418 | 0.12380 |  |
| 4    | 0.12892 | 0.13547 | 0.13572 | 0.12471 | 0.12420 | 0.12440 |  |

Table 2: RMSFE of the ALTO and TAR models for the Foursquare data with different lag and Tucker rank specifications.

The result suggests that the TAR model overperforms the ALTO at any lag specification and in all the tensor rank specifications considered. This can be attributed to two main reasons. The first one is the fact that the ALTO is a 2-steps procedure in which in the first step the model is run as 15 independent VAR models and then, after concatenating the coefficient matrices, a tensor low-rank constraint is implemented. This modelling procedure do not exploits the full potential of the reduced rank regression since the low-rank constraint is imposed only after a full rank model is estimated. The second issue is that the 2-steps procedure do not take into account the interconnections between the 15 locations since those are run independently. The post-estimation low-rank constraint tackle only marginally the problem. In contrast, the Tensor Autoregressive model exploits all the interdimensional information estimating a low-rank model in the first stage.

The data and the Matlab code are available at the webpage of Rose Yu, [http://roseyu.com/code.html](http://roseyu.com/code.html)

The training set is the subsample corresponding to the first portion of the data that is used to estimate the model coefficients. The remaining part, the test set, is used to compute the forecasting error.
5.3.2 Macroeconomic panel data

In this section we compare the forecasting performance of the TAR with respect to the VAR model. For this purpose we test the two model performances with the use of the Diebold-Mariano (DM) test [Diebold and Mariano (1995)] over four forecasting steps (four quarter ahead). The Diebold-Mariano test is performed between the forecast errors of the TAR and VAR to assess if there is statistical difference in terms of forecasting between the two models. The data is split in 3 samples. The training set corresponds to 70% of the sample, 20% is devoted to the parameters optimization for the forecasting model (optimization sample) and the final 10% of the sample (test sample) is used to compute the forecast errors. For this empirical study, we use as baseline VAR model a VAR(1) which is estimated for each country separately. For the TAR, we used a 1 lag specification and the optimal Tucker rank $R_*$ and shrinkage parameter $\lambda$ are found in the parameters optimization step. In particular, we estimated the model parameter for a grid of values of the Tucker rank $R_*$ and shrinkage parameter $\lambda$ and then used the optimization sample to infer which specification had better performances. The parameters values we used are:

- Tucker rank $R_* = [f, g, f, g]$ with $f = 1, \ldots, 6$ and $g = 1, \ldots, 19$.
- Shrinkage parameter $\lambda = 0, 0.5, 1, 2.5, 5$.

Table 3 shows the results of the analysis. We can notice that the BIC computed on the optimization sample is minimized for the specification $R_* = [1, 1, 1, 1]$ and $\lambda = 5$. This correspond to a one-factor model with a strong shrinkage and can be attributed to the fact that there is a strong latent driving force within and between dimensions. For the computation of the forecasting errors, we use the forecasting recursion:

$$\hat{Y}_{t+h,i} = \chi_i(\hat{Y}_{t+h-1,i}, \Theta_i)$$

where $h$ is the forecasting horizon, $\chi_i$ is model $i$ used to forecast and $\Theta_i$ is the set of parameters of model $i$. We then compute the forecast errors for model $i$ as

$$FE_{h,i} = Y_{t+h,i} - \hat{Y}_{t+h,i}$$

and compute the Diebold-Mariao test under the assumption null hypothesis that $E[FE_{h,1} - FE_{h,2}] = 0$ against the alternative $E[FE_{h,1} - FE_{h,2}] \neq 0$. We refer to the original paper for the technical details of the test. We show the results of the Diebold-Mariano test for 1, 2, 3 and 4 steps ahead forecasts in fig. 3 and 4. The figures show the p-values associated to the modified Diebold-Mariano test proposed.

| $\lambda$ | BIC   | $f$ | $g$ |
|-----------|-------|-----|-----|
| 0         | -597.82 | 1   | 1   |
| 0.5       | -777.22 | 1   | 1   |
| 1         | -788.94 | 1   | 1   |
| 2.5       | -807.13 | 1   | 1   |
| 5         | -822.79 | 1   | 1   |

**Table 3:** Table for the BIC computed on predicted values for the optimization sample.
As explained above, the Null hypothesis states that the two competing models have the same predictive power against the alternative hypothesis for which they have different forecasting accuracy. Values lower than $\alpha = 0.05$ represents a rejection of the Null hypothesis of equal predictive ability. It is possible to notice that the two models have similar predictive ability in the majority of the cases. Exceptions to this behaviour are in favour of the TAR. In fact, which roughly 70% of the rejections comes from a lower RMSFE of the TAR model.

![Figure 3](image)

**Figure 3:** P-values of the DM test between TAR\{1; [1, 1, 1, 1]\} and the VAR(1). Yellow rectangles denote p-values lower than 0.05. Red crosses (×) refer to cases in favour of the TAR model while black circles (●) refer to cases in favour of the VAR.

We finally analyse the dependency structure over the different modes of the regression residuals. In order to do so, we compute the covariance over the different modes using the flip-flop algorithm proposed in [Hoff et al. (2011)](Hoff). The procedure is described in Algorithm 2 below. This algorithm is based on the assumption of the array Normal distribution for the residual, assumption generally satisfied for Macroeconomic data.

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5Their test corrects the standard DM test in order to be robustly used in the cases in which the number of forecasting errors is small.
Algorithm 2 Alternating least squares

1: Initialize the algorithm to some $\Sigma_1 \cdots \Sigma_{M+1}$
2: Compute $\mathcal{E} = \mathbf{y} - \hat{\mathbf{A}} - (\mathbf{X}, \hat{\mathbf{B}})(\mathcal{G}_X, \mathcal{G}_B)$
3: repeat for $m = 1, \ldots, M + 1$
4: Compute $\mathcal{E}(m) = \mathcal{E} \times_1 \Sigma_1 \cdots \times_{m-1} \Sigma_{m-1} \times_m \mathbf{I}_m \times_{m+1} \Sigma_{m+1} \cdots \times_M \Sigma_M$
5: Compute $\hat{\Sigma}_m = \mathbb{E}[(\mathcal{E}(m) \mathcal{E}(m))^T]$
6: :
7: Compute $\mathcal{E}(m+1) = \mathcal{E} \times_1 \Sigma_1 \cdots \times_m \hat{\Sigma}_m \times_{m+1} \mathbf{I}_{m+1} \times_{m+2} \Sigma_{m+2} \cdots \times_M \Sigma_M$
8: :
9: until Convergence or Maximum iterations reached.
10: Return $\hat{\Sigma}_1 \cdots \hat{\Sigma}_{M+1}$

However, $\hat{\Sigma}_m$ are not identifiable because if we multiply one of the covariance matrices for a scalar $w$ and another covariance matrix for the inverse value $w^{-1}$, the optimization logarithm reaches the same value. For this reason, we work with the correlation matrices rather than with the covariance matrices. To analyse the dependency structure between the modes, we perform a PCA on the correlation matrices and plot the first and second components in a biplot. Results are presented in fig. 5.

As it is possible to notice from the results, we can easily cluster the countries in 5 groups. We have the group of emerging countries (Korea, India and South Africa), the group of Pacific countries (Australia, Canada, New Zealand and USA), the group of strong non-Euro countries with stable inflation (Japan, Norway and UK), the group of European countries with high public debt (Belgium, France, Italy, Sweden, Spain) and the group of strong European Economies (Austria, Germany,
Netherlands and Switzerland) with a strong German business culture. Regarding the variables, as we could expect, the interest rates are closely related as are the FX log-returns and the Equity log-returns as both come from the financial system. GDP growth and inflation rate are in the same quadrant, making them related in the sense that they both come from real economy but depicts different aspect of an economy growth. These results highlight the good inferential ability of the model and that it can be used also as an inferential tools in addition to as forecasting model.

6 Conclusions

In this paper, we have introduced a tensor (Auto)regression (TAR) in which both the dependent and independent variables are tensors. To estimate the huge dimensional coefficient tensor, a Tucker structure is imposed in order to handle big (possibly dimensionally skewed) data. A penalized tensor regression is also implemented to deal with over-fitting and collinear data. An ALS algorithm is built and the model tested in two different simulation experiments. The simulations results are robust and confirm the ability of the model to deal with structured tensor coefficients and with collinearity. The TAR is then applied to the Foursquare spatio-temporal dataset and to a panel of macroeconomic time series. The forecasting ability of the model is tested against two different leading models: the ALTO model developed in Yu, Cheng, and Liu (2015) and the Vector Autoregression Karlsson (2013). Results showed an over-performance of the TAR model over the ALTO model applied to the Foursquare dataset over all the possible scenarios considered. The application to the Macroeconomic data confirms that, even if in the majority of the cases the TAR and VAR have the same statistical accuracy in forecasting, the proposed model has a better performance in the cases in which a statistical difference is detected (roughly 70% of the cases). These results certify the robust
nature of the predictions the model is able to produce. We finally analysed the dependency structure over different dimensions of the tensor, finding interesting results in the clustered nature of the data over both the variable and countries dimensions. There are still open research questions related to the use of such models. In particular, it is possible to impose a structure on the coefficient tensor, leading to the Structural Tensor Autoregression (STAR) model. This would be useful for both inference and forecasting when information on the structure of $\mathcal{B}$ comes from a theoretical motivation. A second and related research topic is to use local tensor projection to implement Impulse Response Functions (IRF) and analyse the effect and propagation of shocks from one variable (and possibly nation) to the whole system. Finally, since Tensor regression can be seen as a generalization of the Vector Error Correction Model (VECM) [Johansen (1991)], it would be of huge research interest the generalization of the VECM specification to the tensor context and analyse its performance.

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