Formal verification of higher-order probabilistic programs

Reasoning about approximation, convergence, bayesian inference, and optimization.

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Probabilistic programming provides a convenient lingua franca for writing succinct and rigorous descriptions of probabilistic models and inference tasks. Several probabilistic programming languages, including Anglican, Church or Hakaru, derive their expressiveness from a powerful combination of continuous distributions, conditioning, and higher-order functions. Although very important for practical applications, these combined features raise fundamental challenges for program semantics and verification. Several recent works offer promising answers to these challenges, but their primary focus is on semantical issues.

In this paper, we take a step further and we develop a set of program logics, named PPV, for proving properties of programs written in an expressive probabilistic higher-order language with continuous distributions and operators for conditioning distributions by real-valued functions. Pleasingly, our program logics retain the comfortable reasoning style of informal proofs thanks to carefully selected axiomatizations of key results from probability theory. The versatility of our logics is illustrated through the formal verification of several intricate examples from statistics, probabilistic inference, and machine learning. We further show the expressiveness of our logics by giving sound embeddings of existing logics. In particular, we do this in a parametric way by showing how the semantics idea of (unary and relational) $\top$-$\top$-lifting can be internalized in our logics. The soundness of PPV follows by interpreting programs and assertions in quasi-Borel spaces (QBS), a recently proposed variant of Borel spaces with a good structure for interpreting higher order probabilistic programs.

Additional Key Words and Phrases: probabilistic programming, formal reasoning, relational type systems

1 INTRODUCTION

Probabilistic programming is en vogue in statistics and machine learning, where modern probabilistic programming languages are viewed as a convenient lingua franca for writing classical statistical estimators, and for describing probabilistic models and performing probabilistic inference. A key strength of many modern probabilistic programming languages is their expressiveness, which allows programmers to give succinct descriptions for a broad range of probabilistic models, and to program specialized inference algorithms when generic algorithms do not perform well. While practically essential, expressiveness comes with significant theoretical challenges. Specifically, these languages adopt a combination of programming language features that goes beyond standard program semantics and program verification. In this paper, we consider the case of functional programming languages and focus on the following elements:

- **sampling**: the first key ingredient of a probabilistic programming language is a construct to sample from (continuous) distributions. We follow a monadic approach where probabilities are modelled as effects. Concretely, our language features a type constructor $M$ for
probability measures and monadic operations for sampling from continuous distributions or composing probabilistic computations.

- **conditioning**: the second key ingredient of probabilistic programming languages is a conditioning operator, which can be used to build a conditional distribution that incorporates observations from the real world. Conditioning is often performed through a specific construct, called observe, whose semantics is to first scale a distribution to a measure according to a likelihood function, and then normalize the resulting measure back to a distribution.

- **higher-order functions**: probabilistic models and statistical tasks are often described in a natural way by means of functional higher-order programs. The modularity that higher-order functions provide is useful to write likelihood functions, weighting functions, parametric models, etc. These components facilitate writing concise and expressive probabilistic computations.

Examples of probabilistic programming languages that incorporate the features above include Anglican, Church, and Hakaru. For example, Anglican [Wood et al. 2014] extends Scheme with constructs for basic probability distributions and an operation observe, which is used to build conditional distributions with respect to a predicate representing an observation of random variables. Church [Goodman et al. 2008] supports a similar operation but in a simply typed lambda calculus, Hakaru [Narayanan et al. 2016] supports them as a domain-specific language embedded in Haskell.

Despite their popularity, higher-order probabilistic programming languages pose significant challenges for semantics and verification. In particular, a classical result [Aumann 1961] shows that the category of measurable spaces is not cartesian closed, and thus it cannot be used to give denotational models for higher-order probabilistic languages. Aumann’s negative result has triggered a long line of research, which has culminated with recent proposals for semantic models for higher-order probabilistic languages. One such proposal, relevant to our work, is the notion of the quasi-Borel space (QBS) [Heunen et al. 2017], which has a rich categorical structure and yields an elegant denotational model for higher-order probabilistic programs.

A negative consequence of the difficulties of building sound semantic models for higher-order probabilistic programming languages is the scarcity of tools for reasoning about programs written in these languages. Several recent papers have started to look at this. For instance, Staton [2017] and Culpepper and Cobb [2017] have recently proposed equational methods for proving equivalences between higher-order probabilistic programs. While useful, the reasoning principles that these techniques support are limited and applicable only to simple examples. For more complex examples, the only currently viable approach is to resort directly to the denotational semantics; for instance, Ścibior et al. [2017] use semantic methods to prove the correctness of higher-order Bayesian inference. This stands in sharp contrast to non-probabilistic higher-order programs, where one has available a wide array of reasoning principles and logical tools for verifying programs.

**Our work**

The long-term goal of our research is to build practical verification tools for higher-order probabilistic programs, and to leverage these tools for building libraries of formally verified algorithms from machine learning and statistics. This paper makes an initial step towards this goal and justifies its feasibility by introducing a powerful framework, called the Probabilistic Programming Verification framework (PPV), for proving (unary and relational) properties of probabilistic higher-order programs with discrete and continuous distributions. PPV is:
• **expressive**: it can reason about different properties of probabilistic programs, including approximation, convergence, probabilistic inference and optimization.

• **practical**: it supports lean derivations that are not cluttered with measurability issues.

• **sound**: it can be soundly interpreted in the category of quasi-Borel spaces.

PPV’s design is based on three different logics: PL, UPL and RPL. These logics are presented in the style of [Aguirre et al. 2017]: PL is a higher-order logic which manipulates judgments of the form $\Gamma \vdash_\text{PL} \phi$; UPL is a unary program logic which manipulates judgments of the form $\Gamma \vdash_\text{UPL} e : \tau \mid \phi$, and finally RPL is a relational program logic which manipulates judgments of the form $\Gamma \vdash_\text{RPL} e : \tau \sim e' : \tau' \mid \phi'$. Here $\Gamma$ is a simple typing context; $\tau$ and $\tau'$ are the simple types of the expressions $e$ and $e'$; $\Psi$ is a set of assumed assertions; $\psi$ is a postcondition; and $\psi'$ is a relational postcondition. The proof systems are equi-expressive, but the unary and relational systems are closer to the syntax-directed style of reasoning generally favoured in program verification. We define an interpretation of assertions in the category of QBS predicates and prove that the logics are sound with respect to the interpretation. This interpretation guarantees that every subset of a quasi-Borel space yields an object in the category. As a consequence, assertions of the logic are interpreted set-theoretically, and extensionality is valid. This facilitates formal reasoning and thus program verification.

To further ease program verification, we define carefully crafted axiomatizations of fundamental probabilistic definitions and results, including expectations as well as concentration bounds. Following Scibior et al. [2017], we validate the soundness of these axiomatizations using synthetic measure theory for the QBS framework. This ensures that a derivation based on our proof system and axioms is valid in quasi-Borel spaces. A pleasant consequence of this approach is that in order to verify programs a user of PPV does not need to be familiar with QBS.

We validate our design through a series of examples from statistics, Bayesian inference and machine learning. We also demonstrate that our systems can be used as a framework where other program logics can be embedded. We show this in a parametrized way by using PPV to define a family of graded $\top\top$-liftings, a logical relation like technique to construct predicates/relations over probability distributions, starting from predicates/relations over values. As a concrete application, we use this definition to embed in our logic a union bound logic for reasoning about accuracy (on given predicates) [Barthe et al. 2016], and a logic for reasoning about probability distributions through relational couplings [Aguirre et al. 2018].

Overall, our work provides a fresh, verification-oriented, perspective on quasi-Borel spaces, and contributes to establish their status as a sound theoretical framework for practical verification of higher-order probabilistic programs.

2 **PPV BY EXAMPLE**

In this section we will introduce the general ideas behind PPV by presenting two examples.

*Continuous Observations: Two Uniform samples.* This warm-up example serves as an introduction to Bayesian conditioning and how we can reason about it in our system. Let us consider the following program `twoUs`:

```plaintext
let u1 = Uniform(0, 1) in let u2 = Uniform(0, 1) in mlet y = u1 \otimes u2 in observe y as \lambda x.(if \pi_1(x) < .5 \lor \pi_2(x) > .5 then 1 else 0)
```

The first line samples uniformly two reals in the unit interval, $u_1$ and $u_2$, while the second one puts them on the same probabilistic space over $\mathbb{real} \times \mathbb{real}$. Then, the third line introduces a bayesian conditioning. The prior $y$ gets conditioned by the likelihood function corresponding to the observation $\pi_1(x) < .5 \lor \pi_2(x) > .5$, and a posterior is computed. In this simple example, this is
morally equivalent to give score 1 to the traces that do satisfy the assertion, and score 0 to the ones that do not satisfy it, and rescaling the distribution. In general, we can use the observe construct with an arbitrary likelihood function to perform more general inference. After the observation, the posterior is a uniform distribution over the set \{(x_1, x_2) | x_1 < .5 \lor x_2 > .5\}.

The simple property we will show is that \(\Pr_{x \leftarrow r}[\pi_1(x) > .5] = 1/3\). This is expressed in the unary language UPL, since this is a unary property, through the following judgment:

\[
\Gamma \vdash_{\text{UPL}} \text{twoUs} : M[\text{real} \times \text{real}] \mid \Pr_{x \leftarrow r}[\pi_1(z) > .5] = 1/3
\]

where the distinguished variable \(r\) in the logical assertion represents the term that is being typed, that is, \(\text{twoUs}\). We show informally how to derive this assertion. The system UPL allows us to reason in a syntax-directed manner. Since the program starts with three let bindings (two non-monadic and one monadic), the first step will be to apply the rule for let bindings three times. This rule, which we will present formally in Section 6, moves into the context \(u_1, u_2\) and \(y\) plus the logical assertions about them. The resulting judgement is:

\[
\Gamma, x : \tau \vdash e' : \text{bool} \quad \Gamma, x : \tau \vdash e'' : \text{bool} \quad \Gamma \vdash e : M[\tau]
\]

\[
\Gamma \mid \Psi \vdash_{\text{UPL}} \text{observe } e \text{ as } \lambda x.(\text{if } e' \text{ then } 1 \text{ else } 0) : M[\tau] \mid \Pr_{y \leftarrow r}[e''[y/x]] = \frac{\Pr_x[e''[x]]}{\Pr_x[e']}
\]

This rule corresponds to a natural reasoning principle (derived by Bayes’ theorem) for \text{observe} when we have a boolean condition as likelihood function: the probability of an event \(e''\) under the posterior distribution is equal to the probability of the intersection of the event \(e''\) and the observation \(e'\), under the prior distribution \(e\), divided by the probability of \(e'\) under the prior distribution \(e\).

To apply this rule we need to rewrite the postcondition into the appropriate shape: a fraction that has on the numerator the probability of a conjunction of events and on the denominator the probability of the observed event. This can be done in UPL through subtyping which lets us reason directly in the logic PL. In PL we can prove the following judgment:

\[
\Gamma, x : \tau \vdash e' : \text{bool} \quad \Gamma, x : \tau \vdash e'' : \text{bool} \quad \Gamma \vdash e : M[\tau]
\]

\[
\Gamma \mid \Psi \vdash_{\text{PL}} \text{observe } e \text{ as } \lambda x.(\text{if } e' \text{ then } 1 \text{ else } 0) : M[\tau] \mid \Pr_{y \leftarrow r}[e''[y/x]] = \frac{\Pr_x[e''[x]]}{\Pr_x[e']}
\]

and this can be proved by applying the \(\text{Bayes}\) rule above, concluding the proof. We saw here at work different components of PPV: unary rules, subtyping, and a special rule for \text{observe}. All these components can be assembled in more complex examples as we will show in Section 8.

\[\text{We introduce the rule here to give some intuition, but this is also discussed in Section 6 after introducing PPV.}\]
**Monte Carlo Approximation.** As a second example we show how to use PPV to reason about other classical applications that do not use observations. We consider reasoning about expected value and variance of distributions. We will show convergence in probability of an implementation of the naive Monte Carlo approximation. This algorithm considers a distribution \( d \), and tries to approximate its expected value by sampling a number \( i \) of values and computing their mean.

Consider the following implementation of Monte Carlo approximation:

\[
\text{MonteCarlo} \equiv \text{letrec } f(i : \text{nat}) = \text{if } (i \leq 0) \text{ then return}(0) \text{ else mlet } m = f(i - 1) \text{ in mlet } x = d \text{ in return}((1/i) \ast (h(x) + m \ast (i - 1)))
\]

Our goal is to prove the convergence in probability of this algorithm, that is, the result can be made as accurate as desired by increasing the sample size \( n \), as described by the following UPL judgment (we omit the typing context for simplicity):

\[
\begin{align*}
(\mathbb{E}_{x \sim d}[1] = 1), (\sigma^2 = \text{Var}_{x \sim d}[h(x)]), (\mu = \mathbb{E}_{x \sim d}[h(x)]), (\epsilon > 0) & \vdash_{\text{UPL}} \text{MonteCarlo : nat } \rightarrow \text{M[real]} \mid \forall n, (n > 0) \implies \text{Pr}_{y \sim \text{rn}}[|y - \mu| \geq \epsilon] \leq \sigma^2/n\epsilon^2
\end{align*}
\]

(1)

Formally, we are showing that the probability that the computed mean \( y \) differs from the actual mean \( \mu \) by more than \( \epsilon \) is upper bounded by a value that depends inversely on \( n \). To derive (1) in UPL we need to perform two steps:

- Calculating the mass, mean, and variance of \( \text{MonteCarlo} \) in UPL:

\[
\begin{align*}
(\mathbb{E}_{x \sim d}[1] = 1), (\sigma^2 = \text{Var}_{x \sim d}[h(x)]), (\mu = \mathbb{E}_{x \sim d}[h(x)]), (\epsilon > 0) & \vdash_{\text{UPL}} \text{MonteCarlo : nat } \rightarrow \text{M[real]} \mid \\
\forall n : \text{nat}, (n > 0) & \implies (\mathbb{E}_{y \sim \text{rn}}[1] = 1) \land (\mathbb{E}_{y \sim \text{rn}}[y] = \mu) \land (\text{Var}_{y \sim \text{rn}}[y] = \sigma^2/n)
\end{align*}
\]

(2)

- Applying Chebyshev inequality (which can be proved in PL) to (2) by using subtyping.



We focus on the proof of (2), which is done is by induction on \( n \). In our system, the rule for \( \text{letrec} \) lets us prove inductive properties of (terminating) recursive functions by introducing an inductive hypothesis into the set of assertions that can only be instantiated for smaller arguments. After applying this rule, the new goal is:

\[
\phi_H \equiv \forall n : \text{nat}, (n < i) \implies (n > 0) \implies (\mathbb{E}_{y \sim f(n)}[1] = 1) \land (\mathbb{E}_{y \sim f(n)}[y] = \mu) \land (\text{Var}_{y \sim f(n)}[y] = \sigma^2/n)
\]

On this, we can apply a rule for case distinction accordingly to the two branches of the if-then-else, which gives us the following two premises:

\[
\begin{align*}
\Psi, (i \leq 0) & \vdash \text{return}(0) \mid \psi \\
\Psi, (i > 0) & \vdash \text{mlet } m = f(i - 1) \text{ in return}((1/i) \ast (h(x) + m \ast (i - 1))) \mid \psi
\end{align*}
\]

where \( \Psi = (\mathbb{E}_{x \sim d}[1] = 1), (\mu = \mathbb{E}_{x \sim d}[h(x)]), (\sigma^2 = \text{Var}_{x \sim d}[h(x)]), (i > 0), \phi_H \) and \( \psi = (\mathbb{E}_{y \sim r}[1] = 1) \land (\mathbb{E}_{y \sim r}[y] = \mu) \land (\text{Var}_{y \sim r}[y] = \sigma^2/i) \). The first premise is obvious: the preconditions \( (i > 0) \) and \( (i \leq 0) \) are contradicting. The second premise is proved by applying subtyping to a PL-judgment proved by instantiating the induction hypothesis with \( i - 1 \) and applying axioms on expected values. This allows us to conclude.

Again, we saw how at work different components of PPV: unary rules (including rules for inductive reasoning), subtyping, and the use of equation and axioms. All these components ease verification. We will use these components and the other components of PPV to verify more involved examples (including relational examples) in Section 8.
3  PCFₚ: A PROBABILISTIC EXTENSION OF PCF

We present now the probabilistic language PCFₚ we will consider in this paper, which is probabilistic extension of Plotkin’s PCF. The types come from the following grammar.

\[
\begin{align*}
\tilde{\tau} &::= \text{unit} \mid \text{bool} \mid \text{nat} \mid \text{real} \mid \text{pReal} \mid \tilde{\tau} \times \tilde{\tau} \mid \text{list}(\tilde{\tau}) \quad \text{(Basic Types)} \\
\tau &::= \tilde{\tau} \mid M[\tau] \mid \tau \rightarrow \tau \mid \tau \times \tau \mid \text{list}(\tau) \quad \text{(Types)}
\end{align*}
\]

We distinguish two sorts of types: Basic Types and Types. The former as the name suggests include standard basic types (where pReal is the type of positive real numbers), and products and lists of them, the latter include a monadic type \( M[\tau] \) for general measures on \( \tau \), as well as function and product types. As we will see later in Section 7, Basic Types will be interpreted in standard Borel spaces, while for general Types we will need quasi-Borel spaces. The language of PCFₚ expressions is defined by the following grammar.

\[
e ::= x \mid c \mid f \mid e \mid e \mid \lambda x. e \mid \langle e, e \rangle \mid \pi_i(e) \mid \text{case } e \text{ with } [d, \overline{\tau} \Rightarrow e_i]_i \mid \text{letrec } f x = e \mid \text{return } e \mid \text{bind } e \mid \text{observe } e \text{ as } e \mid \text{Uniform}(e, e) \mid \text{Bern}(e) \mid \text{Gauss}(e, e)
\]

Most of the constructions in the language are standard. We use \( c \) to range over a set of basic constants and \( f \) to range over a set of basic functions. We have monadic constructions \( \text{return } e \) and \( \text{bind } e_1 \). \( e_2 \) for the monadic type \( M[\tau] \), an observe construction \( \text{observe } e_1 \) as \( e_2 \) for computing the posterior distribution given a prior distribution \( e_1 \), and a likelihood function \( e_2 \), and primitives representing basic probability distributions.

\[
\begin{array}{c}
\Gamma \vdash e : M[\tau] \\
\Gamma \vdash e' : \tau \rightarrow \text{pReal} \\
\end{array} \quad \quad \quad
\begin{array}{c}
\Gamma, f : I \rightarrow \sigma, x : I \vdash e : \sigma \\
I \in \{\text{nat}, \text{list}(\tau)\} \\
\end{array}
\]

Here, \( \text{Terminate}(f, x, e) \) is a termination criterion which ensures that all recursive calls are smaller arguments. PCFₚ expressions are typed accordingly to simply typing rules which are rather standard and that we omit here. We also consider a basic equational theory for expressions based on \( \beta \)-reduction, extensionality and monadic rules. Also these are standard and we omit them here. However, we will enrich this equational theory in Section D with axioms and equations reflecting common reasoning principles for probabilistic programming.

4  A LOGIC FOR PROBABILISTIC PROGRAMS

In this section we introduce a logic, named PL, for reasoning about probabilistic programs which forms the basis for PPV. This is an higher order logic with basic predicates over expressions of PCFₚ. To support more natural verification in PPV we enrich PL with a set of axioms allowing a wide variety of reasoning principles over probabilistic programs.

For convenience, we will also use some syntactic sugar: \( \text{let } x = e_1 \text{ in } e_2 \equiv (\lambda x. e_2)e_1, (\text{mlet } x = e_1 \text{ in } e_2) \equiv \text{bind } e_1 \lambda x. e_2, \) and \( e_1 \otimes e_2 \equiv \text{bind } e_1 \lambda x. (\text{bind } e_2 \lambda y. \text{return}(x, y)) \).

4.1  The PL Logic

The first component of PPV is an higher-order logic, named PL, useful to reason about probabilistic programs in PCFₚ. Logical formulas of PL are defined by the following two-level grammar:

\[
t ::= e \mid \mathbb{E}_{x: \tau}(t(x)) \mid \text{scale}(t, t) \mid \text{normalize}(t) \quad \text{enriched expressions} \\
\phi ::= (t = t) \mid (t < t) \mid T \mid \bot \mid \phi \land \phi \mid \phi \implies \phi \mid \neg \phi \mid \forall x : \tau. \phi \mid \exists x : \tau. \phi \quad \text{logical formulas}
\]

Enriched expressions enrich arbitrary PCFₚ expressions \( e \) with constructions for expectations \( \mathbb{E}_{x: \tau}(t(x)) \), rescaling of measures \( \text{scale}(t, t) \), and normalization \( \text{normalize}(t) \). A logical formula
Similarly to expressions in PCFP, we will consider only enriched expressions which are well-typed. The typing rules for the additional constructions in PL are the following.

\[
\Gamma \vdash t_1 : M[\tau] \quad \Gamma \vdash t_2 : \tau \rightarrow p\text{Real} \quad \Gamma \vdash t_1 : M[\tau] \quad \Gamma \vdash t_2 : \tau \rightarrow p\text{Real} \quad \Gamma \vdash t : M[\tau]
\]

Intuitively, \(\mathbb{E}_{x \sim t}(t_2(x))\) is the expected value of the function \(t_2\) over the distribution \(t_1\); \(\text{scale}(t_1, t_2)\) is a distribution obtained from an underlying measure \(t_1\) by rescaling its components by means of the density function \(t_2\); \(\text{normalize}(t)\) is the normalization of a measure \(t\) to a probability distribution (a measure with mass 1). Expectations of real-valued functions are defined by the difference of positive and negative parts. Precisely, for given \(\Gamma \vdash t_1 : M[\tau]\) and \(\Gamma \vdash t_2 : \tau \rightarrow \text{real}\), we define the expectation as the following syntactic sugar \(^2\): 

\[
\mathbb{E}_{x \sim t_1}(t_2(x)) \equiv \mathbb{E}_{x \sim t_1}(\text{if } t_2(x) > 0 \text{ then } |t_2(x)| \text{ else } 0) - \mathbb{E}_{x \sim t_1}(\text{if } t_2(x) < 0 \text{ then } |t_2(x)| \text{ else } 0).
\]

We can also define variance and probability in terms of expectation:

\[
\Pr_{x \sim e'}[e'] \equiv \mathbb{E}_{x \sim e}(\text{if } e' \text{ then } 1 \text{ else } 0) \quad \text{Var}_{x \sim e_1}[e_2] \equiv \mathbb{E}_{x \sim e_1}[(e_2)^2] - \mathbb{E}_{x \sim e_1}[e_2]^2.
\]

A PL judgment is a judgment of the form \(\Gamma \vdash \Psi \vdash \phi\) where \(\Gamma\) is a context assigning types to variables, \(\Psi\) is a set of formulas well formed in the context \(\Gamma\), and \(\phi\) is a formula also well formed in \(\Gamma\). Rules to derive well-formedness judgments \(\Gamma \vdash \phi\) are rather standard and we omit them here. We will often refer to \(\Psi\) as the precondition. The proof rules for PL are rather standard, so we give just a selection of them in Figure 1. We stress that we do not introduce special rules for enriched expressions, but this are treated as standard expressions in higher order logic. However, we will introduce some axioms on enriched expressions in Section D.

5 AXIOMS AND EQUATIONS OF ASSERTIONS FOR STATISTICS

We introduce axioms and equations in the logic PL. First, we have the standard equational theory for expressions based on \(\alpha\)-conversion, \(\beta\)-reduction, extensionality, and the monadic rules of the monadic type \(M\) (we omit here). The monadic type \(M\) also has the commutativity (Fubini-Tonelli equality), written as the following equation:

\[
(b \text{bind } e_1 \lambda x. (b \text{bind } e_2 \lambda y. e(x,y))) = (b \text{ bind } e_2 \lambda y. (b \text{ bind } e_1 \lambda x. e(x,y)) \quad (x, y: \text{ fresh})
\]

We introduce some equalities around expected values. We have the monotonicity and linearity of expected values (axioms 49, 50), and we also have Cauchy-Schwartz inequality (axiom 51). We are

\(^2\) We use absolute values \(| - | : \text{real} \rightarrow p\text{Real}\) to adjust the typing. The right-hand side is undefined if both expectations are infinity. We could avoid this kind of undefinedness by stipulating \(\infty - \infty = -\infty\), but we leave it undefined since this actually never shows up in our concrete examples.
able to transform the variables in the expression of expected values.

\[ (\forall x: \tau . \ e' \geq 0) \implies \mathbb{E}_{x \sim e}[e'] \geq 0 \]  
\[ \mathbb{E}_{x \sim e}[e_1 + e_2] = \mathbb{E}_{x \sim e}[e_1] + \mathbb{E}_{x \sim e}[e_2] \]  
\[ (\mathbb{E}_{x \sim e}[e_1])^2 \leq \mathbb{E}_{x \sim e}[e_1^2] \]  
\[ \mathbb{E}_{x \sim \text{bind} \ e \ \lambda y. \ return(e')}[e'''] = \mathbb{E}_{y \sim e}''[x'/x] \]

We also introduce some basic equalities on observations, rescaling, and normalizations.

\[ \mathbb{E}_{x \sim d'}[h(x) \cdot g(x)] = \mathbb{E}_{x \sim \text{scale}(d',g)[h(x)].} \]  
\[ (\text{scale}(\text{scale}(e_1, e_2), e_3) = (\text{scale}(e_1, \lambda x. (e_2(x) \ast e_3(x))), \quad e = \text{scale}(e, \lambda_.1) \]  
\[ (\text{mlet}\ x = \text{scale}(e_1, e_2) \text{ in } e_3(x)) = (\text{mlet}\ x = e_1 \text{ in } \text{scale}(e_3(x), \lambda u. e_2(x)) \]  
\[ \text{scale}(e_1, e_2) \ast \text{scale}(e_3, e_4) = \text{scale}(e_1 \ast e_2, \lambda w. e_2(\pi_1(w)) \ast e_4(\pi_2(w))) \]  
\[ \mathbb{E}_{y \sim e}[1] < \infty \implies (\text{binde'} \ ' \lambda x.e) = (\text{scale} (e, \mathbb{E}_{y \sim e}[1])) \quad (x \notin \text{FV}(e)) \]  
\[ \text{observe}_e \quad \text{as} \quad e_2 = \text{normalize}(\text{scale}(e_1, e_2)) \]  
\[ \text{normalize}(e) = \text{scale}(e, \lambda u.1/\mathbb{E}_{x \sim e}[1]) \quad (u \notin \text{FV}(\mathbb{E}_{x \sim e}[1])) \]  
\[ 0 < \alpha < \infty \implies \text{normalize}(\text{scale}(e_1, e_2)) = \text{normalize}(\text{scale}(e_1, \alpha \ast e_2)) \]

We may introduce the axioms for particular distributions such as \( \mathbb{E}_{x \sim \text{Bern}(e)}[\text{if } x \text{ then } 1 \text{ else } 0] = e \) \( (0 \leq e \leq 1), \mathbb{E}_{x \sim \text{Gauss}(e_1,e_2)}[x] = e_1 \), and etc. We omit them right now.

5.1 Markov and Chebyshev inequalities

The axioms in PL that we introduced above are quite standard, but we already enjoy meaningful discussions in probability theory. For instance, we can prove Markov inequality (61) and Chebyshev inequality (62) in PL.

\[ d : M[\text{real}] , a : \text{real} \vdash_{\text{PL}} (a > 0) \implies \Pr_{x \sim d} [x \geq a] \leq \mathbb{E}_{x \sim d}[x]/a. \]  
\[ d : M[\text{real}], b : \text{real}, \mu : \text{real} \vdash_{\text{PL}} [\mathbb{E}_{x \sim d}[1] = 1 \wedge \mu = \mathbb{E}_{x \sim d}[x] \wedge b^2 > 0 \implies \Pr_{x \sim d} [|x - \mu| \geq b] \leq \text{Var}_{x \sim d}[x]/b^2. \]  

6 UNARY/RELATIONAL HIGHER-ORDER LOGIC

In this section, we introduce the two program logics which constitute the core of PPV. We first introduce a unary higher-order logic UPL supporting the verification of unary properties of probabilistic programs. Then, we introduce a relational higher-order logic RPL supporting the verification of relational properties of probabilistic programs. Both these program logics use PL as assertion logic.

6.1 The Unary Logic UPL

Judgments in the unary logic UPL have the shape: \( \Gamma \mid \Psi \vdash_{\text{UPL}} e : \tau \mid \phi \) where \( \Gamma \) is a context, \( \Psi \) is a set of assertions on the context variables, \( e \) is a PCF_{\text{P}} expression, \( \tau \) a type, and \( \phi \) is an assertion (possibly) containing a distinguished variable \( r \) of type \( \tau \) which is used to refer to the value of the expression \( e \) in the formula \( \phi \). We give in Figure 2 a selection of proof rules in UPL. We have two groups of rules, rules for pure computations and rules for probabilistic computations. The rules are mostly syntax-directed, with the exception of the rule \( [u-\text{SUB}] \). We present a selection of the pure rules, the rest of them are as in UHOL ([Aguirre et al. 2017]). The rule \( [u-\text{ABS}] \) turns an assertion about the bound variable into a precondition of its lambda abstraction. The \( [u-\text{APP}] \) rule shows a postcondition of a function application provided that the argument satisfies the precondition of
6.2 The Relational Logic RPL

Judgments in the relational logic RPL have the shape: \( \Gamma | \Psi \vdash_{\text{UPL}} e_1 : \tau_1 \sim e_2 : \tau_2 | \phi \) where \( \Gamma \) is a context, \( \Psi \) is a set of assertions on the context, \( e_1 \) and \( e_2 \) are PCF expressions, \( \tau_1 \) and \( \tau_2 \) are types, and \( \phi \) is an assertion (possibly) containing two distinguished variables \( r_1 \) of type \( \tau_1 \) and \( r_2 \) of type \( \tau_2 \) which are used to refer to the value of the expressions \( e_1 \) and \( e_2 \) in the formula \( \phi \). We give in Figure 3 a selection of proof rules in RPL. We present three groups of rules. The first consists of relational rules for pure computations. The second one consists of relational rules for probabilistic computations that are two-sided, meaning that the term on both side of the judgment have the same top-level constructor. Finally, the third group consists of relational rules for probabilistic

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The function. The \([u\text{-LETREC}]\) rule allows to prove properties of terminating recursive functions by introducing an induction hypothesis in the context.

In the case of monadic computations we have rules for monadic return, binding and observation. It is worth noticing that in the second premise of both the rules \([u\text{-BIND}]\) and \([u\text{-OBS}]\), the assertion quantifies over elements in \( M[\tau_1] \), while the input type of the function is just \( \tau_1 \). This follows the spirit of the interpretation (see Section 7), where the Kleisli lifting \((-)^*\) is used to lift a function \( \tau_1 \to M[\tau_2] \) to a function \( M[\tau_1] \to M[\tau_2] \). The quantification over distributions, rather than over elements, is essential to establish a connection with the assertion on the first premise. This will be useful to simplify the verification of our examples.

We can prove that we do not lose in expressivity with respect to PL. We have that the unary logic UPL is sound and complete with respect to the underlying logic PL.

**Theorem 6.1 (Equi-Derivability of PL and UPL).** The judgment \( \Gamma | \Psi \vdash_{\text{PL}} \phi[e/r] \) is derivable if and only if the judgment \( \Gamma | \Psi \vdash_{\text{UPL}} e : \tau | \phi \) is derivable.

---

The Rules for pure constructions.

\[
\frac{\Gamma \vdash x : \tau}{\Gamma | \Psi \vdash_{\text{UPL}} x : \tau | \phi} \quad [\text{VAR}]
\]

\[
\frac{\Gamma | \Psi \vdash_{\text{PL}} \lambda x : \tau. t : \sigma | \phi}{\Gamma | \Psi \vdash_{\text{UPL}} \lambda x : \tau. t : \sigma | \phi} \quad [\text{ABS}]
\]

\[
\frac{\Gamma | \Psi \vdash_{\text{UPL}} \phi[t/r][\tau]}{\Gamma | \Psi \vdash_{\text{UPL}} \phi[t/r]} \quad [\text{SUB}]
\]

\[
\frac{\Gamma | \Psi \vdash_{\text{UPL}} \phi[t/r][\tau]}{\Gamma | \Psi \vdash_{\text{UPL}} \phi[t/r]} \quad [\text{APP}]
\]

\[
\frac{\Gamma, \tau : \tau \to \sigma \mid \forall \phi'. \phi[\tau] \Rightarrow \phi[\tau]}{\Gamma | \Psi \vdash_{\text{UPL}} \phi[t/r][\tau]} \quad [\text{LETREC}]
\]

The Rules for probabilistic constructions.

\[
\frac{\Gamma | \Psi \vdash e : \tau | \phi[\text{return}(r)/r]}{\Gamma | \Psi \vdash \text{return}(e) : M[\tau] | \phi} \quad [\text{RET}]
\]

\[
\frac{\Gamma | \Psi \vdash e : M[\tau_1] | \phi_1 \quad \Gamma | \Psi \vdash e' : \tau_1 \to M[\tau_2] | Vs : M[\tau_1],(\phi_1[s/r] \Rightarrow \phi_2[\text{binds} s/r])}{\Gamma | \Psi \vdash \text{bind} e e' : M[\tau_2] | \phi_2} \quad [\text{BIND}]
\]

\[
\frac{\Gamma | \Psi \vdash e : M[\tau] | \phi_1 \quad \Gamma | \Psi \vdash e' : \tau \to \text{pReal} | Vs : M[\tau],(\phi_1[s/r] \Rightarrow \phi_2[\text{observes} s/r])}{\Gamma | \Psi \vdash \text{observe} e e' : M[\tau] | \phi_2} \quad [\text{OBS}]
\]

Fig. 2. A selection of UPL rules.
computations that are one-sided, meaning that they have a specific top-level constructor on one side and an arbitrary expression on the other. Here we just show the left-sided rules that have the constructor on the left, right-sided rules are symmetrical. As in the unary case, we use an approach that is mostly syntax-directed except for the [r-SUB] rule.

The rules for pure computations are similar to the ones from RHOL [Aguirre et al. 2017] and we just present a selection. For the probabilistic constructions, we have relational rules for the monadic return and bind, and for observe. These rules are the natural generalization of the unary rules to the relational case. In particular, in all the rules for bind and observe we use assertions quantifying over distributions, similarly to what we have in UPL, to establish a connection between the different assertions.

The equi-derivability result for UPL can be lifted to the relational setting: RPL is also sound and complete with respect to the logic PL.
Theorem 6.2 (Equi-derivability of PL and RPL). The judgment $\Gamma \vdash_\text{PL} \phi[e_1/r_1, e_2/r_2]$ is derivable if and only if $\Gamma \vdash_\text{RPL} e_1 : \tau_1 \sim e_2 : \tau_2 | \phi$ is derivable.

6.3 Special Rules
As already discussed in the introduction, we enrich PPV also with special rules that can ease verification. One example is the use the following Bayesian law expressing a general fact about the way we can reason about probabilistic inference when the observation is a boolean:

$$\begin{align*}
\Gamma, x : \tau \vdash e' : \text{bool} & \quad \Gamma, x : \tau \vdash e'' : \text{bool} \\
\Gamma \vdash e : M[r] & \quad \frac{\text{Pr}_{x \sim e'}[e''[y/x]] = \frac{\text{Pr}_{x \sim e'}[e''][y/x]}{\text{Pr}_{x \sim e'}[e']}}{	ext{Bayes}}
\end{align*}$$

This rule can be derived by first using [u-OBS], and then reasoning in PL through the [u-SUB] rule, which is why the premises are just simply typed assumptions. In particular, in PL we use the characterization of observe given in Section D.

We also introduce a [LET] rule, which can be derived by desugaring the let notation:

$$\Gamma \vdash e : \tau | \phi_1 \quad \Gamma, x : \tau_1 | \Psi, \phi_1[x/r] \vdash e' : \tau_2 | \phi_2$$

$$\Gamma \vdash e \text{ let } x = e \text{ in } e' : \tau_2 | \phi_2$$

[LET]

Notice that Theorem 6.1 can be used to convert UPL derivation trees into PL ones and vice versa. Similarly, Theorem 6.2 is used to convert RPL to PL. These conversions are useful to switch between the different levels of our system and reason in the more convenient one. To this end, we introduce the following admissible rules:

$$\frac{\Gamma \vdash_\text{UPL} e : \tau | \phi}{\Gamma \vdash_\text{PL} e[r]/\phi}$$

[conv-UPL]

$$\frac{\Gamma \vdash_\text{RPL} e_1 : \tau_1 \sim e_2 : \tau_2 | \phi}{\Gamma \vdash_\text{PL} e_1[r_1] \sim e_2[r_2]/\phi}$$

[conv-RPL]

7 SEMANTICS
7.1 Background
In this section we introduce the semantics ideas giving the ground on which PPV is designed. We will start by recalling the definition of quasi-Borel spaces [Heunen et al. 2017] and by showing how we can use them to define a monads for probabilistic measures [Ścibior et al. 2017]. These constructions will be then used in the next section to give the semantics of programs on which we will build an higher order logic.

Quasi-Borel Spaces. We introduce here the category QBS of quasi-Borel spaces. Intuitively, the category QBS is a relaxation of the category Meas of measurable spaces which has a nice categorical structure, i.e. it is cartesian closed, and retains the important properties coming from measure theory. Before introducing quasi-Borel spaces, we fix some notation. We will use $\mathbb{R}$ to denote the of the real line equipped with the standard Borel algebra. We will denote $\coprod_{i \in \mathbb{N}} S_i$ the coproduct of a countable family of sets $\{S_i\}_{i \in \mathbb{N}}$, and we will use $[\alpha_i]_{i \in \mathbb{N}}$ for the copairing of functions $\alpha_i$ for $i \in \mathbb{N}$.

Definition 7.1 (Heunen et al. [2017]). The category QBS is the category of quasi-Borel spaces and morphism between them, where a quasi-Borel space $(X, M_X)$ (with respect to $\mathbb{R}$) is a set $X$ equipped with a subset $M_X$ of functions in $\mathbb{R} \to X$ such that (1) If $\alpha : \mathbb{R} \to X$ is constant then $\alpha \in M_X$. (2) If $\alpha \in M_X$ and $f : \mathbb{R} \to \mathbb{R}$ is measurable then $\alpha \circ f \in M_X$. (3) If the family $\{S_i\}_{i \in \mathbb{N}}$ is a countable partition of $\mathbb{R}$, i.e. $\mathbb{R} = \coprod_{i \in \mathbb{N}} S_i$, with each set $S_i$ Borel, and if $\alpha_i \in M_X$ ($\forall i \in \mathbb{N}$) then the copairing $[\alpha_i]_{i \in \mathbb{N}}$ of $\alpha_i |_{S_i}$, $S_i \to X$ belongs to $M_X$.

A morphism from a quasi-Borel space $(X, M_X)$ to a quasi-Borel space $(Y, M_Y)$ is a function $f : X \to Y$ such that $f \circ \alpha \in M_Y$ holds for any $\alpha \in M_X$.
As shown by Heunen et al. [2017], the category QBS has a convenient structure to interpret probabilistic programs. That is, it is well-pointed and cartesian closed and we have the usual structure for currying and uncurrying functions; it has products and coproducts with distributivity between them; every standard Borel space \( \Omega \) is converted to a quasi-Borel space and every measurable function \( f : \Omega_1 \rightarrow \Omega_2 \) is exactly a morphism \( f : \Omega_1 \rightarrow \Omega_2 \) in QBS. Hence, it can be used to interpret a probabilistic functional language, see [Heunen et al. 2017; Ścibior et al. 2017] for more details.

The category QBS has also a convenient structure to reason about probabilistic programs. In particular, the forgetful functor \([-\cdot] : \text{QBS} \rightarrow \text{Set}\) erasing the quasi-Borel structure does not change the underlying structure of functions. This property is fundamental for the design the category \text{Pred(QBS)} of predicates on quasi-Borel spaces.

**Measures on quasi-Borel spaces.** Quasi-Borel spaces were introduced to support measure theory in a cartesian closed category. In particular, given a measure on some standard Borel space \( \Omega \) we can define a measure over quasi-Borel spaces.

**Definition 7.2 (Ścibior et al. [2017]).** A measure on a quasi-Borel space \((X, M_X)\) is a triple \((\Omega, \alpha, \nu)\) where \( \Omega \) is a standard Borel space, \( \alpha : \Omega \rightarrow X \) is a morphism in QBS, and \( \nu \) is a \( \sigma \)-finite measure over \( \Omega \).

For a measure \( \mu = (\Omega, \alpha, \nu) \) on \( X \) and a function \( f : X \rightarrow \mathbb{R} \) in QBS, we define integration over quasi-Borel spaces in terms of integration over Borel spaces: \( \int_X f \, d\mu \overset{def}{=} \int_\Omega (f \circ \alpha) \, d\nu \)

Equivalence of measures in QBS is defined in terms of equality of integrations:
\[
(\Omega, \alpha, \nu) \simeq (\Omega', \alpha', \nu') \overset{def}{=} \forall f : X \rightarrow \mathbb{R} \text{ in QBS. } \int_\Omega (f \circ \alpha) \, d\nu = \int_{\Omega'} (f \circ \alpha') \, d\nu'.
\]

In the following it will be convenient to work with equivalence classes of measures which we denote by \([\Omega, \alpha, \nu]\). Every equivalence class for a measure \((\Omega, \alpha, \nu)\) also contains a measure over \( \mathbb{R} \) defined in the appropriate way [Heunen et al. 2017]. We are now ready to define a monad for measures.

**Definition 7.3 (Ścibior et al. [2017]).** The monad of \( \sigma \)-finite measures \( \mathcal{M} \) is defined by

- For any \( X \) in QBS, \( \mathcal{M}X \) is the set of equivalence classes of \( \sigma \)-finite measures equipped with the quasi-Borel structure given by the following definition

\[
M_{\mathcal{M}X} = \left\{ \lambda r.[D_r, (\alpha, r, \mu)] \middle| D \subseteq_{\text{measurable}} \mathbb{R} \times \Omega, \mu : \sigma \text{-finite measure on } \Omega, \alpha : D \rightarrow X, D_r = \{ \omega \mid (r, \omega) \in D \}, \mu_r = \mu|_{D_r} \right\}
\]

- The unit \( \eta_X : X \rightarrow \mathcal{M}X \) is defined by \( \eta_X(x) = [1, \lambda \ast x, d_x] \).
- The Kleisli lifting is defined as for any \( f : X \rightarrow \mathcal{M}Y \) and \([\Omega, \alpha, \nu] \in \mathcal{M}X \)

\[
f'[\Omega, \alpha, \nu] = [D, \beta, (v \otimes v')|_D]
\]

where \( D = \{ (r, \omega) \mid \omega \in D_r \} \) and \( \beta(-) = \lambda r. \beta(r, -) \) are defined for every \( \gamma : \Omega \rightarrow \mathbb{R} \) and \( \gamma' : \mathbb{R} \rightarrow \Omega \), satisfying \( \gamma \circ \nu = \text{id}_{\Omega} \) through \( (f \circ \alpha)(\gamma'(r)) = [D_r, \beta(r, -), v'] \).

Let us unpack in part this definition. The set of functions \( M_{\mathcal{M}X} \) can be seen as a set of (uncountable) families of measures, indexed by \( r \), supporting infinite measures. The Kleisli lifting uses the fact that each \( (f \circ \alpha)(\gamma'(-)) \) is a function in \( M_{\mathcal{M}Y} \), that \( D \) built as a product measure starting from \( r \) and \( D_r \) is measurable, and \( \beta \) is a morphism from \( D \) to \( Y \).

Thanks to Fubini-Tonelli theorem, the monad \( \mathcal{M} \) on QBS is commutative strong with respect to the cartesian products. We can also use the structure of QBS to define the product measure of \([\Omega, \alpha, \nu]\) and \([\Omega', \alpha', \nu']\) as \([((\Omega \times \Omega'), (\alpha \times \alpha'), (\nu \otimes \nu'))\]. Using the isomorphism \( \mathcal{M}1 \cong [0, \infty) \), usual
We can now interpret observe as follows:

$$QBS \eta$$

where $$P$$ in PCF corresponds to $$f^\#(\mu)$$. We can define the mass $$|\mu|$$ of measure $$\mu = [\Omega, \alpha, v]$$ by $$\int_X 1 d\mu$$ which is the same as the mass $$|v|$$ of base measure $$v$$. The monad $$M$$ captures general measures, for example we can define a null measure as $$\theta = [\Omega, \alpha, 0]$$.

In the sequel, we will also use a commutative monad $$\Psi$$ on $$QBS$$ obtained by restricting the monad $$M$$ to subprobability measures. We have the canonical inclusion $$\Psi X \subseteq M X$$.

### 7.2 Semantics for PPV

In order to give meaning to the logical formulas of PL, we first need to give meaning to expressions in PCF and to enriched expressions in PL. We do this by interpreting types as objects in $$QBS$$ as follows:

$$\text{unit} \mathrel{\Downarrow} 1, \text{bool} \mathrel{\Downarrow} 1 + 1, \text{nat} \mathrel{\Downarrow} \mathbb{N}, \text{real} \mathrel{\Downarrow} \mathbb{R}, \text{pReal} \mathrel{\Downarrow} [0, \infty], \text{list}(\tau) \mathrel{\Downarrow} [\tau_1] \times [\tau_2]$$

where 1 is the terminal object in $$QBS$$, $$\bigl[ \tau \bigr]$$ is the coproduct of the countable family $$\bigl[ \tau \bigr]^n = \bigl[ \tau \bigr] \times \cdots \times \bigl[ \tau \bigr]$$ (n times); $$\bigl[ \tau_1 \bigr] \Rightarrow \bigl[ \tau_2 \bigr]$$ is the exponential object in $$QBS$$.

We interpret each term $$\Gamma \vdash e : \tau$$ as a morphism $$\Gamma \rightarrow [\tau]$$ in $$QBS$$, where as usual the interpretation $$[\Gamma]$$ of a context $$\Gamma$$ is the product of the interpretations of its components. Pure computations are interpreted using the cartesian closed structure of $$QBS$$ where we can interpret recursive terms—thanks to the termination criterion—by means of a least fixed point operator.

$$\Gamma \vdash \text{let rec } fx = e : I \rightarrow \sigma \mathrel{\Downarrow} \text{fix}([\Gamma \vdash e : \sigma. \lambda x. I. e : (I \rightarrow \sigma) \rightarrow (I \rightarrow \sigma)])$$

We interpret return and bind using the structure of the monad $$M$$ of measures on $$QBS$$.

$$\Gamma \vdash e : M[\tau] \mathrel{\Downarrow} m \Gamma \mathrel{\Downarrow} M[\tau]$$

where $$\eta$$, $$(-)^\#$$, and $$s$$ are the unit, the Kleisli lifting, and the tensorial strength of the commutative monad $$M$$. To interpret the other constructions we first introduce two semantics constructions for scaling and normalizing:

$$\text{scale}(v, f) \mathrel{\Downarrow} (\Psi[\tau_2] \circ \text{dst}_{1,X} \circ (f, \eta_X))^\#(v)$$. $$\text{normalize}(v) \mathrel{\Downarrow} \begin{cases} 0 & |v| = 0, \infty \vspace{1em} \\ |v| & \text{otherwise} \end{cases}$$

where $$\text{dst}$$ is the double strength of the commutative monad $$M$$, and $$|v|$$ is the mass of $$v$$. In the definition of $$\text{scale}(v, f)$$, the construction $$\Psi[\tau_2] \circ \text{dst}_{1,X} \circ (f, \eta_X)$$ corresponds to a function mapping an element $$x \in X$$ to a Dirac distribution centered in $$x$$ and scaled by $$f(x)$$, whose domain is then lifted to measures using the Kleisli lifting. To achieve this, we use the equivalence $$[\text{pReal} \mathrel{\Downarrow} [0, \infty]] \equiv M[1]$$, and pairing and projection constructions to manage the duplication of $$\times$$. The definition of $$\text{scale}(v)$$ is more straightforward and reflect the semantics we described before.

Using these constructions we can interpret the corresponding syntactic constructions.

$$\Gamma \vdash \text{scale}(t, t') \mathrel{\Downarrow} \text{scale}(\Gamma \vdash t : M[\tau], \Gamma \vdash t' : \tau \rightarrow \text{pReal}))$$

$$\Gamma \vdash \text{normalize}(t) \mathrel{\Downarrow} \text{normalize}(\Gamma \vdash t : M[\tau])$$

We can now interpret observe as follows:

$$\Gamma \vdash \text{observe } e \text{ as } e' \mathrel{\Downarrow} \text{normalize}(\text{scale}(\Gamma \vdash e : M[\tau], \Gamma \vdash e' : \tau \rightarrow \text{pReal})))$$

Using again the equivalence $$[\text{pReal} \mathrel{\Downarrow} [0, \infty]] \equiv M[1]$$, we interpret expectation as:

$$\Gamma \vdash \mathbb{E}_{X \sim t'}(t'(x)) : \text{pReal} \mathrel{\Downarrow} \lambda y \in \Gamma \cdot \text{(\Gamma \vdash t' : \tau \rightarrow \text{pReal}(y))}^\#(\Gamma \vdash t : M[\tau](y))$$
The primitives of basic probability distributions \( \text{Uniform}, \text{Bern}, \text{Gauss} \) are interpreted by rescaling a measure (given as a constant) with density functions (cf. Section 8.2), and the usual operations on real numbers are given by embedding measurable real function to \( \text{QBS} \).

To interpret formulas in PL we use the category \( \text{Pred}(\text{QBS}) \) of predicates on quasi-Borel spaces. This will be useful to see these formulas as assertions in the unary logic \( \text{UPL} \) and the relational logic \( \text{RPL} \). This is actually the main reason why we use quasi-Borel spaces: we want an assertion logic whose predicate support both higher-order computations and continuous probability. The structure of the category \( \text{Pred}(\text{QBS}) \) is the following:

- An object is a pair \( (X, P) \) where \( X \in \text{QBS} \) and \( P \subseteq X \).
- A morphism \( f: (X, P) \to (Y, Q) \) is \( f: X \to Y \in \text{QBS} \) such that \( \forall x \in P, f(x) \in Q \).

An important property of this category is that every arbitrary subset \( P \) of a quasi-Borel space \( X \) forms an object \( (X, P) \) in \( \text{Pred}(\text{QBS}) \). This allows us to interpret all logical operations, including universal quantifiers, in a set-theoretic way.

We are now ready to interpret formulas in PL. We interpret a typed formula \( \Gamma \vdash \phi \) as an object \( \llbracket \Gamma \vdash \phi \rrbracket \) in \( \text{Pred}(\text{QBS}) \) where the predicate part \( \llbracket \Gamma \vdash \phi \rrbracket \) is interpreted inductively. We give here a selection of the inductive rules defining it:

\[
\llbracket \Gamma \vdash \top \rrbracket \overset{\text{def}}{=} \llbracket \Gamma \rrbracket, \quad \llbracket \Gamma \vdash \exists x: \tau, \phi \rrbracket \overset{\text{def}}{=} \bigcap_{y \in X} \{ y \in \llbracket \Gamma \rrbracket \mid (\Gamma, x: \tau \vdash \phi) \}, \\
\llbracket \Gamma \vdash \bot \rrbracket \overset{\text{def}}{=} \emptyset, \quad \llbracket \Gamma \vdash t_1 = t_2 \rrbracket \overset{\text{def}}{=} \{ y \in \llbracket \Gamma \rrbracket \mid \llbracket \Gamma \vdash t_1 \rrbracket \models y = \llbracket \Gamma \vdash t_2 \rrbracket \models y \}, \\
\llbracket \Gamma \vdash \phi_1 \land \phi_2 \rrbracket \overset{\text{def}}{=} \llbracket \Gamma \vdash \phi_1 \rrbracket \cap \llbracket \Gamma \vdash \phi_2 \rrbracket, \quad \llbracket \Gamma \vdash \neg \phi \rrbracket \overset{\text{def}}{=} \llbracket \Gamma \rrbracket \setminus \llbracket \Gamma \vdash \phi \rrbracket.
\]

This interpretation is well-behaved with respect to substitution. In particular, the substitution \( \phi[t/x] \) of \( x \) by an enriched expression \( t \) can be interpreted by the inverse image \( \llbracket \Gamma \vdash \phi[t/x] \rrbracket = \langle \text{id}_{\llbracket \Gamma \rrbracket}, \llbracket \Gamma \vdash t: \tau \rrbracket \rangle^{-1}(\llbracket \Gamma, x: \tau \vdash \phi \rrbracket) \). Using this property, we can show that the logic PL is sound with respect to the semantics that we defined above.

**Theorem 7.4 (PL Soundness).** If a judgment \( \Gamma \models_{\text{PL}} \phi \) is derivable then we have the inclusion \( \llbracket \phi \rrbracket \subseteq \llbracket \Gamma \vdash \phi \rrbracket \), which is equivalent to having a morphism: \( \text{id}_{\llbracket \Gamma \rrbracket} : \llbracket \Gamma \vdash \land_P \psi \rrbracket \to \llbracket \Gamma \vdash \phi \rrbracket \) in the category \( \text{Pred}(\text{QBS}) \).

Here, the soundness of the axioms in PL introduced in Section D is proved from the basic facts discussed by Scibior et al. [2017], in particular, the isomorphism \( \mathcal{M}_1 \simeq [0, \infty] \), the commutativity of the monad \( \mathcal{M} \), the correspondence between \( f^\mathcal{M}(\mu) \) and usual integration \( \int f \, d\mu \) for any \( f : \mathbb{R} \to [0, \infty] \) and \( \mathcal{M}(\mathbb{R}) \), and that every measurable functions between standard Borel spaces are exactly morphisms in \( \text{QBS} \).

Using Theorem 6.1 and Theorem 7.4, we can prove the semantics soundness of UPL.

**Corollary 7.5 (UPL Semantics Soundness).** If \( \Gamma \models_{\text{UPL}} e : \tau \mid \phi \) then \( \langle \text{id}_{\llbracket \Gamma \rrbracket}, \llbracket \Gamma \vdash e : \tau \rrbracket \rangle : \llbracket \Gamma \vdash \land_P \psi \rrbracket \to \llbracket \Gamma, r : \tau \vdash \phi \rrbracket \) in \( \text{Pred}(\text{QBS}) \).

Using Theorem 6.2 and Theorem 7.4, we can prove the semantics soundness of RPL.

**Corollary 7.6 (RPL Semantics Soundness).** If \( \Gamma \models_{\text{RPL}} e_1 : \tau_1 \sim e_2 : \tau_2 \mid \phi \) then \( \langle \text{id}_{\llbracket \Gamma \rrbracket}, \llbracket \Gamma \vdash e_1 : \tau_1 \rrbracket, \llbracket \Gamma \vdash e_2 : \tau_2 \rrbracket \rangle : \llbracket \Gamma \vdash \land_P \psi \rrbracket \to \llbracket \Gamma, r_1 : \tau_1, r_2 : \tau_2 \vdash \phi \rrbracket \) in \( \text{Pred}(\text{QBS}) \).

### 8 Examples

In section 2 we show two examples showing how to use PPV to reason about probabilistic inference, and Monte Carlo approximation. In this section, we will demonstrate further how PPV can be used to verify a wide range of properties of probabilistic programs. We will start with showing how to reason formally about probabilistic program slicing for continuous random variables as a
relational property. We will then consider an example showing how to use PPV to reason about the *convergence* of probabilistic inference. We will then move to some statistical applications: we will show how to reason about mean estimation of distributions, about the *approximation properties* of importance sampling. Finally, we will show how to use PPV for a proper machine learning task by showing how one can reason about the *Lipschitz continuity* of a generalized iteration algorithm useful for reinforcement learning.

### 8.1 Slicing of probabilistic programs

In this example, we show how PPV can be used to reason about relational properties of probabilistic programs with continuous random variables. Specifically, we show that a combination of relational reasoning in RPL, and equational reasoning in PL allow us to reason about slicing of probabilistic programs [Amtoft and Banerjee 2016]. Slicing is a program analysis technique that can be used to speed up probabilistic inference tasks. Previous works have shown how to slice probabilistic programs with discrete random variables in an efficient way, here we consider the problem of checking the correctness of a slice, when the program contains continuous random variables. Let us look at an example adapted from: consider the following two programs `left` and `right`:

\[
\begin{align*}
\text{left} & \equiv \text{let } x = \text{Uniform}(0,1) \text{ in let } y = \text{Uniform}(0,1) \text{ let } z = x \otimes y \text{ in } \\
& \quad \text{mlet } v = (\text{observe } z \text{ as } \lambda w. \text{if } \pi_2(w) > 0.5 \text{ then } 1 \text{ else } 0) \text{ in return}(\pi_1(v)) \\
\text{right} & \equiv \text{let } x = \text{Uniform}(0,1) \text{ in } x
\end{align*}
\]

Intuitively, the observation in `left` is on `y`, so it does not affect the distribution of `x`, which was sampled independently. Indeed, `right` is a correct slice of `left`—notice that the observation is on the product measure, and not just on the measure of `z`. We can show this in RPL by proving the following judgment.

\[\vdash_{\text{RPL}} \text{left}: M[\text{real}] \sim \text{right}: M[\text{real}] \mid r_1 = r_2\]

To prove this judgment, we first apply the relational [LET] rule, which allow us to introduce an assumption about `x` on both sides, then we apply a sequence of asynchronous [LET-L] rules on the program on the left, which introduce in the context the refinements of `y` and `z`:

\[
\begin{align*}
x & = \text{Uniform}(0,1), y = \text{Uniform}(0,1), z = x \otimes y \vdash_{\text{RPL}} \\
\text{mlet } v & = (\text{observe } z \text{ as } \lambda w. \text{if } \pi_2(w) > 0.5 \text{ then } 1 \text{ else } 0) \text{ in return}(\pi_1(v)) \sim x \mid r_1 = r_2
\end{align*}
\]

To prove this judgement we rely on the equalities on monadic bind, rescaling, and observations in Section D. Starting from the PCFp term on the left, by applying the equations (58), (59), and (54), we reduce it to \(\text{mlet } v = (x \otimes X \text{ in return}(\pi_1(v)))\) where \(X\) is a *normalized* distribution defined by the term \(\text{observe } y \text{ as } \lambda w_2. \text{if } w_2 > 0.5 \text{ then } 1 \text{ else } 0\). We then conclude this is equal to `x` by applying the equalities (54) and the equality

\[\text{mlet } w = e_1 \otimes e_2 \text{ in return } \pi_1(w) = \text{scale}(e_1, \mathbb{E}_{x-e_2}[1])\]

proved from the equalities (57), (52) and monadic laws.

Using RPL we can also reason about situations where we cannot slice a program. Adapting again from Amtoft and Banerjee [2016], let us consider the following two programs `left` and `right`:

\[
\begin{align*}
\text{left} & \equiv \text{let } x = \text{Uniform}(0,1) \text{ in let } y = \text{Uniform}(0,1) \text{ let } z = x \otimes y \text{ in } \\
& \quad \text{mlet } v = (\text{observe } z \text{ as } \lambda w. \text{if } \pi_1(w) + \pi_2(w) > 0.5 \text{ then } 1 \text{ else } 0) \text{ in return}(\pi_1(v)) \\
\text{right} & \equiv \text{let } x = \text{Uniform}(0,1) \text{ in } x
\end{align*}
\]

Now we prove that it is not correct to slice `left` into `right` by means of below judgment:

\[\vdash_{\text{RPL}} \text{left}: M[\text{real}] \sim \text{right}: M[\text{real}] \mid r_1 \neq r_2\]
We now want to show how PPV can be used to reason about this process. In particular, we show that left and right are different, we use the probabilistic inference in the first example to prove \( \vdash _{\text{RPL}} \left( \text{left} \right) \Rightarrow \text{left} \) using the theorem rule and the following calculation:

\[
\begin{align*}
\Pr & \left[ \pi_1 (w) > 0.5 \right] \\
& \geq \frac{1 - \Pr \left[ \pi_1 (x) > 0.5 \right]}{1 - \Pr \left[ \pi_1 (y) > 0.5 \right]}
\end{align*}
\]

Similarly, we can look at the following two programs:

\[
\begin{align*}
\text{left} & \equiv \text{mlet } x = \text{Uniform}(0, 1) \\
\text{mlet } & = (\text{if } x > 0.5 \text{ then } \text{mlet } y = \text{Uniform}(0, 1) \text{ in } \text{mlet } z = x \otimes y \text{ in } \text{observe } z \text{ as } \lambda w. \text{if } \pi_2 (w) > 0.5 \text{ then } 1 \text{ else } 0 \text{ else return } (x \otimes x) ) \text{ in return } (x)
\end{align*}
\]

\[
\begin{align*}
\text{right} & \equiv \text{mlet } x = \text{Uniform}(0, 1) \text{ in return } (x)
\end{align*}
\]

and show that we can slice left into right.

Since we renormalize in observations, we can slice the algorithm left into right. The proof starts by using relational reasoning, and afterwards reuses the proof of the first example. This shows that reasoning relationally about slicing can be better than reasoning directly about equivalence by computing the two distributions.

The proof for this judgment follows the structure of the proof of the previous example, the main difference is that now we need to see the first coordinate of the variable \( w \) in the observation. To prove that left and right are different, we use the probabilistic inference in the first example to prove \( \vdash _{\text{RPL}} \left( \text{left} \right) \Rightarrow \text{left} \) using the theorem rule and the following calculation:

\[
\begin{align*}
\Pr & \left[ \pi_1 (w) > 0.5 \right] \\
& \geq \frac{1 - \Pr \left[ \pi_1 (x) > 0.5 \right]}{1 - \Pr \left[ \pi_1 (y) > 0.5 \right]}
\end{align*}
\]

Similarly, we can look at the following two programs:

\[
\begin{align*}
\text{left} & \equiv \text{mlet } x = \text{Uniform}(0, 1) \\
\text{mlet } & = (\text{if } x > 0.5 \text{ then } \text{mlet } y = \text{Uniform}(0, 1) \text{ in } \text{mlet } z = x \otimes y \text{ in } \text{observe } z \text{ as } \lambda w. \text{if } \pi_2 (w) > 0.5 \text{ then } 1 \text{ else } 0 \text{ else return } (x \otimes x) ) \text{ in return } (x)
\end{align*}
\]

\[
\begin{align*}
\text{right} & \equiv \text{mlet } x = \text{Uniform}(0, 1) \text{ in return } (x)
\end{align*}
\]

A key point in deriving the slicing property of the above examples is the equation \( \text{mlet } w = e_1 \otimes e_2 \text{ in return } \pi_1 (w) = \text{scale}(e_1, \Xi - e_2 [1]) \) of splitting product measure, which is obtained by applying the axioms in Section D. When \( e_2 \equiv \text{observe } e_1 \) as \( e_4 \), we have \( \text{mlet } w = e_1 \otimes e_2 \) in return \( \pi_1 (w) = e_1 \) since our observation is normalized, and hence \( \Xi - e_2 [1] = 1 \). On the other hand, when \( e_2 \) consists of unnormalized observation, we may have the non-slicing \( \text{mlet } w = e_1 \otimes e_2 \) in return \( \pi_1 (w) \neq e_1 \) because \( \Xi - e_2 [1] < 1 \). This is an advantage of our normalized observation. Since we renormalize in observations, we can slice the algorithm left into right in the third example.

Putting the first and the third example together we can consider the following two programs left and right:

\[
\begin{align*}
\text{left} & \equiv \text{mlet } x = \text{Uniform}(0, 1) \\
\text{mlet } & = (\text{if } x > 0.5 \text{ then } \text{mlet } y = \text{Uniform}(0, 1) \text{ in } \text{mlet } z = x \otimes y \text{ in } \text{observe } z \text{ as } \lambda w. \text{if } \pi_2 (w) > 0.5 \text{ then } 1 \text{ else } 0 \text{ else return } (x \otimes x) ) \text{ in return } (x)
\end{align*}
\]

\[
\begin{align*}
\text{right} & \equiv \text{mlet } x = \text{Uniform}(0, 1) \text{ in return } (x)
\end{align*}
\]

Again, we want to show that right is a correct slice of left by proving that: \( \vdash _{\text{RPL}} \left( \text{left} \right) \Rightarrow \text{right} \) : \( M[\text{real}] \sim \text{right} : M[\text{real}] \mid r_1 = r_2 \). The proof of this judgment can be carried out mostly in RPL, by using the similarity between the two programs left and right. The proof starts by using relational reasoning, and afterwards reuses the proof of the first example. This shows that reasoning relationally about slicing can be better than reasoning directly about equivalence by computing the two distributions.

\[
\begin{align*}
\vdash _{\text{RPL}} \text{mlet } v = (\text{observe } \text{Uniform}(0, 1) \otimes \text{Uniform}(0, 1) \\
& \text{as } \lambda w. \text{real } \times \text{real}. \text{if } (\pi_1 (w) + \pi_2 (w) > 0.5) \text{ then } 1 \text{ else } 0 \text{ in return } (\pi_1 (v))) : M[\text{real}] \\
& \text{mlet } x = \text{Uniform}(0, 1) \text{ in return } (x) : M[\text{real}] \mid r_1 = r_2
\end{align*}
\]

### 8.2 Gaussian Mean Learning: convergence and stability

Probabilistic programs are often used as models for probabilistic inference tasks in data analysis. We now want to show how PPV can be used to reason about this process. In particular, we show how to use PPV to reason about two quite common properties: convergence of closed-form bayesian update, and stability of this process under changes in priors. These two properties allow us to...
illustrate two different aspects of PPV. First, the support it offers for reasoning about iterative probabilistic tasks and for reasoning about densities of random variables. Second, the support it offers for relational reasoning about measures of divergences of one distribution with respect to the other. To show this, we first prove the convergence of the iterative closed-form learning of the mean of a Gaussian distribution (with fixed variance). We then prove this process also stable for a precise notion of stability formulated in terms of Kullback-Leibler (KL) divergence.

Let us start by considering the following implementation GaussLearn of an algorithm for Bayesian learning of mean of a Gaussian distribution with known variance $\sigma^2$ from a sample list $L$:

$$\text{GaussLearn} \equiv \lambda p. \text{letrec } f(L) = \text{case } L \text{ with } [] \Rightarrow p, y :: ls \Rightarrow \text{observe } f(ls) \text{ as GPDF}(y, \sigma^2)$$

where GPDF($y, \sigma^2$) is a shorthand for the density function $\lambda r. \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{(r-y)^2}{2\sigma^2}\right)$ of a Gaussian distribution Gauss($y, \sigma^2$) with mean $y$ and variance $\sigma^2$. This algorithm starts by assuming a prior $p$ on the unknown mean. Then, on each iteration, a sample $y$ is read from the list and the prior gets updated by observing it as a Gaussian with mean $y$ and variance $\sigma^2$.

We now want to show two properties of this algorithm. The first property we show is convergence: the mean of the posterior should roughly converge to the mean of the data, but we need to take into account that the posterior also depends on the prior. More precisely, when the prior is also a Gaussian, we can show that:

$$(\sigma > 0), (\xi > 0) \vdash_{\text{RPL}} \text{GaussLearn} \sim \text{Total} \mid \forall L': \text{list}(\text{real}). \forall n: \text{nat.}(n = |L'|)$$

$$\implies r_1(\text{Gauss}(\delta, \xi^2))(L') = \text{Gauss}(\frac{1}{n\xi^2 + \sigma^2}, \frac{n\xi^2 + \sigma^2}{n\xi^2 + \sigma^2}).$$

where Total is an algorithm summing all the elements of a list $L$.

This judgement states that, if the prior on the mean is a Gaussian of mean $\delta$ and variance $\xi^2$, then the posterior is a Gaussian with mean close to the mean of Total($L$) and variance close to 0, but that they are still influenced by the parameters $\delta, \xi^2$ of the prior.

The proof of this judgment can proceed relationally by first applying the one-sided [ABS-L] rule to introduce the prior in the context. Then the proof continues synchronously by applying the [r-LETREC] and [r-LISTCASE] rule. To conclude the proof we need to show the following two premises corresponding to the base case and to the inductive step:

$$(\sigma > 0), (\xi > 0), \phi_{\text{ind.hyp.}}, (L = []), d_{\text{prior}} = \text{Gauss}(\delta, \xi^2), (n = |L|)$$

$$\vdash_{\text{RPL}} d_{\text{prior}} \sim 0 \mid r_1 = \text{Gauss}(\frac{\delta^2 + n\xi^2 + \sigma^2}{n\xi^2 + \sigma^2}, \frac{n\xi^2 + \sigma^2}{n\xi^2 + \sigma^2}).$$

$$(\sigma > 0), (\xi > 0), \phi_{\text{ind.hyp.}}, (L = y :: ls), d_{\text{prior}} = \text{Gauss}(\delta, \xi^2), (n = |L|)$$

$$\vdash_{\text{RPL}} \text{observe } f_1(ls) \text{ as Gauss}(y, \sigma^2) \sim y + f_2(ls) \mid r_1 = \text{Gauss}(\frac{\delta^2 + n\xi^2 + \sigma^2}{n\xi^2 + \sigma^2}, \frac{n\xi^2 + \sigma^2}{n\xi^2 + \sigma^2}).$$

The first premise is obvious. The second premise requires a little more work, and can be proved by applying [r-OBS-L] and [r-SUB] rules and the several equations in PL. We first show in PL that Gaussian distributions are conjugate prior with respect to Gaussian likelihood function by applying the equations on rescaling, normalization, and observation.

$$\vdash_{\text{PL}} (\sigma > 0) \land (\xi > 0) \implies \text{observe Gauss}(\delta, \xi^2) \text{ as GPDF}(z, \sigma^2) = \text{Gauss}(\frac{\xi^2 + \delta^2}{\xi^2 + \sigma^2}, \frac{\xi^2 \sigma^2}{\xi^2 + \sigma^2}).$$

Then, we apply [r-OBS-L] and [r-SUB] to the premise (8.2) to introduce the observations in the precondition, and apply the above fact and the induction hypothesis.

The second property we show is stability. If we run GaussLearn twice with different prior Gaussian distributions, we can show that the posteriors will be close if the list of samples is long enough and not diverging. This closeness is defined in terms of the Kullback-Leibler (KL) divergence. The
KL divergence of two distributions with known density functions, can be defined by expectations: 
\((d_1 = \text{scale}(d_2, f)) \implies (KL(d_1 \| d_2) = \mathbb{E}_{x \sim d_1} [\log f(x)])\). In particular, the KL divergence of two Gaussian distributions can be calculated as follows:

\[
KL(\text{Gauss}(\mu_1, \sigma_1^2) \| \text{Gauss}(\mu_2, \sigma_2^2)) = (\log |\sigma_2| - \log |\sigma_1|) + (\sigma_1^2 + (\mu_1 - \mu_2)^2)/\sigma_2^2 - 1/2.
\]  

(19)

Formally, we want to prove the following judgment:

\[
\sigma, \delta: \text{real}, \xi, \xi_2: \text{real}, \xi_2: \text{real} \mid (\sigma > 0), (\xi > 0), (\xi_2 > 0) \\
\frac{\text{RPL}}{\text{GaussLearn} \sim \text{GaussLearn} \mid \forall L': \text{list}(\text{real}), \forall \epsilon: \text{real}, \forall C: \text{real}.

(\epsilon > 0) \implies \exists N: \text{nat.}(|L'| > N) \land |\text{Total}(L')| < C \ast |L'| \\
\implies KL(r_1(\text{Gauss}(\delta, \xi_2^2))(L') \mid r_1(\text{Gauss}(\delta_2, \xi_2^2))(L')) < \epsilon
\]

(20)

Intuitively, this states that if the algorithm is run twice with different Gaussian priors, and the mean of the data is bounded by some \(C\), then the KL divergence of the posteriors can be made as small as desired by increasing the size of the data. In other words, the effect of the prior on the posterior can be minimized by having enough samples.

By simple calculations, we can prove in PL the following assertion in a similar way as proofs of convergence of sequence using the epsilon-delta definition of limit.

\[
\frac{\text{PL}}{\forall L': \text{list}(\text{real}). \forall \epsilon: \text{real}. \forall C: \text{real}.

(\epsilon > 0) \implies \exists N: \text{nat.}(|L'| > N) \land |\text{Total}(L')| < C \ast |L'| \\
\implies \frac{|\text{Total}(L') \ast \xi^2 + \delta \ast \sigma^2|}{|L'| \ast \xi^2 + \sigma^2} - \frac{|\text{Total}(L') \ast \xi_2^2 + \delta_2 \ast \sigma^2|}{|L'| \ast \xi_2^2 + \sigma^2} < \epsilon
\]

(21)

To prove (20), we want to combine the previous verification (18), with the calculations (19) and (21). To do this, we apply the relational [r-SUB] rule to the judgment (20), which have the following PL premise:

\[
\frac{\text{PL}}{\exists L': \text{list}(\text{real}). \forall \epsilon: \text{real}. \forall C: \text{real}.

(\epsilon > 0) \implies \exists N: \text{nat.}(|L'| > N) \land |\text{Total}(L')| < C \ast |L'| \\
\implies KL(\text{GaussLearn}(\text{Gauss}(\delta, \xi^2))(L') \mid \text{GaussLearn}(\text{Gauss}(\delta_2, \xi_2^2))(L')) < \epsilon
\]

Then we prove it in PL by applying the PL judgment obtained by applying [conv-RPL] to the previous derivation (18), and the PL judgments (19) and (21).

### 8.3 Sample Size Required in Importance Sampling

As another example of common statistical task we use PPV to show the correctness of self-normalizing importance sampling. Importance sampling is an efficient variant of Monte Carlo approximation to estimate the expected value \(\mathbb{E}_{x \sim d}[h(x)]\) when sampling from \(d'\) is not convenient. The idea is to sample from a different distribution \(d\) and then rescale the samples by using the density function \(g\) of \(d'\). The most interesting aspect of this example is that correctness is formulated as a probability bound on the difference between the mean of the distribution \(d'\) and the empirical mean. This shows once again that in PPV we can support reasoning about this kind of probabilistic bounds which are quite widespread in statistical applications. However, here we want to go a step further and show that we can reason about probability bounds that are parametric in the number of data samples available. This quantity is often crucial for both theoretical understanding and practical reasons, since data are an expensive resource. For our specific example, we rely on a recent work by Chatterjee and Diaconis [2015] and use their theorem as correctness statement. This example also show the usefulness of the equations we have identified in Section D to support high-level reasoning.
The following algorithm SelfNormIS is an implementation of self-normalizing importance sampling.

\[
\text{SelfNormIS} \equiv \lambda n: \text{nat}. (\text{mlet } z = \text{SumLoop}(n)(g)(h) \text{ in return } (\pi_1(z) / \pi_2(z)))
\]

\[
\text{SumLoop} \equiv \text{letrec } f(i: \text{nat}) = \lambda g: \tau \to \text{real}. \lambda h: \tau \to \text{real}. \left\{ \begin{array}{l}
\text{if } (i \leq 0) \text{ then return } (0, 0) \text{ else mlet } x = d \text{ in mlet } m = f(i - 1)(g)(h) \text{ in return } ((1/i)(\pi_1(m) + (i - 1) * h(x) * g(x)), (1/i)(\pi_2(m) + (i - 1) * g(x))).
\end{array} \right.
\]

This algorithm approximates \( \mathbb{E}_{x \sim \delta}[h(x)] = \int h(x)g(x) \, dx \) by taking samples \( X_1 \ldots X_n \sim \delta \) and computing the ratio \( (\frac{1}{n} \sum_{i=1}^{n} g(X_i)h(X_i)) / (\frac{1}{n} \sum_{i=1}^{n} g(X_i)) \) of weighed sum instead. Note that SumLoop is the subroutine calculating the numerator \( \frac{1}{n} \sum_{i=1}^{n} g(X_i)h(X_i) \) and denominator \( \frac{1}{n} \sum_{i=1}^{n} g(X_i) \) of empirical expected value from the same samples \( X_1 \sim \delta \).

We verify a recent result on the sample size required in self-normalizing importance sampling. The goal is to prove the following implementation of Chatterjee and Diaconis [2015, Theorem 1.2]:

\[
d: M[\tau], g: \tau \to \text{real}, h: \tau \to \text{real} \vdash \text{UPL SelfNormIS: } \text{nat} \to M[\text{real}] |
\]

\[
\forall d': M[\tau], \forall \mu: \text{real}. \forall \sigma: \text{real}. \forall C: \text{real}. \forall t: \text{real}. \forall \epsilon: \text{real}. \phi \land (\epsilon > \sqrt{t} \exp(-t/4) + 2\sqrt{t} \Pr_{y \sim \delta}[\log(g(y)) > L + t/2])
\]

\[
\implies \forall k: \text{nat}. k > \exp(L + t) \implies \Pr_{y \sim \delta}[h(k)(h)] [\| y - \mu \| \geq \frac{2\epsilon \sqrt{\sigma^2 + \mu^2}}{1 - \epsilon}] \leq 2\epsilon
\]

Here, \( C \) is supposed a unknown normalization factor of \( g \). The following assertion \( \phi \) gives the assumption to give the required sample size.

\[
\phi \equiv \left( \mathbb{E}_{x \sim \delta}[1] = 1 \right) \land (\sigma^2 = \text{Var}_{x \sim \delta}[h(x) * g(x)]) \land (\mu = \mathbb{E}_{y \sim \delta}[h(y)]) \land (t \geq 0) \land (d' = \text{scale}(d, g/C)) \land (C > 0) \land (\mathbb{E}_{x \sim \delta}[1] = 1) \land (L = \mathbb{E}_{x \sim \delta}[\log(g(y))]).
\]

The previous theorem gives a bound on the probability that the estimation differs too much from the actual expected value \( \mu \). The proof of the judgment (22) is involved, and proceeds on various steps. First, we will prove a version of the theorem for naive (non self-normalizing) importance sampling [Chatterjee and Diaconis 2015, Theorem 1.1]. Then, this result will be extended to self-normalizing importance sampling. Naive importance sampling is defined as:

\[
\text{Naive} \equiv \text{letrec } f(i: \text{nat}) = \lambda g: \tau \to \text{real}. \lambda h: \tau \to \text{real}. \left\{ \begin{array}{l}
\text{if } (i \leq 0) \text{ then return } (0, 0) \text{ else mlet } x = d \text{ in mlet } m = f(i - 1)(g)(h) \text{ in return } ((1/i)(m + (i - 1) * h(x) * g(x))).
\end{array} \right.
\]

Here Naive computes \( \frac{1}{n} \sum_{i=1}^{n} g(X_i)h(X_i) \). We want to show:

\[
\vdash \text{UPL Naive: } \text{nat} \to (\tau \to \text{real}) \to (\tau \to \text{real}) \to M[\text{real}] |
\]

\[
\forall d': M[\tau], \forall \mu: \text{real}. \forall \sigma: \text{real}. \forall C: \text{real}. \forall t: \text{real}. \forall \epsilon: \text{real}. \phi \implies \forall k: \text{nat}. k > \exp(L + t)
\]

\[
\implies \mathbb{E}_{x \sim \delta}[h(k)(h)] [\| y - \mu \| \leq \sqrt{t} \sigma^2 + \mu^2] * (\exp(-t/4) + 2\sqrt{t} \Pr_{y \sim \delta}[\log(g(y)) > L + t/2]) \leq 2\epsilon
\]

Notice that we need the normalization factor \( C \).

The main tricks in the proof are the Cauchy-Schwartz inequality and introducing the function \( h_2 = \lambda x: \tau. (\text{if } g(x) \leq k * \exp(-t/2) \text{ then } 1 \text{ else } 0) * h(x) \). To view how our tricks work, we check the following calculation in PL in the proof. We write \( \mu' \equiv \mathbb{E}_{y \sim \delta}[h_2(y)] \).
The main part of the proof is the following calculation in PL.

Finally, we prove our goal (22) from the previous derivation of (23). Let $b \equiv \exp(-t/4) + 2\sqrt{t}\Pr_{y \sim q^*}[\log(g(y)) > L + t/2]$ and $\delta \equiv \sqrt{t}(b + \sqrt{t(\sigma^2 + \mu^2)})$, and assume $\epsilon > \sqrt{t}(b)$. The main part of the proof is the following calculation in PL.

$$\Pr_{\tau \sim \text{SumLoop}(k)}(g|h) \left[\left| \frac{\pi_i(z)}{\pi_i(z)} - \mu \right| \geq \frac{2\sqrt{t}(\sigma^2 + \mu^2)}{1 - \epsilon} \right]$$

$$\leq \Pr_{\tau \sim \text{Naive}(k)}(g|h) \left[|w - \mu| \leq \delta \right] + \Pr_{\tau \sim \text{Naive}(k)}(g|h) \left[|w - 1| \leq \sqrt{t}(b) \right]$$

$$\leq \mathbb{E}_{\tau \sim \text{Naive}(k)}(g|C(h)) \left[|w - \mu|/\delta \right] + \mathbb{E}_{\tau \sim \text{Naive}(k)}(g|C(h)) \left[|w - 1|/\sqrt{t}(b) \right]$$

$$\leq \frac{b + \sqrt{t}(\sigma^2 + \mu^2)}{\delta} \quad \frac{b}{\sqrt{t}(b)} \leq 2\epsilon.$$ 

The first step is proved by switching from Naive to SumLoop which requires some structural reasoning, and calculations on real number supported in PL. The second is applying Markov inequality, and the last is proved by definition of $b$ and $\delta$.

### 8.4 Verifying Lipschitz GVI Algorithm

As our final example, we want to show that PPV can be used to reason about a reinforcement learning task through relational reasoning about Lipschitz continuity and about statistical distances. With this example we want to show the usefulness of relational reasoning in another application domain, and how the expressivity of PL allows us to reason about different notions like Lipschitz continuity and statistical distances.

GVI (Generalized Value Iteration) is a reinforcement learning algorithm to optimize a value function on a Markov Decision Process ($\tau_S, \tau_A, R, T, \gamma$) where $\tau_S$ is a space of states, $\tau_A$ is a set of actions, $R: \tau_S \times \tau_A \rightarrow \text{real}$ is a reward function, $T: \tau_S \times \tau_A \rightarrow D[\tau_S]$ is a transition dynamic and $\gamma$ is a discount factor. Our assumption is that the optimal value function satisfies a specific condition, called a Bellman equation: $Q(s, a) = R(s, a) + \mathbb{E}_{s' \sim T(s, a)}[f(Q(s'))]$, where $f: (\tau_S \times \tau_A \rightarrow \text{real}) \rightarrow (\tau_S \rightarrow \text{real})$ is a backup operator (usually we take $\max_{a \in \tau_A}: \tau_A$).

Asadi et al. [2018] show that, under some constraints, the GVI algorithm returns Lipschitz-continuous value functions, which are convenient for model learning algorithms over the MDP. The following program LipGVI is an implementation of GVI algorithm:

$$\text{LipGVI} \equiv 1\text{etrec}\ h(k: \text{nat}) = \lambda Q': \tau_S \times \tau_A \rightarrow \text{real}.(\lambda(s, a): \tau_S \times \tau_A, R(s, a) + \gamma g(Q')(s))h(k - 1)$$

The algorithm receives an estimation $Q'$ of the value function and updates it using a function $g$ which is assumed to be an approximation of $\lambda Q', \lambda S. \mathbb{E}_{s' \sim T(s, a)}[f(Q'(s'))]$. What we want to verify is the Lipschitz continuity of the result of the algorithm LipGVI. Before stating this result, we first need to add to PL the operators and metrics we need. A function $f : X \rightarrow \text{real}$ is Lipschitz continuous if there exists a finite $K(f)$ such that $K(f) = \sup_{x_1, x_2 \in X}(|f(x_1) - f(x_2)|/\text{dist}(x_1, x_2))$.

To define this concept in PL we start by defining the sup operator:

$$a = \sup_{x: \tau \text{ s.t.} \phi(x)} e(x) \equiv \forall x: \tau. \phi(x) \Rightarrow (e(x) \leq a) \wedge \forall b: \tau. (\forall x: \tau. \phi(x) \Rightarrow e(x) \leq b) \Rightarrow a \leq b.$$ 

Next, we implement the notions of Lipschitz constant and Wasserstein metric (sometimes known as the Kantorovic metric):

$$(a = K_{d_1, d_2}(f)) \equiv a = \sup_{(s_1, s_2) : \tau_1 \times \tau_1} d_2(f(s_1), f(s_2))/d_1(s_1, s_2)$$

$$(a = W_{d_1}(\mu_1, \mu_2)) \equiv a = \sup_{f : \tau_1 \rightarrow \text{real}} \text{s.t.} K_{d_1, d_2}(f) \leq 1 (\mathbb{E}_{\tau_1} f(s) - \mathbb{E}_{\tau_1} f(s))$$

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where \( d_S : \text{real} \times \text{real} \rightarrow \text{pReal} \) is the usual metric in the real line. The standard lemmas on summation and composition for Lipschitz constants (see, e.g., [Asadi et al. 2018, Lemmas 1 and 2]) can be proved in PL by unfolding.

Now we are in condition to state the main theorem by Asadi et al. [2018] in PPV:

\[
\forall Q : \tau_S \times \tau_A \rightarrow \text{real}. K_{d_S,d_b}(f(Q)) \leq \sup_{\alpha : \tau_A} K_{d_S,d_b}(\lambda s : \tau_S.Q(s,a))
\]

\[
g = \lambda Q'. \lambda s. \mathbb{E}_{s'-T(s,a)}[f(Q'(s'))], \forall s : \tau_S. \forall a : \tau_A. \mathbb{E}_{s'-T(s,a)}[1] = 1, \gamma K_{d_S,W}(T) < 1
\]

\[\exists (\alpha : \tau_A. K_{d_S,d_b}(Q(a)) \land (K_2 = \sup_{\alpha : \tau_A} K_{d_S,d_b}(K_4 = K_{d_S,W}(T)) \implies K_2 \leq K_4/(1 - \gamma * K_4) + \epsilon)\tag{24}\]

Here \( d_S \) is a distance function on the state space. The logical assumptions are the losslessness of the transition dynamics \( T \), and the definition \( g = \lambda Q'. \lambda s. \mathbb{E}_{s'-T(s,a)}[f(Q'(s'))] \). We also introduce four slack variables \( K_1, K_2, K_3, \) and \( K_4 \) to use the above syntactic sugar for Lipschitz constants. The judgment (24) itself is proved inductively as in the paper [Asadi et al. 2018]. The key part of the proof is showing the inequality:

\[
K_{d_S,d_b}(\lambda s : \tau_S. \mathbb{E}_{s'-T(s,a)}[f(Q'(s'))]) \leq K_{d_S,d_b}(\lambda s : \tau_S. f(Q'(s'))) \cdot K_{d_S,W}(\lambda s : \tau_S.T(s,a))
\]

Suppose \( K_1 = K_{d_S,d_b}(\lambda s' : \tau_S. f(Q'(s'))) \) and \( K_2 = K_{d_S,W}(\lambda s : \tau_S.T(s,a)) \). What we prove in our framework is that \( z = K_{d_S,d_b}(\lambda s : \tau_S. \mathbb{E}_{s'-T(s,a)}[f(Q'(s'))]) \) implies \( z \leq K_1 + K_2 \). By unfolding and applying linearity of expectation, we obtain:

\[
z = K_{d_S,d_b}(\lambda s : \tau_S. \mathbb{E}_{s'-T(s,a)}[f(Q'(s'))]) \iff z = \sup_{s_1, s_2 : \tau_S} K_1 \cdot (\mathbb{E}_{s'-T(s_2,a)}[f(Q'(s'))]/K_1 - \mathbb{E}_{s'-T(s_1,a)}[f(Q'(s'))]/K_1).
\]

Here \( 1 = K_{d_S,d_b}(\lambda s : \tau_S. f(Q'(s'))/K_1) \) holds from the property \( d_{\mathbb{R}}(\alpha \cdot x, \alpha \cdot y) = \alpha \cdot d_{\mathbb{R}}(x, y) \) of \( d_{\mathbb{R}} \) \((0 \leq \alpha)\) and the losslessness \( \forall s : \tau_S. \forall a : \tau_A. \mathbb{E}_{s'-T(s,a)}[1] = 1 \) of the function \( T \). Hence, we conclude \( z \leq K_1 + K_2 \).

9 DOMAIN-SPECIFIC REASONING PRINCIPLES

Paper proofs of randomized algorithms typically use proof techniques to abstract away unimportant details. In this section, we show how PPV can support custom proof techniques in the form of domain-specific higher order logic. Specifically, we show that the \( T \)-lifting construction by Katsumata [2014]—roughly, a categorical construction useful for building different refinements of the probability distribution monad—can be smoothly incorporated in PPV. As concrete examples, we instantiate the unary \( T \)-lifting construction to a logic for reasoning about the probability of failure using the so-called union bound [Barthe et al. 2016b], and the binary \( T \)-lifting construction to a logic for reasoning about probabilistic coupling.

9.1 Embedding Unary Graded \( T \)-lifting

Roughly speaking, \( T \)-lifting of a monad is given by a large intersection of inverse images of some predicate, called lifting parameters. We can internalize this construction of \( T \)-lifting in PPV using a large intersection of assertions as \( \forall x : \tau. \phi_x \), and the inverse image \( \phi[e/y] \) of an assertion \( \phi \) along an expression \( e \). First, we internalize general construction of graded \( T \)-lifting in the unary logic UPL. Then we instantiate it to construct a unary graded \( T \)-lifting for reasoning about the probability of failure using a union bound.

These instantiations of \( T \)-liftings will need to use subprobability measures. Hence, in this section, we introduce a new monadic type \( D[\tau] \) describing the set of subprobability measures over
We interpret $D$ by $\|D[\tau]\| \overset{def}{=} \|\tau\|$, and interpret monadic structures in the same way as ones on the monadic type $M$. Furthermore, we assume that for every type $\tau$, $D[\tau]$ is a subtype of $M[\tau]$. We introduce the following axioms (25) enabling syntactic conversions from distributions in $D[\text{unit}]$ to real numbers in $[0, 1]$.

\[
\begin{align*}
\Gamma &\vdash e : D[\text{unit}] \\
\Gamma &\vdash e = \text{scale}(\text{return}(*), \lambda z : \text{unit}. \mathbb{E}_{\gamma\in[1]}) \\
\Gamma &\vdash e : D[\tau] \\
\Gamma &\vdash 0 \leq \mathbb{E}_{\gamma\in[1]}[1] \leq 1
\end{align*}
\]  

(25)

**General Construction.** We define a graded $\mathbb{T}\mathbb{T}$-lifting for the monadic type $D$. Consider a type $\zeta$ equipped with preordered monoid structure $\langle \zeta, 1, \leq, \sim \rangle$. A **lifting parameter** is a well-typed formula of form $\Gamma, k : \zeta, 1 : D[\theta] \vdash S$ satisfying the following monotonicity:

\[
\Gamma, k : \zeta, 1 : D[\theta] \vdash \forall x : \tau. \phi[x/r'] \Rightarrow S[\beta/k, f(x)/l].
\]

Notice that the parameters $\beta$ and $f$ ranges over all elements in the types $\zeta$ and $\tau \rightarrow D[\theta]$ respectively. The following formulas require $r : D[\tau]$ to satisfy the bind law. We regard $U_S$ as a constructor of graded $\mathbb{T}\mathbb{T}$-lifting. We then obtain the following graded monadic laws:

**Theorem 9.1 (Graded Monadic Laws of $U_S$).** The following rules are derivable:

\[
\begin{align*}
\Gamma &\vdash \Psi \uplus \Gamma \forall x : \tau. \phi_1[x/r'] \Rightarrow \phi_2[x/r'] \\
\Gamma &\vdash \Psi \uplus \Gamma \forall x : \tau. \phi_1[x/r'] \Rightarrow \phi_2[x/r'] \\
\Gamma &\vdash \Psi \uplus \Gamma \forall x : \tau. \phi_1[x/r'] \Rightarrow \phi_2[x/r']
\end{align*}
\]

The proofs follow by unfolding the constructor $U_S$. Furthermore, the graded monadic laws (Theorem B.3) are proved only using the structure of preordered monoid for grading, the monotonicity of the lifting parameter, $\alpha$-conversions, $\beta\eta$-reductions, the monadic laws of $D$, and applying proof rules in PL. Moreover, the construction of $\mathbb{T}\mathbb{T}$-lifting can be applied to any monadic type.

**Embedding the Union Bound Logic.** We show that the predicate lifting given in the semantic model of the union bound logic [Barthe et al. 2016b] can be implemented as a graded unary $\mathbb{T}\mathbb{T}$-lifting in PPV. Concretely, we give a lifting parameter $S$ such that the graded $\mathbb{T}\mathbb{T}$-lifting $U_S$ corresponds to the predicate lifting for the union bound logic.

Consider the additive monoid structure with usual ordering ($\text{pReal}, 0, +, \leq$). We define the lifting parameter $k : \text{pReal}, 1 : D[\text{unit}] \vdash S$ by $S = (\mathbb{E}_{\gamma\in[1]}[1] \leq k)$. The monotonicity of $S$ is obvious. As we proved above, we have the monadic rules for the graded $\mathbb{T}\mathbb{T}$-lifting $U_S$. Next, we prove that the graded $\mathbb{T}\mathbb{T}$-lifting $U_S$ describes the probability of failure:

**Proposition 9.2.** The following reduction is derivable in PL.

\[
\begin{align*}
\Gamma, r' : r + e : \text{bool} &\quad \Gamma, r' : r \mid \Psi \uplus \Gamma \neg \phi \iff (e = \text{true}) \\
\Gamma, r : D[r] &\quad \Gamma, r : D[r] \mid \Psi \uplus \Gamma U_S^r(\neg \phi) \iff \text{Pr}_{X\rightarrow r}[e[X/r']] \leq \alpha
\end{align*}
\]

Intuitively, this proposition holds because $U_S^r(\neg \phi) \iff \text{Pr}[\phi] \leq \alpha$. The second premise requires $\phi$ to be a measurable assertion, i.e., there is an indicator function $\lambda r' : r.e$ of $\phi$. 

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9.2 Embedding Relational $\top\top$-lifting

Similar to unary graded $\top\top$-lifting, we can also define relational graded $\top\top$-lifting. As a concrete example, we instantiate the (non-graded) relational $\top\top$-lifting for reasoning about probabilistic coupling. Consider a preordered monoid $(\xi, 1_\xi, \cdot_\xi, \preceq_\xi)$. A lifting parameter for relational $\top\top$-lifting is a well-typed formula of the form $\Gamma, k : \xi, l_1 : D[\theta_1], l_2 : D[\theta_2] \vdash S$ wf satisfying the following monotonicity condition:

$$\Gamma, k : \xi, l_1 : D[\theta_1], l_2 : D[\theta_2] \vdash_{\text{PL}} \forall \alpha : \xi, \forall \beta : \xi. (\alpha \preceq_\xi \beta \implies (S[\alpha/k] \implies S[\beta/k])).$$

For any assertion $\Gamma, r' : r_1, l' : l_2 \vdash \phi$ wf and an expression $\Gamma \vdash \alpha : \xi$ we define its relational lifting $\Gamma, r_1 : D[\tau_1], r_2 : D[\tau_2] \vdash S_\xi^2 \phi$ wf for the lifting parameter $S$ as the following assertion:

$$\forall \beta : \xi. \forall f_1 : \tau_1 \rightarrow D[\theta_1]. \forall f_2 : \tau_2 \rightarrow D[\theta_2].
(\forall x_1 : \tau_1, \forall x_2 : \tau_2, \phi[x_1/l_1', x_2/l_2'] \implies S[\beta/k, f_1(x_1)/l_1, f_2(x_2)/l_2])
\implies S[\alpha/\beta/k, \text{bind } r_1, f_1/l_1, \text{bind } r_2/f_2/l_2]).$$

We also have the two-sided graded monadic laws of $S_\xi$. We omit these structure, but they are given in an analogous way to graded unary $\top\top$-lifting.

Embedding the Modality for Relational Coupling of Distributions. As an example of this relational construction, we show how to internalize in our framework the modality for relational probabilistic coupling defined by Aguirre et al. [2018]. We say that two probability distributions $\mu_1$ and $\mu_2$ are coupled by a distribution $\mu$ over a relation $R \subseteq X \times Y$ if $\forall S \subseteq X. \Pr_{x \sim \mu_1}[S] \leq \Pr_{y \sim \mu_2}[R(S)]$. To internalize this construction we now need to supply the appropriate lifting parameters. First, we take the grading monoid to be the trivial one on the unit type $\text{unit}$. Then, we set the lifting parameter $k : \text{unit}, l_1 : D[\text{unit}], l_2 : D[\text{unit}] \vdash S$ by $S = (\Xi_{y \sim 1}[2] \leq \Xi_{y \sim 1}[1])$, which is equivalent to the usual inequality on $[0, 1]$. The assertion $S$ obviously satisfies the monotonicity of lifting parameter. Hence we obtain the $\top\top$-lifting $\Gamma, r_1 : D[\tau_1], r_2 : D[\tau_2] \vdash S_\xi \phi$ for the lifting parameter $S$. What we need to prove is that the lifting $S_\phi$ actually describes the above inequality of probabilistic dominance. In other words, we need to prove the following fundamental property in PL.

Proposition 9.3 (Aguirre et al. [2018, Lemma 2]).

$$\Gamma, r_1 : D[\tau_1], r_2 : D[\tau_2] \vdash_{\text{PL}} S_\xi \phi \implies \forall f_1 : \tau_1 \rightarrow \text{bool}. \forall f_2 : \tau_2 \rightarrow \text{bool}.
(\forall y : \tau_2. (f_2(y) = \text{true})) \implies \forall x : \tau_1. \phi[x/l_1', y/r_2'] \implies (f_1(x) = \text{true}))
\implies \Pr_{x \sim r_1}[f_1(x)] \leq \Pr_{y \sim r_2}[f_2(y)]$$

Intuitively $f_1$ and $f_2$ encode indicator functions $\chi_A$ and $\chi_B$ respectively, where $\phi(A) \subseteq B$. The proof follows Katsumata and Sato [2015, Theorem 12], again using the equivalence $D[\text{unit}] \equiv [0, 1]$ axiomatized in (25) and axioms on scaling of measures.

Specializing the assertion $\phi$ can establish useful probabilistic properties. For instance, taking $\phi$ to be the equality relation yields the following property.

Corollary 9.4 (Aguirre et al. [2018, Corollary 1]). If $\tau_1 = \tau_2 = \tau$ then

$$\Gamma, r : D[\tau] \vdash_{\text{PL}} S_\xi(r_1' = r_2') \iff (\forall g : \tau \rightarrow \text{real}. (\forall x : \tau. 0 \leq g(x) \leq 1) \implies \Xi_{x \sim r_1}[g(x)] \leq \Xi_{y \sim r_2}[g(y)])$$

If we take $g$ to be the indicator function of a (measurable) set $A$, the conclusion shows that the measure of $A$ in $r_1$ is smaller than the measure of $A$ in $r_2$. Since the assertion $\phi$ is symmetric, we can also conclude the inequality in the other direction, hence showing that the measure of $A$ must be equal in $r_1$ and in $r_2$. Since equality holds for all measurable $A$, $r_1$ and $r_2$ must denote equal probability measures.
10 RELATED WORK

Semantics of probabilistic programs. The semantics of probabilistic programs has been extensively studied starting from the seminal work of Kozen [1981]. Imperative first-order programs with continuous distributions have a well-understood interpretation based on the Giry monad [Giry 1982] over the category Meas of measurable spaces and measurable functions [Panangaden 1999]. However, this approach does not naturally extend to the higher-order setting since Meas is not cartesian closed [Aumann 1961]. In addition, although Meas has a symmetric monoidal closed structure [Culbertson and Sturtz 2013], the Giry monad is not strong with respect to the canonical one [Sato 2018].

The category QBS [Heunen et al. 2017] of quasi-Borel spaces was introduced as an “extension” of Meas that is cartesian closed and that can be used to interpret higher-order probabilistic programs with continuous distributions. The category of s-finite kernels [Staton 2017] gives a denotational semantics to observe-like statements in these models. In particular, it supports infinite measures and rescaling of measures. The monad \( \mathcal{M} \) of measures on quasi-Borel spaces we use in this paper was introduced by Ścibior et al. [2017] based on these constructions. One reason we chose QBS is that it has an obvious forgetful functor QBS \( \rightarrow \) Set giving the identity on functions. This is a key property to allow set-theoretic reasoning in PPV.

An alternative approach has been proposed by Ehrhard et al. [2017]. This work uses a domain-theoretic approach based on the category Cstab of cones and stable functions, and it extends previous works based on probabilistic coherent spaces [Ehrhard et al. 2014]. For our work, QBS is a more natural choice than Cstab for two reasons. First, the categorical structure needed for observe-like statements has already been studied in QBS. Second, we are interested in terminating programs and so we do not need the domain-theoretic structure of Cstab. Other models related to both QBS and Cstab that one could consider are the ones by Keimel and Plotkin [2017]; Tix et al. [2009]. Several other papers have studied models for higher-order probabilistic programming starting from the seminal papers on probabilistic powerdomains by Jones and Plotkin [1989] and Saheb-Djahromi [1980]. A non-exhaustive list includes Castellan et al. [2018]; Goubault-Larrecq and Jung [2014]; Jung and Tix [1998]; Mislove [2017]; Varacca et al. [2004]. Many of these model only partially support the features we need. There are also recent work studying the semantics of probabilistic programming from an operational perspective. Borgström et al. [2016] propose distribution-based and sample-based operational semantics for an untyped lambda calculus with continuous random variables. Their calculus also contains primitives for scaling and failing which allow them to model different kinds of observe-like constraints. Culpepper and Cobb [2017] propose an entropy-based operational semantics for a simply typed lambda calculus with continuous random variables and propose an operational theory for it based on logical relations.

Verification of probabilistic programs. Starting from the seminal work on Probabilistic Propositional Dynamic Logic by Kozen [1985], several papers have proposed program logics for the verification of imperative probabilistic programs. McIver and Morgan [2005]; Morgan et al. [1996] propose a predicative logic to reason about an imperative language with probabilistic and non-deterministic choice. Both these program logics allow reasoning about the expected value of a single real-valued function on program states. Many subsequent papers build on this idea [Audebaud and Paulin-Mohring 2009; Gretz et al. 2013; Hurd et al. 2005; Kaminski et al. 2016; Katoen et al. 2010]. Other papers focus on program logics where the pre-condition and post-condition are probabilistic assertions about the input and output distributions Chadha et al. [2007]; den Hartog [2002]; Ramshaw [1979]; Rand and Zdancewic [2015]. Barthe et al. [2018] propose an assertion-based logic, named ELLORA, using expectation for verifying properties of imperative probabilistic programs with discrete random variables. Our assertion logic PL is similar in spirit to the one of ELLORA, but it further
supports continuous distributions and the verification of higher-order programs. On the other hand, ELLORA has powerful rules for probabilistic while loops that PL does not support. It would be interesting to explore if similar rules can also be added to PPV. Formalizations of measure and integration theory in general purpose interactive theorem provers have been considered in many papers [Audebaud and Paulin-Mohring 2009; Coble 2010; Hötzl and Heller 2011; Hurd 2003; Richter 2004]. Avigad et al. [2014] recently completed a proof of the Central Limit theorem, which is the principle underlying concentration bounds. Hötzl [2016] formalized discrete-time Markov chains and Markov decision processes. These and other existing formalizations have been used to verify several case studies, but they are scattered and not easily accessible for our purposes.

Relational Verification. Several papers have explored relational program logics or relational type systems for the verification of different relational properties. Aguirre et al. [2017] propose UHOL/RHOL for the unary and relational verification of higher-order, non-probabilistic, terminating programs. UHOL and RHOL are based on a combination of a higher-order logic for expressing (unary and relational) postconditions, and syntax-directed proof rules for establishing them. Since only terminating, non-probabilistic programs are considered, the higher-order logic and the proof rules can be shown sound in set-theory. Our broad approach to setting up PPV is directly inspired from this work, but we work with probabilistic programs and, therefore, introduce a new monadic type for general/continuous measures along with constructs for conditioning. As a result, we have to interpret the logic and proof system in QBS, not set theory, and had to re-work the entire soundness proof from scratch.

The framework UC/RC [Radiček et al. 2017] is an extention of Aguirre et al. [2017] for reasoning about costs of non-probabilistic, terminating programs. This work introduces a monad, but this monad merely pairs a computation with its cost. The entire development still has a simple model in set theory. The GUHOL and GRHOL [Aguirre et al. 2018] are extensions of Aguirre et al. [2017] to reason about unary and relational properties of Markov chains. These systems include a monad for distributions, but the development is limited to discrete distributions, and relational probabilistic reasoning is limited to coupling. The framework has an interpretation in the topos of trees (which is an extension of set theory with step counting) extended with a Giry monad. In contrast, we handle continuous distributions. As we have shown, the probabilistic coupling of GRHOL can be embedded in PPV by $\top\top$-lifting, but PPV does not cover all features of GRHOL. The reason is the difference in the goals of verification: PPV verifies the static behavior of probabilistic programs such as expected values and equality between probability measures. In contrast, GRHOL verifies behaviors of entire Markov chains.

Barthe et al. [2016a] study a relational type system PrivInfer for Bayesian inference on a functional programming language. Our framework PPV is more flexible since it supports continuous probability distributions while PrivInfer supports only discrete probabilities. In the future, we expect to internalize in PPV the continuous variant of $(f, \delta)$-lifting proposed in PrivInfer, in a manner similar to $\top\top$-lifting.

11 CONCLUSION

In this paper we have introduced a framework PPV supporting the (unary and relational) verification of probabilistic programs including constructions for higher order computations, continuous distributions and conditioning. PPV combines axiomatizations of basic probabilistic constructions with rules of three different logics in order to ease the verification of examples from probabilistic inference, statistics, and machine learning. The soundness of our approach relies on quasi-Borel spaces, a recently proposed semantics framework for probabilistic programs. All these components make PPV a useful framework for the practical verification of higher-order probabilistic programs.
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A PROOFS IN PPV

A.1 More Detailed Proof on Monte Carlo Approximation

We first show the following judgment and then we apply Chebyshev’s inequality.

\[ d : M[\tau], \ h : \tau \rightarrow \text{real} \vdash_{\text{UPL}} \]
\[
\begin{align*}
\text{letrec } f (i : \text{nat}) & = \text{if } (i \leq 0) \text{ then return}(0) \\
& \quad \text{else mlet } m = f(i - 1) \text{ in mlet } x = d \text{ in return}(\frac{1}{i}(h(x) + m \times (i - 1))) \\
\end{align*}
\]
\[
\text{: nat } \rightarrow M[\text{real}] \\
\forall n : \text{nat}. \forall \sigma : \text{real}. \forall \mu : \text{real}. \\
(\mathbb{E}_{x \sim d}[1] = 1) \land (n > 0) \land (\sigma^2 = \text{Var}_{x \sim d}[h(x)]) \land (\mu = \mathbb{E}_{x \sim d}[h(x)]) \\
\implies (\mathbb{E}_{y \sim rn}[1] = 1) \land (\text{Var}_{y \sim rn}[y] = \frac{\sigma^2}{n})
\]
We split the program and postcondition as follows

\[
\text{MonteCarlo} \equiv \text{letrec } f(i) = e_{\text{body}} \\
\quad e_{\text{body}} \equiv \text{if } (i \leq 0) \text{ then } e_{\text{body}0} \text{ else } e_{\text{body}1} \\
\quad e_{\text{body}0} \equiv \text{return}(0) \\
\quad e_{\text{body}1} \equiv \text{mlet } m = f(i-1) \text{ in } (\text{mlet } x = d \text{ in return}(\frac{1}{l}(h(x) + m \ast (i-1))))
\]

\[
\phi \equiv \forall \sigma : \text{real. } \forall \mu : \text{real.} (\phi_0 \implies \phi_1) \\
\phi_0 \equiv (\mathbb{E}_{x \sim d}[1] = 1) \land (i > 0) \land (\sigma^2 = \text{Var}_{x \sim d}[h(x)]) \land (\mu = \mathbb{E}_{x \sim d}[h(x)]) \\
\phi_1 \equiv (\mathbb{E}_{y \sim \text{r}[1]} = 1) \land (\text{Var}_{y \sim \text{r}}[y] = \frac{\sigma^2}{i}).
\]

What we want to show is:

\[
d : M[\tau], h : \tau \rightarrow \text{real} \vdash_{\text{UPL}} \text{MonteCarlo} : \text{nat} \rightarrow M[\text{real}] \mid \forall n : \text{nat. } \phi[n/i]. \quad (26)
\]

To show (26) by applying [u-RETREC] rule, which we have the following premise:

\[
d : M[\tau], h : \tau \rightarrow \text{real, } f : \text{nat} \rightarrow M[\text{real}], i : \text{nat} \mid \phi_{\text{ind.hyp}} \vdash_{\text{UPL}} \text{if } (i \leq 0) \text{ then } e_{\text{body}0} \text{ else } e_{\text{body}1} : M[\text{real}] \\
\forall \sigma : \text{real. } \forall \mu : \text{real.} (\phi_0 \implies \phi_1). \quad (27)
\]

Here, we write

\[
\phi_{\text{ind.hyp}} = (\forall l : \text{nat. } l < i \implies (\sigma : \text{real. } \forall \mu : \text{real.} (\phi_0 \implies \phi_1))[l/i, f(l)/r])
\]

To show (27) by applying [u-CASE], and we need to show

\[
d : M[\tau], h : \tau \rightarrow \text{real, } f : \text{nat} \rightarrow M[\text{real}], i : \text{nat} \mid (i \leq 0) \land \phi_{\text{ind.hyp}} \vdash_{\text{UPL}} \text{return}(0) : M[\text{real}] \mid \forall \sigma : \text{real. } \forall \mu : \text{real.} (\phi_0 \implies \phi_1). \quad (28)
\]

\[
d : M[\tau], h : \tau \rightarrow \text{real, } f : \text{nat} \rightarrow M[\text{real}], i : \text{nat} \mid (i > 0) \land \phi_{\text{ind.hyp}} \vdash_{\text{UPL}} \text{mlet } m = f(i-1) \text{ in } (\text{mlet } x = d \text{ in return}(\frac{1}{l}(h(x) + m \ast (i-1)))) : M[\text{real}] \mid \forall \sigma : \text{real. } \forall \mu : \text{real.} (\phi_0 \implies \phi_1). \quad (29)
\]

The judgment (28) is shown by applying [u-SUB] having the following PL-premise:

\[
d : M[\tau], h : \tau \rightarrow \text{real, } f : \text{nat} \rightarrow \text{list}(M[\text{real}]), i : \text{nat} \mid (i \leq 0) \land \phi_{\text{ind.hyp}} \vdash_{\text{PL}} \forall \sigma : \text{real. } \forall \mu : \text{real.} (\phi_0 \implies \phi_1).
\]

In fact, this is a tautology since

\[
\phi_0 = (\mathbb{E}_{x \sim d}[1] = 1) \land (i > 0) \land (\sigma^2 = \text{Var}_{x \sim d}[h(x)]) \land (\mu = \mathbb{E}_{x \sim d}[h(x)]).
\]

The premise (29) is proved by applying [u-SUB] rule having the following premise:

\[
d : M[\tau], h : \tau \rightarrow \text{real, } f : \text{nat} \rightarrow M[\text{real}], i : \text{nat} \mid (i > 0) \land \phi_{\text{ind.hyp}} \vdash_{\text{PL}} \forall \sigma : \text{real. } \forall \mu : \text{real.} \\
\phi_0 \implies \mathbb{E}_{y \sim \text{r} \cdot (\text{mlet } m = f(i-1) \text{ in } \text{mlet } x = d \text{ in return}(\frac{1}{l}(h(x) + m(i-1))))}[1] = 1 \\
\land \text{Var}_{y \sim \text{r} \cdot (\text{mlet } m = f(i-1) \text{ in } \text{mlet } x = d \text{ in return}(\frac{1}{l}(h(x) + m(i-1))))}[y] = \frac{\sigma^2}{i}.
\]

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To show this by applying [∀] rule twice and [⇒ i] rule, it suffices to show
\[ d: M[\tau], \ h: \tau \to \text{real}, \ \sigma: \text{real}, \ \mu: \text{real}, \ f: \text{nat} \to M[\text{real}], \ i: \text{nat} \ |
\phi_0 \land (i > 0) \land \phi_{\text{ind.hyp}} \vdash_{\text{MHOL}}
\frac{1}{i} \in \mathbb{R} \quad \text{and} \quad \text{new sample } x.
\]
\[ \mathbb{E}_{y \sim \text{mlet } m = f(i - 1) \text{ in } \text{mlet } x = d \text{ in } \text{return}(\frac{1}{i}(h(x) + m(i - 1))))[1] = 1 \]
\[ \wedge \text{Var}_{y \sim \text{mlet } m = f(i - 1) \text{ in } \text{mlet } x = d \text{ in } \text{return}(\frac{1}{i}(h(x) + m(i - 1))))[y] = \sigma^2/i. \]

This is proved by applying (50), (52), (48), and elementary calculations.

For example, to show \( \text{Var}_{y \sim \text{mlet } m = f(i - 1) \text{ in } \text{mlet } x = d \text{ in } \text{return}(\frac{1}{i}(h(x) + m(i - 1))))[y] = \sigma^2/i, \) we calculate by applying [SUBST] rule in PL with equations as follows:

\[ \text{Var}_{y \sim \text{mlet } m = f(i - 1) \text{ in } \text{mlet } x = d \text{ in } \text{return}(\frac{1}{i}(h(x) + m(i - 1))))[y] \]
\[ = \text{Var}_{y \sim \text{mlet } w = d \circ f(i - 1) \text{ in } \text{return}(\frac{1}{i}(h(\pi_1(w)) + \pi_2(w)(i - 1)))}[y] \]
\[ = \text{Var}_{w \sim d \circ f(i - 1)}[\frac{1}{i}(h(\pi_1(w)) + \pi_2(w)(i - 1))] \]
\[ = \text{Var}_{x \sim d}[\frac{1}{i}h(x)] + \text{Var}_{m \sim f(i - 1)}[\frac{i - 1}{i}m] \]
\[ = \frac{1}{i^2} \text{Var}_{x \sim d}[h(x)] + \frac{(i - 1)^2}{i^2} \text{Var}_{m \sim f(i - 1)}[w] \]
\[ = \frac{1}{i^2} \sigma^2 + \frac{(i - 1)^2}{i^2} \frac{\sigma^2}{i - 1} \] (‡)
\[ = \frac{\sigma^2}{i} \] (†)

To obtain (†), precisely, we need further case analysis with \( i > 1 \) and \( i = 1 \). If \( i = 1 \) then we have (†) without (‡). This can be done by [⇒ i] rule in PL and a basic tautology.

Similarly, \( \mathbb{E}_{y \sim \text{mlet } m = f(i - 1) \text{ in } \text{mlet } x = d \text{ in } \text{return}(\frac{1}{i}(h(x) + m(i - 1))))[1] = 1 \) is proved by applying [SUBST] rule with (50), (52), (48) and elementary calculations. To sum up, we obtain

\[ \{ d: M[\tau], \ h: \tau \to \text{real}, \ e: \text{nat} \to M[\text{real}] \ |
\forall n: \text{nat}. \ \sigma: \text{real}, \ \mu: \text{real} \ |
\mathbb{E}_{x \sim d}[1] = 1 \land (n > 0) \land (\sigma^2 = \text{Var}_{x \sim d}[h(x)]) \land (\mu = \mathbb{E}_{x \sim d}[h(x)]) \] \[
\implies \mathbb{E}_{x \sim n}[1] = 1 \land \text{Var}_{y \sim d}[y] = \sigma^2/n. \]

We also have Chebyshev’s inequality (we prove later):

\[ \{ d: M[\text{real}], \ e: \text{real}, \ \mu: \text{real} \} \vdash_{\text{MHOL}} (\mu = \mathbb{E}_{y \sim d}[y]) \land (\mathbb{E}_{x \sim d}[1] = 1) \land (e^2 > 0) \] \[
\implies \Pr_{y \sim d}[|y - \mu| \geq e] \leq \text{Var}_{y \sim d}[y]/e^2. \]

By combining them we conclude what we desired.
A.2 Verification Example: Mean Estimation of Gaussian Distributions

So far, we have shown how to use PPV to reason about probabilistic programs using observe statements to describe Bayesian models. We now want to show it useful also to reason about statistical tasks that are not based on Bayesian update.

As a first example, we show that we can use PPV to prove the correctness of iterative mean estimation for Gaussian distributions. Here mean estimation is formulated in terms of a list of confidence intervals over the empirical mean observed over a set of sample. The correctness guarantees that these are indeed the right confidence intervals if the data actually come from the distribution. This example shows that we can use PPV to reason naturally also about unary iterative properties, and that we can use it to reason about standard statistical tools like confidence intervals.

First, we consider the following implementation GaussMean of the iterative mean estimation of Gaussian distribution with given variance $s$. The algorithm GaussMean receives an integer $i$ indicating the number of iterations and returns a list of length $i$, containing at each position $j$ the estimation after the first $j$ samplings. On each iteration, the algorithm samples an $x$ from $d$ (supposed to be a Gauss($\mu$, $s$) with unknown $\mu$) and updates the previous estimation $\langle \overline{x}, l, u \rangle$ of the empirical mean $\overline{x}$ and confidence interval $[l, u]$.

\begin{verbatim}
letrec f(i: nat) = if(i \leq 0) then[]
else (case f(i-1) with
    [] \Rightarrow mlet x = d in return(\langle \frac{x}{i}, \frac{x}{i} - z\sqrt{s/i}, \frac{x}{i} + z\sqrt{s/i} \rangle)
  r :: \xi' \Rightarrow mlet m = (mlet y = r in return(\langle \xi(y) \rangle)) in mlet x = d in
  return(\langle \frac{1}{i}(x + m \cdot (i - 1)), \frac{1}{i}(x + m \cdot (i - 1)) - z\sqrt{s/i}, \frac{1}{i}(x + m \cdot (i - 1)) + z\sqrt{s/i} \rangle)
) :: (r :: \xi')
\end{verbatim}

We show that $[l, u]$ forms an actually confidence interval of $\mu$ (i.e. $\Pr[l \leq \mu \leq u] \geq a$) for each step update of $\langle \overline{x}, l, u \rangle$. We will prove the following unary UPL judgment.

\begin{verbatim}
(d = Gauss(\mu, s), (s > 0), \Pr_{w \sim Gauss(0, 1)}[-z \leq w \leq z] \geq a, (z > 0) \vdash_{UPL} GaussMean(n): list(M[real \times real \times real]) |
\forall i: nat. 1 \leq i \leq n \Rightarrow \Pr_{(m, l, u)-r[i]}[l \leq \mu \leq u] \geq a.
\end{verbatim}

Here, the role of the assertion ($\Pr_{w \sim Gauss(0, 1)}[-z \leq w \leq z] \geq a$) is referring a table of Z-score of standard Gaussian distribution. We mainly use the reproductive property of Gaussian distributions and conversions of Gaussian distributions through the standard Gaussian distribution Gauss(0, 1). To prove $\Pr_{(m, l, u)-r[i]}[l \leq \mu \leq u] \geq a$, thanks to the equality (64), it suffices to prove

\begin{verbatim}r[i] = (mlet m' = Gauss(\mu, s/i) in return(\langle m', m' - z\sqrt{s/i}, m' + z\sqrt{s/i} \rangle)).\end{verbatim}

To prove this, we apply [u-APP], [u-LETREC], [u-CASE], [u-CONS], and [u-LISTCASE] rules in UPL to GaussMean. We then have the following main premise:

\begin{verbatim}(i > 0) \land (f(i-1) = r :: \xi) \land \phi_{\text{ind.hyp}} \vdash
mlet m = (mlet y = r in return(\langle \xi(y) \rangle)) in mlet x = d in
return(\langle \frac{1}{i}(x + m \cdot (i - 1)), \frac{1}{i}(x + m \cdot (i - 1)) - z\sqrt{s/i}, \frac{1}{i}(x + m \cdot (i - 1)) + z\sqrt{s/i} \rangle) \mid
\phi_{\xi} \Rightarrow r = mlet m' = Gauss(\mu, s/i) in return(\langle m', m' - z\sqrt{s/i}, m' + z\sqrt{s/i} \rangle)
\end{verbatim}

(30)
where $\phi_{\text{ind.hyp}}$ is the induction hypothesis obtained by applying [u-LETREC] rule, and $\phi_0'$ is the assumptions on the sample $d$ and the parameter $z$ in the postcondition of the initial judgment. We first show $\langle \text{mlet } y = r \text{ in return}(\pi_1(y)) = \text{Gauss}(\mu, s/(i-1)) \rangle$ from the preconditions and axioms on monadic type $M$. Then we calculate the result $r$ by applying the equation on Gaussian distributions (63).

A.2.1 More Detailed Proof. Since discussing intervals are easy

$$
\begin{align*}
\Pr_{ \langle m, l, u \rangle} [l \leq \mu \leq u] &= \Pr_{ \langle m, l, u \rangle, m' = \text{Gauss}(\mu, \frac{s}{i})} [l \leq \mu \leq u] \\
&= \Pr_{ m = \text{Gauss}(\mu, \frac{s}{i})} [m - z\sqrt{s/i} \leq \mu \leq m + z\sqrt{s/i}] \\
&= \Pr_{ m = \text{Gauss}(\mu, \frac{s}{i})} [-z\sqrt{s/i} \leq \mu - m \leq z\sqrt{s/i}] \\
&= \Pr_{ m = \langle \text{mlet } x = \text{Gauss}(0, 1) \text{ in return}(x\sqrt{s/(1+\mu)}) \rangle} [-z\sqrt{s/i} \leq -x\sqrt{s/i} \leq z\sqrt{s/i}] \\
&= \Pr_{ x = \text{Gauss}(0, 1)} [-z \leq x \leq z] \geq a
\end{align*}
$$

it suffices to prove the following judgment in UPL:

$$
\vdash_{\text{UPL}} \text{GaussMean: list}(M[\text{real} \times \text{real} \times \text{real}]) | \\
(d = \text{Gauss}(\mu, s)) \land (s > 0) \land (z > 0) \\
\implies \forall i: \text{nat}. \ i \leq n \implies r[i] = \text{mlet } m = \text{Gauss}(\mu, \frac{1}{i}s) \text{ in return}((m, m - z\sqrt{s/i}, m + z\sqrt{s/i})).
$$

We separate the expression $\Gamma \vdash e: \text{list}(M[\text{real} \times \text{real} \times \text{real}])$ as follows:

$$
\begin{align*}
\text{GaussMean} &\equiv \langle \text{letrec } f(i) = e_{\text{body}}(i) \rangle (n) \\
&= \langle \text{if } (i \leq 0) \text{ then } e_{\text{body}0} \text{ else } e_{\text{body}1} \rangle \\
&= \langle [] \rangle \\
&= \langle \text{case } f(i-1) \text{ with } [] \Rightarrow e_{\text{body}10}, r :: \xi' \Rightarrow e_{\text{body}11} \rangle :: f(i-1) \\
&= \langle \text{mlet } x = d \text{ in return}((\frac{x}{i}, \frac{x}{i} - z\sqrt{s/i}, \frac{x}{i} + z\sqrt{s/i})) \rangle \\
&= \langle \text{mlet } m = \langle \text{mlet } y = r \text{ in return}(\pi_1(y)) \rangle \text{ in mlet } x = d \text{ in return}((\frac{1}{i}(x + m(i-1)), \frac{1}{i}(x + m(i-1) - z\sqrt{s/i}, \frac{1}{i}(x + m(i-1)) + z\sqrt{s/i}))) \rangle
\end{align*}
$$

We introduce the following assertions:

$$
\begin{align*}
\phi' &\equiv \phi_0' \implies (\phi'_1 \land \phi'_2) \\
\phi_0' &\equiv \langle d = \text{Gauss}(\mu, s) \rangle \land (i > 0) \land (s > 0) \land (z > 0) \\
\phi'_1 &\equiv \langle r[i] = i \rangle \\
\phi'_2 &\equiv \langle \forall j: \text{nat}. \ 1 \leq j \leq i \implies r[j] = \text{mlet } m = \text{Gauss}(\mu, \frac{s}{j}) \text{ in return}((m, m - z\sqrt{s/j}, m + z\sqrt{s/j}))) \rangle.
\end{align*}
$$
The goal is to prove \( \Gamma \vdash \text{UPL GaussMean: list}(M[\text{real} \times \text{real} \times \text{real}]) \mid \phi'[n/i] \). To show this by applying [u-APP] rule, which have the following premise (\( \Gamma \) is a context):

\[
\begin{align*}
\Gamma & \vdash \text{UPL (letrecf i = e_{body1}: nat \to list}(M[\text{real} \times \text{real} \times \text{real}]) \mid \\
& \forall i: \text{nat}. (\phi'_0 \Rightarrow (\phi'_1 \land \phi'_2))[r_i/r].
\end{align*}
\]

To show this by applying [u-LETREC] rule, which has the premise:

\[
\Gamma, f: \text{nat} \to \text{list}(M[\text{real} \times \text{real} \times \text{real}]), i: \text{nat} | \\
\forall l: \text{nat}. l < i \implies (\phi'_0 \implies (\phi'_1 \land \phi'_2))[l/i, f(l)/r] \vdash \text{UPL}
\]

if \((i \leq 0)\) then \(e_{body0}\) else \(e_{body1}: \text{list}(M[\text{real} \times \text{real} \times \text{real}])\) |

\(\phi'_0 \implies (\phi'_1 \land \phi'_2)\). (31)

To show this by applying [u-CASE], which has the premises:

\[
\begin{align*}
\Gamma, f: \text{nat} \to \text{list}(M[\text{real} \times \text{real} \times \text{real}]), i: \text{nat} | \\
(i \leq 0) \land \forall l: \text{nat}. l < i \implies (\phi'_0 \implies (\phi'_1 \land \phi'_2))[l/i, f(l)/r] \vdash \text{UPL} \tag{31}
\]

[]: list(M[\text{real} \times \text{real} \times \text{real}]) |

\(\phi'_0 \implies (\phi'_1 \land \phi'_2)\)

\[
\begin{align*}
\Gamma, f: \text{nat} \to \text{list}(M[\text{real} \times \text{real} \times \text{real}]), i: \text{nat} | \\
(i > 0) \land \forall l: \text{nat}. l < i \implies (\phi'_0 \implies (\phi'_1 \land \phi'_2))[l/i, f(l)/r] \vdash \text{UPL} \\
\text{(case } f(i - 1) \text{ with } [] \to e_{body10}, r \Rightarrow \text{e}_{body11} : f(i - 1): \text{list}(M[\text{real} \times \text{real} \times \text{real}]) | \\
\phi'_0 \implies (\phi'_1 \land \phi'_2)\). \tag{32}
\end{align*}
\]

The premise (31) is derivable by applying [NIL] rule in UPL, which have the following premise:

\[
\begin{align*}
\Gamma, f: \text{nat} \to \text{list}(M[\text{real} \times \text{real} \times \text{real}]), i: \text{nat} | \\
(i \leq 0) \land \forall l: \text{nat}. l < i \implies (\phi'_0 \implies (\phi'_1 \land \phi'_2))[l/i, f(l)/r] \vdash \text{PL} \\
\top \implies (\phi'_0 \implies (\phi'_1 \land \phi'_2))[[]/r].
\end{align*}
\]

This is derivable in PL because the assertion \((i \leq 0) \implies \neg \phi'_0\) is obviously a tautology.

To show the premise (32) by applying [u-CONS] rule, which have the following premises:

\[
\begin{align*}
\Gamma, f: \text{nat} \to \text{list}(M[\text{real} \times \text{real} \times \text{real}]), i: \text{nat} | \\
(i > 0) \land \forall l: \text{nat}. l < i \implies (\phi'_0 \implies (\phi'_1 \land \phi'_2))[l/i, f(l)/r] \vdash \text{UPL} \\
\text{(case } f(i - 1) \text{ with } [] \to e_{body10}, r \Rightarrow \text{e}_{body11} : M[\text{real} \times \text{real} \times \text{real}] | \\
\phi'_0 \implies r = m \text{let } m = \text{Gauss}(\mu, s/i) \text{ in return}((m, m - z \sqrt{s/i}, m + z \sqrt{s/i})) \tag{33}
\end{align*}
\]

\[
\begin{align*}
\Gamma, f: \text{nat} \to \text{list}(M[\text{real} \times \text{real} \times \text{real}]), i: \text{nat} | \\
(i > 0) \land \forall l: \text{nat}. l < i \implies (\phi'_0 \implies (\phi'_1 \land \phi'_2))[l/i, f(l)/r] \vdash \text{UPL} \\
f(i - 1): \text{list}(M[\text{real} \times \text{real} \times \text{real}]) | \\
(\phi'_0 \implies (\phi'_1 \land \phi'_2))[(i - 1)/i] \tag{34}
\end{align*}
\]

Here, the judgment (34) is easily proved by applying [u-SUB], and [AX], [AX], \([\Rightarrow_E]\) rules in PL. Intuitively we instantiate the assertion \((\phi'_0 \implies (\phi'_1 \land \phi'_2))[l/i, f(l)/r]\) in the precondition by \(l = i - 1\), and then apply [u-SUB] rule.
By definition of the length $|−|$ and reference of components $(-)[i]$ of lists (we need to introduce equations for length of lists $|\xi| + 1 = |r :: \xi|$ and $(r :: \xi)[r :: \xi] = r$) and definition of assertions themselves, we have the following assertion in PL.

$\Gamma, f : \text{nat} \rightarrow \text{list}(M[\text{real} \times \text{real} \times \text{real}]), i : \text{nat} |$

$(i > 0) \land \forall l : \text{nat}. l < i \implies (\phi'_0 \implies (\phi'_1 \land \phi'_2))[l/i, f(l)/r] \vdash_{\text{PL}}$

$\forall \rho : M[\text{real} \times \text{real} \times \text{real}]. \forall \xi : \text{list}(M[\text{real} \times \text{real} \times \text{real}]).$

$(\phi'_0 \implies \rho = \text{mlet } m = \text{Gauss}(\mu, s/i) \text{ in return}(\langle m, m - z\sqrt{s/i}, m + z\sqrt{s/i} \rangle)[r/r])$

$\implies (\phi'_1 \implies (\phi'_1 \land \phi'_2))[i - 1/i, \xi/r] \implies (\phi'_0 \implies (\phi'_1 \land \phi'_2))[r :: \xi/r]$

To show the premise (33) by applying [u-LISTCASE] rule, we need to derive

$\Gamma, f : \text{nat} \rightarrow \text{list}(M[\text{real} \times \text{real} \times \text{real}]), i : \text{nat} \vdash$

$f(i - 1) : \text{list}(M[\text{real} \times \text{real} \times \text{real}])$  \hspace{1cm} (35)

$\Gamma, f : \text{nat} \rightarrow \text{list}(M[\text{real} \times \text{real} \times \text{real}]), i : \text{nat} |$

$(i > 0) \land (f(i - 1) = []) \land \forall l : \text{nat}. l < i \implies (\phi'_0 \implies (\phi'_1 \land \phi'_2))[l/i, f(l)/r] \vdash_{\text{UPL}}$

$m\text{let } x = d \text{ in return}(\langle \frac{x}{i}, \frac{x}{i} - z\sqrt{s/i}, \frac{x}{i} + z\sqrt{s/i} \rangle) : M[\text{real} \times \text{real} \times \text{real}] |$

$(\phi'_0 \implies \rho = \text{mlet } m = \text{Gauss}(\mu, s/i) \text{ in return}(\langle m, m - z\sqrt{s/i}, m + z\sqrt{s/i} \rangle))$  \hspace{1cm} (36)

$\Gamma, f : \text{nat} \rightarrow \text{list}(M[\text{real} \times \text{real} \times \text{real}]), i : \text{nat},$

$r : M[\text{real} \times \text{real} \times \text{real}], \xi : \text{list}(M[\text{real} \times \text{real} \times \text{real}]) |$

$(i > 0) \land (f(i - 1) = r :: \xi) \land \forall l : \text{nat}. l < i \implies (\phi'_0 \implies (\phi'_1 \land \phi'_2))[l/i, f(l)/r] \vdash_{\text{UPL}}$

$m\text{let } m = (\text{mlet } y = r \text{ in return}(\pi_1(y))) \text{ in mlet } x = d \text{ in return}(\frac{1}{i}(x + m(i - 1)), \frac{1}{i}(x + m(i - 1)) - z\sqrt{s/i}, \frac{1}{i}(x + m(i - 1)) + z\sqrt{s/i}) : M[\text{real} \times \text{real} \times \text{real}] |$

$(\phi'_0 \implies \rho = \text{mlet } m = \text{Gauss}(\mu, s/i) \text{ in return}(\langle m, m - z\sqrt{s/i}, m + z\sqrt{s/i} \rangle))$  \hspace{1cm} (37)

The typing judgment (35) is obvious. For the premise (36), we first need to show $i = 1$ from $i > 0$ and $0 = |[]| = |f(i - 1)| = i - 1$. Technically we show by applying [u-SUB] rule,
Formal verification of higher-order probabilistic programs

For the premise (37), since \(|f(i - 1)| = i - 1 > 0\), we must have \(i > 1\) and \(f(i - 1)[i - 1] = r\). Hence the following assertion is derivable:

\[
\Gamma, f : \text{nat} \rightarrow \text{list}(M[\text{real} \times \text{real} \times \text{real}]), i : \text{nat}, \\
\quad r : M[\text{real} \times \text{real} \times \text{real}], \xi : \text{list}(M[\text{real} \times \text{real} \times \text{real}]) \mid \\
\quad (i > 0) \land (f(i - 1) = r :: \xi) \land \forall l : \text{nat}. l < i \implies (\phi_0' \implies (\phi_1' \land \phi_2')[l/i], f(l)/r) ^ \text{PL} \\
\quad r = \text{mlet} \hat{m} = \text{Gauss}(\mu, \frac{s}{i - 1}) \text{ in return}((\hat{m}, \hat{m} - z\sqrt{s/(i - 1)}, \hat{m} + z\sqrt{s/(i - 1)})).
\]

By using this, and monadic laws, \textit{laws of projections}, and assumption on \(d\), we can do the following reduction in the assertion.

\[
\text{mlet } m = (\text{mlet } y = r \text{ in return}(\pi_1(y))) \text{ in mlet } x = d \text{ in} \\
\quad \text{return}(\frac{1}{i}(x + m(i - 1)), \frac{1}{i}(x + m(i - 1)) - z\sqrt{s/i}, \frac{1}{i}(x + m(i - 1)) + z\sqrt{s/i})
\]

{Substituting the above \(r\) and \(d = \text{Gauss}(\mu, s)\) and applying monadic and projection laws.}

\[
\quad = \text{mlet } m = \text{Gauss}(\mu, \frac{s}{i - 1}) \text{ in mlet } x = \text{Gauss}(\mu, s) \text{ in} \\
\quad \quad \text{return}(\frac{1}{i}(x + m(i - 1)), \frac{1}{i}(x + m(i - 1)) - z\sqrt{s/i}, \frac{1}{i}(x + m(i - 1)) + z\sqrt{s/i})
\]

{Applying monadic laws.}

\[
\quad = \text{mlet } \hat{m} = (\text{mlet } m = \text{Gauss}(\mu, \frac{s}{i - 1})) \text{ in mlet } x = \text{Gauss}(\mu, s) \text{ in} \\
\quad \quad \text{return}(\frac{1}{i}(x + m(i - 1)))
\]

{Applying the reproducing property of Gauss}

\[
\quad = \text{mlet } \hat{m} = \text{Gauss}(\mu, \frac{s}{i - 1}) \cdot \frac{(i - 1)^2}{i^2} + s \cdot \frac{1}{i^2}) \text{ in return}(\langle \hat{m}, \hat{m} - z\sqrt{s/i}, \hat{m} + z\sqrt{s/i} \rangle)
\]

{Just calculations.}

\[
\quad = \text{mlet } \hat{m} = \text{Gauss}(\mu, s/i) \text{ in return}(\langle \hat{m}, \hat{m} - z\sqrt{s/i}, \hat{m} + z\sqrt{s/i} \rangle)
\]

{\(\alpha\)-conversion}

\[
\quad = \text{mlet } m = \text{Gauss}(\mu, s/i) \text{ in return}(\langle m, m - z\sqrt{s/i}, m + z\sqrt{s/i} \rangle)
\]

At all, we obtain the following assertion in PL:

\[
\Gamma, f : \text{nat} \rightarrow \text{list}(M[\text{real} \times \text{real} \times \text{real}]), i : \text{nat}, \\
\quad r : M[\text{real} \times \text{real} \times \text{real}], \xi : \text{list}(M[\text{real} \times \text{real} \times \text{real}]) \mid \\
\quad (i > 0) \land (f(i - 1) = r :: \xi) \land \forall l : \text{nat}. l < i \implies (\phi_0' \implies (\phi_1' \land \phi_2')[l/i], f(l)/r) ^ \text{PL} \\
\quad mlet m = (\text{mlet } y = r \text{ in return}(\pi_1(y))) \text{ in mlet } x = d \text{ in} \\
\quad \quad \text{return}(\frac{1}{i}(x + m(i - 1)), \frac{1}{i}(x + m(i - 1)) - z\sqrt{s/i}, \frac{1}{i}(x + m(i - 1)) + z\sqrt{s/i}))
\]

Using this, by applying \([u\text{-}\text{SUB}]\) rule, we complete the proof.
We also have:

\[ \begin{align*}
\forall x : \tau. \ e_2 - e_1 &\geq 0 \implies \mathbb{E}_{x \sim e}[e_2 - e_1] \geq 0, \\
\forall x : \tau. \ e_1 &\leq e_2 \implies \mathbb{E}_{x \sim e}[e_1] \leq \mathbb{E}_{x \sim e}[e_2].
\end{align*} \tag{38} \]

The statement of Markov’s inequality is:

\[
\{d : M[\text{real}], \ a : \text{real} \} \vdash_{\text{PL}} \Pr_{x \sim d}[|x| \geq a] \leq \mathbb{E}_{x \sim d}[|x|]/a.
\]

For any \( a > 0 \), we have \( |x| \geq a \cdot (\text{if } |x| \geq a \text{ then } 1 \text{ else } 0) \). Hence the monotonicity and linearity of expected values: we calculate in PL:

\[
\mathbb{E}_{x \sim d}[|x|] \geq \mathbb{E}_{x \sim d}[a \cdot (\text{if } |x| \geq a \text{ then } 1 \text{ else } 0)]
\]

\[
= a\mathbb{E}_{x \sim d}[\text{if } |x| \geq a \text{ then } 1 \text{ else } 0]
\]

\[
= a \Pr_{x \sim d}[|x| \geq a].
\]

To show the first inequality, it suffices to show

\[
\{a : \text{real, } x : \text{real} \} \vdash_{\text{PL}} |x| \geq a \cdot (\text{if } |x| \geq a \text{ then } 1 \text{ else } 0)
\]

To prove this we show by analyzing if-else expression inside PL:

\[
\{d : M[\text{real}], \ a : \text{real} \} \mid |x| \geq a \vdash_{\text{MHOL}} |x| \geq (a \cdot 1)
\]

\[
\{d : M[\text{real}], \ a : \text{real} \} \mid |x| < a \vdash_{\text{MHOL}} |x| \geq (a \cdot 0).
\]

A.4 Chebyshev Inequality

By applying \( a = b^2 \), \( d = (\text{mlet } x = d' \text{ in return } (x - \mu)^2) \), (52), and \( \alpha \)-conversion to Markov’s inequality we have

\[
\{d : M[\text{real}], \ b : \text{real, } \mu : \text{real} \} \vdash_{\text{PL}} \Pr_{x \sim d}[|x - \mu| \geq b] \leq \mathbb{E}_{x \sim d}[|x - \mu|^2]/b^2.
\]

Hence,

\[
\{d : M[\text{real}], \ b : \text{real, } \mu : \text{real} \} \vdash_{\text{PL}} \mu = \mathbb{E}_{x \sim d}[x] \wedge 1 = \mathbb{E}_{x \sim d}[1] \wedge b^2 > 0
\]

\[
\implies \Pr_{x \sim d}[|x - \mu| \geq b] \leq \mathbb{E}_{x \sim d}[|x - \mu|^2]/b^2.
\]

We also have:

\[
\{d : M[\text{real}], \ b : \text{real, } \mu : \text{real} \} \vdash_{\text{PL}} \mu = \mathbb{E}_{x \sim d}[x] \wedge 1 = \mathbb{E}_{x \sim d}[1] \wedge b^2 > 0
\]

\[
\implies \mathbb{E}_{x \sim d}[|x - \mu|^2] \geq b = \text{Var}_{x \sim d}[x].
\]

Combining the previous two derivations, we conclude the Chebyshev’s inequality:

\[
\{d : M[\text{real}], \ b : \text{real, } \mu : \text{real} \} \vdash_{\text{PL}} \mu = \mathbb{E}_{x \sim d}[x] \wedge 1 = \mathbb{E}_{x \sim d}[1] \wedge b^2 > 0
\]

\[
\implies \Pr_{x \sim d}[|x - \mu| \geq b] \leq \text{Var}_{x \sim d}[x]/b^2.
\]
A.5 Omitted Calculations in the Example of Importance Sampling

The expressions SumLoop2 and Naive are introduced in the verification example of importance sampling is defined by

\[
\begin{align*}
\text{SumLoop2} & \equiv \text{letrec } f(i: \text{nat}) = \lambda g: \tau \to \text{real}. \lambda h: \tau \to \text{real}. h_2: \tau \to \text{real}. \\
& \quad \text{if}(i \leq 0) \text{ then return}(0,0) \text{ else mlet } x = d \text{ in mlet } m = f(i-1)(g)(h)(h_2)\text{in} \\
& \quad \text{return}\left(\frac{1}{i}(\pi_1(m) + (i-1) \ast h(x) \ast g(x)), \frac{1}{i}(\pi_2(m) + (i-1) \ast h_2(x) \ast g(x))\right) \\
\text{Naive} & \equiv \text{letrec } f(i: \text{nat}) = \lambda g: \tau \to \text{real}. h: \tau \to \text{real}. h_2: \tau \to \text{real}. \\
& \quad \text{if}(i \leq 0) \text{ then (return 0) else mlet } x = d \text{ in mlet } m = f(i-1)(g)(h)\text{in} \\
& \quad \text{return}\left(\frac{1}{i}(\pi_1(m) + (i-1) \ast h(x) \ast g(x))\right)
\end{align*}
\]

We have the following structural equalities:

\[
\begin{align*}
\vdash_{\text{RPL}} \text{SumLoop2} & \sim \text{Naive} \mid \text{mlet } z = r_1(k)(g)(h)(h_2) \text{ in return}(\pi_1(z)) = r_4(k)(g)(h) \\
\vdash_{\text{RPL}} \text{SumLoop2} & \sim \text{Naive} \mid \text{mlet } z = r_1(k)(g)(h)(h_2) \text{ in return}(\pi_2(z)) = r_4(k)(g)(h_2) \\
C > 0 \vdash_{\text{RPL}} \text{SumLoop2} & \sim \text{SumLoop2} \mid \text{mlet } z = r_1(k)(g)(h)(1) \text{ in return}(\pi_1(z)/\pi_2(z)) \\
& \quad = \text{mlet } z = r_1(k)(g/C)(h)(1) \text{ in return}(\pi_1(z)/\pi_2(z)) \\
\vdash_{\text{RPL}} \text{SumLoop2} & \sim \text{SumLoop} \mid r_1(k)(g)(h)(1) = r_2(k)(g)(h)
\end{align*}
\]

We will see the most complicated calculation in the verification example of importance samplings.

We set \(h_2 = \lambda x. \tau. (\text{if } g(x) \leq k \ast \exp(-t/2) \text{ then } 1 \text{ else } 0) \ast h(x)\). We first compute:

\[
\begin{align*}
\mathbb{E}_w(\text{mlet } z = \text{SumLoop2}(k)(g)(h)(h_2) \text{ in return}(\pi_1(z))) & \left[|w - \mu|\right] \\
& \quad \text{Variable transformation in expectation values} \\
& = \mathbb{E}_z(\text{SumLoop2}(k,g,h,h_2))\left[|\pi_1(z) - \mu|\right] \\
& \quad \text{Triangle inequality on absolute values and monotonicity of expectations} \\
& \leq \mathbb{E}_z(\text{SumLoop2}(k)(g)(h)(h_2))\left[|\pi_1(z) - \pi_2(z)| + |\pi_2(z) - \mu'\right] + |\mu - \mu'|\right] \\
& \quad \text{additivity pf expectations} \\
& = \mathbb{E}_z(\text{SumLoop2}(k)(g)(h)(h_2))\left[|\pi_1(z) - \pi_2(z)|\right] + \mathbb{E}_z(\text{SumLoop2}(k)(g)(h)(h_2))\left[|\pi_2(z) - \mu'|\right] \\
& \quad + \mathbb{E}_z(\text{SumLoop2}(k)(g)(h)(h_2))\left[|\mu - \mu'|\right]
\end{align*}
\]
Next, we apply Cauchy-Schwartz inequality to each expected values. We denote $a \equiv k \cdot \exp(-t/2) \geq \exp(L + t/2)$, and $\mu' \equiv \mathbb{E}_{y \sim d}[h_2(y)]$. Then we compute:

$$
\mathbb{E}_{y \sim d}[k(h_2)] [\pi_2(z) - \mu'] \leq \sqrt{t} \mathbb{E}_{y \sim d}[|h_2(z)|] \leq \sqrt{t} \|h_2\|
$$

{Applying definition of variance, and definition of $h_2$}

$$
\sqrt{t} \mathbb{E}_{y \sim d}[k(h_2)] [\pi_2(z) - \mu'] \leq \sqrt{t} \mathbb{E}_{y \sim d}[|h_2(z)|] \leq \sqrt{t} \|h_2\|
$$

(Reusing the above calculation)

$$
\mathbb{E}_{y \sim d}[k(h_2)] [\pi_2(z) - \mu'] \leq \sqrt{t} \mathbb{E}_{y \sim d}[|h_2(z)|] \leq \sqrt{t} \|h_2\|
$$

{Applying Cauchy-Schwartz inequality}

$$
\mathbb{E}_{y \sim d}[k(h_2)] [\pi_2(z) - \mu'] \leq \sqrt{t} \mathbb{E}_{y \sim d}[|h_2(z)|] \leq \sqrt{t} \|h_2\|
$$

{Applying definition of variance, and definition of $h_2$}

$$
\sqrt{t} \mathbb{E}_{y \sim d}[k(h_2)] [\pi_2(z) - \mu'] \leq \sqrt{t} \mathbb{E}_{y \sim d}[|h_2(z)|] \leq \sqrt{t} \|h_2\|
$$

A.6 Derivations of Several (in) Equalities.

Several derivations of equalities are bit complicated, so we show some of them.

A.6.1 Marginal Law of Product Measures. Let $\Gamma \vdash e_1 : M[\tau_1]$ and $\Gamma \vdash e_2 : M[\tau_2]$. Then the following equalities are derivable in PL:

$mlet w = e_1 \otimes e_2 \in return \pi_1(w)$

= bind(bind $e_1 \lambda w_1.(bind e_2 \lambda w_2. \ \ \ return(w_1, w_2)) \ \ \ \ \lambda w. \ \ return\pi_1(w)$) (Syntactic sugar)

= bind $e_1 \lambda w_1.(bind e_2 \lambda w_2. \ \ \ return(w_1, w_2)) \ \ \ \ \lambda w. \ \ return\pi_1(w)$ (associativity of bind)

= bind $e_1 \lambda w_1.(bind e_2 \lambda w_2. \ \ \ return(w_1, w_2)) \ \ \lambda w. \ \ return\pi_1(w)$ (associativity of bind)

= bind $e_1 \lambda w_1.(bind e_2 \lambda w_2. \ \ \ return(w_1))$ (Monadic law (unit law))

= (scale($e_1, \lambda w_1.\mathbb{E}_{y \sim d}\[h_2(w_1)][1] \)) (equation 57)

= (scale($e_1, \lambda w_1.\mathbb{E}_{w_2 \sim e_2}[1] \)) (equation 52)
Similarly we have
\[ \vdash_{\text{PL}} (\text{mlet } w = e_1 \otimes e_2 \text{ in return } \pi_2(w)) = (\text{scale}(e_2, \lambda w. \mathbb{E}_{w_1 \sim e_1[1]})). \]

A.6.2 Independence for Product Measures. Let \( \Gamma \vdash d_1 : M[\tau_1], \Gamma \vdash d_2 : M[\tau_2], \Gamma \vdash f : \tau_1 \rightarrow p\text{Real} \) and \( \Gamma \vdash g : \tau_2 \rightarrow p\text{Real} \). Then the following equalities are derivable in PL:

\[ \mathbb{E}_{w \sim d_1 \otimes d_2}[f(\pi_1(w)) \ast g(\pi_2(w))] \]
\[ = \mathbb{E}_{w \sim \text{scale}(d_1 \otimes d_2, \lambda w.f(\pi_1(w)))}[g(\pi_2(w))]) \]  
\[ = \mathbb{E}_{w \sim \text{scale}(d_1,f) \otimes \text{scale}(d_2,1)}[g(\pi_2(w))] \]  
\[ = \mathbb{E}_{w \sim \text{scale}(d_1,f) \otimes d_2}[g(\pi_2(w))] \]  
\[ = \mathbb{E}_{y \sim \text{scale}(d_1,f) \otimes d_2} \lambda w. \text{return } \pi_2(w)[g(y)] \]  
\[ = \mathbb{E}_{y \sim \text{scale}(d_1,f)}[1] \ast \mathbb{E}_{y \sim d_2}[g(y)] \]  
\[ = \mathbb{E}_{x \sim d_1}[f(x)] \ast \mathbb{E}_{y \sim d_2}[g(y)] \]  

A.6.3 Slicing Law on Simple Observations. Let \( \Gamma \vdash x : M[\tau_1], \Gamma \vdash y : M[\tau_2], \Gamma \vdash f : \tau_2 \rightarrow p\text{Real}, \) and assume \( \mathbb{E}_{\_ \sim x[1]} = 1 \). Then the following equalities are derivable in PL:

\[ \text{mlet } v = (\text{observe } x \otimes y \text{ as } \lambda w. f(\pi_2(w)) \text{ in return } \pi_1(v)) \]
\[ = \text{mlet } v = \text{normalize(\text{scale}(x \otimes y, \lambda w.f(\pi_2(w))) \text{ in return } \pi_1(v))} \]  
\[ = \text{mlet } v = \text{scale}(x \otimes y, \lambda w.f(\pi_2(w))), \lambda \_1/K) \text{ in return } \pi_1(v)) \]  
\[ = \text{mlet } v = ((\text{scale}(x, \lambda \_1) \otimes \text{scale}(y, f/K)) \text{ in return } \pi_1(v)) \]  
\[ = \text{mlet } v = (x \otimes (\text{scale}(y, f/K))) \text{ in return } \pi_1(v)) \]  
\[ = \text{scale}(x, \lambda \_1, \mathbb{E}_{\_ \sim \text{scale}(y,f/K)}[1]) \]  
\[ = \text{scale}(x, \lambda \_1, \mathbb{E}_{\_ \sim \text{observe gpdf}}[1]) \]  
\[ = \text{scale}(x, \lambda \_1) = x \]  

Where \( K \equiv \mathbb{E}_{\_ \sim \text{scale}(x \otimes y, \lambda w.f(\pi_2(w)))}[1] \). The equality (‡) is derived as follows. By the independence of product measure, we have \( K = \mathbb{E}_{\_ \sim \text{scale}(y,f)}[1] \ast \mathbb{E}_{\_ \sim x}[1] \). Thanks to \( \mathbb{E}_{\_ \sim x}[1] = 1 \) we have \( K = \mathbb{E}_{\_ \sim \text{scale}(y,f)}[1] \). Hence, we conclude \( \text{scale}(y, f/K) = \text{observe } y \) as \( f \).

A.7 Gaussian are Conjugate Prior wrt Gaussian Likelihood functions

We assumed the definition

\[ \text{Gauss}(x, \sigma^2) = \text{scale} (\text{Lebesgue}, \text{GPDF}(x, \sigma^2)) \]

From the probability of Gaussian distribution and applying (equations 54, 59, and 60), we have

\[ \text{Gauss}(x, \sigma^2) = \text{normalize(\text{scale} (\text{Lebesgue}, \lambda r. \exp(\frac{(r-x)^2}{2\sigma^2})))} \]  

(39)
Using this we calculate,
\[
\text{observe } \text{Gauss}(\delta, \xi^2) \text{ as } \text{GPDF}(z, \sigma^2)
\]
\[
= \text{normalize}(\text{scale}(\text{Gauss}(\delta, \xi^2), \text{GPDF}(z, \sigma^2))) \quad \text{(equation 58)}
\]
\[
= \text{normalize}(\text{scale}(\text{scale}(\text{Lebesgue}, \text{GPDF}(\delta, \xi^2)), \text{GPDF}(z, \sigma^2))) \quad \text{(Axiom on Gauss)}
\]
\[
= \text{normalize}(\text{scale}(\text{Lebesgue}, \lambda r.\text{GPDF}(\delta, \xi^2)(r) * \text{GPDF}(z, \sigma^2)(r))) \quad \text{(equation 56)}
\]
\[
= \text{normalize}(\text{scale}(\text{Lebesgue}, \lambda r.\exp(\frac{(r - \delta)^2}{2\xi^2}) * \exp(\frac{(r - z)^2}{2\sigma^2}))) \quad \text{(equations 59 and 60)}
\]
\[
= \text{normalize}(\text{scale}(\text{Lebesgue}, \lambda r.\exp(\frac{(r - z + \delta)^2}{2\xi^2 + \sigma^2})) \quad \text{(calculation)}
\]
\[
= \text{Gauss}(\frac{z\xi^2 + \delta\sigma^2}{\xi^2 + \sigma^2}, \frac{\xi^2\sigma^2}{\xi^2 + \sigma^2}) \quad \text{(equation 39)}
\]

B \ \text{PROOFS AND SKETCHES ON GRADED } \text{T T-LIFTINGS}

**Theorem B.1 (Graded Monadic Laws of } U_S\).** The following rules are derivable:

\[
\Gamma | \Psi |-\text{PL } \forall \alpha: \xi. \exists \beta: \xi. U^\alpha_S \phi \implies U^\beta_S \phi
\]

\[
\Gamma | \Psi |-\text{UPL } \forall \alpha: \xi. (\forall x: \tau. \phi(x/x')) \implies U^\alpha_S \phi \implies U^\alpha_S \phi_1 \implies U^\alpha_S \phi_2
\]

\[
\Gamma | \Psi |-\text{UPL } e: \tau | \phi(e/x')
\]

\[
\Gamma | \Psi |-\text{UPL } \text{return}(e): D[\tau] | U^\xi_S \phi
\]

\[
\Gamma | \Psi |-\text{UPL } e': \tau \rightarrow D[\tau'] | \forall x: \tau. \phi(x/r) \implies (U^\alpha_S \phi')[rx/r]
\]

\[
\Gamma | \Psi |-\text{UPL } \text{bind } e' : D[\tau'] | U^{\alpha \cdot \beta}_S \phi'
\]

**Proof Sketch.** The proofs are straightforward. For example, the proof of (??) begins with applying the [u-BIND] rule, which has the following premise:

\[
\vdash_{\text{UPL}} e': \tau \rightarrow D[\tau'] | \forall d: D[\tau].(U^\alpha_S \phi)[d/r] \implies (U^\alpha_S \phi'[\text{bind } d \; e'/r]).
\]

We then apply [u-SUB] rule, which has the following PL-premise:

\[
\vdash_{\text{PL}} (\forall x: \tau. \phi(x/r)) \implies (U^\alpha_S \phi')(e'/x'/r)
\]

\[
\implies (\forall d: D[\tau].(U^\alpha_S \phi)[d/r] \implies (U^\alpha_S \phi'[\text{bind } d \; e'/r]).
\]

To prove this premise, consider \( f: \tau' \rightarrow D[\emptyset] \) satisfying \( \phi'[y/r] \implies S[y/k], f(y)/l \) and \( d: D[\tau] \) such that \( (U^\alpha_S \phi)[d/r] \). First, from the assumption on \( e' \) and \( f \), we obtain \( \forall x: \tau. \phi(x/r) \implies S[\beta \cdot y/k, \text{bind } e' \cdot f/l]. \) Hence, by the assumption on \( d \), we obtain \( S[\alpha \cdot \beta \cdot y/k, \text{bind } d \cdot \lambda x: \tau. (\text{bind } e' \cdot f/l]. \) By the associativity of bind, this is equivalent to \( S[\alpha \cdot \beta \cdot y/k, \text{bind } \text{bind } d \; e'] \cdot f/l]. \) Since \( f \) is arbitrary, we conclude \( (U^\alpha_S \phi'[\text{bind } d \; e'/r]). \)

**Proposition B.2.** In the setting in Section 9.1, the following reduction is derivable in PL.

\[
\Gamma, \tau': \tau \vdash e: \text{bool } \quad \Gamma, \tau: \tau | \Psi |-\text{PL } \neg \phi \iff (e = \text{true})
\]

**Proof Sketch.** We observe the following:

\[
U^\alpha_S(\neg \phi) \equiv \begin{cases}
\forall f: \tau \rightarrow D[\text{unit}]. \forall \beta: \text{pReal}.
& \left( \forall x: \tau. \neg \phi(x/x') \implies E_y-f(x)[1] \leq \beta \right) \implies \left( E_y-(\text{bind } r f)[1] \leq \alpha + \beta \right)
\end{cases}
\]
The forward direction of the conclusion is proved by the equality (†) derived from the axioms on scaling in PL:

\[ \Pr_{\mathcal{X}\sim\tau}[e[X/r']] (\triangleright) \mathbb{E}_{y\sim\epsilon}[\lambda X. \text{scale}(\text{return}(\epsilon), \text{unit. if } e[X/r']) \text{then else } 0][1] \leq \alpha + 0. \]

For the converse direction, we need the equivalence between \( D[\text{unit}] \) and the unit interval \([0, 1]\). To realize this we apply the axioms (25) in PL. Combining the axioms (25) and other axioms on scaling of measures, we conclude the equivalence between a distribution \( e : D[\text{unit}] \) and its mass \( \mathbb{E}_{y\sim\epsilon}[1] \). The proof follows by showing that the function \( f : \tau \rightarrow D[\text{unit}] \) such that \( (\forall x. \tau. \neg \phi[x/r'] \implies \mathbb{E}_{y\sim f(x)}[1] \leq \beta) \).

For relational \( \top \top \) lifting, we have the following derivable graded monadic laws:

**Theorem B.3 (Graded Monadic Laws of \( \mathcal{R}_S \)).** The following rules are derivable:

\[
\Gamma \vdash_{\text{PL}} \forall \alpha : \zeta. \forall \beta : \zeta. (\alpha \leq \zeta \beta \implies \mathcal{R}_S^\alpha \phi \implies \mathcal{R}_S^\beta \phi) \tag{40}
\]

\[
\Gamma \vdash_{\text{RPL}} \forall \alpha : \zeta. (\forall x_1 : \tau_1. \forall x_2 : \tau_2. \phi_1[x_1/r', x_2/r'_2] \implies \phi_2[x_1/r'_1, x_2/r'_2]) \implies (\mathcal{R}_S^\alpha \phi_1 \implies \mathcal{R}_S^\alpha \phi_2) \tag{41}
\]

\[
\Gamma \vdash_{\text{RPL}} \text{e}_1 : \tau_1 \sim e_2 : \tau_2 \mid \phi[r_1/r'_1, r_2/r'_2] \tag{42}
\]

\[
\Gamma \vdash_{\text{RPL}} \text{e}_1' : \tau_1 \rightarrow D[\tau_1] \sim e_2' : \tau_2 \rightarrow D[\tau_2] \mid (\mathcal{R}_S^\beta \phi')[r_1 x_1/r_1, r_2 x_2/r_2] \tag{43}
\]

\[
\Gamma \vdash_{\text{RPL}} \text{bind } \text{e}_1 : \tau_1 \sim \text{bind } e_2' : \tau_2 \rightarrow D[\tau_2] \mid \mathcal{R}_S^\beta \phi' \tag{44}
\]

**Proof.** The rule (40) is obvious from the monotonicity of lifting parameter. The rule (41) proved from the fact that formulas in the following form is tautology:

\[
(\phi_A \implies \phi_B) \implies ((\phi_B \implies \phi_C) \implies (\phi_A \implies \phi_C)).
\]

We prove (42). To prove this by applying two-sided \([r\text{-RETURN}]\) rule, having the premise

\[
\Gamma \vdash_{\text{RPL}} \text{e}_1 : \tau_1 \sim e_2 : \tau_2 \mid (\mathcal{R}_S^\lambda \phi)[\text{return}(r_1)/r_1, \text{return}(r_2)/r_2].
\]

To prove this by relational \([r\text{-SUB}]\) rule, having the premise:

\[
\Gamma \vdash_{\text{PL}} \phi[r_1/r'_1, r_2/r'_2] \implies (\mathcal{R}_S^\lambda \phi)[\text{return}(r_1)/r_1, \text{return}(r_2)/r_2].
\]

By the monadic unit law, we obtain the following equality in PL-formulas:

\[
(\mathcal{R}_S^\lambda \phi)[\text{return}(r_1)/r_1, \text{return}(r_2)/r_2]
\]

\[
= \forall \beta : \zeta. \forall f_1 : \tau_1 \rightarrow D[\theta_1], \forall f_2 : \tau_2 \rightarrow D[\theta_2].
\]

\[
(\forall x_1 : \tau_1. \forall x_2 : \tau_2. \phi[x_1/r'_1, x_2/r'_2] \implies S[\beta/k, f_1(x_1)/l_1, f_2(x_2)/l_2])
\]

\[
\implies S[\beta/k, \text{bind return}(r_1) f_1)/l_1, \text{bind return}(r_2) f_2/l_2])
\]

\[
= \forall \beta : \zeta. \forall f_1 : \tau_1 \rightarrow D[\theta_1], \forall f_2 : \tau_2 \rightarrow D[\theta_2].
\]

\[
(\forall x_1 : \tau_1. \forall x_2 : \tau_2. \phi[x_1/r'_1, x_2/r'_2] \implies S[\beta/k, f_1(x_1)/l_1, f_2(x_2)/l_2])
\]

\[
\implies S[\beta/k, f_1(r_1)/l_1, f_2(r_2)/l_2])
\]

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The second formula

\[
\phi[r_1/r'_1, r_2/r'_2] \\
\iff \forall \beta: \zeta. \forall f_1: \tau_1 \rightarrow D[\theta_1]. \forall f_2: \tau_2 \rightarrow D[\theta_2]. \\
(\forall x_1: \tau_1, \forall x_2: \tau_2. \phi[x_1/r'_1, x_2/r'_2] \implies S[\beta/k, f_1(x_1)/l_1, f_2(x_2)/l_2]) \\
\implies S[\beta/k, f_1(r_1)/l_1, f_2(r_2)/l_2])
\]

is a tautology. Hence, we conclude (44).

Next we show (43). To prove this by applying [r-BIND] and [r-SUB] rules, having the following PL-premise

\[
\Gamma | \Psi \vdash_{PL} \forall r_1: \tau_1 \rightarrow D[\tau'_1]. \forall r_2: \tau_2 \rightarrow D[\tau'_2]. \\
(\forall x_1: \tau_1, \forall x_2: \tau_2. \phi[x_1/r'_1, x_2/r'_2] \implies (\mathcal{R}_S^{\alpha, \beta, \phi'}[r_1 x_1/r_1, r_2 x_2/r_2])) \\
\implies (\forall s_1: D[\tau_1]. \forall s_2: D[\tau_2]. \mathcal{R}_S^{\alpha, \beta, \phi}[s_1/r_1, s_2/r_2])
\]

To prove this by applying \(\Rightarrow_I\) and \(\forall_I\), having the following PL-premise

\[
\Gamma, r_1: \tau_1 \rightarrow D[\tau'_1], r_2: \tau_2 \rightarrow D[\tau'_2], s_1: D[\tau_1], s_2: D[\tau_2] \mid \Psi, \\
(\forall x_1: \tau_1, \forall x_2: \tau_2. \phi[x_1/r'_1, x_2/r'_2] \implies (\mathcal{R}_S^{\alpha, \beta, \phi'}[r_1 x_1/r_1, r_2 x_2/r_2])) \\
(\mathcal{R}_S^{\alpha, \beta, \phi}[s_1/r_1, s_2/r_2]) \vdash_{PL} \mathcal{R}_S^{\alpha, \beta, \phi'}[\text{bind } s_1 r_1/r_1, \text{ bind } s_2 r_2/r_2]
\]

We unfold the macro \(\mathcal{R}_S^{\alpha, \beta, \phi'}[\text{bind } s_1 r_1/r_1, \text{ bind } s_2 r_2/r_2]\) to:

\[
\forall \delta: \zeta. \forall f_1: \tau'_1 \rightarrow D[\theta_1]. \forall f_2: \tau'_2 \rightarrow D[\theta_2]. \\
(\forall x'_1: \tau'_1, \forall x'_2: \tau'_2. \phi'[x'_1/r'_1, x'_2/r'_2] \implies S[\delta/k, f_1(x'_1)/l_1, f_2(x'_2)/l_2]) \\
\implies S[\alpha \cdot \beta \cdot \delta/k, \text{bind } s_1 \lambda x_1. (\text{bind } r_1(x_1)) f_1/l_1, \text{bind } s_2 \lambda x_2. (\text{bind } r_2(x_2)) f_2/l_2]
\]

By the associativity of monadic bind, the formula (46) is equivalent to:

\[
\forall \delta: \zeta. \forall f_1: \tau'_1 \rightarrow D[\theta_1]. \forall f_2: \tau'_2 \rightarrow D[\theta_2]. \\
(\forall x'_1: \tau'_1, \forall x'_2: \tau'_2. \phi'[x'_1/r'_1, x'_2/r'_2] \implies S[\delta/k, f_1(x'_1)/l_1, f_2(x'_2)/l_2]) \\
\implies S[\alpha \cdot \beta \cdot \delta/k, \text{bind } s_1 \lambda x_1. (\text{bind } r_1(x_1)) f_1/l_1, \text{bind } s_2 \lambda x_2. (\text{bind } r_2(x_2)) f_2/l_2]
\]

Hence to prove (46) by applying [SUBST], rule with the associativity of monadic bind, and applying \(\Rightarrow_I\) and \(\forall_I\), we need to prove the following PL-premise:

\[
\Gamma, r_1: \tau_1 \rightarrow D[\tau'_1], r_2: \tau_2 \rightarrow D[\tau'_2], s_1: D[\tau_1], s_2: D[\tau_2] \\
(\mathcal{R}_S^{\alpha, \beta, \phi}[s_1/r_1, s_2/r_2]) \\
\Rightarrow_{PL} S[\alpha \cdot \beta \cdot \delta/k, \text{bind } s_1 \lambda x_1. (\text{bind } r_1(x_1)) f_1/l_1, \text{bind } s_2 \lambda x_2. (\text{bind } r_2(x_2)) f_2/l_2]
\]
To prove this judgment by applying \([Ax]\) and \([\forall x]\) rules to the precondition \((\forall \phi \phi[s_1/r_1, s_2/r_2])\) of (47) and applying \([\Rightarrow E]\) rule, we need to prove:
\[
(\ldots) \vdash_\text{PL} \forall x_1 : \tau_1, \forall x_2 : \tau_2, \phi[x_1/r'_1, x_2/r'_2] \implies S[\beta : \delta/k, \text{bind } r_1 x_1 f_1/l_1, \text{bind } r_2 x_2 f_2/l_2]
\]
Similarly, to prove this judgment by instantiating the precondition
\[
(\forall x_1 : \tau_1, \forall x_2 : \tau_2, \phi[x_1/r'_1, x_2/r'_2] \implies (\forall \phi \phi'[r_1 x_1/r_1, r_2 x_2/r_2])
\]
of (47), we need to prove the following judgment:
\[
(\ldots) \vdash_\text{PL} \forall x_1 : \tau_1, \forall x_2 : \tau_2, (\forall \phi \phi'[r_1 x_1/r_1, r_2 x_2/r_2])
\]
\[
\implies S[\beta/k, \text{bind } r_1 x_1 f_1/l_1, \text{bind } r_2 x_2 f_2/l_2].
\]
Similarly, to prove this judgment by instantiating \((\forall x_1 : \tau_1, \forall x_2 : \tau_2, \phi[x_1/r'_1, x_2/r'_2])\), we need to prove the following judgment:
\[
(\ldots) \vdash_\text{PL} \forall y_1 : \tau'_1, \forall y_2 : \tau'_2, \phi'[y_1/r'_1, y_2/r'_2] \implies S[\delta/k, f_1(y_1)/l_1, f_2(y_2)/l_2].
\]

However it is already in the precondition hence we have it by applying \([Ax]\) rule.  

\[\square\]

C DISCUSSION ON THE CORRECTNESS OF GAUSSIAN LEARNING

By using the \(\top\top\)-lifting for the union bound logic, we can sketch the convergence of Gaussian Learning algorithm. We change the typing of the primitive of Gaussian distributions from \(M[\text{real}]\) to \(D[\text{real}]\). Let \(\text{Gauss}(\mu, \sigma)^N : D[\text{list}(\text{real})]\) be a distribution of lists generated from the list \(\{d, d, \ldots, d\}\) with length \(N\) consists of the Gaussian distribution \(d = \text{Gauss}(\mu, \sigma)\).

We show the following UPL-judgment through the \(\top\top\)-lifting for the union bound logic.

\[\vdash_\text{UPL} \text{bind Gauss}(\mu, \sigma)^N \text{GaussLearn}(\text{Gauss}(0, 1)) \mid \text{Pr}_{r^{-1}}[|r - \mu| \geq \varepsilon] \leq \delta(\varepsilon, N) + \frac{4\sigma^2}{N\varepsilon^2}\]

First, in a similar way as the Monte Carlo approximation,

\[\vdash_\text{UPL} \text{Gauss}(\mu, \sigma)^N : D[\text{list}(\text{real})]\mid \text{Pr}_{L \sim \text{Total}(L)}[||\text{Total}(L)/N - \mu| | \geq \varepsilon/2] \leq \frac{4\sigma^2}{N\varepsilon^2}\]

This is interpreted by \(\top\top\)-lifting \((S = (\exists y_{1-1}[1] \leq k))\) for the union bound logic to:

\[\vdash_\text{UPL} \text{Gauss}(\mu, \sigma)^N : D[\text{list}(\text{real})]\mid \exists_S^{\delta(\varepsilon', L)}(|\text{Total}(r')/N - \mu| \leq \varepsilon/2).\]

Since \(\text{GaussLearn}(\text{Gauss}(0, 1))(L) = \text{Gauss}(\text{Total}(L))/(|L| + \sigma^2), \sigma^2/(|L| + \sigma^2))\), there is a function \(\delta : \text{real} \times \text{nat} \rightarrow \text{real}\) such that \(\delta(\varepsilon, |L|)\) satisfies

\[\vdash_\text{UPL} \text{GaussLearn}(\text{Gauss}(0, 1)) : \text{list}(\text{real}) \rightarrow D[\text{real}]\mid \forall L : \text{list}(\text{real})]|\text{Total}(L)/|L| - \mu| \leq \frac{\varepsilon}{2}\]

\[\implies \exists_S^{\delta(\varepsilon, |L|)}(|\mu - \text{Total}(L)/|L| | \leq \frac{\varepsilon}{2} \land |\text{Total}(L)/|L| - r' | \leq \frac{\varepsilon}{2}))[r(s)/r]\]

We also have by the monotonicity of unary graded \(\top\top\)-lifting:

\[\vdash_\text{UPL} \exists_S^{\delta(\varepsilon, |L|)}(|\mu - \text{Total}(L)/|L| | \leq \frac{\varepsilon}{2} \land |\text{Total}(L)/|L| - r' | \leq \frac{\varepsilon}{2}))[r(s)/r]\]

\[\implies \exists_S^{\delta(\varepsilon, |L|)}(|\mu - r' | \leq \frac{\varepsilon}{2})[r(s)/r]\]

Then we apply the weakening and bind rule on unary graded \(\top\top\)-lifting, we conclude

\[\vdash_\text{UPL} \text{bind Gauss}(\mu, \sigma)^N \text{GaussLearn}(\text{Gauss}(0, 1)) \mid \exists_S^{\delta(\varepsilon, N) + \frac{4\sigma^2}{N\varepsilon^2}}(|r' - \mu| \leq \varepsilon).\]
This is equivalent to:

$$\vdash_{\text{UPL}} \text{bind Gauss}(\mu, \sigma)^N \text{GaussLearn}(\text{Gauss}(0,1)) \mid \Pr[r - \mu \geq \epsilon] \leq \delta(\epsilon, N) + \frac{4\sigma^2}{N\epsilon^2}.$$ 

The term $\delta(\epsilon, N) + \frac{4\sigma^2}{N\epsilon^2}$ converges to 0 as $N \to \infty$. Here $\delta$ is calculated by an upper bound under the condition $|\text{Total}(L)/|L| - \mu| > \frac{\epsilon}{2}$ of the following probability:

$$\Pr_{r \sim \text{Gauss}(\frac{\text{Total}(L)}{|L|}, \frac{\sigma^2}{|L| + \sigma^2})}(|r - \text{Total}(L)/|L|| \geq \epsilon) = \Pr_{r \sim \text{Gauss}(\frac{\sigma^2}{|L| + \sigma^2}, \frac{\text{Total}(L)}{|L|})}(|r| \geq \epsilon).$$

Actually, the proof of convergence of $\delta$ in the logic PL is quite complicated, and need to introduce more terminologies of calculations on integrations in PL, and the proof of convergence itself is far from program verification. Hence we omit this discussion from the main body of this paper.

D RECALL: AXIOMS AND EQUATIONS OF ASSERTIONS FOR STATISTICS

We introduce axioms and equations in the logic PL. First, we have the standard equational theory for expressions based on $\alpha$-conversion, $\beta$-reduction, extensionality, and the monadic rules of the monadic type $M$ (we omit here). The monadic type $M$ also has the commutativity (Fubini-Tonelli equality), written as the following equation:

$$(\text{bind } e_1 \lambda x. (\text{bind } e_2 \lambda y.e(x,y))) = (\text{bind } e_2 \lambda y. (\text{bind } e_1 \lambda x.e(x,y)) \quad (x, y: \text{ fresh})$$

We introduce some equalities around expected values. We have the monotonicity and linearity of expected values (axioms 49, 50), and we also have Cauchy-Schwartz inequality (axiom 51). We are able to transform the variables in the expression of expected values.

$$(\forall x: \tau. \epsilon' \geq 0) \implies E_{x \sim e}[\epsilon'] \geq 0$$

We also introduce some basic equalities on observations, rescaling, and normalizations.

We may introduce the axioms for particular distributions such as $E_{x \sim \text{Bern}(e)}[\text{if } x \text{ then } 1 \text{ else } 0] = e$ ($0 \leq e \leq 1$), $E_{x \sim \text{Gauss}(e_1, e_2)}[x] = e_1$, and etc. We omit them right now.

D.1 Markov and Chebyshev inequalities

The axioms in PL that we introduced above are quite standard, but we already able to enjoy meaningful discussions in probability theory. For instance, we can prove Markov inequality (61) and
Chebyshev inequality (62) in PL.

\[ d : M[\text{real}], a : \text{real} \vdash_{\text{PL}} (a > 0) \implies \Pr_{x \sim d}[|x| \geq a] \leq \mathbb{E}_{x \sim d}[|x|]/a. \quad (61) \]

\[ d : M[\text{real}], b : \text{real}, \mu : \text{real} \vdash_{\text{PL}} \mathbb{E}_{x \sim d}[1] = 1 \land \mu = \mathbb{E}_{x \sim d}[x] \land b^2 > 0 \implies \Pr_{x \sim d}[|x - \mu| \geq b] \leq \text{Var}_{x \sim d}[x]/b^2. \quad (62) \]

D.2 The Reproductive Property and Conversions of Gaussian distributions

We can introduce in PL the following equalities of the reproductive property of Gaussian distributions and two equalities converting from Gaussian distribution to the standard Gaussian distribution Gauss(0, 1) and vice versa.

\[
\text{(bindGauss}(\mu_1, \sigma_1^2)\text{)} \in \lambda x. (\text{bindGauss}(\mu_2, \sigma_2^2) \ \lambda y. \text{return}(px + (1-p)y)) = \text{Gauss}(p\mu_1 + (1-p)\mu_2, p^2\sigma_1^2 + (1-p)^2\sigma_2^2).
\]

\[
\text{(bindGauss}(0,1) \ \lambda x. \text{return}(x\sqrt{\sigma^2} + \mu)) = \text{Gauss}(\mu, \sigma^2)
\]

D.3 Soundness of Axioms in PL

Soundness of many axioms are proved by using the equations in the toolbox for synthetic measure theory given in [Ścibior et al. 2017, Figure 14 (the last page)]. Roughly speaking, they consist of notations of the structures relating the commutative monad \( \mathcal{M} \) on the cartesian closed category QBS. The notations of synthetic measure theory for the commutative monad \( \mathcal{M} \) and semiring \( R = [0, \infty] \) is unfolded as follows:

\[
\int_X f(x) \ d\mu(x) \overset{\text{def}}{=} f^\#(\mu) \quad \text{and} \quad w \odot \mu \overset{\text{def}}{=} \int_X w(x) \cdot d_x \ d\mu(x) = \mathcal{M}(\pi_2) \circ (\text{dst}_{\text{in}, x} \circ (w, \eta))^\#
\]

Here, \( w(x) \cdot d_x \) is a scalar multiplication of the Dirac measure \( d_x \) with \( w(x) \), and the projection \( \pi_2 : 1 \times X \to X \) is also the left unitor of Cartesian product (isomorphism). We can then formalize the semantics of the monadic bind, expectation, and rescaling as follows:

\[
\llbracket [\Gamma \vdash \text{bind} \ e \ f] \rrbracket = \lambda y. \int (\llbracket [\Gamma \vdash f] \rrbracket(y))(x) \ d(\llbracket [\Gamma \vdash e] \rrbracket(y))(x)
\]

\[
\llbracket [\Gamma \vdash \mathbb{E}_{x \sim e}[f(x)]] \rrbracket = \lambda y. (\approx^{-1} \circ \int (\approx \circ [\Gamma \vdash f] \rrbracket(y))(x) \ d(\llbracket [\Gamma \vdash e] \rrbracket(y))(x)
\]

\[
\llbracket [\Gamma \vdash \text{scale}(e, f)] \rrbracket = \lambda y. (\llbracket [\Gamma \vdash f] \rrbracket(y) \odot \llbracket [\Gamma \vdash e] \rrbracket(y))
\]

where \( \approx \) is the isomorphism \( \mathcal{M}1 \overset{\approx}{=} [0, \infty] \). The second reformulation of expectation is actually integration in quasi-Borel space. The equality (†) is given from the fact that the correspondence between \( f^\# \mu = \int f \ d\mu \) in the case of \( f : X \to [0, \infty] \) and \( \mu \in \mathcal{M}X \).

Since QBS is well-pointed, to prove the soundness of equalities on PCF_P probabilistic terms, it suffices to show the semantic equation holds for any snapshot \( \gamma \) of environment \( \Gamma \) satisfying the precondition.

- Soundness of equalities (48), (52), (53), (54), and (57) are derived from the equations given in the toolbox for synthetic measure theory [Ścibior et al. 2017, Figure 14 (the last page)]. Notice that \( \alpha \cdot \beta \cdot d_{(x, y)} = (\alpha \cdot d_x) \otimes (\beta \cdot d_y) \) holds by definition of Dirac distribution.
For the monotonicity (inequality 49), linearity (equalities 50), and Cauchy-Schwartz inequality (inequality 51) of expectations, we use the second reformulation of expectation (†). Integrations for measures on qbs are converted into the usual Lebesgue integration, so we may apply the existing lemmas on usual measure theory. For an expectation of a real function which may take negative values, we can easily check that the interpretation of the syntactic sugar corresponds to the actual expected value which is given by an integration directly.

• The soundness of the equality observe e as \( f = \text{normalize}(\text{scale}(e, f)) \) (equality 58) is obvious from the definition.

• The soundness of the equality \( \text{normalize}(e) = \text{scale}(e, \lambda x.1/E_{x \sim e}[1]) \) (equality 59) and normalizing constant-rescaled distribution (60) are proved by the equivalence of scalar multiplication and rescaling with constant scalar function, and the equivalence of the mass of a measure and the expectation \( E_{x \sim e}[1] \). Both are easily proved.