Absence of wave operators for one-dimensional quantum walks

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Abstract

We show that there exist pairs of two time evolution operators which do not have wave operators in a context of one-dimensional discrete time quantum walks. As a consequence, the borderline between existence and nonexistence of wave operators is decided.

Keywords Scattering theory · Wave operator · Quantum walk

Mathematics Subject Classification 46N50 · 47A40

1 Introduction

We consider a discrete time quantum walk on \( \mathbb{Z} \). Let \( \mathcal{H} := l^2(\mathbb{Z}; \mathbb{C}^2) \) be a Hilbert space and \( U := SC \) be the unitary time evolution operator of quantum walks. Here, \( S \) is a shift operator and \( C \) is a coin operator. The axiom of quantum walks is introduced in [23] and the classification of one-dimensional quantum walks is considered in [17]. A comprehensive review of quantum walks until 2012 is summarized in [25]. Quantum walks have been introduced as a quantum counter part of classical random walks [1,10]. It is known that the behavior of quantum walks is different from classical random walks. One of differences appears in a weak limit theorem which is regarded as a quantum walk version of central limit theorem. Konno firstly proved this theorem if a coin operator is position independent of \( \mathbb{Z} \) [13]. An interesting consequence is that the shape of a limit distribution in quantum walks is different from the normal distribution which can be derived from central limit theorem for classical random walks. After that, several researchers extend his result [2,3,5–11,14–16,20–22,24,26–28]. In particular, Grimmett et al. [9] showed that the asymptotic velocity operator plays important roles to get weak limit theorems. Moreover in [9], the explicit form of...
the asymptotic velocity operator of position-independent quantum walks is established through discrete Fourier transforms.

In this paper, we mainly consider a position dependent quantum walk. Namely, $C$ is a multiplication operator by a unitary matrix $C(x) \in U(2)$, $x \in \mathbb{Z}$. If $C$ depends on a position $x \in \mathbb{Z}$, it is difficult to know the form of asymptotic velocity operator since the discrete Fourier transform does not work. To overcome this difficulty, Suzuki introduced the discrete time wave operator for quantum walks in [24]. Suppose that there exist $C_0 \in U(2)$ and constants $\epsilon, \kappa > 0$ such that

$$\|C(x) - C_0\|_{\mathcal{B}(\mathbb{C}^2)} \leq \kappa (1 + |x|)^{-1-\epsilon}, \quad x \in \mathbb{Z}, \quad (1.1)$$

where $\| \cdot \|_{\mathcal{B}(\mathbb{C}^2)}$ is the operator norm on $\mathbb{C}^2$. We set $U_0 := SC_0$. In [20,21,24], the above type condition is called the short-range condition. Under this condition, following wave operators exist and are complete:

$$W_{\pm}(U, U_0) := \lim_{t \to \pm \infty} U^{-t} U_0 \Pi_{ac}(U_0), \quad (1.2)$$

where $s$-lim denotes the strong limit and $\Pi_{ac}(U_0)$ denotes the orthogonal projection onto the absolutely continuous subspace of $U_0$. Moreover in [24], Suzuki introduced the asymptotic velocity operator by using above wave operators and derived the weak limit theorem for position dependent cases. This result is extended to several models (see, e.g., [8,20,21,27]).

The main problem in this paper is the existence or nonexistence of wave operators if $C$ and $C_0$ satisfy

$$\|C(x) - C_0\|_{\mathcal{B}(\mathbb{C}^2)} \leq \kappa (1 + |x|)^{-\gamma}, \quad x \in \mathbb{Z}, \quad (1.3)$$

for some $\kappa > 0$ and $\gamma \in (0, 1]$. We would say $C$ satisfies the long-range condition. In a context of Schrödinger operators, it is known that if a potential slowly converges to 0 at infinity, then the wave operator does not exist in general [4,12,18,19]. From this fact, it is expected that similar situations occur in a context of quantum walks. Consequently, this expectation is true. In other words, there exist examples of $U$ and $U_0$ such that wave operators do not exist. Therefore, we can conclude that the borderline between existence and nonexistence of wave operators is $\gamma = 1$. In this sense, it is reasonable that we call the condition (1.3) the long-range condition. Some results related to nonexistence of wave operators are known in a context of Schrödinger operators [4,12,18,19]. In these cases, we can expect the borderline between existence and nonexistence of wave operators from the large time behavior of a classical orbit of a particle. For these heuristic arguments, see, e.g., [12].

To show the nonexistence of wave operators, we employ the argument introduced by Ozawa [18]. We need careful treatments since the time evolution is discrete. If $C_0$ is diagonal, then the proof is quite simple since the motion of a quantum walker is simple (Remark 3.1). On the other hand, if $C_0$ is not diagonal, then the proof is complicated. We make use of some estimates related to the asymptotic velocity operator of $U_0$ (see Lemmas 3.1, 3.2 and 3.3). The author’s best knowledge, the proof which applies the asymptotic velocity operator is new.
If $C$ and $C_0$ satisfy the long-range condition, we have to introduce modified wave operators instead of standard wave operators. It is an interesting problem how do we introduce the modifier for a pair of $(C, C_0)$. For a class of long-range type quantum walks, we can introduce the modifier and derive the weak limit theorem through modified wave operators [27].

Contents of this paper are as follows. In Sect. 2, we review notation for quantum walks and state the main result. In Sect. 3, we give a proof of the main result.

2 Main result

In this section, we review notation for quantum walks and state the main result in this paper. The Hilbert space is given by

$$\mathcal{H} := l^2(\mathbb{Z}; \mathbb{C}^2) = \left\{ \Psi : \mathbb{Z} \rightarrow \mathbb{C}^2 \mid \sum_{x \in \mathbb{Z}} \| \Psi(x) \|_{\mathbb{C}^2}^2 < \infty \right\},$$

(2.1)

where $\| \cdot \|_{\mathbb{C}^2}$ is the norm on $\mathbb{C}^2$. We denote its inner product and norm by $\langle \cdot, \cdot \rangle_\mathcal{H}$ (linear in the right vector) and $\| \cdot \|_\mathcal{H}$, respectively. If there is no danger of confusion, then we omit the subscript $\mathcal{H}$ of them. We introduce the following dense subspace of $\mathcal{H}$:

$$\mathcal{H}_0 := \{ \phi \in \mathcal{H} \mid \exists N \in \mathbb{N} \text{ such that } \phi(x) = 0 \text{ for all } |x| \geq N \}. \quad (2.2)$$

Next, we introduce two unitary operators $U$ and $U_0$. The action of $S$ is given by (2.3) for each $\Psi \in \mathcal{H}$:

$$(S\Psi)(x) := \begin{bmatrix} \Psi^{(1)}(x + 1) \\ \Psi^{(2)}(x - 1) \end{bmatrix}, \quad x \in \mathbb{Z}. \quad (2.3)$$

For $C_0 \in U(2)$ and $\gamma > 0$, we introduce the following coin operator $C$:

$$(C\Psi)(x) := C(x)\Psi(x), \quad C(x) := e^{i(1 + |x|)^{-\gamma}}C_0, \quad x \in \mathbb{Z}, \quad (2.4)$$

where $i$ is the imaginary unit. Throughout this paper, we identify $C_0$ as a unitary operator on $\mathcal{H}$ such that $(C_0\Psi)(x) = C_0\Psi(x), x \in \mathbb{Z}$. We set $U := SC$ and $U_0 := SC_0$.

Let $\| \cdot \|_{\mathcal{B}(\mathbb{C}^2)}$ be the operator norm on $\mathbb{C}^2$. For any $x \in \mathbb{Z}$, it is seen that

$$\frac{1}{2}(1 + |x|)^{-\gamma} \leq \| C(x) - C_0 \|_{\mathcal{B}(\mathbb{C}^2)} \leq (1 + |x|)^{-\gamma}. \quad (2.5)$$

For any $C_0 \in U(2)$, $C_0$ has a form of

$$C_0 = \begin{bmatrix} a & b \\ -e^{i\delta}b^* & e^{i\delta}a^* \end{bmatrix}, \quad (2.6)$$

where $e^{i\delta} (\delta \in [0, 2\pi))$ is the determinant of $C_0$ and for $z \in \mathbb{C}$, $z^*$ denotes the complex conjugate of $z$. We note that $a$ and $b$ satisfy $|a|^2 + |b|^2 = 1$. 

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Remark 2.1 In this paper, our goal is to find the example of \((U, U_0)\) such that wave operators do not exist. Thus, we only consider the coin operator introduced by (2.4).

Let \(A\) be a unitary or self-adjoint operator on \(\mathcal{H}\). The sets \(\sigma(A)\), \(\sigma_p(A)\), \(\sigma_c(A)\) and \(\sigma_{ac}(A)\) are called spectrum, pure point spectrum, continuous spectrum and absolutely continuous spectrum of \(A\), respectively. For spectral properties of \(U_0\), following facts are known:

Proposition 2.1 [20, Lemma 4.1 and Proposition 4.5]

1. If \(|a| = 1\), then \(U_0\) has purely absolutely continuous spectrum and \(\sigma(U_0) = \sigma_{ac}(U_0) = \{e^{i\tau} | \tau \in [0, 2\pi)\}\).
2. If \(0 < |a| < 1\), \(U_0\) has purely absolutely continuous spectrum and

\[
\sigma(U_0) = \sigma_{ac}(U_0) = \{e^{i\tau} | \tau \in [\delta/2 + \theta, \pi + \delta/2 - \theta] \cup [\pi + \delta/2 + \theta, 2\pi + \delta/2 - \theta]\},
\]

where \(\theta := \arccos |a|\).
3. If \(a = 0\), then \(U_0\) has pure point spectrum and \(\sigma(U_0) = \sigma_p(U_0) = \{ie^{i\delta/2}, -ie^{i\delta/2}\}\).

We are interested in cases \(|a| = 1\) and \(0 < |a| < 1\). The main result is as follows:

Theorem 2.1 For any \(a \in \mathbb{C}\) with \(0 < |a| \leq 1\) and \(\gamma \in (0, 1]\), \(\lim_{t \to \pm \infty} U^{-t}U_0^t\) does not exist.

From Theorem 2.1, we can conclude that the borderline between existence and nonexistence of wave operators is \(\gamma = 1\).

3 Proof of Theorem 2.1

In this section, we prove Theorem 2.1. We divide a proof into two cases (\(|a| = 1\) and \(0 < |a| < 1\)) since methods are quite different.

3.1 Case 1: \(|a| = 1\)

We assume that \(|a| = 1\) throughout in this subsection. Then \(b = 0\) since \(C_0\) is unitary matrix. Thus, \(C_0\) has a form of

\[
C_0 = \begin{bmatrix}
    a & 0 \\
    0 & e^{i\delta}a^*
\end{bmatrix}.
\]

Remark 3.1 Since \(C_0\) is diagonal, the motion of a quantum walker by \(U_0\) is as follows:

1. An element of a set \(\{\Psi \in \mathcal{H} | \Psi(x) = \begin{bmatrix} \Psi^{(1)}(x) \\ 0 \end{bmatrix}, x \in \mathbb{Z}\}\) only moves to left.
2. An element of a set \(\{\Psi \in \mathcal{H} | \Psi(x) = \begin{bmatrix} 0 \\ \Psi^{(2)}(x) \end{bmatrix}, x \in \mathbb{Z}\}\) only moves to right.
**Proof of Theorem 2.1** \((|a| = 1)\). We only consider the case \(t \to \infty\). The other case is also proven by the similar manner. We take \(\phi \in \mathcal{H}_0\). Then, there exists \(M \in \mathbb{N}\) such that \(\phi(x) = 0\) if \(|x| > M\). Suppose that \(\phi_+ := \lim_{t \to \infty} U^{-t} U_0^t \phi\) exists. Since \(\|U^t \phi_+ - U_0^t \phi\| = \|\phi_+ - U^{-t} U_0^t \phi\| \to 0\) (as \(t \to \infty\)), we can take \(N \in \mathbb{N}\) so that \(\|U^t \phi_+ - U_0^t \phi\| \leq \|\phi\|^2 / 4\) if \(t \geq N\). We set \(W(t) := U^{-t} U_0^t\). Then, it follows that

\[
W(t_2) - W(t_1) = \sum_{t = t_1 + 1}^{t_2} U^{-t}(U_0 - U)U_0^t, \quad t_2 > t_1 > 0. \tag{3.1}
\]

For \(t_2 > t_1 > \max\{2M, N\} + 1\), we have

\[
\Im \langle\{W(t_2) - W(t_1)\}\phi, \phi_+\rangle = \sum_{t = t_1 + 1}^{t_2} \Im \langle(U_0 - U)U_0^{t-1} \phi, U_0^t \phi\rangle + \sum_{t = t_1 + 1}^{t_2} \Im \langle(U_0 - U)U_0^{t-1} \phi, U_0^t \phi_+ - U_0^t \phi\rangle
\]

\[
= \sum_{t = t_1 + 1}^{t_2} \Im \langle(C_0 - C)U_0^{t-1} \phi, C_0 U_0^{t-1} \phi\rangle + \sum_{t = t_1 + 1}^{t_2} \Im \langle(U_0 - U)U_0^{t-1} \phi, U_0^t \phi_+ - U_0^t \phi\rangle
\]

\[
= \sum_{t = t_1 + 1}^{t_2} \sum_{x \in \mathbb{Z}} \sin(1 + |x|)^{-\gamma}\|\langle(U_0^{t-1} \phi)(x)\|_{C^2}^2
\]

\[
+ \sum_{t = t_1 + 1}^{t_2} \Im \langle(U_0 - U)U_0^{t-1} \phi, U_0^t \phi_+ - U_0^t \phi\rangle,
\]

where \(\Im z\) is the imaginary part of \(z \in \mathbb{C}\). By \(t \geq 2M + 1\) and Remark 3.1, the intersection of a support of \(U_0^{t-1} \phi\) and \(-t + M, \ldots, -M\) is empty. Thus, we have

\[
\Im \langle\{W(t_2) - W(t_1)\}\phi, \phi_+\rangle \geq \frac{\|\phi\|^2}{2} \sum_{t = t_1 + 1}^{t_2} (1 + t - M)^{-\gamma} - \sum_{t = t_1 + 1}^{t_2} \|\langle(C_0 - C)U_0^{t-1} \phi\|\|U_0^t \phi_+ - U_0^t \phi\|
\]

\[
\geq \frac{\|\phi\|^2}{2} \sum_{t = t_1 + 1}^{t_2} (1 + t - M)^{-\gamma} - \frac{\|\phi\|^2}{4} \sum_{t = t_1 + 1}^{t_2} (1 + t - M)^{-\gamma}
\]

\[
= \frac{\|\phi\|^2}{4} \sum_{t = t_1 + 1}^{t_2} (1 + t - M)^{-\gamma} \to \infty \quad \text{(as } t_2 \to \infty).\]
On the other hand, \( \text{Im}(W(t_2) - W(t_1))\phi, \phi_+ \) is bounded by \( 2\|\phi\|^2 \). This is a contradiction.

\[ \square \]

### 3.2 Case 2: \( 0 < |a| < 1 \)

Hereafter, we assume that \( 0 < |a| < 1 \). In this case, we need more preparations. We set the Hilbert space \( \mathcal{K} := L^2([0, 2\pi), \frac{dk}{2\pi}; \mathbb{C}^2) \) and \( \mathcal{F} : \mathcal{H} \to \mathcal{K} \) be the discrete Fourier transform which is the unitary operator defined as the unique continuous extension of the following operator:

\[
(F\phi)(k) := \sum_{x \in \mathbb{Z}} \phi(x)e^{-ikx}, \quad \phi \in \mathcal{H}_0, \quad k \in [0, 2\pi). \tag{3.2}
\]

In what follows, we denotes the image of the discrete Fourier transform of \( \phi \in \mathcal{H} \) by \( \hat{\phi} \). We define \( \hat{U}_0 := \mathcal{F}U_0\mathcal{F}^{-1} \). \( \hat{U}_0 \) is decomposable and it follows that

\[
(\hat{U}_0f)(k) = \hat{U}_0(k)f(k) \quad \text{with} \quad \hat{U}_0(k) = \begin{bmatrix} e^{ik} & 0 \\ 0 & e^{-ik} \end{bmatrix} C_0, \quad f \in \mathcal{K},
\]

a.e. \( k \in [0, 2\pi) \). \tag{3.3}

We denote an eigenvalue and a correspond normalized eigenvector by \( \lambda_j(k) \) and \( u_j(k) \) \((j = 1, 2)\), respectively. We set

\[
\tau(k) := |a| \cos(k + \arg(a) - \delta/2),
\]

\[
\eta(k) := \sqrt{1 - \tau(k)^2},
\]

\[
\zeta(k) := |a| \sin(k + \arg(a) - \delta/2),
\]

where \( \arg(z) \in [0, 2\pi) \) is the argument of \( z \in \mathbb{C} \). It is known that \( \lambda_j(k) \) and \( u_j(k) \) can be expressed as

\[
\lambda_j(k) = e^{i\delta}\{\tau(k) + i(-1)^{j-1}\eta(k)\},
\]

\[
u_j(k) = \frac{\sqrt{\eta(k) + (-1)^{j-1}\zeta(k)}}{|b|\sqrt{2\eta(k)}} \begin{bmatrix} i|b|e^{i(k+\arg(b)-\delta/2)} \\ \zeta(k) + (-1)^{j-1}\eta(k) \end{bmatrix}, \quad (j = 1, 2). \tag{3.4}
\]

For details, see, e.g., [20]. From (3.4), \( \hat{U}_0(k) \) is expressed as

\[
\hat{U}_0(k) = \sum_{j=1,2} \lambda_j(k) \langle u_j(k), \cdot \rangle_{\mathbb{C}^2} u_j(k), \quad k \in [0, 2\pi), \tag{3.5} \]

\[ \square \]

**Remark 3.2** It is seen that \( \lambda_j(k) \) is a \( 2\pi \) periodic \( C^\infty \) function in the variable \( k \) and \( \mathbb{C}^2 \)-valued function \( u_j(k) \) is also a \( 2\pi \) periodic \( C^\infty \) function in the variable \( k \). Moreover, following quantities are finite:

\[
\sup_{0 \leq k < 2\pi} |\lambda'_j(k)|, |\lambda''_j(k)| < \infty, \quad \sup_{0 \leq k < 2\pi} \|u'_j(k)\|_{\mathbb{C}^2} < \infty, \quad (j = 1, 2),
\]

\[ \square \]
where $\lambda_j'(k)$ and $u_j'(k)$ are derivatives of $\lambda_j(k)$ and $u_j(k)$, respectively, and $\lambda_j''(k)$ is the second derivative of $\lambda_j(k)$. These facts are used in latter lemmas.

Let $V_0$ be the asymptotic velocity operator of $U_0$ given by

$$(\hat{V}_0 f)(k) := \sum_{j=1,2} i\frac{\lambda_j'(k)}{\lambda_j(k)} \langle u_j(k), f(k) \rangle_{\mathbb{C}^2} u_j(k), \quad f \in \mathcal{H}, \quad k \in [0, 2\pi).$$

We note that $V_0$ is bounded and self-adjoint on $\mathcal{H}$.

**Proposition 3.1** [20, Lemma 4.2 (b)] If $0 < |a| < 1$, then $\sigma_p(V_0) = \emptyset$ and $\sigma(V_0) = \sigma_c(V_0) = [-|a|, |a|]$.

Let us denote a subspace of vectors $\phi \in \mathcal{D}$ whose discrete Fourier transform $\hat{\phi}$ is differentiable in a variable $k$ and

$$\sup_{k \in [0, 2\pi]} \left\| \frac{d}{dk} \hat{\phi}(k) \right\| < \infty.$$ 

Let $Q$ be a position operator defined by

$$\text{dom}(Q) := \left\{ \phi \in \mathcal{H} \mid \sum_{x \in \mathbb{Z}} x^2 \| \phi(x) \|_{\mathbb{C}^2}^2 < \infty \right\},$$

$$(Q\phi)(x) := x\phi(x), \quad x \in \mathbb{Z}, \quad \phi \in \text{dom}(Q),$$

where $\text{dom}(Q)$ is the domain of $Q$. We set $Q(t) := U_0^{-t} Q U_0^t$ and $D := \mathcal{F} Q \mathcal{F}^{-1}$.

For $\phi \in \mathcal{D}$, it is seen that $(D\hat{\phi})(k) = i \frac{d}{dk} \hat{\phi}(k)$. Following lemmas are important in our analysis:

**Lemma 3.1** For any $\phi \in \mathcal{D}$, there exists a constant $\kappa_1 > 0$ which is independent of $t$ such that

$$\left\| \frac{Q(t)}{t} \phi - V_0 \phi \right\| \leq \kappa_1 |t|^{-1}, \quad t \in \mathbb{Z} \setminus \{0\}.$$ 

**Proof** Although it is established in the proof of Theorem 4.1 in [24], we give a proof for completeness. By the discrete Fourier transforms, it is seen that

$$\left\| \left( \frac{Q(t)}{t} - V_0 \right) \phi \right\|^2 = \int_{0}^{2\pi} \left\| \hat{U}(k)^{-t} i \frac{d}{dk} \left( \hat{U}(k)^t \hat{\phi}(k) \right) \right\|_{\mathbb{C}^2}^2 d\frac{dk}{2\pi} - \sum_{j=1,2} i\frac{\lambda_j'(k)}{\lambda_j(k)} \langle u_j(k), \hat{\phi}(k) \rangle_{\mathbb{C}^2} u_j(k) \right\|_{\mathbb{C}^2}^2.$$
From (3.5), it is seen that

$$\hat{U}(k)^{-t} \frac{i}{t} \frac{d}{dk} \left\{ \hat{U}(k) \hat{\phi}(k) \right\} = \hat{U}(k)^{-t} \frac{i}{t} \frac{d}{dk} \left( \sum_{j=1,2} \lambda_j(k)^t \langle u_j(k), \hat{\phi}(k) \rangle_{\mathbb{C}^2} u_j(k) \right)$$

$$= \sum_{j=1,2} \frac{i\lambda_j'(k)}{\lambda_j(k)} \langle u_j(k), \hat{\phi}(k) \rangle_{\mathbb{C}^2} u_j(k)$$

$$+ i \frac{1}{t} \hat{U}(k)^{-t} \sum_{j=1,2} \lambda_j(k)^t \frac{d}{dk} \left( \langle u_j(k), \hat{\phi}(k) \rangle_{\mathbb{C}^2} u_j(k) \right).$$

Therefore, we have

$$\left\| \left( \frac{Q(t)}{t} - V_0 \right) \phi \right\|^2 = \frac{1}{t^2} \int_0^{2\pi} \left\| \sum_{j=1,2} \lambda_j(k)^t \frac{d}{dk} \left( \langle u_j(k), \hat{\phi}(k) \rangle_{\mathbb{C}^2} u_j(k) \right) \right\|^2 \frac{dk}{\mathbb{C}^2 2\pi}.$$

By the definition of $\mathcal{D}$ and Remark 3.2, we have

$$\sup_{0 \leq k < 2\pi} \left\| \frac{d}{dk} \left( \langle u_j(k), \hat{\phi}(k) \rangle_{\mathbb{C}^2} u_j(k) \right) \right\|_{\mathbb{C}^2} < \infty.$$

Thus, we have the desired inequality. \qed

**Lemma 3.2** For any $\phi \in \mathcal{D}$, there exist positive constants $L_1$ and $L_2$ such that for any $z \in \mathbb{C}$ with $\text{Im} z \neq 0$,

$$\left\| \left( V_0 - \frac{Q(t)}{t} \right) (z - V_0)^{-1} \phi \right\| \leq \left( L_1 |\text{Im} z|^{-1} + L_2 |\text{Im} z|^{-2} \right) |t|^{-1}, \quad t \in \mathbb{Z} \setminus \{0\}.$$

**Proof** By the discrete Fourier transform, it is seen that

$$\left\| \left( V_0 - \frac{Q(t)}{t} \right) (z - V_0)^{-1} \phi \right\|^2 = \frac{1}{t^2} \int_0^{2\pi} \left\| \sum_{j=1,2} \lambda_j(k)^t \frac{d}{dk} \left( \left( z - \frac{i \lambda_j'(k)}{\lambda_j(k)} \right)^{-1} \langle u_j(k), \hat{\phi}(k) \rangle_{\mathbb{C}^2} u_j(k) \right) \right\|^2 \frac{dk}{\mathbb{C}^2 2\pi}.$$ (3.6)

A direct calculation yields that

$$\frac{d}{dk} \left( \left( z - \frac{i \lambda_j'(k)}{\lambda_j(k)} \right)^{-1} \langle u_j(k), \hat{\phi}(k) \rangle_{\mathbb{C}^2} u_j(k) \right)$$

$$= - \left( z - \frac{i \lambda_j'(k)}{\lambda_j(k)} \right)^{-2} \frac{d}{dk} \left( \frac{i \lambda_j'(k)}{\lambda_j(k)} \langle u_j(k), \hat{\phi}(k) \rangle_{\mathbb{C}^2} u_j(k) \right)$$

$$+ \left( z - \frac{i \lambda_j'(k)}{\lambda_j(k)} \right)^{-1} \frac{d}{dk} \left( \langle u_j(k), \hat{\phi}(k) \rangle u_j(k) \right).$$

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\[= -i \left( z - \frac{i \lambda_j'(k)}{\lambda_j(k)} \right)^{-2} \frac{\lambda''(k) \lambda_j(k) - \left( \lambda_j'(k) \right)^2}{\lambda_j(k)^2} \langle u_j(k), \hat{\phi}(k) \rangle_{\mathbb{C}^2} u_j(k) \]
\[+ \left( z - \frac{i \lambda_j'(k)}{\lambda_j(k)} \right)^{-1} \frac{1}{dk} \langle u_j(k), \hat{\phi}(k) \rangle u_j(k) \].

By the definition of \( D \) and Remark 3.2, there exist constants \( C_1 \) and \( C_2 \) such that
\[
\sup_{0 \leq k < 2\pi} \left\| \frac{d}{dk} \left( z - \frac{i \lambda_j'(k)}{\lambda_j(k)} \right)^{-1} \langle u_j(k), \hat{\phi}(k) \rangle_{\mathbb{C}^2} u_j(k) \right\|_{\mathbb{C}^2}^2 \leq C_1 |\text{Im} z|^{-1} + C_2 |\text{Im} z|^{-2}.
\]
(3.7)

From (3.6) and (3.7), we have the desired result. \( \square \)

We introduce the following set of functions:
\[
C_0^\infty(\mathbb{R}) := \{ f \in C^\infty(\mathbb{R}) | f \text{ has a compact support} \}.
\]

**Lemma 3.3** For any \( G \in C_0^\infty(\mathbb{R}) \) and \( \phi \in D \), there exists a constant \( \kappa_2 > 0 \) which is independent of \( t \) such that
\[
\left\| G\left( \frac{Q(t)}{t} \right) \phi - G(V_0)\phi \right\| \leq \kappa_2 |t|^{-1}, \quad t \in \mathbb{Z}\setminus\{0\}.
\]

**Proof** We apply the Helffer–Sjöstrand formula [4]. For a self-adjoint operator \( A \), it follows that
\[
G(A) = \frac{1}{2\pi i} \int_{\mathbb{C}} (\overline{\partial} \tilde{G})(z)(z - A)^{-1}dzd\overline{z},
\]
(3.8)
where \( z = x + iy, \overline{\partial} = \frac{1}{2}(\partial_x + i \partial_y) \) and \( \tilde{G} \) is the almost analytic extension of \( G \) which satisfies following properties:
1. \( \tilde{G}(x) = G(x) \) if \( x \in \mathbb{R} \),
2. \( \tilde{G} \) is infinitely many differentiable in \( x \) and \( y \),
3. A support of \( \tilde{G} \) is compact in \( \mathbb{C} \),
4. For any \( N \in \mathbb{N} \), there exists a constant \( C_N \) such that \( |\overline{\partial} \tilde{G}(z)| \leq C_N |\text{Im} z|^N \).

We note that the integral on the right hand side of (3.8) is taken in the sense of operator norm topology. By using it, we have
\[
\left\| G\left( \frac{Q(t)}{t} \right) \phi - G(V_0)\phi \right\|
\leq \frac{1}{2\pi} \int_{\mathbb{C}} |(\overline{\partial} \tilde{G})(z)||z - Q(t)\rangle^{-1}\left\| (V_0 - \frac{Q(t)}{t})(z - V_0)^{-1}\phi \right\|dzd\overline{z}
\leq \frac{1}{2\pi} \int_{\mathbb{C}} |(\overline{\partial} \tilde{G})(z)||\text{Im} z|^{-1}\left\| (V_0 - \frac{Q(t)}{t})(z - V_0)^{-1}\phi \right\|dzd\overline{z}.
\]

From Lemma 3.2, there exist positive constants \( L_1 \) and \( L_2 \) such that
\[
\left\| (V_0 - \frac{Q(t)}{t})(z - V_0)\phi \right\| \leq \left( L_1 |\text{Im} z|^{-1} + L_2 |\text{Im} z|^{-2} \right)|t|^{-1}, \quad t \neq 0.
\]
By the property of $\tilde{G}$, there exists a constant $C_3 > 0$ such that $|\overline{\partial}G(z)| \leq C_3|\text{Im}z|^3$. Since the support of $\tilde{G}$ is compact, we have

$$\|G\left(\frac{Q(t)}{t}\right)\phi - G(V_0)\phi\| \leq \frac{1}{2\pi} \int_{\mathbb{C}} |\overline{\partial}G(z)||\text{Im}z|^{-1}(L_1|\text{Im}z|^{-1} + L_2|\text{Im}z|^{-2})|t|^{-1}dzd\bar{z} \leq |t|^{-1} \times \frac{C_3}{2\pi} \int_{\text{supp}G} (L_1|\text{Im}z| + L_2)dzd\bar{z},$$

where $\text{supp}\tilde{G}$ is the support of $\tilde{G}$. Thus, the lemma follows.

**Lemma 3.4** For $\phi \in \mathcal{D}$, there exists a constant $\kappa_3 > 0$ such that for any $t \in \mathbb{Z}\setminus\{0\}$,

$$\text{Im}((U_0 - U)U_0^{-1}t^{-1}\phi, U_0^t\phi) \geq \frac{1}{2}(1 + |2t|)^{-\gamma}(1 - |\alpha|^2)\|\phi\|^2 - \kappa_3 t^{-2}. \quad (3.9)$$

**Proof** It follows that

$$\text{Im}((U_0 - U)U_0^{-1}t^{-1}\phi, U_0^t\phi) = \sum_{|x| < 2|t|} \sin(1 + |x|)^{-\gamma} \|U_0^{-1}t^{-1}\phi(x)\|^2_{C^2} + \sum_{|x| \geq 2|t|} \sin(1 + |x|)^{-\gamma} \|U_0^{-1}t^{-1}\phi(x)\|^2_{C^2} \geq \frac{1}{2}(1 + |2t|)^{-\gamma}\|\phi\|^2 - (1 + |2t|)^{-\gamma} \sum_{|x| \geq 2|t|} \|U_0^{-1}t^{-1}\phi(x)\|^2_{C^2} \geq \frac{1}{2}(1 + |2t|)^{-\gamma}\|\phi\|^2 - (1 + |2t|)^{-\gamma} \frac{x^2}{4t^2} \|U_0^{-1}t^{-1}\phi(x)\|^2_{C^2} \geq \frac{1}{2}(1 + |2t|)^{-\gamma}\|\phi\|^2 - \frac{(1 + |2t|)^{-\gamma} - \frac{1}{4t^2}}{2} \left\| \frac{Q(t)}{t} U_0^{-1}\phi \right\|^2.$$

Since $U_0^{-1}\mathcal{D} \subset \mathcal{D}$, we can apply Lemma 3.1. Hence, it follows that

$$\text{Im}((U_0 - U)U_0^{-1}t^{-1}\phi, U_0^t\phi) \geq \frac{1}{2}(1 + |2t|)^{-\gamma}\|\phi\|^2 - \frac{1}{2} \left\| \frac{Q(t)}{t} U_0^{-1}\phi - V_0U_0^{-1}\phi \right\|^2 - \frac{1}{2} \|V_0U_0^{-1}\phi\|^2 \geq \frac{1}{2}(1 + |2t|)^{-\gamma}(1 - |\alpha|^2)\|\phi\|^2 - \frac{\kappa_1^2}{2} t^{-2},$$

where we used that $U_0^{-1}V_0 = V_0U_0^{-1}$ and Proposition 3.1. By setting $\kappa_3 := \kappa_1^2/2$, we have a desired inequality. \qed

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In what follows, we set \( \epsilon \) as \( 0 < \epsilon < |a|/6 \). We choose \( G_\epsilon \in C_0^\infty(\mathbb{R}) \) such that 
\[ 0 \leq G_\epsilon \leq 1, \quad G_\epsilon(s) = 1 \text{ if } |s| \leq 2\epsilon \text{ and } G_\epsilon(s) = 0 \text{ if } |s| \geq 3\epsilon. \]

**Lemma 3.5** For any \( \phi \in \mathcal{D} \), there exists a constant \( \kappa_4 > 0 \) such that for any \( t \in \mathbb{Z} \setminus \{0\} \),

\[
\| (U_0 - U)U_0^{-1}\phi \| \leq \kappa_4(1 + |2t|)^{-\gamma} + 2\| G_\epsilon(V_0)\phi \|. \tag{3.10}
\]

**Proof** It follows that

\[
\| (U_0 - U)U_0^{-1}\phi \|^2 = \sum_{|x| \geq 2|t|\epsilon} |1 - e^{i(1+|x|)^{-\gamma}}|^2 \| (U_0^{-1}\phi)(x) \|^2_{L^2} + \sum_{|x| < 2|t|\epsilon} |1 - e^{i(1+|x|)^{-\gamma}}|^2 \| (U_0^{-1}\phi)(x) \|^2_{L^2}
\]

\[
\leq 2 \sum_{|x| \geq 2|t|\epsilon} \| (U_0^{-1}\phi)(x) \|^2_{L^2} + (1 + |2\epsilon t|)^{-2\gamma} \sum_{|x| \geq 2|t|\epsilon} \| (U_0^{-1}\phi)(x) \|^2_{L^2}
\]

\[
\leq 2\| E_{Q/t}((-2\epsilon, 2\epsilon))U_0^{-1}\phi \|^2 + \epsilon^{-2\gamma}(1 + |2t|)^{-2\gamma} \| \phi \|^2,
\]

where for a self-adjoint operator \( A \), \( E_A(\cdot) \) is the spectral measure of \( A \). Since \( \| E_{Q/t}((-2\epsilon, 2\epsilon))\phi \| \leq \| G_\epsilon(Q/t)\phi \| \), it follows that

\[
\| (U_0 - U)U_0^{-1}\phi \|^2 \leq 2\| G_\epsilon(Q/t)U_0^{-1}\phi \|^2 + \epsilon^{-2\gamma}(1 + |2t|)^{-2\gamma} \| \phi \|^2
\]

\[
\leq 4\| G_\epsilon(Q/t)U_0^{-1}\phi - G_\epsilon(V_0)U_0^{-1}\phi \|^2 + 4\| G_\epsilon(V_0)\phi \|^2 + \epsilon^{-2\gamma}(1 + |2t|)^{-2\gamma} \| \phi \|^2
\]

\[
\leq 4\kappa_2^{-2} + \epsilon^{-2\gamma}(1 + |2t|)^{-2\gamma} \| \phi \|^2 + 4\| G_\epsilon(V_0)\phi \|^2,
\]

where we used Lemma 3.3 in the last inequality. We note that for any \( t \in \mathbb{Z} \setminus \{0\} \),

\( t^{-2} \leq 9(1 + |2t|)^{-2\gamma} \) follows. Hence, it is seen that

\[
\| (U_0 - U)U_0^{-1}\phi \|^2 \leq (36\kappa_2^{-2} + \epsilon^{-2\gamma} \| \phi \|^2)(1 + |2t|)^{-2\gamma} + 4\| G_\epsilon(V_0)\phi \|^2.
\]

We choose \( \kappa_4 \) as \( \kappa_4 := (36\kappa_2^{-2} + \epsilon^{-2\gamma} \| \phi \|^2)^{1/2} \). Then, the lemma follows. \( \square \)

**Proof of Theorem 2.1** \( (0 < |a| < 1) \). We can take \( 0 \neq \phi \in \mathcal{D} \) such that \( E_{V_0}((-3\epsilon, 3\epsilon))\phi = 0 \) in the following way. We take a function \( f \in C_0^\infty(\mathbb{R}) \) such that \( \text{supp} f \cap [-|a|, |a|] \neq \emptyset \) and \( \text{supp} f \cap (-3\epsilon, 3\epsilon) = \emptyset \), where \( \text{supp} f \) is the support of \( f \). Moreover, we take a vector \( \psi \in \mathcal{D} \) such that \( \psi(k) \neq 0 \) for a.e. \( k \in [0, 2\pi) \). For example, the following vector

\[ \psi(x) = \begin{cases} 
  s_1 
 & (x = j), \\
  s_2 
 & (x \neq j),
\end{cases} \quad j \in \mathbb{Z}, x \in \mathbb{Z}, \quad s_1, s_2 \in \mathbb{C} \]

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belongs to $D$ and $\hat{\psi}(k) \neq 0$ for a.e. $k \in [0, 2\pi)$ if $s_1 \neq 0$ or $s_2 \neq 0$. Then $\phi = f(V_0)\psi$ is a desired nonzero vector belongs to $D$ and $E_{V_0}((-3\epsilon, 3\epsilon))\phi = 0$ since supp$f \cap (-3\epsilon, 3\epsilon) = \emptyset$. We note that $E_{V_0}((-3\epsilon, 3\epsilon))\phi = 0$ implies $G_\epsilon(V_0)\phi = 0$. We suppose that the limit $\phi_+ = \lim_{t \to \infty} U^{-t}U_0^t\phi$ exists. Since $\|U^t\phi_+ - U_0^t\phi\| = \|\phi_+ - U^{-t}U_0^t\phi\| \to 0$ (as $t \to \infty$), we can take $N \in \mathbb{N}$ so that $\|U^t\phi_+ - U_0^t\phi\| \leq (1 - |a|^2)(4\kappa_4)^{-1}\|\phi\|^2$ if $t \geq N$. For $t_2 > t_1 > N$, an application of Lemmas 3.4 and 3.5 yields that

$$
\text{Im}(\langle W(t_2) - W(t_1) \rangle \phi, \phi_+)
= \sum_{t=t_1+1}^{t_2} \text{Im}(\langle (U_0 - U)U_0^{t-1}\phi, U_0^t\phi_+ \rangle)
+ \sum_{t=t_1+1}^{t_2} \text{Im}(\langle (U_0 - U)U_0^{t-1}\phi, U^t\phi_+ - U_0^t\phi \rangle)
\geq \sum_{t=t_1+1}^{t_2} \left\{ \frac{1}{2}(1 + |2t|)^{-\gamma} (1 - |a|^2)\|\phi\|^2 - \kappa_3 t^{-2} \right\}
- \|U^t\phi_+ - U_0^t\phi\| \sum_{t=t_1+1}^{t_2} \{\kappa_4(1 + |2t|)^{-\gamma} + 2\|G_\epsilon(V_0)\phi\| \}
\geq \frac{1}{4}(1 - |a|^2)\|\phi\|^2 \sum_{t=t_1+1}^{t_2} (1 + |2t|)^{-\gamma} - \kappa_3 \sum_{t=t_1+1}^{t_2} t^{-2}
\to \infty \text{ (as } t_2 \to \infty).$$

On the other hand, $\text{Im}(\langle W(t_2) - W(t_1) \rangle \phi, \phi_+)$ is bounded by $2\|\phi\|^2$. This is a contradiction. The case of $t \to -\infty$ is proven similarly. □

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