THE STRESS STRAIN STATE OF THE OXIDE WALL IN TWO-DIMENSIONAL TEMPERATURE FIELD IN A SOFC SINGLE CELL

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ABSTRACT

Analysis of the stress strain state in the ceramic plate are reported. The analysis use a um-scale numerical solutions of the two-dimensional temperature field in the cell. It is first shown that the two-dimensional character of the problem is important. A complete analytical solution of the normal and tangential stresses in two-dimensional temperature field is then presented. The importance of the longitudinal temperature gradient on the material is discussed.

INTRODUCTION

Operational properties of the hard oxide electrolyte of fuel cell are related closely to stress state of the material and the deformations regimes at temperature distribution depending on two spase coordinate and time T = T (r, z, t).

If the temperature depends on the one coordinate "r" and time "t", T = T (r, t), so analytical solutions of the plane problem of the thermoelectrivity are commonly used (1). We will point out that the resolution the models of that theory is achieved at the expense of hypothesis of plane sections. In the framework of this hypothesis it is possible to determine normal stresses for symmetric stress-deformed state (SDS) about the Z-axis and
tangential stresses are taken as zero.

The bending plane sections arises from irregular thermal acting on cylinder lengthwise. In this case, tangential stresses can be significant and they can't be neglected. In the number of papers [2] there shown that thermal processes in fuel cell go on in combinations of irregular temperature field just as in cross direction, so lengthwise. The problem is complicated by the inclusion of two-dimensionality of temperature field and for its solution are usually used the numerical methods. However for quantitative and qualitative analysis of the number of practical problems it would be useful to have analytical solution of the space problem.

**MATHEMATICAL MODEL**

1. Consider SDS of a long thick-walled cylinder of interior radius "a" and exterior radius "b", exposed to temperature. The exterior forces will be considered to be lack, and the temperature distribution, responsible for the stress state of the material to be symmetric about the centreline of the cylinder and to depend on two coordinate and time: $T = T(r, z, t)$. We use the quasi static method [1], such that the time is involved as parameter, the mechanical characteristics are change with the temperature that assumed to be specified. On this approach the thermal part of the problem is solved independent of mechanical one, that is true, if the temperature field is independent of deformations produced by it.

Assume that the deformation pattern is only slightly changing lengthwise, that is the deformation distribution throughout the height of the cylinder is to be governed by the next conditions:

$$\frac{\partial \varepsilon}{\partial z} = \varepsilon_1(r, z), \quad \frac{\partial \varepsilon}{\partial z} = \varepsilon_2(r, z), \quad \frac{\partial \varepsilon}{\partial z} = \varepsilon_3(r, z), \quad [1]$$

were $\varepsilon_1(r, z)$ - are small functions. The order infinitesimal of $\varepsilon_1$ will be defined below. Let us take up the fundamental distinction between the assumptions defined by equality [1] and the hypothesis of a plane sections used in the analytical model of "endless cylinder" [1]. The perpendicular sections to the axes "z" stands plane if the next conditions should be fulfilled:

- the deformations are independent of coordinate "z", that is:
\[
\frac{\partial \varepsilon_z}{\partial z} = 0, \quad \frac{\partial \varepsilon_\theta}{\partial z} = 0, \quad \frac{\partial \varepsilon_r}{\partial z} = 0; \quad [2]
\]

- all the points of the fixed sections \( z = \text{const} \) are of identical longitudinal displacement \( w (\partial w / \partial r = 0) \), but the component of \( \tau_{r\theta} \) tangential stress are zero (the other components of tangential stress are zero by symmetry).

Nonuniformity of thermal action along the length of the cylinder leads to deformation of the plane sections, and to their bending. Then the tangential stresses can be significant and they can’t be neglected. Conditions [1] encloses that ones where shear deformation and tangential stresses are nonzero, what permit to investigate their action to SBS of material. Let us point out that the fulfillment of these conditions can be provided at the expense of the definite restrictions on the type of the temperature function \( T(r, z, t) \).

Let the material be homogeneous and isotropical, and its rheological behaviour having regard to the temperature deformation is subject to the Hooke’s law, that we will write as Lame’s form:

\[
\sigma_r = \lambda (\varepsilon_r + \varepsilon_\theta + \varepsilon_z) + 2G\varepsilon_r - \frac{\alpha E}{1 - 2\nu} \Delta T,
\]

\[
\sigma_\theta = \lambda (\varepsilon_r + \varepsilon_\theta + \varepsilon_z) + 2G\varepsilon_\theta - \frac{\alpha E}{1 - 2\nu} \Delta T, \quad [3]
\]

\[
\sigma_z = \lambda (\varepsilon_r + \varepsilon_\theta + \varepsilon_z) + 2G\varepsilon_z - \frac{\alpha E}{1 - 2\nu} \Delta T, \quad \tau_{r\theta} = G\gamma_{r\theta},
\]

were \( \sigma_r, \sigma_\theta, \sigma_z \) - normal stress components in the radial, tangential and longitudinal direction, respectively, \( \tau_{r\theta} \) - tangential stress component, \( \lambda, G \) - Lame elasticity constants, \( \nu \) - Poisson coefficient, \( \alpha \) - coefficient of linear thermal expansion, \( \Delta T \) - temperature increase, i.e. a difference between the initial temperature \( T_0 \) and the given moment of time \( T(t) \).

Generally speaking, the moduli \( E, G \) and coefficients \( \nu, \alpha \) vary with time. The changeability of \( E, G, \nu, \alpha \) is
conditioned by the unsteady and nonuniform heating of the material at high temperatures. All these parameters will be assumed as a constant at some average temperature of the process, which is considered as known.

Taking into account the Koshy’s equations linking the components of deformation with displacement ones:

\[
\varepsilon_r = \frac{\partial u}{\partial r}, \ \varepsilon_\theta = \frac{\partial u}{\partial r}, \ \varepsilon_z = \frac{\partial w}{\partial z}, \ \gamma_{rz} = \frac{\partial w}{\partial z} + \frac{\partial w}{\partial r}, \ \gamma_{r\theta} = \gamma_{z\theta} = 0 \ [4]
\]

and substituting [4] into [3] we obtain the expression of Hook’s law in displacements:

\[
\sigma_r = \lambda \left( \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} \right) + 2G \frac{\partial u}{\partial r} - \frac{\alpha E}{1 - 2v} \Delta T,
\]

\[
\sigma_\theta = \lambda \left( \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} \right) + 2G \frac{\partial w}{\partial r} - \frac{\alpha E}{1 - 2v} \Delta T, \ [5]
\]

\[
\sigma_z = \lambda \left( \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} \right) + 2G \frac{\partial w}{\partial z} - \frac{\alpha E}{1 - 2v} \Delta T,
\]

\[
\tau_{rz} = G \left( \frac{\partial w}{\partial z} + \frac{\partial w}{\partial r} \right),
\]

\[u, w \] - displacement vector components in the radial, vertical and tangential directions, respectively (tangential displacements are zero because of the symmetry conditions).

The equilibrium equations being written for the case of lacking of external forces, takes the form:

\[
\frac{\partial \sigma_r}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\partial \sigma_r}{\partial z} = 0, \ \frac{\partial \sigma_r}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{rz}}{\partial r} = 0, \ [6]
\]

Substituting [5] into [6] we obtain the equilibrium in displacements:
\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} = g_1 \frac{\partial T}{\partial r} - g_3 \frac{\partial^2 w}{\partial r \partial z} - g_4 \frac{\partial^2 u}{\partial z^2}, \tag{7}
\]

\[
\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} = g_2 \frac{\partial T}{\partial z} - g_5 \left( \frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r} \frac{\partial u}{\partial z} \right) - \frac{1}{g_4} \frac{\partial^2 w}{\partial z^2}, \tag{8}
\]

were \( g_1, \ g_2, \ g_3, \ g_4, \ g_5 \) - constants defined by the following expressions. \( g_1 = \alpha (1+\nu)/(1-\nu), \ g_2 = 2\alpha(1+\nu)/(1-2\nu), \ g_3 = 1/2(1-\nu), \ g_4 = (1-2\nu)/2(1-\nu), \ g_5 = 1/(1-2\nu). \)

Taking into account in [1] the Koshy's equations for displacements [4] we obtain:

\[
\frac{\partial \varepsilon_z}{\partial z} = \frac{\partial^2 w}{\partial z^2} = \varepsilon_1(r,z), \quad \frac{\partial \varepsilon_\theta}{\partial z} = \frac{1}{r} \frac{\partial u}{\partial \theta} = \varepsilon_2(r,z),
\]

\[
\frac{\partial \varepsilon_r}{\partial z} = \frac{\partial^2 u}{\partial r \partial z} = \varepsilon_3(r,z). \tag{9}
\]

Let us determine the order infinitesimal \( \varepsilon_i \) in [9]. Assume that the terms of an equation [8] involved are the next order infinitesimal:

\[
|\varepsilon_i| \ll \left| g_2 \frac{\partial T}{\partial z} \right|. \tag{10}
\]

In addition to that let us assume the condition that the second derivative making an equation [7] is infinitesimal:
Neglecting by infinitesimal terms [10], [11] in equations [7], [8] we have:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} = g_1 \frac{\partial T}{\partial r} - g_3 \frac{\partial^2 w}{\partial r \partial z}, \quad [12]$$

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} = g_2 \frac{\partial T}{\partial z}. \quad [13]$$

On integrating twice the equations [12], [13] we obtain the next expressions for "u" and "w":

$$u(r,z) = g_1 \int_a^r T(r,z) \frac{r}{a} \, dr - g_3 \int_a^r \frac{\partial w}{\partial z} \frac{r}{a} \, dr + \frac{r^2 a_0^2}{2r - a_0^2} + \frac{b(z)}{r} + a(z), \quad [14]$$

$$w(r,z) = g_2 \int_a^r \frac{1}{r} \frac{\partial T}{\partial z} \, dr \, dr + \ln \left[ \frac{r}{a_0} \right] c(z) + d(z). \quad [15]$$

The unknown functions a (z), b (z), c (z), d (z) that arise from the integration, deduce from the boundary conditions on the counter cylinder. Substituting the expressions for "u" and "w" into [5], we obtain the formulae for stresses:
\[
\sigma_r = - \frac{2G g_1}{r^2} \int_{a_0}^{r} \tau dr + \frac{2G g_3}{r^2} \int_{a_0}^{r} \frac{\partial w}{\partial z} \tau dr - G \frac{\partial w}{\partial z} + \\
+ (\lambda + G) a(z) + G a_0^2 \frac{\partial w}{\partial z} a(z) - \frac{2G b(z)}{r^2},
\]

\[
\sigma_\theta = \frac{2G g_1}{r^2} \int_{a_0}^{r} \tau dr - 2G g_1 T(r, z) - \frac{2G g_3}{r^2} \int_{a_0}^{r} \frac{\partial w}{\partial z} \tau dr + [16]
\]

\[
+ \lambda g_4 \frac{\partial w}{\partial z} + (\lambda + G) a(z) - G a_0^2 \frac{\partial w}{\partial z} a(z) + \frac{2G b(z)}{r^2},
\]

\[
\sigma_z = (\lambda g_4 + 2G) \frac{\partial w}{\partial z} - 2G g_1 T(r, z) + \lambda a(z),
\]

\[
\tau_{rz} = \frac{G (g_1 + g_2)}{r} \int_{a_0}^{r} \frac{\partial T}{\partial z} \tau dr + G \frac{r^2 - a^2}{2r} - a(z).
\]

Compared to the calculation the stresses based on initial system of differential equations [7], [8], the calculation by formulae [16] more simply, though the using of found analytical solution involves some simplifying assumptions (see [1]).

2. The possibility of the conditions [1] realizing presents some restrictions on the character of the outside actions, in this case, on the temperature function.

It is obvious that the temperature gradient throughout the height of the cylinder can't be arbitrarily large. Let us found such restrictions.

In essence, the relationships [1] may be considered
as the differential equations for determination of displacements "u" and "w", but in this case, they do not need to be satisfying the initial equilibrium equations [7], [8]. Thus the problem in full volume may be formulated as follows: to identify the displacements "u" and "w" satisfying the equilibrium equations [7], [8] as well as equation [1]. Since the number of equations, in this case, more than the number of unknown terms, this problem at first sight would seem overdefined. However, if the equation [1] to consider as the additional conditions which must be satisfied at the expense of definition of type of the temperature function, so the discussed system of equations [1], [7], [8] will be closed-loop.

It can be shown that for satisfying the conditions of the model [1] the temperature function is to have the next appearance:

\[ |T(r,z)| \leq |T_*(r,z)|, \]
\[ T_*(r,z) = \frac{M_1^2 + M_2^2 + M_3 + \varphi(r)}{2}, \]

were \( M_1, M_2, M_3 \) - are arbitrary constants, a \( \varphi(r) \) - are arbitrary functions.

Thus, given the experimental array \( T(r, z, t) \) (it can be determined using the numerical calculations based on heat model (2)), the problem stands to approximate it closely with function of type (17).

If such problem can to be performed, so the assumptions of the model of small deformation gradients in longitudinal direction will true and the represented analytical solution may be used.

EXAMPLES NUMERICAL CALCULATIONS

Consider the examples of thermoelastic stresses (the computing are performed for modeling material) for two type of boundary conditions:
- the side surface of the cylinder is free from loads;
- the side surface of the cylinder is stiff restraint.

The calculations have been performed for next conditions: \( \Delta T_z / \Delta T_r = \text{grad } T_z / \text{grad } T_r = 0.1 \) (\( \Delta T_z \), \( \Delta T_r \) - temperature difference, \( \text{grad } T_z, \text{grad } T_r \) - temperature gradient respectively across length and radius. The calculations of temperature fields performed previously (2) have been shown that the temperature gradients for real work conditions of the oxide fuel cell were in such
Figure 1. Stress distribution through the cylinder wall:
A - side surfaces are free of loads
- - - $\Delta T_z = 0$, $\Delta T_z / \Delta T_r = 0.05$, $b_o = 1.5a_o$;
B - stiff restraint
- - - $\Delta T_z / \Delta T_r = 0.1$, $\Delta T_z / \Delta T_r = 0.05$, $b_o = 1.25a_o$
ratio \( \text{grad} \ T = 6000\ldots 9000^\circ \text{C}, \ \text{grad} \ T_z = 0\ldots 900^\circ \text{C} \).

Assuming a free from loads side surface of the cylinder the results obtained based on the suggested model with assumption \( \Delta T_z = 0 \) and on the "endless cylinder" (1) are fully identical. Using the solution obtained, we can consider the action of temperature gradient across the length of the cylinder on its SDS.

In Fig. 1a it is shown the stress distribution for \( \Delta T_z / \Delta T_r = 0.05 \). In this case the essence of temperature gradient across the cylinder height, as is seen from Fig. 1a tends to increase tangential stresses \( \tau_{rz} \) internal surface of the cylinder, but they change the signs (became compressible) on the external one.

In the case of stiff restraint of the lateral cylinder surface the pattern of the stress distribution (Fig 1b) differs essentially from the stress distribution for the "endless cylinder". In this case the role of the tangential stresses increases with increasing temperature gradient across \( Z - \text{axis} \).

CONCLUSION

The number of questions, connecting with the analytical solution obtained has remained off the our consideration at this part of work. Mention was made of the necessity to perform the calculation and analysis of SDS of real SOFC material and to study the effect of the tangential stresses on it.

The questions of finding the criterion conditions for the known analytical solutions of the plane problems of the thermoelasticity being applied are off our consideration here.

From the results obtained it may be deduced that calculations of strength for the solid oxide electrolyte can be performed by using the suggested above model. This enables the real recommendations for construction of fuel cell to be received.

REFERENCES

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