Entanglement, number fluctuations and optimized interferometric phase measurement

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Abstract. We derive a phase-entanglement criterion for two bosonic modes that is immune to number fluctuations, using the generalized Moore–Penrose inverse to normalize the phase-quadrature operator. We also obtain a phase-squeezing criterion that is immune to number fluctuations using similar techniques. These are used to obtain an operational definition of relative phase-measurement sensitivity via the analysis of phase measurement in interferometry. We show that these criteria are proportional to the enhanced phase-measurement sensitivity. The phase-entanglement criterion is the hallmark of a new type of quantum-squeezing, namely planar quantum-squeezing. This has the property that it squeezes simultaneously two orthogonal spin directions, which is possible owing to the fact that the SU(2) group that describes spin symmetry has a three-dimensional parameter space of higher dimension than the group for photonic quadratures. A practical advantage of planar quantum-squeezing is that, unlike conventional spin-squeezing, it allows noise reduction over all phase angles simultaneously. The application of this type of squeezing is to the quantum measurement of an unknown phase. We show that a completely unknown

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phase requires two orthogonal measurements and that with planar quantum-squeezing it is possible to reduce the measurement uncertainty independently of the unknown phase value. This is a different type of squeezing compared to the usual spin-squeezing interferometric criterion, which is applicable only when the measured phase is already known to a good approximation or can be measured iteratively. As an example, we calculate the phase entanglement of the ground state of a two-well, coupled Bose–Einstein condensate, similarly to recent experiments. This system demonstrates planar squeezing in both the attractive and the repulsive interaction regime.

1. Introduction

Entanglement criteria are widely used for identifying non-classical resources for potential applications in quantum technology. One application is enhancement of measurement sensitivity. In practice, the highest sensitivity measurements are very often interferometric. Hence, the measurement of an unknown quantity is reduced to the measurement of a phase shift. In this paper, we analyze how non-classical, entangled states can increase phase-measurement sensitivity. To achieve this, we will introduce both an operational measure of relative phase, and a corresponding signature of phase entanglement between two Bose fields, using well-defined interferometric particle-counting procedures. This is shown to quantitatively measure the

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sensitivity enhancement of an interferometric measurement. It is the interferometric equivalent of the spin-squeezing criterion [1–3], which is known to measure the non-classical precision of a clock [3].

We introduce a relative phase operator, which is well defined in the case of variable total particle number, using the generalized inverse method to prevent singularities in the inverse number operator. It is the simplest relative phase operator that is measurable interferometrically. Number fluctuations are always found in current experimental photonic and atom interferometer phase measurements. Hence, we clarify the operational phase-measurement procedure already used heuristically to analyze experiments [4–7]. Most importantly, we show that when the measured phase is unknown prior to measurement, a state preparation that involves mode entanglement is optimal and is closely related to the relative-phase operator. This complements previous studies, which generally assume either that the phase is already known to a good approximation or that the phase shift remains constant during repeated measurements. Here, we use the minimal number of measurements possible. This is inherently different from strategies employed to estimate phase through sequential or multiple measurements, which assume that the phase shift is a classical, time-invariant quantity.

Our entanglement criterion is a normalized form of the recently introduced Hillery–Zubairy (HZ) non-Hermitian operator product criterion [8], similar to that introduced in a previous paper [9]. We prove that this normalized form is a phase-entanglement signature for two Bose fields, and has an advantage of being almost immune to total number fluctuations. We show how this criterion can be interpreted as a variance measure that signifies entanglement, and has a direct physical interpretation as the enhancement of phase measurement sensitivity in an interferometer. This is directly related to the idea of planar-quantum squeezing (PQS), in which the quantum noise is simultaneously reduced in two orthogonal directions of phase measurement.

The present analysis focuses on linear, two-mode interferometry with particle-counting detectors and an arbitrary input number distribution. This technique is by far the most commonly used technique for phase measurement and therefore deserves a careful analysis. Our results are capable of being implemented immediately, since two-mode interferometers are widely available for both photonic and atomic fields. We discuss techniques for generating the required entangled input fields through the creation of a correlated ground state in a coupled, two-mode Bose–Einstein condensate (BEC) with either attractive or repulsive S-wave scattering interactions. We show that as well as giving sub-shot noise (squeezed) phase noise in two orthogonal phase directions simultaneously, it is possible to obtain nearly Heisenberg-limited performance in one of two phase directions, which is useful when the phase is known approximately.

The main results of this paper are briefly given below.

1. **Phase-entanglement criterion.** We introduce a phase-entanglement criterion that is robust against number fluctuations. Entanglement between modes \( \hat{a} \) and \( \hat{b} \) is confirmed by

\[
E_{ph} = \frac{\Delta^2 \tilde{J}^X + \Delta^2 \tilde{J}^Y}{\langle \hat{N}^+ \rangle / 2} < 1,
\]

where \( \tilde{J}^{X,Y} = \tilde{J}^{X,Y} \hat{N}^+ \) are normalized spin operators, \( \Delta^2 \tilde{J} \) is the variance and \( \hat{N}^+ \) is the Moore–Penrose generalized inverse [10] of the total number \( \hat{N} = \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} \). Here we
2. Quantum phase measurements and interferometry

Interferometers are designed to measure a relative phase shift $\phi$, typically between two beams. The phase information must first be encoded on quantum fields before it is measured. Hence, there is a close relationship between interferometry, which measures a phase shift in a medium, and the measurement of phase of a quantum field, which is where the interferometric phase information is stored. The relationship is more important than meets the eye, because an experimentally measured phase shift can be neither truly classical nor time invariant, which are commonly used assumptions. Since repeated measurement is not always possible, one must regard interferometric measurement as primarily a quantum measurement problem. For this reason, we briefly review earlier approaches to the phase measurement of quantum fields in the appendix.

An operational analysis of interferometric measurements can be reduced in the simplest case to a measurement of outputs from the final beam splitter as shown in figure 1, with one mode $b$ experiencing an unknown phase shift of $\phi$, while the other mode $a$ is shifted by a fixed reference phase $\theta$. The other components of the interferometer are then part of the quantum

Figure 1. Interferometric measurements can be reduced in the simplest case to a measurement of outputs from a beam splitter.

employ the Schwinger effective spin operators,

$$\hat{J}^X = (\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger)/2, \quad \hat{J}^Y = (\hat{a}^\dagger \hat{b} - \hat{a} \hat{b}^\dagger)/(2i). \quad (2)$$

2. Phase-squeezing criterion. We show that if $\hat{J}^{X,Y}$ are obtained from $N$ spin-1/2 systems, then entanglement between them is confirmed by a spin-squeezing criterion (with $\Delta \hat{J} = \sqrt{\Delta^2 \hat{J}}$):

$$\xi_{S, \text{ph}}^{Y/X} = \sqrt{\langle \hat{N} \rangle \Delta \hat{J}^Y} / |\langle \hat{J}^X \rangle| < 1. \quad (3)$$

3. Phase-sensitivity criterion. We find that if modes $\hat{a}$ and $\hat{b}$ are used as the input to a generic interferometer (see figure 1), the phase sensitivity $\Delta \phi$ in a single measurement is closely related to the signature of entanglement, equation (1):

$$(\Delta \phi)^2 = (\Delta^2 \hat{J}^X \cot^2(\varphi) + \Delta^2 \hat{J}^Y)/|\langle \hat{J}^X \rangle|^2, \quad (4)$$

where $\varphi = \phi - \theta'$, $\phi$ is the unknown phase and $\theta'$ is the reference phase. This sensitivity is optimized by the use of planar-squeezing, in which both the orthogonal spin variances $\Delta^2 \hat{J}^{X,Y}$ are reduced simultaneously below the usual shot noise level.
state preparation that determines the output expectation values. This is shown schematically as the input beam splitter with input modes $a_i, b_i$, in the schematic diagram of figure 1.

In the following, we shall mostly focus on the generic scheme in which the intermediate modes $a, b$ have an arbitrary state preparation. However, we shall also treat particular examples where the state preparation is obtained through the Mach–Zehnder protocol. In this case we analyze the state preparation of modes $a_i, b_i$, as a practical route toward preparing the intermediate modes, and also introduce an additional phase shift on the input to the MZ, for reasons explained in the last section.

2.1. Quantum limits to classical phase estimation

The problem of measuring a quantum phase $\hat{\phi}$ is related, but not identical, to the analysis of quantum limits to estimation of a classical phase. There is an essential difference, since classical phase estimation usually assumes that there is a phase-shifting element that produces a well-defined classical phase shift $\phi$, which is supposed to be time independent. Theoretical treatments often assume that this phase can be measured iteratively without disturbing it, to improve the accuracy. This fixed-phase assumption rules out many situations where the phase evolves in time, or where the phase experiences a back-action which changes the phase after measurement. Other approaches to the problem make the assumption that it is possible to construct arbitrary quantum states and measuring devices. As a result, these general treatments are not directly applicable to two-mode interferometry, although they may be applicable to some future phase-measuring device.

We first review earlier approaches to phase estimation. A pioneering work by Caves [11], who treated bosons in the context of gravity-wave detection, showed that two-mode interferometry sensitivity could be improved above the shot-noise level, i.e. the standard quantum limit (SQL) of $\Delta \phi = 1/\sqrt{\bar{N}}$:

$$\Delta \phi < 1/\sqrt{\bar{N}}.$$ (5)

This required non-classical, ‘squeezed state’ input radiation, reaching a maximum sensitivity near the Heisenberg limit of

$$\Delta \phi = 1/\bar{N}$$ (6)

for an input state with $\bar{N}$ average particle number. Squeezed states allow the uncertainty of one observable to be reduced below the SQL, at the expense of the complementary observables, so that the Heisenberg uncertainty relation is still satisfied [12]. Thus, for single-mode optical amplitudes where $\hat{X} = (\hat{a} + \hat{a}^\dagger)/2$ and $\hat{P} = (\hat{a} - \hat{a}^\dagger)/(2i)$, for which $\Delta \hat{X} \Delta \hat{P} \geq 1/4$, squeezing of $\hat{X}$ occurs when $\Delta \hat{X} < 1/2$. This is clearly very closely related to the quadrature phase-operator [13] approach of equation (A.8).

Paradoxically, the usual squeezed state technologies of parametric down-conversion are rather inefficient. The total resources employed, in terms of boson number prior to down-conversion, are generally no better than coherent interferometry [14] for a given phase sensitivity. As pointed out by Caves, there is still an advantage of this method for gravity-wave detectors, as it reduces the back-action caused by radiation pressure.

The treatment of Caves also assumed prior knowledge of the unknown phase, to allow a linearized treatment in the limit of small fluctuations. This type of analysis was applied to fermion interferometry by Yurke [15, 16], and was later extended to multiple
measurements [17], with similar conclusions, except for the replacement of \( \bar{N} \) by the total number of particles involved, \( \bar{N}_{\text{TOT}} \).

Squeezed quantum fluctuations have been shown to enhance the sensitivity of other sorts of measurements [18, 19]. Wineland et al [3], in the context of atomic clock measurements, showed that there was a close relationship between interferometry and the concept of spin-squeezing [1]. The uncertainty relation for spin is \( \Delta \hat{J}_Y \Delta \hat{J}_Z \geq |\langle \hat{J}_X \rangle|/2 \), where \( \hat{J}_Z = (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})/2 \), and spin-squeezing exists when the variance of one spin is reduced below the SQL [1]:

\[
\Delta \hat{J}_Y < \sqrt{|\langle \hat{J}_X \rangle|}/2. \tag{7}
\]

The spin-squeezing factor

\[
\xi_S = \frac{\sqrt{2J} \Delta \hat{J}_Y}{|\langle \hat{J}_X \rangle|} \tag{8}
\]

was introduced for a collection of \( N \) two-level atoms or, equivalently, for two occupied modes for which a collective pseudo-spin is defined, and \( J = N/2 \). In spectroscopy or interferometry, the final measurement is that of a spin component of the effective Schwinger spin operators defined in equation (2) above. These are measured as the population difference between the two quantum states, using Bloch rotations or beam splitters to make the final measurements.

Since we wish to consider phase in the initial state prior to a beam splitter, we consider that the \( X \) direction is chosen to be that of the large spin vector, so that \( \langle \hat{J}_X \rangle \sim J \). Fluctuations are thus spin-squeezed when \( \Delta^2 \hat{J}_Y < J/2 = N/4 \). The precision of the quantum measurement is given as [3, 20]

\[
\Delta \phi = \xi_S / \sqrt{N}, \tag{9}
\]

which is enhanced over the SQL (5) when

\[
\xi_S < 1 \tag{10}
\]

and reaches the Heisenberg limit as \( \xi_S \to 1/\sqrt{N} \). This implies that there are large fluctuations in \( \hat{J}_Z \). The most extreme case of this is when we choose the input state to be the eigenstate of the two-mode phase operator defined in the appendix, equation (A.6), which establishes a connection between these two approaches. In this case, one finds that the non-squeezed quadrature has very large fluctuations of \( \Delta^2 \hat{J}_Z = N [N + 2]/12 \approx J^2/3 \).

Further analysis has suggested ways of reaching these Heisenberg limits (6) through macroscopic superpositions or ‘N00N’ states [20]. More recently, nonlinear interferometry [21] was demonstrated as a route to go beyond the Heisenberg limit, although this requires specific nonlinear couplings. We note that all these techniques use a linearized approach to phase estimation. This means that the phase must be already known to an excellent approximation prior to the measurement. In practice, this implies repeated measurements to refine the estimation of the phase until the final, high-precision measurement is made. In particular, we show below that a two-mode phase eigenstate only gives a low interferometric measurement variance when the phase is known almost perfectly prior to the measurement.

There are general treatments that typically involve more than single-pass, two-mode interferometry [22]. Sanders and Milburn [23] found the optimal measurement and state to determine the phase \( \phi \), based on the two-mode phase-operator approach of equation (A.6). Berry and Wiseman [24] and Berry et al [25], showed that this canonical measurement cannot
be realized by counting particles in an interferometer (figure 1), and proposed alternative iterative schemes. Such idealized measurements have also been analyzed by many authors using techniques such as the quantum Fisher information \[26, 27\]. An analysis of these gives an asymptotic Heisenberg limit (at large \(N\)) of \(\Delta \phi \geq 1.376/N\). \(11\)

The assumption of a time-invariant, classical phase shift used in these iterative schemes rules out such techniques for many applications. Phase shifts usually vary in time, and one cannot always avoid quantum back-action. The physical reason for this is that a phase shift corresponds to an energy term in the radiation-matter Hamiltonian \[29\]. This therefore has dynamical consequences for the object being measured, and may change the phase. Hence, the use of iterative or repeated measurement schemes is not always feasible.

Strategies of this type are different from the theory of the present paper, which focuses on minimal numbers of measurements using a two-mode linear interferometer.

2.2. Phase-measurement operator

We now return to the fundamental question of how to measure a phase shift using a quantum phase-measurement operator. We wish to use a strictly operational definition, solely utilizing the interferometer outputs. Our typical interferometer, shown in figure 1, has three stages, with the corresponding mode operators. The first, denoted by \(\hat{a}_i, \hat{b}_i\), are the input modes. These may undergo further phase shifts and beam-splitter operations, as shown in a Mach–Zehnder configuration in the diagram. The second, denoted by \(\hat{a}, \hat{b}\), are the internal modes. Since there are many ways to prepare these—not all using a Mach–Zehnder configuration—we mainly focus on the entanglement properties of these operators in this paper. The third set is the output modes, \(\hat{c}, \hat{d}\), which we assume are always obtained through an unknown phase shift \(\phi\), a reference phase shift \(\theta\) and a lossless, symmetric beam splitter.

If we define the operator phases so that \(\hat{c}, \hat{d} = [\hat{a} e^{-i\theta} \pm \hat{b} e^{-i\phi}] / \sqrt{2}\), then the measured outputs of the quantum interferometer in terms of the internal modes \(\hat{a}, \hat{b}\) are

\[
\hat{N}_\pm = \hat{c}^\dagger c \pm \hat{d}^\dagger d = \frac{1}{2} \hat{N} \pm \frac{1}{2} [\hat{a}^\dagger \hat{b} e^{-i(\phi-\theta)} + \hat{b}^\dagger \hat{a} e^{i(\phi-\theta)}].
\]

(12)

Comparing these quantities with the equivalent angular momentum operator approach from equation (A.5), we see that these quantities can be rewritten as \(\hat{N}_\pm = \hat{N}/2 \pm \hat{J}^\phi\), where \(\hat{N}\) is the total number operator, and we define a general spin projection

\[
\hat{J}^\phi = \hat{J}^X \cos(\phi - \theta) + \hat{J}^Y \sin(\phi - \theta).
\]

(13)

In any interferometric experiment, the two observable number outputs, \(\hat{N}_\pm\), can be always measured simultaneously. The quantum phase of the output field—which is also related to the unknown phase shift \(\phi\)—can then be estimated from the normalized spin projection \(\hat{j} (\phi)\), where

\[
\hat{j} (\phi) = \lim_{\epsilon \to 0^+} \frac{\hat{N}(\hat{N}_+ - \hat{N}_-)}{2(\epsilon + \hat{N}^2)} = \hat{J}^\phi \hat{N}^+.
\]

(14)

Here \(\hat{N}^+\) is the Moore–Penrose generalized inverse \[10\] of \(\hat{N}\). This is a well-defined Hermitian observable that commutes with \(\hat{J}^\phi\), and gives the least-squares solution of any inversion or
variational problem involving $\hat{N}$. The generalized inverse $\hat{N}^+$ has many of the properties of a standard inverse, including the property that its eigenvalues are $N^{-1}$ for number states with the total number $N > 0$. However, it is zero (not infinity) for the vacuum state. This means that, unlike the standard inverse, it has a well-defined value for all quantum states. More details can be found in section 3.

In experiments designed for accurate phase measurement with large average particle numbers, events with zero total particle number occur with vanishingly small probability, and there is no inversion problem. In this case the Moore–Penrose inverse behaves in exactly the same way as one would expect for an operator with a well-behaved inverse. In all cases, the quantity $\hat{j}(\phi)$ is measurable and Hermitian. Hence, we call $\hat{j}(\phi)$ the relative phase quadrature operator, as it is fundamental to phase-sensitive interferometric experiments.

It is vital to normalize by the particle number at each measurement—as indicated by the above operator—for the simple reason that, in general, the particle number is not known in advance. It is often theoretically assumed that the total particle number is known prior to measurement. This is rarely found in real experiments, especially when the number is increased. An inspection of the experimental protocols actually used in recent BEC interferometry experiments [4, 6, 7] shows that the operator given above corresponds rather closely to the way those data are analyzed in practice. Our analysis therefore provides a theoretical justification for these operational procedures. We will treat the effects of typical Poissonian particle number fluctuations in a later section.

The operator $\hat{j}(\phi)$ is different from both the complex phase-difference quadrature operators of Nieto and Carruthers [30] and of Leggett [31], described in the appendix, owing to the normalization chosen here. If we compare the current approach of equation (14) with these earlier suggestions in equations (A.9) and (A.10), there is a very important difference. The operator $\hat{j}(\phi)$ given above is uniquely defined for all inputs, and can be measured completely from a single, combined measurement of the two interferometer outputs. It is not clear how one can measure the earlier proposed phase-measurement operators in practical interferometry experiments, since they require the simultaneous measurement of non-commuting output operators such as $\hat{J}_X$ and $\hat{J}_Z$. Of course, this does not rule out more sophisticated operational measurements, as the combined operators are Hermitian; but these measurements do not appear feasible with simple beam-splitters and photodetectors.

If multiple measurements are made sequentially, then more sophisticated iterative phase-estimation techniques are possible [17, 22, 24, 32, 33]. As pointed out above, this does not help in experiments where the phase is changing. Often, only a single measurement is possible. We also recall that many previous analyses are conditioned on having a priori approximate knowledge of the unknown phase shift $\phi$. In the following, we focus instead on optimizing the sensitivity of the operational phase measurement equation (14), for a range of unknown phase shifts. In other words, we assume that the phase is known to lie in a given interval that is not vanishingly small.

2.3. Entanglement and squeezing

Interferometric sensitivity and particle entanglement have previously been linked through criteria involving Fisher information [26]. Sorenson et al [34] have shown that a signature of particle entanglement is the spin-squeezing criterion (10): a fixed number $N$ of two-level systems (spin-1/2 particles) are separable when $\rho = \sum_R P_R \rho_0^R \ldots \rho_0^R$ and hence entangled.
when

\[ 0 < \xi_S^{YZ} = \frac{\sqrt{N} \Delta J^Y}{|\langle \hat{J}^X \rangle|} < 1. \quad (15) \]

Here we define the collective spins associated with \( N \) spin-1/2 systems: \( \hat{J}^\theta = \sum_{k=1}^N \hat{J}^\theta_k \) (\( \theta = X, Y, Z \)), and \( \hat{J}^\theta_k \) is the spin of the \( k \)th particle. The Heisenberg uncertainty principle places a lower bound on \( \xi_S \) because of the finite size of the Hilbert space. The precise values of the lower bound for fixed \( N \) have been determined by Sorensen and Mølmer in \[35\], and decrease with increasing spin \( J \). The spin-squeezing criterion has been measured experimentally in BEC interferometry \[4–6\], and is related to phase-measurement efficiency when the phase value is approximately known in advance \[3\], as summarized by (9). We note that this criterion is valid for spin projections and variances measured in any two orthogonal directions.

In this paper, we will generalize this criterion to include number fluctuations and also treat a very different type of entanglement, namely that between two distinct spatially separated locations, rather than between many qubits. We will relate the sensitivity of the phase measurement (14) of an unknown phase to a special type of two-mode entanglement between \( a \) and \( b \). Two-mode entanglement is defined as a failure of the separable model

\[ \rho = \sum_R P_R \rho_a^R \rho_b^R, \quad (16) \]

where \( P_R > 0, \sum_R P_R = 1 \) and \( \rho_a^R/\rho_b^R \) are density operators for states at \( a/b \).

Many criteria for two-mode entanglement exist, but we are interested only in those interferometric measures of entanglement that can enhance the phase measurement task of figure 1 \[26, 36, 37\]. The observables that can be measured are given in equation (14). These expressions are written in terms of the internal modes \( \hat{a} \) and \( \hat{b} \) that experience the phase shift prior to the final beam splitter.

In some cases the \( \hat{a} \) and \( \hat{b} \) modes are the input modes. One may, alternatively, consider an MZ-type experiment with an additional phase rotation and two beam splitters as is illustrated in figure 1. In this case, the MZ internal modes are related to the input modes \( \hat{a}_i, \hat{b}_i \) by

\[ \hat{a} = (\hat{a}_i + \hat{b}_i)/\sqrt{2}, \quad \hat{b} = (i \hat{a}_i + \hat{b}_i)/\sqrt{2}, \]

so that the internal spin operators are rotated versions of the Mach–Zehnder input operators, with

\[ \hat{J}^X = \hat{J}_i^X, \]

\[ \hat{J}^Y = -i \hat{J}_i^X. \quad (17) \]

We see from equation (14) that the sensitivity of the phase measurement will depend on the noise levels of the two orthogonal components, \( \hat{J}^X \hat{\hat{N}}^+ \) and \( \hat{J}^Y \hat{\hat{N}}^+ \) (in terms of internal modes \( \hat{a} \) and \( \hat{b} \)). It makes sense then to choose an input state for the interferometer that will maximally reduce the noise in both these components simultaneously. In fact, this requirement is closely related to an entanglement measure. Hillery and Zubairy (HZ) showed \[8\] that for any separable state (16),

\[ \Delta^2 \hat{J}^X + \Delta^2 \hat{J}^Y \geq \langle \hat{\hat{N}} \rangle/2. \quad (18) \]

Entanglement between modes \( \hat{a} \) and \( \hat{b} \) is thus detected when (18) fails:

\[ 0 < E_{HZ} = \frac{\Delta^2 \hat{J}^X + \Delta^2 \hat{J}^Y}{\langle \hat{\hat{N}} \rangle/2} < 1. \quad (19) \]
$E_{HZ} = 1$ gives the SQL noise level, which is the noise level $E_{HZ}$ obtained when the modes $\hat{a}$ and $\hat{b}$ are in the separable product of coherent states, $|\alpha\rangle|\beta\rangle$. It is not possible, however, to choose a state so that both variances $\Delta^2 \hat{J}^X$ and $\Delta^2 \hat{J}^Y$ are zero.

2.4. Planar quantum-squeezing

If we consider the Heisenberg uncertainty principle in the $X$–$Y$ plane, we see that it has the form $\Delta \hat{J}^Y \Delta \hat{J}^X \geq |\langle \hat{J}^Z \rangle|/2$. Here the optimal situation is obtained for equal beam intensities entering the beam splitter, so that $\langle \hat{J}^Z \rangle = 0$. This appears to provide no lower bound to the measured quadrature variances and hence to the phase noise. However, appearances can be very misleading. In fact, there is a non-zero lower bound to $E_{HZ}$, because the variances of $\hat{J}^X$ and $\hat{J}^Y$ cannot be simultaneously zero.

For fixed $N = 2J$, this bound has been determined. It is known that

$$C_J / J \leq E_{HZ},$$

(20)

where the coefficients $C_J \sim 3(2J)^{2/3}/8$ as $J \to \infty$ [38]. This means, however, that both the orthogonal variances in a phase measurement can be simultaneously reduced below the shot-noise level, since we are minimizing the sum of the phase variances, $\Delta^2 \hat{J}^X + \Delta^2 \hat{J}^Y$. In general, a noise reduction of the sum of two variances below the shot-noise level is called PQS [38], as it minimizes quantum noise in a plane, rather than just in one direction on the Bloch sphere. It has the advantage that noise reduction for phase measurement occurs regardless of the value of the unknown phase.

It is instructive to compare the optimal PQS state with the relative phase eigenstate described in the Appendix, equation (A.6). The minimal variance PQS state is

$$|\psi\rangle = \frac{1}{\sqrt{2J + 1}} \sum_{m=-J}^{J} R_m e^{im\theta} |J, m\rangle.$$  

(21)

The asymptotic limit of the optimal coefficient $R_m$, which minimizes the sum of the quadrature variances, is then a Gaussian of the form

$$R_m = (-1)^m e^{-m^2/(4\sigma_m)} / (2\pi \sigma_m)^{1/4},$$

(22)

where the variance in the space of $\hat{J}^Z$ eigenvalues is $\Delta^2 \hat{J}^Z = (J^2/2)^{2/3}$.

We can, similarly, consider the Heisenberg uncertainty principle in the $X$–$Z$ plane with rotated coordinates (17), with a repulsive interaction. More details are discussed for the case of the ground state of a coupled BEC for both attractive interactions ($g < 0$) and repulsive interactions ($g > 0$), in section 5.

The important properties of the two states are shown in table 1. The optimal PQS state reduces the variance in $\hat{J}^X$ and $\hat{J}^Y$ simultaneously, with both variances well below the shot-noise level. In contrast, in a relative phase state, quantum-squeezing only occurs in the $Y$ spin direction, while in both the other two spin directions the noise is greatly increased above the shot-noise level.

In all cases the Heisenberg uncertainty principle in the $Z$–$Y$ plane is obeyed, since $\Delta \hat{J}^Y \Delta \hat{J}^Z = |\langle \hat{J}^X \rangle|/2$. However, while the PQS state is able to reduce phase noise in both quadratures, in an interferometric measurement using a relative phase state, the reference phase...
Table 1. A comparison of the asymptotic (large $J$) spin variances for the ideal PQS and relative phase states. PQS states reduce the variance in $\hat{J}^x$ and $\hat{J}^y$ simultaneously. A relative phase state only reduces the noise in $\hat{J}^y$ and has greatly enhanced noise in both the $\hat{J}^x$ and $\hat{J}^z$ quadratures.

| Observable  | PQS          | Relative phase |
|-------------|--------------|----------------|
| $|\langle \hat{J}^x \rangle|$ | $J$ | $\pi J/4$ |
| $\Delta^2 \hat{J}^x$ | $(2J)^{2/3}/8$ | $(2/3 - \pi^2/16)J^2$ |
| $\Delta^2 \hat{J}^y$ | $(2J)^{2/3}/4$ | $\sqrt{\ln(J)}$ |
| $\Delta^2 \hat{J}^z$ | $(J^2/2)^{2/3}$ | $J^2/3$ |

offset $\theta$ must be adjusted to match $\phi$ with high precision. The difficulty is that $\phi$ may be unknown prior to measurement. This adjustment is necessary to avoid contamination of the results with high levels of noise from the $X$ spin direction, which are well above the shot-noise or Poissonian level. The underlying cause is that interferometric measurements do not project out the relative phase eigenstates.

We see that the main advantage of PQS states in interferometry is that it is possible to have sub-shot precision in both the measured spin directions simultaneously. This is advantageous when the measured phase is truly unknown. At first, it may seem that this is less than optimal as a squeezing strategy when the phase is known approximately. For the optimal PQS state described above, which minimizes the variance sum, neither of the variances are close to the Heisenberg limit. Importantly, there is a range of possible PQS states in which the relative variances in the $X$ and $Y$ directions can be adapted to the desired measurement strategy, including states in which PQS—with the variance sum below the shot-noise level—is combined with nearly Heisenberg-limited variance reduction in one of the two directions. This possibility is discussed in the last section, together with practical techniques for achieving it.

3. Entanglement and number fluctuations

In practical interferometry, the total number of input bosons usually changes at each measurement. Hence, the ensemble used for averaging has a finite distribution over the particle number. This is caused by a number of factors. In optical lasers, it is caused by technical noise in the optical pumping process, as well as well-known quantum noise effects during stimulated emission and out-coupling [39]. In BEC and atom lasers, the factors involved range from fluctuations in the initial atomic density distribution in the magneto-optical trap, to quantum noise due to the atomic collisions that occur in the evaporative cooling process [40]. These number fluctuations are due to the non-equilibrium mechanisms that generate a laser or BEC, respectively, and there is no reason to assume either a canonical or a grand canonical ensemble.

The direct use of the HZ spatial entanglement criterion is highly sensitive to total number fluctuations. For this reason, we will introduce entanglement and spin-squeezing definitions that are normalized by the total number. We refer to such normalized entanglement criteria as phase-entanglement and phase-squeezing criteria, as they signify correlated and reduced phase noise. There is another possible strategy, which is to simply reject all measurements that have the ‘wrong’ particle number. This allows a conditional number state measurement to be obtained.
Although this is feasible, it is also extremely inefficient, since most attempted measurements yield no information at all about the phase with this strategy.

3.1. Experimental number fluctuations

In theoretical treatments which use symmetry-breaking ideas, the original BEC is sometimes regarded as having a well-defined overall phase common to both input modes, even though this is not measurable. As shown in the phase-state example in equation (A.3), a boson field with a well-defined overall phase is a superposition of number states. This is usually not the case in practice. However, just as with optics [41], the idea of an overall phase of a BEC can be justified as corresponding to a probabilistic mixture of coherent states, in which the (unmeasured) coherent state phase changes at each realization. The corresponding number fluctuations are then Poissonian, as in the examples we treat in this paper.

In the cases when the number fluctuations are Poissonian, the probability that there are exactly $N$ bosons is

$$P(N) = \frac{1}{N!} (\hat{N})^N e^{-(\hat{N})}. \quad (23)$$

While this may not be the best input state for a phase measurement, this distribution does give a number standard deviation of $\sigma_N = \sqrt{N}$, which is a typical order of magnitude for the number fluctuations in a well-stabilized photonic laser or BEC. This fluctuation is in the total number $N$. Interferometric beam splitters, of course, introduce relative number fluctuations in the output beams, in addition to the total number fluctuations.

To judge the realism of the Poissonian ansatz, we note that highly stabilized semiconductor lasers have reached slightly lower number variances than Poissonian, in a restricted frequency range [42]. For an atomic BEC experiment, atom number statistics are difficult to measure accurately to this level of precision for large $\bar{N}$. Number fluctuations of at least the Poissonian level are found in almost all current BEC experiments [6] where data are available. Just as with lasers, it is possible to obtain slightly lower variances than this, with some restrictions. In the best results to date, standard deviations as low as $0.6\sqrt{\bar{N}}$ (below the Poissonian level) were observed at very small atom numbers of $\bar{N} \approx 60$. Super-Poissonian variances were found for larger atom numbers of $N > 500$ [43]. Poissonian fluctuations appear to be reasonably typical of well-stabilized current experiments with either photon or atom lasers.

When calculating the entanglement parameter $E_{HZ}$ for states with Poissonian fluctuations, we find that the entanglement appears to be easily destroyed when there are number fluctuations. This is misleading: total number fluctuations by themselves do not destroy entanglement. Accordingly, it is important to use entanglement and phase-sensitivity measures that allow for number fluctuations. Our general criteria therefore include number fluctuations with an arbitrary variance. These criteria can still be used to describe idealized experiments without number fluctuations, even though this is not very realistic. In section 5, we will make use of a Poissonian distribution to model the behavior of typical BEC experiments, with a low atom number of $\bar{N} \sim 100$.

To treat phase measurement including total number fluctuations, we have introduced normalized spin operators: $\tilde{j}^{\theta} = J^{\theta} \hat{N}^+$. Here, $\hat{N}^+$ is the Moore–Penrose generalized inverse of the number operator. We can now use these normalized operators to derive general operational criteria for entanglement and squeezing, which extend the results obtained above to
a realistic environment with number fluctuations. Somewhat different results have been obtained previously with number fluctuations, in work that used un-normalized operators \[44\].

3.2. Phase-entanglement criterion

We now introduce a phase-entanglement criterion that is robust against number fluctuations. We show that in a number-fluctuating environment, entanglement between modes \(\hat{a}\) and \(\hat{b}\) is confirmed via a phase-entanglement criterion that uses the generalized Moore–Penrose inverse of the number operator, \(\hat{N}^+\):

\[
E_{\text{ph}} = \frac{\Delta^2 \hat{j}^X + \Delta^2 \hat{j}^Y}{\langle \hat{N}^+ \rangle / 2} < 1. \tag{24}
\]

**Proof.** We wish to show that entanglement between modes \(\hat{a}\) and \(\hat{b}\) is confirmed via an entanglement phase criterion, equation (24). First, we note the general result that if \(\hat{N}\) commutes with an arbitrary Hermitian operator \(\hat{O}\) having eigenvalues \(m\), we can introduce a limiting procedure to obtain a normalized mean value, \(\langle \tilde{o} \rangle = \langle \hat{O} \hat{N}^+ \rangle\), where \(\hat{N}^+\) is the generalized Moore–Penrose \[10\] inverse of the number operator \(\hat{N}\), so that

\[
\langle \hat{O} \hat{N}^+ \rangle = \lim_{\epsilon \to 0} \sum_{n,m} \frac{nm}{n^2 + \epsilon} P(n, m) = \sum_{n,m} n^+ m P(n, m). \tag{25}
\]

Here, \(P(n, m)\) is the probability for simultaneous outcomes \(n, m\) for \(\hat{N}\) and \(\hat{O}\) respectively, and the eigenvalues of the generalized inverse operator \(\hat{N}^+\) are \(n^+ = n^{-1}\) for \(n > 0\), with \(n^+ = 0\) for \(n = 0\). Here we have used the notation of \(n^+\) to indicate the Moore–Penrose inverse of a \(c\)-number. Hence, the expectation value for the ratio becomes

\[
\langle \tilde{o} \rangle = \sum_{n \geq 0} \langle \tilde{o} \rangle_n P_n, \tag{26}
\]

where \(\langle \hat{O} \rangle_n = \sum_m P(m|n) m\), \(P_n = \sum_m P(n, m)\), \(P(m|n) = P(n, m) / P_n\) and we define \(\langle \tilde{o} \rangle_n = \langle \hat{O} \rangle_n n^+\). Similarly, the corresponding variances, \(\Delta^2 \tilde{o} = \langle \tilde{o}^2 \rangle - \langle \tilde{o} \rangle^2\), can be expanded as

\[
\Delta^2 \tilde{o} = \sum_n \langle \tilde{o}^2 \rangle_n P_n - \left[ \sum_n \langle \tilde{o} \rangle_n P_n \right]^2. \tag{27}
\]

However, we know from elementary variance properties that \(\sum_n \langle \tilde{o} \rangle_n^2 P_n \geq \left[ \sum_n \langle \tilde{o} \rangle_n P_n \right]^2\); hence, we can write that

\[
\Delta^2 \tilde{o} \geq \sum_n \langle \tilde{o}^2 \rangle_n P_n - \sum_n \langle \tilde{o} \rangle_n^2 P_n = \sum_n [\langle \tilde{o}^2 \rangle_n - \langle \tilde{o} \rangle_n^2] P_n = \sum_n [\Delta_n^2 \hat{O}] (n^+)^2 P_n. \tag{28}
\]

Here, as usual, we have defined \(\Delta_n^2 \hat{O} = \sum_m P(m|n) m^2 - \langle \hat{O} \rangle_n^2\).

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Next, we apply this result to normalized spin variances, giving
\[ (\Delta \tilde{j}_X)^2 + (\Delta \tilde{j}_Y)^2 \geq \sum_n (n^+)^2 P_n [\Delta_n^2 \tilde{j}_X + \Delta_n^2 \tilde{j}_Y]. \] (29)
where we have used the definitions that
\[ \Delta_n^2 \tilde{j}_X = \sum_n P_n (\tilde{j}_X^2)_n - (\tilde{j}_X)^2. \] (30)
Finally, if we assume the separability equation (18), we see that for a separable density matrix with a fixed total particle number:
\[ \Delta_n^2 \tilde{j}_X + \Delta_n^2 \tilde{j}_Y \geq n/2, \] (31)
which means that for normalized operators,
\[ (\Delta \tilde{j}_X)^2 + (\Delta \tilde{j}_Y)^2 \geq \frac{1}{2} \sum_n n(n^+)^2 P_n \]
\[ = \frac{1}{2} \sum_n n^+ P_n \]
\[ = \frac{1}{2} \langle \hat{N}^+ \rangle. \] (32)
Here we note that the generalized inverse has many of the properties of the standard inverse, in particular that \( \hat{N}(\hat{N}^+)^2 = \hat{N}^+ \), and that of course no singularities occur with this criterion. When this condition is violated, we must have an entangled state, which leads to (24). \( \square \)

3.3. Phase-squeezing criterion

Similarly, we show that entanglement between \( N \) spin-1/2 systems is confirmed by a normalized spin-squeezing criterion, which we term phase-squeezing:
\[ \xi_{S, ph}^{Y/X} = \frac{\sqrt{\langle \hat{N} \rangle \Delta \tilde{j}_Y}}{|\langle \tilde{j}_X \rangle|} < 1. \] (33)

**Proof.** We wish to show that on entanglement between \( N \) spin-1/2 systems, where the number of systems can fluctuate, is confirmed by the normalized spin-squeezing criterion of equation (33). For \( N \) spin-1/2 separable systems, where \( N \) is fixed and non-zero, one finds that [34]
\[ \Delta^2 \tilde{j}_Z \geq \frac{1}{N} [\langle \tilde{j}_X \rangle^2 + \langle \tilde{j}_Y \rangle^2] \]
\[ \geq N^+ \langle \tilde{j}_X^2 \rangle. \] (34)
The last expression, using the generalized inverse of \( N \), holds even when \( N = 0 \). Thus, using the normalized variance inequality (28), we obtain
\[ \Delta^2 \tilde{j}_Z \geq \sum_n P_n (n^+)^2 [\Delta_n^2 \tilde{j}_Z] \]
\[ \geq \sum_n P_n n^+ \langle \tilde{j}_X \hat{N}^+ \rangle_n^2. \] (35)
Therefore, using the result that \( n^2 n^+ = n \), we see that
\[
\langle \hat{N} \rangle \Delta^2 \tilde{\gamma}^2 \geq \left\{ \sum_{n>0} P_n n^2 n^+ \right\} \sum_{n>0} P_n n^+ (\hat{J}^X \hat{N}^+)_n^2.
\] (36)

Next, we use the Cauchy–Schwarz inequality: \( \{ \sum_{n>0} x_n^2 \} \{ \sum_{n>0} y_n^2 \} \geq | \sum_{n>0} x_n y_n |^2 \), where \( x_n = \sqrt{n} P_n \) and \( y_n = \sqrt{P_n n^+ (\hat{J}^X \hat{N}^+)_n} \), and hence
\[
\langle \hat{N} \rangle \Delta^2 \tilde{\gamma}^2 \geq \left[ \sum_{n>0} P_n (\hat{J}^X \hat{N}^+)_n \right]^2 = |(\hat{J}^X \hat{N}^+)|^2 = |\langle \hat{J}^X \rangle|^2.
\] (37)

This proves the phase-squeezing criterion, equation (33).

Criteria (24) and (33) and the application of them to determine the enhanced sensitivity of a two-mode atom interferometer, in particular a BEC atom interferometer in which incoming number fluctuations are included, constitute the major results of this paper.

4. Phase sensitivity

Next, we will obtain a detailed understanding of the relationship between our phase-entanglement measure and phase-measurement sensitivity. The crucial issue in phase measurement is the measurement sensitivity, or the smallest measurable phase shift. This is related to the differential signal-to-noise ratio, given as [15, 16]
\[
\frac{dS}{d\phi} \equiv (\Delta \phi)^{-1} = \frac{1}{\sqrt{(\Delta \tilde{\gamma})^2}} \left| \frac{d(\tilde{\gamma})}{d\phi} \right|.
\] (38)

4.1. Linearized phase estimation

The smallest measurable change in phase in a single measurement is \( \Delta \phi \). Figure 1 depicts an unknown phase shift \( \phi \) (to be measured) relative to a fixed phase shift \( \theta \). We suppose for simplicity that \( \langle \hat{a}^\dagger \hat{b} \hat{N}^+ \rangle = |\langle \hat{a}^\dagger \hat{b} \hat{N}^+ \rangle| \), so that the direction of the mean spin of the state to be used in the interferometer will be along the x-axis: i.e. when \( \phi = 0 \): \( \langle \hat{J}^X \hat{N}^+ \rangle = |\langle \hat{a}^\dagger \hat{b} \hat{N}^+ \rangle| \), \( \langle \hat{J}^Y \hat{N}^+ \rangle = \langle \hat{J}^Z \hat{N}^+ \rangle = 0 \). For a controlled reference phase shift \( \theta' \), two successive orthogonal measurement settings, \( \theta' + \pi/2 \), will allow determination of the unknown phase \( \phi \):
\[
\begin{align*}
\langle \tilde{j}(\phi, \theta') \rangle &= \cos(\phi - \theta') |\langle \hat{a}^\dagger \hat{b} \hat{N}^+ \rangle|, \\
\langle \tilde{j}(\phi, \theta' + \pi/2) \rangle &= -\sin(\phi - \theta') |\langle \hat{a}^\dagger \hat{b} \hat{N}^+ \rangle|.
\end{align*}
\] (39)

A single measurement setting \( \theta' \) cannot determine the unknown phase completely, since the information given is regarding \( \cos \phi \) only. The mean differential signal for measurement \( \tilde{j}(\phi, \theta') = \hat{J}^\theta \hat{N}^+ \) is \( -\langle \hat{J}^X \hat{N}^+ \rangle \sin \phi \), and \( \Delta \phi \) as given by (38) for this measurement is \( \varphi = \phi - \theta' \)
\[
(\Delta \phi)^2 = [(\Delta \tilde{j})^2 \cot^2(\varphi) + (\Delta j)^2] / |\langle \tilde{j} \rangle|^2
\] (40)
together with a similar expression obtained in the orthogonal direction. The objective is to determine the conditions on the interferometric state so that the uncertainty in the phase estimation is minimized.

The SQL sensitivity $\Delta \phi = 1/\sqrt{N}$, as given by equation (5), is obtained when fields $a$ and $b$ are formed from a number state $|N\rangle$ incident at one port of a beam splitter, with a vacuum state input at the second port [5, 15, 37]. An entangled state results [45], for which $\langle \hat{J}^X \rangle = N/2$, $(\Delta \hat{J}^X)^2 = 0$ and $(\Delta \hat{J}^Y)^2 = N/4$. Then, for all phases $\phi$, it is readily shown that $\Delta \phi \sim 1/\sqrt{N}$.

For some entangled states, it is well known [36, 46] that the phase sensitivity can be enhanced below the SQL. The most well-studied cases, however, consider a small phase shift about a fixed phase reference [3, 37]. It is evident from (40) that the maximum differential for $\tilde{j}(\phi, \theta')$ is at $\phi = \pi/2$, for which $(\Delta \phi)_{\pi/2} = \Delta \tilde{J}^Y/|\langle \tilde{j}^X \rangle|$. The sensitivity at this point is thus given by the normalized spin-squeezing parameter (33), which reduces to (15) for fixed number inputs. Sub-shot noise sensitivity is achieved when $\Delta \phi < 1/\sqrt{\langle N \rangle}$, so by the definition (33), sub-shot noise enhancement occurs for interferometric states, satisfying

$$\xi_{\text{S, norm}} < 1.$$  \hspace{1cm} (41)

The technique relies on an accurate estimate $\theta_X$ of the unknown phase, combined with setting $\theta$ to $\phi_X - \pi/2$, so that subsequent measurements detect small shifts near the optimal phase. We will show in the next section that a near Heisenberg-limited sensitivity of $(\Delta \phi)_{\pi/2} \sim O(\sqrt{2}/N)$ is predicted for this case, when the two- modes $\hat{a}$ and $\hat{b}$ of an atom interferometer are prepared from a two-mode double-potential well BEC ground state.

### 4.2. Estimation of an unknown phase

The question of phase estimation with an unknown phase and a limited number of measurements is a different issue [23]. Where we estimate phase via the interferometric scheme figure 1 based on the number difference measurements (12)–(14), we see, from (40), that a noise-reduction enhancement over a range of angles with a reduced variance in both $\Delta \hat{J}^X$ and $\Delta \hat{J}^Y$ is needed. This is essential where there is no prior knowledge of the phase $\phi$ and successive adaptive phase measurements [24] are not possible. We note that the sensitivity of the measurement $\tilde{j}(\phi, \theta')$ is destroyed by the divergent contribution evident in (40) at $\phi \sim 0, \pi$, unless $\Delta \hat{J}^X = 0$, which places a severe limit on the interferometric state. This is not a necessary consideration, however, if the full phase is to be measured via both orthogonal measurement settings given by (39). The settings in the ‘quiet’ quadrants $\theta' = (\phi + \pi/2) \pm \pi/4$ and $\theta' = (\phi - \pi/2) \pm \pi/4$ have enhanced the sensitivity over those in the ‘noisy’ quadrants $\theta' = \phi \pm \pi/4$ and $\theta' = (\phi + \pi) \pm \pi/4$, and for any unknown phase $\phi$ one of the orthogonal settings $\theta'$ or $\theta' + \pi/2$ will be in the useful quadrants. Least-squares estimation is an obvious strategy here.

We thus consider the following strategy. The first reading of the pair $\tilde{j}(\phi, 0)$ or $\tilde{j}(\phi, \pi/2)$ determines the values for $\cos \phi$ and $\sin \phi$, and thus the location of the phase in the plane. In this way, it can be determined which one of $\tilde{j}(\phi, 0)$ or $\tilde{j}(\phi, \pi/2)$ has been measured in the quiet quadrants. Sub-shot noise sensitivity is then guaranteed at all unknown angles $\phi$ for this preferred measurement, provided it can be shown that $\Delta \phi < 1/\sqrt{\langle N \rangle}$ across the entire two quiet quadrants. According to (40), the worse-case sensitivity for these quiet quadrants is at $\phi = \pm \pi/4, \pm 3\pi/4$, and is given by $(\Delta \phi)_{\text{max}}^2 = (\Delta^2(\tilde{J}^X) + \Delta^2(\tilde{J}^Y))/|\langle \tilde{j}^X \rangle|^2$. The condition for

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to be sub-shot noise is $(\Delta \phi)_{\text{max}} \leq 1/\sqrt{\langle N \rangle}$ which is quantified by a phase-sensitivity measure

$$\eta_{\text{ph}} = \frac{\sqrt{\langle N \rangle \langle N^* \rangle} E_{\text{ph}}/2}{|\langle \hat{j}^X \rangle|} < 1.$$  \hspace{1cm} (42)

When the interferometric fields satisfy (42), sub-shot noise sensitivity for all angles $\phi$ is guaranteed for any fixed measurement setting $\theta$ within the two quiet quadrants for the measurement of $\phi$.

The fundamental quantum limit for (42) is given by the smallness of $\eta_{\text{ph}}$, which is linked to the uncertainty relation for the sum of the two spins $\hat{j}^X$ and $\hat{j}^Y$. It is therefore important to determine a tight lower bound on this sum in order to obtain the ultimate phase interferometric sensitivity. The real question becomes: to what extent can we still minimize $\Delta^2 \hat{j}^Y$, given that the sum $\Delta^2(\hat{j}^X) + \Delta^2(\hat{j}^Y)$ is also to be minimized? The answer is not the same as that for two complementary observables such as two optical quadratures, or position and momentum, for which the commutator is a constant. The uncertainty relation for spin operators has a state-dependent form

$$\Delta \hat{j}^X \Delta \hat{j}^Y \geq |\langle \hat{j}^Z \rangle|/2,$$  \hspace{1cm} (43)

which means the two variances $\Delta \hat{j}^X$ and $\Delta \hat{j}^Y$ can both be reduced below the shot-noise level. For $\langle \hat{j}^Z \rangle = 0$ the Heisenberg uncertainty principle is unable to give any bound at all. This is possibly misleading, since the Heisenberg uncertainty principle is simply a bound that does not guarantee that it can be saturated. As discussed in section 2.4, a recent analysis of the spin-variance uncertainty leads to a tight bound on the variances:

$$\Delta^2 \hat{j}^X + \Delta^2 \hat{j}^Y \geq C_J,$$  \hspace{1cm} (44)

where $C_J$ is a function of the total spin $J$ with an asymptotic limit of $3(2J)^{2/3}/8$. For $J = N/2$ one finds that $\Delta \phi \geq \sqrt{C_J}/J \rightarrow O(\sqrt{1.5}/N^{2/3})$, which, as we will show in the next section, is predicted for the two-well BEC ground state, in both the attractive and repulsive regimes. The fundamental limit is below the SQL of $O(1/\sqrt{N})$ over all the phase angles in the half-plane, but can only reach the Heisenberg limit of $O(1/N)$ over part of the range. We next present the details of how these levels of sensitivity can be realized in a BEC interferometer.

5. Bose–Einstein condensate (BEC) interferometry

We consider how criteria (24), (33) and (42) for entanglement and phase-measurement sensitivity are satisfied in a typical cold-atom experiment. The mechanism for two-mode squeezing and entanglement here is similar to that first realized for optical modes using four-wave mixing [47], except for employing the ground state of an interacting BEC. We consider an idealized two-mode or two-well BEC with a normalized self-interaction coefficient $g$, and a linear tunneling coupling $\kappa$ between two modes with boson operators $\hat{a}$ and $\hat{b}$, described by the Hamiltonian [48]

$$\hat{H}/\hbar = \kappa (\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger) + \frac{g}{2} [\hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} + \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b}].$$  \hspace{1cm} (45)

Solutions for the two-mode entanglement of the ground state have been presented in [9, 45] for the case of an initial state of $N$ atoms distributed evenly between the wells. In order to interpret
Figure 2. The attractive interaction case for $N = 100$, showing individual spin variances versus shot noise $N/4$, and mean spin $|\langle \hat{J}^X \rangle|$, for the ground state solution of the Hamiltonian (46). We find that the sum of spin variances $\Delta^2 \hat{J}^Y + \Delta^2 \hat{J}^X$ is minimized by a critical value of coupling $N/J_0 = Ng/\kappa \simeq -2.1$.

It should be kept in mind that an attractive two-well BEC in the ground state generates a perfect PQS state. These graphs are obtained using a full numerical diagonalization of the Hamiltonian. We note that after transformation to a spin-operator form, this Hamiltonian is identical to the Lipkin–Meshkov–Glick model found in nuclear and condensed matter physics, where it is applied to fermions [49]. This suggests that our analysis has wider applicability to fermionic as well as BEC systems.

5.1. Attractive regime ($g < 0$)

This is easily understood by rewriting the Hamiltonian using spin operators, and introducing $J_0 = \kappa/g$, to give

$$\hat{H}/\hbar = g \left\{ \frac{1}{2} \hat{N}^2 + J_0^2 - \left[ (\hat{J}^X - J_0)^2 + (\hat{J}^Y)^2 \right] \right\}. \quad (46)$$

The first two terms are conserved and have no effect. We choose a negative (attractive) $g$, with a critical value of the coupling so that $\langle \hat{J}^X \rangle = J_0$. The non-constant terms are then simply proportional to the sum of the spin variances $\Delta^2 \hat{J}^X + \Delta^2 \hat{J}^Y$ (with a positive constant of $|g|$). Further details can be found in [38]. This must be minimized in the ground state, thus reducing phase noise in both quadratures, as shown in figure 2. The resulting state can be used in direct interferometry, with a single beam splitter. That is, we apply the Hamiltonian to the internal ($\hat{a}, \hat{b}$) modes in a phase measurement.

5.2. Repulsive regime ($g > 0$)

The repulsive BEC case also produces an asymptotic PQS state with a critical value of the coupling $J_0$, but has a different characteristic. The minimum variance spin operators are $not$ the
Figure 3. The repulsive interaction case for $N = 100$, showing individual spin variances versus shot noise $N/4$, and the mean spin $|\langle \hat{J}_i^X \rangle|$, for the ground state solution of the Hamiltonian (47). We find that the sum of input spin variance $\Delta^2 \hat{J}_i^Z + \Delta^2 \hat{J}_i^X$ is minimized (red point) by a critical value of coupling $N/J_0 = gN/\kappa \simeq 40$. After phase shifting and beam splitter operations, from equation (17) this becomes a PQS state with a strongly reduced variance in the internal interferometer spin operators, $\Delta^2 \hat{J}_i^X + \Delta^2 \hat{J}_i^Y$.

ones measured in direct interferometry with the internal modes. Therefore, a different strategy is required, and this Hamiltonian is used to prepare the states of the input modes $\hat{a}_i, \hat{b}_i$. This state can be used in optimal Mach–Zehnder interferometry, with an additional phase rotation and beam-splitter stage as shown in figure 1, to generate the state with the optimal characteristics for phase measurement. We rewrite the Hamiltonian using the input mode spin operators, which are related to the internal operators by a rotation as explained in equation (17):

$$\hat{H}/\hbar = g \left\{ \frac{1}{4} \hat{N}^2 - \frac{\hat{N}}{2} + (\hat{J}_i^Z)^2 + 2J_0 \hat{J}_i^X \right\}.$$  

(47)

From figure 3, we see that one of the quadratures ($\hat{J}_i^Z = \hat{J}_i^X$) has a phase-noise level reduced almost to the Heisenberg limit at large $g$, while the other phase quadrature ($\hat{J}_i^X = -\hat{J}_i^Y$) is (necessarily) not at the Heisenberg limit. To understand this behavior analytically from the Hamiltonian, equation (47), we first note that the first two terms in the Hamiltonian are conserved and so have no effect. As $g$ is positive (repulsive), we need to minimize $\langle (\hat{J}_i^Z)^2 + 2J_0 \hat{J}_i^X \rangle$ to obtain the lowest energy. The value of the spin variance in each direction in the plane still needs to be calculated in order to define the properties of the resulting planar squeezed stated, as obtained for the attractive case in our previous work [38]. For the general state expansion (21), in the large-$J$ limit we can still assume a symmetric amplitude distribution with $R_m = R_{-m}$, so that $\langle \hat{J}_i^Y \rangle = \langle \hat{J}_i^Z \rangle = 0$. The mean value of $\hat{J}_i^X$ is negative to
minimize the energy
\[
\langle \hat{J}_i^X \rangle = -\langle \hat{J}_i^Y \rangle \approx - \left( J + \frac{1}{2} \right) \left[ 1 - \frac{\sigma}{2} - \frac{3\sigma^2}{8} - \frac{1}{8\sigma (J + 1/2)^2} \right].
\]  (48)

Similarly, on evaluating the square of the spin vector, we find that
\[
\langle (\hat{J}_i^X)^2 \rangle \approx J \left[ (1 - \sigma)J + 1 - \frac{1}{4\sigma J} + \frac{1}{4J} \right] + J \left[ \frac{3}{16\sigma^2 J^3} + \frac{3}{16\sigma^2 J^3} - \frac{1}{\sigma J^2} + \frac{1}{16\sigma J^5} \right],
\]
\[
\langle (\hat{J}_i^Y)^2 \rangle \approx J \left[ \frac{1}{4\sigma J} - \frac{1}{4J} + \frac{1}{4\sigma J^2} + \frac{1}{8\sigma J^3} \right],
\]
\[
\langle (\hat{J}_i^Z)^2 \rangle \approx \left( J + \frac{1}{2} \right)^2 \sigma + \frac{1}{4}.
\]  (49)

Applying variational calculus so that \(d(\hat{H}/\hbar)/d\sigma = 0\) and solving in the limit of large \(J\), we find that \(\sigma \approx \sqrt{J_0/[4(J + 1/2)^3]}\). This gives the result that to leading order
\[
\Delta^2 \hat{J}_i^X = \Delta^2 \hat{J}_i^Y \approx J/8J_0,
\]
\[
\Delta^2 \hat{J}_i^Y = \Delta^2 \hat{J}_i^Z \approx J^{3/2}/2\sqrt{J_0},
\]
\[
\Delta^2 \hat{J}_i^Z = \Delta^2 \hat{J}_i^X \approx \sqrt{J_0 J}/2.
\]  (50)

Noting that after the rotations of equation (17), the mean spin direction is now in the \(Y\) direction in the internal modes, we can see that the phase noise directly orthogonal to this is in the \(X\) direction. This means that the linearized phase sensitivity is enhanced in the strong interaction limit of \(J_0 = \kappa/\gamma \rightarrow 0\), scaling as \(\sqrt{J_0}\). The complementary phase-measurement quadrature sees an increased noise level, scaling as \(1/J_0\). We see that it is important to rotate the quadratures so that the even larger fluctuations in \(\Delta^2 \hat{J}_i^Y\) act in the internal mode \(Z\) direction, where they do not enter the phase-measurement plane leading to output number differences.

Finally, we can set \(d(\Delta^2 \hat{J}_i^X + \Delta^2 \hat{J}_i^X)/d\sigma = 0\) to obtain the optimal value of \(J_0\) that minimizes the PQS sum of spin variances, \(\Delta^2 \hat{J}_i^X + \Delta^2 \hat{J}_i^X = \Delta^2 \hat{J}_i^X + \Delta^2 \hat{J}_i^Y\). We obtain \(J_0 \sim (\sqrt{J}/4)^{2/3} \approx 2.32\) for the parameter values in figure 3. This is very close to the numerical value for the ground state solution of the Hamiltonian (47), as shown in figure 3.

5.3. Entanglement criteria

We present solutions for the ground state of (45), including number fluctuations as described in section 3.1. The dashed curves in figure 4 show that the normalized phase-entanglement criteria equation (24) \(E_{ph} < 1\) detects the two-mode entanglement of the ground state of a two-well BEC [4–6, 9] in a way that is almost immune to Poissonian number fluctuations. On the other hand, the solid curves for \(E_{Hz}\) reveal that the entanglement detected via the un-normalized HZ criterion \(E_{Hz} < 1\) is very easily destroyed by number fluctuations.

Figure 5 shows that the normalized spin-squeezing parameter also detects squeezing and particle entanglement in a way that is insensitive to number fluctuations. The solid curves plot the squeezing predicted for a fixed \(N = 100\), where we recall that this parameter is not defined except in the case of a fixed \(N\), but that according to equation (15) will detect both squeezing

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Figure 4. Entanglement of a ground state BEC including Poissonian atom number fluctuations. Here $N = 100$. (i) Entanglement between wells $a$ and $b$ is detected if $E_{ph} < 1$ (blue and red dashed curves) or $E_{HZ} < 1$ (equation (19)) (purple and green solid curves). Curves $a, b$ (purple solid and blue dashed) are for two-well BEC modes $\hat{a}, \hat{b}$; curves $a_i, b_i$ (green solid and red dashed) are for $\hat{a}, \hat{b}$ formed from BEC modes $\hat{a}_i, \hat{b}_i$ input to the M-Z interferometer sequence depicted in figure 1.

Figure 5. Spin-squeezing and phase measurement sensitivity parameters for ground state of a two-well BEC with $\langle \hat{N} \rangle = 100$. Plot of spin-squeezing parameters $\xi_{S, ph}^Y, \xi_{S, ph}^Z$ (purple and green solid curves) and the normalized parameters $\xi_{S, ph}^Y, \xi_{S, ph}^Z$ (blue and red dashed curves), for Poissonian number fluctuations. States with $\xi_{S, ph}^Y, \xi_{S, ph}^Z < 1$ show sub-shot noise enhanced phase sensitivity for measurements of small rotations about a fixed phase. Inset shows the asymptotic $\xi_{S}^{Y/Z} \sim \sqrt{2/N} (\Delta \phi)_{1/2} \sim O(\sqrt{2/N})$ behavior with increasing $N$ for fixed $g/\kappa = 10^3$. The Heisenberg limit is $\xi_{S}^{Y/Z} \sim O(\sqrt{1/N})$ [26].
and entanglement among the $N$ particles when $\xi_S < 1$. The dashed curves plot the results for the squeezing of the normalized parameter, which requires $\xi_{S,\text{norm}} < 1$ for the detection of entanglement but in the presence of an arbitrary number $N$, showing a perfect overlay. Our proof justifies the normalization procedure used in recent experiments that report spin-squeezing and particle entanglement [4–6], for a repulsive BEC with fluctuating total numbers.

The criteria equations (24) and (33) enable an unambiguous detection of entanglement in the presence of number fluctuations. We note that a similar immunity of the Peres positive partial transpose (PPT) entanglement criterion to loss was shown in [50], and for other entanglement measures to loss in [51].

The figures include both attractive ($g < 0$) and repulsive ($g > 0$) interactions. In the attractive case, criteria (19)–(24) for entanglement are satisfied when applied directly to the modes of the two wells, then described by $a$, $b$. In the repulsive case, an interferometric sequence is necessary: two-well modes $\hat{a}_i, \hat{b}_i$ are phase shifted and placed through an MZ beam splitter. The reason for this is that while an attractive BEC reduces fluctuations directly in the plane of $\hat{J}^{x,y}$, a repulsive BEC reduces fluctuations in a different plane, namely $\hat{J}^{x,z}$. Without the additional input phase shifter, the phase measurement is in the $Y-Z$ plane, where only one phase has reduced fluctuations, as observed in [5].

5.4. BEC interferometric phase measurement

Now we turn to the question of using the BEC two-mode states for the purpose of interferometric phase measurement. The spin-squeezing criterion $\xi_S$ in the presence of a fixed $N$, and the normalized parameter $\xi_{S,\text{norm}}$ in the presence of fluctuating numbers, give the sensitivity for measurements of small rotations about a fixed phase below the SQL when $\xi_S < 1$ and $\xi_{S,\text{norm}} < 1$ (equation (41)). Figure 5 reveals that spin-squeezing is predicted for a wide range of parameters of the ground state solution. The inset shows the reduction in noise to be near Heisenberg limited, with a scaling $\xi_S^{1/2} \sim O(\sqrt{2/N})$ evident.

The worse-case sensitivity of the interferometer to an arbitrary angle defined within the two quiet quadrants of measurement for an unknown phase $\phi$ is given by the $\eta_{\text{ph}}$ of equation (42), which is minimized according to the phase entanglement criterion $E_{\text{ph}}$ of figure 6. The best scaling of $E_{\text{ph}}$ with $N$ is given as $J^{2/3}$, that of the $C_J$ coefficients, and is achieved at the critical value of $Ng/\kappa$. This implies, from the phase sensitivity measure $\eta_{\text{ph}}$, a sensitivity of $(\Delta \phi)_{\text{worse}} \sim O(N^{-2/3})$.

For the proposed BEC interferometer, one can evaluate the actual range of sensitivities for the unknown incoming phase $\phi$ using equation (40) directly. The different range of phase-noise reduction as a function of measured phase angle and BEC interaction strength is shown in figure 6. The best sensitivity is obtained at $\phi = \pi/2$ and the value for $\Delta \phi$ is determined by the spin-squeezing parameter equation (33) $\xi_{S,\text{norm}}$, which reduces to $\xi_S$ for a fixed number $N$. Where one measures an unknown phase $\phi$ using only two orthogonal measurements ($\theta = 0$ and $\theta' = \pi/2$), only the sensitivity over the ‘quiet’ quadrant indicated on the graph by the region $\phi/\pi = 1/2 \pm 1/4$ becomes relevant. In this case, the worse-case sensitivities are at the edges ($\phi = \pi/4$ and $3\pi/4$) and determined by the value of $\eta_{\text{ph}}$, equation (41). This parameter is optimized by minimizing the two-mode entanglement parameter $E_{\text{ph}}$, which reduces to the HZ entanglement parameter when $N$ is fixed. This demonstrates the trade-off between noise reduction and the range of measurable phase.
Figure 6. The measured phase uncertainty $\Delta \phi$ (40) for Poissonian fluctuations. Phase sensitivity is better than shot noise level if $\Delta \phi < 0.1 = 1/\sqrt{\langle \hat{N} \rangle}$. The red dashed curve corresponds to the critical value of $\langle \hat{N} \rangle g/\kappa \simeq 43.6$, which gives the lowest value of $E_{\text{ph}}$ for the repulsive regime. Increasing $\langle \hat{N} \rangle g/\kappa$ to $10^3$ (dotted) and $10^4$ (solid), improves the optimum phase uncertainty, but the useful region of phase angles become narrower. Optimum sensitivity is at $\phi \pm \pi/2$ and is determined by the spin-squeezing parameter $\xi_{S, \text{norm}}$. For detecting an unknown phase using only two orthogonal measurements, the sensitivity over the quiet quadrant $\phi/\pi = 1/2 \pm 1/4$ becomes relevant.

6. Conclusions

In summary, we have introduced the normalized relative phase quadrature operator $\tilde{j}(\phi)$ as the most direct operational expression of how interferometric measurements give rise to phase information. A corresponding phase-entanglement measure, as quantified by $E_{\text{ph}}$, describes a useful physical resource for phase measurement. This directly quantifies the measurement sensitivity increase above the SQL, as $E_{\text{ph}}$ decreases towards a maximally entangled state. We also introduce a normalized phase-squeezing measure, $\xi_{S, \text{ph}}$, which signifies entanglement between qubits or particles. Both measures are normalized in terms of the total particle number using the generalized Moore–Penrose inverse. We show how it is possible to improve BEC phase measurements so that a range of unknown phases possess reduced phase noise, rather than just one pre-selected phase.

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Appendix. Quantum phase measurements

In this appendix, we briefly review earlier approaches to phase measurement of quantum fields in order to explain and motivate the approach used in this paper.

A.1. Phase operators

We start with a generic problem in quantum phase measurement: how can one measure the phase of a quantum field or mode ($\hat{a}$). Classically, one divides up a field amplitude into intensity ($N_\hat{a}$) and phase ($\phi_\hat{a}$) by introducing

$$ a = \sqrt{N_\hat{a}} e^{i\phi_\hat{a}}. \quad \text{(A.1)} $$

Next, if one measures the complex amplitude $a$, one simply classically normalizes to obtain the classical phase,

$$ \phi_\hat{a} = -i \ln\left[ a / \sqrt{N_\hat{a}} \right]. $$

We note that this gedanken experiment for classical phase measurement involves measuring both real and imaginary components of the amplitude.

Quantum studies of this generic problem date back to the early attempt of Dirac [52] to define a quantum phase operator from a canonical commutation relation of the form $[\hat{\phi}_\hat{a}, \hat{N}_\hat{a}] = -i$. The underlying mathematical problem is that a Hermitian quantum operator $\hat{\phi}_\hat{a}$ for the phase of a single harmonic oscillator operator $\hat{a}$ strictly does not exist, which is the topic of many previous studies [13, 30, 53]. This can be seen most easily from the commutation relations $[\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}] = 1$. If the phase is Hermitian, then $\hat{\phi} = \sqrt{\hat{N}} \hat{U}$, where $\hat{U}$ is a unitary operator such that $\hat{U}^\dagger \hat{U} = \hat{U} \hat{U}^\dagger = 1$ and $\hat{N} = \hat{a}^\dagger \hat{a}$. From the commutators, $\hat{N} - 1 = \hat{U} \hat{N} \hat{U}^\dagger$. This violates unitarity, since a unitary transformation cannot change an operator’s eigenvalues.

This problem does not, of course, occur classically, and is at the root of quantum phase-measurement problems. However, the idea of a phase operator with canonical commutators is approximately valid at a large particle number and suggests the existence of a fundamental uncertainty principle called the Heisenberg limit:

$$ \Delta \phi \Delta N \geq \frac{1}{2}. \quad \text{(A.2)} $$

Given that $\Delta N \leq N_{\text{max}}/2$, this has led to the idea of a Heisenberg limit of $\Delta \phi \geq 1/N_{\text{max}}$ on phase measurement with at most $N_{\text{max}}$ particles. Sometimes one interchanges $N$ and $N_{\text{max}}$, as they are often related.

A.2. Truncated Hilbert space methods

One resolution of the lack of a Hermitian phase operator due to Pegg and Barnett is to truncate the Hilbert space to a maximum boson number of $s$ [54]. Next, one defines a phase eigenstate $|\theta\rangle_p$ as a discrete Fourier transform of number states $|n\rangle$, using

$$ |\theta\rangle_p = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{i\theta_n} |n\rangle. \quad \text{(A.3)} $$

From this starting point, it is clear that a Hermitian phase operator can simply be obtained from the definition

$$ \hat{\phi}_p = \sum_{m=0}^{s} \theta_m |\theta_m\rangle_p \langle \theta_m|_p. \quad \text{(A.4)} $$
Here $\theta_m = \theta_0 + 2\pi m / (s + 1)$, and $\theta_0$ is a reference phase. This is a mathematically consistent approach, which resolves the issue of Hermiticity, but leaves a number of practical questions unanswered.

In particular, what is the physical meaning of the maximum number $s$ and the reference phase $\theta_0$? Is it possible to take the limit of large $s$ in a unique way? From an operational perspective, what (if any) is the relationship between the abstract operator $\hat{\phi}_a$ and an interferometric measurement? One of the aims of this paper is to understand how these questions can be answered and implemented using interferometry.

A.3. Relative phase operator

Another solution along these directions is to define a relative phase operator, $\hat{\phi}_r$, for two modes $\hat{a}$ and $\hat{b}$ [55]. This has the conceptual advantage that it corresponds to operational procedures—which always involve relative phase measurement—and clarifies the meaning of the reference phase. A relative phase operator is consistent philosophically with the fundamental idea of relative measurements in physics, and is the most operationally meaningful way of defining quantum phase in many cases.

With this approach, one works in a space whose algebra is defined by the equivalent angular momentum operators in the Schwinger representation:

\[
\hat{J}^X = \frac{1}{2}(\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger),
\]

\[
\hat{J}^Y = \frac{1}{2i}(\hat{a}^\dagger \hat{b} - \hat{a} \hat{b}^\dagger),
\]

\[
\hat{J}^Z = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b} \hat{b}^\dagger),
\]

\[
\hat{N} = \hat{a}^\dagger \hat{a} + \hat{b} \hat{b}^\dagger.
\]

(A.5)

It is usual to assume that one has an eigenstate of the total number $\hat{N}$, with eigenvalue $N$ and hence an equivalent angular momentum eigenstate of $J = N/2$. As we show in the main text, this assumption is questionable in real experimental measurements. The states of a well-defined phase then correspond to linear combinations of angular momentum eigenstates with $\hat{J}^Z |J, m\rangle = m |J, m\rangle$, and one then has a physical phase basis of

\[
|\theta\rangle_r = \frac{1}{\sqrt{2J + 1}} \sum_{m=-J}^{J} e^{im\theta} |J, m\rangle.
\]

(A.6)

This allows the definition of a relative phase-difference operator which is Hermitian, following equation (A.4) above, and has a discrete spectrum. The fact that the angular momentum Hilbert space is finite provides a natural explanation of the truncation parameter $s$ in the Pegg–Barnett approach. Despite these advantages, we show in the main text that these relative phase states do not lead to ideal interferometric phase measurements, and that for an unknown phase, planar squeezed states perform better.

We also note that in practical interferometry experiments, it is nearly impossible have an input state that has a well-defined total particle number, especially at large mean particle number. Instead, the most common situation is that there is an initial mixture of particle numbers,
with number fluctuations that are typically at least Poissonian. These issues of interferometric measurement and number fluctuations are studied in detail in the main text.

A.4. Quantum sine and cosine operators

An alternative resolution of the phase-measurement question is to define quantum sine and cosine operators. This is a way of reaching the phase through the measurement of the real or imaginary part of the amplitude, an idea that has a clear analogue in the classical world, as described above. Operationally, this is the quantum version of a proposal by Zernike [56] to analyze coherence in classical interferometry. The original idea of Zernike was to relate coherence properties directly to the measured classical fringe visibility.

Extending this to the quantum theory of a single quantized mode [13], one can define a normalized amplitude

$$\hat{E} = [1 + \hat{N}]^{-1/2} \hat{a}$$

from which the sine and cosine operators are obtained via

$$\hat{C} = \frac{1}{2} [\hat{E} + \hat{E}^\dagger],$$

$$\hat{S} = \frac{1}{2} i [\hat{E} - \hat{E}^\dagger].$$

This is still operationally somewhat unclear, except in the limit where one arm of an interferometer is a large ‘classical-like’ local oscillator. In this limit, the approach is closely related to the theory of optical quadratures, where the two operators involved have commutators similar to the quantum position and momentum operators: for a field mode $\hat{a}$, the quadrature amplitudes are defined as $\hat{X} = (\hat{a} + \hat{a}^\dagger)/2$, and $\hat{P} = (\hat{a} - \hat{a}^\dagger)/(2i)$.

A drawback of the local oscillator approach implicit in this method is that in practical terms it is highly resource-hungry. A classical local oscillator must necessarily involve a very large boson number in the local oscillator mode, even though this mode is not formally included as part of the measured system. We note that there is a generic issue, which is that since these operators do not commute, it is not possible to measure both quadratures simultaneously. This issue remains in some form for most interferometric methods.

A.5. The carruthers and Nieto phase quadrature operator

Alternatively, the same general idea of quadrature measurement can be applied to the relative phase between two modes. This also allows for a better understanding of the actual resources involved in the measurement, since it effectively combines both the measured beam and the local oscillator into the measured operator. The combined approach is studied in an early review of Carruthers and Nieto [30] and reduces to defining the cosine and sine operators of the phase difference as

$$\hat{C}_{12} = \hat{C}_1 \hat{C}_2 + \hat{S}_1 \hat{S}_2,$$

$$\hat{C}_{12} = \hat{S}_1 \hat{C}_2 - \hat{S}_2 \hat{C}_1. $$

This has the virtue that these operators commute with the total particle number, $\hat{N} = \hat{N}_1 + \hat{N}_2$. This number is kept finite, and provides an indication of the true resource needed for the measurement.
A.6. The Leggett phase quadrature operator

A related definition for a two-mode BEC was proposed by Leggett [31], who suggested defining a phase operator following the approach of Carruthers and Nieto, except that

$$\hat{E} = \frac{(N/2 - \hat{J}^Z)(N/2 + \hat{J}^Z + 1)^{-1/2}}{(\hat{J}^X + i \hat{J}^Y)}.$$  \hspace{1cm} (A.10)

Here the phase is encoded in the rotations in the $X$–$Y$ plane, a convention we follow in this paper unless otherwise noted. In both the last two approaches, the proposed quadrature operators were not analyzed from the perspective of interferometric fringes and their noise properties. We show in the main text that the operationally most relevant approach to interferometry measurements is to use a different normalization to either of the last two suggestions.

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