ELECTROMAGNETIC STEKLOFF EIGENVALUES: EXISTENCE AND BEHAVIOR IN THE SELFADJOINT CASE

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Abstract. In [Camano, Lackner, Monk, SIAM J. Math. Anal., Vol. 49, No. 6, pp. 4376-4401 (2017)] it was suggested to use Stekloff eigenvalues for Maxwell equations as target signature for nondestructive testing via inverse scattering. The authors recognized that in general the eigenvalues due not correspond to the spectrum of a compact operator and hence proposed a modified eigenvalue problem with the desired properties. The Fredholmness and the approximation of both problems were analyzed in [Halla, arXiv:1909.00689 (2019)].

The present work considers the original eigenvalue problem in the selfadjoint case. We report that apart for a countable set of particular frequencies, the spectrum consists of three disjoint parts: The essential spectrum consisting of the point zero, an infinite sequence of positive eigenvalues which accumulate only at infinity and an infinite sequence of negative eigenvalues which accumulate only at zero.

The analysis is based on a representation of the operator as block operator. For small/big enough eigenvalue parameter the Schur-complements with respect to different components can be build. For each Schur-complement the existence of an infinite sequence of eigenvalues is proved via a fixed point technique similar to [Cakoni, Haddar, Applicable Analysis, 88: 4, 475-493 (2009)].

The modified eigenvalue problem considered in the above references arises as limit of one of the Schur-complements.

1. Introduction

Novel nondestructive evaluation methods based on inverse scattering [6] give rise to a multitude of new eigenvalue problems. Among these are so-called transmission eigenvalue problems [7] and Stekloff eigenvalue problems [5]. Not all of these eigenvalue problems fall into classes which are covered in classical literature. Among the important questions on these eigenvalue problems are

- Fredholm properties (which imply the discreteness of the spectrum),
- the existence of eigenvalues,
- properties of the eigenvalues
- and reliable computational approximations.

The electromagnetic Stekloff eigenvalue problem to find \((\lambda, u)\) so that

\[
\begin{align*}
\text{curl} \text{curl} u - \omega^2 \varepsilon u &= 0 \quad \text{in } \Omega, \\
\nu \times \text{curl} u + \lambda \nu \times u &\times \nu = 0 \quad \text{at } \partial \Omega.
\end{align*}
\]

was considered in the recent publication [9]. Therein the authors of [9] considered the case that \(\Omega\) is a ball and the material parameter \(\varepsilon\) is constant. For this setting they proved the existence of two infinite sequences of eigenvalues, one converging to zero and one converging to infinity. Consequently the eigenvalue problem can’t be transformed to an eigenvalue problem for a compact operator. This observation led the authors of [9] to discard the original eigenvalue problem and to modify instead...
the boundary condition to obtain a different eigenvalue problem. The approximation of both eigenvalue problems is discussed in the companion article \cite{13} by means of \cite{14}.

In this article we consider the original electromagnetic Stekloff eigenvalue problem in the selfadjoint case. We give a complete description of the spectrum (see Proposition 3.2): The spectrum consists of three disjoint parts: The essential spectrum consisting of the point zero, an infinite sequence of positive eigenvalues which accumulate only at infinity and an infinite sequence of negative eigenvalues which accumulate only at zero.

As a side result, we also analyze the spectrum of the modified electromagnetic Stekloff eigenvalue problem, see Section 6. Our analysis reveals that the modified eigenvalue problem arises as asymptotic limit of the original eigenvalue problem for large spectral parameter. Though, this doesn’t yield any non-trivial asymptotic statement on the eigenvalues.

The remainder of this article is organized as follows. In Section 2 we set our notation and formulate our assumptions on the domain and the material parameters. We also recall some classic regularity, embedding and decomposition results which will be essential for our analysis. In most cases the respective references don’t apply directly to our setting and hence we formulate adapted variants. In Section 3 we introduce the considered electromagnetic Stekloff eigenvalue problem and define the associated holomorphic operator function \( A_X(\cdot) \). We report in Theorem 3.2 that the spectrum of \( A_X(\cdot) \) is real and that \( A_X(\lambda) \) is Fredholm if and only if \( \lambda \neq 0 \). In Section 4 we analyze the spectrum in a neighborhood of zero. We report in Theorem 4.4 that there exists \( c_0 > 0 \) so that \( \sigma(A_X(\cdot)) \cap (0, c_0) = \emptyset \). We report in Theorem 4.7 the existence of an infinite sequence of negative eigenvalues which accumulate at zero. In Section 5 we analyze the spectrum in a neighborhood of infinity. We report in Theorem 5.3 that there exists \( c_\infty > 0 \) so that \( \sigma(A_X(\cdot)) \cap (-\infty, -c_\infty) = \emptyset \). We report in Theorem 5.15 the existence of an infinite sequence of positive eigenvalues which accumulate at \( +\infty \). In Section 6 we collect our results in Proposition 6.1 and comment on the connection between the original and the modified electromagnetic Stekloff eigenvalue problems.

2. General setting

In this section we set our notation and formulate assumptions on the domain and material parameters. We also recall necessary results from different literature and adapt them to our setting.

2.1. Functional analysis. For generic Banach spaces \((X, \| \cdot \|_X), (Y, \| \cdot \|_Y)\) denote \( L(X, Y) \) the space of all bounded linear operators from \( X \) to \( Y \) with operator norm \( \|A\|_{L(X,Y)} := \sup_{u\in X\setminus \{0\}} \|Au\|_Y/\|u\|_X \). \( A \in L(X,Y) \). We further set \( L(X) := L(X,X) \). For generic Hilbert spaces \((X, \langle \cdot, \cdot \rangle_X), (Y, \langle \cdot, \cdot \rangle_Y)\) and \( A \in L(X,Y) \) we denote \( A^* \in L(Y,X) \) its adjoint operator defined through \( \langle u, A^*u' \rangle_X = \langle Au, u' \rangle_Y \) for all \( u \in X, u' \in Y \). Let \( K(X,Y) \subset L(X,Y) \) be the space of compact operators and \( K(X) := K(X,X) \).

We say that an operator \( A \in L(X) \) is coercive, if \( \inf_{u \in X \setminus \{0\}} |\langle Au, u \rangle_X|/\|u\|_X^2 > 0 \). We say that \( A \in L(X) \) is weakly coercive, if there exists \( K \in K(X) \) so that \( A + K \) is coercive. Let \( \Lambda \subset \mathbb{C} \) be open and consider an operator function \( A(\cdot) : \Lambda \to L(X) \). We call \( A(\cdot) \) (weakly) coercive if \( A(\lambda) \) is (weakly) coercive for all \( \lambda \in \Lambda \). We denote the spectrum of \( A(\cdot) \) as \( \sigma(A(\cdot)) := \{ \lambda \in \Lambda : A(\lambda) \) is not bijective\} and the resolvent set as \( \rho(A(\cdot)) := \Lambda \setminus \sigma(A(\cdot)) \). We denote \( \sigma_{ess}(A(\cdot)) := \{ \lambda \in \Lambda \setminus \sigma(A(\cdot)) \) is not bijective\} and the resolvent set as \( \rho(A(\cdot)) := \Lambda \setminus \sigma(A(\cdot)) \). We denote \( \sigma_{ess}(A(\cdot)) := \{ \lambda \in \})
2.2. Lebesgue and Sobolev spaces. Let \( \Omega \subset \mathbb{R}^3 \) be a bounded path connected open Lipschitz domain and \( \nu \) the outer unit normal vector at \( \partial \Omega \). Let \( C^\infty_0(\Omega) \) be the space of infinitely many times differentiable functions from \( \Omega \) to \( \mathbb{R} \), \( L^2(\Omega) \) be the space of square-integrable functions on \( \Omega \), \( W^{1,\infty}(\Omega) \), \( H^s(\Omega) \) defined on the domain \( \Omega \) and \( L^2(\partial \Omega) \). We recall the continuity of the trace operator \( \text{tr} \in L(H^s(\Omega), H^{s-1/2}) \) for all \( s > 1/2 \). For a vector space \( X \) of scalar valued functions we denote its bold symbol as space of three-vector valued functions \( X : X^3 = X \times X \times X \), e.g. \( L^2(\Omega), H^s(\Omega), L^2(\partial \Omega), H^s(\partial \Omega) \). For \( L^2(\partial \Omega) \) or a subspace, e.g. \( H^s(\partial \Omega) \), the subscript \( s \) denotes the subspace of tangential fields. In particular \( L^2(\partial \Omega) = \{ u \in L^2(\partial \Omega) : \nu \cdot u = 0 \} \) and \( H^s(\partial \Omega) = \{ u \in H^s(\partial \Omega) : \nu \cdot u = 0 \} \). For a vector space \( X \) of vector valued functions \( \nu \) \( H^s(\partial \Omega) \) be the subspace of \( H^s(\Omega) \) of all functions with vanishing mean \( \langle u, 1 \rangle_{L^2(\Omega)} = 0 \) and \( H^s(\partial \Omega) \) be the subspace of \( H^s(\partial \Omega) \) of all functions with vanishing mean \( \langle u, 1 \rangle_{L^2(\partial \Omega)} = 0 \). Additional function spaces. Denote \( \partial_x u \) the partial derivative of a function \( u \) with respect to the variable \( x_i \). Let

\[
\nabla u := (\partial_{x_1} u, \partial_{x_2} u, \partial_{x_3} u),
\]

\[
\text{div}(u_1, u_2, u_3)^\top := \partial_{x_3} u_1 + \partial_{x_2} u_2 + \partial_{x_1} u_3,
\]

\[
\text{curl}(u_1, u_2, u_3)^\top := (\partial_{x_2} u_3 - \partial_{x_3} u_2, \partial_{x_3} u_1 - \partial_{x_1} u_3, \partial_{x_1} u_2 - \partial_{x_2} u_1)^\top.
\]

For \( \epsilon \in (L^\infty(\Omega))^{3 \times 3} \) let \( \text{div} u := \text{div}(\epsilon u) \). For a bounded Lipschitz domain \( \Omega \) let \( \nabla \rho, \text{div} \rho \) and \( \text{curl} \rho = \nu \times \nabla \rho \) be the respective differential operators for functions defined on \( \partial \Omega \). We recall that for \( u \in L^2(\Omega) \) with \( \text{curl} u \in L^2(\Omega) \), the tangential trace \( \text{tr}_{\nu \times} u \in H^{-1/2}(\text{div} \nu ; \partial \Omega) \) is well defined and \( \| \text{tr}_{\nu \times} u \|_{H^{-1/2}(\text{div} \nu ; \partial \Omega)} \) is bounded by a constant times \( \| u \|_{L^2(\Omega)}^2 + \| \text{curl} u \|_{L^2(\Omega)}^2 \). Likewise \( \forall u \in L^2(\Omega) \) with \( \text{div} u \in L^2(\Omega) \) the normal trace \( \text{tr}_{\nu} u \in H^{-1/2}(\partial \Omega) \) is well defined and \( \| \text{tr}_{\nu} u \|_{H^{-1/2}(\partial \Omega)} \) is bounded by a constant times \( \| u \|_{L^2(\Omega)}^2 + \| \text{div} u \|_{L^2(\Omega)}^2 \). Let

\[
L^2(d) := \begin{cases}
L^2(\Omega), & d = \text{curl}, \\
L^2(\Omega), & d = \text{div}, \\
L^2(\partial \Omega), & d = \text{tr}_{\nu \times}, \\
L^2(\partial \Omega), & d = \text{tr}_{\nu},
\end{cases}
\]

(1a)

Let

\[
H(d; \Omega) := \{ u \in L^2(\Omega) : du \in L^2(d) \},
\]

(1b)

\[
\langle u, u' \rangle_{H(d; \Omega)} := \langle u, u' \rangle_{L^2(\Omega)} + \langle du, du' \rangle_{L^2(d)},
\]

(1c)

\[
H(d^0; \Omega) := \{ u \in H(d; \Omega) : du = 0 \}.
\]

Also for

\[
d_1, d_2, d_3, d_4 \in \{ \text{curl}, \text{div}, \text{div}, \text{tr}_{\nu \times}, \text{tr}_{\nu}, \text{tr}_{\nu \times}, \text{tr}_{\nu}, \text{tr}_{\nu \times}, \text{tr}_{\nu}, \text{tr}_{\nu \times} \}
\]
let
\[
H(d_1, d_2; \Omega) := H(d_1; \Omega) \cap H(d_2; \Omega),
\]
\[
\langle u, u' \rangle_{H(d_1, d_2; \Omega)} := \langle u, u' \rangle_{L^2(\Omega)} + \langle d_1 u, d_1 u' \rangle_{L^2(d_1)} + \langle d_2 u, d_2 u' \rangle_{L^2(d_2)},
\]
\[
H(d_1, d_2, d_3; \Omega) := H(d_1; \Omega) \cap H(d_2; \Omega) \cap H(d_3; \Omega),
\]
\[
\langle u, u' \rangle_{H(d_1, d_2, d_3; \Omega)} := \langle u, u' \rangle_{L^2(\Omega)} + \langle d_1 u, d_1 u' \rangle_{L^2(d_1)} + \langle d_2 u, d_2 u' \rangle_{L^2(d_2)} + \langle d_3 u, d_3 u' \rangle_{L^2(d_3)},
\]
\[
H(d_1, d_2, d_3, d_4; \Omega) := H(d_1; \Omega) \cap H(d_2; \Omega) \cap H(d_3; \Omega) \cap H(d_4; \Omega),
\]
\[
\langle u, u' \rangle_{H(d_1, d_2, d_3, d_4; \Omega)} := \langle u, u' \rangle_{L^2(\Omega)} + \langle d_1 u, d_1 u' \rangle_{L^2(d_1)} + \langle d_2 u, d_2 u' \rangle_{L^2(d_2)} + \langle d_3 u, d_3 u' \rangle_{L^2(d_3)} + \langle d_4 u, d_4 u' \rangle_{L^2(d_4)}.
\]

2.4. Assumption on the domain and material parameters.

Assumption 2.1 (Assumption on \(\epsilon\)). Let \(\epsilon \in (L^\infty(\Omega))^{3 \times 3}\) be a real, symmetric matrix function so that there exists \(c_\epsilon > 0\) with
\[
(2a) \quad c_\epsilon |\xi|^2 \leq \xi^H \epsilon(x) \xi
\]
for all \(x \in \Omega\) and all \(\xi \in \mathbb{C}^3\). We further assume that there exists a Lipschitz domain \(\Omega \subset \Omega\) so that the closure of \(\bar{\Omega}\) is compact in \(\Omega\) and \(\epsilon|_{\Omega,\bar{\Omega}}\) equals the identity matrix \(I_{3 \times 3} \in \mathbb{C}^{3 \times 3}\).

We note that a generalization of Assumption 2.1 to \(\epsilon|_{\Omega,\bar{\Omega}} \in W^{1,\infty}(\Omega \setminus \bar{\Omega})\) seems possible. Let \(\Omega \subset \Omega\) be a Lipschitz domain so that the closure of \(\bar{\Omega}\) is compact in \(\Omega\) and the closure of \(\Omega \subset \Omega\) is compact in \(\bar{\Omega}\). Let \(\chi\) be infinitely many times differentiable, so that \(\chi|_{\Omega,\bar{\Omega}} = 1\) and \(\chi|_{\Omega} = 0\).

Assumption 2.2 (Assumption on \(\mu\)). Let \(\mu^{-1} \in (L^\infty(\Omega))^{3 \times 3}\) be a real, symmetric matrix function so that there exists \(c_\mu > 0\) with
\[
(3a) \quad c_\mu |\xi|^2 \leq \xi^H \mu^{-1}(x) \xi
\]
for all \(x \in \Omega\) and all \(\xi \in \mathbb{C}^3\). We further assume that \(\mu|_{\Omega,\bar{\Omega}}\) equals the identity matrix \(I_{3 \times 3} \in \mathbb{C}^{3 \times 3}\).

Assumption 2.3 (Assumption on \(\Omega\)). Let \(\Omega \subset \mathbb{R}^3\) be a bounded path connected Lipschitz domain so that there exists \(\delta > 0\) and the following shift theorem holds on \(\Omega\): Let \(f \in L^2(\Omega),\ g \in H^{1/2}(\partial \Omega)\) with \(\langle g, 1 \rangle_{L^2(\partial \Omega)} = 0\) and \(w \in H^1(\Omega)\) be the solution to
\[
(4a) \quad \Delta w = f \quad \text{in } \Omega,
\]
\[
(4b) \quad n \cdot \nabla w = g \quad \text{at } \partial \Omega.
\]
Then the linear map \((f, g) \mapsto w: L^2(\Omega) \times H^{1/2}(\partial \Omega) \to H^{3/2+\delta}(\Omega)\) is well defined and continuous.

The above assumption holds e.g. for smooth domains and Lipschitz polyhedral [11, Corollary 23.5].

Assumption 2.4 (Assumption on \(\Omega, \epsilon\) and \(\mu^{-1}\)). Let \(\epsilon, \mu^{-1}\) and \(\Omega\) be so that a unique continuation principle holds, i.e. if \(u \in H(\text{curl}; \Omega)\) solves
\[
(5a) \quad \text{curl} \mu^{-1} \text{curl} u - \omega^2 \epsilon u = 0 \quad \text{in } \Omega,
\]
\[
(5b) \quad \text{tr}_{\nu \times} u = 0 \quad \text{at } \partial \Omega,
\]
\[
(5c) \quad \text{tr}_{\nu \times} \mu^{-1} \text{curl} u = 0 \quad \text{at } \partial \Omega,
\]
then \( u = 0 \).

To our knowledge the most general today available result on the unique continuation principle for Maxwell’s equations is the one of Ball, Capdeboscq and Tsering-Xiao [2]. It essentially requires \( \epsilon \) and \( \mu^{-1} \) to be piece-wise \( W^{1,\infty} \).

2.5. Trace regularities and compact embeddings. We recall a classical result from Costabel [10]:

\[
\begin{align}
(6a) \quad & \text{tr}_{\nu'} \in L(H(\text{curl}, \text{div}, \text{tr}_{\nu'}; \Omega), L^2(\partial\Omega)), \\
(6b) \quad & \text{tr}_{\nu \times} \in L(H(\text{curl}, \text{div}, \text{tr}_{\nu \times}; \Omega), L^4_1(\partial\Omega)),
\end{align}
\]

and

\[
\text{The embeddings from } H(\text{curl}, \text{div}, \text{tr}_{\nu'}; \Omega) \text{ and } H(\text{curl}, \text{div}, \text{tr}_{\nu \times}; \Omega) \text{ to } \mathbf{H}^{1/2}(\Omega) \text{ are bounded.}
\]

We adapt the trace results of Costabel to our setting in the next lemmata.

**Lemma 2.5.** Let \( \epsilon \) suffice Assumption 2.1. Thence

\[
\text{tr}_{\nu'} \in L(H(\text{div}; \Omega), H^{-1/2}(\partial\Omega)) \text{ and } \text{tr}_{\nu'} = \text{tr}_{\nu' \epsilon}.
\]

and

\[
\text{tr}_{\nu \epsilon} \in L(H(\text{div}; \Omega), H^{-1/2}(\partial\Omega)) \text{ and } \text{tr}_{\nu \epsilon} = \text{tr}_{\nu}.
\]

**Proof.** If \( u \in H(\text{div}; \Omega) \) then \( \chi u \in H(\text{div}; \Omega) \). Since \( \chi |_{\partial\Omega} = \epsilon |_{\partial\Omega} = 1 \) it follows \( \text{tr}_{\nu \times} u = \text{tr}_{\nu \times} \chi u = \text{tr}_{\nu \times} \epsilon \chi u \). The reverse direction follows the same way. \( \square \)

**Lemma 2.6.** Let \( \epsilon \) suffice Assumption 2.1. Thence

\[
\begin{align}
(8a) \quad & \text{tr}_{\nu \epsilon} \in L(H(\text{curl}, \text{div}, \text{tr}_{\nu \epsilon}; \Omega), L^2(\partial\Omega)), \\
(8b) \quad & \text{tr}_{\nu \times} \in L(H(\text{curl}, \text{div}, \text{tr}_{\nu \times}; \Omega), L^4_1(\partial\Omega)),
\end{align}
\]

**Proof.** Apply (6) to \( \chi u \) and employ Lemma 2.5. \( \square \)

We deduce the next lemma from Amrouche, Bernardi, Dauge and Girault [1].

**Lemma 2.7.** Let \( \epsilon \) suffice Assumption 2.1 and \( \Omega \) suffice Assumption 2.3. Thence

\[
\text{tr}_{\nu \times} \in L(H(\text{curl}, \text{div}, \text{tr}_{\nu \times}^0; \Omega), H^1(\partial\Omega)).
\]

In particular \( \text{tr}_{\nu \times} \in L(H(\text{curl}, \text{div}, \text{tr}_{\nu \times}^0; \Omega), L^4_2(\partial\Omega)) \) is compact.

**Proof.** Apply the proof of [1, Proposition 3.7] to \( \chi u \) and employ Assumption 2.3 to obtain \( \chi u \in H^{1/2+\delta}(\Omega) \). Employ \( \text{tr} \in L(H^{1/2+\delta}(\Omega), H^1(\partial\Omega)) \) and the compact embedding \( H^1(\partial\Omega) \to L^2(\partial\Omega) \). \( \square \)

We recall from Weber [21]:

**Lemma 2.8.** Let \( \epsilon \) suffice Assumption 2.1. Thence the embedding

\[
H(\text{curl}, \text{div}, \text{tr}_{\nu \times}; \Omega) \to L^2(\Omega)
\]

is compact.
Proof. Let $E : H(\text{curl}, \text{div}, \text{tr}_{\nu \times}; \Omega) \to L^2(\Omega) : u \mapsto u$. Let $M(\alpha)$ be the multiplication operator with symbol $\alpha$. We split the identity operator in two parts $I = M(\chi) + M(1-\chi)$. Hence $EM(\chi)$ is compact due to (7) and $EM(1-\chi)$ is compact due to (10). Hence $E = EM(\chi) + EM(1-\chi)$ is compact too. \hfill \square

2.6. Helmholtz decomposition on the boundary. We recall fromBuffia, Costabel and Sheen [4, Theorem 5.5]:

$$L^2_0(\partial\Omega) = \nabla_\partial H^1(\partial\Omega) \oplus \text{curl } H^1(\partial\Omega).$$

and denote the respective orthogonal projections by

$$P_{\nabla_\partial} : L^2_0(\partial\Omega) \to \nabla_\partial H^1(\partial\Omega), \quad P_{\text{curl}} : L^2_0(\partial\Omega) \to \text{curl } H^1(\partial\Omega).$$

Recall $\text{div } \text{tr}_{\nu \times} \in L(H(\text{curl}; \Omega), H^{-1/2}(\partial\Omega))$. So for $u \in H(\text{curl}; \Omega)$ let $z$ be the solution to find $z \in H^1_0(\partial\Omega)$ so that

$$\langle \nabla_\partial z, \nabla_\partial z'\rangle_{L^2_0(\partial\Omega)} = -\langle \text{div } \text{tr}_{\nu \times} u, z'\rangle_{H^{-1}(\partial\Omega) \times H^1(\partial\Omega)}$$

for all $z' \in H^1_0(\partial\Omega)$ and set

$$Su := \nabla_\partial z.$$

From the construction of $S$ it follows $S \in L(H(\text{curl}; \Omega), L^2_0(\partial\Omega))$ and further

$$Su = P_{\nabla_\partial} \text{tr}_{\nu \times} u$$

for $u \in H(\text{curl}, \text{tr}_{\nu \times}; \Omega)$.

3. The electromagnetic Stekloff eigenvalue problem

Let $\omega > 0$ be fixed. For $\lambda \in \mathbb{C}$ let $A(\lambda) \in L(H(\text{curl}, \text{tr}_{\nu \times}; \Omega))$ be defined through

$$\langle A(\lambda)u, u'\rangle_{H(\text{curl}, \text{tr}_{\nu \times}; \Omega)} := \langle \mu^{-1} \text{curl } u, \text{curl } u'\rangle_{L^2(\Omega)} - \omega^2 \langle \epsilon u, u'\rangle_{L^2(\Omega)} - \lambda \langle \text{tr}_{\nu \times} u, \text{tr}_{\nu \times} u'\rangle_{L^2(\partial\Omega)}$$

for all $u, u' \in H(\text{curl}, \text{tr}_{\nu \times}; \Omega)$. The electromagnetic Stekloff eigenvalue problem which we investigate in this note is to find $(\lambda, u) \in \mathbb{C} \times H(\text{curl}, \text{tr}_{\nu \times}; \Omega) \setminus \{0\}$ so that $A(\lambda)u = 0$.

We note that the sign of $\lambda$ herein is reversed compared to [9]. Let

$$\langle u, u'\rangle_\chi := \langle \mu^{-1} \text{curl } u, \text{curl } u'\rangle_{L^2(\Omega)} + \langle \epsilon u, u'\rangle_{L^2(\Omega)} + \langle \text{tr}_{\nu \times} u, \text{tr}_{\nu \times} u'\rangle_{L^2(\partial\Omega)}$$

for all $u, u' \in H(\text{curl}, \text{tr}_{\nu \times}; \Omega)$. It is straight forward to see that the norms induced by $\langle \cdot, \cdot \rangle_\chi$ and $\langle \cdot, \cdot \rangle_{H(\text{curl}, \text{tr}_{\nu \times}; \Omega)}$ are equivalent. To analyze the operator $A(\lambda)$ we introduce the following subspaces of $H(\text{curl}, \text{tr}_{\nu \times}; \Omega)$:

$$V := H(\text{curl}, \text{div}_{t \times}, \text{tr}_{\nu \times}; \Omega),$$

$$W_1 := H(\text{curl}_{0\times}, \text{div}_{0\times}, \text{tr}_{\nu \times}; \Omega) \cap W_{2\times}^{-1},$$

$$W_2 := H(\text{curl}_{0\times}, \text{tr}_{\nu \times}; \Omega).$$

We recall [18, Theorem 4.3 and Remark 4.4]:

$$K_N(\Omega) := \{ \nabla h : u \in H^1(\Omega), \text{div } u = 0 \text{ in } \Omega, \text{tr } u \text{ is constant on each of the connected parts of } \partial\Omega \}$$

and $\dim K_N(\Omega) = \text{number of connected parts of } \partial\Omega - 1 < \infty$. It holds

$$W_2 = \nabla H^1_0(\Omega) \oplus \perp \chi K_N(\Omega).$$
Thus \( W_1 = \{ \nabla u : u \in H^1(\Omega), \ \text{div} \ u = 0 \ in \ \Omega, \ \text{tr}_{\nu \cdot e} \nabla u \in L^2(\partial \Omega) \}, \)

\[ \langle \text{tr}_{\nu \cdot e} \nabla u, 1 \rangle_{L^2(\Gamma)} = 0 \] for each \( \Gamma \) of the connected parts of \( \partial \Omega \).

We continue with a decomposition of \( H(\text{curl}, \text{tr}_{\nu \times} ; \Omega) \), which is similar but different to [13, Theorem 3.1].

**Theorem 3.1.** Let \( \varepsilon \) suffice Assumption 2.1 and \( \mu \) suffice Assumption 2.2. Then

\[ H(\text{curl}, \text{tr}_{\nu \times} ; \Omega) = (V \oplus W_1) \oplus \tilde{X} W_2 \]

in the following sense. There exist projections \( P_V, P_{W_1}, P_{W_2} \in L(H(\text{curl}, \text{tr}_{\nu \times} ; \Omega)) \) with \( \text{ran} \ P_V = V, \text{ran} \ P_{W_1} = W_1, \text{ran} \ P_{W_2} = W_2, W_1, W_2 \subset \ker P_V, V, W_2 \subset \ker P_{W_1}, V, W_1 \subset \ker P_{W_2} \) and \( u = P_V u + P_{W_1} u + P_{W_2} u \) for each \( u \in H(\text{curl}, \text{tr}_{\nu \times} ; \Omega) \).

Thus, the norm induced by

\[ \langle u, u \rangle_X := \langle P_V u, P_V u \rangle_{\tilde{X}} + \langle P_{W_1} u, P_{W_1} u \rangle_{\tilde{X}} + \langle P_{W_2} u, P_{W_2} u \rangle_{\tilde{X}} \]

\( u, u' \in H(\text{curl}, \text{tr}_{\nu \times} ; \Omega) \), is equivalent to \( \| \cdot \|_{H(\text{curl}, \text{tr}_{\nu \times} ; \Omega)} \).

**Proof.** 1. **Step:** Let \( P_{W_2} \) be the \( \tilde{X} \)-orthogonal projection onto \( W_2 \). Hence \( P_{W_2} \in L(H(\text{curl}, \text{tr}_{\nu \times} ; \Omega)) \) is a projection with range \( W_2 \) and kernel

\[ W_2 \cap H(\text{curl}, \text{tr}_{\nu \times} ; \Omega) \supset V, W_1. \]

2a. **Step:** Let \( u \in H(\text{curl}, \text{tr}_{\nu \times} ; \Omega) \). Note that due to \( \text{div}(u - P_{W_2} u) = 0 \) and Lemma 2.6 it holds \( \text{tr}_{\nu \cdot e}(u - P_{W_2} u) \in L^2(\partial \Omega) \) and \( \langle \text{tr}_{\nu \cdot e}(u - P_{W_2} u), 1 \rangle_{L^2(\Gamma)} = 0 \) for each \( \Gamma \) of the connected parts of \( \partial \Omega \). Let \( w_* \in H^1(\Omega) \) be the unique solution to

\[ -\text{div} \ \nabla w_* = 0 \ in \ \Omega, \ \ \ \ \nu \cdot \nabla w_* = \text{tr}_{\nu \cdot e}(u - P_{W_2} u) \ at \ \partial \Omega. \]

Let \( P_{W_1} u := \nabla w_* \). By construction of \( P_{W_1} \) and due to Lemma 2.6 it holds \( P_{W_1} \in L(H(\text{curl}, \text{tr}_{\nu \times} ; \Omega)) \) and ran \( P_{W_1} \subset W_1 \). Let \( u \in W_1 \). Then \( P_{W_2} u \) is 0 and hence \( P_{W_1} u = u \). Thus \( P_{W_1} \) is a projection and ran \( P_{W_1} = W_1 \).

2b. **Step:** If \( u \in W_2 \) then \( u - P_{W_2} u = 0 \), further \( \text{tr}_{\nu \cdot e}(u - P_{W_2} u) = 0 \) and thus \( P_{W_1} u = 0 \). Thus \( W_2 \subset \ker P_{W_1} \). If \( u \in V \) then \( P_{W_2} u = 0 \), further \( \text{tr}_{\nu \cdot e}(u - P_{W_2} u) = \text{tr}_{\nu \cdot e} u = 0 \) and thus \( P_{W_1} u = 0 \). Hence \( V \subset \ker P_{W_1} \).

3. **Step:** Let \( u \in H(\text{curl}, \text{tr}_{\nu \times} ; \Omega) \) and \( P_V u := u - P_{W_1} u - P_{W_2} u \). It follows \( P_V \in L(H(\text{curl}, \text{tr}_{\nu \times} ; \Omega)) \). \( P_V u \in V \) and \( P_V P_V u = P_V u \). If \( u \in V \) then \( P_V u = u \) and hence ran \( P_V = V \). It follows further \( W_1, W_2 \subset \ker P_V \).

4. **Step:** By means of the triangle inequality and a Young inequality it holds

\[ \| u \|_X^2 = \| P_V u + P_{W_1} u + P_{W_2} u \|_X^2 \leq 3(\| P_V u \|_X^2 + \| P_{W_1} u \|_X^2 + \| P_{W_2} u \|_X^2) \]

\[ = 3\| u \|_X^2. \]

On the other hand due to the boundedness of the projections

\[ \| u \|_X^2 = \| P_V u \|_X^2 + \| P_{W_1} u \|_X^2 + \| P_{W_2} u \|_X^2 \]

\[ \leq (\| P_V \|_L^2(\tilde{X}) + \| P_{W_1} \|_L^2(\tilde{X}) + \| P_{W_2} \|_L^2(\tilde{X})) \| u \|_X^2. \]

Thus \( \| \cdot \|_X \) is equivalent to \( \| \cdot \|_{\tilde{X}} \). Since \( \| \cdot \|_{\tilde{X}} \) is equivalent to \( \| \cdot \|_{H(\text{curl}, \text{tr}_{\nu \times} ; \Omega)} \),

\[ \| \cdot \|_X \] is also equivalent to \( \| \cdot \|_{H(\text{curl}, \text{tr}_{\nu \times} ; \Omega)}. \]

Let us look at \( A(\lambda) \) in light of this substructure of \( H(\text{curl}, \text{tr}_{\nu \times} ; \Omega) \). To end this we consider the space

\[ X := H(\text{curl}, \text{tr}_{\nu \times} ; \Omega), \ \ \ \langle \cdot, \cdot \rangle_X \ as \ defined \ in \ (24). \]
It follows that $P_V, P_{W_1}$ and $P_{W_1}$ are even orthogonal projections in $X$. Let further $A_X(\lambda), A_c, A_{tr}, A_{tr} \in L(X)$ be defined through

\begin{align}
(26a) & \langle A_X(\lambda)u, u'\rangle_X := \langle A(\lambda)u, u'\rangle_{H(curl, \triangledown \times \omega)} \quad \text{for all } u, u' \in X, \lambda \in \mathbb{C} \\
(26b) & \langle A_{tr}u, u'\rangle_X := \langle A_{tr}u, u'\rangle_{L^2(\omega)} \quad \text{for all } u, u' \in X, \\
(26c) & \langle A_{tr}u, u'\rangle_X := \langle \epsilon u, u'\rangle_{L^2(\Omega)} \quad \text{for all } u, u' \in X,
\end{align}

From the definitions of $V,W_1$ and $W_2$ we deduce that

\begin{align}
A_X(\lambda) & = (P_V + P_{W_1} + P_{W_2})(A_c - \omega^2 A_c - \lambda A_{tr})(P_V + P_{W_1} + P_{W_2}) \\
& = P_V A_c P_V - \omega^2 (P_V A_c P_V + P_{W_1} A_{tr} P_{W_1} + P_{W_2} A_{tr} P_{W_2}) \\
& \quad - \lambda(P_V + P_{W_1}) A_{tr}(P_V + P_{W_1}) \\
& = P_V A_c P_V - \omega^2 (P_V A_c P_V + P_{W_1} A_{tr} P_{W_1} + P_{W_2} A_{tr} P_{W_2}) \\
& \quad - \lambda(P_V A_{tr} P_{W_1} + P_{W_1} A_{tr} P_{W_1} + P_{W_2} A_{tr} P_{W_2}).
\end{align}

If we identify $X \sim V \times W_1 \times W_2$ and $X \ni u \sim (v,w_1,w_2) \in V \times W_1 \times W_2$, we can identify $A_X(\lambda)$ with the block operator

\begin{align}
\begin{pmatrix}
P_V (A_c - \omega^2 A_c - \lambda A_{tr})|_V \\
-\lambda P_{W_1} A_{tr}|_{W_1} \\
-\lambda P_{W_1} A_{tr}|_V \\
-\lambda P_{W_1} A_{tr}|_{W_1} \\
-\lambda P_{W_1} A_{tr}|_V \\
-\omega^2 P_{W_2} A_{tr}|_{W_2}
\end{pmatrix}
\end{align}

**Theorem 3.2.** Let $\epsilon$ suffice Assumption 2.1, $\mu$ suffice Assumption 2.2 and $\Omega$ suffice Assumption 2.3. Thence $A_X(\lambda)$ is Fredholm if and only if $\lambda \in \mathbb{C} \setminus \{0\}$.

If in addition Assumption 2.4 holds true, then $\sigma(A(\cdot)) \subset \mathbb{R}$ and $\sigma(A(\cdot)) \setminus \{0\}$ consists of an at most countable set of eigenvalues with finite algebraic multiplicity which have no accumulation point in $\mathbb{R} \setminus \{0\}$.

**Proof.** The first statement follows from Theorem 3.2 and Corollary 3.4 of [13]. The second statement can be seen as in the proof of Corollary 3.3 of [13]. \hfill $\Box$

From (27) or (28) we recognize that any eigenfunction $u \in X$ satisfies $P_{W_1} u = w_2 = 0$. Hence to study the eigenvalues of $A_X(\cdot)$ it suffices to study

\begin{align}
(P_V + P_{W_1})A_X(\lambda)|_{V \oplus W_1} \\
\sim \begin{pmatrix}
P_V (A_c - \omega^2 A_c - \lambda A_{tr})|_V \\
-\lambda P_{W_1} A_{tr}|_{W_1} \\
-\lambda P_{W_1} A_{tr}|_V \\
-\lambda P_{W_1} A_{tr}|_{W_1} \\
-\lambda P_{W_1} A_{tr}|_V \\
-\omega^2 P_{W_2} A_{tr}|_{W_2}
\end{pmatrix}
\end{align}

4. **Spectrum in the neighborhood of zero**

First, we establish in Theorem 4.4 the absence of eigenvalues of $A_X(\cdot)$ in $(0,c)$ for sufficiently small $c > 0$. Later on in Theorem 4.7, we establish the existence of an infinite sequence of negative eigenvalues of $A_X(\cdot)$ which accumulate at zero.

4.1. **Spectrum right of zero.** We will require in this section the following additional assumption.

**Assumption 4.1** ($\omega^2$ is no Neumann eigenvalue).

\begin{align}
P_V A_c|_V - \omega^2 P_V A_c|_V \in L(V) \quad \text{is bijective.}
\end{align}

Due to Assumption 4.1 we know that $P_V (A_c - \omega^2 A_c)|_V$ is invertible. Thus by a Neumann series argument $P_V (A_c - \omega^2 A_c - \lambda A_{tr})|_V \in L(V)$ is invertible too for all

\begin{align}
|\lambda| < \frac{1}{\|(P_V (A_c - \omega^2 A_c)|_V)^{-1} P_V A_{tr}|_V\|_{L(V)}}
\end{align}
and hence it holds
\begin{equation}
\| (P_V(A_c - \omega^2 A_c - \lambda A_{tr})|_{V})^{-1} \|_{L(V)} \leq \frac{1}{1 - \lambda \| (P_V(A_c - \omega^2 A_c)|_{V})^{-1} P_V A_{tr} |_{V} \|_{L(V)}}
\end{equation}

For \( \lambda \) satisfying (30) we build the Schur-complement of \( (P_V + P_{W_1}) A_X(\lambda)|_{V \oplus W_1} \) with respect to \( P_V u = v \):
\begin{align}
(32a) & \quad A_{W_1}(\lambda) := -\omega^2 P_{W_1} A_c|_{W_1} - \lambda (P_{W_1} A_{tr}|_{W_1} + H_{W_1}(\lambda)) \in L(W_1), \\
(32b) & \quad H_{W_1}(\lambda) := \lambda P_{W_1} A_{tr}(P_V(A_c - \omega^2 A_c - \lambda A_{tr})|_{V})^{-1} P_V A_{tr}|_{W_1} \in L(W_1).
\end{align}

It is straight forward to see, that for \( \lambda \) satisfying (30), \( \lambda \) is an eigenvalue to \( A_X(\cdot) \) if and only if \( \lambda \) is an eigenvalue to \( A_{W_1}(\cdot) \). Hence to study the eigenvalues of \( A_X(\cdot) \) in a neighborhood of zero, it completely suffices to study the eigenvalues of \( A_{W_1}(\cdot) \) in a neighborhood of zero. For
\begin{equation}
|\lambda| < \frac{1}{2 \| (P_V(A_c - \omega^2 A_c)|_{V})^{-1} P_V A_{tr} |_{V} \|_{L(V)}}
\end{equation}
we deduce
\begin{equation}
\| H_{W_1}(\lambda) \|_{L(W_1)} \leq 2 \| P_V |_{L(X)} \| \| P_W |_{L(X)} \| \| A_{tr} \|_{L(X)}^2.
\end{equation}
Let
\begin{equation}
B_{\lambda} \in L\left( X, L^2_\nu(\partial \Omega) \right): u \mapsto \text{tr}_{\nu \times} u
\end{equation}
so that
\begin{equation}
A_{tr} = B^*_\lambda B_{\lambda}.
\end{equation}

**Lemma 4.2.** Let Assumptions 2.1 hold true. Hence \( P_{W_1} A_{tr}|_{W_1} \) is strictly positive definite, i.e.
\begin{equation}
\inf_{w_1 \in W_1 \setminus \{0\}} \frac{(P_{W_1} A_{tr}|_{W_1} w_1, w_1)_X}{\| w_1 \|_X^2} > 0.
\end{equation}

**Proof.** \( A_{tr} \) is selfadjoint and positive semi definite by (36) and hence so is \( P_{W_1} A_{tr}|_{W_1} \). \( P_{W_1} A_{tr}|_{W_1} \) is weakly coercive by Lemma 2.8 and hence coercive by Lemma 2.8 and \( \text{curl}(P_{W_1} A_{tr}|_{W_1}) \) implies \( w_1 \in W_2 \) and hence \( w_1 = 0 \). Since \( P_{W_1} A_{tr}|_{W_1} \) is selfadjoint, positive semi definite and bijective, it is already strictly positive definite. \( \Box \)

**Lemma 4.3.** Let Assumptions 2.1, 2.2, 2.3 and 4.1 hold true. Hence there exists \( c_0 > 0 \) so that \( P_{W_1} A_{tr}|_{W_1} + H_{W_1}(\lambda) \) is strictly positive definite, i.e.
\begin{equation}
\inf_{w_1 \in W_1 \setminus \{0\}} \frac{(P_{W_1} A_{tr}|_{W_1} + H_{W_1}(\lambda) w_1, w_1)_X}{\| w_1 \|_X^2} > 0,
\end{equation}
for each \( \lambda \in (-c_0, c_0) \).

**Proof.** It is straight forward to see that \( H_{W_1}(\lambda) \) selfadjoint for \( \lambda \in \mathbb{R} \) satisfying (30). The inverse triangle inequality, Lemma 4.2 and (33), (34) yield the claim. \( \Box \)

**Theorem 4.4.** Let Assumptions 2.1, 2.2, 2.3, 2.4 and 4.1 hold true and \( c_0 \) be as in Lemma 4.3. Hence \( \sigma(A_X(\cdot)) \cap (0, c_0) = \emptyset \).

**Proof.** For \( \lambda \in (0, c_0) \), we can build the Schur complement \( A_{W_1}(\lambda) \) of \( A_X(\lambda) \) with respect to \( P_V u = v \) and \( A_X(\lambda) \) is bijective if and only if \( A_{W_1}(\lambda) \) is so. It follows from the definition (32a) of \( A_{W_1}(\lambda) \) and Lemma 4.3 that \( A_{W_1}(\lambda) \) is strictly positive definite for \( \lambda \in (0, c_0) \) and hence bijective. \( \Box \)
4.2. Spectrum left of zero. To study the eigenvalues of $A_{W_1}(\cdot)$ in $(-c_0,0)$ we introduce
\begin{equation}
A_{W_1}(\tau,\lambda) := -\omega^2 P_{W_1} A_{\tau}|_{W_1} - \tau (P_{W_1} A_{\tau}|_{W_1} + H_{W_1}(\lambda)).
\end{equation}
We notice that $\lambda \in (-c_0,0)$ is an eigenvalue of $A_{W_1}(\cdot)$, if and only if $\tau$ is an eigenvalue of $A_{W_1}(\cdot,\lambda)$ and $\tau = \lambda$. We prove the existence of infinite eigenvalues of $A_{W_1}(\cdot)$ in $(-c_0,0)$ by the fixed point technique outlined in [8].

Lemma 4.5. Let Assumptions 2.1, 2.2, 2.3, 2.4 and 4.1 hold true and $c_0$ be as in Lemma 4.3. Let $\lambda \in (-c_0,c_0)$. The spectrum of $A_{W_1}(\cdot,\lambda)$ consists of $\sigma_{\text{ess}}(A_{W_1}(\cdot,\lambda)) = \{0\}$ and an infinite sequence of negative eigenvalues $(\tau_n(\lambda))_{n \in \mathbb{N}}$ which accumulate at zero.

Proof. Due to Lemma 4.3 $(P_{W_1} A_{\tau}|_{W_1} + H_{W_1}(\lambda))^{-1/2}$ is well defined and selfadjoint. It holds $\dim W_1 = \infty$ due to (22). The spectra of $A_{W_1}(\cdot,\lambda)$ and $(P_{W_1} A_{\tau}|_{W_1} + H_{W_1}(\lambda))^{-1/2} A_{W_1}(\cdot,\lambda)(P_{W_1} A_{\tau}|_{W_1} + H_{W_1}(\lambda))^{-1/2} = -\omega^2 (P_{W_1} A_{\tau}|_{W_1} + H_{W_1}(\lambda))^{-1/2} P_{W_1} A_{\tau}|_{W_1} (P_{W_1} A_{\tau}|_{W_1} + H_{W_1}(\lambda))^{-1/2} - \lambda W_1$ coincide. The latter is the pencil of a standard eigenvalue problem for a compact selfadjoint non-positive injective operator on an infinite dimensional Hilbert space and respective properties follow. \hfill \Box

Lemma 4.6. Let Assumptions 2.1, 2.2, 2.3, 2.4 and 4.1 hold true and $c_0$ be as in Lemma 4.3. Let the sequence of negative eigenvalues $(\tau_n(\lambda))_{n \in \mathbb{N}}$ to the operator function $A_{W_1}(\cdot,\lambda)$ be ordered non-decreasingly with multiplicity taken into account. The function $(-c_0,c_0) \to \mathbb{R} : \lambda \mapsto \tau_n(\lambda)$ is continuous for each $n \in \mathbb{N}$.

Proof. Follows from the ordering of $(\tau_n(\lambda))_{n \in \mathbb{N}}$ and [15, § 3] or [19, Proposition 5.4]. \hfill \Box

Theorem 4.7. Let Assumptions 2.1, 2.2, 2.3, 2.4 and 4.1 hold true. Then there exists an infinite sequence $(\lambda_n)_{n \in \mathbb{N}}$ of negative eigenvalues to $A_X(\cdot)$ which accumulate at zero.

Proof. Let $(\tau_n(\lambda))_{n \in \mathbb{N}}$ be as in Lemma 4.6. Let $\lambda \in (-c_0,0)$. Let $n_1 \in \mathbb{N}$ be so that $\lambda < \tau_{n_1}(\lambda)$. Consider the function $f_1(t) := \tau_{n_1}(t) - t$. It hold: $f_1$ is continuous on $(-c_0,c_0)$ due to Lemma 4.6, $f_1(\lambda) > 0$ and $f_1(0) = \tau_{n_1}(0) < 0$. It follows from the Intermediate Value Theorem that there exists $\lambda_1 \in (\lambda,0)$ with $f_1(\lambda_1) = 0$, i.e. $\lambda_1$ is an eigenvalue to $A_{W_1}(\cdot)$. Let now $\lambda \in (\lambda_1,0)$ and $n_2 \in \mathbb{N}$ be so that $\lambda < \tau_{n_2}(\lambda)$. We can repeat the former procedure to construct a second eigenvalue $\lambda_2 \in (\lambda_1,0)$ to $A_{W_1}(\cdot)$. Since $\lambda_2 \in (\lambda_1,0)$, $\lambda_2$ is distinct from $\lambda_1$. We can repeat the former procedure inductively to construct a sequence $(\lambda_n \in (-c_0,0))_{n \in \mathbb{N}}$ of pairwise distinct eigenvalues to $A_{W_1}(\cdot)$.

As already discussed, the spectra of $A_{W_1}(\cdot)$ and $A_X(\cdot)$ coincide on the ball (30). Since $[-c_0,0]$ is compact and the sequence $(\lambda_n \in (-c_0,0))_{n \in \mathbb{N}}$ has an infinite index set, $(\lambda_n)_{n \in \mathbb{N}}$ admits a cluster point in $[-c_0,0]$. Due to Theorem 3.2 $\sigma(A_X(\cdot))$ admits no cluster points in $C \setminus \{0\}$. Thus $(\lambda_n)_{n \in \mathbb{N}}$ accumulate at zero. The claim is proven. \hfill \Box

5. Spectrum in the neighborhood of infinity

First, we establish in Theorem 5.3 the absence of eigenvalues of $A_X(\cdot)$ in the interval $(-\infty,-c)$ for sufficiently large $c > 0$. Later on in Theorem 5.15, we establish the existence of an infinite sequence of positive eigenvalues of $A_X(\cdot)$ which accumulate at $+\infty$. 
5.1. The spectrum near negative infinity. We require the following additional assumption for Theorem 5.3.

**Assumption 5.1** \((\omega^2)\) is not a Dirichlet eigenvalue. There exists no non-trivial solution \(u \in H(\text{curl}, tr^0_{\nu,x}; \Omega)\) to \(\text{curl} \mu^{-1} \text{curl} u - \omega^2 eu = 0\) in \(\Omega\).

**Lemma 5.2** (Nitsche penalty technique). Let Assumptions 2.1, 2.2, 2.3 hold true. Let \(f \in L^2(\Omega)\) and \(u \in H(\text{curl}, tr^0_{\nu,x}; \Omega)\) be the solution to \(\text{curl} \mu^{-1} \text{curl} u + eu = f\) in \(\Omega\). Let \(\lambda > 0\) and \(u_{\lambda} \in H(\text{curl}, tr_{\nu,x}; \Omega)\) be the solution to
\[
(\mu^{-1} \text{curl} u_{\lambda}, \text{curl} u')L^2(\Omega) + \langle \text{curl} u, u' \rangle_{L^2(\Omega)} = \langle f, u' \rangle_{L^2(\Omega)}
\]
for all \(u' \in H(\text{curl}, tr_{\nu,x}; \Omega)\). Then there exist \(C, \lambda_0 > 0\) so that
\[
\|u - u_{\lambda}\|_{H(\text{curl}, tr_{\nu,x}; \Omega)} \leq C/\lambda
\]
for all \(\lambda > \lambda_0\).

**Proof.** We are not aware of a direct appropriate reference for this lemma. Although we believe that the technique applied in this proof is common knowledge. We introduce mixed equations for \(u\) and \(u_{\lambda}\) as e.g. in [20] as follows. Let \(f \in X\) be so that \((f, u')_X = (f, u')_{L^2(\Omega)}\) for all \(u' \in X\). Due to \(u \in H(\text{curl}, tr^0_{\nu,x}; \Omega)\) and Assumption 2.2 it follows \(\phi := \nu \times tr_{\nu,x} \mu^{-1} \text{curl} u \in L^2(\partial \Omega)\). It holds \(\phi_{\lambda} := \nu \times tr_{\nu,x} \mu^{-1} \text{curl} u_{\lambda} = \lambda tr_{\nu,x} u_{\lambda} \in L^2(\partial \Omega)\). Integration by parts yields that \((u, \phi), (u_{\lambda}, \phi_{\lambda}) \in X \times L^2(\partial \Omega)\)
solve
\[
\begin{pmatrix}
A_c + A_c & B_{tr}^* \\
B_{tr} & 0
\end{pmatrix}
\begin{pmatrix}
u \\
\phi
\end{pmatrix}
= \begin{pmatrix}
\langle f, \phi \rangle \\
0
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
A_c + A_c & B_{tr}^* \\
B_{tr} & -\lambda^{-1} I_{L^2(\partial \Omega)}
\end{pmatrix}
\begin{pmatrix}
u_{\lambda} \\
\phi_{\lambda}
\end{pmatrix}
= \begin{pmatrix}
\langle f, \phi \rangle \\
0
\end{pmatrix}
\]
respectively. Both (40) and (41) are stable saddle point problems [3, Theorem 4.3.1]. Since (41) is a perturbation of (40) by magnitude \(\lambda^{-1}\), the claim follows. \(\square\)

**Theorem 5.3.** Let Assumptions 2.1, 2.2, 2.3, 2.4 and 5.1 hold true. Thence there exists \(c > 0\) so that \(A_X(\lambda)\) is bijective for all \(\lambda \in (-\infty, -c)\).

**Proof.** Assume the contrary. Thus there exists a sequence \((\lambda_n < 0)_{n \in \mathbb{N}}\) with \(\lambda_n \to -\infty\), so that \(A_X(\lambda_n)\) is not bijective. Due Theorem 3.2 \((\lambda_n)_{n \in \mathbb{N}}\) are eigenvalues of \(A_X(\cdot)\). Hence let \((u_n \in X)_{n \in \mathbb{N}}\) be a corresponding sequence of normalized eigenfunctions: \(A_X(\lambda_n)u_n = 0\) and \(\|u_n\|_X = 1\) for each \(n \in \mathbb{N}\). It follows
\[
u_n = (\omega^2 + 1)(A_c + A_c + |\lambda_n|A_{tr})^{-1}A_{tr}u_n.
\]
As already discussed at the end of Section 3, it holds \(u_n \in V \oplus W_1\) for each \(n \in \mathbb{N}\). Denote \(E \in L(X, L^2(\Omega))\) the embedding operator and \(M_c \in L(\text{curl}L^2(\Omega))\) the multiplication operator with symbol \(c\). Thus \(A_c = E^*M_cE\). Due to Lemma 2.8 there exist \(f \in L^2(\Omega)\) and a subsequence \((n(m))_{m \in \mathbb{N}}\) so that \(\lim_{m \in \mathbb{N}} Eu_{n(m)} = f\). Let \(u \in H(\text{curl}, tr^0_{\nu,x}; \Omega)\) be the solution to \(\text{curl} \mu^{-1} \text{curl} u + eu = f\) in \(\Omega\). It follows from Lemma 5.2 and (42) that \(\lim_{m \in \mathbb{N}} u_{n(m)} = (\omega^2 + 1)u\) in \(X\). Since
\[
\text{curl} \mu^{-1} \text{curl} u_{n(m)} - \omega^2 eu_{n(m)} = 0 \quad \text{in} \ \Omega
\]
for each \(m \in \mathbb{N}\), it follows that
\[
\text{curl} \mu^{-1} \text{curl} u - \omega^2 eu = 0 \quad \text{in} \ \Omega
\]
as well. Due to Assumption 5.1 it holds \(u = 0\), which is a contradiction to \(\|u_{n(m)}\|_X = 1\) for each \(m \in \mathbb{N}\). The claim is proven. \(\square\)
5.2. The spectrum near positive infinity. $P_{W_1}A_{tr}|W_1 \in L(W_1)$ is strictly positive definite due to Lemma 4.2. Hence there exists $c_\infty > 0$ so that

\begin{equation}
(P_{W_1} \omega^2 A + \lambda A_{tr})|_{W_1} = \lambda P_{W_1} \omega^2 \lambda^{-1} A + A_{tr})|_{W_1}
\end{equation}

is coercive and thus bijective for each $\lambda \in \mathbb{C}$ with $|\lambda| > c_\infty$. (Since $A_t$ is positive semi definite, it follows even that $P_{W_1} \omega^2 A + \lambda A_{tr})|_{W_1}$ is coercive for each $\lambda \in \mathbb{C} \setminus \mathbb{R}^-$. However, we will not use this fact.) Hence for $|\lambda| > c_\infty$ we build and study the Schur complement of $(P_{W_1} + P_{W_1}) A_X(\lambda)|_{V \in W_1}$ with respect to $P_{W_1}u = w_1$:

\begin{align}
A_V(\lambda) &:= P_V(A_e - \omega^2 A_s)|_V - \lambda K_V(\lambda) \in L(V), \\
K_V(\lambda) &:= P_V(A_{tr} - A_{tr} S_V(\lambda) P_{W_1} A_{tr})|_V \in L(V), \\
S_V(\lambda) &:= \left(P_{W_1} \omega^2 \lambda^{-1} A + A_{tr})|_{W_1} \right)^{-1} \in L(W_1).
\end{align}

It is straightforward to see, that for $\lambda$ satisfying $|\lambda| > c_\infty$, $\lambda$ is an eigenvalue to $A_X(\cdot)$ if and only if $\lambda$ is an eigenvalue to $A_V(\cdot)$. Hence to study the eigenvalues of $A_X(\cdot)$ in a neighborhood of infinity, it completely suffices to study the eigenvalues of $A_V(\cdot)$ in a neighborhood of infinity. It will be more convenient to work with $\lambda^{-1}$ instead of $\lambda$. Hence let

\begin{align}
\hat{A}_V(\hat{\lambda}) &:= \hat{\lambda} A_V(\hat{\lambda}^{-1}) = \hat{\lambda} P_V(A_e - \omega^2 A_s)|_V - \hat{K}_V(\hat{\lambda}) \in L(V), \\
\hat{K}_V(\hat{\lambda}) &:= K_V(\hat{\lambda}^{-1}) = P_V(A_{tr} - A_{tr} S_V(\hat{\lambda}) P_{W_1} A_{tr})|_V \in L(V), \\
\hat{S}_V(\hat{\lambda}) &:= S_V(\hat{\lambda}^{-1}) = \left(P_{W_1} \omega^2 \hat{\lambda}^{-1} A + A_{tr})|_{W_1} \right)^{-1} \in L(W_1),
\end{align}

for $\hat{\lambda} \in \mathbb{C}$ with $|\hat{\lambda}| < c_\infty^{-1}$. Again, it is straightforward to see that $\hat{\lambda}$ with $0 < |\hat{\lambda}| < c_\infty^{-1}$ is an eigenvalue to $A_V(\cdot)$ if and only if $\hat{\lambda}^{-1}$ with $|\hat{\lambda}^{-1}| > c_\infty$ is an eigenvalue to $A_V(\cdot)$. Thus we study the eigenvalues of $A_V(\cdot)$ in the ball

\begin{equation}
B_{c_\infty^{-1}} := \{ z \in \mathbb{C} : |z| < c_\infty^{-1} \}.
\end{equation}

To this end we introduce

\begin{equation}
\hat{A}_V(\hat{\tau}, \hat{\lambda}) := \hat{\tau} P_V(A_e - \omega^2 A_s)|_V - \hat{K}_V(\hat{\lambda})
\end{equation}

We note that $\hat{\lambda} \in B_{c_\infty^{-1}}$ is an eigenvalue of $\hat{A}_V(\cdot)$, if and only if $\hat{\tau}$ is an eigenvalue of $\hat{A}_V(\cdot)$ and $\hat{\tau} = \hat{\lambda} \in B_{c_\infty^{-1}}$.

We would like to proceed as in Section 4. Operator $\hat{K}_V(\hat{\lambda})$ is compact due Lemma 2.7. However different to Section 4, $P_V(A_e - \omega^2 A_s)|_V$ is (for arbitrary $\omega > 0$) not definite! Moreover, $\hat{K}_V(\hat{\lambda})$ is not injective! Indeed $\{ \hat{f} : f \in (C^\infty_0(\Omega \setminus \Omega)\} \subset \ker \hat{K}_V(\hat{\lambda})$. Therefore, we introduce the abstract Lemma 5.4. Subsequently, we prove that the conditions of Lemma 5.4 are satisfied and the lemma can be employed for our particular application. We derive the results aimed at in Lemma 5.13 and consequently continue the analysis in the same manner as in Section 4.

**Lemma 5.4.** Let $Y$ be a separable Hilbert space. Let $G \in L(Y)$ be compact, selfadjoint and $I + G$ be bijective. Let $K \in L(Y)$ be compact, selfadjoint, positive semi definite and so that $\ker K = \ker (K^{1/2}(I + G)K^{1/2})$ and $\dim(\ker K)^+ = \infty$. Let $P_{(\ker K)^+}$ be the orthogonal projection onto $(\ker K)^+$ and $P_{(\ker K)^+}(I + G)|_{(\ker K)^+}$ be bijective.

Then the spectra of $(I + G)K$ and $K^{1/2}(I + G)K^{1/2}$ coincide and consist of the essential spectrum $\{0\}$ and an infinite sequence $(\tau_n \in \mathbb{R})$ of non-zero eigenvalues. Apart from a finite set all $(\tau_n)_{n \in \mathbb{N}}$ are positive and it holds $\lim_{n \in \mathbb{N}} \tau_n = 0$.

**Proof.** 1. Step: If $(\tau, y) \in \mathbb{C} \times \{0\} \times Y \setminus \{0\}$ solves

$$
(\tau I - (I + G)K)y = 0,
$$

then...
then $K^{1/2}y \neq 0$ and

$$0 = K^{1/2} \left( \tau I - (I + G)K \right) y = \left( \tau I - K^{1/2}(I + G)K^{1/2} \right) K^{1/2} y.$$ 

Vice-versa, if $(\tau, y') \in \mathbb{C} \setminus \{0\} \times Y \setminus \{0\}$ solves

$$\left( \tau I - K^{1/2}(I + G)K^{1/2} \right) y' = 0,$$

then $(I + G)K^{1/2}y' \neq 0$ and

$$0 = \left( I + G \right) K^{1/2} \left( \tau I - K^{1/2}(I + G)K^{1/2} \right) y' = \left( \tau I - (I + G)K \right) (I + G)K^{1/2} y'.$$

By assumption, $(I + G)K y = 0$ if and only if $K^{1/2}(I + G)K^{1/2} y = 0$. Thus the spectra of $(I + G)K$ and $K^{1/2}(I + G)K^{1/2}$ coincide.

2. Step: Since $K^{1/2}(I + G)K^{1/2}$ is compact and selfadjoint and $Y$ is separable with $\dim Y \geq \dim(\ker K)^+ = \infty$, the Spectral Theorem for compact, selfadjoint operators yields: The spectrum of $K^{1/2}(I + G)K^{1/2}$ consists of the essential spectrum $\{0\}$ and an infinite sequence of eigenvalues $(\tau_n \in \mathbb{R})_{n \in \mathbb{N}}$ (with multiplicity taken into account), $\lim_{n \in \mathbb{N}} \tau_n = 0$ and there exists an orthonormal basis $(y_n)_{n \in \mathbb{N}}$ of corresponding eigenvalues. Due to $\dim(\ker K)^+ = \infty$, there exists an infinite index set $\mathbb{M} \subset \mathbb{N}$ so that $\tau_m \neq 0$ for each $m \in \mathbb{M}$.

3. Step: It remains to prove that all $(\tau_m)_{m \in \mathbb{M}}$ apart from a finite set are positive. To this end we apply a technique which is inspired by [17, §3]. Let

$$\tilde{Y} := \overline{\text{span}} \{y_m : m \in \mathbb{M}\} = (\ker K)^+ = (\ker K)^+$$

and denote $P_{\tilde{Y}}$ the orthogonal projection onto $\tilde{Y}$. We note that for each $y \in Y$, $y^0 \in \ker K$ it holds

$$\langle K^{1/2} y, y^0 \rangle_Y = \langle y, K^{1/2} y^0 \rangle_Y = 0.$$

Thus ran $K^{1/2} \subset (\ker K)^{\bot} = \tilde{Y} = (\ker K)^\bot$ and so $(\tau I + K)^{1/2} \tilde{Y} \subset \tilde{Y}$. Let $G = G_+ - G_-$ so that $G_+$ and $G_-$ are compact, selfadjoint and positive semi definite, i.e. a decomposition of $G$ in the positive and the negative part. For $\tau > 0$ we compute

$$\left( \tau I + K^{1/2}(I + G)K^{1/2} \right) |_{\tilde{Y}} = \left( \tau I + K + K^{1/2}GK^{1/2} \right) |_{\tilde{Y}}$$

$$= \left( \tau I + K \right)^{1/2} \left( I - (P_{\tilde{Y}}(\tau I + K))^{-1/2} \right) K^{1/2} \left( G_{+}^{1/2}G_{+}^{1/2} - G_{-}^{1/2}G_{-}^{1/2} \right) K^{1/2} \left( P_{\tilde{Y}}(\tau I + K)(\tau I + K)^{-1/2} \right) (\tau I + K)^{1/2} |_{\tilde{Y}}.$$

By means of the Spectral Theorem for compact, selfadjoint operators we deduce that $(P_{\tilde{Y}}(\tau I + K))^{-1/2} K^{1/2}$ converges point-wise to $P_{\tilde{Y}}$ for $\tau \to 0+$. Since $G_{\pm}^{1/2}$ is compact it follows that $(P_{\tilde{Y}}(\tau I + K))^{-1/2} K^{1/2} G_{\pm}^{1/2}$ converges to $P_{\tilde{Y}} G_{\pm}^{1/2}$ in $L(Y)$ for $\tau \to 0+$. Hence

$$\left( P_{\tilde{Y}}(\tau I + K)(\tau I + K)^{-1/2} G_{\pm}^{1/2} \right)^* = G_{\pm}^{1/2} K^{1/2} (P_{\tilde{Y}}(\tau I + K)(\tau I + K)^{-1/2}) P_{\tilde{Y}}$$

converges to $(P_{\tilde{Y}} G_{\pm}^{1/2})^* = G_{\pm}^{1/2} P_{\tilde{Y}}$ in $L(Y)$. Thus

$$P_{\tilde{Y}} \left( I - (P_{\tilde{Y}}(\tau I + K)(\tau I + K)^{-1/2}) K^{1/2} G_{\pm}^{1/2} (P_{\tilde{Y}}(\tau I + K)(\tau I + K)^{-1/2}) \right) |_{\tilde{Y}}$$

converges in norm to $P_{\tilde{Y}} (I - G)|_{\tilde{Y}}$. Hence there exists $\epsilon > 0$ so that (48) is bijective for all $\tau \in (0, \epsilon)$. Since for each $\tau \in (0, \epsilon)$, $(\tau I + K^{1/2}(I + G)K^{1/2}) |_{\tilde{Y}} \in L(\tilde{Y})$ is a composition of three bijective operators in $L(\tilde{Y})$, it is bijective. Due to $\lim_{m \in \mathbb{M}} \tau_m = 0$ there can only exist a finite number of $m \in \mathbb{M}$ with $\tau_m < 0$. \hfill \Box
Lemma 5.5. Let Assumptions 2.1, 2.2, 2.3 hold true. Then \( \tilde{K}_V(\tilde{\lambda}) \) is compact, selfadjoint and positive semi definite for each \( \tilde{\lambda} \in [0, c\_\infty^{-1}) \). It holds further \( \text{ker} \tilde{K}_V(\tilde{\lambda}) = \text{ker} B_{tr} \) for each \( \tilde{\lambda} \in (0, c\_\infty^{-1}) \).

Proof. Let \( \tilde{\lambda} \in [0, c\_\infty^{-1}) \). \( \tilde{K}_V(\tilde{\lambda}) \) is compact due Lemma 2.7. It follows from the definition of \( \tilde{K}_V(\tilde{\lambda}) \), that \( \tilde{K}_V(\tilde{\lambda}) \) is selfadjoint. Let \( v \in V \) and \( w_1 := \tilde{S}_V(\tilde{\lambda}) P_{W_1} B_{tr}^* B_{tr} v \). We compute

\[
\langle B_{tr} w_1, B_{tr} w_1 \rangle_{L^2_\om} \leq \langle B_{tr} w_1, B_{tr} w_1 \rangle_{L^2_\om} + \omega^2 \tilde{\lambda} (\epsilon v, w_1)_{L^2_\Omega} = \langle (A_{tr} + \omega^2 \tilde{\lambda} A_v) w_1, w_1 \rangle_X \\
= \langle (A_{tr} + \omega^2 \tilde{\lambda} A_v) \tilde{S}_V(\tilde{\lambda}) P_{W_1} B_{tr}^* B_{tr} v, w_1 \rangle_X \\
= \langle B_{tr} v, B_{tr} w_1 \rangle_{L^2_\om} \leq \| B_{tr} v \|_{L^2_\om} \| B_{tr} w_1 \|_{L^2_\om}
\]

and hence \( \| B_{tr} \tilde{S}_V(\tilde{\lambda}) P_{W_1} B_{tr}^* B_{tr} v \|_{L^2_\om} = \| B_{tr} w_1 \|_{L^2_\om} \leq \| B_{tr} v \|_{L^2_\om} \). Thus

\[
\langle B_{tr} \tilde{S}_V(\tilde{\lambda}) P_{W_1} B_{tr}^* B_{tr} v, B_{tr} v \rangle_{L^2_\om} \leq \| B_{tr} \tilde{S}_V(\tilde{\lambda}) P_{W_1} B_{tr}^* B_{tr} v \|_{L^2_\om} \| B_{tr} v \|_{L^2_\om} \leq \| B_{tr} v \|_{L^2_\om} \| B_{tr} v \|_{L^2_\om}.
\]

Hence

\[
\langle \tilde{K}_V(\tilde{\lambda}) v, v \rangle_X = \langle (A_{tr} - A_{tr}) \tilde{S}_V(\tilde{\lambda}) P_{W_1} A_{tr} v, v \rangle_X = \langle B_{tr} v, B_{tr} v \rangle_{L^2_\om} - \langle B_{tr} \tilde{S}_V(\tilde{\lambda}) P_{W_1} B_{tr}^* B_{tr} v, B_{tr} v \rangle_{L^2_\om} \\
\geq 0.
\]

Let \( \tilde{\lambda} \in (0, c\_\infty^{-1}) \). Let \( B_{tr} v \neq 0 \). If \( P_{W_1} B_{tr}^* B_{tr} v = 0 \) it follows \( \tilde{S}_V(\tilde{\lambda}) P_{W_1} B_{tr}^* B_{tr} v = 0 \) and (49) is strict. So let \( P_{W_1} B_{tr}^* B_{tr} v \neq 0 \). It follows \( w_1 \neq 0 \) and hence \( (\epsilon v, w_1)_{L^2_\om} > 0 \). Since \( w_1 \in W_1 \), it also holds \( \| B_{tr} w_1 \|_{L^2_\om} \neq 0 \). In this case \( \| B_{tr} \tilde{S}_V(\tilde{\lambda}) P_{W_1} B_{tr}^* B_{tr} v \|_{L^2_\om} < \| B_{tr} v \|_{L^2_\om} \) and (49) is strict too. Thus \( \tilde{K}_V(\tilde{\lambda}) v \neq 0 \). On the other hand: If \( B_{tr} v = 0 \), then also \( \tilde{K}_V(\tilde{\lambda}) v = 0 \) due to the definition of \( \tilde{K}_V(\tilde{\lambda}) \). Thus \( \text{ker} \tilde{K}_V(\tilde{\lambda}) = \text{ker} B_{tr} \) for each \( \tilde{\lambda} \in (0, c\_\infty^{-1}) \). \( \square \)

Lemma 5.6. Let Assumptions 2.1, 2.2, 2.3 hold true. Then

\[
\tilde{K}_V(0) = B_{tr}^* P_{\nabla_\theta} B_{tr}.
\]

Proof. Let \( P \in L(L^2_\om) \) be the \( L^2_\om \)-orthogonal projection onto the closure of \( \text{ran} B_{tr}|W_1| \). It follows from the definition of \( \tilde{K}_V(0) \) that \( \tilde{K}_V(0) = B_{tr}^* (I - P) B_{tr} \). The claim is proven, if we show that \( \text{ran} B_{tr}|W_1| = \text{curl}_\theta H^1(\om) \). It follows from the definition of \( W_1 \) that \( \text{ran} B_{tr}|W_1| \subset \text{curl}_\theta H^1(\om) \). Let \( \phi \in \text{curl}_\theta H^1(\om) \) and \( \psi \in H^1(\om) \) so that \( \phi = \text{curl}_\theta \psi = \nu \times \nabla_\theta \psi \). Let \( \tilde{w} \in H^1(\Omega) \) solve \( \text{div} \nabla \tilde{w} = 0 \) in \( \Omega \) and \( \text{tr} \tilde{w} = \psi \) at \( \partial \Omega \). With (21) it follow \( \nabla \tilde{w} - P_{W_1} \nabla \tilde{w} =: w \in W_1 \) and \( \text{tr} w = \phi \). Thus \( \text{ran} B_{tr}|W_1| = \text{curl}_\theta H^1(\om) \) and

\[
\tilde{K}_V(0) = B_{tr}^* (I - P) B_{tr} = B_{tr}^* (I - P \nabla_\theta) B_{tr} = B_{tr}^* P_{\nabla_\theta} B_{tr}.
\]

\( \square \)

Lemma 5.7. Let Assumptions 2.1, 2.2, 2.3 hold true. Then

\[
\dim(\text{ker} B_{tr}|V|)^{1/V} = \dim(\text{ker} P_{\nabla_\theta} B_{tr}|V|)^{1/V} = \infty.
\]

Proof. Let \( (f_n)_{n \in \mathbb{N}} \) be an orthonormal basis of \( \text{curl}_\theta H^1(\om) \subset L^2_\om(\om) \). Let \( u_n \in H(\text{curl}; \Omega) \) be so that \( \text{tr}_{\nu; \theta} u_n = f_n \). Hence \( u_n \in X \). It follows

\[
P_{\nabla_\theta} B_{tr}(P_{\nabla_\theta} u_n + \text{ker} P_{\nabla_\theta} B_{tr}|V|) = f_n.
\]
Thus if \( \sum_{n=1}^{N} c_n (P_V u_n + \ker P_{\nabla} B_{tr}|_V) \) would be a non-trivial linear combination of zero in \( V/(\ker P_{\nabla} B_{tr}|_V) \), then \( \sum_{n=1}^{N} c_n f_n \) would be a non-trivial linear combination of zero in \( \nabla_{\partial} H^1(\partial \Omega) \). Hence \( \dim V/(\ker P_{\nabla} B_{tr}|_V) = +\infty \). Since \( \ker B_{tr} \subset \ker P_{\nabla} B_{tr} \) it follows \( \dim V/(\ker B_{tr}|_V) \geq \dim V/(\ker P_{\nabla} B_{tr}|_V) \) and thus the dimension of \( \dim V/(\ker B_{tr}|_V) \) is infinite too. The claim follows from \( \dim V/Z = \dim Z^{-\nu} \) for any closed subspace \( Z \subset V \). \( \square \)

We require the following additional assumption for Lemma 5.9.

**Assumption 5.8** (\( \omega^2 \) is no “Dirichlet” eigenvalue). Let

\[
Z_1 := \{ z \in V : B_{tr} z = 0 \} = H(\text{curl}, \text{div}^0, \text{tr}^0_{\nu \times}, \text{tr}^0_{\nu \cdot} ; \Omega)
\]

and denote \( P_{Z_1} \) the \( X \)-orthogonal projection onto \( Z_1 \). The operator

\[
P_{Z_1} A_c |_{Z_1} - \omega^2 P_{Z_1} A_c |_{Z_1} \in L(Z_1)
\]

is bijective.

**Lemma 5.9.** Let Assumptions 2.1, 2.2, 2.3, 4.1 and 5.8 hold true. Let \( \tilde{\lambda} \in (0, c_{\infty}^{-1}) \). Thence

\[
\ker \left( \tilde{K}_V(\tilde{\lambda}) \right)^{1/2} (P_V (A_c - \omega^2 A_c)|_V)^{-1} \tilde{K}_V(\tilde{\lambda})^{1/2} = \ker \tilde{K}_V(\tilde{\lambda}).
\]

**Proof.** Let \( v \in \ker \left( \tilde{K}_V(\tilde{\lambda}) \right)^{1/2} (P_V (A_c - \omega^2 A_c)|_V)^{-1} \tilde{K}_V(\tilde{\lambda})^{1/2} \) and

\[
z := (P_V (A_c - \omega^2 A_c)|_V)^{-1} \tilde{K}_V(\tilde{\lambda})^{1/2} v.
\]

It follows \( B_{tr} z = 0 \) due to \( \ker K_V(\tilde{\lambda})^{1/2} = \ker K_V(\tilde{\lambda}) \) and Lemma 5.5. Due to the definitions of \( z \) and \( Z_1 \), \( z \in Z_1 \) solves

\[
(P_{Z_1} A_c - \omega^2 P_{Z_1} A_c) z = 0.
\]

It follows from Assumption 5.8 that \( z = 0 \). Thus \( v \in \ker K_V(\tilde{\lambda})^{1/2} = \ker K_V(\tilde{\lambda}) \). \( \square \)

We require the following additional assumption for Lemma 5.11.

**Assumption 5.10** (\( \omega^2 \) is no “hybrid” eigenvalue). Let

\[
Z_2 := \{ z \in V : P_{\nabla_{\partial}} B_{tr} z = 0 \}
\]

and denote \( P_{Z_2} \) the \( X \)-orthogonal projection onto \( Z_2 \). The operator

\[
P_{Z_2} A_c |_{Z_2} - \omega^2 P_{Z_2} A_c |_{Z_2} \in L(Z_2)
\]

is bijective.

**Lemma 5.11.** Let Assumptions 2.1, 2.2, 2.3, 4.1 and 5.10 hold true. Thence

\[
\ker \left( \tilde{K}_V(0) \right)^{1/2} (P_V (A_c - \omega^2 A_c)|_V)^{-1} \tilde{K}_V(0)^{1/2} = \ker \tilde{K}_V(0).
\]

**Proof.** Let \( v \in \ker \left( \tilde{K}_V(0) \right)^{1/2} (P_V (A_c - \omega^2 A_c)|_V)^{-1} \tilde{K}_V(0)^{1/2} \) and

\[
z := (P_V (A_c - \omega^2 A_c)|_V)^{-1} \tilde{K}_V(0)^{1/2} v.
\]

It follows \( P_{\nabla_{\partial}} B_{tr} z = 0 \) due to \( \ker K_V(0)^{1/2} = \ker K_V(0) \) and Lemma 5.6. Due to the definitions of \( z \) and \( Z_2 \), \( z \in Z_2 \) solves

\[
(P_{Z_2} A_c - \omega^2 P_{Z_2} A_c) z = 0.
\]

It follows from Assumption 5.10 that \( z = 0 \). Thus \( v \in \ker K_V(0)^{1/2} = \ker K_V(0) \). \( \square \)

We require the following additional assumption for Lemma 5.13.
Assumption 5.12 \((\omega^2\text{ is no "projected" eigenvalue})\). The operators
\[
P_{Z_1}(P_V(A_c - \omega^2 A_c)|V)^{-1}|Z_1 \in L(Z_1)
\]
and
\[
P_{Z_2}(P_V(A_c - \omega^2 A_c)|V)^{-1}|Z_2 \in L(Z_2)
\]
are bijective.

We note that \(P_{V}A_{c}|V = P_{V}(I - A_{c} - A_{tr})|V\)
and consequently
\[
P_{V}(A_{c} - \omega^2 A_{c})|V)^{-1} = I - P_{V}(A_{c} - \omega^2 A_{c})|V)^{-1}P_{V}((\omega^2 + 1)A_{c} + A_{tr})|V
\]
\[
= I_{V} + G.
\]

Lemma 5.13. \(\text{Let Assumptions 2.1, 2.2, 2.3, 2.4 and 4.1, 5.8, 5.10, 5.12 hold true. Let} \lambda \in [0, c^{-1}_{\omega}]\). The spectrum of \(\hat{A}_V(\cdot, \lambda)\) consists of \(\sigma_{ess}(\hat{A}_V(\cdot, \lambda)) = \{0\}\) and an infinite sequence of non-zero eigenvalues \((\tau_n(\lambda))_{n \in \mathbb{N}}\) with \(\lim_{n \in \mathbb{N}} \tau_n(\lambda) = 0\).

\text{Apart from a finite number, all non-zero eigenvalues} \((\tau_n(\lambda))_{n \in \mathbb{N}}\) are positive.

Proof. We note that \(\hat{A}_V(\cdot, \lambda)\) and \(P_V(A_c - \omega^2 A_c)|V)^{-1}\hat{A}_V(\cdot, \lambda)\) have the very same spectral properties. We aim to apply Lemma 5.4 to
\[
P_{V}(A_{c} - \omega^2 A_{c})|V)^{-1} \hat{A}_V(\cdot, \lambda) = \tau I - (I + G)\hat{K}_V(\lambda)
\]
with \(G\) defined as in (51). \(G\) is compact due to Lemma 2.8 and Lemma 2.7. Since \(P_{V}(A_{c} - \omega^2 A_{c})|V)^{-1}\) and the identity are selfadjoint, the selfadjointness of \(G\) follows from (51). \(I + G\) is bijective due to its definition and Assumption 4.1. \(\hat{K}_V(\lambda)\) is compact, selfadjoint and positive semi definite due to Lemma 5.5. It holds \(\ker(\hat{K}_V(\lambda)^{1/2}(I + G)\hat{K}_V(\lambda)^{1/2}) = \ker \hat{K}_V(\lambda)\) due to Lemma 5.9 and Lemma 5.11.

It holds \(\dim(\ker\hat{K}_V(\lambda))^{+} = \infty\) due to Lemma 5.7. \(P_{\ker \hat{K}_V(\lambda)^{+}}(I + G)|_{\ker \hat{K}_V(\lambda)^{+}}\) is bijective due to Assumption 5.12. Hence the conditions of Lemma 5.4 are satisfied and the claim follows.

Lemma 5.14. \(\text{Let Assumptions 2.1, 2.2, 2.3 and 4.1, 5.8, 5.10, 5.12 hold true. For} \lambda \in [0, c^{-1}_{\omega}]\) let \((\hat{\tau}_n(\lambda))_{n \in \mathbb{N}}\) be a non-increasing ordering with multiplicity taken into account of the positive eigenvalues of \(\hat{A}_V(\cdot, \lambda)\). Hence for each \(n \in \mathbb{N}\) the function \(\hat{\tau}_n^{+} : [0, c^{-1}_{\omega}] \to \mathbb{R}^{+}\) is continuous.

Proof. We note that for each \(n \in \mathbb{N}\) it holds \(\inf_{\lambda \in [0, c^{-1}_{\omega}]} \hat{\tau}_n(\lambda) > 0\): Indeed the existence of \(\tilde{\lambda}_0 \in [0, c^{-1}_{\omega}]\), \(n \in \mathbb{N}\) so that \(\lim_{\lambda \to \tilde{\lambda}_0} \hat{\tau}_n^{+}(\lambda) = 0\) would imply that for \(\hat{\lambda} = \tilde{\lambda}_0\) there would exist only a finite number of positive eigenvalues, which is a contradiction to Lemma 5.13. The continuity of \(\hat{\tau}_n^{+}\) follows with [19, Proposition 5.4]. We note that a delicate part of [19, Proposition 5.4] is the existence of eigenvalues. However, the existence of eigenvalues is already established by Lemma 5.13. We only require the continuity result of [19, Proposition 5.4].

Theorem 5.15. \(\text{Let Assumptions 2.1, 2.2, 2.3, 2.4 and 4.1, 5.8, 5.10, 5.12 hold true. Hence there exists an infinite sequence} (\lambda_n)_{n \in \mathbb{N}} \text{ of positive eigenvalues to} A_X(\cdot) \text{ which accumulate at} +\infty.\)

Proof. Proceed as in the proof of Theorem 4.7.
We conclude with a summary of Theorems 3.2, 4.4, 4.7, 5.3 and 5.15 and some remarks on assumptions and the relation to the modified electromagnetic Stekloff eigenvalue considered in [13], [9].

6.1. Main result. We formulate the individual results of the previous sections in the following proposition.

**Proposition 6.1.** Let Assumptions 2.1, 2.2, 2.3, 2.4 and 4.1, 5.1, 5.8, 5.10, 5.12 be satisfied. Then it hold

\[
\sigma(A_X(\cdot)) = \sigma_{\text{ess}}(A_X(\cdot)) \cup \bigcup_{n \in \mathbb{N}} \{\lambda_n^0\} \cup \bigcup_{n \in \mathbb{N}} \{\lambda_n^{+\infty}\}
\]

and \(\sigma_{\text{ess}}(A_X(\cdot)) = \{0\}\). The sequence \((\lambda_n^0)_{n \in \mathbb{N}}\) consists of pair-wise distinct negative eigenvalues with finite algebraic multiplicity so that \(\lim_{n \in \mathbb{N}} \lambda_n^0 = 0\). The sequence \((\lambda_n^{+\infty})_{n \in \mathbb{N}}\) consists of pair-wise distinct positive eigenvalues with finite algebraic multiplicity so that \(\lim_{n \in \mathbb{N}} \lambda_n^{+\infty} = +\infty\).

**Proof.** Follows from Theorems 3.2, 4.4, 4.7, 5.3 and 5.15. □

6.2. Remarks to the assumptions. The condition in Assumptions 2.1 and 2.2 that \(\mu\) and \(\epsilon\) equal the identity matrix in a neighborhood of the boundary is used to obtain extra regularity of traces. If this extra regularity can be derived by other means, then the mentioned assumption becomes obsolete.

Each of the Assumptions 4.1, 5.1, 5.8, 5.10, 5.12 can be formulated in the following manner: \(Y\) is a Hilbert space, \(A \in L(Y)\) is weakly coercive, \(K(\cdot) : \Lambda \subset \mathbb{C} \rightarrow K(\mathbb{Y})\) is holomorphic and it is imposed that \(A - K(\omega^2)\) is bijective. Consequently for fixed domain \(\Omega\) and fixed material parameters \(\mu^{-1}, \epsilon\) there exists only a countable set of frequencies \(\omega\) for which the Assumptions 4.1, 5.1, 5.8, 5.10, 5.12 are not satisfied (see e.g. [16, Proposition A.8.4]).

6.3. Modified electromagnetic Stekloff eigenvalues. The modified electromagnetic Stekloff eigenvalue problem considered in [13] is to find \((\lambda, u) \in \mathbb{C} \times H(curl; \Omega)\setminus\{0\}\) so that

\[
\mu^{-1} \text{curl} u, \text{curl} u'_{L^2(\Omega)} - \omega^2 (e u, u'_{L^2(\Omega)}) - \lambda(S u, S u')_{L^2(\partial\Omega)} = 0
\]

for all \(u' \in H(curl; \Omega)\) (with \(S\) defined as in (14)). It can easily be seen that the eigenvalue problem decouples with respect to the decomposition \(H(curl; \Omega) = H(curl, \text{div}^e_0, tr_{\nu_e; \Omega}) \oplus \nabla H^1(\Omega)\). Thus the eigenvalue problem can be reformulated to find \((\lambda, u) \in \mathbb{C} \times H(curl, \text{div}^e_0, tr_{\nu_e; \Omega})\setminus\{0\}\) so that

\[
0 = \mu^{-1} \text{curl} u, tr^e_0 u'_{L^2(\Omega)} - \omega^2 (e u, u'_{L^2(\Omega)}) - \lambda(P_{\nu_0} tr_{\nu_e} u, tr_{\nu_e} u')_{L^2(\partial\Omega)}
\]

\[
= (\lambda A_V(\nu^{-1}, 0) u, u')_{L^2(\Omega)}
\]

for all \(u' \in H(curl, \text{div}^e_0, tr_{\nu_e; \Omega})\). Thence if the respective assumptions are satisfied, Lemma 5.13 yields that the spectrum consists of an infinite sequence of eigenvalues \((\lambda_n)_{n \in \mathbb{N}}\) which accumulate only at \(+\infty\). A similar existence result has been reported in [9, Theorem 3.6]. Though it seems to us that the proof of [9, Theorem 3.6] requires \(\text{dim}(\text{ker} T) = +\infty\) which the authors don’t elaborate on.

The former observation admits to interpret the modified electromagnetic Stekloff eigenvalue problem as asymptotic limit of the original electromagnetic Stekloff eigenvalue problem for large eigenvalue parameter \(\lambda\). Though, this doesn’t yield any non-trivial asymptotic statement on the eigenvalues.
We have seen that (at least in the selfadjoint case) the original electromagnetic Stekloff eigenvalue problem yields two kind of spectra. Contrary the modified electromagnetic Stekloff eigenvalue problem yields only one kind of spectrum. This suggests that for inverse scattering applications the original version is more advantageous than the modified version, because it contains more information. Though the approximation of the modified eigenvalue problem is better understood than for the original version [13].

It would be further interesting to consider a far field measurement procedure which relates to a second kind of modified electromagnetic Stekloff eigenvalues. Namely to the spectrum of the asymptotic limit of $A_X(·)$ for small spectral parameter $\lambda$: $A_{W_1}(·,0)$. This eigenvalue problem can also be formulated as to find $(\lambda,u) \in \mathbb{C} \times \{u \in H^1(\Omega): \text{tr}_{\nu \times \nabla u} \in L^2(\partial\Omega)\} \setminus \{0\}$ so that

\begin{align}
(55a) & \quad \text{div} \nabla u = 0 \quad \text{in } \Omega, \\
(55b) & \quad \nu \cdot \epsilon \nabla u - \lambda \Delta_{\partial\Omega} u = 0 \quad \text{at } \partial\Omega.
\end{align}

The special feature of this eigenvalue problem is that it is independent of $\mu$ and $\omega$!

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