Solitons of the midpoint mapping and affine curvature

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Abstract. For a polygon $x = (x_j)_{j \in \mathbb{Z}}$ in $\mathbb{R}^n$ we consider the midpoints polygon $(M(x))_j = (x_j + x_{j+1})/2$. We call a polygon a soliton of the midpoints mapping $M$ if its midpoints polygon is the image of the polygon under an invertible affine map. We show that a large class of these polygons lie on an orbit of a one-parameter subgroup of the affine group acting on $\mathbb{R}^n$. These smooth curves are also characterized as solutions of the differential equation $\dot{c}(t) = Bc(t) + d$ for a matrix $B$ and a vector $d$. For $n = 2$ these curves are curves of constant generalized-affine curvature $k_{ga} = k_{ga}(B)$ depending on $B$ parametrized by generalized-affine arc length unless they are parametrization of a parabola, an ellipse, or a hyperbola.

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1. Introduction

We consider an infinite polygon $(x_j)_{j \in \mathbb{Z}}$ given by its vertices $x_j \in \mathbb{R}^n$ in an $n$-dimensional real vector space $\mathbb{R}^n$ resp. an $n$-dimensional affine space $\mathbb{A}^n$ modelled after $\mathbb{R}^n$. For a parameter $\alpha \in (0, 1)$ we introduce the polygon $M_\alpha(x)$ whose vertices are given by

$$(M_\alpha(x))_j := (1 - \alpha)x_j + \alpha x_{j+1}.$$ 

For $\alpha = 1/2$ this defines the midpoints polygon $M(x) = M_{1/2}(x)$. On the space $\mathcal{P} = \mathcal{P}(\mathbb{R}^n)$ of polygons in $\mathbb{R}^n$ this defines a discrete curve shortening process $M_\alpha : \mathcal{P} \rightarrow \mathcal{P}$, already considered by Darboux [4] in the case of a closed resp.
periodic polygon. For a discussion of this elementary geometric construction see Berlekamp et al. [1].

The mapping $M_\alpha$ is invariant under the canonical action of the affine group. The affine group $\text{Aff}(n)$ in dimension $n$ is the set of affine maps $(A, b) : \mathbb{R}^n \longrightarrow \mathbb{R}^n, x \mapsto Ax + b$. Here $A \in \text{Gl}(n)$ is an invertible matrix and $b \in \mathbb{R}^n$ a vector. The translations $x \mapsto x + b$ determined by a vector $b$ form a subgroup isomorphic to $\mathbb{R}^n$. Let $\alpha \in (0, 1)$. We call a polygon $x_j$ a soliton for the process $M_\alpha$ (or affinely invariant under $M_\alpha$) if there is an affine map $(A, b) \in \text{Aff}(n)$ such that

$$\left(M_\alpha(x)\right)_j = Ax_j + b$$

for all $j \in \mathbb{Z}$. In Theorem 1 we describe these solitons explicitly and discuss under which assumptions they lie on the orbit of a one-parameter subgroup of the affine group acting canonically on $\mathbb{R}^n$. We call a smooth curve $c : \mathbb{R} \longrightarrow \mathbb{R}^n$ a soliton of the mapping $M_\alpha$ resp. invariant under the mapping $M_\alpha$ if there is for some $\epsilon > 0$ a smooth mapping $s \in (-\epsilon, \epsilon) \longmapsto (A(s), b(s)) \in \text{Aff}(n)$ such that for all $s \in (-\epsilon, \epsilon)$ and $t \in \mathbb{R}$:

$$\tilde{c}_s(t) := (1 - \alpha)c(t) + \alpha c(t + s) = A(s)c(t) + b(s).$$

Then for some $t_0 \in \mathbb{R}$ and $s \in (-\epsilon, \epsilon)$ the polygon $x_j = c(js + t_0), j \in \mathbb{Z}$ is a soliton of $M_\alpha$. The parabola is an example of a soliton of $M = M_{1/2}$, cf. Fig. 1 and Example 1, Case (e). We show in Theorem 2 that the smooth curves invariant under $M_\alpha$ coincide with the orbits of a one-parameter subgroup of the affine group $\text{Aff}(n)$ acting canonically on $\mathbb{R}^n$. For $n = 2$ we give a characterization of these curves in terms of the general-affine curvature in Sect. 5.

Figure 1 The parabola $c(t) = (t^2/2, t)$ as soliton of the midpoints map $M$
The authors discussed \textit{solitons}, i.e. curves affinely invariant under the curve shortening process $T : \mathcal{P}(\mathbb{R}^n) \rightarrow \mathcal{P}(\mathbb{R}^n)$ with

$$
(T(x))_j = \frac{1}{4} \left\{ x_{j-1} + 2x_j + x_{j+1} \right\}
$$

in [9]. The solitons of $M = M_{1/2}$ form a subclass of the solitons of $T$, since $(T(x))_j = (M^2(x))_{j-1}$. Instead of the discrete evolution of polygons one can also investigate the evolution of polygons under a linear flow, cf. Viera and Garcia [11] and [9, Sec. 4] or a non-linear flow, cf. Glickenstein and Liang [5].

2. The affine group and systems of linear differential equations of first order

The affine group $\text{Aff}(n)$ is a semidirect product of the general linear group $\text{Gl}(n)$ and the group $\mathbb{R}^n$ of translations. There is a linear representation

$$
(A, b) \in \text{Aff}(n) \rightarrow \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \in \text{Gl}(n+1),
$$

of the affine group in the general linear group $\text{Gl}(n+1)$, cf. [8, Sec. 5.1]. We use the following identification

$$
\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + b \\ 1 \end{pmatrix}.
$$

Hence we can identify the image of a vector $x \in \mathbb{R}^n$ under the affine map $x \mapsto Ax + b$ with the image $\begin{pmatrix} Ax + b \\ 1 \end{pmatrix}$ of the extended vector $\begin{pmatrix} x \\ 1 \end{pmatrix}$. Using this identification we can write down the solution of an inhomogeneous system of linear differential equations with constant coefficients using the power series $F_B(t)$ which we introduce now:

\textbf{Proposition 1.} For a real $(n,n)$-matrix $B \in M_\mathbb{R}(n)$ we denote by $F_B(t) \in M_\mathbb{R}(n)$ the following power series:

$$
F_B(t) = \sum_{k=1}^{\infty} \frac{t^k}{k!} B^{k-1}.
$$

(a) We obtain for its derivative:

$$
\frac{d}{dt} F_B(t) = \exp(Bt) = B F_B(t) + 1.
$$

The function $F_B(t)$ satisfies the following functional equation:

$$
F_B(t+s) = F_B(s) + \exp( Bs ) F_B(t),
$$

resp. for $j \in \mathbb{Z}, j \geq 1$:

$$
F_B(j) = \{ 1 + \exp(B) + \exp(2B) + \cdots + \exp((j-1)B) \} F_B(1) = (\exp(B) - 1)^{-1} (\exp(jB) - 1) F_B(1).
$$
The solution \( c(t) \) of the inhomogeneous system of linear differential equations
\[
\dot{c}(t) = Bc(t) + d
\]
with constant coefficients (i.e. \( B \in \mathbb{M}_\mathbb{R}(n,n) \), \( d \in \mathbb{R}^n \)) and with initial condition \( v = c(0) \) is given by:
\[
c(t) = v + F_B(t)(Bv + d) = \exp(Bt)(v) + F_B(t)(d). \tag{9}
\]

Proof. (a) Equation (6) follows immediately from Eq. (5). Then we compute
\[
\frac{d}{dt} (F_B(t + s) - \exp(Bs)F_B(t)) = \exp(B(t + s)) - \exp(Bs)\exp(Bt) = 0.
\]
Since \( F_B(0) = 0 \) Eq. (7) follows. And this implies Eq. (8).

(b) We can write the solution of the differential equation (8)
\[
\frac{d}{dt} \left( \frac{c(t)}{1} \right) = \left( \frac{Bd}{0} \right) \left( \frac{c(t)}{1} \right)
\]
as follows:
\[
\begin{align*}
\left( \frac{c(t)}{1} \right) &= \exp \left( \left( \frac{Bd}{0} \right) t \right) \left( \frac{v}{1} \right) \\
&= \left( \frac{\exp(Bt)F_B(t)(d)}{0} \right) \left( \frac{v}{1} \right) = \left( \frac{\exp(Bt)(v) + F_B(t)(d)}{1} \right)
\end{align*}
\]
which is Eq. (9). One could also differentiate Eq. (9) and use Eq. (6) □

Remark 1. Equation (2) shows that \( c(t) \) is the orbit
\[
t \in \mathbb{R} \mapsto c(t) = \exp \left( \left( \frac{Bd}{0} \right) t \right) \left( \frac{v}{1} \right) \in \mathbb{R}^n.
\]
of the one-parameter subgroup
\[
t \in \mathbb{R} \mapsto \exp \left( \left( \frac{Bd}{0} \right) t \right) \in \text{Aff}(n)
\]
of the affine group \( \text{Aff}(n) \) acting canonically on \( \mathbb{R}^n \).

3. Polygons invariant under \( M_\alpha \)

Theorem 1. Let \( (A, b) : x \in \mathbb{R}^n \mapsto Ax + b \in \mathbb{R}^n \) be an affine map and \( v \in \mathbb{R}^n \).
Assume that for \( \alpha \in (0,1) \) the value \( 1 - \alpha \) is not an eigenvalue of \( A \), i.e. the matrix \( A_\alpha := \alpha^{-1}(A + (\alpha - 1)1) \) is invertible. Then the following statements hold:

(a) There is a unique polygon \( x \in \mathcal{P}(\mathbb{R}^n) \) with \( x_0 = v \) which is a soliton for \( M_\alpha \) resp. affinely invariant under the mapping \( M_\alpha \) with respect to the affine map \( (A, b) \), cf. Eq. (1). If \( b_\alpha = \alpha^{-1}b \), then for \( j > 0 \):
\[
x_j = A_\alpha^j(v) + A_\alpha^{j-1}(b_\alpha) + \cdots + A_\alpha(b_\alpha) + b_\alpha \\
= v + (A_\alpha^j - 1) \left( v + (A_\alpha - 1)^{-1}(b_\alpha) \right).
\]

(b) The solution \( c(t) \) of the inhomogeneous system of linear differential equations
\[
\dot{c}(t) = Bc(t) + d
\]
with constant coefficients (i.e. \( B \in \mathbb{M}_\mathbb{R}(n,n) \), \( d \in \mathbb{R}^n \)) and with initial condition \( v = c(0) \) is given by:
\[
c(t) = v + F_B(t)(Bv + d) = \exp(Bt)(v) + F_B(t)(d). \tag{9}
\]
and for \( j < 0 \):
\[
x_j = A_j^j(v) - A_j^j(b_\alpha) + \cdots + A_\alpha^{-1}(b_\alpha)
\]
\[
= v + (A_j^j - I) \left( v - (A_\alpha^{-1} - I)^{-1}(A_\alpha^{-1}(b_\alpha)) \right).
\]

(b) If \( A_\alpha = \exp(B_\alpha) \) for a \((n,n)\)-matrix \( B_\alpha \) and if \( b_\alpha = F_{B_\alpha}(1)(d_\alpha) \) for a vector \( d_\alpha \in \mathbb{R}^n \) then the polygon \( x_j \) lies on the smooth curve
\[
c(t) = v + F_{B_\alpha}(t)(B_\alpha v + d_\alpha)
\]
i.e. \( x_j = c(j) \) for all \( j \in \mathbb{Z} \).

Proof. (a) By Eq. (1) we have
\[
(1 - \alpha)x_j + \alpha x_{j+1} = Ax_j + b
\]
for all \( j \in \mathbb{Z} \). Hence the polygon is given by \( x_0 = v \) and the recursion formulae
\[
x_{j+1} = A_\alpha(x_j) + b_\alpha; \quad x_j = A_\alpha^{-1}(x_{j+1} - b_\alpha).
\]
for all \( j \in \mathbb{Z} \). Then Eqs. (11) and (12) follow.

(b) For \( A_\alpha = \exp(B_\alpha); b_\alpha = F_{B_\alpha}(d_\alpha) \) we obtain from Eq. (6) for all \( j \in \mathbb{Z} : \)
\[
A_\alpha - I = B_\alpha F_{B_\alpha}(1) \quad \text{and} \quad A_\alpha^{-1} - I = B_\alpha F_{B_\alpha}(j).
\]
Hence for \( j > 0 \):
\[
x_j = v + (A_j^j - I) \left( v + (A_\alpha^{-1} - I)^{-1}b_\alpha \right)
\]
\[
= v + B_\alpha F_{B_\alpha}(j) \left( v + (B_\alpha F_{B_\alpha}(1))^{-1}(b_\alpha) \right)
\]
\[
= v + F_{B_\alpha}(j)(B_\alpha v + d_\alpha) = c(j).
\]
The functional Eq. (7) for \( F_B(t) \) implies
\[
0 = F_B(0) = F_B(-1 + 1) = F_B(-1) + \exp(-B)F_B(1),
\]
hence
\[
F_B(-1) = -\exp(-B)F_B(1); \quad F_B(-1)^{-1} = -\exp(B)F_B(1)^{-1}.
\]
Note that the matrices \( B, F_B(t), F_B(t)^{-1} \) commute. With this identity we obtain for \( j < 0 \):
\[
x_j = v + (A_j^j - I) \left( v - (A_\alpha^{-1} - I)^{-1}(A_\alpha^{-1}b_\alpha) \right)
\]
\[
= v + B_\alpha F_{B_\alpha}(j) \left( v - (B_\alpha F_{B_\alpha}(1))^{-1}\exp(-B_\alpha)(b_\alpha) \right)
\]
\[
= v + B_\alpha F_{B_\alpha}(j) \left( v - F_{B_\alpha}(1)^{-1}B_\alpha^{-1}\exp(-B_\alpha)F_{B_\alpha}(1)(d_\alpha) \right)
\]
\[
= v + F_{B_\alpha}(j)(B_\alpha v + d_\alpha) = c(j).
\]

\[\square\]

Remark 2. (a) Using the identification Eq. (4) we can write
\[
\left( \begin{array}{c} x_{j+1} \\ 1 \end{array} \right) = \left( \begin{array}{cc} A_\alpha & b_\alpha \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} x_j \\ 1 \end{array} \right); \quad \left( \begin{array}{c} x_j \\ 1 \end{array} \right) = \left( \begin{array}{cc} A_\alpha & b_\alpha \\ 0 & 1 \end{array} \right)^{-1} \left( \begin{array}{c} v \\ 1 \end{array} \right)
\]
for all \( j \in \mathbb{Z} \).
(b) If $A_\alpha = \exp(B_\alpha)$ for a $(n,n)$-matrix $B_\alpha$ and if $b_\alpha = F_{B_\alpha}(1)(d_\alpha)$ for a vector $d_\alpha \in \mathbb{R}^n$ then we obtain from Eq. (10):

$$\left(\frac{c(t)}{1}\right) = \exp\left(\begin{pmatrix} B_\alpha & d_\alpha \\ 0 & 0 \end{pmatrix} t\right) \left(\frac{v}{1}\right) = \left(\frac{\exp(B_\alpha t)(F_{B_\alpha}(t)(d_\alpha))}{1} \right) \left(\frac{v}{1}\right)$$

$$= \left(\frac{\exp(B_\alpha t)(v) + F_{B_\alpha}(t)(d_\alpha)}{1}\right) = \left(\frac{v + F_{B_\alpha}(t)(B_\alpha v + d_\alpha)}{1}\right)$$

Hence $t \in \mathbb{R} \mapsto c(t) \in \mathbb{R}^n$ is the orbit of a one-parameter subgroup of the affine group applied to the vector $v$.

4. Smooth curves invariant under $M_\alpha$

For a smooth curve $c : \mathbb{R} \rightarrow \mathbb{R}^n$ and a parameter $\alpha \in (0,1)$ we define the one-parameter family $\tilde{c}_s : \mathbb{R} \rightarrow \mathbb{R}^n, s \in \mathbb{R}$ by Eq. (2). And we call a smooth curve $c : \mathbb{R} \rightarrow \mathbb{R}^n$ a soliton of the mapping $M_\alpha$ (resp. affinely invariant under $M_\alpha$) if there is $\epsilon > 0$ and a smooth map $\epsilon \in (-\epsilon, \epsilon) \rightarrow (A, b) \in \text{Aff}(n)$ such that

$$\tilde{c}_s(t) = (1 - \alpha)c(t) + \alpha c(t + s) = A(s)(c(t)) + b(s).$$

Then we obtain as an analogue of [9, Thm.1]:

**Theorem 2.** Let $c : \mathbb{R} \rightarrow \mathbb{R}^n$ be a soliton of the mapping $M_\alpha$ satisfying Eq. (14). Assume in addition that for some $t_0 \in \mathbb{R}$ the vectors $\dot{c}(t_0), \ddot{c}(t_0), \ldots, c^{(n)}(t_0)$ are linearly independent.

Then the curve $c$ is the unique solution of the differential equation

$$\dot{c}(t) = Bc(t) + d$$

for $B = \alpha^{-1}A'(0), d = \alpha^{-1}b'(0)$ with initial condition $v = c(0)$.

And $A(s) = (1 - \alpha)\mathbb{1} + \alpha \exp(Bs), b(s) = \alpha F_B(s)(d)$.

Hence the curve $c(t)$ is the orbit of a one-parameter subgroup

$$t \in \mathbb{R} \mapsto B(t) := \exp\left(\begin{pmatrix} B & d \\ 0 & 0 \end{pmatrix} t\right) = (\exp(Bt), F_B(t)(d)) \in \text{Aff}(n)$$

of the affine group, i.e.

$$c(t) = B(t) \left(\begin{array}{c} v \\ 1 \end{array}\right) = v + F_B(t)(Bv + d),$$

cf. Remark 1.

**Remark 3.** For an affine map $(A, b) \in \text{Gl}(n), b \in \mathbb{R}^n$ the linear isomorphism $A$ is called the linear part. For $n = 2$ we discuss the possible normal forms of $A \in \text{Gl}(2)$ resp. the normal forms of the one-parameter subgroup $\exp(tB)$ and of the one-parameter family $A(s) = (1 - \alpha)\mathbb{1} + \exp(Bs)$ introduced in Theorem 2. This will be used in Sect. 5.
1. $A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ for $\lambda, \mu \in \mathbb{R} - \{0\}$, i.e. $A$ is diagonalizable (over $\mathbb{R}$), then $A$ is called scaling, for $\lambda = \mu$ it is called homothety. For an endomorphism $B$ which is diagonalizable over $\mathbb{R}$ the one-parameter subgroup $B(t) = \exp(Bt)$ as well as the one-parameter family $A(s) = (1 - \alpha)I + \alpha \exp(Bs)$ consists of scalings.

2. $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for $a, b \in \mathbb{R}, b \neq 0$, i.e. $A$ has no real eigenvalues. Then $A$ is called a similarity, i.e. a composition of a rotation and a homothety. For an endomorphism $B$ with no real eigenvalues the one-parameter subgroup $B(t) = \exp(Bt), t \neq 0$ as well as the one-parameter family $A(s) = (1 - \alpha)I + \alpha \exp(Bs), s \neq 0$ consist of affine mappings without real eigenvalues, i.e. compositions of non-trivial rotations and homotheties.

3. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is called shear transformation. Hence the matrix $A$ has only one eigenvalue 1 and is not diagonalizable. If $B$ is of the form $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, i.e. $B$ is nilpotent, then the one-parameter subgroup $B(t) = \exp(Bt), t \neq 0$ as well as the one-parameter family $A(s) = (1 - \alpha)I + \alpha \exp(Bs), s \neq 0$ consist of shear transformations.

4. $A = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$ with $\lambda \in \mathbb{R} - \{0, 1\}$. Then $A$ is invertible with only one eigenvalue $\lambda \neq 1$ and not diagonalizable. This linear map is a composition of a homothety and a shear transformation. The one-parameter subgroup $B(t) = \exp(Bt), t \neq 0$ as well as the one-parameter family $A(s) = (1 - \alpha)I + \alpha \exp(Bs), s \neq 0$ consist of linear mappings with only one eigenvalue different from 1 which are not diagonalizable. Hence they are compositions of non-trivial homotheties and shear transformations, too.

We use the following convention: For a one-parameter family $s \mapsto c_s$ of curves or a one-parameter family $s \mapsto A(s), s \mapsto b(s)$ of affine maps we denote the differentiation with respect to the parameter $s$ by $'$. On the other hand we use for the differentiation with respect to the curve parameter $t$ of the curves $t \mapsto c(t), t \mapsto c_s(t)$ the notation $\dot{c}, \dot{c}_s$.

**Proof.** The proof is similar to the Proof of Theorem [9, Thm.1]: Let

$$c_s(t) = A(s)c(t) + b(s) = (1 - \alpha)c(t) + \alpha c(t + s). \quad (15)$$

For $s = 0$ we obtain $c(t) = c_0(t) = A(0)c(t) + b(0)$ for all $t \in \mathbb{R}$, resp. $(A(0) - I)(c(t)) = -b(0)$ for all $t$. We conclude that

$$(A(0) - I) \left( c^{(k)}(t) \right) = 0 \quad (16)$$

for all $k \geq 1$. Since for some $t_0$ the vectors $\dot{c}(t_0), \ddot{c}(t_0), \ldots, c^{(n)}(t_0)$ are linearly independent by assumption we conclude from Eq. (16): $A(0) = I, b(0) = 0$. 

Eq. (15) implies for \( k \geq 1 \):
\[
A(s)c^{(k)}(t) = (1 - \alpha)c^{(k)}(t) + \alpha c^{(k)}(t + s)
\]
and hence
\[
A'(s)c^{(k)}(t) = \alpha c^{(k+1)}(t + s).
\]
We conclude from Eq. (15):
\[
\frac{\partial c_s(t)}{\partial s} = A'(s)c(t) + b'(s)
\]
\[
= \frac{\partial c_s(t)}{\partial t} -(1 - \alpha)\dot{c}(t) = (A(s) - (1 - \alpha)I)\dot{c}(t).
\]

Since \( A(0) = I \) the endomorphisms \( A(s) + (\alpha - 1)I \) are isomorphisms for all \( s \in (0, \epsilon) \) for a sufficiently small \( \epsilon > 0 \). Hence we obtain for \( s \in (0, \epsilon) \):
\[
\dot{c}(t) = (A(s) + (\alpha - 1)I)^{-1}A'(s)c(t) + (A(s) + (\alpha - 1)I)^{-1}b'(s). \tag{17}
\]

Differentiating with respect to \( s \):
\[
\left((A(s) + (\alpha - 1)I)^{-1}A'(s)\right)'(c(t)) + \left((A(s) + (\alpha - 1)I)^{-1}b'(s)\right)' = 0
\]
and differentiating with respect to \( t \):
\[
\left((A(s) + (\alpha - 1)I)^{-1}A'(s)\right)'c^{(k)}(t) = 0; \quad k = 1, 2, \ldots, n.
\]

By assumption the vectors \( \dot{c}(t_0), \ddot{c}(t_0), \ldots, c^{(n)}(t_0) \) are linearly independent. Therefore we obtain \( \left((A(s) + (\alpha - 1)I)^{-1}A'(s)\right)' = 0 \). Let \( B = \alpha^{-1}A'(0), d = \alpha^{-1}b'(0) \). Then we conclude
\[
A'(s) = (A(s) + (\alpha - 1)I)B; \quad b'(s) = (A(s) + (\alpha - 1)I)(d), \tag{18}
\]
We obtain from Eq. (17):
\[
\dot{c}(t) = Bc(t) + d.
\]

Equation (18) with \( A(0) = I \) implies \( A(s) = (1 - \alpha)I + \alpha \exp(Bs) \). And we obtain \( b'(s) = \alpha \exp(Bs)(d) = \alpha F_B(s)(d) \). Hence \( b(s) = \alpha F_B(s)(d) \) since \( b(0) = 0 \). \( \square \)

As a consequence we obtain the following

**Theorem 3.** For a \((n, n)\)-matrix \( B \) and a vector \( d \) any solution of the inhomogeneous linear differential equation \( \dot{c}(t) = Bc(t) + d \) with constant coefficients is a soliton of the mapping \( M_\alpha \). These solitons are orbits of a one-parameter subgroup of the affine group, i.e. they are of the form given in Eq. (9).

**Proof.** Any solution of the equation \( \dot{c}(t) = Bc(t) + d \) has the form
\[
c(t) = v + F_B(t)(Bv + d)
\]
with \( v = c(0) \), cf. Proposition 1. Then with \( A(s) = (1 - \alpha) \mathbb{1} + \alpha \exp(Bs) \) and \( b(s) = \alpha F_B(s)(d) \) we conclude from Eqs. (6) and (7):

\[
\tilde{c}_s(t) = (1 - \alpha) c(t) + \alpha c(t + s)
\]

\[
= v + (1 - \alpha) F_B(t)(Bv + d) + \alpha F_B(t + s)(Bv + d)
\]

\[
= v + (1 - \alpha)(c(t) - v) + \alpha (F_B(s) + \exp(Bs)F_B(t))(Bv + d)
\]

\[
= (1 - \alpha)(c(t) + \alpha \exp(Bs)(v) + \alpha F_B(s)(d) + \alpha \exp(Bs)(c(t) - v)
\]

\[
= ((1 - \alpha) \mathbb{1} + \alpha \exp(Bs))(c(t)) + \alpha F_B(s)(d)
\]

\[
= A(s)c(t) + b(s).
\]

Hence \( c \) is a soliton of the mapping \( M_\alpha \), cf. Eq. (14).

\[ \square \]

**Remark 4.** In [9] the authors consider the curve shortening process \( T : \mathcal{P}(\mathbb{R}^n) \rightarrow \mathcal{P}(\mathbb{R}^n) \), \( T(x)_j = M^2(x)_{j-1} \) satisfying Eq. (3). Hence the midpoints mapping \( M \) is applied twice followed by an index shift. The smooth curves \( c = c(t) \) invariant under this process can be characterized as solutions of a inhomogeneous linear system of second order differential equations

\[
\ddot{c}(t) = Bc(t) + d
\]

for a \((n, n)\)-matrix \( B \) and a vector \( d \), cf. [9, Thm. 2]. These solutions can be reduced to a system of first order differential equations, cf. [9, Rem. 1]. Explicit formulas for these solutions can be written down in terms of power series in \( t \) whose coefficients are expressed in terms of powers of \( B \). If \( B = B_1^2, d = B_1(d_1) \) for a \((n, n)\)-matrix \( B_1 \) and a vector \( d_1 \in \mathbb{R}^n \), then the orbits of one-parameter subgroups of the affine group \( \text{Aff}(\mathbb{R}^n) \) acting on \( \mathbb{R}^n \) satisfying

\[
\dot{c}(t) = B_1 c(t) + d_1
\]

are particular solutions. In the next section we will see that for \( n = 2 \) the solitons of the midpoints mapping have constant affine curvature. On the other hand not all solitons of the process \( T \) have constant affine curvature, cf. [9, Sec. 5].

## 5. Curves with constant affine curvature

The orbits of one-parameter subgroups of the affine group \( \text{Aff}(2) \) acting on \( \mathbb{R}^2 \) can also be characterized as curves of constant general-affine curvature parameterized proportional to general-affine arc length unless they are parametrizations of a parabola, an ellipse or a hyperbola. This will be discussed in this section. The one-parameter subgroups are determined by an endomorphism \( B \) and a vector \( d \). We describe in Proposition 2 how the general-affine curvature can be expressed in terms of the matrix \( B \).

For certain subgroups of the affine group \( \text{Aff}(2) \) one can introduce a corresponding *curvature* and *arc length*. One should be aware that sometimes in the literature the curvature related to the equi-affine subgroup \( S\text{Aff}(2) \) generated by the special linear group \( \text{SL}(2) \) of linear maps of determinant one and
the translations is also called affine curvature. We distinguish in the following between the equi-affine curvature \( k_{ea} \) and the general-affine curvature \( k_{ga} \) as well as between the equi-affine length parameter \( s_{ea} \) and the general-affine length parameter \( s_{ga} \).

We recall the definition of the equi-affine and general-affine curvature of a smooth plane curve \( c: I \rightarrow \mathbb{R}^2 \) with \( \det(\dot{c}(t) \hat{c}(t)) = |\dot{c}(t) \hat{c}(t)| \neq 0 \) for all \( t \in I \).

By eventually changing the orientation of the curve we can assume \( |\dot{c}(t) \hat{c}(t)| > 0 \) for all \( t \in I \). A reference is the book by P. and A. Schirokow [10, §10] or the recent article by Kobayashi and Sasaki [7]. Then \( s_{ea}(t) := \int |\dot{c}(t) \hat{c}(t)|^{1/3} \, dt \) is called equi-affine arc length. We denote by \( t = t(s_{ea}) \) the inverse function, then \( \tilde{c}(s_{ea}) = c(t(s_{ea})) \) is the parametrization by equi-affine arc length. Then \( \tilde{c}''(s_{ea}), \tilde{c}'(s_{ea}) \) are linearly dependent and the equi-affine curvature \( k_{ea}(s_{ea}) \) is defined by

\[
\tilde{c}'''(s_{ea}) = -k_{ea}(s_{ea}) \tilde{c}'(s_{ea})
\]
resp.

\[
k_{ea}(s_{ea}) = |\tilde{c}''(s_{ea})\tilde{c}'''(s_{ea})|.
\]

Assume that \( c = c(s_{ea}), s_{ea} \in I \) is a smooth curve parametrized by equi-affine arc length for which the sign \( \epsilon = \text{sign}(k_{ea}(s_{ea})) \in \{0, \pm 1\} \) of the equi-affine curvature is constant. If \( \epsilon = 0 \) then the curve is up to an affine transformation a parabola \((t, t^2)\). Now assume \( \epsilon \neq 0 \) and let \( K_{ea} = |k_{ea}| = \epsilon k_{ea} \). Then the general-affine arc length \( s_{ga} = s_{ga}(s_{ea}) \) is defined by

\[
s_{ga} = \int \sqrt{K_{ea}(s_{ea})} \, ds_{ea}.
\]

(19)

We call a curve \( c = c(t) \) parametrized proportional to general-affine arc length if \( t = \lambda_1 s_{ga} + \lambda_2 \) for \( \lambda_1, \lambda_2 \in \mathbb{R} \) with \( \lambda_1 \neq 0 \). The general-affine curvature \( k_{ga} = k_{ga}(s_{ea}) \) is defined by

\[
k_{ga}(s_{ea}) = K_{ea}'(s_{ea})K_{ea}(s_{ea})^{-3/2} = -2 \left( K_{ea}^{-1/2}(s_{ea}) \right)'.
\]

(20)

If the general-affine curvature \( k_{ga} \) (up to sign) and the sign \( \epsilon \) is given with respect to the equi-affine arc length parametrization, then the equi-affine curvature \( k_{ea} = k_{ea}(s_{ea}) \) is determined up to a constant by Eq. (20). Hence the curve is determined up to an affine transformation. The invariant \( k_{ga} \) already occurs in Blaschke’s book [2, §10, p.24]. Curves of constant general-affine curvature are orbits of a one-parameter subgroup of the affine group. These curves already were discussed by Klein and Lie [6] under the name W-curves.

**Proposition 2.** For a non-zero matrix \( B \in M_\mathbb{R}(2,2) \) and vectors \( d, v \in \mathbb{R}^2 \) where \( Bv + d \) is not an eigenvector of \( B \) let \( c : \mathbb{R} \rightarrow \mathbb{R}^2 \) be the solution of the differential equation \( c'(t) = Bc(t) + d; c(0) = v \), i.e. \( c(t) = v + F_B(t)(Bv+d) = \exp(tB)(v)+F_B(t)(d) \). We assume that \( \beta = |c'(0) c''(0)|^{1/3} = |Bv + d B(Bv + d)|^{1/3} > 0 \). Define
\[ k = k(B) = -2 + 9 \det(B)/\tr^2(B); \quad K = K(B) = |k(B)|^{-1/2}. \]  

(a) If \( \tr(B) = 0 \) then the curve is parametrized proportional to equi-affine arc length and the equi-affine curvature is constant \( k_{ea} = \det(B)/\beta^2 \) and \( \epsilon = \text{sign}(\det(B)) \), the curve is a parabola, if \( \epsilon = 0 \), an ellipse, if \( \epsilon > 0 \), or a hyperbola, if \( \epsilon < 0 \), cf. Remark 5.

(b) If \( \tr(B) \neq 0 \) then we can choose a parametrization by equi-affine arc length \( s_{ea} \) such that the equi-affine curvature \( k_{ea} \) is given by:

\[ k_{ea}(s_{ea}) = k(B)s_{ea}^{-2}. \]  

If \( k(B) = 0 \) the curve has vanishing equi-affine curvature and is a parametrization of a parabola, cf. the Remark 5. If \( k(B) \neq 0 \) then the general-affine curvature is defined and constant:

\[ k_{ga}(s_{ea}) = -2K(B). \]  

Up to an additive constant the general-affine arc length parameter \( s_{ga} \) is given by:

\[ s_{ga} = \frac{\tr B}{3K(B)} t. \]

Hence the curve \( c(t) \) is parametrized proportional to general-affine arc length.

**Remark 5.** It is well-known that the curves of constant equi-affine curvature are parabola, hyperbola or ellipses, cf. [2, §7]. For \( k_{ea} = 0 \) we obtain a parabola: \( c(t) = c(0) + c'(0)s_{ea} + c''(0)s_{ea}^2/2 \), for \( k_{ea} > 0 \) the ellipse \( c(s_{ea}) = (a \cos(\sqrt{k_{ea}}s_{ea}), b \sin(\sqrt{k_{ea}}s_{ea})) \) with \( k_{ea} = (ab)^{-2/3} \) and for \( k_{ea} < 0 \) the hyperbola \( c(s_{ea}) = (a \cosh(\sqrt{-k_{ea}}s_{ea}), b \sinh(\sqrt{-k_{ea}}s_{ea})) \) with \( k_{ea} = -(ab)^{-2/3} \). Here \( a, b > 0 \).

**Proof.** Following Proposition 1 we obtain as solution of the differential equation: \( c(t) = v + F_B(t)(Bv + d) \), hence for the derivatives: \( c^{(k)}(t) = B^{k-1} \exp(tB)(Bv + d) \). Then:

\[ |\dot{c}(t)\ddot{c}(t)| = |\exp(Bt)||bv + d\ B(Bv + d)| \]

\[ = \exp(\tr(B)t)|bv + d\ B(Bv + d)|. \]

Let \( \beta = (|Bv + d\ B(Bv + d)|)^{1/3} \) and \( \tau = \tr(B) \). Then

\[ |\dot{c}(t)\ddot{c}(t)| = \beta^3 \exp(\tau t). \]

(a) If \( \tau = 0 \) then \( s_{ea} = t\beta \), i.e. the curve is parametrized proportional to equi-affine arc length and

\[ \ddot{c}(s_{ea}) = c(t(s_{ea})) = c(s_{ea}/\beta) = v + F_B(s_{ea}/\beta)(Bv + d). \]

Then

\[ \ddot{c}'(s_{ea}) = \beta^{-1} \exp(Bs_{ea}/\beta)(Bv + d) \]

\[ \ddot{c}''(s_{ea}) = \beta^{-3}B^2 \exp(Bs_{ea}/\beta)(Bv + d) = -\det(B)\beta^{-2}c'(s_{ea}). \]
Here we use that by Cayley–Hamilton $B^2 - \tau B = B^2 = -\det(B) \cdot 1$. Hence we obtain $k_{ea}(s_{ea}) = \det(B)/\beta^2$ and $\epsilon = \text{sign}(\det(B))$. Then the claim follows from Remark 5.

(b) Assume $\tau \neq 0$. Then the equi-affine arc length $s_{ea} = s_{ea}(t)$ is given by

$$s_{ea}(t) = \beta \int \exp(\tau t/3) dt = \frac{3\beta}{\tau} \exp(\tau t/3). \quad (24)$$

Hence the equi-affine arc length parametrization of $c$ is given by

$$\tilde{c}(s_{ea}) = v + F_B \left( \frac{3}{\tau} \ln \left( \frac{\tau}{3\beta} s_{ea} \right) \right) (Bv + d).$$

Then we can express the derivatives:

$$\tilde{c}'(s_{ea}) = \frac{3}{\tau} \frac{1}{s_{ea}} \exp \left( \frac{3}{\tau} B \ln \left( \frac{\tau}{3\beta} s_{ea} \right) \right) (Bv + d)$$

$$\tilde{c}''(s_{ea}) = \left( \frac{3}{\tau} B - 1 \right) \frac{1}{s_{ea}} \tilde{c}'(s_{ea})$$

$$\tilde{c}'''(s_{ea}) = \left( \frac{3}{\tau} B - 1 \right) \left( \frac{3}{\tau} B - 2 \right) \frac{1}{s_{ea}^2} \tilde{c}'(s_{ea})$$

$$= - \left( \frac{9 \det(B)}{\tau^2} - 2 \right) \frac{1}{s_{ea}^2} \tilde{c}'(s_{ea}).$$

Here we used that by Cayley–Hamilton $B^2 - \tau B = -\det B \cdot 1$. Hence we obtain for the equi-affine curvature

$$k_{ea}(s_{ea}) = \frac{k(B)}{s_{ea}^2}. \quad (25)$$

Then $\epsilon = \text{sign}(k(B))$ and for $k(B) \neq 0$ we obtain from Eqs. (20) and (25):

$$k_{ga}(s_{ea}) = -\frac{2}{K(B)}.$$

And for the general-affine arc length we obtain

$$s_{ga} = \ln(|s_{ea}|)/K(B),$$

resp. up to an additive constant:

$$s_{ga} = \frac{\tau t}{3K(B)}$$

using Eq. (24).

The parametrization by general-affine arc length is given by

$$c^*(s_{ga}) = v + F_B \left( 3K(B) s_{ga}/\tau \right) (Bv + d).$$

Example 1. Depending on the real Jordan normal forms of the endomorphism $B$ we investigate the solitons $c(t)$, their special and general affine curvature. The normal forms of the corresponding one-parameter subgroup $B(t) = \exp(Bt)$ as well of the one-parameter family $A(s) = (1 - \alpha)1 + \exp(Bs)$ follow from
Remark 3. Since \( c(\mu t) = \exp(\mu B t) \) the multiplication of \( B \) with a non-zero real \( \mu \) corresponds to a linear reparametrization of the curve. If \( B \) has a non-zero real eigenvalue we can assume without loss of generality that it is 1 and in the case of a non-real eigenvalue we can assume that it has modulus 1.

(a) Let \( B = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \), \( d = (0, 0) \), \( c(0) = (1, 1) \) and \( \lambda \neq 0, 1 \). Then \( \beta = (\lambda(1 - \lambda))^{1/3} \neq 0 \), \( \text{tr} B = 1 + \lambda \) and \( c(t) = (\exp(t), \exp(\lambda t)) \). Up to parametrization we have \( c(u) = (u, u^\lambda) \).

If \( \lambda = -1 \) then \( c \) is a parametrization of a hyperbola, \( \text{tr} B = 0 \) and \( k_{ea} = -2^{-2/3} \), cf. Remark 5.

If \( \lambda \neq -1 \) we obtain for the equi-affine curvature with respect to a equi-affine parametrization \( s_{ea} \) from Eq. (22):

\[
k_{ea}(s_{ea}) = \left(9 \det B - 2 \right) \frac{1}{s_{ea}^2} = \frac{(\lambda - 2)(2\lambda - 1)}{(\lambda + 1)^2} \frac{1}{s_{ea}^2}.
\]

For \( \lambda = 1/2 \) we obtain a parametrization of a parabola with vanishing equi-affine curvature, cf. Remark 5. Now we assume \( \lambda \neq 1/2, 2 \). Hence \( \epsilon = 1 \) if and only if \( 1/2 < \lambda < 2 \). The affine curvature \( k_{ga} \) is constant:

\[
k_{ga} = -\frac{1}{\sqrt{|(\lambda - 2)(2\lambda - 1)|}},
\]

cf. [7, Ex.2.14]. We have \( \epsilon = 1 \) if and only if \( 1/2 < \lambda < 2 \), then \( k_{ga} \in (-\infty, -4) \). And \( \epsilon = -1 \) if and only if \( \lambda < 1/2, \lambda \neq 0 \) or \( \lambda > 2 \), then \( k_{ga} \in (-\infty, -\sqrt{2}) \cup (-\sqrt{2}, 0) \).

Hence in this case the corresponding one-parameter subgroup

\[
B(t) = \begin{pmatrix} \exp(t) & 0 \\ 0 & \exp(\lambda t) \end{pmatrix}
\]

as well as the one-parameter family

\[
A(s) = \begin{pmatrix} 1 - \alpha + \alpha \exp(s) & 0 \\ 0 & 1 - \alpha + \alpha \exp(\lambda s) \end{pmatrix}
\]

consist of scalings.

(b) \( B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \) and \( d = (1, 0) \). Then the solution of the Equation \( \dot{c}(t) = Bc(t) + d \) with \( c(0) = (0, 1) \) is of the form \( c(t) = (t, \exp(t)) \). Then we obtain \( \epsilon = -1 \) and \( k_{ga} = -\sqrt{2} \). The corresponding one-parameter subgroup \( B(t) \) as well as the one-parameter family \( A(s) \) consist of scalings, the affine transformation \( (A(s), b(s)) \) is given by \( (A(s), b(s)) = \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \alpha \begin{pmatrix} s \\ 0 \end{pmatrix} \right) \), i.e. a composition of scalings and translations.
Figure 2 The soliton \( c(t) = ((t + 1) \exp(t), \exp(t)) \) with the family \( c_s(t) = A(s)c(t) \)

(c) If \( B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, d = (0,0), c(0) = (1,1) \) then \( c(t) = ((t + 1) \exp(t), \exp(t)) \)
i.e. up to an affine transformation and a reparametrization the curve is of the form \( c(u) = (u, u \ln(u)) \). Then \( \epsilon = 1 \) and \( k_{ga} = -4 \). The corresponding one-parameter subgroup \( B(t) \) as well as the one-parameter family \( A(s) \) consist of compositions of a homothety and a shear transformations:

\[
B(t) = \exp(t) \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}; A(s) = \begin{pmatrix} 1 - \alpha + \alpha \exp(s) & \alpha s \exp(s) \\ 0 & 1 - \alpha + \alpha \exp(s) \end{pmatrix},
\]
cf. Fig. 2.

(d) If \( B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \) with \( b \neq 0, a^2 + b^2 = 1, d = 0, c(0) = (1,0) \) then \( c(t) = \exp(at) (\cos(bt), \sin(bt)) \). For \( a = 0 \), this is a circle with \( k_{ea} = 1 \). Now we assume \( a \neq 0 \) : Then \( \epsilon = \text{sign}(9 \det(B) - 2tr^2(B)) = \text{sign}(a^2 + 9b^2) = 1 \)
and we obtain for the general-affine curvature \( k_{ga} = -4|a|/\sqrt{a^2 + 9b^2} = -4|a|/\sqrt{9 - 8a^2}, \) i.e. \( k_{ga} \in (-4,0) \). The corresponding one-parameter subgroup \( B(t) \) as well as the one-parameter family \( A(s) \) consist of similarities.

(e) If \( B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) one can choose \( c(0) = (0,0), d = (0,1) \) and obtain \( c(t) = (t^2/2, t) \), i.e. a parabola. In this case the one parameter subgroup \( B(t) = \exp(tB) \) consists of shear transformations. The one-parameter family \( (A(s), b(s)) = \left( \begin{pmatrix} 1 & \alpha s \\ 0 & 1 \end{pmatrix}, \alpha \begin{pmatrix} s^2/2 \\ s \end{pmatrix} \right) \) consists of a composition of a shear transformation and a translation, cf. Fig. 1. For the affine curve shortening flow the parabola is a translational soliton. Therefore it is also called the affine analogue of the grim reaper, cf. [3, p. 192]. For the curve
shortening process $T$ defined by Eq. (3) the parabola is also a transla-
tional soliton, cf. [9, Sec. 5, Case (5)].

Note that the parabola occurs twice, in Case (a) it occurs with the parametriza-
tion $c(t) = (\exp(t), \exp(2t))$, in Case (e) it occurs with a parametrization pro-
portional to equi-affine arc length. Summarizing we obtain from Theorems 2 and 3 together with Proposition 2 resp. Example 1 the following

**Theorem 4.** Let $c : \mathbb{R} \to \mathbb{R}^2$ be a smooth curve for which $\dot{c}(0), \ddot{c}(0)$ are linearly independent. Then $c$ is a soliton of the mappings $M_\alpha, \alpha \in (0, 1)$, in particular of the midpoints mapping $M = M_{1/2}$, if it is a curve of constant equi-affine curvature parametrized proportional to equi-affine arc length, or a parabola with the parametrization $c(t) = (\exp(t), \exp(2t))$ up to an affine transformation, or if it is a curve of constant general-affine curvature parametrized proportional to general-affine arc length.

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