QUALITATIVE PROPERTIES OF SOLUTIONS TO AN INTEGRAL SYSTEM ASSOCIATED WITH THE BESSEL POTENTIAL

LU CHEN AND ZHAO LIU*
Laboratory of Mathematics and Complex Systems,
School of Mathematical Sciences, Beijing Normal University,
Beijing 100875, P. R. China

GUOZHEN LU
Department of Mathematics, Wayne State University,
Detroit, MI 48202, U. S. A.

(Communicated by Wenxiong Chen)

Abstract. In this paper, we study a differential system associated with the Bessel potential:
\[
\begin{align*}
(I - \Delta)^\alpha_2 u(x) &= f_1(u(x), v(x)), \\
(I - \Delta)^\beta_2 v(x) &= f_2(u(x), v(x)),
\end{align*}
\]
where \( f_1(u(x), v(x)) = \lambda_1 u^{p_1}(x) + \mu_1 v^{q_1}(x) + \gamma_1 u^{\alpha_1}(x) v^{\beta_1}(x), \)
\( f_2(u(x), v(x)) = \lambda_2 u^{p_2}(x) + \mu_2 v^{q_2}(x) + \gamma_2 u^{\alpha_2}(x) v^{\beta_2}(x), \)
\( I \) is the identity operator and \( \Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} \) is the Laplacian operator in \( \mathbb{R}^n \). Under some appropriate conditions, this differential system is equivalent to an integral system of the Bessel potential type. By the regularity lifting method developed in [4] and [18], we obtain the regularity of solutions to the integral system. We then apply the moving planes method to obtain radial symmetry and monotonicity of positive solutions. We also establish the uniqueness theorem for radially symmetric solutions. Our nonlinear terms \( f_1(u(x), v(x)) \) and \( f_2(u(x), v(x)) \) are quite general and our results extend the earlier ones even in the case of single equation substantially.

1. Introduction. The main purpose of this paper is to obtain the regularity of solutions to the integral and differential systems of the Bessel potential type and establish the radial symmetry and monotonicity of positive solutions to such systems. We also establish the uniqueness theorem for radially symmetric solutions.

More precisely, in this paper, we consider the following semi-linear fractional partial differential equations in \( \mathbb{R}^n \):
\[
\begin{align*}
(I - \Delta)^\alpha_2 u(x) &= f_1(u(x), v(x)), \\
(I - \Delta)^\beta_2 v(x) &= f_2(u(x), v(x)),
\end{align*}
\]

2000 Mathematics Subject Classification. Primary: 35J48; Secondary: 35B06, 45G15.

Key words and phrases. Bessel potential, method of moving planes in integral forms, radial symmetry, regularity, uniqueness.

The first two authors were partly supported by grant from the NNSF of China (No.11371056), the third author was partly supported by a US NSF grant DMS #1301595 and a Simons Fellowship from the Simons Foundation.

*Corresponding Author: Zhao Liu at liuzhao@mail.bnu.edu.cn
where \( f_1(u(x), v(x)) = \lambda_1 u^{p_1}(x) + \mu_1 v^{q_1}(x) + \gamma_1 u^{\alpha_1}(x) v^{\beta_1}(x), \)
\( f_2(u(x), v(x)) = \lambda_2 u^{p_2}(x) + \mu_2 v^{q_2}(x) + \gamma_2 u^{\alpha_2}(x) v^{\beta_2}(x), \)
the identity operator and \( \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \) is the Laplacian operator in \( \mathbb{R}^n \).

Under some appropriate decay conditions on the solutions \((u, v)\) at infinity, system \((1.1)\) is equivalent to the following integral system
\[
\begin{aligned}
&\begin{cases}
  u(x) = \int_{\mathbb{R}^n} g_\alpha(x-y) f_1(u(y), v(y)) dy, \\
  v(x) = \int_{\mathbb{R}^n} g_\alpha(x-y) f_2(u(y), v(y)) dy,
\end{cases}
\end{aligned}
\tag{1.2}
\]
where
\[
g_\alpha(x) = \frac{1}{(4\pi)^{\frac{\alpha}{4}} \Gamma\left(\frac{\alpha}{2}\right)} \int_{0}^{\infty} e^{-\frac{\|x-y\|^2}{s}} e^{-\frac{\delta}{\pi} \frac{\alpha-\alpha-2}{2}} d\delta
\]
is the \( \alpha \)-order Bessel kernel.

When \( \lambda_1 = \gamma_1 = \mu_2 = \gamma_2 = 0 \) and \( \mu_1 = \lambda_2 = 1 \), system \((1.1)\) reduces to the following equations
\[
\begin{aligned}
&\begin{cases}
  (I - \Delta)^{\frac{\alpha}{2}} u(x) = v^{q_1}(x), \\
  (I - \Delta)^{\frac{\alpha}{2}} v(x) = u^{p_2}(x).
\end{cases}
\end{aligned}
\tag{1.3}
\]

For \( \alpha = 1 \), system \((1.3)\) is the stationary Dirac-Schrödinger system. For \( \alpha = 2 \), system \((1.3)\) is the stationary Schrödinger system and the stationary model of a system of reaction-diffusion equations was studied in \([22]\). In particular, for the single semi-linear fractional partial differential equation with Bessel potential
\[
(I - \Delta)^{\frac{\alpha}{2}} u = u^{p},
\tag{1.4}
\]
Kwong \([15]\) established the uniqueness of the positive, radially symmetric solution to the differential equation \((1.4)\) for \( \alpha = 2 \). Ma and Chen \([19]\) exploited a new Hardy-Littlewood-Sobolev type inequality for the Bessel potential and proved that every positive solution of corresponding integral equation of \((1.4)\) is radially symmetric and strictly decreasing about some point for any \( \alpha > 0 \). Lam and Lu established in \([16]\) the existence of positive solutions to a class of polyharmonic equation of Bessel potential type on bounded domains when the nonlinearity has the exponential growth and doesn’t satisfy the Ambrosetti-Rabinowitz condition. Bao, Lam and Lu \([1]\) further studied the Bessel type poly-harmonic equation on the entire space when the nonlinear terms have the critical exponential growth in the sense of Adams’ inequalities. In \([1]\), the authors show that solutions are uniformly bounded and Lipschitz continuous. They also proved that positive solutions are radially symmetric and monotone decreasing about some point. For more results about Bessel potential, please see \([12, 13, 20, 23]\) and the references therein.

Recently, Chen, Li and Ma \([18]\) developed further some new methods to prove the regularity of the solutions to a system of integral equations associated with the Wolff potentials by combinations of contracting and shrinking operators. We use their methods to prove the first two theorems for the integral system \((1.2)\).

In this paper, we always assume that \( \lambda_i, \mu_i, \gamma_i \ (i = 1, 2) \) are nonnegative constants and they are not equal to zero simultaneously. We also use the following notations
\[
\Pi_1 = \{ f(x) : f \in L^{1,1}(\mathbb{R}^n) \cap L^{1,2}(\mathbb{R}^n) \cap L^{h_0}(\mathbb{R}^n) \}, \\
\Pi_2 = \{ f(x) : f \in L^{2,1}(\mathbb{R}^n) \cap L^{2,2}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n) \},
\]
Theorem 1.4. For \( \alpha_i, \beta_i, \alpha_1, \beta_1 > 1 \), let \((u, v) \in \Pi_1 \times \Pi_2\) be a pair of nonnegative solutions of integral system (1.2). Then \( u, v \in L^s(\mathbb{R}^n) \) for any \( \max\{p_i, q_i, \alpha_i + \beta_i\} < s \leq \infty \).

We note that one cannot use interpolation to conclude what the spaces \( \Pi_1 \) or \( \Pi_2 \) are because \( t_{1i}, t_{2i} \) and \( k_0 \) cannot be compared easily. Therefore, it is highly nontrivial to derive the \( L^s \) integrability of the solutions \( u \) and \( v \) in Theorem 1.1.

Theorem 1.2. For \( \alpha_i, \beta_i, \alpha_1, \beta_1 > 1 \), let \((u, v) \in \Pi_1 \times \Pi_2\) be a pair of nonnegative solutions of integral system (1.2). Then \( u(x) \) and \( v(x) \) are Lipschitz continuous.

It is well known that the method of moving plane was invented by the Soviet mathematician Alexandrov in the 1950s. Then it was further developed by Serrin [21], Gidas, Ni and Nirenberg [10, 11], Caffarelli, Gidas and Spruck [2], Chen and Li [5], Chang and Yang [3] and many others. However, their methods of moving plane are mainly based on the maximum principle, and hence one is not able to apply it in the absence of the maximum principle. In [7, 8], Chen, Li and Ou developed the method of moving plane in integral forms. The moving plane in integral forms can be easily applied to higher order equations without maximum principles. To establish the symmetry of the solutions, we use the moving plane in integral forms to prove the following theorem. For more results related to the moving plane in integral forms, please see [4, 5, 6, 9, 14, 17] and the references therein.

Theorem 1.3. For \( \alpha_i, \beta_i, \alpha_1, \beta_1 > 1 \), assume that \((u, v) \in \Pi_1 \times \Pi_2\) is a pair of positive solutions of integral system (1.2), then \( u \) and \( v \) are radially symmetric and strictly decreasing about some point.

We also can obtain the following uniqueness theorem.

Theorem 1.4. Let \( \alpha = 2 \) and \( \lambda_1 = \mu_2 = 0 \), \( \mu_1 = \lambda_2 = \gamma_1 = \gamma_2 \), \( q_1 = p_2 \), \( \alpha_1 < \alpha_2 \), \( \alpha_1 + \beta_1 = \alpha_2 + \beta_2 \). Then any pair of radially symmetric solutions \((u, v)\) satisfying integral system (1.2) is unique, moreover \( u = v \).

This paper is organized as follows. In Section 2, we apply regularity lifting by contracting operators to prove Theorem 1.1. In Section 3, we obtain every pair of nonnegative solutions of integral system (1.2) is Lipschitz continuous, thus the proof of Theorem 1.2 is complete. In Section 4 and Section 5, we prove the radial symmetry and uniqueness of positive solutions of integral system (1.2), thus we complete the proofs of Theorem 1.3 and Theorem 1.4.

2. The proof of Theorem 1.1. In order to prove Theorem 1.1, we give some lemmas.

Let \( V \) be a topological vector space. Suppose there are two extended norms (i.e., the norm of an element in \( V \) might be infinity) defined on \( V \),

\[ \| \cdot \|_X, \| \cdot \|_Y : V \rightarrow [0, \infty]. \]

Let

\[ X := \{ f \in V : \| f \|_X < \infty \} \quad \text{and} \quad Y := \{ f \in V : \| f \|_Y < \infty \}. \]
Lemma 2.1 (Regularity lifting I). Let $T$ be a contraction map from $X$ into itself and from $Y$ into itself. Assume that for any $f \in X$, there exists a function $g \in Z := X \cap Y$ such that $f = Tf + g \in X$. Then $f \in Z$.

Lemma 2.2 (Decay estimate of $g_\alpha$). Given $\alpha > 0$, the Bessel kernel $g_\alpha(x)$ has the decay estimate

$$g_\alpha(x) \sim e^{-\frac{|x|}{\alpha}}$$

when $|x| \geq 2$, and when $|x| \leq 2$,

$$g_\alpha(x) \sim \begin{cases} |x|^\alpha & 0 < \alpha < n, \\ \log \frac{2}{|x|^\alpha} & \alpha = n, \\ 1 & \alpha > n. \end{cases}$$

Lemma 2.3 (see [19]). Suppose $s > \max\{\beta, \frac{n(\beta - 1)}{\alpha}\}$ and $f \in L^s$, then $\|B_\alpha f\|_{L^s} \leq C\|f\|_{L^s}$.

Now, we start our proof.

Denote

$$u_\alpha(x) = \begin{cases} u(x), & |u(x)| > a \text{ or } |x| > a, \\ 0, \text{ otherwise.} \end{cases}$$

Set $u_\alpha(x) = u(x) - u_\alpha(x)$, then $u_\alpha$ is uniformly bounded by $a$ in $B_a(0)$. Assume that $(\phi, \varphi) \in L^s(\mathbb{R}^n) \times L^s(\mathbb{R}^n)$ for $\max\{p_1, q_1, \alpha_1 + \beta_1\} \leq s \leq \infty$. Since $(u, v)$ satisfies (1.2), we have

$$u = g_\alpha \ast (\lambda_1(u_{a_1}^{p_1} + u_{a_1}^{p_1}) + \mu_1(v_{a_1}^{q_1} + v_{a_1}^{q_1}) + \gamma_1(u_{a_1}^{r_1} + u_{a_1}^{r_1} + u_{a_1}^{r_1} + u_{b_1}^{r_1} + u_{b_1}^{r_1} + u_{b_1}^{r_1} + u_{b_1}^{r_1} + u_{b_1}^{r_1}))$$

$$= g_\alpha \ast (\lambda_1(u_{a_1}^{p_1} - 1 + \gamma_1 u_{a_1}^{r_1} - 1 + u_{a_1}^{r_1} + u_{a_1}^{r_1} + u_{b_1}^{r_1} + u_{b_1}^{r_1} + u_{b_1}^{r_1} + u_{b_1}^{r_1}))$$

$$+ g_\alpha \ast (\lambda_1 u_{a_1}^{p_1} + \mu_1 v_{a_1}^{q_1} + \gamma_1 u_{a_1}^{r_1} + 1)$$

$$= g_\alpha \ast (H^1 u_a + H^2 v_a + H^3), \quad (2.1)$$

where $H^1 = \lambda_1 u_{a_1}^{p_1} + \gamma_1 u_{a_1}^{r_1} - 1 + u_{a_1}^{r_1} + u_{a_1}^{r_1} + u_{a_1}^{r_1} + u_{b_1}^{r_1} + u_{b_1}^{r_1} + u_{b_1}^{r_1}$ and

$H^2 = \mu_1 v_{a_1}^{q_1} + \gamma_1 u_{a_1}^{r_1} - 1 + u_{a_1}^{r_1} + u_{a_1}^{r_1} + u_{b_1}^{r_1} + u_{b_1}^{r_1} + u_{b_1}^{r_1} + u_{b_1}^{r_1}$

Similarly, we also have

$$v = g_\alpha \ast (\lambda_2(u_{a_2}^{p_2} + u_{a_2}^{p_2}) + \mu_2(v_{a_2}^{q_2} + v_{a_2}^{q_2}) + \gamma_2(u_{a_2}^{r_2} + u_{a_2}^{r_2} + u_{a_2}^{r_2} + u_{b_2}^{r_2} + u_{b_2}^{r_2} + u_{b_2}^{r_2}))$$

$$= g_\alpha \ast (\lambda_2(u_{a_2}^{p_2} - 1 + \gamma_2 u_{a_2}^{r_2} - 1 + u_{a_2}^{r_2} + u_{a_2}^{r_2} + u_{b_2}^{r_2} + u_{b_2}^{r_2} + u_{b_2}^{r_2}))$$

$$+ g_\alpha \ast (\lambda_2 u_{a_2}^{p_2} + \mu_2 v_{a_2}^{q_2} + \gamma_2 u_{a_2}^{r_2} + 1)$$

$$= g_\alpha \ast (K^1 v_a + K^2 v_a + K^3), \quad (2.2)$$

where $K^1 = \mu_1 v_{a_1}^{r_2} - 1 + \gamma_2 u_{a_1}^{p_2} + 1 + u_{a_1}^{r_2} + u_{a_1}^{r_2} + u_{b_1}^{r_2} + u_{b_1}^{r_2}$ and

$K^2 = \mu_2 v_{a_2}^{r_2} + 1 + \gamma_2 u_{a_2}^{p_2} + 1 + u_{a_2}^{r_2} + u_{a_2}^{r_2} + u_{b_2}^{r_2} + u_{b_2}^{r_2}$

$K^3 = \mu_2 v_{a_2}^{r_2} + 1 + \gamma_2 u_{a_2}^{p_2} + 1 + u_{a_2}^{r_2} + u_{a_2}^{r_2} + u_{b_2}^{r_2} + u_{b_2}^{r_2}$

$$= g_\alpha \ast (H^1 u_a + H^2 v_a + H^3), \quad (2.1)$$

Define the bilinear operator

$$T^1_{(u_a, v_a)}(\phi, \varphi) = \int_{\mathbb{R}^n} g_\alpha(x - y)H^1(y)\phi(y)dy + \int_{\mathbb{R}^n} g_\alpha(x - y)H^2(y)\varphi(y)dy,$$

$$T^2_{(u_a, v_a)}(\phi, \varphi) = \int_{\mathbb{R}^n} g_\alpha(x - y)K^1(y)\phi(y)dy + \int_{\mathbb{R}^n} g_\alpha(x - y)K^2(y)\varphi(y)dy.$$

Combining (2.1) and (2.2), we can write

$$u_a = u - u_b = T^1_{(u_a, v_a)}(u_a, v_a) + g_\alpha * H^3 - u_b = T^1_{(u_a, v_a)}(u_a, v_a) + F^1,$$
\[ v_a = v - v_b = T^2_{(u_a, v_a)}(u_a, v_a) + g_a * K^3 - v_b = T^2_{(u_a, v_a)}(u_a, v_a) + F^2, \]

where \( F^1 = g_a * H^3 - u_b \) and \( F^2 = g_a * K^3 - v_b \).

Denote the norm in the cross product space \( L^s(\mathbb{R}^n) \times L^s(\mathbb{R}^n) \) by
\[
\|(u, v)\|_{L^s \times L^s} = \|u\|_{L^s} + \|v\|_{L^s},
\]
and define the mapping \( T_{(u_a, v_a)}: L^s(\mathbb{R}^n) \times L^s(\mathbb{R}^n) \rightarrow L^s(\mathbb{R}^n) \times L^s(\mathbb{R}^n) \) by
\[
T_{(u_a, v_a)}(\phi, \varphi) = (T^1_{(u_a, v_a)}(\phi, \varphi), T^2_{(u_a, v_a)}(\phi, \varphi)).
\]

Consider the integral system
\[
(\phi, \varphi) = T_{(u_a, v_a)}(\phi, \varphi) + (F^1, F^2). \tag{2.3}
\]

It is easy to verify that \( (u_a, v_a) \) is a pair of solutions of integral system (2.3).

In order to prove Theorem 1.1, it suffices to verify that \( (u_a, v_a) \in L^s(\mathbb{R}^n) \times L^s(\mathbb{R}^n) \) for \( \max\{p_i, q_i, \alpha_i + \beta_i\} < s < \infty \). We divide the proof into two steps.

Step 1. We show \( (u_a, v_a) \in L^s(\mathbb{R}^n) \times L^s(\mathbb{R}^n) \) for \( \max\{p_i, q_i, \alpha_i + \beta_i\} < s < \infty \). For this purpose, we only need to prove that

(i) \( T_{(u_a, v_a)} \) is a contracting map from \( L^s(\mathbb{R}^n) \times L^s(\mathbb{R}^n) \) to itself for sufficiently large \( a \) for \( \max\{p_i, q_i, \alpha_i + \beta_i\} < s < \infty \).

(ii) \( (F^1, F^2) \) belongs to \( L^s(\mathbb{R}^n) \times L^s(\mathbb{R}^n) \) for \( \max\{p_i, q_i, \alpha_i + \beta_i\} < s < \infty \).

We first show (i). For any \( (\phi, \varphi) \in L^s(\mathbb{R}^n) \times L^s(\mathbb{R}^n) \), we use Lemma 2.3 and Minkowski inequality to write
\[
\|T^1_{(u_a, v_a)}(\phi, \varphi)\|_{L^s} \leq C \left( \left\|u^1_{a, 1} - \phi\right\|_{L^s} + \left\|u^1_{a, 1} - \phi\right\|_{L^s} \right)
\]
\[
+ C \left( \left\|v^1_{a, 1} - \varphi\right\|_{L^s} + \left\|v^1_{a, 1} - \varphi\right\|_{L^s} \right) \tag{2.4}
\]
for all \( s > \max\{p_i, q_i, \alpha_i + \beta_i\} \), where \( \theta_1 = \frac{(p_1 - 1)s + t_1}{t_1}, \theta_2 = \frac{(q_1 - 1)s + t_2}{t_2} \) and \( \theta_3 = \theta_4 = \theta_5 = \alpha_1 + \beta_1 + s + k_0 \).

Similarly, we can also obtain
\[
\|T^2_{(u_a, v_a)}(\phi, \varphi)\|_{L^s} \leq C \left( \left\|v^2_{a, 1} - \phi\right\|_{L^s} + \left\|v^2_{a, 1} - \phi\right\|_{L^s} \right) + \left\|v^2_{a, 1} - \phi\right\|_{L^s} + \left\|v^2_{a, 1} - \phi\right\|_{L^s} \tag{2.5}
\]
for any \( s \geq \max\{p_i, q_i, \alpha_i + \beta_i\} \), where \( \theta_4 = \theta_5 = \alpha_1 + \beta_1 + s + k_0 \).

Since \( u \in L^{t_1}(\mathbb{R}^n) \cap L^{k_0}(\mathbb{R}^n), v \in L^{t_2}(\mathbb{R}^n) \cap L^{k_0}(\mathbb{R}^n) \), we may choose sufficiently large \( a \) such that
\[
C \left( \left\|u^2_{a, 1} - v_b\right\|_{L^{k_0}} \right) \leq \frac{1}{4},
\]
\[
C \left( \left\|v^2_{a, 1} - v_b\right\|_{L^{k_0}} \right) \leq \frac{1}{4},
\]
\[
C \left( \left\|v^2_{a, 1} - u^2_{a, 1}\right\|_{L^{k_0}} \right) \leq \frac{1}{4},
\]
\[
C \left( \left\|u^2_{a, 1} - v_b\right\|_{L^{k_0}} \right) \leq \frac{1}{4}. \tag{2.6}
\]
Hence, combining (2.4), (2.5) and (2.6), we obtain
\[ \|T_{(u,v)}^1(\phi,\varphi)\|_{L^s} \leq \frac{1}{4} (\|\phi\|_{L^s} + \|\varphi\|_{L^s}), \quad \|T_{(u,v)}^2(\phi,\varphi)\|_{L^s} \leq \frac{1}{4} (\|\phi\|_{L^s} + \|\varphi\|_{L^s}). \]

(2.7)

By (2.7), we have
\[ \|T_{(u,v)}(\phi,\varphi)\|_{L^s} \leq \frac{1}{2} (\|\phi\|_{L^s} + \|\varphi\|_{L^s}). \]

(2.8)

It turns out that \( T \) is contracting from \( L^s(\mathbb{R}^n) \times L^s(\mathbb{R}^n) \) to itself.

Next, we estimate \( F^1(x) \) and \( F^2(x) \). Since the estimates of \( F^1(x) \) and \( F^2(x) \) are similar, we only consider \( F^1(x) \). We write
\[ F^1(x) = g_a \ast H^3(x) - u_b(x) \]
\[ = \int_{\mathbb{R}^n} g_a(x-y) \left( \lambda_1 u_b^{p_1}(y) + \mu_1 v_b^{q_1}(y) + \gamma_1 u_b^{q_1} (y) v_b^{r_1}(y) \right) dy - u_b(x). \]

Then for all \( s > 1 \),
\[ \|F^1\|_{L^s} \leq C \left( \|u_b\|_{L^{p_1}} + \|v_b\|_{L^{q_1}} + \|u_b\|_{L^{q_1} L^{r_1}} \right) \]
\[ < \infty. \]

(2.9)

Combining (2.8), (2.9) and Lemma 2.1, we accomplish the proof of Step 1.

**Step 2.** We show \((u, v) \in L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)\). One only needs to prove that \( u \in L^\infty(\mathbb{R}^n) \). By integral system (1.2), we have
\[ |u(x)| = |g_a \ast f_1 (u, v)(x)| \]
\[ \leq C \left( |g_a \ast u^{p_1}(x)| + |g_a \ast v^{q_1}(x)| + |g_a \ast (u^{r_1} v^{r_1})(x)| \right) \]
\[ \leq C \left( \|g_a\|_{L^{m_1}} \|u\|_{L^{p_1}} + \|g_a\|_{L^{m_2}} \|v\|_{L^{q_1}} \right) \]
\[ + \|g_a\|_{L^{m_3}} \|u\|_{L^{q_1} L^{q_1}} \|v\|_{L^{r_1} L^{r_1}} \],

(2.10)

where \( \frac{1}{m_i} + \frac{1}{m_j} = 1, i = 1, 2, 3 \). We choose sufficiently large \( n_1, n_2, n_3 \) such that
\[ \|u\|_{L^{p_1} L^{q_1}}, \|v\|_{L^{q_1}}, \|u\|_{L^{q_1} L^{r_1}}, \|v\|_{L^{r_1} L^{r_1}} < \infty \]

Therefore, one only needs to verify that \( \|g_a\|_{L^{m_i}} < \infty \) when \( m_i \) is sufficiently close to \( 1 \). Since \( g_a \) has exponential decay at infinity, we only need to deal with \( g_a \) at origin. According to Lemma 2.2, we separate three cases.

(i) If \( 0 < \alpha < n \), then
\[ \int_{|x| \leq 2} |g_a(x)|^{m_i} dx \leq C \int_{|x| \leq 2} |x|^{m_i(\alpha-n)} dx < \infty. \]

(ii) If \( n = \alpha \), then
\[ \int_{|x| \leq 2} |g_a(x)|^{m_i} dx \leq C \int_{|x| \leq 2} |\log \frac{2}{|x|}|^{m_i} dx < \infty. \]

(iii) If \( \alpha > n \), then
\[ \int_{|x| \leq 2} |g_a(x)|^{m_i} dx \leq C \int_{|x| \leq 2} dx < \infty. \]

Therefore, we accomplish the proof of Step 2.
3. The proof of Theorem 1.2. In this section, we need the following regularity lifting II to prove Theorem 1.2.

**Lemma 3.1** (Regularity lifting II). Let \( X \) and \( Y \) be an “XY-pair”, and assume that \( X,Y \) are both complete. Let \( X \) and \( Y \) be closed subsets of \( X \) and \( Y \) respectively, and \( T \) be an operator, which is contracting from \( X \) to \( X \) and shrinking from \( Y \) to \( Y \). Define \( Sw = Tw + g \) for some \( g \in X \cap Y \), and assume that \( T : X \cap Y \to X \cap Y \). Then there exists a unique solution \( u \) of the equation \( w = Tw + g \) in \( X \), and \( u \in Y \).

**Remark 1.** \( X \) and \( Y \) are called an “XY-pair” if whenever the sequence \( \{u_n\} \subseteq X \) with \( u_n \to u \) in \( X \) and \( \|u_n\|_Y \leq C \) will imply \( u \in Y \). In Theorem 1.2, we choose \( X = L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \) and \( Y = \Lambda^1(\mathbb{R}^n) \times \Lambda^1(\mathbb{R}^n) \), where \( \Lambda^1(\mathbb{R}^n) \) denotes the space of Lipschitz continuous functions.

Now, we start the proof of Theorem 1.2. Let

\[
X = \{ (\phi, \varphi) \in L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) : \|\phi\|_{L^\infty} \leq 2\|u\|_{L^\infty}, \|\varphi\|_{L^\infty} \leq 2\|v\|_{L^\infty} \}
\]

and

\[
Y = \{ (\phi, \varphi) \in \Lambda^1(\mathbb{R}^n) \times \Lambda^1(\mathbb{R}^n) : \|\phi\|_{L^\infty} \leq 2\|u\|_{L^\infty}, \|\varphi\|_{L^\infty} \leq 2\|v\|_{L^\infty} \}.
\]

Note that

\[
u(x) = g_\alpha * f_1(u,v)(x)
\]

\[
= \int_{\mathbb{R}^n} g_\alpha(x - y)f_1(u(y),v(y))dy
\]

\[
= \int_0^m \int_{\partial B_e(x)} g_\alpha(x - y)f_1(u(y),v(y))dydr
\]

\[
+ \int_m^\infty \int_{\partial B_e(x)} g_\alpha(x - y)f_1(u(y),v(y))dydr
\]

and

\[
v(x) = g_\alpha * f_2(u,v)(x)
\]

\[
= \int_{\mathbb{R}^n} g_\alpha(x - y)f_2(u(y),v(y))dy
\]

\[
= \int_0^m \int_{\partial B_e(x)} g_\alpha(x - y)f_2(u(y),v(y))dydr
\]

\[
+ \int_m^\infty \int_{\partial B_e(x)} g_\alpha(x - y)f_2(u(y),v(y))dydr.
\]

For a fixed real number \( m > 0 \), define

\[
T_m^1(\phi, \varphi)(x) = \int_0^m \int_{\partial B_e(x)} g_\alpha(x - y)f_1(\phi, \varphi)(y)dydr,
\]

\[
G^1(x) = \int_m^\infty \int_{\partial B_e(x)} g_\alpha(x - y)f_1(u, v)(y)dydr,
\]

and

\[
T_m^2(\phi, \varphi)(x) = \int_0^m \int_{\partial B_e(x)} g_\alpha(x - y)f_2(\phi, \varphi)(y)dydr,
\]

\[
G^2(x) = \int_m^\infty \int_{\partial B_e(x)} g_\alpha(x - y)f_2(u, v)(y)dydr.
\]
Consider the following integral system

\[(\phi, \varphi) = (T_m^1(\phi, \varphi), T_m^2(\phi, \varphi)) + (G^1, G^2). \tag{3.1}\]

It is easy to verify that \((u, v)\) is a pair of nonnegative solutions of integral system (3.1). In order to show \((u, v) \in \Lambda^1(\mathbb{R}^n) \times \Lambda^1(\mathbb{R}^n)\), we carry out the proof by four steps

(i) \(T_m = (T_m^1, T_m^2)\) is contracting from \(\mathcal{X}\) to \(L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)\).
(ii) \(T_m = (T_m^1, T_m^2)\) is shrinking from \(\mathcal{Y}\) to \(\Lambda^1(\mathbb{R}^n) \times \Lambda^1(\mathbb{R}^n)\).
(iii) \((G^1, G^2) \in \mathcal{X} \cap \mathcal{Y}\).
(iv) \(S : \mathcal{X} \cap \mathcal{Y} \to \mathcal{X} \cap \mathcal{Y}\).

**Step 1.** For any \((\phi_1, \varphi_1), (\phi_2, \varphi_2) \in \mathcal{X}\),

\[
\|T_m(\phi_1, \varphi_1) - T_m(\phi_2, \varphi_2)\|_{L^\infty \times L^\infty}
\leq \| \int_0^m \int_{\partial B_r(x)} g_\alpha(x - y)f_1(\phi_1, \varphi_1)(y) - f_1(\phi_2, \varphi_2)(y) \, dydr \|_{L^\infty}
+ \| \int_0^m \int_{\partial B_r(x)} g_\alpha(x - y)f_2(\phi_1, \varphi_1)(y) - f_2(\phi_2, \varphi_2)(y) \, dydr \|_{L^\infty}
\leq C \left(\|u\|_{L^\infty}^{p_1-1} + \|u\|_{L^\infty}^{p_2-1} + \|u\|_{L^\infty}^{\alpha_1} \|v\|_{L^\infty}^{\beta_1} + \|u\|_{L^\infty}^{\alpha_2} \|v\|_{L^\infty}^{\beta_2}\right).
\tag{3.2}
\]

According to Lemma 2.2, we can choose sufficiently small \(m\) such that

\[
\left(\|u\|_{L^\infty}^{p_1-1} + \|u\|_{L^\infty}^{p_2-1} + \|u\|_{L^\infty}^{\alpha_1} \|v\|_{L^\infty}^{\beta_1} + \|u\|_{L^\infty}^{\alpha_2} \|v\|_{L^\infty}^{\beta_2}\right) \|g_\alpha\|_{L^1(B_m(0))} < \frac{1}{4}, \tag{3.3}
\]

Combining (3.2) and (3.3), we have

\[
\|T_m(\phi_1, \varphi_1) - T_m(\phi_2, \varphi_2)\|_{L^\infty \times L^\infty} \leq \frac{1}{2}\|\phi_1 - \phi_2\|_{L^\infty}, \tag{3.4}
\]

which implies that \(T_m\) is contracting from \(\mathcal{X}\) to \(L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)\).

**Step 2.** For any \((\phi, \varphi) \in \mathcal{Y}\),

\[
|T_m(\phi, \varphi)(x) - T_m(\phi, \varphi)(z)|
\leq \int_0^m \int_{\partial B_r(x)} g_\alpha(x - y)f_1(\phi, \varphi)(y) - f_1(\phi, \varphi)(y + z - x) \, dydr
+ \int_0^m \int_{\partial B_r(x)} g_\alpha(x - y)f_2(\phi, \varphi)(y) - f_2(\phi, \varphi)(y + z - x) \, dydr
\leq C|x - z| \left(\|u\|_{L^\infty}^{p_1-1} + \|u\|_{L^\infty}^{p_2-1} + \|u\|_{L^\infty}^{\alpha_1} \|v\|_{L^\infty}^{\beta_1} + \|u\|_{L^\infty}^{\alpha_2} \|v\|_{L^\infty}^{\beta_2}\right) \|\phi\|_{\Lambda^1}
+ \left(\|v\|_{L^\infty}^{q_1-1} + \|v\|_{L^\infty}^{q_2-1} + \|u\|_{L^\infty}^{\alpha_1} \|v\|_{L^\infty}^{\beta_1} + \|u\|_{L^\infty}^{\alpha_2} \|v\|_{L^\infty}^{\beta_2}\right) \|\varphi\|_{\Lambda^1} \|g_\alpha\|_{L^1(B_m(0))}. \tag{3.5}
\]
Similarly, we can choose sufficiently small $m$ such that
\[
C\left(\|u\|_{L^1\infty}^{\beta_1} + \|v\|_{L^1\infty}^{\beta_2} + \|w\|_{L^1\infty}^{\beta_3} + \|u\|_{L^2\infty}^{\beta_4} + \|v\|_{L^2\infty}^{\beta_5} + \|w\|_{L^2\infty}^{\beta_6}\right)\|g_0\|_{L^1(B_m(0))} < \frac{1}{8},
\]
which implies that $T_m$ is shrinking from $Y$ to $\Lambda^1(\mathbb{R}^n) \times \Lambda^1(\mathbb{R}^n)$.

**Step 3.** We show that $(G^1, G^2) \in X \cap Y$. Observe that
\[
\left\| (G^1, G^2) \right\|_{L^{\infty} \times L^{\infty}} = \left\| \int_y^\infty \int_{\partial B_r(y)} g(y, x) f_1(u, v)(y) dy dr \right\|_{L^{\infty}}
\]
\[
+ \left\| \int_y^\infty \int_{\partial B_r(y)} g(y, x) f_2(u, v)(y) dy dr \right\|_{L^{\infty}}
\]
\[
\leq \left\| (u, v) \right\|_{L^{\infty} \times L^{\infty}}.
\]
In order to show that $(G^1, G^2) \in Y$, we only need to show that $G^1$ and $G^2$ are Lipschitz continuous because of (3.8). Since the estimates of $G^1$ and $G^2$ are similar, we only prove that $G^1$ is Lipschitz continuous.

Considering that $g_\alpha(x)$ is bounded for $|x| \geq m$, we have
\[
|G^1(x) - G^1(z)|
\]
\[
= \left| \int_y^\infty \int_{\partial B_r(y)} g(y, x) f_1(u, v)(y) dy dr - \int_y^\infty \int_{\partial B_r(z)} g(y, z) f_1(u, v)(y) dy dr \right|
\]
\[
\leq C \left| \int_y^\infty \int_{\partial B_r(x)} f_1(u, v)(y) dy dr - \int_y^\infty \int_{\partial B_r(z)} f_1(u, v)(y) dy dr \right|
\]
\[
\leq C \left\| f_1(u, v) \right\|_{L^{\infty}} |x - z|.
\]

Therefore, $G^1$ is Lipschitz continuous.

**Step 4.** To show that $S: X \cap Y \to X \cap Y$. For $(\phi, \varphi) \in X \cap Y$, since $T_m$ is contracting from $X$ to $L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ and shrinking from $Y$ to $\Lambda^1(\mathbb{R}^n) \times \Lambda^1(\mathbb{R}^n)$, it is easy to verify that $T_m(\phi, \varphi) \in (L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)) \cap (\Lambda^1(\mathbb{R}^n) \times \Lambda^1(\mathbb{R}^n))$. It follows that $S(\phi, \varphi) = T_m(\phi, \varphi) + (G^1, G^2) \in (L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)) \cap (\Lambda^1(\mathbb{R}^n) \times \Lambda^1(\mathbb{R}^n))$. By (3.4) and (3.7), we have
\[
\left\| S(\phi, \varphi) \right\|_{L^{\infty} \times L^{\infty}} \leq \left\| T_m(\phi, \varphi) \right\|_{L^{\infty} \times L^{\infty}} + \left\| (G^1, G^2) \right\|_{L^{\infty} \times L^{\infty}} \leq \frac{3}{2} \left\| (u, v) \right\|_{L^{\infty} \times L^{\infty}},
\]
which implies $S(\phi, \varphi) \in X \cap Y$. Therefore, we can conclude that $u$ and $v$ are Lipschitz continuous. This completes the proof of Theorem 1.2.

4. **The proof of Theorem 1.3.** In this section, we use the method of moving plane in integral forms introduced by Chen, Li and Ou [7] to prove that each pair of positive solutions $(u, v)$ of (1.1) is radially symmetric and strictly decreasing about some point. In order to prove our theorem, we first introduce some notations.

For any real number $\lambda$, let the moving plane be $T_\lambda = \{ x : x_1 = \lambda \}$ and denote
\[
\Sigma_\lambda = \{ x = (x_1, x_2, \ldots, x_n) : x_1 \leq \lambda \}.
\]
Let $x_\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$ be the reflection of the point $x$ about the moving plane $T_\lambda$, and define $u_\lambda(x) = u(x_\lambda)$ and $v_\lambda(x) = v(x_\lambda)$.
Lemma 4.1. If \((u, v)\) is a pair of positive solutions of (1.2), for any \(x \in \Sigma_\lambda\) and \(x \neq y\), we have

\[
\begin{align*}
    u(x) - u(x^\lambda) &= \int_{\Sigma_\lambda} \left( g_\alpha(x - y) - g_\alpha(x^\lambda - y) \right) \{f_1(u(y), v(y)) - f_1(u_\lambda(y), v_\lambda(y))\} \, dy \\
    v(x) - v(x^\lambda) &= \int_{\Sigma_\lambda} \left( g_\alpha(x - y) - g_\alpha(x^\lambda - y) \right) \{f_2(u(y), v(y)) - f_2(u_\lambda(y), v_\lambda(y))\} \, dy.
\end{align*}
\]

Proof. By (1.2), we have

\[
\begin{align*}
    u(x) &= \int_{\Sigma_\lambda} g_\alpha(x - y) f_1(u(y), v(y)) \, dy + \int_{\mathbb{R}^n \setminus \Sigma_\lambda} g_\alpha(x - y) f_1(u(y), v(y)) \, dy \\
    &= \int_{\Sigma_\lambda} g_\alpha(x - y) f_1(u(y), v(y)) \, dy
\end{align*}
\]

and

\[
\begin{align*}
    u(x^\lambda) &= \int_{\Sigma_\lambda} g_\alpha(x^\lambda - y) f_1(u(y), v(y)) \, dy + \int_{\Sigma_\lambda} g_\alpha(x^\lambda - y) f_1(u_\lambda(y), v_\lambda(y)) \, dy.
\end{align*}
\]

Hence we use the fact that \(|x - y^\lambda| = |x^\lambda - y|\) and \(|x^\lambda - y^\lambda| = |x - y|\), we obtain

\[
\begin{align*}
    u(x) - u(x^\lambda) &= \int_{\Sigma_\lambda} \left( g_\alpha(x - y) - g_\alpha(x^\lambda - y) \right) f_1(u(y), v(y)) \, dy \\
    &+ \int_{\Sigma_\lambda} \left( g_\alpha(x - y^\lambda) - g_\alpha(x^\lambda - y^\lambda) \right) f_1(u_\lambda(y), v_\lambda(y)) \, dy \\
    &= \int_{\Sigma_\lambda} \left( g_\alpha(x - y) - g_\alpha(x^\lambda - y) \right) \{f_1(u(y), v(y)) - f_1(u_\lambda(y), v_\lambda(y))\} \, dy.
\end{align*}
\]

Similarly, we can obtain the second formula. This completes the proof of Lemma 4.1. \(\square\)

We start to prove Theorem 1.3. One can split the proof into two steps.

Step 1. For \(\lambda < 0\), we define

\[
    \Sigma^u_\lambda = \{x \in \Sigma_\lambda : u(x) > u_\lambda(x)\} \quad \Sigma^v_\lambda = \{x \in \Sigma_\lambda : v(x) > v_\lambda(x)\}.
\]

We want to show that both \(\Sigma^u_\lambda\) and \(\Sigma^v_\lambda\) are empty for \(\lambda\) sufficiently negative. By Lemma 4.1, for any \(x \in \Sigma^u_\lambda\), we have

\[
\begin{align*}
    u(x) - u(x^\lambda) &= \int_{\Sigma_\lambda} \left( g_\alpha(x - y) - g_\alpha(x^\lambda - y) \right) \{f_1(u(y), v(y)) - f_1(u_\lambda(y), v_\lambda(y))\} \, dy \\
    &\leq \lambda_1 p_1 \int_{\Sigma^u_\lambda} g_\alpha(x - y) u^{p_1 - 1}(y)(u(y) - u_\lambda(y)) \, dy \\
    &+ \mu_1 q_1 \int_{\Sigma^u_\lambda} g_\alpha(x - y) v^{q_1 - 1}(y)(v(y) - v_\lambda(y)) \, dy \\
    &+ \gamma_1 \alpha_1 \int_{\Sigma^u_\lambda} g_\alpha(x - y) u^{\alpha_1 - 1}(y)v^{\beta_1}(y)(u(y) - u_\lambda(y)) \, dy \\
    &+ \gamma_1 \beta_1 \int_{\Sigma^u_\lambda} g_\alpha(x - y) u^{\alpha_1}(y)v^{\beta_1 - 1}(y)(v(y) - v_\lambda(y)) \, dy \\
    &=: A_1(x) + A_2(x) + A_3(x) + A_4(x).
\end{align*}
\]
For any $s > \max\{p_i, q_i, \alpha_i + \beta_i\}$ ($i = 1, 2$), we apply Lemma 2.3 and Minkowski inequality to (4.1) and obtain,

$$
\|u - u_\lambda\|_{L^s(\Sigma^\lambda)} \leq C\left(\|A_1\|_{L^s(\Sigma^\lambda)} + \|A_2\|_{L^s(\Sigma^\lambda)} + \|A_3\|_{L^s(\Sigma^\lambda)} + \|A_4\|_{L^s(\Sigma^\lambda)}\right)
$$

$$
\leq C\left(\|u^{\alpha_1-1}(u - u_\lambda)\|_{L^s(\Sigma^\lambda)} + \|u^{\alpha_1-1}(v - v_\lambda)\|_{L^s(\Sigma^\lambda)}
+ \|u^{\alpha_1}v^{\beta_1}(u - u_\lambda)\|_{L^s(\Sigma^\lambda)} + \|u^{\alpha_1}v^{\beta_1}(v - v_\lambda)\|_{L^s(\Sigma^\lambda)}\right)
$$

$$
\leq C\left(\|u\|_{L^{p_1-1}(\Sigma^\lambda)}^2\|u - u_\lambda\|_{L^s(\Sigma^\lambda)} + \|v\|_{L^{q_1-1}(\Sigma^\lambda)}^2\|v - v_\lambda\|_{L^s(\Sigma^\lambda)}
+ \|u\|_{L^{p_2-1}(\Sigma^\lambda)}^2\|u - u_\lambda\|_{L^s(\Sigma^\lambda)} + \|v\|_{L^{q_2-1}(\Sigma^\lambda)}^2\|v - v_\lambda\|_{L^s(\Sigma^\lambda)}\right),
$$

(4.2)

where $\theta_1 = \frac{(p_1-1)s}{t_1} + 1$, $\theta_2 = \frac{(q_1-1)s}{t_2} + 1$ and $\theta_3 = \theta_4 = \frac{(\alpha_1+\beta_1-1)s}{\mu_0} + 1$.

Similarly, we can also deduce that

$$
\|v - v_\lambda\|_{L^s(\Sigma^\lambda)} \leq C\left(\|v\|_{L^{p_1-1}(\Sigma^\lambda)}^2\|u - u_\lambda\|_{L^s(\Sigma^\lambda)} + \|v\|_{L^{q_1-1}(\Sigma^\lambda)}^2\|v - v_\lambda\|_{L^s(\Sigma^\lambda)}
+ \|u\|_{L^{p_2-1}(\Sigma^\lambda)}^2\|u - u_\lambda\|_{L^s(\Sigma^\lambda)} + \|v\|_{L^{q_2-1}(\Sigma^\lambda)}^2\|v - v_\lambda\|_{L^s(\Sigma^\lambda)}\right),
$$

(4.3)

In virtue of the conditions $(u, v) \in \Pi_1 \times \Pi_2$, we can choose sufficiently negative $\lambda$ such that

$$
C\|u\|_{L^{p_1-1}(\Sigma^\lambda)}^2 + C\|u\|_{L^{p_2-1}(\Sigma^\lambda)}^2 < \frac{1}{4},
$$

$$
C\|v\|_{L^{q_1-1}(\Sigma^\lambda)}^2 + C\|v\|_{L^{q_2-1}(\Sigma^\lambda)}^2 < \frac{1}{4},
$$

(4.4)

By (4.2), (4.3) and (4.4), we have

$$
\|u - u_\lambda\|_{L^s(\Sigma^\lambda)} = 0, \quad \|v - v_\lambda\|_{L^s(\Sigma^\lambda)} = 0,
$$

which implies that $\Sigma^u_\lambda$ and $\Sigma^v_\lambda$ must be measure zero. Therefore, both $\Sigma^u_\lambda$ and $\Sigma^v_\lambda$ must be empty sets.

Step 2. For sufficiently negative $\lambda$, we have shown that

$$
u(x) \leq u_\lambda(x), \quad v(x) \leq v_\lambda(x), \quad \forall x \in \Sigma_\lambda.
$$

(4.5)

The inequality (4.5) provides a starting point to move the plane $T_\lambda = \{x \in \mathbb{R}^n : x_1 = \lambda\}$. Now we start from the negative infinity of $x_1$-axis and move the plane to the right as long as (4.5) holds. Define

$$
\lambda_0 = \sup\{\lambda : u_\mu(x) \geq u(x), \quad v_\mu(x) \geq v(x), \quad \mu \leq \lambda, \quad \forall x \in \Sigma_\mu\},
$$

then we have $u(x) \leq u_{\lambda_0}(x)$ and $v(x) \leq v_{\lambda_0}(x)$.

We show that if $\lambda_0 < 0$, both $u$ and $v$ are symmetric and strictly decreasing about the plane $x_1 = \lambda_0$. If not, we may assume that $u_{\lambda_0}(x) \neq u(x)$. By Lemma 4.1, we can obtain $v_{\lambda_0}(x) \neq v(x)$. Furthermore, we can also derive $u_{\lambda_0}(x) > u(x)$ and $v_{\lambda_0}(x) > v(x)$ in the interior of $\Sigma_{\lambda_0}$. 


Next, we will show that the plane can be moved further to the right. More precisely, there exists an \( \varepsilon > 0 \) such that for any \( \lambda \in [\lambda_0, \lambda_0 + \varepsilon) \),

\[
    u(x) \leq u_\lambda(x), \quad v(x) \leq v_\lambda(x), \quad \forall x \in \Sigma_\lambda.
\]

(4.6)

Let

\[
    \Sigma^u_{\lambda_0} = \{ x \in \Sigma_{\lambda_0} : u(x) \geq u_{\lambda_0}(x) \}, \quad \Sigma^v_{\lambda_0} = \{ x \in \Sigma_{\lambda_0} : v(x) \geq v_{\lambda_0}(x) \}.
\]

We can derive that \( \Sigma^u_{\lambda_0} \) and \( \Sigma^v_{\lambda_0} \) are measure zero, and \( \lim_{\lambda \to \lambda_0} \Sigma^u_\lambda \subset \Sigma^u_{\lambda_0} \), \( \lim_{\lambda \to \lambda_0} \Sigma^v_\lambda \subset \Sigma^v_{\lambda_0} \). This together with integrability conditions \((u, v) \in \Pi_1 \times \Pi_2\) ensures that one can choose \( \varepsilon \) small enough such that for all \( \lambda \in [\lambda_0, \lambda_0 + \varepsilon) \)

\[
    C\|u\|_{L^1(t_1, t_2)} + C\|u\|_{L^6(\Sigma^u_\lambda)} \|v\|_{L^6(\Sigma^v_\lambda)} < \frac{1}{4},
\]

\[
    C\|v\|_{L^1(t_1, t_2)} + C\|u\|_{L^6(\Sigma^u_\lambda)} \|v\|_{L^6(\Sigma^v_\lambda)} < \frac{1}{4},
\]

\[
    C\|u\|_{L^1(t_1, t_2)} + C\|u\|_{L^6(\Sigma^u_\lambda)} \|v\|_{L^6(\Sigma^v_\lambda)} < \frac{1}{4},
\]

\[
    C\|v\|_{L^1(t_1, t_2)} + C\|u\|_{L^6(\Sigma^u_\lambda)} \|v\|_{L^6(\Sigma^v_\lambda)} < \frac{1}{4}.
\]

Using the similar estimates as (4.2) and (4.3), we can derive that \( \|u - u_\lambda\|_{L^6(\Sigma^u_\lambda)} = \|v - v_\lambda\|_{L^6(\Sigma^v_\lambda)} = 0 \), thus \( \Sigma^u_{\lambda} \) and \( \Sigma^v_{\lambda} \) must be measure zero, which verifies (4.6).

If \( \lambda_0 \geq 0 \), we can move the plane in the negative \( x_1 \) direction from positive infinity toward the origin. If the plane \( T_\lambda \) stops somewhere before the origin, we conclude that \( u \) and \( v \) are symmetric and strictly decreasing about the plane. If they stop at the origin, we also obtain that \( u \) and \( v \) are symmetric and strictly decreasing in the \( x_1 \) direction by combining the two inequalities obtained in the two opposite directions. Since we can choose any direction to start the process, we conclude that \( u \) and \( v \) are radially symmetric and strictly decreasing about some point.

5. The proof of Theorem 1.4. By Theorem 1.3, we know that \( u \) and \( v \) are radially symmetric about some point \( x_0 \), we may assume that \( x_0 = 0 \). For \( \alpha = 2 \), \( \lambda_1 = \mu_2 = 0, \mu_1 = \lambda_2, \gamma_1 = \gamma_2, q_1 = p_2 \), by (1.1), we have

\[
    r^{\alpha_n+1}u(r) - (r^{\alpha_n-1}u'(r))' = r^{\alpha_n+1}(\mu_1 v^{q_1}(r) + \gamma_1 u^{\alpha_1}(r)v^{\beta_1}(r))
\]

and

\[
    r^{\alpha_n+1}v(r) - (r^{\alpha_n-1}v'(r))' = r^{\alpha_n+1}(\mu_1 u^{q_1}(r) + \gamma_1 u^{\alpha_2}(r)v^{\beta_2}(r)).
\]

Then, integrating both sides of the above two equations two times, we have

\[
    u(r) = u(0) + \int_0^r \int_0^\tau \rho^{\alpha_n-1}(u - \mu_1 v^{q_1} - \gamma_1 u^{\alpha_1} v^{\beta_1})d\rho d\tau,
\]

(5.1)

\[
    v(r) = v(0) + \int_0^r \int_0^\tau \rho^{\alpha_n-1}(v - \mu_1 v^{q_1} - \gamma_1 u^{\alpha_2} v^{\beta_2})d\rho d\tau.
\]

(5.2)

To prove the uniqueness, we only need to show that \( u(0) = v(0) \). Otherwise, suppose that \( u(0) < v(0) \). By the continuity, for small \( r > 0 \) and \( \alpha_1 < \alpha_2 \), we have

\[
    u(r) < v(r).
\]

Therefore, there exists an \( \epsilon \) such that

\[
    u(r) < v(r), \quad \forall r \in (0, \epsilon).
\]

(5.3)
Let $u_0$ be the supreme value of $u$ such that (5.3) holds, so we have $u(t_0) = v(t_0)$. Since $\alpha_1 + \beta_1 = \alpha_2 + \beta_2$ and $\alpha_1 < \alpha_2$, by (5.1) and (5.2), we have
\[ u(0) - v(0) = \int_0^{t_0} \tau^{1-n} \int_0^\tau \rho^{n-1} (v - u + \mu_1 v^q_1 - \mu_1 u_1 + \gamma_1 u^{a_1} v^{b_1} - \gamma_1 u^{a_2} v^{b_2}) d\rho d\tau. \]
This is impossible. Similarly, we also can prove that $u(0) > v(0)$ is impossible. Then we obtain that $u(0) = v(0)$. Furthermore, by the standard ODE theory, we derive that $u(r) = v(r)$. This completes the proof of Theorem 1.4.

Acknowledgments. The authors wish to thank the referees for their careful reading which improve the exposition of the paper.

REFERENCES

[1] J. Bao, N. Lam and G. Lu, Polyharmonic equations with critical exponential growth in the whole space $\mathbb{R}^n$, *Discrete Contin. Dyn. Syst.*, 36 (2016), 577–600.
[2] L. Caffarelli, B. Gidas and J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equation with critical Sobolev growth, *Commun. Pure Appl. Math.*, 42 (1989), 271–297.
[3] A. Chang and P. Yang, On uniqueness of solutions of nth order differential equations in conformal geometry, *Math. Res. Lett.*, 4 (1997), 91–102.
[4] W. Chen and C. Li, Methods on Nonlinear Elliptic Equations, AIMS Book Series on Diff. Equa. and Dyn. Sys., Vol. 4, 2010.
[5] W. Chen and C. Li, Classification of solutions of some nonlinear elliptic equations, *Duke Math. J.*, 63 (1991), 615–622.
[6] W. Chen and C. Li, Regularity of solutions for a system of integral equations, *Commun. Pure Appl. Math.*, 59 (2006), 330–343.
[7] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation, *Comm. Pure Appl. Math.*, 59 (2006), 330–343.
[8] W. Chen, C. Li and B. Ou, Classification of solutions for a system of integral equations, *Comm. Partial Differential Equations*, 30 (2005), 59–65.
[9] Y. Fang and W. Chen, A Liouville type theorem for poly-harmonic Dirichlet problems in a half space, *Adv. Math.*, 229 (2012), 2835–2867.
[10] B. Gidas, W. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.*, 68 (1979), 209–243.
[11] B. Gidas, W. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, *Mathematical Analysis and Applications*, vol. 7a of the book series Advances in Mathematics, Academic Press, New York, 1981.
[12] X. Han and G. Lu, Regularity of solutions to an integral equation associated with Bessel potential, *Commun. Pure Appl. Anal.*, 10 (2011), 1111–1119.
[13] X. Han, G. Lu and J. Zhu, Characterization of balls in terms of Bessel-potential integral equation, *J. Differential Equations*, 252 (2012), 1589–1602.
[14] C. Jin and C. Li, Symmetry of solution to some systems of integral equations, *Proc. Amer. Math. Soc.*, 134 (2006), 1661–1670.
[15] M. K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$, *Arch. Ration. Mech. Anal.*, 105 (1989), 243–266.
[16] N. Lam and G. Lu, Existence of nontrivial solutions to polyharmonic equations with subcritical and critical exponential growth, *Discrete Contin. Dyn. Syst.*, 32 (2012), 2187–2205.
[17] C. Li and J. Lim, The singularity analysis of solutions to some integral equations, *Commun. Pure Appl. Anal.*, 6 (2007), 453–464.
[18] C. Ma, W. Chen and C. Li, Regularity of solutions for an integral system of Wolff type, *Adv. Math.*, 226 (2011), 2676–2699.
[19] L. Ma and D. Chen, Radial symmetry and monotonicity for an integral equation, *J. Math. Anal. Appl.*, 342 (2008), 943–949.
[20] W. Reichel, Characterization of balls by Riesz-potentials, *Ann. Mat.*, 188 (2009), 235–245.
[21] J. Serrin, A symmetry problem in potential theory, *Arch. Ration. Mech. Anal.*, 43 (1971), 304–318.
[22] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Grundlehren der Mathematischen Wissenschaften, vol. 258, Springer-Verlag, New York, Berlin, 1983.

[23] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Ser. Appl. Math., vol. 32, Princeton Univ. Press, Princeton, NJ, 1970.

Received August 2015; revised November 2015.

E-mail address: luchen@mail.bnu.edu.cn
E-mail address: liuzhao@mail.bnu.edu.cn
E-mail address: gzlu@wayne.edu