Local Linearizability

Andreas Haas, Thomas A. Henzinger, Andreas Holzer, Christoph M. Kirsch, Michael Lippautz, Hannes Payer, Ali Sezgin, Ana Sokolova, and Helmut Veith

1 Google Inc.
2 IST Austria, Austria
3 University of Toronto, Canada
4 University of Salzburg, Austria
5 University of Cambridge, UK
6 Vienna University of Technology, Austria

Abstract

The semantics of concurrent data structures is usually given by a sequential specification and a consistency condition. Linearizability is the most popular consistency condition due to its simplicity and general applicability. Nevertheless, for applications that do not require all guarantees offered by linearizability, recent research has focused on improving performance and scalability of concurrent data structures by relaxing their semantics.

In this paper, we present local linearizability, a relaxed consistency condition that is applicable to container-type concurrent data structures like pools, queues, and stacks. While linearizability requires that the effect of each operation is observed by all threads at the same time, local linearizability only requires that for each thread T, the effects of its local insertion operations and the effects of those removal operations that remove values inserted by T are observed by all threads at the same time. We investigate theoretical and practical properties of local linearizability and its relationship to many existing consistency conditions. We present a generic implementation method for locally linearizable data structures that uses existing linearizable data structures as building blocks. Our implementations show performance and scalability improvements over the original building blocks and outperform the fastest existing container-type implementations.

1 Introduction

Concurrent data structures are pervasive all along the software stack, from operating system code to application software and beyond. Both correctness and performance are imperative for concurrent data structure implementations. Correctness is usually specified by relating concurrent executions, admitted by the implementation, with sequential executions, admitted by the sequential version of the data structure. The latter form the sequential specification of the data structure. This relationship is formally captured by consistency conditions, such as linearizability, sequential consistency, or quiescent consistency [25].

Linearizability [26] is the most accepted consistency condition for concurrent data structures due to its simplicity and general applicability. It guarantees that the effects of all
operations by all threads are observed consistently. This global visibility requirement imposes the need of extensive synchronization among threads which may in turn jeopardize performance and scalability. In order to enhance performance and scalability of implementations, recent research has explored relaxed sequential specifications [23, 40, 2], resulting in well-performing implementations of concurrent data structures [2, 18, 23, 28, 38, 6]. Except for [27], the space of alternative consistency conditions that relax linearizability has been left unexplored to a large extent. In this paper, we explore (part of) this gap by investigating local linearizability, a novel consistency condition that is applicable to a large class of concurrent data structures that we call container-type data structures, or containers for short. Containers include pools, queues, and stacks. A fine-grained spectrum of consistency conditions enables us to describe the semantics of concurrent implementations more precisely, e.g., we show in our appendix that work stealing queues [35] which could only be proven to be linearizable wrt pool are actually locally linearizable wrt double-ended queue.

Local linearizability is a (thread-)local consistency condition that guarantees that insertions per thread are observed consistently. While linearizability requires a consistent view over all insertions, we only require that projections of the global history—so called thread-induced histories—are linearizable. The induced history of a thread $T$ is a projection of a program execution to the insert-operations in $T$ combined with all remove-operations that remove values inserted by $T$ irrespective of whether they happen in $T$ or not. Then, the program execution is locally linearizable iff each thread-induced history is linearizable. Consider the example (sequential) history depicted in Figure 1. It is not linearizable wrt a queue since the values are not dequeued in the same order as they were enqueued. However, each thread-induced history is linearizable wrt a queue and, therefore, the overall execution is locally linearizable wrt a queue. In contrast to semantic relaxations based on relaxing sequential semantics such as [23, 2], local linearizability coincides with sequential correctness for single-threaded histories, i.e., a single-threaded and, therefore, sequential history is locally linearizable wrt a given sequential specification if and only if it is admitted by the sequential specification.

Local linearizability is to linearizability what coherence is to sequential consistency. Coherence [22], which is almost universally accepted as the absolute minimum that a shared memory system should satisfy, is the requirement that there exists a unique global order per shared memory location. Thus, while all accesses by all threads to a given memory location have to conform to a unique order, consistent with program order, the relative ordering of accesses to multiple memory locations do not have to be the same. In other words, coherence is sequential consistency per memory location. Similarly, local linearizability is linearity per local history. In our view, local linearizability offers enough consistency for the correctness of many applications as it is the local view of the client that often matters. For example, in a locally linearizable queue each client (thread) has the impression of using a perfect queue—no reordering will ever be observed among the values inserted by a single thread. Such guarantees suffice for many e-commerce and cloud applications. Implementations of locally linearizable data structures have been successfully applied for managing free lists in the design of the fast and scalable memory allocator scalloc [5]. Moreover, except for fairness, locally linearizable queues guarantee all properties required from Dispatch Queues [1], a common concurrency programming mechanism on mobile devices.
In this paper, we study theoretical and practical properties of local linearizability. Local linearizability is compositional—a history over multiple concurrent objects is locally linearizable iff all per-object histories are locally linearizable (see Thm. 12) and locally linearizable container-type data structures, including queues and stacks, admit only “sane” behaviours—no duplicated values, no values returned from thin air, and no values lost (see Prop. 4). Local linearizability is a weakening of linearizability for a natural class of data structures including pools, queues, and stacks (see Sec. 4). We compare local linearizability to linearizability, sequential, and quiescent consistency, and to many shared-memory consistency conditions.

Finally, local linearizability leads to new efficient implementations. We present a generic implementation scheme that, given a linearizable implementation of a sequential specification \( S \), produces an implementation that is locally linearizable wrt \( S \) (see Sec. 6). Our implementations show dramatic improvements in performance and scalability. In most cases the locally linearizable implementations scale almost linearly and even outperform state-of-the-art pool implementations. We produced locally linearizable variants of state-of-the-art concurrent queues and stacks, as well as of the relaxed data structures from [23, 28]. The latter are relaxed in two dimensions: they are locally linearizable (the consistency condition is relaxed) and are out-of-order-relaxed (the sequential specification is relaxed). The speedup of the locally linearizable implementation to the fastest linearizable queue (LCRQ) and stack (TS Stack) implementation at 80 threads is 2.77 and 2.64, respectively. Verification of local linearizability, i.e. proving correctness, for each of our new locally linearizable implementations is immediate, given that the starting implementations are linearizable.

2 Semantics of Concurrent Objects

The common approach to define the semantics of an implementation of a concurrent data structure is (1) to specify a set of valid sequential behaviors—the sequential specification, and (2) to relate the admissible concurrent executions to sequential executions specified by the sequential specification—via the consistency condition. That means that an implementation of a concurrent data structure actually corresponds to several sequential data structures, and vice versa, depending on the consistency condition used. A (sequential) data structure \( D \) is an object with a set of method calls \( \Sigma \). We assume that method calls include parameters, i.e., input and output values from a given set of values. The sequential specification \( S \) of \( D \) is a prefix-closed subset of \( \Sigma^* \). The elements of \( S \) are called \( D \)-valid sequences. For ease of presentation, we assume that each value in a data structure can be inserted and removed at most once. This is without loss of generality, as we may see the set of values as consisting of pairs of elements (core values) and version numbers, i.e. \( V = E \times \mathbb{N} \). Note that this is a technical assumption that only makes the presentation and the proofs simpler, it is not needed and not done in locally linearizable implementations. While elements may be inserted and removed multiple times, the version numbers provide uniqueness of values. Our assumption ensures that whenever a sequence \( s \) is part of a sequential specification \( S \), then, each method call in \( s \) appears exactly once. An additional core value, that is not an element, is \( \text{empty} \). It is returned by remove method calls that do not find an element to return. We denote by \( \text{Emp} \) the set of values that are versions of \( \text{empty} \), i.e., \( \text{Emp} = \{\text{empty}\} \times \mathbb{N} \).

**Definition 1** (Appears-before Order, Appears-in Relation). Given a sequence \( s \in \Sigma^* \) in which each method call appears exactly once, we denote by \( \prec_s \) the total appears-before order over method calls in \( s \). Given a method call \( m \in \Sigma \), we write \( m \in s \) for \( m \) appears in \( s \).

Throughout the paper, we will use pool, queue, and stack as typical examples of containers. We specify their sequential specifications in an axiomatic way [24], i.e., as sets of
Definition 2 (Pool, Queue, & Stack). A pool, queue, and stack with values in a set $V$ have the sets of methods $\Sigma_P = \{\text{ins}(x), \text{rem}(x) \mid x \in V\} \cup \{\text{rem}(e) \mid e \in \text{Emp}\}$, $\Sigma_Q = \{\text{enq}(x), \text{deq}(x) \mid x \in V\} \cup \{\text{deq}(e) \mid e \in \text{Emp}\}$, and $\Sigma_S = \{\text{push}(x), \text{pop}(x) \mid x \in V\} \cup \{\text{pop}(e) \mid e \in \text{Emp}\}$, respectively. We denote the sequential specification of a pool by $S_P$, the sequential specification of a queue by $S_Q$, and the sequential specification of a stack by $S_S$. A sequence $s \in \Sigma_P^*$ belongs to $S_P$ iff it satisfies axioms (1) - (3) in Table 1—the pool axioms—when instantiating $i()$ with $\text{ins}()$ and $r()$ with $\text{rem}()$. We keep axiom (1) for completeness, although it is subsumed by our assumption that each value is inserted and removed at most once. Specification $S_Q$ contains all sequences $s$ that satisfy the pool axioms and axiom (4)—the queue order axiom—after instantiating $i()$ with $\text{enq}()$ and $r()$ with $\text{deq}()$. Finally, $S_S$ contains all sequences $s$ that satisfy the pool axioms and axiom (5)—the stack order axiom—after instantiating $i()$ with $\text{push}()$ and $r()$ with $\text{pop}()$.

We represent concurrent executions via concurrent histories. An example history is shown in Figure 1. Each thread executes a sequence of method calls from $\Sigma$; method calls executed by different threads may overlap (which does not happen in Figure 1). The real-time duration of method calls is irrelevant for the semantics of concurrent objects; all that matters is whether method calls overlap. Given this abstraction, a concurrent history is fully determined by a sequence of invocation and response events of method calls.

We distinguish method invocation and response events by augmenting the alphabet. Let $\Sigma_i = \{m_i \mid m \in \Sigma\}$ and $\Sigma_r = \{m_r \mid m \in \Sigma\}$ denote the sets of method-invocation events and method-response events, respectively, for the method calls in $\Sigma$. Moreover, let $I$ be the set of thread identifiers. Let $\Sigma'_i = \{m^i_k \mid m \in \Sigma, k \in I\}$ and $\Sigma'_r = \{m^r_k \mid m \in \Sigma, k \in I\}$ denote the sets of method-invocation and -response events augmented with identifiers of executing threads. For example, $m^i_k$ is the invocation of method call $m$ by thread $k$. Before we proceed, we mention a standard notion that we will need in several occasions.

Definition 3 (Projection). Let $s$ be a sequence over alphabet $\Sigma$ and $M \subseteq \Sigma$. By $s|_M$ we denote the projection of $s$ on the symbols in $M$, i.e., the sequence obtained from $s$ by removing all symbols that are not in $M$.

Definition 4 (History). A (concurrent) history $h$ is a sequence in $(\Sigma'_i \cup \Sigma'_r)^*$ where (1) no invocation or response event appears more than once, i.e., if $h = m_1 \ldots m_n$ and $m_h = m^i_k(x)$ and $m_j = m^i_l(x)$, for $s \in \{i, r\}$, then $h = j$ and $k < l$, and (2) if a response event $m^r_k$ appears in $h$, then the corresponding invocation event $m^i_k$ also appears in $h$ and $m_i \preceq_m h_m_r$.

Example 5. A queue history (left) and its formal representation as a sequence (right):

Table 1 The pool axioms (1), (2), (3); the queue order axiom (4); the stack order axiom (5)
A history is sequential if every response event is immediately preceded by its matching invocation event and vice versa. Hence, we may ignore thread identifiers and identify a sequential history with a sequence in $\Sigma^*$. The linearization of a sequential history is a sequential history with a sequence in $\Sigma^*$. For example, the linearization of a sequential history $\text{seq}(1)\text{seq}(2)\text{deq}(2)\text{deq}(1)$ identifies the sequential history in Figure 1.

A history $h$ is well-formed if $h[k]$ is sequential for every thread identifier $k \in I$ where $h[k]$ denotes the projection of $h$ on the set $\{m^k_\Sigma \mid m \in \Sigma\} \cup \{m^k_m \mid m \in \Sigma\}$ of events that are local to thread $k$. From now on we will use the term history for well-formed history. Also, we may omit thread identifiers if they are not essential in a discussion.

A history $h$ determines a partial order on its set of method calls, the precedence order:

Definition 6 (Appears-in Relation, Precedence Order). The set of method calls of a history $h$ is $M(h) = \{m \mid m_\Sigma \in h\}$. A method call $m$ appears in $h$, notation $m \in h$, if $m \in M(h)$. The precedence order for $h$ is the partial order $\prec_h$ such that, for $m, n \in h$, we have that $m \prec_n n$ iff $m \prec_n n$. By $\prec_h$ we denote $\prec_{h[k]}$, the subset of the precedence order that relates pairs of method calls of thread $k$, i.e., the program order of thread $k$.

We can characterize a sequential history as a history whose precedence order is total. In particular, the precedence order $\prec_s$ of a sequential history $s$ coincides with its appears-before order $\prec_s$. The total order for history $s$ in Fig. 1 is $\text{seq}(1) \prec_s \text{seq}(2) \prec_s \text{deq}(2) \prec_s \text{deq}(1)$.

Definition 7 (Projection to a set of method calls). Let $h$ be a history, $M \subseteq \Sigma$, $M_I = \{m^k_\Sigma \mid m \in M, k \in I\}$, and $M_I^f = \{m^k_m \mid m \in M, k \in I\}$. Then, we write $h|M$ for $h|(M_I \cup M_I^f)$.

Note that $h|M$ inherits $h$’s precedence order: $m \prec_{h|M} n \iff m \in M \land n \in M \land m \prec_h n$.

A history $h$ is complete if the response of every invocation event in $h$ appears in $h$. Given a history $h$, Complete$(h)$ denotes the set of all completions of $h$, i.e., the set of all complete histories that are obtained from $h$ by appending missing response events and/or removing pending invocation events. Note that Complete$(h) = \{h\}$ iff $h$ is a complete history.

A concurrent data structure $D$ over a set of methods $\Sigma$ is a (prefix-closed) set of concurrent histories over $\Sigma$. A history may involve several concurrent objects. Let $O$ be a set of concurrent objects with individual sets of method calls $\Sigma_q$ and sequential specifications $S_q$ for each object $q \in O$. A history $h$ over $O$ is a history over the (disjoint) union of method calls of all objects in $O$, i.e., it has a set of method calls $\bigcup_{q \in O}\{q.m \mid m \in \Sigma_q\}$. The added prefix $q$ ensures that the union is disjoint. The projection of $h$ to an object $q \in O$, denoted by $h|_q$, is the history with a set of method calls $\Sigma_q$ obtained by removing the prefix $q$, in every method call in $h|_q\{q.m \mid m \in \Sigma_q\}$.

Definition 8 (Linearizability [26]). A history $h$ is linearizable wrt the sequential specification $S$ if there is a sequential history $s \in S$ and a completion $h_c \in \text{Complete}(h)$ such that (1) $s$ is a permutation of $h_c$, and (2) $s$ preserves the precedence order of $h_c$, i.e., if $m \prec_h n$, then $m \prec_s n$. We refer to $s$ as a linearization of $h$. A concurrent data structure $D$ is linearizable wrt $S$ if every history $h$ of $D$ is linearizable wrt $S$. A history $h$ over a set of concurrent objects $O$ is linearizable wrt the sequential specifications $S_q$ for $q \in O$ if there exists a linearization $s$ of $h$ such that $s|q \in S_q$ for each object $q \in O$.

3 Local Linearizability

Local linearizability is applicable to containers whose set of method calls is a disjoint union $\Sigma = \text{Ins} \cup \text{Rem} \cup \text{D0b} \cup \text{S0b}$ of insertion method calls $\text{Ins}$, removal method calls $\text{Rem}$, data-observation method calls $\text{D0b}$, and (global) shape-observation method calls $\text{S0b}$. Insertions (removals) insert (remove) a single value in the data set $V$ or empty; data observations return
Local Linearizability

a single value in $V$; shape observations return a value (not necessarily in $V$) that provides information on the shape of the state, for example, the size of a data structure. Examples of data observations are head$(x)$ (queue), top$(x)$ (stack), and peek$(x)$ (pool). Examples of shape observations are empty$(b)$ that returns true if the data structure is empty and false otherwise, and size$(n)$ that returns the number of elements in the data structure.

Even though we refrain from formal definitions, we want to stress that a valid sequence of a container remains valid after deleting observer method calls:

$$S \setminus (\text{Ins} \cup \text{Rem}) \subseteq S.$$  

There are also containers with multiple insert/remove methods, e.g., a double-ended queue (deque) is a container with insert-left, insert-right, remove-left, and remove-right methods, to which local linearizability is also applicable. However, local linearizability requires that each method call is either an insertion, or a removal, or an observation. As a consequence, set is not a container according to our definition, as in a set ins$(x)$ acts as a global observer first, checking whether (some version of) $x$ is already in the set, and if not inserts $x$. Also hash tables are not containers for a similar reason.

Note that the arity of each method call in a container being one excludes data structures like snapshot objects. It is possible to deal with higher arities in a fairly natural way, however, at the cost of complicated presentation. We chose to present local linearizability on simple containers only. We present the definition of local linearizability without shape observations here and discuss shape observations in Appendix A.

**Definition 9 (In- and out-methods).** Let $h$ be a container history. For each thread $T$ we define two subsets of the methods in $h$, called in-methods $I_T$ and out-methods $O_T$ of thread $T$, respectively:

$$I_T = \{m | m \in M(h) \cap \text{Ins}\}$$

$$O_T = \{m(a) \in M(h) \cap \text{Rem} | \text{ins}(a) \in I_T\} \cup \{m(e) \in M(h) \cap \text{Rem} | e \in \text{Emp}\}$$

$$\cup \{m(a) \in M(h) \cap \text{DOb} | \text{ins}(a) \in I_T\}.$$

Hence, the in-methods for thread $T$ are all insertions performed by $T$. The out-methods are all removals and data observers that return values inserted by $T$. Removals that remove the value empty are also automatically added to the out-methods of $T$ as any thread (and hence also $T$) could be the cause of “inserting” empty. This way, removals of empty serve as means for global synchronization. Without them each thread could perform all its operations locally without ever communicating with the other threads. Note that the out-methods $O_T$ of thread $T$ need not be performed by $T$, but they return values that are inserted by $T$.

**Definition 10 (Thread-induced History).** Let $h$ be a history. The thread-induced history $h_T$ is the projection of $h$ to the in- and out-methods of thread $T$, i.e., $h_T = h | (I_T \cup O_T)$. 

**Definition 11 (Local Linearizability).** A history $h$ is locally linearizable wrt a sequential specification $S$ if (1) each thread-induced history $h_T$ is linearizable wrt $S$, and (2) the thread-induced histories $h_T$ form a decomposition of $h$, i.e., $m \in h \Rightarrow m \in h_T$ for some thread $T$. A data structure $D$ is locally linearizable wrt $S$ if every history $h$ of $D$ is locally linearizable wrt $S$. A history $h$ over a set of concurrent objects $O$ is locally linearizable wrt the sequential specifications $S_q$ for $q \in O$ if each thread-induced history is linearizable over $O$ and the thread-induced histories form a decomposition of $h$, i.e., $q.m \in h \Rightarrow q.m \in h_T$ for some thread $T$.

Local linearizability is sequentially correct, i.e., a single-threaded (necessarily sequential) history $h$ is locally linearizable wrt a sequential specification $S$ iff $h \in S$. Like linearizabil-
local linearizability is compositional. The complete proof of the following theorem and missing or extended proofs of all following properties can be found in Appendix B.

**Theorem 12** (Compositionality). A history $h$ over a set of objects $O$ with sequential specifications $S_q$ for $q \in O$ is locally linearizable iff $h|q$ is locally linearizable wrt $S_q$ for every $q \in O$.

**Proof (Sketch).** The property follows from the compositionality of linearizability and the fact that $(h|q)_T = h_T|q$ for every thread $T$ and object $q$.

The Choices Made. Splitting a global history into subhistories and requiring consistency for each of them is central to local linearizability. While this is common in shared-memory consistency conditions [22, 31, 32, 3, 16, 4, 20], our study of local linearizability is a first step in exploring subhistory-based consistency conditions for concurrent objects.

We chose thread-induced subhistories since thread-locality reduces contention in concurrent objects and is known to lead to high performance as confirmed by our experiments. To assign method calls to thread-induced histories, we took a data-centric point of view by (1) associating data values to threads, and (2) gathering all method calls that insert/return a data value into the subhistory of the associated thread (Def. 9). We associate data values to the thread that inserts them. One can think of alternative approaches, for example, associate with a thread the values that it removes. In our view, the advantages of our choice are clear:

First, by assigning inserted values to threads, every value in the history is assigned to some thread. In contrast, in the alternative approach, it is not clear where to assign the values that are inserted but not removed. Second, assigning inserted values to the inserting thread enables eager removals and ensures progress in locally linearizable data structures.

In the alternative approach, it seems like the semantics of removing `empty` should be local.

An orthogonal issue is to assign values from shape observations to threads. In Appendix A, we discuss two meaningful approaches and show how local linearizability can be extended towards shape and data observations that appear in insertion operations of sets.

Finally, we have to choose a consistency condition required for each of the subhistories. We chose linearizability as it is the best (strong) consistency condition for concurrent objects.

### 4 Local Linearizability vs. Linearizability

We now investigate the connection between local linearizability and linearizability.

**Proposition 1** (Lin 1). In general, linearizability does not imply local linearizability.

**Proof.** We provide an example of a data structure that is linearizable but not locally linearizable. Consider a sequential specification $S_{\text{NearlyQ}}$ which behaves like a queue except when the first two insertions were performed without a removal in between—then the first two elements are removed out of order. Formally, $s \in S_{\text{NearlyQ}}$ iff (1) $s = s_1\text{enq}(a)\text{enq}(b)s_2\text{deq}(b)s_3\text{deq}(a)s_4$ where $s_1\text{enq}(a)\text{enq}(b)s_2\text{deq}(a)s_3\text{deq}(b)s_4 \in S_Q$ and $s_1 \in \{\text{deq}(e) | e \in \text{Emp}\}^*$ for some $a,b \in V$, or (2) $s \in S_Q$ and $s \neq s_1\text{enq}(a)\text{enq}(b)s_2$ for $s_1 \in \{\text{deq}(e) | e \in \text{Emp}\}^*$ and $a,b \in V$. The example below is linearizable wrt $S_{\text{NearlyQ}}$. However, $T_1$’s induced history \text{enq}(1)\text{enq}(2)\text{deq}(1)\text{deq}(2)$ is not.

```
T1  enq(1)------------------- enq(2)-------------------
      enq(3)------------------- deq(2)---------
T2  ---------------------- enq(3)-------------------
      deq(1)-------------------
```

The following condition on a data structure specification is sufficient for linearizability to imply local linearizability and is satisfied, e.g., by pool, queue, and stack.
Definition 13 (Closure under Data-Projection). A seq. specification $S$ over $\Sigma$ is closed under data-projection if for all $s \in S$ and all $V' \subseteq V$, $s\{m(x) \in \Sigma \mid x \in V' \cup \text{Emp}\} \in S$. 

For $s = \text{enq}(1)\text{enq}(3)\text{deq}(3)\text{deq}(1)\text{deq}(2)$ we have $s \in S_{\text{NearlyQ}}$, but $s\{\text{enq}(x), \text{deq}(x) \mid x \in \{1, 2\} \cup \text{Emp}\} \notin S_{\text{NearlyQ}}$, i.e., $S_{\text{NearlyQ}}$ is not closed under data-projection.

Proposition 2 (Lin 2). Linearizability implies local linearizability for sequential specifications that are closed under data-projection.

Proof (Sketch). The property follows from Definition 13 and Equation (1).

There exist corner cases where local linearizability coincides with linearizability, e.g., for $S = \emptyset$ or $S = \Sigma^*$, or for single-producer/multiple-consumer histories.

We now turn our attention to pool, queue, and stack.

Proposition 3. The seq. specifications $S_P$, $S_Q$, and $S_S$ are closed under data-projection.

Proof (Sketch). Let $s \in S_P$, $V' \subseteq V$, and let $s' = s\{\text{ins}(x), \text{rem}(x) \mid x \in V' \cup \text{Emp}\}$. Then, it suffices to check that all axioms for pool (Definition 2 and Table 1) hold for $s'$.

Theorem 14 (Pool & Queue & Stack, Lin). For pool, queue, and stack, local linearizability is (strictly) weaker than linearizability.

Proof. Linearizability implies local linearizability for pool, queue, and stack as a consequence of Proposition 2 and Proposition 3. The history in Figure 2 is locally linearizable but not linearizable wrt pool, queue and stack (after suitable renaming of method calls).

Although local linearizability wrt a pool does not imply linearizability wrt a pool (Theorem 14), it still guarantees several properties that ensure sane behavior as stated next.

Proposition 4 (LocLin Pool). Let $h$ be a locally linearizable history wrt a pool. Then:
1. No value is duplicated, i.e., every remove method appears in $h$ at most once.
2. No out-of-thin-air values, i.e., $\forall x \in V. \text{rem}(x) \in h \Rightarrow \text{ins}(x) \in h \land \text{rem}(x) \not<_{h} \text{ins}(x)$.
3. No value is lost, i.e., $\forall x \in V. \forall e \in \text{Emp}. \text{rem}(e) <_{h} \text{rem}(x) \Rightarrow \text{ins}(x) \not<_{h} \text{rem}(e)$ and $\forall x \in V. \forall e \in \text{Emp}. \text{ins}(x) <_{h} \text{rem}(e) \Rightarrow \text{rem}(x) \in h \land \text{rem}(e) \not<_{h} \text{rem}(x)$.

Proof. By direct unfolding of the definitions.

Note that if a history $h$ is linearizable wrt a pool, then all of the three stated properties hold, as a consequence of linearizability and the definition of $S_P$.

5 Local Linearizability vs. Other Relaxed Consistency Conditions

We compare local linearizability with other classical consistency conditions to better understand its guarantees and implications.

1 The same notion has been used in [7] under the name closure under projection.
Sequential Consistency (SC). A history $h$ is *sequentially consistent* [25, 30] wrt a sequential specification $S$, if there exists a sequential history $s \in S$ and a completion $h_c \in \text{Complete}(h)$ such that (1) $s$ is a permutation of $h_c$, and (2) $s$ preserves each thread’s program order, i.e., if $m <_T n$, for some thread $T$, then $m <_s n$. We refer to $s$ as a *sequential witness* of $h$. A data structure $D$ is sequentially consistent wrt $S$ if every history $h$ of $D$ is sequentially consistent wrt $S$.

Sequential consistency is a useful consistency condition for shared memory but it is not really suitable for data structures as it allows for behavior that excludes any coordination between threads [39]: an implementation of a data structure in which every thread uses a dedicated copy of a sequential data structure without any synchronization is sequentially consistent. A sequentially consistent queue might always return *empty* in one (consumer) thread as the point in time of the operation can be moved, e.g., see Figure 3. In a producer-consumer scenario such a queue might end up with some threads not doing any work.

▶ Theorem 15 (Pool, Queue & Stack, SC). For pool, queue, and stack, local linearizability is incomparable to sequential consistency. ◀

Figures 2 and 3 give example histories that show the statement of Theorem 15. In contrast to local linearizability, sequential consistency is not compositional [25].

(Quantitative) Quiescent Consistency (QC & QQC). Like linearizability and sequential consistency, quiescent consistency [13, 25] also requires the existence of a sequential history, a *quiescent witness*, that satisfies the sequential specification. All three consistency conditions impose an order on the method calls of a concurrent history that a witness has to preserve. Quiescent consistency uses the concept of *quiescent states* to relax the requirement of preserving the precedence order imposed by linearizability. A quiescent state is a point in a history at which there are no pending invocation events (all invoked method calls have already responded). In a quiescent witness, a method call $m$ has to appear before a method call $n$ if and only if there is a quiescent state between $m$ and $n$. Method calls between two consecutive quiescent states can be ordered arbitrarily. *Quantitative quiescent consistency* [27] refines quiescent consistency by bounding the number of reorderings of operations between two quiescent states based on the concurrent behavior between these two states.

The next result about quiescent consistency for pool is needed to establish the connection between quiescent consistency and local linearizability.

▶ Proposition 5. A pool history $h$ satisfying 1.-3. of Prop. 4 is quiescently consistent. ◀

From Prop. 4 and 5 follows that local linearizability implies quiescent consistency for pool.

▶ Theorem 16 (Pool, Queue & Stack, QC). For pool, local linearizability is (strictly) stronger than quiescent consistency. For queue and stack, local linearizability is incomparable to quiescent consistency. ◀

Local linearizability also does not imply the stronger condition of quantitative quiescent consistency. Like local linearizability, quiescent consistency and quantitative quiescent consistency are compositional [25, 27]. For details, please see Appendix D.

Consistency Conditions for Distributed Shared Memory. There is extensive research on consistency conditions for distributed shared memory [3, 4, 8, 16, 20, 22, 30, 31, 32]. In Appendix E, we compare local linearizability against coherence, PRAM consistency, processor consistency, causal consistency, and local consistency. All these conditions split a history into subhistories and require consistency of the subhistories. For our comparison,
we first define a sequential specification $S_M$ for a single memory location. We assume that each memory location is preinitialized with a value $v_{init} \in V$. A read-operation returns the value of the last write-operation that was performed on the memory location or $v_{init}$ if there was no write-operation. We denote write-operations by $\text{ins}$ and read-operations by $\text{head}$. Formally, we define $S_M$ as

$$S_M = \{\text{head}(v_{init})\} \cdot \{\text{ins}(v)\text{head}(v)^i \mid i \geq 0, v \in V\}^*.$$

Note that read-operations are data observations and the same value can be read multiple times. For brevity, we only consider histories that involve a single memory location. In the following, we summarize our comparison. For details, please see Appendix E.

While local linearizability is well-suited for concurrent data structures, this is not necessarily true for the mentioned shared-memory consistency conditions. On the other hand, local linearizability appears to be problematic for shared memory. Consider the locally linearizable history in Figure 4. There, the read values oscillate between different values that were written by different threads. Therefore, local linearizability does not imply any of the shared-memory consistency conditions. In Appendix E, we further show that local linearizability is incomparable to all considered shared-memory conditions.

6 Locally Linearizable Implementations

In this section, we focus on locally linearizable data structure implementations that are generic as follows: Choose a linearizable implementation of a data structure $\Phi$ wrt a sequential specification $S_\Phi$, and we turn it into a (distributed) data structure called LLD $\Phi$ that is locally linearizable wrt $S_\Phi$. An LLD implementation takes several copies of $\Phi$ (that we call backends) and assigns to each thread $T$ a backend $\Phi_T$. Then, when thread $T$ inserts an element into LLD $\Phi$, the element is inserted into $\Phi_T$, and when an arbitrary thread removes an element from LLD $\Phi$, the element is removed from some $\Phi_T$ eagerly, i.e., if no element is found in the attempted backend $\Phi_T$ the search for an element continues through all other backends. If no element is found in one round through the backends, then we return $\text{empty}$.

Proposition 6 (LLD correctness). Let $\Phi$ be a data structure implementation that is linearizable wrt a sequential specification $S_\Phi$. Then LLD $\Phi$ is locally linearizable wrt $S_\Phi$.

Proof. Let $h$ be a history of LLD $\Phi$. The crucial observation is that each thread-induced history $h_T$ is a backend history of $\Phi_T$ and hence linearizable wrt $S_\Phi$. ▷

Any number of copies (backends) is allowed in this generic implementation of LLD $\Phi$. If we take just one copy, we end up with a linearizable implementation. Also, any way of choosing a backend for removals is fine. However, both the number of backends and the backend selection strategy upon removals affect the performance significantly. In our LLD $\Phi$ implementations we use one backend per thread, resulting in no contention on insertions, and always attempt a local remove first. If this does not return an element, then we continue a search through all other backends starting from a randomly chosen backend.

LLD $\Phi$ is an implementation closely related to Distributed Queues (DQs) [18]. A DQ is a (linearizable) pool that is organized as a single segment of length $\ell$ holding $\ell$ backends. DQs come in different flavours depending on how insert and remove methods are distributed.
across the segment when accessing backends. No DQ variant in \[18\] follows the LLD approach described above. Moreover, while DQ algorithms are implemented for a fixed number of backends, LLD Φ implementations manage a segment of variable size, one backend per (active) thread. Note that the strategy of selecting backends in the LLD Φ implementations is similar to other work in work stealing \[35\]. However, in contrast to this work our data structures neither duplicate nor lose elements. LLD (stack) implementations have been successfully applied for managing free lists in the fast and scalable memory allocator scallo\[5\]. The guarantees provided by local linearizability are not needed for the correctness of scallo\[c\], i.e., the free lists could also use a weak pool (pool without a linearizable emptiness check). However, the LLD stack implementations provide good caching behavior when threads operate on their local stacks whereas a weak pool would potentially negatively impact performance.

We have implemented LLD variants of strict and relaxed queue and stack implementations. None of our implementations involves observation methods, but the LLD algorithm can easily be extended to support observation methods. For details, please see App. F.4. Finally, let us note that we have also experimented with other locally linearizable implementations that lacked the genericity of the LLD implementations, and whose performance evaluation did not show promising results (see App. F.4). As shown in Sec. 4 a locally linearizable pool is not a linearizable pool, i.e., it lacks a linearizable emptiness check. Indeed, LLD implementations do not provide a linearizable emptiness check, despite of eager re-moves. We provide LL + D Φ, a variant of LLD Φ, that provides a linearizable emptiness check under mild conditions on the starting implementation Φ (see App. F.4 for details).

**Experimental Evaluation.** All experiments ran on a uniform memory architecture (UMA) machine with four 10-core 2GHz Intel Xeon E7-4850 processors supporting two hardware threads (hyperthreads) per core, 128GB of main memory, and Linux kernel version 3.8.0. We also ran the experiments without hyper-threading resulting in no noticeable difference. The CPU governor has been disabled. All measurements were obtained from the artifact-evaluated Scal benchmarking framework \[12, 19, 11\], where you can also find the code of all involved data structures. Scal uses preallocated memory (without freeing it) to avoid memory management artifacts. For all measurements we report the arithmetic mean and the 95% confidence interval (sample size=10, corrected sample standard deviation).

In our experiments, we consider the linearizable queues Michael-Scott queue (MS) \[34\] and LCRQ \[36\] (improved version \[37\]), the linearizable stacks Treiber stack (Treiber) \[12\] and TS stack \[14\], the k-out-of-order relaxed \(k\)-FIFO queue \[28\] and \(k\)-Stack \[23\] and linearizable well-performing pools based on distributed queues using random balancing \[18\] (1-RA DQ for queue, and 1-RA DS for stack). For each of these implementations (but the pools) we provide LLD variants (LLD LCRQ, LLD TS stack, LLD \(k\)-FIFO, and LLD \(k\)-Stack) and, when possible, LL + D variants (LL + D MS queue and LL + D Treiber stack). Making the pools locally linearizable is not promising as they are already distributed. Whenever LL + D is achievable for a data structure implementation Φ we present only results for LL + D Φ as, in our workloads, LLD Φ and LL + D Φ implementations perform with no visible difference.

We evaluate the data structures on a Scal producer-consumer benchmark where each producer and consumer is configured to execute \(10^6\) operations. To control contention, we add a busy wait of 5\(\mu\)s between operations. This is important as too high contention results in measuring hardware or operating system (e.g., scheduling) artifacts. The number of threads ranges between 2 and 80 (number of hardware threads) half of which are producers and half consumers. To relate performance and scalability we report the number of data
structure operations per second. Data structures that require parameters to be set are configured to allow maximum parallelism for the producer-consumer workload with 80 threads. This results in $k = 80$ for all $k$-FIFO and $k$-Stack variants (40 producers and 40 consumers in parallel on a single segment), $p = 80$ for 1-RA-DQ and 1-RA-DS (40 producers and 40 consumers in parallel on different backends). The TS Stack algorithm also needs to be configured with a delay parameter. We use optimal delay ($7\mu s$) for the TS Stack and zero delay for the LLD TS Stack, as delays degrade the performance of the LLD implementation.

Figure 5 shows the results of the producer-consumer benchmarks. Similar to experiments performed elsewhere \cite{14, 23, 28, 36}, the well-known algorithms MS and Treiber do not scale for 10 or more threads. The state-of-the-art linearizable queue and stack algorithms LCRQ and TS-interval Stack either perform competitively with their $k$-out-of-order relaxed counter parts $k$-FIFO and $k$-Stack or even outperform and outscale them. For any implementation $\Phi$, LLD $\Phi$ and LL$^+\Phi$ (when available) perform and scale significantly better than $\Phi$ does, even slightly better than the state-of-the-art pool that we compare to. The best improvement show LLD variants of MS queue and Treiber stack. The speedup of the locally linearizable implementation to the fastest linearizable queue (LCRQ) and stack (TS Stack)
implementation at 80 threads is 2.77 and 2.64, respectively. The performance degradation for LCRQ between 30 and 70 threads aligns with the performance of \texttt{fetch-and-inc}—the CPU instruction that atomically retrieves and modifies the contents of a memory location—on the benchmarking machine, which is different on the original benchmarking machine [36]. LCRQ uses \texttt{fetch-and-inc} as its key atomic instruction.

7 Conclusion & Future Work

Local linearizability splits a history into a set of thread-induced histories and requires consistency of all such. This yields an intuitive consistency condition for concurrent objects that enables new data structure implementations with superior performance and scalability. Local linearizability has desirable properties like compositionality and well-behavedness for container-type data structures. As future work, it is interesting to investigate the guarantees that local linearizability provides to client programs along the line of [15].

Acknowledgments

This work has been supported by the National Research Network RiSE on Rigorous Systems Engineering (Austrian Science Fund (FWF): S11402-N23, S11403-N23, S11404-N23, S11411-N23), a Google PhD Fellowship, an Erwin Schrödinger Fellowship (Austrian Science Fund (FWF): J3696-N26), EPSRC grants EP/H005633/1 and EP/K008528/1, the Vienna Science and Technology Fund (WWTF) trough grant PROSEED, the European Research Council (ERC) under grant 267989 (QUAREM) and by the Austrian Science Fund (FWF) under grant Z211-N23 (Wittgenstein Award).

References

1 URL: https://developer.apple.com/library/ios/documentation/General/Conceptual/ConcurrencyProgrammingGuide/OperationQueues/OperationQueues.html.

2 Y. Afek, G. Korland, and E. Yanovsky. Quasi-Linearizability: Relaxed Consistency for Improved Concurrency. In OPODIS, pages 395–410, 2010.

3 M. Ahamad, R.A. Bazzi, R. John, P. Kohli, and G. Neiger. The Power of Processor Consistency. In SPAA, pages 251–260, 1993.

4 M. Ahamad, G. Neiger, J.E. Burns, P. Kohli, and P.W. Hutto. Causal memory: definitions, implementation, and programming. Distributed Computing, 9(1):37–49, 1995.

5 M. Aigner, C. M. Kirsch, M. Lippautz, and A. Sokolova. Fast, multicore-scalable, low-fragmentation memory allocation through large virtual memory and global data structures. In OOPSLA, pages 451–469, 2015.

6 D. Alistarh, J. Kopinsky, J. Li, and N. Shavit. The SprayList: A Scalable Relaxed Priority Queue. In PPoPP, pages 11–20, 2015.

7 A. Bouajjani, M. Emmi, C. Enea, and J. Hamza. On Reducing Linearizability to State Reachability. In ICALP, pages 95–107, 2015.

8 S. Burckhardt, A. Gotsman, H. Yang, and M. Zawirski. Replicated Data Types: Specification, Verification, Optimality. In POPL, pages 271–284, 2014.

9 A. Cerone, A. Gotsman, and H. Yang. Parameterised Linearisability. In ICALP, pages 98–109, 2014.

10 S. Chakraborty, T. A. Henzinger, A. Sezgin, and V. Vafeiadis. Aspect-Oriented Linearisability Proofs. Logical Methods in Computer Science, 11(1:20):1–33, 2015.
Local Linearizability

11 POPL 2015 Artifact Evaluation Committee. POPL 2015 Artifact Evaluation. Accessed on 01/14/2015. URL: [http://popl15-aec.cs.umass.edu/home/](http://popl15-aec.cs.umass.edu/home/)

12 Computational Systems Group, University of Salzburg. Scal: High-Performance Multicore-Scalable Computing. URL: [http://scal.cs.uni-salzburg.at](http://scal.cs.uni-salzburg.at).

13 J. Derrick, B. Dongol, G. Schellhorn, B. Tofan, O. Travkin, and H. Wehrheim. Quiescent Consistency: Defining and Verifying Relaxed Linearizability. In *FM*, pages 200–214, 2014.

14 M. Dodds, A. Haas, and C.M. Kirsch. A Scalable, Correct Time-Stamped Stack. In *POPL*, pages 233–246, 2015.

15 I. Filipovic, P.W. O’Hearn, N. Rinetzky, and H. Yang. Abstraction for concurrent objects. *Theor. Comput. Sci.*, 411(51-52):4379–4398, 2010.

16 J.R. Goodman. *Cache consistency and sequential consistency*. University of Wisconsin-Madison, Computer Sciences Department, 1991.

17 A. Haas, T.A. Henzinger, A. Holzer, C.M. Kirsch, M. Lippautz, H. Payer, A. Sezgin, A. Sokolova, and H. Veith. Local Linearizability for Concurrent Container-Type Data Structures. In *CONCUR*, 2016.

18 A. Haas, T.A. Henzinger, C.M. Kirsch, M. Lippautz, H. Payer, A. Sezgin, and A. Sokolova. Distributed Queues in Shared Memory: Multicore Performance and Scalability through Quantitative Relaxation. In *CF*, 2013.

19 A. Haas, T. Hütter, C.M. Kirsch, M. Lippautz, M. Preishuber, and A. Sokolova. Scal: A Benchmarking Suite for Concurrent Data Structures. In *NETYS*, pages 1–14, 2015.

20 A. Heddaya and H. Sinha. Coherence, Non-coherence and Local Consistency in Distributed Shared Memory for Parallel Computing. Technical report, Computer Science Department, Boston University, 1992.

21 S. Heller, M. Herlihy, V. Luchangco, M. Moir, W.N. Scherer, and N. Shavit. A Lazy Concurrent List-based Set Algorithm. In *OPODIS*, 2005.

22 J.L. Hennessy and D.A. Patterson. *Computer Architecture, Fifth Edition: A Quantitative Approach*. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 5th edition, 2011.

23 T.A. Henzinger, C.M. Kirsch, H. Payer, A. Sezgin, and A. Sokolova. Quantitative relaxation of concurrent data structures. In *POPL*, pages 317–328, 2013.

24 T.A. Henzinger, A. Sezgin, and V. Vafeiadis. Aspect-Oriented Linearizability Proofs. In *CONCUR*, pages 242–256, 2013.

25 M. Herlihy and N. Shavit. *The Art of Multiprocessor Programming*. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 2008.

26 M. Herlihy and J.M. Wing. Linearizability: A Correctness Condition for Concurrent Objects. *ACM Trans. Program. Lang. Syst.*, 12(3):463–492, 1990.

27 R. Jagadeesan and J. Riely. Between Linearizability and Quiescent Consistency - Quantitative Quiescent Consistency. In *ICALP*, pages 220–231, 2014.

28 C.M. Kirsch, M. Lippautz, and H. Payer. Fast and Scalable, Lock-free k-FIFO Queues. In *PaCT*, pages 208–223, 2013.

29 A. Kogan and E. Petrank. Wait-free queues with multiple enqueuers and dequeuers. In *PPoPP*, pages 223–234, 2011.

30 L. Lamport. How to Make a Multiprocessor Computer That Correctly Executes Multiprocess Programs. *IEEE Trans. Comput.*, 28(9):690–691, September 1979.

31 R.J. Lipton and J.S. Sandberg. PRAM: A Scalable Shared Memory. Technical Report Nr. 180, Princeton University, Department of Computer Science, 1988.

32 R.J. Lipton and J.S. Sandberg. Oblivious memory computer networking, September 28 1993. CA Patent 1,322,609.

33 M.M. Michael. Hazard Pointers: Safe Memory Reclamation for Lock-Free Objects. *IEEE Trans. Parallel Distrib. Syst.*, 15(6):491–504, 2004.
M.M. Michael and M.L. Scott. Simple, fast, and practical non-blocking and blocking concurrent queue algorithms. In PODC, pages 267–275, 1996.

M.M. Michael, M.T. Vechev, and V.A. Saraswat. Idempotent Work Stealing. In PPoPP, pages 45–54, 2009.

A. Morrison and Y. Afek. Fast Concurrent Queues for x86 Processors. In PPoPP, pages 103–112, 2013.

Multicore Computing Group, Tel Aviv University. Fast Concurrent Queues for x86 Processors. Accessed on 01/28/2015. URL: http://mcg.cs.tau.ac.il/projects/lcrq/

H. Rihani, P. Sanders, and R. Dementiev. MultiQueues: Simpler, Faster, and Better Relaxed Concurrent Priority Queues. CoRR, 2014. arXiv:1411.1209

A. Sezgin. Sequential Consistency and Concurrent Data Structures. CoRR, abs/1506.04910, 2015.

N. Shavit. Data Structures in the Multicore Age. CACM, 54(3):76–84, March 2011.

R.C. Steinke and J.G. Nutt. A Unified Theory of Shared Memory Consistency. J. ACM, 51(5):800–849, September 2004.

R.K. Treiber. Systems Programming: Coping with Parallelism. Technical Report RJ-5118, IBM Research Center, 1986.
There are two possible ways to deal with shape observers: treat them locally, in the thread-induced history of the performing thread, or treat them globally. While a local treatment is immediate and natural to a local consistency condition, a global treatment requires care. We present both solutions next.

**Definition 17** (Local Linearizability LSO). A history $h$ is locally linearizable with local shape observers (LSO) wrt a sequential specification $S$ if it is locally linearizable according to Definition 11 with the difference that the in-methods (Definition 9) also contain all shape observers performed by thread $T$, i.e., $I_T = \{ m \mid m \in M(h|T) \cap (\text{Ins} \cup \text{SOb})\}$.

Global observations require more notation and auxiliary notions. Let $s_j$ for $j \in J$ be a collection of sequences over alphabet $\Sigma$ with pairwise disjoint sets of symbols $M(s_j)$. A sequence $s$ is an interleaving of $s_j$ for $j \in J$ if $M(s) = \bigcup_j M(s_j)$ and $s|M(s_j) = s_j$ for all $j \in J$. We write $\prod_j s_j$ for the set of all interleavings of $s_j$ with $j \in J$.

Given a history $h$ and a method call $m \in h$, we write $h^{\leq m}$ for the (incomplete) history that is the prefix of $h$ up to and without $m_r$, the response event of $m$. Hence, $h^{\leq m}$ contains all invocation and response events of $h$ that appear before $m_r$.

**Definition 18.** Let $S$ denote the sequential specification of a container $D$. A shape observer $m$ in a history $h$ has a witness if there exists a sequence $s \in \Sigma^*$ such that $sm \in S$ and $s \in \prod_T s_T$ for some $s_T$ that is a linearization of the thread-induced history $(h^{\leq m})_T$.

Informally, the above definition states that a global shape observer $m$ must be justified by a (global) witness. Such a global witness is a sequence that (1) when extended by $m$ belongs to the sequential specification, and (2) is an interleaving of linearizations of the thread-induced histories up to $m$.

**Definition 19** (Local Linearizability GSO). A history $h$ is locally linearizable with global shape observers (GSO) wrt a sequential specification $S$ if it is locally linearizable and each shape observer $m \in \text{SOb}$ has a witness.

We illustrate the difference in the local vs. the global approach for shape observers with the following example.

**Example 20.** Consider the following queue history with global observer $\text{size()}$

```
T_1 ... \text{enq(1)} \text{......} \text{enq(2)} \text{..........} \text{......} \\
T_2 \text{......} \text{deq(1)} \text{......} \text{size(n)} \text{......}
```

where $n$ is just a placeholder for a concrete natural number. For $n = 0$, the history $h$ is locally linearizable LSO, but not locally linearizable GSO. For $n = 1$, the history $h$ is locally linearizable GSO, but not locally linearizable LSO.

Global observers and non-disjoint operations are expected to have negative impact on performance. If one cares for global consistency, local linearizability is not the consistency condition to be used. The restriction to containers and disjoint operations specifies, in an informal way, the minimal requirements for local consistency to be acceptable.

Neither sets nor maps are containers according to our definition. However, it is possible to extend our treatment to sets and maps similar to our treatment of global observers. Locally
linearizable sets and maps will be weaker than their linearizable counterparts, but, due to
the tight coupling between mutator and observer effects, the gain in performance is unlikely
to be as substantial as the one observed in other data structures. The technicalities needed
to extend local linearizability to sets and maps would complicate the theoretical development
without considerable benefits and we, therefore, excluded such data structures.

B Additional Results and Proofs

Theorem 12 (Compositionality). A history \( h \) over a set of objects \( O \) with sequential
specifications \( S_q \) for \( q \in O \) is locally linearizable if and only if \( h|q \) is locally linearizable with
respect to \( S_q \) for every \( q \in O \).

Proof. The property follows from the compositionality of linearizability and the fact that
\((h|q)_T = h_T|q\) for every thread \( T \) and object \( q \). Assume that \( h \) over \( O \) is locally linearizable.
This means that all thread-induced histories \( h_T \) over \( O \) are linearizable. Hence, since
linearizability is compositional, for each object \( q \in O \) the history \( h_T|q \) is linearizable with
respect to \( S_q \). Now from \((h|q)_T = h_T|q\) we have that for every object \( q \) the history \((h|q)_T\)
is linearizable for every thread \( T \).

Similarly, assume that for every object \( q \in O \) the history \( h|q \) is locally linearizable. Then,
for every \( q \), \((h|q)_T = h_T|q\) is linearizable for every thread \( T \). From the compositionality of
linearizability, \( h_T \) is linearizable for every thread \( T \). This proves that \( h \) is locally linearizable.

Proposition 2 (Lin vs. LocLin 2). Linearizability implies local linearizability for sequential
specifications that are closed under data-projection.

Proof. Assume we are given a history \( h \) which is linearizable with respect to a sequential
specification \( S \) that is closed under data-projection. Further assume that, without loss of
generality, \( h \) is complete. Then there exists a sequential history \( s \in S \) such that (1) \( s \) is a permutation of \( h \), and (2) if \( m \prec_h n \), then also \( m \prec_s n \). Given a thread \( T \), consider
the thread-induced history \( h_T \) and let \( s_T = s|I_T \cup O_T \). Then, \( s_T \) is a permutation of \( h_T \) since
\( h_T \) and \( s_T \) consist of the same events. Furthermore, \( s_T \in S \) since \( S \) is closed under data-
projection and since Equation (1) holds for containers. Finally, we have for each \( m \in h_T \) and \( n \in h_T \) that, if \( m \prec_h n \), then also \( m \prec_s n \) since \( m \prec_h n \) and therefore \( m \prec_s n \) which
implies \( m \prec_s n \). Thereby, we have shown that \( h_T \) is linearizable with respect to \( S \), for an
arbitrary thread \( T \). Hence \( h \) is locally linearizable with respect to \( S \).

Proposition 3 (Data-Projection Closedness). The sequential specifications of pool,
queue, and stack are closed under data-projection.

Proof. Let \( s \in S_P \), \( V' \subseteq V \), and let
\[
\{x \in \text{ins}(x), \text{rem}(x) | x \in V' \cup \text{Emp} \}
\]
Then, it suffices to check that all axioms for pool (Definition 2 and Table 1) hold for \( s' \).
Clearly, all methods in \( s' \) appear at most once, as they do so in \( s \). If \( \text{rem}(x) \in s' \), then
\( \text{rem}(x) \in s \) and, since \( s \in S_P \), \( \text{ins}(x) \prec_s \text{rem}(x) \). But then also \( \text{rem}(x) \in s' \) and
\( \text{ins}(x) \prec_s \text{rem}(x) \). Finally, if \( \text{ins}(x) \prec_e \text{rem}(e) \) for \( e \in \text{Emp} \), then \( \text{ins}(x) \prec_s \text{rem}(e) \) implying
that \( \text{rem}(x) \in s \) and \( \text{rem}(x) \prec_s \text{rem}(e) \). But then \( \text{rem}(x) \in s' \) as well and \( \text{rem}(x) \prec_s \text{rem}(e) \).
This shows that \( S_P \) is closed under data-projection.
Assume now that \( s \in S_Q \) and \( s' \) is as before (with \( \text{enq}() \) and \( \text{deq}() \) for \( \text{ins}() \) and \( \text{rem}() \), respectively). Then, as \( S_P \) is closed under data-projection, \( s' \) satisfies the pool axioms. Moreover, the queue-order axiom (Definition 2 and Table 1) also holds: Assume \( \text{enq}(y) \prec_s \text{enq}(y) \) and \( \text{deq}(y) \in s' \). Then \( \text{enq}(x) \prec_s \text{enq}(y) \) and \( \text{deq}(y) \in s \). Since \( s \in S_Q \) we get \( \text{deq}(x) \in s \) and \( \text{deq}(x) \prec_s \text{deq}(y) \). But this means \( \text{deq}(x) \in s' \) and \( \text{deq}(x) \prec_s \text{deq}(y) \). Hence, \( S_Q \) is closed under data-projection.

Finally, if \( s \in S_S \) and \( s' \) is as before (with \( \text{push}() \) and \( \text{pop}() \) for \( \text{ins}() \) and \( \text{rem}() \), respectively), we need to check that the stack-order axiom (Definition 2 and Table 1) holds. Assume \( \text{push}(x) \prec_s \text{push}(y) \prec_s \text{pop}(x) \). This implies \( \text{push}(x) \prec_s \text{push}(y) \prec_s \text{pop}(x) \) and since \( s \in S_S \) we get \( \text{pop}(y) \in s \) and \( \text{pop}(y) \prec_s \text{pop}(x) \). But then \( \text{pop}(y) \in s' \) and \( \text{pop}(y) \prec_s \text{pop}(x) \). So, \( S_S \) is closed under data-projection.

**Proposition 4 (LocLin Pool).** Let \( h \) be a locally linearizable history wrt a pool. Then:
1. No value is duplicated, i.e., every remove method appears in \( h \) at most once.
2. There are no out-of-thin-air values, i.e.,
   \[
   \forall x \in V. \text{rem}(x) \in h \Rightarrow \text{ins}(x) \in h \land \text{rem}(x) \not\prec_h \text{ins}(x).
   \]
3. No value is lost, i.e., \( \forall x \in V. \forall e \in \mathit{Emp}. \text{ins}(x) \not\prec_h \text{rem}(e) \Rightarrow \text{rem}(x) \in h \land \text{rem}(e) \not\prec_h \text{rem}(x) \)
   and \( \forall x \in V. \forall e \in \mathit{Emp}. \text{rem}(e) \not\prec_h \text{rem}(x) \Rightarrow \text{ins}(x) \not\prec_h \text{rem}(e) \).

**Proof.** Note that if a history \( h \) is linearizable wrt a pool, then all of the three stated properties hold, as a consequence of linearizability and the definition of \( S_P \). Now assume that \( h \) is locally linearizable wrt a pool.

If \( \text{rem}(x) \) appears twice in \( h \), then it also appears twice in some thread-induced history \( h_T \) contradicting that \( h_T \) is linearizable with respect to a pool. This shows that no value is duplicated.

If \( \text{rem}(x) \in h \), then \( \text{rem}(x) \in h_T \) for some \( T \) and, since \( h_T \) is linearizable with respect to a pool, \( \text{ins}(x) \in h_T \) and \( \text{rem}(x) \not\prec_{h_T} \text{ins}(x) \). This yields \( \text{ins}(x) \in h \) and \( \text{rem}(x) \not\prec_h \text{ins}(x) \). Hence, there are no thin-air values.

Finally, if \( \text{rem}(e) \in h \) for \( e \in \mathit{Emp} \) then \( \text{rem}(e) \in h_T \) for all \( T \). Let \( \text{ins}(x) \not\prec_h \text{rem}(e) \) and let \( T' \) be such that \( \text{ins}(x) \in h_{T'} \). Then \( \text{ins}(x) \not\prec_{h_{T'}} \text{rem}(e) \) and since \( h_{T'} \) is linearizable with respect to a pool, \( \text{rem}(x) \in h_{T'} \) and \( \text{rem}(e) \not\prec_{h_{T'}} \text{rem}(x) \). This yields \( \text{rem}(x) \in h \) and \( \text{rem}(e) \not\prec_h \text{rem}(x) \). Similarly, the other condition holds. Hence, no value is lost.

**Theorem 25 (Queue Local Linearizability).** A queue concurrent history \( h \) is locally linearizable with respect to the queue sequential specification \( S_Q \) if and only if
1. \( h \) is locally linearizable with respect to the pool sequential specification \( S_P \), and
2. \( \forall x, y \in V. \forall i. \text{enq}(x) \prec_h \text{enq}(y) \land \text{deq}(y) \in h \Rightarrow \text{deq}(x) \in h \land \text{deq}(y) \not\prec_h \text{deq}(x). \)

**Proof.** Assume \( h \) is locally linearizable with respect to \( S_Q \). Since \( S_Q \subseteq S_P \) (with suitably renamed method calls), \( h \) is locally linearizable with respect to \( S_P \). Moreover, since all \( h_i \) are linearizable with respect to \( S_Q \), by Theorem 24 for all \( i \) we have \( \forall x, y \in V. \text{enq}(x) \prec_h \text{enq}(y) \land \text{deq}(y) \in h_i \Rightarrow \text{deq}(x) \in h_i \land \text{deq}(y) \not\prec_h \text{deq}(x). \)

Assume \( x, y \in V \) are such that \( \text{enq}(x) \prec_h \text{enq}(y) \) and \( \text{deq}(y) \in h \). Then \( \text{enq}(x) \prec_h \text{enq}(y) \) and \( \text{deq}(y) \in h_i \) so \( \text{deq}(x) \in h_i \) and \( \text{deq}(y) \not\prec_h \text{deq}(x) \). This implies \( \text{deq}(x) \in h \) and \( \text{deq}(y) \not\prec_h \text{deq}(x) \).

For the opposite, assume that conditions 1. and 2. hold for a history \( h \). We need to show that (1) \( h_i \) form a decomposition of \( h \), which is clear for a queue, and (2) each \( h_i \) is linearizable with respect to \( S_Q \).
By 1., each \( h_i \) is linearizable with respect to a pool. Assume \( \text{enq}(x) <_h \text{enq}(y) \) and \( \text{deq}(y) \in h_i \). Then \( \text{enq}(x) <_h \text{enq}(y) \land \text{deq}(y) \in h \) and hence by 2., \( \text{deq}(x) \in h \land \text{deq}(y) \not\prec_h \text{deq}(x) \). Again, as \( \text{enq}(x), \text{deq}(x) \in h_i \) we get \( \text{deq}(x) \in h_i \land \text{deq}(y) \not\prec h \text{deq}(x) \). According to Theorem 24 this is enough to conclude that each \( h_i \) is linearizable with respect to \( S_Q \). ▶

**Theorem 15 (Pool, Queue, & Stack, SC).** For pool, queue, and stack, local linearizability is incomparable to sequential consistency.

**Proof.** The following histories, when instantiating \( i() \) with \( \text{ins}() \), \( \text{enq}() \), and \( \text{push}() \), respectively, and instantiating \( r() \) with \( \text{rem}() \), \( \text{deq}() \), and \( \text{pop}() \), respectively, are sequentially consistent but not locally linearizable wrt pool, queue and stack:

(a) Pool:

\[
\begin{array}{c}
T_1 \quad \cdots \quad \text{i(1)} \quad \cdots \quad \text{r(1)} \\
T_2 \quad \text{r(empty)} \\
\end{array}
\]

(b) Queue:

\[
\begin{array}{c}
T_1 \quad \text{r(2)} \\
T_2 \quad \text{i(1)} \quad \text{i(2)} \quad \text{r(1)} \\
\end{array}
\]

(c) Stack:

\[
\begin{array}{c}
T_1 \quad \text{i(1)} \quad \text{i(2)} \\
T_2 \quad \text{r(1)} \quad \text{r(2)} \\
\end{array}
\]

History (a) is already not locally linearizable wrt pool, queue, and stack, respectively, histories (b) and (c) provide interesting examples. The history in Figure 2 is locally linearizable but not sequentially consistent wrt a pool. The following histories are locally linearizable but not sequentially consistent wrt a queue and a stack, respectively:

(d) Queue:

\[
\begin{array}{c}
T_1 \quad \text{i(1)} \quad \text{i(2)} \quad \text{i(3)} \quad \text{r(1)} \\
T_2 \quad \text{r(2)} \quad \text{i(4)} \quad \text{r(4)} \quad \text{r(3)} \\
\end{array}
\]

The two thread-induced histories \( i(1)i(2)i(3)r(1)r(2)r(3) \) and \( i(4)r(4) \) are both linearizable with respect to a queue. However, the overall history has no sequential witness and is therefore not sequentially consistent: To maintain the queue behavior, the order of operations \( r(1) \) and \( r(2) \) cannot be changed. However, this implies that the value 3 instead of the value 4 would have to be removed directly after \( i(4) \).

(e) Stack:
Local Linearizability

$q_i = \text{ins}(x_i,1)\text{rem}(x_i,1)\ldots\text{ins}(x_i,p)\text{rem}(x_i,p)\text{rem}(x_i,p+1)\text{rem}(x_i,q)\text{rem}(\text{empty})^r\text{ins}(x_i,q+1)\ldots\text{ins}(x_i,m)$.

**Figure 6** Sequential history $q_i$.

The two thread-induced histories $i(1)i(2)r(2)i(1)$ and $i(3)r(3)$ are both linearizable with respect to a stack. The operations $i(2)$ and $r(2)$ prevent the reordering of operations $i(1)$ and $i(3)$. Therefore, the overall history has no sequential witness and hence it is not sequentially consistent.

**Proposition 5** (Pool, QC). Let $h$ be a pool history in which no data is duplicated, no thin-air values are returned, and no data is lost, i.e., $h$ satisfies 1.-3. of Proposition 4. Then $h$ is quiescently consistent.

**Proof.** Assume $h$ is a pool history that satisfies 1.-3. of Proposition 4. Let $h_1,\ldots,h_n$ be histories that form a sequential decomposition of $h$. That is $h = h_1\cdots h_n$ and the only quiescent states in any $h_i$ are at the beginning and at the end of it. Note that this decomposition has nothing to do with a thread-local decomposition. Let $M_i = M_{h_i}$ be the set of methods of $h_i$, for $i \in \{1,\ldots,n\}$. Note that the sanity conditions 1.-3. ensure that none of the following two situations can happen:

- $\text{rem}(x) \in M_i, \text{ins}(x) \in M_j, j > i$,
- $\text{ins}(x) \in M_i, \text{rem}(\text{empty}) \in M_j, \text{rem}(x) \in M_k, k > j > i$.

Let $V_i = \{x_{i,1},\ldots,x_{i,m}\}$ denote the set of values in $M_i$ ordered in a way that there is a $p$ and $q$ such that

- $\text{ins}(x_{i,j}), \text{rem}(x_{i,j}) \in M_i$ for $j \leq p$;
- $\text{rem}(x_{i,j}) \in M_i$ for $j > p, j \leq q$; and
- $\text{ins}(x_{i,j}) \in M_i$ for $j > q$.

Moreover, let $r_i$ be the number of occurrences of $\text{rem}(\text{empty})$ in $h_i$.

We now construct a sequential history for $h$, which has the form $q = q_1\cdots q_n$ where each sequential history $q_i$ is a permutation of $M_i$ shown in Figure 6. Using the observations above, it is easy to check that $q$ is indeed a quiescent witness for $h$.

**Theorem 16** (Pool, Queue, & Stack, QC). For pool, local linearizability is stronger than quiescent consistency. For queue and stack, local linearizability is incomparable to quiescent consistency.

**Proof.** The following histories are quiescently consistent but not locally linearizable wrt pool, queue, and stack, respectively:

(a) Pool:

\[
\begin{align*}
T_1 & \quad \text{rem}(\text{empty}) \quad \cdots \rightarrow \\
T_2 & \quad \cdots \rightarrow \text{ins}(1) \quad \text{rem}(\text{empty}) \quad \text{rem}(1) \quad \cdots \rightarrow
\end{align*}
\]
(b) Queue:

\[
\begin{align*}
T_1 & \quad \text{enq(1)} \\
T_2 & \quad \text{enq(2)} \quad \text{enq(3)} \quad \text{deq(3)} \quad \text{deq(2)}
\end{align*}
\]

(c) Stack:

\[
\begin{align*}
T_1 & \quad \text{push(1)} \\
T_2 & \quad \text{push(2)} \quad \text{push(3)} \quad \text{pop(2)} \quad \text{pop(3)}
\end{align*}
\]

In all three histories, the only quiescent states are before and after the longest operation. Therefore, all operations in thread \(T_2\) can be reordered arbitrarily, in particular in a way such that they satisfy the sequential specification of the respective concurrent data structure. However, each of the thread-induced histories for thread \(T_2\) are not linearizable with respect to pool, queue, and stack, respectively. Therefore, none of these histories is locally linearizable. Also here history (a) suffices.

On the other hand, the following histories are not quiescently consistent but locally linearizable wrt queue, and stack, respectively:

(d) Queue:

\[
\begin{align*}
T_1 & \quad \text{enq(1)} \quad \text{deq(2)} \\
T_2 & \quad \text{enq(2)} \quad \text{deq(1)}
\end{align*}
\]

(e) Stack:

\[
\begin{align*}
T_1 & \quad \text{push(1)} \quad \text{pop(1)} \\
T_2 & \quad \text{push(2)} \quad \text{pop(2)}
\end{align*}
\]

In histories (d) and (e), between each two operations, the concurrent data structure is in a quiescent state. Therefore, none of the operations can be reordered and, hence, no sequential witness exists. However, all thread-induced histories are linearizable and, therefore, the overall histories are locally linearizable. In particular, on a history where each pair of operations is separated by a quiescent state, i.e., there is no overlap of operations, a quiescent consistent data structure behaves as it would be linearizable with respect to its sequential specification and we see the same semantic differences to local linearizability as we see between linearizability and local linearizability.

C Case Study: Work Stealing Queues

Consider a data structure \(D\) which admits two operation types: \texttt{ins}(x), which inserts the element \(x\) into the container, and \texttt{rem}(), which returns and removes an element from the container. Now imagine that the implementation uses a Work Stealing Queue (WSQ) \[35\]. Every thread \(T\) that uses \(D\) has its unique designated buffer \(Q_T\) in the WSQ. Whenever thread \(T\) calls \texttt{ins}(x), \(x\) is appended to the tail of \(Q_T\). When \(T\) calls \texttt{rem}(), WSQ first
Local Linearizability

![Figure 7](image)

**Figure 7** History that is QQC but not LL.

checks whether $Q_T$ is non-empty; if it is, then it returns the element at the tail of $Q_T$ (LIFO semantics) and removes it. Otherwise, it chooses some other $Q_{T'}$ and tries to return an element from that buffer. But any time a different thread’s buffer is checked, the element to be removed is taken from the head (FIFO semantics). If $T$ and $T'$ are both trying to access the same buffer at the same time, then usual synchronization measures are taken to ensure that exactly one thread removes one element.

Given this implementation, the developer of $D$ wants to write a specification for the potential users of $D$. Since $D$ is essentially a collection of deques, the developer is tempted to state that $D$ is a deque with a particular consistency condition. However, $D$ is not a linearizable deque because $\text{ins}(x)$ by $T$ followed by $\text{ins}(y)$ by $T'$ followed by $\text{rem}()$ returns either $x$ or $y$ depending on whether $T$ or $T'$ calls it; i.e. $\text{rem}()$ has ambiguous semantics. $D$ can be seen as a sequentially consistent (SC) deque but then $D$ does not allow many behaviors that an SC deque would allow; i.e. SC does not capture the behaviors of $D$ tightly. Relaxed sequential specifications will not work either since $D$ does converge to sequential semantics (of a LIFO stack) when a single thread uses it. In short, the developer will fail to capture the semantics of $D$ in a satisfactory manner.

$D$ on the other hand is a locally linearizable deque in which $\text{rem}()$ by $T$ from $Q_T$ is treated as FIFO removal whenever $T \neq T'$ and as LIFO removal whenever $T = T'$. In other words, local linearizability provides a succinct and clean representation of a well-known implementation framework (WSQ) hiding away implementation details. Compare this with the fact that even though WSQ has a queue in it, to argue its correctness it is proved to be a linearizable pool even though it has stronger semantics than a pool; i.e. linearizable pool semantics is too weak for $D$. Observe also that since what we have described in the example is essentially providing the illusion of using a monolithic structure which is implemented in terms of distributed components (shared memory is typically implemented on message passing), we expect local linearizability to be widely applicable.

### Quiescent Consistency & Quantitative Quiescent Consistency

Without going into the details of the definition of quantitative quiescent consistency we give a history in Figure 7 that is quantitatively quiescently consistent but not locally linearizable wrt a queue. Quantitative quiescent consistency allows to reorder the two insert-operations in thread $T_2$ and thereby violates local linearizability.

### Consistency Conditions for Distributed Shared Memory
| Consistency Condition | Decomposition per |
|-----------------------|-------------------|
| LL                    | thread            |
| Coherence             | memory location   |
| PRAM                  | thread            |
| PC                    | thread            |
| CC                    | thread & memory location |
| LC                    | thread & memory location |

| #SHs | Write-Operations $I_h(i)$ | Read-Operations $O_h(i)$ | CCfSH | LoD |
|------|---------------------------|--------------------------|-------|-----|
| $n$  | $\{\text{ins}(v) \in h|T_i | v \in V\}$ | $\{\text{head}(v_{\text{init}}) \in h\}$ \cup $\{\text{head}(v) \in h | \text{ins}(v) \in I_h(i)\}$ | Lin.  | no  |
| $k$  | $\{\text{ins}(v) \in h | v \in V\}$ | $\{\text{head}(v) \in h | v \in V\}$ | SC    | yes |
| $n$  | $\{\text{ins}(v) \in h | v \in V\}$ | $\{\text{head}(v) \in h|T_i | v \in V\}$ | SC    | yes |
| $n$  | $\{\text{ins}(v) \in h | v \in V\}$ | $\{\text{head}(v) \in h|T_i | v \in V\}$ | SC$^a$| yes |
| $n \cdot k$ | $\{\text{ins}(v) \in h|T_i | v \in V\}$ \cup $\{\text{ins}(v) \in h | \text{head}(v) \in h|T_i\}$ | $\{\text{head}(v) \in h|T_i | v \in V\}$ | SC$^b$| yes |
| $n \cdot k$ | $\{\text{ins}(v) \in h|T_i | v \in V\}$ \cup $\{\text{ins}(v) \in h | \text{head}(v) \in h|T_i\}$ | $\{\text{head}(v) \in h|T_i | v \in V\}$ | SC$^c$| yes |

SC$^a$: SC and ins-operations are in the same order for each witness.
SC$^b$: SC and ins-operations are ordered by the transitive closure of the thread program orders and write-read pairs.
SC$^c$: SC and ins-operations from threads other than $T_i$ can be reordered even if they are from the same thread and only logical contradictions in the local history are considered for consistency.

$n$: number of threads, $k$: number of memory locations,
#SHs: number of subhistories, CCfSH: consistency condition for subhistories, LoD: loss of data

Table 2 Comparison of consistency conditions for a single distributed shared memory location, i.e., $k = 1$
In Table 2 we compare local linearizability (LL) against the consistency conditions coherence [3], pipelined RAM (PRAM) consistency [31, 32, 11, 4], processor consistency (PC) [3, 10], causal consistency (CC) [4], and local consistency (LC) [20]. Local linearizability shares with all these consistency conditions the idea of decomposing a concurrent history into several subhistories.

Coherence projects a concurrent history to the operations on a single memory location and each resulting history has to be sequentially consistent. Since sequential consistency is not compositional, coherence does not imply sequential consistency for the overall history [3] whereas local linearizability for each single memory location implies local linearizability for the overall history.

In contrast to coherence and local consistency, local linearizability, PRAM consistency, PC, and CC all decompose the history into per-thread subhistories, i.e., if there are \( n \) threads then these conditions consider \( n \) subhistories and need \( n \) sequential witnesses. Coherence requires one witness per memory location and local consistency requires one witness per thread and memory location.

For determining the subhistory for a thread \( T_i \), coherence, PRAM consistency, PC, and CC consider all write-operations in a given history, i.e., \( I^h(i) = \{ \text{ins}(v) \in h \mid v \in V \} \). In contrast, local linearizability only considers the write-operations in thread \( T_i \), i.e., \( I^h(i) = \{ \text{ins}(v) \in h | T_i \mid v \in V \} \) and local consistency considers all write-operations in thread \( T_i \) as well as all write-operations whose values are read in thread \( T_i \), i.e., \( I^h(i) = \{ \text{ins}(v) \in h | T_i \mid v \in V \} \). Regarding read-operations, PRAM consistency, PC, CC, and LC consider only the read-operations in thread \( T_i \). Coherence considers all read-operations in a given history and local linearizability only considers read-operations that read the initial value \( v_{\text{init}} \) and read-operations that read values that were written by a write-operation in thread \( T_i \). Reading the initial value is analogous to returning empty in a data structure.

Local linearizability requires that each subhistory, i.e., thread-induced history, is linearizable with respect to the sequential specification under consideration. In contrast, coherence, PRAM consistency, PC, and LC require that each subhistory is sequentially consistent (or a variant thereof) with respect to the sequential specification. However, the variants of sequential consistency that are used by these consistency conditions are vulnerable to a loss of data as discussed in Section 5 and, therefore, make these consistency conditions unsuitable for concurrent data structures.

When considering PRAM consistency, the sequentialization of the write-operations of different threads might be observed differently by different threads, e.g., a thread \( T_1 \) might observe all write-operations of thread \( T_2 \) before the write operations of thread \( T_3 \) but a thread \( T_4 \) might observe all write operations of \( T_3 \) before the write operations of \( T_2 \). In contrast, thread-induced histories as defined by local linearizability do not involve write-operations from other threads but involve (some) read-operations performed by other threads. Like PRAM consistency, processor consistency requires for each thread \( T_i \) that the read- and write-operations performed by \( T_i \) are seen in \( T_i \)'s program order and that the write-operations performed by other threads are seen in their respective program order. Furthermore, processor consistency also requires that two write-operations to the same memory location appear in the same order in each sequential witness of each thread even if they are from different threads [3, 10]. This additional condition makes processor consistency strictly stronger than PRAM consistency [3]. This condition also creates a similar effect as the consideration of read-operations in different threads when forming the thread-induced history in local linearizability. Causal consistency considers a causal order instead of the thread program orders alone.
linearizability, causal consistency matches write-read pairs across different threads. In particular, the causal order is the transitive closure of the thread program orders and write-read pairs. By considering the causal order, writes from different threads can become ordered which is not the case for local linearizability.

\section*{F \quad LLD and LL$^+$D Implementation Details}

As already mentioned, each thread inserts elements into a local backend and removes elements either from its local backend (preferred) or from other backends (fall-back) accessed through a single segment (thread-indexed array), effectively managing single-producer/multiple-consumer backends for a varying number of threads.

The segment is dynamic in length (with a predefined maximum). A slot in this segment refers to a node that consists of a backend and a flag indicating whether the corresponding thread is alive or has terminated. Similar to other work \cite{2, 21} the flag is used for logically removing the node from the segment (it stays in the segment until its backend is empty). Additionally, a (global) version number keeps track of all changes in the segment. The algorithm is divided into two parts: (1) maintaining the segment, and (2) adding and removing elements to backends.

In the following we refer to the segment as $s$, a thread’s $T_i$ local node as $n_i$, the version number of the segment as $v$ and the current length of the segment as $\ell$. The range of indices $r$ is then defined as $0 \leq r < \ell$.

For maintaining the segment we provide two methods \texttt{announce\_thread()} and \texttt{cleanup\_thread(node)} that are used to add and remove nodes to the segment. Upon removal of a node the segment is also compacted, i.e., the hole that is created by removing a node pointer is filled with the last node pointer in the segment. As nodes are added and removed the length of the segment $\ell$ and thus the range of valid indices $i$ of the segment, $0 \leq i < \ell$, is updated. All changes to the segment involve incrementing the version number.

More detailed, the operations for maintaining the segment and compacting it as nodes are cleaned up are:

- \texttt{announce\_thread()}: Allocates a node for the thread as follows: searches for an existing node of a terminated thread and reuses it if it finds one; otherwise it creates a new node, adds the node to $s$, and adjusts $\ell$. In both cases it then increments $v$ and returns the node. The creation of new node is illustrated in Figure 8.

- \texttt{cleanup\_thread(Node n)}: Searches for the node $n$ in $s$ using linear search. If it finds $n$ at slot $j$, it copies the pointer of $s[j]$ to $s[j+1]$, decrements $\ell$, increments $v$, and resets $s[\ell]$ to \emph{null} using the new $\ell$. If $n$ is not found, then a concurrent thread has already performed the cleanup and the operation just returns. Figure 9 illustrates an example
Local Linearizability

where initially $\ell = 5$, the thread owning the node at $s[0]$ is dead and the corresponding backend is empty.

Note that updating the segment state is only needed when threads are joining or when backends of terminated threads become empty. We consider both scenarios as infrequent and implement the corresponding operations using locks. Alternatively those operations can be implemented using helping approaches, similar to wait-free algorithms [29]. Also note that although operations on segments are protected by locks, partial changes can be observed, e.g., a remove operation (as defined below) can observe a segment in an intermediate state with two pointers pointing to a node during cleanup. The invariant is that no change can destroy the integrity of the segment within the valid range, i.e., all slots within the range either point to a valid node or nothing (null).

The actual algorithm for adding and removing elements is then defined as follows:

- **ins()**: Upon first insertion, a thread $T_i$ gets assigned a node $n_i$ (containing backend $b_i$) using `announce_thread()`. The element is then inserted into $b_i$. Subsequent insertions from this thread will use $n_i$ throughout the lifetime of the thread.

- **rem()**: The remove operation consists of two parts: (a) finding and removing an element and (b) cleaning up nodes of terminated threads. For (a) a thread $T_i$ tries to get an element from its own backend in $n_i$. If $n_i$ does not exist (because the thread has not yet performed a single `ins()` operation) or the corresponding backend is empty, then a different node $n$ is selected randomly within the valid range. If the backend contained in $n$ is empty, the operation scans all other nodes’ backends in linear fashion. However, if the version number changed during the round of scanning through all backends, the operation is restarted immediately. Note that since $\ell$ is dynamic a remove operation may operate on a range that is no longer valid. Checking the version number ensures that the operation is restarted in such a case. For (b) a thread calls `cleanup_thread(n)` upon encountering a node $n$ that has its alive-flag set to false (dead) and contains an empty backend. A cleanup also triggers a restart of the remove operation.

- **terminate()**: Upon termination a thread $T_i$ changes the alive flag of $n_i$ to false (dead).

Dynamic memory used for nodes is susceptible to the ABA problem and requires proper handling to free memory. Our implementations use 16-bit ABA counters to avoid the ABA problem and refrain from freeing memory. Hazard pointers [33] can be used for solving the ABA problem as well as for freeing memory.

**F.1 LL+D: LLD with Linearizable Emptiness Check**

We call a data structure implementation $\Phi$ *stateful* if the remove methods of $\Phi$ can be modified to return a so-called *state* that changes upon an insert or a remove of an element,
but does not change between two removes that return empty unless an element has been inserted in the data structure in the meantime. For stateful implementations \( \Phi \) we can create the locally linearizable version with linearizable emptiness check \( LL^+D \Phi \). Michael-Scott queue \[31\] and Treiber stack \[12\] are stateful implementations, whereas LCRQ \[36\] is not. Also TS stack \[14\], and \( k \)-FIFO \[28\] and \( k \)-Stack \[23\] are stateful implementations, but the notion of a state in these data structures is huge making it unsuitable for \( LL^+D \).

For \( LL^+D \) implementations, linearizable emptiness checks are achieved via an atomic snapshot \[25\], just like for DQs. A detailed description of the LLD and \( LL^+D \) implementations, as well as the pseudo code, can be found in the appendix. Here, we only present the results of the experimental performance evaluation.

### F.2 Correctness of \( LL^+D \)

**Proposition 7 (LLD and \( LL^+D \)).** Let \( \Phi \) be a stateful data structure implementation that is linearizable with respect to a sequential specification \( S_\Phi \). Then \( LL^+D \Phi \) is linearizable with respect to a pool.

**Proof.** Proving that \( LL^+D \Phi \) is linearizable with respect to pool, in particular that it has a linearizable emptiness check, follows the proof for DQ in general, see \[18\]: The emptiness check is performed by creating an atomic snapshot \[25\] of the states of all backends (stored in the \texttt{states} array) using the first loop (lines 28-43). If the atomic snapshot is valid (checked via the second loop, lines 46-52, in particular line 48) and all backends are empty in this atomic snapshot, then there existed a point in time during the creation of the atomic snapshot where all backends were indeed empty.

Notice that since the segment is dynamic in length it can happen that some backends are not contained in the atomic snapshot. To guarantee that no elements are missed in the emptiness check the atomic snapshot is extended by the version number \( v \) of the segment. If a new backend is added to the segment during the generation of the atomic snapshot, then the version number is increased and the atomic snapshot becomes invalid (line 45).

The linearization point of the remove operation that returns empty is inbetween the two loops (the last remove attempt of the first loop) if the version check and second loop go through.

### F.3 LLD Pseudo Code

All implementations use the interfaces depicted in Listing 1. For simplicity, the interface only mentions pool, queue, and stack. The highlighted code refers to linearizable emptiness check, i.e., it is only part of the \( LL^+D \) implementations: Methods retrieving elements (e.g. \texttt{rem}) are assumed (or modified when possible) to also return a \texttt{State} object that uniquely identifies the state of the data structure with respect to methods inserting elements (e.g. \texttt{ins}). The same state can be accessed via the \texttt{get_state()} observer method.

Listing 2 illustrates the pseudo-code for maintaining the segment. The backend on line 2 can either be declared as \texttt{Stack} or \texttt{Queue} as defined in Listing 1 (or any other linearizable data structure).

Listing 3 shows the pseudo-code for \( LL^+D \). When removing the highlighted code, we obtain the code for LLD. Each thread maintains its own backend, enclosed in a thread-local node (line 3), for insertion. The local backend is always accessed through \texttt{get_local_node} (line 5). This method also makes sure that a thread is announced (line 7) upon first insertion and acquires a node. An \texttt{ins()} operation then always uses a thread’s local backend.
Local Linearizability

Listing 1 Pool, queue, and stack interfaces

(line 13 and 14) for insertion. For removing an element in `rem()`, a thread tries to remove an element from its local backend first (line 19-23). If no element can be found, all backends in the valid range are searched in a linear fashion, starting from a random index. The highlighted code (lines 46-52) illustrates checking the atomic snapshot for LLD D.

F.4 LLD with Observer Methods

We have implemented LLD variants of (strict and relaxed) queue and stack implementations. None of our LLD implementations involves observer methods, but the LLD algorithm can easily be extended to support observer methods:

- A data observer on LLD $\Phi$ (independently of which thread performs it) amounts to a data observer on any $\Phi_T$.
- A local shape observer on LLD $\Phi$ performed by thread $T$ executes the shape observer on $\Phi_T$.
- A global shape observer on LLD $\Phi$ executes the shape observer on each backend $\Phi_T$ and produces an aggregate value.

G Additional Implementations

We now present and evaluate additional algorithms that provide locally linearizable variants of queues and stacks, obtained by modifying relaxed $k$-out of order queues and stacks in a way that makes them sequentially correct. We have also tried another generic implementation, related to the construction in [9], that implements a flat-combining wrapper with sequential (to be precise, single-producer multiple-consumer) backends. In our initial experiments the performance of such an implementation was not particularly promising.

G.1 Locally Linearizable $k$-FIFO Queue and $k$-Stack

$k$-FIFO queues and $k$-Stacks are relaxed queues and stacks based on lists of segments where each segment holds $k$ slots for elements, effectively allowing reorderings of elements of up to $k - 1$. The list of segments is implemented by a variant of Michael-Scott queue for
Node {
  Pool backend; // Any linearizable data structure.
  Bool alive;
}

Segment {
  Node nodes[MAX_THREADS];
  Int l = 0;
  Int version = 0;

  // Returns all indexes between 0 and l (exclusive) in random order.
  [Int] range();

  // Announces a node in the buffer, effectively adding it to nodes_,
  // adjusting l, and changing the version.
  Node announce_thread() {
    segment_lock(); // Protecting against concurrent announce or cleanup operations.
    Node n = find_dead_node();
    if (n == null) {
      n = Node(b: Backend());
      nodes[l] = n;
      l ++;
    }
    n.alive = true;
    version ++;
    segment_unlock();
    return n;
  }

  // Removes a node from the buffer, effectively removing it from nodes_,
  // adjusting l, and changing the version.
  void cleanup_thread(Node n, Int old_version) {
    segment_lock(); // Protecting against concurrent announce or cleanup operations.
    <j, error> = find_node_in_segment(n);
    if (error || n.alive) {
      old_version != version {
        segment_unlock();
        return;
      }
      nodes[j] = nodes[l-1];
      l --;
      version ++;
      nodes[l] = null;
      segment_unlock();
    }
  }
}

Listing 2 Node and segment structure for LLD and LL+D (queue or stack)

\[k\]-FIFO and a variant of Treiber stack \[42\] for \(k\)-Stack. Insert and remove methods operate on the segments ignoring any order of elements within the same segment. Segments used for insertion and removal are identified by insertion and removal pointers, respectively.

For queues, elements are removed from the oldest segment and inserted into the most-recent not-full segment. Upon trying to remove an element from an empty segment the segment is removed and the removal pointer advanced to the next segment. Upon trying to insert an element into a full segment a new segment is appended and the insertion pointer is advanced to this new segment. Similarly (but different) for a stack, removal and insertion operate on the most-recent segment, i.e., removal and insertion pointer are synonyms and identify the same segment at all times. Again, upon trying to remove an element from an empty segment the segment is removed and the removal pointer advanced to the next segment. Upon trying to insert an element into a full segment a new segment is prepended and the insertion pointer is set to this new segment.

\[k\]-FIFO queues and \(k\)-Stacks are relaxed queues and stacks that are: (1) linearizable with respect to \(k\)-out-of-order queue and stack \[23\], respectively; (2) linearizable with respect to a pool \[23, 28\]; (3) not locally linearizable with respect to queue and stack, respectively, for \(k \geq 1\) since reordering elements that are inserted in the same segment (even sequentially by
DynamicLocallyLinearizableDQ {  
  Segment s;  
  thread_local Node local_node;  

  Node get_local_node(Bool create_if_absent) {  
    if (create_if_absent) && (local_node == null) {  
      local_node = s.announce_thread();  
    }  
    return local_node;  
  }  

  void ins(Element e) {  
    n = get_local_node(create_if_absent: true);  
    n.backend.ins(e);  
  }  

  Element rem() {  
    // Fast path of retrieving an element from the thread-local backend.  
    n = get_local_node(create_if_absent: false);  
    if (n != null) {  
      <e, state> = n.backend.rem();  
      if (e != null { return e; }  
    }  
    while true {  
      retry = false;  
      old_version = s.version;  
      range = s.range();  
      for i in range {  
        n = s.nodes[i];  
        if old_version != s.version {  
          retry = true; break; }  
        Bool alive = n.alive;  
        <e, state> = n.backend.rem();  
        if e == null {  
          states[i] = state;  
          if !alive {  
            s.cleanup_thread(n, old_version);  
            retry = true; break; }  
        } else {  
          return e;  
        }  
      }  
      if retry { continue; }  
      if old_version != s.version { continue; }  
    }  
    return null; // Empty case.  
  }  

}  

// Called upon thread termination.  
void terminate() {  
  n = get_local_node(create_if_absent: false);  
  if n != null { n.alive = false; }  
}  

Listing 3 LLD and LL+D (queue and stack)

a single thread) is allowed, see the histories (b) and (c) in the proof of Theorem 15 and (4) not sequentially consistent with respect to queue and stack, as shown by the histories (d) and (e) in the proof of Theorem 15 that are k-FIFO and k-Stack histories, respectively, for \( k \geq 1 \).
We now present LL $k$-FIFO and LL $k$-Stack, modifications of $k$-FIFO and $k$-Stack, that enforce local linearizability by ensuring that no thread inserts more than once in a single segment. Assuming that segments are unique (by tagging pointers), LL $k$-FIFO remembers the last used insertion pointer per thread. For LL $k$-Stack the situation is more subtle as (due to the stack semantics) segments can be reached multiple times for insertion and removal. Figure 10 illustrates an example where the top segment of a $k$-Stack is reached multiple times by the same thread ($T_1$). Since in the general case all segments could be reached multiple times by a single thread it is required to maintain the full history of each thread’s insertions. Assuming the maximum number of threads is known in advance, a bitmap is used to maintain the information in which segment a thread has already pushed a value. One can similarly implement a locally linearizable version of the Segment Queue [2].

![Figure 10](image)

**Figure 10** LL $k$-Stack run ($k = 2$). $T_1$ can only insert in uncolored segments and needs to prepend a new segment (for insertion) otherwise.

### G.1.1 $k$-FIFO Queue and LL $k$-FIFO Queue Pseudo Code.

Listing 4 shows the pseudo code for LL $k$-FIFO queue. Again we highlight the code we added to the original pseudo code [28]. Similar to the locally linearizable $k$-Stack each thread inserts at most one element into a segment. However, in the $k$-FIFO queue we do not need flags in each segment to achieve this property. It is sufficient to remember the last segment used for insertion for each thread (set_last_tail; line 15). For each enqueue the algorithm checks whether the executing thread has already used this segment for enqueueing an element (get_last_tail; line 5). If the segment has already been used, the thread tries to append a new segment (effectively adding a new tail).

### G.1.2 Correctness Proof of LL $k$-FIFO Queue.

Having Theorem 25, the proof of correctness of LL $k$-FIFO queue is easy.

**Theorem 21** (Correctness of LL $k$-FIFO). LL $k$-FIFO queue presented in Listing 4 is locally linearizable.

**Proof.** Using Theorem 25 as a first proof obligation we have to show that any history $h$ of the LL $k$-FIFO queue is locally linearizable with respect to the pool sequential specification $S_P$. This proof is analogous to the proof that any history of the LL $k$-Stack is locally linearizable with respect to the pool sequential specification $S_P$, and is therefore postponed until the corresponding LL $k$-Stack theorem.

What remains to show is that

$$\forall x, y \in V. \forall i. \text{enq}(x) <_h \text{enq}(y) \& \text{deq}(y) \in h \Rightarrow \text{deq}(x) \in h \& \text{deq}(y) \notin_h \text{deq}(x)$$


Assume $\text{enq}(x) <_h \text{enq}(y)$. This means that $x$ and $y$ were enqueued by the same thread $i$ and therefore inserted into different segments. Moreover, the segment of $x$ is closer to the head of the list than the segment of $y$. A $\text{deq}(y)$ method call can remove $y$ only if the segment of $y$ is the head segment. The segment of $y$ can only become the head segment if all segments closer to the head of the list get empty. This means that also the segment of $x$ has to become empty. Therefore there has to exist a $\text{deq}(x)$ method call which removes $x$ from the segment, and $\text{deq}(x) \not<_h \text{deq}(y)$.
G.1.3 $k$-Stack and LL $k$-Stack Pseudo Code.

Listing 5 shows the pseudo code for LL $k$-Stack. The highlighted code is the code we added to the original pseudo code [23] to achieve local linearizability. The difference to the original algorithm is that a thread inserts at most one element into a segment. To achieve this property each segment in the k-stack contains a flag per thread which is set when an element is inserted into the segment (mark_segment_as_used; line 14 and line 60). If a thread encounters a segment where its flag is already set, the thread does not insert its element into that segment but tries to prepend a new segment (is_segment_marked; line 50). Otherwise the element is inserted into the existing segment and the flag of the thread in that segment is set.

```c
LocallyLinearizableKStack {
    SegmentPtr top;
    void init():
        new_ksegment = calloc (sizeof(ksegment));
        top = atomic_value(new_ksegment, 0);
    bool try_add_new_ksegment(top_old, item):
        if top_old == top:
            new_ksegment = calloc (sizeof(ksegment));
            new_ksegment->next = top_old;
            new_ksegment->s[0] = atomic_value(item, 0); // Use first slot for item.
            top_new = atomic_value(new_ksegment, top_old.ver+1);
            mark_segment_as_used(top_new);
            if CAS(&top, top_old, top_new):
                return true;
            return false;
    void try_remove_ksegment(top_old):
        if top_old == top:
            if top_old->next != null:
                atomic_increment(&top_old->remove);
            if empty(top_old):
                top_new = atomic_value(top_old->next, top_old.ver+1);
                if CAS(&top, top_old, top_new):
                    return;
                atomic_decrement(&top_old->remove);
    bool committed(top_old, item_new, index):
        if top_old->s[index] != item_new:
            return true;
        else if top_old->remove == 0:
            return true;
        else: //top_old->remove >= 1
            item_empty = atomic_value(EMPTY, item_new.ver+1);
            if top_old != top:
                if !CAS(&top_old->s[index], item_new, item_empty):
                    return true;
            else:
                top_new = atomic_value(top_old.val, top_old.ver+1);
                if CAS(&top, top_old, top_new):
                    return true;
                if !CAS(&top_old->s[index], item_new, item_empty):
                    return true;
                return false;
    void push(item):
        while true:
            top_old = top;
            if segment_is_marked(top_old):
                if try_add_new_ksegment(top_old, item):
                    return true;
            continue; // Restart while loop.
    item_old, index = find_empty_slot(top_old);
    if top_old == top:
        if item_old.val == EMPTY:
            item_new = atomic_value(item, item_old.ver+1);
            if CAS(&top_old->s[index], item_new, item_old, item_new):
                if committed(top_old, item_new, index):
```
The local linearizability proof of LL $k$-Stack is more involved, but very interesting. We use a theorem from the published artifact of [13], which has been mechanically proved in the Isabelle HOL theorem prover.

**Theorem 22 (Empty Returns for Stack).** Let $h$ be a history, and let $h'$ be the projection of $h$ to $\Sigma \setminus \text{pop(\text{empty})}$. If $h$ is linearizable with respect to the sequential specification $S_P$ of a pool (see Definition 2), and $h'$ is linearizable with respect to the sequential specification $S_S$ of a stack (see Definition 3), then $h$ is linearizable with respect to $S_S$.

**Proof.** Here we repeat the key insights of the proof and leave out technical details. A complete and mechanized version of the proof is available in the published artifact of [13].

As $h$ is linearizable with respect to $S_P$, and $h'$ is linearizable with respect to $S_S$, there exists a sequential history $s \in S_P$ such that $s$ is a linearization of $h$, and there exists a sequential history $s' \in S_S$ such that $s'$ is a linearization of $h'$. We show that we can construct a sequential history $t \in S_S$ such that $t$ is a linearization of $h$.

The linearization $t$ is constructed as follows: the position of pop(\text{empty}) in $s$ is preserved in $t$. This means for any method call $m \in s$ that if $\text{pop(\text{empty})} \prec_s m$, then also $\text{pop(\text{empty})} \prec_t m$, and if $m \prec_s \text{pop(\text{empty})}$, then also $m \prec_t \text{pop(\text{empty})}$. Moreover, if two method calls $m, n \in s$ are ordered as $m \prec_s \text{pop(\text{empty})}$, then $m \prec_t \text{pop(\text{empty})}$ and therefore by transitivity it holds that $m \prec_s n$, then also $m \prec_t n$.

For all other method calls the order of $s'$ is preserved. This means for any two method calls $m, n \in s$ with $m \prec_{s'} n$, that if for all $\text{pop(\text{empty})}$ it holds that $\text{pop(\text{empty})} \prec_s m$ if and only if $\text{pop(\text{empty})} \prec_t m$, then $m \prec_t n$.

By construction, the history $t$ is sequential and a permutation of $h$. Next we show that $t$ is a linearization of $h$ by showing that $t$ preserves the precedence order of $h$. Also by construction, it holds that if $m \prec_t n$ for any two method calls $m, n \in t$, then also either $m \prec_s n$ or $m \prec_{s'} n$. Both $s$ and $s'$ are linearizations of $h$ and $h'$, respectively. Therefore it cannot be for any $m, n$ with $m \prec_h n$ that $n \prec_s m$ or $n \prec_{s'} m$, it can also not be that $n \prec_t m$. Since $t$ is sequential, this means that $t$ preserves the precedence order of $h$.

Next we show that $t \in S_P$ according to Definition 2.
(1) Every method call, but pop(empty), appears in s at most once: This is guaranteed since t is a permutation of s, and s ∈ S_P.

(2) If pop(x) appears in t, then also push(x) does and push(x) ≺_t pop(x): again, since t is a permutation of s and s ∈ S_P, if pop(x) ∈ t, then also push(x) ∈ t. Since push(x) ≺_s pop(x) and push(x) ≺_s' pop(x) (because both s and s’ are in S_P) it also holds that push(x) ≺_t pop(x), as we argued already above.

(3) ∀x ∈ V. push(x) ≺_t pop(empty) ⇒ pop(x) ≺_t pop(empty): this property is satisfied trivially as all pop(empty) operations are ordered the same in t as in s, and s ∈ S_P.

It only remains to check that all elements are removed in a stack fashion. We have to show the following:

∀x, y ∈ V. push(x) ≺_t push(y) ≺_t pop(x) ⇒ pop(y) ∈ t ∧ pop(y) ≺_t pop(x)

First we show that if push(x) ≺_t push(y) ≺_t pop(x), then also push(x) ≺_s push(y) ≺_s pop(x). We do this by showing that there cannot exist a pop(empty) such that push(x) ≺_t pop(empty) ≺_t push(y) or push(y) ≺_t pop(empty) ≺_t pop(x).

Assume, towards a contradiction, push(x) ≺_t pop(empty) ≺_t push(y). By the transitivity of ≺_t this implies that push(x) ≺_t pop(empty) ≺_t pop(x), which contradicts our observation above that t ∈ S_P. Therefore push(x) ≺_t pop(empty) ≺_t push(y) is not possible, and for the same reason also push(y) ≺_t pop(empty) ≺_t pop(x) is not possible.

Now, as s’ ∈ S_S and push(x) ≺_s’ push(y) ≺_s’ pop(x), there has to exist a pop(y) ∈ s’ with pop(y) ≺_s’ pop(x). For the same reason as above it cannot be that pop(x) ≺_s pop(empty) ≺_s pop(y). Therefore pop(y) and pop(x) are ordered in t the same as in s’, i.e. pop(y) ≺_t pop(x), and therefore t ∈ S_S.

▸ Theorem 23 (Correctness of LL k-Stack). The LL k-Stack algorithm presented in Listing 5 is locally linearizable.

Proof. We have to show that every history h of LL k-Stack is locally linearizable with respect to the sequential specification S_S defined in Definition 2. This means that we have to show that every thread-induced history h_i of h is linearizable with respect to S_S for any thread i.

Having Theorem 22 we only have to show that h_i is linearizable with respect to the sequential specification S_P of a pool (defined in Definition 2), and that h_i’, the projection of h_i to Σ \ pop(empty), is linearizable with respect to the sequential specification S_S of a stack.

We start with the proof that h_i is linearizable with respect to S_P. We construct a sequential history s_i from h_i by identifying the linearization points of the push and pop method calls of the LL k-Stack. This means that two method calls m,n are ordered in s_i, m ≺_s n if the linearization point of m is executed before the linearization point of n in h_i.

The linearization point of push method calls is either the successful insertion of a new element from its segment slot in line 15 or the last successful CAS which writes the element into a segment slot in line 58. The linearization point of pop method calls is the successful CAS which removes an element from its segment slot in line 25.

For the linearization point of pop(empty) we take the linearization point of the call to empty in line 77. The empty method creates an atomic snapshot 25 of the top segment. This atomic snapshot is the state of the top segment at some point (i.e. linearization point of empty) within the execution of empty. If empty returns true, then there exists no element in the atomic snapshot of the segment.
Next we show that \( s_i \) is in \( S_P \) as defined in Definition 2.

1. Since there exists exactly one linearization point per method call, every method call, but \( \text{rem}() \), appears in \( s_i \) at most once.
2. If \( \text{pop}(x) \) appears in \( s_i \), then it reads \( x \) in a slot of the top segment before its linearization point. Since only push method calls write their elements into segment slots, there has to exist a \( \text{push}(x) \) which wrote \( x \) into that slot. Therefore the linearization point of \( \text{push}(x) \) is always before the linearization point of \( \text{pop}(x) \), and therefore \( \text{push}(x) \prec_{s_i} \text{pop}(x) \).
3. Segments are only removed from the list of segments when they become empty. The call to \text{committed} \ guarantees that elements are not inserted into segments which are about to be removed.

A pop method calls \text{empty} \ only if there is a single segment left in the LL \( k \)-Stack and no element was found in that segment in \text{find_item}.

Now assume a \( \text{push}(x) \) method call inserts an element \( x \) which is missed by \text{find_item}.

If \( \text{push}(x) \) wrote \( x \) into a segment before the linearization point of \( \text{pop}() \) and the segment was not the last segment, then the top segment changed since \( \text{pop}() \) searched for an element and therefore the check in line 70 would fail. If \( \text{push}(x) \) wrote \( x \) into the last segment of the LL \( k \)-Stack, then a \( \text{pop}(x) \) method call removed \( x \) from the segment because otherwise \( x \) would be in the atomic snapshot of \text{empty} \ and therefore \text{empty} \ would return \text{false}. Therefore, if \( \text{push}(x) \prec_{s_i} \text{pop}() \), then also \( \text{pop}(x) \prec_{s_i} \text{pop}(\text{empty}) \).

Therefore \( s_i \) is in the sequential specification \( S_P \) of a pool.

Next we show that \( h'_i \) is linearizable with respect to \( S_g \). We construct again a sequential history \( s'_i \) from \( h'_i \) by identifying the linearization points of the push and pop method calls of LL \( k \)-Stack.

The linearization point of the push operations is the successful insertion of a new segment in line 15 if it is executed, or the reading of the empty slot (line 54) in the last (and therefore successful) iteration of the main loop. The linearization point of a pop operation is the reading of a non-empty slot (line 69) in the last (and therefore successful) iteration of the main loop. There do not exist any \( \text{pop}(\text{empty}) \) method calls in \( s'_i \). Since we assume a sequentially consistent memory model, these read operations define a total order on the LL \( k \)-Stack method calls in \( h'_i \).

First we show that \( s'_i \) is in the sequential specification \( S_P \) of a pool as defined in Definition 2.

1. Since there exists exactly one linearization point per method call, every method call appears in \( s'_i \) at most once.
2. If \( \text{pop}(x) \) appears in \( s'_i \), then it read \( x \) in a slot of the top segment at its linearization point. Since only push operations write their elements into segment slots, there has to exist a \( \text{push}(x) \) which wrote \( x \) into that slot. The linearization point of \( \text{push}(x) \) is always before \( x \) is written into a segment slot. Therefore \( \text{push}(x) \prec_{s'_i} \text{pop}(x) \).
3. Since there exist no \( \text{pop}(\text{empty}) \) operations in \( s'_i \), the third pool condition is trivially correct.

Next we show that \( s'_i \) also provides a stack order, which means that we have to show that

\[
\forall x, y \in V. \, \text{push}(x) \prec_{s'_i} \text{push}(y) \prec_{s'_i} \text{pop}(x) \Rightarrow \text{pop}(y) \in s'_i \land \text{pop}(y) \prec_{s'_i} \text{pop}(x).
\]

We start by observing some invariants.
1. A thread never inserts elements into the same segment twice. This is guaranteed by the call to \texttt{segment_is_marked}.

2. Between the linearization point of a push and the time it writes its element into a segment the segment the element gets written into is not removed: if the push operation inserts a new segment this is trivially correct. If the push operation writes the element into an existing segment, then the call to \texttt{committed} in line 59 guarantees that the segment was not removed.

3. At the time of the linearization point of the pop, which is the time when the pop reads the non-empty slot (line 69) in the last (and therefore successful) iteration, the pop reads the non-empty slot from the top segment. This is guaranteed by the check in line 70.

Now assume there exist the operations \texttt{push(x)}, \texttt{push(y)} and \texttt{pop(x)} in \textit{s′}, and \texttt{push(x)} ≺ \textit{s′}, \texttt{push(y)} ≺ \textit{s′}, \texttt{pop(x)}. Since \texttt{push(x)} and \texttt{push(y)} are both in \textit{s′}, this means that both operations are executed by the same thread. Therefore, according to Invariant 1., \texttt{x} and \texttt{y} get inserted into different segments, with the segment \texttt{y} on top of the segment of \texttt{x}.

The linearization point of \texttt{pop(x)} cannot be before \texttt{y} is written into its segment because according to Invariant 2. the segment \texttt{y} gets inserted into does not get removed between the linearization point of \texttt{push(y)} and the time \texttt{y} is written into the segment. With Invariant 3. this means that \texttt{x} is unaccessible for \texttt{pop(x)} before \texttt{y} gets written into a segment. Also because of the third invariant the top segment changes between the insertion of \texttt{y} and the linearization point of \texttt{pop(x)}.

Next we observe that as long as \texttt{y} is not removed, no segment below the segment of \texttt{y} can become the top segment. Therefore for the segment of \texttt{x} to become the top segment so that \texttt{pop(x)} can remove it, \texttt{y} has to be removed first. Only a \texttt{pop(y)} can remove \texttt{y}, and therefore there exists a \texttt{pop(y)} and the linearization point of \texttt{pop(y)} is before the linearization point of \texttt{pop(x)}.

Hence \textit{s′} is in the sequential specification of a stack. Using Theorem 22 this means that \texttt{LL k-Stack} in listing 5 is locally-linearizable with respect to the sequential specification of a stack.

**H Additional Experiments**

We also evaluate the implementations on another Scal workload, the sequential alternating workload. However, we note that in this workload in the locally linearizable implementations threads only access their local backends, so no wonder they perform perfectly well.

**Mixed Workload.** In order to evaluate the performance and scalability of mixed workloads, i.e., workloads where threads produce and consume values, we exercise the so-called sequential alternating workload in Scal. Each thread is configured to execute $10^6$ pairs of insert and remove operations, i.e., each insert operation is followed by a remove operation. As in the producer-consumer workload, the contention is controlled by adding a busy wait of $5\mu s$. The number of threads is configured to range between 1 and 80. Again we report the number of data structure operations per second.

Data structures that require parameters to be set are configured like in the producer-consumer benchmark. Figure 11 shows the results of the mixed workload benchmark for all considered data structures.

The MS queue and Treiber stack do not perform and scale for more than 10 threads. As in the producer-consumer benchmark, LCRQ and TS Stack either perform competitively
Local Linearizability

Queues, LL queues, and “queue-like” pools

Stacks, LL stacks, and “stack-like” pools

Figure 11 Performance and scalability of sequential alternating microbenchmarks with an increasing number of threads on a 40-core (2 hyperthreads per core) machine

with their $k$-out-of-order relaxed counter parts $k$-FIFO and $k$-Stack or even outperform and outscale them (in the case of LCRQ, that even outperforms the pool).

LL$^+$D MS queue, LLD LCRQ, and LL$^+$D Treiber stack perform very well and scale (nearly) linearly in the number of threads. A surprising result is that LLD $k$-FIFO performs poorly in this experiment. The reason is that $k$-FIFO performs poorly when it is almost empty, and in this experiment each backend instance of LLD $k$-FIFO contains at most one element at any point in time. The $k$-Stack performs better on a nearly-empty state. The benefit of trying to perform a local operation first in the LLD algorithms is visible when comparing to 1-RA DQ and DS that do not utilize a local fast path.

1 Verifying Local Linearizability

In general, verifying local linearizability amounts to verifying linearizability for a set of smaller histories. This might enable verification in a modular/compositional way. Aside from this, it is important to mention (again) that for our locally linearizable data structures in Section 6 built from linearizable building blocks, the correctness proofs are straightforward assuming the building blocks are proven to be linearizable. In addition, for queue we can state an “axiomatic” verification theorem for local linearizability in the style of [24, 10], whose main theorem we recall next (with a slight reformulation).

Theorem 24 (Queue Linearizability). A queue concurrent history $h$ is linearizable wrt the queue sequential specification $S_Q$ if and only if

1. $h$ is linearizable wrt the pool sequential specification $S_P$ (with suitable renaming of method calls), and
2. $\forall x, y \in V. \ enq(x) \prec_h \ enq(y) \land \ deq(y) \in h \ \Rightarrow \ deq(x) \in h \ \land \ deq(y) \nprec_h \ deq(x)$.

We note that an analogous change to the axioms in the sequential specification of a pool and a stack does not lead to a characterisation of linearizability for pools and stacks, cf. [13]. An axiomatic characterisation of linearizability for pools and stacks would involve an infinite number of axioms/infinite axioms, due to the need to prohibit infinitely many problematic shapes, cf. [7].

We are now able to state the queue-local-linearizability-verification result.
Theorem 25 (Queue Local Linearizability). A queue concurrent history $h$ is locally linearizable wrt the queue sequential specification $S_Q$ if and only if

1. $h$ is locally linearizable wrt the pool sequential specification $S_P$ (after suitable renaming of method calls), and
2. $\forall x, y \in V. \forall T. enq(x) <^T_h enq(y) \land deq(y) \in h \Rightarrow deq(x) \in h \land deq(y) \not<^h_h deq(x)$.