A simple approximate expression for the Apéry’s constant accurate to 21 digits

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Abstract

In this note, I present a simple approximate expression for the number \( \zeta(3) \), known as the Apéry’s constant, which is accurate to 21 digits. This finite closed-form expression has been found experimentally via the PSLQ algorithm, with a suitable search basis involving the numbers \( \pi \), \( \ln 2 \), \( \ln (1 + \sqrt{2}) \), and \( G \) (the Catalan’s constant). The short Maple code written for finding the rational coefficients is also shown.

Key words: Apéry’s constant, Integer relation detection

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“In fact, numerical experimentation is crucial to Number Theory, perhaps more so than to other areas of mathematics.”

F. R. Villegas

1. Introduction

The Apéry’s constant is defined as the number to which the series $\sum_{n=1}^{\infty} 1/n^3$ converges and it is so designated in honor to R. Apéry, who proved in 1978 that this number is irrational [1]. It is not known if it is a transcendental number. The convergence of that series to a real number between 1 and 2 is guaranteed by the Cauchy’s integral test, a result that remains valid for $\sum_{n=1}^{\infty} 1/n^s$, for any real $s > 1$. In fact, convergence is also found for $\Re(s) > 1$, a complex domain in which this series is defined as $\zeta(s)$, the Riemann’s zeta function. Apéry’s constant is then identified with $\zeta(3) = 1.202056903\ldots$

For positive integer values of $s$ ($s > 1$), it was Euler (1734) the first to obtain an exact closed-form result for $\zeta(s)$, namely $\zeta(2) = \pi^2/6$, the solution of the so-called Basel’s problem [2]. Some years later, he did generalize this result for all (positive) even values of $s$ [3, 4]:

$$\zeta(2n) = (-1)^{n-1} \frac{2^{2n-1} B_{2n} \pi^{2n}}{(2n)!},$$

where $n$ is a positive integer and $B_{2n} \in \mathbb{Q}$ are Bernoulli numbers. For odd values of $s$, closed-form expressions are not known. In a recent work, it is even claimed that $\zeta(2n+1)$ is not a rational multiple of $\pi^{2n+1}$ [23].

The increase of interest in $\zeta(3)$, which comes from both pure and applied mathematics [30] has stimulated its high-precision numerical computation, as

\footnote{For instance, given three integers chosen at random, the probability that no common
well as the search for *simple* approximate expressions for it [5]. Let us adopt a reasonable criterium for the adjective “simple,” in the context of finite approximate expressions. Here, this will designate closed-form expressions containing only *a few* terms/factors composed by other mathematical constants and *a few* integers with *small* absolute values. This criterium is, of course, vague due to the forms “a few” and “small”. To make it *not-so-imprecise*, “a few” will mean less than, say, ten (certainly not twenty), and “small” will mean less than, say, five hundreds (certainly not a thousand). A such approximation was presented by Galliani (2002), namely [6]

\[
\zeta(3) \approx \frac{1}{\sqrt[3]{\gamma}} + \frac{1}{\pi^4} \left(1 + 2\gamma - \frac{2}{130 + \pi^2}\right)^{-3},
\]

(1)

where \(\gamma\) is the Euler-Mascheroni constant, which is accurate to 4 digits (typo corrected). Another nice simple approximation is

\[
\zeta(3) \approx 4\sqrt{\gamma + \frac{71}{47}},
\]

(2)

due to Hudson (2004), which is accurate to 7 digits [6]. Among the many approximations for \(\zeta(3)\) presented by Hudson, the most accurate is [6]

\[
\zeta(3) \approx 525587^{1/\sqrt{5123}},
\]

(3)

which is accurate to 12 digits and, clearly, is not a *simple* approximate expression (in our terminology). The same for \(\zeta(3) \approx \frac{97525}{2815594}\pi^3\), which I found

factor will divide them all is \(1/\zeta(3)\). Also, if \(n\) is a power of 2, then the number \(#(n)\) of distinct solutions for \(n = p + xy\) with \(p\) prime and \(x, y\) positive integers obeys the asymptotic relation \(#(n)/n \sim 105 \zeta(3) / (2\pi^4)\). It also arises in a number of physical problems, including the computation of electron’s gyromagnetic ratio.
by searching for an integer relation between $\zeta(3)$ and $\pi^3$, before the appearance of Ref. [23]. In trying to reach greater accuracy, however, one soon observes that it is very difficult to avoid the appearance of large integers.

In this short note, I present a simple approximate expression for $\zeta(3)$ which is accurate to 21 digits, thus triply most accurate than the best previous simple approximation. This closed-form expression involves only small integers and the constants $\pi$, $\ln 2$, $\ln (1 + \sqrt{2})$, and $G$ (the Catalan’s constant) and it has been found experimentally by employing a search basis with six elements in the PSLQ algorithm. The short Maple code I have written for finding such expression is also shown in view to stimulate people to develop their own computational experiments.

2. The search for integer relations and the PSLQ algorithm

An important task in experimental mathematics is to search for integer relations involving a finite set of computed numbers. An integer relation algorithm is a computational scheme that, for a given real vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $n$ being a positive integer, $n > 1$, it either finds a nonnull vector of integers $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ such that $a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = 0$ or else establishes that there is no such integer vector within a ball of some radius about the origin.

Presently, the best algorithm for detecting integer relations is the PSLQ algorithm (acronym for Partial Sum of Least Squares) introduced by Ferguson and Bailey (1992) [7]. A simplified formulation for this algorithm, mathematically equivalent to the original one, was subsequently developed by Ferguson and co-workers (1999) [8]. This more efficient version of PSLQ
is currently implemented in both Maple and Mathematica, two of the most popular mathematical softwares. It was this version, together with some reduction schemes, that was named one of the ‘ten algorithms of the century’ [9].

In short, PSLQ operates as follows. Given a vector \( \mathbf{x} \) of \( n \) given real numbers, input as a list of floating-point (FP) numbers, the algorithm uses QR decomposition in order to construct a series of matrices \( A_m \) such that the absolute values of the entries of the vector \( \mathbf{y}_m = A_m^{-1} \cdot \mathbf{x} \) decrease monotonically. At any given iteration, the largest and smallest entries of \( \mathbf{y}_m \) usually differ by no more than a few orders of magnitude. When the desired integer relation is detected, the smallest entry of \( \mathbf{y}_m \) abruptly decreases to roughly the computer working precision \( \epsilon \) and the relation is given by the corresponding column of \( A_m^{-1} \). This numerically stable matrix reduction procedure, together with some techniques that allow machine arithmetic to be used in many intermediate steps, usually yields a rapid convergence, which makes PSLQ faster than other concurrent algorithms [10]. When a relation is detected, the ratio between the smallest and the largest entry of the vector \( A^{-1} \cdot \mathbf{x} \) can be taken as a “confidence level” that the relation is true (i.e., exact) and not an artifact of insufficient precision. Very small ratios at detection certainly indicate the result is probably true.2

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2In addition to possessing good numerical stability, PSLQ is guaranteed to find an integer relation in a number of iterations bounded by a polynomial in \( n \).
3. Maple code for finding an approximate expression via PSQL

Since an efficient PSLQ routine is available as part of a Maple package named IntegerRelations, then simple short codes can be written in this system. For illustrating this, let me list just the source code I have written for finding an approximate expression for $\zeta(3)$.

```maple
restart; # Clear memory
Digits := 24: # The number of digits for FP numbers
with(IntegerRelations): # Call the package containing PSLQ
xSymb:=[Zeta(3),1,Pi^2*ln(2),Pi*ln(2)^2,ln(2)^3,ln(1+sqrt(2))^3,Pi*Catalan];
xSymb := [ζ(3), 1, π^2 ln(2), π ln(2)^2, ln(2)^3, ln(1 + √2)^3, π Catalan]
n := nops(xSymb); # The number of elements in xSymb
n := 7
x:=evalf(xSymb): # Convert to FP numbers
a:=PSLQ(x); # Applies PSLQ algorithm to x
a := [10, 394, -11, 283, -472, -209, -186]
soma:=0:
for k from 1 to n do soma:=soma+a[k]*xSymb[k]; end do:
aprox:=solve(soma=0, Zeta(3));
aprox := -5/197 + 11/394 * π^2 * ln(2) - 283/394 * π * ln(2)^2 + 236/197 * ln(2)^3 + 209/394 * ln(1 + √2)^3 + 93/197 * π Catalan
evalf( aprox, 30 ); # Our approximation
1.20205690315959428539958993430

evalf( Zeta(3), 30 ); # Exact value (for a better comparison)
1.20205690315959428539973816151
```
With this Maple routine, I have found the following simple approximate expression for the Apéry’s constant:

$$\zeta(3) \approx -\frac{5}{197} + \frac{11}{394} \pi^2 \ln 2 - \frac{283}{394} \pi \ln^2 2 + \frac{236}{197} \ln^3 2 + \frac{209}{394} \ln^3 (1 + \sqrt{2}) + \frac{93}{197} \pi G,$$

(4)

which is accurate to 21 digits, as the reader can check by comparing the last two outputs of the program. Equation (4) is the main result of this paper.

All that rests is to explain the motivation for choosing the vector

$$\mathbf{x} = \left(1, \pi^2 \ln 2, \pi \ln^2 2, \ln^3 2, \ln^3 (1 + \sqrt{2}), \pi G\right)$$

as the search basis for $\zeta(3)$. The motivation comes primarily from a conjecture by Euler (1785) that $\zeta(3) = \alpha \ln^3 2 + \beta \pi^2 \ln 2$, for some $\alpha, \beta \in \mathbb{Q}$.

After many numerical experiments, I could not find any such pair of rational coefficients for composing an exact closed-form expression for $\zeta(3)$. Not even a simple approximate expression was found with only these two terms. I am currently testing the conjecture by Connon (2008) that either $\alpha$ or $\beta$ could contain a factor of $\sqrt{2}$, or another small surd, or perhaps $\ln(2\pi)$. I was then inclined to improve the basis by including other weight-3 constants such as $\pi \ln^2 2$ and $\pi G$, which arise in many exact results involving $\zeta(3)$, as for instance:

(i) Special values of non-elementary functions:

$$\text{Li}_3\left(\frac{1}{2}\right) = \frac{7}{8} \zeta(3) - \frac{1}{12} \pi^2 \ln 2 + \frac{1}{6} \ln^3 2, \quad \Re\left[\text{Li}_3\left(\frac{1}{2} + i\right)\right] = \frac{35}{64} \zeta(3) + \frac{1}{48} \ln^3 2 - \frac{5}{192} \pi^2 \ln 2$$

(15), $\pi^2 \psi^{(-4)}(1) = \frac{1}{2} \pi^2 \ln A + \frac{1}{12} \pi^2 \ln 2 + \frac{1}{12} \pi^2 \ln \pi + \frac{1}{8} \zeta(3)$

(16)

(typo corrected);

\footnote{\textit{Weight}, here, is according to the definition by Boros and Moll (see Ref. \textit{[14]}, p. 203).}
(ii) Series:
\[
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 2^k} = \frac{1}{16} \zeta(3) - \frac{1}{6} \ln^3 2, \quad \pi^2 \sum_{k=1}^{\infty} \frac{\zeta(2k)}{(k+1) 2^k} = \frac{1}{8} \pi^2 - \frac{1}{2} \pi^2 \ln 2 + \frac{35}{4} \zeta(3) - 4 \pi G, \quad \pi^4 \sum_{m=0}^{\infty} \frac{(-1)^m \pi^{2m} E_{2m+1}(1)}{(2m+5)!} = \frac{1}{3} \pi^2 \ln 2 - \frac{3}{2} \zeta(3).
\]

(iii) Definite integrals:
\[
\int_0^{\pi/2} x \ln(\sin x) \, dx = \frac{7}{16} \zeta(3) - \frac{1}{8} \pi^2 \ln 2, \quad \int_0^{\pi/4} x \ln(\cos x) \, dx = -\frac{21}{128} \zeta(3) + \frac{1}{8} \pi G - \frac{1}{32} \pi^2 \ln 2, \quad \int_0^{\pi/2} x^2 \tan x \, dx = -\frac{21}{64} \zeta(3) + \frac{1}{4} \pi G - \frac{1}{32} \pi^2 \ln 2, \quad \int_0^{\pi/2} x^2 / \sin x \, dx = 2 \pi G - \frac{7}{8} \zeta(3), \quad \int_0^1 (\arcsin t)^2 \, dt = \frac{1}{4} \pi^2 \ln 2 - \frac{7}{8} \zeta(3).
\]

(see Eq.(6.6.25) in Ref. [14]).

Here, \( \text{Li}_3(z) := \sum_{n=1}^{\infty} z^n / n^3 \) is the trilogarithm function, \( \psi^{(-4)}(z) \) is the polygamma function (extended to negative indexes, according to Ref. [21]), and \( E_{2m+1}(x) \) are Euler polynomials.

With respect to \( \ln^3 (1 + \sqrt{2}) \), which is the cube of a logarithm of an algebraic number, thus a weight-3 constant, I have included it in the search basis because, curiously, the number \( \text{arcsinh}(1) = \ln (1 + \sqrt{2}) \) emerges in the coordinates of the vertices of a cusped hyperbolic cube whose volume is \( \frac{7}{8} \zeta(3) \), as I have found in a recent work on multiple integrals related to \( \zeta(2) \) and \( \zeta(3) \).

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