1+3 Covariant Cosmic Microwave Background anisotropies II: 
The almost–Friedmann Lemaître model

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This is the second of a series of papers extending the 1+3 Covariant and Gauge-Invariant treatment of kinetic theory to an examination of Cosmic Microwave Background temperature anisotropies arising from inhomogeneities in the early universe. The first paper (Paper I) dealt with algebraic issues, representing anisotropies in a covariant and gauge invariant way by means of Projected Symmetric and Trace-Free tensors.

Here we derive the mode form of the Integrated Boltzmann Equations, first, giving a covariant version of the standard derivation using the mode recursion relations, second, demonstrating the link to the the Multipole Divergence Equations and finally various analytic ways of solving the resulting equations are discussed.

A general integral form of solution is obtained for the equations with Thomson scattering. The covariant Friedmann-Lemaître multipole form of the transport equations are found near tight-coupling using the covariant and gauge-invariant generalization of the Peebles and Yu expansion in Thompson scattering time. The dispersion relations and damping scale are then obtained from the covariant approach. The equations are integrated to give the covariant and gauge-invariant equivalent of the canonical scalar sourced anisotropies in the $K = 0$ (flat background) case.

We carry out a simple treatment of the matter dominated free-streaming projection, slow decoupling, and tight-coupling cases in covariant and gauge-invariant theory, with the aim both giving a unified transparent derivation of this range of results and clarifying the formal connection between the usual approaches (for example Hu & Sugiyama) and the covariant and gauge-invariant like treatments for scalar perturbations (for example of Challinor & Lasenby).

1. INTRODUCTION

The present series of papers (Paper I \cite{18}) establishes the 1+3 covariant kinetic theory formalism of Ellis, Treciokas and Matravers \cite{29, 30} in a form that makes possible an investigation of the Cosmic Microwave Background anisotropies in the non-local context of emission of radiation at the surface of last scattering in the early universe, and its reception here and now (the ‘Sachs-Wolfe effect’ and its later extensions). These papers aims to provide the link between the exact (non-linear) theory \cite{59} and the linearised threading formalism (this paper and to a lesser degree \cite{10}), to the linearised foliation formalism, based on Bardeens’ approach \cite{3} to cosmological perturbations \cite{39}.

The Ellis, Treciokas and Matravers papers introduced a covariant approach to kinetic theory based on a 1+3 covariant representation of Cosmic Microwave Background anisotropies in terms of Projected Symmetric Trace-Free (PSTF) tensors orthogonal to a preferred time-like vector field $u^a$ \cite{76, 67}. The benefits of the approach have been briefly summarized in Paper I \cite{18} (see also Challinor & Lasenby \cite{10, 11}) – we seek clarity by providing a direct formal derivation of the standard results as well as providing the background necessary to include
the non-perturbative corrections discussed by Maartens, Gebbie and Ellis [59]. In essence, we provide (a) clear definitions of the variables used, (b) 1+3 Covariant and Gauge-Invariant (CGI) variables and equations, (c) a sound basis from which to proceed to non-linear calculations (as introduced in [59]), and (d) the possibility of using any desired coordinate and tetrad system for evaluating the variables and solving the equations in specific cases (because the general equations and variables used are covariant).

This approach has been used in a previous series of papers [57, 73, 58] to look at the local generation of anisotropies in freely-propagating radiation caused by anisotropies and inhomogeneities in any universe model, providing a proof that near-spatial homogeneity in a region \( U \) follows from radiation near-isotropy in that region (an ‘Almost-Ehlers, Geren and Sachs’ theorem, generalizing the important paper by Ehlers, Geren, and Sachs [23]). By contrast, the present series looks at non-local (integrated) anisotropy effects in the context of the standard model of cosmology – given the observational justification provided by the almost-Ehlers, Geren and Sachs theorem [72].

Paper I [18] of the present series of papers [18] dealt with the CGI irreducible representation of cosmic background radiation anisotropies by PSTF tensors, and their relation to observable quantities (specifically, the angular correlation functions), first within a general framework and then and specialized to almost-Friedmann-Lemaître universe models [57], as well as dealing with multipole and mode expansions and the relation to the usual formalisms in the literature [82, 35, 39, 40, 20].

In this paper (Paper II), we use the CGI approach to study cosmic background radiation anisotropies in almost-Friedmann-Lemaître models in an analytic manner, by time-like integration of the almost-Friedmann-Lemaître differential relations. Our emphasis is on the canonical linearised model for cosmic background radiation anisotropies [68, 81, 1, 75, 35, 39, 40, 79], systematically developing the CGI approach and providing a comprehensive and transparent analytic link to the alternative analytic gauge-invariant treatments based on the Bardeen gauge-invariant variables. We develop these results both in mode and multipole form, emphasizing the different physical processes and assumptions and demonstrating how these are dealt with in the CGI context. This requires the covariant mode form of the integrated Boltzmann equations, based on the recursion relations for almost-Friedmann-Lemaître mode functions [18], enabling a direct mirroring of standard treatments based on Wilson’s seminal paper [81, 35, 39, 40, 79], except carried out in a CGI fashion, thus forming a sound basis for extension to non-linear effects.

Indeed one of the advantages of the CGI approach in the context of the generic multipole divergence equations is the ability to include non-linear corrections to the almost-Friedmann-Lemaître treatment. Towards this end, the relationship between the covariant mode formulations and the almost-Friedmann-Lemaître covariant multipole treatment are given with this in mind, based on the relations in Paper I [18]. A key point here is that there is nothing new about the linear formulation itself, however recovering the standard analytic linear results from the the exact theory [59] in a straightforward way, is new.

Moreover, the results presented here provide the foundation for a non-linear extension of this approach which is outlined in a paper by Maartens, Gebbie and Ellis [59].

We emphasize that in our treatment, \( \langle \tau_A \tau_A^4 \rangle \) (the multipole form of the angular correlation function) is given for small temperature anisotropies irrespective of the form of the geometry [59], making it the natural representation for the inclusion of non-linear dynamic effects, while the analysis for \( |\tau|_2 \) (the mode coefficient mean square) is specifically for almost-Friedmann-Lemaître models [18]. The non-linear extension of the almost-Friedmann-Lemaître treatment given here will be based on the multipole-to-mode relations, leaving the use of mode expansion to the latest possible stage.

\[1\text{In order to be clear on the use of these names, Robertson-Walker refers to the Robertson-Walker geometry whatever the dynamics, while Friedmann-Lemaître refers to such a geometry which additionally obeys the Friedmann-Lemaître dynamics implied by imposing the Einstein Field Equations. An almost-Friedmann-Lemaître universe is a universe model governed by the Einstein field equations, whose difference from a Friedmann-Lemaître universe is at most } O(\epsilon) \text{ in terms of a small parameter } \epsilon [25].\]
Our focus is on the era following spectral decoupling (near 500 eV). A complex series of interactions take place at the various epochs of the expansion of the universe. The kinetic equations developed in Paper I and [59] can represent almost any such interactions provided we use appropriate interaction (‘collision’) terms; the issue is how to obtain simplified models that are reasonably accurate in the various epochs. We will consider only two kinds of representation here: namely

1. Thomson scattering, valid at late times when particle and photon numbers are conserved and the matter is non-relativistic (during decoupling, an alternative approach is to use a visibility function).

2. An effective two-fluid approach, obtained by truncation of the hierarchy and valid at earlier times when strong interactions take place establishing equilibrium or close to equilibrium conditions between the components, i.e. when the interaction time-scale is much less than the expansion time-scale; an alternative description is to use a single imperfect fluid [60].

Both descriptions are valid when the matter is relativistic. The detailed form of interactions does not need to be represented in this case, because the state of the matter depends only on its equilibrium condition, characterized by its equation of state.

At some times either form is valid and they can then be related to each other. We do not attempt here to give a description of earlier non-equilibrium eras when processes such as pair production, nucleosynthesis, etc, occur, nor do we consider issues such as inflation and reheating after inflation, and the differences between the inflation sourced perturbations as opposed to those based on other phase transitions. Thus our models will be valid after the end of any period of inflation that may have occurred and after strong non-equilibrium processes have ceased. During this era the processes which result from initial fluctuations left over from earlier non-equilibrium epochs, determine the final cosmic background radiation anisotropy.

Specifically, we deal with four eras of interest. Going backwards in time from the present, they are, firstly, free streaming from last scattering to here and now, in the matter dominated almost-Friedmann-Lemaître context; secondly, slow decoupling during a matter dominated era, which is when the cosmic background radiation spectrum freezes out; thirdly, the late tight-coupling era after matter-radiation equality, during which structure formation begins; and fourthly, the main tight-coupling era after any inflationary epoch and before matter-radiation equality, during which acoustic modes occur in the tightly-coupled fluid, the initial matter perturbations having been seeded by earlier conditions (for example, inflation). We then show how to put these CGI results together to determine the major features of the expected anisotropy spectrum. We develop sound models of the dominant effects in each of the eras we consider, but there will always be a need for refinement of these models by taking into account further effects (in particular polarisation and the effect of neutrinos).

Although the effect of the neutrinos is crucial, and can be subsumed into the gravitational variables. We do not provide nor discuss further the additional hierarchy of neutrino moment equations that would then need to be included. The exact massless neutrino evolution equations are given in a previous paper [59] and it is easy to show that the resulting linearised equations are essentially the same as those for massless radiation without collisionals, hence in this paper we focus on the photon equations.

In more detail:

- **Free-Streaming**: We find the CGI integral solutions to the almost-Friedmann-Lemaître multipole divergence equations with no collision term, and use them to project the initial data from decoupling to the current sky. We explicitly do this for instant decoupling. Neither the Vishniac, Rees-Sciama, thermal Sunyaev-Zel’dovich nor lensing effects are considered here – we will focus on the CGI model of the dominant processes, and these further effects will introduce detailed modifications. However, a comprehensive understanding of such secondary higher order effects relies on a derivation of the anisotropy effects given here.

- **Slow-Decoupling**: Here we consider modification of the previous results when slow decoupling of the interactions due to Thomson scattering is taken into account. We consider the
damped integral solution for slow recombination, and as an alternative description modify the integral solutions appropriately with a visibility function carrying the functional dependence of the varying electron fraction, in a matter dominated context. Effectively, recombination is complete before the radiation decouples. This means that the surface of last scattering is found a little after the end of recombination. It is during this era that photon diffusion damping scale effects become important – the damping scale is affected by the duration of this era. This will be investigated in the context of almost-Friedmann-Lemaître universes after matter radiation equality.

• **Tight-coupling:** This is the key to the entire treatment. We give the CGI version of the tight-coupling approximation of Hu *et al* [39, 40, 44]. In the almost-Friedmann-Lemaître treatment, the slow decoupling and free-streaming era’s will only ‘damp’ and ‘project’ the spectrum formed at the end of tight-coupling onto the current sky.

We consider two different tight-coupling regimes.

– The *late tight coupling era* is separated, conceptually, from the early tight coupling era by matter-radiation equality, after which time the matter perturbations have effectively decoupled and (CDM-based) structure formation begins. In this era, strong interactions have ceased and a Thomson scattering description can be used. We first carry out the near tight-coupling treatment of this era based on Peebles & Yu [66, 47], and then reduce these equations to the covariant tight-coupling equations equivalent to those of Hu *et al* [39, 40, 44]. This provides the basis for understanding the acoustic signatures in the temperature anisotropies within the CGI approach.

– The *early tight coupling era* occurs between the matter and radiation eras (after strong interactions have ceased), when a Thomson scattering description will also be sufficient. This era is characterized by acoustic oscillations in a tightly coupled fluid; for calculation convenience this can be represented as a single dissipative fluid [60], or for a slightly more sophisticated treatment by tightly coupled two-fluid models [14, 13]. We give a CGI derivation of the harmonic oscillator equation providing the primary source terms in the standard model of Doppler peak formation by acoustic oscillations.

We put these results together in sections 7 and 8, where the equations are integrated to give the CGI treatment in the $K = 0$ (flat background) case in terms of an integral solution.

The primary sources of the temperature anisotropies (the acoustic and Doppler contributions near last scattering resulting in ‘Doppler Peaks’ today) are demonstrated. This recovers the Sachs-Wolfe family of effects for flat background Robertson-Walker geometries, but derived from a CGI kinetic theory viewpoint as opposed to the photon-propagation description used in the original Sachs-Wolfe paper.

The form of the angular correlation function is determined for the primary effects (although not given explicitly in terms of the matter power spectrum). The normalization to standard CDM (Adiabatic CDM) is presented in terms of CGI variables. This demonstrates the basic effects in the CGI formalism, and links our approach to the standard literature, see for example [79] and references therein, where further details of this era are given.

It is important to note that the integrations considered here are carried out along time-like curves, even though the cosmic background radiation reaches us along null curves. These are alternative approaches that are equivalent in the context of linearisations about Friedmann-Lemaître models; differences will however occur if we include non-linear corrections. Briefly, the key point about cosmic background radiation integrations is that there are two ways in

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2 A note on nomenclature: By *decoupling* we have in mind the situation when the interaction rate per particle becomes less that the expansion rate. By *last scattering surface* we mean the surface upon which the diffusion scale is equal to the horizon scale, after which it is larger than that scale and the free-streaming approximation is sufficient. The photons will decouple from the thermal plasma near 0.2 eV, and from the matter after recombination has effectively ended, near 0.3 eV. Free-streaming is considered to be a good approximation from about 0.26 eV.

3 This useful feature of the almost-Friedmann-Lemaître models is due to the homogeneity and isotropy conditions in the background Friedmann-Lemaître universe and is not generic [59].

4 following the analytic treatment of Hu & Sugiyama [39, 40]
which to proceed: Firstly a null-cone integration, following the radiation from last scattering to the present day\(^5\), and secondly a time-like integration along the matter flow lines (as here). In the latter case one is (at least implicitly) thinking of a small comoving box containing matter and radiation [36] which is similar to all other small boxes at the same time, and where one has assumed that the radiation leaving is exactly balanced by the radiation entering (from neighbouring boxes), whether in tight-coupling (when it is a local assumption) or in the free-streaming era (when it is a non-local assumption). In effect one integrates behaviour in such a box in a small domain about our own world-line from tight-coupling through decoupling to the present day; to do this, one does not need to know about the behaviour of null-geodesics (the integration is along time-like geodesics). Before decoupling the matter and radiation evolve as a unit while after they need to be integrated separately (giving the corresponding transfer functions in each case). Then this is related to observations by, first, conceptually shifting copies of the small box at the time of last scattering from our world line to all points where the past null-cone intersects the surface of last scattering at that time; this can be done because spatial homogeneity says that these boxes are essentially the same (a Copernican assumption is used here, justified by the almost-Ehlers-Geren-Sachs theorem [57, 58]); second, by then relating distances on the last scattering surface to observed angles by using the area distance relation, relating physical distances at last scattering to angular size in the sky.

The next paper in the series, Paper III [15], deals with the explicit relationship between null-cone and time-like integrations. Further papers will look at non-linear extensions of the results given here.

2. LINEARISED COVARIANT MODE EQUATIONS

To study details of cosmic background radiation anisotropy generation we need both a spatial Fourier decomposition, defining wavelengths of perturbations, together with the angular harmonic decomposition relating anisotropies to observed angles in the sky. The CGI versions of both decompositions were given in Paper I [18], giving the relationship between the mode and multipole variables.

The dynamic relations obeyed by these quantities, determining the cosmic background radiation spatial and angular structure, can be obtained from the Boltzmann equation in two ways: via multipole divergence equations or via the integrated Boltzmann equations. In each case the general equation needs to be mode-analyzed to obtain the spatial structure.

In the first case, the almost-Friedmann-Lemaître multipole divergence equations are obtained by systematically linearising the non-linear multipole divergence equations for small temperature anisotropies given in Maartens, Gebbie and Ellis [59]. The mode form of these equations (16-18) [59] can then be obtained by mode analysis, see (19-20) below.

By contrast, the more common procedure is to directly construct the mode form of the integrated Boltzmann equations from the linearised integrated Boltzmann equations by substituting (B.3) into (B.1) and integrating over the energy shell with respect to \(E^2dE\) (For a more detailed relativistic kinetic theory description of these equations see [59] and [11]):

\[
\int_0^\infty E^2dE \left[ E(u^a + e^a) \nabla_a f - \left( \frac{1}{3} \Theta + A_a e^a + \sigma_{ab} e^a e^b \right) E^2 \frac{\partial f}{\partial E} \right] \approx \int_0^\infty E^2dE \mathcal{C} f, \quad (1)
\]

Upon using the CGI definition of directional bolometric brightness [59]:

\[
T(x) [1 + \tau(x, e)] = \left[ \frac{4\pi}{r} \int E^3 f(x^i, E, e^a) dE \right]^{1/4}, \quad (2)
\]

the covariant equivalent of the standard formulae given in the Wilson-Silk approach [81] can be found, where as usual the partial energy derivative is removed by an integration by parts and

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\(^{5}\)which can be parametrized either by a null cone parameter, a projected spatial coordinate, or a projected time coordinate
application of the regularity conditions. This is the approach we develop now, showing later the relation to the approach based on the multipole divergence equations.

2.1. The Mode Equations

The optical depth $\kappa$ is given in terms of the Thompson scattering cross-section $\sigma_T$, the number density of electrons $n_e$, and the fraction of electrons ionized $x_e$:

$$\kappa(t_0, t) = \int_{t_0}^{t} \sigma_T n_e(t') x_e(t') dt' = \int \kappa dt', \quad (3)$$

where $\eta$ is the conformal time ($dt = ad\eta$) in the Friedmann-Lemaître background. Starting with the almost-Friedmann-Lemaître integrated Boltzmann equations (1) for the temperature anisotropies (2) with the collisional term for isotropic scattering in terms of the optical depth, and the expansion replaced by substituting from the radiation energy conservation equation (G.11,15), we obtain the linearised integrated Boltzmann equations:

$$-\dot{\tau} \approx e^a D_a \tau - \frac{4}{3} D_a \tau^a + (D_a \ln T + A_a) e^a + \sigma_{ab} e^a e^b - \dot{\kappa} (e^a v_B^a - \tau). \quad (4)$$

We take a mode expansion (see Paper I [18]) for $\tau$ (the temperature anisotropy), $A_a$ (the acceleration), $D_a \ln T$ (the spatial-tempature perturbation), $\sigma_{ab}$ (the shear), and the gradient of the radiation dipole, $D^a \tau_a$, based on solutions $Q(x)$ of the scalar Helmholtz equation:

$$D^a D_a Q = -\frac{k^2}{a^2} Q \quad (5)$$

in the (background) space sections, where the $Q$'s are covariantly constant scalar functions (i.e. to linear order $Q \approx 0$) corresponding to a wavenumber $k$. The physical wavenumber is defined by $k_{\text{phys}}(t) = k/a(t)$. These functions define tensors $Q_{A\ell}(k, x^i)$ that are PSTF (in the case of scalar perturbations they are given by the PSTF covariant derivatives of the eigenfunctions $Q$) and so allow us to define the mode functions [18]:

$$Q_{A\ell} = \left(-\frac{k}{a}\right)^{-\ell} D_{(A\ell)} Q \quad \text{and} \quad G_{\ell}[Q] = O^{A\ell} Q_{A\ell}, \quad (6)$$

where $O^{A\ell} = e^{(A\ell)}$ is the trace-free part of $e^{A\ell}$. We can expand any given function $f(x, e)$ in terms of these functions, thus for scalar perturbations (see (G.20-G.23), the mode coefficients of the temperature anisotropy are constructed as follows:

$$\tau(x, e) = \sum_{\ell=1}^{\infty} \sum_k \tau_{\ell}(t, k) G_{\ell}[Q]. \quad (7)$$

Note that $\tau_0$ is identically zero, because (2) defines the temperature $T$ gauge-invariantly as the all-sky average in the real (lumpy) universe (it is not defined in terms of a background model). We can then write:

$$D^a \tau_a = \sum_k \frac{k}{a} \tau_1(k, t) Q, \quad D_a \ln T + A_a = \sum_k [\frac{k}{a} \delta T(k, t) + A(k, t)] Q_a, \quad (8)$$

and

$$\sigma_{ab} = \sum_k \sigma(k, t) Q_{ab}. \quad (9)$$

The mode coefficient of the velocity of the baryons $v_B$ relative to the reference frame $u^a$ is given by

$$v_B^a = \sum_k v_B(k, t) Q_a. \quad (10)$$
The equations are linearised at $O(u^2)$, $O(\epsilon v)$ and $O(\epsilon^2)$ [59], so we can use, for example, $u_B^\alpha = u^\alpha + \dot{v}_B^\alpha$ to give the baryon relative velocity.

The equations here are gauge-invariant (given relative to a unique physically-based choice of the 4-velocity vector $u^\alpha$) and valid for any choice of mode functions, but the detailed result of their translation back into the spacetime representation $\tau(x, \epsilon)$ via (7) will depend on the harmonic functions $Q(x^i)$ chosen.\(^6\)

We substitute these expansions into the linearised integrated Boltzmann equations and then use the recursion relation [81, 35, 39, 18] for mode functions $G_L[Q](\ell^a, x^i)$ with wave-number $k$:

$$e^\alpha \mathcal{D}_\alpha[G_L[Q]] = k \left( \frac{\ell^2}{(2\ell + 1)(2\ell - 1)} \left( 1 - \frac{K}{2\ell^2} \right) \right) G_{L-1}[Q] - G_{L+1}[Q] ,$$

(10)

where $K$ is the curvature constant of the background space sections.

With the mode decompositions of each term in (4) for each wave number\(^7\), on using the recursion relation (10) and separating out the different harmonic components, the almost-Friedmann-Lemaître mode equations are found from (3)\(^8\):

$$-\dot{\tau}_L \approx k \left( \frac{(\ell + 1)^2}{2(2\ell + 1)^2} \left( 1 - \frac{K}{2\ell^2} \right) \right) \tau_{\ell+1} - \tau_{\ell-1} + \kappa \tau_L , \quad \ell \geq 3 ,$$

(11)

$$-\dot{\tau}_2 \approx k \left( \frac{9}{35} \left( 1 - \frac{8K}{K^2} \right) \right) \tau_3 - \tau_1 + [\sigma] + k\tau_2 ,$$

(12)

$$-\dot{\tau}_1 \approx k \left( \frac{4}{15} \left( 1 - \frac{3K}{K^2} \right) \right) \tau_2 + \frac{1}{3} \dot{\Theta} + A - \kappa(v_B - \tau_1) .$$

(13)

The above equations demonstrate the up and down cascading effect whereby lower order terms generate anisotropies in the higher order terms, and vice versa, in a wavelength-dependent way; curvature affects the down-cascade but not the up one. These equations can be compared to the equations of Hu & Sugiyama, in particular (eqn. 6, p. 2601) [40]. They are identical if we use the Newtonian frame (discussed in the following sections), and so have the same physical content, but are more general since they are valid with respect to a general frame $u^\alpha$.

### 2.2. From Multipole Equations to Mode Equations

The relationship between the angular harmonic and mode expansions are given in Paper I [18]. We start by writing the CGI harmonic coefficients $\tau_{A_L}$ in terms of the mode functions (6):

$$\tau_{A_L} = \sum_k \tau_k(k, l)Q_{A_L} \approx \sum_k \tau_k(l, k)(-\frac{k}{a})^{-\ell}\mathcal{D}(A_L)Q .$$

(14)

Then the angular harmonic expansion for $f$ becomes the mode expansion (7). On taking the multipole integrals of $f$ as in Paper I [18], they too are then mode-expanded by (14); so the linearised divergence relations for these multipoles given in [59] become mode equations, equivalent to the almost-Friedmann-Lemaître mode equations (11-13) derived above.

In detail: The almost-Friedmann-Lemaître multipole divergence equations are [59]:

$$-\left( \frac{\dot{T}}{T} + \frac{1}{3}\dot{\Theta} \right) \approx \frac{1}{3}\mathcal{D}^c\tau_c ,$$

(15)

\(^6\)In effect there are two major choices, namely plane wave solutions and spherical solutions; the former occur naturally in galaxy formation studies and the latter in observational analysis, so the relation between the two (see Paper I [18]) is a central feature of analyzing null-cone observations.

\(^7\)There should be a summation over wave numbers in the following equations. However we follow the established custom in omitting this summation and any explicit reference to the assumed wave number $k$.

\(^8\)Note that these equations are valid for any choice of $Q$, including both spherical and plane wave harmonic functions.
\begin{align}
(-\tau_a) & \simeq D_a \ln T + A_a + \frac{2}{\ell} D^\ell \tau_{ac} - \sigma_T n_e (v^B_a - \tau_a), \\
(-\dot{\tau}_{ab}) & \simeq \sigma_{ab} + D_{(a} \tau_{b)} + \frac{3}{2} D^\ell \tau_{abc} + \sigma_T n_e \tau_{ab}, \\
(-\dot{\tau}_{A_k}) & \simeq D_{(a_k} \tau_{A_{k-1})} + \frac{(\ell + 1)}{2(\ell + 3)} D^\ell \tau_{A_{k-1}c} + \sigma_T n_e \tau_{A_k}. 
\end{align}

Now the following identities are used (dropping the k-summation):
\begin{align}
O^{A_i} D^\ell \tau_{A_{k-1}c} & \approx \tau_{\ell + 1} \left( \frac{\ell + 1}{2(\ell + 1)} \right) \left( + \frac{k^3}{a^2} \right) \left[ 1 - \frac{K}{k^2} (\ell + 2) \right] O^{A_i} Q_{\ell}, \\
O^{A_i} D_{(a_k} \tau_{A_{k-1})} & \approx \tau_{\ell - 1} \left( -\frac{k^3}{a^2} \right) O^{A_i} Q_{\ell},
\end{align}
where the first relation arises from the use of the identity \(^9\):
\[
D^\ell D_{(a_k} Q = \frac{\ell + 1}{2(\ell + 1)} \left( -\frac{k^3}{a^2} \right) \left[ 1 - \frac{K}{k^2} (\ell + 2) \right] D_{(a_k} Q.
\]
These are substituted directly into the multipole equations after taking a mode expansion of those equations and then dropping the k-space summation, leading again to the equations (13). The point to note is that while one does not explicitly need the multipole equations in order to find the almost-Friedmann-Lemaître mode equations (which can be derived from the linearised integrated Boltzmann equations as shown above), in order to examine non-linear effects one can obtain the necessary equations by proceeding as here from the non-linear multipole divergence equations, to obtain higher approximations of the mode equations and the mode-mode couplings.

### 2.3. The Einstein Equations

The key quantities which link the radiation evolution through to the matter in the spacetime geometry are the shear \(\sigma_{ab} = u_{(ab)}\), the acceleration \(A_a = u_{ab} u^b\) and the expansion \(\Theta\). These couple the multipole divergence equations to the Einstein field equations (which are given in Appendix G, see (G.3)-(G.18)).

The shear and acceleration, are related to the electric part of the Weyl tensor \(E_{(ab)}\), the anisotropic pressure \(\pi_{(ab)}\) and matter spatial gradients (see (G.3)-(G.18)) while the CGI spatial gradient of the expansion is linked to the divergence of the shear, heat flux vector \(q_a\) and the vorticity vector \(\omega_a\):
\begin{align}
-\frac{1}{2} (\rho + p) \sigma_{ab} & \approx (\dot{E}_{ab} + \frac{1}{2} \dot{\pi}_{ab}) + 3H (E_{ab} + \frac{1}{2} \pi_{ab}) - H \pi_{ab} - \left\{ \frac{1}{2} D_{(a} q_{b)} \right\}, \\
(\rho + p) A_a & \approx -D_a p - D^b \pi_{ab} - \left\{ q_a + 4H q_a \right\}, \\
\frac{1}{2} D_a \Theta & \approx \frac{1}{3} (D^b \sigma_{ab}) - \left\{ \frac{1}{2} q_a + \text{curl} \omega_a \right\}.
\end{align}
These equations are valid for general almost-Friedmann-Lemaître perturbations. In the restricted case of scalar perturbations, we set the vorticity to zero\(^10\) and non-zero quantities can be written in terms of potentials \([71]\). In particular
\begin{align}
E_{ab} & \approx D_{(a} D_b) \Phi_E = \frac{2}{3} D_{(a} D_b) (\Phi_A - \Phi_H), \\
\pi_{ab} & \approx D_{(a} D_b) \Phi_\pi = -D_{(a} D_b) (\Phi_H + \Phi_A),
\end{align}
\(^9\)This has also been derived by Challinor and Lasenby \([11]\) and is found from the PSTF recursion relations and the generalized Helmholtz equation (which is in turn found from the constant curvature Ricci identity \([18]\). \(^{10}\)We can obtain scalar equations even when the vorticity is not zero, by taking the total divergence of these equations; we will not pursue that case here.
where the potentials $\Phi_A$ and $\Phi_H$ are analogous to the GI potentials used by Bardeen [8]. The following useful combinations can be found:

$$E_{ab} - \frac{1}{2} \pi_{ab} \approx D_{(a} D_{b)} \Phi_A, \quad \text{and} \quad E_{ab} + \frac{1}{2} \pi_{ab} \approx -D_{(a} D_{b)} \Phi_H. \quad (26)$$

Using the Einstein field equations, the total flux, $q_a$, can also be expressed covariantly in terms of these potentials:

$$H q_a \approx D^b D_{(a} D_{b)} \Phi_H - \frac{1}{3} D_a \rho, \quad (27)$$

$$H D_{(a} q_{b)} \approx D_{(a} D_{b)} \left[ \frac{1}{3} (D^2 \Phi_H + (\rho - 3H^2) \Phi_H) - \frac{1}{4} p \right]. \quad (28)$$

This then allows us to write the shear and acceleration in terms of the scalar potentials and perturbation variables:

$$\frac{1}{2} (\rho + p) \sigma_{ab} \approx (D_{(a} D_{b)} \Phi_H) + 3 H D_{(a} D_{b)} \Phi_H - H D_{(a} D_{b)} (\Phi_H + \Phi_A) + \left\{ \frac{1}{2} D_{(a} q_{b)} \right\}, \quad (29)$$

$$(\rho + p) A_a \approx -D_a \rho - D^b D_{(a} D_{b)} (\Phi_H + \Phi_A) - \{ \dot{q}_a + 4 H q_a \}. \quad (30)$$

### 2.4. Frame Transformations and Gauge Fixing

There is freedom associated with the choice of reference velocity $u^a$, which we call a frame choice. This is to be distinguished from the choice of coordinates in the realistic universe model, which can be done independently of the choice of $u^a$. It is equivalent to the choice of time-like world-lines mapped into each other by the perturbation gauge chosen (i.e. the mapping between the background model and the realistic lumpy universe model, see for example Bruni and Ellis [25]), but is independent of the choice of time surfaces in that mapping. Given a particular covariantly defined choice for this velocity, the frame choice is physically specified and the equations are covariantly determined and gauge invariant under the remaining gauge freedom.

In simple situations this choice will be unique, however in more complex situations several choices of this velocity are possible, each leading to a somewhat different CGI description.

When we restrict ourselves to a particular frame in order to simplify calculations, we can straightforwardly make the appropriate simplifications in the general equations to see what the implications are (for example setting $q^a = 0$ for the energy frame, the quantities in the braces in equations (29,30) above vanish). However it is also useful to explicitly transform from one frame to another and examine the resulting effect on dynamic and kinematic quantities.

Under a frame transformation $\tilde{u}^a \approx u^a + v^a$, $|v^a| \ll 1$ (restricting our analysis to non-relativistic relative velocities), the following relations [59] hold:

$$\sigma_{ab} \approx \sigma_{ab} + D_{(a} v_{b)}, \quad (31)$$

$$A_a \approx A_a + \dot{v}_a + H v_a, \quad (32)$$

$$\Theta \approx \Theta + \text{div} v, \quad (33)$$

$$q_a \approx q_a - (\rho + p) v_a, \quad (34)$$

$$\omega_a \approx \omega_a - \frac{1}{2} \text{curl} v_a. \quad (35)$$

The quantities $\rho$, $p$, $\pi_{ab}$, $E_{ab}$ and $H_{ab}$, remain unchanged to linear order in almost Friedmann-Lemaître universes (e.g. $\pi_{ab} \approx \pi_{ab}$ and $E_{ab} \approx E_{ab}$), and the temperature anisotropies ($T_{Ak}$) for

---

11Gauge fixing requires in addition a specification of the correspondence of time surfaces in the realistic and background models (effectively specified by determining the choice of surfaces of constant time in the realistic universe model) and of points in initial space-like surfaces.

12It is important to recall that gauge invariance is only guaranteed if the choice of velocity $u_a$ coincides exactly with its value in the background spacetime. This is not difficult in practice, as appropriate physically defined frames $\tilde{u}_a$ will necessarily obey this condition because of the Robertson-Walker symmetries.
\( \ell > 1 \) are similarly invariant for the small velocity transformations. The baryon and radiation (dipole) relative velocities change according to:

\[
\tilde{v}_B^a \approx v_B^a - v_a, \quad (36)
\]

\[
\tilde{\tau}_a \approx \tau_a - v_a. \quad (37)
\]

Also, the projection tensor \( h_{ab} \) changes if we boost to the frame \( \tilde{u}_a \) giving \( \tilde{h}_{ab} \), hence any spatial gradients need to be modified and \( \tilde{D}_a \) will be the totally projected derivative in that frame. The consequence is that the perturbation variables change accordingly: Thus for any species \( I \), \[59\]

\[
\tilde{D}_a \ln \rho_I \approx D_a \ln \rho_I - 3Hv_a (\rho^I + p^I)/\rho^I. \quad (38)
\]

For example \( I = R \) and \( I = B \) give the equations for radiation and baryons respectively, implying:

\[
\tilde{D}_a \ln T \approx D_a \ln T - v_a H, \quad \tilde{D}_a \ln \rho_M \approx D_a \ln \rho_M - 3Hv_a. \quad (39)
\]

These equations allow us to determine the required transformation to obtain desired properties of a particular choice \( \tilde{u}^a \). The almost-Friedmann-Lemaître multipole divergence equations (16-15) are valid in any frame; in particular, if a frame transformation is performed as above, they can be given in terms of the resulting variables in the new frame, \( \tilde{u}_a \), with whatever restrictions result.

While various choices of \( \tilde{u}^a \) are available in a multi-fluid description of the early universe [59], there are three particularly useful choices.

1. The energy frame: \( \tilde{q}_a = 0 \) is preferred when dealing with two coupled particle species, as in the two fluid scenario [13]. This is useful as the Einstein field equations are simplified to a form which takes on a similar structure to the matter dominated equations, and even in the strong interaction case may be expected to be unaffected by collisions because of energy-momentum conservation (this choice is dealt with in more detail below in the context of scalar perturbations).

2. The zero acceleration frame (or CDM frame): \( \tilde{u}_a = u_C^a \approx u_a + v_C^a \). From the CDM velocity equation [11, 59] and (32) we then find: \( \dot{v}_a^C + Hv_a^C + A_a \approx 0 \Rightarrow A_a \approx 0 \) [56, 11, 59]. This choice is particularly useful in multi-species situations, as this frame will be geodesic right through tight-coupling, slow decoupling and into the free-streaming era.

3. The Newtonian frame: \( \tilde{u}_a = n_a \) in which the vorticity and shear of the reference frame vanishes: \( \tilde{\sigma}_{ab} \approx D_{(a}n_{b)} = 0 \), when such a frame can be found. This frame is only consistent in restricted cases [31], but is particularly useful in making comparisons with much of the analytic literature [39, 40, 35] and in making connections with the local physics in terms of Newtonian analogues in Eulerian coordinates. For example, the matter shear can be then thought of in terms of distortion due to the relative velocities (31): \( \sigma_{ab} \approx -D_{(a}v_{N}^{b)}. \)

4. The constant expansion frame: \( \tilde{D}_a \dot{\Theta} = 0 \). This choice is sometimes useful when discussing perturbations on small scales.

These various choices will simplify the equations in significant ways, and enable us to recover many of the standard results. It should be noted however, that the covariant equations we have given above are general and do not require either gauge or coordinate restrictions to be meaningful, and the covariant quantities have in them a natural invariant geometric meaning. We will therefore retain the covariant form and do not restrict ourselves to any particular frame nor gauge choice for most of this paper, however we retain the freedom to make such a choice when useful. If and when we do pick a particular frame, this will be explicitly stated along with the reason for doing so.

2.4.1. The Newtonian Frame Link to the Bardeen Variables
Here we demonstrate the direct link between our variables in the scalar case, and those used in the Newtonian gauge, in terms of the Bardeen variables. From (31) and (32) we find easily that for \( \tilde{u}^a = n^a \) where \( D(\alpha n_b) = 0 \), \( n_a = u_a + v_a^N \) (the consistency of this choice is discussed in [31]):

\[
0 \approx \sigma_{ab} - D(\alpha v_b^N), \tag{40}
\]
\[
\tilde{A}_a \approx A_a + \dot{v}_a^N + H v_a^N, \tag{41}
\]
\[
\hat{D}_a \ln T \approx D_a \ln T - H v_a^N, \tag{42}
\]
\[
\tilde{\tau}_a \approx \tau_a - v_a^N, \tag{43}
\]
\[
\hat{\Theta} \approx \Theta + \text{div} v^N, \tag{44}
\]
\[
\tilde{q}_a \approx q_a + (\rho + p)v_a^N. \tag{45}
\]

The effect of this frame transformation is to modify the \( \ell = 1 \) and 2 multipole divergence equations (15-17):

\[
-\tilde{\dot{\tau}}_a \approx \hat{D}_a \ln T + \hat{A}_a + \frac{2}{3}D^c \tau_{ab} - \sigma_T n_c (v_a^H - \tau_a), \tag{46}
\]
\[
-\tilde{\dot{\tau}}_{ab} \approx D(\alpha \tilde{q}_{ab}) + \frac{2}{3}D^c \tau_{abc} + \sigma_T n_c \tau_{ab} . \tag{47}
\]

The \( \ell > 2 \) equations (18) remain unchanged, however the field equations as well the perturbation equations need to modified, if necessary using the transformation relations (32 -37). For example, (15) becomes

\[
(\tilde{D}_a \ln T)' + H(D_a \ln T + \hat{A}_a) \approx -\frac{4}{3}D_a \hat{\Theta} - \frac{4}{3}D_a (D^c \tau_c), \tag{48}
\]

which can easily be checked to be invariant under the frame transformations \( \tilde{u}_a \approx u_a + v_a \).

From the shear evolution equation (G.6):

\[
D(\alpha \hat{A}_b) \approx E_{ab} + \frac{1}{3} \pi_{ab}, \quad \Rightarrow \quad \tilde{A}_a \approx D_a \Phi_A , \tag{49}
\]

the momentum constraint (G.7) and the propagation equation for the electric part of the Weyl tensor (G.12) one finds respectively that:

\[
\text{3}D(\alpha \tilde{q}_b) \hat{\Theta} \approx -\frac{1}{2}D(\alpha \tilde{q}_b), \quad \frac{1}{3}D(\alpha \tilde{D}_b) \hat{\Theta} \approx D(\alpha D_b) \hat{\Phi}_H - HD(\alpha D_b) \Phi_A . \tag{50}
\]

This gives the scalar monopole equation for the temperature perturbation:

\[
D(\alpha D_b) (\ln T)' \approx -D(\alpha D_b) \hat{\Phi}_H - \frac{4}{3}D(\alpha D_b) (D^c \tau_c) . \tag{51}
\]

It is important to note here is that in the shear-free frame we can interpret the acceleration directly in terms of the \( \Phi_A \) potential, in other words, in terms of its Newtonian analog, while \( \Phi_H \) can be interpreted as a curvature perturbation. In terms of the potentials used by [39, 16] one can identify \( \Psi = \Phi_H \) and \( \Phi = \Phi_A \).

The above formulation is useful in linking the covariant work to the usual GI treatments. So for example we take the mode expansion of the potentials, one finds on dropping the k-index on the right,

\[
A_{\alpha} \approx (\Phi_A|_{\alpha} + V'_S|_{\alpha} + HV_S|_{\alpha}) = (V'_S(0) + \frac{a'}{a}V_S(0) - k\Phi_A)N_{\alpha}, \tag{52}
\]

\[
\sigma_{\alpha\beta} \approx a(D(\alpha D_b) V_S) = -akV_S(0)N_{\alpha\beta}, \tag{53}
\]
where the prime (′) denotes the time-derivative with respect to the conformal time, $\Phi$ are the eigenfunctions of $Y|_{\alpha} = -k^{2}Y$ and following Kodama and Sasaki [50, 8], the bar (\(\bar{\alpha}\)) denotes spatial derivatives with respect to surfaces of constant curvature in the background. Furthermore, one can identify $V_{S}$ as a relative velocity.

### 2.4.2. The Energy Frame

In order to be clear on the consequences and subtleties involved in fixing the frame, here we give the source terms in the energy frame ($\bar{u}^{a} = u_{a}^{E} \Rightarrow \bar{q}_{a} = 0$) for scalar perturbations. The important point is that this is a physical frame, uniquely defined by the local physics. The equations (29) and (30) then take on the form:

\[
(\rho + p)\sigma_{ab} \approx 2(D(aD_{b})\Phi_{H}) + 4HD(aD_{b})\Phi_{H} - 2HD(aD_{b})\Phi_{A} + D(a\nu^{E}_{b}),
\]

\[
(\rho + p)\hat{A}_{a} \approx -D_{a}p - D^{0}D(aD_{b})(\Phi_{H} + \Phi_{A}) + \dot{\nu}^{E}_{a} + Hv^{E}_{a}.
\]

Many CGI treatments use this frame [13], and have the advantage that the equations take on a form which is similar to those for the matter dominated case, but can still be used near to matter-radiation equality.

### 2.4.3. Matter Domination

During matter domination (we have in mind the CDM dominated case) there is a unique physically relevant frame defined by the matter 4-velocity, hence without restricting the almost-Friedmann-Lemaître universe further there is natural frame, $u^{a}$, in which the variables will be gauge invariant and the above relations hold.

In this frame, equations (29) and (30) become

\[
a^{3}\rho_{a}\sigma_{ab} \approx -2(a^{3}D(aD_{b})\Phi_{H})^{\dot{\dot{}}} + A_{a} \approx 0.
\]

Here $\rho \approx \rho_{M}$ is now the density of the matter content only. The key point is that to retain a consistent linearisation scheme as well as retaining gauge invariance, we now have three smallness parameters: $\nu$ (non-relativistic relative velocities), $\eta$ (radiation-baryon ratio is at least $10^{-5}$), $\epsilon$ (the universe is almost Friedmann-Lemaître when $\epsilon$ is at least $10^{-5}$) [57, 59, 72]. It follows that $\rho_{0}$ (the radiation energy density) is now $O(\eta)$ and we can drop all terms at least $O(\eta^{2})$, $O(\epsilon^{2})$ and $O(\eta^{2}$) such as, for example, $\rho \sigma_{ab} \approx 0$ or $\frac{4}{3}\rho_{0}\tau_{ab} \approx \pi_{ab} \approx 0$. This is how the anisotropic pressure is eliminated to the order of the calculation in this scheme.

The link to the matter distribution in the spacetime comes through the mode coefficients $\sigma(k, t)$ (of the shear), $A(k, t)$ (of the acceleration), and $\delta T(k, t)$ (of the temperature perturbation). A mode analysis leads to a particular solution of the linearised Einstein field equations due to scalar modes as in (G.20 - G.23):

\[
E_{ab} \approx \Phi Q_{ab} = \frac{k^{2}}{a^{2}}\Phi_{H}(t, k)Q_{ab},
\]

\[
\sigma_{ab} \approx -\frac{2}{3}(H_{0}^{2}\Omega_{0})^{-1}k^{2}(a\Phi_{H}(k, t))Q_{ab},
\]

\[
D_{0} \ln \rho_{a} \approx \frac{2}{3}(H_{0}^{2}\Omega_{0})^{-1}\Phi_{H}(k, t) [k^{2} - 3K] Q_{a}.
\]

For a matter dominated open model ($K \neq 0$) where $a_{0} = +1$ we have $H_{0}^{2} \approx K_{0}/(\Omega_{0} - 1)$. If we add the adiabatic assumption (see Appendix G.4) we find

\[
D_{a} \ln T \approx \frac{4}{3}D_{a} \ln \rho_{M},
\]

where we have used that $\rho_{a} \approx 3H_{0}^{2}\Omega_{0}a^{-3}$ in the background. One can then put the mode coefficients, in the matter dominated scalar adiabatic almost-Friedmann-Lemaître models, into
the form:

\[
\delta T(k, t) \approx \frac{1}{3}(H_0^2 \Omega_0)^{-1} (a \Phi_H) \left[ \frac{2}{3} (k^2 - 3 K) \right],
\]

(61)

\[
A(k, t) \approx 0 \approx v_a(k, t),
\]

(62)

\[
\sigma(k, t) \approx -\frac{2}{3}(H_0^2 \Omega_0)^{-1} (a \Phi_H) \left[ k^2 \right].
\]

(63)

The first expression gives the direct effect of the gravitational potential on the cosmic background radiation anisotropies (Sachs-Wolfe effect), and the third the effect of the time variation of the potential on these anisotropies. These are investigated in detail in later sections. The matter dominated Einstein field equations are at least $O(\epsilon^2)$ and fix the form of the shear, the acceleration and the temperature perturbations $D_a \ln T$ as they enter the integrated Boltzmann equations (which is how the geometry enters into these equations). The hierarchy itself is $O(\epsilon)$ and although the radiation variables do not enter the almost-Friedmann-Lemaître (matter) Einstein field equations, they remain non-zero, and therefore there are still temperature anisotropies, $\tau_A$. This is an important but subtle point – matter domination implies the radiation moves as a test field over the geometry.

2.5. Linearisations, Approximations and Scales

In this section we discuss the various linearisation and approximations (already mentioned in the last section), that will be used in this paper.

2.5.1. Almost-Friedmann-Lemaître Linearisation

Here we drop all terms that are at least $O(\epsilon^2)$. The implication of this is that one can only consider small velocity boosts, large ones would break the linearity about the Friedmann-Lemaître background – hence we include $v^2 = |v^a v_a| \ll 1$ as an almost-Friedmann-Lemaître limit dropping the additional terms that are at least $O(\epsilon^2)$.

The important subtlety here is that $\epsilon$ is in fact the temperature anisotropy smallness parameter as related to the temperature moment mean squares $|\tau_A| \propto \epsilon$. The almost-Friedmann-Lemaître limits on the geometry, $\epsilon$, (which define $\sigma_{ab}$, $A_a$ and $D_a \Theta$ (for example) as $O(\epsilon)$ in appropriate dimensionless units [73, 72] are related to $\epsilon$ via the almost-Ehlers-Geren-Sachs theorem. In other words, limits on the temperature anisotropies, $\epsilon$, put bounds on the size of the smallness parameter $\epsilon$, given that a weak Copernican principle holds. Furthermore, the limits on $\epsilon$ in turn place consistency limits on the size of the $v/c$ boosts that are applicable (here in units of $c = 1$). Thus at least almost-Friedmann-Lemaître means keeping terms that are at most:

Almost-Friedmann-Lemaître $\approx O(\epsilon, v)$. 

(64)

2.5.2. Matter Dominated Linearisation

This is based on the radiation-baryon ratio, $\eta \propto \frac{\rho_R}{\rho_M}$. We keep every $O(\eta)$ but in the almost-Friedmann-Lemaître case of matter domination we then drop everything that is at least $O(\eta \epsilon, \eta \epsilon^2, \eta \epsilon^2, \epsilon^2)$. We then have that matter dominated almost-Friedmann-Lemaître means keeping terms that are at most:

Matter dominated almost-Friedmann-Lemaître $\approx O(\epsilon, v, \eta)$. 

(65)
2.5.3. Expansion in Thompson Scattering Time

We will introduce a perturbative scheme in the Thompson scattering time, \( t_c = (\sigma_T n_e)^{-1} \), and will consider terms up to \( O(t_c^3) \) during the tight-coupling calculation – such an expansion will be used to generate equations near to tight-coupling, the limiting case being when \( t_c = 0 \). Additionally an equivalent scheme can be constructed in terms of the differential optical depth \( \kappa' \). This scheme is useful in the slow-decoupling era, i.e. in expansions where \( \kappa' \) and \((\kappa')^2\) are sufficiently small to be ignored when compared to terms of order \( \kappa \). This approximation allows one to additionally consider the case when \( \kappa' e^{-\kappa} \ll e^{-\kappa} \).

2.5.4. Small and Large Scales

We will find it convenient to introduce the notion of small and large scales. We will do this in two heuristically equivalent ways. The first scheme is based on the parameter \( \epsilon_H \), where the Hubble expansion is of order \( \epsilon_H \), and is used when considering situations outside the Hubble flow; thus in the almost-Friedmann-Lemaître small-scale case one would ignore all terms at least \( O(\epsilon_H^2, \epsilon^2, \nu^2, \epsilon \nu, \epsilon \nu H) \). This scheme is useful since it can be used without a mode expansion. It is ideal for making qualitative statements without the details which arise when introducing mode functions; specifically avoiding the complexity of mode-mode coupling in the small scale non-linear situation. By small-scale almost-Friedmann-Lemaître we have in mind keeping only terms that are at most:

\[
\text{Small-scale almost-Friedmann-Lemaître} \approx O(\epsilon_H, \epsilon, \nu) . \tag{66}
\]

A second and more precise scheme is based on the Hubble scale \( \lambda_H = a_E/k_H \) defined near the time of emission (E), allowing one to use \( k/k_H > 1 \) and \( k/k_H < 1 \) as characterizing large and small scales respectively.

3. COVARIANT INTEGRAL SOLUTIONS

This section has three aims: (i) Reproducing the integral solution of the free-streaming mode equations and modifying them in order to take into account Thomson scattering, using the CGI variables [40, 69]. We carry out a time-like integration, instead of a null-cone integration corresponding to the original Sachs-Wolfe paper [68], restricting ourselves to scalar perturbations with adiabatic modes only and assuming for the most part a \( K = 0 \) almost-Friedmann-Lemaître background universe. (ii) Showing in the CGI formulation how fluctuations at last scattering time result in measurable cosmic background radiation anisotropies. (iii) Demonstrating how the solution can be related to standard formalisms by choosing specific frames; in particular we consider the Newtonian frame based on a shear free congruence.

The basic equation we are concerned with in this section is the integrated Boltzmann equation (4). In covariant form it is given by

\[
\dot{\tau}(x,e) + e^a D_a \tau(x,e) + B(x,e) \approx C[x,e] , \tag{67}
\]

where the gravitational source term, \( B \), and Thomson source term, \( C \), for damping by Thompson scattering are respectively given by:

\[
B = -\frac{1}{3} D^a \tau_a + (D_a \ln T + A_a) e^a + \sigma_{ab} e^a e^b ; \quad C[x,e] \approx k(e^a v_a B - \tau) , \tag{68}
\]
and in the almost-Friedmann-Lemaître situation in mind\textsuperscript{13}, the Einstein field equations give:

\[ 3\rho_m^{-1}(\text{div } E)_a \approx D_a \ln \rho_m, \quad \sigma_{ab} \approx -2[(\rho_m^{-1} E_{ab}) - \text{curl } (\rho_m^{-1} H_{ab})]. \] (69)

The mode expanded form of this equation for the flat case \((K = 0)\) can be written in the compact form as follows:

\[ \tau'_\ell + k \left[ \frac{(\ell + 1)^2}{(2\ell + 1)(2\ell + 3)} \tau_{\ell+1} - \tau_{\ell-1} \right] + \kappa' \tau_\ell \approx S_B, \] (70)

where

\[ S_B = -[aB_0 \delta_{\ell 0} + (aB_1 + \kappa' v_B) \delta_{\ell 1} + aB_2 \delta_{\ell 2}]. \] (71)

This combines (11-13) in a single equation for \(\tau_\ell(\eta, k)\) (see \([18, 59]\)), where \(\tau = \sum \tau G_\ell[Q]\) is written in terms of conformal time \(\eta (dt = a d\eta)\), rather than proper time \(t\). It is valid for all \(\ell \geq 0\), with \(\tau_0 = 0\) a solution as required, consistent with the definitions we introduced above \((B_0\) cancels the dipole term on the right in this case).

In what follows, we will deal with the integral solutions to (70) given the source terms (71). The paradigm is to match an (almost-Friedmann-Lemaître) era of free-streaming to one of tight-coupling. We will construct the homogeneous solution (without gravity or scattering) first, then include the gravitational effects to construct the free-streaming solution (i.e. after decoupling to the present day) and finally include Thompson scattering to find the integral solution including scattering (which can be used during slow decoupling or to include effects of reionization). Diffusion damping is included in this full solution with Thompson scattering, which in general has to be solved numerically, however it is helpful to introduce various analytical approximations for the different stages described by the solution; this will be done later, where the visibility function approximation is used and the damping scale derived. Additional effects, such as the anisotropic correction and polarization correction, have to be dealt with separately.

### 3.1. Integral Solutions (Flat Almost-Friedmann-Lemaître Case)

Here we wish to find the general solution to (70) without collision terms, i.e. with \(\kappa' = 0\), integrating along time-like curves using conformal variables. In order to do this we first find the solution to the homogeneous version of the above equation (i.e. for no gravity and no scattering), second, an integral solution of the inhomogeneous equation with gravity taken into account from the homogeneous solution, and third, the general solution for free-streaming. In the next sections we consider the effect of Thompson scattering \((\kappa \neq 0)\), and the transition from tight coupling to free-streaming.

The approach here is similar to the Seljak-Zaldariagga treatment \([69, 83]\), however, they have taken the Sachs-Wolfe like formulation of the integrated Boltzmann equations, which is an integration down the null cone, and integrated out the angular dependence over Legendre polynomials (angular averaging) in order to construct the mode coefficients; then the conformal radial distance, \(\chi\), is written in terms of the conformal time \(\eta\), leading to an integral solution dependent only on the conformal time. Thus, formally they have carried out a null-integration. By contrast, what is carried out here is in effect an integration of the integrated Boltzmann equations down the matter world-lines, thus this is a time-like integration, onto an initial surface (‘last scattering’). The corresponding initial data near our past world line on that surface can

\textsuperscript{13}One can compare this to the formulation of Durrer \([16, 17]\) (eqns 3.5 and 18). To see how this is linked to the Bardeen potentials \(\Psi\) and \(\Phi\), we can use \(E_{ab} = \frac{1}{2} D_a D_b (\Psi - \Phi)\) \([17]\). Notice that Durrer’s integral solutions take on the Sachs-Wolfe form and can be compared with the treatments in \([13, 61]\), while ours follow the form of \([81, 69]\).
then be related to data on the intersection of the past null-cone with the surface by means of a suitable homogeneity assumption. The time-like nature of the integration is often not made particularly clear in the literature, but solutions of the generic multipole divergence equations (from which the almost-Friedmann-Lemaître mode hierarchy of temperature anisotropies are derived) are usually based on time-like integrations in the relativistic kinetic theory [59].

There is no effective difference between the time-like and null-integrations at linear order. This is because, at linear order, one needs to only integrate up one null geodesic, and then decompose the resulting temperature into its various moments. This is equivalent to making the multipole decomposition first and integrating these up one time-like world-line. This equivalence does not hold in the exact case, so when trying to include the effects of non-linearity, assuming such an equivalence could lead to misleading results.

3.1.1. Finding the Homogeneous Solution

The $\ell = 0, 1$ and 2 multipole divergence equations and hence mode form of the integrated Boltzmann equations are exceptional, given that $\tau_0 = 0$. The point of the integral solutions is to cast the exceptional equations $\ell = 0, 1, 2$ into a form that allows analytic investigation. We now find the covariant homogeneous solutions.

Consider the homogeneous equation (valid for $\ell \geq 1$):

$$\tau^{(0)'}_\ell + k \left[ \frac{(\ell + 1)^2}{(2\ell + 1)(2\ell + 3)} \tau^{(0)}_{\ell+1} - \tau^{(0)}_{\ell-1} \right] = 0,$$

(72)

for the background $K = 0$ case\textsuperscript{14} without damping. The functions

$$\tau^{(0)}_\ell(k, \eta) = (2\ell + 1)\beta^{-1}_\ell j_\ell(k\eta)$$

(73)

are solutions of (86), if the coefficients $\beta_\ell$ obey the recursion relations [18]:

$$(\ell + 1)\beta_\ell = (2\ell + 1)\beta_{\ell+1}, \beta_\ell(2\ell - 1) = \ell\beta_{\ell-1}.$$  

(74)

This can be shown by multiplying (72) through by $\alpha_\ell = \beta_\ell(2\ell + 1)^{-1}$ and comparing the result with the recursion relation for spherical Bessel functions. If the function $j_\ell(k\eta)$ satisfies the equation (72) for $\ell = 0$ (there are of course no terms with $\ell < 1$) then the rest of the equations ($\ell \geq 1$) will be satisfied because of the recursion relations:

$$-(2\ell + 1)(\alpha_\ell\tau_\ell)' \simeq k[(\ell + 1)(\alpha_{\ell+1}\tau_{\ell+1}) - \ell(\alpha_{\ell-1}\tau_{\ell-1})].$$

(75)

The freedom in $\beta_\ell(k)$ occurs in $\beta_0(k)$ and $\beta_1(k)$. Given that the Bessel function is finite at the origin: $j_\ell(0) = \delta_{0\ell}$, these can be chosen to satisfy $\beta_0 = \beta_1 = +1$, the rest are generated through the recursion relations on $\beta_\ell$ and then determine the solution $\tau^{(0)}_\ell(k, \eta)$. The arbitrarily specifiable initial data is later fixed by introducing an integral solution ((79) below) containing arbitrary functions $C_A(\eta)$ (see (80)) which are determined by the Einstein field equations through $B_I(\eta)$.

The corresponding mode functions are

$$\tau^{(0)}(x, e) = \sum_{\ell=1}^{\infty} \beta^{-1}_\ell(2\ell + 1)j_\ell(k\eta)O^{A_\ell}Q_{A_\ell|\text{FLAT}},$$

(76)

(cf. (7)) and the corresponding multipole coefficients can then be found:

$$\tau_{A_\ell|\text{FLAT}} \simeq \beta^{-1}_\ell(2\ell + 1)j_\ell(k\eta)Q_{A_\ell|\text{FLAT}}.$$  

(77)

\textsuperscript{14}Or for the open or closed cases by following [39] [35].
Notice that this differs by a factor of $i^{-\ell}$ from Wilson [81] since we are using plain mode functions instead of plane waves, although these can be easily related. Note there is no explicit mode mixing in this approximation, but such mixing is implicitly determined by the recursion relations (74). This shows that we should be careful with any truncation procedure we propose (see Appendix E). This procedure can be easily extended to the open case using the recursion relations for the open mode functions (see Appendix K).

3.1.2. Construction of the Integral Solution

Given that we have the solution $\tau^{(0)}_{\ell}(k, \eta)$ to the homogeneous equation of the form (86), now consider the equation with given gravitational source terms, but still without damping:

$$\tau'_{\ell} + k \left[ \frac{(\ell + 1)^2}{(2\ell + 1)(2\ell + 3)} \tau_{\ell+1} - \tau_{\ell-1} \right] = - \left[ aB_1 \delta_{\ell1} + aB_2 \delta_{\ell2} \right]. \quad (78)$$

What is important to notice here, is that this equation is valid for $\ell \geq 1$, not $\ell \geq 0$ as in (70); indeed $\tau_0 = 0$. We need to find a particular solution to this equation.

We proceed as follows. Consider the ansatz in terms of $A_\ell$ using $\delta \eta = \eta - \eta'$, along with the Liebnitz rule for differentiation of integrals:

$$\tau^P_{\ell}(\eta) = \int_0^{\eta} d\eta' A_\ell(\eta, \eta') \Rightarrow \frac{\partial \tau^P_{\ell}}{\partial \eta}(\eta) = \int_0^{\eta} d\eta' \frac{\partial}{\partial \eta} A_\ell(\eta, \eta') + A_\ell(\eta, \eta). \quad (79)$$

Now we define the kernel, $A_\ell$, as in [81, 83]:

$$A_\ell(\eta, \eta') = C_0(\eta) \tau^{(0)}_{\ell}(\delta \eta) + C_1(\eta) \frac{\partial}{\partial \eta} \tau^{(0)}_{\ell}(\delta \eta) + C_2(\eta) \frac{\partial^2}{\partial \eta^2} \tau^{(0)}_{\ell}(\delta \eta). \quad (80)$$

It can then be shown from (79) and (80) that:

$$\tau^P_{\ell} + k \left[ \frac{(\ell + 1)^2}{(2\ell + 1)(2\ell + 3)} \tau^P_{\ell+1} - \tau^P_{\ell-1} \right] = S_C, \quad (81)$$

where

$$S_C = C_0(\eta) \tau^{(0)}_{\ell}(0) + C_1(\eta) \tau^{(0)}_{\ell}(0) + C_2(\eta) \tau^{(0)}_{\ell}(0), \quad (82)$$

given $\tau^P_{\ell}(\eta)$ as in (79-80). Thus (78) is satisfied by our ansatz provided the coefficients $C_0(\eta), C_1(\eta),$ and $C_2(\eta)$ in the integral solution are found in terms of the CGI variables $B_0(\eta), B_1(\eta),$ and $B_2(\eta)$ determined by the Einstein field equations. This will be considered next, when we put the parts of the solution together to obtain (90).

3.1.3. Inclusion of Damping

Here we extend the previous solution (78), where the relationship between the coefficients in the integrated Boltzmann equations (D.1) can be read off from (82), including damping through $\kappa'$. We notice that if $\tau^{(0)}_{\ell}$ is a solution to (86), then

$$\tau^*_\ell(\eta) = e^{-\kappa(\eta)} \tau^{(0)}_{\ell}(\eta) \quad (83)$$

will be a solution to

$$\tau^*_{\ell} + k \left[ \frac{(\ell + 1)^2}{(2\ell + 1)(2\ell + 3)} \tau^*_{\ell+1} - \tau^*_{\ell-1} \right] + \kappa' \tau^*_{\ell} \simeq 0. \quad (84)$$
Similarly we find that for the integral solution (79), of (78), the expressions

\[
\tau_\ell^P(\eta) = \int_0^\eta d\eta' e^{-\kappa(\eta')} A_\ell(\eta, \eta') \quad \text{or} \quad \tau_\ell^P(\eta) = e^{-\kappa(\eta)} \int_0^\eta d\eta' A_\ell(\eta, \eta') ,
\]

will be particular solutions to (70), given the correct choice of \(C_0, C_1\) and \(C_2\). Hence we can modify the solutions of the previous section to include Thompson scattering by simply including the damping terms as in these equations. Hence the extended equations include the special case of free streaming, when for some interval of time \(\kappa = 0\); thus they can extend all the way from late tight coupling to the present day, if we include a suitably time-dependent scattering coefficient \(\kappa\).

In more detail: we have that \(\tau_\ell^{(0)}(\eta_r) = 0\). This is simply due to the fact that during tight-coupling there are no higher moments, just the monopole. Here we assume that free-streaming begins after some \(\eta_r\). The slow decoupling solution will modify this assumption. As before we have (now using \(\delta \eta^* = \eta - \eta_r\))

\[
\tau_\ell^{(0)}(\eta) = (2\ell + 1)\beta^{-1}_\ell j_\ell(k\delta \eta^*) \Rightarrow \tau_\ell^P(\eta) \simeq \int_0^{\delta \eta^*} d\eta' A_\ell(\eta, \eta') ,
\]

where the initial data is now given by \(C_i(\eta' - \eta_r)\); notice that we do not introduce an additional \(\eta_r\) as we will be using the solution of \(\tau^{(0)}\) already including the initial conditions \(^\dagger\). Once again we have the integral solution \(\tau_\ell^P(\eta)\) integrated from 0 to \(\delta \eta^*\) such that the anisotropies are now determined by \(\tau_\ell^P(\eta_0)\). Evaluating the integral from 0 to \(\Delta \eta_r\) (for \(\Delta \eta = \eta_0 - \eta_r\)), we find

\[
\tau_\ell(\eta) = \int_0^{\eta_0} d\eta' e^{-\kappa(\eta' - \eta_r)} \left[ C_0(\eta' - \eta_r) \tau_\ell^{(0)}(\Delta \eta) + C_1(\eta' - \eta_r) \tau_\ell^{(0)'}(\Delta \eta) + C_2(\eta' - \eta_r) \tau_\ell^{(0)''}(\Delta \eta) \right] ,
\]

where the damping term now enters explicitly.

3.1.4. The Complete Solution

We can now construct the general solution to (70) with \(\kappa' \neq 0\) by putting the previous results together. The homogeneous seed solution \(\tau_\ell^{(0)}(\eta)\) is given by (73). The particular integral solution \(\tau_\ell^P(\eta)\) is given, in terms of \(\delta \eta = \eta - \eta_r\) by (85). The general solution is then given by

\[
\tau_\ell(\eta) = e^{-\kappa(\eta)} \tau_\ell^{(0)}(\eta) + \tau_\ell^P(\eta) .
\]

Substituting this into the general equation (70) and using the radial eigenfunctions evaluated at zero (in particular \(j_\ell(0) = \delta \ell_0\) along with the recursion relation) gives

\[
C_0\delta \ell_0 + C_1 \beta^{-1}_1 k\delta \ell_1 + C_2 k^2 5\beta^{-1}_2 \left( \frac{2}{15} \delta \ell_2 + \frac{1}{3} \delta \ell_0 \right) = -[(aB_1 + \kappa' v_B)\delta \ell_1 + aB_2\delta \ell_2] ,
\]

relating the functions determining the solution to the time-dependent coefficients in the equation. From (89) the functions in the integral solution are found in terms of the dynamical CGI variables:

\[
-C_0(\eta) \simeq +\frac{5}{2} aB_2 , \quad -C_1(\eta) \simeq +\frac{1}{k} (aB_1 + \kappa' v_B) , \quad -C_2(\eta) \simeq +\frac{1}{k^2} aB_2 .
\]

\(^\dagger\)We could have used \(\tau_\ell^{(0)}(\delta \eta) \rightarrow \tau_\ell^{(0)}(\delta \eta + \eta_r)\) along with the original homogeneous solution unchanged.
For scalar perturbations, the term $C_2$ is effectively the coefficient of the shear; the term $C_1$ is the coefficient of the gradient of the temperature and the acceleration and the coefficient $C_0$ also contributes to the shear (see (G.19)). Note that these are CGI with respect to $u^a$.

These functions are all evaluated at a time $\eta$, which takes all values from $\eta_d$, the time of decoupling, to $\eta_0$, the time of observation. In the null-cone formulation of the problem this input of new information corresponds to the way the null geodesics from the point of emission to the observer keep crossing new matter and hence encounter new information. Because we are integrating on a time-like curve, this information is represented here as varying with time along that curve; and in some simple circumstances, the values at later times are determined fully by the values at earlier times (as happens, for example, in the original Sachs-Wolfe case: $K = 0$, $p = 0$, and only growing scalar modes are considered).

### 3.2. Integration by Parts

In order to deal easily with the initial data it is now useful to write the general solution for $K = 0$ in terms of the present time, $\eta_0$, and the initial time, $\eta_*$, by integrating with respect to conformal time and defining $\Delta \eta_0 = \eta_0 - \eta_*$.\footnote{The relationship between the conformal time $\eta$ and the radial distance $\chi$ is $d\chi = -d\eta$ so $\chi = \eta_0 - \eta$ which follows for the homogeneity and isotropy in the background.} We could fix the conformal time by setting $\eta_0 = 0$ here. In order to recover the results of [10, 83] one would take $\eta_* \to 0$, however, we would like to recover the results as close to [39] and so retain their conventions where possible.

Notice that from $d\tau^{(0)}/dt = 0$, we have $\tau^{(0)}(x(\eta), e(\eta)) = \tau^{(0)}(x(\eta_*), e(\eta_*))$, and $j_0(0) = \delta_{\eta_0}$ (we have chosen the solution to be finite at origin). We choose the initial conditions $\tau_0(\eta_*) = 0$ [81] and $\tau_0(\eta) = 0$ and using the parameter freedom in the homogeneous solution, set $\beta_0(k) = +1$ and $\beta_1(k) = +1$. The homogeneous solution is now fixed as in (86), for $\delta \eta^* = \eta - \eta_*$ and this in turn sets the integral solution to (85) [81]:

$$
\tau_\ell^*(\eta) = \int_0^{\delta \eta^*} e^{-\kappa(\eta')} A_\ell(\eta, \eta') d\eta',
$$

where we still have the freedom of setting the initial data for the integral solution from the $C_1(\eta)$’s which are fixed by the Einstein field equations. Putting this all together we find

$$
\tau_\ell(\eta_0) = \tau_\ell^{(0)}(\eta_*) + \tau_\ell^*(\eta_0) = \tau_\ell^*(\eta_0) = \int_0^{\Delta \eta_0} e^{-\kappa(\eta_0)} A_\ell(\eta_0, \eta') d\eta'.
$$

On changing the integration to from $\eta_*$ to $\eta_0$ in (87), integrating by parts, and using the initial conditions (once again $\tau_\ell^{(0)}(\eta_*) = 0$) we find:

$$
\tau_\ell(\eta_0) \simeq [C_1(\eta_0) - C_1(\eta_*)] e^{-\kappa(\eta_0)} - C_2(\eta_0) e^{-\kappa(\eta_0)} \tau_\ell^{(0)}(\eta_0),
$$

$$
+ \int_{\eta_*}^{\eta_0} d\eta' e^{-\kappa} [C_0(\eta') - C_1(\eta')] + C_2(\eta') \tau_\ell^{(0)}(\eta_0 + \eta_* - \eta'),
$$

$$
+ \int_{\eta_*}^{\eta_0} d\eta' (\kappa' e^{-\kappa'}) [C_1(\eta') - 2C_2(\eta')] \tau_\ell^{(0)}(\eta_0 + \eta_* - \eta'), \ldots
$$

$$
+ \int_{\eta_*}^{\eta_0} d\eta' ((\kappa')^2 e^{-\kappa'}) e^{-\kappa} C_2(\eta') \tau_\ell^{(0)}(\eta_0 + \eta_* - \eta').
$$

The initial data for the solution $\tau_\ell^{(0)}(k, \eta)$ is the set of constants $C_1(\eta_*)$ which are determined by $B_A(k) = \{B_0(k), B_1(k), B_2(k)\}$; these must be matched to the initial distribution function on an appropriate initial surface $\Sigma$ (for example, the ‘surface of last scattering’ which can be
covariantly and gauge invariantly defined). This then determines the solution up to the present day (and after). We are free to chose any \( Q \)'s as long as they solve the Helmholtz equation in the background. The choice of \( Q \) then explicitly determines \( G_\ell[Q] \), for example we are free to choose \( Q \) to be the spherical or plane-wave basis. In practice we naturally use two sets of mode functions \( G_\ell[Q] \), matching those for the null-cone (given in a spherical basis) to those in some initial surface (given in terms of a plane-wave basis). The matching of these two sets of harmonics is then given by the relations usually written into the construction of the mode coefficients (see (76)). This matching is based on mode functions \( G_\ell[Q] \) in the Robertson-Walker background, which is acceptable because of the homogeneity assumption. By using \( G_\ell[Q] \) we do not actually need the explicit form of the \( Q \)'s.

Equation (93) shows (r.h.s. of the first line) how major parts of the cosmic background radiation anisotropy are determined directly from the set of initial conditions (at last scattering, for the freely propagating radiation). The integrated effect arises through the coefficients \( C_\ell(\eta) \) as integrated down time-like geodesics in the remaining terms on the right hand side. In general there is a non-linear coupling through the field equations between the matter, the radiation and the acceleration and shear terms that arise in the integrated part. The situation is much simpler when this back-reaction can be neglected; for this reason it is convenient to consider the case of matter domination, during which the radiation can be considered as a test-field propagating on the background determined by the matter content. However we can also consider the general set of linearised field equations (see Appendix G.1) and the coupling to the radiation via the source terms, first the gravitational source, \( B \), and second, the scattering source, \( C[\tau] \), (68), in (67). In the following sections we look at the various approximations that can be applied at different epochs.

4. FREE-STREAMING

Using the integral solution (93) we construct the almost-Friedmann-Lemaître free-streaming projection of the initial conditions near last scattering to here and now (the determination of these initial conditions is demonstrated in latter sections) and the integrated secondary contributions arising during the period after last-scattering until now (we have dropped the baryon relative velocity effect using the instantaneous decoupling assumption):

\[
\frac{\tau_\ell(\eta_0)\beta_\ell}{(2\ell+1)} \simeq \left[ \frac{1}{k} [aB_1](\eta_\ast) - \frac{5}{3} \frac{1}{k^2} [aB_2]'(\eta_\ast) - \frac{1}{k} [aB_2](\eta_\ast) \frac{\partial}{\partial \eta_0} \right] j_\ell(k\Delta \eta_\ast) \\
- \int_{\eta_\ast}^{\eta_0} d\eta \left\{ \frac{5}{6} aB_2 - \frac{1}{k} (aB_1)' + \frac{5}{3} \frac{1}{k^2} (aB_2)'' \right\} j_\ell(k\Delta \eta) ,
\]

(94)

Where we used as final conditions:

\[
[aB_1]'(\eta_0) = [aB_2](\eta_0) = 0 .
\]

(95)

The first term on the right, the \( B_1 \) term, will generate the acoustic primary effect on the anisotropies, the second term is the Doppler contribution due to the radiation dipole (the baryon velocity contribution which would arise through \( C_1 \) (90)), the third and fourth terms give the effect of any shear, near last scattering (through the initial conditions of \( B_2 \)). The remaining terms represent the integrated Sachs-Wolfe effect.

The above equation will be modified in the following section to include slow decoupling, but first we demonstrate how to recover the basic Sachs-Wolfe effect.
4.1. The Almost-Friedmann-Lemaître Sachs-Wolfe Effect

We now find the solutions corresponding to the matter dominated, free-streaming era, with adiabatic modes only, using the Newtonian frame treatment (see section 7.1.2). Using the field equations from [59], or from the Appendix G.1 and 59, we can find the source terms $aB_I^A(k, t)$ for the free-streaming projection: eqns. G.24 through G.26 in Appendix G.2. This applies to the case of instant decoupling.

During matter domination the dipole is negligible, so we ignore it. The shear contribution is small on large scales, hence we can also ignore it.

On substituting these equations into the flat almost-Friedmann-Lemaître integral solution (93) with $K = 0$ in the source terms (eqns. G.24 through G.26) we can find the free-streaming almost-Friedmann-Lemaître solutions for the temperature anisotropies (94). Using $(\Phi \rho^{-1}_a)' \sim 0$ we find the CGI kinetic theory equivalent of the Sachs-Wolfe formula for cosmic background radiation anisotropies in terms of matter inhomogeneities at last scattering at various wavelengths, together with an integral term. In various cases (see [41] for references to more general treatments), in particular the matter-dominated spatially flat solutions with only growing scalar modes, the integral terms vanish and we obtain:

$$\frac{\tau^{SW}_\ell(\eta_0, k)\beta_\ell}{(2\ell + 1)} \approx \frac{2}{3} |\Phi \rho^{-1}_a| (\eta_*) j_\ell(k \Delta \eta_*).$$

This gives the (approximate) projection of the large scale potential inhomogeneities at last scattering onto the sky today; the mean-squares $|\tau_\ell|^2$ can then constructed, using the results from Paper I [18]. The effect arise from the terms $\Delta_a \ln T \approx \frac{1}{3} \Delta_a \ln \rho \approx \rho^{-1}_a D^b E_{ab}$, having used the adiabatic assumption.

This recovers the standard Sachs-Wolfe result [68, 13] on large scales, where the potential fluctuations are just due to primordial initial inhomogeneities that are unchanged by intervening physics.

We discuss the origin of these fluctuations in later sections – they are given by solving these equations before decoupling, which for example implies the existence of acoustic oscillations. These potential fluctuations are what seed structure formation through the production of matter perturbations undergoing gravitational collapse beneath the Jeans scale. The matter perturbations effectively decouple near matter-radiation equality, making the large scale temperature anisotropies the key link between the radiation anisotropies now to the potential fluctuations then (near radiation decoupling), and so to the matter power spectrum both on large and small scales today.

Notice that these equations will hold for any choice of 4-velocity that is close to the matter 4-velocity, i.e. there is still frame-freedom associated with this freedom of choice. As has been remarked various times, there are several possible physical choices for this 4-velocity (which will all agree at late times); the interpretation of the physical meaning of the cosmic background radiation anisotropy sources will change depending on this choice. The important difference about the derivation of the Sachs-Wolfe effect here as opposed to other treatments is that (i) this result is found in the total matter frame, (ii) the integration is explicitly time-like, so it is not treated mathematically as a projection along null rays but rather as the evolution of the anisotropies of radiation in a small comoving box, as explained in the introduction. Thus the initial data here is not at the intersection of the past light cone and the last scattering surface, but rather at the intersection of the world line of the observer and the last scattering surface.

This analysis can be compared to primary anisotropy source term of the gauge-invariant treatment used in [40]. The subtle difference between the Bardeen variable gauge-invariant approach and the CGI approach used here is that the Doppler source, which in their case
arises through $B_0$, now enters through the integral terms only; there is no direct Doppler contribution at last scattering from the first term in the integrated solution.

5. SLOW DECOUPLING

To deal with slow-decoupling, we return to the general damped solution, (93), and introduce the slow decoupling approximation

$$\kappa'' \ll 1, \quad (\kappa')^2 \ll 1,$$

(97)
to find, on substituting in from the coefficient relations (90):

$$\frac{\tau_\ell(y_0)b_\ell}{(2\ell + 2)} \approx e^{-\kappa} \left[ \frac{1}{k} [aB_1] \left( \tau_\ell \right) + \frac{1}{k^2} \frac{5}{3} [aB_2]'(\tau_\ell) - \frac{1}{k} [aB_2] \left( \tau_\ell \right) \frac{\partial}{\partial y_0} \right] j_\ell(k\Delta \eta_*),$$

$$- \int_{y_*}^{y_0} dy' e^{-\kappa} \left\{ \frac{5}{6} aB_2 - \frac{1}{k} (aB_1)' + \frac{5}{3} k^2 (aB_2)'' \right\} j_\ell(k\Delta \eta)$$

$$+ \frac{1}{k} (\kappa' e^{-\kappa}) v_B(y_0) j_\ell(k\Delta \eta_*) + \int_{y_*}^{y_0} dy' (v_B' \kappa' + \kappa'' v_B e^{-\kappa}) \frac{1}{k} j_\ell(k\Delta \eta)$$

$$+ \int_{y_*}^{y_0} dy' (\kappa' e^{-\kappa}) \left[ \frac{1}{k} (aB_1) + \frac{2}{k^2} (aB_2) \right] j_\ell(k\Delta \eta) .$$

(98)

We see that damping effects are controlled by $e^{-\kappa}$ and $\kappa' e^{-\kappa}$. A further approximation would be to take $e^{-\kappa} \gg \kappa' e^{-\kappa}$, so that we need only consider the free-streaming like solutions, which we then convolve with the damping factor, defined as a combination of the visibility function and the diffusion damping envelope – this is done later using the damping envelope as derived from the dispersion relations in section (5.2). Later we will explicitly recover these equations in the Newtonian frame of section (2.4.1).

5.1. Silk Damping

Diffusion damping will occur and introduce a damping scale, the Silk scale, giving a cut-off in the matter perturbations, and there will be a corresponding diffusion damping effect in the photons. This is naturally included in our general damped solutions in terms of the exponential envelope implied by the equations, which can be demonstrated heuristically.

The cut-off arising through photon diffusion occurs when the term involving $\kappa'$ in (70) dominates the other terms; that is, when for any $\ell$, $k$ is large enough that

$$k \left[ \frac{1}{4} \tau_\ell+1 - \tau_\ell-1 \right] \approx \frac{3}{4} k\tau_\ell \ll \kappa' \tau_\ell,$$

(99)

the approximation assuming that the damped modes are roughly of the same magnitude (independent of $\ell$ when this condition is satisfied). This then implies an exponential decay in the relevant modes:

$$k \ll \frac{4}{3} \kappa' \Rightarrow \tau_\ell' \approx -\kappa' \tau_\ell \Rightarrow \tau_\ell(\eta) \approx \exp(-\kappa' \eta) \tau_\ell(0) .$$

(100)

Thus small scales will be heavily damped by this process and long wavelengths unaffected, leading to a wavelength-dependent damping envelope. The resulting cut-off in perturbation amplitude at a critical wavelength at last scattering will result in a corresponding cut-off in cosmic background radiation anisotropy amplitudes observed at a critical angular scale.

A more detailed examination undertaken later will show the explicit wavelength dependence of this cut-off effect.
5.2. The Visibility Function

An alternative approach to the slow decoupling solution (98), is to argue that the dominant contribution during slow-decoupling arises from the visibility function defined by $\mathcal{V}(k, \eta) \approx \kappa' e^{-\kappa}$ as convolved with the free-streaming integral solution. The visibility function gives the probability of a photon last scattering during a small time interval $d\eta$. From Hu & Sugiyama [39] it is useful to define the damping factor (now including diffusion damping, which will be derived from the coupled baryon-photon equations in (169)):

$$D(\eta_0, k) = \int_{\eta_*}^{\eta_0} d\eta C(\eta, k)e^{-(k/k_D)^2} \approx e^{-(k/k_D)^2} . \quad (101)$$

The visibility function will model the changing ionization fraction, this does not include the diffusion damping, which is added in by hand above, through the damping scale. It should be realized here that the Gaussian diffusion damping is naturally included in the original Friedmann-Lemaître integral solution (93). However, given that we will use solutions that are first-order (in the scattering time) and then recover an explicit dispersion relation for the damping scale, at second-order in the scattering time, it is convenient to modify the damping factors such that they are re-written in terms of the visibility function 101. The second order damping scale of the form used here is explicitly derived in section 6.2.5.

Now, we can modify the free-streaming projection by including the damping factor $D$ and the baryon velocity effect (which must be put in from (93)). In this approximation, we can effectively drop the last two lines of (93) except for the initial baryon velocity contribution, to obtain:

$$\frac{\tau_{\ell}(\eta_0) \beta_{\ell}}{(2\ell + 1)} \simeq [C_2'(\eta_\ast) - C_1(\eta_\ast)] D(\eta_0, k) j_\ell(k \Delta \eta_\ast)$$

$$+ k C_2(\eta_\ast) D(\eta_0, k) \left[ \frac{\ell}{(2\ell + 1)} j_{\ell-1}(k \Delta \eta_\ast) - \frac{(\ell + 1)}{(2\ell + 1)} j_{\ell+1}(k \Delta \eta_\ast) \right]$$

$$+ \int_{\eta_*}^{\eta_0} d\eta C(\eta, k) e^{-(k/k_D)^2} \left\{ C_0(\eta) - C_1'(\eta) + C_2''(\eta) \right\} j_\ell(k \Delta \eta) , \quad (102)$$

where the coefficients $C_1$, $C_2$ and $C_3$ are given by (90). This has the effect of taking the previous more general solution (93) and specializing it to the most important regime as far as decoupling is concerned, thus giving a major improvement on the sharp decoupling approximation, while avoiding the complications of the complete integral solution given above. A similar correction is made using the visibility function in the integrated part of the solution, in order to best deal with a changing ionization fraction, given that we will once again only be using almost-Friedmann-Lemaître solutions, that are either first order or zero-th order in the scattering times (discussed below).

5.3. Slow Decoupling in the Conformal Newtonian Frame

Here we cast the above derived solutions (94,98, 102) based on the integral solution (93) into the CGI Newtonian frame (based on the shear-free frame described in section 2.4.1) for the case of scalar perturbations, in terms of the Bardeen like scalar potentials $\Phi_H$ and $\Phi_A$.

The vanishing shear condition $\sigma_{ab} \approx 0$ implies that $\hat{C}_0(\eta) \approx 0$ and $\hat{C}_2(\eta) \approx 0$, hence we can use these conditions directly to find the slow-decoupling anisotropy solution for an almost-Friedmann-Lemaître model in the Newtonian frame:

$$\frac{\tau_{\ell}(\eta_0) \beta_{\ell}}{(2\ell + 1)} \simeq -\hat{C}_1(\eta_\ast) D(\eta_0, k) j_\ell(k \Delta \eta_\ast) - \int_{\eta_*}^{\eta_0} d\eta C(\eta, k) e^{-(k/k_D)^2} \hat{C}_1'(\eta) j_\ell(k \Delta \eta) . \quad (103)$$
Using the results from section (2.4.1) and the almost-Friedmann-Lemaître relations (given in Appendix G), it can be shown that the key quantity of interest $\tilde{B}_a \approx \tilde{D}_a \ln T + \tilde{D}_a \Phi_A$, in the case of scalar perturbations, obeys the following relation in the Newtonian frame.

$$D_a \tilde{B}_b \approx D_a D_b (\Phi_A - \Phi_H) - 2H D_a D_b \Phi_A - \frac{1}{3} D_a D_b (D^c \tilde{\tau}_c)$$  \hfill (104)

Now we need to find the mode coefficients for $\mathcal{B}$ in terms of the quantities defined in the Newtonian frame. Writing

$$\tilde{D}_a \ln T \approx \frac{k}{a} \delta \tilde{T} Q_a, \quad D_a \Phi_A \approx \frac{k}{a} \Phi_A Q_a, \quad \text{and} \quad D_a \Phi_H \approx \frac{k}{a} \Phi_H Q_a,$$  \hfill (105)

we find that

$$\tilde{B}_a \approx \frac{k}{a} (\delta \tilde{T} + \Phi_A) Q_a, \quad \Rightarrow \quad \tilde{B}_1 \approx \frac{k}{a} (\delta \tilde{T} + \Phi_A).$$  \hfill (106)

On mode expanding (104) and transforming to the conformal time derivative we obtain:

$$(a \tilde{B}_1)' \approx -a^2 H \tilde{B}_1 + k (\Phi_A - \Phi_H) - 2H ak \Phi_A + \frac{1}{3} k^2 \tilde{\tau}_1.$$  \hfill (107)

Now using (90) we find that:

$$-\tilde{C}_1' (\eta) - (\kappa' \tilde{v}_B)' \approx + \frac{1}{k} (a \tilde{B}_1)'.$$  \hfill (108)

We can now put this all together, first, from (90) and (106) to find:

$$-\tilde{C}_1 (\eta) \approx \kappa' \tilde{v}_B + (\delta \tilde{T} + \Phi_A).$$  \hfill (109)

and second from (108), (109) and (107) to find:

$$-\tilde{C}_1' (\eta) \approx \kappa' \tilde{v}_B + \left( (\Phi_A - \Phi_H) - aH (\delta \tilde{T} + \Phi_A) - 2aH \Phi_A + \frac{1}{3} k \tilde{\tau}_1 \right).$$  \hfill (110)

Substituting these results into the integral solution (103) we obtain:

$$\frac{\tau_{c}(\eta_0) \beta_{\ell}^{c}}{(2\ell + 1)} \approx \left[ (\kappa' \tilde{v}_B) \right] \left[ \left( \tau_{s}(\eta_0, k) j_{\ell}(k \Delta \eta_s) + \int_{\eta_s}^{\eta_0} d\eta \mathcal{V} e^{-\frac{k}{k_{D}}} \left( (\kappa' \tilde{v}_B') + \frac{1}{3} k \tilde{\tau}_1 \right) j_{\ell}(k \Delta \eta) \right] \right] (2\ell + 1) \approx \left[ (\Phi_A - \Phi_H) - aH (\delta \tilde{T} + \Phi_A) - 2aH \Phi_A + \frac{1}{3} k \tilde{\tau}_1 \right).$$  \hfill (111)

Here the second order terms (both in terms of the scattering time and in the almost-Friedmann-Lemaître sense) have been dropped. We can then pull out the canonical solution when we ignore the Doppler contribution, the initial baryon relative velocity at last scattering (it is tightly coupled to the radiation velocity and is thus small already). We also ignore the intermediate scale integrated effect which contributes to the early-integrated Sachs Wolfe effect. The result is:

$$\frac{\tau_{c}(\eta_0) \beta_{\ell}^{c}}{(2\ell + 1)} \approx \left[ (\Phi_A - \Phi_H) + \int_{\eta_s}^{\eta_0} d\eta \mathcal{V} e^{-\frac{k}{k_{D}}} \left( \Phi_A - \Phi_H \right) j_{\ell}(k \Delta \eta) \right].$$  \hfill (112)

---

18 We will use $D_a D_b \Phi_A \approx - \Phi_A D_a D_b Q$, $D^a \tau_c \approx + \frac{k}{a} \tau_0 Q$ and $D_a \tilde{B}_b \approx - \frac{1}{a} (\tilde{B}_1 + 2HB_1) D_a D_b Q$. 
This completes the recovery of the standard integral solution results using the 1+3 CGI approach at linear order – we have notationally suppressed the \( k \)-dependence of the temperature anisotropy \( \tau(\eta_0) \equiv \tau(\eta_0, k) \). It corroborates the standard anisotropy derivations based on a 3+1 hypersurface foliation, which uses the Bardeen formalism in the conformal Newtonian gauge.

### 6. LATE TIGHT-COUPLING

Here we extend the Thomson scattering analysis of the previous sections to include a simple model of late-tight coupling and hence of fast decoupling.

We aim to reproduce in covariant form the Peebles and Yu near-tight coupling [66] and Hu and Sugiyama tight-coupling approximation [40] treatments, valid for the period of late tight-coupling, up to and including decoupling \(^{19}\). Remember that we are ignoring the anisotropic and polarization effects as these can be corrected at the level of the damping scale.

#### 6.1. Integrated Boltzmann Equation: Near-tight Coupling

Here the almost-Friedmann-Lemaître integrated Boltzmann equations is used to construct a set of multipole divergence equations that describe the radiation near tight-coupling. These are the intermediate scale equations, valid in the tight-coupling era. This is done by carrying out a CGI version of perturbation theory in terms of the scattering time.

##### 6.1.1. The Scattering-strength Expansion

The solutions we have considered so far are linearised through a small-parameter expansion in terms of the anisotropy parameter \( \tau \). The basic idea now, following the method of Peebles and Yu [66], is that additionally a second expansion is constructed in terms of the collision parameter \( t_c = (\sigma_T n_e)^{-1} \), without truncating the Boltzmann hierarchy at the order of the calculation, and thus avoiding the problems inherent in exact truncation [29] (see Appendix E). We thus find the evolution equations for the energy density, momentum flux, and the anisotropic flux of the radiation close to tight coupling.

Consider the almost-Friedmann-Lemaître integrated Boltzmann equations (4) and (67) for isotropic (in the baryon frame) Thompson scattering (68) [59]; this is inverted to find:

\[
\tau(x^i, e^a) = v_B^a e_a - t_c [B + \dot{\tau} + e^a D_a \tau] .
\]  

We now systematically approximate (113) in terms of the smallness parameter \( t_c \). The right-hand side (the scattering term) is used to find the zero-th order collision-time correction to the total bolometric temperature, with corresponding temperature anisotropy given by

\[
\tau_{(0)}(x^i, e^a) \approx v_B^a e_a .
\]  

The equation is now perturbed about the zero order velocity perturbation and one can then recover the first and second order corrections in \( t_c \) to the zero-th order temperature anisotropies, to find, \( \tau_{(1)} \) and \( \tau_{(2)} \) respectively, where the n-th. order correction is denoted by \( \tau_{(n)} \). We obtain an almost-Friedmann-Lemaître perturbative expansion in \( t_c \):

\[
\tau_{(n)} \approx v_B^a e_a - t_c [B + \dot{\tau}_{(n-1)} + e^a D_a \tau_{(n-1)}] .
\]  

\(^{19}\)Here we are explicitly making a distinction between the treatment [40] (what we call tight-coupling approximation) and that in [66] (what we call near-tight coupling. By tight-coupling approximation we mean that \( \dot{k}^{-1} \) is sufficiently small that it can be ignored (inducing a contribution of the order of magnitude (say) of at least \( 10^{-6} \)) when multiplying quantities of linear order such as the shear) The near-tight coupling includes the radiation quadrupole in the case of isotropic Thompson scattering (in the matter frame).
The tight-coupling limit is recovered when \( t_c = 0 \). This treatment is then a consistent (in the sense of the truncation conditions described in Appendix E) near-to-tight-coupling treatment in almost-Friedmann-Lemaître universes. (The first, in \( n \), three temperature anisotropies, are given in Appendix C).

6.1.2. Solid Angle Integration

Now the temperature anisotropy is integrated over the solid sphere to ensure the condition that there is no contribution to the bolometric average \( T_b (x^i) \),

\[
\int_{4\pi} \tau(x^i, e^a) d\Omega = 0 .
\]

It should be clear why the second order correction to the temperature anisotropy is needed even though we intend to keep the expansion only to first order in \( t_c \); the integrations over term the \( e^a v_a \) will vanish. Now by integrating \( \tau (2) \) (C.6) over the solid angle and using (116) and orthogonality of \( O^{A\ell} \) the gradient of the radiation flux is found:

\[
D_a \tau^a \simeq D_a v_B^a - \alpha_c \left[ (D_a v_B^a) - (D_a \tau^a) + \frac{1}{3} D^2 (\ln \rho R) + (D_a v_B^a) + (D_a A^a) \right] .
\]

By taking spatial gradients of the radiation flux (121) we find on comparing with (117), that in order for there to be no contributions to scalars,

\[
(\text{div } v)_B \approx (\text{div } \tau) .
\]

The above equation (117) then becomes

\[
(\text{div } \tau) \simeq (\text{div } v)_B - t_c \left[ \frac{4}{3} D^2 \ln \rho R + (D_a v_B^a) + (\text{div } A) \right] .
\]

6.1.3. The Transport Equations

Finally the individual PSTF multipoles are recovered at a given order

\[
\tau_{A\ell} = \Delta^{-1}_\ell \int_{4\pi} O_{A\ell} \tau_{(n)} (x^i, e^a) d\Omega.
\]

Here \( \Delta_\ell \) is defined as before in Paper I [18]. The second order temperature multipoles are now found from (120) and integrating \( \tau_{(2)} (x, e) \) (C.5) after the combination of direction vectors has been replaced by PSTF tensors (one can use C.7-C.9):

\[
\tau^b \approx v_B^b - t_c \left[ D^b \ln T + A^b + \dot{v}_B^b \right] + t_c^2 \left[ (D^b \ln T + A^b) + \dot{v}_B^b - \frac{1}{3} D^b D_c \tau^c \right],
\]

\[
\tau^{ab} \approx - t_c \left[ \sigma^{ab} + D^{(a} v^{b)}_B \right] + t_c^2 \left[ (D^{(a} v^{b)}_B) + D^{(a} (D^b \ln T + A^b) + D^{(a} \dot{v}^{b)}_B + \ddot{\sigma}^{ab} \right],
\]

\[
\tau^{abc} \approx + t_c^2 \left[ D^{(ab} v^{c)}_B \right],
\]

\[
\tau^{A\ell} \approx 0 \quad \forall \ell > 3,
\]

where we have dropped terms of \( O(t_c^3) \). These are the key results of this section. They are the appropriate transport equations for the late-tight coupling era, i.e. up to last scattering, and are essentially equivalent to the causal transport equations given by causal relativistic thermodynamics [60].
What we have shown here is that if we are interested in the behaviour of the photon-baryon systems to first order in the scattering time, a dissipative fluid approximation is sufficient to describe the radiation (cf. the papers by Israel and Stewart [45]), and will not result in an explicit truncation of the Boltzmann multipole hierarchy, rather it gives a systematic approximation scheme where we can, to the appropriate accuracy, ignore the third order and higher terms. This is significant; one cannot merely drop the higher order moments and truncate to a fluid description, as the kinetic theory treatment fixes the transport equations. Here we have consistently decoupled the $l < 3$ multipole equations from the rest of the hierarchy and the consistency of this decoupling is maintained through (117) and (121 - 124).

The solutions to these equations, which lead to acoustic oscillations during this period, will then affect the cosmic background radiation anisotropies by setting initial conditions for the free-streaming solution discussed in the previous section. We give a derivation of these results in the following section.

### 6.2. Late Tight-coupling and the Oscillator Equation

Here we derive the CGI equivalent of the analytic tight-coupling approximation used by Hu & Sugiyama [39, 40]. This approach uses the tight-coupling limit in order to get rid of the radiation quadrupole during late tight-coupling and covariantly reproduces Hu and Sugiyama’s conclusions about the cosmic background radiation anisotropy due to inhomogeneities, acoustic and Doppler sources (what they call ‘primary sources’). This gives the ‘Sachs-Wolfe effect’ due to the Newtonian potential near last scattering, but not the ‘integrated Sachs-Wolfe effect’ due to changing potentials after tight coupling (resulting from more complex matter models and/or spatial curvature).

#### 6.2.1. Near Tight-coupling Equations

We start with the near-tight coupling equations (121, 122) and (124) except rewritten to first order and in terms of the optical depth so as to be closer to the notation of the better known treatments [40]:

\[
\tau_a \simeq v^B_a - \kappa^{-1} \left[ D_a (\ln T) + A_a + \dot{v}^B_a \right],
\]

\[
\tau_{ab} \simeq -\kappa^{-1} \left[ \sigma_{ab} + D_{\langle a} \dot{v}_{b \rangle} \right],
\]

\[
\tau^{A \ell} \simeq 0 \quad \forall \ell > 2.
\]

Here we have assumed that the collisions are dominated by Thompson scattering and is therefore isotropic in the matter frame.

The relative velocity of the matter with respect to the preferred reference frame is $v^a_B \simeq u^2_B - u^a$. Rewriting (126) in terms of the shear of the baryon frame, we have $\tau_{ab} \simeq -\kappa^{-1} D_{\langle a} u^B_{b \rangle}$, so the quadrupole is given entirely by the shear of the matter.

Notice that $\pi_{ab} = \rho B \tau_{ab} \approx 0$, as the case of matter domination. This condition is not sufficient to ensure that $\tau_{ab}$ can be ignored in equations when it appears on its own, even though the quadrupole is small. The key point, which was discussed in section 2.5, is that there are four principal linearisations: the almost-Friedmann-Lemaître one at least $O(\epsilon^2)$, the almost-Friedmann-Lemaître radiation isotropy one $O(\epsilon \eta)$, $O(\eta^2)$, (implying the previous by the almost-Ehlers-Geren-Sachs theorem), the non-relativistic assumption $O(\nu \eta)$, $O(\nu^2)$, $O(\nu \epsilon)$.

---

20If there are any anisotropic contributions such as anisotropic scattering (in the matter frame) or large shear (from gravitational waves) at that time this sort of approximation should be considered with care - such phenomena would break tight-coupling.
and the linearisation scheme based on the differential optical depth. Hence care must be taken when approximations are made to the equations.

6.2.2. Tight coupling: Momentum Equations

The tight-coupling calculation is now carried out, assuming (125-127) hold. Consider once again the radiation energy and momentum conservation equations ($\ell = 0$ and $\ell = 1$ multipole divergence equations):

\begin{align*}
\dot{(\ln T)} + \frac{1}{3} \Theta &\simeq -\frac{1}{3} \text{div} \tau, \\
\dot{\tau}_a &\simeq A_a + D_a (\ln T) - \dot{\kappa} (v^B_a - \tau_a) + \frac{2}{5} D^c \tau_{ac},
\end{align*}

and the baryon energy and momentum conservation equations

\begin{align*}
\dot{\ln \rho} + \Theta &\simeq -D^a v^B_a, \\
\dot{v}^B_a &\simeq +H v^B_a + A_a + R^{-1} \dot{\kappa} (v^B_a - \tau_a).
\end{align*}

Here $\dot{\kappa}$ is the optical depth, and the radiation-baryon ratio in the real universe is given by (using the enthalpy $h = \rho + p$) by

\begin{equation}
R(x^i) = \frac{\rho_M(x) + \rho_B(x)}{\rho_R(x)} \approx \frac{3}{4} \rho(x^i) \Rightarrow \ddot{R} \simeq HR.
\end{equation}

This is related to the speed of sound in the background via $c^2_s = (1/3)(R_0 + 1)$. The matter momentum equations, (131), give

\begin{equation}
v^B_a \simeq \tau_a - \frac{R}{\dot{\kappa}} [\dot{v}_a^B + H v^B_a + A_a].
\end{equation}

Substituting (125) into (133) and retaining all terms up to linear order (in the relaxation time) we obtain

\begin{equation}
v^B_a \simeq \tau_a - R \dot{\kappa}^{-1} [\dot{v}_a^B + A_a + H \tau_a] + O(\dot{\kappa}^{-2}).
\end{equation}

This is then substituted into (129) in order to remove the velocity terms, and with a little algebra, we find

\begin{equation}
-\dot{\tau}_a \simeq \dot{u}_a + \frac{1}{1 + R} D_a \ln T + \frac{\dot{R}}{(1 + R)} \tau_a - \frac{2}{5} \frac{\dot{\kappa}^{-1}}{(1 + R)} D^b D_{(a} u^m_b),
\end{equation}

where the last term has been written in terms of the matter shear. We can now consider the situation where $\dot{\kappa}^{-1}$ becomes sufficiently small (but non-zero) so that the last term can be ignored. This is possible as the matter shear is already at least first order. We then find (cf [40] eqn. (B2 b)):

\begin{equation}
\dot{\tau}_a + \frac{\dot{R}}{(1 + R)} \tau_a + \frac{1}{1 + R} D_a (\ln T) \simeq -A_a.
\end{equation}

\[21\] This is found to $O[1]$ by substituting the matter energy conservation equation (dust part of G.10) into matter momentum equation (dust part of G.11) all to $O[1]$.

\[22\] By speed of sound we mean adiabatic sound speed: $c^2_s = \frac{\dot{\rho}_M}{\rho_M + \rho_R} = \frac{\dot{\rho}_M}{\rho_M + \rho_R} = \frac{\dot{\rho}_R}{\rho_M + \rho_R} = \frac{1}{3} \left( \frac{\dot{\rho}_M}{\rho_M + \rho_R} + 1 \right)$, for matter domination $c_s \approx 0$. 

This is the momentum flux equation for the radiation and is a key result. It can be rewritten as

\[ [(1 + R) \tau_a] + D_a(\ln T) \simeq -(1 + R) A_a , \]  

or on taking its divergence as

\[ [a(1 + R)(D^a \tau_a)] + a(D^2 \ln T) \simeq -(1 + R)a(D^a A_a) . \]  

### 6.2.3. Spatial Gradients and the Oscillator Equation on Small Scales

The basis of this derivation is the ‘small-scale’ assumption which effectively means that on small enough scales we can ignore the expansion (see section 2.5). This is just the statement that the scale of inhomogeneity is less than the Hubble scale (66), so we drop all terms of \( O(\epsilon_H) \) (see section 2.5.4).

Our aim is to recover the standard oscillator equation (the source equation for the acoustic oscillations) using the 1+3 CGI formalism. Note however that we still have the freedom to set the relative velocity in the small boost equations (which we will do in the next section).

Taking the spatial gradient\(^{23}\) of the radiation energy conservation equation, (128), we find

\[ -\frac{1}{3} D_a(D_c \tau^c) \simeq (D_a \ln T) + \frac{1}{3} D_a \Theta + H(D_a \ln T + A_a) , \]  

and taking the divergence of the resulting equation above gives

\[ -\frac{1}{3} (D^2(D_c \tau^c)) \simeq (D^2 \ln T)' + 2H(D^2 \ln T) + \frac{1}{3} D^2 \Theta + H(div A) , \]

and this can be written as

\[ -\frac{1}{3} (D^2(D_c \tau^c)) \simeq (a^2 D^2 \ln T)' + \frac{1}{3} (a^2 D^2 \Theta) + H(a^2 div A) . \]  

This is analogous to equation (B3) in [40]. Then using (138) we find

\[ [(1 + R)aD_c(D^a \tau_a)] + H(1 + R)aD_c(D^a \tau_a) + aD_c(D^2 \ln T) \simeq -(1 + R)aD_c(div A) . \]  

Substituting (139) into (142) and using the fact that \( HD^a A_a \approx O(\epsilon_H) \), we obtain

\[ -3[a(1 + R)(D_c \ln T)'][a^2D_c(D^2 \ln T) + aD_c(D^2 \ln T) \simeq [a(1 + R)(D_c \Theta)] - a(1 + R)D_c(div A) . \]  

Finally, transforming to conformal time, \( dt = ad\eta \) gives

\[ 3[(1 + R)(aD_c \ln T)'][a^2D_c(D^2(aD_a \ln T))] \simeq -[a(1 + R)(D_c \Theta)'][a^2(1 + R)D_c(div A) . \]  

On using the small-scale linearisation scheme described in section (66) we find, on dividing through by \( 3(1 + R) \), the oscillator equation:

\[ (aD_c \ln T)'' \simeq \frac{a^3}{3(1 + R)}D_c(D^2 \ln T) - a^2(D_c \Theta)' + \frac{a^2}{3}D_c(D^a A_a) . \]  

\(^{23}\)Using the identity \( (D_a f) \simeq D_a f - HD_a f + A_a f \) we find that \( (D_a \ln T) \simeq D_a \ln T - H(D_a \ln T + A_a) \) from the almost-Friedmann-Lemaître \( \ell = 0 \) multipole divergence equations and \( H = \frac{\dot{a}}{a} \) [59].
This is the covariant harmonic-oscillator equation which describes the acoustic modes \(^{24}\). We can compare it to the usual gauge invariant result found in the Newtonian gauge by transforming to the shear-free frame \(\tilde{u}_a = n^a\) where \(D_{(a)n_b)} = 0\).

We will investigate this equation in more detail in the next section and relate our results to those in the standard literature (which are expressed in the Newtonian gauge).

6.2.4. The Newtonian Frame Oscillator Equation

In this section we recover the harmonic oscillator equation of Hu and Sugiyama in the Newtonian gauge \([39]\) from the CGI formalism. The difference between this and the previous section is that here we apply the small scale approximation at the very end of the calculation.

Using the Newtonian frame choice \(n^a \approx u^a + \nu^a_N\), \(D_{(a)n_b)} = 0\), equations (136) and (15) become

\[
\begin{align*}
\dot{\tilde{r}}_a + \frac{\dot{R}}{1 + R} \tilde{r}_a + \frac{1}{1 + R} \tilde{D}_a (\ln T) & \approx -\tilde{A}_a \approx -D_a \Phi_A , \quad (146) \\
(\tilde{D}_a \ln T) + H(\tilde{D}_a \ln T + \tilde{A}_a) + \frac{1}{3} \tilde{D}_a \tilde{\Theta} & \approx -\frac{1}{3} D_a (D_c \tilde{r}_c) . \quad (147)
\end{align*}
\]

These can be re-written and put into the following form by using the almost-Friedmann-Lemaître Einstein field equations, together with the transformation relations given in Appendix G:

\[
\begin{align*}
(D_a \tilde{u}^a) + H(D_a \tilde{u}^a) & \approx -\frac{\dot{R}}{1 + R} (D_a \tilde{u}^a) - \frac{1}{1 + R} (\tilde{D}_a \ln T) - (D^2 \Phi_A) , \quad (148) \\
D_{(a)D_c} (\ln T) & \approx -D_{(a)D_c} \Phi_H - \frac{1}{3} D_{(a)D_c} (D_c \tilde{r}_c) . \quad (149)
\end{align*}
\]

Taking the time derivative of (149) and substituting into equation (148) after first taking PSTF derivatives we obtain the full equation for \(D_{(a)D_b} \ln T\):

\[
\begin{align*}
(D_{(a)D_b} \ln T) + \dot{H} (D_{(a)D_b} \Phi_A + 2 D_{(a)D_b} \ln T) + H \left[(D_{(a)D_b} \Phi_A) + 2 (D_{(a)D_b} \ln T)\right] & \\
& \approx -(D_{(a)D_b} \Phi_H) - 2 \dot{H} D_{(a)D_b} \Phi_H - 2 H (D_{(a)D_b} \Phi) + \frac{1}{1 + R} D_{(a)D_b} D^2 \ln T \\
& - D_{(a)D_b} D^2 \Phi_A + \left[\frac{\dot{R}}{1 + R} - 3 H\right] \left(D_{(a)D_b} (\ln T) + D_{(a)D_b} \Phi_H\right) . \quad (150)
\end{align*}
\]

On dropping all terms \(O(\epsilon H)\), we once again obtain the 1+3 covariant form of the small scale Newtonian frame oscillator equation (without using a mode expansion):

\[
\begin{align*}
(D_{(a)D_b} \ln T) + \frac{\dot{R}}{(1 + R)} D_{(a)D_b} (\ln T) + \frac{1}{1 + R} D_{(a)D_b} D^2 \ln T & \\
& \approx -(D_{(a)D_b} \Phi_H) + \frac{\dot{R}}{(1 + R)} D_{(a)D_b} \Phi_H - D_{(a)D_b} D^2 \Phi_A . \quad (151)
\end{align*}
\]

The techniques used to derive the above equation become useful latter when dealing with the non-linear terms as they avoid the complication of mode-mode couplings when understanding the qualitative features of various effects \([59]\).

\[^{24}\]This can also be obtained from a two-fluid CGI description \([13]\), as well as from the imperfect-fluid description \([60]\) – the point here is that we have derived it from a self consistent kinetic theory approach, listing along the way the necessary physical approximation required to reduce it to the standard acoustic oscillator form.
Upon using the mode decomposition definitions for the temperature perturbations, radiation dipole and the scalar Newtonian and curvature perturbations respectively \(^{25}\), we can write the mode decomposition of (148) and (149) as:

\[
\dot{\tilde{T}}_1 \approx -\frac{\dot{R}}{1 + R} \tilde{T}_1 - \frac{k}{1 + R} \frac{1}{a} \Phi - \frac{k}{a} \Phi A,
\]

(152)

\[
\delta T \approx -H \Phi_A - \dot{\Phi}_H + \frac{k}{a} \frac{1}{3} \tilde{T}_1.
\]

(153)

Upon ignoring the expansion coupled term (i.e, the small scale approximation) and including the curvature fluctuations, since this makes the resulting equations applicable up to the Jeans length (above which the matter would not be gravitationally bound), and substituting the second equation (153) into (152) we obtain the well-known equation describing the acoustic oscillations in the radiation [39]:

\[
\delta T'' + \frac{R'}{1 + R} \delta T' + k^2 c^2 S \delta T \approx -\Phi'' \frac{H}{a} - \frac{R'}{1 + R} \Phi'_H - \frac{k^2}{3} \Phi A.
\]

(154)

Here we have used conformal time (since we are now working in the conformal Newtonian frame).

6.2.5. The Dispersion Relations and Photon Damping Scale

In this section we derive the dispersion relations for small scale anisotropies, where the focus is once again on developing generic covariant equations in parallel to the usual gauge invariant treatments [47, 51, 41]. To this end, we begin by iterating the baryon velocity equation in much the same manner as we iterated the integrated Boltzmann equations for the radiation.

We begin with the the baryon relative velocity equation (131) which is once again inverted in order for it to take the form in (133). This is then turned into the basis of an iteration scheme in terms of the scattering time:

\[
v_{(n)}^a \approx \tau_a - \frac{R}{k} \left[ \dot{v}_{(n-1)}^a + H v_{(n-1)}^a + A^a \right].
\]

(155)

Using this equation and the zero-th order tight-coupling approximation, \(v_{(0)}^a \approx \tau_a\), the covariant second order baryon velocity equation is found:

\[
v_{\alpha}^B \approx \tau_a - \frac{R}{k} \left[ \dot{v}_{\alpha}^B + H v_{\alpha}^B + A_a \right] + \frac{R^2}{k^2} \left[ \ddot{v}_{\alpha}^B + (\dot{H} + H^2) v_{\alpha}^B + \dot{A}_a + H A_a + 2H \dot{v}_{\alpha}^B \right],
\]

(156)

where we have dropped terms of \(O(\tilde{\kappa}^{-3})\). We now consider the small scale version of this equation, by ignoring terms scaled by the Hubble parameter \(H\) and effects due to the gravitational potentials (see (66)):

\[
v_{\alpha}^B \approx \tau_a - \frac{R}{k} \left[ \dot{v}_{\alpha}^B + A_a \right] + \frac{R^2}{k^2} \ddot{v}_{\alpha}.
\]

(157)

Expanding the transport equation for the second order radiation quadrupole (122) to first order and ignoring the shear contribution which is negligible in the almost-Friedmann-Lemaître small scale limit, we obtain

\[
\tau^{ab} \approx -t_v D^{(a)}_{\nu B} \approx -\tilde{\kappa}^{-1} D^{(a)}_{\nu B},
\]

(158)

\(^{25}\)We use, as before, \(D_{\alpha} \ln T \approx \frac{2}{3} \delta T Q_a\), \(\tau_a \approx \tilde{T}_a Q_a\), \(D_{\alpha} \Phi_A = \frac{1}{a} \Phi_A Q_a\) and \(D_{\alpha} \Phi_H \approx \frac{1}{a} \Phi H Q_a\).
where again we retain only first order terms. We now substitute (157) and (158) into the radiation dipole evolution equation (129) to find:

\[-\dot{\tau}_a = (A_a + D_a \ln T) - R(\dot{\tau}_a + A_a) + R^2 \dot{\kappa}^{-1} \dot{\tau}_a - \frac{2}{5} \dot{\kappa}^{-1} D^c D_a \tau_c ,\]  

(159)

which can be compared to the mode equation A-11 in Hu and Sugiyama [41]). This is the key equation from which we will now proceed to recover the dispersion relations and hence standard damping scale results. The key-point here is that the diffusion damping is second order in the scattering time, while the acoustic oscillations are first order.

We now take a covariant mode expansion of the necessary quantities: \(\tau_a = \tau_1 Q_a, A_a = A_1 Q_a\) and \(D_a \ln T = (\frac{k}{a} \delta T) Q_a,\) together with the well known result (see Paper I [18] for the general case): \(D^b Q_{ab} = -\frac{2}{3}(ak)^{-1}(-k^2+3K)Q_a.\) We consider only the flat case here, so we set \(K = 0.\)

Putting these all into the second order radiation dipole equation (159) we obtain:

\[-\dot{\tau}_1 \approx (A + (\frac{k}{a} \delta T)) + R(\dot{\tau}_1 + A) + R^2 \dot{\kappa}^{-1} \dot{\tau}_1 + \frac{4}{15} \dot{\kappa}^{-1} \frac{k^2}{a^2} \tau_1 .\]  

(160)

Now we use the WKB approximation:

\[\tau_1 \propto \exp i \int (\omega/a) dt ,\]  

(161)

and drop terms scaled by \(H\) and \(a \approx R.\) This gives

\[-i(1 + R) \frac{1}{a} \tau_1 \approx (\frac{k}{a} \delta T) + (1 + R)A - R^2 \dot{\kappa}^{-1} \frac{\omega^2}{a^2} \tau_1 + \frac{4}{15} \dot{\kappa}^{-1} \frac{k^2}{a^2} \tau_1 .\]  

(162)

Now, in order to deal with the terms arising from \(A_a\) and \(D_a \ln T\) we consider the covariant radiation monopole perturbation equation (15):

\[(D_a \ln T) + \frac{1}{3} D_a \Theta + H(D_a \ln T + A_a) \simeq -\frac{1}{3} D_a(D_c \tau^c) .\]  

(163)

We find, on taking a mode expansion and applying the WKB approximation again, dropping terms scaled by \(H\) and the expansion gradient (the small scale approximation described in section 2.5), that

\[(\frac{k}{a} \delta T) \approx \frac{1}{3} \frac{k^2}{a^2} (-i\omega) \tau_1 .\]  

(164)

Upon substituting (164) into (162), factoring out the dipole coefficient \(\tau_1\), writing the differential optical in terms of conformal time: \(\dot{\kappa}^{-1} = (ak')^{-1}\) and multiplying through by \(i\omega a\) we get:

\[\omega^2 (1 + R) + R^2 \dot{\kappa}^{-1} \omega^2 (i\omega) - \frac{4}{15} \dot{\kappa}^{-1} k^2 (i\omega) \approx \frac{4}{3} k^2 .\]  

(165)

On re-arranging terms we finally obtain

\[\omega^2 \approx \frac{R^2}{3(1+R)} + k^2 (i\omega k')^{-1} \left( \frac{R^2}{3(1+R)} + \frac{4}{15} \right) .\]  

(166)

Splitting \(\omega\) up into its natural frequency \(\omega_0\) and the diffusion damping term \(\gamma\) and then solving the quadratic for the frequency \(\omega\) we obtain

\[\omega \approx \pm \omega_0 + i\gamma ,\]  

(167)

where

\[\omega_0 \approx \frac{k}{\sqrt{3(1+R)}} \approx c_s k, \quad \text{and} \quad \gamma \approx k^2 \left( \frac{k^{-1}}{6} \right) \left[ \frac{R^2 + \frac{4}{15}(1+R)}{(1+R)^2} \right] \approx k^2 / k_D^2 .\]  

(168)
Here $k_D$ is the diffusion damping scale and $c_s$ the baryotropic speed of sound in the matter. We get oscillations when $k < k_D$ (see the next section), and diffusion damping when $k > k_D$, leading to a damping envelope.

The above covariant result is equivalent to the analytic small scale approximation scheme of [39, 40, 41], who have demonstrated its robustness and closeness to more precise numerical studies\footnote{For numerical integrations of the covariant scalar temperature anisotropy equations we would refer the reader to [11].}

This derivation can be easily corrected to include the effect of anisotropic scattering (which breaks the tight-coupling approximation) and polarization. This is done by correcting the scattering terms in the calculation of the damping scale. We follow the approach of Kaiser [47, 39, 40, 41] and include the correction $f_2$ to find the modified damping factor:

$$
\frac{18}{25} = k^2 (k^2)^{-1} f_2 (1 + R)^{-1} R^2 (1 + R)^{-1} (1 + R)^{-1} \approx \frac{k^2}{k^*_D},
$$

where, first, for the anisotropic effect, $f_2 = 9/10$, and second, to compensate for the polarization, $f_2 = 3/4$. Hence we get the diffusion damping envelope for $\tau_1$ when $k > k_D$.

### 6.2.6. The Temperature Oscillations

To examine the solution when $k < k_D$, we take a mode expansion, using the $Q_{A,\ell}$'s defined in Paper I [18]:

$$
a(D_c \ln T)^k = k(\delta T)Q_c \Rightarrow a(D_cD_a \ln T) = (-\frac{k^3}{a})(\delta T)Q_c, \tag{170}
$$

where the driving term is written in terms of a generic potential $\Phi_F$ defined by

$$
-\frac{a^3}{3}D_c(D^2\Phi_F) \approx -a^2(D_c\Theta)' + a^2D_c(D^\alpha A_\alpha) \text{ and } a^3D_c(D^2\Phi_F) \approx \frac{k^3}{3}\Phi_F Q_c. \tag{171}
$$

From (145) we find the oscillator equation (once again working in the Newtonian frame cf. [40] eqn. (B3) and equation (154)).

The solution at first order must be convolved with the damping envelope, found from the dispersion relations, in order to include the damping cut-off. We have waited until the form of this damping is known, so that it can be easily included by simply replacing the natural oscillator frequency, $\omega_0$, with $\omega = \omega_0 + i\gamma$, which now includes the damping. We use equation (145) in a manifestly gauge invariant form, to find upon mode expanding

$$
(\delta T)'' + k^2 c_s^2 (\delta T) \approx -\frac{k^2}{3}\Phi_F, \tag{172}
$$

where the sound speed $c_s$ is given by $[3(1 + R)]^{-\frac{1}{2}}$. This gives the well known solution (see for example [39, 44])

$$
(\delta T)(\eta) \approx [(\delta T)(0) + (1 + R)\Phi_F] \cos(kr_s) - (1 + R)\Phi_F, \tag{173}
$$

where the sound horizon scale is given by $r_s = \int c_s d\eta \approx c_s \eta$. This describes the source term for the acoustic oscillations with the isocurvature term dropped\footnote{The other part of this solution comes from $\sin(kr_s)$; this is the isocurvature part ($\delta T'(0) \neq 0$), giving the full solution:

$$
(\delta T)(\eta) = [(kr_s)^{-1}(\delta T)'(0)] \sin(kr_s) + [(\delta T)(0) + (1 + R)\Phi_F] \cos(kr_s) - (1 + R)\Phi_F.
$$

In this paper we consider adiabatic perturbations only so $(\delta T)'(0) = 0$.}.
gives the Sachs-Wolfe effect, due to the potential; the other terms give the adiabatic acoustic oscillations.

By putting this back into the momentum flux equation (136) in mode form we obtain
\[ \tau'_1(\eta, k) \simeq -\frac{1}{1 + R}(k \delta T) - ak\Phi_F , \]
and using the temperature anisotropy mode expansion that results when (126) holds, the Doppler contribution to the temperature anisotropies at decoupling can be found. These are given by
\[ \tau_1(\eta, k) \approx (-k^{-1})3[(a \ln T)(0) + (1 + R)\Phi_u]c_4 \sin(kr_\lambda) . \]

7. TEMPERATURE ANISOTROPIES FROM INTEGRAL SOLUTIONS

In this section we derive the explicit form of the temperature anisotropies, using the general \( u^a \)-frame integral solutions and the tight-coupling approximation solutions.

Before we do this, there are two important points that need to be discussed. Firstly, where does the high-\( \ell \) cut-off come from? The natural frequency of the system is \( \omega_0 \) and is set by the sound speed in the tight-coupled system to first order in Thompson scattering time – the oscillator equation. The diffusion damping is a second order effect whose correction is found by deriving the dispersion relations at second order in the Thompson expansion for the baryon and photon momentum equations and using the WKB approximation to find the new oscillator frequency to be: \( \omega \approx \omega_0 + i\gamma \). The assumption that the anisotropies are sourced by these oscillations in the radiation is the key to the physics; the initial conditions before free-streaming are \( \tau_{A\ell} \approx 0 \) for \( \ell > 2 \) (where the anisotropic correction at \( \ell = 2 \) is included as a correction to the damping scale). The free-streaming radiation transfer function (the spherical Bessel function in k-space) is then convolved with the initial power sourced by the oscillations in the average temperature and the dipole (essentially a cosine and sine function in k-space respectively).

As free-streaming continues power is shifted up into the higher \( \ell \)'s, as the radiation transfer functions’ maximum is near \( \ell \propto k\Delta \eta \) (where \( \Delta \eta \) is the elapsed time since last scattering). This maximum moves into higher \( \ell \) for longer times to give one a sense for how the power is distributed through the \( \ell \pm 1 \) moment couplings in the almost-Friedmann-Lemaître integrated Boltzmann equations. Obviously at high-\( \ell \) (which corresponds to high-\( k \)) the amount of power surviving the \((k/k_D)^2\) damping will be very small, and hence as time progresses the peak in the transfer functions drops off for higher-k, thus giving the high-\( \ell \) cut-off.

Given that in free-streaming there is no diffusion damping to cut-off the high-\( k \) power, the truncation of the hierarchy is only consistent and meaningful if the significant anisotropy signal is sourced from the initial conditions at low-\( \ell \) near to tight-coupling.

Secondly, why do we find angular variations in the temperature anisotropies that are entirely due to oscillations in the dipole and monopole of the radiation – particularly given that in relativistic kinetic theory one has time-like integrations? 28

This is important for non-linear extensions of the almost-Friedmann-Lemaître model [59], since care should be taken when treating ensemble averaged quantities integrated down the null-cone as being generally equivalent to those integrated down a single time-like world-line. The issue will become even more complex once the Gaussian averaging necessary for the construction of the angular correlation functions is relaxed. It is this, along with the weak Copernican principle that makes possible the extension of the data here and now, as integrated in a little box down a time-like world-line to last scattering, to global statements.

28 Only in the almost-Friedmann-Lemaître sub-case can one think of the averaged relationship between the null-projection
Now we return to the issue of sourcing the temperature anisotropies from oscillations in the dipole and monopole. The integral solutions give the full almost-Friedmann-Lemaître solution to the radiation anisotropies given the appropriate initial data. Generically it is the transfer of power from low-$\ell$ initial data up into high-$\ell$, with the accompanying reverse transfer of power; the $\ell \pm 2, \ell \pm 1$ transfer of power where there is a wall at $\ell = 0$ but none at high-$\ell$. There is strictly no $\ell_{\text{max}}$ unless the geometry is exactly Robertson-Walker [30]. The tight-coupling approximation gives the monopole and dipole at the $\ell = 0$ initial wall. Diffusion damping in the initial power near last scattering as well as additional damping through slow decoupling cuts off the transfer of power. This power is sourced by those initial conditions and therefore give the temperature anisotropies in terms of the dipole and monopole alone. Additional integrated effects will only modify the primary projection, effectively leaving the cut-off unchanged.

7.1. Temperature Anisotropies

The integral solution gives the projection of these conditions at last scattering onto the current sky. These results are covariant.

The diffusion damping can be found from the dispersion relations arising from the coupled baryon-photon equations using the WKB method in tight-coupling which was described in section 6.2. We can now construct the primary source temperature anisotropies, by first considering the acoustic and Doppler contributions in the free-streaming projection (94). Our solutions for $B_0$ and $B_1$ are first order in scattering time expansion hence to include the second order diffusion cut off, we multiply the solution through by $\exp \gamma$ (or $\exp \gamma^*$ if we want to include the corrections for the anisotropic and polarization effects).

We can re-write $\gamma$ or $\gamma^*$ in terms of the damping scale, $k_D$ (cf [40] eqn (A7)-(A8)).

7.1.1. Sources of Temperature Anisotropies

In the manifestly gauge invariant integral solution of the temperature anisotropies in almost-Friedmann-Lemaître spacetime, using the slow decoupling approximation with scalar perturbations (98,102), there is additional complexity of having to deal with shear terms. We therefore choose the Newtonian frame in which the shear contribution does not appear\(^\text{29}\). We are then able to recover the canonical scalar treatment [81, 75, 35, 39, 40]. This can be written out in terms of the scalar perturbation potentials $\Phi_A$ and $\Phi_H$, the temperature perturbation, $\delta T$, the radiation dipole $\tau_1$ (through slow decoupling) and the baryon relative velocity, $v_B$, all within the covariant approach (including Doppler contributions (111)):

$$\frac{\beta \tau_1(\eta_0)}{2(2\ell + 1)} = S_P(k, \eta_0) j_\ell(k \Delta \eta_0) + \int^{\eta_0}_{\eta_0} d\eta (S_{DISW} + S_{ISW}) j_\ell(k \Delta \eta) .$$

Here the source terms are given by:

$$S_P(\eta, k) = D(\eta_0, k) \left(\delta T + \Phi_A + \kappa' v_B\right),$$

$$S_{DISW}(\eta, k) = V e^{-(k/k_D)^2} \left[\frac{1}{3} \kappa \tau_1 + (\kappa' \tilde{v}_B + \kappa'' \tilde{v}_B)\right],$$

$$S_{ISW}(\eta, k) = V e^{-(k/k_D)^2} \left[\Phi_A' - \Phi_H' \right](k, \eta) - aH[\delta T + 3\Phi_A](k, \eta).$$

In the first line, on the RHS, we have the Sachs-Wolfe and acoustic sourced projection effects which are the primary sources. The next term describe secondary Doppler effects during slow

\(^{29}\)Incidently this would also remove any problems we may have with the introduction of high-$\ell$ truncation as discussed in Appendix E
decoupling. The last term models the integrated Sachs-Wolfe (ISW), the late-ISW and early-ISW effects respectively. We have used $\Delta \eta_a = \eta_0 - \eta_*$ (the conformal time difference between the surface of last scattering and the time of reception, here and now) and $\Delta \eta = \eta_0 - \eta$.

In this way we have recovered the solution from the exact equations in a systematic manner using the 1+3 CGI formalism rather than recovering the solution from a perturbation theory about a foliation of Robertson-Walker surfaces of homogeneity. Note that there is no monopole temperature anisotropy in the CGI approach while there is one in the canonical treatment.

The angular correlation functions, $C_{\ell}$, for the general small temperature anisotropy case have been derived in terms of a superposition of homogeneous and isotropic Gaussian random fields with respect to the temperature anisotropy multipoles to find the multipole mean-squares, $\langle \tau_{\alpha\beta}\tau^{\alpha\beta}\rangle$, in terms of the an ensemble average (see Paper I [18]). The relationship between the multipole mean-squares (in the almost-Friedmann-Lemaître case) and the mode coefficient mean-squares $|\tau_{\ell}|^2$ in terms of the Robertson-Walker mode functions $G_{\ell}(Q)$ are also given in Paper I [18]. We do not discuss these further here. It must be emphasized that the multipole mean-squares are given for general geometries, while the mode means squares are only for almost-Robertson-Walker geometries. This is why the application of the non-linear extension is possible; where the corrections to the almost-Friedmann-Lemaître standard model are calculated using the multipole formalism of [59].

7.1.2. Sachs-Wolfe Effect and the Acoustic Source

In the Newtonian frame (177) we find that the Sachs-Wolfe effect arises from a combination of the $\delta T(0,k)$ and $\Phi_A(0,k)$ along with $r_s$ $\sim$ $c_s\eta$ and ignoring the time evolution of the potential (for the equivalent canonical version see [39]) we find

$$\delta \tilde{T}(\eta, k) + \Phi_A(\eta, k) \approx [\delta \tilde{T}(0, k) + (1 + R)\Phi_A(0, k)] \cos(kr_s) - R\Phi_A(\eta, k). \quad (180)$$

Upon taking the matter dominated limit ($R \approx O(\epsilon^2)$) and then using the adiabatic assumption in tight-coupling $\delta T(0,k) \sim \frac{1}{2} \Delta(0,k) \sim -\frac{2}{3} \Phi_A(0,k)$ (adiabatic flat CDM model) and finally taking $r_s \sim 0$ (by using that $kn_s \ll 0$) we recover the usual results as in [39, 43]

$$[\delta \tilde{T} + \Phi_A](\eta, k) \approx +\frac{1}{3} \Phi_A(0,k)(1 + 3R) \cos(kr_s) + R\Phi_A(0,k) \sim \frac{1}{4} \Phi_A(0,k). \quad (181)$$

Here we have used that $\Phi_A(\eta, k) \approx \Phi_A(0,k)$ from the Einstein de-Sitter result that the potential is constant if we drop the decaying mode. The physics of the Sachs-Wolfe effect as opposed to the acoustic oscillations is then quite clear. There are no oscillations just an imprint due to an acceleration potential. Although the adiabatic assumption is invariant to order $O(\epsilon)$, the relationship between the Electric part of the Weyl tensor and the perturbations are not – so we will always use the matter perturbation in the total frame where it can be easily related to the Newtonian potential and the temperature perturbation in the Newtonian frame where the oscillator equation takes on a useful form. In a similar manner we find the explicit form of the radiation dipole and acoustic oscillations in the slow decoupling era:

$$\delta \tilde{T}(k, \eta) \approx \frac{1}{4} \Phi_A(0,k)(1 + 3R) \cos(kc_s\eta) + (1 + R)\Phi_A(0,k),$$

$$\tau_1(k, \eta) \approx \Phi_A(0,k)(1 + 3R)c_s \sin(kc_s\eta). \quad (182)$$

7.1.3. Weak-coupling for the Small Scale Solution

Here we briefly consider the integrated Sachs-Wolfe effect in connection with the weak-coupling approximation. The idea is that the anisotropies fall with $\ell$ more rapidly than a
simple projection would imply. What one has in mind is the situation where the anisotropy contributes across many wavelengths of the fluctuation allowing cancellations on small scales; the secondary sources, in particular the term \((\Phi'_A - \Phi'_H)\), varies slowly on small scales [43]. Specifically we consider the situation where

\[
\int_{\eta_*}^{\eta_0} d\eta e^{-(k/k_D)^2} \{ (\Phi'_A - \Phi'_H) \} j_\ell(k\Delta\eta) \approx \sqrt{\pi/2k} (\Phi'_A - \Phi'_H)(\Delta\eta_*) D(k, \eta_0),
\]

(183)

where we used \(\int_0^\infty j_\ell(k\Delta\eta)d\eta = [\sqrt{\pi/2k}] \Gamma((\ell+1)/2)/\Gamma((\ell+2)/2)\). The weak-coupling solution is implied by the assumption that:

\[
(\ddot{\Phi}_H - \ddot{\Phi}_A) \ll k(\dot{\Phi}_H - \dot{\Phi}_A).
\]

(184)

This is nothing more than a useful approximation allowing the direct construction of analytic solutions. Some care should be taken when using the weak-coupling approximation when trying to estimate the early-ISW (the free-streaming analogue of the acoustic driving effect) and late-ISW (due to non-vanishing curvature or cosmological constant, which will dominate the expansion rate at latter time) effects. Ideally one should evaluate the slowly-varying function, which has been taken out of the integral, at the \(\ell\)-th peak; \(\eta_\ell = \eta_0 - (\ell + \frac{1}{2})/k\) rather than at \(\Delta\eta_* = \eta_0 - \eta_*\).

In the case of the early-ISW effect, since it satisfies neither the tight-coupling nor weak-coupling criteria, our approximations schemes here are not entirely appropriate; its decay time and wavelength are comparable [39, 40, 43]. However we can use the weak-coupling in the case of the late-ISW effect because cancellation effects lead to damping on small scales. The temperature perturbation decays, and hence the potential decays on the order of the expansion time near the end of the matter dominated era. The photons will free-stream across many wavelengths of the perturbations below the Hubble scale, leading to cancellation and damping effects.

The important point about the late-ISW effect is that there will be an imprint due to the exit from the matter dominated era into a \(\Lambda\) or curvature dominated one. In the context of the equations here, we can consider \(\Lambda\)-dominated effects by investigating the evolution equations for the potentials in a \(\Lambda\)-dominated case.

In this case the effective expansion changes, given here for late-times (well after matter-radiation equality):

\[
H^2 = a^{-3}\Omega_0 H_0^2 + \frac{1}{3}\Lambda
\]

(185)

for the \(\Omega_0 + \Omega_\Lambda = +1\) model. This has however been dealt with in depth in the literature (see for example [43, 40]).

7.1.4. The Mode Coefficients

The angular correlation functions, measured here and now \(\langle x_\ell^* \rangle\) can then be computed using the the mode coefficients from Paper I [18]:

\[
C_\ell = \frac{2}{\pi} \frac{\beta_\ell^2}{(2\ell + 1)^2} \int_0^\infty \frac{dk}{k} k^3 |\tau_\ell(k, \eta_0)|^2.
\]

(186)

The final step is to construct the temperature anisotropy solutions in terms of the matter power spectra. We discuss this in the next section. At linear order, there is no important difference in our solutions from those found in the canonical treatment [39, 40], however the advantage is that we can easily relate our formulation of the temperature anisotropies to the
mean-squares of the multipoles ($\tau_{A\ell}$) and hence to the mean-square of generalized temperature anisotropies, ($\Pi_{A\ell}$) [59]. This was not attainable in the canonical treatment.

7.1.5. The Multipole Coefficients

The mean-square of the multipole moments are related to the almost-Friedmann-Lemaître mode coefficients by (see Paper I [18])

$$\langle \tau_{A\ell}^2 \rangle \approx \frac{1}{2\pi^2} \beta_\ell \int k^2 dk |\tau_\ell(k, \eta_0)|^2,$$

(187)

giving the angular correlation function (see Paper I [18]):

$$C_\ell = \Delta\ell (2\ell + 1)^{-1} \langle \tau_{A\ell}^2 \rangle.$$

(188)

This relates the matter fluctuation amplitude at last scattering to the present day via the Newtonian like potential, which in turn may be related to the matter power spectrum directly. A plethora of numerical studies of Doppler peak features exist in the gauge-invariant literature (see for example [42] and [74]).

We will now summarize the standard picture of acoustic peak characteristics before discussing the matter power spectrum.

7.2. Some ‘Acoustic peak’ Characteristics

The key features of the standard model of cosmic background radiation primary sourced Doppler peaks are listed below [39, 40, 43, 44, 79, 46]. The standard model of acoustic peak formation has been given in its analytic form above.

1. The $j$th peak positions (as given in the flat adiabatic case) is given by

$$\ell_j \approx k_J |r_\theta(\eta)| = \frac{r_\theta}{r_s} j\pi,$$

(189)

where $r_\theta$ is the comoving angular diameter and $r_s$ is the sound-horizon near decoupling. Notice that $j\ell(k\Delta\eta)$ peaks near $\ell \sim k\Delta\eta$, where free-streaming projects this physical scale onto the angular scale $\theta\Delta\eta$ on the current sky. The first peak is dependent on the driving force which is model independent. It provides a way of fixing the angular diameter distance – when using the almost-Friedmann-Lemaître assumptions.

2. The acoustic relative peak spacings are given by

$$k_{j+1} - k_j = k_A \approx \frac{\pi}{r_s} \iff \Delta\ell \sim k_A r_\theta.$$

(190)

The peak spacing is fixed by the natural frequency of the oscillator: $\omega = kc_s$, which is independent of the driving force. The factor $c_s$ is the photon-baryon sound speed.

3. The peak ratios arise from the angular power spectrum ratio $C_{jk} = C_j/C_k$ found from $C_\ell$ in terms of $|\tau_\ell|^2$ or $\langle \tau_{A\ell}^2 \rangle$.

4. The damping tail provides yet another angular diameter distance test of curvature [43] via the damping scale $k_D$. The peak spacings $\Delta\ell$ and the damping tail location $\ell_D$ depend only on the background quantities, they are robust to model changes, assuming that secondary effects do not overwhelm the signal. The diffusion scale near decoupling is the angular scale of the wavenumber $k_D \sim \sqrt{k/\Delta\eta} \sim \sqrt{a/\Delta\eta a} \sim \Delta\eta^{-1}$. Given that the damping tail of the acoustic oscillations takes on the form $e^{-(k/k_D)^2}$, $k_D$ can be found from (as derived previously):

$$k_D^{-2} \approx \frac{1}{6} \int_{\eta_s}^{\eta_0} d\eta ' \frac{1}{k'} R^2 + \frac{4}{7} f_2^{-1}(1 + R) \frac{(1 + R)}{(1 + R)^2},$$

(191)
for no anisotropic stress [43, 47]. This can be used to find that the damping tail location is $\ell_D = k_D r_\theta$.

5. The damping tail position to peak scale, $\ell_D/\ell_A$, is a good measure of the number of peaks and gives an indication of the delay in recombination independently of the area distance.

8. THE MATTER POWER SPECTRUM

In this section we related the temperature anisotropies in the case of the Sachs-Wolfe effect to the matter power spectrum using the 1+3 CGI formalism. We also deal with basics of the normalization of the cosmic background radiation angular power spectrum to the matter power spectrum.

Up to this point everything we have derived has been found covariantly from general relativity and from relativistic kinetic theory, however on small scales we do not have a covariant analytic derivation of the transfer function, although it is well known for large scales, so this is the only gap in our treatment \(^3\). The aim here is to try to predict the matter distribution today from the cosmic background radiation spectrum. From the cosmic background radiation spectrum one finds the matter power spectrum (as fixed by the cosmic background radiation data today) from which the current distribution of matter is found (in terms of $\sigma_8$).

Briefly, in the literature there have been two methods for normalizing the cosmic background radiation data in relation to COBE: firstly the $\sigma(10^0)$ normalization where the r.m.s temperature fluctuation was averaged over a $10^0$ FWHM beam, and secondly the $Q_{rms-PS}$ normalization which uses the best fitting amplitude for a $n = 1$ Harrison-Zel’dovich (HZ) spectrum quoted for the quadrupole, $\langle Q \rangle$. The first method had the advantage of being a model independent way of fitting the data - it is observationally determined, however, because of this it does not provide the most accurate normalization for a specific model and care must be taken in order to properly deal with cosmic variance. The second approach is model dependent and works well only for the HZ spectrum [20, 9].

Improved, more general normalization schemes have now been developed and there currently seem to be two favoured ones, both having variations of the HZ spectrum in mind. It is these two schemes we now briefly consider here. We emphasis the so called CDM-models with $\Omega_0 = 1$. These models have a baryon fraction $\Omega_B$ with the rest of the matter being made up of massive Cold Dark Matter (CDM). Of these the most easily dealt with is the so called standard-CDM model, where initial fluctuations are assumed to have a Gaussian distribution and have adiabatic, scalar density fluctuations with a HZ spectrum on large scales:

$$P(k) \propto k^{n-1}, \quad n = 1, \quad H_0 = 50 \text{km}^{-1}\text{Mpc}^{-1}, \quad \Omega_B = 0.05.$$ (192)

There are three other popular models: the Adiabatic CDM model of Peebles, which is a version of standard-CDM; the ICDM model (which is an isocurvature version of the Adiabatic CDM model; and lastly the CMD model which is the large cosmological constant version of standard-CDM. More recently Hu has introduce a Generalized Dark Matter model (GDM) [37].

\(^3\) There is still some hope that the almost Friedmann-Lemaître Quasi-Newtonian models, will be useful in this regard [31] providing us a covariant derivation of the Harrison-Zel’dovich spectrum.

\(^31\) $Q_{rms-PS} \approx \frac{5}{4\pi} C_2$ and $C_\ell^{SW} \approx \frac{6}{\pi(\ell+1)} C_2$. 


8.1. The Power Spectrum

The Power Spectrum $\mathcal{P}(k)$ is fully specified by a shape and normalization (see Appendix J for additional details and references). The normalization is fixed by the amplitude of the temperature fluctuations in the cosmic background radiation\textsuperscript{32}.

There are two normalization scales, large and small (see Appendix J). On small scales the normalization is expressed in terms of $\sigma_8$, the ratio of the r.m.s mass fluctuations to the galaxy number fluctuations - both averaged over randomly located spheres of radius $8h^{-1}$ Mpc; this is the variance of the density field in these spheres.

On large scales the power spectrum is found by fixing the shape function [19] which is a measure of the horizon scale at matter radiation equality. We look at these now for the case of a flat matter dominated almost-Friedmann-Lemaître model (the open case is considered in Appendix K).

In the 1+3 CGI approach the emphasis is on the quantity $aD_a \ln \rho_\delta \approx a\delta_\delta Q_a$. Here $\delta_\delta = a\delta_\delta(0, k)$ and $\langle \delta_\delta(k, 0)\delta_\delta(k', 0) \rangle = (2\pi)^3 P(k)\delta(k-k')/k^2$ for $|\delta_\delta(k, 0)|^2 = \mathcal{P}(k)$. It becomes preferable in the context here to use the variable $\Delta$: $\frac{\pi}{2} \Delta(k, t) = \delta_\delta(k, t)$\textsuperscript{33}. We will, however, be using the power spectrum for the gauge invariant acceleration potential because we consider the situation of matter dominated adiabatic perturbations. In the general situation it is best to use $\bar{\Phi}$. Note, in the matter dominated situation we can use that $\Phi = 0$ and write everything in terms of either $\Phi_H$ or $\Phi_A$.

The relationship between the power spectrum of the acceleration potential and the power spectrum of the matter perturbation is

$$|\Phi_A(k, 0)|^2 = P_{\Phi A}(k), \quad \text{and} \quad |\Delta(k, 0)|^2 = P(k),$$

(193)

where these are then related to each other in the total frame via the constraints for the electric part of the Weyl tensor for the flat case:

$$P_{\Phi A}(k, 0) = \left(\frac{3}{2}H_0^2\Omega_0 \frac{D}{a} \right)^2 k^{-4} P(k).$$

(194)

Here $\bar{\Phi} = -\frac{k^2}{\omega^2}\Phi_A$ and as noted before:

$$(\bar{\Phi} \rho_{\delta A}^{-1})(k, t) \approx -(3H_0^2\Omega_0)^{-1} D(t) [k^2\Phi_A(k, 0)].$$

(195)

and $D \approx a \approx \eta^2$ for a flat almost-Friedmann-Lemaître dust model. We will also use $a_0 = +1$.

8.2. Relating the Power Spectrum to Temperature Anisotropies

We now combine the CGI Sachs-Wolfe effect (due to potential fluctuations) and the matter power spectrum to express the anisotropies in terms of the matter power spectrum near decoupling.

8.2.1. Flat Matter Dominated Almost-Friedmann-Lemaître

We first consider the flat almost-Friedmann-Lemaître model. Using (193) and (181, 177 and 176) (that is with $K = 0$) we obtain

$$\frac{\tau^w_{\ell} (\eta_R) \beta}{(2\ell + 1)} \approx +\frac{4}{3} \Phi_A(k, \eta) j_\ell(k\Delta \eta) ,$$

(196)

Here we find the angular power spectrum from (186), (196) and (194):

$$C^w_\ell = \frac{2}{\pi} \left(\frac{H_0^2\Omega_0 \frac{D}{a}}{\pi} \right)^2 \frac{1}{4} \frac{1}{k^2} P(k) j_\ell^2(k\Delta \eta) .$$

(197)

\textsuperscript{32}It seems that models normalized to COBE which are fixed in both the amplitude at small scales and by the shape predicted by standard-CMD, are inconsistent with observations [49, 9].

\textsuperscript{33} $\Delta(k)^2 \approx \frac{4}{2\pi} |\delta_\delta(k)|^2 = \frac{8}{2\pi} A k^{n+1}$
where \( P(k) \) is given by either (J.1) or (J.3). Now if use that \( (\Omega_0 D_*/a_*)^2 \approx \Omega_0^{1.54} \) we find the standard result \([20]\):

\[
C^{\text{SW}}_\ell = \frac{1}{2\pi} H_0^4 \Omega_0^{1.54} \int \frac{dk}{k^2} P(k) j_\ell^2(k \Delta \eta_*) .
\]  

(198)

The next step is to normalize the matter power spectrum to any given structure formation theory on both small and large scales, this is done in the next section in the case of the standard-CDM model.

8.3. Setting the CDM Normalization

From the previous section we can now relate the angular power spectrum for the matter dominated flat \((K = 0)\) almost-Friedmann-Lemaître model to the current matter power spectrum on small and large scales using the two different normalizations to the standard-CDM model. In what follows we will use the useful result:

\[
\int_0^\infty dz \frac{z^m j_\ell^2(z)}{z^{m+2}} = \frac{\pi}{2^{m+2}} \left( \frac{m}{m/2} \right)! \left( \frac{\ell + m}{2} + \frac{1}{2} \right)! .
\]  

(199)

8.3.1. Large Scales

From (198) and \( P(k) = A k^{n-1} \) (J.1) we obtain

\[
C_\ell = \frac{1}{2\pi} A H_0^4 \Omega_0^{1.54} \int \frac{dk}{k^2} k^{n-1} j_\ell^2(k \chi) ,
\]  

(200)

and on using (199) for \( m = 2 \) \((n = 1)\) we find that

\[
C_\ell = \frac{A}{2} H_0^4 \Omega_0^{1.54} \frac{1}{(2\ell + 3)(2\ell + 1)(2\ell - 1)} .
\]  

(201)

Of more immediate interest is the angular correlation function for matter below the horizon scale near decoupling, as this is what is used to normalize the angular correlation function.

8.3.2. Small Scales

For small scales one can use the \( \sigma_8 \) normalization via (198) and \( P(k) = B T^2(k) k^n \) (J.3) and the parametrized transfer function (J.5). This gives

\[
C_\ell \simeq \frac{1}{2\pi} H_0^4 \Omega_0^{1.54} \int \frac{dk}{k^2} B k^n T^2(k) j_\ell^2(k \Delta \eta_*) .
\]  

(202)

For example using the standard-CDM model with \( n = 1, h = 0.5, \Omega_B = 0.05 \) and \( \Gamma = 0.48 \) we can relate \( B \) and \( A \) to the scale of fluctuations near horizon crossing and the quadrupole:

\[
B = 2\pi^2 A = (6\pi^{2/5}) \langle Q \rangle / T_0^2 .
\]  

(203)

To normalize to \( \sigma_8 \) we can choose \( \sigma_8 \approx 1.3 \). This then allows us, in principle, to invert the angular correlation function to find the matter power spectrum once the initial spectrum is known.

We are however more concerned with normalizing the radiation angular correlation function to CDM on horizon scales near decoupling. In this regard we once again begin with the matter power spectrum \( (T(k) = 1) \) and on using (199) for \( m = 1 \) we find that:

\[
C_\ell^{\text{(CDM)}} = \left[ \frac{1}{2\pi} H_0^4 \Omega_0^{1.54} B \frac{\pi^2}{8} \right] \frac{1}{\ell(\ell + 1)} .
\]  

(204)
8.3.3. $C_\ell$ Normalized to Adiabatic CDM

Finally we normalize the angular correlation function $C_\ell$ (found from 102) to the potential fluctuations normalized to Adiabatic CDM (204) above:

$$D_\ell = \frac{C_\ell}{C_\ell^{(CDM)}} = \left[ \frac{1}{2\pi} H_0^4 \Omega_0^{1.54} B \frac{\pi \Omega}{8} \right]^{-1} \ell (\ell + 1) C_\ell.$$

(205)

The angular correlation function $C_\ell$ can be found from the primary (177) and secondary (the integrated part of 102) source that make up the total: $C_\ell = C_\ell^{(P)} + C_\ell^{(S)}$. This then allows one to remove that part of the angular correlation function arising from the standard Sachs-Wolfe potential fluctuations leaving a signal which is dominated by the photon primary and secondary sourced physics. The convention is to use $D_\ell$ rather than $C_\ell$ [46], so we have from (198 and 204) for the flat case:

$$D_\ell = \ell (\ell + 1) C_\ell = \int \frac{dk}{k^2} P^*(k) |j_\ell(k\chi)|^2 \text{ using } P^*(k) = \left( \frac{k}{k_s} \right)^{n_s + \alpha \ln(k/k_s)},$$

(206)

where $k_s$ is the normalization scale, $\alpha$ gives the deviation from the power law, and $n_s$ give the scalar power law index. The angular power per $\ln \ell$ is $(\ell (\ell + 1)/4\pi) C_\ell$.

9. CONCLUSIONS

In this paper we have carried out a covariant analytic time-like integration reproducing the well known primary effects generating the ‘acoustic’ peaks measured here and now for an almost-Friedmann-Lemaître universe with adiabatic scalar perturbations. We have also demonstrated how, in the CGI formalism, the angular correlation functions are constructed in terms of the matter power spectrum and normalized on large and small scales for standard-CDM, given appropriate approximations for the transfer functions.

As pointed out initially, the aim of this paper was to clarify the link between the standard gauge-invariant and CGI treatments of cosmic background radiation anisotropies and provide a strong basis from which to tackle non-linear and gravitational wave effects using CGI methods.

Some of the key outstanding issues are:

1. How does one deal with anisotropic scattering and anisotropic stresses before and during decoupling within the covariant approach, specifically in such a manner so that consistency is maintained when using general relativity, its covariant linearisations and relativistic kinetic theory.

2. The small anisotropy equations developed in [59] with the application to spacetimes with arbitrary anisotropy and inhomogeneity have yet to be properly investigated; these become applicable when one ignores the Copernican principle that underlies the almost-Ehlers-Geren-Sachs theorem, which in turn provides the theoretical basis for using almost-Friedmann-Lemaître spacetime dynamics. An investigation of their consequences on the cosmic background radiation may provide an alternative method of testing the Copernican principle other than the Sunyeav-Zel’dovich effect or via the use of polarization maps.

3. There is a need to find a working non-Gaussian treatment from which one can construct a generic characterization of observables here and now (other than the angular power spectrum alone (see Paper I [18]) and, second, finding an \textit{ab initio} covariant analysis of transfer functions extending the post-Newtonian treatments which use periodic boundary conditions.
In the next paper in the series, Paper III [15], we hope to establish, in a complete fashion, the relationship between the null-cone integrations (favoured in the literature) and the time-like integrations (found in the exact relativistic kinetic theory).

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APPENDIX A

Some Useful Almost-Friedmann-Lema
tre relations

Some useful covariant linearised differential identities are given below [60]:

\[ \text{curl } D_a \psi = -2 \dot{\psi} \omega_a, \quad (A.1) \]

\[ D^2 (D_a \psi) \approx D_a (D^2 \psi) + \frac{2}{3} (\rho - 3H^2) D_a \psi + 2 \dot{\psi} \text{curl } \omega_a, \quad (A.2) \]

\[ (D_a \psi) \approx D_a \dot{\psi} - HD_a \psi + \dot{\psi} A_a, \quad (A.3) \]

\[ (aD_a J^{Bm} \ell_A) \approx \frac{1}{3} (3H^2 - \rho) V_{[a} h_{b]c}, \quad (A.4) \]

\[ \left( \frac{D_a D_b V_c}{c} \right) \approx \frac{1}{3} (3H^2 - \rho) V_{[a} h_{b]c}, \quad (A.6) \]

\[ D_a \text{curl} V_a \approx -\frac{1}{2} \text{curl } (D^b S_{ab}), \quad (A.9) \]

\[ \text{curl curl } V_a \approx D_a \left( D^b V_b \right) - D^2 V_a + \frac{2}{3} (\rho - 3H^2) V_a, \quad (A.10) \]

\[ \text{curl curl } S_{ab} \approx \frac{3}{2} D_a D^c S_{bc} - D^2 S_{ab} + (\rho - 3H^2) S_{ab}, \quad (A.11) \]

\[ D^a D^b D_{(a} V_{b)} \approx \frac{2}{3} D^2 \text{div } V + (\rho - 3H^2) \text{div } V, \quad (A.12) \]

where the vectors and tensors are \( O(\epsilon) \) and \( S_{ab} = S_{(ab)} \).

APPENDIX B

Integrated Boltzmann Equation Relations

Here we repeat some useful results from [59]. The integrated Boltzmann equation is:

\[ \int_0^\infty L(f) E^2 dE = \int_0^\infty E^2 dE \left[ p^a \partial_a f - \Gamma_{bc}^a p^b p^c \frac{\partial f}{\partial p^a} \right] = \int_0^\infty E^2 dE. \quad (B.1) \]

Here \( f \) is the single particle distribution function, \( C[f] \) the scattering correction. The Liouville operator for the photons, \( L(f) = \frac{df}{dv} \) and \( d/dv = p^a \nabla_a \) is the null derivative. Hence we find

\[ L(f(E, x^i, e^a)) = \frac{df}{dv} = p^a \nabla_a f + \frac{dE}{dv} \frac{\partial f}{\partial E}, \quad (B.2) \]

as in [59], so that given the identity (see [59])

\[ \frac{dE}{dv} = -E^2 \left[ \frac{1}{3} \Theta + A_a e^a + \sigma_{ab} e^a e^b \right], \quad \text{and} \quad p^a = E(u^a + \epsilon^a), \quad (B.3) \]

we have covariant derivation of the almost-Friedmann-Lema
tre integrated Boltzmann equations.

APPENDIX C

Scattering Strength Expansion

The almost-Friedmann-Lema
tre integrated Boltzmann equation for Thompson scattering is

\[ B + \tau + e^a D_a \tau \approx t_c^{-1} [\epsilon^a e_a - \tau]. \quad (C.1) \]
This enables us to find
\[ \tau(x^i, e^a) = v^ae_a - t_c [B + \dot{\tau} + e^a D_a \tau] . \]

We now systematically approximate (C.2) in terms of the smallness parameter \( t_c \):
\[ \tau(n) \approx v^ae_a - t_c [B + \dot{\tau}(n-1) + e^a D_a \tau(n-1)] . \]

Up to second order the following anisotropies are recovered:
\[ \tau(0)(x^i, e^a) \approx v^ae_a , \]
\[ \tau(1)(x^i, e^a) \approx v^ae_a - t_c [B + \dot{\tau}^a(\dot{a}) + e^a e^b D_a v_b] , \]
\[ \tau(2)(x^i, e^a) \approx v^ae_a - t_c [B + \dot{\tau}^a(\dot{a}) + e^a e^b D_a v_b] + t_c^2 [B + e^a \ddot{v}_a + e^a e^b D_a v_b] . \]

In order to carry out the solid angle integration over the sky (116), the following results will be useful. From the normalization of the direction vectors, \( e^a \), along with the recursive definition of \( O^A \), it can be demonstrated that
\[ e^a e^b D_a v_b = O^{ab} D_a v_b + \frac{1}{3} \delta^{ab} D_a v_b = O^{ab} D_a v_b + \frac{1}{3} D_a v^a , \]
\[ e^a e^b D_a \dot{v}_b = O^{ab} D_a \dot{v}_b + \frac{1}{3} D_a \dot{v}^a , \]
\[ e^a e^b e^c D_a D_b v_c = O^{abc} D_a D_b v_c - \frac{1}{5} \left( O^a D_b v^b + O^b D_a v^a \right) . \]

Using the orthogonality conditions we find, first, from (68):
\[ \int d\Omega O^A B \approx \delta_0^e \left[ -\frac{4\pi}{3} D_a \tau^a \right] + \delta^e \left[ \frac{4\pi}{3} \left( \frac{1}{4} D^a \ln \rho_R + A^a \right) \right] + \delta_2^e \left[ \frac{8\pi}{15} \sigma^{a_1 a_2} \right] , \]
and second,
\[ \int \tau^a D_a B d\Omega = + \frac{4\pi}{3} \left( \frac{1}{4} D^a \ln \rho_R + D_a A^a \right) . \]

The integration over the solid angle can now carried out resulting in an equation for the gradient of the radiation flux.

**APPENDIX D**

**Integral Solutions**

We use the following notation: \( \Delta \eta = \eta_0 - \eta_0, \Delta \eta = \eta_0 - \eta \) and \( \delta \eta = \eta - \eta' \) such that \( \delta \eta_a = \eta_a - \eta'_a \) and \( \delta \eta_b = \eta - \eta_0 \).

Beginning with an integral ansatz of the form:
\[ \tau_P^e (\eta) = \int_0^\eta d\eta' \left[ C_0(\eta') \tau_P^{(0)}(\eta) + C_1(\eta') \frac{\partial}{\partial \eta} \tau_P^{(0)}(\eta) + C_2(\eta') \frac{\partial^2}{\partial \eta^2} \tau_P^{(0)}(\eta) \right] , \]
Using the Leibniz rule for differentiation of integrals we obtain:
\[ \frac{\partial \tau_P^e (\eta)}{\partial \eta} = \int_0^\eta d\eta' \frac{\partial}{\partial \eta} \left[ C_0(\eta') \tau_P^{(0)}(\eta) + C_1(\eta') \tau_P^{(0)}(\eta) + C_2(\eta') \tau_P^{(0)}(\eta) \right] . \]
\[ + \left[ C_0(\eta)\tau_{\ell}^{(0)}(0) + C_1(\eta)\tau_{\ell}^{(0)'}(0) + C_2(\eta)\tau_{\ell}^{(0)''}(0) \right]. \quad (D.2) \]

If we hold \( \eta' \) constant in the partial derivatives, it follows from (86) and (D.1) that

\[ k \left[ \frac{(\ell + 1)^2}{(2\ell + 1)(2\ell + 3)} \tau_{\ell+1}^P(\delta \eta) - \tau_{\ell-1}^P(\delta \eta) \right] = -\frac{\partial}{\partial \eta} \tau_{\ell}^P(\delta \eta). \quad (D.3) \]

This gives us

\[ k \left[ \frac{(\ell + 1)^2}{(2\ell + 1)(2\ell + 3)} \tau_{\ell+1}^P(\delta \eta) - \tau_{\ell-1}^P(\delta \eta) \right] \]

\[ = -\int_0^\eta d\eta' \left[ C_0(\eta') \frac{\partial}{\partial \eta} \tau_{\ell}^{(0)}(\delta \eta) + C_1(\eta') \frac{\partial}{\partial \eta} \tau_{\ell}^{(0)'}(\delta \eta) + C_2(\eta') \frac{\partial}{\partial \eta} \tau_{\ell}^{(0)''}(\delta \eta) \right]. \quad (D.4) \]

Putting (D.1), (D.2) and (D.4) together, we find :

\[ \tau_{\ell}^{P'} + k \left[ \frac{(\ell + 1)^2}{(2\ell + 1)(2\ell + 3)} \tau_{\ell+1}^P - \tau_{\ell-1}^P \right] = C_0(\eta)\tau_{\ell}^{(0)}(0) + C_1(\eta)\tau_{\ell}^{(0)'}(0) + C_2(\eta)\tau_{\ell}^{(0)''}(0). \quad (D.5) \]

**APPENDIX E**

**Truncation Conditions**

The Ellis-Treciokas-Matravers treatment [30] makes various general statements about exact relativistic kinetic theory in the free streaming case. Of these probably the most important are:

Firstly, if any four consecutive harmonics vanish, say those with \( l = L + 1, L + 2, L + 3, L + 4 \), but those for \( l = L \) are non-zero, then

\[ F_{(A_l \sigma_{L+1} a_{L+2})} = 0. \quad (E.1) \]

This means, as \( F_{A_l} \) is non-zero, that the shear must vanish exactly: \( \sigma_{ab} = 0 \). This results arises from the requirement that \( \lim_{E \to \infty} F_{A_l} = 0 \).

Secondly, it can also be shown that if the first 3 multipole harmonics are zero, i.e., \( l = 1, 2, 3 \), once again the shear is necessarily zero: \( \sigma_{ab} = 0 \):

\[ \sigma_{ab} \int_m^\infty E^5 \frac{\partial F}{\partial E} dE = 0 \quad (E.2) \]

Thus in both cases the resulting spacetimes are highly restricted, and do not include generic perturbations.

A simplistic approach to linearisation will suggest that these relations and their implications can be ignored in (linearised) almost-Friedmann-Lemaître universes, because the equations leading to these results are second-order relations and so can be dropped when linearising. However that argument is not correct, if one carries out a careful linearisation procedure: indeed both these statements will hold in almost-Friedmann-Lemaître universes also. This can be seen as follows: although both \( F_{A_l} \) and \( \sigma_{ab} \) are at most \( O[1] \) (or \( O(\epsilon) \)) in the almost-Ehlers-Geren-Sachs sense [73], there are no zero or first order terms in the relevant equations leading to the above results, to explicitly linearise with respect to. Thus they cannot be dropped relative to larger (first order) terms in these equations, as there are no such terms; the first non-zero terms are second-order, and hence these equations with these terms must be obeyed.
even if we carry out a (first-order) linearisation. Thus they are both at most $O(\varepsilon^2)$ equations, but are both still valid in the almost-Friedmann-Lemaître universes\(^1\).

What this means is that one must be very careful about any kind of truncation in the multipole hierarchy, even in the almost-Friedmann-Lemaître universes - this includes not only the free-streaming case but the case with a generalized Krook equation for the scattering term [30]. What should be of interest is that in the matter dominated almost-Friedmann-Lemaître models with scalar perturbations, any truncation leading to zero shear would suppress the perturbations, reducing the dynamics to that of an exact Friedmann-Lemaître model. This doesn’t mean that one cannot consistently damp the higher moments out, it just means that they cannot be formally truncated – when higher moments are ignored the consistency condition in the exact theory should still be taken into account. Where this makes a difference is, for example, on intermediate scales where one is tempted to drop everything with $l \geq 3$, i.e., when one is close to tight-coupling. However, this is, strictly speaking a truncation and hence problematic. This is why a perturbative analysis in the Thompson scattering time is important; one can in this way consistently, without truncation, build up the entire multipole divergence equation hierarchy perturbatively - provided one has a meaningful sense of smallness, in this case the relaxation time. We do this in the case of small scales, in this way decoupling a subset of the hierarchy from the full set of multipole divergence equations, as in the gauge-invariant formulation of Hu-Sugiyama [39, 40].

**APPENDIX F**

**Linking Different Expansions**

The gauge invariant and covariant mode expansion (6) and (7) [18, 11] in the almost-flat-Friedmann-Lemaître case ($K = 0$) give the mode coefficient recursion relations for $\ell \geq 2$:

\[
-\dot{\tau}_\ell \simeq \frac{k}{a} \left[ \frac{(\ell + 1)^2}{(2\ell + 3)(2\ell + 1)} \tau_{\ell+1} - \tau_{\ell-1} \right].
\]  
(F.1)

Multiplying through by $\beta_\ell$, one finds

\[
-(\beta_\ell \dot{\tau}_\ell) \simeq \frac{k}{a} \left[ \frac{(\ell + 1)}{(2\ell + 3)} (\beta_{\ell+1} \tau_{\ell+1}) - \frac{\ell}{(2\ell - 1)} (\beta_{\ell-1} \tau_{\ell-1}) \right].
\]  
(F.2)

Changing from the proper time derivative in (F.2) to the conformal time derivative $'$ using $dt = a d\eta$ we find

\[
-(\beta_\ell \tau_\ell') \simeq k \left[ \frac{(\ell + 1)}{(2\ell + 3)} (\beta_{\ell+1} \tau_{\ell+1}) - \frac{\ell}{(2\ell - 1)} (\beta_{\ell-1} \tau_{\ell-1}) \right].
\]  
(F.3)

This can be immediately seen to be the same mode equation for $\ell > 2$ as in Hu & Sugiyama [39] and Wilson & Silk [81] (eqn 7).

If now $\beta_\ell$ is replaced with $\alpha_\ell^{-1}(2\ell + 1) = \beta_\ell$, we find that (F.3) can be rewritten (on first multiplying through by $(2\ell + 1)^{-1}$) as

\[
-(2\ell + 1)(\alpha_\ell^{-1} \tau_\ell') \simeq k \left[ (\ell + 1)(\alpha_{\ell+1}^{-1} \tau_{\ell+1}) - \ell (\alpha_{\ell-1}^{-1} \tau_{\ell-1}) \right].
\]  
(F.4)

This can be recognized as the form of the $\ell > 2$ free-streaming integrated Boltzmann equation of Ma & Bertschinger [56] (cf. $\ell$-th mode equation, 49 or 50) or Seljak & Zaldarriaga [69] equation 3d. Once again, it may be useful to remind the reader of our nomenclature, $\tau_\ell$ are

\(^1\)Although $O(\varepsilon^2) \ll O(\varepsilon)$, this doesn’t mean that $O(\varepsilon^2) = 0$ on its own, even though formally the notation $O(\varepsilon^2) \simeq 0$ is often adopted. These are not equivalent.
the mode coefficients (from the mode expansion) while $\tau_{A_\ell}$ are the multipole coefficients (from the multipole expansion). This distinction is not made in the Bardeen-variable gauge-invariant treatments based on the Friedmann-Lemaître mode expansion.

The solution to the mode coefficients with respect to the time-like integration in the flat ($K = 0$) almost-Friedmann-Lemaître case, are spherical Bessel functions. This should not be surprising given that the recursion relation in the linear Friedmann-Lemaître case is merely a projection from an initial section onto a sphere around here and now, modified as a result of Robertson-Walker expansion.

**APPENDIX G**

*Almost-Friedmann-Lemaître Equations*

Here some of the results arising from the Einstein field equations are listed, these are presented in [59].

**G.1. MAIN ALMOST-FRIEDMANN-LEMAÎTRE EQUATIONS**

We give the almost-Friedmann-Lemaître constraint, propagation and perturbation equations [57, 28]. Using the $1 + 3$ covariant dynamical equations [32] [24], in geometrized units: $c = 8\pi G = 1$. With the projection tensors $U^a_b = -u^a u_b$, $h^a_b = \delta^a_b + u^a u_b$, $\dot{h}^{ab} = 0$ and $0 = D_a U_{bc} = D_a h_{bc}$, where the totally projected spatial derivative is $D_a$. The covariant derivative of $u^a$ is

$$\nabla_b u_a = -u_a A_b + D_a u_b := -u_a A_b + \sigma_{ab} + H h_{ab} + \epsilon_{abc} \omega^c,$$

where we have written the vorticity in terms of the vorticity vector, $\epsilon_{abc} = \eta_{abcd} u^d$, and the shear $\sigma_{ab} = \sigma_{<ab>}$. We consider a cosmological model with matter (dust) and radiation. Then the stress-tensor given by

$$T^{ab} = (\rho_M + \rho_R) u^a u^b + 2 q^{(a} u^{b)} + \pi^{ab} + ph^{ab},$$

where energy densities $\rho_M$ and $\rho_R$ are the energy densities of the matter and radiation respectively, $q^a$ is the total energy flux, $\pi^{ab}$ is the anisotropic pressure of the radiation.

We linearise about a Robertson-Walker model as explained previously (see section 3.2 and [25]). The convention $A = B$ implies that $A$ equals $B$ in the exact theory, while $A \approx B$ indicates that $A$ equals $B$ to at least $O[1]$ in the almost-Friedmann-Lemaître sense, and lastly $A \approx B$ is retained to indicate that $A$ equals $B$ in relation to some other specific smallness parameter such as the relaxation time or ratio of radiation density to matter density. The almost-Friedmann-Lemaître Gauss-Codacci relation (Hamiltonian constraint) is

$$\mathcal{R} \simeq 2 \rho + \frac{2}{3} \Theta^2.$$

Using the Ricci identities, the linearised *propagation equations* are

$$\dot{\Theta} + \frac{1}{3} \Theta^2 - (\text{div} \, A) + \frac{1}{2} (\rho + 3p) \simeq -\frac{3}{2} B,$$

$$\dot{\omega}_{(a)} + 2 H \omega_{(a)} + \frac{1}{2} \text{curl} A_a \simeq 0,$$

$$\dot{\sigma}_{(ab)} + 2 H \sigma_{(ab)} - D_{(a} A_{b)} + E_{(ab)} \simeq \frac{1}{2} \pi_{(ab)},$$
and the constraint equations are given by

\begin{align}
\text{(div } \sigma\text{)}^a - \frac{2}{3} D^a \Theta - \text{curl} \omega^a &\simeq q^a, \quad (G.7) \\
D_a \omega^a &\simeq 0, \quad (G.8) \\
H_{(ab)} - \text{curl } \sigma_{(ab)} - D_{(a} \omega_{b)} &\simeq 0. \quad (G.9)
\end{align}

Using the second Bianchi identities, the linearised propagation equations are

\begin{align}
\dot{\rho} + (\rho + p) \Theta &\simeq -3 HB - D^a q_a, \quad (G.10) \\
(\rho + p) A_a + D_a p &\simeq -D_a B - \dot{q}_a - 4 H q_a - D^b \pi_{ab}, \quad (G.11)
\end{align}

\begin{align}
\dot{E}^{(ab)} + 3 H E^{(ab)} - \text{curl} H^{(ab)} + \frac{1}{2} (\rho + p) \sigma^{(ab)} &\simeq -\frac{1}{2} \pi^{(ab)} - \frac{1}{2} H \pi^{(ab)} - \frac{1}{2} D^a q^b, \quad (G.12) \\
\dot{H}^{(ab)} + 3 H H^{(ab)} + \text{curl} E^{(ab)} &\simeq \frac{1}{2} \text{curl} \pi^{(ab)}, \quad (G.13)
\end{align}

and the constraint equations are:

\begin{align}
\text{(div } E\text{)}_a - \frac{1}{3} D_a \rho &\simeq -H q_a - \frac{1}{2} D^b \pi_{ab}, \quad (G.14) \\
\text{(div } H\text{)}_a - (\rho + p) \omega_a &\simeq -\frac{1}{2} \text{curl} q_a. \quad (G.15)
\end{align}

The background Friedmann-Lemaître equations are:

\begin{align}
\rho &\quad= 3 H^2 + \frac{3 K}{a^2}, \quad (G.16) \\
\dot{H} &\quad= -H^2 - \frac{1}{3}(\rho + 3 p), \quad (G.17) \\
\dot{\rho} &\quad= -3 H (\rho + p). \quad (G.18)
\end{align}

### G.2. SOURCE TERMS

From [59], the source terms in the almost-Friedmann-Lemaître multipole divergence equations are:

\begin{align}
B_0(x) &\simeq -\frac{1}{3} D^a \tau_a(x), \quad B_a(x) \simeq D_a \ln T(x) + A_a(x), \quad B_{ab}(x) \simeq \sigma_{ab}(x). \quad (G.19)
\end{align}

In the case of scalar perturbations in a matter dominated almost-Friedmann-Lemaître universe [59], using the mode expansion [18], the electric part of the Weyl tensor \( E_{ab} \) (gravitational tidal effects), the acceleration \( A_a \) and anisotropic pressure \( \pi_{ab} \) can be related to their corresponding potentials:

\begin{align}
E_{ab} &= \Phi(k, t) Q_{ab}, \quad A_a = \Phi_u(k, t) Q_a, \quad \pi_{ab} = \Phi_\pi(k, t) Q_{ab}. \quad (G.20)
\end{align}
Using the Mode functions $Q_a$ and $Q_{(ab)}$, taking care of the additional wavenumber factors which get introduced \(^1\) \([18]\), the following decomposition will be used:

\[
(D^b E_{ab})^k = \Phi(k, t) D^b Q_{ab} \quad \text{and} \quad (D^b \pi_{ab})^k = \Phi_x D^b Q_{ab},
\]

and the matter density gradients:

\[
(D_a \ln \rho_{\alpha})^k = \delta^\alpha (k, t) Q_a = +\frac{k}{a} \Delta(k, t) Q_a, \quad \text{(G.22)}
\]

\[
(D_a \ln \rho_{\bar{\alpha}})^k = 4\frac{k}{a} \delta T(k, t) Q_a, \quad \text{(G.23)}
\]

It then follows that \(^2\)

\[
B_0(k, t) \approx -\frac{1}{3} \frac{k}{a} \tau_1(k, t), \quad \text{(G.24)}
\]

\[
B_1(k, t) \approx \frac{2}{3} (\rho^{-1}(t) \Phi(k, t)) \frac{k}{a} \left(\frac{k^2 - 3K}{k^2}\right), \quad \text{(G.25)}
\]

\[
B_2(k, t) \approx -2[\rho^{-1}(t) \Phi(k, t)]^t. \quad \text{(G.26)}
\]

In the case of matter domination we then find the relationship between the Newtonian potential and the matter density gradients:

\[
2\frac{d}{k^2} (k^2 - 3K) \Phi(k, t) \approx (a^2 \rho_{\alpha}) \delta^\alpha \approx \frac{k}{a} (a^2 \rho_{\alpha}) \Delta(k, t). \quad \text{(G.27)}
\]

Replacing $a^2 \rho_{\alpha}$ in terms of the density parameter and the curvature constant from the Friedmann equation (See Appendix A) for both the $K = 0$ ($\Omega_0 = +1$) and $K < 0$ ($\Omega_0 < 1$) matter dominated cases respectively, we obtain

\[
\Phi(k, t) \approx \frac{3}{2} \left(H_0^2 \Omega_0 D(\eta) \Delta(k, \eta_0) \right), \quad \text{(G.28)}
\]

\[
\Phi(k, t) \approx \frac{3}{2} \frac{a}{k} H_0^2 \delta^\alpha, \quad \text{(G.29)}
\]

\[
\Phi(k, t) \approx \frac{3}{2} \frac{k}{a} \left[\frac{k}{(k^2 - 3K)(\Omega_0 - 1)}\right] D(\eta) \Delta(k, \eta_0). \quad \text{(G.30)}
\]

These are then the equations that will be used for the slow decoupling and free-streaming eras.

**G.3. THE TEMPERATURE MONOPOLE**

Using the energy conservation equations ($\ell = 0$ multipole divergence equation) \(^3\)

\[
(D_a \ln T) + H(D_a \ln T + A_a) + \frac{1}{3} D_a \Theta \simeq +\frac{1}{3} D_a (D^c \tau_c), \quad \text{(G.30)}
\]

\(^1\)Once again one should not confuse the Fourier coefficients of the potentials here with those defined from 
\(E_{ab} = D_{(ab)} \Phi(x^i), A_a = D_a \Phi(x^i)\) and \(\pi_{ab} = D_{(ab)} \Phi_x (x^i)\).

\(^2\)To see where these factors come from notice that $a^2 D^\alpha D_{αc} Q = (-k^2 + 2K) D_a Q \Rightarrow a^2 D^\alpha D_{(αc)} Q = \frac{2}{k} (-k^2 + 3K) D_a Q$ (after removing the trace) for the general $l - \text{th order relations see} \ [18]$. (-λ)\(-1\) \(D_{(α)} Q = Q_{α}\) to find that 
\(D^b Q_{ab} = -\frac{4}{k} (ak)^{-1} (-k^2 + 3K) Q_{α}\), where as before \(D^a D_a Q = -λ^2 Q_{α}\) and \(λ = \frac{2}{k}\) \([8, 18]\).

\(^3\)Notice that $D_a (\ln T) \simeq (D_a \ln T) + H (D_a (\ln T) + A_a)$ where $D_a = h_{ab} \nabla b$ and $-\dot{u}_a (\ln T) \simeq H \dot{u}_a$. It is also well known that 
\(D^α (D_a \ln T) \simeq (D^2 \ln T) + H (D^2 \ln T)\).
and taking another spatial covariant derivative, we obtain one of key equations in the primary source calculation:

\[(D^2 \ln T) + 2H(D^2 \ln T) - \frac{1}{3} D^2(D^c \tau_c) \simeq - \frac{1}{2} (D_a D_b \sigma^{ab} - D_a q^a) - H(D^a A_a). \quad (G.31)\]

### G.4. THE ADIABATIC CONDITION

The entropy perturbation \( S_a \), for a radiation-dust almost-Friedmann-Lemaître universe, is given by

\[ S_a \simeq \frac{1}{4} D_a \ln \rho_R - \frac{1}{4} D_a \ln \rho_M, \quad (G.32) \]

and this gives the relation used in [13]:

\[ D^b E_{ba} \simeq (\frac{1}{4} D_a \ln \rho_R - S_a) \rho_M. \quad (G.33) \]

The adiabatic condition is then characterized by \( S_a = 0 \). This can either be written as

\[ D^a \rho_R = R D^a \rho_R, \quad \frac{1}{a} \delta T(k, t) = + \frac{1}{4} \delta^a D^a Q_a \] and \( \delta T(k, t) = \frac{1}{4} \Delta(k, t) \).

Provided the initial conditions produced, for example after inflation, lead to adiabatic perturbations, these perturbations will remain adiabatic until decoupling, however after decoupling generic density perturbations do not satisfy the adiabatic condition \( S_{(rm)} = 0 \). This is due to the fact that the average velocity of the radiation does not proceed along geodesics, while the matter does. Thus any perturbation that starts off adiabatic at last scattering will not remain so [13].

### G.5. ALMOST–FRIEDMANN–LEMAÎTRE SCALAR PERTURBATIONS

In the case of scalar perturbations, the almost-Friedmann-Lemaître Einstein field equations for matter domination reduce to the following CGI perturbation equations:

\[ \dot{\sigma}_{ab} + 2H \sigma_{ab} + E_{ab} \approx 0 \quad \iff \quad (a^2 \sigma_{ab}) \approx - a^2 E_{ab}, \quad (G.34) \]

\[ \dot{E}_{ab} + 3HE_{ab} + \frac{1}{2} \rho_{sd} \sigma_{ab} \approx 0, \quad \iff \quad (\rho_{sd}^{-1} E_{ab}) \approx - \frac{1}{2} \sigma_{ab}. \quad (G.35) \]

Taking the time derivative of the above equations, we obtain

\[ (a^2 \sigma_{ab}) + H(a^2 \sigma_{ab}) \approx - \frac{1}{2} a^2 \rho_{sd} \sigma_{ab}, \quad (G.36) \]

\[ (\rho_{sd}^{-1} E_{ab}) + 2H(\rho_{sd}^{-1} E_{ab}) \approx \frac{1}{2} E_{ab}. \quad (G.37) \]

To see how these relate to evolution equations for the density gradient, we take the spatial divergence of (G.37) [8].

A useful consequence of the above relations is that for matter domination\(^5\), the monopole equations take on a simple form:

\[ (D_a \ln T) + H(D_a \ln T) + \frac{1}{3} D_a(D^c \tau_c) \approx - \frac{1}{2} D^b \sigma_{ab}. \quad (G.38) \]

\(^4\)By adiabatic we mean that the comoving entropy density is constant.

\(^5\)A further consequence of \( a^3 E_{ab} \approx - \frac{1}{a^2} a^2 \rho_{sd} \sigma_{ab} \) and \( \rho = 3H^2 \Omega a^{-3} \) for \( a_0 = +1 \) is that \( a^3 E_{ab} \approx - \frac{1}{a} 3H^2 \Omega a^{-3} \sigma_{ab} \).
To summarize, the matter dominated limit lead to the following simple relationships between the dynamical variables and the electric part of the Weyl tensor:

\[
A_a \approx 0, \quad \text{(G.39)}
\]

\[
\sigma_{ab} \approx -\frac{2}{3}(H_0^2\Omega_0)^{-1}a^c \nabla_c (a^3E_{ab}), \quad \text{(G.40)}
\]

\[
D_a \ln \rho \approx (H_0^2\Omega_0)D^b (a^3E_{ab}), \quad \text{(G.41)}
\]

\[
\frac{2}{3}D^a \Theta \approx -\frac{2}{3}(H_0^2\Omega_0)^{-1}\left[(a^3D_bE^{ab}) + H(a^3D_bE^{ab})\right]. \quad \text{(G.42)}
\]

Using

\[
E_{ab} = \sum_k \Phi Q_{ab} \equiv D_a D_b \Phi(x) \quad \text{and} \quad D_a \Phi_E(x) = \sum_k \Phi_{E}(k,t)Q_a \quad \text{(G.43)}
\]

we obtain

\[
\Phi(k, t) \approx -\frac{k^2}{a^2} \Phi_E(k, t). \quad \text{(G.44)}
\]

Note that in [13], \(E_{ab} = \sum_k (k^2/a^2) \Phi_k Q_{ab} \) which is based on the notation of Kodama and Sasaki [50]. The relationship between \(\Phi_k \) and the potential used here is \(\Phi = (k^2/a^2) \Phi_k \).

The evolution equation for the Newtonian like potential follows from (G.37) [26, 28]:

\[
(\rho^{-1}_m \Phi) + 2H(\rho^{-1}_m \Phi)' \approx \frac{4}{3} \Phi. \quad \text{(G.45)}
\]

In terms of the conformal time derivative (\(dt = a \ln a \)) which we denote by a prime 'prime' this becomes:

\[
(\rho^{-1}_m \Phi)'' + H(\rho^{-1}_m \Phi)' \approx \frac{3}{2} a \Phi. \quad \text{(G.46)}
\]

On rearranging (G.45) using (G.44) and (G.16-G.18) we find the evolution equation for \(\Phi_E\):

\[
\left[\Phi_E + 4H\Phi_E\right] \approx \left[\frac{1}{2} \rho_m - \dot{H} - 3H^2\right] \Phi_E \approx \left(\frac{2K}{a^2}\right) \Phi_E. \quad \text{(G.47)}
\]

which can be simplified to give

\[
(a\Phi_E)''(t, k) + 2H(a\Phi_E)'(t, k) \approx \frac{3}{2} H^2_0 \Omega_0 a^{-2} \Phi_E(t, k). \quad \text{(G.48)}
\]

Notice that for \(K = 0\) we obtain the usual equation \(\Phi_E \approx -4H \Phi_E\) for dust as used in [13]. This gives the well known result

\[
\Phi_k(k, t) \approx \Phi^+(k, 0) + \Phi^-(k, t)t^{-5/3}. \quad \text{(G.49)}
\]

Considering the constant mode only, we then have:

\[
(\Phi\rho_m^{-1}) \approx (3H_0^2\Omega_0)^{-1}a[k^2\Phi^+_a(k, 0)] \approx (3H_0^2\Omega_0)^{-1}a[k^2\Phi_A(k, 0)], \quad \text{(G.50)}
\]

It then follows that \((\Phi\rho_m^{-1}) \sim k^2 \dot{a} \) and \(\Phi'_A \approx -\Phi'_{H} \approx 0). Hence we recover the standard result that the ‘potential fluctuations’ \(\Phi_E\) are time independent for the \(K = 0\) matter dominated dust scenario.

We can then write \(a\Phi_A(k, 0) = D/a(a\Phi_A(k, 0)) = (D/a)\Phi_A(k, t)\) where \(D\) is the the linear growth factor. The flat dust case is recovered using \(D = a\).

\[
(\Phi\rho_m^{-1}) \approx (3H_0^2\Omega_0)^{-1}D_k a[k^2\Phi_A(k, t)]. \quad \text{(G.51)}
\]
APPENDIX H
The Einstein Field Equations in the Energy Frame

In the energy frame $\tilde{q}_a = 0$, one finds the following useful constraints and a simple form for the evolution equation for the electric part of the Weyl tensor:

\[
\text{(div } \tilde{\sigma})_a \simeq \frac{2}{3} D^a \tilde{\Theta}, \quad (H.1)
\]
\[
\text{(div } E)_a \simeq \frac{1}{3} D_a (\rho + \rho_R), \quad (H.2)
\]
\[
(\rho_M + \rho_R + p) \tilde{A}_a \simeq -D_a p, \quad (H.3)
\]
\[
\dot{E}^{ab} + 3H E^{ab} \simeq -\frac{1}{2} (\rho_M + \rho_R + p) \tilde{\sigma}^{ab}. \quad (H.4)
\]

These are very similar to those found in the case of matter domination [59]. The useful features that arises when using the energy frame are: Firstly, we have a simply relationship between the expansion perturbation and the shear (H.1). Secondly, the Newtonian like potential can be related to the density of the matter and radiation content (H.2). Thirdly, by taking the divergence of (H.3) we find that

\[
(\rho_M + \frac{4}{3} \rho_R) \text{(div } \tilde{\sigma}) \simeq -D^2 \rho_R \Rightarrow D^2 \rho_R \simeq -(1 + R^{-1}) \rho^{-1} \text{(div } \tilde{\sigma}), \quad (H.5)
\]

from which we can find an evolution equation for (div \(A\)) from the perturbation equations for the radiation energy density. Finally, equation (H.4) gives the relationship between the shear and the Newtonian potential.

H.1. ON RELATING $A^2 \rho$ TO THE CURVATURE AND $\Omega$

The Friedmann equation in a Friedmann-Lemaître universe is

\[
H^2 + \frac{K}{a^2} \simeq \frac{1}{3} \rho \Rightarrow a^2 H^2 + K = \frac{1}{3} a^2 \rho. \quad (H.6)
\]

Using definition of the density parameters:

\[
\Omega = \frac{\rho}{3H^2}, \quad \text{and} \quad \Omega_\lambda = \frac{\rho_\lambda}{3H^2}. \quad (H.7)
\]

we can show that

\[
a^2 \rho \simeq 3a^2 H^2 \Omega, \quad \text{and} \quad a^2 \rho_\lambda \simeq 3a^2 H^2 \Omega_\lambda. \quad (H.8)
\]

Eq. (H.6) and eqn.(H.8) can be used to deduce that

\[
a^2 H^2 + K = a^2 H^2 \Omega, \Rightarrow a^2 H^2 = \frac{K}{(\Omega - 1)}. \quad (H.9)
\]

Using this and (H.8) we obtain

\[
a^2 \rho \simeq \frac{3K \Omega}{(\Omega - 1)}. \quad (H.10)
\]

\footnote{Note that this can also be written as $D^2 (\ln \rho_R) \simeq \frac{4}{3} (1 + R)(\text{div } \tilde{\sigma})$ using $D^2 (\ln \rho_R) = 4D^2 (\ln T)$.}
APPENDIX I
The Correlation Function

If \( \langle (\delta N/N)^2 \rangle \) is the square of the variance in the number of objects in a volume \( V \), then the correlation function, the excess probability over a random variable of finding an object within a distance \( \chi \) of a given object \([64]\), is given by

\[
\xi(\chi) = \frac{d}{dV} \left[ V(\chi) \langle \left( \frac{\delta N}{N} \right) \rangle_{V(\chi)} \right],
\]

where \( V(\chi) \) is the volume enclosed with a radius \( \chi \) for flat universe \((K = 0)\). The power spectrum \( P(k) \) is related \(^1\) to the correlation function for a given distribution \([49]\):

\[
\xi(\chi) = \frac{1}{2\pi^2} \int k^2 dk P(k) \frac{\sin k\chi}{k\chi} \quad \iff \quad P(k) = 4\pi \int \chi^2 d\chi \xi(\chi) \frac{\sin k\chi}{k\chi}.
\]

In an open universe the definition for the correlation function can still be retained. To see how, one first notices that in a flat universe \( \langle (\delta N/N)^2 \rangle_{V(\chi)} \sim \langle N \rangle^{-1} \sim V^{-\alpha} \), where \( \langle N \rangle_V \) is the average number of objects in a volume \( V \). In a space of constant negative curvature the volume enclosed by a sphere of radius \( \chi \) is \( V(\chi) = \pi (\sinh(2\chi) - 2\chi) \) so that

\[
\xi(\chi) \sim V^{-\alpha} \sim (\sinh(2\chi) - 2\chi)^{-\alpha}.
\]

The power spectrum for a power law correlation function in the volume in an open universe is then

\[
P(k) = \frac{1}{2\pi^2} B \int \sinh^2 \chi d\chi \frac{\sin k\chi}{k\sinh(2\chi) - 2\chi} \frac{1}{(\sinh(2\chi) - 2\chi)^\alpha},
\]

where \( B \) is the normalization constant. This diverges for small \( \chi \) so a small scale cut-off is necessary \([49]\). To relate this to the power spectrum today on scales measured by galaxy surveys, the power spectrum is multiplied by the square of a transfer function \( T^2(k) \). The power spectrum can then be normalized to \( \sigma_8 \).

APPENDIX J
Power Spectrum Normalization

There are two possible normalization schemes which one can follow.

1. On considering a power spectrum of the form

\[
P(k) = A(k\eta_0)^{-n-1},
\]

where \( \eta_0 \simeq 3t_0 \simeq 2H_0^{-1} \) for \( \Omega_0 = 1 \) gives the conformal time today, the scale factor can be normalized and \( A \) is one way of expressing the amplitude of scalar perturbations since it is related to the dimensionless scale of matter fluctuations at horizon crossing, \( \lambda_H \), by \([63, 2, 7, 80]\):

\[
\lambda_H^2 = \frac{4}{\pi} A.
\]

2. The alternative scheme is to consider a power spectrum of the form

\[
P(k) = Bk^n T^2(k),
\]

\(^1\)Care should be taken with the normalization convention.
where the transfer function $T(k) \sim +1$ on large scales. This means that if the fluctuations arise purely from the Sachs-Wolfe effect (potential fluctuations) near decoupling, $B$ and $A$ can be related at $n = +1$ [80]

$$P(k) = 2\pi^2 \eta_0^4 A k T^2(k). \quad (J.4)$$

What is often used is $P(k) = 2.5 \times 10^{16} A T^2(k)$. The units of $A$ are Mpc$^2$ and of $T^2$ (Mpc$^3$). For standard - CMD it is the convention to use the parametrized transfer function [6]:

$$T(k) = \left[1 + \left(ak + (bk)^{3/2} + (ck)^2\right)\right]^{3/2}, \quad (J.5)$$

where

$$a = 6.4\Gamma^{-1}h^{-1}\text{Mpc}, \quad b = 3.0\Gamma^{-1}h^{-1}\text{Mpc}, \quad c = 1.7\Gamma^{-1}h^{-1}\text{Mpc}, \quad \nu = 1.13. \quad (J.6)$$

Now the shape function $\Gamma$ can be given as approximately $\Gamma \simeq \Omega_0 h$. (by choosing $h = 0.5$ and $\Omega_B = 0.05$ the shape function is given as $\Gamma = 0.48$).

Large scale flows provide a measure of the power spectrum in as much as the variance of the velocity field sphere of radius $x_f$, $v_{rms}^2(x_f)$ can be expressed as an integral over the power spectrum. On small scales (clusters of galaxies) the power spectrum is normalized to $\sigma^2_8$, the variance of the galaxy distribution on scales of $x_f = 8h^{-1}\text{Mpc}$ [12]:

$$\sigma^2_8 = \frac{1}{b^2} = \frac{1}{2\pi^2} \int k^2 dk P(k) T^2(k) W^2(k x_f), \quad (J.7)$$

where $b_\rho$ is the $x_f$ scale ‘bias’ such that $\sigma^2_8 = \sigma_8^2/b_\rho$ and the appropriate variance and $W(x) = 3(\sin x - x \cos x)/x^3$ is the top-hat function. It should also be pointed out that it is more convenient to use the form

$$\sigma^2_8 = \int_0^\infty \frac{dk}{k} A(k\eta_0)^{n+3} T^2(k) \left(\frac{3j_1(k x_f)}{k x_f}\right). \quad (J.8)$$

The variance of galaxies possibly biased to the matter ($\delta_{gal} = b\delta_\rho$) is roughly unity on the scale of $8h^{-1}$ Mpc. Interestingly enough the standard-CDM normalization from COBE seems to give $\sigma_8 \approx 1.3$ which is seems to imply that it is not correct to assume a pure Sachs-Wolfe-HZ power spectrum or even a $n = 1.15$ Sachs-Wolfe one. An appropriate table of standard-CDM normalizations is provided in [9].

**APPENDIX K**

The Open Almost-Friedmann-Lemaître Case

**K.1. EXTENDING THE INTEGRAL SOLUTION TO THE OPEN CASE**

In the main body of this paper we keep to the almost-flat-Friedmann-Lemaître case for clarity; the extension to the open case is straightforward. The point is that we shown that the standard results can be recovered from the CGI approach. Given that the generic linear Friedmann-Lemaître models are well treated in the standard literature, we provide only an outline for the open case. The essence of the open solution is given via:

$$\tau_\ell' + k \left[\frac{(\ell + 1)^2}{(2\ell + 3)(2\ell + 1)} \left(1 - \frac{K}{k^2}((\ell + 1)^2 - 1)\right)\tau_{\ell+1} - \tau_{\ell-1}\right]$$

$$\approx -\kappa' \tau_\ell - [a B_0 \delta_{\ell 0} + (a B_1 + \kappa' v_\rho) \delta_{\ell 1} + a B_2 \delta_{\ell 2}]. \quad (K.1)$$
The left hand side, the homogeneous case, is solved in very much that same way as for the flat case.

The flat radial eigenfunctions are found from (K.3):

$$\frac{d}{d\chi} j_{\ell}(k\chi) = \frac{\ell}{(2\ell + 1)}k j_{\ell-1}(k\chi) - \frac{\ell + 1}{(2\ell + 1)}k j_{\ell+1}(k\chi), \quad (K.2)$$

where $Q_{lm} = j_{\ell}(k\chi) Y_{lm}^{\ast}(\hat{\Omega})$. For the general almost-Friedmann-Lemaître (linearised Friedmann-Lemaître) models [39, 35, 83]:

$$\frac{d}{d\chi} X_{\nu}^{\ell} = \frac{\ell}{(2\ell + 1)}k X_{\nu}^{\ell-1} - \frac{(\ell + 1)}{(2\ell + 1)}k \left[ 1 - \ell(\ell + 2)\frac{K}{k^2} \right] X_{\nu}^{\ell+1}. \quad (K.3)$$

Thus, in order to construct the open solution, we just replace the homogeneous solution for $\nu^2 + 1 = k^2/(-K)$ [35, 81] after reading off the solution using the recursion relation:

$$\tau_0^{(0)}(k, \eta) = (2\ell + 1)\beta_\ell^{-1} X_{\nu}^{\ell}(\eta). \quad (K.4)$$

Of course one needs to be careful to redefine the wavenumber. The integral solution (87) is then used in (K.1), i.e., the solutions (K.4) are substituted into (93) to recover the open integral solutions.

One carries out the same treatment as for $K = 0$ but using (K.3) and (11-13). Alternatively one can redefine the mode expansion $M_\ell |Q| [18].$

All that remains is to solve the evolution equations for the scalar perturbations, the coefficients $C_I(\eta, k)$ now include curvature terms when written out in terms of the $B$ terms (one uses the open recursion relation instead of the flat). In turn, these terms, $B$, will also pick up curvature terms when written out in terms of the perturbation variables. When the curvature starts to dominate the evolution, one gets an additional ISW contribution (which will be similar to the late ISW effect for a $\Lambda$ dominated flat model).

**K.1.1. Extending the Power Spectrum to the Open Matter-dominated Case**

We consider the primary anisotropy term:

$$\tau_0^{SW}(\eta_0) \beta_\ell \approx \frac{a}{k} B_1(\eta_\ast) X_{\nu}^{\ell}(\Delta \eta_\ast), \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \}
Then we can use
\[
\langle \tau_{A\ell} \tau_{A\ell} \rangle = \frac{2}{\pi} \beta_\ell \int k^2 dk \xi_n^2 |\tau_{\ell}|^2 j_\ell^2(k\chi), \tag{K.10}
\]
for
\[
\xi_\ell^2 = \prod_{n=1}^{\ell} \frac{n^2}{(2n + 1)(2n - 1)} \left[ 1 - \frac{K}{k^2}(n^2 - 1) \right], \tag{K.11}
\]
and (198) to find the angular correlation function. It is more straight-forward to redefine the mode function \(G_\ell[Q]\) in the mode function expansion, that is we use \(M_\ell[Q] = (\xi_\ell)^{-1}G_\ell[Q]\) instead of \(G_\ell[Q]\). This then allows us to retain the flat like form for the angular correlation function (198), however we must retain the flat mode eigenfunction normalization:
\[
C_\ell = 16\pi^2 \left[ \frac{K}{\rho^{-1}(\Omega_m - 1)} \right]^2 \int \frac{d\nu}{\nu^2} P^*(\nu) j_\ell^2(\nu\chi), \tag{K.12}
\]
where \(\nu^2 = k^2 - 1\).

The angular power spectrum can then be normalized to a given structure formation theory such as standard-CDM. It would seem that the favoured model (by current observational limits) is the \(\Lambda CDM\) model [54].

Also, in the open case from (K.12 and 204):
\[
D_\ell = \left( \frac{K\Omega_m}{a^2 H^2(\Omega_m - 1)} \right)^2 \int \frac{d\nu}{\nu^2} P^*(\nu) j_\ell^2(\nu\chi). \tag{K.13}
\]

Here \(P^*(206)\) is defined as before, while the mode expansion has been carried out in terms of \(M_\ell[Q]\) rather than \(G_\ell[Q]\) [18].