On the Beer Index of Convexity and Its Variants

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Measuring convexity

How to measure convexity of a given polygon?

There are (at least) two known approaches.
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More convex

Less convex
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Measuring convexity via a largest convex subset
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Measuring convexity via visibility

Points $A, B \in S$ see each other in $S$ if we have $AB \subseteq S$.

The Beer index of convexity $b(S)$ of $S$ is the probability that two randomly chosen points from $S$ see each other in $S$.

That is, $b(S) := \frac{\lambda^4(\{A, B \in S \times S : AB \subseteq S\})}{\lambda^2(S)^2} \in [0, 1]$.

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A lower bound on $b(S)$

We are interested in a relationship between $c(P)$ and $b(P)$.

Observation
For every $n \in \mathbb{N}$ there is a simple polygon $P$ satisfying $c(P) \leq \frac{1}{n}$ and $b(P) \geq \frac{1}{n} - \varepsilon$ for any $\varepsilon > 0$.

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Upper bounds on \( b(S) \)

Theorem (Cabello et al., 2014)
Every weakly star-shaped polygon \( P \) satisfies \( b(P) \leq 18 \ c(P) \).

In a weakly star-shaped set \( S \), there is a line segment in \( S \) that sees the entire \( S \).
Up to a constant, this is the best possible.

Theorem (Cabello et al., 2014)
Every simple polygon \( P \) satisfies \( b(P) \leq 12 \ c(P) \left( 1 + \log \frac{1}{c(P)} \right) \).

Conjecture (Cabello et al., 2014)
There is a constant \( \alpha > 0 \) so that for every simple polygon \( P \) we have \( b(P) \leq \alpha \ c(P) \).
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Theorem

Every set \( S \subseteq \mathbb{R}^2 \) with simply connected components satisfies \( b(S) \leq 180c(S) \).

Gives a positive answer to the previous conjecture.

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In fact, \( S := [0,1]^2 \setminus Q^2 \) gives \( c(S) = 0 \) and \( b(S) = 1 \).
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- In fact, $S := [0, 1]^2 \setminus \mathbb{Q}^2$ gives $c(S) = 0$ and $b(S) = 1$. 
Sketch of the proof

Main idea: assign a set $R(A) \subseteq \mathbb{R}^2$ of measure $O(c(S) \lambda_2(S))$ to every $A \in S$ such that for every $B \subseteq S$ we have $B \in R(C)$ or $C \in R(B)$.

Step 1: Reduce $S$ to a bounded simply-connected open set.

Step 2: Partition $S$ into weakly star-shaped sets.

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Higher-order Beer index

The main result fails in higher dimensions.

We introduce the following new parameter.

For \( k \in \mathbb{N} \) and \( S \subseteq \mathbb{R}^d \), let the \( k \)-index of convexity \( b_k(S) \) of \( S \) be the probability that the convex hull of randomly chosen \( k+1 \) points from \( S \) is contained in \( S \).

That is, \( b_k(S) := \frac{\lambda_{d+1}}{\lambda_d} \left( \frac{{\binom{k+1}{1}}}{\binom{d+k+1}{d}} \right) \left( \frac{1}{\lambda_d(S)} \right)^{k+1} \).

Note that \( b_k(S) \in [0, 1] \) and \( b_1(S) = b(S) \).
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Few observations

We have $b_1(S) \geq b_2(S) \geq \cdots \geq b_d(S)$.

For general set $S$ only $b_d(S)$ admits a nontrivial upper bound in $c(S)$.

The set $S' := [0,1]_d \setminus Q_d$ satisfies $c(S') = 0$ and $b_1(S') = b_2(S') = \cdots = b_{d-1}(S') = 1$.

Is there a nontrivial upper bound on $b_d(S)$? Lower bounds?

In the plane this is not the original problem.
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- We have $b_1(S) \geq b_2(S) \geq \cdots \geq b_d(S)$.
- For general set $S$ only $b_d(S)$ admits a nontrivial upper bound in $c(S)$.
  - The set $S' := [0,1]^d \setminus \mathbb{Q}^d$ satisfies $c(S') = 0$ and
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Our results for $b_d(S)$

Theorem
For every $d \geq 2$, there is $\beta = \beta(d) > 0$ such that every $S \subseteq \mathbb{R}^d$ satisfies $b_d(S) \leq \beta c(S)$.

We do not know whether this upper bound is the best possible. It is optimal up to a logarithmic factor.

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For every $d \geq 2$, there is $\gamma = \gamma(d) > 0$ such that for every $\varepsilon \in (0, 1]$, there is a set $S \subseteq \mathbb{R}^d$ satisfying $c(S) \leq \varepsilon$ and $b_d(S) \geq \gamma \varepsilon \log_2 \frac{1}{\varepsilon}$, and in particular, we have $b_d(S) \geq \gamma c(S) \log_2 \frac{1}{c(S)}$. 
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Open problems

Is there a linear upper bound on $b^{d-1}(S)$ for 'topologically nice' sets $S$?

Conjecture
For every $d \geq 2$, there is $\alpha = \alpha(d) > 0$ such that if $S \subseteq \mathbb{R}^d$ is a set whose every component is contractible, then $b^{d-1}(S) \leq \alpha c(S)$.

Does large $b(S)$ imply existence of large triangle with boundary in $S$?

More generally, is this true for $b^k(S)$ and $k$-skeletons $\text{Skel}_k(T)$?

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