Quantum jumps and attractors of Maxwell–Schrödinger equations

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Annotacija

We suggest a novel mathematical interpretation of basic postulates (or “principles”) of Quantum Mechanics (transitions to quantum stationary orbits, the wave-particle duality, and the probabilistic interpretation) as inherent properties of the coupled (self-consistent) nonlinear Maxwell–Schrödinger equations. The interpretation relies on our novel general mathematical conjecture on global attractors of G-invariant nonlinear Hamiltonian partial differential equations with a Lie symmetry group G.

This conjecture is confirmed by our results on global attractors of nonlinear Hamiltonian PDEs obtained since 1990 for a list of model equations with three basic symmetry groups: the trivial group, the group of translations, and the unitary group U(1). We present a brief survey of these results.

However, for the Maxwell–Schrödinger equations the global attraction remains an open problem.

Keywords: attractors; stationary states; solitons; stationary orbits; Hamiltonian equations; nonlinear partial differential equations; symmetry group; Lie group; Lie algebra; Maxwell–Schrödinger equations; quantum transitions; wave-particle duality; electron diffraction; probabilistic interpretation.

Содержание

1 Quantum postulates and Maxwell-Schrödinger equations 2
2 Bohr’s transitions and attractors 3
  2.1 Schrödinger theory of stationary orbits 4
  2.2 Bohr’ postulates by perturbation theory 4
  2.3 Bohr’ postulates as global attraction 5
  2.4 The Einstein–Ehrenfest paradox 5
  2.5 Attractors of dissipative and Hamiltonian PDEs 5
3 Conjecture on attractors of G-invariant Hamilton nonlinear PDEs 6
4 Results on global attractors for nonlinear Hamilton PDEs 6
  4.1 Global attraction to stationary states 7
  4.2 Global attraction to solitons 7
  4.3 Global attraction to stationary orbits 8
5 De Broglie’ wave-particle duality 9
6 Born’ probabilistic interpretation 10
  6.1 Diffraction of electron beams 10
  6.2 Discrete registration of electrons 10

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1 Quantum postulates and Maxwell-Schrödinger equations

The present paper is inspired by the problem of a mathematical description of “quantum jumps”, i.e., of transitions between quantum stationary orbits. We suggest new conjectures on the mathematical description of the following basic postulates of quantum theory:

I. Transitions between quantum stationary orbits.
II. Wave-particle duality.
III. Probabilistic interpretation.

A rigorous dynamical description of these postulates is still unknown. This lack of theoretical clarity hinders the progress in the theory (e.g., in superconductivity and in nuclear reactions), and in numerical simulation of many engineering processes (e.g., of laser radiation and quantum amplifiers) since a computer can solve dynamical equations but cannot take into account postulates.

Transitions between stationary orbits of atoms and accompanying radiation (postulate I) were postulated by Bohr in 1913. The wave-particle duality (postulate II) was conjectured by de Broglie in 1923, and the probabilistic interpretation (postulate III) was introduced by Born in 1927. On the other hand, after the discovery of Schrödinger’s quantum mechanics, the question arose on the validity of these postulates I–III in new theory—this still remains an open problem. These and other questions have been frequently addressed in the 1920s and 1930s in discussions of Bohr, Schrödinger, Einstein and others [1]. However, a satisfactory solutions were not achieved. We propose a novel approach to these problems relying on recent progress in the theory of attractors for Hamilton nonlinear PDEs.

The main purposes of this paper are i) to suggest a relation of these postulates to the theory of attractors, ii) to survey related results on global attractors for nonlinear Hamiltonian PDEs, iii) to formulate novel general conjectures on the attractors, and iv) to apply these conjectures to a mathematical interpretation of the postulates I–III in the context of coupled nonlinear Maxwell–Schrödinger equations.

The Maxwell equations in the entire space of $\mathbb{R}^3$ in the Heaviside–Lorentz units read ([7, p. 781])

$$\text{div } E(x,t) = \rho(x,t), \quad \text{rot } E(x,t) = -\frac{1}{c} B(x,t), \quad \text{div } B(x,t) = 0, \quad \text{rot } B(x,t) = \frac{1}{c} [j(x,t) + \dot{E}(x,t)].$$

(1.1)

The second and third equations imply the Maxwell representations $B(x,t) = \text{rot } A(x,t)$ and $E(x,t) = -\frac{1}{c} A(x,t) - \nabla A^0(x,t)$. Then in the Coulomb gauge $\text{div } A(x,t) \equiv 0$ the Maxwell equations (1.1) are equivalent to the system

$$\frac{1}{c^2} \ddot{A}(x,t) = \Delta A(x,t) + \frac{1}{c} P j(x,t), \quad \Delta A^0(x,t) = -\rho(x,t), \quad x \in \mathbb{R}^3,$$

(1.2)

where $\rho(x,t)$ and $j(x,t)$ are the charge and current densities, respectively, and $P$ denotes the orthogonal projection onto free-divergent vector fields from the Hilbert space $L^2(\mathbb{R}^3) \otimes \mathbb{R}^3$. The coupled Maxwell–Schrödinger equations read as (cf. [11])

$$\begin{cases}
\begin{aligned}
\frac{1}{c^2} \ddot{A}(x,t) &= \Delta A(x,t) + \frac{1}{c} P j(x,t), \\
\Delta A^0(x,t) &= -\rho(x,t)
\end{aligned}
\end{cases}$$

$$ih\dot{\psi}(x,t) = \frac{1}{2m} [-i\hbar \nabla - \frac{e}{c}(A(x,t) + A_{\text{ext}}(x,t))]^2 \psi(x,t) + e(A^0(x,t) + A^0_{\text{ext}}(x,t))\psi(x,t)$$

(1.3)

where $A_{\text{ext}}(x,t)$ and $A_{\text{ext}}^0(x,t)$ are some external Maxwell potentials, $e < 0$ is the electron charge and $c$ is the speed of light in a vacuum. The coupling is defined by expressing the charge and current densities in the wave function:

$$\rho(x,t) = e|\psi(x,t)|^2, \quad j(x,t) = \frac{e}{m} \text{Re } \{\bar{\psi}(x,t)[-i\hbar \nabla - \frac{e}{c}(A_{\text{ext}}(x,t) + A(x,t))]\psi(x,t)\}.$$

(1.4)

These densities satisfy the continuity identity

$$\dot{\rho}(x,t) + \text{div } j(x,t) \equiv 0.$$

(1.5)

The system (1.3) is formally Hamiltonian, with the Hamilton functional (which is the energy up to a factor)

$$\mathcal{H}(\Pi, A, \psi, t) = \frac{1}{2} \left[ \| -\frac{1}{c} \Pi \|^2 + \| \text{rot } A \|^2 \right] + (\psi, H_A(t)\psi),$$

(1.6)
where $\| \cdot \|$ stands for the norm in the real Hilbert space $L^2(\mathbb{R}^3) \otimes \mathbb{R}$ and the brackets $(\cdot, \cdot)$ stand for the inner product in $L^2(\mathbb{R}^3) \otimes \mathbb{C}$. The Schrödinger magnetic operator

$$H_A(t) := \frac{1}{2m}[-i\hbar \nabla - \frac{e}{c}(A(x,t) + A_{\text{ext}}(x,t))]^2 + e(\frac{1}{2}A^0(x,t) + A^0_{\text{ext}}(x,t)),$$

where $A^0(x,t) := (-\Delta)^{-1}\rho(t)$ and $\rho(x,t) := |\psi(x,t)|^2$. The system (1.3) can be written in the Hamilton form as

$$\begin{cases}
\frac{1}{c}\dot{A}(t) = D_t\mathcal{H}(\Pi(t), A(t), \psi(t), t), \\
\dot{\psi}(t) = \frac{1}{\hbar}D_t\mathcal{H}(\Pi(t), A(t), \psi(t), t),
\end{cases}$$

(1.7)

taking into account that $(\psi e^{A^0}, \psi) = (A^0, \rho) = (\Delta^{-1}\rho, \rho)$, and hence, $D_\psi(\psi e^{A^0}, \psi) = 4eA^0_\psi$.

Therefore, the energy is conserved in the case of static external potentials

$$A_{\text{ext}}(x,t) = A_{\text{ext}}(x), \quad A^0_{\text{ext}}(x,t) = A^0_{\text{ext}}(x).$$

(1.8)

For instance, in the case of an atom, $A^0_{\text{ext}}(x)$ is the nucleus Coulomb potential, while $A_{\text{ext}}(x)$ is the vector potential of the nucleus magnetic field. On the other hand, the total charge $Q(t) := \int \rho(x,t)dx$ formally is conserved for arbitrary time-dependent external potentials.

The Hamiltonian (1.6) is invariant with respect to the action of the group $U(1)$,

$$(A(x), \Pi(x), \psi(x)) \mapsto (A(x), \Pi(x), \psi(x)e^{i\theta}), \quad \theta \in (0, 2\pi).$$

(1.9)

This invariance implies the charge continuity equation (1.5) by the general Noether theorem on invariants [3, Section 13.4.3]. One can also check (1.5) by direct differentiation [3, Section 3.4]. Moreover, for any solution $A(x,t), \Pi(x,t), \psi(x,t)$ the functions $A(x,t), \Pi(x,t), \psi(x,t)e^{i\theta}$ are also a solution.

**Remark 1.1.** The existence of global solutions to the Cauchy problems for systems (1.3) in the entire space $\mathbb{R}^3$ without external potentials was proved in [9] for all finite energy initial states (1.6). The uniqueness of the solutions has so far been proved only in narrower classes of functions, [11, 12].

Taking into account the electron spin, the Maxwell-Schrödinger system (1.3) should be replaced by the Maxwell–Pauli system with current density

$$j(x,t) = \frac{e}{m}\text{Re}\sigma\{\overline{\psi}(x,t)\sigma \cdot [-i\hbar \nabla - \frac{e}{c}(A(x,t) + A_{\text{ext}}(x,t))]\psi(x,t)\}. $$

(1.10)

where $\sigma := (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. In this case, the Schrödinger equation is replaced by the Pauli equation

$$i\hbar \dot{\psi}(x,t) = H_P(t)\psi(x,t) = \frac{1}{2m}[\sigma \cdot (-i\hbar \nabla - \frac{e}{c}(A(x,t) + A_{\text{ext}}(x,t))]^2 \psi(x,t) + e(A^0(x,t) + A^0_{\text{ext}}(x,t))\psi(x,t),$$

and the Schrödinger operator $H_A(t)$ in the Hamiltonian (1.6) must be replaced by the Pauli magnetic operator

$$H_A^P(t) := \frac{1}{2m}[\sigma \cdot (-i\hbar \nabla - \frac{e}{c}(A(x,t) + A_{\text{ext}}(x,t))]^2 + e(\frac{1}{2}A^0(x,t) + A^0_{\text{ext}}(x,t)).$$

**Remark 1.2.** The system (1.3) was introduced essentially by Schrödinger in his first articles [6], and it underlies the entire theory of laser radiation [8].

## 2 Bohr’s transitions and attractors

In 1913, Bohr formulated two fundamental postulates of quantum theory of atoms:

**I.** An atom is always in one of quantum stationary orbits, and sometimes it jumps from one stationary state to another: in the Dirac notation

$$|E_n\rangle \mapsto |E_{n'}\rangle. $$

(2.1)

**II.** The atom does not radiate in stationary orbits. Every jump is followed by a radiation of an electromagnetic wave with the frequency

$$\omega_{nn'} = \frac{E_{n'} - E_n}{\hbar} = \omega_{n'} - \omega_n, \quad \omega_n := \frac{E_n}{\hbar},$$

(2.2)

With the discovery of the Schrödinger theory in 1926 the question arose about the validity of these Bohr’s axioms in the new theory.
2.1 Schrödinger theory of stationary orbits

Besides the equation for the wave function, the Schrödinger theory contains quite nontrivial definition of stationary orbits in the case when the Maxwell external potentials do not depend on time: stationary orbits are solutions of the form

\[ \psi(x,t) \equiv \psi(x)e^{-i\omega t}, \quad \omega \in \mathbb{R}. \] (2.3)

Substitution into the Schrödinger equation (2.5) leads to the famous eigenvalue problem.

From a formal analytical point of view, such a definition is traditional and rather natural, since then \(|\psi(x,t)|\) does not depend on time. Most likely, this definition was suggested by the de Broglie wave function \(\psi(x,t) = Ce^{i(kx-\omega t)}\) for free particles, which factorises as \(Ce^{ikx}\) since the spatial properties have changed and ceased to be homogeneous. On the other hand, the homogeneous time factor \(e^{-i\omega t}\) must be preserved, since the external potentials are independent of time. However, these "algebraic" arguments do not remove the question on agreement of the Schrödinger definition with the Bohr postulate (2.1)!

Thus, a problem arises on a mathematical interpretation of the Bohr postulate (2.1) in the Schrödinger theory. One of the simplest interpretation of the jump (2.1) is the asymptotics for each finite energy solution

\[ \psi(x,t) \sim \psi_{\pm}(x)e^{-i\omega_{\pm}t}, \quad t \to \pm \infty, \] (2.4)

where \(\omega_- = \omega_n\) and \(\omega_+ = \omega_{n'}\). However, such asymptotics are obviously wrong due to the superposition principle for linear Schrödinger equation (2.5): for example, for solutions of the form \(\psi(x,t) \equiv \psi_1(x)e^{-i\omega_1 t} + \psi_2(x)e^{-i\omega_2 t}\) with \(\omega_1 \neq \omega_2\). It is exactly this contradiction which shows that the linear Schrödinger equation alone cannot serve as a basis for the theory compatible with the Bohr postulates. Our main conjecture is that these postulates are inherent properties of the nonlinear Maxwell–Schrödinger equations (1.3). This conjecture is suggested by the following perturbative arguments.

2.2 Bohr’ postulates by perturbation theory

The remarkable success of the Schrödinger theory was the explanation of the Bohr’ postulates in the case of static external potentials by perturbation theory applied to the system (1.3). Namely, as a first approximation, the fields \(A(x,t)\) and \(A^0(x,t)\) in the Schrödinger equation of the system (1.3) can be neglected:

\[ i\hbar \psi(x,t) = H\psi(x,t) := \frac{1}{2m}[-i\hbar \nabla - \frac{e}{c}A_{ext}(x)]^2\psi(x,t) + eA_{ext}(x)\psi(x,t), \] (2.5)

For "sufficiently good" external potentials and initial conditions any finite energy solution can be expanded in eigenfunctions

\[ \psi(x,t) = \sum_n C_n \psi_n(x)e^{-i\omega_n t} + \psi_c(x,t), \quad \psi_c(x,t) = \int C(\omega)e^{-i\omega t}d\omega, \] (2.6)

where integration is performed over the continuous spectrum of the Schrödinger operator \(H\), and the integral decays as \(t \to \infty\) in each bounded domain \(|x| \leq R\), see, for example, [4, Theorem 21.1]. The substitution of this expansion into the expression for currents (1.4) gives the series

\[ j(x,t) = \sum_{nn'} j_{nn'}(x)e^{-i\omega_{nn'} t} + c.c. + j_c(x,t), \] (2.7)

where \(j_c(x,t)\) contains a continuous frequency spectrum. Therefore, the currents on the right hand side of the Maxwell equation from (1.3) contains, besides the continuous spectrum, only discrete frequencies \(\omega_{nn'}\). Hence, the discrete spectrum of the corresponding Maxwell field also contains only these frequencies \(\omega_{nn'}\). This proves the Bohr rule (2.2) in the first order of perturbation theory, since this calculation ignores the inverse effect of radiation onto the atom.

Moreover, these arguments also clarify the asymptotics (2.1). Namely, the currents (2.7) on the right hand of the Maxwell equation from (1.3) produce the radiation when nonzero frequencies \(\omega_{nn'}\) are present. However, this radiation cannot last forever since the total energy is finite. Hence, in the long-time limit should remain only \(\omega_{nn'} = 0\) which means exactly one-frequency asymptotics (2.4) and the limiting stationary Maxwell field.
2.3 Bohr’ postulates as global attraction

As a result, we expect the long-time asymptotics

\[
(A(x,t), \psi(x,t)) \sim (A_{\pm}(x), e^{-i\omega t} \psi_{\pm}(x)), \quad t \to \pm\infty
\]

for all finite-energy solutions to the Maxwell–Schrödinger equations (1.3). The asymptotics should hold in \(H^1\)-norms on every bounded region of \(\mathbb{R}^3\).

**Remark 2.1.** Experiments show that the time of transitions (2.1) is of order \(10^{-8}\) s, though the asymptotics (2.8) requires an infinite time. We suppose that this discrepancy can be explained by the fact that \(10^{-8}\) s is the time when the atom emits an overwhelming part of the radiated energy.

Such asymptotics are still open problems for the Maxwell–Schrödinger system (1.3). On the other hand, similar asymptotics are now proved for a list of model Hamilton nonlinear PDEs with symmetry groups \(U(1)\). In next section we state a general conjecture which reduces to the asymptotics (2.8) in the case of the Maxwell–Schrödinger system.

**Definition 2.2.** Stationary orbits of the Maxwell-Schrödinger nonlinear system (1.3) are finite energy solutions of the form \((A(x), e^{-i\omega t} \psi(x))\).

Existence of stationary orbits for the system (1.3) in the whole space was proved in [10] under conditions

\[
A_{\text{ext}}(x,t) \equiv 0, \quad A_0^{\text{ext}}(x,t) = -\frac{eZ}{|x|}, \quad \int |\psi_{\pm}(x)|^2 dx \leq Z.
\]

The asymptotics (2.8) mean global attraction to the set of stationary orbits. We suggest similar attraction for Maxwell–Dirac, Maxwell–Yang–Mills and other coupled equations. In other words, we suggest to interpret quantum stationary states as the points and trajectories lying on the global attractor of the corresponding quantum dynamical equations.

2.4 The Einstein–Ehrenfest paradox

An instant orientation of atomic magnetic moment during \(\sim 10^{-4}\) s when turning on the magnetic field in the Stern–Gerlach experiments caused a discussion in the "Old Quantum Mechanics," because classical model gave relaxation time \(\sim 10^9\) s taking into account the moment of inertia of the atom [2]. In the linear Schrödinger’s theory, this phenomenon also did not find a satisfactory explanation.

However, this instantaneous orientation is exactly in line with asymptotics (2.8) for solutions to the coupled Maxwell-Schrödinger system. Namely, in the absence of the magnetic field, the ground states (with a fixed charge) form a two-dimensional manifold. When the magnetic field is turned on, the structure of the attractor (i.e. the set of corresponding stationary orbits \((A, \psi)\)) instantly changes: the two-dimensional manifold bifurcates in two one-dimensional manifolds with a certain spin value. This bifurcation is not related to any moment of inertia and corresponds to the "alternative A" in the terminology of Einstein–Ehrenfest [2]: "... atoms can never fall into the state in which they are quantised not fully".

2.5 Attractors of dissipative and Hamiltonian PDEs

Such interpretation of the Bohr transitions as a global attraction is rather natural. On the other hand, the existing theory of attractors of dissipative systems [28]–[31] does not help in this case since all fundamental equations of quantum theory are Hamiltonian. The global attraction for dissipative systems is caused by energy dissipation. However, such a dissipation in the Hamilton systems is absent.

This is why we have developed in the 1990–2019s together with our collaborators a novel theory of global attractors for Hamilton PDEs, especially for application to the problems of Quantum Theory. Our results [32]–[49] for the Hamilton equations rely on energy radiation, which irrevocably carries the energy to infinity and plays the role of energy dissipation. A brief survey of these results can be found in Section 4, and the detailed survey – in [50].

The results obtained so far indicate an explicit correspondence between the type of long-time asymptotics of finite energy solutions and the symmetry group of the equation. We formalize this correspondence in our general conjecture (3.2).
3 Conjecture on attractors of $G$-invariant Hamilton nonlinear PDEs

Let us consider $G$-invariant Hamilton nonlinear PDEs of type
\[ \Psi(x, t) = F(\Psi(x, t)), \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}, \] (3.1)
with a Lie symmetry group $G$. Here $\Psi(\cdot, t)$ belongs to the Hilbert space $L^2(\mathbb{R}^3) \otimes \mathbb{R}^d$, and the Hamilton structure means that $F(\Psi) = JD\mathcal{H}(\Psi)$, where $J^* = -J$. $G$-invariance means that $F(g\Psi) = gF(\Psi)$ for all states $\Psi$ and all transformations $g \in G$ (more precisely, $g$ belong to a representation of the Lie group $G$). In this case, for any solution $\Psi(t)$ to equations (3.1) the trajectory $g\Psi(t)$ is also a solution.

Let us note that the theory of elementary particles deals systematically with the symmetry groups $SU(2)$, $SU(3)$, $SU(5)$, $SO(10)$ and other, and $G := SU(4) \times SU(2) \times SU(2)$ is the symmetry group of "Grand Unification see [26].

**Conjecture A.** For "generic" $G$-invariant equations (3.1) any finite energy solution $\Psi(t)$ admits a long-time asymptotics
\[ \Psi(t) \sim e^{\hat{g}_\pm t} \Psi_{\pm}, \quad t \rightarrow \pm \infty, \] (3.2)
where the generators $\hat{g}_\pm$ belong to the corresponding Lie algebra $\mathfrak{g}$ (more precisely, $\hat{g}_\pm$ belong to a representation of $\mathfrak{g}$), and the asymptotics holds in some local seminorms.

In other words, all $G$-orbits form a global attractor for "generic" $G$-invariant Hamilton nonlinear PDEs of type (3.1). This conjecture is a generalization of rigorous results [32]–[50] obtained since 1990 for a list of model equations of type (3.1) with three basic symmetry groups: the trivial group, the group of translations, and the unitary group $U(1)$. We give a brief survey of these results in Section 4.

For the case of Maxwell-Schrödinger system (1.3) with the symmetry group $U(1)$, the conjecture (3.2) reduces to the asymptotics (2.8).

**Empirical evidence.** Conjecture (3.2) agrees with the Gell-Mann–Ne’eman theory of baryons [24, 25]. Namely, in 1961 Gell-Mann and Ne’eman suggested the symmetry group $SU(3)$ and other ones for the strong interaction of baryons relying on the discovered parallelism between empirical data for the baryons, and the “Dynkin scheme” of Lie algebra $\mathfrak{g} = su(3)$ with 8 generators (the famous ‘eightfold way’). This theory resulted in the scheme of quarks and in the development of the quantum chromodynamics [26], and in the prediction of a new baryon with prescribed values of its mass and decay products. This particle, the $\Omega^-$-hyperon, was promptly discovered experimentally [27]. The elementary particles seem to describe long-time asymptotics of quantum fields. Hence, the empirical correspondence between elementary particles and generators of the Lie algebras presumably gives an evidence in favour of our general conjecture (3.2) for equations with Lie symmetry groups.

Conjecture (3.2) suggests to define stationary “$G$-orbits" for equations (3.1) as solutions of the type
\[ \Psi(t) = e^{\hat{g} t} \Psi, \quad t \in \mathbb{R}, \] (3.3)
where $\hat{g}$ belongs to the corresponding Lie algebra $\mathfrak{g}$ (more precisely, $\hat{g}$ belong to a representation of the Lie algebra $\mathfrak{g}$). This definition leads to the corresponding "$\hat{g}$-eigenvalue problem"
\[ \hat{g}\Psi = F(\Psi). \] (3.4)
In particular, for the linear Schrödinger equation with the symmetry group $U(1)$, stationary orbits are solutions of the form $e^{i\omega x} \psi(x)$, where $\omega \in \mathbb{R}$ is an eigenvalue of the Schrödinger operator, and $\psi(x)$ is the corresponding eigenfunction. However, Conjecture (3.2) fails for linear equations, i.e., linear equations are exceptional, not “generic"! In the case of the symmetry group $G = SU(3)$, the generator ("eigenvalue") $\hat{g}$ is $3 \times 3$-matrix, and solutions (3.3) are quasiperiodic in time.

4 Results on global attractors for nonlinear Hamilton PDEs

Here we describe rigorous results [32] - [49] obtained since 1990 on the corresponding asymptotics for a number of Hamiltonian nonlinear partial differential equations of type (3.1). We give only a brief listing of the results, see the details in [50]. The results obtained confirm the existence of finite-dimensional attractors in the Hilbert phase space, and demonstrate an explicit correspondence between the long-time asymptotics and the symmetry group $G$ of equations.
The results were obtained so far for model equations with three basic groups of symmetry: the trivial symmetry group \( G = \{e\} \), the translation group \( G = \mathbb{R}^n \) for translation-invariant equations, and the unitary group \( G = U(1) \) for phase-invariant equations.

4.1 Global attraction to stationary states

For “generic” equations with trivial symmetry group \( G \) the long-time asymptotics of all finite energy solutions is the convergence to stationary states (see Fig. 1)

\[
\psi(x, t) \to S_{\pm}(x), \quad t \to \pm \infty \tag{4.1}
\]

which is proved for a variety of model equations in [32]–[36]: i) for a string coupled to nonlinear oscillators, ii) for a three-dimensional wave equation coupled to a charged particle and for the Maxwell–Lorentz equations, and also iii) for wave equation, and Dirac and Klein–Gordon equations with concentrated nonlinearities.

Here \( S_{\pm}(x) \) are some stationary states depending on the considered trajectory \( \psi(x, t) \), and the convergence holds in local seminorms of type \( L^2(|x| < R) \) for any \( R > 0 \). The convergence (4.1) in global norms (i.e., corresponding to \( R = \infty \)) cannot hold due to energy conservation.

Example 4.1. Nonlinear Huygens Principle. Consider solutions to 3D wave equation with a unit propagation velocity and initial data with support in a ball \( |x| < R \). The corresponding solution is concentrated in spherical layers \( |t| - R < |x| < |t| + R \). Therefore, the solution converges everywhere to zero as \( t \to \pm \infty \), although its energy remains constant. This convergence to zero is known as the strong Huygens principle. Thus, attraction to stationary states (4.1) is a generalization of this Huygens principle to nonlinear equations. The difference is that for a linear wave equation the limit behind the wave front is always zero, while for nonlinear equations the limit can be any stationary solution.

The proofs in [34] and [35] rely on the relaxation of the acceleration

\[
\ddot{q}(t) \to 0, \quad t \to \pm \infty. \tag{4.2}
\]

Such relaxation has been known for a long time in classical electrodynamics as “radiation damping”, but was first proved in [34] and [35] for charged relativistic particle in a scalar field and in the Maxwell field under the Wiener Condition on the particle charge density. This condition is an analogue of the “Fermi Golden Rule”, first introduced by Sigal in the context of nonlinear wave- and Schrödinger equations [37]. The proof of the relaxation (4.2) relies on a novel application of the Wiener Tauberian theorem.

4.2 Global attraction to solitons

For “generic” translation-invariant equations, the long-time asymptotics of all finite energy solutions is the convergence to solitons

\[
\psi(x, t) \sim \psi_{\pm}(x - v_{\pm} t), \quad t \to \pm \infty, \tag{4.3}
\]
where the convergence holds in local seminorms in the comoving frame of reference, that is, in $L^2(|x - v_{\pm}t| < R)$ for any $R > 0$. Such soliton asymptotics were proved in [39]–[42] for three-dimensional wave equation coupled to a charged particle and for the Maxwell–Lorentz equations. These results gave the first rigorous proof of “radiation damping” in classical electrodynamics, which has been an open problem for about 100 years. The proofs in [39] and [40] rely on variational properties of solitons and their orbital stability, as well as on the relaxation of the acceleration (4.2) under the Wiener condition on the particle charge density.

More accurate soliton asymptotics in global norms with several solitons were first discovered in 1965 by Zabuzhsky and Kruskal in numerical simulation of the Korteweg–de Vries equation (KdV): it is the decay to solitons

$$\psi(x,t) \sim \sum_k \psi_{\pm}^k(x) e^{-i\omega_{\mp}k t} + w_{\pm}(x,t), \quad t \to \pm \infty,$$

where $w_{\pm}$ are some dispersion waves. In [43] the results of numerical simulation were presented to confirm the soliton asymptotics (4.4) with many solitons for 1D relativist-invariant nonlinear wave equations. Later on, such asymptotics were proved by the method of inverse scattering problem for nonlinear integrable Hamiltonian translation-invariant equations (KdV, etc.) in the works of Ablowitz, Segur, Eckhaus, van Harten and others [38]. A trivial example is provided by the d’Alembert equation $\ddot{\psi}(x,t) = \psi''(x,t)$, for which any solution reads

$$\psi(x,t) = f(x-t) + g(x+t).$$

### 4.3 Global attraction to stationary orbits

For “generic” equations with unitary symmetry group $G = U(1)$, the long-time asymptotics are global attraction to “stationary orbits” (see Fig. 2)

$$\psi(x,t) \sim \psi_{\pm}(x)e^{-i\omega_{\mp}t}, \quad t \to \pm \infty;$$

they were proved in [44]–[48] for the Klein–Gordon and Dirac equations coupled to $U(1)$-invariant nonlinear oscillators, and in [49], for discrete in space and time difference approximations of such coupled systems, i.e., for the corresponding difference schemes.

The global attraction was proved under assumption that the equations are “strictly nonlinear”. For linear equations the attraction obviously fails if the discrete spectrum consist at least of two points.

**Remark 4.2.** Let us comment on the term generic in the results of the previous section and in Conjecture (3.2). Namely, this conjecture means that the asymptotics (3.2) hold for all solutions for an open dense
set of $G$-invariant equations.

i) For example, asymptotics (4.1), (4.3), (4.5) hold under appropriate conditions, which define some “open dense set” of $G$-invariant equations with three types of the symmetry group $G$: either under the Wiener condition or under the strict nonlinearity condition, etc. The asymptotics may break down if these conditions fail—this corresponds to some “exceptional” equations: for example, asymptotics (4.5) break down for the linear Schrödinger equations with at least two different eigenvalues.

ii) General situation is the following. Let a Lie group $G_1$ is a (proper) subgroup of some larger Lie group $G_2$. So, the $G_2$-invariant equations form an “exceptional subset” among all $G_1$-invariant equations, and the corresponding asymptotics (3.2) may be completely different. For example, the trivial group $\{e\}$ is a subgroup in $U(1)$ and in $\mathbb{R}^n$, and asymptotics (4.3) and (4.5) may differ significantly from (4.1).

5 De Broglie’ wave-particle duality

In 1923, de Broglie suggested in his PhD to identify the beam of particles with a harmonic wave:

- a beam of particles with moment $p$ and energy $E = \frac{p^2}{2m} \leftrightarrow \psi(x, t) = Ce^{i(kx - \omega t)}, \ (p, E) = \hbar(k, \omega). \ \ (5.1)$

This identification was suggested as a counterpart to the Einstein corpuscular treatment of light as a beam of photons. The duality (5.1) was the key source for the Schrödinger quantum mechanics.

We suggest a mathematical description of the wave-particle duality relying on a generalization of the conjecture (3.2) for the case of translation-invariant Maxwell–Schrödinger system (1.3) without external potentials, i.e., $A_{\text{ext}}(x, t) = 0$, $A^0_{\text{ext}}(x, t) = 0$. In this case the Schrödinger equation of (1.3) becomes

$$i\hbar\psi(x, t) = \frac{1}{2m}[-i\hbar\nabla - \frac{e}{c}A(x, t)]^2\psi(x, t) + eA^0(x, t)\psi(x, t), \ x \in \mathbb{R}^3. \ \ (5.2)$$

Now the symmetry group of the system (1.3) becomes $G = \mathbb{R}^3 \times U(1)$, and our general conjecture (3.2) should be strengthened similarly to (4.4):

$$A(x, t) \sim \sum_k A^k_{\pm}(x - v^k_{\pm}t) + A_{\pm}(x, t), \ \psi(x, t) \sim \sum_k \psi^k_{\pm}(x - v^k_{\pm}t)e^{\pm i\Phi^k_{\pm}(x, t)} + \psi_{\pm}(x, t), \ t \to \pm \infty \ (5.3)$$

for each finite energy solution, where $A_{\pm}(x, t)$ and $\psi_{\pm}(x, t)$ stand for the corresponding dispersion waves. The solitons (traveling wave solutions) $(A(x - vt), \psi(x - vt))$ for (1.3) were constructed in [10]. These asymptotics suggest to treat the solitons as electrons and provisionally correspond to the reduction (or collapse) of wave packets.

The asymptotics (5.3) suggest a mathematical description of the wave-particle duality under several assumptions. Namely, let us consider the wave function $\psi(x, t) = Ce^{i(kx - \omega t)}$ as initial data. Then initially the corresponding charge and current densities

$$\rho(x, t) = e|\psi(x, t)|^2 \equiv e|C|^2, \quad j(x, t) = \frac{e}{m}\text{Re}\{\overline{\psi(x, t)}[-i\hbar\nabla\psi(x, t)]\} \equiv \frac{e}{m}|C|^2\hbar k \ \ (5.4)$$

are uniform. Hence, initially the Maxwell field $E(x, t)$ and $B(x, t)$ vanish, as well as the potentials $A(x, t)$ and $A^0(x, t)$. Therefore, the Schrödinger equation (5.2) implies that the wave function $\psi(x, t) = Ce^{i(kx - \omega t)}$ satisfies initially the free Schrödinger equation which implies $\hbar\omega = \frac{\hbar^2 k^2}{2m}$.

Further, we expect that averaged in space charge, momentum and energy densities do not depend on time due to the corresponding conservation laws. Then the density of the electrons (solitons) should be $n = |C|^2$. Similarly, the density of momentum and energy of the solitons should be, respectively, $P = -i\hbar\overline{\psi(x, t)}\nabla\psi(x, t) = \hbar|C|^2k$, and $E = \frac{\hbar^2}{2m}(|\nabla\psi(x, t)|^2 = \frac{\hbar^2}{2m}|C|^2k^2$. Finally, it is natural to assume that the velocities of the solitons should be identical by translation homogeneity. Then the momentum $p = P/n$ and energy $E = E/n$ of one electron are equal, respectively, to $p = \hbar k$ and $E = \frac{\hbar^2 k^2}{2m} = \frac{p^2}{2m} = \hbar\omega$ which agrees with (5.1).
6 Born’ probabilistic interpretation

In 1927, Born suggested the probabilistic interpretation of the wave function:

\[ \text{The probability of detecting an electron at a point } x \text{ at the time } t \text{ is proportional to } |\psi(x, t)|^2. \]

We suggest below an interpretation of this postulate relying on asymptotics (5.3) for the coupled Maxwell–Schrödinger equations (1.3). However, the corresponding rigorous justification for the nonlinear equations (1.3) is still an open problem.

6.1 Diffraction of electron beams

Born proposed the probabilistic interpretation to describe the diffraction experiments of Davisson and Germer of 1924–1927s. In these experiments the electron beam was scattered by a nickel crystal, and the reflected beam was fixed on a photo-film. The resulting images are similar to “Lauegrams”, first obtained in 1912 by the method of Laue. Later on, such experiments were also carried out with transmitted electron beams scattered by thin gold and platinum crystalline films (G. P. Thomson, Nobel Prize 1937). Recently Bach & al. for the first time observed double-slit diffraction of electrons [17], which is the first realization for electronic waves of classical Young’s experiment of 1803.

The electron diffraction was considered for a long time as a paradoxical phenomenon incompatible with the concept of a point elementary particle. On the other hand, the diffraction phenomena are inherent properties of the linear Schrödinger equation which is commonly recognized for a long time [20–23]. We showed in [18] that there is a fine quantitative agreement of solutions to the linear Schrödinger equation for two-slit screen with the results of recent diffraction experiments [17]. Namely, according to the principle of limiting amplitude, the diffracted wave admits the asymptotics

\[ \psi_d(x, t) \sim a_\infty(x)e^{-i\omega t}, \quad t \to \infty, \]

where \( \omega \) is the frequency of the incident wave. The calculation of the diffraction amplitudes \( a_\infty(x) \) in [18] using the Kirchoff approximation demonstrates that the maxima of \( a_\infty(x) \) on the screen agree very well with that of the diffraction pattern in experiments [17]. Thus, the diffraction of electron beams finds a natural basis in the linear Schrödinger theory.

6.2 Discrete registration of electrons

However, in 1948 the probabilistic interpretation received new content and confirmation after the experiments of Biberman, Sushkin and Fabrikant [13]. In these experiments with very low electron beam intensity, the diffraction pattern is created as an averaging of random discrete registration of individual electrons. Later on similar experiments were carried out by Chambers, Tonomura, Frabboni, Bach & al. [14, 15, 16, 17]. To explain this phenomenon, there are at least two possibilities, which are both related to random fluctuations:

i) Random interaction with counters. One possibility to explain the discrete registration is a random triggering either a) of registration counters located at the screen points, or b) of atoms of the photo emulsion. We suppose that the probability of triggering is proportional to the current, which is given by

\[ j(x, t) = \frac{e}{m} \text{Re}\left\{ \bar{\psi}(x, t)[-i\hbar \nabla \psi(x, t)] \right\} \approx \frac{\hbar}{m} k |a_\infty(x)|^2, \quad t \to \infty, \]

according to (1.4) and (6.1). Let us note that we set \( A_{\text{ext}}(x, t) = 0 \) in the formula (1.4) since there is no external fields between the scatterer screen and the screen of observation. The term with \( A(x, t) \) in (1.4) is also neglected since its contribution is relatively small.

Therefore, the averaged diffraction pattern should have maxima at the screen points with maximal electric current, which coincide with the screen points with maximal amplitude \( |a_\infty(x)| \) by (6.2). This coincidence is confirmed in [18] by calculations of \( a_\infty(x) \) and by comparison with experiments [17]. Thus, the discrete registration of electrons also allows an interpretation in the linear Schrödinger theory.

ii) Random reduction of wave packets. Another possibility to explain the discrete registration is the soliton-conjecture (5.3) for translation-invariant Maxwell–Schrödinger system (1.3). This conjecture is inspired by the asymptotics (4.4), which was proved for translation-invariant integrable nonlinear PDEs. Respectively, we suppose that the decay (5.3) should hold between the scatterer screen and the screen of observation,
where the external fields vanish, and hence, the system (1.3) is translation-invariant, see Fig. 3. Such a decay into solitons should be considered as random process, as it is subject to microscopic fluctuations.

An averaged registration rate of electrons at a point of the screen should be proportional to the current (6.2) if the contribution of the dispersion waves $\psi_{\pm}$ is negligible. This follows from the charge conservation law. Therefore, the averaged diffraction pattern again should have maxima at the screen points with maximal amplitude $|a_\infty(x)|$. Thus, this treatment of the discrete registration of electrons requires the soliton-conjecture (5.3) for nonlinear Maxwell–Schrödinger equations.

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