Quantum fluctuations in FRLW space-time

Yevgeniya Rabochaya

Dipartimento di Fisica, Università di Trento,
Centro INFN-TIFPA, Trento
Via Sommarive 14, 38123 Povo, Italia
*E-mail: yevgeniya.rabochay@unitn.it

In this paper we study a quantum field theoretical approach, where a quantum probe is used to investigate the properties of generic non-flat FRLW space-time. The fluctuations related to a massless conformal coupled scalar field defined on a space-time with horizon is identified with a probe and the procedure to measure the local temperature is presented.

Keywords: Quantum fluctuation; Temperature; Unruh effect.

1. Introduction

The Hawking radiation\textsuperscript{1} is one of the most robust and important predictions of quantum field theory in curved space-time. Here we would like to study some (local) properties of a generic Friedmann-Lemaître-Robertson-Walker (FLRW) space-time with non-flat topology.

Let us remind some basic facts about the formalism. Any spherically symmetric four dimensional metric can be expressed in the form:

$$ds^2 = \gamma_{ij}(x^i)dx^i dx^j + R^2(x^i)d\Omega^2,$$

with $\gamma_{ij}(x^i)$ a tensor describing a two-dimensional space-time with coordinates $x^i$, $R(x^i)$ being the “areal radius” and $d\Omega^2$ encoding the metric of a two-dimensional sphere orthogonal respect to the first one.

The dynamical trapping horizon -if exists- is located in the correspondence of

$$\chi(x^i)|_H = 0, \quad \partial_i \chi(x^i)|_H \geq 0, \quad \chi(x^i) = \gamma^{ij}(x^i)\partial_j R(x^i) \partial_j R(x^i).$$

Thus, one may define the quasi-local Misner-Sharp gravitational energy as

$$E_{MS}(x^i) := \frac{1}{2G_N}R(x^i) \left[ 1 - \chi(x^i) \right].$$

For example, the mass of a black hole described by this formalism results to be $E = \mathcal{R}_H/(2G_N)$. The Killing vector fields $\xi_\mu(x^\nu)$ are the generators of the isometries with $\nabla_\mu \xi_\nu(x^\nu) + \nabla_\nu \xi_\mu(x^\nu) = 0$: in the static case, with the time-like Killing vector field $K^\mu = (1, 0, 0, 0)$, the Killing surface gravity $\kappa_K$ is given by

$$\kappa_K K^\mu(x^\nu) = K^\nu \nabla_\nu K^\mu(x^\nu).$$

In the dynamical case, the real geometric object which generalizes the Killing vector field is the Kodama vector field\textsuperscript{2},

$$\kappa^i(x^i) := \frac{1}{\sqrt{-\gamma}} \varepsilon^{ij} \partial_j R(x^i), \quad i = 0, 1; \quad \kappa^i := 0, \quad i \neq 0, 1.$$
Thus, the Hayward surface gravity associated with dynamical horizon is
\[ \kappa_H := \frac{1}{2} \Box_x R(x) \bigg|_H. \] (6)

The Hawking radiation is a thermal radiation of the black holes due to quantum effects. In the static case, all derivations of Hawking radiation lead to a semi-classical expression for the radiation rate \( \Gamma \) in terms of the exchange \( \Delta E_K \) of the Killing energy \( E_K \) and the Killing/Hawking temperature \( T_K \),
\[ \Gamma \equiv e^{-\frac{2\pi \Delta E_K}{\kappa}}, \quad T_K := \frac{\kappa_K}{2\pi}. \] (7)

In the dynamical case one may suggest the Kodama/Hayward temperature:
\[ T_H := \frac{\kappa_H}{2\pi}. \] (8)

An important example that demonstrates the covariance of the formalism is given by the de Sitter space-time. The static patch reads
\[ ds^2 = -dt^2(1 - H_0^2 r^2) + \frac{dr^2}{(1 - H_0^2 r^2)} + r^2 d\Omega^2, \] (9)
where \( R = r \) and the horizon is located at \( r_H = 1/H_0 \) with surface gravity \( \kappa_H = H_0 \). The second patch is given by the expanding coordinates of the flat FLRW metric,
\[ ds^2 = -dt^2 + e^{2H_0 t} \left( dr^2 + r^2 d\Omega^2 \right), \] (10)
where \( R = e^{H_0 t}r \) and the dynamical (cosmological) horizon is \( r_H = 1/H_0 \) with \( \kappa_H = H_0 \). Finally, the global patch in non-flat FLRW metric is given by
\[ ds^2 = -dt^2 + \cosh^2(H_0 t) \left( \frac{dr^2}{(1 - H_0^2 r^2)} + r^2 d\Omega^2 \right), \] (11)
with \( R = r \cosh(H_0 t) \), and \( r_H = 1/H_0 \) and \( \kappa_H = H_0 \) again. Now we will see how it is possible to associate a temperature to the dynamical horizon of flat and non-flat de Sitter space-time in [10]–[11].

2. Quantization of massless field in FLRW metric

We recall the quantization of a conformal coupled massless scalar field in the FLRW space-time. The metric reads
\[ ds^2 = a^2(\eta)(-d\eta^2 + d\Sigma^2_3), \quad d\Sigma^2_3 = \frac{dr^2}{1 - k h_0^2 r^2} + r^2 dS^2_2. \] (12)
where \( d\eta = dt/a(t) \) is the conformal time, \( h_0 \) is a mass scale and the topology of the spacial section can be flat, spherically or hyperbolic for \( k = 0, 1, -1 \), respectively.

Given a massless scalar field,
\[ \phi(x) = \sum_{\alpha} f_\alpha(x) a_\alpha + f_\alpha^\dagger(x) a_\alpha^\dagger, \] (13)
such that the modes are conformal invariant, namely $(\Box - R/6)f_\alpha(x) = 0$, the associated Wightman function $W(x, x') = \langle \phi(x)\phi(x') \rangle$ results to be

$$W(x, x') = \sum_\alpha f_\alpha(x)f_\alpha^*(x'), \quad \left(\Box - \frac{R}{6}\right) W(x, x') = 0.$$  \hfill (14)

The Wightman function satisfies the following rule for the conformal transformations of the metric:

$$ds^2 = \Omega(x)^2 ds_0^2, \quad \phi = \frac{1}{\Omega(x)} \phi_0, \quad W(x, x') = \frac{1}{\Omega(x)\Omega(x')} W_0(x, x').$$  \hfill (15)

We may also take $W(x, x') = W(\eta - \eta', r - r')$ due to the homogeneity and isotropy of FLRW space-times.

Let us consider the spherical case ($k = 1$) in (12),

$$ds^2 = a^2(\eta) \left( -d\eta^2 + d\chi^2 + \frac{1}{h_0^2} \sin^2 h_0 \chi dS_2^2 \right), \quad h_0 \chi = \arcsin h_0 r.$$  \hfill (16)

This metric is conformally related to the Minkowski space-time,

$$ds^2 = a^2(\eta) 4 \cos^2 \left( h_0 \frac{\eta + \chi}{2} \right) 4 \cos^2 \left( h_0 \frac{\eta - \chi}{2} \right) \left( -dt^2 + dr^2 + r^2 dS_2^2 \right),$$  \hfill (17)

with

$$t \pm r = \frac{1}{h_0} \tan \left( h_0 \frac{\eta \pm \chi}{2} \right).$$  \hfill (18)

Thus, by starting from the well-known Wightman function in Minkowski space-time, one can use (15) and derive for the spherical FLRW metric

$$W(x, x') = \frac{h_0^2}{8\pi^2 a(\eta)a(\eta')} \frac{1}{\cos(h_0(\eta - \eta')) - \cos(h_0(\chi - \chi'))}.$$  \hfill (19)

The hyperbolic case $k = -1$ is obtained with the substitution $h_0 \rightarrow ih_0$, while the flat case $k = 0$ corresponds to the limit $h_0 \rightarrow 0$.

### 3. Quantum fluctuations in flat space-time

Let us consider a free massless quantum scalar field $\phi(x)$ in thermal equilibrium at the temperature $T$ in flat space-time. We know that finite temperature field theory effects of this kind can be investigated by given that the scalar field defined in the Euclidean manifold $S_1 \times R^3$, where the imaginary time is $\tau = -it$, compactified in the circle $S_1$ with period $\beta = 1/T$.

We briefly review the local quantity $\langle \phi(x)^2 \rangle$, which is a divergent quantity due to the product of valued operator distribution in the same point $x$. By making use of the zeta-function regularization procedure, the quantum fluctuations read

$$\langle \phi(x)^2 \rangle = \zeta(1)L_\beta(x), \quad L_\beta = -\partial_t^2 - \nabla^2,$$  \hfill (20)
where \( \zeta(z|L_\beta)(x) \) is the local zeta-function associated with the operator \( L_\beta \). It is easy to see that the analytic continuation of the local zeta-function is regular at \( z = 1 \) and finally one gets
\[
< \phi(x)^2 > = \frac{1}{12\beta^2} = \frac{T^2}{12}.
\]
In this way we obtain the temperature of the quantum field in thermal equilibrium from the zeta-function renormalized vacuum expectation value, namely we have a quantum thermometer.

### 3.1. Quantum fluctuations in FLRW space-time

Now we extend the argument to generic FLRW metric. The off-diagonal Wightman function
\[
W(x, x') = < \phi(x)\phi(x') > = \frac{1}{4\pi^2} \frac{1}{\Sigma^2(x, x')},
\]
with
\[
\Sigma^2(\tau, \tau - s) = a(\tau)a(\tau - s)\frac{2}{h_0^2} (\cos h_0(\Delta \chi(s)) - \cos h_0(\Delta \eta(s))),
\]
where \( a(\tau) \) is the conformal factor. Thus, in the limit \( s \to 0 \), one has
\[
< \phi(x)^2 > = W(\tau, \tau).
\]
It is possible to show that
\[
W(\tau, \tau - s) = -\frac{1}{4\pi s^2} + \frac{B}{48\pi^2} + O(s^2),
\]
where
\[
B = H^2 + A^2 + 2\dot{H}t + \frac{h_0^2}{a^2}(1 - 2\dot{t}^2), \quad A^2 = \frac{1}{t^2 - 1} (\dot{t}^2 + H(\dot{t}^2 - 1)),
\]
the dot being the derivative with respect to the proper time, \( H = (da(t)/dt)/a(t) \) being the usual Hubble parameter, and \( A^2 \) the radial acceleration. Therefore, after the regularization for the divergent part at \( s \to 0 \),
\[
< \phi^2 > |_{R} = \frac{1}{48\pi^2} \left( H^2 + A^2 + 2\dot{H}t \pm \frac{h_0^2}{a^2}(1 - 2\dot{t}^2) \right).
\]
This result is quite general and it is valid also for spatial curvature \( k \neq 0 \).

### 3.2. Quantum fluctuations in non flat de Sitter space-time

In the case of de Sitter space-time with \( k = 1 \), we may put \( H_0 = h_0 \) and the expression for quantum fluctuations reads
\[
< \phi^2 > |_{R} = \frac{1}{48} \left( H_0^2 + A^2 \right) = \frac{1}{48\pi^2} \frac{H_0^2}{1 - R_0H_0^2}.
\]
where $R_0 = \text{const}$ is the areal radius of the Kodama observer and the acceleration has been computed as

$$A^2 = \frac{R_0^2 H_0^4}{1 - R_0^2 H_0^2}. \quad (29)$$

For a Kodama observer with $R_0 = 0$ we recover the Gibbons-Hawking temperature associated with de Sitter space-time,

$$T = \frac{H_0}{2\pi}. \quad (30)$$

This is an important check of our approach, since it shows the coordinate independence of the result for the important case of de Sitter space-time.

### 3.3. Quantum fluctuations in FRLW form of Minkowski space-time

The Minkowski space-time may be written in a FRLW form with hyperbolic section $k = -1$ (Milne universe),

$$ds_M^2 = -dt^2 + t^2 \left( \frac{dr^2}{1 + r^2} + r^2 d\Omega^2 \right), \quad h_0 = 1. \quad (31)$$

Making use of the Hayward formalism, it is easy to verify that there is no dynamical horizon and the surface gravity is vanishing. In this case we obtain

$$< \phi^2 >_R = \frac{A^2}{48\pi^2}, \quad (32)$$

namely only the radial acceleration $A^2$ is present and the temperature is defined as

$$T_U = \frac{A}{2\pi}, \quad (33)$$

recovering the well known Unruh effect.

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