Kundt Three Dimensional Left Invariant Spacetimes

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Abstract

Kundt spacetimes are of great importance to General Relativity. We show that a Kundt spacetime is a Lorentz manifold with a non-singular isotropic geodesic vector field having its orthogonal distribution integrable and determining a totally geodesic foliation. We give the local structure of Kundt spacetimes and some properties of left invariant Kundt structures on Lie groups. Finally, we classify all left invariant Kundt structures on three dimensional simply connected unimodular Lie groups.

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1. Introduction

Kundt spacetimes are of great importance to General Relativity, as well as alternative gravity theories. To begin with, let us say that Kundt spacetimes constitute a natural generalization of pp-wave spacetimes. Roughly speaking, a Kundt spacetime is defined by the fact that it supports a vector field all of its scalar invariants vanish, without being Killing (see for instance [19, Chapter 6]). One of our motivations here is to provide a coordinate-free treatment of Kundt spacetimes, which is hard to find in the General Relativity literature. More precisely, let us define a Kundt spacetime as a Lorentz manifold \((M, g)\) having the following property (see for instance [7]):

(K1) There exists a non-singular vector field \(V\) on \(M\) such that

\[ g(V, V) = g(\nabla_X V, \nabla_X V) = \text{tr}(A^V) = g(B^V, B^V) = g(C^V, C^V) = 0, \]

where \(\nabla\) is the Levi-Civita connection, \(A^V : TM \rightarrow TM\) denotes the \((1, 1)\) tensor field given by \(A^V(X) = \nabla_X V\), \(B\) its symmetric part, \(C\) its skew-symmetric part and if \(F : TM \rightarrow TM\) is a \((1, 1)\)-tensor field then \(g(F, F) = \text{tr}(F \circ F^\ast)\) where \(F^\ast\) is its adjoint with respect to \(g\).

It turns out (see Proposition 2.1) that this property is equivalent to:

(K2) There exists a non-singular vector field \(V\) on \(M\) and a differential 1-form \(\alpha\) such that

\[ g(V, V) = 0, \nabla_X V = \alpha(X)V \quad \text{and} \quad \nabla_Y V = 0 \]

for any vector field \(X\) orthogonal to \(V\).

A fundamental observation for us was (see Proposition 2.1 and more details in Sections 2, 3 and 4) that Kundt properties (K1) or (K2) imply the following property:
(LK) There exists on $M$ a codimension one totally geodesic foliation which is degenerate with respect to $g$. More precisely, there exists a codimension one foliation $\mathcal{F}$ such that each leaf $L$ of $\mathcal{F}$ is a lightlike (totally) geodesic hypersurface (that is $T^2L \subset TL$ and any geodesic $\gamma : [a,b] \to M$, $a < 0 < b$, somewhere tangent to $L$, is locally contained in $L$ if $\gamma'(0) \in T_{\gamma(0)}L$ then there exists $\epsilon > 0$ such that $\gamma([-\epsilon, +\epsilon]) \subset L$).

Actually, up to assuming the direction field $T^2\mathcal{F}$ orientable (which is always possible up to passing to a double covering), the property (LK) is equivalent to:

(LKbis) there exists a non-singular vector field $V$ on $M$ and a differential $1$-form $\alpha$ such that

$$g(V, V) = 0, \nabla_X V = \alpha(X)V$$

for any vector field $X$ orthogonal to $V$.

We will refer to a Lorentz manifold satisfying (LK) as a local Kundt spacetime. Indeed, (LK) implies (K1) locally near any point in $M$. In other words, a spacetime admitting a codimension one lightlike geodesic foliation is locally Kundt.

Another major motivation to study Kundt Lorentz manifolds lies in their relation with CSI-spaces, those having all of their scalar curvature invariants are constant. The simplest one is the scalar curvature $\text{Scal}_g$, but one can also consider eigenvalues of the Ricci operator $\text{Ric}_g$ or the $g$-norm of the Riemann tensor $\text{Rm}_g$. All those are scalar curvature invariants of order $1$. Higher order ones are obtained by considering covariant derivatives of $\text{Rm}_g$. So CSI means, in particular, that all these quantities are constant functions on $M$.

Locally homogeneous spaces are CSI, and the existence of CSI spaces that are not locally-homogeneous spaces is a non-Riemannian phenomena which makes a one major difference between the positive and the non-definite cases in pseudo-Riemannian structures. A typical example is given by the (conformally flat) plane wave metric $g = dx^2 + dy^2 - 2dvdu - 2f(u)(x^2 + y^2)du^2$, which is VSI (i.e. has vanishing scalar invariants) for any $f$, but locally homogeneous for only few $f$’s. This example is Kundt, in fact $V = \frac{\partial}{\partial u}$ is a parallel vector field.

It is believed that a CSI Lorentz space, if it is not locally homogeneous, must be of Kundt type! This conjecture has been proved in some cases, e.g. in the lower dimensions $3$ and $4$, see for instance [6, 8]. In another direction, there is a notion of $I$-degenerate metrics, meaning that they have non-trivial (i.e. non-isometric) deformations keeping all the scalar curvature invariant functions the same (non-depending of the deformation parameter, but maybe depending on the point of $M$). Those are believed to be Kundt too.

Not all locally homogeneous spacetimes are Kundt, neither all Kundt spacetimes are locally homogeneous, but it is worthwhile to consider locally homogeneous Kundt spaces as a special class of both the homogeneous and the Kundt categories! Our project is to study Kundt structures on three dimensional Lie groups $G$ endowed with a left invariant Lorentzian metric $g$. It is natural, in this special homogeneous framework, to introduce a stronger Kundt property as follows. We call $(G, g)$ a Kundt Lie group (resp. locally Kundt Lie group) if it admits a non-singular left invariant vector field $V$ satisfying (K2) (resp. (LKbis)). In other words, we assume here compatibility between the (K2) or the (LKbis) property and the algebraic structure of $G$.

1.1. Results

One of our principal results, Theorem 4.1, states, essentially, that a three dimensional Lorentz group which is Kundt as a spacetime, is in fact a Kundt group. We also classify, up to isometric isomorphism, all unimodular three dimensional Kundt groups.

The paper is organized as follows. In Section 2, we provide a synthetic (coordinate-free) account on Kundt spacetimes emphasising on their relationship with lightlike geodesic foliations. We introduce Kundt groups and general facts about them in Section 3. The proof of Theorem 4.1 as well as further results are given in Section 4. Section 5 contains the classification up to automorphism of Kundt Lorentz groups.
2. Kundt spacetimes and geodesic foliations

Recall from the introduction that a Kundt spacetime is a Lorentz manifold satisfying the property (K1). Let us prove that this property is equivalent to (K2) and implies (LK). Moreover, (LK) implies (K1) locally near any point of the Lorentz manifold.

Proposition 2.1. Let \((M, g)\) be a Lorentz manifold. Consider the following assertions:

(i) \((M, g)\) is a Kundt spacetime.
(ii) There exists on \(M\) an isotropic non-singular vector field \(V\) and a differential 1-form \(\alpha\) such that, for any \(X \in \Gamma(V^\perp),\)

\[
\nabla_X V = \alpha(X) V \quad \text{and} \quad \nabla_V V = 0.
\]

(iii) There exists on \(M\) a totally geodesic codimension one foliation which is degenerate with respect to \(g\). This means that there exists a vector sub-bundle \(F \subset TM\) of rank \((\dim M) - 1\), where the restriction of \(g\) to \(F\) is degenerate and, for any \(X, Y \in \Gamma(F), \nabla_X Y \in \Gamma(F)\) where \(\nabla\) is the Levi-Civita connection of \(g\).

Then (i) and (ii) are equivalent and both imply (iii). Moreover, (iii) implies that (ii) holds in a neighbourhood of any point in \(M\).

Proof. (i) \(\implies\) (ii). Assume that \((M, g)\) is a Kundt spacetime. This means that there exists a non-singular vector field \(V\) satisfying (1). Fix a point \(p \in M\) and denote by \(A\) to the endomorphism given by \(Au = \nabla_u V\) for any \(u \in T_p M\).

Choose an isotropic vector \(U \in T_p M\) such that \(g(U, V) = 1\) and an orthonormal basis \((e_1, \ldots, e_{n-2})\) of \(\{U, V_p\}^\perp\). Note that

\[
g(A(V), A(V)) = g(V_p, V_p) = g(A(V), V) = 0.
\]

Thus the vector subspace \(\text{span}\{A(V), V\}\) is totally isotropic, hence its dimension equals 1 which means \(A(V) = \alpha_0 V\) for some \(\alpha_0 \in \mathbb{R}\). On the other hand, since \(g(V, V) = 0\), for any \(u \in T_p M\), \(g(A(u), V) = g(\nabla_u V, V) = 0\) hence \(A(T_p M) \subset V^\perp\).

This implies that for any \(u \in T_p M\), \(g(A(u), A(u)) \geq 0\) and \(g(A(u), A(u)) = 0\) if and only if \(A(u) = \alpha(u)V\). Now, from (1),

\[
0 = \text{tr}(A^* A) = 2g(A(V), AU) + \sum_{i=1}^{n-2} g(A(e_i), A(e_i)) = \sum_{i=1}^{n-2} g(A(e_i), A(e_i)).
\]

Moreover, we have

\[
0 = \text{tr}(A) = g(AV, U) + g(AU, V) + \sum_{i=1}^{n-2} g(Ae_i, e_i) = \alpha_0
\]

hence \(\nabla_V V = 0\). This completes the proof of (i) \(\implies\) (ii).

Let us prove (ii) \(\implies\) (i). Fix a point \(p\) and consider \(A\) and \((U, V, e_2, \ldots, e_{n-2})\) as defined above. We have \(A(T_p M) \subset \mathbb{R}V\) hence \(V^\perp \subset \ker A^*\). With this fact in mind, we get

\[
\text{tr}(A) = g(AU, V) + g(AV, U) + \sum_{i=2}^{n} g(Ae_i, e_i) = 0,
\]

\[
\text{tr}(BB^*) = 2g(BU, BV) + \sum_{i=2}^{n} g(Be_i, Be_i) = 0,
\]

\[
\text{tr}(CC^*) = 2g(CU, CV) + \sum_{i=2}^{n} g(Ce_i, Ce_i) = 0.
\]

This completes the proof of (i) \(\implies\) (ii).

Let us prove now that (ii) \(\implies\) (iii). Let \(F \subset V^\perp\). We have, for any \(X, Y \in \Gamma(F),\)

\[
g(\nabla_X Y, V) = -g(Y, \nabla_X V) = -\alpha(X)g(Y, V) = 0
\]
hence $\nabla_x Y \in \Gamma(F)$. This shows that $F$ is integrable and defines a degenerate codimension one totally geodesic foliation.

Now, we prove that if (iii) holds then, for any $p \in M$, there exists a vector field $V$ near $p$ satisfying (i). So, suppose that there exists an integrable degenerate codimension one sub-bundle $F \subset TM$ which defines a totally geodesic foliation. Fix a point $p \in M$ and choose a non-singular vector field $V \in \Gamma(F^\perp) \subset \Gamma(F)$ near $p$. It is obvious that $V$ is isotropic. Moreover, since $\nabla V \in \Gamma(F)$ and $g(\nabla V, V) = 0$ then $\nabla V = aV$. As above, denote by $A$ the endomorphism $A^i_v$ and choose a basis $(u, V_p, e_1, \ldots, e_{n-2})$. We have, obviously, that $A(T_p M) \subset F_p$. Moreover, since $F$ is totally geodesic, for any $X, Y \in \Gamma(F)$,

$$0 = g(\nabla X, Y) = -g(Y, \nabla X)$$

which implies that $A(F_p) \subset \mathbb{R}V$. So far, we have shown that locally near $p$, for any $X \in \Gamma(F)$,

$$AV = a_0 V, A(X) = a(X)V.$$

To finish the proof, we look for a vector field $V' = e^f V$ where $f$ is a function such that $V'$ satisfies (ii). This is equivalent to $V(f) = -a_0$ and such a function exists locally. 

\[ \Box \]

2.1. Kundt coordinates

Another way to compare the Kundt property with the existence of a codimension one lightlike geodesic foliation is given by the following fact which asserts the existence of adapted local coordinates associated to lightlike geodesic foliations, where the metric has a special form. The same adapted coordinates are known to characterize Kundt spacetimes.

Proposition 2.2. Let $(M, g)$ be a Lorentz manifold satisfying (iii) of Proposition 2.1. Then near any point in $M$ there exists a local coordinates system $(v, u, x = (x^2, \ldots, x^8))$ where the metric has the form:

$$g = 2dudv + H(v, u, x)du^2 + \sum_{i=2}^n W_i(v, u, x)du dx^i + \sum_{i,j} h_{ij}(u, x)dx^i dx^j.$$ 

Remarks 2.1. - Observe that the functions $h_{ij}$ do not depend on $v$.

- The foliation in Proposition 2.1 corresponds to the (local) $u$-levels.

- One can also show the converse, that a foliation admitting an adapted chart where the metric has such a form is lightlike geodesic.

Proof. Suppose that there exists a vector sub-bundle $F$ of $TM$ of rank $(\dim M) - 1$ such that the restriction of the metric to $F$ is degenerate and the $\Gamma(F)$ is stable by the Levi-Civita product.

Fix a point $p \in M$ and let $\Sigma$ be a local hypersurface containing $p$ and transversal to $F^\perp$. Then $F_\Sigma = F \cap T\Sigma$ determines a foliation on $\Sigma$ hence there exists a coordinates system $(x^2, \ldots, x^8, u)$ on $\Sigma$ such that the leaves of $F_\Sigma$ are the $u$-levels. There is a section $T : \Sigma \rightarrow (F^\perp)_\Sigma$ such that $g(T, \frac{\partial}{\partial u}) = 2$. Choose an injective immersion $\phi : \mathbb{R}^{n-1} \rightarrow M$ such that $\phi(\mathbb{R}^{n-1}) = \Sigma$. Then there exists $\varepsilon > 0$ such that the map $\Phi : \mathbb{R}^{n-1} \times (-\varepsilon, \varepsilon) \rightarrow M$ given by $\Phi(t, s) = \exp_{\phi(t)}(sT)$ is a diffeomorphism into its image. Denote by $V$ the image by $\Phi$ of the vector field $\frac{\partial}{\partial u}$. Since $F^\perp$ is totally geodesic, then $V$ is tangent to $F^\perp$ hence satisfies $g(V, V) = 0$. By construction, we have $\nabla V = 0$ and according to the proof of Proposition 2.1, for any $X \in \Gamma(F)$, $\nabla_x V = a(X)V$.

On the other hand, the vector fields $\frac{\partial}{\partial v}, \frac{\partial}{\partial u}, \ldots, \frac{\partial}{\partial x}$ on $\Sigma$ define a family of vector fields on $\mathbb{R}^{n-1}$ hence define a family of vector fields on $\mathbb{R}^{n-1} \times (-\varepsilon, \varepsilon)$ which commute with $\frac{\partial}{\partial w}$. Let $U, X_2, \ldots, X_n$ be their images by $\Phi$. We deduce that $V, U, X_2, \ldots, X_n$ are commuting, and give rise to a local coordinate system $(v, u, x^2, \ldots, x^n)$ on $M$ such that

$$V = \frac{\partial}{\partial v}, \quad U = \frac{\partial}{\partial u} \quad \text{and} \quad X_i = \frac{\partial}{\partial x_i}, \quad i = 2, \ldots, n.$$
Observe now that, for any vector field $Z$ commuting with $V$, the scalar product $g(V, Z)$ is constant along the $V$-trajectories. Indeed, since $g(V, V) = 0$ and $[Z, V] = 0$ we get

$$V_g(V, Z) = g(\nabla_V V, Z) + g(V, \nabla_V Z) = g(V, \nabla_Z V) = 0.$$ 

We deduce that for any $i = 2, \ldots, n$, $g(V, X_i)$ and $g(U, V)$ are constant along the trajectories of $V$ and since they are constant along $\Sigma$ we get that $g(U, V) = 2$ and $g(V, X_i) = 0$. Moreover, we have

$$V_g(X_i, X_j) = g(\nabla_V X_i, X_j) + g(X_i, \nabla_V X_j) = g(\nabla_{X_i} V, X_j) + g(X_i, \nabla_{X_j} V) = \alpha(X_i)g(V, X_j) + \alpha(X_j)g(X_i, V) = 0.$$ 

This completes the proof. \[\square\]

**Remark 1.** For a general codimension one lightlike foliation, not necessarily geodesic, we have similar adapted coordinates, but with the functions $h_{ij}$ depending also on $v$.

### 3. Kundt Groups

A **Lorentz Lie group** is a Lie group $G$ endowed with a left invariant Lorentzian metric $g$. Denote by $\mathfrak{g}$ the Lie algebra of $G$ and $(\cdot, \cdot) = g(e)$. We call $(\mathfrak{g}, (\cdot, \cdot))$ a Lorentz Lie algebra. The Levi-Civita product is the product $\bullet$ on $\mathfrak{g}$ given by

$$2(u \bullet v, w) = \langle [u, v], w \rangle + \langle [w, u], v \rangle + \langle [w, v], u \rangle, \quad u, v, w \in \mathfrak{g}. \tag{4}$$

A **Kundt Lie group** (resp. **locally Kundt Lie group**) is a Lorentz Lie group $(G, g)$ having an isotropic left invariant vector field satisfying (2) (resp (3)).

**Proposition 3.1.** Let $(G, g)$ be a connected Lorentz Lie group. Then the following are equivalent:

(i) $(G, g)$ is a locally Kundt Lie group.

(ii) There exists a codimension one subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which is degenerate and stable by the Levi-Civita product.

Moreover, if $(G, g)$ is a locally Kundt Lie group then its is a Kundt Lie group if and only if, for any generator $e$ of $\mathfrak{h}^\perp$, $e \bullet e = 0$.

**Proof.** Let us prove that (i) implies (ii). $(G, g)$ is a locally Kundt Lie group if and only if there exists a left invariant vector field $V$ satisfying (3). The codimension one subspace $\mathfrak{h} = V(e) \perp$ is degenerate and, for any $u, v \in \mathfrak{h}$, denote by $u'$ and $v'$ the corresponding left invariant vector fields. Then

$$g(\nabla_{u'} V, V) = u' \cdot g(v', V) - g(v', \nabla_{u'} V) = -\alpha(u')g(v', V) = 0$$

and hence $\nabla_{u'} V(e) = u \bullet v \in \mathfrak{h}$. This means that $\mathfrak{h}$ is stable by the Levi-Civita product, which completes the proof of (i) $\implies$ (ii).

Let us show now that (ii) $\implies$ (i). Suppose there exists $\mathfrak{h}$ a codimension one degenerate subalgebra of $\mathfrak{g}$ which is stable by the Levi-Civita product and consider $v$ a generator of $\mathfrak{h}^\perp$. Denote by $V$ the left invariant vector field associated to $v$. Then, according to the proof of Proposition 2.1, we have, for any $x \in \mathfrak{h}$,

$$\nabla_x V = \alpha(x) V \quad \text{and} \quad \nabla_V V = \alpha_0 V$$

where $\alpha_0$ is a constant. The last assertion is obvious. \[\square\]

**Definition 3.1.** Let $\mathfrak{g}$ be a Lie algebra. A **Kundt pair** on $\mathfrak{g}$ is a pair $(\langle \cdot, \cdot \rangle, \mathfrak{h})$ where $(\langle \cdot, \cdot \rangle)$ is a Lorentzian product on $\mathfrak{g}$ and $\mathfrak{h}$ is a $\langle \cdot, \cdot \rangle$-degenerate codimension one subalgebra stable by the Levi-Civita product $\bullet$ given by (4) and for any $e \in \mathfrak{h}^\perp$, $e \bullet e = 0$.

There is a large class of Lie groups which cannot carry a locally Kundt group structure.

**Lemma 3.1.** Let $\mathfrak{g}$ be a semi-simple compact Lie algebra. Then $\mathfrak{g}$ cannot have a codimension one subalgebra.
Theorem 4.1. Let $G$ be a Lie group of dimension 3 endowed with a left invariant Lorentz metric $g$. Assume there exists a degenerate totally geodesic hypersurface $\Sigma$ in $(G, g)$ (not necessarily complete).

(i) Then, either $(G, g)$ has a constant sectional curvature, or $(G, g)$ is a locally Kundt Lie group.
Lemma 4.1. Let \((G, g)\) be a Lorentz Lie group of dimension 3. If there are three non-tangent (i.e. having different tangent planes at 1) lightlike geodesic hypersurfaces through 1, then \((G, g)\) has constant curvature.

Proof. Assume there exist three different lightlike geodesic hypersurfaces \(\Sigma_1, \Sigma_2\) and \(\Sigma_3\), through 1 \(\in G\). Let \(T_1, T_2, T_3\) be their tangent spaces and \(V_1, V_2, V_3\), non-vanishing vectors in their orthogonal \(T^*_1, T^*_2\) and \(T^*_3\).

Since the \(\Sigma_i\)'s are geodesic, each \(T_i\) is invariant under the Riemann curvature: if \(u, v, w \in T_i\), then \(R(u, v)w \in T_i\).

Let \(e_i\) be a unit (spacelike) vector generating \(T_1 \cap T_2\, and consider \(A_e : V \to R(e_3, V)e_3\). Then \((A_e(V_i), e_3) = \langle R(e_3, V_i)e_3, e_3 \rangle = 0\), that is \(A_e(V_i)\) is orthogonal to \(e_3\). But it is also orthogonal to \(V_1\), by curvature-invariance of \(V_i^\perp\).

Therefore, \(A_e(V_i)\) is collinear to \(V_1\), say \(A_e(V_i) = \lambda_{12} V_1\). Similarly \(A_e(V_2) = \lambda_{23} V_2\).

Consider \((R(e_3, V_i)e_3, V_j)\), which equals \((R(e_3, V_2)e_3, V_1)\). It also equals \(\lambda_{21}(V_1, V_2) = \lambda_{21}(V_1, V_2)\). Since \(V_1\) and \(V_2\) are both null, \(\langle V_1, V_2 \rangle = 0\) then \(\lambda_{12} = \lambda_{21}\). In conclusion \(A_e\) is a homothety on \((e_3)^2\), of ratio, say \(\lambda = \lambda_{12} = \lambda_{21}\).

Since \(A_e(e_3) = 0\), we conclude that \(R(e_2, W)e_3 = \lambda(W(e_3, e_3) - \langle W, e_3 \rangle e_3)\), for any \(W\).

Consider now \(X = (R(e_i, e_j)e_k, e_i)\), with \(k \neq i, j\) and \(i \neq j\). Then \(\langle X, e_k \rangle = 0\), and \(\langle X, e_i \rangle = \langle R(e_i, e_j)e_i, e_k \rangle\) and, hence, computable. Similarly for \((X, e_j)\), and therefore \(X\) is computable.

From all this, it follows that all the curvatures \(R(e_i, e_j)e_k\) are computable, exactly as in the case of a space of constant curvature \(\lambda\).

Proof of Theorem 4.1.

(i) It follows form the lemma that if \((G, g)\) does not have constant curvature, then, through any point pass exactly one or two (germs of) lightlike geodesic hypersurfaces.

For the sake of clarity, let us consider first the case where there exists exactly one germ of such hypersurfaces.

More precisely, there exists a geodesic lightlike hypersurface \(\Sigma\) containing 1. Uniqueness means that for any \(S\) a geodesic lightlike hypersurface, with \(1 \in S\), then \(\Sigma \cap S\) is a neighbourhood of 1 in both \(\Sigma\) and \(S\).

For any \(x \in G\), the translated hypersurface \(\Sigma x\) is the unique geodesic lightlike hypersurface passing through \(x\).

Let us see that the tangent space of \(\Sigma\) is left invariant: if \(x \in \Sigma\), then \(T_x \Sigma = xT_1 \Sigma\) (the last notation means the left translation by \(x\) of \(T_1 \Sigma\)). Indeed, both \(\Sigma\) and \(x\Sigma\) are geodesic lightlike hypersurfaces containing \(x\) hence they coincide near \(x\), and thus have same tangent space: \(T_x \Sigma = T_x(x \Sigma) = xT_1 \Sigma\).

This left invariance of \(T_1 \Sigma\), means that \(\Sigma\) is a “local subgroup”. Its “maximal extension” will be a subgroup which is a geodesic lightlike hypersurface. To be more formal, one defines a plane field \(E\) on \(G\), with \(E(x)\) being the tangent space of the unique geodesic lightlike hypersurface through \(x\). So, \(E(x) = T_x(x \Sigma)\) which implies \(E\) is left invariant. Uniqueness implies \(E\) is integrable: \(\Sigma\) is a local leaf of \(E\) through 1. The global leaf of \(E\) is a subgroup.

Let us now consider the case where we have two geodesic lightlike hypersurfaces \(\Sigma^1\) and \(\Sigma^2\) through 1, which do not coincide near 1, this is equivalent to \(E(1) = T_1 \Sigma^1 \neq T_1 \Sigma^2 = F(1)\).

Let \(x \in \Sigma^1\). Then through \(x\), we have three geodesic lightlike hypersurfaces: \(\Sigma^1, x \Sigma^1\) and \(x \Sigma^2\). Therefore, two among them coincide locally. Let us assume \(x\) is close to 1 and deduce that \(\Sigma^1\) and \(x \Sigma^2\) can not coincide near \(x\).

For this, it is easy to show they have different tangent spaces \(A = T^1 \Sigma^1 \neq T^1(x \Sigma^2) = B\). If they were equal, they would have the same translation to 1, \(x^{-1}A = x^{-1}B\). On one hand, \(x^{-1}B = T^1 \Sigma^2 = F(1)\), and on the other
hand, \( x^{-1}A = x^{-1}T_x \Sigma \) is close to \( E(1) = T_1 \Sigma^1 \), by continuity of the tangent space of \( \Sigma^1 \) and the fact that \( x \) is close to 1. Since \( E(1) \) and \( F(1) \) are transversal, the same is true for \( x^{-1}A \) and \( x^{-1}B \), for \( x \) sufficiently close to 1, and in particular they are not equal, hence \( A \neq B \). From all of this, we infer that \( \Sigma^1 \) and \( x \Sigma^1 \) coincide near \( x \).

As in the case of a unique geodesic lightlike hypersurface, one deduces that \( \Sigma^1 \) is a “local group”. More precisely, one defines two left invariant plane fields \( E \) and \( F \), extending \( E(1) \) and \( F(1) \) respectively. Our previous argument implies that \( \Sigma^1 \) is a local leaf at 1 of \( E \), and so \( E \) is integrable, and the same applies for \( F \). The \( E \) and \( F \)-leaves of 1 are therefore two subgroups which are geodesic lightlike hypersurfaces.

(ii) Now, assume, \( I \), the isotropy group of 1 in the isometry group \( \text{Isom}(G, g) \) is non-compact. Let \( f_n \in I \) be a diverging sequence (it has non-convergent sub-sequence) and consider their graphs \( F_n = \text{Gr}(f_n) \subset G \times G \). Endow \( G \times G \) with the metric \( g \oplus (-g) \), then the \( F_n \)'s are isotropic and totally geodesic. Then we will consider a limit \( L \) of a subsequence of the \( F_n \). To give a formal meaning of this, consider a small convex neighbourhood \( C \) of \((1,1) \) in \( G \times G \). This means any two points of \( C \) can be joined by a unique geodesic segment contained in \( C \). Consider \( F_n \cap C \) and note \( F_n^{\text{tr}} \) the connected component of \((1,1) \) in \( F_n \cap C \). Now, one can give sense to convergence of \( F_n \), exactly as in the situation of affine subspaces in an affine flat space. More precisely, \( F_n \) converge to \( L \), if the tangent spaces \( T_{(1,1)}F_n^{\text{tr}} \) converge to \( T_{(1,1)}L \). Such a limit \( L \) is a geodesic isotropic submanifold in \( G \times G \) of dimension equal to \( \dim G \), but it is no longer a graph of some map \( f : G \to G \), since otherwise, \( f_n \) will converge to \( f \). Thus \( L \) intersects non-trivially the vertical \((1) \times G \), and hence projects onto a degenerate geodesic submanifold \( \Sigma \) in \( G \times \{1\} \). The intersection \( L \cap (\{1\} \times G) \) has dimension one, since it is isotropic, therefore \( \Sigma \) is a hypersurface.

4.1. Comments on the constant curvature case

It is natural to ask what happens if \((G, g)\) has constant sectional curvature, say \( c \)? Let us give here some examples and hints, details will appear elsewhere. The proof of Theorem 4.1 does not apply since there are infinitely many “germs” of lightlike geodesic hypersurfaces through 1: any lightlike hyperplane in \( T_1G \) is tangent to a lightlike geodesic hypersurface exactly as in the universal Lorentz space \( M(c) \) of constant curvature \( c \).

- It turns out that if the isotropy group \( I \) is non-compact and has dimension 1 or 2, then \((G, g)\) is a Kundt group. Indeed, the proof of Theorem 4.1 can be adapted well to this situation where \( \dim I = 1 \) or 2. Consider for this the derivative action of \( I \) on \( g = T_1G \). It preserves exactly one or two hyperplanes which are in fact lightlike. Indeed, let \( L \) be a closed connected non-compact subgroup of \( \text{O}(1,2) \). If \( \dim L = 1 \), then it is either a hyperbolic one parameter group, in this case it preserves exactly two lightlike hyperplanes, or it is a unipotent one parameter group and in this case it preserves exactly one lightlike hyperplane. In case \( \dim L = 2 \), it is conjugate to the triangular subgroup of \( \text{SL}(2, \mathbb{R}) \) and preserves exactly one lightlike hyperplane. All these claims can be confirmed by a direct check-in. In summary, if \( \dim I = 1, 2 \), the argument in the proof of Theorem 4.1 can be adapted and yields a left invariant plane field \( E \) and, thus, a lightlike geodesic subgroup \( H \).

Let us give the following example with \( c = 0 \). Consider on \( \mathbb{R}^3 \), the Lorentz metric \( dx^2 + dydz \). The plane \( E = \{z = 0\} \) is lightlike. Its linear stabilizer is a subgroup \( S \) of dimension 2 in \( \text{SO}(1,2) \) and, hence, its stabilizer in the full Poincaré group \( \text{SO}(1,2) \ltimes \mathbb{R}^3 \) is \( S \ltimes E \). It contains in particular the 3-dimensional (non-unimodular) group \( G \) of elements \((t, a, b) \in \mathbb{R} \times E \) acting by \((x, y, z) \to (x + a, e^t y + b, e^t z) \). This action is free and transitive on the upper half space \( \{z > 0\} \) and, hence, \( G \) inherits a left invariant (non-complete) flat metric. It is Kundt, since its (abelian) subgroup \( E \) has lightlike geodesic orbits. Observe here that the full isometry group of this left invariant metric is \( S \ltimes E \). In particular the isotropy group has dimension one.

- Assume now that the isotropy group has dimension 3. Thus \( \dim \text{Isom}(G, g) = 6 \), and \((G, g)\) is locally isometric to the universal space \( M(c) \) of constant curvature \( c \). However, a subgroup of dimension 3 in \( \text{O}(1,2) \) contains at least its identity component, and also, a subgroup of dimension 6 in \( \text{Isom}(M(c)) \) contains at least its identity component \( \text{Isom}^0(M(c)) \). Therefore, as a homogeneous space \((G, g)\) is globally isometric to \( M(c) = \text{Isom}^0(M(c))/\text{O}^0(1, 2) \). In other words, \( G \) acts transitively and freely on \( M(c) \) (or equivalently, \((G, g)\) has constant curvature \( c \) and is complete). Let us give the example of the Euclidean group \( \text{Euc}_2 \). Its universal cover \( \tilde{\text{Euc}}_2 \) acts simply transitively isometrically on \((\mathbb{R}^3, dx^2 + dy^2 - dz^2) \) by: \((x, y, z) \to ((R(x, y) + (a, b)), z + t)\), where \( R_t \) is the rotation of angle \( t \) and \((t, a, b) \in \tilde{\text{Euc}}_2 \).
Its unique 2-dimensional subgroup is $\mathbb{R}^2$, which acts by translations $(x, y, z) \to (x+a, y+b, z)$. It has spacelike geodesic orbits. Therefore $E_{1\overset{\perp}{,}2}$ is a flat complete Lorentz group that is not a Kundt group.

We believe this is the unique complete Lorentz group of constant curvature which is not a Kundt group?

- Finally, there are examples of flat groups $(G, g)$ with isotropy group $I$ of dimension 0, which are not Kundt groups. To see an example, consider as above the metric $dx^2 + dydz$. Let $S \subset SO(1, 2)$ be the stabilizer of the isotropic direction $\mathbb{R}^2_x$ and $T$ the subgroup of translations in this direction. Take $G = S \times T$. It has an open orbit on which it acts freely which allows one to endow it with a flat (non-complete) metric. One can show it is not a Kundt group.

5. Classification of three dimensional unimodular simply-connected Kundt Lie groups

In this section, we give a complete classification of Kundt Lie group structures on three dimensional unimodular Lie groups. According to Proposition 3.1, the classification of Kundt Lie group structures on a simply connected Lie group $G$ is equivalent to the classification of Kundt pairs $(h, (\cdot, \cdot))$ on its Lie algebra $\mathfrak{g}$. Two Kundt pairs $(h_1, (\cdot, \cdot))$ and $(h_2, (\cdot, \cdot))$ are called equivalent if there exists an automorphism of Lie algebra $\phi : \mathfrak{g} \to \mathfrak{g}$ such that $\phi(h_1) = h_2$ and $\phi(h_1) = (\cdot, \cdot)_1$.

In dimension 3, we have the following useful characterization of Kundt pairs.

**Proposition 5.1.** Let $(\mathfrak{g}, (\cdot, \cdot))$ be a Lorentzian Lie algebra and let $\mathfrak{h}$ be a codimension one subalgebra. Then:

(i) If $\mathfrak{h}$ is abelian then $(\mathfrak{h}, (\cdot, \cdot))$ is a Kundt pair if and only if $\mathfrak{h}$ is degenerate and if $e$ is a generator of $\mathfrak{h}^\perp$ then $ad_e(\mathfrak{g}) \subset \mathfrak{h}$.

(ii) If $\mathfrak{h}$ is non-abelian then $(\mathfrak{h}, (\cdot, \cdot))$ is a Kundt pair if and only if $\mathfrak{h}^\perp = [\mathfrak{h}, \mathfrak{h}]$ and if $e$ is a generator of $\mathfrak{h}^\perp$ then $ad_e(\mathfrak{g}) \subset \mathfrak{h}$.

**Proof.** Let $e$ a generator of $\mathfrak{h}^\perp$. Then $(\mathfrak{h}, (\cdot, \cdot))$ is a Kundt pair if and only if $e \in \mathfrak{h}^\perp$, $(e, e) = 0$, $e \bullet e = 0$ and for any $u, v \in \mathfrak{h}$,

$$0 = 2(u \bullet v, e) = \langle [e, u], v \rangle + \langle [e, v], u \rangle. \quad (7)$$

Note first that $e \bullet e = 0$ and $(e, e) = 0$ if and only if, for any $u \in \mathfrak{g}$,

$$0 = \langle e \bullet e, x \rangle = \langle [u, e], e \rangle$$

which is equivalent to $ad_e(\mathfrak{g}) \subset \mathfrak{h}$.

If $\mathfrak{h}$ is abelian then (7) holds trivially.

Suppose now that $\dim \mathfrak{g} = 3$ and $\mathfrak{h}$ is not abelian. Then there exists a basis $(u, v)$ of $\mathfrak{h}$ such $\langle u, v \rangle = 0$ and $[u, v] = u$.

Put $e = au + bv$. Then from the relation above, we get

$$0 = -b(u, v) + a(u, u) = a(u, u) \quad \text{and} \quad 0 = \langle [e, u], u \rangle = b(u, u).$$

This implies that $\langle a, u \rangle = 0$ and hence $\mathfrak{h}^\perp = [\mathfrak{h}, \mathfrak{h}]$. The converse is obviously true.

There are five simply connected three dimensional unimodular non abelian Lie groups:

1. The nilpotent Lie group $\text{Nil}$ known as Heisenberg group whose Lie algebra will be denoted by $\mathfrak{n}$. We have

$$\text{Nil} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R} \right\} \quad \text{and} \quad \mathfrak{n} = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{n}$ has a basis $\mathfrak{B}_0 = (X_1, X_2, X_3)$ where

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where the non-vanishing Lie bracket is $[X_1, X_2] = X_3$. 

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2. $\text{SU}(2) = \left\{ \begin{pmatrix} a+bi & -c+di \\ c-di & a-bi \end{pmatrix} \mid a^2 + b^2 + c^2 + d^2 = 1 \right\}$ and $\text{su}(2) = \left\{ \begin{pmatrix} iz & y+ix \\ -y+xi & -zi \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$. The Lie algebra $\text{su}(2)$ has a basis $\mathbb{B}_0 = \{X_1, X_2, X_3\}$

\[ X_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad X_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad X_3 = \frac{1}{2} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \]

where the non-vanishing Lie brackets are

\[ [X_1, X_2] = X_3, \quad [X_2, X_3] = X_1 \quad \text{and} \quad [X_3, X_1] = X_2. \]

3. The universal covering group $\text{PSL}(2, \mathbb{R})$ of $\text{SL}(2, \mathbb{R})$ whose Lie algebra is $\text{sl}(2, \mathbb{R})$. The Lie algebra $\text{sl}(2, \mathbb{R})$ has a basis $\mathbb{B}_0 = \{e, f, h\}$ where

\[ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

where the non-vanishing Lie brackets are

\[ [e, f] = h, \quad [h, e] = 2e \quad \text{and} \quad [h, f] = -2f. \]

4. The solvable Lie group $\text{Sol} = \left\{ \begin{pmatrix} e^x & 0 & y \\ 0 & e^{-x} & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$ whose Lie algebra is $\text{sol} = \left\{ \begin{pmatrix} x & 0 & y \\ 0 & -x & z \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$. The Lie algebra $\text{sol}$ has a basis $\mathbb{B}_0 = \{X_1, X_2, X_3\}$ where

\[ X_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \]

where the non-vanishing Lie brackets are

\[ [X_1, X_2] = X_3 \quad \text{and} \quad [X_1, X_3] = -X_3. \]

5. The universal covering group $\tilde{\text{E}_0}(2)$ of the Lie group

\[ \tilde{\text{E}_0}(2) = \left\{ \begin{pmatrix} \cos(\theta) & \sin(\theta) & x \\ -\sin(\theta) & \cos(\theta) & y \\ 0 & 0 & 1 \end{pmatrix} \mid \theta, x, y \in \mathbb{R} \right\} \].

Its Lie algebra is

\[ \mathfrak{e}_0(2) = \left\{ \begin{pmatrix} 0 & \theta & x \\ -\theta & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid \theta, x, y \in \mathbb{R} \right\} \].

The Lie algebra $\mathfrak{e}_0(2)$ has a basis $\mathbb{B}_0 = \{X_1, X_2, X_3\}$ where

\[ X_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \]

where the non-vanishing Lie brackets are

\[ [X_1, X_2] = X_3 \quad \text{and} \quad [X_1, X_3] = -X_2. \]

Let us find the 2-dimensional subalgebras of the 3-dimensional unimodular Lie algebras.

**Proposition 5.2.** 1. Let $\mathfrak{h}$ be a 2-dimensional subalgebra of $\mathfrak{n}$. Then $\mathfrak{h} = \text{span}[X_3, aX_1 + bX_2], \ (a, b) \neq (0, 0).$
2. \( \text{su}(2) \) has no subalgebra of dimension 2.
3. Let \( \mathfrak{h} \) be a 2-dimensional subalgebra of \( \text{so}(3) \) then either \( \mathfrak{h} = \text{span}[X_2, X_3], \) \( \mathfrak{h} = \text{span}[X_2, X_1 + aX_3] \) or \( \mathfrak{h} = \text{span}[X_3, X_1 + aX_2] (a \in \mathbb{R}) \)
4. Let \( \mathfrak{h} \) be a 2-dimensional subalgebra of \( \text{e}_0(2) \) then \( \mathfrak{h} = \text{span}[X_2, X_3]. \)
5. Let \( \mathfrak{h} \) be 2-dimensional subalgebra of \( \text{sl}(2, \mathbb{R}) \). Then there exists an automorphism of \( \text{sl}(2, \mathbb{R}) \) which sends \( \mathfrak{h} \) to \( \text{span}[h, e]. \)

**Proof.**
1. A 2-dimensional subalgebra \( \mathfrak{h} \) of \( \mathfrak{n} \) must be abelian and contains the center. So \( \mathfrak{h} = \text{span}[X_1, aX_1 + bX_2] \) and \((a, b) \neq (0, 0). \)
2. It is a consequence of Lemma 3.1.
3. Denote by \( \mathfrak{h}_0 = \text{span}[X_2, X_3]. \) If \( \mathfrak{h} \) is abelian and \( \mathfrak{h} \neq \mathfrak{h}_0 \) then \( \mathfrak{h} = \text{span}[X_1 + U, V] \) where \( U, V \in \mathfrak{h}_0 \) hence \([X_1, V] = 0\) which is impossible. So if \( \mathfrak{h} \) is abelian then \( \mathfrak{h} = \mathfrak{h}_0. \)
If \( \mathfrak{h} \) is not abelian then \( \mathfrak{h} \neq \mathfrak{h}_0. \) Thus \( \mathfrak{h} = \text{span}[X_1 + U, V] \) where \( U, V = aX_2 + bX_3 \in \mathfrak{h}_0 \) and \([\mathfrak{h}, \mathfrak{h}] = \mathbb{R} V. \) Now
\[
[X_1 + U, V] = aX_2 - bX_3.
\]
So the vectors \( aX_2 + bX_3, aX_2 - bX_3 \) must be linearly dependent, hence \( ab = 0. \) Which complete the proof.
4. We can use the same argument as above and get \( a^2 + b^2 = 0. \)
5. Note first that \([e, f] = h, [h, e] = 2e, [h, f] = -2f. \) Let \( \mathfrak{h} \) be a 2-dimensional subalgebra of \( \text{sl}(2, \mathbb{R}). \) Then there exists a basis \((u, v)\) of \( \mathfrak{h} \) such that \([u, v] = 2v. \) The endomorphism \( \text{ad}_h \) is skew-symmetric with respect to the Killing form hence \( \text{tr} (\text{ad}_h) = 0. \) It has 2 and 0 as eigenvalues so the third eigenvalue is \(-2. \) So there exists \( w \in \text{sl}(2, \mathbb{R}) \) such that \([u, w] = -2w. \) Now
\[
[u, v, w] = 2[v, w] - 2[v, w] = 0
\]
and hence \([v, w] = au. \) By replacing \( w \) by \( \frac{1}{a} w \) we get that the automorphism \( \phi \) which sends \((u, v, w) \) to \((h, e, f) \) sends \( \mathfrak{h} \) to \( \text{span}[h, e] \) which completes the proof.

**Theorem 5.1.** Let \((\mathfrak{h}, \langle , \rangle)\) be a Kautz pair of \( \mathfrak{n}. \) Then \((\mathfrak{h}, \langle , \rangle)\) is equivalent to \((\mathfrak{h}_0, \langle , \rangle_0)\) where either:

1. \( \langle , \rangle_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \mu \end{bmatrix}, \mu > 0 \) and \( \mathfrak{h}_0 = \text{span}[X_1 \pm X_2, X_3]. \)
2. \( \langle , \rangle_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \) and \( \mathfrak{h}_0 = \text{span}[X_1, X_3]. \)

**Proof.** Let \((\mathfrak{h}, \langle , \rangle)\) be a Kautz pair on \( \mathfrak{n}. \) According to [1, Theorem 3.1], there exists an automorphism \( \phi \) of \( \mathfrak{n} \) such that the matrix of \((\phi^{-1})^\ast(\langle , \rangle)\) in the basis \((X_1, X_2, X_3))\) has one of the following forms:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & \mu
\end{bmatrix}, \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -\mu
\end{bmatrix}, \mu > 0.
\]

By virtue of Proposition 5.2, \( \phi(\mathfrak{h}) = \mathfrak{h}_0 = \text{span}[X_3, aX_1 + bX_2] \) with \((a, b) \neq (0, 0). \) According to Proposition 5.1, \( (\phi(\mathfrak{h}), (\phi^{-1})^\ast(\langle , \rangle)) \) is a Kautz pair if and only if \( \phi(\mathfrak{h}_0) \) is degenerate and \( \text{ad}_e(\mathfrak{n}) \subseteq \mathfrak{h}_0 \) where \( e \) is a generator of \( \mathfrak{h}_0. \) Since \([\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{h}_0 \) then the last condition holds.

Now, \( \mathfrak{h}_0 \) cannot be \( \mathfrak{n}_2 \)-degenerate and it is \( \mathfrak{n}_3 \)-degenerate if and only if \( b = 0. \) Finally, \( \phi(\mathfrak{h}) \) is \( \mathfrak{n}_1 \)-degenerate if and only if \( a^2 - b^2 = 0 \) which completes the proof.

**Theorem 5.2.** Let \((\mathfrak{h}, \langle , \rangle)\) be a Kautz pair of \( \text{so}(3) \) Then \((\mathfrak{h}, \langle , \rangle)\) is equivalent to \((\mathfrak{h}_0, \langle , \rangle_0)\) where either:
1. \( \langle \cdot, \cdot \rangle_0 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \), \( \lambda > 0 \) and \( b_0 = \text{span}[X_2, X_1] \) or \( b_0 = \text{span}[X_3, X_1] \).

2. \( \langle \cdot, \cdot \rangle_0 = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 1 & 0 \end{pmatrix} \) and \( b_0 = \text{span}[X_1, X_1] \).

3. \( \langle \cdot, \cdot \rangle_0 = \begin{pmatrix} 0 & 0 & -\frac{2}{\lambda} \\ 0 & 1 & 1 \\ -\frac{2}{\lambda} & 1 & 1 \end{pmatrix} \), \( b > 0 \) and \( b_0 = \text{span}[X_2, X_3] \).

4. \( \langle \cdot, \cdot \rangle_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \) and \( b_0 = \text{span}[X_2, X_3] \).

**Proof.** Let \( (h, \langle \cdot, \cdot \rangle) \) be a Kundt pair on sol. Then according to [1, Theorem 3.4], there exists an automorphism \( \phi \) of sol such that the matrix of \((\phi^{-1})'(\langle \cdot, \cdot \rangle)\) in the basis \((X_1, X_2, X_3)\) has one of the following forms:

\[
\begin{align*}
\text{sol}_1 &= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & u & 1 \end{pmatrix}, \quad v > 0, u < v, \\
\text{sol}_2 &= \begin{pmatrix} 4 & 0 & 0 \\ \frac{4}{v-u} & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad v > 0, u < v, \\
\text{sol}_3 &= \begin{pmatrix} 0 & 0 & -\frac{2}{\lambda} \\ 0 & 1 & 1 \\ -\frac{2}{\lambda} & 1 & 1 \end{pmatrix}, \quad b > 0, \\
\text{sol}_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda \neq 0, \\
\text{sol}_5 &= \begin{pmatrix} \frac{1}{u-v} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad u > 0, v > 0,
\end{align*}
\]

By virtue of Proposition 5.2, \( \phi(h) = b_0 \) where either \( b_0 = \text{span}[X_2, X_3] \), \( b_0 = \text{span}[X_2, X_1 + aX_2] \) or \( b_0 = \text{span}[X_3, X_1 + aX_2] \) (\( a \in \mathbb{R} \)).

If \( b_0 = \text{span}[X_2, X_3] \) then it is abelian and it is sol$_3$-degenerate and sol$_7$-degenerate. For sol$_3$, \( \text{ad}_{X_2,X_1}^{X_3}(\text{sol}) \subset b_0 \). We have the same situation for sol$_7$. Thus \( (b_0, \text{sol}_3) \) and \( (b_0, \text{sol}_7) \) are Kundt pairs.

If \( b_0 = \text{span}[X_2, X_1 + aX_2] \) then \( [b_0, b_0] = \mathbb{R}X_2 \). We have obviously, \( \text{ad}_{X_2,X_1}^{X_2}(\text{sol}) \subset b_0 \) and, according to Proposition 5.1, \( (b_0, \langle \cdot, \cdot \rangle) \) is a Kundt pair if and only if \( \langle X_2, X_2 \rangle = \langle X_2, X_1 + aX_3 \rangle = 0 \). This is possible if and only if \( \langle \cdot, \cdot \rangle = \text{sol}_2 \) with \( u = 0 \) and \( a = 0 \).

If \( b_0 = \text{span}[X_3, X_1 + aX_2] \) then \( [b_0, b_0] = \mathbb{R}X_1 \). We have obviously, \( \text{ad}_{X_2,X_1}^{X_1}(\text{sol}) \subset b_0 \) and, according to Proposition 5.1, \( (b_0, \langle \cdot, \cdot \rangle) \) is a Kundt pair if and only if \( \langle X_1, X_3 \rangle = \langle X_3, X_1 + aX_2 \rangle = 0 \). This is possible if and only if \( \langle \cdot, \cdot \rangle = \text{sol}_2 \), \( u = 0 \) and \( a = 0 \) or \( \langle \cdot, \cdot \rangle = \text{sol}_4 \) and \( a = 0 \).

**Theorem 5.3.** Let \( (h, \langle \cdot, \cdot \rangle) \) be a Kundt structure on \( e_0(2) \). Then \( (h, \langle \cdot, \cdot \rangle) \) is equivalent to \( (b_0, \langle \cdot, \cdot \rangle_0) \) where

\[
\begin{pmatrix}
1 & 0 \\
0 & 0 \\
\mu & 0
\end{pmatrix}, \quad \mu > 0 \quad \text{and} \quad b_0 = \text{span}[X_2, X_3].
\]

**Proof.** Let \( (h, \langle \cdot, \cdot \rangle) \) be a Kundt pair on \( e_0(2) \). Then according to [1, Theorem 3.5], there exists an automorphism \( \phi \) of \( e_0(2) \) such that the matrix of \((\phi^{-1})'(\langle \cdot, \cdot \rangle)\) in the basis \((X_1, X_2, X_3)\) has one of the following forms:

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & u & 0 \\
1 & 0 & \nu
\end{pmatrix}, \quad u > 0, \nu > 0, \quad \text{and} \quad
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
0 & 0 & \nu
\end{pmatrix}, \quad u > 0, \nu > 0, \quad \text{and} \quad
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & \nu
\end{pmatrix}, \quad \nu > 0.
\]

By virtue of Proposition 5.2, \( \phi(h) = b_0 = \text{span}[X_2, X_1] \). According to Proposition 5.1, \( (\phi(b), (\phi^{-1})'(\langle \cdot, \cdot \rangle)) \) is a Kundt pair if and only if \( b_0 \) is degenerate and \( \text{ad}_{X_i}(\text{sol}) \subset b_0 \) where \( e \) is a generator of \( b_0 \). Now, \( b_0 \) cannot be \( \langle \cdot, \cdot \rangle_1 \)-degenerate or \( \langle \cdot, \cdot \rangle_2 \)-degenerate. Finally, \( b_0 \) is \( \langle \cdot, \cdot \rangle_3 \)-degenerate, \( b_0^3 = \mathbb{R}X_2 \) and \( \text{ad}_{X_i}(e_0(2)) \subset b_0 \).

**Theorem 5.4.** Let \( (h, \langle \cdot, \cdot \rangle) \) be a Kundt pair on \( \text{sl}(2, \mathbb{R}) \). Then there exists an automorphism \( \phi \) of \( \text{sl}(2, \mathbb{R}) \) such that \( \phi(h) = \text{span}[e, h] \) and the matrix of \( \phi'(\langle \cdot, \cdot \rangle) \) in the basis \( (e, f, h) \) has one of the following forms:
We consider the isomorphism $A$. Denote by $P$ that $\alpha$ are known. We give here the normal form of those having an isotropic eigenvector. According to $\langle e, e \rangle = (e, h) = 0$ and $\text{Ad}(\text{sl}(2, \mathbb{R})) \subset \mathfrak{h}$. This is equivalent to $ae$ is a generator of the orthogonal of $\mathfrak{h}$ with respect to $B$ which is equivalent to the existence of $\alpha \neq 0$ such that $ae = \alpha e$. The normal form of isomorphisms which are symmetric with respect to a Lorentzian scalar product are known. We give here the normal form of those having an isotropic eigenvector. According to [16], there exits a basis $\mathbb{B} = (f_1, f_2, f_3)$ of $\text{sl}(2, \mathbb{R})$ such that:

$$
\begin{bmatrix}
0 & 4\alpha & 0 \\
4\alpha & 0 & 0 \\
0 & 0 & 8\beta
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 4\alpha & 0 \\
4\alpha & 1 & 0 \\
0 & 0 & 8\beta
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 4\alpha & 0 \\
4\alpha & 0 & 2\sqrt{2} \\
0 & 2\sqrt{2} & 8\alpha
\end{bmatrix}, \quad \beta > 0, \ \alpha \in \mathbb{R}^+.
$$

Proof. Let $(\mathfrak{h}, \langle , \rangle)$ be a Kundt pair on $\text{sl}(2, \mathbb{R})$. According to Proposition 5.2, we can suppose that $\mathfrak{h} = \text{span}(e, h)$. A direct computation using the software Maple shows that the automorphisms of $\text{sl}(2, \mathbb{R})$ leaving $\mathfrak{h}$ invariant are of the form

$$
T = \begin{bmatrix}
a & -ab^2 & -2ab \\
0 & a^{-1} & 0 \\
0 & b & 1
\end{bmatrix}, \quad a, b \in \mathbb{R}.
$$

Denote by $B$ the Killing form of $\text{sl}(2, \mathbb{R})$. Its matrix in the basis $(e, f, h)$ is given by

$$
M = \begin{bmatrix}
0 & 4 & 0 \\
4 & 0 & 0 \\
0 & 0 & 8
\end{bmatrix}.
$$

We consider the isomorphism $A$, symmetric with respect to $B$, and given by $B(Au, v) = \langle u, v \rangle$ for any $u, v \in \text{sl}(2, \mathbb{R})$. According to Proposition 5.1, the pair $(\mathfrak{h}, \langle , \rangle)$ is Kundt if and only if $\langle e, e \rangle = (e, h) = 0$ and $\text{Ad}(\text{sl}(2, \mathbb{R})) \subset \mathfrak{h}$. This is equivalent to $ae$ is a generator of the orthogonal of $\mathfrak{h}$ with respect to $B$ which is equivalent to the existence of $\alpha \neq 0$ such that $ae = \alpha e$. The normal form of isomorphisms which are symmetric with respect to a Lorentzian scalar product are known. We give here the normal form of those having an isotropic eigenvector. According to [16], there exits a basis $\mathbb{B} = (f_1, f_2, f_3)$ of $\text{sl}(2, \mathbb{R})$ such that:

(i) $\text{Mat}(A, \mathbb{B}) = \begin{bmatrix}
\beta & 0 & 0 \\
0 & \alpha & 0 \\
0 & 0 & \alpha
\end{bmatrix}$ and $\text{Mat}(B, \mathbb{B}) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}$.

(ii) $\text{Mat}(A, \mathbb{B}) = \begin{bmatrix}
\beta & 0 & 0 \\
0 & \alpha & 1 \\
0 & 0 & \alpha
\end{bmatrix}$ and $\text{Mat}(B, \mathbb{B}) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}$.

(iii) $\text{Mat}(A, \mathbb{B}) = \begin{bmatrix}
\alpha & 1 & 0 \\
0 & \alpha & 1 \\
0 & 0 & \alpha
\end{bmatrix}$ and $\text{Mat}(B, \mathbb{B}) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}$.

Let $P$ be the passage matrix from $\mathbb{B}_0 = (e, f, h)$ to $\mathbb{B}$.

\begin{itemize}
  \item The case (i). The relation $A(e) = ae$ implies that we can choose $f_2 = e$ and the relation $PMP = \text{Mat}(B, \mathbb{B})$ gives that $P = \begin{bmatrix}
-\sqrt{2} & 1 & -1 \\
0 & 0 & 1/4 \\
1/4 & \sqrt{2} & 0
\end{bmatrix}$. The matrix of $\langle , \rangle$ in the basis $B_0$ is given by $\text{Mat}(A, \mathbb{B}_0)^T M$ and $\text{Mat}(A, \mathbb{B}_0) = \begin{bmatrix}
0 & 4\alpha & 0 \\
4\alpha & 32\beta - 32\alpha & -16\beta + 16\alpha \\
0 & -16\beta + 16\alpha & 8\beta
\end{bmatrix}$. So we get
\end{itemize}
Now the automorphism

\[ T_1 = \begin{bmatrix} 1 & -4 & -4 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \]

satisfies

\[ T'_1 \text{Mat}((\cdot,\cdot),\mathbb{B}_0)T_1 = \begin{bmatrix} 0 & 4\alpha & 0 \\ 4\alpha & 0 & 0 \\ 0 & 0 & 8\beta \end{bmatrix}. \]

- The case (ii). We have

\[ P = \begin{bmatrix} -\sqrt{2} & 1 & -1 \\ 0 & 0 & 1/4 \\ 1/4 \sqrt{2} & 0 & 1/2 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 4 & -1 & -4 \\ 0 & 1/4 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}, \]

\[ T'_2 \text{Mat}((\cdot,\cdot),\mathbb{B}_0)T_2 = \begin{bmatrix} 0 & 4\alpha & 0 \\ 4\alpha & 1 & 0 \\ 0 & 0 & 8\beta \end{bmatrix}. \]

- The case (iii). We have

\[ P = \begin{bmatrix} 1 & -\sqrt{2} & -1 \\ 0 & 0 & 1/4 \\ 0 & 1/4 \sqrt{2} & 1/2 \end{bmatrix} \quad \text{and} \quad T_3 = \begin{bmatrix} 4 & -1 & -4 \\ 0 & 1/4 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}, \]

\[ T'_3 \text{Mat}((\cdot,\cdot),\mathbb{B}_0)T_3 = \begin{bmatrix} 0 & 4\alpha & 0 \\ 4\alpha & 0 & 2 \sqrt{2} \\ 0 & 2 \sqrt{2} & 8\alpha \end{bmatrix}. \]

\[ \square \]

5.1. Kundt vs Locally Kundt Lie Groups

In fact, it turns out from the previous proofs, we have shown that any 3-dimensional unimodular locally Kundt Lie group is in fact a Kundt Lie group. This result is not true in general as the following example shows.

**Example 2.** Consider \( \mathbb{R}^4 \) endowed with the Lie algebra structure where the only non vanishing Lie bracket is given by \([e_1,e_2] = e_2\) and the Lorentzian scalar product given by

\[ \langle \cdot, \cdot \rangle = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \]

The Lie subalgebra \( \mathfrak{b} = \text{span}[e_1,e_3,e_4] \) is abelian and satisfies \( \mathfrak{b}^+ = \mathbb{R}e_1 \) and hence, according to Proposition 3.1, defines a local Kundt Lie group structure on the corresponding simply connected Lie group. However, \( \text{ad}_{e_1}(\mathbb{R}^4) \not\subset \mathfrak{b} \) and hence, according to Proposition 5.1 this structure is not global.

We think however, it is worthwhile to investigate the natural question: is a locally Kundt Lie group, without being a Kundt group, still a (globally) Kundt spacetime?
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