CALDERÓN-ZYGMUND CAPACITIES AND WOLFF POTENTIALS ON CANTOR SETS

XAVIER TOLSA

Abstract. We show that, for some Cantor sets in $\mathbb{R}^d$, the capacity $\gamma_s$ associated to the $s$-dimensional Riesz kernel $x/|x|^{s+1}$ is comparable to the capacity $\dot{C}_2^{(d-s),\frac{3}{2}}(d-s)$ from non linear potential theory. It is an open problem to show that, when $s$ is positive and non integer, they are comparable for all compact sets in $\mathbb{R}^d$. We also discuss other open questions in the area.

1. Introduction

In the first part of this paper we show that, for some Cantor sets in $\mathbb{R}^d$, the capacity $\gamma_s$ associated to the $s$-dimensional Riesz kernel $x/|x|^{s+1}$ is comparable to the capacity $\dot{C}_2^{(d-s),\frac{3}{2}}(d-s)$ from non linear potential theory. It is an open problem to show that, when $s$ is a positive and non integer, they are comparable for all compact sets in $\mathbb{R}^d$. In the last part of the paper, we discuss other related open questions.

To state our results in detail we need to introduce some notation. For $0 < s < d$, the $s$-dimensional Riesz kernel is defined by

$$K^s(x) = \frac{x}{|x|^{s+1}}, \quad x \in \mathbb{R}^d, \quad x \neq 0.$$ Notice that this is a vectorial kernel. The $s$-dimensional Riesz transform (or $s$-Riesz transform) of a real Radon measure $\nu$ with compact support is

$$R^s\nu(x) = \int K^s(y-x)\,d\nu(y), \quad x \notin \text{supp}(\nu).$$

Although the preceding integral converges a.e. with respect to Lebesgue measure, the convergence may fail for $x \in \text{supp}(\nu)$. This is the reason why one considers the truncated $s$-Riesz transform of $\nu$, which is defined as

$$R^s_\varepsilon \nu(x) = \int_{|y-x|>\varepsilon} K^s(y-x)\,d\nu(y), \quad x \in \mathbb{R}^d, \quad \varepsilon > 0.$$ These definitions also make sense if one consider distributions instead of measures. Given a compactly supported distribution $T$, set

$$R^s(T) = K^s * T.$$ Partially supported by grant MTM2007-62817 (Spain) and 2009-SGR-420 (Catalonia). Part of this paper was done during the attendance to the research semester “Harmonic Analysis, Geometric Measure Theory and Quasiconformal Mappings” at the CRM (Barcelona), in 2009.
(in the principal value sense for \( s = d \)), and analogously

\[
R_s^\varepsilon(T) = K_s^\varepsilon * T,
\]

where \( K_s^\varepsilon(x) = \chi_{|x|>\varepsilon} x/|x|^{s+1} \).

Given a positive Radon measure with compact support and a function \( f \in L^1(\mu) \), we consider the operators \( R_s^\mu(f) := R_s^\varepsilon(f \, d\mu) \) and \( R_s^\mu,\varepsilon(f) := R_s^\varepsilon(f \, d\mu) \). We say that \( R_s^\mu \) is bounded on \( L^2(\mu) \) if \( R_s^\mu,\varepsilon \) is bounded on \( L^2(\mu) \) uniformly in \( \varepsilon > 0 \), and we set

\[
\|R_s^\mu\|_{L^2(\mu) \to L^2(\mu)} = \sup_{\varepsilon>0} \|R_s^\mu,\varepsilon\|_{L^2(\mu) \to L^2(\mu)}.
\]

Given a compact set \( E \subset \mathbb{R}^d \), the capacity \( \gamma_s \) of \( E \) is

\[
(1.1) \quad \gamma_s(E) = \sup \{|T, 1|\},
\]

where the supremum is taken over all distributions \( T \) supported on \( E \) such that \( \|R_s^\varepsilon(T)\|_{L^\infty(\mathbb{R}^d)} \leq 1 \). Following [Vol03], we call \( \gamma_s \) the \( s \)-dimensional Calderón-Zygmund capacity. The case \( s = d-1 \) is particularly relevant: \( \gamma_{d-1} \) coincides with the capacity \( \kappa \) introduced by Paramonov [Par93] in order to study problems of \( C^1 \) approximation by harmonic functions in \( \mathbb{R}^d \) (the reader should notice that \( \kappa \) is called \( \kappa' \) in [Par93]). When \( d = 2 \) and \( s = 1 \), \( z/|z|^{s+1} \) coincides with the complex conjugate of the Cauchy kernel \( 1/z \). Thus, if one allows \( T \) to be a complex distribution in the supremum above, then \( \gamma_1 \) is the analytic capacity.

If we restrict the supremum in (1.1) to distributions \( T \) given by positive Radon measures supported on \( E \), we obtain the capacities \( \gamma_s,+,E \). Clearly, we have \( \gamma_s(E) \geq \gamma_{s,+}(E) \). On the other hand, the opposite inequality also holds (up to a multiplicative absolute constant \( c_s \)):

\[
\gamma_s(E) \leq c_s \gamma_{s,+}(E).
\]

This was first shown for \( s = 1, d = 2 \) by the author [Tol03], and it was extended to the case \( s = d-1 \) by Volberg [Vol03]. For other values of \( s \), this can be proved by combining the techniques from [Vol03] with others from [MPV05] (see [Pra09]).

Now we turn to non linear potential theory. Given \( \alpha > 0 \) and \( 1 < p < \infty \) with \( 0 < \alpha p < 2 \), the capacity \( \hat{C}_{\alpha,p} \) of \( E \subset \mathbb{R}^d \) is defined as

\[
\hat{C}_{\alpha,p}(E) = \sup \mu(E)^p,
\]

where the supremum runs over all positive measures \( \mu \) supported on \( E \) such that

\[
I_\alpha(\mu)(x) = \int \frac{1}{|x-y|^{2-\alpha}} d\mu(x)
\]

satisfies \( \|I_\alpha(\mu)\|_{p'} \leq 1 \), where as usual \( p' = p/(p-1) \).

For our purposes, the characterization of \( \hat{C}_{\alpha,p} \) in terms of Wolff potentials is more useful than its definition above. Consider

\[
\hat{W}^\mu_{\alpha,p}(x) = \int_0^\infty \left( \frac{\mu(B(x,r))}{r^{2-\alpha p}} \right)^{p'-1} dr.
\]
A well known theorem of Wolff asserts that
\begin{equation}
\dot{C}_{\alpha,p}(E) \approx \sup_{\mu} \mu(E),
\end{equation}
where the supremum is taken over all measures \( \mu \) supported on \( E \) such that \( \dot{W}_{\alpha,p}^\mu(x) \leq 1 \) for all \( x \in E \) (see [AH96, Chapter 4], for instance). The notation \( A \approx B \) means that there is an absolute constant \( c > 0 \), or depending on \( d \) and \( s \) at most, such that \( c^{-1}A \leq B \leq cB \).

Mateu, Prat and Verdera showed in [MPV05] that if \( 0 < s < 1 \), then
\[ \gamma_s(E) \approx \dot{C}_{2(d-s)\frac{1}{2}}(E). \]
Notice that the Wolff’s potential for the capacity \( \dot{C}_{2(d-s)\frac{1}{2}}(E) \) is
\[ \dot{W}_{2(d-s)\frac{1}{2}}^\mu(x) = \int_0^\infty \left( \frac{\mu(B(x,r))}{r^s} \right)^2 \frac{dr}{r}. \]
When \( s = 1 \) and \( d = 2 \), from the characterization of \( \gamma_{1,+} \) in terms of curvature of measures, one easily gets \( \gamma_{1}(E) \geq \dot{C}_{2(d-s)\frac{1}{2}}(E) \). Using analogous arguments (involving a symmetrization of the kernel and the \( T(1) \) theorem), in [ENV08] it has been shown that this also holds for all indices \( 0 < s < d \):
\[ \gamma_s(E) \geq \dot{C}_{2(d-s)\frac{1}{2}}(E), \]
for any compact set \( E \subset \mathbb{R}^d \). The opposite inequality is false when \( s \) is integer (for instance, if \( E \) is contained in an \( s \)-plane and has positive \( s \)-dimensional Hausdorff measure, then \( \gamma_s(E) > 0 \), but \( \dot{C}_{2(d-s)\frac{1}{2}}(E) = 0 \)). When \( 0 < s < d \) is non integer, it is an open problem to prove (or disprove) that
\[ \gamma_s(E) \lesssim \dot{C}_{2(d-s)\frac{1}{2}}(E). \]
See Section 6 for more details and related questions.

In the present paper we show that the comparability \( \gamma_s(E) \approx \dot{C}_{2(d-s)\frac{1}{2}}(E) \) holds for some Cantor sets \( E \subset \mathbb{R}^d \), which are defined as follows. Given a sequence \( \lambda = (\lambda_n)_{n=1}^\infty \), \( 0 \leq \lambda_n < 1/2 \), we construct \( E \) by the following algorithm. Consider the unit cube \( Q^0 = [0,1]^d \). At the first step we take \( 2^d \) closed cubes inside \( Q^0 \), of side length \( \ell_1 = \lambda_1 \), with sides parallel to the coordinate axes, such that each cube contains a vertex of \( Q^0 \). At the second step 2 we apply the preceding procedure to each of the \( 2^d \) cubes produced at step 1, but now using the proportion factor \( \lambda_2 \). Then we obtain \( 2^{2d} \) cubes of side length \( \ell_2 = \lambda_1 \lambda_2 \). Proceeding inductively, we have at the \( n \)-th step \( 2^{nd} \) cubes \( Q^0_j \), \( 1 \leq j \leq 2^{nd} \), of side length \( \ell_n = \prod_{j=1}^n \lambda_j \). We consider
\[ E_n = E(\lambda_1, \ldots, \lambda_n) = \bigcup_{j=1}^{2^{nd}} Q^n_j, \]
and we define the Cantor set associated to \( \lambda = (\lambda_n)_{n=1}^{\infty} \) as
\[
E = E(\lambda) = \bigcap_{n=1}^{\infty} E_n.
\]

For example, if \( \lim_{n \to \infty} \ell_n / 2^{-nd/s} = 1 \), then the Hausdorff dimension of \( E(\lambda) \) is \( s \). If moreover \( \ell_n = 2^{-nd/s} \) for each \( n \), then \( 0 < \mathcal{H}^s(E(\lambda)) < \infty \), where \( \mathcal{H}^s \) stands for the \( s \)-dimensional Hausdorff measure. In the planar case \( (d=2) \), this class of Cantor sets first appeared in [Gar72] (as far as we know), and its study has played a very important role in the last advances concerning analytic capacity.

Our result reads as follows.

**Theorem 1.1.** Assume that, for all \( n, 0 < \lambda_n \leq \tau_0 < \frac{1}{2} \). Denote \( \theta_n = 2^{-nd/\ell_n^s} \). For any \( N = 1, 2, \ldots \) we have

\[
\gamma_s(E_N) \approx \dot{C}^{s/(d-s)}_{2,d-s}(E_N) \approx \left( \sum_{n=1}^{N} \theta_n^2 \right)^{-1/2},
\]

where the constants involved in the relationship \( \approx \) depend on \( d, s \) and \( \tau_0 \), but not on \( N \).

Observe that if \( \mu \) is for the probability measure on \( E_N \) given by \( \mu = \frac{\mathcal{L}^d|E_N}{\mathcal{L}^d(E_N)} \), where \( \mathcal{L}^d \) stands for the Lebesgue measure in \( \mathbb{R}^d \), then \( \theta_n = \mu(Q^n_j)/\ell_n^s \). So \( \theta_n \) is the \( s \)-dimensional density of \( \mu \) on a cube from the \( n \)-th generation.

Showing that \( \dot{C}^{s/(d-s)}_{2,d-s}(E_N) \approx \left( \sum_{n=1}^{N} \theta_n^2 \right)^{-1/2} \) is not difficult, using the characterization of \( \dot{C}^{s/(d-s)}_{2,d-s} \) in terms of Wolff’s potentials (see Section 2). The difficult part of the theorem consists in showing that

\[
(1.3) \quad \gamma_s(E_N) \approx \left( \sum_{n=1}^{N} \theta_n^2 \right)^{-1/2}.
\]

The main step in proving this result consists in estimating the \( L^2(\mu) \) norm of the \( s \)-dimensional Riesz transform \( R^s_\mu \).

Let us remark that (1.3) has been proved for analytic capacity \( (s = 1, d = 2) \) in [MTV03] (using previous results from Mattila [Mat96] and Eiderman [Eıd98]). The arguments in [MTV03] (as well as the ones in [Mat96] and Eiderman [Eıd98]) rely heavily on the relationship between the Cauchy transform and curvature of measures. See [Mel95] and [MV95] for more details on this relationship.

In the case \( s = d - 1 \), the comparability (1.3) was proved by Mateu and the author [MT04] under the additional assumption that \( \lambda_n \geq 2^{-d/s} \) for all \( n \), which is equivalent to saying that the sequence \( \{\theta_n\} \) is non increasing. It is not difficult to show that the arguments in [MT04] extend to all indices \( 0 < s < d \). However, getting rid of the assumption \( \lambda_n \geq 2^{-d/s} \) is much more delicate. This is what we carry out in this paper.
Let us also mention that in [GPT06] it was shown that the estimate (1.3) also holds if one replaces $E_N$ by some bilipschitz image of itself, also under the assumption $\lambda_n \geq 2^{-d/s}$. On the other hand, recently in [ENV08] some examples of random Cantor sets where the comparability $\gamma_s \approx \dot{C}_{2(d-s),3}^{\frac{2}{3}}(d-s,3)^2$ holds have been studied.

The plan of the paper is the following. In Section 2 we show that $\dot{C}_{2(d-s),3}^{\frac{2}{3}}(d-s,3)^2(E_N) \approx \left(\sum_{n=1}^{N} \theta_n^2\right)^{-1/2}$. The proof of (1.3) is contained in Sections 3, 4, and 5. In the final Section 6 we discuss open problems in connection with Calderón-Zygmund capacities, Riesz transforms, and Wolff potentials.

Throughout all the paper, the letters $c,C$ will stand for absolute constants (which may depend on $d$ and $s$) that may change at different occurrences. Constants with subscripts, such as $C_1$, will retain their values, in general.

2. PROOF OF $\dot{C}_{2(d-s),3}^{\frac{2}{3}}(E_N) \approx \left(\sum_{n=1}^{N} \theta_n^2\right)^{-1/2}$

The proof of this result is essentially contained in [AH96, Section 5.3]. However, for the reader’s convenience we give a simple and almost self-contained proof.

Recall that $\mu$ stands for the probability measure on $E_N$ defined by $\mu = \frac{\mathcal{L}^d|E_N|}{\mathcal{E}(E_N)}$. Given $x \in E_N$, let $Q^n(x)$ denote the cube $Q^n_j$ from the $n$-th generation in the construction of $E_N$ that contains $x$, so that $\ell(Q^n(x)) = \ell_n$ is its side length. It is straightforward to check that for all $x \in E_N$,

$$\dot{W}^{\mu}_{2(d-s),3}(x) = \int_{0}^{\infty} \left(\frac{\mu(B(x,r))}{r^s}\right)^2 \frac{dr}{r} \approx \sum_{n \geq 0} \left(\frac{\mu(Q^n(x))}{\ell(Q^n(x))}\right)^2 = \sum_{n \geq 0} \theta_n^2.$$ 

Thus, if we consider the measure

$$\nu = \left(\sum_{n \geq 0} \theta_n^2\right)^{-1/2} \mu,$$

we have $\dot{W}^{\nu}_{2(d-s),3}(x) \leq 1$ for all $x \in E_N$. From (1.2) we infer that

$$\dot{C}_{2(d-s),3}^{\frac{2}{3}}(E_N) \gtrsim \nu(E_N) = \left(\sum_{n \geq 0} \theta_n^2\right)^{-1/2}.$$ 

To prove the converse inequality, we recall that given any Borel measure $\sigma$ on $\mathbb{R}^d$, for any capacity $\dot{C}_{a,p}$,

$$\dot{C}_{a,p}(\{x \in \mathbb{R}^d : W_{a,p}(x) > \lambda\}) \leq c_{a,p} \frac{\sigma(\mathbb{R}^d)}{\lambda^{a-1}}, \text{ for all } \lambda > 0.$$
See Proposition 6.3.12 of [AH96]. If we apply this estimate to
\( \dot{C}_2(\sigma) \), \( \sigma = \mu \), and \( \lambda \approx \sum_{n \geq 0} \theta_n^2 \), we get
\[
\dot{C}_2(\sigma)(E_N) \leq \dot{C}_2(\sigma) \left( \{ x \in \mathbb{R}^d : W_\sigma^s(\sigma)(x) > \lambda \} \right) \leq \frac{1}{\left( \sum_{n \geq 0} \theta_n^2 \right)^{1/2}}.
\]

3. Preliminaries for the proof of \( \gamma_s(E_N) \approx \left( \sum_{n=1}^{N} \theta_n^2 \right)^{-1/2} \)

To simplify notation, to denote the \( s \)-dimensional Riesz transform of \( \mu \), we will write \( R\mu \) instead of \( R^s\mu \), and also \( K(x) \) instead of \( K^s(x) = x/|x|^{s+1} \). Moreover, \( \| \cdot \| \) stands for the \( L^2(\mu) \) norm.

Arguing as in [MT04, Lemma 4.2], it turns out that the estimate
\[
(3.1) \quad \gamma_s(E_N) \approx \left( \sum_{n=1}^{N} \theta_n^2 \right)^{-1/2}
\]
follows from the next result.

**Theorem 3.1.** Let \( \mu \) be the preceding probability measure supported on \( E_N \). We have
\[
\| R\mu \|^2 \approx \sum_{j=0}^{N} \theta_j^2.
\]

We will skip the arguments that show that (3.1) can be deduced from this theorem, which the interested reader can find in the aforementioned reference.

Sections 4 and 5 of this paper are devoted to the proof of Theorem 3.1. In the remaining part of the current section, we introduce some additional notation that we will use below, and we prove a technical estimate.

Denote \( \tilde{\Delta} = \{ Q_j^n : n \geq 0, 1 \leq j \leq 2^{nd} \} \), where the \( Q_j^n \)'s are the cubes which appear in the construction of the \( E(\lambda) \). Let \( \Delta_n \) be the family of cubes in \( \tilde{\Delta} \) from the \( n \)-th generation. That is, \( \Delta_n = \{ Q_j^n \}_{j=1}^{2^{nd}} \). For a fixed \( N \geq 1 \), we set \( \Delta = \bigcup_{n=1}^{N} \Delta_n \) (so \( E_N \) is constructed using the cubes from \( \Delta_N \)).

Given a cube \( Q \subset \mathbb{R}^d \), we set
\[
\theta(Q) := \frac{\mu(Q)}{f(Q)^s},
\]
i.e. \( \theta(Q) \) is the average \( s \)-dimensional density of \( \mu \) over \( Q \). Thus \( \theta_n = \theta(Q) \) if \( Q \in \Delta_n \).

Given a cube \( Q \in \Delta \) and a function \( f \in L^1_{\text{loc}}(\mu) \), we define
\[
S_Qf(x) = \frac{1}{\mu(Q)} \int_Q f \, d\mu \chi_Q(x).
\]
Also, for \(0 \leq j \leq N\), we set \(S_j f = \sum_{Q \in \Delta_j} S_Q f\). If we denote by \(\mathcal{F}(Q)\) the cubes from \(\Delta\) which are sons of \(Q\), we set

\[
D_Q f(x) = \sum_{P \in \mathcal{F}(Q)} S_P f(x) - S_Q f(x),
\]

and for \(0 \leq j \leq N\) we denote \(D_j f = \sum_{Q \in \Delta_j} D_Q f = S_{j+1} f - S_j f\).

Let \(\Delta^0 = \Delta \setminus \Delta_N\). Notice that the functions \(D_Q f\) and \(D_P f\) are orthogonal for \(P \neq Q\). If \(\int f \, d\mu = 0\), then

\[
S_N f = \sum_{j=0}^{N-1} D_j f = \sum_{Q \in \Delta^0} D_Q f,
\]

and thus

\[
\|f\|^2 \geq \|S_N f\|^2 = \sum_{Q \in \Delta^0} \|D_Q f\|^2.
\]

In particular, if we take \(f = R\mu\), by antisymmetry \(\int R\mu \, d\mu = 0\), and thus

\[
(3.2) \quad \|R\mu\|^2 \geq \|S_N(R\mu)\|^2 = \sum_{Q \in \Delta^0} \|D_Q(R\mu)\|^2.
\]

Given cubes \(Q, R \in \Delta\), we denote

\[
(3.3) \quad p(Q) := \sum_{P \in \Delta: Q \subset P} \theta(P) \frac{\ell(Q)}{\ell(P)}, \quad p(Q, R) := \sum_{P \in \Delta: Q \subset P \subset R} \theta(P) \frac{\ell(Q)}{\ell(P)}.
\]

For \(0 \leq j \leq N\), we denote \(p_j := p(Q)\), for \(Q \in \Delta_j\).

**Lemma 3.2.** Let \(Q \in \Delta\) and \(x, x' \in Q\). Let \(\hat{Q}\) the parent of \(Q\). Then we have

\[
\left| R(\chi_{\mathbb{R}^d \setminus Q}\mu)(x) - R(\chi_{\mathbb{R}^d \setminus Q}\mu)(x') \right| \leq C_1 \frac{\ell(Q)}{\ell(\hat{Q})} p(\hat{Q}).
\]

Thus,

\[
\left| R(\chi_{\mathbb{R}^d \setminus Q}\mu)(x) - R(\chi_{\mathbb{R}^d \setminus Q}\mu)(x') \right| \leq C_1 p(\hat{Q}) \leq C_2 p(Q).
\]

**Proof.** We have

\[
\left| R(\chi_{\mathbb{R}^d \setminus Q}\mu)(x) - R(\chi_{\mathbb{R}^d \setminus Q}\mu)(x') \right|
\]

\[
\leq \int_{\mathbb{R}^d \setminus Q} |K(x-y) - K(x'-y)| \, d\mu(y)
\]

\[
\leq C|x-x'| \int_{\mathbb{R}^d \setminus Q} \frac{1}{|x-y|^{s+1}} \, d\mu(y)
\]

\[
\leq C|x-x'| \sum_{P \in \Delta: Q \subset P} \frac{\mu(P)}{\ell(P)^{s+1}} \leq C \frac{\ell(Q)}{\ell(\hat{Q})} p(\hat{Q}).
\]

\( \Box \)
4. Proof of $\|R\mu\|^2 \lesssim \sum_{j=0}^{N} \theta_j^2$.

**Lemma 4.1.** If $Q \in \Delta^0$ and $P$ is a son of $Q$, then

\begin{equation}
|S_P(R\mu) - S_Q(R\mu)| \lesssim p(Q).
\end{equation}

As a consequence,

\begin{equation}
\|D_Q(R\mu)\|^2 \lesssim p(Q)^2 \mu(Q).
\end{equation}

**Proof.** It is clear that (4.2) follows from (4.1). To prove (4.1), we use the antisymmetry of the kernel $K(x)$:

\begin{align}
S_P(R\mu) - S_Q(R\mu) &= S_P(R(\chi_{\mathbb{R}^d \setminus P} \mu)) - S_Q(R(\chi_{\mathbb{R}^d \setminus Q} \mu)) \\
&= S_P(R(\chi_Q \setminus P \mu)) + S_P(R(\chi_{\mathbb{R}^d \setminus Q} \mu)) - S_Q(R(\chi_{\mathbb{R}^d \setminus Q} \mu)).
\end{align}

From Lemma 3.2 it follows that

$$|S_P(R(\chi_{\mathbb{R}^d \setminus Q} \mu)) - S_Q(R(\chi_{\mathbb{R}^d \setminus Q} \mu))| \lesssim p(Q).$$

To estimate $S_P(R(\chi_Q \setminus P \mu))$ we take into account that $\text{dist}(Q \cap E_N \setminus P, P) \approx \ell(Q)$, and so for every $x \in P$,

$$|R(\chi_Q \setminus P \mu)(x)| \lesssim \frac{\mu(Q)}{\ell(Q)^s} = \theta(Q) \leq p(Q).$$

From the preceding estimates and (4.3), we get (4.1). \qed

**Lemma 4.2.** We have

$$\|S_N(R\mu)\|^2 \lesssim \sum_{j=0}^{N-1} \theta_j^2$$

and

$$\|R\mu\|^2 \lesssim \sum_{j=0}^{N} \theta_j^2.$$

**Proof.** By (3.2) and Lemma (4.1),

$$\|S_N(R\mu)\|^2 = \sum_{Q \in \Delta^0} \|D_Q(R\mu)\|^2 \lesssim \sum_{Q \in \Delta^0} p(Q)^2 \mu(Q) = \sum_{j=0}^{N-1} p_j^2.$$

On the other hand, by Lemma 3.2, for each $Q \in \Delta_N$ and $x \in Q$,

$$|S_Q(R\mu) - R(\chi_{\mathbb{R}^d \setminus Q} \mu)(x)| = |S_N(R\chi_{\mathbb{R}^d \setminus Q} \mu)) - R(\chi_{\mathbb{R}^d \setminus Q} \mu)(x)| \lesssim p(Q).$$

Using also that

$$\|\chi_Q R(\chi_Q \mu)\| \leq \theta(Q) \mu(Q)^{1/2},$$
we obtain
\[ \| R \mu \|^2 = \sum_{Q \in \Delta_N} \| \chi_Q R(\mu) \|^2 \leq 2 \sum_{Q \in \Delta_N} \left( \| \chi_Q R(\chi_Q \mu) \|^2 + \| \chi_Q R(\chi_{R^c} \mu) \|^2 \right) \]
\[ \leq 2 \sum_{Q \in \Delta_N} \left( \| \chi_Q R(\chi_Q \mu) \|^2 + \| R(\chi_{R^c} \mu) - S_N(\chi_{R^c} \mu) \|^2 + \| S_N(\mu) \|^2 \right) \]
\[ \lesssim \sum_{Q \in \Delta_N} \theta(Q)^2 \mu(Q) + \sum_{Q \in \Delta_N} p(Q)^2 \mu(Q) + \sum_{Q \in \Delta_0} p(Q)^2 \mu(Q) \]
\[ \lesssim \sum_{Q \in \Delta} p(Q)^2 \mu(Q) = \sum_{j=0}^{\infty} p_j^2. \]

It only remains to show that \( \sum_{j=0}^{M} p_j^2 \lesssim \sum_{j=0}^{M} \theta_j^2 \) both for \( M = N - 1 \) and \( M = N \). This follows easily from the definition of \( p_j \) and Cauchy-Schwartz:
\[ \sum_{j=0}^{M} p_j^2 = \sum_{j=0}^{M} \left( \sum_{k=0}^{j} \theta_k \ell_j \ell_k \right)^2 \leq \sum_{j=0}^{M} \left( \sum_{k=0}^{j} \theta_k^2 \ell_j \ell_k \right) \left( \sum_{k=0}^{j} \ell_j \ell_k \right) \]
\[ \leq 2 \sum_{j=0}^{M} \sum_{k=0}^{j} \theta_k^2 \ell_j \ell_k = 2 \sum_{k=0}^{M} \theta_k^2 \sum_{j=k}^{M} \ell_j \ell_k \leq 4 \sum_{k=0}^{M} \theta_k^2. \]

\[ \square \]

5. Proof of \( \| R \mu \|^2 \gtrsim \sum_{j=0}^{N} \theta_j^2 \)

5.1. The main lemma. The main lemma to prove the estimate
\[ \| R \mu \|^2 \gtrsim \sum_{j=0}^{N} \theta_j^2 \]
is the following.

Lemma 5.1. We have
\[ \| D_Q(R \mu) \|^2 \gtrsim \sum_{j=0}^{N-1} \theta_j^2. \]

Let us see how one deduces (5.1) from the preceding inequality.

Proof of (5.1) using Lemma 5.1. From (3.2) and (5.2) we infer that
\[ \| R \mu \|^2 \geq \| S_N(R \mu) \|^2 \geq C_3^{-1} \sum_{j=0}^{N-1} \theta_j^2. \]

So we only have to show that \( \| R \mu \|^2 \gtrsim \theta_N^2. \)
Consider \( Q \in \Delta_N \) and \( x \in Q \). We split \( R\mu(x) \) as follows:
\[
R\mu(x) = R(\chi_Q\mu)(x) + R(\chi_{\mathbb{R}^d \setminus Q}\mu)(x)
= R(\chi_Q\mu)(x) + S_N(R\mu)(x) + (R(\chi_{\mathbb{R}^d \setminus Q}\mu)(x) - S_N(R\mu)(x)).
\]

So we get
\[
\|R\mu\| \geq \left\| \sum_{Q \in \Delta_N} \chi_Q R(\chi_Q\mu) \right\| - \|S_N(R\mu)\| - \left\| \sum_{Q \in \Delta_N} \chi_Q R(\chi_{\mathbb{R}^d \setminus Q}\mu) - S_N(R\mu) \right\|.
\]

(5.4)

It is easy to check that
\[
\left\| \sum_{Q \in \Delta_N} \chi_Q R(\chi_Q\mu) \right\| \geq C_4^{-1} \theta_N.
\]

To deal with \( S_N(R\mu) \) we simply use the fact that
\[
\|S_N(R\mu)\| \leq \|R\mu\|.
\]

On the other hand, by Lemma 3.2, if \( x \in Q \in \Delta_N \),
\[
|R(\chi_{\mathbb{R}^d \setminus Q}\mu)(x) - S_N(R\mu)(x)| = |R(\chi_{\mathbb{R}^d \setminus Q}\mu)(x) - S_Q(R(\chi_{\mathbb{R}^d \setminus Q}\mu))(x)| \lesssim p_{N-1}.
\]

By Cauchy-Schwartz, it follows easily that \( p_{N-1} \leq C \left( \sum_{j=0}^{N-1} \theta^2_j \right)^{1/2} \). Then we deduce
\[
\left\| \sum_{Q \in \Delta_N} \chi_Q R(\chi_{\mathbb{R}^d \setminus Q}\mu) - S_N(R\mu) \right\|^2 \leq C \sum_{j=0}^{N-1} \theta^2_j.
\]

Then, by (5.4) and the estimates above, we get
\[
\|R\mu\| \geq C_4^{-1} \theta_N - \|R\mu\| - C_5 \left( \sum_{j=0}^{N-1} \theta^2_j \right)^{1/2}.
\]

From (5.3), we infer that
\[
C_4^{-1} \theta_N \leq 2\|R\mu\| + C_5 \left( \sum_{j=0}^{N-1} \theta^2_j \right)^{1/2} \leq 2\|R\mu\| + C_3^{1/2} C_5 \|R\mu\|,
\]
and thus the lemma follows.

\[\square\]

5.2. The stopping scales and the intervals \( I_k \). To prove Lemma 5.1 we need to define some stopping scales on the squares from \( \Delta \). Let \( B \) be some big constant (say, \( B > 100 \)) to be fixed below. We proceed by induction to define a subset \( \text{Stop} := \{s_0, \ldots, s_m\} \subset \{0, 1, \ldots, N\} \). First we set \( s_0 = 0 \). If, for some \( k \geq 0 \), \( s_k \) has already been defined and \( s_k < N - 1 \), then \( s_{k+1} \) is the least integer \( i > s_k \) which verifies at least one of the following conditions:

(a) \( i = N \), or
(b) \( \theta_i > B \theta_{s_k} \), or
(c) \( \theta_i < B^{-1} \theta_{s_k} \).

We finish the construction of Stop when we find some \( s_{k+1} = N \). Notice that we have
\[
[0, N - 1] \cap \mathbb{Z} = \bigcup_{k=0}^{m-1} [s_k, s_{k+1}) \cap \mathbb{Z} =: \bigcup_{k=0}^{m-1} I_k.
\]
Moreover, the intervals \( I_k \) are pairwise disjoint.

If \( s_k \) satisfies the condition (a) above, then we say that \( I_k \) is terminal (in this case \( k + 1 = m \)). If \( s_k \) satisfies (b) but not (a), then we say that \( I_k \) is an interval of increasing density, \( I_k \in ID \). If (c) holds for \( s_k \), but not (a) nor (b), then we say that \( I_k \) is an interval of decreasing density, \( I_k \in DD \). We denote its length by \( |I_k| \).

Notice that it coincides with \#\( I_k \).

For \( 0 \leq k \leq m \), we denote
\[
T_k \mu = \sum_{j:s_k \leq j < s_{k+1}} D_j(R \mu).
\]
In this way,
\[
S_N(R \mu) = \sum_{k=0}^{m-1} T_k \mu,
\]
and since the functions \( D_j(R \mu) \) are pairwise orthogonal,
\[
\|S_N(R \mu)\|^2 = \sum_{k=0}^{m-1} \|T_k \mu\|^2.
\]

To simplify notation, given \( A \subset \{0, \ldots, N\} \), we denote
\[
\sigma(A) := \sum_{j \in A} \theta_j^2.
\]
So \( \sigma \) can be thought as a measure on \( \{0, \ldots, N\} \).

5.3. Good and bad scales. We say that \( j \in \{0, N - 1\} \) is a good scale, and we write \( j \in \mathcal{G} \), if
\[
p_j \leq 40 \theta_j.
\]
Otherwise, we say that \( j \) is a bad scale and we write \( j \in \mathcal{B} \).

Lemma 5.2. We have
\[
\sigma(\mathcal{B}) \leq \frac{1}{10} \sigma([0, N - 1]).
\]

Proof. As in (4.4) (replacing \( M \) by \( N - 1 \)),
\[
\sum_{j=0}^{N-1} p_j^2 \leq 4 \sum_{k=0}^{N-1} \theta_k^2 = 4 \sigma([0, N - 1]).
\]
Thus,
\[
\sigma(\mathcal{B}) \leq \frac{1}{40} \sum_{j=0}^{N-1} p_j^2 \leq \frac{1}{10} \sigma([0, N - 1]).
\]

5.4. **Good and bad intervals** \(I_k\). We also say that an interval \(I_k\) is good if
\[
\sigma(I_k \cap \mathcal{G}) \geq \frac{1}{10} \sigma(I_k).
\]
Otherwise we say that it is bad.

**Lemma 5.3.**
\[
\sigma([0, N - 1]) \leq \frac{9}{8} \sum_{k: I_k \text{ good}} \sigma(I_k).
\]

**Proof.** If \(I_k\) is bad, then
\[
\sigma(I_k \cap \mathcal{B}) \geq \frac{9}{10} \sigma(I_k).
\]
Thus,
\[
\sum_{k: I_k \text{ bad}} \sigma(I_k) \leq \frac{10}{9} \sigma(\mathcal{B}) \leq \frac{10}{9} \frac{1}{10} \sigma([0, N - 1]) = \frac{1}{9} \sigma([0, N - 1]).
\]

Therefore,
\[
\sigma([0, N - 1]) = \sum_{k: I_k \text{ good}} \sigma(I_k) + \sum_{k: I_k \text{ bad}} \sigma(I_k)
\]
\[
\leq \sum_{k: I_k \text{ good}} \sigma(I_k) + \frac{1}{9} \sigma([0, N - 1]),
\]
and so
\[
\sigma([0, N - 1]) \leq \frac{9}{8} \sum_{k: I_k \text{ good}} \sigma(I_k).
\]

5.5. **Long and short intervals** \(I_k\). Let \(N_L\) be some (big) integer to be fixed below. We say that an interval \(I_k\) is long if
\[
|I_k| = s_{k+1} - s_k \geq N_L.
\]
Otherwise we say that \(I_k\) is short.
5.6. Estimates for long good intervals $I_k$. The key lemma.

**Lemma 5.4.** Let $I_k$ be good, and set $j_0 = \min(I_k \cap \mathcal{G})$. Then,

$$j_0 - s_k \leq \frac{10B^4}{1 + 10B^4} (s_{k+1} - s_k).$$

**Proof.** We denote $\ell = s_{k+1} - s_k$ and $\lambda = j_0 - s_k$. Then we have

$$\sigma(I_k \cap \mathcal{G}) \leq B^2 \theta_{s_k}^2 (\ell - \lambda),$$

and also

$$\sigma(I_k \cap \mathcal{B}) \geq B^{-2} \theta_{s_k}^2 \lambda.$$

Since $I_k$ is good, we have $\sigma(I_k \cap \mathcal{B}) \leq 10 \sigma(I_k \cap \mathcal{G})$, and so we infer that

$$\lambda \leq 10B^4 (\ell - \lambda),$$

and the lemma follows. \qed

**Lemma 5.5.** Let $0 \leq k \leq N - 1$. There exists some absolute constant $C_6$ such that if

$$\sum_{j=k}^{k+h} \|D_j(R\mu)\|^2 \geq C_7^{-1} 2^{-hd} (\theta_k + \theta_{k+1} + \ldots + \theta_{k+h})^2,$$

then

$$\|f(x)\| \geq C_9^{-1} \theta_k + \theta_{k+1} + \ldots + \theta_{k+h} - C_8 \frac{s_k}{s_{k-1}} p_{k-1}.$$

**Proof.** Denote $f = \sum_{j=k}^{k+h} D_j(R\mu)$. Take $P \in \Delta_{k+h+1}$ and $Q \in \Delta_k$ containing $P$. Then, for $x \in P$ we have

$$f(x) = S_P(R\mu)(x) - S_Q(R\mu)(x).$$

By antisymmetry, as in (4.3), we get

$$f(x) = S_P(R(\chi_{Q \setminus P} \mu))(x) + S_P(R(\chi_{\mathbb{R}^d \setminus Q} \mu))(x) - S_Q(R(\chi_{\mathbb{R}^d \setminus Q} \mu))(x).$$

From Lemma 3.2 it follows that

$$|S_P(R(\chi_{\mathbb{R}^d \setminus Q} \mu))(x) - S_Q(R(\chi_{\mathbb{R}^d \setminus Q} \mu))(x)| \leq C_8 \frac{s_k}{s_{k-1}} p_{k-1}.$$

On the other hand, if $P \in \Delta_{k+h+1}$ is a cube containing a corner of $Q$, then it is easy to check that

$$\left| \frac{1}{\mu(P)} \int_P R(\chi_{Q \setminus P} \mu) \, d\mu \right| \geq C_9^{-1} (\theta_k + \theta_{k+1} + \ldots + \theta_{k+h}).$$

Therefore,

$$|f(x)| \geq C_9^{-1} (\theta_k + \theta_{k+1} + \ldots + \theta_{k+h}) - C_8 \frac{s_k}{s_{k-1}} p_{k-1}.$$
As a consequence, if $C_6 \leq C^{-1}_9 C^{-1}_8 / 2$, then
\[
\| \chi_Q f \|^2 \geq C^{-1} \left( \theta_k + \theta_{k+1} + \ldots + \theta_{k+h} \right)^2 \mu(P) \\
= 2^{-(h+1)d} C^{-1} \left( \theta_k + \theta_{k+1} + \ldots + \theta_{k+h} \right)^2 \mu(Q).
\]
Summing over all the cubes $Q \in \Delta_k$, the lemma follows. \qed

Lemma 5.6. ([Key lemma]) Let $A, c_0$ be positive constants, and $r, q \in [0, N - 1] \cap \mathbb{Z}$ such that $q \leq r$, $\frac{t_m}{\ell_{q-1}} p_{q-1} \leq c_0 \theta_q$ and, for all $j$ with $q \leq j \leq r$,
\[
A^{-1} \theta_q \leq \theta_j \leq A \theta_q.
\]
There exists $N_1 = N_1(c_0, A)$ such that if $|q - r| > N_1$, then
\[
\sum_{j=q}^{r} \| D_j(R\mu) \|^2 \geq C |q - r| \theta_q^2,
\]
where $C$ is some positive constant depending on $c_0$ and $A$.

Proof. Set $f = \sum_{j=q}^{r} D_j(R\mu)$. We have to show that $\| f \|^2 \geq C |q - r| \theta_q^2$.

Let $M_0$ some positive integer depending on $c_0, A$ to be fixed below. We decompose $f$ as follows
\begin{equation}
(5.6) \quad f = \sum_{j=q}^{q + tM_0 - 1} D_j(R\mu) + \sum_{j=q + tM_0}^{r} D_j(R\mu),
\end{equation}
where $t$ is the biggest integer such that $q + t M_0 - 1 \leq r$. Assuming $N_1$ big enough we will have $t \approx |q - r|$, with constants depending on $M_0$, and so on $c_0, A$.

We write the first sum on the right side of (5.6) as follows:
\[
\sum_{j=q}^{q + tM_0 - 1} D_j(R\mu) = \sum_{h=0}^{t-1} \sum_{j=q + hM_0}^{q + (h+1)M_0 - 1} D_j(R\mu) =: \sum_{h=0}^{t-1} U_h(\mu).
\]
By orthogonality, we have
\[
\| f \|^2 \geq \sum_{h=0}^{t-1} \| U_h(\mu) \|^2.
\]
We will show below that if the parameter $M_0 = M_0(c_0, A)$ is chosen big enough, then
\begin{equation}
(5.7) \quad \| U_h(\mu) \|^2 \geq C(c_0, A) \theta_q^2 \quad \text{for all } 0 \leq h \leq t - 1,
\end{equation}
and thus
\[
\| f \|^2 \geq C(c_0, A) |q - r| \theta_q^2,
\]
if $N_1 \geq 2 M_0$, say.
To prove (5.7) we intend to apply Lemma 5.5. Recall that \( \frac{\ell}{q-1} p_{q-1} \leq c_0 \theta_q \), and since
\[
\frac{\ell}{q-1} p_{q-1} = \sum_{i \leq q+h_{M_0}} \frac{\ell}{\ell_i} \theta_i = \sum_{q-1 < i \leq q+h_{M_0}} \frac{\ell}{\ell_i} \theta_i + \frac{\ell}{q-1} p_{q-1},
\]
we infer that
\[
p_{q+h_{M_0}} \leq 2A \theta_q + \frac{\ell}{q-1} p_{q-1}.
\]
Therefore,
\[
\frac{\ell}{q+h_{M_0}} p_{q+h_{M_0}} \leq 2A \theta_q + \frac{\ell}{q-1} p_{q-1} \leq 2A \theta_q + \frac{\ell}{q-1} p_{q-1} \leq (2A + c_0) \theta_q.
\]
On the other hand,
\[
\sum_{j=q+h_{M_0}}^{q+(h+1)M_0} \theta_j \geq M_0 A^{-1} \theta_q.
\]
If \( M_0 \) is big enough then \( 2A + c_0 \leq C_6 M_0 A^{-1} \) and so the assumption (5.5) in Lemma 5.5 is satisfied. Thus
\[
\|U_k \mu\|^2 \geq C_7^{-1} 2^{-M_0} \left( \sum_{j=q+h_{M_0}}^{q+(h+1)M_0} \theta_j \right)^2 \geq C_7^{-1} 2^{-M_0} M_0^2 A^{-2} \theta_q^2,
\]
and so our claim (5.7) follows. \( \square \)

**Lemma 5.7.** Suppose that the constant \( N_L \) is chosen big enough (depending on \( B \)). If \( I_k \) is long and good, then
\[
\sigma(I_k) \leq C(B) \|T_k \mu\|^2.
\]

Recall that \( T_k \mu = \sum_{j:s_k \leq j < s_{k+1}} D_j(R \mu) \).

**Proof.** Set \( \ell = s_{k+1} - s_k \). Notice that
\[
\sigma(I_k) \leq \ell B^2 \theta_{s_k}^2.
\]
Let \( j_0 = \min(I_k \cap \mathcal{G}) \). We suppose that \( N_L \gg B^4 \), so that by Lemma 5.4,
\[
s_{k+1} - j_0 \geq \frac{1}{10B^4} \ell \gg 1.
\]
We split \( T_k \mu \) as follows
\[
T_k \mu = \sum_{j=s_k}^{j_0-1} D_j(R \mu) + \sum_{j=j_0}^{s_{k+1}-1} D_j(R \mu),
\]
Now we apply Lemma 5.6, with $A = B$, $c_0 = 40$, and we deduce that if $N_L$ is big enough, then

$$\sum_{j=j_0}^{s_{k+1} - 1} \|D_j(R\mu)\|^2 \geq C(B)^{-1}|s_{k+1} - j_0|\theta_{s_k}^2$$

By orthogonality,

$$\|T_k\mu\|^2 \geq \sum_{j=j_0}^{s_{k+1} - 1} \|D_j(R\mu)\|^2,$$

and thus the lemma follows. □

5.7. **The intervals** $J_h$. By Lemmas 5.3 and 5.7, to finish our proof of $\sigma([0,N-1]) \lesssim \sum_j \|D_j(R\mu)\|^2$, it is enough to show that

$$\sum_{k: I_k \text{ short good}} \sigma(I_k) \lesssim \sum_j \|D_j(R\mu)\|^2. \quad (5.8)$$

To this end, we have to define some auxiliary intervals $J_h$.

We consider the following partial ordering in the family of intervals contained in $\mathbb{R}$: if $I, J$ are disjoint intervals such that all $x \in I$, $y \in J$ satisfy $x < y$, then we write $I \prec J$.

An interval $J_h$, $h \geq 1$, is the union of two intervals $I_k, I_{k+1}$, so that $I_k$ is of type $ID$ and $I_{k+1}$ is either of type $DD$ or it is the terminal interval $I_m$. Then $\{J_h\}_{1 \leq h \leq m_L}$ is the collection of all these intervals. We assume that $J_h \prec J_{h+1}$ for all $h$. Moreover, for convenience, if $I_0$ is of type $DD$, we set $J_0 = I_0$.

**Remark 5.8.** Of course, there may be intervals $I_k$ which are not contained in any interval $J_h$. Suppose that, for some $0 \leq h \leq m_L$, there are intervals $I_k$ such that

$$J_h \prec I_k \prec I_{k+1} \prec \ldots \prec I_{k+r} \prec J_{h+1}.$$

Then, from the definition of the intervals $J_h$, it turns out that either all the intervals $I_k, \ldots, I_{k+r}$ are of type $ID$, or all are of type $DD$, or there exists $1 \leq s \leq r$ such that $I_k, \ldots, I_{k+s-1}$ are of type $DD$, and $I_{k+s}, \ldots, I_{k+r}$ are of type $ID$.

Given an interval $I \subset [0, N]$, we denote

$$\theta_{\max}(I) = \max_{j \in I} \theta_j.$$

**Lemma 5.9.** Let $J_h$, $0 \leq h \leq m_L - 1$, be such that

$$J_h \prec I_k \prec I_{k+1} \prec \ldots \prec I_{k+r} \prec J_{h+1},$$

or in the case $h = m_L$, 

$$J_h \prec I_k \prec I_{k+1} \prec \ldots \prec I_{k+r}.$$

Then,

$$\sum_{k \leq i \leq k+r \atop I_i \text{ short}} \sigma(I_i) \leq C(B, N_L) \left[\theta_{\max}(J_h)^2 + \theta_{\max}(J_{h+1})^2\right]. \quad (5.9)$$
where, for convenience, we set $\theta_{m+1} = 0$.

**Proof.** Notice that any short interval $I_k$ satisfies
\begin{equation}
\sigma(I_k) \leq B^2 N_L \theta_{s_k}^2.
\end{equation}
If there is some $q \geq 1$ such that the intervals $I_k, \ldots, I_{k+q-1}$ are of type $DD$, then
\begin{equation}
\theta_{s_{k+q-1}} \leq B^{-1} \theta_{s_{k+q-2}} \leq \ldots \leq B^{1-q} \theta_{s_k} \leq B^{-q} \theta_{\text{max}}(J_h).
\end{equation}
Thus,
\begin{equation}
\sum_{k \leq i \leq k+q-1 \atop I_i \text{ short}} \sigma(I_i) \leq C(B, N_L) \theta_{\text{max}}(J_h)^2.
\end{equation}
Analogously, one deduces that
\begin{equation}
\sum_{k+q \leq i \leq k+r \atop I_i \text{ short}} \sigma(I_i) \leq C(B, N_L) \theta_{\text{max}}(J_{h+1})^2,
\end{equation}
and the lemma follows. \hfill \square

**Lemma 5.10.** We have
\begin{equation}
\sum_{k: I_k \text{ short}} \sigma(I_k) \leq C(B, N_L) \sum_{h=0}^{m_f} \theta_{\text{max}}(J_h)^2.
\end{equation}

**Proof.** This is a direct consequence of Lemma 5.9. \hfill \square

5.8. **The standard intervals $J_h$.** By Lemma 5.10, in order to prove (5.8), it is enough to show that
\begin{equation}
\sum_{h} \theta_{\text{max}}(J_h)^2 \lesssim \sum_{j} \|D_j(R\mu)\|^2.
\end{equation}
To this end, we need to distinguish different types of intervals $J_h$. For $h \geq 1$, let $t_h \in J_h$ be the least integer such that
\begin{equation}
\theta_{t_h} > B^{-1/2} \theta_{\text{max}}(J_h).
\end{equation}
Notice that, if $J_h = I_k \cup I_{k+1}$, then $\theta_{\text{max}}(J_h) \leq B \theta_{s_{k+1}}$. However we cannot ensure that $\theta_{\text{max}}(J_h) \leq B^2 \theta_{s_k}$ because it may happen that $\theta_{s_{k+1}} \gg B \theta_{s_k}$.

We say that $J_h$ is **standard** if
\begin{equation}
\frac{\ell_{t_h}}{\ell_{t_h-1}} p_{t_h-1} \leq C_{10} \theta_{\text{max}}(J_h),
\end{equation}
where $C_{10} = C_6/2$ (with $C_6$ from (5.5)). For convenience, if $J_0$ exists (and thus $J_0 = I_0 \in DD$) we also say that $J_0$ is standard.

**Lemma 5.11.** If $J_h$ is standard, then
\begin{equation}
\theta_{\text{max}}(J_h)^2 \leq C(B) \sum_{j \in J_h} \|D_j(R\mu)\|^2.
\end{equation}
Proof. In the special case $h = 0$ (with $J_0 = I_0$), it is immediate to check that $\| D_0(R\mu) \|^2 \geq C^{-1} \theta_0^2 \geq C^{-1} B^{-2} \theta_{\text{max}}(J_0)^2$ (for instance, one can apply Lemma 5.5 with $p_\gamma = 0$), and thus the lemma holds.

For $h \geq 1$, we set $J_h = [s_k, t_h - 1) \cup [t_h, s_{k+2}) =: J_h^a \cup J_h^b$.

Observe that $\theta_{\text{max}}(J_h)$ is attained at some scale from $J_h^b$, and $\theta_j \leq B^{-1/2} \theta_{\text{max}}(J_h)$ for $j \in J_h^b$.

We distinguish two cases:

**Case 1.** Suppose first that the length $|J_h^b|$ is big. That is, $|J_h^b| = s_{k+2} - t_h > N_2$, where $N_2 = N_2(C_10, B)$ is some big integer. By (5.12), we have

$$\ell_{t_h} \ell_{t_h-1} \leq C_{10} \theta_{\text{max}}(J_h) \leq C(B) \theta_{t_h},$$

and thus from Lemma 5.6 we infer that if $N_2$ is chosen big enough, then

$$\theta_{\text{max}}(J_h)^2 \leq C(B) \sum_{j \in J_h^b} \| D_j(R\mu) \|^2,$$

and so the lemma holds in this case.

**Case 2.** Assume that $|J_h^b| \leq N_2$. From (5.12), recalling that $C_{10} = C_6/2$, we infer that

$$\ell_{t_h} \ell_{t_h-1} \leq C_{10} \theta_{\text{max}}(J_h) \leq C_6 \sum_{j \in J_h^b} \theta_j,$$

and then, by Lemma 5.5,

$$\sum_{j \in J_h^b} \| D_j(R\mu) \|^2 \geq C_7^{-1} 2^{-N_2d} \left( \sum_{j \in J_h^b} \theta_j \right)^2 \geq C_7^{-1} 2^{-N_2d} \theta_{\text{max}}(J_h)^2,$$

and so the lemma also holds in this situation. \qed

5.9. The non standard intervals $J_h$.

**Lemma 5.12.** Suppose that $B$ has been chosen big enough. We have

$$\sum_{h: J_h \text{ non standard}} \theta_{\text{max}}(J_h)^2 \leq C(B) \sum_{h: J_h \text{ standard}} \theta_{\text{max}}(J_h)^2.$$

**Proof.** Denote by $\{J_{n,1}^{\text{st}}\}_n$ the subfamily of the standard intervals from $\{J_h\}_h$, ordered so that $J_{n,1}^{\text{st}} < J_{n+1,1}^{\text{st}}$ for all $n$. For a fixed $n$, denote by $\Lambda_1, \ldots, \Lambda_m$ the collection of all non standard intervals from the family $\{J_h\}$ such that either

$$J_{n,1}^{\text{st}} < \Lambda_1 < \Lambda_2 < \ldots < \Lambda_m < J_{n+1,1}^{\text{st}}$$

if $J_{n+1,1}^{\text{st}}$ exists,

or

$$J_{n,1}^{\text{st}} < \Lambda_1 < \Lambda_2 < \ldots < \Lambda_m$$

if $J_{n+1,1}^{\text{st}}$ does not exist.
We will prove that
\begin{equation}
\theta^{\max}(\Lambda_i) \leq B^{-i/8s} \theta^{\max}(J^s_n) \quad \text{for } i \geq 1,
\end{equation}
by induction on \(i\). The lemma follows easily from this estimate.

To simplify notation, we set \(\Lambda_0 = J^s_n\) and \(\theta^{\max}_i = \theta^{\max}(\Lambda_i)\). Also, if \(\Lambda_i = I_k \cup I_{k+1}\),
we denote by \(Q_i\) a cube from \(\Delta_{s_k}\), by \(\tilde{Q}_i\) a cube from \(\Delta_{s_{k-1}}\) (see (5.12)), and by \(Q_{i}^{\max}\) a cube from \(\bigcup_{j \in I_{k+1}} \Delta_j\) such that \(\theta^{\max}_i = \theta_j\). Moreover, we assume that
\(Q_i \supset \tilde{Q}_i \supset Q_{i+1}^{\max} \supset \tilde{Q}_{i+1} \supset Q_{i+1}^{\max} \supset \ldots\)

First we prove (5.13) for \(i = 1\). Since \(\Lambda_1\) is not standard,
\begin{equation}
\theta^{\max}_1 \leq C_{10}^{-1} \frac{\ell(s(\tilde{Q}_1))}{\ell(Q_1)} p(\tilde{Q}_1),
\end{equation}
where \(s(\tilde{Q}_1)\) stands for a son of \(\tilde{Q}_1\). To estimate \(p(\tilde{Q}_1)\) (recall the notation in (3.3)),
we decompose it as follows:
\[
p(\tilde{Q}_1) \leq p(\tilde{Q}_1, Q_1) + \frac{\ell(\tilde{Q}_1)}{\ell(Q_1)} p(Q_1, Q_0^{\max}) + \frac{\ell(\tilde{Q}_1)}{\ell(Q_0^{\max})} p(Q_0^{\max}).
\]
Now observe that
\begin{equation}
p(\tilde{Q}_1, Q_1) \leq 2B^{-1/2} \theta^{\max}_1,
\end{equation}
since \(\theta(\tilde{Q}_1) \leq 2B^{-1/2} \theta^{\max}_1\) for \(\tilde{Q}_1 \subset P \subset Q_1\). Also,
\begin{equation}
p(Q_1, Q_0^{\max}) \leq 2\theta(Q_1) + 2\theta^{\max}_0,
\end{equation}
because \(\theta(Q_1) \leq \theta(Q_1) + \theta^{\max}_0\) for \(Q_1 \subset P \subset Q_0^{\max}\), taking into account Remark 5.8.
And finally,
\begin{equation}
p(Q_0^{\max}) \leq p(Q_0^{\max}, \tilde{Q}_0) + \frac{\ell(Q_0^{\max})}{\ell(\tilde{Q}_0)} p(\tilde{Q}_0)
\leq p(Q_0^{\max}, \tilde{Q}_0) + \frac{\ell(s(\tilde{Q}_0))}{\ell(Q_0)} p(\tilde{Q}_0) \leq 4\theta^{\max}_0,
\end{equation}
because \(\theta(\tilde{Q}_1) \leq \theta^{\max}_0\) for \(Q_0^{\max} \subset P \subset \tilde{Q}_0\) and moreover \(\Lambda_0\) is standard (we assume \(C_6 \leq 1\), say). Thus we infer that
\[
p(\tilde{Q}_1) \leq 2B^{-1/2} \theta^{\max}_1 + 2\frac{\ell(\tilde{Q}_1)}{\ell(Q_1)} (\theta(Q_1) + \theta^{\max}_0) + 4\frac{\ell(\tilde{Q}_1)}{\ell(Q_0^{\max})} \theta^{\max}_0
\leq 4B^{-1/2} \theta^{\max}_1 + \frac{6\ell(\tilde{Q}_1)}{\ell(Q_1)} \theta^{\max}_0,
\]
using that \( \theta(Q_1) \leq B^{-1} \theta_1^{\max} \leq B^{-1/2} \theta_1^{\max} \) in the second inequality. If we plug this estimate into (5.14) we deduce
\[
\theta_1^{\max} \leq 4C_{10}^{-1} B^{-1/2} \theta_1^{\max} + 6C_{10}^{-1} \frac{\ell(s(\tilde{Q}_1))}{\ell(Q_1)} \theta_0^{\max}.
\]
If we assume \( B \) big enough, so that \( 4C_{10}^{-1} B^{-1/2} \leq 1/2 \) (recall that \( C_{10} = C_6/2 \) does not depend on \( B \)), we obtain
\[
\theta_1^{\max} \leq 12C_{10}^{-1} \frac{\ell(s(\tilde{Q}_1))}{\ell(Q_1)} \theta_0^{\max}.
\]
On the other hand, since \( \theta(s(\tilde{Q}_1)) > B^{1/2} \theta(Q_1) \) (by the definition of \( \tilde{Q}_1 \)), we infer that
\[
(5.18) \quad \ell(s(\tilde{Q}_1)) \leq B^{-1/2} \ell(Q_1),
\]
and so
\[
\theta_1^{\max} \leq 12C_{10}^{-1} B^{-1/2} \theta_0^{\max}.
\]
If we suppose \( B \) big enough again, (5.13) follows in the particular case \( i = 1 \).

The proof of (5.13) for an arbitrary integer \( i \geq 2 \) when we assume that it holds for \( 1, \ldots, i-1 \) is analogous to the one for the case \( i = 1 \). For the sake of completeness we will show the detailed arguments. As in (5.14), we have
\[
(5.19) \quad \theta_i^{\max} \leq C_{10}^{-1} \frac{\ell(s(\tilde{Q}_i))}{\ell(Q_i)} p(Q_i),
\]
because \( \Lambda_i \) is not standard. Now we split \( p(\tilde{Q}_i) \) as follows:
\[
p(\tilde{Q}_i) \leq p(\tilde{Q}_i, Q_i) + \frac{\ell(Q_i)}{\ell(Q_i)} p(Q_i, Q_i^{\max})
+ \sum_{j=1}^{i-1} \frac{\ell(Q_i)}{\ell(Q_j^{\max})} p(Q_j^{\max}, Q_j^{\max}) + \frac{\ell(Q_i)}{\ell(Q_0^{\max})} p(Q_0^{\max}).
\]
We will estimate each of the terms in the preceding inequality separately. As in (5.15), we have
\[
p(\tilde{Q}_i, Q_i) \leq 2B^{-1/2} \theta_i^{\max},
\]
and as in (5.16),
\[
p(Q_i, Q_i^{\max}) \leq 2\theta(Q_i) + 2\theta_i^{\max} \leq 2B^{-1/2} \theta_i^{\max} + 2\theta_i^{\max}.
\]
By analogous arguments,
\[
p(Q_j^{\max}, Q_{j-1}^{\max}) \leq 2\theta_j^{\max} + 2\theta_{j-1}^{\max}.
\]
On the other hand, the term $p(Q_0^{\text{max}})$ has been estimated in (5.17). By the preceding inequalities and the induction hypothesis, we obtain

\[
p(\tilde{Q}_i) \leq 2B^{-1/2}\theta_i^{\text{max}} + \frac{\ell(s(\tilde{Q}_i))}{\ell(Q_i)} (2B^{-1/2}\theta_i^{\text{max}} + 2\theta_{i-1}^{\text{max}}) + \sum_{j=1}^{i-1} \frac{\ell(s(\tilde{Q}_i))}{\ell(Q_j^{\text{max}})} (2\theta_j^{\text{max}} + 2\theta_{j-1}^{\text{max}}) + \frac{4\ell(s(Q_i))}{\ell(Q_0^{\text{max}})} \theta_0^{\text{max}}
\]

\[
\leq 4B^{-1/2}\theta_i^{\text{max}} + 2\frac{\ell(s(\tilde{Q}_i))}{\ell(Q_i)} B^{-(i-1)/8s} \theta_0^{\text{max}} + 4\frac{\ell(s(\tilde{Q}_i))}{\ell(Q_0^{\text{max}})} \theta_0^{\text{max}}.
\]

If we plug this inequality into (5.19) and we assume $B$ big enough, we deduce that

\[
(5.20) \quad \theta_i^{\text{max}} \leq C \left[ \frac{\ell(s(\tilde{Q}_i))}{\ell(Q_i)} B^{-(i-1)/8s} \theta_0^{\text{max}} + \sum_{j=1}^{i-1} \frac{\ell(s(\tilde{Q}_i))}{\ell(Q_j^{\text{max}})} B^{-(j-1)/8s} \theta_0^{\text{max}} + \frac{\ell(s(Q_i))}{\ell(Q_0^{\text{max}})} \theta_0^{\text{max}} \right],
\]

with $C$ independent of $B$. As in (5.18), we have

\[
\frac{\ell(s(\tilde{Q}_i))}{\ell(Q_i)} \leq B^{-1/2s},
\]

and for $0 \leq j \leq i - 1$,

\[
\frac{\ell(s(\tilde{Q}_i))}{\ell(Q_j^{\text{max}})} \leq \frac{\ell(s(\tilde{Q}_i))}{\ell(Q_i)} \leq B^{j-i/2s}.
\]

From the latter estimates and (5.20) we obtain

\[
\theta_i^{\text{max}} \leq C \left[ B^{-1/2s} B^{-(i-1)/8s} \theta_0^{\text{max}} + \sum_{j=1}^{i-1} B^{(j-i)/2s} B^{-(j-1)/8s} \theta_0^{\text{max}} + B^{-i/2s} \theta_0^{\text{max}} \right]
\]

\[
\leq C B^{-1/4s} B^{-i/8s} \theta_0^{\text{max}},
\]

and so (5.13) holds if we assume $B$ big enough. $\square$
5.10. **Proof of Lemma 5.1.** From Lemmas 5.3, 5.7, and 5.10, we get
\[
\sum_{j=0}^{N-1} \theta_j^2 = \sum_k \sigma(I_k) \leq C \sum_{k: I_k \text{ good}} \sigma(I_k)
= C \sum_{k: I_k \text{ long good}} \sigma(I_k) + C \sum_{k: I_k \text{ short good}} \sigma(I_k)
\leq C \sum_{j=0}^{N-1} \| D_j(R\mu) \|^2 + C \sum_h \theta_{\max}^h (J_h)^2.
\]

By Lemmas 5.12 and 5.11,
\[
\sum_{h: J_h} \theta_{\max}^h (J_h)^2 \lesssim \sum_{h: J_h \text{ standard}} \theta_{\max}^h (J_h)^2 \lesssim \sum_{j=0}^{N-1} \| D_j(R\mu) \|^2.
\]
We are done. \(\square\)

6. **Open problems**

In this section we discuss some open problems in connection with Riesz transforms and Wolff potentials.

1) **Riesz transforms and rectifiability.**

Let \( E \subset \mathbb{R}^d \) be a compact set with \( 0 < H^n(E) < \infty \), for some integer \( 0 < n < d \), and set \( \mu = H^n|E \). If \( R^n_\mu \) is bounded in \( L^2(\mu) \), is then \( E \) \( n \)-rectifiable? Recall that \( E \) is called \( n \)-rectifiable if there exist Lipschitz mappings \( g_i: \mathbb{R}^n \to \mathbb{R}^d \) such that
\[
\mu \left( \mathbb{R}^d \setminus \bigcup_{i=1}^{\infty} g_i(\mathbb{R}^n) \right) = 0.
\]

When \( n = 1 \), David and Léger [Lég99] answered the question in the affirmative, using the relationship between curvature and the Cauchy kernel. By [Vol03], when \( n = d - 1 \) this question is equivalent to the following: is it true that \( \kappa(E) = 0 \) if and only if \( E \) is purely \( (d-1) \)-unrectifiable? \( (E \) is called purely \( (d-1) \)-unrectifiable if it does not contain any \( n \)-rectifiable subset \( F \) with \( H^{d-1}(F) > 0 \)).

A partial result was obtained in [Tol08], where it was shown that the existence of the principal values \( \lim_{r \to 0} R^n_\mu(x) \) for \( \mu \)-a.e. \( x \in \mathbb{R}^d \) implies \( E \) to be \( n \)-rectifiable. Under the additional assumption
\[
\theta^n_{\mu,*,}(x) := \liminf_{r \to 0} \frac{\mu(B(x,r))}{r^n} > 0 \quad \mu\text{-a.e. on } \mathbb{R}^d,
\]
this had been proved previously by Mattila and Preiss in [MP95]. Unfortunately, it is not known if the \( L^2(\mu) \) boundedness of the Riesz transform \( R^n_\mu \) implies the existence of principal values, and so the results in [Tol08] and [MP95] do not help to solve the problem above.
Another related result is given in [MP95, Theorem 5.5], where it is proved that if (6.1) holds and all the operators
\[ T_{f}(x) = \int K(x - y)f(y) \, d\mu(x), \]
with kernel of the form \( K(x) = \varphi(|x|)x/|x|^{n+1} \) satisfying \( |\nabla^{j}K(x)| \leq \frac{C(j)}{|x|^j} \) for \( j \geq 0 \) are bounded in \( L^2(\mu) \), then \( E \) is \( n \)-rectifiable.

A variant of this problem, posed by David and Semmes, consists in taking \( E \) Ahlfors-David regular and \( n \)-dimensional. That is,
\[ \mathcal{H}^{n}(E \cap B(x,r)) \approx r^{n} \quad \text{for all } x \in E, \ 0 < r \leq \text{diam}(E). \]
Again, set \( \mu = \mathcal{H}^{n}_{E} \). If \( R_{\mu}^{n} \) is bounded in \( L^2(\mu) \), is then \( E \) uniformly \( n \)-rectifiable?

For the definition of uniform rectifiability, see [DS91] and [DS93] (for the reader’s convenience let us say that, roughly speaking, uniform rectifiability is the same as rectifiability plus some quantitative estimates). For \( n = 1 \) the answer is true again, because of curvature. The result is from Mattila, Melnikov and Verdera [MMV96]. For \( n > 1 \), in [DS91] and [DS93] some partial answers are given. In particular, it is shown that if all the operators \( T \) with kernel \( K \) as above are bounded in \( L^2(\mu) \), then \( E \) is uniformly rectifiable.

2) Calderón-Zygmund capacities and Wolff potentials of non integer dimension.

This problem was already mentioned in the Introduction: is it true that for \( 0 < s < d \) non integer we have
\[ \gamma_{s}(E) \approx C_{\frac{d}{d-s}, \frac{d}{2}}(E) \]
with constants independent of \( E \)? Recall that this was shown to be true when \( 0 < s < 1 \) by Mateu, Prat and Verdera [MPV05]. In Theorem 1.1 we have proved that (6.2) holds for the particular case of the Cantor sets \( E_{N} = E_{N}(\lambda) \) associated to a sequence \( \lambda = (\lambda_{n})_{n=1}^{\infty} \), with \( \lambda_{n} \leq \tau_{0} < \frac{1}{2} \). It might be also interesting to consider more general Cantor type sets. For instance, for \( n \geq 0 \) and \( 1 \leq j \leq 2^{nd} \) take dilation factors \( \lambda_{n,j} \), with \( 0 < \lambda_{n,j} \leq \tau < 2 \), and construct a Cantor type set analogous to the one in Theorem 1.1, but allowing different values \( \lambda_{n,j}, \ 1 \leq j \leq 2^{nd} \), for the different squares from a fixed scale \( n \). Does (6.2) hold for these new Cantor sets? To solve this more general case, one might try to implement the techniques used in Theorem 1.1. However, several non trivial difficulties arise. First, one has to take into account that in this case, if one considers a probability measure \( \mu \) such that \( \mu(Q_{n,j}) = 2^{-nd} \) for each square \( Q_{n,j} \) from the corresponding set \( E_{N} \), then both estimates
\[ \gamma_{s}(E_{N}) \approx \left( \sum_{Q \in \Delta} \theta(Q_{n,j})^{2} \right)^{-1/2}, \quad C_{\frac{d}{2}(d-s), \frac{d}{2}}(E_{N}) \approx \left( \sum_{Q \in \Delta} \theta(Q_{n,j})^{2} \right)^{-1/2} \]
are false in general (as in Theorem 1.1, \( \Delta \) stands for the collection of all squares \( Q_{n,j} \) in the construction of \( E_{N} \)). Thus \( \mu \) should be replaced by another measure.
On the other hand, for some of the arguments involved in the estimates of the \( L^2(\mu) \) norm of \( R(\mu) \) in Theorem 3.1, the homogeneity of the set \( E_N \) (i.e. the fact that \( \ell(Q_{n,j}) \) only depends on \( n \)) is essential.

As mentioned in the Introduction, it is proved in [ENV08] that the estimate \( \gamma_s(E) \geq C_s(d-s)\frac{1}{\delta}(E) \) holds for \( 0 < s < d \). The main obstacle to prove the opposite inequality is the following. It is not known if, for \( s \in \mathbb{Z} \), there are sets \( E \) with \( 0 < \mathcal{H}^s(E) < \infty \) such that the Riesz transform \( R^s_\mu \), with \( \mu = \mathcal{H}^s_{|E} \), is bounded in \( L^2(\mu) \). If (6.2) holds, then such sets do not exist. This is the case for \( 0 < s < 1 \), as shown by Prat [Pra04] using the curvature method, and for other \( s \notin \mathbb{Z} \) by Vihtila [Vih96] under the additional assumption that \( \theta^s_{\mu,\ast}(x) > 0 \) for \( \mu \)-a.e. \( x \in \mathbb{R}^d \), where \( \theta^s_{\mu,\ast}(x) \) is defined in (6.1).

On the other hand, in [RdVT] it has been proved that, for \( 0 < s < d \) and \( \mu = \mathcal{H}^s_{|E} \), with \( 0 < \mathcal{H}^s(E) < \infty \), the existence of the principal values \( \lim_{\varepsilon \to 0} R^s_\mu(x) \) for \( \mu \)-a.e. \( x \in \mathbb{R}^d \) forces \( s \) to be integer. Notice that if one combines the results on principal values from [Tol08] mentioned above with the ones from [RdVT], then one gets:

**Theorem.** For \( 0 < s \leq d \), let \( E \subset \mathbb{R}^d \) be a set satisfying \( 0 < \mathcal{H}^s(E) < \infty \). The principal value

\[
\lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{x-y}{|x-y|^{s+1}} d\mathcal{H}^s_{|E}(y)
\]

exists for \( \mathcal{H}^s \)-almost every \( x \in E \) if and only if \( s \) is integer and \( E \) is \( s \)-rectifiable.

It is interesting to compare the last theorem with well known results in geometric measure theory due essentially to Marstrand [Mar64] and Preiss [Pre87]:

**For** \( 0 < s \leq m \), let \( E \subset \mathbb{R}^m \) be a set satisfying \( 0 < \mathcal{H}^s(E) < \infty \). The density \( \theta^s_{\mathcal{H}^s_{|E}}(x) \) exists for \( \mathcal{H}^s \)-almost every \( x \in E \) if and only if \( s \) is integer and \( E \) is \( s \)-rectifiable.

**3) \( L^2 \) boundedness of Riesz transforms and square functions.**

Given a non-increasing radial \( C^\infty \) function \( \psi \) such that \( \chi_{B(0,1/2)} \leq \psi \leq \chi_{B(0,2)} \), for each \( j \in \mathbb{Z} \), we set \( \psi_j(z) := \psi(2^j z) \) and \( \varphi_j := \psi_j - \psi_{j+1} \), so that each function \( \varphi_j \) is non-negative and supported in the annulus \( A(0,2^{-j-2},2^{-j+1}) \), and moreover we have \( \sum_{j \in \mathbb{Z}} \varphi_j(x) = 1 \) for all \( x \in \mathbb{R}^d \setminus \{0\} \). For each \( j \in \mathbb{Z} \) we denote \( K_j^\ast(x) = \varphi_j(x) x/|x|^{s+1} \) and

\[
(6.3) \quad R^s_j \mu(x) = \int K^\ast_j(x-y) d\mu(y).
\]

Notice that, at a formal level, we have \( R\mu = \sum_{j \in \mathbb{Z}} R_j \mu \), and so

\[
\|R^s \mu\|_{L^2(\mu)}^2 = \sum_{j \in \mathbb{Z}} \|R^s_j \mu\|_{L^2(\mu)}^2 + \sum_{j \neq k} \langle R^s_j \mu, R^s_k \mu \rangle.
\]
Consider the square function
\[ Q^s \mu(x) = \left( \sum_{j \in \mathbb{Z}} |R^s_j \mu(x)|^2 \right)^{1/2}, \]
and set \( Q^s_\mu(f) = Q^s(f \, d\mu). \) Notice that
\[ \|Q^s_\mu(f)\|_{L^2(\mu)}^2 = \sum_{j \in \mathbb{Z}} \|R^s_j(f \, d\mu)\|_{L^2(\mu)}^2. \]
One should view \( Q^s_\mu(f) \) as a square function associated to the Riesz transform \( R^s_\mu(f). \)

When \( s \) is integer and \( E \subset \mathbb{R}^d \) uniformly rectifiable, with \( \mu = \mathcal{H}^s|_E, \) then \( Q^s_\mu \) is bounded in \( L^2(\mu). \) Moreover, the converse is also true: if \( E \) is Ahlfors-David regular, the \( L^2(\mu) \) boundedness of \( Q^s_\mu \) implies that \( E \) is uniformly rectifiable (at least for an appropriate choice of the function \( \psi \) above), as shown in [Tol09]. In the non Ahlfors-David regular case it is also true that the boundedness of \( Q^s_\mu \) implies the rectifiability of \( E \) [MV09a].

On the other hand, given \( E \subset \mathbb{R}^d \) such that \( 0 < \mathcal{H}^s(E) < \infty, \) \( 0 < s < d, \) and \( \mu = \mathcal{H}^s|_E, \) if \( Q^s_\mu \) is bounded in \( L^2(\mu), \) then \( s \in \mathbb{Z}. \) This follows easily from the results of [RdVT], as shown in [MV09b]. Thus the following question arises naturally:

Let \( 0 < s < d \) and let \( \mu \) be a Radon measure on \( \mathbb{R}^d \) with no atoms. Is it true that \( R^s_\mu \) is bounded in \( L^2(\mu) \) if and only if \( Q^s_\mu \) is bounded in \( L^2(\mu) \)?

As remarked above, solving this question would be a fundamental contribution for the solution of the problems explained above in 1) and 2).

4) Bilipschitz and affine invariance, and other problems.

Let \( \mu \) be a Radon measure on \( \mathbb{C} \) such that the Cauchy transform \( C_\mu \) is bounded in \( L^2(\mu). \) Recall that
\[ C_\mu f(z) = \int \frac{1}{z - \xi} f(\xi) \, d\mu(\xi). \]
In [Tol05] it has been shown that if \( \varphi : \mathbb{C} \to \mathbb{C} \) is a bilipschitz map and \( \sigma = \varphi\#\mu \) is the image measure of \( \mu, \) then \( C_\sigma \) is bounded in \( L^2(\sigma). \) The analogous problem for the \( (d - 1) \)-dimensional Riesz transform \( R^{d-1}_\mu \) in \( \mathbb{R}^d \) is open, and it seems that before trying to solve it, one should understand better the relationship between the \( L^2 \) boundedness of the Riesz transforms and rectifiability [i.e. one should first solve the questions in 1)], since this is a basic ingredient in the proof of the analogous result for the Cauchy transform in [Tol05]. However, in the case \( d > 2, \) the problem is open even when \( \varphi \) is an affine map. For instance, let
\[ \varphi(x_1, x_2, x_3, \ldots, x_d) = (2x_1, x_2, x_3, \ldots, x_d). \]
If \( R^{d-1}_\mu \) is bounded in \( L^2(\mu) \) and we set \( \sigma = \varphi\#\sigma, \) is then \( R^{d-1}_\sigma \) bounded in \( L^2(\sigma)? \)

A similar question in terms of the capacity \( \kappa \) is the following. Is it true that for any compact set \( E \subset \mathbb{R}^d, \) \( \kappa(E) \approx \kappa(\varphi(E))? \) Analogous questions can be posed for the other capacities \( \gamma_s \) and the Riesz transforms of codimension different from 1.
Let us discuss another problem whose solution may help to understand the relationship between the $L^2$ boundedness of Riesz transforms and geometry. Let $R_{(j)}^s$, $0 \leq j \leq d$, denote the scalar components of the (vectorial) Riesz transform $R^s$. Let $\mu$ be a Radon measure on $\mathbb{R}^d$ such that $\mu(B(x,r)) \leq Cr^s$ for all $x \in \mathbb{R}^d$, $r > 0$. Suppose that $d - 1$ components of $R^s_{\mu}$, say $R^s_{(1),\mu}, \ldots, R^s_{(d-1),\mu}$ are bounded in $L^2(\mu)$. Is then $R^s_{\mu}$ bounded in $L^2(\mu)$? When $s = 1$ the answer is yes, because of the curvature method. However, for other values of $s$, the problem is open again.

An analogous question can be posed in terms of the capacities associated to these kernels. That is, for a compact set $E \subset \mathbb{R}^d$, let $\tilde{\gamma}_s(E) = \sup |\nu(E)|$, where the supremum is taken over signed measures (or distributions) supported on $E$ such that $\|R^s_{(j)}v\|_{L^\infty(\mathbb{R}^d)} \leq 1$ for $0 \leq j \leq d - 1$ and $|\nu(B(x,r))| \leq r^s$ for all $x \in \mathbb{R}^d$, $r > 0$ (in case $\nu$ is a distribution the latter condition should be reformulated appropriately). Is $\tilde{\gamma}_s(E) \approx \gamma_s(E)$? It is shown in [MPV10] that the answer is affirmative for $s = 1$ and negative for $0 < s < 1$, while it is unknown when $s > 1$.

References

[AH96] David R. Adams and Lars Inge Hedberg. *Function spaces and potential theory*, volume 314 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1996.

[DS91] G. David and S. Semmes. Singular integrals and rectifiable sets in $\mathbb{R}^n$: Beyond Lipschitz graphs. *Astérisque*, (193):152, 1991.

[DS93] Guy David and Stephen Semmes. *Analysis of and on uniformly rectifiable sets*, volume 38 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1993.

[Ëíd98] V. Ya. Èıderman. Hausdorff measure and capacity associated with Cauchy potentials. *Mat. Zametki*, 63(6):923–934, 1998.

[ENV08] V. Èiderman, F. Nazarov, and A. Volberg. Vector-valued riesz potentials: Cartan type estimates and related capacities, 2008.

[Gärt72] John Garnett. *Analytic capacity and measure*. Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 297.

[GPT06] John Garnett, Laura Prat, and Xavier Tolsa. Lipschitz harmonic capacity and bilipschitz images of Cantor sets. *Math. Res. Lett.*, 13(5-6):865–884, 2006.

[Léger99] J. C. Léger. Menger curvature and rectifiability. *Ann. of Math. (2)*, 149(3):831–869, 1999.

[Mar64] John M. Marstrand. The $(\varphi, s)$ regular subsets of $n$-space. *Trans. Amer. Math. Soc.*, 113:369–392, 1964.

[Mat96] Pertti Mattila. On the analytic capacity and curvature of some Cantor sets with non $\sigma$-finite length. *Pub. Mat.*, 40, 1996.

[Mel95] M. S. Mel'nikov. Analytic capacity: a discrete approach and the curvature of measure. *Mat. Sb.*, 186(6):57–76, 1995.

[MMV96] Pertti Mattila, Mark S. Melnikov, and Joan Verdera. The Cauchy integral, analytic capacity, and uniform rectifiability. *Ann. of Math. (2)*, 144(1):127–136, 1996.

[MP95] Pertti Mattila and David Preiss. Rectifiable measures in $\mathbb{R}^n$ and existence of principal values for singular integrals. *J. London Math. Soc.* (2), 52(3):482–496, 1995.

[MPV05] Joan Mateu, Laura Prat, and Joan Verdera. The capacity associated to signed Riesz kernels, and Wolff potentials. *J. Reine Angew. Math.*, 578:201–223, 2005.
[MPV10] Joan Mateu, Laura Prat, and Joan Verdera. Capacities associated with scalar signed Riesz kernels, and analytic capacity. Preprint, 2010.

[MT04] Joan Mateu and Xavier Tolsa. Riesz transforms and harmonic Lip$_1$-capacity in Cantor sets. *Proc. London Math. Soc.* (3), 89(3):676–696, 2004.

[MTV03] Joan Mateu, Xavier Tolsa, and Joan Verdera. The planar Cantor sets of zero analytic capacity and the local $T(b)$-theorem. *J. Amer. Math. Soc.*, 16(1):19–28 (electronic), 2003.

[MV95] Mark S. Melnikov and Joan Verdera. A geometric proof of the $L^2$ boundedness of the Cauchy integral on Lipschitz graphs. *Internat. Math. Res. Notices*, (7):325–331, 1995.

[MV00a] Svitlana Mayboroda and Alexander Volberg. Boundedness of the square function and rectifiability. *C. R. Math. Acad. Sci. Paris*, 347(17-18):1051–1056, 2009.

[MV00b] Svitlana Mayboroda and Alexander Volberg. Square function and riesz transform in non-integer dimensions. Preprint, 2009.

[Par93] P. V. Paramonov. $C^m$-approximations by harmonic polynomials on compact sets in $\mathbb{R}^n$. *Mat. Sb.*, 184(2):105–128, 1993.

[Pra04] Laura Prat. Potential theory of signed Riesz kernels: capacity and Hausdorff measure. *Int. Math. Res. Not.*, (19):937–981, 2004.

[Pra09] Laura Prat. Personal communication. 2009.

[Pre87] David Preiss. Geometry of measures in $\mathbb{R}^n$: distribution, rectifiability, and densities. *Ann. of Math.* (2), 125(3):537–643, 1987.

[RdT] Aleix Ruiz de Villa and Xavier Tolsa. Non existence of principal values of signed riesz transforms of non integer dimension. *To appear in Indiana Univ. Math. J.*

[Tol03] Xavier Tolsa. Painlevé’s problem and the semiadditivity of analytic capacity. *Acta Math.*, 190(1):105–149, 2003.

[Tol05] Xavier Tolsa. Bilipschitz maps, analytic capacity, and the Cauchy integral. *Ann. of Math.* (2), 162(3):1243–1304, 2005.

[Tol08] Xavier Tolsa. Principal values for Riesz transforms and rectifiability. *J. Funct. Anal.*, 254(7):1811–1863, 2008.

[Tol09] Xavier Tolsa. Uniform rectifiability, Calderón-Zygmund operators with odd kernel, and quasiorthogonality. *Proc. Lond. Math. Soc.* (3), 98(2):393–426, 2009.

[Vih96] Merja Vihtilä. The boundedness of Riesz $s$-transforms of measures in $\mathbb{R}^n$. *Proc. Amer. Math. Soc.*, 124(12):3797–3804, 1996.

[Vol03] Alexander Volberg. Calderón-Zygmund capacities and operators on nonhomogeneous spaces, volume 100 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2003.

Institució Catalana de Recerca i Estudis Avançats (ICREA) and Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Catalonia

E-mail address: xtolsa@mat.uab.cat

URL: http://mat.uab.cat/~xtolsa