ABSTRACT: The Feynman-Vernon influence functional formalism is used to derive the effect of a black body, treated as an environment, on a massless scalar field in 1+1 dimensions. The black body is modeled as a finite region of space in which an independent ohmic heat bath is weakly coupled to the field at every point. A weak coupling approximation is developed which implements the concept of an ideal black body in the context of quantum field theory. The calculation takes advantage of the suitability to harmonic oscillators and free fields of Bargmann-Fock coherent state variables, whose convenience is illustrated by a preliminary derivation of the thermalization of a single harmonic oscillator by a heat bath with slowly varying spectral density. The black body model exhibits absorption, thermal equilibrium, and emission consistent with classical results for black bodies. It is argued that this model is in fact a realistic description of a very fine, granular medium, such as lampblack.

Pedagogically, quantum theory typically begins with quantum systems coupled to some sort of environment, not itself directly under investigation, which is coupled to the system under observation. One effectively ignores the environmental degrees of freedom, and replaces the environmental part of the interaction Hamiltonian with its expectation value. A thorough understanding of the role of classical concepts in quantum mechanics is still being sought, of course, but part of the explanation for the successes of external source models is that this external source approximation is accurate to first order in the coupling strength, and in the most common case of electromagnetic interactions, this coupling is weak enough that higher order effects may often be neglected.

There are some important situations, however, in which the quantum fluctuations around the expectation value of the coupling operator produce effects on the observed system that are not small, despite being of second order in a small coupling constant $g$. This happens because other parameters, which are large enough to outweigh the smallness of $g^2$, may enter the problem. In cases such as these there may be two kinds of second order terms — non-negligible terms containing the large parameter, as well as negligible contributions which do not. For example, when a previously uncorrelated system and environment are suddenly made to interact, the second order effects include quantum measurement, in which the rate at which the wave function collapses is set by the ultraviolet cut-off of the unobserved environment. On the other hand, a sufficiently long time scale can also enable the second order terms to drive a system into thermodynamic equilibrium with its environment.
It is a fact that should be more surprising than it is, that the Stefan-Boltzmann constant $\sigma = \frac{2\pi^5 k^4}{15h^3c^2}$ is independent of the elementary charge $e$.  

II. OSCILLATOR WITH A HEAT BATH IN BARGMANN-FOCK VARIABLES

A. The Bargmann-Fock Path Integral

When using path integrals to obtain the time evolution of a quadratic quantum mechanical system (i.e., one whose action has canonical kinetic terms, quadratic potentials, and bilinear interactions), the problem may be solved in two stages. First, one solves an equation of motion with initial and final conditions; and second, one averages over the initial and final wave functions as weights. These averages for initial and final states have particularly simple forms in a representation in which the path integral is constructed over position eigenstates. However, an ohmic environment with constant spectral density, and consider a massless quantum field coupled to a heat bath restricted to a finite region of space. Here the coherent state variables are particularly advantageous, allowing us to clearly see that this model exhibits uniform absorption and black body radiation. We then conclude with a discussion of the realism of the model chosen, the issues raised by our analysis, and some related problems for future study. An appendix reviews the important justification, originally given by Feynman and Vernon, for treating a generic weakly-coupled environment as a bath of independent harmonic oscillators.
Consider the following coherent states as basis elements for bras and kets:

\[ |\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad \langle \bar{\alpha}| = \sum_{n=0}^{\infty} \frac{\bar{\alpha}^n}{\sqrt{n!}} \langle n| \]

where the \(|n\rangle\) are harmonic oscillator energy eigenstates, and \(\alpha\) and \(\bar{\alpha}\) are complex numbers (not necessarily complex conjugates of each other). Since a whole 2-plane of parameter space is larger than needed to span the states of an oscillator, we constrain \(\alpha\) to lie on a line \(C\) through the origin, and \(\bar{\alpha}\) to lie on another line \(\bar{C}\) through the origin, such that \(\bar{C}\) is perpendicular to \(C\). We then have an expression for the identity operator,

\[
\frac{1}{2\pi i} \int_{C} d\alpha \int_{\bar{C}} d\bar{\alpha} e^{-\bar{\alpha}\alpha} |\alpha\rangle \langle \bar{\alpha}| = 1 .
\]

(This relation may be verified by inserting the definitions (1) and using differentiation under the integral sign, treating the expression as a distribution.)

Since \(|\alpha\rangle\) is an eigenstate of the annihilation operator \(\hat{a}\), any operator expressed as a normal-ordered function of creation and annihilation operators has a simple matrix form in this representation:

\[
\langle \bar{\alpha}| : A(\hat{a}^\dagger, \hat{a}) : |\alpha\rangle = A(\bar{\alpha}, \alpha) .
\]

In particular, for a Hamiltonian \(\hat{H} = :H(\hat{a}^\dagger, \hat{a})::\), we have the following matrix elements of the infinitesimal time evolution operator:

\[
\langle \bar{\alpha}| e^{-i\hat{H}\delta t} |\alpha\rangle = e^{-i\delta t h(\bar{\alpha}, \alpha)} + O(\delta t^2) .
\]

(Throughout this paper we set \(\hbar = c = 1\).) Using (2) and (4), we can derive the path integral for a transition amplitude in the usual way, by inserting identity operators between an infinite succession of infinitesimal time intervals. We obtain the Bargmann-Fock path integral

\[
\langle \bar{\alpha}| e^{-i\int_{0}^{\tau} ds \hat{H}(s)} |\alpha\rangle = \int \mathcal{D}\alpha \mathcal{D}\bar{\alpha} \delta(\alpha(0) - \alpha_i) \delta(\bar{\alpha}(\tau) - \bar{\alpha}_f) \times \exp \left[ \frac{\bar{\alpha}_f \alpha_i + \bar{\alpha}(0) \alpha_i}{2} + i \int_{0}^{\tau} ds \left( \frac{\bar{\alpha}_i \alpha_i - \bar{\alpha}(\tau) \alpha_i}{2} - H(\bar{\alpha}, \alpha) \right) \right] ,
\]

\(s\) being used as a time parameter, with \(\bar{\alpha} \equiv \frac{d}{ds}\alpha(s)\). Note that, according to the delta functions in (5), \(\alpha(s)\) and \(\bar{\alpha}(s)\) have only one boundary condition each, at the initial time \(s = 0\), and the final time \(s = \tau\), respectively.

In the case where the Hamiltonian \(H\) is up to a normalization factor, to the value of extremized. As usual, this condition is satisfied equations. In this case, the Euler-Lagrange equations that only the boundary terms do not vanish therefore simplifies to

\[
\langle \bar{\alpha}| e^{-i\int_{0}^{\tau} ds \hat{H}(s)} |\alpha\rangle = \frac{1}{Z} e^{i\bar{\alpha}_f \alpha_i + i \int_{0}^{\tau} ds H(\bar{\alpha}, \alpha)} ,
\]

for some normalization factor \(Z\). The subscript endpoint values of the solutions of the Euler-lagrange function, in the case of a harmonic oscillator driven by

\[
\hat{H}(s) = \Omega(\hat{a}^\dagger \hat{a} + \frac{1}{2}) + f(s)
\]

where \(f\) and \(\bar{f}\) are c-number functions representing the driving sources, as well as on the initial and final conditions:

\[
\alpha(t) = \alpha_i e^{-i\Omega t} + i \int_{0}^{t} ds \bar{f} e^{-i\Omega s} + i \int_{0}^{t} ds f e^{-i\Omega s} .
\]

While this paper deals with interactions that have no external sources of this type, similar calculations of the Euler-Lagrange equations, and boundary values below.

Because \(|n\rangle\langle \alpha| = \frac{\alpha^n}{\sqrt{n!}}\), it is easy to obtain these states, without integrating, from (6). One simply

\[
\langle \bar{\alpha}_f| e^{-i\int_{0}^{\tau} ds \hat{H}(s)} |\alpha_i\rangle = \sum_{m,n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \bar{\alpha}^m \Omega^{m+n} \]

and extracts the co-efficients of the powers of the harmonic oscillator, this gives a clear interpretation of the formula.

The terms in \(\alpha(t)\) proportional to \(\alpha_i\), and in particular, at time \(t = \tau\), are energy-conserving evolution of initial states \(\alpha_i\) to final states \(\alpha_f\).
\( \alpha(t) \) describes excitation of the system, by the driving source, while the other term in \( \bar{\alpha}(0) \) implies absorption by the source. This interpretation is emphasized here in order to provide some intuition for the results of Section III, in which a quantum mechanical black body will appear instead of a classical driving force, but in which the action will still be quadratic. All of the system’s dynamics will still be expressed in the boundary values of \( \alpha \) and \( \bar{\alpha} \) variables.

The basis in which a path integral is expressed determines the type of boundary conditions which the paths must satisfy. For given boundary conditions, though, one is free to change variables in the path integral. For example, we are free to re-express (5) as

\[
(\bar{\alpha}_f | e^{-i \int_0^t ds \mathcal{H}(s)} | \alpha_i) =
\int D\!P D\!Q \delta(\Omega Q(0) + iP(0) - \sqrt{2\Omega} \alpha_i) \delta(\Omega Q(t) - iP(t) - \sqrt{2\Omega} \bar{\alpha}_f)
\times \exp \left[ \frac{\bar{\alpha}_f [\Omega Q(t) + iP(t)] + \alpha_i [\Omega Q(0) - iP(0)]}{2\sqrt{2\Omega}} \right] 
\times \exp \left[ \frac{\bar{\alpha}_f [\Omega Q(t) + iP(t)] + \alpha_i [\Omega Q(0) - iP(0)]}{2\sqrt{2\Omega}} \right] 
\]

using the transformation

\[
\alpha = \frac{1}{\sqrt{2\Omega}} (\Omega Q + iP) \quad \bar{\alpha} = \frac{1}{\sqrt{2\Omega}} (\Omega Q - iP)
\]

and absorbing the constant Jacobian in the new measure. While the integral now involves familiar-looking \( P \) and \( Q \) variables, the transformation we have used will in general make \( P \) and \( Q \) complex, and the usual \( P(t) \) and \( Q(0) \) boundary conditions do not apply. They are replaced by conditions on the linear combinations \( \alpha \) and \( \bar{\alpha} \), reflecting the fact that (10) is still a transition amplitude between coherent states. The \( P, Q \) type of variables will be more convenient for solving the equations of motion of Section III, while \( \alpha, \bar{\alpha} \) variables will still be better suited to the initial and final states. In the path integral formalism, there will be no reason for us not to take the best of both worlds.

**B. Oscillator coupled to a heat bath**

Consider now a system consisting of a simple harmonic oscillator of natural frequency \( \Omega \), minimally coupled to a heat bath made up of environmental oscillators. This is a toy model, but it is very relevant to more significant problems involving free quantum fields. We assume that

\[
\hat{H} = \hat{H}_{\text{SHO}} + \hat{H}_{\text{ENV}}
\]

\[
\hat{H}_{\text{SHO}} = \frac{1}{2} (\dot{P}^2 + \Omega^2 \dot{Q}^2) = \Omega \sqrt{\omega} (\dot{Q} + \Omega \alpha)
\]

\[
\hat{H}_{\text{ENV}} = \frac{1}{2} \int_0^\infty d\omega \, I(\omega) \left( \frac{\hat{P}}{\sqrt{\omega}} \right)^a
\]

where \( \hat{a} \equiv \frac{1}{\sqrt{2\Omega}} (\Omega \hat{Q} + i\hat{P}) \), and \( g \) is a coupling constant, \( \Omega \), and will use this fact extensively below. The density, whose properties will be further developed, is a positive definite and of reducing the size normalization of \( \Omega \).

We are interested in the behaviour of the observed oscillator alone, and so we must integrate out of the problem all of the environmental degrees of freedom.

Assume that the initial density matrix for the complete system factorizes:

\[
\rho(0) = \bar{\rho}(0) \otimes \rho_{\text{env}}(0)
\]

where \( \bar{\rho}(0) \) acts in the Hilbert space of the observed oscillator at time \( t \) is then given, as follows.

The propagator equation

\[
\rho(\bar{\alpha}_f, \alpha'_i; t) \equiv \langle \bar{\alpha}_f | \rho(t) | \alpha'_i \rangle
\]

is given by

\[
\int \frac{d\alpha_d d\bar{\alpha}_d d\alpha'_d d\bar{\alpha}'_d}{(2\pi)^2} J(\bar{\alpha}_j, \alpha'_j, \alpha'_i, \bar{\alpha}'_i)\rho_d(\alpha_d, \bar{\alpha}_d; 0)\rho_{\text{env}}(\alpha'_d, \bar{\alpha}'_d; 0)
\]

This equation may be written in the normal propagator basis, by defining the propagator \( K_{klmn} \) in terms that

\[
J(\bar{\alpha}_j, \alpha'_j, \alpha'_i, \bar{\alpha}'_i) \equiv \sum_{k l m n} K_{klmn}
\]

which implies that

\[
\rho_{\text{env}}(t) = \sum_{k l m n} K_{klmn} \rho_d(0)
\]

\[
\int d\alpha_d d\bar{\alpha}_d d\alpha'_d d\bar{\alpha}'_d
\]

\[
\int d\alpha_d d\bar{\alpha}_d d\alpha'_d d\bar{\alpha}'_d
\]
\[
J(\tilde{\alpha}_j, \alpha'_j; \alpha'_i, \alpha_i; t) \text{ may be obtained explicitly from a path integral similar to (10):}
\]
\[
J(\tilde{\alpha}_j, \alpha'_j; \alpha'_i, \alpha_i; t) = \int DQ DQ' D\alpha D\alpha' \delta(\alpha(0) - \alpha_i) \delta(\tilde{\alpha}(t) - \tilde{\alpha}_j) \\
\times \delta(\alpha'(t) - \alpha'_j) \delta(\tilde{\alpha}'(0) - \tilde{\alpha}_i) e^{i(A + B + V)}.
\]

In this expression, \(A\) is the action term
\[
A \equiv \frac{1}{2} \int_0^t ds P\tilde{Q} - Q\tilde{P} - P'\tilde{Q}' + Q'\tilde{P}' - (P^2 - P'^2) - \Omega^2(Q^2 - Q'^2); \tag{18}
\]
\(B\) is the boundary term
\[
B \equiv \frac{\tilde{\alpha}_j \alpha(t) + \alpha(0) + \tilde{\alpha}'(0) + \alpha'(t) + \alpha'_j}{2t}, \tag{19}
\]
where the \(\alpha\) variables are given in terms of the \(Q\)'s and \(P\)'s by (11); and \(V = V[Q, Q']\) is the influence phase. (The quantity \(e^{iV}\) is known as the influence functional.)

\(V\) contains all the information about the heat bath that is relevant to the observed oscillator. It may be computed in a straightforward manner. (This has been done using the Lagrangian form of the environmental path integral: the Lagrangian obtained from \(H_{ENV}\) is
\[
L_{ENV} = \frac{1}{2} \int_0^\infty d\omega I(\omega) \delta \left( \eta^2 + 2gQ_\omega - \omega^2 \eta_\omega^2 \right), \tag{20}
\]
which is that of the velocity coupled model analysed in Reference [11].) In the case where the initial state of the bath is thermal at inverse temperature \(\beta\), one finds that
\[
V[Q, Q'] = \frac{ig^2}{2} \int_0^\infty d\omega I(\omega) \int_0^t ds \int_0^s ds' \left( \left( Q - Q' \right) \cos \omega(s - s') - i \left( Q + Q' \right) \sin \omega(s - s') \right) \\
- \frac{g^2}{2} \int_0^\infty d\omega I(\omega) \int_0^t ds \left( Q^2 - Q'^2 \right)_s \\
\equiv \frac{ig^2}{2} \int_0^t ds \int_0^s ds' \left( \left( Q - Q' \right) \nu(s - s') - i \left( Q + Q' \right) \eta(s - s') \right) \\
- \frac{\mu^2}{2} \int_0^t ds \left( Q^2 - Q'^2 \right). \tag{21}
\]

We use the notation \((Q - Q')_s = (Q(s) - Q'(s))_s\) to define the quantity \(\mu^2\) and the functions \(\eta\) and \(\nu\).

The exponent \(i(A + B + V)\) in (17) is quite constant to the extremized value of the partition function subject to the constraints imposed by the boundary conditions
\[
\Omega_\pm = 0, \quad P_\pm = P \pm P';
\]
\[
\dot{Q}_\pm(s) = P_\pm(s)
\]
\[
\dot{P}_\pm(s) = -\Omega^2 Q_\pm(s) - g^2.
\]
These equations together imply that \(A + V \pm B\) reduces to one of endpoint values at the extremum,
\[
J(\tilde{\alpha}_j, \alpha'_j; \alpha'_i, \alpha_i; t) = \frac{1}{Z} \exp \left( iA + B + V \right) \bigg|_0 = \frac{1}{Z} \exp \left( i\tilde{\alpha}_j \alpha(t) + \alpha(0) \right)
\]
where \(Z\) is a normalization constant. Thus (25), subject to the boundary conditions
\[
\frac{1}{2} \left( \Omega(Q_+ + Q_-) + i(P_+ + P_-) \right) = 0
\]
\[
\frac{1}{2} \left( \Omega(Q_+ - Q_-) - i(P_+ - P_-) \right) = 0
\]
\[
\frac{1}{2} \left( \Omega(Q_+ + Q_-) + i(P_+ + P_-) \right) = 0
\]
imposed by the delta functions in (17), and subject to the boundary conditions, these solutions determine.

The easiest way to solve these coupled equations is to work in Fourier space. The problem then reduces to the solution of a pair of linear partial differential equations, which can be solved by means of Fourier transforms. The solution can then be inverted to obtain the time-dependent correlation functions.
of a complex resonant frequency, as a pole in Fourier space. To begin, we differen-
tiate (22) and substitute it into (24), obtaining a decoupled, second order integro-
differential equation for \( Q_- \):

\[
\dot{Q}_- + (\Omega^2 + \mu^2)Q_- = -g^2 \int_\infty^t ds' Q_-(s')\eta(s - s').
\]  
(28)

We then define the Fourier transform \( Q_-(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dw Q_- e^{i\omega(t-s)} \), and use the definition \( \eta(s - s') = \int_0^{\infty} dw \omega I(\omega) \sin(\omega(s - s')) \) (implicit in Eq. (21)) to obtain

\[
\int_{-\infty}^{\infty} d\omega \, e^{i\omega(t-s)} \left[ Q_-(\Omega^2 - \omega^2 (1 + g^2\lambda(\omega^2))) - \frac{g^2}{2}\omega I(\omega)P \right] \int_{-\infty}^{\infty} d\omega' \frac{Q_-}{\omega' - \omega} = 0,
\]  
(29)

where we have combined the \( \mu^2 = \int_0^{\infty} d\omega I(\omega) \) term with the contribution from the lower limit of the \( s' \) integral to form the frequency re-normalization term

\[
\lambda(\omega^2) \equiv P \int_{0}^{\infty} d\omega' \frac{I(\omega^2)}{\omega^2 - \omega^2}.
\]  
(30)

The \( P \) in front of the integral sign denotes the Cauchy principal value. We assume that \( \lambda(\omega^2) \ll \frac{1}{\gamma^2} \).

The crux of the problem now appears to be the Hilbert transform in the rightmost term of (29). The Hilbert transform has the property that

\[
P \int_{-\infty}^{\infty} \frac{d\omega'}{\omega' - \omega - (x + iy)} = -i\pi \text{sgn}(y) \int_{-\infty}^{\infty} \frac{1}{\omega - (x + iy)}.
\]  
(31)

With this in mind, we choose the ansatz

\[
Q_- = \frac{A}{\omega - (\Omega + i\gamma)}
\]  
(32)

for some \( \Omega \) and \( \gamma > 0 \) to be determined, and \( A \) a constant to be fitted to boundary conditions. Equation (29) then becomes

\[
\int_{-\infty}^{\infty} d\omega \, e^{i\omega(t-s)} \left( \omega^2 (1 + g^2\lambda) - \Omega^2 - i\frac{g^2}{2}\omega I(\omega) \right) = 0.
\]  
(33)

We now invoke our assumption that \( \frac{\gamma^2}{\Omega^2} \) is a small quantity, and require further that the spectral density \( I(\omega) \) is always much smaller than \( \frac{1}{\gamma^2} \), and is slowly varying near \( \omega = \Omega \). With these conditions, we can choose

\[
\frac{\gamma}{\Omega} = \frac{\frac{\pi g^2}{4} I(\Omega)}{\sqrt{1 + g^2\lambda(\Omega^2)}},
\]  
(34)

and take the \( Q_- \) chosen in (32) as the leading \( \frac{\gamma}{\Omega} \). The first order correction to \( Q_-(s) \) will

\[
\frac{Ag^2}{4\pi i\Omega} \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega(t-s)}}{\omega - \Omega - i\gamma} \left( \frac{\pi g^2}{2} I(\Omega) - I(\omega) \right)
\times \left( \frac{1}{\omega - \Omega - i\gamma} - \frac{1}{\omega + \Omega - i\gamma} \right) \lambda(\omega^2)
\]

The poles at \( \pm \Omega + i\gamma \) contribute a correction to the zeroth order solution. For a wide range of realistic spectral densities.

We can substitute \( \Omega \rightarrow -\Omega \) and the perform the change of variable \( \omega \rightarrow -\omega \) in (31) to get

\[
Q_-(s) \approx Ac e^{i\Omega(t-s)} e^{\gamma(s-t)} + \ldots
\]

valid up to corrections of order \( \frac{\gamma}{\Omega} \).

This solution is anti-damped, growing exponential behaviour is usually considered unphysical, but such solutions do not apply in this case. Despite being, similarly to a position variable, it is really a measure of the nonlinearity in the position basis of the oscillator, and not a position variable, it represents the oscillator’s mean displacement from the classical equation of motion, but rather a tool for computing the evolution of a quantum mechanical system. In classical mechanics, conditions must apply at the initial time, this results from a “runaway” solution; but in quantum mechanics, final time, with its own boundary conditions, must be made to conform. And in this case, the essential feature responsible for establishing exponential solutions has long been known to occur in the limit when

\[
\frac{\gamma}{\Omega} \ll \frac{1}{\gamma^2}.
\]  

1 This series may be constructed by replacing \( \delta \) delta functions. (A continuous spectral density for such a discrete spectrum, anyway.)
We now turn to Eq. (23), differentiating it and applying (25). We obtain the damped and driven oscillator equation
\[ \ddot{Q}_+ + (\Omega^2 + \mu^2)Q_+ - g^2 \int_0^t ds' Q_+(s')\eta(s-s') = ig^2 \int_0^t ds' Q_-(s')\nu(s-s') . \]

The homogeneous part of (37) is simply the equation formed by substituting \((t-s) \to \infty\). We therefore have the free (approximate) solutions
\[ Q_+^0(s) \simeq B e^{-i\Omega s} e^{-\gamma s} + \bar{B} e^{i\Omega s} e^{-\gamma s} . \tag{38} \]

The source on the RHS of (37) is given by (36) as
\[ \int_0^t ds' Q_-(s')\nu(s-s') \simeq \frac{g^2}{2} \int_{-\infty}^{\infty} d\omega \omega \coth \frac{\beta \omega}{2} I(|\omega|) e^{i\omega s} \left( A e^{(i\Omega - \gamma)t} - e^{-i\omega t} + A e^{(-i\Omega - \gamma)t} - e^{-i\omega t} \right) . \tag{39} \]

For the particular solution, we presume that \(Q_+(s)\) does behave as a damped oscillator, and therefore consider the candidate
\[ Q_+^p(s) = \frac{g^2}{2} \int_{-\infty}^{\infty} d\omega \omega \coth \frac{\beta \omega}{2} I(|\omega|) e^{i\omega s} \left( A e^{(i\Omega - \gamma)t} - e^{-i\omega t} + A e^{(-i\Omega - \gamma)t} - e^{-i\omega t} \right) \tag{40} \]

We examine first the term proportional to \(A\) in the LHS of (37):
\[ LHS(A) = A \int_{-\infty}^{\infty} d\omega e^{i\omega s} \left( Q_+^p \left( \Omega^2 - \omega^2 (1 + g^2 \lambda) \right) - \frac{g^2}{2} \omega I(|\omega|) P \int_{-\infty}^{\infty} d\omega' \frac{Q_+^{0'}}{\omega' - \omega} \right) \]
\[ = A \int_{-\infty}^{\infty} d\omega e^{i\omega s} \left[ Q_+^p \left( \Omega^2 - \omega^2 (1 + g^2 \lambda) \right) - \frac{g^2}{2} \omega I(|\omega|) \coth \frac{\beta \Omega}{2} I(|\Omega|) \right] \]
\[ \times \left( e^{-i\omega t} - e^{(i\Omega - \gamma)t} \right) \]
\[ = A \int_{-\infty}^{\infty} d\omega e^{i\omega s} \left[ Q_+^p \left( \Omega^2 - \omega^2 (1 + g^2 \lambda) \right) + ig^2 \gamma \omega \omega I(|\omega|) \coth \frac{\beta \Omega}{2} (e^{(i\Omega - \gamma)t} - e^{-i\omega t}) \right] \]
\[ = A \int_{-\infty}^{\infty} d\omega e^{i\omega s} \left[ Q_+^p \left( \Omega^2 - \omega^2 (1 + g^2 \lambda) \right) + ig^2 \gamma \omega \omega I(|\omega|) \coth \frac{\beta \Omega}{2} (e^{(i\Omega - \gamma)t} - e^{-i\omega t}) \right] \]
\[ + \left( B e^{-i\Omega s} + \bar{B} e^{i\Omega s} \right) e^{-\gamma s} - \coth \frac{\beta \Omega}{2} \left( A e^{-i\Omega s} + \bar{B} e^{i\Omega s} \right) e^{-\gamma s} \tag{41} \]

neglecting terms of \(O(g^2)\). In (41) we have two sharply peaked around \(\omega = -\Omega\), and that \(\omega \to \infty\) region. We have also invoked similar approximations (36).

Adding the similar term proportional to \(\bar{B}\), we can see that (40) does indeed satisfy (37) (40), once again using the fact that \(\coth \frac{\beta \Omega}{2} = \frac{1}{\beta \Omega} \\coth \frac{\beta \Omega}{2}\) is odd), we have the approximate solution
\[ Q_+(s) \simeq (B e^{-i\Omega s} + \bar{B} e^{i\Omega s}) e^{-\gamma s} - \coth \frac{\beta \Omega}{2} \left( A e^{-i\Omega s} + \bar{B} e^{i\Omega s} \right) e^{-\gamma s} \tag{42} \]

Using (36) and (43) in (22) and (23), we have
\[ iP_-(s) \simeq \Omega (A e^{i\Omega(t-s)} e^{\gamma(s-t)} - \bar{A} e^{i\Omega(t-s)}) \]
\[ iP_+(s) \simeq -\Omega \coth \frac{\beta \Omega}{2} (A e^{i\Omega(t-s)} + \bar{B} e^{i\Omega(t-s)}) \]

We can now impose the boundary conditions
\[ \sqrt{\Omega} \alpha \simeq \Omega \left( 1 - \coth \frac{\beta \Omega}{2} \right) \]
\[ \sqrt{\Omega} \alpha' \simeq \Omega \left( -1 + \coth \frac{\beta \Omega}{2} \right) \]
\[ \sqrt{2\Omega} \alpha' \simeq \Omega \left( 1 + \coth \frac{\beta \Omega}{2} \right) \]
\[ \sqrt{2\Omega} \alpha' \simeq \Omega \left( (\coth \frac{\beta \Omega}{2} - 1) \right) \]
In the thermal limit where \( e^{-\gamma t} \to 0 \) and \( \frac{g^2}{\Omega} \to 0 \), these imply

\[
A = -\sqrt{\frac{2}{\Omega}} \coth \frac{\beta \Omega}{2} + 1
\]

\[
\bar{A} = \sqrt{\frac{2}{\Omega}} \coth \frac{\alpha'}{2} + 1
\]

\[
B = \sqrt{\frac{2}{\Omega}} \alpha_f
\]

\[
\bar{B} = \sqrt{\frac{2}{\Omega}} \alpha_f'.
\]  

(46)

Using (46) and (11), we now obtain

\[
\alpha(t) = e^{-\beta \bar{\Omega} \alpha_j'} \quad \alpha'(0) = \alpha_i,
\]

\[
\bar{\alpha}'(t) = e^{-\beta \bar{\Omega} \bar{\alpha}_j} \quad \bar{\alpha}(0) = \bar{\alpha}_i'.
\]  

(47)

Applying these results to (26), we find the propagator

\[
J = \frac{1}{Z} \exp \left( \bar{\alpha}' \bar{\alpha} + e^{-\beta \bar{\Omega} \bar{\alpha}} \alpha \right),
\]  

(48)

where \( Z \) will turn out to be the partition function of the canonical ensemble.

Using (14), we find that the final state of the oscillator has the density matrix

\[
\rho(\bar{\alpha}, \alpha) = \frac{1}{Z} \exp \left( e^{-\beta \bar{\Omega} \bar{\alpha}} \alpha \right),
\]

which is the canonical ensemble in coherent state variables, regardless of the initial state. We may translate this result into the standard energy representation, using (15) to discern

\[
K_{klmn}(t) = (1 - e^{-\beta \bar{\Omega}}) \delta_{kl} e^{-m \beta \bar{\Omega}} \delta_{mn} + \mathcal{O}(\frac{g^2}{\Omega}) + \mathcal{O}(e^{-\gamma t})
\]

which implies that

\[
\rho_{mn}(t) \simeq (1 - e^{-\beta \bar{\Omega}}) \delta_{mn} e^{-m \beta \bar{\Omega}}.
\]  

(49)

(Although \( \bar{\Omega} = \Omega + \mathcal{O}(\frac{g^2}{\Omega}) \)), one might be concerned that for low temperatures, where \( \beta \) becomes arbitrarily large, \( \beta g^2 \lambda(\Omega^2) \) would not be small, so that it would be wrong to write \( \Omega \) instead of \( \bar{\Omega} \) in (49). But because \( g^2 \ll \Omega \), regardless of temperature, \( \beta g^2 \) can only be non-negligible when \( \beta \Omega \) is so large that \( e^{-\beta \bar{\Omega}} \simeq e^{-\beta \Omega} \simeq 0 \). Hence the distinction between \( \Omega \) and \( \bar{\Omega} \) is not discernible in the final state, within the limits of our approximations.)

We therefore conclude that a harmonic oscillator coupling to an environment with slowly varying spectral density is driven to thermal equilibrium on the thermal time scale \( t_\gamma \), according to the result of Reference [7], where the path integral in position variables was used to obtain thermalization for a weakly coupled environment. In our velocity coupled model, this would be equivalent to solving Reference [7] notes that more general dissipative environments can, in principle, induce thermalization, but does not treat them explicitly.)

We conclude this section by emphasizing that, in the case of the single oscillator coupled to a bath, weak coupling and slowly varying spectral density effectively prevent the environment from communicating to the system any information other than its temperature, and so guarantee the canonical thermal behavior.

III. BLACK BODY RADIATION

A. Setting up the model

We now present our main result. When a massless field is weakly coupled to an environment which is confined to a finite region of space, modes with wavelengths long compared to the coupling tend to be reflected at the boundaries of this region, but shorter wavelength quanta may experience negligible reflection.

When the thickness of the region is large compared to the coupling, transmission through it is also negligible, and the quanta are absorbed — and emitted as thermal radiation. After the initial time \( \tau = 0 \), when the field and its environment are uncorrelated, this radiation propagates outwards from the absorbing region; but behind a wavefront region whose width is on the order of the inverse square of the coupling, it is time-independent.

† Cooling of the emitting medium is not considered.
coupled to the field $\Lambda \phi$ will only be touched on in this paper. We let our spatial coordinate extrapolation to a vector field in $3 + 1$ dimensions is straightforward enough, but indiscriminately.

Also for simplicity, we consider a scalar field in $1 + 1$ dimensions; the extrapolation to a vector field in $3 + 1$ dimensions is straightforward enough, but will only be touched on in this paper. We let our spatial co-ordinate $x \in [-\Lambda, \Lambda]$, and impose periodic boundary conditions on the field $\phi$. (We will eventually let $\Lambda \rightarrow \infty$, of course, but this infrared regulator will turn out to be convenient.) The black body is placed at the origin, filling the region $-L < x < L$ with a uniform medium consisting of independent ohmic heat baths at every point, minimally coupled to the field\footnote{Attaching an independent bath to every point is an approximation, valid for sufficiently long wavelengths, as discussed in Section IV. Similar models, at zero temperature, have been discussed in References [12] and [13].}.

The Hamiltonian is

$$\hat{H} = \frac{1}{2} \int_{-\Lambda}^{\Lambda} dx (\hat{\Pi}(x) + \partial_x \hat{\phi}^2(x)) + \int_{-L}^{L} dx \int_{0}^{\infty} d\omega \left( \left( \hat{p}_\omega(x) - g \hat{\phi}(x) \right)^2 + \omega^2 \hat{q}_\omega^2(x) \right).$$

Once again, we assume that $g^2$ is very small compared to an infrared cut-off frequency $\Omega$.

Define creation and annihilation operators by

$$\hat{a}_n = \int_{-\Lambda}^{\Lambda} dx \frac{dx}{2|m|\pi} e^{-i\frac{n\pi}{\Lambda}} \left( \frac{n\pi}{\Lambda} \hat{\phi}(x) + i\hat{\Pi}(x) \right),$$

so that $[\hat{a}_n, \hat{a}_m^\dagger] = \delta_{m,n}$, and then introduce the Bargmann-Fock variables $\alpha_n(s)$ and $\bar{\alpha}_n(s)$. (For the mode $n = 0$, we are proceeding as though the field had an infinitesimal IR regulating mass $\epsilon << \frac{n\pi}{\Lambda}$, so that

$$\frac{n\pi}{\Lambda} \rightarrow \begin{cases} \frac{n\pi}{\Lambda}, & n \neq 0 \\ \epsilon, & n = 0 \end{cases},$$

but we will keep this replacement implicit for brevity.) Introduce as well the primed variables $\alpha'_n(s), \bar{\alpha}'_n(s)$ needed to express a mixed state of the field.

In this section we once again use a path to derive a propagator, similar to that in (14), for the scalar field:

$$\rho[\alpha'_m, \alpha'^*_l; t] \equiv \langle \alpha'^*_m \vert \hat{\rho}(t) \vert \alpha'^*_l \rangle = \prod_{k,l=-\infty}^\infty \int_{k,l=0}^\infty \frac{d\alpha_k d\bar{\alpha}_l}{2\pi i} \rho_{k,l}(t) \mathcal{J}[\alpha'_k, \bar{\alpha}'_l],$$

where $|\alpha'_l\rangle$ is shorthand for the infinite tensor likewise for the bra.

The propagator $\mathcal{J}$ may be computed from (17):

$$\mathcal{J}[\alpha'_k, \bar{\alpha}'_l, \alpha'_m, \bar{\alpha}'_n; t] = \int D\phi D\bar{\phi} \rho_{k,l}(t) \mathcal{J}[\alpha'_k, \bar{\alpha}'_l, \alpha'_m, \bar{\alpha}'_n; t],$$

where the field variables $\phi(x,s)$ and $\Pi(x,s)$ a new variables according to

$$\alpha_m(s) = \int_{-\Lambda}^{\Lambda} \frac{dx}{2|m|\pi} e^{-im \frac{\pi}{\Lambda} x} \phi(x,s),$$

$$\bar{\alpha}_m(s) = \int_{-\Lambda}^{\Lambda} \frac{dx}{2|m|\pi} e^{im \frac{\pi}{\Lambda} x} \bar{\phi}(x,s).$$

The primed version of (53) defines $\phi'$ and $\Pi'$ variables in (52) are as follows:

$$\alpha'_m(0) = \int_{-\Lambda}^{\Lambda} \frac{dx}{2|m|\pi} e^{-i\frac{m\pi}{\Lambda} x} \phi'(x,0),$$

$$\bar{\alpha}'_m(0) = \int_{-\Lambda}^{\Lambda} \frac{dx}{2|m|\pi} e^{i\frac{m\pi}{\Lambda} x} \bar{\phi}'(x,0),$$

$$\alpha'_m(t) = \int_{-\Lambda}^{\Lambda} \frac{dx}{2|m|\pi} e^{-i\frac{m\pi}{\Lambda} x} \phi'(x,t),$$

$$\bar{\alpha}'_m(t) = \int_{-\Lambda}^{\Lambda} \frac{dx}{2|m|\pi} e^{i\frac{m\pi}{\Lambda} x} \bar{\phi}'(x,t).$$

The boundary term $B$ in (52) is most briefly written as

$$B = \frac{1}{2i} \sum_{m,n=-\infty}^{\infty} \left( \hat{a}_m^\dagger \alpha_m(t) + \hat{a}_n(0) \bar{\alpha}_n^\dagger \right).$$
where the boundary values of the $\alpha$'s may be found from the boundary values of the $\phi$'s and $\Pi$'s using (53). As in Section II, the action term in the path integral is of the usual $p\dot{q} - H$ form (up to a boundary term), minus a like term in primed variables:

$$
A = \frac{1}{2} \int_0^t ds \int_{-\Lambda}^\Lambda dx \left( \Pi_\phi - \phi \Pi - \Pi' \dot{\phi}' + \phi' \Pi' - \Pi^2 - \Pi'^2 - \partial_x \phi^2 + \partial_x \phi'^2 \right)
$$

$$= \frac{1}{2} \int_0^t ds \int_{-\Lambda}^\Lambda dx \left( \Pi_\phi + \Pi_\phi - \phi \Pi_\phi - \phi \Pi_\phi - \Pi_\phi - \Pi' \partial_x \phi \partial_x \phi' \right),
$$

where we introduce the linear combinations $\phi_\pm = \phi \pm \phi'$ and $\Pi_\pm = \Pi \pm \Pi'$.

Because the heat baths coupled to the field at each point are independent, the influence phase $\mathcal{V}$ is simply an integral of influence phases similar to (21) (with $I(\omega)$ set equal to one and some simplification via integration by parts, as in Reference [11]):

$$
\mathcal{V} = \frac{ig^2}{8} \int_0^t ds \int_0^t ds' \int_{-L}^L dx \phi_-(x,s) \phi_-(x,s') \int_{-\infty}^\infty d\omega \coth \frac{\beta \omega}{2} e^{i\omega(s-s')} - \frac{\pi g^2}{4} \int_0^t ds \int_{-L}^L dx \dot{\phi}_+(x,s) \phi_-(x,s) - \frac{\pi g^2}{4} \int_{-L}^L dx \phi_+(x,0) \phi_-(x,0).
$$

Since we will once again be interested in the case where $g^2$ is small, we will neglect the last term in (57), on the grounds that it will produce only negligible perturbations on the solutions to the field equations. The black body is here assumed to have a uniform initial temperature $\frac{1}{\cosh \omega}$, and the limits on the integral over $\omega$ are meant to be taken to infinity only after all other integrations have been performed.

The exponent of the integrand in (52) is thus quadratic, and the propagator $\mathcal{J}$ is proportional to the value of the integrand when this exponent is extremized, subject to the boundary conditions. Once again, $\mathcal{A} + \mathcal{V}$ vanishes on shell, and the propagator is given by

$$
\mathcal{J}[\alpha^i, \alpha'^i, \alpha^f, \alpha'^f; t] = \frac{1}{Z} e^{i\mathcal{S}}|_0^t.
$$

The Euler-Lagrange equations for this extremum are

$$
\dot{\phi}_- (x,s) = \Pi_-, \quad \dot{\Pi}_- (x,s) = -\frac{ig^2}{2} \theta(L - |x|) \phi_- = 0
$$

$$
\dot{\phi}_+ (x,s) = \Pi_+, \quad \dot{\Pi}_+ (x,s) = -\frac{ig^2}{2} \theta(L - |x|) \dot{\phi}_+ = \frac{ig^2}{2} \int_0^t ds' \phi_-(x,s').
$$

B. Solving the equations

We will find solutions to (63) and (64) that are complex exponentials in time and space. We begin with (63), and consider

$$
\phi_-(x,s) = e^{-iks} \begin{cases} 
L_+ e^{iks} + L_- \\
M_+ e^{iks} + M_- \\
R_+ e^{iks} + R_-
\end{cases}
$$

where $\kappa$ is a complex frequency to be determined, and

$$
\kappa^2 = k^2 - \frac{\pi g^2}{2} k = (k - i \frac{\pi g^2}{4}).
$$

To be a saddlepoint of the path integral, the solution prescribed by (65) must be $C^1$ everywhere. We must therefore
approach \( x = \pm L \) from either side. It is easy to see from the discontinuity in the wavenumber implied by (66) that there will be reflection at the boundary, with reflection co-efficient of order \( \frac{\gamma^2}{\Omega} \). This implies that there is really no such thing as a perfect absorber, since below some threshold frequency there will always be significant reflection. Nevertheless, if we restrict our attention to field modes with frequencies above an infrared cut-off \( \Omega \), such that \( \gamma^2 << \Omega \), then we can neglect reflection, and consider our black body to absorb perfectly at all frequencies of interest. Satisfying the periodic boundary conditions at \( x = \pm \Lambda \) forces \( k \) to assume a discrete set of values. Imposing these constraints on \( L_\pm, M_\pm, \) and \( R_\pm \), we find the most convenient parametrization of the general solution, above the IR cut-off, to be given approximately by

\[
\phi_-(x, s) = \sum_{|\frac{\pi x}{L}| > \Omega} e^{i \frac{2 \pi x}{L}} (B_n f_n(x) e^{-i |\frac{2 \pi x}{L}|} + B_n f_n(-x) e^{i |\frac{2 \pi x}{L}|} ) e^{\gamma s} + O(\frac{\gamma^2}{\Omega}),
\]

where

\[
f_n(x) = f(x \text{ sgn}(n)) = \begin{cases} \frac{1}{f(-x)}, & L < x < \Lambda \\ e^{i \gamma x}, & -\Lambda < x < -L \\ e^{-i \gamma x}, & \frac{L}{2} < x < L \\ e^{i (\Gamma + \gamma) x} e^{-\gamma x}, & L < x < \Lambda \\ e^{-i (\Gamma + \gamma) x} e^{\gamma x}, & -L < x < \frac{L}{2} \\ \end{cases}
\]

The attenuation co-efficients \( \gamma \equiv \frac{\pi^2 \Lambda}{L} \) and \( \Gamma \equiv \frac{\pi^2 \Lambda}{L} \) are defined so that \( f \) is indeed periodic.

The general solution to the homogeneous part of (64) is of the same form as \( \phi_-(x, s) \). The driven solutions, using the source \( \phi_-(x, s') \) as given by (67), may be found by the usual Fourier technique. Once again discarding from this particular solution terms which are actually free solutions, we obtain the general approximate solution

\[
\phi_+(x, s) \simeq \sum_{|\frac{\pi x}{L}| > \Omega} e^{i \frac{2 \pi x}{L}} (B_n f_n(-x) e^{-i |\frac{2 \pi x}{L}|} + B_n f_n(x) e^{i |\frac{2 \pi x}{L}|} ) e^{-\gamma s} - \sum_{|\frac{\pi x}{L}| > \Omega} e^{i \frac{2 \pi x}{L}} \frac{\beta |n| \pi}{2 \Lambda} \left( A_n f_n(x) e^{-i |\frac{2 \pi x}{L}|} - \bar{A}_n f_n(-x) e^{i |\frac{2 \pi x}{L}|} \right) e^{-\gamma s},
\]

where the “\( \simeq \)” means that we have neglected all modes with frequency less than \( \Omega \).

Using (69) in (59) and (60), and dropping terms of order \( \gamma \), this will cancel in our final results, leaving the contributions from the absorbed modes, and as long as we restrict our attention to field modes with frequency less than \( \Omega \), we find also

\[
i \Pi_+(x, s) \simeq \sum_{|\frac{\pi x}{L}| > \Omega} \frac{\beta |n| \pi}{2 \Lambda} \left( A_n f_n(x) e^{-i |\frac{2 \pi x}{L}|} - \bar{A}_n f_n(-x) e^{i |\frac{2 \pi x}{L}|} \right) e^{-\gamma s} - \sum_{|\frac{\pi x}{L}| > \Omega} \frac{\beta |\pi|}{2 \Lambda} e^{i |\frac{2 \pi x}{L}|} \coth \left[ \frac{\beta |n| \pi}{2 \Lambda} \right] (A_n f_n(x) e^{-i |\frac{2 \pi x}{L}|} + \bar{A}_n f_n(-x) e^{i |\frac{2 \pi x}{L}|}) e^{-\gamma s}.
\]

With regard to the neglect of modes below the cut-off, modes still have solutions of the form (65), and the non-negligible reflection. When we return to (53), therefore, their contribution to the Fourier transform will be negligible, for \( |\frac{\pi x}{L}| > \Omega \). Hence the infrared modes are effectively decoupled from the absorbed modes, and as long as we restrict our attention to field modes with frequency less than \( \Omega \), we can use (69) and (70) without worrying about the infrared contributions.

This last argument hinges on the fact that \( |\eta| << |k| \) so that the growth or decay of the function is only significant over very many wavelengths, is primarily close to \( k \). This statement may be made more explicit by considering the matrix

\[
d_{mn} \equiv \int_{-\Lambda}^{\Lambda} \frac{dx}{2 \Lambda} e^{i (n-m) \frac{2 \pi x}{L}} f_n(x)
\]

where we take the upper (lower) of the \( \pm \) signs for \( \eta \), quasi-diagonal — predominantly concentrated within a distance \( \Lambda \) of the diagonal. If we take the upper (lower) of the \( \pm \) signs for \( \eta \), the large factor \( e^{i L} \) will cancel in our final results, and the off-diagonal elements of \( d_{mn} \) still suppressed.
reflection terms ignored in (67) remain negligible, despite the fact that some will have amplitudes of order $e^{\ell L}$.

The quasi-diagonality of $d_{nm}$ will be of crucial importance in the remainder of this paper. We will use it to construct an approximation whereby modes of all frequencies above the IR cut-off are slowly modulated, so that they effectively vanish in large regions of space, but retain well-defined wavelengths in the regions where they do not. It is in this manner that we will identify thermal radiation of all frequencies propagating outwards from our model black body. The "no reflection" condition that $f_{nn} << 1$ thus turns out to have additional simplifying consequences far beyond making reflection negligible. As in Section II, it is weak coupling and slowly varying spectral density that lead to canonical thermal behaviour; the non-reflectivity of a black body is simply a sign that the weak coupling limit applies.

We can now re-combine $\phi_{x}$ and $\Pi_{x}$ into the Bargmann-Fock $\alpha$ variables, having effectively used the other variables to de-couple the Bargmann-Fock equations of motion. Combining (69) and (70) with the definition (53), assuming $|\frac{m x}{A}| > \Omega$, and then using $|n|d_{mn} \simeq |m|d_{mn}$, we deduce the following approximate solutions for the Bargmann-Fock variables:

$$\alpha_{n}(s) \simeq \sqrt{|m|\pi} \sum_{n=-\infty}^{\infty} e^{-i \frac{\pi m x}{A}|s|} \left( -d_{mn}(\coth \frac{\beta |n| \pi}{2\Lambda} - 1) e^{\gamma s} A_{n} + d_{mn}^{*} e^{-\gamma s} B_{n} \right)$$

$$\bar{\alpha}_{n}(s) \simeq \sqrt{|m|\pi} \sum_{n=-\infty}^{\infty} e^{-i \frac{\pi m x}{A}|s|} \left( -d_{mn}(\coth \frac{\beta |n| \pi}{2\Lambda} + 1) e^{\gamma s} A_{n} + d_{mn}^{*} e^{-\gamma s} B_{n} \right)$$

$$\bar{\alpha}_{n}^{*}(s) \simeq \sqrt{|m|\pi} \sum_{n=-\infty}^{\infty} e^{i \frac{\pi m x}{A}|s|} \left( d_{mn}(\coth \frac{\beta |n| \pi}{2\Lambda} - 1) e^{\gamma s} A_{n} + d_{mn}^{*} e^{-\gamma s} B_{n} \right)$$

$$\alpha_{n}^{*}(s) \simeq \sqrt{|m|\pi} \sum_{n=-\infty}^{\infty} e^{i \frac{\pi m x}{A}|s|} \left( d_{mn}(\coth \frac{\beta |n| \pi}{2\Lambda} + 1) e^{\gamma s} A_{n} + d_{mn}^{*} e^{-\gamma s} B_{n} \right).$$

Once again, the dotted equality sign indicates that these equations are valid up to corrections of order $\frac{\gamma^{2}}{\Omega}$.

C. Boundary conditions

We must now constrain $A_{n}$, $\bar{A}_{n}$, $B_{n}$, $\bar{B}_{n}$ to meet the boundary conditions of
where $G_m(x, t) \equiv e^{\gamma t} f_m(x) f_m(-x + t \, \text{sgn}(m))$ (continuing $f$ periodically beyond $[-\Lambda, \Lambda]$). These last two lines are obtained via another quasi-diagonal approximation: because the dominant contributions to (75) come from the terms where $g$ can vary significantly over its otherwise negligible width. For sufficiently small $G$, the quasi-diagonal approximation is accurate and natural. At the quantum level, the quasi-diagonal approximation is the ideal black body approximation.

Applying this approximation again allows us to invert (76), obtaining

$$
C_k \simeq \sum_m \frac{\alpha_m^f}{\sqrt{|m|\pi}} \int_{-\Lambda}^{\Lambda} dx \, \frac{e^{i(k-m)\frac{x}{\Lambda}} e^{-i|k|\frac{x}{\Lambda}}}{\sqrt{|m|\pi} G_m(x, t) - \frac{\gamma t}{\alpha_m^f(x, t)}}
$$

$$
D_k \simeq \sum_m \frac{\alpha_m^e}{\sqrt{|m|\pi}} \int_{-\Lambda}^{\Lambda} dx \, \frac{e^{-i(k-m)\frac{x}{\Lambda}} e^{i|k|\frac{x}{\Lambda}}}{\sqrt{|m|\pi} G_m(x, t) - \frac{\gamma t}{\alpha_m^e(x, t)}}
$$

(77)

Note that the function $G_m(x, t) = G(x \, \text{sgn}(m), t)$, where for $t > 2L$

$$
G(x, t) = f(x)f(-x + t)e^{\gamma t} = \begin{cases} 
1, & x \in [-\Lambda, -L] \\
\frac{1}{\sqrt{2\pi}} \gamma^2 (L + x), & x \in [-L, L] \\
\frac{1}{\sqrt{2\pi}} \gamma^2 L, & x \in [L, t-L] \\
\frac{1}{\sqrt{2\pi}} \gamma^2 (L-t+x), & x \in [t-L, t+L] \\
1, & x \in [t+L, \Lambda] 
\end{cases}
$$

(78)

For $t < 2L$, we have instead

$$
G(x, t) = f(x)f(-x + t)e^{\gamma t} = \begin{cases} 
1, & x \in [-\Lambda, -L] \\
\frac{1}{\sqrt{2\pi}} \gamma^2 (L + x), & x \in [-L, t-L] \\
\frac{1}{\sqrt{2\pi}} \gamma^2 t, & x \in [t-L, L] \\
\frac{1}{\sqrt{2\pi}} \gamma^2 (L-t+x), & x \in [L, t+L] \\
1, & x \in [t+L, \Lambda] 
\end{cases}
$$

(79)

As long as $L$ and $t$ are both sufficiently large, the differences between the two cases (78) and (79) turn out to be insignificant.

Applying another quasi-diagonal approximation in (77), we obtain the boundary values

$$
\alpha_m(t) \simeq e^{-i|\lambda| t} \sqrt{|m|\pi} \sum_k \frac{\alpha_m^f}{\sqrt{|k|\pi}} \int_{-\Lambda}^{\Lambda} dx \, e^{i(k-m)\frac{x}{\Lambda}}
$$

$$
+ \sqrt{|m|\pi} \sum_k \frac{\alpha_m^f}{\sqrt{|k|\pi}} \int_{-\Lambda}^{\Lambda} dx \, e^{i(k-m)\frac{x}{\Lambda}}
$$

$$
\alpha_m'(0) \simeq e^{-i|\lambda| t} \sqrt{|m|\pi} \sum_k \frac{\alpha_m^f}{\sqrt{|k|\pi}} \int_{-\Lambda}^{\Lambda} dx \, e^{i(k-m)\frac{x}{\Lambda}}
$$

In a similar manner, we also obtain

$$
\bar{\alpha}_m(t) \simeq e^{-i|\lambda| t} \sqrt{|m|\pi} \sum_k \frac{\bar{\alpha}_m^f}{\sqrt{|k|\pi}} \int_{-\Lambda}^{\Lambda} dx \, e^{i(k-m)\frac{x}{\Lambda}}
$$

$$
+ \sqrt{|m|\pi} \sum_k \frac{\bar{\alpha}_m^f}{\sqrt{|k|\pi}} \int_{-\Lambda}^{\Lambda} dx \, e^{i(k-m)\frac{x}{\Lambda}}
$$

$$
\bar{\alpha}_m'(0) \simeq e^{-i|\lambda| t} \sqrt{|m|\pi} \sum_k \frac{\bar{\alpha}_m^f}{\sqrt{|k|\pi}} \int_{-\Lambda}^{\Lambda} dx \, e^{i(k-m)\frac{x}{\Lambda}}
$$

Using (80) and (81) in (55) and (58), we find $J$ to be given by

$$
J[\bar{\alpha}_m', \bar{\alpha}_m, \alpha'_m, \alpha_m'; t] = \frac{1}{Z} \exp \sum_{l, m} (\bar{\alpha}_m' T_{lm} + \alpha_m' T_{lm}')
$$
where

\[
T_{lm} \simeq e^{-\frac{|m| t}{A}} \int_{-A}^{A} dx \ e^{i(x - (m - l) \frac{\pi}{A})} \left( 1 - e^{-\frac{|m| t}{A}} \right) \frac{G_m(x, t) - \frac{1}{G_m(x, t)}}{G_m(x, t) - \frac{e^{-\beta |m| t}}{G_m(x, t)}}
\]

\[
T'_{lm} \simeq e^{-\frac{|m| t}{A}} \int_{-A}^{A} dx \ e^{i(x - (m - l) \frac{\pi}{A})} \left( 1 - e^{-\frac{|m| t}{A}} \right) \frac{G_m(x, t) - \frac{1}{G_m(x, t)}}{G_m(x, t) - \frac{e^{-\beta |m| t}}{G_m(x, t)}}
\]

\[
E_{lm} \simeq \int_{-A}^{A} dx \ e^{i(x - (m - l) \frac{\pi}{A})} \left( 1 - e^{-\beta |m| t} \right) \frac{G_m(x, t) - \frac{1}{G_m(x, t)}}{G_m(x, t) - \frac{e^{-\beta |m| t}}{G_m(x, t)}}
\]

\[
A_{lm} \simeq \int_{-A}^{A} dx \ e^{i(x - (m - l) \frac{\pi}{A})} \left( 1 - e^{-\beta |m| t} \right) \frac{G_m(x, t) - \frac{1}{G_m(x, t)}}{G_m(x, t) - \frac{e^{-\beta |m| t}}{G_m(x, t)}}
\]

These matrices are obtained using one new kind of quasi-diagonal approximation, which may be illustrated by writing

\[
T_{lm} \simeq e^{-\frac{|m| t}{A} \text{sgn}(m)} \int_{-A}^{A} dx \ e^{i(x - (m - l) \frac{\pi}{A})} \left( 1 - e^{-\beta |m| t} \right) \frac{G_m(x, t) - \frac{1}{G_m(x, t)}}{G_m(x, t) - \frac{e^{-\beta |m| t}}{G_m(x, t)}}
\]

\[
E_{lm} \simeq \int_{-A}^{A} dx \ e^{i(x - (m - l) \frac{\pi}{A})} \left( 1 - e^{-\beta |m| t} \right) \frac{G_m(x, t) - \frac{1}{G_m(x, t)}}{G_m(x, t) - \frac{e^{-\beta |m| t}}{G_m(x, t)}}
\]

by substituting \( x \to x - (x - t \text{sgn}(m)) \) and observing that \( G_m(-x + t \text{sgn}(m)) = G_m(x, t) \). (The last step in (84) invokes the kind of quasi-diagonal approximation already introduced in (74).)

Equations (82) and (83) provide us with the propagator for the density matrix of the modes above \( \Omega \) of the massless scalar field, in the presence of a black body at inverse temperature \( \beta \), initially uncorrelated with the field. The integrals in (83) do not seem to have closed form evaluations, but since \( T, T', A, E \) are all matrices formed by adding and multiplying matrices which are quasi-diagonal, they must themselves also be quasi-diagonal. (The inverse of a quasi-diagonal matrix is quasi-diagonal; one way of showing that this statement applies in our case is to consider the massless field as the continuum limit of a lattice, in which case all the matrices become finite dimensional, and the proposition is then elementary. One might be concerned that diagonal matrices could cancel when they are of the matrices in (83) can be computed, and quasi-diagonality of the matrices in (83) with body propagator \( \mathcal{J} \), by calculating some significant quantities to zeroth order in \( g^2 \).

C. Physical Interpretation

For convenience in explaining the significance to some important spatial regions; these regions overlapping intervals \( [-(L + t), L] \) and \( [-(L + t - \lambda), \nu] \) left and right shadows of the black body. We define \( \lambda \) be the smallest length for which \( e^{-\beta \frac{\lambda^2}{A}} \) is not \( [-(L + t), -(L + t - \lambda)] \) and \( [(L + t - \lambda), (L + t)] \) while \( [-(L + t - \lambda), -L] \) and \( [L, (L + t - \lambda)] \) call \( [-(L + t), -(L - \lambda)] \) and \( [(L - \lambda), L] \) are referred to \( [-(L - \lambda), (L - \lambda)] \) forms the core. As long as the differences between the two forms of \( G(x, t) \) no effect on our identification of these regions insignificant in (83).

With these relevant regions in mind, a propagator is immediately suggested by the form \( E_{mk} \) observe that \( G_k(x, t) = \frac{1}{G_k(x, t)} \) vanishes for body; it has support within the left shadow for \( k > 0 \). \( E_{mk} \) therefore seems to from the black body, and \( A_{mk} \) seems related into it. As well, \( 1 - e^{-\beta \frac{|m| t}{A}} \right) \frac{G_m(x, t) - \frac{1}{G_m(x, t)}}{G_m(x, t) - \frac{e^{-\beta |m| t}}{G_m(x, t)}} \) umbral and black body zones, but equals unit are therefore suggestive of ordinary, free field disconnected from the black body.

These impressions can be confirmed by involving \( \mathcal{J} \). Firstly, we can consider the fields evolves under \( \mathcal{J} \), and show that all \( N \)-point
are the same as those for uniform, unidirectional thermal radiation, as long as all the \( N \) points are within the umbra of the same side of the black body’s shadow. Secondly, we can show that the uniform thermal state at the temperature of the black body evolves under \( \mathcal{J} \) into itself. Finally, we can set the temperature of the black body to zero, and obtain the “quantum wall effect” whereby quanta are absorbed by the black body, without reflection. These three special cases suffice to illustrate that (82) is indeed the correct quantum mechanical description of the effect of a black body on a massless field.

The final state evolving from initial vacuum is found simply by setting \( \alpha_n = \bar{\alpha}_n^\dagger = \bar{0} \) in (82):

\[
\rho_0[\bar{\alpha}_m, \alpha_n; t] = \frac{1}{\mathcal{Z}} \exp \sum_{km} \bar{\alpha}_m \alpha_k E_{nk} \tag{85}
\]

The correlation functions for products of operators \( \hat{\phi}(x_1) \ldots \hat{\phi}(x_N) \hat{\Pi}(y_1) \ldots \hat{\Pi}(y_M) \) may be formed from the expectation values

\[
\left\langle \prod_{n=1}^N \left( \sum_m e^{-i \alpha_m x_n \bar{\alpha}_n^\dagger} \frac{1}{2\Lambda} \right) \prod_{m=1}^M \left( \sum_n e^{i \alpha_n y_n \bar{\alpha}_n^\dagger} \frac{1}{2\Lambda} \right) \right\rangle = \langle \hat{\mathcal{O}} \rangle \tag{86}
\]

where \( u_m, v_m \) are the factors \( \left( \frac{|m|}{\Lambda} \right)^{\frac{1}{2}} \) applicable to either \( \hat{\phi} \) or \( \hat{\Pi} \). This expectation value may be computed by twice completing “squares”\(^\dagger\) in a Gaussian integral:

\[
\langle \hat{\mathcal{O}} \rangle = \int \prod_j d\alpha_j d\bar{\alpha}_j d\beta_j d\bar{\beta}_j e^{\sum_j (\bar{\alpha}_j \beta_j + \bar{\beta}_j \alpha_j)} \rho_0[\bar{\alpha}_1, \alpha_n; t] \langle \beta_j | \hat{\mathcal{O}} | \beta_j \rangle
\]

\[
= \left[ \prod_{n=1}^N \frac{d}{dK_n} \right] \left( \prod_{m=1}^M \frac{d}{dK_m} \right) \exp \frac{1}{2\Lambda} \sum_{n=1}^N \sum_{m=1}^M \bar{K}_m U_{mn} K_n \right]_{K_n = \bar{K}_n = 0} \tag{87}
\]

where

\[
U_{mn} = \sum_{kl} e^{-\frac{i}{\Lambda} \bar{\alpha}_l \alpha_k} u_{kn} \left( 1 - E \right)_{kl} \left[ 1 - G_E(x_n, t) \right] \frac{1 - e^{-\beta_k \bar{\alpha}_l}}{1 - e^{-\beta_k \alpha_k}} \tag{88}
\]

\(^\dagger\) Actually, bilinears in Bargmann-Fock variables are not squares, since \( \bar{\alpha} \) and \( \alpha \) are not complex conjugates, but the procedure is exactly analogous to the usual method of manipulating quadratic exponents.

We derive this by using quasi-diagonality to

\[
(1 - E_{kl})^{-1} \approx \int_{-\Lambda}^{\Lambda} \frac{dx}{2\Lambda} e^{\left( (k - l)^2 \right) \frac{x}{\Lambda}}
\]

The expectation value (87) can be computed only if the states of uniform, isotropic thermal radiation travelling in one direction only. The expression to that of (88), which are

\[
U_{mn} = \sum_{k=0}^\infty e^{\frac{i}{\Lambda} \bar{\alpha}_k (y_m - x_n)} \frac{1}{e^M - 1}
\]

\[
U_{mn} = \sum_{k=0}^\infty e^{\frac{i}{\Lambda} \bar{\alpha}_k (y_m - x_n)} \frac{1}{e^M - 1}
\]

By comparing (88) with (90) and referring to (78), we can therefore see that, as far as measurements made outside the shadow of the black body are concerned, the state represented by \( \rho_0[\bar{\alpha}_1, \alpha_n; t] \) is indistinguishable from the state where only left-moving (right-moving) measurements made in the left (right) umbra, it is indistinguishable from the vacuum; with the asymmetry therefore contributes terms of order \( \frac{1}{(x-y)} \). The asymmetry therefore contributes to our ideal black body.

For our second illustration, we compute the thermal state at the black body temperature operator

\[
\rho_{\beta}[\bar{\xi}_m, \xi_n; 0] = \frac{1}{\mathcal{Z}} \exp \sum_{m} \bar{\xi}_m \bar{T} \cdot (\beta T) \tag{51}
\]

and using (51), integrating by completing the square

\[
\rho_{\beta}[\bar{\alpha}_m, \alpha_n; t] = \frac{1}{\mathcal{Z}} \exp \sum_{mn} \bar{\alpha}_m T : (e^\beta T) \tag{89}
\]

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We can approximate
\[
(e^{-\frac{\beta |\pi x|}{\Lambda}} - A)_{lm}^{-1} \simeq \int_{-\Lambda}^{\Lambda} dx \frac{e^{i(l-m)\frac{x}{\Lambda}}}{2\Lambda} \frac{G_m(x,t) - e^{-\frac{\beta |\pi x|}{\Lambda}}}{G_m(x,t)(e^{-\frac{\beta |\pi x|}{\Lambda}} - 1)},
\]
which leads to
\[
[T \cdot (e^{-\frac{\beta |\pi x|}{\Lambda}} - A)^{-1} \cdot T']_{lm} \simeq e^{i(m-t)} \operatorname{sgn}(m) \frac{\Lambda}{\pi} \int_{-\Lambda}^{\Lambda} dx \frac{e^{i(l-m)\frac{x}{\Lambda}} e^{-\frac{\beta |\pi x|}{\Lambda}} (1 - e^{-\frac{\beta |\pi x|}{\Lambda}})}{G_m(x,t) - e^{-\frac{\beta |\pi x|}{\Lambda}}},
\]
We make this last step by again changing variables \(x \rightarrow -(x-t \operatorname{sgn}(m))\) wrapping around periodically in the domain \([-\Lambda, \Lambda]\). It immediately follows that
\[
[T \cdot (e^{-\frac{\beta |\pi x|}{\Lambda}} - A)^{-1} \cdot T' + E]_{lm} \simeq e^{-\frac{\beta |\pi x|}{\Lambda}} \delta_{lm},
\]
which establishes the fact that
\[
\rho_{\beta}[\bar{\alpha}_m, \alpha_n; t] = \rho_{\beta}[\bar{\alpha}_m, \alpha_n; 0] + \mathcal{O} \left( \frac{\beta^2}{\Omega} \right),
\]
as far as modes above the IR cut-off \(\Omega\) are concerned. The canonical ensemble at the temperature of the ideal black body is indeed the equilibrium state.

Finally, we can consider the evolution of an one-particle initial pure state, in the non-emitting case, so that \(E_{lm} \rightarrow 0\). Changing to the \(N\)-particle basis by extracting the co-efficients of \(\bar{\alpha}_m^{\dagger} \alpha_n^{\dagger}\) and \(\bar{\alpha}_m \alpha_n\) in \(\mathcal{J}_{\beta \rightarrow \infty}\), we find that the final state evolving from the initial state
\[
\hat{\rho}_f = \sum_{l,m} \psi_l^{\dagger} \psi_m \left| \frac{m\pi}{\Lambda} \right\rangle \left\langle \frac{m\pi}{\Lambda} \right|,
\]
where \(|\frac{m\pi}{\Lambda}\rangle\) denotes the one particle state with momentum \(\frac{m\pi}{\Lambda}\) has the density operator
\[
\hat{\rho}_f = F |0\rangle \langle 0| + (1 - F) \sum_{l,m} \psi_l^{\dagger} \psi_m \left| \frac{m\pi}{\Lambda} \right\rangle \left\langle \frac{m\pi}{\Lambda} \right|,
\]
where \(|0\rangle\) denotes the vacuum and
\[
F \simeq \frac{1}{\sqrt{1 - F}} \sum_{m} \psi_m^{\dagger} A \psi_m \simeq \frac{1}{\sqrt{1 - F}} \int_{-\Lambda}^{\Lambda} dx e^{-\frac{\beta |\pi x|}{\Lambda}} T_{lm} \left( e^{-\frac{\beta |\pi x|}{\Lambda}} - A \right)_{lm}^{-1} \psi_{lm}^{\dagger} \psi_{lm} \simeq \frac{1}{\sqrt{1 - F}} \int_{-\Lambda}^{\Lambda} dx e^{-\frac{\beta |\pi x|}{\Lambda}} \psi_{lm}^{\dagger} \psi_{lm}.
\]
In the limit \(\beta \rightarrow \infty\), we have
\[
T_{lm} \left| \beta \rightarrow \infty \right. \simeq e^{-\frac{\beta |\pi x|}{\Lambda}} \int_{-\Lambda}^{\Lambda} dx
\]
This ensures that an initial state localized within the black body evolves into a state in which the IR cut-off are negligibly excited. A black body evolves into one in which no modes are excited. This then implies that the probability of the black body is negligible. Note that, when \(\beta \rightarrow \infty\),

This shows that the total probability is conserved, while the particle being present decreases from one.

Our model and the quasi-diagonal approximation thereupon exhibit absorption, thermodynamic equilibrium, and black body radiation. We have presented, in an idealized macroscopic black body with a massless quantum field, a description from quantum mechanical first principles of the interaction of a unobserved environment has not received its

\[
\text{IV. DISCUSSION}
\]

\[
\text{A. Summary}
\]

The use of influence functionals to describe the second order effects of an interaction therefore exhibit absorption, thermodynamic equilibrium, and black body radiation. We have presented, in an idealized macroscopic black body with a massless quantum field, a description from quantum mechanical first principles of the interaction of a unobserved environment has not received its
Among second order effects, there can be some that are enhanced by large parameters that do not appear at lower orders. These are very important, since they are apparently responsible for the significant phenomena of quantum measurement and thermal dissipation. As discussed by Feynman and Vernon in their classic paper introducing influence functionals\cite{9}, independent harmonic oscillators can be used to describe any environment, as long as we are only interested in effects up to second order in the coupling between environment and observed system. To this order, a general quantum mechanical environment having allowed transitions with energy differences $\omega_k$ is equivalent to a set of harmonic oscillators with those natural frequencies, with effective coupling strengths proportional to the matrix elements of the interaction Hamiltonian $h$.

This result is derived in the Appendix. To first order in the coupling, a generic quantum environment may be treated as a classical source; the first improvement beyond this treatment, valid to second order in the coupling, is to model a generic environment as a collection of independent oscillators. Our treatment of matter as a collection of baths of harmonic oscillators is therefore not merely a gross idealization, but is actually accurate up to second order in perturbation theory.

From another viewpoint, though, our model of a free quantum field interacting with first-quantized matter can actually be considered as a way of approaching some non-perturbative physics. The atoms and molecules represented by heat baths are in reality bound states involving the electromagnetic field. Interacting fields can effectively have many more degrees of freedom than free fields, because their Hilbert spaces include arbitrary numbers of bound states. These bound states couple to the unbound modes, and can have significant effects if they are present in macroscopic quantities — as in a black body of macroscopic size.

That these effects can be significant despite weak coupling can be starkly demonstrated by calculating the energy radiated by a black body. From our one dimensional model in Section III, we find (in the limit where the infrared cutoff is removed) that the outgoing radiation carries energy $\mathcal{R}$

$$\mathcal{R} = \frac{\pi^2}{6\lambda^2} = \frac{\pi}{6\hbar^2}$$

making $\hbar$ and the Boltzmann constant $k_B$ explicit; this is the one-dimensional, spin zero Stefan-Boltzmann law,

$$\mathcal{R}_B = \sigma T^4$$

The Stefan-Boltzmann constant $\sigma$ is independent of the electromagnetic coupling; yet it governs electromagnetic radiation generated by matter. As discussed by Feynman and Vernon in their classic paper introducing influence functionals\cite{9}, independent harmonic oscillators can be used to describe any environment, as long as we are only interested in effects up to second order in the coupling between environment and observed system. To this order, a general quantum mechanical environment having allowed transitions with energy differences $\omega_k$ is equivalent to a set of harmonic oscillators with those natural frequencies, with effective coupling strengths proportional to the matrix elements of the interaction Hamiltonian $h$.

The condition that the black body has a vanishing reflectivity of a black body can thus be thought of as a mere effect by macroscopically large parameters such as our electromagnetic coupling. This is the one-dimensional, spin zero Stefan-Boltzmann law,

$$\mathcal{R}_B = \sigma T^4$$

The vanishing reflectivity of a black body can thus be thought of as a mere "invisible" charge-dependent factor of a form like $e^{-g^2/\lambda}$; and the Stefan-Boltzmann constant might therefore be said to contain an "invisible" charge-dependent factor of a form like $e^{-g^2/\lambda}$, which would vanish if the electromagnetic coupling were negligibly different from unity for macroscopic black bodies.

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familiar physical phenomenon. Yet it also permits a powerful type of approximation, elucidating “qualitative” properties which, like black body radiation and quantum measurement, have traditionally been treated by axiom rather than by perturbation. The opacity of a black body can be considered as an indicator that a “large parameter” approximation is appropriate.

With two very different, very drastic approximations applying at once, it is little wonder that a non-reflective, opaque object is such a clean theoretical subject, and why its behaviour is so universal (depending only on temperature). The highly non-trivial implications of total absorptivity, first deduced from thermodynamic axioms, have now been traced to their microscopic origins in quantum theory. The results of our seemingly very idealized model, with uncountably many independent harmonic oscillators coupled to the field at every point, are thus quite universal. Is this model then more realistic than it at first seems?

B. Justifying the model

Apart from the reduction to one dimension, this model is in fact quite realistic. In a disordered collection of many molecules there are very many allowed transition energies, and the continuous spectrum \( I(\omega) \) becomes a reasonable approximation to a discrete spectrum of equivalent oscillators, that will not break down until one considers a time scale on the order of the inverse of the spectral spacing. Since a continuous spectrum of harmonic oscillators has infinite heat capacity, one would expect this time scale to be associated with heating and cooling of the black body. These important phenomena are not described by our model, because of our assumption of a continuous spectrum.

In Section III, however, we assumed that there is an independent heat bath interacting with the field at every point in the black body, so that our model is continuously dense in space, as well as in frequency. This is a valid approximation for a discrete model, as long as we consider wavelengths of light long compared with the lattice spacing. We are therefore certainly only modeling the effect of matter on light with wavelengths above the X-ray range. In fact, we are assuming that each heat bath represents a grain or clump of molecules or atoms that effectively couples to the field at a single point, and so must be of a size less than the wavelengths of interest. For wavelengths as small as the near ultraviolet range, this permits us to have on the order of a thousand atoms per heat bath. This number of degrees of freedom must still yield a bath cooling time scale much longer than our relaxation time scale much longer than our relaxation time scale, because of our assumption of continuity in space and frequency?

We can estimate the cooling time scale for its classical thermal energy \( \frac{2}{3} \sigma k_B T \) by the radiative black body of radius two to three nanometers. Using the Stefan-Boltzmann law (100), we find the cooling time to be

\[
\tau_c \sim \frac{1500k_B T}{4\pi^2 \sigma T^4} \sim 10^4 \text{ seconds.}
\]

We can bound \( g^2 \) phenomenologically by assuming infrared, at around \( 10^2 \text{ Hz} \), and by knowing that \( g \) is a thickness of around \( 10^{-4} \) metres, giving a range of seconds. For temperatures ranging into the hundreds of Kelvin, therefore, we have \( \tau_c >> g^2 \). The continuous spectral density may not break down for our local heat baths until the processes described in Section III are well established. At this point issues arise, concerning the medium, that are beyond the scope of this paper and relate to a further limitation on our model: independent of all the others, we are explicitly excluding any effects involving longer range correlations in real, weak dielectrics, on the other hand, should be adequately represented by our model.

The implicit infrared and ultraviolet cut-off in conflict with the ohmic spectral density assumption of the spectral density is not essential for the results of that Section, since the slowly varying \( I(\omega) \) leads to similar results with a frequency dependent decay constant \( \Gamma(\omega) \). As long as \( e^{-2\pi\Gamma(\omega)} \) is negligible for the black absorptive.

As far as infrared to near ultraviolet light is concerned, this paper is actually a fairly accurate representation of a medium composed of closely packed grains, whose size is on the order of a few nanometers, each of which consists of fine-grained deposits of carbon atoms or soot. This turns out to be about soot.
C. Issues raised by the model

The problem we have studied presents a new face of the basic quantum dilemma regarding localization in position and momentum. One may often think of a black body as thermally exciting outgoing field modes, and absorbing incident modes, but this is misleading. The positional and directional localization involved in identifying outgoing and incoming radiation is more subtle than the one-to-one mapping by which we associate Fourier modes with particle momenta.

This subtlety can be illustrated by sketching the extension of our work to a black body which is a solid sphere in three dimensions. Choosing spherical polar co-ordinates, the radial eigenfunctions of the free field Hamiltonian are spherical Bessel and Neumann functions. While these can be combined into spherical Hankel functions, describing incoming and outgoing waves, the Neumann functions diverge at the origin, and the only physical modes are pure Bessels, representing standing waves. There are not enough field degrees of freedom, then, to be decomposed into orthogonal outgoing and incoming modes.

The black body environment surrounding the origin changes the path integral saddlepoint, and modifies the Euler-Lagrange equations. Taking the zero angular momentum mode as an example, we find \( \hat{f}_0(r) = \frac{e^{ikr + \Gamma r}}{\sqrt{r}} - \frac{e^{-ikr - \Gamma r}}{\sqrt{r}} \) appearing in solutions, instead of \( \frac{\sin kr}{r} \). Radial functions that are predominantly incoming or outgoing (outside the black body) will therefore appear in the three-dimensional analogue of (83). But the number of orthogonal field modes is not suddenly doubled, and each of the incoming and outgoing sets of eigenfunctions could in fact be expanded in terms of the other. There is only a half-line \( k \in [0, \infty] \) worth of radial degrees of freedom to be described.

This point is one more illustration among many that the wave-particle duality of quantum fields is not completely explained by referring to the expansion of the field operator in plane waves. While it is convenient for scattering processes in negligible gravity, this notion of particles fails to be physical in curved spaces or accelerating frames; it also requires elaboration (via our quasi-diagonal approximation) to describe emission and absorption from macroscopic bodies at rest in flat space. A really satisfactory formulation of particle states would be of great value, and it has yet to be found.

A second issue is raised by our zero-temperature, purely absorptive limit. The “quantum wall effect” blocks field excitations from passing through the black body, but it does not affect the ground state. But there is no Casimir effect, at zeroth order in a dielectric[13]. The case of a black body should also admit a similar kind of point heat baths replaced by a gas of free electrons would be interesting to study such a model without simply imposing boundary conditions. A really satisfactory formulation of particle states is common to both black hole and thermal radiation.

Finally, we could relax the assumption of an implied infinite heat capacity and discarded radiation. As the black body cooled to low temperatures, we would expect that information from the field would be re-emitted, as it approaches the ground state. A really satisfactory formulation of particle states would be of great value, and it has yet to be found.

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APPENDIX: INFLUENCE FUNCTIONAL FOR A WEAKLY COUPLED ENVIRONMENT

Consider an environment represented by the single degree of freedom $q$, with conjugate momentum $p$. Let this environment be coupled to an observed system, such that the total Hamiltonian operator is

$$\hat{H} = \hat{H}_S + \hat{H}_E + \epsilon \hat{A}(p, q) \hat{Q}. \quad (102)$$

The coupling constant $\epsilon$ is assumed to be small and dimensionless. $\hat{H}_S$ and $\hat{Q}$ are operators on the observed system, while $\hat{H}_E$ and $\hat{Q}$ are operators in the unobserved sector.

Let $|K\rangle$ denote eigenstates of the environmental Hamiltonian, such that $\hat{H}_E|K\rangle = E_K|K\rangle$, assuming no special form for $\hat{H}_E(p, q)$. We can then in principle define the $q$ and $p$ representation wave functions $\psi_K(q) = \langle q|K\rangle$ and $\Psi_K(p) = \langle p|K\rangle$. Let the mixed initial state of the environment at time $s = 0$ be described by the density operator $\hat{R} = \mathcal{R}(\hat{\rho}, \hat{q})$, where the colons imply normal ordering by placing all $\hat{p}$'s to the left of all $\hat{q}$'s, so that $R(p, q)$ is the Wigner function $\langle p|\mathcal{R}|q\rangle$. $\hat{R}$ has the matrix elements

$$\langle K|\hat{R}|L\rangle \equiv R_{KL}$$

in the basis of energy eigenstates. The Wigner function may be expressed in terms of these matrix elements using wave functions:

$$R(p, q) = \sum_{K, L} \Psi_K(p) R_{KL}(0) \psi_L^*(q). \quad (103)$$

For the environment with this Hamiltonian and initial state, the influence phase $V[Q, Q']$ is given by the path integral

$$e^{iV[Q, Q']} = \int dp dq dq' dp' d\dot{q} d\dot{p} R(q, p, q', p') e^{-\hat{S}_P + i \Delta}$$

$$\times \int_{p, q, 0}^{p', q'} Dp Dq \int_{p', q'}^{q, p} Dp' Dq' e^{\frac{i}{\hbar} \int_{0}^{t} ds \left((p-q') - (p'-q) - (H_E - H'_E) - \epsilon (A - A') \right)}.$$  

Here $A'$ and $H'_E$ are short for $A(p', q')$ and $H_E(p', q')$.  

This path integral may be computed explicitly for $H_E(p, q) = \frac{i}{2}(p^2 + \omega^2 q^2)$ and $A(p, q) = q$. We find

$$V[Q, Q'] \equiv \frac{i e^2}{2\hbar \omega} \int_{0}^{t} ds \int_{0}^{s} ds' (Q - Q') \times \left((Q - Q'), \coth \frac{\beta \omega}{2} \cos \omega(s - s') \right)$$

where $s$ and $s'$ are time variables, $(Q - Q')$, time.

The influence phases for any number of uncoupled and initiall uncorrelated environmental degrees of freedom simply add, and if a number of uncoupled and initiall uncorrelated systems are in a thermal state, each will be separated from the other systems. Consider, therefore, an environment consisting of a number $N$ of independent harmonic oscillators, each with natural frequencies $\omega_k$ and coupling strengths $\epsilon_k$. Assume the environment is in a thermal state and uncorrelated with the observed system. The influence functional of the form

$$V[Q, Q'] = \frac{i}{\hbar} \int_{0}^{t} ds \int_{0}^{s} ds' (Q - Q') \times \left((Q - Q'), \coth \frac{\beta \omega}{2} \cos \omega(s - s') \right)$$

where

$$I(\omega) = \frac{\hbar}{2} \sum_{k=1}^{N} \delta(\omega - \omega_k)$$

As we have seen in Section II, above, the influence phase of the same form as (106).

Returning to the environment with only one environment degree of freedom, we consider the case of general $H_E(p, q)$ and $A(p, q)$. If $\epsilon_k$ is small enough, equation (104) may be approximated.
in $\epsilon$, we have

$$e^{i V[Q,Q']} = \int dp_1 dq_1 dp'_1 dq'_1 R(q_1,p_1') e^{-iA_{1}q'_1} \times \int_{q_1,0}^{p_1,0} DpDq \int_{q'_1,0}^{p'_1,0} Dp'Dq' e^{i \frac{1}{\hbar} \int_{0}^{t} ds \left( H_{F} - H_{F}^{\prime} \right)} \times \left( 1 - i \frac{\epsilon}{\hbar} \int_{0}^{t} ds \left( AQ - A'Q' \right) + O(\epsilon^2) \right)$$

$$= 1 - \frac{\epsilon}{\hbar} \int_{0}^{t} ds \sum_{L,M} R_{LM} A_{LM} e^{-i\omega_{LM}s} (Q - Q') + O(\epsilon^2)$$

$$= \exp \left( -\frac{\epsilon}{\hbar} \int_{0}^{t} ds \left( A(s) \right)(Q - Q') \right) + O(\epsilon^2) .$$

In (108) we use the following notation:

$$A_{LM} = \langle L | \hat{A} | M \rangle$$

$$\omega_{LM} = \frac{E_{L} - E_{M}}{\hbar}$$

$$\langle A(s) \rangle = \sum_{L,M} R_{LM} A_{LM} e^{-i\omega_{LM}s} .$$

When we recall that the influence phase appears in any path integral over observed degrees of freedom as an additional term in the exponent,

$$e^{i \frac{V}{\hbar} (S[Q] - S[Q'])} \rightarrow e^{i \frac{V}{\hbar} (S[Q] - S[Q'] + V[Q,Q'])} ,$$

we can see that the first order term in (108) clearly constitutes a perturbation to the observed sector action $S[Q]$, with the expectation value $\langle A(s) \rangle$ acting as a classical external source.

Since the first order term leads to effects that are already well studied in the regime of closed system quantum mechanics, we generally wish to concentrate on the specifically open quantum behaviour produced by the higher order terms. If we restrict our attention to environments that are approximately in equilibrium, so that $R_{MN} = \frac{1}{2} e^{-\beta E_{M}} \delta_{MN}$, and assume that the expectation value of $A$ vanishes\(^{\dagger}\), then the first non-trivial term in (109) is

$$e^{i V[Q,Q']} = \int dp_1 dq_1 dp'_1 dq'_1 R(q_1,p_1') e^{-iA_{1}q'_1} \times \int_{q_1,0}^{p_1,0} DpDq \int_{q'_1,0}^{p'_1,0} Dp'Dq' e^{i \frac{1}{\hbar} \int_{0}^{t} ds \left( H_{F} - H_{F}^{\prime} \right)} \times \left( 1 - \frac{\epsilon^2}{2\hbar^2} \int_{0}^{t} ds \left( AQ - A'Q' \right) \right)$$

$$= 1 - \frac{\epsilon^2}{2\hbar^2} \int_{0}^{t} ds \int_{0}^{s} ds' \sum_{L,M} e^{-i\beta E_{M}} e^{i\omega_{LM}(s-s')} (Q - Q')$$

$$\times \left( (Q - Q')_{s'} \coth \frac{\beta \hbar \omega_{LM}}{2} - i(Q + Q')_{s'} \right) ,$$

where the effective spectral density is

$$G(\omega; \beta) = \frac{1}{2} \sum_{L > M} \delta(\omega - \omega_{LM}) A_{LM} .$$

The restriction of the environment to physically irrelevant to the above derivation, and may be considered a mere notation.

\(^{\dagger}\) By considering a slightly generalized thermal state, we can in fact include first order external sources together with the second order term.
convenience. Comparing (110) and (106), therefore, we see that any weakly coupled environment that is initially uncorrelated with the system and in a thermal state is equivalent, at a fixed temperature, to a bath composed of independent harmonic oscillators. It is worth emphasizing that the effective spectral density $G(\omega; \beta)$ can be temperature dependent, as this fact seems to have escaped comment heretofore.

In fact, (110) can also apply in a situation where the weak coupling approximation is not as straightforward, namely the case of a large number $\varepsilon$ where all of the degrees of freedom, each of which is weakly coupled to the observed system, but not so weakly that the total effect of the environment has to be small. In this case the Hamiltonian may be written as

$$\hat{H} = \hat{H}_S + \sum_{k=1}^{N} \left( \hat{H}_k + \epsilon_k \hat{A}_k (p_k, q_k) \hat{Q} \right),$$

where all of the $\epsilon_k$ are much less than unity.

Assuming again the initial thermal state, with vanishing expectation values $\langle A_k(s) \rangle$, we find

$$e^{iV[Q,Q'] = \int dp dq dq' dp' \left[ e^{i \bar{p} q - i \bar{q}' p'} \right] \times \prod_{j=1}^{p_{j,t}} Dp Dq \int_{q_{j,0}}^{q_{j,t}} Dp Dq' \times N \left( 1 - \frac{\varepsilon_j^2}{h} \right) \left[ \int_0^t ds \left( A_k Q - A_{k} Q' \right) \right] + O(\varepsilon_j^3)$$

$$= \prod_{k=1}^{N} \left( 1 - \frac{\varepsilon_k^2}{h} \right) \int_0^t ds \int_0^{s'} ds' \int_0^\infty d\omega G_k(\omega; \beta) \times \left[ \langle (Q - Q')_s \coth \frac{\beta h \omega}{2} \cos \omega(s - s') \right. \left. - i(Q + Q')_s \sin \omega(s - s') \rangle + O(\varepsilon_k^3) \right].$$

Here $G_k(\omega; \beta)$ is the effective spectral density for the $k$th environmental degree of freedom, defined similarly to $G(\omega; \beta)$ in (111).

In the case of large $N$, we can now set $1/N$. The leading order term may be found by taking the limit $N \to \infty$. In this limit, the perspective

$$\lim_{N \to \infty} \prod_{k=1}^{N} \left( 1 - \frac{\varepsilon_k}{N} \right)$$

and so we see that the influence phase for the system of weakly coupled degrees of freedom is given by

$$V[Q, Q'] = \left( 1 + O(\varepsilon_k) + O(\frac{1}{N}) \right) \times \int \left( (Q - Q')_s \coth \frac{\beta h \omega}{2} \right.$$

$$\left. - i(Q + Q')_s \right),$$

where the effective spectral density is

$$G(\omega; \beta) = \sum_{k=1}^{N} \epsilon_k^2 G_k(\omega; \beta).$$

Equation (114) is, to leading order in $\epsilon_k$ and $1/N$, a generalization of (109) and (106).

For large enough $N$, quantities such as the derivations of wave function collapse[6]—most of the $\epsilon_k$ are all small. It is important to note that the coupling of the observed system to each environmental degree of freedom may be weak, there may be so many such degrees that the total effect may be large. Equation (114) shows that a generic environment by a bath of independent oscillators provides an adequate model for a large environment, capable of inducing drastic effects like quantum measurement.
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Figure 1: Significant regions in space
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