HALL POLYNOMIALS FOR REPRESENTATION-FINITE
CLUSTER-TILTED ALGEBRAS

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ABSTRACT. We show the existence of Hall polynomials for representation-finite cluster-
tilted algebras.

1. INTRODUCTION

1.1. Let $k$ be a finite field and $\Lambda$ a locally bounded $k$-algebra, that is, $\Lambda$ is an associative algebra and $\Lambda$ has a set of primitive orthogonal idempotents $\{e_i\}_{I}$ such that $\Lambda = \bigoplus_{i,j \in I} e_i \Lambda e_j$, and both $\dim_k e_i \Lambda$ and $\dim_k e_i$ are finite for all $i \in I$. Let $\text{mod} \Lambda$ be the category of right $\Lambda$-modules with finite length. For $L, M, N \in \text{mod} \Lambda$, we denote by $F^{M}_{N,L}$ the number of submodules $U$ of $M$ such that $U \cong L$ and $M/U \cong N$.

Let $E$ be a field extension of $k$. For any $k$-space $V$, we denote by $V^E$ the $E$-space $V \otimes_k E$. Clearly, $\Lambda^E$ naturally becomes an $E$-algebra. The field $E$ is called conservative [12] for $\Lambda$ if for any indecomposable $M \in \text{mod} \Lambda$, $(\text{End}_E M/\text{rad End}_E M)^E$ is a field. Set

$$\Omega = \{E|E \text{ is a finite field extension of } k \text{ which is conservative for } \Lambda\}.$$ 

For a given $\Lambda$ with $\Omega$ infinite, the algebra $\Lambda$ has Hall polynomials provided that for any $L, M, N \in \text{mod} \Lambda$, there exists a polynomial $\phi^{M}_{N,L} \in \mathbb{Z}[T]$ such that for any conservative finite field extension $E$ of $k$ for $\Lambda$,

$$\phi^{M}_{N,L}(|E|) = F^{M}_{N,E,L}.$$ 

We call $\phi^{M}_{N,L}$ the Hall polynomial associated to $L, M, N \in \text{mod} \Lambda$. Note that if $\Lambda$ is representation-finite, then $\Omega$ is an infinite set.

It has been conjectured by Ringel [12] that any representation-finite algebra has Hall polynomials. This conjecture has been verified for representation-directed algebras by Ringel [12], cyclic serial algebras by Guo [6] and Ringel [13] and some other classes of algebras (cf. eg. [7]).

1.2. Let $A$ be a finite-dimensional hereditary algebra over a field $k$. Let $\text{mod} A$ be the finitely generated right $A$-modules and $D^{b}(\text{mod} A)$ the bounded derived category with suspension functor $\Sigma$. The cluster category $\mathcal{C}(A)$ associated with $A$ was introduced in [4] (independently in [5] for $A_n$ case) as the orbit category $D^{b}(\text{mod} A)/\tau^{-1} \circ \Sigma$, where $\tau$ is the Auslander-Reiten translation of $D^{b}(\text{mod} A)$. A cluster-tilting object $T$ in $\mathcal{C}(A)$ is an object such that $\text{Ext}^{1}_{\mathcal{C}(A)}(T, T) = 0$ and it is maximal with this property. The endomorphism algebra $\text{End}_{\mathcal{C}(A)}(T)$ of a cluster-tilting object $T$ is called the cluster-tilted algebra of $T$, which were first introduced by Buan, Marsh and Reiten in [3]. Among others, they showed that cluster-tilted algebras are Gorenstein of dimension 1. This has been further generalized to a more general setting by Keller-Reiten [9] and König-Zhu [10]. Moreover, Keller and
Reiten [9] proved that the stable Cohen-Macaulay category of a given cluster-tilted algebra (more generally, 2-Calabi-Yau tilted algebra) is 3-Calabi-Yau. A direct consequence of the stably Calabi-Yau property is that the Auslander-Reiten conjecture holds true for cluster-tilted algebras. Namely, let $B$ be a cluster-tilted algebra over a field $k$, if $M$ is a finitely generated right $B$-module such that $\text{Ext}^i_{\text{mod} \ B}(M, M \oplus B) = 0$ for all $i \geq 1$, then $M$ is a projective $B$-module.

In [14], Zhu has introduced certain Galois coverings for cluster categories and cluster-tilted algebras (cf. also [1]), called repetitive cluster categories and repetitive cluster-tilted algebras respectively (for the precisely definition, cf. Section 2). We refer to [2] for the notions of (Galois) covering functors. The aim of this note is to show that Ringel’s conjecture holds true for representation-finite repetitive cluster-tilted algebras. In particular, representation-finite cluster-tilted algebras have Hall polynomials.

**Theorem 1.1.** Let $\Lambda$ be a representation-finite repetitive cluster-tilted algebra over a finite field $k$. Then $\Lambda$ has Hall polynomials.

Let us mention here that a variant proof of this theorem may be applied to generalized cluster-tilted algebras of higher cluster categories of type ADE. In order to make this note concise, we restrict ourselves to the case of cluster-tilted algebras. After recall some basic definitions and properties of repetitive cluster-tilted algebras in Section 2, we will give the proof of Theorem 1.1 in Section 3.

2. **Repetitive cluster categories and repetitive cluster-tilted algebras**

2.1. Let $D$ be a $k$-linear triangulated category with suspension functor $\Sigma$ and $\mathcal{T}$ a functorially finite subcategory of $D$. The subcategory $\mathcal{T}$ is called a **cluster-tilting subcategory**, if the followings are equivalent:

- $X \in \mathcal{T}$;
- $\text{Ext}^1_D(X, \mathcal{T}) = 0$;
- $\text{Ext}^1_D(\mathcal{T}, X) = 0$.

An object $T \in D$ is a **cluster-tilting object** if and only if $\text{add} T$ is a cluster-tilting subcategory. A cluster-tilting object $T \in D$ is called basic provided that $T = \bigoplus_{i=1}^n T_i$, where $T_i, i = 1, \ldots, n$ are indecomposable and $T_i \not\cong T_j$ whenever $i \neq j$.

Let $\mathcal{T}$ be a cluster-tilting subcategory of $D$ and $\text{mod} \mathcal{T}$ the category of finitely presented right $\mathcal{T}$-modules. It has been proved by König and Zhu in [10] (cf. also [9]) that the functor $\text{Hom}_D(\mathcal{T}, -) : D \to \text{mod} \mathcal{T}$ induces an equivalence $D/\text{add} \Sigma \mathcal{T} \cong \text{mod} \mathcal{T}$. Moreover, if $D$ has Auslander-Reiten triangles, then the Auslander-Reiten sequences of $\text{mod} \mathcal{T}$ is induced from the Auslander-Reiten triangles of $D$. In this case, we have $\tau^{-1} \Sigma \mathcal{T} = \mathcal{T}$, where $\tau$ is the Auslander-Reiten translation of $D$.

2.2. Let $A$ be a finite-dimensional hereditary algebra over a field $k$. Let $\text{mod} A$ be the category of finitely generated right $A$-modules and $D^b(\text{mod} A)$ the bounded derived category with suspension functor $\Sigma$. Let $\tau$ be the Auslander-Reiten translation of $D^b(\text{mod} A)$. In the following, we fix a positive integer $m$ and set $F := \tau^{-1} \circ \Sigma$. The **repetitive cluster category** $C_{F^m}(A)$ introduced by Zhu [14] is the orbit category of $D^b(\text{mod} A)/ < F^m >$, which is by definition a $k$-linear category whose objects are the same as $D^b(\text{mod} A)$, and whose morphisms are given by

$$C_{F^m}(A)(X, Y) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{D^b(\text{mod} A)}(X, F^m Y),$$

where $X, Y \in D^b(\text{mod} A)$. 

When \( m = 1 \), we get the cluster category \( \mathcal{C}(A) \). By the main theorem of Keller [8], we know that \( \mathcal{C}_{\text{FM}}(A) \) admits a canonical triangle structure such that the canonical projection functor \( \pi_m : \mathcal{D}^b(\text{mod} \ A) \to \mathcal{C}_{\text{FM}}(A) \) is a triangle functor. Moreover, by the universal property of the orbit category \( \mathcal{C}_{\text{FM}}(A) \), we have a triangle functor \( \rho_m : \mathcal{C}_{\text{FM}}(A) \to \mathcal{C}(A) \) such that \( \pi_A = \rho_m \circ \pi_m \), where \( \pi_A \) is the canonical projection functor.

It has been shown in [14] that there exists bijections between the following three sets: the set of cluster-tilting subcategory of \( \mathcal{D}^b(\text{mod} \ A) \), the set of cluster-tilting subcategories of \( \mathcal{C}_{\text{FM}}(A) \) and the set of cluster-tilting subcategories of \( \mathcal{C}(A) \), via the triangle functors: \( \pi_m : \mathcal{D}^b(\text{mod} \ A) \to \mathcal{C}_{\text{FM}}(A) \) and \( \rho_m : \mathcal{C}_{\text{FM}}(A) \to \mathcal{C}(A) \). In particular, the repetitive cluster categories have cluster-tilting objects.

Let \( \tilde{T} \) be a basic cluster-tilting object in the repetitive cluster category \( \mathcal{C}_{\text{FM}}(A) \), the endomorphism algebra \( \text{End}_{\mathcal{C}_{\text{FM}}(A)}(\tilde{T}) \) is called the repetitive cluster-tilted algebra of \( T \). We have the following main results of [14] (cf. Theorem 3.7 and Theorem 3.8 in [14]).

**Theorem 2.1.** Let \( T \) be a basic cluster-tilting object in \( \mathcal{C}(A) \) and \( A = \text{End}_{\mathcal{C}(A)}(\tilde{T}) \) the cluster-tilted algebra of \( T \). Let \( \tilde{T} \) be the corresponding basic cluster-tilting object in \( \mathcal{C}_{\text{FM}}(A) \) of \( T \) via the triangle functor \( \rho_m \) and \( \tilde{A} = \text{End}_{\mathcal{C}_{\text{FM}}(A)}(\tilde{T}) \) the associated repetitive cluster-tilted algebra. Then we have the followings:

1. the restriction of \( \pi_m : \mathcal{D}^b(\text{mod} \ A) \to \mathcal{C}_{\text{FM}}(A) \) induces a Galois covering \( \pi_m : \pi_A^{-1}(\text{add} \ T) \to \text{add} \tilde{T} \) of \( \tilde{A} \). Moreover, the projection functor \( \pi_m : \mathcal{D}^b(\text{mod} \ A) \to \mathcal{C}_{\text{FM}}(A) \) induces a push-down functor \( \tilde{\pi}_m : \mathcal{D}^b(\text{mod} \ A)/\pi_A^{-1}(\text{add} \Sigma T) \to \text{mod} \tilde{A} \);
2. the functor \( \rho_m : \mathcal{C}_{\text{FM}}(A) \to \mathcal{C}(A) \) restricted to the cluster-tilting subcategory \( \text{add} \tilde{\pi}_m \) is a Galois covering of \( A \). Moreover, the functor \( \rho_m \) also induces a push-down functor \( \tilde{\rho}_m : \text{mod} \tilde{A} \to \text{mod} A \).

**Remark 2.2.** Set \( \mathcal{T} = \pi_A^{-1}(\text{add} \ T) \) and let \( \text{ind} \mathcal{T} \) be a set of representatives of the isoclasses of all indecomposable objects in \( \mathcal{T} \). Set

\[
\text{End}(\mathcal{T}) := \bigoplus_{T_i, T_j \in \text{ind} \mathcal{T}} \text{Hom}_{\mathcal{D}^b(\text{mod} \ A)}(T_i, T_j),
\]

which is an associative algebra without units. It is not hard to see that \( \text{End}(\mathcal{T}) \) is locally bounded. On the other hand, the finitely presented \( \mathcal{T} \)-modules coincides with \( \text{End}(\mathcal{T}) \)-modules of finite length. Hence, we have an equivalence of categories

\[
\text{mod} \text{End}(\mathcal{T}) \cong \mathcal{D}^b(\text{mod} \ A)/\Sigma \mathcal{T}
\]

by [10], which implies that \( \text{End}(\mathcal{T}) \) is directed.

**Remark 2.3.** One can verify that the functor \( \tilde{\pi}_m \) coincides with the restriction of the push-down functor induced by the Galois covering \( \pi_m : \pi_A^{-1}(\text{add} \ T) \to \text{add} \tilde{T} \). Hence, \( \tilde{\pi}_m \) is an exact functor. On the other hand, any exact sequence of \( \mathcal{D}^b(\text{mod} \ A)/\Sigma \mathcal{T} \) can be lifted to a triangle in \( \mathcal{D}^b(\text{mod} \ A) \) (cf. Lemma 8 of [11]). Then one can also prove the exactness directly in this setting.

Note that a Galois covering functor will not induce a Galois covering for the corresponding categories of modules in general. However, for the Galois coverings in the above theorem, we have the following observation.

**Proposition 2.4.** Keep the notations in Theorem 2.1, we have the followings:

1. the push-down functor \( \tilde{\pi}_m : \mathcal{D}^b(\text{mod} \ A)/\pi_A^{-1}(\text{add} \Sigma T) \to \text{mod} \tilde{A} \) is a Galois covering of \( \text{mod} \tilde{A} \);
2. the push-down functor \( \tilde{\rho}_m : \text{mod} \tilde{A} \to \text{mod} A \) is a Galois covering of \( \text{mod} A \).
Proof. We will only prove the first statement and the second one follows similarly.

Let $\mathcal{T} = \pi_{A}^{-1}(\text{add } T)$. Since $\Sigma \mathcal{T}$ is also a cluster-tilting subcategory of $\mathcal{D}^b(\text{mod } A)$, we have $F \Sigma \mathcal{T} = \Sigma \mathcal{T}$. One shows that $F^m : \mathcal{D}^b(\text{mod } A) \to \mathcal{D}^b(\text{mod } A)$ induces a $k$-linear equivalence $F^m : \mathcal{D}^b(\text{mod } A) / \Sigma \mathcal{T} \to \mathcal{D}^b(\text{mod } A) / \Sigma \mathcal{T}$. Let $G$ be the infinite cyclic group generated by $F^m$ which is acting freely on the objects of $\mathcal{D}^b(\text{mod } A) / \Sigma \mathcal{T}$. Note that by Proposition 2.4, we have the following commutative diagram

$$\mathcal{D}^b(\text{mod } A) \xrightarrow{\pi_m} \mathcal{C}_{F^m}(A)$$

$$\xrightarrow{Q_1} \mathcal{D}^b(\text{mod } A) / \Sigma \mathcal{T} \xrightarrow{\pi_m} \mathcal{C}_{F^m}(A) / \Sigma \mathcal{T} = \text{mod } \tilde{A}$$

where $Q_1$ and $Q_2$ are natural quotient functors. Then for any indecomposable objects $X, Y$ in $\mathcal{D}^b(\text{mod } A) / \Sigma \mathcal{T}$, one can show that $\pi_m(X) \cong \pi_m(Y)$ if and only if there exists $i \in \mathbb{Z}$ such that $Y \cong F^m X$ in $\mathcal{D}^b(\text{mod } A) / \Sigma \mathcal{T}$. On the other hand, for any objects $M, N$ in $\text{mod } \tilde{A}$ with preimages $\tilde{M}, \tilde{N}$ in $\mathcal{D}^b(\text{mod } A) / \Sigma \mathcal{T}$ respectively. We clearly have

$$\text{Hom}_{\text{mod } \tilde{A}}(M, N) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(\text{mod } A) / \Sigma \mathcal{T}}(\tilde{M}, (F^m)^i \tilde{N}) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(\text{mod } A) / \Sigma \mathcal{T}}((F^m)^i \tilde{M}, \tilde{N}).$$

This particular implies that $\text{mod } \tilde{A}$ identifies the orbit category of $\mathcal{D}^b(\text{mod } A) / \Sigma \mathcal{T}$ by $G$. It is easy to see that the functor $\pi_m$ coincides with the quotient functor. □

3. PROOF OF THE MAIN THEOREM

To prove our main result, we need the following result of Guo and Peng [7], which gives a sufficient condition for the existence of Hall polynomials.

**Lemma 3.1.** Let $k$ be a finite field. Let $\Lambda$ be a finite-dimensional $k$-algebra of representation-finite type and there is a locally bounded $k$-algebra $R$ which is directed, such that there exists a covering functor $F : \text{mod } R \to \text{mod } \Lambda$, and for any $M, N \in \text{mod } \Lambda$, there exist $X, Y \in \text{mod } R$ with $F X = M$ and $F Y = N$ such that $F$ induces the $k$-isomorphism

$$\text{Ext}^1_R(X, Y) \cong \text{Ext}^1_\Lambda(M, N).$$

Then $\Lambda$ has Hall polynomials.

**Remark 3.2.** According to Theorem 5.1 of [7], the last condition in Lemma 3.1 can be weakened to the following: for any $M, N \in \text{mod } \Lambda$ with $N$ indecomposable, there exist $X_i, Y_i \in \text{mod } R, i = 1, 2$ with $F X_i \cong M$ and $F Y_i \cong N$ such that $F$ induces the $k$-isomorphisms

$$\text{Ext}_R(X_1, Y_1) \cong \text{Ext}^1_\Lambda(M, N) \text{ and } \text{Ext}_R(X_2, Y_2) \cong \text{Ext}^1_\Lambda(N, M).$$

Now we are in a position to prove the main theorem of this note.

**Proof of Theorem 3.1.** By the definition of repetitive cluster-tilted algebras, there is a finite-dimensional hereditary algebra $A$ such that $\Lambda$ is the endomorphism algebra of a basic cluster-tilting object $\tilde{T}$ in a repetitive cluster category $\mathcal{C}_{F^m}(A)$ of $A$. Note that we have an equivalence of categories $\text{Hom}_{\mathcal{C}_{F^m}(A)}(\tilde{T}, -) : \mathcal{C}_{F^m}(A) / \Sigma \tilde{T} \to \text{mod } \Lambda$. Hence, $\Lambda$ is representation-finite implies that $A$ is representation-finite. Let $\mathcal{T} = \pi_{m}^{-1}(\text{add } \tilde{T})$, by Remark 2.2, we know that $\text{End}(\mathcal{T})$ is locally bounded and directed. By part (1) of Proposition 2.4, we deduce that

$$\tilde{\pi}_m : \text{mod } \text{End}(\mathcal{T}) \to \text{mod } \Lambda$$

is a directed Galois covering of $\text{mod } \Lambda$. 

According to Lemma \[3.1\] and Remark \[3.2\] it suffices to show that for any $M, N \in \text{mod } \Lambda$ with $N$ indecomposable, there exist $X_i, Y_i \in \text{mod } \text{End}(T)$, $i = 1, 2$ with $FX_i \cong M$ and $FY_i \cong N$ such that $F$ induces the $k$-isomorphisms

$$\text{Ext}^1_{\text{End}(T)}(X_1, Y_1) \cong \text{Ext}^1_A(M, N) \quad \text{and} \quad \text{Ext}^1_{\text{End}(T)}(Y_2, X_2) \cong \text{Ext}^1_A(N, M).$$

We will only prove the existence for the first isomorphism, where the second one follows similarly. We may and we will assume that $M$ is also indecomposable and $\text{Ext}^1_A(M, N) \neq 0$.

Recall that we have the following commutative diagram

$$
\begin{array}{ccc}
D^b(\text{mod } A) & \xrightarrow{\pi_m} & C_{Fm}(A) \\
\text{mod } \text{End}(T) & \cong & \text{mod } D^b(\text{mod } A)/\Sigma T \\
\downarrow Q_1 & & \downarrow Q_2 \\
\text{mod } \text{End}(T) & \cong & C_{Fm}(A)/\Sigma\hat{T} \cong \text{mod } \Lambda
\end{array}
$$

where $Q_1$ and $Q_2$ are natural quotient functors. Since $\pi_m$ is a Galois covering, there exists $\hat{M}, \hat{N} \in \text{mod } \text{End}(T)$ such that $\pi_m(\hat{M}) = M$ and $\pi_m(\hat{N}) = N$. Moreover, $\{F^i m \hat{M} | i \in \mathbb{Z}\}$ forms a complete set of preimages of $M$ in $\text{mod } \text{End}(T)$. For simplicity, we set $\hat{M}_i = F^i m \hat{M}$ in the following. By abuse of notations, we still denote by $\hat{M}_i, \hat{N} \in D^b(\text{mod } A)$ the unique indecomposable preimages of $\hat{M}_i$ and $\hat{N}$ respectively. Without loss generality, we may assume that $\hat{N} \in \text{mod } A$. By the definition of covering functor and the fact that $\pi_m$ is an exact functor preserving projectivity, we have the following $k$-isomorphism

$$\bigoplus_{i \in \mathbb{Z}} \text{Ext}^1_{\text{End}(T)}(\hat{M}_i, \hat{N}) \cong \text{Ext}^1_A(M, N).$$

Since $0 \neq \dim_k \text{Ext}^1_A(M, N) < \infty$, the vector space $\text{Ext}^1_{\text{End}(T)}(\hat{M}_i, \hat{N})$ vanishes for all but finitely many $i$. Let $t \in \mathbb{Z}$ such that $\text{Ext}^1_{\text{End}(T)}(\hat{M}_i, \hat{N}) \neq 0$ and $\text{Ext}^1_{\text{End}(T)}(\hat{M}_j, \hat{N}) = 0$ for $j > t$.

We claim that $\text{Ext}^1_{\text{End}(T)}(\hat{M}_i, \hat{N}) \cong \text{Ext}^1_A(M, N)$. It suffices to show that $\text{Ext}^1_{\text{End}(T)}(\hat{M}_i, \hat{N}) = 0$ for $i < t$. By the Auslander-Reiten translation formula, we have

$$\text{Ext}^1_{\text{End}(T)}(\hat{M}_i, \hat{N}) \cong D\text{Hom}^1_{\text{End}(T)}(\hat{N}, \tau \hat{M}_i),$$

where $\tau$ is the Auslander-Reiten translation of $\text{mod } \text{End}(T)$ which is induced by the Auslander-Reiten translation of $D^b(\text{mod } A)$. On the other hand, we have

$$D\text{Hom}^1_{\text{End}(T)}(\hat{N}, \tau \hat{M}_i) = \frac{\text{Hom}^{D^b(\text{mod } A)}(\hat{N}, \tau \hat{M}_i)}{\{f : \hat{N} \to \tau \hat{M}_i \text{ factoring through add } \Sigma T \text{ or add } \Sigma^2 T\}}.$$

Note that $\text{Ext}^1_{\text{End}(T)}(\hat{M}_i, \hat{N}) \neq 0$ implies that $\text{Hom}^{D^b(\text{mod } A)}(\hat{N}, \tau \hat{M}_i) \neq 0$. Since $\hat{N} \in \text{mod } A$, we deduce that $\tau \hat{M}_i \in \text{mod } A$ or $\tau \hat{M}_i \in \Sigma \text{mod } A$. Recall that $\hat{M}_i = F^i m \hat{M}$ if $\tau \hat{M}_i \in \text{mod } A$, then we have

$$\text{Hom}^{D^b(\text{mod } A)}(\hat{N}, \tau \hat{M}_i) = \text{Hom}^{D^b(\text{mod } A)}(\hat{N}, F^{(i-t)m} \tau \hat{M}_i) = 0 \text{ for } i < t.$$

Now assume that $\tau \hat{M}_i \in \Sigma \text{mod } A$ and let $\tau \hat{M}_i = \Sigma L$, where $L \in \text{mod } A$. From

$$0 \neq \text{Hom}^{D^b(\text{mod } A)}(\hat{N}, \tau \hat{M}_i) = \text{Hom}^{D^b(\text{mod } A)}(\hat{N}, \Sigma L),$$
we deduce that \( L \) is a predecessor of \( \widehat{N} \) in \( D^b(\text{mod } A) \). In this case, we have

\[
\text{Hom}_{D^b(\text{mod } A)}(\widehat{N}, \tau \widehat{M}_t) = \begin{cases} 
\text{Hom}_{D^b(\text{mod } A)}(\widehat{N}, F^{(i-t)m}\tau \widehat{M}_t) 
& \text{if } i < t - 1; \\
\text{Hom}_{D^b(\text{mod } A)}(\widehat{N}, \tau^{m\Sigma^{1-m}L}) & \text{if } i = t - 1.
\end{cases}
\]

It is clear that \( \text{Hom}_{D^b(\text{mod } A)}(\widehat{N}, \tau^{m\Sigma^{1-m}L}) = 0 \) if \( m \geq 2 \). Now suppose that \( m = 1 \) and \( \text{Hom}_{D^b(\text{mod } A)}(\widehat{N}, \tau L) \neq 0 \), then \( \widehat{N} \) is a predecessor of \( \tau L \) and hence a predecessor of \( L \), which contradicts to the fact that \( D^b(\text{mod } A) \) is directed. We have proved that \( \text{Hom}_{D^b(\text{mod } A)}(\widehat{N}, \tau \widehat{M}_t) = 0 \) for \( i \neq t \) and hence \( \text{Ext}^1_{\text{End}(\mathcal{T})}(\widehat{M}_t, \widehat{N}) = 0 \) for \( i \neq t \), which completes the proof.

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