Overlap for Majorana–Weyl fermions.

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The power of the overlap formalism is illustrated by regularizing theories based on Majorana-Weyl fermions. The results reported in this contribution were obtained in collaboration with Patrick Huet \cite{1}.

More than half of the talk will be independent of the overlap formalism. To present our ideas in the simplest context possible we start directly from an example. Consider a $U(1)$ chiral model, defined on a two dimensional torus:

$$\mathcal{L} = \sum_{\alpha} \int \bar{\psi}_{L}^{\alpha} (\sigma \cdot (\partial + iq_{L}^{\alpha} A)) \psi_{L}^{\alpha} +$$

$$\sum_{\beta} \int \bar{\psi}_{R}^{\beta} (\sigma^{\ast} \cdot (\partial + iq_{R}^{\beta} A)) \psi_{R}^{\beta} + \frac{1}{4g^{2}} \int F_{\mu\nu} F^{\mu\nu}.$$ 

The $q_{R,L}^{\alpha,\beta}$ are integers subjected to the anomaly cancelation condition

$$\sum_{\alpha} (q_{L}^{\alpha})^{2} = \sum_{\beta} (q_{R}^{\beta})^{2}.$$ 

Since we are on a torus,

$$\frac{1}{2\pi} \int F_{12} = 0, \pm 1, \pm 2, \ldots$$

All integers can appear, but no other values are allowed. This implies that the scale of the gauge field is fixed and consequently the magnitudes of the charges have absolute meaning. Focus on a single right-handed Weyl fermion of unit charge. The Dirac-Weyl differential operator is:

$$W_{R} \equiv \sigma^{\mu}_{\rho} (\partial_{\rho} + iA_{\mu}).$$

Suppose the gauge background carries some topological charge,

$$\frac{1}{2\pi} \int F_{12} \neq 0.$$ 

This information alone implies

$$\dim[Ker(W^{\dagger}_{R})] \neq \dim[Ker(W_{R})].$$

If one thinks in matrix terms the differential operator $W_{R}$ must be viewed as a rectangular (as opposed to square) matrix. While the matrix is infinite the distinction still makes sense since it refers to the difference between the number of rows and columns. When $W_{R}$ is replaced by $W^{\dagger}_{R}$ the rows and columns are interchanged. Suppose we tried to force, by picking a lattice regularization (or any other method of truncating the space $W_{R}$ acts on) a square shape on $W_{R}$. The most likely outcome of this brute force approximation of $W_{R}$ would be that the square shape would be used to accommodate representatives of not one continuum $W_{R}$ but two continuum differential operators, one of type $W_{R}$ and the other of type $W^{\dagger}_{R}$. The two rectangular shapes would combine to fit snuggly into the allotted square.

In the action $W_{R}$ appears sandwiched between a $\bar{\psi}$ and a $\psi$. A square shape for a regularized $W_{R}$ implies simply an equal number of $\bar{\psi}$’s and $\psi$’s. The previous paragraph indicates that in a topologically nontrivial background one will most likely find two sets of Weyl fields (or a larger even number) making up together a Dirac fermion (or several). So in nontrivial topology we are led to expect (in)famous doubling.

On the other hand, on a lattice the link variables can be smoothly deformed to connect “zero topology” to “non–zero topology” gauge configurations. Thus doubling is there, no matter what the background is.

Of course, doubling is an old story; what we wanted to show is that as long as one keeps an equal number of $\bar{\psi}$ and $\psi$ Grassmann integration

\textsuperscript{*}Research supported in part by the DOE under grants # DE-FG06-91ER40614 and # DE-FG06-90ER40561.

\textsuperscript{†}Research supported in part by the DOE under grant #DE-FG05-90ER40559.
variables on the lattice the number of physical fermions is likely to be doubled. This cannot be fixed by simply including one \( \bar{\psi} \) more than the \( \psi \)'s, say; the number of \( \psi \) excess we need is background dependent. It looks like we are in trouble because we cannot write down a functional integral as long as we don’t know how many integration variables we have to use. Clearly this story hinges on us being able to tell apart a \( \bar{\psi} \) from a \( \psi \). This is related to being able to distinguish a particle from its anti-particle. But, we could use Majorana-Weyl fermions instead, and in this basis the differences between particles and anti-particles are not notationally apparent.

Let us change variables from one Weyl fermion, \( \psi_L \) (left-handed this time) to two Majorana-Weyl fermions:

\[
\bar{\psi}_L = \frac{\xi + i\eta}{\sqrt{2}} \psi_L = \frac{\xi - i\eta}{\sqrt{2}}
\]

\[
\int \bar{\psi}_L \sigma \cdot (\partial + iA) \psi_L = \int \bar{\psi}_L \sigma \cdot \left[ \partial + \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \right] \psi_L.
\]

We get an \( SO(2) \) doublet interacting with an \( SO(2) \) gauge field. However, now, \( W_{mL}^\dagger = -W_{mL}^* \) and \( \dim[Ker(W_{mL})] = \dim[Ker(W_{mL}^\dagger)] \).

It would seem that \( W_{mL} \) can be represented by a square and our problem went away by changing notation, but this cannot be true. The fermion number conservation in the classical Lagrangian has just been obscured by the new notation.

The above exercise indicates that the argument we used to convince ourselves that an indefinite number of \( \bar{\psi}, \psi \) integration variables is needed might not be that general. In particular, what would happen if we had some other irreducible Majorana-Weyl multiplet in a (real) representation of some gauge group? We shall see that even then there exists an argument forcing us to look for a representation of the functional integral containing an indefinite number of fermion fields.

The basic point is that the parity of the number of zero modes of \( W_{mL} \) is a topological invariant. Somewhere in the vast index literature this mod(2) index should be mentioned, but we have been unable to locate a specific source. We can therefore refer only to our own paper [3].

\[
Ind \equiv (-1)^{\dim[Ker(W_{mL})]} \text{ is a topological invariant essentially because } W_{mL} \text{ is skew symmetric. If we again visualize the differential operator } W_{mL} \text{ as a matrix the latter would be antisymmetric. The mod(2) index simply tells us whether the matrix has an odd or even dimension. This makes sense even for an infinite matrix because the rank of an antisymmetric matrix is always even, so the parity of its dimension is the parity of the dimension of the kernel of the operator and the kernel is finite dimensional. The index monitors whether the number of Grassmann integration variables is even or odd. This parity depends on the gauge field. So we again face a difficulty because we can’t decide a priori how many Grassmann variables there are.}

In the continuum, for a multiplet of Weyl fermions carrying a real representation, the fermion integral factorizes for any background into two isomorphic Majorana multiplets: The Majorana-Weyl determinant becomes the fourth power of the corresponding Dirac determinant.

\[
\int \bar{\psi}_L \sigma \cdot (\partial + iA) \psi_L = \frac{1}{2} \int (\xi, \eta) \left[ \sigma \cdot (\partial + iA) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \psi_L.
\]

The above works because \( \Delta = -\Delta^\dagger \).

The overlap exactly preserves this factorization at the regularized level. This is not entirely trivial since the overlap uses operators of Dirac type to represent Weyl fermions. The factorization in the overlap is proven by carrying out a canonical Bogolyubov transformation.

We first review the Weyl overlap [2]: The exponent of the effective action is given by

\[
\gamma_w[U] = U < L - |L+ > U,
\]

where the two states are ground states of two Hamiltonians, each describing a system of non-interacting fermions:

\[
\mathcal{H}^\pm = \sum_{x,\alpha, y, \beta} a_{x,\alpha}^* H^\pm(x, y; U) a_{y, \beta}.
\]
The single particle Hamiltonians are parametrically dependent on the gauge fields:

\[ H^\pm = \begin{pmatrix} B^\pm & C \\ C^\dagger & -B^\pm \end{pmatrix}, \]

\[ C(x, y) = \frac{1}{2} \sum_{\mu} \sigma_{\mu} (\delta_{y, x+\mu} U_{\mu}(x) - \delta_{x, y+\mu} U^\dagger_{\mu}(y)), \]

\[ B^\pm(x, y) = \pm m \delta_{x, y} + \frac{1}{2} \sum_{\mu} (2 \delta_{x, y} - \delta_{y, x+\mu} U_{\mu}(x) - \delta_{x, y+\mu} U^\dagger_{\mu}(y)). \]

To go to the Majorana representation we define the transformation:

\[ a_1 = \frac{\alpha_1 - i \beta_1}{\sqrt{2}}, \quad a_2 = \frac{\alpha_2 - i \beta_2}{\sqrt{2}} \]

\[ \alpha_1 = \alpha_2^\dagger, \quad \beta_1 = \beta_2^\dagger. \]

In terms of the new creation/annihilation operators we obtain:

\[ H^\pm = \frac{1}{2} a_1^\dagger H^+ a + \frac{1}{2} b_1^\dagger H^\pm b. \]

Both Hamiltonians split into two terms which commute with each other and act in separate subspaces. The split into subspaces is kinematical. Therefore the overlap factorizes and so would any fermionic expectation values. Hence, we can restrict our attention to only one of the two terms. This reduction leads us to the overlap formulation of a system containing a single Majorana-Weyl multiplet. The overlap will be between the ground states of the Hamiltonians

\[ \mathcal{H}^\pm_{mw} = \frac{1}{2} \Gamma H^\pm_{mw} \Gamma, \]

where the \( \Gamma \) operators are hermitian versions of fermionic creation/annihilation operators:

\[ \Gamma_1 = \frac{\alpha_1 + \alpha_1^\dagger}{\sqrt{2}}, \quad \Gamma_2 = \frac{\alpha_2 - \alpha_2^\dagger}{i \sqrt{2}}. \]

The mixing of creation with annihilation operators reflects the indistinguishableness of Majorana particles from their anti-particles. The single particle Hamiltonians are real.

Analogously to the Weyl case, the overlap preserves the topological properties of the differential operator it represents. The regularized version of the mod(2) index is obtained as follows: The Hamiltonians are operators in a Clifford Algebra on a space \( \mathbf{V} \). In this Algebra, \( \Gamma_\infty \equiv \prod \Gamma \), \( \Gamma_{\infty}^2 \propto 1 \), \( |\Gamma_\infty, H| = 0 \). One can split the space into two subspaces \( \mathbf{V} = \mathbf{V}_+ \oplus \mathbf{V}_- \) that are invariant under the “chirality” operator: \( \Gamma_\infty (\mathbf{V}_\pm) = \mathbf{V}_\pm \). All the eigenstates of the Hamiltonians we are dealing with can be chosen to be entirely contained in either subspace.

Now the source of the index \( \text{Ind} \) becomes evident: \( \Gamma \propto L > U = \zeta|L > U \Rightarrow \text{Ind} = 1 \), \( \Gamma \propto L > U = \pm \zeta|L > U \Rightarrow \text{Ind} = -1 \). \( \zeta \) is some fixed constant of unit modulus. It is also clear that when the index is \( -1 \) the chiral Majorana determinant is exactly zero. This exact “zero mode” can be “soaked up” by the insertion of a string containing an odd number of \( \Gamma \)’s.

More explicitly, the index is given by the determinant of an orthogonal matrix \( O \). This matrix is such that \( OH^+ O^T \) can be smoothly deformed to \( H^- \) by varying the mass parameter \( m \). Only an even number of generic eigenvalue-crossings should occur throughout the deformation. One way to find an orthogonal matrix with the above property is to rotate each \( H^\pm \) to a common canonical form with the help of two corresponding orthogonal matrices \( O^\pm \). Then, define \( O \) by \( O = O^+ O^- \).

REFERENCES

1. P. Huet, R. Narayanan, H. Neuberger, Phys. Lett. B380 (1996) 291.
2. R. Narayanan, H. Neuberger, Nucl. Phys. B 443 (1995) 305.