MODULAR INVARIANT OF QUANTUM TORI II: THE GOLDEN MEAN

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ABSTRACT. In [2] a modular invariant $j^{qt}(\theta)$ of the quantum torus $T(\theta)$ was defined. In this paper, we consider the case of $\theta = \phi =$ the golden mean. We show that $j^{qt}(\phi) = 9538.249655644$ by producing an explicit formula for it involving weighted versions of the Rogers-Ramanujan functions.

INTRODUCTION

In [2], using nonstandard methods, the following definition of modular invariant $j^{qt}(\theta)$ of the quantum torus $T(\theta) = \mathbb{R}/(1, \theta)$ was presented. Let $\| \cdot \|$ = the distance-to-the-nearest-integer function and for each $\varepsilon > 0$ let

$$B_{\varepsilon}(\theta) = \{ n \in \mathbb{N} | \| n \theta \| < \varepsilon \}.$$

Define

$$j_{\varepsilon}(\theta) := \frac{12^3}{1 - J_{\varepsilon}(\theta)}; \quad J_{\varepsilon}(\theta) := \frac{49}{40} \left( \frac{\sum_{n \in B_{\varepsilon}(\theta)} n^{-6}}{\sum_{n \in B_{\varepsilon}(\theta)} n^{-4}} \right)^2$$

and

$$j^{qt}(\theta) := \lim_{\varepsilon \to 0} j_{\varepsilon}(\theta)$$

provided the limit exists; if not, we define $j^{qt}(\theta) = \infty$. In [2] it was shown that $j^{qt}(\theta) = \infty$ for all $\theta \in \mathbb{Q}$.

In this paper we study the case of $\theta = \phi =$ the golden mean. We will show that $j^{qt}(\phi) \approx 9538.249655644 = 12^3 \times 5.5198204025717$ by providing an explicit formula for $j^{qt}(\phi)$, in which

$$J^{qt}(\phi) = \frac{49}{40} \left( \frac{G_6(\phi) + H_6(\phi)}{G_4(\phi) + H_4(\phi)} \right)^3$$

and where $G_M(x), H_M(x)$ are weighted variants of the Rogers-Ramanujan functions.

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For example, the function $G_M(x)$ is defined as follows: for each partition $I = i_1 \leq i_2 \leq \cdots \leq i_k$ of $n$, consider the weighting polynomial $f_I(x) = x^{i_1} + \cdots + x^{i_k}$. Let $P(n)$ be the set of partitions of $n$ whose parts are distinct and whose differences are at least 2 and write $C_{x,M}(n) := x^{Mn} \sum_{I \in P(n)} f_I(x)^{-M}$. Then

$$G_M(x) = \sum_{C_{x,M}(n)} x^n.$$ 

If one replaces $f_I(x)$ by the equiweight $x^n$ for all $I \in P(n)$, one recovers the variable part of the function appearing on the left hand side of the first Rogers-Ramanujan identity.

The classical Rogers-Ramanujan functions appear in Baxter's solution \cite{Baxter} to the hard hexagon model of statistical mechanics; in view of the quantum statistical mechanical treatment of Complex Multiplication produced in \cite{complex_multiplication_1}, \cite{complex_multiplication_2}, it would seem not unreasonable to ask that the weighted Rogers-Ramanujan functions or $j^{qt}(\varphi)$ appear as partition function or internal energy of some quantum statistical mechanical system.

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1. **The Golden Mean**

Let

$$\varphi := \frac{1 + \sqrt{5}}{2}$$

be the golden mean. In this section we recall some facts about $\varphi$ and its diophantine approximations, see for example \cite{Landau}, \cite{Mahler}.

The minimal polynomial of $\varphi$ is $X^2 - X - 1$ and $\varphi$ is a unit in $\mathbb{Q}(\sqrt{5})$, whose inverse is $-1$ times its conjugate:

$$\varphi^{-1} = -\varphi' = \frac{\sqrt{5} - 1}{2}.$$ 

The discriminant of $\varphi$ is $\sqrt{5}$, and the class number of $\mathbb{Q}(\sqrt{5})$ is one. The pseudo lattice $\langle 1, \varphi \rangle$ has endomorphism ring equal to $O_K$, hence has conductor $f = 1$.

If we denote by $[a_0, a_1, \ldots]$ the sequence of partial quotients of a real number $\theta$ then for $\vartheta = \varphi$, $a_i = 1$ for all $i$. It follows that the sequence of best approximations $(p_m, q_m)$ of $\varphi$ is given by $(F_{m+1}, F_m)$, where $(F_m) = (1, 1, 2, 3, 5, 8, \ldots)$, $m \geq 1$, denotes the Fibonacci sequence:

$$F_{m+1} = F_m + F_{m-1}, \quad m \geq 1.$$ 

See for example \cite{Mahler}. This means that as $m \to \infty$,

$$\varepsilon_m := F_m \varphi - F_{m+1} \to 0$$

and that for all $0 < n < F_m$,

$$\|n \varphi\| > \|F_m \varphi\| = |\varepsilon_m|,$$

where as before $\|x\|$ is the distance of $x$ to the nearest integer.

We recall Binet’s formula \cite{Binet}:

$$F_m = \frac{\varphi^m - (\varphi')^m}{\sqrt{5}} = \frac{\varphi^m - (-1)^m \varphi^{-m}}{\sqrt{5}} = \begin{cases} \frac{\varphi^m}{\sqrt{5}} & \text{if } m \text{ is even} \\ \frac{\varphi^m}{\sqrt{5}} & \text{if } m \text{ is odd} \end{cases}$$
Using Binet’s formula, we may obtain the following explicit expression for $\varepsilon_m$ of (2):

$$\varepsilon_m = (-1)^{m+1} \varphi^{-m}.$$  

Indeed, for each integer $m$ we have

$$F_m \varphi - F_{m+1} = \left(\frac{\varphi^m + (-1)^{m+1} \varphi^{-m}}{\sqrt{5}}\right) \varphi - \left(\frac{\varphi^{m+1} + (-1)^m \varphi^{-m-1}}{\sqrt{5}}\right)$$

$$= \frac{1}{\sqrt{5}} (\varphi^{m+1} + (-1)^m \varphi^{m-1} - \varphi^m + (-1)^{m+1} \varphi^{-m-1})$$

$$= \frac{1}{\sqrt{5}} (\varphi + \varphi^{-1})(-1)^m \varphi^{-m} = (-1)^m \varphi^{-m}.$$

Notice then that for $m \geq 2$, we have

$$\|F_m \varphi\| = |\varepsilon_m|.$$

For $m$ large, $\sqrt{5} F_m \approx \varphi^m$, with an error term $= \pm \varphi^{-m}$ that decays exponentially as $m \to \infty$.

Finally, we recall Zeckendorf’s representation (which is actually a special case of a more general result of Ostrowski [10]):

**Theorem 1** (Zeckendorf, [14]). Every natural number $n \in \mathbb{N}$ may be written uniquely as a sum of non-consecutive Fibonacci numbers:

$$n = F_I := F_{i_1} + \cdots + F_{i_k}, \quad 2 \leq i_1, i_1 + 2 \leq i_2, \ldots, i_k - 1 + 2 \leq i_k, \quad 1 \leq k.$$

**Note 1.** The condition that $i_1 \geq 2$ is to ensure uniqueness in the decomposition, otherwise the value 1 could occur in two different ways, as $F_1$ or $F_2$.

**2. AN EXPLICIT FORMULA**

In this section we will produce, assuming that $j^q(\varphi)$ converges, an explicit formula for $\varphi$ obtained by evaluating at $\varphi$ a certain rational expression involving weighted variants of the Rogers-Ramanujan functions. The convergence of $j^q(\varphi)$ will be then proved in §3.

Recall the standard formula (1). Write $\varepsilon = |\varepsilon_m|$ and $B = B_m(\varphi) = \{n \in \mathbb{N} | \|n \varphi\| < \varepsilon\}$ so that

$$j^q_\varepsilon(\varphi) = \frac{49}{40} \left(\frac{\sum_{n \in B} n^{-6}}{\sum_{n \in B} n^{-4}}\right)^2.$$

The first step is to determine the elements of $B$ in terms of their Zeckendorf representations. In what follows, for a multi-index $I = (i_1, \ldots, i_k)$, define $|I| = k$.

**Lemma 1.** Let $n = F_I = F_{i_1} + \cdots + F_{i_k}$ written in its unique Zeckendorf form. Then $n \in B$ if and only

I. $|I| \geq 1, \ i_1 \geq m + 1$ or

II. $|I| \geq 2, \ i_1 = m$ and $i_2 - m$ is odd.

**Note 2.** Since the Zeckendorf form consists of sums of nonconsecutive Fibonacci numbers, we must have that $i_2 - m \geq 3$ in II.

**Proof.** First note that we have trivially by (3) that $F_{m+1} \in B$ for $i \geq 1$. Suppose that $n = F_I$ is a sum of more than one non-consecutive Fibonacci numbers and $i_1 \geq m + 1$. Then we have

$$\|n \varphi\| < \varphi^{-(m+1)} + \varphi^{-(m+3)} + \cdots = \varphi^{-(m+1)} (1 - \varphi^{-2})^{-1}.$$
Since \( \varphi = \varphi^2 - 1 \) it follows that \((1 - \varphi^{-2})^{-1} = \varphi\). Then \( \|n\varphi\| < \varphi^{-m} \) which implies that \( n \in B \). Thus every element of the type described in I. belongs to \( B \). On the other hand, if \( i_1 \leq m - 1 \), then we claim that
\[
\varphi^{-m} = \varepsilon < \|n\varphi\| < 1 - \varphi^{-1} = \varphi^{-2}.
\]
Indeed, if \( n = F_I \), the associated error term sum
\[
\varepsilon_I := \pm \varepsilon_{i_1} \pm \cdots \pm \varepsilon_{i_k}
\]
is minimized in absolute value by taking \( i_1 = m - 1 \) and assuming that the remaining indices \( i_2, \ldots \) are such that the signs of the associated error terms \( \varepsilon_{i_2}, \ldots \) are different from the sign of the error term \( \varepsilon_{m-1} \). More precisely,
\[
|\varepsilon_I| > \varphi^{-(m-1)} - (\varphi^{-(m+2)} + \varphi^{-(m+4)} + \cdots) = \varphi^{-m} (\varphi - \varphi^{-2} (1 - \varphi^{-2})^{-1}).
\]
Since \( \varphi^{-2}(1 - \varphi^{-2})^{-1} = \varphi^{-1} \) and \( \varphi - \varphi^{-1} = 1 \), it follows that \( |\varepsilon_I| > \varphi^{-m} = \varepsilon \). In addition \( |\varepsilon_I| \) is maximized by taking \( i_1 = 2, i_2 = 4, \ldots \), so that
\[
|\varepsilon_I| < \varphi^{-2} + \varphi^{-4} + \cdots = \frac{1}{\varphi^2 - 1} = \varphi^{-1}.
\]
Note that the distance of the latter bound \( \varphi^{-1} \) to the nearest integer is \( 1 - \varphi^{-1} = \varphi^{-2} \). It follows then from the definition of \( \| \cdot \| \) and the fact that we are assuming that \( m > 2 \) that \( \|n\varphi\| > \varphi^{-m} = \varepsilon \) and \( n \notin B \). Now if \( i_1 = m \) and \( i_2 - m \) is even, then the error terms \( \varepsilon_m \) and \( \varepsilon_{i_2} \) share the same sign, and we have
\[
\|n\varphi\| > \varphi^{-m} + \varphi^{-i_2} - \left( \varphi^{-(i_2+3)} + \varphi^{-(i_2+5)} + \cdots \right) = \varepsilon + (\varphi^{-i_2} - \varphi^{-(i_2+3)}(1 - \varphi^{-2})^{-1}) > \varepsilon
\]
Indeed, the last inequality follows since
\[
\varphi^{-i_2} - \varphi^{-(i_2+3)}(1 - \varphi^{-2})^{-1} = \varphi^{-i_2}(1 - \varphi^{-3}(1 - \varphi^{-2})^{-1}) = \varphi^{-i_2}(1 - \varphi^{-2}) > 0.
\]
On the other hand, if \( i_1 = m \) and \( i_2 = m+k, k \) odd, then the sign of the corresponding error terms differ, and we have
\[
\|n\varphi\| < \varphi^{-m} - \varphi^{-m-k} + \varphi^{m-k-3} + \varphi^{m-k-5} + \cdots = \varphi^{-m} - \varphi^{-m-k} (1 - (\varphi^{-3} + \varphi^{-5} + \cdots)) \]
\[
= \varphi^{-m} - \varphi^{-m-k} (1 - \varphi^{-3}(1 - \varphi^{-2})^{-1}) \]
\[
= \varphi^{-m} - \varphi^{-m-k} (1 - \varphi^{-2}) < \varepsilon
\]
so that \( n \in B \). \hfill \square

Let \( \mathcal{N} \) be the set of increasing, finite tuples \( I = (i_1, \ldots, i_l) \) of natural numbers with \( |I| = l \geq 2 \) and which are not consecutive i.e. \( i_1 + 2 \leq i_2, \ldots, i_{l-1} + 2 \leq i_l \). Denote by
\[
(4) \quad \mathcal{N}(m) = \{ I = (i_1, \ldots, i_l) \in \mathcal{N} | i_1 \geq m \}.
\]
Also denote by
\[
(5) \quad \mathfrak{M}(m) = \{ I \in \mathcal{N}(m) | i_1 = m \text{ and } i_2 = m+k \text{ for } k \text{ odd} \}.
\]
Consider $B_m$ for $m > 2$. Then by the Lemma we have

$$J_{B_m}^q (\varphi) := \frac{49}{40} \left( \frac{\sum_{n \in B_m} n^{-6}}{\sum_{n \in B_m} n^{-4}} \right)^2 = \frac{49}{40} \left( \frac{\sum_{i=1}^{\infty} F_{m+i}^{-6} + \sum_{i \in \mathbb{N}(m+1)} F_{m+i}^{-6} + \sum_{i \in \mathbb{N}(m)} F_{m+i}^{-6}}{\sum_{i=1}^{\infty} F_{m+i}^{-4} + \sum_{i \in \mathbb{N}(m+1)} F_{m+i}^{-4} + \sum_{i \in \mathbb{N}(m)} F_{m+i}^{-4}} \right)^3,$$

an expression whose status is still only formal. Consider also the formal expression

$$J^q := \frac{49}{40} \left( \frac{\sum_{i=1}^{\infty} \varphi^{-6i} + \sum_{i \in \mathbb{N}(1)} \varphi_i^{-6} + \sum_{i \in \mathbb{N}(0)} \varphi_i^{-6}}{\sum_{i=1}^{\infty} \varphi^{-4i} + \sum_{i \in \mathbb{N}(1)} \varphi_i^{-4} + \sum_{i \in \mathbb{N}(0)} \varphi_i^{-4}} \right)^3,$$

where

$$\varphi_i := \varphi^{i1} \cdots \varphi^{i\ell}.$$

**Theorem 2.** If $J^q (\varphi)$ converges then so does $J_{B_m}^q (\varphi)$ for each $m$ and

$$J_{B_m}^q (\varphi) \rightarrow J^q (\varphi),$$

as $m \rightarrow \infty$.

**Proof.** Multiply the numerator and denominator of $J_{B_m}^q (\varphi)$ by $F_{m+1}^{12}$ to obtain

$$J_{B_m}^q (\varphi) := \frac{49}{40} \left( \frac{\sum_{i=1}^{\infty} (F_{m+i}/F_{m})^6 + \sum_{i \in \mathbb{N}(m+1)} (F_{m+i}/F_{m})^6 + \sum_{i \in \mathbb{N}(m)} (F_m/F_{m+i})^6}{\sum_{i=1}^{\infty} (F_{m+i}/F_{m+i})^4 + \sum_{i \in \mathbb{N}(m+1)} (F_{m+i}/F_{m+i})^4 + \sum_{i \in \mathbb{N}(m)} (F_m/F_{m+i})^4} \right)^3.$$

It will suffice to show that each term $T_{m+6} = T_{m+6}^6 (T_m^{-4} = T_m^{-4})$ appearing in a sum contained in the numerator (denominator) of (7) satisfies

$$C_m^{-6} T^{-6} < C_m^{-6} T^{-6} \quad \left( C_m^{-4} T^{-4} < C_m^{-4} T^{-4} \right)$$

where $T = T_I$ is the correspondingly indexed term of $J^q (\varphi)$ and

$$C_m = \frac{1 + \varphi^{-2m}}{1 - \varphi^{-2m}}.$$

This will give convergence of each $J_{B_m}^q (\varphi)$, as well as the bound

$$\left( \frac{1 - \varphi^{-2m}}{1 + \varphi^{-2m}} \right)^{24} J_{B_m}^q (\varphi) < J_{B_m}^q (\varphi) < \left( \frac{1 + \varphi^{-2m}}{1 - \varphi^{-2m}} \right)^{24} J^q (\varphi),$$

which implies that $J_{B_m}^q (\varphi) \rightarrow J^q (\varphi)$.

We will now make use of Binet’s formula, $\sqrt{5}F_m = (\varphi^m + \varphi^{-m})$. Note that the $\sqrt{5}$ factors drop out and so we may simply replace every Fibonacci term $F_m$ appearing by $\varphi^m + \varphi^{-m}$.

We consider first the numerator of (7), treating each of the three sums there separately. The first sum may be written

$$\sum_{i=1}^{\infty} (F_{m+i}/F_m)^6 = \sum_{i=1}^{\infty} \left( \varphi^m + \varphi^{-m} \right)^6 \left( 1 \pm (-1)^i \varphi^{-(m+i)} \right)^6 = \sum_{i=1}^{\infty} \varphi^{-6i} \left( 1 \pm (-1)^i \varphi^{2m-2i} \right)^6.$$ 

Note that

$$\left( \frac{1 - \varphi^{-2m}}{1 + \varphi^{-2m}} \right)^6 < \left( \frac{1 + \varphi^{-2m}}{1 - \varphi^{-2m}} \right)^6 < \left( \frac{1 + \varphi^{-2m}}{1 - \varphi^{-2m}} \right)^6.$$
The next sum is

\[ \sum_{I \in \mathcal{G}(m+1)} (F_m/F_I)^6 = \sum_{I \in \mathcal{G}(m+1)} \left( \frac{\varphi^m + \varphi^{-m}}{(\varphi^{m+i_1} + \varphi^{-m-i_1} + \cdots + (\varphi^{m+i_k} + \varphi^{-m-i_k}) \pm \varphi^{-m-i_k})^6} \right), \]

where we are writing our generic \( I \in \mathcal{G}(m+1) \) in the form \( I = (i_1 + m, \ldots, i_k + m) \) with \( 1 \leq i_1 < i_2 < \cdots < i_k \). Letting \( I_0 = (i_1, \ldots, i_k) \), then each term of the sum in (8) may be re-written

\[ \left( \frac{1 \pm \varphi^{-2m}}{\varphi I_0 + (\pm \varphi^{-I_0-2m})} \right)^6 = \varphi I_0^{-6} \left( \frac{1 \pm \varphi^{-2m}}{1 + (\pm \varphi^{-I_0-2m})/\varphi I_0} \right)^6 \]

where

\[ \pm \varphi^{-I_0-2m} = \pm \varphi^{-i_1-2m} \pm \cdots \pm \varphi^{-i_k-2m}, \]

the signs determined as in Binet’s formula by the parities of the powers. It is easy to see that

\[ \left( \frac{1 - \varphi^{-2m}}{1 + \varphi^{-2m}} \right)^6 < \left( \frac{1 \pm \varphi^{-2m}}{1 + (\pm \varphi^{-I_0-2m})/\varphi I_0} \right)^6 < \left( \frac{1 + \varphi^{-2m}}{1 - \varphi^{-2m}} \right)^6 \]

indeed, both inequalities in (10) follow since

\[ \varphi^{-2m} > (\pm \varphi^{-I_0-2m})/\varphi I_0 > -\varphi^{-2m}, \]

true as

\[ (\pm \varphi^{-I_0-2m})/\varphi I_0 = \varphi^{-2m} \left( \frac{\pm \varphi^{-i_1} \cdots \pm \varphi^{-i_k}}{\varphi I_1 \cdots \varphi I_k} \right). \]

What remains is the sum over \( \mathcal{G}(m) \): the analysis here is essentially the same as that made for the sum over \( \mathcal{G}(m+1) \), only we take into account that \( I = (m, m+j, m+i_3, \ldots, m+i_k) \) where \( j \) is odd. Writing \( I_0 = (0, j, i_3, \ldots, i_k) \), then we have the equation (9) with

\[ \pm \varphi^{-I_0-2m} = \pm \varphi^{-2m} \mp \varphi^{-j-2m} \pm \cdots \pm \varphi^{-i_k-2m}, \]

where the \( \mp \) sign of \( \varphi^{-j-2m} \) indicates that this sign is opposite to that of \( \varphi^{-2m} \), as \( j \) is odd. The analogue of (11) is then

\[ (\pm \varphi^{-I_0-2m})/\varphi I_0 = \varphi^{-2m} \left( \frac{\pm 1 \mp \varphi^{-j} \pm \cdots \pm \varphi^{-i_k}}{1 + \varphi^j \cdots + \varphi^{i_k}} \right), \]

which yields the analogue of (10) in this case. This completes our bounding of the numerator. Analogous bounds, with the exponent 6 replaced by 4, may be found for the corresponding sums in the denominator of \( d_{B_m}^{\text{ct}} \). The result now follows. \( \square \)

Let \( P(n) \) be the set of partitions of \( n \) into into distinct parts whose differences are at least 2, and let \( c(n) = |P(n)| \). The generating function

\[ F(x) = \sum c(n)x^n = \sum \frac{x^n}{(1-x) \cdots (1-x^n)} \]

is of substantial combinatorial interest: it is the left-hand side of the first Rogers-Ramanujan identity [5].

For each partition \( I \in P(n) \), let \( f_I(x) = x^{i_1} + \cdots + x^{i_k} \) be the associated weighting polynomial. Define

\[ C_{x,M}(n) = x^M \sum_{I \in P(n)} f_I(x)^{-M}. \]

Considering the generating function

\[ G_M(x) = \sum C_{x,M}(n)x^n. \]
Clearly we have
\[ G_M(\phi) = \sum_{i=1}^{\infty} \phi^{-M_i} + \sum_{I \in \mathfrak{N}(1)} \phi^{-M_I}. \]

Similarly, let \( Q(n) \subset P(n) \) be the set of those partitions \( I = i_1 < i_2 < \cdots < i_k \) in \( P(n) \) for which \( i_1 \) is odd and \( \geq 3 \). Let
\[ D_{x,M}(n) := x^{Mn} \sum_{I \in Q(n)} (1 + f_I(x))^{-M} \]
and define
\[ H_M(x) := \sum_{I \in \mathfrak{N}(0)} x^{-M_I}. \]

Then
\[ H_M(\phi) = \sum_{I \in \mathfrak{N}(0)} \phi^{-M_I}. \]

The following is then immediate:

**Corollary 1.** Let \( J^q_\ell(\phi) \) be as above. Then
\[ J^q_\ell(\phi) = 49 \left[ G_6(\phi) + H_6(\phi) \right]^2 \]
\[ 40 \left[ G_4(\phi) + H_4(\phi) \right]^3. \]

**Note 3.** If one replaces in the formula for \( C_{x,M}(n) \) the weighting polynomial \( f_I(x)^{-M} \) by the equiweight \( x^{-Mn} \) one recovers \( c(n) \). Thus the functions \( G_M(x), H_M(x) \) may be viewed as weighted variants of the variable part of the Rogers-Ramanujan function.

### 3. Convergence

In this section we will show that \( j(\phi)^q < \infty \). As before we write \( j^q_\ell(\phi) := 12^3/(1 - J^q_\ell(\phi)) \).

**Theorem 3.** \( j(\phi)^q \) converges with the bounds
\[ 9150 < j^q_\ell(\phi) < 9840. \]

**Proof.** To prove the convergence of \( J^q_\ell(\phi) \), it is enough to prove convergence of the explicit formula \( J^q_\ell(\phi) \) obtained from (5). Observe first that
\[ \sum_{i=1}^{\infty} \phi^{-6i} = (\phi^6 - 1)^{-1}, \quad \sum_{i=1}^{\infty} \phi^{-4i} = (\phi^4 - 1)^{-1} \]
so we may write
\[ J^q_\ell(\phi) = 49 \frac{\left( (\phi^6 - 1)^{-1} + \sum_{I \in \mathfrak{N}(1)} \phi^{-6_I} + \sum_{I \in \mathfrak{N}(0)} \phi^{-6_I} \right)^2}{40 \left( (\phi^4 - 1)^{-1} + \sum_{I \in \mathfrak{N}(1)} \phi^{-4_I} + \sum_{I \in \mathfrak{N}(0)} \phi^{-4_I} \right)^3}. \]

We now find an explicit approximation and an upper bound for the sum \( \sum_{I \in \mathfrak{N}(1)} \phi^{-M_I} \) where \( M \) is a positive integer. In fact, we will show that
\[ \sum_{I \in \mathfrak{N}(1)} \phi^{-M_I} = \frac{1}{(\phi^M - 1)(\phi^2 + 1)^M} + C(M) \]
where
\[ C(M) < \tilde{C}(M) := \frac{1}{\phi^{2M}(\phi^M - 1)^2} + \frac{1}{\phi^M(\phi^M - 1)^2(\phi^{2M} - \phi^M - 1)}. \]
Consider first the sum of those $I$ with $|I| = 2$:

$$\sum_{i_1 \geq 1, i_2 \geq i_1 + 2} \frac{1}{(\varphi^{i_1} + \varphi^{i_2})^M} = \sum_{i_1 \geq 1, i_2 \geq i_1 + 2} \frac{\varphi^{-M i_1}}{(1 + \varphi^{i_1})^M} \cdot \frac{1}{(1 + \varphi^{i_2})^M}$$

(14)

$$< \frac{1}{\varphi^M - 1} \left\{ \frac{1}{(1 + \varphi^2)^M} + \sum_{k=3}^{\infty} (1 + \varphi^k)^{-M} \right\}$$

(15)

The equality (14) produces the explicit term $1/((\varphi^M - 1)(\varphi^2 + 1)^M)$ appearing in (12); the second term in (15) is the first bounding term in (13).

For $|I| = 3$ we have

$$\sum_{i_1 \geq 1, i_2 \geq i_1 + 2, i_3 \geq i_2 + 2} \frac{1}{(\varphi^{i_1} + \varphi^{i_2} + \varphi^{i_3})^M} = \sum_{i_1 \geq 1, i_2 \geq i_1 + 2, i_3 \geq i_2 + 2} \varphi^{-M i_1} \cdot \frac{1}{(1 + \varphi^{i_2 - i_1} + \varphi^{i_3 - i_1})^M}$$

$$< \sum_{i_1 \geq 1, i_2 \geq i_1 + 2, i_3 \geq i_2 + 2} \varphi^{-M i_1} \cdot \varphi^{-M(i_2 - i_1)} \frac{1}{(1 + \varphi^{i_2 - i_1})^M}$$

$$= \sum_{i_1 \geq 1, i_2 \geq i_1 + 2, i_3 \geq i_2 + 2} \varphi^{-M i_1} \sum_{j=2}^{\infty} \varphi^{-M j} \sum_{k=2}^{\infty} \varphi^{-M k}$$

$$= \left( \frac{\varphi^{-M}}{\varphi^M - 1} \right)^3.$$  

Inductively, for the terms with $|I| = l \geq 3$ we have the bound

$$\left( \frac{\varphi^{-M}}{\varphi^M - 1} \right)^{l-1}.$$  

Summing these bounds from $l = 3$ to $\infty$ gives the second term in (13):

$$\sum_{l=3}^{\infty} \left( \frac{\varphi^{-M}}{\varphi^M - 1} \right)^{l-1} = \varphi^M \sum_{l=3}^{\infty} \frac{1}{(\varphi^M - 1)^l} = \frac{1}{\varphi^M - 1} \left( \frac{1}{(\varphi^2 M - \varphi^M - 1)} \right)$$

We now bound the second type of sum appearing in $J^q(\varphi)$, $\sum_{I \in \mathcal{S}(0)} \varphi^{-M}$. We will show here that

$$\sum_{I \in \mathcal{S}(0)} \varphi^{-M} = \frac{1}{(1 + \varphi^M)^M} + D(M)$$

(16)

where

$$D(M) < \tilde{D}(M) := \frac{1}{\varphi^3 M (\varphi^2 M - 1)} + \frac{1}{\varphi^M (\varphi^2 M - 1)(\varphi^2 M - \varphi^M - 1)}$$

(17)
When $|I| = 2$ we have, since $i_1 = 0$, that $i_2 = 2j + 1$ is odd, where $j \geq 1$ (recall the definition of $\mathcal{M}(m)$ found in (5)). For such $I$ we have the contribution

\begin{equation}
\sum_{j \geq 1} \frac{1}{(1 + \varphi^j)^M} = \frac{1}{(1 + \varphi^3)^M} + \sum_{j = 2}^{\infty} \frac{1 + \varphi^{(2j+1)-1}M}{(1 + \varphi^3)^M} - \varphi^{-2Mj} - \frac{1}{1 - \varphi^{-2M}} = \frac{1}{(1 + \varphi^3)^M} + \varphi^{-5M} \sum_{j = 2}^{\infty} \varphi^{-2Mj} \frac{1}{1 - \varphi^{-2M}} = \frac{1}{(1 + \varphi^3)^M} + \varphi^{-5M} \sum_{j = 2}^{\infty} \varphi^{-2Mj} \frac{1}{1 - \varphi^{-2M}}
\end{equation}

\begin{equation}
\sum_{j \geq 1} \frac{1}{(1 + \varphi^{2j+1} + \varphi^k)^M} < \sum_{j \geq 1, k \geq (2j+1)+2} \frac{1}{(1 + \varphi^{2j+1} + \varphi^k)^M} < \sum_{j = 1}^{\infty} \varphi^{-M(2j+1)} \frac{1}{1 - \varphi^{-M}} \frac{1}{\varphi^M(\varphi^{2M} - 1)} = \frac{1}{\varphi^M + 1} \left( \frac{\varphi^{-M}}{\varphi^M - 1} \right)^{l-1}
\end{equation}

For the sum over $I$ with $|I| = l$, we obtain inductively the bound

\begin{equation}
\frac{1}{\varphi^M + 1} \left( \frac{\varphi^{-M}}{\varphi^M - 1} \right)^{l-1}
\end{equation}

and summing these from $l = 3$ to $\infty$ gives

\begin{equation}
\frac{1}{\varphi^M(\varphi^{2M} - 1)(\varphi^{2M} - \varphi^M - 1)}.
\end{equation}

It follows then that

\begin{equation}
J^q(\varphi) < 49 \frac{\{(\varphi^6 - 1)^{-1} + ((\varphi^6 - 1)(\varphi^2 + 1)^6)^{-1} + (1 + \varphi^3)^{-6} + \tilde{C}(6) + \tilde{D}(6)\}^2}{\{(\varphi^4 - 1)^{-1} + ((\varphi^4 - 1)(\varphi^2 + 1)^4)^{-1} + (1 + \varphi^3)^{-4}\}^3}
\end{equation}

\begin{equation}
\approx 0.824376700276.
\end{equation}

A lower bound may be given by

\begin{equation}
0.81115979990388 \approx 49 \frac{\{(\varphi^6 - 1)^{-1} + ((\varphi^6 - 1)(\varphi^2 + 1)^6)^{-1} + (1 + \varphi^3)^{-6}\}^2}{\{(\varphi^4 - 1)^{-1} + ((\varphi^4 - 1)(\varphi^2 + 1)^4)^{-1} + (1 + \varphi^3)^{-4} + \tilde{C}(4) + \tilde{D}(4)\}^3}
\end{equation}

\begin{equation}
< J^q(\varphi)
\end{equation}
which give the bounds presented in the statement of the theorem.

\( \square \)

**Note 4.** Using a program such as PARI one can calculate using the explicit formula of Corollary [1] that \( j^{\phi}(\rho) \approx 9538.249655644 \).

**REFERENCES**

[1] Baxter, R.J., *Exactly Solved Models in Statistical Mechanics*. Reprint of the 3rd edition of the work originally published by Academic Press. Dover Books on Physics, Dover Publications, Mineola, NY, 2008.

[2] Castaño Bernard, C. & Gendron, T., Modular Invariant of Quantum Tori I: Nonstandard and Standard. [arXiv:0909.0143](http://arxiv.org/abs/0909.0143)

[3] Connes A., Marcolli, M. & Ramakrishnan, D., KMS states and complex multiplication I, *Selecta Math.* (New Ser.) **11** (2005), no. 3-4, 325-347.

[4] Connes A., Marcolli, M. & Ramakrishnan, D., KMS states and complex multiplication II, in “Operator Algebras: The Abel Symposium 2004”, pp. 15-59, Abel Symp. 1, Springer-Verlag, New York, 2006.

[5] Hardy, G.H. & Wright, E.M., *An Introduction to the Theory of Numbers*. 5th edition. Oxford Science Publications, Oxford University Press, Oxford, UK, 1980.

[6] Manin, Yu. I., Real multiplication and noncommutative geometry. in "The Legacy of Niels Henrik Abel", pp. 685-727, Springer-Verlag, New York, 2004.

[7] Marcolli, M., *Arithmetic Noncommutative Geometry*, University Lecture Series **36**, AMS, Providence, RI, 2005.

[8] Marker, D., *Model Theory: An Introduction*, Graduate Texts in Mathematics **217**, Springer-Verlag, New York, 2002.

[9] Niven, I., Zuckerman, H.S. & Montgomery, H.L., *An Introduction to the Theory of Numbers*, 5th edition, John Wiley & Sons, Inc., New York, 1991.

[10] Ostrowski, V.A., Bemerkungen zur theorie der diophantischen approximationen, *Abh. Math. Semin. Hamburg Univ.* **1** (1922), 77–98.

[11] Schmidt, W.M., *Diophantine Approximation*. Lecture Notes in Mathematics **785**, Springer-Verlag, New York, 1980.

[12] Serre, J.-P., Complex multiplication. in ‘Algebraic Number Theory: Proceedings of an Instructional Conference Organized by the London Mathematical Society” (ed. by J. W. S. Cassels and Frölich), pp. 190D197.

[13] Vorobiev, Nicolai N., *Fibonacci Numbers*, Birkhäuser Verlag, Basel, 2002.

[14] Zeckendorf, E., Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas. *Bull. Soc. R. Sci. Liège** 41** (1972), 179–182.

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