Abelian Chern-Simons field theory

and anyon equation on a torus

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Abstract

We quantize the abelian Chern-Simons theory coupled to non-relativistic matter field on a torus without invoking the flux quantization. Through a series of canonical transformations which is equivalent to solving the Gauss constraint, we obtain an effective Hamiltonian density with periodic matter field. We also obtain the many-anyon Schrödinger equation with periodic Aharonov-Bohm potentials and analyze the periodic property of the wave-function. Some comments are given on the different features of our approach from the previous ones.
I. INTRODUCTION

Chern-Simons theory \([1,2]\) in 1+2 dimensions has attracted much interest because it exhibits various theoretically interesting aspects. The particles of fractional statistics (anyons) \([3]\) are described as particles of ordinary statistics interacting with a Chern-Simons gauge field \([4–8]\). It presents an effective description of the statistical transmutation. When the theory is coupled to a certain scalar field theory, the system possesses stable topological and non-topological vortex solutions \([9]\) whose interpretation \([10]\) as anyons is also suggested.

Quantum mechanics of many anyons is actively investigated last few years in relation with its direct relevance to the fractional quantum Hall effect and the related hierarchical models and its speculative role in high \(T_c\) super-conductivity \([11]\). The construction of many anyon wavefunction is not easy since even its free form can not be written as a product of free single-body wavefunctions, although there are some analytical methods \([12]\) found for simple models on an infinite plane. In addition, the analysis of many anyons is plagued with the hard-core boundary condition and the idea of smooth statistical interpolation of the Hilbert space from boson to fermion (or fermion to boson) waits further clarification \([13]\) whose numerical try \([14]\) has been done for a few-body system.

Anyons on a compact space need extra-care since they require consistent boundary conditions at the edge or some invariance property on a closed surface. In addition, zero-modes of the gauge field complicate the analysis. It is well known \([15]\) that the vacuum on a torus is degenerate due to the zero-modes of the gauge fields and the degeneracy is finite if the Chern-Simons coefficient is rational.

Many particle quantum mechanics on the torus is also constructed \([16–19]\) from the abelian Chern-Simons field theory like in the infinite plane case \([21–22]\), and it is demonstrated that the degeneracy structure of the vacuum can affect the quantum mechanics and give rise to a multi-component anyon wavefunction \([18–20]\). It seems, however, that at least two essential points, quantization and periodic property of the matter field, are to be clarified. The problem related with the quantization can arise when one solves the
Gauss constraint before quantization. It is noted \cite{23} that treating the number operator as a c-number by restricting the Hilbert space on the fixed number of particles may lead to subtleties. Closely related with this is the quasi-periodic boundary condition on the fields and the statistical flux quantization. To avoid the possible subtleties, it seems desirable to quantize all the fields before solving the Gauss constraint and analyze the translation invariance of the system on the torus to discover the periodic property of the fields.

In this paper, we re-analyze the Chern-Simons theory coupled to non-relativistic matter fields on a torus by following the procedure done in Refs. \cite{7,8} for the infinite plane case. Namely, we quantize the theory first and through canonical transformations we obtain an effective theory of matter fields only by removing the gauge fields including the zero-modes, which is equivalent to solving the Gauss constraint. In this canonical formalism, it turns out that the statistical flux quantization is not necessary. The translation invariance along the non-contractible loops is manifest as well as the gauge invariance irrespective of the particle numbers. The transparent periodic property of fields enables us to deduce that of the original fields. From this, we also obtain the periodic property of the gauge invariant wavefunction of many anyons and construct the Schrödinger equation for that.

This paper is organized as follows. In section II, we briefly describe the quantization of abelian pure Chern-Simons gauge field on a finite plane (with the periodic boundary condition) with an eye to the zero-modes of the gauge field. In section III, we obtain an effective hamiltonian of matter field in the fundamental domain of the torus by defining new fields through a series of canonical transformations. In section IV, we construct the many-body Schrödinger equation in the fundamental domain of the torus. The equation becomes the free one except the Aharonov-Bohm potential which is responsible for the statistical transmutation. In section V, we consider the periodic property of the system. We construct the effective hamiltonian density such that it is translation invariant along the non-contractible loops by defining a simple periodic matter field in consistency with two commuting translation operators. The periodic property of the original fields is also presented through canonical transformations. We obtain the Schrödinger equation of many
anyons with periodic Aharonov-Bohm potential. In section VI, we summarize our results and discuss the different aspects of our analysis from the previous ones.

II. MODE EXPANSION OF GAUGE FIELD

Lagrangian density of abelian pure Chern-Simons theory on an infinite plane is given as

\[ \mathcal{L} = \frac{\mu}{2} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho. \]  

(2.1)

The gauge field does not propagate and the hamiltonian system becomes a constraint one. The phase space variables are \( a_1 \) and \( \pi_1 \), where

\[ \pi_1 = \frac{\partial \mathcal{L}}{\partial a_1} = \frac{\mu}{c} a_2, \]

(2.2)

and the gauge-field itself constitutes the phase space. Its equal-time commutation relation is given as

\[ [a_1(x), a_2(y)] = i \frac{\hbar}{\mu} \delta(\vec{x} - \vec{y}), \]

(2.3)

where \( x \) denotes the three vector \((ct, \vec{x})\) and \( \vec{x} = (x^1, x^2) = (x_1, -x_2) \). \( a_0(x) \) is treated as a lagrange multiplier and commutes with \( a_i \)'s.

The hamiltonian density becomes (ignoring the space-time boundary terms),

\[ \mathcal{H} = \mu a_0 b, \]

(2.4)

where \( b = -(\partial_1 a_2 - \partial_2 a_1) \) is the statistical magnetic field. This shows that \( b \) is the gauge generator. The physical state \( |\text{phys}> \) should be annihilated by \( b, b|\text{phys}>=0 \).

To understand the physical state more deeply, we recall that the gauge operators \( a_i \)'s on an infinite plane is expanded as \[ \]

\[ a_i(x) = \int \frac{d\vec{p}}{2\pi} \frac{1}{\sqrt{2p_0}} \left[ \frac{p_i}{\mu} g(\vec{p}) + i \frac{\epsilon_{ij} p_j}{p^0} h(\vec{p}) e^{-ip \cdot x} + c.c \right]_{\rho = |\vec{p}|}, \]

\[ a_0(x) = \int \frac{d\vec{p}}{2\pi} \frac{1}{\sqrt{2p_0}} \left[ \frac{p_0}{\mu} g(\vec{p}) e^{-ip \cdot x} + c.c \right]_{\rho = |\vec{p}|}. \]

(2.5)
\(g(\vec{p})\) and \(h(\vec{q})\) satisfy the commutation relations,

\[
[g(\vec{p}), h^+(\vec{q})] = \hbar \delta(\vec{p} - \vec{q})
\]

\[
[g(\vec{p}), h(\vec{q})] = [g(\vec{p}), g^+(\vec{q})] = [h(\vec{p}), h^+(\vec{q})] = 0.
\] (2.6)

The statistical magnetic field \(b\) is given as

\[
b(x) = \int \frac{d\vec{p}}{2\pi} \frac{1}{\sqrt{2p_0}} [p^0 h(\vec{p}) e^{-ip \cdot x} + c.c]_{p^0 = |\vec{p}|},
\] (2.7)

This representation is constructed such that the Lorentz gauge fixing condition is satisfied identically, \(\partial_\mu a^\mu = 0\) and \([a_0, a_i] = 0\). We note that there is still left a residual gauge degree of freedom with the Lorentz gauge fixing condition maintained: \(a_\mu \to a_\mu + \partial_\mu \Lambda\) where \(\partial^2 \Lambda = 0\). This merely redefines \(g(\vec{p})\) in Eq. (2.4) satisfying the same commutation relation in Eq. (2.6). In our analysis, we do not need the specific form of \(g(\vec{p})\) and therefore, we can proceed in the residual gauge independent manner.

On a finite plane, the gauge field is given in terms of summation of discrete modes, which should include non-commuting zero-modes \(\theta_1\) and \(\theta_2\) to satisfy the commutation relation Eq. (2.3) on the box. For definiteness, we will consider the finite plane (the fundamental domain of the torus) as the rectangular box with (period) length \(L_1\) and \(L_2\) in the following. If \(b\) is periodic with a mean magnetic field, then \(a_i\) is given as

\[
a_i(x) = \frac{\theta_i(t)}{L_i} + \bar{x}_i \xi(t)
\]

\[
+ \frac{1}{\sqrt{L_1 L_2}} \sum_{\vec{p} \neq 0} \frac{1}{2p^0} [(\frac{p_i}{\mu} g(\vec{p}) + i \frac{\epsilon_{ij} p_j}{p^0} h(\vec{p})) e^{-ip \cdot x} + c.c]_{p^0 = |\vec{p}|},
\] (2.8)

where \(p_i = \frac{2\pi n_i}{L_i}\) with \(i = 1, 2\). \(\theta_i\)'s are two non-commuting real zero-modes;

\[
[\theta_1, \theta_2] = \frac{\hbar}{\mu}.
\] (2.9)

\(\theta_i\)'s commute with the rest of the other modes. The \(\theta_i\) modes are essential to satisfy the commutation relation in Eq. (2.3) on the box. We introduced the mean magnetic flux mode \(\xi\), which commutes with the other modes and
\[ \tilde{x}_1 \equiv \phi \frac{x^2}{L_1 L_2} ; \quad \tilde{x}_2 \equiv 0, \tag{2.10} \]

where \( \phi \) is the mean statistical magnetic flux. \( g \) and \( h \) satisfy the discrete version of the commutation relation, Eq. (2.4).

On the other hand, \( a_0(x) \) does not contain the zero-modes and is expressed in the discrete form of Eq. (2.5) with an addition of a time independent constant Lagrangian multiplier \( \lambda \) (which is convenient to describe the system coupled with matter field),

\[ a_0(x) = \lambda + \frac{1}{\sqrt{L_1 L_2}} \sum_{\vec{p} \neq 0} \frac{1}{\sqrt{2p_0}} \left[ \frac{p_0}{\mu} g(\vec{p}) e^{-ip \cdot x} + c.c \right]. \tag{2.11} \]

\( a_0 \) commutes with other fields. We note that the Lorentz gauge fixing condition is satisfied identically even though the zero-modes, \( \theta_i \) are time-dependent.

Since there are two non-commuting zero-modes, which commute with the Hamiltonian density \([\theta_i, \mathcal{H}] = 0\) (recall \( b \) and \( a_0 \) do not contain the zero-modes), the physical state \( |\text{phys} > \) can be degenerate. The degeneracy structure is interesting in itself [13,17–19] but we do not elaborate on this. As is seen in section III and V, the gauge sector of the vacuum can be decoupled from the matter sector and one can choose many anyon wavefunction such that the details of the vacuum degeneracy do not enter. Other choice of the wavefunction and its relation with the vacuum degeneracy is discussed in section VI.

### III. MATTER-COUPLED SYSTEM

Let us consider the system in the fundamental domain of the torus, \( D_{(0,0)}, (0 \leq x^1 < L_1 \) and \( 0 \leq x^2 < L_2 \). The lagrangian density of matter field and Chern-Simons gauge field is given as

\[ \mathcal{L} = \frac{\mu}{2} \epsilon^{\mu \nu \rho} a_\mu \partial_\nu a_\rho + \psi^+ i \hbar D_0 \psi - \frac{\hbar^2}{2m} |\vec{D}_\psi|^2, \tag{3.1} \]

where \( i \hbar D_\mu \psi = (i \hbar \partial_\mu - e a_\mu) \psi \). We will choose the matter field \( \psi \) as a fermion-field for definiteness. The following analysis goes parallel with bosonic one for the above lagrangian. The Hamiltonian density is written as
\[ \mathcal{H} = a_0(\mu b + \frac{e}{c} J_0) + \frac{1}{2m}|i\hbar D_i\psi|^2, \]  

(3.2)

where \( J_0 = \psi^+\psi \). The operators satisfy the equal-time (anti-)commutation relations;

\[ \{\psi(x), \psi^+(y)\} = \delta(x - y) \]

\[ [J_0(x), \psi^+(y)] = \psi^+(x)\delta(x - y) \]

\[ [a_1(x), a_2(y)] = i\frac{\hbar}{\mu}\delta(x - y). \]  

(3.3)

The gauge field \( a_i \) has the mode expansion in Eq.(2.8) and \( a_0 \) is given in Eq. (2.11). The constraint is given as the Gauss law, \( \Gamma = 0 \) where

\[ \Gamma = b + \frac{e}{\mu c} J_0 \]  

(3.4)

is the gauge generator. (Note that due to the \( \lambda \) in Eq. (2.11) the Gauss law includes the mean field component.) Therefore, a physical state should be annihilated by \( \Gamma \), \( \Gamma|_{\text{phys}} = 0 \).

A commonly adopted procedure for this system is to solve the Gauss constraint explicitly before quantizing the fields. On an infinite plane, this method is equivalent to the one with the reverse process, \( i.e. \) quantizing first and requiring constraint on the Hilbert space. On a finite plane, however, solving the constraint first can lead to subtleties as mentioned in the Introduction. To avoid this, we quantize the field first as in Eq. (3.3) and perform a canonical transformation of this system to get a new effective hamiltonian density in terms of newly defined fields and the physical states are explicitly constructed. To give an overview on our method, we summarize our procedure as follows.

The hamiltonian density \( \mathcal{H} \) is the functional of \( \psi(x), a_i(x), \) and \( a_0(x) \).

\[ \mathcal{H} = \mathcal{H}(\psi(x), a_i(x), a_0(x)). \]

We will define new fields with a canonical transformation operator \( V(t) \) which commutes with \( a_0(x) \) as

\[ \psi(x) = V(t)\psi^{(N)}(x)V^+(t) \]

\[ a_i(x) = V(t)a_i^{(N)}(x)V^+(t). \]  

(3.5)
It is obvious that the new fields satisfy the same commutation relations in Eq. (3.3). Then \( \mathcal{H} \) is given as canonical transformation of the hamiltonian density with the new fields

\[
\mathcal{H}(\psi(x, a_i(x), a_0(x))) = V(t) \mathcal{H}^{(N)}(x, a^{(N)}_i(x), a^{(N)}_0(x)) V^+(t).
\]

Finally we express the same hamiltonian as a new functional of the new fields as

\[
\mathcal{H}(\psi(x, a_i(x), a_0(x))) = \mathcal{H}^{(N)}(\psi^{(N)}(x), a^{(N)}_0(x)). \tag{3.6}
\]

The merit of this procedure lies in that we can choose \( V(t) \) such that \( \psi^{(N)}(x) \) is gauge invariant

\[
[\psi^{(N)}(x), \Gamma(y)] = 0, \tag{3.7}
\]

and the effective hamiltonian density contains the gauge invariant fields only. In fact, we choose the newly defined statistical magnetic field as the gauge generator. Therefore, the Gauss constraint is satisfied automatically if the vacuum is defined as the (local) gauge invariant state (which has neither flux nor particle in the original picture). The physical states are obtained if the product of the newly defined matter field \( \psi^{(N)}+ \) is applied on the vacuum.

To substantiate the above procedure clearly, we will take the three steps of canonical transformations, \( V = V_1 V_2 V_3 \). Let us consider a canonical transformation operator

\[
V_1 = \exp i \left[ Q \eta^{(1)} + \frac{e}{\hbar c} \int \int d\vec{x} d\vec{y} J_0^{(1)}(\vec{x}, t) G_p(\vec{x}, \vec{y}) \partial_k a_k^{(1)}(\vec{y}, t) \right]. \tag{3.8}
\]

\( Q \equiv \int d\vec{x} J_0(x) \) is the number operator. \( G_p \) is the periodic (single-valued on the torus) Green’s function satisfying

\[
\nabla^2 G_p(\vec{x}, \vec{y}) = \delta(\vec{x}, \vec{y}) - \frac{1}{L_1 L_2}; \quad G_p(\vec{x}, \vec{y}) = G_p(\vec{y}, \vec{x}), \tag{3.9}
\]

and is given by

\[
G_p(\vec{x}, \vec{y}) = \frac{1}{4\pi} \ln \left| \frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)} \right|^2 - \frac{(\text{Im} z)^2}{2\text{Im} \tau}, \tag{3.10}
\]
where $z = [(x^1 - y^1) + i(x^2 - y^2)]/L_1$, $\tau = iL_2/L_1$ and $\theta_1(z|\tau)$ is the odd Jacobi theta function. We note that it is regularized such that

$$\epsilon_{ij}\partial_x^iG_p(\bar{x}, \bar{y})|_{\bar{x}=\bar{y}} = 0.$$  \hspace{1cm} (3.11)

$\eta^{(1)}$ in Eq. (3.8) is a conjugate to $\xi^{(1)}$ satisfying equal-time commutation relation,

$$[\eta^{(1)}, \xi^{(1)}] = i.$$  \hspace{1cm} (3.12)

The original fields are given in terms of the new fields as

$$\psi(x) = V_1(t)\psi^{(1)}(x)V_1^+(t)$$
$$= \psi^{(1)}(x)\exp -i\{\eta^{(1)} + \frac{e}{\hbar c} \int d\bar{y}G_p(\bar{x}, \bar{y})\partial_{\bar{y}}a^{(1)}_i(y)\},$$

$$a_i(x) = V_1(t)a^{(1)}_i(x)V_1^+(t)$$
$$= a^{(1)}_i(x) - Q\bar{x}_i - \frac{e}{\mu c}\epsilon_{ij}\partial_x^j \int d\bar{y}G_p(\bar{x}, \bar{y})\xi_{0}^{(1)}(y),$$

$$\xi(t) = V_1(t)\xi^{(1)}(t)V_1^+(t) = \xi^{(1)}(x) - Q,$$

$$\eta(t) = V_1(t)\eta^{(1)}(t)V_1^+(t) = \eta^{(1)}(t),$$  \hspace{1cm} (3.13)

$\theta_i(t) = \theta^{(1)}_i(t)$, and $J_0(x) = J_0^{(1)}(x)$. To have the relation $\Gamma(x) = b^{(1)}(x)$, we choose the flux $\phi = \frac{e}{\mu c}$, which simplifies the analysis for the constraint system. We replaced the superscript $(N)$ in Eq. (3.3) by (1) for an obvious reason. The hamiltonian density $H^{(1)}$ is given in terms of the new fields,

$$H^{(1)} = \mu a_0 b^{(1)} + \frac{1}{2m}\left|(i\hbar \partial_i - \frac{e}{c}a^{(eff)}_i)\psi^{(1)}\right|^2.$$  \hspace{1cm} (3.14)

Implicit ordering of $\psi^{(1)+}$ to the left of $a^{(eff)}_i$ is assumed. Here $a^{(eff)}_i$ is an effective gauge-like term,

$$a^{(eff)}_i(x) = \frac{\theta_i}{L_i} - (Q + 1)\bar{x}_i + \epsilon_{ij}\partial_x^j \int d\bar{y}G(\bar{x}, \bar{y})b^{(1)}(y) - \frac{e}{\mu c}\epsilon_{ij}\partial_x^j \int d\bar{y}G_p(\bar{x}, \bar{y})J_0(y).$$  \hspace{1cm} (3.15)

This is obtained by noting that (see Eq. (2.8))

$$a^{(1)}_i(x) = \frac{\theta_i}{L_i} + \epsilon_{ij}\partial_x^j \int d\bar{y}G(\bar{x}, \bar{y})b^{(1)}(x) + \partial_x^j \int d\bar{y}G_p(\bar{x}, \bar{y})\partial_x^j a^{(1)}_j(y),$$  \hspace{1cm} (3.16)
where

\[ G(\vec{x}, \vec{y}) = G_p(\vec{x}, \vec{y}) + G_{np}(\vec{x}, \vec{y}) . \]

\( G_{np} \) satisfies

\[ \nabla^2 G_{np}(\vec{x}, \vec{y}) = \frac{1}{L_1 L_2} . \]

If one chooses \( G_{np} \) (there is a gauge-like degree of freedom for \( G_{np} \)) as

\[ G_{np}(\vec{x}, \vec{y}) = \frac{(\text{Im } z)^2}{2 \text{Im } \tau} = \frac{(x_2 - y_2)^2}{2L_1 L_2} , \tag{3.17} \]

then

\[ G(\vec{x}, \vec{y}) = \frac{1}{4\pi} \ln \left| \frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)} \right|^2 . \tag{3.18} \]

The gauge generator of this hamiltonian system is

\[ B \equiv i\frac{\mu}{\hbar} \int d\vec{x} b^{(1)}(x) \Lambda(x) , \tag{3.19} \]

where \( \Lambda(x) \) is an arbitrary real periodic function. \( a_i^{(1)}(x) \), \( a_i(x) \) and \( \psi(x) \) are gauge-dependent since

\[ [B, a_i^{(1)}(x)] = \partial_i \Lambda(x) , \quad [B, a_i(x)] = \partial_i \Lambda(x) , \]

\[ [B, \psi(x)] = -ie\frac{\hbar}{\mu c} \Lambda(x) \psi(x) \tag{3.20} \]

whereas \( \psi^{(1)}(x) \) is gauge invariant

\[ [B, \psi^{(1)}(x)] = 0 . \tag{3.21} \]

We emphasize that the readers should not confuse the gauge field \( a_i^{(1)}(x) \) with \( a_i^{(\text{eff})}(x) \). \( a_i^{(1)}(x) \)'s are satisfying the commutation relation given in Eq. (3.3). However, the commutation relation between \( a_i^{(\text{eff})}(x) \)'s is given as

\[ [a_1^{(\text{eff})}(x), a_2^{(\text{eff})}(y)] = i\frac{\hbar}{\mu L_1 L_2} , \tag{3.22} \]

which is coordinate-independent.
Since \( b^{(1)}(x) \) commutes with any other operators in \( \mathcal{H}^{(1)} \), and since we consider the gauge invariant operators and physical states, we may drop \( b^{(1)}(x) \) completely. Therefore, the local gauge fields disappear from the hamiltonian density,

\[
\mathcal{H}^{(1)}(x) = \frac{1}{2m} \left| (\hbar \partial_i - \frac{e}{c} A_i^{[1]}(x)) \psi^{(1)}(x) \right|^2 ,
\]

where

\[
A_i^{[1]}(x) = \frac{\theta_i}{L_i} - (Q + 1) \bar{x}_i - \frac{e}{\mu c} \varepsilon_{ij} \partial_{x_j} \int d\bar{y} G_p(\bar{x}, \bar{y}) J_0(y) .
\]

The effective hamiltonian density and the newly defined fields enable us to construct the physical state by simply applying operators consisting of \( \psi^{(1)+} \) and \( \theta_i \)'s on the vacuum \( |0> \) which satisfies \( b^{(1)}|0> = J_0|0> = 0 (\xi|0> = 0) \). The presence of non-commuting zero-modes \( \theta_i \)'s however, prohibits simple interpretation for the effective hamiltonian density. Furthermore, the system can possess a large gauge transformation invariance. Suppose we transform the zero-modes by a constant \( \Lambda_i \);

\[
\theta_i \to \theta_i + \Lambda_i
\]

and the fermi-field by

\[
\psi^{(1)}(x) \to \exp(-i\phi_i(\bar{x}))\psi^{(1)}(x)
\]

where \( \phi_i(\bar{x}) = \frac{e\Lambda_i}{\hbar c L_i} \) \((i = 1, 2, \text{not sum on } i)\). This transformation leaves \( \mathcal{H}^{(1)} \) invariant. This transformation is, however, not a gauge one since \( \phi_i(\bar{x}) \) does not satisfy the periodic boundary condition. On the other hand, for the “compact” \( U(1) \) gauge group (equivalent to a circle) \( \phi_i(\bar{x}) \) is defined modulo \( 2\pi \). In this case, this transformation becomes the large gauge transformation if \( \frac{e\Lambda_i}{\hbar c L_i} = 2\pi n_i \), where \( n_i \) is an integer. We, therefore, perform another canonical transformations and obtain an effective hamiltonian density which has no explicit dependence of the zero-modes. In this picture, the invariance under the possible large gauge transformation is manifest.

The canonical transformation is explicitly given as

\[
V_2(t) = \exp \left[ \frac{ie\theta_2^{(2)}}{\hbar c} \int d\bar{y} J_0^{(2)}(y) y^2 \frac{L_2}{L_2} \right] ,
\]

\[11\]
and the new fields are given as
\[
\psi^{(1)}(x) = V_2(t)\psi^{(2)}(x)V_2^{+}(t) = \psi^{(2)}(x) \exp\{-i\frac{e}{\hbar cL_2}x^2\theta_2^{(2)}\}
\]
\[
\theta_1(t) = V_2(t)\theta_1^{(2)}(t)V_2^{+}(t) = \theta_1^{(2)}(t) + \phi \int d\vec{y}J_0^{(2)}(y)\frac{y^2}{L_2}
\]
\[
\theta_2(t) = V_2(t)\theta_2^{(2)}(t)V_2^{+}(t) = \theta_2^{(2)}(t),
\]
(3.26)
and \(J_0^{(2)}(x) = J_0(x)\). The hamiltonian density becomes
\[
\mathcal{H}^{(2)}(x) = \frac{1}{2m}\left|i\hbar \partial_i - \frac{e}{c}A_1^{[2]}(x)\psi^{(2)}(x)\right|^2,
\]
(3.27)
where we have new effective gauge-like terms
\[
A_1^{[2]}(x) = \theta_1^{(2)}(t) - \vec{x}Q + \int d\vec{y}J_0(y)\vec{y} - \phi \partial_{x^2} \int d\vec{y}G_p(\vec{x}, \vec{y})J_0(y)
\]
\[
= \theta_1^{(2)}(t) - \phi \partial_{x^2} \int d\vec{y}G(\vec{x}, \vec{y})J_0(y),
\]
\[
A_2^{[2]}(x) = \phi \partial_{x^1} \int d\vec{y}G(\vec{x}, \vec{y})J_0(y),
\]
(3.28)
noting that \(\partial_{x^1}G_{np}(\vec{x}, \vec{y}) = 0\) from Eq. (3.17).

We remark that \(\psi^{(2)}\) is invariant under \(\theta_2 \rightarrow \theta_2 + 2\pi n_2 \frac{\hbar c}{e}\) whereas \(\psi^{(1)}\) is not:
\[
\psi^{(2)}(x) \rightarrow \psi^{(2)}(x); \quad \psi^{(1)}(x) \rightarrow \exp\{-i2\pi n_2 \frac{x^2}{L_2}\} \psi^{(1)}(x).
\]
Not only for that, \(\theta_2\) disappears in \(\mathcal{H}^{(2)}\) and the resulting effective gauge-like term \(A_1^{[2]}\) commutes each other. With one more transformation using
\[
V_3(t) = \exp \left[\frac{ie\theta_1^{(3)}(t)}{\hbar c} \int d\vec{y}J_0^{(3)}(y)\frac{y^1}{L_1}\right],
\]
(3.29)
we get
\[
\mathcal{H}^{(3)}(x) = \frac{1}{2m}\left|i\hbar \partial_i - \frac{e}{c}A_1^{[3]}(x)\psi^{(3)}(x)\right|^2,
\]
(3.30)
where
\[
\psi^{(2)}(x) = V_3(t)\psi^{(3)}(x)V_3^{+}(t) = \psi^{(3)}(x) \exp\{-i\frac{e}{\hbar cL_1}\theta_1^{(3)}\}
\]
\[
\theta_1^{(3)}(t) = V_3(t)\theta_1^{(3)}(t)V_3^{+}(t) = \theta_1^{(3)}(t)
\]
\[
\theta_2^{(3)}(t) = V_3(t)\theta_2^{(3)}(t)V_3^{+}(t) = \theta_2^{(3)}(t) - \phi \int d\vec{y}J_0(y)\frac{y^1}{L_1}
\]
\[
A_1^{[3]}(x) = -\phi\epsilon_{ij}\partial_{x^j} \int d\vec{y}G(\vec{x}, \vec{y})J_0(y),
\]
(3.31)
and \( J_0^{(3)}(x) = J_0(x) \).

This is the desired form of the Hamiltonian density we are looking for, which amounts to the one what one usually obtains by solving the Gauss constraint if one neglects the \( x \)-independent term in \( A_i^{[3]}(x) \), \( \int d\vec{y} J_0(y) \vec{y}_i \) in Eq. (3.31). In this canonical formalism, the effective gauge fields, \( A_i^{[3]}(x) \) satisfy the Gauss constraint equation naturally,

\[
b^{(\text{eff})}(x) \equiv -\epsilon_{ij} \partial_i A_j^{[3]}(x) = -\phi J_0(x).
\]

It is interesting to see that the commutation relation for \( \Pi_i^{[N]}(x) \equiv i\hbar \partial_i - \frac{e}{c} A_i^{[N]}(x) \) \((N = 1, 2, 3)\) is given as

\[
[\Pi_1^{[N]}(x), \Pi_2^{[N]}(x)] = -i\hbar \frac{e\phi}{c} J_0(x).
\]

The crucial point of our canonical formalism lies in introducing the (local and large) gauge-invariant (composite) matter field \( \psi^{(3)}(x) \). \( \psi^{(3)}(x) \) is related with the original matter field \( \psi(x) \) as

\[
\psi(x) = \exp -i(\eta + \frac{e}{\hbar c} \int d\vec{y} G_{\rho}(\vec{x}, \vec{y}) \partial_y a_i^{(3)}(y)) \\
\times \exp \left( i \frac{e^2 x^2}{\mu \hbar c^2 L_2} \int d\vec{y} J_0^{(3)}(y) \frac{y_1}{L_1} \right) \\
\times \exp \left( -i \frac{e x^2 \theta_i^{(3)}}{\hbar c L_2} \right) \exp \left( -i \frac{e x^1 \theta_i^{(3)}}{\hbar c L_1} \right) \psi^{(3)}(x),
\]

where \( J_0^{(3)}(y) = J_0(y) \). From this, we can deduce the gauge transformation property of \( \psi(x) \). Since \( a_i^{(3)}(x) \) is decoupled from the Hamiltonian density \( \mathcal{H}^{(3)}(x) \), we can allow the gauge transformation for \( a_i^{(3)}(x) \) without transforming \( \psi^{(3)}(x) \),

\[
a_i^{(3)}(x) \to a_i^{(3)}(x) + \partial_i \Lambda(x) \\
\theta_i^{(3)}(t) \to \theta_i^{(3)}(t) + 2\pi n_i \frac{\hbar c}{e}.
\]

This induces the gauge transformation for \( \psi(x) \) and \( a_i(x) \) as

\[
a_i(x) = a_i^{(3)}(x) - \phi \epsilon_{ij} \partial_j x \int d\vec{y} J_0(y) G(\vec{x}, \vec{y}) - \delta_{lj} \frac{\phi L_2}{L_1} \int d\vec{y} J_0(y) \frac{y_1}{L_1} \\
\to a_i(x) + \partial_i \Lambda(x),
\]
\[ \theta_i(t) = \theta_i^{(3)}(t) + \phi_{ij} \int d\vec{y} J_0(y) \frac{y^j}{L_j} \]
\[ \rightarrow \theta_i(t) + 2\pi n_i \frac{\hbar c}{e}, \]
\[ \psi(x) \rightarrow \psi(x) \exp -i \left[ \frac{eA(x)}{\hbar c} + 2\pi \frac{n_1 x^1}{L_1} + 2\pi \frac{n_2 x^2}{L_2} \right]. \quad (3.35) \]

The physical operators are products of \( \psi^{(3)}(x) \)'s and \( \psi^{(3)+}(x) \)'s, and the vacuum \(|0>\) satisfies \( b^{(3)}|0> = J_0|0> = 0 \). In evaluating the expectation value of the operators between the physical states, we do not need the gauge sector of the vacuum and therefore, degeneracy structure resulting from the zero-modes \( \theta_1 \) and \( \theta_2 \) does not contribute as far as the matter field \( \psi^{(3)}(x) \) is concerned.

**IV. MANY-BODY QUANTUM MECHANICS**

The \( N \)-particle wavefunction is defined as
\[ \Phi(1, \ldots, N) \equiv <0|\psi^{(3)}(x^{(1)}) \cdots \psi^{(3)}(x^{(N)})|N> \quad (4.1) \]

We assume that all the coordinates of the particles lie in a fundamental domain. \(|0>\) is the vacuum which satisfies \( J_0|0> = 0 \) and \(|N>\) is the \( N \)-body Heisenberg state vector. We suppress the vacuum degeneracy index due to the gauge sector. The wavefunction is gauge invariant since \( \psi^{(3)}(x) \) is (local and large) gauge invariant.

The Schrödinger equation is given as
\[ i\hbar \frac{\partial \Phi}{\partial t}(1, \ldots, N) = \sum_{p=1}^{N} <0|\psi^{(3)}(x^{(1)}) \cdots i\hbar \frac{\partial \psi^{(3)}(x^{(p)})}{\partial t} \cdots \psi^{(3)}(x^{(N)})|N>. \quad (4.2) \]

To evaluate this, we need the Heisenberg equation for the fermi-field \( \psi^{(3)}(x) \),
\[ i\hbar \frac{\partial \psi^{(3)}(x)}{\partial t} = [\psi^{(3)}(x), H] \]
\[ = \frac{1}{2m} \left[ (i\hbar \partial_x - eA_i^{[3]}(x))^2 + \left( \frac{e}{c} \right)^2 \int d\vec{y} \psi^{(3)+}(y) K_i(y, x) K_i(y, x) \psi^{(3)}(y) \right. \]
\[ - \left( \frac{e}{c} \right)^2 \int d\vec{y} \psi^{(3)+}(y) \left( i\hbar \partial_y - eA_i^{[3]}(y) \right) K_i(y, x) \psi^{(3)}(y) \]
\[- \left( \frac{e}{c} \right) \int d\vec{y} \psi^{(3)+}(y) \right) K_i(y, x) \left( i\hbar \partial_{\vec{y}} - \left( \frac{e}{c} \right) A_i^{[3]}(y) \right) \psi^{(3)}(y) \bigg] \psi^{(3)}(x) , \tag{4.3} \]

where

\[ K_i(x, y) = -\phi \epsilon_{ij} \partial_{x_j} G(\vec{x}, \vec{y}) . \tag{4.4} \]

We put $\psi^{(3)+}$ and $A_i^{[3]}$ to the far left in each term such that the operators vanish when they act on the vacuum $\langle 0 |$ and used the identity

\[ [\psi^{(3)}(x), A_i^{[3]}(y)] = K_i(y, x) \psi^{(3)}(x) . \]

The one-particle wavefunction satisfies the Schrödinger equation

\[ i\hbar \partial_t \Phi(x) = \langle 0 | i\hbar \frac{\partial \psi^{(3)}(x)}{\partial t} | 1 \rangle = -\hbar^2 \frac{\partial^2}{2m} \Phi(x) . \tag{4.5} \]

The equation does reduce to the free one. For the two-body case, we have

\[ i\hbar \partial_t \Phi(1, 2) = \langle 0 | \psi^{(3)}(x^{(1)}) i\hbar \frac{\partial \psi^{(3)}(x^{(2)})}{\partial t} + i\hbar \frac{\partial \psi^{(3)}(x^{(1)})}{\partial t} \psi^{(3)}(x^{(2)}) | 2 \rangle = \frac{1}{2m} \{ (i\hbar \partial_t^{(1)} - \frac{e}{c} A_i(1, 2))^2 + (i\hbar \partial_t^{(2)} - \frac{e}{c} A_i(2, 1))^2 \} \Phi(1, 2) , \tag{4.6} \]

where

\[ A_i(p, r) = -\phi \epsilon_{ik} \hat{\partial}_k^{(p)} G(p, r) . \tag{4.7} \]

In general for the $N$-particle case, we have

\[ i\hbar \partial_t \Phi(1, \cdots, N) = \frac{1}{2m} \sum_{p=1}^{N} \{ i\hbar \partial_t^{(p)} - \frac{e}{c} \sum_{r=1}^{N} \epsilon_{r(\neq p)} A_i(p, r) \} \Phi(1, \cdots, N) . \tag{4.8} \]

Finally, the Aharonov-Bohm gauge potential, $A_i(\vec{x}, \vec{y})$ can be transformed away through the singular gauge transformation as in the infinite plane case. Explicitly, using the Green’s function in Eq. (3.18) we have

\[ A_i(p, r) = -\partial_i \left[ \frac{i\phi}{4\pi} \ln \frac{\theta(z_{pr}|\tau)}{\theta^*(z_{pr}|\tau)} \right] , \]

if we neglect the singular parts at the coincident points. As the result, the Schrödinger equation becomes the free one, whereas the transformed wave-function (anyonic wavefunction) is multivalued

\[ \Phi^{(\text{anyon})}(1, \cdots, N) = \prod_{p>r}^{N} \left( \frac{\theta(z_{pr}|\tau)}{\theta^*(z_{pr}|\tau)} \right)^{\frac{1}{2}} \Phi(1, \cdots, N) , \]

where $\nu = e^2/(2\pi\hbar c^2)$ and $\phi = \nu \phi_0$ with the unit flux quantum $\phi_0 = hc/e$. 

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V. PERIODIC PROPERTY

To describe a closed surface such as a torus, one usually adopts the covering space, which consists of the repeated domains of the fundamental one, and identifies the edges of the fundamental domain with each other. Then the non-contractible loop on the closed surface is identified with a line on the fundamental domain from one edge to the other.

We have considered so far the hamiltonian system on the fundamental domain, \( D(0,0) \) \((0 \leq x^1 < L_1, 0 \leq x^2 < L_2)\). To describe the hamiltonian density outside the fundamental domain, let us introduce the domain \( D(m,n) \) for definiteness, \((x'^1 = x^1 + mL_1, x'^2 = x^2 + nL_2)\) where \(x^i\) lies on \( D(0,0) \) and \(m, n\) are integers. (In the following, we reserve the unprimed coordinates for the ones in the fundamental domain \( D(0,0) \) and primed for the domain \( D(m,n) \)).

Obviously, the effective hamiltonian density on \( D(m,n) \) can be written as

\[
\mathcal{H}^{(3)}(x') = \frac{1}{2m} \left| \left( i\hbar \partial_t - \frac{e}{c} A_i^{[3]}(x') \right) \psi^{(3)}(x') \right|^2 ,
\]

where

\[
A_i^{[3]}(x') = -\phi\epsilon_{ij}\partial_j \int d\vec{y} G(\vec{x}', \vec{y}') J_0(y') .
\]

Noting the relation, \( G(\vec{x}, \vec{y}) = G(\vec{x}', \vec{y}') \), we have

\[
A_i^{[3]}(x') = -\phi\epsilon_{ij}\partial_j \int d\vec{y} G(\vec{x}, \vec{y}) J_0(y') .
\]

This hamiltonian density on \( D(m,n) \) is canonically related with the one on \( D(0,0) \)

\[
\mathcal{H}^{(3)}(x') = T_{(m,n)} \mathcal{H}^{(3)}(x) T_{(m,n)}^+ ,
\]

where

\[
J_0(x') = T_{(m,n)} J_0(x) T_{(m,n)}^+ ,
\]
\[
b^{(3)}(x') = T_{(m,n)} b^{(3)}(x) T_{(m,n)}^+ ,
\]
\[
\psi^{(3)}(x') = T_{(m,n)} \psi^{(3)}(x) T_{(m,n)}^+ .
\]

The explicit translation operator along the non-contractible loop is given as
\[ T_{(m,n)} = T_1^m T_2^n \]

where

\[ T_1 \equiv \exp -L_1 \int d\vec{y}\{ \psi^{(3)+} \partial_\nu \psi^{(3)}(y) + \frac{i\mu}{\hbar} a_1^{(3)}(y) \partial_\nu a_2^{(3)}(y) \} \]

\[ T_2 \equiv \exp -L_2 \int d\vec{y}\{ \psi^{(3)+} \partial_\nu \psi^{(3)}(y) + \frac{i\mu}{\hbar} a_1^{(3)}(y) \partial_\nu a_2^{(3)}(y) \} . \]

(5.6)

We also note that the translation operators connect the gauge field as

\[ a_i^{(3)}(x') = T_{(m,n)} a_i^{(3)}(x) T_{(m,n)}^+ . \]

(5.7)

The operators \( T_1 \) and \( T_2 \) commute each other,

\[ T_1 T_2 = T_2 T_1 , \]

(5.8)

without the flux quantization in contrast with the previous analyses. The difference originates from the way to treat the Gauss constraint. We introduced the mean field mode \( \xi \) in \( a_i(x) \) to yield a simple periodic condition (see Eq. (5.18) below) whereas the previous analyses require the quasi-periodic condition on the gauge field \( a_i(x) \) and the matter field \( \psi(x) \), which results in non-commuting translation operators unless the flux is quantized.

On the torus, one can leave the hamiltonian density invariant under the translation along the non-contractible loops as given in Eq. (5.4) if one defines the matter field as \( \psi^{(3)}(x') = \psi^{(3)}(x) \exp iC_{mn} \), where \( C_{mn} \) is a constant on \( D_{(m,n)} \). We can choose the constant \( C_{mn} = 0 \) without losing any generality.

\[ \psi^{(3)}(x') = \psi^{(3)}(x) . \]

(5.9)

This definition leads to the periodic condition on \( J_0 \) and \( b^{(3)} \) as

\[ J_0(x') = J_0(x) ; \quad b^{(3)}(x') = b^{(3)}(x) . \]

(5.10)

This makes the effective gauge field \( A_i^{[3]}(x') \) manifestly translation-invariant,

\[ A_i^{[3]}(x') = A_i^{[3]}(x) , \]

(5.11)
and the translation-invariance of the hamiltonian density follows:

\[ [T_i, \mathcal{H}(x)] = 0. \] (5.12)

We note that the vacuum defined on the fundamental domain remains the same on the covering space. Therefore, the physical state is the same on all the covering space and the Gauss constraint is identically satisfied.

The equal-time commutation relation of \( J_0(x) \) with \( \psi^{(3)}(y') \) is given as

\[ [J_0(x), \psi^{(3)}(y')] = [J_0(x), \psi^{(3)}(y)] = \psi^{(3)}(y)\delta(\vec{x} - \vec{y}). \] (5.13)

One can also define the periodic property of the gauge field \( a_i^{(3)} \) up to a gauge transformation, which leaves the field strength \( b^{(3)} \) invariant as given in Eq.\((5.10)\),

\[ a_i^{(3)}(x') = a_i^{(3)}(x) + \partial_i \Omega(x), \] (5.14)

where \( \Omega(x) \) is an arbitrary periodic function.

The original hamiltonian density \( \mathcal{H}(x') \) on \( D_{(m,n)} \) can be obtained from the effective one \( \mathcal{H}^{(3)}(x') \) if one uses the canonical operators,

\[
V_{(m,n)1}(t) = \exp \left[ i \frac{Q\eta^{(1)}}{\hbar c} \int \! dx \! dy J_0(x') G_p(x', y) \partial_y a_k^{(1)}(y') \right], \\
V_{(m,n)2}(t) = \exp \left[ i \frac{e\theta_2^{(2)}}{\hbar c} \int \! dy J_0(y') \frac{y^2}{L_2} \right], \\
V_{(m,n)3}(t) = \exp \left[ i \frac{e\theta_1^{(3)}}{\hbar c} \int \! dy J_0(y') \frac{y^1}{L_1} \right].
\] (5.15)

where the integration \( \int dx \) denotes for \( 0 \leq x^1 < L_1, 0 \leq x^2 < L_2 \). The operators are the same as the ones given in Eqs. \((3.30), (3.25), (3.29)\) except that the field operators are replaced by the ones on \( D_{(m,n)} \). The hamiltonian density becomes

\[ \mathcal{H}(x') = \frac{1}{2m} \left| \left( \frac{i}{\hbar} \partial_t - \frac{e}{c} a_i(x') \right) \psi(x') \right|^2, \] (5.16)

where
\[ \psi(x') = \exp \left( -i \left( \eta + \frac{e}{\hbar c} \int d\bar{y} G_p(\bar{x}, \bar{y}) \partial_{\bar{y}} a_i^{(3)}(y') \right) \right) \]
\[ \times \exp \left( \frac{ie^2 x^2}{\mu \hbar c^2 L_2^2} \int d\bar{y} J_0^{(3)}(y) \frac{y^1}{L_1} \right) \exp \left( -i \frac{ex^2 \theta_{i}^{(3)}}{\hbar c L_2} \right) \exp \left( -i \frac{ex \theta_{i}^{(3)}}{\hbar c L_1} \right) \psi^{(3)}(x), \]
\[ a_i(x') = a_i^{(3)}(x') - \phi e_{ij} \partial_{x_j} \int d\bar{y} G(\bar{x}, \bar{y}) J_0(y) - \delta_{i2} \frac{\phi}{L_2} \int d\bar{y} J_0(y) \frac{y^1}{L_1}. \quad (5.17) \]

The periodic property of the fields are easily deduced using Eqs. (5.9, 5.14),
\[ \psi(x') = \psi(x) \exp \left( -i \frac{e \Omega(x)}{\hbar c} \right), \]
\[ a_i(x') = a_i(x) + \partial_i \Omega(x), \quad (5.18) \]

which demonstrates that the periodic property for \( \psi(x) \) and \( a_i(x) \) is given as a local gauge transformation. The commutation relations of the fields are given as
\[ \{ \psi(x'), \psi(y'') \} = 0 \]
\[ \{ \psi(x'), \psi^+(y'') \} = \delta(\bar{x} - \bar{y}) \]
\[ [a_1(x'), a_2(y'')] = i \frac{\hbar}{\mu} \delta(\bar{x} - \bar{y}), \]

where \( x' \) and \( y'' \) may lie in different domains. We leave the explicit construction of the two commuting translation operators \( VT_i V^+(i = 1, 2) \) for the original picture in terms of the original fields to the devoted reader.

We emphasize that the canonical operators in Eq. (5.15) are not the unique choice. One may adopt a different choice for the canonical operators as,
\[ V'_{(m,n)}(t) = \exp \left[ i \left( Q \eta^{(1)} + \frac{e}{\hbar c} \int d\bar{x}d\bar{y} J_0(x') G_p(x', y') \partial_{\bar{x}} a_k^{(1)}(y') \right) \right], \]
\[ V'_{(m,n)}(t) = \exp \left[ \frac{ie \theta_{2}^{(2)}}{\hbar c} \int d\bar{y} J_0(y') \frac{y^2}{L_2} \right], \]
\[ V'_{(m,n)}(t) = \exp \left[ \frac{ie \theta_{1}^{(3)}}{\hbar c} \int d\bar{y} J_0(y') \frac{y^1}{L_1} \right]. \quad (5.19) \]

Here, the difference lies in that the coordinates are also primed. In this case, the hamiltonian density becomes the same form \( \mathcal{H}(x') \) in Eq. (5.10). The difference comes to the periodic property of the matter field \( \psi'(x') \) (we put the prime to distinguish this from \( \psi(x') \) in Eq. (5.17)) since the matter field is given as
\[
\psi'(x') = \exp \left( -i \left( \eta + \frac{e}{\hbar c} \int d\vec{y} G_p(\vec{x}, \vec{y}) \partial_p a_i^{(3)}(y') \right) \right) \times \exp \left( \frac{ic^2 x'^2}{\mu \hbar c^2 L_2} \int d\vec{y} J_0^{(3)}(y) \psi^{(1)}_1 \exp \left( -i \frac{e x'^2 \theta_2^{(3)}}{\hbar c L_2} \right) \exp \left( -i \frac{e x'^1 \theta_1^{(3)}}{\hbar c L_1} \right) \psi^{(3)}(x) \right).
\]

(5.20)

It shows that the periodic property is as complicated as can be and it has not only the number operator dependent phase but also the \( \theta_i \) dependent one \( \exp \left\{ -i \frac{e \theta_i}{\hbar c} \right\} \) in addition to the phase factor \( \exp \left\{ -i \frac{e \Omega(x)}{\hbar c} \right\} \) in Eq. (5.18). However, the local and large gauge transformation property of the two matter fields in Eqs. (5.17, 5.20) is exactly the same.

Let us consider the many anyon wavefunctions whose coordinates are located in different domains. Similarly in Eq. (4.1), we define the wavefunction as

\[
\Phi(\{1\}, \ldots, \{N\}) \equiv \langle 0 | \psi^{(3)}(\{1\}) \cdots \psi^{(3)}(\{N\}) | N \rangle \tag{5.21}
\]

where the coordinates \( \{i\} \equiv (x^{(i)}_1 + m^{(i)} L_1, x^{(i)}_2 + n^{(i)} L_2) \) with \( m^{(i)} \) and \( n^{(i)} \) integers may lie in any mixed domain. The Schrödinger equation is the same as the one given in Eq. (4.8) since the hamiltonian density \( \mathcal{H}^{(3)}(x') \) is translation-invariant:

\[
i \hbar \frac{\partial \Phi}{\partial t}(\{1\}, \ldots, \{N\}) = \frac{1}{2m} \sum_{p=1}^N \{i \hbar \partial_i^{(p)} \} - \frac{e}{c} \sum_{r=1(\neq p)}^N A_i(p, r) \}^2 \Phi(\{1\}, \ldots, \{N\}) \tag{5.22}
\]

The hamiltonian has the periodic gauge-like potential since it contains the coordinate function of \( x^{(i)} \) rather than \( x^{(i)'} \). One should note that the wavefunction is trivially periodic

\[
\Phi(\{1\}, \ldots, \{N\}) = \Phi(1, \ldots, N), \tag{5.23}
\]

and it maintains the fermionic exchange property

\[
\Phi(\ldots, \{i\}, \ldots, \{j\}, \ldots) = -\Phi(\ldots, \{j\}, \ldots, \{i\}, \ldots), \tag{5.24}
\]

due to the periodic property of the matter field in Eq. (5.9).

**VI. SUMMARY AND DISCUSSION**

We have analyzed the Chern-Simons theory coupled to non-relativistic matter field on a torus. Quantizing the field first and performing canonical transformation we obtain an
effective theory in terms of matter field \( \psi^{(3)}(x) \) only such that it manifests the gauge invariance and the translation invariance along the non-contractible loops. The periodic property of the new fields are explicitly obtained. This assigns the periodic property of the original fields through the canonical transformation. It is noted, however, the periodic property of the original fields may not be uniquely determined, which depends on the choice of the canonical transformation. From the effective hamiltonian we construct the Schrödinger equation for many anyons. It has the periodic Aharonov-Bohm potential of the singular gauge form adapted to the boundary condition and when transformed away it reduces to the free equation with multi-valued (anyon) wavefunction.

Our analysis does not need the flux quantization. In the literature, one attempts to solve the Gauss constraint on the \( N \)-particle state, \( \int d\vec{y}(b(y) + \phi J_0(y))|N > = 0 \), and requires \( \nu(\equiv \frac{e^2}{2\pi\hbar c^2}) \) be a rational value since \( \int d\vec{y}b(x)/\phi_0 \) (\( \phi_0 \) is the unit flux quantum) and the number operator \( Q \) are assumed to be independent integers. In our analysis, however, the two eigenvalues are related each other, \( \int d\vec{y}b(y)|N > = -\phi N|N > \) due to the mean field mode \( \xi \) in the gauge field and \( Q|N > = N|N > \) since

\[
|N > \sim \int d\vec{x}^{(1)} \cdots d\vec{x}^{(N)} \psi^{(3)+}(1) \cdots \psi^{(3)+}(N)|0 > ,
\]

and the Gauss law is automatically satisfied as mentioned following Eq. (3.7). In addition, the translation operators we constructed on the torus commute each other. The effective gauge-like potential can be defined as a periodic one from the beginning (See Eq. (5.11)) and therefore, the translation operators automatically commute each other as given in Eqs. (5.8, 5.12) for any value of \( \nu \).

We also demonstrate that the local and large gauge transformation property for the matter field does not fix the periodic condition for the matter field in section V. In addition, the periodic property of the gauge invariant wavefunction may depend on the choice of its definition. Suppose one assigns the periodic property of \( \psi(x) \) as given in Eq. (5.18) and choose the local and large gauge invariant wavefunction as

\[
\Phi_h(\{1\}, \cdots, \{N\})
\]
\[ \equiv 0 | \exp \left( i \sum_{p=1}^{N} e^{\{x^{(p)}\} \theta_1} \right) \exp \left( i \sum_{p=1}^{N} e^{\{x^{(p)}\} \theta_2} \right) \psi^{(1)}(\{1\}) \cdots \psi^{(1)}(\{N\}) | N > , \]

where \( \psi^{(1)}(\{i\}) \) is locally gauge invariant. We can rewrite \( \Phi_h \) as

\[ \Phi_h(\{1\}, \cdots, \{N\}) \]

\[ = C_{(\{m\}, \{n\})}(1, \cdots, N) < 0 | W_{(\tilde{m}, \tilde{n})}(\theta_1, \theta_2) \psi^{(3)}(\{1\}) \cdots \psi^{(3)}(\{N\}) | N > , \]

where \( C_{(\{m\}, \{n\})}(1, \cdots, N) \) is a phase factor. \( W_{(\tilde{m}, \tilde{n})}(\theta_1, \theta_2) \) is the Wilson loop operator defined as

\[ W_{(\tilde{m}, \tilde{n})}(\theta_1, \theta_2) \equiv \exp \left( i \tilde{m} e \theta_1 \right) \exp \left( i \tilde{n} e \theta_2 \right) , \]

with \( \tilde{m} = \sum_{i=1}^{N} m^{(i)} \) and \( \tilde{n} = \sum_{i=1}^{N} n^{(i)} \). The Schrödinger equation for \( \Phi_h \) looks the same as that of \( \Phi \) in Eq. (5.22). However, the Wilson loop will detect the vacuum degeneracy due to the zero-modes of the gauge fields independently of the matter sector. Therefore, the periodic condition can be non-trivial as given in Ref. [18] due to the Wilson loop and the wavefunction can behave as the multi-component one whose component is finite when \( \nu \) (related with the Chern-Simons coefficient \( \mu \)) is rational. It is not clear yet to us which choice of the definition of the wavefunction is physically relevant.

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