Grassmannians, Calibrations and Five-Brane Intersections

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ABSTRACT

We present a geometric construction of a new class of hyper-Kähler manifolds with torsion. This involves the superposition of the four-dimensional hyper-Kähler geometry with torsion associated with the NS-5-brane along quaternionic planes in $\mathbb{H}^k$. We find the moduli space of these geometries and show that it can be constructed using the bundle space of the canonical quaternionic line bundle over a quaternionic projective space. We also investigate several special cases which are associated with certain classes of quaternionic planes in $\mathbb{H}^k$. We then show that the eight-dimensional geometries we have found can be constructed using quaternionic calibrations. We generalize our construction to superpose the same four-dimensional hyper-Kähler geometry with torsion along complex planes in $\mathbb{C}^{2k}$. We find that the resulting geometry is Kähler with torsion. The moduli space of these geometries is also investigated. In addition the applications of these new geometries to M-theory and sigma models are presented. In particular, we find new solutions of IIA supergravity with the interpretation of intersecting NS-5-branes at $Sp(2)$-angles on a string and show that they preserve $3/32$, $1/8$, $5/32$ and $3/16$ of supersymmetry. We also show that two-dimensional sigma models with target spaces the above manifolds have $(p,q)$ extended supersymmetry.
1. Introduction

Much insight into the classical and quantum structure of field theories, strings and M-theory can be obtained by investigating the classical solutions of these systems. In particular, the study of superstring dualities and M-theory involves the study of configurations that preserve some spacetime supersymmetry. There is a large number of such configurations which can be put into two classes. The configurations of the first class are ‘elementary’. The other class contains configurations that are constructed by composing or superposing elementary ones. The elementary configurations typically preserve 1/2 of spacetime supersymmetry whereas the composed ones can preserve 1/2 or less.

In M-theory, the class of ‘elementary’ configurations is rather small but they lead to a large number of composite configurations. These can be constructed using a few simple rules. In particular, M-theory has two ‘elementary’ brane configurations: the membrane and the M-5-brane. Using the M-brane intersection rules, one is able to construct a large number of M-theory configurations that have the interpretation of intersecting M-branes [1]. These have found many applications in black holes (for a review see [2] and references within) and Yang-Mills theories [3]. Another application of the above set of ideas is to the construction of a large class of intersecting brane solutions of D=11 supergravity theory starting from the membrane and M-5-brane [1-7]. The first configurations that were constructed preserving some spacetime supersymmetry had the interpretation of orthogonal brane intersections. Later it was realized that non-orthogonal brane intersections can also preserve some spacetime supersymmetry [8-17]. Representing branes as planes embedded in a vector space, supersymmetry is preserved only if the embeddings are chosen in a particular way. In fact it was found in [18,19, 20, 21] that there is a correspondence between supersymmetric intersecting brane configurations and calibrations [22-24]. In this context the branes that are involved in an intersection must lie along calibrated planes.

In general, it appears that one can begin from supersymmetric or BPS-like
elementary configurations and construct many others by superposing them along particular planes in a higher-dimensional vector space. The properties of the resulting configuration depend on the planes that are chosen. A typical choice of planes are those associated with calibrations. These lead to composite configurations that preserve some spacetime supersymmetry or satisfy a BPS-like condition. These methods apply to various theories. For example, the relation between calibrations and Yang-Mills instanton-like configurations in various dimensions has been examined in [25]. Later in [26], a class of the $4k$-dimensional instanton solutions of [27] were interpreted as four-dimensional instantons superposed along quaternionic lines in $\mathbb{H}^k$. Another application is in the context of hyper-Kähler geometry. The $4k$-dimensional toric hyper-Kähler geometries constructed in [10] can be thought of as superpositions of four-dimensional Taub/NUT geometries along three-planes in $\mathbb{R}^{3k}$. Similarly, the $4k$-dimensional hyper-Kähler geometries with torsion (HKT) of [28] can be constructed as superpositions of the four-dimensional HKT geometry associated with the NS-5-branes along planes in $\mathbb{H}^k$. Both the toric hyper-Kähler and HKT geometries in $4k$ dimensions have found applications in the study of the moduli spaces of BPS monopoles [29,30] and five-dimensional black-holes [28], respectively.

In this paper, we shall investigate various superpositions of the geometry associated with the NS-5-brane. The non-trivial part of the metric of the NS-5-brane in the string frame defines a four-dimensional HKT geometry. We shall first superpose this four-dimensional HKT geometry using quaternionic maps $\tau$ from $\mathbb{H}^k$ into $\mathbb{H}$. This will lead to new HKT geometries in $4k$-dimensions which generalize those of [28]. We shall then describe several special cases by choosing different coefficients for the maps $\tau$. We shall show that the eight-dimensional HKT geometries can be constructed using quaternionic calibrations. In addition, we shall describe the moduli space of the HKT geometries. We shall show that it is constructed from the canonical bundle over the Grassmannian of quaternionic lines in $\mathbb{H}^k$. The eight-dimensional HKT geometries ($k = 2$) have already been used to construct solutions of IIA supergravity with the interpretation of NS-5-branes in-
tersecting on a string [10, 11]. Here we shall show that these solutions preserve the
fractions 3/32, 1/8, 5/32 and 3/16 of spacetime supersymmetry depending on the
choice of coefficients for the maps. Then we shall investigate other superpositions
of the four-dimensional HKT geometry associated with the NS-5-brane using holo-
morphic maps. This will lead to the construction of new Kähler geometries with
torsion (KT) in every even dimension. We shall also find the moduli space of these
KT geometries and discuss some of their applications in the context of supersym-
metric two-dimensional sigma models. In particular we find new two-dimensional
(2,0)-supersymmetric sigma models. We shall also construct new two-dimensional
sigma models with (4,2) and (2,2) supersymmetry.

This paper has been organized as follows: In section two, we give the con-
struction of 4k-dimensional HKT geometries. In section three, we investigate the
structure of their moduli space and explain the relation to calibrations. In section
four, we present the applications of these geometries to intersecting NS-5-branes.
In section five, we give the construction of new KT geometries in every even dimen-
sion and describe their moduli space. In section six, we present the application to
two-dimensional supersymmetric sigma models. In section seven, we describe some
other ways to superpose the four-dimensional HKT geometry and in section eight
we give our conclusions. Finally in appendix A, we construct five-brane solutions
that preserve less supersymmetry.

2. HKT geometries in 4k dimensions

2.1. HKT GEOMETRIES

A 4k-dimensional hyper-complex manifold $M$ with a tri-hermitian metric $g$
adsmits an HKT structure if the holonomy of one of the metric connections

$$\nabla^{(\pm)} = \nabla \pm H$$

(2.1)
is $Sp(k)$ with respect to the hyper-complex structure, where $\nabla$ is the Levi-Civita
connection and $H(X, Y, Z) = g(X, H(Y, Z))$ is a closed three-form on $M$. 

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For later use, we shall give the definition of a Kähler manifold with torsion (KT). Let \((M, g, H)\) be a hermitian manifold with metric \(g\), complex structure \(J\) and a closed 3-form \(H\). Then \(M\) admits a Kähler structure with torsion if the complex structure \(J\) is covariantly constant with respect to one of the connections \(\nabla^{(\pm)}\) defined as in (2.1).

Let \(\omega_1, \omega_2\) be the Kähler forms of the complex structures \(J_1, J_2\), i.e.

\[
\omega_1(X, Y) = g(X, J_1Y) \quad (2.2)
\]

and similarly for \(\omega_2\). The covariant constancy condition of the complex structures with respect to one of the connections \(\nabla^{(\pm)}\), say the \(\nabla^{(+)}\) one, can be replaced [31] by

\[
\begin{align*}
d\omega_1 - 2i_1H &= 0 \\
d\omega_2 - 2i_2H &= 0 \tag{2.3}
\end{align*}
\]

where \(i_1, i_2\) are the inner derivations with respect to the complex structures \(J_1\) and \(J_2\), respectively\(^*\). We remark that the two conditions in (2.3) imply a similar condition for the third complex structure \(J_3\). To find the associated conditions for the \(\nabla^{(-)}\) connection, we simply set \(H \rightarrow -H\) in (2.3).

An example of a four-dimensional HKT geometry is

\[
\begin{align*}
\frac{ds^2}{a} &= \frac{|dq|^2}{|q|^2} \\
\frac{a}{f_3} &= \frac{1}{3!} \text{Re} \left( \frac{d\bar{q} \land dq \land (d\bar{q}q - \bar{q}dq)}{2|q|^4} \right) \tag{2.4}
\end{align*}
\]

on \(\mathbb{H} - \{0\} = \mathbb{R} \times S^3\), where \(q \in \mathbb{H}\) is a quaternion\(^†\). The metric is complete and \(f_3\) is the volume form on \(S^3\). In fact this geometry admits two HKT structures. One

\(^*\) Our conventions for a p-form are \(\omega_p = \frac{1}{p!} \omega_{M_1, \ldots, M_p} dx^{M_1} \land \ldots \land dx^{M_p}\) and the action of the inner derivation of the complex structure \(J_1\) on \(dx^M\) is \(i_1(dx^M) = J_1^{MN} dx^N\).

\(^†\) This geometry is closely related to the NS-5-brane solution of IIA supergravity in the string frame.
is with respect to the pair \((\nabla^+, J_r)\), where the complex structures are defined by right-multiplication with the imaginary unit quaternions \(i, j, k\) as

\[
\begin{align*}
J_1 & : dq \rightarrow -dq i \\
J_2 & : dq \rightarrow -dq j \\
J_3 & : dq \rightarrow -dq k.
\end{align*}
\] (2.5)

The other HKT structure is with respect to the pair \((\nabla^-, I_r)\), where

\[
\begin{align*}
I_1 & : dq \rightarrow i dq \\
I_2 & : dq \rightarrow j dq \\
I_3 & : dq \rightarrow k dq.
\end{align*}
\] (2.6)

In what follows, we shall appropriately superpose this four-dimensional HKT geometry to construct new HKT and KT geometries. We shall then present some of the applications of these geometries in M-theory and supersymmetric sigma models.

### 2.2. HKT geometries and quaternionic maps

We consider the maps

\[
\tau : \mathbb{H}^k \rightarrow \mathbb{H}
\] (2.7)

such that

\[
q \equiv \tau(u) = p_i u^i - a
\] (2.8)

where \((u^1, \ldots, u^k) \in \mathbb{H}^k, q \in \mathbb{H}, \) and \(\{p_1, \ldots, p_k; a\}\) are quaternions that parameterize the map. This map has \(4k\) rotational \(\{p_1, \ldots, p_k\}\) and four translational \(\{a\}\) parameters.
The 4k-dimensional HKT geometry \((ds^2_{(4k)}, H)\) which arises from superposing the four-dimensional HKT geometry above, (2.4), is

\[
ds^2_{(4k)} = ds^2_\infty + \sum_{\tau} \mu(\tau)\tau^* d\tilde{s}^2,
\]

\[
H = \sum_{\tau} \mu(\tau)\tau^* f_3, \tag{2.9}
\]

where \(ds^2_\infty\) is a flat metric on \(\mathbb{H}^k\), \(\mu(\tau)\) are real constants and the sum is over different choices of the map \(\tau\). Writing the \(ds^2_{(4k)}\) and \(H\) explicitly, we have

\[
ds^2_{(4k)} = ds^2_\infty + \sum_{\{p,a\}} \mu(p,a) \frac{|p_i du^i|^2}{|p_i u^i - a|^4}.
\]

\[
H = \frac{1}{3!} \sum_{\{p,a\}} \mu(p,a) \text{Re} \left[ \frac{d\bar{u}^j \bar{p}_j \wedge p_\ell du^\ell \wedge (\bar{u}^m \bar{p}_m - \bar{a}) p_n du^n}{2|p_i u^i - a|^4} \right]. \tag{2.10}
\]

The geometry (2.9) on \(\mathbb{H}^k - \cup \tau^{-1}(0)\) admits an HKT structure with respect to the connection \(\Gamma^{(+)}\) and the complex structures

\[
J_1 : \quad du^i \rightarrow -du^i i
\]

\[
J_2 : \quad du^i \rightarrow -du^i j \tag{2.11}
\]

\[
J_3 : \quad du^i \rightarrow -du^i k.
\]

To show this, we first observe that \(H\) in (2.9) is a closed three-form as required. Also, \(J_1, J_2, J_3\) are integrable complex structures since they are constant. A straightforward computation reveals that the metric (2.9) is hermitian with respect to \(J_1, J_2, J_3\) provided that \(ds^2_\infty\) is chosen to be hermitian with respect to these complex structures. We remark that it is always possible to find such a flat metric on \(\mathbb{H}^k\). It remains to show that (2.9) satisfies condition (2.3) with respect to the complex structures (2.11). For this, we observe that

\[
d\tau J_1 = J_1 d\tau
\]

\[
d\tau J_2 = J_2 d\tau \tag{2.12}
\]

\[
d\tau J_3 = J_3 d\tau.
\]
Next we remark that (2.3) is linear in the metric and torsion. Using the commutativity of the complex structures with $d\tau$, (2.3) can be written as

$$\sum_{\tau} \tau^*(d\omega_1 - 2i_1f_3) = 0 \quad (2.13)$$

for the first complex structure, where $\omega_1$ is the Kähler form of the complex structure $J_1$ with respect to the metric $ds^2$ in (2.4), and similarly for the other two complex structures. However, each part of the sum over $\tau$ vanishes since the geometry that we are pulling-back from $\mathbb{H} - \{0\}$ is an HKT geometry. Thus (2.3) is also satisfied, and so (2.9) admits an HKT structure.

2.3. Special Cases

We shall consider the following three special cases.

(i) : We introduce another triplet of complex structures on $\mathbb{H}^k$ as follows:

$$I_1 : \quad du^i \to i \, du^i$$
$$I_2 : \quad du^i \to j \, du^i$$
$$I_3 : \quad du^i \to k \, du^i \quad (2.14)$$

For generic choices of parameters $\{p_1, \ldots, p_k\}$ the maps $\tau$ do not commute with the complex structures $I_1, I_2$ and $I_3$. However if the parameters $\{p_1, \ldots, p_k\}$ of $\tau$ are real numbers, i.e.

$$\quad (p_1, \ldots, p_k) \in \mathbb{R}^k \quad (2.15)$$

instead of quaternions, then $d\tau$ also commutes with the pairs of complex structures $(I_1, I_1), (I_2, I_2)$ and $(I_3, I_3)$. As a result repeating the argument mentioned above, (2.9) admits another HKT structure with respect to the pair $(\nabla^r, I_r)$ provided that the asymptotic metric $ds^2_\infty$ is chosen to be hermitian with respect to all complex structures*. These geometries with two HKT structures have already been investigated in [10,28].

* There is always such a metric on $\mathbb{H}^k$. 

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Another special case is to choose the parameters \( \{p_1, \ldots, p_k\} \) of the maps \( \tau \) such that \( d\tau \) commute with only one of the pairs of complex structures \( \{(I_r, I_r)\} \), say \( (I_1, I_1) \). This implies that the parameters are \textit{complex} numbers, i.e.

\[
(p_1, \ldots, p_k) \in \mathbb{C}^k,
\]

(2.16)

and so

\[
p_i = a_i + ib_i
\]

where \( \{a_1, \ldots, a_k; b_1, \ldots, b_k\} \) are real numbers. For such a choice of maps, (2.9) admits an HKT structure with respect to \((\nabla^+, J_r)\). In addition, (2.9) admits a Kähler structure with torsion with respect to \((\nabla^-, I_1)\), provided the asymptotic metric \( ds_\infty^2 \) is chosen to be hermitian with respect to \( J_1, J_2, J_3 \) and \( I_1 \). Therefore the holonomy of \( \nabla^- \) is a subgroup of \( U(2k) \). In fact, we shall show later that the holonomy of \( \nabla^- \) is a subgroup of \( SU(2k) \). The proof of this statement will be described for \( k = 2 \) when we give the applications of these geometries to strings but it generalizes trivially to any even dimension.

For the third special case, we take \( k = 2 \). The rotational parameters of the maps \( \tau \) are two quaternions \( \{p_1, p_2\} \). The restriction on the rotational parameters is

\[
\text{Re}(\bar{p}_1 p_2) = 0,
\]

(2.18)
i.e. \( \bar{p}_1 p_2 \in \text{Im} \mathbb{H} \). The resulting eight-dimensional geometry admits an HKT structure with respect to \((\nabla^+, J_r)\) as in all the above cases. In addition, it turns out that the holonomy of \( \nabla^- \) is \( \text{Spin}(7) \) provided \( ds_\infty^2 \) is chosen appropriately. For example one can choose \( ds_\infty^2 \) to be the Euclidean metric, but more general asymptotic metrics are possible.

The reason for the restriction (2.18) on \( \tau \) will become clear later. In particular, using an equivalence relation on the space of parameters of \( \tau \) that we shall give
in the next section, it is easy to see that the previous two conditions (2.15) and (2.17) can be rewritten as

\[ \bar{p}_1 p_2 \in \mathbb{R} \]  

and

\[ \bar{p}_1 p_2 \in \mathbb{C} , \]  

respectively. We remark that other similar conditions to that of (2.18) can be imposed on the rotational parameters of \( \tau \). For example, we can set

\[ \text{Re}(i\bar{p}_1 p_2) = 0 , \]

and similarly for \( j \) and \( k \). However, the properties of HKT geometries that result from these conditions are similar to those associated with (2.18) and we shall not investigate them further here.

3. Grassmannians and Calibrations

3.1. The Moduli Space

The HKT geometry (2.10) is determined solely by the arrangement of quaternionic \((k-1)\)-planes in \( \mathbb{H}^k \) given by the kernels of the maps \( \tau \). To see this, we note that the HKT geometries of section (2.2) are invariant on replacing each map \( \tau \) with parameters \( \{p_1, p_2, \ldots, p_k; a\} \) with another map \( \hat{\tau} \) with parameters \( \{sp_1, sp_2, \ldots, sp_k; sa\} \) where \( s \in \mathbb{H} - 0 \). This is because both the metric and torsion of the HKT geometry are invariant under a rescaling of the parameters of the maps with a quaternion from the left. Therefore, we define an equivalence relation
on the space of maps $\tau$, so that $\hat{\tau} \sim \tau$ i.e.

$$(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_k; \hat{a}) \sim (p_1, p_2, \ldots, p_k; a)$$ (3.1)

if there exists an $s \in \mathbb{H} - 0$ for which $\hat{\tau} = s\tau$, i.e.

$$(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_k; \hat{a}) = (sp_1, sp_2, \ldots, sp_k; sa) .$$ (3.2)

The space of equivalence classes of maps is the bundle space $E(\gamma_1^{k-1})$ of the canonical quaternionic line bundle $\gamma_1^{k-1}$ over the Grassmannian

$$Gr(1, \mathbb{H}^k) = Sp(k)/Sp(1) \times Sp(k-1) .$$ (3.3)

The fiber directions of $\gamma_1^{k-1}$ are associated with the translation parameters whereas the rest are associated with the rotational parameters of the maps. In particular, $a$ is the fibre coordinate of $\gamma_1^{k-1}$ and $(p_1, \ldots, p_k)$ are the homogeneous coordinates of $Gr(1, \mathbb{H}^k)$. Observe that the maps that lie within the same equivalence class have the same kernel.

The moduli space, $\mathcal{M}_N(\mathbb{H})$, of HKT geometries associated with $N$ distinct maps $\tau$, i.e. maps that are not equivalent in the sense of (3.1), is

$$\mathcal{M}_N(\mathbb{H}) = D_N \times \left( \times^N E(\gamma_1^{k-1}) - \Delta \right) / S_N ,$$ (3.4)

where $D_N$ is a domain in $\mathbb{R}^N$ which parameterizes the scale factor of each term in the sum for the metric (and torsion), $\Delta$ is a diagonal term in $\times^N E(\gamma_1^{k-1})$ and $S_N$ is the permutation group of $N$ points. $D_N$ includes the $(\mathbb{R}^+)^N$ subspace of $\mathbb{R}^N$.

The moduli space, $\tilde{\mathcal{M}}_N(\mathbb{H})$, of rotation and translation parameters of such HKT geometry is

$$\tilde{\mathcal{M}}_N(\mathbb{H}) = \frac{\times^N E(\gamma_1^{k-1}) - \Delta}{S_N} .$$ (3.5)

We remark that these moduli spaces are different from the ones that arise in the study of extreme black holes. The moduli space of $N$-indistinguishable black
holes is isomorphic to the configuration space of N-indistinguishable particles. The coordinates of the moduli space of extreme black holes specify their location and in our terminology consist of translational parameters. However, as we have seen, the moduli spaces of our configurations contain also rotational parameters.

The simplest case to consider is \( N = 2 \) and \( k = 2 \). The diagonal term in this case is

\[
\Delta = \{ (x_1, x_2) \in \times^2 E(\gamma_1^1) : x_1 = x_2 \}
\]  

(3.6)

It is clear that if \( x_1 = x_2 \), then the metric degenerates, ignoring the asymptotic part, to the four-dimensional HKT one.

It remains to investigate the moduli of these geometries in the three special cases of section (2.3).

\( (i) \) If the rotational parameters of the maps \( \tau \) are real numbers, then the equivalence relation is as in (3.1) and (3.2) but \( s \) in this case is a real number. Therefore, the space of parameters is the bundle space \( E(\oplus^4 \zeta_{\frac{k-1}{1}}) \), where \( \zeta_{\frac{k-1}{1}} \) is the canonical real line bundle over the real projective space

\[
\mathbb{R}P^{k-1} = \frac{SO(k)}{\mathbb{Z}_2 \times SO(k - 1)} = \frac{S^{k-1}}{\mathbb{Z}_2}.
\]  

(3.7)

The moduli space, \( \mathcal{M}_N(\mathbb{R}) \), of HKT geometries associated with \( N \) distinct maps \( \tau \) with real rotational parameters is

\[
\mathcal{M}_N(\mathbb{R}) = \frac{D_N \times (\times^N E(\oplus^4 \zeta_{\frac{k-1}{1}}) - \Delta)}{S_N},
\]  

(3.8)

where \( D_N \), \( \Delta \) and \( S_N \) are defined as in (3.4).

\( (ii) \) : Now, if the rotational parameters of the maps are complex numbers, the space of parameters of \( \tau \) is the bundle space \( E(\zeta^{k-1}_{1} \oplus \zeta^{k-1}_{1}) \), where \( \zeta^{k-1}_{1} \) is the
canonical complex line bundle over the complex projective space

\[ \mathbb{C}P^{k-1} = \frac{U(k)}{U(1) \times U(k-1)}. \] (3.9)

The moduli space, \( \mathcal{M}_N(\mathbb{C}) \), of HKT geometries associated with \( N \) distinct maps \( \tau \) is

\[ \mathcal{M}_N(\mathbb{C}) = \frac{D_N \times \left( \times^N E(\xi^{k-1}_1 \oplus \xi^{k-1}_1) - \Delta \right)}{S_N} . \] (3.10)

(iii) : To investigate the moduli space for the third case in section (2.3), we observe that the relation \( \text{Re} \bar{p}_1 p_2 = 0 \) is invariant under the equivalence relation (3.1). Then it is easy to show that the space of equivalence classes of linear maps is isomorphic to the bundle space \( E \) of a quaternionic line bundle \( \eta \) over \( S^3 \). The moduli space, \( \mathcal{M}_N(\text{Im}\mathbb{H}) \), of HKT geometries associated with \( N \) distinct maps \( \tau \) is then

\[ \mathcal{M}_N(\text{Im}\mathbb{H}) = \frac{D_N \times \left( \times^N E(\eta) - \Delta \right)}{S_N} . \] (3.11)

3.2. Calibrations and HKT Geometries

In this section we shall see that the new \( k = 2 \) HKT geometries that we have constructed above are related to calibrations in eight dimensions. We shall find that they are “calibrated” geometries in the sense that they are superpositions of a model four-dimensional geometry along planes in the contact set of some calibration. Below we give a short summary of some of the main properties of calibrations that we shall use here. For a detailed account of calibrations and their main application to the construction of minimal surfaces we refer the reader to [22, 23].

A degree \( \ell \) calibration in \( \mathbb{R}^n \) is associated with a certain closed \( \ell \)-form \( \phi \) defined on \( \mathbb{R}^n \). In most applications \( \phi \) is a constant form on \( \mathbb{R}^n \). In addition, \( \phi \) is chosen such that if the co-volume form \( \zeta \) of any \( \ell \)-plane is evaluated on \( \phi \), then \( \phi(\zeta) \leq 1 \).
We say that an $\ell$-plane with co-volume form $\zeta$ is calibrated by $\phi$, if $\phi(\zeta) = 1$. The set of calibrated planes of $\phi$ is a subset of the Grassmannian of oriented $\ell$-planes in $\mathbb{R}^n$, $Gr(\ell, \mathbb{R}^n)$. We shall refer to the set of calibrated planes of $\phi$ as the contact set of the calibration and denote it with $G_\phi$. In most cases, the contact set is a homogeneous space $G/H$.

A large class of calibrations in $\mathbb{R}^8$ were investigated in [24]. Here we shall use those associated with constant self-dual forms in $\mathbb{R}^8$. We begin by choosing on $\mathbb{R}^8$ the hyper-complex $\{J_1, J_2, J_3\}$ of section two. Let $\{\omega_{J_1}, \omega_{J_2}, \omega_{J_3}\}$ be the associated Kähler forms with respect to the flat metric. A calibrating 4-form is

$$\Phi_J = \frac{1}{6}(\omega_{J_1}^2 + \omega_{J_2}^2 + \omega_{J_3}^2),$$

with contact set the grassmannian

$$Gr(1, \mathbb{H}^2) = Sp(2)/Sp(1) \times Sp(1) = S^4,$$

where $\omega_{J_1}^2 = \omega_{J_1} \wedge \omega_{J_1}$ and similarly for $J_2$ and $J_3$. Note that $\Phi_J$ is the $Sp(2) \cdot Sp(1)$-invariant quaternionic four-form associated with $\{J_1, J_2, J_3\}$.

There are special cases of the above calibration for which the contact set is a subspace of (3.13):

(i) : Let $\Phi_I$ be the quaternionic four-form associated with $\{I_1, I_2, I_3\}$ of section two. A calibration form is then

$$\Theta = \frac{1}{2}(\Phi_I + \Phi_J),$$

with contact set given by

$$G_\Theta = S^1.$$

(ii) : Let $\omega_{I_1}$ be the Kähler form of the $I_1$ complex structure of section two
with respect to the flat metric. A calibration form is then

$$\Lambda = \frac{1}{5} \omega_{\mathbf{I} i} + \frac{3}{5} \Phi_J ,$$

(3.16)

and the contact set is

$$G_{\Lambda} = SU(2)/S(U(1) \times U(1)) = S^2 .$$

(3.17)

(iii) : Let \( \Omega \) be a certain self-dual \( Spin(7) \)-invariant four-form. A calibration form is

$$\Psi = \frac{1}{4} \Omega + \frac{3}{4} \Phi_J ,$$

(3.18)

with contact set given by

$$G_{\Psi} = \frac{Sp(1) \times Sp(1)}{Sp(1)} = S^3 .$$

(3.19)

We proceed now to establish the correspondence of our HKT geometries with the above calibrations. As shown in the previous section our HKT geometries are determined by the maps \( \tau \). The kernels of the maps \( \tau \) are

$$p_i u^i - a = 0 ,$$

(3.20)

where \( i = 1, 2 \). We find that if the rotational parameters \( \{ p_1, p_2 \} \) are quaternions, then the kernels of maps \( \tau \) are calibrated by \( \Phi_J \). To see this, we observe that their tangent spaces are stabilized by the action of \( J_1, J_2 \) and hence also by \( J_3 \), and so they are quaternionic lines in \( \mathbb{H}^2 \). Thus they are calibrated by \( \Phi_J \).

Let us investigate now the special cases where we restrict the rotational parameters to be (i) real, (ii) complex and (iii) satisfy \( Re(\bar{p}_1 p_2) = 0 \).

(i) : If the rotational parameters are real numbers, then the tangent spaces of the kernels of \( \tau \) are stabilized by the additional hyper-complex structure \( \{ I_1, I_2, I_3 \} \). So they are calibrated simultaneously by \( \Phi_J \) and \( \Phi_I \), and thus also by \( \Theta \).
(ii) : Choosing, as in section (2.3), $p_i = a_i + ib_i$, we observe that the tangent spaces of the kernels of $\tau$ are stabilized by $I_1$. So they are calibrated simultaneously by $\Phi_J$ and $\frac{1}{2}\omega_{I_1}^2$, and thus also by $\Lambda$.

(iii) : Restricting, as in section (2.3), $\text{Re}(\bar{p}_1 p_2) = 0$, we find that the kernels of $\tau$ are calibrated by the self-dual $\text{Spin}(7)$-invariant form $\Omega$ with non-vanishing components

$$
\Omega_{1234} = \Omega_{1265} = \Omega_{1287} = \Omega_{1375} = \Omega_{1476} = \Omega_{1368} = \Omega_{1485} = 1.
$$

To see this, we evaluate the co-volume form $\eta$ of the kernels on $\Psi$ and find

$$
\Psi(\eta) = \frac{(\bar{p}_1 p_1 + \bar{p}_2 p_2)^2 - 2(\text{Re}(\bar{p}_1 p_2))^2}{(\bar{p}_1 p_1 + \bar{p}_2 p_2)^2}.
$$

So $\Psi(\eta) = 1$, iff $\text{Re}(\bar{p}_1 p_2) = 0$. It then follows that the kernels of the maps $\tau$ are also calibrated by $\Omega$ iff $\text{Re}(\bar{p}_1 p_2) = 0$.

In all the above cases, the only planes that are calibrated by the corresponding four-forms are those given by the associated maps $\tau$ (see [24]). This explains why the moduli spaces of the eight-dimensional HKT geometries can be constructed using projective spaces which are isomorphic to the contact sets of the above calibrations. It would be of interest to extend this result to $4k, k > 2$, dimensions.

3.3. Angles

The asymptotic angles between the planes defined by the kernels of two maps $\rho$ and $\tau$ can be computed using the normal vectors $n$ and $m$ of these planes, respectively*. These are given by

$$
\langle n, X \rangle = d\rho(X)
$$

$$
\langle m, X \rangle = d\tau(X)
$$

where the inner product $\langle \cdot, \cdot \rangle$ is with respect to the metric at infinity. The

* Each plane has four such normal vectors.
angles-matrix is then defined as

\[ A = \frac{\langle n, m \rangle}{|n||m|} \] (3.24)

where \( |n|^2 = \langle n, n \rangle \) and similarly for \( |m| \).

Choosing as asymptotic metric the flat metric on \( \mathbb{H}^k - \cup \tau^{-1}(0) \), we find that

\[ A = \frac{p_i \bar{q}_j \delta^{ij}}{\sqrt{\delta^{ij}} p_i \bar{p}_j \sqrt{\delta^{kl}} q_k \bar{q}_l} \] (3.25)

where \( \{p_1, \ldots, p_k\} \) and \( \{q_1, \ldots, q_k\} \) are the rotational parameters of the maps \( \rho \) and \( \tau \), respectively. The explicit expression of the angles-matrix for a general asymptotic metric is a straightforward generalization of that given in [11] for the intersecting five-branes and so it will not be presented here.

The angles-matrix \( A \) depends on the choice of representative in the equivalence relation (3.1). In particular, let \( \rho \to e \rho \) and \( \tau \to \ell \tau \). Then

\[ A \to \frac{e}{|e|} A \frac{\bar{\ell}}{|\ell|}, \] (3.26)

where \( e, \ell \in \mathbb{H} \). Therefore it depends on some of the data of the parameterization of the maps \( \tau \) unlike the underlying HKT geometry which depends only on the planes defined by the kernels of these maps.
4. Applications

4.1. Intersecting IIA five-branes on a string

The eight-dimensional HKT geometries found in the previous section admit a brane interpretation and lead to the supergravity solutions constructed in [11]. To relate the two, we express the quaternionic coordinates \( \{ u^i; i = 1, 2 \} \) used in the previous section in terms of the real coordinates \( \{ x^{i \mu}; i = 1, 2; \mu = 0, 1, 2, 3 \} \) used in [11] as \( u^i = x^{i0} + ix^{i1} + jx^{i2} + kx^{i3} \). Let \((ds^2_{(8)}, H_{(8)})\) be the metric and closed three-form of the HKT geometry. The IIA supergravity solution in the string frame is

\[
\begin{align*}
   ds^2 &= ds^2(E^{(1,1)}) + ds^2_{(8)} \\
   H &= H_{(8)} \\
   e^{8\phi} &= g_{(8)},
\end{align*}
\]

where \( ds^2 \) is the ten-dimensional supergravity metric, \( \phi \) is the dilaton, \( H \) is the \( \text{NS} \otimes \text{NS} \) three-form field strength and \( g_{(8)} \) is the determinant of the HKT metric \( ds^2_{(8)} \). This solution has the interpretation of NS-5-branes intersecting on a string. The positions of the NS-5-branes are determined by the kernels of the maps \( \tau \).

We proceed to determine the proportion of spacetime supersymmetry preserved by the above solution. Let \( \Gamma \) be the Levi-Civita connection of the supergravity metric. The IIA Killing spinor equations in the string frame are

\[
\begin{align*}
   \nabla^{(+)\ M}\epsilon &= 0, \\
   \nabla^{(-)\ M}\eta &= 0 \\
   (\Gamma^{M}\partial_M\phi - \frac{1}{6}H_{MNP}\Gamma^{MNP})\epsilon &= 0, \\
   (\Gamma^{M}\partial_M\phi + \frac{1}{6}H_{MNP}\Gamma^{MNP})\eta &= 0,
\end{align*}
\]

where \( \epsilon \) and \( \eta \) are the chiral and anti-chiral Majorana-Weyl Killing spinors, respec-
tively, $\nabla^{(\pm)}$ are the covariant derivatives of the connection

$$\Gamma^{(\pm)MNP} = \Gamma_{NP}^M \pm H_{NP}^M , \quad (4.3)$$

and $M, N, P = 0, \ldots , 9$. It is clear that the first two Killing spinor equations have solutions provided that the decompositions of the Majorana-Weyl (chiral and anti-chiral) representations of $\text{Spin}(1,9)$ have singlets under the holonomy groups of the connections $\nabla^{(\pm)}$. The number of linearly independent spinors determines the proportion of supersymmetry preserved by the solution.

The holonomies of the $\nabla^{(\pm)}$ connections of the solution (4.1) are those of the associated connections of the HKT geometry in (4.1). It follows from the results in [10] that the holonomy of the connection $\nabla^{(+)}$ is exactly $\text{Sp}(2)$ for a generic choice of maps $\tau$. Moreover, we have verified by an explicit computation of the curvature tensor, that for a generic choice of maps $\tau$, the holonomy group of $\nabla^{(-)}$ is exactly $\text{SO}(8)$. Therefore, the only solution of the second Killing spinor equation is the trivial one, i.e. $\eta = 0$. With the above choice of dilaton the third Killing spinor equation is satisfied without any additional conditions. In fact, the first Killing spinor equation implies the third provided the holonomy of the connection $\nabla^{(+)}$ is a subgroup of $\text{SU}(4)$, [32, 33]. In our context the condition for $SU(4)$ holonomy can be expressed, in a coordinate system relative to which the associated complex structure is constant, as a condition on the metric,

$$g^{ab} \partial_a g_{bc} = \frac{1}{4} g^{ab} \partial_c g_{ab} , \quad (4.4)$$

where $a, b, c = 1, \ldots , 8$. This is precisely the condition needed to satisfy the third Killing spinor equation.

To determine the proportion of supersymmetry preserved by the solution, we have to find the decomposition of the Majorana-Weyl representation of $\text{Spin}(9,1)$ under the holonomy group of $\nabla^{(+)}$. We first decompose the Majorana-Weyl rep-
presentation of Cliff(9,1) under the Spin(8) subgroup of Spin(9,1) as

\[ 16 \rightarrow 8_s \oplus 8_c , \]  

(4.5)

where \(8_s\) is the spinor representation of Spin(8) and \(8_c\) is its conjugate. Next we decompose \(8_s\) and \(8_c\) under \(Sp(2) \subset \text{Spin}(8)\) as

\[ 8_s \rightarrow 5 \oplus 1 \oplus 1 \oplus 1 \]
\[ 8_c \rightarrow 4 \oplus 4 . \]

(4.6)

Note that there are three singlets in the decomposition of \(8_s\). Hence, we conclude that our solution has three Killing spinors and therefore it preserves 3/32 of supersymmetry.

4.2. Special Cases

There are three special cases of intersecting five-brane configurations to consider. These are associated with eight-dimensional HKT geometries for which the rotational parameters \(\{p_1, p_2\}\) of the maps \(\tau\) are (i) real, (ii) complex and (iii) satisfy \(\text{Re}(\bar{p}_1 p_2) = 0\). In case (i), the solution has already been investigated in [10]. It has been found that the configuration admits six Killing spinors and therefore preserves 3/16 of spacetime supersymmetry. This analysis will not be repeated here.

Next let us take the rotational parameters of the maps to be complex numbers. The analysis of the first and third Killing spinor equations in (4.2) is as before. This implies that our solutions admit three Killing spinors associated with the Majorana-Weyl (sixteen component) supersymmetry parameter \(\epsilon\). The analysis of the second and fourth Killing spinor equations associated with the other Majorana-Weyl supersymmetry parameter \(\eta\) of opposite chirality is different. For this, we note that these HKT geometries also admit a KT structure with respect to the pair \((\nabla^{(-)}, I_1)\). This implies that the holonomy of the connection \(\nabla^{(-)}\) is a subgroup of
In fact, it turns out that it is a subgroup of $SU(4)$. One way to show this is to verify that the curvature $R^{(-)}$ of $\nabla^{(-)}$ takes values in the Lie algebra of $SU(4)$. However since the complex structure $I_1$ is constant in the natural coordinate system that we have introduced, this condition on the curvature is implied by

$$\text{Tr}(\Gamma^{(-)}_a I_1) = 0.$$  \hspace{1cm} (4.7)

After some computation, one can verify that (4.7) holds using (4.4). The fourth Killing spinor equation then follows without further restrictions.

To determine the proportion of supersymmetry preserved by the solution, we have to find the decomposition of the Majorana-Weyl representation of $\text{Spin}(9,1)$ under the holonomy group $SU(4)$ of $\nabla^{(-)}$. We first use the decomposition $16 \rightarrow 8_s \oplus 8_c$ of the Majorana-Weyl representation of $\text{Spin}(1,9)$ under $\text{Spin}(8)$ as in (4.5). Next we decompose $8_s$ and $8_c$ under $SU(4)$ as

$$8_s \rightarrow 6 \oplus 1 \oplus 1,$$

$$8_c \rightarrow 4 \oplus \bar{4}.$$  \hspace{1cm} (4.8)

Hence, we conclude that our solution has two additional Killing spinors associated with the supersymmetry parameter $\eta$. Therefore, the IIA intersecting five-brane configurations given by the HKT geometries with complex rotational parameters have five Killing spinors and therefore preserve $5/32$ of spacetime supersymmetry.

It remains to investigate the intersecting five-brane configurations associated with an eight-dimensional HKT metric for which the rotational parameters of the maps satisfy $\text{Re}(\bar{p}_1 p_2) = 0$. The holonomy of the $\nabla^{(+)}$ connection is $Sp(2)$ as in the previous cases. The holonomy of $\nabla^{(-)}$ is $\text{Spin}(7)$ provided that $ds^2$ is chosen appropriately. For this, we have verified after some computation that

$$\frac{1}{2} \Omega_{\hat{a} \hat{b}} \hat{c} \hat{d} R^{(-)}_{\hat{c} \hat{d}} = -R^{(-)}_{\hat{a} \hat{b}},$$  \hspace{1cm} (4.9)

where $\Omega$ is the $\text{Spin}(7)$ invariant form given in section (3.2) and $\hat{a}, \hat{b}, \hat{c}, \hat{d} = 1, \ldots, 8$ are tangent space indices. As remarked previously we can choose an Euclidean
asymptotic metric on the eight-dimensional space transverse to the string, although more general flat tri-hermitian (relative to $J_1, J_2, J_3$) asymptotic metrics are possible. We shall not discuss this further here. Under the decomposition of the Majorana-Weyl representation of Spin(9, 1) under $Spin(7)$, we find an additional singlet. Furthermore, the last Killing spinor equation in (4.2), i.e. the dilatino equation associated with the $\nabla^-$ connection, is satisfied for this spinor. This follows from

$$\partial_d \phi = -\frac{1}{6} \Omega^{\hat{a}\hat{b}\hat{c}} \partial_d H_{\hat{a}\hat{b}\hat{c}}$$

which can be verified after some careful computation. In the above $H_{\hat{a}\hat{b}\hat{c}}$ correspond to the tangent frame components of the NS 3-form field strength. We thus conclude that this solution preserves $1/8$ of supersymmetry.

We summarize some of our results of the previous two sections in the table below.

| $\bar{p}_1 p_2$ | $\nabla^{(+)}$ | $\nabla^{(-)}$ | Susy | Contact Set |
|-----------------|----------------|----------------|------|-------------|
| $\mathbb{R}$    | $Sp(2)$        | $Sp(2)$        | $\frac{3}{16}$ | $S^1$        |
| $\mathbb{C}$    | $Sp(2)$        | $SU(4)$        | $\frac{5}{32}$ | $S^2$        |
| Im$\mathbb{H}$  | $Sp(2)$        | $Spin(7)$      | $\frac{1}{8}$  | $S^3$        |
| $\mathbb{H}$    | $Sp(2)$        | $SO(8)$        | $\frac{3}{32}$ | $S^4$        |

**Table 1: Intersecting NS-5-branes on a string** This table contains (i) the type of rotational parameters $\{p_1, p_2\}$ of the eight-dimensional HKT geometries, (ii) and (iii) the holonomy of the associated $\nabla^{(+)}$ and $\nabla^{(-)}$ connections, respectively, (iv) the fraction of supersymmetry preserved by the corresponding intersecting NS-5-brane configurations and (v) the contact sets of the associated calibrations.
5. KT geometries in diverse dimensions

5.1. Holomorphic maps and KT structures

New KT geometries can be constructed by superposing the four-dimensional HKT geometry (2.4) using holomorphic maps. Let \( J \) be one of the complex structures on \( E_4 \) associated with an anti-self dual two-form. Using \( J \), we identify \( E_4 \) with \( \mathbb{C}^2 \). The HKT geometry in (2.4) is now rewritten as

\[
\frac{d s^2}{|z|^2} = \left| \frac{dz}{|z|^2} \right|^2 \delta_{\alpha \bar{\beta}} \delta_{\gamma \bar{\epsilon}} z^\alpha z^\beta \bar{z}^\gamma \bar{z}^\bar{\epsilon} + c.c.
\]

where \( \{z^1, z^2\} \) are the complex coordinates on \( \mathbb{C}^2 \).

To proceed, we identify \( E_{4k} = \mathbb{C}^{2k} \) and introduce the holomorphic maps \( \tau \): \( \mathbb{C}^{2k} \to \mathbb{C}^2 \)

\[
\tau : \mathbb{C}^{2k} \to \mathbb{C}^2
\]

such that

\[
z^\alpha \equiv \tau^\alpha (w) = p^{\alpha \beta} w^\beta - a^\alpha
\]

where \( \{w^\beta; i = 1, \ldots, k; \alpha = 1, 2\} \) are holomorphic coordinates of \( \mathbb{C}^{2k} \) and \( \{p_1, \ldots, p_k; a\} \) are the complex parameters of the map \( \tau \). These maps \( \tau \) have \( 8k \) rotational parameters \( \{p_1, \ldots, p_k\} \) and four translational parameters \( \{a^\alpha; \alpha = 1, 2\} \). Following the construction of \( 4k \)-dimensional HKT geometries in section two, we find a new geometry on \( \mathbb{C}^{2k} - \cup \tau^{-1}(0) \) by pulling-back (5.1) with generic maps \( \tau \) and then summing up over the different choices of \( \tau \). The resulting geometry is

\[
ds^2 = ds^2_{\infty} + \sum_{\{p,a\}} \mu(\{p, a\}) \left| \frac{d(pw)}{|pw - a|^2} \right|^2
\]

\[
H = \sum_{\{p,a\}} \mu(\{p, a\}) \left[ \frac{\delta_{\alpha \rho} \delta_{\beta \gamma} \bar{p}^{\rho \mu} \bar{w}^\mu \bar{p}^{\beta \sigma} \bar{p}^{\gamma \bar{\epsilon}} dw^\nu \wedge dw^{k \sigma} \wedge dw^{\bar{\epsilon} \bar{\tau}}}{|pw - a|^4} \right] + c.c.,
\]

where \( ds^2_{\infty} \) is a constant metric on \( \mathbb{C}^{2k} \).
It turns out that (5.4) admits a KT structure. To see this, we introduce a complex structure $J$ on $\mathbb{C}^{2k} - \cup_t \tau^{-1}(0)$ as

$$J : dw^i \alpha \rightarrow dw^{i\alpha}.$$  

(5.5)

This complex structure is integrable since it is constant. The metric in (5.4) is hermitian with respect to $J$ provided that $ds^2_\infty$ is also hermitian relative to this complex structure*. So it remains to show that (5.4) satisfies (2.3) with respect to $J$. The proof of this is similar to that given for the HKT case in section two. The key point is that the map $\tau$ is holomorphic and so $d\tau$ commutes with the complex structures $J$ and $J$ on $\mathbb{C}^2$ and $\mathbb{C}^{2k}$, respectively.

As we have seen, the method described above has led to the construction of KT geometries in $4k$ dimensions. A simple modification of this method can also lead to new KT geometries in all even dimensions. For this, we choose maps $\tau : \mathbb{C}^{2k} \rightarrow \mathbb{C}^2$ which restrict to a subspace $V$ of $\mathbb{C}^{2k}$, i.e.

$$\tau : V \subset \mathbb{C}^{2k} \rightarrow \mathbb{C}^2.$$  

(5.6)

The vector space $V$ is complex with complex structure $J_V$ induced by $J$ of $\mathbb{C}^{2k}$. Repeating the procedure for constructing KT geometries above, we find a new KT geometry on $V - \cup_t \tau^{-1}(0)$. In particular, $Jd\tau = d\tau J_V$ is satisfied because $Jd\tau = d\tau J$. Since we are allowed to choose any complex subspace $V$ of $\mathbb{C}^{2k}$ and so subspaces with dimensions which are not multiple of four, this construction will give new KT geometries in all even dimensions.

* It is always possible to choose $ds^2_\infty$ to be hermitian with respect to a complex structure.
5.2. Moduli space and angles

The maps $\tau$ used in the previous section for the construction of KT geometries are parameterized by $\{p_1, \ldots, p_k; a\}$, where $\{p_1, \ldots, p_k\}$ are $2 \times 2$ complex matrices and $\{a\}$ is a complex two-vector. Let $G$ be the group of $2 \times 2$ complex matrices such that

$$g^\dagger g = r \mathbf{1}$$

(5.7)

for some $r \in \mathbb{R} - 0$. Then, it is straightforward to see that the maps $\tau$ and $\hat{\tau}$ with parameters $\{p_1, \ldots, p_k; a\}$ and $\{\hat{p}_1, \ldots, \hat{p}_k; \hat{a}\}$, respectively, that satisfy

$$(\hat{p}_1, \ldots, \hat{p}_k; \hat{a}) = (g p_1, \ldots, g p_k; g a),$$

(5.8)

where $g \in G$, lead to the same KT geometry. So we define the equivalence relation on the space of maps

$$\tau \sim \hat{\tau} ,$$

(5.9)

iff there is $g \in G$ such that their parameters are related as in (5.8) or simply $\hat{\tau} = g \tau$. The group $G$ is isomorphic to $(\mathbb{R} - \{0\}) \times U(2)$. To see this, let us first take $r > 0$. In this case,

$$\Phi : G \to \mathbb{R}^+ \times U(2)$$

$$g \to (|\det g|^\frac{1}{2}, |\det g|^{-\frac{1}{2}} g)$$

(5.10)

with inverse

$$\Phi^{-1} : \mathbb{R}^+ \times U(2) \to G$$

$$(r, U) \to rU ,$$

(5.11)

is a group isomorphism. Finally if $r < 0$, it is easy to show that $G$ is diffeomorphic to $\mathbb{R}^- \times U(2)$.
Let us denote with $E$ the set of equivalence classes of maps. It is clear that, unlike the quaternionic case in sections of (2.1)-(2.3), the above KT geometries are not only specified by the kernels of the maps $\tau$ in $\mathbb{C}^{2k}$ but in fact also depend on the actual parameterization of the maps. The moduli space $\mathcal{M}_N$ of KT geometries associated with $N$-distinct maps $\tau$, i.e. maps that are not equivalent with respect to the above relation, is

$$\mathcal{M}_N = \frac{D_N \times \left( \times^N E - \Delta \right)}{S_N},$$

(5.12)

where $D_N$ is a domain in $\mathbb{R}^N$ associated with the conformal factors of every term in the sum over the maps in (5.4) and $\Delta$ is the diagonal.

The asymptotic angles between the planes defined by the kernels of two holomorphic maps $\rho$ and $\tau$ can be computed using the normal vectors $n$ and $m$. We define the normal vectors and the angles matrix as in the quaternionic case. The angle matrix is

$$A = \frac{\langle n, m \rangle}{|n||m|}$$

(5.13)

where $\langle \cdot, \cdot \rangle$ is the inner product with respect to the asymptotic metric and $|n|^2 = \langle n, n \rangle$, and similarly for $|m|.$

Choosing as asymptotic metric the flat metric on $\mathbb{C}^{2k} - \cup_{\tau=1}^{N} 0$, we find that

$$A = \frac{p_i q_j \delta^{ij}}{\sqrt{\delta^{ij} p_i p_j \sqrt{\delta^{k\ell} q_k q_\ell}}}$$

(5.14)

where $\{p_1, \ldots, p_k\}$ and $\{q_1, \ldots, q_k\}$ are the rotational parameters of the maps $\rho$ and $\tau$, respectively. The explicit expression of the angles-matrix for general asymptotic metric is

$$A^{\alpha\bar{\beta}} = \frac{g^{\gamma\bar{\delta} p^{\alpha} i\gamma q^{\bar{\beta}} j\bar{\delta}}}{\sqrt{g^{i\epsilon j\zeta} p^{\alpha i\epsilon} j\zeta q^{\bar{\beta} i\epsilon} j\zeta}}$$

(5.15)

For this reason, there is not a direct relation between these KT geometries and Kähler calibrations.
5.3. Special Cases

There are special cases of the geometries (5.4) which admit two commuting KT structures. To find these, we introduce a complex structure

\[
I(dz^1) = dz^2 \\
I(dz^2) = -dz^1 \\
I(d\bar{z}^1) = d\bar{z}^2 \\
I(d\bar{z}^2) = -d\bar{z}^1
\]  

(5.16)
on \mathbb{C}^2 and the induced complex structure \( I \) on \( \mathbb{C}^{2k} \) as in previous sections. The complex structure \( J \) commutes with \( I \). We have seen that (5.4) admits a KT structure with respect to \((\nabla^+, J)\). For this geometry to admit another KT structure with respect to \((\nabla^-, I)\), the asymptotic metric should be also hermitian with respect to \( I \) and

\[
I \, d\tau = d\tau \, I .
\]  

(5.17)
The former condition is easily met for a suitable choice of asymptotic metric. The latter condition is satisfied provided that

\[
I \, p_i = p_i \, I ,
\]  

(5.18)
where the rotational parameters of the maps \( \tau \) have been expressed in terms of complex \( 2 \times 2 \) matrices \( \{p_i; i = 1, \ldots, k\} \). This condition implies that

\[
p_i = \begin{pmatrix} u_i & v_i \\ -v_i & u_i \end{pmatrix}
\]  

(5.19)
where \( \{u_i, v_i; i = 1, \ldots, k\} \) are complex numbers. Therefore, all the geometries (5.4) associated with maps that have rotational parameters (5.19) admit two commuting KT structures. We remark that due to the choice of the complex structure \( I \), the dimension of these geometries is always a multiple of four.
6. Supersymmetric Sigma Models

Most of the applications of the geometries that we have constructed in the previous sections are in the context of two-dimensional supersymmetric sigma models [34, 35]. It is well known that bosonic two-dimensional sigma models with target space manifolds that have one KT structure admit an extension with off-shell (2,0) or (2,1) supersymmetry. If the target space of such a sigma model admits another KT structure that commutes with the first, then the bosonic sigma model admits an extension with off-shell (2,2) supersymmetry. Next, bosonic sigma models with target space manifolds that have an HKT structure admit an extension with off-shell (4,0) or (4,1) supersymmetry. If the target space also admits a KT structure that commutes with the HKT structure, then the sigma model admits an extension with off-shell (4,2) supersymmetry. Finally, bosonic sigma models with target space manifolds that have two commuting HKT structures admit a supersymmetric extension with off-shell (4,4) supersymmetry. For example, sigma models with target spaces the HKT geometries constructed in sections (2.1)-(2.3) admit a (4,0)&(4,1), (4,2) or (4,4) supersymmetric extension depending on whether the rotational parameters of the maps are quaternions, complex or real numbers, respectively. Sigma models with target spaces the KT geometries constructed in sections (5.1)-(5.3) admit a (2,0)&(2,1) or (2,2) supersymmetric extension depending on whether the rotational parameters are generic or satisfy (5.19), respectively.

All supersymmetric sigma models that we have described above admit an off-shell constrained superfield formulation. The construction of these superfields as well as their actions can be easily done following the results of [35]. Quantum mechanically, the off-shell (4,q)-supersymmetric two-dimensional sigma models are ultraviolet finite. Therefore the eight-dimensional HKT geometries that we have constructed describe consistent string backgrounds. However, it is expected that the sigma models with (4,0) supersymmetry may receive $\alpha'$ corrections due to the cancellation of the sigma model anomaly [36]. The (2,q)-supersymmetric, $q \leq 2$, sigma models have non-vanishing $\beta$-function even at one loop. Related to this, the
eight-dimensional \((k = 2)\) KT geometries constructed in (5.1)-(5.3) do not seem to have a brane interpretation. This is because they do not solve the IIA supergravity field equations which are the \(\beta\)-function vanishing conditions at one loop. It would be of interest to find the Kähler-like potentials [34, 37] associated with the above KT and HKT geometries.

We summarize some of our results on the applications of our geometries to sigma models in the table below.

| Supersymmetry | \(\nabla^{(+)}\)  | \(\nabla^{(-)}\)  | Geometry       |
|--------------|-----------------|-----------------|----------------|
| \((4, 4)\)   | \(Sp(k)\)       | \(Sp(k)\)       | HKT & HKT      |
| \((4, 2)\)   | \(Sp(k)\)       | \(SU(2k)\)      | HKT & KT       |
| \((4, 0) & (4, 1)\) | \(Sp(k)\)       | \(SO(4k)\)      | HKT            |
| \((2, 2)\)   | \(U(2k)\)       | \(U(2k)\)       | KT & KT        |
| \((2, 0) & (2, 1)\) | \(U(2k)\)       | \(SO(4k)\)      | KT             |

**Table 2: Sigma Model Geometries** This table contains (i) the supersymmetry preserved by the two-dimensional sigma models, (ii) & (iii) the holonomy of the \(\nabla^{(+)}\) and \(\nabla^{(-)}\) connections, and (iv) the type of geometry that the sigma model target space has with respect to \(\nabla^{(+)}\) and \(\nabla^{(-)}\), respectively.

7. Other superpositions

7.1. KT geometries and quaternions

There are many other ways to superpose the four-dimensional HKT geometry (2.4) to construct new geometries in \(d\)-dimensions. We can achieve this by appropriately choosing maps \(\tau : E^d \rightarrow E^4\). However, a generic choice of maps \(\tau\) will lead to geometries which do not admit KT or HKT structures (see also the conclusions). However, it is possible to choose the maps \(\tau\) in such a way that the moduli space
of the resulting geometries is constructed using certain Grassmannians. To present an example of such geometry, we consider the maps

$$\tau : \mathbb{H}^k \rightarrow \mathbb{H}$$

such that

$$q \equiv \tau(u) = \sum_i p_i u^i \bar{c}_i - a$$

where $$(u^1, \ldots, u^k) \in \mathbb{H}^k$$, $$q \in \mathbb{H}$$, and $$\{p_1, \ldots, p_k; c_1, \ldots, c_k; a\}$$ are quaternions that parameterize the map. The geometry that describes the superposition of four-dimensional HKT geometries in this case is again obtained by pulling back (7.4) and summing over the maps $$\tau$$. We can also add $$ds^2_\infty$$ in the metric to control its asymptotic behaviour. For generic choices of the parameters $$\{p_1, \ldots, p_k; c_1, \ldots, c_k\}$$, the geometry that describes the superposition will not admit any KT structures.

(i) : A special case of the above geometry is to allow either the rotational parameters $$\{p_1, \ldots, p_k\}$$ or the rotational parameters $$\{c_1, \ldots, c_k\}$$ of the maps $$\tau$$ to be complex numbers. Let us suppose that $$\{c_i = c_i^0 + ic_i^1; i = 1, \ldots, k\}$$; the other case is symmetric. Then the geometry which is associated with this superposition will admit a KT structure with respect to $$(\nabla^+, J_1)$$. Two-dimensional sigma models with bosonic couplings determined by this geometry admit a (2,0) off-shell supersymmetric extension.

(ii) : Next, let us take all the rotational parameters $$\{p_1, \ldots, p_k; c_1, \ldots, c_k\}$$ of the maps $$\tau$$ to be complex numbers, i.e. $$\{p_i = p_i^0 + ip_i^1; i = 1, \ldots, k\}$$ and $$\{c_i = c_i^0 + ic_i^1; i = 1, \ldots, k\}$$. We remark that this construction can be done with other combinations of complex structures. In this case, we can show that the geometry that describes the superposition admits two KT structures, $$(\nabla^+, J_1)$$ and $$(\nabla^-, I_1)$$. This geometry is related to that given in section (5.3) but the choice of complex structures is different. Since the complex structures $$I_1$$ and $$J_1$$ commute, two-dimensional sigma models with bosonic couplings determined by this geometry admit a (2,2) off-shell supersymmetric extension.
7.2. Moduli and Calibrations

Maps $\tau$ and $\hat{\tau}$ that are equivalent under the relation

$$\tau \sim \hat{\tau}, \quad (7.3)$$

where

$$\{\hat{p}_1, \ldots, \hat{p}_k; \hat{c}_1, \ldots, \hat{c}_k; \hat{a}\} = \{sp_1, \ldots, sp_k; vc_1, \ldots, vc_k; sa\bar{v}\}, \quad (7.4)$$

$s, v \in \mathbb{H}$, lead to the same geometries. The space of equivalence classes of such maps is the bundle space $E(\theta_1^{k-1})$ of a quaternionic line bundle over

$$Gr(1, \mathbb{H}^k) \times Gr(1, \mathbb{H}^k). \quad (7.5)$$

The quaternionic line bundle $\theta_1^{k-1}$ restricted to the first quaternionic projective space is isomorphic to the associated canonical quaternionic line bundle, and $\theta_1^k$ restricted to the second quaternionic projective space is isomorphic to the associated conjugate canonical quaternionic line bundle. The moduli space of the above geometries associated with $N$ distinct linear maps $\tau$ is

$$\mathcal{M}_N = \frac{D_N \times \left( \times^N E(\theta_1^{k-1}) - \Delta \right)}{S_N}. \quad (7.6)$$

(i) In the case above for which half of the rotational parameters of the maps $\tau$ are complex, the equivalence classes of maps is the bundle space $E(\kappa)$ of a quaternionic line bundle $\kappa$ over

$$Gr(1, \mathbb{H}^k) \times \mathbb{C}P^{k-1}. \quad (7.7)$$

The moduli space of the above geometries associated with $N$ distinct linear maps $\tau$ is

$$\mathcal{M}_N = \frac{D_N \times \left( \times^N E(\kappa) - \Delta \right)}{S_N}. \quad (7.8)$$

It is clear that this geometry is associated with the Kähler calibration constructed out of $J_1$. 

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Finally, in the case for which all the rotational parameters of $\tau$ are complex numbers, the equivalence classes of maps is the bundle space $E(\lambda)$ of a quaternionic line bundle $\kappa$ over
\[ \mathbb{C}P^{k-1} \times \mathbb{C}P^{k-1}. \] (7.9)
The bundle $\lambda$ can be constructed using the canonical complex line bundle of $\mathbb{C}P^{k-1}$. The moduli space of the above geometries associated with $N$ distinct linear maps $\tau$ is
\[ \mathcal{M}_N = \frac{D_N \times (\times^N E(\lambda) - \Delta)}{S_N}. \] (7.10)

The eight-dimensional ($k = 2$) KT geometry above is also associated with one of the calibrated geometries of [24]. To see this, we write the kernel of the maps $\tau$ as
\[ \sum_{i=1}^{2} p_i u_i \bar{c}_i - a = 0. \] (7.11)
We observe that the tangent spaces of the kernels are stabilized both by the action of $I_1$ and $J_1$, so they are complex relative to these holomorphic structures. Thus they are simultaneously calibrated by the Kähler 4-forms associated with $I_1$ and $J_1$, and so also by
\[ \tilde{\Theta} = \frac{1}{4} (\omega_{I_1}^2 - \omega_{J_1}^2). \] (7.12)
The contact set of this calibration is
\[ G_A = \frac{SU(2)}{S(U(1) \times U(1))} \times \frac{SU(2)}{S(U(1) \times U(1))} = S^2 \times S^2. \] (7.13)
Note that again the moduli space of these KT geometries is constructed out of the contact set of the calibration.
8. Conclusions

We have presented new HKT and KT geometries by superposing the four-dimensional HKT geometry associated with the NS-5-brane. The $4k$-dimensional HKT geometries have been constructed using superpositions along quaternionic planes in $\mathbb{H}^k$. We have found that the moduli space of these HKT geometries is constructed from the canonical quaternionic line bundle over the projective space $Gr(1,\mathbb{H}^k)$. Several special cases of these geometries were also considered and their moduli spaces have been given. In addition, we have shown that our eight-dimensional HKT geometries are superpositions along calibrated quaternionic planes in $\mathbb{H}^2$. Their moduli spaces can be constructed using a quaternionic line bundle over the contact set of the associated calibration. All these eight-dimensional geometries are solutions of supergravity theories with the interpretation of intersecting branes on a string preserving $3/32$, $1/8$, $5/32$ and $3/16$ of spacetime supersymmetry. The proportion of supersymmetry preserved is directly related to the class of calibrated planes used to construct the superposition. The $4k$-dimensional HKT geometries have also applications in the context of two-dimensional sigma models. In particular we have found new sigma models with $(4,0)$, $(4,1)$ and $(4,2)$ supersymmetry generalizing the $(4,4)$-supersymmetric sigma models of [28]. We have also constructed new KT geometries using superpositions of the same four-dimensional HKT geometry along complex planes in $\mathbb{C}^{2k}$. The resulting KT geometries are associated with, but not uniquely determined by, arrangements of complex planes in $\mathbb{C}^{2k}$. In addition we investigated several other aspects of these geometries leading, for example, to the construction of geometries associated with two-dimensional sigma models with $(2,0)$ and $(2,2)$ supersymmetry. Several special cases have also been examined. In particular, we have found that some of the eight-dimensional KT geometries are also associated with calibrated planes in $\mathbb{C}^4$.

The investigation of the superposition of four-dimensional HKT geometries may be extended further in two ways. First, there may be calibrations in $4k$-dimensions, $k > 2$, for which the planes, that are used in the construction of the
$4k$-dimensional, $k > 2$, HKT geometries, are calibrated. These calibrations will generalize those of [24] and some of the results we have found for the eight-dimensional HKT geometries. Alternatively, we may be able to find new superpositions of four-dimensional HKT geometries by choosing planes in $\mathbb{E}^{4k}$ which are associated with other calibrations like for example the Special Lagrangian calibration or the calibrations associated with the exceptional groups $G_2$ and Spin(7) [22, 23]. It would be of interest to study the properties of the geometries that result from such superpositions and see whether they admit a brane interpretation.

In eight dimensions, it is clear that there is a correspondence between the type of calibration used to construct the HKT geometry and the holonomies of the $\nabla^{(+)}$ and $\nabla^{(-)}$ connections. In particular the holonomy of $\nabla^{(+)}$ is in all cases $Sp(2)$. This $Sp(2)$ fixes (up to the trivial $Sp(1)$-factor) a quaternionic calibrating four-form. Adding to this quaternionic four-form a certain $Spin(7)$, $SU(4)$ or $Sp(2)$ invariant calibrating four-form results in $\nabla^{(-)}$ having $Spin(7)$, $SU(4)$ or $Sp(2)$ holonomy, respectively. The understanding of this correspondence is a key for studying the geometric properties of the superposed geometry and it may lead to many more applications in string and M-theory.

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APPENDIX

10. Symmetry, Supersymmetry and Five-branes

The HKT geometry on $\mathbb{H} - \{0\}$ associated with the NS-5-brane can be rewritten as

$$ds^2_{(4)} = (1 + \frac{\mu}{|q|^2})|dq|^2$$

$$H = \frac{\mu}{3!} \text{Re} \left( \frac{dq \wedge dq \wedge (d\bar{q}q - \bar{q}dq)}{2|q|^4} \right), \quad (10.1)$$

where $q \in \mathbb{H}$, $\bar{q}$ is the conjugate of $q$ and $|q|^2 = \bar{q}q$. We can introduce a pair of hyper-complex structures using left and right multiplication by the imaginary unit quaternions on $dq$ as in (2.6) and (2.5). Since the left and right actions commute, the triplets of complex structures also commute leading to a commuting pair of HKT structures. The manifold $\mathbb{H} - \{0\}$ with (10.1) can be the target space of a two-dimensional sigma model with (4,4) supersymmetry.

The metric and torsion of the HKT geometry (10.1) are invariant under the $SO(4) = Sp(1) \times Sp(1)/\mathbb{Z}_2$ action

$$q \rightarrow aq \bar{b}, \quad (10.2)$$

where $(a, b)$ are quaternions and $(a, b) \in Sp(1) \times Sp(1)$. This group action does not commute with the maps associated with the complex structures, i.e. the above isometries are not holomorphic.

Using this group action, we can construct new HKT geometries from that of (10.1). This can be achieved by factoring $\mathbb{H} - \{0\}$ with a discrete subgroup, $G$, of $Sp(1) \times Sp(1)$. Similar orbifold-like constructions have been done for D-branes in [38]. There are many choices for the subgroup $G$. However we shall require the resulting manifold to admit at least one HKT structure; there are other choices of $G$ that lead to manifolds with one or two Kähler structures with torsion. This
restricts the choice of $G$ to be a subgroup either of the $Sp(1)$ that acts from the left or the $Sp(1)$ that acts from the right. If we choose $G$ to be a subgroup of $Sp(1)$ that acts from the left, then the manifold $\mathbb{H} - \{0\}/G$ will admit an HKT structure with respect to the pair $(\nabla^{(+)}, J_r)$. We remark that $G$ acts on $\mathbb{H} - \{0\}$ freely since the only fixed point of the left $Sp(1)$ action on $\mathbb{H}$ is $q = 0$ but this point is not part of the space $\mathbb{H} - \{0\}$. The manifold $\mathbb{H} - \{0\}/G$ can be the target space of a two-dimensional sigma model with $(4,0)$ or $(4,1)$ off-shell supersymmetry.

A special case of the above construction arises whenever the discrete subgroup, $G$, of $Sp(1)$ is chosen such that it preserves one of the $I_1, I_2$ or $I_3$ complex structures. If $G$ preserves say the $I_1$ complex structure, then the action of $G$ commutes with the map induced by the $I_1$ complex structure. This requires that the elements $a \in G \subset Sp(1)$ are of the form

$$a = a_0 + i a_1 .$$  \quad (10.3)

Next since $\bar{a} a = 1$, then

$$a = \cos \theta + i \sin \theta ,$$  \quad (10.4)

where $\theta$ is an angle. For example, we choose $G = \mathbb{Z}_N$ acting on $\mathbb{H} - \{0\}$ with $e^{2\pi ki/N}$. The manifold $\mathbb{H} - \{0\}/G$ admits an HKT structure with respect to the pair $(\nabla^{(+)}, J_r)$. In addition it admits a Kähler structure with torsion with respect to the complex structure $I_1$. The manifold $\mathbb{H} - \{0\}/G$ can be the target space of a two-dimensional sigma model with $(4,2)$ off-shell supersymmetry.
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