Design framework for optimizing waypoints of vehicle trajectory considering terminal velocity and impact angle constraints

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ABSTRACT
Ballistic missiles often require the terminal velocity and impact angle to be confined to a certain region around a desired value considering a wide variety of uncertain initial conditions and operational ranges. This study presents a design framework to determine optimum waypoints that satisfy constraints for given launch conditions and mission profiles. As a systematic approach to this kinodynamic motion planning problem, the proposed framework deterministically samples the waypoints. The trajectory generated by the determined waypoints in consideration of various conditions satisfies the terminal constraint. Numerical simulations are performed to demonstrate the effectiveness of the proposed method.

1. Introduction
A missile should satisfy the terminal velocity and impact angle constraint to enhance the lethality of the warhead and enable the seeker to keep track of the target. For example, a missile should fly more slowly than a specified velocity to protect the fuse from excessive aerodynamic heating when striking a target. In addition, a missile should have a sufficiently high velocity to destroy a target effectively in a multistory shelter. As mechanical energy is the sum of kinetic energy and potential energy, the guidance design problem for striking a target with a desired velocity is basically a problem of energy management.

Numerous studies have been performed on the guidance problem with terminal velocity constraints in various applications, each with unique design concerns, while approaching the problem from different perspectives. The velocity of a launch vehicle with a solid rocket motor (SRM) should satisfy the specified terminal velocity to deliver the payload. After the report by Perkins (1966), various methods were proposed to modify the trajectory considering the energy of the vehicle (Chandler and Smith 1967; McHenry et al. 1979; White 1993; Kim and Um 2015). Additionally, re-entry vehicles should carefully dissipate their energy for the landing phase; in this context, Kraemer and Ehlers (1975) proposed a guidance scheme to change the energy dissipation rate by modifying the trajectory or modulating the speed brake for the landing phase of re-entry vehicles; the trajectory was divided into several sections to increase the efficiency of the proposed scheme. Williamson, Greene, and Hull (1976) proposed the perturbation method to generate the optimized shuttle trajectory. Wang, Tang, and Zhang (2019) proposed a re-entry guidance law with angle and velocity guidance for...
hyper-sonic gliding vehicle; the vehicle manoeuvre was based on conical motion to deceleration. Lan, Xu, and Wang (2020) proposed a guidance law for controlling the angle of attack and the bank angle to reduce the errors in the terminal area energy management phase. Predictive methods were effective at satisfying the terminal velocity constraint. For example, the terminal velocity was estimated using open-loop guidance profiles (Kuroda and Imado 1997) and the specific trajectory shape (Zhou, Rahman, and Chen 2015). Chen and Xia (2016) and Tahk, Moon, and Shim (2019) proposed polynomial form expressions for the flight altitude to predict the terminal velocity, and Moon, Park, and Ryoo (2015) utilized neural networks to obtain the terminal velocity constraint. Note that predictive methods have been shown to be particularly effective in satisfying the terminal velocity constraint.

Unlike previous studies that have dealt only with the impact angle constraint, satisfying the terminal velocity constraint of a ballistic missile is very challenging because no direct control effector exists over the velocity of the missile. A ballistic missile usually employs an SRM to improve the usability and strategic effectiveness. However, an SRM is sensitive to the surrounding environment. A change in the operational environment may lead to deviations in thrust and moment of burn-out, and in velocity and altitude at burn-out, among other consequences. Therefore, a missile exhibiting uncertain burn-out conditions should adjust its energy during the gliding phase to meet the terminal velocity constraint. Additionally, a missile should be able to attack targets located within a wide range of distances from the launch point.

It is almost impossible to design an explicit feedback guidance law that satisfies the terminal velocity constraint. The difficulty is mainly due to the inherent nature of the energy management problem. Unlike the position or the flight path angle, which describe the kinematics of a missile, the variables related to the dynamics are the velocity and energy, and the time-domain solution for these variables associated with the dynamics is hardly available in an analytic form. The lack of a closed-form expression that describes the evolution of the predicted final velocity in relation to an input variable is the main difficulty complicating an analytical approach to this problem. The terminal velocity constraint problem has not been studied as extensively as the impact angle control (IAC). Zhou, Rahman, and Chen (2015) presented a review of previous research covering the last few decades, but most previous studies were limited to the investigations of IAC.

Considering these difficulties, a waypoint optimization approach to feasible trajectory generation may provide an effective solution method. Rafique et al. (2011) and Shirazi (2016) proposed the heuristic method to solve the waypoint optimization problem. One workaround in the heuristic method was to carefully construct the locations and incidence angles of waypoints so that the resultant trajectory terminated at the target with the desired velocity. Although this solution worked in an open-loop fashion, it could be implemented in practice. However, finding the waypoint parameters to cover the entire set of boundary conditions required a considerable amount of time and effort. The optimization method to obtain waypoints for the Mars landing mission was investigated by Guo, Hawkins, and Wie (2013). The zero-effort-miss/zero-effort-velocity scheme to obtain the intermediate waypoints was proposed within the thrust limit environment.

Once the position and the incidence angle of the waypoints necessary to accomplish the mission are determined by an appropriate optimization framework, various existing guidance methods can be applied for flight through the waypoints. Jorris and Cobb (2009) proposed a method to generate a constrained optimal trajectory with the specified waypoints and no-fly zones as constraints. Because a powered hypersonic cruise vehicle was the platform, constraints could be satisfied while flying over long distances. Liang, Li, and Ren (2016) described the review of research for the entry guidance. A guidance algorithm with the atmospheric entry condition for the waypoint constraint was proposed to reach the waypoints and target. He et al. (2019) proposed waypoint guidance for an unmanned vehicle. The guidance command was calculated from waypoint passing constraints and flight path angle constraints to satisfy the arrival angle of the vehicle.

The approach for waypoint optimization of a ballistic missile propelled by an SRM is still relatively new because, for this scenario, it is difficult to apply the methods previously developed for other types of flight vehicles. A ballistic missile inevitably flies through the atmosphere along a considerable part
of its flight path to the target, which is typically located on the ground. In addition, a missile does not have a propulsion system nor a braking system to directly control its velocity after the engine burns out. Therefore, trajectory shaping is the only method for optimized trajectory with the terminal constraints.

The aim of this study is to propose a design framework to optimize waypoints for ballistic missiles with the terminal velocity and impact angle constraints. The proposed method enhances the robustness of the design against uncertainties in the conditions at burn-out, making this framework widely applicable to missions with different target distances. Moreover, the proposed approach is useful in practice because it is obtained by a closed-loop simulation that includes guidance and control schemes.

The remainder of this article is organized as follows. Sections 2 and 3 describe the problem statement and equations of motion. Section 4 introduces the proposed design framework for obtaining an optimized trajectory to satisfy the terminal constraints. In Section 5, simulation results are shown to demonstrate the effectiveness of the proposed framework. Finally, the concluding remarks are given in Section 6.

2. Problem statement

The objective of this study is to generate a feasible trajectory that satisfies terminal velocity and flight path angle constraints to define an optimization problem. Consider the following performance index $J$:

$$\min J = \omega \times V_{err} + (1 - \omega) \times \gamma_{err}, \quad (1)$$

where $V_{err}$ and $\gamma_{err}$ are the velocity error and the flight path angle error evaluated at the moment of impact, respectively, and $\omega$ is the weighting factor. The optimization framework developed in this study benefits from two central elements. First, a classical optimal guidance law for IAC is employed to account for the reduction of $\gamma_{err}$. Secondly, an heuristic search method is devised to find feasible trajectories effectively. A deterministic search method considering trajectory characteristics with waypoints is developed to satisfy the terminal velocity constraint because stochastic optimization methods may yield one of many local solutions.

Normally, a ballistic missile moves almost entirely in the vertical plane to strike the target as fast as possible; it does not manoeuvre in the horizontal plane unless there exists a specific mission or the survivability of the missile is desired. The effectiveness of the aerodynamic control surfaces decreases at high altitudes because of low air density, which means that the energy dissipation effect is not significant. Accordingly, the missile should dissipate its energy at low altitudes. Because of the conditions imposed on the manoeuvre plane and the altitude, the ballistic missile under consideration preferably performs pull-up manoeuvres. To satisfy terminal velocity constraints, it is necessary properly to determine the waypoints that generate a trajectory with the characteristics. Figure 1 shows the schematic configuration of the trajectory and waypoints considered in this study for a ballistic missile. In Figure 1, $P$ is the position, and $V$ and $\gamma$ are the velocity and flight path angle, respectively. The subscript BO represents the burn-out time of the missile, the subscript WP$i$ represents the $i$th waypoint, and the subscript TGT represents the target of the missile.

To establish the proposed method and to analyse the results reflecting those waypoints, a closed-loop simulator describing the dynamics and control of the missile is required. It is assumed that the guidance algorithm for the precision-guided munition considered in this study uses the waypoints’ information. The terminal conditions computed in the simulator are passed to the framework to determine whether those conditions satisfy the given constraints. Figure 2 shows a block diagram of the closed-loop simulator illustrating how it exchanges information with the framework. As shown in Figure 2, the closed-loop simulator includes the plant, i.e. the missile dynamics and the guidance, navigation and control subsystem.
3. Dynamic modelling of guided ballistic missiles

Equations of motion for the missile can be represented as (Lu 2014)

\[
\dot{V} = -\frac{D}{m} - g \sin \gamma + \Omega_e^2 r \cos \phi (\sin \gamma \cos \phi - \cos \gamma \sin \phi \sin \gamma) \tag{2}
\]

\[
\dot{\gamma} = \frac{L \cos \sigma}{mV} - \frac{g}{V} \cos \gamma + \frac{V}{r} \cos \gamma + 2 \Omega_e^2 \cos \phi \cos \psi
\]

\[
+ \frac{\Omega_e^2 r}{V} \cos \phi (\cos \gamma \cos \phi + \sin \gamma + \sin \phi \sin \psi) \tag{3}
\]

\[
\dot{x} = V \cos \gamma \tag{4}
\]

\[
\dot{h} = V \sin \gamma \tag{5}
\]

where \( V \) is the velocity of the missile, \( \gamma \) is the vertical flight path angle, \( \sigma \) is the bank angle, \( \phi \) is the roll angle, \( \psi \) is the yaw angle, \( x \) is the horizontal displacement (defined to be positive in the direction from the launch point to the target), \( h \) is the altitude of the missile, \( m \) denotes the vehicle mass, \( \Omega_e \) is the Earth’s rotation rate, and \( g \) is the gravitational acceleration constant (9.8 m/s²). \( D \) and \( L \) are the drag force and the lift force, respectively, which can be represented as

\[
D = C_D \left( \frac{1}{2} \rho V^2 \right) S \tag{6}
\]

\[
L = C_L \left( \frac{1}{2} \rho V^2 \right) S, \tag{7}
\]

where \( \rho \) is the density of the atmosphere, \( S \) is the reference area of the missile, and \( C_D \) and \( C_L \) are the aerodynamic coefficients.
In this study, a three-degrees-of-freedom (3-DOF) point-mass model is used in the closed-loop system, and the flat earth and true navigation models are used. Note that although a high-fidelity model (such as a full 6-DOF model) will provide more accurate results, the 3-DOF model employed herein is sufficient to reflect the performance of the proposed framework because all the simulation results, including the waypoints’ information, are calculated under the same simulation conditions. Note that the waypoint optimization framework is developed to find the waypoints on the vertical plane since most ballistic missiles tend to avoid excessive manoeuvres in the horizontal direction. Nevertheless, the proposed framework can be extended in a similar manner to find waypoints in the horizontal plane as well, provided that the dynamic model is supplemented with the equations related to the lateral motion. The 3-DOF models have become widely common for the optimization of trajectories and the development of guidance algorithms for rocket and missile systems (Zarchan 1990). In the 3-DOF simulation, the response lag due to autopilot delay is neglected so that the actual lift acceleration is considered to be the same as the acceleration command given by a guidance law. Therefore, the lift force generated by the guidance algorithm can be defined as

\[ L = ma_{cmd}, \]  

where \( a_{cmd} \) is the acceleration command calculated by the guidance law in Figure 2. In more detail, assuming that the mass is exactly known and the vehicle achieves the commanded normal acceleration without any lag or error, the acceleration command generated by the guidance law itself is considered to be the actual lift. Note that the angle of attack should be limited within some flight boundaries during the flight. In this study, the maximum limit of the acceleration command is considered, which is related to the angle of attack. In this article, the optimal IAC guidance law of Ryoo, Cho, and Tahk (2005) is adopted to generate the acceleration command considering the flight path angle constraint as

\[ a_{cmd} = C_R t_{go} + C_S, \]  

where \( t_{go} \) is the time-to-go estimate. The constant coefficients of the ramp response \( (C_R) \) and the step response \( (C_S) \) are defined as

\[ C_R = \frac{6V}{t_{go}^3} (-2t_{go}\theta + t_{go}\gamma + t_{go}\gamma_{TGT}) \]  

\[ C_S = -\frac{2V}{t_{go}^4} (3t_{go}\theta + t_{go}\gamma + 2t_{go}\gamma_{TGT}), \]  

where \( \theta \) is the line-of-sight angle in the pitch plane, which can be represented as

\[ \theta = \arctan \frac{h - h_{TGT}}{x - x_{TGT}}. \]  

Using Equations (10)–(12) in Equation (9), \( a_{cmd} \) can be rewritten as

\[ a_{cmd} = \frac{V}{t_{go}} (-6\theta + 4\gamma + 2\gamma_{TGT}). \]  

Considering the 3-DOF simulation, \( t_{go} \) and \( p_{go} \) are obtained as follows (Ryoo, Cho, and Tahk 2005):

\[ t_{go} = \frac{p_{go}}{V} \]  

\[ p_{go} = \sqrt{(x - x_{TGT})^2 + (h - h_{TGT})^2}. \]
4. Framework for optimization of waypoints

4.1. Overview

The objective of the design framework is considered in this study to determine the proper locations of the waypoints and the incidence angles relative to those waypoints. A change in either the waypoint location or the incidence angle leads to a change in the final energy through the generation of different trajectories. The waypoint design framework may provide a solution to satisfy the terminal velocity constraint. In particular, the proposed framework imposes some conditions on the burn-out time and the terminal time because the waypoints are determined with respect to the energy consumption. The proposed framework consists of four components, each of which is responsible for different tasks, namely gravity turn planning, waypoint candidate generation, candidate pruning and waypoint determination.

The proposed framework determines the first waypoint by taking the point with the maximum altitude from the trajectory of the gravity turn, reducing the influences of the burn-out conditions. The subsequent waypoints are determined by searching over a group of candidates while satisfying the given constraints. In the waypoint candidate generation step, a range of points is generated for the candidates to be considered. Then, incorrect candidates are excluded in consideration of the mission constraints; this step is called candidate pruning. The best waypoint among the remaining candidates is finally selected through a comparison of the values of the performance index; this step is called waypoint determination. The first waypoint is calculated from the burn-out conditions by open-loop propagation, whereas the others are determined sequentially in reverse order from the last waypoint to the second waypoint using the trajectories obtained from the closed-loop simulator.

The waypoint selection processes are performed using the closed-loop simulator, which incorporates the missile dynamics and control systems, as shown in Figure 2. A motion planning method based on the simulated closed-loop dynamics has the advantage of containing fewer discrepancies between the model and reality in comparison to methods based solely on simple kinematics. Hence, the planning result will show superior dynamic feasibility as a natural consequence of incorporating a more accurate dynamic model.

Figure 3 shows an overview of the framework proposed in this study. The subscript WP1 refers to the first waypoint, \( V_{\text{fin}} \) is the estimated terminal velocity obtained by the closed-loop simulator, and Candidate A and Candidate B are the results from the candidate generation subroutine and the candidate pruning subroutine, respectively.

4.2. First waypoint planning: forward propagation

A missile and the environment in which the missile flies do not always behave as predicted. Furthermore, differences in the trajectory also cause the missile conditions at the burn-out time to vary, which
greatly affects the ability of the missile to manoeuvre to the target. Figure 4 shows the changes in the trajectory as a result of variations in the thrust and drag during the thrust phase. In Figure 4, Case 1 shows the trajectory in which the thrust is increased by 1% and the drag coefficient is decreased by 5% compared to the nominal case, while Case 2 shows the trajectory in which the thrust is decreased by 1% and the drag coefficient is increased by 5% compared to the nominal case.

The first waypoint is selected to reduce the influences of the initial conditions and to facilitate the manoeuvring of the missile beyond the burn-out point. It is known that, by performing a gravity turn, the angle of attack is kept near zero, beneficially minimizing the energy wasted due to the induced drag (Callaway 2004). In this regard, the proposed framework employs a gravity turn to determine the first waypoint that comes immediately after the boosting stage.

In this study, the first waypoint is selected where either the flight path angle of the missile or the flight altitude has a specific predefined value due to the system operating conditions; for example, the zero flight path angle of the missile or the maximum altitude from the burn-out point can be considered. Table 1 shows the pseudocode of the gravity turn subroutine used to determine the first waypoint, where $\epsilon$ is the threshold value used in the stop condition based on violating the terminal velocity constraint and the subscript $WP_{1des}$ represents the desired value for the first waypoint.

### 4.3. Planning the $i$th waypoint: backward propagation

Planning the $i$th waypoint in the framework is achieved by three steps. The candidate generation process produces candidates for the $i$th waypoint from the given conditions. The candidate pruning step excludes all candidates that violate the constraints. The waypoint determination procedure selects the best candidate waypoint in consideration of the performance index; choosing the optimal index value is critical to the success of the given mission of the missile. The waypoints are determined sequentially in reverse order, i.e. from the last waypoint to the second waypoint.

#### 4.3.1. Candidate generation

In general, the combination of waypoints that satisfies the terminal velocity constraint is not unique. During the candidate generation step, all candidates that satisfy the given waypoint conditions are found.
Table 1. Pseudocode: planning the first waypoint.

| Procedure | Gravity turn |
|-----------|-------------|
| **Input** | $P_{BO}, V_{BO}, \gamma_{BO}$ |
| **Output** | $P_{WP_1}, V_{WP_1}, \gamma_{WP_1}$ |
| Initialization parameters; | |
| **While** $(|\gamma_{M} - \gamma_{WP_{des}}| < \epsilon_{\gamma}) \& (|h_{M} - h_{WP_{des}}| < \epsilon_{h})$ do | |
| Closed-loop simulator; | |
| **return** $P_{WP_1}, V_{WP_1}, \gamma_{WP_1}$ | |

There are two scenarios in this process depending on the existence of the initial condition. The first scenario is the determination of the second waypoint. In this scenario, the first waypoint should be considered as the initial condition, and the second waypoint is treated as an intermediate condition between the first waypoint and the third waypoint. In the second scenario, the initial condition is not given; that is, only the final condition is given, and the $i$th waypoint condition should be determined by considering the $(i+1)$th waypoint. The second waypoint is determined from the third waypoint by backward propagation. The waypoints are searched one by one; finally, the second waypoint is found considering the first waypoint and the third waypoint. Figure 5 shows a schematic diagram of the candidate generation step.

If the terminal conditions are given, the candidate waypoints are obtained during the candidate generation step. Note that the waypoints' information can be easily stored in the onboard computer and implemented for online waypoint look-up tables. Figure 6 shows a flowchart of the candidate generation step, and the detailed procedure is given in Table 2. The subscripts $WP_{i-1}$, $WP_i$, and $WP_{i+1}$ denote the $(i-1)$th, $i$th, and $(i+1)$th waypoints, respectively, and the subscript $WP_iC$ represents a candidate of the $i$th waypoint.

4.3.2. Candidate pruning

Other constraints, such as limitations on the lateral acceleration or altitude, may exist. Some candidates obtained from the candidate generation step should be excluded by considering additional constraints. The remaining candidates are then passed to the waypoint decision process, as shown in Figure 3. Figure 7 shows a schematic diagram of the candidate pruning step.

In Figure 7, the waypoints represented by circles are the remaining candidates, while the waypoints represented by crosses are the excluded candidates that are not feasible with respect to the constraints considered. The position represented by the star is the target location. Only the remaining candidates are passed to the waypoint decision process.

Table 3 shows the pseudocode of the candidate generation step. In Table 3, the subscript $WP_iC'$ represents a remaining candidate of the $i$th waypoint following the candidate pruning step.
4.3.3. Waypoint determination

This step selects the best waypoint from among the results of the candidate pruning step. Various criteria can be considered to make a determination among the waypoint candidates. For example, the flight time is a good performance index that takes into account the mission time as well as the battery capacity of electronic devices in the missile or the uncertainty of the environment. In terms of the survivability of a missile with respect to an interceptor, the altitude can be considered as a performance index. Aerodynamic heating is also a factor that can be incorporated into the planning stage to reduce the effects of abrasion (Eggers 1957). The best performance index should be adopted considering the given mission of the missile. Additionally, multiple objectives can be considered by combining them into a single index by, for example, using the weighted sum of several performance indices.

5. Numerical examples

To demonstrate the effectiveness of the proposed energy management waypoint framework, numerical simulations of various example cases are performed. A ballistic missile with an SRM is considered for the stationary target. In all simulation cases, the number of waypoints is fixed to three. Waypoints are an effective means of modifying the trajectory to achieve the given mission, and the number of
**Table 2.** Pseudocode: generating the candidates for the $i$th waypoint.

**Procedure** Candidate generation

**Input**
\[ P_{WP_{i-1}}, V_{WP_{i-1}}, \gamma_{WP_{i-1}} \]
\[ P_{WP_{i+1}}, V_{WP_{i+1}}, \gamma_{WP_{i+1}} \]

**Output**
\[ P_{WP_i}^C, V_{WP_i}^C, \gamma_{WP_i}^C \]

Initialization parameters;
foreach $P_{WP_i}, V_{WP_i}, \gamma_{WP_i}$ in $R_{WP_i}$ do
if Initial condition exists then
  Initial conditions ← $P_{WP_{i-1}}, V_{WP_{i-1}}, \gamma_{WP_{i-1}}$;
  Intermediate conditions ← $P_{WP_i}, V_{WP_i}, \gamma_{WP_i}$;
  Final conditions ← $P_{WP_{i+1}}, V_{WP_{i+1}}, \gamma_{WP_{i+1}}$;
else
  Initial conditions ← $P_{WP_i}, V_{WP_i}, \gamma_{WP_i}$;
  Final conditions ← $P_{WP_{i+1}}, V_{WP_{i+1}}, \gamma_{WP_{i+1}}$;
While Missile manoeuvring stops do
  Closed-loop simulator;
  if $|V_M| < V_{des} < \epsilon$ then
    $P_{WP_i}^C, V_{WP_i}^C, \gamma_{WP_i}^C ← P_{WP_i}, V_{WP_i}, \gamma_{WP_i}$;
return $P_{WP_i}^C, V_{WP_i}^C, \gamma_{WP_i}^C$

**Figure 7.** Schematic diagram of the candidate pruning step.

**Table 3.** Pseudocode: pruning the candidates for the $i$th waypoint.

**Procedure** Candidate pruning

**Input**
\[ P_{WP_i}^C, V_{WP_i}^C, \gamma_{WP_i}^C \]

**Output**
\[ P_{WP_i}^C, V_{WP_i}^C, \gamma_{WP_i}^C \]

foreach $P_{WP_i}^C, V_{WP_i}^C, \gamma_{WP_i}^C$ do
  Closed-loop simulator;
  if satisfying the mission conditions AND missile conditions then
    $P_{WP_i}^C, V_{WP_i}^C, \gamma_{WP_i}^C ← P_{WP_i}, V_{WP_i}, \gamma_{WP_i}$;
return $P_{WP_i}^C, V_{WP_i}^C, \gamma_{WP_i}^C$
Figure 8. Energy management waypoint framework for three waypoints.

Table 4. Simulation case setup.

| Parameters       | Description                          |
|------------------|--------------------------------------|
| Case 1           | Initial condition                    |
| Case 2           | Drag                                 |
| Case 3           | Flight path angle at burn-out         |
|                  | Drag uncertainty                      |

Table 5. Initial conditions.

| Scenario | x_{BO} (km) | h_{BO} (km) | V_{BO} (m/s) | γ_{BO} (°) |
|----------|-------------|-------------|--------------|------------|
| A        | 10          | 9           | 1200         | 15         |
| B        | 17.5        |             |              |            |
| C        | 20          |             |              |            |

Waypoints can be treated as a variable. The waypoints can be reduced or increased depending on the energy developed from the SRM, range to target and operating environment. WP₁ is calculated first, and then WP₃ and WP₂ are determined in the reverse order. WP₃ and WP₂ are selected from the feasible area of each waypoint. Figure 8 shows the steps of the energy management waypoint framework for three waypoints.

Case 1 considers different initial conditions to the target. Case 2 considers the uncertainty in the dynamic model used in the closed-loop simulator. Table 4 summarizes the simulation setup for each case.

In this study, the grid method is adopted to investigate the proposed candidate pruning step. The grid method, which divides a given area into a rectangular grid with Δx and Δh intervals and checking parameters with Δν and Δγ intervals. To reduce the computation time, the algorithm terminates the process if the estimated velocity reaches (V_{des} ± ϵ_v) m/s. Considering a conventional missile engagement, Δx, Δh, ΔV, Δγ, and ϵ_v are set as 0.4 km, 0.4 km, 5 m/s, 1° and 5 m/s, respectively.

5.1. Case 1: different initial conditions

To demonstrate the effectiveness of the proposed framework for different initial conditions, three scenarios are considered, as summarized in Table 5. Each scenario has the same down range, altitude and velocity. However, the initial flight path angle at the burn-out time is different.

The terminal conditions at the target are set as follows: x_{TGT} = 120 km, h_{TGT} = 0 km, V_{TGT} = 350 m/s and γ_{TGT} = −90°. The chosen terminal constraints are close to those of typical tactical ballistic missiles such as the Army Tactical Missile System (ATACMS) and the Russian mobile short-range ballistic missile system ISKANDER, which adopt solid rocket motor fuel and hit a specified target with the trajectory that can deliver the highest attack effectiveness. The time step of the simulation
is 0.1 s, and the mass of the missile is 1000 kg. To reflect a realistic environment, the missile is limited to a minimum altitude of 2 km until it reaches waypoint 3, and an altitude of 5 km above the target is considered for waypoint 3. The maximum lateral acceleration is set as 10 g, and the altitude of waypoint 2 does not exceed the maximum altitude of the whole trajectory.

### 5.1.1. Waypoint 1
Waypoint 1 is determined as the point where the flight path angle of the missile becomes less than or equal to zero for the first time. The obtained results for waypoint 1 are summarized in Table 6.

### 5.1.2. Waypoint 3
The criterion for waypoint 3 is set as the minimum error between the estimated velocity at the target and the desired velocity, $V_{\text{TGT}} = 350$ m/s. The candidates for waypoint 3 and the selected point are shown in Figure 9. The star is the target, and the circles are the candidates for waypoint 3 obtained by the candidate generation and candidate pruning steps; in addition, the square is the waypoint selected by the waypoint determination step, and the solid line is the corresponding trajectory using the selected point that satisfies the terminal velocity condition. Note that one target condition has many solutions satisfying the terminal velocity constraint, and the decision criterion determines the shape of the trajectory. The selected points are $x_{\text{WP3}} = x_{\text{TGT}} - 15.2$ km, $h_{\text{WP3}} = h_{\text{TGT}} + 5.8$ km, $V_{\text{WP3}} = 365$ m/s and $\gamma_{\text{WP3}} = -28^\circ$. The errors of the terminal states are $|x_{\text{err}}| = 2.5 \times 10^{-4}$ km, $|h_{\text{err}}| = 5.4 \times 10^{-4}$ km, $|V_{\text{err}}| = 1.0 \times 10^{-4}$ m/s and $|\gamma_{\text{err}}| = 5.86^\circ$. The flight path angle at the target point has some error because only the terminal velocity error is considered in this case.

| Scenario   | $x_{\text{WP1}}$ (km) | $h_{\text{WP1}}$ (km) | $V_{\text{WP1}}$ (m/s) | $\gamma_{\text{WP1}}$ (°) |
|------------|------------------------|------------------------|-------------------------|-----------------------------|
| Scenario A | 35.24                  | 12.74                  | 834.37                  | -0.05                       |
| Scenario B | 38.37                  | 13.95                  | 819.89                  | -0.02                       |
| Scenario C | 41.50                  | 15.33                  | 809.96                  | -0.04                       |

**Figure 9.** Candidates and selected point for waypoint 3.
Table 7. Waypoint 2 conditions and terminal errors at $x_{\text{TGT}} = 120 \text{ km}$.

| Scenario | $x_{\text{WP}_2}$ (km) | $h_{\text{WP}_2}$ (km) | $V_{\text{WP}_2}$ (m/s) | $\gamma_{\text{WP}_2}$ (°) | $|V_{\text{err}}|$ (m/s) | $|\gamma_{\text{err}}|$ (°) |
|----------|-------------------------|-------------------------|-------------------------|--------------------------|-------------------------|--------------------------|
| A        | 98.84                   | 7.14                    | 384.29                  | -28                      | 0.30                    | 4.92                     |
| B        | 100.37                  | 6.75                    | 381.79                  | -21                      | 0.07                    | 5.17                     |
| C        | 74.70                   | 12.93                   | 379.48                  | 73                       | 0.49                    | 4.8                      |

Figure 10. Final trajectory of example Case 1.

5.1.3. Waypoint 2

Waypoint 2 is determined through the following process. The candidates for waypoint 2 are determined from the points selected for waypoint 1 and waypoint 3. After the candidates for waypoint 2 are obtained, simulation is performed for every candidate from the burn-out point to the target point. Then, the errors between the terminal conditions and simulation results are calculated. These errors are analysed to determine the optimal point for waypoint 2 using Equation (1) with an inequality constraint $h_{\text{WP}_1\rightarrow\text{WP}_3} \geq h_{\text{lim}}$. In this example, the weighting factor is chosen as $\omega = 0.9$ to increase the priority of the velocity constraint over the impact angle constraint. Table 7 summarizes the point selected for waypoint 2 and errors at the target location in each scenario.

Considering $|V_{\text{err}}| = 1.0 \times 10^{-4}$ m/s and $|\gamma_{\text{err}}| = 5.86$° for waypoint 3, the results for various initial conditions are similar to the expected values. These findings indicate that the framework can provide the expected result. Moreover, a missile with an initial velocity of 1200 m/s that travels a distance of 120 km hits the target precisely with a terminal velocity error of less than 0.5 m/s. The proposed framework yields good performance by managing the energy of the missile. Figure 10 shows the trajectory of each scenario.
Table 8. Terminal errors at $x_{\text{TGT}} = 120$ km with unexpected drag uncertainty of $-10\%$ and lift uncertainty of $\pm 10\%$.

| Scenario | $|V_{\text{err}}|$ (m/s) | $|\gamma_{\text{err}}|$ ($^\circ$) | Lift $-10\%$ | Lift $+10\%$ |
|----------|-----------------|-----------------|--------------|--------------|
| Scenario A | 33.54 (9.58%) | 0.22 (12.41%) | 43.43 | 4.46 |
| Scenario B | 35.86 (10.24%) | 0.76 (12.84%) | 44.94 | 11.70 |
| Scenario C | 45.86 (13.10%) | 2.04 (13.76%) | 48.17 | 8.61 |

Table 9. Terminal errors at $x_{\text{TGT}} = 120$ km with unexpected drag uncertainty of $+10\%$ and lift uncertainty of $\pm 10\%$.

| Scenario | $|V_{\text{err}}|$ (m/s) | $|\gamma_{\text{err}}|$ ($^\circ$) | Lift $-10\%$ | Lift $+10\%$ |
|----------|-----------------|-----------------|--------------|--------------|
| Scenario A | 30.43 (8.69%) | 2.82 (5.30%) | 18.54 | 3.43 |
| Scenario B | 31.67 (9.05%) | 3.02 (5.66%) | 19.82 | 4.11 |
| Scenario C | 48.06 (13.73%) | 3.28 (9.51%) | 33.28 | 6.37 |

5.2. Case 2: influence of uncertainty in the models

The actual environment in which a missile operates is different from the models used for the closed-loop simulation. Terminal errors may increase in the presence of modelling uncertainties. In this case, a variation of $\pm 10\%$ in the drag and lift models are considered. The nominal values of the models are used to find the waypoints, and the models with uncertainty are used to calculate the terminal errors. Tables 8 and 9 summarize the simulation results with the uncertainties in the drag model.

The terminal velocity error increases by a factor of 100 compared to the previous cases. However, a model uncertainty of 10%, which is large, is not easily considered in the development phase. With this consideration, a terminal velocity error near a Mach number of 0.1 shows that the framework has the capability to manage the energy while reducing the error, even under abnormal circumstances. Using a realistic simulation model, the framework can effectively manage the energy of the missile to satisfy the terminal velocity constraint and provide more accurate results.

6. Conclusions

A design framework for optimizing waypoints to satisfy the terminal velocity and impact angle constraints is proposed. The proposed framework employs the optimal IAC guidance law to achieve the impact angle constraint, and the heuristic search method for determining waypoints to satisfy the velocity constraint, considering a typical mission profile of a ballistic missile that flies in the vertical plane whenever possible and performs a pull-up manoeuvre. The proposed framework consists of several distinct steps: gravity turn planning, candidate generation, candidate pruning and waypoint determination. The candidate generation step generates the waypoint candidates, while the candidate pruning step constructs the feasible area of each waypoint with respect to the given constraints, and waypoint determination selects the best waypoint from among a group of candidates. Simulation results show that the proposed framework provides the optimal trajectory satisfying the terminal constraints.
Disclosure statement

No potential conflict of interest was reported by the author(s).

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