Quantum pumping: Coherent Rings versus Open Conductors

M. Moskalets\textsuperscript{1} and M. Büttiker\textsuperscript{2}

\textsuperscript{1}Department of Metal and Semiconductor Physics, National Technical University "Kharkov Polytechnical Institute", 61002 Kharkov, Ukraine
\textsuperscript{2}Département de Physique Théorique, Université de Genève, CH-1211 Genève 4, Switzerland

(Dated: March 22, 2022)

We examine adiabatic quantum pumping generated by an oscillating scatterer embedded in a one-dimensional ballistic ring and compare it with pumping caused by the same scatterer connected to external reservoirs. The pumped current for an open conductor, paradoxically, is non-zero even in the limit of vanishing transmission. In contrast, for the ring geometry the pumped current vanishes in the limit of vanishing transmission. We explain this paradoxical result and demonstrate that the physics underlying adiabatic pumping is the same in open and in closed systems.

PACS numbers: 72.10.-d, 73.23.-b, 73.40.Ei

Adiabatic particle transport under slow cyclic evolution of an internal potential has a long history\cite{1}. However, only recently was such adiabatic transport investigated experimentally in open phase coherent mesoscopic conductors\cite{2-4}. This has stimulated increasing interest in this subject\cite{3, 4, 6, 7, 8}. If the scatterer is connected to the external reservoirs Fig.1a then the scattering matrix approach to low frequency ac transport is small compared to the inverse time \(\tau^{-1}\) taken for carriers to traverse the scatterer, then such a pump can be termed adiabatic. Brouwer Ref.\cite{3} gave an elegant formulation of adiabatic (\(\omega \tau \ll 1\)) quantum pumping based on the scattering matrix approach to low frequency ac transport in phase coherent mesoscopic systems\cite{3}.

In open systems the electron spectrum is continuous and even a slowly oscillating scatterer induces transitions between electron states. Therefore a purely quantum-mechanical adiabaticity condition is always violated. However if the oscillation frequency \(\omega\) is small compared to the inverse time \(\tau^{-1}\) for the current in the open and closed cases differ significantly. To break the time reversal invariance\cite{3, 20, 22}.

Interestingly, we find that the expressions for the pumped current in the open and closed cases differ significantly. To illustrate this difference we consider a simple specific model: A scatterer with two one-channel leads. In the absence of magnetic fields such a model is described by the symmetric 2 \(\times\) 2 scattering matrix

\[
\hat{S} = \begin{pmatrix} \sqrt{R} e^{-i\theta} & i\sqrt{T} \\ i\sqrt{T} & \sqrt{R} e^{i\theta} \end{pmatrix}.
\] (1)

Here \(R\) and \(T\) are the reflection and the transmission probability, respectively (\(R+T = 1\)). The phase \(\theta\) characterizes the asymmetry of particle reflection to the left and to the right.

We assume the quantities \(R, T = 1 - R, \theta\) to be functions of external parameters varying with frequency \(\omega\). If the scatterer is connected to the external reservoirs Fig.1 then the adiabatically pumped current \(I_{dc}\) is

\[
I_{dc}^{(open)} = \frac{e \omega}{4\pi^2} \int_0^T dt R \frac{\partial \theta}{\partial t}.
\] (2)

Here \(T = 2\pi/\omega\) is the period of a pumping cycle. For the closed ring-geometry Fig.1b, we will show below that each energy level \(E^{(i)}\) can carry a pumped current \(I_{dc}^{(i)}\) given by

\[
I_{dc}^{(i)} = \frac{e \omega}{4\pi} (-1)^i \int_0^T dt \sqrt{R} \frac{\partial \theta}{\partial t}.
\] (3)

The full current circulating in a ring is given by the sum over all occupied levels. Eq. (3) is valid only if \(R \neq 0\).

There is a striking difference between Eq. (2) and Eq. (3): Eq. (2) predicts pumping even in the limit of \(R = 1\) if only the phase \(\theta\) changes by \(2\pi\) during a pump cycle. This result is paradoxical because at \(R = 1\) the two reservoirs are in fact completely decoupled from each other.

In contrast, for

\[ V_2(t) \]
\[ V_1(t) \]
\[ \alpha=1 \]
\[ \alpha=2 \]
\[ x=L \]
\[ x=0 \]
\[ S(t) \]
\[ I_{dc} \]
\[ I_{dc} \]
\[ I_{dc} \]

FIG. 1: A quantum dot with scattering matrix \(\hat{S}\) and two leads. Two nearby metallic gates modulate the shape and hence the scattering properties of the dot. If the gate potentials \(V_1\) and \(V_2\) change cyclically but shifted in phase then a current \(I_{dc}\) can arise in the leads. (a) - in an open conductor the current \(I_{dc}\) flows between the external reservoirs; (b) - in a closed conductor the current \(I_{dc}\) flows along a ring of length \(L\) formed by the leads. The Greek letter \(\alpha\) numbers the scattering channels.
the ring, the expression for the current Eq. (3) seems to be more reasonable because it gives no pumped current at \( R = 1 \) (when the ring is transformed into a wire disconnected from a cavity).

We now first discuss the resolution of this puzzling difference and only subsequently discuss the derivation of Eq. (3). To resolve the paradox we analyze the topology of adiabatic pump cycles and show that not each cycle is a genuine pump cycle. Moreover for a true pump cycle both Eq. (2) and Eq. (3) simultaneously either give a pumped current or give no pumped current.

From Eq. (2) it follows that the charge \( Q = (2\pi/\omega) I_{dc}^{\text{(open)}} \) pumped during the LPC in the open case is exactly quantized \( Q[(\text{LPC})_n] = ne \). In some papers \([8, 14, 21]\) this, in fact, topological result was used to analyze the conditions for quantization of the pumped charge. However, we can ask: How can a charge \( ne \) be pumped between reservoirs if during the cycle under consideration the reservoirs are completely decoupled from each other since \( R = 1 \)?

If the sample is characterized by the scattering matrix Eq. (1) then any pump cycle can be represented by some closed curve in the plane with \( \sqrt{R} \) and \( \theta \) being the polar coordinates. Because the maximum value for \( R \) is unity each pump cycle lies inside the circle of radius \( R = 1 \). This circle (shown in Fig. 2a) itself represents a pump cycle. We call this cycle a "limiting pump cycle" (LPC).

In fact there is a set of cycles which differ from each other by how many times \( n \) the curve encircles the origin. We will use this winding number \( n \) to distinguish different LPC's. During the (LPC)_n the parameters of the scattering matrix change as follows: \( R = 1 \), \( 0 \leq \theta < 2\pi n \). Note that any pump cycle with \( R(t) \leq 1 \) characterized by the winding number \( n \) lies inside the (LPC)_n.

The answer is the following. During the LPC the charge \( ne \) comes from the left reservoir and accumulates on the left side of the sample. In addition the same charge flows from the right side of the sample to the right reservoir. As a result the charge \( ne \) is effectively transferred between the reservoirs. But this is not only the result of the LPC. There is an unavoidable (dipole) charge accumulation inside the sample during the LPC. Formally we can show this as follows. Since the direct transmission through the sample is prohibited, \( S_{12} = S_{21} = 0 \), the sample can effectively be viewed as a mesoscopic capacitor \([23, 24]\). The left and the right sides of a sample are the plates of a capacitor which connect to the left and to the right reservoirs, respectively. We can define the (one-channel) scattering matrices \( S_L \) and \( S_R \) for the left and for the right plates, respectively: \( S_L \equiv S_{11} = e^{-i\theta} \), \( S_R \equiv S_{22} = e^{i\theta} \). According to the Friedel sum rule \([25]\) the variation of the scattering matrix defines the variation of the charge on the scatterer: \( \delta Q = \frac{\delta}{\delta\theta} \ln(\det(S)) \). Therefore the charge variation on the plates of a capacitor is

\[
\delta Q_L = -\frac{\delta\theta}{2\pi} e, \quad \delta Q_R = \frac{\delta\theta}{2\pi} e.
\]

Although formally the scattering matrices \( S_L \) and \( S_R \) are periodic in \( \theta \) with the period of \( 2\pi \), the absolute value of \( \theta \) has nevertheless a strict physical meaning: The change of \( \theta \) determines the change of the charge of a capacitor. Thus we can conclude that after each LPC the sample does not return to its initial state but rather the sample accumulates some dipole charge inside: \( Q_R[(\text{LPC})_n] = -Q_L[(\text{LPC})_n] = ne \). Note that the same amount of charge \( ne \) is effectively transferred between the reservoirs (during this cycle). Due to the build-up of a dipole charge the scatterer cannot operate for an infinitely long time and therefore the LPC is not a "true" pump cycle.

To obtain a true pump cycle (with no dipole charge accumulation inside the scatterer) we have to return the sample to its initial state. To this end we need to discharge the capacitor. Formally this means that during such a process (discharging) the parameter \( \theta \) has to change from \( 2\pi n \) to zero (if the cycle starts with \( \theta = 0 \)). Physically this means that we have to make an electrical contact between the plates. In other words, the sample has to become (at least partially) transmitting for a moment.

The discharging can be realized in a number of ways. For instance, we can transform any \( (\text{LPC})_n \) into a "true limiting pump cycle" (TLPC)_n as shown in Fig. 2b for \( n = 1 \). In this case the overall pumped charge remains the same \( Q = en \). Importantly, the system now returns to its initial state after the completion of each pump cycle. Hence the TLPC can be repeated as many times as desired.

From this discussion one can see that in the integral representation Eq. (2) generally consists of two parts. The first is a true pumped current which results from the direct charge exchange between the outside reservoirs. The second is a pseudo pumped current which is a consequence of a charge exchange between the scatterer and each of the reservoirs separately. Strictly speaking this last part does not follow from the calculations of the pumped current (see e.g., Ref. \([3]\)) and it arises exclusively due to the representation of the pumped current as a contour integral in the scattering matrix space. To be consistent we can use the integral representation Eq. (2) only with the restriction that any cycle showing a pseudo pump effect must be excluded. Thus the (true) pumped current has no contribution coming from the topology. This is in agreement with Ref. \([10]\).

We can therefore conclude that for any true pump cycle Eqs. (2) and (3) both give either zero or give a pumped current. Thus the same scatterer subject to the same (true) pump cycle produces current in the open case Fig. 1a as well as in the closed case Fig. 1b. Therefore the physics responsible for generating a pump effect is the same in open and in closed geometries. Of course because of the different spectra (con-

![FIG. 2:](image)
The pumping currents in an open and in a closed system can be of very different magnitudes.

We use the scattering matrix approach to pumping in closed systems developed in Ref. 22. This allows us to consider the pump effect in closed and open cases on the same footing. To clarify the essential physics of an adiabatic quantum pump effect in closed systems we consider a simple model: A one-dimensional ring of length $L$ with embedded scatterer (a quantum dot) of a small size $w \ll L$ (see Fig. 11). The quantum dot is characterized by the $2 \times 2$ scattering matrix $S$. We are interested in dc current arising in a ring under the slow cyclic evolution of the scattering properties of a quantum dot. We assume that there are no other effects which could generate circulating currents. In particular, (i) there is no magnetic flux through the ring; (ii) the stationary properties of an adiabatic quantum dot are completely described by the stationary scattering matrix $S$ with parameters depending on time $S(t) = S(p_i(t))$ 24. For instance, the Fourier coefficients $S_{nm}$ of this scattering matrix define the amplitudes $A_n = \sqrt{k/k_n}S_{nm}$ for scattering (transmission or reflection) of an electron with energy $E = h^2k^2/(2m_e)$ with the emission $(n < 0)$ or the absorption $(n > 0)$ of $n$ energy quanta $\hbar\omega$.

In the adiabatic limit 3 24 knowledge of the solution of a scattering problem with small oscillating amplitudes is sufficient to calculate the pumped current in the lowest (first) order in $\omega$ at arbitrary oscillating strength (amplitudes). Therefore, first, we consider the case when the parameters oscillate with small amplitudes: $p_i(t) \ll p_{0i}$, $\forall i$. We calculate the current in the lowest nonvanishing order in the oscillating amplitudes. In this case it is enough to take into account only the first sidebands 18. Thus in the expansion Eq. 4 we keep only the terms with $n = 0, \pm 1$ (we put all the coefficients $A_n, B_n$ for $|n| > 1$ equal to zero). The scattering matrix relates the incoming waves $A_n$, $B_n$ to outgoing ones $A_n e^{-ik_nL}$, $B_n e^{-ik_nL}$. We number the scattering channels as shown in Fig. 11. Thus the scattering matrix defines the boundary conditions for an electron wave function Eq. 5 $(n = 0, \pm 1)$ as follows 22:

$$A_n e^{-ik_nL} = \sum_{m=0,\pm 1} \frac{\sqrt{k_m}}{k_n} \times (A_{n-m}S_{21,m} + B_{n-m}S_{22,m})$$

$$(9)$$

$$B_n e^{-ik_nL} = \sum_{m=0,\pm 1} \frac{\sqrt{k_m}}{k_n} \times (A_{n-m}S_{11,m} + B_{n-m}S_{12,m}).$$

Note that on the RHS of the above equations for $n = \pm 1$ we have to put $A_{1\pm 2} = 0$ and $B_{0\pm 2} = 0$. To obtain the current Eq. 7 to first order in $\omega$ we expand $e^{-i\Delta k_{\pm 1}L}$ in Eq. 10 as follows:

$$e^{-i\Delta k_{\pm 1}L} \approx e^{-i\Delta k_{\pm 1}L} \left(1 + i\frac{\omega}{\omega_0} - \frac{1}{2}\left(\frac{\omega}{\omega_0}\right)^2 \pm \frac{i}{6}\left(\frac{\omega}{\omega_0}\right)^3\right),$$

$$(10)$$

where $\omega_0 = v/L$, and $v = \hbar k/m_e$ is an electron velocity. In the above expansion we ignore all the terms containing additional small factors $\omega/E$.

Using Eq. 5 after a lengthy but rather straightforward calculation we obtain the circulating current Eq. 6 (we restore the upper index $^{(i)}$)

$$I^{(i)}_{dc} = e\omega Im \left[\Gamma^{(i)}(\theta^{(i)}_\theta^{(i)})\right].$$

$$(11)$$

Here we have introduced two real quantities. The first one is characteristic of the spatial asymmetry of the scatterer: $\theta = \frac{i}{2}Im(n_{11}/s_{22})$. This quantity is real since $|s_{11}|^2 = |s_{22}|^2$. The second one is $\Gamma^{-1} = -i(e^{-ikL}s_{12} - 1)$, where $K = k(p_{0i})$ is the solution of the dispersion equation for the stationary problem (with $p_i = p_{0i}$). This dispersion equation reads: $|e^{-ikL}s_{12} - 1| = 0$. From the dispersion equation it follows that the imaginary part of $\Gamma$ vanishes. In particular, for the scattering matrix Eq. 11 we have $K = k^{(0)} = \frac{i}{2}(m - (-1)^i \arcsin(\sqrt{T}))$ and $\Gamma^{(i)}(\theta^{(i)}) = (-1)^i \sqrt{T/R}$. Equations 11 determines the current carried by the particular energy level $E^{(i)}$. To find the full circulating current we have to sum Eq. 11 over all the occupied levels in the ring. Equation 11 shows that the adiabatically pumped current

\[\text{\textit{continuously and discrete}}\]
In equation (12) the integrand should be considered as a function of the time-dependent parameters $p_i = p_i(t)$ and the eigenenergy $E_i(t) = E(t)\{p_i(t)\}$ which adiabatically follows them.

The integral representation Eq. (12) allows us to calculate the circulating current for the pump cycle of an arbitrary strength. The only necessary condition is that the adiabaticity conditions Eq. (5) must hold at any point of a pump cycle. Note that the level spacing depends on time since each eigenenergy is a function of time. Therefore our consideration is valid if there is no level crossing ($\Delta(E) \neq 0$) during the pump cycle.

We would like to stress that the dual representation for the pumped current in Eq. (12) shows clearly that only true pump cycles Fig. 2 contribute to the calculated quantity - the pumped current. This is in full agreement with the Floquet scattering matrix approach to the pump effect in the open case [21]. The integral representation (see Eq. (18) in Ref. [24]) for the pumped current is a direct consequence of a differential representation (see Eq. (17) in Ref. [24]). The latter does not support a pseudo pump effect.

In conclusion, we have developed the scattering matrix approach to adiabatic quantum pumping in closed mesoscopic systems such as a ring with an embedded quantum dot. This formulation permits a direct comparison of pumping in open and closed systems. We have discussed the seemingly paradoxical nature of the result for open systems. Closer inspection of the two results demonstrates that the physics underlying the adiabatic quantum pump effect in closed systems is very similar to that in open systems coupled to external reservoirs. The approach presented can be generalized to many channel rings and to closed systems with a more complicated topology. Experimental comparisons of pumping in open and closed systems would be very desirable.

This work is supported by the Swiss National Science Foundation.

[1] D.J. Thouless, Phys. Rev. B 27, 6083 (1983).