Remarks on Haar meager sets and Haar null sets in spaces of sequences

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Abstract

In the paper we will show how to construct a Haar meager set (consequently meager) which is not Haar null, and conversely, a meager Haar null set which is not Haar meager in spaces of sequences: $l^p$ with $p \geq 1$, $c_0$ or $c$. It refers to the paper [2].

Keywords: Haar meager set, Haar null set, meager set

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1. Introduction

In 1972 J.P.R. Christensen defined "Haar null" sets in an abelian Polish group (a topological abelian group with a complete separable metric) in such a way that in a locally compact group it is equivalent to the notion of Haar measure zero set. More precisely, in a fixed abelian Polish group $X$ a set $A \subset X$ is called Haar null if there is a Borel probability measure $\mu$ on $X$ and a Borel set $B \subset X$ such that $A \subset B$ and $\mu(x + B) = 0$ for all $x \in X$. These definition has been extended further to nonabelian groups by J. Mycielski [7]. Unaware of the result of Christensen, B.R. Hunt, T. Sauer and J.A. Yorke [3]-[4] found this notation again, but in a topological abelian group with a complete metric (not necessary separable).

In 2013 U.B. Darji introduced another family of "small" sets in an abelian Polish group, which is equivalent to the notion of meager sets in a locally compact group. In an abelian Polish group $X$ he called a set $A \subset X$ Haar meager if there is a Borel set $B \subset X$ with $A \subset B$, a compact metric space $K$ and a continuous function $f : K \rightarrow X$ such that $f^{-1}(B + x)$ is meager in $K$ for all $x \in X$.

The main aim of the paper is to show easy constructions of a Haar meager, but not Haar null set and, conversely, a meager Haar null set which is not Haar meager in spaces of sequences: $l^p$ with $p \geq 1$, $c_0$ or $c$.
2. The main results

**Definition 1.** Let $X$ be an abelian Polish group, $\mathcal{B}(X)$ be the Borel $\sigma$–algebra on $X$ and denote by $\mathcal{F}(X)$ the family of all sets $A \subset X$ such that

$$\forall_{K \subset X\text{-compact}} \exists_{x_K \in X} K + x_K \subset A.$$ 

In fact $\mathcal{F}(X)$ is a proper linearly invariant $\sigma$–filter. What is interesting, each set $A \in \mathcal{F}(X) \cap \mathcal{B}(X)$ is neither Haar null (in view of the Ulam theorem), nor Haar meager.

S. Solecki [8], and also E. Matoušková and M. Zelený [5], showed how to find a closed nowhere dense set from the family $\mathcal{F}$ in any abelian non-locally compact Polish group. We use this fact to construct a Haar meager, but not Haar null set, as well as a meager Haar null set which is not Haar meager in spaces of sequences: $l_p$ with $p \geq 1$, $c_0$ or $c$.

First we prove two theorems, which we will use in further considerations.

**Theorem 1.** Let $X, Y$ be an abelian Polish group. If $A \subset X$ is Haar meager and $B \subset B_0$ for some $B_0 \in \mathcal{B}(Y)$, then the set $A \times B \subset X \times Y$ is Haar meager.

**Proof.** Assume that $A \subset X$ is Haar meager in $X$, i.e. there are a set $A_0 \in \mathcal{B}(X)$ with $A \subset A_0$, a compact metric space $K$ and a continuous function $f : K \rightarrow X$ such that $f^{-1}(A_0 + x)$ is meager in $K$ for each $x \in X$. Take any compact set $L \subset Y$ and define a continuous function $g : K \times L \rightarrow X \times Y$ in the following way:

$$g(k, l) = (f(k), l) \text{ for every } (k, l) \in K \times L.$$ 

Then,

$$g^{-1}((A_0 \times B_0) + (x, y)) = g^{-1}((A_0 + x) \times (B_0 + y))$$

$$= f^{-1}(A_0 + x) \times [(B_0 + y) \cap L]$$

for each $(x, y) \in X \times Y$. Since the set $f^{-1}(A_0 + x)$ is meager in $K$, by the Kuratowski-Ulam theorem the set $g^{-1}((A_0 \times B_0) + (x, y))$ is meager in $K \times L$. Clearly, $A \times B \subset A_0 \times B_0$ and $A_0 \times B_0 \in \mathcal{B}(X \times Y)$, what ends the proof.

**Theorem 2.** Let $X, Y$ be an abelian Polish group. For every set $A \in \mathcal{B}(X) \cap \mathcal{F}(X)$ and non-Haar meager $B \in \mathcal{B}(Y)$ the set $A \times B \subset X \times Y$ is not Haar meager.

**Proof.** Clearly, $A \times B \in \mathcal{B}(X \times Y)$. Take a compact metric space $K$ and a continuous function $f : K \rightarrow X \times Y$. Then there are continuous functions $f_X : K \rightarrow X$ and $f_Y : K \rightarrow Y$ such that

$$f(z) = (f_X(z), f_Y(z)) \text{ for each } z \in K.$$
The set $f_X(K)$ is compact in $X$ and $A \in \mathcal{F}(X)$, so $A \supset f_X(K) + x_K$ for some $x_K \in X$. Hence $f_X^{-1}(A - x_K) \supset K$. Since $B$ is not Haar meager in $Y$, $f_Y^{-1}(B + y_K)$ is comeager in $K$ for some $y_K \in Y$. Moreover:

$$f^{-1}((A \times B) + (-x_K, y_K)) = f^{-1}((A - x_K) \times (B + y_K)) = f_X^{-1}(A - x_K) \cap f_Y^{-1}(B + y_K) \supset K \cap f_Y^{-1}(B + y_K).$$

Thus $f^{-1}((A \times B) + (-x_K, y_K))$ is comeager in $K$ and, consequently, the set $A \times B$ is not Haar meager in $X \times Y$.

The above theorem suggest the following

**Problem 1.** Let $X$, $Y$ be an abelian Polish group. Is it rue or false that for every non-Haar meager sets $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ the set $A \times B \subset X \times Y$ is not Haar meager?

A negative answer implies the same question under additional assumption that one of abelian Polish group $X$, $Y$ is locally compact.

3. Applications

Now, consider the space $X$ as one of the following spaces of sequences: $c_0$, $c$ or $l_p$ with $p \geq 1$. Such spaces have a very nice property: $X = \mathbb{R} \times X$.

Fix $A \in \mathcal{B}(X) \cap \mathcal{F}(X)$. Let $S := B \times A \subset \mathbb{R} \times X$, where $B \subset \mathbb{R}$ is a meager set of positive Lebesgue measure.

By the analogue of the Fubini theorem [1, Theorem 6] it is easy to observe that $S$ is not Haar null, since $S(a) = B$ is the set of the positive Lebesgue measure for each $a \in A$ and $A$ is not Haar null. Moreover, in view of Theorem [1] the set $S$ is Haar meager. In this way we constructed the set $S$, which is Haar meager (consequently meager), but not Haar null in $X$.

Now, fix $A \in \mathcal{B}(X) \cap \mathcal{F}(X)$ once again. Let $T := C \times A \subset \mathbb{R} \times X$, where $C \subset \mathbb{R}$ is a comeager set of the Lebesgue measure zero.

By an analogue of the Fubini theorem [1, Theorem 6] we can easy deduce that $T$ is Haar null in $X$, because $T(a) = C$ is the set of the Lebesgue measure zero for each $a \in A$ and $T(a) = \emptyset$ for each $a \in X \setminus A$. Moreover, by Theorem [2] the set $T$ is not Haar meager in $\mathbb{R} \times X = X$. Finally, $T$ is meager according to the Kuratowski-Ulam theorem, because the set $A$ is meager. In this way we constructed the set $T$, which is Haar null, meager, but not Haar meager in $X$.

**Example 1.** Define the set $A = \left\{(x_n)_{n \in \mathbb{N}} \in c_0 : \forall n \in \mathbb{N} \; x_n \geq 0\right\}$. Such set is a closed nowhere dense set (see [6, Example 6.2]), which belongs to the filter $\mathcal{F}(c_0)$.

Let $S := B \times A \subset c_0$, where $B \subset \mathbb{R}$ is a meager set of the positive Lebesgue measure. Then the set $S$ is Haar meager, but not Haar null in $c_0$. 

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Let $T := C \times A \subset c_0$, where $C \subset \mathbb{R}$ is a comeager set of the Lebesgue measure zero. Such set $T$ is Haar null, meager, but not Haar meager in $c_0$.

**Problem 2.** Let $X$ be any abelian Polish group, which is not locally compact. How to find a Haar meager but not Haar null set and, conversely, how to construct a Haar null, meager but not Haar meager set in $X$?

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