AX-SCHANUEL AND EXCEPTIONAL INTEGRABILITY

JONATHAN PILA AND JACOB TSIMERMAN

Abstract. When can a primitive of a given algebraic function be con-
structed by iteratively solving algebraic equations and composing with
the primitives of some other given algebraic functions or their inverses?
We establish some results in this direction. Specifically, we establish
decision procedures for determining whether a given primitive can be
expressed in terms of finitely many others, or in terms of elliptic inte-
grals.

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1. Introduction and main results

This paper is concerned with the following question. Given an algebraic
function \( \alpha(x) \), when can a primitive (i.e. an anti-derivative) of it be con-
structed by iteratively solving algebraic equations and composing with
the primitives of some other given algebraic functions or their inverses? This is
the exceptional integrability of the title.

Definition. Let \( B = (B_1, \ldots, B_k) \in \mathbb{C}[X,Y]^k \) be a sequence of irreducible
polynomials. We say that a function \( z(x) \), regular on some disk \( \Delta \subset \mathbb{C} \), is
\( B \)-strictly elementary if there is an open disk \( \Delta^* \subset \Delta \) and a sequence of
pairs of functions \((x_1, z_1), \ldots, (x_k, z_k)\), all regular in \( \Delta^* \) (as functions of \( x \)),
such that:

1. For each \( i = 1, \ldots, k \), \( B_i(x_i, \frac{dz_i}{dx_i}) = 0 \);
2. For each \( i \), either \( x_i \) or \( z_i \) is algebraic over \( \mathbb{C}(x, x_1, z_1, \ldots, x_{i-1}, z_{i-1}) \);
3. \( z \) is algebraic over \( \mathbb{C}(x, x_1, z_1, \ldots, x_k, z_k) \).
Let $S \subset \mathbb{C}[X,Y]$ be a set of irreducible polynomials. We say that a function $z$ is $S$-elementary if there is a non-negative integer $k$ and a tuple $(B_1, \ldots, B_k) \in S^k$ such that $z$ is $(B_1, \ldots, B_k)$-strictly elementary.

By an algebraic function we mean a function $\alpha(x)$, regular on an open disk $\Delta \subset \mathbb{C}$, such that there is an algebraic relation $A(x, \alpha(x)) = 0$ for $x \in \Delta$. Here $A \in \mathbb{C}[X,Y]$ is non-zero and irreducible. Thus an $\emptyset$-elementary function is simply an algebraic function of $x$.

An $(XY - 1)$-elementary function is what is classically known as an elementary function, as we have $z_i = \log x_i + c$ or $x_i = c \exp z_i$, and the question of whether a given function has an elementary primitive is the question of elementary integrability. Elementary integrability is characterized by a classical theorem of Liouville (see e.g. [19]; for extensions see e.g. [23]). Based on this result, Risch [18] gave an algorithm to decide, in various situations, whether a given elementary function has an elementary primitive (see also [5] [10]).

Definition. Suppose that $w(x)$ is a function regular on an open disk $\Delta \subset \mathbb{C}$, and $B = (B_1, \ldots, B_k) \in \mathbb{C}[X,Y]^k$ and $S \subset \mathbb{C}[X,Y]$ as before. We say that $w$ is $B$-strictly integrable if some (equivalently every) primitive $z$ of $w$ is $B$-strictly elementary. We say that $w$ is $S$-integrable if some (equivalently every) primitive of $w$ is $S$-elementary, and we will say alternatively that $w$ is integrable in terms of $S$.

We are concerned then with the question of when an algebraic function $\alpha(x)$, defined by some $A(x, \alpha(x)) = 0$, is integrable in terms of some given set $S$. We will see that $(B_1, \ldots, B_k)$-strict integrability is really a property of $A$ and the $B_i$, not the specific branch $\alpha$. More precisely, if some branch $\alpha(x)$ of $A(x, y) = 0$ on some disk $\Delta$ is $(B_1, \ldots, B_k)$-strictly integrable then, given any other branch $\alpha'$ and disk $\Delta'$, $\alpha'$ is $(B_1, \ldots, B_k)$-strictly integrable on (some open subdisk $\Delta'_*$ of) $\Delta'$. In fact, suitably formulated, it is a property of $A, B_1, \ldots, B_k$ in any differential field.

Our first main result is the following, which we will establish in a more precise version in the setting of a differential field.

**Theorem 1.1.** Suppose that $\alpha(x)$ is an algebraic function on a disk $\Delta$ which is $(B_1, \ldots, B_k)$-strictly integrable. Then there is a regular function $x_1, \ldots, x_k, z_1, \ldots, z_k$ and $z$ on $\Delta_*$, constants $c_1, \ldots, c_k$, and an algebraic function $\gamma(x)$, such that:

1. $z$ is a primitive of $\alpha$
2. For $i = 1, \ldots, k$, $B_i(x_i, \frac{dz}{dx_i}) = 0$
3. For $i = 1, \ldots, k$, $x_i$ is algebraic over $x$
4. $z = \sum_{i=1}^k c_i z_i + \gamma$.

This can be seen as a generalization (to arbitrary $B_i$) of the restriction of Liouville’s theorem to algebraic functions $\alpha$. Liouville’s Theorem implies (in particular) the same condition for the elementary integrability of any elementary function. The converse clearly holds.
The key observation behind the theorem is that $B$-strict integrability implies that we are in the exceptional case of the Ax-Schanuel theorem for a suitable connected commutative complex algebraic group $G$ (more specifically, a product of generalized Jacobians). We will see further that the linear relation afforded by the theorem is of a particular form: it corresponds to a defining equation of a coset of a suitable algebraic subgroup of $G$. Looking for such a connection was suggested by the results of Masser-Zannier [13] (see also [25]) connecting questions of elementary integrability in a pencil of meromorphic differentials on curves with problems of Zilber-Pink type.

Let $\alpha(x)$ be an algebraic function defined by $A(x, \alpha(x)) = 0$. By supplying some additional algebraic functions $\xi = (x, \ldots)$, algebraic over $x$, we can associate with $A$ a smooth projective curve $X$ and meromorphic differential $\omega$ on $X$ with $\omega = \alpha(x)dx$. We can then recast the integrability problem in terms of pairs $(X, \omega)$ of smooth projective curves and meromorphic differentials. This will lead us to a geometric reformulation and refinement of the above result. We will write $z = \int^{x} \omega$ to mean that $z$ is a multi-valued function on $X$ obtained by integrating the form $\omega$. If $x$ is a uniformizing function on $X$ then there is some associated polynomial $A \in \mathbb{C}[X, Y]$ (depending on $X$, $\omega$, and $x$) such that $\omega = \alpha(x)dx$ where $A(x, \alpha(x)) = 0$. Thus considering $z$ as a function of $x$ we have $A(x, dz/dx) = 0$.

**Definition.** Let $(X_0, \omega_0)$ and $(X_1, \omega_1), \ldots, (X_k, \omega_k)$ be pairs of smooth projective curves over $\mathbb{C}$ and meromorphic differentials. We say that $(X_0, \omega_0)$ is $((X_1, \omega_1), \ldots, (X_k, \omega_k))$-strictly integrable if for some (or equivalently any) choice of non-constant rational functions $x_0, x_i$ of the curves, with associated polynomials $A, B_i \in \mathbb{C}[X, Y]$, setting $z_0 = \int^{x_0} \omega, z_i = \int^{x_i} \omega_i$, we have that $z_0$ is $(B_1, \ldots, B_k)$-strictly integrable. We define $S$-integrability for finite sets of curves and differentials in analogy with the definition for algebraic functions.

We will prove a reformulation of Theorem 1.1 in terms of curves and differentials. This result (Theorem 3.3) is stated and proved in section 3. The algebraic relations among the uniformizing variables gives a curve $Z \subset X_0 \times X_1 \times \ldots X_k$ giving a correspondence between $X_0$ and each $X_i$. This allows us to pull back differentials from $X_i$ to $Z$ and to express integrability in terms of their traces to $X_0$.

**Definition.** Let $(X_i, \omega_i), i = 1, 2$ be curves with differentials. Let $Z \subset X_1 \times X_2$ be a curve with no fibral-components. Then if $\omega_2 = \pi_{2*} \pi_1^{*} \omega_1$ we say that $\omega_2$ is a trace-image of $\omega_1$.

**Theorem 1.2.** Let $(X_i, \omega_i), 0 \leq i \leq k$ be smooth projective curves with meromorphic differentials. The following conditions are equivalent.

1. $(X_0, \omega_0)$ is integrable in terms of $\{(X_1, \omega_1), \ldots, (X_k, \omega_k)\}$
2. $\omega_0$ is in the linear span of differentials on $X_0$ which are trace-images of the $\omega_i$, and exact differentials.
This theorem can then serve as the basis of a decision procedure when all the curves and differentials are defined over an explicitly given finite type field. This generalizes to general sets $S$ the special case of Risch’s algorithm in which $\alpha$ is algebraic rather than just elementary (on this see [5]).

**Theorem 1.3.** Let $K$ be a finite type field of characteristic zero. There is a decision procedure with the following property. Given pairs $(X_0, \omega_0)$ and $(X_1, \omega_1), \ldots, (X_k, \omega_k)$ of smooth projective complex curves and meromorphic differentials on them, defined over $K$, the procedure decides whether $(X_0, \omega_0)$ is integrable in terms of $\{(X_1, \omega_1), \ldots, (X_k, \omega_k)\}$.

Hardy, in his book [7] on the integration of univariate functions, discusses (p9-10) the known results concerning the integration of algebraic functions, in particular concerning whether they are or are not elementary. He notes (p10) that there are cases in which “integrals associated with curves whose deficiency [i.e genus] is greater than unity are in reality reducible to elliptic integrals”, and observes that “no general method has been devised by which we can always tell, after a finite series of operations, whether any given integral is really elementary, or elliptic, or belongs to a higher order of transcendents”. A precise definition of “reducible to elliptic integrals” is not given, but the following could be considered as providing a procedure for a natural formulation of this question.

**Theorem 1.4.** Let $K$ be a finite type field of characteristic zero. Let $(X, \omega)$ be a curve with a rational differential form, defined over $K$. Then there is a decision procedure for determining whether $(X, \omega)$ is integrable in terms of the set of all elliptic curves (over $\overline{K}$) with all rational differentials (over $\overline{K}$) on them.

The plan of the paper is as follows. In §2 we prove Ax-Schanuel for a commutative complex algebraic group, generalizing Bertrand’s generalization [3] of results of Ax [1, 2] (see also Kirby [9]). In §3 we recall the definition and some properties of generalized Jacobians and reformulate exceptional integrability in terms of differentials on curves. In §4 we recall the statement of the Seidenberg Elimination Theorem. Then in §5 we reformulate both exceptional integrability and Ax-Schanuel in the setting of a differential field and prove a version of Theorem 1.1 in that setting, making extensive use of §4. §6 interprets meromorphic forms on a curve in terms of cohomology. Theorem 1.2 is then proved in §7 followed by the description of the decision procedures. The final section §8 connects integrability questions for curves with regular differentials in a pencil with certain problems of unlikely intersections.

2. **Ax-Schanuel for a connected commutative complex algebraic group**

Let $G$ be a connected commutative complex algebraic group. Recall that such a group is an extension of a semiabelian variety by a vector group.
Following Bertrand [3], say that such a group has no vectorial quotients if there do not exist non-trivial maps $G \rightarrow \mathbf{G}_a$. We begin with the following simple (and certainly well known) lemma:

**Lemma 2.1.** Let $G$ be a connected commutative complex algebraic group. There is a canonical connected subgroup $H$ of $G$ without vectorial quotients, such that $G \cong H \times V$ for some (non-canonical) vector subgroup $V < G$.

**Proof.** We first establish uniqueness. Indeed, such an $H$ would necessarily have to be the maximal subgroup of $G$ which lies in the kernel of every map $G \rightarrow \mathbf{G}_a$. To show existence, define $H$ to be the maximal subgroup of $G$ which lies in the kernel of every map to $\mathbf{G}_a$. Note that since $G$ is connected and $\mathbf{G}_a$ is simply connected, $H$ must also be connected.

Next, let $\phi : G \rightarrow V$ be a map onto a vector group such that $H = \text{Ker}\phi$. By replacing $V$ by the image of $\phi$ we may assume $\phi$ is surjective. It remains to construct a section of $\phi$.

Let $U$ be the unipotent radical of $G$. We claim $\phi(U) = V$. Indeed, if not, we would obtain a surjection map from the semiabelian variety $G/U$ to the vector group $V/\phi(U)$. Finally, as maps between vector spaces split we may find a section of $\phi|_U$, as desired. □

We will call $H$ the maximal NVQ subgroup of $G$. Note that, if $H_i$ are the maximal NVQ subgroups of $G_i$, $i = 1, \ldots, k$, then $H_1 \times \ldots \times H_k$ is the maximal NVQ subgroup of $G_1 \times \ldots \times G_k$. We proceed to prove our main theorem of this section:

**Theorem 2.2** (Ax-Schanuel for a connected commutative complex algebraic group). Let $H$ be a complex connected commutative algebraic group without vectorial quotient, and let $V$ be a vector group. Let $\pi : T \rightarrow H$ be the universal cover of $H$ with graph $D$. Let $Z \subset V \times T \times H$ be an algebraic variety and $U \subset Z \cap (V \times D)$ be an analytically irreducible component. If $\dim U > \dim Z - \dim H$ then the projection of $U$ to $H$ lies in the translate of a proper subgroup (i.e. a proper weakly special).

**Proof.** The case where $\dim V = 0$ is proven by Bertrand [3] We shall reduce to that case. Without loss of generality we can assume that $Z$ is the Zariski closure of $U$.

Let $Z'$ be the projection of $Z$ to $T \times H$ and let $U'$ be the projection of $U$ to $D$. Then $U'$ is a component of $Z' \cap D$. Now since $Z$ is the Zariski closure of $U$, the generic dimension of the fibers of the projection $Z \rightarrow Z'$ is the same as the generic dimension of the fibers of the projection $U \rightarrow U'$. Thus $\dim Z - \dim U = \dim Z' - \dim U'$. The claim therefore follows immediately from the case where $\dim V = 0$. □

It might seem that the above is artificial, and so we explain how to relate it to the usual geometric Ax-Schanuel statement. Indeed, let $G$ be an arbitrary
Now let \( T_G, T_H, T_V \) be the tangent spaces of \( G, H, V \) respectively. The key point is that the map \( \pi_V : T_V \to V \) is an algebraic isomorphism. Therefore, if we let \( D_H, D_G, D_V \) be the graphs, then while \( D_{\text{zar}} H = T_H \times H \), we have \( D_{\text{zar}} V = D_V \) and therefore

\[
D_{\text{zar}} G \cong D_{\text{zar}} H \times D_{\text{zar}} V \cong T_H \times H \times V.
\]

Now if we are to set up the usual Ax-Schanuel as \( Z \subset T_G \times G \) and \( U \) a component of \( Z \cap D_G \), then the assumption of \( Z = U^{\text{zar}} \) would entail that \( Z \subset D_{\text{zar}} G \) and thus this is the natural space to work in.

The projection to \( H \) is non-canonical, but the dimension of a minimal coset containing the image of \( U \) is independent of the choice of projection. The coset can (non-canonically) be given a group structure and we can reframe the theorem in the following way.

**Corollary 2.3.** Let \( H \) be a complex connected commutative algebraic group without vectorial quotient, and let \( V \) be a vector group. Let \( \pi : T \to H \) be the universal cover of \( H \) with graph \( D \). Let \( Z \subset V \times T \times H \) be an algebraic variety and \( U \subset Z \cap (V \times D) \) be an analytically irreducible component. Let \( K \) be a minimal coset containing the image of \( U \) under projection to \( H \). Then

\[
\dim U \leq \dim Z - \dim K.
\]

3. **Ax-Schanuel for complex curves and differentials**

Much of the following follows \cite{22}, but we work throughout over \( \mathbb{C} \).

Let \( X = X(\mathbb{C}) \) be a smooth projective complex algebraic curve, \( \Sigma \subset X \) a finite set of (distinct) points, and \( m \) a modulus with support \( \Sigma \) (i.e. an assignment of a positive integer \( n(P) \) to each \( P \in \Sigma \)).

Associated with this data is the generalized Jacobian \( J_m = J_m(X) \), with \( g_m = \dim J_m = \dim H^0(X, \Omega(\mathfrak{m})) \). We have a map

\[
(X - \Sigma)^{\mathfrak{m}} \to J_m
\]

sending \((x_1, \ldots, x_g)\) to the corresponding divisor class (following the choice of a suitable base point), and which factors through the symmetric product \((X - \Sigma)^{\mathfrak{m}} \to J_m\). The map \((X - \Sigma)^{\mathfrak{m}} \to J_m\) is birational. If we map \( X - \Sigma \to J_m \) sending \( x \) to the corresponding divisor class then the image generates \( J_m \) as a group.

Let \( T_m \) be the lie algebra of \( J_m \), identified with its tangent space at the origin. Here see particularly \cite{22} pages 100–101. See also \cite{13}, pages 46–47. We have the map

\[
(X - \Sigma)^{\mathfrak{m}} \to T_m
\]

given by integrating the differentials in \( H^0(X, \Omega(\mathfrak{m})) \) (again following the choice of a suitable base point in \( X - \Sigma \)), and this map composed with the Lie exponential recovers the divisor class map, if we stipulate that the
tuple of base points maps to the identity. The generalised Jacobian $J_m$ is analytically isomorphic to the quotient of $T_m$ by the group $\Pi_m$ of periods of the differentials.

**Lemma 3.1.** The group $J_m$ has a non-trivial vectorial quotient if and only if there exists a non-trivial exact differential in $H^0(X, \Omega(m))$.

**Proof.** Let $V = H^0(X, \Omega(m))$. We have $J_m(\mathbb{C}) = V^\vee / \Pi$ where $\Pi$ is the group of periods. Then $J_m$ having a non-trivial vectorial quotient is the same as saying the complex span of $\Pi$ is not all of $V^\vee$, which is the same as saying there is a non-zero $v \in V$ which vanishes on $\Pi$, which is the same as saying there is a non-zero differential whose periods vanish. But given such a differential, we can integrate it globally and therefore its exact. $\Box$

Further, the dimension of the largest vectorial quotient is equal to the dimension of the subspace $E_m$ of $H^0(X, \Omega(m))$ of exact $m$-differentials. Let $H_m$ denote the maximal NVQ subgroup of $J_m$, of dimension $h_m$.

Now we consider a finite sequence of pairs $(X_1, \omega_1), \ldots, (X_k, \omega_k)$ where $X_i$ is a smooth projective complex curve and $\omega_i$ is a non-zero meromorphic differential. For $i = 1, \ldots, k$, let $m_i$ be a modulus on $X_i$ with $\omega_i \in H^0(X_i, \Omega(m_i))$. We adopt the above notation, but replace subscripts $m_i$ by $i$. Thus we have the corresponding generalised Jacobians $J_i = J_{m_i}(X_i)$, of dimension $g_i$, their maximal NVQ subgroups $H_i = H_{m_i}$, etc.

Let $G = J_1 \times \ldots \times J_k$ be the product of generalised Jacobians. Let $T = T_1 \times \ldots \times T_k$ its Lie algebra. Then $H = H_1 \times \ldots \times H_k$ is the maximal NVQ subgroup of $G$, of dimension $h = \sum h_i$.

We choose a basis $\omega_{ij}, j = 1, \ldots, g_i$ of $H^0(X_i, \Omega(m_i))$, with $\omega_i = \omega_{i1}$. Pick base points and define locally

$$z_{ij} = \int_{x_i}^{x_i} \omega_{ij}, \quad i = 0, \ldots, k, \quad j = 1, \ldots, g_i$$

where $x_i \in X_i(\mathbb{C})$. We will also write $z_i = z_{i1}$.

By a locus in a variety (or complex space) $W$ we will mean a regular map $w : \Delta \to W$ on some open disk in the complex plane. If we have loci $x_i : \Delta \to X_i - \Sigma_i$ for $i = 1, \ldots, k$ then the divisor class map to the generalized Jacobian, and integration, give a locus

$$(x, z) : \Delta \to G \times T.$$

Here $x = (x_1, \ldots, x_k)$ while $z = (\overline{Z_1}, \ldots, \overline{Z_k})$ with $\overline{Z_i} = (z_{i1}, \ldots, z_{ig_i})$. Let $U$ denote the image of the locus and $Z$ its Zariski closure. Then

$$\dim Z = \text{tr.deg.}(\mathbb{C}(x_i, z_{ij}, i = 1, \ldots, k, j = 1, \ldots, g_i)/\mathbb{C}).$$

Later we will assume that each $\omega_i$ is not exact. It will then be convenient to choose a basis $\omega_{ij}$ of $H^0(X_i, \Omega(m_i))$ in such a way that $\omega_{i1}, \ldots, \omega_{ig_i}$ is a basis of $H^0(X_i, \Omega(m_i))/E_i$. With such a choice, we note that the condition that the projection $U$ to $H$ lies in a coset of a proper subgroup is equivalent
to the existence of constants $c_{ij}$, not all zero, primitives $\gamma_i$ of exact $m_i$-differentials for $i = 1, \ldots, k$, and a constant $c$, such that

$$\sum_{i=1}^{k} h_i \sum_{j=1}^{h_i} c_{ij} z_{ij} + \sum_{i=1}^{k} \gamma_i = c,$$

where this corresponds to a defining relation of a coset of an irreducible algebraic subgroup of $H$. Thus the linear relations given by the theorems are of a particular type, corresponding to suitable “bi-algebraic” varieties under the uniformisation of the group $G$.

**Theorem 3.2** (Ax-Schanuel Theorem for complex curves and differentials). Let $(x, z)$ be a locus in $G \times T$ with image $U$ as above with each $x_i$ non-constant. Suppose that

$$\text{tr.deg.}(\mathbb{C}(x_1, \ldots, x_k, z_1, \ldots, z_k)/\mathbb{C}) \leq k$$

Then there is a non-empty subset $I \subset \{1, \ldots, k\}$, non-zero complex constants $c_i$ and primitives $\gamma_i$ of exact $m_i$-differentials on $X_i$ for $i \in I$, such that, as functions of $t \in \Delta$,

$$\sum_{i \in I} c_i (z_i + \gamma_i) = 0.$$

(Note that the degree of $\gamma_i$ is bounded in terms of $X_i, m_i$.) Further, this relation corresponds to a defining relation of a coset of an algebraic subgroup of $H$.

Suppose $I$ above has $\#I = \ell$ and rename the corresponding curves and differentials as $(Y_i, \eta_i), i = 1, \ldots, \ell$. Then there exists an algebraic curve in $Y_1 \times \ldots \times Y_\ell$, dominant to each factor, such that, writing the generic point as $(\xi_1, \ldots, \xi_\ell)$, and writing $\zeta_i = \int \xi_i \eta_i, i = 1, \ldots, \ell$, we have

$$\sum_{i = 1}^{\ell} c_i \zeta_i = \gamma(\xi_1)$$

with the same complex constants $c_i$ as above and an algebraic function $\gamma(\xi_1)$.

**Proof.** If some $\omega_i$ is exact then we immediately get the conclusion with $I = \{i\}$. So we assume that all the $\omega_i$ are inexact.

Then we can complete each $\omega_{i1}$ to a basis $H^0(X_i, \Omega(m_i))/E_i$ by adjoining a further $h_i - 1$ of the $\omega_{ij}$. The remaining $z_{ij}$ are algebraic over these and the $x_i$. We thus find

$$\text{tr.deg.}(\mathbb{C}(x_i, z_{ij})/\mathbb{C}) \leq h.$$

By Ax-Schanuel, since $\dim U = 1$, the locus $U$ projects into a proper coset in $H$. As noted above, this is equivalent to the $z_{ij}$ satisfying a linear relation of a specific type: there exist constants $c_{ij}$, not all zero, primitives $\gamma_i$ of exact $m_i$-differentials for $i = 1, \ldots, k$, and a constant $c$, such that

$$\sum_{i=1}^{k} h_i \sum_{j=1}^{h_i} c_{ij} z_{ij} + \sum_{i=1}^{k} \gamma_i = c.$$
where this relation corresponds to a defining relation of a coset in $H$.

Let $K$ be a minimal such proper coset, whose codimension in $H$ is $\ell$. Then the $z_{ij}$ satisfy $\ell$ independent relations as above (meaning that the vectors $(c_{ij})$ are linearly independent over $\mathbb{C}$).

Let us suppose that $z_{11}, \ldots, z_{k1}$ are linearly independent modulo primitives of exact $m_i$-differentials. Then we can complete $z_{11}, \ldots, z_{k1}$ to a basis of the $z_{ij}$ modulo such relations of size $h-\ell$, where $h = \sum_{i=1}^{k} h_i$, by including a further $h - \ell - k$ of the $z_{ij}$.

We now apply Ax-Schanuel to the group $G' = K \times V$ and its Lie algebra $L' \times M$. We have $\dim U = 1$ and

$$\text{tr.deg.}(\mathbb{C}(x_1, \ldots, x_k, z_{11}, \ldots, z_{k_{k}}/\mathbb{C}) \leq h - \ell$$

and we conclude that the locus projects into a proper coset in $K$. This contradicts the minimality of $K$, and we conclude that $z_1, \ldots, z_k$ cannot be linearly independent modulo primitives of exact $m_i$-differentials. Therefore we get some relation $R$ of the required form. The degree of $\gamma_i$ is bounded, once $X_i$ and $m_i$ are specified, as this fixes the spaces of exact differentials.

Let $I$ be the set of indices for which $c_i$ in $R$ is non-zero. Now $G'$ is dominant to each factor $J_i, i = 1, \ldots, k$, as $x_i$, being non-constant, is a generic point of $X_i$ over $\mathbb{C}$ and $X_i$ generates $J_i$ as a group. Fix $i \in I$ and fix points $y_j \in X_j$ for $I \ni j \neq i$. Consider the coset $K_i \subset J_i$ of points $y_i$ such that $\gamma \in G'$. If $K_i$ contains $X_i$ then it contains all $J_i$. But this is impossible, as fixing $y_j, j \neq i$ entails a restriction on $z_i$. Thus we see that, for each $i \in I$, $x_i$ is algebraically dependent on $\{x_j : j \in I, j \neq i\}$.

Now taking $I$ as above, and renumbering, we can take an algebraic curve $Z$ in $G' \cap Y$, where $Y = \prod Y_i$, such that each $Y_i$ coordinate is algebraic over each other $Y_j$. \hspace{1cm} \square

Now we consider exceptional integrability. Suppose that $(X_0, \omega_0)$ and $(X_1, \omega_1), \ldots, (X_k, \omega_k)$ are smooth projective complex curves with meromorphic differentials as above. We keep all the above notation, in particular we have moduli $m_i$ on $X_i$ such that $\omega_i \in H^0(X_i, \Omega(m_i))$, $i = 0, \ldots, k$.

**Theorem 3.3** (Exceptional integrability for complex curves, differentials). Suppose that $(X_0, \omega_0)$ is $((X_1, \omega_1), \ldots, (X_k, \omega_k))$-strictly integrable. Then (after possibly replacing $((X_1, \omega_1), \ldots, (X_k, \omega_k))$ with a subsequence) there exists a curve $Z \subset X_0 \times X_1 \times \ldots \times X_k$, irreducible and dominant to each factor such that, writing $(\xi_0, \xi_1, \ldots, \xi_k)$ for a point of $Z$ avoiding the supports of all the $m_i$, and $z_0 = \int \omega_0, z_i = \int \xi_i \omega_i$, there exist constants $c_i$ and primitives $\gamma_i(x_i)$ of exact $m_i$-differentials such that, locally,$$
z_0 = \sum_{i=1}^{k} c_i z_i + \gamma_i.$$
Proof. If $\omega_0$ is exact the conclusion is immediate. So we assume it is inexact. If some $\xi_i$ is constant then $\xi_i, z_i$ are algebraic over $C$, and can be removed from the sequence preserving the integrability. So we may assume all $\xi_i$ are non-constant. If $\omega_i$ is exact for some $i \in \{1, \ldots, k\}$ then both $\xi_i, z_i$ are algebraic over previous and we can omit them from the sequence. So we may assume all $\omega_i$ are inexact.

The transcendence degree of $C(x_0, \xi_1, z_1, \ldots, \xi_k, z_0)$ over $C$ is at most $k + 1$. Thus, by Theorem 3.2, we get a non-empty set of indices $I \subset \{0, 1, \ldots, k\}$ with the properties given there.

Suppose $I$ does not include the index 0. Then, taking the highest index $i \in I$, we see that $z_i$ and $\xi_i$ are both algebraic over the $\{\xi_j, z_j, j \in I, j \neq i\}$. So we can omit $\xi_i, z_i$ from the sequence.

So eventually we find that $I$ does include 0. Then renumbering the curves and taking the curve in the product, dominant to each factor, as in Theorem 3.2, gives the required conclusion. □

Again, the linear relation corresponds to a defining relation of a coset of an irreducible algebraic subgroup of the maximal NVQ subgroup $H$ of $G$.

The idea will be to leverage the above theorem into a general statement in a differential field (and with rationality consequences) using the Seidenberg Elimination Theorem together with the Seidenberg Embedding Theorem.

4. The Seidenberg Elimination Theorem

Let $K$ be a differential field with commuting derivations $D_1, \ldots, D_m$ and constant field $C$. The differential polynomial ring $K\{U_1, \ldots, U_n\}$ in $n$ differential indeterminates $U_i$ is the differential ring obtained by the adjunction of the $U_i$. Each differential polynomial $f \in K\{U_1, \ldots, U_n\}$ gives rise to a function $f : K^n \to K$ by substitution.

**Theorem 4.1** (Seidenberg Elimination Theorem [20], as in §1.6 of [4]). Given a finite system

1. $f_1(a_1, \ldots, a_N, y_1, \ldots, y_n) = 0, \ldots, f_s(a_1, \ldots, a_N, y_1, \ldots, y_n) = 0,$
   
   where $f_1, \ldots, f_s, g \in Z\{a_1, \ldots, a_N, y_1, \ldots, y_n\}$, there are a finite number of finite systems

1’. $f_{i_1}(a_1, \ldots, a_N) = 0, \ldots, f_{i_s}(a_1, \ldots, a_N) = 0,$
   
   where $f_{ij}, g_i \in Z\{a_1, \ldots, a_N\}$, such that for any differential field $K$, and for any $\overline{a}_1, \ldots, \overline{a}_N \in K$, the system

1 T

   $f_1(\overline{a}_1, \ldots, \overline{a}_N, y_1, \ldots, y_n) = 0, \ldots, f_s(\overline{a}_1, \ldots, \overline{a}_N, y_1, \ldots, y_n) = 0,$
   
   $g(\overline{a}_1, \ldots, \overline{a}_N, y_1, \ldots, y_n) \neq 0$

has a solution in some differential extension field of $K$ if and only if there is an $i$ such that

   $(\overline{a}_1, \ldots, \overline{a}_N)$
solves $1'$. Further, there is an algorithm that, given the system $1$, produces the finite systems $1'$.

On the complexity of this algorithm see [11] and the references therein. In the case of one derivation see [16].

5. Exceptional integrability in a differential field

Let $K$ be a differential field of characteristic zero with derivation $D$ and constant field $C$. Let $x \in K$ be non-constant. We assume $Dx = 1$ (this can always be achieved by replacing the given derivation $D$ by $1/D$).

Let $\xi, \alpha \in K$. By saying that $\alpha$ is an algebraic function of $\xi$ we mean that $A(\xi, \alpha) = 0$ for some non-zero absolutely irreducible $A \in C[X,Y]$. By saying that $\alpha$ is exact (relative to $x$) we mean that there exists $z$ algebraic over $x$, perhaps in some differential extension field, with $Dz/Dx = \alpha$.

The next result is a version of Theorem 3.2 in a differential field, and it will include the statement of Theorem 1.2.

**Definition.** Let $K$ be a differential field of characteristic 0 with constant field $C$ and $x \in K$ with $Dx = 1$. Let $A, B_1, \ldots, B_k \in C[X,Y]$ be absolutely irreducible. Let $\alpha \in K$ with $A(x, \alpha) = 0$. We say that $\alpha$ is $(B_1, \ldots, B_k)$-strictly integrable if, in some differential extension field $\tilde{K}$ (with constant field $\tilde{C}$), there exist elements $x_1, \ldots, x_k, z_1, \ldots, z_k, z$ with the following properties:

1. $Dz = \alpha$
2. For $i = 1, \ldots, k$ we have $B_i(x_i, \frac{1}{Dx_i}Dz_i) = 0$
3. For $i = 1, \ldots, k$ either $x_i$ or $z_i$ is algebraic over $\tilde{C}(x, x_1, z_1, \ldots, x_{i-1}, z_{i-1})$
4. $z$ is algebraic over $\tilde{C}(x, x_1, z_1, \ldots, x_k, z_k)$.

**Theorem 5.1** (Theorem on exceptional integrability in a differential field). Let $K$ be a differential field of characteristic 0 with constant field $C$ and $x \in K$ with $Dx = 1$. Let $A, B_1, \ldots, B_k \in C[X,Y]$ be absolutely irreducible.

Let $\alpha \in K$ with $A(x, \alpha) = 0$.

Suppose $\alpha$ is $(B_1, \ldots, B_k)$-strictly integrable.

Then there exists a differential extension field $\tilde{K}$ of $K$ (with constants $\tilde{C}$) containing elements $x_i$ algebraic over $x$, constants $c_i$, and algebraic functions $\gamma_i(x_i)$ of degree bounded as in Theorem 3.2 such that

1. $Dz = \alpha$
2. For each $i = 1, \ldots, k$, $B_i(x_i, \frac{1}{Dx_i}Dz_i) = 0$
3. For each $i = 1, \ldots, k$, $x_i$ is algebraic over $C(x)$
4. $z = \sum c_i z_i + \gamma_i(x)$.

Moreover, the constants, the $x_i$ as algebraic functions of $x$ and the $\gamma_i(x_i)$ can be taken to be defined over an algebraic extension of $\mathbb{Q}(\overline{a}, \overline{b}_i)$, where these are the tuples of coefficients of $A, B_i$. 
Proof. After extending the system with some further elements algebraic over $x$ and each $x_i$ to get tuples $\overline{x}, \overline{x}_i$, they describe algebraic curves $X_i$ whose projective closures are smooth. After a Seidenberg embedding, the $z, z_i$ (perhaps adjusted by complex constants) are integrals of meromorphic differentials $\omega, \omega_i$ on these curves.

So we are in the situation of Theorem 3.2, and we conclude that there is a linear relation as in 3.2, with some algebraic $\gamma_i$ of degree at most $d_i$ over $x_i$. This is then a linear relation in some differential extension field of $K$.

Let $a, b_i$ be the coefficient vectors of $A, B_i$. Under the hypotheses, there is a choice of degrees of the algebraic relations at stage $i$, whether it is $x_i$ or $z_i$ that is algebraic over previous elements, and the degree of the final algebraic relation of $z$ over the other elements. Call this choice of discrete data a shape. Given a shape $\tau$, the connecting algebraic relations $G_i$, have some coefficients $g_i$ and the final algebraic relation for $z$ we call $G_0$ with coefficients $\overline{g}_0$.

Then the hypothesis implies that, for some shape $\tau$, we have the existence in a differential extension of a solution to a suitable differential system $\Sigma_\tau$. Let’s write $DA = 0$ as a short-hand for $\sum a_{ij}X^iY^j$, and $A(x, \alpha) = 0$ as a shorthand for $\sum a_{ij}x^i\alpha^j = 0$, etc. Then the system $\Sigma_\tau$ has the form:

\[
DA = 0, \quad DB_i = 0, i = 1, \ldots, k, \quad Dx = 1, \quad A(x, \alpha) = 0
\]

\[
Dz = \alpha, \quad B_i(x_i, \frac{1}{Dx_i}Dz_i) = 0, i = 1, \ldots, k,
\]

\[
DG_i = 0, i = 1, \ldots, k, \quad DG = 0,
\]

\[
G_i(x, x_1, \ldots, z_i-1, y_i) = 0, \quad G(x, x_1, \ldots, x_k, z_k, z) = 0,
\]

where $y_i = x_i$ or $z_i$ is dictated by the shape.

We can further augment this system with variables $c_i, \gamma_i$ for elements as above (here each $\gamma_i$ is a tuple of constants giving the coefficients in a linear combination of elements in a basis of the primitives of exact $m_i$-differentials) in the linear relation to get a system $\Sigma^+_\tau$ including in addition

\[
\sum_{i=1}^k c_i(z_i + \gamma_i) = 0.
\]

By the Seidenberg Elimination Theorem, the existence of solutions (in some differential extension field) is characterized by some differential algebraic constructible systems over $\mathbb{Z}$ on the non-eliminated variables.

Let us first eliminate all the way down to $\overline{a}, \overline{b}_i, i = 1, \ldots, k$. Then the existence of a solution $(x, \alpha, \ldots)$ is characterized by some algebraic constructible conditions $\text{Cond}(\overline{a}, \overline{b}_i)$.

Next consider the elimination down to $\overline{a}, \overline{b}_i, c_i, \gamma_i$. As these elements are all constant, this is again a constructible algebraic condition over $\mathbb{Z}$, and the theorem guarantees that suitable $c_i, w_i$ exists if $\text{Cond}(\overline{a}, \overline{b}_i)$ holds. Thus,
given $\overline{a}, \overline{b}$, the admissible $c_i$ may be described in constructible algebraic terms over $\overline{a}, \overline{b}$, and we conclude the rationality statement.

Let us consider the elimination down to $\overline{a}, \overline{b}, c_i, w_i, x$. The constructible differential algebraic system cannot distinguish the different solutions to $Dx = 1$, as $x$ can only enter via $Dx$ in any equality $f(\overline{a}, \ldots, x) = 0$, while any inequality $f(\overline{a}, \ldots, x) \neq 0$ will be automatically satisfied if such $x$ appears. So if, for some $\overline{a}, \overline{b}$, there is a solution for some $x$ with $Dx = 1$ then there is a solution for any such $x$.

Next we include $\alpha$. The system $\text{Cond}(\overline{a}, \overline{b}, c_i, w_i, x, \alpha)$ can’t distinguish between roots of $A(x, \alpha)$ over $\mathbb{Q}(\overline{a}, \overline{b}, x)$, so if there is a solution for some branch then there is one for any branch. To see this, we note that if $A(x, \alpha) = 0$ then $D\alpha$ can be expressed as a ratio of polynomials (depending on $A$) in $x$ and $\alpha$. Thus if $f(\overline{a}, \ldots, x, \alpha) = 0$ holds for some root $\alpha$ it holds for all its conjugates $\alpha'$ over $\mathbb{C}(x)$ as well, and likewise for $g(\overline{a}, \ldots, x, \alpha) \neq 0$.

Now in the situation of Theorem 3.2 we also conclude the existence of an algebraic curve in the base $X \times X_1 \times \ldots \times X_k$, of suitable degrees over $X$ in each $X_i$.

In general there is no bound on the degrees of the $x_i$ over $x$ in terms of the given data, as these represent the equations of weakly special subvarieties. But, in any case, for some degree there are such $x_i$ algebraic over $x$ given by 3.2, and so they appear in some elimination system, and can be taken to be defined by (constructible) algebraic equations over the base. This completes the proof. \hfill $\Box$

There is a more general version of Theorem 5.1 analogous to Theorem 3.2. Here we do not require a tower of fields with the specific form as in $B$-strict integrability, but only a sequence of tuples $(x_i, z_i)$ with $B_i(x_i, \frac{1}{Dx_i}Dz_i) = 0$ whose total transcendence degree is too small.

**Theorem 5.2** (Ax-Schanuel Theorem in a differential field). Let $K$ be a differential field of characteristic 0 with constant field $C$ and $x \in K$ with $Dx = 1$. Let $B_1, \ldots, B_k \in C[X,Y]$ be absolutely irreducible.

Suppose that there are elements $x_1, \ldots, x_k, z_1, \ldots, z_k$ in $K$ such that

$$B_i(x_i, \frac{1}{Dx_i}Dz_i) = 0$$

for each $i$ and

$$\text{tr.deg.}(C(\overline{x}, \overline{z})/C) \leq k.$$

Then there is a non-empty set $I \subset \{1, \ldots, k\}$, a differential extension $\tilde{K}$ (with constants $\tilde{C}$), non-zero constants $c_i \in \tilde{C}$, and algebraic functions $\gamma_i$ (primitives of suitable $m_i$-exact differentials) such that

$$\sum_{i \in I} c_i(z_i + \gamma_i) = 0.$$

Moreover, we can find $x_i^*$ mutually algebraic over each other and $z_i^*$ with $B_i(x_i^*, \frac{1}{Dx_i^*}Dz_i^*) = 0$ for which the same relation holds, and we can take the
Then one has a corresponding version of Theorem 1.2, and a decision procedure for this more general notion of “weak integrability” positing that $x_0, z_0$ solves some “over-determined” algebraic system on some $x_i, z_i$.

**Proof of Theorem 1.1.** This follows immediately from Theorem 5.1. □

6. HODGE THEORETIC PERSPECTIVE ON MEROMORPHIC DIFFERENTIALS

We establish here some well-known and less well-known relations on how to think of meromorphic forms on a curve cohomologically. We let $X$ denote a smooth compact complex curve and $U$ a Zariski-open set of $X$. Recall that $H^1(X, \mathbb{C})$ has a natural subspace isomorphic to $H^0(U, \Omega^1_U)$ which we will denote by $F_1H^1_0(X, \mathbb{C})$. The complex Jacobian $J_X$ can be naturally identified with $H^1(X, \mathbb{C})/F_1H^1_0(X, \mathbb{C})$.

To relate this to the usual construction of $H^0_0(U, \Omega^1_U)$ modulo periods, we simply use the identification $H^1(X, \mathbb{C})/F_1H^1_0(X, \mathbb{C}) \cong H^0_0(U, \Omega^1_U)$ given by the cup product.

For this section, $R$ will denote a sub-ring of $\mathbb{C}$, with the key cases being $\mathbb{Z}, \mathbb{Q}, \mathbb{C}$.

6.1. Residue-less forms. For simplicity, and because this will come up as a special case, we first consider the case of meromorphic forms without residues. To understand this space better, let $U = X - D$ be an open set, and let $R(D)$ denote the free $R$ module on $D$ and $R^0_D$ the submodule of coefficients that sum to 0. Then we have a natural exact sequence

\[
0 \to H^1(X, R) \to H^1(U, R) \to R^0_D \to 0.
\]

Now, we can consider the infinite-dimensional space $H^0(U, \Omega_U')$ which has a natural map to $H^1(U, \mathbb{C})$. There is a natural subspace $H^0(U, \Omega_U')'$ of forms whose residues at all points of $D$ are 0. Equivalently, this is the kernel of the map to $R^0_D$. By the exact sequence above, there is a natural map $H^0(U, \Omega_U')' \to H^1(X, R)$. Note that if a form is exact then it is in the kernel of the above map.

**Theorem 6.1.** If $U \neq X$, then there are isomorphisms

\[
(1) \quad H^0(U, \Omega_U)/dH^0(U, \mathcal{O}_X) \cong H^1(U, R),
\]

and

\[
(2) \quad H^0(U, \Omega_U')/dH^0(U, \mathcal{O}_X) \cong H^1(X, R),
\]

**Proof.** We first prove injectivity. Suppose $\omega$ is in the kernel of the above map. Then the integral of $\omega$ along any closed loop is 0, which means $\omega$ has a well defined integral, or in other words $\omega = dg$ for a meromorphic function.
g on U. Moreover, g is meromorphic since ω is, and hence is algebraic (by
Riemann Existence, for example).

For surjectivity, we simply compute dimensions. We want to work with
finite dimensional spaces, so we notice that for each positive integer m,
\( H^0(X, \Omega_X(mD))/dH^0(X, \mathcal{O}((m-1)D)) \hookrightarrow H^0(U, \Omega_U)/dH^0(U, \mathcal{O}_X) \). Com-
puting dimensions using Riemann-Roch for m large, we see that
\[
\dim H^0(X, \Omega_X(mD)) = g + m|D| - 1,
\]
and
\[
\dim H^0(X, \Omega_X(mD))' = g + (m - 1)|D|,
\]
and also
\[
\dim H^0(X, \mathcal{O}((m - 1)D)) = (m - 1)|D| + (1 - g);
\]
and since the kernel of d is 1-dimensional,
\[
\dim dH^0(X, \mathcal{O}((m - 1)D)) = (m - 1)|D| - g.
\]
Hence, finally,
\[
dim \left( H^0(X, \Omega_X(mD))/dH^0(X, \mathcal{O}((m - 1)D)) \right) = |D| + 2g - 1,
\]
which gives surjectivity of the first map, and
\[
\dim \left( H^0(X, \Omega_X(mD))/dH^0(X, \mathcal{O}((m - 1)D)) \right) = 2g,
\]
which gives surjectivity of the second map. \( \square \)

6.2. **Forms with Residues.** We let \( U = X - D \) as in the previous subsec-
tion. Now we set \( F^1H^1(U, \mathbb{C}) \) to be the image of \( H^0(X, \Omega_X(D)) - \) in other
words, differential forms which have simple residues along D. We

**Lemma 6.2.** \( H^1(U, \mathbb{C})/F^1H^1(U, \mathbb{C}) \cong H^1(X, \mathbb{C})/F^1H^1(X, \mathbb{C}) \)

**Proof.** Riemann-Roch shows that \( F^1H^1(U, \mathbb{C}) \) surjects onto \( \mathbb{C}^0_U \). The claim
thus follows by computing dimensions. \( \square \)

We thus obtain a canonical map \( \phi_U : H^1(U, \mathbb{Z}) \to H^1(X, \mathbb{C})/F^1H^1(X, \mathbb{C}) \)
by composing the map in lemma 6.2 with the natural map \( H^1(U, \mathbb{Z}) \to H^1(U, \mathbb{C})/F^1H^1(U, \mathbb{C}) \). Tensoring up with \( \mathbb{C} \) we obtain a map
\[
\phi_{U, \mathbb{C}} : H^1(U, \mathbb{C}) \to H^1(X, \mathbb{C})/F^1H^1(X, \mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{C}.
\]
Finally, taking a direct limit gives a map
\[
\phi_{\mathbb{C}} : \lim_{U} H^1(U, \mathbb{C}) \to H^1(X, \mathbb{C})/F^1H^1(X, \mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{C}.
\]
Note that \( \frac{df}{dz}f \) is in the kernel of \( \phi_U \) and hence of \( \phi_{\mathbb{C}} \). Let \( \Omega_{\text{X,dlog}} \) denote
the complex vector space spanned by all such differentials, which naturally
injects into \( \lim_{U} H^1(U, \mathbb{C}) \).

**Theorem 6.3.** The map \( \phi_{\mathbb{C}} \) induces an isomorphism between \( \frac{\lim_{U}}{\Omega_{\text{X,dlog}}} H^1(U, \mathbb{C}) \)
and \( H^1(X, \mathbb{C})/F^1H^1(X, \mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{C} \).
Notation: In the following, we write $R_{X}^{X(C)}$ to mean the $R$-module of functions $X(C) \to R$ with finite support, and $R_{0}^{X(C)}$ to mean the submodule of functions whose sum over all values is 0.

Proof. We first establish surjectivity. We first observe the exact sequence

$$0 \to H^{1}(X, \mathbb{Z}) \to H^{1}(X, \mathbb{C})/F^{1}H^{1}(X, \mathbb{C}) \to J_{X} \to 0$$

where $J_{X}(\mathbb{C})$ is the complex points of the Jacobian of $X$. We thus get induced maps $\tilde{\Phi}_{U} : \mathbb{Z}_{0}^{D} \to J_{X}(\mathbb{C})$ and $\tilde{\Phi} : \mathbb{Z}_{0}^{X(C)} \to J_{X}(\mathbb{C})$ which is the usual map sending degree 0 divisors to the Jacobian [21, Prop 12.7]. This is well known to be surjective, and therefore $C_{0}^{X(C)}$ surjects onto $J_{X}(\mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{C}$. This is equivalent to $\phi_{\mathbb{C}}$ being surjective after quotienting out by $H^{1}(X, \mathbb{C})$. However, $\phi_{\mathbb{C}}$ induces an isomorphism $H^{1}(X, \mathbb{C}) \to H^{1}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^{1}(X, \mathbb{C})$. Thus we have established surjectivity.

Next we establish injectivity. Suppose that $\gamma \in \lim_{\rightarrow U} H^{1}(U, \mathbb{C})$ is in the kernel of $\phi_{\mathbb{C}}$. Then as we established above, the image $\tilde{\gamma} \in C_{0}^{X(C)}$ in $J_{X}(\mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{C}$ is 0. Since tensoring with $\mathbb{C}$ over $\mathbb{Z}$ is exact, this means that $\tilde{\gamma}$ is a $\mathbb{C}$-linear combination of principal divisors $(f)$. Since $\frac{d}{2\pi i} = (f)$, we may subtract an element $\tau$ of $\Omega_{X,d\log}$ from $\gamma$ to obtain an element $\gamma'$ in the kernel of $\phi_{U}$ whose image in $C_{0}^{D}$ is 0. However, that means that $\gamma' \in H^{1}(X, \mathbb{C})$, and since $\phi_{U} |_{H^{1}(X, \mathbb{C})}$ is injective, we must have that $\gamma' = 0$. Thus $\gamma = \tau \in \Omega_{X,d\log}$ as desired.

Finally, we obtain the following:

Theorem 6.4. Let $M_{X}^{1}$ denote the group of all meromorphic 1-forms on $X$, and $M_{X}$ denote the field of meromorphic functions on $X$. Then

$$\frac{M_{X}^{1}}{dM_{X} + d\log M_{X}} \cong H^{1}(X, \mathbb{C})/F^{1}H^{1}(X, \mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Proof. Taking the direct limit over $U$ of the first isomorphism in Theorem 6.1, we see that $\frac{M_{X}^{1}}{dM_{X}} \cong \lim_{\rightarrow U} H^{1}(U, \mathbb{C})$. The result now follows from Theorem 6.3.

\[\square\]

Remark. From a Hodge theory perspective, the $\mathbb{Z}$ extension (1) along with the $\mathbb{C}$-subspace $F^{1}H^{1}(U, \mathbb{C})$ constitutes the mixed Hodge structure on $H^{1}(U, \mathbb{C})$. The forms $\frac{1}{2\pi i} d\log(f)$ are precisely the Hodge vectors. The extension class is determined by the subgroup generated by $\mathbb{Z}_{0}^{D}$ in the Jacobian. The universal extension is then naturally constructed the direct limit $\lim H^{1}(U, \mathbb{C})$ quotiented out by all trivial extensions, which are just the $d\log(f)$. Thus, the above result is essentially saying that

$$H^{1}(X, \mathbb{Z}) \to H^{1}(X, \mathbb{C})/F^{1}H^{1}(X, \mathbb{C}) \to J_{X}$$

is the universal extension by powers of $\mathbb{Z}(-1)$ of the pure Hodge structure on $H^{1}(X, \mathbb{C})$. 

\[\square\]
7. Decision Procedures

The purpose of this section is to provide an algorithm to fully solve the decision problem of whether one period-function can be expressed by finitely many others.

7.1. Algorithms from algebraic-geometry. Following [17] we use finitely generated fields (fields of finite type) but we restrict to characteristic 0. We work over the algebraic closure of a finite type field $K$, presented as the fraction field of a finitely-generated $\mathbb{Z}$-algebra.

The following result is well-known to experts but we couldn’t find a clean reference, so we record it in the form we need:

**Lemma 7.1.** Let $A/K$ be an abelian variety, and let $\phi : \mathbb{Z}^n \to A(K)$ be a homomorphism, presented by specifying the image of a basis. Then there is a decision procedure which returns the image $\phi(\mathbb{Z}^k)$ as $F + T$ where $T$ is a torsion subgroup and $F$ is a free abelian group. Equivalently, the decision procedure returns generators for the kernel of $\phi$.

**Proof.** First, by increasing $K$ we may assume that $A$ and $\phi$ are defined over $K$ itself.

Next, assume $K$ is a number field. Then using the theory of the Néron-Tate height we can compute the torsion subgroup $A(K)_{tor}$ of $A(K)$. Set $A(K)' := A(K)/A(K)_{tor}$. If we compute the kernel of $\psi : \mathbb{Z}^n \to A(K)'$ then finding the kernel of $\phi$ is simply a matter of checking the finitely many sublattices of $\text{Ker} \psi$ finite index at most $\#A(K)_{tor}$. We thus focus on finding $\text{Ker} \psi$.

Moreover, again using the theory of the Néron-Tate height we may compute the image of $\psi$ in $A(K)'/mA(K)'$ for every positive integer $m$, as this just amounts to checking which of finitely many elements are $m$'th powers in $A(K)$. Note that if $P_1, \ldots, P_d$ are independent in $A(K)'$ then they are independent in $A(K)/pA(K)$ for a large enough prime $p$. Thus, by day we can look for elements in the kernel of $\phi$, and by night we can try to prove it by showing independence modulo large primes $p$.

Finally, assume $K$ is a finite type field. Recall we may write $K$ as the fraction field of a finitely-generated $\mathbb{Q}$-algebra $R$. We may assume $A$ spreads out over $R$ such that $A(K) = A(R)$. Then by the main theorem of [12] there are specializations $R \to L$ where $L$ is a number field which induce isomorphisms $A(R) \to A(L)$. Thus, we may by day look for elements in $\text{Ker} \phi$ and by night try to prove we’ve found them all by computing the same for all specializations. $\square$

7.2. Rephrasing through traces.

**Definition.** Let $(X_i, \omega_i), i = 1, 2$ be curves with differentials. Let $D \subset X_1 \times X_2$ be a curve with no fibral-components. Then if $\omega_2 = \pi_{2*} \pi_1^* \omega_1$ we say that $\omega_2$ is a trace-image of $\omega_1$. 
Lemma 7.2. Let \((X_i, \omega_i), 0 \leq i \leq m\) be curves with meromorphic differentials over \(\overline{K}\). TFAE:

- \((X_0, \omega_0)\) is integrable in terms of \((X_1, \omega_1)\)
- \(\omega_0\) is in the linear span of differentials on \(X_0\) which are trace-images of the \(\omega_i\), and exact differentials.

Proof. By Theorem 3.3 condition (i) is equivalent to the existence of a curve \(C\) with maps \(\phi_j\) to the \(C_{f(j)}, j > 0\) and a single map \(f\) to \(C_0\) such that \(f^*\omega_0\) is in the linear span of the \(\phi_j^*\omega_{f(j)}\), for some function \(f\). Assuming this is the case, taking the trace from \(C\) to \(X_0\) yields (ii).

Now suppose that (ii) holds, so that there are correspondences \(D_i \subset X_0 \times X_{f(i)}\) such that the push-pull of \(\omega_{f(i)}\) contain \(\omega_0\) in their linear span.

Let \(d_i\) be the degree of \(d_i\) over \(X_0\). We first let \(E_i\) be the maximal closed reduced subscheme of the \(d_i\) fiber product of \(D_i\) over \(X_0\) whose generic points admit \(d_i\) distinct maps to \(D_i\). We then let \(C\) be the fiber product of all of the \(E_i\) over \(X_0\). Note that for each \(i\), \(C\) has \(d_i\) distinct maps to \(C_{f(i)}\). Moreover, there is an action of \(G := \prod S_{d_i}\) action on \(C\) making it a torsor over \(X_0\). We may thus identify forms on \(X_0\) with \(G\)-invariant forms on \(C\).

Now consider a fixed \(i\), and the form \(\pi_2^*\pi_1^*\omega_{f(i)}\). If we pull this form back to \(E_i\), it becomes \(S_{d_i}\)-invariant and may be expressed as a sum over \(\pi^*\omega_{f(i)}\) where \(\pi\) ranges over all \(d_i\) maps to \(X_0\). Thus the same is true once this form is pulled back all the way to \(E\).

Finally, we simply write the linear relation guaranteed by (ii) and pull it back to \(E\), obtaining (i). Note that our curve \(E\) is not irreducible, but we may restrict to any irreducible component. \(\square\)

7.3. Decision procedures. We are now ready to give an algorithmic procedure for deciding whether \((X_0, \omega_0)\) is integrable in terms of the \((X_i, \omega_i)\). The procedure deals with residue-less forms and forms-with-residues separately, though the spirit is similar.

7.3.1. None of the \(\omega_i, i > 0\) have any residues.

Step 1: Reduction to computing a basis for trace images

We claim that it is sufficient to obtain a basis for the linear span of trace-images of each \(\omega_i\) for each \(i\), modulo exact forms. Indeed, suppose we have obtained such a basis \(L\). Then by lemma 7.2 it is sufficient to decide whether \(\omega_0\) is in the linear span of \(L\) and exact-forms. Now, suppose \(m\) is a large enough modulus such that \(\omega_0\) and \(L\) have their poles in \(m\). Then it is enough to restrict to exact-differential forms whose poles are in \(m\). These are finite dimensional vector spaces that can be easily computed, and the decidability question can be answered from here.

Step 2: Computing correspondences between curves

By Theorem 6.3 there is a natural isomorphism between reside-less forms modulo exact ones and \(H^1\) of the curve. These isomorphisms are just given by integration alon cycles and so are compatible with push-forwards and pull-backs. Thus, given a curve \(D \subset X_0 \times X_1\) we obtain a natural map
\( \phi_D : H^1(X_1, \mathbb{C}) \to H^1(X_0, \mathbb{C}) \), and the image of \([\omega_1]\) under this map will be the class of the trace-image of \(\omega_1\) under \(C\). Hence, it is enough to obtain a basis for all maps \(\phi_D\) corresponding to all correspondences \(D\).

This is done by Theorem 8.15 of [17]

The above returns a set of cycles \(D_i\), which finishes the algorithm.

7.3.2. At least one of the \(\omega_i, i > 0\) has a non-zero residue.

**Step 1: Getting \(\log(x)\)**

We first show that we have access to the logarithm function. Suppose wlog that \(\omega_1\) have at least one residue. We construct a map \(f : C \to \mathbb{P}^1\) which sends all the points at which \(\omega_1\) has a residue to 0 except one which gets sent to \(\infty\). Then the trace of \(\omega_1\) will be some multiple of \(\frac{dx}{x}\) and so by summing up the primitives we obtain the logarithm function. Crucially, this gives us access to all differentials of the form \(d\log f\) where \(f\) is a meromorphic function.

**Step 2: Reduction to Computing a basis for trace images**

We now proceed very much like before. We claim that it is sufficient to obtain a basis for the linear span of trace-images of each \(\omega_i\), modulo exact and log-exact differential forms. Indeed, suppose we have obtained such a basis \(L\). Then Theorem \(7.2\) says that it is sufficient to check whether \(\omega_0\) is in the span of \(L\), and exact and log-exact differential forms.

Now, by Theorem \(7.3\) we may work in \(H^1(X_0, \mathbb{C})/F^1 H^1(X_0, \mathbb{C}) \otimes_{\mathbb{Q}} \mathbb{C}\). So we may first take the image in \(J_0 \otimes_{\mathbb{Q}} \mathbb{C}\). Thus \(L\) and \(\omega_0\) gives us a morphism \(G : \mathbb{Z}^M \to J_0 \otimes_{\mathbb{Q}} \mathbb{C}\). Moreover, this morphism is very explicit, given by just taking the residues of our forms. By taking bases for the residues over \(\mathbb{Q}\), this becomes a question of identifying the subgroup in \(J_0\), which is Lemma \(7.4\). Thus we may obtain the kernel of \(G\).

In that case that the image of \(\omega_0\) in \(J_0 \otimes_{\mathbb{Q}} \mathbb{C}\) is already not expressible in terms of \(L\), we can stop. Otherwise, by subtracting off the appropriate element in \(L\), we may assume the image of \(\omega_0\) is 0, and we may restrict to the subgroup \(L'\) of \(L\) in the kernel of \(G\). Now adding log-exact forms will introduce a non-zero residue, and so we are reduced to checking whether \(\omega_0\) is in the span of \(L'\) and exact forms, which we do as in the previous subsubsection.

7.4. Elliptic integrals. In this section we explain how to generalize the above algorithm the following question: *When is a period-function expressible in terms of elliptic integrals?*

Recall that an elliptic integral is defined as

\[
\int_c^x R(t, \sqrt{P(t)}) dt
\]

where \(P(t)\) is a polynomial of degree 3 or 4, and \(R\) is a rational function. If we consider the elliptic curve \(E_P := y^2 = P(x)\) then \(\omega = R(x,y) dx\) is a rational differential form on \(E_P\), and its integral is precisely the pullback of \(f(x)\). Thus, in answering the above question, we have the following:
Theorem 7.3. Let \((X, \omega)\) be a curve with a rational differential form. Then there is a decision procedure for determining where \((X, \omega)\) is integrable in terms of the set of all elliptic curves with all rational differentials on them.

Proof. We proceed as in the previous two sections. Note that \(\omega\) gives us a class \([\omega]\) in \(H^1(X, \mathbb{C})/F^1H^1(X, \mathbb{C}) \otimes_{\mathbb{Q}} \mathbb{C}\) by Theorem 6.3. The key to our proof is first decomposing the Jacobian \(J_X\) into its elliptic and non-elliptic part. Note that if \(J_X\) contains an elliptic curve then there is a map from \(X\) to that elliptic curve. Thus we must find all elliptic curves with a map from \(X\). These all show up inside \(X \times X\) via their induced correspondences, and thus we may find them all as before by computing the Neron-Severi group of \(X \times X\). Having found all maps from \(X\) to elliptic curves, we may write \(J_X \sim A \times B\) via an explicit isogeny such that \(A\) is a power of elliptic curves and \(B\) does not have elliptic factors. Note that this induces natural isomorphisms

\[
H^1(X, \mathbb{C})/F^1H^1(X, \mathbb{C}) \cong \tilde{A} \times \tilde{B}
\]

where \(\tilde{A}\) denotes the universal cover, and also

\[
H^1(X, \mathbb{C})/F^1H^1(X, \mathbb{C}) \otimes_{\mathbb{Q}} \mathbb{C} \cong \tilde{A} \otimes_{\mathbb{Q}} \mathbb{C} \times \tilde{B} \otimes_{\mathbb{Q}} \mathbb{C}.
\]

Finally, the question is whether \([\omega]\) has trivial image in \(\tilde{B} \otimes_{\mathbb{Q}} \mathbb{C}\). As before, we first check whether the image of \([\omega]\) is 0 in \(B \otimes_{\mathbb{Q}} \mathbb{C}\). This is merely a question about the Mordell-Weil group of \(B\) and can therefore be answered as before.

Supposing this is the case, we then adjust \(\omega\) by the image of an elliptic form so that \([\omega]\) has trivial image also in \(\tilde{A} \otimes_{\mathbb{Q}} \mathbb{C}\) and thus in \(J_X \otimes_{\mathbb{Q}} \mathbb{C}\) (and is therefore residueless). Thus, \([\omega]\) is now merely a class in \(H^1(X, \mathbb{C})\) and we must check if its contained in \(H^1(A, \mathbb{C})\). But this is easy, as we may generate a basis for \(H^1(A, \mathbb{C})\) using correspondences of \(X\) with elliptic curves. The proof is therefore complete.

Remark. (1) Note that in the above proof we handled the case of all elliptic integrals, but it is easy to adjust the proof so as to allow only consider \((E, \omega)\) where \(\omega\) is regular, or only having simple poles, or being residue-less. This would correspond to periods of differentials of the first, second, or third kind.

(2) Since (almost) every Abelian variety of dimension 2 or 3 is the Jacobian of a curve, one may similarly make an argument for the family of all differentials on all genus \(g\) curves, where \(g = 2, 3\).

8. A Connection with unlikely intersections

For a fully general pencil of curves and differentials, Masser and Zannier are able to precisely (and effectively) describe when the generic fibre in the pencil is not elementary integrable, but that elementary integrability of fibres is not “unlikely” in the Zilber-Pink sense (and indeed occurs for
infinite many values of the parameter). Further, in the “unlikely” case, they are able to prove the finiteness statement via their results on Relative Manin-Mumford.

Here we restrict to regular differentials and observe that the question of exceptional integrability again leads to questions of Zilber-Pink type, and that results in the literature answer them in some cases.

**Theorem 8.1.** Fix some smooth projective curve $Y$ of genus $g \geq 1$ and a non-zero regular differential $\eta$, and suppose the Jacobian $\text{Jac}(Y)$ is simple.

Suppose that $(X_t, \omega_t)$ is a pencil of curves of genus $g$ and regular differentials over some quasi-projective base curve $B$ (so we have removed finitely many points where the fibre $X_t$ is not smooth or the differential not regular). We can assume that $B \subset A_g$.

Then there are only finitely many $t$ such that $z_t = \int x^t \omega_t$ is $\{w = \int y \eta\}$-integrable if $B$ is not a weakly special subvariety.

The same holds (with same proof) if we assume several differentials on $Y$ are available, on several different (simple Jacobian) curves $Y$.

**Proof.** The integrability condition entails that there is a non-product weakly special subvariety of

$$\text{Jac}(X_t) \times \text{Jac}(Y)^n$$

which is dominant to both factors. But $Y$ is simple, so this can only happen if $X_t$ is isogenous to $Y$. If there are infinitely many such specialisations we find that $B$ has infinitely many points in the isogeny class of the moduli point $[Y] \in A_g$ of $Y$. This is a problem of Zilber-Pink type, more specifically of André-Pink-Zannier type: one expects this can only happen if $B$ is a proper weakly special subvariety, and for curves this is a theorem of Orr [15, Theorem 1.2].

Note that if $X_t$ is isogenous to $Y$ this does not in general lead to integrability (the differentials might not satisfy a suitable linear relation), so our theorem is not sharp. But isogeny does lead to integrability in the case $g = 1$, since elliptic curves have only 1 regular differential (up to scale). However Theorem 8.1 is uninteresting in that case as the moduli space is one-dimensional, so $B$ is always weakly special.

We can consider however the question of when two given elliptic logarithms $z_1 = \int x^1 \omega_1$ on an elliptic curve $X_1$ and $z_2 = \int x^2 \omega_2$ on an elliptic curve $X_2$ are simultaneously integrable by means of a third elliptic logarithm $z_3 = \int x^3 \omega_3$ on an elliptic curve $X_3$. If now $X_1, X_2, X_3$ vary in a pencil then we may assume that the pencil is parameterised by the points of a curve $V \subset Y(1)^3$. Then the Zilber-Pink conjecture predicts that the set of $t \in V$ for which two functions are integrable in terms of the third (i.e. the three elliptic curves are pairwise isogenous) is finite unless $V$ is contained in a proper special subvariety. Some partial results on this problem are in [6].
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JP: Mathematical Institute, University of Oxford, Oxford, UK.
   pila@maths.ox.ac.uk

JT: Department of Mathematics, University of Toronto, Toronto, Canada.
   jacobt@math.toronto.edu