Comparison of categorical characteristic classes of transitive Lie algebroid with Chern-Weil homomorphism

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Abstract

Transitive Lie algebroids have specific properties that allow to look at the transitive Lie algebroid as an element of the object of a homotopy functor. Roughly speaking each transitive Lie algebroids can be described as a vector bundle over the tangent bundle of the manifold which is endowed with additional structures. Therefore transitive Lie algebroids admits a construction of inverse image generated by a smooth mapping of smooth manifolds.

Due to K.Mackenzie ([1]) the construction can be managed as a homotopy functor $\mathcal{T}LA_{g}$ from category of smooth manifolds to the transitive Lie algebroids. The functor $\mathcal{T}LA_{g}$ associates with each smooth manifold $M$ the set $\mathcal{T}LA_{g}(M)$ of all transitive algebroids with fixed structural finite dimensional Lie algebra $g$. Hence one can construct ([2],[4]) a classifying space $B_{g}$ such that the family of all transitive Lie algebroids with fixed Lie algebra $g$ over the manifold $M$ has one-to-one correspondence with the family of homotopy classes of continuous maps $[M,B_{g}]$: $\mathcal{T}LA_{g}(M) \approx [M,B_{g}]$.

It allows to describe characteristic classes of transitive Lie algebroids from the point of view a natural transformation of functors similar to the classical abstract characteristic classes for vector bundles and to compare them with that derived from the Chern-Weil homomorphism by J.Kubarski([3]). As a matter of fact we show that the Chern-Weil homomorphism does not cover all characteristic classes from categorical point of view.
1 Basic definitions and functor $\mathcal{T}LA_\g(\bullet)$

1.1 Definitions

**Definition 1.1.1.** (See [1], Definition 3.1.1) A Lie algebroid $A$ over a smooth manifold $M$ is a vector bundle $p : A \rightarrow M$ together with a Lie algebra structure $\{\bullet\}$ on the space of $\Gamma^\infty(A; M)$ and a bundle map $a : A \rightarrow TM$ called the anchor, such that

(i) the induced map $a : \Gamma(A; M) \rightarrow \Gamma(TM; M) = \mathfrak{X}^1(M)$ is a Lie algebra homomorphism

(ii) for any sections $\alpha, \beta \in \Gamma(A; M)$ and smooth function $f \in C^\infty(M)$ we have

\[
\{\alpha, f \cdot \beta\} = f \cdot \{\alpha, \beta\} + a(\alpha)(f) \cdot \beta
\]

We call $A$ a regular Lie algebroid if the rank of $a$ is locally constant and $A$ a transitive Lie algebroid if $a$ is surjective. The Lie algebroid homomorphism and isomorphism is defined in [1]. And we often use the Atiyah exact sequence $0 \rightarrow L \xrightarrow{j} A \xrightarrow{a} TM \rightarrow 0$ to denote a transitive Lie algebroid. Here $L = \text{Ker} a$ is called the adjoint bundle. Sometimes we use $(A, M, \{\text{bullet}\}, a)$ to note Lie algebroid in order to highlight the bracket. All transitive Lie algebroids (isomorphic class) and homomorphisms between them form a category that is fundamental in our considerations.

**Example 1.1.2.** (See [1]) The followings are important examples of transitive Lie algebroid.

1. Let $M$ be a manifold and let $\g$ be a Lie algebra. On $TM \bigoplus (M \times \g)$ define $a : TM \bigoplus (M \times \g) \rightarrow TM$ by $a : (X, \mu) \mapsto X$. And a bracket

\[
\{(X, \mu), (Y, \nu)\} = ([X, Y], X(\nu) - Y(\nu) + [\mu, \nu]).
\]

for $(X, \mu), (Y, \nu) \in \Gamma(TM \bigoplus (M \times \g); M)$.

Then $TM \bigoplus (M \times \g)$ is a transitive Lie algebroid on $M$, called the trivial Lie algebroid on $M$ with structural Lie algebra $\g$.

2. Let $L$ be a Lie algebra bundle on smooth manifold $M$. The Lie algebroid $\mathcal{D}_{\text{Der}}(L)$ of covariant derivatives on $\Gamma^\infty(L)$ is a transitive Lie algebroid on $M$.

3. The Lie algebroid $\mathcal{D}(E)$ of covariant differential operators on the space of sections of vector bundle $E$.

As vector space is commutative Lie algebra, vector bundle $E$ is also commutative Lie algebra bundle. Thus $\mathcal{D}(E)$ and $\mathcal{D}_{\text{Der}}(E)$ are identical in this case. In the following part of this article we use $\g$ to note Lie algebra and $\h$ to note commutative Lie algebra. All the Lie algebras we consider in this article are finite dimensional.
1.2 Functor $\mathcal{TA}_g(\bullet)$

In [1], K. Mackenzie defines pullback of transitive Lie algebroid over smooth map $f : M' \rightarrow M$. It means that given a Lie algebra $g$ there is the functor $\mathcal{TA}_g(\bullet)$ such that with any manifold $M$ it assigns the family $\mathcal{TA}_g(M)$ of all transitive Lie algebroid with structural Lie algebra $g$.

**Lemma 1.2.1.** (See [1], page 248) Let $0 \rightarrow L \xrightarrow{\beta} A \xrightarrow{\alpha} TM \rightarrow 0$ be a transitive Lie algebroid on smooth manifold $M$. Then $L$ is a Lie algebra bundle with respect to the braces structure on $\Gamma(L;M)$ induced from the braces on $\Gamma(A;M)$.

**Lemma 1.2.2.** (See [1], page 100) Let $A$ be a transitive Lie algebroid on $M$ and let $U \subset M$ be an open subset. Then the braces $\{,\} : \Gamma(A;M) \times \Gamma(A;M) \rightarrow \Gamma(A;M)$ restricted to $\Gamma(A_U;U) \times \Gamma(A_U;U) \rightarrow \Gamma(A_U;U)$ make $A_U$ be a Lie algebroid on $U$ called the restriction of $A$ to $U$.

**Lemma 1.2.3.** (See [1], page 317) Consider a transitive Lie algebroid $0 \rightarrow L \xrightarrow{\beta} A \xrightarrow{\alpha} TM \rightarrow 0$ on $M$ with fixed structural Lie algebra $g$. Given any open covering $\{U_\alpha\}$ of $M$ by contractible sets, for arbitrary $\alpha$, there is an Lie algebroid isomorphism

$$S_\alpha : TU_\alpha \bigoplus (U_\alpha \times g) \rightarrow A_{U_\alpha}$$

where $TU_\alpha \bigoplus (U_\alpha \times g)$ is trivial Lie algebroid on $U_\alpha$.

By using Lemma 1.2.1, Lemma 1.2.2, Lemma 1.2.3 and the method used in [2], we get the following theorem.

**Theorem 1.2.4.** Let $M$ and $N$ be smooth manifolds. Given an arbitrary transitive Lie algebroid $A$ on $N$. Let $f,g : M \rightarrow N$ are homotopic smooth maps. Then the pullback of $A$ over $f$ and $g$ are Lie algebroid isomorphic, that is $f^!A \approx g^!A$.

Hence the functor $\mathcal{TA}_g(\bullet)$ is homotopy functor for fixed structural Lie algebra $g$. There exists a classifying space $\mathcal{B}_g$ such that $\mathcal{TA}_g(M)$ has one to one correspondence with the family of homotopy classes of continuous maps $[M;\mathcal{B}_g]$. Here $\mathcal{B}_g$ is abstract and can be described in more or less understandable way (see [3]).
2 Obstruction

2.1 Cohomology

Definition 2.1.1. (see [1], page 107) Let $A$ be an arbitrary Lie algebroid on a smooth manifold $M$ and $E$ is a vector bundle on $M$. Let $\mathcal{D}(E)$ be the Lie algebroid of covariant derivative on $\Gamma^\infty(E)$. A representation of $A$ on $E$ is a Lie algebroid homomorphism

$$\rho : A \to \mathcal{D}(E).$$

The cohomology space $\mathcal{H}^n(A, \rho, E), n \geq 0$ can be defined when the representation $\rho$ is given (see [1], page 260). When $A$ is $TM$, we denote the representation by $\nabla : TM \to E$. Then there is $\mathcal{H}^n(M, \nabla, E), n \geq 0$. The representation $\nabla : TM \to E$ can be regard as a flat connection on $E$ (see [1], page 109, page 186.). Due to Lemma 1.1.6 and Lemma 1.2.2 in [3], the following theorem holds.

Theorem 2.1.2. Let $E$ be a vector bundle on smooth manifold $M$ and $\nabla : TM \to E$ be a representation of $TM$ on $E$. Let $f : M' \to M$ be a smooth map between smooth manifold $M'$ and $M$. Let $E' = f^*E$ be the pullback of vector bundle over $f$. Then

(i) the representation $\nabla$ induces a representation of $TM'$ on $E'$ noted by $\nabla' : TM' \to \mathcal{D}(E')$.

(ii) the map $f$ induces a homomorphism between cohomologies

$$f^* : \mathcal{H}^*(M, \nabla, E) \to \mathcal{H}^*(M', \nabla', E'),$$

where

$$\mathcal{H}^*(M, \nabla, E) = \bigoplus_{n=0}^\infty \mathcal{H}^n(M, \nabla, E), \quad \mathcal{H}^*(M', \nabla', E') = \bigoplus_{n=0}^\infty \mathcal{H}^n(M', \nabla', E').$$

From fundamental differential geometry, the following theorem holds.

Theorem 2.1.3. Let $E$ be a commutative Lie algebra bundle with fiber $\mathfrak{h}$. Let $\nabla$ be a flat connection on it. Then $\nabla$ induces the system of transition functions $\{\varphi_{\alpha\beta}\}$ for $E$ that are locally constant. Then $E$ can be seen as vector bundle with discrete structural group $\text{Aut}(\mathfrak{h})_d$, and denoted by $E^\nabla \to M$. Here $\text{Aut}(\mathfrak{h})_d$ is the group of all automorphisms of $\mathfrak{h}$, that is $\text{Aut}(\mathfrak{h})$, with discrete topology.
2.2 Obstruction class

Let $L$ be a Lie algebra bundle on smooth manifold $M$ with fiber $\mathfrak{g}$. There is a commutative diagram (see [1]).

\[
\begin{array}{cccccccc}
0 & 0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & 0 \\
ZL & = & ZL & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & 0 \\
L & = & L & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & 0 \\
\text{Der}(L) & \rightarrow & \text{Der}(L) & \rightarrow & TM & \rightarrow & 0 \\
\alpha & \rightarrow & \alpha & \rightarrow & \alpha & \rightarrow & \alpha & \rightarrow \\
\text{Out}_{\text{Der}}(L) & \rightarrow & \text{Out}_{\text{Der}}(L) & \rightarrow & TM & \rightarrow & 0 \\
\end{array}
\]

in which both rows and columns are exact.

Consider a coupling $\Xi : TM \rightarrow \text{Out}_{\text{Der}}(L)$, that is the curvature tensor

\[ R^\Xi : \Lambda^2(TM) \rightarrow \text{Out}_{\text{Der}}(L) \]

defined by

\[ R^\Xi(X,Y) = [\Xi(X), \Xi(Y)] - \Xi([X,Y]) \]

for $X,Y \in \mathfrak{X}(M)$ is zero.

There is a lifting $\nabla_\Xi : TM \rightarrow \text{Der}(L)$ of the coupling $\Xi$:

\[
\begin{array}{cccccccc}
L & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & 0 \\
0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & 0 \\
\end{array}
\]

in which $\nabla$ is vector bundle map.
Then for curvature tensor $R^\nabla : \Lambda^2(TM) \to \text{Der}(L)$ defined by $R^\nabla(X, Y) = [\nabla_X(Y), \nabla_Y(X)] - \nabla_{[X, Y]}$, the following diagram is commutative.

\[
\begin{array}{ccc}
L & \xrightarrow{\text{ad}} & \text{Der}(L) \\
\downarrow & & \downarrow \\
\Lambda^2(TM) & \xrightarrow{R^\nabla} & \Lambda^2(TM) \\
\downarrow & & \downarrow \\
\text{Out}_{\text{Der}}(L) & \xleftarrow{\phi^0} & \text{Out}_{\text{Der}}(L)
\end{array}
\]

Since vertical column is exact there is a lifting of $R^\nabla$ that is a bundle map $\Omega : \Lambda^2(TM) \to L$ such that the diagram

\[
\begin{array}{ccc}
L & \xrightarrow{\text{ad}} & \text{Der}(L) \\
\downarrow & & \downarrow \\
\Lambda^2(TM) & \xrightarrow{R^\nabla} & \Lambda^2(TM) \\
\downarrow & & \downarrow \\
\text{Out}_{\text{Der}}(L) & \xleftarrow{\phi^0} & \text{Out}_{\text{Der}}(L)
\end{array}
\]

is commutative.

Define $d^\nabla : \Gamma(\Omega^n(M, L); M) \to \Gamma(\Omega^{n+1}(M, L); M)$ by

\[
d^\nabla f(X_1, X_2, ..., X_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} \nabla(X_i)(f(X_1, X_2, ..., \hat{X}_i, ..., X_{n+1})) + \sum_{i<j} (-1)^{i+j} f([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_{n+1})
\]

here $f \in \Gamma(\Omega^n(M, L); M)$ and $X_1, X_2, ..., X_{n+1} \in \mathfrak{x}^1$.

For $\Omega$ in diagram (1), $d^\nabla = 0 \in \Omega^3(M, ZL)$ and $d^\nabla \circ (d^\nabla) = 0$ where $\nabla^\nabla L$ is induced by $\nabla_\Xi$ (see [1]). Then define $\text{Obs}(\nabla_\Xi) = [d^\nabla(\Omega)] \in H^3(M, \nabla^\nabla L, ZL)$. The connection $\nabla^\nabla L$ and cohomology class $\text{Obs}(\nabla_\Xi)$ depend only on $\Xi$ (see [1], page 273 and Theorem 7.2.12). Then the class $\text{Obs}(\nabla_\Xi)$ is called the \textit{obstruction class} of the coupling $\Xi$, and is denoted by $\text{Obs}(\Xi)$.

**Theorem 2.2.1. (The functorial property)** Let $L$ be a finite dimensional Lie algebra bundle on a smooth manifold $M$. Let $M'$ be a smooth manifold and $f : M' \to M$ is smooth map. Let $L' = f^*L$ be the pullback of Lie algebra bundle
over $f$. Consider a coupling $\Xi : TM \to \text{Out}_{\mathcal{D}_{\text{Der}}} L$. Then $\Xi$ induces a coupling $\Xi' : TM' \to \text{Out}_{\mathcal{D}_{\text{Der}}} L'$ and $f$ induces a homomorphism

$$f^* : H^*(M, \Xi, ZL) \to H^*(M', \Xi', ZL').$$

Further more the obstruction class $\text{Obs}(\Xi') \in H^3(M', \Xi', ZL')$ satisfies the condition

$$f^*(\text{Obs}(\Xi)) = \text{Obs}(\Xi').$$

**Definition 2.2.2.** An extension of $TM$ by Lie algebra bundle $L$ is an exact sequence of Lie algebroid over $M$

$$0 \to L \xrightarrow{j} A \xrightarrow{\alpha} TM \to 0.$$  

**Theorem 2.2.3.** (see [1], corollary 7.3.9) Let $L$ be a Lie algebra bundle on $M$. Let $\Xi : TM \to \text{Out}_{\mathcal{D}_{\text{Der}}} (L)$ be a coupling. Then, if $\text{Obs}(\Xi) = 0$, there is a Lie algebroid extension

$$0 \to L \xrightarrow{j} A \xrightarrow{\alpha} TM \to 0$$

of $TM$ by $L$ inducing the coupling $\Xi$.

**Corollary 2.2.4.** Let $E$ be a vector bundle over $M$ (that is the Lie algebra bundle with commutative Lie algebra). There is a Lie algebroid extension

$$0 \to E \xrightarrow{j} A \xrightarrow{\alpha} TM \to 0$$

if and only if the bundle $E$ is flat.

**Proof.** Suppose that the extension exists

$$0 \to E \xrightarrow{j} A \xrightarrow{\alpha} TM \to 0$$

Let $\lambda : TM \to A$ be a splitting. Define

$$\nabla^\lambda : \mathfrak{X}^1(M) \times \Gamma^\infty(E; M) \to \Gamma^\infty(E; M)$$

by the formula

$$\nabla^\lambda_X(\mu) = \{\lambda(X), \mu\}.$$  

Then

$$R^\nabla_X(\nabla^\lambda, \nabla^\lambda_Y)(\mu) = [\nabla^\lambda_X, \nabla^\lambda_Y](\mu) - \nabla^\lambda_{[X,Y]}(\mu) =$$

$$= \{[\lambda(X), \lambda(Y)] - \lambda([X,Y]), \mu\} = 0$$

for arbitrary $X, Y \in \mathfrak{X}^1(M), \mu \in \Gamma(E; M)$ since $a([\lambda(X), \lambda(Y)] - \lambda([X,Y])) = 0$ that is $[\lambda(X), \lambda(Y)] - \lambda([X,Y]) \in \Gamma(E; M)$ and the structural Lie algebra is commutative.

Conversely. If $E$ is flat, there is a flat connection $\nabla$ on $E$ which also is a representation of the Lie algebroids

$$\nabla : TM \to \mathcal{D}(E),$$

that is $R^\nabla(X,Y) = 0$.

By definition of obstruction class this means that $\text{Obs}(\nabla) = 0 \in H^3(M, \nabla, E)$. Then there exist Lie algebroid extensions. \qed
3 Characteristic Classes

In this section a system of characteristic classes of transitive Lie algebroid with commutative adjoint bundle will be described. Then they will be compared with characteristic classes derived from Chern-Weil homomorphism by J. Kubarski ([3]). As a matter of fact we show that the Chern-Weil homomorphism does not cover all characteristic classes from categorical point of view.

3.1 A system of characteristic classes for commutative case

Let $\mathfrak{h}$ be a finite dimensional commutative Lie algebra. Let $\text{Aut}(\mathfrak{h})_d$ be the group $\text{Aut}(\mathfrak{h})$ with discrete topology. The functor $\text{Vector}_d^b(\bullet)$ associates with each paracompact topology space $X$ the set $\text{Vector}_d^b(X)$ of all vector bundle with structural group $\text{Aut}(\mathfrak{h})_d$. Let $E^\infty \to B\text{Aut}(\mathfrak{h})_d$ be universal bundle with group $\text{Aut}(\mathfrak{h})_d$ and let $B\text{Aut}(\mathfrak{h})_d$ be the classifying space.

**Lemma 3.1.1.** (See [6], Definition 11.1, Theorem 11.2, Theorem 12.2) There is a bijection between $\text{Vector}_d^b(X)$ and the homotopy classes of continuous maps $[X; B\text{Aut}(\mathfrak{h})_d]$.

Let $M$ be a smooth manifold and

$$0 \to E \xrightarrow{j} A \xrightarrow{\alpha} TM \to 0$$

(2)

be a transitive Lie algebroid with fixed structural commutative Lie algebra $\mathfrak{h} = \mathbb{R}^n$. Let $\lambda : TM \to A$ be a splitting. Define $\nabla = \nabla^\lambda$ by a formula $\nabla^\lambda_\mu(\nu) = \{\lambda(\nu), \mu\}$. The bundle $E$ possesses a flat structure $E^\nabla \in \text{Vector}_d^b(M)$. Let $f : M' \to M$ be a smooth map and $f^!A$ be the pullback of Lie algebroid $A$ over $f$, that is

$$0 \to f^*E \xrightarrow{j'} f^!A \xrightarrow{\alpha'} TM' \to 0.$$  

(3)

Let $\lambda' : TM' \to f^!A$ be a splitting. Define $\nabla' = \nabla^\lambda'$ on $f^*E$ and $f^*E$ is corresponding to $(f^*E)^{\nabla'}$.

**Lemma 3.1.2.** (i) $\nabla$ and $\nabla'$ are independent of the choice of $\lambda$ and $\lambda'$,

(ii) The bundle $(f^*E)^{\nabla'}$ is the pullback of $E^\nabla$ over $f : M' \to M$ in the category of vector bundle with discrete structural group $\text{Aut}(\mathfrak{h})_d$.

**Proof.** Statement (i) is obvious.

(ii) : Consider the splitting of transitive Lie algebroid $[3]$

$$\lambda' : TM' \to f^!A$$

by the formula

$$\lambda'(X') = (X', \lambda(Tf(X'))),$$

$X' \in TM'$.
Let \( \sum_i h_i \cdot (\mu_i \circ f) \in \Gamma(f^* E; M) \), here \( h_i \in C^\infty(M'), \mu_i \in \Gamma(E; M) \). Then

\[
\nabla^X_{\lambda_i} \left( \sum_i h_i \cdot (\mu_i \circ f) \right) = \sum_i X' \left( \nabla^X_{\lambda_i} \right) \cdot (\mu_i \circ f) + \sum_i h_i \cdot (\nabla^Y_{f(X')}(\mu_i) \circ f)
\]

(4)

As \( \nabla^X \) is flat connection, there exist chart \( \{ \varphi_{\alpha} : E_U \to U \times h \}_{\alpha \in \Delta} \) which satisfies the condition

\[
\varphi_{\alpha} (\nabla^X_{\lambda}(\mu_{\alpha})) = X(\varphi_{\alpha}(\mu_{\alpha}))
\]

(5)

for arbitrary \( \mu_{\alpha} \in \Gamma(E_{U_{\alpha}}; U_{\alpha}), X \in \mathfrak{X}(M) \).

Consider \( \mu \in \Gamma(E_{U_{\alpha}} \cap U_{\beta}; U_{\alpha} \cap U_{\beta}) \). Then

\[
X(\varphi_{\beta} \circ \varphi_{\alpha}^{-1} \circ \varphi_{\alpha}(\mu)) = X(\varphi_{\beta}(\mu)) = \varphi_{\beta}(\nabla^X_{\lambda}(\mu)) = \varphi_{\beta} \circ \varphi_{\alpha}^{-1}(\varphi_{\alpha}(\nabla^X_{\lambda}(\mu))).
\]

Then

\[
X(\varphi_{\beta} \circ \varphi_{\alpha}^{-1} \circ \varphi_{\alpha}(\mu)) = \varphi_{\beta} \circ \varphi_{\alpha}^{-1} \circ X(\varphi_{\alpha}(\mu))
\]

(6)

Thus the transition functions \( \{ \varphi_{\alpha\beta} \}_{\alpha, \beta \in \Delta} \) are all locally constant.

Let \( \{ V'_{\alpha} = f^{-1}(U_{\alpha}) \}_{\alpha \in \Delta} \) be atlas of charts on \( M' \). Define the homomorphism of \( C^\infty(V'_{\alpha}) \)-modules

\[
\psi_{\alpha} : \Gamma(f^* E|_{V'_{\alpha}}; V'_{\alpha}) \to \Gamma(V'_{\alpha} \times h; V'_{\alpha})
\]

defined by the formula

\[
\psi_{\alpha}(h_{\alpha, i} \cdot (\mu_{\alpha} \circ f)) = h_{\alpha, i} \cdot \varphi_{\alpha}(\mu_{\alpha}) \circ f
\]

for \( h_{\alpha, i} \cdot (\mu_{\alpha} \circ f) \in \Gamma(f^* E|_{V'_{\alpha}}; V'_{\alpha}) \), where \( h_{\alpha, i} \in C^\infty(V'_{\alpha}), \mu_{\alpha} \in \Gamma(E_{U_{\alpha}}; U_{\alpha}) \).

As \( \varphi_{\alpha} \) is vector bundle isomorphism, \( \psi_{\alpha} \) induces a vector bundle isomorphism. Then \( \{ V'_{\alpha}, \psi_{\alpha} : f^* E|_{V'_{\alpha}} \to V'_{\alpha} \times h \}_{\alpha \in \Delta} \) is a chart for \( f^* E \). Consider a vector field \( X' \in \mathfrak{X}(M') \). Then

\[
\psi_{\alpha}(h_{\alpha, i} \cdot (\mu_{\alpha} \circ f)) = \psi_{\alpha}(X'(h_{\alpha, i}) \cdot (\mu_{\alpha} \circ f) + h_{\alpha, i} \cdot (T f(X'))(\varphi_{\alpha}(\mu_{\alpha})) \circ f) = X'(h_{\alpha, i} \cdot (\varphi_{\alpha}(\mu_{\alpha})) \circ f) = \psi_{\alpha}(h_{\alpha, i} \cdot (\mu_{\alpha} \circ f))
\]

The transition functions

\[
\psi_{\alpha\beta} : V'_{\alpha} \cap V'_{\beta} \to \text{Aut}(h)_d
\]

are defined by

\[
\psi_{\alpha\beta}(x') = \varphi_{\alpha\beta}(f(x'))
\]

for \( x' \in V'_{\alpha} \cap V'_{\beta} \).

So \( (f^* E)' \) is the pullback of \( E' \) over \( f : M' \to M \) in the category of vector bundle with discrete structural group \( \text{Aut}(h)_d \). \qed
The Lemma 3.1.2 shows that the following definition is corrected.

**Definition 3.1.3.** Let \( \mathfrak{h} \) be a commutative Lie algebra and \( M \) be a smooth manifold. Let \( A \in \mathcal{T}LA(h)(M) \), with splitting \( \lambda \). Let \( E^{\boxtimes \lambda} \) be the correspondent Lie algebra bundle with flat structure. Let \( \theta : \text{Vector}_h^M(M) \rightarrow [M; B\text{Aut}_h] \) be the bijection defined in Lemma 3.1.1. Then \( \theta(E^{\boxtimes \lambda}) = [f] \in [M; B\text{Aut}_h] \) induces a homomorphism

\[
f^* : H^*(B\text{Aut}_h; R) \rightarrow H^*(M; R).
\]

The class \( f^*(c) \in H^*(M; R) \) is characteristic class of \( A \), for arbitrary \( c \in H^*(B\text{Aut}_h; R) \).

### 3.2 Chern-Weil Homomorphism

**Definition 3.2.1.** (see [3], page17) Given a transitive Lie algebroid \( (A,q,M,\{\},a) \) with adjoint bundle \( L \). The adjoint representation of a transitive Lie algebroid \( A \) is

\[
ad : A \rightarrow \mathcal{D}(L)
\]

defined by

\[
ad(\xi)(\nu) = \{\xi,\nu\}
\]

for \( \xi \in \Gamma(A; M), \nu \in \Gamma(L; M) \). Let \( L^* \) be dual bundle of \( L \) and \( \bigvee^k L^* \) is \( k \)-th symmetric power of \( L^* \) (see [7], page 191). The adjoint representation \( ad \) can rise to

\[
\bigvee^k ad^\delta : A \rightarrow \mathcal{D}(\bigvee^k L^*)
\]

such that

\[
< \bigvee^k ad^\delta(\xi)(\varphi), \nu^1 \vee \nu^2 \vee ... \vee \nu^k > = \sum_{i=1}^k < \varphi, \nu^i \vee ... \vee \{\xi,\nu^1\} \vee ... \vee \nu^k >
\]

for \( \xi \in \Gamma(A; M), \varphi \in \Gamma(\bigvee^k L^*; M), \nu^i \in \Gamma(L; M) \).

**Remark 3.2.2.** Here we only consider the vector bundle structure of \( L \) that is commutative Lie algebra structure. Hence we use notation \( \mathcal{D}(L) \) and \( \mathcal{D}(\bigvee^k L^*) \).

**Definition 3.2.3.** (see [3], Definition 2.3.1) Given an arbitrary transitive Lie algebroid \( 0 \rightarrow L \rightarrow A \xrightarrow{\pi} TM \rightarrow 0 \). Let \( L^* \) be dual bundle of \( L \). A section \( \varphi \in \Gamma(\bigvee^k L^*; M) \) is called \( \bigvee^k ad^\delta \)-invariant if \( \bigvee^k ad^\delta(\xi)(\varphi) = 0 \) for all \( \xi \in \Gamma(A; M) \). The space of all \( \bigvee^k ad^\delta \)-invariant sections of \( \bigvee^k L^* \) is denoted by \( \Gamma^{\text{inv}}(\bigvee^k L^*; M) \).

**Definition 3.2.4 (Chern-Weil homomorphism).** (see [3], page 29) Given a transitive Lie algebroid \( (A,q,M,\{\},a) \) with adjoint bundle \( L \). Let \( \lambda : TM \rightarrow \)
A be a splitting and $R^\lambda \in \Omega^2(M; L)$ be the curvature tensor, $R^\lambda(X, Y) = \{\lambda(X), \lambda(Y)\} - \lambda([X, Y])$.

Define a homomorphism of $C^\infty(M)$–modules

$$\chi_{(A, \lambda), I} : \Gamma^I(\bigwedge^k L^*; M) \to \Omega^{2k}(M)$$

by the formula

$$\chi_{(A, \lambda), I} = \frac{1}{k!} <\varphi, R_\lambda \vee R_\lambda \vee ... R_\lambda >$$

for $\varphi \in \Gamma(\bigwedge^k L^*; M)$. Here

$$<\varphi, R_\lambda \vee ... \vee R_\lambda > (X_1, X_2, ..., X_{2k}) =
= <\varphi, \frac{1}{k!} \sum_{\sigma}^{} (-1)^{\sigma} R_\lambda(X_{\sigma(1)}, X_{\sigma(2)}) \vee R_\lambda(X_{\sigma(3)}, X_{\sigma(4)}) \vee ... \vee R_\lambda(X_{\sigma(2k-1)}, X_{\sigma(2k)}) >$$

The forms from the image of $\chi_{(A, \lambda), I}$ is closed (see [3], proposition 4.1.2). Then Chern-Weil homomorphism is defined by the composition

$$h_{(A, \lambda)} : \bigoplus_{k \geq 0}^{} \Gamma^I(\bigwedge^k L^*; M) \xrightarrow{\chi_{(A, \lambda), I}} \text{Ker } d^\nabla^\lambda \xrightarrow{i} H^*_{D\text{-Ram}}(M; R).$$

The Chern-Weil homomorphism has functorial property and is independent of the choice of splitting (see [3], theorem 4.2.2, theorem 4.3.7). Then $h_{(A, \lambda)}$ can be denoted as

$$h_A : \bigoplus_{k \geq 0}^{} \Gamma^I(\bigwedge^k L^*; M) \to H^*_{D\text{-Ram}}(M; R).$$

### 3.3 Example

The following example shows that the Chern-Weil homomorphism does not cover all categorical characteristic classes.

Consider a flat 1–dimensional vector bundle $E$ over a torus $T^2 = S^1 \times S^1$. We will consider $E$ as a Lie algebra bundle with commutative Lie algebra $\mathfrak{h} \approx \mathbb{R}^1$. The structural group of the bundle $E$ is the group $R^* = R \setminus \{0\}$ with discrete topology. The flat structure on $E$ is defined by an atlas of charts $\{U_\alpha\}$ with trivialization of the bundle $E$ on each chart $U_\alpha$ such that all transition function are locally constant. Transition functions are fully defined by a representation of the fundamental $\pi_1(T^2)$ in the structural group $\text{Aut} (\mathfrak{h})_d$, $\rho : \pi_1(T^2) \to \text{Aut} (\mathfrak{h})_d$.

There is a flat connection $\nabla$ on $E \to T^2$ which corresponds to the flat structure on $E$. This means that the connection on each chart $U_\alpha$ (after trivialization of the bundle $E$) coincides with usual derivative ($\nabla_X = \frac{\partial}{\partial X}$).

Construct a Lie algebroid $\mathcal{A} :$

$$0 \to E \to T(T^2) \bigoplus E \to T(T^2) \to 0$$
with bracket

\[
\{(X, \mu), (Y, \nu)\} = ([X, Y], \nabla_X (\nu) - \nabla_Y (\mu) + \Omega(X, Y))
\]

for \((X, \mu), (Y, \nu) \in \Gamma(T(T^2) \bigoplus E; T^2)\). Here \(\Omega \in \text{Ker} \ d^\nabla \subset \Omega^2(T^2, E)\). Let \(E^*\) be the bundle dual to \(E\). Let \(f \in \Gamma^\prime(E^*; T^2)\). Then

\[
ad^\nabla((X, \nu))(f)(\mu) = X(f(\mu)) - f([X \oplus \nu, 0 \oplus \mu]) = X(f(\mu)) - f(\nabla_X (\mu)) = 0 \tag{7}
\]

for arbitrary \(\mu \in \Gamma(E; T^2)\), \((X, \nu) \in \Gamma(T(T^2) \bigoplus E; T^2)\). Hence locally on the chart \(U_\alpha\) the function \(f\) is constant.

This means that in the case of nontrivial representation the space \(\Gamma^\prime(E^*; T^2)\) has only trivial element. Thus the characteristic class for \(A\) defined by Chern-Weil homomorphism by J.Kubarski is trivial.

On the other hand the characteristic classes due to definition 3.1.3 are not trivial. Namely the structural group \(\text{Aut}(\mathfrak{h})_d\) is isomorphic to \(\mathbb{Z}_2 \times \mathbb{R}\). Hence the classifying space \(B_\mathbb{Z}_2 \times B_\mathbb{R}\) can be represent as a direct limits \(B_\mathbb{R} = \lim_{\leftarrow \mathbb{R}}(b \subset B; b)\), where each \(b \in B\) is a finite collection of indexes

\[
b = \{ \alpha_1, n_1, \alpha_2, n_2, \ldots, \alpha_k, n_k \}, \alpha_j \in A, n_j \in \mathbb{Z}
\]

that are ordered in the natural way, \(T_b = \prod_{j=1}^{k} S_{\alpha_j, n_j} \approx \mathbb{T}^k\).

The cohomology group \(H^*(B_{\text{Aut}(\mathfrak{h})_d}; R)\) can be describe in the following way:

\[
H^*(B_{\mathbb{Z}_2}; R) \approx \mathbb{R}; \quad H^*(B_{\mathbb{R}}; R) \approx \lim_{\leftarrow \mathbb{R}} H^*(\mathbb{T}_b; R).
\]

The representation \(\rho : \pi_1(T^2) \rightarrow \text{Aut}(\mathfrak{h})_d\) induces the map

\[
B_\rho : \mathbb{T}_2 \rightarrow B_{\mathbb{Z}_2} \times B_{\mathbb{R}},
\]

and the homomorphism in cohomology

\[
B_\rho^* : H^*(B_{\mathbb{Z}_2} \times B_{\mathbb{R}}; R) \rightarrow H^*(\mathbb{T}_2; R).
\]

**Lemma 3.3.1** (Key lemma). The homomorphism \(B_\rho^*\) is surjective.
The example show that Chern-Weil homomorphism cannot define all characteristic classes for transitive Lie algebroid.

**Remark 3.3.2.** This example show that there is a natural problem to generalize the Chern-Weil homomorphism for non trivial flat bundle $ZL$ of local coefficients for cohomologies that contain characteristic classes.

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