A SELECTIVE VERSION OF LIN'S THEOREM

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Abstract. We prove a selective version of Lin’s Theorem for nearly commuting operators. This is accomplished by establishing the tracial stability of a certain class of graph products of $C^*$-algebras. This result involves the development of the “pincushion class” of finite graphs. We are further motivated by some approximation properties of right-angled Artin groups. In particular, we show that the full $C^*$-algebra of any right-angled Artin group is quasidiagonal and thus has an amenable trace. Tracial stability is then applied in showing when these amenable traces are in fact locally finite dimensional.

1. Introduction

Lin’s Theorem resolved a long-standing problem posed by Halmos in [30] and can be stated as follows.

**Theorem 1.1** ([34]). For any $\varepsilon > 0$ there is a $\delta > 0$ such that for any $n \in \mathbb{N}$ and any two contractive self-adjoint elements $A, B \in M_n$ with $\|[A, B]\| < \delta$ there exist commuting self adjoints $A', B' \in M_n$ with $\|A - A'\| + \|B - B'\| < \varepsilon$. Here, $M_n$ denotes the $n \times n$ matrices with complex entries, and $[a, b] := ab - ba$.

In English, this says “a pair of nearly commuting self-adjoint matrices is near a pair of commuting self-adjoint matrices.” This is a widely celebrated result in the operator theory community and frequently has been the subject of extension and variation. Friis-Rørdam in [21] extended Lin’s Theorem to more operator algebras, including factor von Neumann algebras. Further quantitative refinements can be found in [20, 32]. A number of counterexamples are also present throughout the literature; in particular, it has been shown that the above statement does not apply to unitaries ([42]), or to more than two self-adjoints ([15, 14, 35, 36]).

Halmos’s problem is often posed in different contexts. When we consider the Hilbert-Schmidt norm given by a trace, we have a positive result for an $n$-tuple of normal, self-adjoint, or unitary elements. To put a finer point on it, Hadwin gave the following Hilbert-Schmidt variation of Lin’s Theorem in [25].

**Theorem 1.2** ([25]). For every $n \in \mathbb{N}$ and each $\varepsilon > 0$, there is a $\delta > 0$ such that for any finite factor von Neumann algebra $M$ with tracial state $\tau$, if $A_1, \ldots, A_n$ are contractions in $M$ such that

$$\|[A_i, A_j^*]\|_{2,\tau}, \|[A_i, A_j]\|_{2,\tau} < \delta$$

for $1 \leq i, j \leq n$, then there is a commuting family $\{B_1, \ldots, B_n\}$ of normal elements in $M$ such that

$$\sum_{i=1}^n \|A_i - B_i\|_{2,\tau} \leq \varepsilon.$$
If \( \|A_i - A_i^*\|_{2,\tau} < \delta \) for \( 1 \leq i \leq n \), then the \( B_i \)'s can be taken to be self-adjoint; and if \( \|1 - A_iA_i^*\|_{2,\tau} < \delta \), then the \( B_i \)'s can be taken to be unitaries. Here, \( \|x\|_{2,\tau} = \sqrt{\tau(x^*x)} \).

See \[22, 19\] for more on the Hilbert-Schmidt version. Theorem 1.2 is proved in \[25\] using an ultraproduct argument. In \[27\], Hadwin-Shulman consider the “ultra-mathematical” notion of tracial stability—see \S\S 2.1. Upon revisiting Hadwin’s proof of Theorem 1.2 in \[25\] one can see how it utilizes the notion of tracial stability two decades before its definition. This usage is to be discussed in more detail in \S\S 2.1.

It should also be noted that Arzhantseva-Păunescu recently proved an analogous result for symmetric groups in \[2\]: with respect to the Hamming metric, nearly commuting permutations are near commuting permutations.

As the title suggests, the main result of this article is a selective version of Lin’s Theorem. As in \[25, 22, 19\] we consider the Hilbert-Schmidt norm from a tracial state; so a more accurate description would be a selective version of Hadwin’s tracial variation of Lin’s Theorem. The main result of this article shows that the statement of Theorem 1.2 holds when the near-commuting hypothesis is assumed only for a selected sub-collection of pairs of the \( A_i \)'s. Using the concept of graph products, the selectivity of the near commuting relations can be encoded using a graph with the vertices representing the \( n \) elements and the edges connecting the selected pairs of elements which nearly commute. We introduce the pincushion class of finite graphs and show that under the right conditions, graph products over such graphs preserve tracial stability—see Theorem 3.9. This result introduces a wide class of new examples of tracially stable algebras. Then as a direct consequence we can show that for graphs in the pincushion class, a version of Theorem 1.2 holds for elements that nearly commute according to those graphs. A precise statement is given in Theorem 3.10.

We can apply this result to establish an approximation property of certain right-angled Artin groups (see Definition 4.2). In particular, we will first show that all right-angled Artin groups have quasidiagonal full group \( C^* \)-algebras. This fact guarantees the existence of amenable traces (cf. Definition 4.11) on these algebras. Our result then shows that for a right-angled Artin group \( A \) coming from a graph in the pincushion class, all amenable traces on \( C^*(A) \) are in fact locally finite dimensional (Definition 4.13).

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2. Preliminaries

2.1. Tracial stability. Let \( \mathcal{U} \) denote a free ultrafilter on \( \mathbb{N} \). If for each \( k \in \mathbb{N} \) \( A_k \) is a unital \( C^* \)-algebra and \( \tau_k \) is a tracial state on \( A_k \), we let \( (A_k, \tau_k)^U \) denote the tracial ultraproduct of the \( A_k \)'s with respect to the \( \tau_k \)'s. Let \( a_k \in A_k \) for each \( k \in \mathbb{N} \); then we let \( (a_k)_U \in (A_k, \tau_k)^U \) denote the coset of the sequence \( (a_k)_{k \in \mathbb{N}} \) in \( \prod_{k \in \mathbb{N}} A_k \). We will sometimes suppress the \( \tau_k \) notation when the context is clear. See Appendix A of \[9\] for an introduction to ultrafilters and ultraproducts.

Definition 2.1 (\[27\]). Let \( A, A_k \) be unital \( C^* \)-algebras, and for each \( k \in \mathbb{N} \) let \( \tau_k \) be a tracial state on \( A_k \). A unital \( * \)-homomorphism \( \pi: A \rightarrow (A_k, \tau_k)^U \) is approximately liftable if there is a set \( E \in \mathcal{U} \) such that for each \( k \in E \) there is a
von Neumann algebras are $\mathcal{C}$. In particular, if $\pi_k$ is approximately liftable by Theorem 2.4. Hence, for each $\pi_k \in \mathcal{C}$, there exists a homomorphism $\pi_k : \mathcal{A} \to \mathcal{A}_k$ such that $\pi_k(a) = (\pi_k(a))_{U_k}$ for every $a \in \mathcal{A}$ where $\pi_k(a) = 0$ for $k \notin E$. Such a sequence $\{\pi_k\}$ is called a lifting of $\pi$.

The following is a general lemma about approximate liftings that will be a valuable utility. It first appeared as Lemma 2.2 in [27].

**Lemma 2.2** ([27]). Let $\mathcal{A}$ be a separable unital $C^*$-algebra with generators $\{b_j\}_{j \in \mathbb{N}}$. For each $k \in \mathbb{N}$, let $\mathcal{A}_k$ be a unital $C^*$-algebra and $\tau_k$ be a tracial state on $\mathcal{A}_k$. A $*$-homomorphism $\pi : \mathcal{A} \to (\mathcal{A}_k, \tau_k)^{U_k}$ with $\pi(b_j) = (b_j(k))_{U_k}$ is approximately liftable if and only if for every $\varepsilon > 0$ and every $N \in \mathbb{N}$, there is a set $E \in U_k$ with the property that for every $k \in E$ there is a unital $*$-homomorphism $\pi_k : \mathcal{A} \to \mathcal{A}_k$ such that for every $1 \leq j \leq N$ and every $k \in E$,

$$||\pi_k(b_j) - b_j(k)||_{2, \tau_k} < \varepsilon$$

where $|| \cdot ||_{2, \tau_k}$ denotes the Hilbert-Schmidt norm coming from the trace $\tau_k$.

**Definition 2.3** ([27]). Let $\mathcal{C}$ denote a class of unital $C^*$-algebras closed under $*$-isomorphism. A unital $C^*$-algebra $\mathcal{A}$ is $\mathcal{C}$-tracially stable if every unital $*$-homomorphism $\pi : \mathcal{A} \to (\mathcal{A}_k, \tau_k)^{U_k}$ is approximately liftable whenever $\mathcal{A}_k \in \mathcal{C}$ and $\tau_k$ is a tracial state on $\mathcal{A}_k$ for every $k \in \mathbb{N}$.

Next, we discuss some results that give some examples of tracially stable $C^*$-algebras. Recall that a $C^*$-algebra is real rank zero (RR0) if its self-adjoint elements can be approximated by self-adjoint elements with finite spectrum.

**Theorem 2.4** ([27]). Let $\mathcal{C}$ be a class of RR0 $C^*$-algebras. Then any separable unital commutative $C^*$-algebra $C(X)$ is $\mathcal{C}$-tracially stable.

We now show how we can use Theorem 2.4 to rewrite Hadwin’s proof of Theorem 1.2 using the language of tracial stability.

**Proof of Theorem 1.2**. Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be given. Suppose for the sake of contradiction that for every $k \in \mathbb{N}$ there is a finite factor $M_k$ with tracial state $\tau_k$ and elements $A_{1,k}, \ldots, A_{n,k}$ such that

$$||[A_{i,k}, A_{i,k}^*]||_{2, \tau_k}, ||[A_{i,k}, A_{j,k}]||_{2, \tau_k} < \frac{1}{k}$$

with the property that for any commuting family of normal elements $\{B_{1,k}, \ldots, B_{n,k}\}$ in $M_k$, we have

$$\sum_{i=1}^{n} ||A_{i,k} - B_{i,k}||_{2, \tau_k} > \varepsilon.$$  \hspace{1cm} (2.1)

Now consider the ultraproduct $(M_k, \tau_k)^{U_k}$. For $1 \leq j \leq n$, let $X_j = (A_{j,k})_{U_k}$. Then the elements $X_1, \ldots, X_n$ are normal and pairwise commute in $(M_k, \tau_k)^{U_k}$. Thus $C^*(X_1, \ldots, X_n)$ is a commutative $C^*$-algebra. Consider the identity $\pi : C^*(X_1, \ldots, X_n) \to (M_k, \tau_k)^{U_k}$. Since finite factor von Neumann algebras are RR0 $C^*$-algebras (by Borel functional calculus), then $\pi$ is approximately liftable by Theorem 2.4. Hence, for each $k \in \mathbb{N}$, there is a $*$-homomorphism $\pi_k : C^*(X_1, \ldots, X_n) \to M_k$ such that $(\pi_k(a))_{U_k} = \pi(a)$ for every $a \in C^*(X_1, \ldots, X_n)$. For each $1 \leq j \leq n$ and each $k \in \mathbb{N}$, let $B_{j,k} = \pi_k(X_j)$. Then for each $k \in \mathbb{N}$, $\{B_{1,k}, \ldots, B_{n,k}\}$ is a collection of commuting normal elements in $M_k$ with $(A_{j,k})_{U_k} = (B_{j,k})_{U_k}$ for every $1 \leq j \leq n$. This contradicts (2.1). \hspace{1cm} \square
Let $\mathbb{II}_1$ denote the class of $\mathbb{II}_1$-factor von Neumann algebras. Hadwin-Shulman studied the tracial stability of finitely presented groups in [28]; the main result of that paper is the following theorem.

**Theorem 2.5 ([28]).** If $G$ is a one-relator group with nontrivial center, then $C^*(G)$ is $\mathbb{II}_1$-tracially stable.

A $C^*$-algebra $A$ is tracially amenable if for every tracial state $\tau$ on $A$, $\pi_\tau(A)'$ is hyperfinite where $\pi_\tau$ denotes the GNS representation induced by $\tau$. The following theorem first appeared as Theorem 4.10 of [26] for nuclear algebras.

**Theorem 2.6 ([26]).** Any unital separable tracially amenable $C^*$-algebra is $\mathbb{II}_1$-tracially stable.

See Theorem 5.15 of [3] for the tracially amenable case and a concise proof. We can use tracial stability to characterize the separably acting hyperfinite $\mathbb{II}_1$-factor in the following way.

**Theorem 2.7.** Let $N$ be a separably acting $\mathbb{II}_1$-factor satisfying the Connes Embedding Problem. That is, there exists a unital trace preserving embedding $N \to R^d$ where $R$ denotes the separably acting hyperfinite $\mathbb{II}_1$-factor. Then $N$ is hyperfinite if and only if $N$ is $\mathbb{II}_1$-tracially stable.

**Proof.** ($\Rightarrow$): This follows from Theorem 2.6.

($\Leftarrow$): If $N$ is not hyperfinite, then it does not embed into $R$. Thus there is no approximately liftable $*$-homomorphism $N \to R^d$. □

This result leads to the following open question.

**Question 2.8.** In [31], Jung showed that a (finitely generated) separably acting $\mathbb{II}_1$-factor $N$ satisfying the Connes Embedding Problem is hyperfinite if and only if any two embeddings $N \to R^d$ are unitarily equivalent. Theorem 2.7 shows that the separably acting hyperfinite $\mathbb{II}_1$-factor $R$ has the weaker property that any embedding $R \to R^d$ is approximately liftable. Does this property characterize $R$? That is, if $N$ is a separably acting $\mathbb{II}_1$-factor such that every embedding $N \to N^d$ is approximately liftable, then is $N$ necessarily hyperfinite?

The next result follows directly from the universal property of free products.

**Proposition 2.9.** Let $\mathcal{C}$ be any class of unital $C^*$-algebras closed under $*$-isomorphisms. The full/universal free product of finitely many $\mathcal{C}$-tracially stable $C^*$-algebras is $\mathcal{C}$-tracially stable.

The corresponding tensor product result takes more care.

**Theorem 2.10 ([27]).** Let $\mathcal{C}$ be a class of $\mathcal{RR}_0$ $C^*$-algebras closed under taking direct sums and unital corners. If $A$ is $\mathcal{C}$-tracially stable and $X$ is a compact Hausdorff space, then $A \otimes C(X)$ is $\mathcal{C}$-tracially stable.

**Theorem 2.10** yields an interesting corollary about relative commutants of tracially stable subalgebras of ultraproducts which may be known to experts but has not appeared in the literature.

**Corollary 2.11.** Let $\mathcal{C}$ be a class of $\mathcal{RR}_0$ $C^*$-algebras closed under taking direct sums and unital corners. Let $A$ be a unital $\mathcal{C}$-tracially stable $C^*$-algebra. For each $k \in \mathbb{N}$, let $A_k$ be a member of $\mathcal{C}$ and $\tau_k$ be a tracial state on $A_k$. Fix a $*$-homomorphism $\pi : A \to (A_k, \tau_k)^d$. 

2.12 Remark
The reverse containment is obvious. □

(1) If \( x \in \pi(A)' \cap (A_k, \tau_k)^U \) is normal (self-adjoint, a projection, a unitary), then there is a lifting \( \{\pi_k : A \to A_k\} \) of \( \pi \) such that \( x \in (\pi_k(A)' \cap A_k, \tau_k)^U \).
That is, \( x = (x_k)_U \) where \( x_k \in \pi_k(A)' \cap A_k \) is normal (respectively self-adjoint, a projection, a unitary) for every \( k \in E \) for some \( E \in U \).

(2) From (1) we have
\[
\pi(A)' \cap (A_k, \tau_k)^U = \bigvee_{\{\pi_k\} \text{ is a lifting of } \pi} (\pi_k(A)' \cap A_k, \tau_k)^U
\]

where \( \bigvee \) denotes the join.

**Proof.** (1): We prove the normal case; the other cases follow *mutatis mutandis.*
Let \( x \in \pi(A)' \cap (A_k, \tau_k)^U \) be normal, and let \( X \) denote the spectrum of \( x \). By Theorem 2.10, \( A \otimes C(X) \) is \( C \)-tracially stable. Let \( \rho : C(X) \to (A_k, \tau_k)^U \) denote the \( * \)-homomorphism given by \( \text{id} \mapsto x \). By hypothesis, the images of \( \pi \) and \( \rho \) commute. Thus we can form the approximately liftable \( * \)-homomorphism \( \pi \circ \rho : A \otimes C(X) \to (A_k, \tau_k)^U \). So there is a lifting \( \{\theta_k\} \) of \( \pi \otimes \rho \) such that for every \( y \in A \otimes C(X) \), \( (\pi \otimes \rho)(y) = (\theta_k(y))_U \). Put \( \pi_k := \theta_k|_{A \otimes 1} \). Then \( \{\pi_k\} \) is a lifting of \( \pi \) and \( x = \rho(\text{id}) = (\theta_k(1 \otimes \text{id}))_U \in (\pi_k(A)' \cap A_k, \tau_k)^U \).

(2): The algebra \( \pi(A)' \cap (A_k, \tau_k)^U \) is spanned by its self-adjoints. By (1), for every self-adjoint \( x \in \pi(A)' \cap (A_k, \tau_k)^U \), there is a lifting \( \{\pi_k\} \) such that \( x \in (\pi_k(A)' \cap A_k, \tau_k)^U \). Thus
\[
\pi(A)' \cap (A_k, \tau_k)^U \subset \bigvee_{\{\pi_k\} \text{ is a lifting of } \pi} (\pi_k(A)' \cap A_k, \tau_k)^U.
\]

The reverse containment is obvious. □

**Remark 2.12.** Some concrete classes of RR0 \( C^* \)-algebras of interest are \( \text{fvNa} \): finite von Neumann algebras, \( \text{ff} \): finite factor von Neumann algebras, \( \Pi_1 \): \( \Pi_1 \)-factors, and \( \text{matrix} \): matrix algebras. The class \( \text{fvNa} \) is a class of RR0 \( C^* \)-algebras closed under taking direct sums and unital corners. This class contains matrix algebras and \( \Pi_1 \)-factors. So any algebra that is \( C \)-tracially stable for any class \( C \) of RR0 \( C^* \)-algebras closed under taking direct sums and unital corners is \( \text{fvNa} \), \( \text{ff} \), \( \Pi_1 \)-, and \( \text{matrix} \)-tracially stable.

2.2 Graph products. In operator algebras, graph products unify the notion of free and tensor products. In particular, given a simplicial graph \( \Gamma \) assign an algebra to each vertex. If there is an edge between two vertices then the two corresponding algebras commute with each other in the graph product; if there is no edge between two vertices then the two corresponding algebras have no relations with each other within the graph product. Thus free products are given by edgeless graphs, and tensor products are given by complete graphs. Such products initially appeared in the group theory context, and the most well-known examples are right-angled Artin groups (graph products of \( \mathbb{Z} \)) and right-angled Coxeter groups (graph products of \( \mathbb{Z}/2\mathbb{Z} \)). See [6, 12, 18, 17, 16, 23, 11, 14]. Interest in graph products has recently entered the realm of operator algebras. See [37, 10, 39, 33, 11, 13].

**Definition 2.13.** A simplicial graph \( \Gamma \) is a graph with the following properties

1. undirected;
2. no single-vertex loops;
3. there is at most one edge between vertices.
We denote the set of vertices and the set of edges of \( \Gamma \) as \( V \Gamma \) and \( E \Gamma \) respectively. We can consider \( E \Gamma \) as a symmetric subset of \( V \Gamma \times V \Gamma \) that does not intersect the diagonal, and when convenient, we will identify a pair \( (v, w) \in E \Gamma \) with \( (w, v) \).

Per usual, there are reduced and universal versions of graph products of \( C^* \)-algebras. In this article, we will consider the universal graph product of \( C^* \)-algebras defined as follows.

**Definition 2.14.** Given a simplicial graph \( \Gamma \) and unital \( C^* \)-algebras \( A_v \) for every \( v \in V \Gamma \), the universal graph product \( C^* \)-algebra is the unique unital \( C^* \)-algebra \( \bigstar \Gamma A_v \) together with unital \(*\)-homomorphisms \( \iota_v : A_v \to \bigstar \Gamma A_v \) satisfying the following universal property.

1. \( [\iota_v(a), \iota_w(b)] = 0 \) whenever \( a \in A_v, b \in A_w, (v, w) \in E \Gamma \);
2. For any unital \( C^* \)-algebra \( B \) with \(*\)-homomorphisms \( \{ f_v : A_v \to B \}_{v \in V \Gamma} \) such that \( [f_v(a), f_w(b)] = 0 \) whenever \( a \in A_v, b \in A_w, (v, w) \in E \Gamma \), there exists a unique \(*\)-homomorphism \( \bigstar \Gamma f_v : \bigstar \Gamma A_v \to B \) such that \( \bigstar \Gamma f_v \circ \iota_v = f_v \) for every \( v \in V \Gamma \).

**Definition 2.15.** Let \( \Gamma \) be a simplicial graph, and let \( A \) be a \( C^* \)-algebra. For each \( v \in V \Gamma \), let \( A_v \) be a \( C^* \)-subalgebra of \( A \). If \( [a, b] = 0 \) whenever \( a \in A_v \) and \( b \in A_w \) with \( (v, w) \in E \Gamma \), then we say the subalgebras \( \{ A_v \}_{v \in V \Gamma} \) commute according to \( \Gamma \). If \( \{ A_v \}_{v \in V \Gamma} \) are elements of \( A \) indexed by \( \Gamma \), then the elements \( \{ A_v \}_{v \in V \Gamma} \) commute according to \( \Gamma \) if \( [A_v, A_w] = 0 \) whenever \( (v, w) \in E \Gamma \).

For the remainder of this section, fix a simplicial graph \( \Gamma \), and for each \( v \in V \Gamma \), let \( A_v \) be a unital \( C^* \)-algebra. When working with graph products, the bookkeeping can be done by considering words with letters from the vertex set \( V \Gamma \). Such words are given by finite sequences of elements from \( V \Gamma \) and will be denoted with bold letters. In order to encode the commuting relations given by \( \Gamma \), we consider the equivalence relation generated by the following relations.

\[
\begin{align*}
(v_1, \ldots, v_i, v_{i+1}, \ldots, v_n) & \sim (v_1, \ldots, v_i, v_{i+2}, \ldots, v_n) \quad \text{if} \quad v_i = v_{i+1} \\
(v_1, \ldots, v_i, v_{i+1}, \ldots, v_n) & \sim (v_1, \ldots, v_{i+1}, v_i, \ldots, v_n) \quad \text{if} \quad (i, i+1) \in E \Gamma.
\end{align*}
\]

The concept of a reduced word is central to the theory of graph products. The following definition is Definition 3.2 of [38] in graph language; the equivalent definition in [10] appears differently.

**Definition 2.16.** A word \( \mathbf{v} = (v_1, \ldots, v_n) \) is reduced if whenever \( v_k = v_l, k < l \), then there exists a \( p \) with \( k < p < l \) such that \( (v_k, v_p) \notin E \Gamma \). Let \( W_{\text{red}} \) denote the set of all reduced words. We take the convention that the empty word is reduced.

**Proposition 2.17** ([23], [10]).

1. Every word \( \mathbf{v} \) is equivalent to a reduced word \( \mathbf{w} = (w_1, \ldots, w_n) \). (We let \( |\mathbf{w}| = n \) denote the length of the reduced word.)
2. If \( \mathbf{v} \sim \mathbf{w} \sim \mathbf{w'} \) with both \( \mathbf{w} \) and \( \mathbf{w'} \) reduced, then the lengths of \( \mathbf{w} \) and \( \mathbf{w'} \) are equal and \( \mathbf{w'} = (w_{\sigma(1)}, \ldots, w_{\sigma(n)}) \) is a permutation of \( \mathbf{w} \). Furthermore, this permutation \( \sigma \) is unique if we insist that whenever \( w_k = w_l, k < l \) then \( \sigma(k) < \sigma(l) \).
3. Lifting selective commuting relations

3.1. The pincushion class. We begin this section by defining the pincushion class of graphs to which our main result applies. Let \( \mathcal{G} \) denote the collection of all finite simplicial graphs.

**Definition 3.1.** Let \( \mathcal{G}^{(1)} \) denote the class of finite simplicial graphs where each \( \Gamma \in \mathcal{G}^{(1)} \) is obtained by recursively adding a vertex and adhering to one of the following two rules.

1. The new vertex is isolated;
2. The new vertex is adjacent to exactly one vertex.

**Definition 3.2.** Given \( \Gamma \in \mathcal{G} \), a pin of \( \Gamma \) is a vertex \( v \in V \) that is adjacent to exactly one other vertex \( w \). That is, there is a unique \( w \in V \Gamma \) such that \( (v, w) \in E \Gamma \).

Let \( P \Gamma \) denote the (possibly empty) set of all pins of \( \Gamma \).

So rule (2) above describes forming a pin. For example, the following graph is contained in \( \mathcal{G}^{(1)} \), and the gray vertices are pins of the graph.

\[ \begin{align*}
\text{To define the pincushion class of graphs, we generalize the construction of } \mathcal{G}^{(1)} \text{ by introducing the following “pinning” operation on } \mathcal{G}. & \\
\text{Definition 3.3. Given } \Gamma \in \mathcal{G}, \text{ fix } v \in V \Gamma \text{ and define } \Phi_{(\Gamma, v)} : \mathcal{G} \to \mathcal{G} \text{ as follows. For } \Gamma' \in \mathcal{G}, \text{ put } \& \\
& V \Phi_{(\Gamma, v)}(\Gamma') := V \Gamma \sqcup V \Gamma' \\
& E \Phi_{(\Gamma, v)}(\Gamma') := E \Gamma \sqcup E \Gamma' \sqcup \{(w, v) : w \in V \Gamma'\} & \\
\end{align*} \]

We call this pinning \( \Gamma' \) to \( \Gamma \) at \( v \).

**Example 3.4.** We illustrate this pinning operation in this example. Let \( \Gamma \) be given by the following graph, and let \( v \in V \Gamma \) be as indicated below.

\[ \begin{align*}
\text{Let } \Gamma' \text{ be given by the following graph.} & \\
\text{Then } \Phi_{(\Gamma, v)}(\Gamma') \text{ is given as follows.} & \\
\end{align*} \]
Using this pinning operation, we recursively construct the classes \( \mathcal{G}(m) \) for \( m \in \mathbb{N} \).

**Definition 3.5.** Assume \( \mathcal{G}(m) \) has been formed; define \( \mathcal{G}(m+1) \) as follows. A graph \( \Gamma \in \mathcal{G}(m+1) \) is obtained by recursively appending a graph from \( \mathcal{G}(m) \) and adhering to one of the following two rules.

1. The appended graph is isolated from the graph obtained in the previous step;
2. The appended graph is pinned to the graph from the previous step at some vertex.

**Remark 3.6.** We observe here that if we define \( \mathcal{G}(0) \) to be the set of finite simplicial graphs consisting only of the single vertex graph, then the above recursive definition for \( m = 0 \) recovers the definition of \( \mathcal{G}(1) \). So the collections \( \mathcal{G}(m) \) can be considered as graded generalizations of \( \mathcal{G}(1) \) via the pinning operation.

Note that \( \mathcal{G}(m-1) \subseteq \mathcal{G}(m) \), and the following proposition shows that this containment is strict.

**Proposition 3.7.** For every \( m \in \mathbb{N} \), \( \mathcal{G}(m) \setminus \mathcal{G}(m-1) \neq \emptyset \). In particular, if \( K_{m+1} \) denotes the complete graph with \( m+1 \) vertices, then \( K_{m+1} \in \mathcal{G}(m) \setminus \mathcal{G}(m-1) \).

**Proof.** We proceed by induction on \( m \). The base case of \( m = 1 \) is clear. Let \( m > 1 \) and assume that for \( 1 \leq k < m \), we have \( K_{k+1} \in \mathcal{G}(k) \setminus \mathcal{G}(k-1) \). Since \( K_{m+1} = \Phi(K_1,v_1)(K_m) \) where \( VK_1 = \{v_1\} \) and \( K_m \in \mathcal{G}(m-1) \) by the induction hypothesis, we have that \( K_{m+1} \in \mathcal{G}(m) \).

Suppose for the sake of contradiction that \( K_{m+1} \in \mathcal{G}(m-1) \). Since every pair of vertices in \( K_{m+1} \) is adjacent, \( K_{m+1} \) is obtained by using rule (2) of Definition 3.5 exclusively. Let \( v_0 \in VK_{m+1} \) denote the vertex of \( K_{m+1} \) to which a graph \( \Gamma_0 \in \mathcal{G}(m-2) \) was pinned in the last step of the construction of \( K_{m+1} \). The graph \( \Gamma_0 \) is a subgraph of \( K_{m+1} \) so it must be complete too. Since \( K_m \notin \mathcal{G}(m-2) \) by the induction hypothesis, then \( \Gamma_0 \neq K_m \). Thus there must be a vertex \( v_1 \in VK_{m+1} \) different from \( v_0 \) such that \( v_1 \notin V \Gamma_0 \). Hence \( v_1 \) must have been present in the construction process before \( \Gamma_0 \) was pinned to \( v_0 \). But this means that no vertex in \( \Gamma_0 \) is adjacent to \( v_1 \), contradicting the completeness of \( K_{m+1} \). Thus \( K_{m+1} \notin \mathcal{G}(m-1) \). \( \square \)

We are now prepared to define the pincushion class of finite simplicial graphs.

**Definition 3.8.** The **pincushion class** of finite simplicial graphs is denoted \( \mathcal{G}(\infty) \) and is given by

\[
\mathcal{G}(\infty) := \bigcup_{m \in \mathbb{N}} \mathcal{G}(m).
\]

In addition to containing all complete graphs, the pincushion class also contains all finite trees. This is a straightforward exercise left to the reader.
3.2. Main results. Proposition 2.9 and Theorem 2.10 show that in certain circumstances, tracial stability is preserved under free products in general and tensor products with commutative $C^*$-algebras. When a property behaves well with free and tensor products, it is natural to ask about how it behaves with graph products. If the graph comes from the pincushion class and certain component algebras are commutative, then the corresponding graph product indeed preserves tracial stability. For the remainder of this section, let $\mathcal{G}$ be a class unital $RR0$ $C^*$-algebras closed under taking direct sums and unital corners. The following theorem can be considered as a generalization of Theorem 2.10.

**Theorem 3.9.** Let $\Gamma \in \mathcal{G}(\infty)$ be a pincushion class graph. For each $v \in \Gamma$, let $A_v$ be a separable unital $\mathcal{G}$-tracially stable by Theorem 2.4, Proposition 2.9, and the induction hypothesis. Consider the graph $\Gamma$ be a pin of $\Gamma$. Let $\Gamma_1 \in \mathcal{G}(\infty)$ with $|\Gamma_1| > 1$ to a vertex $v_1$. By the induction hypothesis, $\Gamma_1, A_{v_1}$ is $\mathcal{G}$-tracially stable. Consider the graph $\Gamma'$ given by taking $\Gamma$ and collapsing $\Gamma_1$ to a single vertex $v'$ with an edge connecting $v'$ to $v_1$. Thus, $v'$ is a pin of $\Gamma'$. For each $v \in \Gamma'$, put

$$A'_v := \begin{cases} A_v & \text{if } v \neq v' \\ \star_{\Gamma_1} A_w & \text{if } v = v'. \end{cases}$$

Then by the induction hypothesis, we have

$$\star_{\Gamma} A_v = \star_{\Gamma'} A'_v$$

is $\mathcal{G}$-tracially stable.

**Case I:** $P\Gamma = \emptyset$. There is a construction of $\Gamma$ in which the last step is to pin a subgraph $\Gamma_1 \in \mathcal{G}(\infty)$ with $|\Gamma_1| > 1$ to a vertex $v_1$. By the induction hypothesis, $\star_{\Gamma_1} A_{v_1}$ is $\mathcal{G}$-tracially stable. Consider the graph $\Gamma'$ given by taking $\Gamma$ and collapsing $\Gamma_1$ to a single vertex $v'$ with an edge connecting $v'$ to $v_1$. Thus, $v'$ is a pin of $\Gamma'$. For each $v \in \Gamma'$, put

$$A'_v := \begin{cases} A_v & \text{if } v \neq v' \\ \star_{\Gamma_1} A_w & \text{if } v = v'. \end{cases}$$

Then by the induction hypothesis, we have

$$\star_{\Gamma} A_v = \star_{\Gamma'} A'_v$$

is $\mathcal{G}$-tracially stable.

**Case II:** $P\Gamma \neq \emptyset$. Let $v_0 \in P\Gamma$ be a pin of $\Gamma$. Let $v_1$ be the (unique) vertex adjacent to $v_0$. If $v_1$ is also a pin, then the subgraph $\Gamma_{01}$ formed by $v_0, v_1$, and the edge joining them is isolated from the complementary subgraph $\Gamma_2$ of $\Gamma$. Then

$$\star_{\Gamma} A_v = (\star_{\Gamma_{01}} A_v) \ast (\star_{\Gamma} A_{v_1}) = (C(X_{v_0}) \otimes C(X_{v_1})) \ast (\star_{\Gamma_{01}} A_v)$$

is $\mathcal{G}$-tracially stable by Theorem 2.4, Proposition 2.9 and the induction hypothesis.

If $v_1$ is not a pin, then $A_{v_0}$ is $\mathcal{G}$-tracially stable, and $A_{v_0} = C(X_{v_0})$ for some compact Hausdorff $X_{v_0}$. For each $k \in \mathbb{N}$ let $M_k$ be a $C^*$-algebra in $\mathcal{G}$, and let $\tau_k$ be a tracial state on $M_k$. Fix a unital $*$-homomorphism $\pi : \star_{\Gamma} A_v \rightarrow (M_k, \tau_k)^{\mathbb{N}}$. As in the proof of Theorem 2.10 in [27], we will use Lemma 2.2 to show that $\pi$ is approximately liftable. Fix $\varepsilon > 0$ and contractions $x_1, \ldots, x_n \in \star_{\Gamma} A_v$. We set out
to find \( \pi_k : \bigstar_{1} A_v \to M_k \) such that
\[
||((\pi_k(x_j))_{U} - \pi(x_j))||_U < 3\varepsilon
\]
for every \( 1 \leq j \leq n \). For each \( 1 \leq j \leq n \) there are elements \( w_j^{(j)} \in \bigstar_{1} A_v \) of the form
\[
w_j^{(j)} = a(j, i, 1) \cdots a(j, i, s_i^{(j)})
\]
where \( a(j, i, l) \in (A_v(j, i, l)) \leq 1 \) for \( 1 \leq l \leq s_i^{(j)} \) and \( (v(j, i, l), \ldots, v(j, i, s_i^{(j)}) ) \) is reduced with the property that
\[
\left\| x_j - \sum_{i=1}^{N_j} w_i^{(j)} \right\| < \varepsilon.
\]
The image \( \pi(A_v) = \pi(C(X_v)) \) is commutative, so let \( \pi(C(X_v)) = C(\Omega) \) for some compact Hausdorff \( \Omega \). Let \( a_1(j, i_1, l_1), \ldots, a_1(j_T, i_T, l_T) \) be the elements of \( A_v \) appearing in the above decompositions of the \( w_j^{(j)} \)'s, and for each \( 1 \leq j \leq n, 1 \leq i \leq N_j \) let \( S_j^{(j)} \) denote the number of elements in \( \{a_1(j_1, i_1, l_1), \ldots, a_1(j_T, i_T, l_T)\} \) appearing in the above decomposition of \( w_j^{(j)} \). Put
\[
N := \max_{1 \leq j \leq n} N_j
\]
and
\[
S := \max_{1 \leq i \leq N_j} S_j^{(j)}.
\]
We can approximate each of \( a_1(j_1, i_1, l_1), \ldots, a_1(j_T, i_T, l_T) \) using simple functions. That is, there is a disjoint collection \( \{E_1, \ldots, E_m\} \) of Borel subsets of \( \Omega \) whose union is \( \Omega \), and for each \( 1 \leq d \leq m \), there is an element \( \omega_d \in E_d \) such that
\[
\left\| \pi(a_1(j, i, l)) - \sum_{d=1}^{m} \pi(a_1(j, i, l))(\omega_d)\chi_{E_d} \right\| < \frac{\varepsilon}{NS}
\]
f or \( 1 \leq t \leq T \).

Let \( \Gamma_2 \) denote the subgraph of \( \Gamma \) given by \( V_{\Gamma_2} = V_\Gamma \setminus \{v_0\} \) and \( E_{\Gamma_2} = E_\Gamma \setminus \{(v_0, v_1)\} \). For each \( v \in V_{\Gamma_2} \), let \( B_v \) be defined as follows.
\[
B_v := \left\{ \begin{array}{ll} A_v & \text{if } v \neq v_1 \\ C^* (\chi_{E_1}, \ldots, \chi_{E_d}) & \text{if } v = v_1 \end{array} \right.
\]
It is clear that \( \Gamma_2 \in \mathcal{G}(\infty) \). Indeed, in the construction of \( \Gamma \in \mathcal{G}(\infty) \), after \( v_1 \) appears, the step of pinning \( v_0 \) to \( v_1 \) is independent of the rest of the steps in the construction. So by the induction hypothesis, \( \bigstar_{1} B_v \) is \( \mathcal{G} \)-tracially stable. Let \( \rho = \bigstar_{1} \rho_v : \bigstar_{1} B_v \to (M_k, \tau_k)^H \) where
\[
\rho_v = \left\{ \begin{array}{ll} \pi_v & \text{if } v \neq v_1 \\ \text{id} & \text{if } v = v_1. \end{array} \right.
\]
Note that \( \rho \) is well-defined because \( C^* (\chi_{E_1}, \ldots, \chi_{E_d}) \subset W^* (\pi(C(X_v))) \), thus \( \{\rho_v(B_v)\}_{v \in V_{\Gamma_2}} \) satisfies all requisite commuting relations. Hence by \( \mathcal{G} \)-tracial stability, for each \( k \in \mathbb{N} \), we can find a unital \( \star \)-homomorphism \( \rho_k : \bigstar_{1} B_v \to M_k \) such that \( (\rho_k(a))_U = \rho(a) \) for every \( a \in \bigstar_{1} B_v \), and for every \( v \in V_{\Gamma_2} \), let \( \rho_{v,k} = \rho_k|_{B_v} \).
We now construct the desired sequence of \(*\)-homomorphisms \(\{\pi_k\}\) satisfying (3.1). This follows an argument parallel to the one found in the proof of Theorem 2.7 of [27]. For each \(1 \leq d \leq m\) and each \(k \in \mathbb{N}\) let \(P_{d,k} := \rho_{v_1,k}(\chi_{E_d})\). Then we have \(\chi_{E_d} = (P_{d,k})_U\) and \(\{P_{1,k}, \ldots, P_{m,k}\}\) is a pairwise orthogonal partition of unity in \(M_k\). Observe that

\[
C^*(\chi_{E_1}, \ldots, \chi_{E_m}) \subset \pi(C^*(A_{v_0}, C(X_{v_1})))' \cap (M_k, \tau_k)^U.
\]

Therefore

\[
\pi(C^*(A_{v_0}, C(X_{v_1}))) \subset \left( \sum_{d=1}^{m} P_{d,k}M_kP_{d,k}, \tau_k \right)^U = \bigoplus_{d=1}^{m} (P_{d,k}M_kP_{d,k}, \tau_k)^U.
\]

For \(1 \leq d \leq m\), let \(\varphi_d\) denote the projection onto the \(d\)-th summand of \(\bigoplus_{d=1}^{m} (P_{d,k}M_kP_{d,k}, \tau_k)^U\).

Since \(A_{v_0}\) is \(\mathcal{G}\)-tracially stable, for each \(k \in \mathbb{N}\) there is a unital \(*\)-homomorphism

\[
\pi_{v_0,d,k} : A_{v_0} \to P_{d,k}M_kP_{d,k}
\]

such that \((\pi_{v_0,d,k}(a))_U = \varphi_d \circ \pi(a)\) for every \(a \in A_{v_0}\). Define

\[
\pi_{v_0,k} := \bigoplus_{d=1}^{m} \pi_{v_0,d,k}.
\]

For each \(k \in \mathbb{N}\), define

\[
\pi_{v_1,k}(a) = \sum_{d=1}^{m} \pi(a)(\omega_d)P_{d,k},
\]

and for \(v \in VT \setminus \{v_0, v_1\}\), set

\[
\pi_{v,k} = \rho_{v,k}.
\]

By construction, for each \(k \in \mathbb{N}\), the images of the \(\pi_{v,k}\)'s satisfy the commuting relations prescribed by \(\Gamma\). So for each \(k \in \mathbb{N}\), we can define

\[
\pi_k = \star \Gamma \pi_{v,k}.
\]

We claim that these \(\pi_k\)'s satisfy (3.1). For \(1 \leq j \leq n\) we have

\[
\|(\pi_k(x_j))_U - \pi(x_j)\|_2 \leq 2\varepsilon + \left\| \sum_{i=1}^{N_j} \left( \pi_k \left( w_i^{(j)}(j) \right) \right)_U - \pi \left( w_i^{(j)}(j) \right) \right\|_2
\]

\[
< 2\varepsilon + \sum_{i=1}^{N_j} \left\| \left( \pi_k \left( w_i^{(j)}(j) \right) \right)_U - \pi \left( w_i^{(j)}(j) \right) \right\|_2
\]

\[
< 2\varepsilon + \sum_{i=1}^{N_j} S_{ij}^{(j)} \varepsilon \frac{\varepsilon}{NS}
\]

\[
\leq 3\varepsilon.
\]

(3.4)
There is a \( \varepsilon > 0 \) and \( \Gamma \) such that for every \( v \in \mathcal{V} \mathcal{T} \), \( A_v \) is a contraction in \( M \) such that \( \|[A_v, A_v^*]_2,\tau < \delta \) for every \( v \in \mathcal{V} \mathcal{T} \) and
\[
\sum_{v \in \mathcal{V} \mathcal{T}} \|[A_v, B_v]_2,\tau \leq \varepsilon.
\]

If \( \|A_v - A_v^*\|_{2,\tau} < \delta \) for \( v \in \mathcal{V} \mathcal{T} \), then the \( B_v \)'s can be taken to be self-adjoint; and if \( \|1 - A_v A_v^*\|_{2,\tau} < \delta \), then the \( B_v \)'s can be taken to be unitaries.

Proof. As in the proof of Theorem 1.2, we will apply tracial stability. Let \( \Gamma \in \mathcal{G}(\infty) \) be a pincushion graph, and let \( \varepsilon > 0 \) be given. Suppose for the sake of contradiction that for every \( k \in \mathbb{N} \), there is a \( C^* \)-algebra \( M_k \in \mathcal{C} \) with tracial state \( \tau_k \) and elements \( \{A_{v,k}, A_{v,k}^*\} \) such that \( \|[A_{v,k}, A_{v,k}^*]_{2,\tau_k} < \frac{1}{k} \) for every \( v \in \mathcal{V} \), and \( \|[A_{v,k}, A_{w,k}]_{2,\tau_k} < \frac{1}{k} \) whenever \( (v, w) \in \mathcal{E} \Gamma \) with the property that for any family of normal elements \( \{B_{v,k}\} \subset M_k \) commuting according to \( \Gamma \), we have
\[
\sum_{v \in \mathcal{V} \mathcal{T}} \|[A_{v,k}, B_{v,k}]_{2,\tau_k} > \varepsilon.
\]

Now consider the ultraproduct \( (M_k, \tau_k)^U \). For \( v \in \mathcal{V} \mathcal{T} \), let \( X_v = (A_{v,k})_U \). Then the elements \( \{X_v\} \) are normal in \( (M_k, \tau_k)^U \) and commute according to \( \Gamma \). For each \( v \in \mathcal{V} \) consider the identity \( \star \)-homomorphism denoted \( \pi_v : C^*(X_v) \to (M_k, \tau_k)^U \). Since the images of the \( \pi_v \)'s commute according to \( \Gamma \), we can form the \( \star \)-homomorphism \( \pi = \bigstar \pi_v : \bigstar C^*(X_v) \to (M_k, \tau_k)^U \). Since \( C^*(X_v) \) is commutative for every \( v \in \mathcal{V} \) and \( \Gamma \) is in the pincushion class, \( \pi \) is approximately liftable by Theorem 3.9. Hence, for each \( k \in \mathbb{N} \), there is a \( \star \)-homomorphism \( \pi_k : \bigstar C^*(X_v) \to M_k \) such that \( (\pi_k(a))_U = \pi(a) \) for every \( a \in \bigstar C^*(X_v) \). For each \( v \in \mathcal{V} \) and each \( k \in \mathbb{N} \), let \( B_{v,k} = \pi_k(X_v) \). Then for each \( k \in \mathbb{N} \), \( \{B_{v,k}\} \) is a collection of normal elements in \( M_k \) commuting according to \( \Gamma \) with \( (A_{v,k})_U = (B_{v,k})_U \) for every \( v \in \mathcal{V} \). This contradicts (3.5).

4. Approximation Properties of RAAGs

While the main result is motivation enough for this article, we are further motivated by the consideration of certain approximation properties of right-angled Artin groups (see Definition 4.2). This section lays out the narrative of how the
tracial stability results in this article impact certain approximation properties of right-angled Artin groups.

To begin, we show that if $A$ is a right-angled Artin group (RAAG), its full group $C^*$-algebra $C^*(A)$ is quasidiagonal (QD) by extending Brown-Ozawa’s proof of the fact that $C^*(F_n \times F_n)$ is QD (Proposition 7.4.5 in [9]). Quasidiagonality is an approximation property for $C^*$-algebras that has been the subject of intense study for several decades now. Group $C^*$-algebras are natural examples for which one can consider quasidiagonality. In the context of the reduced $C^*$-algebra of a countable discrete group $G$, the score is settled: $C^*_r(G)$ is QD if and only if $G$ is amenable. The “only if” direction has been known for over thirty years now thanks to Rosenberg in [24], whereas the “if” direction (Rosenberg’s conjecture) was only recently resolved by Tikuisis-White-Winter in [41]. For full/universal $C^*$-algebras of (non-amenable) groups the question of quasidiagonality is not as well-understood, but there are some surprising results nonetheless. For example, due to Choi’s residual finite dimensional result in [13], $C^*(F_n)$ is QD. We already mentioned Brown-Ozawa’s result showing that $C^*(F_n \times F_n)$ is QD. Since these groups are examples of RAAGs, Theorem 4.9 generalizes these two examples. In the wake of the resolution of Rosenberg’s conjecture in [41], it is worth recording that if we combine the fact that amenable groups have QD $C^*$-algebras with the fact that the universal free product of QD $C^*$-algebras is QD (Proposition 13 of [7]), then we obtain the following theorem.

**Theorem 4.1.** Let $\{G_i\}_{i \in I}$ be a collection of countable discrete amenable groups. Then $C^*(\ast_{i \in I} G_i) \cong \ast_{i \in I} C^*(G_i)$ is QD.

Theorems 4.1 and 4.9 add to the list of groups with QD universal $C^*$-algebras.

RAAGs were first introduced by Baudisch in [6]. Since their introduction, such groups have been heavily studied in the group theory literature—see [11] for a survey and repository of references. In addition to the attention earned by their subgroups, the relation of RAAGs with special cube complexes in Agol’s resolution of the virtual Haken conjecture has further enriched their cachet—see [1, 29, 44].

**Definition 4.2.** A group $A$ with presentation

\[
A = \langle s_1, \ldots, s_n | s_is_j = s_js_i \text{ if } (i, j) \in E \rangle
\]

for some symmetric subset $E \subseteq \{1, \ldots, n\}^2$, is called a right-angled Artin group (or RAAG).

RAAGs are specific examples of graph products of groups. We can define the graph product of groups using a universal property analogous to the $C^*$-algebraic one in Definition 2.14. Alternatively, one can define a graph product of groups as follows.

**Definition 4.3.** Fix a simplicial (undirected, no single-vertex loops, at most one edge between vertices) graph $\Gamma$. For each $v \in V\Gamma$, let $G_v$ be a group. The graph product group $\ast_{\Gamma} G_v$ is given by the free product $\ast_{v \in V\Gamma} G_v$ modulo the relations $[g, h] = 1$ whenever $g \in G_v, h \in G_w$ and $(v, w) \in E\Gamma$.

A RAAG can be perceived as a graph product of copies of $\mathbb{Z}$; that is, $A = \ast_{\Gamma} \mathbb{Z}$ where $V\Gamma = \{1, \ldots, n\}$ and $E\Gamma = E$ for $E$ as in (4.1). For instance, $\mathbb{F}_2 \times \mathbb{F}_2 \cong \ast_{\Gamma} \mathbb{Z}$ where $\Gamma$ is the following graph.
Here, the unshaded vertices generate one copy of $\mathbb{F}_2$ and the shaded vertices generate the second copy commuting with the first.

The following proposition shows that graph products are durable under the $C^*$ functor from groups to $C^*$-algebras.

**Proposition 4.4** ([11]). $C^*(\bigstar G_v) \cong \bigstar C^*(G_v)$.

Thus, given a RAAG $A$, we have that $C^*(A) = C^*(\bigstar \mathbb{Z}) \cong \bigstar C^*(\mathbb{Z})$. We record the following well-known fact about these component $C^*$-algebras $C^*(\mathbb{Z}) \cong C(\mathbb{T})$.

**Proposition 4.5.** $C^*(\mathbb{Z})$ enjoys the following universal property. Given a Hilbert space $\mathcal{H}$ and a unitary $u \in B(\mathcal{H})$, there exists a unital *-homomorphism $k : C^*(\mathbb{Z}) \to C^*(u)$ such that $k(s) = u$ where $s$ is the generator of $\mathbb{Z}$ and $C^*(u)$ denotes the $C^*$-algebra generated by $u$.

**Definition 4.6.** A $C^*$-algebra $\mathcal{A}$ is quasidiagonal (or QD) if for every finite subset $F \subset \mathcal{A}$ and every $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and a contractive completely positive map $\varphi : \mathcal{A} \to \mathbb{M}_n$ such that

$$||\varphi(ab) - \varphi(a)\varphi(b)|| < \varepsilon$$

and

$$||\varphi(a)|| > ||a|| - \varepsilon$$

for every $a, b \in F$. We will say that a group $G$ is QD if $C^*(G)$ is QD. So, for instance, Theorem [4.4] says that the free product of amenable groups is QD.

The last ingredient we need before we give the next main result is a version of Voiculescu’s homotopy invariance property for quasidiagonality from [43].

**Definition 4.7.** Let $\mathcal{A}$ and $\mathcal{B}$ be two $C^*$-algebras. Two *-homomorphisms $\varphi, \psi : \mathcal{A} \to \mathcal{B}$ are homotopic if there are *-homomorphisms $\sigma_t : \mathcal{A} \to \mathcal{B}$ for $t \in [0, 1]$ such that $\sigma_0 = \varphi, \sigma_1 = \psi$, and for every $a \in \mathcal{A}$, $\sigma_t(a)$ is a norm-continuous path.

**Proposition 4.8** ([9]). Let $\varphi, \psi : \mathcal{A} \to \mathcal{B}$ be homotopic *-homomorphisms such that $\varphi$ is injective and $\psi(A)$ is QD. Then $\mathcal{A}$ is QD.

**Theorem 4.9.** Let $A$ be a RAAG. Then $A$ is QD.
are homotopic for \(1 \leq i \leq n\). The universal property for graph products of \(C^*\)-algebras then tells us that since \(\pi(C^*(s_1))'', \ldots, \pi(C^*(s_n))''\) commute according to \(\Gamma, \pi = \bigstar_1 \pi_i\), and \(\sigma = \bigstar_1 \sigma_i\) are homotopic \(*\)-homomorphisms. Since \(\sigma(C^*(A)) = \mathbb{C}\) is QD, then by the above proposition, so is \(\pi(C^*(A)) \cong C^*(A)\).

Since subgroups of RAAGs are of major interest, we observe the following immediate corollary.

**Corollary 4.10.** Let \(A\) be a RAAG and \(B \leq A\) be a subgroup. Then \(B\) is QD.

**Proof.** This follows from the facts that \(C^*(B) \subset C^*(A)\) and quasidiagonality is preserved under taking subalgebras.

A consequence of quasidiagonality is the existence of an amenable trace, defined as follows.

**Definition 4.11.** [8] Let \(A\) be a unital \(C^*\)-algebra, and let \(T(A)\) denote the collection of all tracial states on \(A\). A trace \(\tau \in T(A)\) is amenable if there exists a sequence of u.c.p. maps \(\varphi_k : A \to M_{n(k)}\) such that \(\lim_{k \to \infty} \|\varphi_k(ab) - \varphi_k(a)\varphi_k(b)\|_{2, \text{tr}_n(k)} = 0\) and \(\tau(a) = \lim_{k \to \infty} \text{tr}_n(k) \circ \varphi_k(a)\) for every \(a, b \in A\). Let \(\text{AT}(A)\) denote the set of amenable traces on \(A\).

So clearly we have the following corollary.

**Corollary 4.12.** Let \(A\) be a RAAG and \(B \leq A\) be a subgroup. Then \(C^*(B)\) has an amenable trace.

Now that we know the set of amenable traces on a RAAG is nonempty, we wish to further analyze such amenable traces on RAAGs. In [8], Brown studies amenable traces and specific subclasses of amenable traces with stronger approximation properties. A subclass of interest for this article is the following.

**Definition 4.13** ([8]). A trace \(\tau \in T(A)\) is called locally finite dimensional if there exist u.c.p. maps \(\varphi_k : A \to M_{n(k)}\) such that \(\text{tr}_n(k) \circ \varphi_k \to \tau\) in the weak-* topology and \(\lim_{k \to \infty} d(a, A_{\varphi_k}) = 0\) for every \(a \in A\). Here \(d(a, A_{\varphi_k}) = \inf_{b \in A_{\varphi_k}} \|a - b\|\) and \(A_{\varphi_k}\) is the multiplicative domain of \(\varphi_k\). Let \(\text{AT}_{\text{LFD}}(A)\) denote the set of locally finite dimensional traces on \(A\).

Consider the following fact for free groups.

**Theorem 4.14** ([8]). \(\text{AT}(C^*(F_n)) = \text{AT}_{\text{LFD}}(C^*(F_n))\).

This result also clearly holds for any abelian RAAG (i.e. \(\mathbb{Z}^n\)); thus it is reasonable to ask if such a result holds for any RAAG. Theorem 4.14 is proved in [8] essentially by using the fact that \(C^*(F_n)\) is tracially stable for the class of finite factor von Neumann algebras. Thus, applying Theorem 3.1.7 of [8], there is a \(*\)-homomorphism \(\pi : C^*(\bigstar_1 \mathbb{Z}) \to R^d\) (\(R\) denotes the separably acting hyperfinite II\(_1\)-factor) such that \(\tau = \pi_U \circ \pi\). By standard approximation arguments, we can assume that \(\pi\) is a
\( C^*(\mathbb{Z}) \cong \mathbb{Z} \cong C((T)) \)

is \( f\text{vNa} \)-tracially stable by Theorem 3.9, we have that

\[
\pi : C^*(\mathbb{Z}) \to (M_{n(k)})^U \subset R^U
\]

is approximately liftable. So there is a sequence of *-homomorphisms

\[
\pi_k : C^*(\mathbb{Z}) \to M_{n(k)}
\]

such that \((\pi_k(a))_U = \pi(a)\) for every \(a \in C^*(\mathbb{Z})\). Then it is clear to see that \(\tau\) satisfies Definition 4.13. 

While the pincushion class of finite simplicial graphs is a large subclass that includes complete graphs, trees, and many more, it is still of significant interest to extend the results of this article to all finite simplicial graphs. For instance, the smallest non-pincushion graph is the square.

Thus we wish to resolve the following question.

**Question 4.16.** Let \(\mathcal{C}\) be a class of RR0 \(C^*\)-algebras closed under taking direct sums and unital corners, and let \(\Gamma\) be a finite simplicial graph. For each \(v \in V\Gamma\), let \(X_v\) be a compact Hausdorff space; is \(\mathbb{Z}\)-tracially stable?

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