NORMALOID WEIGHTED COMPOSITION OPERATORS
ON $H^2$

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Abstract. When $\varphi$ is an analytic self-map of the unit disk with Denjoy-Wolff point $a \in \mathbb{D}$, and $\rho(W_{\psi, \varphi}) = \psi(a)$, we give an exact characterization for when $W_{\psi, \varphi}$ is normaloid. We also determine the spectral radius, essential spectral radius, and essential norm for a class of non-power-compact composition operators whose symbols have Denjoy-Wolff point in $\mathbb{D}$. When the Denjoy-Wolff point is on $\partial \mathbb{D}$, we give sufficient conditions for several new classes of normaloid weighted composition operators.

1. Introduction

In this paper, we are interested in weighted composition operators on the classical Hardy space $H^2$, the Hilbert space of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the open unit disk $\mathbb{D}$ such that

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$ 

A composition operator $C_\varphi$ on $H^2$ is given by $C_\varphi f = f \circ \varphi$. When $\varphi$ is an analytic self-map of $\mathbb{D}$, the operator $C_\varphi$ is necessarily bounded. A Toeplitz operator $T_\psi$ on $H^2$ is given by $T_\psi f = P(\psi f)$ where $P$ is the projection back to $H^2$. When $\psi \in H^\infty$, the space of bounded analytic functions on $\mathbb{D}$, we simply have $T_\psi f = \psi f$, since $\psi f$ is guaranteed to be in $H^2$, and all such Toeplitz operators are bounded. Throughout this paper, we will assume $\psi \in H^\infty$. We write $W_{\psi, \varphi} = T_\psi C_\varphi$ and call such an operator a weighted composition operator. We are interested in when such operators are normaloid.

For a bounded operator $T$, we have the following definitions:

- $\sigma(T)$ is the spectrum of $T$.
- $\rho(T)$ is the spectral radius of $T$.
- $\rho_e(T)$ is the essential spectral radius of $T$.

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• $W(T)$ is the numerical range of $T$.
• $\|T\|_e$ is the essential norm of $T$.
• $r(T)$ is the numerical radius of $T$.

An operator $T$ is:

1. **self-adjoint** if $T = T^*$.
2. **normal** if $T^*T = TT^*$.
3. **hyponormal** if $T^*T \geq TT^*$.
4. **cohyponormal** if $T^*T \leq TT^*$.
5. **normaloid** if $\|T\| = \rho(T)$.
6. **convexoid** if the closure of $W(T)$ is the convex hull of $\sigma(T)$.
7. **spectraloid** if $\rho(T) = r(T)$.
8. **power-compact** if $T^n$ is compact for some positive integer $n$.

Here is a list of well-known facts which we will use repeatedly:

1. We have the following sequences of implications:
   self-adjoint $\Rightarrow$ normal $\Rightarrow$ (co)hyponormal $\Rightarrow$ normaloid / convexoid $\Rightarrow$ spectraloid
2. if $\psi \in H^\infty$, then $\|T\psi\| = \|\psi\|_\infty$.
3. For a bounded operator $T$, we have
   $$\|T\|_e = \inf \{\|T - Q\| : Q \text{ is compact}\}$$
   $$\rho_e(T) = \lim_{k \to \infty} (\|T^k\|_e)^{1/k}$$
   $$\rho(T) = \lim_{k \to \infty} (\|T^k\|)^{1/k}.$$

Throughout this paper, we will focus on weighted composition operators where $\rho(W_\psi, \varphi) = |\psi(a)|\rho(C_\varphi)$, where $a$ is the Denjoy-Wolff point of $C_\varphi$. This is a large class, including every power-compact weighted composition operator [7, Theorem 4.3], and many weighted composition operators whose compositional symbol converges uniformly to its Denjoy-Wolff point [5, Corollary 10]. Due to the norm inequality $\|W_\psi, \varphi\| \leq \|T\psi||C_\varphi|| = \|\psi\|_\infty\|C_\varphi\|$, we will often assume $|\psi(a)| = \|\psi\|_\infty$. It is unclear whether this is necessary, but we do have $\|\psi\|_2$ as a lower bound for $|\psi(a)|$ when $\rho(C_\varphi) = 1$.

**Proposition 1.1.** Suppose $\varphi$ is an analytic self-map of $\mathbb{D}$ with Denjoy-Wolff point $a$, $\psi \in H^\infty$, $W_\psi, \varphi$ is normaloid, and $\rho(W_\psi, \varphi) = |\psi(a)|$. Then $\|\psi\|_2 \leq |\psi(a)|$.

**Proof.**

$$|\psi(a)| = \rho(W_\psi, \varphi) = \|W_\psi, \varphi\| \geq \|W_\psi, \varphi 1\|_2 = \|\psi\|_2.$$

The organization of the rest of the paper is as follows. In Section 2, we consider the case when the Denjoy-Wolff point $a$ of $\varphi$ belongs to $\mathbb{D}$. In Section 3, we show that if $a$ belongs to $\partial \mathbb{D}$, the set of operators for which $\rho(W_\psi, \varphi) = |\psi(a)|\rho(C_\varphi)$ is non-trivial. For such operators, we discover new classes of normaloid weighted composition operators in Section 4. We end
with further questions about normaloid weighted composition operators in Section 5.

2. $a \in \mathbb{D}$

When the Denjoy-Wolff point $a$ of $\varphi$ is in $\mathbb{D}$, $C_\varphi$ is rarely normaloid, as the next theorem shows.

**Theorem 2.1.** If the Denjoy-Wolff point of $\varphi$ is in $\mathbb{D}$, then $C_\varphi$ is normaloid if and only if $\varphi(0) = 0$.

**Proof.** By [3, Theorem 3.9], the spectral radius of $C_\varphi$ is 1. By [3, Corollary 3.7], we have

$$\left( \frac{1}{1 - |\varphi(0)|^2} \right)^{1/2} \leq \|C_\varphi\| \leq \left( \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{1/2}$$

and we have $\|C_\varphi\| = 1$ if and only if $\varphi(0) = 0$. $\square$

Unsurprisingly, then, we show that if $\rho(W_\psi,\varphi) = |\psi(a)|\rho(C_\varphi)$, then $W_\psi,\varphi$ is normaloid if and only if $\psi$ has a particular form. For the interior fixed point case, since $\rho(C_\varphi) = 1$, that assumption is really $\rho(W_\psi,\varphi) = |\psi(a)|$. Since this case also always has $|\psi(a)| \leq \rho(W_\psi,\varphi) \leq \|W_\psi,\varphi\|$, the next theorem focuses on when $|\psi(a)| = \|W_\psi,\varphi\|$.

**Theorem 2.2.** Suppose $\varphi$ is an analytic self-map of the disk with Denjoy-Wolff point $a \in \mathbb{D}$, and $\psi \in H^\infty$. Then $|\psi(a)| = \|W_\psi,\varphi\|$ if and only if $\psi$ has the form

$$\psi = \psi(a) \frac{K_a}{K_a \circ \varphi}.$$

**Proof.** First, assume the two values are equal. Suppose $\varphi(0) = 0$ so that $\|W_\psi,\varphi\| = |\psi(0)|$. However,

$$|\psi(0)| = \|W_\psi,\varphi\| \geq \|W_\psi,\varphi 1\| = \|\psi\| = \sqrt{|\psi(0)|^2 + |\psi'(0)|^2 + \frac{|\psi''(0)|^2}{2} + \ldots}$$

and we only have equality if every derivative of $\psi$ at 0 is 0, making $\psi$ constant (equal to $\psi(0)$), which trivially fits the required form, since $K_0 = 1$.

Now, suppose $\varphi(a) = a$, for some $a \in \mathbb{D}$ other than 0. The weighted composition operator $W_{\zeta,\tau}$ where $\zeta = \sqrt{1 - |a|^2} \frac{1}{1 - az}, \tau = \frac{a - \bar{z}}{1 - az}$ is unitary by [1] Theorem 6. Note that $\tau$ switches $a$ and 0 and is an involution and $W_{\zeta,\tau}$ is its own inverse. Therefore $W_{\zeta,\tau}W_\psi,\varphi W_{\zeta,\tau}$ is unitarily equivalent to $W_\psi,\varphi$, and it is again a weighted composition operator $W_f, g$, where $f = (\zeta)(\psi \circ \tau)(\zeta \circ \varphi \circ \tau)$ and $g = \tau \circ \varphi \circ \tau$. Since $g(0) = 0$, by the same logic as above, $f$ is a constant function, and the constant is $f(0) = \psi(a)$. Therefore, we have

$$\psi \circ \tau = \frac{\psi(a)}{(\zeta)(\zeta \circ \varphi \circ \tau)}.$$
and now composing both sides with \( \tau \), and recalling \( \tau \circ \tau = z \), we have

\[
\psi = \frac{\psi(a)}{(\zeta \circ \tau)(\zeta \circ \varphi)} = \psi(a) \left( \frac{1}{1 - \overline{a}z} \right) (1 - \overline{a}\varphi)
\]

\[
= \psi(a) \frac{K_a}{K_a \circ \varphi}.
\]

For the other direction, suppose \( \psi \) has the form given in equation (2.2), and we will show that \( |\psi(a)| = \|W_{\psi,\varphi}\| \). By the same logic as the other direction, \( W_{\psi,\varphi} \) is unitarily equivalent to \( W_{\zeta,\tau}W_{\psi,\varphi}W_{\zeta,\tau} \), and again, this is a weighted composition operator of the form \( W_{f,g} \), \( f = (\zeta \circ \varphi \circ \tau) \), \( g = \tau \circ \varphi \circ \tau \). Using the form for \( \psi \) from Equation (2.1), we see that \( f = (\zeta) (\psi(a)) (\zeta \circ \varphi \circ \tau) = \psi(a) \).

Then \( W_{f,g} = \psi(a)C_g \), and since \( g(0) = 0 \), \( ||C_g|| = 1 \), so we have \( ||W_{\psi,\varphi}|| = ||W_{f,g}|| = ||\psi(a)|| ||C_g|| = ||\psi(a)|| \), \( \square \).

From this, we have an immediate corollary about when \( W_{\psi,\varphi} \) is normaloid.

**Corollary 2.3.** Suppose \( \varphi \) is an analytic self-map of the disk with Denjoy-Wolff point \( a \in \mathbb{D} \), \( \psi \in H^\infty \), and \( \rho(W_{\psi,\varphi}) = |\psi(a)| \). Then \( W_{\psi,\varphi} \) is normaloid if and only if \( \psi \) has the form

\[
\psi = \psi(a) \frac{K_a}{K_a \circ \varphi}.
\]

**Proof.** Note that we have \( |\psi(a)| \leq \rho(W_{\psi,\varphi}) \leq ||W_{\psi,\varphi}|| \) since \( a \in \mathbb{D} \). Then if \( W_{\psi,\varphi} \) is normaloid, we have \( ||W_{\psi,\varphi}|| = \rho(W_{\psi,\varphi}) = |\psi(a)| \), where the second equality is by hypothesis. If instead we assume \( \psi \) has the given form, by Theorem 2.2 we have \( |\psi(a)| = ||W_{\psi,\varphi}|| \), therefore \( |\psi(a)| = \rho(W_{\psi,\varphi}) = ||W_{\psi,\varphi}|| \), so \( W_{\psi,\varphi} \) is normaloid. \( \square \)

In the previous corollary, we assumed that \( \rho(W_{\psi,\varphi}) = |\psi(a)| = |\psi(a)| \rho(C_{\varphi}) \).

The next corollary shows this case includes all power-compact weighted composition operators with \( \psi \in H^\infty \), and Section 3 shows it includes several different classes of \( W_{\psi,\varphi} \) with Denjoy-Wolff point of \( \varphi \) on \( \partial \mathbb{D} \).

**Corollary 2.4.** Suppose \( \varphi \) is an analytic self-map of \( \mathbb{D} \) with Denjoy-Wolff point \( a \in \mathbb{D} \), \( \psi \in H^\infty \), and \( W_{\psi,\varphi} \) is power-compact. Then \( W_{\psi,\varphi} \) is normaloid if and only if \( \psi \) has the form

\[
\psi = \psi(a) \frac{K_a}{K_a \circ \varphi}.
\]
Proof. By [7, Proposition 4.3], if $W_{\psi,\varphi}$ is compact, then $\rho(W_{\psi,\varphi}) = |\psi(a)|$, so Theorem 2.2 applies. If $W_{\psi,\varphi}$ is power-compact, we still have that the spectral radius is the absolute value of its largest eigenvalue, and $W^*_{\psi,\varphi}(K_a) = \overline{\psi(a)K_a}$, so $\rho(W_{\psi,\varphi}) = |\psi(a)|$. \hfill \Box

This form for $\psi$ is not unexpected, since the same form is required for $W_{\psi,\varphi}$ to be normal when the fixed point of $\varphi$ belongs to $\mathbb{D}$ [1, Proposition 8], or even for $W_{\psi,\varphi}$ to be cohyponormal (in [4], they are shown to be equivalent when $a \in \mathbb{D}$). However, while normality requires a much stricter characterization for $\varphi$, here we show that this form for $\psi$ is sufficient for $W_{\psi,\varphi}$ to be normaloid, while allowing for many different forms for $\varphi$.

At the end of [3], the authors ask how often we have $\sigma(W_{\psi,\varphi}) = \psi(a)\sigma(C_\varphi)$. The work above shows that there are weighted composition operators with $\varphi$ having Denjoy-Wolff point $a$ where this is false.

Example 2.5. In [6, Theorem 3.7], examples are given of weights $\psi \in H^\infty$ such that $W_{\psi,\varphi}$ is hyponormal when $\varphi(z) = \frac{sz}{1-(1-s)z}$, $0 < s < 1$. Every hyponormal operator is normaloid, but the weights are not as prescribed in Corollary 2.3. Therefore, it must be that $\rho(W_{\psi,\varphi}) > |\psi(0)|$.

When $\varphi$ has Denjoy-Wolff point $a \in \mathbb{D}$, the primary differentiation of spectrum comes from whether or not $\varphi$ has a fixed point or a periodic point on $\partial \mathbb{D}$. Here, we obtain a partial result, that gives the spectral radius in Example 2.5. The proof of the following theorem is heavily borrowed from theorems about $C_\varphi$ in [3].

Remark 2.6. Let $\varphi_n$ denote the $n$th iterate of $\varphi$, i.e. $\varphi_n = \varphi \circ \varphi \cdots \circ \varphi$, $n$ times. By the discussion ahead of [3, Theorem 7.36], when $\varphi$ has Denjoy-Wolff point in $\mathbb{D}$, is analytic in a neighborhood of the closed disk, and is not an inner function, there is an integer $n$ so that the set $S_n = \{ w : |w| = 1 \text{ and } |\varphi_n(w)| = 1 \}$ is either empty or consists only of the finitely many fixed points of $\varphi_n$ on the circle. The essential spectral radius of $C_\varphi$ is

$$\rho_e(C_\varphi) = \max\{\varphi_n'(w)^{-1/2n} : w \in S_n\}.$$ 

If $S_n$ is empty, then $C_\varphi$ is power-compact, which we have covered, so we will assume $S_n$ is nonempty. We will say that the chosen element $b$ of $S_n$ establishes $\rho_e(C_\varphi)$.

Theorem 2.7. Suppose $\varphi$, not an inner function, is an analytic self-map of $\mathbb{D}$ which is univalent on $\mathbb{D}$ and analytic in a neighborhood of $\partial \mathbb{D}$, with Denjoy-Wolff point $a \in \mathbb{D}$. Suppose $b \in \partial \mathbb{D}$ establishes $\rho_e(C_\varphi)$. Let $\psi \in H^\infty$ be continuous at $b$ and let $|\psi(b)| = ||\psi||_\infty$. Then

(1) $|W_{\psi,\varphi}|_e = |\psi(b)||C_\varphi|_e$,
(2) $\rho_e(W_{\psi,\varphi}) = |\psi(b)\rho_e(C_\varphi)|$, and
(3) $\rho(W_{\psi,\varphi}) = \max\{|\psi(a)|, |\psi(b)|\rho_e(C_\varphi)|\}$. 

Proof. By [3, Theorem 7.31], we have that

$$\rho_e(C_\varphi) = \lim_{k \to \infty} \left( \limsup_{|w| \to 1} \frac{\|K\varphi_k(w)\|}{\|Kw\|} \right)$$

and this happens in particular as \( w \) approaches the element \( b \) of \( \partial \mathbb{D} \) that gives the maximum value in the definition in Remark 2.6.

Now, we adapt the proof from [3, Proposition 3.13.] to show that\[\|W_{\psi,\varphi}\|_e \geq |\psi(b)|\|C_\varphi\|_e.\]

Let \( w_j \) be a sequence in \( \mathbb{D} \) tending to \( \partial \mathbb{D} \). Then the normalized weight sequence\[k_j = \frac{Kw_j}{\|Kw_j\|}\]tends to 0 weakly as \( j \) approaches infinity. If \( Q \) is an arbitrary compact operator on \( H^2 \), then \( Q^*(k_j) \to 0 \).

Now, \( \|W_{\psi,\varphi}\|_e = \inf\{\|W_{\psi,\varphi} - Q\| : Q \text{ is compact}\} \), and for \( Q \) compact,

\[
\|W_{\psi,\varphi} - Q\| \geq \limsup_{j \to \infty} \|(W_{\psi,\varphi} - Q)^*k_j\|
= \limsup_{j \to \infty} \|W_{\psi,\varphi}^*k_j\|
= \limsup_{j \to \infty} |\psi(w_j)|\|C_\varphi^*k_j\|.
\]

Since \( \|C_\varphi\|_e = \limsup_{j \to \infty} \|C_\varphi^*k_j\| \) is achieved by taking \( w_j \) tending towards \( b \), and likewise \( \limsup_{j \to \infty} |\psi(w_j)| = |\psi(b)| = \|\psi\|_\infty \) is achieved by taking \( w_j \) towards \( b \), we have \( \limsup_{j \to \infty} |\psi(w_j)|\|C_\varphi^*k_j\| = |\psi(b)|\|C_\varphi\|_e \) and \( \|W_{\psi,\varphi}\|_e \geq |\psi(b)|\|C_\varphi\|_e. \)

For the other direction, note that since the compact operators are an ideal, \( T_{\psi}Q \) is compact for any compact operator \( Q \), and if \( B \subseteq A \), then \( \inf A \leq \inf B \). Then

\[
\|W_{\psi,\varphi}\|_e = \inf\{\|W_{\psi,\varphi} - Q\| : Q \text{ is compact}\}
\leq \inf\{\|W_{\psi,\varphi} - T_{\psi}Q\| : Q \text{ is compact}\}
= \inf\{\|T_{\psi}(C_\varphi - Q)\| : Q \text{ is compact}\}
\leq \inf\{\|T_{\psi}\|\|C_\varphi - Q\| : Q \text{ is compact}\}
= \inf\{\|\psi\|_\infty\|C_\varphi - Q\| : Q \text{ is compact}\}
= \inf\{|\psi(b)|\|C_\varphi - Q\| : Q \text{ is compact}\}
= |\psi(b)|\inf\{\|C_\varphi - Q\| : Q \text{ is compact}\}
= |\psi(b)|\|C_\varphi\|_e.
\]

Therefore we have \( \|W_{\psi,\varphi}\|_e = |\psi(b)|\|C_\varphi\|_e.\)

Suppose momentarily that \( b \) is a fixed point of \( \varphi \). Since \( \varphi \) is analytic in a neighborhood of \( \partial \mathbb{D} \) (i.e. continuous at \( b \)), the above gives the same result if \( \psi \) is replaced by \( \psi \circ \varphi_k \) for any \( k \). Then,
\[ \rho_e(W_{\psi, \varphi}) = \lim_{k \to \infty} \left( \|W_{\psi, \varphi}^k\|_e \right)^{1/k} \]

\[ = \lim_{k \to \infty} \left( \|T_{(\psi)(\psi_o\varphi)(\psi_o\varphi_2)\ldots(\psi_o\varphi_{k-1})C\varphi_k}\|_e \right)^{1/k} \]

\[ = \lim_{k \to \infty} \left( |\psi(b)|^k \|C\varphi_k\|_e \right)^{1/k} \]

\[ = |\psi(b)| \lim_{k \to \infty} \left( \|C\varphi_k\|_e \right)^{1/k} \]

\[ = |\psi(b)| \rho_e(C\varphi). \]

Now, while \( b \) may not be a fixed point of \( \varphi \), we know it is the fixed point of some \( n \)th iterate \( \varphi_n \). Furthermore, we know that \( \rho_e(W_{\psi, \varphi}) = \lim_{k \to \infty} \left( \|W_{\psi, \varphi}^k\|_e \right)^{1/k} \) is a convergent sequence. Therefore, every subsequence converges to \( \rho_e(W_{\psi, \varphi}) \).

By taking the subsequence with indices \( nk \) values of finite multiplicity. By [7, Proposition 4.3], any eigenvalue of \( W_{\psi, \varphi} \) must be of the form \( \sigma \) in \( \mathbb{C} \) and the largest of those values in magnitude is \( |\psi(b)| \). Furthermore, we know that \( |\psi(b)| \rho_e(C\varphi) \).

The complement in \( \sigma(W_{\psi, \varphi}) \) of the essential spectrum consists of eigenvalues of finite multiplicity. By [7, Proposition 4.3], any eigenvalue of \( W_{\psi, \varphi} \) must be of the form

\[ \{0, \psi(a), \psi(a)\varphi'(a), \psi(a)(\varphi'(a))^2, \psi(a)(\varphi'(a))^3, \ldots \} \]

and the largest of those values in magnitude is \( \psi(a) \). This value is necessarily in \( \sigma(W_{\psi, \varphi}) \), since \( W_{\psi, \varphi}^* (K_a) = \overline{\psi(a)K_a} \).

Therefore, \( \rho(W_{\psi, \varphi}) = \max\{|\psi(a)|, |\psi(b)|\rho_e(C\varphi)\} \).

\[ \square \]

**Corollary 2.8.** Suppose \( \varphi \), not an inner function, is an analytic self-map of \( \mathbb{D} \) which is univalent on \( \mathbb{D} \) and analytic in a neighborhood of \( \overline{\mathbb{D}} \), with Denjoy-Wolff point \( a \in \mathbb{D} \). Suppose \( b \in \partial \mathbb{D} \) establishes \( \rho_e(C\varphi) \). Let \( \psi \in H^\infty \) be continuous at \( b \) and let \( |\psi(b)| = \|\psi\|_\infty \).

Suppose further that \( W_{\psi, \varphi} \) is normaloid. Then, either \( \|W_{\psi, \varphi}\| = \rho(W_{\psi, \varphi}) = |\psi(a)| \) and \( \psi = \psi(a)K_a^{\psi(1)} \), or \( \|W_{\psi, \varphi}\| = \rho(W_{\psi, \varphi}) = \rho_e(W_{\psi, \varphi}) = |\psi(b)|\rho_e(C\varphi) \).

**Proof.** By Theorem 2.7, \( \rho(W_{\psi, \varphi}) = \max\{|\psi(a)|, |\psi(b)|\rho_e(C\varphi)\} \). If \( \rho(W_{\psi, \varphi}) = |\psi(a)| \), by Theorem 2.2 \( \psi \) has the form \( \psi = \psi(a)K_a^{\psi(1)} \). Otherwise, \( \rho(W_{\psi, \varphi}) = |\psi(b)|\rho_e(C\varphi) \).

\[ \square \]

**Example 2.9.** Let \( W_{\psi, \varphi} \) be the hyponormal operator given by \( \psi(z) = \frac{2z}{2z-1} \), \( \varphi(z) = \frac{z}{z-1} \) [4, Example 3.8]. Then \( \rho(W_{\psi, \varphi}) = |\psi(1)|\varphi'(1)^{-1/2} = \sqrt{2}e \).

While neither Theorem 2.7 nor Corollary 2.8 gives sufficient conditions for \( W_{\psi, \varphi} \) to be normaloid, Theorem 2.7 does accomplish something else. An operator \( T \) is said to be essentially normaloid if \( \rho_e(T) = \|T\|_e \).

**Corollary 2.10.** Suppose \( \varphi \), not an inner function, is an analytic self-map of \( \mathbb{D} \) which is univalent on \( \mathbb{D} \) and analytic in a neighborhood of \( \overline{\mathbb{D}} \),
with Denjoy-Wolff point \( a \in D \). Let \( b \) be a fixed point of \( \varphi \) on \( \partial D \) such that establishes \( \rho_\varphi(C_\varphi) \). Let \( \psi \in H^\infty \) be continuous at \( b \) and let \( |\psi(b)| = \|\psi\|_\infty \). Then \( W_{\psi,\varphi} \) is essentially normaloid if and only if \( C_\varphi \) is essentially normaloid.

**Proof.** This is an immediate consequence of (1) and (2) in Theorem 2.7. \( \square \)

**Example 2.11.** Let \( \varphi(z) = e^{-z}, \psi = e^{-z} \). Then \( |\psi(0)| = 1 \) and \( |\psi(1)| \varphi'(1)^{-1/2} = \sqrt{2} < 1 \). Since \( \psi \) is not of the form \( |\psi(0)| K_0 \), \( W_{\psi,\varphi} \) is not normaloid. However, since \( C_\varphi \) is essentially normaloid ([3, Theorem 7.31, 7.36]), therefore so is \( W_{\psi,\varphi} \).

### 3. Uniformly Convergent Iteration (UCI)

We now turn our attention to when \( \varphi \) has Denjoy-Wolff point \( a \in \partial D \).

We wish to continue to assume that \( \rho(W_{\psi,\varphi}) = |\psi(a)| \rho(C_\varphi) \). Our goal in this section, before determining when such operators are normaloid, is to show that this class is non-trivial. To do so, we will put restrictions on how the iterates of \( \varphi \) converge to the Denjoy-Wolff point. This definition is from [5], where this hypothesis is used to determine the spectrum of weighted composition operators in this setting.

The Denjoy-Wolff Theorem [3, Theorem 2.51] states that all analytic self-maps of \( D \) other than elliptic automorphisms have a point in \( \overline{D} \) that they converge to under iteration on compact subsets of \( D \). Here, we ask the convergence to be stronger.

**Definition 3.1 (Uniformly Convergent Iteration).** We say \( \varphi \) is UCI if \( \varphi \) is an analytic self-map of \( D \) and the iterates of \( \varphi \) converge uniformly to the Denjoy-Wolff point uniformly on all of \( D \), rather than compact subsets of \( D \).

If \( \varphi \) is UCI and the Denjoy-Wolff point \( a \) of \( \varphi \) belongs to \( D \), then \( W_{\psi,\varphi} \) is power-compact [5, Corollary 2], so we have already covered that scenario in Section 2 without requiring this additional hypothesis.

Analytic self-maps of \( D \) that exhibit UCI while having Denjoy-Wolff point \( a \) on \( \partial D \) are a non-trivial set, and include maps whose derivative at the Denjoy-Wolff point are both less than 1 (e.g. \( \varphi(z) = (z + 1)/2 \)) and equal to 1 (e.g. \( \varphi(z) = 1/(2 - z) \)) [5, Example 5]. A simple sufficient condition for UCI when \( \varphi'(a) < 1 \) is that \( \varphi_N(D) \subseteq D \cup \{a\} \) for some \( N \) [5, Theorem 4]. This includes, then, any linear-fractional map with Denjoy-Wolff point on the boundary and \( \varphi'(a) < 1 \).

The main reason to now introduce this definition is the following theorem, proved in [5].

**Theorem 3.2.** Suppose \( \varphi \) is UCI with Denjoy-Wolff point \( a \in \partial D \), \( \psi \in H^\infty \) is continuous at \( a \), and \( \psi(a) \neq 0 \). Then:

- (1) \( \sigma_p(\psi(a)C_\varphi) \subseteq \sigma_{ap}(T_\psi C_\varphi) \subseteq \sigma(T_\psi C_\varphi) \subseteq \sigma(\psi(a)C_\varphi) \),
(2) If $\sigma_p(C_{\varphi}) = \sigma(C_{\varphi})$, then $\sigma(T_{\psi}C_{\varphi}) = \sigma(\psi(a)C_{\varphi})$.

(3) If $\varphi'(a) < 1$, then $\sigma(T_{\psi}C_{\varphi}) = \sigma(\psi(a)C_{\varphi})$ and $\sigma_p(W_{\psi,\varphi}) = \sigma_p(\psi(a)C_{\varphi})$.

We have an immediate corollary regarding the spectral radius.

**Corollary 3.3.** Suppose $\varphi$ is UCI with Denjoy-Wolff point $a \in \partial \mathbb{D}$, $\psi \in H^\infty$ is continuous at $a$, and $\psi(a) \neq 0$. Then $\rho(W_{\psi,\varphi}) = |\psi(a)|\rho(C_{\varphi})$.

**Proof.** If $\varphi'(a) < 1$, then (3) of the Theorem 3.2 makes this clear. If $\varphi'(a) = 1$, note that $1 \in \sigma_p(C_{\varphi})$, so $\psi(a) \in \sigma_p(\psi(a)C_{\varphi})$, and therefore by (1) of Theorem 3.2 $\rho(W_{\psi,\varphi}) \geq |\psi(a)|$. Again by (1), we also have $\sigma(W_{\psi,\varphi}) \subseteq \sigma(\psi(a)C_{\varphi})$, so $\rho(W_{\psi,\varphi}) \leq \rho(\psi(a)C_{\varphi}) = |\psi(a)|\rho(C_{\varphi}) = |\psi(a)|$, since $\rho(C_{\varphi}) = 1$ [3, Theorem 3.9]. Therefore $\rho(W_{\psi,\varphi}) = |\psi(a)| = |\psi(a)|\rho(C_{\varphi})$. \hfill \qed

4. $a \in \partial \mathbb{D}$

Here we follow the same path as Section 2. We will continue to assume that we have $\rho(W_{\psi,\varphi}) = |\psi(a)|\rho(C_{\varphi})$, and will seek to determine conditions for which $\|W_{\psi,\varphi}\|$ is the same. We will also obtain a few corollaries for when $\varphi$ is explicitly UCI.

**Theorem 4.1.** Suppose $\varphi$ is an analytic self-map of $\mathbb{D}$ with Denjoy-Wolff point $a \in \partial \mathbb{D}$, $\psi \in H^\infty$, and $\psi$ is continuous at the Denjoy-Wolff point $a$ of $\varphi$, with $\|\psi\|_\infty = |\psi(a)|$. Furthermore, assume $\rho(W_{\psi,\varphi}) = |\psi(a)|\rho(C_{\varphi})$. If $C_{\varphi}$ is normaloid, then $W_{\psi,\varphi}$ is normaloid.

**Proof.** Note that

$$
\|W_{\psi,\varphi}\| \leq \|T_{\psi}\|\|C_{\varphi}\|
= |\psi|\|C_{\varphi}\|
= |\psi(a)|\|C_{\varphi}\|
= |\psi(a)|\rho(C_{\varphi})
= \rho(W_{\psi,\varphi}) \leq \|W_{\psi,\varphi}\|.
$$

\hfill \qed

**Example 4.2.** If $\psi = e^z$ and $\varphi = (z + 1)/2$, then $\|\psi\|_\infty = e = \psi(1)$. Since $C_{\varphi}$ is cohyponormal and therefore normaloid [3, Theorem 8.7], $W_{\psi,\varphi}$ is normaloid and also convexoid.

While examples generated by Theorem 4.1 are reasonable to come by when $\varphi'(a) < 1$, they are actually impossible to come by when $\varphi'(a) = 1$.

**Theorem 4.3.** Suppose $\varphi$ is an analytic self-map of $\mathbb{D}$ with Denjoy-Wolff point $a \in \partial \mathbb{D}$ and $\varphi'(a) = 1$. Then $C_{\varphi}$ is not normaloid.

**Proof.** The proof is analogous to Theorem 2.1. The spectral radius for $C_{\varphi}$ is $\varphi'(a)^{-1/2}$ when $a \in \partial \mathbb{D}$ [3, Theorem 3.9], so here we have $\rho(C_{\varphi}) = 1$. Since $\varphi(0) \neq 0$, we know $\|C_{\varphi}\| > 1$, therefore $C_{\varphi}$ is not normaloid. \hfill \qed
However, there are known weighted composition operators where $\varphi'(a) = 1$ and $W_{\psi,\varphi}$ is normaloid - even self-adjoint [2]. We make a minor adjustment to Theorem 4.1 to generate new examples of normaloid weighted composition operators in this setting.

**Corollary 4.4.** Suppose $\varphi$ is an analytic self-map of $D$ with Denjoy-Wolff point $a$, $\psi \in H^\infty$ is continuous at $a$, and $f \in H^\infty$ is also $f$ is continuous at $a$, with $\|f\|_\infty = |f(a)|$. If $W_{\psi,\varphi}$ is normaloid and $\rho(W_{\psi,\varphi}) = |f(a)|\rho(W_{\psi,\varphi})$, then $W_{f\psi,\varphi}$ is normaloid.

**Proof.** The proof is identical to Theorem 4.1, with a mere adjustment of symbols:

$$\|W_{f\psi,\varphi}\| \leq \|T_f\|\|W_{\psi,\varphi}\| = \|f\|_\infty\|W_{\psi,\varphi}\| = |f(a)|\|W_{\psi,\varphi}\| = |f(a)|\rho(W_{\psi,\varphi}) = \rho(W_{f\psi,\varphi}) \leq \|W_{f\psi,\varphi}\|.$$ 

□

**Example 4.5.** Suppose $\psi(z) = \frac{1}{z^2}, f(z) = e^z$. Then $W_{\psi,\psi}$ is self-adjoint and therefore normaloid by [2, Theorem 6]. Since $\psi$ is UCI [5, Example 5], we have $\rho(W_{f\psi,\psi}) = |f(1)|\|\psi(1)\| = |f(1)|\rho(W_{\psi,\psi})$. Since $|f(1)| = e = \|f\|_\infty$, we have that $W_{f\psi,\psi}$ is normaloid.

We end this section with a few extra facts for when $\varphi$ is UCI and $\varphi'(a) < 1$.

**Theorem 4.6.** Suppose $\varphi$ is UCI, the Denjoy-Wolff point $a$ of $\varphi$ is on $\partial D$, and $\varphi'(a) < 1$. Then $W_{\psi,\varphi}$ is convexoid if and only if $W_{\psi,\varphi}$ is spectraloid.

**Proof.** Every convexoid operator is spectraloid. In the other direction, assume $W_{\psi,\varphi}$ is spectraloid, so that $\rho(W_{\psi,\varphi}) = r(W_{\psi,\varphi})$. Note that by (3) of Theorem 3.2, the spectrum of $W_{\psi,\varphi}$ is a closed disk centered at the origin, completely filling in the set $\{\lambda \in \mathbb{C} : |\lambda| \leq \rho(W_{\psi,\varphi})\}$. Since $\rho(W_{\psi,\varphi}) = r(W_{\psi,\varphi})$, this set is necessarily also the closure of the numerical range. Therefore, $W_{\psi,\varphi}$ is convexoid.

□

**Corollary 4.7.** Suppose $\varphi$ is UCI, the Denjoy-Wolff point $a$ of $\varphi$ is on $\partial D$, and $\varphi'(a) < 1$. If $W_{\psi,\varphi}$ is normaloid, then $W_{\psi,\varphi}$ is convexoid.

**Proof.** Every normaloid operator is spectraloid, so $W_{\psi,\varphi}$ is spectraloid. By Theorem 4.6, if $W_{\psi,\varphi}$ is spectraloid, it is also convexoid.

□

5. Further Questions

Here we summarize the questions raised by the work of this paper.

1. If $W_{\psi,\varphi}$ is normaloid and $\rho(W_{\psi,\varphi}) = |\psi(a)|\rho(C_\varphi)$, is it necessary that $|\psi(a)| = \|\psi\|_\infty$?
(2) Can the many hypotheses of Theorem 2.7 be weakened, to identify \( \rho_e \) in the general setting when \( \varphi \) has interior Denjoy-Wolff point and \( C_\varphi \) is not power-compact?

(3) Can we then characterize all normaloid weighted composition operators where \( \varphi \) has Denjoy-Wolff point in \( \mathbb{D} \)?

(4) What are the necessary conditions for \( W_{\psi,\varphi} \) to be normaloid when the Denjoy-Wolff point of \( \varphi \) is on \( \partial \mathbb{D} \)?

(5) Ultimately, can we get an exact characterization of when \( W_{\psi,\varphi} \) is normaloid?

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