Distributed Stochastic Bandit Learning with Delayed Context Observation

Jiabin Lin and Shana Moothedath, Member, IEEE

Abstract—We consider the problem where M agents collaboratively interact with an instance of a stochastic K-armed contextual bandit, where K ∝ M. The goal of the agents is to simultaneously minimize the cumulative regret over all the agents over a time horizon T. We consider a setting where the context is observed after a delay and at the time of choosing the action the agents are unaware of the context and only a distribution on the set of contexts is available. Such a situation arises in different applications where at the time of the decision the context needs to be predicted (e.g., weather forecasting or stock market prediction), and the context can be estimated once the reward is obtained. We propose an Upper Confidence Bound (UCB)-based distributed algorithm and prove regret and communications bounds for linearly parametrized reward functions. We validated the performance of our algorithm via numerical simulations on synthetic data and real-world Movielens data.

I. INTRODUCTION

Sequential decision making is a common problem in many applications, including control and robotics [1], [2], communications [3], and ecology [4]. Bandit algorithms provide a learning framework to model the sequential decision making problem where the learner interacts with the environment in several rounds and the goal of the learner is to choose the best action in each round to maximize the cumulative reward over a period of time [5]. A popular variant of bandit algorithms is contextual bandits. In the standard contextual bandit model, the learner observes a context/feature vector, chooses an action and receives a reward based on the context and the chosen action. One of the key challenges in bandits is to balance the trade-off between exploring new actions in the pursuit of finding the best action and exploiting the known actions [5], [6].

Recently, many papers studied MAB problems with multiple agents, where a set of agents/learners face the same MAB problem. Collaboration among multiple agents expedites the learning process in many applications that use contextual bandit algorithms, such as recommender systems, clinical trials, control and robotics, and cognitive radio [7], [8], [9]. However, often the contexts are noisy or represent predictive measures, e.g., weather prediction or stock market prediction. In such scenarios, the exact contexts are not available and learners only observe a distribution on the set of contexts. There are many applications where the actions/decisions are made based on a prediction/distribution and the exact contexts are observed after choosing an action (e.g., we decide whether to take an umbrella based on the weather forecast and we know if it rained later in the day). In such situations, the exact context is available to the learner after a delay and we refer to this MAB problem as contextual bandits with delay.

Our goal in this paper is to propose a communication cost-effective algorithm for distributed bandit learning with M agents and delayed context observation. Bandit learning with delayed contexts is more challenging due to the fact that the learner does not have access to the context information while choosing the action. In order to address this difficulty, we convert the problem using a feature mapping that is used in [10] for a single agent bandit problem. After modifying the problem, we add a new set of feature vectors such that the reward under this set of d-dimensional context feature vectors is an unbiased observation for the action selection. We propose a UCB-based distributed bandit algorithm with regret bound $O(d\sqrt{MT\log^2(T)})$ for linearly parameterized reward functions; the order of our regret bound coincides with the regret bound of the distributed bandit algorithm in [8] ([8] assumed the exact contexts are known). Our setting recovers the distributed bandit algorithm with known contexts in [8] when the context distribution is a Dirac delta distribution.

We note that there is a straightforward communication protocol for distributed bandit learning is immediate sharing where each agent shares every new sample immediately with the other agents as noted in [8]. While the agents can achieve near-optimal regret under this protocol, the amount of communication data is directly proportional to the total size of gathered samples, rendering the problem non-scalable over large time horizons. To minimize the communication cost while retaining optimum regret, we use the observation in [11] and execute synchronization between agents only when the extra information accessible to the agents is significant when compared to the last synchronization.

This paper makes the following contributions.

- We model a distributed stochastic linear bandits (LBs) problem where M agents collaborate to minimize their total regret under the coordination of a central server when the contexts are observed with a delay and are unknown while choosing the action. We refer to this problem as the distributed LBs with delayed contexts.
- We present a UCB-based algorithm that achieves a $O(d\sqrt{MT\log^2(T)})$ high probability regret bound for distributed LBs with delayed contexts.
- We validated the performance of our approach via simulations on synthetic data and on the real world Movielens data.

The rest of the paper is organized as follows. In Section II we present the notations and the problem formulation. In Section III we present the related work. In Section IV we present the algorithm and regret analysis. In Section V we present the simulation results and in Section VI we present the conclusion.

II. PROBLEM SETTING AND NOTATIONS

Notations: The norm of a vector $z \in \mathbb{R}^d$ with respect to a matrix $V \in \mathbb{R}^{d\times d}$ is defined as $\|z\|_V := \sqrt{z^\top V z}$ and $|z|$ for a vector $z$ denotes element-wise absolute values. Further, $\top$ denotes matrix or vector transpose and $(\cdot, \cdot)$ denotes inner product. For an integer $N$, we define $[N] := \{1, 2, \ldots, N\}$.

Problem Setting: Distributed Linear Stochastic Bandits with Context Distribution: In this section, we first specify the standard linear bandit problem below and then explain the distributed stochastic bandit setting studied in this paper. Let $X$ be the action set, $C$ be the context set, and the environment is defined by a fixed and unknown reward function $y : X \times C \rightarrow \mathbb{R}$. In linear bandit setting, at any time $t \in \mathbb{N}$, the agent observes a context $c_t \in C$ and has to choose an action $x_t \in X$. Each context-action pair $(x, c)$, $x \in X$ and $c \in C$, is associated with a feature vector $\phi_{x,c} \in \mathbb{R}^d$, i.e., $\phi_{x,c} = \phi(x, c_t)$. Upon selection of an action $x_t$, the agent observes a reward $y_t \in \mathbb{R}$

$$y_t := \langle \theta^*, \phi_{x_t,c_t} \rangle + \eta_t, \quad (1)$$

where

- $\theta^*$ is the true reward function.
- $\eta_t$ is the stochastic noise.
- $\phi_{x,c}$ is the feature vector associated with action $x$ and context $c$.
where $\theta^* \in \mathbb{R}^d$ is the unknown reward parameter, $(\theta^*, \phi_{x_t, c_t}) = r(x_t, c_t)$ is the expected reward for action $x_t$ at time $t$, i.e., $r(x_t, c_t) = E[y_t]$, and $\eta_t$ is $\sigma$-subGaussian, additive noise. The goal is to choose optimal actions $x_t^*$ for all $t \in T$ such that the cumulative reward, $\sum_{t=1}^{T} y_t$, is maximized. This is equivalent to minimizing the cumulative (pseudo)-regret denoted as

$$R_T = \sum_{t=1}^{T} (\theta^*, \phi'_{x_t, c_t}) - \sum_{t=1}^{T} (\theta^*, \phi_{x_t, c_t}).$$

Here $x_t^*$ is the optimal best action for context $c_t$ and $x_t$ is the action chosen by the agent for context $c_t$.

In this work, we consider a distributed stochastic linear bandit setting with context distribution and unknown contexts. The communication network consists of a server and a set of $M$ agents, and the agents can communicate with the server by sending and receiving packets. We assume that the communication between the server and the agents have zero latency. We consider a setting where the context at time $t$, $c_t$, is unobservable rather only a distribution of the context denoted as $\mu_t$ is observed by the agents. At round $t$, the environment chooses a distribution $\mu_t \in \mathcal{P}(\mathcal{C})$ over the context set and samples a context realization $c_t \sim \mu_t$. The agents observe only $\mu_t$ and not $c_t$ and each agent selects an action, say action chosen by agent $i$ is $x_{t,i}$, and receive reward $y_{t,i}$, where $y_{t,i} = (\theta^*, \phi_{x_{t,i}, c_t}) + \eta_t$. Our aim is to learn an optimal mapping/policy $\mathcal{P}(\mathcal{C}) \rightarrow \mathcal{X}$ of contexts to actions such that the cumulative reward, $\sum_{t=1}^{T} \sum_{i=1}^{M} y_{t,i}$ is maximized. Formally, our aim is to minimize the cumulative regret

$$R(T) = \sum_{i=1}^{M} \sum_{t=1}^{T} (\theta^*, \phi'_{x_t, c_t}) - \sum_{i=1}^{M} \sum_{t=1}^{T} (\theta^*, \phi_{x_t, c_t}).$$

Here, $x^*_t = \arg\max_{x \in \mathcal{X}} E_{c_t \sim [\mu_t]} [r(x, c_t)]$ is the best action provided we know $\mu_t$, but not $c_t$, and $T$ is the total number of rounds.

Consider the set $\mathcal{I} = \{(1,2,\ldots,T) \times (1,2,\ldots,M)\} = \{I_1, I_2, \ldots, I_{MT}\}$, which is the set of all possible $(t,i)$ pairs for $t \in [T]$ and $i \in [M]$. For $I_{j-1} = (t,i) \in \mathcal{I}$, we have $i = j-1 \mod M$, $t = \lfloor j-1/M \rfloor$, and we define $F_{j-1} := \{(x_t, c_t, y_{t,i}) \mid (x_t, c_t) \in \mathcal{C}, (t,i) \in \mathcal{I}_{j-1}, H_t\}$. We note that in (3), we compete with the best possible mapping $\pi^* : \mathcal{P}(\mathcal{C}) \rightarrow \mathcal{X}$ from the observed context distribution to actions, that maximizes the expected reward $\sum_{i=1}^{M} \sum_{t=1}^{T} E_{c_t \sim \mu_t} [r(x_t, c_t)]$. $F_{j-1}$ is the filtration that contains all information available at the end of round $j-1$.

Our goal is to develop a distributed multi-armed bandit algorithm with the least possible communication cost to solve this problem. We define the communication cost as a protocol as the number of integers or real numbers communicated between the server and the agents [8]. We make the standard assumptions on the additive noise $\eta_t$ and the unknown parameter $\theta^*$ [10], [12].

**Assumption 1.** Each element $\eta_t$ of the noise sequence $\{\eta_t\}_{t=1}^{\infty}$ is conditionally $\sigma-$subGaussian, i.e.,

$$E[\phi^2 (\eta_{F_{j-1}, \mu_t})] \geq \exp(\lambda^2 \sigma^2/2).$$

**Assumption 2.** There exist constants $S,D \geq 0$ such that $\|\theta^*\|_2 \leq S$, $\|\phi_{x_t, c_t}\|_2 \leq D$, and $\phi^\top_{x_t, c_t} \theta^* \in [0,1]$, for all $t$ and all $x \in \mathcal{X}$.

### III. Related Work

MAB algorithms are well studied and various solution methods were suggested, for a survey see [5] and [6]. Our work deals with the class of linear contextual MABs with unknown contexts. Linear contextual bandit problems with context-dependent uncertainty are studied in [10], [13], [14], [15]. In [14], a scenario was explored in which contexts are disturbed by noise and the goal is to compete with the optimal policy that can access the undisturbed feature vector. Reference [10] studied a setting in which only a distribution on the context is known, as opposed to the exact context, and the goal is to pick the optimal action according to the distribution function. The model in [10] is closely related to ours, and the primary distinction is that while [10] considered a single-agent MAB problem we study a multi-agent MAB problem. In our initial work [15] we studied a single-agent conservative contextual MAB problem where the contexts are unknown and the learner is constrained to satisfy certain performance criteria.

Multiple-player MAB has gained more attention recently [16]. One class of problem study distributed MABs with collisions, where the reward for an arm reduces or is set to zero if a player chooses that action in [17], [18], [19], [3]. In the following work of [18], [20] investigated a context in which regret rises as a result of agent communication. A collision-based approach is associated with problems in cognitive radio networks, where the goal is to learn through action collisions rather than communication. This is in stark contrast to the setting considered in our work. In [8], agents face the same bandit model and communicate with a central server by sending and acquiring information in order to learn concurrently and collaboratively. Our model is similar to the time-variation action set case considered in [8]. In our scenario, however, contexts are observed after a delay, whereas in [8] the contexts are observed before choosing the action.

### IV. DISTRIBUTED UCB FOR LINEAR STOCHASTIC BANDITS WITH CONTEXT OBSERVATION

#### A. Proposed Algorithm and Guarantee

In this section, we present our algorithm and regret bound for the setting where the actual context $c_t$ (e.g., actual weather measurements) is observable to the agents after they choose their actions. We note that with the context observation the agents have $\{(x_{t,i}, c_t, y_{t,i})\}_{i=1}^{M}$ available to them while estimating $\hat{\theta}_t$ although not for selecting the action. The pseudocode of our algorithm is given in Algorithm 1.

Given the distribution $\mu_t$, we first construct the feature vectors $\Psi_t = \{\psi_{x_t, \mu_t} : x \in \mathcal{X}\}$, where $\psi_{x_t, \mu_t} = E_{c_t \sim \mu_t}[\phi_{x_t, c_t}]$ is the expected feature vector of action $x$ under $\mu_t$. Each feature $\psi_{x_t, \mu_t}$ corresponds to exactly one action $x \in \mathcal{X}$ and $\Psi_t$ denotes the feature context set at time $t$. Algorithm 1 is based on the optimism in the face of uncertainty principle, where at each time $t \in [T]$, each agent $i \in [M]$ maintains a confidence set $B_{t,i} \subseteq \mathbb{R}^d$ that contains the unknown parameter vector $\theta^*$ with high probability. Each agent then chooses an optimistic estimate $\hat{\theta}_{t,i} = \arg\max_{\theta \in B_{t,i}} E_{c_t \sim \mu_t}[\psi^\top_{x_t, \mu_t} \theta]$ and chooses an action $x_{t,i} = \arg\max_{x \in \mathcal{X}} \psi^\top_{x_t, \mu_t} \hat{\theta}_{t,i}$. Equivalently the agent chooses the pair $(x_{t,i}, \hat{\theta}_{t,i}) = \arg\max_{(x, \theta) \in \mathcal{X} \times B_{t,i}} \psi^\top_{x_t, \mu_t} \theta$ which jointly maximizes the reward. The agents now play their respective optimistic actions, $x_{t,i}$’s, and receive rewards $y_{t,i}$’s and utilize the reward observations and the now observable context to update their individual confidence set.

We note that while choosing the action the agents are unaware of the context and hence the decisions are made using $\psi$ rather than $\phi$ (line 9). In line 10 $y_{t,i}$ is a noisy observation of $\phi_{x_{t,i}, \theta^*}$ and the algorithm expects the reward $\psi_{x_{t,i}, \mu_t} \theta^*$. To address this, we construct a feature set $\Psi_t$ in such a way that $y_{t,i}$ is an unbiased observation for the action choice $\psi_{x_t, \mu_t}$ similar to the technique in [10] for single agent bandits. After the actions are chosen, the agents receive the respective rewards and the contexts are observable now. Hence in the estimation we utilize the information about the context. We denote $\sum \phi_{x_{t,i}, \mu_t} \theta^\top_{x_t, \mu_t}$ and $\sum \phi_{x_{t,i}, \mu_t} y_{t,i}$ for each agent $i \in [M]$.
Theorem 4.1. The cumulative regret of Algorithm 1 with expected feature set $\Psi$, and $\beta_{i,j} = \beta_{i,j}(\sigma, \delta)$ is bounded at time $T$ with probability (w. p.) at least $1 - M\delta$ by

$$R(T) \leq 4BT\sqrt{MTd\log(MT)(1+\log(MT))} + 4BT + V + 2M\log \frac{3}{\delta}.$$ 

Further, for $\delta = \frac{1}{MT}$, Algorithm 1 achieves a regret of $O(d\sqrt{MT\log^2(T)})$ with $O(M^{1.5}t^3)$ communication cost.

Proof. See Section IV-B.

The significance of Theorem 4.1 is that it allows us to use a smaller scaling $\beta_{i,j}$ for the confidence set, which indeed affects the action chosen by the algorithm. It is known that in practice $\beta_{i,j}$ has a large impact on the amount of exploration, and a tighter choice of $\beta_{i,j}$ can result in a significant reduction of the regret bound [10], which we validate through the experiments.

B. Regret Analysis

Proof of Theorem 4.1 relies on Lemma 4.2, Proposition 4.3, and the two main results we present below, Theorems 4.8 and 4.9.

Lemma 4.2. For any $\delta > 0$ w.p. $1 - M\delta$, $\theta^*$ always lies in the constructed $B_{i,j}$ for all $i$ and $t$.

Proof. The proof follows using Theorem 2 in [11] and union bound over all agents.

For a positive definite matrix $V$, we have the result below [11].

Proposition 4.3 (Lemma 11, [11]). Let $\{X_t\}_{t=1}^T$ be a sequence in $\mathbb{R}^d$ of $d \times d$ positive definite matrices and define $V_t = V + \sum_{s=1}^{t}X_sX_t^T$. We have that log det $V_t < \sum_{i=1}^{n}||X_i||^2_{F}$. Further, if $||X_i||_{F} < L$ for all $t$, then

$$\sum_{i=1}^{n} \min \left\{ 1, ||X_i||_{F}^{-1} \right\} \leq 2 \left( \log \det(V_n) - \log \det(V) \right) / d - \log \det(V).$$

The lemma above proves a bound on the per-step regret of the protocol.

Lemma 4.4. In Algorithm 1, with probability $1 - \delta$, the single step pseudo-regret $r_{i,t}(\theta^*, \phi_{i,t}^c, \phi_{i,t}^u)$ with $\beta_{i,j} = \beta_{i,j}(\sigma, \delta)$ is bounded by

$$r_{i,t} \leq 2\theta_{i,j}(\phi_{i,t}^c, \phi_{i,t}^u) + S_{ij} + D_{ij},$$

where $S_{ij} = \theta^*, \phi_{i,t}^c, \phi_{i,t}^u, \phi_{i,t}^u$ and $D_{ij} = \theta^*, \phi_{i,t}^c, \phi_{i,t}^u, \phi_{i,t}^u$ and $S_{ij}$ and $D_{ij}$ are defined as

$$S_{ij} = \theta^*, \phi_{i,t}^c, \phi_{i,t}^u, \phi_{i,t}^u, \phi_{i,t}^u$$

and $D_{ij} = \theta^*, \phi_{i,t}^c, \phi_{i,t}^u, \phi_{i,t}^u, \phi_{i,t}^u$.

Proof. Let us assume that $\theta^* \in B_{i,j}$. Then we have

$$r_{i,t} = \theta^*, \phi_{i,t}^c, \phi_{i,t}^u, \phi_{i,t}^u.$$

Further, for $\delta = \frac{1}{MT}$, Algorithm 1 achieves a regret of $O(d\sqrt{MT\log^2(T)})$ with $O(M^{1.5}t^3)$ communication cost.

Proof. See Section IV-B.
Below we present the Azuma-Hoeffdings inequality.

**Proposition 4.5. (Azuma-Hoeffdings)** Let \( M_t \) be a martingale on a filtration \( \mathcal{F}_t \) with almost surely bounded increments \( |M_t - M_{t-1}| < Q \). Then
\[
\Pr[|M_t - M_0| > b] \leq \exp \left( - \frac{b^2}{2Q^2} \right).
\]

**Lemma 4.6.** \( D_j := \left( \theta_t^r \phi_{\delta_t^r, x_t} - \phi_{\delta_t^r, x_t} - \psi_{\delta_t^r, x_t} + \psi_{\delta_t^r, x_t} \right) \) is a martingale difference sequence with \( |D_j| \leq 4 \) and \( \sum_j D_j \) is a martingale. Further, \( \sum_j D_j \) is bounded w.p. at least \( 1 - \delta \) as
\[
\sum_j D_j \leq 4 \sqrt{2n \log \frac{1}{\delta}}.
\]

**Proof.** Recall that \( D_j := \left( \theta_t^r \phi_{\delta_t^r, x_t} - \phi_{\delta_t^r, x_t} - \psi_{\delta_t^r, x_t} + \psi_{\delta_t^r, x_t} \right) \) is a martingale difference sequence with \( |D_j| \leq 4 \) and \( \sum_j D_j \) is a martingale. Using Proposition 4.5 with \( Q = 4 \sum_j D_j \leq 4 \sqrt{2n \log \frac{1}{\delta}} \).

Below we prove \( \sum_{j \in [MT]} S_j \) is a supermartingale.

**Lemma 4.7.** \( S_j := \| \psi_{\delta_t^r, x_t} \|_{\tau_j} - \| \phi_{\delta_t^r, x_t} \|_{\tau_j} \). Then, \( \sum_{j \in [MT]} S_j \) is a supermartingale with probability at least \( 1 - \delta \) as \( \sum_j S_j \leq 2 \lambda^{-1/2} \sqrt{2n \log \frac{1}{\delta}} \).

**Proof.** Recall that \( S_j := \| \psi_{\delta_t^r, x_t} \|_{\tau_j} - \| \phi_{\delta_t^r, x_t} \|_{\tau_j} \) with \( i = j - 1 \) mod \( M_t \), \( i = \lfloor j - 1/M_t \rfloor \), and \( \mathcal{F}_{j-1} := \{ \{ x \sigma_q, y \}_{q \in \mathbb{Q}} \} \} \} \). Thus we have
\[
E_{\psi_{\delta_t^r, x_t}} | D_j - \mathcal{F}_{j-1}, \mu_{t-M_t} | = E_{\psi_{\delta_t^r, x_t}} ( \| \psi_{\delta_t^r, x_t} \|_{\tau_j} - \| \phi_{\delta_t^r, x_t} \|_{\tau_j} ) | D_j - \mathcal{F}_{j-1}, \mu_{t-M_t} | = E_{\psi_{\delta_t^r, x_t}} ( \| \psi_{\delta_t^r, x_t} \|_{\tau_j} - \| \phi_{\delta_t^r, x_t} \|_{\tau_j} ) | D_j - \mathcal{F}_{j-1}, \mu_{t-M_t} | = 0.
\]

Eq. (7) follows from \( \psi_{\delta_t^r, x_t} = E_{\psi_{\delta_t^r, x_t}} ( \| \psi_{\delta_t^r, x_t} \|_{\tau_j} - \| \phi_{\delta_t^r, x_t} \|_{\tau_j} ) | D_j - \mathcal{F}_{j-1}, \mu_{t-M_t} | = 0 \). Thus \( D_j \) is a martingale difference sequence with \( |D_j| \leq 4 \) and \( \sum_j D_j \) is a martingale. Using Proposition 4.5 with \( Q = 4 \sum_j D_j \leq 4 \sqrt{2n \log \frac{1}{\delta}} \).

The last step follows from \( \| V_{i,j}^{-1} \| \leq \| V_{i,j}^{-1} \| \). Therefore \( \sum_{j \in [MT]} S_j \) is a supermartingale with \( |S_j| = 2 \lambda^{-1/2} \). Now from Proposition 4.5 we have \( \sum_j S_j \leq 2 \lambda^{-1/2} \sqrt{2n \log \frac{1}{\delta}} \).

Consider an arbitrary epoch in Algorithm 1, say the \( p \)-th epoch. Let \( \mathcal{E}_t \) be the set of all \((t,i)\) pairs in epoch \( p \) and \( \mathcal{V}_p \) be the \( V_{last} \) in epoch \( p \). Then we know \( \frac{\det(V_p)}{\det(V_{p-1})} = 1 + \sum_{(t,i) \in A_p} \| \psi_{\delta_t^r, x_t} \|_{\tau_j} \). Assume \( \sum_{(t,i) \in A_p} \| \psi_{\delta_t^r, x_t} \|_{\tau_j} \leq 1 \). Then we have
\[
1 \leq \frac{\det(V_p)}{\det(V_{p-1})} \leq 2.
\]

All the epochs that satisfy (10) are referred to as the good epochs. Similarly, all the epochs that do not satisfy (10) are referred to as the bad epochs. To prove Theorem 4.1 we first prove bounds for good epochs and bad epochs separately. Let us denote the number of timesteps that belong to good (bad) epochs as \( T_g \) (\( T_b \)) and the cumulative regret in all good (bad) epochs until time \( T \) as \( R_g(T) \) (\( R_b(T) \)). We present bounds for \( R_g(T) \) and \( R_b(T) \) separately and then use those bounds to prove Theorem 4.1. Our approach uses similar argument in Theorem 4 in [11].

**Theorem 4.8.** The cumulative regret of all good epochs in Algorithm 1 with expected feature set \( \Psi \) and \( \beta_j = \beta_j | \sigma, \delta/3 \) is bounded at time \( T \) with probability at least \( 1 - M \delta \) by
\[
R_g(T) \leq 4 BT \sqrt{MTd \log(MT)} + 4(\beta T + 1) \sqrt{2MT \log \frac{3}{\delta}}.
\]

**Proof.** To bound the cumulative regret of good epochs we use Theorem 4 in [11]. Assume that the MT pulls are all made by one agent in a round-robin fashion (i.e., the agent takes \( x_1, x_2, \ldots, x_{M-1}, x_{2M-1}, \ldots, x_{T-1}, \ldots, x_{T/2, T} \)). We define \( \mathcal{V}_{ij} := \lambda I + \sum_{(t,i) \in \mathcal{E}_t} \langle \phi_{\delta_t^r, x_t} \rangle \rangle \langle \phi_{\delta_t^r, x_t} \rangle \rangle \). Thus \( \mathcal{V}_{ij} \) denotes the imaginary agent calculates when the agent gets to \( x_{ij} \). Since \( x_{ij} \) is in a good epoch (say the \( p \)-th epoch), we have
\[
1 \leq \frac{\det(\mathcal{V}_{ij})}{\det(\mathcal{V}_{p-1})} \leq \frac{\det(\mathcal{V}_p)}{\det(\mathcal{V}_{p-1})} \leq 2
\]

(11)

Eq. (11) uses \( \det(\mathcal{V}_{ij}) \geq \det(\mathcal{V}_{p-1}) \) and the fact that for good epochs \( \det(\mathcal{V}_{ij}) = \det(\mathcal{V}_p) \). Thus from Lemma 4.4 we have
\[
r_{ij} \leq 2B \beta_i \mathcal{V}_{ij}^{-1} \mathcal{V}_{ij}^{-1} \phi_{\delta_t^r, x_t} + 2B \phi_{\delta_t^r, x_t} + \mathcal{D}_j
\]

(12)
\[
\begin{align*}
    &\left( \sum_{p \in P_g} \sum_{j \in J_p} \left( \ell_{t,i} - 2 \beta_t \sigma_j - D_j \right) \right)^2 \\
    &\leq \sqrt{MT_g} \sum_{p \in P_g} \sum_{j \in J_p} \min \left( \left\| \Phi_{t,i,c} \right\|_{\nu_{t,i}^{-1}}, 1 \right)
\end{align*}
\]
and compared the results as shown in Figure 1b. We ran for 500 time-period and 20 independent trails.

**Movielens data:** We used MovieLens data to evaluate the performance of our algorithm. For the rating matrix $R = \{r_{ij}\} \in \mathbb{R}^{943 \times 1682}$ of the data, we first obtained a non-negative matrix factorization $R = W H$, where $W \in \mathbb{R}^{943 \times 6}$, $H \in \mathbb{R}^{6 \times 1682}$ [21]. Each row of $W$, $\{W_j\}_{j=1}^{943}$, represents a context and each column of $H$, $\{H_k\}_{k=1}^{1682}$, represents an action. The feature vector for a given context $W_j \in \mathbb{R}^6$ and action $H_k \in \mathbb{R}^6$ is given by the diagonal of the matrix $W_j H_k^\top$. Hence the feature vector is of dimension 6 and $\theta^* = [1, 1.1, 1.1, 1.1]$. We chose 100 actions randomly from the action set. We present the plots showing the variation of the cumulative regret with respect to the execution time for the MovieLens data for different settings in Figure 1c. The reward $r(t, c_t)$ is bounded above by 1, and the observation noise $\eta_t$ is set as Gaussian with zero mean and standard deviation $10^{-3}$. In this experiment, as expected, the exact setting outperforms the delayed setting. We ran for a time-period of 1000 and the plots are shown in Figure 1c.

**VI. CONCLUSION**

In this work, we studied distributed stochastic multi-arm contextual bandit problem when the contexts are observed after a delay and only a distribution on the contexts is available at the time of decision. In our distributed setting, $M$ agents face the same MAB problem and work collaboratively to choose optimal actions to minimize the total cumulative regret. We leveraged the feature vector transformation in [10] and proposed a UCB-based algorithm and proved the regret bound for linearly parametrized reward functions. To validate the performance of our approach we performed numerical simulations on synthetic data and on Movielens data set.

**REFERENCES**

[1] M. Y. Cheung, J. Leighton, and F. S. Hover, “Autonomous mobile acoustic relay positioning as a multi-armed bandit with switching costs,” in IEEE/RSJ International Conference on Intelligent Robots and Systems, 2013, pp. 3368–3373.

[2] V. Srivastava, P. Reverdy, and N. E. Leonard, “Surveillance in an abruptly changing world via multiarmed bandits,” in IEEE Conference on Decision and Control (CDC), 2014, pp. 692–697.

[3] A. Anandkumar, N. Michael, A. K. Tang, and A. Swami, “Distributed algorithms for learning and cognitive medium access with logarithmic regret,” IEEE Journal on Selected Areas in Communications, vol. 29, no. 4, pp. 731–745, 2011.

[4] V. Srivastava, P. Reverdy, and N. E. Leonard, “On optimal foraging and multi-armed bandits,” in Annual Allerton Conference on Communication, Control, and Computing (Allerton), 2013, pp. 494–499.

[5] S. Bubeck and N. Cesa-Bianchi, “Regret analysis of stochastic and nonstochastic multi-armed bandit problems,” arXiv preprint arXiv:1204.5721, 2012.

[6] T. Lattimore and C. Szepesvári, Bandit algorithms. Cambridge University Press, 2020.

[7] R. Huang, W. Wu, J. Yang, and C. Shen, “Federated linear contextual bandits,” Advances in Neural Information Processing Systems, vol. 34, 2021.

[8] Y. Wang, J. Hu, X. Chen, and L. Wang, “Distributed bandit learning: Near-optimal regret with efficient communication,” arXiv preprint arXiv:1904.06309, 2019.

[9] P. Landgren, V. Srivastava, and N. E. Leonard, “Distributed cooperative decision making in multi-agent multi-armed bandits,” Automatica, vol. 125, p. 109445, 2021.

[10] J. Kirschner and A. Krause, “Stochastic bandits with context distributions,” Advances in Neural Information Processing Systems, vol. 32, pp. 14113–14122, 2019.

[11] Y. Abbasi-Yadkori, D. Pál, and C. Szepesvári, “Improved algorithms for linear stochastic bandits,” Advances in Neural Information Processing Systems, vol. 24, pp. 2312–2320, 2011.

[12] A. Kazemipour, M. Olavizadeh, Y. Abbasi-Yadkori, and B. Van Roy, “Conservative contextual linear bandits,” Advances in Neural Information Processing Systems, 2017.

[13] S. Lamprier, T. Gisselbrecht, and P. Gallinari, “Profile-based bandit with unknown profiles,” The Journal of Machine Learning Research, vol. 19, no. 1, pp. 2060–2099, 2018.

[14] S.-Y. Yun, J. H. Nam, S. Mo, and J. Shin, “Contextual multi-armed bandits under feature uncertainty,” arXiv preprint arXiv:1703.01347, 2017.

[15] J. Lin, Y. X. Lee, T. Juberj, S. Moothethad, S. Sarkar, and B. Ganapathysubramanian, “Stochastic conservative contextual linear bandits,” IEEE Conference on Decision and Control, 2022.

[16] X. Yi, X. Li, T. Yang, L. Xie, T. Chai, and K.-H. Johansson, “Distributed bandit online convex optimization with time-varying coupled inequality constraints,” IEEE Transactions on Automatic Control, vol. 66, no. 10, pp. 4620–4635, 2020.

[17] I. Bistritz and A. Leshem, “Distributed multi-player bandits—a game of thrones approach,” Advances in Neural Information Processing Systems, vol. 31, 2018.

[18] D. Kalathil, N. Nayyar, and R. Jain, “Decentralized learning for multiplayer multiarmed bandits,” IEEE Transactions on Information Theory, vol. 60, no. 4, pp. 2331–2345, 2014.

[19] J. Rosenski, O. Shamir, and L. Szlak, “Multi-player bandits—a musical chairs approach,” in International Conference on Machine Learning, PMLR, 2016, pp. 155–163.

[20] N. Nayyar, D. Kalathil, and R. Jain, “On regret-optimal learning in decentralized multiplayer multiarmed bandits,” IEEE Transactions on Control of Network Systems (IEEE-T CNS), vol. 5, no. 1, pp. 597–606, 2016.

[21] I. Bogunovic, A. Loskota, A. Krause, and J. Scarlett, “Stochastic linear bandits robust to adversarial attacks,” in International Conference on Artificial Intelligence and Statistics, 2021, pp. 991–999.