Minimum degree conditions for the existence of a sequence of cycles whose lengths differ by one or two

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Abstract
Gao, Huo, Liu, and Ma proved a result on the existence of paths connecting specified two vertices whose lengths differ by one or two. By using this result, they settled two famous conjectures due to Thomassen. In this paper, we improve their result, and obtain a generalization of a result of Bondy and Vince.

KEYWORDS
cycle length, minimum degree, path length

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1 | INTRODUCTION

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. For a graph $G$, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively, and $\deg_G(v)$ denotes the degree of a vertex $v$ in $G$.

In 1983, Thomassen proposed the following two conjectures.

**Conjecture A** (Thomassen [6]). For a positive integer $k$, every graph of minimum degree at least $k + 1$ contains cycles of all even lengths modulo $k$.

**Conjecture B** (Thomassen [6]). For a positive integer $k$, every 2-connected nonbipartite graph of minimum degree at least $k + 1$ contains cycles of all lengths modulo $k$. 
The above conjectures originated from the conjecture of Burr and Erdős concerning the extremal problem for the existence of cycles with prescribed lengths modulo $k$ (see [2]). We refer the reader to [5] for more details. In 2018, Liu and Ma proved that Conjectures A and B are true for all even integers $k$ by considering the existence of a sequence of paths whose lengths differ by two in bipartite graphs, see [5].

Recently, Gao, Huo, Liu, and Ma [4] announced that they had confirmed Conjectures A and B for all integers $k$ by using the following theorem. Here, we say that a sequence of $k$ paths (or $k$ cycles) $P_1, \ldots, P_k$ is admissible if $|E(P_i)| \geq 2$, and either $|E(P_i)| - |E(P_i)| = 1$ for $1 \leq i \leq k - 1$ or $|E(P_i)| - |E(P_i)| = 2$ for $1 \leq i \leq k - 1$.

**Theorem C** (Gao et al. [4]). Let $k$ be a positive integer, and let $G$ be a 2-connected graph, and $x, y$ be two distinct vertices of $G$. If $\deg_G(v) \geq k + 1$ for each $v \in V(G) \setminus \{x, y\}$, then $G$ contains $k$ admissible paths from $x$ to $y$.

In this paper, we show that the degree condition in Theorem C can be relaxed as follows.

**Theorem 1.** Let $k$ be a positive integer, and let $G$ be a 2-connected graph, $x, y$ be two distinct vertices of $G$, and $z$ be a vertex of $G$ (possibly $z \in \{x, y\}$) such that $V(G) \setminus \{x, y, z\} \neq \emptyset$. If $\deg_G(v) \geq k + 1$ for each $v \in V(G) \setminus \{x, y, z\}$, then $G$ contains $k$ admissible paths from $x$ to $y$.

This study also originated from the question of whether every graph of a minimum degree at least three contains two admissible cycles, which was raised by Erdős (see [1]). In 1998, Bondy and Vince answered this question by proving the following stronger theorem.

**Theorem D** (Bondy and Vince [1]). Every graph of order at least three, having at most two vertices of degree less than three, contains two admissible cycles.

They also conjectured that for a given integer $m \geq 3$, if $n$ is sufficiently large compared to $m$, then every graph of order $n$ having at most $m$ vertices of degree less than three, contains two admissible cycles. In 2020, Gao and Ma [3] settled the conjecture in the affirmative for $n \geq 5m^2$.

We give the following another generalization of Theorem D by using Theorem 1.

**Theorem 2.** For an integer $k \geq 2$, every graph of order at least three, having at most two vertices of degree less than $k + 1$, contains $k$ admissible cycles.

Note that the two exceptional vertices in Theorem 2 may be nonadjacent, and so Theorem C does not imply Theorem 2. To settle Conjectures A and B for all integers $k$, we only need a weaker version of Theorem 2, which can be obtained from Theorem C (see [4]).

To show Theorem 1, in Section 2, we give an equivalent formulation of Theorem 1 by considering the existence of admissible paths in “rooted graphs”; see Theorem 3 in Section 2. We also extend the concept of “cores” which was used in the argument of [4, 5] in preparation for the proof of Theorem 3. In Section 3, we prove Theorem 3 and also give the proof of Theorem 2 at the end of Section 3.
2 | PRELIMINARIES

2.1 | Admissible paths in rooted graphs

Let $G$ be a graph. A cut-vertex of $G$ is a vertex whose removal increases the number of components of $G$. A block of $G$ is a maximal connected subgraph of $G$ which has no cut-vertex, and a block $B$ of $G$ is called an end-block if $B$ has at most one cut-vertex of $G$. If $G$ itself is connected and has no cut-vertex, then $G$ is a block and is also an end-block.

For distinct vertices $x$ and $y$ of $G$, $(G, x, y)$ is called a rooted graph. A rooted graph $(G, x, y)$ is 2-connected if

(R1) $G$ is a connected graph of order at least three with at most two end-blocks, and

(R2) every end-block of $G$ contains at least one of $x$ and $y$ as a non-cut-vertex of $G$.

Note that $(G, x, y)$ is 2-connected if and only if $G + xy$ (i.e., the graph obtained from $G$ by adding the edge $xy$ if $xy \notin E(G)$) is 2-connected. We denote by $(G, x, y; z)$ a rooted graph $(G, x, y)$ with a specified vertex $z$ (this includes the case where $z \in \{x, y\}$ or $z \notin V(G)$). For a rooted graph $(G, x, y; z)$, we define $\delta(G, x, y; z) = \min\{\deg_G(v) : v \in V(G) \setminus \{x, y, z\}\}$ if $V(G) \setminus \{x, y, z\} \neq \emptyset$; otherwise, let $\delta(G, x, y; z) = -\infty$.

In this paper, we prove the following theorem, which immediately implies Theorem 1. (Note that Theorem 3 is also implied by Theorem 1, so Theorems 1 and 3 are actually equivalent.)

**Theorem 3.** Let $k$ be a positive integer, and let $(G, x, y; z)$ be a 2-connected rooted graph. If $\delta(G, x, y; z) \geq k + 1$, then $G$ contains $k$ admissible paths from $x$ to $y$.

2.2 | Terminology and notation

In this subsection, we prepare terminology and notation which will be used in the proof of Theorem 3.

Let $G$ be a graph. We denote by $N_G(v)$ the neighborhood of a vertex $v$ in $G$. For $S \subseteq V(G)$, we define $N_G(S) = (\bigcup_{v \in S} N_G(v)) \setminus S$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of $G$ induced by $S$, and let $G - S = G[V(G) \setminus S]$. A set $S$ of vertices in $G$ is called a clique (resp., an independent set) of $G$ if $G[S]$ is complete (resp., $G[S]$ is edgeless). We denote by $\text{dist}(u, v)$ the length of a shortest path from a vertex $u$ to a vertex $v$ in $G$. For $U, V \subseteq V(G)$ with $U \cap V = \emptyset$, a path in $G$ is a $(U, V)$-path if one end-vertex of the path belongs to $U$, the other end-vertex belongs to $V$, and the internal vertices do not belong to $U \cup V$. We write a path $P$ with a given orientation as $\overrightarrow{P}$. For an oriented path $\overrightarrow{P}$ and $u, v \in V(P)$, a path from $u$ to $v$ along $\overrightarrow{P}$ is denoted by $u\overrightarrow{P}v$.

For $t (\geq 2)$ pairwise disjoint subsets $V_1, \ldots, V_t$ of $V(G)$, we define the new graph $V_1 \cup \cdots \cup V_t$ to be the graph obtained from $G[V_i] \cup \cdots \cup G[V_i]$ by adding all possible edges between $V_i$ and $V_{i+1}$ for $1 \leq i \leq t - 1$. For convenience, we let $V_i \cup \cdots \cup V_t \cup \emptyset = V_i \cup \cdots \cup V_t$.

Let $D$ be a connected graph and $v$ be a vertex of degree one. The $v$-end-block of $D$ is an end-block $B_v$ with cut-vertex $b_v$ in $D$ such that $V(B_v) = \{v, b_v\}$. The $v$-end-block of $D$, if exists, is unique, and so we always denote it by $B_v$ for a vertex $v$. We also denote by $b_v'$ the unique cut-vertex of $D$ which is contained in $B_v$. If $\deg_D(b_v) = 2$, then let $b_v'$ denote the unique neighbor of $b_v$ in $D$ which is not $v$; otherwise, let $b'_v = b_v$. See Figure 1.
For notational simplicity, throughout the rest of this paper, we often denote the singleton set \( \{v\} \) by \( v \), and for a subgraph \( H \) of \( G \), we often write \( H \) to denote its vertex set \( V(H) \). For example, we write \( N_H(x) \) instead of \( N_{V(H)}(x) \).

### 2.3 The concept of cores

In this subsection, we extend the concept of cores which was used in the argument of \([4, 5]\).

Let \( \ell \) be an integer. Let \( x \) be a vertex of a graph \( G \), and let \( H \) be an induced subgraph of \( G \) whose vertex set is \( x \cup S \cup T \), where \( S \) and \( T \) are disjoint subsets of \( V(G) \setminus \{x\} \) (\( S \) may be an emptyset).

- \( H \) is called an \( \ell \)-core of type 1 with respect to \( x \) in \( G \) if \( H = x \cup T \cup S \), where \( \ell \geq 1 \), \( S = \emptyset \) and \( T \) is a clique of size exactly \( \ell + 1 \) of \( G \).
- \( H \) is called an \( \ell \)-core of type 2 with respect to \( x \) in \( G \) if \( H = x \cup S \cup T \), where \( \ell \geq 2 \), \( S \) is an independent set of size exactly 2 of \( G \), and \( T \) is a clique of size exactly \( \ell \) of \( G \).
- \( H \) is called an \( \ell \)-core of type 3 with respect to \( x \) in \( G \) if \( H = x \cup T \cup S \), where \( \ell \geq 0 \) and, \( S \) and \( T \) are independent sets of sizes exactly \( \ell \) and at least \( \max\{\ell + 1, 2\} \) of \( G \), respectively.

See Figure 2. We say that \( H \) is an \( \ell \)-core with respect to \( x \) when there is no need to specify the type. We also say that \( H \) is an \( \ell \)-core with respect to \( (x, y) \) if \( H \) is an \( \ell \)-core with respect to \( x \), and \( y \) is a vertex of \( V(G) \setminus V(H) \). In what follows, “a core” always means an \( \ell \)-core for some integer \( \ell \), and \( S \) and \( T \) denote the same ones as in the definition of an \( \ell \)-core depending on the type number of an \( \ell \)-core \( H \).
Remark 1. Let $G$ be a graph, and $x, y$ be two distinct vertices of $G$. If $\deg_G(x) \geq 2$ and $xy \notin E(G)$, then there always exists a core of type 1 or type 3 with respect to $(x, y)$ in $G$.

We give three facts which will be used frequently in the proof of Theorem 3. Here, an admissible sequence in which the condition $|E(P_i)| \geq 2$ is changed by $E(P_i)| \geq 1$, is said to be semi-admissible.

Fact 1 (Gao et al. [4]). Let $s, t$ be positive integers. Let $G$ be a graph, $x, y$ be two distinct vertices and $U \subseteq V(G) \setminus \{x, y\}$. Assume that $G - x$ contains $s$ semi-admissible $(U, y)$-paths $P_1, ..., P_s$, and let $u_i$ be the unique vertex of $V(P_i) \cap U$ for $1 \leq i \leq s$. Assume further that for each $1 \leq i \leq s$, $G - (V(P_i) \setminus \{u_i\})$ contains $t$ semi-admissible $(x, u_i)$-paths $Q_{i,1}, ..., Q_{i,t}$. If $|V(Q_{1,j})| = |V(Q_{2,j})| = \cdots = |V(Q_{s,j})|$ for $1 \leq j \leq t$, then $G$ contains $s + t - 1$ admissible $(x, y)$-paths.

Fact 2. Let $H$ be an $\ell$-core with respect to $x$. Then the following hold.

1. If $H$ is of type 2, then for any $s \in S$, $H$ contains $\ell$ admissible $(x, s)$-paths of lengths $3, 4, ..., \ell + 2$; if $H$ is of type 3, then for any $s \in S$ and $t \in T$, $H - t$ contains $\ell$ admissible $(x, s)$-paths of lengths $2, 4, ..., 2\ell$.
2. If $H$ is of type 2, then for any $t \in T$, $H$ contains $\ell + 1$ admissible $(x, t)$-paths of lengths $2, 3, ..., \ell + 2$; if $H$ is of type 1 or type 3, then for any $T' \subseteq T$ with $|T'| = \ell' + 1 \leq \ell + 1$ and any $t \in T'$, $H - (T \setminus T')$ contains $\ell' + 1$ semi-admissible $(x, t)$-paths of lengths $1, 2, ..., \ell' + 1$ (if $H$ is of type 1) or $1, 3, ..., 2\ell' + 1$ (if $H$ is of type 3).

Proof. The fact is straightforward from the definition of an $\ell$-core.

Fact 3. Let $k$ be a positive integer and $(G, x, y)$ be a 2-connected rooted graph. Let $H$ be an $\ell$-core with respect to $(x, y)$ and $C$ be the component of $G - V(H)$ such that $y \in V(C)$. Assume that $G$ does not contain $k$ admissible $(x, y)$-paths. Then (1) $\ell \leq k - 1$, and (2) if $N_G(C) \cap T \neq \emptyset$, then $\ell \leq k - 2$.

Proof. Since $(G, x, y)$ is 2-connected, $G - x$ contains an $(S \cup T, y)$-path $\overrightarrow{P}$. Let $u$ be the unique vertex of $V(P) \cap (S \cup T)$. If $u \in S$, then by Fact 2(1), $H$ contains an $\ell$ admissible $(x, u)$-paths $\overrightarrow{P_1}, ..., \overrightarrow{P_\ell}$, and so $x\overrightarrow{P_i}uy$ $(1 \leq i \leq \ell)$ are $\ell$ admissible $(x, y)$-paths; if $u \in T$, then by Fact 2(2), $H$ contains an $\ell + 1$ semi-admissible $(x, u)$-paths $\overrightarrow{P_1}, ..., \overrightarrow{P_{\ell+1}}$, and so $x\overrightarrow{P_i}uy$ $(1 \leq i \leq \ell + 1)$ are $\ell + 1$ admissible $(x, y)$-paths. Note that if $N_G(C) \cap T \neq \emptyset$, then we can choose $P$ so that $u \in T$. This together with the assumption of Fact 3 immediately implies (1) and (2).

3 PROOF OF THEOREM 3

Proof of Theorem 3. We prove it by induction on $|V(G)| + |E(G)|$. Let $(G, x, y; z)$ be a minimum counterexample with respect to $|V(G)| + |E(G)|$. 
Claim 3.1. (1) \( k \geq 2 \) (and so \( \delta(G, x, y; z) \geq 3 \)), and (2) \(|V(G)| \geq 5\).

Proof. (1) If \( k = 1 \), then by (R1) and (R2), we can easily see that \( G \) contains an \((x, y)\)-path of length at least 2, a contradiction. Thus \( k \geq 2 \), and so \( \delta(G, x, y; z) \geq k + 1 \geq 3 \).

(2) Since \( \delta(G, x, y; z) \geq k + 1 \geq 3 \), we have \(|V(G)| \geq 4\). Suppose that \(|V(G)| = 4\). Note that, then \( \delta(G, x, y; z) = k + 1 = 3 \). Let \( u \) and \( v \) be two distinct vertices of \( V(G) \setminus \{x, y\} \) such that \( u \neq z \). Since \( \text{deg}_G(u) = 3 \), we have \( N_G(u) = \{x, y, v\} \). By (R2), we also have \( N_G(v) \cap \{x, y\} \neq \emptyset \), say \( xv \in E(G) \) up to symmetry, and then \( xuv \) and \( xuvy \) are \( k (=2) \) admissible \((x, y)\)-paths, a contradiction. \( \square \)

Claim 3.2. (1) \( G \) is 2-connected and (2) \([x, y, z] \) is independent.

Proof. (1) Suppose that \( G \) is not 2-connected. Then by (R1), \( G \) has a cut-vertex \( c \) and \( G \setminus c \) has exactly two components \( C_1 \) and \( C_2 \). By (R2), without loss of generality, we may assume that \( x \in V(C_1) \) and \( y \in V(C_2) \). Since \( V(G) \setminus \{x, y, z\} \neq \emptyset \) by Claim 3.1(2), and by the symmetry of \( x \) and \( y \), we may assume that \( V(C_i) \setminus \{x, z\} \neq \emptyset \). Let \( G_i = G[C_i \cup c] \) for \( i \in \{1, 2\} \). Then \((G_1, x, c; z)\) is a 2-connected rooted graph such that \( \delta(G_1, x, c; z) \geq \delta(G, x, y; z) \). Hence by the induction hypothesis, \( G_1 \) contains \( k \) admissible \((x, c)\)-paths \( \vec{P}_1, \ldots, \vec{P}_k \). Let \( \vec{Q} \) be a \((c, y)\)-path in \( G_2 \). Then \( x\vec{P}_1c\vec{Q}y \) \((1 \leq i \leq k)\) are \( k \) admissible \((x, y)\)-paths in \( G \), a contradiction.

(2) Suppose that \( xv \in E(G) \) for some \( v \in \{y, z\} \), and choose such a vertex \( v \) so that \( v = y \) if possible. If \( G - xv \) (i.e., the graph obtained from \( G \) by deleting the edge \( xv \)) is 2-connected, then by the induction hypothesis, it follows that \( G - xv \) (and also \( G \)) contains \( k \) admissible \((x, y)\)-paths, a contradiction. Thus \( G - xv \) is not 2-connected. Since \( G \) is 2-connected by Claim 3.2(1), this implies that \((G - xv, x, v)\) is a 2-connected rooted graph with exactly two end-blocks.

Let \( B_1, \ldots, B_t \) \((t \geq 2)\) be all the blocks of \( G - xv \) such that \( V(B_i) \cap V(B_{i+1}) \neq \emptyset \) for \( 1 \leq i \leq t - 1 \), say \( V(B_i) \cap V(B_{i+1}) = \{b_i\} \) for \( 1 \leq i \leq t - 1 \). Without loss of generality, we may assume that \( x \in V(B_1) \setminus \{b_1\} \) and \( v \in V(B_t) \setminus \{b_{t-1}\} \). Then \( y \in V(B_p) \setminus \{b_{p-1}\} \) for some \( p \) with \( 1 \leq p \leq t \), where we let \( b_0 = x \).

Suppose that \( p = t \). Then \((G - xv, x, y; z)\) is a 2-connected rooted graph such that \( \delta(G - xv, x, y; z) = \delta(G, x, y; z) \). Hence, by the induction hypothesis, \( G - xv \) (and also \( G \)) contains \( k \) admissible \((x, y)\)-paths, a contradiction. Thus \( p \leq t - 1 \). This implies that \( v \neq y \), that is, \( v = z \). Then by the choice of \( v \), we have \( xy \notin E(G) \).

Let \( G' = G[\bigcup_{1 \leq i \leq p} V(B_i)] \), and let \( z' = b_p \). Note that \( z \notin V(G') \). Note also that if \( p = 1 \), then since \( xy \notin E(G) \), \( V(G') \setminus \{x, z'\} = V(B_1) \setminus \{x, b_1\} \neq \emptyset \) holds; if \( p \geq 2 \), \( V(G') \setminus \{x, y, z'\} \neq \emptyset \) clearly holds. Then \((G', x, y; z')\) is a 2-connected rooted graph such that \( \delta(G', x, y; z') \geq \delta(G, x, y; z) \), and so the induction hypothesis yields that \( G' \) (and also \( G \)) contains \( k \) admissible \((x, y)\)-paths, a contradiction. Thus \( xv \notin E(G) \) for each \( v \in \{y, z\} \). By the symmetry of \( x \) and \( y \), we also have \( yz \notin E(G) \). \( \square \)

By Remark 1 and Claim 3.2, there exist cores with respect to \((x, y)\) and \((y, x)\), respectively, in \( G \). By the symmetry of \( x \) and \( y \), we can rename the vertices \( x \) and \( y \) so that

\[(XY1)\] there exists a core with respect to \((x, y)\) whose type number is at most the type number of any core with respect to \((y, x)\),
(XY2) subject to (XY1), rename $x$ and $y$ so that $\deg_G(x) \leq \deg_G(y)$ if possible, and

(XY3) subject to (XY1) and (XY2), if $z \in V(G)$, then rename $x$ and $y$ so that $\dist_G(x,z) \leq \dist_G(y,z)$ if possible.

Let $H'$ be an $\ell'$-core with respect to $(x,y)$ in $G$ for some integer $\ell'$, and let $S'$ and $T'$ be sets of vertices of $G$ which play the same role as $S$ and $T$ in the definition of an $\ell$-core, respectively, depending on the type number of $H'$. Let $C'$ be the component of $G - V(H')$ such that $y \in V(C')$. Choose $H'$ so that

(H1) the type number of $H'$ is as small as possible, and

(H2) subject to (H1),

(H2-1) if $H'$ is of type 1 or type 2, then $|T'|$ is maximum;

(H2-2) if $H'$ is of type 3, then (i) $S'$ is maximum, and (ii) $|T'|$ is maximum, subject to (i).

We now define the induced subgraph $H$ of $G$ and the component $C$ of $G - V(H)$ by modifying $H'$ if the following condition (T) holds:

(T) $H'$ is of type 3, $|T'| \geq 3$, $V(C') = \{y\}$, $N_G(x) = N_G(y) = T'$, and there exists a component $D_0$ of $G - V(H')$ such that $D_0 \neq C'$, $V(D_0) \setminus \{z\} \neq \emptyset$, and $N_G(D_0) \cap T' \neq \emptyset$.

Let $t_0 \in N_G(D_0) \cap T' = N_G(D_0) \cap N_G(y)$, and we define $H$, $\ell$, $S$, $T$ depending on the following two cases (M1) and (M2), and finally let $C$ be the component of $G - V(H)$ such that $y \in V(C)$ (see also Figure 3).

(M1) if $|T'| = |S'| + 1$, then $H := H' - \{s_0, t_0\}$, $\ell := \ell' - 1$, $S := S' \setminus \{s_0\}$, and $T := T' \setminus \{t_0\}$, where $s_0$ is an arbitrary fixed vertex in $S'$;

(M2) if $|T'| \geq |S'| + 2$, then $H := H' - \{t_0\}$, $\ell := \ell'$, $S := S'$, and $T := T' \setminus \{t_0\}$.

If (T) does not hold, then we define $H = H'$, $\ell = \ell'$, $S = S'$, $T = T'$, $t_0 = y$, and $C = C'$.

\footnote{If the condition obtained by swapping the role of $(x,y)$ in (XY1) also holds, then we assume $\deg_G(x) \leq \deg_G(y)$. In addition in (XY3), if $\deg_G(x) = \deg_G(y)$ and $z \in V(G)$, then we assume $\dist_G(x,z) \leq \dist_G(y,z)$.}
If (T) holds, then since \(|T'| \geq 3\), we have \(|T| \geq \max\{\ell + 1, 2\}\); thus the new graph \(H' = x \vee T \vee S\) is an \(\ell\)-core of type 3 with respect to \((x, y)\). Moreover, it follows from the definition that \(H\) is an \(\ell\)-core with respect to \((x, y)\) whether or not (T) holds and, \(S\) and \(T\) have the same role as \(S\) and \(T\) in the definition of an \(\ell\)-core, respectively, depending on the type number of \(H\). To make the difference clear, we sometimes say that

- \(H\) is of type 3\(^{\flat}\) if (T) holds (i.e., \(H \neq H'\)),
- \(H\) is of type 3\(^{\sharp}\) if (T) does not hold (i.e., \(H = H'\)) and \(H'\) is of type 3.

Note also the following: \(H\) is of type 1, type 2, or type 3\(^{\flat}\) if and only if (T) does not hold (i.e., \(H = H'\), \(\ell = \ell'\), \(S = S'\), \(T = T'\), \(t_0 = y\), and \(C = C'\)).

**Claim 3.3.** If \(v\) is a vertex of \(V(G) \setminus (V(H) \cup \{v, t_0\})\), then \(|N_G(v) \cap V(H)| \leq \ell + 1\). Moreover, if the equality holds, then \(N_G(v) \cap T \neq \emptyset\).

**Proof.** Let \(v\) be a vertex of \(V(G) \setminus (V(H) \cup \{v, t_0\})\). We show the claim as follows.

Assume first that \(H\) is of type 1. If \(|N_G(v) \cap V(H)| \geq \ell + 2\), then we have \(N_G(v) \cap V(H) = x \cup T = x \cup T'\), which contradicts the maximality of \(T'\) (see (H2-1)). Thus \(|N_G(v) \cap V(H)| \leq \ell + 1\). If the equality holds, we clearly have \(N_G(v) \cap T \neq \emptyset\), since \(\ell + 1 \geq 2\).

Assume next that \(H\) is of type 2, and suppose that \(|N_G(v) \cap V(H)| \geq \ell + 2\). Since there exist no cores of type 1 with respect to \((x, y)\) by (H1), we have \(x \notin N_G(v)\) or \(N_G(v) \cap S = \emptyset\). Since \(|N_G(v) \cap V(H)| \geq \ell + 2\), \(|S| = 2\) and \(|T| = \ell\), this yields that \(N_G(v) \cap V(H) = S \cup T = S' \cup T'\), which contradicts the maximality of \(T'\) (see (H2-1)). Thus \(|N_G(v) \cap V(H)| \leq \ell + 1\). If the equality holds, then since \(x \notin N_G(v)\) or \(N_G(v) \cap S = \emptyset\) holds, it follows that \(|N_G(v) \cap T| = (\ell + 1) - |N_G(v) \cap (x \cup S)| \geq (\ell + 1) - 2 = \ell - 1 \geq 1\). Thus we have \(N_G(v) \cap T \neq \emptyset\).

Assume finally that \(H\) is of type 3. Then the following facts hold:

**Fact 3.3.1.** (1) \(x \notin N_G(v)\) or \(N_G(v) \cap T = \emptyset\), (2) \(|N_G(v) \cap T| \leq \ell + 1\) or \(N_G(v) \cap S = \emptyset\), (3) \(|N_G(v) \cap T| \leq \ell + 1\), and (4) \(x \cup S \notin N_G(v)\).

**Proof.** Since there exist no cores of type 1 by (H1), (1) holds. Since there exist no cores of type 2 by (H1), (2) holds. If \(H\) is of type \(3^{\flat}\) by (H2-2)-(i), (3) holds; if \(H\) is of type \(3^{\flat}\) in (M1), then since \(|T| = \ell + 1\), (3) holds; if \(H\) is of type \(3^{\flat}\) in (M2), then by (H2-2)-(i) and, since \(|T| \geq \ell + 1 = \ell' + 1\) and \(|S| = \ell = \ell'\), (3) holds. If \(H\) is of type \(3^{\sharp}\), then by (H2-2)-(ii), (4) holds; if \(H\) is of type \(3^{\flat}\), then since \(N_G(x) = T \cup \{t_0\}\) (by (T)) and \(v \neq t_0\), (4) holds.

Suppose that \(|N_G(v) \cap V(H)| \geq \ell + 1\). Then Fact 3.3.1(1) and (4) yield that \(N_G(v) \cap T \neq \emptyset\) and \(x \notin N_G(v)\). If \(N_G(v) \cap S = \emptyset\), then Fact 3.3.1(3) yields that \(|N_G(v) \cap V(H)| = |N_G(v) \cap T| \leq \ell + 1\); if \(N_G(v) \cap S \neq \emptyset\), then Fact 3.3.1(2) yields that \(|N_G(v) \cap V(H)| = |N_G(v) \cap S| + |N_G(v) \cap T| \leq \ell + 1\). In either case, the equality \(|N_G(v) \cap V(H)| = \ell + 1\) holds. Therefore we have \(|N_G(v) \cap V(H)| = \ell + 1\) and \(N_G(v) \cap T \neq \emptyset\).

This completes the proof of Claim 3.3.
By Claim 3.2(2) and (T), we can easily obtain the following.

**Claim 3.4.** If $H$ is of type $\mathbb{3}^3$, then $\lvert V(C) \setminus \{y, z\} \rvert \geq 2$.

**Proof.** By (T), $V(D_0) \setminus \{z\} \neq \emptyset$. Since $t_0 \in N_C(y)$, it follows from Claim 3.2(2) that $t_0 \neq z$. Hence we have $\lvert V(C) \setminus \{y, z\} \rvert \geq \lvert V(D_0) \setminus \{z\} \rvert + \lvert \{t_0\} \rvert \geq 2$. □

We now divide the proof into two cases according to $V(C) = \{y\}$ or $V(C) \neq \{y\}$.

**Case 1.** $V(C) = \{y\}$.

By Claim 3.4, $H$ is not of type $\mathbb{3}^3$, which implies that (T) does not hold (i.e., $H = H', \ell = \ell'$, $S = S', T = T', t_0 = y$, and $C = C'$).

**Claim 3.5.** If $H$ is of type 1 or type 3, then $N_G(x) = N_G(y) = T$. If $H$ is of type 2, then $N_G(x) = N_G(y) = S$.

**Proof.** Note that by Claim 3.2 and the assumption of Case 1, $N_G(y) \cap V(H - x) = \deg_G(y) \geq 2$.

Assume first that $H$ is of type 1. Then $N_G(y) \cap T \geq 2$, and so there exists a core of type 1 with respect to $(y, x)$. This implies that we can use the inequality in (XY2). Since $N_G(y) \subseteq T \subseteq N_G(x)$, it follows from (XY2) that $N_G(x) = N_G(y) = T$. Thus the claim follows.

Assume next that $H$ is of type 2. Since $|N_G(y) \cap V(H - x)| \geq 2$, and since there exist no cores of type 1 with respect to $(y, x)$ by (XY1) and (H1), we have $N_G(y) = S$. This in particular implies that $y \vee S \vee T$ is an $\ell$-core of type 2 with respect to $(y, x)$. Thus we can use the inequality in (XY2). Since $N_G(y) = S \subseteq N_G(x)$, it follows from (XY2) that $N_G(x) = N_G(y) = S$. Thus the claim follows.

Assume finally that $H$ is of type 3. By (XY1) and (H1), there exist no cores of type 1 or type 2 with respect to $(y, x)$, and so any core with respect to $(y, x)$ is of type 3. This implies that we can use the inequality in (XY2). This also implies that $N_G(y) \subseteq T$ or $N_G(y) \subseteq S$. Since $|T| \geq \max\{\ell + 1, 2\} > \ell = |S|$ and $T \subseteq N_G(x)$, it follows from (XY2) that $N_G(x) = N_G(y) = T$. Thus the claim follows. □

**Claim 3.6.** Assume that $H$ is of either type 1 or type 3. Let $D$ be a component of $G - V(H)$ such that $D \neq C$ and $V(D) \setminus \{z\} \neq \emptyset$. Then $N_G(D) \cap \mathbb{S} \neq \emptyset$. (This in particular implies that, if $H$ is of type 1, then $G - V(H)$ does not have a component $D$ such that $D \neq C$ and $V(D) \setminus \{z\} \neq \emptyset$.)

**Proof.** Suppose that $N_G(D) \subseteq T$. By Claim 3.5, there exists a vertex $t_{cd} \in N_G(y) \cap N_G(D) \cap T$. Let $D^*$ be the graph obtained from $G[D \cup N_G(D)]$ by contracting $N_G(D) \setminus \{t_{cd}\}$ into a new vertex $t^*$. Since $N_G(D) \subseteq T$ and $G$ is 2-connected, it follows that $(D^*, t^*, t_{cd}, z)$ is a 2-connected rooted graph. Since $V(D) \setminus \{z\} \neq \emptyset$, we also have $\emptyset \neq V(D^*) \setminus \{t^*, t_{cd}, z\} \subseteq V(G) \setminus \{x, y, z\}$. Let

$$\epsilon = \begin{cases} 1 & \text{if } |T| = \ell + 1, \\ 0 & \text{if } |T| \geq \ell + 2. \end{cases}$$

We claim that $\delta(D^*, t^*, t_{cd}, z) \geq (k - \ell + \epsilon) + 1$. Let $v$ be a vertex in $V(D^*) \setminus \{t^*, t_{cd}, z\}$. We note that $\ell - \epsilon \geq 0$, for otherwise $\ell = 0$ and $\epsilon = 1$, implying
that |T| = ℓ + 1 = 1, which contradicts the definition of H. If NG(v) ∩ (T \ {t_{cd}}) = ∅, then deg_{D'}(v) = deg_G(v) ≥ deg_G(v) − ℓ + ε. If NG(v) ∩ (T \ {t_{cd}}) ≠ ∅, then by Claim 3.3, we can easily see that |NG(v) ∩ (T \ {t_{cd}})| ≤ ℓ + 1 − ε. Then, deg_{D'}(v) ≥ deg_G(v) − |NG(v) ∩ (T \ {t_{cd}})| + 1 ≥ deg_G(v) − ℓ + ε. Thus, δ(D*, t*, t_{cd}; z) ≥ δ(G, x, y; z) − ℓ + ε ≥ (k − ℓ + ε) + 1 as claimed.

By the induction hypothesis, D* contains k − ℓ + ε admissible (t*, t_{cd})-paths. This implies that G[T ∪ D] contains k − ℓ + ε admissible (T \ {t_{cd}}, t_{cd})-paths \( \overrightarrow{P}_1, \ldots, \overrightarrow{P}_{k−ℓ+ε} \). Let \( t_i \) be the unique vertex of \( V(P_i) \cap (T \setminus \{t_{cd}\}) \) for 1 ≤ i ≤ k − ℓ + ε. Then \( t_i \overrightarrow{P}_1 t_{cd}y \) is k − ℓ + ε admissible (T \ {t_{cd}}, y)-paths in G[T ∪ D ∪ C]. On the other hand, it follows from Fact 2(2) that for each 1 ≤ i ≤ k − ℓ + ε, H − t_{cd} contains \( ℓ − ε + 1 \) (x, \( t_i \))-paths of lengths 1, 2, ..., ℓ − ε + 1 (if H is of type 1) or 1, 3, ..., 2(ℓ − ε) + 1 (if H is of type 3). Hence by Fact 1, we obtain k = (k − ℓ + ε) + (ℓ − ε + 1) − 1 admissible (x, y)-paths in G, a contradiction. Thus \( N_G(D) \not\subseteq T \). Combining this with Claim 3.5, we have \( N_G(D) \cap S = N_G(D) \setminus (T \cup x) = N_G(D) \setminus T \neq ∅ \).

Case 1.1. H is of type 1.

By Claim 3.5, \( N_G(x) = N_G(y) = T \). By Claim 3.6, we also have \( V(G) \subseteq T \cup \{x, y, z\} \). Since |T| = ℓ + 1 ≤ k − 1 by Fact 3(2), and since T \ {z} ≠ ∅, the degree condition yields that ℓ + 1 = |T| = k − 1, \( z \in V(G) \setminus (T \cup \{x, y\}) \), and \( N_G(v) = (T \setminus \{v\}) \cup \{x, y, z\} \) for all \( v \in T \). This implies that G contains (x, y)-paths of lengths 2, 3, ..., k + 1. Thus G contains k admissible (x, y)-paths, a contradiction.

Case 1.2. H is of type 2.

By Claim 3.5, we have \( N_G(x) = N_G(y) = S \), say \( N_G(x) = N_G(y) = \{s_1, s_2\} \). Let \( G' = G \setminus \{x, y\} \). Since G and H − x are 2-connected, respectively, and \( |V(H \setminus x)| \geq 4 \), it follows that \( (G', s_1, s_2; z) \) is a 2-connected rooted graph such that \( δ(G', s_1, s_2; z) \geq δ(G, x, y; z) \). Therefore, by the induction hypothesis, we obtain k admissible (s_1, s_2)-paths \( \overrightarrow{P}_1, \ldots, \overrightarrow{P}_k \) in G'. Then \( x_{s_1} \overrightarrow{P}_1 s_2 y \) (1 ≤ i ≤ k) are k admissible (x, y)-paths in G, a contradiction.

Case 1.3. H is of type 3.

Claim 3.7. There exists a component D of \( G \setminus V(H) \) such that D ≠ C, \( V(D) \setminus \{z\} \neq ∅ \), and \( N_G(D) \cap T \neq ∅ \).

Proof. If there exists \( t \in T \) such that \( N_G(t) \setminus (V(H) \cup \{y, z\}) \neq ∅ \), then the assertion clearly holds. Thus, we may assume that \( N_G(t) \setminus (V(H) \cup \{y, z\}) = ∅ \) for all \( t \in T \). Since \( N_G(y) \cap T \neq ∅ \) by Claim 3.5, Fact 3(2) yields |S| = ℓ ≤ k − 2. Then for a vertex \( t \in T \setminus \{z\} \neq ∅ \), we have

\[
0 = |N_G(t) \setminus (V(H) \cup \{y, z\})| ≥ (k + 1) - (|S| + |\{x, y, z\}|) ≥ (k + 1) - (k + 1) = 0.
\]

Thus all equalities hold, which implies that |S| = ℓ = k − 2, \( z \in V(G) \setminus (V(H) \cup \{y\}) \) and \( tz \in E(G) \) for all \( t \in T \). By Claim 3.5 and since there exist no cores of type 2 with respect to \( (x, y) \) by (H1), we also have \( N_G(x) = N_G(y) = T = N_G(z) \cap V(H) \).
If \( N_G(z) \setminus V(H) \neq \emptyset \), then since \( T \subseteq N_G(z) \), the claim follows. So we may assume that \( N_G(z) \setminus V(H) = \emptyset \), that is, \( N_G(z) = T \). Recall that \( \ell = \ell' \) in Case 1. If \( |T| \geq \ell + 2 \), then \( x \lor T \lor (S \cup z) \) is an \((\ell' + 1)\)-core of type 3 in \( G \), contradicting to (H2-2)-(i). Thus we have \( |T| = \ell + 1 = k - 1 \), which also implies that \( |s| = \ell \geq 1 \). Since \( N_G(x) = N_G(z) = N_G(y) = T \), a vertex \( s \in S \) satisfies

\[
|N_G(s) \setminus (V(H) \cup \{y, z\})| = |N_G(s)| - |T| \geq (k + 1) - |T| = (k + 1) - (k - 1) > 0.
\]

Hence there exists a component \( D \) of \( G - V(H) \) such that \( D \neq C, V(D) \setminus \{z\} \neq \emptyset \), and \( N_G(D) \subseteq S \). Let \( s_d \in N_G(D) \) and \( D^* \) be the graph obtained from \( G[D \cup N_G(D)] \) by contracting \( N_G(D) \setminus \{s_d\} \) into a new vertex \( s^* \). Since \( N_G(D) \subseteq S \) and \( G \) is 2-connected, it follows that \((D^*, s^*, s_d; z)\) is a 2-connected rooted graph. Since \( V(D) \setminus \{z\} \neq \emptyset \) and \( |s| = \ell = k - 2 \), we also have \( \delta(D^*, s^*, s_d; z) \geq \delta(G, x, y; z) - (k - 2) + 1 \geq 3 + 1 \).

Therefore, by the induction hypothesis, \( D^* \) contains three admissible \((s^*, s_d)\)-paths. This implies that \( G[S \cup D] \) contains three admissible \((S \setminus \{s_d\}, s_d)\)-paths \( \overrightarrow{P_1}, \overrightarrow{P_2}, \) and \( \overrightarrow{P_z} \). Let \( s_i \) be the unique vertex of \( V(P_i) \cap (S \setminus \{s_d\}) \) for \( 1 \leq i \leq 3 \), and let \( t_c \in N_G(y) \cap T \). Then \( s_1 \overrightarrow{P_1}, s_2 \overrightarrow{t_c}, y \) \((1 \leq i \leq 3)\) are three admissible \((S \setminus \{s_d\}, y)\)-paths in \( G[t_c \cup S \cup D \cup C] \). On the other hand, since \( |T| \setminus \{t_c\} | = |(S \setminus \{s_d\}) \cup \{z\}| = \ell = k - 2 \) and \( N_G(z) = T \), it follows that for each \( 1 \leq i \leq 3 \), \( G[(V(H) \setminus \{t_c, s_d\}) \cup \{z\}] \) contains \( k - 2 \) \((x, s_i)\)-paths of lengths \( 2, 4, ..., 2(k - 2) \). Hence by Fact 1, we obtain \( k = 3 + (k - 2) - 1 \) admissible \((x, y)\)-paths in \( G \), a contradiction. 

Let \( D \) be a component of \( G - V(H) \) as in Claim 3.7. Recall that (T) does not hold in Case 1, and \( H = H', S = S', T = T', \) and \( C = C' \). Then, taking into account the conditions obtained in Claim 3.5, we have \( |T| = |T'| = 2 \), say \( T = T' = \{t_1, t_2\} \). Since \( N_G(D) \cap S \neq \emptyset \) by Claim 3.6, we have \( S \neq \emptyset \). Since \( G \) is 2-connected and \( N_G(x) = N_G(y) = \{t_1, t_2\} \) by Claim 3.5, \( G' = G - \{x, y\} \) is connected. Also, it is easy to check that \((G', t_1, t_2; z)\) is a 2-connected rooted graph. Since \( \emptyset \neq V(D) \setminus \{z\} \subseteq V(G') \), we also have \( \delta(G', t_1, t_2; z) \geq \delta(G, x, y; z) \). Therefore, by the induction hypothesis, we obtain \( k \) admissible \((t_1, t_2)\)-paths \( \overrightarrow{P_1}, ..., \overrightarrow{P_k} \) in \( G' \). Then \( x \overrightarrow{P_i} t_2 y \) \((1 \leq i \leq k)\) are \( k \) admissible \((x, y)\)-paths in \( G \), a contradiction.

This completes the proof of Case 1.

**Case 2.** \( V(C) \neq \{y\} \).

**Claim 3.8.** Assume that \( H \) is of type 3. If \( |s| = 1 \), then \( N_G(C) \cap T \neq \emptyset \).

**Proof.** Suppose that \( |s| = 1 \), say \( S = \{s\} \), and \( N_G(C) \cap T = \emptyset \). Let \( G' = G - V(C) \). Since \( G \) and \( H \) are 2-connected, \( y \not\in V(G') \) and \( |V(H)| \geq |\{x\}| + |T| + |s| \geq 1 + 2 + 1 \geq 4 \), it follows that \((G', x, s, z)\) is a 2-connected rooted graph such that \( \delta(G', x, s, z) \geq \delta(G, x, y; z) \). By the induction hypothesis, \( G' \) contains \( k \) admissible \((x, s)\)-paths \( \overrightarrow{P_1}, ..., \overrightarrow{P_k} \).

Since \( G \) is 2-connected and \( N_G(C) \cap T = \emptyset \), we have \( s \in N_G(C) \), and so there exists an \((s, y)\)-path \( Q \) in \( G[C \cup s] \). Then \( x \overrightarrow{P_i} sQ y \) \((1 \leq i \leq k)\) are \( k \) admissible \((x, y)\)-paths in \( G \), a contradiction.
In this case, we will apply the induction hypothesis for new graphs obtained from $H$ and blocks with at most two cut-vertices of $C$. However, the $z$-end-block of $C$ will not help us to find admissible paths in the argument. So, in the following two claims, we study the structure for the case where $C$ contains the $z$-end-block, and show that $C$ is not a $(y, z)$-path of order exactly 3. (See Section 2.2 for the definitions of the $z$-end-block $B_z$ and the vertices $b_z, b'_z$)

Claim 3.9. Assume that there exists the $z$-end-block $B_z$ with cut-vertex $b_z$ in $C$ such that $y \notin \{z, b_z\}$. Assume further that $\deg_C(b_z) = 2$. Then the following hold.

1. $\ell = k - 2$.
2. $|N_G(b_z) \cap V(H)| = \ell + 1$.
3. $(N_G(z) \cup N_G(b'_z)) \cap T = \emptyset$.
4. If $b'_z \neq y$, then $\deg_C(b'_z) \geq 3$.

Proof. By our assumption, $\deg_G(b_z) \geq k + 1$ and there exists a $(b_z, y)$-path $R$ in $C - z$. If $H$ is of type $3^b$, then since $y \neq z$, $N_C(y) = \{t_0\}$ and by Claim 3.4, we have $b_z \neq t_0$; if $H$ is not of type $3^b$, then $b_z \neq y = t_0$. In either case, we have $b_z \neq t_0$.

((1), (2)) To show (1) and (2), we first prove that

$$\ell \leq k - 2. \tag{3.1}$$

Since $\ell \leq k - 1$ by Fact 3(1), it suffices to show that $\ell \neq k - 1$. Suppose to the contrary that $\ell = k - 1$. Then it follows from Fact 3(2) that $N_G(C) \cap T = \emptyset$. Combining this with Claim 3.2, we have $N_G(z) \cap S \neq \emptyset$, say $s_z \in N_G(z) \cap S$. This in particular implies that $H$ is of type 2 or type 3.

Suppose that $N_G(b_z) \cap S \neq \emptyset$, say $s_b \in N_G(b_z) \cap S$. Then $s_b b_z R y$ and $s_z z b_z R y$ are two admissible $([s_b, s_z], y)$-paths in $G[S \cup C]$. On the other hand, it follows from Fact 2(1) that for each $s \in \{s_b, s_z\}$, $H$ contains $k - 1 (= \ell)$ admissible $(x, s)$-paths. Hence by Fact 1, we obtain $k = 2 + (k - 1 - 1)$ admissible $(x, y)$-paths in $G$, a contradiction. Thus $N_G(b_z) \cap S = \emptyset$, that is, $N_G(b_z) \cap (S \cup T) = \emptyset$. Then $1 \leq k - 1 \leq \deg_G(b_z) - \deg_C(b_z) = |N_G(b_z) \cap V(H)| \leq |\{x\}| = 1$. Thus all equalities hold, which implies that $\ell = k - 1 = 1$ and $N_G(b_z) \cap V(H) = \{x\}$. If $H$ is of type 2, then $x b_z R y$ and $x z z b_z R y$ are $k (= 2)$ admissible $(x, y)$-paths in $G$, a contradiction; if $H$ is of type 3, then since $S| = \ell = 1$ and $N_G(C) \cap T = \emptyset$, this contradicts Claim 3.8. Thus (3.1) is proved.

Now, by Claim 3.3 and (3.1), we have

$$k - 1 \leq \deg_G(b_z) - \deg_C(b_z) = |N_G(b_z) \cap V(H)| \leq \ell + 1 \leq k - 1.$$  

Thus all equalities hold, which implies that $\ell = k - 2$ and $|N_G(b_z) \cap V(H)| = \ell + 1$.  

(3) Note that by Claims 3.3 and 3.9(2), $N_G(b_z) \cap T \neq \emptyset$, say $t_b \in N_G(b_z) \cap T$. To show (3), suppose that $N_G(v) \cap T \neq \emptyset$ for some $v \in \{z, b'_z\}$, and let $t_v \in N_G(v) \cap T$.

Since $N_C(b_z) = \{z, b'_z\}$, it follows that $G[\{z, b_z, b'_z, t_b\}]$ contains a $(t_v, b'_z)$-path $P$ of length 1 or 3. Hence $P$ and $t_b b_z b'_z$ are $(t_v, t_b), (b'_z)$-paths of lengths 1 and 2, respectively, or lengths 3 and 2, respectively. By adding $b'_z R y$ to each of the two paths, we obtain two semi-admissible $(\{t_v, t_b\}, y)$-paths in $G[C \cup \{t_v, t_b\}]$. On the other hand, it follows from Fact 2(2) and Claim 3.9(1) that for each $t \in \{t_v, t_b\}$, $H$ contains $k - 1 (= \ell + 1)$
semi-admissible \((x, t)\)-paths. Hence by Fact 1, \(G\) contains \(k (= 2 + (k - 1) - 1)\) admissible \((x, y)\)-paths, a contradiction.

(4) Assume that \(b'_c \neq y\) and \(\deg_C(b'_c) \leq 2\). We first claim that \(b'_c \neq t_0\). If \(H\) is not of type 3\(^b\), then since \(t_0 = y\), we have \(b'_c \neq y = t_0\). So we may assume that \(H\) is of type 3\(^b\). Suppose to the contrary that \(b'_c = t_0\). (See Figure 3.) If \(H\) is of type 3\(^b\) in (M1), then \(\deg_C(b'_c) = \deg_C(t_0) \geq |N_G(t_0) \cap V(D_0)| + |\{y, s_0\}| \geq 1 + 2 = 3\), a contradiction. Thus \(H\) is of type 3\(^b\) in (M2). Note that, in this case, \(|S| = \ell = \ell'\). Note also that \((b'_2 b'_c) = b_c t_0 \in E(G)\) and \(N_G(t_0) \cap V(H) = x \cup S\). If \(xb_z \in E(G)\), then \(x \vee \{b_z, t_0\} \cup \emptyset\) is a core of type 1 with respect to \((x, y)\) in \(G\); if \(xb_z \notin E(G)\), \(N_G(b_z) \cap S \neq \emptyset\), and \(N_G(b_z) \cap T \neq \emptyset\), say \(s_b \in N_G(b_z) \cap S\) and \(t_b \in N_G(b_z) \cap T\), then \(x \vee \{t_0, t_b\} \cup \{s_b, b_z\}\) is a core of type 2 with respect to \((x, y)\) in \(G\). Since there exist no cores of type 1 or type 2 with respect to \((x, y)\) by (H1), we have \(xb_z \notin E(G)\), and \(N_G(b_z) \cap S = \emptyset\) or \(N_G(b_z) \cap T = \emptyset\). This together with Claim 3.9(2) implies that \(N_G(b_z) \cap T = |N_G(b_z) \cap V(H)| = \ell + 1 = \ell' + 1\). Hence \(H'' := x \vee ((N_G(b_z) \cap T) \cup t_0) \cup (S \cup b_z)\) is an \((\ell' + 1)\)-core of type 3 in \(G\), which contradicts (H2-2)-(i). Thus \(b'_c \neq t_0\), and hence we have \(b'_c \notin V(G) \cap V(H) \cup \{y, t_0\}\).

By Claim 3.9(3), \(N_G(b'_c) \cap T = \emptyset\), and so Claims 3.3 and 3.9(1) yield that \(|N_G(b'_c) \cap V(H)| \leq \ell = k - 2\). Then we obtain

\[
\deg_C(b'_c) \leq \deg_C(b'_c) + |N_G(b'_c) \cap V(H)| \leq 2 + \ell = k,
\]

a contradiction.

This completes the proof of Claim 3.9.

Claim 3.10. \(C\) is not a \((y, z)\)-path of order exactly 3.

Proof. Suppose that \(C\) is a \((y, z)\)-path of order exactly 3. By Claims 3.2 and 3.9(3), we have \(N_G(z) \cap S \neq \emptyset\), say \(s_z \in N_G(z) \cap S\). This in particular implies that \(H\) is not of type 1.

Suppose that \(H\) is of type 2. Note that then \(b_z \in V(G) \setminus (V(H) \cup \{y, t_0\})\). Since \(N_G(b_z) \cap T \neq \emptyset\) by Claims 3.3 and 3.9(2), it follows from Fact 2(2) and Claim 3.9(1) that \(G[H \cup b_z] \subset \) contains \(k - 1 (= \ell + 1)\) admissible \((x, b_z)\)-paths \(\overrightarrow{P}_1, \ldots, \overrightarrow{P}_{k-1}\) of lengths \(3, 4, \ldots, \ell + 3\). On the other hand, by Fact 2(1), \(H\) contains an \((x, s_z)\)-path \(\overrightarrow{Q}\) of length \(\ell + 2\), and so \(\overrightarrow{P}_k := x \overrightarrow{Q} s_z \overrightarrow{z} b_z\) is an \((x, b_z)\)-path of length \(\ell + 4\). Then \(x \overrightarrow{Q} b_z y (1 \leq i \leq k)\) are \(k\) admissible \((x, y)\)-paths in \(G\), a contradiction. Thus \(H\) is not of type 2, that is, \(H\) is of type 3.

Since \(N_G(y) \cap T = N_G(b'_c) \cap T = \emptyset\) by Claim 3.9(3), and since \(xy \notin E(G)\) by Claim 3.2(2), it follows that \(N_G(y) \subseteq S \cup b_z\). By (XY1) and (H1), any core with respect to \((x, y)\) or \((y, x)\) is of type 3. This implies that we can use the inequality in (XY2). Note that \(\ell \geq 1\), since \(S \neq \emptyset\). Hence

\[
\ell + 1 = \max\{\ell + 1, 2\} \leq |T| \leq \deg_G(x) \leq \deg_G(y) \leq |S \cup b_z| = \ell + 1.
\]

Thus all equalities hold, which implies that \(\deg_G(x) = \deg_G(y)\) and \(N_G(x) = T\). The first equality implies that we can use the inequality in (XY3), and so \(\text{dist}_G(x, z) \leq \text{dist}_G(y, z) = 2\) holds. On the other hand, since \(xz \notin E(G)\) by Claim
3.2(2), and since \( N_G(z) \cap N_G(x) = N_G(z) \cap T = \emptyset \) by Claim 3.9(3), it follows that \( \text{dist}_G(x, z) \geq 3 \). This is a contradiction.

Let \( V_c \) be the set of cut-vertices of \( C \). A block \( B \) of \( C \) is said to be feasible if \( B \) satisfies the following condition (F).

\[
(F) \quad V(B) \cap (V_c \cup \{y, z\}) \not\subseteq 2 \quad \text{and} \quad V(B) \setminus (V_c \cup \{y, z\}) \neq \emptyset.
\]

Note that by the assumption of Case 2 and Claim 3.2(2), if \( C \) itself is a block, then \( C \) is feasible.

Claim 3.11. There exists a feasible block of \( C \).

Proof. Suppose that there exists no feasible block of \( C \). Then the condition (F) yields the following: \( C \) is not a block; its block-tree is a path; one of the two end-blocks of \( C \) is the \( y \)-end-block and the other is the \( z \)-end-block. By the definition of \( b_z \) and \( b'_z \), if \( \text{deg}_C(b_z) \geq 3 \) then \( b_z = b'_z \) and so \( \text{deg}_C(b'_z) \geq 3 \); if \( \text{deg}_C(b_z) = 2 \), then it follows from Claims 3.9(4) and 3.10 that \( \text{deg}_C(b'_z) \geq 3 \). In either case, \( \text{deg}_C(b'_z) \geq 3 \) holds. Hence there exists a block \( B \) of \( C \) which is not an end-block and satisfies (F).

In the rest of the proof, \( B, b, \) and \( z' \) denote any one of the following (B1), (B2)-(i), (B2)-(ii), and (B3) (note that by Claim 3.11 and (F), such a tuple \((B, b, z')\) exists, see also Figure 4):

(B1) \( B \) is a feasible block of \( C \) such that \( |V(B) \cap V_c| = 0 \) (i.e., \( C \) itself is a block and \( B = C \)), \( b := y \), and \( z' := z \).

(B2) \( B \) is a feasible block of \( C \) such that \( |V(B) \cap V_c| = 1 \), say \( V(B) \cap V_c = \{b'\} \), and

(i) if \( y \in V(B) \setminus \{b'\} \), then \( b := y \) and \( z' := b' \);

(ii) if \( y \notin V(B) \setminus \{b'\} \), then \( b := b' \) and \( z' := z \).

(B3) \( B \) is a feasible block of \( C \) such that \( |V(B) \cap V_c| = 2 \), and \( b \) is the unique vertex of \( V(B) \cap V_c \) such that \( C - (V(B) \setminus V_c) \) contains a \((b, y)\)-path (possibly \( b = y \)) and \( \{z'\} := (V(B) \cap V_c) \setminus \{b\} \).

Note that \( \emptyset \neq V(B) \setminus \{b, z', b'\} \subseteq V(G) \setminus \{x, y, z\} \) and \( N_G(v) \subseteq V(B) \cup V(H) \) for \( v \in V(B) \setminus \{b, z', b'\} \). Note also that \( t_0 \notin V(B) \setminus \{b, z'\} \), since \( t_0 \) is a cut-vertex of \( C \) (if \( H \) is of type \( 3^3 \)) or \( t_0 = y \) (if \( H \) is not of type \( 3^3 \)). Let \( \vec{R} \) be a \((b, y)\)-path in \( C \) such that \( V(R) \cap V(B - b) = \emptyset \).

FIGURE 4 The definitions of \( B, b, \) and \( z' \)
Claim 3.12. (1) \( N_G(B - b) \cap V(H) \subseteq x \cup S \) and (2) \( N_G(B - b) \cap (x \cup S) \) \( \geq 2 \).

**Proof.** (1) Suppose that \( N_G(B - b) \cap T \neq \emptyset \). Let \( B^* \) be the graph obtained from \( G[B \cup (N_G(B - b) \cap T)] \) by contracting \( N_G(B - b) \cap T \) into a new vertex \( t^* \). Since \( \emptyset \neq V(B) \setminus \{b, z^*\} \subseteq V(G) \setminus \{x, y, z\}, (B^*, t^*, b; z^*) \) is a 2-connected rooted graph such that \( \emptyset \neq V(B^*) \setminus \{t^*, b, z^*\} \subseteq V(G) \setminus \{x, y, z\}. Then, it follows from Claim 3.3 that for a vertex \( v \) of \( V(B^*) \setminus \{t^*, b, z^*\} \), the following hold:

- If \( \left| N_G(v) \cap T \right| = 0 \), then \( \deg_{B^*}(v) \geq \deg_G(v) - \ell \).
- If \( \left| N_G(v) \cap T \right| \geq 1 \), then \( \deg_{B^*}(v) \geq \deg_G(v) - (\ell + 1) + 1 = \deg_G(v) - \ell \).

Thus, the definition of \( B^* \) yields that

\[
\delta(B^*, t^*, b; z^*) \geq \delta(G, x, y; z) - \ell \geq (k - \ell) + 1.
\]

By the induction hypothesis, \( B^* \) contains \( k - \ell \) admissible \((t^*, b)\)-paths. Thus \( G \) contains \( k - \ell \) admissible \((T, b)\)-paths \( \vec{P}_1, \ldots, \vec{P}_{k-\ell} \). Let \( t_i \in V(P_i) \cap T \) for \( 1 \leq i \leq k - \ell \). Then \( t_i \vec{P}_i b \vec{R} y \) is a \((t_i, y)\)-path in \( G[C \cup t_i] \) for \( 1 \leq i \leq k - \ell \). On the other hand, by Fact 2(2), \( H \) contains \( \ell + 1 \) semi-admissible \((x, t_i)\)-paths for \( 1 \leq i \leq k - \ell \). Hence by Fact 1, \( G \) contains \( k = (k - \ell) + (\ell + 1) - 1 \) admissible \((x, y)\)-paths, a contradiction. Thus (1) holds.

(2) Suppose that either (i) \( \left| N_G(B - b) \cap (x \cup S) \right| = 1 \) or (ii) \( N_G(B - b) \cap (x \cup S) = \emptyset \) holds. Note that if \( B \) satisfies (B1) or (B2)-(ii), then the 2-connectivity of \( G \) implies that (i) holds; that is to say, if (ii) holds, then \( B \) satisfies (B2)-(i) or (B3). If (i) holds, say \( N_G(B - b) \cap (x \cup S) = \{v\} \), then let \( B' = G[B \cup v] \); if (ii) holds, then let \( v = z' \) and \( B' = B \). Since \( \emptyset \neq V(B) \setminus \{b, z'\} \subseteq V(G) \setminus \{x, y, z\}, (B', v, b; z') \) is a 2-connected rooted graph such that \( \delta(B', v, b; z') \geq \delta(G, x, y; z) \). By the induction hypothesis, \( B' \) contains \( k \) admissible \((v, b)\)-paths \( \vec{P}_1, \ldots, \vec{P}_k \). If (i) holds, then let \( \vec{Q} \) be an \((x, v)\)-path in \( H \); if (ii) holds, then by the 2-connectivity of \( G \), there exists an \((x, v)\)-path \( \vec{Q} \) in \( G[H \cup (V(C) \setminus (V(B' - v) \cup V(R)))] \). Then \( x \vec{Q} \vec{P}_i b \vec{R} y (1 \leq i \leq k) \) are \( k \) admissible \((x, y)\)-paths in \( G \), a contradiction. \( \square \)

**Case 2.1.** \( H \) is of type 1.

By Claim 3.12(2), we have \( N_G(B - b) \cap S \neq \emptyset \), which contradicts \( S = \emptyset \).

**Case 2.2.** \( H \) is of type 2.

By Claim 3.12(2), \( N_G(B - b) \cap S \neq \emptyset \). Let \( B^* \) be the graph obtained from \( G[B \cup (N_G(B - b) \cap S)] \) by contracting \( N_G(B - b) \cap S \) into a new vertex \( s^* \). Then \( (B^*, s^*, b; z') \) is a 2-connected rooted graph such that \( V(B^*) \setminus \{s^*, b, z'\} = V(B) \setminus \{b, z'\} \neq \emptyset \). Since there exist no cores of type 1 with respect to \((x, y)\) by (H1), it follows that \( x \notin N_G(v) \) or \( N_G(v) \cap S = \emptyset \) for \( v \in V(B) \setminus \{b\} \). This together with the definition of \( B^* \) and Claim 3.12(1) implies that \( \delta(B^*, s^*, b; z') \geq \delta(G, x, y; z) - 1 \geq (k - 1) + 1 \). Hence, by the induction hypothesis, \( B^* \) contains \( k - 1 \) admissible \((s^*, b)\)-paths, and so \( G[S \cup B] \) contains \( k - 1 \) admissible \((S, b)\)-paths \( \vec{P}_1, \ldots, \vec{P}_{k-1} \). Let \( s_i \in V(P_i) \cap S \) for \( 1 \leq i \leq k - 1 \). Then \( s_i \vec{P}_i b \vec{R} y \) is an
(s_i, y)-path in G[C ∪ s_i] for 1 ≤ i ≤ k − 1. On the other hand, by Fact 2(1), H contains two admissible (x, s_i)-paths for 1 ≤ i ≤ k − 1. By Fact 1, G contains k (= (k − 1) + 2 − 1) admissible (x, y)-paths, a contradiction.

Case 2.3. H is of type 3.

Note that, by Claim 3.12(2), ℓ = |S| ≥ 1. Let

\[ V_{nc} = V(C) \setminus (V_c \cup \{y, z\}). \]

We divide H into three types as follows:

1. H is of type I if ℓ = 1 and \(|N_G(v_0) \cap T| = 2\) for some v_0 ∈ V_{nc}.
2. H is of type II if ℓ = 1 and \(|N_G(v_0) \cap S| = |N_G(v_0) \cap T| = 1\) for some v_0 ∈ V_{nc}.
3. H is of type III if H is of neither type I nor type II.

If H is of type I or type II, then let v_0 be a vertex as described above, and let \(H^\# = G[x \cup T \cup S]\) and \(\ell^\# = |S^\#|\). We then let

\[ H^\# = G[x \cup T \cup S] \text{ and } \ell^\# = |S^\#|. \]

Then the following (i) and (ii) hold: (i) If H is of type I or type II, then \(\ell^\# = \ell + 1 = 2\); if H is of type III, then \(\ell^\# = \ell \geq 1\). (ii) If H is of type I or type II, then by the definitions of V_{nc} and the types, and by Claim 3.12(1), \(v_0 \notin V(B)\) and \(v_0\) does not separate B and y in C. This implies that there exists a (b, y)-path internally disjoint from B in \(G - V(H^\#)\). Moreover, by (ii), B is still a block of a component of \(G - V(H^\#)\). Recall that by Claim 3.12(1), \(N_G(v) \subseteq V(B) \cup (x \cup S)\) for \(v \in V(B) \setminus \{b, z\}\).

Claim 3.13. \(\ell^\# = 1\).

Proof. Suppose that \(\ell^\# \geq 2\). Let \(B^*\) be the graph obtained from \(G[B \cup (N_G(B - b) \cap S)]\) by contracting \(N_G(B - b) \cap S\) into a new vertex \(s^*\). By Claim 3.12(2), \((B^*, s^*, b; z')\) is a 2-connected rooted graph such that \(V(B^*) \setminus \{s^*, b, z'\} \neq \emptyset\). Recall that \(t_0 \notin V(B) \setminus \{b, z'\}\).

For a vertex \(v \in V(B^*) \setminus \{s^*, b, z'\}\), it follows from Claims 3.3, 3.12(1) and \(\ell^\# \geq 2\) that

\[
\deg_B(v) = \begin{cases} 
\deg_G(v) - |x| & \geq k \geq (k - \ell^\# + 1) + 1 \text{ if } N_G(v) \cap S = \emptyset, \\
\deg_G(v) - \ell + 1 & \geq \deg_G(v) - \ell^\# + 1 \text{ otherwise}, \\
& \geq (k - \ell^\# + 1) + 1
\end{cases}
\]

and thus \(\delta(B^*, s^*, b; z') \geq (k - \ell^\# + 1) + 1\). By the induction hypothesis, \(B^*\) contains \(k - \ell^\# + 1\) admissible \((s^*, b)\)-paths. Therefore \(G[B \cup (N_G(B - b) \cap S)]\) contains \(k - \ell^\# + 1\) admissible \((S, b)\)-paths \(P_1, \ldots, P_{k - \ell^\# + 1}\). Let \(s_i \in V(P_i) \cap S\) for \(1 \leq i \leq k - \ell^\# + 1\), and let \(\overrightarrow{R'}\) be a \((b, y)\)-path internally disjoint from B in \(G - V(H^\#)\). Then \(s_i \overrightarrow{P_i} b \overrightarrow{R'} y\) is an \((s, y)\)-path in \(G[(V(G) \setminus V(H^\#)) \cup \{s_i\}]\) for \(1 \leq i \leq k - \ell^\# + 1\). On the other hand, it follows from Fact 2(1) and the definitions of types I, II that for each \(s_i\), \(H^\#\) contains \(\ell^\#\) admissible \((x, s_i)\)-paths of lengths \(2, \ldots, 2\ell^\#\) (if \(H\)
is of type III) or 2, 4 (if $H$ is of type I) or 2, 3 (if $H$ is of type II). Hence by Fact 1, $G$ contains $k = (k - \ell^H + 1) + \ell^H - 1$ admissible $(x, y)$-paths, a contradiction. □

Since $\ell \geq 1$, it follows from Claim 3.13 that

$$S^H = S,$$

that is, $H$ is of type III, $\ell^H = \ell = |S| = 1$, say $S = \{s\}$, $H^H = H$.

Then the following hold (note that $t_0 \notin V_{nc}$, since $t_0$ is a cut-vertex of $C$ or $t_0 = y$):

$$N_G(B - b) \cap V(H) = \{x, s\} \quad \text{(by Claim 3.12 (2))},$$

$$|N_G(v) \cap V(H)| \leq 1 \text{ for each } v \in V_{nc} \quad \text{(by Claim 3.3)}.$$  \hfill (3.2)

**Claim 3.14.** (1) $k \geq 3$, and (2) if $z \in V(C)$, then $N_G(T) \cap V(C - y) \neq \emptyset$.

**Proof.** By Claim 3.8, we have $N_G(T) \cap V(C) \neq \emptyset$. Therefore, it follows from Fact 3(2) that $k \geq 3$. Thus (1) holds. To show (2), suppose that $z \in V(C)$ and $N_G(T) \cap V(C - y) = \emptyset$. Since $N_G(T) \cap V(C) \neq \emptyset$, we have $N_G(T) \cap V(C) = \{y\}$. Let $G' = G - V(C - y)$. Since $G$ and $H$ are 2-connected, $z \in V(C)$ and $|V(H)| \geq 4$, it follows that $(G', x, y; s_i)$ is a 2-connected rooted graph such that $\delta(G', x, y; s_i) \geq \delta(G, x, y; z)$. Therefore, by the induction hypothesis, $G'$ (and also $G$) contains $k$ admissible $(x, y)$-paths in $G$, a contradiction. Thus (2) also holds. □

**Claim 3.15.** $C$ has exactly two end-blocks, and each end-block of $C$ contains exactly one of $z$ and $y$ as a non-cut-vertex of $C$. (This in particular implies that $(C, z, y)$ is a 2-connected rooted graph.)

**Proof.** We first show that

$$z' \in V(B - b).$$  \hfill (3.4)

Suppose to the contrary that $z' \notin V(B - b)$. (Note that then $B$ satisfies (B1) or (B2)-(ii).) Recall that (3.2) holds. If $|N_G(s_i) \cap V(B)| \geq 2$, then let $B' = G[B \cup \{x, s_i\}]$ and $z_B = s_i$; if $|N_G(s_i) \cap V(B)| = 1$, say $N_G(s_i) \cap V(B) = \{v\}$, then let $B' = G[B \cup x]$ and $z_B = v$. Note that $|V(B)| \geq 3$, since $V(B) \setminus \{b, z'_B\} \neq \emptyset$ and $\delta(G, x, y; z) \geq k + 1 \geq 4$ by Claim 3.14(1). Then $(B', x, b; z_B)$ is a 2-connected rooted graph and $\delta(B', x, b; z_B) \geq \delta(G, x, y; z)$. Hence by the induction hypothesis, $B'$ contains $k$ admissible $(x, b)$-paths $\overrightarrow{P_1}, ..., \overrightarrow{P_k}$. Then $xP_i bR_i y$ $(1 \leq i \leq k)$ are $k$ admissible $(x, y)$-paths in $G$, a contradiction. Thus (3.4) is proved.

Suppose now that $C$ itself is a block. Then (B1) holds, so $B = C$, $b = y$, and $z' = z$. Moreover, (3.4) implies that $z = z' \in V(C)$. Then by Claim 3.14(2), $N_G(T) \cap V(B - b) = N_G(T) \cap V(C - y) \neq \emptyset$. But, this contradicts Claim 3.12(1). Thus $C$ is not a block, which implies that $C$ has at least two end-blocks. Recall that $(B, b, z'_B)$ is defined in one of (B1), (B2)-(i), (B2)-(ii), and (B3). Therefore if Claim 3.15 does not hold, then it is easy to check
that there is a feasible end-block $B$ of $C$ which satisfies (B2)-(ii), but does not satisfy (3.4), a contradiction.

Let $B_1, \ldots, B_t$ be all the blocks of $C$. By Claim 3.15, we have $t \geq 2$ and $|V_i| = t - 1$, and we may assume that $V (B_i) \cap V (B_{i+1}) \neq \emptyset$ for $1 \leq i \leq t - 1$, say $V (B_i) \cap V (B_{i+1}) = \{b_i\}$ for $1 \leq i \leq t - 1$. We may also assume that $z \in V (B_1) \setminus \{b_1\}$ and $y \in V (B_t) \setminus \{b_{t-1}\}$, and let $b_0 = z$ and $b_t = y$. Then $B = B_p$ for some $p$ with $1 \leq p \leq t$. Note that $B_p, b_{p-1}, b_p$ satisfy (B2)-(i) (if $p = t$) or (B2)-(ii) (if $p = 1$) or (B3) (if $2 \leq p < t$) as $(B, b, z') = (B_p, b_p, b_{p-1})$, and so it follows from Claim 3.12(1) that

$$N_G (T) \cap V (B_p - b_p) = \emptyset. \quad (3.5)$$

Claim 3.16. $N_G (T) \cap (\bigcup_{1 \leq p \leq t} V (B_i)) = \emptyset$.

Proof. Suppose that $N_G (T) \cap (\bigcup_{1 \leq p \leq t} V (B_i)) = \emptyset$. Then it follows from Fact 2(2) that $G [H \cup (\bigcup_{1 \leq p \leq t} V (B_i))]$ contains two admissible $(x, b_{p-1})$-paths. On the other hand, since $v \in V_e$ for $v \in V (B_p) \setminus \{b_{p-1}, b_p\}$, it follows from (3.3) that $(B_p, b_{p-1}, b_p; z)$ is a 2-connected rooted graph such that $\delta (B_p, b_{p-1}, b_p; z) \geq \delta (G, x, y; z) - 1 \geq (k - 1) + 1$, and hence the induction hypothesis yields that $B_p$ contains $k - 1$ admissible $(b_{p-1}, b_p)$-paths. Let $\vec{R} \rightarrow$ be a $(b_p, y)$-path in $G [\bigcup_{1 \leq p \leq t} V (B_i)]$. Then $b_{p-1} \vec{P}_1 \vec{P}_1 \vec{R} \vec{y}$ ($1 \leq i \leq k - 1$) are $k - 1$ admissible $(b_{p-1}, y)$-paths. Therefore, by Fact 1, $G$ contains $k = (k - 1) + 2 - 1$ admissible $(x, y)$-paths, a contradiction.

Choose $B = B_p$ so that $p$ is as large as possible. If $p = t$, then by (3.5) and Claim 3.16, we have $N_G (T) \cap V (C - y) = \emptyset$; since $z \in V (B_1) \setminus \{b_1\}$, this contradicts Claim 3.14(2). Thus $p < t$ and the choice of $B = B_p$ implies that $|V (B_i)| = 2$, that is, $V (B_i) = \{b_{i-1}, y\} (= \{b_{i-1}, b_i\})$.

Note that we can use the inequality in (XY2), since any core with respect to $(x, y)$ or $(y, x)$ is of type 3 by (XY1) and (H1). By (3.2), $\deg_G (x) \geq |T| + |N_G (x) \cap (B_p - b_p)| \geq |T| + 1$, and so (XY2) yields that $\deg_G (y) \geq |T| + 1$. Since $N_G (y) \subseteq H \cup b_{i-1}$, we obtain $|N_G (y) \cap V (H)| \geq |T| + 2$. Since $xy \notin E (G)$ and there exist no cores of type 1 with respect to $(y, x)$, we have $N_G (y) \cap V (H) = T$ and $N_G (b_{i-1}) \cap T = \emptyset$. If $p = t - 1$, then by the same argument as the case $p = t$, we get a contradiction to Claim 3.14(2). Thus $p < t - 1$ and the choice of $B = B_p$ implies that $|V (B_{i-1})| = 2$, that is, $V (B_{i-1}) = \{b_{i-2}, b_{i-1}\}$. Since $\deg_G (y) = |T| + 1$, and by (3.2), we have $N_G (x) = T \cup (N_G (x) \cap (B_p - b_p))$, and so $x \notin N_G (b_{i-1})$ because $b_{i-1} \notin V (B_p)$. Therefore $N_G (b_{i-1}) \subseteq \{y, b_{i-2}, s_1\}$. Since $\deg_G (b_{i-1}) \geq k + 1$, we obtain $k \leq 2$, contradicting to Claim 3.14(1).

This completes the proof of Theorem 3.

We finally give the proof of Theorem 2.

Proof of Theorem 2. It suffices to show the case where a given graph is connected. Let $k \geq 2$ be an integer, and let $G$ be a connected graph of order at least three having at most two vertices of degree less than $k + 1$. Then we can take two vertices $x$ and $z$ of $G$ such that all the vertices except $x$ and $z$ have a degree at least $k + 1$ in $G$. Suppose now that $G$ is a counterexample.
We first consider the case where $G$ is 2-connected. Choose arbitrary edge $xy$ in $G$ (possibly $y = z$). Since $V(G) \geq 3$ and $\deg_G(v) \geq k + 1 \geq 3$ for $v \in V(G) \setminus \{x, z\}$, we have $V(G) \setminus \{x, y, z\} \neq \emptyset$ and $\deg_G(v) \geq k + 1$ for $v \in V(G) \setminus \{x, y, z\}$. Hence Theorem 1 yields that $G$ contains $k$ admissible $(x, y)$-paths. By adding $xy$ to each of the $k$ paths, we obtain $k$ admissible cycles, a contradiction. Thus $G$ is not 2-connected.

We next consider the case where there exists an end-block $B$ with cut-vertex $b$ such that $|V(B)| \geq 3$ and $|V(B-b) \cap \{x, z\}| \leq 1$. Let $x' \in V(B-b) \cap \{x, z\}$ if exists; otherwise, let $x'$ be a vertex in $B-b$. Then the same argument as in the case where $G$ is 2-connected can work with $G_{xz}$ and $B_{b_1, b_2}$ in the above argument for the case where $G$ is 2-connected, we obtain $k$ admissible cycles in $B$, a contradiction again.

\[ \square \]

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