Monotone gradient dynamics and the location of stationary \((p, q)\)-configurations

Emilia Petrisor

Department of Mathematics, Polytechnic University of Timisoara, Pta Victoriei 2, 300006 Timisoara, Romania

E-mail: emilia.petrisor@mat.upt.ro

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Abstract
By exploiting the monotone property of the gradient dynamics of the Frenkel–Kontorova model, we are able to locate the ordered and unordered stationary states in the space of \((p, q)\)-configurations, as well as forbidden regions for such states. Moreover, we will show that some generalized Frenkel–Kontorova models (which are associated with multiharmonic standard maps) can have ordered \((p, q)\)-configurations that are neither action minimizing nor mini-maximizing, and we give their location with respect to the set of \((p, q)\)-minimizers and mini-maximizers.

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(Some figures may appear in colour only in the online journal)

1. Introduction
Area preserving twist maps have been extensively studied as being typical examples of dynamical systems that exhibit a full range of behavior, from regular to chaotic motions. Their study is mainly concerned with the existence/breakup of rotational invariant circles (RICs). These invariant sets are barriers to transport and their breakup leads to a loss of stability. A criterion that gives conditions ensuring that no RIC of given rotation number can exist or no RIC can pass through a point or region of the phase space is called a converse KAM criterion [12]. A survey of the known converse KAM criteria to date can be found in [6].

One of the first such criteria was formulated by Boyland and Hall [4, 5], which states that if a twist map has an unordered periodic orbit, \(\mathcal{O}\), then there are no invariant circles whose rotation number belongs to the rotation band of \(\mathcal{O}\).
The search for unordered periodic orbits using classical methods can be difficult due to the instability of the system in the region of the phase space where they lie. It is much more simple to detect them looking for their representatives in the space of \((p, q)\)-configurations; that is, as zeros of the periodic action gradient.

The gradient of the action functional of an area preserving twist map of the infinite annulus (cylinder) is in fact the gradient of the energy of the Frenkel–Kontorova (FK) model, which is studied in solid-state physics [7, 8]. The standard FK model is associated with an infinite one-dimensional chain of atoms connected by harmonic springs and subjected to an external potential [2, 8]. If the potential is multiharmonic, then the corresponding FK model is also called a generalized FK model.

Using an interplay between the variational approach to the dynamics of area preserving twist maps and the gradient dynamics of the periodic action, we locate the regions in the space of \((p, q)\)-configurations where ordered or unordered stationary states of the action gradient can lie, as well as the forbidden regions for such states. More precisely, we show that if \(x, x'\) are two consecutive \((p, q)\)-minimizers and \(y\) is a \((p, q)\)-mini-maximizer, such that \(x < y < x'\), then no other stationary \((p, q)\)-configuration can belong to the intervals \([x, y], [y, x']\) (proposition 2.3). However, in the complement of the set \([x, y] \cup [y, x']\), with respect to the interval \([x, x']\), there can exist ordered \((p, q)\)-stationary configurations. We give an example of a three harmonic standard map, exhibiting an ordered \((1, 2)\)-orbit which is neither minimizing nor mini-maximizing and whose corresponding configuration lies in such a complement. As far as we are aware, no such periodic orbit was revealed in the theoretical and numerical investigations of the twist maps that are reported in the literature.

The location of the \((p, q)\)-stationary states has both a theoretical and practical importance. In an attempt to implement the Hall and Boyland criterion, first of all we need to know where in the space of \((p, q)\)-configurations it is advisable to look for stationary states, and then to decide whether an identified configuration is ordered or not.

A general presentation of the gradient dynamics of the FK model can be found in [3, 7]. Angenent [1] was the first to exploit the monotonicity of this gradient semiflow in the study of periodic orbits of twist maps. Golé [10], Mramor and Rinky [15] and Sljepčević [17] have also used the monotonicity of the gradient semiflow of the FK-model to prove the existence of ghost circles and to prove the existence of Mather’s shadowing orbits of twist maps, respectively.

We consider an exact \(C^{r}\)-area preserving positive twist diffeomorphism \((r \geq 1)\), \(f\), of the infinite annulus \(\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}\), as represented by a lift, \(F\) (for basic properties of such maps the reader is referred to [10]). \(\pi : \mathbb{R}^2 \to \mathbb{A}\) denotes the covering projection.

The dynamics of \(F\) has a variational formulation. Its orbits are in a one-to-one correspondence with the critical points of an action.

Under the hypotheses on \(f\), it follows that the lift \(F\) admits a \(C^2\)-generating function \(h : \mathbb{R}^2 \to \mathbb{R}\) (which is unique up to additive constants), such that \(h_{12}(x, x') = \partial h/\partial x \partial x' < 0\) on \(\mathbb{R}^2\), and \(F(x, y) = (x', y')\) iff \(y = -h_1(x, x')\), \(y' = h_2(x, x')\) \((h_1 = \partial h/\partial x, h_2 = \partial h/\partial x')\).

For \(p, q\) coprime integers, \(q > 0\), the \(F\)-orbit of a point \((x_0, y_0)\) in \(\mathbb{R}^2\) is called a \((p, q)\)-type orbit, and its projection onto annulus is called \((p, q)\)-periodic orbit if \(F^q(x_0, y_0) = (x_0 + p, y_0)\).

We denote by \(X_{pq}\), the space of \((p, q)\)-type configurations; that is, sequences of real numbers, \(\mathbf{x} = (x_n)_{n \in \mathbb{Z}}\), such that \(x_{n+q} = x_n + p\), for all \(n \in \mathbb{Z}\). Being an affine subspace of \(\mathbb{R}^{q+1} = \{(x_0, x_1, \ldots, x_q)\}\), of equation \(x_q = x_0 + p\), \(X_{pq}\) can be identified with \(\mathbb{R}^q\).

The generating function \(h\) defines the action \(W_{pq}\), on the space \(X_{pq}\) of \((p, q)\)-configurations:

\[ W_{pq}(\mathbf{x}) = \sum_{k=0}^{q-1} h(x_k, x_{k+1}). \]
By Aubry–Mather theory, \([2, 14]\), for each pair of relative prime integers \((p, q)\), \(q > 0\), \(F\) has at least two orbits of type \((p, q)\). One orbit corresponds to a non–degenerate minimizing \((p, q)\)-configuration of the action. The second orbit corresponds to a mini-maximizing configuration. The corresponding \(F\)-orbits and \(f\)-orbits are called in the sequel \((p, q)\)-minimizing and \((p, q)\)-mini-maximizing orbits, respectively. These orbits are well ordered.

More precisely, an invariant set \(M\) of a positive twist map, \(f\), is well ordered if for every \((x, y), (x', y') \in \pi^{-1}(M)\) we have \(x < x'\) if \(F_1(x, y) < F_1(x', y')\), where \(F_1\) is the first component of the lift \(F\).

A well-ordered \((p, q)\)-orbit is also called Birkhoff orbit, while a badly-ordered orbit is called a non-Birkhoff or unordered orbit.

An appropriate framework to study the order properties of periodic orbits is defined as follows.

The space \(\mathcal{X}_{pq}\) of \((p, q)\)-configuration is partially ordered with respect to an order relation inherited from \(\mathbb{R}^2\). \(x = (x_i), y = (y_k) \in \mathbb{R}^2\) are related, and we write:

\[
x \leq y \iff x_k \leq y_k, \forall k \in \mathbb{Z}.
\]

One also defines:

\[
x < y \iff x \leq y, \text{ but } x \neq y
\]

\[
x < y \iff x_k < y_k, \forall k \in \mathbb{Z}.
\]

If \(x < y\) one says that \(x\) and \(y\) are weakly ordered, while if \(x < y\) one says that they are strictly ordered. \((\mathcal{X}_{pq}, \leq)\) is a lattice.

To each configuration \(x \in \mathcal{X}_{pq}\) one associates the positive order cone \(V_+(x) = \{y \in \mathcal{X}_{pq} \mid x \leq y\}\) and the negative order cone \(V_-(x) = \{z \in \mathcal{X}_{pq} \mid z \leq x\}\). Any two \((p, q)\)-configurations, \(x, y \in \mathcal{X}_{pq}\), that are comparable with respect to the relation \(\leq\), or \(\prec\), define the intervals:

\[
[x, y] = \{z \in \mathcal{X}_{pq} \mid x \leq z \leq y\}, \quad [[x, y]] = \{z \in \mathcal{X}_{pq} \mid x < z < y\}.
\]

Let \(\tau_{ij} : \mathbb{R}^2 \to \mathbb{R}^2\) be the translation map defined by:

\[
(\tau_{ij}x)_k = x_{k+j} + j, \quad \forall x = (x_k) \in \mathbb{R}^2, \quad i, j, k \in \mathbb{Z}
\]

\(\mathcal{X}_{pq}\) is invariant to any integer translation, \(\tau_{ij}\).

A \((p, q)\)-configuration, \(x\), such that:

\[
\forall i, j \in \mathbb{Z}, \text{ either } x \leq \tau_{ij}x \quad \text{or} \quad \tau_{ij}x \leq x
\]

is called cyclically ordered, as well as the corresponding \((p, q)\)-orbit of \(F\):

\[
(x_k, y_k) = (x_k, -h_1(x_k, x_{k+1})), \quad \forall k \in \mathbb{Z}
\]

Let \(x\) be a \((p, q)\)-configuration. The piecewise affine function that interpolates linearly the points \((k, x_k), k \in \mathbb{Z}\) is called an Aubry function.

Let \(M_{pq}\) be the subset of \((\mathcal{X}_{pq}, \leq)\) consisting in all \(W_{pq}\)-globally minimizing configurations. Any \(x \in M_{pq}\) is cyclically ordered. By the Aubry–Mather theory \([14]\) \(M_{pq}\) is a completely ordered subset of the set \(\mathcal{X}_{pq}\). In the sequel, the elements in \(M_{pq}\) are referred to as \((p, q)\)-minimizers.

\(x, x' \in M_{pq}\) are called consecutive if \(x \prec x'\) and there is no \(x'' \in M_{pq}\) such that \(x \prec x'' \prec x'\).

Besides the variational interpretation, the \((p, q)\)-orbits of the twist map \(F\) can be also associated to the stationary states (equilibrium points) of the gradient of the action, \(\nabla W_{pq}\). If the second derivative of the generating function \(h\) of the twist map \(F\) is bounded, then the system of differential equations \(\dot{x} = -\nabla W_{pq}\) is a cooperative dynamical system on \((\mathcal{X}_{pq}, \leq)\) (in fact \(\mathcal{X}_{pq}\) is invariant to the semiflow of the minus gradient of the energy of the FK model; for more details see \([3, 10]\)). Its \(C^1\)-semiflow, \(\xi_t, t \geq 0\), commutes with the group of translations.
Remark 2.1. From the Aubry–Mather theory we know that the \( M_{pq} \) is a critical \((p, q)\)-configuration of the action and the linearized gradient at \( x \), \( D^2W_{pq}(x) \) is the Hessian of \( W_{pq} \) at \( x \).

An important property of the gradient of the action relevant for my approach is that the semiflow of the minus gradient is strictly monotone ([3, 10]); that is,

\[
x < y \Rightarrow \xi_t x < \xi_t y, \forall t > 0.
\]  

(5)

With these results in mind, in the next section we detect the regions in the space of \((p, q)\)-configurations where no stationary state can lie, respectively, where cyclically ordered or unordered \((p, q)\)-stationary states are possible to be located.

2. Location of stationary \((p, q)\)-configurations

We denote by \( S_{pq} = \{z \in X_{pq} \mid \nabla W_{pq}(z) = 0 \} \) the set of stationary states in the space of \((p, q)\)-configurations.

**Proposition 2.1.** If \( x \) is a minimizer or a mini-maximizer of \( W_{pq} \) then no stationary state \( z \in X_{pq}, z \neq x \), can belong to the boundaries of the cones \( V_{\pm}(x) \).

**Proof.** Suppose \( z \in S_{pq}, z \neq x \), and \( z \in \partial V_{+}(x) \) (boundary of \( V_{+}(x) \)). This means that \( x < z \) and from the strict monotonicity of the flow it follows that \( \xi_t x < \xi_t z \). But the last relation is impossible because \( \xi_t x = x, \xi_t z = z, \forall t \geq 0 \). Similarly, one shows that \( z \notin \partial V_{-}(x) \). \( \square \)

**Proposition 2.2.** If \( z \in S_{pq} \setminus M_{pq} \) then there exist two consecutive \((p, q)\)-minimizers, \( x < x' \), such that \( x_0 < z_0 < x'_0 \). If \( z \in \text{Int}(V_{+}(x) \cap V_{-}(x')) = [[x, x']] \) then \( z \) is cyclically ordered, while if \( z \in \text{Int}(V_{+}(x)) \setminus V_{-}(x) \) or \( z \in \text{Int}(V_{-}(x')) \setminus V_{+}(x) \) then \( z \) is unordered.

**Proof.** Since to each \( x \in S_{pq} \) one associates the \( F \)-orbit of the point \((x_0, -h_1(x_0, x_1)) \in \mathbb{R}^2 \), the existence of the two consecutive minimizers is obviously as stated.

If \( z \in [[x, x']] \), then for any \( i, j \in \mathbb{Z}, \tau_i x < \tau_j z < \tau_j x' \). Since \( x, x' \) are consecutive elements in \( M_{pq} \), it follows that no translation \( \tau_k \), can exist, such that \( \tau_k x \in [[x, x']] \) or \( \tau_k x' \in [[x, x']] \). Thus, except for \( i = 0, j = 0 \) the intervals \([[x, x']], [[\tau_j x, \tau_j x']] \) have no point in common. Hence, \( x \leq \tau_j x \) implies \( z \leq \tau_j z \), and analogously \( \tau_j x \leq x \) implies \( \tau_j z \leq z \). Since \( x \) is cyclically ordered, it follows that \( z \) is also cyclically ordered. One also says that \( z \) is cyclically ordered with respect to \( M_{pq} \).

If \( z \notin [[x, x']] \), but \( x_0 < z_0 < x'_0 \), then either \( z - x \) or \( z - x' \) does not have coordinates of the same sign. Since the Aubry functions associated to \( x \) and \( x' \) are monotone, it follows that the Aubry function of \( z \) is not monotone; consequently, the configuration \( z \) cannot be cyclically ordered. \( \square \)

**Corollary 2.1.** A stationary state \( y \in X_{pq} \) that is ordered with respect to \( M_{pq} \) is comparable with each \( x \in M_{pq} \) while an unordered one is incomparable with at least a \((p, q)\)-mini-minimizer.

**Remark 2.1.** From the Aubry–Mather theory we know that the \((p, q)\)-mini-maximizing configurations are cyclically ordered with respect to elements in \( M_{pq} \) [14]. More precisely, if the action \( W_{pq} \) is a Morse function (i.e. all its critical points are nondegenerate), then for any consecutive \((p, q)\)-minimizers, \( x < x' \), there exists a mini-maximizing sequence \( y \in X_{pq} \), such that \( x < y < x' \). From the above proposition, it follows that it is cyclically ordered with respect to \( M_{pq} \).

It needs to be asked if there are other cyclically ordered \((p, q)\)-stationary states with respect to \( M_{pq} \) besides the \((p, q)\)-mini-maximizers. In order to answer this question, let us first analyze where a cyclically ordered stationary state that is different from a \((p, q)\)-mini-maximizer can lie within an interval \([[x, x']] \) of ends \( x, x' \) that are consecutive \((p, q)\)-mini-maximizers.
**Proposition 2.3.** If \( x, x' \) are two consecutive \((p, q)\)-minimizers and \( y \in X_{pq} \) is a mini-maximizer, such that \( x < y < x' \), then no stationary state of the gradient semiflow, \( \xi_t \), can belong to the intersection of order cones \( V_+(x) \cap V_-(y), V_+(y) \cap V_-(x') \).

**Proof.** The derivative \( -D^2 W_{pq} \) at \( x' \) has \( q \) negative eigenvalues, while at \( y \) there is one positive eigenvalue. Thus, the unstable manifold of \( y \) is one-dimensional. Moreover, there is a strictly ordered heteroclinic connection, \( \gamma \), between the two hyperbolic equilibria (an arc of ghost circle) \([9, 10, 15]\)). (A detailed proof of the existence of this heteroclinic connection is given in \([15]\), lemma 8.7.) For every \( v \in \gamma \), \( \xi_t v \to x' \), as \( t \to \infty \).

Suppose that that there is a \((p, q)\)-stationary state \( z \in V_+(y) \cap V_-(x') \). Since \( z \) cannot belong to the boundaries of the two cones, it is interior to \( V_+(y) \cap V_-(x') \), and we can choose a point \( v \in \gamma \) such that \( v < z \). It follows that \( \xi_t v < \xi_t z, \forall t > 0 \), which is impossible because \( \xi_t v \to x' \), as \( t \to \infty \), and \( \xi_t z = z, \forall t \geq 0 \). Similarly, one shows that \( z \) cannot belong to the intersection \( V_+(y) \cap V_-(x') \).

Figure 1 illustrates a part of the space of \((1, 2)\)-configurations. \( x \) and its integer translates \( \tau_{i0}, i = 1, 2, \) are \((1, 2)\)-minimizers, \( y \) and \( \tau_{i0}(y) \) are mini-maximizers. The gray semi-lines starting from these points are boundaries for the associated order cones. The semi-lines, as well as the gray regions, cannot contain other stationary \((1, 2)\)-configurations. Theoretically, in the green regions one can find ordered \((1, 2)\)-stationary states, while in the white regions one can find unordered ones.

Figure 2 gives the contour plot of the action \( W_{12} \), associated to the generating function \( h(x, x') = \frac{1}{2}(x - x')^2 + \frac{\epsilon}{(2\pi)^2} \cos(2\pi x) \) of the standard map. The red points are \((1, 2)\)-minimizers, those colored in blue are \((1, 2)\)-mini-maximizers, while the lateral points are unordered \((1, 2)\)-stationary configurations. One can observe that all of these points have the relative positions that have been deduced above.

We note that in \( X_{pq}/\tau_{0,1} := X_{pq}/G \) (\( G \) is a group isomorphic with \( \mathbb{Z} \), generated by the translation \( \tau_{0,1} \)), there exist at least \( q \) minimizers. If a positive twist map exhibits only a \((p, q)\)-minimizing orbit (this is a case in the classical standard map) then in \( X_{pq}/\tau_{0,1} \) there exist
exactly $q$ minimizers. Actually, if $x = (x_0, x_1, \ldots, x_{q-1})$ is a $(p, q)$-minimizer then there exist $q$ pairs $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ such that $(t_j, x)_0 \in [0, 1)$. Figure 1 is representative for such a case.

Computer experiments reveal that a multiharmonic standard map, that is, a twist map defined by a generating function:

$$h(x, x') = \frac{1}{2} (x - x')^2 + \epsilon V(x), \quad V(x) = \sum_{k=1}^{\ell} \frac{a_k}{(2\pi k)^2} \cos(2\pi k x), \quad a_k \in \mathbb{R}, \quad \ell \geq 2,$$

can exhibit in some range of parameters two $(p, q)$-minimizing orbits. Such maps are reversible; that is, there exists an involution $R$ such that $f^{-1} = R \circ f \circ R$. $f$ factorizes as $f = I \circ R$, where $I$ is also an involution. The fixed point sets of the two involutions consist of two components. $\text{Fix}(R) = \Gamma_0 \cup \Gamma'_0$, $\text{Fix}(I) = \Gamma_1 \cup \Gamma'_1$, where:

$$\Gamma_0 : x = 0, \quad \Gamma'_0 : x = 1/2, \quad \Gamma_1 : x = y/2 \pmod{1}, \quad \Gamma'_1 : x = (y - 1)/2 \pmod{1}$$

Symmetric $(p, q)$-periodic orbits (i.e. orbits that intersect $\text{Fix}(R)$) of a reversible twist map can undergo a Rimmer bifurcation [13]. Namely, if for fixed $a_k$, $k = 1, \ell$, at some threshold $\epsilon = \epsilon_r$, one of the two symmetric $(p, q)$-orbits changes its extremal type either from minimizing to mini-maximizing or conversely, and the two asymmetric $(p, q)$-orbits are born then, for $\epsilon$ in an interval $(\epsilon_r, \epsilon_c)$ the corresponding twist map has two minimizing and two mini-maximizing $(p, q)$-orbits. To each of the two $(p, q)$-minimizing orbits starting at $(x_0, y_0)$, respectively at $(x'_0, y'_0)$, corresponds a $(p, q)$-minimizer, $x = (x_0, x_1, \ldots, x_{q-1})$, respectively $x' = (x'_0, x'_1, \ldots, x'_{q-1}) \in X_{pq}$. The set of $(p, q)$-minimizers is in this case the union of two completely ordered subsets of $X_{pq}, M_{pq} = \{t_j(x), i, j \in \mathbb{Z}\} \cup \{t_{\ell k}(x'), k, \ell \in \mathbb{Z}\}$. The elements in the second subset interlace those of the first subset. Thus, in $X_{pq}/\tau_{01}$ there exist $2q$ minimizers. Between two consecutive $(p, q)$-minimizers there exists a $(p, q)$-mini-maximizer.

Figure 2. The contour plot of the action $W_{12}$ defined by the generating function of the standard map, corresponding to $\epsilon = 12$, and ordered and unordered $(1, 2)$-stationary configurations.
Figure 3. The four (1, 2)-periodic orbits of a three harmonic standard map after a Rimmer bifurcation.

\[ W_{12}(x, x') \]

Figure 4. The contour plot of the action \( W_{12} \) associated to a generating function defined by a three harmonic potential. The associated twist map has four (1, 2)-minimizers and four (1, 2)-mini-maximizers within \( X_{12}/\tau_{01} \).

Such a case is shown in figure 4, which illustrates the contour plot of the action \( W_{12} \) defined by the generating function corresponding to \( \epsilon = 1.2 \), and the three harmonic potential:

\[
V(x) = \sum_{k=1}^{3} a_k \left( \frac{2k\pi}{\epsilon} \right)^2 \cos(2k\pi x), \quad a_1 = 1, \quad a_2 = -0.3, \quad a_3 = 0.2.
\]

The corresponding twist map exhibits two (1, 2)-minimizing asymmetric orbits, respectively, two (1, 2)-mini-maximizing symmetric orbits (figure 3).

The red and dark red crosses in figure 4 represent distinct (1, 2)-minimizers \( x, x' \) and a few of their integer translates, while the blue and light blue points are the interlacing distinct (1, 2)-mini-maximizers.

The asymmetric \((p, q)\)-configurations present interest and are also studied in condensed-matter physics [18].

From proposition 2.3 it follows that if \( x, x' \) are two consecutive \((p, q)\)-minimizers and \( y \) is a mini-maximizer, \( x < y < x' \), then any stationary state, \( z \), ordered with respect to \( M_{pq} \), and different from \( y \), can only lie within \([x, x'] \setminus ([x, y] \cup [y, x'])\).
In many theoretical and numerical investigations of the standard map and the two-harmonic standard map reported in the literature, no ordered \((p, q)\)-orbit different from the minimizing and mini-maximizing orbit has been revealed. This is due to the fact that the main tool used to decide the stability type of a \((p, q)\)-periodic orbit is its residue. In [16] it was pointed out through a few examples that residue can be a misleading quantity in the characterization of a periodic orbit. It allows us only to deduce the stability type of a periodic orbit (i.e. regular hyperbolic, elliptic or inverse hyperbolic) but not the right extremal type of the \((p, q)\)-stationary state corresponding to that orbit. Instead, the twist number of a periodic orbit is much more relevant. The twist number measures the average rotation of tangent vectors under the action of the derivative of the twist map along a periodic orbit [16].

In the following, we show that the three harmonic standard map from the example 4 [16] has a \((1, 2)\)-periodic orbit starting at a point \((0, 0.5)\), whose corresponding \((1, 2)\)-configuration \(z = (z_0, z_1)\) is ordered with respect to \(M_{12}\), and it is not a mini-maximizer.

The generating function that defines the map of interest has the potential:

\[
V(x) = -\frac{0.18}{2\pi} \cos(2\pi x) + \frac{0.42}{4\pi} \cos(4\pi x) + \frac{0.11}{6\pi} \cos(6\pi x).
\]

The associated three harmonic standard map has four ordered \((1, 2)\)-orbits: a minimizing one which is symmetric of twist number \(\tau = 0\), two asymmetric mini-maximizing orbits of twist number \(\tau = -1/2\) (they are inverse hyperbolic orbits), and a symmetric orbit of twist number \(-1\) (for details see [16]). Connecting this three harmonic standard map with the integrable map one has deduced in [16] that the mini-maximizing orbit intersecting \(\Gamma_0\) undergoes a sequence of bifurcations that leads to the decrease of its twist number. The last bifurcation is a Rimmer-type bifurcation. At the bifurcation threshold, it turns from an orbit of \(\tau \in (-1, -1/2)\) into an orbit of \(\tau = -1\), and two asymmetric orbits of \(\tau = (-1, -1/2)\) are born. This means that one of the two classical scenarios of Rimmer bifurcation illustrated in figure 5(a) occurs in our case, as in figure 5(b). Moreover the asymmetric orbits bifurcates further from \(\tau \in (-1, -1/2)\) to \(\tau = -1/2\). Each bifurcation is a bifurcation of an ordered orbit and, thus, after each threshold the old and the new born orbits are also ordered.

The corresponding \((1, 2)\)-configurations are shown in figure 6. To the symmetric orbit of twist number \(\tau = -1\) corresponds the dark red \((1, 2)\)-configurations. One can see that such a configuration that lies in an interval \([x, x']\) having as ends two consecutive \((1, 2)\)-minimizers (the red points) is incomparable with the \((1, 2)\)-mini-maximizers (colored in blue and green).

We note that the existence of such unusual ordered \((p, q)\)-configurations is ensured by the property of the analyzed three harmonic standard map; namely, it is a two-component strong folding region map [16].

All properties deduced in this note are illustrated in the space \(X_{12}\) because it has small dimension and the stationary states can be easily visualized.
The space of (1, 2)-configurations is somewhat particular because if \( \mathbf{x} = (x_0, x_1) \) is a stationary configuration, then both semi-lines of the lines \( X_0 = x_0, X_1 = x_1 \) (in the system of coordinates \( (X_0, X_1) \)) starting from \( \mathbf{x} \) are boundaries for one of the order cones \( V_{+}^{q}(\mathbf{x}) \). For \( q > 2 \) this is not the case and only a part from a hyperplan \( X_i = x_i, i = 0, q - 1 \) is a boundary for an order cone.

More precisely, if \( \mathbf{x} = (x_0, x_1, \ldots, x_{q-1}) \) is a stationary state then the boundary of the cones \( V_{+}^{q}(\mathbf{x}) \) is the union of the sets:

\[
B_{i}(\pm) = \{ \mathbf{z} = (z_0, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots, z_{q-1}) \mid z_j - x_j \geq 0, \quad \text{for } V_{+}, z_j - x_j \leq 0 \text{ for } V_{-}, j \neq i \} \]

for \( i = 0, q - 1 \).

This remark is exploited in the following proposition, which gives a sufficient condition ensuring that a \((p, q)\)-periodic orbit is unordered.

**Proposition 2.4.** Let \( (x_i, y_i), i = 0, q - 1, \) be a \((p, q)\)-minimizing orbit of a positive twist map \( f \). If there is another \((p, q)\)-periodic orbit that intersects one of the vertical lines \( x = x_j, j = 0, q - 1 \) then that orbit is unordered.

**Proof.** Let \( \mathbf{x}' = (x'_0, x'_1, \ldots, x'_{q-1}) \) be a \((p, q)\)-minimizer in \( \mathcal{X}_{pq} \) with \( x'_j = x_j \), and \( \mathbf{z} = (z_0, z_1, \ldots, z_{q-1}) \), \( z_0 = x_j \) a stationary state associated to the second orbit. \( \mathbf{z} \) belongs to the hyperplane of equation \( X_j = x_j \), but by proposition 2.1 \( \mathbf{z} \not\in V_{+}(\mathbf{x}') \). Hence \( \mathbf{z} \) is unordered. \( \square \)

**Corollary 2.2.** If \( f \) is a multi-harmonic standard map having a \((p, q)\)-orbit that intersects the symmetry line \( \Gamma_0 \) or \( \Gamma_1 \) that is also intersected by the \((p, q)\)-minimizing orbit, then it is unordered.
3. Conclusions

Using the duality between an area preserving twist map of the infinite cylinder and the FK model, we have deduced the location of stationary \((p, q)\)-configurations in the space of all such configurations. We have exploited the strict monotone property of the semiflow of the action gradient in order to locate the regions where no equilibrium point of the gradient can lie. Moreover, we have pointed out through an example that a twist map can exhibit ordered periodic orbits whose corresponding stationary configurations are neither action minimizing nor mini-maximizing. As far as we are aware, no such orbit has yet been identified in the dynamics of a twist map.

The knowledge of forbidden subsets for stationary states, as well as the subsets where ordered or unordered configurations can lie, is useful in numerical search for zeros of the action gradient because the success of a numerical method for detecting zeros of a vector field depends on the appropriate choice of the initial condition. The existence of ordered stationary \((p, q)\)-configurations, whose associated periodic orbits have large absolute twist numbers can also be of physical interest from the point of view of the theory of commensurate–incommensurate phase transitions.

The numerical assessment of the Boyland and Hall criterion \([11]\) has led to the conclusion that it is not as successful as other converse KAM criteria because unordered periodic orbits of small periods were detected only for large values of the perturbation parameter, for which other methods ensured the breakup of all invariant circles of the standard map. In our theoretical and numerical study of uni-component and two-component strong folding region twist maps, started in \([16]\), it appears that just after the breakdown of an invariant circle, the nearby unordered orbits have a large, not small, period. Thus, one expects that this criterion can work well if we look for unordered periodic orbits of large period. In order to confirm and illustrate this behavior we need to know where in the space of \((p, q)\)-configurations we must look for unordered stationary configurations.

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