ON POISSON TRANSFORMS OF DIFFERENTIAL FORMS ON REAL HYPERBOLIC SPACES

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Abstract. This paper is concerned with the Poisson transform of differential forms on the hyperbolic space $H^n(\mathbb{R})$. Consider an integer $p$ such that $1 \leq p \leq n$ and let $q$ be either $p - 1$ or $p$. For $1 < r < \infty$, we prove that the Poisson transform is a topological isomorphism from the space of $L^r$-differential $q$-forms on the boundary $\partial H^n(\mathbb{R})$ onto a Hardy-type subspace of $p$-eigenforms of the Hodge-de Rham Laplacian on $H^n(\mathbb{R})$.

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1. Introduction

Let $G/K$ be a Riemannian symmetric space of non-compact type and let $G/P$ be its Furstenberg boundary. In [13], S. Helgason claimed that all eigenfunctions of $G$-invariant differential operators on $G/K$ are Poisson transforms of hyperfunctions on $G/P$. This conjecture was initially proved in [13] for the case where $G/K$ has rank one, and was later established in its entirety by Kashiwara et al. [19]. Since then, this conjecture has received significant attention across various settings (see, e.g., [2–5, 7, 17, 21, 22, 26, 29–31]).

This problem has been extended to include Poisson transforms for homogeneous vector bundles over $G/K$, (see, e.g., [9, 10, 18, 25, 33, 37]). In this paper, we will focus on studying the Poisson transform for the bundle of differential forms on the real hyperbolic space $H^n(\mathbb{R})$.

To elaborate further, we will suppose $G = \text{SO}_0(n, 1)$, $K = \text{SO}(n)$ and $H^n(\mathbb{R}) = G/K$. The boundary is realized as $\partial H^n(\mathbb{R}) = G/P$ where $P = MAN$ with $M = \text{SO}(n - 1)$, $A \simeq \mathbb{R}$ and $N \simeq \mathbb{R}^{n-1}$.

Date: November 11, 2024.
Keywords: Real hyperbolic space, Poisson transform, Eigenforms, Differential forms.
2010 AMS Classification: 58A10, 43A85, 53C35, 22E30
Let $0 \leq p \leq n$ be an integer, and consider the space $C^\infty(\bigwedge^p H^n(\mathbb{R}))$ of smooth differential $p$-forms on the hyperbolic space, i.e. smooth sections of the bundle $\bigwedge^p H^n(\mathbb{R}) := \bigwedge^p T^*_c H^n(\mathbb{R}))$.

Let $\mathcal{D}(\bigwedge^p H^n(\mathbb{R}))$ be the algebra of left-invariant differential operators, acting on the space $C^\infty(\bigwedge^p H^n(\mathbb{R}))$. To keep this introduction simple for the reader, we suppose $p$ generic, i.e. $1 \leq p < \frac{n-1}{2}$ (but we get similar results for general $p$). We exclude the case $p = 0$, which corresponds to the well-known case of functions. A result of Gaillard [11] states that in this case $\mathcal{D}(\bigwedge^p H^n(\mathbb{R}))$ is a commutative algebra generated by $dd^*$ and $d^*d$, where $d$ is exterior differentiation operator and $d^*$ is the co-differentiation operator. Let us further denote $\Delta = dd^* + d^*d$ the Hodge de-Rham operator. Therefore, any character $\chi$ of $\mathcal{D}(\bigwedge^p H^n(\mathbb{R}))$ can be only of type $\chi_{p,\lambda}(dd^*) = \lambda^2 + (\rho - p)^2$ or $\chi_{p-1,\lambda}(d^*d) = \lambda^2 + (\rho - p + 1)^2$, where $\rho = \frac{n-1}{2}$.

Let
\[ \mathcal{E}^p_{q,\lambda} = \left\{ F \in C^\infty(\bigwedge^p H^n(\mathbb{R})) \mid DF = \chi_{q,\lambda}(D)F, \ \forall D \in \mathcal{D}(\bigwedge^p H^n(\mathbb{R})) \right\}, \quad q = p, p-1 \ (1.1) \]

Note that $\mathcal{E}^p_{q,\lambda}$ (resp. $\mathcal{E}^p_{p-1,\lambda}$) represents the space of coclosed (resp. closed) eigenforms of $\Delta$ associated with the eigenvalue $\lambda^2 + (\rho - p)^2$ (resp. $\lambda^2 + (\rho - p + 1)^2$). Gaillard [9, 10] proved that for regular values of $\lambda$, there exist $G$-isomorphisms
\[ C_q^{-\omega} \xrightarrow{\sim} \mathcal{E}^p_{q,\lambda} \]
realized by Poisson transformations $\mathcal{P}^p_{q,\lambda}$, where $C_q^{-\omega} = C_q^- \left( \bigwedge^n T^*_c \partial H^n(\mathbb{R}) \right)$ denotes the space of hyperfunction vectors of an induced representation from $P = MAN$ to $G$, constructed using the natural representation of $M$ on $\bigwedge^n \mathbb{C}^{n-1}$, a suitable character of $A$, and the trivial representation of $N$ (see Theorem 3.1, Corollary 3.2 and Corollary 3.3).

The purpose of this paper is to study the Poisson transforms on the space $L^r(\bigwedge^n T^*_c \partial H^n(\mathbb{R}))$, for $1 < r < \infty$, and to precisely characterize its image. Let us now provide further details.

Let $\tau_p$ be the $p$-th exterior power of the coadjoint representation of $K$ acting on $V_{\tau_p} := \bigwedge^p (g^*_C/t_C)^* \simeq \bigwedge^p \mathbb{C}^n$, with the associated vector bundle $G \times_K V_{\tau_p}$. Since $\bigwedge^n H^n(\mathbb{R})$ is canonically isomorphic to $G \times_K V_{\tau_p}$, we identify $C^\infty(\bigwedge^n H^n(\mathbb{R}))$ with the space $C^\infty(G/K; \tau_p)$ of smooth $\tau_p$-equivariant functions $f : G \to V_{\tau_p}$.

With our assumption, $p$ is generic ($1 \leq p < \frac{n-1}{2}$), the representation $\tau_p$ is irreducible and the restriction of $\tau_p$ to $M$ decomposes as $\tau_p|_M = \sigma_{p-1} \oplus \sigma_p$, where the representation $\sigma_q$, for $q = p - 1, p$, is the $q$-th exterior power of the coadjoint representation of $M$ on $V_{\sigma_q} \simeq \bigwedge^q \mathbb{C}^{n-1}$. Denote by $\hat{M}(\tau_p)$ the set of irreducible components of $\tau_p|_M$. So in this case (generic $p$), $\hat{M}(\tau_p) = \{ \sigma_{p-1}, \sigma_p \}$ (see (2.3) for the general case, $1 \leq p \leq n$).

Denote by $a$ the Lie algebra of $A$. For $\lambda \in a^*_C$ and $\sigma \in \hat{M}(\tau)$, we consider the representation $\pi_{\sigma,\lambda}$ of $P = MAN$ acting on $V_{\sigma}$ by $\pi_{\sigma,\lambda}(man) = e^{(\rho - i\lambda)t} \sigma(m)$ where $m \in M$, $a_t \in A$, $n \in N$. Let $F_{\pi_{\sigma,\lambda}}$ be the associated homogeneous vector bundle over $G/P$, and let $C^{-\omega}(F_{\pi_{\sigma,\lambda}})$ be the space of its hyperfunctional sections, which will be identified with the space $C^{-\omega}(G/P; \pi_{\sigma,\lambda})$ of hyperfunctions $f : G \to V_{\sigma}$ satisfying $f(gma_n) = e^{i(\lambda - \rho)t} \sigma(m^{-1})f(g)$ for every $g \in G$, $m \in M$, $n \in N$ and $a_t \in A$. Let $C^{-\omega}(T^*_c \bigwedge^n \partial H^n(\mathbb{R}))$ be the space of $p$-forms with hyperfunction coefficients on $\partial H^n(\mathbb{R})$. Then we have $C^{-\omega}(T^*_c \bigwedge^n \partial H^n(\mathbb{R})) \simeq C^{-\omega}(K/M, \sigma)$, the space of hyperfunctions $F : K \to V_{\sigma}$ such that $F(km) = \sigma(m)^{-1}F(k)$ ($m \in M, k \in K$), see background for more details.

Now, using the fact that, as a $K$ module, $C^{-\omega}(G/P; \pi_{\sigma,\lambda})$ is isomorphic to the space $C^{-\omega}(K/M; \sigma)$, we define (in the compact picture) the Poisson transform of any hyperform
\[ f \in C^{-\omega}(K/M; \sigma) \]

\[
\mathcal{P}_{\sigma,\lambda}^\tau f(g) = \sqrt{\frac{\text{dim } \tau}{\text{dim } \sigma}} \int_{K} e^{-(i\lambda + \rho)H(g^{-1}k)\tau(\kappa(g^{-1}k))} \nu^\tau_\sigma(f(k)) \, dk, \quad g \in G,
\]

where \( \nu^\tau_\sigma \) is the natural embedding of \( V_\sigma \) into \( V_\tau \).

Taking the above identifications into account, we regard the algebra \( \mathbb{D}(\mathcal{L}^p H^n(\mathbb{R})) \) as the algebra \( \mathbb{D}(G/K; \tau) \) of \( G \)-invariant differential operators acting on the space \( C^\infty(G/K; \tau) \). Furthermore, we continue to treat \( dd^* \) and \( d^*d \) as the fundamental generators (for generic \( p \)).

The previously mentioned eigenspaces (1.1) are now (for \( p \) generic)

\[
\mathcal{E}_{\sigma,\lambda}(G/K; \tau_p) = \{ F \in C^\infty(G/K; \tau_p) : \Delta F = (\lambda^2 + (\rho - p)^2)F \text{ and } d^*F = 0 \},
\]

and

\[
\mathcal{E}_{\sigma,\lambda}(G/K; \tau_{p-1}) = \{ F \in C^\infty(G/K; \tau_{p-1}) : \Delta F = (\lambda^2 + (\rho - p + 1)^2)F \text{ and } dF = 0 \}.
\]

For a general integer \( 1 \leq p \leq n \), the Hodge \( \ast \)-operator implies that \( \tau_p \) and \( \tau_{n-p} \) are equivalent, allowing us to restrict our focus to \( 1 \leq p \leq \frac{n}{2} \). It is well known that \( \tau_p \) is irreducible if and only if \( p \neq \frac{n}{2} \). Moreover, when \( p = \frac{n}{2} \), \( \tau_{\frac{n}{2}} \) decomposes into two irreducible subrepresentations, \( \tau_{\frac{n}{2}} = \tau_{\frac{n}{2}}^+ + \tau_{\frac{n}{2}}^- \), where \( \tau_{\frac{n}{2}}^\pm \) correspond to the eigenspaces of the Hodge \( \ast \)-operator and its restriction to \( M \) is given by \( \tau_{\frac{n}{2}}^\pm |_M = \sigma_{\frac{n}{2}}^\pm \). Furthermore, for \( p = \frac{n-1}{2} \), we have

\[
\tau_{\frac{n-1}{2}} |_M = \sigma_{\frac{n-1}{2}-1}^+ \oplus \sigma_{\frac{n-1}{2}+1}^- \oplus \sigma_{\frac{n-1}{2}}.
\]

Based on this branching law, for pairs \((\tau, \sigma) = (\tau^\pm, \sigma^\pm)\) with \( q = p \) or \( p - 1 \), we define the Poisson transforms \( \mathcal{P}_{\sigma,\lambda}^\tau \) and the eigenspaces \( \mathcal{E}_{\sigma,\lambda}(G/K; \tau) \), in analogy with the generic case. For a full description, refer to Section 3.

For \( 1 \leq r < \infty \), let \( L^r(K/M; \sigma) \) be the subspace of \( L^r \)-hyperforms in \( C^{-\omega}(K/M; \sigma) \) and recall that the central goal of this paper is to characterize the image \( \mathcal{P}_{\sigma,\lambda}^\tau(L^r(K/M; \sigma)) \). To accomplish this, we introduce the space \( \mathcal{E}_{\sigma,\lambda}(G/K; \tau) \) of all \( F \) in \( \mathcal{E}_{\sigma,\lambda}(G/K; \tau) \) satisfying

\[
\|F\|_{r,\lambda} := \sup_{t > 0} e^{t(\rho - \Re(i\lambda))} \left( \int_{K} \|F(ka_t)\|^r_{V_\tau} \, dk \right)^{\frac{1}{r}} < \infty,
\]

where \( dk \) is the normalized Haar measure of \( K \). The main result is:

**Theorem** (See Theorem 6.1). Let \( \tau = \tau_1, \ldots, \tau_{\frac{n-1}{2}}, \tau_{\frac{n}{2}}^\pm \) and \( \sigma = \sigma_q \in \widehat{M}(\tau) \) accordingly.

Consider \( \lambda \in \mathbb{C} \) such that

\[
\begin{cases}
\Re(i\lambda) > 0, & \text{if } q = p \\
\Re(i\lambda) > 0, & \text{if } q = p - 1
\end{cases}
\]

where \( \rho = \frac{n-1}{2} \). Then the Poisson transform \( \mathcal{P}_{\sigma,\lambda}^\tau \) is a topological isomorphism from the space \( L^r(K/M; \sigma) \) onto the space \( \mathcal{E}_{\sigma,\lambda}(G/K; \tau) \). Moreover, for every \( f \in L^r(K/M; \sigma) \), we have

\[
|c_\sigma(\lambda, \tau)| \|f\|_{L^r(K/M; \sigma)} \leq \sqrt{\frac{\text{dim } \sigma}{\text{dim } \tau}} \|\mathcal{P}_{\sigma,\lambda}^\tau f\|_{r,\lambda} \leq \gamma_\lambda \|f\|_{L^r(K/M; \sigma)},
\]

where \( \gamma_\lambda \) is a positive constant and \( c_\sigma(\lambda, \tau) \) denotes the scalar component of the vector-valued Harish-Chandra c-function \( c(\lambda, \tau) \). Refer to Proposition 4.5 for its explicit expression.

As an immediate consequence of the main theorem, we obtain a characterization of closed harmonic \( p \)-forms via the Poisson transform, corresponding to the following description: generic \( p, q = p \) and \( i\lambda = \rho - p \) (see Corollary 6.2). Moreover, if we additionally assume \( p = 0, \)
we recover the well-known result that the Poisson transform is an isometric isomorphism from
$L^2(\mathcal{L}^p(\partial H^n(\mathbb{R})))$ onto a Hardy-harmonic space on $H^n(\mathbb{R})$ (see [32]).

The paper is organized as follows: Section 2 provides the necessary background, while
Section 3 introduces the Poisson transform on the space of differential forms on the boundary
$\partial H^n(\mathbb{R})$. In Section 4, we establish a Fatou-type theorem for Poisson integrals, a crucial tool
in obtaining the explicit formulas for the scalar component $c_\sigma(\lambda, \tau)$ that appear in the main
result. Section 5 makes fundamental use of this Fatou-type theorem to establish the main
result for $r = 2$, and also provides an $L^2$-inversion formula for the Poisson transform. Finally,
in Section 6, we synthesize all preceding results to prove the main theorem for all $1 < r < \infty$.

2. Background

2.1. The real hyperbolic space. Let $H^n(\mathbb{R})$ be the $n$-dimensional real hyperbolic space
$(n \geq 2)$ viewed as the rank one symmetric space of the noncompact type $G/K$ where $G = SO_0(n, 1)$
and $K = SO(n)$.

Let $\mathfrak{g} \simeq \mathfrak{so}(n, 1)$ and $\mathfrak{k} \simeq \mathfrak{so}(n)$ be the Lie algebras of $G$ and $K$ respectively and write
$\mathfrak{g} = \mathfrak{e} \oplus \mathfrak{p}$ for the Cartan decomposition of $\mathfrak{g}$. The tangent space $T_{o}(G/K) \simeq \mathfrak{p}$ of $G/K = H^n(\mathbb{R})$
at the origin $o = eK$ will be identified with the vector space $\mathbb{R}^n$.

There exists an element $H_0 \in \mathfrak{p}$ such that $a = RH_0$ is a Cartan subspace in $\mathfrak{p}$. Let
$A = \exp \mathfrak{a} = \{a_t = e^{tH_0}, \ t \in \mathbb{R}\}$ be the corresponding analytic Lie subgroup of $G$. We
define $\alpha \in \mathfrak{a}^*$ by $\alpha(H_0) = 1$. Then the positive restricted root subsystem is $\Sigma^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha\}$. We
will identify $\mathfrak{a}_\alpha^*$ and $\mathbb{C}$ via the map $\lambda \alpha \mapsto \lambda$. Then the half-sum of positive roots is
$\rho = (n - 1)\alpha/2 = (n - 1)/2$.

Let $n = \mathfrak{g}_\alpha \simeq \mathbb{R}^{n-1}$ be the positive root subspace and $N = \exp(\mathfrak{n})$ the corresponding
analytic subgroup of $G$. The groupe $G$ has an Iwasawa decomposition $G = KAN$. Thus, each
each $g \in G$ can be uniquely written as $g = k(g)e^{H(g)n(g)}$, where $k(g) \in K, H(g) \in \mathfrak{a}$ and
$n(g) \in N$.

Let $P = MAN$ be the standard minimal parabolic subgroup of $G$, where $M = SO(n - 1)$ is
the centralizer of $A$ in $K$. Then the boundary $\partial H^n(\mathbb{R})$ is realized as the flat space $\partial H^n(\mathbb{R}) =
G/P = K/M$.

2.2. Differential forms on $H^n(\mathbb{R})$ and $\partial H^n(\mathbb{R})$. Let $\langle \cdot, \cdot \rangle$ be the standard Euclidean scalar
product in $\mathbb{R}^n$. Let $(e_1, e_2, \ldots, e_n)$ be the standard orthonormal basis of $\mathbb{R}^n$ and denote
$(e_1^*, e_2^*, \ldots, e_n^*)$ its dual basis.

For an integer $p$ such that $0 \leq p \leq n$, let $\bigwedge^p(\mathbb{C}^n)^* = \bigwedge^p(\mathbb{R}^n)^* \otimes \mathbb{C}$ be the space of complex-valued alternating linear $p$-forms on $\mathbb{R}^n$.

For the reader’s convenience and to keep the notations simple, we will identify $(\mathbb{C}^n)^*$ with
$\mathbb{C}^n$ and $\bigwedge^p(\mathbb{C}^n)^*$ with $\bigwedge^p \mathbb{C}^n$.

We define an inner product $\langle \cdot, \cdot \rangle_{\bigwedge^p \mathbb{C}^n}$ on $\bigwedge^p \mathbb{C}^n$ as an extension of the one on $\mathbb{C}^n$ by setting
$$
\langle v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p \rangle_{\bigwedge^p \mathbb{C}^n} = \text{det}(\langle v_i, w_j \rangle)_{i,j}.
$$
(2.1)

It is easy to show that the basis of $\bigwedge^p \mathbb{C}^n$ consisting of the $p$-vectors $e_I := e_{i_1} \wedge \cdots \wedge e_{i_p}$, where
$I = \{i_1, \ldots, i_p\}$, with $1 \leq i_1 < \cdots < i_p \leq n$, is orthonormal basis of $\bigwedge^p \mathbb{C}^n$ with respect to
the inner product (2.1).

In this section, we choose and fix an integer $p$, such that $0 \leq p \leq n$. Let $\bigwedge^p H^n(\mathbb{R}) :=
\bigwedge^p T_{eK}H^n(\mathbb{R})$ be the $p$-th exterior power of the complexified cotangent bundle of $H^n(\mathbb{R})$. A
differential $p$-form on $H^n(\mathbb{R})$ is a section on $\bigwedge^p H^n(\mathbb{R})$. 

Let $\tau_p$ be the standard representation of $K$ on $V_{\tau_p} = \wedge^p \mathbb{C}^n$. Notice that $\tau_p$ is equivalent to the $p$-th exterior power of the coadjoint representation $Ad^*$ of $K$ on $\mathfrak{p}_C$. Let $G \times_K V_{\tau_p}$ be the $G$-homogeneous vector bundle associated to $\tau_p$. Since $T_{eK}H^n(\mathbb{R}) \simeq \mathfrak{p} \simeq \mathbb{R}^n$, then as $G$-homogeneous vector bundles we have $\wedge^p H^n(\mathbb{R}) \simeq G \times_K V_{\tau_p}$. As usual we shall identify the space $C^\infty(\wedge^p H^n(\mathbb{R}))$ of smooth functions $F : G \to V_{\tau_p}$ which are right $K$-covariant, i.e.,

$$F(gk) = \tau_p(k^{-1})F(g) \quad \text{for all } g \in G, k \in K.$$

(2.2)

It is known that the representation $\tau_p$ is irreducible unless $n$ is even and $p = \frac{n}{2}$ in which case it decomposes as $\tau_{n/2} = \tau_{n/2}^+ \oplus \tau_{n/2}^-$ with the corresponding decomposition of the representation space $\wedge^n_+ \mathbb{C}^n = \wedge^p_+ \mathbb{C}^n \oplus \wedge^p_- \mathbb{C}^n$, where $\wedge^p_\pm \mathbb{C}^n = \{ \alpha \in \wedge^p \mathbb{C}^n ; \star \alpha = \mu \pm \alpha \}$. In this case we have $\wedge^n_+ H^n(\mathbb{R}) = \wedge^p_+ H^n(\mathbb{R}) \oplus \wedge^p_- H^n(\mathbb{R})$ with $\wedge^p_+ H^n(\mathbb{R}) = G \times_K \wedge^p_- \mathbb{C}^n$. Here $\star$ is the Hodge operator and $\mu \pm = \pm 1$ if $\frac{n}{2}$ is even and $\mu \pm = \pm i$ if $\frac{n}{2}$ is odd. Notice that the Hodge operator induces the equivalence $\tau_p \sim \tau_{n-p}$, hence, hereafter we shall restrict our discussion to $0 \leq p \leq \frac{n}{2}$.

To distinguish between representations of $K$ and representations of $M$, we will use the Greek letter $\sigma$ to denote those representations of $M$. This notation will help clearly differentiate the representations associated with these two groups throughout our discussion, ensuring that the analysis remains precise and unambiguous.

For $0 \leq q \leq n - 1$, let $\sigma_q$ be the standard representation of $M$ on $V_{\sigma_q} = \wedge^q(\mathbb{C}\mathfrak{e}_2 \oplus \cdots \oplus \mathbb{C}\mathfrak{e}_n) = \wedge^q \mathbb{C}^{n-1}$, where $(e_j)_{j=1}^n$ is the natural basis of $\mathbb{C}^n$. Then the branching rules for $(K,M) = (\text{SO}(n),\text{SO}(n-1))$ is given as follows (see e.g. [1, 16, 27]):

**Lemma 2.1.**

1. If $p < \frac{n-1}{2}$, then $\tau_p|_M = \sigma_{p-1} \oplus \sigma_p$, with

   $$\wedge^p \mathbb{C}^n = e_1 \wedge \wedge^{p-1} \mathbb{C}^{n-1} \oplus \wedge^p \mathbb{C}^{n-1} \simeq \wedge^{p-1} \mathbb{C}^{n-1} \oplus \wedge^p \mathbb{C}^{n-1};$$

2. If $p = \frac{n-1}{2}$, then $\tau_p|_M = \sigma_{p-1} \oplus \sigma_p^\pm \oplus \sigma_p^\mp$, with

   $$\wedge^p \mathbb{C}^n = e_1 \wedge \wedge^{p-1} \mathbb{C}^{n-1} \oplus \wedge^p \mathbb{C}^{n-1} \simeq \wedge^{p-1} \mathbb{C}^{n-1} \oplus \wedge^p \mathbb{C}^{n-1} \oplus \wedge^p \mathbb{C}^{n-1};$$

3. If $p = \frac{n}{2}$, then $\tau_{\frac{n}{2}}|_M = \tau_{\frac{n}{2}-1}|_M \oplus \tau_{\frac{n}{2}}^\pm|_M = \sigma_{\frac{n}{2}-1} \oplus \sigma_{\frac{n}{2}}^\pm$, with

   $$\wedge^\frac{n}{2} \mathbb{C}^n = e_1 \wedge \wedge^{\frac{n}{2}-1} \mathbb{C}^{n-1} \oplus \wedge^\frac{n}{2} \mathbb{C}^{n-1} \simeq \wedge^{\frac{n}{2}-1} \mathbb{C}^{n-1} \oplus \wedge^\frac{n}{2} \mathbb{C}^{n-1}.$$

In particular, $\tau_{\frac{n}{2}}^\pm|_M = \sigma_{\frac{n}{2}}^\pm$.

Above, we used the tilde symbol in the direct sum $R_1 \tilde{\oplus} R_2$ to indicate that the representations $R_1$ and $R_2$ are equivalent.

We will refer to *generic case* for $1 \leq p \leq \frac{n}{2}$ with $p \neq \frac{n-1}{2}, \frac{n}{2}$ (we will not deal with the well-known case $p = 0$) and *special cases* for the cases where $n$ is odd and $p = \frac{n-1}{2}$ or $n$ even and $p = \frac{n}{2}$.

For $\tau = \tau_1, \cdots, \tau_{n-1}, \tau_{\frac{n}{2}}$, we denote by $\widehat{M}(\tau)$ the set of representations in $\widehat{M}$ that occur in the restriction of $\tau$ to $M$. According to the above branching rules we have,

$$\widehat{M}(\tau) = \begin{cases} 
\widehat{M}(\tau_p) = \{ \sigma_{p-1}^-, \sigma_p \} & \text{for } p < \frac{n-1}{2}, \\
\widehat{M}(\tau_p) = \{ \sigma_{p-1}^+, \sigma_p^-, \sigma_p^+ \} & \text{for } p = \frac{n-1}{2}, \\
\widehat{M}(\tau_p^+) = \{ \sigma_p \} & \text{for } p = \frac{n}{2}.
\end{cases}$$

(2.3)
For $1 \leq p \leq \frac{n}{2}$ and $\sigma_q \in \hat{M}(\tau_p)$, the pair of representations $(\tau_p, \sigma_q)$ will stand for
\[
(\tau_p, \sigma_q) = \begin{cases} 
(\tau_p, \sigma_p) & \text{if } p \text{ is generic} \\
(\tau_{\frac{n-1}{2}}, \sigma_{\frac{n+1}{2}}) & \text{if } p = \frac{n-1}{2} \\
(\tau^{\frac{1}{2}}, \sigma_{\frac{n}{2}}) & \text{if } p = \frac{n}{2}
\end{cases}
\]
while the pair $(\tau_p, \sigma_{p-1})$ represents
\[
(\tau_p, \sigma_{p-1}) = \begin{cases} 
(\tau_p, \sigma_{p-1}) & \text{if } p \text{ is generic} \\
(\tau_{\frac{n-1}{2}}, \sigma_{\frac{n-3}{2}}) & \text{if } p = \frac{n-1}{2}
\end{cases}
\]

Let $\sigma \in \hat{M}(\tau)$. Since $T_{eM}K/M = \mathfrak{t}/\mathfrak{m} \simeq \mathfrak{a}^\perp \simeq \mathbb{R}^{n-1}$ then the space $\bigwedge^q \partial H^n(\mathbb{R}) := \bigwedge^q \mathbb{T}^* \partial H^n(\mathbb{R})$ of differential $q$-forms on $\partial H^n(\mathbb{R})$ is canonically isomorphic to the homogenous bundle $K \times_M V_q = K \times_M \bigwedge^q \mathbb{C}^{n-1}$, with $q = p, p - 1$.

For $1 < r < \infty$, the space $L^r(\bigwedge^q \partial H^n(\mathbb{R}))$ of $L^r$ $q$-forms on $\partial H^n(\mathbb{R})$ will be identified with the space $L^r(K/M; \sigma)$ of vector valued functions $f : K \to V_\sigma$ satisfying the identity
\[
f(km) = \sigma(m)^{-1} f(k), \quad \text{for all } k \in K, \ m \in M
\]
and such that
\[
\| f \|_{L^r(K/M; \sigma)} = \left( \int_K \| f(k) \|^r \, dk \right)^{\frac{1}{r}} < \infty.
\]
Similarly, we will identify the space $C^{-\infty}(\bigwedge^q \partial H^n(\mathbb{R}))$ of $q$-hyperforms on $\partial H^n(\mathbb{R})$ with the space $C^{-\infty}(K/M; \sigma)$ of vector-valued hyperfunctions $f : K \to V_\sigma$ satisfying the identity (2.6).

### 3. Poisson transform on differential forms

We will now introduce the Poisson transform for differential forms on $\partial H^n(\mathbb{R})$. Let $\tau = \tau_1, \ldots, \tau_{\frac{n-1}{2}}, \tau^{\frac{1}{2}}, \tau^{\frac{n}{2}}$ and $\sigma \in \hat{M}(\tau)$. Let $\iota_\sigma : V_\sigma \to V_\tau$ the natural embedding. For any $\lambda \in \mathbb{C}$, the Poisson transform
\[
\mathcal{P}_{\sigma, \lambda}^r : C^{-\infty}(K/M; \sigma) \to C^\infty(G/K; \tau)
\]
is given by
\[
\mathcal{P}_{\sigma, \lambda}^r f(g) = d_{\tau, \sigma} \int_K dk e^{-(i\lambda + \rho)H(g^{-1}k)\tau(\kappa(g^{-1}k))} \iota_\sigma f(k),
\]
where
\[
d_{\tau, \sigma} = \sqrt{\frac{\dim \tau}{\dim \sigma}}.
\]

Notice that, for $\tau_p \in \{\tau_1, \ldots, \tau_{\frac{n-1}{2}}, \tau^{\frac{1}{2}}\}$ and $\sigma = \sigma_p$, the Poisson transform $\mathcal{P}_{\sigma_p, \lambda}^r$ is up to a constant the Poisson transform $\Phi_{\rho - i\lambda}^{r}$ investigated by Gaillard in [9].

To present the main result from [9], let us review the description of the algebra $\mathbb{D}(G/K; \tau_p)$ of $G$-invariant differential operators acting on differential forms on $H^n(\mathbb{R})$.

Let $d : C^\infty \bigwedge^p H^n(\mathbb{R}) \to C^\infty \bigwedge^{p+1} H^n(\mathbb{R})$ be the exterior differentiation operator, $d^* = (-1)^{n(p+1)+1} \ast d \ast$ is the co-differentiation operator, and $\Delta = dd^* + d^*d$ is the Hodge-de
Rham Laplacian. Here $\star$ is the Hodge $\star$-operator. Then it is established in [11, Theorem 3.1] and [28, Corollary 2.2] that

$$
\mathbb{D}(G/K; \tau) = \begin{cases}
\mathbb{C}[dd^*, d^*d] & \text{if } \tau = \tau_p, \ p < \frac{n-1}{2} \\
\mathbb{C}[dd^*, d] & \text{if } \tau = \tau_p, \ p = \frac{n-1}{2}, \\
\mathbb{C}[\Delta] & \text{if } \tau = \tau_p^\pm, \ p = \frac{n}{2}, \\
\mathbb{C}[\star, d \star d] & \text{if } \tau = \tau_p, \ p = \frac{n}{2},
\end{cases}
$$

In particular, $\mathbb{D}(G/K; \tau)$ is a commutative algebra except for $\tau = \tau_{n/2}$ (last case).

Above, we have identified the operators $\Delta$, $d^*$, $d$ and $\star$ with their counterpart under the identification $C^\infty(\wedge^n T^*(\mathbb{R})) \simeq C^\infty(G/K; \tau_p)$. This convention will be used throughout the paper.

Let $\tau_p \in \{\tau_1, \cdots, \tau_{n-1}, \tau_{n/2}\}$ and consider the component $\sigma_p$ of $\tau_p|_{K/M}$. For $\lambda \in \mathbb{C}$, we consider the following space of coclosed $p$-eigenforms of the Hodge-de Rham Laplacian:

$$
\mathcal{E}_{\sigma_p, \lambda}(G/K; \tau_p) = \{F \in C^\infty(G/K; \tau_p) : \Delta F = (\lambda^2 + (\rho - p)^2)F, \text{ and } d^*F = 0\}.
$$

Then Gaillard’s result may be stated as follows:

**Theorem 3.1** ([9, Theorem 2]). Let $\tau_p = \tau_1, \cdots, \tau_{n-1}, \tau_{n/2}$ and $\lambda \in \mathbb{C}$. The Poisson transform

$$
\mathcal{P}_{\sigma_p, \lambda}^\tau : C^{-\omega}(K/M; \sigma_p) \to \mathcal{E}_{\sigma_p, \lambda}(G/K; \tau_p)
$$

is a $G$-isomorphism if and only if

$$
i\lambda \notin \mathbb{Z}_{\leq 0} - \rho \cup \{p - \rho\}. \quad (3.3)
$$

A similar result was announced in [18, Theorem 4].

To get the analog to Theorem 3.1 for the pairs $(\tau_p, \sigma_{p-1})$ with $1 \leq p \leq \frac{n-1}{2}$, and $(\tau_p, \sigma_p^\pm)$ with $p = \frac{n-1}{2}$, we consider the following eigenspaces:

$$
\mathcal{E}_{\sigma_{p-1}, \lambda}(G/K; \tau_p) = \{F \in C^\infty(G/K; \tau_p) : \Delta F = (\lambda^2 + (\rho - p + 1)^2)F \text{ and } dF = 0\}, \ 1 \leq p \leq \frac{n-1}{2},
$$

and

$$
\mathcal{E}_{\sigma_p^\pm, \lambda}(G/K; \tau_p) = \{F \in C^\infty(G/K; \tau_p) : \star dF = \pm \rho^2 \lambda F \text{ and } d^*F = 0\}, \ p = \frac{n-1}{2}.
$$

**Corollary 3.2.** Let $\lambda \in \mathbb{C}$.

1. For $p \leq \frac{n-1}{2}$, the Poisson transform

$$
\mathcal{P}_{\sigma_{p-1}, \lambda}^\tau : C^{-\omega}(K/M; \sigma_{p-1}) \to \mathcal{E}_{\sigma_{p-1}, \lambda}(G/K; \tau_p)
$$

is a $G$-isomorphism if and only if

$$
i\lambda \notin \mathbb{Z}_{\leq 0} - \rho \cup \{\rho - p + 1\}. \quad (3.4)
$$

2. For $p = \frac{n-1}{2}$, the Poisson transform

$$
\mathcal{P}_{\sigma_p^\pm, \lambda}^\tau : C^{-\omega}(K/M; \sigma_p^\pm) \to \mathcal{E}_{\sigma_p^\pm, \lambda}(G/K; \tau_p)
$$

is a $G$-isomorphism if and only if

$$
i\lambda \notin \mathbb{Z}_{\leq 0} - \rho.
$$
Proof. Notice that the Hodge operator induces the equivalences \( \tau_p \sim \tau_{n-p} \) and \( \sigma_q \sim \sigma_{n-1-q} \).

(1) Using the identity
\[
\star (P_{\tau_p}^p f) = (-1)^p (p-1) P_{\tau_n-p}^n (\star f),
\]
as well as the relations
\[
d^\star = (-1)^{n+1-p^2} d
\]
\[
\Delta \star = \star \Delta,
\]
we easily see that the following diagram
\[
\begin{array}{ccc}
C^{-\omega}(K/M; \sigma_{n-p}) & \xrightarrow{P_{\tau_n-p}^n} & E_{\sigma_{n-p},\lambda}(G/K; \tau_{n-p}) \\
\downarrow \star & & \downarrow \star \\
C^{-\omega}(K/M; \sigma_p) & \xrightarrow{P_{\tau_p}^p} & E_{\sigma_p,\lambda}(G/K; \tau_p)
\end{array}
\]
is commutative and the first part of the corollary follows from Theorem 3.1.

(2) Let \( p = \frac{n-1}{2} \). Then \( \sigma_p = \sigma_p^+ \oplus \sigma_p^- \). By using [27, (4.24) and (4.38)] we have
\[
\text{Im} P_{\sigma_{\frac{n-1}{2}},\lambda} \subset E_{\sigma_{\frac{n-1}{2}},\lambda}(G/K; \tau_{\frac{n-1}{2}}).
\]
Since \( (\star d)^2 = (-1)^{\frac{n(n-1)}{2}} d^\star d \), it follows that \( E_{\sigma_p,\lambda}(G/K; \tau_p) = E_{\sigma_p^+,\lambda}(G/K; \tau_p) \oplus E_{\sigma_p^-,\lambda}(G/K; \tau_p) \).

This together with Theorem 3.1 give the desired result. \( \square \)

It remains to prove the analog of Theorem 3.1 for the pair \( (\tau_p^\pm, \sigma_p) \) with \( p = \frac{n}{2} \). Since we could not obtain this result directly from Gaillard’s theorem, we will instead turn to a more general result provided by Olbrich in [25]. Notice that we could have applied Olbrich’s result to the other cases, but we intentionally chose to remain within the framework of differential forms.

Let
\[
E_{\sigma_p,\lambda}(G/K; \tau_p^\pm) = \left\{ F^\pm \in C^\infty(G/K; \tau_p^\pm) : \Delta F^\pm = \left( \lambda^2 + \frac{1}{4} \right) F^\pm \right\}, \quad p = \frac{n}{2}.
\]

Then the corollary below follows from [25, Theorem 4.16].

Corollary 3.3. For \( p = \frac{n}{2} \), the Poisson transform
\[
P_{\sigma_p,\lambda}^\pm : C^{-\omega}(K/M; \sigma_p) \rightarrow E_{\sigma_p,\lambda}(G/K; \tau_p^\pm)
\]
is a \( G \)-isomorphism if and only if
\[
i\lambda \notin \mathbb{Z}_{\leq 0} - \rho \cup \left\{ -\frac{1}{2} \right\}.
\]

It follows from Theorem 3.1, Corollary 3.2 and Corollary 3.3 that, for \( 1 \leq p \leq \frac{n}{2} \) and \( q = p - 1, p \), the Poisson transform \( P_{\sigma_q,\lambda}^p \) maps \( L^r(K/M; \sigma_q) \) into \( E_{\sigma_q,\lambda}(G/K; \tau_p) \). We aim to precisely characterize the image of \( L^r(K/M; \sigma_q) \). This will be the focus of the following sections.
4. Fatou-type theorem and the Harish-Chandra c-function

The following fact about the Jacobi functions will be necessary for subsequent sections (see, e.g., [20]). The Jacobi function is defined as

\[ \phi^{(\alpha,\beta)}_\mu(t) = {}_2F_1 \left( \frac{i\mu + \alpha + \beta + 1}{2}, \frac{-i\mu + \alpha + \beta + 1}{2}; \alpha + 1; -\sinh^2 t \right), \]

with \( \Re(\alpha+1) > 0 \) and \( {}_2F_1 \) is the classical hypergeometric function. We shall need the following asymptotic behavior of Jacobi functions. The asymptotic behavior of \( \phi^{(\alpha,\beta)}_\mu(t) \) as \( t \to \infty \) is given by

\[ \phi^{(\alpha,\beta)}_\mu(t) = e^{(i\mu-\alpha-\beta-1)t}(c_{\alpha,\beta}(\mu) + o(1)) \quad \text{as} \quad t \to \infty \]

for \( \Re(i\mu) > 0 \), where

\[ c_{\alpha,\beta}(\mu) = \frac{2^{\alpha+\beta+1}-i\mu \Gamma(\alpha+1)\Gamma(i\mu)}{\Gamma \left( \frac{i\mu+\alpha+\beta+1}{2} \right) \Gamma \left( \frac{i\mu+\alpha-\beta+1}{2} \right)}. \]

Let \( \tau \in \{ \tau_1, \cdots, \tau_{q-p-1}, \tau_{\frac{q-p}{2}} \} \) and \( \sigma \in \hat{M}(\tau) \). We define the Hardy-type space \( \mathcal{E}_{\sigma,\lambda}^r(G/K; \tau) \), as the subspace of \( F \in \mathcal{E}_{\sigma,\lambda}^\infty(G/K; \tau) \) such that

\[ \|F\|_{r,\lambda} := \sup_{t>0} e^{(\rho-\Re(i\lambda))t} \left( \int_K \|F(ka_t)\|_{V_{rt}}^r \, dk \right)^{\frac{1}{r}} < \infty. \]

**Proposition 4.1.** Let \( \tau \in \{ \tau_1, \cdots, \tau_{q-p-1}, \tau_{\frac{q-p}{2}} \} \) and \( \sigma \in \hat{M}(\tau) \). For \( \lambda \in \mathbb{C} \) such that \( \Re(i\lambda) > 0 \), and for \( r > 1 \), the Poisson transform of every \( f \in L^r(K/M; \sigma) \) belongs to \( \mathcal{E}_{\sigma,\lambda}^r(G/K; \tau) \). More precisely,

\[ \|\mathcal{P}^\tau_{\sigma,\lambda} f\|_{r,\lambda} \leq \gamma_{\lambda} d_{r,\sigma} \|f\|_{L^r(K/M; \sigma)}, \]

for some positive constant \( \gamma_{\lambda} \), where \( d_{r,\sigma} \) as in (3.2).

**Proof.** We prove the inequality (4.5) for \( \tau = \tau_p \), with generic \( p \), and \( \sigma = \sigma_q \) with \( q = p, p-1 \). We will use the notation \( \mathcal{P}^\tau_{\sigma,\lambda} = \mathcal{P}^p_{\sigma,\lambda} \) and \( d_{r,\sigma} = d_{r,\sigma_q} \). The special cases \( p = \frac{q+1}{2} \) and \( p = \frac{q}{2} \) can be similarly established.

By (3.1) we have

\[ \|\mathcal{P}^p_{\sigma,\lambda} f(ka_t)\|_{\Lambda^p C^n} \leq d_{p,q} \int_K e^{-(\Re(i\lambda)+\rho)H(\alpha_i^{-1}k^{-1}h)} \| \tau_p(k(\alpha_i^{-1}k^{-1}h)) \|_{V_p(f(h))} \|_{\Lambda^p C^n} \, dh \]

\[ \leq d_{p,q} \int_K e^{-(\Re(i\lambda)+\rho)H(\alpha_i^{-1}k^{-1}h)} \| \nu^p_q(f(h))\|_{\Lambda^p C^n} \, dh, \]

where the last inequality follows from the unitarity of the representation \( \tau_p \). Since \( \nu^p_q \) is an isometric embedding, it follows that

\[ \|\mathcal{P}^p_{\sigma,\lambda} f(ka_t)\|_{\Lambda^p C^n} \leq d_{p,q} \int_K e^{-(\Re(i\lambda)+\rho)H(\alpha_i^{-1}k^{-1}h)} \| f(h)\|_{\Lambda^q C^{n-1}} \, dh \]

\[ = d_{p,q} e_{\lambda,t} \| f(\cdot) \|_{\Lambda^q C^{n-1}(k)}, \]
where $e_{\lambda,t}(g) := e^{-(\Re(i\lambda)+\rho)H(a_{i}^{-1}g^{-1})}$ and $*$ is the standard convolution over $K$. Therefore, for $r > 1$, Young’s inequality implies
\[
\left( \int_{K} \| P_{q,\lambda} f(k a_{t}) \|_{L^{p}(C^{n})} \frac{dk}{k} \right)^{1/r} \leq d_{p,q} \| e_{\lambda,t} \|_{L^{r}(K)} \| f \|_{L^{r}(K/M; \sigma_{q})}.
\]
Now,
\[
\| e_{\lambda,t} \|_{L^{r}(K)} = \int_{K} e^{-(\Re(i\lambda)+\rho)H(a_{i}^{-1}k^{-1})} \frac{dk}{k} = \phi^{(\Re(i\lambda)-\rho)t}_{\pm}(t),
\]
where $\phi^{(\alpha,\beta)}$ is the Jacobi function (4.1). Since $\Re(i\lambda) > 0$, the asymptotic behavior (4.2) gives
\[
\| e_{\lambda,t} \|_{L^{r}(K)} = e^{(\Re(i\lambda) - \rho)t} \left( c_{p-\frac{1}{2} - \frac{1}{2}}(-i\Re(i\lambda)) + o(1) \right) \text{ as } t \to \infty,
\]
where the constant $c_{p-\frac{1}{2} - \frac{1}{2}}(-i\Re(i\lambda))$ is given by (4.3). This finishes the proof of (4.5) for generic $p$.

Our next objective is to prove an analog of Proposition 4.1 where the inequality is reversed. We will establish a Fatou-type theorem for the Poisson transform to achieve this.

Let $\tilde{N} = \theta(N)$, where $\theta$ is the Cartan involution of $G$. For $\lambda \in \mathbb{C}$, the generalized Harish-Chandra $c$-function is defined as
\[
c(\lambda, \tau_{p}) = \int_{\tilde{N}} e^{-(i\lambda + \psi)H(\tilde{\eta})} \tau_{p}(\kappa(\tilde{\eta})) d\tilde{\eta} \in \End(V_{\tau_{p}}).
\]
Here $d\tilde{\eta}$ is the Haar measure on $\tilde{N}$ with the normalization
\[
\int_{\tilde{N}} e^{-2\psi(H(\tilde{\eta}))} d\tilde{\eta} = 1.
\]
The integral (4.6) converges for $\lambda$ such that $\Re(i\lambda) > 0$ and has a meromorphic continuation to $\mathbb{C}$ (see, e.g. [36]).

Let $\tau_{p} \in \{ \tau_{1}, \ldots, \tau_{\frac{n-1}{2}}, \tau_{\frac{n}{2}} \}$ and let $\sigma_{q} \in \tilde{M}(\tau_{p})$. Since the restriction $c(\lambda, \tau_{p})|_{V_{\sigma_{q}}}$ commutes with $\sigma_{q}$, Schur’s lemma implies that there exists a complex scalar $c_{\sigma_{q}}(\lambda, \tau_{p})$ such that $c(\lambda, \tau_{p})|_{V_{\sigma_{q}}} = c_{\sigma_{q}}(\lambda, \tau_{p}) \text{Id}_{V_{\sigma_{q}}}$. Therefore, for generic $p$, we have
\[
c(\lambda, \tau_{p}) = c_{\sigma_{q}}(\lambda, \tau_{p}) \text{Id}_{\Lambda_{-\rho}^{n-1}C^{n-1}} + c_{\sigma_{p}}(\lambda, \tau_{p}) \text{Id}_{\Lambda_{\rho}^{n+1}C^{n-1}}.
\]
For simplicity, we shall denote the scalar component $c_{\sigma_{q}}(\lambda, \tau_{p})$ by $c_{q}(\lambda, \tau_{p})$.

Similarly, for $p = \frac{n-1}{2}$ (when $n$ is odd), there exist three scalar coefficients $c_{\frac{n-1}{2}}(\lambda, \frac{n-1}{2}), c_{\frac{n-1}{2}}(\lambda, \frac{n-1}{2})$, and $c_{\frac{n-1}{2}}(\lambda, \frac{n-1}{2})$ such that
\[
c(\lambda, \tau_{\frac{n-1}{2}}) = c_{\frac{n-1}{2}}(\lambda, \frac{n-1}{2}) \text{Id}_{\Lambda_{-\rho}^{n-1}C^{n-1}} + c_{\frac{n-1}{2}}(\lambda, \frac{n-1}{2}) \text{Id}_{\Lambda_{\rho}^{n+1}C^{n-1}} + c_{\frac{n-1}{2}}(\lambda, \frac{n-1}{2}) \text{Id}_{\Lambda_{\rho}^{n+1}C^{n-1}}
\]
and for $p = \frac{n}{2}$ (when $n$ is even), we have
\[
c(\lambda, \tau_{\frac{n}{2}}) = c_{\frac{n}{2}}(\lambda, \frac{n}{2}) \text{Id}_{\Lambda_{\rho}^{n}C^{n}}.
\]
To simplify our notation, we will use the unified symbol $c_{q}(\lambda, \tau_{p})$ to represent all scalar components appearing in the identities (4.7), (4.8), and (4.9). Although the same symbol will
be employed, the value and interpretation of each constant will be distinguishable based on the specific context in which it is used.

For generic \( p \), explicit expressions of the scalars \( c_q(\lambda, p) \) are provided in [34] through a direct computation of the integral (4.6). Using a different approach, we will calculate the scalar components \( c_q(\lambda, p) \) in both generic and special cases (see Proposition 4.5 and Proposition 4.6).

The following lemma is needed for later use.

**Lemma 4.2.** Let \( \tau_p \in \{\tau_1, \ldots, \tau_{\frac{n-1}{2}}, \tau_{\frac{n+1}{2}}\} \) and \( \sigma_q \in \hat{M}(\tau_p) \).

1. For every \( v \in V_{\sigma_q} \),
   \[
   \|c(\lambda, \tau_p)\psi_q^p(v)\|_{V_{\tau_p}} = |c_q(\lambda, p)|\|v\|_{V_{\sigma_q}},
   \] (4.10)
2. For every linear operator \( L \) from a vector space \( V \) to \( V_{\sigma} \),
   \[
   \|c(\lambda, \tau_p)\psi_q^pL\|_{HS} = |c_q(\lambda, p)|\|L\|_{HS},
   \] (4.11)

where \( \|\cdot\|_{HS} \) is the Hilbert-Schmidt norm.

**Proof.** Follows immediately from (4.7), (4.8) and (4.9). \( \Box \)

**Proposition 4.3** (Fatou-type theorem). Let \( \tau_p \in \{\tau_1, \ldots, \tau_{\frac{n-1}{2}}, \tau_{\frac{n+1}{2}}\} \) and \( \sigma_q \in \hat{M}(\tau_p) \). Let \( \lambda \in \mathbb{C} \) such that \( \Re(\lambda) > 0 \). Then

\[
\lim_{t \to \infty} e^{(\rho - \lambda)t}\mathcal{P}_q^p f(k_\lambda t) = d_{p,q}c(\lambda, \tau_p)\psi_q^p(f(k)),
\]

(i) uniformly on \( K \) for \( f \in C^\infty(K/\mathbb{M}; \sigma_q) \),

(ii) in the \( L^r(K; \Lambda^p \mathbb{C}^n) \)-sense, for every \( f \in L^r(K/\mathbb{M}; \sigma_q) \).

**Proof.** Statement (i) has been proved previously; see, for instance, [33] and [37].

(ii) Assume \( f \in L^r(K/\mathbb{M}; \sigma_q) \) and \( \varepsilon > 0 \). By density, we can find a \( K \)-finite vector \( \varphi \) in \( C^\infty(K/\mathbb{M}; \sigma_q) \) such that \( \|f - \varphi\|_{L^r(K/\mathbb{M}; \sigma_q)} < \varepsilon \). Put \( \mathcal{P}_q^p f(k) = \mathcal{P}_q^p \varphi(k) \), then

\[
\|e^{-\lambda t}\mathcal{P}_q^p f(k) - d_{p,q}c(\lambda, \tau_p)\psi_q^p f(k)\|_{\Lambda^p \mathbb{C}^n} \leq \|e^{-\lambda t}\mathcal{P}_q^p (f - \varphi)(k)\|_{\Lambda^p \mathbb{C}^n} + \|e^{-\lambda t}\mathcal{P}_q^p (\varphi)(k) - d_{p,q}c(\lambda, \tau_p)\psi_q^p \varphi(k)\|_{\Lambda^p \mathbb{C}^n} + d_{p,q}^r\|c(\lambda, \tau_p)\psi_q^p \varphi(k) - c(\lambda, \tau_p)\psi_q^p f(k)\|_{\Lambda^p \mathbb{C}^n}.
\]

From Proposition 4.1 we obtain

\[
\int_{K} \|e^{-\lambda t}\mathcal{P}_q^p (f - \varphi)(k)\|_{\Lambda^p \mathbb{C}^n} \, dk \leq \int_{\Lambda^p \mathbb{C}^n} \|f - \varphi\|_{L^r(K/\mathbb{M}; \sigma_q)} \, d\mu_{\Lambda^p \mathbb{C}^n},
\]

and form part (i) above it follows that

\[
\lim_{t \to \infty} \int_{K} \|e^{-\lambda t}\mathcal{P}_q^p (\varphi)(k) - d_{p,q}c(\lambda, \tau_p)\psi_q^p \varphi(k)\|_{\Lambda^p \mathbb{C}^n} \, dk = 0.
\]

Further, according to (4.10) we obtain

\[
\int_{K} \|c(\lambda, \tau_p)\psi_q^p \varphi(k) - c(\lambda, \tau_p)\psi_q^p f(k)\|_{\Lambda^p \mathbb{C}^n} \, dk \leq |c_q(\lambda, p)|\|f - \varphi\|_{L^r(K/\mathbb{M}; \sigma_q)}.
\]

In conclusion, we have

\[
\lim_{t \to \infty} \int_{K} \|e^{-\lambda t}\mathcal{P}_q^p f(k) - d_{p,q}c(\lambda, \tau_p)\psi_q^p f(k)\|_{\Lambda^p \mathbb{C}^n} \, dk \leq \varepsilon^r d_{p,q}^r(c_q(\lambda, p))r,
\]
and this proves the desired statement.

The following inequalities are crucial.

**Proposition 4.4.** Let \( \tau_p \in \{ \tau_1, \cdots, \tau_{\frac{m+1}{2}}, \tau_{\frac{m+1}{2}}^\pm \} \) and \( \sigma_q \in \widehat{M}(\tau_p) \). For every \( \lambda \in \mathbb{C} \) such that \( \Re(\lambda) > 0 \), there exists a positive constant \( \gamma_\lambda \) such that for all \( f \in L^r(K/M; \sigma_q) \), 1 < \( r < \infty \), we have

\[
d_{p,q}^r c_q(\lambda, p) ||f||_{L^r(K/M; \sigma_q)} \leq ||P_{q,\lambda}^r f||_{r,\lambda} \leq d_{p,q} \gamma_\lambda ||f||_{L^r(K/M; \sigma_q)}. \tag{4.12}
\]

**Proof.** The right-hand side inequality is nothing but the estimate \((4.5)\). For the left-hand side inequality, by Proposition 4.3[(ii)], there exists a sequence \((t_j)\) with \( t_j \to \infty \) such that

\[
\lim_{j \to \infty} ||e^{(\rho - i\lambda)t_j}P_{q,\lambda}^r f(k_\tau t_j)||_{\Lambda^p \mathcal{C}^n} = ||d_{p,q} c(\lambda, \tau_p)tau_q^p(f(k))||_{\Lambda^p \mathcal{C}^n}
\]

almost everywhere in \( K \). Consequently, by the classical Fatou theorem and \((4.10)\) we get

\[
d_{p,q}^r c_q(\lambda, p) ||f||_{L^r(K/M; \sigma_q)} \leq ||P_{q,\lambda}^r f||_{r,\lambda}.
\]

\( \square \)

In the rest of this section, we will see how the asymptotic behavior formula given in Proposition 4.3 will allow us to give explicitly all the scalar components appearing in \((4.7)\), \((4.8)\), and \((4.9)\).

Let us recall that a continuous function \( F: G \to \text{End}(V_\tau) \) is called elementary \( \tau_p \)-spherical if \( F \) satisfies:

(i) \( F \) is a \( \tau_p \)-radial function, i.e., \( F(k_1 g k_2) = \tau_p(k_2)^{-1} F(g) \tau_p(k_1)^{-1} \), for all \( g \in G \) and \( k_1, k_2 \in K \).

(ii) For all \( v \in V_{\sigma_q} \), \( g \mapsto F(g)v \) is a joint-eigenfunction of all \( D \in \mathbb{D}(G/K; \tau_p) \) with \( F(e) = \text{Id} \).

It is known that a \( \tau_p \)-radial function \( F: G \to \text{End}(V_{\tau_p}) \) is determined by its restriction \( F|_\lambda \) to the subgroup \( A \) of \( G \). Since \( A \) and \( M \) commute, \( F|_\lambda \) becomes an \( M \)-morphism of \( V_\tau \). Furthermore, \( \tau_p|_M \) decomposes with multiplicity one, therefore by Schur’s lemma, \( F|_\lambda \) is scalar on each \( M \)-irreducible component \( V_{\sigma_q} \), for \( \sigma_q \in \widehat{M}(\tau_p) \). Thus

\[
F|_\lambda (a_t) = \sum_{\sigma_q \in \widehat{M}(\tau_p)} f_{\sigma_q}(t) \text{Id}_{V_{\sigma_q}},
\]

the coefficients \( f_{\sigma_q}(t) \) are called the scalar components of \( F \).

It is also known that any \( \tau_p \)-spherical function is given by the following Eisenstein integral by

\[
\Phi_{\sigma_q, \lambda}^p(g) = d_{p,q}^r \int_K e^{-((\lambda + \rho)H(g^{-1}k)\tau_p(\kappa(g^{-1}k))\tau_p^p(\pi^q_p(\tau_p(k)^{-1}))dk}, \tag{4.13}
\]

where \( \pi^q_p \) is the dual endomorphism of \( \tau^p_q \). By [28], the scalar components of \( \Phi_{\sigma_q, \lambda}^p \) are given in terms of the Jacobi function \( \phi^{(\alpha, \beta)}_\mu \) as follows:

(1) When \( p \) is generic:
(a) The scalar components $\varphi_{p-1,\lambda}, \varphi_{p,\lambda}$ of $\Phi^{\tau_p}_{\sigma_p,\lambda}$ are given by

$$\varphi_{p-1,\lambda}(t) = \phi_{\lambda}^{(\frac{p}{2},-\frac{1}{2})}(t), \quad (4.14)$$

$$\varphi_{p,\lambda}(t) = \frac{n}{n-p} \phi_{\lambda}^{(\frac{p}{2}-1,-\frac{1}{2})}(t) - \frac{p}{n-p} \cosh(t) \phi_{\lambda}^{(\frac{p}{2},-\frac{1}{2})}(t). \quad (4.15)$$

(b) The scalar components $\varphi_{p-1,\lambda}, \varphi_{p,\lambda}$ of $\Phi^{\tau_p}_{p-1,\lambda}$ are given by

$$\varphi_{p-1,\lambda}(t) = \frac{n}{n-p} \phi_{\lambda}^{(\frac{p}{2}-1,-\frac{1}{2})}(t) - \frac{n-p}{n-p} \cosh(t) \phi_{\lambda}^{(\frac{p}{2},-\frac{1}{2})}(t), \quad (4.16)$$

$$\varphi_{p,\lambda}(t) = \phi_{\lambda}^{(\frac{p}{2},-\frac{1}{2})}(t). \quad (4.17)$$

(2) In the special case $p = \frac{n+1}{2}$:

(a) The scalar components $\varphi_{p-1,\lambda}, \varphi_{p,\lambda}, \varphi_{p-1,\lambda}$ of $\Phi^{\tau_p}_{\sigma_p-1,\lambda}$ are given by

$$\varphi_{p-1,\lambda}(t) = \frac{2n}{n-1} \phi_{\lambda}^{(\frac{n}{2},-\frac{1}{2})}(t) - \frac{n-1}{n-1} \cosh(t) \phi_{\lambda}^{(n/2,-1/2)}(t) \pm \frac{i2\lambda}{n+1} \sinh(t) \phi_{\lambda}^{(\frac{n}{2},-\frac{1}{2})}(t), \quad (4.18)$$

$$\varphi_{p,\lambda}(t) = \phi_{\lambda}^{(\frac{n}{2},-\frac{1}{2})}(t). \quad (4.19)$$

(b) The scalar components $\varphi_{p-1,\lambda}, \varphi_{p,\lambda}, \varphi_{p-1,\lambda}$ of $\Phi^{\tau_p}_{\sigma_p^\pm,\lambda}$ are given by

$$\varphi_{p-1,\lambda}(t) = \phi_{\lambda}^{(\frac{n}{2},-\frac{1}{2})}(t), \quad (4.20)$$

$$\varphi_{p,\lambda}(t) = \frac{2n}{n+1} \phi_{\lambda}^{(\frac{n}{2}-1,-\frac{1}{2})}(t) - \frac{n-1}{n+1} \cosh(t) \phi_{\lambda}^{(n/2,-1/2)}(t) \pm \frac{i2\lambda}{n+1} \sinh(t) \phi_{\lambda}^{(\frac{n}{2},-\frac{1}{2})}(t), \quad (4.21)$$

$$\varphi_{p,\lambda}(t) = \phi_{\lambda}^{(\frac{n}{2},-\frac{1}{2})}(t) - \frac{n-1}{n+1} \cosh(t) \phi_{\lambda}^{(\frac{n}{2}-\frac{1}{2})}(t) \pm \frac{i2\lambda}{n+1} \sinh(t) \phi_{\lambda}^{(\frac{n}{2},-\frac{1}{2})}(t). \quad (4.22)$$

(3) In the special case $p = \frac{n}{2}$, the scalar component $\varphi_{p,\lambda}$ of $\Phi^{\tau_p}_{\sigma_p^\pm,\lambda}$ is given by

$$\varphi_{p,\lambda}(t) = \varphi_{p,\lambda}(t) = \cosh \left( \frac{t}{2} \right) \phi_{\lambda}^{(\frac{n}{2},-\frac{1}{2})}(t). \quad (4.23)$$

Below we will give explicitly the scalar components of the generalized Harish-Chandra c-function appearing in (4.7).

**Proposition 4.5.** Let $\lambda \in \mathbb{C}$ such that $\Re(i\lambda) > 0$. For generic $p$, the generalized Harish-Chandra c-function is given by

$$c(\lambda, \tau_p) = c_{p-1}(\lambda, p) \text{Id}_{\mathbb{A}^{p-1} \otimes \mathbb{C}^{n-1}} + c_p(\lambda, p) \text{Id}_{\mathbb{A}^p \otimes \mathbb{C}^{n-1}},$$

where the scalar coefficients are explicitly given by

$$c_{p-1}(\lambda, p) = \frac{i\lambda - \rho + p - 1}{i\lambda + p} c(\lambda), \quad c_p(\lambda, p) = \frac{i\lambda + \rho - p}{i\lambda + p} c(\lambda),$$

with

$$c(\lambda) = c_{\frac{n}{2}-1,\frac{1}{2}}(\lambda) = 2^{\rho - i\lambda} \frac{\Gamma(i\lambda) \Gamma \left( \frac{\rho + \frac{1}{2}}{2} \right)}{\Gamma \left( \frac{i\lambda + \rho + 1}{2} \right) \Gamma \left( \frac{i\lambda + \rho + 1}{2} \right)}.$$
Proof. Assume that $p$ is generic and $q = p - 1, p$. For $\lambda \in \mathbb{C}$ such that $\Re(i\lambda) > 0$, we have
\[
\Phi_{q,\lambda}^p(k\alpha_t) = d_{p,q}P_{q,\lambda}^p \left( \pi_p^q(\tau_p(k^{-1})) \right) (a_t).
\]
Proposition 4.3 implies
\[
\Phi_{q,\lambda}^p(a_t) = d_{p,q}^2 c(\lambda, \tau_p) e^{i(\lambda - \rho) t} \left( \pi_p^q + o(1) \right) \quad \text{as } t \to \infty,
\]
with
\[
d_{p,q}^2 = \begin{cases} 
\frac{n}{n-p} & \text{if } q = p, \\
\frac{n}{p} & \text{if } q = p - 1.
\end{cases}
\]
Let us first consider the case $q = p$. Using the asymptotic behavior (4.2) of Jacobi functions together with the relation
\[
c_{\frac{q}{2},-\frac{1}{2}}(\lambda) = \frac{2n}{i\lambda + p} c_{\frac{q}{2},-\frac{1}{2}}(\lambda),
\]
we obtain
\[
\varphi_{p,p}(\lambda, t) = \frac{1}{n - p} e^{i(\lambda - \rho) t} \left( n c_{\frac{q}{2},-\frac{1}{2}}(\lambda) - \frac{p}{2} c_{\frac{q}{2},-\frac{1}{2}}(\lambda) + o(1) \right) \quad \text{as } t \to \infty
\]
\[
= e^{i(\lambda - \rho) t} \frac{n}{n - p} c_{\frac{q}{2},-\frac{1}{2}}(\lambda) \left( \frac{i\lambda + \rho - p}{i\lambda + \rho} + o(1) \right) \quad \text{as } t \to \infty.
\]
Similarly, we get
\[
\varphi_{p,p-1}(\lambda, t) = e^{i(\lambda - \rho - 1) t} \left( c_{\frac{q}{2},-\frac{1}{2}}(\lambda) + o(1) \right) \quad \text{as } t \to \infty.
\]
Thus
\[
\Phi_{\sigma_p,\lambda}^p(a_t) = e^{i(\lambda - \rho - 1) t} \left( c_{\frac{q}{2},-\frac{1}{2}}(\lambda) + o(1) \right) \text{Id}_{\Lambda^p \mathbb{C}^{n-1}}
\]
\[
\quad + e^{i(\lambda - \rho) t} \frac{n}{n - p} c_{\frac{q}{2},-\frac{1}{2}}(\lambda) \left( \frac{i\lambda + \rho - p}{i\lambda + \rho} + o(1) \right) \text{Id}_{\Lambda^p \mathbb{C}^{n-1}},
\]
from which we deduce that
\[
\lim_{t \to \infty} e^{(\rho - i\lambda)t} \Phi_{\sigma_p,\lambda}^p(a_t) = \frac{n}{n - p} \left( \frac{i\lambda + \rho - p}{i\lambda + \rho} \right) c_{\frac{q}{2},-\frac{1}{2}}(\lambda) \text{Id}_{\Lambda^p \mathbb{C}^{n-1}}.
\]
Finally, by identification of (4.24) and (4.25) it follows that
\[
c_p(\lambda, p) = \frac{i\lambda + \rho - p}{i\lambda + \rho} c_{\frac{q}{2},-\frac{1}{2}}(\lambda) = \frac{i\lambda + \rho - p}{i\lambda + \rho} c(\lambda).
\]
Similarly, for $q = p - 1$ we can prove that
\[
\lim_{t \to \infty} e^{i(\lambda - \rho) t} \Phi_{\sigma_{p-1},\lambda}^p(a_t) = \frac{n}{n - p} \left( \frac{i\lambda - p + 1}{i\lambda + 1} \right) c_{\frac{q}{2},-\frac{1}{2}}(\lambda) \text{Id}_{\Lambda^{p-1} \mathbb{C}^{n-1}},
\]
from which we deduce that
\[
c_{p-1}(\lambda, p) = \frac{i\lambda - p + 1}{i\lambda + \rho} c_{\frac{q}{2},-\frac{1}{2}}(\lambda) = \frac{i\lambda - p + 1}{i\lambda + \rho} c(\lambda).
\]
\hfill \Box

The scalar components appearing in (4.8) and (4.9) are given below.

**Proposition 4.6.** Let $\lambda \in \mathbb{C}$ such that $\Re(i\lambda) > 0$. In the special cases, the generalized Harish-Chandra c-function is given as follows:
(1) For \( p = \frac{n-1}{2} \) (when \( n \) is odd), we have
\[
c(\lambda, \tau_{n \pm 1}) = c_{n-2}^{(1)}(\lambda, \frac{n-1}{2}) \text{Id}_{\Lambda_{n-2}^{\pm} \mathbb{C}^{n-1}} + \\
c_{n-1}^{(1)}(\lambda, \frac{n-1}{2}) \text{Id}_{\Lambda_{n-1}^{\pm} \mathbb{C}^{n-1}} + c_{n-1}^{(2)}(\lambda, \frac{n-1}{2}) \text{Id}_{\Lambda_{n-1}^{\pm} \mathbb{C}^{n-1}}
\]
where the scalar coefficients are explicitly given by
\[
c_{n-2}^{(1)}(\lambda, \frac{n-1}{2}) = \frac{2n}{n-1} \frac{i\lambda-1}{i\lambda+\rho} c(\lambda), \\
c_{n-1}^{(2)}(\lambda, \frac{n-1}{2}) = \frac{2n}{n+1} \frac{i\lambda}{i\lambda+\rho} c(\lambda),
\]
with
\[
c(\lambda) = c_{\frac{n}{2} - 1, \frac{n}{2}}(\lambda) = 2^{\rho-i\lambda} \frac{\Gamma(i\lambda)\Gamma(\frac{\rho}{2})}{\Gamma\left(\frac{i\lambda+i\rho}{2}\right)\Gamma\left(\frac{i\lambda+i\rho+1}{2}\right)}.
\]

(2) For \( p = \frac{n}{2} \) (when \( n \) is even), we have
\[
c(\lambda, \tau_{n \pm \frac{1}{2}}) = c_{n}^{(\pm)}(\lambda, \frac{n}{2}) \text{Id}_{\Lambda_{n}^{\pm} \mathbb{C}^{n}}.
\]
where the scalar coefficient is given by
\[
c_{n}^{(\pm)}(\lambda, \frac{n}{2}) = c_{n-1}^{(1)}(2\lambda) = 2^{n+1-2\lambda} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma(2i\lambda)}{\Gamma\left(\frac{2i\lambda+n+1}{2}\right)\Gamma\left(\frac{2i\lambda+1}{2}\right)}.
\]

Proof. The proof is similar to the generic case. \(\square\)

5. The \( L^2 \)-range of the Poisson transform

Recall that our primary objective is to describe the image of the space \( L^r(K/M; \sigma_q) \) under the Poisson transform \( \mathcal{P}_{q,\lambda}^\rho \) for \( 1 < r < \infty \). To achieve this, we will first examine the case when \( r = 2 \).

Let \( \tau_\rho \in \{\tau_1, \cdots, \tau_{n-1}, \tau_{n \pm \frac{1}{2}}\} \) and \( \sigma_q \in \hat{K}(\tau_\rho) \) with \( q = p, p-1 \). Let \( (\delta, V_\delta) \) be an element in \( \hat{K}(\sigma_q) \), where \( \hat{K}(\sigma_q) \subset \hat{K} \) denotes the subset of those representations that include \( \sigma_q \) when restricted to \( K \). By the branching law, \( \sigma_q \) occurs in the restriction \( \delta \mid_M \) with multiplicity one and therefore \( \dim \text{Hom}_M(V_\delta, V_{\sigma_q}) = 1 \). We choose the orthogonal projection \( P_\delta : V_\delta \rightarrow V_{\sigma_q} \) as a generator of \( \text{Hom}_M(V_\delta, V_{\sigma_q}) \).

Let \( (v_j)_{j=1}^{d_\delta} \) be an orthonormal basis for \( V_\delta \), where \( d_\delta = \dim V_\delta \). Then the functions
\[
k \mapsto \phi_j^\delta(k) = P_\delta(\delta(k^{-1})v_j), \quad 1 \leq j \leq d_\delta, \quad \delta \in \hat{K}(\sigma)
\]
define an orthogonal basis of the space \( L^2(K/M; \sigma_q) \), see, e.g. [35]. Thus, the Fourier expansion of any \( f \in L^2(K/M; \sigma_q) \) is given by
\[
f(k) = \sum_{\delta \in \hat{K}(\sigma_q)} \sum_{j=1}^{d_\delta} a_j^\delta \phi_j^\delta(k),
\]
with
\[
\| f \|_{L^2(K/M; \sigma_q)}^2 = \sum_{\delta \in \hat{K}(\sigma_q)} d_\delta \sum_{j=1}^{d_\delta} | a_j^\delta |^2.
\]
Theorem 5.1. Let $\tau = \tau_p \in \{\tau_1, \ldots, \tau_{n-1}, \tau_n^+\}$ and $\sigma = \sigma_q \in \widehat{\mathcal{M}}(\tau)$ with $q = p-1, p$. Assume $\lambda \in \mathbb{C}$ such that
\[
\left\{ \begin{array}{ll}
\Re(i\lambda) > 0 & \text{if } q = p, \\
\Re(i\lambda) > 0 \text{ and } i\lambda \neq \rho - p + 1 & \text{if } q = p-1.
\end{array} \right.
\]

The Poisson transform $\mathcal{P}^{\tau}_{\sigma,\lambda}$ is a topological isomorphism of the space $L^2(K/M; \sigma)$ onto the space $\mathcal{E}^2_{\sigma,\lambda}(G/K; \tau)$. Moreover, for every $f \in L^2(K/M; \sigma)$, there exists a positive constant $\gamma_\lambda$ such that
\[
|c_{\sigma}(\lambda, \tau)| \|f\|_{L^2(K/M; \sigma)} \leq \sqrt{\frac{\dim \sigma}{\dim \tau}} \|\mathcal{P}^{\tau}_{\sigma,\lambda}f\|_{L^2(K/M; \sigma)} \leq \gamma_\lambda \|f\|_{L^2(K/M; \sigma)}, \tag{5.2}
\]
where $|c_{\sigma}(\lambda, \tau)|$ are given explicitly in Proposition 4.5 and Proposition 4.6.

Proof. We will present the proof only when $p$ is generic; the special cases $p = (n - 1)/2$ and $p = n/2$ are analogous and are left to the reader.

Assume $p$ generic and $q = p-1$ or $q = p$. Recall the notation $\mathcal{P}^{\tau}_{q,\lambda} = \mathcal{P}^{\tau}_{\sigma_q,\lambda}$. From Proposition 4.4 it follows that the right-hand side of the estimate (5.2) holds and that $\mathcal{P}^{\tau}_{q,\lambda}$ is a an injective continuous map from $L^2(K/M; \sigma_q)$ into $\mathcal{E}^2_{\sigma_q,\lambda}(G/K; \tau_p)$.

On the other hand, for $F \in \mathcal{E}^2_{\sigma_q,\lambda}(G/K; \tau_p)$, by Theorem 3.1, Corollary 3.2 and Corollary 3.3, there exists a hyperform $f \in C^{\omega}(K/M; \sigma_q)$ such that $F = \mathcal{P}^{\tau}_{\sigma_q,\lambda}f$. We can write $F(k)$ as
\[
f(k) = \sum_{j=1}^{d_1} \sum_{\delta \in \hat{\mathcal{K}}(\sigma_q)} a^\delta_j P_\delta(\delta(k^{-1})v_j),
\]
then
\[
F(g) = d_{p,q} \sum_{\delta \in \hat{\mathcal{K}}(\sigma_q)} \sum_{j=1}^{d_1} a^\delta_j \int_K e^{-(i\lambda + \rho)H(g^{-1}k)} \tau_p(\kappa(g^{-1}k)) t^\rho_q P_\delta(\delta(k^{-1})v_j)dk,
\]
where $d_{p,q} = \sqrt{\frac{\dim \tau_p}{\dim \sigma_q}}$. Define $\Phi_{\lambda,\delta}$ by
\[
\Phi_{\lambda,\delta}(g)(v) = d_{p,q} \int_K e^{-(i\lambda + \rho)H(g^{-1}k)} \tau_p(\kappa(g^{-1}k)) t^\rho_q P_\delta(\delta(k^{-1})v_j)vk, \tag{5.3}
\]
for $g \in G$ and $v \in V_\delta$. Clearly $\Phi_{\lambda,\delta}(k_1gk_2) = \tau_p(k_2^{-1})\Phi_{\lambda,\delta}(g)\delta(k_1^{-1})$ for every $g \in G$ and $k_1, k_2 \in K$. Further
\[
\int_k \langle F(\kappa a_t), F(ka_t) \rangle_{L^p\mathcal{C}^n} dk = \sum_{\delta, \delta'} \sum_{j, \ell} a^\delta_j a^{\delta^\ell}_{j, \ell} \int_k \langle \Phi_{\lambda,\delta}(ka_t)v_j, \Phi_{\lambda,\delta'}(ka_t)v_{\ell} \rangle_{L^p\mathcal{C}^n} dk.
\]
By the covariance property and Schur’s lemma, we obtain
\[
\int_k \langle \Phi_{\lambda,\delta}(ka_t)v_j, \Phi_{\lambda,\delta'}(ka_t)v_{\ell} \rangle_{L^p\mathcal{C}^n} dk = \begin{cases} 
0 & \text{if } \delta' \sim \delta \\
\frac{1}{d_\delta} \text{tr}\left(\Phi_{\lambda,\delta}(a_t)^*\Phi_{\lambda,\delta}(a_t)\right) \langle v_j, v_{\ell} \rangle & \text{otherwise}
\end{cases}
\]
Thus
\[
\int_K \| F(ka_t) \|_{\mathcal{L}^p, C_n}^2 \, dk = \sum_{\delta \in \hat{K}(\sigma)} \frac{1}{d_\delta} \sum_{j=1}^{d_\delta} |a_\delta^j|^2 \text{tr} (\Phi_{\lambda, \delta}(a_t)^* \Phi_{\lambda, \delta}(a_t)),
\]
\[
= \sum_{\delta} \frac{1}{d_\delta} \| \Phi_{\lambda, \delta}(a_t) \|_{\text{HS}}^2 \sum_j |a_\delta^j|^2,
\]
where $\| \cdot \|_{\text{HS}}$ is the Hilbert-Schmidt norm. Hence, for a finite subset $\Omega \subset \hat{K}(\sigma_q)$ we get
\[
\sum_{\delta \in \Omega} \frac{1}{d_\delta} \sum_j |a_\delta^j e^{(\rho - i\lambda)t} \Phi_{\lambda, \delta}(a_t)|^2 \leq \sup_{t>0} e^{2(\rho - \Re(i\lambda))t} \int_K \| F(ka_t) \|_{\mathcal{L}^p, C_n}^2 \, dk,
\]
\[
= \| F \|_{2, \lambda}^2.
\]
Under the assumption $\Re(i\lambda) > 0$, we may use Proposition 4.3, i.e.,
\[
\lim_{t \to \infty} e^{(\rho - i\lambda)t} \Phi_{\lambda, \delta}(a_t) = d_{p, q} \mathbf{c}(\lambda, \tau_p) \delta^p \delta,
\]
and (4.11) to obtain
\[
d_{p, q}^2 |c_q(\lambda, p)|^2 \sum_{\delta \in \Omega} \frac{1}{d_\delta} \sum_j |a_\delta^j P_\delta|_{\text{HS}}^2 \leq \| F \|_{2, \lambda}^2.
\]
That is
\[
d_{p, q}^2 |c_q(\lambda, p)|^2 \sum_{\delta \in \Omega} \frac{1}{d_\delta} \sum_j d_{\sigma_q} |a_\delta^j|^2 \leq \| F \|_{2, \lambda}^2.
\]
Since the subset $\Omega \subset \hat{K}(\sigma_q)$ is arbitrary, it follows that
\[
d_{p, q}^2 |c_q(\lambda, p)|^2 \sum_{\delta \in \hat{K}(\sigma_q)} \frac{d_{\sigma_q}}{d_\delta} \sum_j |a_\delta^j|^2 \leq \| F \|_{2, \lambda}^2 < \infty.
\]
This shows that $f \in L^2(K/M; \sigma_q)$ with
\[
d_{p, q} |c_q(\lambda, p)| \| f \|_{L^2(K/M; \sigma_q)} \leq \| P_{\lambda, \delta} \|_{2, \lambda}.
\]

**Lemma 5.2.** For any $\lambda \in \mathbb{C}$ such that $\Re(i\lambda) > 0$ we have
\[
\sup_{t>0} e^{(\rho - \Re(i\lambda))t} \| \Phi_{\lambda, \delta}(a_t) \|_{\text{HS}} \leq \gamma_{\lambda} d_{p, q} \| P_\delta \|_{\text{HS}} = \gamma_{\lambda} d_{p, q} \sqrt{d_{\sigma_q}}.
\]

**Proof.** Let $v \in V_{\tau_p}$. By Proposition 4.1 we have
\[
\sup_{t>0} e^{(\rho - \Re(i\lambda))t} \left( \int_K \| P_{\lambda, \delta}(P_\delta(\delta^{-1}(\cdot)v))(ka_t) \|_{\mathcal{L}^p}^2 \, dk \right)^{1/2} \leq \gamma_{\lambda} d_{p, q} \| P_\delta(\delta^{-1}(\cdot)v) \|_{L^2(K/M; \sigma_q)}.
\]
Proof. Now the desired inequality follows from

\[
\int_K \| \mathcal{P}_{q, \lambda} (\delta^{-1} \cdot v) (ka_t) \|_{V_{p}}^2 \, dk = \int_K \langle \Phi_{\lambda, \delta} (a_t) \delta (k^{-1} v), \Phi_{\lambda, \delta} (a_t) \delta (k^{-1} v) \rangle_{V_{p}} \, dk,
\]

\[
= \frac{1}{d_{\delta}} \text{tr} (\Phi_{\lambda, \delta} (a_t)^* \Phi_{\lambda, \delta} (a_t)) \| v \|_{V_{\delta}}^2,
\]

\[
= \frac{1}{d_{\delta}} \| \Phi_{\lambda, \delta} (a_t) \|_{V_{H} S} \| v \|_{V_{\delta}}^2.
\]

Now the desired inequality follows from

\[
\| P_{\delta} (\delta^{-1} \cdot v) \|_{L^2 (K/M; \sigma_q)}^2 = \frac{d_{\sigma_q}}{d_{\delta}} \| v \|_{V_{\delta}}^2.
\]

\[\square\]

Lemma 5.3. For any \( \delta \in \tilde{K} (\sigma_q) \) and any \( \lambda \in \mathbb{C} \) such that \( \Re (i \lambda) > 0 \) we have

\[
\lim_{t \to \infty} e^{2 (\rho - \Re (i \lambda)) t} \| \Phi_{\lambda, \delta} (a_t) \|_{V_{H S}}^2 = d_{p, q}^2 | c_q (\lambda, p) |^2 d_{\sigma_q}.
\]

Proof. Recall that \( \Phi_{\lambda, \delta} (a_t) = \mathcal{P}_{q, \lambda} (\delta^{-1} \cdot v) (a_t) \). Then

\[
e^{2 (\rho - \Re (i \lambda)) t} \| \Phi_{\lambda, \delta} (a_t) \|_{V_{H S}}^2 = \sum_{j=1}^{d_{\delta}} \| e^{(\rho - \Re (i \lambda)) t} \Phi_{\lambda, \delta} (a_t) v_j \|_{V_{p}}^2,
\]

\[
= \sum_{j=1}^{d_{\delta}} \| e^{(\rho - \Re (i \lambda)) t} \mathcal{P}_{q, \lambda} (\delta^{-1} \cdot v_j) (a_t) \|_{V_{p}}^2.
\]

Using Proposition 4.3 and (4.11), we obtain

\[
\lim_{t \to \infty} e^{2 (\rho - \Re (i \lambda)) t} \| \Phi_{\lambda, \delta} (a_t) \|_{V_{H S}}^2 = d_{p, q}^2 \sum_{j=1}^{d_{\delta}} \langle c (\lambda, \tau_p) t_{q}^P P_{\delta} v_j, c (\lambda, \tau_p) t_{q}^P P_{\delta} v_j \rangle_{V_{p}}.
\]

\[
= d_{p, q}^2 | c_q (\lambda, p) |^2 d_{\sigma_q}.
\]

\[\square\]

Theorem 5.4 (Inversion formula). Let \( \tau_p \in \{ \tau_1, \ldots, \tau_{\frac{p-1}{2}}, \tau_{\frac{p+1}{2}} \} \) and \( \sigma_q \in \tilde{M} (\tau_p) \). Assume \( \lambda \in \mathbb{C} \) such that

\[
\begin{cases}
\Re (i \lambda) > 0 & \text{if } q = p, \\
\Re (i \lambda) > 0 \text{ and } i \lambda \neq p - 1 & \text{if } q = p - 1.
\end{cases}
\]

Let \( F \in \mathcal{E}_{\sigma_q}^2 (G/K; \tau_p) \) and let \( f \in L^2 (K/M; \sigma_q) \) be its boundary value. Then the following inversion formula holds in \( L^2 (K/M; \sigma_q) \)

\[
f (k) = d_{p, q}^{-1} | c_q (\lambda, p) |^{-2} \lim_{t \to \infty} e^{2 (\rho - \Re (i \lambda)) t} \pi_{q}^F \left( \int_K P_{p, \lambda} (ha_t, k) \ast F (ha_t) \, dh \right),
\]

where \( P_{p, \lambda} \) is the Poisson kernel given by \( P_{p, \lambda} (g, k) = e^{-(i \lambda + \rho) H (g^{-1} k)} = \Phi_{\lambda, \delta} (a_t) \).
Proof. Let \( F \in \mathcal{E}^2_{\sigma_q}(G/K; \tau_p) \). By Theorem 5.1, there exists a unique \( f \in L^2(K/M; \sigma_q) \) such that \( F = \mathcal{P}_{q,\lambda}^p f \). Write

\[
 f(k) = \sum_{\delta \in \mathcal{R}(\sigma_q)} \sum_{j=1}^{d_\delta} a_{\delta j}^p \delta(k^{-1})v_j.
\]

Then

\[
 F(k \alpha_t) = \sum_{\delta} \sum_{j} a_{\delta j}^p \Phi_{\lambda,\delta}(a_t) \delta(k^{-1})v_j,
\]

and therefore

\[
 \int_K \| F(k \alpha_t) \|^2_{V_{\tau_p}} \, dk = \sum_{\delta} \sum_{j} \frac{|a_{\delta j}^p|^2}{d_\delta} \| \Phi_{\lambda,\delta}(a_t) \|^2_{\mathcal{H}^2}.
\]

From Lemma 5.3 we deduce

\[
 \lim_{t \to \infty} e^{2(\rho - R(\lambda))t} \int_K \| \mathcal{P}_{q,\lambda}^p f(k \alpha_t) \|^2_{V_{\tau_p}} \, dk = d_{p,q}^2 |c_q(\lambda, p)|^2 \| f \|^2_{L^2(K/M; \sigma_q)},
\]

which implies

\[
 \lim_{t \to \infty} (g_t, \varphi)_{L^2(K/M; \sigma_q)} = (f, \varphi)_{L^2(K/M; \sigma_q)}, \quad \forall \varphi \in L^2(K/M; \sigma_q),
\]

where \( g_t \) is the \( V_{\sigma_q} \)-valued function defined by

\[
 g_t(k) = d_{p,q}^{-1} |c_q(\lambda, p)|^{-2} e^{2(\rho - R(\lambda))t} \pi_p^q \int_K \mathcal{P}_{p,\lambda}(h \alpha_t, k)^* F(h \alpha_t) \, dh.
\]

To obtain the inversion formula, it is only required to show that

\[
 \lim_{t \to \infty} \| g_t \|_{L^2(K/M; \sigma_q)} = \| f \|_{L^2(K/M; \sigma_q)}.
\]

To do so, let us first compute the Fourier coefficients \( c_{\delta j}^q(g_t) \) of \( g_t \):

\[
 c_{\delta j}^q(g_t) = \frac{d_\delta}{d_{\sigma_q}} \int_K \langle g_t(k), \mathcal{P}_\delta(k^{-1})v_j \rangle_{V_{\sigma_q}} \, dk
\]

\[
 = d_{p,q}^{-1} |c_q(\lambda, p)|^{-2} e^{2(\rho - R(\lambda))t}

\times \frac{d_\delta}{d_{\sigma_q}} \sum_{\delta' \ell} a_{\delta' \ell}^q \int_K \left\langle \pi_i^q \int_K \mathcal{P}_{p,\lambda}(h \alpha_t, k)^* \Phi_{\lambda,\delta'}(a_t) \delta'(h^{-1})v_\ell dh, P_\delta(k^{-1})v_j \right\rangle_{V_{\sigma_q}} \, dk.
\]

Since \( (\pi_i^q)^* = \pi_i^q \), we get

\[
 c_{\delta j}^q(g_t) = d_{p,q}^{-1} |c_q(\lambda, p)|^{-2} e^{2(\rho - R(\lambda))t}

\times \frac{d_\delta}{d_{\sigma_q}} \sum_{\delta' \ell} a_{\delta' \ell}^q \int_K \left\langle \Phi_{\lambda,\delta'}(a_t) \delta'(h^{-1})v_\ell, P_{p,\lambda}(h \alpha_t, k)^* \mathcal{P}_\delta(k^{-1})v_j \right\rangle_{V_{\tau_p}} \, dh \, dk.
\]
As \( \int_K P_{\lambda, \delta}(ha_t, k)Q^P P_\delta \delta(k^{-1}) \, dk = d^{-1}_{p, q} \Phi_{\lambda, \delta}(ha_t) \), we obtain
\[
\frac{c_j^\delta(g_t)}{c_j^\delta} = d^{-2}_{p, q} |c_q(\lambda, p)|^{-2} e^{2(\rho - R(i\lambda))t} \sum_{\delta, \ell} a_{\ell}^\delta \int_K \langle \Phi_{\lambda, \delta}(a_t) \delta'(h^{-1}) v_\ell, \Phi_{\lambda, \delta}(a_t) \delta'(h^{-1}) v_j \rangle V_{\rho, \ell} \, dh,
\]
\[
= d^{-2}_{p, q} |c_q(\lambda, p)|^{-2} e^{2(\rho - R(i\lambda))t} \sum_{\delta, \ell} a_{\ell}^\delta \int_K \langle \delta(h) \Phi_{\lambda, \delta}(a_t) \Phi_{\lambda, \delta}(a_t) \delta'(h^{-1}) v_\ell, v_j \rangle V_{\rho, \ell} \, dh.
\]
By the Schur lemma, we get
\[
\frac{c_j^\delta(g_t)}{c_j^\delta} = d^{-2}_{p, q} |c_q(\lambda, p)|^{-2} e^{2(\rho - R(i\lambda))t} \frac{d_{\sigma_q} \sum_{\ell} a_{\ell}^\delta}{d_{\sigma_q} a_{\ell}^\delta} \frac{1}{d_{\sigma_q}} \text{tr} (\Phi_{\lambda, \delta}(a_t)^* \Phi_{\lambda, \delta}(a_t)) \langle v_\ell, v_j \rangle V_q \, dh,
\]
\[
= d^{-2}_{p, q} |c_q(\lambda, p)|^{-2} e^{2(\rho - R(i\lambda))t} \frac{1}{d_{\sigma_q}} \| \Phi_{\lambda, \delta}(a_t) \|_{\text{HS}}^2.
\]
From all the above computations, we conclude that,
\[
\| g_t \|_{L^2(K/M, \sigma_q)}^2 = \left( e^{2(\rho - R(i\lambda))t} |d_{p, q} c_q(\lambda, p)|^{-2} \right) \sum_{\delta} \frac{d_{\sigma_q}}{d_{\delta}} \sum_{j} |a_j^\delta|^2 \| \Phi_{\lambda, \delta}(a_t) \|_{\text{HS}}^4,
\]
and by Lemma 5.3 we get
\[
\lim_{t \to \infty} \| g_t \|_{L^2(K/M, \sigma_q)}^2 = \sum_{\delta} \frac{d_{\sigma_q}}{d_{\delta}} \sum_{j} |a_j^\delta|^2 = \| f \|_{L^2(K/M, \sigma_q)}^2.
\]
To finish the proof, we have to justify that we can reverse \( \lim_{t \to \infty} \) and \( \sum_{\delta} \) by proving that the series
\[
\sum_{\delta \in K(\sigma_q)} \frac{1}{d_{\delta}} \sum_{j=1} |a_j^\delta|^2 \| \Phi_{\lambda, \delta}(a_t) \|_{\text{HS}}^4,
\]
is uniformly convergent. This follows easily from Lemma 5.2. 

\[\square\]

6. The \( L^r \)-range of the Poisson transform

In this section, we will use the \( L^2 \)-characterization established in Theorem 5.1 and the inversion formula proved in Theorem 5.4 to prove our main theorem :

**Theorem 6.1.** Let \( \tau = \tau_p \in \{ \tau_1, \ldots, \tau_{q-1}, \tau_{q+1} \} \) and \( \sigma = \sigma_q \in M(\tau) \) with \( q = p - 1, p \). Assume \( \lambda \in \mathbb{C} \) such that

\[
\begin{cases}
R(i\lambda) > 0 & \text{if } q = p, \\
R(i\lambda) > 0 \text{ and } i\lambda \neq \rho - p + 1 & \text{if } q = p - 1.
\end{cases}
\]

For \( 1 < r < \infty \), the Poisson transform \( P^r_{\sigma, \lambda} \) is a topological isomorphism mapping the space \( L^r(K/M; \sigma) \) onto \( E^r_{\sigma_q, \lambda}(G/K; \tau) \). Furthermore, there exists a positive constant \( \gamma_\lambda \) such that for every \( f \in L^r(K/M; \sigma) \),
\[
d_{\tau, \sigma} |c_{\sigma}(\lambda, \tau)| \| f \|_{L^r(K/M; \sigma)} \leq \| P^r_{\sigma, \lambda} f \|_{r, \lambda} \leq d_{\tau, \sigma} \gamma_\lambda \| f \|_{L^r(K/M; \sigma)},
\]
where \( |c_{\sigma}(\lambda, \tau)| \) are given explicitly in Proposition 4.5 and Proposition 4.6.
Proof. We will present the proof specifically for the generic case; similar arguments apply to the special cases, which we leave to the reader.

The necessary condition follows from Theorem 3.1, Corollary 3.2, Corollary 3.3 and Proposition 4.4. For the sufficiency condition, let \( F \in \mathcal{E}_{\sigma,\lambda}^p(G/K; \tau_p) \) and write \( F(g) = \sum_{i} F_i(g)u_i \) where \((u_i)_i\) is an orthonormal basis of \( \bigwedge^p \mathbb{C}^n \). Fix \((\chi_m)_m\) to be an approximation of the identity in \( C^\infty(K) \) and let \( F_{i,m}(g) = \int_K \chi_m(k)F_i(k^{-1}g)dk \). Then \((F_{i,m})_m\) converges point-wise to \( F_i \). Define \( F_m : G \to \bigwedge^p \mathbb{C}^n \) by \( F_m(g) = \sum_{i} F_{i,m}(g)u_i \). Then

\[
F_m(g) = \sum_{i} \left( \int_K \chi_m(k)F_i(k^{-1}g)dk \right) u_i,
\]

\[
= \int_K \chi_m(k) \sum_{i} F_i(k^{-1}g)u_i dk,
\]

\[
= \int_K \chi_m(k)F(k^{-1}g)dk.
\]

We have \( \|F_m(g) - F(g)\|_{\bigwedge^p \mathbb{C}^n}^2 \to 0 \) as \( m \to \infty \), then \( F_m \in \mathcal{E}_{\sigma,\lambda}(G/K; \tau_p) \) for every \( m \). Further,

\[
F_m(ka_t) = \int_K \chi_m(h)F(h^{-1}ka_t)dh,
\]

\[
= (\chi_m * F^t)(k),
\]

where \( F^t : K \to \bigwedge^p \mathbb{C}^n \) is defined for any \( t > 0 \) by \( F^t(g) = F(ga_t) \). Since

\[
\| (\chi_m * F^t)(k) \|_{\bigwedge^p \mathbb{C}^n} \leq \int_K |\chi_m(h)||F^t(h^{-1}k)||\bigwedge^p \mathbb{C}^n| dh,
\]

it follows

\[
\| F^t_m(k) \|_{\bigwedge^p \mathbb{C}^n} \leq \| \chi_m(\cdot) * F^t(\cdot) \|_{\bigwedge^p \mathbb{C}^n}(k).
\]

Therefore

\[
\| F^t_m \|_{L^r(K; \bigwedge^p \mathbb{C}^n)} \leq \| \chi_m(\cdot) * F^t(\cdot) \|_{\bigwedge^p \mathbb{C}^n} \|_{L^r(K)}.
\]

Applying Young’s involution inequality, we obtain

\[
\| F^t_m \|_{L^r(K; \bigwedge^p \mathbb{C}^n)} \leq \| \chi_m \|_{L^1(K)} \| F^t(\cdot) \|_{\bigwedge^p \mathbb{C}^n} \|_{L^r(K)} = \| F^t \|_{L^r(K; \bigwedge^p \mathbb{C}^n)},
\]

and

\[
\| F^t_m \|_{L^2(K; \bigwedge^p \mathbb{C}^n)} \leq \| \chi_m \|_{L^2(K)} \| F^t(\cdot) \|_{\bigwedge^p \mathbb{C}^n} \|_{L^2(K)} = \| \chi_m \|_{L^2(K)} \| F^t \|_{L^2(K; \bigwedge^p \mathbb{C}^n)}.
\]

The inequality \( 6.2 \) implies

\[
\sup_{t > 0} e^{(\rho - 2\Re(\lambda))t} \left( \int_K \| F_m(ka_t) \|_{\bigwedge^p \mathbb{C}^n}^2 \right)^{1/2} \leq \| \chi_m \|_{L^2(K)} \| F \|_{r,\lambda} < \infty.
\]

Hence, for each \( m \), \( F_m \in \mathcal{E}_{q,\lambda}^2(G/K; \tau_p) \) and from Theorem 5.1 it follows that there exists \( f_m \in L^2(K/M; \sigma_q) \) such that \( F_m = \mathcal{P}_{q,\lambda} f_m \). Now, we need to prove that \( f_m \in L^r(K/M; \sigma_q) \).
According to Theorem 5.4 we have, for any \( \varphi \in C^\infty(K/M; \sigma_q) \),
\[
\int_K \langle f_m(k), \varphi(k) \rangle_{L^p} \, dk = \lim_{t \to \infty} \int_K \langle g_m^t(k), \varphi(k) \rangle_{L^p} \, dk,
\]
where
\[
g_m^t(k) := d_{p,q}^{-2} |c_q(\lambda, p)|^{-2} e^{2(\rho - R(i\lambda))t} \pi_p(t) \int_K P_{p,\lambda}(ha_t, k)^* F_m(ha_t) \, dh.
\]
Further,
\[
\int_K \langle g_m^t(k), \varphi(k) \rangle_{L^p} \, dk
= d_{p,q}^{-2} |c_q(\lambda, p)|^{-2} e^{2(\rho - R(i\lambda))t} \int_K \langle \pi_p(t) \int_K P_{p,\lambda}(ha_t, k)^* F_m(ha_t) \, dh, \varphi(k) \rangle_{L^p} \, dk,
\]
\[
= d_{p,q}^{-2} |c_q(\lambda, p)|^{-2} e^{2(\rho - R(i\lambda))t} \int_K \langle F_m(ha_t), P_{p,\lambda}(ha_t, k)^* P_{p,\lambda}(ha_t, k) \rangle_{L^p} \, dh,
\]
\[
= d_{p,q}^{-3} |c_q(\lambda, p)|^{-2} e^{2(\rho - R(i\lambda))t} \int_K \langle F_m(ha_t), (P_{p,\lambda}^p)^* (ha_t) \rangle_{L^p} \, dh.
\]
It follows then that
\[
\left| \int_K \langle g_m^t(k), \varphi(k) \rangle_{L^p} \, dk \right| \leq d_{p,q}^{-3} |c_q(\lambda, p)|^{-2} e^{2(\rho - R(i\lambda))t} \int_K \left\| F_m(ha_t) \right\|_{L^p(K/M; \sigma_q)} \left\| (P_{p,\lambda}^p)^* (ha_t) \right\|_{L^p(K/M; \sigma_q)} \, dh.
\]
By Hölder’s inequality (with \( \frac{1}{r} + \frac{1}{s} = 1 \), we deduce
\[
\left| \int_K \langle g_m^t(k), \varphi(k) \rangle_{L^p} \, dk \right| \leq d_{p,q}^{-3} |c_q(\lambda, p)|^{-2} e^{2(\rho - R(i\lambda))t} \left\| F_m \right\|_{L^r(K/M; \sigma_q)} \left\| (P_{p,\lambda}^p)^* \right\|_{L^s(K/M; \sigma_q)},
\]
where \((P_{p,\lambda}^p)^* \)(k) = \((P_{p,\lambda}^p) \varphi \)(ha_t). Using (6.1) and Proposition 4.1 we get
\[
\left| \int_K \langle f_m(k), \varphi(k) \rangle_{L^p} \, dk \right| \leq \gamma \lambda d_{p,q}^{-2} |c_q(\lambda, p)|^{-2} \left\| F_m \right\|_{L^r(K/M; \sigma_q)} \left\| \varphi \right\|_{L^s(K/M; \sigma_q)},
\]
\[
\leq \gamma \lambda d_{p,q}^{-2} |c_q(\lambda, p)|^{-2} \left\| F_m \right\|_{L^r(K/M; \sigma_q)} \left\| \varphi \right\|_{L^s(K/M; \sigma_q)}.
\]
By taking the supremum over all \( \varphi \in C^\infty(K/M; \sigma_q) \) with \( \| \varphi \|_{L^s(K/M; \sigma_q)} = 1 \) we obtain
\[
\left\| f_m \right\|_{L^r(K/M; \sigma_q)} \leq \gamma \lambda d_{p,q}^{-2} |c_q(\lambda, p)|^{-2} \left\| F \right\|_{L^r(K/M; \sigma_q)},
\]
which implies \( f_m \), initially belongs to \( L^2(K/M; \sigma_q) \), is in fact in \( L^r(K/M; \sigma_q) \).

For every \( m \), define the linear form \( T_m \) on \( L^r(K/M; \sigma_q) \) by
\[
T_m(\varphi) = \int_K \langle f_m(k), \varphi(k) \rangle_{L^p} \, dk.
\]
Clearly, \( T_m \) is continuous and
\[
\left| T_m(\varphi) \right| \leq \gamma \lambda d_{p,q}^{-2} |c_q(\lambda, p)|^{-2} \left\| F \right\|_{L^r(K/M; \sigma_q)} \left\| \varphi \right\|_{L^s(K/M; \sigma_q)}.
\]
This shows that \((T_m)_m\) is uniformly bounded in \( L^s(K/M; \sigma_q) \), with
\[
\sup_m \| T_m \|_{op} \leq \gamma \lambda d_{p,q}^{-2} |c_q(\lambda, p)|^{-2} \left\| F \right\|_{L^r(K/M; \sigma_q)}.
\]
The Banach-Alaouglu-Bourbaki theorem will then ensures the existence of a subsequence of bounded operators \( (T_{m_j}) \) which converges to a bounded operator \( T \) under the weak-* topology, with
\[
\|T\|_{op} \leq \gamma d_{p,q}^{-2} c_q(\lambda, p)^{-2} \|F\|_{r,\lambda}.
\]
Thus, Riesz’s representation theorem guarantees the existence of a unique \( f \in L^r(K/M; \sigma_q) \) such that
\[
T(\varphi) = \int_K \langle \varphi(k), f(k) \rangle \Lambda^q C_n - 1 \, dk.
\]
We consider the test function \( \varphi_g(k) = P_{p,\lambda}(g,k)v \) with \( v \in \Lambda^p C^n \) and where \( P_{p,\lambda}(g,k) = e^{-(i\lambda + \rho)H(g^{-1}k)} \tau_p (\kappa(g^{-1}k)) \) is the Poisson kernel. Then
\[
T(\varphi_g) = \langle v, P_{q,\lambda}^p f(g) \rangle \Lambda^p C^n.
\]
On the other hand
\[
T_{m_j}(\varphi_g) = \langle v, P_{q,\lambda}^p f_{m_j}(g) \rangle \Lambda^p C^n = \langle v, F_{m_j}(g) \rangle \Lambda^p C^n.
\]
Taking the limit of the above identity when \( j \to \infty \) we conclude that \( F(g) = P_{q,\lambda}^p f(g) \) for every \( g \in G \).

As an immediate consequence of Theorem 6.1 we obtain the following characterization of co-closed harmonic \( p \)-forms on \( H^n(\mathbb{R}) \):

**Corollary 6.2.** Let \( p \) be an integer with \( 0 \leq p < (n-1)/2 \). For \( 1 < r < \infty \), the Poisson transform \( P_{p,i(p-r)} \) is a topological isomorphism from the space \( L^r(K/M; \sigma_p) \) onto the space \( \mathcal{E}_{p,i(p-r)}(G/K; \tau_p) \). Moreover, for every \( f \in L^r(K/M; \sigma_p) \) the following estimates hold,
\[
\frac{2(p-r)}{2p-r} c_p(\rho) \|f\|_{L^r(K/M; \sigma_p)} \leq \|P_{p,i(p-r)}^p f\|_{r,i(p-r)} \leq c_p(\rho) \|f\|_{L^r(K/M; \sigma_p)},
\]
where
\[
c_p(\rho) = d_{p,p} \frac{2^p \Gamma(p + \frac{1}{2}) \Gamma(p - r)}{\Gamma(p - \frac{r}{2}) \Gamma(p - \frac{r}{2} + \frac{1}{2})}.
\]
In the case where \( p = 0 \), we recover the classical fact that the Poisson transform is an isometric isomorphism from \( L^r(\partial H^n(\mathbb{R})) \) onto the Hardy-harmonic space on \( H^n(\mathbb{R}) \) (see [32]).

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