On Braid Words and Irreflexivity

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A left-distributive algebra is a set \( B \) equipped with a binary operation (here written as concatenation) such that \( a(bc) = (ab)(ac) \) for all \( a, b, c \in B \). The free left-distributive algebra on \( n \) letters is denoted \( A_n \), and we write \( A \) for \( A_1 \).

If \( P, Q \in B \) write \( P <_L Q \) iff one can write \( P \) as a strict prefix of \( Q \), i.e., \( Q = ((PQ_1) \ldots)Q_k \) for some \( Q_1, \ldots, Q_k, k \geq 1 \). Then a proof that \( <_L \) is irreflexive (that is, that \( P \neq ((PQ_1) \ldots)Q_k \) for all \( P, Q_1, \ldots, Q_k \in A \)) on \( A \) was found by R. Laver (see [Lav 92]), under large cardinal assumptions, as part of a theorem that \( A \) is isomorphic to a certain algebra of elementary embeddings from set theory.

It was also proved in [Lav 92] that \( <_L \) linearly orders \( A \), the part that for all \( P, Q \in A \) at least one of \( P <_L Q, P = Q, Q <_L P \) holds being proved independently and by a different method by P. Dehornoy ([Deh 89a, Deh 89b]). The linear ordering of \( A \) gives left cancellation, the solvability of the word problem, and other consequences. Left open was whether irreflexivity, and hence the linear ordering, can be proved in ZFC.

Recently, Dehornoy ([Deh 92]) has found such a proof, involving an extension of the infinite braid group but without invoking axioms extending ZFC. The purpose of this note is to prove the result without the additional machinery of this extended group, and at shorter length.

1 The \( \sigma_1 \)-proposition implies irreflexivity

We recall from [Deh 92] the connection between the braid group and \( A \).
Note that since $A$ is free, if for some left-distributive algebra $B$, $<_L$ is irreflexive on $B$, then $<_L$ is irreflexive on $A$. Dehornoy found such a $B$, where $B$ is a subset of the infinite dimensional braid group endowed with a bracket operation.

**Definition 1** The infinite braid group $\langle B_\infty, \varepsilon, \cdot \rangle$ is given by generators $\{\sigma_i : i \in \omega^+\}$ and relations $\sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i$ when $|i - j| > 1$ and $\sigma_i \cdot \sigma_j \cdot \sigma_i = \sigma_j \cdot \sigma_i \cdot \sigma_j$ when $|i - j| = 1$.

**Definition 2** $s$ is the endomorphism of $B_\infty$ which extends the shift map $s(\sigma_i) = \sigma_{i+1}$ (possible because $s$ preserves the defining relations of $B_\infty$).

**Definition 3** (The Dehornoy bracket) For $p, q \in B_\infty$ define a binary operation $[\cdot]$:

$$p[q] = p \cdot s(q) \cdot \sigma_1 \cdot s(p)^{-1}$$

Motivation: Suppose that $B$ is a left-distributive algebra with left cancellability (as $A$ turns out to be from the linearity of $<_L$). Then the braid group has a partial action on $B^\omega$ defined by, for $\vec{b} \in B^\omega$,

$$((\vec{b})\sigma_i)_j = \begin{cases} 
\vec{b}_j \vec{b}_{i+1} & \text{if } j = i \\
\vec{b}_i & \text{if } j = i + 1 \\
\vec{b}_j & \text{else.}
\end{cases}$$

with left cancellability making the partial actions of the inverses of braid generators well defined. Thus $(P, Q, R, S, \ldots)\sigma_2 = (PQR, Q, S, \ldots)$ for $P, Q, R, S, \ldots \in B$.

Then one is led to the Dehornoy bracket in the braid group by noting that, for $x, P, Q \in B$ and $p, q \in B_\infty$, if $(x, x, x, \ldots)p = (P, x, x, \ldots)$ and $(x, x, x, \ldots)q = (Q, x, x, \ldots)$ then

$$(x, x, x, \ldots)p[q] = (x, x, x, \ldots)p \cdot s(q) \cdot \sigma_1 \cdot s(p)^{-1}$$

$$(x, x, x, \ldots)p[q] = (P, x, x, \ldots)s(q) \cdot \sigma_1 \cdot s(p)^{-1}$$

$$(x, x, x, \ldots)p[q] = (P, Q, x, x, \ldots)\sigma_1 \cdot s(p)^{-1}$$

$$(x, x, x, \ldots)p[q] = (PQ, P, x, x, \ldots)s(p)^{-1}$$

$$(x, x, x, \ldots)p[q] = (PQ, x, x, \ldots)$$
which suggests, in order that \((x,x,x,\ldots)p[q] = (PQ,x,x,\ldots)\), the above definition of \(p[q]\).

Dehornoy proved as a corollary of his irreflexivity result the following theorem:

**Theorem 4 (Dehornoy—the \(\sigma_n\)-proposition)** If \(p \in B_\infty\) is written as a product of generators and their inverses, including at least one \(\sigma_n\) and no \(\sigma_{-1}^n\), then \(p \neq \varepsilon\).

He also showed that a direct proof of theorem 4 would give a relatively short proof of irreflexivity, namely:

**Theorem 5 (Dehornoy)**

1. The Dehornoy bracket \([\_]\) on the braid group \(B_\infty\) is left distributive;

2. If the \(\sigma_1\)-proposition holds then irreflexivity holds in \(B_\infty\) and so also in \(\mathcal{A}\); furthermore, the closure under the Dehornoy bracket of any element of the braid group is isomorphic to \(\mathcal{A}\).

Proof of 5.1: Compute

\[ p[q[r]] = p \cdot s(q) \cdot s^2(r) \cdot \sigma_2 \cdot \sigma_1 \cdot s^2(q)^{-1} \cdot s(p)^{-1} = p[q][p[r]] \]

to get left-distributivity.

Proof of 5.2: Suppose \(p,q_1,\ldots,q_k \in B_\infty\) satisfy \(p = p[q_1] \ldots [q_k]\). The right hand expands to \(p \cdot (r_1 \cdot \sigma_1 \cdot s_1) \cdot (r_2 \cdot \sigma_1 \cdot s_2) \cdot \ldots \cdot (r_k \cdot \sigma_1 \cdot s_k)\) with \(r_i, s_i \in s(B_\infty)\). Then multiplying by \(p^{-1}\) one obtains a contradiction to the \(\sigma_1\)-proposition. Therefore irreflexivity holds in \(B_\infty\), and hence in \(\mathcal{A}\).

Let \(x\) be the generator of the free left-distributive algebra \(\mathcal{A}\). For \(r \in B_\infty\) define the homomorphism \(\chi^r : \mathcal{A} \to \langle B_\infty, [\_] \rangle\) inductively by \(\chi^r_x = r\) and \(\chi^{pq} = \chi^q[P]\). This is well defined as \([\_]\) is left-distributive and \(\mathcal{A}\) is free. To show that \(\chi^r\) is injective, if \(P \neq Q\) in \(\mathcal{A}\), by trichotomy suppose \(P <_L Q\). Conclude \(\chi^r_P <_L \chi^r_Q\), and so by irreflexivity in \(B_\infty\) that \(\chi^r_P \neq \chi^r_Q\). Hence \(\langle r, [\_] \rangle\) is isomorphic to \(\mathcal{A}\).
2 Proof of the $\sigma_1$-proposition

We will prove the $\sigma_1$-proposition first, using the action of the braid group $B_\infty$ as automorphisms of the free group on countably many generators, getting as quickly as possible to the minimum needed for irreflexivity. Subsequently, we will prove the full $\sigma_n$-proposition, for which we use a somewhat non-standard action, and have to deal with a greater number of cases.

**Definition 6** $\langle F_G, \varepsilon, \cdot \rangle$ is the free group on generators $G = \{g_i : i \in \omega\}$.

**Definition 7**

$$
\begin{align*}
(g_i)\sigma_i &= g_i \cdot g_{i+1} \cdot g_i^{-1} \\
(g_{i+1})\sigma_i &= g_i \\
(g_j)\sigma_i &= g_j \text{ if } j \neq i, i + 1.
\end{align*}
$$

The following is well known, see for example [Bir 75, cor. 1.8.3, pg. 25] (although our group has an extra generator $g_0$ for convenience later, and we do not require faithfulness).

**Lemma 8** The action of the $\sigma_i$s on the $g_j$s extends to a faithful action of $B_\infty$ on $F_G$.

We record for convenience the action of $\sigma_i^{-1}$:

$$
\begin{align*}
(g_i)\sigma_i^{-1} &= g_{i+1} \\
(g_{i+1})\sigma_i^{-1} &= g_{i+1}^{-1} \cdot g_i \cdot g_{i+1} \\
(g_j)\sigma_i^{-1} &= g_j \text{ if } j \neq i, i + 1.
\end{align*}
$$

In passing we observe without proof that if $\xi$ is the antiautomorphism of $B_\infty$ obtained by reversing products of generators and their inverses (so $\xi(\sigma_1 \cdot \sigma_2^{-1}) = \sigma_2^{-1} \cdot \sigma_1$), then

$$(g_1, g_2, \ldots)\xi(p) = ((g_1)p, (g_2)p, \ldots),$$

where the left is the action previously defined of a braid on an element of $F_G$, where $F_G$ is equipped with the left distributive operation of conjugation $f_1 f_2 = f_1 \cdot f_2 \cdot f_1^{-1}$, and the right are the actions of braids on elements of $F_G$ just defined in 8.

The main property about this action that we need is the following observation:
Lemma 9 If a reduced word $f$ in $F_G$ begins with $g_1$, and $\sigma \in B_\infty$ is a generator or the inverse of a generator, but not $\sigma^{-1}$, then the reduced form of $(f)\sigma$ also begins with $g_1$.

⊢ Assume that the action of $\sigma$ is applied to each generator or its inverse in the reduced form of $f$, and then a fixed reduction is applied to the term that results to produce the reduced form of $(f)\sigma$.

There are two cases where the reduced form of $(f)\sigma$ may fail to begin with $g_1$:

Case 9.1 $\sigma = \sigma_i^{\pm 1}$ with $i > 1$.

In this case, all $g_i^{\pm 1}$'s are unchanged by the action of $\sigma$ on $f$, and none are produced. Therefore the only way that the leading $g_1$ could be cancelled in the reduction of $(f)\sigma$ is by a $g_1^{-1}$ already present in the reduced form of $f$.

Let $f = g_1 \cdot f_1 \cdot g_1^{-1} \cdot f_2$, in reduced form, display that instance of $g_1^{-1}$. Then $(f)\sigma = g_1 \cdot (f_1)\sigma \cdot g_1^{-1} \cdot (f_2)\sigma$, so $(f_1)\sigma = \varepsilon$. But then $f_1 = \varepsilon$, so that $f$ was not in reduced form, contradiction.

Case 9.2 $\sigma = \sigma_1$.

In this case, $g_1^{-1}$'s are produced by $(g_1^{\pm 1})\sigma_1 = g_1 \cdot g_2^{\pm 1} \cdot g_1^{-1}$ and by $(g_2^{-1})\sigma_1 = g_1^{-1}$.

There are two possibilities for the leading $g_1$ produced by the action of $\sigma_1$ on $f$ to be cancelled in the reduction:

SubCase 9.2.1 The leading $g_1$ in the unreduced form of $(f)\sigma_1$ is cancelled by a $g_1^{-1}$ produced by the action of $\sigma_1$ on a $g_1^{\pm 1}$.

First we note that the $g_1^{\pm 1}$ occurrence of $f$ which produces the $g_1^{-1}$ which cancels the leading $g_1$ of $(f)\sigma_1$ is distinct from the leading $g_1$ of $f$.

As above, let $f = g_1 \cdot f_1 \cdot g_1^{\pm 1} \cdot f_2$, in reduced form, display that instance of $g_1^{\pm 1}$. Then $(f)\sigma_1 = g_1 \cdot g_2 \cdot g_1^{-1} \cdot (f_1)\sigma_1 \cdot g_1 \cdot g_2^{\pm 1} \cdot g_1^{-1} \cdot (f_2)\sigma_1$, so $g_2 \cdot g_1^{-1} \cdot (f_1)\sigma_1 \cdot g_1 \cdot g_2^{\pm 1} = \varepsilon$. But then $f_1 = g_1^{-1} \cdot g_2^{\pm 1}$, so that $f$ was not in reduced form, contradiction.

SubCase 9.2.2 The leading $g_1$ in the unreduced form of $(f)\sigma_1$ is cancelled by a $g_1^{-1}$ produced by the action of $\sigma_1$ on a $g_2^{-1}$.
Again, let \( f = g_1 \cdot f_1 \cdot g_2^{-1} \cdot f_2 \), in reduced form, display that instance of \( g_2^{-1} \). Then \((f)\sigma_1 = g_1 \cdot g_2 \cdot g_1^{-1} \cdot (f_1)\sigma_1 \cdot g_1^{-1} \cdot (f_2)\sigma_1 \), so \( g_2 \cdot g_1^{-1} \cdot (f_1)\sigma_1 = \varepsilon \). But then \( f_1 = g_1^{-1} \cdot g_2 \), so that \( f \) was not in reduced form, contradiction. \( \dashv \)

To prove the \( \sigma_1 \)-proposition, assume that \( p \in B_\infty \) is formed from a product of generators of \( B_\infty \) and their inverses, with at least one \( \sigma_1 \) and no \( \sigma_1^{-1} \). Write \( p = p_1 \cdot \sigma_1 \cdot p_2 \) where \( \sigma_1^\pm 1 \) doesn’t occur in \( p_1 \) and \( \sigma_1^{-1} \) doesn’t occur in \( p_2 \).

Then 
\[
(g_1^{-1}) p = (g_1^{-1}) p_1 \cdot \sigma_1 \cdot p_2 = (g_1^{-1}) \sigma_1 \cdot p_2 = (g_1 \cdot g_2^{-1} \cdot g_1^{-1}) p_2
\]

Since \( g_1 \cdot g_2^{-1} \cdot g_1^{-1} \) is in reduced form and begins with \( g_1 \), and since \( \sigma_1^{-1} \) does not appear in \( p_2 \), then by repeated application of lemma 9 it follows that the reduced form of \((g_1 \cdot g_2^{-1} \cdot g_1^{-1}) p_2 \), and hence of \((g_1^{-1}) p \), must begin with \( g_1 \). Therefore \((g_1^{-1}) p \neq g_1^{-1} \), and so \( p \neq \varepsilon \). \( \dashv \)

### 3 Proof of the \( \sigma_n \)-proposition

We now prove the full \( \sigma_n \) proposition. It can be done by the braid action defined above, using as an invariant that the reduced form begins with \( \prod_{i=0}^n g_i \cdot g_m^\pm 1 \) for some \( m > n \), and showing that words retains that property when acted upon by a braid generator or its inverse, except for \( \sigma_n^{-1} \), obtaining the desired result by starting with \( \prod_{i=0}^n g_i \).

We make use of a slightly different group action which decreases the number of cases to check. The map \( \phi : \mathcal{F}_X \rightarrow \mathcal{F}_G \) given by \((x_i)\phi = \prod_{i=0}^n g_n\), with domain the free group \( \langle \mathcal{F}_X, \varepsilon \cdot \rangle \) on generators \( X = \{x_i : i \in \omega\} \), extends to an isomorphism with inverse \((g_0)\phi^{-1} = x_0\), \((g_{i+1})\phi^{-1} = x_i^{-1} \cdot x_{i+1}\).

Then the action of \( B_\infty \) on \( \mathcal{F}_G \) induces an action on \( \mathcal{F}_X \) by, for \( f \in \mathcal{F}_X, p \in B_\infty \), defining \((f)p = (f)\phi p \phi^{-1} \).

We record the actions of generators and their inverses of \( B_\infty \) on generators of \( \mathcal{F}_X \):
\[
(x_i)\sigma_i^{\pm 1} = x_{i \pm 1} \cdot x_i^{-1} \cdot x_{i \mp 1}
\]
\[
(x_j)\sigma_i^{\pm 1} = x_j \text{ if } i \neq j.
\]
We note the possible ways a generator or its inverse of the braid group can change the leading variable of an element of the free group written in reduced form:

**Lemma 10** If the reduced form of a word \( f \in F_X \) begins with \( x_m \), then the reduced form of \( (f)\sigma_k^{\pm 1} \) begins with (this same) \( x_m \) except in the following two cases (with the displayed words in reduced form):

1. \((x_m \cdot x_{m\pm 1}^{-1} \cdot \ldots )\sigma_{m\pm 1}^{\pm 1} = x_{m\pm 1} \cdot \ldots \)
2. \((x_m \cdot x_{m\pm 1}^{-1} \cdot \ldots )\sigma_m^{\pm 1} = x_{m\pm 1} \cdot x_{m}^{-1} \cdot \ldots \)

†Write \( \sigma \) for \( \sigma_k^{\pm 1} \).

In each of the following cases we assume \( f \) is reduced, that the braid action is applied to \( f \), and then some unspecified but fixed reduction is applied to the word in \( F_X \) that results.

**Case 10.1** \( m \neq k \), and the leading \( x_m \) is cancelled out in the reduction of \( (f)\sigma \) by an \( x_m^{-1} \) which is already present in the reduced form of \( f \).

This means that \( f \) had the reduced form \( f = x_m \cdot f_1 \cdot x_m^{-1} \cdot f_2 \). Then \( (f)\sigma = x_m \cdot (f_1)\sigma \cdot x_m^{-1} \cdot (f_2)\sigma \). But then \((f_1)\sigma = \varepsilon\), so \( f_1 = \varepsilon \), and \( f \) was not in reduced form, contradiction.

**Case 10.2** \( m \neq k \), and the leading \( x_m \) is cancelled out by an \( x_m^{-1} \) which was produced by the braid action on \( f \).

An examination of cases shows that the only ways that an \( x_m^{-1} \) can be so produced are:

1. \((x_m^{-1})\sigma_{m\pm 1}^{\pm 1} = x_m^{-1} \cdot x_{m\pm 1} \cdot x_m^{-1} \cdot x_{m\pm 2}
2. \((x_m^{-1})\sigma_{m\pm 1}^{\mp 1} = x_m^{-1} \cdot x_{m\pm 1} \cdot x^{-1} \cdot x_{m\pm 2}

This means that \( f \) had the reduced form \( f = x_m \cdot f_1 \cdot x_{m\pm 1}^{-1} \cdot f_2 \) and that \( k = m \pm 1 \).

**SubCase 10.2.1** \( \sigma = \sigma_{m\pm 1}^{\pm 1} \).
The first of these two possibilities develops as: \((f)\sigma_{m+1}^{\pm 1} = x_m \cdot (f_1)\sigma_{m+1}^{\pm 1} \cdot (x_{m-1} \cdot x_m \cdot x_{m+1} \cdot x_{m+2}^{-1}) \cdot (f_2)\sigma_{m+1}^{\pm 1}\). In order that this \(x_m^{-1}\) should be the one which cancels the leading \(x_m\), it must be that \((f_1)\sigma = \varepsilon\), and so \(f_1 = \varepsilon\). Hence the reduced form of \(f\) must be \(f = x_m \cdot x_{m+1}^{-1} \cdot f_2\).

This is the lefthand side of the first possibility of the lemma. To finish this case we need to ensure that the \(x_{m+1}\) produced above, which is the leading factor of \((f)\sigma\) before reducing, remains the leading factor after reducing.

If it were to cancel out in the reduction, then it must do so by cancelling with an \(x_{m+1}\) produced by the action of \(\sigma\) on an \(x_{m+1}\) inside \(f_2\).

Hence \(f_2 = f_3 \cdot x_{m+1} \cdot f_4\) in reduced form, so that \(f = x_m \cdot x_{m+1}^{-1} \cdot f_3 \cdot x_{m+1} \cdot f_4\). Then \((f)\sigma = x_{m+1} \cdot x_{m+2}^{-1} \cdot (f_3)\sigma \cdot x_{m+2} \cdot x_{m+1}^{-1} \cdot x_m \cdot (f_4)\sigma\). In order that this \(x_{m+1}\) be the one to cancel the leading \(x_{m+1}\), it must be that \(x_{m+2}^{-1} \cdot (f_3)\sigma \cdot x_{m+2} = \varepsilon\), and so \(f_3 = \varepsilon\). But then \(f\) was not in reduced form.

**SubCase 10.2.2** \(\sigma = \sigma_{m+1}^{\pm 1}\).

The second of these two possibilities develops as: \((f)\sigma_{m+1}^{\pm 1} = x_m \cdot (f_1)\sigma_{m+1}^{\pm 1} \cdot (x_{m+2}^{-1} \cdot x_{m+1}^{-1} \cdot x_m^{-1}) \cdot (f_2)\sigma_{m+1}^{\pm 1}\). In order that this \(x_m^{-1}\) be the one to cancel the leading \(x_m\), it must be that \(x_{m+2}^{-1} \cdot (f_3)\sigma \cdot x_{m+2} = \varepsilon\), and so \(f_3 = \varepsilon\). But then \(f\) was not in reduced form.

**Case 10.3** \(k = m\).

Let \(f = x_m \cdot f_1\) in reduced form. Then \((f)\sigma = x_{m+1} \cdot x_{m+1}^{-1} \cdot x_{m+1} \cdot (f_1)\sigma_{m+1}^{\pm 1}\).

**SubCase 10.3.1** The second factor of \((f)\sigma\), \(x_m^{-1}\), does not cancel in the reduction of \((f)\sigma\).

If this \(x_m^{-1}\) does not cancel in the reduction, then this case leads to the second possibility stated in the lemma.

**SubCase 10.3.2** The second factor of \((f)\sigma\), \(x_m^{-1}\), does cancel in the reduction of \((f)\sigma\).

In order that this \(x_m^{-1}\) cancel, there must be an \(x_m^{-1}\) in the reduced form of \(f_1\), i.e., \(f_1 = f_2 \cdot x_m^{-1} \cdot f_3\) and so \(f = x_m \cdot f_2 \cdot x_m^{-1} \cdot f_3\) in reduced form.

Hence \((f)\sigma = x_{m+1} \cdot x_{m+1}^{-1} \cdot x_m \cdot (f_2)\sigma \cdot x_{m+1}^{-1} \cdot x_m \cdot x_{m+1}^{-1} \cdot (f_3)\sigma\). In order that this \(x_m\) be the one to cancel the second \(x_m^{-1}\), it must be that \(x_{m+1} \cdot (f_2)\sigma \cdot x_{m+1}^{-1} = \varepsilon\), and so \(f_2 = \varepsilon\). But then \(f\) was not in reduced form.

This completes the proof of the lemma.

\(\square\)
Definition 11 Say that a reduced word in $F_X$ leans right at $n$ if it begins with either $x_m$ or $x_n \cdot x_m^{-1}$ with $m > n$.

Lemma 12 If $f \in F_X$ leans right at $n$, then so does $(f)\sigma_i^{\pm 1}$, where $\sigma_i^{\pm 1} \neq \sigma_n^{-1}$.

We first verify that if $f$ leans right at $n$, then the leading factor of the reduced form of $(f)\sigma_i^{\pm 1}$ is some $x_m$ with $m \geq n$.

By lemma 10, the index of the leading factor can change by at most 1.

Hence assume by way of contradiction that $m = n$ and the action of $\sigma_i^{\pm 1}$ on $f$ leaves it with a leading factor of $x_{n-1}$.

The first possibility afforded by lemma 10 is

$$(x_n \cdot x_{n-1}^{-1} \cdot \ldots)\sigma_{n-1}^{-1} = x_{n-1} \cdot \ldots$$

but this is ruled out by the requirement that the index of the second factor exceed $n$.

The second possibility is $(x_n \cdot \ldots)\sigma_n^{-1} = x_{n-1} \cdot x_n^{-1} \cdot \ldots$ and this is ruled out by $\sigma_i^{\pm 1} \neq \sigma_n^{-1}$.

Hence if $f$ leans right at $n$ then $(f)\sigma_i^{\pm 1}$ leads with $x_n$ or $x_m$ with $m > n$.

Now we verify that if $f$ leans right at $n$, and $(f)\sigma_i^{\pm 1}$ starts with $x_n$, then its second factor is $x_n^{-1}$ with $m > n$.

Two possibilities arise. Either $f$ starts with $x_{n+1}$ or with $x_n$.

In the first case, where $f$ starts with $x_{n+1}$, the lemma gives two possibilities where $(f)\sigma_i^{\pm 1}$ should start with $x_n$.

The first is $(x_{n+1} \cdot x_{n-1}^{-1} \cdot \ldots)\sigma_n^{-1} = x_n \cdot \ldots$, but this is ruled out by $\sigma_i^{\pm 1} \neq \sigma_n^{-1}$.

The second is $(x_{n+1} \cdot \ldots)\sigma_{n+1}^{-1} = x_n \cdot x_{n+1}^{-1} \cdot \ldots$, and this result (which is in reduced form) leans right at $n$.

In the second case, where $f$ starts with $x_n$, since $f$ leans right at $n$, $f$ must start with $x_n \cdot x_m^{-1}$ with $m > n$. We note that a dual version of the lemma shows that a reduced word starting with $x_m^{-1}$ and acted upon by a braid generator or its inverse either does not change its leading factor, or the leading factor becomes $x_{m\pm 1}^{-1}$. Hence the only concern is $f = x_n \cdot x_{n+1}^{-1} \cdot \ldots$ acted on by a braid generator or its inverse that does not change the leading $x_n$ and does change $x_{n+1}^{-1} \cdot \ldots$ into $x_n^{-1} \cdot \ldots$, which contradicts that the leading $x_n$ is not cancelled.
Therefore if \( f \) leans right at \( n \), and \( (f)\sigma_i^{\pm 1} \) leads with \( x_n \), then its second factor is \( x_m^{-1} \) with \( m > n \).

This completes the proof. \( \sqcup \)

To prove the \( \sigma_n \)-proposition, assume that \( p \in B_\infty \) is formed from a product of generators of \( B_\infty \) and their inverses, with at least one \( \sigma_n \) and no \( \sigma_n^{-1} \).

Write \( p = p_1 \cdot \sigma_n \cdot p_2 \) where \( \sigma_n^{\pm 1} \) doesn’t occur in \( p_1 \) and \( \sigma_n^{-1} \) doesn’t occur in \( p_2 \).

Then

\[
(x_n)p = (x_n)p_1 \cdot \sigma_n \cdot p_2 \\
= (x_n)\sigma_n \cdot p_2 \\
= (x_{n+1} \cdot x_{n-1} \cdot x_{n-1})p_2
\]

But \( x_{n+1} \cdot x_{n-1} \cdot x_{n-1} \) leans right at \( n \), and hence so does \( (x_{n+1} \cdot x_{n-1} \cdot x_{n-1})p_2 \).

Since \( x_n \) does not lean right at \( n \), they cannot be equal, and so \( p \neq \varepsilon \). \( \sqcup \)

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