0. Introduction.

Let $X$ be a projective $K3$ surface and $\mathcal{M}(r,c_1,c_2)$ be the moduli space of semistable (with respect to the polarization $\mathcal{O}_X(1)$) torsion-free sheaves on $X$ of rank $r$, with Chern classes $c_1$, $c_2$. If it so happens that every semistable sheaf is actually stable, e.g. when $c_1$ is non-divisible or when $r = 2$, $c_1 = 0$ and $c_2$ is odd (the polarization must be “generic”), then $\mathcal{M}(r,c_1,c_2)$ is smooth and holomorphically symplectic [Mk]. It has been proved [O1,O3] that in both of the above mentioned cases the moduli space is equivalent, up to deformation of complex structure and birational modifications, to a Hilbert scheme parametrizing zero-dimensional subschemes of $X$; one expects that the same statement is true whenever semistability implies stability (Yoshioka [Y] proved this in many cases). The present paper deals with the “opposite” case, that is when there do exist strictly semistable (i.e. non stable) sheaves. We will analyze the moduli space $\mathcal{M}_c$ of rank-two torsion-free sheaves with $c_1 = 0$ and $c_2 = c$ even. The sheaf $I_W \oplus I_Z$, where $2\ell(W) = 2\ell(Z) = c$, is strictly semistable for any choice of polarization; conversely if the polarization is generic (see (0.2) for the precise definition) these are the only strictly semistable sheaves. When $c = 0$ or $c = 2$ no stable sheaves exist, thus $\mathcal{M}_0$ is a point and $\mathcal{M}_2 \cong X^{(2)}$: in short not much is going on. Instead if $c \geq 4$ the moduli space $\mathcal{M}_c$ is interesting: it is singular exactly along the locus parametrizing strictly semistable sheaves, and the smooth locus is symplectic. A natural question to ask is the following:

(0.1) Does there exist a symplectic desingularization (or smooth model) of $\mathcal{M}_c$?

If $c = 4$ the answer is “yes”: we will construct a projective symplectic desingularization $\widetilde{\mathcal{M}}_4$ of $\mathcal{M}_4$; in another paper [O4] we will prove that $\widetilde{\mathcal{M}}_4$ is a “new” irreducible symplectic variety, even up to deformation of complex structure and birational modifications. The first step in the construction of $\widetilde{\mathcal{M}}_4$ is to apply Kirwan’s procedure for desingularizing G.I.T. quotients. This gives a desingularization $\widehat{\mathcal{M}}_4$ with a two-fold degenerating on a single divisor $\widehat{\Omega}_4$, namely the inverse image of the locus parametrizing sheaves equivalent to $I_Z \oplus I_Z$. The divisor $\widehat{\Omega}_4$ is a $\mathbf{P}^2$-fibration, and the normal bundle has degree $-1$ on the $\mathbf{P}^2$’s. Hence we can contract $\mathcal{M}_4$ along this fibration, and we get a smooth complex manifold $\mathcal{M}_4$ with a holomorphic symplectic form. We identify this contraction with the contraction of a certain $K_{\mathcal{M}_4}$-negative extremal ray, hence $\widehat{\mathcal{M}}_4$ is projective by Mori theory. We also show that the a priori rational map $\widehat{\mathcal{M}}_4 \to \mathcal{M}_4$ is regular, thus $\mathcal{M}_4$ is a symplectic desingularization of $\mathcal{M}_4$. If $c \geq 6$ the picture is similar, but we do not succeed in constructing a symplectic desingularization of $\mathcal{M}_c$: in fact we suspect that there is no smooth symplectic model of $\mathcal{M}_c$. (This would imply that $\mathcal{M}_c$ is never birational to a Hilbert scheme parametrizing zero-dimensional subschemes of a $K3$.) As in the case $c = 4$, we first construct Kirwan’s desingularization $\widetilde{\mathcal{M}}_c$. There is a holomorphic two form on $\widetilde{\mathcal{M}}_c$, degenerating on the three exceptional divisors of $\widetilde{\mathcal{M}}_c \to \mathcal{M}_c$. We describe a $K_{\mathcal{M}_c}$-negative face of $\overline{NE}_1(\mathcal{M}_c)$, generated by three integral classes $\partial_c, \check{c}_c, \check{\gamma}_c$. Contracting $\mathbf{R}^+\check{\gamma}_c$, then the image of $\mathbf{R}^+\check{c}_c$, and finally the image of $\mathbf{R}^+\check{\sigma}_c$, one gets the desingularization map $\widetilde{\mathcal{M}}_c \to \mathcal{M}_c$. What if we reverse the order of the contractions? We will show that contracting first $\mathbf{R}^+\check{\sigma}_c$ and then the image of $\mathbf{R}^+\check{c}_c$ (which has negative intersection with the canonical bundle of the contracted scheme) we get a smooth desingularization $\mathcal{M}_c$ of $\mathcal{M}_c$. Unfortunately, if we go all the way and contract the image of $\mathbf{R}^+\check{\gamma}_c$ (this is a $K_{\mathcal{M}_c}$-negative ray) we get a singular space. Although $\mathcal{M}_c$ is not symplectic, we hope that it will be helpful in answering (0.1). We close this discussion by mentioning one of our motivations for posing Question (0.1). Vafa and Witten [VW] have proposed formulae for the Euler characteristics (suitably interpreted?) of moduli spaces of semistable sheaves on surfaces. If the answer to (0.1) is affirmative, the Euler characteristic of any smooth symplectic model of $\mathcal{M}_c$ (which is independent of the model chosen) should be equal to Vafa-Witten’s characteristic. If on the contrary the answer to (0.1) is negative it is not clear which “mathematical” Euler characteristic should equal the “physicists” characteristic.
Organization of the paper.

§1. F. Kirwan [K] defined a procedure for partially desingularizing G.I.T. quotients: one blows up certain loci consisting of strictly semistable points. Since the moduli space $M_c$ is the G.I.T. quotient of a Quot-scheme $Q_c$, we can follow Kirwan’s procedure. To do this we need to analyze the local structure of $Q_c$ at points corresponding to semistable sheaves $F = I_W \oplus I_Z$; this is equivalent to studying the versal deformation space of such sheaves. We will see that $Q_c$ is singular at such points. Since Kirwan assumes that the semistable locus is smooth, the general results of [K] do not apply. However we are lucky: we show that Kirwan’s blow-ups, dictated by the need to eliminate strictly semistable points, give also a desingularization of the semi-stable locus. At this stage the quotient is not yet smooth (except when $c = 4$), but a last blow-up gives a space with smooth quotient $\hat{M}_c$. We show that there is a regular two-form on $\hat{M}_c$ extending the symplectic form on the smooth locus of $M_c$.

§2. We construct the symplectic desingularization $\tilde{M}_4$ of $M_4$.

§3. We describe a $K$-negative extremal face of $NE_1(\hat{M}_c)$, and we indicate how to contract extremal rays in order to obtain a desingularization of $M_c$ which is “nearer” to being symplectic than $\hat{M}_c$ is.

Notation used throughout the paper.

We let $c$ be an even integer with $c \geq 4$, and we set $c = 2n$.

We let $X$ be a projective $K3$ surface and $H = O_X(1)$ be a $c$-generic polarization, i.e. an ample divisor class such that for $D \in \text{Div}(S)$,

\[(0.2) \quad \text{if } D \cdot H = 0 \text{ and } -c \leq D \cdot D, \text{ then } D \sim 0.\]

There exist $c$-generic polarizations for any choice of $c$, because the collection of hyperplanes $D^\perp$, for $D \in \text{Div}(X)$ with

\[-c \leq D \cdot D < 0,\]

defines a set of locally finite walls in the ample cone of $X$.

A torsion-free sheaf $F$ on $X$ is Gieseker-Maruyama semistable (with respect to the polarization $H$) if for every exact sequence

\[0 \to L \to F \to Q \to 0,\]

\[(\text{rk}Q) \cdot \chi(L(n)) \leq (\text{rk}L) \cdot \chi(Q(n)), \text{ for } n \gg 0.\]

If strict inequality holds (when $n \gg 0$) for all such sequences with $\text{rk}L \neq 0 \neq \text{rk}Q$ then $F$ is Gieseker-Maruyama stable. A semistable non-stable sheaf is strictly semistable.

We let $\mathcal{M}_c$ be the moduli space of rank-two (Gieseker-Maruyama) semistable torsion-free sheaves $F$ on $X$ with $c_1(F) = 0$, $c_2(F) = c$; this is a projective scheme whose closed points are in one-to-one correspondence with $S$-equivalence classes of such sheaves [G, Ma].

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1. Kirwan’s desingularization.

1.1. The Quot-scheme and Kirwan’s desingularization.

We briefly recall the construction of $\mathcal{M}_c$ according to Simpson [S,Le]. By Serre’s F.A.C. theorems if $k \gg 0$ the following holds. Let $F$ be a sheaf parametrized by $\mathcal{M}_c$; then $H^p(F(k)) = 0$ for $p > 0$, and $F$ can be realized as a quotient

\begin{equation}
\mathcal{O}_X(-k)^N \to F,
\end{equation}

so that the induced map $\mathbb{C}^N \to H^0(F(k))$ is an isomorphism. Let $Quot(k)$ be the Quot-scheme parametrizing quotients (1.1.1) whose Hilbert polynomial is that of rank-two sheaves with $c_1 = 0, c_2 = c$; if $x \in Quot(k)$ we let $F_x$ be the quotient sheaf parametrized by $x$. Then $G$ acts on $Quot(k)$ and also on some positive multiple of the “Plücker” line-bundle over $Quot(k)$, i.e. the action is linearized. Hence it makes sense to speak of $G$-linearized (semi)stable points: let $Q^ss, Q^s \subseteq Quot(k)$ be the open subsets consisting of $G$-semistable (respectively stable) points $x$ such that $F_x$ is torsion-free and $rk(F_x) = 2, c_1(F_x) = 0, c_2(F_x) = c$. Let $Q_c$ be the schematic closure of $Q^ss$ in $Quot(k)$. Simpson proves that for $k$ sufficiently large a point $x \in Q_c$ is $G$-semistable (stable) if and only if $F_x$ is Gieseker-Maruyama semistable (respectively stable), and that

$\mathcal{M}_c = Q_c//PGL(N)$.

Kirwan’s partial desingularization will be the $PGL(N)$-quotient of a variety obtained by successively blowing up $Q_k$ along loci parametrizing strictly semistable points: the idea is that strictly semistable points will gradually disappear and in the end all semistable points will be stable (in particular their stabilizers will be finite). A key ingredient is a theorem of Kirwan relating stability on a $G$-scheme to stability on the blow-up of a $G$-invariant subscheme. More precisely, let $G$ be a reductive group acting linearly on a complex projective scheme $Y$ (linearly means: the $G$-action has been lifted to an action on $\mathcal{O}_Y(1)$), let $V$ be a $G$-invariant closed subscheme of $Y$, and $\pi: \tilde{Y} \to Y$ be the blow-up of $V$. Then $G$ acts on $\tilde{Y}$, and also on

$D_\ell := \pi^*\mathcal{O}_Y(\ell) \otimes \mathcal{O}_{\tilde{Y}}(-E)$,

where $E$ is the exceptional divisor of $\pi$. Thus the action on $\tilde{Y}$ is linearized. Let $Y^ss \subseteq Y, Y^s \subseteq Y$ be the loci of semistable (stable) points with respect to $\mathcal{O}_Y(1)$, and let $\tilde{Y}^ss(\ell) \subseteq \tilde{Y}$ and $\tilde{Y}^s(\ell) \subseteq \tilde{Y}$ be the loci of semistable (stable) points with respect to $D_\ell$.

(1.1.2) Theorem (Kirwan [K, 3.1-3.2-3.11]). Keep notation as above. For $\ell \gg 0$ the loci $\tilde{Y}^ss(\ell)$ and $\tilde{Y}^s(\ell)$ are independent of $\ell$: denote them by $\tilde{Y}^ss$ and $\tilde{Y}^s$ respectively. The following holds:

\begin{align}
\pi(\tilde{Y}^ss) \subset Y^ss, \\
\pi^{-1}(Y^s) \subset \tilde{Y}^s.
\end{align}

In particular $\pi$ induces a morphism

$\pi: \tilde{Y}//G \to Y//G$.

If $\ell$ is also sufficiently divisible, this morphism is identified with the blow-up of $V//G$.

In our case $PGL(N)$ acts on $Q_c$, and we will blow up loci parametrizing strictly semistable sheaves.

(1.1.5) Lemma. A point $x \in Q_c$ is strictly semistable (i.e. $x \in Q^ss \setminus Q^s$) if and only if $F_x$ fits into an exact sequence

\begin{equation}
0 \to I_Z \to F_x \to I_W \to 0,
\end{equation}

where $Z, W$ are zero-dimensional subschemes of $X$ of length $\ell(Z) = \ell(W) = n$, and $I_Z, I_W$ are their ideal sheaves. Furthermore the orbit $PGL(N)x$ is closed in $Q^ss_c$ if and only if the exact sequence above is split.

Proof. A straightforward computation shows that if $F_x$ fits into Exact sequence (1.1.6) then it is strictly Gieseker-Maruyama semistable, hence $x$ is strictly semistable. Now assume $x \in Q_c$ is strictly semistable. Then $F_x$ is strictly Gieseker-Maruyama semistable, i.e. it fits into an exact sequence

\begin{equation}
0 \to I_Z(D) \to F_x \to I_W(-D) \to 0,
\end{equation}

\begin{equation}
0 \to I_Z(D) \to F_x \to I_W(-D) \to 0,
\end{equation}
where $Z$, $W$ are zero-dimensional subschemes of $X$, and $D \in \text{Div}(X)$, with

\[(\dagger) \quad \chi(I_Z(D) \otimes O_X(n)) = \chi(I_W(-D) \otimes O_X(n)) \text{ for } n \gg 0.\]

Applying Whitney’s formula to $(\ast)$ and writing out explicitly $(\dagger)$ we get

\[-c + \ell(Z) + \ell(W) = D \cdot D \text{ and } D \cdot H = 0,
\]

respectively. Since $H$ is $c$-generic (see (0.2)) we conclude that $D \sim 0$, hence the destabilizing subsheaf and quotient sheaf are $I_Z$ and $I_W$ respectively. Equality $(\dagger)$ then gives $\ell(Z) = \ell(W)$, hence $F_x$ fits into Exact sequence (1.1.6). Let $e \in \text{Ext}^1(I_W, I_Z)$ be the extension class of (1.1.6), and assume $e \neq 0$. One can construct a family of extensions $\{E_t\} \in \mathcal{A}_t$ of $I_W$ by $I_Z$ with extension class $te$: for $t \neq 0$ the sheaves $E_t$ are all isomorphic non-split extensions, while $E_0 \cong I_Z \oplus I_W$. From this it follows that if (1.1.6) is non-split the orbit $\text{PGL}(N)x$ is not closed. Since there must be a closed orbit in $Q^{ss}$ which corresponds to the S-equivalence class of the semistable sheaf appearing in (1.1.6), this orbit must parametrize split extensions. q.e.d.

Let

\[
\Omega^0_Q := \{x \in Q| F_x \cong I_Z \oplus I_Z, \ [Z] \in X^{[n]} \},
\]

\[
\Gamma^0_Q := \{x \in Q| F_x \text{ is a non-trivial extension of } I_Z \text{ by } I_Z, \ [Z] \in X^{[n]} \},
\]

\[
\Sigma^0_Q := \{x \in Q| F_x \in I_Z \oplus I_W, \ [Z], [W] \in X^{[n]}, \ Z \neq W \},
\]

\[
\Lambda^0_Q := \{x \in Q| F_x \text{ is a non-trivial extension of } I_Z \text{ by } I_W, \ [Z], [W] \in X^{[n]}, \ Z \neq W \}
\]

Here and in the rest of the paper we drop the subscript $c$ from $\mathcal{M}, Q_c, \text{ etc. whenever this causes no confusion.}$ We let $\Omega_Q, \Gamma_Q, \Sigma_Q, \Lambda_Q$ be the closures in $Q$ of $\Omega^0_Q, \Gamma^0_Q, \Sigma^0_Q$ and $\Lambda^0_Q$ respectively. By Lemma (1.1.5),

\[(1.1.7) \quad Q^{ss} \setminus Q^* = \bar{\Omega}_Q^0 \amalg \bar{\Gamma}_Q^0 \amalg \bar{\Sigma}_Q^0 \amalg \bar{\Lambda}_Q^0.
\]

If $G$ is a group acting on a set $A$, and $x \in A$, we let $St(x)$ be the stabilizer of $x$.

(1.1.8) Corollary. Let $x \in Q^{ss}$. Then

\[St(x) \cong \begin{cases} 
\text{PGL}(2) & \text{if } x \in \Omega^0_Q, \\
\mathbb{C}^* & \text{if } x \in \Gamma^0_Q, \\
\{1\} & \text{if } x \in \Lambda^0_Q \amalg Q^*.
\end{cases}
\]

Proof. For $x \in Q$, one has $St(x) \cong \text{Aut}(F_x)/\text{scalars}$; the result follows easily. q.e.d.

By the above corollary the points of $Q^{ss}$ with non-trivial reductive stabilizers are parametrized by $\Omega^0_Q$ and $\Sigma^0_Q$. Thus Kirwan’s method for (partially) desingularizing $\mathcal{M}_c$ is the following. Let

\[(1.1.9) \quad \pi_R: R \to Q
\]

be the blow-up of $\Omega_Q$. We let $\Sigma_R \subset R$ be the strict transform of $\Sigma_Q$. (Notice that $\Sigma_Q \supset \Omega_Q$.) Let

\[(1.1.10) \quad \pi_S: S \to R
\]

be the blow-up of $\Sigma_R$. The linear action of PGL$(N)$ on $Q$ lifts to linear actions on $R$ and $S$. Applying Theorem (1.1.2) to $\pi_R$ and $\pi_S$ we get a morphism

\[S_c/\text{PGL}(N) \to Q_c/\text{PGL}(N) = \mathcal{M}_c.
\]

We will prove that $\hat{\mathcal{M}}_4 := S_c/\text{PGL}(N)$ is smooth. If $c \geq 6$ then $S_c/\text{PGL}(N)$ has quotient singularities; a last blow up will produce a smooth desingularization of $\mathcal{M}_c$, which we denote $\hat{\mathcal{M}}_c$. Both of these statements are proved in Subsection (1.8). In Subsection (1.9) we will define a regular two-form on $\hat{\mathcal{M}}_c$ (for any $c \geq 4$) which extends Mukai’s symplectic form on the smooth locus of $\mathcal{M}_c$.

1.2. Luna’s étale slice.

If $W$ is a subscheme of a scheme $Z$ we let $C_W Z$ be the normal cone to $W$ in $Z$ [Fu]. Since the exceptional divisor of $\pi_R$ is equal to $\text{Proj}(C_{\Sigma_R} Q)$, we will need to determine the normal cone to $\Omega_Q$ in $Q$ (at semistable points); similarly we will need to know $C_{\Sigma_R} R$. Luna’s étale slice theorem reduces this to a problem about deformations of sheaves. We recall Luna’s theorem. Let $G$ be a reductive group acting linearly on a quasi-projective scheme $Y$. For $y \in Y$ we let $O(y)$ be its orbit. If $y \in Y^{ss}$ and $O(y)$ is closed (in $Y^{ss}$) then $St(y)$ is reductive.
(1.2.1) Luna’s étale slice Theorem [Lu]. Keeping notation as above, suppose \( y_0 \in Y^{ss} \) and \( O(y_0) \) is closed in \( Y^{ss} \). Then there exists a slice normal to \( O(y_0) \), i.e. an affine subscheme \( V \hookrightarrow Y^{ss} \), containing \( y_0 \) and invariant under the action of \( St(y_0) \), such that the following holds. The (multiplication) morphism

\[
G \times_{St(y_0)} V \xrightarrow{\phi} Y^{ss}
\]

has open image, and is étale over its image. (Here \( St(y_0) \) acts on \( G \times V \) by \( h(g, y) := (gh^{-1}, hy) \).) Furthermore \( \phi \) is \( G \)-equivariant (the \( G \)-action on \( G \times_{St(y_0)} V \) is induced by left multiplication on the first factor). The quotient map

\[
\overline{\phi} : V/\!/St(y_0) \rightarrow Y^{ss}/\!/G
\]

has open image and is étale over its image. If \( Y^{ss} \) is smooth at \( y_0 \), then \( V \) is also smooth at \( y_0 \).

Now assume \( W \subset Y^{ss} \) is a locally closed subset containing \( y_0 \), stable for the action of \( G \); for example \( W \) could be \( \Omega^0_Q \subset Q^{ss} \). Set

\[
W := W \cap V.
\]

(1.2.2) Corollary. Keep notation and hypotheses as above. There is a \( St(y_0) \)-equivariant isomorphism

\[
(C_W Y^{ss})_{y_0} \cong (C_W V)_{y_0}.
\]

Proof. We work throughout in neighborhoods of \( (1, y_0) \in G \times_{St(y_0)} V \) and of \( y_0 \in Y \). Set

\[
\overline{W} := G \times_{St(y_0)} W.
\]

Since \( \phi^{-1}W = \overline{W} \), and since \( \phi \) is étale there is an isomorphism

\[
C_W Y^{ss} \cong C_{\overline{W}} (G \times_{St(y_0)} V).
\]

The projections

\[
(G \times_{St(y_0)} V) \rightarrow G/St(y_0) \quad \overline{W} = (G \times_{St(y_0)} W) \rightarrow G/St(y_0)
\]

are fibrations étale locally trivial, with fibers \( V \) and \( W \) respectively. Taking the fiber over the coset \( [St(y_0)] \) we get the corollary. q.e.d.

Let’s go back to our case: \( \text{PGL}(N) \) acting on \( Q^{ss} \). The following result identifies the normal slice with a versal deformation space. (See also Wehler [W].)

(1.2.3) Proposition. Let \( x \in Q^{ss} \) be a point such that \( O(x) \) is closed (in \( Q^{ss} \)). Let \( V \) be a normal slice (see (1.2.1)), and \( (V, x) \) be the germ of \( V \) at \( x \). Let \( \mathcal{F} \) be the restriction to \( X \times (V, x) \) of the tautological quotient sheaf on \( X \times Q \). The couple \( ((V, x), \mathcal{F}) \) is a versal deformation space of \( F_x \).

Proof. We must prove two things: that the family \( \mathcal{F} \) is complete and that the Kodaira-Spencer map

\[
\pi : T_x V \rightarrow \text{Ext}^1(F_x, F_x)
\]

is an isomorphism. Completeness follows easily from the universal property of the Quot-scheme. Let’s prove that \( \pi \) is an isomorphism. Since \( \mathcal{F} \) is complete \( \pi \) is surjective, thus it suffices to show that \( \pi \) is injective. Letting

\[
\kappa : T_x Q \rightarrow \text{Ext}^1(F_x, F_x)
\]

be the Kodaira-Spencer map at \( x \) of the tautological quotient on \( X \times Q \), we must show that

\[
\ker(\kappa) = T_x O(x).
\]

Letting \( E_x \) be the kernel of the map (1.1.1) for \( F = F_x \), we have

\[
0 \rightarrow E_x \rightarrow O_X(-k)^{(N)} \rightarrow F_x \rightarrow 0.
\]

(\*)
This gives the exact sequence
\[
0 \to \text{Hom}(F_x, F_x) \to \text{Hom}(O_X(-k)^{(N)}, F_x) \supset \text{Hom}(E_x, F_x) \supset \text{Ext}^1(F_x, F_x),
\]
where \( \text{Hom}(E_x, F_x) = T_xQ \) and \( \kappa \) is Kodaira-Spencer. Thus we are reduced to proving that \( \text{Im} \alpha = T_xO(x) \).

Let
\[
\beta: \text{Hom}(O_X(-k)^{(N)}, O_X(-k)^{(N)}) \to \text{Hom}(O_X(-k)^{(N)}, F_x)
\]
be the obvious map. Since \( T_xO(x) = \text{Im}(\alpha \circ \beta) \), it suffices to check that \( \beta \) is surjective. This follows at once from the isomorphism \( H^0(O_X^{(N)}) \cong H^0(F_x(k)) \).

**q.e.d.**

### 1.3. Normal cones and deformations of sheaves.

We will need to describe \( C_Q^{ss} \) and similar normal cones. By Corollary (1.2.2) and Proposition (1.2.3) this will be equivalent to describing the normal cone of certain loci in the versal deformation space of semistable sheaves; in this subsection we provide the necessary tools.

**The Hessian cone.** Let \( Y \) be a scheme, and \( B \hookrightarrow Y \) a locally-closed subscheme such that

\[
(1.3.1) \quad \text{B is smooth and dim } T_bY \text{ is constant for every } b \in B.
\]

By (1.3.1) we have a normal vector-bundle \( N_BY \). Let \( I_B \) be the ideal sheaf of \( B \) in \( Y \): the graded surjection
\[
\bigoplus_{d=0}^{\infty} S^d(I_B/I_B^d) \to \bigoplus_{d=0}^{\infty} (I_B^d/I_B^{d+1})
\]
defines an embedding of cones \( : C_BY \hookrightarrow N_BY \). The (homogeneous) ideal \( I(I(C_BY)) \) contains no linear terms. We let the **Hessian cone** of \( B \) in \( Y \) be the subscheme \( H_BY \hookrightarrow N_BY \) whose homogeneous ideal is generated by the quadratic terms \( I(I(C_BY))_2 \). Thus we have a chain of cones over \( B \):
\[
C_BY \subset H_BY \subset N_BY.
\]

Notice that if \( b \in B \), then

\[
(1.3.2) \quad \text{P}(H_BY) \text{ is the cone over } \text{P}(H_BY)_b \text{ with vertex } \text{P}(T_bB).
\]

Let \( I_m := \text{Spec} \left( C[t]/(t^{m+1}) \right) \); thus tangent vectors to \( Y \) at \( b \) are identified with pointed maps \( I_1 \to (Y, b) \).

Then
\[
(1.3.3) \quad (H_BY)_{\text{red}} = \{ f_1: I_1 \to (Y, b) \mid \text{there exists } f_2: I_2 \to (Y, b) \text{ extending } f_1 \}.
\]

**Deformations of sheaves.** Let \( \mathcal{E} \) be a coherent sheaf on a projective scheme, and let \( (\text{Def}(\mathcal{E}), 0) \) be the parameter space of the versal deformation space of \( \mathcal{E} \). Thus

\[
(1.3.4) \quad T_0\text{Def}(\mathcal{E}) \cong \text{Ext}^1(\mathcal{E}, \mathcal{E}).
\]

We will give explicit equations of the reduced Hessian cone of \( \text{Def}(\mathcal{E}) \) at the origin. Let
\[
\text{Ext}^p(\mathcal{E}, \mathcal{E})^0 := \ker (\text{Tr}: \text{Ext}^p(\mathcal{E}, \mathcal{E}) \to H^p(O_X)),
\]
where the trace \( \text{Tr} \) is defined as in [DL]. The composition of Trace with Yoneda product
\[
\text{Ext}^p(\mathcal{E}, \mathcal{E}) \times \text{Ext}^q(\mathcal{E}, \mathcal{E}) \xrightarrow{\text{Yon}} \text{Ext}^{p+q}(\mathcal{E}, \mathcal{E}) \xrightarrow{\text{Tr}} H^{p+q}(O_Y)
\]
is a bilinear map, symmetric if \((-1)^pq = 1\), anti-symmetric if \((-1)^pq = -1\). We are particularly interested in the Yoneda square map

\[
\Ext^1(\mathcal{E}, \mathcal{E}) \xrightarrow{\mathcal{Y}_e} \Ext^2(\mathcal{E}, \mathcal{E})^0 \xrightarrow{e \cup e} \Ext^3(\mathcal{E}, \mathcal{E})^0.
\]

**Proposition.** Keep notation as above. Then

\[(1.3.5) \quad (\text{H}_0\text{Def}(\mathcal{E}))_{\text{red}} = (\mathcal{Y}^{-1}(0))_{\text{red}}.
\]

**Proof.** An \(m\)-th order deformation of \(\mathcal{E}\) is a sheaf \(\mathcal{E}_m\) on \(Y \times I_m\), flat over \(I_m\), such that \(\mathcal{E}_m \otimes \mathcal{C} \cong \mathcal{E}\). By (1.3.3), the left-hand side of (1.3.5) consists of first order deformations of \(\mathcal{E}\) which can be extended to second order deformations. Let \(e \in \Ext^1(\mathcal{E}, \mathcal{E})\), and let

\[(*) \quad 0 \rightarrow t\mathcal{E} \xrightarrow{\alpha} \mathcal{E}_1 \xrightarrow{\beta} \mathcal{E} \rightarrow 0
\]

be the first-order deformation of \(\mathcal{E}\) corresponding to \(e\). Here \(t\mathcal{E}\) means that the \(\mathcal{C}[t]/(t^2)\)-module structure of \(\mathcal{E}_1\) is described as follows: if \(\sigma\) is a local section of \(\mathcal{E}_1\), then \(t\sigma := \alpha(t\beta(\sigma))\). From \((*)\) we get

\[
\Ext^1(\mathcal{E}, \mathcal{E}_1) \rightarrow \Ext^1(\mathcal{E}, \mathcal{E}) \xrightarrow{\partial} \Ext^2(\mathcal{E}, \mathcal{E}).
\]

Since \(\partial\) is Yoneda product with \(e\) we must prove that \(\mathcal{E}_1\) can be extended to a second order deformation if and only if \(\partial(e) = 0\). By the above exact sequence \(\partial(e) = 0\) if and only if Extension \((*)\) is the push-out (via \(\beta\)) of an extension

\[(†) \quad 0 \rightarrow t\mathcal{E}_1 \xrightarrow{\gamma} \mathcal{F} \rightarrow \mathcal{E} \rightarrow 0.
\]

So let’s assume that \(\mathcal{E}_1\) can be extended to a second order deformation \(\mathcal{E}_2\): thus \(\mathcal{E}_2\) is an extension

\[
0 \rightarrow t\mathcal{E}_1 \xrightarrow{\gamma} \mathcal{F} \rightarrow \mathcal{E} \rightarrow 0.
\]

Let’s check that the push-out of \(\mathcal{E}_2\) is isomorphic to \((*)\). Since \(\beta\) is surjective and \(\ker(\beta) = \alpha(t\mathcal{E})\),

\[
\text{push-out of } \mathcal{E}_2 = \mathcal{E}_2/\iota \circ (\alpha(t\mathcal{E})) = \mathcal{E}_2/t^2\mathcal{E} \cong \mathcal{E}_1.
\]

Now let’s prove the converse: assuming there is an extension \((†)\) whose push-out is \((*)\) we will give \(\mathcal{F}\) a structure of \(\mathcal{C}[t]/(t^2)\)-module making it a second order extension of \(\mathcal{E}_1\). Since the push-out of \(\mathcal{F}\) is equal to \(\mathcal{F}/\gamma(t\alpha(t\mathcal{E}))\), we have

\[
0 \rightarrow t^2\mathcal{E} \xrightarrow{\gamma t\alpha t} \mathcal{F} \xrightarrow{\delta} \mathcal{E}_1 \rightarrow 0.
\]

If \(\sigma\) is a local section of \(\mathcal{F}\) we set \(t\sigma := \gamma(t\delta(\sigma))\).

q.e.d.

We will need the following result.

\[(1.3.6) \quad \text{Proposition. Let } x \in Q^{ss} \text{ be a point with closed orbit, and let } \mathcal{V} \ni x \text{ be slice normal to the orbit PGL}(N). \text{ Then}
\]

\[
\dim \mathcal{V} \geq \dim \Ext^1(F_x, F_x) - \dim \Ext^2(F_x, F_x)^0.
\]

More precisely, every irreducible component of the germ \((\mathcal{V}, x)\) has dimension at least equal to the right-hand side of the above inequality.

**Proof.** By Proposition (1.2.3) there is an identification between \((\mathcal{V}, x)\) and \(\text{Def}(F_x)\). It is known [Fr] that \(\text{Def}(F_x)\) is the zero-locus of an obstruction map with domain a smooth germ of dimension \(\dim \Ext^1(F_x, F_x)\), and codomain \(\Ext^2(F_x, F_x)^0\): this implies the proposition. Alternatively, one can use the following result about the local structure of the Hilbert scheme \(Q\) at \(x\). Let \(A\) be the sheaf on \(X\) defined by the exact sequence

\[(1.3.7) \quad 0 \rightarrow A \rightarrow \mathcal{O}_X(-k)^{(N)} \rightarrow F_x \rightarrow 0.
\]
(See (1.1.1).) Then by Lemmas (1.4)-(1.8) of [Li] we have
\[ \dim_x Q \geq \dim T_x Q - \dim \text{Ext}^2(F_x, F_x) = \dim \text{Hom}(A, F_x) - \dim \text{Ext}^2(F_x, F_x), \]
or more precisely every irreducible component of the germ \((Q, x)\) has dimension at least equal to the right-hand side of the above inequality. Applying the functor \(\text{Hom}(\cdot, F_x)\) to (1.3.7) one gets
\[ \dim \text{Hom}(A, F_x) = \dim \text{Ext}^1(F_x, F_x) + (N^2 - 1) - \dim St(x). \]
On the other hand, by Luna’s étale slice Theorem (1.2.1) we have
\[ \dim_x Q = \dim \text{PGL}(N) + \dim_x V - \dim St(x). \]
Putting together the above (in)equalities one gets the proposition. \(\text{q.e.d.}\)

1.4. The normal cone to \(\Sigma^0_Q\).

Let \(x \in \Sigma^0_Q\), and set
\[ F_x = I_{Z_1} \oplus I_{Z_2}, \quad \ell(Z_i) = n, \quad Z_1 \neq Z_2. \]
Recall (1.1.8) that \(St(x) \cong C^*.\) To simplify notation we often set \(\Sigma = \Sigma^0_Q\).

1.4.1 Proposition. Keep notation as above. Then \(\Sigma^0_Q\) is smooth, and its normal cone in \(Q\) is a locally-trivial fibration over \(\Sigma^0_Q\), with fiber the affine cone over a smooth quadric in \(\mathbb{P}^{2e-5}\). More precisely, for \(x \in \Sigma^0_Q\) there is a canonical isomorphism
\[ (C_2Q)_x \cong \{(e_{12}, e_{21}) \in \text{Ext}^1(I_{Z_1}, I_{Z_2}) \oplus \text{Ext}^1(I_{Z_2}, I_{Z_1}) \mid e_{12} \cup e_{21} = 0\}. \]
Furthermore the action of \(St(x)\) on \((C_2Q)_x\) is given by
\[ \lambda(e_{12}, e_{21}) = (\lambda e_{12}, \lambda^{-1} e_{21}). \]

The proposition will be proved in several steps.

I. Yoneda square. Since
\[ \text{Ext}^p(F_x, F_x) = \bigoplus_{i,j} \text{Ext}^p(I_{Z_i}, I_{Z_j}) \]
we can write Yoneda square \(\Upsilon := \Upsilon_{F_x}\) as
\[ \Upsilon(\sum e_{ij}) = \sum e_{1k} \cup e_{k1} + \sum e_{1k} \cup e_{k2} + \sum e_{2k} \cup e_{k1} + \sum e_{2k} \cup e_{k2}, \quad e_{ij} \in \text{Ext}^1(I_{Z_i}, I_{Z_j}). \]
By Serre duality
\[ \text{Ext}^2(I_{Z_i}, I_{Z_j}) \cong \text{Hom}(I_{Z_j}, I_{Z_i})^\vee, \]
hence
\[ \text{Ext}^2(I_{Z_i}, I_{Z_j}) = 0 \text{ if } i \neq j \text{ and } \text{Tr}: \text{Ext}^2(I_{Z_i}, I_{Z_i}) \rightarrow H^2(O_X) \text{ is an isomorphism.} \]
In particular
\[ \text{Ext}^2(\mathcal{E}, \mathcal{E}) = \{ (f_1, f_2) \in \text{Ext}^2(I_{Z_1}, I_{Z_1}) \oplus \text{Ext}^2(I_{Z_2}, I_{Z_2}) \mid f_1 + f_2 = 0\}. \]
Furthermore
\[ \Upsilon(e) = (e_{11} \cup e_{11} + e_{12} \cup e_{21}, e_{21} \cup e_{21} + e_{12} \cup e_{22}). \]
By skew-commutativity (see (1.3)) \(\text{Tr}(e_{ii} \cup e_{ii}) = 0\), hence (1.4.4) implies that \(e_{ii} \cup e_{ii} = 0\). This gives
\[ \Upsilon(e) = (e_{12} \cup e_{21}, e_{12} \cup e_{12}). \]
Let
\[ \Psi: \text{Ext}^1(F_x, F_x) \rightarrow \text{Ext}^2(I_{Z_1}, I_{Z_1}) \]
\[ e \mapsto e_{12} \cup e_{12}. \]
By (1.4.6)-(1.4.5) we can identify \(\Upsilon\) with \(\Psi\), in particular \(\Upsilon^{-1}(0) = \Psi^{-1}(0)\). Let
\[ \overline{\Psi}: \text{Ext}^1(I_{Z_1}, I_{Z_2}) \oplus \text{Ext}^1(I_{Z_2}, I_{Z_1}) \rightarrow \text{Ext}^1(I_{Z_1}, I_{Z_1}) \]
\[ (e_{12}, e_{21}) \mapsto e_{12} \cup e_{21}. \]
Thus \(\overline{\Psi}\) is identified with the map induced by \(\Psi\) on \(\text{Ext}^1(F_x, F_x)/\text{ker} \Psi\).
(1.4.8) Claim. $P\Psi^{-1}(0)$ is a smooth quadric hypersurface in $P^{2c-5}$. In particular, since $c \geq 4$, $P\Psi^{-1}(0)$ is a reduced irreducible quadric.

Proof. By Serre duality, Yoneda product

$$\text{Ext}^1(I_{Z_1}, I_{Z_2}) \times \text{Ext}^1(I_{Z_2}, I_{Z_1}) \to \text{Ext}^2(I_{Z_1}, I_{Z_1})$$

is a perfect pairing. Hence

$$P\Psi^{-1}(0) \subset P(\text{Ext}^1(I_{Z_1}, I_{Z_2}) \oplus \text{Ext}^1(I_{Z_2}, I_{Z_1}))$$

is a smooth quadric hypersurface. The claim follows from the equalities

$$-\dim \text{Ext}^1(I_{Z_1}, I_{Z_2}) = \chi(I_{Z_1}, I_{Z_2}) = \chi(I_{Z_1}, I_{Z_1}) = 2 - c.$$  

q.e.d.

II. The cone at the origin of the deformation space. Let $\mathcal{V}$ be a slice normal to the (closed) orbit $\text{PGL}(N)x$: by Proposition (1.2.3) there is a natural isomorphism of germs $(\mathcal{V}, x) \cong \text{Def}(F_x)$. In particular we have an embedding

$$C_x \mathcal{V} \subset \text{Ext}^1(F_x, F_x).$$

(1.4.10) Proposition. Keep notation as above. There are natural isomorphisms of schemes

$$C_x \mathcal{V} = H_x \mathcal{V} \cong \Psi^{-1}(0).$$

Proof. By (1.3.5) and Claim (1.4.8), we have

$$(PH_x \mathcal{V})_{\text{red}} = \Psi^{-1}(0) = P\Psi^{-1}(0).$$

Since $P\Psi^{-1}(0)$ is a reduced irreducible quadric hypersurface and $PH_x \mathcal{V}$ is cut out by quadrics, it follows that

$$PH_x \mathcal{V} = P\Psi^{-1}(0).$$

Next, consider the inclusion

$$C_x \mathcal{V} \subset H_x \mathcal{V} = \Psi^{-1}(0).$$

By Proposition (1.3.6) and by (1.4.5)-(1.4.4) we have

$$\dim C_x \mathcal{V} = \dim \mathcal{V} \geq \dim \text{Ext}^1(F_x, F_x) - 1 = \dim \Psi^{-1}(0).$$

Since $\Psi^{-1}(0)$ is reduced irreducible, we must have $C_x \mathcal{V} = \Psi^{-1}(0)$. q.e.d.

III. The normal cone. Let

$$\mathcal{W} := \mathcal{V} \cap \Sigma^0_Q.$$ 

By Corollary (1.2.2) there is a $St(x)$-equivariant isomorphism

$$\text{Ext}^1(I_{Z_1}, I_{Z_1}) \cong \mathcal{W} \mathcal{V}.$$

(1.4.11)

(1.4.12) Claim. Keeping notation as above, $\mathcal{W}$ is smooth at $x$ and

$$T_x \mathcal{W} \cong \text{Ext}^1(I_{Z_1}, I_{Z_1}) \oplus \text{Ext}^1(I_{Z_2}, I_{Z_3}).$$

Furthermore, shrinking $\mathcal{V}$ if necessary, we can assume that

$$\dim T_x \mathcal{V} = \dim T_x \mathcal{W} \text{ for all } x' \in \mathcal{W}.$$
Proof. We continue identifying \((V, x)\) with Def\((F_x)\). First we prove (1.4.13). Let \(F\) be a first order deformation of \(F_x\) and let \(e = \sum_{i,j} e_{ij} \in \text{Ext}^1(F_x, F_x)\) be the corresponding extension class. Then \(e\) is tangent to \(W\) if and only if the \((2)\) exact sequences

\[0 \to I_{Z_i} \to F_x \to I_{Z_j} \to 0, \quad i \neq j,\]

lift to \(F\). This condition is equivalent [O2, (1.17)] to

\[e_{12} = e_{21} = 0.\]

This proves (1.4.13). To prove smoothness of \(W\), notice that \(W\) parametrizes all sheaves of the form \(I_{Z'} \oplus I_{W'}, \text{for}\) \(Z'\) near \(Z\) and \(W'\) near \(W\); this implies that \(\dim_x W \geq 2c\). On the other hand the right-hand side of (1.4.13) has dimension \(2c\), hence \(W\) is smooth at \(x\). To prove the last statement, notice that the family \(F\) of sheaves parametrized by \(V\) is complete at all \(x'\) in a neighborhood of \(x\), and that \(\dim \text{Ext}^1(F_{x'}, F_{x'})\) is constant for \(x' \in W\).

q.e.d.

Now let’s prove Proposition (1.4.1), except for (1.4.3). First we show \(\Sigma^0\) is smooth. Let \(x \in \Sigma^0\), \(V\) be a slice normal to \(O(x)\), and \(W := V \cap \Sigma^0\). By the Étale slice Theorem (1.2.1), a neighborhood of \(x\) in \(\Sigma^0\) is isomorphic to a neighborhood of \((1, x)\) in \(\text{PGL}(N) \times_{\text{St}(x)} W\). This last space is smooth at \((1, x)\) because by (1.4.12) \(W\) is smooth at \(x\). Thus \(x\) is a smooth point of \(\Sigma^0\). Now let’s prove (1.4.2). By (1.4.11), in order to prove (1.4.2) we must give an isomorphism

\[(C_W V)_x \cong \overline{\Psi}^{-1}(0),\]

where \(\overline{\Psi}\) is as in (1.4.7). By Claim (1.4.12) the normal bundle \(N_W V\) is defined, hence we have inclusions of cones

\[(C_W V)_x \subset (H_W V)_x \subset (N_W V)_x \cong \text{Ext}^1(I_{Z_1}, I_{Z_2}) \oplus \text{Ext}^1(I_{Z_2}, I_{Z_1}).\]

(The last isomorphism follows from (1.4.13).) Equality (1.3.2) together with Proposition (1.4.10) gives an isomorphism

\[(H_W V)_x \cong \overline{\Psi}^{-1}(0).\]

Arguing as in the proof of (1.4.10), we conclude that \((C_W V)_x = (H_W V)_x\). In detail:

\[\dim(C_W V)_x \geq \dim V - \dim W \geq \dim(H_W V)_x = \dim \overline{\Psi}^{-1}(0),\]

where the second inequality follows from (1.3.6). Since \((C_W V)_x \subset (H_W V)_x = \overline{\Psi}^{-1}(0)\) and \(\overline{\Psi}^{-1}(0)\) is reduced irreducible, we get (1.4.15). The rest of Proposition (1.4.1), except for (1.4.3), follows at once from (1.4.2).

q.e.d.

IV. Action of \(\text{St}(x)\). Let \(x \in Q^{ss}\) be a point with closed orbit \((x \text{ need not be in } \Sigma^0)\), and let \(V\) be a slice normal to \(O(x)\). Since \(\text{St}(x) = \text{Aut}(F_x)/\{\text{scalars}\}\), the action of \(\text{St}(x)\) on \(V\) defines also an action of \(\text{Aut}(F_x)\) on \(V\). For \(g \in \text{Aut}(F_x)\) we let

\[g_*: T_x V \to T_x V\]

be the differential at \(x\) of the map corresponding to \(g\).

(1.4.16) Lemma. Keeping notation as above, let

\[e \in T_x V \cong T_0 \text{Def}(F_x) \cong \text{Ext}^1(F_x, F_x).\]

Then \(g_*(e) = g \cup e \cup g^{-1}\).

Proof. Set \(F_0 = F_x\). Let \(F_1\) and \(E_1\) be the first order deformations of \(F_0\) corresponding to \(e\) and \(g_*e\) respectively. The lemma follows from the existence of an isomorphism \(\alpha_g: F_1 \to E_1\) fitting into a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & tF_0 & \to & F_1 & \to & F_0 & \to & 0 \\
& & \uparrow g^{-1} & & \alpha_g & \downarrow g & & \\
0 & \to & tF_0 & \to & E_1 & \to & F_0 & \to & 0.
\end{array}
\]
The isomorphism $\alpha_g$ exists because $\text{Aut}(F_0)$ also acts on the restriction to $X \times V$ of the tautological quotient on $X \times Q$, compatibly with the action on $V$. \hfill q.e.d.

Now let’s assume $x \in \Sigma^0_Q$, and let
\[ g = (\alpha, \beta) \in C^* \times C^* = \text{Aut}(F_x) \quad e = \sum_{i,j} e_{ij} \in \text{Ext}^1(F_x, F_x). \]

By Lemma (1.4.16) we have
\[ g^*(e) = g e g^{-1} = e_{11} + \alpha \beta^{-1} e_{12} + \alpha^{-1} \beta e_{12} + e_{22}. \]
Equation (1.4.3) follows at once from the above formula.

1.5. The normal cone to $\Omega^0_Q$.

Let $x \in \Omega^0_Q$. If $V$ is a two-dimensional complex vector space, then
\[ F_x \cong I_Z \otimes_C V, \quad \ell(Z) = n \quad St(x) = \text{Aut}(F_x)/\{ \text{scalars} \} \cong \text{PGL}(V). \]

Let $W := \text{sl}(V)$ be the Lie algebra of $\text{PGL}(V)$, and let
\[ (m, n) = 4\text{Tr}(mn) \]
be the Killing form. The adjoint representation gives an identification
\[ \text{PGL}(V) \cong \text{SO}(W). \]

Let
\[ E_Z = \text{Ext}^1(I_Z, I_Z). \]

Choose a non-zero two-form $\omega \in H^0(K_X)$: the composition
\[ E_Z \times E_Z \xrightarrow{\text{Yon}} \text{Ext}^2(I_Z, I_Z) \xrightarrow{\nabla} H^2(\mathcal{O}_X) \xrightarrow{i_!} H^2(K_X) \xrightarrow{\int_X} \mathbb{C} \]
defines a skew-symmetric form $\langle , \rangle$, non-degenerate by Serre duality. Set
\[ \text{Hom}^\omega(W, E_Z) := \{ \varphi : W \to E_Z | \varphi^* \langle , \rangle = 0 \}. \]
Then $St(x) = \text{SO}(W)$ acts (by composition) on $\text{Hom}^\omega(W, E_Z)$. In this subsection we will prove the following result.

(1.5.1) Proposition. Keep notation as above. Then $\Omega^0_Q$ is a smooth locally closed subset of $Q$, and its normal cone is a locally trivial bundle over $\Omega^0_Q$. If $x \in \Omega^0_Q$ there is a natural $St(x)$-equivariant isomorphism

\[ (C_{11}Q)_x \cong \text{Hom}^\omega(W, E_Z). \]

1. Yoneda square. There is a natural isomorphism
\[ \text{Ext}^p(F_x, F_x) \cong \text{Ext}^p(I_Z, I_Z) \otimes gl(V), \]
and Yoneda product is the tensor product of Yoneda product on $\text{Ext}^p(I_Z, I_Z)$ times the composition on $gl(V)$. Hence if $\Upsilon : E_Z \otimes gl(V) \to \text{sl}(V)$ is Yoneda square,
\[ \Upsilon \left( \sum_i e_i \otimes m_i \right) = \sum_{i,j} (e_i \cup e_j) \otimes (m_i m_j). \]
Since the composition
\[ \text{Ext}^2(I_Z, I_Z) \xrightarrow{\mathrm{Tr}} H^2(O_X) \xrightarrow{\cup} H^2(K_X) = \mathbb{C} \]
is an isomorphism we can write, with a slight abuse of notation,
\[ (1.5.3) \quad \Psi \left( \sum_i e_i \otimes m_i \right) = \frac{1}{2} \sum_{i,j} \langle e_i, e_j \rangle [m_i, m_j], \]
where \([\cdot, \cdot]\) is the commutator (bracket). Now consider the decomposition
\[ (1.5.4) \quad E_Z \otimes gl(V) = E_Z \otimes s(V) \oplus E_Z \otimes \text{Cld}_V = E_Z \otimes W \oplus E_Z \otimes \text{Cld}_V, \]
and let \( \Psi \dagger := \Psi|_{E_Z \otimes W} \). Since bracket in \( s(V) \) corresponds to wedge product in \( W \), we have
\[ \Psi \left( \sum_i e_i \otimes v_i \right) = \frac{1}{2} \sum_{i,j} \langle e_i, e_j \rangle v_i \wedge v_j. \]
The map \( \Psi \) has the following geometric interpretation. The Killing form \((\cdot, \cdot)\) on \( W \) gives isomorphisms
\[ E_Z \otimes W \cong \text{Hom}(W, E_Z) \quad W \cong \bigwedge^2 W^*. \]
Let \( \varphi \in \text{Hom}(W, E_Z) \); a straightforward computation shows that, via the above identifications,
\[ (1.5.5) \quad \Psi^{-1}(0) = \text{Hom}^\omega(W, E_Z). \]
\( (1.5.6) \) Lemma. \( \mathbb{P} \Psi^{-1}(0) \) is a reduced irreducible complete intersection of three quadrics in \( \mathbb{P}(E_Z \otimes W) \).
Since \( \mathbb{P} \Psi^{-1}(0) \) is a cone over \( \mathbb{P} \Psi^{-1}(0) \) with vertex \( \mathbb{P}(E_Z \otimes \text{Cld}_V) \), it follows that \( \mathbb{P} \Psi^{-1}(0) \) is a reduced irreducible complete intersection of three quadrics in \( \mathbb{P}(E_Z \otimes gl(V)) \).
\( \textbf{Proof.} \) The symmetric bilinear form
\[ \Psi \left( \sum_i e_i \otimes v_i, \sum_j f_j \otimes w_j \right) := \frac{1}{2} \sum_{i,j} \langle e_i, f_j \rangle v_i \wedge w_j \]
is the polarization of \( \Psi \), hence the differential of \( \Psi \) at \( \varphi = \sum_i e_i \otimes v_i \) is given by
\[ (1.5.7) \quad d \Psi(\varphi) : \text{Hom}(W, E_Z) \rightarrow \bigwedge^2 W^* \cong W \\
\sum_j f_j \otimes w_j \mapsto \frac{1}{2} \sum_{i,j} \langle e_i, f_j \rangle v_i \wedge w_j. \]
From the above formula one easily gets
\[ (1.5.8) \quad \text{rk}(d \Psi(\varphi)) = \begin{cases} 3 & \text{if } \text{rk} \varphi \geq 2, \\ 2 & \text{if } \text{rk} \varphi = 1, \\ 0 & \text{if } \varphi = 0. \end{cases} \]
Let \( \text{cr} \Psi \) be the set of critical points of \( \Psi \). Since \( c \geq 4 \), Equation (1.5.8) gives
\[ \dim \mathbb{P}(\text{cr} \Psi) = c + 2 < 3c - 4 = \dim \mathbb{P}(E_Z \otimes W) - 3. \]
This proves \( \mathbb{P} \Psi^{-1}(0) \) is a reduced complete intersection of three quadrics. To prove irreducibility, notice that by the above formulae
\[ (1.5.9) \quad \text{cod} \left( \text{sing} \mathbb{P} \Psi^{-1}(0), \mathbb{P} \Psi^{-1}(0) \right) \geq 2. \]
On the other hand, Equation (1.5.8) shows that
\[ \dim T_p \mathbb{P} \Psi^{-1}(0) = \dim \mathbb{P} \Psi^{-1}(0) + 1 \]
at every \( p \in \text{sing} \mathbb{P} \Psi^{-1}(0) \). Now assume \( \mathbb{P} \Psi^{-1}(0) \) is reducible. Since \( \mathbb{P} \Psi^{-1}(0) \) is connected, there are two irreducible components which meet. The above equality shows that the intersection of these components is locally the intersection of two divisors in a smooth ambient space, hence it has codimension one in \( \mathbb{P} \Psi^{-1}(0) \). This contradicts (1.5.9).
\( \textbf{q.e.d.} \)

**II. The cone at the origin of the deformation space.** Let \( V \) be a slice normal to the (closed) orbit \( \text{PGL}(N)x \): by Proposition (1.2.3) there is a natural isomorphism of germs \((V, x) \cong \text{Def}(F_x)\).
Proposition. Keeping notation as above, there are natural isomorphisms of schemes
\[ C_x V = H_x V \cong \Upsilon^{-1}(0). \]

Proof. The proof is similar to that of Proposition (1.4.10). By (1.3.5) and Lemma (1.5.6), we have
\[ (PH_x V)_{red} = P\Upsilon^{-1}(0). \]
By (1.5.6), \( P\Upsilon^{-1}(0) \) is a reduced irreducible complete intersection of quadrics. Since \( PH_x V \) is cut out by quadrics, it follows that
\[ PH_x V = P\Upsilon^{-1}(0). \]
Next, consider the inclusion
\[ C_x V \subset H_x V = \Psi^{-1}(0). \]
By Proposition (1.3.6) and (1.5.6) we have
\[ \dim C_x V = \dim V \geq \dim \text{Ext}^1(F_x,F_x) - 3 = \dim \Upsilon^{-1}(0). \]
By (1.5.6) \( \Upsilon^{-1}(0) \) is reduced irreducible, hence \( C_x V = \Upsilon^{-1}(0) \). q.e.d.

III. Proof of Proposition (1.5.1). Let \( V \) be a slice normal to the (closed) orbit \( \text{PGL}(N)x \), and let
\[ W := V \cap \Omega_0^0. \]
By Corollary (1.2.2) there is a \( \text{St}(x) \)-equivariant isomorphism
\[ (C_Q)_x \cong (C_W V)_x. \]
The following result is analogous to Claim (1.4.12).

Claim. Keeping notation as above, \( W \) is smooth at \( x \) and
\[ T_x W = E_Z \otimes \text{Cl}d_V. \]
(See Decomposition (1.5.defdec).) Furthermore, shrinking \( V \) if necessary, we can assume that
\[ \dim T_{x'} V = \dim T_x V \text{ for all } x' \in W. \]

Proof. Identifying the germ \((V,x)\) with \((\text{Def}(F_x),0)\) (see Proposition (1.2.3)), we see that a neighborhood of \( x \in W \) parametrizes all sheaves of the form \( I_{Z'} \oplus I_{Z''} \), for \( Z' \) near \( Z \). In particular
\[ \dim W \geq c = \dim E_Z. \]
Hence Equation (1.5.13) implies the first statement. We prove (1.5.13). Let
\[ \epsilon \in T_0\text{Def}(F_x) = \text{Ext}^1(F_x,F_x) = E_Z \otimes \text{gl}(V), \]
and let \( \mathcal{F} \) be the corresponding first order deformation of \( F_x \). Then \( \epsilon \) is tangent to \( W \) if and only if, for every exact sequence
\[ 0 \to C \to V \to C \to 0 \]
the following holds: the exact sequence
\[ 0 \to I_Z \to F_x \to I_Z \to 0 \]
obtained tensoring (e) by L, lifts to \( \mathcal{F} \). Choosing a basis of \( V \) adapted to (e), write \( \epsilon \) as a \( 2 \times 2 \)-matrix with entries in \( E_Z \), then by [O2, (1.17)] Exact sequence (f) lifts to \( \mathcal{F} \) if and only if \( \epsilon \) is upper triangular. Since this must hold for any choice of (e), the matrix \( \epsilon \) must be a scalar, i.e. \( \epsilon \in E_Z \otimes \text{Cl}_V \). To prove the second statement, notice that the family \( \mathcal{F} \) of sheaves parametrized by \( V \) is complete at all \( x' \) in a neighborhood of \( x \), and that \( \dim \text{Ext}^1(F_{x'}, F_{x'}) \) is constant for \( x' \in \mathcal{W} \).

Using (1.5.12) and arguing as in the proof of Proposition (1.4.1) we see that \( \Omega^0_Q \) is smooth. Let’s prove Isomorphism (1.5.2) (we leave it to the reader to verify that the normal cone of \( \Omega^0_Q \) is locally trivial). By (1.5.11)-(1.5.5) we must give an isomorphism

\[
(C_WV)_x \cong \mathbf{T}^{-1}(0).
\]

We proceed as in the proof of (1.4.15). By (1.5.12) the normal bundle \( N_WV \) is defined, hence we have inclusions of cones

\[
(C_WV)_x \subset (H_WV)_x \subset (N_WV)_x \cong E_Z \otimes W.
\]

(The last isomorphism follows from (1.5.13).) By (1.3.2) and (1.5.10) we have

\[
(H_WV)_x \cong \mathbf{T}^{-1}(0).
\]

Furthermore

\[
\dim (C_WV)_x \geq \dim V - \dim W \geq \dim (H_WV)_x = \dim \mathbf{T}^{-1}(0),
\]

where the second inequality follows from (1.3.6). Since \( (C_WV)_x \subset (H_WV)_x = \mathbf{T}^{-1}(0) \) and \( \mathbf{T}^{-1}(0) \) is reduced irreducible, we get (1.5.14). Finally, to prove that Isomorphism (1.5.2) is \( \text{St}(x) \)-equivariant, we apply Lemma (1.4.16) to \( (V, x) = \text{Def}(F_x) \). If \( g \in \text{Aut}(F_x) = \text{GL}(V) \), and \( \epsilon \in \text{Ext}^1(F_x, F_x) = E_Z \otimes \text{gl}(V) \),

\[
g_*\epsilon = g \cup \epsilon \cup g^{-1}.
\]

Since Yoneda products are given by composition in \( \text{gl}(V) \), the automorphism group \( \text{GL}(V) \) acts by conjugation on \( E_Z \otimes \text{gl}(V) \), i.e. we get the standard action of \( \text{SO}(W) \) on \( W \).

### 1.6. Semistable points in the exceptional divisor of \( \pi_R \)

Recall that \( \pi_R: R \to Q \) is the blow-up of \( \Omega_Q \). Let \( \Omega_R \subset R \) be the exceptional divisor. By Theorem (1.1.2) we have

\[
\pi_R(\Omega_R^{ss}) \subset \Omega_Q^{ss} = \Omega_Q^0.
\]

In order to describe \( \Omega_R^{ss} \) we will need a general result in the style of Theorem (1.1.2). Let \( G \) be a reductive group acting linearly on a projective scheme \( Y \), let \( W = Y \) be a \( G \)-invariant closed subscheme, let \( \pi: \widetilde{Y} \to Y \) be the blow-up of \( W \). Then \( G \) acts also on \( D_t := \pi^*\mathcal{O}_Y(t)(-E) \), where \( E \) is the exceptional divisor. The following lemma can be extracted from [K]; we sketch a proof for the reader’s convenience.

#### (1.6.1) Lemma. Keep notation as above. If \( \ell \gg 0 \) then the following holds. Let \( \mathbf{y} \in \widetilde{Y} \) be such that \( y = \pi(\mathbf{y}) \) is semistable with orbit closed in \( Y^{ss} \). Then \( \mathbf{y} \) is \( G \)-(semi)stable if and only if it is (semi)stable for the action of \( \text{St}(y) \) on \( \pi^{-1}(y) \) (with the linearization obtained by restriction).

**Proof.** First we prove the lemma when \( G \) is a torus \( T \). Let \( \ell_0 \) be such that \( \pi^*\mathcal{O}_Y(\ell_0)(-E) \) is very ample. Let \( \{\sigma_i\}, \{\tau_j\} \) be diagonal bases for the action of \( T \) on \( H^0(D_{\ell_0}) \) and \( H^0(\mathcal{O}_Y(1)) \) respectively. Let \( p_i, q_j \) be the corresponding characters of \( T \). Letting \( \Phi \) be the lattice of characters of \( T \), and \( \Phi_R := \Phi \otimes \mathbb{R} \), we set

\[
\Delta_\sigma := \text{convex hull in } \Phi_R \text{ of the } p_i \text{ such that } \sigma_i(\mathbf{y}) \neq 0,
\]

\[
\Delta_\tau := \text{convex hull in } \Phi_R \text{ of the } q_j \text{ such that } \tau_j(\mathbf{y}) \neq 0,
\]

\[\Lambda := \text{linear span of } \Delta_\tau.\]

Since \( O(y) \) is closed in \( Y^{ss} \), there is an open \( U \subset \Lambda \) such that \( 0 \in U \subset \Delta_\tau \). Thus there exists \( m_0 \) such that the following holds for all \( m \geq m_0 \). If \( V \subset \Phi_R \) is a codimension-one subspace not containing \( \Lambda \), the image
of \((\Delta_\sigma + m\Delta_r)\) in \(\Phi_R/V\) contains an open neighborhood of the origin. We claim that if \(\ell = (\ell_0 + m)\), with \(m \geq m_0\), the lemma holds for \(\tilde{y}\). By the numerical criterion for (semi)stability [Mm, (2.1)] it suffices to show that if \(\lambda\) is a one-parameter subgroup of \(G\) not contained in \(St(y)\) then \(\lambda\) does not desemistabilize \(\tilde{y}\). Semistability with respect to the complete linear system \(H^0(D_\ell)\) is the same as with respect to the sublinear system

\[ H^0(D_{\ell_0}) \otimes H^0(\pi^*O_Y(m)). \]

Identifying one-parameter subgroups of \(T\) with \(\text{Hom}(\Phi, \mathbb{Z})\), we get a one-to-one correspondence

\[ \{\lambda: \mathbb{C}^* \to T| \lambda(\mathbb{C}^*) y = y\} \leftrightarrow \{f \in \text{Hom}(\Phi, \mathbb{Z})| \Delta_r \subset \ker(f \otimes \mathbb{R})\}. \]

Hence, with our choice of \(m\), if \(\lambda\) does not fix \(y\) then \(\lambda\) does not desemistabilize \(\tilde{y}\) with respect to the above sublinear system. This proves the result for a single \(y\), but in fact we can choose an \(m_0\) which works for every \(y\). Now let \(G\) be an arbitrary reductive group. Let \(T < G\) be a maximal torus. Since the lemma holds for the action of \(T\), and since every one-parameter subgroup of \(G\) is conjugate to a subgroup of \(T\), the numerical criterion for (semi)stability shows that the lemma holds also for the action of \(G\).  

**Remark.** If \(y\) is stable, the proof above is exactly Kirwan’s proof of (1.1.4).

Let \(x \in \Omega^0_Q\), and set \(F_x = (I_Z \oplus I_Z)\), where \([Z] \in X^{[n]}\). Let

\[
(1.6.2) \quad \text{Hom}_k(W, E_Z) := \{\varphi \in \text{Hom}(W, E_Z)| \, \text{rk} \varphi \leq k\},
\]

\[
\text{Hom}_k^\omega(W, E_Z) := \text{Hom}_k(W, E_Z) \cap \text{Hom}^\omega(W, E_Z).
\]

By Proposition (1.5.1) there is a locally trivial fibration

\[
\text{PHom}^\omega(W, E_Z) \longrightarrow \pi^{-1}_R(\Omega^0_Q) \quad \text{with fiber the cone over a smooth quadric in } \mathbb{P}^{2c-5}.
\]

**Proposition.** Keeping notation as above, a point \([\varphi] \in \text{PHom}^\omega(W, E_Z)\) is PGL\((N)\)-semistable if and only if:

\[
\text{rk} \varphi \begin{cases} \geq 2, & \text{or} \\ = 1 & \text{and } \ker \varphi^\perp \text{ is non-isotropic}. \end{cases}
\]

**Proof.** By Lemma (1.6.1) \([\varphi]\) is PGL\((N)\)-semistable if and only if it is semistable for the SO\((W)\)-action on \(\text{PHom}^\omega(W, E_Z)\). The proposition follows easily from the numerical criterion for semistability.  

**q.e.d.**

1.7. Description of \(\Sigma^\text{ss}_R\).

Recall that \(\Sigma_R \subset R\) is the strict transform of \(\Sigma_Q\) under the blow-up \(\pi_R: R \to Q\). We are interested in \(\Sigma^\text{ss}_R\), a locally closed subset of \(R\). The main result of this subsection is the following.

**Proposition.** Keeping notation as above,

1. \(\Sigma^\text{ss}_R\) is smooth,
2. The scheme-theoretic intersection of \(\Sigma^\text{ss}_R\) and \(\Omega_R\) is smooth and reduced.
3. The normal cone of \(\Sigma^\text{ss}_R\) in \(R\) is a locally trivial bundle over \(\Sigma^\text{ss}_R\), with fiber the cone over a smooth quadric in \(\mathbb{P}^{2c-5}\).

I. Description of \(\Sigma^\text{ss}_R \setminus \Omega_R\). We will prove that

\[
(1.7.2) \quad \Sigma^\text{ss}_R \setminus \Omega_R = \pi^{-1}_R(\Sigma_Q^0).
\]

By Theorem (1.1.2) we know

\[
\Sigma^\text{ss}_R \setminus \Omega_R \subset \pi^{-1}_R(\Sigma^\text{ss}_Q - \Omega_Q).
\]
From (1.1.7) one gets
\[ \Sigma_Q^0 \subset \left( \Sigma_Q^{ss} - \Omega_Q \right) \subset \Sigma_Q^0 \cup \Gamma_Q^0. \]
In fact the middle term is equal to the third term, but we will not need this because

\[ (1.7.3) \quad \pi_R^{-1}(\Gamma_Q^0) \cap R^{ss} = \emptyset. \]

Before proving (1.7.3) we give a general lemma. We assume \( G \) is a reductive group acting linearly on a complex projective scheme \( Y \), and \( V \subset Y \) is a \( G \)-invariant closed subscheme. Let \( \pi: \bar{Y} \to Y \) be the blow-up of \( V \).

\[ \textbf{(1.7.4) Lemma.} \quad \text{Keep notation as above. Let } \bar{x} \in \bar{Y} \text{ be a point such that } x := \pi(\bar{x}) \text{ satisfies:} \]
\[ x \notin V; \quad \bar{O}(x) \cap V^{ss} \neq \emptyset. \]

Then \( \bar{x} \) is not semistable.

\[ \textbf{Proof.} \quad \text{It follows from our hypothesis that there exists a one-parameter subgroup } \lambda: \mathbb{C}^* \to G \text{ such that} \]
\[ \lim_{t \to 0} \lambda(t)x = y \in V^{ss}. \]

Let \( E \) be the exceptional divisor of \( \pi \), and
\[ \sigma_0, \ldots, \sigma_r \in H^0(\pi^*\mathcal{O}_Y(\ell)(-E)) \]
be a diagonal basis for the action of \( \lambda \); set \( \lambda(t)\sigma_i = t^{n_i}\sigma_i \). We make the identification
\[ H^0(\pi^*\mathcal{O}_Y(\ell)(-E)) \cong H^0(I_V(\ell)). \]

Since \( y \) is semistable, \( n_i \geq 0 \) for all \( i \) such that \( \sigma_i(\bar{x}) = \sigma_i(x) \neq 0 \). We will show that in fact \( n_i > 0 \) for all such \( i \); this will prove \( \bar{x} \) is not semistable. Suppose \( n_i = 0 \); then \( \sigma_i \) is \( \lambda \)-invariant, hence \( \sigma_i(x) \neq 0 \) implies \( \sigma_i(y) \neq 0 \). This is absurd because \( \sigma_i \) vanishes on \( V \).

Let’s prove (1.7.3). Assume \( \pi_R(y) = x \in \Gamma_Q^0 \). Then \( \bar{O}(x) \cap \Omega_Q^0 \neq \emptyset \), hence by Lemma (1.7.4) \( y \) is not semistable. This proves the left-hand side of (1.7.2) is contained in the right-hand side. Let’s show that \( \pi_R^{-1}(\Sigma_Q^0) \) is contained in \( R^{ss} \). For \( y \in \pi_R^{-1}(\Sigma_Q^0) \), let \( x = \pi_R(y) \). Since \( O(x) \) is closed in \( Q^{ss} \), and disjoint from the closed \( \text{PGL}(N) \)-invariant subset \( \Omega_Q \), there exists a \( \text{PGL}(N) \)-invariant section \( \sigma \in H^0(\mathcal{O}_Q(\ell)) \) such that \( \sigma(x) \neq 0 \) and \( \sigma \) vanishes on \( \Omega_Q \). Viewing \( \sigma \) as a (invariant) section of \( \pi_R^*\mathcal{O}_Q(\ell)(-E) \) we see that \( y \) is semistable.

\[ \textbf{II. Description of } \Sigma_R^{ss} \cap \Omega_R. \quad \text{Of course } \Sigma_R^{ss} \cap \Omega_R \text{ is contained in } \Omega_R^{ss} \subset \pi_R^{-1}\Omega_Q^{ss} = \pi_R^{-1}\Omega_Q^0. \]

\[ \textbf{(1.7.5) Lemma.} \quad \text{Let } x \in \Omega_Q^0, \text{ and set } F_x = (I_Z \oplus I_Z). \text{ Then} \]
\[ (1.7.6) \quad \pi_R^{-1}(x) \cap \Sigma_R^{ss} = \text{PHom}_{1}^{ss}(W, E_Z). \]

\[ \textbf{Proof.} \quad \text{If } y \in \Sigma_Q^0 \text{ then } St(y) \cong \mathbb{C}^*. \text{ Thus } \text{dim } St(\bar{y}) \geq 1 \text{ for all } \bar{y} \in \Sigma_R. \text{ In particular, if} \]
\[ [\phi] \in \pi_R^{-1}(x) \cap \Sigma_R^{ss}, \]
the stabilizer \( St([\phi]) \) (\( < \text{SO}(W) \)) is positive dimensional. An easy analysis shows that \( St([\phi]) \) is positive dimensional if and only if \( \text{rk} \phi = 1 \). By Proposition (1.6.3) we conclude that the left-hand side of (1.7.6) is contained in the right-hand side. Let’s prove inclusion in the other direction. Assume \( [\phi] \in \text{PHom}_{1}^{ss}(W, E_Z) \). The identifications
\[ W^* \cong \text{sl}(V)^\vee \cong \text{sl}(V) \]

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(the second isomorphism is given by the Killing form), allow us to write
\[ \varphi = m \otimes \alpha, \quad m \in \mathfrak{sl}(V), \quad \alpha \in E_Z, \quad \text{Tr}(m^2) \neq 0. \]

Since \( \text{Tr}(m^2) \neq 0 \) we can diagonalize \( m \), hence using a basis of eigenvectors we can write
\[ (1.7.7) \quad \varphi = \begin{bmatrix} e & 0 \\ 0 & -e \end{bmatrix} \quad e \in E_Z. \]

Since \( X^{[n]} \) is smooth there exist sheaves \( \mathcal{L}, \mathcal{L}' \) on \( X \times C \), flat over \( C \), where \( C \) is a smooth curve, such that the following holds. For a certain point \( 0 \in C \)
\[ \mathcal{L}_0 \cong \mathcal{L}'_0 \cong I_Z, \]
and furthermore, if \( \kappa, \kappa' \) are the Kodaira-Spencer maps of \( \mathcal{L}, \mathcal{L}' \) at 0, respectively, then
\[ \kappa(\partial/\partial t) = e, \quad \kappa'(\partial/\partial t) = -e, \quad \partial/\partial t \in T_0 C. \]

Lastly we can assume \( \mathcal{L}_p \not\cong \mathcal{L}'_p \) for all \( p \neq 0 \). Set \( G := \mathcal{L} \oplus \mathcal{L}' \). If \( \mathcal{V} \) is a slice normal to \( O(x) \) at \( x \) then, by versality of the tautological quotient on \( X \times \mathcal{V} \) (Proposition (1.2.3)), there exists a map
\[ f: C \to \mathcal{V} \quad f(0) = x \]
such that \( \varphi = f(0) \in \Sigma^{ss}_R \cap \Omega_R \).

q.e.d.

III. Explicit construction of \( \Sigma^{ss}_R \) and proof of Items (1)-(2) of (1.7.1). Let
\[ \beta: \mathcal{X}^{(n)} \to X^{[n]} \times X^{[n]} \]
be the blow-up of the diagonal. Set \( N := h^0(I_Z(k) \oplus I_W(k)) \), where \( [Z], [W] \in X^{[n]} \). Let
\[ \alpha: \tilde{U} \to \mathcal{X}^{(n)} \]
be the principal \( \text{PGL}(N) \)-fibration whose fiber over \( y \in \mathcal{X}^{(n)} \) is
\[ (1.7.8) \quad \text{P} \text{Isom} \left( \mathcal{C}^N, H^0 \left( I_Z(k) \oplus I_W(k) \right) \right), \]
where \( ([Z], [W]) = \beta(y) \). Over \( \tilde{U} \times X \) there is a tautological family of quotients
\[ (1.7.9) \quad \mathcal{O}^{(N)}_{\tilde{U} \times X} \to \mathcal{L}_1(k) \oplus \mathcal{L}_2(k) \to 0. \]

Here \( \mathcal{L}_i \) is defined as follows. Let
\[ p_i: X^{[n]} \times X^{[n]} \times X \to X^{[n]} \times X, \quad i = 1, 2, \]
be the projection which forgets the \( i \)-th factor, and let \( Z \subset X^{[n]} \times X \) be the tautological subscheme; then
\[ \mathcal{L}_i(k) := ((\beta \circ \alpha) \times \text{id}_X)^* p_i^* (I_Z \otimes \mathcal{O}_X(k)). \]
The family of quotients (1.7.9) defines a morphism
\[ \tilde{h}: \tilde{U} \to Q \quad \tilde{h}(\tilde{u}) = (\Sigma^0_Q \cup \Omega^0_Q). \]

There is an action of O(2) on \( \tilde{U} \). In fact realize O(2) as the subgroup of PGL(2) generated by
\[ \text{SO}(2) = \left\{ \theta_\alpha := \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \right\}/\{\pm \text{Id}\}, \quad \tau := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \]

Then each \( \theta_\alpha \) can be viewed as an isomorphism of \( L_1(k) \oplus L_2(k) \), hence it acts on \( \tilde{U} \). Similarly, let \( \iota: U \to U \) be the map induced by the involution interchanging the factors of \( X^{[n]} \times X^{[n]} \); viewing \( \tau \) as an isomorphism between \( L_1(k) \oplus L_2(k) \) and \( \iota^* (L_1(k) \oplus L_2(k)) \) we get an action of \( \iota \) on \( \tilde{U} \). Consider the G.I.T. quotient
\[ \mathcal{U} := \tilde{U}/O(2). \]

A word about the linearization. Taking first the quotient by SO(2) and then by O(2)/SO(2) we see that a choice of linearization is relevant only for the quotient \( \mathcal{U}/\text{SO}(2) \). Since SO(2) preserves the fibers of \( \alpha \) we need only specify what is the linearization on each fiber (1.7.8); but the fiber is affine, thus a linearization is “not needed”, or more precisely we choose the trivial linearization of the trivial line-bundle. The action of O(2) is free, hence \( \mathcal{U} \) is an orbit space, and since \( \mathcal{U} \) is smooth, also \( \mathcal{U} \) is smooth. The map \( \tilde{h} \) is constant on O(2)-orbits, hence it factors through a map
\[ h_Q: U \to (\Sigma^0_Q \cup \Omega^0_Q). \]

The following proposition proves Items (1)-(2) of (1.7.1).

**Proposition.** Keep notation as above. The map \( h_Q \) lifts to a map \( h_R: U \to R \), whose image is \( \Sigma^s_R \). Furthermore
\[ h_R: U \to \Sigma^s_R \]
is an isomorphism, in particular \( \Sigma^s_R \) is smooth. Finally, the scheme-theoretic intersection of \( \Sigma^s_R \) and \( \Omega_R \) is smooth and reduced.

**Proof.** First let’s prove that \( h_Q \) lifts. Let \( D^{[n]} \subset X^{[n]} \times X^{[n]} \) be the diagonal, and set
\[ D^{(n)} := \text{exc. div. of } \beta = \beta^{-1}D^{[n]}, \quad \tilde{D} := \alpha^*D^{(n)}, \quad D := \tilde{D}/\text{O}(2). \]
Notice that \( \tilde{D} \) is an O(2) invariant divisor, hence \( D \) is a (Cartier) divisor. We will show that
(\dag) \quad \tilde{h}^*I_{\Omega_Q} = O_{\tilde{U}}(-\tilde{D}).

This implies that \( h_Q^*I_{\Omega_Q} = O_U(-D) \), and hence, by the universal property of blow-up, \( h_Q \) lifts. As sets \( \tilde{h}^{-1}\Omega_Q = \tilde{D} \), hence to prove (\dag) it will suffice to show the following: for any \( u \in \tilde{D} \), there exists a tangent vector \( v \in T_u\tilde{U} \) such that
(\ast) \quad (\tilde{h})_*(v) \notin T_{\tilde{u}(u)}\Omega_Q.

To prove it, let \( y = \alpha(u) \), \( ([Z], [Z]) = \beta(y) \), and \( x = \tilde{h}(y) \). The normal bundle of \( D^{[n]} \) in \( X^{[n]} \times X^{[n]} \) is canonically identified with the \((-1)\)-eigenbundle for the action on
\[ (T_{X^{[n]} \times X^{[n]}})|_{D^{[n]}} \]
of the involution interchanging the factors of \( X^{[n]} \times X^{[n]} \), hence
\[ \beta^{-1}([Z], [Z]) = P \{ (e, e') \in \text{Ext}^1(I_Z, I_Z) \oplus \text{Ext}^1(I_Z, I_Z) \mid e = -e' \}. \]
Thus, if \( w \in T_y \mathcal{U}^{(n)} \) is transversal to \( T_y D^{(n)} \),

\[
\beta_s(w) = (e, -e) \quad 0 \neq e \in \text{Ext}^1(I_Z, I_Z).
\]

Since \( \alpha \) is a smooth fibration there exists \( v \in T_u \mathcal{U} \) such that \( \alpha_s(v) = w \). Identifying \((C_\Omega Q)_x\) with \( \text{Hom}^\omega(W, E_Z) \) as in (1.5.2) we see that

\[
(1.7.11)
\]

\[
\bar{h}_s(v) = e \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

In particular, since the right-hand side is non-zero, we have proved \((*)\). Thus the lift \( h_R \) exists. Now let’s prove that \( h_R \) is an isomorphism between \( \mathcal{U} \) and \( \Sigma_R^{ss} \). Clearly \( h_R(\mathcal{U} - D) = \pi^{-1}_R \Sigma_Q^{ss} \), and hence by (1.7.2) we get a surjective map

\[
h_R|_{(\mathcal{U} - D)}: (\mathcal{U} - D) \to (\Sigma_R^{ss} \setminus \Omega_R).
\]

The above map is clearly one-to-one, and since \( (\Sigma_R^{ss} \setminus \Omega_R) \) is smooth it must be an isomorphism. Next, let’s examine \( h_R|D \). The restriction

\[
\bar{h}|_D: \bar{D} \to \Omega_R^0
\]

is surjective; if \( x \in \Omega_Q^0 \), and \( F_x = V \otimes I_Z \), then

\[
\bar{h}^{-1}(x) = \text{PGL}(V) \times \mathbf{P}(\text{Ext}^1(I_Z, I_Z))
\]

and thus

\[
h_Q^{-1}(x) = (\text{PGL}(V)/O(2)) \times \mathbf{P}(E_Z) = \mathbf{P}(\{\varphi \in W \mid \varphi \text{ is non-isotropic}\}) \times \mathbf{P}(E_Z).
\]

On the other hand, by (1.7.6) the term on the right is identified, via the Segre embedding, with \( \pi^{-1}_R(x) \cap \Sigma_R^{ss} \). It follows from (1.7.11) that

\[
h_R|_{h_Q^{-1}(x)}: h_Q^{-1}(x) \to \pi^{-1}_R(x) \cap \Sigma_R^{ss}
\]

is the Segre isomorphism. Since \( \Sigma_R^{ss} \cap \Omega_R \) is smooth, we get that

\[(\bullet) \quad h_R|D: D \to \Sigma_R^{ss} \cap \Omega_R
\]

is an isomorphism. Now we can finish proving that \( h_R \) is an isomorphism onto \( \Sigma_R^{ss} \). First notice that \( h_R \) is a homeomorphism, hence \( \Sigma_R^{ss} \) is unibranch at every point of \( \Sigma_R^{ss} \cap \Omega_R \). Thus it remains only to show that the differential \( dh_R(u) \) is an isomorphism for all \( u \in D \). Since \((\bullet)\) is an isomorphism it is sufficient to verify that \( (h_R)_* (v) \notin T_{h_R(y)} \Omega_R \) for some \( v \in T_u \mathcal{U} \): look at \((*)\). The proof just given also shows that the scheme-theoretic intersection of \( \Sigma_R^{ss} \) and \( \Omega_R \) is smooth and reduced. \textbf{q.e.d.}

\textbf{IV. Proof of Item (3) of Proposition (1.7.1).} Since \( \pi_R \) is an isomorphism outside \( \Omega_R \), it follows from (1.7.2) that the normal cone of \( (\Sigma_R^{ss} \setminus \Omega_R) \) in \( R \) is isomorphic to that of \( \Sigma_Q^{ss} \) in \( Q \). Thus “outside \( \Omega_R \)” Item (3) of (1.7.1) follows from Proposition (1.4.1). Now let \( y \in \Sigma_R^{ss} \cap \Omega_R \), and set \( x = \pi_R(y) \). Following the notation of Lemma (1.7.5) we let \( \varphi = [\varphi] \), where \( \varphi \in \text{Hom}_1(W, E_Z) \). Let \( \omega_\varphi \) be the symplectic form induced by \( \omega \) on \( (\text{Im} \varphi^+/\text{Im} \varphi) \). We let \( \text{Hom}^\omega(\ker \varphi, \text{Im} \varphi^+/\text{Im} \varphi) \) be the set of homomorphisms whose image is \( \omega_\varphi \)-isotropic.

\textbf{Proposition.} Keep notation as above. If \( [\varphi] \in \Sigma_R^{ss} \cap \Omega_R \) there is a \( \text{St}([\varphi]) \)-equivariant isomorphism

\[
(1.7.12) 
\]

\[
(C_{\Sigma R})_{[\varphi]} \cong \text{Hom}^\omega(\ker \varphi, \text{Im} \varphi^+/\text{Im} \varphi).
\]

The above proposition implies that Item (3) of (1.7.1) holds also “over \( \Sigma_R^{ss} \cap \Omega_R \).” In fact the right-hand side of (1.7.12) embeds into the \( 2(c - 2) \)-dimensional vector space \( \text{Hom}(\ker \varphi, \text{Im} \varphi^+/\text{Im} \varphi) \), and since \( \omega_\varphi \) is non-degenerate the image is the affine cone over a smooth projective quadric. Let’s prove (1.7.12). By Item (2) of (1.7.1) the Cartier divisor \( \Omega_R \) intersects transversely \( \Sigma_R^{ss} \), hence

\[
(C_{\Sigma R})_{[\varphi]} \cong (C_{\Sigma \cap \Omega R})_{[\varphi]}.
\]

Furthermore \( \Omega_R^{ss} \to \Omega_Q^{ss} \) is a locally-trivial fibration with smooth base, hence we can replace the right-hand side of the above equation by the normal in the fiber through \([\varphi]\). More precisely, if \( x = \pi_R([\varphi]) \),

\[
(C_{\Sigma \cap \Omega R})_{[\varphi]} \cong (C_{\text{Hom}_1(W, E_Z)} \text{Hom}^\omega(W, E_Z))_{[\varphi]}.
\]

Thus (1.7.12) follows from the following result.
Lemma. Let $[\varphi] \in \text{PHom}_1(W, E_Z)$ (not necessarily semistable). There is a $\text{St}([\varphi])$-equivariant isomorphism

\begin{equation}
(C_{\text{PHom}_1(W, E_Z)} \text{PHom}^\omega(W, E_Z))_{[\varphi]} \cong \text{Hom}^{\omega \cdot \varphi}(\text{Ker} \varphi, \text{Im} \varphi^*/\text{Im} \varphi).
\end{equation}

Proof. Clearly

\begin{equation}
(C_{\text{PHom}_1(W, E_Z)} \text{PHom}^\omega(W, E_Z))_{[\varphi]} \cong (C_{\text{Hom}_1(W, E_Z)} \text{Hom}^\omega(W, E_Z))_{\varphi},
\end{equation}

and we will work with the right-hand side. We will show that the Hessian cone of $\text{Hom}_1(W, E_Z)$ in $\text{Hom}^\omega(W, E_Z)$ satisfies the hypothesis of Lemma (1.3.6), hence the normal cone will be equal to the Hessian cone. First we check that the Hessian cone is defined, i.e. that (1.3.1) is satisfied. First $\text{Hom}_1(W, E_Z)$ is smooth, and secondly, since $\text{Hom}^\omega(W, E_Z)$ is the zero-scheme of $\Upsilon_0$, and the differential $d \Upsilon_0$ has constant rank along $\text{Hom}_1(W, E_Z)$ by (1.5.6), the tangent space to $\text{Hom}^\omega(W, E_Z)$ has constant rank along $\text{Hom}_1(W, E_Z)$. By (1.3.5) the Hessian cone is given by the zeroes of the Hessian map along $\text{Hom}^\omega(W, E_Z)$.

Let's compute the Hessian cone. For this we will choose bases $\{e_1, \ldots, e_{2n}\}$ of $E_Z$ and $\{v_1, v_2, v_3\}$ of $W$ such that $\varphi = e_1 \otimes v_1$, and so that

\begin{equation}
\langle e_i, v_j \rangle = \begin{cases} 
1 & \text{if } i = 2q - 1, j = 2q, \\
-1 & \text{if } i = 2q, j = 2q - 1, \\
0 & \text{otherwise}.
\end{cases}
\end{equation}

Using Formula (1.5.7) for the differential $d \Upsilon_0$ we get

\begin{equation}
d \Upsilon_0(\varphi) \left( \sum_{ij} Z_{ij} e_i \otimes v_j \right) = -\frac{1}{2} Z_{2,2} v_2 \wedge v_3 - \frac{1}{2} Z_{2,3} v_1 \wedge v_3,
\end{equation}

hence

\begin{equation}
(T \text{Hom}^\omega(W, E_Z))_{\varphi} = \left\{ \sum_{ij} Z_{ij} e_i \otimes v_j \middle| Z_{2,2} = Z_{2,3} = 0 \right\}.
\end{equation}

Furthermore

\begin{equation}
(T \text{Hom}_1(W, E_Z))_{\varphi} = \left\{ \sum_{ij} Z_{ij} e_i \otimes v_j \middle| Z_{ij} = 0 \text{ if } i \geq 2, j \geq 2 \right\}.
\end{equation}

Thus we have an isomorphism

\begin{equation}
(N_{\text{Hom}_1} \text{Hom}^\omega(W, E_Z))_{\varphi} \cong \left\{ \sum_{3 \leq i} Z_{ij} e_i \otimes v_j \right\}.
\end{equation}

To give an intrinsic formulation notice that there is a natural isomorphism

\begin{equation}
(N_{\text{Hom}_1} \text{Hom}(W, E_Z))_{\varphi} \cong \text{Hom}(\text{Ker} \varphi, E_Z/\text{Im} \varphi),
\end{equation}

hence Isomorphism (1.7.15) can be read as

\begin{equation}
(N_{\text{Hom}_1} \text{Hom}^\omega(W, E_Z))_{\varphi} \cong \left\{ \alpha : \text{Ker} \varphi \rightarrow E_Z/\text{Im} \varphi \middle| \text{Im} \alpha \subset (\text{Im} \varphi^*/\text{Im} \varphi) \right\}.
\end{equation}

Referring to (1.3.4), it follows from (1.7.14) that $p_{E}(\bullet, v_1) = (\bullet, v_1)$: a computation gives the following equation (see (1.3.4)) for the Hessian cone of $\text{Hom}_1(W, E_Z)$ in $\text{Hom}^\omega(W, E_Z)$ at $\varphi$:

\begin{equation}
\sum_{2 \leq q \leq n} (Z_{2q-1,2} Z_{2q,3} - Z_{2q,2} Z_{2q-1,3}) = 0.
\end{equation}
The following is the main result of this section.

Let \( \Delta > \) only if \( \hat{\Delta} = \Omega \).

In particular \( \Delta \) is induced by the PGL(\( \pi \))\). Set \( \Omega := \Omega_{\pi} \). 

\[ \begin{align*}
    (v_1, v_i) &= -\delta_{i1}, \\
    (v_j, v_j) &= 0, \\
    (v_2, v_3) &= 1,
\end{align*} \]

and \( v_1 \wedge v_2 \wedge v_3 \) is the volume form. Easy considerations show that \( St([\varphi]) = O(\ker \varphi) \), and more precisely \( St([\varphi]) \) is generated by

\[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^{-1} \end{bmatrix} \quad \tau := \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \].

The action of \( St([\varphi]) \) on \( (C_S R)_{[\varphi]} \) is given by:

\[ \begin{align*}
    \theta_{\varphi} \left( \sum_{3 \leq i} (Z_{i,2} e_i \otimes v_2 + Z_{i,3} e_i \otimes v_3) \right) &= \sum_{3 \leq i} (\alpha Z_{i,2} e_i \otimes v_2 + \alpha^{-1} Z_{i,3} e_i \otimes v_3), \\
    \tau \left( \sum_{3 \leq i} (Z_{i,2} e_i \otimes v_2 + Z_{i,3} e_i \otimes v_3) \right) &= \sum_{3 \leq i} (-Z_{i,3} e_i \otimes v_2 - Z_{i,2} e_i \otimes v_3).
\end{align*} \] (1.7.16)

1.8. Analysis of Kirwan’s desingularization.

Recall that \( \pi_S: S \rightarrow R \) is the blow-up of \( R \) along \( \Sigma_R \) (see (1.1.10)). Let \( \Omega_S \subset S \) be the strict transform of \( \Omega_R \), and \( \Sigma_S \subset S \) be the exceptional divisor (i.e. the inverse image of \( \Sigma_R \)). Let \( x \in \Omega_S^0 \), and set \( F_x = I_Z \oplus I_{\bar{Z}} \).

By Item (2) of (1.7.1) and by (1.7.6) we have

\[ (\pi_R \circ \pi_S)^{-1}(x) = Bl_{\text{Hom}_1} \text{Hom}^\omega(W, E_Z). \] (1.8.1)

Thus

\[ \bigcup_{x \in \Omega_S^0} Bl_{\text{Hom}_1} \text{Hom}^\omega(W, E_Z) \subset \Omega_S. \]

Let \( \Delta_S \subset \Omega_S \) be the closure of the left-hand side. Notice that

\[ \text{cod}(\Delta_{S_c}, S_c) = c - 3. \] (1.8.2)

In particular \( \Delta_{S_c} = \Omega_{S_c} \) if and only if \( c = 4 \). Let \( \pi_{T_c}: T_c \rightarrow S_c \) be the blow-up of \( \Delta_{S_c} \) of course \( T_c \neq S_c \) only if \( c > 4 \). Let \( \Delta_{T_c} \subset T_c \) be the exceptional divisor, and \( \Omega_{T_c}, \Sigma_{T_c} \subset T_c \) be the proper transforms of \( \Omega_{S_c}, \Sigma_{S_c} \) respectively. Let

\[ \hat{M}_c := T_c/\text{PGL}(N), \quad \hat{\pi}: \hat{M}_c \rightarrow M_c, \]

where \( \hat{\pi} \) is induced by the PGL(\( N \))-equivariant map \( (\pi_R \pi_S \pi_T) \). Set

\[ \hat{\Omega}_c := \Omega_{T_c}/\text{PGL}(N), \quad \hat{\Sigma}_c := \Sigma_{T_c}/\text{PGL}(N), \quad \hat{\Delta}_c := \Delta_{T_c}/\text{PGL}(N), \]

\[ \Omega_c := \Omega_{Q_c}/\text{PGL}(N) \cong X^{[n]}, \quad \Sigma_c := \Sigma_{Q_c}/\text{PGL}(N) \cong \left( X^{[n]} \times X^{[n]} \right)/\text{involution}. \]
(1.8.3) Proposition. Keep notation as above. Then \( \hat{M}_c \) is a desingularization of \( M_c \).

The proof of the proposition will be given after some preliminary results.

Analysis of \( \Omega_S^{ss} \). We will prove the following result.

(1.8.4) Proposition.

1. \( \Omega_S^{ss} \) is smooth.
2. \( \Omega_S^{ss} = \Omega_S^s \).

We start with some preliminary lemmas.

(1.8.5) Lemma. Let \([Z] \in X^{[n]}\). Then the blow-up

\[ \text{Bl}_{\text{PHom}_1} \text{PHom}^\omega(W, EZ) \]

is smooth.

Proof. By (1.7.13) the exceptional divisor is a (locally-trivial) fibration over \( \text{PHom}_1(W, EZ) \), and the fiber over \([\varphi]\) is

\[ \text{PHom}^\omega(\text{Ker}\varphi, \text{Im}\varphi^\perp / \text{Im}\varphi), \]

i.e. a smooth quadric in \( \mathbb{P}^{2c-5} \). Since \( \text{PHom}_1(W, EZ) \) is smooth, it follows that the exceptional divisor is smooth. Thus the blow-up is smooth along the exceptional divisor. The complement of the exceptional divisor is smooth by (1.5.6). q.e.d.

(1.8.6) Lemma. All \( \text{SO}(W) \)-semistable points of \( \text{Bl}_{\text{PHom}_1} \text{PHom}^\omega(W, EZ) \) are \( \text{SO}(W) \)-stable. Explicitly:

1. Semistable points in the exceptional divisor are given by \((\text{referring to (1.7.12)}) \)

\[ \{([\varphi], [\alpha]) | [\varphi] \in \text{PHom}_1^{ss}(W, EZ), [\alpha] \in \text{PHom}^\omega(\text{Ker}\varphi, \text{Im}\varphi^\perp / \text{Im}\varphi) \alpha(v_2) \neq 0 \neq \alpha(v_3)\}. \]

Furthermore, for \([[\varphi],[\alpha]]\) in the above set,

\[ \text{St}([\varphi],[\alpha]) \cong \begin{cases} \mathbb{Z}/(2) & \text{if } \text{rk} \alpha = 2, \\ \mathbb{Z}/(2) \oplus \mathbb{Z}/(2) & \text{if } \text{rk} \alpha = 1. \end{cases} \]

2. Semistable points not in the exceptional divisor are given by

\[ \{[\varphi] \in \text{PHom}^\omega(W, EZ) | \text{rk} \varphi = 3 \text{ or } \text{rk} \varphi = 2 \text{ and } \text{ker} \varphi \text{ non-isotropic} \}. \]

For \([\varphi]\) belonging to the above set, \( \text{St}([\varphi]) \) is trivial if \( \text{rk} \varphi = 3 \), and \( \text{St}([\varphi]) \cong \mathbb{Z}/(2) \) if \( \text{rk} \varphi = 2 \).

Proof. Let’s prove Item (1). By (1.1.2) semistable points of the exceptional divisor are contained in the inverse image of \( \text{PHom}_1^{ss}(W, EZ) \). Applying Lemma (1.6.1) we get that a point in the exceptional divisor lying over \([\varphi]\) is \( \text{SO}(W) \)-(semi)stable if and only if it is \( \text{St}([\varphi]) \)-(semi)stable. One easily verifies that the points described in Item (1) are exactly the \( \text{St}([\varphi]) \)-semistable points, and that in fact they are all stable. The computation of stabilizers is an easy exercise. Let’s prove Item (2). Applying the numerical criterion for (semi)stability one checks that the points described are \( \text{SO}(W) \)-stable. By Proposition (1.1.2) they remain stable in the blow-up. Now let’s show that if \( \text{rk} \varphi = 2 \) and \( \text{ker} \varphi \) is isotropic, then \([\varphi]\) is not semistable in the blow-up. Choose \( v \in W \) with \( v \perp \text{ker} \varphi \) and \( v \notin \text{ker} \varphi \). Then there exists a one-parameter subgroup \( \lambda: \mathbb{C}^* \to \text{SO}(W) \) such that

\[ \lim_{t \to 0} \lambda(t) \varphi = \psi, \]

where \( \text{rk} \psi = 1 \) and \( \text{ker} \psi^\perp = v \). Thus \([\psi] \in \text{PHom}_1^{ss}(W, EZ) \). Since \( \text{PHom}_1 \) is the center of the blow-up, Lemma (1.7.4) tells us that \([\varphi]\) becomes non-semistable in the blow-up, as claimed. The stabilizers are easily computed. q.e.d.

Proof of (1.8.4). By (1.1.2) we know that \((\pi_R \circ \pi_S)(\Omega_S^{ss}) \subset \Omega_Q^0 \). Let \( x \in \Omega_Q^0 \), and set \( F_x = I_Z \oplus I_Z \).

By (1.8.1)-(1.8.5) the fiber \((\pi_R \circ \pi_S)^{-1}(x)\) is smooth. Since, by (1.5.1), \( \Omega_Q^0 \) is smooth, and since semistability is an open condition, we get that \( \Omega_S^{ss} \) is smooth. This proves Item (1). The second item follows at once from (1.6.1) and (1.8.6).

Analysis of \( \Sigma_S^{ss} \). We will prove the following result.
(1.8.7) Proposition.

1. \( \Sigma_{ss}^s \) is smooth.
2. \( \Sigma_{ss}^s = \Sigma_s \).

Proof. By (1.1.2) we know that
\[
\Sigma_{ss}^s \subset \pi_s^{-1}(\Sigma_R^s) = \mathcal{P}(C_{\Sigma_R^s}) R.
\]
Let \( y \in \Sigma_R^s \), and let \( x = \pi_R(y) \). By (1.7.2) either \( x \in \Sigma_Q^s \) or \( x \in \Omega_Q^0 \). In the latter case \( \pi_s^{-1}(y)^s \) is described in (1.8.6). In the former case, an easy computation together with (1.6.1) gives the following.

(1.8.8) Claim. Keep notation as above. If \( x \in \Sigma_Q^0 \) then (referring to (1.4.2))
\[
\Sigma_{ss}^s \cap (\pi_R \circ \pi_s)^{-1}(x) = \mathcal{P} \{ (e_{12}, e_{21}) | e_{12} \cup e_{21} = 0 \text { or } e_{12} \neq 0 \}
\]
and all semistable points are stable. The stabilizer of any point in the above set is \( \mathbb{Z}/(2) \).

Thus for every \( y \in \Sigma_R^s \), \( \pi_s^{-1}(y) \) is a smooth quadric in \( \mathbb{P}^{2c-5} \) (see Item (3) of (1.7.1)). By Item (1) of (1.7.1) \( \Sigma_R^s \) is smooth. Since semistability is an open condition, we conclude that \( \Sigma_{ss}^s \) is smooth. This proves Item (1) of (1.8.7). The second Item follows from (1.6.1) and (1.8.8).

q.e.d.

Analysis of \( S_{ss}^s \). Let’s show that
\[
S_{ss}^s = \Sigma_{ss}^s \cup \Omega_{ss}^s \cup (\pi_s \circ \pi_R)^{-1}(Q^s).
\]
By (1.1.4) the term on the right is contained in the term on the left. On the other hand, by (1.1.2)
\[
S_{ss}^s \subset \Sigma_{ss}^s \cup \Omega_{ss}^s \cup (\pi_s \circ \pi_R)^{-1}(Q^s \cup \Gamma_Q^0 \cup \Lambda_Q^0).
\]
By (1.7.3) and (1.1.3) there are no semistable points in \( (\pi_s \circ \pi_R)^{-1}(\Gamma_Q^0) \). Now apply (1.7.4) to \( Y = R \), \( \bar{Y} = S \), and \( V = \Sigma_R \); since for \( y \in \pi_s^{-1}(\Gamma_Q^0) \) we have \( O(y) \cap \Sigma_R^s \neq \emptyset \), there are no semistable points in \( (\pi_s \circ \pi_R)^{-1}(\Lambda_Q^0) \). This finishes the proof of (1.8.9).

(1.8.10) Claim.

1. \( S_{ss}^s = S^s \).
2. \( S^s \) is smooth.

Proof. Let’s prove Item (1). By (1.1.4) \( (\pi_s \circ \pi_R)^{-1}(Q^s) \) is in the stable locus. By (1.8.7)-(1.8.4) \( \Omega_{ss}^s = \Omega_s^s \) and \( \Sigma_{ss}^s = \Sigma_s^s \). Thus Item (1) follows from (1.8.9). Let’s prove Item (2). First of all we show \( Q^s \) is smooth: by (1.2.1)-(1.2.4) it suffices to prove that for \( x \in Q^s \) the deformation space \( \text{Def}(F_x) \) is smooth, and this follows (see (1.3.8)) from
\[
\text{Ext}^2(F_x, F_x)^0 \cong (\text{Hom}(F_x, F_x)^0)^{\vee} = 0.
\]
Since \( (\pi_s \circ \pi_R) \) is an isomorphism outside \( \Sigma_S \cup \Omega_S \), we get that \( (\pi_s \circ \pi_R)^{-1}(Q^s) \) is smooth. Secondly \( \Sigma_s^s \) and \( \Omega_s^s \) are smooth by (1.8.7)-(1.8.4); since they are Cartier divisors \( S^s \) is smooth along \( \Sigma_s^s \) and \( \Omega_s^s \). By (1.8.9) we conclude that \( S^s \) is smooth.

q.e.d.

Analysis of \( \Delta_s^2 \). We will prove the following.

(1.8.11) Proposition. Keeping notation as above, \( \Delta_s^2 \) is smooth.

First we need a preliminary result. For \( [Z] \in X[n] \) let
\[
\text{Gr}^\omega(k, E_Z) := \{ [A] \in \text{Gr}(k, E_Z) | A \text{ is } \omega\text{-isotropic} \},
\]
\[
\text{PHom}^\omega_2(W, E_Z) := \langle ([K], [A], [\varphi]) \in \mathcal{P}(W) \times \text{Gr}^\omega(2, E_Z) \times \text{PHom}^\omega_2(W, E_Z) | K \subset \ker \varphi, \text{Im} \varphi \subset A \rangle,
\]
and let \( g: \text{PHom}^\omega_2(W, E_Z) \to \text{PHom}^\omega_2(W, E_Z) \) be the map which forgets the first two “entries”.

q.e.d.
(1.8.12) Lemma. Keeping notation as above, there exists an $SO(W)$-equivariant isomorphism

$$f: \tilde{\text{PHom}}_2(W, E_Z) \cong \text{Bl}_{\text{PHom}_1} \text{PHom}_2(W, E_Z).$$

The map $g$ corresponds to the blow-down map.

Proof. The ideal $I_{\text{PHom}_1}$ of $\text{PHom}_1$ is generated by $2 \times 2$ minors (Second Fundamental Theorem of Invariant Theory). Thus $g^* I_{\text{PHom}_1}$ is locally generated by the “determinant” of $\mathfrak{g}: W/K \to A$, hence it is locally principal. By the universal property of the blow-up, there exists a map $f$ as in the statement of the claim. Let’s prove $f$ is an isomorphism. Choose bases of $W$ and $E_Z$, and realize the blow-up as the closure in $\text{PHom}_2(W, E_Z) \times \mathbf{P}^{17}$ of

$$\{([\varphi], \ldots, [m_{IJ}(\varphi)], \ldots) | \varphi \in (\text{Hom}_2(W, E_Z) \setminus \text{Hom}_1(W, E_Z)), m_{IJ}(\varphi) = (I \times J)\text{-minor of } \varphi, |J| = |J| = 2\}.$$

A computation shows that $f$ is given by

$$([K], [A], [\varphi]) \mapsto ([\varphi], \ldots, [p_I(K)q_J(A)], \ldots),$$

where $p_I(K)$ are Plücker coordinates of $[K^{-1}] \in \text{Gr}(2, W)^*$, and $q_J(A)$ are Plücker coordinates of $[A]$. This proves $f$ is an isomorphism. Clearly $f$ is $SO(W)$-equivariant. q.e.d.

Proof of (1.8.11). By (1.1.2) we have $\pi R \pi_S(\Delta_S^x) \subset \Omega_{Q}^0$. If $x \in \Omega_{Q}^0$ and $F_x = I_Z \oplus I_Z$, then

$$\Delta_S^x \cap (\pi R \pi_S)^{-1}(x) \subset \text{Bl}_{\text{PHom}_1} \text{PHom}_2(W, E_Z).$$

The right-hand side is smooth by (1.8.12). Since $\Omega_{Q}^0$ is smooth by (1.5.1), and since stability is an open condition, we conclude that $\Delta_S^x$ is smooth. q.e.d.

Proof of Proposition (1.8.3). By Item (1) of (1.8.10) we have $S^{ss} = S^*$, hence (1.1.2) implies that

$$T^{ss} = T^* = \pi_T^{-1}(S^*) = \text{Bl}_{\Delta_S^x}(S^*).$$

By (1.8.11) $\Delta_S^x$ is smooth; since $S^x$ is smooth (by (1.8.10)) we get that $T^x$ is smooth. Let $x \in \Omega_{Q}^0$ and let $F_x = I_Z \oplus I_Z$; one easily proves that

$$\Delta_S \cap \Sigma_S \cap (\pi R \pi_S)^{-1}(x) = \{([\varphi], [\alpha]) | [\varphi] \in \text{PHom}_1(W, E_Z), [\alpha] \in \text{PHom}^{aw}(\text{Ker} \varphi, \text{Im} \varphi^+, /\text{Im} \varphi), \text{rk} \alpha = 1\},$$

where notation is as in (1.8.6). Hence if $z \in T^x$ we get by (1.8.6)-(1.8.8) that

$$\text{St}(z) = \begin{cases} \{1\} & \text{if } z \notin \Sigma_T^* \cup \Delta_T^x, \\ \mathbf{Z}/(2) & \text{if } z \in (\Sigma_T^* \cup \Delta_T^x) \setminus (\Sigma_T^* \cap \Delta_T^x), \\ \mathbf{Z}/(2) \oplus \mathbf{Z}/(2) & \text{if } z \in (\Sigma_T^* \cap \Delta_T^x). \end{cases}$$

Since $\Sigma_T^*$ and $\Delta_T^x$ are divisors, we conclude that $\widehat{\mathcal{M}} = T^*/\text{PGL}(N)$ is smooth.

1.9. The two-form on the moduli space.

Let $B$ be a smooth scheme, and $E$ be a sheaf on $X \times B$, flat over $B$. The Mukai-Tyurin form $\omega_E \in \Gamma(\Omega_B^2)$ is defined as follows [Mk,T]: if $v, w \in T_bB$

$$\langle \omega_E(b), v \wedge w \rangle := \int_X \text{Tr} (\kappa_E(b)(v) \cup \kappa_E(b)(w)) \wedge \omega,$$

where $\kappa_E: T_bB \to \text{Ext}^1(E_b, E_b)$ is the Kodaira-Spencer map at $b$. We apply this construction to $B = T_c^*$, and $E$ the pull-back via $X \times T_c^* \to X \times Q_c$ of the tautological quotient sheaf on $X \times Q_c$. 24
(1.9.1) **Claim.** The two-form $\omega_\xi$ on $T^*_c$ is $\text{PGL}(N)$-invariant.

**Proof.** The group $\text{GL}(N)$ acts on $T^*_c$ and on $\xi$. For $g \in \text{GL}(N)$ we have $g^*\omega_\xi = \omega_{g^*\xi}$. Thus

$$\langle g^*\omega_\xi(b), v \wedge w \rangle = \langle \omega_{g^*\xi}(b), v \wedge w \rangle = \int_X \text{Tr} (\kappa_{g^*\xi}(b)(v) \cup \kappa_{g^*\xi}(b)(w)) \wedge \omega$$

$$= \int_X \text{Tr} (g^{-1}\kappa_{\xi}(b)(v) \cup \kappa_{\xi}(b)(w)g) \wedge \omega$$

$$= \int_X \text{Tr} (\kappa_{\xi}(b)(v) \cup \kappa_{\xi}(b)(w)) \wedge \omega$$

$$= \langle \omega_\xi(b), v \wedge w \rangle,$$

where the third equality is proved similarly to Lemma (1.4.16). \[\text{q.e.d.}\]

Applying Claim (1.9.1) and the étale slice Theorem (1.2.1) one sees that the two-form $\omega_\xi$ on $T^*_c$ descends to a holomorphic two-form $\tilde{\omega}_c$ on $\tilde{M}_c$. By a theorem of Mukai [Muk] we get the following.

(1.9.2) **Proposition.** The two-form $\tilde{\omega}_c$ on $\tilde{M}_c$ is non-degenerate outside $\tilde{\Omega}_c \cup \tilde{\Sigma}_c \cup \tilde{\Delta}_c$.

**Proof.** The map $\tilde{\pi}$ gives an isomorphism between the complement of $\tilde{\Omega}_c \cup \tilde{\Sigma}_c \cup \tilde{\Delta}_c$ and $(M_c \setminus \Omega_c \setminus \Sigma_c)$. If $[F]$ is either $M_c$ or $\Omega_c \setminus \Sigma_c$ and then by (1.2.3) $T_{[F]}M_c \cong \text{Ext}^1(F, F)$, thus $\tilde{\omega}_c$ is non-degenerate at $[F]$ by Serre duality. \[\text{q.e.d.}\]

2. A symplectic desingularization of $M_4$.

Let $\text{Gr}_2(2, X^{[2]})$ be the relative symplectic Grassmannian over $X^{[2]}$, with fiber $\text{Gr}_2(2, E_Z)$ over $[Z] \in X^{[2]}$, and let $\mathcal{A}$ be the tautological $G^2$-bundle over $\text{Gr}_2(2, X^{[2]})$. We will prove (later) the following.

(2.0.1) **Proposition.** Keep notation as above. Then $\tilde{\Omega}_4$ is isomorphic to $P(S^2A)$; under this isomorphism the map $\tilde{\pi}\big|_{\tilde{\Omega}_4} : \tilde{\Omega}_4 \to \Omega_4 \cong X^{[2]}$

corresponds to the natural projection $P(S^2A) \to X^{[2]}$.

For $[Z] \in X^{[2]}$, set $\tilde{\Omega}_Z := \tilde{\pi}^{-1}([I_Z \oplus I_Z])$.

We define classes $\tilde{\epsilon}_Z, \tilde{\gamma}_Z \in NE_1(\tilde{\Omega}_Z)$ as follows. Proposition (2.0.1) gives an isomorphism $\tilde{\Omega}_Z \cong P(S^2A_Z)$, where $A_Z$ is the restriction of $A$ to $\text{Gr}_2(2, E_Z)$ (i.e. the tautological vector-bundle). We let $\tilde{\epsilon}_Z$ be the class in $N_1(\tilde{\Omega}_Z)$ of a line in a fiber of $P(S^2A_Z) \to \text{Gr}_2(2, E_Z)$. Next choose $[L] \in P(E_Z)$, $[q_L] \in P(S^2L)$. Let $\{[A_t] \in \text{Gr}_2(2, E_Z)\}_{t \in \mathbb{P}^1}$ be a line through $[L]$, i.e. for every $t \in \mathbb{P}^1$ we have an inclusion $t : L \to A_t$ and $[A_t/L] \in P(L^+/L)$ varies in a line. We set $\tilde{\gamma}_Z := \text{class in } N_1(\tilde{\Omega}_Z)$ of $\{[A_t], [t^*q_L]\}$.

Letting $i^Z : \tilde{\Omega}_Z \to \tilde{M}_4$ be inclusion, we set $\tilde{\epsilon}_4 := i^Z_4 \tilde{\epsilon}_Z$, $\tilde{\gamma}_4 := i^Z_4 \tilde{\gamma}_Z$.

Since the right-hand sides of the above equalities are independent of $[Z]$, the classes $\tilde{\epsilon}_4$, $\tilde{\gamma}_4$ are well-defined elements of $NE_1(\tilde{M}_4)$. The main result of this section is the following.
(2.0.2) Proposition. Keep notation as above. Then:

1. $R^+\hat{e}_4$ is a $K_{\hat{M}_4}$-negative extremal ray; let $\hat{M}_4$ be the scheme obtained contracting $R^+\hat{e}_4$.
2. $\hat{M}_4$ is a smooth projective symplectic desingularization of $M_4$.

(2.0.3) Remark. Recall that $T_4 = S_4$, hence $\hat{M}_4 = S_4//PGL(N)$. We let $q: S^1_4 \to \hat{M}_4$ be the quotient map.

2.1. The divisor $\hat{\Omega}_4$.

Proof of (2.0.1). Since $\hat{\Omega}_4$ and $\Omega_4$ are orbit spaces, the fiber of $\hat{\pi}$ over $[Z] \in X^{[2]}$ is given by (see (2.0.3)-(1.8.1)-(1.8.12))

$$\hat{\Omega}_4 = Bl_{\text{phom}_1}\text{Phom}_2^e(W, E_Z)//SO(W) = \text{Phom}_2^e(W, E_Z)//SO(W).$$

Since $SO(W)$ acts trivially on $\text{Gr}^\omega(2, E_Z)$, we get a map

$$f: \text{Phom}_2^e(W, E_Z)//SO(W) \to \text{Gr}^\omega(2, E_Z).$$

It follows easily from (1.8.12), (1.8.6) and (1.6.1) that

$$(2.1.1) \quad \text{Phom}_2^e(W, E_Z)^s = \text{Phom}_2^e(W, E_Z)^s = \{(K, [A], [\varphi]) | [K] \text{ is non-isotropic}\},$$

hence the projection $\text{Phom}_2^e(W, E_Z) \to P(W)$ maps the stable locus to the complement of the isotropic conic, i.e. $P(W)^s$. Since $SO(W)$ acts transitively on $P(W)^s$, we get that

$$f^{-1}([A]) \cong \text{Phom}(K^\perp, A)//O(K^\perp),$$

where $[K] \in P(W)^s$ is any chosen point. As is easily verified, the map

$$(2.1.2) \quad \text{Phom}(K^\perp, A) \to P(S^2A) \quad [\alpha] \mapsto [\alpha \circ^t \alpha]$$

is the quotient map for the $O(K^\perp)$-action. This proves Proposition (2.0.1).

The effective cone of $\hat{\Omega}_4$. We will need the following result.

Lemma. Keep notation as above. Let $[Z] \in X^{[2]}$. Then

$$(2.1.3) \quad NE_1(\hat{\Omega}_4) = R^+\hat{e}_Z \oplus R^+\hat{\gamma}_Z.$$ 

Proof. We have two projections

$$\text{Gr}^\omega(2, E_Z) \xleftarrow{f} \hat{\Omega}_4 \xrightarrow{g} \text{Phom}^e(W, E_Z)//SO(W).$$

As is easily verified, the maps $f$, $g$ are the contractions of $R^+\hat{e}_Z$, $R^+\hat{\gamma}_Z$ respectively. Thus each of $R^+\hat{e}_Z$, $R^+\hat{\gamma}_Z$ is an extremal ray. On the other hand, since $f$ is a $P^2$-fibration over a smooth quadric threefold, $N_1(\Omega_Z)$ has rank two. Hence (2.1.3) holds. q.e.d.

2.2. The normal bundle of $\hat{\Omega}_4$.

Let $[Z] \in X^{[2]}$ and $[A] \in \text{Gr}^\omega(2, E_Z)$, so that $([Z], [A]) \in \text{Gr}^\omega(2, T_X^{[2]})$. Proposition (2.0.1) gives an embedding $P(S^2A) \hookrightarrow \hat{\Omega}_4$; we will prove that

$$(2.2.1) \quad \hat{\Omega}_4|_{P(S^2A)} \cong O_{P(S^2A)}(-1).$$
Claim. Keeping notation as above,

\[ q^*\hat{\Omega}_4 \sim 2\Omega_{S_4}^* \]

Proof. Since \( q^{-1}\hat{\Omega}_4 = \Omega_{S_4}^* \), all we have to do is determine the multiplicity of \( q^*\hat{\Omega}_4 \) at a generic point of \( \Omega_{S_4}^* \). Let \( z \in (\Omega_{S_4}^* \setminus S_4) \); by (1.8.8) \( St(z) = \mathbb{Z}/(2) \). Let \( \mathcal{V} \subset S_4^* \) be a slice normal to \( O(z) \). By (1.2.1)

\[ \mathcal{V}/(\mathbb{Z}/(2)) \cong \text{neighborhood of } q(z) \in \hat{\mathcal{M}}_4. \]

Since the fixed locus for the action of \( \mathbb{Z}/(2) \) is a perfect pairing, one gets \( \hat{\mathcal{M}}_4 \). As is easily checked \( q \mathcal{V} = \mathcal{V} \). Let \( \hat{\mathcal{M}}_4 \). It follows from (2.1.1) that there exists a straight line \( \Lambda \subset \text{PHom}(K^\perp,A)^s \subset \text{PHom}^2(W,E_Z)^s \subset S_4^* \).

Claim. Keeping notation as above,

\[ \Omega_{S_4}^* \cdot \Lambda = -1. \]

Proof. We have \( \Omega_{S_4} \sim \pi_{S_4}^* \Omega_{R_4} \), and

\[ [\Omega_{R_4}]|_{\text{PHom}^2(W,E_Z)} \cong \mathcal{O}_{\text{PHom}^2(W,E_Z)}(-1). \]

Since the restriction of \( \pi_S \) to \( \Lambda \) is an isomorphism to a straight line in \( \text{PHom}^2(W,E_Z) \), Equation (2.2.3) follows at once.

Let’s prove (2.2.1). Let \( [\hat{\Omega}_4]|_{\text{P}^2(S^2A)} \) be a \( \mathcal{O}_{\text{P}^2(S^2A)}(a) \). By (2.1.2) \( q \) maps \( \Lambda \) one-to-one onto a conic \( \Gamma \subset \mathbb{P}^2(S^2A) \). Using (2.2.2)-(2.2.3) we get

\[ 2a = \hat{\Omega}_4 \cdot \Gamma = q^*\hat{\Omega}_4 \cdot \Lambda = 2\Omega_{S_4} \cdot \Lambda = -2. \]

Thus \( a = -1 \); this proves (2.2.1).

2.3. Digression on \( \hat{\Sigma}_4^* \).

For \( [Z],[W] \in X^{|2|} \) with \( Z \neq W \) we set

\[ \hat{\Sigma}_{Z,W} := \pi^{-1}( [I_Z \oplus I_W] ). \]

Thus \( \hat{\Sigma}_{Z,W} \subset (\hat{\Sigma}_4 \setminus \hat{\Omega}_4) \).

(2.3.1) Proposition. Keep notation as above. Let \( [Z],[W] \in X^{|2|} \), with \( Z \neq W \). There is an isomorphism \( \hat{\Sigma}_{Z,W} \cong \mathbb{P}^1 \), and furthermore

\[ \hat{\Sigma}_4 \cdot \hat{\Sigma}_{Z,W} = -2. \]

Proof. By (1.4.2),

\[ \hat{\Sigma}_{Z,W} = \text{P}\{ (e_{12}, e_{21}) \mid e_{12} \cup e_{21} = 0 \}\slash\mathcal{C}^*, \]

where \( e_{12} \in \text{Ext}^1(I_Z, I_W), e_{21} \in \text{Ext}^1(I_W, I_Z) \), and \( \mathcal{C}^* \) acts as in (1.4.3). Taking into account that, by Serre duality,

\[ \text{Ext}^1(I_Z, I_W) \times \text{Ext}^1(I_W, I_Z) \rightarrow \text{Ext}^2(I_Z, I_Z) \]

is a perfect pairing, one gets \( \hat{\Sigma}_{Z,W} \cong \mathbb{P}^1 \). Now let’s prove (2.3.2). Let \( f: \text{Ext}^1(I_Z, I_W) \rightarrow \text{Ext}^1(I_W, I_Z) \) be a skew-symmetric isomorphism, and let

\[ \Lambda := \{ [e,f(e)] \subset \text{P}\{ (e_{12}, e_{21}) \mid e_{12} \cup e_{21} = 0 \}^s. \]

As is easily checked \( q(\Lambda) = \hat{\Sigma}_{Z,W} \), and \( q|_\Lambda: \Lambda \rightarrow \hat{\Sigma}_{Z,W} \) is an isomorphism. Thus

\[ \hat{\Sigma}_4 \cdot \hat{\Sigma}_{Z,W} = q^*\hat{\Sigma}_4 \cdot \Lambda. \]

Arguing as in the proof of (2.2.2), one sees that \( q^*\hat{\Sigma}_4 \sim 2\Omega_{S_4}^* \). Furthermore, since \( \Lambda \) is a line in

\[ \text{P}\{ (e_{12}, e_{21}) \mid e_{12} \cup e_{21} = 0 \}, \]

we have \( \Sigma_{S_4} \cdot \Lambda = -1 \). Thus

\[ \hat{\Sigma}_4 \cdot \hat{\Sigma}_{Z,W} = q^*\hat{\Sigma}_4 \cdot \Lambda = 2\Omega_{S_4} \cdot \Lambda = -2. \]

q.e.d.

Let \( k^Z,W: \hat{\Sigma}_{Z,W} \rightarrow \hat{\mathcal{M}}_4 \) be inclusion. We will need the following result.
Lemma. Keeping notation as above,

\[(2.3.3) \quad k_*^{Z,W} \mathcal{N}E_1(\hat{\Sigma}_{Z,W}) = R^+\hat{\gamma}_4.\]

Proof. By (2.3.1) we know the left-hand side of (2.3.3) equals \(R^+[\hat{\Sigma}_{Z,W}]\). Letting \([W]\) approach \([Z]\) we see that \([\hat{\Sigma}_{Z,W}]\) can be represented by a one-cycle \(\Gamma\) on \(\hat{\Omega}_Z \cap \hat{\Sigma}_c\). The cycle \(\Gamma\) must be mapped to a single point by the map induced from \(\pi_{S_4}\)

\[\hat{\Omega}_Z = \tilde{P}\text{Hom}_W(W,E_Z)/\text{SO}(W) \rightarrow \text{PHom}_W(W,E_Z)/\text{SO}(W).\]

This implies \(\Gamma\) is multiple of the cycle defining \(\hat{\gamma}_Z\). q.e.d.

2.4. Proof of Proposition (2.0.2).

First we prove the formula

\[(2.4.1) \quad K_{\hat{\mathcal{M}}_4} \sim 2\hat{\Omega}_4.\]

By (1.9.2) the two-form \(\hat{\omega}_4\) is non-degenerate outside \(\hat{\Omega}_4 \cup \hat{\Sigma}_4\), hence

\[(\wedge^5 \hat{\omega}_4) = x\hat{\Omega}_4 + y\hat{\Sigma}_4\]

for non-negative integers \(x, y\). Applying adjunction to \(\hat{\Sigma}_4\), and using (2.3.1), we get that \(y = 0\). Applying adjunction to \(\hat{\Omega}_4\), and using (2.2.1) one gets \(x = 2\), and thus

\[(2.4.2) \quad (\wedge^5 \hat{\omega}_4) = 2\hat{\Omega}_4.\]

This proves (2.4.1).

Proof of Proposition (2.0.2)-Item(1). From (2.4.1) and (2.2.1) we get

\[K_{\hat{\mathcal{M}}_4} \cdot \hat{\epsilon}_4 = -2,\]

hence \(R^+\hat{\epsilon}_4\) is \(K_{\hat{\mathcal{M}}_4}\)-negative. Before proving that \(R^+\hat{\epsilon}_4\) is extremal we give a preliminary result.

(2.4.3) Claim. Let \([Z] \in X^{[2]}\). Then:

a. The classes \(\hat{\epsilon}_4, \hat{\gamma}_4 \in N_1(\hat{\mathcal{M}}_4)\) are linearly independent.

b. The map \(i^\sharp: \mathcal{N}E_1(\hat{\Omega}_Z) \rightarrow \mathcal{N}E_1(\hat{\mathcal{M}}_4)\) is injective, with image \(R^+\hat{\epsilon}_4 \oplus R^+\hat{\gamma}_4\).

Proof. From (2.2.1) we get

\[\hat{\Omega}_4 \cdot \hat{\epsilon}_4 = (i^\sharp)^*[\hat{\Omega}_4] \cdot \hat{\epsilon}_4 = -1.\]

By (2.3.3) we get \(\hat{\Omega}_4 \cdot \hat{\gamma}_4 = 0\). Item (a) follows at once from these two formulae. Item (b) follows from (2.1.3). q.e.d.

The following result shows that \(R^+\hat{\epsilon}_4\) is extremal.

Lemma. Keeping notation as above, \(R^+\hat{\epsilon}_4 \oplus R^+\hat{\gamma}_4\) is an extremal face of \(\mathcal{N}E_1(\hat{\mathcal{M}}_4)\).

Proof. Assume

\[\sum_{\alpha \in \tilde{I}} m_\alpha [\Gamma_\alpha] \in R^+\hat{\epsilon}_4 \oplus R^+\hat{\gamma}_4,\]

where, for each \(\alpha \in \tilde{I}\), \(m_\alpha > 0\) and \(\Gamma_\alpha\) is an irreducible curve on \(\hat{\mathcal{M}}_4\). We must show that

\[(2.4.4) \quad [\Gamma_\alpha] \in R^+\hat{\epsilon}_4 \oplus R^+\hat{\gamma}_4 \text{ for all } \alpha \in \tilde{I}.\]
We begin by stating some auxiliary results. If \( V \hookrightarrow P \) and \( CC \) we will prove (later) the following. Hence we can partition the indexing set as \( I = I_\Omega \cup I_\Sigma \), so that
\[
\begin{cases}
\alpha \in I_\Omega, & \text{then } \Gamma_\alpha \subset \widehat{\Omega}_{Z_\alpha}, \text{ for some } Z_\alpha \in X^{[2]}, \\
\alpha \in I_\Sigma, & \text{then } \Gamma_\alpha \subset \widehat{\Sigma}_{Z_\alpha, W_\alpha} \text{ for } Z_\alpha, W_\alpha \in X^{[2]} \text{ with } Z_\alpha \neq W_\alpha.
\end{cases}
\]
Statement (2.4.4) follows from Claim (2.4.3)-Item (b) if \( \alpha \in I_\Omega \), and from (2.3.3) if \( \alpha \in I_\Sigma \).

**Proof of Proposition (2.0.2)-Item(2).** That \( \hat{M}_4 \) is projective follows from Mori theory. Let’s prove \( \hat{M}_4 \) is smooth. By (2.0.1) we have \( P^2 \)-fibration
\[
P^2 \to \hat{\Omega}_4 \to \text{Gr}^\omega(2, T_{X^{[2]}}),
\]
where the fiber over \( ([Z], [A]) \) is canonically isomorphic to \( P(S^2 A) \).

(2.4.6) **Claim.** The contraction of \( R^+ \hat{c}_4 \) is identified with the contraction of \( \hat{M}_4 \) along Fibration (2.4.5).

**Proof.** Let \( L \) be a line in a fiber of (2.4.5); then \([L] = \hat{c}_4\). Hence we must prove that if \( C \subset \hat{M}_4 \) is an irreducible curve such that \([C] \in R^+ \hat{c}_4\), then \( C \) belongs to a fiber of (2.4.5). By (2.1.1), \( C \cdot \hat{\Omega}_4 < 0 \), hence \( C \subset \hat{\Omega}_4 \). Furthermore, since \( \hat{\pi}_1 C \equiv 0 \), there exists \([Z] \in X^{[2]}\) such that \( C \subset \hat{\Omega}_2 \). By (2.4.3)-Item (b), we have the following relation in \( N_1(\hat{\Omega}_2) \):
\[
[C] \in R^+ \hat{c}_2.
\]
This implies that \( C \) belongs to a fiber of (2.4.5).

By Claim (2.4.6) \( \hat{\pi}_4 \) is smooth. Let \( \bar{\omega}_4 \) be the two-form on \( \hat{\pi}_4 \) induced by \( \omega_4 \); we will show that \( \bar{\omega}_4 \) is non-degenerate. Let \( \Omega_4 \) be the image of \( \hat{\Omega}_4 \) under the contraction map \( \hat{\pi}_4 \to \hat{M}_4 \). By Claim (2.4.6)
(2.4.7)
\[
\text{cod}(\bar{\Omega}_4, \hat{M}_4) = 3.
\]
By (2.4.2) we conclude that \( \bar{\omega}_4 \) is non-degenerate, i.e. it gives a symplectic form. Finally we prove that the rational map \( \hat{\pi}: \hat{M}_4 \to \hat{M}_4 \) induced by \( \hat{\pi} \) is in fact regular. Let \( \Gamma \subset \hat{\pi}_4 \) be the graph of \( \hat{\pi} \), and let \( \rho_1, \rho_2 \) be the projections of \( \Gamma \) on \( \hat{\pi}_4, \hat{M}_4 \) respectively. Since \( \hat{\pi}_4 \) is smooth, it suffices, by Zariski’s Main Theorem to prove that there are no exceptional divisors of \( \rho_1 \). Suppose \( E \) is such an exceptional divisor. Since \( \hat{\pi} \) gives an isomorphism between \( (\hat{M}_4 \setminus \hat{\Omega}_4) \) and \( (\hat{M}_4 \setminus \Omega_4) \), we have
\[
j: E \to \hat{\Omega}_4 \times \hat{\Omega}_4.
\]
The \( P^2 \)-fibers of \( \hat{\Omega}_4 \) which have been contracted are contained in the fibers of \( \hat{\pi} \), hence \( j \) factors through an inclusion
\[
j_0: E \to \hat{\Omega}_4 \times_{\Omega_4} \Omega_4 = \hat{\Omega}_4.
\]
This is absurd by (2.4.7).

3. **Towards a smooth minimal model of \( \mathcal{M}_c \), for \( c \geq 6 \).**

We begin by stating some auxiliary results. If \( V \) is a three-dimensional vector space, let
\[
\text{CC}(V) := \text{closure of } \{(C, D) \in P(S^2 V) \times P(S^2 \tilde{V}) \mid C, D \text{ are smooth conics dual to each other}\},
\]
i.e. the space of complete conics in \( P(V) \). Let \( \text{Gr}^\omega(3, T_{X^{[n]}}) \) be the relative symplectic Grassmannian over \( X^{[n]} \), with fiber \( \text{Gr}^\omega(3, E_Z) \) over \([Z] \in X^{[n]}\). Let \( B \) be the tautological rank-three bundle over \( \text{Gr}^\omega(3, T_{X^{[n]}}) \), and \( \text{CC}(B) \) be the tautological family of complete conics over \( \text{Gr}^\omega(3, T_{X^{[n]}}) \). Thus we have a locally-trivial fibration
\[
\begin{array}{ccc}
\text{CC}(B) & \to & \text{CC}(B) \\
([Z], [B]) & \in & \text{Gr}^\omega(3, T_{X^{[n]}}).
\end{array}
\]
We will prove (later) the following.
(3.0.1) Proposition. Keep notation as above, and assume \( c \geq 6 \). Then \( \hat{\Omega}_c \) is isomorphic to \( \text{CC}(B) \); under this isomorphism the map
\[
\tilde{\pi}|_{\hat{\Omega}_c} : \hat{\Omega}_c \to \Omega_c \cong X^{[n]}
\]
corresponds to the natural projection \( \text{CC}(B) \to X^{[n]} \).

For \( [Z] \in X^{[n]} \), let
\[
\hat{\Omega}_Z := \text{CC}(B_Z) \cong \tilde{\pi}^{-1}(I_Z) \cap \hat{\Omega}_c,
\]
where \( B_Z \) is the restriction of \( B \) to \( \text{Gr}^{\omega}(3, E_Z) \). Let’s define classes \( \hat{\sigma}_Z, \hat{\epsilon}_Z, \hat{\gamma}_Z \in NE_1(\hat{\Omega}_Z) \) as follows. If \( [B] \in \text{Gr}^{\omega}(3, E_Z) \) let \( \hat{\Omega}_B \) be the fiber over \( [B] \) of the natural fibration \( \hat{\Omega}_Z \to \text{Gr}^{\omega}(3, E_Z) \); thus \( \hat{\Omega}_B \cong \text{CC}(B) \).

We have two projections
\[
P(S^2B) \xrightarrow{\Phi} \text{CC}(B) \xrightarrow{\hat{\phi}} P(S^2B).
\]
Each of \( \Phi_B, \hat{\Phi}_B \) is the blow-up of the locus parametrizing conics of rank one. Set
\[
\hat{\sigma}_Z := \text{class in } N_1(\hat{\Omega}_Z) \text{ of a line in a } P^2\text{-fiber of } \Phi_B,
\]
\[
\hat{\epsilon}_Z := \text{class in } N_1(\hat{\Omega}_Z) \text{ of a line in a } P^2\text{-fiber of } \Phi_B.
\]
To define \( \hat{\gamma}_Z \), choose \( [A] \in \text{Gr}^{\omega}(2, E_Z) \), \( q \in S^2A \) of rank two, and a line \( \Lambda \subset P(A^\perp/A) \). For \( t \in \Lambda \), let \( B_t \subset E_Z \) be the three-dimensional subspace containing \( A \) and projecting to the line corresponding to \( t \); clearly \( [B_t] \in \text{Gr}^{\omega}(3, E_Z) \). Furthermore, let \( q_t \in S^2B_t \) be the image of \( q \) under the inclusion \( A \hookrightarrow B_t \). We set
\[
\hat{\gamma}_Z := \text{class in } N_1(\hat{\Omega}_Z) \text{ of } \{ \Phi_{B_t}^{-1}([q_t]) \}_{t \in \Lambda}.
\]
Notice that since \( q_t \) has rank two for all \( t \), \( [q_t] \) is never in the exceptional locus of \( \Phi_{B_t} \), and thus the right-hand side of the above equality is indeed a curve. Now let \( i^2 : \hat{\Omega}_Z \hookrightarrow \hat{M}_c \) be inclusion, and set
\[
\hat{\sigma}_c := i_*^2 \hat{\sigma}_Z,
\]
\[
\hat{\epsilon}_c := i_*^2 \hat{\epsilon}_Z,
\]
\[
\hat{\gamma}_c := i_*^2 \hat{\gamma}_Z.
\]
Notice that the right-hand sides of the above equalities are independent of \( Z \), thus \( \hat{\sigma}_c, \hat{\epsilon}_c, \hat{\gamma}_c \) are well-defined elements of \( NE_1(\hat{M}_c) \). Later we will prove the following.

(3.0.2) Proposition. Keep notation as above, and assume \( c \geq 6 \). Then:

1. \( \hat{\sigma}_c, \hat{\epsilon}_c, \hat{\gamma}_c \) are linearly independent.
2. \( R^+\hat{\sigma}_c \oplus R^+\hat{\epsilon}_c \oplus R^+\hat{\gamma}_c \) is a \( K_{\hat{M}_c} \)-negative extremal face of \( NE_1(\hat{M}_c) \).

Let \( \overline{M}_c \) be the scheme obtained contracting the \( K_{\hat{M}_c} \)-negative extremal ray \( R^+\hat{\sigma}_c \).

(3.0.3) Proposition. Keep notation as above. Then \( \overline{M}_c \) is a smooth projective desingularization of \( M_c \).

Let \( \overline{\epsilon}_c \in N_1(\overline{M}_c) \) be the image of \( \hat{\epsilon}_c \). We will prove the following.

(3.0.4) Proposition. Keep notation as above. Then \( R^+\overline{\epsilon}_c \) is a \( K_{\overline{M}_c} \)-negative extremal ray of \( NE_1(\overline{M}_c) \).

The scheme \( \overline{M}_c \) obtained contracting \( R^+\overline{\epsilon}_c \) is a smooth projective desingularization of \( M_c \). It carries a holomorphic two-form, degenerate on a single irreducible divisor (the image of \( \hat{\Sigma}_c \) under the map \( \hat{M}_c \to M_c \)).

We stop at \( \overline{M}_c \); it is the best we can do in trying to find a smooth symplectic model of \( M_c \). If \( \hat{\gamma}_c \in NE_1(\overline{M}_c) \) denotes the image of \( \hat{\gamma}_c \), then \( R^+\hat{\gamma}_c \) is a \( K_{\overline{M}_c} \)-negative extremal ray; let \( \overline{M}_c^0 \) be the scheme obtained contracting \( R^+\hat{\gamma}_c \). Then \( \overline{M}_c^0 \) is a minimal model (the canonical bundle is trivial) of \( M_c \), but it is not smooth. In fact
\[
(M_c \setminus \Omega_c) \cong (M_c^0 \setminus \Omega_c^0),
\]
where $\Omega^S_c$ is the image of $\hat{\Omega}_c$. As a last observation, we remark that $\mathcal{M}_c$ is obtained from $\hat{\mathcal{M}}_c$ as follows: first we contract $R^+\hat{\sigma}_c$ to get $S_c//\text{PGL}(N)$, then we contract the image of $R^+\hat{\sigma}_c$ in $N\mathcal{E}_1(S_c//\text{PGL}(N))$ to get $R_c//\text{PGL}(N)$, finally the contraction of the image of $R^+\sigma_c$ in $N\mathcal{E}_1(R_c//\text{PGL}(N))$ gives $\mathcal{M}_c$. In other words, $\mathcal{M}_c$ is obtained from $\hat{\mathcal{M}}_c$ by reversing the order of the contractions.

### 3.1. Proof of Proposition (3.0.1).

Let $x \in \Omega^0_Q$: set $F_x = I_Z \otimes V$ and $W = \text{sl}(V)$. (Here $V \cong \mathbb{C}^2$.) Consider the following fiber bundles over $\text{Gr}^\omega(3, E_Z)$:

$$
P\text{Hom}(W, B_Z) := P(W \otimes B_Z),$$

$$P\text{Hom}_k(W, B_Z) := \{ \varphi \in \text{PHom}(W, B_Z) \mid \text{rk}\varphi \leq k \},$$

$$\hat{\text{PHom}}(W, B_Z) := \text{blow-up of PHom}(W, B_Z) \text{ along PHom}_1(W, B_Z).$$

Let $h: \hat{\text{PHom}}(W, B_Z) \to \text{PHom}(W, B_Z)$ be the blow-down map, and

$$f: \hat{\text{PHom}}(W, B_Z) \to \text{PHom}^\omega(W, E_Z)$$

be the composition of $h$ with the obvious map $\text{PHom}(W, B_Z) \to \text{PHom}^\omega(W, E_Z)$. Proposition (3.0.1) will be a straightforward consequence of the following result.

#### (3.1.1) Proposition

Let $x \in \Omega^0_Q$, and keep notation as above. There is an isomorphism

$$\tilde{f}: \hat{\text{PHom}}(W, B_Z) \sim (\pi_R \pi_S \pi_T)^{-1}(x)$$

such that $\pi_S \pi_T \tilde{f} = f$. (Recall (1.5.2) that $\pi_R^{-1}(x) \cong \text{PHom}^\omega(W, E_Z)$.)

**Proof.** We break up the proof into various steps.

1. **The map $\tilde{f}$.** We will prove there exists a map

$$\tilde{f}: \hat{\text{PHom}}(W, B_Z) \to (\pi_R \pi_S)^{-1}(x)$$

lifting $f$. Let $D_1 \subset \hat{\text{PHom}}(W, B_Z)$ be the exceptional divisor of $h$. Equality (1.8.1) gives that $(\pi_R \pi_S)^{-1}(x)$ is the blow-up of $\pi_R^{-1}(x) \cong \text{PHom}^\omega(W, E_Z)$ along $\text{PHom}_1(W, E_Z)$; hence to prove $\tilde{f}$ exists it is sufficient to verify that

$$(3.1.2) \quad f^*\mathcal{O}_{\text{PHom}(W, B_Z)}(-D_1).$$

Since we have an equality of sets $f^{-1}(\text{PHom}_1(W, E_Z)) = D_1$, we must show that given any $p \in D_1$, there exists $w \in T_p \hat{\text{PHom}}(W, B_Z)$ such that

$$(3.1.3) \quad f_*(w) \notin T_{f(p)} \text{PHom}_1(W, E_Z).$$

Let $[B] \in \text{Gr}^\omega(3, E_Z)$ be the image of $p$ under the bundle projection. Thus $h(p) \in \text{PHom}(W, B)$, and $p$ is in the image of the inclusion

$$(3.1.4) \quad \iota: \text{Bl}_{\text{PHom}_1(W, B)} \text{PHom}(W, B) \hookrightarrow \hat{\text{PHom}}(W, B_Z).$$

The intersection of $D_1$ with the left-hand side is smooth, thus there exists

$$v \in T_p \left( \text{Bl}_{\text{PHom}_1(W, B)} \text{PHom}(W, B) \right)$$

transverse to $D_1$. The vector $w := \iota_*(v)$ satisfies (3.1.3).

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II. The restriction of $\overline{f}$ to $D_1$. For vector spaces $A$, $B$, let $\text{Hom}_k(A, B)$ be the determinantal variety of linear maps $A \to B$ of rank at most $k$, and let $\text{PHom}_k(A, B)$ be its projectivization. Let $\varphi \in \text{Hom}_k(A, B)$ be of rank exactly $h$; recall that the natural map

$$T_\varphi \text{Hom}(A, B) = \text{Hom}(A, B) \to \text{Hom}(\text{Ker}\varphi, B/\text{Im}\varphi)$$

induces an isomorphism

$$(C_{\text{Hom}_h} \text{Hom}_k(A, B)) \varphi \cong \text{Hom}_{k-h}(\text{Ker}\varphi, B/\text{Im}\varphi),$$

Projectivizing we get an isomorphism

$$(3.1.5) \quad (C_{\text{PHom}_h} \text{PHom}_k(A, B))_{\varphi} \cong \text{Hom}_{k-h}(\text{Ker}\varphi, B/\text{Im}\varphi),$$

canonical up to scalars. Applying this isomorphism to $\text{PHom}_1(W, B) \subset \text{PHom}(W, B)$, for $[B] \in \text{Gr}^w(3, E_Z)$, we get

$$D_1 \cong \{([B], [\varphi], [\alpha]) | [B] \in \text{Gr}^w(3, E_Z), [\varphi] \in \text{PHom}_1(W, B), [\alpha] \in \text{PHom}(\text{Ker}\varphi, B/\text{Im}\varphi)\}.$$ 

By (3.1.2) we know $\overline{f}$ maps $D_1$ to $\Sigma_S$. By (1.7.12) there is a canonical isomorphism

$$\Sigma_S \cap (\pi_R \pi_S)^{-1}(x) \cong \{([\varphi], [\alpha]) \mid [\varphi] \in \text{PHom}_1(W, E_Z), [\alpha] \in \text{PHom}_{w^*}(\text{Ker}\varphi, \text{Im}\varphi^\perp/\text{Im}\varphi)\}.$$

For $([B], [\varphi], [\alpha]) \in D_1$, let $j: B \hookrightarrow E_Z$, $\overline{f}: B/\text{Im}\varphi \hookrightarrow \text{Im}\varphi^\perp/\text{Im}\varphi$ be the inclusion maps; one verifies easily that

$$(3.1.6) \quad \overline{f}([B], [\varphi], [\alpha]) = ([j \circ \varphi], [j \circ \varphi \circ \alpha]).$$

This describes the restriction of $\overline{f}$ to $D_1$. Let $D_2 \subset \text{PHom}^w(W, B_Z)$ be the strict transform of $\text{PHom}_2(W, B_Z)$; applying (3.1.5) one gets

$$D_1 \cap D_2 = \{([B], [\varphi], [\alpha]) \mid \text{rk}\alpha = 1\},$$

$$(\Sigma_S \cap \Delta_S) \cap (\pi_R \pi_S)^{-1}(x) = \{([\varphi], [\alpha]) \mid \text{rk}\alpha = 1\}.$$ 

In particular we have an isomorphism

$$(3.1.7) \quad \overline{f}|_{(D_1 \setminus D_2)}: (D_1 \setminus D_2) \xrightarrow{\sim} (\Sigma_S \setminus \Delta_S) \cap (\pi_R \pi_S)^{-1}(x).$$

III. The map $\hat{f}$. We will lift $\overline{f}$ to a map

$$\hat{f}: \text{PHom}(W, B_Z) \to (\pi_R \pi_S \pi_T)^{-1}(x).$$

Let

$$\Delta := \Delta_S \cap (\pi_R \pi_S)^{-1}(x) \cong \text{PHom}_2^w(W, E_Z).$$

(See (1.8.12).) Since $(\pi_R \pi_S \pi_T)^{-1}(x)$ is the blow-up of $(\pi_R \pi_S)^{-1}(x)$ along $\Delta$, the existence of a lift $\hat{f}$ will follow from

$$(3.1.8) \quad \overline{f} I_\Delta = O_{\text{PHom}(W, B_Z)}(-D_2).$$

To prove this equality, first notice that set-theoretically $\overline{f}^{-1}(\Delta) = D_2$. Thus it suffices to show that for any $p \in D_2$, there exists $w \in T_p \text{PHom}(W, B_Z)$ such that $\overline{f}_* w /\not\in T_\Delta$. Let $[B] \in \text{Gr}^w(3, E_Z)$ be the image of $p$ under the bundle projection; thus $p$ is in the image of Inclusion (3.1.4). Since

$$D_2 \cap Bl_{\text{PHom}_1(W, B)} \text{PHom}(W, B) = Bl_{\text{PHom}_1(W, B)} \text{PHom}_2(W, B),$$

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and the right-hand side is smooth (see (1.8.12)), there exists \( v \in T_p (Bl_{\text{PHom}_{1}(W,B)} \text{PHom}(W,B)) \) transverse to \( D_2 \). As is easily checked \( w := \iota_* v \) has the stated property.

**IV. Proof that \( \hat{f} \) is an isomorphism.** Clearly \( \hat{f} \) is birational. Since \( (\pi_R \pi_S \pi_T)^{-1} (x) \) is smooth, it suffices by Zariski’s Main Theorem to show that \( \hat{f} \) is an isomorphism in codimension one. One checks easily that the restriction of \( \hat{f} \) to the complement of \( \{ D_1 \cap D_2 \} \) is an isomorphism onto its image. \( \text{q.e.d.} \)

Now we prove Proposition (3.0.1). Let \([Z] \in X^{[\nu]} \), and let \( x \in \Omega_Q^0 \) be such that \( F_x \cong I_Z \otimes V \). By Proposition (3.1.1)

\[
\hat{\pi}^{-1}([I_Z \otimes V]) = \hat{\text{PHom}}(W,B_Z)//\text{SO}(W).
\]

The projection \( \hat{\text{PHom}}(W,B_Z) \rightarrow \text{Gr}^{\omega}(3, E_Z) \) is \( \text{SO}(W) \)-invariant, hence it descends to

\[
\lambda: \hat{\text{PHom}}(W,B_Z)//\text{SO}(W) \rightarrow \text{Gr}^{\omega}(3, E_Z).
\]

For \([B] \in \text{Gr}^{\omega}(3, E_Z) \) we have

\[
\lambda^{-1}([B]) = Bl_{\text{PHom}_{1}} \text{PHom}(W,B)//\text{SO}(W).
\]

The map

\[
\text{PHom}(W,B)^{ss} \rightarrow P(S^2B)
\]

\[
\alpha \mapsto \alpha \circ \alpha^t,
\]

identifies \( \text{PHom}(W,B)//\text{SO}(W) \) with \( P(S^2B) \). Since \( \text{PHom}_{1}(W,B)//\text{SO}(W) \) is the locus of conics of rank one, and since taking the quotient commutes with blowing up (1.1.2), we conclude that \( \lambda^{-1}([B]) = \text{CC}(S^2B) \). Proposition (3.0.1) follows at once.

**3.2. Proof of Proposition (3.0.2)-Item (1).**

Let \([Z] \in X^{[\nu]} \): we will introduce a basis of \( N^1(\hat{\Omega}_Z) \). Letting \( \rho: P(S^2B_Z) \rightarrow \text{Gr}^{\omega}(3, E_Z) \) be bundle projection, and \( \theta: \hat{\Omega}_Z \rightarrow P(S^2B_Z) \) be the blow-down map (see (3.0.1)), set

\[
h := c_1(\Theta_Z), \quad x := c_1(\mathcal{O}_{P(S^2B_Z)}(1)), \quad e := \text{class of the exceptional divisor of } \theta.
\]

Abusing notation we will denote with the same symbols the classes obtained pulling back \( h \) and \( x \) to \( \hat{\Omega}_Z \).

(3.2.1) **Claim.** The classes \( h, x, e \) form a basis of \( N^1(\hat{\Omega}_Z) \).

**Proof.** For \([B] \in \text{Gr}^{\omega}(3, E_Z) \) the restrictions of \( x, e \) to \( \hat{\Omega}_B \) give a basis of \( H^2(\hat{\Omega}_B) \), hence it suffices to prove that \( h \) generates \( H^2(\text{Gr}^{\omega}(3, E_Z)) \). This follows by applying Sommese’s generalization of Lefschetz’ hyperplane section theorem [La,(1.8)] to the embedding \( \text{Gr}^{\omega}(3, E_Z) \hookrightarrow \text{Gr}(3, E_Z) \). \( \text{q.e.d.} \)

(3.2.2) **Corollary.** Keeping notation as above,

\[
N_1(\hat{\Omega}_Z) = \mathbb{R}^+ \Theta_Z \oplus \mathbb{R}^+ \hat{\epsilon}_Z \oplus \mathbb{R}^+ \hat{\gamma}_Z.
\]

**Proof.** As is easily checked the intersection matrix of \{\( h, x, e \)\} with \{\( \Theta_Z, \hat{\epsilon}_Z, \hat{\gamma}_Z \)\} is non-singular (see (3.4.8)). Hence the result follows from duality together with (3.2.1). \( \text{q.e.d.} \)

We will prove the following formulae:

\[
c_1(\Theta_c \cap \hat{\Omega}_Z) = e, \quad (3.2.3)
\]

\[
c_1(\Delta_c \cap \hat{\Omega}_Z) = -2e - 2h + 3x, \quad (3.2.4)
\]

\[
c_1(K_{\hat{\Omega}_Z}) = 2e - (c - 6)h - 6x. \quad (3.2.5)
\]

Before proving the formulae we draw some consequences.
By Corollary (3.2.2) it suffices to show that each of

\[ (3.2.6) \]

**Proof.** By (3.2.3)-(3.2.5) we get that \( N^1(\mathcal{M}_e) \to N^1(\hat{\Omega}_Z) \) is surjective. Dualizing we conclude \( i^*_Z \) is injective. q.e.d.

In particular, since by (3.2.2) \( \hat{\sigma}_Z, \hat{\epsilon}_Z, \hat{\gamma}_Z \) are linearly independent, we see that \( \hat{\sigma}_e, \hat{\epsilon}_e, \hat{\gamma}_e \) are linearly independent; this proves Item (1) of Proposition (3.0.2). Now let’s prove Formulae (3.2.3)-(3.2.5). The first one is the easiest: it follows immediately from (3.1.2).

**Proof of (3.2.4).** Let \( D_Z \subset \mathbf{P}(S^2B_Z) \) be the locus parametrizing singular conics. Then \( \hat{\Delta}_e \cap \hat{\Omega}_Z = \) the strictly transfrom of \( D_Z \) under \( \theta \). Since, for \( [B] \in \text{Gr}^{w}(3, E_Z) \), the locus of singular conics in \( \mathbf{P}(S^2B) \) has multiplicity two along the locus of rank-one conics, we get that

\[ c_1(\hat{\Delta}_e \cap \hat{\Omega}_Z) = \theta^*c_1(D_Z) - 2e. \]

hence Formula (3.2.4) will follow from the following equation:

\[ (3.2.7) \quad c_1(D_Z) = -2h + 3x. \]

To prove it, we observe that \( D_Z \) is the degeneracy locus of the tautological map

\[ \rho^*\mathbf{B}_Z \otimes \mathcal{O}_{\mathbf{P}(S^2B_Z)}(-1) \xrightarrow{\Phi} \rho^*\mathbf{B}_Z. \]

Since \( \text{Det}\Phi \in \Gamma(\wedge^3\rho^*\mathbf{B}_Z \otimes \wedge^3\rho^*\mathbf{B}_Z \otimes \mathcal{O}_{\mathbf{P}(S^2B_Z)}(3)) \), Equation (3.2.7) follows at once.

**Proof of (3.2.5).** First we prove that

\[ (3.2.8) \quad c_1(K_{\text{Gr}^{w}(3, E_Z)}) = -(c - 2)h. \]

Consider the exact sequence

\[ 0 \to T_{\text{Gr}^{w}} \to T_{\text{Gr}^{w}}|_{\text{Gr}^{w}} \xrightarrow{\rho^*} N_{\text{Gr}^{w}/\text{Gr}} \to 0. \]

Since \( \text{Gr}^{w} \) is the zero-locus of a section of \( \wedge^2B''_Z \),

\[ c_1(N_{\text{Gr}^{w}/\text{Gr}}) = c_1(\wedge^2B''_Z) = 2h. \]

Equation (3.2.8) follows from this together with the formula for the canonical bundle of a Grassmannian. Next, we get

\[ (3.2.9) \quad c_1(K_{\mathbf{P}(S^2B_Z)}) = -(c - 6) - 6x \]

by considering the exact sequence

\[ 0 \to \text{Ker}\rho_* \to T_{\mathbf{P}(S^2B_Z)} \xrightarrow{\rho^*} \rho^*T_{\text{Gr}^{w}} \to 0. \]

Finally, Formula (3.2.5) follows from (3.2.9) because \( \hat{\Omega}_Z \) is obtained by blowing up a codimension-two subset of \( \mathbf{P}(S^2B_Z) \), and \( e \) is the class of the exceptional divisor.

**Claim.** Keeping notation as above, we have

\[ (3.2.10) \quad \overline{\text{NE}}_1(\hat{\Omega}_Z) = \mathbf{R}^+\hat{\sigma}_Z \oplus \mathbf{R}^+\hat{\epsilon}_Z \oplus \mathbf{R}^+\hat{\gamma}_Z. \]

**Proof.** By Corollary (3.2.2) it suffices to show that each of \( \mathbf{R}^+\hat{\sigma}_Z, \mathbf{R}^+\hat{\epsilon}_Z, \mathbf{R}^+\hat{\gamma}_Z \) is extremal. The maps

\[ \mathbf{P}^2(S^2B_Z) \leftarrow \text{CC}(B_Z) \to \mathbf{P}^2(S^2\hat{B}_Z) \]

are extremal. This is clear.
can be identified with the contraction of $R^+\hat{c}_Z$ and $R^+\hat{\sigma}_Z$ respectively. Thus $R^+\hat{c}_Z$ and $R^+\hat{\sigma}_Z$ are extremal rays of $\mathbb{NE}_1(\hat{\Omega}_c)$. Next notice that the contraction of $R^+\hat{\gamma}_c$ can be identified with the map $\hat{\mathcal{M}}_c \to S_c//\mathrm{PGL}(N)$, hence $R^+\hat{\gamma}_c$ is an extremal ray; by Lemma (3.2.6) we conclude that $R^+\hat{\gamma}_Z$ is an extremal ray. \hfill q.e.d.

3.3. Digression on $\hat{\Sigma}_c$.

For $[Z], [W] \in X^{[n]}$, with $Z \neq W$, set

$$\hat{\Sigma}_{Z,W} := \hat{\pi}^{-1}([I_Z \oplus I_W]).$$

Thus

$$\hat{\Sigma}_{Z,W} \in \hat{\Sigma}_c \setminus (\hat{\Omega}_c \cup \hat{\Delta}_c).$$

For $k$ a positive integer, let $I_k := \{(p, H) \in \mathbf{P}^k \times \hat{\mathbf{P}}^k \mid p \in H\}$.

(3.3.2) Proposition. Keep notation as above. Let $[Z], [W] \in X^{[n]}$, with $Z \neq W$. There is an isomorphism

$$\hat{\Sigma}_{Z,W} \cong I_{c-3}.$$  

Letting $r: \hat{\Sigma}_{Z,W} \to \mathbf{P}^{c-3}$ and $\hat{r}: \hat{\Sigma}_{Z,W} \to \hat{\mathbf{P}}^{c-3}$ be the maps determined by the above isomorphism,

$$[\hat{\Sigma}_c]|_{\hat{\Sigma}_{Z,W}} \cong r^*\mathcal{O}_{\mathbf{P}^{c-3}}(-1) \otimes \hat{r}^*\mathcal{O}_{\hat{\mathbf{P}}^{c-3}}(-1).$$

Proof. Isomorphism (3.3.3) is an easy consequence of Proposition (1.4.1). Let’s prove (3.3.4). By a monodromy argument,

$$[\hat{\Sigma}_c]|_{\hat{\Sigma}_{Z,W}} \cong r^*\mathcal{O}_{\mathbf{P}^{c-3}}(a) \otimes \hat{r}^*\mathcal{O}_{\hat{\mathbf{P}}^{c-3}}(a)$$

for some integer $a$. Copying the proof of (2.3.2) one gets $a = -1$. \hfill q.e.d.

3.4. The canonical class, and intersection numbers.

We will prove the following formula:

$$K_{\hat{\mathcal{M}}_c} \sim (3c - 7)\hat{\Omega}_c + (c - 4)\hat{\Sigma}_c + (2c - 6)\hat{\Delta}_c.$$  

First notice that there exist non-negative integers $\alpha_c, \beta_c, \gamma_c$ such that

$$K_{\hat{\mathcal{M}}_c} \sim \alpha_c\hat{\Omega}_c + \beta_c\hat{\Sigma}_c + \gamma_c\hat{\Delta}_c.$$  

In fact by (1.9.2) the canonical form $\wedge^{2c-3}\hat{\omega}_c$ is non-zero on the complement of $(\hat{\Omega}_c \cup \hat{\Sigma}_c \cup \hat{\Delta}_c)$.

(3.4.2) Lemma. Keeping notation as above, we have $\beta_c = (c - 4)$.

Proof. Let $[Z], [W] \in X^{[n]}$, with $Z \neq W$. Applying adjunction to $\hat{\Sigma}_c$ we get that

$$K_{\hat{\Sigma}_{Z,W}} \cong [K_{\hat{\mathcal{M}}_c} + \hat{\Sigma}_c]|_{\hat{\Sigma}_{Z,W}} = [\beta_c + 1]\hat{\Sigma}_c|_{\hat{\Sigma}_{Z,W}}.$$  

By (3.3.3) we know $\hat{\Sigma}_{Z,W} \cong I_{c-3}$, hence

$$K_{\hat{\Sigma}_{Z,W}} \cong r^*\mathcal{O}_{\mathbf{P}^{c-3}}(-(c - 3)) \oplus \hat{r}^*\mathcal{O}_{\hat{\mathbf{P}}^{c-3}}(-(c - 3)).$$  

By (3.3.4) we conclude that $\beta_c = (c - 4)$. \hfill q.e.d.
(3.4.3) Lemma. Let $[Z] \in X^{[n]}$ and $[B] \in \text{Gr}^c(3, E_Z)$. Then

$$c_1(\Omega_c)|_{\Omega_B} = (-2x + e)|_{\Omega_B}.$$  

(see (3.2) for the definition of $x$, e.)

Proof. Let $\hat{\text{Hom}}(W, B)$ be the blow up of $\text{Hom}(W, B)$ along the locus of rank-one homomorphisms. Then

$$\hat{\Omega}_B = \hat{\text{Hom}}(W, B)/\text{SO}(W) = \text{Hom}(W, B)^*/\text{SO}(W).$$

Let $\hat{f}: \hat{\text{Hom}}(W, B)^* \to \hat{\Omega}_B$ be the quotient map; we have

$$\hat{f}^*[\Omega]_{\hat{\Omega}_B} \cong [\Omega_T]_{\text{Hom}(W, B)^*}.$$ 

By (1.8.4)-(1.8.10)-(1.8.11), $\Omega_T^3$, $S^*$, and $\Delta_s^3$ are all smooth, hence

$$\Omega_T^3 \sim (\pi_S \pi_T)^* \Omega_R^3 - \Delta_T^*.$$ 

Now let $\lambda: \hat{\text{Hom}}(W, B)^* \to \text{Hom}(W, B)$ be the blow-down map, i.e. the restriction of $(\pi_S \pi_T)$ (see (3.1.1)). By the previous linear equivalence we have

$$[\Omega_T] \cong \lambda^*[\Omega_R] \otimes [-\Delta_T] \text{ in Pic} \left( \text{Hom}(W, B)^* \right).$$

Clearly $\lambda^*[\Omega_R] \cong \lambda^* \mathcal{O}_{\text{Hom}(W, B)}(-1)$. On the other hand, $\Delta_T|_{\text{Hom}(W, B)}$ is the strict transform under $\lambda$ of the locus parametrizing morphisms of rank at most two; an easy computation gives

$$[\Delta_T]|_{\text{Hom}(W, B)} \cong \lambda^* \mathcal{O}_{\text{Hom}(W, B)}(3) \otimes [-2E],$$

where $E$ is the exceptional divisor of $\lambda$. Thus (3.4.4) becomes

$$\hat{f}^*[\Omega] \cong \lambda^* \mathcal{O}_{\text{Hom}(W, B)}(-4) \otimes [2E].$$

Now consider the commutative diagram

$$\begin{array}{ccc}
\hat{\text{Hom}}(W, B)^* & \xrightarrow{\hat{f}} & \hat{\Omega}_B \\
\downarrow{\lambda} & & \downarrow{\theta_B} \\
\text{Hom}(W, B)^* & \xrightarrow{f} & \mathbb{P}(S^2 B),
\end{array}$$

where $f$ is the quotient map, and $\theta_B$ is the blow-up of conics of rank one. Since $f([\alpha]) = [\alpha \theta^*], \lambda$, we have

$$\hat{f}^* \theta_B^* x = \lambda^* f^* x = \lambda^* c_1(\mathcal{O}_{\text{Hom}(W, B)}).$$

Furthermore, since the generic point of $E$ has stabilizer of order two, $\hat{f}^* e = 2c_1(E)$. Feeding these equalities into (3.4.5) we get

$$\hat{f}^* c_1(\Omega_c) \cong \hat{f}^* (-2\theta_B^* x + e).$$

Since the pull-back map $\hat{f}^*: \text{Pic}(\hat{\Omega}_B) \to \text{Pic}(\hat{\text{Hom}}(W, B)^*)$ is injective [DN, Lemme (3.2)], this proves the lemma. 

q.e.d.

Now we prove Formula (3.4.1). Writing out adjunction for $\hat{\Omega}_Z$ and applying Lemma (3.4.2), we get

$$c_1(K_{\hat{\Omega}_Z}) \cong \left( c_1(K_{\hat{\lambda}_c}) + c_1(\Omega_c) \right)|_{\hat{\Omega}_Z} = \left( (\alpha_c + 1)c_1(\hat{\Omega}_c) + (c - 4)c_1(\hat{\lambda}_c) + \gamma_c c_1(\hat{\Delta}_c) \right)|_{\hat{\Omega}_Z}.$$

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By (3.2.1) together with Lemma (3.4.3) we can write
\[ c_1(\widehat{\Omega}_c)|_{\widehat{\Omega}_Z} = k_c h - 2x + e, \]

for some integer \( k_c \). Feeding this equality, together with (3.2.3)-(3.2.4)-(3.2.5), into Formula (3.4.6), we get three equations in the unknowns \( k_c, \alpha_c, \gamma_c \). The equations uniquely determine the unknowns, and we get (3.4.1). We also get the formula
\[ (3.4.7) \quad c_1(\widehat{\Omega}_c)|_{\widehat{\Omega}_Z} = h - 2x + e. \]

We close this subsection with tables of intersection numbers to be used later on. A straightforward computation gives the first table:

| \( \epsilon_z \) | \( h \) | \( x \) | \( e \) |
| --- | --- | --- | --- |
| \( 0 \) | \( 0 \) | \( -1 \) |
| \( 0 \) | \( 1 \) | \( 2 \) |
| \( 1 \) | \( 0 \) | \( 0 \) |

(3.4.8)

Formulae (3.2.3)-(3.2.4)-(3.4.7), together with the table above, give the following intersection matrix:

| \( \widehat{\Omega}_c \) | \( \widehat{\Sigma}_c \) | \( \widehat{\Delta}_c \) |
| --- | --- | --- |
| \( -1 \) | \( -1 \) | \( 2 \) |
| \( 0 \) | \( 2 \) | \( -1 \) |
| \( 1 \) | \( 0 \) | \( -2 \) |

(3.4.9)

3.5. Digression on \( \widehat{\Delta}_c \).

For \( [Z] \in X^{[n]} \) set \( \widehat{\Delta}_Z := \widehat{\pi}^{-1}([I_Z \oplus I_Z]) \cap \widehat{\Delta}_c \).

We will describe \( \widehat{\Delta}_Z \) quite explicitly. Let \( \mathcal{A}_Z \) be the tautological rank-two vector bundle on \( \text{Gr}^{\omega}(2, E_Z) \).

(3.5.1) Proposition. The image of the map \( f: \widehat{\Delta}_Z \to S_c//\text{PGL}(N) \) is naturally identified with \( \text{P}(S^2 \mathcal{A}_Z) \). The map \( f \) is a \( \text{P}^{c-4} \)-fibration. There is an identification

\[ (3.5.2) \quad \widehat{\Delta}_Z \cap \widehat{\Omega}_c \cong \{ ([A], [B], [q]) \mid [A] \in \text{Gr}^{\omega}(2, E_Z), [B] \in \text{Gr}^{\omega}(3, E_Z), [q] \in \text{P}(S^2 A), \ A \subset B \}, \]

such that the restriction of \( f \) is identified with the forgetful map \( ([A], [B], [q]) \mapsto ([A], [q]) \).

Proof. By definition, \( \widehat{\Delta}_c = \Delta_{T_c}//\text{PGL}(N) \), where \( \Delta_{T_c} \) is the exceptional divisor of \( \pi_T: T_c \to S_c \), the blow-up of \( \Delta_{S_c} \). By (1.8.10)-(1.8.13) there are no strictly semistable points to consider, hence we get a map

\[ \varphi: \widehat{\Delta}_c = \Delta_{T_c}^*/\text{PGL}(N) \to \Delta_{S_c}^*/\text{PGL}(N), \]

where the single slash is a reminder that the quotients are orbit spaces. Since \( \Delta_{S_c}^* \) and \( \Delta_{S_c}^* \) are both smooth (1.8.10)-(1.8.11), and since by (1.8.2) we have \( \text{cod}(\Delta_{S_c}^*, S_{c}^*) = (c - 3) \), \( \Delta_{T_c}^* \) is a \( \text{P}^{c-4} \) bundle over \( \Delta_{S_c}^* \). If \( x \in \Delta_{S_c}^* \), the stabilizer of \( x \) acts trivially on \( \pi_{T_c}^{-1}(x) \), hence \( \varphi \) is also a \( \text{P}^{c-4} \)-fibration. Now let’s show that the fiber of

\[ \psi: \Delta_{S_c}^*/\text{PGL}(N) \to \Omega_{Q_c}^{\omega}//\text{PGL}(N) = \Omega_c \cong X^{[n]} \]

over \([I_Z \oplus I_Z]\) is isomorphic to \( \text{P}(S^2 \mathcal{A}_Z) \). In fact, by (1.8.12)

\[ \psi^{-1}([I_Z \oplus I_Z]) \cong \text{P} \text{Hom}_g(W, E_Z)//\text{SO}(W). \]
The projection \( \overline{\text{Hom}}^*_n(W, E_Z) \to \text{Gr}^n(2, E_Z) \) is \( SO(W) \)-invariant, hence it descends to a map

\[
\psi^{-1}([I_Z \oplus I_Z]) \to \text{Gr}^n(2, E_Z).
\]

One checks easily that the fiber over \([A]\) is naturally isomorphic to \( \mathbb{P}(S^2A) \); this gives the isomorphism

\[
f(\hat{\Delta}_Z) = \psi^{-1}([I_Z \oplus I_Z]) \cong \mathbb{P}(S^2A_Z).
\]

Hence \( \hat{\Delta}_Z \) is indeed a \( \mathbb{P}^{c-4} \)-fibration over \( \mathbb{P}(S^2A_Z) \). To finish the proof of the proposition we define a map from \( \hat{\Delta}_Z \cap \hat{\Omega}_c \) to the right-hand side of (3.5.2). If \([B] \in \text{Gr}^n(3, E_Z)\), then

\[
\hat{\Delta}_Z \cap CC(B) = \text{closure of } \{(C, D) \in CC(B) | C \text{ has rank two}\}.
\]

Thus for every \((C, D) \in \hat{\Delta}_Z \cap CC(B)\) the conic \( D \subset \mathbb{P}(B) \) has rank one, i.e. it is the projectivization of a codimension one linear subspace \( A_D \subset B \). Thus we get a map

\[
\begin{array}{c}
\hat{\Delta}_Z \cap CC(B) \\
(C, D)
\end{array} \mapsto \begin{array}{c}
\text{Gr}(2, B) \\
[A_D].
\end{array}
\]

The fiber over \([A_D]\) is naturally identified with \( \mathbb{P}(S^2A_D) \); let \([q_{C,D}] \in \mathbb{P}(S^2A_D)\) be the point corresponding to \((C, D)\). We set

\[
\hat{\Delta}_Z \cap \hat{\Omega}_c \mapsto \text{right-hand side of (3.5.2)}
\]

\[
([B], C, D) \mapsto ([A_D], [B], [q_{C,D}]).
\]

This gives Isomorphism (3.5.2). \( \text{q.e.d.} \)

We continue examining \( \hat{\Delta}_Z \). Let \([A] \in \text{Gr}^n(2, E_Z)\) and consider \( \mathbb{P}(S^2A) \hookrightarrow \mathbb{P}(S^2A_Z) \); restricting the \( \mathbb{P}^{c-4} \)-fibration to \( \mathbb{P}(S^2A) \) we get a fibration

\[
\mathbb{P}^{c-4} \to f^{-1}\mathbb{P}(S^2A) \\
\downarrow
\mathbb{P}(S^2A).
\]

(3.5.3) Lemma. Fibration (3.5.3) is trivial.

Proof. The intersection \( f^{-1}\mathbb{P}(S^2A) \cap \hat{\Omega}_c \) is a divisor restricting to a hyperplane section (embedded linearly) on each \( \mathbb{P}^{c-4} \) fiber, and furthermore by (3.5.2) it is isomorphic to \( \mathbb{P}(S^2A) \times \mathbb{P}^{c-5} \). This implies that

\[
f^{-1}\mathbb{P}(S^2A) \cong \mathbb{P}(V),
\]

where \( V \) is a vector-bundle fitting into an exact sequence

\[
0 \to \mathcal{O}_{\mathbb{P}(S^2A)}(a)^{(c-4)} \to V \to \mathcal{O}_{\mathbb{P}(S^2A)}(b) \to 0,
\]

with \( f^{-1}\mathbb{P}(S^2A) \cap \hat{\Omega}_c = \mathbb{P}(\mathcal{O}_{\mathbb{P}(S^2A)}(a)^{(c-4)}) \). Now let \( \mathbb{P}(S^2A) \times [B] \in \mathbb{P}(\mathcal{O}_{\mathbb{P}(S^2A)}(a)^{(c-4)}) \). By (3.5.5) we have

\[
[\hat{\Omega}_c]|_{\mathbb{P}(S^2A) \times [B]} \cong \mathcal{O}_{\mathbb{P}(S^2A)}(b-a).
\]

Hence to prove the lemma it suffices to check that the left-hand side of the above equality is trivial. Let \( L \subset \mathbb{P}(S^2A) \times [B] \) be a line. In \( N_1(\hat{M}_c) \) we have \([L] = \hat{\sigma}_c\), so that by the entry on the first column and second row of (3.4.9) we get \([\hat{\Omega}_c]|_{L} = \mathcal{O}_L\). This proves \( a = b \). \( \text{q.e.d.} \)

We will need to know \( N_1(\hat{\Delta}_Z) \) and \( \overline{\text{NE}1}(\hat{\Delta}_Z) \). Let \( j^2: \hat{\Delta}_Z \hookrightarrow \hat{\Delta}_Z \) be Inclusion.
(3.5.6) Lemma. The map \( j^Z_s : N_1(\hat{\Delta}_Z) \to N_1(\hat{M}_c) \) is injective, and its image equals \( R\hat{\sigma}_c \oplus R\hat{\epsilon}_c \oplus R\hat{\gamma}_c \).

Proof. The map \( N_1(\hat{\Delta}_Z \cap \hat{\Omega}_c) \to N_1(\hat{\Delta}_Z) \) is an isomorphism. Since also \( N_1(\hat{\Delta}_Z \cap \hat{\Omega}_c) \to N_1(\hat{\Omega}_Z) \) is an isomorphism, the result follows from (3.2.6).

By the above lemma we can define \( \hat{\sigma}'_Z, \hat{\epsilon}'_Z, \hat{\gamma}'_Z \in N_1(\hat{\Delta}_Z) \) as the classes such that

\[
(3.5.7) \quad j^Z_s \hat{\sigma}'_Z = \hat{\sigma}_c, \quad j^Z_s \hat{\epsilon}'_Z = \hat{\epsilon}_c, \quad j^Z_s \hat{\gamma}'_Z = \hat{\gamma}_c.
\]

Let's give explicit representatives of the above classes. All representatives will be contained in \( \hat{\Delta}_Z \cap \hat{\Omega}_c \), so we refer to (3.5.2) for the description of the latter. Choose \([L] \in P(E_Z), [A] \in Gr^w(2, E_Z), [B] \in Gr^w(3, E_Z), [q^L] \in P(S^2L), [q^4] \in P(S^2A)\), with \( A \subset B \). Let \( \Lambda_1, \Lambda_2, \Lambda_3 \subset \hat{\Delta}_Z \cap \hat{\Omega}_c \) be the curves defined by

\[
\Lambda_1 := \{(A, [B], [q]) | [q] \in P^2(S^2A) \text{ varies in a line}\},
\]
\[
\Lambda_2 := \{(A, [B], [q^4]) | [B_t/A] \text{ varies in a line}\},
\]
\[
\Lambda_3 := \{(A_t, [B], [q^4]) | i^t : L \to A_t, A_t/L \text{ varies in a line}\}.
\]

It follows easily from (3.5.2) that

\[
\hat{\sigma}_Z = [\Lambda_1], \quad \hat{\gamma}_Z = [\Lambda_2], \quad 2\hat{\epsilon}_Z = [\Lambda_3].
\]

(3.5.8) Lemma. Keeping notation as above, we have

\[
\overline{NE}_1(\hat{\Delta}_Z) = R^+\hat{\sigma}'_Z \oplus R^+\hat{\epsilon}'_Z R^+\hat{\gamma}'_Z.
\]

Proof. By (3.5.6) it suffices to prove that each of \( R^+\hat{\sigma}'_Z, R^+\hat{\epsilon}'_Z, R^+\hat{\gamma}'_Z \) is extremal. By Lemma (3.5.4) there is a fibration

\[
P^2 \times P^{c-4} \to \hat{\Delta}_Z \quad \to \quad \text{Gr}^w(2, E_Z).
\]

Correspondingly we have two maps of \( \hat{\Delta}_Z \), the first contracting the \( P^2 \)'s, the second contracting the \( P^{c-4} \)'s. As is easily checked the first map can be identified with the contraction of \( R^+\hat{\sigma}'_Z \), and the second map can be identified with the contraction of \( R^+\hat{\gamma}'_Z \). Thus \( R^+\hat{\sigma}'_Z \) and \( R^+\hat{\gamma}'_Z \) are both extremal rays. To prove \( R^+\hat{\epsilon}'_Z \) is extremal, consider the natural map

\[
P(S^2A_Z) \xrightarrow{\phi} P(S^2E_Z) \quad \quad ([A], [q]) \mapsto [i^4_*q],
\]

where \([A] \in \text{Gr}^w(2, E_Z), [q] \in P(S^2A), \) and \( i^4_* : S^2A \to S^2E_Z \) is the map induced by inclusion. As is easily checked, \( \phi \) is the contraction of \( R^+[\Gamma] \), where \( \Gamma \subset P(S^2A_Z) \) is defined as follows: fix \([L] \in P(E_Z), [q^L] \in P(S^2L)\), and set

\[
\Gamma := \{([A_t], [i^t_*q_L]) | i^t : L \to A_t, A_t/L \text{ varies linearly in } P(L^+ /L)\}.
\]

Thus \( R^+[\Gamma] \) is an extremal ray of \( \overline{NE}_1(P(S^2A_Z)) \). Now consider the map \( f : \hat{\Delta}_Z \to P(S^2A_Z); \) then

\[
f_*\hat{\epsilon}'_Z = f_*[\Lambda_3] = [\Gamma].
\]

Since \([\Gamma] \) generates an extremal ray, and since \( f \) is the contraction of \( R^+\hat{\epsilon}'_Z \), we see that if \( R^+\hat{\epsilon}'_Z \) is not extremal, there exists an irreducible curve \( C \subset \hat{\Delta}_Z \) such that in \( N_1(\hat{\Delta}_Z) \)

\[
[C] = s\hat{\epsilon}'_Z - t\hat{\gamma}'_Z, \quad s > 0, t > 0.
\]

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Intersecting with $\tilde{\Omega}$, and applying (3.4.9) we get that $C \cdot \tilde{\Omega} < 0$, hence $C \subset \tilde{\Omega}$. By (3.5.7) and (3.2.6), we conclude that for some $s > 0, t > 0$, $[C] = (s\tilde{e} - t\tilde{c})$ in $N_1(\tilde{\Omega}Z)$, contradicting (3.2.10). \textbf{q.e.d.}

3.6. Proof of Proposition (3.0.2)-Item (2).

By Formula (3.4.1) and Table (3.4.9) we get

$$
\begin{align*}
K_{\tilde{\cal M}_c} \cdot \tilde{\sigma}_c &= -2, \\
K_{\tilde{\cal M}_c} \cdot \tilde{\epsilon}_c &= -1, \\
K_{\tilde{\cal M}_c} \cdot \tilde{\gamma}_c &= -(c - 5).
\end{align*}
$$

Thus we are left with the task of proving $R^+\tilde{\sigma}_c \oplus R^+\tilde{\epsilon}_c \oplus R^+\tilde{\gamma}_c$ is an extremal face. First we give a preliminary result. Let $[Z], [W] \in X^{[n]}$ with $Z \neq W$, and let $k^Z,W: \Sigma_{Z,W} \rightarrow \tilde{\cal M}_c$ be inclusion.

(3.6.1) \textbf{Lemma.} \textit{Keeping notation as above,}

$$
k^Z,W_*(NE_1(\Sigma_{Z,W})) = R^+(\tilde{\epsilon}_c + \tilde{\gamma}_c).
$$

\textbf{Proof.} Letting $[W]$ approach $[Z]$ we see that

$$
k^Z,W_*(NE_1(\Sigma_{Z,W})) \subset i^Z_*(NE_1(\tilde{\Omega}))
$$

Furthermore by (3.3.1)

$$
k^Z,W_*(NE_1(\Sigma_{Z,W})) \subset \left(Rc_1(\tilde{\Omega}) \oplus Rc_1(\tilde{\Delta})\right).
$$

Applying Table (3.4.9) we get the lemma. \textbf{q.e.d.}

Now assume that

$$
\sum_{\alpha \in I} m_{\alpha} [\Gamma_{\alpha}] \in R^+\tilde{\sigma}_c \oplus R^+\tilde{\epsilon}_c \oplus R^+\tilde{\gamma}_c,
$$

where, for each $\alpha \in I$, $m_{\alpha} > 0$ and $\Gamma_{\alpha}$ is an irreducible curve on $\tilde{\cal M}_c$; we must show that

$$
[\Gamma_{\alpha}] \in R^+\tilde{\sigma}_c \oplus R^+\tilde{\epsilon}_c \oplus R^+\tilde{\gamma}_c \text{ for each } \alpha \in I.
$$

From $\tilde{\pi}_s \tilde{\sigma}_c = \tilde{\pi}_s \tilde{\epsilon}_c = \tilde{\pi}_s \tilde{\gamma}_c = 0$, we get $\tilde{\pi}_s \Gamma_{\alpha} \equiv 0$ for all $\alpha$, and since $\tilde{\cal M}_c$ is projective this implies $\tilde{\pi}(\Gamma_{\alpha})$ is a point. Thus we can partition the indexing set as $I = I_\Omega \cap I_\Sigma \cap I_\Delta$ so that

$$
\begin{cases}
\alpha \in I_\Omega, & \text{then } \Gamma_{\alpha} \subset \tilde{\Omega}Z_{\alpha} \text{ for some } \Omega_{\alpha} \in X^{[n]}, \\
\alpha \in I_\Sigma, & \text{then } \Gamma_{\alpha} \subset \tilde{\Sigma}Z_{\alpha}W_{\alpha} \text{ for } \Omega_{\alpha}, W_{\alpha} \in X^{[n]} \text{ with } \Omega_{\alpha} \neq W_{\alpha}, \\
\alpha \in I_\Delta, & \text{then } \Gamma_{\alpha} \subset \tilde{\Delta}Z_{\alpha} \text{ for some } \Omega_{\alpha} \in X^{[n]}.
\end{cases}
$$

Statement (3.6.3) follows from (3.2.10) if $\alpha \in I_\Omega$, from (3.6.1) if $\alpha \in I_\Sigma$, and from (3.5.8) if $\alpha \in I_\Delta$.

3.7. Proof of Proposition (3.0.3).

By Proposition (3.0.2)-Item(2) and Mori theory, $\overline{\tilde{\cal M}_c}$ is projective. Let’s prove $\overline{\tilde{\cal M}_c}$ is smooth. Fibration (3.5.9) shows that $\tilde{\Delta}$ is a $\mathbb{P}^2$-fibration (with base a $\mathbb{P}^{c-4}$-bundle over $\text{Gr}^{ think}(2, E^c)$), hence $\tilde{\Delta}$ is a $\mathbb{P}^2$-fibration

$$
\mathbb{P}^2 \rightarrow \tilde{\Delta},
$$

(3.7.1)
where $\Lambda_c$ fibers over $X^{[n]}$, the fiber over $[Z]$ being a $\mathbb{P}^{c-4}$-bundle over $\text{Gr}^ω(2, E_Z)$. Let $\mathbb{P}^2$ be a fiber of (3.7.1) and $L \subset \mathbb{P}^2$ be a line. By (3.5) we have $[L] = \tilde{\sigma}_c$ in $N_1(\tilde{M}_c)$, hence (3.4.9) gives

\[(3.7.2) \quad [\tilde{\sigma}_c]|_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-1).\]

Claim. Keep notation as above. The contraction of $R^+\tilde{\sigma}_c$ is identified with the contraction of $\tilde{M}_c$ along Fibration (3.7.1).

Proof. If $L$ is line in a fiber of (3.7.1), then $[L] = \tilde{\sigma}_c$. Hence we must prove that if $Γ \subset \tilde{M}_c$ is an irreducible curve such that $[Γ] \in R^+\tilde{\sigma}_c$, then $Γ$ lies in a fiber of (3.7.1). Since $Γ \cdot \tilde{\sigma}_c < 0$, $Γ$ is contained in $\tilde{\Delta}_c$. Furthermore $\tilde{\pi}_cΓ \equiv 0$, hence there exists $[Z] \in X^{[n]}$ such that $Γ \subset \tilde{\Delta}_Z$. Applying Lemma (3.5.6), we get the following relation in $N_1(\tilde{\Delta}_Z)$:

\[Γ \in R^+\tilde{\sigma}_Z^{−}.\]

This implies $Γ$ is contained in a fiber of (3.7.1).

The above claim together with (3.7.2) proves that $\tilde{M}_c$ is smooth. Finally we must show that the rational map $\tilde{M}_c \cdots > M_c$ induced by $\tilde{\pi}$ is regular. One proceeds as in the proof that the analogous map $\tilde{M}_4 \cdots > M_4$ is regular (see (2.4)); the point is that $\tilde{\pi}$ is constant on the $\mathbb{P}^5$’s we have contracted.

3.8. Proof of Proposition (3.0.4).

Let $\tilde{θ}: \tilde{M}_c \to \tilde{M}_c$ be the contraction map, and $\tilde{π}: \tilde{M}_c \to M_c$ be the map induced by $\tilde{π}$; thus $\tilde{π} = π \circ \tilde{θ}$. Since, by Proposition (3.0.2), $R^+\hat{\epsilon}_c$ is extremal, so is $R^+\tilde{ε}_c$. Applying (3.4.9) we get

\[(3.8.1) \quad K_{\tilde{M}_c} \cdot \tilde{ε}_c = \tilde{θ}^* K_{\tilde{M}_c} \cdot \hat{ε}_c = (K_{\tilde{M}_c} - 2\tilde{∆}_c) \cdot \hat{ε}_c = -5.\]

Now let $\tilde{θ}: \tilde{M}_c \to \tilde{M}_c$ be the contraction of $R^+\tilde{ε}_c$; by Mori theory $\tilde{M}_c$ is projective. To prove $\tilde{M}_c$ is smooth, consider $\Omega_c := \tilde{θ}(\Omega_c)$. Clearly we have a fibration

\[(3.8.2) \quad \mathbb{P}^5 \to \Omega_c \quad \text{Gr}^ω(3, T_X^{[n]}),\]

where the fiber over $([Z], B)$ is canonically identified with $\mathbb{P}(S^2 B)$. If $L$ is a line in a fiber of the above fibration, then $[L] = τ_c$, hence by (3.8.1) together with adjunction we get

\[[\Omega_c]|_{\mathbb{P}^5} \cong \mathcal{O}_{\mathbb{P}^5}(-1).\]

Claim. Keeping notation as above, $\tilde{M}_c$ is obtained contracting $\tilde{M}_c$ along Fibration (3.8.2). In particular $M_c$ is smooth.

Proof. For $[Z] \in X^{[n]}$, let $\Omega_Z := \pi^{-1}([I_Z \oplus I_Z])$; we have a fibration

\[\mathbb{P}(S^2 B) \to \Omega_Z \quad \text{Gr}^ω(3, E_Z).\]

It follows from (3.2.6) that the map $N_1(\Omega_Z) \to N_1(\tilde{M}_c)$ induced by inclusion is injective. Arguing as in (3.7) we get the claim. q.e.d.

The a priori rational map $\tilde{M}_c \cdots > M_c$ is seen to be regular by an argument similar to that given in (2.4); the point is that $\tilde{π}$ is constant on the $\mathbb{P}^5$’s which have been contracted. Finally, let $\tilde{ω}_c$ be the two-form on $M_c$ induced by $\tilde{Ω}_c$; clearly $\tilde{ω}_c$ is non-degenerate outside $\tilde{Σ}_c := (\tilde{θ} \circ \tilde{θ})(\tilde{Σ}_c)$, in fact by (3.4.1)

\[(\lambda^{2c-3} \tilde{ω}_c) = (c - 4) \tilde{Σ}_c.\]
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