Graphical Nonbinary Quantum Error-Correcting Codes

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In this paper, based on the nonbinary graph state, we present a systematic way of constructing good non-binary quantum codes, both additive and nonadditive, for systems with integer dimensions. With the help of computer search, which results in many interesting codes including some nonadditive codes meeting the Singleton bounds, we are able to construct explicitly four families of optimal codes, namely, $[[6, 2, 3]]_p$, $[[7, 3, 3]]_p$, $[[8, 2, 4]]_p$, and $[[8, 4, 3]]_p$ for any odd dimension $p$ and a family of nonadditive code $[[5, p, 3]]_p$ for arbitrary $p > 3$. In the case of composite numbers as dimensions, we also construct a family of stabilizer codes $[[6, 2 \cdot p^2, 3]]_{2p}$ for odd $p$, whose coding subspace is not of a dimension that is a power of the dimension of the physical subsystem.

I. INTRODUCTION

Noises are inevitable and they cause errors in quantum informational processes. One active way of dealing with errors is provided by the quantum error-correcting codes (QECCs) [1, 2, 3, 4], which have found many applications in quantum computations and quantum communications, such as the fault-tolerant quantum computation [5], the quantum key distributions [6], and the entanglement purification [7, 8]. Roughly speaking, a QECC is a subspace of the Hilbert space of a system of many physical subsystems with the property that the quantum data encoded in this subspace can be recovered faithfully, even though a certain number of physical subsystems may suffer arbitrary errors, by suitable syndrome measurements followed by corresponding unitary transformations.

An important family of QECCs is the stabilizer code [9, 10, 11], which is specified by the joint +1 eigenspace of a stabilizer, an Abelian group of tensor products of Pauli operators. The stabilizer formalism has also been established in the nonbinary case [12, 13, 14, 15, 16] and many good codes for systems of a prime or a power of prime dimension have been constructed [15, 16, 17, 18], including the well-known perfect code with five registers [17] for all dimensions. Though the majority of QECCs constructed so far are stabilizer codes, including the CSS codes [19], the topological codes [20], color codes [21], and also the recently introduced entanglement-assisted codes [22], there are a few exceptions called as nonadditive codes [23, 24, 25, 26].

The nonadditive code does not admit a stabilizer structure and it should not be a subcode of some larger stabilizer code with the same distance otherwise it will be a trivial nonadditive code. Though difficult to construct and identify the nonadditive codes promise a larger coding subspace since less structured than the stabilizer codes. For qubits the nonadditive error-correcting code that outperforms the stabilizer codes has been constructed [26] based on the binary graph states. A graphical approach [27], as well as a codeword stabilized code approach [28], to the construction of binary additive and nonadditive codes has been developed based on the binary graph states.

The graph states [29, 30] are useful multipartite entangled states that are essential resources for the one-way computing [31] and can be experimentally demonstrated [32]. The binary graph state proves to an extremely effective tool [27, 30, 33] in the construction of QECCs. The nonbinary graph states were introduced first in [30] and discussed in details in the case of systems of an odd prime dimension [35]. It is also investigated in the context of universal quantum computation [36] and other applications [37]. Recently an approach to construct QECCs based on nonbinary graph states has been introduced in [38] and some new codes are found via computer search for qubits and qudits.

Here we shall generalize the graphical construction of QECCs [27] to the nonbinary case based on the nonbinary graph states from which some analytic constructions are attainable. In Sec.II the nonbinary graph states are introduced. In Sec.III we introduce the concept of coding clique for a weighted graph and show how it is related to the construction of both stabilizer and nonadditive QECCs. In Sec.IV we present some codes found via numerical searches and the graphical versions of some known codes as illustrations. In Sec.V we construct analytically four families of optimal stabilizer codes that saturate the quantum Singleton bound for any odd dimension as well as a family of nonadditive codes $[[5, p, 3]]_p$ for all $p > 3$. In Sec.VI we investigate the graphical codes arising from composite systems and construct a family of stabilizer codes $[[6, 2 \cdot p^2, 3]]_{2p}$ with $p$ being odd. The graph states of systems composed of coprime subsystems are in a one-to-one correspondence with the direct product of the graph states of subsystems.
II. NONBINARY GRAPH STATES

Here we shall consider a general system with \( p \) levels, a \textit{qupit} for short, where \( p \) is arbitrary. We denote by \( \mathbb{Z}_p = \{0, 1, \ldots, p-1\} \) the ring with addition modulo \( p \). Under the computational basis \( \{|i\}; i \in \mathbb{Z}_p \) of a qupit, the generalized bit shift and phase shift operators read

\[
X = \sum_{l \in \mathbb{Z}_p} |l + 1\rangle\langle l|, \quad Z = \sum_{l \in \mathbb{Z}_p} \omega^l |l\rangle\langle l|, \quad (\omega = e^{i \pi / p}).
\]

Obviously \( X \mathbb{P} = \mathbb{Z}_p = \mathbb{I} \) and \( ZX = \omega XZ \). Here we denote by \( X \) the Hermitian conjugate of \( X \). The computational basis \( |l\rangle \) is the eigenstate of \( Z \) with eigenvalue \( \omega^l \) while the eigenstate of \( X \) with eigenvalue \( \omega^l \) reads

\[
|\theta_j\rangle = \frac{1}{\sqrt{p}} \sum_{l \in \mathbb{Z}_p} \omega^{-l} |l\rangle.
\]

A \( \mathbb{Z}_p \)-weighted graph \( G = (V, \Gamma) \) is composed of a set \( V \) of \( n \) vertices and a set of weighted edges specified by the \textit{adjacency matrix} \( \Gamma \in \mathbb{Z}_p^{\times n} \times n \), an \( n \times n \) matrix with zero diagonal entries and the matrix element \( \Gamma_{ab} \in \mathbb{Z}_p \) denoting the weight of the edge connecting vertices \( a \) and \( b \). The graph state associated with a given weighted graph \( G = (V, \Gamma) \) of a system of \( n \) qubits labeled with \( V \) reads \cite{30}

\[
|\Gamma\rangle = \frac{1}{\sqrt{p^n}} \sum_{s \in \mathbb{Z}_p^n} \omega^{|s| \cdot \Gamma \cdot s} |s\rangle = \prod_{a, b \in V} (U_{ab})^{\Gamma_{ab}} |\theta_0\rangle^V.
\]

Here we have denoted by \( \mathbb{Z}_p^n \) the set of all the vectors \( s = (s_1, s_2, \ldots, s_n) \) with \( n \) components \( s_a \in \mathbb{Z}_p \) \((a \in V)\), by \( |s\rangle \) the common eigensate of all the phase shifts \( \mathcal{Z}_a \) \((a \in V)\) with eigenvalue \( \omega^s \), by

\[
|\theta_0\rangle^V = |\theta_0\rangle^1 \otimes |\theta_0\rangle^2 \otimes \ldots |\theta_0\rangle^n
\]

the joint +1 eigenstate of all bit shifts \( \mathcal{X}_a \) \((a \in V)\), and by

\[
U_{ab} = \sum_{i, j \in \mathbb{Z}_p} \omega^{ij} |i\rangle_{a} \otimes |j\rangle_{b}
\]

the non-binary controlled phase gate between two qupits \( a \) and \( b \). The non-binary graph state \( |\Gamma\rangle \) is also the unique (up to a global phase factor) joint +1 eigenstate of the following \( n \) vertex stabilizers

\[
\mathcal{G}_a = \mathcal{X}_a \prod_{b \in V} (Z_b)^{\Gamma_{ab}}, \quad a \in V.
\]

For \( s \in \mathbb{Z}_p^n \) we denote \( \mathcal{X}^s = \mathcal{X}_1^{s_1} \mathcal{X}_2^{s_2} \ldots \mathcal{X}_n^{s_n} \) and similarly for the phase shift operator \( Z^s \). Obviously

\[
\mathcal{G}_s \equiv \prod_{a \in V} (\mathcal{G}_a)^{s_a} = \omega^{s \cdot \Gamma \cdot s} \mathcal{X}^s Z^s \Gamma
\]

is also a stabilizer of the graph state for arbitrary \( s \in \mathbb{Z}_p^n \), i.e., \( \mathcal{G}_s |\Gamma\rangle = |\Gamma\rangle \). All the stabilizers of the graph state belong to the generalized Pauli group for qupits

\[
P_n = \{ e^{-i (p-1)/2} \mathcal{X}^s Z^t, s, t \in \mathbb{Z}_p^n \} \times \{ |\omega^l\rangle \in \mathbb{Z}_p \}. \quad (8)
\]

The \textit{graph-state basis} of the \( n \)-qupit Hilbert space \( \mathcal{H}_n \) refers to \( \{|\Gamma\rangle \equiv \mathcal{Z}^c |\Gamma\rangle |c\in \mathbb{Z}_p^n \} \). Under the computational basis a graph-state basis looks like

\[
|\Gamma_c\rangle = \mathcal{Z}^c |\Gamma\rangle = \frac{1}{\sqrt{p}} \sum_{s \in \mathbb{Z}_p^n} \omega^{|s| \cdot \Gamma \cdot s + c \cdot s} |s\rangle.
\]

A collection of \( K \) different vectors \( \{c_1, c_2, \ldots, c_K\} \) in \( \mathbb{Z}_p^n \) specifies a \( K \) dimensional subspace of \( \mathcal{H}_n \) that is spanned by \( K \) graph-state basis \( \{|\Gamma_{c_k}\rangle\}_{k=1}^K \).

For an example the graph state corresponding to the star graph \( S_3 \) on 3 vertices with all edges weighted 1 as shown in Fig[1](a) represents the GHZ state

\[
|S_3\rangle = \frac{1}{\sqrt{p}} \sum_{j \in \mathbb{Z}_p} |\theta_{p-j}\rangle^1 \otimes |j\rangle^2 \otimes |\theta_{p-j}\rangle^3.
\]

An edge with weight 1 will be represented by a black line and an edge with weight \( p-1 \) will be represented by a red thick line as in Fig[2]. In addition we will also indicate a vector in \( \mathbb{Z}_p^n \) via colored vertices with white, black, blue, and red vertices representing weights 0, 1, 2, \( p-1 \) respectively. For example in the graph shown in Fig[1](b) a vector \( (1, 0, 2) \in \mathbb{Z}_p^3 \) is indicated via the colored vertices and therefore we have a graph-state base

\[
S_1 Z_2^3 |S_3\rangle = \frac{1}{\sqrt{p}} \sum_{j \in \mathbb{Z}_p} |\theta_{p-j}\rangle^1 \otimes |j\rangle^2 \otimes |\theta_{p-j}\rangle^3.
\]
### III. CODING CLIQUES AND QECCS

From the standard theory of the QECC we know that if a set of Pauli errors can be corrected then all the errors can also be corrected. The situation in non-binary case is the same, because the nonbinary error basis defined by $\mathcal{E} = \{X^s Z^t | s,t \in Z_2 \}$ is a nice basis [12] [10], i.e., the set $\mathcal{E}$ forms a basis for the operators acting on the $n$-qupit Hilbert space $\mathcal{H}_n$.

For a given weighted graph $G = (V, \Gamma)$, from the definition the vertex stabilizers of a graph state, we have

$$X^s Z^t = \omega^{-\frac{1}{2} s \Gamma \cdot s} g_s Z^{t - s} \Gamma. \quad (12)$$

That is to say every non-binary Pauli error acting on the graph state $\ket{\Gamma}$ can be equivalently replaced by some qupit phase flip errors, up to some phase factors. For convenience we refer that the vector $t - s \cdot \Gamma$ is covered by the error $X^s Z^t$.

Given an integer $d$ we introduce a $d$-uncoverable set as

$$D_d = Z^t_p - \{ t - s \cdot \Gamma | 0 < |\sup(s) \cup \sup(t)| < d \}, \quad (13)$$

where we have denoted by $\sup(s) = \{ a \in V | s_a \neq 0 \}$ the support of a vector $s \in Z^t_p$ and by $|C|$ the number of the elements in $C \subseteq V$. That is to say $D_d$ is the set of all the $d$-uncoverable vectors, i.e., vectors cannot be covered by Pauli errors acting nontrivially on less than $d$ qubits, in $Z^t_p$. In addition we define the $d$-parity set as

$$S_d = \{ s \in Z^t_p | |\sup(s) \cup \sup(s \cdot \Gamma)| < d \}. \quad (14)$$

It is obvious that if $s \in S_d$ then the graph stabilizer $g_s$ has a support less than $d$, i.e., acts nontrivially on less than $d$ qubits.

A **coding clique** $C^K_d$ of a given graph $G = (V, \Gamma)$ is a collection of $K$ different vectors in $Z^t_p$ that satisfy:

i) $0 \in C^K_d$;

ii) $s \cdot c = 0$ for all $s \in S_d$ and every $c \in C^K_d$;

iii) $c - c' \in D_d$ for all $c, c' \in C^K_d$.

If the coding clique $C^K_d$ forms a group with respect to the addition modulo $p$, then it will be referred to as a **coding group**. In words, a coding clique $C^K_d$ is a collection of $K$ $d$-uncoverable vectors in $Z^t_p$ that is orthogonal to the vectors in $S_d$ and the difference between any two vectors in $C^K_d$ is also $d$-uncoverable. We can build a super graph $G$ with vertices being the vectors in $D_d \cap S_d$, i.e., $d$-uncoverable vectors that are orthogonal to all the vectors in $S_d$, and two vertices in the super graph are connected by an edge iff their difference is also $d$-uncoverable. Then the coding clique $C^K_d$ is exactly a $K$-clique, a subset of the vertices which are pairwise connected, of the super graph $G$. This fact justifies the nomenclature clique.

**Theorem 1** Given a graph $G = (V, \Gamma)$ and one of its coding clique $C^K_d$, the subspace $(G, K, d)_p$ spanned by the graph-state basis $\{ \Gamma_c | c \in C^K_d \}$ is an $((n, K, d),p)$ code, which is a stabilizer code if $C^K_d$ is a group and a nonadditive code if $C^K_d$ is neither a group nor a subset of a coding group $C^K_d'$ of $G$ with $d' \geq d$.

**Proof.** To prove that the subspace spanned by the graph-state basis $\{ \Gamma_c | c \in C^K_d \}$ is an $((n, K, d),p)$ code we have only to show that the condition [3] [30]

$$\langle \Gamma_c | \Gamma_{c'} \rangle = f(\mathcal{E}_d) \delta_{cc'}$$

is fulfilled for all $c, c' \in C^K_d$ and all the error $E_d = X^s Z^t$ that acts nontrivially on a number of qupits that is less than $d$, i.e., $|\sup(s) \cup \sup(t)| < d$.

If the error is proportional to a stabilizer of the graph state, i.e., $E_d = f(s) g_s$ for some $s \in Z^t_p$ and phase $f(s)$, then $s \in S_d$. Because of condition ii) of the coding clique the error acts like a constant operator on the subspace $(G, K, d)$ so that the condition Eq. (15) is fulfilled with a $f(\mathcal{E}_d) = f(s)$ that is independent of $c$. If the error $E_d$ is neither one of the stabilizers of the graph state $\ket{\Gamma}$ nor the identity operator, i.e., $E_d = X^s Z^t \propto g_s Z^{t - s} \Gamma$ with $t \neq s \cdot \Gamma$, then $\langle \Gamma_c | E_d | \Gamma_{c'} \rangle \propto \langle \Gamma | E_d | \Gamma \rangle$ with $e = c' - c + t - s \cdot \Gamma$. By virtue of condition iii) of the coding clique it is ensured that $e \neq 0$ for all $c, c' \in C^K_d$ which gives rise to Eq. (15) with $f(\mathcal{E}_d) = 0$. Thus we have proved that $(G, K, d)_p$ is an $((n, K, d),p)$ code.

By definition, a nonbinary stabilizer code is the joint +1 eigenspace of an Abelian subgroup of generalized Pauli group $P_n$, given in Eq. (8). The subspace $(G, K, d)_p$ spanned by the graph-state basis $\{ \Gamma_c | c \in C^K_d \}$ is stabilized by an element $S$ of $P_n$ if and only if $S$ is one of the stabilizers of the graph state $g_s$ with $s \in S$ where

$$S = \{ s \in Z^t_p | s \cdot c = 0, \forall c \in C^K_d \}. \quad (16)$$

If $C^K_d$ is a coding group then it is obviously an Abelian subgroup and can be generated by a set of vectors $(c_1, c_2, \ldots, c_k)$ in $Z^t_p$ with degrees $[\mu_1, \mu_2, \ldots, \mu_k]$, i.e., the minimal positive number such that $\mu_i c_i = 0 \pmod{p}$ ($i = 1, 2, \ldots, k$). It is easy to see that all $\mu_i$'s divide $p$ so that $K = \prod_{i=1}^k \mu_i$ divides $p^n$. Furthermore ($\omega = e^{i2\pi/p}$)

$$p^n = \sum_{c \in C^K_d} \sum_{s \in Z^t_p} \omega^{s \cdot c} = \sum_{s \in Z^t_p} \omega^{s \cdot c} = \|S\| K \quad (17)$$

because $0 \in C^K_d$ for the first equality and

$$\sum_{c \in C^K_d} \omega^{s \cdot c} = \prod_{i=1}^k \sum_{m_i=0}^{\mu_i-1} \omega^{m_i s c_i} = \begin{cases} K & \text{if } s \in S, \\ 0 & \text{if } s \notin S, \end{cases} \quad (18)$$

(since $\mu_i c_i = 0 \pmod{p}$) for the last equality. That is to say if $C^K_d$ is a group then we can find exactly a number $p^n/K$ of stabilizers $\{g_s | s \in S\}$ whose joint +1 eigenspace is exactly $(G, K, d)_p$. Thus the code $(G, K, d)_p$ is a stabilizer code.

If $C^K_d$ is not a group then we denote by $C$ the group generated by $C^K_d$, i.e., the smallest group that contains...
Obviously the subspace $Q$ spanned by $\{||\Gamma_c||c \in C\}$ is the joint $+1$ eigenspace of stabilizer $\langle G_\delta|s \in S\rangle$ and $|S| < p^d/K$. If the subspace $Q$ can detect $d - 1$ or more errors then $C$ is a coding group of $G$. If the subspace $Q$ cannot detect $d - 1$ errors then any subset of the stabilizer cannot either. On the other hand every stabilizer code that contains $(G, K, d)_p$ must have a stabilizer that is a subset of the stabilizer of $Q$. Therefore if $C^K_d$ is not a subset of some coding group $C^K_d'$ of $G$ with $d' \geq d$ then the code $(G, K, d)$ is a nonadditive code. Q.E.D.

Similar conclusions about the stabilizer codes appeared also in [38]. If all the generators of a coding group have the maximal degree $p$ then corresponding stabilizer code can be denoted as $[[n, k, d]]_p$. However there are cases where the generators of the coding group are not all of maximal degree. Then we have still a stabilizer code but the dimension of the code subspace may not be a power of $p$ and we shall denote such a stabilizer code as $((n, \mu_1, \mu_2, \ldots, \mu_k), d)_p$. A family of such kind of stabilizer codes will be provided in Sec. VI.

**IV. GRAPHICAL NONBINARY QECCS VIA NUMERICAL SEARCH**

According to Theorem 1 we can use the systematic algorithm developed for binary case in [27] to do a systematic search for the non-binary quantum codes, i.e.

i) To input a $Z_p$-weighted graph $G = (V, \Gamma)$ on $n$ vertices;

ii) To choose a distance $d$ and compute the $d$-purity set $S_d$ and the $d$-uncoverable set $D_d$ so that a super $G$ can be built;

iii) To find all the $K$-clique $C^K_d$ of the super graph $G$;

iv) To output a $(G, K, d)_p$, i.e., an $((n, K, d), p)$ code that is spanned by the basis $\{||\Gamma_c||c \in C^K_d\}$.

In practice, we have used the clique finding program clique [29] to search for the cliques for the super graph. Within our present computation power systematic search for graphical codes can be done up to $p = 6$ and $n = 6$ and some tentative searches have been done for $n = 8$. In what follows a quantum code will be specified by a weighted graph together with a coding clique or the generators of a coding group. Though the search for cliques are hard, the verifications of them are relative easy.

**A. The code $[[3, 1, 2]]_3$**

The first example is the stabilizer code $[[3, 1, 2]]_3$, which is known and has been constructed, e.g., in [15]. We consider the star graph $S_3$ on 3 vertices with two edges all weighted 1 as shown in Fig. II(a). A coding group generated by $(1, 0, 2) \in Z^3_3$ as shown in Fig. II(b) provides the code $[[3, 1, 2]]_3$, i.e., a 3-dimensional subspace spanned by the graph-state basis

$$\{Z^1_1 Z^2_3 |S_3\} |a \in Z_3\}. \quad (19)$$

The stabilizer of this code is generated by $\langle G_2, G_1 G_3 \rangle$ with $G_3$ being defined in [3]. It is easy to see that the stabilizer is equivalent to the stabilizer $\langle X_{123}, Z_{123} \rangle$ appeared in $\{8\}$ under a local unitary transformation. It is not difficult to see that we can have a $[[3, 1, 2]]_p$ for any odd $p$ with the same graph and the same coding group generated by $(1, 0, -1) \in Z^3_p$ with stabilizer $\langle G_2, G_1 G_3 \rangle$.

**B. The code $((3, 3, 2))_4$**

For an odd number of qupits with $p$ being even there is no code of distance 2 that saturates the Singleton bound so far. We have found a suboptimal code $((3, 3, 2))_4$ instead. For the equal weighted star graph on 3 vertices as shown in Fig. II(a) we have found a coding clique: $\{(0, 0, 0), (1, 0, 2), (2, 0, 1)\}$, with corresponding graph-state basis shown in Fig. II(a-c) which span the code subspace of a nonadditive $((3, 3, 2))_4$ code. In fact, as we see later, we can construct a code $((3, p - 1, 2))_p$ with even $p$.

**C. The code $[[4, 2, 2]]_6$**

The next example we concerned is also a 1-error detecting code $[[4, 2, 2]]_6$ which can be constructed from the star graph $S_4$ on 4 vertices as shown in Fig. II(d). From this graph a 2 dimensional coding group can be found to be generated by vectors $(1, -1, 0, 0)$ and $(1, 0, -1, 0)$ in $Z_2^4$. That is to say the code $[[4, 2, 2]]_6$ is the 36-dimensional subspace spanned by the basis

$$\{Z^1_1 Z^2_3 Z^4_3 |S_4\} |a, b \in Z_6\}. \quad (20)$$

whose stabilizer is generated by $\langle G_1 G_2 G_3, G_4 \rangle$. On the same graph we also find another coding clique as

$$\{(a + b, -a, -b, \delta_{a1} \delta_{b1}) \in Z^4_6 |a, b \in Z_6\} \quad (21)$$

which can be stabilized by $\langle G_1 G_2 G_3 \rangle$ only thus we have a nonadditive $((4, 36, 2))_6$ code. The nonadditive codes meeting the Singlet Bound is a very common situations in the nonbinary graphical codes, almost every stabiliser codes will have a clique set which are not a group, which means we can always construct a nonadditive code from a graphical nonbinary stabilizer code.

**D. The code $[[5, 1, 3]]_3$**

The first example of 1-error-correcting code is the well-known $[[5, 1, 3]]_3$ code, which is the only error-correcting
To input a

To find all the

for dimension 4 we also found a nonadditive code

that is spanned by graph-state basis

\[
\{ Z^a_1 Z^b_2 Z^c_3 Z^d_4 | L_5 \} \quad (a \in \mathbb{Z}_p) .
\]

E. The code \(((5,4,3))\)

For dimension 4 we also found a nonadditive code \(((5,4,3))\) that saturates the quantum Singleton bound. The graph we considered is still the loop graph \(L_5\) on 5 vertices with all edges weighted 1. The coding clique contains the following 4 vectors in \(Z_2^5\)

\[
\{(0,0,0,0,0), (1,1,1,1,1), (2,3,3,2,3),(3,2,2,3,2)\}
\]

among which one basis is shown in Fig[1](e) and two bases are shown in Fig[1](d). Since there is a stabilizer of the code containing 4^4 < 4^5 elements only and it satuates the Singleton bound, the code is not a subcode of any 1-error correcting code and is therefore nonadditive. In fact we can construct a nonadditive \(((5,p,3))\) for any \(p > 3\) as will be shown later.

F. The code \(((6,12,3))\)

We also found a nonadditive code \(((6,12,3))\) on the loop graph \(L_6\) on 6 vertices with one edge weighted 3 represented by a thick red edge as in Fig[2]. The coding clique \(C_6^2\) of \(L_6\) with 12 vectors reads

\[
\begin{align*}
(0, 0, 0, 0, 0) & (0, 1, 0, 2, 1) (0, 2, 3, 0, 1, 3) \\
(1, 0, 3, 1, 0) & (1, 2, 1, 3, 3) (1, 3, 3, 2, 3) \\
(2, 0, 3, 1, 0) & (2, 1, 2, 1, 1) (2, 2, 2, 1, 2) \\
(2, 3, 0, 2, 3) & (3, 0, 1, 2, 0) (3, 1, 2, 2, 2)
\end{align*}
\]

which is represented in Fig[2] with white, black, blue, and red vertices having weights 0,1,2,3 respectively. There are only 4 stabilizers can be found for this code, namely

\[
\{ I, g_2^a g_6^b, (g_1 g_2 g_4 g_5)^2 g_3 g_6, (g_1 g_2 g_4 g_5)^2 g_3 g_6 \},
\]

whose joint +1 eigenspace cannot be any 1-error correcting code because of the Singleton bound. Therefore the code is nonadditive.

G. The codes \([7,3,3]_3\) and \([8,4,3]_3\)

We found also two families of optimal qutrit codes, namely the codes \([7,3,3]_3\) and \([8,4,3]_3\), which can be constructed from the loop graphs \(L_7\) and \(L_8\) respectively. The generators of the coding groups are indicated in Fig[3](a) and Fig[3](b). For \(d = 3\) the blue and red vertices coincide. However those generators of the coding groups are also valid for the codes \([7,3,3]_p\) and \([8,4,3]_p\) for all odd \(p > 3\) which will be discussed in details in the next section.

H. The code \([8,2,4]_3\)

The last example is the code \([8,2,4]_3\) which is found only recently [35] from the hype cube graph. The graph considered here is the equal-weighted wheel graph \(W_6\) as shown in Fig[3](c) and the corresponding graph state is
TABLE I: The codes for p = 3.

| d=2   | d=3   | d=4   |
|-------|-------|-------|
| n=3   | [3,1,2] |   |   |
| n=4   | [4,2,2] |   |   |
| n=5   | [5,3,2] | [5,1,3] |   |
| n=6   | [6,4,2] | [6,2,3] |   |
| n=7   | [7,3,3] |   |   |
| n=8   | [8,4,3] | [8,2,4] |   |

denoted as |W₃⟩. The coding group is generated by

(0,1,1,0,2,2,0,0) and (1,0,2,2,0,1,1,1) \hspace{1cm} (23)

and the graphical code \( W₈; 2, 4 \)₃ is spanned by the graph-state basis

\[ \{ Z₃^a Z₄^b Z₅^{a+b} Z₆^a Z₇^b Z₈^b | W₈ \} \forall a, b \in \mathbb{Z}_3 \] \hspace{1cm} (24)

The stabilizer of the code has the following set of generators

\[ \langle G₁G₄, G₂G₅, G₃G₆, G₄G₇, G₅G₆, G₆G₇, G₁G₂G₃ \rangle. \] \hspace{1cm} (25)

To conclude this section we list all the stabilizer codes that saturate the Singleton bound, i.e., \( k + 2d \leq n + 2 \).

In Table I it should be emphasized that the absence of some codes in Table I does not imply that these codes do not exist. To find some of these codes are beyond our computers’ capacity and some of them will be constructed in the following sections. Furthermore, we can find the codes with a length \( n \geq 6 \) via a random searching method, thus some of absent codes may still be found by using our systematic algorithm developed above.

V. GRAPHICAL NONBINARY QECCS VIA ANALYTICAL CONSTRUCTIONS

Because the clique finding problem is intrinsically an NP-complete problem, it is not plausible to rely on the numerical search for codes with larger length or higher dimension. Here we shall provide some analytical constructions of good codes for higher dimension, which is based on the graphs and their coding cliques in lower dimensions found via computer research. In practice, we start from a graphical code found for qutrit and then generalize the subspace to arbitrary dimension by adopting the same graph and similar coding clique, and finally we prove that the subspace provides a code in arbitrary dimension.

A. The nonadditive code \(((3, p - 1, 2))_p\) with even \( p \)

Description of the code—Suppose that \( p = 2q \). We consider the star graph \( S₃ \) labeled with \( V = \{ A, B, C \} \) as shown in Fig.4(a). The \( p - 1 \) dimensional subspace spanned by the basis

\[ Z₄^a Z₅^b | S₃ \rangle, \ Z₆^{a+b} Z₇^b | S₃ \rangle, \] \hspace{1cm} (26)

with \( 0 \leq l \leq q - 1 \) and \( 0 \leq j \leq q - 2 \) is a nonadditive code \(((3, p - 1, 2))_p\).

Proof—It is enough to demonstrate that a subset \( \mathbb{C}^p \) composed of \( 2q - 2 \) vectors of forms \( (l, 0, 2l) \) and \( (q + j, 0, 2j + 1) \) satisfies all three conditions of coding clique. Since the 2-purity set is empty and \( l \) and \( j \) is straightforward to show that all vectors in \( \mathbb{C}^p \) and pairwise difference cannot be covered by single qubit error. Thus we have a code \(((3, p - 1, 2))_p\). The stabilizer of the code turns out to be generated by \( G₈ \). Therefore it is a nonadditive code.

Via a similar construction by Rains [24] we are able to construct the code \(((2n + 3, p^{2n}(p - 1, 2))_p\) for even \( p \). We consider the graph on \( 2n + 3 \) vertices composed of the star graph \( S₃ \) as shown in Fig.4(a) and the graph \( B_{2n} \) as shown in Fig.4(b) and denote the corresponding graph state as \( |S₃⟩ ⊗ |B_{2n}⟩ \). Let us denote by \{\|v\|\} the basis in Eq. (26) for the \(((3, p - 1, 2))_p\) code constructed above then the code subspace is spanned by the basis

\[ (ZₐZ₈)_{\sum_{i=1}^{2^{2n}} (Z₉)_{\sum_{i=1}^{2^{2n-1}} |v⟩} \otimes Z^s | B_{2n}⟩) \] \hspace{1cm} (27)

with \( s \in \mathbb{Z}^n_2 \) being arbitrary. We notice that the phase flips acting on \(|v⟩\) is a single qubit error on qupit \( B \).

B. The code \([2n, 2n - 2, 2]\)

We consider the graph on \( 2n \) vertices with all edges weighted 1 as shown in Fig.4(b) and denote the corresponding graph state as \( |B_{2n}⟩ \). The subspace spanned by the graph-state basis

\[ Z₁^{a₁} Z₂^{a₂} Z₃^{a₃} Z₄^{a₄} \cdots Z₂^{a₂n-1} Z₁^{a₁} | B_{2n}⟩ \] \hspace{1cm} (28)

with \( a_j, b_j \in \mathbb{Z}_p \) for \( j = 1, 2, \ldots, n - 1 \) is the code \([2n, 2n - 2, 2]\) whose stabilizer is generated by

\[ X₁Z₂X₃Z₄ \cdots X_{2n-1}Z_{2n}, \]

\[ Z₁X₂Z₃X₄ \cdots Z_{2n-1}X_{2n}. \] \hspace{1cm} (29)

It is straightforward to see from the stabilizer that every single qubit error can be detected, i.e., not commute with at least one of two generators defined above. In comparison in [35] the star graph has been used to construct the code \([n, n - 2, 2]\)ₚ.
The total number of the solutions to Eq. (15) is not a multiple of 3 and 3q^3 if p is not 3 but can be divided by 3. In either case the number of stabilizers is less than p^4 if p > 3 and since the code saturates the Singleton bound we conclude that the code Eq. (32) is nonadditive.

### E. The code \([6, 2, 3]_p\)

**Description of the code**—Consider the weighted loop graph on 6 vertices as shown in Fig. 4(c) and the corresponding graph state \(|L_6\rangle\). The \(p^2\)-dimensional subspace spanned by the graph-state basis

\[
\{ Z_a^a Z_2^a Z_3^a Z_4^a Z_5^a Z_6^a | L_6\rangle | a, b \in \mathbb{Z}_p \}
\]

is a \([6, 2, 3]_p\) code for all odd \(p > 2\). A set of the generators of its stabilizer is listed in Table II.

**Proof**—Because of Theorem 1 we have only to prove that the subset of vectors

\[
\{ c = (a, a + b, b, -a, a - b, b)| a, b \in \mathbb{Z}_p \}
\]

satisfies all three conditions of the coding clique. Conditions i) and ii) are obviously satisfied since \(a, b\) can assume value zero and the 3-purity set is empty for the loop graph for any dimension \(p\). To prove condition iii) we have to show that \(c\) cannot be covered by single and 2-qubit errors for all \(a, b \in \mathbb{Z}_p\). Obviously \(c\) cannot be covered by any single vertex error, we consider only 2-qubit errors in what follows.

i) \(b = 0, a \neq 0\) with \(c = (a, a, 0, -a, a, 0)\). If a 2-qubit error is supported on vertices \(\{1, 2, 4, 5\}\) then it can only takes form

\[
X_1^n Z_1^s X_3^n Z_3^s Z_5^s Z_6^s,
\]

which covers \((s, n, 0, n, t, 0), (n, s, 0, t, -n, 0)\) and \((-s, t, 0, t, s, 0)\) respectively with \(n, s, t \in \mathbb{Z}_p\) being arbitrary. All these 3 types of vectors cannot be identified with \(c\) provided \(p > 2\) and odd. For example if \((s, n, 0, n, t, 0) = c\) then \(2a = 0 = 0\) which is impossible for odd \(p\).

ii) \(a = 0, b \neq 0\) with \(c = (0, b, b, 0, -b, b)\). In this case it can be proved in exactly the same manner as the first case that \(c\) cannot be covered by 2-qubit error.
iii) $a = b \neq 0$ with $c = (a, 2a, a, -a, 0, a)$. Any 2-qupit error that covers $\{1, 2, 3, 4, 6\}$ can only take form $X_1^{a}X_4^{a}Z_{1}^{b}Z_{4}^{b}$ or $X_4^{a}Z_1^{b}X_3^{a}Z_4^{b}$, which covers 

$$(s, m + n, t, n, 0, -m)$$

respectively with $m, n, s, t \in \mathbb{Z}_p$ being arbitrary. To cover $c$ we have to require $2a = -2a$ or $a = -a$ which are impossible for odd $p$.

iv) $a = -b \neq 0$ with $c = (-b, 0, b, -2b, b)$. Any 2-qupit error that covers 5 vertices $\{1, 3, 4, 5, 6\}$ can only be $X_1^{a}X_4^{a}Z_{1}^{b}Z_{4}^{b}$ or $X_4^{a}Z_1^{b}X_3^{a}Z_4^{b}$, which covers 

$$(m, 0, m, n, s, n)$$

respectively with $m, n, s, t \in \mathbb{Z}_p$ being arbitrary. To cover $c$ we have to require $b = -b$ or $2b = -2b$ which is impossible for odd $p$.

v) $a \neq \pm b, a \neq 0$ with $c = (a, a + b, b, -a, a - b, b)$.

To cover all the 6 vertices a 2-qupit error can only takes form

$$X_1^{a}X_4^{a}Z_{1}^{b}Z_{4}^{b}, X_4^{a}Z_1^{b}X_3^{a}Z_4^{b}, X_4^{a}Z_1^{b}X_3^{a}Z_4^{b}$$

which covers $(s, m, n, t, n, -m), (m, s, m, n, t, n),$ and $(-n, m, s, m, n, t)$ respectively. However all these vectors can not be identified with $c$ if $p$ is odd. For example if $(s, m, n, t, n, -m) = c$ then $2a = 0$ which is impossible for odd $p$.

In summary, for any $a, b \in \mathbb{Z}_p$ the vector $c$ cannot be covered by any single or 2-qupit errors so that is a coding clique and corresponding subspace is the $[[6, 2, 3], p]$ code for all odd $p > 2$.

F. The code $[[7, 3], p]$

Description of the code—Consider the equal weighted loop graph on 8 vertices as shown in Fig.4(d) and the corresponding graph state $|L_7\rangle$. The $p^4$-dimensional subspace spanned by the graph-state basis

$$\{Z_1^{a+b+c}, Z_2^{a-b+c}, Z_3^{a-c}, Z_4^{a+c}, Z_5^{b-c}, Z_6^{b-c}, Z_7^{a+b+c}, Z_8^{a+b+c}|L_7\rangle\}$$

is a $[[7, 3], p]$ code for all odd $p > 2$. A set of the generators of its stabilizer is listed in Table III.

| TABLE III: The stabilizer of the code $[[7, 3], p]$ |
|--------------------------------------------------|
| $G_1G_6$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $G_1G_7$ | Z | Z | X | Z | Z | X | Z |
| $G_2G_3$ | Z | I | Z | X | Z | Z | X |
| $G_1G_2G_4$ | Z | Z | X | Z | 2 | X | Z |
| $G_1G_2G_3G_4$ | Z | X | X | X | Z | Z | I | Z |

Proof—Because of Theorem 1, we have to show that the subset of $Z_8^p$ composed of vectors of form

$$c = (e, b-c, c-e, c-a, a+b, a-b+c+e, 2b, a-c+e)$$

is a coding clique. Since the 3-purity set is empty for the loop graphs with any dimension $p$, we need only to show that any nonzero $c$ cannot be covered by any single or 2-qupit error.

It is not difficult to see that $c$ cannot be covered by any single qupit error, which covers a vectors with 4 consecutive components being zero. Because of the symmetry of the loop graph, all 2-qupit errors can be classified into 3 types $T_{12}, T_{13},$ and $T_{14}$ where $T_{ij}$ denotes two errors occur on qubits $a + l$ and $a + j$ with $n \in \mathbb{Z}_7$. Each type of errors covers the vector $(c_1, c_2, \ldots, c_7)$ with properties

- $T_{12}$: $c_{n} = c_{n+1} = c_{n+2} = 0$;
- $T_{13}$: $c_{n-1} + c_{n+3} = c_{n+1}$ and $c_{n-2} = c_{n-3} = 0$;
- $T_{14}$: $c_{n-1} = c_{n+1}, c_{n-2} = 0$, and $c_{n-3} = c_{n+3}$.

It can be easily checked that as given in Eq. (36) for arbitrary $a, b, c \in \mathbb{Z}_p$ does not belong to all these 3 types of vectors when $p$ is odd. For example a $T_{12}$ type of error covers vector with 3 consecutive components being zero, which is impossible to be identified with a nonzero $c$. That is to say $c$ cannot be covered by any 2-qupit errors and therefore we have a coding clique and a $[[7, 3], p]$ code for all odd $p$.

G. The code $[[8, 4], p]$

Description of the code—Consider the equal weighted loop graph on 8 vertices as shown in Fig.4(e) and the corresponding graph state $|L_8\rangle$. The $p^4$-dimensional subspace spanned by the graph-state bases

$$\{Z_1^{a+b+c}, Z_2^{a-b+c}, Z_3^{a-c}, Z_4^{a+c}, Z_5^{b-c}, Z_6^{b-c}, Z_7^{a+b+c}Z_8^{a+b+c}|L_8\rangle\}$$

with $a, b, c \in \mathbb{Z}_p$ is a $[[8, 4], p]$ code for all odd $p > 2$. A set of the generators of its stabilizer is listed in Table IV.

| TABLE IV: The stabilizer of the code $[[8, 4], p]$ |
|-----------------------------------------------|
| $G_1G_3G_4G_6$ | XZ | Z | X | Z | X | Z |
| $G_2G_3G_6G_7$ | $Z^2$ | XZ | $Z^2$ | XZ | Z |
| $G_1G_2G_3G_4G_6$ | $Z^2X^2$ | $ZX^2$ | $Z^2$ | X | $Z^3$ |
| $G_1G_2G_3G_4G_6$ | $ZX^2$ | $ZX^2$ | $Z^2X^2$ | X | $Z^3$ |

Proof—Because of Theorem 1, we have to show that the subset of $Z_8^p$ defined as

$$\{c = (a + b + c, a, b - c, b) | a, b, c \in \mathbb{Z}_p\}$$

is a coding clique. Since the 3-purity set is empty for the loop graphs with any dimension $p$, we need only to show that any nonzero $c$ cannot be covered by any single or 2-qupit error.
which with \( a, b, c, e \in \mathbb{Z}_p \) is a coding clique of the loop graph. Because of the symmetry of the loop graph, all 2-qupit errors can be classified into 4 types \( T_{12}, T_{13}, T_{14}, \) and \( T_{15} \). Each type of errors covers the vector \((c_1, c_2, \ldots, c_8)\) with properties \((n \in \mathbb{Z}_q)\):

\[
\begin{align*}
T_{12}: & \ c_{n+1} = c_{n+2} = c_{n+3} = c_{n+4} = 0; \\
T_{13}: & \ c_{n+2} = c_{n+3} = c_{n+4} = 0 \text{ and } c_{n+1} + c_{n-3} = c_{n-1}; \\
T_{14}: & \ c_{n+2} = c_{n+3} = 0, \ c_{n+1} = c_{n-1}, \text{ and } c_{n-2} = c_{n-4}; \\
T_{15}: & \ c_{n+4} = 0, \ c_{n+1} = c_{n+3}, \text{ and } c_{n-1} = c_{n-3}.
\end{align*}
\]

As an example we consider a \( T_{12} \) type of errors which covers a vector with 4 consecutive components being zero. A nonzero vector \( c \) as defined in Eq.\( [38] \) is not possible to have such a property, e.g., the last 4 components being zero. In the same manner it can be checked that \( c \) cannot belong to any of those 4 types of vectors above for arbitrary \( a, b, c, e \in \mathbb{Z}_p \) when \( p \) is odd. Since all single qupit errors cover vectors with the same property as in type \( T_{12} \), we conclude that \( c \) cannot be covered by any single or 2-qupit errors so that we have a coding clique and therefore a \([8, 4, 3]_p\) code for all odd \( p \).

### H. The code \([8, 2, 4]_p\)

**Description of the code**— Consider the equal weighted wheel graph on 8 vertices as shown in Fig.3(d) and the corresponding graph state \( |W_8\rangle \). The \( p^2 \)-dimensional subspace spanned by the graph-state basis

\[
\{ z_1^b z_2^a z_3^{-b} z_4^{2a} z_5^{2b} z_6^b z_7^b z_8^b | \Gamma \} | a, b \in \mathbb{Z}_p \}
\]

is a \([8, 2, 4]_p\) code for all odd \( p > 2 \). A set of the generators of its stabilizer is listed in Table [V]

| Table V: The stabilizer of the code \([8, 2, 4]_p\) |
|-----------------|-----|-----|-----|-----|-----|-----|-----|
| \( \mathcal{G}_1 \mathcal{G}_2 \) 2 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| \( \mathcal{G}_1 \mathcal{G}_4 \) 2 | X | Z | Z | Z | Z | X | I |
| \( \mathcal{G}_1 \mathcal{G}_4 \) 2 | X | Z | Z | Z | Z | Z | X | I |
| \( \mathcal{G}_1 \mathcal{G}_4 \) 2 | Z | X | Z | Z | Z | Z | X | I |
| \( \mathcal{G}_1 \mathcal{G}_4 \) 2 | Z | X | Z | Z | Z | Z | X | I |
| \( \mathcal{G}_1 \mathcal{G}_4 \) 2 | Z | X | Z | Z | Z | Z | X | I |
| \( \mathcal{G}_1 \mathcal{G}_4 \) 2 | Z | X | Z | Z | Z | Z | X | I |
| \( \mathcal{G}_1 \mathcal{G}_4 \) 2 | Z | X | Z | Z | Z | Z | X | I |

**Proof**— Because of Theorem 1 we have to show that the subset of \( \mathbb{Z}_p^9 \) defined as

\[
\{(c = (b, a, a - b, 2b, 2a, b - a, b, b))| a, b \in \mathbb{Z}_p \}\]

is a coding clique of the wheel graph. We have only to prove condition iii of the coding clique since the 4-purity set is empty.

A single qupit error on vertex \( n \) can cover a vector with property \( c_{n+2} = c_{n+3} = 0 \) which is impossible for \( c \) defined above. Because of symmetry all 2-qupit errors can be classified into 4 types \( T_{12}, T_{13}, T_{14}, \) and \( T_{15} \) with each type of errors covering the vector \((c_1, c_2, \ldots, c_8)\) with properties \((n \in \mathbb{Z}_q)\):

\[
\begin{align*}
T_{12}: & \ c_{n+1} = c_{n+4}, c_{n+2} = c_{n+5} = 0, \text{ and } c_{n+3} = c_{n+6}; \\
T_{13}: & \ c_{n+3} + c_{n+1} = c_{n+4}; \\
T_{14}: & \ c_{n} = c_{n+3} \text{ and } c_{n+1} + c_{n+2} = 0; \\
T_{15}: & \ c_{n} = c_{n+6}, c_{n+2} = c_{n+4} \text{ and } c_{n+1} = c_{n+5} = 0.
\end{align*}
\]

It can be checked in a straightforward manner that none of the above equalities can be satisfied by \( c \) as defined in Eq.\( [40] \). For example we consider error of type \( T_{12} \) with \( n = 0 \). In this case we have constraints \( b = 2b = a = 2a \), and \( a-b = b-a \) which are impossible for nonzero \( c \) when \( p \) is odd. This is true for all \( n \in \mathbb{Z}_8 \). Thus any 2-qupit error cannot cover \( c \).

All 3-qupit errors can be classified into 7 types \( T_{123}, T_{124}, T_{125}, T_{126}, T_{127}, T_{135}, \) and \( T_{136} \) with each type of errors covering vectors with the following properties \((n \in \mathbb{Z}_q)\):

\[
\begin{align*}
T_{123}: & \ c_{n+2} = c_{n+5} \text{ and } c_{n-2} = c_{n-5}; \\
T_{124}: & \ c_{n} = c_{n+3} \text{ and } c_{n+2} = 0; \\
T_{125}: & \ c_{n} + c_{n+1} = c_{n+3} + c_{n+4} = 0; \\
T_{126}: & \ c_{n} + c_{n-1} = c_{n-3} \text{ and } c_{n-4} = 0; \\
T_{127}: & \ c_{n} = c_{n-3} \text{ and } c_{n-2} = 0; \\
T_{135}: & \ c_{n} + c_{n+1} = c_{n+3} + c_{n+1} = c_{n-3}; \\
T_{136}: & \ c_{n} = c_{n+2} + c_{n+3} = c_{n-2} + c_{n-3}.
\end{align*}
\]

It can be checked in a tedious but straightforward manner that for every \( n \in \mathbb{Z}_q \) none of those equalities above can be satisfied by \( c \) as defined in Eq.\( [40] \). For example we consider the 3-qupit error of type \( T_{126} \) with \( n = 1 \). In this case we have \( b + b = a - a = 0 \) which are impossible for nonzero \( c \).

To summarize, the vector \( c \) defined in Eq.\( [40] \) for all \( a, b \in \mathbb{Z}_p \) is 4-uncoverable. Thus we have proved that the subspace defined in is a \([8, 4, 2]_p\) code for all odd dimension \( p \).

### VI. CODES FROM COMPOSITE SYSTEMS

Consider a system with \( pq \) levels whose computational bases are denoted by \([|l\rangle|pq\rangle]_{l=0}^{p-1}\). We can also regard this system as a composite system of a \( p \)-level system and a \( q \)-level system, whose computational bases are denoted as \([|s\rangle|p\rangle]_{s=0}^{q-1}\) and \([|l\rangle|q\rangle]_{l=0}^{q-1}\) respectively. If there are \( n \) copies of \( pq \)-level systems, we also have \( n \) copies of \( p \)-level and \( q \)-level systems. On the other hand if we have two groups
of $p$-level and $q$-level systems we can also obtain $n$ copies of $pq$-level system by pairing up one $p$-level system and one $q$-level system to make up a composite system.

Given a $\mathbb{Z}_p$-weighted graph $(V, \Gamma_p)$ and a $\mathbb{Z}_q$-weighted graph $(V, \Gamma_q)$ on the same vertex set $V$ and corresponding coding cliques $C_d^K$ and $\tilde{C}_d^K$, the subspace spanned by the basis

$$\{ Z^{\alpha}_p | \Gamma_p \} \otimes Z^{\beta}_q | \Gamma_q \} \in \mathbb{C}_d^K, \in \mathbb{C}_\tilde{d}^K \} \quad (41)$$

is an $((n, K \tilde{K}, d))_{pq}$ code with $n = |V|$. This is because all the direct products of Pauli errors of the $p$-level systems and $q$-level systems form a nice error basis for the $pq$-level system. That is to say, via a direct product of two graphical codes $(G, K, d)_p$ and $(G, \tilde{K}, d)_q$ we can construct a code $((n, K \tilde{K}, d))_{pq}$, which is however not necessarily to be another graphical code. As will be shown below this construction will yield a graphical code of a higher dimension if $p$ and $q$ are coprime, in which case there exist two integers $\alpha$ and $\beta$ such that

$$\alpha p + \beta q = 1. \quad (42)$$

Given a $\mathbb{Z}_{pq}$-weighted graph $(V, \Gamma_{pq})$ we can build a $\mathbb{Z}_p$-weighted graph $(V, \Gamma_p)$ and a $\mathbb{Z}_q$-weighted graph $(V, \Gamma_q)$ whose adjacency matrices are given by

$$p \Gamma_{pq} \equiv \Gamma_q (mod q), \quad q \Gamma_{pq} \equiv \Gamma_p (mod p). \quad (43)$$

On the other hand, given a $\mathbb{Z}_p$-weighted graph $(V, \Gamma_p)$ and a $\mathbb{Z}_q$-weighted graph $(V, \Gamma_q)$ on the same vertex set $V$, we can also build a $\mathbb{Z}_{pq}$-weighted graph $(V, \Gamma_{pq})$ with adjacency matrix given by

$$\Gamma_{pq} \equiv p \alpha \Gamma_q + q \beta \Gamma_p (mod pq). \quad (44)$$

By relabeling the bases of a $pq$-level system according to $|s\rangle_p \otimes |t\rangle_q \mapsto |pt + qs\rangle_{pq}$, we can define an isometry between $n$ $pq$-level systems and $n$ pairs of $p$-level subsystems and $q$-level subsystems as

$$\mathcal{R} = \sum_{s \in \mathcal{Z}_p, t \in \mathcal{Z}_q} |pt + qs\rangle_{pq} \langle s|_p \otimes \langle t|_q, \quad (45)$$

which is only possible when $p$ and $q$ are coprime, we have

$$|\Gamma_{pq}\rangle = \mathcal{R} |\Gamma_q\rangle. \quad (46)$$

Accordingly the bit flips and phase flips are related to each other via

$$X_{pq} = \mathcal{R} (X_p \otimes X_q) \mathcal{R}^\dagger, \quad Z_{pq} = \mathcal{R} (Z_p \otimes Z_q) \mathcal{R}^\dagger. \quad (47)$$

**Theorem 2** If $p, q$ are coprime then the graph state on the subsystems of dimension $p$-level graph is in a one-to-one correspondence with the direct product of two graph states on a $\mathbb{Z}_p$-weighted graph and a $\mathbb{Z}_q$-weighted graph whose adjacency matrices are related via Eqs. (43)-(44).

According to the fundamental theorem of arithmetics any integer $p$ can be expressed as $p = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_L^{\alpha_L}$ for some primes $p_i$ ($i = 1, \ldots, L$). Therefore a graph state on the subsystems of dimension $p$ will be in a one-to-one correspondence of a direct product of graph states on the subsystems of dimension $p_1^{\alpha_1}$, as a consequence we have only to consider the graph states of systems of a dimension that is prime or a power of prime. So far the definition of graph state is unique in the case of prime dimension and there is a second definition of graph states $\mathbb{Z}_{pq}$ in the case of prime power dimension.

As an example we consider the case where $q = 2$ and there 6 copies of 2$^p$-level composite systems and accordingly 6 pairs of qubits and qupits. A direct product of a graphical code $[[6, 1, 3]]_p$ on qubits constructed from the loop graph with a leaf $\tilde{C}_1$ as shown in Fig.5(a) and the graphical code $[[6, 2, 3]]_p$ arising from the graph $L_6$ as constructed in Sec.V (E) will result a code $((6, 2p^2, 3))$. It is in fact a stabilizer code and we should note that the dimension of the coding subspace is not a power of 2p.

Via the relabeling of the bases of the $pq$-level systems as specified $\mathcal{R}$ in Eq. (15) we obtain a $\mathbb{Z}_{2p}$-weighted graph $\mathcal{H}_6$ as shown in Fig. 5(b). From this graph we consider a subset $\mathbb{C}$ of $2p^2$ vectors in $\mathbb{Z}_{2p}$ of form

$$(p + 1)(a, a + b, -a - b, a + b, + p(c, c, c, c, 0)) \quad (48)$$

with $a, b \in \mathbb{Z}_p$ and $c \in \mathbb{Z}_2$. The subspace spanned by the graph-state basis $\{Z^e |\mathcal{H}_6\} (e \in \mathbb{C})$ is exactly the stabilizer code $((6, 2p^2, 3))$ constructed from composite systems. A set of the generators of its stabilizer is listed in Table VI.

Because the code $[[6, 2, 3]]_p$ does not exist we cannot remove any stabilizers from the lower half part of the above table to obtain a $(2p)^2$-dimensional code space, therefore the code $((6, 2p^2, 3))_p$ is not a subcode of a $[[6, 2, 3]]_2p$, if there is any.

**VII. DISCUSSION**

We have generalized the graphical construction $\mathcal{C}$ of QECCs to the nonbinary case to find both additive and
TABLE VI: The stabilizer of the code \((6, 2p^2, 3)_{2p}\).

| \[G_i^G_j\] | \(X^2\) | \(Z^{p+1}\) | \(Z^{p+1}\) | \(X^2\) | \(Z^{p+1}\) | \(Z^{2p+1}\) |
|\(G_i^G_j\) | \(Z^{p+1}\) | \(Z^{p+1}\) | \(Z^{p+1}\) | \(X^2\) | \(Z^{p+1}\) | \(Z^{2p+1}\) |
| \(G_{i1}^G_{j1}\) | \(Z^{p+1}\), \(X^2\) | \(Z^{2p+1}\), \(X^2\) | \(Z^{p+1}\) | \(I\) | \(Z^{2p+1}\) |
| \(G_{i1}^G_{j2}\) | \(Z^{2p+1}\) | \(Z^{2p+1}\) | \(Z^{2p+1}\) | \(X^2\) | \(Z^{2p+1}\) |
| \(G_{i2}^G_{j2}\) | \(Z^{2p+1}\) | \(Z^{2p+1}\) | \(Z^{2p+1}\) | \(X^2\) | \(Z^{2p+1}\) |
| \(G_i^X\) | \(I\) | \(I\) | \(I\) | \(Z^{p}\) | \(X^p\) |
| \(G_i^X_i\) | \(X^p\) | \(Z^p\) | \(Z^p\) | \(I\) | \(Z^{2p}\) |
| \(G_i^X_j\) | \(X^p\) | \(Z^p\) | \(Z^p\) | \(I\) | \(I\) |
| \(G_i^X_j\) | \(X^p\) | \(Z^p\) | \(Z^p\) | \(I\) | \(I\) |

nonadditive codes based on nonbinary graph states. The advantages of our graphical approach lies in the fact that we are able to construct codes on physical systems of arbitrary dimension, prime or prime power, to construct both additive or nonadditive codes, pure or impure. In addition the basis for all codes are explicitly constructed. In principle for prime dimension our method exhausts all the stabilizer codes, which can be demonstrated in exactly the same manner as in binary case [27].

Via numerical search we have found many optimal codes including two optimal codes \([8, 3, 3]_3\) and \([8, 2, 4]_3\) which have been found in [38] on a hyper cube graph. Since the clique finding problem for non-binary case is even more difficult and is intrinsically a NP-complete problem, our computational result is mainly limited in codes with \(n \leq 8\) and small distance \(d \leq 4\).

With the help of the codes found numerically we also manage to construct analytically some families of optimal stabilizer codes that saturating the Singleton bound such as \([6, 2, 3]_p, [7, 3, 3]_p, [8, 4, 3]_p\) and \([8, 2, 4]_p\) for any odd \(p\). We have also explicitly constructed the code \([n, n-2, 2]_p\) except the case of even \(p\) and odd \(n\). There exist stabilizer codes whose code subspace is not a power of the dimension of physical systems such as the code \((6, 2 \cdot 2^2, 3)_{2p}\) which is constructed via a composite system approach. Furthermore we have constructed a family of nonadditive codes \((3, p - 1, 2)_p\) with even \(p\), a nonadditive optimal code \((5, p, 3)_p\) for all \(p > 3\).

Finally, we briefly address some questions which are still open. Although our method can also be used to find additive and nonadditive codes for non-prime dimension, we can not exhaust all the non-prime stabilizer codes because of our definition of graph states. Because of Theorem 2 we have to consider the graph states on system of prime and prime power dimension. There are at least two different definitions of the graph state on \(n\) subsystems of a dimension \(p = q^n\) being a power of prime: one is as defined in this paper and in [35] and the other one is as defined in [33]. A different definition of graph states may result in some new families of codes.

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