Four-derivative couplings
via T-duality constraint

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Abstract

We examine the proposal that the dimensional reduction of the effective action of perturbative string theory on a circle, should be invariant under T-duality transformations. The T-duality transformations are the standard Buscher rules plus some higher covariant derivatives. By explicit calculations at order $\alpha'$ for metric, dilaton and B-field, we show that the T-duality constraint can fix both the effective action and the higher derivative corrections to the Buscher rules up to an overall factor. The corrections depend on the scheme that one uses for the effective action. We have found the effective action and its corresponding T-duality transformations in an arbitrary scheme.
1 Introduction

One of the most exciting discoveries in perturbative string theory is T-duality \[1, 2\]. This duality may be used to construct the $D$-dimensional effective field theory at any order of $\alpha'$. One approach for constructing this effective action is the Double Field Theory (DFT) \[3, 4, 5, 6, 7\] in which the effective action in 2$D$-space is invariant under T-duality and a gauge transformation. The T-duality is the standard $O(D, D)$ transformation whereas the gauge transformation is non-standard and receives $\alpha'$-corrections \[8, 7, 9, 10\]. Another proposal for constructing the $D$-dimensional effective action is to use the T-duality constraint on the reduction of the effective action on a circle \[11\]. In this approach one reduces the standard gauge invariant effective action on a circle to produce the corresponding $(D - 1)$-dimensional effective action. Up to some boundary terms, this action should be invariant under the T-duality transformations which is the standard Buscher rules \[12, 13\] plus some $\alpha'$-corrections \[14, 15, 16\]. Using this proposal, the known gravity and dilaton couplings in the effective actions at orders $\alpha', \alpha'^2, \alpha'^3$ have been found up to some overall factors \[17, 18\]. The corrections to the Buscher rules, however, could not be fixed in the case that $B$-field is zero. For the effective action at order $\alpha'$ that has been found in \[19\], the form of $\alpha'$-corrections to the Buscher rules have been found in \[16\] for the case that $B$-field is non-zero.

In this paper we speculate that in the presence of $B$-field, the T-duality constraint may fix both the effective action and the $\alpha'$-corrections to the Buscher rules. We have done this calculation explicitly at order $\alpha'$. Using the Bianchi identities and the field redefinition freedom, one can write the most general $D$-dimensional covariant action at four-derivative level in a specific scheme which has 8 parameters \[20\]. We then reduce it on a circle to find its corresponding $(D - 1)$-dimensional action which should be invariant under the T-duality transformations up to some boundary terms. Constraining this action to be invariant under the Buscher rules makes all parameters to be zero unless one adds some corrections to the Buscher rules. We then write the most general covariant corrections at two-derivative level to the Buscher rules and impose the $(D - 1)$-dimensional action to be invariant under this deformed T-duality transformations. Interestingly, the T-duality constraint fixes all parameters in the effective actions and in the deformed T-duality transformations, up to an overall factor. The effective action is exactly the standard action that has been found by the S-matrix calculation \[20\]. The T-duality transformations, however, are not the same as the T-duality transformations that have been found in \[16\] because the effective action that we have found and the effective action that has been used in \[16\] are in different schemes.

Since the T-duality transformations depend on the scheme that one uses for the effective action, it would be desirable to find the T-duality transformations for the effective action in an arbitrary scheme. We will show that the T-duality constrain can fix the effective action even if one does not use the field redefinition. If fact, using the Bianchi identity and removing total derivative terms, one finds that the most general $D$-dimensional effective action at order $\alpha'$ has 20 parameters \[20\]. 3 of them are unambiguous as they are not changed under field redefinitions and the other 17 parameters which are ambiguous are changed under the field redefinitions. However, there are 5 combinations of these parameters that remain invariant under the field
redefinitions. To have the minimum number of couplings, one should keep 5 parameters which are called essential parameter, and remove all other parameters [20]. In general, one may keep all ambiguous parameters. In this case, the S-matrix calculations should fix the 3 unambiguous and the 5 essential parameters. The other 12 parameters should remain arbitrary. We will show that the T-duality constrain on the most general effective action with the 20 parameters, fixes the the effective action up to 12 arbitrary parameters, one of them is unambiguous parameter and all other 11 parameters are ambiguous parameters. The T-duality transformations are also found in terms of these parameters. Any choice for these 11 parameters gives the effective action and its corresponding T-duality transformations in a specific scheme. We will show that the effective action for a specific choice for these parameters becomes the action that has been found in [19], and the corresponding T-duality transformations are exactly the one that has been found in [16].

The outline of the paper is as follows: In section 2, we perform the calculations at order $\alpha'^0$. In particular, we write the most general $D$-dimensional effective action at two-derivative level which has 3 parameters. We then reduce it on a circle to find its corresponding $(D-1)$-dimensional effective action. Constraining it to be invariant under the Buscher rules up to some boundary terms, the 3 parameters are fixed up to an overall factor.

In section 3, we perform the calculations at order $\alpha'$. In subsection 3.1, we consider the 8-parameter effective action in the specific field variables studied in [20]. We show that the reduction of this action on a circle is invariant under the Buscher rules when all parameters in the effective action are zero. To have non-zero effective action at order $\alpha'$, we then deform the Buscher rules by some terms at order $\alpha'$ with arbitrary parameters. Some relations between these parameters are found by the constraint that the T-duality transformations must form a $Z_2$-group. Constraining the reduction of the effective actions at orders $\alpha'^0$ and $\alpha'$ to be invariant under the deformed T-duality transformations, fixes all independent parameters in the deformed T-duality transformations and in the effective action. Up to an overall factor, the effective action is the one has been found in [20] by the S-matrix method. In subsection 3.2, we consider the 20-parameter effective action in which the field redefinitions are not used. We then impose the T-duality constraint on this action. We find the effective action and the corresponding T-duality transformations in terms of one unambiguous parameters and 11 ambiguous parameters. A specific choice for these 11 parameters, gives the effective action and the T-duality transformations found in [19][16]. In this subsection, we have also shown that the Chern-Simons couplings in the heterotic theory which is resulted from the non-standard gauge transformation of $B$-field is also invariant under the T-duality and found its corresponding T-duality transformations.

2 Effective action at order $\alpha'^0$

We now construct the most general $D$-dimensional action at two-derivative level which is invariant under the coordinate transformations and under the standard gauge transformation of
B-field, i.e., $B_{ab} \rightarrow B_{ab} + \partial_i \lambda_j$. Up to total derivative terms, it has the following three terms:

$$S_0 = -\frac{2}{k^2} \int d^D x e^{-2\phi} \sqrt{-G} \left( c_1 R + c_2 \nabla_a \Phi \nabla^a \Phi + c_3 H^2 \right). \quad (1)$$

where the three-form $H$ is field strength of the two-form $B$, i.e., $H_{abc} = \partial_a B_{bc} + \partial_b B_{ca} + \partial_c B_{ab}$, and $c_1, c_2, c_3$ are three constants.

To impose abelian T-duality constraint on this action, we have to consider a background with $U(1)$ isometry. It is convenient to use the following background for metric and Kalb-Ramond field:

$$G_{ab} = \begin{pmatrix} \bar{g}_{\mu \nu} + e^\varphi g_{\mu \nu} & e^\varphi g_{\mu} \\ e^\varphi g_{\nu} & e^\varphi \end{pmatrix}, \quad B_{ab} = \begin{pmatrix} \bar{b}_{\mu \nu} + \frac{1}{2} b_{\mu} g_{\nu} - \frac{1}{2} b_{\nu} g_{\mu} & b_{\mu} \\ 0 & 0 \end{pmatrix} \quad (2)$$

where $\bar{g}_{\mu \nu}, \bar{b}_{\mu \nu}$ are the metric and the B-field, and $g_{\mu}, b_{\mu}$ are two vectors in the $(D-1)$-dimensional base space. Inverse of the above $D$-dimensional metric is

$$G^{ab} = \begin{pmatrix} \bar{g}^{\mu \nu} & -g^{\mu} \\ -g^{\nu} & e^{-\varphi} + g_{\alpha} g^{\alpha} \end{pmatrix} \quad (3)$$

where $\bar{g}^{\mu \nu}$ is inverse of the $(D-1)$-dimensional metric which raises the index of the vectors. In this parametrization, the $(D-1)$-dimensional dilaton is $\bar{\phi} = \Phi - \varphi/4$. The Buscher rules \[12, 13\] in this parametrization are the following linear transformations:

$$\varphi' = -\varphi, \quad g_{\mu}' = b_{\mu}, \quad b_{\mu}' = g_{\mu}, \quad \bar{g}_{\alpha \beta}', \quad g_{\alpha \beta}' = \bar{g}_{\alpha \beta}, \quad \bar{b}_{\alpha \beta}' = \bar{b}_{\alpha \beta}, \quad \bar{\phi}' = \bar{\phi} \quad (4)$$

They form a $Z_2$-group, i.e., $(x')' = x$ where $x$ is any field in the base space.

To simplify the calculations, we assume that the base space is flat, i.e., $g_{\mu \nu} = \eta_{\mu \nu}$. As long as the T-duality constraint fixes all coefficients in the effective action in this case, we do not need to consider the general case of curved base space. If the effective action contains terms with at most second derivative, i.e., $R, \nabla \Phi, \nabla H$, the covariant derivatives in the $(D - 1)$-dimensional base space can be written as ordinary derivatives in the local frame in which $\Gamma_{\mu \nu}^{\alpha} = 0$. However, the curvature terms in the base space are not zero in the local frame. One expects the coefficients of these terms appears in many other terms like $\nabla \nabla \varphi$ which might be fixed by the T-duality constraint when the base space is flat.

In order to reduce $R$, one should write the curvature in terms of metric $G_{ab}$ and then use the reductions (2) and (3). When the base space is flat it becomes

$$R = -\partial_{\mu} \partial_{\mu} \varphi - \frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi - \frac{1}{4} e^{\varphi} V^2 \quad (5)$$

where $V_{\mu \nu}$ is field strength of the $U(1)$ gauge field $g_{\mu}$, i.e., $V_{\mu \nu} = \partial_{\mu} g_{\nu} - \partial_{\nu} g_{\mu}$. For curved base space, the ordinary derivatives in (5) become covariant derivatives and there is also the scalar curvature of the base space. The reduction of the overall factor and the second term in (1) when the base space is flat, are

$$e^{-2\Phi} \sqrt{-G} = e^{-2\bar{\phi}}$$

$$\nabla_a \Phi \nabla^a \Phi = \partial_{\mu} \bar{\phi} \partial^{\mu} \bar{\phi} + \frac{1}{2} \partial_{\mu} \bar{\phi} \partial^{\mu} \varphi + \frac{1}{16} \partial_{\mu} \varphi \partial^{\mu} \varphi \quad (6)$$
For curved base space, there is a factor of $\sqrt{-g}$ in the right-hand side of the first equation. Reduction of the third term in (1) is

$$H^2 = \tilde{H}_{\mu\nu\alpha} \tilde{H}^{\mu\nu\alpha} + 3e^{-\phi}W^2$$

where $W_{\mu\nu}$ is field strength of the $U(1)$ gauge field $b_{\mu}$, i.e., $W_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$. The three-form $\tilde{H}$ is defined as $\tilde{H}_{\mu\nu\alpha} = \tilde{H}_{\mu\nu\alpha} - g_{\mu}W_{\nu\alpha} - g_{\alpha}W_{\mu\nu} - g_{\nu}W_{\alpha\mu}$ where the three-form $\tilde{H}$ is field strength of the two-form $\tilde{b}_{\mu\nu} + \frac{1}{2}b_{\mu}g_{\nu} - \frac{1}{2}b_{\nu}g_{\mu}$ in (2). The three-form $\tilde{H}$ is not the field strength of a two-form. It satisfies the following Bianchi identity [16]:

$$\partial_{[\mu} \tilde{H}_{\nu\alpha]\beta] = -\frac{3}{2}V_{[\mu\nu}W_{\alpha]\beta]}$$

To find the T-duality transformation of the three-form $\tilde{H}$, one can rewrite it as

$$\tilde{H}_{\mu\nu\alpha} = \hat{H}_{\mu\nu\alpha} - \frac{1}{2}g_{\mu}W_{\nu\alpha} - \frac{1}{2}g_{\alpha}W_{\mu\nu} - \frac{1}{2}b_{\mu}V_{\nu\alpha} - \frac{1}{2}b_{\alpha}V_{\mu\nu} - \frac{1}{2}b_{\nu}V_{\alpha\mu}$$

where $\hat{H}$ is field strength of the T-duality invariant two-form $\hat{b}_{\mu\nu}$. It is evident that $\tilde{H}$ is invariant under the T-duality transformations (4). Using the above relation, one may rewrite $H^2$ in (7) in terms of $\hat{H}$ which satisfies the standard Bianchi identity $d\hat{H} = 0$, and some other terms that are not $U(1) \times U(1)$ gauge invariant. However, it is more convenient to write $H^2$ in terms of $\tilde{H}$ which satisfies the anomalous Bianchi identity (8), and some gauge invariant terms as in (7).

The reduction of (11) when the base space is flat then becomes

$$S_0 = -\frac{2}{\kappa^2} \int d^{D-1}xe^{-2\phi} \left[ (\frac{1}{2}c_1 + \frac{1}{16}c_2) \partial_\mu \phi \partial^\mu \phi + c_2 \tilde{\rho} \partial^\mu \tilde{\phi} \partial_\mu \tilde{\phi} + c_3 \tilde{H}^2 
- c_1 \partial^\mu \partial_\mu \phi + \frac{1}{2}c_2 \partial^\mu \tilde{\phi} \partial_\mu \tilde{\phi} - \frac{1}{4}c_1 e^{-\phi}V^2 + 3c_3 e^{-\phi}W^2 \right]$$

For curved base space, there is the factor $\sqrt{-g}$, the scalar curvature term $c_1 \tilde{R}$ and the partial derivatives become covariant derivatives. The terms in the first line are invariant under the Buscher rules.

The T-duality constraint is that the reduced action (10) must be invariant under T-duality up to some boundary terms, i.e.,

$$\delta S_0 \equiv S_0 - S'_0$$

$$= -\frac{2}{\kappa^2} \int d^{D-1}xe^{-2\phi} \left[ -2c_1 \partial^\mu \partial_\mu \phi + c_2 \partial^\mu \tilde{\phi} \partial_\mu \tilde{\phi} + (\frac{1}{4}c_1 + 3c_3)(e^{-\phi}W^2 - e^{\phi}V^2) \right]$$

must be a boundary term. Note that $\delta S_0$ is odd under the T-duality transformations and is invariant under the $U(1) \times U(1)$ gauge transformations. One can easily observe that $\delta S_0$ is a boundary term when

$$c_3 = -\frac{1}{12}c_1 \quad ; \quad c_2 = 4c_1$$

(12)
This fixes the $D$-dimensional effective action to be

$$S_0 = -\frac{2c_1}{\kappa^2} \int dx e^{-2\Phi} \sqrt{-G} \left( R + 4 \nabla_a \Phi \nabla^a \Phi - \frac{1}{12} H^2 \right).$$

which is the standard effective action at order $\alpha'^0$, up to an overall factor. The overall factor must be $c_1 = 1$ to be the effective action of string theory. In the next section, we extend these calculations to the order $\alpha'$.

3 Effective action at order $\alpha'$

The most general $D$-dimensional effective action at order $\alpha'$ which is invariant under the coordinate transformation and under the $B$-field gauge transformation has three classes. One class contains terms that are zero by Bianchi identities, one class contains terms that are total derivative terms, and all other terms belong to the third class. There are 20 such terms in which the field variables are arbitrary [20]. Using the field redefinition freedom, however, one may write the effective action in specific field variables. In this case there are 8 independent couplings [20]. The T-duality constraint may be used for both specific field variables and for arbitrary field variables. In the next subsection we use the T-duality constraint for specific field variables.

3.1 Effective action in a specific scheme

Using the field redefinition freedom, one can write the 20-parameter effective action at order $\alpha'$ in terms of independent couplings. There are also choices for these minimal couplings. One may choose the couplings to be [20]

$$S_1 = -\frac{2}{\kappa^2} \alpha' \int d^Dx e^{-2\Phi} \sqrt{-G} \left[ b_1 R_{abcd} R^{abcd} + b_2 R_{abcd} H^{abe} H^{cd} e + b_3 H_{fgb} H^f_a H^{gb}_c H^{ha} + b_4 H_{fgb} H^{fch} H^{gbh} + b_5 H_{abcd} H^{cd} \partial^a \Phi \partial^b \Phi + b_6 (H^2)^2 + b_7 H^2 \partial_a \Phi \partial^a \Phi + b_8 (\partial_a \Phi \partial^a \Phi)^2 \right]$$

where $b_1, b_2, \ldots, b_8$ are eight parameters. The field redefinitions freedom allows us to choose the eight arbitrary couplings in many different schemes. The above is one particular scheme. The above parameters have been found in [20] by the S-matrix method. We are going to show that the proposed T-duality constraint can fix these parameters up to an overall factor.

To impose the T-duality constraint on the couplings in (14), one should reduce it on the background with the $U(1)$ isometry as in the previous section. The reduction of the terms in the last line can easily be read from the reductions of the corresponding terms in (4) and (7). When the base space is flat, the reduction of the first, the second, the third, the fourth and the fifth terms in (14) are

$$\text{first} = \partial_\mu \partial_\nu \varphi (\partial^\mu \partial^\nu \varphi + \partial^\mu \varphi \partial^\nu \varphi) + \frac{1}{4} (\partial_\mu \varphi \partial^\mu \varphi)^2 + e^{2\varphi} \left( \frac{5}{8} V_{\mu} V^{\mu} V^{\mu} V_{\nu} V_{\nu} + \frac{3}{8} (V^2)^2 \right)$$
As expected, all terms on the right-hand side are invariant under $U(1) \times U(1)$ gauge transformations. Under parity $\tilde{H}$ and $W$ are odd and all other fields are even. All above terms are even under the parity because the original terms in (14) are even. Note that each $V$ has a factor of $e^{\varphi/2}$ and each $W$ has a factor of $e^{-\varphi/2}$.

The T-duality constraint is that the the reduction of the effective action (14) must be invariant under T-duality transformation up to some boundary terms. If the T-duality transformations at order $\alpha'$ are only the Buscher rules (4), then one finds

$$
\delta S_1 \equiv S_1 - S'_1
= -\frac{2}{\kappa^2} \int d^{D-1}x e^{-2\tilde{\phi}} \left[ b_1 \partial_{\mu} \phi \partial_{\nu} \phi \partial^\mu \phi \partial^\nu \phi + b_8 \partial_{\mu} \phi \partial_{\nu} \phi \partial^\mu \phi \partial^\nu \phi + \frac{1}{16} b_8 \partial_{\mu} \phi \partial_{\nu} \phi \partial^\mu \phi \partial^\nu \phi
+ \frac{1}{2} b_7 \partial_{\mu} \phi \partial_{\nu} \phi \tilde{H}^2 + \frac{1}{2} b_5 \partial_{\mu} \phi \partial_{\nu} \phi \tilde{H}^{\mu\nu} \tilde{H}^{\alpha\beta} - 6 \beta_6 \alpha e^{\varphi} \tilde{H}_{\alpha\beta} V^\mu V^\nu + 3 b_1 e^{\varphi} \partial_{\mu} \phi \partial_{\nu} \phi \partial^\mu \phi \partial^\nu \phi + \frac{3}{8} b_7 \partial_{\mu} \phi \partial_{\nu} \phi \partial^\mu \phi \partial^\nu \phi + \frac{3}{16} (8 b_1 - b_7) \partial_{\mu} \phi \partial^\mu \phi \partial^\nu \phi \right] V^2 + 2 b_2 \partial_{\mu} \phi \tilde{H}_{\alpha\beta} V^\alpha V^\beta W^\mu
+ e^{1/2} \left( \frac{5}{8} b_1 - 3 b_3 - 4 b_4 \right) \tilde{H}^\mu V^\nu V^\alpha V^\beta + \frac{3}{8} b_1 - 9 b_6 - b_4 \right) (V^2)^2
- e^{1/2} \left( \frac{b_2}{2} + 6 b_3 \right) \tilde{H}^\mu V^\nu + (\frac{b_2}{2} + 2 b_4) \tilde{H}^\mu V^\nu \tilde{H}^{\alpha\beta} W^\nu W^\nu V^\alpha - \left[ \nu \rightarrow -\nu, \nu \mu \rightarrow W \nu \right] \right) (16)
$$

Note that $\delta S_1$ is odd under the Buscher rules. One observes that constraining the integrand to be a total derivative term, would constrain all coefficients in (14) to be zero. To clarify this point, we note that the total derivative terms in $(D-1)$-spacetime must have the following structure

$$
J = \int d^{D-1}x \partial_{\mu} \left( e^{-2\tilde{\phi}} J^\mu \right) \quad (17)
$$
where the vector $J^\mu$ should be a parity invariant, it should be odd under the Buscher rules, and it should be invariant under the $U(1) \times U(1)$ gauge transformations. This vector which is at three-derivative order, is made of $\partial \phi, \partial \bar{\phi}, H, e^{\phi/2}V, e^{-\phi/2}W$ and their derivatives. Hence, the four-derivative terms in (16) which contain only $H, e^{\phi/2}V, e^{-\phi/2}W$, have no contribution from the total derivative terms. The coefficients of these terms must be zero. Moreover, since there is no term with derivative of $\bar{H}$ in (16), the total derivative terms can not produce the terms with $\bar{H}$, hence, the coefficients of the term in (16) which have $\bar{H}$ must be zero. These two constraints force the coefficients $b_1, \ldots, b_7$ to be zero. Removing these coefficients from (16), there remains two terms with coefficient $b_8$ which contains only first derivatives $\partial \phi$ and $\partial \bar{\phi}$. They can not be written as total derivative term because total derivative terms must include at least one term with second derivative. Hence, $b_8$ is also zero.

Therefore, to have non-zero effective action, one has to assume the T-duality transformations (18) receive higher-derivative corrections [14, 15, 16]. At order $\alpha'$, the T-duality transformations should be

$$
\phi' = -\phi + \alpha' \Delta \phi , \quad g'_\mu = g_\mu + \alpha' \frac{e^{\phi/2} \Delta g_\mu}{2} , \quad b'_\mu = b_\mu + \alpha' \frac{e^{-\phi/2} \Delta b_\mu}{2} ,
$$

$$
\bar{g}'_{\alpha \beta} = \bar{g}_{\alpha \beta} + \alpha' \Delta \bar{g}_{\alpha \beta} , \quad \bar{H}'_{\alpha \beta \gamma} = \bar{H}_{\alpha \beta \gamma} + \alpha' \Delta \bar{H}_{\alpha \beta \gamma} , \quad \bar{\phi}' = \bar{\phi} + \alpha' \Delta \bar{\phi}
$$

(18)

where $\Delta \phi, \ldots, \Delta \bar{\phi}$ contains some contractions of $\partial \phi, \partial \bar{\phi}, e^{\phi/2}V, e^{-\phi/2}W, H$ and their derivatives at order $\alpha'$. We have multiplied the factors of $e^{\phi/2}$ and $e^{-\phi/2}$ to $\Delta g_\mu$, and $\Delta b_\mu$, respectively. As we will see, this makes it explicit to have a factor of $e^{\phi/2}$ in front of each $V$ and a factor of $e^{-\phi/2}$ in front of each $W$ in the T-duality transformation of (10).

Since $\bar{H}$ is not field strength of a two-form, it is convenient to consider the T-duality transformation of the three form $\bar{H}$. The deformation $\Delta \bar{H}_{\mu \nu \alpha}$ however, is not independent of the the deformations $\Delta g_\mu$ and $\Delta b_\mu$ [16]. The T-dual field $\bar{H}'$ must satisfy the same Bianchi identity as $\bar{H}$, i.e.,

$$
\partial_{[\mu} \bar{H}'_{\nu \alpha \beta]} = -\frac{3}{2} V'_{[\mu \nu} W'_{\alpha \beta]}
$$

(19)

Using the T-duality transformations (18), one finds at order $\alpha'$ the corrections $\Delta \bar{H}, \Delta g, \Delta b$ satisfy the following differential equation:

$$
\partial_{[\mu} \Delta \bar{H}_{\nu \alpha \beta]} = -3 \partial_{[\mu} (V_{\nu \alpha} e^{\phi/2} \Delta g_{\beta]} - 3 \partial_{[\mu} (W_{\nu \alpha} e^{-\phi/2} \Delta b_{\beta]})
$$

(20)

where we have used the fact that the exterior derivative of $V$ and $W$ are zero. This leads to the following relation between $\Delta \bar{H}$ and $\Delta g, \Delta b$:

$$
\Delta \bar{H}_{\mu \nu \alpha} = \alpha_{19} \partial_{[\mu} (W_{\nu \alpha} V_{\alpha \beta]} - 3 e^{\phi/2} V_{[\mu \nu} \Delta g_{\alpha]} - 3 e^{-\phi/2} W_{[\mu \nu} \Delta b_{\alpha]} - 3 e^{\phi/2} V_{[\mu \nu} \Delta g_{\alpha]} - 3 e^{-\phi/2} W_{[\mu \nu} \Delta b_{\alpha]}
$$

(21)

where $\alpha_{19}$ is an arbitrary parameter.

The T-duality transformations (18) should form a $\mathbb{Z}_2$ group [17]. This indicates that the corrections $\Delta \phi, \Delta \bar{\phi}, \Delta \bar{g}, \Delta g, \Delta b, \Delta \bar{H}$ must satisfy the following constraints:

$$
\Delta \phi - \Delta \phi_{\phi \rightarrow -\phi, V \rightarrow W, W \rightarrow V} = 0
$$
\[ \Delta \bar{\phi} + \Delta \bar{\phi}|_{\varphi \to \varphi, V \to W, W \to V} = 0 \]
\[ \Delta g + \Delta g|_{\varphi \to \varphi, V \to W, W \to V} = 0 \]
\[ \Delta b + \Delta g|_{\varphi \to -\varphi, V \to W, W \to V} = 0 \]
\[ \Delta H + \Delta \bar{H}|_{\varphi \to \varphi, V \to W, W \to V} = 0 \]  \hspace{1cm} (22)

Now, we consider the T-duality constraint on the effective actions (11) and (14) using the T-duality transformations (18). The T-duality transformation of action (1) is now

\[ \delta S_0 \equiv S_0 - S'_0 \]
\[ = -\frac{2\alpha'}{k^2} \int d^{D-1}x e^{-2\phi} \left[ -\left( -2 \partial^\mu \partial^\nu \phi + \frac{1}{4} \partial^\mu \varphi \partial^\nu \varphi + \frac{1}{4} \bar{H}^\alpha_{\mu\beta} \bar{H}^\nu_{\alpha\beta} + \frac{1}{2} \epsilon^\varphi V^\mu V^\nu \right) \right. \]
\[ + \frac{1}{2} e^{-\varphi} W^{\mu\alpha} W^\nu_{\alpha} \right] \Delta \bar{g}_{\mu\nu} - \left( \frac{1}{2} \partial_\alpha \partial^\alpha \phi - \partial_\alpha \varphi \partial^\alpha \varphi - \frac{1}{4} \bar{H}^2 - \frac{1}{2} \epsilon^\varphi V^2 \right) \Delta \varphi \]
\[ - e^{-\varphi/2} \left( - \partial_\nu W^{\mu\nu} + 2 \partial_\nu \phi W^{\mu\nu} + \partial_\nu \varphi W^{\mu\nu} \right) \Delta b_{\mu} + \frac{1}{6} \bar{H}^\mu_{\nu\alpha} \Delta \bar{H}^\mu_{\nu\alpha} \]

where we have used the relations (22), removed some total derivative terms, and used the leading order T-duality constraint (12). We have also absorbed the overall coefficient \( c_1 \) in the arbitrary parameters in \( \Delta \varphi, \Delta \bar{\phi}, \Delta \bar{g}, \Delta g, \Delta b, \Delta \bar{H} \). In finding the above result for \( \Delta \bar{g}_{\mu\nu} \) we assumed the metric of the base space is \( g_{\mu\nu} \), and then use the T-duality transformations (18). At the end we set \( \bar{g}_{\mu\nu} = \eta_{\mu\nu} \). Note that the extra factors of \( e^{\varphi/2} \) and \( e^{-\varphi/2} \) in (18) make it possible to have a factor of \( e^{\varphi/2} \) in front of each \( V \) and a factor of \( e^{-\varphi/2} \) in front of each \( W \). Note that \( \delta S_0 \) is odd under the Buscher rules.

Since the terms in (16) are all invariant under the parity, the most general form of the corrections \( \Delta \varphi, \Delta \bar{\phi}, \Delta \bar{g}, \Delta g, \Delta b \) satisfying the constraints (22) and making the terms in (23) to be even under the parity, are:

\[ \Delta \varphi = \alpha_1 \partial_\mu \partial^\mu \phi + \alpha_2 \partial_\mu \phi \partial^\mu \bar{\phi} + \alpha_3 \partial_\mu \bar{\phi} \partial^\mu \phi + \alpha_4 \bar{H}^2 + \alpha_5 (e^\varphi V^2 + e^{-\varphi} W^2) \]
\[ \Delta \bar{\phi} = \alpha_6 \partial_\mu \phi \partial^\mu \bar{\phi} + \alpha_7 \partial_\mu \bar{\phi} \partial^\mu \phi + \alpha_8 (e^\varphi V^2 + e^{-\varphi} W^2) \]
\[ \Delta g_{\mu\nu} = \alpha_9 \partial_\mu \partial_\nu \varphi + \alpha_{10} (\partial_\mu \varphi \partial_\nu \bar{\phi} + \partial_\nu \bar{\phi} \partial_\mu \varphi) + \alpha_{11} (e^\varphi V^\alpha V_\nu - e^{-\varphi} W^\alpha W_\nu) \]
\[ + \eta_{\mu\nu} [\alpha_{12} \partial_\alpha \partial^\alpha \phi + \alpha_{13} \partial_\alpha \partial^\alpha \bar{\phi} + \alpha_{14} (e^\varphi V^2 - e^{-\varphi} W^2)] \]
\[ \Delta g_{\mu\nu} = \alpha_{15} e^{-\varphi/2} \partial_\nu W^\mu_{\nu} + \alpha_{16} e^{\varphi/2} \bar{H}^\mu_{\nu\alpha} V^\nu_{\alpha} + \alpha_{17} e^{-\varphi/2} \partial^\mu \bar{\phi} W^\nu_{\mu \nu} + \alpha_{18} e^{-\varphi/2} \partial^\nu \varphi W^\mu_{\mu \nu} \]
\[ \Delta b_{\mu} = -\alpha_{15} e^{\varphi/2} \partial^\nu V^\mu_{\nu} - \alpha_{16} e^{-\varphi/2} \bar{H}^\mu_{\nu\alpha} W^\nu_{\alpha} - \alpha_{17} e^{\varphi/2} \partial^\nu \bar{\phi} V^\mu_{\mu \nu} + \alpha_{18} e^{\varphi/2} \partial^\nu \varphi V^\mu_{\mu \nu} \]  \hspace{1cm} (24)

where \( \alpha_1, \ldots, \alpha_{18} \) are arbitrary parameters. If \( \delta S_1 \) were odd under the parity, then the corrections \( \Delta \varphi, \Delta \bar{\phi}, \Delta \bar{g}, \Delta g, \Delta b \) would contain terms that have opposite parity.

When one replaces (24) into (23), one would find that for some specific relations between the parameters, \( \delta S_0 \) becomes zero. That indicates that not all the parameters in (24) produce
non-zero \( \delta S_0 \). We are not going to write (24) in terms of the parameters that produce non-zero \( \delta S_0 \) and then impose the T-duality constraint. Instead, we first impose the T-duality constraint on all parameters and then remove the terms that produce zero \( \delta S_0 \).

The T-duality transformation of action (14) under (18) produce the same terms as in (16) plus some terms at higher order of \( \alpha' \) in which we are not interested. Hence, the T-duality constraint at order \( \alpha' \) requires \( \delta S_0 + \delta S_1 \) where \( \delta S_0 \) is given in (23) and \( \delta S_1 \) is given in (16), to be a boundary term. This constraint produces some algebraic equations that their solution fixes the coefficients of both the effective action (14) and the corrections to the Buscher rules. We have found that for the following parameters:

\[
\begin{align*}
 b_2 &= -\frac{b_1}{2}, \\
b_3 &= \frac{b_1}{24}, \\
b_4 &= -\frac{b_1}{8}, \\
b_5 &= b_6 = b_7 = b_8 = 0 \\
\alpha_2 &= 8\alpha_{14} - \alpha_1, \\
\alpha_3 &= 2b_1 + \frac{-\alpha_{14}}{2} - \frac{\alpha_1}{16}, \\
\alpha_4 &= -\frac{5\alpha_{14}}{6} - \frac{\alpha_1}{48}, \\
\alpha_5 &= 2b_1 - \frac{3\alpha_{14}}{2} - \frac{\alpha_1}{16}, \\
\alpha_6 &= -12\alpha_{14} - \frac{\alpha_1}{16}, \\
\alpha_7 &= 24\alpha_{14} + \frac{\alpha_1}{8}, \\
\alpha_8 &= \frac{b_1}{2} + 6\alpha_{14} + \frac{\alpha_1}{32}, \\
\alpha_9 &= 0, \\
\alpha_{10} &= 0, \\
\alpha_{11} &= 2b_1, \\
\alpha_{12} &= -2\alpha_{14}, \\
\alpha_{13} &= 4\alpha_{14}, \\
\alpha_{15} &= 2b_1, \\
\alpha_{16} &= b_1, \\
\alpha_{17} &= -4b_1, \\
\alpha_{18} &= 0, \\
\alpha_{19} &= -12b_1
\end{align*}
\]

The T-duality transformation \( \delta S_0 + \delta S_1 \) is a total derivative (17) with the following vector:

\[
J^\mu = -b_1 \partial^\mu \varphi \partial_\nu \varphi \partial^\nu \varphi + b_1 e^\varphi \left( 2\partial^\alpha V_\alpha V^\nu V_\mu - 2\partial^\alpha V^\mu V_\nu V_\alpha - 4\partial^\alpha \tilde{\varphi} V^\mu V_\nu V_\alpha - \partial^\mu \varphi V^2 \right)
+ b_1 e^{-\varphi} \left( -2\partial^\alpha W_{\alpha \beta} W^\mu W^\beta + 2\partial^\alpha W^\mu W^\beta W_{\alpha \beta} + 4\partial^\alpha \tilde{\varphi} W^\mu W^\beta W_{\alpha \beta} - \partial^\mu \varphi W^2 \right)
\]

which is odd under the Buscher rules and is even under the parity, as expected because \( \delta S_0 + \delta S_1 \) is also odd under the Buscher rules and is even under the parity.

The most important part of the results (25) is that they fix uniquely all eight parameters in the \( D \)-dimensional action (14) in terms of \( b_1 \), i.e.,

\[
S_1 = \frac{-2b_1}{\kappa^2} \alpha' \int d^Dx e^{-2\varphi} \sqrt{-G} \left( R_{abcd} R^{abcd} - \frac{1}{2} R_{abcd} H^{abe} H^{cd e} + \frac{1}{24} H_{f h g} H^f_a H^h_b H^g_c - \frac{1}{8} H_f^{a b} H_{a b c} H^{f c g} H^{b c g} \right)
\]

Up to the overall factor \( b_1 \), the above couplings are the standard effective action of the bosonic string theory which has been found in [20] by the S-matrix calculations. This action now is invariant under T-duality.

When replacing the relations (25) into (13), one finds the following corrections to the Buscher rules:

\[
\begin{align*}
\Delta \tilde{g}_{\mu \nu} &= 2b_1 \left( e^\varphi V_\mu^\alpha V_{\nu \alpha} - e^{-\varphi} W_\mu^\alpha W_{\nu \alpha} \right), \\
\Delta \tilde{\varphi} &= \frac{b_1}{2} \left( e^\varphi V^2 - e^{-\varphi} W^2 \right), \\
\Delta \varphi &= 2b_1 \left( \partial_\mu \varphi \partial^\mu \varphi + e^\varphi V^2 + e^{-\varphi} W^2 \right)
\end{align*}
\]
\[ \Delta g_{\mu} = b_1 \left( 2e^{-\varphi/2}\partial^\nu W_{\mu\nu} + e^{\varphi/2}H_{\mu\nu\alpha}V^{\nu\alpha} - 4e^{-\varphi/2}\partial^\nu \phi W_{\mu\nu} \right) \]
\[ \Delta b_{\mu} = -b_1 \left( 2e^{\varphi/2}\partial^\nu V_{\mu\nu} + e^{-\varphi/2}H_{\mu\nu\alpha}W^{\nu\alpha} - 4e^{\varphi/2}\partial^\nu \phi V_{\mu\nu} \right) \]
\[ \Delta H_{\mu\nu\alpha} = 12b_1 \partial_{[\mu}(W_{\nu\alpha]} - 3e^{\varphi/2}V_{[\mu\nu}\Delta g_{\alpha]} - 3e^{-\varphi/2}W_{[\mu\nu}\Delta b_{\alpha]} \] (28)

The correction \( \Delta \tilde{g}_{\mu\nu}, \Delta \tilde{\phi}, \Delta \varphi \) have also some terms that depend on \( \alpha_1, \alpha_{14} \). They are
\[ \Delta \tilde{g}_{\mu\nu} = \alpha_{14} \left( -2\partial_\mu \partial^\nu \varphi + 4\partial_\mu \phi \partial^\nu \varphi + e^\varphi V^2 - e^{-\varphi}W^2 \right) \eta_{\mu\nu} \] (29)
\[ \Delta \tilde{\phi} = (6\alpha_{14} + \frac{1}{32}\alpha_1) \left( -2\partial_\mu \partial^\mu \varphi + 4\partial_\mu \phi \partial^\mu \varphi + e^\varphi V^2 - e^{-\varphi}W^2 \right) \]
\[ \Delta \varphi = \alpha_{14} \left( 8\partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} - \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{5}{6} \tilde{H}^2 - \frac{1}{2} e^\varphi V^2 - \frac{3}{2} e^{-\varphi}W^2 \right) \]
\[ + \alpha_1 \left( \partial_\mu \varphi \partial^\mu \phi - \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} - \frac{1}{16} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{48} \tilde{H}^2 - \frac{1}{16} e^\varphi V^2 - \frac{1}{16} e^{-\varphi}W^2 \right) \]

However, replacing \( \Delta \tilde{g}_{\mu\nu}, \Delta \tilde{\phi}, \Delta \varphi \) into (28), one would find \( \delta S_0 \) becomes zero. That is the reflection of the fact that the parameters \( \alpha_1, \ldots, \alpha_{14} \) in (23) are not all produce non-zero \( \delta S_0 \). To consider the parameters that produce non-zero \( \delta S_0 \), one has to set these two parameters to zero. Hence,
\[ \Delta \tilde{g}_{\mu\nu} = \Delta \tilde{\phi} = \Delta \varphi = 0 \] (30)

This ends our illustration that the T-duality constraint on the effective action (14) can fix both the effective action and the corresponding corrections to the Buscher rules up to the overall factor of \( b_1 \).

Similar T-duality constraint has been used in [21] by reducing the effective action (14) to one dimension. In that approach, however, not all parameters in (14) are fixed up to an overall factor because some of the terms in (14) become zero when reducing them to one dimension [21].

### 3.2 Effective action in arbitrary scheme

The corrections to the Buscher rules depend on the scheme that one uses for the effective action. The corrections (28) correspond to the effective action (27). If we had started with the effective action (14) in a different scheme, then the T-duality constraint would fix the eight arbitrary parameters in the action and the corresponding corrections to the Buscher rules up to an overall factor.

The field redefinitions have been used to write the effective action (14) in terms of only eight parameters. If one does not use the field redefinition to reduce the independent couplings, then the effective action would have the following 20 terms [20]:

\[ S_1 = \frac{-2\alpha'}{\kappa^2} \int d^D x e^{-2\Phi} \sqrt{-G} \left[ a_1 R_{abcd} R^{abcd} + a_2 (H^2)^2 + a_3 H_{fgh} H^f_a H^g_b H^h_c + a_4 R_{abcd} H^{abcd} + a_5 R_{ab} R^{ab} + a_6 RH^2 + a_7 R^2 + a_8 R_{abcd} H^{abc} + a_9 H_{ab} \right] + a_{10} R_{ab} \partial^a \Phi \partial^b \Phi + a_{11} R_{ab} \partial^a \Phi \partial^b \Phi + \cdots \]
Apart from the unambiguous coefficients \(a_1, a_3, a_8\) which are not changed under field redefinitions, all other coefficients are ambiguous because they are changed under field redefinitions. There are 5 parameters in the ambiguous parameters which are essential and all other are arbitrary parameters. If one does not use the field redefinitions, one would not be able to distinguish between the essential and the arbitrary parameters. This distinction, however, can be found by imposing the T-duality constraint on (31). We find that the T-duality constraint can fix the 3 unambiguous parameters in terms of one of them, and the 17 ambiguous parameters in terms of 11 arbitrary parameters.

Using the same steps as in the previous subsection, one finds that the T-duality constraint on the above action produces the following relations:

\begin{align}
 a_{17} &= -\frac{a_{16}}{4}, a_3 = \frac{a_1}{24}, a_4 = -\frac{a_{10}}{16} + \frac{3a_{12}}{4} + \frac{3a_{13}}{32} + \frac{5a_{14}}{64} + \frac{9a_{15}}{10} + \frac{36a_2}{5} - \frac{24a_{20}}{5} - \frac{3a_1}{5} - \frac{a_{19}}{10}, \\
 a_5 &= \frac{a_{10}}{4} - 3a_{12} + \frac{2a_{19}}{5} + \frac{16a_{20}}{5} + \frac{18a_{15}}{5} - \frac{14a_{12}}{5} + 2a_1 - \frac{3a_{13}}{5} - \frac{5a_{14}}{16}, \\
 a_6 &= -\frac{a_{10}}{4} + \frac{a_{12}}{4} + \frac{a_{13}}{12} - \frac{96}{2} - \frac{8}{2} - \frac{a_{18}}{48} - \frac{5a_{11}}{4}, a_7 = \frac{a_{11}}{4} + \frac{a_{18}}{2} - \frac{a_{13}}{8} - \frac{a_{14}}{16}, \\
 a_8 &= -\frac{a_1}{2}, a_9 = -3a_{12} - a_{16} + \frac{2a_{19}}{5} + \frac{16a_{20}}{5} + \frac{18a_{15}}{5} - \frac{14a_{12}}{5} + 2a_1 - \frac{3a_{13}}{5} - \frac{5a_{14}}{16}, \\
 a_{15} &= \frac{a_{10}}{8} + \frac{a_{16}}{5} + \frac{a_{19}}{5} + \frac{8a_{20}}{2} - \frac{3a_{12}}{5} + \frac{9a_{15}}{5} - \frac{72a_2}{5} + \frac{11a_1}{5} - \frac{3a_{13}}{5} - \frac{5a_{14}}{16}, \\
 a_{16} &= -\frac{a_{16}}{4} - 4a_{20} + \frac{a_1}{2}, a_{17} = -a_{16} + 2a_{19} + 16a_{20} - 2a_1, \\
 a_{18} &= -\frac{3a_{10}}{16} + \frac{9a_{12}}{4} + \frac{9a_{13}}{32} + \frac{15a_{14}}{64} + \frac{27a_{15}}{10} + \frac{10a_{12}}{5} + \frac{a_{16}}{2} - \frac{4a_{19}}{5} - \frac{32a_{20}}{5} - \frac{4a_1}{5}, \\
 a_{11} &= \frac{a_{10}}{8} + \frac{a_{19}}{5} - \frac{3a_{12}}{2} + \frac{9a_{15}}{5} - \frac{72a_2}{5} + \frac{32a_{20}}{6a_1} - \frac{3a_{13}}{5} - \frac{5a_{14}}{16}, \\
 a_9 &= \frac{a_{10}}{4} - 3a_{12} + \frac{12a_{19}}{5} + \frac{96a_{20}}{5} + \frac{18a_{15}}{5} - \frac{14a_{12}}{5} + \frac{12a_1}{5} - \frac{3a_{13}}{5} - \frac{5a_{14}}{16}, \\
 a_{12} &= \frac{5a_{10}}{16} + \frac{5a_{11}}{5} + \frac{5a_{18}}{4} + \frac{2a_{19}}{5} + \frac{16a_{20}}{5} + \frac{15a_{12}}{5} + \frac{32a_2}{5} + \frac{2a_1}{5} + \frac{51a_{15}}{32} - \frac{35a_{13}}{5} - \frac{45a_{14}}{64}, \\
 a_{10} &= -6a_{12} + \frac{4a_{19}}{5} + \frac{32a_{20}}{5} - \frac{3a_{13}}{5} - \frac{36a_{15}}{5} - \frac{288a_2}{5} + \frac{4a_1}{5} - \frac{5a_{14}}{8}, \\
 a_{13} &= \frac{a_{10}}{8} + \frac{15a_{12}}{2} + \frac{11a_{13}}{16} + \frac{13a_{14}}{32} + \frac{51a_{15}}{5} + \frac{64a_{20}}{5} + \frac{a_{11}}{2} + \frac{a_{18}}{2} - \frac{4a_{19}}{5} - \frac{32a_{20}}{5} - \frac{4a_1}{5}, \\
 a_6 &= \frac{15a_{10}}{8} + \frac{15a_{11}}{2} + \frac{31a_{18}}{4} + \frac{12a_{19}}{5} + \frac{96a_{20}}{5} + \frac{45a_{12}}{2} - \frac{153a_{15}}{5} - \frac{1944a_2}{5} + \frac{12a_1}{5} + \frac{105a_{13}}{16} - \frac{135a_{14}}{32}, a_7 = \frac{3a_{10}}{4} - 3a_{11} + \frac{81a_{12}}{2} + \frac{55a_{13}}{16} + \frac{63a_{14}}{32} + \frac{279a_{15}}{5}.
\end{align}
\[-3a_{18} + \frac{3672a_2}{5} - \frac{16a_{19}}{5} - \frac{128a_{20}}{5} - \frac{16a_1}{5},\]  
\[\alpha_2 = \frac{a_{10}}{4} - a_{11} - 9a_{12} + \frac{3a_{13}}{8} + \frac{5a_{14}}{16} - a_{18} + \frac{8a_{19}}{5} + \frac{144a_2}{5} + \frac{64a_{20}}{5} - \frac{42a_{15}}{5} + \frac{8a_3}{5},\]  
\[\alpha_3 = -\frac{15a_{10}}{64} + \frac{33a_{12}}{16} + \frac{69a_{13}}{128} + \frac{99a_{14}}{99} + \frac{93a_{15}}{40} + \frac{63a_2}{5} - \frac{52a_{20}}{5} - \frac{13a_{19}}{10} - \frac{7a_1}{10} - \frac{7a_{11}}{16} - \frac{7a_{18}}{16},\]  
\[\alpha_4 = \frac{5a_{10}}{192} + \frac{5a_{11}}{48} + \frac{5a_{18}}{48} - \frac{123a_2}{128} - \frac{4a_{20}}{15} - \frac{5a_{12}}{16} + \frac{a_{19}}{30} - \frac{a_{15}}{30} - \frac{a_{19}}{40} + \frac{15a_{14}}{256} - \frac{35a_{13}}{384},\]  
\[\alpha_5 = \frac{5a_{10}}{64} + \frac{a_{11}}{16} + \frac{a_{18}}{16} - \frac{45a_2}{4a_{20}} - \frac{3a_{1}}{5} - \frac{15a_{15}}{16} - \frac{15a_{12}}{16} - \frac{19a_{13}}{128} - \frac{29a_{14}}{256},\]  
\[\alpha_8 = \frac{3a_{10}}{128} + \frac{3a_{15}}{80} + \frac{a_{18}}{32} + \frac{a_{19}}{10} - \frac{6a_{20}}{10} + \frac{27a_2}{16} - \frac{7a_{1}}{8} + \frac{9a_{12}}{16} - \frac{9a_{13}}{256} - \frac{15a_{14}}{512}, a_{19} = -12a_1,\]

where the unambiguous parameters \(a_1\) is not fixed, and the 11 ambiguous parameters \(a_{2}, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}, a_{18}, a_{19}, a_{20}\) remain arbitrary. Any specific value for these parameters, gives the effective action in one particular scheme. For the special case that the parameters are

\[a_{2} = \frac{a_{1}}{144}, a_{10} = -16a_1, a_{11} = 8a_1, a_{12} = \frac{2a_{1}}{3}, a_{13} = 16a_1, a_{14} = -16a_1,\]

\[a_{15} = -\frac{2a_{1}}{3}, a_{16} = 0, a_{18} = 0, a_{19} = 2a_1, a_{20} = -\frac{a_{1}}{8},\]

one finds the effective action\(^{[31]}\) becomes

\[\mathcal{S}_1 = 2\alpha' a_1 \int d^D x e^{-2\phi} \sqrt{-G} \left[ -R_{GB}^2 + 16(R^{ab} - \frac{1}{2}g^{ab} R) \partial_a \phi \partial_b \phi - 16\nabla^2 \phi (\partial \phi)^2 + 16(\partial \phi)^4 \right.\]

\[\left. + \frac{1}{2}(R^{abcd} H^{ab} H^{cd} - 2R^{ab} H^{2}_a + \frac{1}{2} 3R H^2)^2 - 2(\nabla \partial \phi H^{2}_a - \frac{1}{2} \nabla^2 \phi H^2)^2 - \frac{2}{3}(\partial \phi)^2 H^2 \right.\]

\[\left. - \frac{1}{144} H^{2}_{ab} H^{2}_{ab} - \frac{1}{144}(H^2)^2 \right],\]

where \(R_{GB}^2 = R_{abcd} R^{abcd} - 4R_{ab} R^{ab} + R^2 \) and \(H_{ab} = H_{a}^{cd} H_{cd}.\) This action has been found in\(^{[19]}\). Replacing \((33)\) into \((32)\), one finds the corresponding T-duality transformations to be

\[\Delta \bar{g}_{\mu \nu} = 0,\]

\[\Delta \bar{\phi} = 0,\]

\[\Delta \varphi = a_1 \left( \partial_{\mu} \varphi \partial^{\mu} \varphi + e^\varphi V^2 + e^{-\varphi} W^2 \right),\]

\[\Delta g_{\mu} = a_1 \left( 2e^{-\varphi/2} \partial_\nu \varphi W_{\mu \nu} + e^{\varphi/2} H_{\mu \nu \alpha} V^{\nu \alpha} \right),\]

\[\Delta b_{\mu} = a_1 \left( 2e^{\varphi/2} \partial_{\nu} \varphi V_{\mu \nu} - e^{-\varphi/2} H_{\mu \nu \alpha} W^{\nu \alpha} \right),\]

\[\Delta H_{\mu \nu \alpha} = 12a_1 \partial_{\mu} (W_{\nu \beta} V_{\lambda \nu}) - 3e^{\varphi/2} V_{[\mu \nu]} \Delta g_{\alpha} - 3e^{-\varphi/2} W_{[\mu \nu] \Delta b_{\alpha}}\]

These transformation are exactly those have been found in\(^{[16]}\).

We have seen that the T-duality constraint can not fix the overall factor \(b_1\) in \((27)\) or \(a_1\) in \((34)\). This is as expected because the bosonic, the heterotic and superstring theories all have
the same T-duality but they have different overall factor. In fact, $a_1 = b_1 = -1/16$ for bosonic theory, $a_1 = b_1 = -1/32$ for heterotic theory and $a_1 = b_1 = 0$ for superstring theory. If the T-duality constraint could fix the overall factor, then the effective action that the T-duality constraint generated would not be the correct effective action of all bosonic, heterotic and superstring theories at order $\alpha'$.

The heterotic theory has another term at four-derivative level, i.e.,

$$S_1 \supset -\frac{2c_1\alpha'}{\kappa^2} \int d^{10} x e^{-2\phi} \sqrt{-G} \left(-\frac{1}{6}H_{abc}\Omega^{abc}\right)$$

This term is resulted from the Green-Schwarz anomaly cancellation mechanism [22] which requires the non-standard gauge transformation of the $B$-field, i.e.,

$$B_{ab} \rightarrow B_{ab} + \partial_{[a} \lambda_{b]} + \alpha' \partial_{[a} \Lambda_{b]}^i \omega_{b]i}$$

where $\Lambda_{b]}^i$ is the matrix of the Lorentz transformations and $\omega_{b]i}$ is spin connection. Under this transformation the 3-form $H_{abc} + \alpha' \Omega^{abc}$ is invariant, i.e., $H_{abc} + \alpha' \Omega^{abc} \rightarrow H_{abc} + \alpha' \Omega^{abc}$. The Chern-Simons three-form $\Omega$ is

$$\Omega_{abc} = \omega_{[a i} \partial_{b} \omega_{c]i} + \frac{2}{3} \omega_{[a i} \omega_{b]k} \omega_{c]i} ; \quad \omega_{a i} = \partial_{a} e_{b}^{i} e_{c}^{b} - \Gamma_{ab}^{c} e_{c}^{i} e_{i}$$

where $e_{a}^{i} e_{b}^{j} \eta_{ij} = G_{ab}$. We have imposed the T-duality constraint on this action and found that it is invariant under the T-duality transformation (37) provided that $\Delta \bar{g}_{\mu\nu} = \Delta \bar{\phi} = 0$ and

$$\Delta \phi = \frac{c_1}{6} V_{\mu\nu} W^{\mu\nu}$$

$$\Delta g_{\mu} = -\frac{c_1}{12} \left(e^{\phi/2} \partial^\nu \phi V_{\mu\nu} - \frac{1}{2} e^{-\phi/2} \bar{H}_{\mu\nu\alpha} W^{\nu\alpha} + e^{\phi/2} \bar{\omega}_{\mu\alpha\nu} V^{\nu\alpha}\right)$$

$$\Delta b_{\mu} = -\frac{c_1}{12} \left(e^{-\phi/2} \partial^\nu \phi W_{\mu\nu} + \frac{1}{2} e^{\phi/2} \bar{H}_{\mu\nu\alpha} V^{\nu\alpha} - e^{-\phi/2} \bar{\omega}_{\mu\nu\alpha} W^{\nu\alpha}\right)$$

$$\Delta \bar{H}_{\mu\nu\alpha} = -3e^{\phi/2} V_{[\mu\nu} \Delta g_{\alpha]} - 3e^{-\phi/2} W_{[\mu\nu} \Delta b_{\alpha]}$$

where $\bar{\omega}_{\mu\nu\alpha}$ is 9-dimensional spin connection. In finding the above result, we did not assume that the base space is flat. As expected, the parity of above terms are different from the corresponding terms in (28) because the parity of their corresponding actions are different.

We have shown in this paper that the T-duality constraint when the $B$-field is not zero, can be used to find both the effective action and its corresponding T-duality transformations at order $\alpha'$. It would be interesting to extend this calculations to the orders $\alpha'^2, \alpha'^3$ that their effective actions are not known in the literature. When $B$-field is zero it has been shown in [17, 18] that the T-duality constraint reproduces the known couplings in the literature.

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References

[1] A. Giveon, M. Porrati and E. Rabinovici, Phys. Rept. 244, 77 (1994) doi:10.1016/0370-1573(94)90070-1 [hep-th/9401139].

[2] E. Alvarez, L. Alvarez-Gaume and Y. Lozano, Nucl. Phys. Proc. Suppl. 41, 1 (1995) doi:10.1016/0920-5632(95)00429-D [hep-th/9410237].

[3] W. Siegel, Phys. Rev. D 47, 5453 (1993) doi:10.1103/PhysRevD.47.5453 [hep-th/9302036].

[4] W. Siegel, Phys. Rev. D 48, 2826 (1993) doi:10.1103/PhysRevD.48.2826 [hep-th/9305073].

[5] W. Siegel, “Manifest duality in low-energy superstrings,” [hep-th/9308133].

[6] C. Hull and B. Zwiebach, JHEP 0909, 099 (2009) doi:10.1088/1126-6708/2009/09/099 [arXiv:0904.4664 [hep-th]].

[7] G. Aldazabal, D. Marques and C. Nunez, Class. Quant. Grav. 30, 163001 (2013) doi:10.1088/0264-9381/30/16/163001 [arXiv:1305.1907 [hep-th]].

[8] O. Hohm, C. Hull and B. Zwiebach, JHEP 1008, 008 (2010) doi:10.1007/JHEP08(2010)008 [arXiv:1006.4823 [hep-th]].

[9] O. Hohm and B. Zwiebach, JHEP 1411, 075 (2014) doi:10.1007/JHEP11(2014)075 [arXiv:1407.3803 [hep-th]].

[10] D. Marques and C. A. Nunez, JHEP 1510, 084 (2015) doi:10.1007/JHEP10(2015)084 [arXiv:1507.00652 [hep-th]].

[11] M. R. Garousi, Phys. Rept. 702, 1 (2017) doi:10.1016/j.physrep.2017.07.009 [arXiv:1702.00191 [hep-th]].

[12] T. H. Buscher, Phys. Lett. B 194, 59 (1987). doi:10.1016/0370-2693(87)90769-6

[13] T. H. Buscher, Phys. Lett. B 201, 466 (1988). doi:10.1016/0370-2693(88)90602-8

[14] A. A. Tseytlin, Mod. Phys. Lett. A 6, 1721 (1991). doi:10.1142/S021773239100186X

[15] E. Bergshoeff, B. Janssen and T. Ortin, Class. Quant. Grav. 13, 321 (1996) doi:10.1088/0264-9381/13/3/002 [hep-th/9506156].

[16] N. Kaloper and K. A. Meissner, Phys. Rev. D 56, 7940 (1997) doi:10.1103/PhysRevD.56.7940 [hep-th/9705193].

[17] H. Razaghian and M. R. Garousi, JHEP 1802, 056 (2018) doi:10.1007/JHEP02(2018)056 [arXiv:1709.01291 [hep-th]].
[18] H. Razaghian and M. R. Garousi, Phys. Rev. D 97, 106013 (2018) doi:10.1103/PhysRevD.97.106013 [arXiv:1801.06834 [hep-th]].

[19] K. A. Meissner, Phys. Lett. B 392, 298 (1997) doi:10.1016/S0370-2693(96)01556-0 [hep-th/9610131].

[20] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. B 293, 385 (1987). doi:10.1016/0550-3213(87)90077-0

[21] O. Hohm and B. Zwiebach, JHEP 1604, 101 (2016) doi:10.1007/JHEP04(2016)101 [arXiv:1510.00005 [hep-th]].

[22] M. B. Green and J. H. Schwarz, Phys. Lett. 149B, 117 (1984). doi:10.1016/0370-2693(84)91565-X