ON GENERAL EXTENSION FIELDS FOR THE CLASSICAL GROUPS IN DIFFERENTIAL GALOIS THEORY

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Abstract. Let $G$ be one of the classical groups of Lie rank $l$. We make a similar construction of a general extension field in differential Galois theory for $G$ as E. Noether did in classical Galois theory for finite groups. More precisely, we build a differential field $E$ of differential transcendence degree $l$ over the constants on which the group $G$ acts and show that it is a Picard-Vessiot extension of the field of invariants $E^G$. The field $E^G$ is differentially generated by $l$ differential polynomials which are differentially algebraically independent over the constants. They are the coefficients of the defining equation of the extension. Finally we show that our construction satisfies generic properties for a specific kind of $G$-primitive Picard-Vessiot extensions.

1. Introduction

In classical Galois theory there is a well-known construction of the general equation with Galois group the symmetric group $S_n$. As a starting point one takes $n$ indeterminates $T = (T_1, \ldots, T_n)$ and considers the rational function field $\mathbb{Q}(T)$. The group $S_n$ acts on $\mathbb{Q}(T)$ by permuting the indeterminates and one can show that $\mathbb{Q}(T)$ is a Galois extension of the field of invariants $\mathbb{Q}(T)^{S_n}$ for a polynomial equation of degree $n$ whose coefficients are the elementary symmetric polynomials. The field of invariants is generated by these polynomials and they are algebraically independent over $\mathbb{Q}$. A generalization of this idea leads to the so called Noether problem which would solve the inverse problem over $\mathbb{Q}$ for finite groups. Unfortunately, the answer to the Noether problem is not affirmative.

In differential Galois theory one can construct in a similar way a general equation with differential Galois group the full general linear group $GL_n(\mathbb{C})$. For an algebraically closed field $\mathbb{C}$ of characteristic zero and $n$ differential indeterminates $y = (y_1, \ldots, y_n)$ one begins with the differential field $\mathbb{C}^\langle y \rangle$, that is the field differentially generated by the indeterminates $y$ over $\mathbb{C}$. The action of the group $GL_n(\mathbb{C})$ on $\mathbb{C}^\langle y \rangle$ is induced by linear transformations from the right on the wronskian matrix in $y$. As in classical Galois theory one can show that $\mathbb{C}(y)$ is a Picard-Vessiot extension of the field of invariants $\mathbb{C}(y)^{GL_n(\mathbb{C})}$ with differential Galois group $GL_n(\mathbb{C})$. As in the case of the symmetric group this extension is defined by a linear differential equation whose coefficients are differentially algebraically independent over $\mathbb{C}$ and differentially generate the field of invariants. This idea was generalized in [3] by L. Goldman and he applied the techniques developed there to some connected subgroups $G \subset GL_n(\mathbb{C})$ obtaining explicit equations.

In the present work we perform a similar construction in differential Galois theory for the classical groups. For this purpose let $G$ be one of these groups and denote by $l$ its Lie rank. As in the above example we start our construction with a differential field $\mathbb{C}(\eta)$ which is differentially generated by $l$ differential indeterminates $\eta = (\eta_1, \ldots, \eta_l)$ over the constants $\mathbb{C}$. We use this purely differential transcendental extension to build our final general extension field $E \supset \mathbb{C}(\eta)$ by taking into account the structure of $G$-primitive Picard-Vessiot extensions. These are Picard-Vessiot extensions with differential Galois group $G$ whose fundamental solution matrices
satisfy the algebraic relations defining the group $G$. The structural information of these extensions is obtained by connecting results from the theory of reductive groups with differential Galois theory. More precisely, the Bruhat decomposition provides a normal form of elements of $G$ parameterized by a fixed Borel subgroup $B$ and the Weyl group. It turns out that the Bruhat decomposition of a fundamental matrix $\bar{w}$, stemming from the fact that $B\bar{w}B$ is a rational variety. Using the normal form $\bar{Y} = \bar{u}_1 \bar{u}_2$, where $\bar{u}_1$, $\bar{u}_2$ are elements of the maximal unipotent subgroup of $B$ and $t$ lies in the torus, we found out that the entries of $t$ and $u_2$ generate a Liouvillian extension over the fixed field under the Borel subgroup, which on his part is generated by the entries of $u_1$. In a specific case when the defining matrix has shape as constructed in [9], the fixed field is differentially generated by Lie rank many elements $x = (x_1, \ldots, x_l)$ and the Liouvillian extension can be parameterized by these elements, that is, it is generated by $l$ exponentials, whose logarithmic derivatives are $x$, and successive integrals of these exponentials. As a consequence the whole Picard-Vessiot extension is determined by the $l$ elements $x$. According to these observations we construct our general extension field $E$. We use the indeterminates $\eta$ to parameterize a general element of the Cartan subalgebra and this element together with the basis elements of root spaces belonging to the negative simple roots define the Liouvillian extension $E$ of $C(\eta)$ with fundamental matrix $tu_2$ which lies in the Borel group. We show that we can build an element $u_1$ in the maximal unipotent subgroup with entries in $C(\eta)$ such that the logarithmic derivative of $Y = u_1 \bar{u}_2$ has shape as the matrix equations constructed in [9]. Similar as in the case of $GL_n(C)$ the group $G$ acts on $E$ by right multiplication on $Y$, but now we take the Bruhat decomposition to obtain the effect on the generators of $E$. We prove that $E$ is a Picard-Vessiot extension of the field of invariants $E^G$ with differential Galois group $G$ and that $E^G$ is generated by $l$ invariants $h = (h_1(\eta), \ldots, h_l(\eta))$ which are differentially algebraically independent over the constants. The coefficients of the linear differential equation defining our extension are differential polynomials in these invariants. In opposition to the above approach we started in [9] with a differential field $C(t)$ generated by $l$ differential indeterminates $t = (t_1, \ldots, t_l)$ as our differential base field. Under the consideration of the geometrical structure of the group $G$, we defined a parameter differential equation $\partial(y) = A_G(t)y$ and showed that it defines a Picard-Vessiot extension of $C(t)$ with differential Galois group $G$. Our extension field $E$ is build in such away that its defining equation is actually $\partial(y) = A_G(h)y$, that is the indeterminates $t$ can be taken to be the differentially algebraically independent invariants $h$. In fact we have shown that for the parameterized equations of [9] we can perform a similar construction as E. Noether did in classical Galois theory.

In contrast to the above example of the general equation for $GL_n(C)$, where the extension field is generated by $n$ differential indeterminates over the constants, the differential transcendence degree of the extension field of our construction is equal to the Lie rank. In the first case the extension and equation are clearly generic. We can substitute any $n$ linearly independent solutions of any scalar linear differential equation of degree $n$ with differential Galois group a subgroup of $GL_n(C)$ into $y$ and the general equation specializes to it. At a first view our construction is only generic for $G$-primitive Picard-Vessiot extensions of a differential field $F$ stemming from differential equations of shape $\partial(y) = A_G(f)y$ with $f \in F^l$ in the sense that we can substitute the $l$ elements $x$ determining the extension into $\eta$ which has the effect that $A_G(h)$ specializes to $A_G(f)$. In order to be generic for a broader range of Picard-Vessiot extension we consider extensions whose defining matrix is gauge equivalent to our equation. In [9] we have seen that every matrix of an open subset
of the direct sum of the Lie algebra of $B^-$ and the root spaces corresponding to the
positive simple roots is gauge equivalent to a matrix of shape $A_G(f)$. For such a
defining matrix the logarithmic derivative of the Liouvillian part $u_2$ has the shape
of a principal nilpotent matrix in normal form (for a definition see Chapter 3). It
turns out that an arbitrary defining matrix is gauge equivalent to a matrix of shape
$A_G(f)$ if and only if the defining matrix of its Liouvillian part is gauge equivalent
by specific transformations to principal nilpotent matrix in normal form. We will
call such extensions $G$-primitive with normalisable unipotent part and our general
extension field will be generic for such extensions.

The paper is organized in the following way. In Chapter 2 we recapitulate some
basic facts about the structure of the classical groups and their Lie algebras and
introduce the corresponding notation. Chapter 3 shortly deals with the decom-
position of the gauge transformation into the adjoint action and the logarithmic
derivative. In Chapter 4 the connection between the group structure and Picard-
Vessiot extensions is established. The normal form decomposition of a fundamental
matrix of a $G$-primitive Picard-Vessiot extension and the resulting normal form
coefficients which generate the extension are given in Proposition 4.5. Using the
results we show in Proposition 4.7 that the normal form coefficients which belong to
the torus and the second unipotent group in the decomposition of the fundamental
matrix generate a Liouvillian extension and that these coefficients are exponentials
and successive integrals respectively. In Chapter 5 we construct our general exten-
sion field. We prove in Theorem 5.13 that it is a Picard-Vessiot extension of the
differential field generated by the coefficients of the logarithmic derivative of the
matrix $Y$ from above and that the differential Galois group of the extension is $G$.
Reinterpreting the result in Remark 5.14 we obtain that $E$ with the action of $G$
has the above outlined properties. Every step in the construction comes along with
an example for the group $SL_4$. At the end of the chapter we consider the case of
the group $G_2$. Chapter 6 deals with the generic properties of our general extension
field. For a full $G$-primitive Picard-Vessiot extension we give in Theorem 6.7 a
criteria for gauge equivalence of its defining matrix to a matrix of shape $A_G(f)$ in
terms of its Liouvillian part. Our final results are stated in Theorem 6.9.

2. The structure of the classical groups

Let $G$ be one of the classical groups of Lie rank $l$ over $C$. Let $\Phi$ be the root
system of $G$ of the respective type and let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ be a basis of $\Phi$. We
write $\Phi^+$ for the set of positive and $\Phi^-$ for the set of negative roots and we set
$m := |\Phi^+| = |\Phi^-|$. Every root $\alpha$ can be written uniquely as
$$\alpha = k_1\bar{\alpha}_1 + \cdots + k_l\bar{\alpha}_l$$
with integer coefficients $k_i$ where all $k_i$ are nonpositive or nonnegative. By the
integer $\text{ht}(\alpha) = k_1 + \cdots + k_l$ we mean the height of the root $\alpha$.

We denote by $W$ the Weyl group of $\Phi$, that is the finite group which consists of the
orthogonal reflections
$$w_{\alpha}(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$$
for $\alpha, \beta \in \Phi$ where $\langle \beta, \alpha \rangle = 2(\beta, \alpha)(\alpha, \alpha)^{-1}$ and $\langle \cdot, \cdot \rangle$ denotes the inner product of the
corresponding ambient real vector space of $\Phi$. The Weyl group is generated by the
reflections $w_{\alpha_i}$ for the simple roots $\bar{\alpha}_i \in \Delta$ and every $w_\alpha \in W$ can be written as
$$w_{\alpha} = w_{\bar{\alpha}_i} \cdots w_{\bar{\alpha}_i}$$
with $k$ minimal. The integer $k$ is called the length of $w_\alpha$. We denote by $\bar{\omega}$ the
unique element of maximal length, that is the element which maps $\Phi^+$ to $\Phi^-$ and
vice versa.
Let \( \mathfrak{h} \) be a maximal Cartan subalgebra of the Lie algebra \( \mathfrak{g} \) of \( G \). Then with respect to \( \mathfrak{h} \) we obtain a root space decomposition
\[
\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}
\]
where we denote by \( \mathfrak{g}_{\alpha} \) the one-dimensional root spaces of \( \mathfrak{g} \). With respect to this root space decomposition we can choose a Chevalley basis
\[
\{ H_i \mid \alpha_i \in \Delta \} \cup \{ X_\alpha \mid \alpha \in \Phi \}
\]
where \( \mathfrak{h} = \langle H_1, \ldots, H_l \rangle \) and \( \mathfrak{g}_{\alpha} = \langle X_\alpha \rangle \). Further we consider in the following the maximal nilpotent subalgebras \( \mathfrak{u} = \sum_{\alpha \in \Phi^+} \mathfrak{g}_{\alpha} \) and \( \mathfrak{u}^- = \sum_{\alpha \in \Phi^-} \mathfrak{g}_{\alpha} \) defined by all positive and all negative roots respectively. Moreover, let \( \mathfrak{b}^+ = \mathfrak{h} + \mathfrak{u}^+ \) be the maximal solvable subalgebra of \( \mathfrak{g} \) which contains the positive maximal nilpotent subalgebra \( \mathfrak{u}^+ \) and the Cartan subalgebra \( \mathfrak{h} \). Analogously let \( \mathfrak{b}^- = \mathfrak{h} + \mathfrak{u}^- \).

For \( X, Y \in \mathfrak{g} \) we denote by \([X, Y]\) the usual bracket product and we write \( \text{ad}(X) \) for the endomorphism \( \text{ad}(X) : Y \mapsto [X, Y] \). For an element \( g \in G \) we denote by
\[
\text{Ad}(g) : \mathfrak{g} \to \mathfrak{g}, \ X \mapsto gXg^{-1}
\]
the adjoint action of \( G \) on \( \mathfrak{g} \).

For \( l \) nonzero elements \( s = (s_1, \ldots, s_l) \) of \( C \) let
\[
A^+_0(s) := \sum_{i=1}^l s_i X_{\overline{\alpha}_i} \quad \text{and} \quad A^-_0(s) := \sum_{i=1}^l s_i X_{-\overline{\alpha}_i}
\]
be the sum of all basis elements which correspond to the positive simple roots (resp. negative simple roots) with coefficients \( s \). We call \( A^+_0(s) \) a principal nilpotent matrix in normal form and to shorten notation we write \( A^+_0 \) and \( A^-_0 \) in case of \( s = (1, \ldots, 1) \).

**Numbering of the Roots.** We order the negative roots \( \Phi^- = \{ \beta_1, \ldots, \beta_m \} \) such that \( \text{ht}(\beta_i) \geq \text{ht}(\beta_{i+1}) \) for all \( i = 1, \ldots, m-1 \). Then the indices of all roots of a given height form an unbroken string \( i_1, \ldots, i_2 \). Let \( \gamma_1, \ldots, \gamma_l \) be complementary roots of \( \Phi^- \) (see \([9\text{, Lemma 6.4 and Definition 6.5}])

For every height we reorder the roots such that the complementary roots have the greatest indices, that is, the indices of the non-complementary roots of a given height form an unbroken string \( i_1, \ldots, i_2 \).

We rename the basis elements \( \{ X_{\beta_i} \mid \beta_i \in \Phi^- \} \) of \( \mathfrak{u}^- \) into \( X_i \). One can decompose \( \mathfrak{g} \) into a direct sum of subspaces \( \mathfrak{g}^{(j)} \) and \( \mathfrak{h} \) where each \( \mathfrak{g}^{(j)} \) is the direct sum of root spaces corresponding to roots of height \( j \) (see \([9\text{, Lemma 6.1}])\). We denote by \( r(i) \) the negative number such that \( X_i \in \mathfrak{g}^{(r(i))} \), that is \( r(i) = \text{ht}(\beta_i) \). Then the basis is ordered such that \( r(i) \geq r(i+1) \) for all \( i = 1, \ldots, m-1 \). For each \( X_i \) we define
\[
W_i := [X_i, A^+_0].
\]
Then \( W_i \in \mathfrak{g}^{(r(i)+1)} \) and the set \( \{ W_i \mid i = 1, \ldots, m \} \) is a basis of \( \text{ad}(A^+_0)(\mathfrak{u}^-) \).

Further by \([9\text{, Lemma 6.4}])\ the set
\[
\{ W_i \mid i = 1, \ldots, m \} \cup \{ X_{\gamma_k} \mid k = 1, \ldots, l \}
\]
is a basis of \( \mathfrak{b}^- \).

We denote by \( T \) the maximal torus of \( G \) whose Lie algebra is \( \mathfrak{h} \). Further by \( U^+ \) and \( U^- \) we mean the maximal unipotent groups whose Lie algebras are \( \mathfrak{u}^+ \) and \( \mathfrak{u}^- \) respectively. Finally we write \( B^+ \) and \( B^- \) for the Borel subgroups of \( G \) with respective Lie algebras \( \mathfrak{b}^+ \) and \( \mathfrak{b}^- \).

For a root \( \alpha \in \Phi \) we denote by \( U_\alpha \) the one-dimensional root group of \( G \) whose Lie algebra coincide with the root space \( \mathfrak{g}_{\alpha} \). There is an isomorphism \( \mathbb{G}_m \to U_\alpha \) and for \( x \in C \) we denote by \( u_\alpha(x) \) the image of \( x \) in \( U_\alpha \). For the negative roots \( \beta_1, \ldots, \beta_m \),
which we ordered in a specific way, we rename the root groups $U_i$, into $U_i$ and their elements into $u_i(x)$. For $i = 1, \ldots, l$ let $T_i$ be the one-dimensional subtorus of $T$ whose Lie algebra coincides with $\langle H_i \rangle \subset \mathfrak{h}$. Then there is an isomorphism $G \to T_i$ and for $z \in C^\times$ we denote by $t_i(z)$ the image of $z$ in $T_i$. For $m$ elements $x = (x_1, \ldots, x_m)$ of $C$ we denote in the following by $u(x)$ the product

$$u(x) := u_1(x_1) \cdots u_m(x_m) \in U^-$$

and for $l$ elements $z = (z_1, \ldots, z_l)$ of $C^\times$ we write $t(z)$ for the product

$$t(z) := t_1(z_1) \cdots t_l(z_l) \in T.$$

An important result in the structure theory of reductive groups is the Bruhat decomposition. It gives a normal form for elements of $G$ parametrized by a fixed Borel subgroup and the Weyl group.

**Theorem 2.1** (Bruhat Decomposition). Fix a Borel subgroup of $G$. Then we have $G = \bigcup_{w \in W} BwB$ (disjoint union) with $BwB = B\bar{w}B$ if and only if $w = \bar{w}$ in $W$.

**Proof.** See [3] Theorem 28.3. □

For $w \in W$ we fix a representative $n(w)$ of $w$ in the normalizer $N_G(T)$ of $T$ in $G$. Following the discussion of [3] Chapter 28.1 we obtain a $T$-stable subgroup $U_w = U \cap n(w)U^-n(w)^{-1}$ of $U$ for $w$. A normal form for elements of $G$ can now be made unique.

**Theorem 2.2.** For each $w \in W$ fix a coset representative $n(w) \in N_G(T)$. Then each element $x \in G$ can be written in the form $x = u'n(w)tu$, where $w \in W$, $t \in T$, $u \in U$ and $u' \in U'_w$ are all determined uniquely by $x$.

**Proof.** See [3] Theorem 28.4. □

Borel subgroups are maximal connected solvable subgroups of $G$ and so by [3] Theorem 19.3 and Exercise 17.7 there is a descending chain of closed subgroups of $B^-$ where each group is a normal subgroup of its predecessor and their quotient is isomorphic to $G_n$ or $G_m$. For $i = 0, \ldots, l-1$ define $T_i := T_{i+1} \cdots T_l$ and denote by $B_i^-$ the subgroup $T_i \times U^- \subset B^-$. Moreover for $i = 0, \ldots, m-1$ denote by $U_i^-$ the subgroup $U_{i+1} \cdots U_m$ and let $U_m^- = \{ e \}$. Following [2] Theorem 5.3.3 these groups form a descending chain of closed subgroups

$$(1) \quad B^- = B_0^- \supset B_1^- \supset \cdots \supset B_{l-1}^- \supset U_0^- \supset U_1^- \supset \cdots U_{m-1}^- \supset U_m^- = \{ e \}$$

where each group is a normal subgroup of its predecessor and their quotient is isomorphic to $G_n$ or $G_m$. More precisely, for $i = 0, \ldots, l-2$ the quotients $B_i^-/B_{i+1}^-$ and $B_{i-1}^-/U_i^-$ are isomorphic to $G_m$ and for $i = 0, \ldots, m-1$ the quotients $U_i^-/U_{i+1}^-$ are isomorphic to $G_n$. We denote the positive roots by $\Phi^+ = \{ \alpha_1, \ldots, \alpha_n \}$ and we order them in the same way as the negative roots. If we define as in the case of $B^-$ and $U^-$ the subgroups $B_i^+ \subset B^+$ and $U_i^+ \subset U^+$, then these subgroups form a chain with the same properties.

3. **The Gauge Transformation**

Let $F$ be a differential field with derivation $\partial$ and field of constants $C$. In the following we mean by $G(F)$ and $\mathfrak{g}(F)$ the group and Lie algebra of $F$-rational points.

**Definition 3.1.** The map

$$\ell \delta : \text{GL}_n(F) \to F^{n \times n}, g \mapsto \partial(g)g^{-1}$$

is called the logarithmic derivative.
A gauge transformation of an element \( A \in \mathfrak{g}(F) \) by an element \( g \) of \( G(F) \) is defined as

\[
\text{Ad}(g)(A) + \ell\delta(g).
\]

We can decompose the gauge transformation of \( A \) into the image \( \text{Ad}(g)(A) \) of \( A \) under the adjoint action by \( g \) and into the image \( \ell\delta(g) \) of \( g \) under the logarithmic derivative. We will mostly look at the two images separately and then afterwards combine the results. This makes it possible to use the root structure of \( G \) and \( g \) to describe the gauge transformation.

Proposition \( \ref{prop:ad} \) below shows that the image of an element of a linear algebraic group under the logarithmic derivative lies in its Lie algebra.

**Proposition 3.2.** Let \( G \subset \text{GL}_n \) be a linear algebraic group. Then the restriction of \( \ell\delta \) to \( G \) maps \( G(F) \) to its Lie algebra \( \mathfrak{g}(F) \), that is

\[
\ell\delta |_G : G(F) \to \mathfrak{g}(F)
\]

**Proof.** A proof can be found in \([8]\). \qed

In order to understand better the adjoint action we will use Remark \( \ref{rem:adjoint} \) below. It gives a description of the image of elements of a Chevalley basis under the adjoint action of a root group element by the root system.

**Remark 3.3.** For \( \alpha, \beta \in \Phi \) linearly independent let \( \alpha - r\beta, \ldots, \alpha + q\beta \) be the \( \beta \)-string through \( \alpha \) for \( r, q \in \mathbb{N} \) and let \( \langle \alpha, \beta \rangle \) be the Cartan integer. We have

\[
\text{Ad}(u_\beta(x))(X_\alpha) = \sum_{i=0}^q c_{\beta, \alpha, i} x^{i} X_{\alpha + i\beta},
\]

\[
\text{Ad}(u_\beta(x))(H_\alpha) = H_\alpha - \langle \alpha, \beta \rangle x X_\beta,
\]

\[
\text{Ad}(u_\beta(x))(X_{-\beta}) = X_{-\beta} + xH_\beta - x^2 X_\beta
\]

where \( c_{\beta, \alpha, 0} = 1 \) and \( c_{\beta, \alpha, i} = \pm \binom{r+i}{i} \).

4. **Connecting the structure of the classical groups with Picard-Vessiot extensions**

In this section we will establish a connection between the geometrical structure of a classical group \( G \) and a \( G \)-primitive Picard-Vessiot extension (see Definition \( \ref{def:PV} \)) of the differential field \( F \). This link will be obtained by applying the Bruhat decomposition to a fundamental solution matrix of the extension. In the following we denote by \( C[\text{GL}_n] \) the ring \( C[X_{ij}, \text{det}(X_{ij})^{-1}] \) in the indeterminates \( X_{ij} \) with \( 1 \leq i, j \leq n \) and for a linear algebraic group \( H \subseteq \text{GL}_n \) we write \( C[H] \) for the coordinate ring of \( H \), that is the quotient ring of \( C[\text{GL}_n] \) by the defining ideal of \( H \). Further we denote by \( \overline{X}_{ij} \) the image of \( X_{ij} \) in \( C[H] \) and we write shortly \( X \) and \( \overline{X} \) for the matrices \( (X_{ij}) \) and \( (\overline{X}_{ij}) \) respectively. Finally, if \( H \) is connected we mean by \( C(H) \) the field of fractions of the coordinate ring \( C[H] \).

**Lemma 4.1.** There are algebraically independent elements \( z = (z_1, \ldots, z_l) \) and \( y = (y_1, \ldots, y_m) \) of \( C[B^{-}] \) such that

\[
\overline{X} = t(z)u(y).
\]

**Proof.** We shortly write \( U \) for the direct product \( U_1 \times \cdots \times U_m \) and analogously \( T \) for the direct product \( T_1 \times \cdots \times T_l \). The product maps \( U \to U^{-} \) and \( T \to T \) are isomorphisms of varieties and since \( T_i \) and \( U_i \) are isomorphic to \( \mathbb{G}_m \) and \( \mathbb{G}_a \), we identify the coordinate ring of \( T_i \) with \( C[\bar{z}_i, \bar{z}_i^{-1}] \) and the coordinate ring of \( U_i \) with \( C[\bar{y}_i] \) for some new indeterminates \( \bar{z}_i \) and \( \bar{y}_i \), respectively. Together we obtain a \( C \)-algebra isomorphism

\[
C[T \times U] \to C[\bar{z}_1, \bar{z}_1^{-1}, \ldots, \bar{z}_l, \bar{z}_l^{-1}, \bar{y}_1, \ldots, \bar{y}_m]
\]
Lemma 4.2. Let \( \bar{z} = (\bar{z}_1, \ldots, \bar{z}_l) \) and \( \bar{y} = (\bar{y}_1, \ldots, \bar{y}_m) \) be algebraically independent. From [1] Theorem 10.6 (4) we obtain that the product map from \( T \times U \to B^- \) is an isomorphism and combined with the above product maps it follows that the map

\[
\varphi : T \times U \to B^-, \quad (t_1, \ldots, t_l, u_1, \ldots, u_m) \mapsto t_1 \cdots t_l u_1 \cdots u_m
\]

is an isomorphism of varieties. Thus its comorphism

\[
\varphi^* : C[B^-] \to C[T \times U], \quad \overline{X}_{ij} \mapsto (t(\bar{z})u(\bar{y}))_{ij}
\]

is an isomorphism of \( C \)-algebras and its inverse sends the algebraically independent elements \( \bar{z} \) and \( \bar{y} \) to algebraically independent elements \( z = (z_1, \ldots, z_l) \) and \( y = (y_1, \ldots, y_m) \) of the coordinate ring of \( B^- \). Putting everything together we have

\[
\overline{X} = \varphi^{-1}(t(\bar{z})u(\bar{y})) = t(z)u(y)
\]

where \( z \) and \( y \) have the required property.

\[\square\]

**Lemma 4.2.** There are algebraically independent elements \( x = (x_1, \ldots, x_m) \), \( z = (z_1, \ldots, z_l) \) and \( y = (y_1, \ldots, y_m) \) of the field of fractions \( C(G) \) of \( C[G] \) such that

\[
\overline{X} = u(x)n(\bar{w})t(z)u(y).
\]

**Proof.** Let \( U \) and \( T \) be as in the proof of Lemma 11. We show that the map

\[
\varphi : U \times T \times U \to G \text{ defined by}
\]

\[
(u_1, \ldots, u_m, t_1, \ldots, t_l, u_1, \ldots, u_m) \mapsto u_1 \cdots u_m n(\bar{w}) t_1 \cdots t_l u_1 \cdots u_m
\]

is an isomorphism onto an open subset of \( G \), that is, we prove that \( U \times T \times U \) and \( G \) are birational equivalent. The image of \( \varphi \) is the variety \( U^- n(\bar{w})B^- \subset G \) and we have to show that it is an open subset of \( G \). By [1] Chapter 14.14 the subset

\[
n(\bar{w})U^- n(\bar{w})B^- = U^+ B^-
\]

is open in \( G \). It is the isomorphic image of \( U^- n(\bar{w})B^- \) under left multiplication with \( n(\bar{w}) \). Since left multiplication is a continuous morphism, we conclude that \( U^- n(\bar{w})B^- \) is an open subset of \( G \). Hence, \( \varphi \) is a birational morphism.

As in the proof of Lemma 11 we identify the coordinate ring \( C[U \times T \times U] \) with

\[
C[\bar{x}_1, \ldots, \bar{x}_m, \bar{z}_1, \bar{z}_l^{-1}, \ldots, \bar{z}_l, \bar{z}_l^{-1}, \bar{y}_1, \ldots, \bar{y}_m]
\]

for algebraically independent elements \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_m) \), \( \bar{z} = (\bar{z}_1, \ldots, \bar{z}_l) \) and \( \bar{y} = (\bar{y}_1, \ldots, \bar{y}_m) \) which represent the coordinates of the first, second and third factor of \( U \times T \times U \) respectively. Since \( \varphi \) is a birational morphism, its comorphism

\[
\varphi^* : C(G) \to C(U \times T \times U), \quad \overline{X}_{ij} \mapsto (u(\bar{x})n(\bar{w})t(\bar{z})u(\bar{y}))_{ij}
\]

is an isomorphism of fields. Its inverse sends the algebraically independent elements \( \bar{x}, \bar{z} \) and \( \bar{x} \) to algebraically independent elements \( x = (x_1, \ldots, x_m) \), \( z = (z_1, \ldots, z_l) \) and \( y = (y_1, \ldots, y_m) \) of \( C(G) \). This yields

\[
\overline{X} = \varphi^{-1}(u(\bar{x})n(\bar{w})t(\bar{z})u(\bar{y})) = u(x)n(\bar{w})t(z)u(y)
\]

with the required properties.

\[\square\]

**Corollary 4.3.** Let

\[
\overline{X} = u(\bar{x})n(\bar{w})t(\bar{z})u(\bar{y})
\]

be the normal form as in Lemma 4.2.

(a) Let \( g \) of \( G(C) \). Then there are over \( C \) algebraically independent elements \( x = (x_1, \ldots, x_m) \), \( z = (z_1, \ldots, z_l) \) and \( y = (y_1, \ldots, y_m) \) of \( C(G) \) such that

\[
\overline{X} g = u(x)n(\bar{w})t(z)u(y).
\]
Proof. (a) Let $g \in G(C)$. The translation map $\psi : G(C) \to G(C)$, $x \mapsto xg$ is an isomorphism of fields

$$\psi^* : G(C) \to G(C), \quad X_{ij} \mapsto (Xg)_{ij}.$$ 

Applying $\psi^*$ to the normal form of $\overline{X}$ we obtain

$$\overline{X}g = \psi^*(u(\overline{x})n(\overline{w})t(\overline{z})u(\overline{y})) = u(x)n(w)\ell(z)u(y)$$

where the images $x = (x_1, \ldots, x_m)$, $z = (z_1, \ldots, z_l)$ and $y = (y_1, \ldots, y_m)$ of $\overline{x}$, $\overline{z}$ and $\overline{y}$ under $\psi^*$ are algebraically independent over $C$.

(b) Let $b \in B_i^+(C)$ and represent it as a product $t_{i+1} \cdots t_lu$ with $u \in U^-(C)$ and $t_{i+1} \in T_{i+1}(C), \ldots, t_l \in T_l(C)$. Using that $U^-(C(G))$ is normal in $B_i^+(C(G))$ we compute

$$t(\overline{z})u(\overline{y})t_{i+1} \cdots t_lu = t(\overline{z})t_{i+1} \cdots t_lu(\overline{y}) = t_1(\overline{z}_1) \cdots t_l(\overline{z}_l)t_{i+1}(z_{i+1}) \cdots t_l(z_l)u(y)$$

where $z_{i+1}, \ldots, z_l$ and $y = (y_1, \ldots, y_m)$ are elements of $C(G)$. From the last equation we obtain the normal form

$$u(\overline{x})n(\overline{w})t(\overline{z})u(\overline{y})b = u(\overline{x})n(\overline{w})t_1(\overline{z}_1) \cdots t_l(\overline{z}_l)t_{i+1}(z_{i+1}) \cdots t_l(z_l)u(y).$$

Since the elements in the normal form are uniquely determined, it follows with (a) that $\overline{x}$, $(\overline{z}_1, \ldots, \overline{z}_l)$ and $\overline{y}$ are algebraically independent over $C$.

(c) For $u \in \hat{U}_i^-(C)$ the subgroup structure gives the identity

$$u(\overline{y})u = u_1(\overline{y}_1) \cdots u_l(\overline{y}_l)u_{i+1}(y_{i+1}) \cdots u_m(y_m)$$

with $y_{i+1}, \ldots, y_m$ in $C(G)$. The rest of the proof works as in (b). \hfill \Box

The next step is to connect our results with Picard-Vessiot theory. We will mainly consider a specific type of Picard-Vessiot extensions.

**Definition 4.4.** A Picard-Vessiot extension $E/F$ is called a full $G$-primitive extension of $F$ if the differential Galois group of $E/F$ is $G(C)$ and if there is a matrix $Y \in G(E)$ whose entries generate $E$ over $F$ and which satisfies $\ell\delta(Y) = A \in g(F)$.

**Proposition 4.5.** Let $E/F$ be a full $G$-primitive Picard-Vessiot extension with matrix $Y \in G(E)$. Then there are over $F$ algebraically independent elements $x = (x_1, \ldots, x_m)$, $z = (z_1, \ldots, z_l)$ and $y = (y_1, \ldots, y_m)$ of $E$ such that

$$Y = u(x)n(\overline{w})t(\overline{z})u(y).$$

**Proof.** The matrix $A = \ell\delta(Y) \in g(F)$ is a defining matrix for the Picard-Vessiot extension $E$ of $F$. We use the standard construction method to build a differentially isomorphic Picard-Vessiot extension for $A$. Since $A \in g(F)$ and the differential Galois group for $A$ is $G(C)$ the defining ideal of $G$ in $C[GL_n]$ extends to a maximal differential ideal of $F[GL_n]$. Its quotient ring $S$ is then a Picard-Vessiot ring for $A$ and the matrix $\overline{X}$ in $G(S)$ is by construction a fundamental solution matrix. Denote by $L$ the field of fractions of $S$. We have the embedding $C(G) \hookrightarrow L$ and so the normal form of $\overline{X}$ in $G(C(G))$ which we obtain from Lemma [12] yields the normal form of the fundamental matrix $\overline{X} \in G(L)$. 

Since $E$ and $L$ are both Picard-Vessiot extensions for $A$, there is a differential $F$-algebra isomorphism 
$$\varphi : L \to E, \quad \mathcal{X} \mapsto Yg$$
where $g \in \text{GL}_n(C)$. Because the matrix $\mathcal{X}$ satisfies the defining conditions for $G$, its image also does and so $Yg$ lies in $G(E)$. We conclude that $g$ also lies in $G(C)$. Applying Corollary 4.3 to $\mathcal{X}g^{-1}$ we obtain that there are over $C$ algebraically independent elements $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m)$, $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_l)$ and $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_m)$ of $C(G)$ such that
$$\mathcal{X}g^{-1} = u(\bar{x})n(w)t(\bar{z})u(\bar{y}).$$
Since $\varphi$ is an $F$-algebra isomorphism there are over $F$ algebraically independent elements $x = (x_1, \ldots, x_m)$, $z = (z_1, \ldots, z_l)$ and $y = (y_1, \ldots, y_m)$ of $E$ such that
$$Y = \varphi(u(\bar{x})n(w)t(\bar{z})u(\bar{y})) = u(x)n(w)t(z)u(y).$$

**Definition 4.6.** For a full $G$-primitive extension $E/F$ with fundamental matrix $Y \in G(E)$ we call the elements $x = (x_1, \ldots, x_m)$, $z = (z_1, \ldots, z_l)$ and $y = (y_1, \ldots, y_m)$ of Proposition 4.3 the coefficients of the normal form of $Y$.

The Borel subgroups, which are by definition maximal solvable subgroups, play an important role in the structure theory of reductive groups. In differential Galois theory Picard-Vessiot extensions with solvable Galois group form a special class of extensions. We will see now how these two concepts are linked in our setting.

**Proposition 4.7.** Let $x$, $z$ and $y$ be the coefficients of a normal form of full $G$-primitive extension $E/F$. Then $L = F(x)$ is a differential field with constants $C$. Moreover, $E/L$ is a Picard-Vessiot extension with differential Galois group $B^{-}(C)$ and $E$ is generated as a field by $z$ and $y$ over $L$. Further $E/L$ is a Liouvillian extension with tower of differential fields
$$L(z, y_1, \ldots, y_{m-1})(y_m) \supset \cdots \supset L(z)(y_1) \supset L(z_1, \ldots, z_{l-1})(z_l) \supset \cdots \supset L(z_1) \supset L,$$
where the elements $z$ are exponentials and the elements $y$ are integrals.

**Proof.** Applying the Galois correspondence we obtain that $E$ is a Picard-Vessiot extension of the fixed field $E^{B^-}$ with differential Galois group $B^{-}(C)$. Since $B^{-}$ is solvable, the extension $E/E^{B^-}$ is a Liouvillian extension by [10, Theorem 1.43]. The field $E$ is generated by $z$ and $y$ over $F(x)$, because we have $E = F(x, z, y)$. We show that $E^{B^-} = F(x)$. This will complete the proof of the lemma except for the statement with the tower of fields.

It follows from Corollary 4.3(b) applied with $i = 0$, that the elements $x$ are left invariant by all $b \in B^{-}$ and so the differential field $E^{B^-}$ contains $F(x)$. We prove that the two fields are actually equal by comparing their transcendence degrees. Because $G$ is connected and of dimension $2m + l$, the field extension $E$ of $F$ is a transcendental extension of degree $2m + l$. As an intermediate field $E^{B^-}$ is also a transcendental extension of $F$. Since $B^{-}$ is connected and has dimension $l + m$, the extension $E$ of $E^{B^-}$ is a transcendental extension of degree $l + m$. We conclude that the transcendence degree of $E^{B^-}$ over $F$ is $m$. Since $x = (x_1, \ldots, x_m)$ are algebraically independent over $F$, the transcendence degree of $F(x)$ is also $m$. It follows that $E^{B^-} = F(x)$ and so $F(x)$ has the required properties.

It is left to show that there is a tower of differential fields as stated and that the elements $z$ are exponentials and the elements $y$ are integrals. Following the notation of Chapter 2 we have a descending chain of subgroups as in [11].
For \( i = 0, \ldots, l - 2 \) the quotients of \( \bar{B}^-_i \) by \( \bar{B}^-_{i+1} \) and the quotient of \( \bar{B}^-_{l-1} \) by \( \bar{U}^-_0 \) are isomorphic to \( \mathbb{G}_m \) and so the fixed fields

\[
E^{\bar{B}^-_{i+1}} \supset E^{\bar{B}^-_i} \quad \text{and} \quad E^{\bar{U}^-_i} \supset E^{\bar{B}^-_{i-1}}
\]

are Picard-Vessiot extensions of transcendence degree one and they are generated by an exponential. From above we already know that \( F(x) = E^{\bar{U}^-_0}(C) \) and from Corollary \([4.3(b)]\) and \([4.3(c)]\) applied with \( i = 0 \), we obtain for \( i = 1, \ldots, l - 1 \) that

\[
F(x, z_1, \ldots, z_i) \subseteq E^{\bar{B}^-_i}(C) \quad \text{and} \quad F(x, z_1, \ldots, z_i) \subseteq E^{\bar{U}^-_i}(C)
\]

One proves now inductively that the inclusions are actually equalities by comparing the transcendence degrees of the corresponding extensions. Summing up we have that for \( i = 1, \ldots, l \) the extensions \( F(x, z_1, \ldots, z_{i-1}) \subseteq F(x, z_1, \ldots, z_i) \) are Liouvillian and \( z_i \) is an exponential.

For \( i = 0, \ldots, m - 1 \) we have that the quotient of \( \bar{U}^-_i \) by \( \bar{U}^-_{i+1} \) is isomorphic to \( \mathbb{G}_a \). Thus the fixed fields

\[
E^{\bar{U}^-_{i+1}} \supset E^{\bar{U}^-_i}
\]

define Picard-Vessiot extensions of transcendence degree one which are obtained by adjoining an integral. From above we know that \( F(x, z_1, \ldots, z_i) = E^{\bar{U}^-_0}(C) \) and trivially \( F(x, z, y_1, \ldots, y_m) = E^{\bar{U}^-_m}(C) \). Moreover, Corollary \([4.3(c)]\) yields for \( i = 1, \ldots, m - 1 \) that

\[
F(x, z, y_1, \ldots, y_i) \subseteq E^{\bar{U}^-_i}(C).
\]

As above if we compare the transcendence degrees of the corresponding extensions we can show that the inclusions are actually equalities. We conclude that for \( i = 1, \ldots, m \) the extension \( F(x, z, y_1, \ldots, y_i) \) of \( F(x, z, y_1, \ldots, y_{i-1}) \) is Liouvillian and that \( y_i \) is an integral. □

**Remark 4.8.** In the notation of Proposition \([4.7]\) the matrix \( \ell \delta(Y) \in \mathfrak{g}(F) \) is a defining matrix for the Picard-Vessiot extension \( E \) of \( L \). It is actually very easy to produce a defining matrix for this extension which lies in the Lie algebra \( \mathfrak{b}^-(L) \). Indeed, the inverse of the matrix \( u(x)u(\bar{w}) \) is an element of \( G(L) \) and so the gauge transformation of \( \ell \delta(Y) \) with this matrix is defined over \( L \) and gives an equivalent defining matrix for the extension. The gauge equivalent defining matrix is the logarithmic derivative of

\[
(u(x)u(\bar{w}))^{-1}Y = t(z)u(y)
\]

which lies in \( \mathfrak{b}^-(L) \) by Proposition \([5.2]\).

5. The construction of the general extension field

In this chapter we construct our general extension field \( E \). It will be a Picard-Vessiot extension with differential Galois group \( G \) of a differential field which has differential transcendence degree \( l \) over \( C \). The differential structure of \( E \) will be associated to a defining matrix of specific form. In \([9]\) we constructed a matrix differential equation depending on \( l \) parameters \( t = (t_1, \ldots, t_l) \) which defines a Picard-Vessiot extension of the differential field generated over \( C \) by the differential indeterminates \( t \) with Galois group \( G(C) \). More precisely, for \( l \) complementary roots \( \gamma_1, \ldots, \gamma_l \) of \( \Phi^- \) and for \( A^+_0 \) the equation is defined by the matrix

\[
A_C(t) = A^+_0 + \sum_{i=1}^{l} t_i X_{\gamma_i}.
\]

The construction of our general extension field will be based on the differential field \( C(\eta) \) where \( \eta = (\eta_1, \ldots, \eta_l) \) are \( l \) differential indeterminates over \( C \). It will be a Liouvillian extension of \( C(\eta) \) whose differential Galois group will be \( B^-(C) \). The
defining matrix of the Liouvillian extension will be the sum of a parametrization of the Cartan subalgebra by $\eta$ and $A^-$. It will turn out that a fundamental matrix of this extension is $t(z)u(y)$ where $z = (z_1, \ldots, z_l)$ are exponentials and $y = (y_1, \ldots, y_l)$ are integrals. Using the root structure we will show that there are $m - l$ differential polynomials

$$f_{l+1}(\eta), \ldots, f_m(\eta)$$

of $C\{\eta\}$ with the property that the logarithmic derivative of

$$u(\eta_1, \ldots, \eta_l, f_{l+1}(\eta), \ldots, f_m(\eta)) n(\bar{w}) t(z)u(y)$$

is $A_C(h)$ where $h = (h_1(\eta), \ldots, h_l(\eta))$ are elements of $C\{\eta\}$. Finally we will prove that $h$ are differentially algebraically independent over $C$ and that $E$ is a Picard-Vessiot extension of $C(h)$ for $C(h)$ with differential Galois group $G(C)$. The degree of freedom of our general extension field $E$ lies in the $l$ differential indeterminates $\eta = (\eta_1, \ldots, \eta_l)$.

For technical reasons we start the construction of $E$ with a tuple of $m$ differential indeterminates $\eta_m = (\eta_1, \ldots, \eta_m)$ which we will reduce later to the tuple of the first $l$ indeterminates $\eta = (\eta_1, \ldots, \eta_l)$. To shorten notation we will in the following denote by $u_i(\eta_m)$ the product $u_1(\eta_1) \cdots u_i(\eta_i)$ for $i = 2, \ldots, m - 1$.

**Lemma 5.1.** The image of $u(\eta_m)$ under the logarithmic derivative is

$$\ell \delta(u(\eta_m)) = \sum_{i=1}^l \eta_i' X_i + \sum_{i=l+1}^m (\eta_i' + v_i(\eta_m)) X_i$$

where $v_i(\eta_m)$ lies in $C\{\eta_1, \ldots, \eta_{s_2}\}$ with $s_2$ maximal such that $r(s_2) = r(i) + 1$ and all its terms are of order one and of degree greater than one.

**Proof.** With the product rule and the definition of the adjoint action we obtain for the image of $u(\eta_m)$ under the logarithmic derivative the sum

$$\ell \delta(u(\eta_m)) = \ell \delta(u_1(\eta_1)) +$$

$$\Ad(u_1(\eta_1))(\ell \delta(u_2(\eta_2))) + \cdots + \Ad(u_{m-1}(\eta_m))(\ell \delta(u_m(\eta_m))).$$

We are going to determine the logarithmic derivatives $\ell \delta(u_i(\eta_i))$ for $i = 1, \ldots, m$ and the images of $\ell \delta(u_{k+1}(\eta_{k+1}))$ under $\Ad(u_k(\eta_m))$ for $k = 1, \ldots, m - 1$. Using the results the statement of the lemma will follow from the last equation.

The logarithmic derivative of $u_i(\eta_i)$ is $\eta_i' X_i$ for $i = 1, \ldots, m$. Indeed, Proposition 4.2 yields that $\ell \delta(u_i(\eta_i))$ lies in the root space $g_i$. Moreover, the root group element $u_i(\eta_i)$ is the image of $\eta_i X_i$ under the exponential map where $X_i$ nilpotent. Using this representation of $u_i(\eta_i)$ one easily checks that the only contribution to $X_i$ in the corresponding product of power series is $\eta_i' X_i$.

For $j = 2, \ldots, m$ let $X$ be a linear combination of basis elements $X_i$ with $r(i) \leq r(j)$ whose the coefficients lie in $C\{\eta_1, \ldots, \eta_{j_2}\}$ with $j_2$ maximal such that $r(j_2) = r(j)$. We claim that for $k = 1, \ldots, j - 1$ the image of $X$ under the adjoint action of $u_k(\eta_k)$ can be represented as a linear combination of the same basis elements and that the coefficients of those $X_i$ with $r(i) = r(j)$ are the same as in $X$ and of those with $r(i) < r(j)$ lie again in $C\{\eta_1, \ldots, \eta_{j_2}\}$. Let $i$ be an index with $r(i) \leq r(j)$ and let $v(\eta_m)$ be a nonzero element of $C\{\eta_1, \ldots, \eta_{j_2}\}$. From the first formula in Remark 3.3 we obtain

$$\text{Ad}(u_k(\eta_k))(v(\eta_m)X_i) = \sum_{s \geq 0} c_{\beta_k, \beta_i, s} \eta_k^s v(\eta_m) X_{\beta_i + s \beta_k}$$

where the coefficient $c_{\beta_k, \beta_i, s}$ is nonzero if $\beta_i + s \beta_k$ is a root and zero otherwise. Clearly if $\beta_i + s \beta_k$ is a root, then it is equal to $\beta_i$ (case $s = 0$) or it is of height less than $\beta_i$ (case $s \geq 1$). Thus the right hand side of (2) is a linear combination of basis elements corresponding to roots of height less or equal than $r(i)$. For $s = 0$
the coefficient of $X_i$ is $v(\eta_m)$ and because the index $k$ is less than $j$, the coefficients in the sum of the right hand side of $[2]$ lie in $C\{\eta_1, \ldots, \eta_s\}$ for $s \geq 1$. The claim now follows from the linearity of the adjoint action.

For $k = 1, \ldots, m - 1$ we apply the claim iteratively to
\[
\text{Ad}(u_k(\eta_m))((\ell\delta(u_{k+1}(\eta_{k+1}))) = \text{Ad}(u_1(\eta_1))\ldots(\text{Ad}(u_k(\eta_k)))u_{k+1}(X_{k+1})\ldots).
\]
Representing the result as a linear combination of $X_1, \ldots, X_m$ we obtain that the coefficients of $X_i$ with $r(i) \geq r(k+1)$ and $i \neq k+1$ are zero, that the coefficient of $X_{k+1}$ is $\eta_k(\eta_{k+1})$ and that the coefficients of $X_i$ with $r(i) < r(k+1)$ are in $C\{\eta_1, \ldots, \eta_s\}$ where $s$ is maximal such that $r(s) = r(k+1)$. In case of the latter coefficients we can adapt the definition of the ring containing them such that it becomes independent of $k$. Let $s_2$ be maximal such that $r(s_2) = r(i) + 1$. For basis elements with $r(i) + 1 = r(k+1)$ the ring $C\{\eta_1, \ldots, \eta_{s_2}\}$ coincides with $C\{\eta_1, \ldots, \eta_s\}$ and for basis elements with $r(i) + 1 < r(k+1)$ the ring $C\{\eta_1, \ldots, \eta_{s_2}\}$ contains the ring $C\{\eta_1, \ldots, \eta_s\}$. 

\textbf{Example 5.2.} We consider the group $\text{SL}_4(C)$. In $[3$, Chapter 7$]$ we worked out the root system of type $A_3$ and presented a corresponding explicit Chevalley basis of the Lie algebra $\mathfrak{sl}_4(C)$. Our computations will be with respect to this basis. We denote and number the negative roots according to Chapter 2 as
\[
\beta_1 = -\alpha_1, \quad \beta_2 = -\alpha_2, \quad \beta_3 = -\alpha_3, \quad \beta_4 = -\alpha_1 - \alpha_2 - \alpha_3, \quad \beta_5 = -\alpha_1 - \alpha_2 - \alpha_3.
\]
so that $r(i) = -1$ for $i = 1, 2, 3$, $r(i) = -2$ for $i = 4, 5$ and $r(i) = -3$ for $i = 6$. We write for the basis element $X_{s_2}$ and for a root group element $u_{s_2}$, shortly $X_i$ and $u_i$ respectively. From the exponential map we obtain that for a negative root $\beta_i$ the element $u_i(\eta_i)$ of the root group $U_i$ is the matrix $u_i(\eta_i) = E + \eta_i X_i$ where $E$ denotes the $4 \times 4$ unit matrix. It follows that $\ell\delta(u_i(\eta_i)) = \eta_i X_i$. We have
\[
\ell\delta(u(\eta_0)) = \eta_1 X_1 + \text{Ad}(u_1(\eta_1))(\eta_2 X_2) + \text{Ad}(u_1(\eta_1)u_2(\eta_2))(\eta_3 X_3) + \text{Ad}(u_1(\eta_1)\cdots u_3(\eta_3))(\eta_4 X_4) + \text{Ad}(u_1(\eta_1)\cdots u_4(\eta_4))(\eta_5 X_5) + \text{Ad}(u_1(\eta_1)\cdots u_5(\eta_5))(\eta_6 X_6).
\]
We determine the terms in the coefficients of the linear representation of the logarithmic derivative of $u(\eta_0)$ with respect to the basis $\{X_i \mid i = 1, \ldots, 6\}$. Obviously for all $i = 1, \ldots, 6$ the coefficient of $X_i$ contains the term $\eta_i$. Since $\beta_2 + \beta_1 = \beta_3$ the second summand contributes the term $\eta_1 \eta_2$ to the coefficient of $X_2$. From the third summand the coefficients of $X_5$ and $X_6$ obtain the terms $\eta_2 \eta_3$ and $\eta_1 \eta_2 \eta_3$ respectively, since $\beta_3 + \beta_2 = \beta_5$ and $\beta_3 + \beta_1 = \beta_6$. The fourth summand contributes the term $\eta_1 \eta_2 \eta_4$ to the coefficient of $X_4$, since $\beta_4 + \beta_2 = \beta_6$. From the next summand the coefficient of $X_6$ obtains the term $\eta_1 \eta_2 \eta_5$, since $\beta_5 + \beta_1 = \beta_6$. Finally the last summand gives no contribution, because the sum of $\beta_6$ and any negative root is not a root. Putting our results together we obtain
\[
\ell\delta(u(\eta_0)) = \eta_1 X_1 + \eta_2 X_2 + \eta_3 X_3 + (\eta_4 + v_4(\eta_0))X_4 + (\eta_5 + v_5(\eta_0))X_5 + (\eta_6 + v_6(\eta_0))X_6
\]
with
\[
v_4(\eta_0) = -\eta_2 \eta_1, \quad v_5(\eta_0) = -\eta_3 \eta_2, \quad v_6(\eta_0) = \eta_2 \eta_4 - \eta_1 \eta_1 + \eta_3 \eta_2 \eta_1
\]
and one easily checks that the coefficients are as in the statement of the lemma.

\textbf{Lemma 5.3.} We have
\[
\text{Ad}(u(\eta_{m}))(A_{0}^+) = A_{0}^+ + \sum_{i=1}^{l} g_i(\eta)H_i + \sum_{i=1}^{m} (\ell_i(\eta_{m}) + p_i(\eta_{m}))X_i
\]

Proof. For $j = 1, \ldots, m$ we prove by induction that

$$\text{Ad}(u_j(\eta_m))(A^+_0) = A^+_0 + \sum_{i=1}^j \eta_i W_i + \sum_{i=1}^m \bar{p}_i(\eta_m) X_i$$

where $\bar{p}_i(\eta_m)$ is zero or $\bar{p}_i(\eta_m) \in C[\eta_1, \ldots, \eta_m]$ with $i_2$ maximal such that $r(i_2) = r(i)$ and each term is of degree greater than one.

Since by definition $W_j = [X_j, A^+_0]$ we have $[\eta_j, X_j, A^+_0] = \eta_j W_j$ and from Remark 5.3 we obtain

$$\text{Ad}(u_j(\eta_j))(A^+_0) = A^+_0 + \eta_j W_j + \sum_{k=1}^{l} \sum_{s \geq 2} c_{j_k, s\beta_j} \eta_j^s X_{\bar{\alpha}_{j_k} + s\beta_j}$$

where the coefficients of the double sum are either zero or trivially a homogeneous polynomial of degree greater than one in $\eta_j$ depending whether $\bar{\alpha}_k + s\beta_j$ is a root or not.

For $j = 1$ the above observations yield

$$\text{Ad}(u_1(\eta_1))(A^+_0) = A^+_0 + \eta_1 W_1 + \bar{p}_1(\eta_m) X_1$$

where $\bar{p}_1(\eta_m) \in C[\eta_1]$ is a homogeneous polynomial of degree two.

For $j > 1$ we obtain with the same observations that

$$\text{Ad}(u_j(\eta_m))(A^+_0) = \text{Ad}(u_{j-1}(\eta_m))(A^+_0 + \eta_j W_j + \sum_{i=j_1}^m \bar{p}_i(\eta_m) X_i)$$

where we take $j_1$ minimal such that $r(j_1) = r(j)$ and the polynomials $\bar{p}_i(\eta_m)$ in $C[\eta_j]$ are either zero or homogeneous of degree greater than one. The polynomial $\bar{p}_i(\eta_m)$ is the coefficient of the basis element which corresponds to the root $\bar{\beta}_i = \bar{\alpha}_k + s\beta_j$ with $s \geq 2$ and so for $i_2$ maximal such that $r(i_2) = r(i)$ we have $\bar{p}_i(\eta_m) \in C[\eta_1, \ldots, \eta_m]$. Since the adjoint action is linear we can consider the three images on the right hand side of the last equation individually and then combine our results. For the image of $A^+_0$ the induction assumption implies

$$\text{Ad}(u_{j-1}(\eta_m))(A^+_0) = A^+_0 + \sum_{i=1}^{j-1} \eta_i W_i + \sum_{i=1}^m \bar{p}_i(\eta_m) X_i$$

where $\bar{p}_i(\eta_m)$ is zero or $\bar{p}_i(\eta_m) \in C[\eta_1, \ldots, \eta_m]$ with $i_2$ maximal such that $r(i_2) = r(i)$ and each term of it is of degree greater than one.

The vector $\eta_j W_j$ is contained in the sum of root spaces corresponding to roots of height $r(j) + 1$, that is in $g^{\nu(j)+1}$, and $\eta_j$ lies in $C[\eta_1, \ldots, \eta_j]$ with $j_2$ maximal
such that \( r(j_2) = r(j) \). The image of \( \eta_i W_j \) computes with the above observations iteratively as

\[
\text{Ad}(u_{j-1}(\eta_m))(\eta_i W_j) = \eta_i W_j + \sum_{i=j_1}^{m} \tilde{p}_i(\eta_m)X_i
\]

where \( j_1 \) is minimal such that \( r(j_1) = r(j) \). Since the entries of \( u_{j-1}(\eta_m) \) are in \( C[\eta_1, \ldots, \eta_{j-1}] \) and \( \eta_j \in C[\eta_1, \ldots, \eta_{j_2}] \) with \( j_2 \) maximal such that \( r(j_2) = r(j) \), we conclude that all \( \tilde{p}_i(\eta_m) \) are contained in \( C[\eta_1, \ldots, \eta_{j_2}] \). Because in each iteration step \( s \geq 1 \) in the above formula and because we apply the adjoint action to basis elements with polynomial coefficients whose terms are of degree at least one, the degree of each term in \( \tilde{p}_i(\eta_m) \) is greater than one. For \( i \geq j_1 \) we have that \( i_2 \) maximal such that \( r(i_2) = r(i) \) satisfies \( i_2 \geq j_2 \) and so \( \tilde{p}_i \) also lies in the eventually larger ring \( C[\eta_1, \ldots, \eta_{j_2}] \).

For the last image the above observations yield

\[
\text{Ad}(u_{j-1}(\eta_m))(\sum_{i=j_1}^{m} \tilde{p}_i(\eta_m)X_i) = \sum_{i=j_1}^{m} \tilde{p}_i(\eta_m)X_i + \sum_{i=j_1}^{m} \tilde{p}_i(\eta_m)X_i
\]

where \( j_1' \) is minimal such that \( r(j_1') = r(j_1) - 1 \). For each \( i \geq j_1' \) the index \( i_2 \) maximal such that \( r(i_2) = r(i) \) is a polynomial whose terms are of degree greater than one. Since \( \{ W_i \mid r(i) = -1 \} \) is a basis of \( \mathfrak{h} \), we obtain that the coefficients of \( H_1, \ldots, H_l \) are nonzero \( C \)-linearly independent homogeneous polynomials of degree one which we denote by \( g_i(\eta) \). Let \( \beta_i \) be a non complementary root and let \( k_1 \) be minimal and \( k_2 \) be maximal such that \( r(k_1) = r(i) - 1 = r(k_2) \). Since \( \beta_i \) and at least one of the roots \( \beta_{k_1}, \ldots, \beta_{k_2} \) differ by a simple root, the coefficient of \( X_i \) in \( (3) \) has a non-zero homogeneous part of degree one in \( C[\eta_{k_1}, \ldots, \eta_{k_2}] \) stemming from \( W_{k_1}, \ldots, W_{k_2} \). We denote this part by \( \ell_i(\eta_m) \) and the remaining part of degree greater than one by \( p_i(\eta_m) \). We prove that

\[
\ell_i(\eta_m) = 0, \ldots, \ell_i(\eta_m) = 0
\]

with \( i_1 \) minimal and \( i_2' \) maximal such that \( r(i_1) = r(i_2') \) and \( i_2' \) does not correspond to a complementary root form a linear system in \( \eta_{k_1}, \ldots, \eta_{k_2} \) of full rank. Indeed, the set \( \{ W_{k_1}, \ldots, W_{k_2} \} \cup \{ X_{\eta_m} \mid r(k) = r(i) \} \) is a basis of \( g^{(r(i))} \) and so the coefficient matrix for these vectors with respect to the basis \( X_{1_1}, \ldots, X_{1_2} \) with \( i_2 \) maximal such that \( r(i) = r(i_2) \) has full rank. Since the two basis coincide for the indices \( k_2 + 1, \ldots, i_2 \), it is a block matrix whose last row consists of a corresponding zero and unit matrix. We conclude that the first block, that is, the matrix without the rows and columns for the complementary roots has also full rank. Identifying \( W_{k_1}, \ldots, W_{k_2} \) with \( \eta_{k_1}, \ldots, \eta_{k_2} \) shows that

\[
\ell_i(\eta_m) = 0, \ldots, \ell_i(\eta_m) = 0
\]

is a linear system in \( \eta_{k_1}, \ldots, \eta_{k_2} \) of full rank. \( \square \)
Example 5.4. We proceed with Example \[\text{Example 5.2}\]. According to Lemma \[\text{Lemma 5.3}\], the image of the matrix $A_0^+$ under the adjoint action with $u(\eta_6)$ can be written as

$$
\text{Ad}(u(\eta_6))(A_0^+) = A_0^+ + \sum_{i=1}^{4} g_i(\eta) H_i + \sum_{i=1}^{6} (\ell_i(\eta_6) + p_i(\eta_6)) X_i.
$$

We determine the differential polynomials $g_i(\eta)$, $\ell_i(\eta_6)$ and $p_i(\eta_6)$ and check that they have the described properties. We have

$$
\text{Ad}(u_0(\eta_6))(A_0^+) = A_0^+ + \eta_6 W_0 = A_0^+ - \eta_6 X_4 + \eta_6 X_5
$$
$$
\text{Ad}(u_5(\eta_6))(A_0^+) = A_0^+ + \eta_5 W_5 = A_0^+ - \eta_5 X_2 + \eta_5 X_3
$$
$$
\text{Ad}(u_4(\eta_4))(A_0^+) = A_0^+ + \eta_4 W_4 = A_0^+ - \eta_4 X_1 + \eta_4 X_2
$$

Indeed, there are no nonlinear parts, since it is not possible to obtain a root by adding any simple root to $2 \beta_i$ for $i = 6, 5, 4$. The components of $W_i$ in the basis \[\{X_i \mid i = 1, \ldots, m\}\] correspond to those roots which are the sum of $\beta_i$ and a simple root. Further with Remark \[\text{Remark 5.3}\] we get

$$
\text{Ad}(u_3(\eta_3))(A_0^+) = A_0^+ + \eta_3 W_3 - \eta_3^2 X_3
$$
$$
\text{Ad}(u_2(\eta_2))(A_0^+) = A_0^+ + \eta_2 W_2 - \eta_2^2 X_2
$$
$$
\text{Ad}(u_1(\eta_1))(A_0^+) = A_0^+ + \eta_1 W_1 - \eta_1^2 X_1
$$

where $W_i = -H_i$ for $i = 3, 2, 1$. The $W_j$ with $j = 1, \ldots, 6$ determine the terms in the homogeneous polynomials of degree one $g_i(\eta)$ and $\ell_i(\eta_6)$. We have

$$
g_1(\eta) = -\eta_1, \quad g_2(\eta) = -\eta_2, \quad g_3(\eta) = -\eta_3
$$

and they clearly satisfy Lemma \[\text{Lemma 5.3(a)}\]. Further we read off

$$
\ell_1(\eta_6) = -\eta_1, \quad \ell_2(\eta_6) = \eta_2 - \eta_3, \quad \ell_3(\eta_6) = \eta_3,
$$
$$
\ell_4(\eta_6) = -\eta_4, \quad \ell_5(\eta_6) = \eta_5, \quad \ell_6(\eta_6) = 0
$$

and they obviously satisfy the first part of statement \[\text{Lemma 5.3(c)}\]. Since the complementary roots are $\beta_3$, $\beta_5$ and $\beta_6$ (see \[\text{Lemma 7.1}\]), we have the two linear systems

$$
\ell_1(\eta_6) = 0, \quad \ell_2(\eta_6) = 0 \quad \text{and} \quad \ell_4(\eta_6) = 0
$$

in the variables $\eta_4$, $\eta_5$ and in the variable $\eta_6$ respectively and both systems have full rank. In order to determine the non-linear parts $p_i(\eta_6)$ one can use the recursion

$$
r_0 := \text{Ad}(u_0(\eta_6))(A_0^+),
$$
$$
r_k := \text{Ad}(u_0(\eta_6))(r_{k-1}) \text{ for } k = 1, \ldots, 5
$$

which computes the whole image of $A_0^+$ under $\text{Ad}(u(\eta_6))$. We sketch how to proceed. We already know that in the steps $k = 3, 4, 5$ we obtain from $A_0^+$ in $r_{k-1}$ the term $\eta_0^2 - \eta_k$ in the coefficient of $X_6 - \eta_k$ in the linear representation of $r_k$. We get further nonlinear terms from the component in the Cartan subalgebra. Since it is only nonzero in the steps 4 and 5, we obtain the new nonlinear terms $\eta_2 \eta_3$ and $\eta_2 \eta_1$ in the coefficients of $X_2$ and $X_1$ in the linear representation of $r_4$ and $r_5$ respectively. Finally in each step $k$ one obtains new nonlinear terms in the coefficient of $X_j$ in the representation of $r_k$ from multiplying the coefficient of that $X_j$ in the representation of $r_{k-1}$ with $\eta_6 - \eta_k$ for which $\beta_i + \beta_6 - \beta_k$ is the root $\beta_j$. One can check that the nonlinear parts compute as

$$
p_1(\eta_6) = -\eta_1^2 + \eta_2 \eta_1, \quad p_2(\eta_6) = -\eta_2^2 + \eta_3 \eta_2, \quad p_3(\eta_6) = -\eta_3^2,
$$
$$
p_4(\eta_6) = -\eta_2 \eta_4 + \eta_1 (\eta_2^2 - \eta_4 - \eta_5 \eta_2 + \eta_5), \quad p_5(\eta_6) = -\eta_5 \eta_2 + \eta_1 (\eta_4 - \eta_5 + \eta_2 \eta_3),
$$
$$
p_6(\eta_6) = -\eta_1 \eta_3 \eta_4 - \eta_1 \eta_6 - \eta_4 \eta_1 + \eta_3 \eta_2 \eta_1 - \eta_3^2 \eta_1 + \eta_3 \eta_5 \eta_1 - \eta_3 \eta_6
$$

and that they have the stated properties of Lemma \[\text{Lemma 5.3(b)}\].
In the next step we construct the Liouvillian extension of $C(\eta)$. For this purpose we fix a representative $n(\bar{w})$ in the normalizer of the torus for the Weyl group element $\bar{w}$ of maximal length. Since $\bar{w}$ sends $\{-\bar{\alpha}_1, \ldots, -\bar{\alpha}_l\}$ bijectively to $\Delta$ and the adjoint action of $n(\bar{w})$ maps $X_1, \ldots, X_l$ bijectively to non zero multiples of $X_{\bar{\alpha}_1}, \ldots, X_{\bar{\alpha}_l}$, there are units $e = (e_1, \ldots, e_l)$ in $C$ such that

$$\text{Ad}(n(\bar{w}))(A_0^- (e)) = A_0^e.$$ 

Further the adjoint action of $n(\bar{w})$ sends each basis element $H_i$ of the Cartan subalgebra to a nonzero multiple. Thus for $y_i(\eta)$ of Lemma 5.3(a) there are nonzero $C$-linear independent homogeneous polynomials $\bar{y}_1(\eta), \ldots, \bar{y}_l(\eta)$ of degree one in $C[\eta]$ such that

$$\text{Ad}(n(\bar{w}))(\sum_{i=1}^l \bar{y}_i(\eta)H_i) = \sum_{i=1}^l -y_i(\eta)H_i.$$ 

With these definitions we set

$$A_L(\eta) = \sum_{i=1}^l \bar{y}_i(\eta)H_i + A_0^- (e) \in b^- (C(\eta))$$

and consider the matrix differential equation defined by $A_L(\eta)$.

**Proposition 5.5.** There is a Picard-Vessiot extension $E$ of $C(\eta)$ for $A_L(\eta)$ with the following properties:

(a) The differential Galois group of $E$ over $C(\eta)$ is $B^-(C)$.

(b) There is a fundamental solution matrix $Y_L \in B^-(E)$ and there are elements $z = (z_1, \ldots, z_l)$ and $y = (y_1, \ldots, y_m)$ of $E$ algebraically independent over $C(\eta)$ with

$$Y_L = t(z)u(y).$$

(c) The extension is Liouvillian and there is a tower of differential fields

$$E_0 = C(\eta) \subset E_1 \subset \cdots \subset E_l \subset E_{l+1} \subset \cdots \subset E_{m+1} = E$$

where $E_i = E_{i-1}(z_i)$ with $z_i$ an exponential for $i = 1, \ldots, l$ and $E_i = E_{i-1}(y_{i-1})$ with $y_{i-1}$ an integral for $i = l + 1, \ldots, m + 1$.

(d) The elements $z$ and $y$ have shape

$$z_1 = e^\int \bar{y}_1(\eta), \ldots, z_l = e^\int \bar{y}_l(\eta), \quad y_1 = \int c_1 \tilde{\alpha}_1(t(z))^{-1}, \ldots, y_l = \int c_l \tilde{\alpha}_l(t(z))^{-1}, \quad y_{l+1} = \int -v_{l+1}(y), \ldots, y_m = \int -v_m(y),$$

where $\bar{y}_i(\eta)$ are as in the definition of $A_L(\eta)$ and $v_{l+1}(y), \ldots, v_m(y)$ are as in Lemma 5.1.

**Proof.** (a) We show that for $n \in \mathbb{N}$ there are nonzero elements $\bar{c}_1, \ldots, \bar{c}_l$ of $C$ such that the differential Galois group of a Picard-Vessiot extension for the matrix

$$A_L = \sum_{i=1}^l \bar{c}_i z^n H_i + c_i X_i$$

over the rational function field $C(z)$ with standard derivation is the group $B^-(C)$. The statement then follows from [9, Theorem 4.3] and [10, Proposition 1.31]. Indeed, since $\bar{g}_1(\eta), \ldots, \bar{g}_l(\eta)$ are $C$-linearly independent homogeneous polynomials of degree one in $C[\eta_1, \ldots, \eta_l]$, the corresponding coefficient matrix is invertible and so there are nonzero elements $\bar{c}_1, \ldots, \bar{c}_l$ of $C$ such that

$$\bar{c}_i z^n = \bar{g}_i(\bar{c}_1 z^n, \ldots, \bar{c}_l z^n).$$
It follows that the differential $C$-Algebra homomorphism $\sigma : C(\eta) \to C[z]$ defined by $\sigma(\eta) = \tilde{c}z^n$ satisfies $\sigma(A_L(\eta)) = A_L$ and is a surjective $R_1$-specialization.

In order to show that there exists $\mathcal{E}$ such that $A_L$ has Galois group $B^-(C)$ we consider the quotient homomorphism

$$\pi : B^- \to B^-/[U^-, U^-]$$

and the corresponding Lie algebra homomorphism $d\pi : b^- \to b^-/[u^-, u^-]$. According to [3] Proposition 16 it is enough to prove that there exists $\mathcal{E}$ such that the differential Galois group of the image $d\pi(A_L)$ is the full group $\pi(B^-)$. The first assumption of the proposition is satisfied by [3] Corollary of Lemma 2. For the remaining assumptions the standard construction method yields a Picard-Vessiot extension $E$ of $F$ with a fundamental solution matrix $Y \in B^-(E)$, since $A_L$ lies in $b^-$. By [3] Proposition 3] the logarithmic derivative of $\pi(Y)$ is $d\pi(A_L)$ and so $\pi(Y)$ is a fundamental solution matrix for $d\pi(A_L)$ and its entries generate a Picard-Vessiot extension. The last condition is satisfied by construction.

A basis of the Lie algebra $d\pi(b^-)$ is given by the images $\mathcal{H}_{1}, \ldots, \mathcal{H}_l$ of the basis $H_1, \ldots, H_l$ of the Cartan subalgebra and the images $\mathcal{X}_{1}, \ldots, \mathcal{X}_l$ of the basis elements $X_1, \ldots, X_l$ corresponding to the negative simple roots and so

$$d\pi(A_L) = \sum_{i=1}^{l} c_i z^n \mathcal{H}_i + c_i \mathcal{X}_i = \mathcal{A}_L.$$ 

We follow the argumentation of [3] Chapter III and use the notation. We need to show that there are $C$ such that the images of

$$\mathcal{E}_z = (\mathcal{H}_1 z^n, \ldots, \mathcal{H}_l z^n)$$

in the quotient $C(z)/\ell\delta(C(z)^*)$ are linearly independent over $\mathbb{Z}$ and that $c_i$ does not reduce to zero modulo $L\mathcal{E}_z, \bar{\alpha}_i(C(z))$. Indeed, if these statements are fulfilled, it follows then from [3] Proposition 16] together with [3] Proposition 15] and [3] Lemma 5] that the differential Galois group of $\mathcal{A}_L$ is $\pi(B^-)$.

For the first statement [3] Lemma 6 yields that for $z^n$ there are infinitely many choices of coefficients $C$ such that $\mathcal{E}_z$ are $\mathbb{Z}$-linearly independent modulo $\ell\delta(C(z)^*)$. For the second statement we need to show for $i = 1, \ldots, l$ that the differential equations

$$L_{\mathcal{E}_z, \bar{\alpha}_i}(\zeta) = \zeta' - \sum_{j=1}^{l} (\bar{\alpha}_j, \bar{\alpha}_i) \mathcal{E}_j z^n \zeta = c_i.$$ 

have no solutions in $C(z)$. According to [113] Exercise 1.36 4.] for any nonzero $\tilde{c} \in C$ and $n \in \mathbb{N}$ the differential Galois group of a Picard-Vessiot extension of $C(z)$ for the differential equation defined by the matrix

$$\begin{pmatrix} 0 & 1 \\ \tilde{c}^2 z^{2n}/4 + \tilde{c} n z^{n-1}/2 & 0 \end{pmatrix}$$

is conjugate to the Borel subgroup of $\text{SL}_2$. It can be easily checked that this matrix is gauge equivalent to

$$\begin{pmatrix} \tilde{c} z^n/2 & c_i \\ 0 & -\tilde{c} z^n/2 \end{pmatrix}$$

and so there are no solutions in $C(z)$ for differential equations of shape

$$\zeta' = \tilde{c} z^n \zeta = c_i.$$ 

(b) We construct a Picard-Vessiot extension $E$ of $C(\eta)$ for $A_L(\eta)$ such that there is a fundamental solution matrix $Y_L$ in $B^-(E)$ and elements $z = (z_1, \ldots, z_l)$ and $y = (y_1, \ldots, y_m)$ of $E$ such that $Y_L$ can be written as a product as stated where we use the notation of Chapter 3. Since $A_L(\eta)$ lies in the Lie algebra of $B^-$ the
defining ideal of $B^-$ in $C[GL_n]$ extends to a differential ideal in $C\langle \eta \rangle [GL_n]$ where the derivation on $X$ is defined by multiplication with $A_L(\eta)$. It follows from (a) that it is a maximal differential ideal and so the quotient ring is a Picard-Vessiot ring $R$. By construction the matrix $Y_L := Y$ is a fundamental solution matrix and lies in $B^-(R)$. Since $C[B^-]$ embeds canonically into $R$, Lemma 4.4 yields that there are over $C\langle \eta \rangle$ algebraically independent elements $z = (z_1, \ldots, z_l)$ and $y = (y_1, \ldots, y_m)$ of $R$ such that
\[
Y_L = t(z)u(y).
\]
The field of fractions $E$ of $R$ has then the desired properties.

(c) The extension is clearly a Liouvillian extension, since its differential Galois group is solvable. We prove that there is a tower of fields as stated for the elements $z$ and $y$ of (b) and that they are respectively exponentials and integrals. The proof is similar to the one of Proposition 1.6. Recall from Chapter 2 that we have a descending chain of normal subgroups for $B$ and that they are respectively exponentials and integrals. We can prove inductively by comparing the transcendence degree of the respective fields that
\[
E^{\bar{B}_{i+1}} \supset E^{\bar{B}_i} \quad \text{and} \quad E^{\tilde{U}_0} \supset E^{\tilde{U}_{i-1}}
\]
are Picard-Vessiot extensions of transcendence degree one and they are generated by an exponential. Corollary 4.3(b) and 4.3(c) (with $i = 0$) implies after multiplication with the inverse of $u(x)\nu(\eta)$ for $i = 1, \ldots, l - 1$ the inclusions
\[
C(\eta)(z_1, \ldots, z_i) \subseteq E^{\bar{B}_i} \quad \text{and} \quad C(\eta)(z_1, \ldots, z_i) \subseteq E^{\tilde{U}_i}.
\]
We can prove inductively by comparing the transcendence degree of the respective fields that
\[
E^{\bar{B}_i} = E^{\bar{B}_{i-1}}(z_i) \quad \text{and} \quad E^{\tilde{U}_i} = E^{\tilde{U}_{i-1}}(z_i)
\]
and it follows with the above that $z_i$ an exponential.

For $i = 0, \ldots, m - 1$ the quotients $\bar{U}_i$ by $\bar{U}_{i+1}$ are isomorphic to $G_a$ and so the corresponding inclusions of fixed fields
\[
E^{\bar{U}_{i+1}} \supset E^{\bar{U}_i}
\]
are Picard-Vessiot extensions of transcendence degree one and the extensions are generated by an integral. We already know that $E^{\tilde{U}_0} = C(\eta)(z)$ and that $E^{\tilde{U}_m} = C(\eta)(z, y)$. For $i = 1, \ldots, m - 1$ we conclude as above with Corollary 4.3(c) that $z, y_1, \ldots, y_i$ are left fixed by $\bar{U}_i$, that is we have
\[
C(\eta)(z, y_1, \ldots, y_i) \subseteq E^{\bar{U}_i}.
\]
Again one proves inductively by comparing the transcendence degrees of the respective fields that the inclusions are actually equalities. Summing up it follows that $E^{\bar{U}_i} = E^{\bar{U}_{i-1}}(y_i)$ with $y_i$ an integral.

(d) We determine the shape of the exponentials $z$ and the integrals $y$. Using the product rule the logarithmic derivative of the fundamental matrix $Y_L = t(z)u(y)$ computes as
\[
\ell \delta(Y_L) = \ell \delta(t(z)) + \Lambda d(t(z)) (\ell \delta(u(y))) = A_L.
\]
By Proposition 3.2 the logarithmic derivative of $t(z)$ and $u(y)$ are elements of $\mathfrak{h}$ and $u^-$ respectively. Moreover the adjoint action of the torus stabilizes the root spaces and so 4.4 implies
\[
\ell \delta(t(z)) = \sum_{i=1}^l \ell \delta(u(y)) H_i.
\]
The logarithmic derivative of \( t_i(z_i) \) computes as \( z_i'/z_iH_i \). Moreover with the product rule and the fact that the adjoint action of the torus stabilizes the Cartan algebra we conclude that \( \bar{\gamma}_i(\eta) = z_i'/z_i \) and so we have

\[
z_i = e^{\int \bar{\gamma}_i(\eta)}.
\]

It is left to determine the integrals. From (4) we know that

\[
\text{Ad}(t(z))(\ell \delta(u(y))) = A_0^\pm(e).
\]

Since the torus acts on \( X_i \) by the roots we conclude with Lemma [5.1] that the coefficients of basis elements corresponding to the negative simple roots satisfy \( c_i = \alpha_i(t(z))y_i' \). Thus for \( i = 1, \ldots, l \) the integral \( y_i \) is

\[
y_i = \int c_i \alpha_i(t(z))^{-1}.
\]

For the remaining indices that is for the roots \( \beta_i \) with \( r(i) \leq -2 \) the component of the logarithmic derivative of \( u(y) \) in \( g_i \) has to be zero. It follows then from Lemma [5.1] that for \( l + 1 \leq i \leq m \) we have

\[
y_i = \int -v_i(y)
\]

where \( v_i(y) \) only depends on integrals \( y_1, \ldots, y_j \) with \( j \) maximal such that \( r(j) = r(i) + 1 \).

\[\begin{align*}
\text{Example 5.6.} & \text{ We continue with the example for SL}_4(C). \text{ For the Weyl group element of maximal length we choose the representative} \\
n(\bar{\omega}) &= E_{14} - E_{23} + E_{32} - E_{41} \\
\text{in the normalizer of the torus of SL}_4(C). \text{ With } \bar{\gamma}_1(\eta) = -\eta_3, \bar{\gamma}_2(\eta) = -\eta_2, \bar{\gamma}_3(\eta) = -\eta_1 \text{ and } c_i = -1 \text{ the defining matrix of the Liouvillian extension becomes} \\
A_L(\eta) &= -A_0^\pm - \eta_3 H_1 - \eta_2 H_2 - \eta_1 H_3
\end{align*}\]

and it satisfies

\[
\text{Ad}(n(\bar{\omega}))(A_L(\eta)) = A_0^\pm - g_1(\eta)H_1 - g_2(\eta)H_2 - g_3(\eta)H_3
\]

where \( g_i(\eta) \) is as in Example [5.2]. With \( \bar{\gamma}_i(\eta) \) we define the exponentials

\[
z_1 = e^{\int -\eta_1}, \quad z_2 = e^{\int -\eta_2}, \quad z_3 = e^{\int -\eta_3}
\]

and obtain for the roots of height \(-1\) the integrals

\[
y_1 = - \int e^{\int(-2\eta_3 + \eta_2)}, \quad y_2 = - \int e^{\int(-2\eta_2 + \eta_1 + \eta_3)}, \quad y_3 = - \int e^{\int(-2\eta_1 + \eta_3)}.
\]

Furthermore with \( v_i(\eta_2) \) as in Example [5.2] we compute for the remaining roots the integrals

\[
y_4 = \int -v_4(y_1, y_2, y_3) = \int \left( \int e^{\int(-2\eta_3 + \eta_2)} \right)e^{\int(-2\eta_2 + \eta_1 + \eta_3)}, \\
y_5 = \int -v_5(y_1, y_2, y_3) = \int \left( \int e^{\int(-2\eta_2 + \eta_1 + \eta_3)} \right)e^{\int(-2\eta_1 + \eta_2)}, \\
y_6 = \int -v_6(y_1, \ldots, y_6) = \int \left( \int e^{\int(-2\eta_1 + \eta_2)} \right) \left( \int e^{\int(-2\eta_2 + \eta_3)} \right)e^{\int(-2\eta_2 + \eta_1 + \eta_3)}.
\]

By construction \( Y_L = t(z)u(y) \) satisfies \( \ell \delta(Y_L) = A_L(\eta) \).
Lemma 5.7. Let $z$, $y$ be as in Proposition 5.4 and define

$$ Y = u(\eta_m)u(\bar{w})t(z)u(y). $$

We have

$$ \ell\delta(Y) = A_0^+ + h_1(\eta_m)X_1 + \cdots + h_m(\eta_m)X_m $$

with coefficients

$$ h_i(\eta_m) = \eta_i' + \ell_i(\eta_m) + q_i(\eta_m) $$

where $\ell_i(\eta_m)$ is as in Lemma 5.4 and $q_i(\eta_m)$ lies in $C[\eta_1, \ldots, \eta_{s_2}, \ldots, \eta_{s_2+1}, \ldots, \eta_{s_2+2}]$ with $s_2$ and $i_2$ maximal such that $r(s_2) = r(i) + 1$ and $r(i_2) = r(i)$. Moreover each term of $q_i(\eta_m)$ is of degree greater than one and the derivatives appearing are at most of order one.

Proof. With the product rule and $\ell\delta(n(\bar{w})) = 0$ the logarithmic derivative of $Y$ computes as

$$ \ell\delta(Y) = \ell\delta(u(\eta_m)) + u(\eta_m)n(\bar{w}) \ell\delta(t(z)u(y)) \ (u(\eta_m)n(\bar{w}))^{-1}. $$

We look at the two summands individually and then combine our results. From Lemma 5.4 we obtain that the first summand is an element of $u^-(C(\eta))$ and that if we represent it as a linear combination of the basis elements $X_1, \ldots, X_m$ then the coefficient of $X_i$ is $\eta_i' + v_i(\eta_m)$ where $v_i(\eta_m) \in C[\eta_1, \ldots, \eta_{s_2}]$ with $s_2$ maximal such that $r(s_2) = r(i) + 1$ and all its terms are of order one and of degree greater than one.

Next we consider the second summand. Since $z$ and $y$ are as in Proposition 5.4 we have

$$ \ell\delta(t(z)u(y)) = A_L(\eta) = A_0^- (c) + \sum_{i=1}^l g_i(\eta)H_i, $$

where $g_i(\eta)$ is as in Lemma 5.3(a) and each term of $g_i(\eta)$ is of degree greater than one. Moreover each term of $q_i(\eta_m)$ is of degree greater than one. By the second formula of Remark 5.3 the adjoint action of $u(\eta_m)$ maps $A_0^+$ by construction to

$$ A_0^+ + \sum_{i=1}^l -g_i(\eta)H_i $$

and so the second summand of (5) becomes

$$ \sum_{i=1}^l -g_i(\eta)\text{Ad}(u(\eta_m))(H_i) = \sum_{i=1}^l -g_i(\eta)H_i + \sum_{i=1}^m a_i(\eta_m)X_i $$

where $g_i(\eta) \in C[\eta_1, \ldots, \eta]$ are nonzero $C$-linear independent homogeneous polynomials of degree one, $\ell_i(\eta_m) \in C[\eta_{k_1}, \ldots, \eta_{k_2}]$ with $k_1$ minimal and $k_2$ maximal such that $r(k_1) = r(i) - 1 = r(k_2)$ and $\ell_i(\eta_m)$ is a homogeneous polynomial of degree one and $p_i(\eta_m) \in C[\eta_1, \ldots, \eta_{s_2}]$ with $i_2$ maximal such that $r(i_2) = r(i)$ and each term of $p_i(\eta_m)$ is of degree greater than one. By the second formula of Remark 5.3 the adjoint action of $u(\eta_m)$ maps $H_i$ to

$$ H_i + (\alpha_i, \beta_k) \eta_k X_k $$

and so the second summand of (5) becomes

$$ \sum_{i=1}^l -g_i(\eta)\text{Ad}(u(\eta_m))(H_i) = \sum_{i=1}^l -g_i(\eta)H_i + \sum_{i=1}^m a_i(\eta_m)X_i $$
where $a_i(\eta_m)$ is a polynomial of $C[\eta_1, \ldots, \eta_n]$ with $i_2$ maximal such that $r(i_2) = r(i)$ and each term of $a_i(\eta_m)$ is of degree greater than one. Indeed, this follows from the fact that $a_i(\eta_m)$ is the product of $\eta_0 \in C[\eta_1, \ldots, \eta_n]$ and the homogeneous polynomial of degree one $g_1(\eta) \in C[\eta_1, \ldots, \eta_n]$.

Combining the results one checks that the coefficient of each $H_i$ in the linear combination of $\delta \delta(Y)$ vanishes and that the coefficient of each $X_i$ is as described in the statement. \qed

**Example 5.8.** We proceed with the example for $S\mathcal{L}_4(C)$. The logarithmic derivative of

$$Y = u(\eta_0)n(\bar{w})t(z)u(y),$$

where $z$ and $y$ are as in Example 5.6 computes as

$$\delta \delta(Y) = \delta(u(\eta_0)) + Ad(u(\eta_0)n(\bar{w}))(\delta(t(z)u(y))) = \delta(u(\eta_0)) + Ad(u(\eta_0))(\eta_0^+ - g_1(\eta)H_1 - g_2(\eta)H_2 - g_3(\eta)H_3).$$

Combining the results of Example 5.2 and Example 5.3 we obtain

$$\delta \delta(Y) = A_0^+ + h_1(\eta_0)X_1 + \ldots + h_6(\eta_0)X_6$$

with coefficients

$$h_1(\eta_0) = \eta_1^2 - \eta_4 + q_1(\eta_0), \quad h_2(\eta_0) = \eta_2^2 + \eta_4 - \eta_5 + q_2(\eta_0),$$

$$h_3(\eta_0) = \eta_3^2 + \eta_5 + q_3(\eta_0), \quad h_4(\eta_0) = \eta_4^2 - \eta_6 + q_4(\eta_0),$$

$$h_5(\eta_0) = \eta_5^2 + \eta_6 + q_5(\eta_0), \quad h_6(\eta_0) = \eta_6^2 + q_6(\eta_0),$$

where

$$q_i(\eta_0) = v_i(\eta_0) + p_i(\eta_0) + a_i(\eta_0)$$

and $a_i(\eta_0)$ is as in the proof of Lemma 5.7. By computation we obtain

$$q_1(\eta_0) = \eta_1^2, \quad q_2(\eta_0) = \eta_2^2, \quad q_3(\eta_0) = \eta_3^2,$$

and

$$q_4(\eta_0) = \eta_5 \eta_1 - (\eta_2 \eta_2 - \eta_1), \quad q_5(\eta_0) = \eta_3 \eta_1 - \eta_3 \eta_4 + v_4(\eta_0),$$

$$q_6(\eta_0) = \eta_5^2 - \eta_5 \eta_2 - \eta_4 + q_5(\eta_0).$$

Recall that the indices 1, 2, 3 correspond to the roots of height $-1$, the indices 4, 5 belong to the roots of height $-2$ and the root belonging to the index 6 is the only root of height $-3$. It is easy to check that the linear and non-linear parts of the coefficients satisfy the statement of the lemma.

**Lemma 5.9.** Let

$$\{i_1, \ldots, i_{m-1} \mid \beta_i, \notin \{\gamma_1, \ldots, \gamma_l\}\} \cup \{j_1, \ldots, j_l \mid \beta_j, \in \{\gamma_1, \ldots, \gamma_l\}\}$$

be the partition of $\{1, \ldots, m\}$ into indices corresponding to non-complementary and complementary roots and let $h_1(\eta_m), \ldots, h_m(\eta_m)$ be the differential polynomials of Lemma 5.7. Then the system of equations

$$h_{i_1}(\eta_m) = 0, \ldots, h_{i_{m-1}}(\eta_m) = 0$$

is equivalent to the system

$$\eta_{i+1} = \bar{\ell}_{i+1}(\eta) + \bar{p}_{i+1}(\eta), \ldots, \eta_m = \bar{\ell}_m(\eta) + \bar{p}_m(\eta)$$

where $\bar{\ell}_i(\eta)$ and $\bar{p}_i(\eta)$ have the following properties:

(a) The differential polynomial $\bar{\ell}_i(\eta) \in C[\eta_1, \ldots, \eta_n]$ is homogeneous of degree one and of order $|r(i)+1|$ with $j$ maximal such that $r(i) = -1$ and $j$ does not correspond to a complementary root.
Proof. We fix the following notation. For a height minimal and index $i$ and $k\le i$ such that each term is of degree greater than one. Setting $\ell$ with $\ell-s$ solving for $\ell$ statement (a) and $C_1$ is a linear system of full rank in \( (9) \) where by assumption the left hand sides of the system define a quadratic linear \( (8) \) We obtained these equations from solving the equations Gaussian elimination we obtain the equations of full rank in the variables $\eta$ on the variables $\bar{\eta}$ $\ell$ be maximal such that $\ell-s$ does not correspond to a complementary root. Further let $k_1$ be minimal and $k_2$ be maximal such that $r(k_1) = q-1 = r(k_2)$. If not otherwise stated the index $i$ runs between $i_1 \le i \le i'_2$ and the index $k$ between $k_1 \le k \le k_2$.

We use induction on the height $q = -1, \ldots, r(m)$. Let $q = -1$. Note that $i_1 = 1$, $i'_2 = j$ with $j$ as in (a) and $k_1 = l + 1$. By assumption we have
\[
h_1(\eta_m) = \eta'_1 + \ell_1(\eta_m) + q_1(\eta), \ldots, h_{i'_2}(\eta_m) = \eta'_{i'_2} + \ell_{i'_2}(\eta_m) + q_{i'_2}(\eta)
\]
with $\ell_i(\eta_m) \in C[\eta_{t+1}, \ldots, \eta_{k_2}]$ homogeneous of degree one and $q_i(\eta) \in C[\eta_1, \ldots, \eta_l]$ such that each term is of degree greater than one. Setting $h_i(\eta_m)$ to zero and then solving for $\ell_i(\eta_m)$ we obtain the system
\[
\ell_1(\eta_m) = -\eta'_1 - q_1(\eta), \ldots, \ell_{i'_2}(\eta_m) = -\eta'_i - q_{i'_2}(\eta)
\]
where by assumption the left hand sides of the system define a quadratic linear system
\[
\ell_1(\eta_m) = 0, \ldots, \ell_{i'_2}(\eta_m) = 0
\]
of full rank in the variables $\eta_{t+1}, \ldots, \eta_{k_2}$. Note that the right hand sides depend only on the variables $\eta_1, \ldots, \eta_l$ and the first order derivatives of $\eta_1, \ldots, \eta_{i'_2}$. Applying Gaussian elimination we obtain the equations
\[
\eta_{t+1} = \tilde{\ell}_{t+1}(\eta) + \tilde{p}_{t+1}(\eta), \ldots, \eta_{k_2} = \tilde{\ell}_{k_2}(\eta) + \tilde{p}_{k_2}(\eta)
\]
where $\tilde{\ell}_i(\eta)$ and $\tilde{p}_i(\eta)$ represent the corresponding results obtained from applying the same row operations to $-\eta'_i$ and to $q_i(\eta)$ respectively. Clearly $\tilde{\ell}_i(\eta)$ satisfies statement (a) and
\[
\tilde{\ell}_{t+1}(\eta) = 0, \ldots, \tilde{\ell}_{k_2}(\eta) = 0
\]
is a linear system of full rank in $\eta'_1, \ldots, \eta'_{i'_2}$. Since the properties of $q_i$ also hold for their $C$-linear combinations, we conclude that $\tilde{p}_i(\eta)$ satisfies (c).

Let $q \le -2$. By the induction assumption we have the equations
\[
\eta_{t+1} = \tilde{\ell}_{t+1}(\eta) + \tilde{p}_{t+1}(\eta), \ldots, \eta_{i-1} = \tilde{\ell}_{i-1}(\eta) + \tilde{p}_{i-1}(\eta),
\]
\[
\eta_i = \tilde{\ell}_i(\eta) + \tilde{p}_i(\eta), \ldots, \eta_{i_2} = \tilde{\ell}_{i_2}(\eta) + \tilde{p}_{i_2}(\eta),
\]
where $\tilde{\ell}_i(\eta)$ and $\tilde{p}_i(\eta)$ with $i = t + 1, \ldots, i_2$ satisfy (a), (b) and (c) respectively. We obtained these equations from solving the equations $h_i(\eta_m) = 0$ with $i = 1, \ldots, i-1$ and $i$ does not correspond to a complementary root. Thus we did not yet use the differential polynomials
\[
h_{i_1}(\eta_m), \ldots, h_{i_2}(\eta_m).
\]
Setting then to zero and solving for $\ell_i(\eta_m)$ we obtain the system of equations
\[
\ell_{i_1}(\eta_m) = -\eta'_{i_1} - q_{i_1}(\eta), \ldots, \ell_{i_2}(\eta_m) = -\eta'_{i_2} - q_{i_2}(\eta_m)
\]
where by assumption the left hand sides define a quadratic linear system of full rank in the indeterminates \( \eta_1, \ldots, \eta_{k_2} \) and the right hand sides depend on the differential polynomials

\[ q_i(\eta_m) \in C[\eta_1, \ldots, \eta_{k_1-1}, \eta_{k_1}, \ldots, \eta_{k_2}] \]

whose terms are of degree greater than one and only contain derivatives of order at most one. Substituting \( \mathcal{S} \) into \( -\eta'_{i_1}, \ldots, -\eta'_{i_2} \) as well as \( \mathcal{G} \) into \( -q_i(\eta_m) \) the equations in \( \mathcal{S} \) become

\[ \ell_{i_1}(\eta_m) = -\ell_{i_1}(\eta)' - \tilde{q}_{i_1}(\eta), \ldots, \ell_{i_2}(\eta_m) = -\ell_{i_2}(\eta)' - \tilde{q}_{i_2}(\eta) \]

where in \( \tilde{q}_i(\eta) \) we collected the terms stemming from \( -\tilde{p}_i(\eta)' \) and \( -q_i(\eta_m) \). Since the linear system

\[ \ell_{i_1}(\eta)' = 0, \ldots, \ell_{i_2}(\eta)' = 0 \]

in the variables \( \eta_1^{(i+1)}, \ldots, \eta_j^{(i+1)} \) has full rank and is equal to the subsystem of \( \mathcal{G} \) obtained by canceling those equations whose index correspond to a complementary root. The properties of \( \tilde{p}_i(\eta) \) and \( \tilde{q}_i(\eta) \) imply that \( \tilde{q}_i(\eta) \) satisfies (c). We apply now Gaussian elimination to \( \mathcal{S} \) and obtain

\[ \eta_{k_1} = \tilde{\ell}_{k_1}(\eta) + \tilde{p}_{k_1}(\eta), \ldots, \eta_{k_2} = \tilde{\ell}_{k_2}(\eta) + \tilde{p}_{k_2}(\eta) \]

where \( \tilde{\ell}_k(\eta) \) and \( \tilde{p}_k(\eta) \) are obtained by applying the same row operations to \( -\tilde{\ell}_k(\eta)' \) and \( -\tilde{q}_k(\eta) \) respectively. Since the above mentioned properties of the last differential polynomials do not change under row operations, they also hold for \( \tilde{\ell}_k(\eta) \) and \( \tilde{p}_k(\eta) \).

**Example 5.10.** We continue with the example for SL\(_4\). The equations corresponding to non-complementary roots of height \(-1\) are

\[ h_1(\eta_6) = \eta_4 - \eta_1 + \eta_2^2 = 0, \quad h_2(\eta_6) = \eta_2^2 + \eta_1 - \eta_2 \eta_1 = 0. \]

Solving for their linear parts yields

\[ \eta_4 = \eta_1' + \eta_2^2, \quad -\eta_2 + \eta_5 = \eta_2^2 + \eta_2(\eta_2 - \eta_1) \]

and if we add the first to the second equation we obtain

\[ \eta_4 = \eta_1' + \tilde{p}_4, \quad \eta_5 = \eta_2' + \eta_1' + \tilde{p}_5 \]

where

\[ \tilde{p}_4(\eta) = \eta_2^2, \quad \tilde{p}_5(\eta) = \eta_2^2 + \eta_2(\eta_2 - \eta_1). \]

One easily checks that the linear and non-linear parts in \( \mathcal{S} \) satisfy respectively the statements of Lemma 5.9(a) and 5.9(c). Furthermore the linear system

\[ \tilde{\ell}_4(\eta) = \eta_4' = 0, \quad \tilde{\ell}_5(\eta) = \eta_2' + \eta_4' = 0 \]

fulfills the first part of 5.9(b). For the non-complementary root of height \(-2\) we consider the equation

\[ h_4(\eta_6) = \eta_4^2 - \eta_6 - (\eta_2(\eta_2 - \eta_1))\eta_1 - \eta_3\eta_4 + \eta_5\eta_1 - \eta_2^2\eta_1 = 0 \]

which we solve for its linear part. We obtain

\[ \eta_6 = \eta_4' - (\eta_2(\eta_2 - \eta_1))\eta_1 - \eta_3\eta_4 + \eta_5\eta_1 - \eta_2^2\eta_1. \]

We use the expressions in \( \mathcal{S} \) for the substitution of \( \eta_4 \) and \( \eta_5 \) and obtain

\[ \eta_6 = \eta_4'' + \tilde{p}_6(\eta) \]
where

\[ \tilde{p}_6(\eta) = 3\eta_1\eta'_1 + \eta_1^2 - \eta_2\eta'_2 - \eta_2^2. \]

One sees that the linear and non-linear part satisfy 5.9(a) and 5.9(c) respectively. The linear system \( \hat{\ell}_6(\eta) = \eta'_1 = 0 \) clearly satisfies the first part of 5.9(b) and canceling the second equation of the system

\[ \hat{\ell}_4(\eta) = 0, \quad \hat{\ell}_5(\eta) = 0, \]

which corresponds to a complementary root, yields the second part of 5.9(b). The unique root of height \(-3\) is a complementary root and so there is no equation to solve.

**Lemma 5.11.** We keep the notation of Lemma 5.9 Using the equivalent system of the differential polynomials \( h_{j_1}(\eta_m), \ldots, h_{j_l}(\eta_m) \) whose indices correspond to the complementary roots reduce to differential polynomials

\[ h_{j_1}(\eta) = \ell_{j_1}(\eta) + \hat{\eta}_{j_1}(\eta) \]

where \( \ell_{j_1}(\eta) \) and \( \hat{\eta}_{j_1}(\eta) \) have the following properties:

(a) The differential polynomial \( \ell_{j_1}(\eta) \in C\{\eta\} \) is homogeneous of degree one and of order \( |r(j_1)| \).

(b) The matrix formed by the coefficients of the linear system

\[ \ell_{j_1}(\eta) = 0, \ldots, \ell_{j_l}(\eta) = 0 \]

in the variables \( \eta_1, \ldots, \eta_l \) obtained by ignoring the derivatives is quadratic and has full rank.

(c) The derivatives appearing in \( \hat{\eta}_{j_1}(\eta) \in C\{\eta\} \) are at most of order \( |r(j_1) + 1| \) and each term is of degree greater than one.

**Proof.** We fix the following notation. For a height \( q = -1, \ldots, r(m) \) let \( j_{r_1} \) and \( j_{r_2} \) be the minimal and maximal index among the indices of the complementary roots \( \{j_1, \ldots, j_l\} \) such that \( q \leq r(j_{r_1}) = r(j_{r_2}) \) and let \( i_1 \) and \( k_1 \) be minimal and \( i_2 \) and \( k_2 \) be maximal such that \( r(i_1) = q = r(i_2) \) and \( r(k_1) = q - 1 = r(k_2) \). Furthermore in the following the index \( j \) (resp. \( i \) and \( k \)) runs between \( j_{r_1} \leq j \leq j_{r_2} \) (resp. \( i_1 \leq i \leq i_2 \) and \( k_1 \leq k \leq k_2 \)). If not otherwise stated we ignore the derivatives of the variables when we consider a linear system.

We prove by induction on the height \( q = -1, \ldots, r(m) \) that using the equations

\[ \eta_{q+1} = \ell_{q+1}(\eta) + \hat{\eta}_{q+1}(\eta), \ldots, \eta_m = \ell_m(\eta) + \hat{\eta}_m(\eta) \]

of Lemma 5.9 the differential polynomials \( h_{j}(\eta_m) \) reduce to

\[ h_{j_1}(\eta) = \ell_{j_1}(\eta) + \hat{\eta}_{j_1}(\eta) \]

where \( \ell_{j_1}(\eta) \) satisfy (a), \( \hat{\eta}_{j_1}(\eta) \) fulfill (c) and the linear system

\[ \ell_{k_1}(\eta) = 0, \ldots, \ell_{k_2}(\eta) = 0, \quad \ell_{j_1}(\eta) = 0, \ldots, \ell_{j_{r_2}}(\eta) = 0 \]

has full rank.

Let \( q = -1 \). In this case \( j_{r_1} = j_1 \) and \( k_1 = l + 1 \). We have the differential polynomials

\[ h_{j_1}(\eta_m) = \eta'_1 + \ell_{j_1}(\eta_m) + q_{j_1}(\eta), \]

where according to Lemma 5.7 the element \( q_{j_1}(\eta) \) lies in \( C[\eta_1, \ldots, \eta_l] \) and each of its terms is of degree greater than one and \( \ell_{j_1}(\eta_m) \) is homogeneous of degree one in the variables \( \eta_{q+1}, \ldots, \eta_k \). From Lemma 5.9 we obtain the equations

\[ \eta_{q+1} = \ell_{q+1}(\eta) + \hat{\eta}_{q+1}(\eta), \ldots, \eta_k = \ell_k(\eta) + \hat{\eta}_k(\eta) \]

where

\[ \ell_{q+1}(\eta) = 0, \ldots, \ell_k(\eta) = 0 \]
is a linear system in the variables $\eta'_1, \ldots, \eta'_{j_2-1}$ of full rank and each term of $\bar{p}_k(\eta)$ which lies in $C[\eta_1, \ldots, \eta]$ is of degree greater than one. We substitute the expressions of (14) into $\ell_j(\eta_m)$ and $q_j(\eta)$ of the right hand sides of (15) and obtain

$$h_j(\eta) = \eta'_j + \bar{\ell}_j(\eta_{j+1}), \ldots, \bar{\ell}_{k_2}(\eta) + \bar{\ell}_j(\bar{p}_{k_2}(\eta), \ldots, \bar{p}_{k_2}(\eta)) + q_j(\eta)$$

where $j_1 \leq j \leq j_{r_2}$. Since the variables $\eta'_1, \ldots, \eta'_{j_2}$ do not appear among the variables $\eta'_1, \ldots, \eta'_{j_1-1}$, we conclude that the linear system

$$\bar{\ell}_{j+1}(\eta) = 0, \ldots, \bar{\ell}_{k_2}(\eta) = 0, \bar{\ell}_j(\eta) = \eta'_j + \bar{\ell}_j(\bar{\ell}_{j+1}(\eta), \ldots, \bar{\ell}_{k_2}(\eta)) = 0$$

with $j_1 \leq j \leq j_{r_2}$ has full rank and all variables in $\bar{\ell}_j(\eta)$ have order one. Since $\ell_j(\eta_m)$ is linear in the variables $\eta_{k_1}, \ldots, \eta_{k_2}$ and $\bar{p}_k(\eta)$ as well as $q_j(\eta)$ are polynomials where each term is of degree greater than one, the same holds for

$$\tilde{p}_j(\eta) = \ell_j(\bar{p}_{k_1}(\eta), \ldots, \bar{p}_{k_2}(\eta)) + q_j(\eta)$$

with $j_1 \leq j \leq j_{r_2}$.

Let $q \leq -2$. We distinguish between the cases when there are complementary roots of height $q$ and when there are not. In the first case we have the equations

$$h_j(\eta_m) = \eta'_j + \bar{\ell}_j(\eta_m) + q_j(\eta_m)$$

with $j_1 \leq j \leq j_{r_2}$ where the non-linear parts $q_j(\eta_m)$ are as in Lemma 5.7 and the linear parts form the linear system

$$\ell_{j_1}(\eta_m) = 0, \ldots, \ell_{j_{r_2}}(\eta_m) = 0$$

of full rank in the variables $\eta_{k_1}, \ldots, \eta_{k_2}$. From Lemma 5.9 we obtain the two systems of equations $\eta_i = \tilde{\ell}_i(\eta) + \bar{p}_i(\eta)$ and $\eta_k = \tilde{\ell}_k(\eta) + \bar{p}_k(\eta)$ with $i_1 \leq i \leq i_2$ and $k_1 \leq k \leq k_2$. The corresponding linear systems

$$\tilde{\ell}_{i_1}(\eta) = 0, \ldots, \tilde{\ell}_{i_2}(\eta) = 0 \quad \text{and} \quad \tilde{\ell}_{k_1}(\eta) = 0, \ldots, \tilde{\ell}_{k_2}(\eta) = 0$$

in the variables $\eta_{l_1}^{(q+1)}, \ldots, \eta_{l_2}^{(q+1)}$ and $\eta_{l_1}^{(q)}, \ldots, \eta_{l_2}^{(q)}$ respectively have full rank where $j'$ is as $j$ in Lemma 5.9. We substitute the corresponding expressions among $\eta_{l_1}, \ldots, \eta_{l_2}$ into $\eta'_{j_1}, \ldots, \eta'_{j_2}$ and the expressions for $\eta_{k_1}, \ldots, \eta_{k_2}$ into $\ell_j(\eta_m)$ of the right hand sides of the equations in (15). We obtain

$$h_j(\eta_m) = \tilde{\ell}_j(\eta) + \bar{p}_j(\eta) + \ell_j(\tilde{\bar{p}}_k(\eta), \ldots, \tilde{\bar{p}}_k(\eta)) + q_j(\eta_m)$$

with $j_{r_1} \leq j \leq j_{r_2}$ where the homogeneous linear parts

$$\tilde{\ell}_j(\eta) = \tilde{\ell}_j(\eta) + \ell_j(\tilde{\bar{p}}_k(\eta), \ldots, \tilde{\bar{p}}_k(\eta))$$

are in variables $\eta_{l_1}^{(q)}, \ldots, \eta_{l_2}^{(q)}$. By induction assumption the linear system

$$\tilde{\ell}_{i_1}(\eta) = 0, \ldots, \tilde{\ell}_{i_2}(\eta) = 0, \tilde{\ell}_{j_1}(\eta) = 0, \ldots, \ell_{j_{r_1}-1}(\eta) = 0$$

has full rank and by Lemma 5.9(b) the linear system

$$\tilde{\ell}_{k_1}(\eta) = 0, \ldots, \tilde{\ell}_{k_2}(\eta) = 0$$

is equivalent to the subsystem

$$\tilde{\ell}_{i_1}(\eta) = 0, \ldots, \tilde{\ell}_{j_{r_1}-1}(\eta) = 0$$

of the system

$$\tilde{\ell}_{i_1}(\eta) = 0, \ldots, \tilde{\ell}_{i_2}(\eta) = 0$$

obtained by canceling the rows which correspond to complementary roots of height $q$. We conclude that the linear system

$$\tilde{\ell}_{k_1}(\eta) = 0, \ldots, \tilde{\ell}_{k_2}(\eta) = 0, \ell_{j_1}(\eta) = 0, \ldots, \ell_{j_{r_1}-1}(\eta) = 0, \tilde{\ell}_{j_1}(\eta) = 0, \ldots, \tilde{\ell}_{j_{r_2}}(\eta) = 0$$
has full rank. It follows from the order of the variables in \( \tilde{e}_i(\eta) \) and \( \tilde{e}_k(\eta) \) that the variables in \( \tilde{e}_j(\eta) \) have order \(|q|\). This proves (a) and (b).

We substitute the expression

\[
\eta = \eta_1, \ldots, \eta_6
\]

for the non-complementary roots into Lemma \[5.9\] into the non-linear parts

\[
\tilde{p}_j = \tilde{p}_j(\eta) + \tilde{e}_j(\eta) \tilde{p}_k(\eta) + q_j(\eta_m)
\]

with \( j_1 \leq j \leq j_2 \). The elements \( q_j(\eta_m) \) lie in \( C[\eta_1, \ldots, \eta_{1-1}] \eta_{1}, \ldots, \eta_{2} \) according to Lemma \[5.7\] and each term of \( q_j(\eta_m) \) is of degree greater than one and the derivatives appearing are at most of order one. Since the derivatives appearing in the expressions of \( \eta_1, \ldots, \eta_2 \) and of \( \eta_1, \ldots, \eta_{1-1} \) are of order \(|q| \) and \(|q + 2| \) respectively, the derivatives appearing in \( q_j(\eta_m) \) are at most of order \(|q + 1| \). The derivatives appearing in \( \tilde{p}_k(\eta) \) and \( \tilde{e}_j(\eta) \) are at most of order \(|q + 1| \) and \(|q + 2| \) respectively. Since \( \tilde{e}_j(\eta_m) \) are homogeneous of degree one in the variables \( \eta_{1}, \ldots, \eta_{3} \), we conclude that the derivatives appearing in \( \tilde{p}_j(\eta) \) are at most of order \(|q + 1| \).

Assume that we are now in the case when there are no complementary roots of height \( q \). By induction assumption we have that the linear system

\[
\tilde{e}_i(\eta) = 0, \ldots, \tilde{e}_k(\eta) = 0, \tilde{e}_j(\eta) = 0, \ldots, \tilde{e}_{j_2}(\eta) = 0
\]

has full rank. Since the two systems

\[
\tilde{e}_i(\eta) = 0, \ldots, \tilde{e}_k(\eta) = 0 \quad \text{and} \quad \tilde{e}_i(\eta) = 0, \ldots, \tilde{e}_k(\eta) = 0
\]

are equivalent by Lemma \[5.9(b)\], it follows that

\[
\tilde{e}_i(\eta) = 0, \ldots, \tilde{e}_k(\eta) = 0, \tilde{e}_j(\eta) = 0, \ldots, \tilde{e}_{j_2}(\eta) = 0
\]

also has full rank. \( \square \)

**Example 5.12.** We continue with the example for SL. From Example \[5.8\] we obtain that the coefficients in the linear representation of \( \ell_0(Y) \) of the basis elements for the non-complementary roots are

\[
h_3(\eta_6) = \eta_1, h_5(\eta_6) = \eta_2, h_6(\eta_6) = \eta_6
\]

(16) \( h_3(\eta_6) = \eta_1^2 + \eta_5 + q_3(\eta_6), h_5(\eta_6) = \eta_2^2 + \eta_5 + q_5(\eta_6), h_6(\eta_6) = \eta_6 + q_6(\eta_6) \).

Example \[5.10\] provides the expressions

\[
\eta_1 = \eta_1^2 + \tilde{p}_4(\eta), \quad \eta_5 = \eta_5 + \eta_1^2 + \tilde{p}_5(\eta), \quad \eta_6 = \eta_6^2 + \tilde{p}_6(\eta).
\]

We substitute these accordingly into the differential polynomials of \[110\] and obtain

\[
h_3(\eta) = \eta_1^2 + \eta_5 + \tilde{p}_3(\eta), \quad h_5(\eta) = \eta_2^2 + 2\eta_5 + \tilde{p}_5(\eta), \quad h_6(\eta) = \eta_6^2 + \tilde{p}_6(\eta)
\]

where

\[
\tilde{p}_3(\eta) = \tilde{p}_5(\eta) + q_3(\eta), \quad \tilde{p}_5(\eta) = \tilde{p}_5(\eta)^2 + \tilde{p}_6(\eta) + q_5(\eta), \quad \tilde{p}_6(\eta) = \tilde{p}_6(\eta)^2 + q_6(\eta).
\]

When we consider in the following a linear systems, we ignore the derivatives of the variables, that is we understand the systems to be in the respective variables \( \eta \).

We follow the argumentation of the proof of Lemma \[5.11\] to show that the linear system

\[
\tilde{e}_3(\eta) = \eta_3^2 + \eta_5 + \eta_1^2 = 0, \quad \tilde{e}_5(\eta) = \eta_5^2 + 2\eta_5 + \eta_6 = 0, \quad \tilde{e}_6(\eta) = \eta_6^2 + 2\eta_5 + \eta_6 = 0
\]

has full rank. The linear system

\[
\tilde{e}_4(\eta) = \eta_4, \quad \tilde{e}_5(\eta) = \eta_5^2 + \eta_6 = 0
\]

which corresponds to the roots of height \(-2\) has full rank. It follows that the linear system

\[
\tilde{e}_4(\eta) = 0, \quad \tilde{e}_5(\eta) = 0, \quad \tilde{e}_6(\eta) = \eta_3^2 + \eta_5 + \eta_6 = 0
\]
has full rank, since the variable \( \eta_3 \) only appears in \( \hat{\ell}_3(\eta) \). For the root of height \(-3\) the linear system \( \bar{\ell}_6(\eta) = \eta_1'' = 0 \) is equivalent to the subsystem \( \ell_4(\eta) = 0 \) of

\[
\ell_4(\eta) = 0, \quad \ell_5(\eta) = \eta_2'' + 2\eta_1'' = 0, \quad \ell_3(\eta) = \eta_3'' + \eta_2'' + \eta_1'' = 0.
\]

It follows that the linear system

\[
\ell_6(\eta) = 0, \quad \ell_5(\eta) = \eta_2'' + 2\eta_1'' = 0, \quad \ell_3(\eta) = \eta_3'' + \eta_2'' + \eta_1'' = 0
\]

has full rank, since \( \hat{\ell}_5(\eta) = 0 \) is obtained from adding \( \bar{\ell}_6(\eta) = 0 \) to \( \ell_5(\eta)'' = 0 \). Because the linear system \( \ell_6(\eta) = \eta_1''' = 0 \) is equal to the linear system \( \ell_6(\eta)' = 0 \) we obtain that the linear system

\[
\ell_6(\eta) = \eta_1''' = 0, \quad \ell_5(\eta) = \eta_2'' + 2\eta_1'' = 0, \quad \ell_3(\eta) = \eta_3'' + \eta_2'' + \eta_1'' = 0
\]

has full rank.

It is left to check that \( \tilde{p}_j(\eta) \) fulfills \( 5.11(c) \) for \( j = 3, 5, 6 \). Since all terms in \( \tilde{p}_3(\eta), \tilde{p}_5(\eta), \tilde{q}_3(\eta), \tilde{q}_5(\eta) \) and \( \tilde{q}_6(\eta) \) are of degree greater than one, the same holds for \( \tilde{p}_j(\eta) \). We determine the maximal order of the derivatives appearing in \( \tilde{p}_3(\eta) \). Since \( \tilde{p}_5(\eta) \) and \( \tilde{q}_3(\eta) \) are polynomials, so is \( \tilde{p}_3(\eta) \). The derivatives appearing in \( \tilde{p}_5(\eta) \) and \( \tilde{q}_3(\eta) \) are at most of order one and so the same holds for \( \tilde{p}_6(\eta) \). Finally the derivatives appearing in \( \tilde{q}_6(\eta) \) are at most of order two and therefore the derivatives appearing in \( \tilde{p}_6(\eta) \) are at most of order two.

**Theorem 5.13.** Let \( E = C(\eta)(z, y) \) and the elements \( h = (h_{j_1}(\eta), \ldots, h_{j_l}(\eta)) \) be as in Lemma 5.11 and 5.12. Then \( E \) is a Picard-Vessiot extension of \( C(h) \) for

\[
A_G(h) = A^l + \sum_{k=1}^{l} h_{jk}(\eta)X_{jk}
\]

with differential Galois group \( G(C) \). The differential field \( C(h) \) is a purely differential transcendental extension of \( C \) of degree \( l \).

**Proof.** We prove that the differential field extension \( C(h) \) of \( C \) is a purely differential transcendental extension of degree \( l \). Clearly \( C(\eta) \) is a purely differential transcendental extension of \( C \) of degree \( l \) and we have the tower of differential fields

\[ C \subset \mathbb{C}(h) \subset C(\eta). \]

From Lemma 5.11 we obtain that

\[
h_{j_1}(\eta) = \hat{\ell}_{j_1}(\eta) + \tilde{p}_{j_1}(\eta), \ldots, h_{j_l}(\eta) = \hat{\ell}_{j_l}(\eta) + \tilde{p}_{j_l}(\eta).
\]

Moreover the properties of \( \hat{\ell}_{j_1}(\eta), \ldots, \hat{\ell}_{j_l}(\eta) \) and \( \tilde{p}_{j_1}(\eta), \ldots, \tilde{p}_{j_l}(\eta) \) imply that after prolonging \( h_{j_1}(\eta), \ldots, h_{j_l}(\eta) \) to the order \( |r(j_i)| = |r(m)| \) the prolongations of \( \hat{\ell}_{j_1}(\eta), \ldots, \hat{\ell}_{j_l}(\eta) \) form a linear system in \( \eta_{i|r(m)|}, \ldots, \eta_{l|r(m)|} \) of full rank and the derivatives appearing in the terms in the prolongations of the non-linear parts \( \tilde{p}_{j_1}(\eta), \ldots, \tilde{p}_{j_l}(\eta) \) are at most of order \( |r(m)| + 1 \). We conclude that the transcendence degree of \( C(\eta) \) over \( C(h) \) is finite. It follows that its differential transcendence degree is zero and so the differential transcendence degree of \( C(h) \) over \( C \) is \( l \) by [9, 11, 9, Corollary 2]. Since the extension is differentially generated by \( l \) elements, it is a purely differential transcendental extension.

We show that \( E \) is Picard-Vessiot extension of \( C(h) \) and that its differential Galois group is \( G(C) \). Since \( E \) is a Picard-Vessiot extension of \( C(\eta) \) by Proposition 5.5 its field of constants is \( C \). Moreover, by construction

\[
Y = u((\eta_1, \ldots, \eta_l, f_{i+1}(\eta), \ldots, f_m(\eta)))u(\bar{w})t(z)u(y)
\]

is a fundamental solution matrix where \( f_{i+1}(\eta), \ldots, f_m(\eta) \) are as in Lemma 5.9. We need to show that the entries of \( Y \) generate \( E \) as field over \( C(h) \). Clearly \( E \) contains \( C(h)(Y_{ij}) \) and we need to prove the other inclusion. Since \( Y \) lies in \( G \) we
can compute its Bruhat decomposition over $C(h)(Yg)$. Using the uniqueness of the elements of the Bruhat decomposition we get that $\eta, f_{l+1}(\eta), \ldots, f_m(\eta), z$ and $y$ can be represented as rational expressions in the entries of $Y$. Since $C(h)(Yg)$ is a differential field it follows that it is equal to $E$ and so $E$ is generated as a field by the entries of $Y$ over $C(h)$. We conclude that $E$ is a Picard-Vessiot extension of $C(h)$. We proved above that $h$ are differentially algebraically independent over $C$ and so it follows from Proposition 5.3, Lemma 6.8 and Theorem 4.3 of [9] applied to $A_G(h)$ that the differential Galois group is $G(C)$. □

**Remark 5.14.** Theorem 5.13 shows that for the differential field $E = C(\eta_1)(z, y)$ and the group $G(C)$ a construction as explained in the introduction is possible. Let

$$Y = u((\eta_1, f_{l+1}(\eta), \ldots, m(\eta)))u(\bar{w})t(z)u(y)$$

and for $g \in G(C)$ consider the Bruhat decomposition of $Yg$. Let $(\bar{x}, \bar{z}, \bar{y})$ be the corresponding normal form coefficients. Then the map

$$\varphi_g : E \to E, \left((\eta_1, \ldots, \eta_l, f_{l+1}(\eta), \ldots, m(\eta)), z, y\right) \mapsto (\bar{x}, \bar{z}, \bar{y})$$

is a differential $C$-automorphism and the group $G(C)$ acts on $E$ by

$$G(C) \times E \to E, \ (g, x) \mapsto \varphi_g(x)$$

where the field of invariants is $E^G = C(h)$. Indeed, it follows from Theorem 5.13 that for $g \in G$ the map $\varphi_g$ is actually the Galois automorphism induced by $Yg$ of the Picard-Vessiot extension $E$ of $C(h)$, since the normal form coefficients of the Bruhat decompositions of $Y$ and $Yg$ can be written as rational expression in the entries of $Y$ and $Yg$ respectively.

**Example 5.15.** In this example we consider the group $G$ and its Lie algebra $\mathfrak{g}$ of type $G_2$. We compute the general extension field $E$ with its Liouvillian solutions and the invariants generating the fixed field. Let $\Delta = \{\tilde{\alpha}_1, \tilde{\alpha}_2\}$ be simple roots and we number the six negative roots according to Chapter 2 as

$$\beta_1 = -\tilde{\alpha}_1, \quad \beta_2 = -\tilde{\alpha}_2, \quad \beta_3 = -3\tilde{\alpha}_1 - \tilde{\alpha}_2, \quad \beta_4 = -2\tilde{\alpha}_1 - \tilde{\alpha}_2, \quad \beta_5 = -3\tilde{\alpha}_1 - 2\tilde{\alpha}_2, \quad \beta_6 = -3\tilde{\alpha}_1 - 2\tilde{\alpha}_2.$$

The complementary roots are $\{\beta_2, \beta_6\}$. We take the representation of $\mathfrak{g}$ as in [9], Section 11]. From the Chevalley basis presented there we compute the root group elements $u_1(\eta_1), \ldots, u_6(\eta_6)$ belonging to the negative roots for indeterminates $\eta_6 = (\eta_1, \ldots, \eta_6)$ and the torus elements $t_1(z_1), t_2(z_2)$ for indeterminates $z = (z_1, z_2)$. For the simple reflection $w_{\tilde{\alpha}_1}$ and $w_{\tilde{\alpha}_2}$ we choose the representatives

$$n(w_{\tilde{\alpha}_1}) = -E_{11} + E_{72} - E_{63} - E_{54} + E_{45} - E_{36} - E_{27}$$

and obtain the representative $n(\bar{w}) = (n(w_{\tilde{\alpha}_1})n(w_{\tilde{\alpha}_2}))^3$ for the longest Weyl group element $\bar{w}$. With further indeterminates $y = (y_1, \ldots, y_6)$ we define

$$Y = u_1(\eta_1) \cdots u_6(\eta_6)n(\bar{w})t_1(z_1)t_2(z_2)u_1(y_1) \cdots u_6(y_6)$$

and compute its logarithmic derivative

$$\ell \delta(Y) = \sum_{i=1}^{6} a_i X_i + d_1 H_1 + d_2 H_2 + \sum_{i=1}^{6} a_{-\beta_i} X_{-\beta_i}$$

where the coefficients are differential rational expressions in the indeterminates $\eta_6, z$ and $y$. In [9], Section 11] the parameter differential equation with Galois group $G$ is defined by the matrix $A_0^+ + t_1 X_2 + t_2 X_6$ with parameters $t_1, t_2$ and so we
determine in the following \( \eta_6 \), \( z \) and \( y \) such that the logarithmic derivative of \( Y \) satisfies

\[
\ell \delta(Y) = A_+^\circ + h_2(\eta)X_2 + h_6(\eta)X_6
\]

where \( \eta = (\eta_1, \eta_2) \). In a first step we require that

\[
a_{-\beta_1} = 0, \ldots, a_{-\beta_3} = 0, a_{-\beta_4} = 1, a_{-\beta_5} = 1, d_1 = 0, d_2 = 0.
\]

It is possible to solve these equations successively for \( y_6, \ldots, y_1 \) and \( z_1/z_1, z_2/z_2 \). Indeed, one can check that in each step we can solve the equation for the first derivative or the logarithmic derivative of the corresponding variable and substitute the obtained expression in the remaining equations. This process yields the solutions

\[
z_1 = e\int \eta_1, \quad z_2 = e\int -\eta_2, \quad y_1 = \int \frac{-1}{z_1^2z_2}, \quad y_2 = \int -z_1^3z_2, \quad y_3 = \int y_1y_2',
\]

\[
y_4 = \int -3y_1y_5 - 3y_2y_5 + 2y_5, \quad y_5 = \int -2y_1y_5' + y_2y_5', \quad y_6 = \int -3y_3y_5 - y_2y_6.
\]

The differential field extension \( E = C(\eta)(z, y) \) of \( C(\eta) \) is a Picard-Vessiot extensions with differential Galois group \( B^- \) according to Proposition 5.5 and it is our general extension field. It is left to determine the two generators of the field of invariants. If we substitute these solutions into the logarithmic derivative of \( Y \) we obtain

\[
\ell \delta(Y) = A_+^\circ + h_1(\eta_6)X_1 + \cdots + h_6(\eta_6)X_6
\]

with

\[
\begin{align*}
\eta_1 &= \eta_2 + 3\eta_3 + \eta_4 \quad h_1(\eta_6) = 0 \\
\eta_2 &= \eta_2 + 3\eta_3 + \eta_2 - 3\eta_1 \eta_2 \quad h_2(\eta_6) = 0 \\
\eta_3 &= \eta_3 - h_2(\eta_6)\eta_1 - 2\eta_{-1} - \eta_3 \eta_3 \quad h_3(\eta_6) = 0 \\
\eta_4 &= \eta_4 + \eta_1 \eta_4 - \eta_4 + \eta_3^2 + h_2(\eta_6)\eta_3^2 \\
\eta_5 &= \eta_4 - h_6(\eta_6)\eta_3^3 - \eta_6 + \eta_3(3\eta_1 - \eta_2) \\
\eta_6 &= \eta_6 + h_2(\eta_6)(\eta_3^2\eta_3 - 3\eta_3^2\eta_1) + \eta_2\eta_6 + 3\eta_2^2 - 2\eta_3^2
\end{align*}
\]

Next we require that the coefficients of basis elements corresponding to the non-complementary roots vanish. This leads to the system of equations \( h_1(\eta_6) = 0 \), \( h_3(\eta_6) = 0 \), \( h_4(\eta_6) = 0 \) and \( h_5(\eta_6) = 0 \). These equations depend linearly on the variables \( \eta_1, \eta_3, \eta_4 \) and \( \eta_6 \). Successively we solve the equations for the corresponding variables and substitute the obtained expressions for the variables into the remaining equations. This process yields the equations

\[
\eta_3 = f_3(\eta), \quad \eta_5 = f_5(\eta), \quad \eta_4 = f_4(\eta), \quad \eta_6 = f_6(\eta)
\]

and the invariants

\[
\begin{align*}
h_1(\eta) &= \eta_2 + 3(\eta_1' + \eta_2') + \eta_2' - 3\eta_1 \eta_2 \\
h_6(\eta) &= -1/4(2\eta_2'' + 4\eta_1\eta_1'' + 6\eta_2' + (4\eta_1^2 - 2h_1)\eta_1'' - 2\eta_1h_1' + 2\eta_1^2 + 2h_1^2)\eta_2' \\
&\quad -2\eta_1(5) - 10\eta_1(4) + (-2\eta_1' + 2\eta_2' + 6\eta_2') - 16\eta_1^2 + 2h_1)\eta_1'' - 19\eta_1^2 \\
&\quad + (6\eta_1' - 90\eta_1'\eta_2' + 4\eta_1\eta_2' - 12h_1\eta_2')\eta_1' - 18\eta_1^3 + (6\eta_2^2 - 18\eta_1\eta_2' + 2h_1)\eta_1'' + 6h_1(1)\eta_1'' \\
&\quad + (6h_1' - 12\eta_1')\eta_1'\eta_2' + 2h_1)\eta_1'' + 12\eta_1\eta_1' + 4\eta_1^2\eta_2' + (-2\eta_1\eta_2' + 6\eta_1\eta_2 - 2\eta_1^2)h_1' \\
&\quad + (2\eta_1^2 + 2h_1\eta_2')\eta_1' + (-6\eta_1'' - 6h_1\eta_1')\eta_2 + 5\eta_1' + 6h_1h_1' - 3h_1^2)\eta_1
\end{align*}
\]

where we write shortly \( h_1 \) for \( h_1(\eta) \).
6. Full G-Primitive Picard-Vessiot Extensions and the General Extension Field

Let $E$ be a full $G$-primitive Picard-Vessiot extension of a differential field $F$ with defining matrix $A \in \mathfrak{g}(F)$. In this chapter we pursue the question whether there exist a specialization of the parameters

$$h = (h_{j_1}, \ldots, h_{j_k}) \mapsto f = (f_1, \ldots, f_l)$$

to elements of $F$ such that $E$ is a Picard-Vessiot extension of $F$ for the specialized matrix $A_G(f)$. If $A$ is gauge equivalent to some $A_G(f)$ by an element of $G(F)$, then the corresponding specialization has trivially the desired properties. We develop in this chapter a criterion depending on the differential subfield $F(x)$ of $E$, where $x$ are the first $m$ normal form coefficients of a fundamental matrix for $A$, to decide whether the orbit of $A$ under gauge transformation intersects the set of matrices of shape $A_G(f)$.

The orbit of a matrix in $\mathfrak{g}(F)$ under $G(F)$ depends on the differential field $F$. In the special case when $F$ is equal to the field constants $C$, the gauge transformation reduces to the adjoint action. The orbit structure in the algebraic case was studied by B. Kostant in [7]. According to [7, Remark 19’ and Theorem 8] the orbits of the matrices of shape $A_G(f)$ are in bijective correspondence to the orbits of maximal dimension in $\mathfrak{g}(F)$. Moreover in the special case when $F$ is differentially closed the logarithmic derivative from $G(F)$ to $\mathfrak{g}(F)$ is surjective and so any two matrices are gauge equivalent, that is there is only a single orbit. In both special cases we have a description of all matrices which are gauge equivalent to matrices of shape $A_G(f)$ and they show how the orbit structure varies with the differential field $F$.

For an algebraically closed differential field $F$ the following lemma shows that matrices of a specific subset of $\mathfrak{g}(F)$ are gauge equivalent to matrices of shape $A_G(f)$.

**Lemma 6.1** (Transformation Lemma). Let $F = \overline{F}$ and $A \in A_+^g(s) + B^-(F)$ with nonzero $s = (s_1, \ldots, s_l)$ in $F$. Then $A$ is gauge equivalent by an element of $B^-(F)$ to a matrix of shape $A_G(f)$.

For proof of Lemma 6.1 see [9] Lemma 6.8]

**Lemma 6.2.** Let $A$ be an element of an orbit of a matrix of shape $A_G(f)$. Then there are $w \in W$, $u \in (U^-)'_w$ and nonzero $s = (s_1, \ldots, s_l)$ of $F$ such that $A$ is an element of the plane

$$\text{Ad}(un(w))(A_+^g(s) + B^-) + \ell \delta(u)$$

where $n(u)$ is a representative for $w$ and $(U^-)'_w$ is as in Theorem 2.2 with the roles of the positive and negative roots interchanged.

**Proof.** Since $A$ is an element of the orbit of $A_G(f)$, there is $g \in G(F)$ such that

$$A = \text{Ad}(g)(A_G(f)) + \ell \delta(g).$$

By Theorem 2.2 where we interchange the role of the positive and negative roots, there exists $w \in W$, $u \in (U^-)'_w(F)$, $t \in T(F)$ and $\bar{u} \in U^-(F)$ such that $g = un(w)t\bar{u}$ with $n(w)$ a representative of $w$ in the normalizer of $T$. It follows that

$$A = \text{Ad}(un(w))(\text{Ad}(t\bar{u})(A_G(f)) + \ell \delta(t\bar{u})) + \ell \delta(un(w)).$$

Finally Remark 6.3 and Proposition 6.3 imply that the gauge-transform of $A_G(f)$ with $t\bar{u}$ lies in the plane $A_+^g(s) + B^-(F)$ for some nonzero $s = (s_1, \ldots, s_l)$ of $F$. □

**Lemma 6.3.** Let $w \in W$ and fix a representative $n(w)$ in the normalizer of $T$. Moreover let $C(x, z, y)$ be the rational function field in the indeterminates $x = (x_1, \ldots, x_m)$, $z = (z_1, \ldots, z_l)$ and $y = (y_1, \ldots, y_m)$. Then there are $\bar{x} = \ldots$
\( (\bar{x}_1, \ldots, \bar{x}_m) \) in \( C(x) \) and \( \bar{v} = (\bar{v}_1, \ldots, \bar{v}_m) \) and nonzero \( \bar{z} = (\bar{z}_1, \ldots, \bar{z}_l) \) in \( C(x, z) \) such that
\[
n(w) \cdot (u(x)n(\bar{w})t(z)u(y)) = u(\bar{x})n(\bar{w})t(\bar{z})u(\bar{v})u(y).
\]

**Proof.** For a simple root \( \bar{\alpha}_i \) let \( G_i \) be the centralizer of \( \ker(\bar{\alpha}_i) \subseteq T \) in \( G \). Then \( G_i \) is of semisimple rank 1 and it is generated according to [1] Chapter 26.2, Corollary B] by the subgroups \( U_{\bar{\alpha}_i}, U_{-\bar{\alpha}_i}, \) and \( T \). We prove that a specific relation holds in \( G_i \). Let
\[
\bar{u}_\alpha(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \bar{u}_{-\alpha}(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \quad \bar{t}(x) = \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix}, \quad \bar{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]
be the standard representation of \( SL_2(C) \) where \( \alpha \) is the unique simple root and \( \bar{n} \) a representative of the unique nontrivial Weyl group element. Then a simple computation shows that
\[
\bar{n}\bar{u}_{-\alpha}(x) = \bar{u}_{-\alpha}(-1/x)\bar{t}(x)\bar{u}_\alpha(1/x).
\]
Applying the injective group homomorphism from \( SL_2(C) \) to \( G_i \) which is defined by
\[
\bar{u}_\alpha(x) \mapsto u_{\alpha_i}(x), \quad \bar{u}_{-\alpha}(x) \mapsto u_{-\alpha_i}(x), \quad \bar{t}(x) \mapsto t_i(x), \quad \bar{n} \mapsto n(w_{\alpha_i})
\]
to the last equation we obtain that the corresponding relation is also true in \( G_i \).

For a simple root \( \bar{\alpha}_i \) let \( \Psi = \Phi^- \setminus \{-\bar{\alpha}_i\} \). By [1] Chapter 14.12, (2) (iii) the root groups \( U_\beta \) with \( \beta \in \Psi \) directly span a subgroup \( U_\Psi \) which is normalized by \( G_i \) and satisfies \( U^- = U_{-\bar{\alpha}_i}U_{\bar{\alpha}_i} \). We conclude that
\[
n(w_{\bar{\alpha}_i})u(x) = n(w_{\bar{\alpha}_i})u_{\bar{\alpha}_i}(\bar{x}_i)u_{-\bar{\alpha}_i}(\bar{x}_i) = u_{\bar{\alpha}_i}(\bar{x}_i)u(w_{\bar{\alpha}_i})u_{-\bar{\alpha}_i}(\bar{x}_i)
\]
where \( \bar{x}_i \) and \( \bar{x}_i = (\bar{x}_1, \ldots, \bar{x}_{i-1}, \bar{x}_{i+1}, \ldots, \bar{x}_m) \) are elements in \( C(x) \) and
\[
u_{\bar{\alpha}_i}(\bar{x}_i) = u_1(\bar{x}_i) \cdots u_{i-1}(\bar{x}_{i-1})u_{i+1}(\bar{x}_{i+1}) \cdots u_m(\bar{x}_m).
\]
Since \( n(w_{\bar{\alpha}_i})u_{-\bar{\alpha}_i}(\bar{x}_i) \in G_i \) we obtain from the above relation that
\[
u_{\bar{\alpha}_i}(\bar{x}_i)u_{-\bar{\alpha}_i}(\bar{x}_i) = nu_{\bar{\alpha}_i}(\bar{x}_i)u_{-\bar{\alpha}_i}(-1/\bar{x}_i)t_i(\bar{x}_i)u_{\bar{\alpha}_i}(1/\bar{x}_i) = u(\bar{x})t_i(\bar{x}_i)u_{\bar{\alpha}_i}(1/\bar{x}_i)
\]
with suitable \( \bar{x} \) in \( C(x) \). The torus is normalized by \( n(\bar{w}) \) and the adjoint action of \( n(\bar{w}) \) maps the positive root groups bijectively to the negative root groups. So we have
\[
u(\bar{x})t_i(\bar{x}_i)u_{\bar{\alpha}_i}(1/\bar{x}_i)nu(\bar{w})t(\bar{z})u(y) = u(\bar{x})n(\bar{w})t(\bar{z})u(\bar{v})t(\bar{z})u(y)
\]
with nonzero \( \bar{z} \) and \( \bar{v} \) in \( C(x) \). Finally, since the torus normalizes \( U^- \), we obtain
\[
u(\bar{x})n(\bar{w})t(\bar{z})u(\bar{v})t(\bar{z})u(y) = u(\bar{x})n(\bar{w})t(\bar{z})u(\bar{v})u(y)
\]
with nonzero \( \bar{z} \) and \( \bar{v} \) in \( C(x, z) \) which completes the proof for \( w_{\bar{\alpha}_i} \in W \).

Now let \( w \in W \) be arbitrary. By [1] Theorem 10.3] it is the product of simple reflections and so the statement of the lemma follows from applying successively the above result.

\[ \square \]

**Corollary 6.4.** Let \( E/F \) be a full \( G \)-primitive Picard-Vessiot extension with matrix \( Y \in G(E) \) and normal form decomposition
\[
Y = u(x)n(\bar{w})t(z)u(y).
\]
Moreover let \( u \in U^-(F) \) and \( w \in W \) with representative \( n(w) \). Then there are \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_m) \) in \( F(x) \) and nonzero \( \bar{z} = (\bar{z}_1, \ldots, \bar{z}_l) \) and \( \bar{v} = (\bar{v}_1, \ldots, \bar{v}_l) \) in \( F(x, z) \) such that
\[
n(w)u \cdot (u(x)n(\bar{w})t(z)u(y)) = u(\bar{x})n(\bar{w})t(\bar{z})u(\bar{v})u(y).
\]
Proof. Since \( u \) and \( u(x) \) are elements of \( U^-(F(x)) \), there are \( \hat{x} = (\hat{x}_1, \ldots, \hat{x}_m) \) in \( F(x) \) such that \( u \cdot u(x) = u(\hat{x}) \). Then the statement follows from applying Lemma \ref{lem:2.2} to

\[
 n(w)u(\hat{x})n(\hat{w})t(z)u(y). 
\]

\[\square\]

**Remark 6.5.** Let \( E \) be a full \( G \)-primitive Picard-Vessiot extension of \( F \) with normal form decomposition

\[
 Y = u(x)n(\hat{w})t(z)u(y). 
\]

The fundamental theorem yields that \( E \) is a Picard-Vessiot extension of \( E U^- \) with differential Galois group \( U^-(C) \) and it follows from Proposition \ref{prop:6.4} that \( E U^- \) is \( F(x, z) \). Since \( u(x)n(\hat{w})t(z) \) lies in \( G(F(x, z)) \), the matrix \( u(y) \) is a fundamental matrix for the extension \( E \) of \( F(x, z) \). Clearly the defining matrix \( \delta(y)u(y) \) for the extension lies in the subalgebra \( u^-(F(x, z)) \). We call \( u(y) \) the unipotent part of the normal form for \( Y \).

**Definition 6.6.** A gauge transformation of the logarithmic derivative of the unipotent part by an element \( u(\tilde{v}) \in U^-(F(x, z)) \) as in Corollary \ref{cor:6.4} is called an equivariant transformation of the unipotent part.

**Theorem 6.7.** Let \( E \) be a full \( G \)-primitive Picard-Vessiot extension of \( F \) with normal form decomposition

\[
 Y = u(x)n(\hat{w})t(z)u(y). 
\]

Then \( \delta(Y) \) is gauge equivalent by an element of \( G(F) \) to a matrix of shape \( A_G(f) \) if and only if there is an equivariant transformation of the unipotent part to a principal nilpotent matrix in normal form \( A_0(s) \) with \( s = (s_1, \ldots, s_l) \) in \( F(x, z) \).

**Proof.** First assume that there is \( g \in G(F) \) and \( f = (f_1, \ldots, f_m) \) in \( F \) such that

\[
 \delta(Y) = Ad(g)(A_G(f)) + \delta(g). 
\]

We conclude with Theorem \ref{thm:2.2} that there are \( w \in W, u \in U^- \) and nonzero \( \tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_l) \) in \( F \) such that

\[
 Ad(n(w)u)((\delta(Y)) + \delta(n(w)u) = \tilde{A} \in A_0^+(\tilde{s}) + b^-(F) 
\]

where \( n(w) \) is a representative of \( w \). The last equation implies that \( n(w)uY \) is a fundamental solution matrix for \( \tilde{A} \). Moreover from Corollary \ref{cor:6.4} we obtain that there are \( \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_m) \) in \( F(x) \) and nonzero \( \tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_l) \) and \( \tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_l) \) in \( F(x, z) \) such that

\[
 n(w)uY = (n(w)u) \cdot (u(x)n(\tilde{w})t(z)u(y)) = u(\tilde{x})n(\tilde{w})t(\tilde{z})u(\tilde{v})u(y) 
\]

and so \( u(\tilde{v}) \) is an equivariant transformation of the unipotent part \( u(y) \). Further the logarithmic derivative of the fundamental matrix \( n(w)uY \) computes as

\[
 \tilde{A} = \delta(\tilde{u}(\tilde{x})) + Ad(\tilde{u}(\tilde{x})n(\tilde{w}))((\delta(\tilde{t}(\tilde{x})) + Ad(\tilde{u}(\tilde{x})n(\tilde{w})t(\tilde{z}))((\delta(\tilde{u}(\tilde{v}))u(y))). 
\]

The first two summands lie in \( b^- \). Since \( \tilde{w} \) sends the negative roots bijectively to the positive roots and in particular the negative simple roots to the simple roots, we conclude that there are \( s = (s_1, \ldots, s_l) \) in \( F(x, z) \) such that

\[
 \delta(u(\tilde{v})u(y)) = A_0^-(s). 
\]

Hence the logarithmic derivative of \( u(y) \) is gauge equivalent by \( u(\tilde{v}) \) to a principal nilpotent matrix in normal form.

Next we assume that there is an equivariant transformation of the logarithmic derivative of \( u(y) \) to a matrix of shape \( A_0^-(s) \) with nonzero \( s = (s_1, \ldots, s_l) \) in
The logarithmic derivative of the right hand side of the last equation is
\[
\ell \delta(\mathbf{u}(\tilde{v})) + \text{Ad}(\mathbf{u}(\tilde{x})n(\tilde{w})t(\tilde{z})) + \text{Ad}(\mathbf{u}(\tilde{x})n(\tilde{w})t(\tilde{z})) = \mathbf{u}(\tilde{x})n(\tilde{w})t(\tilde{z})u(\tilde{v})u(\mathbf{y})
\]
and that
\[
\ell \delta(\mathbf{u}(\tilde{v}))u(\mathbf{y}) = A_0^-(s).
\]
As above the first two summands lie in the subalgebra \( b^- \). Since the logarithmic derivative of \( u(\tilde{v})u(\mathbf{y}) \) is \( A_0^-(s) \), it is send by the adjoint action with \( n(\tilde{w})t(\tilde{z}) \) to a matrix of shape \( A_0^+(\tilde{s}) \) with nonzero \( \tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_l) \) in \( F(x, z) \). Further the adjoint action with \( u(\tilde{x}) \) maps \( A_0^+(\tilde{s}) \) to the plane \( A_0^+(\tilde{s}) + b^- \) and so combining all results we get that the logarithmic derivative of \( n(\tilde{w})u\mathbf{Y} \) lies in the plane \( A_0^+(\tilde{s}) + b^-(F) \) and that \( \tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_l) \) are actually in \( F \). The statement then follows from applying Lemma 6.1.

**Definition 6.8.** A full \( G \)-primitive Picard-Vessiot extension such that there is an equivariant transformation of the unipotent part to a principal nilpotent matrix in normal form is called a full \( G \)-primitive Picard-Vessiot extension with normalisable unipotent part.

**Theorem 6.9.** The Picard-Vessiot extension \( C(\eta)(z, y) \) of \( C(h) \) and the defining matrix \( A_G(h) \) of Theorem 5.13 are generic for every full \( G \)-primitive Picard-Vessiot extension \( L \) of an algebraically closed differential field \( F \) with normalisable unipotent part in the following sense:

(a) There exists a specialization \( \sigma : C(h) \to F \) such that \( L \) is a Picard-Vessiot extension of \( F \) for \( A_G(\sigma(h)) \) and the defining matrix of \( L \) is gauge equivalent to \( A_G(\sigma(h)) \).

(b) There exists a specialization \( \sigma : C(\eta)[z, z^{-1}, y] \to L \) such that
\[
L = F(\sigma(Y))
\]
where \( Y \) is the fundamental matrix for \( A_G(h) \).

(c) For every specialization \( \sigma : C(h) \to F \) the differential Galois group of a Picard-Vessiot extension for \( A_G(\sigma(h)) \) is a subgroup of \( G(C) \).

**Proof.** Let \( \tilde{Y} \in G(L) \) be a fundamental solution matrix for the extension \( L \) of \( F \). Since the extension has a normalisable unipotent part and \( F \) is algebraically closed it follows from Theorem 6.7 that there are \( g \in G(F) \) and \( f = (f_1, \ldots, f_l) \) in \( F \) such that \( \ell \delta(\tilde{Y}) \) can be gauge transformed by \( g \) to \( A_G(f) \). This means that \( L/F \) is a full \( G \)-primitive Picard-Vessiot extension for \( A_G(f) \) with fundamental solution matrix \( g\tilde{Y} \in G(L) \). Furthermore we have a normal form decomposition
\[
g\tilde{Y} = u(\tilde{x}_m)n(\tilde{w})t(\tilde{z})u(\tilde{y})
\]
where \( \tilde{x}_m = (\tilde{x}_1, \ldots, \tilde{x}_m) \), \( \tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_l) \) and \( \tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_m) \) in \( L \) are the normal form coefficients. For the first \( l \) entries \( \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_l) \) in \( \tilde{x}_m \) we define the specialization
\[
\sigma : C(\eta) \to L, \eta \mapsto \tilde{x}.
\]
Since the logarithmic derivative of \( g\tilde{Y} \) is \( A_G(f) \) it follows from Theorem 5.13 that \( h_{1i}(\tilde{x}) = f_i \) and so the restriction
\[
\sigma : C(\eta) : C(h) \to F, h \mapsto f
\]
satisfies statement (a). Furthermore by the same argument we have that
\[
\tilde{x}_{l+1} = f_{l+1}(\tilde{x}), \ldots, \tilde{x}_m = f_m(\tilde{x})
\]
and that $\ell(t(\bar{z})u(\bar{y})) = A_L(\bar{x})$. Thus we can extend $\sigma$ to a specialization

$$\sigma : C\{\eta\}[z, z^{-1}, y] \to L, \ (\eta, z, y) \mapsto (\bar{x}, \bar{z}, \bar{y})$$

which satisfies

$$\sigma(Y) = u(\sigma(\eta, f_{t+1}(\eta), \ldots, f_m(\eta))) n(\bar{w}) t(\sigma(z)) u(\sigma(y)) = u(\bar{x}) n(\bar{w}) t(\bar{z}) u(\bar{y}).$$

This shows the second statement of the theorem. Finally the third statement follows from Poposition 1.31, since for every specialization $\sigma : C\{h\} \to F$ the matrix $A_G(\sigma(h))$ lies in $g$.

\[ \square \]

**Remark 6.10.** The proof of Theorem 6.9 shows that every full $G$-primitive Picard-Vessiot extension with normalisable unipotent part is determined by $l = \text{rank}(G)$ many elements of $L$. More precisely, the elements $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_l)$ differentially generate the intermediate extension $L/B^-$ and, since $L/F$ has normalisable unipotent part, they determine the Liouvillian extension $L/L_B$ by $A_L(\bar{x})$.

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