STATIONARY COCYCLES FOR THE CORNER GROWTH MODEL

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Abstract. We study the directed last-passage percolation model on the planar integer lattice with nearest-neighbor steps and general i.i.d. weights on the vertices, outside the class of exactly solvable models. Stationary cocycles are constructed for this percolation model from queueing fixed points. These cocycles serve as boundary conditions for stationary last-passage percolation, define solutions to variational formulas that characterize limit shapes, and yield new results for Busemann functions, geodesics and the competition interface.

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1. Introduction

We study nearest-neighbor directed last-passage percolation (LPP) on the lattice \( \mathbb{Z}^2 \), also called the corner growth model. Random i.i.d. weights \( \{\omega_x\}_{x \in \mathbb{Z}^2} \) are used to define last-passage times \( G_{x,y} \) between lattice points \( x \leq y \) in \( \mathbb{Z}^2 \) by

\[
G_{x,y} = \max_x \sum_{k=0}^{n-1} \omega_{x_k}
\]

where the maximum is over paths \( x, = \{x = x_0, x_1, \ldots, x_n = y\} \) that satisfy \( x_{k+1} - x_k \in \{e_1, e_2\} \) (up-right paths).

When \( \omega_x \geq 0 \) this defines a growth model in the first quadrant \( \mathbb{Z}^2_+ \). Initially the growing cluster is empty. The origin joins the cluster at time \( \omega_0 \). After both \( x - e_1 \) and \( x - e_2 \) have joined the cluster, point \( x \) waits time \( \omega_x \) to join. (However, if \( x \) is on the boundary of \( \mathbb{Z}^2_+ \), only one of \( x - e_1 \) and \( x - e_2 \) is required to have joined.) The cluster at time \( t \) is \( \mathcal{A}_t = \{x \in \mathbb{Z}^2_+ : G_{0,x} + \omega_x \leq t\} \). Our convention to exclude the last weight \( \omega_{x_n} \) in (1.1) forces the clumsy addition of \( \omega_x \) in the definition of \( \mathcal{A}_t \), but is convenient for other purposes.

The interest is in the large-scale behavior of the model. This begins with the deterministic limit \( \lim_{n \to \infty} n^{-1} G_{0,\lfloor n\xi \rfloor} \) for \( \xi \in \mathbb{R}_2^+ \), the fluctuations of \( G_{0,\lfloor n\xi \rfloor} \), and the behavior of the maximizing paths in (1.1) called geodesics. Closely related are the Busemann functions that are limits of gradients \( G_{x,v_n} - G_{y,v_n} \) as \( v_n \to \infty \) in a particular direction and the competition interface between subtrees of the geodesic tree. To see how Busemann functions connect with geodesics, note that by (1.1) the following identity holds along any geodesic \( x \), from \( u \) to \( v_n \):

\[
\omega_{x_i} = \min(G_{x_i,v_n} - G_{x_i+e_1,v_n}, G_{x_i,v_n} - G_{x_i+e_2,v_n}).
\]

Busemann functions arise also in a limiting description of the \( G_{x,y} \) process locally around a point \( v_n \to \infty \). Take a finite subset \( \mathcal{V} \) of \( \mathbb{Z}^2 \). A natural expectation is that the vector \( \{G_{0,v_n-u} - G_{0,v_n} : u \in \mathcal{V}\} \) converges in distribution as \( v_n \to \infty \) in a particular direction. A shift by \( -v_n \) and reflection \( \omega_x \mapsto -\omega_x \) turn this vector into \( \{G_{u,v_n} - G_{0,v_n} : u \in \mathcal{V}\} \). For this last collection of random gradients we can expect almost sure convergence, in particular if the geodesics from 0 and \( u \) to \( v_n \) coalesce eventually. These types of results will be developed in the paper.

Here are some particulars of what follows, in relation to past work.

In [14] we derived variational formulas for the point-to-point limit \( g_{pp}(\xi) \) and its point-to-line counterpart (introduced in Section 2) and developed a solution ansatz for these variational formulas in terms of stationary cocycles. In the present paper we construct these cocycles for the planar corner growth model with general i.i.d. weights bounded from below, subject to a moment bound. This construction comes from the fixed points of the associated queuing operator. The existence of these fixed points was proved by Mairesse and Prabhakar [23]. These cocycles are constructed on an extended space of weights. The Markov process analogy is a simultaneous construction of processes for all invariant distributions, coupled by common Poisson clocks that drive the evolution. The i.i.d. weights \( \omega \) are the analogue of the clocks and the cocycles the analogues of the initial state variables. With the help of the cocycles we establish new results for Busemann functions and directional geodesics for the corner growth model.
A related recent result is Krishnan’s [20, Theorem 1.5] variational formula for the time constant of first passage bond percolation. His formula is analogous to our (2.15).

Under some moment assumptions on the weights, the corner growth model is expected to lie in the Kardar-Parisi-Zhang (KPZ) universality class. (For a review of KPZ universality see [7].) The fluctuations of $G_{0,\lfloor n\xi \rfloor}$ are expected to have order of magnitude $n^{1/3}$ and limit distributions from random matrix theory. When the weights have exponential or geometric distribution the model is exactly solvable. In these cases the cocycles mentioned above have explicit product form distributions and last-passage values can be described with the Robinson-Schensted-Knuth correspondence and determinantal point processes. These techniques allow the derivation of exact fluctuation exponents and limit distributions [3, 18, 19]. The present paper can be seen as an attempt to begin development of techniques for studying the corner growth model beyond the exactly solvable cases.

On Busemann functions and geodesics, past milestones are the first-passage percolation results of Newman et al. summarized in [26], the applications of his techniques to the exactly solvable exponential corner growth model by Ferrari and Pimentel [13], and the recent improvements to [26] by Damron and Hanson [9]. Coupier [8] further sharpened the results for the exponential corner growth model. Stationary cocycles have not been developed for first-passage percolation. [26] utilized a global curvature assumption to derive properties of geodesics, and then the existence of Busemann functions. [9] began with a weak limit of Busemann functions from which properties of geodesics follow.

In our setting everything flows from the cocycles, both almost sure existence of Busemann functions and properties of geodesics. With a cocycle appropriately coupled to the weights $\omega$, geodesics can be defined locally in a constructive manner, simply by following minimal gradients of the cocycle.

The role of the regularity of the function $g_{pp}$ in our paper needs to be explained. Presently it is expected but not yet proved that under our assumptions (i.i.d. weights with some moment hypothesis) $g_{pp}$ is differentiable and, if $\omega_0$ has a continuous distribution, strictly concave. Our development of the cocycles and their consequences for Busemann functions, geodesics and the competition interface by and large do not rely on any regularity assumptions. Instead the results are developed in a general manner so that points of non-differentiability are allowed, as well as flat segments even if $\omega_0$ has a continuous distribution. After these fundamental but at times technical results are in place, we can invoke regularity assumptions to state cleaner corollaries where the underlying cocycles and their extended space do not appear. We put these tidy results at the front of the paper in Section 2. The real work begins after that. The point we wish to emphasize is that no unrealistic assumptions are made and we expect future work to verify the regularity assumptions that appear in this paper.

**Organization of the paper.** Section 2 describes the corner growth model and the main results of the paper. These results are the cleanest ones stated under assumptions on the regularity of the limit function $g_{pp}(\xi)$. The properties we use as hypotheses are expected to be true but they are not presently known. Later sections contain more general results, but at a price: (a) the statements are not as clean because they need to take corners and flat segments of $g_{pp}$ into consideration and (b) the results are valid on an extended space that supports additional edge weights (cocycles) in addition to vertex weights $\omega$ in (1.1).
Section 3 develops a convex duality between directions or velocities $\xi$ and tilts or external fields $h$ that comes from the relationship of the point-to-point and point-to-line percolation models.

Section 4 states the existence and properties of the cocycles on which all the results of the paper are based. The cocycles define a stationary last-passage model. The variational formulas for the percolation limits are first solved on the extended space of the cocycles.

Section 5 develops the existence of Busemann functions.

Section 6 studies cocycle geodesics and with their help proves our results for geodesics.

Section 7 proves results for the competition interface.

Section 8 discusses examples with geometric and exponential weights $\{\omega_x\}$. These are of course exactly solvable models, but it is useful to see the theory illustrated in its ideal form.

Several appendixes come at the end. Appendix A proves the main theorem of Section 4 by relying on queuing results from [23, 27]. Appendix B proves the coalescence of cocycle geodesics by adapting the first-passage percolation proof of [21]. A short Appendix C states an ergodic theorem for cocycles proved in [15]. Appendix D proves properties of the limit $g_{pp}$ in the case of a percolation cone, in particular differentiability at the edge. The proofs are adapted from the first-passage percolation work of [2, 24]. Appendix E states the almost sure shape theorem for the corner growth model from [25] and proves an $L^1$ version.

Notation and conventions. $\mathbb{R}_+ = [0, \infty)$, $\mathbb{Z}_+ = \{0, 1, 2, 3, \ldots \}$, and $\mathbb{N} = \{1, 2, 3, \ldots \}$.

The standard basis vectors of $\mathbb{R}^2$ are $e_1 = (1, 0)$ and $e_2 = (0, 1)$ and the $\ell^1$-norm of $x \in \mathbb{R}^2$ is $|x| = |x \cdot e_1| + |x \cdot e_2|$. For $u, v \in \mathbb{R}^2$ a closed line segment on $\mathbb{R}^2$ is denoted by $[u, v] = \{tu + (1-t)v : t \in [0, 1]\}$, and an open line segment by $(u, v) = \{tu + (1-t)v : t \in (0, 1)\}$.

Coordinatewise ordering $x \preceq y$ means that $x \cdot e_i \leq y \cdot e_i$ for both $i = 1$ and 2. Its negation $x \not\preceq y$ means that $x \cdot e_1 > y \cdot e_1$ or $x \cdot e_2 > y \cdot e_2$. An admissible or up-right path $x_{0,n} = (x_k)_{k=0}^n$ on $\mathbb{Z}^2$ satisfies $x_k - x_{k-1} \in \{e_1, e_2\}$.

The basic environment space is $\Omega = \mathbb{R}^{\mathbb{Z}^2}$ whose elements are denoted by $\omega$. There is also a larger product space $\hat{\Omega} = \Omega \times \Omega'$ whose elements are denoted by $\hat{\omega} = (\omega, \omega')$ and $\hat{\omega}$.

Parameter $p > 2$ appears in a moment hypothesis $\mathbb{E}[|\omega_0|^p] < \infty$, while $p_1$ is the probability of an open site in an oriented site percolation process.

A statement that contains $\pm$ or $\mp$ is a combination of two statements: one for the top choice of the sign and another one for the bottom choice.

2. Main results

2.1. Assumptions. The two-dimensional corner growth model is the last-passage percolation model on the planar square lattice $\mathbb{Z}^2$ with admissible steps $\mathcal{R} = \{e_1, e_2\}$. $\Omega = \mathbb{R}^{\mathbb{Z}^2}$ is the space of environments or weight configurations $\omega = (\omega_x)_{x \in \mathbb{Z}^2}$. The group of spatial translations $\{T_x\}_{x \in \mathbb{Z}^2}$ acts on $\Omega$ by $(T_x \omega)_y = \omega_{x+y}$ for $x, y \in \mathbb{Z}^2$. Let $\mathcal{B}$ denote the Borel $\sigma$-algebra of $\Omega$. $\mathbb{P}$ is a Borel probability measure on $\Omega$ under which the weights $\{\omega_x\}$ are independent, identically distributed (i.i.d.) nondegenerate random variables with a $2 + \varepsilon$ moment. Expectation under $\mathbb{P}$ is denoted by $\mathbb{E}$. For a technical reason we also assume $\mathbb{P}(\omega_0 \geq c) = 1$ for some finite constant $c$. 

For future reference we summarize our standing assumptions in this statement:

\[ \mathbb{P} \text{ is i.i.d., } \mathbb{E}[|\omega_0|^p] < \infty \text{ for some } p > 2, \ 
\sigma^2 = \text{Var}(\omega_0) > 0, \text{ and } \]

\[ \mathbb{P}(\omega_0 \geq c) = 1 \text{ for some } c > -\infty. \]  

Assumption (2.1) is valid throughout the paper and will not be repeated in every statement. The constant

\[ m_0 = \mathbb{E}(\omega_0) \]

will appear frequently. The symbol \( \omega \) is reserved for these \( \mathbb{P} \)-distributed i.i.d. weights, also later when they are embedded in a larger configuration \( \hat{\omega} = (\omega, \omega') \).

Assumption \( \mathbb{P}(\omega_0 \geq c) = 1 \) is required in only one part of our proofs, namely in Appendix A where we rely on results from queueing theory. In that context \( \omega_x \) is a service time, and the results we use have been proved only for \( \omega_x \geq 0 \). (The extension to \( \omega_x \geq c \) is immediate.) The point we wish to make is that once the queueing results have been extended to general real-valued i.i.d. weights \( \omega_x \) subject to the moment assumption in (2.1), everything in this paper is true for these general real-valued weights.

### 2.2. Last-passage percolation

Given an environment \( \omega \) and two points \( x, y \in \mathbb{Z}^2 \) with \( x \leq y \) coordinatewise, define the point-to-point last-passage time by

\[ G_{x,y} = \max_{x_0,n} \sum_{k=0}^{n-1} \omega_{x_k}. \]

The maximum is over paths \( x_{0,n} = (x_k)_{k=0}^n \) that start at \( x_0 = x \), end at \( x_n = y \) with \( n = |y-x|_1 \), and have increments \( x_{k+1} - x_k \in \{e_1, e_2\} \). We call such paths admissible or up-right.

Given a vector \( h \in \mathbb{R}^2 \), an environment \( \omega \), and an integer \( n \geq 0 \), define the \( n \)-step point-to-line last passage time with tilt (or external field) \( h \) by

\[ G_n(h) = \max_{x_{0,n}} \left\{ \sum_{k=0}^{n-1} \omega_{x_k} + h \cdot x_n \right\}. \]

The maximum is over all admissible \( n \)-step paths that start at \( x_0 = 0 \).

It is standard (see for example [25] or [28]) that under assumption (2.1), for \( \mathbb{P} \)-almost every \( \omega \), simultaneously for every \( \xi \in \mathbb{R}^2_+ \) and every \( h \in \mathbb{R}^2 \), the following limits exist:

\[ g_{pp}(\xi) = \lim_{n \to \infty} n^{-1} G_{0,|n\xi|}, \]

\[ g_{pl}(h) = \lim_{n \to \infty} n^{-1} G_n(h). \]

In the definition above integer parts are taken coordinatewise: \(|v| = ([a], [b]) \in \mathbb{Z}^2 \) for \( v = (a, b) \in \mathbb{R}^2 \).

Under assumption (2.1) the limits above are finite nonrandom continuous functions. In particular, \( g_{pp} \) is continuous up to the boundary of \( \mathbb{R}^2_+ \). Furthermore, \( g_{pp} \) is a symmetric, concave, 1-homogeneous function on \( \mathbb{R}^2_+ \) and \( g_{pl} \) is a convex Lipschitz function on \( \mathbb{R}^2 \). Homogeneity means that \( g_{pp}(c\xi) = cg_{pp}(\xi) \) for \( \xi \in \mathbb{R}^2_+ \) and \( c \in \mathbb{R}_+ \). By homogeneity, for most purposes it suffices to consider \( g_{pp} \) as a function on the convex hull \( \mathcal{U} = \{te_1 + (1-t)e_2 : t \in [0,1]\} \) of \( \mathcal{R} \). The relative interior \( \text{ri} \mathcal{U} \) is the open line segment \( \{te_1 + (1-t)e_2 : t \in (0,1)\} \).
Decomposing according to the endpoint of the path and some estimation (Theorem 2.2 in [28]) give

\[ g_{pl}(h) = \sup_{\xi \in \mathcal{U}} \{ g_{pp}(\xi) + h \cdot \xi \}. \]  

By convex duality for \( \xi \in \text{ri} \mathcal{U} \)

\[ g_{pp}(\xi) = \inf_{h \in \mathbb{R}^2} \{ g_{pl}(h) - h \cdot \xi \}. \]

Let us say \( \xi \in \text{ri} \mathcal{U} \) and \( h \in \mathbb{R}^2 \) are dual if

\[ g_{pp}(\xi) = g_{pl}(h) - h \cdot \xi. \]

Very little is known in general about \( g_{pp} \) beyond the soft properties mentioned above. In the exactly solvable case, with \( \omega_x \) either exponential or geometric, \( g_{pp}(s,t) = (s + t)m_0 + 2\sigma \sqrt{st} \). The Durrett-Liggett flat edge result ([11], Theorem 2.10 below) tells us that this formula is not true for all i.i.d. weights. It does hold for general weights asymptotically at the boundary [25]: \( g_{pp}(1,t) = m_0 + 2\sigma \sqrt{t} + o(\sqrt{t}) \) as \( t \downarrow 0 \).

2.3. Gradients and convexity. Regulation properties of \( g_{pp} \) play a role in our results, so we introduce notation for that purpose. Let

\[ \mathcal{D} = \{ \xi \in \text{ri} \mathcal{U} : g_{pp} \text{ is differentiable at } \xi \}. \]

To be clear, \( \xi \in \mathcal{D} \) means that the gradient \( \nabla g_{pp}(\xi) \) exists in the usual sense of differentiability of functions of several variables. At \( \xi \in \text{ri} \mathcal{U} \) this is equivalent to the differentiability of the single variable function \( s \mapsto g_{pp}(s,1-s) \) at \( s = \xi \cdot e_1/|\xi|_1 \). By concavity the set \( (\text{ri} \mathcal{U}) \setminus \mathcal{D} \) is at most countable.

A point \( \xi \in \text{ri} \mathcal{U} \) is an exposed point if

\[ \forall \zeta \in \text{ri} \mathcal{U} \setminus \{ \xi \} : g_{pp}(\zeta) < g_{pp}(\xi) + \nabla g_{pp}(\xi) \cdot (\zeta - \xi). \]  

The set of exposed points of differentiability of \( g_{pp} \) is \( \mathcal{E} = \{ \xi \in \mathcal{D} : (2.6) \text{ holds} \} \). The condition for an exposed point is formulated entirely in terms of \( \mathcal{U} \) because \( g_{pp} \) is a homogeneous function and therefore cannot have exposed points as a function on \( \mathbb{R}^2_+ \).

It is expected that \( g_{pp} \) is differentiable on all of \( \text{ri} \mathcal{U} \). But since this is not known, our development must handle possible points of nondifferentiability. For this purpose we take left and right limits on \( \mathcal{U} \). Our convention is that a left limit \( \xi \to \zeta \) on \( \mathcal{U} \) means that \( \xi \cdot e_1 \) increases to \( \zeta \cdot e_1 \), while in a right limit \( \xi \cdot e_1 \) decreases to \( \zeta \cdot e_1 \).

For \( \zeta \in \text{ri} \mathcal{U} \) define one-sided gradient vectors \( \nabla g_{pp}(\zeta \pm) \) by

\[ \nabla g_{pp}(\zeta \pm) \cdot e_1 = \lim_{\varepsilon \downarrow 0} g_{pp}(\zeta \pm \varepsilon e_1) - g_{pp}(\zeta) \pm \varepsilon \]

and

\[ \nabla g_{pp}(\zeta \pm) \cdot e_2 = \lim_{\varepsilon \downarrow 0} g_{pp}(\zeta \pm \varepsilon e_2) - g_{pp}(\zeta) \mp \varepsilon. \]

Concavity of \( g_{pp} \) ensures that the limits exist. \( \nabla g_{pp}(\zeta \pm) \) coincide (and equal \( \nabla g_{pp}(\xi) \)) if and only if \( \xi \in \mathcal{D} \). Furthermore, on \( \text{ri} \mathcal{U} \),

\[ \nabla g_{pp}(\zeta -) = \lim_{\xi \downarrow \zeta \setminus \zeta \cdot e_1} \nabla g_{pp}(\xi) \text{ and } \nabla g_{pp}(\zeta +) = \lim_{\xi \uparrow \zeta \setminus \zeta \cdot e_1} \nabla g_{pp}(\xi). \]
For $\xi \in ri \mathcal{U}$ define maximal line segments on which $g_{pp}$ is linear, $\mathcal{U}_{\xi-}$ for the left gradient at $\xi$ and $\mathcal{U}_{\xi+}$ for the right gradient at $\xi$, by

\begin{equation}
\mathcal{U}_{\xi \pm} = \{ \zeta \in ri \mathcal{U} : g_{pp}(\zeta) - g_{pp}(\xi) = \nabla g(\xi \pm) \cdot (\zeta - \xi) \}.
\end{equation}

Either or both segments can degenerate to a point. Let

\begin{equation}
\mathcal{U}_{\xi} = \mathcal{U}_{\xi-} \cup \mathcal{U}_{\xi+} = [\xi, \xi] \quad \text{with} \quad \xi \cdot e_1 \leq \xi \cdot e_1.
\end{equation}

If $\xi \in D$ then $\mathcal{U}_{\xi+} = \mathcal{U}_{\xi-} = \mathcal{U}_{\xi}$, while if $\xi \notin D$ then $\mathcal{U}_{\xi+} \cap \mathcal{U}_{\xi-} = \{ \xi \}$. If $\xi \in E$ then $\mathcal{U}_{\xi} = \{ \xi \}$.

Figure 1 illustrates.

For $\zeta \cdot e_1 < \eta \cdot e_1$ in $ri \mathcal{U}$, $[\zeta, \eta]$ is a maximal linear segment for $g_{pp}$ if $\nabla g_{pp}$ exists and is constant in $[\zeta, \eta]$ but not on any strictly larger open line segment in $ri \mathcal{U}$. Then $[\zeta, \eta] = \mathcal{U}_{\zeta+} = \mathcal{U}_{\eta-} = \mathcal{U}_{\xi}$ for any $\xi \in [\zeta, \eta]$. If $\zeta, \eta \in D$ we say that $g_{pp}$ is differentiable at the endpoints of this maximal linear segment. This hypothesis will be invoked several times.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{A graph of a concave function over $\mathcal{U}$ to illustrate the definitions. $\zeta, \xi$ and $\zeta$ are points of differentiability while $\xi = \xi$ and $\xi$ are not. $\mathcal{U}_{\zeta} = \mathcal{U}_{\xi} = \mathcal{U}_{\xi} = [\zeta, \zeta]$. The red lines represent supporting hyperplanes at $\xi$. The slope from the left at $\xi$ is zero, and the horizontal red line touches the graph only at $\xi$. Hence $\mathcal{U}_{\xi-} = \{ \xi \}$. Points on the line segments $[\zeta, \zeta]$ and $[\xi, \xi]$ are not exposed. $\mathcal{E} = ri \mathcal{U} \setminus ([\zeta, \zeta] \cup [\xi, \xi])$.}
\end{figure}

2.4. Cocycles. The next definition is central to the paper.

\textbf{Definition 2.1 (Cocycle).} A measurable function $B : \Omega \times \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$ is a stationary $L^1(\mathbb{P})$ cocycle if it satisfies the following three conditions.

(a) Integrability: for each $z \in \{ e_1, e_2 \}$, $E[|B(0, z)|] < \infty$.

(b) Stationarity: for $\mathbb{P}$-a.e. $\omega$ and all $x, y, z \in \mathbb{Z}^2$, $B(\omega, z + x, z + y) = B(T_z \omega, x, y)$.

(c) Cocycle property: $\mathbb{P}$-a.s. and for all $x, y, z \in \mathbb{Z}^2$, $B(x, y) + B(y, z) = B(x, z)$.

The space of stationary $L^1(\mathbb{P})$ cocycles on $(\Omega, \mathcal{G}, \mathbb{P})$ is denoted by $\mathcal{C}(\Omega)$.

A cocycle $F(\omega, x, y)$ is centered if $E[F(x, y)] = 0$ for all $x, y \in \mathbb{Z}^2$. The space of centered stationary $L^1(\mathbb{P})$ cocycles on $(\Omega, \mathcal{G}, \mathbb{P})$ is denoted by $\mathcal{C}_0(\Omega)$. 
The cocycle property (c) implies that \( B(x,x) = 0 \) for all \( x \in \mathbb{Z}^2 \) and the antisymmetry property \( B(x,y) = -B(y,x) \). \( \mathcal{C}_0(\Omega) \) is the \( L^1(\mathbb{P}) \) closure of gradients \( F(\omega, x, y) = \varphi(T_y \omega) - \varphi(T_x \omega), \varphi \in L^1(\mathbb{P}) \) (see [29, Lemma C.3]). Our convention for centering a stationary \( L^1 \) cocycle \( B \) is to let \( h(B) \in \mathbb{R}^2 \) denote the vector that satisfies
\[
\mathbb{E}[B(0,e_i)] = -h(B) \cdot e_i \quad \text{for } i \in \{1,2\}
\]
and then define \( F \in \mathcal{C}_0(\Omega) \) by
\[
F(x,y) = h(B) \cdot (x-y) - B(x,y).
\]

2.5. Busemann functions. We can now state the theorem on the existence of Busemann functions. This theorem is proved in Section 5.

**Theorem 2.2.** Let \( \xi \in \text{ri} \mathcal{U} \) with \( \mathcal{U}_\xi = [\xi, \bar{\xi}] \) defined in (2.9). Assume that \( \xi, \xi, \bar{\xi} \) are points of differentiability of \( g_{pp} \). (The degenerate case \( \xi = \xi = \bar{\xi} \) is also acceptable.) There exists a stationary \( L^1(\mathbb{P}) \) cocycle \( \{B(x,y) : x,y \in \mathbb{Z}^2\} \) and an event \( \Omega_0 \) with \( \mathbb{P}(\Omega_0) = 1 \) such that the following holds for each \( \omega \in \Omega_0 \): for each sequence \( v_n \in \mathbb{Z}_+^2 \) such that
\[
|v_n|_1 \to \infty \quad \text{and} \quad \xi \cdot e_1 \leq \lim_{n \to \infty} \frac{v_n \cdot e_1}{|v_n|_1} \leq \lim_{n \to \infty} \frac{v_n \cdot e_1}{|v_n|_1} \leq \xi \cdot e_1,
\]
we have the limit
\[
B(\omega, x, y) = \lim_{n \to \infty} (G_{x,v_n}(\omega) - G_{y,v_n}(\omega))
\]
for all \( x,y \in \mathbb{Z}^2 \). Furthermore,
\[
\nabla g_{pp}(\zeta) = \left( \mathbb{E}[B(x,x+e_1)], \mathbb{E}[B(x,x+e_2)] \right) \quad \text{for all } \zeta \in \mathcal{U}_\xi.
\]

To paraphrase the theorem, Busemann functions \( B_\xi \) exist in directions \( \xi \in \mathcal{E} \), and furthermore, if \( g_{pp} \) is differentiable at the endpoints of a maximal linear segment, then Busemann functions exist and agree in all directions on this line segment. (Note that if \( \zeta \neq \bar{\xi} \), the statement of the theorem is the same for any \( \zeta \in [\xi, \bar{\xi}] \).) In particular, if \( g_{pp} \) is differentiable everywhere on \( \text{ri} \mathcal{U} \), then (i) for each direction \( \xi \in \text{ri} \mathcal{U} \) there is a Busemann function \( B_\xi \) such that, almost surely, \( B_\xi(\omega, x, y) \) equals the limit in (2.13) for any sequence \( v_n/|v_n|_1 \to \xi \) and (ii) the \( B_\xi \)'s match on linear segments of \( g_{pp} \).

We shall not derive the cocycle property of \( B_\xi \) from the limit (2.13). Instead in Section 4 and Appendix A we construct a family of cocycles on an extended space \( \Omega = \Omega \times \Omega' \) and show that one of these cocycles equals the limit on the right of (2.13).

The Busemann limits (2.13) can also be interpreted as convergence of the last-passage process to a stationary last-passage process, described in Section 4.2.

Equation (2.14) was anticipated in [16] (see paragraph after the proof of Theorem 1.13) for Euclidean first passage percolation (FPP) where \( g_{pp}(x,y) = c \sqrt{x^2 + y^2} \). A version of this formula appears also in Theorem 3.5 of [9] for lattice FPP.

2.6. Variational formulas. Cocycles arise in variational formulas that describe the limits of last-passage percolation models. In Theorems 3.2 and 4.3 in [14] we proved these variational formulas: for \( h \in \mathbb{R}^2 \)
\[
g_{pl}(h) = \inf_{F \in \mathcal{C}_0(\Omega)} \mathbb{P}_{-\text{ess sup}} \max_{i \in \{1,2\}} \{\omega_0 + h \cdot e_i + F(\omega, 0, e_i)\}
\]
and for $\xi \in \text{ri} \mathcal{U}$

\begin{equation}
(2.16) \quad g_{pp}(\xi) = \inf_{B \in \mathcal{C}(\Omega)} \mathbb{P}\text{-ess sup}_{\omega} \max_{i \in \{1,2\}} \{\omega_0 - B(\omega,0,e_i) - h(B) \cdot \xi\}.
\end{equation}

The next theorem states that the Busemann functions found in Theorem 2.2 give minimizing cocycles.

**Theorem 2.3.** Let $\xi \in \text{ri} \mathcal{U}$ with $\mathcal{U}_\xi = [\xi,\overline{\xi}]$ defined in (2.9). Assume that $\xi,\xi,\overline{\xi} \in \mathcal{D}$. Let $B^\xi \in \mathcal{C}(\Omega)$ be given by (2.13). We have $h(B^\xi) = -\nabla g_{pp}(\xi)$ by (2.14) and (2.10). Define $F(x,y) = h(B^\xi) \cdot (x-y) - B^\xi(x,y)$ as in (2.11).

(i) Let $h = h(B^\xi) + (t,t)$ for some $t \in \mathbb{R}$. Then for $\mathbb{P}$-a.e. $\omega$

\begin{equation}
(2.17) \quad g_{pl}(h) = \max_{i \in \{1,2\}} \{\omega_0 + h \cdot e_i + F(\omega,0,e_i)\} = t.
\end{equation}

In other words, $F$ is a minimizer in (2.15) and the essential supremum vanishes.

(ii) For $\mathbb{P}$-a.e. $\omega$

\begin{equation}
(2.18) \quad g_{pp}(\xi) = \max_{i \in \{1,2\}} \{\omega_0 - B(\omega,0,e_i) - h(B) \cdot \xi\}.
\end{equation}

In other words, $B^\xi$ is a minimizer in (2.16) and the essential supremum vanishes.

The condition $h = h(B^\xi) + (t,t)$ for some $t \in \mathbb{R}$ is equivalent to $h$ dual to $\xi$. Every $h$ has a dual $\xi \in \text{ri} \mathcal{U}$ as we show in Section 3. Consequently, if $g_{pp}$ is differentiable everywhere on $\text{ri} \mathcal{U}$, each $h$ has a minimizing Busemann cocycle $F$ that satisfies (2.17). Theorem 2.3 is proved in Section 5.

The choice of $i \in \{1,2\}$ in (2.17) and (2.18) must depend on $\omega$. This choice is determined if $\xi$ is not the asymptotic direction of the competition interface (see Remark 2.8 below).

Borrowing from the homogenization literature (see e.g. page 468 of [1]), a minimizer of (2.15) or (2.16) that also removes the essential supremum, that is, a cocycle that satisfies (2.17) or (2.18), is called a corrector.

### 2.7. Geodesics.

For $u \leq v$ in $\mathbb{Z}^2$ an admissible path $x_{0,n}$ from $x_0 = u$ to $x_n = v$ (with $n = |v-u|_1$) is a (finite) geodesic from $u$ to $v$ if

$$G_{u,v} = \sum_{k=0}^{n-1} \omega_{x_k}.$$  

An up-right path $x_{0,\infty}$ is an infinite geodesic emanating from $u \in \mathbb{Z}^2$ if $x_0 = u$ and for any $j > i \geq 0$, $x_{i,j}$ is a geodesic between $x_i$ and $x_j$. Two infinite geodesics $x_{0,\infty}$ and $y_{0,\infty}$ coalesce if there exist $m,n \geq 0$ such that $x_{n,\infty} = y_{m,\infty}$.

A geodesic $x_{0,\infty}$ is $\xi$-directed or a $\xi$-geodesic if $x_n/|x_n|_1 \to \xi$ for $\xi \in \mathcal{U}$, and simply directed if it is $\xi$-directed for some $\xi$. Flat segments of $g_{pp}$ on $\mathcal{U}$ prevent us from asserting that all geodesics are directed. Hence we say more generally for a subset $\mathcal{V} \subset \mathcal{U}$ that a geodesic $x_{0,\infty}$ is $\mathcal{V}$-directed if all the limit points of $x_n/|x_n|_1$ lie in $\mathcal{V}$.

Recall that $g_{pp}$ is strictly concave if there is no nondegenerate line segment on $\text{ri} \mathcal{U}$ on which $g_{pp}$ is linear. Recall also the definition of $\mathcal{U}_\xi$ from (2.8) and $\mathcal{U}_\xi = \mathcal{U}_{\xi+} \cup \mathcal{U}_{\xi-}$. 
Theorem 2.4.  
(i) The following statements hold for $\mathbb{P}$-almost every $\omega$. For every $u \in \mathbb{Z}^2$ and $\xi \in U$ there exist infinite $U_{\xi+}$- and $U_{\xi-}$-directed geodesics starting from $u$. Every geodesic is $U_{\xi}$-directed for some $\xi \in U$.
(ii) If $g_{pp}$ is strictly concave then, with $\mathbb{P}$-probability one, every geodesic is directed.
(iii) Suppose $\mathbb{P}\{\omega_0 \leq r\}$ is a continuous function of $r \in \mathbb{R}$. Fix $\xi \in U$ and assume $\xi, \xi, \xi \in D$. Then $\mathbb{P}$-almost surely there is a unique $U_{\xi}$-geodesic out of every $u \in \mathbb{Z}^2$ and all these geodesics coalesce.

In the next theorem we repeat the assumptions of Theorem 2.2 to have a Busemann function and then show that in a direction that satisfies the differentiability assumption there can be no other geodesic except a Busemann geodesic.

Theorem 2.5. As in Theorem 2.2 let $\xi \in \text{ri} U$ with $U_{\xi} = [\xi, \xi]$ satisfy $\xi, \xi, \xi \in D$. Let $B$ be the limit from (2.13). The following events have $\mathbb{P}$-probability one.
(i) Every up-right path $x_{0,\infty}$ such that $\omega_{x_k} = B(x_k, x_{k+1})$ for all $k \geq 0$ is an infinite geodesic. We call such a path a Busemann geodesic.
(ii) Every geodesic $x_{0,\infty}$ that satisfies
\[
\xi \cdot e_1 \leq \lim_{n \to \infty} \frac{x_n \cdot e_1}{n} \leq \lim_{n \to \infty} \frac{x_n \cdot e_1}{n} \leq \xi \cdot e_1
\]
is a Busemann geodesic.
(iii) For each $m \geq 0$, for any sequence $v_n$ as in (2.12), there exists $n_0$ such that if $n \geq n_0$, then for any geodesic $x_{0,|v_n|}$ from $x_0 = 0$ to $v_n$ we have $B(\omega, x_i, x_{i+1}) = \omega_{x_i}$ for all $0 \leq i \leq m$.

When the distribution of $\omega_0$ is not continuous uniqueness of geodesics (Theorem 2.4(iii)) cannot hold. Hence we consider leftmost and rightmost geodesics. The leftmost geodesic $\xi_k$ (between two given points or in a given direction) satisfies $\xi_k \cdot e_1 \leq x_k \cdot e_1$ for any geodesic $x$, of the same category. The rightmost geodesic satisfies the opposite inequality.

Theorem 2.6. Let $\xi \in \text{ri} U$ with $U_{\xi} = [\xi, \xi]$ satisfying $\xi, \xi, \xi \in D$. The following statements hold $\mathbb{P}$-almost surely.
(i) There exists a leftmost $U_{\xi}$-geodesic from each $u \in \mathbb{Z}^2$ and all these leftmost geodesics coalesce. Same statement for rightmost.
(ii) For any $u \in \mathbb{Z}^2$ and sequence $v_n$ as in (2.12), the leftmost geodesic from $u$ to $v_n$ converges to the leftmost $U_{\xi}$-geodesic from $u$ given in part (i). A similar statement holds for rightmost geodesics.

Theorems 2.4, 2.5, and 2.6 are proved at the end of Section 6.

2.8. Competition interface. For this subsection assume that $\mathbb{P}\{\omega_0 \leq r\}$ is a continuous function of $r \in \mathbb{R}$. Then with probability one no two finite paths of any lengths have equal weight and consequently for any $v \in \mathbb{Z}^2_+$ there is a unique finite geodesic between 0 and $v$. Together these finite geodesics form the geodesic tree $T_0$ rooted at 0 that spans $\mathbb{Z}^2_+$. The two subtrees rooted at $e_1$ and $e_2$ are separated by an up-right path $\varphi = (\varphi_k)_{k \geq 0}$ on the lattice $(\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2_+$ with $\varphi_0 = (\frac{1}{2}, \frac{1}{2})$. The path $\varphi$ is called the competition interface. The term comes from the interpretation that the subtrees are two competing infections on the lattice [12, 13]. See Figure 2.
Figure 2. The geodesic tree $T_0$ rooted at 0. The competition interface (solid line) emanates from $(\frac{1}{2}, \frac{1}{2})$ and separates the subtrees of $T_0$ rooted at $e_1$ and $e_2$.

Adopt the convention that $G_{e_i, ne_j} = -\infty$ for $i \neq j$ and $n \geq 0$ (there is no admissible path from $e_i$ to $ne_j$). Fix $n \in \mathbb{N}$. As $v$ moves to the right along $|v|_1 = n$, the function $G_{e_2,v} - G_{e_1,v}$ is nonincreasing. This is a consequence of Lemma 5.4 below. There is a unique $0 \leq k < n$ such that

$$G_{e_2,(k,n-k)} - G_{e_1,(k,n-k)} > 0 > G_{e_2,(k+1,n-k-1)} - G_{e_1,(k+1,n-k-1)}.$$  

This identifies the point $\varphi_{n-1} = (k + \frac{1}{2}, n - k - \frac{1}{2})$.

**Theorem 2.7.** Assume $\mathbb{P}\{\omega_0 \leq r\}$ is continuous in $r$ and that $g_{pp}$ is differentiable at the endpoints of all its linear segments. Then we have the law of large numbers

$$\xi_*(\omega) = \lim_{n \to \infty} n^{-1} \varphi_n(\omega) \quad \mathbb{P}\text{-a.s.}$$

The limit $\xi_*$ is almost surely an exposed point in $ri\mathcal{U}$ and the support of its distribution intersects every open interval outside the closed line segments on which $g_{pp}$ is linear.

**Remark 2.8.** Assume that $\mathbb{P}\{\omega_0 \leq r\}$ is continuous and that differentiability holds everywhere on $ri\mathcal{U}$ so that no caveats are needed. Connecting back to the variational formulas, the maximum in (2.17) and (2.18) is taken at $i = 2$ if $\xi \cdot e_1 < \xi_* \cdot e_1$ and at $i = 1$ if $\xi \cdot e_1 > \xi_* \cdot e_1$. This is a consequence of the following two facts: (i) $\omega_0 = B(0, e_1) \wedge B(0, e_2)$ as follows from (1.2), and (ii) for $\xi \cdot e_1 < \xi_* \cdot e_1 < \zeta \cdot e_1$ we have $B(0, e_1) > B(0, e_2)$ and $B(0, e_1) < B(0, e_2)$. The second fact will become clear in Section 7.

The competition interface is a natural direction in which there are two geodesics from 0. Note that nonuniqueness in the random direction $\xi_*$ does not violate the almost sure uniqueness in a fixed direction given in Theorem 2.4(iii).

**Theorem 2.9.** Assume $\mathbb{P}\{\omega_0 \leq r\}$ is continuous in $r$.

(i) Assume $g_{pp}$ is differentiable at the endpoints of all its linear segments. Then $\mathbb{P}$-almost surely, for every $x \in \mathbb{Z}^2$, there exist at least two $U_{\xi_*(T_x\omega)}$-geodesics out of $x$ that do not coalesce.
(ii) Assume \( g_{pp} \) is strictly concave. Then with \( \mathbb{P} \)-probability one and for any \( x \in \mathbb{Z}^2 \), there cannot be two distinct geodesics from \( x \) with a common direction other than \( \xi_*(T_x \omega) \).

For the exactly solvable corner growth model with exponential weights Coupier [8] proved that the set of directions with two non-coalescing geodesics in \( \mathbb{Z}_+^2 \) is countable and dense in \( \mathcal{U} \). Here we have a partial result towards characterizing this set as \( \{ \xi_*(T_x \omega) \}_{x \in \mathbb{Z}_+^2} \). Partial, because we consider only pairs of geodesics from a common initial point.

Point (i) of Theorem 2.9 is actually true without the differentiability assumption, but at this stage of the paper we have no definition of \( \xi_* \) without that assumption. This will change in Theorem 7.2 in Section 7. In point (ii) above \( g_{pp} \) has no linear segments and so the differentiability of \( g_{pp} \) at endpoints of linear segments is vacuously true.

Theorems 2.7 and 2.9 are proved in Section 7. An additional fact proved there is that \( \mathbb{P}(\xi_* = \xi) > 0 \) is possible only if \( \xi \notin D \). In light of the expectation that \( g_{pp} \) is differentiable, the expected result is that \( \xi_* \) has a continuous distribution.

When weights \( \omega_0 \) do not have a continuous distribution, there are two competition interfaces: one for the tree of leftmost geodesics and one for the tree of rightmost geodesics. We compute the limit distributions of the two competition interfaces for geometric weights in Sections 2.9 and 8.

2.9. Exactly solvable models. We illustrate our results in the two exactly solvable cases: the distribution of the mean \( m_0 \) weights \( \omega_0 \) is

\[
\begin{align*}
\text{exponential: } & \mathbb{P}\{\omega_x \geq t\} = m_0^{-1}e^{-t/m_0} \text{ for } t \geq 0 \text{ with } \sigma^2 = m_0^2, \\
& \quad \text{or geometric: } \mathbb{P}\{\omega_x = k\} = m_0^{-1}(1 - m_0^{-1})^{k-1} \text{ for } k \in \mathbb{N} \text{ with } \sigma^2 = m_0(m_0 - 1).
\end{align*}
\]

Calculations behind the claims below are sketched in Section 8.

For both cases in (2.22) the point-to-point limit function is

\[
g_{pp}(\xi) = m_0(\xi \cdot e_1 + \xi \cdot e_2) + 2\sigma \sqrt{(\xi \cdot e_1)(\xi \cdot e_2)}.
\]

In the exponential case this formula was first derived by Rost [30] (though without the last-passage formulation) while early derivations of the geometric case appeared in [6, 17, 31]. Convex duality (2.5) becomes

\[
\xi \in \text{ri} \mathcal{U} \text{ is dual to } h \text{ if and only if } h = (m_0 + \sigma^2 \sqrt{\xi \cdot e_1/\xi \cdot e_2} + t, m_0 + \sqrt{\xi \cdot e_2/\xi \cdot e_1} + t), \quad t \in \mathbb{R}.
\]

This in turn gives an explicit formula for \( g_{pl}(h) \).

Since the \( g_{pp} \) above is differentiable and strictly concave, all points of \( \text{ri} \mathcal{U} \) are exposed points of differentiability. Theorem 2.2 implies that Busemann functions (2.13) exist in all directions \( \xi \in \text{ri} \mathcal{U} \). They minimize formulas (2.15) and (2.16) as given in (2.17) and (2.18). For each \( \xi \in \text{ri} \mathcal{U} \) the processes \( \{B^\xi(ke_1, (k + 1)e_1) : k \in \mathbb{Z}_+\} \) and \( \{B^\xi(ke_2, (k + 1)e_2) : k \in \mathbb{Z}_+\} \) are i.i.d. processes independent of each other, exponential or geometric depending on the case, with means

\[
\begin{align*}
\mathbb{E}[B^\xi(ke_1, (k + 1)e_1)] &= m_0 + \sigma \sqrt{\xi \cdot e_2/\xi \cdot e_1} \\
\mathbb{E}[B^\xi(ke_2, (k + 1)e_2)] &= m_0 + \sigma \sqrt{\xi \cdot e_1/\xi \cdot e_2}.
\end{align*}
\]
Section 2.7 gives the following results on geodesics. Every infinite geodesic has a direction and for every fixed direction $\xi \in \text{ri} \mathcal{U}$ there exists a $\xi$-geodesic. In the exponential case $\xi$-geodesics are unique and coalesce. In the geometric case uniqueness cannot hold, but there exists a unique leftmost $\xi$-geodesic out of each lattice point and these leftmost $\xi$-geodesics coalesce. The same holds for rightmost $\xi$-geodesics. Finite (leftmost/rightmost) geodesics from $u \in \mathbb{Z}^2$ to $v_n$ converge to infinite (leftmost/rightmost) $\xi$-geodesics out of $u$, as $v_n/|v_n|_1 \to \xi$ with $|v_n|_1 \to \infty$.

In the exponential case the distribution of the asymptotic direction $\xi_*$ of the competition interface given by Theorem 2.7 can be computed explicitly. For the angle $\theta_* = \tan^{-1}(\xi_* \cdot e_2/\xi_* \cdot e_1)$ of the vector $\xi_*$,

$$\mathbb{P}\{\theta_* \leq t\} = \frac{\sqrt{\sin t}}{\sqrt{\sin t + \sqrt{\cos t}}} \quad t \in [0, \pi/2].$$

In the exponential case these results for geodesics and the competition interface were shown in [13]. This paper utilized techniques for geodesics from [26] and the coupling of the exponential corner growth model with the totally asymmetric simple exclusion process (TASEP). For this case our approach provides new proofs.

The model with geometric weights has a tree of leftmost geodesics with competition interface $\varphi^{(l)} = (\varphi^{(l)}_k)_{k \geq 0}$ and a tree of rightmost geodesics with competition interface $\varphi^{(r)} = (\varphi^{(r)}_k)_{k \geq 0}$. Note that $\varphi^{(r)}$ is to the left of $\varphi^{(l)}$ because in (2.20) there is now a middle range $G_{e_2,(k,n-k)} - G_{e_1,(k,n-k)} = 0$ that is to the right (left) of $\varphi^{(r)}$ ($\varphi^{(l)}$). Strict concavity of the limit $\mathbb{g}_\text{pp}$ implies (with the arguments of Section 7) the almost sure limits

$$n^{-1} \varphi^{(l)}_n \to \xi^{(l)}_* \quad \text{and} \quad n^{-1} \varphi^{(r)}_n \to \xi^{(r)}_* .$$

The angles $\theta^{(a)}_* = \tan^{-1}(\xi^{(a)}_* \cdot e_2/\xi^{(a)}_* \cdot e_1)$ ($a \in \{l, r\}$) have these distributions (with $p_0 = m_0^{-1}$ denoting the success probability of the geometric): for $t \in [0, \pi/2]$

$$\mathbb{P}\{\theta^{(r)}_* \leq t\} = \frac{\sqrt{(1-p_0) \sin t}}{\sqrt{(1-p_0) \sin t + \sqrt{\cos t}}}$$

and

$$\mathbb{P}\{\theta^{(l)}_* \leq t\} = \frac{\sqrt{\sin t}}{\sqrt{\sin t + \sqrt{(1-p_0) \cos t}}} .$$

Taking $p_0 \to 0$ recovers (2.24) of the exponential case. For the details, see Section 8.

2.10. **Flat edge in the percolation cone.** We describe a known nontrivial example where the assumption of differentiable endpoints of a maximal linear segment is satisfied. A short detour into oriented percolation is needed.

In oriented site percolation vertices of $\mathbb{Z}^2$ are assigned i.i.d. $\{0, 1\}$-valued random variables $\{\sigma_z\}_{z \in \mathbb{Z}^2}$ with $p_1 = \mathbb{P}\{\sigma_0 = 1\}$. For points $u \leq v$ in $\mathbb{Z}^2$ we write $u \to v$ (there is an open path from $u$ to $v$) if there exists an up-right path $u = x_0, x_1, \ldots, x_m = v$ with $x_i - x_{i-1} \in \{e_1, e_2\}$, $m = |v-u|_1$, and such that $\sigma_{x_i} = 1$ for $i = 1, \ldots, m$. (The openness of a path does not depend on the weight at the initial point of the path.) The percolation event $\{u \to \infty\}$ is the existence of an infinite open up-right path from point $u$. There exists a critical threshold $\tilde{p}_c \in (0, 1)$ such that if $p_1 < \tilde{p}_c$ then $\mathbb{P}\{0 \to \infty\} = 0$ and if $p_1 > \tilde{p}_c$ then $\mathbb{P}\{0 \to \infty\} > 0$. 


(The facts we need about oriented site percolation are proved in article [10] for oriented edge percolation. The proofs are applicable to site percolation just as well.)

Let \( \mathcal{O}_n = \{ u \in \mathbb{Z}^2_+ : |u|_1 = n, 0 \to u \} \) denote the set of vertices on level \( n \) that can be reached from the origin along open paths. The right edge \( a_n = \max_{u \in \mathcal{O}_n} \{ u \cdot e_1 \} \) is defined on the event \( \{ \mathcal{O}_n \neq \emptyset \} \). When \( p_1 \in (\bar{p}_c, 1) \) there exists a constant \( \beta_{p_1} \in (1/2, 1) \) such that [10, eqn. (7) on p. 1005]

\[
\lim_{n \to \infty} \frac{a_n}{n} \mathbb{1}\{0 \to \infty\} = \beta_{p_1} \mathbb{1}\{0 \to \infty\} \quad \mathbb{P}\text{-a.s.}
\]

Let \( \eta = (\beta_{p_1}, 1 - \beta_{p_1}) \) and \( \bar{\eta} = (1 - \beta_{p_1}, \beta_{p_1}) \). The percolation cone is the set \( \{ \xi \in \mathbb{R}^2_+ : \xi/|\xi|_1 \in [\eta, \bar{\eta}] \} \).

The point of this for the corner growth model is that if the \( \omega \) weights have a maximum that percolates, \( g_{pp} \) is linear on the percolation cone and differentiable on the edges. This is the content of the next theorem.

**Theorem 2.10.** Assume that \( \{ \omega_x \}_{x \in \mathbb{Z}^2} \) are i.i.d., \( \mathbb{E}[|\omega_0|^p] < \infty \) for some \( p > 2 \) and \( \omega_x \leq 1 \). Suppose \( \bar{p}_c < p_1 = P\{\omega_0 = 1\} < 1 \). Let \( \xi \in \mathcal{U} \). Then \( g_{pp}(\xi) \leq 1 \), and \( g_{pp}(\xi) = 1 \) if and only if \( \xi \in [\eta, \bar{\eta}] \). The endpoints \( \eta \) and \( \bar{\eta} \) are points of differentiability of \( g_{pp} \).

The theorem above summarizes a development that goes through papers [2, 11, 24]. The proofs in the literature are for first-passage percolation. We give a proof of Theorem 2.10 in Appendix D, by adapting and simplifying the first-passage percolation arguments for the directed corner growth model.

As a corollary, our results that assume differentiable endpoints of a maximal linear segment are valid for the percolation cone.

**Theorem 2.11.** Assume (2.1), \( \omega_x \leq 1 \) and \( \bar{p}_c < p_1 = P\{\omega_0 = 1\} < 1 \). There exists a stationary \( L^1(\mathbb{P}) \) cocycle \( \{ B(x,y) : x,y \in \mathbb{Z}^2 \} \) and an event \( \Omega_0 \) with \( P(\Omega_0) = 1 \) such that the following statements hold for each \( \omega \in \Omega_0 \). Let \( v_n \in \mathbb{Z}^2_+ \) be a sequence such that

\[
|v_n|_1 \to \infty \quad \text{and} \quad 1 - \beta_{p_1} \leq \lim_{n \to \infty} \frac{v_n \cdot e_1}{|v_n|_1} \leq \lim_{n \to \infty} \frac{v_n \cdot e_1}{|v_n|_1} \leq \beta_{p_1}.
\]

Then

\[
B(\omega, x, y) = \lim_{n \to \infty} \left( G_{x, v_n}(\omega) - G_{y, v_n}(\omega) \right)
\]

for all \( x, y \in \mathbb{Z}^2 \). For each \( m \geq 0 \) there exists \( n_0 \) such that if \( n \geq n_0 \), then any geodesic \( x_0, v_n, x_i \) from \( x_0 = 0 \) to \( v_n \) satisfies \( B(\omega, x_i, x_{i+1}) = \omega_{x_i} \) for all \( 0 \leq i \leq m \).

Furthermore,

\[
\mathbb{E}[B(x, x + e_1)] = \mathbb{E}[B(x, x + e_2)] = 1.
\]

This completes the presentation of results and we turn to developing the proofs.

3. Convex duality

By homogeneity we can represent \( g_{pp} \) by a single variable function. A way of doing this that ties in naturally with the queuing theory arguments we use later is to define

\[
(3.1) \quad \gamma(s) = g_{pp}(1, s) = g_{pp}(s, 1) \quad \text{for} \quad 0 \leq s < \infty.
\]
Function $\gamma$ is real-valued, continuous and concave. Consequently one-sided derivatives $\gamma'(s\pm)$ exist and are monotone: $\gamma'(s_0+) \geq \gamma'(s_1-) \geq \gamma'(s_1+)$ for $0 \leq s_0 < s_1$. Symmetry and homogeneity of $g_{pp}$ give $\gamma(s) = s\gamma(s^{-1})$.

**Lemma 3.1.** The derivatives satisfy $\gamma'(s\pm) > m_0$ for all $s \in \mathbb{R}_+$, $\gamma'(0+) = \infty$, and $\gamma'(-\infty) \equiv \lim_{s \to -\infty} \gamma'(s\pm) = \gamma(0) = m_0$.

**Proof.** The boundary shape universality of J. Martin [25, Theorem 2.4] says that

\begin{equation}
\gamma(s) = m_0 + 2\sigma \sqrt{s} + o(\sqrt{s}) \quad \text{as } s \searrow 0.
\end{equation}

This gives $\gamma(0) = m_0$ and $\gamma'(0+) = \infty$. Lastly,

\begin{equation}
\gamma'(-\infty) = \lim_{s \to -\infty} s^{-1} \gamma(s) = \lim_{s \to -\infty} \gamma(s^{-1}) = \gamma(0) = m_0.
\end{equation}

Martin’s asymptotic (3.2) and $\gamma(s) = s\gamma(s^{-1})$ give

\begin{equation}
\gamma(s) = sm_0 + 2\sigma \sqrt{s} + o(\sqrt{s}) \quad \text{as } s \nearrow \infty.
\end{equation}

This is incompatible with having $\gamma'(s) = m_0$ for $s \geq s_0$ for any $s_0 < 0$. □

The lemma above has two important geometric consequences: (i) any subinterval of $U$ on which $g_{pp}$ is linear must lie in the interior $\text{ri}U$ and (ii) the boundary $\{ \xi : g_{pp}(\xi) = 1 \}$ of the limit shape is asymptotic to the axes.

Define

\begin{equation}
f(\alpha) = \sup_{s \geq 0} \{ \gamma(s) - s\alpha \} \quad \text{for } m_0 < \alpha < \infty.
\end{equation}

**Lemma 3.2.** Function $f$ is a strictly decreasing, continuous and convex involution of the interval $(m_0, \infty)$ onto itself, with limits $f(m_0+) = \infty$ and $f(\infty-) = m_0$. That $f$ is an involution means that $f(f(\alpha)) = \alpha$.

**Proof.** Asymptotics (3.2) and (3.3) imply that $m_0 < f(\alpha) < \infty$ for all $\alpha > m_0$ and also that the supremum in (3.4) is attained at some $s$. Furthermore, $\alpha < \beta$ implies $f(\beta) = \gamma(s_0) - s_0\beta$ with $s_0 > 0$ and $f(\beta) < \gamma(s_0) - s_0\alpha \leq f(\alpha)$. As a supremum of linear functions $f$ is convex, and hence continuous on the open interval $(m_0, \infty)$.

We show how the symmetry of $g_{pp}$ implies that $f$ is an involution. By concavity of $\gamma$,

\begin{equation}
f(\alpha) = \gamma(s) - s\alpha \quad \text{if and only if } \alpha \in [\gamma'(s+), \gamma'(s-)]
\end{equation}

and by Lemma 3.1 the intervals on the right cover $(m_0, \infty)$. Since $f$ is strictly decreasing the above is the same as

\begin{equation}
\alpha = \gamma(s^{-1}) - s^{-1}f(\alpha) \quad \text{if and only if } f(\alpha) \in [f(\gamma'(s-)), f(\gamma'(s+))].
\end{equation}

Differentiating $\gamma(s) = s\gamma(s^{-1})$ gives

\begin{equation}
\gamma'(s\pm) = \gamma(s^{-1}) - s^{-1}\gamma'(s^{-1}\pm).
\end{equation}

By (3.5) and (3.7) the condition in (3.6) can be rewritten as

\begin{equation}
f(\alpha) \in [\gamma(s) - s\gamma'(s-), \gamma(s) - s\gamma'(s+)] = [\gamma'(s^{-1}+), \gamma'(s^{-1}-)].
\end{equation}

Combining this with (3.5) and (3.6) shows that $\alpha = f(f(\alpha))$. The claim about the limits follows from $f$ being a decreasing involution. □
Extend these functions to the entire real line by \( \gamma(s) = -\infty \) when \( s < 0 \) and \( f(\alpha) = \infty \) when \( \alpha \leq m_0 \). Then convex duality gives
\[
(3.9) \quad \gamma(s) = \inf_{\alpha > m_0} \{ f(\alpha) + s\alpha \}.
\]

The natural bijection between \( s \in (0, \infty) \) and \( \xi \in \text{ri} \mathcal{U} \) that goes together with (3.1) is
\[
(3.10) \quad s = \xi \cdot e_1 / \xi \cdot e_2.
\]
Then direct differentiation, (3.5) and (3.7) give
\[
(3.11) \quad \nabla g_{pp}(\xi \pm) = (\gamma'(s\pm), \gamma'(s^{-1}\mp)) = (\gamma'(s\pm), f(\gamma'(s\pm))).
\]
Since \( f \) is linear on \([\gamma'(s+), \gamma'(s-)]\), we get the following connection between the gradients of \( g_{pp} \) and the graph of \( f \):
\[
(3.12) \quad [\nabla g_{pp}(\xi+), \nabla g_{pp}(\xi-)] = \{ (\alpha, f(\alpha)) : \alpha \in [\gamma'(s+), \gamma'(s-)] \} \quad \text{for} \ \xi \in \text{ri} \mathcal{U}.
\]

The next theorem details the duality between tilts \( h \) and velocities \( \xi \).

**Theorem 3.3.** (i) Let \( h \in \mathbb{R}^2 \). There exists a unique \( t = t(h) \in \mathbb{R} \) such that
\[
(3.13) \quad h - t(e_1 + e_2) \in -[\nabla g_{pp}(\xi+), \nabla g_{pp}(\xi-)]
\]
for some \( \xi \in \text{ri} \mathcal{U} \). The set of \( \xi \) for which (3.13) holds is a nonempty (but possibly degenerate) line segment \([\bar{\xi}(h), \bar{\xi}(h)] \subset \text{ri} \mathcal{U} \). If \( \bar{\xi}(h) \neq \bar{\xi}(h) \) then \([\bar{\xi}(h), \bar{\xi}(h)] \) is a maximal line segment on which \( g_{pp} \) is linear.

(ii) \( \xi \in \text{ri} \mathcal{U} \) and \( h \in \mathbb{R}^2 \) satisfy duality (2.5) if and only if (3.13) holds.

*Proof.* The graph \( \{ (\alpha, f(\alpha)) : \alpha > m_0 \} \) is strictly decreasing with limits \( f(m_0+) = \infty \) and \( f(\infty-) = m_0 \). Since every 45 degree diagonal intersects it at a unique point, the equation
\[
(3.14) \quad h = -(\alpha, f(\alpha)) + t(e_1 + e_2)
\]
defines a bijection \( \mathbb{R}^2 \ni h \longleftrightarrow (\alpha, t) \in (m_0, \infty) \times \mathbb{R} \) illustrated in Figure 3. Combining this with (3.12) shows that (3.13) happens for a unique \( t \) and for at least one \( \xi \in \text{ri} \mathcal{U} \).

Once \( h \) and \( t = t(h) \) are given, the geometry of the gradients ((3.11)–(3.12) and limits (2.7)) can be used to argue the claims about the \( \xi \) that satisfy (3.13). This proves part (i).

That \( h \) of the form (3.13) is dual to \( \xi \) follows readily from the fact that gradients are dual and \( g_{pl}(h + t(e_1 + e_2)) = g_{pl}(h) + t \) (this last from Definition (2.3)).

Note the following general facts for any \( q \in [\nabla g_{pp}(\xi+), \nabla g_{pp}(\xi-)] \). By concavity \( g_{pp}(\eta) \leq g_{pp}(\xi) + q \cdot (\eta - \xi) \) for all \( \eta \). Combining this with homogeneity gives \( g_{pp}(\xi) = q \cdot \xi \). Together with duality (2.4) we have
\[
(3.15) \quad g_{pl}(-q) = 0 \quad \text{for} \quad q \in \bigcup_{\xi \in \text{ri} \mathcal{U}} [\nabla g_{pp}(\xi+), \nabla g_{pp}(\xi-)].
\]

It remains to show that if \( h \) is dual to \( \xi \) then it satisfies (3.13). Let \( (\alpha, t) \) be determined by (3.14). From the last two paragraphs
\[
g_{pl}(h) = g_{pl}(-\alpha, -f(\alpha)) + t = t.
\]
Let \( s = \xi \cdot e_1 / \xi \cdot e_2 \) so that
\[
g_{pp}(\xi) + h \cdot \xi = \frac{\gamma(s)}{1 + s} - \frac{\alpha s + f(\alpha)}{1 + s} + t.
\]
Thus duality $g_{pl}(h) = g_{pp}(\xi) + h \cdot \xi$ implies $\gamma(s) = \alpha s + f(\alpha)$ which happens if and only if $\alpha \in [\gamma'(s+), \gamma'(s-)]$. (3.12) now implies (3.13). □

4. Stationary cocycles

In this section we begin with the stationary cocycles, then show how these define stationary last-passage percolation processes and also solve the variational formulas for $g_{pp}(\xi)$ and $g_{pl}(h)$.

4.1. Existence and properties of stationary cocycles. We come to the technical centerpiece of the paper. By appeal to queueing fixed points, in Appendix A we construct a family of cocycles \( \{B^{\xi \pm}\}_{\xi \in \mathcal{U}} \) on an extended space $\hat{\Omega} = \Omega \times \Omega'$. The next theorem gives the existence statement and summarizes the properties of these cocycles. Assumption (2.1) is in force. This is the only place where our proofs actually use the assumption $\mathbb{P}(\omega_0 \geq c) = 1$, and the only reason is that the queueing results we reference have been proved only for $\omega_0 \geq 0$.

The cocycles of interest are related to the last-passage weights in the manner described in the next definition. A potential is simply a measurable function $V : \hat{\Omega} \to \mathbb{R}$. The case relevant to us will be $V(\hat{\omega}) = \omega_0$ where $\hat{\omega} = (\omega, \omega') \in \hat{\Omega}$ is a configuration in the extended space and contains the original weights $\omega$ as a component.

**Definition 4.1.** A stationary $L^1$ cocycle $B$ on $\hat{\Omega}$ recovers potential $V$ if

$$V(\hat{\omega}) = \min_{i \in \{1,2\}} B(\hat{\omega},0,e_i) \text{ for } \hat{\mathbb{P}}\text{-a.e. } \hat{\omega}. \tag{4.1}$$

The extended space is the Polish space $\hat{\Omega} = \Omega \times \mathbb{R}^{(1,2)} \times \mathcal{A}_0 \times \mathbb{Z}^2$ where $\mathcal{A}_0$ is a certain countable subset of the interval $[m_0, \infty)$. A precise description of $\mathcal{A}_0$ appears in the beginning of the proof of Theorem 4.2 on page 46 in Appendix A. Let $\hat{\mathcal{S}}$ denote the Borel $\sigma$-algebra.
of $\Omega$. Generic elements of $\hat{\Omega}$ are denoted by $\hat{\omega} = (\omega, \omega')$ with $\omega = (\omega_x)_{x \in \mathbb{Z}^2} \in \Omega = \mathbb{R}^{\mathbb{Z}^2}$ as before and $\omega' = (\omega^i_{x})_{i \in \{1, 2\}, \alpha \in \mathcal{A}_0, x \in \mathbb{Z}^2}$. Spatial translations act on the $x$-index in the usual manner: $(T_x \hat{\omega})_y = \hat{\omega}_{x+y}$ for $x, y \in \mathbb{Z}^2$ where $\hat{\omega}_x = (\omega_x, \omega'_x) = (\omega_x, (\omega^i_{x})_{i \in \{1, 2\}, \alpha \in \mathcal{A}_0})$.

**Theorem 4.2.** There exist functions $\{B^{\xi+}(\hat{\omega}, x, y), B^{\xi-}(\hat{\omega}, x, y) : x, y \in \mathbb{Z}^2, \xi \in \text{ri}\mathcal{U}\}$ on $\hat{\Omega}$ and a translation invariant Borel probability measure $\hat{\mathbb{P}}$ on the space $(\hat{\Omega}, \hat{\mathcal{S}})$ such that the following properties hold.

(i) For each $\xi \in \text{ri}\mathcal{U}$, $x \in \mathbb{Z}^2$, and $i \in \{1, 2\}$, the function $B^{\xi\pm}(\hat{\omega}, x, x + e_i)$ is a function only of $(\omega_x^\alpha : \alpha \in \mathcal{A}_0)$. Under $\hat{\mathbb{P}}$, the marginal distribution of the configuration $\omega$ is the i.i.d. measure $\mathbb{P}$ specified in assumption (2.1). The $\mathbb{R}^3$-valued process $\{\varphi^{\xi\pm}_x\}_{x \in \mathbb{Z}^2}$ defined by

$$\varphi^{\xi\pm}_x(\hat{\omega}) = (\omega_x, B^{\xi\pm}(x, x + e_1), B^{\xi\pm}(x, x + e_2))$$

is separately ergodic under both translations $T_{e_1}$ and $T_{e_2}$. The same holds with $\xi^+$ replaced by $\xi^-$. For each $z \in \mathbb{Z}^2$, the variables $\{(\omega_x, B^{\xi\pm}(\hat{\omega}, x, x + e_1), B^{\xi-}(\hat{\omega}, x, x + e_1)) : x \leq z, \xi \in \text{ri}\mathcal{U}, i \in \{1, 2\}\}$ are independent of $\{\omega_x : x \leq z\}$.

(ii) Each process $B^{\xi\pm} = \{B^{\xi\pm}(x, y)\}_{x, y \in \mathbb{Z}^2}$ is a stationary $L^1(\hat{\mathbb{P}})$ cocycle (Definition 2.1) that recovers the potential $V(\hat{\omega}) = \omega_0$ (Definition 4.1), and the same is true of $B^{\xi\pm}$. The associated tilt vectors $h(\xi\pm) = h(B^{\xi\pm})$ defined by (2.10) satisfy

$$h(\xi\pm) = -\nabla g_{pp}(\xi\pm)$$

and are dual to velocity $\xi$ as in (2.5).

(iii) No two distinct cocycles have a common tilt vector. That is, if $h(\xi\pm) = h(\zeta\pm)$ then $B^{\xi\pm}(\hat{\omega}, x, x + e_1) = B^{\zeta\pm}(\hat{\omega}, x, x + e_1)$, and similarly $h(\xi\pm) = h(\zeta\pm)$ implies $B^{\xi\pm}(\hat{\omega}, x, x + e_1) = B^{\zeta\pm}(\hat{\omega}, x, x + e_1)$. These equalities hold without any almost sure modifier because they come for each $\hat{\omega}$ from the construction. In particular, if $\xi$ is a point of differentiability for $g_{pp}$ then

$$B^{\xi\pm}(\hat{\omega}, x, x + e_1) = B^{\xi-}(\hat{\omega}, x, x + e_1) = B^{\xi}(\hat{\omega}, x, x + e_1)$$

where the second equality defines the cocycle $B^{\xi}$.

(iv) The following inequalities hold $\hat{\mathbb{P}}$-almost surely, simultaneously for all $x \in \mathbb{Z}^2$ and $\xi, \zeta \in \text{ri}\mathcal{U}$: if $\xi \cdot e_1 < \zeta \cdot e_1$ then

$$B^{\xi-}(x, x + e_1) \geq B^{\xi\pm}(x, x + e_1) \geq B^{\xi-}(x, x + e_1)$$

and

$$B^{\xi-}(x, x + e_2) \leq B^{\xi\pm}(x, x + e_2) \leq B^{\xi-}(x, x + e_2).$$

Fix $\zeta \in \text{ri}\mathcal{U}$ and let $\xi_n \rightarrow \zeta$ in $\text{ri}\mathcal{U}$. If $\xi_n \cdot e_1 \nearrow \zeta \cdot e_1$ then for all $x \in \mathbb{Z}^2$ and $i = 1, 2$

$$\lim_{n \rightarrow \infty} B^{\xi_n\pm}(x, x + e_1) = B^{\xi\pm}(x, x + e_1) \quad \hat{\mathbb{P}}\text{-a.s. and in } L^1(\hat{\mathbb{P}}).$$

Similarly, if $\xi_n \cdot e_1 \searrow \zeta \cdot e_1$ the limit holds $\hat{\mathbb{P}}$-a.s. and in $L^1(\hat{\mathbb{P}})$ with $\zeta$- on the right.
Remark 4.3. The construction of the cocycles has this property: there is a countable dense set $U_0$ of $U$ such that, for $\xi \in U_0$, the cocycles are coordinate projections $B^{\xi\pm}(x, x + e_i) = \omega_{x,\gamma(s\pm)}$ where $s$ is defined by (3.10). A point $\zeta \in U \setminus U_0$ will lie in $D$ and we define $B^\zeta$ through one-sided limits from $B^{\xi\pm}$, $\xi \in U_0$. We comment next on various technical properties of the cocycles that are important for the sequel.

(a) A natural question is whether $B^{\xi\pm}(\hat{\omega}, x, y)$ can be defined as a function of $\omega$ alone, or equivalently, whether it is $\mathcal{G}$-measurable. This is important because we use the cocycles to solve the variational formulas for the limits and to construct geodesics, and it would be desirable to work on the original weight space $\Omega$ rather than on the artificially extended space $\hat{\Omega} = \Omega \times \Omega'$. We can make this $\mathcal{G}$-measurability claim for those cocycles that arise as Busemann functions or their limits. (see Remark 5.3 below).

(b) By part (iii), if $g_{pp}$ is linear on the line segment $[\xi', \xi''] \subset \text{ri}U$ with $\xi' \cdot e_1 < \xi'' \cdot e_1$, then

$$B^{\xi'}(\hat{\omega}, x, x + e_i) = B^\xi(\hat{\omega}, x, x + e_i) = B^{\xi''}(\hat{\omega}, x, x + e_i)$$

$$\forall \hat{\omega} \in \hat{\Omega}, \xi \in [\xi', \xi''], i \in \{1, 2\}.$$ 

The equalities in part (iii) do not extend to $B^{\xi\pm}(x, y)$ for all $x, y$ without exceptional $\hat{\mathbb{P}}$-null sets because the additivity $B^{\xi\pm}(x, z) = B^{\xi\pm}(x, y) + B^{\xi\pm}(y, z)$ cannot be valid for each $\hat{\omega}$, only almost surely.

(c) When we use these cocycles to construct geodesics in Section 6, it is convenient to have a single null set outside of which the ordering (4.3) is valid for all $\xi, \zeta$. For the countable family $\{B^{\xi\pm}\}_{\xi \in U_0}$ we can arrange for (4.3) to hold outside a single $\hat{\mathbb{P}}$-null event. By defining $\{B^{\xi}\}_{\xi \in U \setminus U_0}$ through limits from the left, we extend inequalities (4.3) to all $\xi, \zeta \in \text{ri}U$ outside a single null set. This is good enough for a definition of the entire family $\{B^{\xi\pm}\}_{\xi \in \text{ri}U}$. But in order to claim that limits from left and right agree at a particular $\zeta$, we have to allow for an exceptional $\hat{\mathbb{P}}$-null event that is specific to $\zeta$. Thus the limit (4.4) is not claimed outside a single null set for all $\zeta$.

(d) When $\mathbb{P}\{\omega_0 \leq r\}$ is a continuous function of $r$ it is natural to ask whether $B^\xi(x, y)$ can be modified to be continuous in $\xi$. We do not know the answer.

4.2. Stationary last-passage percolation. Fix a cocycle $B(\hat{\omega}, x, y) = B^{\xi\pm}(\hat{\omega}, x, y)$ from Theorem 4.2. Fix a point $v \in Z^2$ that will serve as an origin. By part (i) of Theorem 4.2, the weights $\{\omega_x : x \leq v - e_1 - e_2\}$ are independent of $\{B(v - (k + 1)e_1, v - ke_i) : k \in Z, i \in \{1, 2\}\}$. These define a stationary last-passage percolation process in the third quadrant relative to the origin at $v$ in the following sense. Define passage times $G_{u,v}^\text{NE}$ that use the cocycle as edge weights on the north and east boundaries and weights $\omega_x$ in the bulk $x \leq v - e_1 - e_2$:

$$G_{u,v}^\text{NE} = B(u, v) \quad \text{for } u \in \{v - ke_i : k \in Z, i \in \{1, 2\}\}$$

and

$$G_{u,v}^\text{NE} = \omega_u + G_{u+e_1,v}^\text{NE} \lor G_{u+e_2,v}^\text{NE} \quad \text{for } u \leq v - e_1 - e_2.$$ 

It is immediate from recovery $\omega_x = B(x, x + e_1) \land B(x, x + e_2)$ and additivity of $B$ that

$$G_{u,v}^\text{NE} = B(u, v) \quad \text{for all } u \leq v.$$
Process \( \{G_{u,v}^{\text{NE}} : u \leq v\} \) is stationary in the sense that the increments
\[
G_{x,v}^{\text{NE}} - G_{x+e_i,v}^{\text{NE}} = B(x, x+e_i)
\]
are stationary under lattice translations and, as the equation above reveals, do not depend on the choice of the origin \( v \) (as long as we stay southwest of the origin).

**Remark 4.4.** In the exactly solvable cases where \( \omega_x \) is either exponential or geometric, more is known. Given the stationary cocycle, define weights

\[
Y_x = B(x - e_1, x) \wedge B(x - e_2, x).
\]

Then the weights \( \{Y_x\} \) have the same i.i.d. distribution as the original weights \( \{\omega_x\} \). Furthermore, \( \{Y_x : x \geq v + e_1 + e_2\} \) are independent of \( \{B(v + ke_i, v + (k+1)e_i) : k \in \mathbb{Z}_+, i \in \{1, 2\}\} \). Hence a stationary last-passage percolation process can be defined in the first quadrant with cocycles on the south and west boundaries:

\[
G_{v,x}^{\text{SW}} = B(v, x) \quad \text{for } x \in \{v + ke_i : k \in \mathbb{Z}_+, i \in \{1, 2\}\}
\]

and

\[
G_{v,x}^{\text{SW}} = Y_x + G_{v,x-e_1}^{\text{SW}} \vee G_{v,x-e_2}^{\text{SW}} \quad \text{for } x \geq v + e_1 + e_2.
\]

This feature appears in [3] as the “Burke property” of the exponential last-passage model. It also works for the log-gamma polymer in positive temperature [15, 32]. We do not know presently if this works in the general last-passage case. It would follow for example if we knew that the distributions of the cocycles of Theorem 4.2 satisfy this lattice symmetry:

\[
\{B(x, y) : x, y \in \mathbb{Z}^2\} \overset{d}{=} \{B(-y, -x) : x, y \in \mathbb{Z}^2\}.
\]

**4.3. Solution to the variational formulas.** We solve variational formulas (2.15)–(2.16) with the cocycles on the extended space \( (\tilde{\Omega}, \tilde{\mathcal{S}}, \tilde{\mathbb{P}}) \). Once we have identified some cocycles as Busemann functions in Section 5, we prove Theorem 2.3 as a corollary at the end of Section 5.

Theorem 3.6 in [14] says that if a cocycle \( B \) recovers \( V(\hat{\omega}), h(B) \) is defined by (2.10), and centered cocycle \( F \) is defined by (2.11), then \( F \) is a minimizer in (2.15) for any \( h \in \mathbb{R}^2 \) that satisfies \( (h - h(B)) \cdot (e_2 - e_1) = 0 \). For such \( h \), the essential supremum over \( \hat{\omega} \) in (2.15) disappears and we have

\[
g_{pl}(h) = \max \{V(\hat{\omega}) + h \cdot e_1 + F(\hat{\omega}, 0, e_1), V(\hat{\omega}) + h \cdot e_2 + F(\hat{\omega}, 0, e_2)\}
\]

\[
= (h - h(B)) \cdot z \quad \text{for } \tilde{\mathbb{P}}\text{-a.e. } \hat{\omega} \text{ and any } z \in \{e_1, e_2\}.
\]

Recall from Theorem 3.3 that \( g_{pp} \) is linear over each line segment \([\xi(h), \overline{\xi}(h)]\) and hence, by part (iii) of Theorem 4.2, cocycles \( B^\xi \) (and hence the tilts \( h(\xi) \) they define) coincide for all \( \xi \in [\xi(h), \overline{\xi}(h)] \).

**Theorem 4.5.** Let \( \{B^\xi\pm\} \) be the cocycles given in Theorem 4.2. Fix \( h \in \mathbb{R}^2 \). Let \( t(h), \xi(h), \overline{\xi}(h) \) be as in Theorem 3.3. One has the following three cases.

(i) \( \xi(h) \neq \overline{\xi}(h) \): For any \( \xi \) (and hence all) \( \xi \in [\xi(h), \overline{\xi}(h)] \) let

\[
F^\xi(x, y) = h(\xi) \cdot (x - y) - B^\xi(x, y).
\]

Then \( F^\xi \) solves (2.15): for \( \tilde{\mathbb{P}}\)-almost every \( \hat{\omega} \)

\[
g_{pl}(h) = \max \{\omega_0 + h \cdot e_1 + F^\xi(\hat{\omega}, 0, e_1), \omega_0 + h \cdot e_2 + F^\xi(\hat{\omega}, 0, e_2)\} = t(h).
\]
(ii) \( \xi(h) = \xi(h) = \xi \in \mathcal{D} \): (4.9) holds for \( F^\xi \) defined as in (4.8).

(iii) \( \xi(h) = \xi(h) = \xi \notin \mathcal{D} \): Let \( \theta \in [0,1] \) be such that

\[
h - t(h)(e_1 + e_2) = \theta h(\xi-) + (1 - \theta)h(\xi+)
\]

and define

\[
F^\xi_{\pm}(x,y) = h(\xi_\pm) \cdot (x - y) - B^\xi_{\pm}(x,y) \quad \text{and} \quad F(x,y) = \theta F^\xi_{\pm}(x,y) + (1 - \theta)F^\xi_{\pm}(x,y).
\]

Then \( F \) solves (2.15): for \( \mathbb{P} \)-almost every \( \omega \)

\[
g_{\text{pl}}(h) = \hat{\mathbb{P}}^{\text{ess sup}} \max \{\omega_0 + h \cdot e_1 + F(\hat{\omega},0,e_1), \omega_0 + h \cdot e_2 + F(\hat{\omega},0,e_2)\} = t(h).
\]

If \( \theta \in \{0,1\} \), then the essential supremum is not needed in (4.10), i.e. (4.9) holds almost surely with \( F \) in place of \( F^\xi \).

Here is the qualitative descriptions of the cases above.

(i) The graph of \( f \) has a corner at the point \((\alpha,f(\alpha))\) where it crosses the 45° diagonal that contains \(-h\). Correspondingly, \( \gamma \) is linear on the interval \([\xi(h),\bar{\xi}(h)]\) and \( g_{\text{pp}} \) is linear on \([\xi(h),\bar{\xi}(h)]\) with gradient \( \nabla g_{\text{pp}}(\xi) = -(\alpha,f(\alpha)) \) at interior points \( \xi \in [\xi(h),\bar{\xi}(h)] \).

(ii) \( g_{\text{pp}} \) is strictly concave and differentiable at \( \xi \) dual to tilt \( h \).

(iii) \( g_{\text{pp}} \) is strictly concave but not differentiable at \( \xi \) dual to tilt \( h \).

**Proof of Theorem 4.5.** By (ii) of Theorem 4.2 the cocycles \( B^\xi \) and \( B^\xi_{\pm} \) that appear in claims (i)–(iii) recover the potential as required by Definition 4.1 and hence conclusion (4.7) is in force.

In cases (i) and (ii) (4.7) implies

\[
g_{\text{pl}}(h) = \max \{\omega_0 + h \cdot e_1 + F(0,e_1), \omega_0 + h \cdot e_2 + F(0,e_2)\} = (h - h(\xi)) \cdot e_1.
\]

The last term equals \( t(h) \), by combining (3.13) and (4.2). The same proof works for (iii) when \( \theta \in \{0,1\} \).

For the last case (iii), (4.2) and (3.15) imply \( g_{\text{pl}}(h(\xi_\pm)) = 0 \). Then Theorem 3.3 implies

\[
g_{\text{pl}}(h) = g_{\text{pl}}(\xi) + h \cdot \xi = t(h) + \theta g_{\text{pl}}(\xi) + h(\xi_\pm) \cdot (1 - \theta)g_{\text{pl}}(\xi)
\]

\[
= t(h) + \theta g_{\text{pl}}(h(\xi_\pm)) = (1 - \theta)g_{\text{pl}}(h(\xi_\pm))
\]

Furthermore, we have for \( \mathbb{P} \)-almost every \( \omega \)

\[
\min \{\theta B^\xi(0,e_1) + (1 - \theta)B^\xi_{\pm}(0,e_1), \theta B^\xi(0,e_2) + (1 - \theta)B^\xi_{\pm}(0,e_2)\} \geq \omega_0.
\]

This translates into

\[
\hat{\mathbb{P}}^{\text{ess sup}} \max \{\omega_0 + h \cdot e_1 + F(0,e_1), \omega_0 + h \cdot e_2 + F(0,e_2)\} \leq t(h) = g_{\text{pl}}(h).
\]

Formula (2.15) implies then that the above inequality is in fact an equality and (4.10) is proved.

We state also the corresponding theorem for the point-to-point case, though there is nothing to prove.
Theorem 4.6. Let $\xi \in \mathrm{ri} \mathcal{U}$. Cocycles $B^{\xi \pm}$ solve (2.16):

$$g_{pp}(\xi) = \max\{\omega_0 - B^{\xi \pm}(\hat{\omega}, 0, e_1) - h(\xi) \cdot \xi, \omega_0 - B^{\xi \pm}(\hat{\omega}, 0, e_2) - h(\xi) \cdot \xi\}$$

$$= -h(\xi) \cdot \xi \quad \text{for } \hat{\mathbb{P}}\text{-a.e. } \hat{\omega}.$$ 

If $\xi \notin \mathcal{D}$ and $\theta \in (0, 1)$, then cocycle $B = \theta B^\infty + (1 - \theta)B^{\xi \pm}$ also solves (2.16):

$$g_{pp}(\xi) = \hat{\mathbb{P}}\text{-ess sup}_{\hat{\omega}} \max\{\omega_0 - B(\hat{\omega}, 0, e_1) - h(B) \cdot \xi, \omega_0 - B(\hat{\omega}, 0, e_2) - h(B) \cdot \xi\}$$

$$= -h(B) \cdot \xi.$$

The above theorem follows directly from (2.5), Theorem 4.2(ii), (4.9), and the fact that cocycles $B^{\xi \pm}$ recover $V(\hat{\omega}) = \omega_0$. The last claim follows similarly to (4.10).

5. Busemann functions

In this section we prove the existence of Busemann functions. As before (2.1) is a standing assumption. Recall the line segment $\mathcal{U}_\xi = [\xi, \xi]$ with $\xi \cdot e_1 \leq \xi \cdot e_1$ from (2.8)–(2.9) and the cocycles $B^{\xi \pm}$ constructed on the extended space $(\hat{\Omega}, \hat{\mathcal{G}}, \hat{\mathbb{P}})$ in Theorem 4.2.

Theorem 5.1. Fix $\xi \in \mathrm{ri} \mathcal{U}$. Then there exists an event $\hat{\Omega}_0$ with $\hat{\mathbb{P}}(\hat{\Omega}_0) = 1$ such that for each $\hat{\omega} \in \Omega_0$ and for any sequence $v_n \in \mathbb{Z}_+^2$ that satisfies

$$|v_n|_1 \to \infty \quad \text{and} \quad \xi \cdot e_1 \leq \lim_{n \to \infty} \frac{v_n \cdot e_1}{|v_n|_1} \leq \lim_{n \to \infty} \frac{v_n \cdot e_1}{|v_n|_1} \leq \xi \cdot e_1,$$

we have

$$B^{\xi \pm}(\hat{\omega}, x, x + e_1) \leq \lim_{n \to \infty} (G_{x,v_n}(\omega) - G_{x+e_1,v_n}(\omega))$$

$$\leq \lim_{n \to \infty} (G_{x,v_n}(\omega) - G_{x+e_1,v_n}(\omega)) \leq B^{\xi \pm}(\hat{\omega}, x, x + e_1)$$

and

$$B^{\xi \pm}(\hat{\omega}, x, x + e_2) \leq \lim_{n \to \infty} (G_{x,v_n}(\omega) - G_{x+e_2,v_n}(\omega))$$

$$\leq \lim_{n \to \infty} (G_{x,v_n}(\omega) - G_{x+e_2,v_n}(\omega)) \leq B^{\xi \pm}(\hat{\omega}, x, x + e_2).$$

The interesting cases are of course the ones where we have a limit. For the corollary note that if $\xi, \xi, \xi \in \mathcal{D}$ then by Theorem 4.2(iii) $B^{\xi \pm} = B^\xi = B^{\xi \pm}$.

Corollary 5.2. Assume that $\xi, \xi$ and $\xi$ are points of differentiability of $g_{pp}$. Then there exists an event $\hat{\Omega}_0$ with $\hat{\mathbb{P}}(\hat{\Omega}_0) = 1$ such that for each $\hat{\omega} \in \Omega_0$, for any sequence $v_n \in \mathbb{Z}_+^2$ that satisfies (5.1), and for all $x, y \in \mathbb{Z}^2$,

$$B^{\xi}(\hat{\omega}, x, y) = \lim_{n \to \infty} (G_{x,v_n}(\omega) - G_{y,v_n}(\omega)).$$

In particular, if $g_{pp}$ is differentiable everywhere on $\mathrm{ri} \mathcal{U}$, then for each direction $\xi \in \mathrm{ri} \mathcal{U}$ there is an event of full $\hat{\mathbb{P}}$-probability on which limit (5.4) holds for any sequence $v_n/|v_n|_1 \to \xi$.

Before turning to proofs, let us derive the relevant results of Section 2 and address the question of measurability of cocycles raised in Remark 4.3(a).
Proof of Theorem 2.2. Immediate consequence of Corollary 5.2. Equation (2.14) follows from (4.2).

Proof of Theorem 2.3. The theorem follows from Theorems 4.5 and 4.6 because the Busemann function $B^\xi$ is the cocycle $B^\xi$ from Theorem 4.2.

Remark 5.3 ($\mathcal{G}$-measurability of cocycles). A consequence of limit (5.4) is that the cocycle $B^\xi(\omega, x, y)$ is actually a function of $\omega$ alone, in other words, $\mathcal{G}$-measurable. Furthermore, all cocycles $B^{\xi\pm}$ that can be obtained as limits, as these $\xi$-points converge to $\zeta\pm$ on $riU$, are also $\mathcal{G}$-measurable. In particular, if $g_{op}$ is differentiable at the endpoints of its linear segments (if any), all the cocycles $\{B^{\xi\pm} : \xi \in riU\}$ described in Theorem 4.2 are $\mathcal{G}$-measurable. At points $\zeta \notin D$ of strict concavity this follows because $\zeta$ can be approached from both sides by points $\xi \in E$ which satisfy (5.4).

The remainder of this section proves Theorem 5.1. We begin with a general comparison lemma. With arbitrary real weights $\{\tilde{Y}_{x}\}_{x \in \mathbb{Z}^2}$ define last passage times

$$
\tilde{G}_{u,v} = \max_{x_0, n} \sum_{k=0}^{n-1} \tilde{Y}_{x_k}.
$$

The maximum is over up-right paths from $x_0 = u$ to $x_n = v$ with $n = |v - u|_1$. The convention is $\tilde{G}_{v,v} = 0$. For $x \leq v - e_1$ and $y \leq v - e_2$ denote the increments by

$$
\tilde{I}_{x,v} = \tilde{G}_{x,v} - \tilde{G}_{x+e_1,v} \quad \text{and} \quad \tilde{J}_{y,v} = \tilde{G}_{y,v} - \tilde{G}_{y+e_2,v}.
$$

Lemma 5.4. For $x \leq v - e_1$ and $y \leq v - e_2$,

$$
\tilde{I}_{x,v+e_2} \geq \tilde{I}_{x,v} \geq \tilde{I}_{x,v+e_1} \quad \text{and} \quad \tilde{J}_{y,v+e_2} \leq \tilde{J}_{y,v} \leq \tilde{J}_{y,v+e_1}.
$$

Proof. Let $v = (m, n)$. The proof goes by an induction argument. Suppose $x = (k, n)$ for some $k < n$. Then on the north boundary

$$
\tilde{I}_{(k,n),(m,n+1)} = \tilde{G}_{(k,n),(m,n+1)} - \tilde{G}_{(k+1,n),(m,n+1)}
$$

$$
= \tilde{Y}_{k,n} + \tilde{G}_{(k+1,n),(m,n+1)} \vee \tilde{G}_{(k,n+1),(m,n+1)} - \tilde{G}_{(k+1,n),(m,n+1)}
$$

$$
\geq \tilde{Y}_{k,n} = \tilde{G}_{(k,n),(m,n)} - \tilde{G}_{(k+1,n),(m,n)} = \tilde{I}_{(k,n),(m,n)}.
$$

On the east boundary, when $y = (m, \ell)$ for some $\ell < n$

$$
\tilde{J}_{(m,\ell),(m,n+1)} = \tilde{G}_{(m,\ell),(m,n+1)} - \tilde{G}_{(m,\ell+1),(m,n+1)}
$$

$$
= \tilde{Y}_{m,\ell} = \tilde{G}_{(m,\ell),(m,n)} - \tilde{G}_{(m,\ell+1),(m,n)} = \tilde{J}_{(m,\ell),(m,n)}.
$$

These inequalities start the induction. Now let $u \leq v - e_1 - e_2$. Assume by induction that (5.5) holds for $x = u + e_2$ and $y = u + e_1$.

$$
\tilde{I}_{u,v+e_2} = \tilde{G}_{u,v+e_2} - \tilde{G}_{u+e_1,v+e_2} = \tilde{Y}_{u} + (\tilde{G}_{u+e_2,v+e_2} - \tilde{G}_{u+e_1,v+e_2})^+
$$

$$
= \tilde{Y}_{u} + (\tilde{I}_{u+e_2,v+e_2} - \tilde{J}_{u+e_1,v+e_2})^+
$$

$$
\geq \tilde{Y}_{u} + (\tilde{I}_{u+e_2,v} - \tilde{J}_{u+e_1,v})^+ = \tilde{I}_{u,v}.
$$
For the last equality simply reverse the first three equalities with \( v \) instead of \( v + e_2 \). A similar argument works for \( \tilde{I}_{u,v} \geq \tilde{I}_{u,v+e_1} \) and a symmetric argument works for the \( \tilde{J} \) inequalities. \( \square \)

The estimates needed for the proof of Theorem 5.1 come from coupling \( G_{u,v} \) with the stationary LPP described in Section 4.2. For the next two lemmas fix a cocycle \( B(\hat{\omega}, x, y) = B^\pm(\hat{\omega}, x, y) \) from Theorem 4.2 and let \( r = \zeta \cdot e_1 / \zeta \cdot e_2 \) so that \( \alpha = \gamma'(r \pm) \) satisfies
\[
\alpha = \hat{E}[B(x, x + e_1)] \quad \text{and} \quad f(\alpha) = \hat{E}[B(x, x + e_2)].
\]
As in (4.5) define
\[
G_{u,v}^{NE} = B(u, v) \quad \text{for} \quad u \in \{v - ke_i : k \in \mathbb{Z}_+, i \in \{1, 2\}\}
\]
and
\[
G_{u,v}^{NE} = \omega + G_{u+e_1,v}^{NE} \lor G_{u+e_2,v}^{NE} \quad \text{for} \quad u \leq v - e_1 - e_2.
\]

Let \( G_{u,v}^{NE}(A) \) denote a maximum over paths restricted to the set \( A \). In particular, below we use
\[
G_{0,v_n}^{NE}(v_n - e_i \in x_i) = \max_{x, x : |x|_{v_n} = v_n - e_i} \sum_{k=0}^{|v_n|_1 - 1} \tilde{Y}_{x_k}
\]
where the maximum is restricted to paths that go through the point \( v_n - e_i \), and the weights are from (5.7): \( \tilde{Y}_x = \omega_x \) for \( x \leq v - e_1 - e_2 \) while \( \tilde{Y}_{v-ke_i} = B(v - ke_i, v - (k-1)e_i) \).

Figure 4 makes the limits of the next lemma obvious. But a.s. convergence requires some technicalities because the north-east boundaries themselves are translated as the limit is taken.

**Figure 4.** Illustration of (5.8). Forcing the last step to be \( e_1 \) restricts the maximization to paths that hit the north boundary instead of the east boundary. The path from 0 to \((s - \tau, t)\) contributes \( g_{pp}(s - \tau, t) \) and the remaining segment of length \( \tau \) on the north boundary contributes \( \alpha \tau \).

**Lemma 5.5.** Assume (2.1). Fix \((s, t) \in \mathbb{R}_+^2 \). Let \( v_n \in \mathbb{Z}_+^2 \) be such that \( v_n / |v_n|_1 \to (s, t) / (s + t) \) as \( n \to \infty \) and \( |v_n|_1 \geq \eta_0 n \) for some constant \( \eta_0 > 0 \). Then we have the following almost sure limits:
\[
|v_n|_1^{-1} G_{0,v_n}^{NE}(v_n - e_1 \in x_i) \to (s + t)^{-1} \sup_{0 \leq \tau \leq s} \{\alpha \tau + g_{pp}(s - \tau, t)\}
\]
and
\[
|v_n|^{-1} G_{0,v_n}^{\text{NE}} (v_n - e_2 \in x) \to (s + t)^{-1} \sup_{0 \leq \tau \leq t} \{ f(\alpha) \tau + g_{pp}(s, t - \tau) \}.
\]

**Proof.** We prove (5.8). Fix \( \varepsilon > 0 \), let \( M = \lceil \varepsilon^{-1} \rceil \), and
\[
q^n_j = j \left\lfloor \frac{\varepsilon |v_n|_1 s}{s + t} \right\rfloor \quad \text{for } 0 \leq j \leq M - 1, \quad \text{and } q^n_M = v_n \cdot e_1.
\]
For large enough \( n \) it is the case that \( q^n_{M-1} < v_n \cdot e_1 \).

Suppose a maximal path for \( G_{0,v_n}^{\text{NE}} (v_n - e_1 \in x, ) \) enters the north boundary from the bulk at the point \( v_n - (\ell, 0) \) with \( q_j^n < \ell \leq q_{j+1}^n \). Then
\[
G_{0,v_n}^{\text{NE}} (v_n - e_1 \in x) = G_{0,v_n} - (\ell, 1) + \omega_{v_n} - (\ell, 1) + B(v_n - (\ell, 0), v_n)
\]
\[
\leq G_{0,v_n} - (q_j^n, 1) + q_j^n \alpha - \sum_{k=q_j^n+1}^{\ell-1} (\omega_{v_n} - (k, 1) - m_0) + (\ell - q_j^n) m_0
\]
\[
+ \left( B(v_n - (\ell, 0), n) - \ell \alpha \right) + (\ell - q_j^n) \alpha.
\]
The two main terms come right after the inequality above and the rest are errors. The inequality comes from
\[
G_{0,v_n} - (\ell, 1) + \sum_{k=q_j^n+1}^{\ell} \omega_{v_n} - (k, 1) \leq G_{0,v_n} - (q_j^n, 1)
\]
and algebraic rearrangement.

Define the centered cocycle \( F(x, y) = h(B) \cdot (x - y) - B(x, y) \) so that
\[
B(v_n - (\ell, 0), v_n) - \ell \alpha = F(0, v_n - (\ell, 0)) - F(0, v_n).
\]
The potential-recovery property (4.1) \( \omega_0 = B(0, e_1) \wedge B(0, e_2) \) gives
\[
F(0, e_i) \leq \alpha \lor f(\alpha) - \omega_0 \quad \text{for } i \in \{1, 2\}.
\]
The i.i.d. distribution of \( \{\omega_x\} \) and \( \mathbb{E}(\lfloor \omega_0 \rfloor^p) < \infty \) with \( p > 2 \) are strong enough to guarantee that Lemma C.1 from Appendix C applies and gives
\[
\lim_{N \to \infty} \frac{1}{N} \max_{x \geq 0 : |x| \leq N} |F(\hat{\omega}, 0, x)| = 0 \quad \text{for a.e. } \hat{\omega}.
\]
Collect the bounds for all the intervals \( (q_j^n, q_j^{n+1}] \) and let \( C \) denote a constant. Abbreviate
\[
S^n_{j,m} = \sum_{k=q_j^n+1}^{q_j^{n+1}} (\omega_{v_n} - (k, 1) - m_0).
\]

\[
G_{0,v_n}^{\text{NE}} (v_n - e_1 \in x) \leq \max_{0 \leq j \leq M - 1} \left\{ G_{0,v_n} - (q_j^n, 1) + q_j^n \alpha + C(q_j^{n+1} - q_j^n)
\right\}
\]
\[
+ \max_{0 \leq m \leq q_j^{n+1} - q_j^n} |S^n_{j,m}| + \max_{q_j^n < \ell \leq q_j^{n+1}} F(0, v_n - (\ell, 0)) - F(0, v_n).
\]
Divide through by \( |v_n|_1 \) and let \( n \to \infty \). Limit (E.1) gives convergence of the \( G \)-term on the right. We claim that the terms on the second line of (5.11) vanish. Limit (5.10) takes care
of the $F$-terms. Combine Doob’s maximal inequality for martingales with Burkholder’s inequality \cite[Thm. 3.2]{4} to obtain, for $\delta > 0$,

$$
P\{ \max_{0 \leq m < q_{j+1} - q_j} |S_{j,m}^n| \geq \delta |v_n|_1 \} \leq \frac{\mathbb{E}[|S_{j+1}^n - g_j^n|_1^p]}{\delta^p |v_n|_1^p} \leq \frac{C}{|v_n|_1^p}.
$$

Thus Borel-Cantelli takes care of the $S_{j,m}^n$-term on the second line of (5.11). (This is the place where the assumption $|v_n|_1 \geq \eta_0 n$ is used.) We have the upper bound

$$
\lim_{n \to \infty} |v_n|_1^{-1} G^{\text{NE}}_{0,v_n}(v_n - e_1 \in x) \leq (s + t)^{-1} \max_{0 \leq j \leq M-1} [g_{\text{pp}}(s - sj \varepsilon, t) + sj \varepsilon \alpha + C \varepsilon s].
$$

Let $\varepsilon \searrow 0$ to complete the proof of the upper bound.

To get the matching lower bound let the supremum $\sup_{\tau \in [0,s]} \{ \tau \alpha + g_{\text{pp}}(s - \tau, t) \}$ be attained at $\tau^* \in [0,s]$. With $m_n = |v_n|_1/(s + t)$ we have

$$
G^{\text{NE}}_{0,v_n}(v_n - e_1 \in x) \geq G^{\text{NE}}_{0,v_n - (\lfloor m_n \tau^* \rfloor \vee 1, 1)} + \omega_{v_n - (\lfloor m_n \tau^* \rfloor \vee 1, 1)} + B(v_n - (\lfloor m_n \tau^* \rfloor \vee 1, 0), v_n).
$$

Use again the cocycle $F$ from above, and let $n \to \infty$ to get

$$
\lim_{n \to \infty} |v_n|_1^{-1} G^{\text{NE}}_{0,v_n}(v_n - e_1 \in x) \geq (s + t)^{-1} [g_{\text{pp}}(s - \tau^*, t) + \tau^* \alpha].
$$

This completes the proof of (5.8). \hfill \square

**Figure 5.** Illustration of Lemma 5.6. With $\alpha$-boundaries geodesics tend to go in the $\alpha$-characteristic direction $\zeta$. If $v_n$ converges in a direction below $\zeta$, maximal paths to $v_n$ tend to hit the north boundary. The dotted path that hits the east boundary is unlikely to be maximal for large $n$.

Continue with the stationary LPP defined by (5.7) in terms of a cocycle $B = B^{\zeta \mp}$, with $r = \zeta \cdot e_1 / \zeta \cdot e_2$ and $\alpha$ as in (5.6). Let us call the direction $\zeta$ *characteristic* for $\alpha$. The next lemma shows that in stationary LPP a maximizing path to a point below the characteristic
direction will eventually hit the north boundary before the east boundary. (Illustration in Figure 5.) We leave to the reader the analogous result to a point above the characteristic line.

**Lemma 5.6.** Let \( s \in (r, \infty) \). Let \( v_n \in \mathbb{Z}^2 \) be such that \( v_n/|v_n|_1 \to (s,1)/(1+s) \) and \( |v_n|_1 \geq \eta_0 n \) for some constant \( \eta_0 > 0 \). Assume that \( \gamma'(r+) > \gamma'(s-) \). Then \( \hat{\mathbb{P}} \)-a.s. there exists a random \( n_0 < \infty \) such that for all \( n \geq n_0 \),

\[
G_{0,v_n}^{\text{NE}} = G_{0,v_n}^{\text{NE}}(v_n - e_1 \in x). 
\]

**Proof.** The right derivative at \( \tau = 0 \) of \( \alpha \tau + g_{pp}(s-\tau,1) = \alpha \tau + \gamma(s-\tau) \) equals

\[
\alpha - \gamma'(s-) > \alpha - \gamma'(r+) \geq 0.
\]

The last inequality above follows from the assumption on \( r \). Thus we can find \( \tau^* \in (0, r) \) such that

\[
(5.13) \quad \alpha \tau^* + g_{pp}(s-\tau^*,1) > g_{pp}(s,1).
\]

To produce a contradiction let \( A \) be the event on which \( G_{0,v_n}^{\text{NE}} = G_{0,v_n}^{\text{NE}}(v_n - e_2 \in x) \) for infinitely many \( n \) and assume \( \hat{\mathbb{P}}(A) > 0 \). Let \( m_n = |v_n|_1/(1+s) \). On \( A \) we have for infinitely many \( n \)

\[
|v_n|^{-1} G_{0,v_n}^{\text{NE}}(v_n - e_2 \in x) = |v_n|^{-1} G_{0,v_n}^{\text{NE}}
\]

\[
\geq |v_n|^{-1} B(v_n - ([m_n \tau^*] + 1)e_1, v_n) + |v_n|^{-1} G_{0,v_n - ([m_n \tau^*]+1,1)}
\]

\[
+ |v_n|^{-1} \omega_{v_n - ([m_n \tau^*]+1,1)}. 
\]

Apply (5.9) to the leftmost quantity. Apply limits (E.1) and (5.10) and stationarity and integrability of \( \omega \) to the expression on the right. Both extremes of the above inequality converge almost surely. Hence on the event \( A \) the inequality is preserved to the limit and yields (after multiplication by \( 1+s \))

\[
\sup_{0 \leq \tau \leq 1} \{ f(\alpha \tau + g_{pp}(s,1-\tau)) \geq \alpha \tau^* + g_{pp}(s-\tau^*,1). 
\]

The supremum of the left-hand side is achieved at \( \tau = 0 \) because the right derivative equals

\[
f(\alpha) - \gamma'(\frac{1-\tau}{s}) \leq f(\alpha) - \gamma'(r^{-1}) \leq 0
\]

where the first inequality comes from \( s^{-1} < r^{-1} \) and the second from (3.8). Therefore

\[
g_{pp}(s,1) \geq \alpha \tau^* + g_{pp}(s-\tau^*,1)
\]

which contradicts (5.13). Consequently \( \hat{\mathbb{P}}(A) = 0 \) and (5.12) holds for \( n \) large. \( \square \)

**Proof of Theorem 5.1.** The proof goes in two steps.

**Step 1.** First consider a fixed \( \xi = (\frac{s}{1+s}, \frac{s}{1+s}) \in \text{ri} \mathcal{U} \) and a sequence \( v_n \) such that \( v_n/|v_n|_1 \to \xi \) and \( |v_n|_1 \geq \eta_0 n \) for some \( \eta_0 > 0 \). We prove that the last inequality of (5.2) holds almost surely. Let \( \zeta = (\frac{\xi}{|\xi|}, \frac{\xi}{|\xi|}) \) satisfy \( \zeta \cdot e_1 < \xi \cdot e_1 \) so that \( \gamma'(r+) > \gamma'(s-) \) and Lemma 5.6 can be applied. Use cocycle \( B^{\xi} \) from Theorem 4.2 to define last-passage times...
$G_{u,v}^{\text{NE}}$ as in (5.7). Furthermore, define last-passage times $G_{u,v}^N$ that use cocycles only on the north boundary and bulk weights elsewhere:

$$
G_{u,v}^N = \omega_u + G_{u+e_1,v}^N \vee G_{u+e_2,v}^N \quad \text{for } u \leq v - e_1 - e_2.
$$

For large $n$ we have

$$
G_{x,v_n} - G_{x+e_1,v_n} \leq G_{x,v_n+e_2} - G_{x+e_1,v_n+e_2} = G_{x,v_n+e_2}(v_n + e_2 \in \mathcal{X}) - G_{x+e_1,v_n+e_2}(v_n + e_2 \in \mathcal{X}) = G_{x,v_n+e_2} - G_{x+e_1,v_n+e_2} = B_{\zeta^+}(x, x + e_1).
$$

The first inequality above is the first inequality of (5.5). The first equality above is obvious. The second equality is Lemma 5.6 and the last equality is (4.6). Thus

$$
\lim_{n \to \infty} (G_{x,v_n} - G_{x+e_1,v_n}) \leq B_{\zeta^+}(x, x + e_1).
$$

Let $\zeta \cdot e_1$ increase to $\xi \cdot e_1$. Theorem 4.2(iv) implies

$$
\lim_{n \to \infty} (G_{x,v_n} - G_{x+e_1,v_n}) \leq B_{\zeta^+}(x, x + e_1).
$$

An analogous argument gives the matching lower bound (first inequality of (5.2)) by taking $\zeta \cdot e_1 > \xi \cdot e_1$ and by reworking Lemma 5.6 for the case where the direction of $v_n$ is above the characteristic direction $\zeta$. Similar reasoning works for vertical increments $G_{x,v_n} - G_{x+e_2,v_n}$.

**Step 2.** We prove the full statement of the theorem. Let $\eta_\ell$ and $\zeta_\ell$ be two sequences in $\mathcal{R}\mathcal{I}\mathcal{U}$ such that $\eta_\ell \cdot e_1 < \zeta_\ell \cdot e_1$, $\bar{\zeta}_\ell \cdot e_1 < \zeta_\ell \cdot e_1$, $\eta_\ell \to \zeta$, and $\zeta_\ell \to \bar{\zeta}$. Let $\hat{\Omega}_0$ be the event on which limits (4.4) holds for directions $\xi$ and $\zeta$ (with sequences $\zeta_\ell$ and $\eta_\ell$, respectively) and (5.2) holds for each direction $\zeta_\ell$ with sequence $\lfloor n\zeta_\ell \rfloor$, and for each direction $\eta_\ell$ with sequence $\lfloor n\eta_\ell \rfloor$. $\hat{\mathbb{P}}(\hat{\Omega}_0) = 1$ by Theorem 4.2(iv) and Step 1.

Fix $\ell$ and a sequence $v_n$ as in (5.1). Abbreviate $a_n = |v_n|_1$. For large $n$

$$
|a_n\eta_\ell| \cdot e_1 < v_n \cdot e_1 < |a_n\zeta_\ell| \cdot e_1 \quad \text{and} \quad |a_n\eta_\ell| \cdot e_2 > v_n \cdot e_2 > |a_n\zeta_\ell| \cdot e_2.
$$

By repeated application of Lemma 5.4

$$
G_{x,[a_n\zeta_\ell]} - G_{x+e_1,[a_n\zeta_\ell]} \leq G_{x,v_n} - G_{x+e_1,v_n} \leq G_{x,[a_n\eta_\ell]} - G_{x+e_1,[a_n\eta_\ell]}.
$$

Take $n \to \infty$ and apply (5.2) to the sequences $\lfloor a_n\zeta_\ell \rfloor$ and $\lfloor a_n\eta_\ell \rfloor$. This works because $\lfloor a_n\zeta_\ell \rfloor$ is a subset of $\lfloor n\zeta_\ell \rfloor$ that escapes to infinity. Thus for $\hat{\omega} \in \hat{\Omega}_0$

$$
B_{\hat{\omega}} \leq \lim_{n \to \infty} (G_{x,v_n}(\omega) - G_{x+e_1,v_n}(\omega)) \leq \lim_{n \to \infty} (G_{x,v_n}(\omega) - G_{x+e_1,v_n}(\omega)) \leq B_{\hat{\omega}} \leq \lim_{n \to \infty} (G_{x,v_n}(\omega) - G_{x+e_1,v_n}(\omega)) \leq B_{\hat{\omega}}.
$$

Take $\ell \to \infty$ and apply (4.4) to arrive at (5.2) as stated. (5.3) follow similarly. \(\square\)
6. Directional geodesics

This section proves the results on geodesics. We work on the extended space \( \hat{\Omega} = \Omega \times \Omega' \) and define geodesics in terms of the cocycles \( B^{\pm} \) constructed in Theorem 4.2. The idea is in the next lemma, followed by the definition of cocycle geodesics.

**Lemma 6.1.** Let \( B \) be any stationary cocycle that recovers potential \( V(\hat{\omega}) = \omega_0 \), as in Definitions 2.1 and 4.1. Fix \( \hat{\omega} \) so that properties (b)–(c) of Definition 2.1 and (4.1) hold for all translations \( T_x \hat{\omega} \).

(a) Let \( x_{m,n} = (x_k)_{k=m}^n \) be any up-right path that follows minimal gradients of \( B \), that is,

\[
\omega_{x_k} = B(\hat{\omega}, x_k, x_{k+1}) \quad \text{for all } m \leq k < n.
\]

Then \( x_{m,n} \) is a geodesic from \( x_m \) to \( x_n \):

\[
G_{x_m,x_n}(\omega) = \sum_{k=m}^{n-1} \omega_{\bar{x}_k} = B(\hat{\omega}, x_m, x_n).
\]

(b) Let \( x_{m,n} = (x_k)_{k=m}^n \) be an up-right path such that for all \( m \leq k < n \)

\[
\text{either } \omega_{x_k} = B(x_k, x_{k+1}) < B(x_k, x_k + e_1) \vee B(x_k, x_k + e_2)
\]

\[
\text{or } x_{k+1} = x_k + e_2 \text{ and } B(x_k, x_k + e_1) = B(x_k, x_k + e_2).
\]

In other words, path \( x_{m,n} \) follows minimal gradients of \( B \) and takes an \( e_2 \)-step in a tie. Then \( x_{m,n} \) is the leftmost geodesic from \( x_m \) to \( x_n \). Precisely, if \( \bar{x}_{m,n} \) is an up-right path from \( x_m = \bar{x}_m \) to \( x_n = \bar{x}_n \) and \( G_{x_m,x_n} = \sum_{k=m}^{n-1} \omega_{\bar{x}_k} \), then \( x_k \cdot e_1 \leq \bar{x}_k \cdot e_1 \) for all \( m \leq k \leq n \).

If ties are broken by \( e_1 \)-steps the resulting geodesic is the rightmost geodesic between \( x_m \) and \( x_n \): \( x_k \cdot e_1 \geq \bar{x}_k \cdot e_1 \) for all \( m \leq k < n \).

**Proof.** Part (a). Any up-right path \( \bar{x}_{m,n} \) from \( \bar{x}_m = x_m \) to \( \bar{x}_n = x_n \) satisfies

\[
\sum_{k=m}^{n-1} \omega_{\bar{x}_k} \leq \sum_{k=m}^{n-1} B(\bar{x}_k, \bar{x}_{k+1}) = B(x_m, x_n) = \sum_{k=m}^{n-1} B(x_k, x_{k+1}) = \sum_{k=m}^{n-1} \omega_{x_k}.
\]

Part (b). \( x_{m,n} \) is a geodesic by part (a). To prove that it is the leftmost geodesic assume \( \bar{x}_k = x_k \) and \( x_{k+1} = x_k + e_1 \). Then \( \omega_{\bar{x}_k} = B(x_k, x_{k+1}) < B(x_k, x_k + e_1) \). Recovery of the weights gives \( G_{x,y} \leq B(x, y) \) for all \( x \leq y \). Combined with (6.1),

\[
\omega_{x_k} + G_{x_k + e_2, x_n} < B(x_k, x_k + e_2 + e_2, x_n) = B(x_k, x_n) = G_{x_k,x_n}.
\]

Hence also \( \bar{x}_{k+1} = \bar{x}_k + e_1 \) and the claim about being the leftmost geodesic is proved. The other claim is symmetric. \( \square \)

Next we define a cocycle geodesic, that is, a geodesic constructed by following minimal gradients of a cocycle \( B^{\pm} \) constructed in Theorem 4.2. Since our treatment allows discrete distributions, we introduce a function \( t \) on \( \mathbb{Z}^2 \) to resolve ties. For \( \xi \in riU \), \( u \in \mathbb{Z}^2 \), and \( t \in \{e_1, e_2\}^{\mathbb{Z}^2} \), let \( x_{0,\infty}^{u,t,\xi+} \) be the up-right path (one path for \( \xi+ \), one for \( \xi- \)) starting at
that under certain

Similarly, if

Lemma 6.1 (a), \( x_{0,\infty}^{u,\xi} \) is an infinite geodesic. Since the cocycles are defined on the space \( \hat{\Omega} \), the geodesics are measurable functions on \( \hat{\Omega} \). Recall from Remark 5.3 that under certain conditions a cocycle \( B^{\xi} \) is \( \mathcal{S} \)-measurable. When that happens, the geodesic \( x_{0,\infty}^{u,\xi} \) can be defined on \( \Omega \) without the artificial extension to the space \( \hat{\Omega} = \Omega \times \Omega' \). In particular, if \( g_{\text{pp}} \) is differentiable at the endpoints of its linear segments (if any), all geodesics \( x_{0,\infty}^{u,\xi} \) are \( \mathcal{S} \)-measurable.

If we restrict ourselves to the event \( \Omega_0 \) of full \( \hat{\mathcal{P}} \)-measure on which monotonicity (4.3) holds for all \( \xi, \zeta \in \text{ri} U \), we can order these geodesics in a natural way from left to right. Define a partial ordering on \( \{ e_1, e_2 \}^{\mathbb{Z}^2} \) by \( e_2 \leq e_1 \) and then \( t \leq t' \) coordinatewise. Then on the event \( \Omega_0 \), for any \( u \in \mathbb{Z}^2, t \leq t', \xi, \zeta \in \text{ri} U \) with \( \xi \cdot e_1 < \zeta \cdot e_1 \), and for all \( n \geq 0 \),

(6.2) \[ x_{n,\xi,\zeta}^{u,t} \cdot e_1 \leq x_{n,u',\xi}^{u,t} \cdot e_1, \quad x_{n,u,\xi}^{u,t} \cdot e_1 \leq x_{n,u,\xi}^{u,t} \cdot e_1, \quad \text{and} \quad x_{n,u,\xi}^{u,t} \cdot e_1 \leq x_{n,u,\xi}^{u,t} \cdot e_1. \]

The leftmost and rightmost tie-breaking rules are defined by \( \mathbf{1}_- = e_2 \) and \( \mathbf{1}_+ = e_1 \) for all \( x \in \mathbb{Z}^2 \). The cocycle limit (4.4) forces the cocycle geodesics to converge also, as the next lemma shows.

**Lemma 6.2.** Fix \( \xi \) and let \( \zeta_n \to \xi \) in \( \text{ri} U \). If \( \zeta_n \cdot e_1 > \xi \cdot e_1 \) \( \forall n \) then for all \( u \in \mathbb{Z}^2 \)

(6.3) \[ \hat{\mathcal{P}} \{ \forall k \geq 0 \exists n_0 < \infty : n \geq n_0 \Rightarrow x_{0,k}^{u,\zeta_n} = x_{0,k}^{u,\xi} \} = 1. \]

Similarly, if \( \zeta_n \cdot e_1 < \xi \cdot e_1 \) the limit holds with \( \xi \) on the right and \( \mathbf{1} \) replaced by \( \mathbf{1}_- \).

**Proof.** It is enough to prove the statement for \( u = 0 \). By (4.4), for a given \( k \) and large enough \( n \), if \( x \geq 0 \) with \( |x_1| \leq k \) and \( B^{\xi}(x, x + e_1) \neq B^{\xi}(x, x + e_2) \), then \( B^{\zeta}(x, x + e_1) = B^{\zeta}(x, x + e_2) \) does not vanish and has the same sign as \( B^{\xi}(x, x + e_1) = B^{\xi}(x, x + e_2) \). From such an \( x \) geodesics following the minimal gradient of \( B^{\zeta} \) or the minimal gradient of \( B^{\xi} \) stay together for their next step. On the other hand, when \( B^{\xi}(x, x + e_1) = B^{\xi}(x, x + e_2) \), monotonicity (4.3) implies

\[ B^{\zeta}(x, x + e_1) \leq B^{\zeta}(x, x + e_1) = B^{\xi}(x, x + e_2) \leq B^{\xi}(x, x + e_2). \]

Once again, both the geodesic following the minimal gradient of \( B^{\zeta} \) and rules \( \mathbf{1} \) and the one following the minimal gradients of \( B^{\xi} \) and rules \( \mathbf{1} \) will next take the same \( e_1 \)-step. This proves (6.3). The other claim is similar. \( \square \)

Recall the line segments \( U_{\xi}, U_{\xi}^{\pm} \) defined in (2.8)–(2.9). The endpoints of \( U_{\xi} = [\xi, \bar{\xi}] \) are given by

\[ \bar{\xi} \cdot e_1 = \sup\{ \alpha : (\alpha, 1 - \alpha) \in U_{\xi}^{+} \} \quad \text{and} \quad \xi \cdot e_1 = \inf\{ \alpha : (\alpha, 1 - \alpha) \in U_{\xi}^{-} \}. \]

By Lemma 3.1 both points are again in \( \text{ri} U \). When needed we extend this definition to the endpoints of \( U \) by \( U_{e_i} = U_{e_i}^{\pm} = \{ e_i \}, i \in \{ 1, 2 \} \).

The next theorem concerns the direction of the cocycle geodesics.
Theorem 6.3. We have these two statements:

\[(6.4) \quad \hat{P}\left\{ \forall \xi \in \text{ri} \mathcal{U}, \forall t \in \{e_1, e_2\}^{\mathbb{Z}^2}, \forall u \in \mathbb{Z}^2 : x_{n,0,\xi}^{u,t} \text{ is } \mathcal{U}_{\xi^-} \text{-directed} \right\} = 1.\]

If \(\xi \in \mathcal{D}\) then the statement should be taken without the \(\pm\).

Proof. Fix \(\xi \in \text{ri} \mathcal{U}\) and abbreviate \(x_n = x_n^{u,t,\xi} \). Since \(B_{\xi^+}\) recovers weights \(\omega\), Lemma 6.1(a) implies that \(G_{u,x_n} = B_{\xi^+}(u, x_n)\). Furthermore, \(B_{\xi^+}(x, y) + h(\xi^+) \cdot (y - x)\) is a centered cocycle, as in Definition 2.1. Theorem C.1 implies then

\[
\lim_{n \to \infty} |x_n|^{-1}(G_{u,x_n} + h(\xi^+) \cdot x_n) = 0 \quad \hat{P}\text{-almost surely.}
\]

Define \(\zeta(\hat{\omega}) \in \mathcal{U}\) by \(\zeta \cdot e_1 = \lim_{n \to \infty} \frac{x_n \cdot e_1}{|x_n|}\). If \(\zeta \cdot e_1 > \xi \cdot e_1\) then \(\zeta \not\in \mathcal{U}_{\xi^+}\) and hence

\[
gpp(\zeta) + h(\xi^+) \cdot \zeta = gpp(\zeta) - \nabla gpp(\xi^+) \cdot \zeta < gpp(\xi) - \nabla gpp(\xi^+) \cdot \xi = 0.
\]

(The first and last equalities come from (4.2) and (4.11).) Consequently, by the shape theorem (limit (E.1)), on the event \(\{\zeta \cdot e_1 > \xi \cdot e_1\}\)

\[
\lim_{n \to \infty} |x_n|^{-1}(G_{u,x_n} + h(\xi^+) \cdot x_n) < 0.
\]

This proves that

\[
\hat{P}\left\{ \lim_{n \to \infty} \frac{x_n \cdot e_1}{|x_n|} \leq \xi \cdot e_1 \right\} = 1.
\]

Repeat the same argument with \(\hat{\xi}\) replaced by \(\bar{\xi}\) and \(\overline{\xi}\) by the other endpoint of \(\mathcal{U}_{\xi^+}\) (which is either \(\xi^+\) or \(\xi^-\)). To capture all \(t\) use geodesics ordering (6.2). An analogous argument works for \(\xi^-\). We have, for a given \(\xi\),

\[(6.5) \quad \hat{P}\left\{ \forall t \in \{e_1, e_2\}^{\mathbb{Z}^2}, \forall u \in \mathbb{Z}^2 : x_{n,0,\xi}^{u,t,\xi} \text{ is } \mathcal{U}_{\xi^\pm} \text{-directed} \right\} = 1.
\]

Let \(\tilde{\mathcal{U}}_0\) be an event of full \(\hat{P}\)-probability on which all cocycle geodesics satisfy the ordering (6.2), and the event in (6.5) holds for \(\xi^\pm\) in a countable set \(\mathcal{U}_0\) that contains all points of nondifferentiability of \(gpp\) and a countable dense subset of \(\mathcal{D}\). We argue that \(\tilde{\mathcal{U}}_0\) is contained in the event in (6.4).

Let \(\zeta \not\in \mathcal{U}_0\) and let \(\overline{\zeta}\) denote the right endpoint of \(\mathcal{U}_\zeta\). We show that

\[
(6.6) \quad \lim_{n \to \infty} \frac{x_n \cdot \zeta \cdot e_1}{|x_n|} \leq \overline{\zeta} \cdot e_1 \quad \text{on the event } \tilde{\mathcal{U}}_0.
\]

(Note that \(\zeta \in \mathcal{D}\) so there is no \(\zeta^\pm\) distinction in the cocycle geodesic.) The limit with \(\hat{\xi}\) and \(\tilde{\xi}\) comes of course with the same argument.

If \(\zeta \cdot e_1 < \zeta \cdot e_1\) pick \(\xi \in \mathcal{D} \cap \mathcal{U}_0\) so that \(\zeta \cdot e_1 < \xi \cdot e_1 < \zeta \cdot e_1\). Then \(\xi = \overline{\xi}\) and (6.6) follows from the ordering.

If \(\zeta = \overline{\zeta}\), let \(\varepsilon > 0\) and pick \(\xi \in \mathcal{D} \cap \mathcal{U}_0\) so that \(\zeta \cdot e_1 < \xi \cdot e_1 \leq \xi \cdot e_1 < \zeta \cdot e_1 + \varepsilon\). This is possible because \(\nabla gpp(\xi)\) converges to but never equals \(\nabla gpp(\zeta)\) as \(\xi \cdot e_1 \searrow \zeta \cdot e_1\). Again by the ordering

\[
\lim_{n \to \infty} \frac{x_n \cdot \zeta \cdot e_1}{|x_n|} \leq \lim_{n \to \infty} \frac{x_n \cdot \zeta \cdot e_1}{|x_n|} \leq \xi \cdot e_1 < \zeta \cdot e_1 + \varepsilon.
\]
This completes the proof of Theorem 6.3.

**Lemma 6.4.** (a) Fix \( \xi \in \text{ri} \mathcal{U} \). Then the following statement holds \( \hat{\mathbb{P}} \)-almost surely. For any geodesic \( x_{0,\infty} \)

\[
(6.7) \quad \lim_{n \to \infty} \frac{x_n \cdot e_1}{|x_n|} \geq \xi \cdot e_1 \quad \text{implies that} \quad x_n \cdot e_1 \geq x_{n \to \infty} \cdot e_1 \quad \text{for all} \quad n \geq 0
\]

and

\[
(6.8) \quad \lim_{n \to \infty} \frac{x_n \cdot e_1}{|x_n|} \leq \xi \cdot e_1 \quad \text{implies that} \quad x_n \cdot e_1 \leq x_{n \to \infty} \cdot e_1 \quad \text{for all} \quad n \geq 0.
\]

(b) Fix a maximal line segment \([\xi, \zeta]\) on which \( g_{pp} \) is linear and such that \( \xi \cdot e_1 < \zeta \cdot e_1 \). Assume \( \zeta \) and \( \hat{\zeta} \) are both points of differentiability of \( g_{pp} \). Then the following statement holds \( \hat{\mathbb{P}} \)-almost surely. Any geodesic \( x_{0,\infty} \) such that a limit point of \( x_n/|x_n| \) lies in \( [\xi, \zeta] \) satisfies

\[
(6.9) \quad x_{n \to \infty} \cdot e_1 \leq x_n \cdot e_1 \leq x_{n \to \infty} \cdot e_1 \quad \text{for all} \quad n \geq 0.
\]

**Proof.** Part (a). We prove (6.7), (6.8) is proved similarly.

Fix a sequence \( \zeta_t \in \mathcal{D} \) such that \( \zeta_t \cdot e_1 \) is of \( \zeta_t \cdot e_1 \) so that, in particular, \( \zeta \notin \mathcal{U}_{\ell_t} \). The good event of full \( \hat{\mathbb{P}} \)-probability is the one on which \( x_{0,\infty}^{x_0} \cdot e_1 \) is \( \mathcal{U}_{\ell_t} \)-directed (Theorem 6.3), \( x_{0,\infty}^{x_0} \cdot e_1 \) is the leftmost geodesic between any two of its points (Lemma 6.1(b) applied to cocycle \( B_{\ell_t} \)) and \( x_{0,\infty}^{x_0} \cdot e_1 \to x_{0,\infty}^{x_0} \cdot e_1 \) (Lemma 6.2).

By the leftmost property, if \( x_{0,\infty}^{x_0} \cdot e_1 \) ever goes strictly to the right of \( x_{0,\infty} \), these two geodesics cannot touch again at any later time. But by virtue of the limit points, \( x_{n \to \infty}^{x_0} \cdot e_1 < x_n \cdot e_1 \) for infinitely many \( n \). Hence \( x_{0,\infty}^{x_0} \cdot e_1 \) stays weakly to the left of \( x_{0,\infty} \). Let \( \ell \to \infty \).

Part (b) is proved similarly. The differentiability assumption implies that the geodesic \( x_{0,\infty}^{x_0} \cdot e_1 \) can be approached from the left by geodesics \( x_{0,\infty}^{x_0} \cdot e_1 \) such that \( \zeta \notin \mathcal{U}_{\ell_t} \).

Recall the set \( \mathcal{E} \) of exposed points of differentiability defined below (2.6). Define sets \( \hat{\mathcal{E}}^+ \) and \( \hat{\mathcal{E}}^- \) of their one-sided limit points by

\[
\hat{\mathcal{E}}^\pm = \{ \zeta \in \text{ri} \mathcal{U} : \exists \xi_n \in \mathcal{E} \text{ such that } \xi_n \to \zeta \text{ and } \pm \xi_n \cdot e_1 > \pm \zeta \cdot e_1 \}.
\]

**Corollary 6.5.** Fix \( \xi \in \text{ri} \mathcal{U} \) such that \( \zeta \in \mathcal{E} \cup \hat{\mathcal{E}}^- \). Then \( \hat{\mathbb{P}} \)-almost surely and for all \( u \in \mathbb{Z}^2 \), \( x_{0,\infty}^{u \cdot \xi \cdot e_1} \) is the leftmost \( \mathcal{U}_{\xi \cdot e_1} \)-geodesic out of \( u \). Similarly, for a fixed \( \xi \) with \( \zeta \in \mathcal{E} \cup \hat{\mathcal{E}}^+ \), \( \hat{\mathbb{P}} \)-almost surely and for all \( u \in \mathbb{Z}^2 \), \( x_{0,\infty}^{u \cdot \xi \cdot e_1} \) is the rightmost \( \mathcal{U}_{\xi \cdot e_1} \)-geodesic out of \( u \).

**Proof.** Theorem 6.3 implies that \( x_{0,\infty}^{u \cdot \xi \cdot e_1} \) is a \( \mathcal{U}_{\xi \cdot e_1} \)-geodesic. If \( \xi \in \mathcal{E} \cup \hat{\mathcal{E}}^- \) then either \( \mathcal{U}_{\xi \cdot e_1} = \mathcal{U}_{\xi \cdot e_1} \) or \( \mathcal{U}_{\xi \cdot e_1} = \{ \zeta \} \subset \mathcal{U}_{\xi \cdot e_1} \). Thus, \( x_{0,\infty}^{u \cdot \xi \cdot e_1} \) is a \( \mathcal{U}_{\xi \cdot e_1} \)-geodesic. \( \xi \in \mathcal{E} \cup \hat{\mathcal{E}}^- \) implies that there cannot be a linear segment adjacent to \( \xi \) to the left, and consequently \( \zeta = \xi \). Lemma 6.4(a) implies that \( x_{0,\infty}^{u \cdot \xi \cdot e_1} \) is to the left of any other \( \mathcal{U}_{\xi \cdot e_1} \)-geodesic out of \( u \). The claim about rightmost geodesics is proved similarly.
The next result concerns coalescence of geodesics \( \{x_{0,\infty}^{\pm} : u \in \mathbb{Z}^2\} \), for fixed \( t \) and \( \xi \in \text{ri} \mathbb{U} \).

**Theorem 6.6.** Fix \( t \in \{e_1, e_2\}^{\mathbb{Z}^2} \) and \( \xi \in \text{ri} \mathbb{U} \). Then \( \hat{\mathbb{P}} \)-almost surely, for all \( u, v \in \mathbb{Z}^2 \), there exist \( n, m \geq 0 \) such that \( x_{n,\infty}^{u,t,\xi^-} = x_{m,\infty}^{v,t,\xi^-} \), with a similar statement for \( \xi^+ \).

Theorem 6.6 is proved by adapting the argument of [21], originally presented for first passage percolation and later ported by [13] to the exactly solvable corner growth model with exponential weights. Briefly, the idea is the following. Stationarity and two nonintersecting geodesics create three nonintersecting geodesics. A modification of the weights turns the middle geodesic of the triple into a geodesic that stays disjoint from all geodesics that emanate from sufficiently far away. Stationarity again gives at least \( \delta L^2 \) such disjoint geodesics emanating from an \( L \times L \) square. This gives a contradiction because there are only \( 2L \) boundary points for these geodesics to exit through. The details are in Appendix B.

To get to uniqueness of geodesics, we show that continuity of the distribution of \( \omega_0 \) prevents ties between cocycle weights. (The construction of the cocycles implies, through eqn. (A.6), that the variables \( B^{\xi^\pm}(x, y) \) have continuous marginal distributions, but here we need a property of the joint distribution.) Consequently for a given \( \xi \), \( \hat{\mathbb{P}} \)-almost surely cocycle geodesics \( x_{0,\infty}^{u,t,\xi^\pm} \) do not depend on \( t \).

**Lemma 6.7.** Assume (2.1) and that \( \mathbb{P}\{\omega_0 \leq r\} \) is a continuous function of \( r \in \mathbb{R} \). Fix \( \xi \in \text{ri} \mathbb{U} \). Then for all \( u \in \mathbb{Z}^2 \),

\[
\hat{\mathbb{P}}\{B^{\xi^+}(u, u + e_1) = B^{\xi^+}(u, u + e_2)\} = \hat{\mathbb{P}}\{B^{\xi^-}(u, u + e_1) = B^{\xi^-}(u, u + e_2)\} = 0.
\]

**Proof.** Due to shift invariance it is enough to prove the claim for \( u = 0 \). We work with the case \( \xi^+ \), the other case being similar.

Assume by way of contradiction that the probability in question is positive. Pick an arbitrary \( t \in \{e_1, e_2\}^{\mathbb{Z}^2} \). By Theorem 6.6, \( x_{0,\infty}^{e_1,t,\xi^+} \) and \( x_{0,\infty}^{e_2,t,\xi^+} \) coalesce with probability one. Hence there exists \( v \in \mathbb{Z}^2 \) and \( n \geq 1 \) such that

\[
\mathbb{P}\{B^{\xi^+}(0, e_1) = B^{\xi^+}(0, e_2), x_n^{e_1,t,\xi^+} = x_n^{e_2,t,\xi^+} = v\} > 0.
\]

Note that if \( B^{\xi^+}(0, e_1) = B^{\xi^+}(0, e_2) \) then both are equal to \( \omega_0 \). Furthermore, by Lemma 6.1(a) we have

\[
B^{\xi^+}(e_1, v) = \sum_{k=0}^{n-1} \omega(x_k^{e_1,t,\xi^+}) \quad \text{and} \quad B^{\xi^+}(e_2, v) = \sum_{k=0}^{n-1} \omega(x_k^{e_2,t,\xi^+}).
\]

(For aesthetic reasons we wrote \( \omega(x) \) instead of \( \omega_x \).) Thus

\[
\omega_0 + \sum_{k=0}^{n-1} \omega(x_k^{e_1,t,\xi^+}) = B^{\xi^+}(0, e_1) + B^{\xi^+}(e_1, v) = B^{\xi^+}(0, v)
\]

\[
= B^{\xi^+}(0, e_2) + B^{\xi^+}(e_2, v) = \omega_0 + \sum_{k=0}^{n-1} \omega(x_k^{e_2,t,\xi^+}).
\]
The fact that this happens with positive probability contradicts the assumption that \( \omega_x \) are i.i.d. and have a continuous distribution. The lemma is proved. \( \square \)

It is known that, in general, uniqueness of geodesics cannot hold simultaneously for all directions. In our development this is a consequence of Theorem 7.2 below.

**Proof of Theorem 2.4.** Part (i). The existence of \( \mathcal{U}_{\xi^\pm} \)-directed geodesics for \( \xi \in \text{ri} \mathcal{U} \) follows by fixing \( t \) and taking geodesics \( x_{0,1}^{u,\xi^\pm} \) from Theorem 6.3. For \( \xi = e_i \) geodesics are simply \( x_{0,\infty} = (ne_i)_{n \geq 0} \).

Let \( \mathcal{D}_0 \) be a dense countable subset of \( \mathcal{D} \). Let \( \hat{\Omega}_0 \) be the event of full \( \hat{\mathbb{P}} \)-probability on which event (6.4) holds and Lemma 6.4(a) holds for each \( u \in \mathbb{Z}^2 \) and \( \zeta \in \mathcal{D}_0 \). We show that on \( \hat{\Omega}_0 \), every geodesic is \( \mathcal{U}_\zeta \)-directed for some \( \xi \in \mathcal{U} \).

Fix \( \hat{\omega} \in \hat{\Omega}_0 \) and an arbitrary geodesic \( x_{0,\infty} \). Define \( \xi' \in \mathcal{U} \) by

\[
\xi' \cdot e_1 = \lim_{n \to \infty} \frac{x_n \cdot e_1}{|x_n|_1}.
\]

Let \( \xi = \xi' \) be the left endpoint of \( \mathcal{U}_{\xi'} \). We claim that \( x_{0,\infty} \) is \( \mathcal{U}_\xi = [\xi, \xi'] \)-directed. If \( \xi' = e_2 \) then \( x_n / |x_n|_1 \to e_2 \) and \( \mathcal{U}_\xi = \{e_2\} \) and the case is closed. Suppose \( \xi' \neq e_2 \).

The definition of \( \xi \) implies that \( \xi' \in \mathcal{U}_{\xi^+} \) and so

\[
\lim_{n \to \infty} \frac{x_n \cdot e_1}{|x_n|_1} = \xi' \cdot e_1 \leq \xi \cdot e_1.
\]

From the other direction, for any \( \zeta \in \mathcal{D}_0 \) such that \( \zeta \cdot e_1 < \xi' \cdot e_1 \) we have

\[
\lim_{n \to \infty} \frac{x_n \cdot e_1}{|x_n|_1} \geq \zeta \cdot e_1
\]

which by (6.7) implies \( x_n \cdot e_1 \geq x_n^{0,\xi} \cdot e_1 \). Then by (6.4)

\[
\lim_{n \to \infty} \frac{x_n \cdot e_1}{|x_n|_1} \geq \lim_{n \to \infty} \frac{x_n^{0,\xi} \cdot e_1}{|x_n|_1} \geq \zeta \cdot e_1
\]

where \( \zeta \) is the left endpoint of \( \mathcal{U}_{\zeta} \). It remains to observe that we can take \( \zeta \cdot e_1 \) arbitrarily close to \( \xi \cdot e_1 \). If \( \xi \cdot e_1 < \xi' \cdot e_1 \) then we take \( \xi \cdot e_1 < \zeta < \xi' \cdot e_1 \) in which case \( \zeta = \xi \) and \( \zeta = \xi' \) then also \( \xi = \xi' = \xi \). In this case, as \( \mathcal{D}_0 \ni \zeta \nless \xi \), \( \nabla g(\zeta) \) approaches but never equals \( \nabla g(\xi^-) \) because there is no flat segment of \( g_{pp} \) adjacent to \( \xi \) on the left. This forces both \( \zeta \) and \( \xi \) to converge to \( \xi \).

Part (ii). If \( g_{pp} \) is strictly concave then \( \mathcal{U}_\xi = \{\xi\} \) for all \( \xi \in \text{ri} \mathcal{U} \) and part (ii) follows from part (i).

Part (iii). By Theorem 4.2(iii) there is a single cocycle \( B^\xi \) simultaneously for all \( \xi \in [\zeta, \zeta'] \). Consequently cocycle geodesics \( x_{0,\infty}^{0,\xi} \) and \( x_{0,\infty}^{\xi_0,\xi} \) coincide for any given tie breaking function \( t \). By Corollary 5.2 this cocycle \( B^\xi \) is \( \mathcal{G} \)-measurable and hence so are the cocycle geodesics. On the event of full \( \mathbb{P} \)-probability on which there are no ties between \( B^\xi(x, x + e_1) \) and \( B^\xi(x, x + e_2) \) the tie breaking function \( t \) makes no difference. Hence the left and right-hand side of (6.9) coincide. Thus there is no room for two \( [\zeta, \zeta'] \)-directed geodesics from any point. Coalescence comes from Theorem 6.6. \( \square \)
Proof of Theorem 2.5. Part (i) follows from Lemma 6.1.

Part (ii). Take sequences $\eta_n, \zeta_n \in \text{ri} U$ with $\eta_n \cdot e_1 < \xi \cdot e_1 \leq \zeta \cdot e_1 < \zeta_n \cdot e_1$ and $\zeta_n \to \xi$, $\eta_n \to \xi$. Consider the full measure event on which Theorem 5.1 holds for each $\zeta_n$ and $\eta_n$ with sequences $v_m = \lfloor m\zeta_n \rfloor$ and $\lfloor m\eta_n \rfloor$, and on which continuity (4.4) holds as $\zeta_n \to \xi$, $\eta_n \to \xi$. In the rest of the proof we drop the index $n$ from $\zeta_n$ and $\eta_n$.

We prove the case of an infinite geodesic $x_{0,\infty}$ that satisfies $x_0 = 0$ and (2.19). For large $m$, $\lfloor m\eta \cdot e_1 \rfloor < x_m \cdot e_1 < \lfloor m\zeta \cdot e_1 \rfloor$.

Consider first the case $x_1 = e_1$. If there exists a geodesic from 0 to $\lfloor m\zeta \rfloor$ that goes through $e_2$, then this geodesic would intersect $x_{0,\infty}$ and thus there would exist another geodesic that goes from 0 to $\lfloor m\zeta \rfloor$ passing through $e_1$. In this case we would have $G_{e_1,\lfloor m\zeta \rfloor} = G_{e_2,\lfloor m\zeta \rfloor}$. On the other hand, if there exists a geodesic from 0 to $\lfloor m\eta \rfloor$ that goes through $e_1$, then we would have $G_{e_1,\lfloor m\eta \rfloor} \geq G_{e_2,\lfloor m\eta \rfloor}$. Thus, in either case, we have

$$G_{0,\lfloor m\zeta \rfloor} - G_{e_1,\lfloor m\zeta \rfloor} \leq G_{0,\lfloor m\eta \rfloor} - G_{e_2,\lfloor m\eta \rfloor}.$$

Taking $m \to \infty$ and applying Theorem 5.1 we have $B^{\bar{x}}(0,e_1) \leq B^{\bar{x}}(0,e_2)$. Taking $\zeta \to \bar{x}$ and applying (4.4) we have $B^x(0,e_1) \leq B^{\bar{x}}(0,e_2)$. Since $\bar{x}$ and $\bar{x}$ are points of differentiability of $g_{pp}$, we have $B^x = B^{\bar{x}}$. Consequently, we have shown $B^x(0,e_1) \leq B^x(0,e_2)$. Since $B^x$ recovers the potential (Definition 4.1), the first step satisfies $\omega_0 = B^x(0,e_1) \wedge B^x(0,e_2) = B^x(0,x_1)$.

When $x_1 = e_2$ repeat the same argument with $\eta$ in place of $\zeta$ to get $B^x(0,e_2) \leq B^x(0,e_1)$. This proves the theorem for the first step of the geodesic and that is enough.

Part (iii). The statement holds if $B^x(0,e_1) = B^x(0,e_2)$, since then both are equal to $\omega_0$ by potential recovery (4.1). If $\omega_0 = B^x(0,e_1) < B^x(0,e_2)$ then convergence (2.13) implies that for $n$ large enough $G_{e_1,v_n} > G_{e_2,v_n}$. In this case any maximizing path from 0 to $v_n$ will have to start with an $e_1$-step and the claim of the lemma is again true. The case $B^x(0,e_1) > B^x(0,e_2)$ is similar.

$\square$

Proof of Theorem 2.6. Part (i) follows from Corollary 6.5 and Theorem 6.6. Part (ii) follows from Theorem 2.5(iii) and the fact that the geodesics in Corollary 6.5 are the Busemann geodesics from Theorem 2.5. $\square$

7. Competition interface

In this section we prove the results of Section 2.8. As before, we begin by studying the situation on the extended space $\Omega$ with the help of the cocycles $B^x$ of Theorem 4.2.

Lemma 7.1. Define $B^{e_1-}$ as the monotone limit of $B^{\bar{x}}$ when $\zeta \to e_1$. Then $\hat{P}$-almost surely $B^{e_1-}(0,e_1) = \omega_0$ and $B^{e_1-}(0,e_2) = \infty$. A symmetric statement holds for the limit as $\zeta \to e_2$.

Proof. The limit in the claim exists due to monotonicity (4.3). Furthermore, by potential recovery we have almost surely $B^{e_1-}(0,e_1) \geq \omega_0$. On the other hand, dominated convergence, (3.11) and Lemma 3.1 give

$$\hat{E}[B^{e_1-}(0,e_1)] = \lim_{\zeta \to e_1} \hat{E}[B^{\bar{x}}(0,e_1)] = \lim_{\zeta \to e_1} e_1 \cdot \nabla g_{pp}(\zeta \pm) = m_0 = \hat{E}[\omega_0].$$
Thus, $B^{e_1^-}(0, e_1) = \omega_0$ almost surely.

The cocycle property (Definition 2.1(c)) and recovery (Definition 4.1), both of which are satisfied by $B^{e_1^-}$, imply the relation
\[
B^{e_1^+}(ne_1, ne_1 + e_2) = \omega_{ne_1} + B^{e_1^-}(ne_1 + e_2, (n + 1)e_1 + e_2) = \omega_{ne_1} + B^{e_1^-}((n + 1)e_1, (n + 1)e_1 + e_2) + \omega_{ne_1 + e_2}^+.
\]
The second equality is from the just proved identity $B^{e_1^-}(x, x + e_1) = \omega_x$.

Repeatedly dropping the outer $+$-part and applying the same formula inductively leads to
\[
B^{e_1^-}(0, e_2) \geq \omega_0 + \sum_{i=1}^n (\omega_{e_i} - \omega_{(i-1)e_1 + e_2}) + \left(B^{e_1^-}((n + 1)e_1, (n + 1)e_1 + e_2) - \omega_{ne_1 + e_2}\right)^+.
\]
Since the summands are i.i.d. with mean 0, taking $n \to \infty$ gives $B^{e_1^-}(0, e_2) = \infty$ almost surely.

Next we use the cocycles to define a random variable on $\hat{\Omega}$ that represents the asymptotic direction of the competition interface. Assume now that $P\{\omega_0 \leq r\}$ is continuous in $r$. By Lemma 6.7, with $\hat{P}$-probability one, $B^{\pm}(0, e_1) \neq B^{\pm}(0, e_2)$ for all rational $\xi \in riU$. Furthermore, monotonicity (4.3) gives that
\[
B^{\xi^+}(0, e_1) - B^{\xi^+}(0, e_2) \leq B^{\xi^-}(0, e_1) - B^{\xi^-}(0, e_2) \leq B^{\eta^+}(0, e_1) - B^{\eta^+}(0, e_2)
\]
when $\zeta \cdot e_1 > \eta \cdot e_1$. Lemma 7.1 implies that $B^{\xi^+}(0, e_1) - B^{\xi^+}(0, e_2)$ converges to $-\infty$ as $\zeta \to e_1$ and to $\infty$ as $\zeta \to e_2$. Thus there exists a unique $\zeta_*(\hat{\omega})$ in $riU$ such that for rational $\zeta \in riU$,
\[
B^{\xi^+}(\hat{\omega}, 0, e_1) < B^{\xi^+}(\hat{\omega}, 0, e_2) \quad \text{if} \quad \zeta \cdot e_1 > \zeta_*(\hat{\omega}) \cdot e_1
\]
and
\[
B^{\xi^+}(\hat{\omega}, 0, e_1) > B^{\xi^+}(\hat{\omega}, 0, e_2) \quad \text{if} \quad \zeta \cdot e_1 < \zeta_*(\hat{\omega}) \cdot e_1.
\]

**Theorem 7.2.** Assume $P\{\omega_0 \leq r\}$ is continuous in $r$. Then on the extended space $(\hat{\Omega}, \hat{\xi}, \hat{P})$ of Theorem 4.2 the random variable $\zeta_*(\hat{\omega}) \in riU$ defined by (7.1) has the following properties.

(i) $\hat{P}$-almost surely, for every $x \in \mathbb{Z}^2$, there exist at least two $\mathcal{U}_{\zeta_*(T_x \hat{\omega})}$-geodesics out of $x$ that do not coalesce.

(ii) Recall $\mathcal{U}_{\zeta_*(\hat{\omega})} = [\bar{\xi}_*(\hat{\omega}), \bar{\xi}_*(\hat{\omega})]$ from (2.9). Then the following holds $\hat{P}$-almost surely. Let $x_{0,\infty}'$ and $x_{0,\infty}''$, be any geodesics with
\[
\lim_{n \to \infty} \frac{x_{n'} \cdot e_1}{n} < \bar{\xi}_*(\hat{\omega}) \quad \text{and} \quad \lim_{n \to \infty} \frac{x_{n''} \cdot e_1}{n} > \bar{\xi}_*(\hat{\omega}).
\]
Then $x_1' = e_2$ and $x_1'' = e_1$.

(iii) $\hat{P}\{\hat{\omega} : \zeta_*(\hat{\omega}) = \xi\} = 0$ for any $\xi \in D$.

(iv) For $\zeta, \eta \in riU$ such that $\zeta \cdot e_1 < \eta \cdot e_1$ and $\nabla g_{pp}(\zeta^+) \neq \nabla g_{pp}(\eta^-)$, with $\mathcal{U}_\zeta = [\bar{\zeta}, \bar{\zeta}]$ and $\mathcal{U}_\eta = [\bar{\eta}, \bar{\eta}]$. Then for $\hat{P}$-almost every $\hat{\omega}$ there exists $z \in \mathbb{Z}^2$ such that $\zeta_*(T_z \hat{\omega}) \in [\bar{\zeta}, \bar{\eta}]$. 


Proof. Define
\[
B^+(\hat{\omega}, x, y) = \lim_{\zeta \to e_1} B^\zeta(\hat{\omega}, x, y)
\]
and
\[
B^-(\hat{\omega}, x, y) = \lim_{\zeta \to e_1} B^\zeta(\hat{\omega}, x, y).
\]

As pointed out in Remark 4.3(c), we have to keep the $B^\pm$ distinction even if $g_{pp}$ is everywhere differentiable, because direction $\xi_*$ is random and continuity (4.4) has not been shown simultaneously for all directions with a single $\overline{P}$-null set.

In any case, $B^\pm$ satisfy the cocycle property (Definition 2.1(c)) and recovery $\omega_x = \min_{n=1,2} B^\pm(\hat{\omega}, x, x + e_1)$ (Definition 4.1). From (7.1) we have $B^+_\star(0, e_1) \leq B^+_\star(0, e_2)$ and $B^-\star(0, e_1) \geq B^-\star(0, e_2)$. By Lemma 6.1 there exists a geodesic from 0 through $e_1$ (by following minimal $B^+_\star$ gradients) and another through $e_2$ (by following minimal $B^-\star$ gradients), with an arbitrary tie breaking function $t$. These two geodesics cannot coalesce because $\omega_0$ has a continuous distribution.

Let $\zeta \cdot e_1 < \xi_* \cdot e_1 < \eta \cdot e_1$. By the limits in (7.2) and monotonicity (4.3),
\[
B^\zeta(\hat{\omega}, x, x + e_1) \geq B^\zeta(\hat{\omega}, x, x + e_1) \geq B^\eta(\hat{\omega}, x, x + e_1)
\]
and
\[
B^\zeta(\hat{\omega}, x, x + e_2) \leq B^\zeta(\hat{\omega}, x, x + e_2) \leq B^\eta(\hat{\omega}, x, x + e_2).
\]

These inequalities imply that the $B^\pm$-geodesics stay to the right of $x_{0,\infty}^{0,\eta-}$ and to the left of $x_{0,\infty}^{0,\xi_*}$. By Theorem 6.3 these geodesics are $U_{\xi_*}$ and $U_{\eta-}$-directed, respectively. Hence the $B^\pm$-geodesics are $U_{\xi_*}$-directed. Part (i) is proved.

In part (ii) we prove the first claim, the other claim being similar. The assumption allows us to pick a rational $\eta \in \text{ri} \mathcal{U}$ such that $\lim n^\eta \cdot e_1/n < \eta \cdot e_1 \leq \eta \cdot e_1 < \xi_* \cdot e_1$. Since $\omega_0$ has a continuous distribution and geodesic $x_{0,\infty}^{0,\eta-}$ is $U_{\eta-}$-directed, geodesic $x_{0,\infty}$ has to stay always to the left of it. (7.1) implies $x_{1,\infty}^{0,\eta-} = e_2$. Hence also $x_1 = e_2$. The claim is proved.

For part (iii) fix $\xi \in \text{ri} \mathcal{D}^*$, which implies $B^\xi = B^\xi$. By Lemma 6.7, $B^\xi(0, e_1) \neq B^\xi(0, e_2)$ almost surely. Let $\zeta \cdot e_1 \not\in \text{ri} \mathcal{U}$ along rational points $\zeta \in \text{ri} \mathcal{U}$. By (4.4), $B^\zeta(0, e_1) \to B^\xi(0, e_1)$ a.s. Then on the event $B^\xi(0, e_1) > B^\xi(0, e_2)$ there almost surely exists a rational $\zeta$ such that $\zeta \cdot e_1 > \xi \cdot e_1$ and $B^\zeta(0, e_1) > B^\xi(0, e_2)$. By (7.1) this forces $\xi_* \cdot e_1 \geq \xi \cdot e_1$. Similarly on the event $B^\xi(0, e_1) < B^\xi(0, e_2)$ we have almost surely $\xi_* \cdot e_1 < \xi \cdot e_1$. The conclusion is that $\mathbb{P}(\xi_* = \xi) = 0$ and part (iii) is proved.

In part (iv), $U_{\zeta_*} \neq U_{\eta-}$ and directedness (Theorem 6.3) force the cocycle geodesics $x_{0,\infty}^{0,\eta-}$ and $x_{0,\infty}^{0,\zeta_*}$ to separate. If $n \geq 0$ is the time after which they separate, then by cocycle geodesics ordering (6.2) there exists $z \in \mathbb{Z}$ such that $x_n^{0,\eta-} = x_n^{0,\zeta_*} = z$, $x_{n+1}^{0,\eta-} = z + e_1$, and $x_{n+1}^{0,\zeta_*} = z + e_2$. Definition (7.1) implies that $\zeta \cdot e_1 \leq \xi_*(T_2 \hat{\omega}) \cdot e_1 \leq \overline{\eta} \cdot e_1$. For suppose $\overline{\eta} \cdot e_1 < \xi_*(T_2 \hat{\omega}) \cdot e_1$. Pick a rational point strictly between $\overline{\eta}$ and $\xi_*(T_2 \hat{\omega})$. The second line of (7.1) and ordering (4.3) imply that $B^{\eta-}(\hat{\omega}, z, z + e_1) > B^{\eta-}(\hat{\omega}, z, z + e_2)$, contradicting the choice $x_{n+1}^{0,\eta-} = z + e_1$.

\begin{corollary}
Assume $\mathbb{P}(\omega_0 \leq r)$ is continuous in $r$ and $g_{pp}$ is differentiable at the endpoints of all its linear segments. Then $\xi_*$ lies almost surely outside the union of the closed linear segments of $g_{pp}$. Equivalently, $\xi_*$ is almost surely an exposed point.
\end{corollary}
Proof. By Theorem 2.4(iii) each linear segment has a unique geodesic from 0 directed into it. Since there are at most countably many linear segments of \( g_{pp} \), Theorem 7.2(i) contradicts \( \xi_* \) lying on a flat segment. Under the differentiability assumption endpoints of flat segments are not exposed. \( \square \)

The next theorem identifies the asymptotic direction of the competition interface \( \phi = (\phi_k)_{0 \leq k < \infty} \) defined in Section 2.8.

**Theorem 7.4.** Assume \( \mathbb{P}\{\omega_0 \leq r\} \) is continuous in \( r \).

(i) All limit points of the asymptotic velocity of the competition interface are in \( U_{\xi_*}(\hat{\omega}) \):

\[
\xi_*(\hat{\omega}) \cdot e_1 \leq \lim_{n \to \infty} n^{-1} \varphi_n(\omega) \cdot e_1 \leq \lim_{n \to \infty} n^{-1} \varphi_n(\omega) \cdot e_1 \leq \xi_*^{\ast}(\hat{\omega}) \cdot e_1.
\]

(ii) If \( g_{pp} \) is differentiable at the endpoints of its linear segments then \( \xi_* \) is \( \mathcal{G} \)-measurable and gives the asymptotic direction of the competition interface: \( \hat{\omega} \)-almost surely

\[
\lim_{n \to \infty} n^{-1} \varphi_n(\omega) = \xi_*^{\ast}(\hat{\omega}).
\]

**Proof.** Fix \( t \in \{e_1, e_2\}^\mathbb{Z}^2 \). By (7.1), if \( \zeta \cdot e_1 < \xi_*^{\ast}(\hat{\omega}) \cdot e_1 < \eta \cdot e_1 \), then \( x_{0,t}^{1,\pm} = e_2 \) and \( x_{1,t}^{1,\pm} = e_1 \). Since the path \( \phi \) separates the geodesics that go through \( e_1 \) and \( e_2 \), it has to stay between \( x_{0,t}^{1,\zeta} \) and \( x_{0,t}^{1,\eta} \). By Theorem 6.3 these geodesics are \( U_{\zeta^+} \) and \( U_{\eta^-} \) directed, and we have

\[ \zeta \cdot e_1 \leq \lim_{n \to \infty} n^{-1} \varphi_n(\omega) \cdot e_1 \leq \lim_{n \to \infty} n^{-1} \varphi_n(\omega) \cdot e_1 \leq \eta \cdot e_1. \]

Claim (7.3) follows by taking \( \zeta \) and \( \eta \) to \( \xi_*^{\ast} \).

If \( g_{pp} \) is differentiable at the endpoints of its linear segments, then cocycles are \( \mathcal{G} \)-measurable and hence so is \( \xi_*^{\ast} \). Furthermore, \( \xi_*^{\ast} \) is an exposed point by Corollary 7.3. In this case, \( \xi_*^{\ast} = \xi_*^{\ast\ast} \) and claim (7.4) is proved. \( \square \)

**Proof of Theorem 2.7.** Limit (2.21) is in (7.4). The fact that the limit lies in \( ri \mathcal{U} \) is in the construction in the paragraph that contains (7.1), and the properties of the limit are in Theorem 7.2(iv) and Corollary 7.3. \( \square \)

**Proof of Theorem 2.9.** Part (i) comes directly from Theorem 7.2(i). For part (ii) assume \( g_{pp} \) strictly concave. Then \( \mathcal{U}_e = \{\xi\} \) and by Theorem 2.4(iv) every geodesic is directed. In this case, Theorem 7.2(ii) implies that with \( \mathbb{P} \)-probability one, there cannot be two distinct geodesics from 0 with a common direction other than \( \xi_*^{\ast} \). \( \square \)

As mentioned at the end of Section 2.8, if \( \mathbb{P}\{\omega_0 \leq r\} \) is not continuous in \( r \), we have competition interfaces \( \phi^{(l)} \) and \( \phi^{(r)} \) for the trees of leftmost and rightmost geodesics. Their limiting directions \( \xi_*^{(r)}(\hat{\omega}), \xi_*^{(l)}(\hat{\omega}) \in ri \mathcal{U} \) are defined by

\[
B^{\zeta^+}(\hat{\omega}, 0, e_1) > B^{\zeta^+}(\hat{\omega}, 0, e_2) \quad \text{if} \quad \zeta \cdot e_1 < \xi_*^{(r)}(\hat{\omega}) \cdot e_1,
\]

\[
B^{\zeta^+}(\hat{\omega}, 0, e_1) = B^{\zeta^+}(\hat{\omega}, 0, e_2) \quad \text{if} \quad \xi_*^{(r)}(\hat{\omega}) \cdot e_1 < \zeta \cdot e_1 < \xi_*^{(l)}(\hat{\omega}) \cdot e_1
\]

and

\[
B^{\zeta^+}(\hat{\omega}, 0, e_1) < B^{\zeta^+}(\hat{\omega}, 0, e_2) \quad \text{if} \quad \zeta \cdot e_1 > \xi_*^{(l)}(\hat{\omega}) \cdot e_1.
\]
With this definition limit (7.3) is valid also with superscripts \((l)\) and \((r)\). Consequently \(n^{-1}\varphi_n^{(a)}(\omega) \to \xi^*(\omega)\) a.s. for \(a \in \{l, r\}\) under the assumption that \(g_{pp}\) is strictly concave.

8. **Exactly solvable models**

We go through some details for the exactly solvable models discussed in Section 2.9.

8.1. **Geometric weights.** The weights \(\{\omega_k\}\) are i.i.d. with \(\mathbb{P}(\omega_k = k) = (1-m_0^{-1})k^{-1}m_0^{-1}\) for \(k \in \mathbb{N}\), mean \(m_0 = \mathbb{E}(\omega_0) > 1\) and variance \(\sigma^2 = m_0(m_0 - 1)\).

Begin by investigating the queueing fixed point. With \(\{S_{n,0}\}\) i.i.d. geometric with mean \(m_0\), let the initial arrival process \(\{A_{n,0}\}\) be i.i.d. geometric with mean \(\alpha\). Let \(J_n = S_{n,0} + W_{n,0}\). Then equations (A.3) and (A.4) show that the process \(\{(A_{n,1}, J_{n+1}) : n \in \mathbb{Z}\}\) is an irreducible aperiodic Markov chain with transition probability

\[
P(A_{n,1} = b, J_{n+1} = j | A_{n-1,1} = a, J_n = i) = P\{(A_{n,0} - i)^+ + S_{n+1,0} = b, (i - A_{n,0})^+ + S_{n+1,0} = j\}.
\]

Note that the equations also show that \(\{A_{n,0}, S_{n+1,0}\}\) are independent of \(\{A_{n-1,1}, J_n\}\). Since \(\{(A_{n,1}, J_{n+1}) : n \in \mathbb{Z}\}\) is stationary, its marginal must be the unique invariant distribution of transition (8.1), namely

\[P(A_{n-1,1} = k, J_n = j) = (1 - \alpha^{-1})^{k-1}\alpha^{-1} \cdot (1 - f(\alpha)^{-1})^{j-1}f(\alpha)^{-1} \quad \text{for } k, j \in \mathbb{N}.
\]

with \(f(\alpha) = m_0^{-1}m_0^{-1}\). This shows that i.i.d. mean \(\alpha\) geometric is a queueing fixed point. Next solve for \(\gamma(s) = \inf_{\alpha > m_0} \{\alpha s + f(\alpha)\}\). The unique minimizing \(\alpha\) in terms of \(s = \xi \cdot e_1/\xi \cdot e_2\) is

\[\alpha = m_0 + \sigma\sqrt{\xi \cdot e_2/\xi \cdot e_1}\]

which defines the bijection between \(\xi \in \mathbb{R}^U\) and \(\alpha \in (m_0, \infty)\). From this

\[f(\alpha) = m_0^{-1}m_0^{-1} = m_0 + \sigma\sqrt{\xi \cdot e_1/\xi \cdot e_2}.
\]

The terms in the sum \(J_n = S_{n,0} + W_{n,0}\) are independent, so we can also find the distribution of the waiting time:

\[P(W_{n,0} = 0) = \frac{\alpha - m_0}{\alpha - 1}, \quad P(W_{n,0} = k) = \frac{m_0 - 1}{\alpha - 1} \cdot \left(1 - \frac{1}{f(\alpha)}\right)^{k-1} \frac{1}{f(\alpha)} (k \geq 1).
\]

The distributions of \(\xi^{(r)}\) and \(\xi^{(l)}\) claimed in (2.25) come from (7.5), knowing that \(B^{(a,1-a)}(0, e_1)\) and \(B^{(a,1-a)}(0, e_2)\) are independent geometrics with means (2.23). The calculation for \(\xi^{(r)}\) goes

\[P\{\xi^{(r)} \cdot e_1 > a\} = P\{B^{(a,1-a)}(0, e_1) > B^{(a,1-a)}(0, e_2)\} = \frac{\alpha - m_0}{\alpha} \cdot \frac{\sqrt{(m_0 - 1)(1 - a)} - \sqrt{m_0a + \sqrt{(m_0 - 1)(1 - a)}}}{\sqrt{m_0 (m_0 - 1)(1 - a)}}.
\]

from which the first formula of (2.25) follows. Similar computation for \(\xi^{(l)}\).
8.2. Exponential weights. The weights \( \{\omega_x\} \) are i.i.d. exponential with mean \( m_0 = \mathbb{E}(\omega_x) > 0 \) and variance \( \sigma^2 = m_0^2 \), with marginal distribution
\[
P(\omega_x > t) = m_0^{-1}e^{-t/m_0} \quad \text{for } t \geq 0.
\]
The queuing fixed point can be derived as in the geometric case. The distribution of \( \xi_* \) comes from knowing that \( B^{(a,1-a)}(0,e_1) \) and \( B^{(a,1-a)}(0,e_2) \) are independent exponentials with parameters \( \sqrt{a}/(\sqrt{a} + \sqrt{1-a}) \) and \( \sqrt{1-a}/(\sqrt{a} + \sqrt{1-a}) \). Hence
\[
P(\xi_* \cdot e_1 > a) = P(B^{(a,1-a)}(0,e_1) > B^{(a,1-a)}(0,e_2)) = \frac{\sqrt{1-a}}{\sqrt{a} + \sqrt{1-a}}.
\]
Equation (2.24) follows.

Appendix A. Cocycles from Queueing Fixed Points

This section proves Theorem 4.2. By shifting the variables \( \{\omega_x, B^{\xi}(x,x + e_j)\} \) in Theorem 4.2 if necessary, we can assume without loss of generality that \( P(\omega_0 > 0) = 1 \). Then the weights \( \omega_x \) can represent service times and we can tap into queueing theory. We switch now to terminology and notation from queueing theory to enable the reader to relate this appendix to the existing queueing literature.

Consider an infinite sequence of \( \cdot / G/1/\infty/\text{FIFO} \) queues in tandem. That is, each queue or service station (these terms are used almost interchangeably) has a general service time distribution (the law of \( \omega_x \) under \( P \)), a single server, unbounded room for customers waiting to be served, and customers obey first-in-first-out discipline. The service stations are indexed by \( k \in \mathbb{Z}_+ \) and a bi-infinite sequence of customers is indexed by \( n \in \mathbb{Z} \). Customers enter the system at station 0 and move from station to station in order. The server at station \( k \) serves one customer at a time. Once the service of customer \( n \) is complete at station \( k \), customer \( n \) moves to the back of the queue at station \( k+1 \) and customer \( n+1 \) enters service at station \( k \) if he was already waiting in the queue. If the queue at station \( k \) is empty after the departure of customer \( n \), then server \( k \) remains idle until customer \( n+1 \) arrives. Each customer retains his integer label as he moves through the system.

Here is the mathematical apparatus. The system needs two ingredients: an initial inter-arrival process \( \{A_{n,0} : n \in \mathbb{Z}\} \) and the service times \( \{S_{n,k} : n \in \mathbb{Z}, k \in \mathbb{Z}_+\} \). \( A_{n,0} \geq 0 \) is the time between the arrival of customer \( n \) and customer \( n+1 \) at queue 0. \( S_{n,k} \geq 0 \) is the amount of time the service of customer \( n \) takes at station \( k \). Let \( \{S_{n,k} : n \in \mathbb{Z}, k \in \mathbb{Z}_+\} \) be i.i.d. such that \( S_{0,0} \) has the distribution of \( \omega_0 \) under \( P \). Assume \( \{A_{n,0} : n \in \mathbb{Z}\} \) is stationary, ergodic, and independent of \( \{S_{n,k} : k \in \mathbb{Z}_+, n \in \mathbb{Z}\} \). Assume
\[
E[S_{0,0}] = m_0 < E[A_{0,0}] < \infty.
\]
This guarantees in particular a stable system where queues do not blow up. The service time distribution is taken to be fixed, while the input \( \{A_{n,0}\} \) varies, by analogy with varying the initial distribution of a Markov process.

As a product of an ergodic process and an i.i.d. process \( (A_{n,0}, S_{n,k})_{n \in \mathbb{Z}, k \in \mathbb{Z}_+} \) is stationary and ergodic under translations of the \( n \)-index. Consequently the entire queueing system is stationary and ergodic under translations of the \( n \)-index. The issue of interest is finding
input processes \( \{A_{n,0}\} \) such that the system is also stationary under translations of the \( k \)-
index. Such a process or its distribution on \( \mathbb{R}^Z \) will be called a fixed point of the queueing
operator.

Next we develop the iterative equations that describe the evolution of the system from
station to station, as \( k \) increases. These are the variables. \( A_{n,k} \) is the inter-arrival time
between customers \( n \) and \( n + 1 \) at queue \( k \), or, equivalently, the inter-departure time between
customers \( n \) and \( n + 1 \) from queue \( k - 1 \). \( W_{n,k} \) is the waiting time of customer \( n \) at queue
\( k \), that is, the time between the arrival of customer \( n \) at queue \( k \) and the beginning of
his service at queue \( k \). The total time customer \( n \) spends at station \( k \) is the sojourn time
\( W_{n,k} + S_{n,k} \).

The development begins with the waiting times. Define the stationary, ergodic process
\( \{W_{n,0}\}_{n \in \mathbb{Z}} \) by

\[
W_{n,0} = \left( \sup_{j \leq n-1} \sum_{i=j}^{n-1} (S_{i,0} - A_{i,0}) \right)^+. 
\]

By the ergodic theorem and \( (A.1) \)
\( W_{n,0} < \infty \) for all \( n \in \mathbb{Z} \).

Process \( \{W_{n,0}\} \) satisfies Lindley’s equation:
\[
W_{n+1,0} = (W_{n,0} + S_{n,0} - A_{n,0})^+. 
\]

This equation agrees naturally with the queuing interpretation. If \( W_{n,0} + S_{n,0} < A_{n,0} \)
then customer \( n \) leaves station 0 before customer \( n + 1 \) arrives, and consequently customer
\( n + 1 \) has no wait and \( W_{n+1,0} = 0 \). In the complementary case customer \( n + 1 \) waits time
\( W_{n+1,0} = W_{n,0} + S_{n,0} - A_{n,0} \) before entering service at station 0.

With some additional work we prove the following.

**Lemma A.1.** \( n^{-1}W_{n,0} \to 0 \) almost surely as \( n \to \infty \).

**Proof.** Abbreviate \( U_n = S_{n,0} - A_{n,0} \). For \( a \geq 0 \) and \( \varepsilon > 0 \) define
\[
W_0^{\varepsilon}(a) = a \quad \quad W_{n+1}^{\varepsilon}(a) = (W_n^{\varepsilon}(a) + U_n - E(U_0) + \varepsilon)^+ \quad \text{for } n \geq 0 .
\]

Check inductively that
\[
W_n^{\varepsilon}(0) = \left( \max_{0 \leq m < n} \sum_{k=m}^{n-1} [U_k - E(U_0) + \varepsilon] \right)^+ .
\]

Consequently
\[
W_n^{\varepsilon}(a) \geq W_n^{\varepsilon}(0) \geq \sum_{k=0}^{n-1} [U_k - E(U_0) + \varepsilon] \to \infty \quad \text{as } n \to \infty .
\]

Thus \( W_n^{\varepsilon}(a) > 0 \) for large \( n \) which implies, from its definition, that for large \( n \)
\[
W_{n+1}^{\varepsilon}(a) = W_n^{\varepsilon}(a) + U_n - E(U_0) + \varepsilon .
\]

Another application of the ergodic theorem gives \( n^{-1}W_n^{\varepsilon}(a) \to \varepsilon \) \( P \)-a.s. as \( n \to \infty \).
Now for the conclusion. Since $W_{0,0} = W_0^\varepsilon(W_{0,0})$, we can check inductively that
\[ W_{n+1,0} = (W_{n,0} + U_n)^+ \leq (W_n^\varepsilon(W_{0,0}) + U_n)^+ \]
\[ \leq (W_n^\epsilon(W_{0,0}) + U_n - E(U_0) + \varepsilon)^+ = W_{n+1}^\varepsilon(W_{0,0}). \]
From this, $0 \leq n^{-1}W_n \leq n^{-1}W_n^\varepsilon(W_{0,0}) \to \varepsilon$, and we let $\varepsilon \searrow 0$. \hfill \qed

The stationary and ergodic process $\{A_{n,1} : n \in \mathbb{Z}\}$ of inter-departure times from queue 0 (equivalently, inter-arrival times at queue 1) is defined by
\begin{equation} \label{A4}
A_{n,1} = (A_{n,0} - S_{n,0} - W_{n,0})^+ + S_{n+1,0},
\end{equation}
again by considering the two cases: either customer $n+1$ arrives before customer $n$ departs ($A_{n,0} < S_{n,0} + W_{n,0}$) and goes into service the moment customer $n$ departs, or server 0 is empty waiting for customer $n+1$ for time $(A_{n,0} - S_{n,0} - W_{n,0})^+$ before service of customer $n+1$ begins. Process $\{A_{n,1} : n \in \mathbb{Z}\}$ is independent of $\{S_{n,k} : k \geq 1, n \in \mathbb{Z}\}$.

Combining equations (A.3) and (A.4) and iterating gives
\[ W_{1,0} + S_{1,0} + \sum_{i=1}^{n} A_{i,1} = W_{n+1,0} + S_{n+1,0} + \sum_{i=1}^{n} A_{i,0} \quad \text{for } n \geq 1. \]
This and Lemma A.1 imply $E[A_{0,1}] = E[A_{0,0}]$. (In the queueing literature, this has been observed in [22].)

These steps are repeated at each queue. At queue $k$ we have the stationary, ergodic arrival process $\{A_{n,k}\}_{n \in \mathbb{Z}}$ that is independent of the service times $\{S_{n,j} : n \in \mathbb{Z}, j \geq k\}$. Waiting times at queue $k$ are defined by
\begin{equation} \label{A5}
W_{n,k} = \left( \sup_{j \leq n-1} \sum_{i=j}^{n-1} (S_{i,k} - A_{i,k}) \right)^+.
\end{equation}
Properties $W_{n,k} < \infty$, Lemma A.1, and $E[A_{n,k}] = E[A_{0,0}]$ are preserved along the way. This procedure constructs the process $\{A_{n,k}, S_{n,k}, W_{n,k} : n \in \mathbb{Z}, k \in \mathbb{Z}_+\}$ that satisfies the following system of equations:
\begin{equation} \label{A6}
\begin{aligned}
W_{n+1,k} + S_{n+1,k} &= S_{n+1,k} + (W_{n,k} + S_{n,k} - A_{n,k})^+, \\
A_{n,k+1} &= (A_{n,k} - S_{n,k} - W_{n,k})^+ + S_{n+1,k}, \\
S_{n+1,k} &= (S_{n+1,k} + W_{n+1,k}) \wedge A_{n,k+1}.
\end{aligned}
\end{equation}
The third equation follows directly from the first two. A useful consequence of (A.6) is the “conservation law”
\begin{equation} \label{A7}
W_{n+1,k} + S_{n+1,k} + A_{n,k} = W_{n,k} + S_{n,k} + A_{n,k+1}.
\end{equation}

The next four statements summarize the situation with fixed points, quoted from articles [23, 27]. Given a stationary ergodic probability measure $\mu$ on $\mathbb{R}^\mathbb{Z}$ consider random variables
\[ \{A_{n,0}, S_{n,0}, W_{n,0}, A_{n,1} : n \in \mathbb{Z}\} \]
where $\{A_{n,0} : n \in \mathbb{Z}\}$ has distribution $\mu$, $\{S_{n,0} : n \in \mathbb{Z}\}$ are i.i.d. with distribution $\mathbb{P}$, the two collections are independent of each other, $W_{n,0}$ are defined via (A.2), and $A_{n,1}$ are defined via (A.4). Let $\Phi(\mu)$ denote the distribution of $\{A_{n,1} : n \in \mathbb{Z}\}$. $\Phi$ is the queueing operator whose fixed points are the focus now.
Let $\mathcal{M}_\alpha^0(\mathbb{R}^2)$ be the space of translation-ergodic probability measures $\mu$ on $\mathbb{R}^2$ with marginal mean $E^\mu[A_{0,0}] = \alpha$. We are mainly interested in ergodic fixed points, so we define

$$A = \{\alpha > m_0 : \exists \mu \in \mathcal{M}_\alpha^0(\mathbb{R}^2) \text{ such that } \Phi(\mu) = \mu\}.$$

**Theorem A.2.** [27, Thm. 1] Let $\alpha \in A$. Then there exists a unique $\mu^\alpha \in \mathcal{M}_\alpha^0(\mathbb{R}^2)$ with $\Phi(\mu^\alpha) = \mu^\alpha$. Furthermore, let $A^0 = \{A_{n,0} : n \in \mathbb{Z}\}$ be ergodic with mean $E[A_{0,0}] = \alpha$ and $\{S_{n,k} : n \in \mathbb{Z}, k \in \mathbb{Z}_+\}$ i.i.d. with distribution $\mathbb{P}$ and independent of $A^0$. Let $A^k = \{A_{n,k} : n \in \mathbb{Z}\}$, $k \in \mathbb{N}$, be defined via inductions (A.2) and (A.6). Then the distributions of $A^k$ converge weakly to $\mu^\alpha$.

**Theorem A.3.** [23, Thm. 5.1 and 6.4 and Lm. 6.3(a)] The set $A$ is closed and nonempty, $\inf A = m_0$, and $\sup A = \infty$. If $\alpha < \beta$ are both in $A$ then $\mu^\alpha \leq \mu^\beta$ in the usual sense of stochastic ordering.

**Lemma A.4.** [23, Lm. 6.3(b)] Let $\alpha \in A$, $A^0 \sim \mu^\alpha$, and $\{S_{n,k}\} \sim \mathbb{P}$ independent of $A^0$. Define $W_{n,0}$ via (A.2). Then

$$E^\mu[\mathcal{P}[W_{0,0} + S_{0,0}] = f(\alpha).$$

Suppose $\alpha \in (m_0, \infty) \cap A^c$. Let

$$\underline{\alpha} = \sup(A \cap (m_0, \alpha)) \in A \quad \text{and} \quad \bar{\alpha} = \inf(A \cap [\alpha, \infty)) \in A,$$

$$t = (\bar{\alpha} - \alpha)/(\bar{\alpha} - \underline{\alpha})$$

and $\mu^\alpha = t\mu^\underline{\alpha} + (1-t)\mu^\bar{\alpha}$. Now $\mu^\alpha$ is a mean $\alpha$ fixed point of $\Phi$. This fixed point is again attractive, in the following sense.

**Theorem A.5.** [23, Prop. 6.5] Let $\alpha > m_0$. Let $\{A_{n,0} : n \in \mathbb{Z}\}$ be ergodic with mean $E[A_{0,0}] = \alpha$ and $\{S_{n,k} : n \in \mathbb{Z}, k \in \mathbb{Z}_+\}$ i.i.d. with distribution $\mathbb{P}$ and independent of the $A$-process. Let $\{A_{n,k} : n \in \mathbb{Z}, k \in \mathbb{N}\}$ be defined via inductions (A.2) and (A.6). The Cesàro mean of the distributions of $\{A_{n,k} : n \in \mathbb{Z}\}$ converges weakly to $\mu^\alpha$.

**Lemma A.6.** (a) Let $\alpha < \bar{\alpha}$ be points in $A$ such that $(\alpha, \bar{\alpha}) \subset A^c$. Then $f$ is linear on the interval $[\alpha, \bar{\alpha}]$.

(b) Let $\xi \in \mathcal{D}$, $s = \xi \cdot e_1/\xi \cdot e_2$ and $\alpha = \gamma'(s)$. Then $\alpha \in A$.

**Proof.** Part (a). Let $0 < t < 1$ and $\alpha = t\underline{\alpha} + (1-t)\bar{\alpha}$. In the notation of [23], consider a sequence of tandem queues $(A^k, S^k, W^k, A^{k+1})_{k \in \mathbb{Z}_+}$ where the initial arrival process $A^0 = (A_{n,0} : n \in \mathbb{Z})$ is ergodic with mean $E[A_{0,0}] = \alpha$, the service times $\{S^k\}_{k \in \mathbb{Z}_+} = \{S_{n,k} : n \in \mathbb{Z}, k \in \mathbb{Z}_+\}$ are independent of $A^0$ and i.i.d. $\mathbb{P}$-distributed, and the remaining variables are defined iteratively. Let $(\hat{A}, \hat{S}, \hat{W}, \hat{D})$ denote a weak limit point of the Cesàro averages of the distributions of $(A^k, S^k, W^k, A^{k+1})$. Then, as shown in [23, eqn. (29)] in the course of the proof of their Theorem 5.1, $\hat{W} = \Psi(\hat{A}, \hat{S})$ where the mapping $\Psi$ encodes definition (A.2). By Theorem A.5 [23, Prop. 6.5] the distribution of $\hat{A}$ is $t\mu^\underline{\alpha} + (1-t)\mu^\bar{\alpha}$. By [23, Theorem 4.1],

$$n^{-1} \sum_{k=0}^{n-1} W_{0,k} \to M(\alpha) \equiv f(\alpha) - m_0 \quad \text{almost surely}.$$
where the last equality comes from [23, Lemma 6.3(b)] restated as Lemma A.4 above. The weak limit, combined with the law of large numbers (A.9) and dominated convergence, gives, for any \( c < \infty \) and along a subsequence,

\[
E(\widetilde{W}_0 \wedge c) = \lim_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} E(W_{0,k} \wedge c) \leq \lim_{n \to \infty} E \left( n^{-1} \sum_{k=0}^{n-1} W_{0,k} \right) \wedge c = M(\alpha) \wedge c \leq M(\alpha).
\]

Letting \( c \nearrow \infty \) gives

\[
tM(\alpha) + (1-t)M(\overline{\alpha}) \leq M(\alpha).
\]

Since \( M \) is convex and \( f \) differs from \( M \) by a constant, this implies \( f(\alpha) = tf(\alpha) + (1-t)f(\overline{\alpha}) \) and completes the proof of part (a).

Part (b). To get a contradiction, suppose \( \alpha \in \mathcal{A}^c \). Then there exist \( \underline{\alpha} < \overline{\alpha} \) in \( \mathcal{A} \) such that \( \alpha \in (\underline{\alpha}, \overline{\alpha}) \subset \mathcal{A}^c \). By part (a) \( f \) is linear on \([\underline{\alpha}, \overline{\alpha}]\). Basic convex analysis implies that \( \gamma \) has multiple tangent slopes at \( s \) and hence cannot be differentiable at \( s \). Here is the argument.

By (3.5) the assumption \( \gamma'(s) = 0 \) implies that \( \gamma(s) = f(\alpha) + \alpha s \). It follows that \( s \) must be the slope of \( f \) on \((\underline{\alpha}, \overline{\alpha})\). For suppose this slope is \( s_1 \) and let \( \alpha_1 \in (\underline{\alpha}, \overline{\alpha}) \). Then by the duality (3.9)

\[
\gamma(s) \leq f(\alpha_1) + \alpha_1 s = f(\alpha) + s_1(\alpha_1 - \alpha) + \alpha_1 s = f(\alpha) + \alpha s + (\alpha_1 - \alpha)(s_1 - s)
\]

which contradicts \( \gamma(s) = f(\alpha) + \alpha s \) unless \( s_1 = s \) because we can make \( \alpha_1 - \alpha \) both positive and negative.

Since \( s \) is the slope of \( f \) on \((\underline{\alpha}, \overline{\alpha})\), we have \( f(t) = f(\alpha) + \alpha s = f(\alpha_1) + \alpha_1 s \) for all \( \alpha_1 \in [\underline{\alpha}, \overline{\alpha}] \). Hence for any \( t \neq s \) and any \( \alpha_1 \in [\underline{\alpha}, \overline{\alpha}] \)

\[
\gamma(t) - \gamma(s) \leq (f(\alpha_1) + \alpha_1 t) - (f(\alpha_1) + \alpha_1 s) = \alpha_1 (t - s)
\]

which contradicts \( \gamma'(s) = 0 \) because we can choose \( \alpha_1 \) smaller and larger than \( \alpha \). \( \square \)

To prepare for the proof of Theorem 4.2, fix \( \alpha > m_0 \). Let \( \{A_{n,0} : n \in \mathbb{Z}\} \) have the \( \Phi \)-invariant distribution \( \mu^\alpha \), let \( \{S_{n,k} : n \in \mathbb{Z}, k \in \mathbb{Z}_+\} \) be i.i.d. with distribution \( \mathbb{P} \), let the two collections be independent, and define \( \{W_{n,k}, A_{n,k+1} : n \in \mathbb{Z}, k \in \mathbb{Z}_+\} \) using (A.2) and (A.6). Since \( \Phi(\mu^\alpha) = \mu^\alpha \), process \( \{A_{n,k}, S_{n,k}, W_{n,k} : n \in \mathbb{Z}, k \in \mathbb{Z}_+\} \) is stationary, both in \( n \) and in \( k \). This allows us to extend the process to entire lattice \( \mathbb{Z}^2 \) and thereby define the \( \mathbb{Z}^2 \)-indexed stationary process \( (A,S,W) = \{A_{n,k}, S_{n,k}, W_{n,k} : n, k \in \mathbb{Z}\} \). Define also another \( \mathbb{Z}^2 \)-indexed stationary process \( (\widetilde{A}, \widetilde{S}, \widetilde{W}) \) by

\[
(\widetilde{A}_{i,j}, \widetilde{S}_{i,j}, \widetilde{W}_{i,j}) = (W_{j-1,i+1} + S_{j-1,i+1}, S_{j,i}, A_{j-1,i+1} - S_{j,i})
\]

**Lemma A.7.** Suppose \( \alpha \in \mathcal{A} \). Then the process \( (A,S,W) \) is ergodic under translation \( T_{e_1} \), and also ergodic under \( T_{e_2} \). Furthermore, \( f(\alpha) \in \mathcal{A} \). \( (\widetilde{A}, \widetilde{S}, \widetilde{W}) \) is a stationary queueing system where \( \{A_{n,0} : n \in \mathbb{Z}\} \) has distribution \( \mu^{f(\alpha)} \), and is also ergodic under both \( T_{e_1} \) and \( T_{e_2} \).

**Proof.** Ergodicity under \( T_{e_1} \) follows from the construction. Process \( (A_{n,0}, S_{n,k})_{n \in \mathbb{Z}, k \in \mathbb{Z}_+} \) is ergodic under \( T_{e_1} \), as a product of an ergodic process and an i.i.d. process. The equations developed above define \( (A_{0,k}, S_{0,k}, W_{0,k})_{k \in \mathbb{Z}_+} \) as a function of the process \( (A_{n,0}, S_{n,k})_{n \in \mathbb{Z}, k \in \mathbb{Z}_+} \),
and for each \( m \in \mathbb{Z} \), \((A_{m,k}, S_{m,k}, W_{m,k})\) is obtained by applying the same function to the \( T_{m} \)-shift of the process \((A_{n,0}, S_{n,k})\) for each \( k \in \mathbb{Z}^+ \). Thus \((A_{n,k}, S_{n,k}, W_{n,k})\) is ergodic under \( T_{n} \).

The same argument gives \( T_{n} \)-ergodicity of \((A_{n,k}, S_{n,k}, W_{n,k})\) for any \( \ell \in \mathbb{Z} \). For the final step, let \( B \) be a \( T_{n} \)-invariant event of the full process \((A_{n,k}, S_{n,k}, W_{n,k})\). Write \( G_{\ell} \) for the \( \sigma \)-algebra generated by \((A_{n,k}, S_{n,k}, W_{n,k})\). The conditional expectations \( E(1_B|G_{\ell}) \) are \( T_{n} \)-invariant, hence a.s. constant by the ergodicity proved thus far. \( E(1_B|G_{\ell}) \rightarrow 1_B \) almost surely as \( \ell \rightarrow -\infty \), and consequently \( 1_B \) is a.s. constant. This completes the proof of ergodicity under \( T_{n} \).

To get ergodicity under \( T_{n} \) we transpose, and that leads us to look at \((\tilde{A}, \tilde{S}, \tilde{W})\). To see that \((\tilde{A}, \tilde{S}, \tilde{W})\) is another queueing system with the same i.i.d. service time distribution \( \tilde{S}_{i,j} = S_{j,i} \), we need to check three items.

(i) Independence of \( \{A_{i,\ell}\}_{\ell \in \mathbb{Z}} \) and \( \{\tilde{S}_{i,j}\}_{i,j \geq \ell} \), for each \( \ell \in \mathbb{Z} \). This follows from the structure of equations (A.6) and the independence of the \( \{S_{i,j}\} \).

(ii) \( \tilde{A}_{i,j+1} = (\tilde{A}_{ij} - \tilde{S}_{ij} - \tilde{W}_{ij}) + \tilde{S}_{i+1,j} \). This follows from the top equation of (A.6).

(iii) The third point needed is

\[
\tilde{W}_{k+1,j} = \left( \sup_{n: n \leq k} \sum_{i=n}^{k} (\tilde{S}_{ij} - \tilde{A}_{ij}) \right)^{+}.
\]

This needs a short argument. Fix \( k, j \). The middle equation of (A.6) gives

\[
\tilde{W}_{ij} = (\tilde{W}_{i-1,j} + \tilde{S}_{i-1,j} - \tilde{A}_{i-1,j})^{+}
\]

which can be iterated to give

\[
\tilde{W}_{k+1,j} = \left( \{\tilde{W}_{\ell j} + \sum_{i=\ell}^{k} (\tilde{S}_{ij} - \tilde{A}_{ij})\} \vee \left\{ \max_{n: n \leq \ell, k} \sum_{i=n}^{k} (\tilde{S}_{ij} - \tilde{A}_{ij}) \right\} \right)^{+} \quad \text{for } \ell \leq k.
\]

Thus (A.10) follows if \( \tilde{W}_{ij} = 0 \) for some \( \ell \leq k \). Suppose on the contrary that \( \tilde{W}_{ij} > 0 \) for all \( i \leq k \). Apply (A.11) to all \( \tilde{W}_{ij} \) for \( n < i \leq k \) and divide by \( |n| \) to get

\[
\frac{\tilde{W}_{kj}}{|n|} = \frac{\tilde{W}_{nj}}{|n|} + \frac{1}{|n|} \sum_{i=n}^{k-1} (\tilde{S}_{ij} - \tilde{A}_{ij})
\]

which is the same as

\[
\frac{A_{j-1,k+1}}{|n|} - \frac{S_{jk}}{|n|} = \frac{A_{j-1,n+1}}{|n|} - \frac{S_{jn}}{|n|} + \frac{1}{|n|} \sum_{i=n}^{k-1} (S_{ji} - W_{j-1,i+1} - S_{j-1,i+1}).
\]

Let \( n \rightarrow -\infty \). The i.i.d. property of the \( \{S_{ij}\} \) and Theorem 4.1 of [23], combined with (A.8) from above, give the a.s. limit

\[
\lim_{n \rightarrow -\infty} \frac{1}{|n|} \sum_{i=n}^{k-1} \tilde{A}_{ij} = \lim_{n \rightarrow -\infty} \frac{1}{|n|} \sum_{i=n}^{k-1} (W_{j-1,i+1} + S_{j-1,i+1}) = f(\alpha).
\]
The four leftmost terms of \((A.12)\) vanish as \(n \to -\infty\) (by stationarity and finite expectations). Hence letting \(n \to -\infty\) in \((A.12)\) leads to \(0 = m_0 - f(\alpha) < 0\) (the last inequality from Lemma 3.2). This contradiction verifies \((A.10)\).

At this point we have shown that the stationary process \(\{\tilde{A}_{n,0}\}_{n \in \mathbb{Z}}\) is a fixed point for \(\Phi\) with the deterministic pathwise limit \((A.13)\). By Prop. 4.4 of [23] the process \(\{\tilde{A}_{n,0}\}_{n \in \mathbb{Z}}\) must be ergodic. We have shown that \(f(\alpha) \in \mathcal{A}\). The part of the lemma already proved gives the ergodicity of the process

\[
\{\tilde{A}_{ij}, \tilde{S}_{ij}, \tilde{W}_{ij}\} = \{W_{j-1,i+1} + S_{j-1,i+1}, S_{j,i}, A_{j-1,i+1} - S_{j,i}\}
\]

under translations of the index \(i\). Since ergodicity is preserved by mappings that respect translations, a suitable mapping of the right-hand side above gives the \(T_{\mathcal{E}_2}\)-ergodicity of \(\{A_{nk}, S_{nk}, W_{nk}\}\).

\[\square\]

**Proof of Theorem 4.2.** We begin by constructing a convenient countable subset \(\mathcal{A}_0\) of \(\mathcal{A}\). Let \(\mathcal{U}_0\) be a dense countable subset of \(\mathbb{R} \times \mathbb{q}\) such that \(\mathcal{U}_0\) contains all (at most countably many) points of nondifferentiability of \(g_{pp}\) and a dense countable subset of points of differentiability of \(g_{pp}\). Then put \(\mathcal{A}_0 = \{\gamma'(s \pm) : (\frac{s}{\sqrt{n}}, \frac{t}{\sqrt{n}}) \in \mathcal{U}_0\}\). \(\mathcal{A}_0 \subset \mathcal{A}\) by virtue of Lemma A.6(b) and the closedness of \(\mathcal{A}\).

We construct a measure \(\tilde{\mu}\) on \(\mathbb{R}^{\mathcal{A}_0 \times \mathbb{Z}}\) that couples together the distributions \(\mu^\alpha\) for \(\alpha \in \mathcal{A}_0\) so that the coordinates \(\{\eta_n^\alpha\}_{\alpha \in \mathcal{A}_0, n \in \mathbb{Z}}\) satisfy \(\{\eta_n^\alpha\}_{n \in \mathbb{Z}} \sim \mu^\alpha\) and \(\eta_n^\alpha \leq \eta_n^\beta\) \(\tilde{\mu}\)-a.s. for \(\alpha < \beta\) in \(\mathcal{A}_0\). This measure \(\tilde{\mu}\) comes from a weak limit of a coupled system of queues. For each \(\alpha \in \mathcal{A}_0\) let an initial inter-arrival process be the deterministic constant process \(A_{n,0}^\alpha = 0\). As before use the iterative equations to construct the variables \((A_{n,k}^\alpha, S_k, W_{n,k}^\alpha)\) for \(k \in \mathbb{Z}_+\). Each process uses the same version of the service times \(\{S_{nk}\}\). According to Theorem A.2 [27, Thm. 1], each \(A_{n,k}^\alpha\) converges weakly to \(\mu^\alpha\). Let \(\tilde{\mu}\) be any weak limit point of the joint distributions of the systems \(\{A_{n,k}^\alpha : \alpha \in \mathcal{A}_0\}\) as \(k \to \infty\). The inequalities

\[A_{n,k}^\alpha \leq A_{n,k}^\beta \quad \text{and} \quad W_{n,k}^\alpha \geq W_{n,k}^\beta \quad \text{for} \quad \alpha < \beta\]

are true for the \(A\)-processes at \(k = 0\) by construction. They are propagated for all \(k\) by equations \((A.5)\) and \((A.6)\). Consequently \(\tilde{\mu}\) has the desired properties.

Next we construct a joint queueing process that couples together the stationary queueing processes for all \(\alpha \in \mathcal{A}_0\). Let the inputs \((\{A_{n,0}^\alpha : \alpha \in \mathcal{A}_0\}, \{S_k : k \in \mathbb{Z}_+\})\) have distribution \(\tilde{\mu} \otimes \mathbb{P}\). Construct again the variables \(A_{n,k}^\alpha, S_{nk}, W_{nk}^\alpha : n \in \mathbb{Z}, k \in \mathbb{Z}_+, \alpha \in \mathcal{A}_0\) with the iterative equations. Use the stationarity under translations of \(k\) to extend the joint distribution to a process indexed by \(\mathbb{Z}^2\), denoted by \(\{(A^\alpha, S, W^\alpha) : \alpha \in \mathcal{A}_0\}\) = \(\{A_{n,k}^\alpha, S_{nk}, W_{nk}^\alpha : n, k \in \mathbb{Z}, \alpha \in \mathcal{A}_0\}\). Then for each \(\alpha \in \mathcal{A}_0\), \((A^\alpha, S, W^\alpha)\) is as described in Lemma A.7: stationary and ergodic under both translations, \(A_{n,0}^\alpha : n \in \mathbb{Z}\) \(\sim \mu^\alpha\), and \(\{S_{0,k} + W_{0,k}^\alpha : k \in \mathbb{Z}\} \sim \mu f(\alpha)\). Furthermore, inequalities \((A.14)\) continue to hold almost surely in this coupling.

Define the following mapping from the coordinates \(\{(A^\alpha, S, W^\alpha) : \alpha \in \mathcal{A}_0\}\) to the coordinates \(\{(\omega_x)_{x \in \mathbb{Z}, \alpha \in \mathcal{A}_0, \epsilon \in \{1,2\}}\) of the space \(\tilde{\Omega} = \Omega \times \mathbb{R}^{(1,2) \times \mathcal{A}_0 \times \mathbb{Z}^2}\); for \((n, k) \in \mathbb{Z}^2\) and \(\alpha \in \mathcal{A}_0\),

\[\omega_{n,k}^{\alpha}, \omega_{n,k}^{1,\alpha}, \omega_{n,k}^{2,\alpha} = (S_{-n,-k}, A_{-n-1,-k+1}^\alpha, W_{-n,-k}^\alpha + S_{-n,-k}).\]
Let \( \hat{\mathbb{P}} \) be the distribution induced on \( \hat{\Omega} \) by this mapping, from the joint distribution of the coupled stationary queueing processes.

The probability space \((\hat{\Omega}, \hat{\mathcal{S}}, \hat{\mathbb{P}})\) of Theorem 4.2 has now been constructed. For \( \xi \in U_0 \) and \( i = 1, 2 \) define the functions \( B^{\xi \pm}(x, x + e_i) \) as the following coordinate projections:

\[
B^{\xi \pm}(\omega, x, x + e_i) = \omega_{x}^{i, \gamma'(s \pm)} \quad \text{for} \ s = \xi \cdot e_1 / \xi \cdot e_2.
\]

The set \( A_0 \) was constructed to ensure \( \gamma'(s \pm) \in A_0 \) for each \( \xi \in U_0 \) so these functions are well-defined.

The remainder of the proof consists of two steps: (a) verification that the processes \( B^{\xi \pm}(x, x + e_i) \) defined in (A.16) for \( \xi \in U_0 \) satisfy all the properties required by Theorem 4.2 and (b) definition of \( B^{\xi \pm}(x, x + e_i) \) for all \( \xi \in \mathbb{R} U \) through monotone limits followed by another verification of the required properties.

In part (i) of Theorem 4.2 the measurability claim comes from the construction. The stationarity and ergodicity of each process \( \varphi^{\xi \pm}_x(\omega) = (\omega_x, B^{\xi \pm}(x, x + e_1), B^{\xi \pm}(x, x + e_2)) \) under both translations \( T_{e_1} \) and \( T_{e_2} \) are a consequence of Lemma A.7. The independence claim follows from the fact that in the queuing construction the triple \( (A^{\alpha}_{n-1-k}, S_{n-k}, W^{\alpha}_{-n-k}) \) is a function of \( \{A_{i,m}^{\alpha}, S_{i,j}: i \leq -n, m \leq j \leq -k \} \) for any \( m < -k \).

Part (ii) of Theorem 4.2 requires the cocycle properties. The conservation law (A.7) of the queuing construction implies that, almost surely, for all \( \alpha \in A_0 \)

\[
W^{\alpha}_{-n-k} + S_{-n-k} + A^{\alpha}_{n-1-k} = A^{\alpha}_{n-1-k+1} + W^{\alpha}_{-n-1-k} + S_{-n-1-k}.
\]

Via (A.15) and (A.16) this translates into the \( \tilde{\mathbb{P}} \)-almost sure property

\[
B^{\xi \pm}(x, x + e_2) + B^{\xi \pm}(x + e_2, x + e_1 + e_2) = B^{\xi \pm}(x, x + e_1) + B^{\xi \pm}(x + e_1, x + e_1 + e_2)
\]

for \( x = (n, k) \) and all \( \xi \in U_0 \). Thus each process \( B^{\xi \pm}(x, x + e_i) \) extends to a cocycle \( \{B^{\xi \pm}(x, y) : x, y \in \mathbb{Z}^2\} \). Stationarity came in the previous paragraph and integrability comes form the next calculation.

The tilt vectors satisfy

\[
h(\xi \pm) = -\left( \mathbb{E}[B^{\xi \pm}(0, e_1)], \mathbb{E}[B^{\xi \pm}(0, e_2)] \right) = -(E[A^{\gamma'_{0,0}}_{0,0}], E[W^{\gamma'_{0,0}}_{0,0} + S_{0,0}] = -(\gamma'(s), f(\gamma'(s))) = -\nabla g_{pp}(\xi \pm).
\]

The fact that one-sided gradients satisfy the duality (2.5) is basic convex analysis.

Via (A.15) and (A.16) the bottom equation of (A.6) translates into the potential-recovery property

\[
\omega_x = B^{\xi \pm}(x, x + e_1) \land B^{\xi \pm}(x, x + e_2) \quad \hat{\mathbb{P}}\text{-a.s.}
\]

Part (ii) of Theorem 4.2 has been verified for \( B^{\xi \pm}(\omega, x, x + e_1) \) for \( \xi \in U_0 \).

Part (iii) of Theorem 4.2 is the equality of cocycles that share the tilt vector. This is clear from definition (A.16) because \( h(\xi \pm) \) determines \( \gamma'(s \pm) \).

For the inequalities of part (iv), let \( s = \xi \cdot e_1 / \xi \cdot e_2 \) and \( t = \xi \cdot e_1 / \zeta \cdot e_2 \) for \( \xi, \zeta \in U_0 \). Then \( \xi \cdot e_1 < \zeta \cdot e_1 \) implies \( s < t \). By concavity \( \gamma'(s \pm) \geq \gamma'(s \pm) \geq \gamma'(t \pm) \) and the first inequality of (A.14) gives \( A^{\gamma_{n,k}}_{n,k} \geq A^{\gamma_{n,k}}_{n,k} \geq A^{\gamma_{n,k}}_{n,k} \) which translates into the first inequality of (4.3). Assuming \( \xi_n \cdot e_1 \searrow \zeta \cdot e_1 \), monotonicity gives a.s. existence of the limit and

\[
\lim_{n \to \infty} B^{\xi_n \pm}(x, x + e_1) \leq B^{\xi \pm}(x, x + e_1) \quad \hat{\mathbb{P}}\text{-a.s.}
\]
Monotonicity of the family of cocycles gives a bound that justifies dominated convergence, and hence
\[
\hat{E}\left[\lim_{n\to\infty} B^{\xi_n}(x, x + e_1)\right] = \lim_{n\to\infty} \gamma'(s_{n\pm}) = \gamma'(t+) = \hat{E}[B^{\xi}(x, x + e_1)].
\]
Equality of expectations forces a.s. equality in (A.17). To complete part (iv) of Theorem 4.2 replace \(e_1\) with \(e_2\), take limits from below, and adapt these arguments.

Theorem 4.2 has now been verified for \(B^{\xi}(x, x + e_i)\) defined in (A.16) for \(\xi \in U_0\). The next step is to define \(B^{\xi}(x, x + e_i) = B^{\xi}(x, x + e_i)\) for \(\xi \in (\text{ri} U) \setminus U_0\). Since all points of nondifferentiability of \(g_{\text{op}}\) were included in \(U_0\), \(\xi\) must be a point of differentiability in which case we define \(B^{\xi}(x, x + e_i)\) as equal and denote the process by \(B^{\xi}(x, x + e_i)\).

In order to secure a single null set for all \(\xi, \xi \in \text{ri} U\) for the monotonicity in (4.3), we define the remaining cocycles as one-sided limits. Hence define
\[
B^{\xi}(\hat{\omega}, x, x + e_1) = B^{\xi}(\hat{\omega}, x, x + e_1) = \inf_{\xi \in U_0 : \xi \preceq \xi_1 < \xi < \xi_1 + \gamma \cdot e_1} B^{\xi}(\hat{\omega}, x, x + e_1)
\]
\[
B^{\xi}(\hat{\omega}, x, x + e_2) = B^{\xi}(\hat{\omega}, x, x + e_2) = \sup_{\xi \in U_0 : \xi \preceq \xi_1 < \xi < \xi_1 + \gamma \cdot e_1} B^{\xi}(\hat{\omega}, x, x + e_2). \tag{A.18}
\]

Fix an event \(\hat{\Omega}_0\) of full \(\hat{P}\)-probability on which cocycles are finite and inequalities (4.3) hold for all \(\xi, \xi \in U_0\). Definition (A.18) extends (4.3) to all \(\xi, \xi\).

Pick sequences \(\xi_n'\) and \(\xi_n''\) in \(U_0\) such that \(\xi_n' \cdot e_1 \rightarrow \xi \cdot e_1\) and \(\xi_n'' \cdot e_1 \rightarrow \xi \cdot e_1\). Let \(s_n' = \xi_n' \cdot e_1/\xi_n' \cdot e_2\) and similarly \(s_n''\). Definition (A.18) implies that on the event \(\hat{\Omega}_0\)
\[
B^{\xi}(\hat{\omega}, x, x + e_1) = \lim_{n\to\infty} B^{\xi_n}(\hat{\omega}, x, x + e_1) \tag{A.19}
\]
and by monotonicity and integrable bounds the limit also holds in \(L^1(\hat{P})\).

Next we argue that for the price of a \(\hat{P}\)-null set that is specific to \(\xi\), we can also take the limit in (A.19) from the right, as \(\xi_n'' \rightarrow \xi\). Consider the edge \((x, x + e_1)\) first. Monotonicity gives
\[
B^{\xi}(x, x + e_1) \geq \lim_{n\to\infty} B^{\xi_n''}(\hat{\omega}, x, x + e_1) \quad \text{on the event } \hat{\Omega}_0.
\]
Again monotonicity and integrability of the cocycles give both almost sure and \(L^1(\hat{P})\) convergence. From differentiability of \(\gamma\) at \(t = \xi \cdot e_1/\xi \cdot e_2\) follows
\[
\mathbb{E}[B^{\xi}(x, x + e_1)] = \lim_{n\to\infty} \mathbb{E}[B^{\xi_n}(x, x + e_1)] = \lim_{n\to\infty} \gamma'(s_n') = \gamma'(t) = \lim_{n\to\infty} \gamma'(s_n'').
\]
Consequently \(B^{\xi}(x, x + e_1) = \lim_{n\to\infty} B^{\xi_n}(x, x + e_1)\) \(\hat{P}\)-a.s. The same argument with reversed inequalities works for \(e_2\). Now we have the limit
\[
B^{\xi}(x, x + e_1) = \lim_{n\to\infty} B^{\xi_n}(x, x + e_1) \quad \text{for } U_0 \ni \xi_n \to \xi \in (\text{ri} U) \setminus U_0 \tag{A.20}
\]
both \(\hat{P}\)-a.s. and \(L^1(\hat{P})\), but with a \(\hat{P}\)-null set that can depend on \(\xi_n\).

We turn to verifying the remaining claims of Theorem 4.2 for the newly defined processes \(B^{\xi}(x, x + e_i)\).

Part (i). The measurability claim again comes from the construction. Stationarity and the independence claim are preserved by limits but ergodicity is not. To verify the ergodicity
of $\varphi_L(\hat{\omega}) = (\omega_x, B^c(x, x + e_1), B^c(x, x + e_2))$ under both translations $T_{e_1}$ and $T_{e_2}$ we return to the queuing picture. The limit (A.18) can also be taken in the queuing processes. First $A_0 \ni \alpha_n = \gamma'(s_n^\omega) \land \gamma'(t) = \beta$. Since $A$ is closed, $\beta \in A$. Hence there is a stationary queueing process $(A^\beta, S, W^\beta)$ that satisfies Lemma A.7 and that we can include in the coupling with the queuing processes indexed by $A_0$. The coordinatewise monotone a.s. limit $\lim_{n \to \infty} (A^\alpha_n, S, W^\alpha_n)$ must coincide with $(A^\beta, S, W^\beta)$ by the same reasoning used above: there are inequalities, namely $\lim_{n \to \infty} A^\alpha_{m,k} \leq A^\beta_{m,k}$ and $\lim_{n \to \infty} W^\alpha_{m,k} \geq W^\beta_{m,k}$, but the expectations agree and hence force agreement. The continuous mapping (A.15) transports the distribution of $\{\omega_x, B^c(x, x + e_1), B^c(x, x + e_2) : x \in \mathbb{Z}^2\}$, which thereby inherits from Lemma A.7 the ergodicity claimed in part (i) of Theorem 4.2.

The cocycle properties of part (ii) are preserved by pointwise limits. The identities of part (iii) continue to hold without null sets if we refine the limit definition (A.20) by defining $B^c(x, x + e_i) = B^c(x, x + e_i)$ whenever $\zeta \in \text{ri} U \setminus \mathcal{U}_0$, $\xi \in \mathcal{U}_0 \cap \mathcal{D}$, and $\nabla \mathcal{g}_{pp}(\zeta) = \nabla \mathcal{g}_{pp}(\xi)$. The inequalities and limits of part (iv) were discussed above.

Appendix B. Coalescence of directional geodesics

This section is dedicated to the proof of Theorem 6.6, namely, that two cocycle geodesics defined by the same cocycle and tie-breaking rule coalesce almost surely. The proof follows closely the ideas in [21] for first passage percolation, ported later by [13] to last passage percolation with exponential $\omega_0$. A key portion of the proof is a modification of the weights to produce an event of positive probability that leads to a contradiction. We present the modification argument first.

Given $\mathcal{V} \subset \mathbb{Z}^2$ let $\mathcal{V}^* = \cap_{x \in \mathcal{V}} \{y \not\in x\}$. Define two complementary projections on $\hat{\Omega}$ by

$$
\phi_\mathcal{V}(\hat{\omega}) = \{\omega_x, \omega^\alpha_y : x \not\in \mathcal{V}, y \in \mathcal{V}^*, i \in \{1, 2\}, \alpha \in A_0\}
$$

and

$$
\psi_\mathcal{V}(\hat{\omega}) = \{\omega_x, \omega^\alpha_y : x \in \mathcal{V}, y \not\in \mathcal{V}^*, i \in \{1, 2\}, \alpha \in A_0\}.
$$

Then $\hat{\omega} = (\phi_\mathcal{V}(\hat{\omega}), \psi_\mathcal{V}(\hat{\omega}))$. Coordinates $\psi_\mathcal{V}(\hat{\omega})$ are to be modified. For each finite subset $\mathcal{V} \subset \mathbb{Z}^2$ we are given a $\sigma(\omega_x : x \in \mathcal{V})$-measurable event $R_\mathcal{V} \in \hat{\mathcal{G}}$ such that $\hat{\mathbb{P}}\{R_\mathcal{V}\} > 0$. This means that $R_\mathcal{V} = \{\hat{\omega} \in \hat{\Omega} : (\omega_x)_{x \in \mathcal{V}} \in H_\mathcal{V}\}$ for a Borel set $H_\mathcal{V} \subset \mathbb{R}^\mathcal{V}$. Event $R_\mathcal{V}$ represents the desirable properties that the modified weights will have. For each $\hat{\omega} \in \hat{\Omega}$ let $\mathcal{W}(\hat{\omega})$ be a finite subset of $\mathbb{Z}^2$ that depends on $\hat{\omega}$ in a $\hat{\mathcal{G}}$-measurable manner. This is the set of sites whose weights will be modified to lie in $H_{\mathcal{W}(\hat{\omega})}$.

Let $\hat{\mathbb{P}}(\cdot | \phi_\mathcal{V} = \hat{\omega})$ denote a regular conditional probability measure of $\hat{\mathbb{P}}$, given $\phi_\mathcal{V} = \hat{\omega}$. For $\hat{\mathbb{P}}$-almost every $\hat{\omega}$ it is supported on the event $\{\hat{\omega} : \phi_\mathcal{V}(\hat{\omega}) = \hat{\omega}\}$. For $A \in \hat{\mathcal{G}}$ define

$$
(\mathcal{B}.1) \quad \Psi(A) = \bigcup_{\mathcal{V}} [R_\mathcal{V} \cap \{\hat{\omega} \in \hat{\Omega} : \hat{\mathbb{P}}(A \cap \{W = \mathcal{V}\} | \phi_\mathcal{V} = \hat{\omega}) > 0\}] \in \hat{\mathcal{G}}.
$$

The next lemma says that almost every $\hat{\omega} \in \Psi(A)$ is a modification of an $\hat{\omega} \in A$ whose coordinates $\{\omega_x : x \not\in \mathcal{W}(\hat{\omega})\}$ and $\{\omega^\alpha_y : y \in \mathcal{W}(\hat{\omega})^*, i \in \{1, 2\}, \alpha \in A_0\}$ were left unchanged but whose weights $\{\omega_x : x \in \mathcal{W}(\hat{\omega})\}$ were replaced by new weights $\{\tilde{\omega}_x : x \in \mathcal{W}(\hat{\omega})\}$ in $H_{\mathcal{W}(\hat{\omega})}$. The association of $\hat{\omega}$ to $\tilde{\omega}$ might not be measurable but that is not a problem.
Lemma B.1. \([21, \text{ Lemma 3.1}]\) \(\hat{\mathbb{P}}(A) > 0\) implies \(\hat{\mathbb{P}}(\Psi(A)) > 0\). For \(\hat{\mathbb{P}}\)-almost every \(\hat{\omega} \in \Psi(A)\) there exists \(\hat{\omega} \in A\) and a finite \(V \subset \mathbb{Z}^2\) such that \(\hat{\omega} \in R_V\), \(W(\hat{\omega}) = V\), and \(\phi_V(\hat{\omega}) = \phi_V(\hat{\omega})\).

Proof. To show that \(\hat{\mathbb{P}}(\Psi(A)) > 0\), fix \(V\) so that \(\hat{\mathbb{P}}(A \cap \{W = V\}) > 0\). By Theorem 4.2(i), \(R_V\) and \(\phi_V\) are independent. Hence

\[
\hat{\mathbb{P}}(\Psi(A)) \geq \hat{\mathbb{P}}(R_V) \cdot \hat{\mathbb{P}}[\hat{\mathbb{P}}(A \cap \{W = V\} \mid \phi_V) > 0] > 0.
\]

Let \(\hat{\omega} \in \Psi(A)\) be such that \(\hat{\mathbb{P}}(\cdot \mid \phi_V = \hat{\omega})\) is supported on the event \(\{\hat{\omega} : \phi_V(\hat{\omega}) = \hat{\omega}\}\). Pick a finite \(V \subset \mathbb{Z}^2\) such that \(\hat{\omega} \in R_V\) and \(\hat{\mathbb{P}}(A \cap \{W = V\} \mid \phi_V = \hat{\omega}) > 0\). A set of positive measure cannot be empty so there exists \(\hat{\omega} \in A \cap \{W = V\} \cap \{\phi_V = \hat{\omega}\}\). □

We turn to the proof of coalescence. Beginning with two geodesics that never intersect, stationarity and the modification argument show that with positive probability the following happens for some fixed rectangle: from the north boundary of the rectangle emanates a geodesic that intersects no geodesic that starts to the west or south of the rectangle. By stationarity this gives at least \(cL^2\) disjoint geodesics that start inside an \(L \times L\) square. For large \(L\) this is a contradiction because there are only \(2L\) north and east boundary points through which these geodesics can exit.

Fix \(\xi \in \text{ri } U\) and \(t \in \{e_1, e_2\}\mathbb{Z}^2\). We consider geodesics \(x^u, t, \xi\) and drop \(\xi, t\) from the notation. Because these geodesics follow the same rule \(t\) and cocycle \(B^\xi\), any two that intersect stay together forever. Therefore, we need to prove only that geodesics eventually intersect. The proof is done by way of contradiction.

Suppose \(\hat{\mathbb{P}}\{x^a_{0,\infty} \cap x^b_{0,\infty} = \emptyset\} > 0\) for some \(a, b \in \mathbb{Z}^2\). These geodesics are directed into the interior of the first quadrant (Theorem 6.3 and \(U_{\xi} \subset \text{ri } U\)) and hence must cross any vertical line to the right of \(a\) and \(b\). By restarting the geodesics and by stationarity we can assume \(a = 0\), \(b = me_2\) for some \(m \in \mathbb{N}\) and \(x_0^b = e_1\). Thus we can take the following assumption as the basis from which a contradiction will come.

\[(B.2) \quad \hat{\mathbb{P}}\{x^0_{0,\infty} \cap x^{me_2}_{0,\infty} = \emptyset, \ x^0_{1,\infty} = e_1\} > 0.\]

Again by shift invariance \(x^{ie_2}_{0,\infty} \cap x^{(i+m)e_2}_{0,\infty} = \emptyset\) for infinitely many \(i \geq 0\), with positive probability. Consequently, there exists \(i > m\) such that

\[
\hat{\mathbb{P}}\{x^0_{0,\infty} \cap x^{me_2}_{0,\infty} = \emptyset, \ x^{ie_2}_{0,\infty} \cap x^{(i+m)e_2}_{0,\infty} = \emptyset, \ x^0_{1,\infty} = e_1\} > 0.
\]

Let \(\ell = i + m\). If \(x^{me_2}_{0,\infty} \cap x^{\ell e_2}_{0,\infty} \neq \emptyset\) then by planarity \(x^{me_2}_{0,\infty}\) coalesces with \(x^{ie_2}_{0,\infty}\) forcing \(x^{ie_2}_{0,\infty}\) to intersect with \(x^{\ell e_2}_{0,\infty}\). So we have \(0 < m < \ell\) such that

\[
\hat{\mathbb{P}}\{x^0_{0,\infty} \cap x^{me_2}_{0,\infty} = \emptyset, \ x^{me_2}_{0,\infty} \cap x^{\ell e_2}_{0,\infty} = \emptyset, \ x^0_{1,\infty} = e_1\} > 0.
\]

The remainder of the proof depends on whether the essential infimum of \( \omega_0 \) is taken with positive probability.

**Case 1.** Suppose \(K = \mathbb{P}-\text{ess inf } \omega_0 > -\infty\) and \(\mathbb{P}\{\omega_0 = K\} > 0\). Since \(n^{-1} \sum_{k=0}^{n-1} \omega_{ke_2} \rightarrow m_0 > K\) a.s. we can follow the geodesic \(x^0_{0,\infty}\) and fix large enough deterministic \(M_1 > 0\).
and $M_2 > \ell$ such that
\[
\tilde{\mathbb{P}}\left\{ \hat{\omega} : x_{0,\infty}^0 \cap x_{0,\infty}^{me_2} = \emptyset, x_{0,\infty}^{me_2} \cap x_{0,\infty}^{\ell e_2} = \emptyset, x_1^0 = e_1, \right. \\
\left. x_{M_1+M_2}^0 = (M_1, M_2), \sum_{j=\ell}^{M_2-1} \omega_{0,j} > K(M_2 - \ell) \right\} > 0.
\]
(B.3)

Denote by $A$ the intersection of the event above with the full probability event on which cocycle $B^\omega$ recovers potential $V(\hat{\omega}) = \omega_0$ (i.e. (4.1) holds for $T_x\hat{\omega}$ for all $x$) and satisfies the additivity condition (c) in Definition 2.1. Let $u_1, u_2$ and $u_3$ be the points where geodesics $x_{0,\infty}^{me_2}, x_{0,\infty}^{me_3}$ and $x_{0,\infty}^0$ (respectively) first intersect the line $M_2e_2 + \mathbb{R}e_1$. By definition $u_3 = (M_1, M_2)$. (See Figure 6.)

The geodesic $x_{0,\infty}^0$ will be the one that does not intersect any geodesic that starts from outside, and to the east and/or south of the rectangle $[0, M_1] \times [0, M_2]$. To make this happen with positive probability, we apply the modification argument to the event $A$ defined above.

Let $\mathcal{R}$ be the lattice region strictly between $x_{0,\infty}^0(\hat{\omega})$ and $x_{0,\infty}^{\ell e_2}(\hat{\omega})$, strictly east of $\mathbb{R}e_2$, and strictly south of $M_2e_2 + \mathbb{R}e_1$ (shaded region in Figure 6). Define $\mathcal{W}(\hat{\omega}) = \{x \in \mathcal{R} : \omega_x > K\}$.

For a finite set $\mathcal{V} \subset \mathbb{Z}^2$ let $R_\mathcal{V} = \{\hat{\omega} : \omega_x \leq K \forall x \in \mathcal{V}\}$. Note that $\mathbb{P}(R_\mathcal{V}) > 0$. Event $\Psi(A)$ is given in (B.1) and by Lemma B.1 $\mathbb{P}(A) > 0$ implies $\mathbb{P}(\Psi(A)) > 0$. The claim to be proved now is this:

**Lemma B.2.** In almost every environment $\hat{\omega} \in \Psi(A)$, geodesic $x_{0,\infty}^{u_2}(\hat{\omega})$ does not intersect any geodesic that starts outside the rectangle $[0, M_1] \times [0, M_2]$ at a point $(a, b)$ with either $a \leq 0$ or $b \leq 0$.

**Proof.** From Lemma B.1 we read that almost every $\hat{\omega} \in \Psi(A)$ is a modification of some $\hat{\omega} \in A$ so that the following items hold.

(i) If $\omega_x > K$ for $x \in \mathcal{R}$ then the modified environment satisfies $\omega_x = K$.

(ii) Weights $\{\omega_x : x \notin \mathcal{R}\}$ as well as the values $\{B^\omega(y, y+e_i) : y \geq u_1, i = 1, 2\}$ remain the same under both $\hat{\omega}$ and $\tilde{\omega}$. In particular, geodesics $x_{0,\infty}^{u_1}(\hat{\omega}), x_{0,\infty}^{u_2}(\hat{\omega})$, and $x_{0,\infty}^{u_3}(\hat{\omega})$ are the same as the ones under $\hat{\omega}$.

Part of the reason that $x_{0,\infty}^{u_2}(\hat{\omega})$ does not intersect any geodesic that starts from east or south of the rectangle is that it is “shielded” by geodesics $x_{0,\infty}^0(\hat{\omega})$ and $x_{0,\infty}^{\ell e_2}(\hat{\omega})$. This is the point of the next lemma.

**Lemma B.3.** Let $\hat{\omega} \in A$ be the configuration associated to $\hat{\omega} \in \Psi(A)$ by Lemma B.1. Then for any $v \in x_{0,\infty}^0(\hat{\omega})$ and $n \geq 0$, $x_n^v(\hat{\omega}) \cdot e_2 \leq x_n^v(\hat{\omega}) \cdot e_2$. Similarly, for any $v \in x_{0,\infty}^{\ell e_2}(\hat{\omega})$ and $n \geq 0$, $x_n^v(\hat{\omega}) \cdot e_2 \geq x_n^v(\hat{\omega}) \cdot e_2$.

We defer the proof of this lemma to the end of the appendix. See Figure 6 for a summary of the construction thus far.

By Lemma B.3, any geodesic $y_{0,\infty}(\hat{\omega})$ that intersects geodesic $x_{0,\infty}^{u_2}(\hat{\omega})$ in violation of Lemma B.2 must (i) enter $\mathcal{R}$ through the vertical line segment $]0, \ell e_2[ and (ii) exit $\mathcal{R}$ through the line segment $]u_1, u_3[$. The reason is that if $y_{0,\infty}(\hat{\omega})$ exits $\mathcal{R}$ through $x_{0,\infty}^{\ell e_2}(\hat{\omega})$ or $x_{0,\infty}^{\ell e_2}(\hat{\omega})$, Lemma B.3 prevents it from ever touching $x_{0,\infty}^{u_2}(\hat{\omega})$.

To rule out this last possibility, we simply observe that in environment $\hat{\omega}$ any path from $]0, \ell e_2[ to ]u_1, u_3[$ through $\mathcal{R}$ is inferior to following the east and north boundaries of the
The shaded region is \( \mathcal{R} \) where the weights are modified to be small. Weights on the thick west and north boundaries are large. The curved lines represent the various geodesics. The middle geodesic starting at \( u_2 \) is shielded by all the other ones around it: after the modification, the geodesic starting at \( \ell \) and going through \( u_1 \) becomes the top geodesic in the picture and the geodesic starting at 0 and going through \( u_3 \) becomes the bottom one in the picture. Geodesics entering from \([0, \ell e_2]\) cannot exit the top between \( u_1 \) and \( u_3 \) and hence cannot touch the middle geodesic starting at \( u_2 \).

rectangle. This is because for \( x \in \mathcal{R} \) each weight \( \tilde{\omega}_x = K \), while along the east and north boundaries each weight \( \tilde{\omega}_x = \omega_x \geq K \) and by (B.3) some weight on the line segment \([\ell e_2, M_2 e_2]\) is \( > K \). Thus no geodesic \( y_{0,\infty}(\tilde{\omega}) \) from outside the rectangle can follow this strategy to intersect \( x_{0,\infty}^{u_2}(\tilde{\omega}) \). Lemma B.2 has been proved.

The Burton-Keane lack of space argument [5] gives a contradiction with Lemma B.2 and \( \hat{P}(\Psi(A)) > 0 \), and thereby contradicts assumption (B.2). By ergodicity and \( \hat{P}(\Psi(A)) > 0 \), for all large enough \( L \) and a small enough fixed \( \delta > 0 \), event \( \Psi(A) \circ T \) occurs for at least \( \delta L^2 \) points \( z \in [0, L]^2 \) such that the rectangles \( z + [0, M_1] \times [0, M_2] \) are pairwise disjoint and lie inside \([0, L]^2\). This gives \( \delta L^2 \) pairwise disjoint geodesics that start inside \([0, L]^2\). Each of these geodesics must exit through a boundary point of \([0, L]^2\), but for large enough \( L \) the number of boundary points is \( < \delta L^2 \). Theorem 6.6 has been proved in Case 1.

Case 2. Assume \( \mathbb{P}\text{-ess inf} \omega_0 \) cannot be taken with positive \( \mathbb{P} \)-probability. By taking \( K \) close enough but strictly above \( \mathbb{P}\text{-ess inf} \omega_0 \) we can again find \( M_1 \) and \( M_2 \) such that

\[
\mathbb{P}\left\{ x_{0,\infty}^{0} \cap x_{0,\infty}^{me_2} = \varnothing, x_{0,\infty}^{me_2} \cap x_{0,\infty}^{\ell e_2} = \varnothing, x_1^{0} = e_1, x_{M_1+M_2}^{0} = (M_1, M_2), \omega_i,M_2 \geq K \forall i \in [0, M_1], \omega_{0,j} > K \forall j \in [0, M_2] \right\} > 0
\]
and \( \mathbb{P}\{\omega_0 \leq K\} > 0 \). Denote again by \( A \) the intersection of the event above with the full measure event on which cocycle \( B^\xi_- \) recovers potential \( V(\hat{\omega}) = \omega_0 \) (i.e. \( (4.1) \) holds for \( T_x \hat{\omega} \) for all \( x \)) and is additive, as in Definition 2.1(c).

The rest of the argument is the same as in Case 1. In particular, again in environment \( \tilde{\omega} \) any path from the east to the north boundary through \( R \) is inferior to following the set and north boundaries. We consider the proof of Theorem 6.6 complete.

It remains to give the proof of Lemma B.3.

**Proof of Lemma B.3.** We do the case \( v \in x_{0,\infty}^0(\tilde{\omega}) \). Let \( z \) be the first point after which \( x_{n}^0(\tilde{\omega}) \cdot e_2 \leq x_{n}^0(\hat{\omega}) \cdot e_2 \) is violated. Then the two geodesics split at \( z \) so that \( x_{1}^1(\tilde{\omega}) = z + e_2 \) and \( x_{1}^1(\hat{\omega}) = z + e_1 \). Point \( z \) lies inside the \([0,M_1] \times [0,M_2]\) rectangle because north and east of this rectangle \( \tilde{\omega} \) geodesics agree with those of \( \hat{\omega} \). Either \( x_{0,\infty}^z(\tilde{\omega}) \) hits the north boundary of the \([0,M_1] \times [0,M_2]\) rectangle, or it hits the path \( x_{0,\infty}^0(\hat{\omega}) \) inside the rectangle. We treat the two cases separately. See Figure 7.

**Case (a).** \( x_{0,\infty}^z(\tilde{\omega}) \) intersects with \([M_2e_2,u_3]\) at some point \( y_1 \). Since weights were not modified on \( x_{0,\infty}^0(\tilde{\omega}) \) and were not increased anywhere, the last passage time \( G_{z,u_3}^z(\tilde{\omega}) \) under \( \tilde{\omega} \) equals \( G_{z,u_3}(\omega) \), the time under the old environment \( \hat{\omega} \). Combine this with Lemma 6.1(a) for \( \omega \) and potential recovery for \( \tilde{\omega} \) to get

\[
B^{\xi^-}(\tilde{\omega}, z, u_3) = G_{z,u_3}(\omega) \leq B^{\xi^-}(\hat{\omega}, z, u_3).
\]

Since \( \hat{\omega} \) agrees with \( \tilde{\omega} \) on the north boundary of the rectangle, we have \( B^{\xi^-}(\tilde{\omega}, y_1, u_3) = B^{\xi^-}(\hat{\omega}, y_1, u_3) \). The additivity of cocycles then implies that \( B^{\xi^-}(\hat{\omega}, z, y_1) \leq B^{\xi^-}(\hat{\omega}, z, y_1) \).

On the other hand, we have

\[
B^{\xi^-}(\tilde{\omega}, z, y_1) = G_{z,y_1}(\tilde{\omega}) \leq G_{z,y_1}(\omega) \leq B^{\xi^-}(\hat{\omega}, z, y_1).
\]

The first equality follows again from Lemma 6.1(a) for the cocycle geodesic \( x_{0,\infty}^z(\hat{\omega}) \) in environment \( \hat{\omega} \). The first inequality comes from the fact that the modified weights are no
larger than the original ones. The second inequality is again due to potential recovery. Combine all the inequalities above to conclude that
\[(B.4) \quad B^\xi(\tilde{\omega}, z, y_1) = B^\xi(\tilde{\omega}, z, y_1)\]
and \(B^\xi(\tilde{\omega}, z, u_3) = G_{z,u_3}(\omega)\). Rewrite the last equality as
\[
\sum_{i=0}^{\lfloor u_3-z \rfloor - 1} B^\xi(\tilde{\omega}, x_i^z(\tilde{\omega}), x_{i+1}^z(\tilde{\omega})) = B^\xi(\tilde{\omega}, z, u_3) = G_{z,u_3}(\omega) = \sum_{i=0}^{\lfloor u_3-z \rfloor - 1} \omega_{x_i^z(\tilde{\omega})}.
\]
Potential recovery under \(\tilde{\omega}\) and \(\tilde{\omega} = \omega\) on the path \(x^0_{0,\infty}(\tilde{\omega})\) now imply that the sums agree term by term. In particular,
\[(B.5) \quad B^\xi(\tilde{\omega}, z, x_1^z(\tilde{\omega})) = \omega_z.\]
In the same manner we deduce the statement
\[(B.6) \quad B^\xi(\tilde{\omega}, z, x_1^z(\tilde{\omega})) = \omega_z.\]
To see this last identity, consider this:
\[
\sum_{i=0}^{\lfloor y_1-z \rfloor - 1} \tilde{\omega}_{x_i^z(\tilde{\omega})} \leq \sum_{i=0}^{\lfloor y_1-z \rfloor - 1} \omega_{x_i^z(\tilde{\omega})} \leq \sum_{i=0}^{\lfloor y_1-z \rfloor - 1} B^\xi(\tilde{\omega}, x_i^z(\tilde{\omega}), x_{i+1}^z(\tilde{\omega}))
\]
\[
= B^\xi(\tilde{\omega}, z, y_1) = B^\xi(\tilde{\omega}, z, y_1) = G_{z,y_1}(\tilde{\omega}) = \sum_{i=0}^{\lfloor y_1-z \rfloor - 1} \tilde{\omega}_{x_i^z(\tilde{\omega})}.
\]
The first two inequalities are valid term by term, by the modification and potential recovery. The third step is cocycle additivity, the fourth is \((B.4)\) from above, and the last two are due to \(\{x_i^z(\tilde{\omega})\}\) being a cocycle geodesic. The upshot is that the second and third sums must agree term by term. The equality of the first terms is \((B.6)\) because \(\tilde{\omega} = \omega\) on the path \(x^0_{0,\infty}(\tilde{\omega})\).

Equations \((B.5)-(B.6)\) are incompatible with \(x_1^z(\tilde{\omega}) \neq x_1^z(\tilde{\omega})\) since both geodesics follow the same tie-breaking rule \(t\). Thus Case (a) led to a contradiction.

**Case (b).** \(x^z_{0,\infty}(\tilde{\omega})\) intersects with \(x^0_{0,\infty}(\tilde{\omega})\) at some point \(y_2 > z\). Start by observing that \(G_{z,y_2}(\tilde{\omega}) = G_{z,y_2}(\omega)\). Hence, \(B^\xi(\tilde{\omega}, z, y_2) = B^\xi(\tilde{\omega}, z, y_2) = G_{z,y_2}(\omega)\). An argument similar to Case (a) shows that Case (b) cannot happen either.

We have proved the part of Lemma B.3 that claims \(x_n^v(\tilde{\omega}) \cdot e_2 \leq x_n^v(\tilde{\omega}) \cdot e_2\) for any \(v \in x^0_{0,\infty}(\tilde{\omega})\). The claim for geodesics starting from \(v \in x^0_{0,\infty}(\tilde{\omega})\) is proved similarly. \(\square\)

### Appendix C. Ergodic theorem for cocycles

Cocycles satisfy a uniform ergodic theorem. The following is a special case of Theorem 9.3 of [15]. Note that a one-sided bound suffices for a hypothesis. Recall Definition 2.1 for the space \(\mathcal{C}_0\) of centered cocycles.

**Theorem C.1.** Assume \(\mathbb{P}\) is ergodic under the transformations \(\{T_{e_i} : i \in \{1,2\}\}\). Let \(F \in \mathcal{C}_0\). Assume there exists a function \(V\) such that for \(\mathbb{P}\)-a.e. \(\omega\)
\[
(C.1) \quad \lim_{\varepsilon \searrow 0} \lim_{n \to \infty} \max_{x : |x|_1 \leq n} \frac{1}{n} \sum_{0 \leq k \leq n} |V(T_{x+ke_i}(\omega))| = 0 \quad \text{for } i \in \{1,2\}
\]
and \( \max_{i \in \{1, 2\}} F(\omega, 0, e_i) \leq V(\omega) \). Then
\[
\lim_{n \to \infty} \max_{x = z_1 + \cdots + z_0, z_1, \ldots, z_n \in \{e_1, e_2\}^n} \frac{|F(\omega, 0, x)|}{n} = 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega.
\]

If the process \( \{V(T_x \omega) : x \in \mathbb{Z}^2\} \) is i.i.d., then a sufficient condition for (C.1) is \( \mathbb{E}(|V|^p) < \infty \) for some \( p > 2 \) [29, Lemma A.4]

**APPENDIX D. PERCOLATION CONE**

In this appendix we prove Theorem 2.10. The proof is divided between two sections. Section D.1 proves that on \( \mathcal{U} \), \( g_{pp} = 1 \) exactly on the percolation cone. Section D.2 proves the differentiability of \( g_{pp} \) at \( \eta = (\beta_{p_1}, 1 - \beta_{p_1}) \). Differentiability at \( \eta = (1 - \beta_{p_1}, \beta_{p_1}) \) comes by symmetry.

The standing assumptions are \( \{\omega_x\}_{x \in \mathbb{Z}^2} \) i.i.d., \( \mathbb{E}|\omega_0|^p < \infty \) for some \( p > 2 \), \( \omega_x \leq 1 \), and \( p_1 = \mathbb{P}\{\omega_0 = 1\} > \overline{p}_c \) where \( \overline{p}_c \) is the critical probability of oriented site percolation. The \( \omega \)-weights are assumed nondegenerate and so \( \mathbb{P}\{\omega_0 < 1\} > 0 \).

The oriented percolation weights are defined by \( \sigma_x = 1{\{\omega_x = 1\}} \). The oriented percolation event \( u \to v \) means that there exists an up-right path \( u = x_0, x_1, \ldots, x_m = v \) with \( x_i - x_{i-1} \in \{e_1, e_2\}, m = |v - u|_1 \), and such that \( \sigma_{x_i} = 1 \) for \( i = 1, \ldots, m \). (The initial point \( u \) may be open or closed.) The percolation event \( u \to \infty \) means that there is an infinite up-right path starting at \( u \) along which all weights \( \sigma_x = 1 \) except perhaps \( \sigma_u \).

**D.1. Flat edge.** This section proves that on \( \mathcal{U} \) the limiting time constant \( g_{pp} \) is equal to one only on the percolation cone.

**THEOREM D.1.** Let \( \xi \in \mathcal{U} \). Then \( g_{pp}(\xi) = 1 \) if and only if \( 1 - \beta_{p_1} \leq \xi \cdot e_1 \leq \beta_{p_1} \).

The rest of the section gives the proof.

**LEMMA D.2.** Fix \( p \in (0, 1) \) and \( L > 0 \). Then there exists \( q_0 = q_0(p, L) \in (0, 1) \) such that the following holds. If \( q \geq q_0 \) and \( \{\tau_z : z \in \mathbb{Z}^2\} \) are stationary \( \{0, 1\} \)-valued random variables with \( \mathbb{P}\{\tau_0 = 1\} = q \) and \( \tau_0 \) independent of \( \{\tau_z : |z|_1 > L\} \), then there are positive constants \( a \) and \( b \) such that, for all \( m \geq 1 \),
\[
P\{\exists \text{ up-right path } z_{0,m} : z_0 = 0, \sum_{i=0}^{m} \tau_{z_i} \leq p(m + 1)\} \leq ae^{-bm}.
\]

**Proof.** First, fix \( m \geq 1 \) and an up-right path \( z_{0,m} \) such that \( z_0 = 0 \). For \( k \leq m + 1 \), on the event \( \sum_{j=0}^{m} (1 - \tau_{z_j}) = k \), we can find, among the \( k \) indices with \( \tau_{z_j} = 0 \), indices \( j_1, \ldots, j_{\lfloor k/(L+1) \rfloor} \) such that \( |z_{j_i} - z_{j_r}|_1 > L \) for all \( r, i \leq \lfloor k/(L+1) \rfloor \). Given these indices, the probability that \( \tau_{z_{j_r}} = 0 \) for all \( r \leq \lfloor k/(L+1) \rfloor \) is bounded above by \( (1 - q)^{k/(L+1)} \). There are at most \( \binom{m+1}{\lfloor k/(L+1) \rfloor} \) many choices of these indices. Consequently,
\[
P\sum_{j=0}^{m} (1 - \tau_{z_j}) = k \leq \binom{m+1}{\lfloor k/(L+1) \rfloor} (1 - q)^{k/(L+1)}.
\]
This implies
\[ P\left\{ \sum_{i=0}^{m} \tau_{z_i} \leq \rho(m+1) \right\} \leq \sum_{(1-\rho)(m+1) \leq k \leq m+1} \left( \frac{m+1}{[k/(L+1)]]} \right) (1-q)^{k/(L+1)}. \]

Since there are \( 2^m \) paths \( z_0,m \), the probability in the claim of the lemma is then bounded above by
\[ 2^m \sum_{(1-\rho)(m+1) \leq k \leq m+1} \left( \frac{m+1}{[k/(L+1)]]} \right) (1-q)^{k/(L+1)} \]
\[ \leq 2^m (1-q)^{(1-\rho)(m+1)/(L+1)} \times (L+1) \sum_{i=0}^{m+1} \left( \frac{m+1}{i} \right) \]
\[ \leq 2(L+1)(1-q)^{(1-\rho)/(L+1)} \exp \left\{ \left[ \log 4 + \frac{1-\rho}{L+1} \log(1-q) \right] m \right\}. \]

This decays exponentially fast as soon as \( q > 1 - 4^{-(L+1)/(1-\rho)}. \)

The main work is in the next proposition.

**Proposition D.3.** Assume \( p_1 = \mathbb{P}\{\omega_0 = 1\} > \bar{p}_c. \) Then for each \( \varepsilon \in (0,1-\beta_{p_1}) \) there exist finite positive constants \( A, B, \) and \( \delta \) such that for all \( \ell, k \in \mathbb{N} \) with \( \ell/k \leq (1-\beta_{p_1} - \varepsilon)/(\beta_{p_1} + \varepsilon) \) we have
\[ \mathbb{P}\{G_{0,(k,\ell)} \geq (1-\delta)(k + \ell)\} \leq Ae^{-Bk}. \]  

**Proof.** For \( N \in \mathbb{N} \) and \( z \in \mathbb{Z}^2 \) define
\[ C_N(0) = \{ x \geq 0 : |x|_1 < N \}, \ C_N(z) = z + C_N(0), \ \text{and} \ \ B_N(z) = C_{2N}(z) \setminus C_N(z). \]

Fix \( \varepsilon_1 \in (0,\varepsilon). \) Fix \( \delta_0 \in (0,1) \) such that \( p_0 = \mathbb{P}\{\omega_0 \geq 1-\delta_0\} > \bar{p}_c \) and \( \beta_{p_1} \leq \beta_{p_0} < \beta_{p_1} + \varepsilon \). Here we used the continuity of \( \beta_p \) as a function of \( p \in [\bar{p}_c,1) \) [10, (3) on p. 1031].

Abbreviate
\[ \lambda_{p_1,\varepsilon_1} = \frac{1-\beta_{p_1} - \varepsilon_1}{\beta_{p_1} + \varepsilon_1}. \]

Given \( N \in \mathbb{N} \) and \( \omega \in \Omega, \) color \( z \in \mathbb{Z}^2 \) black if
\[ G_{u,v} \leq |v - u|_1 - \delta_0 \]
for every \( Nz \leq u \leq v \) with \( |u - Nz|_1 = N, |v - u|_1 = N, \) and
\[ (v-u) \cdot e_2 \]
\[ (v-u) \cdot e_1 \leq \lambda_{p_1,\varepsilon_1}. \]

See the left panel of Figure 9. Color \( z \) white if it is not black. Then
\[ \mathbb{P}\{0 \text{ is white}\} \leq N + 1 \mathbb{P}\{\exists v \geq 0 : |v|_1 = N, v \cdot e_2/v \cdot e_1 \leq \lambda_{p_1,\varepsilon_1}, G_{0,v} > |v|_1 - \delta_0\} \]
\[ = p^{-1}(N + 1) \mathbb{P}\{\exists v \geq 0 : |v|_1 = N, v \cdot e_2/v \cdot e_1 \leq \lambda_{p_1,\varepsilon_1}, G_{0,v} > |v|_1 - \delta_0, \omega_v = 1\}. \]
(For the equality we used the fact that \( G_{0,v} \) is independent of \( \omega_v \).) Define the oriented site percolation weights \( \sigma_x = 1{\omega_x \geq 1-\delta_0}. \) Since \( \omega_x \leq 1 \) for all \( x, \omega_v = 1 \) and \( G_{0,v} > |v|_1 - \delta_0 \)
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imply the existence of an up-right path from 0 to \( v \) with \( \omega_x \geq 1 - \delta_0 \) along the path. In other words \( 0 \to v \) in the oriented percolation process. Thus,

\[
\mathbb{P}\{ \text{0 is white} \} \leq p_1^{-1}(N + 1)\mathbb{P}\{ N^{-1}a_N \geq 1/(\lambda_{p_1, \varepsilon_1} + 1) = \beta_{p_1} + \varepsilon_1 \}.
\]

Since \( \beta_{p_1} + \varepsilon_1 > \beta_{p_0} \), the probability on the right-hand side decays exponentially fast. (See the first remark on p. 1018 of [10].) Consequently, the probability the origin is white vanishes as \( N \to \infty \).

Pick \( \rho \in (0, 1) \) such that

\[
\frac{1 - \beta_{p_1} - \varepsilon}{1 - \beta_{p_1} - \varepsilon_1} < \rho < 1.
\]

Pick \( N \) large enough so that \( \mathbb{P}\{0 \text{ is black} \} \geq q_0 \), where \( q_0 \) is from Lemma D.2. Pick \( \delta > 0 \) small so that

\[
\frac{1 - \beta_{p_1} - \varepsilon}{1 - \beta_{p_1} - \varepsilon_1} + \frac{N\delta}{\delta_0} < \rho.
\]

(D.3)

Given an up-right path \( x_{0,k+\ell} \) from 0 to \( (k, \ell) \) let \( m = \lfloor (k + \ell)/N \rfloor \) and define the up-right path \( z_{0,m} \) by

\[
x_{(j+1)N-1} \in C_N(Nz_j), \quad 0 \leq j \leq m.
\]

Vertices \( Nz_j \) are the south-west corners of the squares \( \{ y : Nz \leq y \leq Nz + (N-1, N-1) \} \), \( z \in \mathbb{Z}_+^2 \), that path \( x_{0,k+\ell} \) enters in succession. See Figure 8.

\[
\text{Figure 8. Up-right path } z_\bullet \text{ constructed from the up-right path } x.
\]

We prove next that for \( k \) and \( \ell \) as in the claim of the proposition

\[
\{ G_{0,(k,\ell)} \geq (1 - \delta)(k + \ell) \}
\]

\[
\subset \left\{ \exists \text{ an up-right path } z_{0,m} : z_0 = 0, \sum_{j=0}^{m} \mathbbm{1}\{z_j \text{ is black} \} \leq \rho(m + 1) \right\}.
\]

(D.4)

The proposition then follows from Lemma D.2 and the fact that \( k + \ell \leq N(m + 1) \).
Fix an up-right path $x_{0,k+\ell}$ such that $x_0 = 0$, $x_{k+\ell} = (k, \ell)$, and

\[ \sum_{i=0}^{k+\ell-1} \omega_{x_i} \geq (1 - \delta)(k + \ell). \]  

Consider $j \leq m$. If $z_j$ is black, then we label $j$ as good if

\[ \frac{(x_{(j+2)N} - x_{(j+1)N}) \cdot e_2}{(x_{(j+2)N} - x_{(j+1)N}) \cdot e_1} \leq \lambda_{p_1, \epsilon_1}. \]

If $z_j$ is black and $j$ is not good, then say $j$ is bad. See Figure 9.

**Figure 9.** A good $j$ (left) and a bad $j$ (right). Here, $u = x_{(j+1)N}$ and $v = x_{(j+2)N}$. The shaded region contains the points $x \geq u$ such that $(x - u) \cdot e_2 / (x - u) \cdot e_1 > \lambda_{p_1, \epsilon_1}$.

If $j$ is good then $x_{(j+1)N}, (j+2)N$, the portion of the path crossing $B_N(z_j)$, has a passage time no larger than $N - \delta_0$ (see (D.2)). Since $\omega_x \leq 1$ for all $x$,

\[ \sum_{i=0}^{k+\ell-1} \omega_{x_i} \leq k + \ell - \delta_0 |\{j \leq m : j \text{ is good}\}|. \]

Together with (D.5) this implies

\[ |\{j \leq m : j \text{ is good}\}| \leq \frac{\delta}{\delta_0} (k + \ell) \leq \frac{\delta N}{\delta_0} (m + 1). \]  

If $j$ is bad, then

\[ (x_{(j+2)N} - x_{(j+1)N}) \cdot e_2 > \lambda_{p_1, \epsilon_1} (x_{(j+2)N} - x_{(j+1)N}) \cdot e_1 = \lambda_{p_1, \epsilon_1} [N - (x_{(j+2)N} - x_{(j+1)N}) \cdot e_2]. \]

This implies

\[ (x_{(j+2)N} - x_{(j+1)N}) \cdot e_2 > \frac{N \lambda_{p_1, \epsilon_1}}{1 + \lambda_{p_1, \epsilon_1}}. \]  

Similarly, $\ell \leq \lambda_{p_1, \epsilon} k$ implies

\[ x_{k+\ell} \cdot e_2 = \ell \leq \frac{(k + \ell) \lambda_{p_1, \epsilon}}{1 + \lambda_{p_1, \epsilon}}. \]
Adding (D.7) over the bad $j$ now leads to
\[
\frac{N \lambda_{p_1,x_1}}{1 + \lambda_{p_1,x_1}} |\{j \leq m : j \text{ is bad}\}| \leq \frac{(k + \ell) \lambda_{p_1,x}}{1 + \lambda_{p_1,x}}.
\]

Consequently,
\[
(D.8) \quad |\{j \leq m : j \text{ is bad}\}| \leq \frac{1 - \beta_{p_1} - \varepsilon}{1 - \beta_{p_1} - \varepsilon_1} \cdot \frac{k + \ell}{N} \leq \frac{1 - \beta_{p_1} - \varepsilon}{1 - \beta_{p_1} - \varepsilon_1} (m + 1).
\]

By the choice of $\delta$ in (D.3), adding (D.6) and (D.8) we see that for a path satisfying (D.5) the proportion of $z_j$ colored black is no more than $\rho(m + 1)$. Inclusion (D.4) has been verified and Proposition D.3 proved. \qed

Proof of Theorem D.1. On the positive probability event \{0 \to \infty\} in the oriented percolation with weights $\sigma_x = \mathbb{1}\{\omega_x = 1\}$ we have $0 \to (a_n, n - a_n)$ and $G_{0,(a_n, n-a_n)} = n$ for all $n$. Since $a_n/n \to \beta_{p_1}$ as $n \to \infty$, the shape theorem (E.1) implies $g_{pp}(\beta_{p_1}, 1 - \beta_{p_1}) = 1$. By symmetry, we have $g_{pp}(1 - \beta_{p_1}, \beta_{p_1}) = 1$ as well. Concavity of $g_{pp}$ and the fact that $g_{pp}(\xi) \leq 1$ for all $\xi \in \mathcal{U}$ imply that $g_{pp}(\xi) = 1$ when $1 - \beta_{p_1} \leq \xi \cdot e_1 \leq \beta_{p_1}$.

For the other direction assume $\xi \cdot e_1 > \beta_{p_1}$ and apply (D.1) to $(k, \ell)$ to $|\{k \ell|n \xi\}|$ in conjunction with (2.2) to deduce that $g_{pp}(\xi) \leq 1 - \delta$ for some $\delta > 0$. The result for $\xi \cdot e_1 < 1 - \beta_{p_1}$ comes by symmetry. \qed

D.2 Differentiability at the endpoints. In this section we prove that $g_{pp}$ is differentiable at $\bar{\eta} = (\beta_{p_1}, 1 - \beta_{p_1})$. It is convenient here to alter the definition (1.1) of the last-passage time $G_{x,y}$ so that $\omega_x$ is excluded and $\omega_y$ is included. This of course makes no difference to the limit $g_{pp}$.

We define the oriented percolation process more generally. The successive levels on which the process lives are denoted by $D_n = \{(i,j) \in \mathbb{Z}^2 : i + j = n\}$. Let $S \subset D_m$ be a given initial occupied set. Then at time $n \in \mathbb{Z}_+$ the occupied set is $\mathcal{O}_n(S) = \{v \in D_{m+n} : \exists u \in S : u \to v\}$. If $S$ is bounded below ($S$ has only finitely many points below the $x$-axis), the lowest point $r_n(S)$ of $\mathcal{O}_n(S)$ is well-defined and satisfies $r_n(S) = (a_n(S), b_n(S))$ where
\[
a_n(S) = \max_{u \in \mathcal{O}_n(S)} \{u \cdot e_1\} \quad \text{and} \quad b_n(S) = \min_{u \in \mathcal{O}_n(S)} \{u \cdot e_2\}.
\]

A particular case of such an initial set is $\tilde{\mathbb{Z}}_- = \{(k,-k) : k \in \mathbb{Z}_+\}$, the antidiagonal copy of $\mathbb{Z}_-$. Occasionally we also use the notation $b(O_n(S)) = b_n(S)$. Let $F_m = \{(k,k) : k = 1, \ldots, m\}$.

Lemma D.4. For infinite sets $A \subseteq B \subseteq \tilde{\mathbb{Z}}_-$,
\[
\mathbb{E}[a_n(A \cup F_m) - a_n(A)] \geq \mathbb{E}[a_n(B \cup F_m) - a_n(B)] \geq m
\]
with equality in the last inequality if $B = \tilde{\mathbb{Z}}_-$. \n
Proof. Since $\mathcal{O}_n(A \cup F_m) = \mathcal{O}_n(A) \cup \mathcal{O}_n(F_m)$,
\[
\begin{align*}
a_n(A \cup F_m) - a_n(A) &= a_n(A) \vee a_n(F_m) - a_n(A) = 0 \vee (a_n(F_m) - a_n(A)) \\
&\geq 0 \vee (a_n(F_m) - a_n(B)) = a_n(B \cup F_m) - a_n(B).
\end{align*}
\]

By a shift of the underlying weights $\omega$,
\[
r_n(\tilde{\mathbb{Z}}_- \cup F_m)(\omega) = r_n(\tilde{\mathbb{Z}}_-)(T_{m,-m}\omega) + (m, -m).
\]
By the shift-invariance of $\mathbb{P}$

$$\mathbb{E}[a_n(\bar{Z}_- \cup F_m)] = \mathbb{E}[a_n(\bar{Z}_-)] = m.$$ □

Let $\ell$ be an integer $\geq 2$. Fix a constant $c_0 < 1$ such that $\mathbb{P}(c_0 \leq \omega_0 < 1) > 0$. To have an $\ell$-triangle (configuration) at $(a, b) \in \mathbb{Z}^2$ means that $\omega_{a+b+i} < 1$ for $i = 1, \ldots, \ell$, $c_0 \leq \omega_{a+1,b} < 1$, and except for $(a+1, b)$ all sites in the triangle $\{(i, j) : i \geq a+1, j \geq b, i-(a+1)+j-b \leq \ell-1\}$ have weight $\omega_{i,j} = 1$. See Figure 10.

**Figure 10.** An $\ell$-triangle at $(a, b)$ with $\ell = 5$. The weight $\omega_{a,b}$ is unrestricted. The column above $(a, b)$ has weights $\omega_{a,b+i} < 1$. Point $(a+1, b)$ has weight $c_0 \leq \omega_{a+1,b} < 1$. In the region inside the dotted boundary all weights are equal to 1.

Let $V_\ell = \{(\ell - i, i) : 0 \leq i \leq \ell - 1\}$. Suppose there is an $\ell$-triangle at $(a, b)$. Then two things happen that are relevant for the sequel.

- There is an up-right path of $\omega$-weight $\ell + \omega_{a+1,b} - 1 \geq \ell + c_0 - 1$ from $(a, b)$ to each of the $\ell$ sites of $(a, b) + V_\ell$, without counting the weight at $(a, b)$.

Furthermore, no open oriented percolation path can go through any of the sites $\{(a+1, b), (a, b+i) : i = 1, \ldots, \ell\}$.

Now let $S$ be an infinite initial set that is bounded below. Set $O_n^0 = O_n(S)$, $r_n^0 = r_n(S) = (a_n^0, b_n^0)$. Define the stopping time

$$\tau_1 = \inf\{n > 0 : \exists \text{ } \ell \text{-triangle at } r_n^0\}.$$

The natural filtration is $\mathcal{H}_m = \sigma\{\omega_{i,j} : i + j \leq m\}$.

Since $r_{\tau_1 \ell - \ell}^0$ is the lowest point of the occupied set $O_{\tau_1 \ell - \ell}$, no open oriented percolation path can reach $r_{\tau_1 \ell - \ell}^0 + V_\ell$:

$$(r_{\tau_1 \ell - \ell}^0 + V_\ell) \cap O_{\tau_1 \ell}^0 = \emptyset.$$

To each point of the set $V_{\tau_1 \ell}^0 = r_{\tau_1 \ell - \ell}^0 + V_\ell \subset D_{\tau_1 \ell}$ there is an up-right path of $\omega$-weight

$$\tau_1 \ell - 1 + \omega_{r_{\tau_1 \ell - \ell} + e_1} \geq \tau_1 \ell - 1 + c_0$$

from some point on $S$.

Start a new process $O_n^1$ at level $\tau_1 \ell$ by joining $V_{\tau_1 \ell}^0$ to the occupied set:

$$O_n^1 = \begin{cases} O_n^0, & n \leq \tau_1 \ell - 1 \\ O_{n-\tau_1 \ell}(O_{\tau_1 \ell}^0 \cup V_{\tau_1 \ell}^0), & n \geq \tau_1 \ell. \end{cases}$$
Let \( r_n^k = (a_n^k, b_n^k) \) be the lowest point of \( \mathcal{O}_n^k \).

Continue in this manner, with
\[
\tau_{k+1} = \inf \{ n > \tau_k : \exists \ell \text{-triangle at } r_{n\ell - \ell}^k \}, \quad V_{\tau_{k+1} \ell}^k = r_{\tau_{k+1} \ell}^k + V_{\ell},
\]

\[
\mathcal{O}_{n}^{k+1} = \begin{cases} 
\mathcal{O}_n^k, & n \leq \tau_{k+1} \ell - 1 \\
\mathcal{O}_{n-\tau_{k+1} \ell}(\mathcal{O}_{\tau_{k+1} \ell}^k \cup V_{\tau_{k+1} \ell}^k), & n \geq \tau_{k+1} \ell,
\end{cases}
\]

with lowest point \( r_{n}^{k+1} = (a_n^{k+1}, b_n^{k+1}) \).

Let \( \rho_\ell > 0 \) be the probability of an \( \ell \)-triangle and \( K_n = \max \{ k : \tau_k \leq n \} \sim \text{Binom}(n, \rho_\ell) \).

**Lemma D.5.** For any infinite initial occupied set that is bounded below,
\[
\mathbb{E}[a_{n\ell}^{K_n} - a_{n\ell}^0] \geq n\ell \rho_\ell \quad \text{and} \quad \mathbb{E}[b_{n\ell}^{K_n} - b_{n\ell}^0] \leq -n\ell \rho_\ell.
\]

**Proof.** The two statements are equivalent since \( a_{n\ell}^{K_n} + b_{n\ell}^{K_n} = n\ell = a_{n\ell}^0 + b_{n\ell}^0 \). Since \( a_{n\ell}^k = a_{n\ell}^{K_n} \) for \( k > K_n \), and by the strong Markov property and Lemma D.4,
\[
\mathbb{E}[a_{n\ell}^{K_n} - a_{n\ell}^0] = \sum_{k=1}^{n} \mathbb{E}[a_{n\ell}^k - a_{n\ell}^{k-1}] = \sum_{k=1}^{n} \mathbb{E}[a_{n\ell}^k - a_{n\ell}^{k-1}, \tau_k \leq n] \\
= \sum_{k=1}^{n} \mathbb{E}[a_{n\ell} - \tau_k \ell (\mathcal{O}_{\tau_k \ell}^{k-1} \cup V_{\tau_k \ell}^{k-1}) - a_{n\ell} - \tau_k \ell (\mathcal{O}_{\tau_k \ell}^{k-1}), \tau_k \leq n] \\
\geq \ell \sum_{k=1}^{n} \mathbb{P}(\tau_k \leq n) \geq n \ell \rho_\ell.
\]

For the remainder of the proof the initial set for oriented percolation is \( S = \overline{Z}_- \).

**Lemma D.6.** With initial set \( S = \overline{Z}_- \),
\[
\lim_{n \to \infty} (n\ell)^{-1} \mathbb{E}[a_{n\ell}^{K_n} | 0 \to D_{n\ell}] \geq \beta_{p_1} + \rho_\ell.
\]

**Proof.** We prove the equivalent statement
\[
\lim_{n \to \infty} (n\ell)^{-1} \mathbb{E}[b_{n\ell}^{K_n} | 0 \to D_{n\ell}] \leq 1 - \beta_{p_1} - \rho_\ell.
\]

Let \( m < n \). Since paths never go in the \(-2\) direction, \( b_{n\ell}^{K_n} \geq 0 \), and so
\[
\mathbb{E}[b_{n\ell}^{K_n} 1\{0 \to D_{n\ell}\}] \leq \mathbb{E}[b_{n\ell}^{K_n} 1\{0 \to D_{m\ell}\}] = \mathbb{E}\left[\mathbb{E}\left\{b(\mathcal{O}_{n\ell}^{K_n}(\overline{Z}_-)) | \mathcal{H}_{m\ell}\right\} 1\{0 \to D_{m\ell}\}\right].
\]

To bound the conditional expectation use the Markov property to restart the evolution at \( \mathcal{O}_{m\ell}^{K_n}(\overline{Z}_-) \) and apply Lemma D.5. Note that \( \mathcal{O}_{m\ell}^{K_n}(\overline{Z}_-) \supseteq \mathcal{O}_{m\ell}^0(\overline{Z}_-) \). Hence if we replace \( \mathcal{O}_{m\ell}^{K_n}(\overline{Z}_-) \) with \( \mathcal{O}_{m\ell}^0(\overline{Z}_-) \) as the initial set of an oriented percolation process, the later occupied set shrinks, which implies that the lowest \(-2\)-coordinate increases.
\[
\mathbb{E}[b(\mathcal{O}_{n\ell}^{K_n}(\overline{Z}_-)) | \mathcal{H}_{m\ell}] = \mathbb{E}\left[b(\mathcal{O}_{(n-m)\ell}^{K_{n-m}}(\mathcal{O}_{m\ell}^{K_n}(\overline{Z}_-))) | \mathcal{H}_{m\ell}\right] \\
\leq \mathbb{E}\left[b(\mathcal{O}_{(n-m)\ell}^0(\mathcal{O}_{m\ell}^{K_n}(\overline{Z}_-))) | \mathcal{H}_{m\ell}\right] - (n-m)\ell \rho_\ell \\
\leq \mathbb{E}[b(\mathcal{O}_{(n-m)\ell}^0(\mathcal{O}_{m\ell}^0(\overline{Z}_-))) | \mathcal{H}_{m\ell}] - (n-m)\ell \rho_\ell.
\]
Substitute this back up in (D.12) to get the bound

\[ \mathbb{E}[b^{K_n}_{nt} \mathbb{1}\{0 \to D_{nt}\}] \]

(D.13)

\[ \leq \mathbb{E}[b(O^0_{nt}(\bar{Z}_-)) \mathbb{1}\{0 \to D_{nt}\}] - (n - m)\rho \mathbb{P}\{0 \to D_{mt}\}. \]

For oriented percolation with \( p_1 > \bar{p}_c \) we have the limits

\[ n^{-1}r(O^0_n(\bar{Z}_-)) = (\beta_{p_1}, 1 - \beta_{p_1}) \quad \text{in } L^1 \quad \text{and} \quad \mathbb{P}\{0 \to D_n\} \to \mathbb{P}\{0 \to \infty\} > 0. \]

The \( L^1 \) convergence of the lowest point is a consequence of the subadditive ergodic theorem and estimate (3) for oriented percolation on p. 1028 pf [10]. To get the conclusion (D.11) divide through (D.13) by \( n\ell \mathbb{P}\{0 \to D_{nt}\} \) and let first \( n \to \infty \) and then \( m \to \infty \). \( \square \)

The final piece of preparation derives a bound on last-passage times.

**Lemma D.7.** Let the initial set for the construction of \( r^k_n \) be \( S = \bar{Z}_- \). Then the oriented percolation event \( 0 \to D_n \) implies that \( G_{0, r^k_n} \geq n + k(c_0 - 1) \) for all \( n, k \geq 0 \).

**Proof.** Induction on \( k \). The case \( k = 0 \) is clear because \( 0 \to D_n \) implies that there is an oriented percolation path from \( 0 \) to \( r^0_n \), which is also an up-right path with \( \omega \)-weight \( n \).

Assume the claim is true for \( k \). For \( n \leq \tau_{k+1} - 1 \) we have \( r^{k+1}_n = r^k_n \) and the claim follows for \( k + 1 \) because \( c_0 - 1 < 0 \).

Suppose \( n \geq \tau_{k+1} \). If \( r^{k+1}_n \) and \( r^k_n \) do not coincide, \( r^{k+1}_n \) must lie below \( r^k_n \) on level \( D_n \). It follows that a path that links level \( D_{\tau_{k+1}} \) to \( r^{k+1}_n \) must originate from \( V^k_{\tau_{k+1}} \). (If not, such a path originates from \( O^k_{\tau_{k+1}} \) which forces \( r^{k+1}_n = r^k_n \).) Now construct a path from \( 0 \) to \( r^{k+1}_n \) as follows. The induction assumption gives a path from \( 0 \) to \( r^k_{\tau_{k+1} - \ell} \) with \( \omega \)-weight \( \geq \tau_{k+1} - \ell + k(c_0 - 1) \). The oriented percolation path from \( V^k_{\tau_{k+1}} \) to \( r^{k+1}_n \) gives \( \omega \)-weight \( n - \tau_{k+1} \). Connect the two paths by taking (D.9) from \( r^k_{\tau_{k+1} - \ell} \) to a point on \( V^k_{\tau_{k+1}} \) with \( \omega \)-weight \( \geq \ell + c_0 - 1 \). Adding up these pieces verifies that \( G_{0, r^{k+1}_n} \geq n + (k + 1)(c_0 - 1) \). \( \square \)

Differentiability of \( g_{\beta_1} \) at \( \eta \) is equivalent to the differentiability of \( \bar{g}(s) = g_{\beta_1}(s, 1 - s) \) at \( s = \beta_{p_1} \). The left derivative \( \bar{g}'(\beta_{p_1} - ) = 0 \) because \( \bar{g}(s) = 1 \) for \( 1 - \beta_{p_1} \leq s \leq \beta_{p_1} \). We show that the right derivative equals zero also. Since \( \bar{g} \) is concave and attains its maximum on \([1 - \beta_{p_1}, \beta_{p_1}]\), it must be strictly decreasing on \([\beta_{p_1}, 1] \).

\[
\bar{g}(\beta_{p_1} + \rho \ell) \geq \bar{g} \left( \lim_{n \to \infty} (n\ell)^{-1} \mathbb{E}[a^{K_n}_{nt} 0 \to D_{nt}] \right)
\]

\[
= \lim_{n \to \infty} \bar{g} \left( \mathbb{E}[ (n\ell)^{-1} a^{K_n}_{nt} 0 \to D_{nt}] \right)
\]

\[ \geq \lim_{n \to \infty} \mathbb{E} \left[ \bar{g}( (n\ell)^{-1} a^{K_n}_{nt} ) 0 \to D_{nt} \right] \]

\[ = \lim_{n \to \infty} \frac{1}{\mathbb{P}\{0 \to D_{nt}\}} \mathbb{E} \left[ \bar{g}( (n\ell)^{-1} a^{K_n}_{nt} ) \mathbb{1}\{0 \to D_{nt}\} \right] \]

\[ = \lim_{n \to \infty} \frac{1}{\mathbb{P}\{0 \to D_{nt}\}} \mathbb{E} \left[ (n\ell)^{-1} g_{\beta_1} \mathbb{1}\{0 \to D_{nt}\} \right] \]

\[ = \lim_{n \to \infty} \frac{1}{\mathbb{P}\{0 \to D_{nt}\}} \mathbb{E} \left[ (n\ell)^{-1} G_{0, r^{K_n}_{nt}} \mathbb{1}\{0 \to D_{nt}\} \right] \]
\[
\geq \lim_{n \to \infty} \frac{1}{\mathbb{P}\{0 \to D_n \ell\}} \mathbb{E}\left[ (1 + (c_0 - 1)(n\ell)^{-1}K_n) \mathbb{1}\{0 \to D_n \ell\}\right] = 1 + (c_0 - 1)\ell^{-1}\rho_\ell.
\]

In the calculation above first use (D.10) and the monotonicity and continuity of \(\tilde{g}\), then concavity to put \(\tilde{g}\) inside the conditional expectation. Homogeneity of \(g_{pp}\) is used. The second-last equality uses the \(L^1\) shape theorem (E.1) from Appendix E. The last inequality uses Lemma D.7. The last equality is from the \(L^1\) limit \(K_n/n \to \rho_\ell\).

From this and \(\tilde{g}(\beta_{p_1}) = 1\) we write
\[
0 \geq \frac{\tilde{g}(\beta_{p_1} + \rho_\ell) - \tilde{g}(\beta_{p_1})}{\rho_\ell} \geq \frac{c_0 - 1}{\ell}.
\]

Letting \(\ell \to \infty\) takes \(\rho_\ell \searrow 0\) and yields \(\tilde{g}'(\beta_{p_1}+) = 0\). Differentiability of \(g_{pp}\) at \(\bar{\eta}\) has been established. This concludes the proof of Theorem 2.10.

**Appendix E. Shape theorem**

**Theorem E.1.** Assume \(\omega_x\) are i.i.d. such that
\[
\int_{-\infty}^{0} \mathbb{P}\{\omega_0 \leq r\}^{1/2} dr < \infty \quad \text{and} \quad \int_{0}^{\infty} \mathbb{P}\{\omega_0 > r\}^{1/2} dr < \infty.
\]

Then
\[
(G.1) \quad \lim_{n \to \infty} n^{-1} \max_{x \in \mathbb{Z}^2 : |x|_1 = n} |G_{0,x} - g_{pp}(x)| = 0 \quad \mathbb{P}\text{-almost surely and in } L^1.
\]

**Proof.** The almost sure limit is in Theorem 5.1(i) of [25]. We prove the \(L^1\) limit.

Fix an integer \(k \geq 2\). Let \(\xi_\ell = (\ell/k, 1 - \ell/k), \ell \in \{0, \ldots, k\}\). Given an integer \(n \geq 2\), let \(m_n^- = \lceil n/(1 + 1/k)\rceil \in [1, n)\) and \(m_n^+ = \lceil n/(1 - 1/k)\rceil \geq n\).

Given \(x \geq 0\) with \(|x|_1 = n\) there exists \(\xi_\ell^-(x) \in \{0, \ldots, k\}\) such that \(m_n^- \xi_\ell^-(x) \leq x\). (This is because \(n - m_n^- \geq m_n^- / k\).) Also, there exists \(\xi_\ell^+(x) \in \{0, \ldots, k\}\) such that \(x \leq m_n^+ \xi_\ell^+(x)\). (This is because \(m_n^+ - n \geq m_n^+ / k\).)

A path from 0 to \(x\) can first go to \([m_n^- \xi_\ell^-(x)]\) and then take at most \(n - m_n^- + 1 \leq n/(k + 1) + 2 e_1\)-steps followed by at most \(n/(k + 1) + 2 e_2\)-steps. Hence,
\[
G_{0,x} \geq G_{0,[m_n^- \xi_\ell^-(x)]} - \sum_{i=0}^{n/(k+1)+1} |\omega_{[m_n^- \xi_\ell^-(x)]+ie_1}|
- \max_{0 \leq i \leq n/(k+1)+1} \sum_{j=0}^{n/(k+1)+1} |\omega_{[m_n^- \xi_\ell^-(x)]+ie_1+je_2}|.
\]
This gives
\[ G_{0,x} - g_{pp}(x) \geq g_{pp}(\lfloor m_n^+ \xi_{\ell(x)} \rfloor) - g_{pp}(x) + G_{0,\lfloor m_n^+ \xi_{\ell(x)} \rfloor} - g_{pp}(\lfloor m_n^+ \xi_{\ell(x)} \rfloor) \]
\[ - \sum_{i=0}^{n/(k+1)+1} |\omega_{\lfloor m_n^+ \xi_{\ell(x)} \rfloor + ie_1}| \]
\[ - \max_{0 \leq i \leq n/(k+1)+1} \sum_{j=0}^{n/(k+1)+1} |\omega_{\lfloor m_n^+ \xi_{\ell(x)} \rfloor + ie_1 + je_2}|. \]

Similarly,
\[ G_{0,x} - g_{pp}(x) \leq g_{pp}(\lfloor m_n^+ \xi_{\ell(x)} \rfloor) - g_{pp}(x) + G_{0,\lfloor m_n^+ \xi_{\ell(x)} \rfloor} - g_{pp}(\lfloor m_n^+ \xi_{\ell(x)} \rfloor) \]
\[ + \sum_{i=0}^{n/(k-1)+1} |\omega_{\lfloor m_n^+ \xi_{\ell(x)} \rfloor - ie_1}| \]
\[ + \max_{0 \leq i \leq n/(k-1)+1} \sum_{j=0}^{n/(k-1)+1} |\omega_{\lfloor m_n^+ \xi_{\ell(x)} \rfloor - ie_1 - je_2}|. \]

Note that \(|\lfloor m_n^+ \xi_{\ell(x)} \rfloor - x| \leq 1 + 2|n - m_n^+| \leq 1 + 2n/(1 \mp k)|. By continuity of \(g_{pp}\)
\[ \lim_{k \to \infty} \lim_{n \to \infty} n^{-1} \max_{x \in n\ell, n} \left| g_{pp}(\lfloor m_n^+ \xi_{\ell(x)} \rfloor) - g_{pp}(x) \right| = \lim_{k \to \infty} \lim_{n \to \infty} \max_{x \in n\ell, n} \left| \left( g_{pp}(n^{-1} \lfloor m_n^+ \xi_{\ell(x)} \rfloor) - g_{pp}(n^{-1} x) \right) \right| = 0. \]

By Proposition 2.1(i) of [25]
\[ \lim_{n \to \infty} n^{-1} \mathbb{E} \left[ \max_{0 \leq \ell \leq k} |G_{0,\lfloor m_n^+ \xi_{\ell} \rfloor} - g_{pp}(\lfloor m_n^+ \xi_{\ell} \rfloor)| \right] = 0. \]

Next we put some distance between the sums to make them independent:
\[ n^{-1} \mathbb{E} \left[ \max_{0 \leq \ell \leq k} \max_{0 \leq i \leq n/(k-1)+1} \sum_{j=0}^{n/(k-1)+1} |\omega_{\lfloor m_n^+ \xi_{\ell} \rfloor + ie_1 + je_2}| \right] \leq n^{-1} \mathbb{E} \left[ \max_{0 \leq \ell \leq k/2} \max_{0 \leq i \leq n/(k-1)+1} \sum_{j=0}^{n/(k-1)+1} |\omega_{\lfloor m_n^+ \xi_{2\ell} \rfloor + ie_1 + je_2}| \right] \]
\[ + n^{-1} \mathbb{E} \left[ \max_{0 \leq \ell \leq k/2} \max_{0 \leq i \leq n/(k-1)+1} \sum_{j=0}^{n/(k-1)+1} |\omega_{\lfloor m_n^+ \xi_{2\ell+1} \rfloor + ie_1 + je_2}| \right]. \]

The proof of the theorem is complete if we prove that the right-hand side vanishes as first \(n \to \infty\) and then \(k \to \infty\). We show the first limit, the second being similar. Centering the \(|\omega_x|\) terms does not change the limit. Hence, we will show that
\[ \lim_{k \to \infty} \lim_{n \to \infty} n^{-1} \mathbb{E} \left[ \max_{0 \leq \ell \leq k/2} \max_{0 \leq i \leq n/(k-1)+1} \sum_{j=0}^{n/(k-1)+1} (|\omega_{\lfloor m_n^+ \xi_{2\ell} \rfloor + ie_1 + je_2} - \mathbb{E}|\omega_0|) \right] = 0. \]
Abbreviate $\sigma^2 = \mathbb{E}[\omega_0^2] - (\mathbb{E}[\omega_0])^2$ and let

$$S_{n,k}^{i,\ell} = \sum_{j=0}^{n/(k-1)+1} (|\omega_{n\xi_2 + ie_1 + je_2}| - \mathbb{E}[\omega_0]).$$

Note that $\{S_{n,k}^{i,\ell}: 0 \leq i \leq n/(k-1)+1, 0 \leq \ell \leq k/2\}$ are i.i.d. By taking $n$ and $k$ large enough and restricting to $t \geq k^{-1/4}$ Chebyshev’s inequality gives $P\{S_{n,k}^{0,0} \geq nt\} \leq 2\sigma^2/( nk t^2 ) < 1/2$. Using $(k/2 + 1)(n/(k-1)+2) \leq 2n$ and $(1-\delta)^n \geq 1 - n\delta$ we bound the expectation:

$$E \left[ \max_{0 \leq \ell \leq k/2} \max_{0 \leq i \leq n/(k-1)+1} n^{-1} S_{n,k}^{i,\ell} \right] = \int_0^\infty \mathbb{P}\left\{ \max_{0 \leq \ell \leq k/2} \max_{0 \leq i \leq n/(k-1)+1} S_{n,k}^{i,\ell} \geq nt \right\} dt \leq k^{-1/4} + \int_{k^{-1/4}}^\infty \left[ 1 - \left( 1 - \frac{2\sigma^2}{nk t^2} \right)^{2n} \right] dt \leq k^{-1/4} + \int_{k^{-1/4}}^\infty \frac{4\sigma^2}{k t^2} dt = k^{-1/4} + 4\sigma^2 k^{-3/4} \to 0$$
as $n \to \infty$ and then $k \to \infty$. The argument can be repeated for the other sums.

**References**

[1] S. N. Armstrong and P. E. Souganidis. Stochastic homogenization of Hamilton-Jacobi and degenerate Bellman equations in unbounded environments. *J. Math. Pures Appl.* (9), 97(5):460–504, 2012.

[2] A. Auffinger and M. Damron. Differentiability at the edge of the percolation cone and related results in first-passage percolation. *Probab. Theory Related Fields*, 156(1-2):193–227, 2013.

[3] M. Balázs, E. Cator, and T. Seppäläinen. Cube root fluctuations for the corner growth model associated to the exclusion process. *Electron. J. Probab.*, 11:no. 42, 1094–1132 (electronic), 2006.

[4] D. L. Burkholder. Distribution function inequalities for martingales. *Ann. Probability*, 1:19–42, 1973.

[5] R. M. Burton and M. Keane. Density and uniqueness in percolation. *Comm. Math. Phys.*, 121(3):501–505, 1989.

[6] H. Cohn, N. Elkies, and J. Propp. Local statistics for random domino tilings of the Aztec diamond. *Duke Math. J.*, 85(1):117–166, 1996.

[7] I. Corwin. The Kardar-Parisi-Zhang equation and universality class. *Random Matrices Theory Appl.*, 1(1):1130001, 76, 2012.

[8] D. Coupier. Multiple geodesics with the same direction. *Electron. Commun. Probab.*, 16:517–527, 2011.

[9] M. Damron and J. Hanson. Busemann functions and infinite geodesics in two-dimensional first-passage percolation. *Comm. Math. Phys.*, 325(3):917–963, 2014.

[10] R. Durrett. Oriented percolation in two dimensions. *Ann. Probab.*, 12(4):999–1040, 1984.

[11] R. Durrett and T. M. Liggett. The shape of the limit set in Richardson’s growth model. *Ann. Probab.*, 9(2):186–193, 1981.
[12] P. A. Ferrari, J. B. Martin, and L. P. R. Pimentel. A phase transition for competition interfaces. *Ann. Appl. Probab.*, 19(1):281–317, 2009.

[13] P. A. Ferrari and L. P. R. Pimentel. Competition interfaces and second class particles. *Ann. Probab.*, 33(4):1235–1254, 2005.

[14] N. Georgiou, F. Rassoul-Agha, and T. Seppäläinen. Variational formulas and cocycle solutions for directed polymer and percolation models. 2013. Submitted (arXiv:1311.0316).

[15] N. Georgiou, F. Rassoul-Agha, T. Seppäläinen, and A. Yilmaz. Ratios of partition functions for the log-gamma polymer. *Ann. Probab.*, 2013. To appear (arXiv:1303.1229).

[16] C. D. Howard and C. M. Newman. Geodesics and spanning trees for Euclidean first-passage percolation. *Ann. Probab.*, 29(2):577–623, 2001.

[17] W. Jockusch, J. Propp, and P. Shor. Random domino tilings and the arctic circle theorem. 1998. arXiv:math/9801068.

[18] K. Johansson. Shape fluctuations and random matrices. *Comm. Math. Phys.*, 209(2):437–476, 2000.

[19] K. Johansson. Random matrices and determinantal processes. In *Mathematical statistical physics*, pages 1–55. Elsevier B. V., Amsterdam, 2006.

[20] A. Krishnan. Variational formula for the time-constant of first-passage percolation I: Homogenization. 2013. arXiv:1311.0316.

[21] C. Licea and C. M. Newman. Geodesics in two-dimensional first-passage percolation. *Ann. Probab.*, 24(1):399–410, 1996.

[22] R. M. Loynes. The stability of a queue with non-independent interarrival and service times. *Proc. Cambridge Philos. Soc.*, 58:497–520, 1962.

[23] J. Mairesse and B. Prabhakar. The existence of fixed points for the $\cdot/GI/1$ queue. *Ann. Probab.*, 31(4):2216–2236, 2003.

[24] R. Marchand. Strict inequalities for the time constant in first passage percolation. *Ann. Appl. Probab.*, 12(3):1001–1038, 2002.

[25] J. B. Martin. Limiting shape for directed percolation models. *Ann. Probab.*, 32(4):2908–2937, 2004.

[26] C. M. Newman. A surface view of first-passage percolation. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 1017–1023, Basel, 1995. Birkhäuser.

[27] B. Prabhakar. The attractiveness of the fixed points of a $\cdot/GI/1$ queue. *Ann. Probab.*, 31(4):2237–2269, 2003.

[28] F. Rassoul-Agha and T. Seppäläinen. Quenched point-to-point free energy for random walks in random potentials. *Probab. Theory Related Fields*, 158(3-4):711–750, 2014.

[29] F. Rassoul-Agha, T. Seppäläinen, and A. Yilmaz. Quenched free energy and large deviations for random walks in random potentials. *Comm. Pure Appl. Math.*, 66(2):202–244, 2013.

[30] H. Rost. Nonequilibrium behaviour of a many particle process: density profile and local equilibria. *Z. Wahrsch. Verw. Gebiete*, 58(1):41–53, 1981.

[31] T. Seppäläinen. Hydrodynamic scaling, convex duality and asymptotic shapes of growth models. *Markov Process. Related Fields*, 4(1):1–26, 1998.

[32] T. Seppäläinen. Scaling for a one-dimensional directed polymer with boundary conditions. *Ann. Probab.*, 40(1):19–73, 2012.
