Evolution equation for a model of surface relaxation in complex networks.

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Abstract

In this paper we derive analytically the evolution equation of the interface for a model of surface growth with relaxation to the minimum (SRM) in complex networks. We were inspired by the disagreement between the scaling results of the steady state of the fluctuations between the discrete SRM model and the Edward-Wilkinson process found in scale-free networks with degree distribution $P(k) \sim k^{-\lambda}$ for $\lambda < 3$ [Pastore y Piontti et al., Phys. Rev. E 76, 046117 (2007)]. Even though for Euclidean lattices the evolution equation is linear, we find that in complex heterogeneous networks non-linear terms appear due to the heterogeneity and the lack of symmetry of the network; they produce a logarithmic divergency of the saturation roughness with the system size as found by Pastore y Piontti et al. for $\lambda < 3$.

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During the last few years the study of complex networks has moved its focus from the study of their topology to the dynamic processes occurring on the underlying network. This is because many physical and dynamic processes use complex networks as substrates. Recently, many studies of dynamic processes on networks, such as epidemic spreading [2], traffic flow [3, 4], cascading failure [5], and synchronization [6, 7], have demonstrated the importance of the topology of the substrate network in the dynamic process. There exists much evidence that many real networks possess a scale-free (SF) degree distribution characterized by a power law tail given by \( P(k) \sim k^{-\lambda} \), where \( k_{\text{max}} \geq k \geq k_{\text{min}} \) is the degree of a node, \( k_{\text{max}} \) is the maximum degree, \( k_{\text{min}} \) is the minimum degree, and \( \lambda \) measures the broadness of the distribution [8]. Almost all the studies on networks regarded the links or nodes as identical. However, in real networks the links or nodes are not identical but have some “weight.” As examples the links between computers in the internet network have different capacities or bandwidths, resistor networks can have different values of resistance [4], and the airline network links connecting pairs of cities in direct flights have different numbers of passengers. Many theoretical studies have been carried out on weighted networks [4, 9]. Recently, several studies on real networks with weights on the links, such as the world-wide airport networks and the Escherichia coli metabolic networks [10], have shown that the weights are correlated with the network topology and this dramatically changes the transport through them [7, 11].

For instance, in synchronization problems, which are very important in brain networks [12], networks of coupled populations in the synchronization of epidemic outbreaks [13], and the dynamics and fluctuations of task completion landscapes in causally constrained queuing networks [14], the weights could have dramatic consequences for the synchronization [7]. Synchronization problems deal with optimization of the fluctuations of some scalar field \( h \). The system will be optimally synchronized when the fluctuations are minimized. The general treatment to analyze the fluctuations of these processes is to map them into a problem of non-equilibrium surface growth via an Edwards-Wilkinson (EW) process on the corresponding network [15]. Given a scalar field \( h \) on the nodes, that represents the interface height at each node, the fluctuations are characterized by the average roughness \( W(t) \) of the interface at time \( t \), given by \( W \equiv W(t) = \left\{1/N \sum_{i=1}^{N}(h_i - \langle h \rangle)^2\right\}^{1/2} \), where \( h_i \equiv h_i(t) \) is the height of node \( i \) at time \( t \), \( \langle h \rangle \) is the mean value on the network, \( N \) is the system size,
and $\{\cdot\}$ denotes an average over configurations. The EW process on networks is given by
\[
\frac{\partial h_i}{\partial t} = \sum_{j=1}^{N} C_{ij}(h_j - h_i) + \eta_i, \tag{1}
\]
where $C_{ij} = A_{ij} w_{ij}$ is a symmetric coupling strength, $\{A_{ij}\}$ is the adjacency matrix ($A_{ij} = 1$ if $i$ and $j$ are connected and zero otherwise), $w_{ij}$ is the weight on the edge connecting $i$ and $j$, and $\eta_i(t)$ is a Gaussian uncorrelated noise with zero mean and covariance $\{\eta_i\eta_j\} = 2D\delta_{ij}\delta(t - t')$. Here $D$ is the diffusion coefficient and is taken in general as a constant. For non-weighted networks $w_{ij} = \nu = \text{const}$ and thus Eq. (1) reduces to the unweighted EW equation on a graph given by $\frac{\partial h_i}{\partial t} = \nu \sum_{j=1}^{N} A_{ij}(h_j - h_i) + \eta_i$. Inspired by the results found for real networks where the weights are correlated with the topology, Korniss [7] studied synchronization for EW processes [see Eq. (1)] on SF networks where $w_{ij} = (k_i k_j)^{\beta}$ and $k_i$ and $k_j$ are the degrees of the nodes connected by a link. Using a mean-field approximation, he found that, subject to a fixed total edge cost, synchronization is optimal when $\beta = -1$, and at that point the performance is equivalent to that of the complete graph with the same edge cost. Pastore y Piontti et. al [1] used a discrete growth model with surface relaxation to the minimum (SRM) in SF networks, which mimics the fluctuation in the task-completion landscapes in certain distributed parallel schemes on computer networks, because it balances the load. They found that in SF networks with $\lambda < 3$ the saturation regime of $W \equiv W_s$ has a logarithmic divergence with $N$ that cannot be explained with the unweighted EW equation in graphs, even though in Euclidean lattices the SRM model belongs to the same universality class as the EW equation [16].

In order to understand this discrepancy, in this paper we derive analytically the evolution equation for the SRM in random unweighted networks [1] and find that the dynamics introduces “weights” on the links. With our evolution equation, which contains non-linear terms in the height differences, we recover the logarithmic divergency of $W_s$ with $N$ found in [1] for SF networks with $\lambda < 3$. Let us first briefly recall the SRM discrete model [16], studied for SF networks by Pastore y Piontti et. al [1]. In this model, at each time step a node $i$ is chosen with probability $1/N$. If we denote by $v_i$ the nearest-neighbor nodes of $i$ and $j \in v_i$, then (1) if $h_i \leq h_j \forall j \in v_i \Rightarrow h_i = h_i + 1$, else (2) if $h_j < h_n \forall n \neq j \in v_i \Rightarrow h_j = h_j + 1$. Next we derive the analytical evolution equation for the local height of the SRM model in random graphs. The procedure chosen here is based on a coarse-grained (CG) version of the discrete Langevin equations obtained from a Kramers-Moyal expansion.
of the master equation [17, 18, 19]. The discrete Langevin equation for the evolution of the height in any growth model is given by [18, 19]

\[
\frac{\partial h_i}{\partial t} = \frac{1}{\tau} G_i + \eta_i, \tag{2}
\]

where \( G_i \) represents the deterministic growth rules that cause evolution of the node \( i \), \( \tau = N \delta t \) is the mean time to grow a layer of the interface, and \( \eta_i \) is a Gaussian noise with zero mean and covariance given by [18, 19]

\[
\{\eta_i(t)\eta_j(t')\} = \frac{1}{\tau} G_i \delta_{ij} \delta(t - t'). \tag{3}
\]

We can write \( G_i \) more explicitly as

\[
G_i = \omega_i + \sum_{j=1}^{N} A_{ij} \omega_j, \tag{4}
\]

where \( \omega_i \) is the growth contribution by deposition on node \( i \) and \( \omega_j \) is the growth contribution to node \( i \) by relaxation from any of its \( j \) neighbors with

\[
\omega_i = \prod_{j \in v_i} \Theta(h_j - h_i),
\]

\[
\omega_j = [1 - \Theta(h_i - h_j)] \prod_{n \in v_j} [1 - \Theta(h_i - h_n)].
\]

Here, \( \Theta \) is the Heaviside function given by \( \Theta(x) = 1 \) if \( x \geq 0 \) and zero otherwise, with \( x = h_t - h_s \equiv \Delta h \). Without lost of generality, we take \( \tau = 1 \) and assume that the initial configuration of \( \{h_i\} \) is random.

In the CG version \( \Delta h \to 0 \); thus after expanding an analytical representation of \( \Theta(x) \) in Taylor series around \( x = 0 \) to second order in \( x \), we obtain

\[
G_i = c_0^{k_i} + C_i + c_1 c_0^{k_i - 1} k_i \left[ \sum_{j=1}^{N} A_{ij} h_j \right] - \frac{c_1}{(1 - c_0)} C_i \left[ \sum_{j=1}^{N} \frac{C_{ij} h_j}{C_i} - h_i \right] + \frac{c_1}{(1 - c_0)} T_i \left[ \sum_{j=1}^{N} \sum_{n=1, n \neq i}^{N} T_{ijn} h_n - h_i \right] - c_2 \sum_{j=1}^{N} A_{ij} \Omega(k_j - 1)(h_j - h_i)^2
\]

\[
- \left[ c_2 + c_2 \frac{c_0^2}{2(1 - c_0)} \right] \sum_{j=1}^{N} A_{ij} \Omega(k_j - 1) \left[ \sum_{n=1, n \neq i}^{N} A_{jn} (h_n - h_i)^2 \right] + c_0^{k_i - 1} \left[ c_2 - c_2 \frac{c_0^2}{2 c_0} \right] \sum_{j=1}^{N} A_{ij} (h_j - h_i)^2 + \frac{c_0^{k_i - 2} c_1^2}{2} \left[ \sum_{j=1}^{N} A_{ij} (h_j - h_i) \right]^2
\]
\begin{equation}
\begin{aligned}
&+ \frac{c_1^2}{(1-c_0)} \sum_{j=1}^{N} A_{ij}\Omega(k_j-1)(h_j-h_i) \left[ \sum_{n=1,n\neq i}^{N} A_{jn}(h_n-h_i) \right] \\
&+ \frac{c_2^2}{2(1-c_0)} \sum_{j=1}^{N} A_{ij}\Omega(k_j-1) \left[ \sum_{n=1,n\neq i}^{N} A_{jn}(h_n-h_i) \right]^2,
\end{aligned}
\end{equation}

where \( c_0, c_1, \) and \( c_2 \) are the first three coefficients of the expansion of \( \Theta(x) \), \( \Omega(k) = (1-c_0)^k \) is the weight on the link \( ij \) introduced by the dynamic process, and

\begin{align}
C_i &= \sum_{j=1}^{N} C_{ij} \\
T_i &= \sum_{j=1}^{N} \sum_{n=1,n\neq i}^{N} T_{ijn},
\end{align}

with \( C_{ij} = A_{ij}\Omega(k_j) \) and \( T_{ijn} = A_{ij}A_{jn}\Omega(k_j) \).

In our equation the non-linear terms in the difference of heights arise as a consequence of the lack of a geometrical direction and the heterogeneity of the underlying network. This result is very different from the one found in Euclidean lattices, where for the SRM model the non-linear terms disappear due to the symmetry of the process and the homogeneity of the lattice.

For the noise correlation [see Eq. (3)], up to zero order in \( \Delta h \) we obtain

\( \{ \eta_i(t)\eta_j(t') \} = 2D(k_i)\delta_{ij}\delta(t-t') \) with

\begin{equation}
D(k_i) = \frac{1}{2}(c_0^k + C_i).
\end{equation}

Notice that all the coefficients of the equation depend on the connectivity of node \( i \), i.e., on the network topology of the underlying network. This dependence on the topology can be thought of as a weight on the links of the unweighted underlying network that appears only due to the dynamics on the heterogeneous network.

Interestingly, the linear terms are different from the EW process as shown below. Keeping only the linear terms in Eq. (5), we numerically integrate our evolution equation in a SF network using the Euler method with the representation of the Heaviside function given by \( \Theta(x) = \{1 + \tanh[U(x+z)]\}/2 \), where \( U \) is the width and \( z = 1/2 \). With this representation \( c_0 = [1 + \tanh(U/2)]/2, c_1 = [1 - \tanh^2(U/2)] U/2, \) and \( c_2 = [-\tanh(U/2) + \tanh^3(U/2)] U^2/2 \). We build the network using the Molloy-Reed (MR) algorithm. In Fig. 1 we plot \( W^2 \) as a function of \( t \), obtained from the integration of Eq. (2) using only the linear terms of Eq. (5) with \( D(k_i) \) given by Eq. (7) for \( \lambda = 3.5 \) and 2.5 and different values of \( N \) with \( k_{\text{min}} = 2 \) in order to ensure that the network is fully connected. For the time step
evolution equation for the heights can be written as

\[ T \frac{\partial W}{\partial t} = \nu_i(k_i) \approx k_i \int_{k_{\min}}^{k_{\max}} P(k|k_i) \Omega(k) \, dk \]

where \( P(k|k_i) \) is the probability that a node with degree \( k_i \) is connected to another with degree \( k \). For uncorrelated networks, \( P(k|k_i) = kP(k)/\langle k \rangle \) does not depend on \( k_i \); then \( C_i(k_i) \approx I_1 k_i/\langle k \rangle \) with \( I_1 = \int_{k_{\min}}^{k_{\max}} P(k) \Omega(k) \, dk \). Making the same assumption for \( T_i \), we obtain \( T_i(k_i) \approx I_2 k_i/\langle k \rangle \) with \( I_2 = \int_{k_{\min}}^{k_{\max}} P(k) k (k-1) \Omega(k) \, dk \). Then the linearized evolution equation for the heights can be written as

\[
\frac{\partial h_i}{\partial t} = F_i(k_i) + \nu_i(k_i) \langle h \rangle - h_i + \eta_i ,
\]  

where \( F_i(k_i) = c_i^{k_i} + k_i I_1/\langle k \rangle \) represents a local driving force, \( \nu_i(k_i) = (c_1 c_i^{k_i-1} + b) k_i \) is a local superficial tension-like coefficient with \( b = c_1 (I_1 + I_2)/\langle k \rangle \), and \( \eta_i \) is a Gaussian noise with covariance \( D(k_i) = F_i(k_i)/2 \). This approximation shows the full topology of the network through \( P(k) \).

Taking the average over the network in Eq. \( \Box \), \( \partial \langle h \rangle /\partial t = 1/N \sum_{i=1}^{N} F_i = F \); then \( \langle h \rangle = F t \) is linear with \( t \). The solution of Eq. \( \Box \) \cite{17} is given by

\[
h_i(t) = \int_0^t e^{-\nu_i(t-s)} \left( F_i + \nu_i \langle h(s) \rangle + \eta_i(s) \right) \, ds
\]

\[
= \left( \frac{F_i - F}{\nu_i} \right) - \left( \frac{F_i - F}{\nu_i} \right) e^{-\nu_i t} \langle h \rangle + \int_0^t e^{-\nu_i(t-s)} \eta_i(s) \, ds .
\]  

Using Eq. \( \Box \), the two-point correlation function for \( t > \text{max} \{1/\nu_i\} \sim 1/k_{\min} \), is

\[
\{ (h_i(t_1) - \langle h \rangle)(h_j(t_2) - \langle h \rangle) \} = \left( \frac{F_i - F}{\nu_i} \right) \left( \frac{F_j - F}{\nu_j} \right)
\]

\[
+ \int_0^{t_2} \int_0^{t_1} e^{-\nu_i(t_1-s_1)} e^{-\nu_j(t_2-s_2)} \{ \eta_i(s_1) \eta_j(s_2) \} \, ds_1 ds_2 .
\]

Then \( W_s \) can be written as

\[
W_s^2 = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{F_i - F}{\nu_i} \right)^2 + \frac{1}{N} \sum_{i=1}^{N} \frac{2D(k_i)}{2\nu_i} .
\]  

\[
(10)
\]
For SF networks it can be shown that $I_1, I_2 \sim \text{const.} + k_{\text{max}} \exp(-k_{\text{max}} \text{const.})$, where $k_{\text{max}} \sim N^{1/(\lambda-1)}$ for MR networks; thus finite-size effects due to the cutoff on these quantities can be disregarded. Replacing in the last equation $D(k_i)$ by $F_i(k_i)/2$, we obtain

$$W_s^2 \sim \left[ 1 - 2\langle k \rangle \frac{1}{\langle k^2 \rangle} + \langle k \rangle \langle k^2 \rangle \right] + \text{const.} .$$

(11)

Notice that, if $F_i = 0$, $D = \text{const}$, and $\nu_i \propto k_i$, we recover the EW equation found in [7]. Using the corrections due to finite-size effects introduced by $k_{\text{max}}$ in Eq. (11),

$$W_s^2 \sim W_s^2(\infty) \left[ 1 + q_1 \frac{1}{N^{\lambda-2}} + q_2 \frac{1}{N} \right] ,$$

(12)

where $W_s^2(\infty) = W_s^2(N \to \infty)$ and $q_1$ and $q_2$ are constants. In the inset of Figs. (a) and (b) we plot $W_s^2$ as function on $N$ and the fitting obtained from Eq. (12). The agreement with the scaling form, Eq. (12), is excellent. Thus, the linear approximation can only explain the finite-size effects due to the MR construction but fails to predict the logarithmic divergency of $W_s$ with $N$ for $\lambda < 3$ found in Ref. [1]. Next we show that the non-linear terms are responsible for this behavior. We integrate our evolution equation for SF networks with the linear terms and only the first non-linear term [see Eq. (5)] due to the numerical instability produced when we try to incorporate all of them. Even with only one non-linear term, we recover the logarithmic divergency of $W_s$ with $N$ for $\lambda < 3$. The results of the integration are shown in Fig. 2, where we plot $W$ as a function of $t$ for (a) $\lambda = 3.5$ and (b) $\lambda = 2.5$ and different values of $N$. In the inset figures we plot $W_s$ as a function of $N$. We can see that, for $\lambda = 3.5$, $W_s$ increases but asymptotically goes to a constant and all the $N$ dependence is due to finite-size effects. However, for $\lambda = 2.5$ we found a logarithmic divergency of $W_s$ with $N$ [1], as shown in the inset of Fig. 2 (b), where we plot $W_s$ as a function of $N$ on a log-linear scale. The fit of $W_s$ with a logarithmic function for $\lambda = 2.5$ shows the agreement between our results and those obtained for the SRM model in SF networks for $\lambda < 3$. Discrepancies between behaviors in regular Euclidean lattices and Euclidean lattices after addition of random links were found before in [22].

In summary, we derived analytically the evolution equation for the SRM model and found, surprisingly, that even when the underlying network is unweighted the dynamics introduces weights on the links that depend on the topology. We also found that the linear terms can explain only finite-size effects due to the MR construction. The linear mean-field approximation shows clearly the effects of the topology on the dynamics and the corrections due
to finite-size effects. When non-linear terms on SF networks are considered, new numerical integration algorithms are needed in order to avoid numerical instabilities. This is still an open problem to be solved in the future. With all the linear terms and one non-linear term, we recovered the logarithmic divergency of $W_s$ with $N$ of the SRM model for $\lambda < 3$. Our analytic procedure can be also applied to any other growth model.

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FIG. 1: \( W^2 \) as a function of \( t \) from the integration of the evolution equation using the linear terms for \( N = 256 \ (\circ), \ 384 \ (\square), \ 512 \ (\diamond), \ 768 \ (\triangle), \) and \( 1024 \ (\triangledown). \) \( \lambda = (a) \ 3.5 \) and \( (b) \ 2.5. \) In the inset figure we plot \( W_s^2 \) vs \( N \) in symbols. The dashed lines represent the fitting with Eq. (12), obtained by considering the finite-size effects introduced by the MR construction. For all the integrations we used \( U = 0.5 \) and typically 10 000 realizations of networks.
FIG. 2: $W$ as a function of $t$ from the integration of the evolution equation using the linear terms and the first non-linear term for $N = 384 (\circ), 512 (\Box), 768 (\diamond), 1024 (\triangle)$, and 1536 (▽). $\lambda = (a)$ 3.5 and (b) 2.5. In the inset figure we plot $W_s$ vs $N$ in symbols. The dashed lines represent the fitting with Eq. (12) in (a) and $W_s \sim \ln N$ in (b).