Agents can differ in many ways, but differences in beliefs or information are perhaps the most interesting. In this paper, we adopt a general diverse beliefs framework that makes such models relatively easy to analyse. Our first result proves that a finite-horizon discrete-time dynamic stochastic general equilibrium (DSGE) where agents have diverse information is observationally equivalent to the one where the agents have diverse beliefs. This is important because diverse beliefs models are quite easy to study, whereas diverse information models are not. The solution to a continuous-time central-planner equilibrium problem is easy to characterize in the framework adopted, and we develop various properties of the solution, including expressions for equilibrium interest rates and stock price dynamics for an economy of constant relative risk aversion (CRRA) agents, and an expression for the volume of trade.

Keywords: diverse beliefs; private information; probability; equilibrium; beauty contest; mistaken beliefs

JEL Classifications: D52; D53

1. Introduction

Dynamic general equilibrium models provide us with perhaps our best hope of understanding how markets and prices evolve, but are often frustratingly difficult to solve. Representative agent models are an exception, but the limitations of the representative agent assumption are only too plain. Stepping up to models with many heterogeneous agents drastically reduces the available range of tractable examples, but is a necessary approach to realism. The simplest form of heterogeneity one could consider is the one in which agents have different preferences, and perhaps different endowments, but such models are not immediately suited to explaining effects arising from different information, or from different beliefs, since the causes are not being modelled.

In a recent survey, Kurz [27] discusses the literature on models with different information or beliefs, presents a compelling critique of models with private information and expounds his own theory of how to handle diverse beliefs. Models where agents receive private signals about random quantities of interest have been extensively studied, but are in general hard to work with; see, for example, Lucas [33], Townsend [39], Grossman and Stiglitz [15], Diamond and Verrecchia [12], Singleton [38], Brown and Jennings [6], Grundy and McNichol [16], Wang [42], He and Wang [19], Judd and Bernardo [24], Morris and Shin [34,35], Hellwig [20,21] and Angeletos and Pavan [1]. Problems such as the Grossman–Stiglitz paradox and the Milgrom–Stokey no-trade theorem necessitate the
introduction of exogenous noise into the models, but nonetheless the treatment of private information is only tractable under very restricted modelling assumptions. There are also problems at a conceptual level, as Kurz points out. Firstly, what is private information? In reality, the majority of agents’ information is common, such as macroeconomic indicators or the past performance of the stock, so we have to accept that a very small amount of private information might have a significant impact. Secondly, if private information does exist, what could we say about it? The private nature of the information would make it very difficult for us to verify any model that relied upon it.

For these reasons, we prefer to examine the class of models where all agents have the same information, but interpret that information differently; agents have different beliefs, but the same information. The main result of this paper is that in a discrete-time finite-horizon single-asset Lucas tree model\(^2\) any private information model is observationally equivalent to a diverse beliefs model. What this means is that if you were only to see the equilibrium price of the asset, and the portfolio/consumption choices of all the agents, you would not be able to tell whether the economy you are observing was a diverse beliefs equilibrium or a private information equilibrium. This is important because it means that private information equilibria offer no empirically testable predictions which could not be explained by diverse beliefs equilibria, and since the latter are much easier to work with, we lose nothing by restricting attention to them.

At first sight, this result seems unremarkable, even obvious. Given a model where different agents receive private signals, we could regard this as a model where all agents receive the same information but interpret it differently: every agent gets to see all the private signals, but believes that the signals received by the others are independent of everything else in the economy! However, this reasoning is fallacious. I may see your signals and believe that there is no information in them, but my actions will not ignore those signals because I know that you rely on them, and that your actions will respond to them. Thus my choices are not functions only of the commonly available information, but also will in general depend on the private signals, which is not possible in the private information equilibrium. The actual statement of the result and its proof are careful and subtle.

In our treatment, the agents’ different beliefs are modelled as different probability measures \(P^j\) defined over the same stochastic base \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})\). Compare this with the situation of private information, where all agents share the same probability \(P\), but work over different stochastic bases \((\Omega, \mathcal{F}', (\mathcal{F}'_j)_{t \geq 0})\). The literature on diverse beliefs is surveyed by Kurz, and includes the papers of Harrison and Kreps [18], Leland [29], Varian [40,41], Harris and Raviv [17], Detemple and Murthy [11], Kandel and Pearson [25], Cabrales and Hoshi [8], Basak [3], Basak [2], Basak and Croitoru [4], Calvet, Grandmont and Lemaire [9], Wu and Guo [44,45], Buraschi and Jiltsov [7], Fan [13], Scheinkman and Xiong [37], Jouini and Napp [23], Gallmeyer and Hollifield [14], Zapatero [46] and Li [30] Kogan et al. [26]. Among these, there are several which use the heterogeneity generated by diverse beliefs to create interesting effects in a portfolio-constrained setting; the papers [2,4,11,14,18,37], are examples. The point is that if (for example) short sales are constrained, the short-sale constraint would never bind in an equilibrium with agents with homogeneous beliefs, because all would agree on the risk premium for all the assets, and if one agent wanted to short a given asset, then so would all the others. In all these papers, with the exception of Refs [23,40], there is a model where agents have to filter a hidden process from observations thereof, using different prior information; the analysis involves quite lengthy and detailed calculations based on some explicit filtering problem formulation. Our second main result, Theorem 3.1, is a general result, stated already in similar form for one period [40] and for finite horizon in continuous time in Ref. [23],
including all these examples, which characterizes the equilibrium state-price density process in a complete market, and hence all equilibrium prices. It is a result which does not require much space to state, or to prove; how can it be so simple? Why are no lengthy calculations required? The answer of course is that we are treating the equilibrium problem at a much greater level of generality than the papers cited earlier; we obtain a general expression for the equilibrium, but for any particular example we would need to specialize and calculate in order to derive explicit solutions. The essential element is to treat different beliefs as different equivalent probability measures; these are characterized by their likelihood-ratio martingales, which are easy to work with. Making the likelihood-ratio martingales explicit in any particular Bayesian learning situation may be quite complicated, but this is not needed to explain the structure of the equilibrium solution.

If the result of Theorem 3.1 is already in Ref. [23], what is added here? The answer is that Jouini and Napp use the characterization of the diverse beliefs equilibrium as the starting point for the construction of what they refer to as a ‘consensus belief’, following on from the work of Calvet et al. [9], but attend less to the properties of the original diverse beliefs equilibrium itself. In contrast, we are interested in how the agents’ different beliefs get combined, and how this shapes the equilibrium riskless rate, the equilibrium stock price, portfolio allocations and volume of trade, which are discussed in Sections 3–5.

In our account, multiple agents take positions in a single asset which pays a continuous dividend stream, and is in unit net supply. There is a riskless asset, in zero net supply. The agents have different beliefs, represented as different probability measures, which we assume with no loss of generality are absolutely continuous with respect to some reference measure. Though we have diverse beliefs, we stress that we do not take a continuum of stochastically identical agents; agents’ diversities do not just get replaced by an average. The form of the agents’ beliefs is otherwise unrestricted:

- the agents could be stubborn bigots who assume they know the true distribution of the processes they observe and never change their views;
- the agents could be Bayesians updating their beliefs as time evolves;
- the evolution of the agents’ beliefs could be interlinked in various ways;

all such structure is irrelevant at the first pass.

Having derived the central-planner equilibrium in Section 3, we immediately show how this framework gives with no effort the result that all agents are ‘rationally overconfident’ – they all think that the particular consumption stream that they have chosen is better than those chosen by the others. We go on to characterize the evolution of the equilibrium interest rate and stock price.

We place in an appendix a very simple-minded model-fitting exercise; this is not because the study is not of intrinsic interest, but rather because it differs in style from the mainly theoretical body of the paper. We take the diverse beliefs model with log agents and try to fit it to various sample moments of the dataset of Shiller, as Kurz et al. [28] do. We find good agreement using a model with just three agents, and having reasonable parameter values. This supports the view that diverse beliefs may be able to resolve the equity premium puzzle, but the ability to match a few moments is not of course sufficient to justify a statistical model. Weizmann [43], Jobert et al. [22] and Li and Rogers [31] analyse the equity premium puzzle from the point of view of a representative Bayesian agent, and find reasonable values for parameter estimates, but do not present evidence that the fitted models do any better than just fitting constants to the data.

Section 6 concludes and maps out directions for future research.
2. Equivalence of private information and diverse beliefs models

The purpose of this section is to show that any private information equilibrium is (in a suitable sense) also a diverse belief equilibrium. It would be impossible to formulate a result broad enough to cover all imaginable instances of this principle, but what we shall do is to prove the result in the context of a discrete-time finite-horizon Lucas tree model with a single asset and multiple agents. To begin with, there is a lengthy, necessary but straightforward statement of notation and definitions. The main result is then expressed quite simply, but its rather lengthy proof is deferred to an appendix.

The time index set is $\mathbb{T} = \{0, 1, \ldots, T\}$ for some positive integer $T$. We suppose there is a single asset which delivers (random) output $\delta_t$ at time $t \in \mathbb{T}$, and there is an $\mathbb{R}^d$-valued process $(X_t)_{t \in \mathbb{T}}$ which we interpret as commonly available information; we suppose that $\delta$ is one of the components of $X$. There are $j$ agents, and in period $t$, agent $j$ receives private signal $z_j^t$; we write $Z_t = (z_1^t, \ldots, z_J^t)$ for the vector of all signals. Agent $j$ has von Neumann–Morgenstern preferences over consumption streams $(c_t)_{t \in \mathbb{T}}$ given by

$$E \left[ \sum_{t=0}^{T} U_j(t, c_t) \right].$$

The functions $U_j(t, \cdot) : (0, \infty) \to \mathbb{R}$ are assumed concave, strictly increasing, $C^2$ and to satisfy the Inada conditions. We write $\mathcal{G}_t = \sigma(X_s, Z_s : s \leq t)$ for the $\sigma$-field of all information at time $t$; all filtrations considered will be sub-filtrations of $\mathcal{G}$.

**Definition 2.1.** A private information equilibrium with initial allocation $y \in \mathbb{R}^J$ is a triple $(\bar{S}_t, \bar{\Theta}_t, \bar{C}_t)_{t \in \mathbb{T}}$ of $\mathcal{G}$-adapted processes, where $\bar{\Theta}_t = (\bar{\theta}_1^t, \ldots, \bar{\theta}_J^t)$, $\bar{C}_t = (\bar{c}_1^t, \ldots, \bar{c}_J^t)$ and $\bar{S}_t$ is real valued, with the following properties:

(i) for all $j$, $\bar{c}_j^t$ is adapted to the filtration $\bar{\mathcal{F}}_j^t = \sigma(X_u, \bar{S}_u, z_u^j : u \leq t)$ and $\bar{\theta}_j^t$ is previsible with respect to $\bar{\mathcal{F}}_j^t$;

(ii) for all $j$ and for all $t \in \mathbb{T}$, the wealth equation

$$\bar{\theta}_j^t(\bar{S}_t + \delta_t) = \bar{\theta}_{j+1}^t \bar{S}_t + \bar{c}_j^t$$

holds, with the convention $\bar{S}_\mathbf{T} = \bar{\theta}_{T+1}^t = 0$;

(iii) for all $t \in \mathbb{T}$, markets clear:

$$\sum_j \bar{\theta}_j^t = 1, \quad \sum_j \bar{c}_j^t = \delta_t;$$

(iv) $\bar{\theta}_0^j = y_j^t$ for all $j$;

(v) For all $j$, $(\bar{\theta}_j^t, \bar{c}_j^t)$ optimizes agent $j$’s objective (2.1) over all choices $(\theta, c)$ of portfolio and consumption which satisfy the wealth equation (2.2), and such that $c$ is $\bar{\mathcal{F}}_j^t$-adapted, $\theta$ is $\bar{\mathcal{F}}_j^t$-previsible and $\theta_0 = y_j^t$.

The notion of a private information equilibrium should be contrasted with the notion of diverse belief equilibrium, where the filtration is common, but the beliefs are not. So we shall suppose that there is some given filtration $(\mathcal{G}_t)_{t \in \mathbb{T}}$ and some probability measure $P^j$ on $(\Omega, \mathcal{G}_T)$ for each $j = 1, \ldots, J$. Agent $j$’s preferences over consumption streams $(c_t)_{t \in \mathbb{T}}$ are...
given by
\[ E^U \left[ \sum_{t=0}^{T} U_j(t, c^j) \right], \]  
(2.3)
where \( E^U \) denotes expectation with respect to \( P^U \).

**Definition 2.2.** A diverse belief equilibrium with initial allocation \( y \in \mathbb{R}^J \) is a triple \((\tilde{S}, \tilde{\Theta}, \tilde{C})_{\in \mathbb{T}}\) of \( \mathcal{G} \)-adapted processes, where \( \tilde{\Theta} = (\tilde{\theta}_j^1, \ldots, \tilde{\theta}_j^J) \), \( \tilde{C} = (\tilde{c}_1^1, \ldots, \tilde{c}_1^J) \) and \( \tilde{S} \) is real-valued, with the following properties.

(i) \( \tilde{\Theta} \) is \( \mathcal{G} \)-previsible;
(ii) for all \( j \) and all \( t \in \mathbb{T} \), the wealth equation
\[ \tilde{\theta}_j^t (\tilde{S}_t + \delta_t) = \tilde{\theta}_j^{t+1} \tilde{S}_t + \tilde{c}_j^t \]  
(2.4)
with the convention \( \tilde{S}_T = \tilde{\theta}_T^{T+1} = 0 \);
(iii) for all \( t \in \mathbb{T} \), markets clear:
\[ \sum_j \tilde{\theta}_j^t = 1, \quad \sum_j \tilde{c}_j^t = \delta_t; \]
(iv) \( \tilde{\theta}_j^0 = y^j \) for all \( j \);
(v) For all \( j \), \((\tilde{\theta}_j^t, \tilde{c}_j^t)\) optimizes agent \( j \)’s objective (2.3) over all choices \((\theta, c)\) of portfolio satisfying the wealth equation (2.4), and such that \( c \) is \( \mathcal{G} \)-adapted, \( \theta \) is \( \mathcal{G} \)-previsible and \( \theta_0 = y^j \).

Now that we have defined our terms, we are ready to state the main result.

**Theorem 2.3.** Suppose that \((\tilde{S}, \tilde{\Theta}, \tilde{C})\) is a private information equilibrium with initial allocation \( y \in \mathbb{R}^J \) for the discrete-time finite-horizon Lucas tree model introduced above. Then, it is possible to construct a filtered measurable space \((\tilde{\Omega}, (\tilde{\mathcal{G}}_t)_{t \in \mathbb{T}})\), carrying \( \tilde{\mathcal{G}} \)-adapted processes \( \tilde{X}, \tilde{S}, \tilde{\Theta}, \tilde{C} \) of dimensions \( d, 1, J \) and \( J \), respectively, and probability measures \( P^j \), \( j = 1, \ldots, J \), on \((\tilde{\Omega}, \tilde{\mathcal{G}}_T)\) such that \((\tilde{S}, \tilde{\Theta}, \tilde{C})_{\in \mathbb{T}}\) is a diverse belief equilibrium with initial allocation \( y \in \mathbb{R}^J \) and beliefs \((P^j)_{j=1}^J\) with the property that
\[ \mathcal{L}(\tilde{X}, \tilde{S}, \tilde{\Theta}, \tilde{C}) = \mathcal{L}(\tilde{X}, \tilde{S}, \tilde{\Theta}, \tilde{C}). \]

**Remark.** Notice that the theorem makes no statement about any analogue on the measurable space \((\tilde{\Omega}, \tilde{\mathcal{G}}_T)\) of the signal process \( Z \) on the measurable space \((\Omega, \mathcal{G}_T)\). There may or may not be one. Without compelling agents in the private information equilibrium to reveal these private signals, the most it would be possible to observe would be the common knowledge \( X \), the equilibrium price \( \tilde{S} \), the portfolio position \( \tilde{\Theta} \) and the consumption choices \( \tilde{C} \). What the theorem says is that the joint law of these processes (that is, the observables) is the joint law of the same observables in a diverse belief equilibrium. So from the point of view of testing model predictions, there are no statistical properties of a private information equilibrium which could not be explained by a diverse belief equilibrium. This justifies the claim that (for at least a finite-horizon Lucas tree model) we may ignore all (complicated) private information models and work only with (easier) diverse belief models; **private information equilibria are contained in diverse belief equilibria.** The proof of Theorem 2.3 is deferred to Appendix B.
3. Diverse beliefs equilibria

We are going to derive a general equilibrium for a dynamic economy with $J \geq 2$ agents, containing a single productive asset, whose output process $(\delta_t)_{t \geq 0}$ is observable to all agents. We shall suppose that time is continuous, and that $\delta$ is adapted to a filtration $(\mathcal{F}_t)_{t \geq 0}$ which is known to all agents. To cover various technical issues, we shall assume that the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}^0)$ satisfies the usual conditions; see Ref. [36] for definitions and further discussion. For simplicity, we shall assume also that $\mathcal{F}_0$ is trivial, so that all $\mathcal{F}_0$-measurable random variables are constant.

Though the $J$ agents all have the same information, they do not share the same beliefs about the distributions of the processes they observe. We suppose that agent $j$ thinks that the true probability is $\mathbb{P}^j$, a measure locally equivalent to $\mathbb{P}^0$, with density process

$$L_j^t = \frac{d\mathbb{P}^j}{d\mathbb{P}^0}_{|\mathcal{F}_t},$$

which is a positive martingale.

Agent $j$ has objective

$$U_j(c) = E^j \int_0^\infty U_j(t, c_t) dt,$$  \hspace{1cm} (3.2)

$$= E^0 \int_0^\infty \Lambda_j^t U_j(t, c_t) dt,$$  \hspace{1cm} (3.3)

defined over consumption streams $c$ which keep the wealth of agent $j$ positive. Here, $U_j$ is some strictly increasing time-dependent utility, such that $U_j(t, \cdot)$ satisfies the Inada conditions. Notice that even if all agents have the same $U_j$, their objective is calculated taking expectations under their different $\mathbb{P}^j$, and so differences in beliefs will result in different optimal behaviour. The two equivalent forms (3.2) and (3.3) of the objective make this difference explicit.

We introduce a central planner whose preferences over consumption streams $c$ are given by the von Neumann–Morgenstern objective\(^6\)

$$U_\nu(c) = \sup \left\{ \sum_j \nu_j \lambda_c U_j(c^j) : \sum_j c^j = c \right\},$$  \hspace{1cm} (3.4)

for positive parameters $\nu_j$ which will be adjusted to achieve the given initial distribution of asset among the agents. In view of the form (3.4) of the central planner’s objective, it is helpful to introduce the notation

$$U_\nu(t, x) = \sup \left\{ \sum_j \nu_j \lambda_c U_j(t, x_j) : \sum_j x_j = x \right\}.$$  \hspace{1cm} (3.5)

The central-planner equilibrium for this market is determined in the following result.

THEOREM 3.1. Suppose that integrability condition (3.10) holds. Then the central-planner equilibrium is determined by the state-price density process $\zeta$, which is related to the individual agents’ optimal consumption processes $c^j$ by

$$\nu_j \zeta_t = U_j^j(t, c^j_t) \Lambda_j^t,$$  \hspace{1cm} (3.6)

for some constants $\nu_j > 0$. The process $\zeta$ is determined from the market-clearing condition
and the $\nu_j$ by
\[
\sum_j I_j \left( t, \xi_j \nu_j \right) = \delta_t,
\]
where $I_j$ is the inverse marginal utility $(U^j_t)^{-1}$ of agent $j$. The stock price is given by
\[
S_t = E^0 \left[ \int_t^\infty \frac{\xi_u \delta_u}{\xi_t} du \bigg| \mathcal{F}_t \right].
\]

Proof. For the time being, we regard the constants $\nu_j$ in Definition (3.4) of the central planner’s objective as being given and fixed; we will explain later how they are to be chosen. The central planner aims to maximize
\[
P_j \nu_2 \nu_1 \left( U(j) \right)
\]
subject to the constraint that
\[
P_j c_t = \delta_t
\]
for all $t$. Absorbing this constraint with Lagrange multiplier process $\xi_t$, the unconstrained equivalent problem is
\[
\sup \{ \nu_1 \nu_2 \nu_1 \left( U(j) \right) + \xi_t \left( \delta_t - \sum_j c_t \right) \} dt.
\]
The first-order condition for optimality is therefore clearly (3.6). Making $c_t$ the subject of this equation yields
\[
I_j \left( t, \xi_t \nu_j \right) = c_t,
\]
and summing over $j$ to apply market clearing gives us (3.7):
\[
\sum_j I_j \left( t, \xi_t \nu_j \right) = \delta_t.
\]
This is an implicit equation for the unknown $\xi$ in terms of the known quantities $\delta$ and $\Lambda^j$, and involving the constants $\nu_j$. We shall assume the integrability condition:
\[
\forall \nu_j > 0, \quad E^0 \left[ \int_0^\infty \xi_t \delta_t dt \right] < \infty,
\]
where $\xi$ is determined from the $\nu_j$ by (3.7). The initial wealth of agent $j$ is then
\[
w^j_0 = \xi_0^{-1} E^0 \int_0^\infty \xi_t c^j_t ds,
\]
and the parameters $\nu_j$ are adjusted to ensure that these values are proportional to the initial allocations of the asset to the agents. The final assertion (3.8) of the theorem is the standard pricing identity, expressing the stock price as the net present value (NPV) of all its future dividends.

Remarks. (i) In the case where all agents have the same beliefs (thus $\Lambda^j = 1$ for all $j$), this reduces to the familiar expression for the state-price density as the marginal utility of optimal consumption.

(ii) Notice that the situation is completely general; there is no assumption about the nature of the stochastic processes, nor is there any assumption about the nature of the diverse beliefs. No such assumption is needed for (3.6).
(iii) Rational overconfidence. Kurz remarks that ‘a majority of people often expect to outperform the empirical frequency measured by the mean or median’. Paraphrasing, each of the agents believes that they will usually do better than the average. In our set-up, this result comes for free. If $\tilde{c}_t$ is any consumption stream and $c^j_t$ is agent $j$’s optimal consumption stream, then we have

$$E_j \int_0^\infty U_j(t, c^j_t)dt \geq E_j \int_0^\infty U_j(t, \tilde{c}_t)dt. \quad (3.12)$$

This follows simply from the fact that $c^j_t$ is agent $j$’s optimal consumption stream. In general, different agents will choose a different consumption stream, even if they have the same utility functions; even if they do have the same utilities, each agent believes that he will do better (on average) than all the other agents.

(iv) Relationship to Jouini–Napp [23]. Jouini and Napp derive Theorem 3.1 in the form appropriate to their slightly different assumptions, and then they develop the state-price density process in terms of a ‘consensus belief’. In more detail, they consider the problem

$$\sup E^0 \int_0^\infty M_t U(t, c_t)dt \text{ subject to } E^0 \int_0^\infty \zeta_t (\delta_t - c_t)dt = 0, \quad (3.13)$$

for some strictly positive continuous semimartingale $M$. This tries to mimic the problem faced by a budget-constrained central planner with beliefs given by the likelihood-ratio martingale $M$; but in general $M$ is not a martingale, and has to be expressed as the product of a positive martingale $\tilde{M}$ and a finite-variation process $B$. While such a representation is possible and quite amusing, we find it to be something of a diversion. The positive martingale $\tilde{M}$ is simply a mathematical construct, it does not represent the beliefs of anyone at all. What is being done is, in effect, some decomposition of the state-price density process, but our view is that the meaningful decomposition is into the product of a change-of-measure martingale times a discount factor. This allows us to identify the equilibrium riskless rate, and the market price of risk, both surely objects of interest. What we shall do therefore is to develop expressions for these processes as far as we can, directly in terms of the likelihood-ratio martingales $\Lambda^j$ and the equilibrium quantities.

4. Agents with linear risk tolerance

For the rest of the paper, we are going to assume that

$$U_j(t, x) = e^{-\rho_j t} u_j(x), \quad (4.1)$$

for some positive $\rho_j$, and $C^2$ functions $u_j$ which are strictly increasing, strictly concave and satisfy the Inada conditions. We shall additionally assume that $u_j$ have linear risk tolerance $-u'/u''$. Such utilities split naturally into the CRRA utilities $u(x) = x^{1-R}/(1-R)$ and the constant absolute risk aversion (CARA) utilities $u(x) = -\exp(-\gamma x)$ for $R > 0$, $R \neq 1$ and $\gamma > 0$; the special case of log utility is the CRRA utility for $R = 1$ and is indeed special as we shall see.

Theorem 3.1 is the starting point; it expresses the state-price density $\zeta$ in terms of the primitives of the problem as at (3.7), and all we have to do is substitute in the explicit form of $I_j$ and do some straightforward calculations. We shall take the CRRA and CARA cases separately.
4.1 CRRA utilities

We shall assume that all the agents have a common coefficient of relative risk aversion \( R > 0 \), so that
\[ I(x) = x^{-1/R} \]
This allows us to exploit scaling in the market-clearing condition (3.7), yielding the following expression for \( \zeta_t \):
\[ \zeta_t = \delta_t^{-R} \left\{ \sum_j \left( \frac{e^{-\rho_j^t \Lambda_j^t}}{v_j} \right)^{1/R} \right\}^R, \]
\[ \equiv \delta_t^{-R} \gamma_t^R, \]
where of course \( y \) is defined as
\[ y_t = \sum_j \left( \frac{e^{-\rho_j^t \Lambda_j^t}}{v_j} \right)^{1/R}. \]

In order to make further progress, we have to start assuming something about \( \Lambda_j^t \) and \( \delta \). Placing ourselves in a Brownian probability space leads to the following result.

**Theorem 4.1.** Suppose that the dividend process satisfies
\[ d \delta_t = \delta_t \sigma_t (dX_t + \alpha_t^* dt), \]
where \( X \) is an \( (\mathcal{F}_t) \)-Brownian motion under \( \mathbb{P}^0 \), and \( \sigma \) is some positive bounded previsible process with bounded inverse. Suppose that the agents’ likelihood-ratio martingales \( \Lambda_j^t \) obey
\[ d \Lambda_j^t = \Lambda_j^t \sigma_t (dX_t + \alpha_t^* dt), \]
where \( \alpha_j^t \) are previsible processes. Then the state-price density process evolves as
\[ d \zeta_t = \zeta_t (-r_t dt - \kappa_t dX_t), \]
where
\[ r_t = \tilde{\rho}_t + R \sigma_t (\alpha_t^* + \bar{\alpha}_t) - \frac{1}{2} \sigma_t^2 R (1 + R) + \frac{R - 1}{2R} v_t, \]
\[ \kappa_t = R \sigma_t - \bar{\alpha}_t. \]

The processes \( \bar{\alpha} \) and \( \tilde{\rho} \) are weighted averages of \( \alpha_j^t \) and \( \rho_j^t \):
\[ \bar{\alpha}_t = \sum_j q_j^t \alpha_j^t, \quad \tilde{\rho}_t = \sum_j q_j^t \rho_j^t, \]
where
\[ q_j^t = y_t^{-1} \left( \frac{e^{-\rho_j^t \Lambda_j^t}}{v_j} \right)^{1/R} \]
and
\[ v_t = \sum_j q_j^t (\alpha_j^t - \bar{\alpha}_t)^2. \]
Proof. The expressions (4.4) for \( y \), (4.6) for \( A \) and (4.5) for \( \delta \) allow us to perform an Itô expansion of \( \zeta \) as given by (4.3). We omit the routine but lengthy details of the calculations.

Notice that from the first-order condition for optimality (3.6) we may deduce that
\[
c_j^t = q_j^t \delta_t,
\]
which implies that
\[
c_j^t = q_j^t \delta_t = q_j^t d_t.
\]
Thus, the split of consumption at time \( t \) is according to the distribution \( (q_j^t) \).

In general, there is little that we can say about the stock price process \( S \) given by (3.8); however, when the utility is log we can go much further.

**Theorem 4.2.** In the case \( R = 1 \) of log utility, the state-price density process is
\[
\zeta_t = \delta_t^{-1} \sum_j \frac{e^{-\rho t} \Lambda_j^t}{v_j}.
\]
The positive constants \( v_j \) are fixed in terms of the initial wealths of the agents by
\[
w_j^0 = \frac{\Lambda_j^0}{v_j \rho_j},
\]
where we make the convention that \( \zeta_0 = 1 \). At all times, the optimal consumption rate processes are related to wealth by
\[
c_j^t = \rho_j w_j^t,
\]
and the stock price is
\[
S_t = \delta_t \frac{\sum_j e^{-\rho t} \Lambda_j^t / \rho_j v_j}{\sum_j e^{-\rho t} \Lambda_j^t / v_j}.
\]

Proof. When \( R = 1 \), relation (3.6) for the state-price density simplifies to
\[
\frac{e^{-\rho t} \Lambda_j^t}{c_j^t} = v_j \zeta_t.
\]
The wealth process of agent $j$ is thus

$$w_j^t = E^0 \left[ \int_t^\infty \frac{\zeta u^{-\mu}}{\xi_t} \, du \bigg| \mathcal{F}_t \right]$$

$$= E^0 \left[ \int_t^\infty \frac{e^{-\rho t \Lambda^j}}{\xi_t} \, du \bigg| \mathcal{F}_t \right]$$

$$= \xi_t^{-1} \left( \frac{e^{-\rho t \Lambda^j}}{\nu_j \rho_j} \right)$$

(4.19)

$$= \frac{c_j^t}{\rho_j}.$$  

(4.20)

The derivation exploits the fact that $\Lambda^j$ is a $\mathbb{F}^0$-martingale. Using (4.18), market clearing gives

$$\delta_t = \sum_j c_j^t = \xi_t^{-1} \sum_j e^{-\rho t \Lambda^j} \frac{\Lambda^j}{\nu_j},$$

and hence by rearrangement

$$\zeta_t = \delta_t^{-1} \sum_j e^{-\rho t \Lambda^j} \frac{\Lambda^j}{\nu_j},$$

which is (4.14). Since the stock is in unit net supply and the bank account in zero net supply, we can quickly identify the stock price using (4.19):

$$S_t = \sum_j w_j^t = \xi_t^{-1} \sum_j e^{-\rho t \Lambda^j} \frac{\Lambda^j}{\rho_j \nu_j}.$$  

(4.21)

Substituting from $\zeta$ from (4.14) leads to

$$S_t = \delta_t \sum_j e^{-\rho t \Lambda^j} \frac{\Lambda^j}{\rho_j \nu_j}. $$

Remarks. (i) Notice that in this case of log utilities, the price–dividend ratio takes a particularly simple form:

$$S_t = \delta_t \sum_j e^{-\rho t \Lambda^j} \frac{\Lambda^j}{\rho_j \nu_j}.$$  

(4.22)

which we shall have need of later when it comes to fitting various moments to the Shiller dataset in Appendix A. If all the agents have the same beliefs, this is just a deterministic function of time, but with heterogeneous beliefs this becomes a random process. Notice also that the price–dividend ratio depends only on the likelihood-ratio martingales, and not on the underlying dividend process, though this property is special to the log case.

(ii) We can also derive the dynamics of the stock price for the log case. After some routine calculations, we arrive at

$$dS_t = S_t \{(\kappa_t + a_t)(dX_t + \kappa_t dt) + r dt\} - \delta_t dt,$$  

(4.23)
where

\[ a_t = \frac{\sum \alpha_j^i e^{-\rho t} \Lambda_j^i / v_j \rho_j}{\sum e^{-\rho t} \Lambda_j^i / v_j \rho_j} \]

is an average of \( \alpha_j^i \) using weights different from \( q_j^i \). This allows us to identify the volatility \( \sigma^S \) of the equilibrium stock price, namely

\[ \sigma^S_t = \kappa_t + a_t = \sigma_t - \bar{\alpha}_t + a_t. \]  \hfill (4.24)

In general, this is different from the volatility \( \sigma_t \) of the dividend process, even if that volatility is constant\(^8\). Observe also that if \( \rho_j = \rho \) is the same for all \( j \), then \( a_t = \bar{\alpha}_t \), and hence \( \sigma^S_t = \sigma_t \). This checks out with what we would get from (4.22), which implies that \( \hat{\delta}_t = \rho \delta_t \) when all the impatience parameters are the same.

(iii) Notice also that if all agents have the same beliefs, \( \alpha_j^i = \alpha_i \) for all \( j \), and \( \alpha^* = 0 \), we see

\[ r_t = -\sigma_t^2 + \frac{\sum e^{-\rho_j t} / v_j}{\sum e^{-\rho_j t} / v_j} \sigma_t \bar{\alpha}_t; \]

thus for constant \( \alpha \) and \( \sigma \), the riskless rate is a smooth deterministic function of time. By contrast, if \( \alpha^i \) are constants but distinct, the agents have different beliefs, and the riskless rate is truly stochastic.

(iv) If all agents agreed, it is also immediate from (4.22) that the volatility of the stock is the same as the volatility of the dividend process; this illustrates again the general principle that heterogeneous beliefs will generate fluctuations which would be absent in a model where all agents agree.

### 4.2 CARA utilities

We shall assume that agent \( j \) has preferences given by

\[ U_j(t, c) = -\gamma_j^{-1} \exp(-\rho_j t - \gamma_j c), \]  \hfill (4.25)

and as befits a study with CARA utilities, we shall suppose that the evolution of the dividend is expressed in additive form:

\[ d\delta_t = \sigma_t (dX_t + \alpha_t^i dt), \]  \hfill (4.26)

while the evolution of \( \Lambda^i \) is exactly as at (4.6). This time we have the following result.

**THEOREM 4.3.** In the CARA case, the state-price density \( z_t \) is given by

\[ \log z_t = -\Gamma \delta_t - \bar{\rho} t + \sum_j p_j \log \Lambda_j^i, \]  \hfill (4.27)

where \( \gamma^{-1} = \sum_j \gamma_j^{-1}, p_j = \Gamma / \gamma_j \) and \( \bar{\rho} = \sum_j \rho_j \rho_j \). Then the evolution of the state-price density process is as

\[ d\xi_t = \xi_t (-\kappa_t dX_t - r_t dt), \]
where

\[ \kappa_t = \Gamma \sigma_t - \bar{\alpha}_t, \]  
\[ r_t = \bar{\rho} + \Gamma \sigma_t (\alpha^*_t + \bar{\alpha}_t) - \frac{1}{2} \Gamma^2 \sigma_t^2 - \frac{1}{2} v_t, \]  

where \( \bar{\alpha}_t = \sum p_j \alpha^j_t \) and \( v_t = \sum p_j (\alpha^j_t - \bar{\alpha}_t)^2 \).

**Proof.** The first-order conditions (3.6) this time become

\[ e^{-\rho t} \Lambda^j_t \exp \{- \gamma_j c^j_t \} = v_j \xi_j, \]  

Taking logs, dividing by \( \gamma_j \) and then summing on \( j \) gives (4.27) when we invoke market clearing. The evolution of all the terms on the right-hand side is known, so routine use of Itô's formula leads after some calculations to conclusions (4.28) and (4.29).

**Remark.** There are formal similarities between this result and the previous result of Theorem 4.1 for the market price of risk process \( \kappa \) and the riskless rate. One point of difference is that the averaging over the agents is with respect to a fixed distribution, not the randomly time-varying distribution \( q^j_t \) that features in the expressions for CRRA agents.

5. Volume of trade

In Section 4, we derived the stock price process, and the individual wealth processes, when all agents had log utility. This simple and explicit set-up allows us to go further, and derive the portfolios held by the individual agents. This is of interest because in the case where there is no diversity of belief, \( \Lambda^j_t = 1 \) for all \( j \), we see from (4.19) to (4.21) that agent \( j \)'s wealth process \( w^j_t \) is of the form \( w^j_t = g_j(t) S_t \) for some smooth deterministic function \( g_j \). This implies that each agent's holding of the stock varies smoothly and deterministically in time; in the extreme case where all \( \rho_j \) are the same, there is no trade at all, and the agents simply stick with their initial holdings of the stock consuming the dividend which it produces. What we shall show in this section is that even when all the agents have identical time preferences, that is, all \( \rho_j \) are the same, diversity of belief generates a considerable amount of trading, and (roughly speaking) the more diverse the beliefs are then, the more trading there is. This compares with Harris and Raviv [17], and De Long et al. [10], who find that (in the context of a private information equilibrium) agent heterogeneity generates trading. Of course, one has to define what is meant by volume of trading, since in the continuous-time setting the portfolio processes are typically of infinite variation. We, therefore, take as our definition of volume of trading the quadratic variation of the agents' portfolios.

**Theorem 5.1.** Suppose that the assumptions of Theorem 4.1 hold. With the notation of that theorem, the number \( \pi^j_t \) of units of the risky asset held by agent \( j \) at time \( t \) is

\[ \pi^j_t = \frac{w^j_t \left( \alpha^j_t + \kappa_t \right)}{\sum_i w^i_t \left( \alpha^i_t + \kappa_t \right)} = \frac{w^j_t \left( \alpha^j_t + \kappa_t \right)}{S_t (a_t + \kappa_t)}. \]  

Assuming further that \( \sigma \) is constant, all \( \alpha^j \) are constant, and that \( \rho_j = \rho \) for all \( j \), the
portfolio amounts $\pi^j$ have stochastic differential expansions

$$d\pi^j = \theta^j_t dX_t + d(\text{finite-variation terms}),$$

where

$$\theta^j_t = -\pi^j_t \tilde{\alpha} + q^j_t \left\{ \alpha^j(\sigma + \alpha^j - \tilde{\alpha}) - \nu_t \right\} / \sigma$$

$$= q^j_t \left[ \frac{\alpha^j - \tilde{\alpha}}{\sigma} - \frac{\nu_t}{\sigma} + \alpha^j - \tilde{\alpha} \right]. \quad (5.2)$$

**Proof.** Starting from expression (4.19) for the wealth, an Ito expansion gives

$$dw^j_t = w^j_t \left\{ -\rho_t dt + (\alpha^j_t + \kappa_t) dX_t + (r_t + \kappa_t^2 + \alpha^j_t \kappa_t) dt \right\}. \quad (5.3)$$

However, the wealth dynamics of agent $j$ can be expressed in terms of the portfolio process $\pi^j$ as

$$dw^j_t = \pi^j_t (dS_t + \delta_t dt) - c^j_t dt + (w^j_t - \pi^j_t S_t) r_t dt. \quad (5.4)$$

Comparing coefficients and using (4.23) leads to the identification

$$\pi^j_t = \frac{w^j_t (\alpha^j_t + \kappa_t)}{\sum_i w^j_i (\alpha^i_t + \kappa_i)} = \frac{w^j_t (\alpha^j_t + \kappa_t)}{S_t (\alpha_t + \kappa_t)},$$

as asserted at (5.1)$^9$.

For the second part of the theorem, we suppose that $\sigma$ is constant, all $\alpha_j$ are constant, and that $\rho_j = \rho$ for all $j$. Expression (5.1) for the proportion held by agent $j$ is now simply

$$\pi^j = \frac{w^j_t (\alpha^j + \sigma - \tilde{\alpha})}{\sigma S_t}$$

$$= \frac{\zeta_j w^j_t (\alpha^j + \sigma - \tilde{\alpha})}{\zeta_j \sigma S_t}$$

$$= \frac{\Lambda^j_t (\alpha^j + \sigma - \tilde{\alpha})}{\sigma \nu_t (\sum \lambda^j_t / \nu_t)}. \quad (5.5)$$

The defining expression for $\tilde{\alpha}_t$, simplified in this situation to

$$\tilde{\alpha}_t = \frac{\sum \alpha_i \lambda^j_t / \nu_t}{\sum \lambda^j_t / \nu_t}, \quad (5.6)$$

leads after some calculations to

$$d\tilde{\alpha}_t = -\tilde{\alpha}^2_t dX_t + \frac{\sum (\alpha^j_t)^2 \lambda^j_t / \nu_t}{\sum \lambda^j_t / \nu_t} dX_t + \text{finite-variation terms}$$

$$= \frac{\sum (\alpha^j_t - \tilde{\alpha}_t)^2 \lambda^j_t / \nu_t}{\sum \lambda^j_t / \nu_t} dX_t + \text{finite-variation terms}$$

$$= \nu_t dX_t + \text{finite-variation terms},$$
say. Suppose that $d\pi^j_t = \theta^j_t dX_t$ with finite-variation terms. Multiplying (5.5) throughout by $\sum \Lambda^j_i / \nu_i$ and expanding give

$$\{ \theta^j_t + \pi^j_t \tilde{\alpha}_i \} \left( \sum \Lambda^j_i / \nu_i \right) = \frac{\Lambda^j_i}{\nu_i} \{ \alpha^j_i (\sigma + \alpha^j_i - \tilde{\alpha}_i) - \nu_i \},$$

after some calculations. Rearranging and recalling (4.11), we obtain the expression

$$\theta^j_t = -\pi^j_t \tilde{\alpha}_i + q^j_i \{ \alpha^j_i (\sigma + \alpha^j_i - \tilde{\alpha}_i) - \nu_i \} / \sigma$$

$$= q^j_i \left[ \frac{(\alpha^j_i - \tilde{\alpha})^2}{\sigma} - \frac{\nu_i}{\sigma} + \frac{\alpha^j_i}{\sigma} - \tilde{\alpha}_i \right],$$

with some calculation, as asserted.

Remarks. Notice that the sum of $\theta^j_t$ is zero, as it must be, since the sum of $\pi^j_t$ is identically 1. The absolute value of $\theta^j_t$ can be interpreted as the volume of trade in the risky stock by agent $j$. Hence, the length of the vector $\theta$ can be interpreted as the total volume of trade. The representation (5.2) shows that in general terms the volume of trade gets bigger with greater diversity of beliefs, though it is hard to make this statement more precise.

6. Conclusions

This paper has shown how to deal with diverse beliefs of agents in a completely general manner; the key observation is that we should model agents’ beliefs as probability measures, whose likelihood-ratio martingales enter naturally into the optimality criterion, and thence into equilibrium prices.

The first consequence of this approach is that we are able to show that equilibria where agents have diverse private information are indistinguishable from equilibria where agents have common information, but different beliefs. This allows us to restrict attention to (analytically simpler) diverse beliefs equilibria. Expressions for the state-price density and for the equilibrium stock price arise simply from the assumptions, and are visibly analogous to (but extensions of) the corresponding expressions with no diversity of belief. An immediate first result is an explanation of the phenomenon of rational overconfidence.

By specializing to the case of CRRA agents, or CARA agents, the equilibrium can be computed quite explicitly and its properties are studied. We find quite simple and explicit expressions for the riskless rate and the market price of risk process, in terms of the fundamentals of the problem, namely, the dynamics of the dividend process and the beliefs of the agents, expressed as likelihood-ratio martingales. Diversity of belief generates an active market, and we are able to find an expression for the volatility of the agents’ holdings of the stock, which we interpret as a proxy for volume of trade. In general, greater diversity of belief generates a larger volume of trade.

There remain many interesting questions to be studied in this area. For example, can diverse beliefs create an economic role for money, by (say) imposing leverage constraints which more money will ease? This paper [32] is a first step down this road. Are there tractable examples where the agents have utilities different from log, and if so, what do the solutions look like? Perhaps most interestingly, can we propose some form for the beliefs of the agents which reflect influence of the market price of the stock on beliefs? If
this can be done, we have a feedback mechanism from prices to beliefs; of course, it can be
done clumsily, but the challenge is to do it in a credible fashion. These and other questions
are, in principle, amenable to a correctly formulated modelling of diverse beliefs, which
this paper has attempted to present.

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Notes
1. Email: A.A. Brown@statslab.cam.ac.uk
2. This specific set-up is not essential to the argument presented, but formulating a very general
statement would be clumsy and obscure the main ideas.
3. The restriction to a single asset is for notational convenience only; the entire analysis works
also for multi-assets situations.
4. If agent \( j \) has probability measure \( \mathbb{P}^j \), we could use the average of \( \mathbb{P}^j \) as a reference
measure.
5. This data set can be downloaded from http://www.econ.yale.edu/~shiller/data.htm.
6. In practice, the potential minor notational clash with (3.2) will be avoided by using roman
letters exclusively in the single-agent context, and greek letters exclusively in the central-
planner context.
7. Thus under the measure \( \mathbb{P}^j \) the process \( X \) becomes a Brownian motion with drift \( \alpha_j \) (by the
Cameron–Martin–Girsanov Theorem; see [36], IV.38 for an account).
8. Compare with Kurz et al. [28].
9. In the case where all the agents have the same belief, we have that

\[
\pi^j_t = \frac{e^{-\rho_j t}}{\sum_i e^{-\rho_i t}/\nu_i \rho_i},
\]

hence there is no volatility in the evolution of \( \pi^j_t \).
10. In the context of a finite-horizon Lucas tree model
11. And (in the log case) the stock price and the volatility of the stock price
12. An interesting extension of the log agent is Ref. [5], where agents are supposed to be CRRA
with an integer coefficient of relative risk aversion.
13. These empirical values are calculated by Kurz and are based on the Shiller data set. They are
based on monthly data from the S&P 500 between 1871 and 1998. See Refs [27,28] for further
details.

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Appendix A: Fitting annual return and consumption data

Kurz [27] uses his model of diverse beliefs to fit various sample moments of the Shiller data set, and we perform a similar study here.

We take a very simple version of the model, with just three agents who never change their beliefs, so we assume that $\alpha^j$ are constant. We also take $\sigma_f$ to be constant.

The quantities of interest are shown in Table A1; we list both the empirical value$^{13}$ and the values as produced by fitting our model.

The results shown were generated by choosing $\sigma = 0.517$, $\alpha^* = -0.01$, $\alpha^1 = 0.210$, $\alpha^2 = 0.727$, $\alpha^3 = -0.05$, $\rho_1 = 0.131$, $\rho_2 = 0.01$, $\rho_3 = 0.443$, $\nu_1 = 14.47$, $\nu_2 = 1.00$, $\nu_3 = 0.174$.

From the table above, we see that the diverse beliefs model with these parameter values gives quite a good fit to the sample moments considered by Kurz et al. Only the standard deviation of the price/dividend ratio is substantially off the empirical value, a sample moment which we note was not fitted very closely by Kurz either, probably because the volatility of recorded annual consumption is, in general, too small to explain the observed volatility in stock returns. Nevertheless, the model seems to be doing a reasonable job explaining these figures given the very specific assumptions made.

Table A1. Simulation results.

|                                | Fitted | Empirical |
|--------------------------------|--------|-----------|
| Mean price/dividend ratio      | 26.06  | 25        |
| Standard deviation of price/dividend ratio | 3.84  | 7.1       |
| Mean return on equity          | 0.077  | 0.07      |
| Standard deviation of return on equity | 0.134 | 0.18      |
| Mean riskless rate             | 0.018  | 0.018     |
| Standard deviation of riskless rate | 0.061 | 0.057     |
| Equity premium                 | 0.059  | 0.06      |
| Sharpe ratio                   | 0.326  | 0.33      |
Appendix B: Proof of Theorem 2.3

There are several steps to the proof.

(i) If we write \( \tilde{\lambda}^j_t = U^j(t, \tilde{c}^j) \), the first thing to prove is that for \( 0 \leq t < T \),

\[
\tilde{\lambda}^j_t S_t = E\left[ \tilde{\lambda}^j_{t+1}(\tilde{S}_{t+1} + \delta_{t+1})|\mathcal{F}^j_t \right].
\]  

(B.1)

Consider a (small) perturbation \( \tilde{\theta}^j \mapsto \tilde{\theta}^j + \eta \) of the portfolio process, with corresponding change \( \tilde{c}^j \mapsto c = \tilde{c}^j + \epsilon \) to the consumption process, where (see (2.2))

\[
\epsilon_t = \eta_t(\tilde{S}_t + \delta_t) - \eta_{t+1}\tilde{S}_t.
\]

Since \( U \) satisfies the Inada condition, \( \tilde{c}^j \) must be strictly positive and so for small enough \( \eta \) the process \( c \) will be strictly positive. To leading order the change in agent \( j \)'s objective is

\[
E\left[ \sum_{t=0}^T U^j_t(t, \tilde{c}^j_t) \epsilon_t \right] = E\left[ \sum_{t=0}^T \tilde{\lambda}^j_t \{ \eta_t(\tilde{S}_t + \delta_t) - \eta_{t+1}\tilde{S}_t \} \right]
\]

\[
= E\left[ \sum_{t=1}^T \eta_t(\tilde{\lambda}^j_t(\tilde{S}_t + \delta_t) - \tilde{\lambda}^j_{t-1}\tilde{S}_{t-1}) \right]
\]

\[
= E\left[ \sum_{t=1}^T \eta_t E(\tilde{\lambda}^j_t(\tilde{S}_t + \delta_t) - \tilde{\lambda}^j_{t-1}\tilde{S}_{t-1}|\mathcal{F}^j_{t-1}) \right],
\]  

(B.2)

using the facts that \( \tilde{S}_T = 0 \), \( \eta_0 = 0 \) (since \( \theta_0^j = y^j \) is fixed), and that the portfolio perturbation must be \( \mathcal{F}^j \)-previsible. This leading-order change in objective must be 0, since \((\tilde{\theta}^j, \tilde{c}^j)\) was optimal; since \( \eta \) is arbitrary, inspection of (B.2) gives (B.1).

(ii) Since \( \tilde{c}^j \) is \( \mathcal{F}^j \)-adapted and \( \tilde{\theta}^j \) is \( \mathcal{F}^j \)-previsible, we have

\[
\mathcal{F}^j_t = \sigma(X_u, \tilde{S}_u, \tilde{c}^j_u : u \leq t) = \sigma(X_u, \tilde{S}_u, \tilde{\theta}^j_{u+1}, \tilde{c}^j_{u+1}, \tilde{c}^j_u : u \leq t)
\]

\[
\supset \sigma(X_u, \tilde{S}_u, \tilde{\theta}^j_{u+1}, \tilde{c}^j : u \leq t) \equiv \mathcal{F}^j_t,
\]

say. Since \( \tilde{\lambda}^j_t \) and \( \tilde{S}_t \) are measurable with respect to \( \mathcal{F}^j_t \), we can refine (B.1) to

\[
\tilde{\lambda}^j_t S_t = E\left[ \tilde{\lambda}^j_{t+1}(\tilde{S}_{t+1} + \delta_{t+1})|\mathcal{F}^j_t \right].
\]  

(B.3)

(iii) We now take a regular conditional distribution \( \kappa^j \) for \( \tilde{S} \) given \((X, \tilde{\theta}^j, \tilde{c}^j)\) – see II.89 in Ref. [36]. To build the sample space \( \tilde{\Omega} \) on which the diverse belief equilibrium will be constructed, the first step is to take \( \Omega_0 \) to be the path space of \((X, \tilde{\Theta}, \tilde{C})\), which is isomorphic to \( \mathbb{R}^{d+j+1} \). This gets its Borel \( \sigma \)-field \( \mathcal{B} \), and canonical filtration; we endow it with a reference probability measure \( \tilde{P}^* \) which is the law of \((X, \tilde{\Theta}, \tilde{C})\). Next, we expand the sample space to \( \tilde{\Omega} \equiv \Omega_0 \times \mathbb{R}^{T+1} \), and we write \((\tilde{X}, \tilde{\Theta}, \tilde{C})\) for the processes defined on \( \tilde{\Omega} \) by

\[
\tilde{X}(\tilde{\omega}) = X(\omega), \quad \tilde{\Theta}(\tilde{\omega}) = \Theta(\omega), \quad \tilde{C}(\tilde{\omega}) = C(\omega),
\]

where \( \tilde{\omega} = (\omega, s) \), \( \omega \in \Omega_0 \) and \( s = (s_0, \ldots, s_T) \in \mathbb{R}^{T+1} \). We define a process \( \tilde{S} \) by

\[
\tilde{S}_t(\tilde{\omega}) = s_t
\]

when \( \tilde{\omega} = (\omega, (s_0, \ldots, s_T)) \). We write \( \tilde{\mathcal{G}}_t = \sigma(\tilde{X}_u, \tilde{\Theta}_{u+1}, \tilde{C}_u, \tilde{S}_u : u \leq t) \) for the filtration generated by these processes.
Now, we specify the probabilities \( P^j \) giving the diverse beliefs of the agents. Firstly, select \((\tilde{X}, \tilde{\theta}^j, \tilde{\xi}^j)\) according to the law \( P^* \) (equivalently, \((\tilde{X}, \tilde{\theta}^j, \tilde{\xi}^j)\) has the same law as \((X, \theta^j, \xi^j)\)). Then conditional on \((\tilde{X}, \tilde{\theta}^j, \tilde{\xi}^j)\) let the law of \( \tilde{S} \) be \( \kappa^j((\tilde{X}, \tilde{\theta}^j, \tilde{\xi}^j); \cdot) \), and let the random variables \( \tilde{\theta}^i, \tilde{\xi}^i, i \neq j \), be independent subject to the constraints \( \sum \tilde{\xi}^i = 1 \), \( \sum \tilde{\xi}^i = \delta_i \). [For example, we could take exponential variables \( V^i, \tilde{V}^j \), and define \( \tilde{\theta}^j_t = (1 - \tilde{\theta}^j) V^j/\sum_{i \neq j} V^i \). This construction has achieved the following properties:

(a) the \( P^j \)-distribution of \((\tilde{X}, \tilde{\theta}^j, \tilde{\xi}^j, \tilde{S})\) is the same as the \( P \)-distribution of \((X, \theta^j, \xi^j, S)\);
(b) \( \tilde{\theta}^j \) is \( \tilde{G} \)-previsible;
(c) \( \tilde{\theta}^j_t (\tilde{S}_t + \delta_t) = \tilde{\theta}^j_{t+1} \tilde{S}_t + \tilde{\xi}^j_t \), with \( P^j \)-probability 1, since this is a statement about the joint law of \((\tilde{X}, \tilde{\theta}^j, \tilde{\xi}^j, \tilde{S})\) and it must, therefore, have the same probability as the corresponding statement about \((X, \theta^j, \xi^j, S)\);
(d) Similarly,

\[
\sum_{j} \tilde{\theta}^j = 1, \quad \sum_{j} \tilde{\xi}^j = \delta_i,
\]

with \( P^* \)-probability 1, and with \( P^k \)-probability 1 for each \( k \).

(iv) Now, we define the filtration \( \tilde{\mathcal{F}}^j_t = \sigma(\tilde{X}_u, \tilde{\theta}^j_u, \tilde{\xi}^j_u; u \leq t) \), and the processes \( \tilde{\lambda}^j_t = U^j_0(t, \tilde{\xi}^j_t) \). Hence, we observe the analogue

\[
\tilde{\lambda}^j_t \tilde{S}_t = E^j \left[ \tilde{\lambda}^j_{t+1} (\tilde{S}_{t+1} + \delta_{t+1}) \mid \tilde{\mathcal{F}}^j_t \right], \quad (B.4)
\]

of \((B.3)\) must hold, because the conditional expedient is determined by the joint law of the conditional and conditioning variables, which (in view of (a)) is the same as the joint law of the corresponding variable in the private information equilibrium for which \((B.3)\) holds.

(v) The next step is to argue that \((B.3)\) holds when we condition on the larger \( \sigma \)-field \( \tilde{G}_t \):

\[
\tilde{\lambda}^j_t \tilde{S}_t = E^j \left[ \tilde{\lambda}^j_{t+1} (\tilde{S}_{t+1} + \delta_{t+1}) \mid \tilde{G}_t \right], \quad (B.5)
\]

But \( \tilde{G}_t = \tilde{\mathcal{F}}^j_t \lor \mathcal{A}^j_t \), where \( \mathcal{A}^j_t = \sigma(\tilde{\xi}^j_u, \tilde{\theta}^j_{u+1}; u \leq t); \quad i \neq j) \) is independent of \( \tilde{\mathcal{F}}^j_t \) so we may apply Proposition 1 to deduce this result.

(vi) The final step is to verify the optimality property (v) in the definition of a diverse belief equilibrium. Suppose for this that \((\theta_t, c_t)\) is any possible investment–consumption pair for agent \( j \) (so \( \theta_0 = y^j \), \( \theta \) is \( \tilde{G} \)-previsible, \( c \) is \( \tilde{G} \)-adapted, \( \theta_t (\tilde{S}_t + \delta_t) = \theta_{t+1} \tilde{S}_t + c_t \), for all \( t \)) and consider the objective

\[
E^j \sum_{t=0}^{T} U^j_t (t, c_t) \leq E^j \sum_{t=0}^{T} \left[ U^j_t (t, \tilde{\xi}^j_t) + \tilde{\lambda}^j_t (c_t - \tilde{\xi}^j_t) \right]
\]

\[
= E^j \sum_{t=0}^{T} \left[ U^j_t (t, \tilde{\xi}^j_t) + \tilde{\lambda}^j_t \left\{ (\theta_t - \tilde{\theta}^j) (\tilde{S}_t + \delta_t) - (\theta_{t+1} - \tilde{\theta}^j_{t+1}) \tilde{S}_t \right\} \right]
\]

\[
= E^j \sum_{t=0}^{T} U^j_t (t, \tilde{\xi}^j_t) + E^j \sum_{t=1}^{T} (\theta_t - \tilde{\theta}^j) \left\{ \tilde{\lambda}^j_t (\tilde{S}_t + \delta_t) - \tilde{\lambda}^j_{t-1} \tilde{S}_t \right\}
\]

\[
= E^j \sum_{t=0}^{T} U^j_t (t, \tilde{\xi}^j_t),
\]
using (respectively) concavity of \( U_j \), the wealth equation, the fact that \( \theta_0 = \tilde{\theta}_0^j = \gamma^j \) and \( \theta_{T+1} = 0 = \tilde{\theta}_{T+1}^j \), and (B.4) together with \( \tilde{G} \)-previsibility of \( \theta, \tilde{\theta}^j \).

We give here a simple and intuitive result that was needed in the proof of Theorem 2.3.

**Proposition 1.** If \( X \) is an integrable random variable, if \( G \) and \( A \) are two sub-\( \sigma \)-fields of \( \mathcal{F} \) such that \( A \) is independent of \( X \) and \( G \), then

\[
E[X|G] = E[X|G \vee A] \quad \text{a.s.} \tag{B.6}
\]

**Proof of Proposition 1.** Consider the collection

\[
\mathcal{C} = \left\{ F \in \mathcal{F} : \int_F E[X|G]dP = \int_F X dP \right\}.
\]

This collection is a \( d \)-system (see [36] Chapter II.1 for definitions and basic results). From the definition of conditional expectation, \( \mathcal{G} \subseteq \mathcal{C} \). Now take any \( G \in \mathcal{G}, A \in \mathcal{A} \) and calculate

\[
\int_{A \cap G} E[X|G]dP = \int_{A \cap G} I_A E[X|G]dP = P(A)E[X : G] \quad (A \text{ is independent of } G)
\]

\[
= \int_{A \cap G} X dP \quad (A \text{ is independent of } G, X).
\]

Thus, \( \mathcal{C} \) contains the \( \pi \)-system \( \mathcal{I} \) consisting of all intersections of the form \( A \cap G, A \in \mathcal{A}, G \in \mathcal{G} \), and by Lemma II.1.8 of Ref. [36], the \( d \)-system \( d(\mathcal{I}) \) generated by \( \mathcal{I} \) equals the \( \sigma \)-field generated by \( \mathcal{I} \), which is \( \mathcal{G} \vee \mathcal{A} \). But \( \mathcal{C} \) is a \( d \)-system, and so contains \( \mathcal{G} \vee \mathcal{A} \). This establishes the result.

**Remark.** Intuitively, it seems plausible that we should not need \( A \) to be independent of \( G \) for this result to hold; after all, what we are adding to the \( \sigma \)-field is independent of the random variable. But this is not true. Take the example of two independent \( B(1, \frac{1}{2}) \) random variables \( X \) and \( Y \), and let \( A = \sigma(Y), \mathcal{G} = \sigma(Z) \) where \( Z = (X + Y) \mod (2) \). Thus \( Z \) is 1 if exactly one of \( X, Y \) is 1, zero otherwise. It is not hard to see that \( \mathcal{G} \) is independent of \( X \), and yet \( A \vee \mathcal{G} = \mathcal{F} \), so the equality (B.6) fails.