Renormalization of Higher Derivative Quantum Gravity Coupled to a Scalar with Shift Symmetry

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Abstract

It has been suggested that higher-derivative gravity theories coupled to a scalar field with shift symmetry may be an important candidate for a quantum gravity. We show that this class of gravity theories are renormalizable in $D = 3$ and 4 dimensions.

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1 Introduction

It is one of the long standing problems in theoretical physics to construct quantum theory of gravity. It has been known for some time that gravity is renormalizable in four dimensions if one includes higher derivative terms [1]. However, the unitarity of the theory, which is one of the most important properties of any physical theory, is not preserved. So the theory has not been taken very seriously. The compatibility of unitarity and renormalizability has also been studied in [2] for three-dimensional theory which could be unitary for judicious choice of parameters. It turned out that the unitarity and the renormalizability are incompatible.

Recently a very interesting suggestion has been made that the time may be an emergent notion [3]. The idea starts with four-derivative theory of gravity coupled to a scalar field with shift symmetry with Euclidean signature. The quadratic terms of the scalar field also have four derivatives, so that its scaling dimension is zero. It was assumed that the theory is renormalizable, but the low-energy effective theory is described by the Einstein theory together with the four-derivative scalar theory. It was then shown that this low-energy effective theory is equivalently described by a Lorentzian action. In this way it was suggested that the low-energy theory becomes Lorentzian but the theory at the short distance is described by a Riemannian (locally Euclidean) theory without the notion of time. If true, this may be a resolution of the ghost problem in the above renormalizable theory of gravity.

It is necessary to consider such higher derivative terms in gravity since the string theory, the possible candidate of quantum gravity, predicts that such terms do exist. Given this fact, we should also take higher derivatives on scalar fields into account if we further consider scalar fields, and naturally we are led to the class of theories we consider.

There are several points that have to be confirmed for the above scenario to work. The obvious and first problem is to explicitly check whether such a theory is really renormalizable or not. Though the above quadratic gravity is shown to be renormalizable even in the presence of a minimal scalar field, i.e. with only the usual kinetic term with second derivative for the scalar [1], it has to be checked if the theory remains renormalizable with additional higher derivative terms. One may think that it is obvious when higher derivative kinetic terms are introduced because they improve the convergence of Feynmann diagrams. However it is not so because such terms also introduce higher derivative interactions in the presence of gravity.

More importantly, if the renormalizability is proved, it has to be seen whether the theory with these higher derivative terms reduces to desirable low-energy effective theory with the above property. For this purpose, one has to study the renormalization group and examine the UV and IR fixed points. For such discussions for higher derivative gravity, see [4, 5, 6, 7, 8, 9, 10, 11]. This would be next step.

In this paper we take the first step in this direction and examine the renormalization property of the theories. There is not much difference between the theories defined for Lorentzian and Euclidean signature in our perturbative approach. We can simply derive propagators and discuss power counting and so on, as usual. We examine whether the theories in $D = 3, 4$ and 5 dimensions are power counting renormalizable or not, and show that these theories are super-renormalizable, renormalizable and one-loop renormalizable, respectively, in these dimensions. Beyond five dimensions, the theory is not renormalizable.

There are several reasons why we consider not only $D = 4$ dimensions but also $D = 3$ and 5 dimensions. First of all, the gravity theories do not have any dynamics in three-dimensional Einstein theory, but acquire interesting dynamics when higher-derivative terms are added. They
are simple but interesting enough since they can be unitary [12, 13, 14, 15, 16]. On the other hand, higher-dimensional gravities are important since string theories live in higher dimensions and there are several interesting subjects explored in the context of extra dimensions. Gravity theories in more than five dimensions cannot be renormalizable in the usual perturbative approach around the flat Minkowski space even if we add further higher order curvature terms. So these are the dimensions we are most interested in.

2 Higher Derivative Gravity

Let us consider the action of higher derivative gravity coupled to a scalar field $\phi$ with shift symmetry $\phi \rightarrow \phi + \text{constant}$. We demand that the theory respect the $Z_2$ symmetry under $\phi \rightarrow -\phi$ as well as the four-dimensional parity $x^\mu \rightarrow -x^\mu$, and contain terms with only up to four derivatives. The action takes the form [3]

$$S = \int d^D x \sqrt{-g} \left[ \frac{1}{\kappa^2} \left( R + \alpha R^2 + \beta R_{\mu\nu}^2 + \gamma R_{\mu\nu\rho\lambda}^2 \right) + Z_0(\nabla_\mu \phi)^2 + Z_1(\nabla_\mu \phi)^2 + Z_2 R^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + Z_3 (g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi)^2 + Z_4 (\Box \phi)^2 + Z_5 (\nabla_\mu \nabla_\nu \phi)^2 \right]$$

$$= \int d^D x \mathcal{L}_{\text{GMG} + \phi},$$

(1)

where $\kappa^2$ is the $D$-dimensional gravitational constant, $\alpha, \beta, \gamma$ and $Z_i$'s $(i = 0, \ldots, 5)$ are constants. The last term can be set to zero because it can be absorbed into other terms upon partial integration. Henceforth we set $Z_5 = 0$.

This is the theory that we examine. Though we should consider the theory in Riemannian geometry with Euclidean signature, there is not much difference if we discuss Lorentzian case in our discussions of renormalization. So in what follows, we discuss this as if the spacetime is the usual Minkowski space. Also, we have written down the above action only for a single scalar field for simplicity, but it is straightforward to extend the theory with several scalar fields.

2.1 Propagator

We define the fluctuation around the Minkowski background by

$$\tilde{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu}.$$  

(2)

For simplicity, we set $\kappa = 1$. Substituting (2) into our action (1), we find the quadratic term is given by

$$\mathcal{L}_2 = \frac{1}{4} h^{\mu\nu} \left[ \left( (\beta + 4\gamma) \Box + 1 \right) P^{(2)} + \frac{(D - 1)(4\alpha + \beta) + \beta + 4\gamma}{(D - 2)^2} \right] - (D - 2) \left\{ P^{(0,s)} \right\} \Box h^{\mu\nu} + \phi (Z_4 \Box + Z_0) \Box \phi,$$

(3)
where we have defined the projection operators as

\[
P_{\mu\nu,\rho\sigma}^{(2)} = \frac{1}{2}\left(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho} - \frac{2}{D-1}\theta_{\mu\nu}\theta_{\rho\sigma}\right),
\]

\[
P_{\mu\nu,\rho\sigma}^{(1)} = \frac{1}{2}\left(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}\right),
\]

\[
P_{\mu\nu,\rho\sigma}^{(0,s)} = \frac{1}{D-1}\theta_{\mu\nu}\theta_{\rho\sigma}, \quad P_{\mu\nu,\rho\sigma}^{(0,w)} = \omega_{\mu\nu}\omega_{\rho\sigma},
\]

\[
P_{\mu\nu,\rho\sigma}^{(0,sw)} = \frac{1}{\sqrt{D-1}}\theta_{\mu\nu}\omega_{\rho\sigma}, \quad P_{\mu\nu,\rho\sigma}^{(0,ws)} = \frac{1}{\sqrt{D-1}}\omega_{\mu\nu}\theta_{\rho\sigma},
\]

with

\[
\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial_{\mu}\partial_{\nu}}{\Box}, \quad \omega_{\mu\nu} = \frac{\partial_{\mu}\partial_{\nu}}{\Box}.
\]

\[
P^{(2)}, P^{(1)}, P^{(0,s)} \quad \text{and} \quad P^{(0,w)}
\]

are the projection operators onto spin 2, 1 and 0 parts, and they satisfy the completeness relation

\[
(P^{(2)} + P^{(1)} + P^{(0,s)} + P^{(0,w)})_{\mu\nu,\rho\sigma} = \frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}),
\]

on the symmetric second-rank tensors. Note that \(\gamma\) can be eliminated by the shift \(\alpha \rightarrow \alpha - \gamma\) and \(\beta \rightarrow \beta - 4\gamma\). (We see that this is true for any dimension in this order, but is valid only for \(D = 3\) and \(4\) at the nonlinear level.) However we keep \(\gamma\) here since \(\gamma\) is expected to be relevant in dimensions higher than 4.

The BRST transformation for the fields is found to be

\[
\delta_B g_{\mu\nu} = -\delta\lambda[g_{\rho\sigma}\partial_{\mu}\phi + g_{\rho\mu}\partial_{\nu}\phi + \partial_{\rho}g_{\mu\nu}\phi],
\]

\[
\delta_B c_\mu = -\delta\lambda\phi c_\mu,
\]

\[
\delta_B b_\mu = i\delta\lambda B_\mu,
\]

\[
\delta_B B_\mu = 0,
\]

\[
\delta_B \phi = -\delta\lambda c_\mu \partial_{\rho}\phi,
\]

which is nilpotent. Here \(\delta\lambda\) is an anticommuting parameter. We use the same gauge fixing as [1] using \(\tilde{g}^{\mu\nu}\), whose BRST transformation is given by

\[
\delta_B \tilde{g}^{\mu\nu} = \delta\lambda(\tilde{g}^{\mu\rho}\partial_{\rho}c^\nu + \tilde{g}^{\nu\rho}\partial_{\rho}c^\mu - \tilde{g}^{\mu\nu}\partial_{\rho}c^\rho - \partial_\rho\tilde{g}^{\mu\nu}c^\rho) \equiv \delta\lambda D^{\mu\nu} \partial_{\rho}c^\rho.
\]

The gauge fixing term and Faddeev-Popov (FP) ghost terms are concisely written as

\[
\mathcal{L}_{GF+FP} = i\delta_B[c_\mu(\partial_{\rho}h^{\mu\nu} - \frac{a}{2}B^\rho)]/\delta\lambda
\]

\[
= -B_\mu\partial_{\rho}h^{\mu\nu} - i\tilde{c}_\mu\partial_{\rho}D^{\mu\nu} \rho\phi + \frac{a}{2}B_\mu B^\rho,
\]

where \(a\) is a gauge parameter and the indices are raised and lowered with the flat metric.

The simplest way to read off the propagator for the graviton is to first eliminate the auxiliary field \(B^\mu\) and look at the quadratic part. We find that it is given by

\[
\mathcal{L}_{2,2} = \frac{1}{4}h^{\mu\nu}\left[((\beta + 4\gamma)\Box + 1)P^{(2)} + \frac{1}{a}P^{(1)} + \frac{4(D - 1)\alpha + D\beta + 4\gamma}{(D - 2)}\frac{\Box}{(D - 2)}\left\{P^{(0,s)} \right\} + (D - 1)P^{(0,w)} + \sqrt{D - 1}\left(\left\{P^{(0,sw)} + P^{(0,ws)}\right\} + \frac{2}{a}P^{(0,w)}\right)\right]_{\mu\nu,\rho\sigma} \Box h^{\rho\sigma}
\]
Using the completeness property (6) and the orthogonality of the projection operators, we find the propagators are given by

\[
D_{h}^{\mu\nu,\rho\sigma}(k) = \frac{4}{(2\pi)^D} \left\{ \frac{P^{(2)}}{k^2((\beta + 4\gamma)k^2 - 1)} + \frac{(D - 2)^2P^{(0,s)}}{k^2[4(D - 1)\alpha + D\beta + 4\gamma]k^2 + D - 2} \right\}
\]

\[
- \frac{a}{2k^2} \left\{ 2P^{(1)} + (D - 1)P^{(0,s)} + P^{0,w} - \sqrt{D - 1} \left( P^{(0,sw)} + P^{(0,ws)} \right) \right\} \right\}_{\mu\nu,\rho\sigma},
\]

We shall take the Landau gauge given by \( a = 0 \). Since the theory is invariant under the general coordinate transformation, this does not cause any problem, but simplifies the discussions considerably [1, 2]. In this gauge, we have

\[
\partial_{\mu}h^{\mu\nu} = 0.
\]

Note that the above propagator satisfies \( k^\mu D_{h}^{\mu\nu,\rho\sigma}(k) = 0 \) in this gauge. Also, these propagators damp as \( k^{-4} \) for large momentum. The ghost propagator damp as \( k^{-2} \), but there is special property in this case. As a result, as we argue later, the theory becomes renormalizable.

### 2.2 Slavnov-Taylor identity

If we introduce the Grassmann-odd source \( K_{\mu\nu} \) and \( M \), and the Grassmann-even source \( L_{\mu} \), we have the BRST-invariant action

\[
I_{\text{sym}}[h_{\mu\nu}, \bar{c}_\alpha, c^\beta, K_{\mu\nu}, L_{\rho}, M] = \int d^Dx [L_{\text{GMG}} + \phi_{\mu\nu}h_{\mu\nu} + L_{\text{GF+FP}} + K_{\mu\nu}D_{\mu\nu}c^\rho - L_{\mu}c^\nu \partial_\nu c^\mu - Mc^\rho \partial_\rho \phi]
\]

\[
\equiv \int d^Dx L_{\text{sym}}.
\]

The BRST invariance follows from (7), (9) and the nilpotency of the BRST transformation.

The generating functional of Green’s functions is given by

\[
Z[J_{\mu\nu}, \bar{\eta}_\alpha, \eta^\beta, N, K_{\mu\nu}, L_{\rho}, M]
\]

\[
= \int [dh][d\phi][d\bar{c}][dc] \exp \left( i \int d^Dx [L_{\text{sym}} + J_{\mu\nu}h^{\mu\nu} + N\phi + \bar{\eta}_\alpha c^\alpha + \bar{c}_\alpha \eta^\alpha] \right)
\]

\[
\equiv \exp \left( iW[J_{\mu\nu}, \bar{\eta}_\alpha, \eta^\beta, K_{\mu\nu}, L_{\rho}, M] \right),
\]

where \( J_{\mu\nu} \) and \( N \) (\( \bar{\eta}_\alpha \) and \( \eta^\alpha \)) are Grassmann-even (Grassmann-odd) sources, respectively. The BRST invariance of the functional (15)

\[
0 = \int [dh][d\phi][d\bar{c}][dc] \delta_B \exp \left( i \int d^Dx [L_{\text{sym}} + J_{\mu\nu}h^{\mu\nu} + N\phi + \bar{\eta}_\alpha c^\alpha + \bar{c}_\alpha \eta^\alpha] \right),
\]

implies that

\[
\left\langle \int d^Dx \left[ J_{\mu\nu}D_{\mu\nu}c^\rho - Nc^\rho \partial_\rho \phi + \bar{\eta}_\alpha c^\alpha \partial_\alpha c^\mu + i\frac{1}{a} \eta_\mu \partial_\mu h^{\mu\nu} \right] \right\rangle = 0,
\]
where the field $B_\mu$ is eliminated by its field equation. This yields the Slavnov-Taylor identity
\[
\int d^Dx \left[ J_{\mu\nu} \frac{\delta W}{\delta K_{\mu\nu}} + N \frac{\delta W}{\delta M} - \bar{\eta}_\mu \frac{\delta W}{\delta L_\mu} + \frac{i}{a} \gamma^\mu \partial_\nu \frac{\delta W}{\delta J_{\mu\nu}} \right] = 0.
\] (18)

The equations of motion for the FP ghost is
\[
\partial_\nu \frac{\delta W}{\delta K_{\mu\nu}} + i \eta_\mu = 0.
\] (19)

As usual, the effective action is defined by
\[
\tilde{\Gamma}[h^{\mu\nu}, \phi, \bar{c}_\alpha, c^\beta, K_{\mu\nu}, L_\rho, M] 
\equiv W[J_{\mu\nu}, \bar{\eta}_\alpha, \eta^\beta, K_{\mu\nu}, L_\rho, M] - \int d^Dx \left[ J_{\mu\nu} h^{\mu\nu} + N \phi + \bar{\eta}_\alpha c^\alpha + \bar{c}_\alpha \eta^\alpha \right].
\] (20)

It follows from (15) that
\[
h^{\mu\nu} = \frac{\delta W}{\delta J_{\mu\nu}}, \quad \phi = \frac{\delta W}{\delta N}, \quad c^\mu = \frac{\delta W}{\delta \bar{\eta}_\mu}, \quad \bar{c}_\mu = -\frac{\delta W}{\delta \eta_\mu}.
\] (21)

The relations dual to these are
\[
J_{\mu\nu} = -\frac{\delta \tilde{\Gamma}}{\delta h^{\mu\nu}}, \quad N = -\frac{\delta \tilde{\Gamma}}{\delta \phi}, \quad \bar{\eta}_\alpha = \frac{\delta \tilde{\Gamma}}{\delta \bar{c}_\alpha}, \quad \eta^\alpha = -\frac{\delta \tilde{\Gamma}}{\delta c^\alpha}.
\] (22)

We further define
\[
\Gamma = \tilde{\Gamma} + \int d^Dx \frac{1}{2a} (\partial_\nu h^{\mu\nu})^2.
\] (23)

With the help of the relations
\[
\frac{\delta \Gamma}{\delta K_{\mu\nu}} = \frac{\delta W}{\delta K_{\mu\nu}}, \quad \frac{\delta \Gamma}{\delta M} = \frac{\delta W}{\delta M}, \quad \frac{\delta \Gamma}{\delta L_\mu} = \frac{\delta W}{\delta L_\mu},
\] (24)

and the ghost field equation
\[
\partial_\nu \frac{\delta \Gamma}{\delta K_{\mu\nu}} - i \frac{\delta \Gamma}{\delta c^\mu} = 0,
\] (25)

the Slavnov-Taylor identity reduces to
\[
\int d^Dx \left[ \frac{\delta \Gamma}{\delta h^{\mu\nu}} \frac{\delta \Gamma}{\delta K_{\mu\nu}} + \frac{\delta \Gamma}{\delta c^\mu} \frac{\delta \Gamma}{\delta L_\mu} + \frac{\delta \Gamma}{\delta \phi} \frac{\delta \Gamma}{\delta M} \right] = 0.
\] (26)

The $n$-loop part of the effective action is denoted by $\Gamma^{(n)}$. The effective action is a sum of these terms:
\[
\Gamma = \sum_{n=0}^{\infty} \Gamma^{(n)}.
\] (27)

Suppose that we have successfully renormalized the effective action up to $(n-1)$-loop order. Write
\[
\Gamma^{(n)} = \Gamma^{(n)}_{\text{finite}} + \Gamma^{(n)}_{\text{div}}.
\] (28)
If we insert this breakup into Eq. (26) and keep only the terms which are of $n$-loop order, we get

$$
\int d^D x \left[ \frac{\delta \Gamma^{(n)}_{\text{div}}}{\delta h^{\mu \nu} \delta K_{\mu \nu}} + \frac{\delta \Gamma^{(0)}_{\text{div}}}{\delta h^{\mu \nu} \delta K_{\mu \nu}} + \frac{\delta \Gamma^{(n)}_\mu \delta \Gamma^{(0)}_\mu}{\delta c^\mu \delta L_\mu} + \frac{\delta \Gamma^{(n)}_{\text{div}}}{\delta \phi \delta M} + \frac{\delta \Gamma^{(0)}_{\text{div}}}{\delta \phi \delta M} \right] = -\int d^D x \sum_{i=0}^n \left[ \frac{\delta \Gamma^{(n-i)}_{\text{finite}}}{\delta h^{\mu \nu} \delta K_{\mu \nu}} + \frac{\delta \Gamma^{(n-i)}_{\text{finite}}}{\delta c^\rho \delta L_\rho} + \frac{\delta \Gamma^{(n-i)}_{\text{finite}}}{\delta \phi \delta M} \right].
$$

(29)

Since each term on the right-hand side of (29) remains finite as $\epsilon \to 0$ in the dimensional regularization, while each term on the left-hand side contains a factor with a pole in $\epsilon$, each side of the equation must vanish separately. This leads to

$$
\int d^D x \left[ \frac{\delta \Gamma^{(0)}_{\text{div}}}{\delta h^{\mu \nu} \delta K_{\mu \nu}} + \frac{\delta \Gamma^{(0)}_{\text{div}}}{\delta h^{\mu \nu} \delta K_{\mu \nu}} + \frac{\delta \Gamma^{(0)}_\mu \delta \Gamma^{(0)}_\mu}{\delta c^\rho \delta c^\lambda} + \frac{\delta \Gamma^{(0)}_\mu \delta L_\rho}{\delta L_\lambda} + \frac{\delta \Gamma^{(0)}_\mu \delta M}{\delta \phi \delta M} + \frac{\delta \Gamma^{(0)}_\mu \delta \phi}{\delta M} \right] \Gamma^{(n)}_{\text{div}} = 0.
$$

(30)

This identity will be used in later discussions of renormalizability.

### 2.3 Renormalizability

Under the expansion (2), the Einstein term gives graviton vertices with two derivatives, and curvature square terms give those with four derivatives. Compared with the theory without scalar [2], we also have scalar vertices with four derivatives as well as scalar-graviton vertices with two and four derivatives.

Consider arbitrary Feynman diagrams. We use the following notations.

- $V_{h,2}$: the number of graviton vertices with two derivatives from the $R$ term.
- $V_{h,4}$: the number of graviton vertices with four derivatives from the $R^2$ term.
- $V_{s,4}$: the number of scalar vertices with four derivatives.
- $V_{h,s,2}$: the number of graviton-scalar vertices with two derivatives.
- $V_{h,s,4}$: the number of graviton-scalar vertices with four derivatives.
- $V_c$: the number of ghost-antighost-graviton vertices with two derivatives.
- $V_K$: the number of $K$-graviton-ghost vertices.
- $V_L$: the number of $L$-ghost-ghost vertices.
- $V_M$: the number of $M$-graviton-ghost vertices.
- $I_h$: the number of internal-graviton propagators.
- $I_s$: the number of internal-scalar propagators.
- $I_c$: the number of internal-ghost propagators.
- $E_h$: the number of external gravitons.
- $E_c$: the number of external ghosts.

Since the graviton and scalar propagators behaves as $k^{-4}$ and the FP ghost propagator as $k^{-2}$, we are led by the standard power counting to the degree of divergence of an arbitrary diagram:

$$
D_{\text{div}} = DL - 4I_h - 4I_s - 2I_c + 4V_{h,4} + 2V_{h,2} + 4V_{s,4} + 4V_{h,s,4} + 2V_{h,s,2} + 2V_c + V_K + V_L + V_M.
$$

(31)
Using the relation
\[ L = I_h + I_c + I_s - (V_{h,4} + V_{h,2} + V_{s,4} + V_{hs,4} + V_{hs,2} + V_c + V_K + V_L + V_M - 1), \]  
we get
\[ D_{\text{div}} = D + (D - 4)(I_h + I_s - V_{h,4} - V_{s,4} - V_{hs,4}) + (D - 2)(V_{h,2} + V_{hs,2} - V_c) \]
\[ - (D - 1)(V_K + V_L + V_M). \]  
(33)

We further use the topological relation
\[ 2V_c + V_K + 2V_L + V_M = 2I_c + E_c + E_c, \]  
(34)
to obtain
\[ D_{\text{div}} = D - (4 - D)(I_h + I_s - V_{h,4} - V_{s,4} - V_{hs,4}) - (D - 2)(V_{h,2} + V_{hs,2}) \]
\[ - \frac{D}{2}(V_K + V_M) - V_L - \frac{D - 2}{2}(E_c + E_c). \]  
(35)

Now the ghost vertex contained in the FP ghost term in (9), upon partial integration, can be rewritten as
\[ i[\partial_\mu \partial_\nu \tilde{c}_\nu \cdot c^\mu h^{\nu \rho} + \partial_\nu \tilde{c}_\nu \cdot c^\nu h^{\mu \rho} + \partial_\rho \tilde{c}_\nu \cdot c^\nu h^{\mu \rho}]. \]  
(36)

In the Landau gauge in which we have (13), the last two terms do not couple to the propagator. Also integration by parts in the remaining term can be used to move the derivative onto the ghost using the gauge condition:
\[ i\partial_\mu \partial_\nu \tilde{c}_\nu \cdot c^\mu h^{\nu \rho} \approx i\tilde{c}_\mu \partial_\mu c^\nu h^{\nu \rho}. \]  
(37)

As a result, in one-particle irreducible (1PI) diagrams, each external ghost and antighost carries two factors of external momentum [1, 2]. The resulting degree of divergence of an arbitrary 1PI diagram is then
\[ D_{\text{div}}^{(1PI)} = D - (4 - D)(I_h + I_s - V_{h,4} - V_{s,4} - V_{hs,4}) - (D - 2)(V_{h,2} + V_{hs,2}) \]
\[ - \frac{D}{2}(V_K + V_M) - V_L - \frac{D + 2}{2}(E_c + E_c). \]  
(38)

We note that \( I_h + I_s - V_{h,4} - V_{s,4} - V_{hs,4} \geq 0 \) for 1PI diagrams, so most of the contributions are negative for \( D \leq 4 \).

First, let us concentrate on \( D = 3 \). The resulting degree of divergence of an arbitrary 1PI diagram is then
\[ D_{\text{div}}^{(1PI)} = 3 - (I_h + I_s - V_{h,4} - V_{s,4} - V_{hs,4}) \]
\[ - (V_{h,2} + V_{hs,2}) - \frac{3}{2}(V_K + V_M) - V_L - \frac{5}{2}(E_c + E_c). \]  
(39)

We find that the possible divergences are restricted; those with external ghosts and antighosts have \( D^{(1PI)} \leq -2 \), those with the external \( K \) and ghost \( D^{(1PI)} \leq -1 \), those with \( L \) and two ghosts have \( D^{(1PI)} \leq -3 \), and those with external \( M \) and ghost have \( D^{(1PI)} \leq -1 \). Hence, we have
\[ \frac{\delta \Gamma^{(n)}_{\text{div}}}{\delta c^\lambda} = \frac{\delta \Gamma^{(n)}_{\text{div}}}{\delta K^{\mu \nu}} = \frac{\delta \Gamma^{(n)}_{\text{div}}}{\delta L^\lambda} = \frac{\delta \Gamma^{(n)}_{\text{div}}}{\delta M} = 0. \]  
(40)
The Slavnov-Taylor identity (30) then reduces to
\[ \int d^3x \left[ \frac{\delta \Gamma^{(0)}}{\delta K^{\mu \nu}} \frac{\delta}{\delta \tilde{g}^{\mu \nu}} + \frac{\delta \Gamma^{(0)}}{\delta M} \frac{\delta}{\delta \phi} \right] \Gamma^{(n)}_{\text{div}} = 0. \] (41)

Together with (40), this implies that \( \Gamma^{(n)}_{\text{div}} \) is gauge invariant. Consequently \( \Gamma^{(n)}_{\text{div}} \) are local gauge-invariant functionals of \( \tilde{g}^{\mu \nu} \) and \( \phi \) with zero and two derivatives (up to three). This allows only the counterterms of the Einstein and cosmological, and \( (\nabla_{\mu} \phi)^2 \) terms. Terms like \( \phi^2 \) are not allowed due to the shift symmetry. Clearly we have divergence at the lower-loop levels. The convergence property improves as more vertices and internal lines are added, i.e. when we go to higher-loop diagrams. Thus there are only finite numbers of divergent diagrams. Hence the theory is super-renormalizable.

Next, let us consider \( D = 4 \). The degree of divergence of an arbitrary 1PI diagram is
\[ D_{\text{div}}^{(1PI)} = 4 - 2(V_{h,2} + V_{hs,2}) - 2(V_K + V_M) - V_L - 3(E_c + E_{\bar{c}}). \] (42)
We find that the possible divergences are again restricted; those with external ghosts and antighosts have \( D^{(1PI)} \leq -2 \), those with the external \( K \) and ghost \( D^{(1PI)} \leq -1 \), those with \( L \) and two ghosts have \( D^{(1PI)} \leq -3 \), and those with external \( M \) and ghost have \( D^{(1PI)} \leq -1 \). Hence, we have
\[ \frac{\delta \Gamma^{(n)}_{\text{div}}}{\delta c^\lambda} = \frac{\delta \Gamma^{(n)}_{\text{div}}}{\delta K^{\mu \nu}} = \frac{\delta \Gamma^{(n)}_{\text{div}}}{\delta L^\lambda} = \frac{\delta \Gamma^{(n)}_{\text{div}}}{\delta M} = 0. \] (43)
The Slavnov-Taylor identity (30) then reduces to
\[ \int d^4x \left[ \frac{\delta \Gamma^{(0)}}{\delta K^{\mu \nu}} \frac{\delta}{\delta \tilde{g}^{\mu \nu}} + \frac{\delta \Gamma^{(0)}}{\delta M} \frac{\delta}{\delta \phi} \right] \Gamma^{(n)}_{\text{div}} = 0. \] (44)
Together with (43), this implies that \( \Gamma^{(n)}_{\text{div}} \) is gauge invariant. Therefore \( \Gamma^{(n)}_{\text{div}} \) are local gauge-invariant functionals of \( \tilde{g}^{\mu \nu} \) and \( \phi \) with zero, two and four derivatives. This allows only the counterterms that are the same as (1) (and cosmological constant). Thus the theory is renormalizable. In this case, we find that the divergence is not necessarily restricted to lower loops since the number of vertices from higher-derivative terms and internal lines do not affect the degree of divergence and hence there may be divergent diagrams involving these in higher loops. In this sense, the theory is only renormalizable.

Finally we turn to \( D = 5 \). The degree of divergence of an arbitrary 1PI diagram is then
\[ D_{\text{div}}^{(1PI)} = 5 + (I_h + I_s - V_{h,4} - V_{s,4} - V_{hs,4}) - 3(V_{h,2} + V_{hs,2}) - \frac{5}{2}(V_K + V_M) - V_L - \frac{7}{2}(E_c + E_{\bar{c}}). \] (45)
Diagrams are more divergent when we go to higher loops since \( I_h + I_s - V_{h,4} - V_{s,4} - V_{hs,4} > 0 \), and the theory in \( D = 5 \) is not renormalizable. However, for the one-loop diagrams, the second term vanishes and we have only the same divergences as for \( D = 4 \). The theory is renormalizable at this level.

Beyond \( D = 5 \), we get more divergences, and the theory is not renormalizable. It is clear that adding higher curvature terms does not help to improve renormalizability, because these do not improve the behavior of propagators around Minkowski space but introduce higher derivative vertices.
3 Conclusions

In this paper we have studied whether higher derivative gravities coupled to a scalar field with shift symmetry in $D = 3, 4, 5$ dimensions are renormalizable or not. We have shown that the general theory is (super-)renormalizable in $D = 3$ and 4, and is not renormalizable in $D = 5$. Theory in $D = 5$ is renormalizable in the one-loop calculations, because $I_h + I_s - V_{h,4} - V_{s,4} - V_{hs,4} = 0$ for one-loop 1PI diagrams. We have noted that theories in further higher dimensions are not renormalizable around Minkowski space even if more higher curvature terms are added. So this analysis exhausts interesting cases in the perturbative approach.

Thus the first step described in the introduction is cleared. The next issue to be studied is the renormalization group properties of these theories. In the four-dimensional case, we expect that the coefficients of the terms in the action have Gaussian fixed point as well as other nontrivial fixed point. The existence of the fixed points is named asymptotic safety, and the Gaussian fixed point corresponds to asymptotic freedom. This has been checked for other type of higher derivative theories [4] – [9], but it should be confirmed in our theory explicitly. On the other hand, in the low energy, we expect that the higher derivative terms become irrelevant and Einstein term has a finite fixed point. It would be interesting to explicitly check if the theory reduces to the Einstein theory together with the four-derivative scalar theory which is equivalently described by a Lorentzian action. We hope to report on this problem elsewhere.

Acknowledgement

We would like to thank Shinji Mukohyama for very useful discussions. This work was supported in part by the Grant-in-Aid for Scientific Research Fund of the JSPS (C) No. 24540290 and (A) No. 22244030.

References

[1] K. S. Stelle, “Renormalization of Higher Derivative Quantum Gravity,” Phys. Rev. D 16 (1977) 953.

[2] K. Muneyuki and N. Ohta, “Unitarity versus Renormalizability of Higher Derivative Gravity in 3D,” Phys. Rev. D 85 (2012) 101501 [arXiv:1201.2058 [hep-th]].

[3] S. Mukohyama and J. -P. Uzan, “From configuration to dynamics – Emergence of Lorentz signature in classical field theory,” Phys. Rev. D 87 (2013) 065020 [arXiv:1301.1361 [hep-th]]; S. Mukohyama, “Emergence of time in power-counting renormalizable Riemannian theory of gravity,” arXiv:1303.1409 [hep-th].

[4] I. G. Avramidi and A. O. Barvinsky, “Asymptotic Freedom In Higher Derivative Quantum Gravity,” Phys. Lett. B 159 (1985) 269.

I. G. Avramidi, “Covariant methods for the calculation of the effective action in quantum field theory and investigation of higher derivative quantum gravity,” hep-th/9510140.

[5] G. de Berredo-Peixoto and I. L. Shapiro, “Higher derivative quantum gravity with Gauss-Bonnet term,” Phys. Rev. D 71 (2005) 064005 [hep-th/0412249].
[6] A. Codello and R. Percacci, “Fixed points of higher derivative gravity,” Phys. Rev. Lett. 97 (2006) 221301 [hep-th/0607128].

[7] A. Codello, R. Percacci and C. Rahmede, “Investigating the Ultraviolet Properties of Gravity with a Wilsonian Renormalization Group Equation,” Annals Phys. 324 (2009) 414 [arXiv:0805.2909 [hep-th]].

[8] O. Lauscher and M. Reuter, “Flow equation of quantum Einstein gravity in a higher derivative truncation,” Phys. Rev. D 66 (2002) 025026 [hep-th/0205062].

[9] D. Benedetti, P.F. Machado and F. Saueressig, “Asymptotic safety in higher-derivative gravity,” Mod. Phys. Lett. A 24 (2009) 2233 [arXiv:0805.2909 [hep-th]].

[10] R. Percacci and E. Sezgin, “One Loop Beta Functions in Topologically Massive Gravity,” Class. Quant. Grav. 27 (2010) 155009 [arXiv:1002.2640 [hep-th]].

[11] N. Ohta, “Beta Function and Asymptotic Safety in Three-dimensional Higher Derivative Gravity,” Class. Quant. Grav. 29 (2012) 205012 [arXiv:1205.0476 [hep-th]].

[12] E. A. Bergshoeff, O. Hohm and P. K. Townsend, “Massive Gravity in Three Dimensions,” Phys. Rev. Lett. 102 (2009) 201301 [arXiv:0901.1766 [hep-th]].

[13] S. Deser, “Ghost-free, finite, fourth order D=3 (alas) gravity,” Phys. Rev. Lett. 103 (2009) 101302 [arXiv:0904.4473 [hep-th]].

[14] I. Gullu, T. C. Sisman and B. Tekin, “Canonical Structure of Higher Derivative Gravity in 3D,” Phys. Rev. D 81 (2010) 104017 [arXiv:1002.3778 [hep-th]].

[15] E. A. Bergshoeff, O. Hohm and P. K. Townsend, “More on Massive 3D Gravity,” Phys. Rev. D 79 (2009) 124042 [arXiv:0905.1259 [hep-th]].

[16] N. Ohta, “A Complete Classification of Higher Derivative Gravity in 3D and Criticality in 4D,” Class. Quant. Grav. 29 (2012) 015002 [arXiv:1109.4458 [hep-th]].