Research Article

Existence of Solutions of Boundary Value Problem for Nonlinear One-Dimensional Wave Equations by Fixed Point Method

Daba Meshesha Gusu and Megersa Danu
Department of Mathematics, Ambo University, Ethiopia

Correspondence should be addressed to Daba Meshesha Gusu; dabam7@gmail.com

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In this article, we investigate the solutions for a class of initial boundary value problems for nonlinear one-dimensional wave equations using a fixed point method. The findings consider initial and boundary data which lead to determining the nature of classes of the nonlinear one-dimensional wave equation. The obtained results indicate the availability of nonnegative solutions which are proved using the method of the fixed point. A new fixed point approach is effective and used as a definitive modeling method to prove the existence of nonnegative solutions for a class of initial boundary value problems based on recent theoretical arguments. The results in this study are considered with examples.

1. Introduction

The nonlinear partial differential equation (NLPDE) has many important applications to study problems of physical phenomena: some of them are mentioned in [1–6]. The attempt to find the exact solutions to NLPDEs is used for understanding the most nonlinear physical phenomena. Among them, the nonlinear wave phenomena appear in various scientific and geometrical physical applications in mathematical physics [1]. A wave is one of the application of partial differential equations used for modeling physical objects based on dependent variables x and time t and one or more independent spatial variables $x \in \mathbb{R}^n$, where $n$ is generally equal to 1 or more than 1 [7]. It has many applications in scientific and engineering disciplines [8]. Moreover, in everyday life, we encounter lots of examples of waves such as a traveling wave along a straight line, the oscillation of strings or an organ pipe, sound waves in the air, and others [9–15]. The wave vibrates as it passes, and an oscillation travels through the medium by transferring energy based on moving one particle to another with an effect on the displacement of the medium [16].

The wave phenomena depend on the existence of solutions for a class of initial boundary value problems for nonlinear one-dimensional wave equations by using a fixed point approach. A fixed point method is used to prove the existence and uniqueness of solutions for nonlinear equations. The one-dimensional wave equation is the first ever partial differential equation (PDE) to be used as a model to study vibrating strings. It has the following form:

$$\frac{\partial^2 u(x, t)}{\partial t^2} = \frac{\partial^2 u(x, t)}{\partial x^2}. \quad (1)$$

The wave equation in two and three dimensions has been studied in [17, 18]. A PDE is subjected to certain conditions in the form of initial or boundary conditions termed as an initial boundary value problem (IBVP). Also for this article, the boundary conditions on the boundary $\partial D$ of the domain $D$ under consideration taken and mixed conditions of the third kind can be considered here in the values of a linear combination of $u$ and its normal derivative prescribed at each point of the boundary $\partial D$ as cited in [17]. For this specific article, we have considered the one-dimensional wave equation restricted to the interval $[0, L]$ where $0 \leq x \leq L$. Hence, a physical problem is to solve for $u = u(x, t)$ the equation $u_t - u_{xx} = f(t, x, u)$ on the region $0 \leq x \leq L$, $t \geq 0$ where $u(0, t) = 0$ and $u(L, t) = 0$ are
boundary conditions; \( u(0, x) = u_0(x) \) is an initial condition at \( t_0 = 0; \) \( u_t(0, x) = u_1(x) \) is also an initial condition which is cited in [19].

Our main concern here is that an IBVP for the nonlinear one-dimensional wave equation defined on the bounded domain \([0, L]\) in one-dimensional space is given by the following equation:

\[
\begin{align*}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= f(t, x, u), \\
\end{align*}
\]

where \( f(t, x, u) \neq 0 \) in the domain \( 0 \leq x \leq L, \ t > 0. \) To describe the wave motion completely, it is necessary to specify suitable initial and boundary conditions for the displacement \( u(x, t) \). The ends are assumed to remain fixed, and therefore, the boundary conditions are as follows:

\[
\begin{align*}
& u(0, t) = 0, \\
& u_x(L, t) = 0, \ t \geq 0. \\
\end{align*}
\]

It is plausible to prescribe two initial conditions, these are the initial position of the string or object.

\[
\begin{align*}
& u(0, x) = u_0(x), \quad 0 \leq x \leq L. \\
\end{align*}
\]

And its initial velocity taken is as follows:

\[
\begin{align*}
& u_t(0, x) = u_1(x), \quad 0 \leq x \leq L, \\
\end{align*}
\]

where \( f \) and \( g \) are given functions. In order for the equations (3)–(5) to be consistent, it is also necessary to require that

\[
\begin{align*}
& u_0(0) = u_0(L) = 0, \\
& u_1(0) = u_1(L) = 0. \\
\end{align*}
\]

The aim of this mathematical problem is to determine the existence of nonnegative solutions for a class of one-dimensional nonlinear wave (2) that satisfies the boundary condition (3) and the initial conditions (4) and (5) via a new fixed point approach. Thus, we are considering some typical initial boundary value problem (IBVP) involving the one-dimensional nonlinear wave equation which is a hyperbolic equation via a new fixed point approach. More precisely, we study the following IBVP.

\[
\begin{align*}
\begin{cases}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= f(t, x, u), \ t \geq 0, \ x \in [0, L], \\
u(0, x) &= u_0(x), \ x \in [0, L], \\
u_t(0, x) &= u_1(x), \ x \in [0, L], \\
u(t, 0) &= u_{\partial x}(t, L) = 0, \ t > 0, \\
u(t, L) &= u_{\partial x}(t, 0) = 0. \\
\end{cases}
\end{align*}
\]

where \( L > 0, \ f : [0, \infty) \times [0, L] \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous, and \( u_0 \in C^0[0, L], \ u_1 \in C^1[0, L] \) are the initial data and satisfy the general conditions. The ubiquitous nature of wave phenomena like water, sound, and electromagnetic waves lead to investigation in more detail in the physical real world. A wave is a time evolution phenomenon that we generally model mathematically using partial differential equations which have a dependent variable \( u(x, t) \) (representing the wave value) and an independent variable time \( t \). The actual form that the wave takes strongly is dependent on the system of initial conditions on the solution domain and any disturbances. Nonlinear waves are described by nonlinear equations, and therefore, the superposition principle does not generally apply, and this leads to investigating nonlinear phenomena.

Global existence for nonlinear wave equations is an important mathematical topic and investigated on nonlinear nature [20, 21]. Here, we proposed another way depending on the method of the new fixed point approach for investigations into the existence of nonnegative solutions for a class of IBVP for nonlinear 1D wave equations via a new fixed point approach. We researched the existence of nonnegative solutions for a class of IBVP for nonlinear 1D wave equations using the so-called new fixed point approach. In this article, we considered the existence of nonnegative solutions for a class of initial boundary value problem (IBVP) for nonlinear 1D wave equation via a new fixed point approach which is given by equation (7). This type of system includes a number of classical systems of equations such as the equations of magnetic hydrodynamics, hydrodynamics of compressible liquids, and gases are equations of the nonlinearity theory of compressible liquids. A transformation which allows certain additional conditions used to derive several existence results by restoring the theory of a new fixed point theory indexed in cones to strict the sect of contraction mappings.

Some of these results have been improved in several directions, and they have been applied to obtain existence results of initial and boundary value problems subject to ordinary and partial differential equations (20)–(23). The existence of solutions (7) when the spatial variable \( x \) ranges over the finite interval of the real line has supplemented with boundary conditions to specify how the solution \( u(x, t) \) is constrained to be at the two end points \( x = 0 \) and \( x = L \). This is linked with the two most commonly arising types of boundary conditions: Dirichlet and Neuman boundary conditions [16]. Therefore, to handle the existence of nonnegative solutions for a class of nonlinear one-dimensional wave equations, using the fixed point approach method is a basic issue for solving such a nonlinear problem.

A fixed point theory is a branch of nonlinear analysis that can be applied successfully to a wide range of contexts in natural sciences [24]. Fixed points of mappings satisfying contractive conditions in generalized metric spaces are highly useful in large numbers of mathematical problems of pure and applied mathematics [25, 26]. While solving the problems of nonlinear partial differential equations, the most important and effective tool is Banach’s fixed point theorem which is used as a definitive modeling method. This is the main ingredient in the implicit function theorem and the inverse function theorem [27]. Fixed point theory plays an important role in the applications of many branches of mathematics. Within the past thirty years, several generalizations of a metric space have been made [28, 29]. As a result of this, the Banach contraction principle is the basic result in the fixed point theory that we have used in this paper.

The analytical solution for a vibrating string wave equation was found by [30]. The classical solution of the wave equation in one dimension with initial and boundary conditions was reported by [31]. Wave equations have
attracted much attention, solving these kinds of equations has been one of the interesting tasks for mathematicians, and the global existence for nonlinear wave equations is an important topic in mathematical sciences [17, 32, 33]. Moreover, the solutions for a class of IBVP for nonlinear one-dimensional wave equations were obtained and reported by several different authors and also very applicable in different mathematical problems and physics [20]. In general, there is a vast literature concerning to existence of nonnegative solutions for a class of IBVP for nonlinear wave equations which is mentioned in [20, 22, 23, 34]. However, they did not apply the fixed point method to determine solutions of the initial boundary value problem for nonlinear one-dimensional wave equations by using the specific method.

2. Mathematical Preliminaries

When we are solving nonlinear partial differential equations, the basic tool is Banach’s fixed point theorem which is sometimes called the contraction mapping principle. The Banach contraction principle is the basic result in fixed point theory. This contraction mapping principle is used to establish the local existence and uniqueness of solutions to various nonlinear equations, and it is also one of the most useful tools in the study of nonlinear equations.

Definition 1. Let $X$ be a real Banach space. Suppose $X$ is a complete metric space with a distance function represented by $d(\ldots)$. A mapping $T: X \longrightarrow X$ is a strict contraction if there exists $0 < a < 1$ such that $d(Tx, Ty) \leq a \, d(x, y)$ for all $x, y \in X$. If $X$ is a complete metric space and $T: X \longrightarrow X$ is a strict contraction, then $T$ has a unique fixed point [34].

Definition 2. Let $f: X \longrightarrow X$ be a map of a metric space to itself. A point $a \in X$ is called a fixed point of $f$ if $f(a) = a$. Let $(X, d)$ be a metric space. A mapping $T: X \longrightarrow X$ is a contraction mapping, or contraction, if there exists a constant $c$, with $0 < c < 1$ such that $d(Tx, Ty) \leq c \, d(x, y)$, for all $x, y \in X$ [20].

Definition 3. Let $X$ be the class of all bounded sets of $X$. The Kuratowski measure of noncompactness, $\alpha: X \longrightarrow [0, \infty)$ is defined by $\alpha(Y) = \inf\{\delta > 0: Y = \bigcup_{j=1}^{m} Y_j \text{ and } \operatorname{diam}(Y_j) \leq \delta, j \in \{1, \ldots, m\}\}$, where $\operatorname{diam}(Y_j) = \sup\{\|x - y\|: x, y \in Y_j\}$ is the diameter of $Y_j$, $j \in \{1, \ldots, m\}$ [35].

Definition 4. A mapping $K: X \longrightarrow X$ is said to be $K$-set contraction if there exists a constant $k \geq 0$ such that $\alpha(K(Y)) \leq k \alpha(Y)$ for any bounded set $Y \subset X$. Obviously, if $K: X \longrightarrow X$ is a completely continuous mapping, then $K$ is 0-set contraction [20].

Definition 5. A mapping $K: X \longrightarrow X$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact (closed and bounded) sets. The concept for $k$-set contraction is related to that of the Kuratowski measure of noncompactness which we recall for completeness [35].

Definition 6. Let $X$ and $Y$ be real Banach spaces. A mapping $K: X \longrightarrow Y$ is said to be expensive if there exists a constant $h > 1$ such that $\|Kx - Ky\|_Y \geq h\|x - y\|_X$, for any $x, y \in X$ [35].

Definition 7. Closed, convex set $P$ in $X$ is said to be cone if

\begin{align*}
(i) & \quad ax \in P \text{ for any } a \geq 0 \text{ and } x \in P \\
(ii) & \quad x, -x \in P \text{ implies } x = 0 \quad [20]
\end{align*}

Definition 8. A mapping $F: X \longrightarrow X$, where $(X, d)$ is a metric space, is a contraction if there exists $k < 1$ such that $d(F(x), F(y)) \leq k \, d(x, y)$, $\forall x, y \in X$. This type of condition is called a Lipschitz condition, where $k \geq 0$ is called the Lipschitz constant. A function $f$ from $S \subset R^n$ into $R^m$ is Lipschitz continuous at $x \in S$ if there is a constant $c$ such that $\|f(y) - f(x)\| \leq c \|y - x\|$, for all $y \in S$ sufficiently near $x$ [35].

3. Main Results

The aim of this section is to prove the existence of solutions of boundary value problems for nonlinear one-dimensional wave equations by applying fixed points. To do that, we consider certain propositions, lemmas, and theorems. Moreover, we investigate some important properties of the solutions related to the concept of nonnegativity of solutions. Finally, we illustrate by example and discussion of the results followed.

3.1. Some Basic Propositions, Lemmas, and Theorems. In this article, we proposed the new fixed point approach on cones using operators. Before calling our main result, we precise the assumptions made on the nonlinearity, initial, and boundary data. We assume that the initial, boundary data, and the function $f$ satisfying the following assumptions of $H_1$ and $H_2$ are given by

\begin{align}
H_1: & \quad u_0, u_1 \in C^2([0, L]), 0 \leq u_0, u_1 \leq r \text{ on } [0, L] \\
& \quad u_0(0) = u_0(L) = 0, u_1(0) = u_1(L) = 0 \\
H_2: & \quad f \in C(0, \infty) \times [0, L] \times \mathbb{R} \text{ is such that} \\
& \quad 0 \leq f(t, x, u) \leq \sum_{j=1}^{l} c_j(t, x)u^{p_j}, \\
& \quad (t, x, u) \in \{0, \infty\} \times [0, L] \times \mathbb{R},
\end{align}

where $p_j > 0$, $c_j \in C([0, \infty) \times n[q0, L])$, $0 \leq c_j \leq d$. On $[0, \infty) \times n[q0, L]$, $j \in \{1, \ldots, l\}$ for some positive constant $d$. Since $l \in \mathbb{N}$, our main result is stated as follows.

Theorem 1. Suppose that $H_1$ and $H_2$ are fulfilled. Then, the initial boundary value problem equation (7) has at least one nonnegative solution $u \in C^2([0, \infty) \times \mathbb{N}[q0, L])$. 
To prove the proposed theorem, we follow the following steps: in the first step, we give some helpful preliminary and auxiliary results which would be used to prove our main result. In the second step, we prove it. In the third step, we give an illustrative using the practical example. Therefore, the following proposition will be used to prove our main result.

**Proposition 1.** Suppose that $Q$ is a cone in Banach space $(X, \| \cdot \|)$. Let $\Omega$ be a subset of $Q$ and $U$ be a bounded open subset of $Q$ with $0 \in U$. Assume that the mapping $T: \Omega \subset Q \rightarrow E$ be such that $(I - T)$ is Lipschitz invertible with constant $\gamma > 0$, $S: \bar{U} \rightarrow E$ is a K-set contraction with $0 \leq K < \gamma^{-1}$, and $S(\bar{U}) \subset (I - T)(\Omega)$. If $Sx \neq (I - T)(\lambda x)$ for all $x \in \partial U \cap \Omega$, $\lambda \geq 1$ and $\lambda x \in \Omega$, then the fixed point index $i_\varepsilon(T + S, U \cap \Omega, Q) = 1$ (which means $T + S$ has at least one fixed point in $U \cap \Omega$ [23].

In order to apply Proposition 1 and prove our main result, we consider the following function:

$$G(t, x, \tau, \xi) = (-1/2\pi) \sqrt{(t - \tau - |x - \xi|)^2},$$

where $H(t - \tau - |x - \xi|)$ is the Green function for the one-dimensional wave equation.

$$u_{tt} - u_{xx} = h(t, x, u, t),$$

$$u(0, x) = u(x, 0),$$

$$u(t, 0) = u_x(t, L),$$

where $H(.)$ denotes the Heaviside function and whose value is defined by $H(.) = \begin{cases} 1 & \text{if}, x > 0 \\ 0 & \text{if}, x \leq 0. \end{cases}$

Let $m \in \mathbb{W}$. Observe that $G(t, x, \tau, \xi) \leq 0$, $t, \tau > 0$, $x, \xi \in [0, L]$.

Note that by [20], we have the following equations:

$$\int_D \int_0^{\infty} (1 + \tau^m)G(t, x, \tau, \xi)d_\tau d_\xi$$

$$\leq \frac{1}{2\pi} \int_{|x-\xi|}^{t_\tau} (1 + \tau^m) \frac{d_\tau d_\xi}{\sqrt{(t - \tau)^2 - |x - \xi|^2}}$$

$$\leq \frac{1 + \tau^m}{2\pi} \left( \int_{|x-\xi|}^{t_\tau} \log t + \sqrt{r^2 - |x - \xi|^2} - \log |x - \xi| d_\xi \right)$$

$$\leq \frac{1 + \tau^m}{2\pi} \left( \log (2t) \int_{|x-\xi|}^{t_\tau} d_\xi - \int_{|x-\xi|}^{t_\tau} \log |x - \xi| d_\xi \right)$$

$$\leq \frac{1 + \tau^m}{2\pi} \left( \log (2t) - \int_{|x-\xi|}^{t_\tau} r_1 \log r_1 dr_1 \right)$$

$$\leq \frac{1 + \tau^m}{2\pi} \left( \pi r^2 \log (2r) - \pi \left( t^2 \log t + \frac{t^2}{2} \right) \right)$$

$$\leq \frac{1 + \tau^m}{2} \left( t^2 \log (1 + 2t) + t^2 \log |t| + \frac{t^2}{2} \right)$$

$$\leq \frac{1 + \tau^m}{2} \left( 2t^2 + t^2 |\log t| + \frac{t^2}{2} \right)$$

$$\leq (1 + \tau^m) \left( t^3 + t^2 (1 + |\log t|) \right), t \geq 0,$$

where $D = [0, L]$ and let $I(t) = t^3 + t^2 (1 + |\log t|), t \geq 0$. Therefore,

$$\int_D \int_0^{\infty} (1 + \tau^m)G(t, x, \tau, \xi)d_\tau d_\xi \leq (1 + \tau^m)I(t), (t, x) \in [0, \infty) \times [0, L].$$

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Now, making transform to \( u = v + u_0 + tu_1 \). This transform allows a certain additional conditions that used to derive the existence of solutions. Hence, we get the new initial boundary value problem. From the above transform, we have \( u_{tt} = v_{tt} + u_0 + tu_1u_{xx} = v_{xx} + u_0 + tu_1 \). Then, by substituting into \( u_{tt} - u_{xx} = f(t, x, u) \), we have the following equation:

\[
\begin{align*}
  u_{tt} - u_{xx} &= f(t, x, u), \\
  v_{tt} + u_0 + tu_1 - (v_{xx} + u_0 + tu_1) &= f(t, x, v + u_0 + tu_1), \\
  v_{tt} + u_0 + tu_1 - v_{xx} - u_0 - tu_1 &= f(t, x, v + u_0 + tu_1), \\
  v_{tt} - v_{xx} &= f(t, x, v + u_0 + tu_1) = f_1(t, x, v + u_0 + tu_1), t > 0, x \in [0, L], \\
  v(0, x) &= v_t(0, x) = 0, x \in [0, L], \\
  v(t, 0) &= v_x(t, L) = 0, t > 0.
\end{align*}
\]  

(12)

There exists a nonnegative function \( g \) such that \( g \in C[0, \infty) \times [0, L] \).

Suppose that \( v \in C^2[0, \infty) \times [0, L] \) is a solution to the integral equation

\[
0 = \frac{1}{8} \int_0^t \int_0^x (x - x_1)^2 (t - t_1)^2 \times g(t_1, x_1) v(t_1, x_1) dx_1 dt_1 - \frac{1}{16\pi} \int_0^t \int_0^x (x - x_1)^2 (t - t_1)^2 \times g(t_1, x_1) \int_0^L \int_0^L \int_0^\infty g(t_1, x_1, t_2, \xi_1, \xi_2) \times f_1(t_2, \xi_1, \xi_2, v(t_2, \xi_1, \xi_2) + u_0(\xi_1, \xi_2) + tu_1(\xi_1, \xi_2)) \\
\times dt_2 d\xi_2 d\xi_1 dx_1 dt_1, t \geq 0, x \in [0, L].
\]  

(13)

By differentiating trice in \( t \) and \( x \), (13) becomes

\[
0 = g(x, t)v(t, x) - \frac{1}{2\pi} g(t, x) \int_0^\infty G(t, x, \tau, \xi)f_1(\tau, \xi, v(\tau, \xi) + u_0(\xi) + tu_1(\xi)) d\tau d\xi,
\]  

(14)

\[
t \geq 0, x \in [0, L],
\]

where upon

\[
0 = v(t, x) - \frac{1}{2\pi} \int_0^\infty G(t, x, \tau, \xi)f_1(\tau, \xi, v(\tau, \xi) + u_0(\xi) + tu_1(\xi)) d\tau d\xi,
\]  

(15)

\[
t \geq 0, x \in [0, L].
\]

Therefore, as indicated above by using the Green function, we conclude that \( v \) is a solution of the initial boundary value problem (12). Thus, any solution \( v \in C^2(0, \infty) \times [0, L] \) of the initial boundary value problem (12)

Let \( E = C^2(0, \infty) \times [0, L] \) be endowed with the norm:

\[
\|u\| = \max \left\{ \|u\|_\infty, \|u_t\|_\infty, \|u_{tt}\|_\infty, \|u_{xx}\|_\infty, \|u_{ttt}\|_\infty, \|u_{xxx}\|_\infty \right\} = \max \left\{ \|u\|_\infty, \|u_t\|_\infty, \|u_{tt}\|_\infty, \|u_{xx}\|_\infty \right\}.
\]  

(16)
Provided it exists, where \( \|v\|_\infty = \sup_{(t,x)} |v(t,x)| \), \( (t,x) \in (0,\infty) \times [0, L] \).

For \( u \in E \), we define the operator as

\[
Fu(t,x) = \frac{1}{8} \int_0^t \int_0^x \text{sign}(x) \times (x - x_j)^2 (t - t_j)^2 \times g(t_j, x_j) v(t_j, x_j) dx_j dt_j
- \frac{1}{16 \pi} \int_0^t \int_0^x \text{sign}(x) (x - x_j)^2 (t - t_j)^2 \times g(t_j, x_j) \int_0^L \int_0^\infty G_{t_j, x_j, t_2, \xi_1, \xi_2} \times f_1(t_2, \xi_1, \xi_2, v(t_2, \xi_1, \xi_2) + u_0(\xi_1, \xi_2) + t_2 u_1(\xi_1, \xi_2))
\times dt_2 d\xi_2 \text{d}x_2 \text{d}t_2, (t,x) \in [0,\infty) \times [0, L].
\]

(17)

Here, sign function is derived, and the signum and piecewise function to define the integration problem verify the existence of solution whose value is defined by

\[
\text{sign}(x) = \begin{cases} 
1, & \text{for, } x > 0, \\
0, & \text{for, } x = 0, \\
-1, & \text{for, } x < 0.
\end{cases}
\]

(18)

Then, \( u \) solves the initial boundary value problem (7) which is more referred from [23].

**Lemma 1.** Suppose \( H_1 \) and \( H_2 \) are fulfilled. Then, for any \( v \in E, \|v\| \leq r \). Then, we have the following estimate:

\[
|f_1(t,x,v(t,x) + u_0(x) + tu_1(x))| \leq d \sum_{j=1}^l 2^{j+1} (1 + t^p) r^p, (t,x) \in [0,\infty) \times [0, L].
\]

(19)

**Proof.** To prove our estimation substituting \( v(t,x) + u_0(x) + tu_1(x) \) into \( \sum_{j=1}^l c_j(t,x)|u|^p \), then we have the following equations:

\[
\begin{align*}
|f_1(t,x,v(t,x) + u_0(x) + tu_1(x))| &\leq \sum_{j=1}^l c_j(t,x)|u|^p = \sum_{j=1}^l c_j(t,x)|v(t,x) + u_0(x) + tu_1(x)|^p \\
&\leq d \sum_{j=1}^l 2^{2p} \left( |v(t,x)^p| + |u_0(x)|^p + |u_1(x)|^p \right) \leq d \sum_{j=1}^l 2^{2p} (r^p + 1 + t^p)^p,
\end{align*}
\]

(20)

where \( p_j > 0, c_j(t,x) \in C[0,\infty) \times [0, L] \), and \( 0 \leq c_j(t,x) \leq d \), on \( [0,\infty) \times \times [0, L] \), \( j \in \{1, \ldots, l\} \) for some positive constant \( d \). This completes the proof.

**Lemma 2.** Suppose \( H_1 \) and \( H_2 \) are fulfilled. Then, for any \( v \in C^2([0,\infty) \times [0, L]) \), \( \forall \epsilon \in E \), \( \|v\| \leq r \) on \( [0,\infty) \times [0, L] \). We have

\[
\left| \int_D \int_0^\infty G(t,x,\tau,\xi)f_1(\tau,\xi,v(\tau,\xi) + u_0(\tau,\xi) + t\epsilon_1(\tau,\xi)) d\tau d\xi \right| \leq d \sum_{j=1}^l 2^{2p_j + 2r_p} (1 + t^p) \epsilon(\tau),
\]

(21)

\( (t,x) \in [0,\infty) \times [0, L] \).
Proof. Let \( v \in C([0, \infty) t \times n[q, 0, L]) \|v\| \leq r \) on \([0, \infty) \times [0, L] \). Then, by Lemma 1, we obtain the following equation:

\[
\int_D \int_0^\infty G(t, x, \tau, \xi) f_1 \left( \tau, \xi, v(\tau, \xi) + u_0(\tau, \xi) + \tau u_1(\tau, \xi) \right) d\tau \ d\xi \leq d \sum_{j=1}^l 2^{p_j+1} r_p \int_D \int_0^\infty G(t, x, \tau, \xi) (1 + r_p^j) d\tau \ d\xi.
\]

Now, applying (11), we get the required result as follows:

\[
\int_D \int_0^\infty G(t, x, \tau, \xi) f_1 \left( \tau, \xi, v(\tau, \xi) + u_0(\tau, \xi) + \tau u_1(\tau, \xi) \right) d\tau \ d\xi \leq d \sum_{j=1}^l 2^{p_j+1} r_p \int_D \int_0^\infty G(t, x, \tau, \xi) (1 + r_p^j) d\tau \ d\xi
\]

\[
\leq d \sum_{j=1}^l 2^{p_j+1} r_p (1 + t^m) I(t), (t, x) \in [0, \infty) \times [0, L].
\]

This completes the proof.

Lemma 3. Suppose \( H_1, H_2, \) and \( H_3 \) are fulfilled. Then, for any \( v \in E, \|v\| \leq r, \) we have \( \|Fv\| \leq A \) \( (r + d \sum_{j=1}^l 2^{p_j+2} r_p) \).

Proof: By applying Lemma 2, we obtain the following estimates:

\[
|Fv(t, x)| \leq r \int_D \int_0^\infty \text{sign}(x) \times (x - x_1)^2 (t - t_1)^2 \times g(t_1, x_1) dx_1 dt_1
\]

\[
+ d \sum_{j=1}^l 2^{p_j+2} r_p \int_D \int_0^\infty \text{sign}(x) \times (x - x_1)^2 (t - t_1)^2 g(t_1, x_1) I(t_1) (1 + r_p^j) dx_1 dt_1
\]

\[
\leq \left( r + d \sum_{j=1}^l 2^{p_j+2} r_p \right) w(t, x) \leq A \left( r + d \sum_{j=1}^l 2^{p_j+2} r_p \right), \quad (t, x) \in [0, \infty) \times [0, L],
\]

and
\[
\frac{\partial}{\partial t} Fu(t, x) \leq r \int_0^t \int_0^x \sigma(x) \times (x - x_1)^2 (t - t_1) \times g(t, x_1) dx_1 dt_1 \\
+ d \sum_{j=1}^I 2^{2p_j^2} r^{p_j} \int_0^t \int_0^x \sigma(x) \times (x - x_1)^2 (t - t_1) g(t, x_1) \\
\times I(t_1) \left(1 + t_1^{p_j}\right) dx_1 dt_1 \leq \left(r + d \sum_{j=1}^I 2^{2p_j^2} r^{p_j}\right) w(t, x) \\
\leq A \left(r + d \sum_{j=1}^I 2^{2p_j^2} r^{p_j}\right), (t, x) \in [0, \infty) \times [0, L],
\]

(26)

and

\[
\left| \frac{\partial^2}{\partial t^2} Fu(t, x) \right| \leq r \int_0^t \int_0^x \sigma(x) \times (x - x_1)^2 g(t, x_1) dx_1 dt_1 \\
+ d \sum_{j=1}^I 2^{2p_j^2} r^{p_j} \int_0^t \int_0^x \sigma(x) \times (x - x_1)^2 g(t_1, x_1) \\
\times I(t_1) \left(1 + t_1^{p_j}\right) dx_1 dt_1 \leq \left(r + d \sum_{j=1}^I 2^{2p_j^2} r^{p_j}\right) w(t, x) \\
\leq A \left(r + d \sum_{j=1}^I 2^{2p_j^2} r^{p_j}\right), (t, x) \in [0, \infty) \times [0, L],
\]

(27)

and

\[
\left| \frac{\partial}{\partial x} Fu(t, x) \right| \leq r \int_0^t \int_0^x \sigma(x) \times (x - x_1) |(x - x_1)| (t - t_1)^2 \times g(t, x_1) dx_1 dt_1 \\
+ d \sum_{j=1}^I 2^{2p_j^2} r^{p_j} \int_0^t \int_0^x \sigma(x) \times (x - x_1) |(x - x_1)| (t - t_1)^2 g(t, x_1) \\
\times I(t_1) \left(1 + t_1^{p_j}\right) dx_1 dt_1 \leq \left(r + d \sum_{j=1}^I 2^{2p_j^2} r^{p_j}\right) w(t, x) \\
\leq A \left(r + d \sum_{j=1}^I 2^{2p_j^2} r^{p_j}\right), (t, x) \in [0, \infty) \times [0, L],
\]

(28)

and
\[ \left| \frac{\partial^2}{\partial x^2} Fu(t, x) \right| \leq r \int_0^x \int_0^x \text{sign}(x) \times (t-t_1)^2 g(t_1, x_1) dx_1 dt_1 + d \sum_{j=1}^l 2^p r_j^2 r_p \int_0^t \int_0^x \text{sign}(x) \times (t-t_1)^2 g(t_1, x_1) \times I(t_1)(1 + r_j^p) dx_1 dt_1 \]

\[ \leq \left( r + d \sum_{j=1}^l 2^p r_j^2 r_p \right) w(t, x) \leq A \left( r + d \sum_{j=1}^l 2^p r_j^2 r_p \right), \quad (t, x) \in [0, \infty) \times [0, L]. \]

Hence, it follows that \( \| Fv \| \leq A (r + d \sum_{j=1}^l 2^p r_j^2 r_p) \). This completes the proof.

For any \( v \in E \), where \( E = C^2 ([0, \infty) \times (0, L]) \) define the operators \( T(v(t, x) = (1 - \epsilon)v(t, x) \) \( S(v(t, x) = \epsilon v(t, x) + \epsilon Fv(t, x), (t, x) \in [0, \infty) \times [0, L] \).

Note that, any fixed point of the operator \( T + S \) is a solution of the integral equation of \( (11) \). Therefore, any fixed point \( v \in E \) of the operator \( T + S \) is a solution of the integral boundary value problem \( (12) \).

Let \( k > 1 \) be an arbitrary set and \( R = r + \left( r + d \sum_{j=1}^l 2^p r_j^2 r_p \right) \) \( w(t, x) = r + A (r + d \sum_{j=1}^l 2^p r_j^2 r_p) \). By choosing \( \epsilon \in (0, 2) \) close enough to 2 such that

\[ R < \frac{r \epsilon}{8k(2 - \epsilon)} f. \quad (30) \]

Define \( \tilde{P} = \{ v \in E: \| v(t, x) \| \leq r, (t, x) \in [0, \infty) \times [0, L] \} \). Let \( \tilde{P} \) be the set of all equicontinuous families in \( \tilde{P} \) (an example for an equicontinuous family in \( P \) is the family \( \{(3 + \sin(t+n))(3 + \cos(x+n)), t \in [0, \infty) \times [0, L] \} \). Let also \( \Omega = \{ v \in \tilde{P}: \| v \| \leq r, \inf v(t, x) \geq 1/k(\| v \|), (t, x) \in [0, \infty) \times [0, L] \} \).

\( (1) \) For any \( v \in \Omega \), by using Lemma 2, we have \( \epsilon/2 \| v \| \leq \| (I - T)v \| = \| v \| \leq 2 \| v \| \). Then, \( I - T: \)

\[ r + A \left( r + d \sum_{j=1}^l 2^p r_j^2 r_p \right) \geq \lambda r \]

\( \lambda \geq \lambda r = \| v \| \geq \| sv \| - (1 - \epsilon) \| v \| \]

\[ \geq \frac{8 \epsilon^2}{2} \int_0^1 \int_0^1 (4 - x_1)^2 (4 - t)^2 g(t, x_1) \times (I)(1 - \epsilon) \| v \| dx_1 dt_1 - (1 - \epsilon) \lambda r \]

\[ \geq \frac{8 \epsilon^2}{2} \int_0^1 \int_0^1 (4 - x_1)^2 (4 - t)^2 g(t, x_1) \times (I)(1 - \epsilon) \| v \| dx_1 dt_1 - (1 - \epsilon) \lambda r \]

\[ \geq \frac{r \epsilon}{8k} \int_0^1 \int_0^1 g(t, x_1) dx_1 dt_1 - (1 - \epsilon) \lambda r = \frac{r \epsilon}{8k} f - (1 - \epsilon) \lambda r. \]

We get \( (2 - \epsilon) \lambda r \geq r/8k \epsilon f \) and \( (2 - \epsilon)(r + A(r + d \sum_{j=1}^l 2^p r_j^2 r_p)) \geq r/8k \epsilon f \) implies \( (2 - \epsilon)R \geq r/8k \epsilon f \). Thus, \( R \geq r \epsilon / 8k (2 - \epsilon) f \) which contradicts with \( (30) \).

\( \Omega \longrightarrow E \) is Lipschitz invertible with a constant \( \epsilon (1/2k, 2) \).

\( (2) \) Again for any \( v \in \tilde{U} \), by Lemma 3, we have \( \| sv \| \leq R + r + d \sum_{j=1}^l 2^p r_j^2 r_p \| w(t, x) \| \leq R + A (r + d \sum_{j=1}^l 2^p r_j^2 r_p) \leq \epsilon R \). Therefore, \( S\tilde{U} \) is uniformly bounded since \( S: \tilde{U} \longrightarrow E \) is a continuous operator, and we have that \( \tilde{S} \) is equicontinuous and \( \tilde{S}: \tilde{U} \longrightarrow E \) is relatively compact and relatively closed and bounded. Therefore, \( S: \tilde{U} \longrightarrow E \) is a 0-set contraction.

\( (3) \) Let \( v \in \tilde{U} \), be arbitrarily chosen taken as \( z = sv/\epsilon \), we have \( z \in \tilde{P} \) and \( \| z \| = \| sv \| / \epsilon \leq r + A (r + d \sum_{j=1}^l 2^p r_j^2 r_p) = R \). Then, \( z \in \tilde{U} \) since \( \tilde{U}: \tilde{U} \longrightarrow \tilde{U} \) is a contraction mapping. Hence, there exists a unique \( z \in \tilde{U} \) such that \( z = Tz = sv \). Therefore, \( S\tilde{U} \subset (I - T)(\tilde{U}) \).

\( (4) \) Assume that there are \( v \in \partial \tilde{U} \) and \( \lambda \geq 1 \) such that \( Sv = (I - T) \lambda v \) and \( \lambda v \in \tilde{P} \). Since we have \( v \in \partial \tilde{U} \) and \( \lambda v \in \tilde{P} \), then we have \( \| v \| = r \) and \( \lambda r = \| v \| \leq r - A (r + d \sum_{j=1}^l 2^p r_j^2 r_p) \). Thus, \( \lambda \leq r + A (r + d \sum_{j=1}^l 2^p r_j^2 r_p) \). This implies \( \lambda \leq R/r \). Next, \( \lambda v = sv + T(\lambda v) = sv + (1 - \epsilon) \lambda v \).

Hence,

\[ \Omega \longrightarrow E \] is Lipschitz invertible with a constant \( \epsilon (1/2k, 2) \).

To do generalization depending on the above discussion and Proposition 3.1, we conclude that the operator \( T + S \) has a fixed point in \( U \). This completes the proof of the main result.
3.2. Example. Consider the following initial boundary value problem to illustrate the obtained result. \( u_t - u_{xx} = |u|^p \), where \( t \geq 0, x \in [0, 1], u(0, x) = (1/25)x(1 - x^2), u(x, 0) = 1/100x(x - 1), u(1, 0) = 0 \), \( t \geq 0 \), where, \( p > 1 \), then we have that \( f(t, x, u) = |u|^p \). Here, taking \( d = r = (1/10) \), \( L = 1 = 1 \), \( C_1(t, x) = 1 \), \( t \in [0, \infty) \times [0, 1] \), \( u(0, x) = u_0(x) = 1/25x(x - 1)^2 \), \( u_1(x) \) is \( 1/100x(1 - x^2), x \in [0, 1] \). Then, \( 0 \leq u_0(x) < r, 0 \leq u_1(x) < r, x \in [0, 1] \), \( u_0(0) = u_1(0) = u_{12}(1) = 0 \) and \( f \in C[0, \infty) \times [0, 1] \times \mathbb{R} \) that means \( H_1 \) and \( H_2 \) hold true.

For instance, by taking any number from the closed bounded domain of \([0, 1]\) and inserting it into the equation of \( u_0(x) = 1/25x(1 - x^2) \) and \( u_1(x) = 1/100x(1 - x^2) \).

Then, we obtain the following results: for \( x = 0 \), \( 1/25 \times 0(1 - 0^2) = 0/25(1 - 0) = 0/25 = 0 \), this implies that the initial condition is zero. For \( x = 1 \), \( 1/25 \times 1(1 - 1^2) = 1/25(1 - 1) = 0 \), which implies that boundary conditions are zero. i.e. \( u_0(0) = u_2(1) = 0 \) and \( u_0(0) = u_0(0) = 0 \).

Again, solving in a similar manner that in \( u_1(x) = 1/100x(1 - x^2) \), we get the following results:

For \( x = 0 \), \( 1/100 \times 0(1 - 0^2) = 0 \) which implies the initial condition is zero. For \( x = 1 \), \( 1/100 \times 1(1 - 1^2) = 0.00234 \). For \( x = 1, \) \( 1/100 \times 1(1 - 1^2) = 0.0029 \). For \( x = 1 \), \( 1/100 \times 1(1 - 1^2) = 0.00375 \). For \( x = 1 \), \( 1/100 \times 1(1 - 1^2) = 0 \), this implies that boundary conditions are zero. i.e. \( u_0(0) = u_2(1) = 0 \) and \( u_0(0) = u_0(0) = 0 \).

This result indicates that when we verify with any number in the interval of \([0, 1]\), the obtained result should be \( 0 \leq u_0(x) \leq r \) which proves that \( 0 \leq u_0(x) < r \) and \( 0 \leq u_1(x) < r \). Hence, \( u_0(0) = u_0(0) = u_{12}(1) = 0 \); \( u_0(0) = u_{12}(0) = u_{12}(0) = 0 \). Therefore, \( H_1 \) and \( H_2 \) hold true.

Now, we are going to construct a nonnegative function \( g \) so that \((13)\) holds true. Let \( h(t) = \log_1(1 + t^5) \), \( t \geq 0 \), \( l(t) = \arctan(t^2/1 - t^0) \). Then, \( d/\sqrt{1 - t^0} = \arctan(t^2/1 - t^0) \).

Now, we take \( g(t, x) = 1/2a_{20}g_1(t, x) \), \( (t, x) \in [0, \infty) \times [0, 1] \). We have that there exists a constant \( A \in [1/2, 1) \) such that

\[
C_2 \geq \int_0^1 \int_0^x \left(\frac{\varphi^2}{1 + \left(\frac{\varphi^2}{1 + t^0}\right)^{1/2}}\right) \left(1 + t^0\right)^{1/2} dx_1 dt_1 \quad \left(\frac{x}{1 + \left(\frac{x}{1 + t^0}\right)^{1/2}}\right)
\]

\[
1 \leq \frac{1}{2} \int_0^1 \int_0^x \varphi^2 \left(\frac{x}{1 + \left(\frac{x}{1 + t^0}\right)^{1/2}}\right) dt_1 \quad \left(\frac{x}{1 + \left(\frac{x}{1 + t^0}\right)^{1/2}}\right)
\]
Also, \( J = \int_0^3 \int_0^3 g(t, x, \hat{x}) dx dt > 0 \). Therefore, \( H_3 \) holds. Consequently, we have that \( H_1, H_2, \) and \( H_3 \) hold. Hence, by Theorem 3.1, it follows that the initial boundary value problem (7) has at least one nonnegative solution \( u \in C^2([0, \infty) \times n_[q, 0, L]) \).

4. Discussion

In this article, we used the method of fixed point approach to solve the existence of nonnegative solutions for a class of initial boundary value problem for nonlinear one-dimensional wave equation. The general form of existence of nonnegative solutions for (7) is restricted on the interval of \( t \in [0, \infty) \) and \( x \in [0, L] \) with initial and boundary data verified by applying the method of fixed point approach. Also, we discussed contraction mapping principle which is a very useful Banach’s fixed point theorem and mathematical tool that is used to prove the existence and nonnegative solutions for (7). For instance, the initial boundary data and the function \( f \) assumed by (8) in order to satisfy (7) with its initial and boundary data are proved and discussed under section 3 as well as applying the operators discussed under subsections 3 which expresses for every \( u \in E \), where \( E = C^2([0, \infty) \times n_[q, 0, L]) \) is proved from (24)–(29), and the fixed point of the operator discussed under subsections 3 is a solution of the integral equation (11). Hence, any fixed point \( v \in E \) of the operator \( T+S \) is a solution of the initial boundary value problem (12), and the operator \( T+S \) has a fixed point in \( U \). In general, to sum up this, the general solutions of (7) are proved by applying the definite method of fixed point approach, and certain examples with analytical method subjected with initial and boundary data on the interval of \( t \in [0, \infty) \) and \( x \in [0, 1] \) are analyzed.

5. Conclusion

The main finding of this article is to show nonnegative solutions for a class of initial boundary value problems for nonlinear one-dimensional wave equations. So in this article, we studied nonlinear wave equations specifically concerning the existence of nonnegative solutions for a class of initial boundary value problems of nonlinear one-dimensional wave equations. The existence of nonnegative solutions for a class of initial boundary value problems for nonlinear one-dimensional wave equations was studied and proved by applying a very effective and powerful mathematical method tool called via fixed point approach. Therefore, the existence of nonnegative solutions for a class of initial boundary value problems for nonlinear one-dimensional wave equations was presented in this research supported by practical examples. In general, in this article, we studied the existence of nonnegative solutions by applying the new fixed point theory which is a very interesting and effective method as well as the easiest and a definitive modeling method to find the existence and nonnegative solutions of nonlinear partial differential equations. In this article, we have shown the existence of nonnegative solutions for a class of initial boundary value problems for nonlinear one-dimensional wave equations using a new fixed point approach, and it solves the problems of determining nonlinear PDEs solutions. We recommend for further work the existence of nonnegative solutions for a class of IVP for nonlinear two-dimensional wave equations using a new fixed point approach.

Data Availability

No data are used in study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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