Quantum Hydrodynamics of Fractional Hall Effect:
Quantum Kirchhoff Equations

P. Wiegmann
Department of Physics, University of Chicago, 929 57th St, Chicago, IL 60637
(Dated: May 5, 2014)

We argue that flows of the quantum electronic liquid in the Fractional Quantum Hall state are
descriptively comprehensively by the hydrodynamics of vortices in the quantum incompressible rotating
liquid. We obtain the quantum hydrodynamics of vortex flow by quantizing Kirchhoff equations for
vortex dynamics. We demonstrate that quantized Kirchhoff equations capture all major features of
FQH states including subtle effects of Lorentz shear force, magneto-roton spectrum, Hall current
in a non-uniform electromagnetic field, thus providing a powerful framework to study FQHE and
superfluids.

PACS numbers: 73.43.Cd, 73.43.Lp

1. Introduction. In Fractional Quantum Hall (FQH)
regime electrons form a peculiar quantum liquid. Characteristic features of the liquid are: flows are incompressible [1], almost dissipation-free [2, 3], the Hall conduc-
tance is quantized [2], vortices are elementary excitations, vortices carry fractionally quantized negative electronic charges [1], they are separated from the ground state by a gap [3, 4]. More subtle features are the universal an-
harmonic term of the structure factor and magneto-roton minimum in excitation spectrum [4] (see (24)), quantized
double layers of a density and shear at boundaries and vortices [5], the Lorentz shear force [6].

Such liquid does not posses linear gapless waves except edge modes propagating along the boundary. Only available
bulk flow is a non-linear flow of vorticity. Since in FQH regime vorticity is linked to the electronic charge,
flows of vorticity are related to a charge flow [25].

A natural approach to study flows of FQH states is
hydrodynamics. Not all quantum liquids are subject of quantum hydrodynamics. Quantum hydrodynamics is
based on a fundamentally restrictive assumption that all quantum states are fully characterized by the density
and velocity. Apart from of superfluids (and superconductors) and Luttinger liquids, FQH liquid is, yet, another
important case.

Hydrodynamics of quantum fluids goes back to Landau
[1] and Feynman [8], but remains in its infancy. A quest
for the hydrodynamics of FQH liquid has been originated
by a seminal paper [4], further discussed in [9–12], it is a
focus of a renewed interest.

As always in hydrodynamics only a few basic principals,
symmetries, and a few phenomenological parameters are sufficient to formulate fundamental equations.
To this aim an underlying microscopic Hamiltonian describing emergent FQH states under a strong Coulomb
interaction is in fact not necessary.

In the letter we attempt to formulate these principals and
to develop hydrodynamics of FQH bulk states in a
close analog of Feynman theory of superfluid helium [8] and magneto-roton theory of collective excitations by [4].

We argue that flows of FQH liquid are equivalent to
flows of vortices in the quantum incompressible rotating
Euler liquid. Based on this conjecture we obtain the major features of FQHE including subtle effects of Lorentz shear force [2] and magneto-roton spectrum [4] missed by the previous hydrodynamics approaches [9–12].

Hydrodynamics of vortex flows (classical and quantum alike) is an interesting subject by its own. Apart from
FQHE it is also relevant to the theory of superfluids and classical hydrodynamics.

Quantization of hydrodynamics is a subtle matter. It is best achieved through quantization of Kirchhoff
equations [13]. Classical Kirchhoff equations describe a motion of vortices in a 2D incompressible isentropic
fluid. We show that, quite remarkable, quantum Kirchhoff equations capture all known features of FQH liquid.
They can be used as a platform for studying FQH Effect.

In this paper we consider only the simplest Laughlin's
cases where fraction ν is an inverse of an odd integer,
say 1/3. In these cases electronic liquids do not posses additional symmetries. Extension to other FQH states,
not discussed here, is possible and interesting.

We start by reminding classical Kirchhoff equations for
rotating incompressible Euler flows (see e.g., [14]).

2. Classical Kirchhoff equations. 2D incompressible isen-
tropic flow (that is a flow where gradient of density are
orthogonal to gradients of pressure, or zero) is fully charac-
terized by its vorticity. Vorticity obeys a single (pseudo)
scalar equation, which in the case of inviscid fluid has a
simple geometrical meaning: the material derivative of
the vorticity vanishes

\[ \nabla \cdot \mathbf{u} = 0 \]  

In other words vorticity \( \varpi = \nabla \times \mathbf{u} \) is transported along divergent-free velocity field \( \mathbf{u} \)

\[ \nabla \cdot \mathbf{u} = 0. \]  

Helmholtz, and later Kirchhoff realized that there is a
class of solutions of the vorticity equation (1) which
consists of a finite number of point-like vortices mov-
ing by Magnus forces. In this case the complex velocity
3. Chiral flow. The flow relevant for FQHE is the chiral flow of vortices. We denote the positions of vortices and use the roman script for complex vectors $a = x - i y$.

A substitution this "pole Ansatz" into the Euler equation yields that the number of vortices $N$ and the circulation $\Gamma$ do not change in time, but moving position of vortices $z_i(t)$ obey the Kirchhoff equations.

If the rotation of fluid is very strong, vortices prefer to be of the same sign opposite to the direction of the rotation. Bearing in mind the quantum case we assume that vortices have the same (minimal) circulation $\Gamma$. Then Kirchhoff equations obtained by localizing the Euler equation to vortex cores at $z = z_i$ expresses velocities $v_i$ of vortex cores through their positions

$$v_i \equiv \dot{z}_i = -i\Omega \xi_i + i \sum_{i \neq j} \frac{\Gamma}{z_i(t) - z_j(t)}.$$  \hspace{1cm} (4)

The Kirchhoff equations replace the non-linear PDE by the dynamical system, reflecting integrability of the Euler flow. They can be used for different aims. Equations describe chaotic motions of a finite number of vortices if $N > 3$. Alternatively they can be used to approximate virtually any flow if $N \to \infty$.

3. Chiral flow. The flow relevant for FQHE is the chiral flow, where a large number of background vortices uniformly distributed with the mean density $\rho = \Omega/(\pi \Gamma)$ largely compensates rotation. This is a very special flow in fluid mechanics. There we distinguish two types of motion: fast motion of the fluid around vortex cores, and a slow motion of vortices. In this respect vortices themselves must be considered as a (secondary) fluid. At the ground state of chiral flow vortices do not move, but the fluid does. Vortices carry an inertia $m_* - a$ phenomenological parameter determined by the equation of state. It is not directly related to the inertia of fluid.

In this paper we propose to model FQHE by a quantized slow flow of vortex fluid.

4. Quantization of Kirchhoff equations. Quantization of the Kirchhoff equations consists of three steps: canonical quantization, a choice of the representation and the inner product.

It is convenient to use a circulation $q = m_* \Gamma$ of momenta of vortices $m_* v_i$. The Poisson brackets followed from canonical Hamiltonian structure of the hydrodynamics. They are equal to the volume per particle per circulation $\{x_i, y_j\}_{P.B.} = 1/(2\pi \rho)$. We replace them by commutators

$$\{\xi_i, \xi_j\}_{P.B.} \to [\xi_i, \xi_j] = 2\ell^2 \delta_{ij}.$$  \hspace{1cm} (5)

Here $2\ell^2 = \nu/(\pi \rho)$ has a dimension of area. Dimensionless number $\nu = h/q$ is a semiclassical parameter.

The next step is a choice of states. We assume that states are holomorphic polynomials of $z_i$. Then operators $\xi_i$ are canonical momenta

$$\xi_i = 2\ell^2 \partial_{z_i}.$$  \hspace{1cm} (6)

The third step is to specify the inner product. We impose the chiral condition: operators $\xi_i$ and $\xi_i$ are assumed to be Hermitian conjugated

$$\text{chiral condition: } \xi_i = \xi_i^\dagger.$$  \hspace{1cm} (7)

This condition combined with representation identifies a set of states with the Bargmann space: the Hilbert space of analytic polynomials $\psi(z_1, \ldots, z_N)$ with the inner product $\langle \psi | \psi \rangle = \int d\mu \psi^* \psi$. A common solution of the set of 1st order PDEs is the holomorphic part of the Laughlin w.f.

Eqs. (10) help to write velocity operators of vortices

$$\Gamma^{-1} v_i = -i2\nu \partial_{z_i} + i \sum_{i \neq j} \frac{1}{z_i - z_j}.$$  \hspace{1cm} (9)

Eqs. (10) are quantum chiral Kirchhoff equations. They are readily generalized to a sphere, or a torus.

5. Quantum Chiral Kirchhoff Equations and FQHE. We identify quantum chiral Kirchhoff equations with FQHE.

First we comment that Bargmann space of analytic polynomials is another way to say that all states belong to the left to anti-holomorphic operators. Eqs. (10) help to write velocity operators of vortices

$$\psi_0 = \prod_{i > j} (z_i - z_j)^\beta, \quad \hbar \beta = q.$$  \hspace{1cm} (10)

The w.f. becomes single valued if $\beta$ is integer, antisymmetric if $\beta$ is an odd-integer.

In this interpretations vortices are identified with "particles" entered into the Laughlin function. Hence a quasi-hole $\psi_h = \prod_{i = 1} (z_i - z_i) \psi_0$ is a hole in the uniform background of vortices - an anti-vortex. Thus we assign the electronic charge to vortices and identify angular velocity with the cyclotron frequency of vortices $\Omega = eB/(m_* c)$. Then the entries of the Kirchhoff equations are the magnetic length $\ell = \sqrt{\hbar c/eB}$ and the filling fraction $\nu = \hbar c/eB = h/q$. (\Gamma \equiv \Omega/(\pi \rho)).

The phenomenological parameter $m_*$ is the inertia of the vortex. It is naturally to assume that the energy
associated with the inertia $\Delta_\nu = h\Omega = \hbar^2/(\ell^2 m_*).$ It is of the same order as the energy of a quasi-hole at rest, or equivalently, is a gap in the excitation spectrum. The latter is of the order of Coulomb energy $e^2/\ell.$ It is known experimentally $\Delta_\nu \sim 10K \rho.$ The very existence of the FQH state requires that this energy scale must be less than the cyclotron frequency $h\omega_c \sim 25meV \gg \hbar\Omega,$ or that $m_*$ exceeds the bare electronic mass $m_e \gg m_\nu.$ An assumption that the energy of the quasi-hole at rest is the same order as the energy of a quasi-hole at rest, or $\Delta_\nu \sim \hbar^2/(\ell^2 m_\nu),$ is often made [26]. From this point of view the density of the liquid with a constituency relation between the flux and the mass flux of the fluid is easy obtain. It is given by $\rho = \rho(u,v) = \rho_\nu u,$ where $u$ and $v$ are coordinates of vortices, not fluid particles. It yields the Lorentz force $\vec{F}_\nu = e\vec{v} \times \vec{B}$ and the identity $2\sum_{i\neq j} \frac{1}{2\pi^2} \int \frac{1}{z - z_i} \delta(z - z_j) = (\frac{1}{z - z_i} + \frac{1}{z - z_j})^2.$

The velocity of the fluid is easy obtain. It is given by \[ \frac{\partial \rho}{\partial t} + \vec{J} = \vec{F}, \] where $\rho$ is the density of the fluid and $\vec{J}$ is the mass flux of the fluid. Using the notation $P_\nu = \{\rho, v\}$ we use: the $\vec{J}$-formula $\vec{J} = \vec{F}$ and the identity $2\sum_{i\neq j} \frac{1}{2\pi^2} \int \frac{1}{z - z_i} \delta(z - z_j) = (\frac{1}{z - z_i} + \frac{1}{z - z_j})^2.$

We obtain the important relation between flux of the vortex flow and the flux of the fluid. Using the notation $a^\mu_\nu = \epsilon_{\mu\alpha\nu\rho} v^\alpha / 2$ for 2-vectors we obtain \[ \vec{J} = \vec{F} + \frac{q}{2} \nabla^\times \rho, \quad \vec{v} = \vec{u} + \frac{\Gamma}{4} \rho^{-1} \nabla^\times \rho. \] The shift (14) holds in the classical and the quantum cases. It has far reaching consequences [27].

The shift (13) can be seen as a similarity transformation. It preserves the volume. Hence the flow of vortices is incompressible $\nabla \cdot \vec{v} = 0$ like the fluid itself. However, velocities of vortices are finite. The shift removes that singularity.

Further meaning of the shift is seen from monodromy of the wave function. Monodromy is the circulation of each particle in units of $\Gamma.$ It is equal to the number of magnetic flux quantum piercing the system $N_\phi.$ The circulation of a particle (a vortex) around the system of remaining $N - 1$ vortices is $\Gamma(N - 1) = 2\pi \sum \vec{u} \cdot \vec{dr}.$ The monodromy is $N_\phi = \beta(N - 1),$ i.e., the number of zeros of the w.f. with respect to each coordinate. On the other hand the circulation $2\pi \sum \vec{u} \cdot \vec{dr}$ gives the total charge $\Gamma N.$ The shift amounts for the difference. It simply means that a vortex does not interfere with itself. Eq. (14) can be seen as a local version of the global condition $N_\phi = \beta N - 2\pi,$ where the shift $h\vec{s} = q/2\rho$ [15].

We summarize the formulas for flux and velocity

\[ P = \frac{1}{2} \{\rho, \nabla\pi_\nu\} - \frac{q}{2} \rho \nabla^\times \varphi + \frac{q}{4} \nabla^\times \rho, \] \[ m_* \vec{v} = \nabla\pi_\nu - \frac{q}{2} \nabla^\times \varphi + \frac{q}{4} \nabla^\times \log \rho. \]
Potential $\varphi$ obeys the Poisson equation $\Delta \varphi = -4\pi (\rho - \bar{\rho})$. It is chosen such that flux vanishes at the ground state.

7. Chiral constituency relation. The next step is to express the flux of the vortices in terms of their density.

Using the formula $[\rho, \partial \pi_\rho] = -i\hbar \partial \rho$ and the chiral relation $2\ell^2 \partial_\pi^1 + z_i$, we write

$$\{\rho, \partial \pi_\rho\} = -2\partial^T \rho + [\rho, \partial \pi_\rho] = -i\hbar (\partial + \frac{\delta}{2\ell^2} \rho).$$

Applying this formula to (15) we obtain the chiral constitutency relation. We write it in two suggestive forms

$$P = -\rho \nabla^* \Psi, \quad \Psi = \frac{\eta}{2} [\varphi - \left(\frac{1}{2} - \nu\right) \log \rho],$$

$$P = \frac{i}{\pi} \partial T, \quad T = \frac{1}{2} (\partial \varphi)^2 - \left(\frac{1}{2} - \nu\right) \partial^2 \varphi. \quad (17)$$

The field $\Psi$ has a meaning of the stream function, hence vorticity is

$$\omega = -\Delta \Psi = 2\pi q [\rho - \bar{\rho} + \frac{1}{4\pi} \left(\frac{1}{2} - \nu\right) \Delta \log \rho], \quad (19)$$

We observe that vorticity of the chiral flow differs from the density of vortices by the term $\Delta \log \rho$. The coefficient comprises of the shift and quantum correction. The source of quantum corrections is the difference of the vorticity of the fluid $\varpi = 2\pi q [\rho - \bar{\rho} - \frac{1}{4\pi} \Delta \log \rho]$ from the density of vortices.

Integration of (17) gives the global version of the chiral condition. It connects the moment of inertia $L = \int (r \times P) d^2 r$ and the gyration $G = \int \rho (\rho - \bar{\rho}) d^2 r$ of the flow:

$$\ell^2 L = h G + N\ell^2 h (\beta - 2)$$

This form generalizes the exact sum rule of the Laughlin state at $P = 0$: $\sum [0| z_i |2^2 |0] = N\ell^2 (N - (\beta - 2))$.

Eq. (17) can be used to find density profiles for various coherent states. Consider e.g., a quasi-hole $\psi_h = \prod_i (z_0 - z_i) \psi_0$. This state describes an anti-vortex (a hole in the sea of vortices). Its flux is $P = \frac{m_e - e}{2\pi}$. Eq (17) expresses its density through the 2-points function. Eq. (17) computes the change of the gyration by a quasi-hole: $N^{-1} \int \rho \delta \rho d^2 r = -\ell^2 = -\frac{\mathcal{E}}{\pi \eta}$. This result often interpreted as a fractional charge of the quasi-hole - it occupies a fraction of volume per particle. Outside of the core, the quasi-hole as a source for vortex in a classical equation

$$-\delta (r - r_0) = \beta [\rho - \bar{\rho} + \frac{1}{4\pi} \left(\frac{1}{2} - \nu\right) \Delta \log \rho].$$

8. Governing equation for the vortex flow. Since the density of vortices determines its flux, the continuity equation $m_e \dot{\rho} + \nabla \cdot P = 0$ is the only governing equation of the chiral flow. Eq (17) provides the close form

$$\dot{\rho} + \frac{\Gamma}{2} \nabla \varphi \times \nabla \rho = 0.$$
The conservation law for the fluid flux \( J \) to FQHE in [20, 21], the Lorentz shear force is in fact, introduced for the integer QHE in [19] and extended the viscous tensor, which we now compute. Initially being introduced for the integer QHE in [19] and extended to FQHE in [21], the Lorentz shear force is in fact, the classical phenomena. It is a feature of vortex flow.

We cast hydrodynamics equations in the form of the conservation law with the Lorentz force \( F = eE - Bv^* \)

\[
\partial_t \rho + \nabla \cdot \rho \Pi = 0.
\]

and compute the momentum flux \( \Pi \).

We start from the first Hamiltonian structure writing conservation law for the fluid flux \( J : \partial_t J_p + \nabla \cdot \Pi = 0 \). In this case the canonical property of the first structure [22] determines form of the momentum flux: \( \Pi = J_p (m, \rho)^{-1} J_p + \rho \delta_{\mu \nu} \), while incompressibility condition determines the intrinsic pressure \( \tilde{p} \) (all operator products are normally ordered).

We obtain the momentum flux of vortices \( \Pi_{\mu \nu} \) by the similarity transformation [14]. Taking time derivative of (14) and using the continuity equation, we write \( \tilde{J}_\mu - \tilde{P}_\mu = \frac{q}{3} \nabla \cdot (v \cdot \nabla) \rho = \nabla \cdot (\Pi_{\mu \nu} - \Pi_{\nu \mu}) \). This gives the transformation of the momentum flux \( \Pi_{\mu \nu} \rightarrow \Pi_{\mu \nu} = \Pi_{\mu \nu} - \Pi_{\nu \mu} \). and the stress tensor \( \sigma_{\mu \nu} \)

\[
\sigma_{\mu \nu} = -\rho \delta_{\mu \nu} - \frac{q}{3} \nabla \cdot (v \cdot \nabla) \rho + \sigma'_{\mu \nu},
\]

\[
\sigma'_{\mu \nu} = -\frac{q}{3} \rho (\nabla_v v^* + \nabla^* v_{\mu} + \nabla^* v_{\nu}).
\]

The viscous part of the stress tensor, \( \sigma'_{11} = -\sigma'_{22} = -\frac{q}{3} \rho (\nabla_x v_y + \nabla_y v_x) \), \( \sigma'_{12} = \sigma'_{21} = -\frac{q}{3} \rho (\nabla_x v_x - \nabla_y v_y) \) is the Lorentz shear force. It is traceless, hence conservative.

The effect can be interpreted in terms of semiclassical motion of electrons. A motion of electrons consists of a fast motion along small orbits and a slow motion of orbits. A shear flow strains orbits elongating them normal to the shear, boundaries and vortices. Elongation yields an addition to the Lorentz, the Lorentz shear force which acts normal to the shear and proportional to the shear.

An important consequence of the Lorentz shear force is accumulation of charges on boundaries and vortices - overshoots. They govern dynamics of edge modes [5].

12. Hall current. Another consequence of the Lorentz shear force is the increase of the Hall current by a non-uniform e.m. fields [22]. It follows from the conservation law [23] that in the linear approximation the Lorentz force equilibrates the Lorentz shear force \( eE - \frac{\pi}{2} Bv^* \approx \frac{1}{4} \Delta v^* \approx \frac{e}{4} E/B \). As the result the Hall current increases. In a uniform magnetic field \( j_{\sigma} = \sigma_{xy} (1 + \frac{1}{4} (k\ell)^2) E_q \).

So far we neglected diamagnetic effects by counting the energy from the ground state. The ground state energy depends on the magnetic field as \( -\frac{1}{4} M h \omega_c \tilde{p} \), where \( M \) is an orbital moment per particle. In a non-uniform density it yields the diamagnetic current \( j_{dia} = -\frac{|e| h}{m_e} M \nabla \rho \). The diamagnetic currents gives a similar contribution to the Hall conductance as the Lorentz shear force. Indeed, the density of the chiral flow is determined by the flux according to the Eq. (17). In the leading order \( m_s \Delta v \approx 2 \pi q \nabla^* \rho \approx m \Delta (eE/B) \), hence \( j_{dia} \approx -\sigma_{xy} \frac{m_s}{m_e} M \ell^2 \Delta E^* \). Thus diamagnetic current merely changes \( \frac{1}{4} \Delta \) to \( \frac{1}{4} + \frac{m_s}{m_e} M \) in the formula for the current. This result has been obtained in [22] where \( m_s \) was set to be equal to bare electronic mass \( m_e \). A factor \( m_s/m_e \) offers an avenue to obtain the inertia \( m_s \) by measuring of the increase of the Hall current by a non-uniform electric or magnetic fields. Comparing it with an independently measured gap allows to check Feynman-Bijl formula [24].

**Acknowledgement.** This study started as a common project with A. G. Abanov. It turned out that our methods and results were complimentary, that we decided to present them in separate publications, see [22]. Discussions of hydrodynamics of quantum liquids with I. Ruskikh, E. Bettelheim and T. Can and their help are acknowledged. The author thanks the International Institute of Physics (Brazil) and Weizmann Institute of Science for the hospitality during the completion of the paper. The works was supported by NSF DMS-1206648, MRSEC DMR-0820054 and BSF-2010345.

[1] R. B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).
[2] D. C. Tsui, H. L. Stormer, A. C. Gossard, Phys. Rev.
I. V. Tokatly, G. Vignale, Phys. Rev. B 76, 161305 (2007)
[16] V. Bargmann, Rev. Mod. Phys. 34, 829 (1962).
[15] A.G.Abanov, P.Wiegmann, Phys. Rev. Lett. 95:076402
[14] V. V. Kozlov, General Theory of Vortices, Springer 2003.
[13] M. Stone, Phys. Rev. B42. 212 (1990).
[12] D-H. Lee, S. C. Zhang, Phys. Rev Lett 66, 1220 (1991).
[11] N. Read, Phys. Rev. Lett. 62, 86 (1989).
[10] R. P. Feynman, Phys. Rev. 91, 1291, 1301 (1953); 94, 262 (1954);
R. P. Feynman, M. Cohen ibid. 102, 1189 (1956).
[9] S. M. Girvin, A. H. MacDonald and P. M. Platzman,
[8] L.D. Landau, JETP11,542(1941);J.Phys. 5,71; 8,1(1941).
[7] S. C. Zhang, T. H. Hansson, S. A. Kivelson, Phys. Rev.
[6] R. Feynman, Phys. Rev. Lett. 49, 1261 (1982).
[5] P. Wiegmann Phys. Rev. Lett. 108, 206810 (2012).
[4] S. M. Girvin, A. H. MacDonald and P. M. Platzman,
[3] R.R. Du, H.L. Stormer, D.C. Tsui, L.N. Pfeiffer, and
697 (1995).
[2] S. M. Girvin, A. H. MacDonald and P. M. Platzman,
[1] R. P. Feynman, Phys. Rev. Lett. 49, 1261 (1982).
[0] R. P. Feynman, M. Cohen ibid 102, 1189 (1956).

Incidentally a similar equation exists inside the core.

The ordering assures that the energy vanishes in the integer case $\beta = 1$.

We would like to clarify the meaning of the Hamiltonian \((20)\). The first quantized version of the Hamiltonian is $H = \frac{1}{\pi \hbar} \sum_{i=1}^{N} a_i^\dagger a_i$ is the second order differential operator, where $a_i = 2\hbar(-\partial_i + \beta \sum_{j \neq i} \frac{1}{z_i-z_j}) = mv_i + \partial_i \log |\psi_i|^2 \sum_{j \neq i} \frac{1}{z_i-z_j}$. It acts in the space of symmetric holomorphic polynomials multiplied by the factor $\prod_{i>j} (z_i-z_j)^{\beta}$ with the inner product \((35)\). The Hamiltonian is a complexified version of the chiral sector of Calogero model. If $Q_\lambda$ is a basis in a ring of polynomials the Hamiltonian can be viewed as a semi-infinite matrix $H_{\lambda\lambda} = (2\hbar^2/m) \int \prod_{i>j} (z_i-z_j)^{\beta} \prod_{i>j} \int d^2 \mu$.

The ordering enforced by the Lagrangian multiplier $a_0$ yields the Chern-Simons term in the Lagrangian: $L = -\rho(\overline{\pi}_x a_0) + a_0 \overline{\pi}_y - \frac{1}{2\pi \hbar} a_0 (\overline{\pi}_x + a - \frac{1}{\pi \hbar} |\rho| \overline{\pi}_x - a)^2 - a q(\nabla \times a) + \frac{1}{2} \nabla \sqrt{|\rho|}$ Two last terms differ this Lagrangian from those of \(6, 10\). They were recovered in \(23\).

The Lagrangian form of the hydrodynamics is not particularly helpful since subtle quantum phenomena and the chiral constraint are hidden in the measure of integration. However, it may help to compare our approach with those of \(6, 10, 23\). Let us choose the density of vortices $\rho$ and its momentum $\pi_\rho$ as canonical variables and represent the velocity as $\nabla \times \pi_\rho = a + \frac{2}{\pi \hbar} \log |\rho|$ with a constraint $-\nabla \times a = 2\pi q(\rho - \bar{\rho})$ to match \(10\). The constraint enforced by the Lagrangian multiplier $a_0$ yields the Chern-Simons term in the Lagrangian: $L = -\rho(\overline{\pi}_x a_0) + a_0 \overline{\pi}_y - \frac{1}{2\pi \hbar} a_0 (\overline{\pi}_x + a - \frac{1}{\pi \hbar} |\rho| \overline{\pi}_x - a)^2 - a q(\nabla \times a) + \frac{1}{2} \nabla \sqrt{|\rho|}$. The differential $\nabla \times P$ yields to the Lorentz shear force.

For Coulomb forces always emerge in flows of electronic liquids. They are essential factor in the bulk, less essential on the edge. In this paper we neglect Coulomb forces in order to unmask laws of quantum hydrodynamics.

In connection to FQHE the relation between density and vorticity in a rotated Euler fluid was discussed in \(12\). Recently the chiral compressible flow has been studied in \(23, 24\). Gapped $\propto q\Omega$ linear waves have been observed.

A similar shift of velocity has been also observed in hydrodynamics of Calogero model \(13\).

Eq. \(17\) expresses flux in terms of the density and the two-potes density correlation function. At the ground state where $P = 0$ the equation establishes a relation between the density and two-point function obtained in A. Zabrodin, P. Wiegmann, J.Phys. A39:8933 (2006).

Incidently a similar equation exists inside the core. There the quantum corrections changes the last term to $-\nabla \pi a \log |\rho|$. Accidentally a similar equation followed from the effective action of Refs.\(3, 10\) erroneously featuring the term $-\frac{1}{\pi \hbar} a \log |\rho|$ inside and outside of the vortex.

Hydrodynamics of a non-chiral version of the same model and with $\beta = 1$ has been studied in relation to normal matrices by J. Feinberg, Nucl. Phys. B 705:103 (2005).

The Hamiltonian proposed by Kirchhoff himself was $H = 2\int \rho \Psi \overline{\Psi} \overline{\Psi} = -q\Omega \sum |z_i - z_j|^2$.$\overline{\Psi}$. In formulas for the energy and momentum flux of Sec. 11 we assume the normal ordering of the density operator as in \(4\) : $\rho := \sum_k e^{-i\hbar \omega_k a_k^\dagger a_k}$. The ordering assures that the energy vanishes in the integer case $\beta = 1$.

We would like to clarify the meaning of the Hamiltonian \((20)\). The first quantized version of the Hamiltonian is $H = \frac{1}{\pi \hbar} \sum_{i=1}^{N} a_i^\dagger a_i$ is the second order differential operator, where $a_i = 2\hbar(-\partial_i + \beta \sum_{j \neq i} \frac{1}{z_i-z_j}) = mv_i + \partial_i \log |\psi_i|^2 \sum_{j \neq i} \frac{1}{z_i-z_j}$. It acts in the space of symmetric holomorphic polynomials multiplied by the factor $\prod_{i>j} (z_i-z_j)^{\beta}$ with the inner product \((35)\). The Hamiltonian is a complexified version of the chiral sector of Calogero model. If $Q_\lambda$ is a basis in a ring of polynomials the Hamiltonian can be viewed as a semi-infinite matrix $H_{\lambda\lambda} = (2\hbar^2/m) \int \prod_{i>j} (z_i-z_j)^{\beta} \prod_{i>j} \int d^2 \mu$.

The differential $\nabla \times P$ yields to the Lorentz shear force.

The Lagrangian form of the hydrodynamics is not particularly helpful since subtle quantum phenomena and the chiral constraint are hidden in the measure of integration. However, it may help to compare our approach with those of \(6, 10, 23\). Let us choose the density of vortices $\rho$ and its momentum $\pi_\rho$ as canonical variables and represent the velocity as $\nabla \times \pi_\rho = a + \frac{2}{\pi \hbar} \log |\rho|$ with a constraint $-\nabla \times a = 2\pi q(\rho - \bar{\rho})$ to match \(10\). The constraint enforced by the Lagrangian multiplier $a_0$ yields the Chern-Simons term in the Lagrangian: $L = -\rho(\overline{\pi}_x a_0) + a_0 \overline{\pi}_y - \frac{1}{2\pi \hbar} a_0 (\overline{\pi}_x + a - \frac{1}{\pi \hbar} |\rho| \overline{\pi}_x - a)^2 - a q(\nabla \times a) + \frac{1}{2} \nabla \sqrt{|\rho|}$ Two last terms differ this Lagrangian from those of \(6, 10\). They were recovered in \(23\).

The Lagrangian can be expressed in terms of the vertex operator $\Phi = \sqrt{\pi} e^{-\frac{1}{\pi \hbar} \overline{\pi}:r}$: $L = a_0 \overline{\Phi} \Phi (\overline{\pi} \overline{\pi} + \frac{1}{\pi \hbar} \int \overline{\pi} \overline{\pi} + q(\nabla \times a) \Phi \Phi + (\frac{1}{\pi \hbar} - 1) (\nabla \Phi) \Phi \Phi)$. Incompressibility condition and the chiral constraint must be added.

The transformation \(13\) is analogues to the Miura transformation of the theory of solitons. Similarly two Hamiltonian structures are analogues to the bi-Hamiltonian structure of theory of solitons C. S. Gardner, J. Math. Phys. 12, 1548 (1971).

Elsewhere we discuss the relation of the extended algebra \(23\) and the Virasoro algebra of conformal field theory with the central charge $c = 1 - \Omega (\sqrt{5} - 1) / \sqrt{5}$ (discussion with E. Betelheim is acknowledged).