THE AUTOMORPHISM GROUPS OF GROUPS OF ORDER $p^2q$

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Abstract. We record for reference a detailed description of the automorphism groups of the groups of order $p^2q$, where $p$ and $q$ are distinct primes.

1. Introduction

Let $p, q$ be distinct primes. O. Hölder classified the groups of order $p^2q$ [Höl93], and also the groups of square-free order [Höl95]. H. Dietrich and B. Eick [DE05] gave a detailed description of the structure of groups of cube-free order, which was complemented by S. Qiao and C.H. Li [QL11]. Groups of order $p^3q$ were classified by A.E. Western [Wes99] and R. Laue [Lau82]. B. Eick [Eic17] has given an enumeration of the groups whose order factorises in at most 4 primes. B. Eick and T. Moede [EM18] have enumerated groups of order $p^nq$, for $n \leq 5$.

For a forthcoming paper of ours, we need a detailed description of the automorphism groups of the groups of order $p^2q$, where $p$ and $q$ are distinct primes. We have recorded these data for reference here.

2. The Groups

With current technology (i.e. Sylow’s theorem), describing the groups $G$ of order $p^2q$, where $p$ and $q$ are distinct primes, is an easy exercise, which we now describe briefly, the basic point being that $G$ has a normal Sylow subgroup.

If there are more than 1, and thus exactly $q$, Sylow $p$-subgroups, then $p \mid q - 1$.

- If the Sylow $p$-subgroups intersect pairwise trivially, counting $p$-elements show that then $G$ has exactly one Sylow $q$-subgroup.
- If there are two distinct Sylow $p$-subgroups $P_1, P_2$ that intersect non-trivially in a subgroup $N$ of order $p$, then $N \trianglelefteq G$, so that $G$ has a subgroup $R$ of order $pq$, which is normal in $G$, as $p < q$.

For the same reason, a Sylow $q$-subgroup of $R$ is normal in $R$. 

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and thus in $G$, so that $G$ has a normal Sylow $q$-subgroup in this
case as well.

We introduce some notation.

- $C_n$: denotes a cyclic group of order $n$.
- $\times$ and $\rtimes$: when they appear without subscripts, they denote the
  unique (up to isomorphism) non-direct, semidirect product that
  is possible in the given situation.
- $(C_p \times C_q) \rtimes_S C_q$: denotes a semidirect product where a generator
  of $C_q$ acts as a non-identity scalar matrix.
- $(C_p \times C_q) \rtimes_{D0} C_q$: denotes a semidirect product where a generator
  of $C_q$ acts as a diagonal, non-scalar matrix with no eigenvalue 1,
  and determinant different from 1.
- $(C_p \times C_q) \rtimes_{D1} C_q$: is the same as above, but the non-scalar matrix
  has determinant 1 (and thus still no eigenvalue 1).
- $(C_p \times C_q) \rtimes_C C_q$: denotes a semidirect product where a generator
  of $C_q$ acts as a suitable power of a Singer cycle; note that the
  determinant of the matrix of a generator of $C_q$ acting on $C_p \times C_q$
  is 1.
- $C_p^3 \rtimes_1 C_q$: denotes a semidirect product with trivial centre.
- $C_p^3 \rtimes_p C_q$: denotes a semidirect product with centre of order $p$.

Considering the possible actions on the normal Sylow subgroup of
another Sylow subgroup, we obtain the following table. The automor-
phism groups are determined in Section 4 on the basis of results
of G.L. Walls [Wal86], J.N.S. Bidwell, M.J. Curran and D.J. McCaughan
[BCM06], and M.J. Curran [Cur08], which we recall in Section 4.

| Type | Conditions | $G$ | Aut$(G)$ | Explanation |
|------|------------|-----|----------|-------------|
| 1    | $p \mid q - 1$ | $C_p^2 \rtimes C_q$ | $C_p \rtimes C_{p-1} \times C_{q-1}$ | Cyclic groups |
| 2    | $p^2 \mid q - 1$ | $C_p^2 \rtimes_C C_q$ | $C_p \rtimes \text{Hol}(C_q)$ | Subs. 4.5 |
| 3    | $q \mid p - 1$ | $C_p^2 \rtimes_1 C_q$ | $\text{Hol}(C_p^2)$ | Thm 3.4 |
| 4    | $q \mid p - 1$ | $C_p \rtimes C_p \rtimes C_q$ | $\text{GL}(2, p) \times C_{q-1}$ | Thm 3.1 |
| 5    | $q \mid p - 1$ | $C_p \times (C_p \rtimes C_q)$ | $C_{p-1} \times \text{Hol}(C_p)$ | Thms 3.1, 3.4 |
| 6    | $2 < q \mid p - 1$ | $(C_p \times C_p) \rtimes_S C_q$ | $\text{Hol}(C_p \times C_p)$ | Subs. 4.1, 4.2 |
| 7    | $3 < q \mid p - 1$ | $(C_p \times C_p) \rtimes_{D0} C_q$ | $\text{Hol}(C_p) \times \text{Hol}(C_p)$ | Subs. 4.1, 4.3 |
| 8    | $2 < q \mid p - 1$ | $(C_p \times C_p) \rtimes_{D1} C_q$ | $C_2 \rtimes (\text{Hol}(C_p) \times \text{Hol}(C_p))$ | Subs. 4.1, 4.3 |
| 9    | $2 < q \mid p + 1$ | $(C_p \times C_p) \rtimes_C C_q$ | $(C_2 \times C_{p-1}) \rtimes (C_p \times C_p)$ | Subs. 4.1, 4.3 |
| 10   | $p \mid q - 1$ | $C_p \times (C_p \rtimes C_q)$ | $\text{Hol}(C_p) \times \text{Hol}(C_q)$ | Subs. 4.6 |

2.1. Isomorphism. It is immediate to see that all types in this table
consist of exactly one isomorphism class of groups, with the exception
of type 8. If $G$ is a group of this type, we can give it a canonical form by
choosing as generators first of all two eigenvectors with respect to dis-
tinct eigenvalues in the normal, elementary abelian Sylow $p$-subgroup
$V$. If $\zeta$ is a fixed element of order $q$ in the multiplicative group of the
field with $p$ elements, we can then choose as a third generator a suitable power $a$ of a $q$-element, so that it has eigenvalues $\{\zeta, \zeta^s\}$ on $V$. The parameter $s \notin \{0, 1, -1\}$ determines $G$. If $t$ is the inverse of $s$ modulo $p$, then $a^t$ has eigenvalues $\{\zeta^t, \zeta\}$ on $V$. It follows that the parameters $s, t$ yield isomorphic groups, so that there are $(q - 3)/2$ isomorphism classes of groups here.

3. Automorphisms of (semi)direct products

We collect here the results we need of [BCM06, Wal86, Cur08]. We write (auto)morphisms as exponents.

**Theorem 3.1** ([BCM06, Theorem 3.2]).

Let $G = H \times K$, where $H, K$ have no common direct factors.

Then $\text{Aut}(G)$ can be described in the natural way via the set of matrices

$$\left\{ \begin{bmatrix} a & c \\ b & d \end{bmatrix} : a \in \text{Aut}(H), d \in \text{Aut}(K), \\
\quad b \in \text{Hom}(K, \mathbb{Z}(H)), c \in \text{Hom}(H, \mathbb{Z}(K)) \right\}.$$ 

**Theorem 3.2.** ([Cur08, Theorem 1])

Let $G = H \rtimes K$ be a semidirect product.

Then the subgroup of $\text{Aut}(G)$ consisting of the automorphisms that leave $H$ invariant can be described in a natural way via the set of matrices

$$\left\{ \begin{bmatrix} a & 0 \\ b & d \end{bmatrix} : a \in \text{Aut}(H), d \in \text{Aut}(K), \\
\quad (h^k)^a = (h^a)^{k^d}, \text{ for } h \in H, k \in K, \\
\quad b : K \rightarrow H, (xy)^b = x^b(y^b)x^d, \text{ for } x, y \in K \right\}.$$ 

**Remark 3.3.** The condition

$$(h^k)^a = (h^a)^{k^d}$$

in Theorem 3.2 can be rewritten as

$$\iota(k)^a = a^{-1}\iota(k)a = \iota(k^d),$$

where

$$\iota : K \rightarrow \text{Aut}(H)$$

$$k \mapsto (h \mapsto h^k).$$

If $\text{Aut}(H)$ is abelian, we get $\iota(k) = \iota(k^d)$, that is, $[k, d] \in C_K(H)$. In particular, if $C_K(H) = 1$, then $d = 1$.

**Theorem 3.4** ([Wal86, Theorem B], [Cur08, Example 1]).

Let $G = C_n \rtimes C_k$, with $Z(G) = 1$. Write $H = C_n$, $K = C_k$. 
Then \( H = G' \) is characteristic in \( G \), and we have
\[
\text{Aut}(G) \cong \text{Hol}(C_n) = C_n \rtimes \text{Aut}(C_n).
\]

**Remark 3.5.** In the matrix terms of Theorem 3.2, Theorem 3.4 can be reformulated as
\[
\text{Aut}(G) = \left\{ \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix} : a \in \text{Aut}(H), b : K \to H \right\},
\]
\[
(xy)^b = x^b (y^b)^x, \text{ for } x, y \in K \}
\]
The \( b \)'s can be described in terms of the image \( b_0 \in H \) of a fixed generator of \( K \): see Subsection 4.1 for the details.

**Theorem 3.6** ([Cur08, Theorem 3 and Example 1]).
Let \( G = C_n \rtimes C_k \), with \( Z(G) \) possibly non-trivial. Write \( H = C_n \), \( K = C_k \).
Assume \( H = G' \).
Then
\[
\text{Aut}(G) \cong H \rtimes (\text{Aut}(H) \times S),
\]
where
\[
S = \left\{ d \in \text{Aut}(K) : [k, d] = k^{-1} k^d \in C_K(H), \text{ for } k \in K \right\}.
\]
In matrix terms, Theorem 3.6 states that
\[
\text{Aut}(G) = \left\{ \begin{bmatrix} a & 0 \\ b & d \end{bmatrix} : a \in \text{Aut}(H), d \in \text{Aut}(K), \right. \]
\[
[k, d] = k^{-1} k^d \in C_K(H), \text{ for } k \in K, \]
\[
b : K \to H, (xy)^b = x^b (y^b)^x, \text{ for } x, y \in K \}
\]

4. **Automorphism groups**

We appeal to the results of Section 3 whose notation we employ.

4.1. **Describing \( b \).** We begin by collecting some facts that hold true for most cases.

Let us first consider the types 7, 8, 9, 10. Write
- \( C_q = \langle z \rangle \),
- \( Z \) for the linear map \( z \) induces on \( H = C_p \rtimes C_p \), and
- \( Y \) for the linear map induced by \( z^d \) on \( H \).

First note that for each \( b_0 \in H \) there exists a unique function \( b \) as in Theorem 3.2 such that \( z^b = b_0 \). In fact, one has for \( j = 1, \ldots, q - 1 \)
\[
(z^j)^b = b_0^{1 + Y + \cdots + Y^{j-1}}.
\]
In \( \text{End}(H) \) we have
\[
0 = Y^q - 1 = (Y - 1)(1 + Y + \cdots + Y^{q-1}).
\]
Now \( Y - 1 \) invertible, as \( Y \) has no eigenvalue 1, so that \( 1 + Y + \cdots + Y^{q-1} = 0 \). It follows that
\[
(z^q)^b = 1 = b_0^1 + Y + \cdots + Y^{q-1},
\]
is also satisfied.

A similar argument holds

- for the types 4, 3, 2,
- for the subgroup \( C_p \rtimes C_q \) of type 6, and
- for the subgroup \( C_p \rtimes C_q \) of type 11.

In these cases \( Y \) is an automorphism of order coprime to \( r \) of a cyclic group \( C \) of order a power of a prime \( r \), so that \( Y - 1 \) is not nilpotent, and thus it is invertible, in \( \text{End}(C) \).

Note that conjugating
\[
\begin{bmatrix}
1 & 0 \\
b & 1
\end{bmatrix}
\]
by
\[
\begin{bmatrix}
a & 0 \\
0 & d
\end{bmatrix}
\]
we get
\[
\begin{bmatrix}
1 & 0 \\
d^{-1}ba & 1
\end{bmatrix},
\]
so that if \( d = 1 \) we have \( z^{ba} = b_0^a \), and thus the group
\[
\begin{bmatrix}
a & 0 \\
b & 1
\end{bmatrix}
\]
is a split extension of \( H \) by the group of the \( a \)'s.

4.1.1. *Between \( d \) and \( a \).* Suppose \( d : z \mapsto z^i \), with \( 0 < i < q \) and \( \gcd(i, q) = 1 \). For \( h \in H \) we have
\[
h^{a-1}Za = hZ^i,
\]
and thus
\[
a^{-1}Za = Z^i. \quad (4.1)
\]

4.2. *Type 7,* \( G = (C_p \times C_p) \rtimes_{S} C_q \). In this case, since \( Z \) is scalar, we have \( Z = Z^i \), thus \( q \mid i - 1 \), that is, \( i = 1 \) and \( d \) is trivial. Since \( a \) is arbitrary, we obtain as the automorphism group the holomorph of \( C_p \times C_p \), that this the affine group in dimension 2 over \( \mathbb{F}_p \).
4.3. **Type 8 and 9**, $G = (C_p \times C_p) \rtimes_{D_0} C_q$ or $(C_p \times C_p) \rtimes_{D_1} C_q$. In this case

$$Z = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix},$$

with $\lambda \neq \mu$, $\lambda, \mu \neq 1$. Then (4.1) yields $\{\lambda, \mu\} = \{\lambda^i, \mu^i\}$. If $\lambda = \lambda^i$ and $\mu = \mu^i$, we obtain that $q \mid i - 1$, and thus $i = 1$ and $d = 1$. From (4.1) and Subsection 4.1, we obtain that $a$ centralizes $Z$, and that the automorphism group contains

$$(C_{p-1} \times C_{p-1}) \rtimes (C_p \times C_p) = \text{Hol}(C_p) \times \text{Hol}(C_p), \quad (4.2)$$

with $C_{p-1} \times C_{p-1}$ acting by diagonal matrices on $C_p \times C_p$, a typical element being

$$\begin{bmatrix} T & 0 \\ b & 1 \end{bmatrix} \quad (4.3)$$

with $T$ diagonal.

If $\lambda = \mu^i$ and $\mu = \lambda^i$, then $\lambda = \lambda^{i^2}$, so that $q \mid (i - 1)(i + 1)$. When $q \mid i - 1$ we get again $d = 1$, whereas when $q \mid i + 1$ we get $z^d = z^{-1}$ and $\lambda = \mu^{-1}$. Thus this case only occurs when $\det(Z) = 1$, that is, when $G$ is of type 9. The inversion $d$ can then be paired with

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

to get $S^{-1}ZS = Z^{-1}$. In this case the automorphism group is the extension of the group (4.2) by the involution

$$\begin{bmatrix} S & 0 \\ 0 & d \end{bmatrix}, \quad (4.4)$$

which acts on (4.3) as

$$\begin{bmatrix} S & 0 \\ 0 & d \end{bmatrix}^{-1} \cdot \begin{bmatrix} T & 0 \\ b & 1 \end{bmatrix} \cdot \begin{bmatrix} S & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} STS & 0 \\ d^{-1}bS & 1 \end{bmatrix}.$$

Now

$$z^{d^{-1}bS} = (z^{-1})^{bS} = (z^{q-1})^{bS} = (b_0^{1+Y+\cdots+Y^{q-2}})^S = (b_0^{-Y^{q-1}})^S = b_0^{-Y^{-1}S},$$

where

$$-Y^{-1}S = \begin{bmatrix} 0 & -\lambda^{-1} \\ -\lambda & 0 \end{bmatrix}$$

is an involution, that acts by exchanging the two copies of Hol($C_p$).
4.4. Type 10, $G = (C_p \times C_p) \rtimes_C C_q$. Note first that the order $q \neq 2$ of $Z$ divides $p + 1$, so it does not divide $p - 1$. It follows that $Z \in \text{SL}(2, p)$, that is, $\det(Z) = 1$.

If $\lambda, \mu = \lambda^{-1}$ are the (distinct) eigenvalues of $Z$ in the field $\mathbb{F}_{p^2}$, then $\{\lambda, \mu\} = \{\lambda^i, \mu^i\}$. If $\lambda = \lambda^i$ and $\mu = \mu^i$, we get once more $d = 1$. Thus in this case $a$ lies in the centralizer of $Z$ in $\text{Aut}(H) = \text{GL}(2, p)$

$$C_{\text{Aut}(H)}(Z) = \{u + vZ : u, v \in \mathbb{F}_p\},$$

which is cyclic, of order $p^2 - 1$.

If $\lambda = \mu^i$ and $\mu = \lambda^i$, then $\lambda = \lambda^2$, so that $q \mid (i - 1)(i + 1)$. When $q \mid i - 1$ we get again $d = 1$, whereas when $q \mid i + 1$ we get $z^d = z^{-1} = z^p$, as $p \equiv -1 \pmod{q}$.

In an appropriate basis of $H$ we have

$$Z = \begin{bmatrix} 0 & 1 \\ -1 & t \end{bmatrix},$$

where $t = \lambda + \lambda^{-1} = \lambda + \lambda^p$.

The equation $\iota(z)^a = \iota(z^d)$ of Remark 3.3 has now become $Z^a = Z^{-1}$.

One solution $a$ for this is

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

as

$$S^{-1}ZS = \begin{bmatrix} t & -1 \\ 1 & 0 \end{bmatrix} = Z^{-1}.$$ 

All other solutions are obtained in the form $a = XC$, where $X \in C_{\text{Aut}(H)}(Z)$. So in this case $\text{Aut}(G)$ is the extension of the subgroup determined by $a = d = 1$, which is isomorphic to $C_p \times C_p$, acted upon by the subgroup determined by $b = 0$. The latter subgroup has a normal subgroup

$$C = \begin{bmatrix} C_{\text{Aut}(H)}(Z) & 0 \\ 0 & 1 \end{bmatrix},$$

which is cyclic, of order $p^2 - 1$ extended by the involution

$$D = \begin{bmatrix} S & 0 \\ 0 & d \end{bmatrix},$$

where $z^d = z^{-1} = z^p$. Now $C_{\text{Aut}(H)}(Z)$ is the multiplicative group of the field with $p^2$ elements; conjugation by $S$ induces an automorphism group of order 2, which is then the Frobenius map. Thus $C_C(D) = C_{\text{Aut}(H)}(S)$ has order $p - 1$, and $g^S = g^p$ for $g \in C$. 
4.5. **Type 2**, $G = C_{p^2} \rtimes_p C_q$. Here $C_{p^2} = \langle x \rangle$ induces on $C_q$ a group of automorphisms of order $p$, and thus the centraliser $C_{\langle x \rangle}(C_q) = \langle x^p \rangle$ has order $p$.

As per Remark 3.3 here

$$S = \left\{ d \in \text{Aut}(C_{p^2}) : [x, d] \in C_{\langle x \rangle}(C_q) = \langle x^p \rangle \right\}.$$  

is a group of order $p$, generated by the automorphism $x \mapsto x^{1+p}$. According to Theorem 3.6 we get that the automorphism group is isomorphic to

$$C_q \rtimes (C_{q-1} \rtimes C_p) \cong C_p \times \text{Hol}(C_q).$$

4.6. **Type 11**, $G = C_p \times (C_p \rtimes C_q)$. According to Theorem 3.1 and Theorem 3.4 we have that the automorphism group has the form

$$\begin{bmatrix} C_{p-1} & 0 \\ C_p & \text{Hol}(C_q) \end{bmatrix}.$$  

To see the structure, let us consider the conjugate

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix},$$  

where $a \in \text{Aut}(C_p)$, $d \in \text{Aut}(C_p \rtimes C_q) \cong \text{Hol}(C_q)$, and $b \in \text{Hom}(C_p \rtimes C_q, C_p)$. The conjugate equals

$$\begin{bmatrix} 1 & 0 \\ d^{-1}ba & 1 \end{bmatrix}.$$  

Since $d$ acts trivially on the quotient $(C_p \rtimes C_q)/C_q$, we get that the automorphism group has structure

$$C_p \rtimes (C_{p-1} \times \text{Hol}(C_q)),$$

with $C_{p-1}$ acting as $\text{Aut}(C_p)$ and $\text{Hol}(C_q)$ acting trivially, that is

$$\text{Hol}(C_p) \times \text{Hol}(C_q).$$

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