THE NÉRON-SEVERI LATTICE OF SOME SPECIAL K3 SURFACES WITH $\mathbb{Z}_2^2$ SYMPLECTIC ACTION

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Abstract. Let $G$ be a finite abelian group which acts symplectically on a K3 surface. Then the Néron-Severi lattice of the general projective K3 surface admitting $G$ symplectic action is computed by Garbagnati and Sarti. In this paper we consider a special 4-dimensional subfamily of the 7-dimensional family of projective K3 surfaces with $\mathbb{Z}_2^2$ symplectic action. If $X$ is one of these special K3 surfaces, then it arises as the minimal resolution of a specific $\mathbb{Z}_2^2$-cover of $\mathbb{P}^2$ branched along six general lines. We show that the Néron-Severi lattice of $X$ is generated by an arrangement of 24 smooth rational curves, and that $X$ specializes to the Kummer surface $\text{Km}(E_i \times E_i)$. We relate this 4-dimensional family to other well known families of K3 surfaces, namely the minimal resolution of the $\mathbb{Z}_2$-cover of $\mathbb{P}^2$ branched along six general lines, and the corresponding Hirzebruch-Kummer covering of exponent 2 of $\mathbb{P}^2$.

1. Introduction

Let $X$ be a K3 surface over $\mathbb{C}$. A subgroup of the automorphism group of $X$ acts symplectically on $X$ if the induced action on $H^0(X, \omega_X)$ is the identity. Finite abelian groups of automorphisms acting symplectically on K3 surfaces were classified by Nikulin in [Nik79], and by Mukai in the non-commutative case (see [Muk88] and also [Xia96]).

Let $G$ be a finite abelian group that acts symplectically on a K3 surface $X$. If $\Omega_G$ denotes the orthogonal complement in $H^2(X; \mathbb{Z})$ of the fixed sublattice $H^2(X; \mathbb{Z})^G$, then $\Omega_G$ is a negative definite primitive sublattice of the Néron-Severi lattice $\text{NS}(X)$. All the possible lattices $\Omega_G$ and their orthogonal complements are computed in [GS07, GS09]. If $X$ is projective, then the Picard number $\rho(X)$ satisfies $\rho(X) \geq \text{rk}(\Omega_G) + 1$.

The projective K3 surfaces $X$ admitting a $G$ symplectic action form a family of dimension $19 - \text{rk}(\Omega_G)$ (see [GS09, Remark 6.2]), and the lattices $\text{NS}(X)$ are computed in [GS09, Proposition 6.2] provided $\rho(X) = \text{rk}(\Omega_G) + 1$. The case $G = \mathbb{Z}_2^2$ received special attention: in [GS16], among other results, the authors compute $\text{NS}(\overline{X/G})$, where $\overline{X/G}$ is the minimal resolution of $X/G$, if $\rho(\overline{X/G}) = 16$.

There is a 7-dimensional family of projective K3 surfaces with $\mathbb{Z}_2^2$ symplectic action, and in this paper we analyze the 4-dimensional subfamily which arises as follows. Consider six general lines in $\mathbb{P}^2$ and divide them into three pairs. Consider the tower of double covers given by

$$X_3 \xrightarrow{z_3} X_2 \xrightarrow{z_2} X_1 \xrightarrow{z_2} \mathbb{P}^2,$$

where the first cover is branched along the first pair of lines, the second cover is branched along the preimage of the second pair of lines, and so on. The minimal resolution of $X_3$ is a projective K3 surface $X$, which we call a triple-double K3 surface (see Section 3.1). Equivalently, $X$ can be

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viewed as an appropriate $\mathbb{Z}_2^3$-cover of $Bl_3 \mathbb{P}^2$, which denotes the blow up of $\mathbb{P}^2$ at three general points. As we show in Section 3, $X$ has Picard number 16, admits a $\mathbb{Z}_2^4$ symplectic action, and has an Enriques involution. Moduli compactifications of the corresponding 4-dimensional family of Enriques surfaces are studied in [Oud10, Sch17]. The main goal of this paper is to compute the lattice $\text{NS}(X)$ and relate it to the geometry of $X$.

**Theorem 1.1.** Let $X$ be a triple-double K3 surface. Then the following hold:

(i) The Néron-Severi lattice $\text{NS}(X)$ is generated by the irreducible components of the preimage of the $(-1)$-curves on $Bl_3 \mathbb{P}^2$. The dual graph of this configuration of 24 smooth rational curves is shown in Figure 2 (see Theorem 4.6);

(ii) $\text{NS}(X)$ has discriminant group $\mathbb{Z}_2^4 \oplus \mathbb{Z}_4^2$ and is isometric to $U \oplus E_8 \oplus Q$, where $Q$ is the lattice in Lemma 4.1 (iii). An explicit $\mathbb{Z}$-basis of $\text{NS}(X)$ which realizes it as a direct sum of $U$, $E_8$, and $Q$ can be found in Remark 4.4;

(iii) The transcendental lattice $T_X$ is isometric to $U \oplus U(2) \oplus \mathbb{Z}_2^2(-4)$ (see Proposition 4.10);

(iv) The Kummer surface $\text{Km}(E_i \times E_i)$ (see [KK01]) appears as a specialization of this 4-dimensional family (see Corollary 4.11). The line arrangement in $\mathbb{P}^2$ that gives rise to this Kummer surface is shown in Figure 3 (see Proposition 4.12);

(v) If $\iota$ is a symplectic involution on $X$ coming from the $\mathbb{Z}_2^2$-action, then the Néron-Severi lattice of $X' = \overline{X}/\iota$ has rank 16 and discriminant group $\mathbb{Z}_4^4 \oplus \mathbb{Z}_4^2$. A $\mathbb{Z}$-basis for $\text{NS}(X')$ is given in Theorem 6.2. The transcendental lattice $T_{X'}$ is isometric to $U(2) \oplus \mathbb{Z}_2^2(-4)$.

Triple-double K3 surfaces are closely related to two classical examples of K3 surfaces. The first one is the Hirzebruch-Kummer covering $Y$ of exponent 2 of $\mathbb{P}^2$ branched along six general lines (see [Cat16, Section 4.3]). The covering map $Y \to \mathbb{P}^2$ is a $\mathbb{Z}_2^2$-cover, and there is a subgroup of $\mathbb{Z}_2^5$ isomorphic to $\mathbb{Z}_2^4$ which acts symplectically on $Y$. The second example is the minimal resolution of the double cover of $\mathbb{P}^2$ branched along six general lines, which we denote by $Z$. Now let $X$ be a triple-double K3 surface. Given the corresponding six lines in $\mathbb{P}^2$, we can consider $Y$ and $Z$ as above. Then in Proposition 5.5 we show that $X$ is isomorphic to $Y/\mathbb{Z}_2^2$ for an appropriate subgroup $\mathbb{Z}_2^2 < \mathbb{Z}_4^4$, and the minimal resolution of the quotient of $X$ by the leftover $\mathbb{Z}_2^4/\mathbb{Z}_2^2 \cong \mathbb{Z}_2^2$ symplectic action gives rise to $Z$. Summarizing, we have the following diagram:

$$
\begin{array}{ccc}
Y & \to & Y/\mathbb{Z}_2^2 \\
\downarrow & \leftarrow & \downarrow \\
X & \to & X/\mathbb{Z}_2^2 \\
\downarrow & \leftarrow & \downarrow \\
\mathbb{P}^2 & \to & Z
\end{array}
$$

The Néron-Severi lattices $\text{NS}(Y), \text{NS}(Z)$ are well known, and the interaction between them is described in [GS16] (see [Klo06, Section 5] for $\text{NS}(Z)$).

The Néron-Severi lattice of a triple-double K3 surface $X$ is not included in the cited work [GS07, GS09, GS10] for the following reasons:

- If $G$ is a finite abelian group acting symplectically on a projective K3 surface, then $16 = \text{rk}(\Omega_G) + 1$ if and only if $G \cong \mathbb{Z}_4^2$ (see [GS09, Proposition 5.1]);
- $X$ does not admit $\mathbb{Z}_4^4$ symplectic action (see Proposition 4.13 (i));
- $X$ is not isomorphic to the minimal resolution of the quotient of a K3 surface by a symplectic action of the group $\mathbb{Z}_4^4$ (see Proposition 4.13 (ii)).

We remark that similar problems are considered in [Bou17], but different families of K3 surfaces are studied there. We also mention [Nik17], where Nikulin classified the main part of the Néron-Severi lattices.
lattice of an arbitrary Kählerian K3 surface (see [Nik17, Section 3]) with large enough group of symplectic automorphisms (see [Nik17, Theorem 2] for a precise statement).

In Section 2 we recall some preliminary facts about K3 surfaces, symplectic automorphisms, and even lattices. In Section 3 we define triple-double K3 surfaces characterizing them in several different ways. We classify the involutions coming from the $\mathbb{Z}_2^2$-action of the cover. In the same section we also analyze the configuration of 24 smooth rational curves on a triple-double K3 surface $X$ coming from the preimage of the $(-1)$-curves under the covering map $X \to \text{Bl}_3 \mathbb{P}^2$. In Section 4 we show that the sublattice of $\text{NS}(X)$ generated by these curves actually equals $\text{NS}(X)$. We also compute $T_X$ and show that $\text{Km}(E_i \times E_i)$ is a specialization of this 4-dimensional family. In Section 5 we relate triple-double K3 surfaces with the Hirzebruch-Kummer covering of exponent 2 of $\mathbb{P}^2$ branched along six general lines. Finally, in Section 6 we compute $\text{NS}(X')$ and $T_{X'}$, where $X'$ is the minimal resolution of the quotient of a triple-double K3 surface by a symplectic involution coming from the $\mathbb{Z}_2^2$ symplectic action. We work over $\mathbb{C}$.

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2. Preliminaries: lattice theory and symplectic automorphisms of K3 surfaces

2.1. Even lattices and the discriminant quadratic form. A lattice $L = (L, b_L)$ is a finitely generated free abelian group $L$ together with a symmetric bilinear form $b_L : L \times L \to \mathbb{Z}$. In what follows, we consider non-degenerate lattices, i.e. $b_L$ is a non-degenerate symmetric bilinear form, and we always assume so. If $e_1, \ldots, e_n$ is a $\mathbb{Z}$-basis for $L$, then the Gram matrix of $L$ associated to the chosen $\mathbb{Z}$-basis is the matrix $(b_L(e_i, e_j))_{1 \leq i, j \leq n}$. The determinant of a Gram matrix of $L$ does not depend on the choice of $\mathbb{Z}$-basis and is called the discriminant of $L$. Lattices of discriminant $\pm 1$ are called unimodular. For the classification of unimodular lattices we refer to [Ser73, Chapter V]. Denote by $U$ (resp. $E_8$) the unique even unimodular lattice of signature $(1, 1)$ (resp. $(0, 8)$).
Let $L$ be a lattice. The dual $L^* = \text{Hom}_\mathbb{Z}(L, \mathbb{Z})$ comes with an induced $\mathbb{Q}$-valued symmetric bilinear form $b_{L^*}$. $L$ embeds into $L^* = \text{Hom}_\mathbb{Z}(L, \mathbb{Z})$ because $b_L$ is non-degenerate, and the quotient $A_L = L^*/L$ is called the discriminant group of $L$. We denote by $\ell(A_L)$ the minimum number of generators of $A_L$. A sublattice $S \subseteq L$ is called primitive if $L/S$ is torsion free. We say that $L$ is even if $b_L(x, x) \in 2\mathbb{Z}$ for all $x \in L$. We recall the following result that guarantees uniqueness up to isometry of certain even unimodular lattices.

**Theorem 2.1 (Nik80 Corollary 1.13.3).** Let $L$ be an even indefinite lattice of signature $(t_+, t_-)$ such that $t_+ + t_- \geq 2 + \ell(A_L)$. Then $L$ is unique up to isometry.

The next theorem gives sufficient conditions for when a copy of $U$ or $E_8$ can be split off from an even indefinite lattice.

**Theorem 2.2 (Nik80 Corollary 1.13.5).** Let $L$ be an even lattice of signature $(t_+, t_-)$.

(i) If $t_+ \geq 1, t_- \geq 8$ and $t_+ + t_- \geq 9 + \ell(A_L)$, then $L \cong E_8 \oplus P$ for some lattice $P$;

(ii) If $t_+ \geq 1, t_- \geq 1$ and $t_+ + t_- \geq 3 + \ell(A_L)$, then $L \cong U \oplus P$ for some lattice $P$.

An important invariant associated to an even lattice is its corresponding discriminant quadratic form, which we now briefly review.

**Definition 2.3.** Let $G$ be a finite abelian group. A quadratic form on $G$ is a map $q: G \to \mathbb{Q}/2\mathbb{Z}$ satisfying the following conditions:

(a) $q( cg) = c^2 q(g)$ for all $g \in G$ and $c \in \mathbb{Z}$;

(b) There exists a symmetric bilinear form $b: G \times G \to \mathbb{Q}/\mathbb{Z}$ such that $q(g_1 + g_2) \equiv q(g_1) + q(g_2) + 2b(g_1, g_2)$ for all $g_1, g_2 \in G$.

**Example 2.4.** If $L$ is an even lattice, then we have a quadratic form $q_L$ on the discriminant group $A_L$ which is defined as follows:

$$q_L: A_L \to \mathbb{Q}/2\mathbb{Z},$$

$$x + L \mapsto b_{L^*}(x, x) \mod 2\mathbb{Z}.$$  

The symmetric bilinear form required by Definition 2.3 (b) is $b_{L^*} \mod \mathbb{Z}$. The quadratic form $q_L$ is called the discriminant quadratic form of $L$.

**Definition 2.5.** Let $L \embed L'$ be an embedding of lattices. Then $L'$ is called an overlattice of $L$ if the quotient $L'/L$ is a finite abelian group.

The following theorem is fundamental in Section 4.

**Theorem 2.6 (Nik80 Proposition 1.4.1 (a)).** Let $L$ be an even lattice. Then there is a 1-to-1 correspondence between overlattices of $L$ and subgroups of $A_L$ which are isotropic with respect to the discriminant quadratic form $q_L$.

We recall the following important result about primitive sublattices of even unimodular lattices.

**Theorem 2.7 (Nik80 Section 1.5).** Let $L$ be an even unimodular lattice and $S \subseteq L$ a primitive sublattice. Then $S^\perp \subseteq L$ is also primitive, and the two discriminant groups $A_S, A_{S^\perp}$ are isomorphic. Moreover, $q_S = -q_{S^\perp}$.

To conclude, we recall another uniqueness criterion which applies to 2-elementary lattices, which are lattices with a 2-elementary discriminant group.
**Definition 2.8.** For a lattice \( L \), define the invariant \( \delta(L) \) as follows:
\[
\delta(L) = \begin{cases} 
0 & \text{if } q_L(x) = 0 \text{ for all } x \in A_L, \\
1 & \text{otherwise}.
\end{cases}
\]

**Theorem 2.9 ([Nik83, Theorem 4.3.2]).** An indefinite 2-elementary even lattice \( L \) is determined, up to isometry, by its rank, \( \ell(A_L) \), and \( \delta(L) \).

### 2.2. K3 surfaces.

A K3 surface \( X \) is a smooth irreducible projective 2-dimensional variety with \( K_X \sim 0 \) and \( h^1(X, \mathcal{O}_X) = 0 \). On a K3 surface \( X \), numerical algebraic, and linear equivalence between divisors coincide (see [Huy16, Chapter 1, Proposition 2.4]), and therefore
\[
\text{Pic}(X) = \text{NS}(X) = \text{Num}(X).
\]

In the case of a K3 surface \( X \) one has that \( \text{NS}(X) \) is a primitive sublattice of \( H^2(X; \mathbb{Z}) \) (the fact that \( h^1(X, \mathcal{O}_X) = 0 \) implies that \( \text{NS}(X) \) embeds into \( H^2(X; \mathbb{Z})/\text{NS}(X) \) embeds into \( H^2(X, \mathcal{O}_X) \cong \mathbb{C} \)). Recall that \( H^2(X; \mathbb{Z}) \) is an even unimodular lattice of signature \((3, 19)\), and is therefore isometric to \( U^{\oplus 3} \oplus E_8^{\oplus 2} \). It follows that the transcendental lattice \( T_X = \text{NS}(X) \perp \) has the same discriminant group as \( \text{NS}(X) \) (see Theorem 2.7). The Picard number of a K3 surface \( X \) is the rank of \( \text{NS}(X) \), and it is denoted by \( \rho(X) \).

### 2.3. Symplectic automorphisms of K3 surfaces.

**Definition 2.10.** Let \( X \) be a K3 surface. An automorphism \( f \) of \( X \) is called *symplectic* if the induced map \( f^* : H^0(X, \omega_X) \to H^0(X, \omega_X) \) is the identity. The effective (left) action of a group \( G \) on the K3 surface \( X \) is called *symplectic* if for all \( g \in G \) the automorphism of \( X \) given by \( x \mapsto g \cdot x \) is symplectic. It is well known that a finite group \( G \) acts symplectically on \( X \) if and only if the minimal resolution \( \tilde{X}/G \) of \( X/G \) is again a K3 surface (see [Muk88, (8.10) Proposition (1)] or [Gar08, Theorem 0.4.2]).

**Definition 2.11.** A symplectic automorphism of order 2 on a K3 surface \( X \) is called *symplectic involution* or *Nikulin involution*. A Nikulin involution on \( X \) has exactly eight fixed points (see [Muk88, Section 0.1]), and in Proposition 2.12 (which is well known) we show that an involution on \( X \) with exactly eight fixed points is necessarily symplectic.

**Proposition 2.12.** Let \( X \) be a K3 surface and let \( \iota \) be an involution on \( X \) with exactly eight fixed points. Then \( \iota \) is a Nikulin involution.

**Proof.** If \( Y = X/\iota \), then we show that \( \tilde{Y} \) is a K3 surface. Let \( X' \to X \) be the blow up at the eight fixed points of \( \iota \) and denote by \( E_1, \ldots, E_8 \) the corresponding exceptional divisors. Then we have the following commutative diagram:
\[
\begin{array}{ccc}
X' & \xrightarrow{\pi} & \tilde{Y} \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y,
\end{array}
\]
where \( \pi \) is the \( \mathbb{Z}_2 \)-cover ramified at \( E_1 + \ldots + E_8 \). Therefore, we have that
\[
E_1 + \ldots + E_8 = K_X \sim \pi^*(K_{\tilde{Y}}) + E_1 + \ldots + E_8 \implies \pi^*(K_{\tilde{Y}}) \sim 0.
\]
We can conclude that \( 2K_{\tilde{Y}} \sim \pi_*\pi^*(K_{\tilde{Y}}) \sim 0 \).
Since $\chi(\mathcal{O}_{X'}) = \chi(\mathcal{O}_X) = 2$, Noether’s formula gives that the topological Euler characteristic $\chi_{\text{top}}(X')$ of $X'$ equals 32. From this we can argue that $\chi_{\text{top}}(\tilde{Y}) = 24$. Noether’s formula again applied to $\tilde{Y}$ implies that $\chi(\mathcal{O}_{\tilde{Y}}) = 2$.

If $B$ denotes the branch locus of $\pi$, then $\pi_*\mathcal{O}_{X'} = \mathcal{O}_{\tilde{Y}} \oplus \mathcal{O}_{\tilde{Y}}(-B/2)$. This guarantees that $q(\tilde{Y}) = 0$, and hence $p_g(\tilde{Y}) = 1$ because $\chi(\mathcal{O}_{\tilde{Y}}) = 2$.

Summing up, $2K_{\tilde{Y}} \sim 0, q(\tilde{Y}) = 0$, and $p_g(\tilde{Y}) = 1$, which is enough to show that $\tilde{Y}$ is a K3 surface.

2.4. Even eights on a K3 surface.

**Definition 2.13.** Let $R_1, \ldots, R_m$ be smooth disjoint rational curves on a K3 surface $X$. Then $\{R_1, \ldots, R_m\}$ is called an even set if $R_1 + \ldots + R_m$ is divisible by 2 in $\text{NS}(X)$.

The following result of Nikulin is a combination of [Nik75, Lemma 3] and [Nik75, Corollary 1].

**Theorem 2.14 (Nik75).** Let $\{R_1, \ldots, R_m\}$ with $m \geq 1$ be an even set on a K3 surface. Then $m = 8$ or 16.

**Observation 2.15.** In Theorem 2.14 it is simple to show that an even set on a K3 surface $X$ cannot have cardinality 4 (this is actually what we need in the current paper). By contradiction, let $\{C_1, C_2, C_3, C_4\}$ be an even set on $X$ and let $C = \frac{1}{2}(C_1 + C_2 + C_3 + C_4)$. Using [Bea96, Theorem I.12], we know that $C$ or $-C$ is effective because $C^2 = -2$. Therefore $C$ is effective, because the ample class on $X$ intersect positively the curves $C_i$. But $C_i \subset \text{Supp}(C)$ for all $i \in \{1, 2, 3, 4\}$ because $C_i \cdot C < 0$. This implies that $C - C_1 - C_2 - C_3 - C_4 = -C$ is also effective. But then $C = 0$, which cannot be.

**Definition 2.16.** An even eight on a K3 surface is an even set of cardinality 8.

**Example 2.17.** In the proof of Proposition 2.12 the irreducible curves in the branch locus of the double cover $\pi: X' \to \tilde{Y}$ form an even eight because they are eight disjoint smooth rational curves and their sum is divisible by 2 in $\text{NS}(\tilde{Y})$.

3. The 4-dimensional subfamily K3 surfaces with $\mathbb{Z}_2^2$ symplectic action

3.1. Triple-double K3 surfaces. Consider six lines $\ell_0, \ldots, \ell_5$ on $\mathbb{P}^2$ without triple intersection points. Divide the six lines into three pairs $(\ell_0, \ell_1), (\ell_2, \ell_3), (\ell_4, \ell_5)$. There exists the double cover $X_1 \to \mathbb{P}^2$ branched along $\ell_0 + \ell_1$ because $\ell_0 + \ell_1 \in 2\text{Pic}(\mathbb{P}^2)$. The pullback to $X_1$ of $\ell_2 + \ell_3$ is also divisible by 2 in $\text{Pic}(X_1)$, so we have another double cover $X_2 \to X_1$. Repeat this construction on $X_2$ with respect to the last pair of lines to obtain a triple-double cover $X_3 \to \mathbb{P}^2$ branched along $\sum_{i=0}^{5} \ell_i$. The preimage of $\ell_0 \cap \ell_1$ under $X_3 \to \mathbb{P}^2$ consists of 4 $A_1$ singularities, and the same applies to $\ell_2 \cap \ell_3$ and $\ell_4 \cap \ell_5$. $X_3$ is smooth away from these 12 singular points. Let $\sigma: X \to X_3$ be the minimal resolution of $X_3$, which is obtained by blowing up the 12 $A_1$ singularities.

**Proposition 3.1.** With the notation introduced above, $X$ is a K3 surface.

**Proof.** A projective realization of $X_3$ can be constructed as follows. Let $[W_0 : W_1 : W_2]$ be coordinates in $\mathbb{P}^2$ and let $L_i = L_i(W_0, W_1, W_2) = 0$ be the equation of the line $\ell_i$. Then $X_3 \subset \mathbb{P}^5$ is given by the vanishing of the following three quadrics:

\[
\begin{align*}
W_0^2 &= L_0 L_1, \\
W_1^2 &= L_2 L_3, \\
W_2^2 &= L_4 L_5.
\end{align*}
\]
From this follows immediately that $K_{X_3} \sim 0$ and $h^1(X_3, \mathcal{O}_{X_3}) = 0$. We conclude that $X$ is a K3 surface because $\sigma^* K_{X_3} \sim K_X$ and $\sigma_* \mathcal{O}_X \cong \mathcal{O}_{X_3}$.

**Definition 3.2.** We call a K3 surface $X$ as in Proposition 3.1 a **triple-double** K3 surface.

**Observation 3.3.** Let $X$ be the triple-double K3 surface as defined above. Let $\text{Bl}_3 \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $\ell_0 \cap \ell_1, \ell_2 \cap \ell_3, \ell_4 \cap \ell_5$. Then $X$ can be viewed as a $\mathbb{Z}_2^3$-cover of $\text{Bl}_3 \mathbb{P}^2$ branched along the strict preimages $\hat{\ell}_0, \ldots, \hat{\ell}_5$ of the six lines in $\mathbb{P}^2$ (see Figure 1). $X$ can also be realized as a hypersurface in $(\mathbb{P}^1)^3$ as follows. The blow up $\text{Bl}_3 \mathbb{P}^2$ can be embedded in $(\mathbb{P}^1)^3$ as a hypersurface given by

$$\sum_{i,j,k=0,1} c_{ijk} X_i Y_j Z_k = 0,$$

where the coefficients $c_{ijk}$ are general, nonzero, and the lines $\hat{\ell}_0, \ldots, \hat{\ell}_5$ are given by the restriction of the toric boundary of $(\mathbb{P}^1)^3$ (see [Sch17, Proposition 3.14]). It follows that $X$ is the hypersurface in $(\mathbb{P}^1)^3$ given by

$$\sum_{i,j,k=0,1} c_{ijk} X_i^2 Y_j^2 Z_k^2 = 0,$$

where the $\mathbb{Z}_2^3$-covering map is given by the restriction to $X$ of

$$([X_0 : X_1], [Y_0 : Y_1], [Z_0 : Z_1]) \mapsto ([X_0^2 : X_1^2], [Y_0^2 : Y_1^2], [Z_0^2 : Z_1^2]).$$

At this point we have several equivalent descriptions of triple-double K3 surfaces. In what follows, we switch from one perspective to another depending on which one is more convenient for our purposes.

**Figure 1.** Toric picture of $\text{Bl}_3 \mathbb{P}^2$ together with the divisor $\hat{\ell}_0 + \ldots + \hat{\ell}_5$

### 3.2. Involutions of a triple-double K3 surface.

**Proposition 3.4.** Let $X$ be a triple-double K3 surface, and view it as a hypersurface in $(\mathbb{P}^1)^3$ of equation

$$\sum_{i,j,k=0,1} c_{ijk} X_i Y_j Z_k = 0,$$

as explained in Observation 3.3. Denote by $i_{ijk}$ the restriction to $X$ of the involution $(\mathbb{P}^1)^3 \to (\mathbb{P}^1)^3$, $([X_0 : X_1], [Y_0 : Y_1], [Z_0 : Z_1]) \mapsto ([X_0 : (-1)^i X_1], [Y_0 : (-1)^j Y_1], [Z_0 : (-1)^k Z_1])$. 


Then the following hold:

(i) \( \iota_{111} \) is an Enriques involution, i.e. \( X/\iota_{111} \) is an Enriques surface;

(ii) If \( \iota \in \{ \iota_{100}, \iota_{101}, \iota_{101} \} \), then \( \iota \) is a rational involution, i.e. \( X/\iota \) is a smooth rational surface. The fixed points locus of \( \iota \) consists of two disjoint smooth genus 1 curves (and hence \( \iota \) is of parabolic type according to [AN06, Section 6.8]). Moreover, \( X/\iota \) is not minimal, and it admits \( \mathbb{P}^1 \times \mathbb{P}^1 \) as a minimal model;

(iii) \( \iota_{110}, \iota_{101}, \iota_{011} \) are Nikulin involutions with pairwise disjoint sets of fixed points.

In particular, the group of automorphisms \( \{ \text{id}_X, \iota_{110}, \iota_{101}, \iota_{011} \} \cong \mathbb{Z}_2^3 \) acts symplectically on \( X \).

**Proof.** \( \iota_{111} \) is an Enriques involution because it has no fixed points (recall that the coefficients \( c_{ijk} \) are nonzero). \( \iota_{110}, \iota_{101}, \iota_{011} \) are symplectic involutions because each one has exactly eight fixed points (see Proposition 2.12). It is simple to observe that \( \iota_{110}, \iota_{101}, \iota_{011} \) have pairwise disjoint sets of fixed points.

Let us prove (ii) for \( \iota_{100} \) (similar arguments hold for \( \iota_{100}, \iota_{001} \)). The fixed points locus of \( \iota_{100} \) is given by

\[
\begin{cases}
X_0 = 0, \\
\sum_{j,k=0,1} c_{1jk} Y_j^2 Z_k^2 = 0
\end{cases}
\]

which are two disjoint smooth genus 1 curves. This is enough to guarantee that \( X/\iota_{100} \) is a smooth rational surface (see [AN06, Section 6]), but let us prove it from scratch. Denote by \( Z \) the quotient \( X/\iota_{100} \), which can be viewed as the following \( (1,2,2) \) hypersurface in \( (\mathbb{P}^1)^3 \):

\[
\sum_{i,j,k=0,1} c_{ijk} X_i Y_j Z_k^2 = 0,
\]

which is smooth because the coefficients \( c_{ijk} \) are general. Consider the short exact sequence

\[
0 \to O(-1,-2,-2) \to O \to O_Z \to 0,
\]

where \( O = O_{(\mathbb{P}^1)^3} \). The associated long exact sequence in cohomology

\[
\ldots \to H^1((\mathbb{P}^1)^3, O) \to H^1(Z, O_Z) \to H^2((\mathbb{P}^1)^3, O(-1,-2,-2)) \to \ldots
\]

gives that \( H^1(Z, O_Z) \cong 0 \) because \( H^1((\mathbb{P}^1)^3, O) \cong H^1((\mathbb{P}^1)^3, O(-2,-2,-2) \otimes O(2,2,2)) \) and \( H^2((\mathbb{P}^1)^3, O(-1,-2,-2)) \cong H^1((\mathbb{P}^1)^3, O(-2,-2,-2) \otimes O(1,2,2)) \) are both isomorphic to zero by Kodaira vanishing. From the adjunction formula we have that

\[
K_Z \sim ((-2,-2,-2) + (1,2,2))|_Z = (-1,0,0)|_Z.
\]

This implies that \( H^0(Z, 2K_Z) \cong 0 \), which together with \( H^1(Z, O_Z) \cong 0 \) guarantees that \( Z \) is rational by Castelnuovo rationality criterion (see [Bea96, Theorem V.1.1]).

Thinking of \( Z \) as \( \mathbb{Z}_2^2 \)-cover of \( \text{Bl}_3 \mathbb{P}^2 \) branched along \( \ell_2, \ldots, \ell_5 \), then \( Z \) contains eight disjoint \((-1\)-curves given by the preimage of the two \((-1\)-curves on \( \text{Bl}_3 \mathbb{P}^2 \) which do not intersect \( \ell_2, \ldots, \ell_5 \). By blowing down these eight disjoint \((-1\)-curves we obtain a surface \( Z' \) with \( K_{Z'}^2 = 8 \). Therefore, \( Z' \) has to be isomorphic to the Hirzebruch surface \( \mathbb{P}_n \) for some \( n \geq 0 \). But it is easy to observe that \( Z' \) has two distinct \( \mathbb{P}^1 \) fibrations, hence \( Z' \cong \mathbb{P}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \).

**Observation 3.5.** A triple-double K3 surface \( X \) has infinite automorphism group because it covers an Enriques surface by Proposition 3.4 (i) (see [Kon86, Page 194]). Moreover, \( X \) contains smooth rational curves (see Proposition 3.6 for a more detailed discussion). Therefore, \( X \) contains infinitely many smooth rational curves by [Huy16, Corollary 4.7].
3.3. A configuration of 24 smooth rational curves on a triple-double K3 surface.

Proposition 3.6. Let $X$ be a triple-double K3 surface and let $X \to \text{Bl}_3 \mathbb{P}^2$ be the corresponding $\mathbb{Z}_2^3$-cover as described in Observation 3.3. Then the preimage of the six $(-1)$-curves on $\text{Bl}_3 \mathbb{P}^2$ under the covering map gives a configuration of 24 smooth rational curves on $X$ whose dual graph is shown in Figure 3.

Proof. Split the $\mathbb{Z}_2^3$-cover $X \to \text{Bl}_3 \mathbb{P}^2$ into three double covers, each one branched along a pair of curves. Then the configuration of 24 smooth rational curves on $X$ can be computed by taking appropriate branched double covers starting from the six $(-1)$-curves on $\text{Bl}_3 \mathbb{P}^2$ as it is shown in Figure 2. The result of the last double cover is shown in Figure 3. \qed

Definition 3.7. Let $X$ be a triple-double K3 surface. Denote by $R_1, \ldots, R_{24}$ the 24 smooth rational curves on $X$ described in Proposition 3.6 and label them as shown in Figure 3. In Section 4 we show that these curves generate $\text{NS}(X)$. 
Figure 3. Dual graph of the smooth rational curves on a triple-double K3 surface arising as the $\mathbb{Z}_3^2$-cover of the six $(-1)$-curves of $\text{Bl}_3 \mathbb{P}^2$.

4. The Néron-Severi lattice of a triple-double K3 surface

Let $X$ be a triple-double K3 surface. In this section, we show that $\text{NS}(X)$ is generated by the 24 smooth rational curves $R_1, \ldots, R_{24}$. Moreover, we give an explicit $\mathbb{Z}$-basis for $\text{NS}(X)$ which decomposes $\text{NS}(X)$ as $U \oplus E_8 \oplus Q$, where $Q$ is described explicitly in Lemma 4.1 (iii).

4.1. The sublattices $S, Q \subset \text{NS}(X)$.

Lemma 4.1. Let $X$ be a triple-double K3 surface. Let $S$ be the sublattice of $\text{NS}(X)$ generated by

$S = \{ R_1, R_5, R_9, R_{13}, R_{17}, R_{23}, R_4, R_{15}, R_8, R_3 \}$.

Let $Q$ be the sublattice of $\text{NS}(X)$ generated by

$Q = \{ R_{16}, R_{14} - R_{21} + R_{22}, R_{11} - R_2 + R_{19} - R_{20}, R_{17} + 2R_{14} - R_{18} - R_{19} + R_{20}, R_{12} - R_{10} + R_{18} + R_{20}, R_3 + 2R_{22} - 2R_6 - R_{12} \}$.

Then the following hold:

(i) $S$ is a $\mathbb{Z}$-basis for $S$ and $S \cong U \oplus E_8$;
(ii) $Q \subset S^\perp$;
(iii) $Q$ is a $\mathbb{Z}$-basis for $Q$ and its corresponding Gram matrix is given by

$$
\begin{pmatrix}
-2 & 0 & 1 & 0 & 2 & -1 \\
0 & -6 & -1 & -4 & 4 & -5 \\
1 & -1 & -8 & 6 & 2 & 0 \\
0 & -4 & 6 & -16 & 4 & -2 \\
2 & 4 & 2 & 4 & -8 & 6 \\
-1 & -5 & 0 & -2 & 6 & -12
\end{pmatrix}.
$$
Figure 4. Subgraph of the dual graph of the curves $R_1, \ldots, R_{24}$ with vertices the $\mathbb{Z}$-basis of the lattice $S$ in Lemma 4.1. These constitute a type $II^*$ singular fiber of an elliptic pencil (see [Kod63, Figure 1]) together with the section $R_3$

Proof. All the statements above can be checked explicitly. We only remark that $S$ is isometric to $U \oplus E_8$ because $S$ is an even unimodular lattice of signature $(1,9)$.

Corollary 4.2. Let $X$ be a triple-double K3 surface. Then $X$ has Picard number 16 and $\text{NS}(X)$ is generated over $\mathbb{Q}$ by $R_1, \ldots, R_{24}$. In particular, given $D_1, D_2 \in \text{NS}(X)$, we have that $D_1 = D_2$ if and only if $D_1 \cdot R_i = D_2 \cdot R_i$ for all $i \in \{1, \ldots, 24\}$.

Proof. The sublattice $S \oplus \mathbb{Q} \subseteq \text{NS}(X)$ has rank 16, hence $\rho(X) = \text{rk}(\text{NS}(X)) \geq \text{rk}(S \oplus \mathbb{Q}) = 16$. We also have that $\rho(X) \leq 16$ because triple-double K3 surfaces form a 4-dimensional family (the inequality follows from the Torelli theorem for K3 surfaces in [PSS71]), therefore we have that $\rho(X) = 16$.

Let $L \subseteq \text{NS}(X)$ be the sublattice of $\text{NS}(X)$ generated by $R_1, \ldots, R_{24}$. Then $S \oplus \mathbb{Q} \subseteq L \subseteq \text{NS}(X)$, which implies that $L$ has rank 16 and hence it generates $\text{NS}(X)$ over $\mathbb{Q}$.

The last statement about $D_1, D_2$ follows from this and from the fact that on a K3 surface numerical equivalence of divisors coincides with linear equivalence (see Section 2.2).

4.2. The discriminant group of $Q$ and proof that $S \oplus \mathbb{Q} = \text{NS}(X)$.

Lemma 4.3. The discriminant group of $Q$ (and hence the discriminant group of $S \oplus \mathbb{Q}$) is isomorphic to $\mathbb{Z}_2^2 \oplus \mathbb{Z}_4^2$, and it is generated by the classes modulo $\mathbb{Q}$ of the following elements in $Q^*$:

$$v_1 = \left(\frac{1}{2}, -\frac{1}{2}, 0, 0, \frac{1}{2}, 0 \right),$$

$$v_2 = \left(-\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0 \right),$$
\[ w_1 = \left( \frac{1}{2}, 0, 0, -\frac{1}{4}, 0, 0 \right), \]
\[ w_2 = \left( 0, 0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right), \]

where the coordinates are with respect to the \( Z \)-basis \( Q \) (see Lemma 4.1).

**Proof.** Let \( B \) be the Gram matrix of \( Q \) associated to the \( Z \)-basis \( Q \) in Lemma 4.1 (iii). Then the lattice \( Q^* \) is generated over \( Z \) by the rows of \( B^{-1} \), and to understand the discriminant group of \( Q \) we compute the Smith normal form of \( B^{-1} \). This can be done using Sage (see [Ste17] function \texttt{smith}\_\texttt{form()}), which gives us \( M_1, M_2 \in \text{SL}_6(\mathbb{Z}) \) such that

\[
M_1 B^{-1} M_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4}
\end{pmatrix}.
\]

This implies that \( A_Q \cong \mathbb{Z}_2^2 \oplus \mathbb{Z}_4^2 \). Moreover, the rows of \( M_1 B^{-1} \) give us an alternative \( Z \)-basis of \( Q^* \) to work with. One has

\[
M_1 B^{-1} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & -\frac{1}{4} & 0 & 0 \\
0 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4}
\end{pmatrix}.
\]

The first 2 rows represent elements in \( Q \) and the last 4 are generators of the discriminant group \( A_Q \), which we denote by \( v_1, v_2, w_1, w_2 \) respectively. \( \square \)

**Remark 4.4.** The matrices \( M_1, M_2 \) in the proof of Lemma 4.3 are

\[
M_1 = \begin{pmatrix}
-1 & -5 & 0 & -2 & 6 & -12 \\
0 & 4 & 3 & 4 & -6 & 5 \\
0 & 5 & 2 & 4 & -5 & 5 \\
1 & -3 & -1 & -2 & 1 & -2 \\
-1 & 1 & -1 & 4 & 0 & 0 \\
0 & 1 & -4 & 5 & 0 & 2
\end{pmatrix}, \quad M_2 = \begin{pmatrix}
0 & 1 & -1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
1 & 2 & -1 & -1 & -1 & 1 \\
0 & 2 & -2 & -2 & -1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

**Lemma 4.5.** Let \( v_1, v_2, w_1, w_2 \in Q^* \) as in Lemma 4.3. Then the values of the symmetric bilinear form \( b_Q^* \) evaluated at pairs of these vectors are shown in the following table.

| \( b_Q^* \) | \( v_1 \) | \( v_2 \) | \( w_1 \) | \( w_2 \) |
|-------------|--------|--------|--------|--------|
| \( v_1 \)   | \( -5 \) | \( \frac{5}{2} \) | \( -1 \) | \( -\frac{1}{2} \) |
| \( v_2 \)   | \( \frac{5}{2} \) | \( -2 \) | \( 1 \) | \( \frac{1}{2} \) |
| \( w_1 \)   | \( -1 \) | \( 1 \) | \( -\frac{3}{2} \) | \( -\frac{5}{4} \) |
| \( w_2 \)   | \( -\frac{1}{2} \) | \( \frac{1}{2} \) | \( -\frac{5}{4} \) | \( -\frac{11}{4} \) |
In particular, the only isotropic elements in the discriminant group $A_Q$ with respect to the discriminant quadratic form $q_Q: A_Q \to \mathbb{Q}/2\mathbb{Z}$ are the classes of

$$2w_1, v_2, v_2 + 2w_1, v_1 + 2w_2, v_1 + 2w_1 + 2w_2, v_1 + 2w_2, v_1 + v_2, v_1 + v_2 + 2w_1.$$  

Moreover, these are not contained in $\text{NS}(X)$.

Proof. A direct calculation gives the values in the table. The listed isotropic vectors are easily obtained after computing all the possible $(av_1 + bw_2 + cw_1 + dw_2)^2$ with $(a, b, c, d) \in \{0, 1\}^2 \times \{0, 1, 2, 3\}^2$.

Given two vectors $u_1, u_2 \in Q^*$, we write $u_1 \approx u_2$ if $u_1 - u_2 \in \text{NS}(X)$. Then we have that

$$2w_1 \approx \frac{R_{17} + R_{18} + R_{19} + R_{20}}{2},$$  

$$v_2 \approx \frac{R_{14} + R_{16} + R_{21} + R_{22}}{2},$$  

$$v_1 + v_2 \approx \frac{R_{10} + R_{12} + R_{18} + R_{20}}{2},$$  

$$v_1 + v_2 + 2w_1 \approx \frac{R_{10} + R_{12} + R_{17} + R_{19}}{2}.$$  

This implies that $2w_1, v_2, v_1 + v_2, v_1 + v_2 + 2w_1 \notin \text{NS}(X)$ by Observation 2.15.

We are left to show that $v_2 + 2w_1, v_1 + 2w_2, v_1 + 2w_1 + 2w_2 \notin \text{NS}(X)$.

- Assume by contradiction that $v_2 + 2w_1 \in \text{NS}(X)$. Then we have that
  $$v_2 + 2w_1 \approx \frac{R_{14} + R_{16} + R_{17} + R_{18} + R_{19} + R_{20} + R_{21} + R_{22}}{2} = \alpha \in \text{NS}(X).$$  

Using Corollary 4.2, it is easy to observe that $R_{13} + R_{14} + R_{17} + R_{18} = R_{15} + R_{16} + R_{19} + R_{20}$ because $(R_{13} + R_{14} + R_{17} + R_{18}) \cdot R_i = (R_{15} + R_{16} + R_{19} + R_{20}) \cdot R_i$ for all $i = 1, \ldots, 24$. In particular, we have that

$$\frac{R_{13} + R_{14} + R_{17} + R_{18} + R_{15} + R_{16} + R_{19} + R_{20}}{2} = \alpha_1 \in \text{NS}(X).$$  

But then $\alpha + \alpha_1 \in \text{NS}(X)$, which contradicts Observation 2.15 because

$$\alpha + \alpha_1 \approx \frac{R_{13} + R_{15} + R_{21} + R_{22}}{2}.$$  

- Similarly, assume that $v_1 + 2w_2 \in \text{NS}(X)$. Then we have that
  $$v_1 + 2w_2 \approx \frac{R_2 + R_3 + R_{11} + R_{12} + R_{14} + R_{16} + R_{17} + R_{18} + R_{21} + R_{22}}{2} = \beta \in \text{NS}(X).$$  

Using Corollary 4.2, we can verify that $R_{11} + R_{12} + R_{14} + R_{16} = R_1 + R_3 + R_{21} + R_{22}$, so that

$$\frac{R_{11} + R_{12} + R_{14} + R_{16} + R_1 + R_3 + R_{21} + R_{22}}{2} = \beta_1 \in \text{NS}(X).$$  

But then $\beta + \beta_1 \in \text{NS}(X)$, which contradicts Observation 2.15 because

$$\beta + \beta_1 \approx \frac{R_1 + R_2 + R_{17} + R_{18}}{2}.$$
Remark 4.8. For a triple-double K3 surface 

\[ v_1 + 2w_1 + 2w_2 \in \text{NS}(X), \]

then

\[ v_1 + 2w_1 + 2w_2 \approx \frac{R_2 + R_3 + R_{11} + R_{12} + R_{14} + R_{16} + R_{19} + R_{20} + R_{21} + R_{22}}{2} = \gamma \in \text{NS}(X). \]

Let \( \beta_1 \) as in the previous point. Then \( \gamma + \beta_1 \in \text{NS}(X) \), which is not allowed by Observation 2.15 because

\[ \gamma + \beta_1 \approx \frac{R_1 + R_2 + R_{19} + R_{20}}{2}. \]

\[ \square \]

Theorem 4.6. Let \( X \) be a triple-double K3 surface. Then the following hold:

(i) \( \text{NS}(X) = S \oplus Q \cong U \oplus E_8 \oplus Q \);

(ii) \( \text{NS}(X) \) is generated by the 24 smooth rational curves \( R_1, \ldots, R_{24} \);

(iii) The discriminant group of \( \text{NS}(X) \) is \( \mathbb{Z}_2^3 \oplus \mathbb{Z}_4^2 \).

Proof. \( \text{NS}(X) \) is an even overlattice of \( S \oplus Q \). Therefore \( \text{NS}(X)/(S \oplus Q) \) corresponds to a subgroup of \( A_{(S \oplus Q)} \cong A_Q \) which is isomorphic with respect to the discriminant quadratic form \( q_Q : A_Q \rightarrow \mathbb{Q}/2\mathbb{Z} \) (see Theorem 2.6). However, Proposition 4.5 shows that no isotropic vector of \( A_Q \) can be contained in \( \text{NS}(X) \). This implies that \( \text{NS}(X) \) has to be equal to \( S \oplus Q \), which recall is isometric to \( U \oplus E_8 \oplus Q \) by Lemma 4.1 (i). It also follows that \( \text{NS}(X) \) is generated by \( R_1, \ldots, R_{24} \) because the sublattice of \( \text{NS}(X) \) generated by these 24 curves contains \( S \oplus Q \). Finally, \( A_{\text{NS}(X)} = A_{(S \oplus Q)} \cong A_Q \), which is isomorphic to \( \mathbb{Z}_2^3 \oplus \mathbb{Z}_4^2 \) by Lemma 4.3.

Remark 4.7. If \( X \) is a triple-double K3 surface, then the following \( \mathbb{Z} \)-basis of \( \text{NS}(X) \) realizes it as a direct sum of \( U, E_8 \), and \( Q \) (see Lemma 4.1):

\[
\text{NS}(X) = S \oplus Q = \langle 2R_1 + 2R_4 + 4R_5 + R_8 + 6R_9 + 5R_{13} + 3R_{15} + 4R_{17} + 3R_{23}, R_3 \rangle \mathbb{Z} \\
\quad \quad \oplus \langle R_1, R_5, R_9, R_{13}, R_{17}, R_{23}, R_4, R_{15} \rangle \mathbb{Z} \\
\quad \quad \oplus \langle R_{16}, R_{14} - R_{21} + R_{22}, R_{11} - R_2 + R_{19} - R_{20}, R_{17} + 2R_{14} - R_{18} - R_{19} + R_{20}, R_{12} - R_{10} + R_{18} + R_{20}, R_3 + 2R_{22} - 2R_6 - R_{12} \rangle \mathbb{Z} \cong U \oplus E_8 \oplus Q.
\]

Remark 4.8. For a triple-double K3 surface \( X \), a splitting \( \text{NS}(X) \cong U \oplus E_8 \oplus P \) for some lattice \( P \) is predicted abstractly by Theorem 2.2 as follows. We know that \( \text{NS}(X) \) is even, of signature (1,15), and discriminant group \( \mathbb{Z}_2^3 \oplus \mathbb{Z}_4^2 \). It follows from Theorem 2.2 (i) that \( \text{NS}(X) \) is isometric to \( E_8 \oplus P' \) for some lattice \( P' \). But then \( P' \) is even, of signature (1,7), and its discriminant group is also \( \mathbb{Z}_2^3 \oplus \mathbb{Z}_4^2 \). In particular, we can apply Theorem 2.2 (ii) to \( P' \) to argue that \( P' = U \oplus P \) for some lattice \( P \). In conclusion, we have that \( \text{NS}(X) \cong U \oplus E_8 \oplus P \), but all we know about \( P \) is that it is even, of signature (0,6), and discriminant group \( \mathbb{Z}_2^3 \oplus \mathbb{Z}_4^2 \). However, in our case we were able to provide an explicit lattice \( Q \) (see Lemma 4.1 (iii)) which realizes the splitting \( \text{NS}(X) \cong U \oplus E_8 \oplus Q \).

Remark 4.9. Let \( X \) be a triple-double K3 surface. The fact that the discriminant group of \( \text{NS}(X) \) is not 2-elementary (see Theorem 4.6 (iii)) gives us another way to show \( \text{Aut}(X) \) is infinite (we already proved this in Observation 3.3). If an algebraic K3 surface has finite automorphism group and the discriminant group of its Néron-Severi lattice is not 2-elementary, then its Picard number cannot equal 16 by [Kon89, Page 4].
4.3. The transcendental lattice of a triple-double K3 surface.

**Proposition 4.10.** Let $X$ be a triple-double K3 surface. Then the transcendental lattice $T_X$ is isometric to $U \oplus U(2) \oplus \mathbb{Z}^2(-4)$.

**Proof.** The transcendental lattice $T_X$ is an even lattice of signature $(2, 4)$. Let us study its discriminant quadratic form. The table in the statement of Lemma 4.5 gives the discriminant quadratic form of the lattice $\text{NS}(X) = S \oplus Q$ with respect to $v_1, v_2, w_1, w_2$, whose classes modulo $Q$ generate the discriminant group. Changing basis to $U$ to $\mathbb{Z}$, we can show that $U$ is isometric to $U \oplus U(2) \oplus \mathbb{Z}^2(-4)$ because $U$ is a primitive sublattice of the even unimodular lattice $U \oplus U(2) \oplus \mathbb{Z}^2(-4)$.

Observe that the lattice $U \oplus U(2) \oplus \mathbb{Z}^2(-4)$ is even, of signature $(2, 4)$, and its discriminant quadratic form equals the discriminant quadratic form of $T_X$. We can conclude by Theorem 2.1 that $T_X$ is isometric to $U \oplus U(2) \oplus \mathbb{Z}^2(-4)$. \qed

Recall that projective Kummer surfaces are special types of projective K3 surfaces obtained as the minimal resolution of the quotient of an abelian surface $A$ by the inversion morphism $a \mapsto -a$. We denote such Kummer surface by $\text{Km}(A)$. A celebrated example of Kummer surface is $\text{Km}(E_i \times E_i)$, where $E_i$ is the elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$. The automorphism group of $\text{Km}(E_i \times E_i)$ is studied in [KK01], and in [GS09] this Kummer surface was used to compute $\Omega_G$ and $\Omega_G^\perp$ for $G = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2^2$, $\mathbb{Z}_4$. In the next corollary we show that $\text{Km}(E_i \times E_i)$ appears as a specialization of the 4-dimensional family of triple-double K3 surfaces.

**Corollary 4.11.** The Kummer surface $\text{Km}(E_i \times E_i)$ is a specialization of the family of triple-double K3 surfaces.

**Proof.** The transcendental lattice of $\text{Km}(E_i \times E_i)$ is $\mathbb{Z}^2(4)$ (see [KK01], Section 1)). First, let us show that $\mathbb{Z}^2(4)$ admits a primitive embedding into the transcendental lattice of a triple-double K3 surface, which is $U \oplus U(2) \oplus \mathbb{Z}^2(-4)$ by Proposition 4.10. Consider the sublattice $L \subset U \oplus U(2) \oplus \mathbb{Z}^2(-4)$ generated by $\alpha = (1, 2, 0, 0, 0, 0, 0), \beta = (0, 0, 1, 1, 0, 0)$. We have that $L$ is isometric to $\mathbb{Z}^2(4)$ because $\alpha^2 = \beta^2 = 4$ and $\alpha \cdot \beta = 0$. To show that $L$ is a primitive sublattice, assume we have $m(x, y, z, w, u, v) \in L$ for some integer $m > 1$. Then $y = 2x, z = w, u = v = 0$, which implies $(x, y, z, w, u, v) \in L$.

Let $P$ be the Néron-Severi lattice of a triple-double K3 surface, which recall is always isometric to $U \oplus E_8 \oplus Q$ with $Q$ as in Lemma 4.1. Then to conclude that $\text{Km}(E_i \times E_i)$ is a specialization of the family of triple-double K3 surfaces we need to show that the moduli space of $P$-polarized K3
surfaces is irreducible. But this is guaranteed by [Dol96] Proposition (5.6), in combination with [Dol96] Lemma (5.4).

The previous corollary guarantees the existence of a configuration of three pairs of lines in \( \mathbb{P}^2 \) such that the minimal resolution of the appropriate \( \mathbb{Z}_2^3 \)-cover of \( \mathbb{P}^2 \) gives \( \text{Km}(E_i \times E_i) \) (by appropriate we mean the usual tower of three double covers). The next proposition describes explicitly this line arrangement.

**Proposition 4.12.** Consider three pairs of lines \((\ell_0, \ell_1), (\ell_2, \ell_3), (\ell_4, \ell_5)\) in \( \mathbb{P}^2 \) such that the resulting line arrangement has exactly four triple intersection points as shown in Figure 6 (this is unique up to an automorphism of \( \mathbb{P}^2 \)). Then the minimal resolution of the appropriate \( \mathbb{Z}_2^3 \)-cover of \( \mathbb{P}^2 \) branched along these three pairs is isomorphic to the singular Kummer surface \( \text{Km}(E_i \times E_i) \).

**Proof.** Let \( \text{Bl}_3 \mathbb{P}^2 \) be the blow up of \( \mathbb{P}^2 \) at the three crossed points in Figure 6. Let \( X \to \text{Bl}_3 \mathbb{P}^2 \) be the appropriate \( \mathbb{Z}_2^3 \)-cover branched along the three pairs of lines \((\ell_0, \ell_1), (\ell_2, \ell_3), (\ell_4, \ell_5)\). In particular, \( X \) has exactly four \( A_1 \) singularities above the four triple intersection points of the line arrangement. Following Observation 4.3, a concrete realization of \( X \) is as the following hypersurface in \((\mathbb{P}^1)^3\):

\[
X_1^2 Y_0^2 Z_2^2 + X_0^2 Y_1^2 Z_0^2 + X_0^2 Y_1^2 Z_1^2 + X_1^2 Y_0^2 Z_1^2 = 0.
\]

The blow up of \( X \) at the four \( A_1 \) singularities is a K3 surface.

Consider the genus 1 fibration on \( X \) given by the restriction to \( X \) of the projection

\[
\pi_3: ([X_0 : X_1], [Y_0 : Y_1], [Z_0 : Z_1]) \mapsto [Z_0 : Z_1].
\]

The general fiber of this fibration is a genus 1 curve in \( \mathbb{P}^1 \times \mathbb{P}^1 \) given by

\[
C: \lambda^2 X_1^2 Y_0^2 + \lambda^2 X_0^2 Y_1^2 + \mu^2 X_0^2 Y_0^2 + \mu^2 X_1^2 Y_1^2 = 0,
\]

for a general \([\lambda : \mu] \in \mathbb{P}^1\). The restriction to \( C \) of the projection \(([X_0 : X_1], [Y_0 : Y_1]) \mapsto [Y_0 : Y_1]\) realizes \( C \) as a double cover of \( \mathbb{P}^1 \) branched along

\[
[i\lambda : \mu], [-i\lambda : \mu], [i\mu : \lambda], [-i\mu : \lambda].
\]

If we set \( \sigma = i(\lambda/\mu) \) in the affine patch of \( \mathbb{P}^1 \) where \( Y_1 \neq 0 \), then the four branch points above become respectively

\[
\sigma, -\sigma, -\frac{1}{\sigma}, \frac{1}{\sigma}.
\]

Using the automorphism of \( \mathbb{P}^1 \) given by \( z \mapsto \left( \frac{z + \sigma^3}{2} \right) \cdot \frac{z - \sigma}{z - 1} \), we can move these branch points to

\[
0, \sigma^2, \frac{(1 + \sigma^2)^2}{4}, \infty,
\]

respectively. But then the elliptic fibration \( \pi_3|_X : X \to \mathbb{P}^1 \) is isomorphic to the elliptic fibration (6) in [GS09] Section 4] (set \( \tau = 1 \)), which gives \( \text{Km}(E_i \times E_i) \).

\( \square \)

4.4. **Further properties of triple-double K3 surfaces.** The next proposition, among other things, shows that the family of triple-double K3 surfaces is disjoint from the family of K3 surfaces with \( \mathbb{Z}_2^3 \) symplectic action.

**Proposition 4.13.** Let \( X \) be a triple-double K3 surface. Then the following hold:

(i) \( X \) does not admit \( \mathbb{Z}_2^3 \) symplectic action. In particular, \( X \) does not admit \( \mathbb{Z}_2^4 \) symplectic action, hence [GS09] Proposition 6.2 cannot be used to compute \( \text{NS}(X) \);
(ii) $X$ is not isomorphic to the minimal resolution of the quotient of a K3 surface by a symplectic action of the group $\mathbb{Z}_2^2$. In particular, $X$ is not isomorphic to the minimal resolution of the quotient of a K3 surface by a symplectic action of the group $\mathbb{Z}_4^2$, hence $\text{NS}(X)$ cannot be computed using [GS16, Theorem 8.3].

Proof. To prove part (i), $X$ admits $\mathbb{Z}_2^3$ symplectic action if and only if the transcendental lattice $T_X$ embeds primitively into $\Omega_{\mathbb{Z}_2^3}^\perp$ (see [Nik79]). But $T_X \cong U \oplus U(2) \oplus \mathbb{Z}^2(-4)$ by Proposition [1.10] and $\Omega_{\mathbb{Z}_2^3}^\perp \cong U(2)^{\oplus 3} \oplus \mathbb{Z}^2(-4)$ by [GS09, Proposition 5.1]. Moreover, $(1, 1, 0, 0, 0) \in U \oplus U(2) \oplus \mathbb{Z}^2(-4)$ squares giving $2$ and $4 \mid x^2$ for all $x \in U(2)^{\oplus 3} \oplus \mathbb{Z}^2(-4)$. Therefore, $T_X$ cannot embed primitively into $\Omega_{\mathbb{Z}_2^3}^\perp$.

For part (ii), we first need some preliminaries. Let $M_{\mathbb{Z}_2^3}$ be the abstract lattice generated by the following vectors:

$$m_1, \ldots, m_{14}, \frac{\sum_{i=1}^{8} m_i}{2}, \frac{\sum_{i=5}^{12} m_i}{2}, \frac{m_1 + m_2 + m_5 + m_6 + m_9 + m_{10} + m_{13} + m_{14}}{2},$$

where $m_i^2 = -2$ and $m_i \cdot m_j = 0$ for all $i, j \in \{1, \ldots, 14\}, i \neq j$. The lattice $M_{\mathbb{Z}_2^3}$ is negative definite of rank 14, has discriminant group $\mathbb{Z}_2^8$, and its discriminant quadratic form is obviously not identically zero (for instance $\frac{m_i + m_j}{2} \in M_{\mathbb{Z}_2^3}$ and it squares giving $-1$). This implies by Theorem [2.9] that $M_{\mathbb{Z}_2^3}$,
the orthogonal complement of $M_{\mathbb{Z}^3}$ in $H^2(X; \mathbb{Z})$, is isometric to the lattice $\mathbb{Z}^2(2) \oplus U(2) \oplus \mathbb{Z}^4(-2)$. Now, $X$ is isomorphic to the minimal resolution of the quotient of a K3 surface by a symplectic action of the group $\mathbb{Z}_2^3$ if and only if the lattice $M_{\mathbb{Z}^3}$ embeds primitively into $\text{NS}(X)$ (see [Gar16, Definition 2.5] and [GS16, Corollary 8.9]), or equivalently if and only if $T_X$ embeds primitively into $M_{\mathbb{Z}^3}$. But $T_X \cong U \oplus U(2) \oplus \mathbb{Z}^2(-4)$, and in particular there are two vectors $x, y \in T_X$ such that $x \cdot y = 1$. However, for any two vectors $x, y \in M_{\mathbb{Z}^3} \cong \mathbb{Z}^2(2) \oplus U(2) \oplus \mathbb{Z}^4(-2)$, the product $x \cdot y$ is even.

**Corollary 4.14.** Let $X$ be a triple-double K3 surface. Then $X$ does not contain 14 disjoint smooth rational curves.

**Proof.** If $X$ contains 14 disjoint smooth rational curves, then the lattice $M_{\mathbb{Z}^3}$ (see the proof of Proposition 4.13(ii)) would be primitively embedded in $\text{NS}(X)$. But this would contradict Proposition 4.13(ii) (see [GS16, Corollary 8.9]).

## 5. $\mathbb{Z}_2^5$-covers of $\mathbb{P}^2$ branched along six lines and K3 surfaces

### 5.1. Kummer coverings.

**Definition 5.1.** Let $n$ be a positive integer and let $V$ be a normal variety. Let $D \subset V$ be a closed subvariety containing the singularities of $V$. Then the **Kummer covering of exponent $n$** of $V$ branched along $D$ is the normal finite covering $Y \to V$ such that its restriction to $V \setminus D$ is the Galois unramified covering associated to the monodromy homomorphism

$$H_1(V \setminus D; \mathbb{Z}) \to H_1(V \setminus D; \mathbb{Z}) \otimes \mathbb{Z}_n.$$  

If $V = \mathbb{P}^2$ and $D \subset \mathbb{P}^2$ is a line arrangement such that no point in $\mathbb{P}^2$ belongs to all the lines, then the minimal resolution of $Y$ is called the **Hirzebruch-Kummer covering of exponent $n$** of $\mathbb{P}^2$ branched along the configuration of lines. An explicit construction of the Hirzebruch-Kummer covering can be found in [Cat16, Definition 63].

**Example 5.2.** We are interested in considering the Hirzebruch-Kummer covering of exponent 2 of $\mathbb{P}^2$ branched along six general lines. This is constructed as follows. Let $[W_0 : \ldots : W_5]$ be coordinates in $\mathbb{P}^5$. Then, for general coefficients $a_i, b_i, c_i, i = 0, \ldots, 5$, let $Y \subset \mathbb{P}^5$ be the following intersection:

$$\begin{cases}
  a_0W_0^2 + \ldots + a_5W_5^2 = 0, \\
  b_0W_0^2 + \ldots + b_5W_5^2 = 0, \\
  c_0W_0^2 + \ldots + c_5W_5^2 = 0.
\end{cases}$$

$Y$ is smooth for a general choice of the coefficients, hence $Y$ is a K3 surface. Consider the restriction to $Y$ of the morphism $\mathbb{P}^5 \to \mathbb{P}^5$ given by

$$[W_0 : \ldots : W_5] \mapsto [W_0^2 : \ldots : W_5^2].$$

Then this realizes $Y$ as a $\mathbb{Z}_2^5$-cover of the linear subspace $H \subset \mathbb{P}^5$ given by

$$\begin{cases}
  a_0W_0 + \ldots + a_5W_5 = 0, \\
  b_0W_0 + \ldots + b_5W_5 = 0, \\
  c_0W_0 + \ldots + c_5W_5 = 0,
\end{cases}$$

which is isomorphic to $\mathbb{P}^2$. The branch locus of this cover is given by the restrictions to $H$ of the six coordinate hyperplanes of $\mathbb{P}^5$, which correspond to six general lines on $\mathbb{P}^2$. Label the six lines
By the properties of the Hirzebruch-Kummer covering, the composition of the double cover \( f: X \to \mathbb{P}^2 \) followed by the minimal resolution \( X \to X_3 \). Let \( R_1, \ldots, R_{24} \) be the smooth rational curves of Definition 3.7. Up to relabeling, we can assume that

\[
\begin{align*}
 f^{-1}(\ell_0 \cap \ell_1) &= \{R_1, R_2, R_3, R_4\}, \\
 f^{-1}(\ell_2 \cap \ell_3) &= \{R_9, R_{10}, R_{11}, R_{12}\}, \\
 f^{-1}(\ell_4 \cap \ell_5) &= \{R_{17}, R_{18}, R_{19}, R_{20}\}.
\end{align*}
\]

Then the proof of Proposition 5.5 together with Example 2.17 guarantee the following three sets are even eights:

\[
\begin{align*}
 R_1, R_2, R_3, R_4, R_9, R_{10}, R_{11}, R_{12} \cup R_1, R_2, R_3, R_4, R_9, R_{10}, R_{11}, R_{12} \cup R_1, R_2, R_3, R_4, R_9, R_{10}, R_{11}, R_{12} \cup R_1, R_2, R_3, R_4, R_9, R_{10}, R_{11}, R_{12}.
\end{align*}
\]

The symmetry of the configuration of curves \( R_1, \ldots, R_{24} \) implies that also the sets

\[
\begin{align*}
 \{R_5, R_6, R_7, R_8, R_{13}, R_{14}, R_{15}, R_{16}\} \cup \{R_{13}, R_{14}, R_{15}, R_{16}, R_{21}, R_{22}, R_{23}, R_{24}\}, \\
 \{R_{21}, R_{22}, R_{23}, R_{24}, R_5, R_6, R_7, R_8\}
\end{align*}
\]

are even eights. Alternatively, one can show these six sets are even eights because the following equalities hold:

\[
\frac{R_1 + R_2 + R_3 + R_4}{2} + \frac{R_6 + R_{10} + R_{11} + R_{12}}{2} = R_3 + R_4 - R_6 + R_7 + R_9 + R_{11},
\]
6.1.1. We need to understand how the involutions act on these curves. In Figure 2, we can see how \( \iota \) acts by base change on the configuration fixing the eight curves with two branches on each one. Therefore, we have that

\[
\iota_0 \cdot (1, \ldots, 24)
\]

6.1.2. Computing the lattices \( \text{NS}(X') \) and \( T_{X'} \). We compute the lattice \( \text{NS}(X') \) in several steps.

5.4. Other K3 surfaces which are \( \mathbb{Z}_2^2 \)-covers of \( \mathbb{P}^2 \) branched along six lines. Consider six general lines in \( \mathbb{P}^2 \) and denote by \( D \) the divisor on \( \mathbb{P}^2 \) given by their sum. Let \( Y \) be the Hirzebruch-Kummer covering of exponent 2 of \( \mathbb{P}^2 \) branched along \( D \) in Example 5.2. Let \( X \) be a triple-double K3 surface obtained as a \( \mathbb{Z}_2^3 \)-cover of \( \mathbb{P}^2 \) branched along \( D \). Lastly, let \( Z \) be the minimal resolution of the double cover of \( \mathbb{P}^2 \) branched along \( D \). As we already remarked in the Introduction, the lattices \( \text{NS}(Y) \) and \( \text{NS}(Z) \) are well known, and the lattice \( \text{NS}(X) \) is computed in the current paper. In the next section, we determine the Néron-Severi lattices of the K3 surface \( X' \), which is the minimal resolution of the quotient of \( X \) by an involution coming from the \( \mathbb{Z}_2^2 \) symplectic action (say, \( \iota_{011} \) in Proposition 3.4). \( X' \) can also be viewed as an appropriate \( \mathbb{Z}_2^2 \)-cover of \( \mathbb{P}^2 \) branched along \( D \). There are other K3 surfaces different from \( X, X', Y, Z \) which are \( \mathbb{Z}_2^3 \)-covers of \( \mathbb{P}^2 \) branched along six general lines with \( n = 3, 4 \). We will investigate their Néron-Severi and transcendental lattices in future work.

Remark 5.6. If \( X' \) is as above, after computing \( \text{NS}(X') \) and \( T_{X'} \) in Section 6 we realize that \( X' \) admits \( \mathbb{Z}_2^4 \) symplectic action (see Proposition 6.4). Since \( \rho(X') = 16 \), this implies that \( \text{NS}(X') \) can be computed using [GS16]. However, in Section 6 we are able to find an explicit \( \mathbb{Z} \)-basis for \( \text{NS}(X') \) and relate it to the geometry of \( X' \).

6. The Néron-Severi lattice of \( X' \)

In this section, \( X \) denotes a triple-double K3 surface, and \( X' \) is the minimal resolution of the quotient of \( X \) by the symplectic involution \( \iota_{011} \) in Proposition 3.4 (\( \iota_{110}, \iota_{101} \) are treated similarly).
smooth rational curves on \( X \).

6.1.2. Even eights on \( X \). Corresponding curve arrangement on \( N \) denote by \( X \) cover of \( R \).

In particular we have the following classes of curves modulo \( \iota \) and hence

\[
\iota_{010} \cdot (1, \ldots, 24) = (2, 1, 4, 3, 5, 6, 7, 8, 11, 12, 9, 10, 14, 13, 16, 15, 17, 18, 19, 20, 23, 24, 21, 22),
\]

and hence

\[
\iota_{011} \cdot (1, \ldots, 24) = (4, 3, 2, 1, 7, 8, 5, 6, 11, 12, 9, 10, 16, 15, 14, 13, 19, 20, 17, 18, 23, 24, 21, 22).
\]

In particular we have the following classes of curves modulo \( \iota_{011} \):

\[
\{R_1, R_4\}, \{R_2, R_3\}, \{R_5, R_7\}, \{R_6, R_8\}, \{R_9, R_{11}\}, \{R_{10}, R_{12}\}, \{R_{13}, R_{16}\}, \{R_{14}, R_{15}\}, \{R_{17}, R_{19}\}, \{R_{18}, R_{20}\}, \{R_{21}, R_{23}\}, \{R_{22}, R_{24}\}.
\]

Consider the commutative diagram (which we discussed in the proof of Proposition 2.12)

\[
\begin{array}{ccc}
\text{Bl}_8 X & \xrightarrow{\pi} & X' \\
\downarrow & & \downarrow \\
X & \longrightarrow & X/\iota_{011},
\end{array}
\]

where \( \text{Bl}_8 X \) is the blow up of \( X \) at the eight fixed point of \( \iota_{011} \). So \( \pi : \text{Bl}_8 X \rightarrow X' \) is the \( \mathbb{Z}_2 \)-cover of \( X' \) branched along the eight exceptional divisors of the resolution \( X' \rightarrow X/\iota_{011} \), which we denote by \( N_1, \ldots, N_8 \). If we set \( C_1 = \pi(R_1) = \pi(R_4), C_2 = \pi(R_2) = \pi(R_3), \) and so on, then the corresponding curve arrangement on \( X' \) is shown in Figure 6. In conclusion, we have two sets of smooth rational curves on \( X' \) given by

\[
\{C_1, \ldots, C_{12}\} \text{ and } \{N_1, \ldots, N_8\}.
\]

6.1.2. Even eights on \( X' \). With the notation introduced above, we know that

\[
N = \frac{N_1 + \ldots + N_8}{2} \in \text{NS}(X'),
\]

and the lattice generated by \( N_1, \ldots, N_8, \frac{N_1 + \ldots + N_8}{2} \) is called the Nikulin lattice. Let us find more sets of curves on \( X' \) whose sum is 2-divisible.

We recall the following result on Nikulin, which can be found in [Gar16, Section 2.3]. We state it in a form that is convenient for us.

**Proposition 6.1.** Let \( L_1, \ldots, L_k \) be smooth disjoint rational curves on a K3 surface \( W \). Then the following hold:

- If \( k = 13 \), then, up to reordering the indices,

\[
\sum_{i=1}^{8} L_i, \quad \sum_{i=5}^{12} L_i \in \text{NS}(W).
\]

- If \( k = 14 \), then, up to reordering the indices,

\[
\sum_{i=1}^{8} L_i, \quad \sum_{i=5}^{12} L_i, \quad \frac{L_1 + L_2 + L_5 + L_6 + L_9 + L_{10} + L_{13} + L_{14}}{2} \in \text{NS}(W).
\]
Let us apply Proposition 6.1 for $k = 13$ to the curves $C_1, C_5, C_6, C_9, C_{10}, N_1, \ldots, N_8$. We already know that $N \in \text{NS}(X').$ This implies that, up to reordering the indices of the curves $N_i$, the sum of $N_1 + N_2 + N_3 + N_4$ with other four curves among $C_1, C_5, C_6, C_9, C_{10}$ is 2-divisible. If we use $C_1$ and $\{C_i, C_j, C_k\} \subset \{C_5, C_6, C_9, C_{10}\}$, then the intersection number 
\[
\left( \frac{C_1 + C_i + C_j + C_k + N_1 + N_2 + N_3 + N_4}{2} \right) \cdot C_3,
\]
is not an integer. Therefore, we have that 
\[
\Lambda_1 = \frac{C_5 + C_6 + C_9 + C_{10} + N_1 + N_2 + N_3 + N_4}{2} \in \text{NS}(X').
\]
But now let us apply Proposition 6.1 for $k = 14$ to the curves $C_1, C_2, C_5, C_6, C_9, C_{10}, N_1, \ldots, N_8$. Then we have that 
\[
\frac{C_1 + C_2 + C_i + C_j + N_1 + N_2 + N_5 + N_6}{2} \in \text{NS}(X'),
\]
where $C_i, C_j$ is a choice of two curves among $C_5, C_6, C_9, C_{10}$. This is true up to permuting $N_1, \ldots, N_4$, and up to permuting $N_5, \ldots, N_8$. Up to permuting $C_5, C_6, C_9, C_{10}$ we can assume that $\{i, j\} = \{5, 6\}$, hence 
\[
\Lambda_2 = \frac{C_1 + C_2 + C_5 + C_6 + N_1 + N_2 + N_5 + N_6}{2} \in \text{NS}(X').
\]
6.1.3. Computation of $\NS(X')$ and $T_{X'}$.

**Theorem 6.2.** Let $X'$ be the minimal resolution of the quotient of a triple-double K3 surface by an involution coming from the $\Z_2^2$ symplectic action. Then the lattice $\NS(X')$ has rank 16 and discriminant group $\Z_2^4 \oplus \Z_2^4$. A $\Z$-basis for $\NS(X')$ is given by

$$\{C_1, C_2, C_3, C_4, C_5, C_7, C_8, C_9, N_1, N_2, N_3, N_5, N_7, N, \Lambda_1, \Lambda_2\}.$$

The discriminant quadratic form of $\NS(X')$ is given by

$$
\begin{pmatrix}
0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4}
\end{pmatrix}.
$$

It follows that the rank 6 transcendental lattice $T_{X'}$ is isometric to $U(2)^{\oplus 2} \oplus \Z^2(-4)$.

**Proof.** Some of the computations that follow are computer assisted (we used [Wol17]). Define

$$M = \langle C_1, \ldots, C_{12}, N_1, \ldots, N_8, N, \Lambda_1, \Lambda_2 \rangle \subset \NS(X').$$

We show that $M = \NS(X')$. The lattice $M$ has rank 16, implying that $\rho(X') \geq 16$. But since the K3 surfaces $X'$ vary in a 4-dimensional family, we have that $\rho(X') = 16$ (see [PSS71]). Therefore, $M$ is a sublattice of $\NS(X')$ of finite index.

A $\Z$-basis for $M$ is given by

$$\{C_1, C_2, C_3, C_4, C_5, C_7, C_8, C_9, N_1, N_2, N_3, N_5, N_7, N, \Lambda_1, \Lambda_2\}.$$

This is true because

$$C_6 = C_3 - C_4 + C_5,$$

$$C_{10} = C_1 + C_2 + C_3 + C_4 - C_7 - C_8 - C_9,$$

$$C_{11} = C_3 + C_5 - C_9,$$

$$C_{12} = -C_1 - C_2 - C_4 + C_5 + C_7 + C_8 + C_9,$$

$$N_4 = -C_1 - C_2 - 2C_3 - 2C_5 + C_7 + C_8 - N_1 - N_2 - N_3 + 2\Lambda_1,$$

$$N_6 = -C_1 - C_2 - C_3 + C_4 - 2C_5 - N_1 - N_2 - N_5 + 2\Lambda_2,$$

$$N_8 = 2C_1 + 2C_2 + 3C_3 - C_4 + 4C_5 - C_7 - C_8 + N_1 + N_2 - N_7 + 2N - 2\Lambda_1 - 2\Lambda_2.$$

To show that $M = \NS(X')$ we use the same strategy we used in the proof of Theorem 4.6; we prove that any isotropic element of the discriminant group $A_M$ cannot be an element in $\NS(X')$. If $B$ is the intersection matrix of the curves in the $\Z$-basis of $M$ above, then the dual $M^*$ is generated over $\Z$ by the rows of $B^{-1}$. Using Sage, we can find matrices $M_1, M_2 \in \SL_{16}(\Z)$ such that

$$M_1 B^{-1} M_2 = \text{diag} \left( 1, \ldots, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4} \right).$$
This tells us that the discriminant group of $M$ is isomorphic to $\mathbb{Z}_2^4 \oplus \mathbb{Z}_4^2$. In particular, the rows of $M_1B^{-1}$ give us an alternative $\mathbb{Z}$-basis of $M^*$ to work with. We have that

$$
M_1B^{-1} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
$$

Denote by $v_1, v_2, v_3, v_4, w_1, w_2$ respectively the last six rows of the matrix above, which generate the discriminant group $A_M$. We can then enumerate the isotropic elements of $A_M$, which are given by the classes modulo $M$ of the following vectors:

$$
(C_1 + C_2 + C_7 + C_8)/2,
(C_1 + C_2 + C_3 + C_4)/2,
(C_3 + C_4 + C_7 + C_8)/2,
(C_5 + C_9 + N_5 + N_7)/2,
(C_1 + C_2 + C_5 + C_7 + C_8 + C_9 + N_5 + N_7)/2,
(C_3 + C_4 + C_5 + C_9 + N_5 + N_7)/2,
(C_1 + C_2 + C_3 + C_4 + C_5 + C_7 + C_8 + C_9 + N_5 + N_7)/2,
(N_1 + N_3 + N_5 + N_7)/2,
(C_1 + C_2 + C_7 + C_8 + N_1 + N_3 + N_5 + N_7)/2,
(C_1 + C_2 + C_3 + C_4 + N_1 + N_3 + N_5 + N_7)/2,
(C_3 + C_4 + C_7 + C_8 + N_1 + N_3 + N_5 + N_7)/2,
(C_5 + C_9 + N_1 + N_3)/2,
(C_1 + C_2 + C_5 + C_7 + C_8 + C_9 + N_1 + N_3)/2,
(C_3 + C_4 + C_5 + C_9 + N_1 + N_3)/2,
(C_1 + C_2 + C_3 + C_4 + C_5 + C_7 + C_8 + C_9 + N_1 + N_3)/2,
(N_2 + N_3 + N_5 + N_7)/2,
(C_1 + C_2 + C_7 + C_8 + N_2 + N_3 + N_5 + N_7)/2,
$$
After these operations, the modified vector is half the sum of four disjoint smooth rational curves.

It is easy to show that all the elements above are not vectors of the lattice $\text{NS}(X')$. To show this, let $v$ be one of these vectors, and assume by contradiction that $v \in \text{NS}(X')$. Then we can

- Add or subtract elements of $M$ to $v$;
- Use the relations

$$
\begin{align*}
C_1 + C_2 + C_3 + C_4 &= C_7 + C_8 + C_9 + C_{10}, \\
C_1 + C_2 + C_{11} + C_{12} &= C_5 + C_6 + C_7 + C_8, \\
C_3 + C_5 &= C_4 + C_6 = C_9 + C_{11} = C_{10} + C_{12}.
\end{align*}
$$

After these operations, the modified vector $v$ is still in $\text{NS}(X')$. However, using these operations we can obtain a vector equal to a half the sum of four disjoint smooth rational curves (see Example 6.3 after the end of this proof), which is impossible by Theorem 2.14 (observe that, in some cases, $v$ is already half the sum of four disjoint smooth rational curves).

Finally, this implies that $M = \text{NS}(X')$. If we choose \{ $v_1, v_2, v_4 + 2w_2, v_3 + v_4, w_1, w_2$ \} as basis for the discriminant group of $\text{NS}(X')$, we obtain the discriminant quadratic form claimed in the statement of Theorem 6.2. By Theorems 2.1 and 2.7 it follows that $T_{X'}$ is isometric to the lattice $U(2)\oplus \mathbb{Z}^2(-4)$. \hfill \Box

**Example 6.3.** Say $v = (C_1 + C_2 + C_7 + C_8 + N_2 + N_3 + N_5 + N_7)/2$. Every time we use one of the operations above, we write “$\approx$”. Then

$$
\begin{align*}
C_1 + C_2 + C_7 + C_8 + N_2 + N_3 + N_5 + N_7 &\approx \frac{C_3 + C_4 + C_9 + C_{10} + N_2 + N_3 + N_5 + N_7}{2}, \\
C_5 + C_6 + C_9 + C_{10} + N_2 + N_3 + N_5 + N_7 &\approx \frac{N_1 + N_4 + N_5 + N_7}{2}.
\end{align*}
$$

6.1.4. **Additional properties of $X'$**. The following proposition is the analogue of Proposition 6.13 for the K3 surfaces $X'$.

**Proposition 6.4.** Let $X'$ be the minimal resolution of the quotient of a triple-double K3 surface by an involution coming from the $\mathbb{Z}_2^2$ symplectic action. Then the following hold:

(i) $X'$ admits $\mathbb{Z}_4^2$ symplectic action;

(ii) $X'$ is not isomorphic to the minimal resolution of the quotient of a K3 surface by a symplectic action of the group $\mathbb{Z}_2^2$;

(iii) $X'$ is isomorphic to the minimal resolution of the quotient of a K3 surface by a symplectic action of the group $\mathbb{Z}_2^2$.

**Proof.** We know from [Nik79] that $X'$ admits $\mathbb{Z}_4^2$ symplectic action if and only if the lattice $T_{X'}$ embeds primitively into $\Omega_{\mathbb{Z}_4^2}$. But $T_{X'} \cong U(2) \oplus \mathbb{Z}(-4)$ by Theorem 6.2, and $\Omega_{\mathbb{Z}_4^2} \cong U(2) \oplus \mathbb{Z}(-8)$ by [GS09] Proposition 5.1. Therefore, it is enough to show that $\mathbb{Z}(-4)$ embeds primitively into $U(2) \oplus \mathbb{Z}(-8)$. The vectors $(1, 1, 1), (1, 1, 0)$ in $U(2) \oplus \mathbb{Z}(-8)$ generate a sublattice $L$ isometric to $\mathbb{Z}(-4)$. To show that $L$ is primitive, assume that $m(x, y, z) = (a-b, a+b, a) \in L$ for some integers $a, b, m$ with $m > 1$. Then $x = 2z - y$, which implies $(x, y, z) = (z)(1, 1, 1) + (y-z)(-1, 1, 0) \in L$, proving part (i).

Part (ii) follows from the fact that $T_{X'}$ is not isometric to any of the transcendental lattices listed in [GS16] Theorem 8.3.

For part (iii), we know from the discussion in Section 5 that $X'$ is the minimal resolution of the quotient of the Hirzebruch-Kummer covering $Y$ (see Example 5.2) by a symplectic action of $\mathbb{Z}_2^3$. □

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