Positive travelling fronts for reaction–diffusion systems with distributed delay

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Received 21 January 2010, in final form 16 July 2010
Published 20 August 2010
Online at stacks.iop.org/Non/23/2457

Recommended by J A Sherratt

Abstract
We give sufficient conditions for the existence of positive travelling wave solutions for multi-dimensional autonomous reaction–diffusion systems with distributed delay. To prove the existence of travelling waves, we give an abstract formulation of the equation for the wave profiles in some suitable Banach spaces and apply known results about the index of some associated Fredholm operators. After a Lyapunov–Schmidt reduction, these waves are obtained via the Banach contraction principle, as perturbations of a positive heteroclinic solution for the associated system without diffusion, whose existence is proven under some requirements. By a careful analysis of the exponential decay of the travelling wave profiles at \(-\infty\), their positiveness is deduced. The existence of positive travelling waves is important in terms of applications to biological models. Our method applies to systems of delayed reaction–diffusion equations whose nonlinearities are not required to satisfy a quasi-monotonicity condition. Applications are given, and include the delayed Fisher–KPP equation.

Mathematics Subject Classification: 34K30, 34K10, 35K57, 35B09, 92D25

1. Introduction

In the last few decades, there has been an increasing number of studies in travelling wave fronts for delayed diffusion equations, and several methods to prove their existence have been developed.

In this paper, we are concerned with the existence and positiveness of travelling waves connecting two equilibria, for a class of \(N\)-dimensional systems of reaction–diffusion equations with distributed delay in the reaction terms, of the form

\[
\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x) + f(u(t, x)), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^p.
\]
Here, \( f : C := C([-\tau, 0]; \mathbb{R}^N) \to \mathbb{R}^N \) is continuous, \( C \) is equipped with the norm \( \|\psi\|_{\infty} := \sup_{t \in [-\tau, 0]} |\psi(t)| \), for some fixed norm \(|\cdot|\) in \( \mathbb{R}^N \), and \( \tau > 0 \). As usual, \( u_t(\cdot, x) \) denotes the restriction of a solution \( u(t, x) \) to the time interval \([t - \tau, t]\), i.e. \( u_t(\theta, x) = u(t + \theta, x) \) for \(-\tau \leq \theta \leq 0, x \in \mathbb{R}^p\). For simplicity, we consider all the diffusion coefficients equal to 1 in (1.1), but all our results apply to the more general case of the diffusion term given by \( D \Delta u(t, x) \), where \( D = \text{diag}(d_1, \ldots, d_N) \) with \( d_i > 0 \).

We are mostly interested in situations where (1.1) represents a population dynamics model or another biological model. Typically, we want to obtain conditions for the existence of a travelling front connecting two steady states, zero and a positive equilibrium \( K \in \mathbb{R}^N \). Due to the biological interpretation of the model, only non-negative solutions are meaningful, therefore we look for \textit{positive} travelling wave solutions, connecting 0 to \( K \) as \( t \) goes from \(-\infty \) to \( \infty \).

With the method presented here, such positive travelling waves are obtained for large wave speeds, as perturbations of a positive heteroclinic solution for the corresponding functional differential equation (FDE) without diffusion,

\[
\dot{u}(t) = f(u_t), \quad t \in \mathbb{R}
\]  

(1.2)

(where \( u_t \in C \) denotes the function \( u_t(\theta) = u(t + \theta) \) for \( \theta \in [-\tau, 0] \)), whose existence we shall prove under some requirements on \( f \). This idea is not original, and has been exploited in the literature (see, e.g., [9, 10]). When compared with [9], in this paper the major novelty is that we give conditions for the travelling waves to be \textit{positive}. We also note that [9] considers delayed reaction–diffusion equations with a global space interaction, a situation not considered here, for the sake of simplicity. Our results can, however, be extended easily, to take into account non-local effects. On the other hand, [10] deals with \textit{scalar} reaction–diffusion equations with one single \textit{discrete} delay of the form

\[
\frac{\partial u}{\partial t}(t, x) = d \frac{\partial^2 u}{\partial x^2}(t, x) + f(u(t, x), u(t - \tau, x)),
\]

(1.3)

where \( f(u(t, x), u(t - \tau, x)) = -u(t, x) + g(u(t - \tau, x)) \) and \( g : [0, \infty) \to [0, \infty) \) is \( C^2 \)-smooth, \( g(0) = 0 \), \( g(K) = K \) for some \( K \geq 0 \), and \( g'(0) > 1 \). Assuming that the two equilibria 0 and \( K \) are hyperbolic, under some further assumptions the existence of positive and in general non-monotone travelling wavefronts connecting 0 to \( K \) was established in [10].

Recently, several techniques have been developed to prove the existence of travelling wave fronts for delayed diffusion equations. They are often based on the application of a fixed point theorem in an adequate Banach space, which requires a \textit{quasi-monotonicity} condition, either for the original equation (1.3) [16, 23] or, more recently, for some auxiliary equations [17]. These methods are usually combined with a monotonic iteration scheme, associated with the construction of a pair of upper and lower solutions. See [16, 17, 23] and references therein.

We emphasize that our method applies to systems (1.1) with non-monotone nonlinearities, in the sense that we do not impose on \( f \) any type of quasi-monotonicity condition, as defined in [20, 23].

Before introducing our hypotheses, we set some standard notation. For \( d = (d_1, \ldots, d_N) \in \mathbb{R}^N \), we say that \( d > 0 \) (respectively \( d \geq 0 \)) if \( d_i > 0 \) (respectively \( d_i \geq 0 \)) for \( i = 1, \ldots, N \). In \( C \), we consider the partial order \( \phi \geq \psi \) if and only if \( \phi(\theta) - \psi(\theta) \geq 0 \) for \( \theta \in [-\tau, 0] \); in a similar way, \( \phi > \psi \) if \( \phi(\theta) - \psi(\theta) > 0 \) for \( \theta \in [-\tau, 0] \). As usual, \( C_+ \) denotes the positive cone \( C([-\tau, 0]; [0, \infty)^N) \).

For \( f \), the following hypotheses will be considered:

(H1) \( f(0) = f(K) = 0 \), where \( K \) is some positive vector;

(H2) (i) \( f \) is \( C^2 \)-smooth; furthermore, (ii) for every \( M > 0 \) there is \( \beta > 0 \) such that \( f_i(\phi) + \beta \phi_i(0) \geq 0, i = 1, \ldots, N \), for all \( \phi \in C \) with \( 0 \leq \phi \leq M \);
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(H3) for equation (1.2), the equilibrium \( u = K \) is locally asymptotically stable and globally attractive in the set of solutions of (1.2) with initial conditions \( \varphi \in C^+, \varphi(0) > 0 \);

(H4) for equation (1.2), its linearized equation about the equilibrium 0 has a real characteristic root \( \lambda_0 > 0 \), which is simple and dominant (i.e. \( \Re z < \lambda_0 \) for all other characteristic roots \( z \)); moreover, there is a characteristic eigenvector \( v > 0 \) associated with \( \lambda_0 \).

We summarize the main results in this paper as follows. In section 2, we assume (H1)–(H4) and establish the existence of a positive heteroclinic solution \( u^*(t) \) to (1.2), with \( u^*(-\infty) = 0, u^*(\infty) = K \), and asymptotic behaviour \( O(e^{\lambda_0 t}) \) at \(-\infty\). In section 3, for large wave speeds we prove the existence of travelling wave solutions for (1.1), connecting 0 to \( K \).

The profiles of these waves are obtained as perturbations of \( u^*(t) \) via a contraction principle argument. For this, we generalize the procedure in [9], and give an abstract formulation of the wave profiles as solutions of an operational equation, acting in suitable Banach spaces, which incorporate a desirable exponential decay \( O(e^{\mu t}) \) at \(-\infty\), \( 0 \leq \mu < \lambda_0 \). Some nice results of Hale and Lin [13] on the index of some associated Fredholm operators are used, and a Lyapunov–Schmidt reduction effected, to set up the right framework for the application of a contraction principle. As mentioned above, an existence result of travelling waves connecting two hyperbolic equilibria was already obtained in [9], for a class of reaction–diffusion equations with global response, but for such waves no exponential decay at \(-\infty\) was derived in [9], nor their positiveness. By a careful analysis of the behaviour of the wave profiles at \(-\infty\), in section 4 we prove that there are positive travelling waves if the wave speed is large enough, and explicitly give their asymptotic decay at \(-\infty\). Section 5 is dedicated to applications, which include the Fisher–KPP equation with delay and a two-dimensional chemostat model. An important theorem on the asymptotic behaviour of solutions of perturbed linear autonomous ordinary FDEs is given in the appendix. This theorem was inspired by a result by Mallet-Paret [18], who only treated the case of FDEs with discrete time-delays (or time-shifts), whereas here we generalize Mallet-Paret’s arguments to the case of FDEs with distributed delay. This result is often used in sections 2 and 4.

2. Positive heteroclinic solution for equation (1.2)

In this section, we prove the existence of a positive solution \( u^*(t) \) of the ordinary FDE (1.2) connecting the equilibrium 0 to the positive equilibrium \( K \). We recall that a function \( u(s) \) defined on a set \( S \) and with values in \( \mathbb{R}^N \) is said to be positive if all its components \( u_1(s), \ldots, u_N(s) \) are positive functions on \( S \).

**Theorem 2.1.** Assume (H1)–(H4). Then

(i) there exists a heteroclinic solution \( u^*(t) \), \( t \in \mathbb{R} \), of equation (1.2), with \( u^*(-\infty) = 0, u^*(\infty) = K \);

(ii) \( u^*(t) \) is positive, \( t \in \mathbb{R} \);

(iii) \( u^*(t) = ce^{\lambda_0 t}v + O(e^{(2\lambda_0-\varepsilon)t}) \) at \(-\infty\), for some \( c > 0 \) and each fixed \( \varepsilon > 0 \).

**Proof.**

(i) Consider the linearization of (1.2) about 0,

\[ u'(t) = Lu, \quad \text{where } L = Df(0), \]

(2.1)

and its characteristic equation

\[ \det \Delta_0(\lambda) = 0, \quad \text{where } \Delta_0(\lambda) = L(e^{\lambda_0 I}) - \lambda I. \]

(2.2)
Recall that \( \lambda \) is a solution of (2.2) if and only if \( \lambda \in \sigma(A) \), where \( A \) is the infinitesimal generator associated with the semi-flow of (2.1).

Let \( \lambda_0 > 0 \) be the leading (simple) eigenvalue of (2.1) given in (H4), and \( v \in \mathbb{R}^n, v > 0 \), such that \( \Delta_0(\lambda_0)v = 0 \). Choose \( \gamma > 0 \) with \( \gamma < \lambda_0 < 2\gamma \) and such that the strip \( \gamma < \Re \lambda < \lambda_0 \) does not contain any root of (2.2). Define \( \chi_0(\theta) = e^{\gamma \theta}v, \theta \in [-r, 0] \), and decompose the phase \( \mathcal{C} \) as \( \mathcal{C} = P \oplus Q \), where \( P = (\chi_0) \) and \( Q \) is the complementary space given by the formal adjoint theory of Hale [14]. Then there are neighbourhoods \( N_0, N_1 \) of 0 in \( P, Q \), respectively, and a \( C^1 \) map \( w : N_0 \to N_1 \) with \( w(0) = 0 \), \( Dw(0) = 0 \) such that the local \( \gamma \)-unstable manifold of 0 for equation (1.2) is given by

\[
W(0) = \{\phi + w(\phi) : \phi \in N_0\}.
\]

Note that \( \varphi \in W(0) \) if and only if there is a full trajectory \( u_t = u_t(\varphi) \) of (1.2) with \( u_0 = \varphi, u_t \in N_0 + N_1 \) for \( t \leq 0 \) and \( u(t)e^{-\gamma t} \to 0 \) as \( t \to -\infty \). See Krisztin et al [15], Hale and Lunel [14, section 10.1-10.2] and Diekmann et al [5, section 8.4].

We now argue as in [19]. Write \( w(t) = (u_1(t), \ldots, u_N(t)) = (v_1, \ldots, v_N) \). Since \( Dw(0) = 0 \), we have \( \lim_{t \to 0} ||w(\phi)||/||\phi|| = 0 \) and \( \lim_{t \to 0} ||w(c\chi_0)||/||c|| = 0 \). Thus, there is \( c_0 > 0 \) such that \( |w_i(c\chi_0)|/|\chi_0| \leq c_i \) for \( c \in (0, c_0), i = 1, \ldots, N \), which implies that for \( 0 < c \leq c_0 \) we have

\[
\inf_{t \leq 0} e^{c\theta t}w_i(\chi_0(\theta)) \geq ce^{-c_i t}v_i/2 > 0, \quad i = 1, \ldots, N,
\]

and therefore \( c\chi_0 + w(c\chi_0) \in W(0) \cap C_i \), for all \( c \in (0, c_0) \). Fix, e.g., \( c = c_0 \), denote \( \phi = \chi_0(\phi) \) and consider the full trajectory \( u_t = u_t(\phi), t \in \mathbb{R} \) and \( \Phi(t) = [0, \ldots, \Phi(t)] = e^{c\theta t}w(c\chi_0) \). This asymptotic result will be crucial to prove the existence of positive on some interval \(-\infty, \gamma\).

We now argue as in [19]. Write \( w(t) = (u_1(t), \ldots, u_N(t)) = (v_1, \ldots, v_N) \). Since \( Dw(0) = 0 \), we have \( \lim_{t \to 0} ||w(\phi)||/||\phi|| = 0 \) and \( \lim_{t \to 0} ||w(c\chi_0)||/||c|| = 0 \). This implies that there is \( T < 0 \) such that \( c(t) \leq c_0 \) for \( t < T \). On the other hand, if \( c(t_0) = 0 \) for some \( t_0 < T \), then \( u_t^* = 0 \), which is not possible by the uniqueness theorem. From (2.3) it follows that \( u_t^* > 0 \) for \( t < T \). Now, from (H3), we have \( u_t^* \to K \) as \( t \to \infty \). This means that \( u_t^* \) is a heteroclinic solution of (1.2) connecting the two equilibria \( 0, K \), with \( u_t^* \) positive on some interval \(-\infty, T\).

(ii) Choose \( M > 0 \) such that \( u_t^* \leq M, t \in \mathbb{R} \), \( i = 1, \ldots, N \). Suppose there is \( t \geq T \) and \( i \in \{1, \ldots, N\} \) such that \( u_t^*(t) = 0 \). Define \( \tau = \min\{t \geq T : u_t^*(t) = 0 \} \) for some \( j \in I \) and \( i \in I \) such that \( u_t^*(\tau) \neq 0 \). For \( M \) as above, let \( b \) be as in (H2), i.e. \( f_i(\varphi) + b \varphi_i(0) \geq 0 \), for \( j = i \) and \( 0 < \varphi_i \leq M \). Since \( u_t^*(t)e^{\varphi t} = [f_i(u_t^*) + b \varphi_i(t)]e^{\varphi t} \geq 0 \), it follows that \( u_t^*(t)e^{\varphi t} \) is non-decreasing, yielding that \( u_t^*(t) > 0 \) for all \( t \in (0, \tau] \), a contradiction.

(iii) We note that \( u_t^* \) belongs to \( W(0) \) for \( t \leq 0 \), thus \( u_t^* = O(e^{\varphi t}) \) at \( -\infty \) and \( u_t^* \) satisfies \( u_t^*(t) = Lu_t^* + h(t) \), with \( h(t) = f(u_t^*) - Lu_t^* = O(e^{\varphi t}) \) at \( -\infty \). From theorem A.2 (see the appendix), for each \( \epsilon > 0 \) we deduce that \( u_t^*(t) = z(t) + O(e^{\varphi t + \epsilon t}) \) at \( -\infty \), where \( z(t) = ce^{\varphi t} \) for some (positive) \( c \in \mathbb{R} \). Thus, \( u_t^*(t) = O(e^{\varphi t}) \) at \( -\infty \).

\[\Box\]

**Remark 2.2.** In fact, one could use [10, lemma 4] and its constructive proof to derive that there is a complete solution \( u_t^*(t) \) of (1.2), with \( u_t^*(-\infty) = 0, u_t^*(\infty) = K \) and \( u_t^*(t) > 0 \) for \( t \leq 0 \). This proves assertion (i) of theorem 2.1. In order to prove that \( u_t^*(t) = O(e^{\varphi t}) \) at \( -\infty \) it is, however, more convenient to explicitly construct \( u_t^*(t) \) as a perturbation of the eigenfunction \( e^{\varphi t} \) as above. This asymptotic result will be crucial to prove the existence of positive travelling waves for (1.1), if the wave speed is sufficiently high. On the other hand, if we assume that the interior of the positive cone \( \mathcal{C}_0 \) is positively invariant for the flow of (1.2), as an alternative to hypothesis (H2)(ii), then the positiveness of \( u_t^*(t) \) on \( \mathbb{R} \) follows immediately from the fact that \( u_t^*(t) \) is positive in the vicinity of \(-\infty \). Instead of (H2)(ii), clearly other assumptions can be found in order to assure the positive invariance of the interior of \( \mathcal{C}_0 \).
3. Existence of travelling waves and their asymptotic decay at \(-\infty\)

Throughout this section, for simplicity we assume that (H1)-(H4) are fulfilled, but in fact some of the hypotheses can be weakened (cf remark 3.12). We shall prove the existence of travelling waves for (1.1) which will be obtained as perturbations of the heteroclinic solution \(u^*(t)\) of (1.2). The asymptotic behaviour at \(-\infty\) of \(u^*(t)\) given in theorem 2.1(iii) will be important to study the asymptotic decay of such waves at \(-\infty\); however, its positiveness is irrelevant here, and will be only used for the analysis in section 4.

For a unit vector \(w \in \mathbb{R}^p\), we look for wave solutions of (1.1) with direction \(w\) and speed \(c > 0\), connecting the equilibria 0 to \(K\), i.e. solutions of the form \(u(t, x) = \phi(ct + w \cdot x)\) with \(\phi(\pm \infty) = 0, \phi(\infty) = K\).

The equation for the travelling wave profile \(\phi\) is given by

\[
\phi''(t) - c\phi'(t) + f_c(\phi_t) = 0, \quad t \in \mathbb{R},
\]

where \(f_c(\phi) = f(\phi(\cdot - c))\), with \(\phi\) subject to the conditions

\[
\phi(\pm \infty) = 0, \quad \phi(\infty) = K.
\]

With \(c = 1/e\) and \(\phi(t) = \phi(ct)\), and dropping the bars for simplicity, (3.1) is equivalent to

\[
\varepsilon^2 \phi''(t) - \phi'(t) + f(\phi_t) = 0.
\]

We also consider equation (3.2) with \(\varepsilon = 0\), in which case it reduces to equation (1.2).

Let \(C_0(\mathbb{R}, \mathbb{R}^N)\) be the space of all continuous and bounded functions from \(\mathbb{R}\) to \(\mathbb{R}^N\), with the supremum norm \(\|y\|_\infty = \sup_{t \in \mathbb{R}} |y(t)|\). As a particular case of the framework in [9], we have the following result:

**Theorem 3.1 ([9])**. Let \(f\) have the form \(f(\phi) = F(\phi(0), g(N\phi))\), \(\phi \in C\), for some bounded linear operator \(N : C \to \mathbb{R}^N\) and \(g : \mathbb{R}^N \to \mathbb{R}^N\), \(F : \mathbb{R}^{2N} \to \mathbb{R}^N\) \(C^2\)-smooth functions. Suppose also that

(i) \(f(0) = f(K) = 0\) for some \(K \in \mathbb{R}^N\),
(ii) for equation (1.2), the equilibrium \(u = 0\) is hyperbolic and unstable, and the equilibrium \(u = K\) is locally asymptotically stable.

Then, if there is a heteroclinic solution \(u^*(t)\) for (1.2) connecting 0 to \(K\), for each unit vector \(w \in \mathbb{R}^p\) there are a neighbourhood \(V\) of \(u^*(t)\) in \(C_0(\mathbb{R}, \mathbb{R}^N)\) and a constant \(c^* > 0\), such that for \(c > c^*\) the set of travelling waves for (1.1) in \(V\), with direction \(w\) and wave speed \(c\), constitutes a \(C^1\)-manifold of dimension \(m\), where \(m\) is the dimension of the unstable space for \(u(t) = Df(0)u_t\).

In this section, the idea is to retrace some arguments in [9] for the proof of theorem 3.1 adapted to the case of (1.1), but in appropriate Banach spaces, which will allow us to deduce not only the existence of travelling wave solutions for (1.1), but also their asymptotic behaviour at \(-\infty\).

This behaviour will be used in section 4, to prove the existence of positive travelling waves.

In addition to \(C_h := C_b(\mathbb{R}, \mathbb{R}^N)\), we introduce the following Banach spaces:

- \(C^1_b := C_b(\mathbb{R}, \mathbb{R}^N) = \{y \in C_b : y' \in C_b\}\) with the norm \(\|y\|_1 = \|y\|_\infty + \|y'\|_\infty\);
- \(C_0 = \{y \in C_b : \lim_{t \to \pm \infty} y(t) = 0\}\) is considered as a subspace of \(C_b\);
- \(C^1_b = \{y \in C^1_b : y, y' \in C_b\}\) is considered as a subspace of \(C^1_b\);
- \(C^1_\mu = \{y \in C_b : \sup_{t \leq 0} e^{-\mu t} |y(t)| < \infty\}\) (for \(\mu > 0\)) with the norm

\[
\|y\|_\mu = \max\{\|y\|_\infty, \|y\|_{\mu}^{-}\} \quad \text{where} \quad \|y\|_{\mu}^{-} = \sup_{t \leq 0} e^{-\mu t} |y(t)|;
\]
\[ C^1_\mu = \{ y \in C^1_b : y, y' \in C_\mu \}, \text{ with the norm } \| y \|_{1,\mu} = \| y \|_\mu + \| y' \|_\mu; \]

\[ C_{\mu,0} = C_\mu \cap C_0 \] is considered as a subspace of \( C_\mu \).

By the change of variables \( \phi(t) = w(t) + u^*(t) \), (3.2) becomes

\[ \varepsilon^2 w''(t) - w'(t) - w(t) = -w(t) - Df(u^*_t)w_t - G(\varepsilon, t, w), \quad t \in \mathbb{R}, \tag{3.3} \]

where

\[ G(\varepsilon, t, w) = f(w_t + u^*_t) - f(u^*_t) - Df(u^*_t)w_t + \varepsilon^2 u^{**}(t), \tag{3.4} \]

subject to the conditions \( w(-\infty) = w(\infty) = 0 \). The roots of the characteristic equation associated with \( \varepsilon^2 w''(t) - w'(t) - w(t) = 0 \) are

\[ \alpha(\varepsilon) = 1 - \sqrt{1 + 4\varepsilon^2}, \quad \beta(\varepsilon) = 1 + \sqrt{1 + 4\varepsilon^2}, \]

and satisfy \( \alpha(\varepsilon) \to -1^* \), \( \beta(\varepsilon) \to \infty \) as \( \varepsilon \to 0^+ \). In the case of different diffusion coefficients \( d_i > 0, i = 1, \ldots, N \), instead of \( \alpha(\varepsilon), \beta(\varepsilon) \) one has to consider \( \alpha_i(\varepsilon), \beta_i(\varepsilon) \), the solutions of the characteristic equations

\[ d_i \varepsilon^2 z^2 - z - 1 = 0, \quad i = 1, \ldots, N, \]

but the arguments are similar (cf [9]).

A bounded function \( w : \mathbb{R} \to \mathbb{R}^N \) is a solution of (3.3) if and only if

\[ Jw(t) = H(\varepsilon, w)(t), \quad t \in \mathbb{R}, \tag{3.5} \]

where

\[ Jw(t) = w(t) - \int_{-\infty}^{t} e^{-(t-s)} [w(s) + Df(u^*_s)w_s] \, ds \]

and

\[ H(\varepsilon, w)(t) = \int_{-\infty}^{t} \left[ \frac{e^{\alpha(\varepsilon)(t-s)}}{\sqrt{1 + 4\varepsilon^2}} - e^{-(t-s)} \right] (w(s) + Df(u^*_s)w_s) \, ds \]

\[ + \frac{1}{\sqrt{1 + 4\varepsilon^2}} \left[ \int_{-\infty}^{t} e^{\alpha(\varepsilon)(t-s)} G(\varepsilon, s, w) \, ds \right. \]

\[ + \left. \int_{t}^{\infty} e^{\beta(\varepsilon)(t-s)} [w(s) + Df(u^*_s)w_s + G(\varepsilon, s, w)] \, ds \right]. \]

Our purpose is to apply a contraction principle argument in order to obtain a solution of equation (3.5), for \( \varepsilon > 0 \) small and \( w \) close to 0, in adequate spaces \( C_\mu \). We first analyse the linearity \( J \), by introducing some auxiliary equations and operators.

Define

\[ (Ty)(t) = y'(t) - Df(u^*_t)y_t, \quad y \in C^1_b, \quad t \in \mathbb{R}. \]

We easily see that \( J : C_0 \to C_0, T : C^1_b \to C_b \) are linear bounded operators and \( w \mapsto H(w, \varepsilon) \) maps \( C_0 \) in \( C_0 \), for \( \varepsilon > 0 \) (cf [9]). For \( \mu > 0 \), we also define

\[ T_\mu := T|_{C^1_\mu} : C^1_\mu \to C_\mu. \]

Lemma 3.2. Let \( \mu > 0 \). Then \( T_\mu \) and \( J|_{C^1_{\mu,0}} : C^1_{\mu,0} \to C_{\mu,0} \) are bounded operators.

Proof. Since the map \( t \mapsto \| Df(u^*_t) \| \) is continuous for \( t \in \mathbb{R} \) and \( \| Df(u^*_t) \| \to \| Df(0) \| \) as \( t \to -\infty \), \( \| Df(u^*_t) \| \to \| Df(K) \| \) as \( t \to \infty \), then \( M := \sup_{t \in \mathbb{R}} \| Df(u^*_t) \| < \infty \). It follows
that \( |(T \gamma)(t)| \leq \max(1, M) \|y\|_{1, \mu} \) for \( t \geq 0 \) and \( e^{-\mu t}|(T \gamma)(t)| \leq \max(1, M) \|y\|_{1, \mu} \) for \( t \leq 0 \), \( y \in C^1_\mu \).

For \( y \in C_\mu \), we now have \( |(J \gamma)(t)| \leq (2 + M) \|y\|_{\mu} \) for \( t \geq 0 \) and \( e^{-\mu t}|(J \gamma)(t)| \leq [1 + (1 + M)(\mu + 1)^{-1}] \|y\|_{\mu} \), hence \( J(C_{\mu, 0}) \subset C_{\mu, 0} \).

Consider the linear variational equation around the heteroclinic solution \( u^\ast(t) \),

\[
y'(t) = Df(u^\ast)y_t.
\]

Define the operators \( L(t) := Df(u^\ast) \) in (3.6), with \( L(-\infty) = Df(0) \) and \( L(\infty) = Df(K) \). Hence, equation (3.6) is asymptotically autonomous, with limiting equations (2.1) and \( y' = Df(K)y_t \), respectively, at \(-\infty \) and \( \infty \).

**Lemma 3.3.** Consider \( \mu \in (0, \lambda_0) \) such that there are no characteristic roots \( \lambda \) of (2.1) with \( \Re \lambda = \mu \). For \( T_\mu : C^1_\mu \to C_\mu \) defined as above,

\[
\text{Ind}(T_\mu) = C_\mu, \quad \text{dim Ker}(T_\mu) = r_\mu.
\]

where \( r_\mu = \#(\lambda \in \mathbb{C} : \det \Delta_0(\lambda) = 0, \Re \lambda > \mu) \). In particular, \( r_\mu = 1 \) for \( \mu \) close to \( \lambda_0 \). Moreover, \( \text{Ker}(T_\mu) \subset C_{\mu, 0} \).

**Proof.** Clearly, equation \( y' = Df(K)y_t \) is asymptotically stable, and the autonomous equation (2.1) admits a ‘shifted exponential dichotomy’ in \( \mathbb{R} \) with the splitting made at \( \mu \) and exponents \( \mu - \delta, \mu + \delta \), for \( \delta > 0 \) small. See Hale and Lin [13] for definitions, and note that \( C_\mu = C^0(\mu, 0) \) in the notation in [13]. From [13, lemma 4.3], there is \( T > 0 \) such that (3.6) has a shifted exponential dichotomy on \((\infty, -T] \) and \([T, \infty) \). We now apply lemma 4.6 of [13] to (3.6). It follows that \( T_\mu \) is a Fredholm operator, with index \( \text{Ind}(T_\mu) \) given by

\[
\text{Ind}(T_\mu) = \text{dim Im}(P^- (-t)) - \text{dim Im}(P^+ (t)), \quad t \geq T,
\]

where \( P^-(t), P^+(-t) \) and \( P^+(t) \) are the projections associated with the (shifted) exponential dichotomies for \( y'(t) = Df(0)y_t \) and \( y'(t) = Df(K)y_t \), respectively. From [13, lemma 4.3], we also have that \( P^-(-t) \to P^-u \), \( P^+(t) \to P^+u \) as \( t \to \infty \), where \( P^-u \) is the canonical projection from \( C \) onto the \( \mu \)-unstable space \( E^-u \); \( P^+u \) is the canonical projection from \( C \) onto the unstable space \( E^+u \); we have \( E^+u = \{0\} \) and \( \text{dim } E^-u = r_\mu \), where \( r_\mu \) is the number of characteristic values for (2.1) (counting multiplicities) with real parts greater than \( \mu \). Hence \( \text{Ind}(T_\mu) = r_\mu \). On the other hand, the index of \( T_\mu \) is defined by \( \text{Ind}(T_\mu) = \text{dim Ker}(T_\mu) - \text{codim Im}(T_\mu) \). Again by [13, lemma 4.6] we have \( \text{dim Ker}(T_\mu) = \text{dim } E^-u = r_\mu \), yielding that \( \text{Im}(T_\mu) = C^1_\mu \).

For \( y \in \text{Ker}(T_\mu) \), from the definition of shifted exponential dichotomy we have \( \lim_{t \to \infty} y(t) = 0 \). Thus, \( \text{Ker}(T_\mu) \subset C_{\mu, 0} \).

Similarly to what was done for \( T \), we now restrict the domain and range of the operator \( J \).

With \( D := d/dt + id \), consider the commutative diagram

\[
\begin{array}{ccc}
C^1_\mu & \xrightarrow[T_\mu]{J} & C_\mu \\
\downarrow & & \downarrow \\
C^1_\mu & \xrightarrow[D]{J} & C_\mu
\end{array}
\]
It is easy to check that this diagram is well defined, and that $D$ is one-to-one and surjective. Since $T_\mu = D \circ J$ is surjective, we may conclude that $J$ is also surjective. Moreover,

**Lemma 3.4.** Consider $\mu \in (0, \lambda_0)$ such that there are no characteristic roots $\lambda$ of (2.1) with $\Im \lambda = \mu$. Then, for the operator $J|_{C_{\mu,0}} : C_{\mu,0} \to C_{\mu,0}$ we have $\text{Ker} (J|_{C_{\mu,0}}) = \text{Ker} (T_\mu)$ and $\text{Im} (J|_{C_{\mu,0}}) = C_{\mu,0}$.

**Proof.** Recall that $\text{Ker} (T_\mu) \subset C_{\mu,0}$. Clearly, for $w \in C_{\mu,0}$ we have $Jw = 0$ if and only if $w'(t) = Df(u^*_\mu)w$, and then $w' \in C_{\mu}$. We therefore deduce that $(\text{Ker} J) \cap C_{\mu} = (\text{Ker} J) \cap C_{\mu,0}$ and $\text{Ker} (J|_{C_{\mu,0}}) = \text{Ker} (T_\mu)$.

We now prove that $\text{Im} (J|_{C_{\mu,0}}) = C_{\mu,0}$. Indeed, for $y \in C_{\mu,0}$ we have that $\xi := y - Jy \in C_{\mu}^1$ and $D\xi(t) = y(t) + Df(u^*_\mu)y_t$, hence $D\xi \in C_{\mu,0}$. Equation $Jw = y$ is equivalent to $J(w - y) = \xi$, and therefore it possesses a solution $\chi = w - y \in C_{\mu}^1$. After applying $D$ to both sides of the latter equation, we get $T_\mu \chi = D\xi \in C_{\mu,0}$. Since the $\omega$-limit operator $T_\mu(\omega)$ is hyperbolic, we may invoke lemma 3.3 from [9] to conclude that $\chi(\omega) = 0$. Thus $w(\omega) = 0$, and $J : C_{\mu,0} \to C_{\mu,0}$ is surjective.

We now focus our attention on the nonlinearity $H$ of equation (3.5). Proceeding as in [9], one sees that $H(w, \varepsilon) \in C_\sigma$ for each $\varepsilon > 0$ and $w \in C_\sigma$. We want, however, to consider the maps $H(\cdot, \varepsilon)$ restricted to some neighbourhood of zero in $C_{\mu,0}$, for $\varepsilon > 0$ and $\mu > 0$. We start with an auxiliary lemma:

**Lemma 3.5.** Let $X$, $Y$ be normed spaces and $C \subset O \subset X$. Suppose that the set $C$ is compact, $O$ is open and $F : O \to Y$ is a continuous map. Then for every $\sigma > 0$ there exists $\delta > 0$ such that

$$|F(x + z) - F(x)| \leq \sigma, \quad x \in C, \ |z| \leq \delta.$$  

**Proof.** By the continuity of $F$, for each $x \in C$ there is $\delta(x) > 0$ such that if $|z| \leq 2\delta(x)$ then $x + z \in O$ and $|F(x + z) - F(x)| \leq \sigma/2$. Since $C \subset \bigcup_{x \in C} B_{\delta(x)}(x)$ is compact, there is a finite subcover $\{B_{\delta(x_j)}(x_j)\}_{j=1}^m$ of $C$. For each $x \in B_{\delta(x_j)}(x_j) \cap C$ and $|z| \leq \delta := \min\{\delta(x_j)\}$, we have $|F(x + z) - F(x)| \leq |F(x + z) - F(x_j)| + |F(x) - F(x_j)| \leq \sigma$, which proves the lemma.

**Lemma 3.6.** Assume (H1)–(H4) and consider $\mu \in (0, \lambda_0)$. For any $\delta > 0$, there are $\varepsilon^\ast > 0$ (independent of $\mu$) and $\sigma > 0$ such that $H(w, \varepsilon) \in C_{\mu,0}$ for any $\varepsilon > 0$ and $w \in C_{\mu,0} \cap B^\mu_\delta(0)$, and

$$\|H(w, \varepsilon)\| \leq \delta(\|w\| + 1),$$

$$\|H(w, \varepsilon) - H(v, \varepsilon)\| \leq \delta \|w - v\|, \quad \text{for } w, v \in C_{\mu,0} \cap B^\mu_\delta(0), \varepsilon \in (0, \varepsilon^\ast) \quad (3.7)$$

where $B^\mu_\delta(0)$ is the $\sigma$-neighbourhood of $0$ in $C_\mu$.

**Proof.** We write $H = H_1 + H_2 + H_3$, where

$$H_1(\varepsilon, w)(t) = \int_\alpha^\beta \left[ \frac{e^{\varepsilon(t-s)}}{\sqrt{1 + 4\varepsilon^2}} - e^{-r(t-s)} \right] w(s) + Df(u^*_\mu)w_s \, ds,$$

$$H_2(\varepsilon, w)(t) = \frac{1}{\sqrt{1 + 4\varepsilon^2}} \int_{-\infty}^t e^{\varepsilon(s-t)} G(\varepsilon, s, w) \, ds,$$

$$H_3(\varepsilon, w)(t) = \frac{1}{\sqrt{1 + 4\varepsilon^2}} \int_t^{\infty} e^{\varepsilon(s-t)} \{w(s) + Df(u^*_\mu)w_s + G(\varepsilon, s, w)\} \, ds,$$
and $G$ is given by (3.4). Let $M = \sup_{t \in \mathbb{R}} \| Df(u^*_t) \|$ as before. For $t \in \mathbb{R}, \epsilon > 0$ and $\mu \geq 0$, we have

$$\left| \int_{-\infty}^{t} \left[ \frac{e^{\beta(t,s)}}{\sqrt{1 + 4\epsilon^2}} - e^{-(t-s)} \right] e^{\alpha s} ds \right|$$

$$\leq \frac{1}{\sqrt{1 + 4\epsilon^2}} \int_{-\infty}^{t} \left( e^{-\alpha t} - e^{-(t-s)} \right) e^{\alpha s} ds$$

$$= \frac{1}{\sqrt{1 + 4\epsilon^2}} \left[ (\sqrt{1 + 4\epsilon^2} - 1) \int_{-\infty}^{t} e^{\beta(t,s)} e^{\alpha s} ds \right]$$

$$\leq \frac{1}{\sqrt{1 + 4\epsilon^2}} \left[ \frac{2\sqrt{1 + 4\epsilon^2} - 1}{\mu - \alpha(\epsilon)} - \frac{1}{\mu + 1} \right]$$

$$= \frac{1}{\sqrt{1 + 4\epsilon^2}} \left[ 1 - \frac{1}{\mu - \alpha(\epsilon)} + \frac{1 + \alpha(\epsilon)}{(\mu - \alpha(\epsilon))(\mu + 1)} \right]$$

where

$$C_1(\epsilon) = -\frac{1}{\alpha(\epsilon)} \left( 1 - \frac{1}{\sqrt{1 + 4\epsilon^2}} + \frac{1 + \alpha(\epsilon)}{\mu - \alpha(\epsilon)} \right) \to 0 \quad \text{as} \quad \epsilon \to 0^+.$$  

From (3.8), we obtain

$$\| H_1(\epsilon, w) - H_1(\epsilon, v) \|_\mu \leq C_1(\epsilon)(1 + \| M \|_\| w - v \|_\mu), \quad w, v \in C_\mu, \epsilon > 0. \quad (3.9)$$

Since $H_1(\epsilon, 0) = 0$, in particular $H_1(\epsilon, w) \in C_\mu$ for $w \in C_\mu$ and $\epsilon > 0$.

For $0 \leq \mu < \beta(\epsilon)$ and $t \in \mathbb{R}$, we now have

$$\int_{-\infty}^{t} e^{\beta(t,s)} e^{\alpha s} ds = \frac{e^{\alpha s}}{\mu - \alpha(\epsilon)} - \frac{e^{\beta(t,s)} e^{\alpha s}}{\beta(\epsilon) - \mu}. \quad (3.10)$$

Consider, e.g., $\mathbb{R}^N$ equipped with the maximum norm. For $t \in \mathbb{R}, \epsilon > 0, w, v \in C_{\mu,0}, i = 1, \ldots, N$, we have

$$|G_i(\epsilon, t, w)| \leq \epsilon^2 |u^{*\alpha}(t)| + |f_i(w_t + u^*_t) - f_i(u^*_t) - Df_i(u^*_t)w_t|$$

$$\leq \epsilon^2 |u^{*\alpha}(t)| + \| Df_i(u^*_t + \xi_{\epsilon,t}w_t) - Df_i(u^*_t) \|_{\| w_t \|_{\infty}} \quad (3.11)$$

and

$$|G_i(\epsilon, t, w) - G_i(\epsilon, t, v)| \leq \| Df_i(v_t + u^*_t + \theta_{t,\epsilon}(w_t - v_t)) - Df_i(u^*_t) \|_{\| w_t - v_t \|_{\infty}}, \quad (3.12)$$

for some $\xi_{\epsilon,t}, \theta_{t,\epsilon} \in (0, 1)$ for $t \in \mathbb{R}$.

Note that $u^{*\alpha} \in C_{\mu}$ for $0 < \mu \leq \lambda_0$. In fact, $u^* \in C_\mu$ from theorem 2.1, hence equation (2.1) and the smoothness of $f$ lead to $|u^{*\alpha}(t)| \leq M_0 \| u^*_t \|_{\infty}$, from which we derive $\| u^{*\alpha} \|_{\| u^* \|_{\| u^* \|_{\infty}}$, for some $M_0 > 0$. By differentiating, we obtain $u^{*\alpha}(t) = Df(u^*_t)(u^{*\alpha})_t$, thus $\| u^{*\alpha} \|_{\mu} \leq M \| u^{*\alpha} \|_{\mu}$.

In order to simplify the notation, for each $\mu, \sigma > 0$ write $C_{\mu,0} \cap B_\sigma(0)$ to denote the $\sigma$-neighbourhood of 0 in $C_{\mu,0}$. Since $f(u^*_t) \to 0$ as $t \to \pm \infty$, then $u^{*\alpha}$ is uniformly bounded on $\mathbb{R}$ and $u^{*}$ uniformly continuous on $\mathbb{R}$. Thus, $\mathcal{K} = [u^*_t, t \in \mathbb{R}] \subset C$ is compact. The continuity of $Df_i : C \to L(C, \mathbb{R})$ and lemma 3.5 imply that, for each $\delta > 0$ fixed,
there is $$\sigma = \sigma(\delta, \mu) > 0$$ such that $$\|Df(\mu, 0, t)\| < \delta$$ for $$\mu, t \in (0, \lambda_0)$$, where $$C(\epsilon), D(\epsilon)$$ do not depend on $$\epsilon$$ and are given by

$$C(\epsilon) = C_1(\epsilon)(1 + M) + \frac{1 + M}{\beta(\epsilon) - \lambda_0} \sqrt{1 + 4\epsilon^2} \|u^{**}\|_{\lambda_0},$$

$$D(\epsilon) = \epsilon^2 \|u^{**}\|_{\lambda_0}.$$  

Since $$C_1(\epsilon) \to 0$$, $$\beta(\epsilon) \to \infty$$ as $$\epsilon \to 0^+$$, by replacing $$\delta$$ by $$\delta/2$$ in (3.15), (3.16), we obtain (3.7) for $$\epsilon > 0$$ sufficiently small.  

We now return to equation (3.5). Let $$0 < \mu < \lambda_0$$. For $$\epsilon > 0$$ small, we look for a solution $$w \in C_{\mu, 0}$$ of (3.5). For the case $$\mu = 0$$, where the space $$C_{0, 0}$$ denotes $$C_0$$, this question was addressed in [9]. Our purpose is to solve this problem for $$\mu \in (0, \lambda_0)$$. We first apply a Lyapunov–Schmidt reduction. From lemmas 3.3 and 3.4, $$X_\mu := \text{Ker}(J|_{C_{\mu}})$$ is finite dimensional, hence there is a complementary subspace $$Y_\mu$$ in $$C_{\mu, 0}$$.

For $$w \in C_{\mu, 0}$$, write $$w = \xi + \phi$$ with $$\xi \in X_\mu, \phi \in Y_\mu$$. Define $$S_\mu := J|_{Y_\mu}$$. Since $$S_\mu : Y_\mu \to C_{\mu, 0}$$ is bounded and bijective, then $$S_\mu^{-1}$$ is bounded. In the space $$C_{\mu, 0}$$, equation (3.5) is equivalent to $$\phi = S_\mu^{-1}H(\epsilon, \xi + \phi)$$, therefore we look for fixed points $$\phi \in Y_\mu$$ of the map

$$F_\mu(\epsilon, \xi, \phi) = S_\mu^{-1}H(\epsilon, \xi + \phi).$$ 

For simplicity, in what follows we write $$S, F, B_\theta(0)$$ instead of $$S_\mu, F_\mu, B_\theta^\mu(0)$$, respectively, when there is no risk of misunderstanding.
Remark 3.7. For $0 < \mu_1 < \mu_2 < \lambda_0$ with $\mu_1, \mu_2 \not\in \Re \sigma(A)$, where $\sigma(A)$ is the set of solutions of (2.2), it is clear that $C_{\mu_2} \subset C_{\mu_1}$ with $\|y\|_{\mu_1} \leq \|y\|_{\mu_2}$, and $X_{\mu_2} \subset X_{\mu_1}$.

Together with lemmas 3.3 and 3.4, this implies that for each interval $I := [\mu_1, \mu_2] \subset (0, \lambda_0) \setminus \Re \sigma(A)$, we have $X_{\mu_2} = X_{\mu_1}$. We now show that the complementary subspaces $Y_{\mu}$ can be chosen so that $Y_{\mu_2} \subset Y_{\mu_1}$ for $\mu \in I$. In fact, let $X_{\mu} = \text{span}\{y_1, \ldots, y_r\}$, where $y_1, \ldots, y_r \in C_{\mu,0}$ and $r = r_{\mu}$ for $\mu \in I$. From the Hahn–Banach theorem, let $h_i \in (C_{\mu,0})'$ be such that $h_i(y_j) = 1$, $h_i(y_j) = 0$ for $j \neq i, i = 1, \ldots, r$. Define the natural injections $i(\mu, \mu_1) : C_{\mu,0} \to C_{\mu_1,0}$, which are continuous, and the subspaces $Y_{\mu} = \{y \in C_{\mu,0} : h_i \circ i(\mu, \mu_1)(y) = 0, i = 1, \ldots, r\}$. Hence $Y_{\mu}$ is a closed subspace of $C_{\mu,0}$, and for $y \in C_{\mu,0}$ we have $\sum_{i=1}^{r} h_i(y) y_i \in X_{\mu}$, $y - \sum_{i=1}^{r} h_i(y) y_i \in Y_{\mu}$, from which the decompositions $C_{\mu,0} = X_{\mu} \oplus Y_{\mu}$ follow, with $Y_{\mu_2} \subset Y_{\mu_1} \subset Y_{\mu}$, for $\mu \in I$.

Theorem 3.8. Assume (H1)–(H4), and denote by $\sigma(A)$ the set of characteristic values for (2.1). Fix an interval $I := [\mu_1, \mu_2] \subset (0, \lambda_0) \setminus \Re \sigma(A)$, and denote $r = r_{\mu}$ for all $\mu \in I$.

Then, there exist $\varepsilon^* > 0$ and $\sigma^* > 0$, such that for $0 < \varepsilon \leq \varepsilon^*$, the following holds: for each unit vector $w \in \Re^p$ and all $\mu \in I$, in a neighbourhood $B_{\varepsilon}^\varepsilon(0)$ of $u^*(t)$ in $C_{\varepsilon}$, the set of all travelling wave solutions $u(t, x) = \psi(\varepsilon t + w \cdot x)$ of (1.1) with speed $c = 1/\varepsilon$ and connecting 0 to $K$ forms a $r$-dimensional manifold (which does not depend on $\mu$), with the profile $\psi \in M_{\mu, \varepsilon}$, where

$$M_{\mu, \varepsilon} = \{\psi : \psi(t/\varepsilon) = u^*(t) + \xi + \phi(\varepsilon, \xi), \quad for \xi \in X_{\mu} \otimes B_{\varepsilon}^\varepsilon(0)\},$$

where $\phi(\varepsilon, \xi) = \phi(\varepsilon, \xi)$ is the fixed point of $T_{\mu}(\varepsilon, \xi, \cdot)$ in $Y_{\mu} \cap B_{\varepsilon}^\varepsilon(0)$, and is continuous on $(\varepsilon, \xi)$.

Proof. In the following, we shall use the simplified notation $S, \mathcal{F}, B_\varepsilon(0), B_{\varepsilon}^\varepsilon(0)$, respectively. Fix $\mu \in I$ and $k \in (0, 1)$. From lemma 3.6 (cf (3.15) and (3.16)), for $\delta > 0$ small there are $\sigma = \sigma(\delta, \mu) > 0$ and $\varepsilon^* = \varepsilon^*(\delta) > 0$ such that for $0 < \varepsilon \leq \varepsilon^*, \xi \in X_{\mu} \cap B_{\varepsilon}^\varepsilon(0)$ and $\phi_1, \phi_2 \in Y_{\mu} \cap B_{\varepsilon}^\varepsilon(0)$ we have

$$\|S^{-1} H(\xi + \phi_1, \varepsilon)\| \leq \|S^{-1} \| (C(e)) \|\xi + \phi_1\| + D(\varepsilon) \leq \varepsilon \|S^{-1}\| (\|\xi + \phi_1\| + 1) \quad (3.18)$$

and

$$\|S^{-1}(H(\xi + \phi_1, \varepsilon) - H(\xi + \phi_2, \varepsilon))\|_\mu \leq \|S^{-1}\| (C(e)) \|\phi_1 - \phi_2\|_\mu \leq \varepsilon \|S^{-1}\| (\|\phi_1 - \phi_2\|_\mu) \quad (3.19)$$

with $\delta(1 + 2\sigma)\|S^{-1}\| \leq \sigma$ and $\sigma\|S^{-1}\| \leq k$. From (3.18) and (3.19), it follows that $\mathcal{F} : (0, \varepsilon^*) \times (Y_{\mu} \cap B_{\varepsilon}^\varepsilon(0)) \times (Y_{\mu} \cap B_{\varepsilon}^\varepsilon(0)) \to Y_{\mu} \cap B_{\varepsilon}^\varepsilon(0)$ is a uniform contraction map of $\phi \in Y_{\mu} \cap B_{\varepsilon}^\varepsilon(0)$, hence for $(\varepsilon, \xi) \in (0, \varepsilon^*) \times (Y_{\mu} \cap B_{\varepsilon}^\varepsilon(0))$ there is a unique solution $\phi(\varepsilon, \xi) = \phi_0(\varepsilon, \xi) \in Y_{\mu}$ of (3.17), with $\phi(\varepsilon, \xi)$ continuous. Define the $r$-dimensional manifold $M_{\mu, \varepsilon} = \{\psi : \psi(t/\varepsilon) = u^*(t) + \xi + \phi(\varepsilon, \xi), \quad for \xi \in Y_{\mu} \cap B_{\varepsilon}^\varepsilon(0)\}$. Choose $\sigma = \sigma(\delta, \mu_2)$, independent of $\mu \in I$. From the uniqueness of the fixed point and remark 3.7, it follows that $\phi_0(\varepsilon, \xi) = \phi_{\mu_2}(\varepsilon, \xi)$ does not depend on $\mu \in I$, as well as $M_{\mu, \varepsilon} := M_{\mu, \varepsilon}$.

We observe that if 0 is a hyperbolic equilibrium of (1.2) and $f$ has the particular form $f(\phi) = F(\phi(0), g(\nabla \phi))$, then theorem 3.1 asserts that the result in theorem 3.8 is valid for $\mu = 0$.

Corollary 3.9. Under the assumptions of theorem 3.8 and with the same notation, for $0 < \mu < \lambda_0$ such that the strip $\{\lambda \in \Re : \Re \lambda \in (\mu, \lambda_0)\}$ does not intersect $\sigma(A)$, the manifold $M_{\mu, \varepsilon}$ is one-dimensional.
Corollary 3.10. Under the assumptions of theorem 3.8 and with the same notation, for an interval $I := \{\mu_1, \mu_2\} \subset (0, \lambda_0) \setminus \sigma(A)$, there are $\varepsilon^* > 0, \sigma > 0$ and $C > 0$ such that the travelling profiles $\psi(\varepsilon, \xi)$ satisfy

$$\|\psi(\varepsilon, \xi)\|_{\mu} \leq C, \quad \|\psi'(\varepsilon, \xi)\|_{\mu} \leq C \quad \text{for} \quad 0 < \varepsilon < \varepsilon^*, \xi \in X_{\mu} \cap \overline{B}_\sigma(0),$$

where $C$ does not depend on $\mu \in I$. In particular $|\psi(\varepsilon, \xi)(t)| \leq Ce^{\mu t}$ for $t \leq 0, 0 < \varepsilon < \varepsilon^*, \xi \in X_{\mu} \cap \overline{B}_\sigma(0)$.

Proof. For all $\mu \in I$, the profiles are given by $\psi(\varepsilon, \xi) = u^* + \xi + \phi(\varepsilon, \xi)$, where $\phi(\varepsilon, \xi) = \phi_{\mu}(\varepsilon, \xi)$ is the fixed point of $F_{\mu}(\varepsilon, \xi, \cdot)$. Since $\|\psi\|_{\mu} \leq \|\xi\|_{\mu}$ for $\xi \in C_{\mu,0}$, we only need to prove the result for $\mu = \mu_2$. In what follows, we write $S_{\mu_2}^{-1} = S^{-1}$.

Fix $k \in (0, 1)$, and consider $\varepsilon_1 > 0$ such that $\|S^{-1}(\varepsilon)\| \leq k$ for $0 < \varepsilon < \varepsilon_1$ where $C(\varepsilon)$ is as in (3.16). From (3.19), for $w_1, w_2 \in C_{\mu_2,0} \cap B_\sigma(0)$ we have

$$\|S^{-1}(H(w_1, \varepsilon) - H(w_2, \varepsilon))\|_{\mu_2} \leq k\|w_1 - w_2\|_{\mu_2},$$

and the contraction principle yields

$$\|\phi(\varepsilon, \xi)\|_{\mu_2} \leq \frac{1}{1 - k}\|F_{\mu_2}(\varepsilon, \xi, 0)\|_{\mu_2} \leq \frac{1}{1 - k}\|S^{-1}\|\|H(\varepsilon, \xi)\|_{\mu_2}, \quad \xi \in X_{\mu} \cap \overline{B}_\sigma(0).$$

For $\varepsilon, \sigma > 0$ small enough, from (3.15) we get

$$\|H(\varepsilon, \xi)(t)\|_{\mu_2} \leq C(\varepsilon)\|\xi\|_{\mu_2} + \varepsilon^2\|u^{\ast\ast}\|_{\mu_2}, \quad \xi \in X_{\mu} \cap \overline{B}_\sigma(0).$$

We thus obtain $\|\psi(\varepsilon, \xi)\|_{\mu_2} \leq \|u^*\|_{\mu_2} + (1 + C(\varepsilon))\sigma + \varepsilon^2\|u^{\ast\ast}\|_{\mu_2} \leq C_1$ for $\varepsilon$ small, where $C_1$ does not depend on $\varepsilon, \xi$.

Now we want to prove a similar estimate for the derivatives $d\psi(\varepsilon, \xi)/dt$. For simplicity, we only prove the result for $\xi = 0$.

Since $\psi(t) := \psi(\varepsilon, 0)(t)$ is a solution of (3.2), then $\psi(t)$ is given by the integral formula

$$\psi(t) = \frac{1}{\sqrt{1 + 4\varepsilon^2}} \left( \int_{-\infty}^t e^{\alpha(s)(t-s)}[\psi(s) + f(\psi_s)] ds + \int_t^{+\infty} e^{\beta(s)(t-s)}[\psi(s) + f(\psi_s)] ds \right),$$

from which we derive

$$\psi'(t) = \frac{1}{\sqrt{1 + 4\varepsilon^2}} \left( \alpha(\varepsilon) \int_{-\infty}^t e^{\alpha(s)(t-s)}[\psi(s) + f(\psi_s)] ds - \beta(\varepsilon) \int_{-\infty}^t e^{\beta(s)(t-s)}[\psi(s) + f(\psi_s)] ds \right).$$

Since $f(\psi_s) \to 0$ as $s \to \pm\infty$, then $|f(\psi_s)|$ is uniformly bounded on $\mathbb{R}$. Thus, there is $\ell$ such that $|f(\psi_s)| \leq \ell$ for $s \in \mathbb{R}$, where $\ell$ does not depend on $\mu, \varepsilon$, and $\|\psi'\|_{\infty} \leq 2(C_1 + \ell)/\sqrt{1 + 4\varepsilon^2}$. From (3.10), the $C^1$-smoothness of $f$ and $f(0) = 0$, we easily deduce that there is $C_2 > 0$ such that $\|\psi'\|_{\mu} \leq C_2$. This completes the proof. □

In fact a stronger result can be proven:

Corollary 3.11. Assume (H1)–(H4), take $\mu \in I := \{\mu_1, \mu_2\} \subset (0, \lambda_0) \setminus \sigma(A)$ and consider the travelling wave profiles $\psi(\varepsilon, \xi) = u^* + \xi + \phi(\varepsilon, \xi)$ for $\varepsilon \in (0, \varepsilon^*), \xi \in X_{\mu} \cap B_\sigma(0)$, given in theorem 3.8. For $\xi = 0$ and $\mu \in I$, the profile $\psi(\varepsilon, 0)$ satisfies

$$\psi(\varepsilon, 0) \to u^* \text{ in } C_\mu \quad \text{as} \quad \varepsilon \to 0^*.$$ (3.20)
Proof. Let \( \varepsilon^*, \sigma > 0 \) be as in the statement of theorem 3.8, and recall that \( \psi(\varepsilon, 0) = \psi_\mu(\varepsilon, 0) \) only depends on \( \varepsilon \). Next, we deduce some estimates as in lemma 3.6, so details are omitted. For \( \varepsilon = 0 \), define

\[
H(0, w)(t) = \int_{-\infty}^{t} e^{-\varepsilon(t-s)} [f(w_s + u*) - f(u*) - Df(u*) w_s] \, ds.
\]

We write \( H(0, w)(t) = \int_{-\infty}^{t} e^{-\varepsilon(t-s)} G(0, t, w) \, ds := H_2(0, w) \), where \( G(0, t, w) \) is given by (3.4). After some computations, we observe that the function \( H \) restricted to \( [0, \varepsilon^*] \times (C_{\mu, 0} \cap \mathcal{B}^\mu_0(0)) \) satisfies

\[
\|H(\varepsilon, w) - H(0, w)\|_{\mu} \leq C_0(\varepsilon) \|w\|_{\mu} + D_0(\varepsilon),
\]

with \( C_0(\varepsilon), D_0(\varepsilon) \) independent of \( \mu, C_0(\varepsilon), D_0(\varepsilon) \to 0 \) as \( \varepsilon \to 0^+ \). This means that the function \( (\varepsilon, w) \mapsto H(\varepsilon, w) \) converges, uniformly on \( w \in C_{\mu, 0} \cap \mathcal{B}^\mu_0(0) \), to \( H(0, \cdot) \) in \( C_{\mu, 0} \) as \( \varepsilon \to 0^+ \).

Moreover, for \( \varepsilon = 0 \) and \( \xi = 0 \) the fixed point of (3.17) is \( \psi(0, 0) = 0 \). Therefore, the application of the contraction principle as in the proof of theorem 3.8 leads to (3.20).

Remark 3.12. As seen in section 2, the existence of a positive eigenvector \( \nu \in \mathbb{R}^N \) associated with the characteristic root \( \lambda_0 \) of (2.2) was crucial to prove the existence of a positive heteroclinic solution \( u^+(t) \) of (1.2), connecting the equilibria \( 0 \) to \( K \). For all the results in this section the positiveness of such heteroclinic solution is irrelevant, and therefore it is not necessary to impose the above requirement in (H4) that \( v \) is positive. For the same reason, in section 3 assumption (H2)(ii) is not needed as well.

4. Positiveness of travelling waves

Consider the characteristic equation for the linearization of (3.2) at 0,

\[
\det \Delta_\varepsilon(z) = 0, \quad \text{where} \quad \Delta_\varepsilon(z) := \varepsilon^2 z^2 I - z I + L(e^{\varepsilon I}),
\]

(4.1)

where \( L = Df(0) \). Recall that for \( \varepsilon = 0 \) the characteristic matrix-valued function \( \Delta_0(z) \) was defined in (2.2). Since \( \lambda_0 > 0 \) is a simple root of the characteristic equation \( \det \Delta_0(z) = 0 \), from the implicit function theorem, for \( \varepsilon > 0 \) small there is a simple real root \( \lambda(\varepsilon) \) of (4.1), with \( \lambda(\varepsilon) \to \lambda_0 \) as \( \varepsilon \to 0^+ \).

Lemma 4.1. For \( \delta > 0 \) sufficiently small and \( \delta_1 > 0 \), there exists \( \varepsilon_0 > 0 \) such that, for \( 0 < \varepsilon < \varepsilon_0 \), \( \lambda(\varepsilon) \) is the only root of the characteristic equation (4.1) on the vertical strip \( 0 - \delta \leq \Re z \leq \lambda_0 + \delta_1 \).

Proof. Let \( \delta > 0 \) be such that \( \lambda_0 \) is the only root of \( \det \Delta_0(z) = 0 \) on the strip \( S = \{z: \lambda_0 - \delta \leq \Re z \leq \lambda_0 + \delta_1 \} \). If \( z(\varepsilon) \in S \) is a root of \( \Delta_\varepsilon(z) = 0 \), then there is a unit vector \( w = w(\varepsilon) \in \mathbb{R}^N \) such that \( (e^{\varepsilon z(\varepsilon)}) w = L(e^{\varepsilon I}) w \), hence

\[
\|L\| \geq |e^{\lambda_0 z(\varepsilon)} - z(\varepsilon)| \geq |\Im(e^{\lambda_0 z(\varepsilon)^2} - z(\varepsilon))| = |\Im z(\varepsilon)||2e^{2\Re z(\varepsilon)} - 1|.
\]

Choose \( \varepsilon_0 > 0 \) such that \( |2e^{2\Re z - 1}| > 1/2 \) for all \( z \in S, |z| < \varepsilon_0 \). For \( |z| < \varepsilon_0 \), we have

\[
|\Im z(\varepsilon)| < 2\|L\|.
\]

Thus, for \( |z| < \varepsilon_0 \) the solutions \( z(\varepsilon) \in S \) of \( \det \Delta_\varepsilon(z) = 0 \) are necessarily inside the rectangle \( \Gamma = [\lambda_0 - \delta, \lambda_0 + \delta_1] \times [-2\|L\|, 2\|L\|] \).

Now, let \( F(z, \varepsilon) := \det \Delta_\varepsilon(z), z \in \mathbb{C}, \varepsilon \in \mathbb{R} \). Clearly, \( F(z, \varepsilon) \to F(z, 0) \) as \( \varepsilon \to 0 \), for all \( z \in \mathbb{C} \). Moreover, since \( \Delta_\varepsilon(z) = \varepsilon^2 z^2 I + \Delta_0(z) \), one easily deduces that the function \( F(\cdot, \varepsilon) \) converges uniformly to \( F(\cdot, 0) \) on bounded sets of \( \mathbb{C} \), as \( \varepsilon \to 0 \).
We now apply Rouché’s theorem on the boundary \(\partial \Gamma\) of \(\Gamma\). Set \(m = \min_{\zeta \in \partial \Gamma} |F(\zeta, 0)| > 0\). For \(|\varepsilon|\) small, we have \(|F(\varepsilon, 0) - F(\zeta, 0)| < m, \zeta \in \partial \Gamma\), hence \(F(\varepsilon, \varepsilon)\) and \(F(0, 0)\) have the same number of zeros inside \(\Gamma\). Thus, for \(\varepsilon > 0\) sufficiently small \(\lambda(\varepsilon)\) is the only solution of (4.1) in the strip \(S\). ■

For (3.2) written as a system in \(\mathbb{R}^{2N}\), its linearized equation at zero is

\[
x'(t) = L_\varepsilon(x),
\]

where

\[
L_\varepsilon \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right) = \left( \begin{array}{c} \phi_2(0) \\ -1 + \frac{1}{\varepsilon^2} L(\phi_1) + \frac{1}{\varepsilon^2} \phi_2(0) \end{array} \right), \quad \phi_1, \phi_2 \in \mathcal{C}([0, 1]; \mathbb{R}^N).
\]

For the linear system (4.2), the characteristic equation is given by

\[
\det D_\varepsilon(s) = 0, \quad \text{where } D_\varepsilon(s) = \left( \begin{array}{cc} sI & -I \\ \frac{1}{\varepsilon^2} L(e^s I) & \left( s - \frac{1}{\varepsilon^2} \right) I \end{array} \right),
\]

and \(I\) is the \(N \times N\) identity matrix. Clearly, \(\det \Delta_\varepsilon(s) = \varepsilon^{2N} \det D_\varepsilon(s)\), hence for \(\varepsilon > 0\) (4.3) is equivalent to (4.1).

**Lemma 4.2.** Consider \(b \in \mathbb{R}\) and \(\varepsilon_0 \in (0, 1)\) such that \(\det \Delta_\varepsilon(s) \neq 0\) for all \(\varepsilon \in [0, \varepsilon_0]\) and \(s\) on the vertical line \(\Sigma = \{s = b + iy : y \in \mathbb{R}\}\). Then there is \(\varepsilon_1 \in (0, \varepsilon_0]\) such that

\[
\sup \left\{ |s| \|\Delta_\varepsilon(s)^{-1}\| : 0 \leq \varepsilon \leq \varepsilon_1, s \in \Sigma \right\} < \infty
\]

and

\[
\sup \left\{ |s| \|D_\varepsilon(s)^{-1}\| : 0 < \varepsilon \leq \varepsilon_1, s \in \Sigma \right\} < \infty.
\]

**Proof.** For \(\varepsilon \in (0, \varepsilon_0], s \in \Sigma\), we have

\[
G(\varepsilon, s) := D_\varepsilon(s)^{-1} = \left( \begin{array}{cc} \Delta_\varepsilon(s)^{-1} & 0 \\ 0 & \Delta_\varepsilon(s)^{-1} \end{array} \right) \left( \begin{array}{cc} (\varepsilon^2 s - 1)I & \varepsilon^2 I \\ -L(e^s I) & \varepsilon^2 s I \end{array} \right).
\]

Clearly,

\[
G(\varepsilon, s) \to \left( \begin{array}{cc} \Delta_0(s)^{-1} & 0 \\ 0 & \Delta_0(s)^{-1} \end{array} \right) \left( \begin{array}{cc} -I \\ -L(e^s I) \end{array} \right) := G(0, s) \quad \text{as } \varepsilon \to 0^+,
\]

with \(G(\varepsilon, s)\) continuous on \([0, \varepsilon_0]\) \(\times\) \(\Sigma\). For \(s \in \Sigma\), we have \(\|L(e^s I)\| \leq \max(1, e^{-bs})\|L\|\).

It follows that

\[
\|G(\varepsilon, s)\| \leq c(e^2|s| + 1)\|\Delta_\varepsilon(s)^{-1}\|, \quad 0 < \varepsilon \leq \varepsilon_0, s \in \Sigma,
\]

for some \(c > 0\). Since \(\Delta_\varepsilon(s) = (\varepsilon^2 s^2 - 1)I + L(e^s I)\), then

\[
\|\Delta_\varepsilon(s)^{-1}\| \leq \frac{1}{|s|\varepsilon^2 s - 1 - \|L(e^s I)\|}
\]

if \(|s|\varepsilon^2 s - 1 - \|L(e^s I)\| > 0\). Choose \(\varepsilon_1 \in (0, \varepsilon_0]\) such that \(1 - \varepsilon_1^2 b \geq 1/2\).

Then for \(s \in \Sigma\) and \(0 < \varepsilon \leq \varepsilon_1\), if \(|s| \geq 4\max(1, e^{-bs})\|L\| =: c_1\), it follows that

\[
|s|\varepsilon^2 s - 1 - \|L(e^s I)\| \geq |s|/4 > 0, \text{ thus } |s|\|\Delta_\varepsilon(s)^{-1}\| \leq 4
\]

and

\[
|s| \|G(\varepsilon, s)\| \leq \frac{c|s|\varepsilon^2 s + 1}{|s|\varepsilon^2 s - 1 - \|L(e^s I)\|} \leq \frac{c\varepsilon^2 s^2}{|s|\varepsilon^2 s - 1 - \|L(e^s I)\|} + 4c
\]
for $\epsilon \in (0, \epsilon_1]$ and $s \in \Sigma$, $|s| \geq c_1$. Now, $|\epsilon^2 x - 1| = \frac{\sqrt{(\epsilon^2 b - 1)^2 + \epsilon^4 y^2}}{\epsilon^2 |x|^2 - 2 \epsilon^2 b + 1} \geq \epsilon^2 |x|$, from the definition of $\epsilon_1$. Hence $|s| |\epsilon^2 x - 1| \geq \epsilon^2 |s|^2$, and

$$|s| |\epsilon^2 x - 1| - \|L(\epsilon^2 I)\| \leq 2$$

if $\epsilon^2 |s|^2 \geq 2 \|L(\epsilon^2 I)\|$, and if $\epsilon^2 |s|^2 \leq 2 \|L(\epsilon^2 I)\|$, then

$$|s| |\epsilon^2 x - 1| - \|L(\epsilon^2 I)\| \leq \frac{2 \|L(\epsilon^2 I)\|}{|s|} \leq 2;$$

hence, $|s| \|G(\epsilon, s)\| \leq 6\epsilon$ if $|s| \geq c_1$.

On the other hand, on the compact set $\{(\epsilon, s) \in [0, \epsilon_1] \times \Sigma : |s| \leq c_1\}$ the continuous functions $|s| \|\Delta_\epsilon(s)^{-1}\|$ and $|s| \|G(\epsilon, s)\|$ attain their suprema, and the conclusion follows. ■

We are finally in a position to prove the main result of this section, on the existence of positive travelling wave solutions of equation (1.1) for large wave speeds.

**Theorem 4.3.** Assume (H1)–(H4). Then, there is $c^* > 0$, such that for $c > c^*$ equation (1.1) has a positive travelling wave solution of the form $u(t, x) = \psi(ct + w \cdot x)$ for each unit vector $w \in \mathbb{R}^p$, with $\psi(-\infty) = 0$, $\psi(\infty) = K$. Moreover, the components of the profile $\psi$ are increasing in the vicinity of $-\infty$ and it satisfies $\psi(t/\epsilon) = O(e^{\epsilon(t/\epsilon)}, \psi(t/\epsilon) = O(e^{\epsilon(t/\epsilon)})$ at $-\infty$, where $\epsilon = c/\mu$ and $\lambda(\epsilon)$ is the real solution of (4.1) with $\lambda(\epsilon) \to \lambda_0$ as $\epsilon \to 0^+$.

**Proof.** Consider equation (3.2), where $\epsilon = 1/c$. Let $\mu \in (0, \lambda_0)$ be as in the statement of corollary 3.9 and satisfy $\lambda_0 < \mu + 2\delta$ for some fixed $\delta \in (0, \mu/4)$. Suppose also that $\lambda(\epsilon)$ is the unique solution of $\det \Delta_\epsilon(s) = 0$ on the strip $\mu - \delta \leq \Re z < 2\mu$ for all $\epsilon \in [0, \epsilon^*]$. Fix the profiles

$$\psi_\epsilon = \psi(0, 0), \quad \psi_0(t) := u^*(t) = e^{\lambda t} v + O(e^{2\mu t}), \quad \epsilon \in [0, \epsilon^*],$$

as in corollary 3.11 and theorem 2.1. Recall that $v$ is a positive eigenvector associated with $\lambda_0$. The proof is now divided in several steps.

**Claim 1.** There is $\epsilon_0 > 0$ such that $(\psi_\epsilon(t), \psi'_\epsilon(t)) = (e^{\lambda(t)\epsilon} v^1(\epsilon), \lambda(\epsilon) e^{\lambda(t)\epsilon} v^1(\epsilon)) + w_\epsilon(t), \epsilon \in [0, \epsilon_0)$, with continuous $v^1(\epsilon) > 0$, $v^1(0) = v$ and $w_\epsilon(t) = O(\epsilon(1+\delta)\mu)$ at $-\infty$.

To prove the above claim, note that $x_\epsilon(t) := (\psi_\epsilon(t), \psi'_\epsilon(t))$ is a solution of the system

$$x'_\epsilon(t) = L x_\epsilon(t), \quad \epsilon^2 x'_\epsilon(t) = x_\epsilon(t) - L x_\epsilon(t), \quad \epsilon \in [0, \epsilon_0]$$

where $L = Df(0)$ and $h_\epsilon(t) = f((\psi_\epsilon)_t) - L((\psi_\epsilon)_t)$. For $\epsilon > 0$, equivalently we write (4.6) as

$$x'_\epsilon(t) = \mathcal{L}_\epsilon x_\epsilon(t) - \frac{1}{\epsilon^2} \left( \begin{array}{c} 0 \\ h_\epsilon(t) \end{array} \right),$$

where $\mathcal{L}_\epsilon$ is as in (4.2). Since $f$ is a $C^2$ function, using the Taylor formula for $f$ (cf e.g. [4, p 23]), we have the estimate

$$|h_\epsilon(t)| \leq \int_0^t (1 - s) \|D^2 f(s(\psi_\epsilon)_t)\| \|\psi_\epsilon\|_{\infty} ds, \quad t \in \mathbb{R}, \epsilon \in [0, \epsilon^*].$$

Since $\|\psi_\epsilon - u^*\|_\mu \to 0$ as $\epsilon \to 0^+$ and $u^*(t) \to 0$ as $t \to -\infty$, the continuity of $D^2 f$ at 0 implies that $\|D^2 f(s(\psi_\epsilon)_t)\|$ is uniformly bounded on $\epsilon \in [0, \epsilon_1] \subset [0, \epsilon^*], t \leq 0, s \in [0, 1]$. Together with corollary 3.10, this leads to

$$|x_\epsilon(t)| \leq Ce^{\mu t}, \quad |h_\epsilon(t)| \leq De^{2\mu t} \quad \text{for } t \leq 0,$$

for some constants $C, D$ independent of $\epsilon \in [0, \epsilon_1]$. 

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Claim 1.
We now apply theorem A.2 to (4.6) at $-\infty$ (see the appendix), and derive that for $\varepsilon > 0$
\begin{equation}
    x_\varepsilon(t) = (\psi_\varepsilon(t), \dot{\psi}_\varepsilon(t)) = z_\varepsilon(t) + u_\varepsilon(t)
\end{equation}
where $z_\varepsilon(t)$ is an eigenfunction for the linear system $x'(t) = L_\varepsilon(x_\varepsilon)$ corresponding to the set $\Lambda_\varepsilon = \{z \in \mathbb{C} : \det \Delta_\varepsilon(z) = 0, \mu \leq \Re z < 2\mu\}$ and $u_\varepsilon(t) = O(e^{(\lambda_0+\delta)t})$ at $-\infty$. From lemma 4.1, let $\varepsilon$ be some interval $(0, \varepsilon_0) \subset (0, \varepsilon_1)$ such that that $\Lambda_\varepsilon = \{\lambda(\varepsilon)\}$. Then, $z_\varepsilon(t)$ is an eigenfunction for $x'(t) = L_\varepsilon(x_\varepsilon)$ associated with the root $\lambda(\varepsilon)$ of (4.3), hence $z_\varepsilon(t) = e^{\lambda(\varepsilon)t}v(\varepsilon)$ with $v(\varepsilon) = (v^1(\varepsilon), v^2(\varepsilon)) \in \mathbb{R}^{2N}$ satisfying $D_\varepsilon(\lambda(\varepsilon))v(\varepsilon) = 0$ for $D_\varepsilon$ as in (4.3). From this we obtain $v^2(\varepsilon) = \lambda(\varepsilon)v^1(\varepsilon)$ and $\Delta_\varepsilon(\lambda(\varepsilon))v^1(\varepsilon) = 0$. Furthermore, from theorem A.2 (with $a = \mu$, $b = \lambda_0 + 2\delta$, $\varepsilon = \delta$) and formulæ (A.5) and (A.7) (adapted to the situation $-\infty$) we get
\begin{equation}
    z_\varepsilon(t) = \text{Res} (e^{t} \tilde{x}_\varepsilon, \lambda(\varepsilon)) = e^{\lambda(\varepsilon)t} \lim_{t \to \lambda(\varepsilon)} (s - \lambda(\varepsilon))\tilde{x}_\varepsilon(s) = e^{\lambda(\varepsilon)t}(v^1(\varepsilon), \lambda(\varepsilon)v^1(\varepsilon))
\end{equation}
and
\begin{equation}
    u_\varepsilon(t) = \frac{1}{2\pi i} \int_{\beta+i\varepsilon}^{-\beta+i\varepsilon} e^{st} \tilde{x}_\varepsilon(s) \, ds,
\end{equation}
with
\begin{equation}
    \tilde{x}_\varepsilon(s) := x_\varepsilon(-s) = G(\varepsilon, s) \left( r_\varepsilon(s) - \left( \begin{array}{c} 0 \\ \tilde{h}_\varepsilon(s) \end{array} \right) \right)
\end{equation}
for $G(\varepsilon, s) = D_\varepsilon(s)^{-1}$ and
\begin{equation}
    r_\varepsilon(s) = x_\varepsilon(0) + L_\varepsilon \left( e^s \int_0^0 e^{-uw}x_\varepsilon(u) \, du \right),
\end{equation}
\begin{equation}
    \tilde{h}_\varepsilon(s) := h_\varepsilon(-s) = \int_0^{-s} e^{-uw}h_\varepsilon(u) \, du, \quad \Re s < \lambda_0 + 2\delta.
\end{equation}
Here, $x_\varepsilon(-s), h_\varepsilon(-s)$ denote the Laplace transforms of the functions $t \mapsto x_\varepsilon(-t), t \mapsto h_\varepsilon(-t)$, respectively. Note that $\tilde{x}_\varepsilon(s)$ is meromorphic for $\Re s < 2\mu$ with a unique singularity at $s = \lambda(\varepsilon)$ which is a simple pole of $\Delta_\varepsilon(s)^{-1}$ (cf appendix).
From (4.8), we get
\begin{equation}
    \psi_\varepsilon(t) = e^{\lambda(\varepsilon)t}v^1(\varepsilon) + w^1_\varepsilon(t), \quad \dot{\psi}_\varepsilon(t) = \lambda(\varepsilon)e^{\lambda(\varepsilon)t}v^1(\varepsilon) + w^2_\varepsilon(t),
\end{equation}
with $w^1_\varepsilon(t) = O(e^{(\lambda_0+\delta)t})$ at $-\infty$ and $w^2_\varepsilon(t) = (w^1_\varepsilon(t))'$. Next, the definition of $r_\varepsilon(s)$ in (4.11) yields
\begin{equation}
    r_\varepsilon(s) = x_\varepsilon(0) + \left( -\frac{1}{\varepsilon}L \left( e^s \int_0^0 e^{-tw}x_\varepsilon(t) \, dt \right) \right),
\end{equation}
hence from (4.4) and (4.10) we obtain
\begin{equation}
    \tilde{x}_\varepsilon(s) = G(\varepsilon, s)x_\varepsilon(0) - \left( \Delta_\varepsilon(s)^{-1}L \left( e^s \int_0^0 e^{-tw}\dot{\psi}_\varepsilon(t) \, dt \right) \right) - \left( \Delta_\varepsilon(s)^{-1}\tilde{h}_\varepsilon(s) \right).
\end{equation}
We now extend naturally this situation for $\varepsilon = 0$. In (4.5), denote $\lambda(0) = \lambda_0$, $v^1(0) = v$.
Write $\tilde{x}_\varepsilon = (\tilde{x}^1_\varepsilon, \tilde{x}^2_\varepsilon)$ for $\varepsilon \in (0, \varepsilon_0)$, and let $\tilde{x}^1_\varepsilon(s)$ be defined by (4.13) for $\varepsilon = 0$. Note that
formula (4.10) can still be used to obtain $\tilde{x}_0^\varepsilon(s)$ (cf appendix for more details),

$$
\tilde{x}_0^\varepsilon(s) = \lambda_0(s)^{-1}[r_0^\varepsilon(s) + h_0(s)],
$$

where $r_0^\varepsilon(s) = u^*(0) + L(e^{s \int_0^0 e^{-st} u^*(t) \, dt})$ is $\lim_{\varepsilon \to 0^+} r_1^\varepsilon(s)$.

For each $\varepsilon \in (0, \varepsilon_0)$, $s = \lambda(\varepsilon)$ is a pole of order one of $G(\varepsilon, s)$, and from (4.10) we deduce that for $\varepsilon \in [0, \varepsilon_0)$ and $\mu \leq \Re s < \lambda_0 + 2\delta$ the function $A(\varepsilon, s)$ defined by $A(\varepsilon, s) = (s - \lambda(\varepsilon))\tilde{x}_0^\varepsilon(s)$ for $s \neq \lambda(\varepsilon)$, $A(\varepsilon, \lambda(\varepsilon)) = v^\varepsilon(\varepsilon)$ is analytic on $s$ and continuous on $(\varepsilon, \sigma)$. In particular, $\lim_{\varepsilon \to 0^+} A(\varepsilon, \lambda(\varepsilon)) = A(0, \lambda_0) = v > 0$, hence $v^\varepsilon(\varepsilon) \to v$ as $\varepsilon \to 0^+$. Moreover, $v^\varepsilon(\varepsilon) > 0$ for $\varepsilon > 0$ sufficiently small. This proves claim 1.

**Claim 2.** For $\varepsilon_0^* > 0$ sufficiently small, there exists a constant $D_0 > 0$ such that

$$
|w_1^\varepsilon(t)| \leq D_0 e^{(\lambda_0 + \delta)t} \quad \text{for all } t \leq 0, 0 < \varepsilon < \varepsilon_0^*.
$$

To prove claim 2, once more we shall use some formulae and estimates in the proof of theorem A.2 in the appendix, changed accordingly to account for the asymptotic behaviour at $-\infty$, rather than $\infty$.

Define $v_i(t) = (v_i^1(t), v_i^2(t)) = e^{-(\lambda_0 + \delta)t} u_i(t)$ and $u_i(t) = (u_i^1(t), u_i^2(t)) = e^{-(\lambda_0 + 3\delta/2)t} u_i(t)$. Note that $v_i(0) = w_i^0(0)$ an $v(t) = w_i^0(0) - \int_0^1 v_i^1(\varepsilon) \, d\varepsilon$, with $w_i(0) = \psi_0(0) - v^\varepsilon(\varepsilon)$. Then $|w_i(0)| \leq C + |v^\varepsilon(\varepsilon)|$, where $C > 0$ is as in (4.7). We need to prove that $v_i(t)$ is uniformly bounded for $t \leq 0$ and $\varepsilon > 0$ small enough. In order to achieve this, we shall show that there are constants $C_0, D_0 > 0$ and $\varepsilon_0 > 0$, such that

$$
\|v_1^\varepsilon\|_{L^1(-\infty, 0)} \leq C_0, \quad \varepsilon \in (0, \varepsilon_0^*),
$$

and

$$
\|(v_1^\varepsilon)'\|_{L^1(-\infty, 0)} \leq D_0/2, \quad \varepsilon \in (0, \varepsilon_0^*),
$$

so that (4.14) follows immediately from (4.16) and $|w_i(0)| \leq D_0/2$ for $\varepsilon \in (0, \varepsilon_0^*)$. These uniform estimates require a careful analysis of the explicit formulae for $w_i$ given in (4.9) and (4.10). We shall prove (4.15) beforehand, and then use (4.15) to prove (4.16).

First, observe that $x_i(t) = z_i(t) + w_i(t)$ is a solution of (4.6), with $x_i(t)$ being an eigenfunction for the linear system $x_i(t) = L_i(x_i)$, hence $w_i(t)$ is a solution of system (4.6) as well. The definition of $v_i(t)$ yields now

$$
(v_i^1)'(t) = - (\lambda_0 + \delta) v_i^1(t) + v_i^2(t)
$$

and

$$
\varepsilon^2 (v_i^1)'(t) - \alpha (v_i^1)'(t) + P_i(t) = 0,
$$

where $\alpha = 1 - 2\varepsilon^2(\lambda_0 + \delta)$ and

$$
P_i(t) = [\varepsilon^2 (\lambda_0 + \delta)^2 - (\lambda_0 + \delta)] v_i^1(t) + L \left( e^{(\lambda_0 + \delta)t} (v_i^1)_t \right) + e^{-(\lambda_0 + \delta)t} h_i(t).
$$

Next, similarly to (A.9) and remark A.3, using (4.9) and the Plancherel theorem, we obtain

$$
\|v_i^1\|_{L^1(-\infty, 0)} = \|w_i^0(t) e^{-\lambda_0 t} \|_{L^1(-\infty, 0)} = \frac{\sqrt{2\pi}}{2\mu} \int_0^{\infty} e^{\mu d(t)/2} d(u) \|\tilde{x}_0^\varepsilon(t)\|_{L^2(\mathbb{R})}
$$

and

$$
\|v_i^1\|_{L^1(-\infty, 0)} \leq C_1(\delta) \|\tilde{x}_0^\varepsilon(\lambda_0 + 3\delta/2 - i)\|_{L^2(\mathbb{R})},
$$

for $\varepsilon \in (0, \varepsilon_0)$.
\[ s = \lambda_0 + 3\delta/2. \] Using now lemma 4.2, from (4.13) we conclude that there is \( K_1 > 0 \) such that
\[
|\tilde{\omega}_x^1(s)| \leq K_1/|s|, \quad \text{for } s = b + \frac{3\delta}{2} + iy, \quad y \in \mathbb{R} \quad \text{and} \quad 0 < \varepsilon < \varepsilon_0,
\]
from which we derive
\[
\|\tilde{\omega}_x^1(\lambda_0 + 3\delta/2 - i\cdot)\|_{L^2(\mathbb{R})}^2 \leq \int_{\mathbb{R}} \frac{K_1}{(b + \frac{3\delta}{2})^2 + y^2} \, dy < \infty.
\]
Together with (4.20), this leads to the estimate (4.15) with \( \varepsilon_0^* = \varepsilon_0 \).

In order to prove (4.16), we now use (4.18) and some ideas from Aguerrea et al [1, lemma 4.1]. After partial integration of equation (4.18), we obtain
\[
(u_v^1)'(t) = e^{(\alpha/\varepsilon)t} (u_v^1)'(0) + \frac{e^{(\alpha/\varepsilon)t}}{\varepsilon^2} \int_t^0 e^{-(\alpha/\varepsilon)s} P_\varepsilon(s) \, ds.
\] (4.21)
Let \( \varepsilon > 0 \) be small, so that \( \alpha > 0 \). From (4.17) and claim 1,
\[
(u_v^1)'(0) = -(\lambda_0 + \delta) v_r^1(0) + v_r^2(0) = -(\lambda_0 + \delta) v_r(0) + (\lambda_0 + \delta - \lambda(\varepsilon)) v^1(\varepsilon) + v_r(0),
\]
with \( v^1(\varepsilon) \to v \) as \( \varepsilon \to 0^+ \). Corollary 3.10 implies that \( |(u_v^1)'(0)| \) is uniformly bounded on \( \varepsilon > 0 \) sufficiently small, hence
\[
|u_v^1)'(0)| \int_{-\infty}^0 e^{(\alpha/\varepsilon)t} \, dt \leq K_2,
\] (4.22)
for some \( K_2 > 0 \) and \( \varepsilon > 0 \) sufficiently small. On the other hand, interchanging the order of integration leads to
\[
\frac{1}{\varepsilon^2} \int_0^\infty e^{(\alpha/\varepsilon)t} \left( \int_t^0 e^{-(\alpha/\varepsilon)s} P_\varepsilon(s) \, ds \right) \, dt = \frac{1}{\varepsilon^2} \int_0^\infty e^{-(\alpha/\varepsilon)t} P_\varepsilon(t) \left( \int_0^t e^{(\alpha/\varepsilon)s} \, ds \right) \, dt,
\]
hence
\[
\frac{1}{\varepsilon^2} \int_{-\infty}^0 e^{(\alpha/\varepsilon)t} \left( \int_t^0 e^{-(\alpha/\varepsilon)s} P_\varepsilon(s) \, ds \right) \, dt \leq \frac{1}{\alpha} \int_{-\infty}^0 |P_\varepsilon(s)| \, ds.
\]
From (4.7), (4.15) and the definition of \( P_\varepsilon(s) \) in (4.19), we easily see that
\[
\|P_\varepsilon\|_{L^1([\varepsilon_1, \infty])} \leq K_3, \quad 0 < \varepsilon < \varepsilon_0^* \]
for some \( \varepsilon_0^* > 0 \). Together with (4.21) and (4.22), this yields the estimate (4.16), and therefore claim 2 is proven.

We finally prove:

**Claim 3.** There is \( \varepsilon_1^* > 0 \) such that \( \psi_\varepsilon(t) > 0 \) for \( t \in \mathbb{R} \) and \( \varepsilon \in (0, \varepsilon_1^*) \).

From claims 1 and 2, for \( \varepsilon > 0 \) small enough
\[
\psi_\varepsilon(t) \geq e^{\lambda(\varepsilon)} v^1(\varepsilon) - D_0 e^{(\lambda_0 + \delta)t} \mathbf{I} \geq e^{\lambda(\varepsilon)} [v^1(\varepsilon) - D_0 e^{(\lambda_0 + \delta)t} \mathbf{I}],
\]
\( t \leq 0, \)
where \( \mathbf{I} = (1, \ldots, 1) \) and \( |\lambda(\varepsilon) - \lambda_0| < \delta/2 \). Choose \( T^* \leq 0 \) and \( \varepsilon_1^* > 0 \) such that
\[
v^1(\varepsilon) > v/2 \quad \text{if } 0 < \varepsilon < \varepsilon_1^* \quad \text{and} \quad D_0 e^{(\lambda_0 + \delta)t} \mathbf{I} \leq v/4.
\]
Then
\[
\psi_\varepsilon(t) \geq e^{\lambda(\varepsilon)} v^1(\varepsilon) > 0, \quad 0 < \varepsilon < \varepsilon_1^*, \quad t \leq T^*.
\]
On the other hand, since \( \|\psi_\varepsilon - u^*\|_\infty \to 0 \) as \( \varepsilon \to 0^+ \), we define \( \eta := \inf_{\varepsilon \geq T^*} u^*(\varepsilon) > 0 \), and suppose that \( \varepsilon_1^* \) was chosen so that \( \|\psi_\varepsilon - u^*\|_\infty < \eta \) for \( 0 < \varepsilon < \varepsilon_1^* \). It follows that \( \psi_\varepsilon(t) > 0 \) for all \( t \in \mathbb{R} \) and \( \varepsilon \in (0, \varepsilon_1^*) \). The proof of the theorem is complete. \( \blacksquare \)
The above proof shows that the requirement that there is a positive eigenvector \( v \in \mathbb{R}^N \) for the dominant characteristic value \( \lambda_0 \) of (2.1) is crucial to deduce the positiveness of the travelling wave fronts, for large wave speeds. Nevertheless, the existence of such waves and their asymptotic behaviour at \(-\infty\) can be deduced from our theorem 3.8, as well as the auxiliary results in section 3, and the proof of claim 1. We summarize these remarks in the following theorem:

**Theorem 4.4.** Assume (H1), (H2)(i) and

(i) for equation (1.2) the equilibrium \( u = K \) is locally asymptotically stable;
(ii) for equation (1.2), the linearized equation about 0 has a real characteristic root \( \lambda_0 > 0 \), which is simple and dominant;
(iii) Equation (1.2) has a heteroclinic solution \( u^*(t), t \in \mathbb{R}, \) with \( u^*(-\infty) = 0, u(\infty) = K \) and \( u^*(t) = O(e^{\lambda_0 t}) \) at \(-\infty\).

Then, there is \( c^* > 0 \), such that for \( c > c^* \), equation (1.1) has a travelling wave solution of the form \( u(t, x) = \psi(ct + w \cdot x) \) for each unit vector \( w \in \mathbb{R}^p \), with \( \psi(\tau/e) = O(e^{\lambda(e)\tau}) \), \( \psi^\prime(\tau/e) = O(e^{\lambda(e)\tau}) \) at \(-\infty\), where \( \epsilon = 1/c \) and \( \lambda(e) \) is the real solution of (4.1) with \( \lambda(e) \to \lambda_0 \) as \( \epsilon \to 0^+ \).

5. Applications

5.1. A diffusive generalized logistic equation with distributed delay

As a first application, we consider a scalar reaction–diffusion equation with distributed delays in the reaction terms, which includes the Fisher–KPP equation with delay as a particular case.

Let \( C = C([-\tau, 0]; \mathbb{R}) \), \( \tau > 0 \), and consider

\[
\frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) + bu(t, x)[1 - L(u(t, \cdot))], \quad t \in \mathbb{R}, \quad x \in \mathbb{R},
\]

(5.1)

where \( b > 0 \) and \( L : C \to \mathbb{R} \) is a nonzero positive linear operator, i.e. \( L \neq 0 \) is linear and \( L(\phi) \geq 0 \) whenever \( \phi \geq 0 \). In particular, \( L \) is bounded and \( \|L\| = L(1) > 0 \).

The corresponding delayed ODE model,

\[
u'(t) = bu(t)[1 - Lu(t)], \quad t \in \mathbb{R},
\]

(5.2)

has two equilibria, \( u = 0 \) and \( u = L(1)^{-1} := K \). Here \( f \) in (1.2) reads as \( f(\phi) = b\psi(0)[1 - L(\phi)], \phi \in C, \) for which it is easy to verify that conditions (H1) and (H2) are satisfied. The linearized equation about zero is the ODE \( u'(t) = bu(t) \), with characteristic value \( b > 0 \).

Now, \( u'(t) = -bKLu(t) \) is the linearized equation about \( K \), with characteristic equation \( P(\lambda) := \lambda + bKL(e^{\lambda}) = 0 \). If \( \lambda = i\omega, \omega > 0 \), is a solution of \( P(\lambda) = 0 \), then \( L(e^{i\omega}) = 0 \) and \( 0 = \omega + bKL(e^{i\omega}) \geq \omega - b \). If \( \omega \tau < \pi/2 \), we deduce that there is \( \delta > 0 \) such that \( \cos(\omega \theta) \geq \delta \) for \( \theta \in [-\tau, 0] \), hence \( L(e^{i\omega}) \geq \delta L(1) > 0 \), which is a contradiction. On the other hand, if \( b\tau \leq 3/2 \), from [8] we conclude that the positive equilibrium \( K \) is globally attractive in the set of all positive solutions of (5.2) with initial conditions \( \phi \in C_+, \phi(0) > 0 \). We thus conclude that (H3) holds if \( b\tau \leq 3/2 \). Therefore, the following result is an immediate consequence of theorem 4.3

**Theorem 5.1.** If \( b\tau \leq 3/2 \), there exists \( c^* > 0 \) such that for \( c > c^* \) equation (5.1) has a positive travelling wave solution \( u(t, x) = \psi(x + ct) \) with \( \psi(-\infty) = 0, \psi(\infty) = K \). Moreover, \( \psi \) is increasing in a vicinity of \(-\infty\) and it has the asymptotic decay \( \psi(t) = O(e^{\phi(c)t}) \), \( \psi'(t) = O(e^{\phi(c)t}) \) at \(-\infty\), where \( b(c) = 2b/(c + \sqrt{c^2 - 4b}) \).
We note that (5.1) includes as a particular case the Fisher–KPP equation with a single delay,
\[
\frac{\partial u}{\partial t}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) + bu(t, x)[1 - u(t - \tau, x)/K], \quad t \in \mathbb{R}, \quad x \in \mathbb{R}.
\] (5.3)
Using a pair of upper-lower solutions and a monotone iterative method, Wu and Zou [23] proved that if \(c > 2\sqrt{b}\), then there exists \(\tau^*(c) > 0\) such that, for a delay \(\tau \leq \tau^*(c)\), (5.3) has a non-decreasing travelling wave front connecting 0 to \(K\) with wave speed \(c\). Our approach does not allow us to determine the minimal wave speed \(c^*\), but, in contrast, we explicitly exhibit an upper bound \(\tau^* = 3/(2b)\) for the delay, under which we can assure the existence of such positive (but not necessarily monotone) wave fronts.

Travelling waves for the Fisher–KPP equation (5.3) with \(b = 1\) were also considered in [9, corollary 6.6], where it was shown that, for \(\tau \leq e^{-1}\), there exists \(c^* > 0\) such that there is a travelling wave front with wave speed \(c > c^*\). We remark that theorem 5.1 applied to (5.3) with \(b = 1\) clearly improves this result, since it guarantees the existence of travelling waves for \(\tau \leq 3/2\), and, most relevant in biological terms, it does assert that such travelling waves are positive.

For some other recent results and references on the Fisher–KPP equation, see [2, 12].

5.2. A chemostat model with delayed growth response

Consider the following model for the growth of bacteria in a well-stirred chemostat supplied by a single essential nutrient (cf Ellermeyer [6] and Ellermeyer et al [7]):
\[
\begin{align*}
S'(t) &= D(S_0 - S(t)) - f(S(t))u(t), \\
u'(t) &= e^{-Dt}f(S(t - \tau))u(t - \tau) - Du(t).
\end{align*}
\] (5.4)
Here \(S(t)\) and \(u(t)\) are the concentration of nutrient in the growth vessel and the biomass concentration of bacteria at time \(t\), respectively, \(D > 0\) is the dilution rate of the chemostat, \(S_0 > 0\) is the input concentration of nutrient and \(\tau \geq 0\) is the delay in the growth response, to account for the lag in the nutrient conversion into biomass due to cellular absorption; \(f\) is the specific functional response for the bacteria, and typically the Michaelis–Menten response is chosen, \(f(s) = ms/(a + s), \quad s \geq 0, \quad m, a > 0\). More generally, one can consider a continuously differentiable and bounded function \(f : [0, \infty) \rightarrow [0, \infty)\) with \(f(0) = 0, \quad f'(s) > 0 \quad \text{for} \quad s \in \mathbb{R}\).

For an unstirred chemostat, the nutrient is added to the vessel but not mixed, so one has to introduce diffusion terms. The diffusion rates \(d_1, d_2 > 0\) for the nutrient and the organisms may be different, and model (5.4) becomes
\[
\begin{align*}
\frac{\partial S}{\partial t}(t, x) &= d_1 \Delta S(t, x) + D(S_0 - S(t, x)) - f(S(t, x))u(t, x), \\
\frac{\partial u}{\partial t}(t, x) &= d_2 \Delta u(t, x) + e^{-Dt}f(S(t - \tau, x))u(t - \tau, x) - Du(t, x),
\end{align*}
\] (5.6)
for \(t \in \mathbb{R}\) and \(x \in (0, L)\) \((L > 0)\) (or more generally \(x \in \Omega\), where \(\Omega \subset \mathbb{R}^3\) is an open domain). There is an extensive literature on chemostat models with ‘delayed growth response’, with and without diffusion. We refer to [3, 6, 7, 11, 21, 22], for results, other related chemostat models, biological explanations, and further references.

For both (5.4) and (5.6), there is always the equilibrium \((S_0, 0)\), corresponding to the ‘washout state’. If
\[
f(S_0) > De^{Dt},
\] (5.7)
there is another non-negative equilibrium \((\bar{S}, \bar{u})\), called the ‘survival state’, given by \((\bar{S}, \bar{u}) = (f^{-1}(D\phi(0)), e^{-D\tau}(S^0 - \bar{S}))\). Condition (5.7) imposes a restriction on the size of the time-delay \(\tau\), which should satisfy \(\tau < D^{-1} \log(f(S^0)D^{-1})\) for \(f\) as in (5.5). Moreover, (5.7) implies that the equilibrium \((S^0, 0)\) of (5.4) is unstable and \((\bar{S}, \bar{u})\) is asymptotically stable and a global attractor of all solutions with initial conditions \((S_0, u_0) = (\phi_1, \phi_2) \in \mathcal{C}_+, \phi_2(0) > 0\) [6, 7].

In order to apply our results, we first observe that both the positive cone \(\mathcal{C}_+\) and the set \[\{(\phi_1, \phi_2) \in \mathcal{C}_+: \phi_2(0) > 0, \phi_1(0) < S^0\}\] are positively invariant for (5.4). Translating the washout state to the origin, by setting \(s(t) = S^0 - S(t)\), we rewrite (5.4) as

\[
s'(t) = -Ds(t) + f(s(t) - v(t)),
\]

\[
u'(t) = e^{-D\tau}f(S^0 - s(t))u(t) - Du(t).
\]

(5.8)

If \(f\) is \(C^2\)-smooth and (5.7) holds, then equation (5.8) satisfies (H1) and (H2). The set \(A = \{(\phi_1, \phi_2) \in \mathcal{C}_+: \phi_2(0) > 0, 0 < \phi_1(0) \leq S^0\}\) is positively invariant, and (5.8) has equilibria \(E_0 = (0, 0)\) and \(K := (\bar{S}, \bar{u}) > 0\), with \(\bar{S} = S^0 - \bar{S}, \bar{u} = e^{-D\tau}\), the first one being unstable and the second one being locally stable and a global attractor of all solutions with initial conditions in \(A\) (cf [6]). It remains to verify that (H4) holds.

The characteristic equation for the linearization of (5.8) at \((0, 0)\) is

\[(\lambda + D)(\lambda + D - e^{-D\tau}f(S^0)e^{-\lambda}) = 0.
\]

(5.9)

Define \(h(x) = (x + D)e^{(x + D\tau)t}, x \in \mathbb{R}\). Under (5.7), there are two real roots of (5.9), \(\lambda_0\) and \(\lambda_0\), where \(\lambda_0 > 0\) is the unique solution of \(h(x) = 0\). For \(\lambda \notin \mathbb{R}\) a solution of (5.9), we have \(\lambda h(\lambda) < f(S^0)\). Since \(h'(x) > 0\) for \(x > 0\), it follows that \(\forall \lambda < \lambda_0\), and we conclude that \(\lambda_0\) is a dominant eigenvalue for the linearization of (5.8) about \((0, 0)\), with \(v = (f(S^0)(\lambda_0 + D))^{-1}, 1 > 0\) as an associated eigenvector. From theorems 2.1 and 4.3, we therefore obtain the following result:

**Theorem 5.2.** Consider equation (5.4), where \(S^0, D, \tau > 0\), the function \(f: [0, \infty) \rightarrow [0, \infty]\) is bounded, \(C^2\)-smooth and satisfies (5.5) and (5.7). Then, there exists a heteroclinic solution \((S^*(t), u^*(t))\) connecting the washout state \((S^0, 0)\) to the survival state \((\bar{S}, \bar{u})\) of (5.4), with \(0 < S^*(t) < S^0, u^*(t) > 0\) for \(t \in \mathbb{R}\) and \((S^*(t), u^*(t)) = (S^0, 0) + O(e^{\omega t})\) at \(-\infty\), where \(\lambda_0 > 0\) satisfies \(\lambda_0 + D = f(S^0)e^{-\lambda_0D\tau}\). For the diffusion model (5.6) with \(d_1, d_2 > 0\), there is \(c^* > 0\) such that for \(c > c^*\) (5.8) has a positive travelling wave solution of the form \((S(t, x), u(t, x)) = (\psi_1(\psi_2(\psi_3(\phi_1(x)), c t + x), x), \psi_2(\psi_3(\phi_1(x)), c t + x), x), \psi_3(\phi_1(x), c t + x), x\), with \(\psi_3(-\infty) = S^0, \psi_3(-\infty) = 0\) and \(\psi_3(\infty) = \bar{S}, \psi_3(\infty) = \bar{u}\); moreover, \(\psi_3(0) < 0, \psi_3(t) > 0\) in the vicinity of \(-\infty\), and \((\psi_3(0), \psi_3(t)) = (S^0, 0) + O(e^{(c^*)t})\) at \(-\infty\), where \(c^*\) is the real solution of

\[
dz^2 - cz - D = e^{-(D+c)t}f(S^0) = 0.
\]

(5.10)

**Proof.** Let \(e = 1/c\), for \(c > 0\) large. With the notation in (4.3), we have that

\[
\det D_t(\lambda) = \frac{1}{e^2d_1d_2}(e^2d_1\lambda^2 - \lambda - D)(e^2d_2\lambda^2 - \lambda - D + e^{-(D+c)t}f(S^0)).
\]

(5.11)

Define \(\lambda(\varepsilon)\) as the real solution of (5.11) such that \(\lambda(\varepsilon) \rightarrow \lambda_0\) as \(\varepsilon \rightarrow 0^+\). Then \(z(c) = e\lambda(\varepsilon)\) satisfies (5.10).

Theorem 5.2 asserts the existence of positive travelling wavefronts for (5.6) with \(S(t, x) < S^0\) for all \(t \in \mathbb{R}, x \in \mathbb{R}\). Here, due to the change of variables \(s = S - S^0\), the positivity of the component \(s(t)\) (or \(s(t, x))\) translates as the nutrient concentration being smaller than \(S^0\). We emphasize that biologically significant solutions of (5.4) and (5.6) must be positive and have a nutrient concentration \(S\) not larger than the input concentration \(S^0\).
Remark 5.3. The existence of a positive eigenvector \( v \) associated with the dominant eigenvalue \( \lambda_0 \), as prescribed in (H4), may seem a quite restrictive requirement, since it is not satisfied by many populations dynamics systems, namely Kolmogorov type models with \( N > 1 \). We, however, observe that if the characteristic matrix for (2.1) at \( \lambda_0, \Delta_0(\lambda_0) \), is an irreducible matrix with non-negative off-diagonal entries, then there is a positive eigenvector for \( \Delta_0(\lambda_0) \) associated with \( \lambda_0 \) (see, e.g., [21, p 258]). This property will be exploited in a forthcoming paper, where theorems 3.8 and 4.3 will be applied to several population models.

Appendix

In this appendix, we extend proposition 7.1 of Mallet-Paret [18] to systems with distributed delays.

Consider the FDE
\[
x'(t) = L_0 x_t + h(t), \quad t \in \mathbb{R}
\]
and the homogeneous system
\[
x'(t) = L_0 x_t,
\]
where \( L_0 : C([-\tau, 0]; \mathbb{R}^N) \to \mathbb{R}^N \) is a bounded linear operator and \( h : \mathbb{R} \to \mathbb{R}^N \) is continuous.

For (A.2), write the characteristic equation
\[
\det \Delta_0(s) = 0, \quad \text{where} \quad \Delta_0(s) = s I - L_0(e^{s} I).
\]
It is well known that the solutions of the characteristic equation are exactly the eigenvalues for the homogeneous system (A.2), i.e. the eigenvalues for the infinitesimal generator \( A \) associated with the semi-flow of (A.2). Furthermore, the spectrum \( \sigma(A) \) of \( A \) is only composed of the point spectrum.

Lemma A.1. If \( f \) is a holomorphic function on a disc \( \{s : |s - \lambda| < \epsilon\} \) where \( \lambda \) is an eigenvalue of (A.2) and \( \epsilon > 0 \) is small, then
\[
x(t) = \text{Res}(e^{s} \Delta_0^{-1} f, \lambda) = \frac{1}{2\pi i} \int_{|s - \lambda| = \epsilon} e^{st} \Delta_0(s)^{-1} f(s) \, ds
\]
is an eigenfunction of (A.2) corresponding to \( \lambda \).

Proof. The proof follows the arguments of Mallet-Paret [18, section 7], so we omit it. \( \square \)

Theorem A.2. Let \( x(t) \) be a solution of (A.1) on \([T, \infty)\) for some \( T \in \mathbb{R} \). Assume there are \( a, b \in \mathbb{R}, a < b \), such that
\[
x(t) = O(e^{-at}), \quad h(t) = O(e^{-bt}) \quad \text{as} \quad t \to \infty.
\]
Then, for every \( \epsilon > 0 \), we have
\[
x(t) = z(t) + O(e^{-(b-\epsilon)t}) \quad \text{as} \quad t \to \infty,
\]
where \( z(t) \) is an eigenfunction of (A.2) associated with the set of eigenvalues \( \Lambda = \{\lambda \in \sigma(A) : -b < \Re \lambda \leq -a\} \). Analogously, if \( x(t) \) is a solution of (A.1) on \((-\infty, T] \) for some \( T \in \mathbb{R} \) and
\[
x(t) = O(e^{at}), \quad h(t) = O(e^{bt}) \quad \text{as} \quad t \to -\infty,
\]
with \( a < b \), then for every \( \epsilon > 0 \) we have
\[
x(t) = z(t) + O(e^{(b-\epsilon)t}) \quad \text{as} \quad t \to -\infty,
\]
where \( z(t) \) is an eigenfunction of (A.2) associated with the set of eigenvalues \( \Lambda = \{\lambda \in \sigma(A) : a \leq \Re \lambda < b\} \).
Proof.} We only prove the result for \( +\infty \); for \(-\infty \) it is analogous. Without loss of generality, take \( T = 0 \). In what follows, for \( f : [0, \infty) \rightarrow \mathbb{C}, f(t) = O(e^{-at}) \) at \(+\infty\), we denote the Laplace transform of \( f \) by
\[
(\mathcal{L}f)(s) = \hat{f}(s) = \int_0^\infty e^{-st} f(t) \, dt, \quad \text{for } \Re s > -a.
\]

Write \( L_0(\varphi) = \int_{-\infty}^0 d\eta(\theta) \varphi(\theta) \), where \( \eta(\theta) \) is an \( N \times N \) matrix-valued function of bounded variation. Applying the Laplace transform to (A.1), we obtain
\[
- x(0) + s \hat{x}(s) = \mathcal{L}(L_0 x_i)(s) + \tilde{h}(s), \quad \Re s > -a,
\]
where
\[
\mathcal{L}(L_0 x_i)(s) = \int_0^\infty e^{-st} \left( \int_{-\tau}^0 \eta(t) x(t + \theta) \, dt \right) \, d\eta(t) = \int_0^\infty \eta(t) \left( \int_0^\infty e^{-s(t+\theta)} x(t+\theta) \, dt \right) \, d\eta(t) = \int_0^\infty \eta(t) \left( \int_0^\infty e^{-s\tau} x(u) \, du + \hat{x}(s) \right) = L_0 \left( e^s \int_0^0 e^{-su} x(u) \, du \right) + L_0(e^s I) \tilde{x}(s).
\]

From (A.3),(A.4), we obtain
\[
\Delta_0(s) \tilde{x}(s) = r(s) + \tilde{h}(s), \quad \text{where } r(s) = x(0) + L_0 \left( e^s \int_0^0 e^{-su} x(u) \, du \right).
\]

We observe that \( r(s) \) is an entire function, \( \tilde{h}(s) \) is defined and holomorphic for \( \Re s > -b \), and \( \tilde{x}(s) = \Delta_0(s)^{-1} [r(s) + \tilde{h}(s)] \) is holomorphic for \( \Re s > -b \) with the exception of finitely many poles.

Take \( \epsilon > 0 \) and \( k > -a \). On any strip of the form \(-b + \epsilon \leq \Re \lambda \leq k\), the functions \( r(s) \) and \( \tilde{h}(s) \) are uniformly bounded. Since \( k \) is greater than the real part of all singularities of \( \tilde{x}(s) \), we can use the inverse formula for the Laplace transform,
\[
x(t) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} e^{st} \tilde{x}(s) \, ds.
\]

Choose \( \epsilon > 0 \) small such that \(-b + \epsilon / 2 < k \) and \( \sigma(A) \cap \{ \lambda : -b \leq \Re \lambda \leq -b + \epsilon / 2 \} = \emptyset \). Note that in the strip \(-b \leq \Re s \leq -a \), the only possible poles of \( \tilde{x} \) lie on \(-b + \epsilon / 2 < \Re s < -a \).

In the strip \(-b + \epsilon / 2 \leq \Re s \leq k \), the functions \(|r(s)|, |\tilde{h}(s)|\) are bounded, and \( \|L_0(e^s I)\| \leq \max(1, e^{-ks}) \|L_0\| \), hence the operator norm for the inverse of \( \Delta_0(s) = sI - L_0(e^s I) \) satisfies
\[
\|\Delta_0(s)^{-1}\| \leq \frac{1}{|s| - \max(1, e^{-ks}) \|L_0\|} \quad \text{for } |s| > \max(1, e^{-ks}) \|L_0\|.
\]

From this estimate and (A.5), we conclude that \( e^{ks} \tilde{x}(s) \rightarrow 0 \) as \( |\Re s| \rightarrow \infty \), uniformly in the strip \(-b + \epsilon / 2 \leq \Re s \leq k \), and that \( \tilde{x}(s) \) is \( L^2 \)-integrable in any straight line \( s = x_0 + iy \), \( y \in \mathbb{R} \), for any fixed \( x_0 \in [-b + \epsilon / 2, k] \). We may shift the path of integration in (A.6) to the left, and obtain
\[
x(t) = z(t) + w(t),
\]
where
\[
z(t) = \sum_{\lambda \in \Lambda} \text{Res} (e^s \tilde{x}, \lambda), \quad w(t) = \frac{1}{2\pi i} \int_{-b + \epsilon / 2 - i\infty}^{-b + \epsilon / 2 + i\infty} e^{st} \tilde{x}(s) \, ds.
\]

From the previous lemma, \( z(t) \) is an eigenfunction of (A.2) associated with the set of eigenvalues in \( \Lambda \). It remains to prove that
\[
w(t) = O(e^{-(b-\epsilon)t}) \quad \text{at } +\infty.
\]

\[\text{[96x656]}\]
Define \( u(t) = e^{(b-\epsilon)/2} w(t) \), \( v(t) = e^{(b-\epsilon)\bar{t}} w(t) \). We first prove that \( v \in L^1[0, \infty) \). Here, \( L^p([0, \infty)) = L^p((0, \infty); \mathbb{C}^N), p = 1, 2 \). We have

\[
u(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{s + b - \epsilon/2} s \hat{x}(s) ds = \frac{1}{2\pi} \int_{\mathbb{R}} e^{is} \hat{x}(-b + \epsilon/2 - is) ds.
\]

By Plancherel theorem, \( u(t) \in L^2([0, \infty)) \), so that \( v(t) = e^{-\epsilon t/2} u(t) \) implies

\[
\|v\|_{L^1([0,\infty))} \leq \|u\|_{L^1([0,\infty))} \|e^{-\epsilon t/2}\|_{L^2([0,\infty))} \leq C \|\hat{x}(-b + \epsilon/2 + i)\|_{L^1(\mathbb{R})} < \infty,
\]

for some \( C > 0 \). Hence, \( v \in L^1([0,\infty)) \). Define now

\[
V(t) = L_0(e^{-(b-\epsilon)t} v_0) = \int_{-\pi}^{\pi} d\eta(\theta) e^{-((b-\epsilon)\theta t)} v(t + \theta).
\]

Then,

\[
\int_{0}^{\infty} |V(t)| dt \leq \max(1, e^{(b-\epsilon)t}) \|L_0\| \|v\|_{L^1([-\pi,\infty))}.
\]

(A.10)

In particular, \( V \in L^1([0,\infty)) \). We now observe that \( x(t) = z(t) + w(t) \), with \( x(t) \) a solution of (A.1) and \( z(t) \) a solution of (A.2). Hence \( w(t) \) satisfies (A.1), \( w'(t) = L_0(w_t) + h(t) \), and we obtain

\[
v'(t) = (b - \epsilon) v(t) + L_0(e^{-(b-\epsilon)t} v_t) + e^{(b-\epsilon)t} h(t),
\]

with \( e^{(b-\epsilon)t} h(t) = O(e^{-at}) \). We conclude that \( v' \in L^1([0,\infty)) \). Since \( |v(t)| \leq |v(0)| + \int_{0}^{t} |v'(s)| ds \) for \( t \geq 0 \), then \( v \) is bounded on \([0,\infty)\), and (A.8) holds.

Remark A.3. For the situation \( x(t) = O(e^{-at}) \), \( h(t) = O(e^{-bt}) \) \((a < b)\) as \( t \to \infty \), denote \( v(t) = (v_1(t), \ldots, v_N(t)) \) as in the above proof. Clearly, one can obtain componentwise estimates similar to (A.9) or (A.10). In fact, one concludes that for \( \epsilon > 0 \) small such that \( \sigma(A) \cap \{s : -b < \Re s \leq -b + \epsilon/2\} = \emptyset \) and \( t \geq 0 \), \( j = 1, \ldots, N \),

\[
\|v_j\|_{L^1([0,\infty))} \leq (2\pi \sqrt{\epsilon})^{-1} \|x_j(-b + \epsilon/2 + i)\|_{L^1(\mathbb{R})}
\]

and \( |v_j(t)| \leq |v_j(0)| + \|v'_j\|_{L^1([0,\infty))} \) with

\[
\|v'_j\|_{L^1([0,\infty))} \leq C \|x_j(-b + \epsilon/2 + i)\|_{L^1(\mathbb{R})} + \|e^{(b-\epsilon)t} h_j\|_{L^1([0,\infty))},
\]

where \( C = (b - \epsilon) + e^{(b-\epsilon)t} \|L_0\|/(2\pi \sqrt{\epsilon}) \). Similar estimates hold for the case \( x(t) = O(e^{bt}) \), \( h(t) = O(e^{bt}) \) at \( -\infty \).

Acknowledgments

This research was supported by FCT (Portugal), Financiamento Base 2009-ISFL-1-209 (Teresa Faria) and by FONDECYT (Chile), projects 7080045 (Teresa Faria) and 1071053 (Sergei Trofimchuk). S. Trofimchuk was also partially supported by CONICYT (Chile) through PBCT program ACT-56 and by the University of Talca, programme ‘Reticulados y Ecuaciones’. This work was initiated while T. Faria was visiting the University of Talca, and she thanks the University for its hospitality. The authors are very grateful to the two referees, for their valuable and helpful comments.
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