Optimal reinsurance using the expected shortfall

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Abstract. The Financial Services Authority Regulation regarding Self-Retention and Domestic Reinsurance Support (Otoritas Jasa Keuangan / OJK Regulation) was issued to establish a more comprehensive framework concerning self-retention and reinsurance support of insurance companies. Insurance Companies must implement and maintain a self-retention limit based on their risk and loss profile. This requires Insurance Companies to make available a certain amount of capital for every risk that they undertake. Insurance companies must develop and implement a reinsurance support strategy to ensure that they have sufficient capacity to meet their liabilities. In this research, we estimate the optimal retention by means of Expected Shortfall which will minimize the risk of individual total based on the quota share reinsurance. We use lognormal bivariate distribution to illustrate the optimal retention in an individual risk model.

1. Introduction
Reinsurance is one of the key strategic variables in risk management of insurance companies, that also a mechanism of transferring risk from an insurer to a second insurance carrier. The former party is referred to as the cedent (or simply the insurer) while the latter is the reinsurer. Reinsurance provides an opportunity for the insurer to reduce the underwriting risk and hence leads to a more effective management of risk [1]. Retention constitutes the amount that an insurer assumes for its own account. In many cases, the choice of retention level is made by the underwriter of the account under consideration. He will use his skill and judgment based on his knowledge of the account, to decide the best retention level. The aim, in deciding on this level is more likely to be to balance the relationship between profits and stability, rather than to reduce the risk that capital is exhausted. The main factors which were then considered when setting retention levels were, level of capital, cost of reinsurance and smoothing of earnings fluctuations.

The insurance company has to set a retention limit which is the amount below which this company will retain the reinsurance and above which it will purchase reinsurance coverage from another (the reinsuring) company. Cai J and KS Tan [1] propose practical solutions for the determination of optimal retentions in a stop-loss reinsurance based on minimizing Value at Risk (VaR) and Expected Shortfall (ES). Tan KS et al. [2] formulate an optimal reinsurance model that minimizes, respectively, the VaR and the ES of the cedent’s total risk exposure, based on quota-share and stop-loss reinsurance. Pressacco F et al. [3] give closed form formulae to express the efficient mean variance retention set both in the retention space and in the mean-variance one. Soleh AZ et al. [4] consider compound Poisson-
Lognormal distribution in determining the retention for stop-loss reinsurance and Noviyanti L and Soleh AZ [5] use Lognormal distribution to illustrate the optimal retention in an individual risk model.

In this research, we estimate the optimal retention by means of the ES risk measurement which will minimize the risk of individual total based on the quota share reinsurance. The insurer will prefer a proportional contract if he wants to reduce his cost. Distribution in nonlife insurance are not symmetrical but show a positive skewness. This means that the variations around the average usually give stronger positive deviation than negative ones. Furthermore, we use lognormal bivariate distribution to illustrate the optimal retention in an individual risk model because based on the data of property insurance, the amount of claims constitute lognormal bivariate distribution. The next section of this paper outline quota-share reinsurance, optimal retention with ES optimization, numerical example in property insurance, and conclusions.

2. Quota-share reinsurance

Quota share reinsurance is a form of pro rata or proportional reinsurance whereby the ceding company is indemnified for a fixed percent of loss on each risk covered by the treaty contract. All liability and premiums are shared from the first dollar. “Quota” or “definite” share relates to the fixed percentage as stated in the treaty. On premiums ceded, the reinsurer pays the ceding company a commission. The commission to the ceding company is an important factor in quota share reinsurance as it provides a financial benefit to the primary company. Proportional forms are often used in property insurance, since this form provides catastrophic protection in addition to individual risk capacity [6].

Let \( X \) be the aggregate loss for an insurance portfolio. We assume that \( X \) is a nonnegative random variable with distribution function \( F_X(x) = \Pr\{X \leq x\} \), survival function \( S_X(x) = \Pr\{X > x\} \), and mean \( E[X] \geq 0 \).

Let \( X_I \) and \( X_R \) be respectively, the loss random variables of the insurer and the reinsurer in the presence of quota-share. The relation of \( X_I \) and \( X_R \) with \( X \) and \( r \in [0,1] \) is

\[
X_I = rX \quad \text{and} \quad X_R = (1-r)X. \quad (1)
\]

A number of premium principles have been proposed for determining the appropriate level at the premium. One of the commonly used principles is the expected value principle in which the reinsurance premium is determined by

\[
\delta(R) = (1 + \rho)E(X_R). \quad (2)
\]

where \( \rho > 0 \) is known as the relative safety loading like add expenses percentage, operational and administration fee. The notation of \( E[X_R] \) constitute the net quota-share premium. The total cost of the insurer in the presence of the quota-share reinsurance \( T \) is captured by two components: the retained loss and the reinsurance premium; that is,

\[
T = X_I + \delta(R) = X_I + (1 + \rho)E(X_R). \quad (3)
\]

A prudent risk management is to ensure that risk measures associated with \( T \) are as small as possible. This information motivates us to consider the following two optimization criteria for seeking the optimal level of retention. The VaR and the ES based on Tan KS et al. [2] optimization is defined as:

\[
VaR_T(R^*, \alpha) = \min_{R>0} \{VaR_T(R, \alpha)\}. \quad (4)
\]

\[
ES_T(R^*, \alpha) = \min_{R>0} \{ES_T(R, \alpha)\},
= \min_{R>0} \{ES_{\alpha}(R, r) + \delta(R)\},
= \min_{R>0} \{(1-r)S_{\alpha}^{-1}(\alpha) + \frac{1}{\alpha} \int_{S_{\alpha}^{-1}(\alpha)}^{\infty} S_X(x)dx + \delta(R)\}. \quad (5)
\]

The \( ES_T(R^*, \alpha) \) will be minimum given by
The relationship between the VaR and the ES is as follow;

\[ \text{ES}_T(R^*, \alpha) = E \left[ T \mid T \geq \text{VaR}_T(R^*, \alpha) \right]. \]  

(7)

The resulting optimal retention \( R^* \) ensures that both the VaR and the ES of the total cost is minimized for a given risk tolerance level \( \alpha \).

3. Optimal retention with ES optimization

This section presents the optimal solution to the VaR and ES optimization in (4) and (5). The survival function of retained loss \( X_i \) is given by

\[
S_{x_i}(x) = \begin{cases} 
S_x(x), & 0 \leq x \leq R, \\
0, & x > R,
\end{cases}
\]  

(8)

and the VaR and the ES of the retained loss \( X_i \) can be represented as

\[
\text{VaR}_{x_i}(R, \alpha) = \begin{cases} 
R, & 0 < R \leq S_x^{-1}(\alpha), \\
S_x^{-1}(\alpha), & R > S_x^{-1}(\alpha).
\end{cases}
\]  

(9)

\[
\text{ES}_\alpha(X_i, r) = (1 - r)S_x^{-1}(\alpha) + \frac{1}{\alpha} \int_{S_x^{-1}(\alpha)}^{\infty} S_x(x) \, dx.
\]  

(10)

A simple relationship between the VaR-ES of the total cost and the VaR-ES of retained risk respectively are

\[
\text{VaR}_T(R, \alpha) = \text{VaR}_{x_i}(R, \alpha) + \delta(R).
\]  

(11)

\[
\text{ES}_T(R, \alpha) = \text{ES}_\alpha(R, r) + \delta(R)
\]  

(12)

By combining (7) and (8), Tan K.S., et al (2007) summarize in the following proposition:

**Proposition 1**

For each \( R > 0 \) and \( 0 \leq \alpha < S_x(0) \), then

\[
\text{VaR}_T(R, \alpha) = \begin{cases} 
R + \delta(R), & 0 < R \leq S_x^{-1}(\alpha), \\
S_x^{-1}(\alpha) + \delta(R), & R > S_x^{-1}(\alpha).
\end{cases}
\]  

(13)

\[
\text{ES}_T(X_i, \alpha) = (1 - r)S_x^{-1}(\alpha) + \delta(R) + \frac{1}{\alpha} \int_{S_x^{-1}(\alpha)}^{\infty} S_x(x) \, dx.
\]  

(14)

Let define \( \rho^* = 1/(1+\rho) \) which plays a critical role in the solutions to our optimization problems. The theorem states the necessary and sufficient conditions for the existence of the optimal retention of the VaR-ES optimization:

**Theorem 1.**

(a) The optimal retention \( R^* > 0 \) exists if and only if both

\[
\alpha < \rho^* < S_x(0)
\]  

(15)

and

\[
S_x^{-1}(\alpha) \geq S_x^{-1}(\rho^*) + \delta(\rho^*)
\]  

(16)

hold.

(b) When the optimal retention \( R^* \) in (4) exists, then \( R^* \) is given by

\[
R^* = S_x^{-1}(\rho^*)
\]  

(17)

and the minimum VaR of \( T \) is given by

\[
\text{VaR}_T(R^*, \alpha) = R^* + \delta(R^*).
\]  

(18)
and the minimum ES of $T$ is given by
\[
ES_T(R^*, \alpha) = R^* + \delta(R^*)
\]

**Corollary 1.**
The optimal retention $R^* > 0$ in (4) exists if both (10) or (11) such as
\[
S_f^2(\alpha) \geq (1 + \rho)E[X]
\]
so the optimal retention $R^*$ and the minimum ES are given by (12) or (14).

### 4. Numerical example in property insurance

In this section, we illustrate how to determine retention value. Let $C_1$ and $C_2$ constitute the benefit and the amount of claim respectively. We have to determine the conditional probability that $C_2$ occurs given that $C_1$ has occurred. If $(C_1, C_2)$ ~ Lognormal Bivariate $((\mu_1, \mu_2), (\sigma_1^2, \sigma_2^2, \rho))$ then the bivariate density function, expectation, and variance [7], are respectively as follow:

\[
f_{C_1,C_2}(C_1, C_2) = \frac{1}{(2\pi)^{1/2} \sigma_1 \sigma_2} e^{-\frac{1}{2} \frac{(\ln C_1 - \mu_1)^2}{\sigma_1^2} - \frac{(\ln C_2 - \mu_2)^2}{\sigma_2^2} - \rho \sigma_1 \sigma_2 \rho \ln C_1 \ln C_2}
\]

(21)

\[
E(C_2|C_1) = e^{\mu_2 - \frac{\sigma_2^2}{2} \rho \ln C_1} + \frac{\sigma_2^2 \rho \ln C_1}{2}
\]

(22)

\[
Var(C_2|C_1) = e^{2\rho \ln C_1} (e^{\sigma_2^2} - 1)
\]

(23)

where

\[
z = \sigma_2^2 (1 - \rho^2)
\]

\[
w = \frac{1}{1 - \rho^2} \left\{ \left( \frac{\ln C_1 - \mu_1}{\sigma_1} \right)^2 - 2 \rho \frac{\ln C_1 - \mu_1}{\sigma_1} \left( \frac{\ln C_2 - \mu_2}{\sigma_2} \right) + \left( \frac{\ln C_2 - \mu_2}{\sigma_2} \right)^2 \right\}
\]

Based on Table 1, we have the expectation and the variance of the amount of claim distribution.

| Benefit (C_1 : 5,000,000,000) | Amount of claim (C_2) |
|--------------------------------|-----------------------|
| $C_1$ ~ Lognormal($\mu_1$, $\sigma_1^2$) | $C_2$ ~ Lognormal($\mu_2$, $\sigma_2^2$) |
| $f_i(C_1) = \frac{1}{1.474 \sqrt{2\pi}} \exp\left(-\frac{1}{2} \left( \frac{\ln C_1 - 22.078}{1.474} \right)^2 \right)$ | $f_i(C_2) = \frac{1}{.7706 \sqrt{2\pi}} \exp\left(-\frac{1}{2} \left( \frac{\ln C_2 - 19.1833}{.7706} \right)^2 \right)$ |
| $\mu_1 = 22.078$ | $\mu_2 = 19.1833$ |
| $\sigma_1 = 1.474$ | $\sigma_2 = 0.7706$ |
| $E(C_2 | C_1) = C_1^{2619} e^{12.79}$ | $E(C_2 | C_1) = C_1^{5971} e^{86.296695448 e^{734}}$ |
| $V(C_2 | C_1) = 360,843,296.35$ | $V(C_2 | C_1) = 203,655,879.89$ |

To find the value of $S_c^{-1}(\rho) ; 0 < \rho < 1$, using the central limit theorem, the distribution of the amount of claims can be approximated using normal distribution. In order to illustrate the results in real terms, let's say the benefits of property insurance due to the disaster, with $\alpha = 0.1$, the weight $r = 0.1$ and the loading premium $\rho = 0.1$. First, it will be checked in advance about the existence of retention.

\[
\rho^* = \left( \frac{1}{1+\rho} \right) = \left( \frac{1}{1+0.1} \right) = 0.9091
\]

Because of $0.0100 < 0.9091 < 1.000$, then the necessary condition is fulfilled. Using equation (6), it will be calculated the retention, multiplied by quota share.

\[
S_i(\rho^*) = P(X > x) = \{1 - P(X \leq x) = 1 - \rho^*\}
\]
\[
\begin{align*}
&= P(X \leq x) = 0.9091 \\
&= \left\{ P \left( z \leq \frac{x - E(C \mid C)}{\sqrt{V(C \mid C)}} \right) = 0.9091 \right\} \\
&= \left\{ P \left( z \leq \frac{x - 360,843,296.35}{203,655,879.89} \right) = 0.9091 \right\} \\
&= \frac{x - 360,843,296.35}{203,655,879.89} = 1.335 \\
x &= S^{-1}_X(\rho^*) = 632,723,896 \\
R^* &= S^{-1}_X(\rho^*) \cdot (1 - r) \\
R^* &= 632,723,896(1 - 0.1) = 569,451,506.40
\end{align*}
\]

Table 2 shows the amounts of retention and reinsurance based on the various scenarios.

| Benefit | r    | ρ    | Claim amount estimation (IDR) | Retention | Reinsurance (min) | Reinsurance (max) |
|---------|------|------|------------------------------|-----------|------------------|------------------|
| 10%     | 10%  | 30%  | 632,723,896.00              | 596,451,506.40 | 63,276,008.50    | 4,930,515,923.51 |
| 20%     | 20%  | 557,864,380.81 | 502,077,942.73 | 55,786,438.08 | 4,997,922,057.27 |
| 30%     | 510,798,357.96 | 459,718,522.16 | 51,079,835.80 | 5,040,281,477.84 |
| 5,500,000,000 | 10%  | 10%  | 632,760,084.99 | 506,208,068.00 | 126,552,017.00 | 4,993,791,932.00 |
| 20%     | 20%  | 557,864,380.81 | 446,291,504.65 | 111,572,876.16 | 5,053,708,495.35 |
| 30%     | 510,798,357.96 | 408,638,686.37 | 102,159,671.59 | 5,091,361,313.63 |
| 5,500,000,000 | 10%  | 10%  | 632,760,084.99 | 442,932,059.50 | 189,828,025.50 | 5,057,067,940.50 |
| 20%     | 20%  | 557,864,380.81 | 390,505,066.57 | 167,359,314.24 | 5,109,494,933.43 |
| 30%     | 510,798,357.96 | 357,558,850.57 | 153,239,507.39 | 5,142,441,149.43 |

5. Conclusion

In this paper we have determined the optimal retention by means of the ES risk measurement which will minimize the risk of individual total based on the quota share reinsurance. We used lognormal bivariate distribution to illustrate the optimal retention in an individual risk model because based on the data of property insurance, the amount of claims constitute lognormal bivariate distribution.

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