Linear metric and temperature fluctuations of a charged plasma in a primordial magnetic field

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Abstract

We discuss tensor metric perturbations in a magnetic field around the homogeneous Jüttner equilibrium of massless particles in an expanding universe. We solve the Liouville equation and derive the energy-momentum tensor up to linear terms in the metric and in the magnetic field. The term linear in the magnetic field is different from zero if the total charge of the primordial plasma is non-zero. We obtain an analytic formula for temperature fluctuations treating the tensor metric perturbations and the magnetic field as independent random variables. Assuming a cutoff on large momenta of the magnetic spectral function we show that the presence of the magnetic field can discriminate only low multipoles in the multipole expansion of temperature fluctuations. In such a case the term linear in the magnetic field may be more important than the quadratic one (corresponding to the fluctuations of the pure magnetic field).

1 Introduction

The magnetic field is ubiquitous in the universe. In particular, the CMB results from quantum thermal fluctuations of the electromagnetic field. It is present in the standard model. Fluctuations of the magnetic field may be expected in any model of the early universe. The non-trivial question concerns the appearance of a macroscopic magnetic field. There are various mechanisms which can be responsible for this phenomenon [1][2]. There is no convincing argument for any of them. Let us mention the one which assumes a non-zero total charge of the primordial plasma[3]; the assumption relevant for this paper.

The fluctuations lead to a diffusion of particle motions [4] and to a random rhs of the Einstein equations resulting from the energy-momentum. The energy-
momentum contains a contribution of the free electromagnetic field. This term has been studied in [5][6][7][8]. When the magnetic field is Gaussian then the noise coming from the energy-momentum of the free electromagnetic field being quadratic is non-Gaussian. Non-Gaussian effects have not been discovered in CMB yet. This may be so because the magnetic field is weak and the quadratic terms are small. We point out in this paper that a particle interaction with the magnetic field leads to a contribution to the energy-momentum of the primordial plasma which is linear in the magnetic field. This happens if the total charge of the primordial plasma is non-zero. In such a case the impact of the linear term may be stronger than the one coming from the energy-momentum quadratic in the magnetic field. The strength of this term depends on the charge. There are strong bounds limiting the charge of the universe [9][10]. We calculate a variation of the metric corresponding to the term depending on the magnetic field. We obtain an analytic formula for the temperature fluctuations resulting from primordial fluctuations of the metric and of the magnetic field. The temperature fluctuations contain an information on structure formation. The impact of the primordial magnetic field on structure formation (and temperature fluctuations) is usually ignored. There are however some arguments (see, e.g. [11],p.575) indicating that the magnetic field should be taken into account in the studies of structure formation.

The plan of this paper is the following. In sec.2 we find a perturbative solution of the Liouville-Vlasov equation describing a stream of particles in an inhomogeneous expanding metric and in the magnetic field. We are interested in the ultrarelativistic limit when all the particles are massless. As a zero order solution we choose the Jüttner distribution [12] with a time dependent temperature. In sec.3 we discuss Einstein equations with the energy-momentum on the rhs which is determined by the solution of the Liouville-Vlasov equation. A perturbative solution of Einstein equations determines a variation of the metric in the magnetic field. Fluctuations of the temperature are calculated in sec.4. Temperature fluctuations are expanded in Legendre polynomials (multipole expansion). We study a dependence of the expansion coefficients on the spectral function of the stochastic primordial magnetic field. In the Appendix we discuss some technical aspects of the estimates on the spectral function of fluctuations of the magnetic field.

2 Liouville-Vlasov equation

In this section we solve perturbatively the Liouville-Vlasov equation describing a distribution of classical trajectories (see [13][14] for its application in general relativity). We decompose

\[ g_{\mu\nu} = \mathcal{T}_{\mu\nu} + h_{\mu\nu}, \]  

(1)
where $\bar{h}_{\mu\nu}$ describes homogenous metric in the conformal time and
\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu = a^2(dt^2 - dx^2 - \gamma_{ij}dx^i dx^j). \] (2)

In eq.(2) we assume that the tensor perturbations $\gamma_{ij}$ are transverse and traceless. We write the Liouville equation in the form
\[ (p^\mu \partial_\mu - \Gamma^k_{\mu\nu} p^\mu p^\nu \partial_k) \Omega_e = \delta \Gamma^k_{\mu\nu} p^\mu p^\nu \partial_k \Omega_e + eF^{\nu\mu}p^\mu \partial_\nu \Omega_e, \] (3)
where $e$ is the electric charge, $\Gamma^\mu_{\nu\rho}$ are Christoffel symbols, $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $x = (t, \mathbf{x})$ (boldface letters denoting the three vectors), $\partial_k = \frac{\partial}{\partial p_k}$ denotes derivatives over momenta.

For massless particles ($m = 0$) and in the homogeneous metric ($h_{\mu\nu} = 0$) any function $f(a^2|p|)$ is a solution of eq.(3) ($\delta \Gamma = 0$)\(^1\). Because of the thermodynamic interpretation we choose the equilibrium distribution $\Omega_E$ [12] as a starting point of the perturbation
\[ \Omega_e^\mu = g(2\pi)^{-3} \left( \exp(a^2\beta|p| - \beta\mu(e)) + q \right)^{-1} \] (4)
with
\[ p^2 = \sum_j p^j p^j. \]

$g$ is the number of particle’s degrees of freedom (we set $g = 1$ from now on), $\beta \equiv \frac{1}{T}$ is the inverse temperature and $\mu(e)$ is the chemical potential for particles of the type $e$. In eq.(4) $q = 1$ for fermions, $q = -1$ for bosons and $q = 0$ for the classical Jüttner distribution. The physical momentum is $a|p|$ and the physical temperature $(\beta a)^{-1}$.

We write
\[ \Omega_e = \Omega_E^\mu (1 + \chi_e) \] (5)
and look for a perturbative solution $\chi$ of eq.(3) in an inhomogeneous metric (2). Then,
\[ \partial_t \chi + n^k \partial_k \chi - 2\mathcal{H}p^k \partial_k \chi + e|p|^{-1} F^{\nu\mu} h_{\nu\mu} p^\mu \partial_\nu \chi \]
\[ = -a^2 \beta f|p|^{-1} \left( \partial_\nu \gamma_{jk} n^j n^k + \frac{1}{2} n^l \partial^\nu \gamma_{ljk} n^k \right) + e a^2 \beta f|p|^{-2} F^{\nu\mu} h_{\nu\mu} p^\mu p^\nu, \] (6)
where $n^k = p^k|p|^{-1}$, $\mathcal{H} = a^{-1} \frac{da}{dt}$ and $f = \hat{\Omega}'(a^2\beta|p|)$ where
\[ \hat{\Omega}'(x) = \exp(x - \mu(e)/\beta) \left( \exp(x - \mu(e)/\beta) + q \right)^{-1}. \]

$f = 1$ for the Jüttner distribution (relativistic equilibrium distribution neglecting the quantum statistics).
We set $f \simeq 1$ and look for a solution of eq.(6) which is of the first order in momentum

$$\chi_\epsilon = a^2 \beta \nu |p| + \beta r_\epsilon.$$  \hspace{1cm} (7)

We assume an infinite conductivity of the primordial plasma. Then, the electric field is zero \[16\]. Inserting (7) in eq.(6) we obtain equations for $\nu$ and $r_e$

$$\partial_t \nu + n^k \partial_k \nu = -\partial_t \gamma_{jk} n^j n^k - \frac{1}{2} n^i \partial_j \gamma_{lk} n^l n^k,$$  \hspace{1cm} (8)

$$\partial_t r_e + n^k \partial_k r_e = e \gamma_{km} \epsilon^{jkl} n^j n^m B_l \equiv \sigma_e,$$ \hspace{1cm} (9)

where we wrote

$$F^{jk} = \epsilon^{jkl} \tilde{B}_l(t)$$

with

$$\tilde{B}(t, x) = a^{-2} B(x)$$

This time-dependence of the magnetic field follows from Maxwell equations in an expanding universe \[17\]. We introduce

$$\Theta = \nu - \frac{1}{2} \gamma_{jk} n^j n^k$$ \hspace{1cm} (10)

Then

$$\partial_t \Theta + n^k \partial_k \Theta = -\frac{1}{2} \partial_t \gamma_{jk} n^j n^k \equiv \mathcal{R}$$ \hspace{1cm} (11)

$\Theta$ as discussed in our earlier paper \[18\] has the meaning of the temperature variation. The solution of eq.(11) reads

$$\Theta (t, x) = \int_0^t ds \mathcal{R}(s, x - (t - s)n),$$ \hspace{1cm} (12)

The Fourier transform of the solution of eq.(9) is

$$\tilde{r}_e(k, t) = \int_0^t ds \tilde{\sigma}_e(k, n, s) \exp(-i kn (t - s))$$ \hspace{1cm} (13)

We calculate (in the conformal time) the energy-momentum tensor till the first order in the metric and the magnetic field perturbation (we preserve the quadratic term $T_{EM}$ of the free electromagnetic field for a later comparison). For this purpose we sum the densities of +1 and −1 particles \[19\]

$$T^{j\ell} = (2\pi)^{-3} \sqrt{\gamma} \int dp \phi_0^{-1} p^j p^\ell (\Omega_+ + \Omega_-) + T^{j\ell}_{EM}$$

$$= \int d\mathbf{n} n^j n^\ell \left((1 + 4\nu)(\Omega_E^{(+)} + \Omega_E^{(-)}) + \beta (r_+ \Omega_E^{(+)} + r_- \Omega_E^{(-)})\right) + T^{j\ell}_{EM}$$

$$\equiv T_0^{j\ell} + \delta T^{j\ell} + T^{j\ell}_{EM}$$ \hspace{1cm} (14)

where

$$T_0^{j\ell} = \frac{1}{8\pi a^6} (2B^j B^\ell - \delta^{ij} B^k B^k)$$

Here, $T_0$ is the energy-momentum of the solution (4) of the Liouville equation on the homogeneous space-time, $\delta T$ denotes the terms linear in the metric $\gamma$
and in the magnetic field $B$. In eq.(14) $\nu$ should still be expressed by $\Theta$ from eq.(10) or determined from eq.(8).

We can calculate the tensor (14) taking multiple derivatives

$$
\int d^m \exp(-im(t-s)m^j \ldots m^r = i^r \frac{\partial}{\partial q^j} \ldots \frac{\partial}{\partial q^r} q^{-1} \sin(q),
$$

where after the calculation of derivatives we should set $q = (t-s)k$.

3 Einstein equations in a magnetic field

We write the spatial part of Einstein equations for traceless transverse metric perturbations in the form ($G^i_j$ is the Einstein tensor, $G$ is the Newton constant)

$$
\delta G^i_j = -2a^{-2} \left( \frac{1}{2} \partial^2_t \gamma_{ij} - \frac{1}{2} \Delta \gamma_{ij} + \mathcal{H} \partial_t \gamma_{ij} \right) = 8\pi a^2 G P_{ij,kl}(\delta T^{kl} + (T_{EM})^{jk}),
$$

where

$$
P_{ij,kl} = \Delta_{ik} \Delta_{lj} + \Delta_{il} \Delta_{kj} - \Delta_{ij} \Delta_{lk},
$$

$$
\Delta_{jk} = \delta_{jk} - \partial_j \partial_k \Delta^{-1}
$$

and $\Delta$ is the three-dimensional Laplacian. The matter energy-momentum $\delta T$ on the rhs is linearly dependent on the magnetic field whereas the free electromagnetic energy-momentum $T_{EM}$ is quadratic in the magnetic field.

We write

$$
\gamma_{jk} = a^{-1} \tilde{\gamma}_{jk}.
$$

Then, eq.(16) takes the form

$$
\mathcal{G} \tilde{\gamma}_{jk} = \partial^2_t \tilde{\gamma}_{jk} - (\Delta + a^{-1} \partial^2 a) \tilde{\gamma}_{jk} = 8\pi a^5 G((\delta T^{TT})_{jk} + (T_{EM}^{TT})_{jk}),
$$

where the rhs still depends on the metric. For a general $a(t)$ it is not simple to solve eq.(18). Let $\mathcal{G}^{-1}(k,t,s)$ be the kernel of the inverse of $\mathcal{G}$. Then, we can solve eq.(18) by iteration (till the first order in $G$)

$$
\tilde{\gamma}_{jk} = \gamma_{jk,grav}^0(t) + 8\pi G \int_0^t ds \mathcal{G}^{-1}(k,t,s)a^5 \delta T_{jk}^{TT}(s, a^{-1} \tilde{\gamma}_{jk,grav}),
$$

where $\gamma_{jk,grav}$ is the solution of the homogeneous equation (at $G = 0$).

In the radiation era $\partial^2_t a = 0$. Then,

$$
\mathcal{G}^{-1}(k,t,s) = k^{-1} \sin(k(t-s)).
$$

In another limit, if $ka << \partial^2_t a$ then the dependence of $\mathcal{G}^{-1}$ on $k$ can be neglected.
Returning to eq.(18) we perform some integrals over $n$ (using eq.(15)) and write it in the form

$$
\partial_2^2 \gamma_{ij} - \Delta \gamma_{ij} + 2\mathcal{H} \partial_t \gamma_{ij} = 8\pi G a^4 P_{ij;kl} T_{EM}^{kl} + 8\pi G (2\pi)^{-3} 96\pi a^{-2} \beta^{-4} P_{ij;kl} (t^k + t^l),
$$

(21)

where

$$
t^k = \int d^n n^j n^l \left( \Theta + \frac{1}{4} \beta (N_+ r_+ + N_- r_- ) \right) \equiv \theta^k + r^k.
$$

(22)

$\Theta$ is determined from eqs.(11)-(12) and $r$ is defined in (13) ($\theta$ and $r$ do not depend on $a$). For the equilibrium distribution $\Omega^E_\mu$ we have assumed the approximate formula (justified for high energies)

$$
\Omega^E_\mu = N_\mu \exp(-\beta a^2 |p|),
$$

(23)

where from eq.(4) at $q \to 0$

$$
N_\mu = \exp(\mu(e) \beta)
$$

In order to write down Einstein equations explicitly we calculate the part of the plasma energy-momentum (22) which is linear in the magnetic field

$$
r^k(t, t) = (N_+ - N_-) \int_0^t ds d m^j m^k \sigma(k, m, s) \exp(-i k m(t - s)) = (N_+ - N_-) \int_0^t ds d m^j \gamma_{ij}(q, s) e^{i p q} B_p(k - q) \exp(-i k m(t - s)) m^j m^r.
$$

(24)

$m$ denotes the directional vector of propagation (which we denoted by $n$ in eq.(9)). So, the part of the metric perturbation coming from the magnetic field is

$$
\partial_2^2 \gamma_{ij} - \Delta \gamma_{ij} + 2\mathcal{H} \partial_t \gamma_{ij} = 8\pi G (2\pi)^{-3} 96\pi a^{-2} \beta^{-4} P_{ij;kl} t^k t^l.
$$

(25)

For the remaining part of eqs.(21)-(22) we have

$$
\delta T^{jk} = (2\pi)^{-3} 96\pi a^{-6} \beta^{-4} t^{jk}
$$

This part of metric fluctuations is discussed in many text-books [20][21](we have calculated it for a diffusive matter in [18]).

4 Temperature fluctuations

There will be temperature fluctuations caused by the density fluctuations (scalar perturbations), gravitational waves (quantum metric fluctuations) as well as fluctuations of the primordial magnetic fields. The solution for the temperature fluctuations is expressed by $\gamma$ (eq.(12))

$$
\Theta(t, n) = -n^l n^j n^i \frac{1}{2} \int_0^t ds \gamma_{ijl}(s, x - (t - s) n),
$$
where $\gamma$ is determined by $t^j$. Now,

$$
\langle \Theta(t, n) \Theta(t, n') \rangle = \frac{1}{(2\pi)^3} \int_0^1 ds \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dk n^s n^r n^p
$$

\[ \partial_s \partial_r \langle \gamma_{jl}(s, k) \rangle \gamma_{rp}(s', k') \rangle \exp(-i(t - s)n k + i(t - s')n' k), \] (26)

In eqs.(25)-(26) we wish to calculate the part of fluctuations coming from the magnetic field. It is determined from the solution of Einstein equations (16).

From the perturbative solution (19) the fluctuations coming from the magnetic field are

$$
\langle \gamma_{jl}(s, k) \gamma_{ab}(s', k') \rangle = \left(8\pi G(2\pi)^{-3} 96\pi a^2 \alpha^4 \right)^{-1} (a(s) a(s'))^{-1} \int_0^s \int_0^{s'} d\tau d\tau' \gamma \gamma^\gamma \gamma - 1 \langle k, s, \tau \rangle \gamma \gamma^\gamma \gamma - 1 \langle k', s', \tau' \rangle P_{jl; pq} P_{ab; mn} (r_{pq} (a^1 \gamma, k, \tau) r_{mn} (a^1 \gamma, k', \tau')), \]

where $r$ is expressed by the metric and by the magnetic field in eq.(24).

We have two random fields in the solution $\Theta$: $\gamma$ and $B$. We assume that $B$ is a Gaussian random field with the covariance

$$
\langle B_i(k) B_j(k') \rangle = \Delta_{ij}(k) \delta(k + k') P_B(k), \] (27)

where

$$
\Delta_{ij}(k) = \delta_{jl} - k_j k_l k^{-2}. \]

$B(k)$ is time-independent as explained below eq.(9) ( we denote a function and its Fourier transform by the same letter; the meaning should follow from the context), $\gamma$ is an independent random field with the covariance

$$
\langle \gamma_{jl}(s, k) \rangle \gamma_{ab}(s', k') \rangle = P_{jl; ab} \delta(k + k') P_{\gamma}(k; s, s'). \]

We wish to calculate the correction to the temperature fluctuations coming from the interaction with the magnetic field. Let us define

$$
P_{\sigma}(q, s, s') \delta(q + q') = \langle \sigma(s, q) \sigma(s', q') \rangle, \] (28)

where

$$
\sigma(k, m, s) = \int d\gamma \gamma_{lk}(p, s) e^{it a} B_a(k - p) m^k m^r \]

Then,

$$
\langle \sigma(k, m, s) \sigma(k', m', s') \rangle = \int d\gamma \gamma_{lk}(p, s) e^{it a} B_a(k - p) m^k m^r \gamma_{l'k'}(p', s') e^{it a'} B_{a'}(k' - p') m'^k m'^r \]

$$

\[ = \delta(k + k') \Delta_{aa'}(k) P_{l'k'}(k) e^{it a} m^i m^r e^{it a'} m'^r m'^i \] (30)

where

$$
w(k, m, m') = \Delta_{aa'}(k) P_{l'k'}(k) e^{it a} m^i m^r e^{it a'} m'^r m'^i. \]

7
The spectral function defining the fluctuations of eq.(26) is determined by

\[ P_{\sigma}(k; s, s') = \int dp P_{\gamma}(p; s, s') P_B(k - p). \] (31)

We consider a simplified version of the graviton correlation function

\[ P_{\gamma}(k; s, s') = f(s, s')k^{-\alpha} \] (32)

with \(0 \leq \alpha \leq 3\) and

\[ P_B(k) = b^2k^\sigma \exp\left(-\frac{k^2}{\lambda}\right) \] (33)

(usually with \(-3 < \sigma \leq 2\)) as a model for the spectral function of a primordial magnetic field with a Debye frequency cutoff for \(k^2 > \lambda\) [17][22]. We discuss the spectral function \(P_{\gamma}\) in the Appendix. We show that for a small \(k\) the spectral function \(P_{\sigma}(k; s, s')\) tends to \(Kf(s, s')\) (with a certain constant \(K\)) whereas for a large \(k\) it behaves like the graviton spectral function, i.e., as \(k^{-\alpha}\). Hence, the magnetic field substantially changes the powerlike behaviour of the power spectrum for a small \(k\) but it does not change the leading behaviour for a large \(k\). For comparison the contribution of the pure electromagnetic energy-momentum tensor to the spectral function of the temperature fluctuations is determined by

\[ P_{2B}(k) = \int dp P_B(p) P_B(k - p) \] (34)

We show in the Appendix that for \(\sigma \geq -1\) it tends to a constant for a small \(k\) and decays exponentially for \(k^2 > \lambda\). So, it does not contribute to high multipoles. For low multipoles its contribution behaves as \(b^4\) (the fourth power of the strength of the magnetic field) whereas the contribution to the temperatures \(24\) (of the term linear in the magnetic field is proportional to \(b^2(N_+ - N_-)^2\). Now, using \(24\),\(26\) and \(30\)-(31)

\[ \langle \partial_{\tau} \gamma_{ij}(k, t)\partial_{\tau'} \gamma_{ij'}(k', t') \rangle = \left(8\pi G(2\pi)^{-3}96\pi a^{-2}b^{-4}\right)^{1/2} \delta(k + k') \delta_{ij} \int_0^t ds \int_0^{t'} ds' (a(t)a(t'))^{-1}G^{-1}(k; t, s)G^{-1}(k'; t', s')a(s)a(s') \int_0^s d\tau \int_0^{s'} d\tau' W_{ij;ij'}(s - \tau, \tau, \tau - \tau', \tau'; k)P_{\sigma}(k; \tau, \tau') \equiv \delta(k + k')P_{ij;ij'}(k)F(t, t'; k), \] (35)

where

\[ W_{ij;ij'}(s - \tau, \tau, \tau - \tau', \tau'; k) = \int d\mathbf{m}d\mathbf{m}' P_{ij;ij'}(k)P_{ij;ij'}(k)m^a m^b m'^a m'^b \exp(ik\mathbf{m}(s - \tau) - i\mathbf{m}'(s' - \tau'))u(k, \mathbf{m}, \mathbf{m}') = \int d\mathbf{m}d\mathbf{m}' P_{ij;ij'}(k)P_{ij;ij'}(k)m^a m^b m'^a m'^b \exp(ik\mathbf{m}(s - \tau) - i\mathbf{m}'(s' - \tau')) \Delta_{aa'}(k)\epsilon^{\tau a'} m^\dagger m^\dagger e^\tau' \epsilon^{\tau' a'} m'^\dagger m'^\dagger. \] (36)

The \(\mathbf{m}\) integrals can be evaluated from the formula
of the primordial magnetic field to the temperature fluctuations (26). We will follow the calculations of our earlier paper [18] (concerning dissipative systems). In the integral described in many textbooks [20][21]. We do not have an explicit formula for $G$. Hence, in eq.(35)

$$P_{ij:ab}P_{ij:ab} = 6.$$ 

(F(t, t'; k) = \frac{1}{\pi} P_{\gamma;j:1} \partial_{t'} \int_0^t ds \int_0^{t'} ds'(a(t)a(t'))^{-1} G^{-1}(k; t, s) G^{-1}(k; t', s') a(s)a(s') \int_0^\tau d\tau \int_0^{\tau'} \int_0^{\tau'} W_{\gamma;j:1'}(s - \tau, \tau, s' - \tau', \tau'; k) P_{\gamma}(k; \tau, \tau'). (37)$$

$F(t, t; k)$ is the spectral function for the magnetic contribution to the temperature fluctuations. For a small $k$ it tends to $g(t)$ (with a certain function $g$) because $P_{\gamma}(k) \rightarrow Kf(s, s')$, and the functions $W_{\gamma;j:1'}$ and $G^{-1}(k; t, s)$ in eq.(37) for a small $k$ also tend to a constant multiplied by a function of time. For a large $k$ the spectrum distribution $P_{\gamma}$ behaves as $P_1$ (the one for gravitons). In the higher orders of the conventional perturbative calculations of the temperature fluctuations for a large $k$ we would obtain (from eq.(19), with the energy-momentum $T$ for matter fields on the rhs) the contribution to temperature fluctuations similar to the one resulting from eq.(37). Hence, we can conclude that the magnetic field does not substantially modify the behaviour of the spectral function for large $k$ in comparison to the one without the magnetic field. Hence, it would not be detectable by a measurement of large multipoles.

We do not have an explicit formula for $G^{-1}$. However, for a small $k$, such that $k << a^{-1}\partial^2 a$, the dependence on $k$ can be neglected. In such a case, eq. (37) gives an analytic formula for metric fluctuations caused by a linear dependence on the primordial magnetic field. After the analytic calculation of the spectral function $F(t, t; k)$ in eq.(37) we are able to derive a formula for the contribution of the primordial magnetic field to the temperature fluctuations (26). We will be brief in the discussion of this derivation because it is already standard and described in many textbooks [20][21]. We follow the calculations of our earlier paper [18] (concerning dissipative systems). In the integral $dk = dk^2d\omega$ we integrate first over $\omega$ in the exponential in eq.(26). We obtain

$$\int d\omega \exp(-i(t-s)nk + i(t-s')n'k) = 2\pi k^{-1}|(t-s)n - (t-s')n'|^{-1}\sin\left(|(t-s)kn - (t-s')kn'|\right). (38)$$

Next, we use the expansion

$$k^{-1}|(t-s)n - (t-s')n'|^{-1}\sin\left(|(t-s)kn - (t-s')qn'|\right) = \sum_{l=0}^{\infty}(2l + 1)\hat{j}_l(k(t-s))\hat{j}_l(k(t-s'))P_l(nn'). (39)$$

We have

$$\int dm \exp(ikm(s-\tau))m^a m^b m^c = (s-\tau)^{-4} \frac{\partial}{\partial m_a} \frac{\partial}{\partial m_b} \frac{\partial}{\partial m_c} (k(s-\tau))^{-1}\sin(k(s-\tau)).$$

...
\( j_i \) is the Bessel spherical function related to the Bessel function \( J \) [23]

\[
j_i(z) = \sqrt{\frac{\pi}{2z}} J_{\frac{1}{2}}(z)
\]

(40)

and \( P_l \) are the Legendre polynomials.

If \( F(s,t) \) is known then owing to eqs. (26) and (34) there remains to perform the integrals over \( s \) and \( k \)

\[
\langle \Theta(t, n) \Theta(t, n') \rangle = \frac{1}{2(2\pi)^{-3}} \sum_{l=0}^{\infty} \int_0^t ds \int_0^{t'} ds' \int dk F(s, s', k) (2(n\Delta(k)n')^2 - (n\Delta(k)n)(n'\Delta(k)n'))
\]

\[
\int_{\delta} j_l(k(t-s)) j_l(k(t-s')) (2l + 1) P_l(nn'),
\]

(41)

where

\[
\begin{align*}
\Delta(k)n & = nn' - k^{-2}(kn)(kn') \equiv \Delta(nn', en, en'), \\
\Delta(k)n & = 1 - k^{-2}(kn)^2 \equiv \delta(en).
\end{align*}
\]

(42)

(43)

This formula is the starting point of calculations in [20] (see also our calculations in [18]). The expansion in Legendre polynomials reads

\[
\langle \Theta(t, n) \Theta(t, n') \rangle = \sum_{l=0}^{\infty} (2l + 1) \tilde{D}_l(t, nn') P_l(nn') = \sum_{l=0}^{\infty} (2l + 1) C_l(t) P_l(nn'),
\]

(44)

(where \( \tilde{D}_l \) is the term in front of \((2l + 1) P_l \) in eq.(41)). In eq.(44) \( \tilde{D}_l P_l \) still must be expanded in Legendre polynomials if the coefficients \( C_l \) are to be independent of the angle. We have from eqs.(26),(36) and (41)

\[
\tilde{D}_l = \frac{1}{16\pi} \int_0^t ds \int_0^{t'} ds' \int_0^{\infty} dk k^2 F(s, s', k)
\]

\[
\left( 2\Delta(nn', -i\partial_s, i\partial_{s'})^2 - \delta(-i\partial_s)\delta(i\partial_{s'}) \right) j_l(k(t-s)) j_l(k(t-s')).
\]

(45)

Let us consider only the term without derivatives in eq.(45) (denoted \( D_l \)) resulting from the expansion

\[
2\Delta(nn', -i\partial_s, i\partial_{s'})^2 - \delta(-i\partial_s)\delta(i\partial_{s'}) = 2(nn')^2 - 1 + O(\partial_s, \partial_{s'})
\]

where \( O \) is a polynomial of at least first order in derivatives. The terms in eq.(45) with derivatives can be calculated when \( D_l \) are known [20]. We have

\[
D_l = \frac{1}{16\pi^2} \int_0^t ds \int_0^{t'} ds' F(s, s', q) j_l(q(t-s)) j_l(q(t-s'))
\]

(46)

We make an approximation for \( F \) in eq.(37) which can be justified on the basis of the discussion following eq.(37)

\[
F(s, s', q) = g(s, s')q^{-3+\epsilon},
\]

(47)

with a certain function \( g \), where \( \epsilon \) is different for a large \( q \) and for a small \( q \). The main contribution to the integrals (45) with the spherical Bessel functions
The magnetic field behaves as $\langle \Theta(t, \mathbf{n})\Theta(t, \mathbf{n}') \rangle_B \simeq \sum_{l=0}^{\infty} (2l+1)(2(\mathbf{n}\mathbf{n}')^2-1) \left(8\pi G(2\pi)^{-3}96\pi \alpha(t_d)^{-2}\beta^{-1}\right)^2 b^2(N_++N_-)^2 d_l P_l(\mathbf{n}\mathbf{n}')$, with a certain slowly varying $d_l$. In eq.(51) $t_d$ is the decoupling time, $d_l$ can be calculated from eq.(37) and (41) by a numerical evaluation of the integrals. It varies slowly with $l$. There is still the contribution from the energy-momentum $T_{EM}$ of the magnetic field (discussed in refs.[5]-[8]). This contribution depends

\begin{align*}
    D_l(t) &= \frac{1}{16\pi^3} (2(\mathbf{n}\mathbf{n}'))^2 - 1 \frac{1}{2^{2l+1} \Gamma(\frac{3}{2}l+\frac{3}{2}) \Gamma(l+\frac{3}{2})} \int_0^t ds \int_0^s ds' \left(\frac{s-s'}{t-s'}\right)^l (t-s)^{\epsilon} \\
    g(s, s') F(l + \frac{\epsilon}{2} - \frac{1}{2} + \frac{\epsilon}{2}, l + \frac{3}{2}, \frac{3(\mathbf{n}\mathbf{n}')^2}{(t-s)^2}).
\end{align*}

(49)

The integral (46) can easily be calculated if $F$ (47) is concentrated at $s = s' = s_d$. This case describes an instantaneous metric perturbation (the metric perturbation is limited to the moment $s_d$) corresponding to a sudden decoupling at $s = s_d$ from the last scattering surface [20][21]. In such a case $s = s' = s_d$ in the argument of the hypergeometric function (49). We can obtain the value of the hypergeometric function at 1 using the formula

$$F(\alpha, \beta, \gamma, 1) = \frac{\Gamma(\gamma) \Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha) \Gamma(\gamma-\beta)}.$$
on $b^4 G^2 a^{-4}$ (the forth power of the magnetic field strength). It is decreasing faster with $l$ because the power spectrum $P_{2B}(k)$ is decreasing exponentially. The ratio of the numerical contributions of the linear and the quadratic ($T_{EM}$) terms depends on several parameters: the total charge $N_+ - N_-$, the strength of the magnetic field $b$ and the temperature $(\beta a(t_d))^{-1}$ at the decoupling. There are no precise estimates of these parameters. However, from the dependence of $d l$ on $l$ we could infer the presence of the primordial magnetic field and the charge of the universe.

5 Summary

We have derived an elementary formula for the particle's density distribution resulting from the perturbative solution of the Liouville-Vlasov equation. We have discussed a variation of the distribution which is linear in the magnetic field. It seems that this term has been ignored in the hitherto studies of the magnetic field in the universe. The linear term is non-zero if the primordial plasma is charged. There are strict estimates on the charge of the universe [9][10]. The most elementary bound results from the argument that the electric repulsion cannot be much bigger than the gravitational attraction. This arguments restrict the ratio of the charge of the universe (in electron units) to the baryonic number to be of the order $10^{-18}$. The strength of the magnetic field is also restricted to be extremely small: 1.0 nG in the epoch of the photon last scattering [1]. As a consequence the linear term gives a small contribution to the temperature fluctuation spectrum. We have calculated its dependence on the magnetic field. There are some undetermined parameters in the formula. However, the functional form of the temperature fluctuations could discriminate between various models of the primordial magnetic field. The usually discussed quadratic perturbation of Einstein equations resulting from the electromagnetic energy-momentum has a contribution to the temperature fluctuations which depends in a different way on the probability distribution of the magnetic field and it does not depend on the temperature at the decoupling.

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6 Appendix

We wish to calculate

$$P_{\sigma}(k) = \int dq P_\gamma(k - q) P_B(q)$$ (1)

with

$$P_\gamma(k) = k^{-\alpha}$$ (2)
as a typical model for a graviton distribution and

\[ P_B(k) = k^\sigma \exp(-\frac{k^2}{\lambda}) \]  \hspace{1cm} (3)

for the magnetic field probability distribution. Then, eq.(1) gives

\[ P_\sigma(k) = 2\pi \int_0^\infty dq \int_{-1}^{+1} dx q^\delta + 2 \left( k^2 + q^2 - 2qkx \right)^{-\alpha} \exp(-\frac{q^2}{\lambda}) \]
\[ = 2\pi k^{-1}(2-\alpha)^{-1} \int_0^k dq q_{\delta+1} \left( (k+q)^{-\alpha} - (k-q)^{-\alpha} \right) \exp(-\frac{q^2}{\lambda}) \]
\[ + 2\pi k^{-1}(2-\alpha)^{-1} \int_k^\infty dq q_{\delta+1} \left( (k+q)^{-\alpha} - (q-k)^{-\alpha} \right) \exp(-\frac{q^2}{\lambda}) \]
\[ = I_1(k) + I_2(k). \]  \hspace{1cm} (4)

It is easy to see that

\[ I_1(k) = k^{-\alpha} g(k) \]  \hspace{1cm} (5)

where \( \lim_{k \to \infty} g(k) = \text{const} \neq 0 \) and \( I_2 \) decays exponentially for a large \( k \).

Moreover, for a small \( k \) we obtain that \( P_\sigma(0) > 0 \) is finite if \( \delta - \alpha > -3 \).

Concerning the convolution (34) of the magnetic spectral function it is useful to consider its Fourier transform \( \tilde{P}_B(y) \)

\[ P_B(k) = \int dy \tilde{P}_B(y) \exp(i ky). \]  \hspace{1cm} (6)

Applying the Fourier transform for the distribution (3) with \( \sigma = 0 \) we obtain

\[ P_B(k) = \exp(-\frac{k^2}{\lambda}) = \int dy \exp(-\frac{\lambda}{4} y^2) \exp(i ky)(\pi \lambda)^{-\frac{3}{2}}. \]  \hspace{1cm} (7)

Hence, the spectral function defined in eq.(34) is

\[ P_{2B}(k) = \int dp P_B(p) P_B(k-p) = \left( \frac{\lambda}{2} \right)^{-3} \int dy \exp(-\frac{\lambda}{4} y^2) \exp(i ky) \]
\[ \int dy' \delta(y - y') \exp(-\frac{\lambda}{4} y'^2) \exp(i ky') = 8(2\pi \lambda)^{\frac{3}{2}} \exp(-\frac{k^2}{2\lambda}). \]  \hspace{1cm} (8)

For general \( \sigma \neq 0 \) it is difficult to obtain explicit formulas for \( P_{2B} \). In general, in eq.(6) if \( P_B \) is decaying faster than any polynomial then also \( \tilde{P}_B \) is decaying in the same way and the convolution (34) of such functions has this property. From eq.(6) we can also derive the relation

\[ P_{2B}(k = 0) = \int dy (\tilde{P}_B(y))^2. \]  \hspace{1cm} (9)

Hence, if \( \tilde{P}_B(y) \) is square integrable then \( P_{2B}(k) \) is regular at \( k = 0 \) and disappears fast at large \( k \). The behaviour for small \( k \) is not generic with respect of convolutions and Fourier transforms. We do not have special reasons to apply
the spectral functions exactly of the form (3). In the literature a sharp cut-off at \( \sqrt{\lambda} \) is also applied \[24\]. For the properties of the stochastic magnetic fields only the behaviour of \( P_B \) for a large \( k \) and a small \( k \) is essential. We suggest to replace the spectral function (3) by

\[
P_B(k) = \int_1^\infty ds s^{-1-\frac{\sigma}{2}} \exp(-s \frac{k^2}{\lambda}). \tag{10}
\]

This function has the same behaviour as the one in eq.(3) for large as well as small \( k \). It has the virtue that its Fourier transform can easily be calculated and we obtain a workable representation of \( P_{2B} \). It can be shown by an explicit calculation that if \( \sigma \geq -1 \) then \( P_{2B}(0) \neq 0 \) is finite. It decays exponentially for a large \( k \). If \( \sigma < -1 \) then \( P_{2B}(k) \) is singular when \( k \to 0 \) but its singularity is less than \( k^{-|\sigma|} \) (e.g., as discussed in \[24\], when \( \sigma = -2 \) then we have \( P_{2B} \simeq k^{-1} \) for a small \( k \)).

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