Experimental solution of Buquoy’s problem

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Abstract. This article addresses the problem in which a chain moves vertically upward from a horizontal plane under constant force – Buquoy’s problem. For the first time, an original technique is proposed for the experimental solution of this problem. The chain’s motion is studied in liquid (water), and the buoyancy force of the float attached to the chain serves as a constant force. The experiment reveals converging oscillation of the system at various displacements of the chain relative to the equilibrium position. The differential equation of chain motion which describes this variable mass system is solved by the numerical technique. A comparison of theoretical and experimental data is made.

1. Introduction

The formation of the dynamics of variable mass systems as an independent section of classical mechanics is usually referred to the second half of the 19th century – see [1].

According to Mikhailov [2] and Šima&Podolský [3], the Czech scientist and inventor von Buquoy “was the first to investigate systems with a varying mass” even before the fundamental work of Meshchersky [1]. In 1812 – 15 Buquoy published three studies [4] in which he formulated the correct dynamical equation of motion for the case when the mass of a moving object is changing.

Considering the motion of a particle under the action of force $F$, Buquoy showed [4] that in the case of a continuous variation of the particle mass, the change in the momentum is not $mdv$, but $mdv + (v-u)dm$, and the basic equation of dynamics takes the form

$$m \frac{dv}{dt} + (v-u) \frac{dm}{dt} = F$$

Although the presented equation was new for that time and preceded the classical Meshchersky equation [1], Buquoy’s ideas did not attract the attention of the scientific community until the mentioned work [2].
Figure 1. The Buquoy’s problem (a); an experiment that implements the Buquoy’s problem (b)

The pioneering works of Buquoy contained a number of examples, among which the following is the subject of the present experimental study:

A heavy homogeneous chain is piled up on a horizontal plane. A constant force $F$ directed vertically upward is suddenly applied to one of its ends. This force carries along the increasing part of the chain (Figure 1, a). The problem is to find the movement of the end of the chain to which the force is applied.

Buquoy obtained a differential equation describing the chain motion and gave an appropriate solution. It is noted that Buquoy’s original solution contains an error (or a typo) [2]. A correct theoretical analysis of the problem of chain motion under the action of a constant vertical force was presented in [3, 5].

Although there are a number of theoretical works [2, 3, 5] devoted to solving Buquoy’s problem, experiments to study the motion of a variable mass system under the action of constant force have not been carried out. The possibility of experimental modelling of the problem under consideration was noted in [2] where it was proposed in laboratory conditions to consider “the vertical motion of a balloon with a heavy rope hanging down”.

The object of this study is to conduct an experiment to solve the Buquoy’s problem and to compare the experimental data with the theory. This work is a logical continuation of our experiments [6] on the falling U-folded chain and with chain fountain.

2. Results and Discussion

2.1. Experimental procedure

In our experiments, the Buquoy’s problem was modelled as the vertical motion of a fishing float with a heavy homogeneous chain hanging down. The constant upward force was the buoyant force exerted on the float in water. Water was the working medium in which the movement of the float-chain system took place.
We used a transparent tank (50×10×50 cm) filled with water, a ruler (Fig. 1, b) and a chain of metal balls. There were 167 metal balls per meter with an average diameter of 4.6 mm (Fig. 2, a). Linear chain density was $\eta = 28$ g/m. The carrying capacity of the float was 4.2 g (it means that the float buoyant force lifts float itself and the chain of 4.2 g excessively). With such a load capacity, the float could “stretch” a chain of 15 cm long. Thus, the value $x_{eq} = 15$ cm corresponded to the static equilibrium of the float-chain system.

The float was attached to the end of the chain (Figure 2, b). The chain and the float were placed on the bottom of the tank and held onto the keel of the float (Figure 1, b). Then the float was released and the movement of the end of the chain was recorded by PowerShotSX50 HS camera (30, 120 and 240 fps). To process of video materials ImageJ program was used. To solve the equations by numerical technique and graphing we used the Wolfram Mathematica 10.0 software.

2.2. Experimental results

Mainly experiments were carried out at $x(0) < x_{eq} = 15$ cm. The float with the chain was placed near the bottom of the tank and released. Then, it rushed up, capturing chain links. After the system had reached the maximum height position $x > x_{eq}$, the float lowered, and damped oscillations were observed (Figure 3). At large times, the coordinate tended to an equilibrium value $-x \rightarrow x_{eq}$. The position $x(t)$ (height above the bottom) of the first (top) chain link as a function of time is shown in Figure 3, a.

![Figure 3. The result of one experiment with $x(0) = 1$ cm (a), and with $x(0) = 27.7$ cm (b)](image-url)
We note that similar fast damped oscillations near $x_{eq}$ were also observed with the initial position of the upper chain link above the equilibrium position – $x(0) > x_{eq}$ – Fig. 3, b.

2.3. Analytical model [3, 5]

To interpret the data of our experiments, we used the differential equation of chain motion obtained in works [3, 5].

![Buquoy's problem – analytic model](image)

The moving chain part with the momentum $Q = m \cdot v = \eta \cdot x \cdot v = \eta \cdot x \cdot \dot{x}$ moves under the action of the external constant force $F$ and the gravity force $mg$ – see Fig. 4. Writing the impulse-momentum theorem

$$\dot{Q} = F - mg$$

we have:

$$\eta(x\dot{x})' = F - mg$$

i.e.

$$\ddot{x} = \frac{F}{x\eta} - g - \frac{(\dot{x})^2}{x}$$

The equilibrium position of the system, in which the gravity force of the chain $mg$ is equal to the buoyant force $F$ acting on the float–chain system is determined as

$$x_{eq} = \frac{F}{g\eta}$$

Then the equation of motion takes the form:

$$\ddot{x} = g \left( \frac{x_{eq}}{x} - 1 \right) - \frac{(\dot{x})^2}{x} \tag{1}$$

According to [3, 5], equation (1) describes the upward chain motion. The inclusion of the next chain link $dx$ to the moving part occurs in an impact way [6]. The speed of this link changes from zero to the velocity $v$ of the moving part during small time $dt$. The momentum of included chain link changes from 0 to

$$\dot{Q} = v \cdot \eta \cdot ds$$

And the momentum change is:

$$d\dot{Q} = \dot{Q} - 0 = \eta \cdot v \cdot ds \tag{2}$$

It occurs under the influence of an elementary impulse of force $N$ applied from the moving part of the chain:
\[ dQ' = N \cdot dt \]  

We substitute eq. (3) in (2) and use \( v = ds/dt \):

\[ N = \eta v^2 \]  

The moving part of the chain does not act on the surface, but on the next included link (Fig. 5, a). By the Third Newton’s Law, the reaction of the included element \(-N\) acts on the moving part of the chain (Fig. 5, b). If we divide eq. (4) by \( \eta \), we get the second term \((\dot{x})^2/x\) of equation (1).

![Figure 5. Upward chain motion: a) the moving part of the chain does not act on the surface, but on the next included element; b) the reaction of the included element acts on the moving part of the chain](image)

![Figure 6. Downward chain motion: a) the force \(N\) acts to the surface from stopping link and the force \(N'\) acts to the stopping link from surface; b) The reaction of the surface \(N'\) does not impact to the moving part of the chain](image)

The situation changes in case of the downward chain motion. The separation of chain links from the moving part also occurs in an impact way. When the link reaches the bottom of the tank, its speed instantly drops to 0. But in this case, the force \(N\) acts between the stopping link and the surface (Figure 6, a). The reaction of the surface \(N'\) acts on the stopping link and does not affect the moving part of the chain (Figure 6, b). It means that the chain moves downward only under the action of the resultant of gravity force \(mg\) and the external force \(F\).

Thus, it is necessary to take into account the second term of equation (1) only when the chain moves upward:

\[ \ddot{x} = g \left( \frac{x_{0u}}{x} - 1 \right) - \frac{1}{2} \left( 1 + \text{sign}(\dot{x}) \right) \left( \frac{\dot{x}}{x} \right)^2 \]  

The eq. (5) is the desired equation of the chain motion. A similar equation was obtained in [3].

We solved the equation (5) by the numerical technique using the initial conditions: \(x(0) = 1\) cm; \(\dot{x}(0) = 0\) cm/s. Figure 7, a shows chain displacement \(x(t)\) obtained by solving differential equation (5) and the data of one experiment. The graph of chain velocity \(\dot{x}(t)\) is shown in Figure 7, b; and the phase trajectory \(\dot{x}(x)\) is presented in Figure 7, c.
2.4. Discussion

The experimental and numerical data in Figures 3 and 7 shows that the float-chain system performs damped oscillations near the equilibrium position. From a comparison of the measured and calculated dependences $x(t)$ in Figure 7, a it follows that the damping of the oscillations of the float-chain system in the experiment occurs much faster than in the theoretical model.

Since water was used as the working fluid in the experiment, and the float had finite dimensions, the corresponding dissipative effects should be taken into account. These effects are associated with the action on the float and the chain of hydrodynamic resistance forces $\sum F_f$ indicated in Table.

The fluid resistance force at low velocity is caused by skin friction drag force (viscous resistance), and at high velocity it is caused by form drag (pressure drag) force. We used the technique described in [7] to estimate the friction and form drag force for both the float and the chain. These forces were presented as functions of velocity $v$ and displacement $x$ (here $x = x(t)$ represents displacement as well as length of the chain moving part, so $x$ shows number of chain links).

**Table.** Fluid resistance force

| Constituent forces                        | Function                                      |
|-------------------------------------------|-----------------------------------------------|
| Friction drag force acting on the chain   | $4,32 \cdot 10^{-5} \cdot (167 \cdot x) \cdot v$ |
| Friction drag force acting on the float   | $14,13 \cdot 10^{-3} \cdot v$                 |
| Form drag force acting on the chain       | $4,12 \cdot 10^{-3} \cdot (167 \cdot x) \cdot v^2$ |
| Form drag force acting on the float       | $2,30 \cdot 10^{-3} \cdot v^2$                |

Thus, equation (5) can be rewritten in the following form:

$$\ddot{x} = g \left( \frac{x_w}{x} - 1 \right) \frac{1 + \text{sign}(\dot{x})}{2} \frac{(\dot{x})^2}{x} - \text{sign}(\dot{x}) \cdot \frac{\sum F_f}{x \eta}$$

(6)

Once again, we solve the eq. (12) by the numerical technique using the initial conditions: $x(0) = 1$ cm; $\dot{x}(0) = 0$ cm/s. The corresponding graphs of chain displacement $x(t)$ (a) and velocity $\dot{x}(t)$ (b), as well as the phase trajectory $\dot{x}(x)$ (c) are shown in Figure 8.

Taking into account the fluid resistance force increases the accuracy of the Buquoy’s problem analytic model (Figure 7, a and 8, a) when comparing with the data of our experiment. However, as
follows from Figure 8, a there is a time shift between the calculated and experimental dependences of the chain displacement. Firstly, it occurs because of the inaccuracy in the description of the fluid resistance force. When determining the fluid resistance force we made an assumption that the fluid is stagnant. In fact, the movement of the float and the chain causes upward and downward fluid flows. The float and the chain move against the flow that they created themselves during the previous half-cycle. The velocity $v = \dot{x}$ in the Table is considered in the absolute coordinate system referred to the tank. In fact, it is the velocity of the chain and the float relative to the liquid [7].

Figure 8. a) The chain displacement $x(t)$: 1 – numerical solution of (5); 2 – experimental data; b) chain velocity $\dot{x}(t)$ and c) the phase trajectory $\dot{x}(x)$ obtained by solving (6)

We varied the values of the coefficients in the fluid resistance forces expressions (Table) and as a result we have achieved a good agreement between the analytical and experimental data (Figure 9).

Figure 9. Analytical and experimental data: dots – experiments, curve – solution of the eq. (6), where the values of the coefficients in the fluid resistance forces expressions were varied for the exact coincidence of analytical and experimental results

In Figure 7, b and 8, b at the initial time there is a jumping in the velocity value. This is due to the fact that an external force $F$ acts for a small period of time only on a small mass of the first chain links, given by the initial condition $x(0) = 1$ cm. Therefore the force causes a large increment of the chain velocity over a small time period.

3. Conclusion
We presented an experimental solution to the classical Buquoy’s problem in which a chain moved vertically upward from a horizontal plane under constant force.

An original experimental technique was developed in which the chain’s motion was studied in liquid (water), and the buoyancy force of the float attached to the chain served as a constant force. Experiments showed the damped nature of the oscillations of the float – chain system when the chain was displaced relative to its equilibrium position.

The experimental data were compared with the available analytical models. For the best agreement between experiment and theory, it is necessary to take into account the peculiarities of the movement of the chain-float system in water.

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