FILTRATIONS OF TILTING MODULES AND COSTALKS OF PARITY SHEAVES
LINYUAN LIU

Abstract. In this article, we proved that the costalks of parity sheaves on the affine Grassmannian correspond to the Brylinski-Kostant filtration of the corresponding weight spaces of tilting modules.

1. Introduction

1.1. Summary. Assume $G$ is a split reductive algebraic group over a field $k$. When $k = \mathbb{C}$, R.K.Brylinski constructed a filtration of weight spaces of a $G$-module, using the action of a principal nilpotent element of the Lie algebra, and proved that this filtration corresponds to Lusztig’s q-analogue of the weight multiplicity (cf. [Bry89]). Later, Ginzburg discovered that this filtration has an interesting geometric interpretation via the geometric Satake correspondence (cf. [Gin89]). The goal of this article is to partially generalise these results to the case where the characteristic of $k$ is positive.

1.2. Main result. In the rest of the article, let $G$ be a reductive group over $k$ and is a product of simply-connected quasi-simple groups and general linear groups. Suppose $k$ algebraically closed such that the characteristic is good for each quasi-simple factor of $G$ in the sense of [JMW]. Suppose there exists a non degenerate $G$-equivariant bilinear form on $g$. When there is no confusion, we write $\otimes$ for $\otimes_k$. Fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Let $X = X^+(T)$ be the weight lattice and $X^+$ be the set of dominant weights with respect to $B$.

Let $G_r$ be the affine Grassmannian variety of the complex Langlands dual group $\hat{G}$ of $G$. Let $\hat{T} \subset \hat{G}$ be the maximal torus. For each $\mu \in X$, let $L_\mu$ be the corresponding $\hat{T}$-fixed point in $G_r$, and let $\mu_\mu$ be the embedding $\{L_\mu\} \to G_r$. When $\mu$ is dominant, denote by $G_r^\mu$ the $\hat{G}(\mathcal{O}) = \hat{G}(\mathbb{C}[[t]])$-orbit of $L_\mu$ in $G_r$, by $E(\mu)$ the indecomposable parity sheaf with respect to the stratum $G_r^\mu$ (cf. [JMW]), and by $T(\mu)$ the indecomposable tilting module of $G$ of highest weight $\mu$.

Denote by $g$, $b$ and $t$ the Lie algebras of $G$, $B$ and $T$. The main result of this article is the following

Theorem 1. Let $e \in b$ be a principal nilpotent element that is $t$-adapted (i.e., there exists $h \in t$ such that $[h,e] = e$). For all $\lambda, \mu \in X^+$, let $F_\bullet(T(\lambda)_\mu)$ be the Brylinski-Kostant filtration of $T(\lambda)_\mu$ defined by $e$, i.e. for all $n \in \mathbb{N}$, we have

$$F_n(T(\lambda)_\mu) = \{v \in T(\lambda)_\mu \mid e^{(i+1)}v = 0 \text{ for all } i \geq n\},$$

and $F_n(T(\lambda)_\mu) = 0$ whenever $n < 0$.

Then we have

$$(1) \quad \dim H^{2n-\dim(Gr \mu)}(i_\mu^!E(\lambda)) = \dim (F_n(T(\lambda)_\mu)/F_{n-1}(T(\lambda)_\mu)).$$
Lemma 1. Let \( V, V' \) be the Grothendieck resolution. Then we have an isomorphism of graded \( k[t^*] \) \( H^*_T(\mathfrak{g}^* \otimes \mathfrak{g}) \), where on the right hand side \( \pi^* \mathcal{L}_{G/P}(V') = \text{Ind}_{\mathfrak{g}}^G(V' \otimes k[V]) \), where \( V' \) is in degree 0 and the global sections are equipped with the grading induced by the \( \mathcal{G} \)-action on \( \mathfrak{g} \) defined by

\[
\gamma \cdot (g \times B x) = g \times B (z^2 x).
\]

Lemma 1. Let \( P \subset G \) be a parabolic subgroup such that \( G \rightarrow G/P \) is locally trivial and \( V, V' \) are \( P \)-modules. Let \( \mathcal{G}_m \) act on \( V \) by \( z : x = z^2 x \). Let \( \pi : G \times P \rightarrow G/P \) be the natural map. Then we have isomorphism of graded \( \mathcal{G}_m \)-modules

\[
\Gamma(G \times P V, \pi^* \mathcal{L}_{G/P}(V')) = \text{Ind}_{\mathfrak{g}}^G(V' \otimes k[V])
\]

where \( \mathcal{L}_{G/P}(V') \) is the associated sheaf induced by the \( P \)-module \( V' \) in the sense of \( \text{Jan03} \). I.5.8 with \( X = G \times V \), the grading on the left hand side is induced by the action of \( \mathcal{G}_m \) on \( V \), and the grading on the right hand side is induced by the grading on \( k[V] \), with \( V' \) placed on degree 2.

Proof. Let \( X = G \times V \), then \( P \) acts on \( X \) by \( (g, x) \cdot p = (gp, p^{-1} x) \) and we have \( G \times P V = X/P \) by definition. Then we have

\[
\pi^*(\mathcal{L}_{G/P}(V')) \cong \mathcal{L}_{X/P}(V')
\]

by \( \text{Jan03} \) I.5.17 (1). Now we have

\[
\Gamma(G \times P V, \pi^* \mathcal{L}_{G/P}(V')) = \Gamma(X/P, \mathcal{L}_{X/P}(V'))
\]

\[
= (V' \otimes k[X])^P
\]

\[
= (V' \otimes k[V])^P
\]

\[
= \text{Ind}_{\mathfrak{g}}^G(V' \otimes k[V])
\]

with the desired gradings.

Apply the lemma to \( P = B, V = (\mathfrak{g}/\mathfrak{n})^* \) and \( V' = k_{-w_0 \mu} \), we get an isomorphism of graded \( G \)-modules

\[
\Gamma(\mathfrak{g}, \mathcal{O}_{\mathfrak{g}}(-w_0 \mu)) \cong \text{Ind}_{\mathfrak{g}}^G(k_{-w_0 \mu} \otimes k[(\mathfrak{g}/\mathfrak{n})^*]).
\]

Hence we have an isomorphism of graded \( k[t^*] \)-modules

\[
H^*_T(\mathfrak{g}^*) (i_{\mu}^1 \mathcal{E})(\lambda) \cong (T(\lambda) \otimes \text{Ind}_{\mathfrak{g}}^G(k_{-w_0 \mu} \otimes k[(\mathfrak{g}/\mathfrak{n})^*]))^G \cong (T(\lambda) \otimes k_{-w_0 \mu} \otimes k[(\mathfrak{g}/\mathfrak{n})^*])^B \otimes k[t^*] k_{\phi},
\]

by tensor identity and Frobenius reciprocity.

Take a regular semisimple element \( \phi \in t^* \). Then we have isomorphisms of filtered vector spaces

\[
H^*_\phi(\mathfrak{g}^*) (i_{\mu}^1 \mathcal{E})(\lambda) \cong (T(\lambda) \otimes k_{-w_0 \mu} \otimes k[(\mathfrak{g}/\mathfrak{n})^*])^B \otimes k[t^*] k_{\phi}.
\]
Identify $g$ with $g^*$ by a non-degenerate $G$-equivariant bilinear form and let $h \in t$ be the image of $\phi$, which is a regular semisimple. Then we have

$$H^*_{T}(-\dim(G^r)) (\iota^!_{\mu} \mathcal{E}(\lambda)) \cong (\mathcal{T}(\lambda) \otimes k_{-w_0 H} \otimes k[b])^B \otimes_{k[H]} k_h$$

To transform the formula above to a form with which is easier to deal, we also need the following three lemmas.

**Lemma 2.** If $h \in t_{rs}$, then we have a $B$-equivariant isomorphism of varieties

$$(h + n) \times t_{rs} \sim \rightarrow b \times t_{rs}$$

such that $(h,h) \mapsto h \times t_{rs}$ and the following diagramme

$$
\begin{array}{ccc}
(h + n) \times t_{rs} & \xrightarrow{\sim} & b \times t_{rs} \\
\downarrow p_2 & & \downarrow p_2 \\
t_{rs} & \xrightarrow{\pi_2} & t_{rs}
\end{array}
$$

is an isomorphism of affine bundles over $t_{rs}$.

**Proof.** First, let us construct a map $\Phi : (h + n) \times t_{rs} \rightarrow b \times t_{rs}$. Let $(X,H) \in (h + n) \times t_{rs}$, then there exists $b \in B$ such that $\text{Ad}(b)(h) = X$. We set $\Phi(X,H) = (\text{Ad}(b)(H),H)$, which indeed lies in $b \times t_{rs}$ since by the projection $b \rightarrow t$, the image of $\text{Ad}(b)(H)$ is $H \in t_{rs}$. We need to check

- $\Phi$ is well-defined, which means it doesn’t depend on the choice of $b \in B$;
- $\Phi$ is $B$-equivariant (obvious);
- $\Phi$ is a morphism of varieties;
- $\Phi$ is bijective;
- $\Phi^{-1}$ is a morphism of varieties;

If $X = \text{Ad}(b)(h) = \text{Ad}(b')(h)$, then $\text{Ad}(b^{-1}b')(h) = h$, hence $b^{-1}b' \in T$ since $h$ is regular semisimple in $t$, and $\text{Ad}(b)(H) = \text{Ad}(b')(H)$ since $H \in t$. To prove that $\Phi$ is a morphism, observe that $\Phi$ is induced by the following commutative diagramme

$$
\begin{array}{ccc}
(h + n) \times t_{rs} & \xrightarrow{\phi \times \text{id}} & U \times t_{rs} \\
\downarrow p_2 & & \downarrow \\
t_{rs} & \xrightarrow{\pi_2} & t_{rs}
\end{array}
$$

where $\psi(u,H) = \text{Ad}(u)(H)$, and $\phi$ is the inverse map of $U \rightarrow h + n : u \mapsto \text{Ad}(u)(h)$ which is a morphism by [Jan04] page 188. Bijectivity is easy to prove. $\Phi^{-1}$ is also a morphism because it is the composition

$$b \times t_{rs} \xrightarrow{\pi_1} b_{t_{rs}} \xrightarrow{f} U \times t_{rs} \xrightarrow{g} (h + n) \times t_{rs}$$

where $f$ is the map on page 188 [Jan04] and $g(u,H) = (\text{Ad}(u)(h),H)$, which are both morphisms.

**Lemma 3.** Let $H$ be an algebraic group over $k$, $A$ a flat $k$-algebra and $M$ an $H$-module. Then the natural map $M^H \otimes_k A \rightarrow (M \otimes_k A)^H$ is an isomorphism of $A$ modules, where the action of $H$ on $M \otimes_k A$ is induced by the action of $H$ on $M$ and the trivial action of $H$ on $A$.

**Remark 1.** The assumption of flatness of $A$ is automatically satisfied, since $k$ is a field. We include this assumption since the statement is also correct even in more general cases, where the flatness will be crucial.

**Proof.** Since $A$ is flat, it is easy to check that the map $M^H \otimes_k A \rightarrow (M \otimes_k A)^H$, $m \otimes a \mapsto m \otimes a$ is a bijection.
Lemma 4. Let $H$ be an algebraic group over $k$ and $A$ a flat $k$-algebra. Let $M$ be an $H$-module and a torsion free $A$-module such that the two actions commute (i.e. $h \cdot (am) = a(h \cdot m)$ for all $m \in M$, $a \in A$ and $h \in H$). Then for any multiplicative subset $S \subset A$, the natural morphism

$$S^{-1}(M^H) \rightarrow (S^{-1}M)^H$$

is an isomorphism of $S^{-1}A$-modules.

Proof. The map is induced by $s^{-1}m \mapsto s^{-1}m$. □

Using the above lemmas, we have isomorphisms of filtered vector spaces

$$(T(\lambda) \otimes k_{-w_0}) \otimes k[b])^B \otimes k[t] k_h$$

$$(T(\lambda) \otimes k_{-w_0} \otimes k[h] \otimes k[t] \otimes k[t_\alpha] k_{h_1}$$

$$(T(\lambda) \otimes k_{-w_0} \otimes k[h + n] \otimes k[t] \otimes k[t_\alpha] k_{h_1}$$

$$(T(\lambda) \otimes k_{-w_0} \otimes k[h + n] \otimes k[t] \otimes k[t_\alpha] k_{h_1}$$

where the second isomorphism is due to Lemma 4, the third is due to Lemma 2 and the fourth is due to Lemma 3.

Hence there is an isomorphism of filtered vector spaces

$$H^\bullet_{\phi}(\Gamma(T^\bullet, E(\lambda))) \cong (T(\lambda) \otimes k_{-w_0} \otimes k[h + n])^B.$$  

On the other hand, by using the geometric Satake equivalence and equivariant localisation, the left-hand side is isomorphic to the vector space $T(\lambda)_\mu$, hence in particular we have

$$\dim T(\lambda)_\mu = \dim (T(\lambda) \otimes k_{-w_0} \otimes k[h + n])^B.$$

Lemma 5. Let $M$ be a $B$-module and $\mu \in X(T)$. Then there exists a natural isomorphism

$$(M \otimes k_{-\mu})^B \cong (M^U)_\mu$$

defined by sending $m \otimes 1$ to $m$.

Proof. 

$$(M \otimes k_{-\mu})^B \cong \text{Hom}_B(k_\mu, M) \cong (M^U)_\mu.$$ □

Lemma 6. The map

$$\Lambda : (T(\lambda) \otimes k[h + n])^U \rightarrow T(\lambda)$$

defined by evaluation on $h$ is an isomorphism of $T$-modules.

In particular, it induces an isomorphism of vector spaces:

$$\Lambda_\mu : (T(\lambda) \otimes k_{-\mu} \otimes k[h + n])^B \cong T(\lambda)_\mu.$$ 

Proof. $\Lambda$ is $T$-equivariant because $h$ is fixed by $T$.

On the other hand, we already have

$$\dim T(\lambda)_\mu = \dim (T(\lambda) \otimes k_{-w_0} \otimes k[h + n])^B = \dim ((T(\lambda) \otimes k[h + n])^U)_\mu$$

because the dimension of the weight spaces with respect to $\mu$ and $w_0\mu$ are the same. By taking the sum over all $\mu$, we have $\dim (T(\lambda) \otimes k[h + n])^U = \dim T(\lambda)$. Hence it suffices to prove that $\Lambda$ is injective because both sides are finite dimensional.

The idea of the proof of injectivity is quite simple. Roughly speaking, an $U$-equivariant function on $h + n$ is zero if it is zero on $h$. The following is just a more rigorous version of this simple idea.
Identify \((T(\lambda) \otimes k[h + n])^U = \text{Hom}_U(T(\lambda)^*, k[h + n])\) and \(T(\lambda) = (T(\lambda)^*)^*\). Then for \(f \in \text{Hom}_U(T(\lambda)^*, k[h + n])\), \(\Lambda(f) : T(\lambda)^* \to k\) is defined by \(\Lambda(f)(\psi) = f(\psi)(h)\). Hence if \(\Lambda(f) = 0\), then for all \(\psi \in T(\lambda)^*\) and \(u \in U\), we have \(f(\psi)(\text{Ad}(u)(h)) = f(u^{-1}\psi)(h) = \Lambda(f)(u^{-1}\psi) = 0\). But since \(h\) is principal semi-simple, we have \(\text{Ad}(U)(h) = h + n\), hence \(f(\psi)(X) = 0\) for all \(X \in h + n\). Since \(k\) is an infinite field, this means that as an element in \(k[h + n]\), we have \(f(\psi) = 0\) (another way to think about this: \(k\) is algebraically closed and \(k[h + n]\) is reduced, then if some function is zero at each closed point, it is zero by Hilbert’s Nullstellensatz.). Since \(\psi\) is arbitrary, we have \(f = 0\). This proves the injectivity. \(\square\)

We conclude the proof of Theorem 1 by the following

**Proposition 1.** Let \(e \in n\) such that \([h, e] = e\). Then \(e\) is a principal nilpotent. Then we have

\[
f \in \text{Hom}_B(T(\lambda)^* \otimes k_\mu, k[h + n]_n) \Leftrightarrow \Lambda(f) \in F_n(T(\lambda)_n)
\]

for all \(n \in N\).

**Remark 2.** Roughly speaking, the idea of the proof is as follows: if a \(B\)-equivariant map from \(T(\lambda)^* \otimes k_\mu\) to \(k[h + n]\) takes any element to a polynomial that has degree \(\leq n\) along the direction \(e \in n\), then it takes any element to a polynomial with degree \(\leq n\), because \(B \cdot e\) is dense. We will make this idea rigorous in the proof.

**Proof.** Denote \(V = T(\lambda)\) Fix \(f \in \text{Hom}_B(V^* \otimes k_\mu, k[h + n])\) and let \(v = \Lambda(f)\). Then \(f \in \text{Hom}_B(V^* \otimes k_\mu, k[h + n]_n)\) if and only if for any \(\psi \in V^*, f(\psi \otimes 1) \in k[h + n]\) has degree \(\leq n\). Since \(k\) is an infinite field, \(f(\psi \otimes 1) \in k[h + n]\) has degree \(\leq n\) if and only if for all \(X \in n\), the polynomial in \(t\)

\[
f(\psi \otimes 1)(h + tX)
\]

has degree \(\leq n\).

Since \(B \cdot e\) is dense in \(n\), we have \(f \in \text{Hom}_B(V^* \otimes k_\mu, k[h + n]_n)\) if and only if it satisfies the following condition \((A)\):

“For all \(\psi \in V^*\) and all \(b \in B\), the polynomial \((\psi \otimes 1)(h + tb \cdot e)\) has degree \(\leq n\).”

Claim: \((A)\) is equivalent to the condition \((B)\):

“For all \(\psi \in V^*\), the polynomial \((\psi \otimes 1)(h + te)\) has degree \(\leq n\).”

Proof of the claim: \((A)\) clearly implies \((B)\). Now suppose \(f\) satisfies \((B)\). Fix \(\psi \in V^*\) arbitrary, choose a \(b_0 \in B\) such that the polynomial \(f(\psi \otimes 1) \in k[h + n]\) reaches maximal degree in the direction \(b_0 \cdot e \in n\) (such a \(b_0\) exists because \(k\) is infinite and \(B \cdot e\) is dense in \(n\)). Then for \(b \in B\) arbitrary, the degree of \(f(\psi \otimes 1)(h + tb \cdot e)\) is no larger than that of \(f(\psi \otimes 1)(h + b0 \cdot e)\). But since the latter is maximal, it is the same with the degree of

\[
f(\psi \otimes 1)(b_0 h + tb_0 \cdot e) = f(b_0^{-1}(\psi \otimes 1))(h + te) = f((\mu(b_0)\cdot b_0^{-1} \cdot \psi) \otimes 1)(h + te),
\]

which is \(\leq n\) by applying \((B)\) to \(\mu(b_0)^{-1}\cdot b_0^{-1} \cdot \psi \in V^*\). This finishes the proof of the claim. Using

\[
f(\psi \otimes 1)(h + te) = f(\psi \otimes 1)(\exp(te) \cdot h) = f(\exp(-te)(\psi \otimes 1))(h).
\]

and the claim, we have \(f \in \text{Hom}_B(V^* \otimes k_\mu, k[h + n]_n)\) if and only if for any \(\psi \in V^*\), the polynomial in \(t\)

\[
f(\exp(-te)(\psi \otimes 1))(h)
\]

has degree \(\leq n\). But the element in \((V^*)^*\) sending \(\psi \in V^*\) to \(f(\exp(-te)(\psi \otimes 1))(h) = f((\exp(-te)\psi) \otimes 1)(h)\) is just \(\exp(te)\Lambda(f) = \exp(te) \cdot v\). \(\square\)
References

[Bry89] Rance-Kathryn Brylinski, Limits of weight spaces, Lusztig’s q-analogs, and fiberings of adjoint orbits, J. Amer. Math. Soc. 2 (1989), 517–533.

[Gin95] Victor Ginzburg, Perverse sheaves on a loop group and Langlands’ duality, preprint arXiv:alg-geom/9511007 (1995).

[GR15] Victor Ginzburg and Simon Riche, Differential operators on G/U and the affine Grassmannian, Journal de l’Institut de Mathématiques de Jussieu 14 (2015), 493-575.

[JMWM] Daniel Juteau, Carl Mautner and Geordie Williamson, Parity sheaves and tilting modules, Ann. Sci. Éc. Norm. Supér. 49 (2016), 257–275.

[MR18] Carl Mautner and Simon Riche, Exotic tilting sheaves, parity sheaves on affine Grassmannians, and the Mirković-Vilonen conjecture. J. Eur. Math. Soc. (JEMS) 20 (2018), no. 9, 2259–2332.

[Jan03] Jens Carsten Jantzen, Representations of algebraic groups. Second edition. Mathematical Surveys and Monographs, 107. American Mathematical Society, Providence, RI, 2003. xiv+576 pp. ISBN: 0-8218-3527-0.

[Jan04] Jens Carsten Jantzen, Nilpotent orbits in representation theory. Lie theory, 1–211, Progr. Math., 228, Birkhäuser Boston, Boston, MA, 2004.

Sydney Mathematical Research Institute, University of Sydney, NSW 2006, Australia
E-mail address: Linyuan.Liu@normalesup.org