Neutron$-^{19}$C scattering near an Efimov state

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Abstract

The low-energy neutron$-^{19}$C scattering in a neutron-neutron-core model is studied with large scattering lengths near the conditions for the appearance of an Efimov state. We show that the real part of the elastic $s$–wave phase-shift ($\delta_R^0$) presents a zero, or a pole in $k\cot\delta_R^0$, when the system has an Efimov excited or virtual state. More precisely the pole scales with the energy of the Efimov state (bound or virtual). We perform calculations in the limit of large scattering lengths, disregarding the interaction range, within a renormalized zero-range approach using subtracted equations. It is also presented a brief discussion of these findings in the context of ultracold atom physics with tunable scattering lengths.

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The experimental observation of an Efimov resonant state [1] in an ultracold gas of Cesium with tunable interactions, performed by the Innsbruck group [2], evidenced the universal properties of large three-body quantum states [3–6]. The observations of [2] gave support to several theoretical analysis on the possibility to identify Efimov states considering precise determinations of three-body recombination rates [7]. They were also consistent with calculations of resonances for the atom-dimer channel [8] and for continuum triatomic Borromean configurations [9]. In this situation, the properties of

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these large systems are critically dependent on the physical scales of the two-body subsystems, fixed by the two-body scattering lengths and one three-body scale [10]. See also the discussions presented in Refs. [11,12] on the scales and the relation between Efimov effect and the Thomas collapse of the three-body ground-state energy [13].

The appearance of resonant three-body states in the maximally symmetric state is controlled by ratios of the two- and three-body scales [9]. For example, the freedom to change the two-body energy ($E_2$), as in ultracold trapped gases with tunable interactions, allowed the study of the behavior of an Efimov state energy ($E_3$) when $E_2$ is changed. For three identical bosons, $E_3$ follows the route virtual-bound-resonance for a large two-body scattering length varying from positive to negative values passing through the infinite (this corresponds to a change from a bound to a virtual state in the two-boson system) [14].

A bi-dimensional map in the parametric space can be defined in the Efimov limit by the critical conditions for an excited state in three-body systems with two-identical particles. The scattering lengths of the two-body subsystems and one three-body scale define the parametric space. The border of the map encloses a region where excited states do exist (see Refs. [3] and [15]). Crossing the critical boundary implies that an excited state becomes a virtual one, when at least one subsystem is bound [16], or a continuum resonance in the Borromean case. These qualitative features are found to be independent on the mass ratios [16].

The Borromean case of a three-boson continuum resonance was recently evidenced [2] in an experiment with trapped ultracold cesium gas near a Feshbach resonance. It is hoped that mixtures of different-mass atoms in traps with tunable interactions allow to check the transition from Borromean to non-Borromean situations where a continuum resonance becomes bound and turns to a virtual state by changing the sign of the scattering length.

In the nuclear context the universal physics associated with the Efimov phenomenon is hoped to be evident in light halo nuclei. Recently it was suggested that the $n-n-^{18}\text{C}$ ($n$ represents a neutron) system would exhibit low-energy resonant properties reminiscent of an Efimov state [17]. The study of the halo nucleus $^{20}\text{C}$ modeled as $n-n-^{18}\text{C}$ system with an $s$–wave short-range interaction between the pairs can be a playground to search for evidences of universal three-body physics. Using a zero-range interaction we have shown that this three-body halo system presents a virtual state that turns into an excited state when the $^{19}\text{C}$ binding is decreased [16]. Close to this condition, it is suggestive to study the low-energy $n-^{19}\text{C}$ elastic scattering in $s$–wave as the Efimov poles of the scattering amplitude are near the elastic scattering threshold.
Indeed, in the neutron-deuteron ($n - d$) doublet $s$--wave elastic scattering, the energy dependence of the phase-shift has to be changed from a simple effective range formula to a more complex one [18,19]. The change in the off-shell behavior of the two-body potential, or three-body forces, modifies the corresponding phase-shift correlated to the triton binding in a way that the scattering length can vanish. This was already seen in the well-known Phillips plot where the doublet scattering length is presented as a function of the triton binding energy [20]. In the case of zero scattering length, the function $k \cot \delta_0$ has a pole at zero relative $n - d$ kinetic energy, where $k$ is the on-energy-shell incoming and final relative momentum related to the energy of the three-body system. The necessity of a pole in $k \cot \delta_0$ has been pointed out in the analysis of the experimental data for the $n - d$ doublet $s$--wave phase-shift [19]. The analytical structure of $k \cot \delta_0$ in the $n - d$ elastic doublet state scattering has been also studied in several works [21–25]. Van Oers and Seagrace proposed to incorporate a pole in the phenomenological effective range formula used to fit the $k \cot \delta_0$ low energy data for the $n - d$ $s$--wave doublet state just below the elastic threshold. The effective range expansion has a form given by [19]

$$ k \cot \delta_0 = -A + Bk^2 - \frac{C}{1 + Dk^2}, \quad (1) $$

where $A$, $B$, $C$, and $D$ are fitted constants. Its position and residue have been calculated from dispersion relations as well as exact solutions of three particle equations with separable interactions by Whiting and Fuda [22]. Later on the existence of the triton virtual state was found on the basis of the effective range expansion [23]. After that, it was suggested from the solution of a three-body model with separable two-body interactions that the triton virtual state appears from an excited Efimov state moving to the non-physical energy sheet through the elastic cut [26]. The same phenomenon is found in the case of the excited Efimov state of the two-neutron halo of $^{20}\text{C}$, where the pole of the $S$-matrix migrates to the second energy sheet through the elastic $n-^{19}\text{C}$ cut when the binding energy of the neutron in $^{19}\text{C}$ is increased.

The understanding of the low-energy properties of $n - n - ^{18}\text{C}$ system also requires the study of the behavior of the scattering amplitude when an Efimov state is near the physical region. That state appears either as a bound excited state or as a virtual state depending on the low-energy parameters of the two-body subsystems as well the three-body scale (e.g., the ground state energy of $^{20}\text{C}$). As we have shown [16], the mass ratios are not relevant for the trajectory of an Efimov excited state turning into a virtual state with the increase of the $^{19}\text{C}$ binding energy, in correspondence to what has been found in the case of the trinucleon system [26]. Our next task is to find if the $n-^{19}\text{C}$ $s$--wave low-energy phase-shift exhibits the same analytical properties as found in the case of the $n - d$ doublet state. It is interesting to find if exists a low-energy pole in $k \cot \delta_0^R$ and to check if that property results to be independent on the
The mass ratios. This is suggested by the $n-^{19}C$ and $n-d$ similar cut structure of the scattering amplitudes, which is already implied by the very same behavior of the excited state turning a virtual Efimov state. We also note that the expression (1) was used to fit the real part of the phase-shift.

To support the possible qualitative and universal common behavior of the elastic $s$-wave scattering amplitudes of $n-^{19}C$ and the $n-d$ doublet state, let us discuss the example of the atom-dimer scattering length as a function of the atom-atom scattering length $a$ and a three-body scale, namely $\Lambda_*$, as considered in Ref. [6]. In the case of three bosons the atom-dimer scattering length is approximately given by

$$a_{AD} = (1.46 - 2.15 \tan[s_0 \ln(a\Lambda_*) + 0.09])a,$$

where $s_0 = 1.00624$. The atom-dimer scattering length is a solution of a one-body problem in the attractive long-range $\rho^{-2}$ potential responsible for the Efimov states. The dimensionless product $a\Lambda^*$ controls the number of three-body bound states, with $a\Lambda^* \rightarrow \infty$ implying in an infinite number of Efimov states. The above limit can be realized either by $a \rightarrow \infty$ for a fixed $\Lambda^*$ or $\Lambda^* \rightarrow \infty$ for a fixed $a$. Efimov states correspond to the first case, whereas the Thomas collapsed states [13] appear in the second case. So, the formation of bound states depending on the dimensionless product $a\Lambda^*$ relates the Thomas collapse and Efimov effect [11]. The expression (2) allows a zero scattering length or a pole of the effective range expansion at zero energy at a value $a = a_0$ given by $a_0 = 1.6543/\Lambda_*$. It is worthwhile to stress that (2) also allows $a_{AD} \rightarrow \pm \infty$, or an Efimov state at the threshold, for a value of the atom-atom scattering length $a = a_B = 4.3563/\Lambda_*$, obtained from $s_0\ln(a_B\Lambda_*) + 0.09 = \pi/2$. So, by changing the two-body scattering length from $a = a_0$ to $a = a_B$, a bound Efimov state is produced at the threshold. In order to change $a_{AD}$ from zero to an infinite value the ratio of atom-atom scattering lengths can be estimated from the above as given by

$$\frac{a_B}{a_0} = \exp\left(\frac{\pi/2 - 0.59654}{s_0}\right) \approx 2.633.$$ 

Therefore, from our qualitative discussion we observe that the physics related to the Efimov effect is also implying a zero in the atom-dimer scattering length, and consequently a pole in the effective range expansion at zero kinetic energy. Our next calculations will substantiate that a pole in the effective range expansion of the $n-^{19}C$ elastic phase-shift appears in a quite good qualitative agreement with the above. In such a case, the three-body system has unequal mass particles, with $m_1 = m_2$ (neutron mass) and $m_3/m_1 = A = 18$, implying in a corresponding variation of $s_0$. Extracted from Fig. 52 of [6], one obtains $s_0 \sim 1.12$ for $A = 18$, such that $a_0/a_B \sim 0.419$. This result will be shown to
be quite close to our numerical results.

The aim of this work is to study the low-energy $s$-wave $^{n-19}$C scattering in a neutron-neutron-core model near the critical condition for the appearance of an excited Efimov state. In this situation it is reasonable to use an $s$-wave zero-range interaction between the pairs, with a three-body scale tuned to the separation energy of the two halo neutrons of $^{20}$C. The three-body scale is introduced through a subtraction in the kernel of the integral equations [12]. Alternatively, another subtraction scheme also allows the introduction of the physical information of the three-body system at the elastic scattering threshold, see e.g. [27]. We will present results for the $s$-wave phase-shift ($\delta_0$) of the $n-^{19}$C elastic scattering for a fixed $^{20}$C energy and singlet scattering length, for different values of the neutron separation energy in $^{19}$C, from 200 up to 850 keV. That range is chosen to study how $\delta_0$ behaves as the excited Efimov state moves into the unphysical energy sheet turning to a virtual state. We will show that the real part of $\delta_0$ presents a zero, or a pole in $k \cot \delta_0$. That pole scales with the energy of the Efimov state (bound or virtual). Our goal is to study the low-energy phase-shift in the limit of large scattering lengths where corrections due to the interaction range are not relevant for the qualitative properties, dominated by a universal scaling behavior (see e.g. [5] and [6]). We will also present a brief discussion on how to apply our findings, through universality, to the context of trapped ultracold atomic systems with different species near a Feshbach resonance.

Next, we present the basic formalism to calculate the $n-^{19}$C elastic scattering amplitude. The input of our subtracted scattering equations are the $n-n$ virtual state energy that will be fixed at $E_{nn} = -143$ keV, the energy of the halo neutron bound in $^{19}$C, $E_{nc}$, and the $^{20}$C that has a ground state energy of $-3.5$ MeV. The neutron separation energy in $^{19}$C has a sizable error with value of $-160\pm110$ keV [28] and $-530\pm130$ keV from Ref. [29]. In our study we will allow wide variations in the $n-^{18}$C energy in a range between 200 up to 850 keV to include the experimental values.

Our units are such that $\hbar = m_n = 1$, where $m_n$ is the mass of the neutron, $m_c = Am_n$ is the core mass ($A$ = the core mass-number) and $\mu_{nc} = Am_n/(A+1)$ is the reduced mass of the $n-c$ system. In the present case, the core is identified with $^{18}$C. In the following formalism, the energies for the two-body ($E_{nn}$ and $E_{nc}$) and three-body ($E_3 \equiv E$) systems are, respectively, redefined to $\varepsilon_{nn}$, $\varepsilon_{nc}$, and $\mathcal{E}$, with the conversion unit factor, $\hbar^2/m_n = 41.47$ MeV fm$^2$, being the same for all the energies: $E_{nn(nc)} = (\hbar^2/m_n)\varepsilon_{nn(nc)}$, and $E_3 \equiv E = (\hbar^2/m_n)\mathcal{E}$. The partial-wave elastic $n-n$ ($nc$) scattering equation has been already discussed in detail in Ref. [16]. To make clear to the reader the main ingredients of the model, we briefly review the formalism in the following. The spectator function $\chi_n(\vec{q})$, which represents the relative motion between the neutron and $^{19}$C target brings the boundary condition of the
elastic scattering as:

\[ \chi_n(q) \equiv (2\pi)^3 \delta(q - \vec{k}_i) + 4\pi \frac{h_n(q; \mathcal{E}(k_i))}{q^2 - k_i^2 - i\epsilon}, \]  

where \( h_n(q; \mathcal{E}(k_i)) \) is the scattering amplitude, and \( q \) is the momentum of the spectator particle \((n)\) with respect to the center-of-mass (CM) of the other two particles \((n - c)\). The on-energy-shell incoming and final relative momentum are related to the three-body energy \( \mathcal{E} \equiv \mathcal{E}_i \equiv \mathcal{E}(k_i) \) by

\[ k_i = |\vec{k}_i| = |\vec{k}_f| = \sqrt{\frac{2(A + 1)}{A + 2}(\mathcal{E}_i - \epsilon_{nc})}. \]

The coupled partial-wave scattering equations can be cast in a single channel Lippmann-Schwinger-type from the relevant amplitude \( h_n^\ell \):

\[ h_n^\ell(q; \mathcal{E}) = \mathcal{V}^\ell(q, k; \mathcal{E}) + \frac{2}{\pi} \int_0^\infty dp^2 \frac{\mathcal{V}^\ell(q, p; \mathcal{E}) h_n^\ell(p; \mathcal{E})}{p^2 - k^2 - i\epsilon}. \]  

The kernel of the integral equation for the \( (n - nc) \) channel amplitude contains the contribution of the coupled \((nn) - c \) channel as given by the integration seen in the expression below:

\[ \mathcal{V}^\ell(q, p; \mathcal{E}) \equiv \pi \frac{(A + 1)}{A + 2} \bar{\tau}_{nc}(q; \mathcal{E}) \times \]

\[ \times \left[ K^\ell_2(q, p; \mathcal{E}) + \int_0^\infty dp' p'^2 K^\ell_1(q, p'; \mathcal{E}) \tau_{mn}(p'; \mathcal{E}) K^\ell_1(p, p'; \mathcal{E}) \right], \]

where

\[ \tau_{mn}(q; \mathcal{E}) \equiv -\frac{2}{\pi} \left[ \sqrt{|\epsilon_{mn}|} + \sqrt{\frac{A + 2}{4A} q^2 - \mathcal{E}} \right], \]

\[ \bar{\tau}_{nc}(q; \mathcal{E}) \equiv -\frac{1}{\pi} \left( \frac{A + 1}{2A} \right)^{\frac{3}{2}} \left[ \sqrt{|\epsilon_{nc}|} + \sqrt{\frac{(A + 2)q^2}{2(A + 1)} - \mathcal{E}} \right], \]

\[ K^\ell_{i=1,2}(q, p; \mathcal{E}) \equiv G^\ell_{i}(q, p; \mathcal{E}) - \delta_{0\ell} G^\ell_{i}(q, p; -\mu^2), \]

\[ G^\ell_{i}(q, p; \mathcal{E}) = \int_{-1}^1 dy \frac{P_{\ell}(y)}{\mathcal{E} - \frac{A+1}{A+1-i}q^2 - \frac{A+1}{2A}p^2 - A^{1-i}pqy + i\epsilon}. \]

The Kronecker delta \( \delta_{0\ell} \) \((= 1 \text{ for } \ell = 0 \text{ and } =0 \text{ for } \ell \neq 0)\) is introduced in order to allow for finite results, using a subtracted kernel, only for \( \ell = 0 \).
where such regularization is necessary. The Thomas collapse is absent for \( \ell > 0 \), due to the centrifugal barrier, and such procedure is not necessary. The subtraction energy \(-\mu^2\) allows to fix the two-neutron separation energy of \(^{20}\text{C}\) to its physical value giving our renormalization procedure.

In this way, the Eq. (6) is renormalized and the three-body observables are completely defined by the two-body energy scales, \( \varepsilon_{nc} \) and \( \varepsilon_{nn} \), and the energy of \(^{20}\text{C}\). The regularization scale \( \mu^2 \), used in the \( s \)–wave [see Eq. (9)], is chosen to reproduce the three-body ground-state energy of \(^{20}\text{C}\) with a value of \(-3.5 \text{ MeV} \) [28]. The scaling function for \( s \)–wave observables has a limit cycle [6,30] evidenced when \( \mu \) is let to be infinity. We point out that a good description of this limit is already reached in the first cycle [31].

The numerical method to calculate the elastic scattering amplitude relies on the use of an auxiliary function [32], which is a solution of an integral equation similar to the original one, but with a kernel without the unitarity cut, due to a subtraction procedure at the momentum \( k \). The scattering amplitude is obtained by evaluating certain integrals over the auxiliary function, when the two-body unitarity cut is introduced back. In the present case, we have the following integral equation for the auxiliary function \( \Gamma \), and the corresponding solution for \( h^\ell_n(q; \mathcal{E}) \):

\[
\Gamma^\ell_n(q, k; \mathcal{E}) = \mathcal{V}^\ell(q, k; \mathcal{E}) + \frac{2}{\pi} \int_0^\infty dp \left[ p^2 \mathcal{V}^\ell(q, p; \mathcal{E}) - k^2 \mathcal{V}^\ell(q, k; \mathcal{E}) \right] \frac{\Gamma^\ell_n(p, k; \mathcal{E})}{p^2 - k^2},
\]

\[
h^\ell_n(q; \mathcal{E}) = \frac{\Gamma^\ell_n(q, k; \mathcal{E})}{1 - \frac{2}{\pi} \int_0^\infty dp \left( \frac{\Gamma^\ell_n(p, k; \mathcal{E})}{p^2 - k^2 - i\epsilon} \right)}. \tag{11}
\]

For the on-shell scattering amplitude, we have

\[
h^\ell_n(k; \mathcal{E}) = [k \cot \delta_\ell - ik]^{-1}, \tag{12}
\]

such that, from Eq. (11), we have

\[
k \cot \delta_\ell = \frac{1}{\Gamma^\ell_n(k, k; \mathcal{E})} \left[ 1 - \frac{2}{\pi} k^2 \int_0^\infty dp \left( \frac{\Gamma^\ell_n(p, k; \mathcal{E}) - \Gamma^\ell_n(k, k; \mathcal{E})}{p^2 - k^2} \right) \right].
\]

Note that the numerical stability and accuracy of the results is delicate when an Efimov state is near the scattering region. In this situation the method from Ref. [32] as outlined above, is far much accurate than the use of contour deformation technique directly in Eq. (6).
To guide the discussion of the results of $k \cot \delta^R_0$ for the $n-^{19}C$ elastic scattering from the numerical solution of Eqs. (11), in analogy with the form given by Eq. (1), we consider an equivalent low energy parametrization of the effective range expansion in terms of the kinetic energy $E_K$:

$$k \cot \delta^R_0 = \frac{-a_{n-^{19}C}^{-1} + \beta E_K + \gamma E_K^2}{1 - E_K/E_0},$$

(13)

where $a_{n-^{19}C}$ is the $n-^{19}C$ scattering length, with $\beta$ and $\gamma$ the effective range parameters to be adjusted. $E_0$ is the position of the pole with respect to the threshold for elastic scattering.

The numerical solution of Eq. (11) gives a zero in $\delta^R_0$, as anticipated by our discussion. In order to present smooth curves, in Fig. 1 we show the results for $(1 - E_K/E_0)k \cot \delta^R_0$ in terms of the CM kinetic energy $E_K$. The calculations were performed for different values of $|E_{^{19}C}|$ below 850 keV. For $|E_{^{19}C}| > 170$ keV, the $^{20}C$ presents an Efimov virtual state [16], which can be seen in Fig. 2, where it is plotted the $^{20}C$ energies against the inverse of the $n-^{18}C$ scattering lengths, $a_{n-^{18}C}$. Below this value, the excited Efimov state turns out to be bound. The given positive $a_{n-^{18}C}$ are related to the $^{19}C$ binding energies by $E_{^{19}C} = [\hbar^2/(2\mu_{n-^{18}C})](1/a_{n-^{18}C}^2)$, where $\mu_{n-^{18}C}$ is the reduced mass.
for the system $n-^{18}\text{C}$, such that $(1/a_{n-^{18}\text{C}}) \approx 0.00676\sqrt{E_{^{19}\text{C}}/\text{keV fm}^{-1}}$. The threshold for $E_{^{20}\text{C}}$ is given by the dashed line. The non-allowed area is the upper-right part of the figure (shadowed region). The arrow marks approximately the point where an excited $E_{^{20}\text{C}}$ state (on the left) becomes a virtual state (on the right). In table 1 we give the effective range parameters from a fit of (13) to the results shown in Fig. 1. The $n-^{19}\text{C}$ scattering length is indeed negative as the nearby Efimov state is virtual, in agreement with the findings of Ref. [16]. The Efimov excited state moves from a bound state (first

| $|E_{^{19}\text{C}}|$(keV) | $(a_{n-^{19}\text{C}})^{-1}$(fm$^{-1}$) | $\beta$(fm.keV)$^{-1}$ | $\gamma$(fm.keV$^2$)$^{-1}$ | $E_0$(keV) |
|---|---|---|---|---|
| 200 | -0.591 $10^{-2}$ | 5.685 $10^{-4}$ | 4.673 $10^{-8}$ | 1442.7 |
| 400 | -0.624 $10^{-1}$ | 6.743 $10^{-4}$ | 8.821 $10^{-8}$ | 823.9 |
| 600 | -2.118 $10^{-1}$ | 9.337 $10^{-4}$ | 1.464 $10^{-7}$ | 451.4 |
| 800 | -1.268 | 3.110 $10^{-3}$ | 4.424 $10^{-7}$ | 115.0 |
| 850 | -5.510 | 1.201 $10^{-2}$ | 1.641 $10^{-6}$ | 28.8 |
energy sheet - left-hand side of the arrow of Fig. 2) to a virtual state (second energy sheet - right-hand side of the arrow of Fig. 2) through the elastic analytical cut, for $|E_{19C}|$ approximately 170 keV ($a_{n-18C} \approx 11.347$ fm). In fact, for 200 keV, the scattering length is very large but still negative indicating the presence of a nearby pole of the S-matrix in the second energy sheet. We also observe the increase of the effective range parameters with the $^{19}$C energy. With the $n-n$ energy fixed to $E_{nn} = -143$ keV and considering the experimental results for the neutron separation energy from $^{19}$C ($-160\pm110$ in [28] and $-530\pm130$ keV in [29]), the existence of excited bound Efimov states in $n-n-^{18}$C is quite doubtful. In order to have a bound excited state, in the present zero-range approach, we need $|E_{19C}| < 170$ keV, which is excluded in [29]. Only the lower limits (absolute values) given in [28] are allowing such possibility. However, we should observe that range corrections will increase the allowed values of $|E_{19C}|$ for a bound $E_{20C}$ excited state [33].

![Fig. 3. Position of the pole in $k \cot \delta_0^R$](image)

The position of the pole in $k \cot \delta_0^R$ is also shown in Fig. 3 as a function of $|E_{19C}|$ (l.h.s.) and the energy of the virtual $^{20}$C state (r.h.s.), respectively. Moving towards small values of $|E_{19C}|$ the pole moves to larger energies. While the increase of $|E_{19C}|$ makes the scattering length goes towards zero somewhat above 850 keV, as indicated by the l.h.s. of Fig. 3. The $s$–wave elastic cross-sections is damped at low-energies for parameters around these values. For $|E_{19C}|$ above 500 keV, the pole is below the breakup threshold and the corresponding $s$–wave cross-section presents a zero. For $|E_{19C}|$ below 500 keV, the pole is above the breakup threshold where absorption occurs, and consequently the $s$–wave cross-section has a minimum, not a zero, at the position of the pole. Clearly, as the Efimov state moves deeper (see r.h.s. of Fig. 3) in the second energy sheet, the pole tends to zero energy. This behavior can be easily verified by using the effective range expansion (13) and calculating the pole of the scattering amplitude for the virtual state energy (see e.g.
As shown by our above numerical results, we observe that in order to go from a zero in the scattering amplitude to the situation where one virtual Efimov state becomes bound, the ratio of the two-body scattering lengths is given by $\sqrt{\frac{170}{850}} \simeq 0.45$, which is quite close to our previous estimative for $A = 18$ ($a_0/a_B \simeq 0.42$), as discussed after Eq. (3). Although the estimative is made using a qualitative approach, it is indicative of the relation between the physics of the Efimov effect and the zero in the scattering amplitude. We should remark that a zero in the scattering length does not imply, in general, the Efimov physics, while the opposite is true, as we have substantiated by the qualitative and quantitative calculations of the $n-^{19}$C elastic phase-shift.

![Fig. 4. s–wave absorption parameter as a function of the CM kinetic energy $E_K$. From left to right the curves correspond to the following $^{19}$C energies: 200, 400, 600, 800 and 850 keV.](image)

The absorption parameter $\eta = |e^{i\delta_0}|$ is shown in Fig. 4. It changes strongly with the binding energy of $^{19}$C. We observe that the absorption increases naturally with the size of $^{19}$C, as we see in the weakly bound case of 200 keV.

In conclusion, we studied the elastic scattering of a neutron on $^{19}$C near the condition for an excited Efimov state of $^{20}$C. In our work the energy of the ground state of $^{20}$C and the neutron-neutron scattering length were fixed to the experimental values. We allowed a change in the absolute value of the neutron binding energy of $^{19}$C up to 850 keV, in order to include the experimental range of values and seek for the interesting Efimov related physics in the $s$–wave elastic scattering amplitude. The effective range expansion of the
$s-$wave phase-shift presents a low-energy pole, that moves toward zero energy as the binding of the neutron in $^{19}$C increases and the virtual Efimov state goes deeper in the second energy sheet. The parametrization of the phase-shift was done by a simple analytical formula of the effective range expansion with a pole, proposed long ago by Van Oers and Seagrave[19] to fit the low-energy experimental data of the doublet $s$-wave neutron-deuteron phase-shift. The three-body $n-n-^{18}$C system shares with the trinucleon system the large subsystem scattering lengths, the small binding energy of the ground state and the proximity of a virtual Efimov state to the scattering region. These common features implies that both systems show analogous universal behaviors [4,34], having qualitatively properties independent on the mass ratios and described within zero range models. Our study extends to the $n-^{19}$C elastic scattering the general properties found in the neutron-deuteron system, in particular the pole in the effective range expansion. Actually, it should be of interest to extend the present analysis of the low-energy elastic scattering to other possible two-body configurations, with different mass relations within the three particle system. Our results, obtained by using a renormalized zero-range approach, which is valid in the limit of large two-particle scattering length, correspond essentially to the dominance of the long range $\rho^{-2}$ interaction, responsible for the Efimov effect, Thomas collapse, as well as the zero in the $n-^{19}$C scattering length. It is expected that a more realistic approach for the neutron-core interaction [33], due to range effects, will increase the possible region of $^{19}$C binding energies for the existence of an excited bound state (moving the arrow mark in the Fig. 2 to the right).

Loosely bound neutron-rich nuclei near the drip-line, such as the carbon and oxygen isotopes with two-neutron halos, are actually the more promising nuclear systems that are being intensively investigated [35] in order to reach a better understanding on the nuclear forces properties and interactions. From another side, the experiments with loosely bound two-neutron halo nuclei can also give relevant informations on the few-body scales and universality. In this respect, as discussed in Ref. [36] for carbon isotopes, the three-body $n-n-^{20}$C system, is probably a more favorable system to study low-energy three-body properties, considering that the two-neutron binding energy in $^{22}$C is close to zero, implying in a very large neutron halo. This is a Borromean case with large possibilities for Efimov resonant states. Another relevant source of information on universality and the corresponding dominance of few scales at very low-energy actually can also be obtained from ultracold atomic physics experiments. The possibility of different two-body scattering lengths can be realized in trapped ultracold atomic systems near a Feshbach resonance. In this situation the environment of ultracold atomic traps allows to follow the scaling of three-body observables with two-body scattering lengths. The atom-dimer scattering length ($a_{AD}$) can change from large and positive to negative and then to zero moving the two-atom scattering length (c.f. Eq. (2)), allowing to control the effective atom-dimer interaction near the Efimov limit. Conse-
quently, the stability of the atom-dimer condensate should be sensitive to an Efimov state that passes from bound to virtual, i.e., the effective interaction proportional to $a_{AD}$ changes from strongly repulsive to attractive, and then to zero by lowering the value of $a$. In particular, when $a_{AD} = 0$, the atom and dimer condensates are invisible to each other, i.e., they decouple. It is conceivable that in an experiment would be possible to dial the coupling between the atom and dimer condensates, and thus controlling the phases of the condensed gas mixture. Interesting phenomena in the condensate due to Efimov states near the scattering threshold have already been discussed in Refs. [37]. The mixed atom-dimer phases near the Efimov limit add to the rich physics of trapped atoms more possibilities to be tested experimentally. In fact, the scaling laws of few-body observables near the Efimov limit is under active experimental investigation, even in more complex situations like that of the four-boson system [38], where the quest for new scales is under debate [39,40]. The recent measurements of the dimer-dimer recombination in ultracold traps near the Feshbach resonance [38] and Efimov limit, asks for further theoretical analysis in order to extend the concepts used here at the three-body level to the four-boson scattering.

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