Rigid $\aleph_\varepsilon$-saturated models of superstable theories

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Abstract

In a countable superstable NDOP theory, the existence of a rigid $\aleph_\varepsilon$-saturated model implies the existence of $2^\lambda$ rigid $\aleph_\varepsilon$-saturated models of power $\lambda$ for every $\lambda > 2^{\aleph_0}$.

1 Introduction

Ehrenfuecht conjectured that given a theory $T$, the class of cardinals for which $T$ has a rigid model is quite well behaved. Shelah refuted Ehrenfuecht’s conjecture showing that this class can be quite complicated.

In this paper we deal with problems related to this question in the context of stability. More specifically, we will study the existence of stable rigid models which satisfy an additional saturation property (note that if no saturation property is required, then a very simple example of a stable rigid model can be found, namely the model whose language is $\{ P_n \mid n < \omega \}$ and consists of the disjoint union of the $P_n$-s, each of which has exactly one element.)

We will give a partial solution to the following questions:

1) What classes of superstable theories have a rigid $\aleph_\varepsilon$-saturated model?

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2) Assuming that there exits a rigid \( \aleph_\bar{\epsilon} \)-saturated model, what can be said about the number of \( \aleph_\bar{\epsilon} \)-saturated models, or perhaps even about the number of rigid \( \aleph_\epsilon \)-saturated models, in large enough cardinality?

In section 3 we consider two properties of a superstable theory \( T \):

1) \( T \) is strongly deep.

2) \( T \) does not admit a nontrivial nonorthogonal automorphism of some saturated model.

We prove that 1) is a necessary condition for the existence of a rigid \( \aleph_\epsilon \)-saturated model, and that 2) is a sufficient condition.

In section 4, we assume that \( T \) is a superstable NDOP theory. We prove that 2) is actually equivalent to the existence of a rigid \( \aleph_\epsilon \)-saturated model.

We can then conclude that the existence of a single rigid \( \aleph_\epsilon \)-saturated model implies that \( T \) has \( 2^\lambda \) such models for every sufficiently large \( \lambda \).

In this paper, the notations will be very similar to Shelah’s notations in [Sh-C]. \( T \) will denote any complete stable theory with no finite models in some language \( L \). \( \kappa, \lambda, \mu \) will denote cardinals.

We work in some huge saturated model \( \mathcal{M} \). Sets \( A, B, C, \ldots \) will be subsets of \( \mathcal{M} \), with cardinality strictly less than the cardinality of \( \mathcal{M} \). \( \bar{a}, \bar{b}, \bar{c}, \ldots \) will denote finite tuples of \( \mathcal{M} \). \( M, N \) will always be elementary submodels of \( \mathcal{M} \). \( p, q, r \) will denote types, usually complete over some set \( A \). \( S(A) \) will denote the set of complete types over \( A \). Also, for a tuple \( \bar{b} \) and a set \( A \), \( \bar{b}/A \) denote the type of \( \bar{b} \) over \( A \).

## 2 Building a dimensionally diverse \( \aleph_\epsilon \)-saturated model

In this section, \( T \) denotes any stable theory. We give a brief outline of some standard constructions following [Sh-C]. \( \kappa(T) \) herein will denote the smallest infinite cardinal \( \kappa \) such that there is no chain \( \{p_\alpha \in S(A_\alpha) : \alpha < \kappa\} \) such that for all \( \alpha < \beta < \kappa \), \( p_\beta \) is a forking extension of \( p_\alpha \). Recall that \( M \) is \( F_\mu^\alpha \)-saturated if for every \( A \subseteq M \) such that \( |A| < \mu \), every strong type over \( A \) is realized in \( M \). \( M \) is \( F_\mu^\alpha \)-prime over \( A \) if \( M \) is \( F_\mu^\alpha \)-saturated and for every \( F_\mu^\alpha \)-saturated model \( N \) such that \( A \subseteq N \) there is an elementary embedding of \( M \) into \( N \) over \( A \). We say that \( M \) is \( \aleph_\epsilon \)-saturated (\( \aleph_\epsilon \)-prime over \( A \)) if \( M \) is \( \aleph_\omega^\alpha \)-saturated (\( \aleph_\omega^\alpha \)-prime over \( A \)).
Definition 1 We say that an $\aleph_\alpha$-saturated model $M$ (of a superstable $T$) is *dimensionally diverse*, if for any two stationary regular types $p, q$ over finite subsets of $M$, $p \perp q$ if and only if $\dim(p, M) \neq \dim(q, M)$.

We also recall the following standard definition:

Definition 2 For superstable $T$, we say that $T$ is *multidimensional* if for every cardinal $\alpha$, there are nonalgebraic $p_i, i < \alpha$ which are pairwise orthogonal.

Our aim in this section is to prove the following standard theorem.

**Theorem 1** Let $T$ be superstable, and let $\mu, \delta$ be cardinals such that $\aleph_\delta = \delta$, $\mu < \delta$ and $2^{[T]} < \delta$, then
1) There exists a $\mu$-saturated model $M$ of cardinality $\delta$ which is dimensionally diverse (in particular if $\mu \geq \aleph_1$, $M$ is $\aleph_\epsilon$-saturated).
2) If $T$ is multidimensional, then for every increasing sequence of cardinals $\bar{\mu} = \langle \mu_\alpha | \alpha < \delta \rangle$, $\mu_\alpha \in (\mu, \delta)$ there exists a $\mu$-saturated and $\aleph_\epsilon$-saturated model $M$ of cardinality $\delta$ which is dimensionally diverse, and such that for every $\alpha < \delta$ there is a stationary and regular (=s.r.) type $p_\alpha$ (over a finite subset of $M$,) such that for every s.r. type $\bar{p}$, $\dim(\bar{p}, M) = \mu_\alpha$ if and only if $\bar{p} \not\perp p_\alpha$. Moreover, for every s.r. type $\bar{p}$ (over a finite subset of $M$) $\dim(\bar{p}, M) = \mu_\beta$ for some $\beta < \delta$.

**Fact 1** Let $M = \bigcup_{i<\alpha} M_i$, where $\langle M_i | i < \alpha \rangle$ is an increasing and continuous sequence of $\kappa$-saturated models, let $A \subseteq M_0$ such that $|A| < \kappa$, and let $p \in S(A)$ be a stationary regular type. Then $\dim(p, M) = \dim(p, M_0) + \sum_{i<\alpha} \dim(p_i, M_{i+1})$, where $p_i$ is the stationarization of $p$ to $M_i$.

**Fact 2** If $cf(\lambda) \geq \kappa(T)$, $M$ is $F^\kappa_\lambda$-prime over $A$, and $I \subseteq M$ is an indiscernible sequence over $A$, then $|I| \leq \lambda$.

Following are two claims which we will use in our proof of Theorem 1 (although weaker versions thereof would suffice.)
**Claim 1** Suppose $\text{cf}(\mu) \geq \kappa = \kappa(T)$ and $M \models T$ is $F_\kappa^\mu$-saturated. Let $A \subseteq M$ such that $|A| \leq \mu$, and suppose $M^+$ is $F_\mu^\alpha$-prime over $M \cup A$. In addition, let $B \subseteq M^+$ such that $|B| < \kappa$, and suppose $p \in S(B)$ is stationary and $\lambda = \dim(p, M^+) > \mu$. Then $p \not\perp M$. Moreover, if $p$ is regular then there is a stationary regular type $q \in S(B^*)$, where $B^* \subseteq M$ and $|B^*| < \kappa$ such that $\dim(q, M) = \lambda$ and $q \not\perp p$.

**Proof of Claim 1** Left to the reader.

**Claim 2** Suppose $\text{cf}(\mu) \geq \kappa = \kappa(T)$ and $M \models T$ is $F_\kappa^\mu$-saturated. Let $p_i \in S(M)$, $i < \alpha$ be pairwise orthogonal and let $E = \bigcup_{i<\alpha} E_i$ where $E_i$ is a Morley sequence of $p_i$. In addition, suppose $N$ is $F_\mu^\alpha$-prime over $M \cup E$, and let $B \subseteq N$ such that $|B| < \kappa$. If $q \in S(B)$ is stationary and regular and $\dim(q, N) > \mu$, then $q \not\perp M$.

**Proof of Claim 2** Assume, for a contradiction, that $q \perp M$. Let $M^+$ be $F_\mu^\alpha$-prime over $M \cup B$ with $M^+ < N$, and let $\bar{q} \in S(M^+)$ be the stationarization of $q$. Then either $\dim(q, M^+) > \mu$ or $\dim(\bar{q}, N) > \mu$, by Fact 1. Now, there exists $S \subseteq \alpha$ with $|S| < \kappa$, such that $p_i \perp tp(M^+/M)$ for all $i \in \alpha \setminus S$. Thus $\dim(q, M^+) > \mu$ contradicts Claim 1, and $\dim(\bar{q}, N) > \mu$ contradicts the above and Fact 2.

**Fact 3** If $\text{cf}(\delta) \geq \kappa(T)$, and $(M_i|i < \delta)$ is an elementary chain of $\lambda$-saturated models, then $M^* = \bigcup_{i<\delta} M_i$ is $\lambda$-saturated. In particular, if $T$ is superstable, then the union of any elementary chain of $\lambda$-saturated models is $\lambda$-saturated.

Theorem 1 easily follows from the above:

**Proof of Theorem 1** Sketch of proof: Define by induction an increasing elementary chain $(M_\alpha|\alpha < \delta)$ of $\mu$-saturated models. At the $\alpha^{th}$ step, choose a non-algebraic stationary type $p_\alpha$ (over a finite set) which is orthogonal to $p_\beta$ for all $\beta < \alpha$, and define $M_\alpha$ to be $\mu$-prime over $\bigcup_{\beta < \alpha} M_\beta \cup \text{Dom}(p_\alpha) \cup I_\alpha$, where $I_\alpha$ is a Morley sequence of $p_\alpha$ with cardinality $\mu_\alpha$. Now let $M^* = \bigcup_{\alpha < \delta} M_\alpha$. From Fact 1, Claim 1 and Claim 2 it follows that $M^*$ realizes the desired dimensions ($\mu_\alpha$), and by Fact 3, $M^*$ is $\mu$-saturated.
The existence of a rigid $\aleph_\epsilon$-saturated model.

In this section $T$ is assumed to be superstable. We say that a superstable theory is strongly deep if the depth of every type is positive (if and only if the depth of every type is infinity). We prove that whenever $T$ has a rigid $\aleph_\epsilon$-saturated model, $T$ is strongly deep. We also introduce the notion of a nontrivial nonorthogonal automorphism and prove that if some saturated model does not have such an automorphism, then in arbitrarily large cardinality, $T$ has a maximal number of rigid $\aleph_\epsilon$-saturated models. (i.e. $2^\lambda$ such models in cardinality $\lambda$.)

Definition 3 We say that $T$ is strongly deep if for every $\aleph_\epsilon$-saturated model $M$, and for every (without loss of generality regular) type $p \in S(M)$, if $N$ is $\aleph_\epsilon$-prime over $M \cup \{a\}$ where $a \models p$, then $q \perp M$ for some $q \in S(N)$.

Lemma 1 Let $N_0$ be $\aleph_\epsilon$-saturated, let $p \in S(N_0)$ be regular, and let $\langle \bar{e}_i | i < \alpha \rangle$ be a Morley sequence of $p$. Suppose $N^+$ is $\aleph_\epsilon$-prime over $N_0 \cup \bigcup_{i<\alpha} \bar{e}_i$. Then the following are equivalent:

(i) There is $p_1 \in S(N_1)$ such that $p_1 \perp N_0$, where $N_1$ is $\aleph_\epsilon$-prime over $N_0 \cup \bar{e}_0$.
(ii) There is $p^+ \in S(N^+)$ such that $p^+ \perp N_0$.

Proof of Lemma 1 (i)$\rightarrow$(ii) We may assume that $N_1 \prec N^+$. If there is such $p_1$, choose $p^+$ to be the nonforking extension of $p_1$ to $N^+$.

(ii)$\rightarrow$(i) Let us assume that such a $p^+$ is given, then $p^+$ is strongly based on some finite subset $A$ of $N^+$. Therefore $tp(A/N_0 \cup \bigcup_{i<\alpha} \bar{e}_i)$ is $F_{\aleph_0}$-isolated, and we may assume $\alpha = n < \omega$. Now, if every $p \in S(N_1)$ is nonorthogonal to $N_0$, then by induction on $n$ we have $p^+ \not\perp N_0$ (recall that the depth of parallel types is the same) which is a contradiction.

Theorem 2 If $T$ has an $\aleph_\epsilon$-saturated model $N$ such that $|Aut(N)| < 2^{\aleph_0}$, then $T$ is strongly deep.
Proof of Theorem 2 Suppose not, then there is some depth 0 type p. Let $N_0 \prec N$ be $\aleph_\epsilon$-prime over $\emptyset$ (without loss of generality $p \in S(N_0)$). Let $I \subseteq N$ be a maximal Morley sequence of p (without loss of generality I is infinite,) and let M be a maximal model such that $N_0 \prec M \prec N$ and $M \not\equiv N_0$ (so M is $\aleph_\epsilon$-saturated). We claim that N is $\aleph_\epsilon$-minimal over $M \cup I$.

Otherwise, let $P \prec N$ be $\aleph_\epsilon$-prime over $M \cup I$, so for some $\bar{b} \in N$, $\bar{b}/P$ is a non-algebraic regular type.

Claim 1 $\bar{b}/P \perp N_0$.

Indeed, if not then (by [Sh-C]) there is $\bar{c} \in N$ such that $\bar{c}/P$ is regular, $\bar{c}/P \not\equiv \bar{b}/P$ and $\bar{c}/P \not\equiv \bar{b}/P$. Thus $\{\bar{c}, M, I\}$ is independent over $N_0$, in contradiction to the definition of M.

Claim 2 For all $\bar{d} \in N$ if $\bar{d}/M \perp N_0$ then $\bar{d} \in M$.

Indeed, let $\bar{d}$ satisfy the above, so $\bar{d} \equiv M$ and $\bar{d} \equiv N_0$. Therefore $M \cup \bar{d} \equiv N_0$.

so by the maximality of M $\bar{d} \in M$.

Claim 3 $\bar{b}/P \perp M$.

Otherwise $\bar{b}/P \not\equiv M$, so there is $\bar{c} \in N$ with $\bar{c}/P$ regular such that $\bar{c}/P \not\equiv \bar{b}/P$. Hence by claim 1 $\bar{c}/P \perp N_0$, so $\bar{c}/M \perp N_0$. But then applying Claim 2, we get $\bar{c} \in M$, which is a contradiction.

Now, continuing to prove the theorem, we recall that $tp(I/M)$ does not fork over $N_0$, so Claim 3 and lemma 1 imply that for all $\bar{e} \in I$, Depth $(\bar{e}/M) > 0$. But $\text{Depth}(\bar{e}/M) = \text{Depth}(\bar{e}/N_0) = 0$, a contradiction. So, we have shown that N is $\aleph_\epsilon$-minimal over $M \cup I$, and therefore $\aleph_\epsilon$-prime over $M \cup I$. By the uniqueness of $\aleph_\epsilon$-prime models and the fact that $M \equiv N_0$, we conclude that every permutation of I induces an automorphism of N, thus $|\text{Aut}(N)| \geq 2^{\aleph_0}$, which is a contradiction.

Conclusion 1 If T has an $\aleph_\epsilon$-saturated model N such that $|\text{Aut}(N)| < 2^{\aleph_0}$, then T is multidimensional. (because even $\text{Depth}(T) > 0$ implies that T is multidimensional.)

Definition 4 We say that $\sigma \in \text{Aut}(M)$ (M is $\aleph_\epsilon$-saturated) is a nontrivial nonorthogonal automorphism (=n.n.a.) if for any nonalgebraic $p \in S(M)$, $p \not\equiv \sigma(p)$, and $\sigma \neq id$. 
Remark 1 $\sigma \in Aut(M)$ ($M$ as above) is a n.n.a. if and only if the unique extension of it $\sigma^{eq}$ to $M^{eq}$ is a n.n.a.

Theorem 3 If $\delta = \aleph_\delta > \beta \geq 2^{|T|}$ and the saturated model of cardinality $\beta$ does not have a nontrivial nonorthogonal automorphism, then $T$ has $2^\delta$ rigid $\beta$-saturated models of cardinality $\delta$.

Proof of Theorem 3 It is enough to show that every dimensionally diverse $\beta$-saturated model of cardinality $\delta$ is rigid. This is indeed enough, because using Theorem 1 we may then take a dimensionally diverse $\beta$-saturated model $N$ of cardinality $\delta$. So $N$ is rigid and by Conclusion 1 $T$ is multidimensional. Then, by Theorem 1, for every subset $D$ of $(\beta, \delta)$ which consists of cardinals, we may choose a $\beta$-saturated model which realizes exactly the dimensions in $D$ (in the sense of Theorem 1, part 2,) from which the theorem follows.

So let $M$ be a dimensionally diverse $\beta$-saturated model of cardinality $\delta$. And let $\sigma \in Aut(M)$. Assume by way of contradiction that $\sigma \neq id$. We define by induction an increasing chain of elementary submodels of $M$: Let $M_0$ be a saturated model of cardinality $\beta$, such that $\sigma|M_0 \neq id$. For all $n < \omega$ let $M_{i+1}$ be $F_\beta^n$-prime over $\bigcup_{i \in \mathbb{Z}} \sigma^i(M_n)$. Now, according to Fact 3, $M_n$ is the saturated model of cardinality $\beta$ or all $n < \omega$. And if $M_\omega = \bigcup_{i < \omega} M_i$, then $M_\omega$ is also the saturated model of cardinality $\beta$. Clearly, $\sigma|M_\omega$ is a nontrivial automorphism of $M_\omega$, so by the assumption of the theorem, there is a regular type $p \in S(M_\omega)$ such that $p \perp \sigma(p)$. Hence for some finite $A \subset M_\omega$, $p$ is strongly based on $A$, so $dim(p|A, M) = dim(\sigma(p|A), M)$, which contradicts the fact that $M$ is dimensionally diverse.

4 A characterization for superstable NDOP countable theories.

In this section $T$ is assumed to be a superstable NDOP countable theory. We will use [SH-401] in order to get the following characterization: such a theory has a rigid $\aleph_\epsilon$-saturated model if and only if every $\aleph_\epsilon$-saturated model does not have a nontrivial nonorthogonal automorphism. Then we will use this characterization to show that the existence of a single rigid $\aleph_\epsilon$-saturated
model implies the existence of a maximal number of such models in every sufficiently large cardinality.

We work in $\mathcal{M}^{eq}$.

**Definition 5** We say that $A$ is *almost finite* if $A$ is contained in the algebraic closure of some finite set.

### 4.1 The $L_{\infty,\aleph_\varsigma}(d.q)$-Characterization Theorem of [SH-401]

Let $M_0, M_1$ be $\aleph_\varsigma$-saturated. We say that they are $L_{\infty,\aleph_\varsigma}(d.q)-$equivalent, if there is a family $\mathcal{F}$ which satisfies the following:

1) Each $f \in \mathcal{F}$ is an elementary partial map from $M_0$ to $M_1$ such that $\text{Dom}(f)$ is almost finite.

2) $\mathcal{F}$ is closed under restriction.

3) For every $f \in \mathcal{F}$, and every $\bar{a}_l \in M_l$ ($\bar{a}_l$ are finite sequences, $l = 0, 1$) there exists $g \in \mathcal{F}$ such that $f \subseteq g$, $\text{acl}(\bar{a}_0) \subseteq \text{Dom}(g)$, and $\text{acl}(\bar{a}_1) \subseteq \text{Rang}(g)$.

4) Whenever $tp(e/A)$ (where $A$ is a almost finite,) is stationary and regular, then for some almost finite $A^* \supseteq A$, if $f \cup \{\langle e_0, e_1 \rangle\} \in \mathcal{F}$ and $tp(e_0/\text{Dom}(f))$ is conjugate to the stationarization of $tp(e/A)$ to $A^*$, then

$$\dim(p, \text{Dom}(f), I_0) = \dim(f(p), \text{Rang}(f), I_1),$$

where $p$ denotes $e_0/\text{Dom}(f)$,

$I_0 = \{e \in M_0 | f \cup \{\langle e, e_1 \rangle\} \in \mathcal{F}\}$, and

$I_1 = \{e \in M_1 | f \cup \{\langle e_0, e \rangle\} \in \mathcal{F}\}$.

Conditions 4) is the main assumption, roughly speaking, it implies that: whenever $\mathcal{F}$ sends a stationary and regular type $p$ to a type $q$, then the full structure of dimensions above $p$ will be the same as the full structure of dimensions above $q$.

We can now state the Characterization Theorem for $\aleph_\varsigma$-saturated models $M_0, M_1$: 

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$M_0, M_1$ are isomorphic if and only if they are $L_{\infty, \aleph_\epsilon}(d.q)$-equivalent.

**Theorem 4** If $T$ has a nontrivial nonorthogonal automorphism of some $\aleph_\epsilon$-saturated model, then every $\aleph_\epsilon$-saturated model of $T$ is not rigid.

**Proof of Theorem 4** By Remark 1, without loss of generality $T = T^{eq}$. Suppose $N_0$ is an $\aleph_\epsilon$-saturated model which has a n.n.a. $\sigma_0$, and let $a_0 \neq a_1$ be in $N_0$ such that $\sigma_0(a_0) = a_1$. Let $N$ be some $\aleph_\epsilon$-saturated model (of $T$), we will show that $N$ is not rigid. Choose $b_0, b_1 \in N$ such that $tp(b_0, b_1) = tp(a_0, a_1)$.

Let $F$ be the family of all partial elementary maps $f$ from $(N, b_0)$ to $(N, b_1)$ with an almost finite domain, such that for some partial elementary map $\tau$, with almost finite domain, $(\sigma_0|A_0)\tau = \tau\sigma$, where $A_0 = rang(\tau)$. By [SH-401] it is sufficient to show that $F$ satisfies 1) - 4) above.

1), 2) are immediate. 3) is also immediate by the fact that $N$ is $\aleph_\epsilon$-saturated. To show 4), we will prove the following claim: if $tp(e/A)$ is stationary and regular, where $A$ is almost finite, then it can be replaced by a nonforking extension of it, denoted again by $tp(e/A)$ (where $A$ is almost finite and $A \cup \{e\} \subseteq N$), such that if $\sigma_i(A) = A'$ and $\sigma_i(e) = e_i$ for $\sigma_i \in F$ ($i = 0, 1$) then $e_0 \not\equiv e_1$. 4) will follow from this, as it implies that for all $f \in F$ with $Dom(f) = A$, if $e_0^* = e$, $e_1^* = f(e), M_0 = M_1 = N$ and $I_0, I_1$ are defined as in 4) of subsection 4.1 (with $e_i^*$ instead of $e_i$) then $dim(tp(e_0^*/A), A, I_0) = dim(tp(e_1^*/f(A)), f(A), I_1) = 1$.

Proceeding to prove this claim, we note that by Theorem 2, $T$ is without loss of generality strongly deep. Let $M_0$ be an $\aleph_\epsilon$-prime model over $\emptyset$, such that $tp(e/M_0)$ is a nonforking extension of $tp(e/A)$, let $M_0^+$ be $\aleph_\epsilon$-prime over $M_0 \cup \{e\}$ and let $e^+$ be such that $tp(e^+/M_0^+) \perp M_0$, with $tp(e^+/M_0^+)$ non algebraic and regular. Now let $B \subseteq M_0^+$ be finite such that $tp(e^+/M_0^+)$ is strongly based on $B$, so there is some finite $C \subseteq M_0$ such that $tp(B/\ acl(C \cup \{e\})) \vdash tp(B/M_0 \cup \{e\})$. We may assume without loss of generality that $M_0^+ \subseteq N$ (because the $\aleph_\epsilon$-prime model over the union of $M_0$ and a countable set is also $\aleph_\epsilon$-prime over $\emptyset$ (see [Sh-C]).

Now, by the way condition 4) was stated we may also assume that $A \supseteq C$. Assume, towards a contradiction, that $e_0 \not\equiv e_1$, and let $\sigma_i^+$ be in $F$ with $A'$
\( \sigma_i \subseteq \sigma_i^+ \) and \( \text{Dom}(\sigma_i^+) \supseteq B \cup \text{acl}(A \cup \{e\}) \), for \( i=0,1 \). Choose \( M'_0 \) to be an \( \aleph_\mu \)-prime model over \( \emptyset \), such that \( A' \subseteq M'_0 \) and \( \{e_0,e_1\} \cup M'_0 \), and set \( B_i = \sigma_i(B) \). Then by the fact that \( tp(e_0,e_1/M'_0) \) determines the type \( tp(\text{acl}(\{e_0\} \cup A), acl(\{e_1\} \cup A')/M'_0) \), and the fact that \( tp(B/\text{acl}(\{e\} \cup A)) \equiv tp(B/\{e\} \cup M_0) \), we conclude that we may choose \( M_0^* \) and \( M_1^* \) which are \( \aleph_\mu \)-prime over \( M'_0 \cup \{e_0\} \) and \( M'_1 \cup \{e_1\} \) respectively, and such that \( B_i \subseteq M_i^* \) for \( i=0,1 \). So if we set \( q = tp(e^+/B) \) and \( q_i = \sigma_i^+(q) \) for \( i = 0,1 \), then by the definition of \( F \), \( q \not\perp q_i \) for \( i = 0,1 \). Let \( \bar{q}_i \in S(M_i^*) \) be a nonforking extension of \( q_i \). Since \( \bar{q}_i \perp M'_0 \) (as \( M_i^* \) is \( \aleph_\mu \)-prime over \( M'_i \cup \{e_i\} \cup B_i \) and unique up to isomorphism over \( M'_i \cup \{e_i\} \cup B_i \)) and since \( M_0^* \cup \nu \sim M_i^* \), we conclude that \( q_0 \perp q_1 \). Thus, \( q_0 \) and \( q_1 \) contradict the fact that nonorthogonality is an equivalence relation on stationary regular types.

**Theorem 5** If \( T \) has a rigid \( \aleph_\mu \)-saturated model then for every cardinal \( \lambda \geq (2^{\aleph_0})^+ \) \( T \) has \( 2^\lambda \) rigid \( \aleph_\mu \)-saturated models of power \( \lambda \).

Before proving the theorem, we will need a combinatorial lemma. To that end, we first introduce the following notations.

**Notations 1**

1) \( \mathcal{T} \) denotes a subtree of \( <\omega\lambda \). Let \( \eta, \nu \in \mathcal{T} \), then
   (i) \( \nu^- = \eta \) if and only if \( \nu \) is a successor of \( \eta \).
   (ii) \( \text{lg}(\eta) \) denotes the length of \( \eta \).
   (iii) \( \mathcal{T}_\nu = \{ \eta \in \mathcal{T}| \nu \lesssim \eta \}, \mathcal{T}_\nu^+ = \{ \eta \in \mathcal{T}| \nu \lesssim \eta \ \text{or} \ \nu \equiv \eta \} \).

2) \( \mathcal{R} = \langle (N_\eta,a_\eta)| \eta \in \mathcal{T} \rangle \) denotes an \( \aleph_\mu \)-representation (see Chapter X, Def. 5.2 in [Sh-C]).

3) For an \( \aleph_\mu \)-representation \( \mathcal{R} = \langle (N_\eta,a_\eta)| \eta \in \mathcal{T} \rangle \) \( E_\mathcal{R} \) denotes the equivalence relation on \( \mathcal{T} \) defined by: \( E_\mathcal{R}(\eta,\eta') \) if and only if there exists \( \nu \in \mathcal{T} \) such that \( \eta = \nu \langle \alpha \rangle, \eta' = \nu \langle \alpha' \rangle \) for some \( \alpha, \alpha' \), and \( tp(a_\eta/N_\nu) = tp(a_{\eta'}/N_\nu) \).

**Definition 6**

1) We say that a subtree \( \mathcal{T} \) of \( <\omega\lambda \) is \( \mu \)-wide if for all \( \eta \in \mathcal{T} \), \( |\{ \nu \in \mathcal{T}| \nu^- = \eta \}| \geq \mu \).
2) Suppose $T_0, T_1 \subseteq <\omega \lambda$ are subtrees, we say that $T_0, T_1$ are $\mu$-equivalent if there exists $\bar{A}_i = \langle A^\rho_i | \rho \in T_i \rangle$ ($i = 0, 1$) such that
   (a) for $i = 0, 1$ and all $\rho \in T_i$, $|A^\rho_i| < \mu$.
   (b) for $i = 0, 1$ $A^\rho_i \subseteq T^\rho_i$, and
   (c) $T_0 \setminus \bigcup_{\tau \in U_0} (T_0)_\tau$, $T_1 \setminus \bigcup_{\tau \in U_1} (T_1)_\tau$ are isomorphic as partial orders, where $U_i = \bigcup_{\rho \in T_i} A^\rho_i$.

3) A tree $T \subseteq <\omega \lambda$ is called $\mu$-strongly rigid if for every $\eta$ and $\alpha_0 < \alpha_1 < \lambda$ (such that $\eta \wedge \langle \alpha_i \rangle \in T$) $T^\eta_\eta^\alpha_0, T^\eta_\eta^\alpha_1$ are not $\mu$-equivalent, and $T$ is $\mu$-wide.

4) We say that an $\aleph_\nu$-representation $R = \langle (N_\eta, a_\eta) | \eta \in T \rangle$ is
   (i) of maximal width, if for every $\nu \in T$, $\{tp(a_\eta/N_\eta^-) | \eta \in T, \eta^- = \nu\}$ is a maximal set of pairwise orthogonal regular types (modulo equality) such that $tp(a_\eta/N_\eta^-) \perp N_\nu^-$.  
   (ii) equally divided, if $|\eta \wedge \langle \alpha \rangle/E_R| = |\eta \wedge \langle \beta \rangle/E_R|$ for all $\eta, \alpha, \beta$ such that $\eta \wedge \langle \alpha \rangle, \eta \wedge \langle \beta \rangle \in T$.
   (iii) $\mu$-wide, if $|\eta/E_R| \geq \mu$ for all $\eta \in T$.

Before stating the following lemma, we would like to point out that much stronger versions of it have been proved by Shelah.

**Lemma 2** For every $\lambda > \mu$ there exist $2^\lambda$ trees $T \subseteq \lambda^{<\omega}$ which are $\mu$-strongly rigid and $\mu$-nonequivalent, of power $\lambda$.

**Proof of Lemma 2** First, it is enough to show the existence of a single such tree, in which the root has $\lambda$ many successors. So, let us construct such a tree:

   a) Let $\langle A_n | n < \omega \rangle$ be a partition of the set of positive natural numbers, such that for all $n < \omega$ $A_n$ is infinite and $\min(A_n) > n$. Also, suppose $A_n = \{k^n_\ell | \ell < \omega\}$, where $k^n_i < k^n_j$ for $i < j$.
   b) Let $h : \lambda \to \lambda$ be surjective, such that for all $\alpha < \lambda$, $|h^{-1}(\alpha)| = \lambda$.
   c) Let $T_0 = <\omega \lambda = \{\eta_i | i < \lambda\}$ ($\eta_i \neq \eta_j$ for $i \neq j$).
d) For every ordinal $i$, let $t_i = \{ \nu | \nu$ is a strictly decreasing sequence of ordinals $< \omega + i \}$.

e) Now, let us define our tree: $T^* = \{ \rho \in \leq \omega \lambda | \text{for all } k < \log(\rho), \text{if } (\ast)(k, \rho) \text{ then } \rho(k) < \mu \}$, where $(\ast)(k, \rho)$ is the following statement:

If $n < \omega$ is the unique natural number such that $k \in A_n$, and $l(\ast), i(\ast)$ are such that $k = k^\alpha_i$ and $\rho|n = \eta_{i(\ast)}$, then $\langle h(\rho(k^\alpha_i)) | l < l(\ast) \rangle \notin t_{i(\ast)}$.

f) The following ranks are useful. Let $(n, S, \eta)$ be a triple, with $n < \omega$, $S \subseteq \leq \omega \lambda$ is a subtree, and $\eta \in S$. We define an ordinal rank $rk^n[\eta, S]$, by:

i) $rk^n[\eta, S] \geq 0$ for every $\eta \in S$.

ii) $rk^n[\eta, S] \geq \alpha + 1$ if there exist $\{ \eta_i^* | i < \mu^+ \}$, where $\eta_i^*$ are distinct elements of $S$, each extending $\eta$ and satisfying $\log(\eta_i^*) \in A_n$, and $rk^n[\eta_i^*, S] \geq \alpha$.

g) It can be easily verified that:

i) If $S \subseteq T \subseteq \leq \omega \lambda$ are $\mu$-equivalent subtrees, $\eta \in S$, and $n < \omega$, then $rk^n[\eta, S] = rk^n[\eta, T]$.

ii) If $i < \lambda$, $n < \omega$ and $\log(\eta_i) = n$, where $\eta_i \in T^*$ then $rk^n[\eta_i, T^*] = \omega + i$.

Now, by g) we conclude that $T^*$ is $\mu$-strongly rigid.

**Fact 5 (Chapter X, [Sh-C])** Suppose that $T$ is a superstable NDOP theory. Let $R_i = \langle (N^i_{\eta}, a^i_{\eta}) | \eta \in T_i \rangle$ be $\aleph_i$-representations which are $\aleph_1$-wide, let $M_i$ be $\aleph_i$-prime over $\bigcup_{\eta \in T_i} N^i_{\eta}$ and let $\sigma : M_0 \to M_1$ be an isomorphism. Then there are $T^*_i \subseteq T_i$ such that $T^*_i, T_i$ are $\aleph_1$-equivalent, for $i = 0, 1$ and an isomorphism $\tilde{\sigma} : T^*_0 \to T^*_1$ (of partial orders) such that

for all $\eta_i \in T^*_i$ $(i = 0, 1)$ if $p^0_i = tp(a^i_{\eta_i}/N^i_{\eta_i})$, then $\sigma(p^0_i) \perp p^1_1$ implies $\tilde{\sigma}((\eta_0/E_{R_0}) \cap T^*_0) = (\eta_1/E_{R_1}) \cap T^*_1$.

**Proof of Theorem 5** By Lemma 2, for every $\lambda > 2^{\aleph_0}$ there exist $2^\lambda$ trees $T \subseteq \leq \omega \lambda$ which are $2^{\aleph_0}$-strongly rigid and $2^{\aleph_0}$-non-equivalent, of power $\lambda$.

We will show that for every such $T$, if $R = \langle (M_0, e_0) | \eta \in T \rangle$ is an equally divided $\aleph_0$-representation of maximal width, and $M$ is $\aleph_0$-prime over it, then $M$ is rigid.

This is sufficient because by Fact 5, any two $\aleph_0$-prime models over $2^{\aleph_0}$-non-equivalent trees are non-isomorphic.

So, assume $\sigma \in \text{Aut}(M)$, we must show $\sigma = \text{id}$: otherwise by Theorem 4, there exists a non-algebraic regular type $p^* \in S(M)$ such that $p^* \perp \sigma p^*$. Since $T$ has NDOP and $M$ is $\aleph_0$-minimal over $\bigcup_{\eta \in T} M_\eta$, we have $p^* \perp M_{m_0}$ for some
By the fact that $M$ is $\aleph_\varepsilon$-prime over an $\aleph_\varepsilon$-representation of maximal width, $p^* \not\perp p_{\eta_0 \wedge (\alpha^*)}$ for some $\alpha^*$. Therefore $\sigma p_{\eta_0 \wedge (\alpha^*)} \not\perp p^* \not\perp p_{\eta_0 \wedge (\alpha^*)}$. In particular, $(*)$ $\sigma p_{\eta_0 \wedge (\alpha^*)} \perp p_{\eta_0 \wedge (\alpha^*)}$. But by Fact 5, there are $T_0^*, T_1^* \subseteq T$ (without loss of generality $\eta_0 \wedge (\alpha^*) \in T_0^*$), and an isomorphism of partial orders $\tilde{\sigma} : T_0^* \rightarrow T_1^*$, such that $T_i^*(i = 0, 1)$ are $\aleph_1$-equivalent to $T$, and $\eta^*$ such that $\sigma p_{\eta_0 \wedge (\alpha^*)} \not\perp p_{\eta^*}$. So by $(*)$ $p_{\eta^*} \neq p_{\eta_0 \wedge (\alpha^*)}$ and by Fact 5, $\tilde{\sigma}(\eta \wedge (\alpha^*)/E_R) = \eta^*/E_R \neq (\eta_0 \wedge (\alpha^*))/E_R$, in contradiction to the fact that $T$ is even $2^{\aleph_0}$-strongly rigid (here we used the fact that $T$ is equally divided).

**Example 1** Let $L = \{E, f\}$ and let $M$ be the following $L$-structure:

$|M| = Z \times \omega$, $E$ is the equivalence relation defined by $E((m_0, k_0), (m_1, k_1))$ if and only if $m_0 = m_1$, and $f : |M| \rightarrow |M|$ is any function with the following properties:

i) For all $(m, k) \in |M|$ $f(m, k) = (m + 1, k')$ for some $k'$.

ii) For all $(m, k) \in |M|$ $f^{-1}(m, k)$ is infinite.

It is not hard to see that $T = Th(M)$ is an $\omega$-stable NDOP theory in which any saturated model does not have a n.n.a., and therefore $T$ has $2^\lambda$ rigid $\aleph_\varepsilon$-saturated models of cardinality $\lambda$ for every $\lambda > 2^{\aleph_0}$.

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