TWISTORIAL STRUCTURES REVISITED

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Abstract. We review the twistorial structures by providing a setting under which
the corresponding (differential) geometry can be described, by involving the $\rho$-con-
nections. This applies, for example, to give new proofs of the existence of the relevant
connections for the projective and the quaternionic geometries. Along the way, we
show that, in this setting, the Ward transformation is a consequence of the good
behaviour of the $\rho$-connections, under pull back.

Introduction

A twistor space is a complex manifold $Z$ endowed with a family $\{Y_x\}_{x \in M}$ of com-
 pact complex submanifolds such that the canonical map $\psi : Y = \bigsqcup_{x \in M} Y_x \to Z$ is
a surjective submersion. The main task is to describe the (differential) geometry induced
on $M$, and to see, to what extent, from this geometry we can retrieve $Z$.

The first examples are provided by the projective and the conformal structures (see
\cite{4, 5}), where $Y$ is $PTM$ and a bundle of hyperquadrics, respectively. On the other
hand, we have the quaternionic geometry, where $Y$ is a Riemann sphere bundle (see
\cite{6}).

To tackle the main task, we, firstly, turn to the canonical projection $\pi : Y \to M$
and observe that, for any $x \in M$, the differential of $\pi$ determines a map from $Y_x$ to
the Grassmannian of $T_xM$. This is how the ‘linear twistorial structures’, which we
consider in Section 1, appear.

Next, we have to understand that the classical connections are not sufficient to model
even the simple examples given by the projective spaces, endowed with the correspond-
ing families of Veronese curves. We are, thus, led to consider the $\rho$-connections \cite{6}.
We have, however, to make some further assumptions, such as ‘regularity’ which means
that the spaces of sections of the duals of $(\ker d\psi)|_{Y_x}, (x \in M)$, have the same di-

This supplies, through direct image, an enriched tangent bundle, that is, a
vector bundle $E$ over $M$ together with a morphism of vector bundles $\rho : E \to TM$.
The relevant notions and results are given in Section 2.

Finally, in Section 3, we study a class of twistor spaces which provides a generaliza-
tion of the $\rho$-quaternionic structures of \cite{6} (Theorem 3.2 and Remark 3.3). The idea,
basic in twistor theory, is that any (fairly good) line bundle over $Z$ induces, through
pull back by $\psi$ and direct image through $\pi$, a vector bundle over $M$ endowed with a

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Along the way, we, also, show that, for regular twistorial structures, the Ward transformation is a consequence of the good behaviour of the $\rho$-connections, under pull back (Corollary 3.1).

### 1. Linear twistorial structures

We work in the category of complex manifolds.

**Definition 1.1.** Let $U$ be a (complex) vector space, let $Y$ be a compact manifold, and let $k \in \mathbb{N}, k \leq \dim U$. A linear twistorial $Y$-structure on $U$ is a map $\varphi : Y \to \text{Gr}_k U$.

Let

\begin{equation}
0 \rightarrow U_- \rightarrow Y \times U \rightarrow U_+ \rightarrow 0
\end{equation}

be the pull back through $\varphi$ of the tautological exact sequence of vector bundles over $\text{Gr}_k U$. We say that (1.1) (which, obviously, determines $\varphi$) is the exact sequence (of vector bundles) of $\varphi$. If the associated cohomology exact sequence gives a linear isomorphism between $U$ and the space of sections of $U_+$ we say that $\varphi$ is maximal (cf. [2]).

For brevity, we shall use the expression ‘linear $Y$-structure’ instead of ‘linear twistorial $Y$-structure’.

The vector spaces endowed with linear $Y$-structures form a category in an obvious way (we can even let $Y$ to change from one vector space to another, if necessary).

Let $\varphi : Y \to \text{Gr}_k U$ be a maximal linear $Y$-structure on $U$. Note that, $\varphi$ must be nonconstant if $Y$ is not a point. Furthermore, $H^0(U_-) = 0$ and, under the further assumption that $H^1(O_Y) = 0$, then, also $H^1(U_-) = 0$. Conversely, the exact sequence (1.1) corresponds to a maximal linear $Y$-structure if $H^0(U_-) = 0$ and $H^1(U_-) = 0$.

Now, dualizing the exact sequence (1.1) of a linear $Y$-structure $\varphi$ on $U$ we obtain the exact sequence of a linear $Y$-structure on $U^*$. However, the dual of a maximal linear $Y$-structure is not necessarily maximal. For this reason, let $E$ be the dual of the space of sections of $U^*_-$ and let $\rho$ be the transpose of the linear map from $U^*$ to $E^*$ given by the cohomology exact sequence of the dual of (1.1). As $U^*_-$ is generated by its (global) sections, the obvious morphism of vector bundles $Y \times E^* \rightarrow U^*_+$ is surjective and therefore we obtain an exact sequence

\begin{equation}
0 \rightarrow U_- \rightarrow Y \times E \rightarrow E \rightarrow 0,
\end{equation}

for some vector bundle $E$ over $Y$. Moreover, $\rho$ induces a morphism from (1.2) to (1.1); in particular, a morphism $\mathcal{R} : E \rightarrow U_+$ from which, if $H^0(U_-) = 0$, we retrieve $\rho$ when passing to the spaces of sections.

**Remark 1.2.** 1) The dual of (1.2) corresponds to a maximal linear $Y$-structure.

2) From the dual of (1.1) we deduce that the cokernel of $\rho$ is the dual of $H^0(U^*_+)$.

Furthermore, if $H^1(O_Y) = 0$ then the kernel of $\rho$ is the dual of $H^1(U_+)$. 

Example 1.3. A vector bundle over $\mathbb{C}P^1$ appears as the corresponding $U_+$ of a linear $\mathbb{C}P^1$-structure if and only if it is nonnegative. If the corresponding map is an embedding, we retrieve the linear quaternionic-like structures of [7], whilst maximal linear $\mathbb{C}P^1$-structure are just the $\rho$-quaternionic structures of [6]. Then the dual of the corresponding (1.1), also, corresponds to a maximal linear $\mathbb{C}P^1$-structure if and only if in the Birkhoff–Grothendieck decomposition of $U_+$ appear only terms of Chern number one; that is, if and only if the given linear $\mathbb{C}P^1$-structure gives a linear (classical, complex) quaternionic structure.

Example 1.4. Let $Y$ be endowed with an ample line bundle $L$. Then from the Kodaira vanishing theorem and Serre duality we obtain $H^0(L^*) = 0$; furthermore, if dim $Y \geq 2$ then, also, $H^1(L^*) = 0$. Therefore if dim $Y \geq 2$, $H^1(O_Y) = 0$ and $L$ has empty base locus, the map from $Y$ into the projectivisation of the dual of the space of sections of $L$ is a maximal linear $Y$-structure. Moreover, the dual of the corresponding (1.1), also, corresponds to a maximal linear $Y$-structure. Furthermore, on tensorising $L$ with a trivial vector bundle then we again obtain the exact sequence of a linear $Y$-structure whose dual, also, corresponds to a linear $Y$-structure.

2. Regular almost twistorial structures

Twistor theory imposes the use of the category whose objects are triples $(M,E,\rho)$, where $M$ is a manifold, $E$ is a vector bundle over $M$, and $\rho : E \to TM$ is a morphism of vector bundles. The morphisms between two such objects $(M,E,\rho)$ and $(M',E',\rho')$ are pairs $(\varphi,\Phi)$, where $\varphi : M \to M'$ is a map, and $\Phi : E \to E'$ is a vector bundles morphism, over $\varphi$, such that $\rho' \circ \Phi = d\varphi \circ \rho$.

The notion of connection adapts, accordingly, to this setting. Let $(P,M,G)$ be a principal bundle, and let $\rho : E \to TM$ be a morphism of vector bundles, where $E$ is a vector bundle over $M$. A principal $\rho$-connection [8] on $P$ is a morphism of vector bundles $\mathcal{C} : E \to TP/G$ such that when composed with the canonical morphism of vector bundles $TP/G \to TM$ gives $\rho$.

We, also, have the notion of associated $\rho$-connections which for vector bundles corresponds to covariant $\rho$-derivations.

Proposition 2.1. Let $(\varphi,\Phi) : (M,E,\rho) \to (M',E',\rho')$ be a morphism and let $(P,M',G)$ be a principal bundle.

For any principal $\rho'$-connection $\mathcal{C}'$ on $P$ there exists a unique principal $\rho$-connection $\mathcal{C}$ on $\varphi^*P$ such that $\mathcal{C}' \circ \Phi = \tilde{\mathcal{C}} \circ \mathcal{C}$, where $\tilde{\mathcal{C}} : T(\varphi^*P)/G \to TP/G$ is the morphism of vector bundles induced by (the differential of) $\varphi$.

Proof. Let $\pi : P \to M'$ be the projection. As $\varphi^*P$ embedded into $M \times P$ is formed of those pairs $(x,u)$ with $\varphi(x) = \pi(u)$, we have that $T(\varphi^*P)/G$ embeds into $TM \times (TP/G)$ such that be formed of those pairs $(X,U)$ satisfying $d\varphi(X) = d\pi(U)$, where $d\pi :
TP/G \to TM' is the morphism of vector bundles induced by dπ.

On the other hand, dπ \circ c' \circ \Phi = \rho' \circ \Phi = d\varphi \circ \rho. Consequently, ρ and c' \circ \Phi uniquely determine a vector bundles morphism c : E \to T(ϕ^*P)/G, as required. □

The principal ρ-connection obtained in Proposition 2.1 is called the pull back by (ϕ, Φ) of c'.

Remark 2.2. The association c' \mapsto c of Proposition 2.1 is a morphism of (possibly empty) affine spaces. The involved vector spaces are the spaces of sections of Hom(E', AdP) and Hom(E, Ad(ϕ'^*P)), respectively. The corresponding linear map is induced by Φ through the identification Ad(ϕ'^*P) = ϕ'^* (AdP).

Now, recall [9] that, an almost twistorial structure on a manifold M is a triple (Y, π, C), where π : Y \to M is a proper surjective submersion, and C is a distribution on Y such that C \cap (ker dπ) = 0.

Let (Y, π, C) be an almost twistorial structure on M. For each x \in M, denote Y_x = π^(-1)(x), and note that we have a map Y_x \to Gr_k(T_x M), y \mapsto dπ(C_y), (y \in Y_x), where k = rank C. If this map is a maximal linear Y_x-structure on T_x M, for any x \in M, we say that (Y, π, C) is maximal.

Let

\[ 0 \to C \to \pi^*(TM) \to T \to 0 \]

be the exact sequence of vector bundles over Y induced, through pull back, by the tautological exact sequence over Gr_k(TM). Obviously, we have an isomorphism between T and the quotient of TY through C + (ker dπ).

We are interested in the direct image through π of the dual of C. This is a vector bundle over M if x \mapsto h^0(C_x^*) is constant (see [11], p. 211), where we have denoted C_x^* = \mathcal{C}|_{Y_x}^*; then we say that (Y, π, C) is regular. Assuming this, on denoting by E the dual of the space of sections of the induced vector bundle, and by dualizing (2.1) we, also, obtain a morphism of vector bundles ρ : E \to TM. Furthermore, in this context, (1.2) becomes

\[ 0 \to C \to \pi^*E \to \mathcal{E} \to 0, \]

for some vector bundle \mathcal{E}. Denote by χ the vector bundles morphism, over π, obtained as the composition of C \to \pi^*E followed by the canonical bundle map π^*E \to E.

Thus, on denoting by ϵ : C \to TY the inclusion, we have obtained a morphism (π, χ) : (Y, C, ϵ) \to (M, E, ρ).

The following result is fundamental in twistor theory.

Proposition 2.3. Let (Y, π, C) be a regular almost twistorial structure on M, let E be the direct image of C^* by π, denote by ρ the induced morphism of bundles from E to TM, and let (π, χ) : (Y, C, ϵ) \to (M, E, ρ) be the resulting morphism, where ϵ : C \to TY is the inclusion.
For any principal bundle \( P \) over \( M \), the pull back by \((\pi,\chi)\) establishes an isomorphism between the following affine spaces:

(i) the space of principal \( \rho \)-connections on \( P \),
(ii) the space of principal partial connections on \( \pi^*P \), over \( C \).

Proof. It is sufficient to consider the case \( P = M \times G \). Then any principal \( \rho \)-connection on \( P \) is given by its (local) connection form which is a section of \( g \otimes E^* \), where \( g \) is the Lie algebra of \( G \). On the other hand, as \( \pi^*P = Y \times G \), any principal partial connection on \( \pi^*P \), over \( C \), is given by a section of \( g \otimes C^* \).

Accordingly, the pull back by \((\pi,\chi)\) associates to any section \( A \) of \( g \otimes E^* \) the restriction to \( C \) of \( \pi^*A \).

Now, as \((Y,\pi,C)\) is regular, and \( E \) is the direct image through \( \pi \) of \( C^* \), to complete the proof, just note that, for any \( x \in M \), the given isomorphism from \( E^*_x \) onto the space of sections of \( C^*_x \) is expressed by \( \alpha \mapsto \alpha|_{C^*_x} \), for any \( \alpha \in E^*_x \).

Let \((Y,\pi,C)\) be a regular almost twistorial structure on \( M \). To try to describe the geometry induced on \( M \), let \( \nabla \) be a \( \rho \)-connection (locally defined) on \( E \). By Proposition \ref{corollary2.6}, \( \nabla \) corresponds to a partial connection on \( \pi^*E \), over \( C \), which we shall denote by \( \pi^*\nabla \). On denoting by \( p : \pi^*E \to E \) the projection, we obtain a section \( b \) of \( E \otimes (\otimes^2C^*) \) given by \( b(s,t) = p((\pi^*\nabla)s, t) \) for any local sections \( s \) and \( t \) of \( C \).

Proposition 2.4. Suppose that the following two conditions are satisfied:

(1) \( H^1(C_x^* \otimes (\otimes^2C^*_x)) = 0 \), for any \( x \in M \);
(2) \( H^0(C_x^*) \otimes H^0(C^*_x) \to H^0(\otimes^2C^*_x) \) is surjective, for any \( x \in M \), and of constant rank.

Then, locally, there exists a \( \rho \)-connection \( \tilde{\nabla} \) on \( E \) such that \( C \) is preserved by \( \pi^*\tilde{\nabla} \) (that is, \((\pi^*\nabla)s, t \) is a local section of \( C \), for any local sections \( s \) and \( t \) of \( C \)).

Proof. Firstly, tensorise (2.2) with \( \otimes^2C^* \), restrict to each fibre of \( \pi \), and then pass to the associated cohomology exact sequences. Then conditions (1), (2), and [3, Theorem 2.3] imply that \( h^0(E_x \otimes (\otimes^2C^*_x)) \) does not depend of \( x \in M \). Consequently, \( b \) is induced (locally) by a section of \( E \otimes (\otimes^2C^*) \), and (applying, again, (2)) we deduce that \( b \) is induced by a section \( B \) of \( E \otimes (\otimes^2E^*) \), that is, \( b = p \circ ((\pi^*B)|_{C^*}) \).

Then the \( \rho \)-connection \( \tilde{\nabla} \) on \( E \) given by \( \tilde{\nabla}s,t = \nabla_s t - B(s,t) \), for any local sections \( s \) and \( t \) of \( E \), is as required. 

Remark 2.5. In Proposition 2.4, if the distribution \( C \) is one-dimensional then (1) implies that \((Y,\pi,C)\) is regular. Also, if \( C^*_x \) is very ample, for any \( x \in M \), then (2) is automatically satisfied, provided that (1) holds.

Corollary 2.6. Let \( \pi : Y \to M \) be a proper surjective submersion, let \( C \) be a distribution on \( Y \), and let \( L \) be a line bundle over \( Y \) such that:

(a) \( C \cap (\ker d\pi) = 0 \);
(b) \( C \otimes L \) restricted to each fibre of \( \pi \) is trivial;
(c) $L|_{Y_x}$ is very ample, with zero first cohomology group, where $Y_x = \pi^{-1}(x)$, for any $x \in M$.

Then $(Y,\pi,\mathcal{C})$ is a regular almost twistorial structure and, locally, there exist $\rho$-connections $\nabla$ on $E$ such that $\mathcal{C}$ is preserved by $\pi^*\nabla$. Furthermore, if some fibre of $Y$ is of dimension at least two and has zero first Betti number then $(Y,\pi,\mathcal{C})$ is maximal.

**Proof.** This follows quickly from Example 1.4 and Proposition 2.4. □

**Example 2.7.** Let $(Y,\pi,\mathcal{C})$ be an almost twistorial structure on $M$ such that $\mathcal{C}$ is one-dimensional. Assume that, for any $x \in M$, the map $\varphi_x : Y_x \to PT_x M$, $\varphi_x(y) = d\pi(C_y)$, $(y \in Y_x)$, is a normal embedding, and $H^1(C_x^*) = 0$, where $C_x = C|_{Y_x}$.

On identifying $Y$ with its image, through $\varphi = (\varphi_x)_{x \in M}$, Corollary 2.6 implies that, locally, there exist partial connections $\nabla$ on some distribution on $M$ such that the projection through $\pi$ of any leaf of $\mathcal{C}$ is a geodesic of $\nabla$. Consequently, the following hold, as well:

1) If the tangent space at some point of a geodesic of $\nabla$ belongs to $Y$ then this holds for all of the points of the geodesic.

2) $\mathcal{C}$ is given by the tautological lifts of the geodesics of $\nabla$ whose tangent spaces belong to $Y$.

For concrete examples, take $Y$ to be $PTM$ or the bundle of isotropic directions of a conformal structure on $M$ (see [4], [5] for different proofs).

3. **Regular twistorial structures**

Recall [9] that, a *twistorial structure* on $M$ is an almost twistorial structure $(Y,\pi,\mathcal{C})$ on $M$ with $\mathcal{C}$ integrable. Suppose, further, that there exists a surjective submersion $\psi : Y \to Z$ satisfying:

1) $\ker d\psi = \mathcal{C}$,

2) $\psi|_{Y_x}$ is injective, for any $x \in M$.

Then $Z$ is the *twistor space* of $(Y,\pi,\mathcal{C})$ and the embedded submanifolds $\psi(Y_x) \subseteq Z$, $(x \in M)$, are the *twistor submanifolds* (cf. [9]).

If $(Y,\pi,\mathcal{C})$ is regular then, as in Section 2, we denote by $E$ the dual of the direct image, through $\pi$, of $\mathcal{C}^*$. We, also, have a canonical morphism of vector bundles $\rho : E \to TM$.

**Corollary 3.1.** Let $(Y,\pi,\mathcal{C})$ be a regular twistorial structure with twistor space given by $\psi : Y \to Z$. If the fibres $\psi$ are simply-connected then, for any Lie group $G$, there exists a functorial correspondence between the following:

(i) Principal bundles $(P,Z,G)$ whose restrictions to the twistor submanifolds are trivial.

(ii) Principal bundles $(P,M,G)$ endowed with principal $\rho$-connections whose pull backs to $(Y,\mathcal{C},\iota)$ are flat (partial connections), where $\iota : \mathcal{C} \to TY$ is the inclusion.
Proof. The correspondence is given by $\psi^*\mathcal{P} = \pi^*P$ and the result is a quick consequence of Proposition 2.3. □

Note that, Corollary 3.1 is just the Ward transformation for regular twistorial structures (compare [8]).

**Theorem 3.2.** Let $(Y, \pi, C)$ be a regular twistorial structure with twistor space given by $\psi: Y \rightarrow Z$, and let $L$ be a line bundle over $Z$ such that:

(a) $C \otimes \psi^*L$ restricted to each fibre of $\pi$ is trivial;
(b) $(\psi^*L)|_{Y_x}$ is very ample, where $Y_x = \pi^{-1}(x)$, for any $x \in M$.

Then, locally, there exist $\rho$-connections $\nabla$ on $E$ such that:

(i) $C$ is preserved by $\pi^*\nabla$;
(ii) $\nabla$ is compatible with the decomposition $E = H \otimes F$, where $H$ is the dual of the direct image (through $\pi$) of $\psi^*L$, and $F$ is the direct image of $C \otimes \psi^*L$.

Furthermore, if some fibre of $Y$ is of dimension at least two and has zero first Betti number then $(Y, \pi, C)$ is maximal.

Proof. Assertion (ii) means that $\nabla = \nabla^H \otimes \nabla^F$, where $\nabla^H$ and $\nabla^F$ are $\rho$-connections on $H$ and $F$, respectively. It is sufficient to prove that we can choose $\nabla^H$ and $\nabla^F$ such that (i) is satisfied.

Let $\nabla^F$ be any $\rho$-connection (locally defined) on $F$ and we define $\nabla^H$ as follows. Firstly, note that, $\psi^*L$ is endowed with a flat partial connection, over $C$, whose covariantly constant local sections are the pull backs by $\psi$ of the local sections of $L$. Consequently, $\pi^*H$ is canonically endowed with a partial connection, over $C$. By Proposition 2.3, this corresponds to a $\rho$-connection $\nabla^H$ on $H$.

Now, the proof follows from the obvious identification $C = \psi^*(L^*) \otimes \pi^*F$ and the fact that, locally, we may choose sections $u$ of $\psi^*(L^*)$ such that $(\pi^*\nabla^H)_{u \otimes v}u = 0$, for any $v \in \pi^*F$ (the latter fact is a consequence of the definition of $\nabla^H$). □

**Remark 3.3.** With the same notations as in the proof of Theorem 3.2, we have an obvious embedding $Y \subseteq PH$. Let $\mathcal{L}$ be the dual of the tautological line bundle over $PH$. On denoting by $\pi$, also, the projection from $PH$ onto $M$, note that, $\mathcal{L}|_Y = \psi^*L$ and $\mathcal{L}^* \otimes \pi^*F$ is a subbundle of $\pi^*E$.

Now, $\nabla^H$ induces a $\rho$-connection on $PH$; that is, a morphism of vector bundles $c : \pi^*E \rightarrow T(PH)$ which when composed with the morphism $T(PH) \rightarrow \pi^*(TM)$, induced by $d\pi$, gives $\pi^*\rho$.

We define $\mathcal{D} = c(\mathcal{L}^* \otimes \pi^*F)$, and we claim that $\mathcal{D}|_Y = C$. Indeed, this follows from the fact that, locally, we may choose local sections $u$ of $\mathcal{L}^*|_Y$ such that $((\pi^*\nabla^H)_{u \otimes v}u = 0$, for any $v \in (\pi^*F)|_Y$.

Therefore $C$ can be retrieved from $(E, \rho, Y, \nabla)$. This applies, for example, when $Z$ is endowed with a locally complete family of Riemann spheres, embedded with nonnegative normal bundles (see [6] for a different proof).
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