The Hénon-Lane-Emden system: 
a sharp nonexistence result

Andrea Carioli\* and Roberta Musina\†

Abstract
We deal with very weak positive supersolutions to the Hénon-Lane-Emden system on neighborhoods of the origin. In our main theorem we prove a sharp nonexistence result.

Keywords: weighted Lane-Emden system, critical hyperbola, distributional solutions.

2010 Mathematics Subject Classification: 35B09, 35B40, 35B33

1 Introduction
The system of elliptic equations

\[
\begin{aligned}
-Δu &= |x|^a v^{p-1} \\
-Δv &= |x|^b u^{q-1}
\end{aligned}
\]  (1.1)

has been largely studied since Mitidieri’s paper \[19\] appeared, in 1990. We focus our attention on the related problem

\[
\begin{aligned}
-Δu &\geq \lambda_1 |x|^a v^{p-1} \\
-Δv &\geq \lambda_2 |x|^b u^{q-1}
\end{aligned}
\]  (\(\mathcal{P}_{a,b}\))

\*SISSA, via Bonomea, 265 – 34136 Trieste, Italy. Email: acarioli@sissa.it. Partially supported by INDAM-GNAMPA.
\†Dipartimento di Matematica e Informatica, Università di Udine, via delle Scienze, 206 – 33100 Udine, Italy. Email: roberta.musina@uniud.it. Partially supported by Miur-PRIN 201274FYK7J04.
on punctured domains $\Omega \setminus \{0\}$, where $p, q > 1$, $\Omega \subset \mathbb{R}^n$ is a neighborhood of the origin and $n \geq 3$. We are interested in nonnegative, distributional (or very weak) solutions to $(P_{a,b})$, accordingly with the next definition.

**Definition 1.1** A nontrivial and nonnegative distributional solution to $(P_{a,b})$ on $\Omega \setminus \{0\}$ is a pair $u, v$ of nonnegative functions satisfying

$$u, v \in L^1_{\text{loc}}(\Omega \setminus \{0\}), u^{q-1}, v^{p-1} \in L^1_{\text{loc}}(\Omega \setminus \{0\}),$$

for which there exist $\lambda_1, \lambda_2 > 0$ such that the inequalities in $(P_{a,b})$ hold in the sense of distributions on $\Omega \setminus \{0\}$.

Problems (1.1) and $(P_{a,b})$ change their nature depending on the sign of the quantity $(p - 1)(q - 1) - 1$. One has to distinguish between the following cases:

- **(AC)** $\frac{1}{p} + \frac{1}{q} < 1$ [Anticoercive case]
- **(H)** $\frac{1}{p} + \frac{1}{q} = 1$ [Homogeneous case]
- **(C)** $\frac{1}{p} + \frac{1}{q} > 1$ [Coercive case]

In the homogeneous case (H) the parameters $\lambda_1, \lambda_2$ have to be regarded as (possibly nonlinear) eigenvalues and can not be a priori prescribed. If (AC) or (C) applies, then one can always assume that $\lambda_1 = \lambda_2 = 1$.

In the present paper we prove a sharp nonexistence result in the spirit of the paper [6] by Brezis and Cabré. More precisely, for fixed $p, q > 1$ we find the region $E_{p,q}$ of parameters $a, b \in \mathbb{R}$, for which there exist positive distributional solutions to $(P_{a,b})$ in neighborhoods of the origin. The set $E_{p,q}$ is defined as follows.

- **Anticoercive case.** If (AC) holds, then

$$E_{p,q} := \left\{(a, b) \mid a, b > -n, \frac{a}{p} + \frac{b}{p'} + 2 > 0, \frac{a}{q'} + \frac{b}{q} + 2 > 0\right\}.$$

- **Homogeneous case.** We put

$$E_{p,p'} := \left\{(a, b) \mid a, b > -n, \frac{a}{p} + \frac{b}{p'} + 2 \geq 0\right\}.$$
• Coercive case. If (C) holds, then

\[
E_{p,q} := \left\{ (a, b) \mid a, b > -n, \quad \frac{a + n}{p} + \frac{b + n}{q(p-1)} > n - 2 \right\}.
\]

We are in position to state our main result.

**Theorem 1.2** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain containing the origin, and let \( p, q > 1 \). Then \( (P_{a,b}) \) has a nontrivial and nonnegative distributional solution on \( \Omega \setminus \{0\} \) if and only if \( (a, b) \in E_{p,q} \).

Trivially, any distributional solution \( u \geq 0 \) to

\[-\Delta u \geq \lambda |x|^a u^{p-1} \tag{1.2}\]

gives rise to the solution \( u, v = u \) to the corresponding system

\[
\begin{align*}
-\Delta u &\geq \lambda_1 |x|^a v^{p-1} \\
-\Delta v &\geq \lambda_2 |x|^a u^{p-1},
\end{align*}
\tag{1.3}
\]

for \( \lambda_1 = \lambda_2 = \lambda \). The converse might not be true, in general. Recall that by [6, Theorem 0.1], the inequality (1.2) has no nontrivial and nonnegative distributional solutions on \( \Omega \setminus \{0\} \) if \( \lambda > 0 \), \( p = 3 \) and \( a \leq -2 \) (see also [8, Theorem 1.2] for \( p > 2 \) and for more general nonlinearities). Thanks to Theorem 1.2, we immediately get the following extension of [6, Theorem 0.1] for systems.

**Corollary 1.3** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain containing the origin, and let \( p > 1 \). Then (1.3) has a nontrivial and nonnegative distributional solution on \( \Omega \setminus \{0\} \) if and only if one of the following conditions is satisfied:

i) \( p > 2 \) and \( a > -2 \)

ii) \( p = 2 \) and \( a \geq -2 \)

iii) \( 1 < p < 2 \), \( a > -n \) and \( p < p_a := \frac{2(n-1) + a}{n-2} \).
Notice that \( p_a \) coincides with Serrin’s critical exponent when \( a = 0 \).

Theorem 1.2 combined with the action of the Kelvin transform

\[
\mathcal{K} : L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \to L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\}), \quad (\mathcal{K}w)(x) = |x|^{2-n}w\left(\frac{x}{|x|^2}\right),
\]

immediately leads to the following sharp Liouville-type result (power-type solutions are computed in the appendix).

**Theorem 1.4** The system of inequalities \((P_{a,b})\) has a positive distributional solution \( u, v \) on \( \mathbb{R}^n \setminus \{0\} \) if and only if the system \((1.1)\) has a positive power-type solution, that is, having the form \( u(x) = c_1|x|^\alpha, v(x) = c_2|x|^\beta \).

Theorems 1.2 and 1.4 are related to some known results. Serrin and Zou [29] constructed positive radial solutions of class \( C^2(\mathbb{R}^n) \) under the assumptions \( a = b = 0, \frac{n}{p} + \frac{n}{q} \leq n - 2 \). One can adapt the shooting method in [29] to find a bounded solution to \((1.1)\) in a ball \( \Omega \) about the origin if \( a, b > -2 \) and \((a, b) \in E_{p,q} \). The restriction \( a, b > -2 \) is necessary to find solutions of class \( C^0(\Omega) \cap C^2(\Omega \setminus \{0\}) \), see for instance [1].

Under the anticoercivity assumption (AC), Bidaut-Veron and Giacomini [2] investigated an equivalent Hamiltonian system to prove the existence of a radial solution \( u, v \in C^0(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\}) \) on \( \mathbb{R}^n \) if and only if \( a, b > -2 \) and

\[
\frac{a + n}{p} + \frac{b + n}{q} \leq n - 2.
\]

To prove the existence part in Theorem 1.2 we write

\[
E_{p,q} = E^+_{p,q} \cup E^-_{p,q},
\]

where \( E^+_{p,q} \) is the set of pairs \((a, b)\) such that

\[
a, b > -n, \quad \frac{a + n}{p} + \frac{b + n}{q} > n - 2,
\]

and \( E^-_{p,q} = E_{p,q} \setminus E^+_{p,q} \). If \((a, b) \in E^-_{p,q}\) then the system \((P_{a,b})\) admits power type solutions, see the explicit computations in the Appendix. For \((a, b) \in E^+_{p,q}\) we take
a large ball $B$ about the origin containing $\Omega$ and we study the system
\begin{equation}
\begin{cases}
-\Delta u = \lambda_1 |x|^a v^{p-1} \\
-\Delta v = \lambda_2 |x|^b u^{q-1} \\
u, v > 0 \quad \text{in } B \\
u = v = 0 \quad \text{on } \partial B
\end{cases}
\end{equation}
(1.6)

The existence of a solution to $(P_{a,b})$ readily follows from the next theorem, that might have an independent interest.

**Theorem 1.5** If (1.5) holds, then there exist $\lambda_1, \lambda_2 > 0$ such that (1.6) has at least a radial solution $u, v$ satisfying
\begin{align*}
u, v &\in W^{2,1} \cap W_0^{1,1}(B) , \quad \int_B |x|^b |u|^q \, dx < \infty , \quad \int_B |x|^a |v|^p \, dx < \infty .
\end{align*}
(1.7)

Theorem 1.5 will be proved in Section 2 via variational arguments. A simple computation shows that $u, v$ can never be a power-type solution. Notice that Theorem 1.5 provides existence also for certain exponents $a, b \neq -2$.

Most of the available nonexistence results concern the system of elliptic equations (1.1) or deal with more regular solutions. First of all, we cite the pioneering paper [18] by Gidas and Spruck, and in particular their Theorem A.3. In [1], Bidaut-Veron used a clever and interesting trick to prove a nonexistence result of classical solutions $u, v$ to (1.1) on the punctured domain $\Omega \setminus \{0\}$. Her argument plainly covers problem $(P_{a,b})$ and can be used to prove Theorem 1.4 under the additional assumption $u, v \in C^2(\mathbb{R}^n \setminus \{0\})$.

D’Ambrosio and Mitidieri [13, Theorem 3.5] used representation formulae to prove the nonexistence result in Theorem 1.4 for locally integrable distributional solutions on $\mathbb{R}^n$. Notice however that [13] include a much larger class of systems.

The nonexistence part of Theorem 1.2 will be proved in Section 3.

The available literature for (1.1) and related problems is very extensive. The interested reader can find exhaustive surveys in [2, 13, 16], besides remarkable results. A number of papers (see for instance [26, 28, 30, 31]) deal with the Hénon-Lane-Emden Conjecture, that originated from the nonexistence results in [19, 21]. We cite also [3, 4, 5, 10, 11, 14, 15, 17, 20, 23, 24, 25, 27], and the references therein.
2 Proof of Theorem 1.5: existence

The homogeneous case (H) is covered by [9, Theorem 1.3]. Thus, we assume that $q \neq p'$. Since the system (1.6) is not homogeneous, we can fix $\lambda_1 = \lambda_2 = 1$. To get existence, we follow the outline of the proof in [9]. For brevity, we will skip some details.

Our approach is based on the formal equivalence, already noticed for instance in [12], between the system (1.6) and the fourth order Navier problem

$$\begin{cases}
-\Delta \left( |x|^{-a(p' - 1)}(-\Delta u)^{p' - 1} \right) = |x|^b u^{p' - 1} \\
u, -\Delta u > 0 & \text{in } B \\
u = \Delta u = 0 & \text{on } \partial B.
\end{cases} \tag{2.1}$$

We use variational methods to show that (2.1) admits a radial weak solution $u$ in a suitably defined energy space. To conclude the proof, one only has to check that the pair $u, v := |x|^{-a(p' - 1)}(-\Delta u)^{p' - 1}$ solves (1.6).

The first step consists in defining

$$W^{2,p'}_{N,\text{rad}}(B; |x|^{-a(p' - 1)}dx)$$

as the completion of the space of radial functions $u \in C^2(\overline{B})$, such that

$$u = 0 \quad \text{on } \partial \Omega, \quad \Delta u \equiv 0 \quad \text{in a neighborhood of the origin},$$

with respect to the norm

$$\|u\| = \left( \int_B |x|^{-a(p' - 1)}|\Delta u|^{p'} dx \right)^{1/p'}.$$

We claim that the infimum

$$m := \inf_{u \neq 0} \frac{\int_B |x|^{-a(p' - 1)}|\Delta u|^{p'} dx}{\left( \int_B |x|^b |u|^q dx \right)^{p'/q}}$$

is positive and achieved by some function $u$. For the sake of clarity, we distinguish the coercive case from the anticoercive one.
Coercive case. If $q < p'$ we take any exponent $b_0$, such that $b_0 > -n$ and

$$\begin{align*}
(i) & \quad \frac{a}{p} + \frac{b_0}{p'} + 2 > 0, \\
(ii) & \quad \frac{b_0 + n}{p'} < \frac{b + n}{q}.
\end{align*}$$

Thanks to (i), we have that $W^{2,p'}_{N,\text{rad}}(B; |x|^{-a(p'-1)} dx)$ is compactly embedded into $L^{p'}(B; |x|^{b_0} dx)$ by [9, Lemma 2.8]. On the other hand, the space $L^{p'}(B; |x|^{b_0} dx)$ is continuously embedded into $L^q(B; |x|^b dx)$ by (ii) and Hölder inequality. The claim follows by standard arguments.

Anticoercive case. Fix exponents $a_0, b_0$ satisfying

$$\frac{a_0 + n}{p} + \frac{b_0 + n}{q} = n - 2, \quad -n < a_0 \leq a, \quad -n < b_0 < b,$$

that is possible as $(a, b) \in E_{p,q}$ and [1.5] holds. By [22, Theorem 4.10], we have that there exists a constant $c > 0$ such that

$$\int_{\mathbb{R}^n} |x|^{-a_0(p'-1)}|\Delta \varphi|^p' \, dx \geq c \left( \int_{\mathbb{R}^n} |x|^{b_0} |\varphi|^q \, dx \right)^{p'/q}$$

for any radially symmetric $\varphi \in C_c^{\infty}(\mathbb{R}^n)$. Since we are dealing with a bounded domain, it is easy to infer that $W^{2,p'}_{N,\text{rad}}(B; |x|^{-a(p'-1)} dx)$ is compactly embedded into $L^q(B; |x|^b dx)$, and this proves the claim.

Next, let $u$ be an extremal for the infimum $m$. Use the arguments in [9, Lemma 3.2] to show that

$$u \in W^{2,1} \cap W^{1,1}_0(B), \quad v := |x|^{-a(p'-1)}|\Delta u|^{p'-2}(-\Delta u) \in W^{2,1} \cap W^{1,1}_0(B),$$

and that, up to a Lagrange multiplier, the pair $u, v$ is a weak solution to the system

$$\begin{align*}
-\Delta u &= |x|^{a} |v|^{p-2} v, \\
-\Delta v &= |x|^{b} |u|^{q-2} u
\end{align*}$$

in the ball $B$. To check that $u, v$ are positive on $B$ use the (standard) argument in [9, Lemma 3.4]. The proof of the existence part is complete.
3 Proof of Theorem 1.2: nonexistence

In this proof we denote by $c$ any inessential positive constant.

Let $u, v$ be a nontrivial and nonnegative distributional solution to $(P_{a,b})$ in $\Omega \setminus \{0\}$. We claim that the following facts hold:

i) $u, v \in L^1_{\text{loc}}(\Omega)$, $|x|^b u^{q-1}, |x|^a v^{p-1} \in L^1_{\text{loc}}(\Omega)$;

ii) $u, v$ solve $(P_{a,b})$ in the sense of distributions on $\Omega$;

iii) $u, v$ are superharmonic and positive on $\Omega$;

iv) $a, b > -n$.

The first two conclusions are immediate consequences of [7, Lemma 1]. Since $u, v \in L^1_{\text{loc}}(\Omega)$ solve $-\Delta u \geq 0$, $-\Delta v \geq 0$, then $u, v$ are superharmonic by well known and classical facts. In particular $u$ and $v$ can be assumed to be lower semicontinuous and positive on $\Omega$, and so iii) holds. Finally, since $v$ is lower semicontinuous and positive, we can find $\delta > 0$ such that $|x|^a v^{p-1} \geq \delta |x|^a$ in a closed ball $\overline{B} \subset \Omega$ about the origin. Now, from $|x|^a v^{p-1} \in L^1(B)$ we infer that the weight $|x|^a$ is locally integrable on $B$, that is, $a > -n$. The conclusion $b > -n$ can be proved in a similar way.

Next, up to dilations we can assume that $\Omega$ contains the closure of the unit ball $B$ about the origin. Let $G^x(\cdot)$ be the Green function for $B$ and let $h^x(\cdot)$ be its regular part, that is,

$$G^x(y) = c_n \left[ \frac{1}{|y-x|^{n-2}} - h^x(y) \right], \quad h^x(y) = \frac{|x|^{n-2}}{|y|^2 - |x|^{n-2}}.$$ 

We claim that

$$u(x) \geq \lambda_1 \int_B G^x_B(y)|y|^a v(y)^{p-1} dy,$$

$$v(x) \geq \lambda_2 \int_B G^x_B(y)|y|^b u(y)^{q-1} dy$$

almost everywhere on $B$. Let us prove the first of the inequalities in (3.1), the second one being similar. For any integer $k \geq 1$, we put

$$f_k = \min\{\lambda_1 |x|^a v^{p-1}, k\}$$
and we introduce the unique solution \( u_k \) to the problem

\[
-\Delta u_k = f_k, \quad u_k \in H^1_0(B).
\]

Green’s representation formula yields

\[
u_k(x) = \int_B G_B^x(y) f_k \, dy,
\]
and the maximum principle for superharmonic functions implies that \( u - u_k \geq 0 \) almost everywhere in \( B \). Thus Fatou’s Lemma gives

\[
u(x) \geq \liminf_{k \to \infty} u_k(x) \geq \int_B G_B^x(y) f(y) \, dy,
\]
for almost every \( x \in B \), as claimed.

We will use (3.1) to estimate the quantities

\[
U_R = \int_{B_R} |x|^b |u|^{q-1} \, dx, \quad V_R = \int_{B_R} |x|^a |v|^{p-1} \, dx
\]
for any \( R > 0 \) small enough (recall that \( U_R, V_R \) are finite). For \( |x| < 1/2 \) and \( y \in B \) we have the uniform lower bound

\[
G^x(y) \geq c_n \left[ \frac{1}{(|x| + |y|)^{n-2}} - M \right], \quad M = \max_{|x| \leq 1/2, |y| < 1} h^x(y).
\]
Therefore, if \( R_0 \) is small enough and \( R \in (0, R_0) \), then \( G_B^x(y) \geq cR^{2-n} \) for \( x, y \in B_R \). Using (3.1), we infer that

\[
u(x) \geq cR^{2-n} \int_{B_R} |y|^a v(y)^{p-1} \, dy, \quad v(x) \geq cR^{2-n} \int_{B_R} |y|^b u(y)^{q-1} \, dy
\]
for almost every \( x \in B_R \), and hence

\[
U_R \geq cR^{(2-n)(q-1)+b+n} V_R^{q-1}, \quad V_R \geq cR^{(2-n)(p-1)+a+n} U_R^{p-1}.
\]

With simple computations we arrive at

\[
U_R^{(p-1)(q-1) - 1} \leq c R^{-(q-1)p \left[ \frac{a+n}{p} + \frac{b+n}{p(q-1)} - n-2 \right]}, \quad (3.2)
\]

\[
V_R^{(p-1)(q-1) - 1} \leq c R^{-(p-1)q \left[ \frac{a+n}{q} + \frac{b+n}{q} - n-2 \right]}, \quad (3.3)
\]
Now we distinguish three cases, depending whether (C), (H) or (AC) is satisfied.

**Coercive case (C).** We have that $\theta := 1 - (p - 1)(q - 1) > 0$, and thus (3.3) gives

$$R^{(p-1)p\left[\frac{a+n}{q(p-1)} + \frac{b+n}{q} - (n-2)\right]} \leq c \mathcal{V}_R.$$ 

But clearly $\mathcal{V}_R \to 0$ as $R \to 0$. Thus

$$\frac{a + n}{q(p-1)} + \frac{b + n}{q} - (n - 2) > 0.$$

A similar argument and (3.2) lead to the conclusion that $(a, b) \in E_{p,q}$.

**Homogeneous case (H).** We have that $q = p'$, and therefore (3.3) gives

$$1 \leq c R^{-p\left[\frac{a}{p} + \frac{b}{p'} + 2\right]}.$$ 

Hence $\frac{a}{p} + \frac{b}{p'} + 2 \geq 0$, that is, $(a, b) \in E_{p,p'}$.

**Anticoercive case (AC).** As $u, v$ are positive and superharmonic, they are uniformly bounded from below on any ball $B_{R_0} \subset \Omega$ about the origin. Hence, for $R \in (0, R_0]$ we get

$$\mathcal{V}_R = \int_{B_R} |x|^p v^{p-1} dx \geq c R^{n+a}, \quad \mathcal{U}_R = \int_{B_R} |x|^q u^{q-1} dx \geq c R^{n+b},$$

that compared with (3.2), (3.3) give

$$c \leq R^{-(q-1)p\left[\frac{a}{p} + \frac{b}{p'} + 2\right]}, \quad c \leq R^{-(p-1)q\left[\frac{a}{q} + \frac{b}{q'} + 2\right]},$$

as $(p - 1)(q - 1) - 1 > 0$. We immediately infer that

$$\frac{a}{q'} + \frac{b}{q} + 2 \geq 0, \quad \frac{a}{p} + \frac{b}{p'} + 2 \geq 0. \quad (3.4)$$

It remains to prove that strict inequalities hold in (3.4). We argue by contradiction. Assume for instance that

$$\frac{a}{q'} + \frac{a}{q} + 2 = 0. \quad (3.5)$$
Then the second inequality in (3.4) and \((p-1)(q-1) - 1 > 0\) imply that \(a \leq -2 \leq b\). In addition, (3.3) becomes
\[
\mathcal{V}_R \leq c R^{n+a}.
\] (3.6)

We first consider the case
\[-n < a < -2, \quad b = -2q - a(q-1) > -2.\]

Since \(v\) is bounded from below on a small ball \(B_{2\sqrt{R}}\), then \(-\Delta u \geq c|x|^a\) on \(B_{2\sqrt{R}}\).

Thus, by the maximum principle,
\[
u(x) \geq c\left(|x|^{a+2} - (2\sqrt{R})^{a+2}\right) \quad \text{on} \quad B_{2\sqrt{R}}.
\]

In particular, \(u(x) \geq c|x|^{a+2}\) on \(B_{\sqrt{R}}\), so that
\[-\Delta v \geq c|x|^{-2} \quad \text{on} \quad B_{\sqrt{R}}.
\]

as \(b + (a + 2)(q - 1) = -2\). Again by the maximum principle, \(v(x) \geq c \log(\sqrt{R}/|x|)\) on \(B_{\sqrt{R}}\), and in particular
\[v(x) \geq c \log|x| \quad \text{on} \quad B_R.
\]

We infer that
\[
\mathcal{V}_R \geq c \int_{B_R} |x|^a \log |x|^{p-1} \, dx = O\left(R^{n+a} |\log R|^{p-1}\right),
\]
in contradiction with (3.6). To exclude the case \(a = -2\), notice that in this case \(b = -2\) by (3.5), hence \(-\Delta v \geq c|x|^{-2}\) on \(\Omega\). Conclude as before. \(\square\)

**Acknowledgements**

The authors are pleased to thank Enzo Mitidieri and Lorenzo D’Ambrosio for useful remarks on how to improve the presentation of this paper.

**Appendix: power-type solutions**

In [1], Bidaut-Veron computed the positive “power-type” solutions \(u, v\) to
\[
\begin{cases}
-\Delta u = \lambda_1 |x|^a v^{p-1} \\
-\Delta v = \lambda_2 |x|^b u^{q-1}
\end{cases}
\] (A.1)
on $\mathbb{R}^n \setminus \{0\}$. Actually we are just interested in finding the set $Q_{p,q}$ of parameters $a, b$, for which $(P_{a,b})$ admits power-type solutions.

Let us start with a few remarks about the Kelvin transform defined in (1.4). A simple computation shows that $K$ maps distributional solutions $u, v \in L^1_{\text{loc}}(\Omega \setminus \{0\})$ of $(P_{a,b})$ into distributional solutions $Ku, Kv \in L^1_{\text{loc}}(\hat{\Omega})$ to $(P_{\kappa(a,b)})$, where $\hat{\Omega}$ is the reflection of $\Omega$ with respect to the unitary sphere and

$$\kappa(a,b) = (-a - 2n + p(n - 2), -b - 2n + q(n - 2)), \quad \kappa : \mathbb{R}^2 \to \mathbb{R}^2.$$

Trivially, $K$ maps power-type solutions into power-type solutions, that is, $Q_{p,q}$ is invariant under the action of $\kappa$. Next we notice that $\kappa$ is a central inversion with respect to its fixed point $F$:

$$\kappa(a,b) = 2F - (a,b), \quad F = \left(\frac{n - 2}{2} - n, \frac{n - 2}{2} - n\right).$$

If the pair $u(x) = |x|^\alpha, v(x) = |x|^\beta$ solves (A.1) with respect to some $\lambda_1, \lambda_2 > 0$, then clearly $\alpha, \beta$ have to satisfy

$$\begin{cases}
(q - 1)\alpha - \beta = -b - 2 \\
-a + (p - 1)\beta = -a - 2.
\end{cases} \quad \text{(A.2)}$$

In the non-homogeneous cases (AC) and (C), we have that the system (A.2) admits the unique solution

$$\alpha = -\frac{a}{p} + \frac{b}{p} + 2 \frac{1}{(p - 1)(q - 1) - 1} p, \quad \beta = -\frac{a}{q} + \frac{b}{q} + 2 \frac{1}{(p - 1)(q - 1) - 1} q.$$

The corresponding pair $u_\alpha, v_\beta$ solves $(P_{a,b})$ with $\lambda_1, \lambda_2$ given, up to positive multipliers, by

$$\lambda_1 = -\left(\frac{a}{p} + \frac{b}{p} + 2\right) \left(\frac{a + n}{q(p - 1)} + \frac{b + n}{q} - (n - 2)\right)$$

$$\lambda_2 = -\left(\frac{a}{q} + \frac{b}{q} + 2\right) \left(\frac{a + n}{p} + \frac{b + n}{p(q - 1)} - (n - 2)\right).$$

In conclusion, nontrivial and positive power-type solutions to (A.1) exist if and only if the couple of exponents $(a, b)$ belongs to the open parallelogram $Q_{p,q}$ whose
vertices are
\[ X = (-n, q(n - 2) - n), \quad X' = \kappa(X) = (p(n - 2) - n, -n), \]
\[ V = (-2, -2), \quad V' = \kappa(V). \]

More explicitly, if (AC) holds we have that
\[
Q_{p,q} = \begin{cases} 
\min \left\{ \frac{a}{p} + \frac{b}{p'} + 2, \frac{a}{q'} + \frac{b}{q} + 2 \right\} > 0 \\
\max \left\{ \frac{a + n}{p} + \frac{b + n}{p(q-1)}, \frac{a + n}{q(p-1)} + \frac{b + n}{q} \right\} < n - 2 
\end{cases}
\]
while in the coercive case (C) we find
\[
Q_{p,q} = \begin{cases} 
\max \left\{ \frac{a}{p} + \frac{b}{p'} + 2, \frac{a}{q'} + \frac{b}{q} + 2 \right\} < 0 \\
\min \left\{ \frac{a + n}{p} + \frac{b + n}{p(q-1)}, \frac{a + n}{q(p-1)} + \frac{b + n}{q} \right\} > n - 2 
\end{cases}
\]
Points in the boundary of \( Q_{p,q} \) correspond to trivial solutions to (A.1) in the sense of Bidaut-Veron [1], that is, at least one of the components is harmonic on \( \mathbb{R}^n \setminus \{0\} \).

The coordinates of the vertices \( X, X' \) satisfy
\[
\frac{a + n}{p} + \frac{b + n}{q} = n - 2. \quad (CL)
\]
The remaining vertices \( V \) and \( V' \) lie on opposite sides of line in the \( a, b \) plane given by \( (CL) \). More precisely, \( V \) is below \( (CL) \) in the anticoercive case (AC), while \( V \) is above \( (CL) \) if (C) holds.

In the homogenous case (H), the line \( (CL) \) becomes
\[
\frac{a}{p} + \frac{b}{p'} + 2 = 0 \quad (CL_H)
\]
and with simple calculations we find that \( Q_{p,q} \) collapses into
\[
Q_{p,p'} = \left\{ (a, b) \in \mathbb{R}^2 : a, b > -n, \frac{a}{p} + \frac{b}{p'} + 2 = 0 \right\}.
\]
that is the open segment of endpoints $X, X'$.

In the next pictures we represent the set $Q_{p,q}$ in the $(a, b)$ plane.

Notice that, in any case, $Q_{p,q} = E_{p,q} \cap \kappa(E_{p,q})$.

In the next pictures we summarize our existence/nonexistence results. We have existence of weak solutions on bounded neighborhoods of the origin in the light gray zone. Power-type solutions correspond to the darker area. The Brezis-Cabr{é} nonexistence result for the inequality (1.2) is related to the vertex $V = (-2, -2)$ in Figure (c) (with $p = q > 2$).

References

[1] M. F. Bidaut-Veron, Local behaviour of the solutions of a class of nonlinear elliptic systems, *Adv. Differential Equations* 5 (2000), no. 1-3, 147–192.

[2] M. F. Bidaut-Veron and H. Giacomini, A new dynamical approach of Emden-Fowler equations and systems, *Adv. Differential Equations* 15 (2010), 1033–1082.
[3] D. Bonheure, E. Moreira dos Santos and M. Ramos, Ground state and non-
ground state solutions of some strongly coupled elliptic systems, *Trans. Amer. 
Math. Soc.* **364** (2012), 447–491.

[4] D. Bonheure, E. Moreira dos Santos and M. Ramos, Symmetry and symmetry 
breaking for ground state solutions of some strongly coupled elliptic systems, *J. 
Funct. Anal.* **264** (2013), 62–96.

[5] J. Busca and R. Manasevich, A Liouville-type theorem for Emden system, *Indi-
an Univ. Math. J.* **51** (2002), 37-51.

[6] H. Brezis and X. Cabré, Some simple nonlinear PDE’s without solutions, *Boll. 
Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)* **1** (1998), no. 2, 223–262.

[7] H. Brezis, L. Dupaigne and A. Tesei, On a semilinear elliptic equation with 
inverse-square potential, *Selecta Math. (N.S.)* **11** (2005), no. 1, 1–7.

[8] P. Caldiroli and R. Musina, On a class of two-dimensional singular elliptic prob-
lems, *Proc. Roy. Soc. Edinburgh Sect. A* **131** (2001), no. 3, 479–497.

[9] A. Carioli and R. Musina, The homogeneous Hénon-Lane-Emden sys-
tem, *NoDEA Nonlinear Differential Equations Appl.*, to appear. Preprint 
[arXiv:1407.1522](http://arxiv.org/abs/1407.1522) (2014).

[10] G. Caristi, L. D’Ambrosio and E. Mitidieri, Representation formulae for so-
lutions to some classes of higher order systems and related Liouville theorems, 
*Milan J. Math.* **76** (2008), 27–67.

[11] Ph. Clément, D. G. de Figueiredo and E. Mitidieri, Positive solutions of semi-
linear elliptic systems, *Comm. Partial Differential Equations* **17** (1992), no. 5-6, 
923–940.

[12] P. Clément, P. Felmer and E. Mitidieri, Homoclinic orbits for a class of infinite-
dimensional Hamiltonian systems, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **24** 
(1997), no. 2, 367–393.

[13] L. D’Ambrosio and E. Mitidieri, Hardy-Littlewood-Sobolev systems and related 
Liouville theorems, *Discrete Contin. Dyn. Syst. Ser. S* **7** (2014), no. 4, 653–671.
[14] D. G. de Figueiredo, I. Peral and J. D. Rossi, The critical hyperbola for a Hamiltonian elliptic system with weights, *Ann. Mat. Pura Appl.* (4) **187** (2008), no. 3, 531–545.

[15] M. Fazly, Liouville type theorems for stable solutions of certain elliptic systems, *Adv. Nonlinear Stud.* **12** (2012), 1–17.

[16] M. Fazly and N. Ghoussoub, On the Hénon-Lane-Emden conjecture, *Discrete Contin. Dyn. Syst.* **34** (2014), no. 6, 2513–2533.

[17] M. García-Huidobro, R. Manasevich, E. Mitidieri and C. Yarur, Existence and nonexistence of positive singular solutions for a class of semilinear elliptic systems, *Arch. Rational Mech. Anal.* **140** (1997), no. 3, 253–284.

[18] B. Gidas and J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, *Comm. Pure Appl. Math.* **34** (1981), no. 4, 525–598.

[19] E. Mitidieri, A Rellich identity and applications, *Rapporti interni dell’Università di Udine* **25** (1990), 1-35.

[20] E. Mitidieri, A Rellich type identity and applications, *Comm. Partial Differential Equations* **18** (1993), 125–151.

[21] E. Mitidieri, Nonexistence of positive solutions of semilinear elliptic systems in $\mathbb{R}^N$, *Differential Integral Equations* **9** (1996), 465–479.

[22] R. Musina, Weighted Sobolev spaces of radially symmetric functions, *Ann. Mat. Pura Appl.* (4) **193** (2014), no. 6, 1629–1659.

[23] R. Musina and K. Sreenadh, Radially symmetric solutions to the Hénon-Lane-Emden system on the critical hyperbola, *Commun. Contemp. Math.* **16** (2014), no. 3, 1350030, 16 pp.

[24] L. A. Peletier and R. C. A. M. Van der Vorst, Existence and nonexistence of positive solutions of nonlinear elliptic systems and the biharmonic equation, *Differential Integral Equations* **5** (1992), no. 4, 747–767.

[25] Q. H. Phan, Liouville-type theorems and bounds of solutions for Hardy-Hénon elliptic systems, *Adv. Differential Equations* **17** (2012), 605–634.
[26] P. Poláčik, P. Quittner and P. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems, Part I: Elliptic systems, Duke Math. J. 139 (2007), 555–579.

[27] J. Serrin and H. Zou, Non-existence of positive solutions of semilinear elliptic systems, in A tribute to Ilya Bakelman (College Station, TX, 1993), 55–68, Discourses Math. Appl., 3 Texas A & M Univ., College Station, TX.

[28] J. Serrin and H. Zou, Non-existence of positive solutions of Hénon-Lane-Emden systems, Differential Integral Equations 9 (1996), 635–653.

[29] J. Serrin and H. Zou, Existence of positive solutions of the Lane-Emden system, Atti Semin. Mat. Fis. Univ. Modena 46 (1998), 369–380.

[30] P. Souplet, The proof of the Hénon-Lane-Emden conjecture in four space dimensions, Adv. Math. 221 (2009), 1409–1427.

[31] M.A.S. Souto; A priori estimates and existence of positive solutions of non-linear cooperative elliptic systems, Differential Integral Equations 8 (1995) 1245-1258.