Length and multiplicity of the local cohomology with support in a hyperplane arrangement

Toshinori Oaku

September 4, 2015

Abstract

Let \( R \) be the polynomial ring in \( n \) variables with coefficients in a field \( K \) of characteristic zero. Let \( D_n \) be the \( n \)-th Weyl algebra over \( K \). Suppose that \( f \in R \) defines a hyperplane arrangement in the affine space \( K^n \). Then the length and the multiplicity of the first local cohomology group \( H_1^{(f)}(R) \) as left \( D_n \)-module coincide and are explicitly expressed in terms of the Poincaré polynomial or the Möbius function of the arrangement.

1 Introduction

Let \( K \) be a field of characteristic zero and \( R = K[x] = K[x_1, \ldots, x_n] \) be the polynomial ring in \( n \) variables \( x = (x_1, \ldots, x_n) \). For a nonzero polynomial \( f \in K[x] \), let us consider the first local cohomology group \( H_1^{(f)}(R) = R[f^{-1}]/R \), where \( R[f^{-1}] = R_f \) is the localization of \( R \) with respect to the multiplicative set \( \{ f^k \mid k \in \mathbb{N} \} \) with \( \mathbb{N} = \{ 0, 1, 2, 3, \ldots \} \).

Let \( D_n = R(\partial) = R(\partial_1, \ldots, \partial_n) \) be the \( n \)-th Weyl algebra, i.e., the ring of differential operators with polynomial coefficients with respect to the variables \( x \), where we denote \( \partial = (\partial_1, \ldots, \partial_n) \) with \( \partial_i = \partial/\partial x_i \) being the derivation with respect to \( x_i \). An arbitrary element \( P \) of \( D_n \) is written in a finite sum

\[
P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x) \partial^\alpha \quad \text{with} \quad a_\alpha(x) \in K[x],
\]

where we denote \( \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \) for a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \).

Then \( H_1^{(f)}(R) \) has a natural structure of left \( D_n \)-module and is holonomic (\([E]\)). We are mainly concerned with the length and the multiplicity of
$H^1_{(f)}(R)$ as a left $D_n$-module in case $f$ defines a hyperplane arrangement $\mathcal{A}$ in the affine space $\mathbb{K}^n$; i.e., $f$ is a multiple of linear (i.e., first-degree) polynomials. In particular, we show that the length and the multiplicity both coincide with $\pi(\mathcal{A}, 1) - 1$, where $\pi(\mathcal{A}, t)$ is the Poincaré polynomial of the arrangement $\mathcal{A}$.

The length of $R[f^{-1}]$ as left $D_n$-module, which equals that of $H^1_{(f)}(R)$ plus one, with $f$ defining a hyperplane arrangement was studied e.g., in [1], [8]. The characteristic cycle of the local cohomology with respect to an arrangement of linear subvarieties was studied in [2]. Although not explicitly stated, Corollary 1.3 of [2] should yield main results of this paper. We give a direct proof for hyperplane arrangements.

2 Length and multiplicity

First let us recall basic facts about the length and the multiplicity of a left $D_n$-module following J. Bernstein ([3]). Let $M$ be a finitely generated left $D_n$-module. A composition series of $M$ of length $k$ is a sequence

$$M = M_0 \supset M_1 \supset \cdots \supset M_k = 0$$

of left $D_n$-submodules such that $M_i/M_{i-1}$ is a nonzero simple left $D_n$-module (i.e. having no proper left $D_n$-submodule other than 0) for $i = 1, \ldots k$. The length of $M$, which we denote by length $M$, is the least length of composition series (if any) of $M$. If there is no composition series, the length of $M$ is defined to be infinite. The length is additive in the sense that if

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$$

is an exact sequence of left $D_n$-modules of finite length, then mult $M = \text{mult } N + \text{mult } L$ holds.

For each integer $k$, set

$$F_k(D_n) = \{ \sum_{|\alpha|+|\beta| \leq k} a_{\alpha\beta} x^\alpha \partial^\beta \mid a_{\alpha\beta} \in \mathbb{K} \}.$$ 

In particular, we have $F_k(D_n) = 0$ for $k < 0$ and $F_0(D_n) = \mathbb{K}$. The filtration $\{F_k(D_n)\}_{k \in \mathbb{Z}}$ is called the Bernstein filtration on $D_n$.

Let $M$ be a finitely generated left $D_n$-module. A family $\{F_k(M)\}_{k \in \mathbb{Z}}$ of $\mathbb{K}$-subspaces of $M$ is called a Bernstein filtration of $M$ if it satisfies

(1) $F_k(M) \subset F_{k+1}(M) \quad (\forall k \in \mathbb{Z}), \quad \bigcup_{k \in \mathbb{Z}} F_k(M) = M$

(2) $F_j(D_n)F_k(M) \subset F_{j+k}(M) \quad (\forall j, k \in \mathbb{Z})$

2
(3) $F_k(M) = 0$ for $k \ll 0$

Moreover, $\{F_k(M)\}$ is called a good Bernstein filtration if

(4) $F_k(M)$ is finite dimensional over $K$ for any $k \in \mathbb{Z}$.

(5) $F_j(D_n)F_k(M) = F_{j+k}(M)$ ($\forall j \geq 0$) holds for $k \gg 0$.

Let $\{F_k(M)\}$ be a good Bernstein filtration of $M$. Then there exists a polynomial $h(T) = h_dT^d + h_{d-1}T^{d-1} + \cdots + h_0 \in \mathbb{Q}[T]$ such that

$$\dim_K F_k(M) = h(k) \quad (k \gg 0)$$

and $d!h_d$ is a positive integer. We call $h(T)$ the Hilbert polynomial of $M$ with respect to the filtration $\{F_k(M)\}$. The leading term of $h(T)$ does not depend on the choice of a good Bernstein filtration $\{F_k(M)\}$. The degree $d$ of the Hilbert polynomial $h(T)$ is called the dimension of $M$ and denoted by $\dim M$. The multiplicity of $M$ is defined to be $d!h_d$, which we denote by $\text{mult } M$.

If $M \neq 0$, then the dimension of $M$ is not less than $n$ (Bernstein’s inequality). We call $M$ holonomic if $M = 0$ or $\dim M = n$. It is known that $H_j^i(R)$ is holonomic for any ideal $I$ of $R$ and any integer $j$ ([6]).

If $M$ is a holonomic left $D_n$-module, we have an equality $\text{length } M \leq \text{mult } M$. Moreover, the multiplicity is additive for holonomic left $D_n$-modules.

Lemma 2.1 Let $h_0 = h_0(x) \in K[x]$ be a linear polynomial and $I$ be an ideal of $R := K[x]$. Let $R' := R/Rh_0$ be the affine ring associated with the hyperplane $h_0(x) = 0$ and set $I' = (I + Rh_0)/Rh_0$. Then we have

$$\text{length } H_{I+Rh_0}(R) = \text{length } H_{I'+1}^{i-1}(R'), \quad \text{mult } H_{I+Rh_0}(R) = \text{mult } H_{I'+1}^{i-1}(R')$$

for any integer $i$.

Proof: Since $H_{I+Rh_0}^i(R) = 0$ for $i \neq 1$, there is an isomorphism

$$H_{I+Rh_0}^1(R) \cong H_{I'+1}^{i-1}(H_{I+Rh_0}^1(R)).$$

We may assume by an affine coordinate transformation, which preserves the Bernstein filtration, that $h_0(x) = x_n$. Then we may regard $R' = K[x_1, \ldots, x_{n-1}]$ and have an isomorphism

$$H_{Rh_0}^1(R) \cong R' \otimes_K H_{[x_n]}^1(K[x_n]),$$

where the tensor product on the right-hand side is a left module over $D_n = D_{n-1} \otimes_K D_1$ with $D_1$ being the ring of differential operators in the variable $x_n$. 

Let \( \{f_1, \ldots, f_r\} \) be a set of generators of \( I \). We may assume that \( f_1, \ldots, f_r \) belong to \( R' \). Then for \( 0 \leq i_1 < \cdots < i_k \leq r \) with \( k \in \mathbb{N} \), the localization by \( f_{i_1} \cdots f_{i_k} \) yields

\[
(H^1_{R_0}(R))_{f_{i_1} \cdots f_{i_k}} = R'_{f_{i_1} \cdots f_{i_k}} \otimes_K H^1_{(x_n)}(K[x_n]).
\]

On the other hand, we have

\[
(H^1_{R_0}(R))_{x_n} = R' \otimes_K (H^1_{(x_n)}(K[x_n]))_{x_n} = 0.
\]

Hence \( H_{I+R_0}^{i-1}(R') \otimes_K H^1_{(x_n)}(K[x_n]) \) is the \((i-1)\)-th cohomology group of the \( \tilde{\text{C}} \)ech complex

\[
0 \to R' \otimes_K H^1_{(x_n)}(K[x_n]) \to \bigoplus_{1 \leq i \leq r} R'_{f_i} \otimes_K H^1_{(x_n)}(K[x_n]) \\
\to \bigoplus_{1 \leq i_1 < i_2 \leq r} R'_{f_1 f_2} \otimes_K H^1_{(x_n)}(K[x_n]) \to \cdots \to R'_{f_1 \cdots f_r} \otimes_K H^1_{(x_n)}(K[x_n]) \to 0,
\]

which is isomorphic to \( H_{I'}^{i-1}(R') \otimes_K H^1_{(x_n)}(K[x_n]) \). This implies

\[
H_{I+R_0}^i(R) \cong H_{I'}^{i-1}(R') \otimes_K H^1_{(x_n)}(K[x_n]) \\
\cong H_{I'}^{i-1}(R') \otimes_K (D_1/D_1 x_n) \cong (D_n/D_n x_n) \otimes_{D_{n-1}} H_{I'}^{i-1}(R'),
\]

where \( D_n/D_n x_n \) is regarded as a \((D_n, D_{n-1})\)-bimodule. The rightmost term is the \( D \)-module theoretic direct image of \( H_{I'}^{i-1}(R') \) with respect to the inclusion \( H_0 := \{ x \in V \mid x_n = 0 \} \to V \). In view of Kashiwara’s equivalence in the category of algebraic \( D \)-modules (see e.g., Theorem 7.11 of \[3\] or Theorem 1.6.1 of \[5\]), there is a one-to-one correspondence between the \( D_{n-1} \)-submodules \( M \) of \( H_{I'}^{i-1}(R') \) and the \( D_n \)-submodules \( M \otimes_K H^1_{(x_n)}(K[x_n]) \) of \( H_{I+R_0}^i(R) \).

This implies

\[
\text{length } H_{I+R_0}^i(R) = \text{length } H_{I'}^{i-1}(R').
\]

Next, let us show

\[
\text{mult } H_{I+R_0}^i(R) = \text{mult } H_{I'}^{i-1}(R').
\]

Let \( \{F_k\} \) be a good Bernstein filtration of \( H_{I'}^{i-1}(R') \) and set \( m = \text{mult } H_{I'}^{i-1}(R') \). We may assume \( F_k = 0 \) for \( k < 0 \). Then there exists a polynomial \( p(k) \) and \( k_1 \in \mathbb{Z} \) such that

\[
\dim_K F_k = p(k) = \frac{m}{(n-1)!} k_1^{n-1} + \text{(terms with degree } < n - 1\text{)}
\]
holds for \( k \geq k_1 \). Define a filtration \( \{G_k\} \) on \( H_i^{-1}(R') \otimes_K H^1_{(x_n)}(K[x_n^{-1}]) \) by

\[
G_k := \sum_{j=0}^{k} F_j \otimes_K (K[x_n^{-1}] + \cdots + K[x_n^{-(k-j)-1}]) = \bigoplus_{j=0}^{k} F_j \otimes_K K[x_n^{-(k-j)-1}].
\]

It is easy to see that \( \{G_k\} \) is a good Bernstein filtration. Hence we have

\[
\dim_K G_k = \sum_{j=0}^{k} \dim_K F_j = \sum_{j=0}^{k_1-1} \dim_K F_j + \sum_{j=k_1}^{k} \dim_K F_j + \sum_{j=k_1}^{k} p(j).
\]

By the assumption, there exists a polynomial \( q(k) \) of degree \( \leq n - 2 \) such that

\[
p(j) = \frac{m}{(n-1)!} j(j+1) \cdots (j+n-2) + q(j).
\]

Since

\[
\sum_{j=k_1}^{k} j(j+1) \cdots (j+n-2) = \frac{1}{n} \{k(k+1) \cdots (k+n-1)-(k_1-1)k_1 \cdots (k_1+n-2)\},
\]

we have

\[
\dim_K G_k = \frac{m}{n!} k^n + (\text{terms with degree } < n) \quad (\forall k \geq k_1).
\]

Thus we also have \( \text{mult } H^1_{I+R_{x_n}}(R) = m \). This completes the proof. \( \square \)

### 3 Hyperplane arrangements

Let \( f \in K[x] \) be a multiple of essentially distinct linear polynomials. Let \( \mathcal{A} \) be the hyperplane arrangement in \( V := K^n \) defined by \( f \).

**Theorem 3.1** Let \( H_0 \) be an element of \( \mathcal{A} \). Set \( \mathcal{A}' := \mathcal{A} \setminus \{H_0\} \) and let \( f' \) be the product of the defining polynomials of hyperplanes belonging to \( \mathcal{A}' \). Let us regard \( \mathcal{A}'' := \{H \cap H_0 \mid H \in \mathcal{A}', H \cap H_0 \neq \emptyset\} \) as a hyperplane arrangement in the affine space \( H_0 \). Let \( R'' = R/Rh_0 \) be the affine ring of \( H_0 \), where \( h_0 \) is a polynomial of first degree defining \( H_0 \). Let \( f'' \in R'' \) be the product of the defining polynomials of the elements of \( \mathcal{A}'' \). Then we have

\[
\begin{align*}
\text{length } H^1_{(f)}(R) &= \text{length } H^1_{(f')} (R) + \text{length } H^1_{(f'')} (R'' ) + 1, \\
\text{mult } H^1_{(f)}(R) &= \text{mult } H^1_{(f')} (R) + \text{mult } H^1_{(f'')} (R'') + 1.
\end{align*}
\]
Proof: By the Mayer-Vietoris exact sequence, we get an exact sequence

\[ 0 \rightarrow H^1_{(f')} (R) \oplus H^1_{(h_0)} (R) \rightarrow H^1_{(f)} (R) \rightarrow H^2_{(f')} + (h_0) (R) \rightarrow 0 \]

of holonomic left \( D_n \)-modules. Since the length and the multiplicity of \( H^1_{(h_0)} (R) \) are both one, it follows that

\[
\begin{align*}
\text{length } H^1_{(f)} (R) &= \text{length } H^1_{(f')} (R) + \text{length } H^2_{(f')} + (h_0) (R) + 1, \\
\text{mult } H^1_{(f)} (R) &= \text{mult } H^1_{(f')} (R) + \text{mult } H^2_{(f')} + (h_0) (R) + 1.
\end{align*}
\]

(1)

Since \( (f'') = R'' f'' \sim (R f' + R h_0) / R h_0 \), Lemma 2.1 implies

\[
\begin{align*}
\text{mult } H^2_{(f')} + (h_0) (R) &= \text{mult } H^1_{(f'')} (R''), \\
\text{length } H^2_{(f')} + (h_0) (R) &= \text{length } H^1_{(f'')} (R'').
\end{align*}
\]

This completes the proof in view of (1).

\[ \square \]

**Corollary 3.1** \( \text{length } H^1_{(f)} (R) = \text{mult } H^1_{(f)} (R) \).

Proof: This can be easily proved by induction on \( \# \mathcal{A} \) by using Theorem 3.1.

\[ \square \]

The intersection poset \( L(\mathcal{A}) \) is the set of the non-empty intersections of elements of \( \mathcal{A} \). For \( X, Y \in L(\mathcal{A}) \), define \( X \leq Y \) if and only if \( X \supset Y \). For \( X, Y \in L(\mathcal{A}) \), the Möbius function \( \mu(X, Y) \) is defined recursively by

\[
\mu(X, Y) = \begin{cases} 
- \sum_{X \leq Z < Y} \mu(X, Z) & \text{if } X < Y \\
1 & \text{if } X = Y \\
0 & \text{otherwise}
\end{cases}
\]

Set \( \mu(X) = \mu(V, X) \). Then \( (-1)^{\text{codim } X} \mu(X) \) is positive (see e.g. Theorem 2.47 of [7]). The Poincaré polynomial of the arrangement \( \mathcal{A} \) is defined by

\[
\pi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X) (-t)^{\text{codim } X}.
\]

**Theorem 3.2**

\[
\text{length } H^1_{(f)} (R) = \pi(\mathcal{A}, 1) - 1 = \sum_{X \in L(\mathcal{A}) \setminus \{V\}} |\mu(X)|.
\]
Proof: Let $H_0$ be an element of $\mathcal{A}$ defined by a first degree polynomial $h_0$. Let us prove the equality by induction on $\# \mathcal{A}$. Since $H_{(h_0)}(R)$ is simple, the equality holds if $\mathcal{A} = \{H_0\}$ with $\pi(\mathcal{A},1) = 2$. Let $\mathcal{A}', \mathcal{A}''$ be as in the proof of Theorem 3.1. By the induction hypothesis, we have

$$
\text{length } H^1_{(f')} \langle R \rangle = \pi(\mathcal{A}',1) - 1, \quad \text{length } H^1_{(f'')} \langle R'' \rangle = \pi(\mathcal{A}'',1) - 1.
$$

Hence by Theorem 3.1 we get

$$
\text{length } H^1_{(f)} \langle R \rangle = \text{length } H^1_{(f')} \langle R \rangle + \text{length } H^1_{(f'')} \langle R'' \rangle + 1 = \pi(\mathcal{A}',1) + \pi(\mathcal{A}'',1) - 1.
$$

On the other hand, $\pi(\mathcal{A},t) = \pi(\mathcal{A}',t) + t\pi(\mathcal{A}'',t)$ holds (see e.g., Theorem 2.56 of [7]). Thus we get

$$
\text{length } H^1_{(f)} \langle R \rangle = \pi(\mathcal{A}',1) + \pi(\mathcal{A}'',1) - 1 = \pi(\mathcal{A},1) - 1.
$$

This completes the proof. □

References

[1] T. Abebaw, R. Bøgvad, Decomposition factors of D-modules on hyperplane configurations in general position. Proceedings of the AMS, 140 (2012), 2699–2711.

[2] J. Álvarez Montaner, R. García López, S. Zarzuela Armengou, Local cohomology, arrangements of subspaces and monomial ideals. Advances in Math. 174 (2003), 35–56.

[3] J. Bernstein, Modules over the ring of differential operators; the study of fundamental solutions of equations with constant coefficients. Functional Analysis and its Applications 5 (1971), 1–16.

[4] A. Borel et al., Algebraic D-modules. Academic Press, 1987.

[5] R. Hotta, K. Takeuchi, T. Tanisaki, D-Modules, Perverse Sheaves, and Representation Theory. Birkhäuser, Boston-Basel-Berlin, 2008.

[6] M. Kashiwara, On the holonomic systems of linear differential equations, II. Invent. Math. 49 (1978), 121–135.

[7] P. Orlik, H. Terao, Arrangements of Hyperplanes, Springer-Verlag, Berlin-Heidelberg, 1992.
[8] U. Walther, Bernstein-Sato polynomial versus cohomology of the Milnor fiber for generic hyperplane arrangements. Compositio Math. 141 (2005), 121–145.