A hidden variables model for interference phenomena based on $p$-adic random dynamical systems

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Abstract

We propose a model based on random dynamical systems (RDS) in information spaces (realized as rings of $p$-adic integers) which supports Buonomano’s non-ergodic interpretation of quantum mechanics. In this model the memory system of an equipment works as a dynamical system perturbed by noise. Interference patterns correspond to attractors of RDS. There exists a large class of $p$-adic RDS for which interference patterns cannot be disturbed by noise. Therefore, if the equipment is described by such a RDS then the result of statistical experiment does not depend on noise in the equipment. On the one hand, we support the corpuscular model, because a quantum particle can be described as a corpuscular object. On the other hand, our model does not differ strongly from the wave model, because a quantum particle interacts with the whole equipment. Hence the interaction has non-local character. For example, in the two slit experiment a quantum particle interacts with both slits (but it passes only one of them).

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1 Introduction

It is well known that the interference phenomenon for quantum particles could not be explained on the basis of the corpuscular model. To find a reasonable description, we have to use the wave picture. This is the root of the *wave-particle dualism*. The wave-particle dualism is one of the cornerstones of quantum mechanics. By this postulate there are physical phenomena which admit only the corpuscular description and there are other physical phenomena which admit only the wave description. An essential part of quantum community is (more or less) satisfied by the wave-particle dualism. On the other hand, other people try to find a hidden basis of this dualism. These attempts generate numerous models with hidden variables (see, for example, []). Bell’s inequality [] was one of the main arguments against theories of hidden variables. There are also numerous arguments against the attempts to use Bell’s inequality as a ”no-go theorem” for theories of hidden variables. We note only that, in principle, Bell’s inequality may be considered as a pure mathematical problem (a consequence of the unlimited use of Kolmogorov’s model of probability theory, []).

In [] one of the authors proposed a *dynamical hidden variables model* which might give an explanation of interference phenomena. By this model it is assumed that statistical interference experiments can be described as functioning of dynamical systems on spaces of hidden variables (information states of an experimental arrangement). In fact, this approach is closely related to *non-ergodic interpretation* of quantum mechanics []. By this interpretation we may not identify time averages and averages with respect to statistical ensembles of independent particles. In particular, our dynamical model with hidden variables does not contradict to Bell’s inequality (because this is the inequality for averages with respect to the statistical ensemble).

By [] we have the following mathematical model for interference experiments. We image all the experimental arrangement $\mathcal{E}$ (an equipment (including a source of radiation), fields, vacuum) as a dynamical system

$$u_{n+1} = f(u_n), \quad u \in U,$$

where $U$ is a space of *information states* of $\mathcal{E}$. These states are related (in some way) to physical observables $A$ of the experiment: $A_n = g(u_n)$, where $A_n$ is the result of the $n$th measurement and $g$ is a ”measurement

\footnote{We do not discuss the question where and how this information is recorded. The simplest way is to reduce this problem to memory effects in the equipment, []. However, at the moment we do not claim this.}
function" which transfers the information state of the equipment to the result of a measurement. Each quantum particle generates a new iteration of (1) (which starts with the result of the previous iteration).

However, this mathematical model seems quite unphysical, because the arrangement $E$ of an experiment is continuously disturbed by a random noise. In principle, this noise must destroy functioning of the $E$-dynamical system (refstar). The natural way to describe effects of a noise is to use the formalism of random dynamical systems (RDS), see, for example, []. Thus it is proposed [] to describe a run of an interference experiment as functioning of the RDS of its arrangement $E$.

Following to [] we describe the information space $U$ by $p$-adic integers (see [] and section of this paper for $p$-adic analysis).

In this paper we show that there exists a large class of RDS over the fields of $p$-adic numbers for which the effect of random perturbations may be automatically eliminated. These RDS have no random attractors i.e., the iterations $x_n(\omega)$ tend to the same value $a$, for a.e. $\omega \in \Omega$, where $(\Omega, F, P)$ is a (Kolmogorov) probability space which describes the noise in $E$. Thus we obtain the same result $A = g(a)$ for any choice of $\omega \in \Omega$. At the same time a "cloud" $A_n(\omega) = g(x_n(\omega))$ appears around $A$ which, of course, depends on $\omega$. Thus pictures are not identical for different $\omega$ (they are only statistically identical). This is our explanation of the interference phenomena.

At the moment the use of $p$-adic numbers is still not standard for quantum physics. Therefore we write the paper in such a way that that all physical ideas can be understand on the elementary level of $p$-adic mathematics.

# 2 Dynamical systems on information spaces of interference experiments

1. **Deterministic model.** We propose the following dynamical model for quantum experiments in which the arrangement $E$ "remembers" previous particles. We assume that the internal state of $E$ (physical characteristics of $E$) is described by some parameter $s$. Denote the space of internal states by $S$. We introduce a space $U$ of information states $u$ of $E$, i.e., $u$ is the information which has been collected in $E$ and would determine a result of the next experiment. We introduce also a "measurement function" $g : U \rightarrow X$, where $x \in X$ are values of physical observables which are measured in the experiment. Finally, we introduce a family of "transformation functions" $f_s : U \rightarrow U, s \in S$, which describe the flow of information in $U$ for different
internal states \( s \) of \( E \).

A run of the quantum experiment is described as functioning of the dynamical system \( (1) \) with \( f = f_s \). Quantum particles play the role of bearers of information for starting a new iteration of \( (1) \). At the first moment \( E \) remembers the initial information \( u_0 \) and the arrival of the first particle is a signal for starting the first iteration of \( (1) \) with \( f = f_s(u_0) \) and we obtain the first result of the measurement \( x_1 = g(u_1) \). This process will give a sequence of information states, \( u_1, u_2, \ldots, u_n, \ldots \), and the corresponding sequence of results of the measurement, \( x_1, \ldots, x_n, \ldots \).

We assume that the dynamical system \( (1) \) has the unique attractor \( a_0 \) and the whole information space \( U \) is its basin of attraction, i.e., for every \( u_0 \in U \) (the initial state of information in \( U \) before the start of the experiment) the iterations \( x_n \) tend to \( a \), when \( n \) goes to \( \infty \). In this case we obtain a statistical sample in \( X \) which has the form of a cloud concentrated around the value \( x_0 = g(a_0) \).

It is easy to demonstrate that in this framework interference pictures appear in a natural way. We can propose many models based on different choices of the information space \( U \) and the measurement map \( g \). Further we consider a \( p \)-adic model.

By using some system of coding we can present the information state \( u \) as the sequence of digits:

\[
u = (\alpha_0, \alpha_1, \ldots, \alpha_m, \ldots), \quad \alpha_j = 0, 1, \ldots, p - 1, \tag{2}\]

where \( p > 1 \) is a prime number. Denote the set of all such sequences by the symbol \( \mathbb{Z}_p \). We introduce the metric on \( \mathbb{Z}_p \) by setting, for \( u = (\alpha_j)_{j=0}^{\infty} \) and \( v = (\beta_j)_{j=0}^{\infty}, \rho_p(u, v) = p^{-k} \) if \( \alpha_j = \beta_j, \ j \leq k - 1, \) and \( \alpha_k \neq \beta_k, \ k = 1, 2, \ldots \) (if \( \alpha_0 \neq \beta_0 \) then \( \rho_p(u, v) = 1 \)). This is a complete metric space which is homeomorphic to the ring of \( p \)-adic integers (see section 1).

Let \( U \subset \mathbb{Z}_p \) be the information space of \( E \) and let \( f_s : U \mapsto U \) be the transformation function (corresponding to the internal state \( s \in S \) of \( E \)). We choose the measurement function \( g : \mathbb{Z}_p \mapsto [0, 1] \subset \mathbb{R} \) in the following way:

\[
g(u) = \frac{\alpha_0}{p} + \frac{\alpha_1}{p^2} + \cdots + \frac{\alpha_m}{p^m} + \cdots \tag{3}\]

for \( u \) defined by (6). We remark that \( g \) is a continuous function [12]. Let the dynamical system have the unique attractor \( a_0 \in U \) and \( U \) be the basin

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2 At the moment we do not discuss a topological structure on the information space \( U \).

3 Of course, we can also use cording systems based on non-prime numbers. The choice of a prime \( p \) simplifies mathematical considerations.
of attraction of \( a_0 \), i.e., iterations \( u_n \) converge to \( a_0 \) for any initial condition \( u_0 \in U \). Thus iterations \( u_n \) in \( U \) induce a convergent sequence of results of measurements \( x_n = g(u_n) \to x_0 = g(a_0) \in [0,1] \). Now we consider an \( \mathcal{E} \) in which the memory effect acts only on the \( x \)-coordinate of the physical observable \( z = (x,y) \) (a point on the plane \( XY \)) and assume that results of measurements of \( y \) are random and have the uniform distribution on the segment \([a,b]\). In this case the statistical sample will have the form of the unsharp vertical strip, \( a \leq y \leq b \), around \( x = x_0 \).

2. Random model. As we have already discussed in the introduction, the main problem of this approach is the presence of noise \( \theta(\omega) \) in the equipment \( \mathcal{E} \). This noise will generate random transformations \( u_n(\omega) \) and in principle the attractor \( a_0 \) may also depend on \( \omega \), i.e., \( a_0 = a_0(\omega) \). This will imply that the resulting picture will also depend on \( \omega \), i.e., for different \( \omega \), there will appear different interference pictures. Another possibility is that stochastics might destroy convergence of iterations. In this case we will observe a random distribution of points on the plane. Therefore, to improve our model, we have to present a random dynamical model for the process of quantum measurements and show that there exist numerous RDS (in the information space \( Z_p \)) which have only deterministic attractors, i.e., in fact, noise could not destroy the memory effect. Such RDS are presented in section.

Moreover, the presence of noise produces interference pictures which are quite realistic. In this way we can obtain arbitrary groups of (unsharp) vertical strips on the plane (see section). Positions of these vertical lines are determined by the form of the dynamical laws \( f_s \). In fact, groups of vertical lines correspond to random mixtures \( s = s(\omega) \) of internal states.

So instead of the deterministic dynamical system (I) we consider RDS in which the result of each transformation depends on \( \omega \), i.e., perturbation by noise which changes the internal state of \( \mathcal{E} \) (its physical characteristics), \( s = s(\omega) \). Moreover, noise also evolves in time, i.e., there is some flow describing the noise process, \( \nu^n(\omega) \), where \( \nu^n \) is the \( n \)th iterate of the noise flow.

As we have told, there is a large class of RDS in \( Z_p \) which have only deterministic sets of attraction. Here \( Z_p = \bigcup_{j=1}^n U_j \) and for each \( j \) there is the attraction set \( A_j = \{a_{j1}, \ldots, a_{jm_j}\} \) such that, for each initial state of information \( u_0 \in U_j \), the orbit \( \{x_n(\omega)\} \) will form a "cloud" around \( A_j \). This cloud will be concentrated around \( A_j \), when \( n \to \infty \). If we apply the measurement map \( g \) we obtain the cloud in \( \mathbb{R} \) which is concentrated around the set \( B_j = g(a_j) = \{x_{j1} = g(a_{j1}), \ldots, x_{jm_j} = g(a_{jm_j})\} \subset [0,1] \). If we again assume that the dynamical system of memory has an influence only in
the $x$-direction and the results of measurement in the $y$-direction are pure random (i.e., there is no dynamical system which controls the results of the experiment), then the statistical sample on the plane $XY$ will have the form of $m$ (unsharp) vertical strips concentrated near lines $x = x_{j1}, \ldots, x = x_{jm_j}$ for all initial conditions $u_0 \in U_j$. The main mathematical result is that the sets of attraction $A_j$ do not depend on $\omega$.

Remark. If $A_j = A_j(\omega)$ then the picture on the $XY$ plane would depend on $\omega$. Thus by repeating the experiment (with the same equipment $E$) we should obtain different interference pictures. Of course, this contradicts the experimental observations.

3 A system of $p$-adic numbers

The system of $p$-adic numbers $\mathbb{Q}_p$ was constructed by K. Hensel [6]. In fact, it was the first example of a commutative number field which was different from the fields of real and complex numbers. Practically during 100 years $p$-adic numbers were only considered as objects in pure mathematics. In recent years these numbers have been intensively used in theoretical physics (see, for example, the books [7][3], [8] and papers [9]-[15]), in the theory of probability [8] as well as in investigations of chaos and dynamical systems [16], [17].

The field of real numbers $\mathbb{R}$ is constructed as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the Archimedean metric $\rho(x, y) := |x - y|$, where $| \cdot |$ is the usual Euclidean norm given by the absolute value. The fields of $p$-adic numbers $\mathbb{Q}_p$ are constructed in a corresponding way, by using another “distance”. For any prime number $p$ the $p$-adic norm $| \cdot |_p$ is defined in the following way. For every nonzero integer $n$ let $\omega_p(n)$ be the highest power of $p$ which divides $n$ (which is well-defined by the unique factorization of $n$ into primes), i.e. $n \equiv 0 \mod p^{\omega_p(n)}$, $n \not\equiv 0 \mod p^{\omega_p(n)+1}$. Then we define $|n|_p := p^{-\omega_p(n)}$, $|0|_p := 0$. For rationals $\frac{n}{m} \in \mathbb{Q}$ we set $|\frac{n}{m}|_p := \frac{|n|_p}{|m|_p} \ (= p^{-\omega_p(n)}+\omega_p(m))$. The completion of $\mathbb{Q}$ with respect to the $p$-adic metric $\rho_p(x, y) := |x - y|_p$ is called the field of $p$-adic numbers $\mathbb{Q}_p$.

We list some important properties of the field $\mathbb{Q}_p$: The metric $\rho_p$ is an ultrametric, i.e. it satisfies the so-called strong triangle inequality

$$|x \pm y|_p \leq \max\{|x|_p, |y|_p\},$$

(4)

where equality holds if $|x|_p \neq |y|_p$. Hence the closed balls $U_r(a) := \{x \in \mathbb{Q}_p : |x - a|_p \leq r\}$ are at the same time open, and every point in $U_r(a)$ is its center. This implies that two balls have nonempty intersection if and only
if one of them is contained in the other. \( S_1(0) := \{ x \in \mathbb{Q}_p : |x|_p = 1 \} \) is called the unit sphere. The unit ball \( U_1(0) \) in \( \mathbb{Q}_p \) is a subring of \( \mathbb{Q}_p \), called the \( p \)-adic integers, and is denoted by \( \mathbb{Z}_p \). It is compact. The unique \( p \)-adic expansion of an element \( x \in \mathbb{Z}_p \) does not involve negative powers of \( p \), that is,
\[
x = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \alpha_3 p^3 + \ldots
\]
where \( \alpha_j \in \{0, 1, \ldots, p-1\} \), \( j \geq 0 \). So we can identify every \( p \)-adic integer with a sequence of digits
\[
x = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \ldots)
\]
and vice versa.

**Lemma 1.** Let \( \gamma \in S_1(0) \) and \( u \in \mathbb{Z}_p, |u|_p \leq \frac{1}{p} \). Then \( |(\gamma + u)^n - \gamma^n|_p = |n|_p |u|_p \) for every \( n \in \mathbb{N} \).

**Proof.** First note that \( |u^k|_p = |u|^k |u|_p \) for \( k \geq 2 \), and that \( |(\binom{n}{k})|_p \leq |n|_p |u|_p \). Then observe
\[
| (\gamma + u)^n - \gamma^n |_p = \left| \sum_{k=1}^{n} \binom{n}{k} \gamma^{n-k} u^k \right|_p \approx \max_k \left| \binom{n}{k} \right|_p | \gamma^{n-k} |_p | u^k |_p = |n|_p |u|_p.
\]

The roots of unity in \( \mathbb{Q}_p \) are essential for the investigation of dynamics of monomial maps in the \( p \)-adic integers. Note that \( x^{p-1} = 1 \) has \( p-1 \) simple solutions. We denote the set of the \( (p-1) \)-th roots of unity by \( \Gamma_p \). There exists a primitive root \( \xi \) such that \( \Gamma_p = \{1, \xi, \xi^2, \ldots, \xi^{p-2}\} \).

For any natural number \( k \), consider the fixed points of the monomial map \( x \mapsto x^k \). They are given by \( x^k = x \), and so besides the points \( x = 0 \) we have the solutions of the equation \( x^{k-1} = 1 \), which we denote by \( \Gamma_k \). Note that \( \Gamma_k = \{1, \xi, \xi^2, \ldots, \xi^{p-2}\} \subseteq \Gamma_p \), with \( m = \left\lceil \frac{p-1}{(k-1)l} \right\rceil \), where \( (\cdot, \cdot) \) denotes the greatest common divisor of the two numbers.

Given two maps \( f_k : x \mapsto x^k \) and \( f_l : x \mapsto x^l \), \( f_l \) maps \( \Gamma_k \) into itself, and we have \( f_l[\Gamma_k] = \Gamma_u \subseteq \Gamma_k \) with \( u = \frac{k-1}{(k-1)l} + 1 \). So the map \( f_l \) acts as permutation on \( \Gamma_k \) iff \( (k-1, l) = 1 \).

Note that \( f_k'(x) = k x^{k-1} \), and so for \( x \in \Gamma_k, |f_k'(x)|_p = |k|_p \), which is less than 1 if and only if \( p \) divides \( k \). Hence the points in \( \Gamma_k \) are attracting if and only if \( p \) divides \( k \). Also note that the monomial maps are isometries on the sphere if \( p \) does not divide the exponent.

4 Random dynamical systems

Random dynamical systems (RDS) describe time evolutions in the presence of noise. The latter is modeled by a measure-preserving transformation \( \theta \)
on a probability space \((\Omega, \mathcal{F}, P)\). For technical reasons one usually assumes that \(\theta\) is invertible and ergodic. The dynamics of the RDS take place on a state space \(X\), which here we assume to be a compact topological space equipped with the corresponding Borel \(\sigma\)-algebra of \(X\). In discrete time an RDS \(\phi\) on \(X\) is then given by products of random continuous mappings \(\phi(\omega), \omega \in \Omega\). These are chosen in a stationary fashion according to the noise model, i.e. the time evolution is given for \(n \in \mathbb{N}\) by

\[
x \mapsto \phi(n, \omega)x = \phi(\theta^{n-1}\omega) \circ \ldots \circ \phi(\omega)x
\]
such that \((\omega, x) \mapsto \phi(\omega)x\) is measurable. \(\phi\) defines a measurable cocycle:

\[
\phi(n + m, \omega) = \phi(n, \theta^m\omega) \circ \phi(m, \omega) \quad \text{for all } \omega \in \Omega, n, m \in \mathbb{N}.
\] (7)

For the description of motion the simplest invariant sets, in particular if they are attracting, are of quite some interest. In the deterministic case these are given by fixed or periodic points. They play a minor role in random dynamical systems. Note for example that a point \(x\) can only be a fixed point of a random dynamical system \(\phi\), if it is a fixed point for all random maps \(\phi(\omega)\), a situation that does not occur in general, but we will meet it soon in \(p\)-adic RDS. The situation for periodic points is even worse. In return there are other notions which gain importance for RDS, namely stationary solutions, which can be seen as random analogues of fixed points. These are given by random variables \(x : \Omega \rightarrow X\) such that \(\phi(\omega)x(\omega) = x(\theta^\omega)\) for all \(\omega \in \Omega\). Another way to look at this phenomenon is to consider at the Dirac measures \(\delta_x(\omega)\) and to integrate them with respect to \(P\) in order to obtain a measure which is invariant for the RDS and hence a very natural object in this theory. Many phenomena in elementary stochastic dynamics can be represented better by such invariant measures than by invariant or stationary subsets of the state space, which in fact correspond to the supports of the measures. The main advantage is that the measures reflect the dynamics, while the invariant sets are static objects. We will encounter this later on in the study of \(p\)-adic RDS.

The invariant sets \(A\) for RDS \(\phi\) are in general random, i.e. they will depend on chance in the sense that they are measurable functions \(A(\omega)\) satisfying \(\phi(\omega)A(\omega) = A(\theta\omega)\). In particular, this makes the introduction of a good notion of attractors very difficult (see Schmalfuß [18] or Schenk [19]), as it requires also random neighborhoods \(U(\omega)\) of these sets that get attracted to \(A(\omega)\) in the sense that

\[
\lim_{n \to \infty} \text{dist}(\phi(n, \theta^{-n}\omega)U(\theta^{-n}\omega), A(\omega)) = 0.
\]
Here we have used the usual Hausdorff metric given by
\[ \text{dist}(D, A) = \sup_{x \in D} \inf_{y \in A} |x - y|_p. \]

We will dispense with the rigorous introduction of this notion, as in the study of \( p \)-adic RDS we will be confronted only with the case of attractors which are able to attract non-random neighborhoods.

We shall study a \( p \)-adic RDS which will be a stochastic generalization of the deterministic dynamical system:
\[ x_{n+1} = f_s(x_n), \quad \text{where} \quad f_s(x) = x^s, \quad s = 2, 3, ..., \quad x \in X, \quad (8) \]
where \( X \) is a subset of \( \mathbb{Q}_p \). First we give some facts \([3], [17]\) about the behaviour of (8). It is evident that the points \( a_0 = 0 \) and \( a_\infty = \infty \) are attractors of (8) with the basins \( D_0 = \bigcup_{1/p}(0) \) and \( D_\infty = \mathbb{Q}_p \setminus \bigcup_{1}(0) \) respectively. We consider now the case \( X = S_1(0) \). First it is evident that the set of fixed points of (8) coincides with \( \Gamma_s \). The behaviour of iterations depends on divisibility of \( s \) by \( p \): (i) if \( s \) is divisible by \( p \) then all points of \( \Gamma_s \) are attractors due to the final remark of the last section; (ii) if \( s \) is not divisible by \( p \) then all points of \( \Gamma_s \) are centers of Siegel disks (see \([3], [17]\) about \( p \)-adic analogues of Siegel disks).

We construct now an RDS corresponding to (8) with randomly changed parameter \( s \). Let \( s(\omega) \) be a discrete random variable that yields values \( s_j \) with probabilities \( q_j > 0 \), \( j = 1, ..., m \), where \( s_j \in \mathbb{N}, s_j \neq s_i \) for \( j \neq i \). We set \( \phi(\omega)x = x^{s(\omega)}, \quad x \in \mathbb{Q}_p \). This random map generates an RDS
\[ \phi(n, \omega)x = x^{S_n(\omega)}, \quad \text{where} \quad S_n(\omega) = s(\omega)s(\theta\omega) \cdots s(\theta^{n-1}\omega), \quad n \geq 1, \quad x \in X, \quad (9) \]
where \( X \) is a subset of \( \mathbb{Q}_p \). Let us introduce the set
\[ O_s(\eta) = \{ a \in \Gamma_p : a = \eta^{k_1} \cdots k_m, \quad k_j = 0, 1, \ldots \} \]
of points which can be reached from \( \eta \) evolving due to the RDS, and the set
\[ O_s^-(\eta) = \{ \gamma \in \Gamma_p : \gamma^{k_1} \cdots k_m = \eta \text{ for some } k_j = 0, 1, \ldots \} \]
of points which can reach \( \eta \) evolving under the RDS. As usual, due to the invertibility of \( \theta \) we can consider \( \phi(n, \theta^{-n}\omega) = x^{S_n(\omega)}, \quad \text{where} \quad S_n(\omega) = s(\theta^{-1}\omega) \cdots s(\theta^{-n}\omega) \). Because of commutativity we have the presentation
\[ S_n(\omega) = \prod_{j=1}^m s_{j,n}(\omega) \] for some \( 0 \leq k_{j,n} \leq n \) with \( \sum_{j=1}^m k_{j,n} = n \). From Poincaré’s Recurrence Theorem we know that
\[ k_{j,n}(\omega) \to \infty, \quad n \to \infty \quad \mathbb{P}\text{-a.e.} \quad (10) \]
In this paper we are only interested in attractors of RDS. Therefore, everywhere below we shall consider the case when at least one of \( s_j, j = 1, 2, \ldots, m \), is divisible by \( p \). As for deterministic systems (\( \mathbb{S} \)), it is easy to prove that \( a_0 = 0 \) and \( a_\infty = \infty \) are attractors of RDS (\( \mathbb{Q} \)) with the basins \( D_0 = U_{1/p}(0) \) and \( D_\infty = \mathbb{Q}_p \setminus U_{1}(0) \) respectively. These attractors are deterministic in the sense that \( \sup_{x \in D_0} |\phi(n, \theta^{-n}\omega)x|_p \to 0, n \to \infty \), and \( \sup_{x \in D_\infty} |\phi(n, \theta^{-n}\omega)x|_p \to \infty, n \to \infty \) \( \mathbb{P} \)-a.e Hence, as in the deterministic case, we have to study the behaviour of (\( \mathbb{Q} \)) only on the unit sphere \( X = S_1(0) \). We shall show that in this case the RDS has also only deterministic invariant sets, but with stochastic dynamics.

A set \( A \subset S_1(0) \) is said to be \( s \)-invariant, if \( f_{s_j}(A) = A \) for all \( j = 1, \ldots, m \).

Define \( \mathcal{I}_s := f_{s_1}^{p-1} \circ \cdots \circ f_{s_m}^{p-1}(\Gamma_p) \). \( \mathcal{I}_s \) is a cyclic subgroup of order \( q \) of \( \Gamma_p \), where \( q \) is the greatest divisor of \( p - 1 \) with \( (q, s_j) = 1 \) for all \( j \), i.e. \( \mathcal{I}_s = \Gamma_q^{+1} \). So this is an \( s \)-invariant set, since \( f_{s_j}(\mathcal{I}_s) = \mathcal{I}_s \), because \( (f_{s_j}(x) = 1 \Leftrightarrow x = 1) \) in this set.

**Example.** Let \( p = 61, s_1 = 61, s_2 = 2 \). Then \( p - 1 = 60 = 2^2 \cdot 3 \cdot 5 \), and \( \Gamma_{61,2} = \Gamma_{61} = \{1, \xi, \ldots, \xi^{60}\} \) for \( \xi \) primitive 60th root of unity. If we now add some exponent \( s_3 \) with \( (s_3, |\Gamma_{61,2}|) = 1 \) (where \( | \cdot | \) denotes the order of the group), then \( \mathcal{I}_{61,2} = \mathcal{I}_{61,2,s_3} \). If we add, e.g., some \( s_3 = 5 \), the set \( \mathcal{I}_{61,2,s_3} \) has order 3 and is equal to \( \{1, \xi^{20}, \xi^{40}\} \) (for further information on this example see also the chapter).

**Theorem 4.1.** The set \( \mathcal{I}_s \) is the attractor for RDS (\( \mathbb{Q} \)) on \( X = S_1(0) \).

**Proof.** By the above, \( \phi(n, \omega)(\mathcal{I}_s) = \mathcal{I}_s \), and \( \mathcal{O}_s^- (\mathcal{I}_s) = \Gamma_p \), by definition. Thus it remains to show

\[
\lim_{n \to \infty} \text{dist}(\phi(n, \theta^{-n}\omega)X, \mathcal{I}_s) = 0 \quad \mathbb{P} \text{- a.e.}
\]

To this end, for every \( x \in S_1(0) \) set \( x := \gamma + u \) for \( \gamma \in \Gamma_p \) and some \( u \) with \( |u|_p \leq \frac{1}{p} \). Note that \( \gamma S_{-n} \in \mathcal{I}_s \) with probability 1 after a finite number of steps, and thus, for \( n \) sufficiently large,

\[
\text{dist}(\phi(n, \theta^{-n}\omega)X, \mathcal{I}_s) = \sup_{x \in S_1(0)} \inf_{z \in \mathcal{I}_s} |\phi(n, \theta^{-n}\omega)x - z|_p
\]

\[
= \sup_{x \in S_1(0)} \inf_{z \in \mathcal{I}_s} |x^{\gamma S_{-n}(\omega)} - z|_p
\]

\[
= \sup_{|u|_p \leq \frac{1}{p}} \inf_{\gamma \in \Gamma_p} |(\gamma + u)^{\gamma S_{-n}(\omega)} - \gamma^{\gamma S_{-n}(\omega)}|_p
\]

\[
= \sup_{|u|_p \leq \frac{1}{p}} |\gamma^{\gamma S_{-n}(\omega)}|_p |u|_p
\]

\[
\to 0 \quad \mathbb{P} \text{- a.e.}
\]
by the Poincaré Recurrence theorem, and the last equality holds by the Lemma 1.

Note that Theorem 4.1 does not make any assertions on the dynamics apart from where this is concentrated. It just describes a static pattern. A more complete picture of the attractors of the RDS can be drawn, if we interpret $A$ as support of an invariant measure $\mu$ which also can be obtained as an attractor for measures. The description of the stochasticity of dynamics can easily be obtained by the upcoming lemma and the invariant measures should be in accordance with this description.

**Corollary 4.2.** The dynamics on $A$ is Markovian with transition probabilities $P_{n,n+1}(a,b,\omega)$ for the transition from $a$ at time $n$ to $b$ at time $n+1$ under the realization $\omega$ of the noise process given by $P_{n,n+1}(a,b,\omega) = P\{\omega \in \Omega : \phi(\theta^n \omega) = f_s, f_s a = b\}$, i.e. on $A$ we have an inhomogeneous Markov chain.

**Proof.** From the presentation of the RDS as products of random maps it is clear that the conditional probability $P(a_k,n_k|a_{k-1},n_{k-1},\ldots,a_0,n_0;\omega)$ for a state $a_k$ at some integer time $n_k$ knowing the previous states $a_{k-1},\ldots,a_0$ at integer times $n_{k-1} > \ldots > n_0 \geq 0$ and the realization $\omega$ of the noise process, is given by

$$P(a_k,n_k|a_{k-1},n_{k-1},\ldots,a_0,n_0;\omega) =$$

$$= P\{\omega \in \Omega : \phi(n_k - n_{k-1},\theta^{n_{k-1}} \omega)a_{k-1} = a_k\}$$

$$= P(a_k,n_k|a_{k-1},n_{k-1};\omega),$$

i.e. the dynamics on $A$ are given by a inhomogeneous Markov chain with transition probabilities $P(a_k,n_k|a_{k-1},n_{k-1},\ldots,a_0,n_0;\omega)$.

Let us mention that in the special case of noise being modeled by a Bernoulli process (see Section 5) the Markov chain becomes homogeneous, as $P\{\omega \in \Omega : \phi(\theta^n \omega) = f_s, f_s a = b\}P\{\omega \in \Omega : \phi(\omega) = f_s, f_s a = b\}$. We can give a qualitative answer of which lengths of invariant sets can be expected. Let $d_a$ be the number of elements in the above orbit. Let $q = p_1^{n_1} \cdots p_u^{n_u}$ be the unique factorization
of \( q \) into primes. Since \( (s_i, q) = 1 \), \( d_a \) is the order modulo \( \frac{q}{(a, q)} \) of \( s_i \), and for this, it divides the number \( q_a \) of multiplicatively invertible elements in the ring \( \mathbb{Z}/\frac{q}{(a, q)}\mathbb{Z} \). Let \( \frac{q}{(a, q)} = p_1^{m_1} \cdots p_u^{m_u} \). Then \( q_a = \Pi_{i=1}^u p_i^{m_i-1}(p_i - 1) \) by well-known number-theoretic considerations. So we know that the length of all orbits divide the numbers \( q_a, a \leq q - 1 \). If for example \( q \) is prime, \( (q, a) = 1 \) for all \( a \), and hence the length of the orbits divide \( q - 1 \). Examples are contained in the next chapter.

The invariant sets of the RDS \( s \) are then appropriate unions of those \( f_{s_i} \)-invariant sets.

It is interesting that the attractor is determined by the greatest common divisors of the exponents \( s_j \) and the number \( (p - 1) \), and the invariant sets and the basins of attraction are determined by the “orders modulo \( q \)” of \( s_j \). So for a given RDS with \( (s_1, \ldots, s_m) \) we can add the numbers \( t \in \mathbb{N} \) with \( t \equiv s_j \mod (p - 1) \) for some \( j \) to the parameter set (or exchange the corresponding parameters). This does not change anything of the structure of invariant sets, but it may change the dynamical behaviour “outside”.

Hence we can extend the class of RDS by considering infinite sets of parameters, i.e., \( s(\omega) = s_j, s_j \neq s_i \) for \( i \neq j, j = 1, 2, \ldots \), with probabilities \( q_j > 0 \) which sum up to 1, and at least one of \( s_j \) is divisible by \( p \). We set \( s = (s_j)_{j \in \mathbb{N}}; \Gamma_s = \cap_{j=1}^\infty \Gamma_{s_j}; \)

\[
O_s(\eta) = \{ a \in \Gamma_p : a = \eta^{k_1 \cdots k_j \cdots}, k_j = 0, 1, \ldots, \sum_{j=1}^\infty k_j < \infty \};
\]

\[
O_s^{-}(\eta) = \{ \gamma \in \Gamma_p : \gamma^{k_1 \cdots k_j \cdots} = \eta \text{ for some } k_j = 0, 1, \ldots, \sum_{j=1}^\infty k_j < \infty \}.
\]

A set \( A \subset S_1(0) \) is said to be \( s \)-invariant, if \( f_{s_j}(A) \subset A \) and \( \bigcup_{j=1}^\infty f_{s_j}(A) = A \). By using Poincaré’s Recurrence Theorem for the random variable \( s(\omega) \) (having an infinite number of values) and repeating the proof of Theorem 4.1 we obtain that this theorem is valid for the RDS generated by \( s(\omega) \).

5 Long-term behaviour, dynamics on the attractor, examples

In this section we consider the long-term behavior of some examples of \( p \)-adic RDS which have an attractor due to Theorem 4.1. Fix a prime number \( p \), denote by \( \xi \) the primitive root of unity of degree \( p - 1 \). By the above said, we only need to consider parameters \( s_j \leq p \). We also leave aside the
parameters $s = 1$ (corresponding to the identity) and $s = p - 1$ (for which the attractor is $\{1\}$). Now let

$$s : \Omega \rightarrow \{s_1, \ldots, s_m\}$$

be a random variable with a distribution given by $(q_1, \ldots, q_m)$, such that $q_i > 0$, $\sum_i q_i = 1$. The RDS $\phi$ is given by

$$\phi(n, \omega)x = \begin{cases} x^{S_n(\omega)}, & n \geq 1, \\ x, & n = 0, \\ x^{S_{-n}(\omega)}, & n \leq -1. \end{cases}$$

For the random selection mechanism we choose for simplicity an $m$ sided dice which is thrown independently in each time step corresponding to the probability distribution $(q_1, \ldots, q_m)$. This type of random influence can be modeled by a so-called Bernoulli shift, which is a measure-preserving transformation $\theta$ on the space of all two-sided sequences consisting of $m$ symbols.

Due to Theorem 4.1 and Corollary 4.2, we can restrict our considerations to the motion of $\phi$ on the attractor $I_s$ where the dynamical behavior of $\phi$ on the attractor can be described by a (possibly inhomogeneous) Markov chain. By the choice of the of the random selection mechanism in our examples the resulting Markov chain is homogeneous, i.e. the transition probability does only depend on the current state and is independent of time and chance. Now, the long-term behavior of this Markov chain is determined by a stationary distribution. Such a stationary distribution always exists due to the fact that the transition matrix of the Markov chain has 1 as an eigenvalue, but it might be not unique if the Markov chain is not irreducible, where irreducibility means that there is a positive probability for each state to reach any other state. It is easy to see that the Markov chain given by $\phi$ on $I_s$ can not be irreducible, since $\xi^0 = 1$ is always a fixed point which is never left if it is hidden once.

If a fixed point is reached, the dynamics of $\phi$ can be considered as a trivial Markov chain on one state, or, as we will see in the following, if there are some $\phi$-invariant subsets of $I_s$ on which $\phi$ acts as a nontrivial Markov chain, we can separate the attractor to components on which the dynamical behaviour of $\phi$ is the one of a irreducible Markov chain. In this case the stationary distribution on such components is unique and determines the motion of $\phi$, but the selection of the components which is finally attained depends on the initial conditions and on chance as well.

Let us look at the RDS $\phi$ with $p = 29$ and $s_1 = 29$, $s_2 = 2$, $s_3 = 3$. Since $p - 1 = 28 = 2^2 \cdot 7$ we obtain the attractor as $I_{(29,2,3)} = \{1, \xi^1, \xi^8, \ldots, \xi^{24}\}$.
consisting of \( q = 7 \) elements where \( \xi \) is the primitive 28\(^{th} \) root of unity. The order of 2 modulo 7 is 3, and the order of 3 is 6. Thus we know that in \( I_{(29, 2, 3)} \) there are 2 \( f_2 \)-invariant sets and 1 \( f_3 \)-invariant set beside \( \{1\} \). This means \( I_{(29, 2, 3)} \) splits into the two \( (29, 2, 3) \)-invariant sets \( \{1, \xi^4, \xi^8, \ldots, \xi^{24}\} \). If we look at the dynamics of \( f_2(x) = x^2 \) on the attractor we see the fixed point 1 with domain of attraction \( \{\xi^7, \xi^{14}, \xi^{21}\} \) and two invariant subsets \( \{\xi^4, \xi^8, \xi^{16}\} \) and \( \{\xi^{12}, \xi^{24}, \xi^{20}\} \) with domains of attraction \( \{\xi, \xi^2, \xi^9, \xi^{18}, \xi^{11}, \xi^{22}, \xi^{15}, \xi^{23}, \xi^{25}\} \) and \( \{\xi^3, \xi^6, \xi^5, \xi^{13}, \xi^{26}, \xi^{17}, \xi^{19}, \xi^{27}\} \), resp. Doing the same for \( f_3 \) we obtain the fixed point 1 and a 6-cycle consisting of \( \mathcal{J} := I_{(29, 2, 3)} \setminus \{1\} \). Due to this 6-cycle for \( f_3 \) both invariant components of \( f_2 \) are merged together such that the attractor of the RDS \( \phi \) consists of two components on which the dynamics is given by an irreducible Markov chain: The set \( \mathcal{J} \) and the fixed point 1. Thus we have the following picture of the Markovian dynamics on the attractor \( I_{(29, 2, 3)} \):

![Markov Chain Diagram](image)

Figure 1: The Markov chain given by \( \phi \) on \( I_{(29, 2, 3)} \), (\( q_1 \) is omitted).

Since the Markov chain on \( \mathcal{J} \) is irreducible, there exist a unique stationary distribution which assigns, by symmetry, probability \( \frac{1}{6} \) to each element of \( \mathcal{J} \) independent of the probability distribution \( (q_1, q_2, q_3) \) of our selection mechanism. If the motion finally reaches the fixed point or if it remains in \( \mathcal{J} \) depends on the initial conditions of the RDS as well as on chance. Thus we determined all the invariant measures of the RDS \( \phi \). First the Dirac measure supported on the fixed point 1, and second the stationary distribution on \( \mathcal{J} \), which are the ergodic invariant measures of \( \phi \). All other invariant measures are convex combinations of these two measures.

Let us now go back to the example with \( p = 61 \) and \( s_1 = 61, \ s_2 = 2 \). As we have seen above the attractor \( I_{(61,2)} \) consists of 15 elements, where we observe the unique fixed point 1, one invariant subset consisting of 2
elements and three subsets consisting of 4 elements each. Again, the ergodic invariant measures of $\phi$ are the unique stationary distributions on these components, which again are all symmetric. As already discussed the size of the attractor shrinks to 5 elements if we add $s_3 = 3$ to the RDS $\phi$. The attractor $I_{(61,2,3)}$ consists of the fixed point 1 and the set $\{\xi^{12}, \xi^{24}, \xi^{48}, \xi^{36}\}$ on which $\phi$ acts as an irreducible Markov chain. Thus the extended $\phi$ again has two ergodic invariant measures, similar to the above example.

In general, these phenomena can be observed if we increase the noise, i.e., if we allow the random variable $s$ to take more different values. But, if the set of values of $s$ becomes too large, everything vanishes to the fixed point 1. Summarizing our experimental results, we can say that more noise decreases the size of the attractor as well as the number of invariant subsets with the fixed point $\xi^0$ remaining if the noise becomes large in some sense. On such invariant subsets $\phi$ acts as an irreducible Markov chain, whose stationary distribution assigns the same probability to all members of this particular subset. The selection of the irreducible component depends on the initial conditions and on chance. Only the time until the irreducible component is reached is affected by the choice of the probabilities $q_i$ for the RDS $\phi$.

6 Examples of interference pictures generated by RDS

Let $S = \{a_1, ..., a_k\}$ be an $s$-invariant subset of $\Gamma_{p-1}$ and $D_S$ be its basin. Then, for any initial state of information $u_0$, iterations $\phi(n, \omega)u_0$ of the RDS of $\mathcal{E}$ will be attracted by points of $S$ (these iterations are distributed uniformly between the points of $S$). The computer simulations demonstrated that the fluctuations $s(\omega)$ of internal states of $\mathcal{E}$ can produce a large number of different configurations for invariant sets.

For example, let $p = 41$, $s_1 = 11$, $s_2 = 41$, then there are 25 invariant subsets (10 fixed points and 15 sets with 2 points). Here the information space $U = \bigcup_{j=1}^{25} U_j$ where $U_j$ are basins of invariant sets. If the initial state of information $u_0 \in U_j$ where $U_j$ corresponds to the fixed point $a$, then the interference pattern will be a single (unsharp) strip around the line $x = g(a)$. If $u_0 \in U_j$ where $U_j$ corresponds to the pair of points $c, d$, then the interference pattern will be two vertical strips around the lines $x = g(c), x = g(d)$. Let $p = 41, s_1 = 17, s_2 = 41$, then there are 16 invariant subsets (8 fixed points and 8 sets with 4 points). There can be interference patterns which are single strips or groups of 4 vertical strips. Let $p = 47, s_1 = 14, s_2 = 47,$
there are two invariant subsets (1 fixed point and one set with 22 points). Thus there can be interference patterns with 22 vertical strips.

Conclusions.

(1) We have presented a model based on RDS in information spaces which supports the non-ergodic interpretation of quantum mechanics [1], [2]. (2) In our model the equipment $E$ involved in the experiment works as a dynamical system which provides iterations of information states. (3) This dynamical system is random, because there is a random noise in $E$. In our model the random noise may be arbitrary strong. Thus we can consider a "macro" noise induced by macro stochastics. (4) The mathematical basis of our model is the use of $p$-adic numbers for coding of information in $E$. There is a large class of $p$-adic random systems in that the random noise does not have strong influence to the final result. Here, in fact, the noise could not destroy the memory effects in $E$. (5) On the one hand, we support the corpuscular picture of quantum mechanics. In our model a quantum particle can be described as a localized object. If we cover one slit then we change the set of possible internal states of the equipment $E$. In fact, we have three different dynamical systems: $(d1)$ the slit No 1 is open, the slit No 2 is closed; $(d2)$ the slit No 2 is open, the slit No 1 is closed; $(d12)$ both slits are open. There are three different random variables $s_1(\omega)$, $s_2(\omega)$, $s_{12}(\omega)$ which describe random fluctuations of internal states of $(d1)$, $(d2)$ and $(d12)$ respectively. There are no reasons that the sum of statistical samples produced by $(d1)$ and $(d2)$ will coincide with the statistical sample produced by $(d12)$. (6) On the other hand, our description does not differ strongly from the description provided by the wave picture of quantum mechanics. We do not claim that the memory effect in $E$ is a local effect. Thus, in fact, a quantum particle interacts with both slits simultaneously. (7) Our model supports investigations for verifying the non-ergodic interpretation of quantum mechanics [1], [2]. Practically each book in quantum mechanics contains the claim that the time average in the two slit experiment coincides with the statistical average. However, this claim has never been verified. In [4], [5] it was proposed to find a statistical pattern on the basis of the average over the ensemble of equipments $\{E_i\}$, i.e., to use a new equipment for each experiment. The present model strongly support this idea. (8) We are able to present a more general interpretation of our model. In fact, we do not need reduce the memory effects to the memory of an equipment. We provided the model for the interference phenomena by assuming that there exists a deterministic flow of information (perturbed by noise) which controls the behaviour of quantum particles. The assumption that it is recorded
in $E$ seems quite natural. However, there might be other possibilities. For example, we might suppose that the information is recorded in vacuum. (9) In fact, we do not need restrict our model to the interference phenomena. We might explain some other (all?) quantum experiments by the memory effect. The set of attraction $A = (a_1, \ldots, a_m)$ determines the values $\Lambda = (x_1, \ldots, x_m), x_j = g(a_j)$, of a physical observable. Hence a quantum state $\Psi$ is described by the domain of attraction $U$ for the set $A$ in the information space and the random fluctuation of internal parameters of the equipment. Here we obtain the explanation of the violation of the classical additive law for quantum probabilities (in the same way as for the two slit experiment).

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