STRONG ATTRACTORS FOR VANISHING VISCOSITY APPROXIMATIONS OF NON-NEWTONIAN SUSPENSION FLOWS

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To Professor Igor Chueshov, in Memoriam

Abstract. In this paper we prove the existence of global attractors in the strong topology of the phase space for semiflows generated by vanishing viscosity approximations of some class of complex fluids. We also show that the attractors tend to the set of all complete bounded trajectories of the original problem when the parameter of the approximations goes to zero.

1. Introduction. Non-Newtonian (or complex) fluids are difficult to model and to analyze because they display essentially nonlinear and even discontinuous flow properties. In this paper we consider an evolution problem which appears in the investigation of the model of concentrated suspensions proposed by Hebraud and Lequex [11]. In this model the system is divided in mesoscopic blocks. Their size is large enough so that the stress and the strain tensors may be defined for each block, but small compared to the characteristic length of the macroscopic stress field. In the mathematical description each block carries a given shear stress $\sigma$ (in the original paper). The stress state in the material is described by the probability

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density $p(x,t)$, which represents the description of shear stress in the assembly of blocks at time $t$ and obeys the Fokker–Planck equation
\[
\frac{\partial p}{\partial t} = -b(t) \frac{\partial p}{\partial x} + D(p) \frac{\partial^2 p}{\partial x^2} - \chi_{\mathbb{R}\setminus[-1,1]}(x)p + \frac{D(p)}{\alpha} \delta_0(x),
\]
where $\alpha > 0$ is a parameter, $\chi_{\mathbb{R}\setminus[-1,1]}$ is the characteristic function of the open set $\mathbb{R}\setminus[-1,1]$, $\delta_0$ is the Dirac delta function with support at the origin,
\[
D(f) = \alpha \int_{|x|>1} f(x)dx,
\]
and the function $b(t)$ corresponds to a variational rate of each block. Moreover, mechanical background of the model requires boundedness with respect to the time of the average stress function
\[
\tau(t) = \int_{\mathbb{R}} xp(t,x)dx.
\]
Existence and uniqueness results for such model were proved in [4]. In [5] the authors investigated the qualitative behavior of a more general model, where the function $b$ was not assumed to be homogeneous in space. Also in [5] it was shown for some values of the parameters that if $b(t) \to b_{\infty}$, as $t \to \infty$, then the corresponding stationary solution is (locally) asymptotically stable.

The theory of global attractors was applied first for (1) in Amigó et al. [1], where the authors proved under the assumption $b(t) \equiv 0$ the existence of global unbounded attractors with respect to the weak topology. Moreover, a lattice dynamical system generated by finite-difference numerical approximations was investigated in [2], [12], [16], whereas numerical simulations were provided in [9].

It is worth noting that in both [4] and [12] the key point was the analysis of the so-called vanishing viscosity approximation system, where the diffusion coefficient was everywhere positive.

In the present work we consider an evolution problem generated by vanishing viscosity approximations and prove the existence of a global attractor for the corresponding semiflow with respect to the strong topology of the phase space. Moreover, we prove the upper semicontinuity of these attractors with respect to the set of bounded complete trajectories of the original problem (1).

2. Setting of the problem and preliminaries. Let $\alpha > 0$ be a positive constant, $0 \leq \varepsilon \ll 1$ be a small parameter, and $b : \mathbb{R}_+ \to \mathbb{R}$ be a measurable function. Consider the following evolution problem with non-degenerate diffusion:
\[
\frac{\partial p}{\partial t} = -b(t) \frac{\partial p}{\partial x} + (D(p) + \varepsilon) \frac{\partial^2 p}{\partial x^2} - \chi_{\mathbb{R}\setminus[-1,1]}(x)p + \frac{D(p)}{\alpha} \delta_0(x), \text{ a.e. in } \mathbb{R} \times \mathbb{R}_+;
\]
\[
p(x,t) \geq 0, \text{ a.e. in } \mathbb{R} \times \mathbb{R}_+; \tag{3}
\]
\[
\int_{\mathbb{R}} p(x,t)dx = 1, \text{ a.e. in } \mathbb{R}_+; \tag{4}
\]
\[
\int_{\mathbb{R}} |x|p(x,t)dx < \infty, \text{ a.e. in } \mathbb{R}. \tag{5}
\]
Suppose that $b$ is an essentially bounded function, that is, there exists a constant $B > 0$ such that
\[ |b(t)| \leq B \text{ for a.e. } t > 0. \]  

Further we will use the following notation:
\[ L^p = L^p(\mathbb{R}), \quad H^1 = H^1(\mathbb{R}), \quad H^{-1} = (H^1)^*, \]
for each $1 \leq p \leq \infty$. Let $\langle \cdot, \cdot \rangle$ be the pairing on $H^{-1} \times H^1$ (on $L^q \times L^p$ respectively with $p \geq 1$ and $1 < q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$) that coincides with the inner product on $L^2$, that is,
\[ \langle f, u \rangle = \int_{\mathbb{R}} f(x)u(x)dx, \]
for each $f \in L^2$ and $u \in H^1$ (for each $f \in L^q$ and $u \in L^p$, respectively).

Let $0 \leq \tau < T < \infty$ be arbitrary fixed. Further, for the sake of simplicity, we will follow the conventions given in the following remark.

**Remark 1.** Let $\gamma \geq 1$, $E$ be a separable Banach space and $I \subseteq \mathbb{R}^+$ be a possibly infinite time interval. As $L_{\text{loc}}^\gamma(I; E)$ we consider the Fréchet space of all locally integrable functions with values in $E$, that is, $\varphi \in L_{\text{loc}}^\gamma(I; E)$ if and only if for any finite interval $[\tau, T] \subseteq \mathbb{R}^+$ the restriction of $\varphi$ on $I_{\tau,T} := I \cap [\tau, T]$ belongs to the space $L^\gamma(I_{\tau,T}; E)$. To simplify further our arguments we denote the restriction of an $E$-valued function $\varphi$ defined on time interval $I$ to a time subinterval $J \subseteq I$ by the same symbol $\varphi$. If, additionally, $E \subseteq L_{\text{loc}}^1(\mathbb{R})$, then any function $\varphi$ from $L_{\text{loc}}^\gamma(I; E)$ can be considered as a measurable mapping that acts from $\mathbb{R} \times I$ into $\mathbb{R}$. Further, we write $\varphi(x, t)$ when we consider this mapping as a function from $\mathbb{R} \times I$ into $\mathbb{R}$, and $\varphi(t)$, if this mapping is considered as an element from $L_{\text{loc}}^\gamma(I; E)$; see, for example, Gajewski et al. [8, Chapter III]; Temam [15]; Babin and Vishik [3]; Chepyzhov and Vishik [6]; Zgurovsky et al. [19] and the references therein. For a Banach space $E$ the notation $E_w$ means that the vector space $E$ is endowed with the standard topology of weak convergence.

A solution of equation (2) on a finite time interval $[\tau, T]$ is defined as follows.

**Definition 2.1.** Let $\varepsilon = 0$. A function $p \in L^\infty(\tau, T; L^1 \cap L^2)$ with $D(p)p \in L^2(\tau, T; H^1)$ and $\frac{\partial p}{\partial t} \in L^2(\tau, T; H^{-1})$ is called a (weak) solution of equation (2) on $[\tau, T]$, if the following equality holds:
\[ \int_{\tau}^{T} \left( \frac{\partial p}{\partial t}, \eta \right) + b(t)(\frac{\partial p}{\partial x}, \eta) + D(p(\cdot, t))(\frac{\partial p}{\partial x}, \frac{\partial \eta}{\partial x}) + \int_{|x|>1} p \cdot \eta \, dx \right) dt = \int_{\tau}^{T} \frac{D(p(\cdot, t))}{\alpha}(\delta_0, \eta) dt, \]  

for each $\eta \in L^2(\tau, T; H^1)$.

Let $0 < \varepsilon \ll 1$. A function $p \in L^\infty(\tau, T; L^1 \cap L^2) \cap L^2(\tau, T; H^1)$ with $\frac{\partial p}{\partial t} \in L^2(\tau, T; H^{-1})$ is called a (weak) solution of equation (2) on $[\tau, T]$, if the equality
\[ \int_{\tau}^{T} \left( \frac{\partial p}{\partial t}, \eta \right) + b(t)(\frac{\partial p}{\partial x}, \eta) + (D(p(\cdot, t)) + \varepsilon)(\frac{\partial p}{\partial x}, \frac{\partial \eta}{\partial x}) + \int_{|x|>1} p \cdot \eta \, dx \right) dt = \int_{\tau}^{T} \frac{D(p(\cdot, t))}{\alpha}(\delta_0, \eta) dt, \]  

holds for each $\eta \in L^2(\tau, T; H^1)$. 


Remark 2. We note that the right hand-side of equality (7) is equal to $\int_\tau^T \frac{D(p(t))}{\alpha} \eta(0,t)\,dt$.

Remark 3. Let $0 < \varepsilon \ll 1$, $0 \leq \tau < T < \infty$, $p_\tau \in L^1 \cap L^2$, and $p$ be a solution of equation (2) on $[\tau, T]$. Since $p \in L^2(\tau, T; H^1)$ and $\frac{\partial p}{\partial t} \in L^2(\tau, T; H^{-1})$, then $p \in C([\tau, T]; L^2)$; Gajewski et al. [8, Chapter III]. Therefore, the following initial condition

$$p|_{t=\tau} = p_\tau(x), \text{ a.e. in } \mathbb{R},$$

makes sense.

Let

$$X := \{p \in L^2(\mathbb{R}) : \int_\mathbb{R} |x||p(x)|\,dx < \infty\},$$

which is a Banach space with the norm

$$\|p\|_X := \|p\|_{L^2} + \int_\mathbb{R} |x||p(x)|\,dx, \text{ } p \in X.$$

Remark 4. Since $\|p\|_{L^1} \leq \sqrt{2} \|p\|_X$ for each $p \in X$, the embedding $X \subset L^1 \cap L^2$ is continuous. Moreover, $X = \mathcal{L}^1 \cap L^2$, where

$$\mathcal{L}^1 := \{p \in L^1 : \int_\mathbb{R} |x||p(x)|\,dx < \infty\}$$

is a Banach space with the following norm:

$$\|p\|_{\mathcal{L}^1} := \int_\mathbb{R} (1 + |x|)|p(x)|\,dx, \text{ } p \in \mathcal{L}^1.$$

Remark 5. Let $I \subseteq \mathbb{R}$ be either a finite or an infinite time interval and $p$ be a measurable function on $\mathbb{R} \times I$. Further $xp$ will denote the measurable function on $\mathbb{R} \times I$ that equals to $xp(x, t)$ for a.e. $x \in \mathbb{R}$ and $t \in I$.

Let $0 \leq \tau < T < \infty$. We understand condition (5) in the sense of the following definition.

Definition 2.2. Let $0 \leq \varepsilon \ll 1$. We recall that the solution $p$ of equation (2) on $[\tau, T]$ satisfies condition (5) on $[\tau, T]$ if $xp \in L^\infty(\tau, T; L^1)$.

Remark 6. Let $0 \leq \varepsilon \ll 1$, $0 \leq \tau < T < \infty$ and $p$ be a solution of equation (2) on $[\tau, T]$. Then $xp \in L^\infty(\tau, T; L^1)$ if and only if $p \in L^\infty(\tau, T; X)$. Moreover, since $p \in L^\infty(0, T; X)$, $p \in C([0, T]; L^2)$, and $X \subset L^2$, we have that $p \in C([0, T]; X_w)$; Temam [14, Chapter III, Lemma 1.4, p. 263]. The topological properties of the space $X$ (including those provided above) are studied by Amigó et al. [1].

Let $0 \leq \tau < T < \infty$ and $0 < \varepsilon \ll 1$ be arbitrary fixed. Cancès et al. [4, Proposition 2.1] proved that for each $p_\tau$ such that

$$p_\tau \in L^1 \cap L^\infty, \text{ } p_\tau \geq 0, \text{ } \int_\mathbb{R} p_\tau(x)\,dx = 1, \text{ } \int_\mathbb{R} |x|p_\tau(x)\,dx < \infty,$$

problem (2)–(5), (9) on $[\tau, T]$ has a unique solution $p$. Moreover, $p \in L^\infty(\mathbb{R} \times (\tau, T))$, $\sigma p \in L^\infty(0, T; L^1)$, $p \in C([\tau, T]; L^2 \cap L^1)$, $D(p) \in C([\tau, T])$ and

$$\int_\mathbb{R} p(t, \sigma)\,d\sigma = 1, \text{ } p(t) \geq 0 \text{ for all } t \geq 0.$$
Therefore, the phase space for this problem is defined as follows:

\[ H := \text{cl}_X E, \quad E := \{ p \in X : p \in L^\infty, p \geq 0, \int p(x)dx = 1 \}, \]

where \( \text{cl}_X \) is the closure in the space \( X \) (see Amigó et al. [1]). Note that the convexity of \( E \) implies the equality \( H = \text{cl}_X E \). The phase space \( H \) is considered as a metric space with the distance \( \rho_H \) endowed from the Banach space \( X \), that is, \( \rho_H(h_1, h_2) := ||h_1 - h_2||_X \), for each \( h_1, h_2 \in H \). The phase space \( H \) endowed with the induced topology from \( X_w \) is denoted by \( H_w \).

**Remark 7.** Let \( 0 < \varepsilon \ll 1 \). Conditions (10) imply that for each \( p_\tau \in E \) there exists a unique solution \( p \) of problem (2)-(5), (9) on \( [\tau, T] \). According to the continuous embedding \( L^1 \cap L^\infty \subset L^2 \), the inclusion \( p \in L^\infty(\tau, T; L^1 \cap L^\infty) \cap C([\tau, T]; L^2) \) yields that \( p \in C([\tau, T]; (L^1 \cap L^\infty)_w) \); Temam [14, Chapter III, Lemma 1.4, p. 263]. In particular, in light of (11), we have that \( p(t) \in E \) for each \( t \in [\tau, T] \). The definition of the phase space \( H \) implies that for each \( p \in H \) the following two conditions hold:

(a) \( p(x) \geq 0 \) for a.e. \( x \in \mathbb{R} \), and
(b) \( \int \mathbb{R} p(x)dx = 1 \) [1, p.212]. Therefore, Cancès et al. [4, Lemma 2.1] proved that for each \( 0 < \varepsilon \ll 1, 0 \leq \tau < T < \infty \), and \( p_\tau \in H \) there exists no more than one solution \( p \) of problem (2)-(4), (9) on \( [\tau, T] \).

The main purpose of this paper is twofold: (i) to establish topological properties of solutions and their a priori estimates; and (ii) to prove in the autonomous case the existence of global attractors in the strong topology of the phase space \( H \) for the semiflow generated by the vanishing viscosity approximations (2)-(5). We also show the upper semicontinuity of these attractors with respect to the set of all complete bounded trajectories of the original problem (1) when \( \varepsilon \to 0 \).

3. **Existence and properties of solutions.** In this section we provide existence of solutions for problem (2)-(5), (9) in the phase space \( H \) and study their topological properties. First, Lemma 3.1 establishes some a priori estimates for solutions. Then, the main result, concerning existence of solutions, is provided in Lemma 3.2, whereas in Lemma 3.3 we are dealing with convergence of solutions when we take approximations of the initial datum. Finally, Lemmas 3.4 and 3.5 prove that the dynamical system is dissipative.

For \( \tau \geq 0 \) and \( 0 \leq \varepsilon \ll 1 \) let \( K_{\tau, \varepsilon} \) denotes the family of all globally defined solutions of problem (2)-(4) on \( [\tau, \infty) \) with \( p(\tau) \in H \), that is, \( p \in K_{\tau, \varepsilon} \) if and only if for each \( T > \tau \) the restriction of \( p \) on \( [\tau, T] \) is a solution of problem (2)-(4) on \( [\tau, T] \) and \( p(\tau) \in H \). Similarly, let \( D_{\tau, \varepsilon} \) be the family of all globally defined solutions of problem (2)-(5) on \( [\tau, \infty) \) with \( p(\tau) \in H \). By definition, the following inclusion holds:

\[ D_{\tau, \varepsilon} \subseteq K_{\tau, \varepsilon}, \]

for each \( \tau \geq 0 \) and \( 0 \leq \varepsilon \ll 1 \). The following Lemma 3.1 and its corollary establishes the converse inclusion in the sense that each globally defined solution \( p \) of problem (2)-(4) with initial datum from \( H \) is a globally defined solution of problem (2)-(5). Furthermore, Lemma 3.1 establishes a priori estimates for the solutions of our problem.

**Lemma 3.1.** There exists a constant \( C > 0 \) such that, if \( 0 \leq \varepsilon \ll 1, \tau \geq 0 \) and \( p \in K_{\tau, \varepsilon} \) with \( p(\tau) \in H \), then \( p \in D_{\tau, \varepsilon} \) and the following inequality holds:

\[ \|p(t)\|_{L^1} \leq \|p(\tau)\|_{L^1} e^{-\frac{1}{2}(t-\tau)} + C, \]
for each $t \geq \tau$. Moreover, for each $\delta > 0$ and a bounded set (in $\mathcal{L}^1$) $K \subset H$ there exist constants $T = T(\delta, K) > 0$ and $\bar{k} = \bar{k}(\delta, K) > 0$ such that for each $0 \leq \varepsilon \ll 1$, $\tau \geq 0$, and $p \in K_{\tau,\varepsilon}$ with $p(\tau) \in K$ the following inequality holds:

$$
\int_{|x| > 2\bar{k}} p(x, t)|x|dx \leq \delta,
$$

(13)
for each $t \geq \tau + T$ and $k \geq \bar{k}$.

Proof. The proof of Lemma 3.1 is provided in Section 5.

Remark 8. The definition of the phase space $H$ and Remark 4 yield that $H \subset \mathcal{L}^1$.

Remark 9. According to Lemma 3.1, each globally defined solution $p$ of problem (2)–(4) on $[\tau, \infty)$ with $\tau \geq 0$, $0 \leq \varepsilon \ll 1$, and $p(\tau) \in H$, belongs to $L^\infty([\tau, \infty); \mathcal{L}^1)$, which implies that $p$ is a globally defined solution of problem (2)–(5) on $[\tau, \infty)$.

Corollary 1. For each $0 \leq \varepsilon \ll 1$ and $\tau \geq 0$ the following equality holds:

$$
\mathcal{D}_{\tau,\varepsilon} = \{p \in K_{\tau,\varepsilon} : p(\tau) \in H\}.
$$

Proof. The statement follows directly from Lemma 3.1 and the definitions of $\mathcal{D}_{\tau,\varepsilon}$ and $K_{\tau,\varepsilon}$ for each $0 \leq \varepsilon \ll 1$ and $\tau \geq 0$.

The existence result has the following formulation.

Lemma 3.2. For each $0 < \varepsilon \ll 1$, $0 \leq \tau < T < \infty$, and $p_\tau \in H$ problem (2)–(5), (9) on $[\tau, T]$ has a unique solution $p$. Moreover, $p \in C([\tau, T]; H)$.

The proof of Lemma 3.2 is provided in Section 5. Its statement follows from Lemma 3.3, that describes some convergence results, in particular, in the strong topology of the phase space $H$.

Lemma 3.3. Let $0 \leq \tau < T < \infty$, $p^n_\tau \in H$, $b_n \in L^\infty(\tau, T)$, and $0 \leq \varepsilon_n \ll 1$ for each $n = 0, 1, \ldots$. Suppose that $|b_n(t)| \leq B$ for a.e. $t \in (\tau, T)$ and $p^n \in C([\tau, T]; H_w)$ be a solution of problem (2)–(5), (9) on $[\tau, T]$ with parameters $p^n_\tau, \varepsilon_n, b_n$, for each $n \geq 1$. If $p^n_\tau \rightarrow p^0_\tau$ weakly-star in $L^\infty(\tau, T)$, then there exists a solution $p \in C([\tau, T]; H_w)$ of problem (2)–(5), (9) on $[\tau, T]$ with parameters $p^0_\tau, \varepsilon_0, b_0$, such that up to a subsequence the following convergence holds:

$$
p^n \rightarrow p \text{ in } C([\tau, T]; H_w).
$$

(14)
Moreover, if $\varepsilon_n > 0$ for each $n = 0, 1, \ldots$ and $p^n_\tau \rightarrow p^0_\tau$ in $H$, then the following statements hold:

(a) $p, p^n \in C([\tau, T]; H)$ for each $n \geq 1$;
(b) the following convergence holds for the entire sequence:

$$
p^n \to p \text{ in } L^2(\tau, T; H^1),
$$

(15)
$$
p^n \to p \text{ in } C([\tau, T]; H).
$$

(16)
If, additionally, $b_n \rightarrow b_0$ in the Lebesgue measure on $[\tau, T]$, then

$$
\frac{\partial p^n}{\partial t} \to \frac{\partial p}{\partial t} \text{ in } L^2(\tau, T; H^{-1}).
$$

(17)

Proof. The proof of Lemma 3.3 is provided in Section 5.

Corollary 2. For $\varepsilon = 0$ and each $\tau \geq 0$ the set $\mathcal{D}_{\tau,0}$ is nonempty. More precisely, for each $p^n_\tau \in H$ there exists $p \in \mathcal{D}_{\tau,0}$ such that $p(\tau) = p_\tau$ and $p \in C([\tau, T]; H_w)$ for each $T > \tau$. 


Theorem 4.1. For each \( \Theta \geq \tau \)
Existence and properties of global attractors in the autonomous case.

**Lemma 3.5.** There exist positive constants \( C_1 \) and \( C_2 \) such that, if \( 0 \leq \varepsilon < 1 \), \( \tau \geq 0 \) and \( p \in D_{\tau, \varepsilon} \) then the following inequality holds:
\[
\|p(t)\|_{L^2}^2 + \int_{\tau}^{t} \exp \left\{ -C_1 \int_{s}^{t} D(p(\cdot, \xi))d\xi \right\} \int_{|x| > 1} (p(x, s))^2 dxds \leq \|p(T)\|_{L^2}^2 + C_2,
\]
for each \( t \geq \tau \). Moreover, there exists \( R_0 > 0 \) such that for each \( \tau \geq 0 \), a bounded in \( X \) set \( K \subset H \), \( 0 \leq \varepsilon < 1 \) and \( p \in D_{\tau, \varepsilon} \) with \( p(\tau) \in K \) the following inequality holds:
\[
\|p(t)\|_X \leq R_0 + \sup_{g \in K} \|g\|_X,
\]
for each \( t \geq \tau \).

**Proof.** The proof of Lemma 3.5 is provided in Section 5.

4. Existence and properties of global attractors in the autonomous case.

Let us consider the autonomous case, that is, when \( b(t) \equiv b \). Then according to Lemma 3.2 for every \( \varepsilon \in (0, 1) \) problem (2)–(5) in the phase space \( H \) generates a classical semigroup
\[
V_\varepsilon : \mathbb{R}_+ \times H \to H,
\]
\[
V_\varepsilon(t, p_0) = p(t), \text{ where } p \text{ is a global solution of (2)–(5) with } p(0) = p_0.
\]

We recall that a function \( \xi : \mathbb{R} \to H \) is a complete trajectory of \( V_\varepsilon \) if \( \xi(t) = V_\varepsilon(t - s, \xi(s)) \) for any \( s \leq t \). In the same way, \( \xi \) is a complete trajectory of the original problem (1) if for any \( \tau \in \mathbb{R} \) the map \( u(\cdot) = \xi(\cdot)_{|_{\tau, +\infty}} \) is a solution of (1).

**Theorem 4.1.** For each \( \varepsilon > 0 \) the semigroup (21) has the connected stable global attractor \( \Theta_\varepsilon \) in the phase space \( H \). Moreover, \( \Theta_\varepsilon \) is bounded in \( H \) uniformly in \( \varepsilon \), it consists of bounded complete trajectories and for any sequences \( \varepsilon_n \searrow 0 \) and \( \xi_n \in \Theta_{\varepsilon_n} \) there exists \( \xi \in \Theta \) such that up to a subsequence \( \xi_n \to \xi \) in \( X \) as \( n \to \infty \), where
\[
\Theta = \{ y(0) : y(\cdot) \text{ is a complete trajectory of problem (1)} \}.
\]

**Proof.** Let us prove the first part of the theorem for fixed \( \varepsilon > 0 \). Due to Lemmas 3.1, 3.3, 3.4 and classical results about existence of global attractors (see [13], [15]) it will be enough to prove that \( V_\varepsilon \) is asymptotically compact, i.e.,
\[
\text{every sequence } \{ \xi_n = V_\varepsilon(t_n, p^0_n) \} \text{ is precompact in } H,
\]
where \( t_n \nearrow +\infty, \|p^0_n\|_X \leq r \).
Let \( \xi_n = V_n(t_n, p_n^0) \). Then \( \xi_n = p_n(t_n) \), \( p_n \) is a solution of (2)–(5) with \( p_n(0) = p_n^0 \). Therefore from Lemma 3.5
\[
\| p_n(t) \|_X \leq R_0 + r, \quad \forall \ n \geq 1, \ t \geq 0.
\]
(22)
So we can claim that \( \{ \xi_n \} \) is precompact in \( H_w \). Indeed, since \( \| \xi_n \|_{L^2} \leq R_0 + r \) then up to subsequence \( \xi_n \to \xi \) in \( \mathcal{L}^2_w \). Let us prove that up to a subsequence \( \xi_n \to \xi \) in \( \mathcal{L}^1_w \). Since \( \xi_n = p_n(t_n) \), then (13) yields that for each \( \delta > 0 \) there exist \( k(\delta) \geq 1 \), \( n(\delta) \geq 1 \) such that
\[
\int_{|x| > k} \xi_n(x)|x|dx < \frac{\delta}{3}, \quad \forall \ k \geq k(\delta), \ n \geq n(\delta).
\]
According to Amigó et al. [1, Lemma 6.1]
\[(L^1)^* = \{ \varphi = (1 + |x|)u : u \in L^\infty \}.
\]
Thus, we set \( d_n(x) = (1 + |x|)\xi_n(x) \) and prove that \( \{ d_n \} \) is a Cauchy sequence in \( \mathcal{L}^1_w \), because
\[
\left| \int \left( d_n(x) - d_m(x) \right) u(x)dx \right| \leq \int_{|x| \leq k} (1 + |x|)(\xi_n(x) - \xi_m(x)) u(x)dx
\]
\[
+ 2\|u\|_{L^\infty} \left( \int_{|x| > k} \xi_n(x)|x|dx + \int_{|x| > k} \xi_m(x)|x|dx \right) < \delta,
\]
for each \( u \in L^\infty \) and \( n, m \geq N = N(\delta, k) \). Since the space \( L^1 \) is weakly complete, then up to a subsequence \( d_n \to d \) in \( \mathcal{L}^1_w \) for some \( d \in L^1 \). Thus \( \xi_n \to \xi = \frac{d}{1 + |x|} \) in \( \mathcal{L}^1_w \). If we consider the restriction of \( \xi_n \) to each interval \([-k, k] \), then we deduce that \( \xi = \xi \) and up to a subsequence \( \xi_n \to \xi \) in \( H_w \).

Now let us prove this convergence in the strong topology of \( H \). Consider a smooth real function \( \theta \) that satisfies the following three conditions:
\[
\begin{align*}
(a) & \quad \theta(s) = 0, \quad |s| \leq 1; \\
(b) & \quad 0 \leq \theta'(s) \leq 1, \quad |s| \in [1, 2]; \\
(c) & \quad \theta(s) = 1, \quad |s| \geq 2,
\end{align*}
\]
(23)
and define for \( k > 1 \)
\[
\rho_k(x) = \theta\left( \frac{x}{k} \right).
\]
According to Amigó et al. [1, pp. 215–216] after multiplying (2) by \( \rho_k(x)p_n \) we obtain
\[
\frac{1}{2} \int \rho_k(x)p_n^2 dx + b \int \rho_k(x)p_n \frac{\partial p_n}{\partial x} dx
\]
\[
+ (D(p_n, \cdot, t) + \varepsilon_n) \int \rho_k(x) \left( \frac{\partial p_n}{\partial x} \right)^2 dx + \frac{1}{k} \int \theta'(\frac{x}{k}) p_n \frac{\partial p_n}{\partial x} dx
\]
\[
= \int \rho_k(x) p_n dx = 0.
\]
(24)
Integrating by parts we deduce
\[
b \int \rho_k(x)p_n \frac{\partial p_n}{\partial x} dx = -b \frac{1}{2k} \int \theta'(\frac{x}{k}) p_n^2 dx,
\]
\[
\frac{1}{k} \int \theta'(\frac{x}{k}) p_n \frac{\partial p_n}{\partial x} dx = -\frac{1}{2k^2} \int \theta''(\frac{x}{k}) p_n^2 dx.
\]
Then from (24) we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \rho_k(x)p_n^2(x)dx + \int_{\mathbb{R}} \rho_k(x)p_n^2(x)dx \leq \left( \frac{|b|}{2k} + \frac{(\alpha + 1)\beta}{2k^2} \right) \int_{\mathbb{R}} p_n^2(x)dx,
\]
where \( \beta := \max \{ |\theta'(s)| + |\theta''(s)| \} \).

Combining (22) and (25) we deduce from Gronwall’s Lemma that for some positive constant \( C = C(r) \)
\[
\int_{|x|>2k} p_n^2(x,t)dx \leq e^{-2t}r^2 + \frac{C(r)}{k}, \quad \forall \, t \geq 0, \, n \geq 1, \, k > 1.
\]

On the other hand, for every solution of (2)-(5) we have the following energy equality (for details see the proof of Lemma 3.3):
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (p(x,t))^2dx + (D(p(\cdot,t)) + \varepsilon) \int_{\mathbb{R}} \left( \frac{\partial p(x,t)}{\partial x} \right)^2 dx + \int_{|x|>1} (p(x,t))^2 dx
\]
\[
= \frac{D(p(\cdot,t))}{\alpha} \delta_0, p(\cdot,t)).\]

Let us consider the functions
\[
\bar{p}_n(t) = V_\varepsilon(t, p_n(t_n - 1)), \quad t \geq 0.
\]
Then \( \bar{p}_n \) is a solution of (2)-(5), \( \bar{p}_n(0) = p_n(t_n - 1) \), \( \bar{p}_n(1) = \xi_n \) and \( \bar{p}_n \) satisfies (22), (24), (27). Moreover, analogously to the previous arguments we deduce that up to subsequence \( \bar{p}_n(0) = p_n(t_n - 1) \rightarrow \bar{p}_0 \) in \( H_w \).

Hence, from Lemma 3.3 we obtain for every \( T > 1 \) that
\[
\bar{p}_n \rightarrow \bar{p} \quad \text{in} \quad C([0,T]; H_w),
\]
where \( \bar{p} \) is a solution of (2)-(5) with \( \bar{p}(0) = \bar{p}_0 \).

Since \( \varepsilon > 0 \) is fixed, we can derive from (22), (27) and the Aubin-Lions theorem [14] (see the proof of Lemma 3.3 for a more detailed explanation of this argument) that for every \( k > 1 \) up to subsequence \( \bar{p}_n \rightarrow \bar{p} \) in \( L^2(0,T; L^2(-k,k)) \).

In particular, \( \bar{p}_n(t) \rightarrow \bar{p}(t) \) in \( L^2(-k,k) \) for a.a. \( t \in (0,T) \).

By a diagonal procedure we obtain that up to a subsequence and for some \( \tau \in (0,1) \),
\[
\bar{p}_n(\tau) \rightarrow \bar{p}(\tau) \quad \text{in} \quad L^2(-k,k), \quad \forall \, k \geq 1.
\]

From (26) we get
\[
\int_{|x|>2k} \bar{p}_n^2(x,\tau)dx \leq e^{-2(\tau+t_n-1)}r^2 + \frac{C(r)}{k}, \quad \forall \, n \geq 1, \, k > 1.
\]

Combining (13), (29), (30) we have
\[
\bar{p}_n(\tau) \rightarrow \bar{p}(\tau) \quad \text{in} \quad X.
\]
Then the second part of Lemma 3.3 guarantees the convergence
\[
\bar{p}_n \rightarrow \bar{p} \quad \text{in} \quad C([\tau,T]; H).
\]

In particular,
\[
\xi_n = \bar{p}_n(1) \rightarrow \bar{p}(1) \quad \text{in} \quad H.
Thus we obtain the required precompactness of \( \{\xi_n\} \) and, therefore, the existence of the connected, stable global attractor \( \Theta \).

Finally, let us prove that for any sequences \( \varepsilon_n \uparrow 0 \) and \( \xi_n \in \Theta_{\varepsilon_n} \) there exists \( \xi \in \Theta \) such that up to a subsequence \( \xi_n \to \xi \) in \( X_w \) as \( n \to \infty \). Let us assume that \( \varepsilon_n \to 0 \). Denote \( V_{\varepsilon_n} = V_n, \Theta_{\varepsilon_n} = \Theta_n \). From (12) and (18) we have
\[
\Theta_n \subset K_0 = \{ p \in H : \| p \|_X \leq R_1 \}, \forall n \geq 1,
\]
where \( R_1 > 0 \) does not depend on \( \varepsilon > 0 \). Thus \( \{\Theta_n\} \) is uniformly bounded in \( H \).

Consider \( p_0^0 \in \Theta_n \). Let us prove that for some \( p_0 \in H \) up to a subsequence \( p_0^0 \to p_0 \) in \( H_w \). Indeed, since \( \| p_0^0 \|_{L^2} \leq R_1 \), then up to a subsequence \( p_0^0 \to p_0 \) in \( L^2_w \). Since \( p_0^0 \in \Theta_{\varepsilon_n} = V_n(t, \Theta_{\varepsilon_n}) \) for each \( t \geq 0 \), we can repeat the previous arguments and obtain from [1, Lemma 6.1] that up to a subsequence \( p_0^0 \to p_0 \) in \( L^1_w \).

Since \( \Theta_n \) consists of connected, stable global attractor \( \Theta \) for each \( t \), the following equality holds:

\[
\begin{align*}
\int_{\mathbb{R}} p(x, t) |x| dx &\leq \| p(t) \|_E e^{-\frac{1}{2}(t-\tau)} + \bar{C}, \\
\end{align*}
\]
for each \( t \geq \tau \), where positive constant \( \bar{C} \) does not depend on \( \varepsilon, \tau, p \), then \( p \in D_{\tau, \varepsilon} \) and inequality (12) holds.

Let us establish inequality (31). Consider the smooth real function \( \theta \) defined in (23) and for arbitrary \( k > 0 \) and \( R \geq 2k \) we set:
\[
\xi(s) := 1 - \theta(s), \quad \rho_k(s) := \theta\left(\frac{s}{k}\right), \quad \tau_R(x) := \xi\left(\frac{x}{R}\right), \quad \eta(s) := \rho_k(s) \tau_R(s), \quad s \in \mathbb{R}.
\]
Denote by \( |x| \eta \) the real function that equals to \( |x| \rho_k(x) \tau_R(x) \) for each \( x \in \mathbb{R} \). Note that \( |x| \eta \in C^\infty(\mathbb{R}) \).

Let us multiply (2) by \( |x| \eta \) and integrate the obtained equality with respect to \( x \) over \( \mathbb{R} \). We obtain:
\[
\begin{align*}
\left\langle \frac{\partial p}{\partial t}, (|x| \eta) \right\rangle + b(t) \left( \frac{\partial p}{\partial x}, (|x| \eta) \right) + (D(p(t)) + \varepsilon) \frac{\partial p}{\partial x} \frac{d}{dx} (|x| \eta) &+ \int_{|x| > 1} p \cdot (|x| \eta) dx = 0, \\
\end{align*}
\]
for a.e. \( t > \tau \).

5. Proofs of Lemmas 3.1, 3.2, 3.3, 3.4, and 3.5. In this section we provide the proofs of Lemmas 3.1, 3.2, 3.3, 3.4, and 3.5.

Proof of Lemma 3.1. We provide the proof in several steps.

Step 1. Let \( 0 \leq \varepsilon < 1, \tau \geq 0 \) and \( p \in K_{\tau, \varepsilon} \) with \( p(\tau) \in H \) be arbitrary fixed. If the following equality holds:

\[
\int_{\mathbb{R}} p(x, t) |x| dx \leq \| p(t) \|_E e^{-\frac{1}{2}(t-\tau)} + \bar{C},
\]
for each \( t \geq \tau \), where positive constant \( \bar{C} \) does not depend on \( \varepsilon, \tau, p \), then \( p \in D_{\tau, \varepsilon} \) and inequality (12) holds.

Let us establish inequality (31). Consider the smooth real function \( \theta \) defined in (23) and for arbitrary \( k > 0 \) and \( R \geq 2k \) we set:
\[
\xi(s) := 1 - \theta(s), \quad \rho_k(s) := \theta\left(\frac{s}{k}\right), \quad \tau_R(x) := \xi\left(\frac{x}{R}\right), \quad \eta(s) := \rho_k(s) \tau_R(s), \quad s \in \mathbb{R}.
\]
Denote by \( |x| \eta \) the real function that equals to \( |x| \rho_k(x) \tau_R(x) \) for each \( x \in \mathbb{R} \). Note that \( |x| \eta \in C^\infty(\mathbb{R}) \).

Let us multiply (2) by \( |x| \eta \) and integrate the obtained equality with respect to \( x \) over \( \mathbb{R} \). We obtain:
\[
\begin{align*}
\left\langle \frac{\partial p}{\partial t}, (|x| \eta) \right\rangle + b(t) \left( \frac{\partial p}{\partial x}, (|x| \eta) \right) + (D(p(t)) + \varepsilon) \frac{\partial p}{\partial x} \frac{d}{dx} (|x| \eta) &+ \int_{|x| > 1} p \cdot (|x| \eta) dx = 0, \\
\end{align*}
\]
for a.e. \( t > \tau \).
According to Amigó et al. [1, pp. 209–210], the following equality and inequality hold:
\[
\langle \frac{\partial p}{\partial t}, (|x|\eta) \rangle = \frac{d}{dt} \langle p, (|x|\eta) \rangle, \quad (34)
\]
\[
|\frac{\partial p}{\partial x} \cdot \frac{d}{dx} (|x|\eta)| \leq \frac{c_1}{k}, \quad (35)
\]
for a.e. \( t > \tau \), where \( c_1 \) is a positive constant that depends only on \( \theta \). Here we note that we understand equality (34) in the sense of distributions. Inequality (35) implies that
\[
(D(p(t)) + \varepsilon)|\frac{\partial p}{\partial x} \cdot \frac{d}{dx} (|x|\eta)| \leq (D(p(t)) + \varepsilon)\frac{c_1}{k} \leq (\alpha + \varepsilon)\frac{c_1}{k}, \quad (36)
\]
for a.e. \( t > \tau \), where the last inequality holds because \( p(t) \) is a probability density and, therefore, \( D(p(t)) \leq \alpha \) for a.e. \( t > \tau \).

Let us estimate the term \( b(t)|\frac{\partial p}{\partial x}, (|x|\eta) \). Direct calculations yield:
\[
|b(t)|\frac{\partial p}{\partial x}, (|x|\eta)| = |b(t) \int_R \frac{\partial p}{\partial x}(x,t)\eta(x)|x|dx| = |b(t) \int_R p(x,t)\frac{d}{dx}(\eta(x)|x|)dx|
\]
\[
\leq |b(t) \int_R p(x,t)|x|\frac{d}{dx}\eta(x)dx| + |b(t) \int_R p(x,t)\eta(x)dx|,
\]
for a.e. \( t > \tau \). The following two inequalities hold:
\[
|b(t) \int_R p(x,t)\eta(x)dx| \leq B \int_R p(x,t)\eta(x)|x|dx, \quad (38)
\]
\[
|b(t) \int_R |x|p(x,t)\frac{d}{dx}\eta(x)dx| \leq B \max_{1 \leq s \leq 2} \frac{d}{ds} \theta(s), \quad (39)
\]
for a.e. \( t > \tau \). Indeed, inequality (38) holds because \( 0 \leq \eta(x) \leq 1 \) for each \( x \in \mathbb{R} \), and \( \eta(x) = 0 \), if \( \frac{|x|}{k} \leq 1 \). Let us verify inequality (39). Since \( |b(t)| \leq B \) for a.e. \( t > \tau \), we have
\[
|b(t) \int_R |x|p(x,t)\eta'(x)dx| \leq B \int_R |x|p(x,t)\eta'(x)dx|,
\]
for a.e. \( t > \tau \). The equality \( \eta'(x) = \frac{1}{k} \theta'(\frac{x}{k}) (1 - \theta(\frac{x}{k})) - \frac{1}{k} \theta(\frac{x}{k}) \theta'(\frac{x}{k}) \) for each \( x \in \mathbb{R} \) yields that
\[
\int_R |x|p(x,t)\eta'(x)dx| \leq \frac{1}{k} \int_R |x|p(x,t)\theta'(\frac{x}{k})(1 - \theta(\frac{x}{k}))dx| + \frac{1}{k} \int_R |x|p(x,t)\theta(\frac{x}{k})\theta'(\frac{x}{k})dx|,
\]
for a.e. \( t > \tau \). Since \( \theta'(\frac{x}{k}) = 0 \) for each \( |x| > 2k \), and \( 0 \leq \theta(\frac{x}{k}) \leq 1 \) for each \( x \in \mathbb{R} \), we have
\[
\frac{1}{k} \int_R |x|p(x,t)\theta'(\frac{x}{k})(1 - \theta(\frac{x}{k}))dx| \leq 2 \int_R p(x,t)\theta'(\frac{x}{k})dx| \leq 2 \max_{1 \leq s \leq 2} \frac{d}{ds} \theta(s), \quad (42)
\]
for a.e. \( t > \tau \), as \( p(t) \) is a probability density for each \( t \geq \tau \). Similarly,
\[
\frac{1}{k} \int_R |x|p(x,t)\theta'(\frac{x}{k})dx| \leq 2 \int_R p(x,t)\theta'(\frac{x}{k})dx| \leq 2 \max_{1 \leq s \leq 2} \frac{d}{ds} \theta(s), \quad (43)
\]
for a.e. \( t > \tau \), because \( \theta'(\frac{x}{k}) = 0 \) for each \( x > 2k \), \( 0 \leq \theta(\frac{x}{k}) \leq 1 \) for each \( x \in \mathbb{R} \), and \( p(t) \) is a probability density for each \( t \geq \tau \). Inequalities (40)–(43) imply inequality (39).
From inequalities (37), (38) and (39) we have that
\[ |b(t)(\frac{\partial p}{\partial x},(|x|\eta)|) \leq 4B \max_{t \leq s \leq t} |\frac{d}{ds}\theta(s)| + \frac{B}{k} \int_{\mathbb{R}} p(x,t)\eta(x)|x|dx, \]
for a.e. \( t > \tau \).

Let us consider the last term in the left hand-side of equality (33). The following equality holds:
\[ \int_{|x| > 1} p(x,t)\eta(x)|x|dx = \int_{\mathbb{R}} p(x,t)\eta(x)|x|dx, \]
for a.e. \( t > \tau \), because (a) \( \eta(x) = 0 \) for each \( |x| \leq k \); and (b) \( k \geq 1 \).

Equalities (33), (34) and (45) and inequalities (36) and (44) give us the following inequality:
\[ \frac{d}{dt} \int_{\mathbb{R}} p(x,t)\eta(x)|x|dx + (1 - \frac{B}{k}) \int_{\mathbb{R}} p(x,t)\eta(x)|x|dx \leq \frac{c_2}{k}(\alpha + \varepsilon) + c_3, \]
for a.e. \( t > \tau \), where the positive constants \( c_2, c_3 \) depend only on \( \theta \) and \( \frac{d}{dt} \) is the derivative in the sense of distributions. Gronwall’s inequality implies that
\[ \int_{\mathbb{R}} p(x,t)\eta(x)|x|dx \leq \int_{\mathbb{R}} p(x,\tau)\eta(x)|x|dx e^{-\frac{1}{2}(t-\tau)} + \frac{2c_2(\alpha + 1)}{k} + 2c_3, \]
for each \( t \geq \tau \), whenever \( k \geq 2B \) and \( R \geq 2k \), because \( 0 \leq \varepsilon \ll 1 \). Since \( \eta(x)|x| \leq |x| \)
for each \( x \in \mathbb{R} \), according to the definition of the norm in \( L^1 \), the following inequality holds:
\[ \int_{\mathbb{R}} p(x,\tau)\eta(x)|x|dx \leq ||p(\tau)||_{L^1}. \]
Moreover,
\[ \int_{2k \leq |x| \leq R} p(x,t)|x|dx \leq \int_{\mathbb{R}} p(x,t)\eta(x)|x|dx, \]
for each \( t \geq \tau \), whenever \( k \geq 2B \) and \( R \geq 2k \), because (a) \( \eta \) is a nonnegative function; and (b) \( \eta(x) = 1 \), if \( 2k \leq |x| \leq R \). Therefore, inequalities (47)–(49) yield that
\[ \int_{2k \leq |x| \leq R} p(x,t)|x|dx \leq ||p(\tau)||_{L^1} e^{-\frac{1}{2}(t-\tau)} + \frac{2c_2(\alpha + 1)}{k} + 2c_3, \]
for each \( t \geq \tau \), whenever \( k \geq 2B \) and \( R \geq 2k \). If we set \( k = 2B \) in inequality (50) and pass in the obtained inequality to the limit as \( R \to \infty \), then we obtain the following inequality:
\[ \int_{|x| \geq 4B} p(x,t)|x|dx \leq ||p(\tau)||_{L^1} e^{-\frac{1}{2}(t-\tau)} + \frac{2c_2(\alpha + 1)}{2B} + 2c_3, \]
for each \( t \geq \tau \).

Since, \( p(x,t) \geq 0 \) for a.e. \( x \in \mathbb{R} \) and \( t > \tau \), and \( \int_{\mathbb{R}} p(x,t)dx = 1 \) for a.e. \( t > \tau \), the following inequalities hold:
\[ \int_{|x| \leq 4B} p(x,t)|x|dx \leq 4B \int_{|x| \leq 4B} p(x,t)dx \leq 4B, \]
for each \( t \geq \tau \).
Inequalities (51) and (52) imply that (31) holds. Therefore, \( p \in D_{\tau,\varepsilon} \) and inequality (12) holds.

**Step 2.** Let \( \delta > 0 \) be a positive number and \( K \subset H \) be a bounded set in \( T^1 \). We will prove that there exist constants \( T = T(\delta, K) > 0 \) and \( \bar{k} = \bar{k}(\delta, K) > 0 \) such that for each \( 0 \leq \varepsilon \ll 1, \tau \geq 0, \) and \( p \in K_{\tau,\varepsilon} \) with \( p(\tau) \in K \) inequality (13) holds for each \( t \geq \tau + T \) and \( k \geq \bar{k} \).

Let us fix arbitrary \( 0 \leq \varepsilon \ll 1, \tau \geq 0, p \in K_{\tau,\varepsilon} \) with \( p(\tau) \in K, k \geq 2B \) and \( R \geq 2k \). Equalities (33), (34) and (45) and inequalities (36), (37), (39) and (40) yield that

\[
\begin{align*}
\frac{d}{dt} & \int_{\mathbb{R}} p(x,t)\eta(x)|x|dx + (1 - \frac{B}{k}) \int_{\mathbb{R}} p(x,t)\eta(x)|x|dx \\
& \leq (\alpha + 1)\frac{C_1}{k} + B \int_{\mathbb{R}} |x|p(x,t)|\eta'(x)|dx,
\end{align*}
\]

for a.e. \( t > \tau \), as \( 0 \leq \varepsilon \ll 1 \). Since the set \( K \subset H \) is bounded in \( T^1 \), inequality (12) implies that

\[
\int_{\mathbb{R}} |x|p(x,t)dx \leq \|p(\tau)\|_{T^1} e^{-\frac{1}{2}(t-\tau)} + C \leq \|K\|_{T^1}^+ + C,
\]

for each \( t \geq \tau \), where \( \|K\|_{T^1}^+ := \sup_{g \in K} \|g\|_{T^1} \). Note that the first inequality in (54) follows from \( \int_{\mathbb{R}} |x|p(x,t)dx \leq \|p(t)\|_{T^1} \). Since \( R > k > 0 \) and \( \eta'(x) = \frac{1}{k} \theta'(\frac{x}{k}) (1 - \theta(\frac{x}{k})) - \frac{1}{k} \theta(\frac{x}{k}) \theta'(\frac{x}{k}) \) for each \( x \in \mathbb{R} \), we have that \( \frac{1}{R} < \frac{1}{k} \) and the following inequality holds:

\[
|\eta'(x)| \leq \frac{2}{k} \max_{1 \leq s \leq 2} |\theta'(s)|,
\]

for each \( x \in \mathbb{R} \), as \( 0 \leq \theta(s) \leq 1 \) for each \( s \in \mathbb{R} \). Inequalities (54) and (55) imply the following inequality:

\[
|B \int_{\mathbb{R}} p(x,t)\eta'(x)|x|dx| \leq \frac{2B}{k} \max_{1 \leq s \leq 2} |\theta'(s)| (\|K\|_{T^1}^+ + C),
\]

for a.e. \( t > \tau \). From inequalities (53), (55) and (56) we obtain that for the constant \( c_4 = c_4(K) := (\alpha + 1)c_1 + 2B \max_{1 \leq s \leq 2} |\theta'(s)| (\|K\|_{T^1}^+ + C) > 0 \) the following inequality is satisfied:

\[
\frac{d}{dt} \int_{\mathbb{R}} p(x,t)\eta(x)|x|dx + \frac{1}{2} \int_{\mathbb{R}} p(x,t)\eta(x)|x|dx \leq \frac{c_4}{k},
\]

for a.e. \( t > \tau \) if \( k \geq 2B \). Grönwall’s inequality and (48) imply that

\[
\int_{\mathbb{R}} p(x,t)\eta(x)|x|dx \leq \|p(\tau)\|_{T^1} e^{-\frac{1}{2}(t-\tau)} + \frac{2c_4}{k},
\]

for each \( t \geq \tau \). Since \( \eta(x) = 1 \) for each \( 2k \leq |x| \leq R \), (58) yields the following inequality:

\[
\int_{2k \leq |x| \leq R} p(x,t)|x|dx \leq \|K\|_{T^1}^+ e^{-\frac{1}{2}(t-\tau)} + \frac{2c_4}{k},
\]

for each \( t \geq \tau \). Let \( T = T(\delta, K) > 0 \) and \( \bar{k} = \bar{k}(\delta, K) \geq 2B > 0 \) satisfy the following inequalities:

\[
\|K\|_{T^1}^+ e^{-\frac{1}{2}T} < \frac{\delta}{2} \text{ and } \frac{2c_4}{k} < \frac{\delta}{2}.
\]

Therefore, since the right hand-side of (59) does not depend on \( R \), inequality (13) holds for each \( 0 \leq \varepsilon \ll 1, \tau \geq 0, p \in K_{\tau,\varepsilon} \) with \( p(\tau) \in K, t \geq \tau + T \), and \( k \geq \bar{k} \). \( \square \)
Proof of Lemma 3.3. We prove this lemma in several steps. We put \( \tau = 0 \) for the sake of simplicity.

**Step 1.** Let \( T > 0, \ p_0^n, p_0 \in H, b_n \in L^\infty(0, T), \) and \( 0 \leq \varepsilon_n \ll 1 \) for each \( n = 0, 1, \ldots, \). Suppose that \( |b_n(t)| \leq B \) for a.e. \( t \in (0, T) \), \( p^n \in C([0, T]; H_w) \) be a solution of problem (2)-(5), (9) on \([0, T]\) with parameters \( p_0^n, \varepsilon_n, b_n \), for each \( n \geq 1, \ p_0^n \to p_0 \) in \( H_w, \varepsilon_n \to \varepsilon_0, \) and \( b_n \to b_0 \) weakly-star in \( L^\infty(0, T) \). We multiply (2) by \( p^n \) with parameters \( \tau = 0, \varepsilon = \varepsilon_n \) and \( b = b_n \) for each \( n \geq 1 \). According to Gajewski et al. [8, Chapter IV, Theorem 1.17] and the equality \( \int_{[0, T]} p^n(x, t) dx = 0 \) for a.e. \( t \in (0, T) \) and each \( n \geq 1 \), we obtain the following equality that holds in the sense of scalar distributions on \((0, T)\):

\[
1 \frac{d}{dt} \int_{\mathbb{R}} (p^n(x, t))^2 dx + \left(D(p^n(\cdot, t)) + \varepsilon_n\right) \int_{\mathbb{R}} \left(\frac{\partial p^n(x, t)}{\partial x}\right)^2 dx + \int_{|x|>1} (p^n(x, t))^2 dx = \frac{D(p^n(\cdot, t))}{\alpha} \langle \delta_0, p^n(\cdot, t) \rangle.
\]

(60)

Since

\[
|\langle \delta_0, p^n(\cdot, t) \rangle| \leq \frac{\alpha}{2} \int_{\mathbb{R}} \left(\frac{\partial p^n(x, t)}{\partial x}\right)^2 dx + \frac{\alpha}{2} \int_{\mathbb{R}} (p^n(x, t))^2 dx + \frac{c^2}{2\alpha},
\]

for a.e. \( t \in (0, T) \), where \( c := ||\delta_0||_{H^{-1}} \), the following inequality holds:

\[
\frac{d}{dt} \int_{\mathbb{R}} (p^n(x, t))^2 dx + \left(D(p^n(\cdot, t)) + \varepsilon_n\right) \int_{\mathbb{R}} \left(\frac{\partial p^n(x, t)}{\partial x}\right)^2 dx \leq D(p^n(\cdot, t)) \left(\int_{\mathbb{R}} (p^n(x, t))^2 dx + \frac{c^2}{\alpha^2}\right),
\]

(61)

in the sense of scalar distributions on \((0, T)\). Therefore, Gronwall’s inequality implies that

\[
\int_{\mathbb{R}} (p^n(x, t))^2 dx \leq \left(\int_{\mathbb{R}} (p^n_0(x))^2 dx + \frac{c^2}{\alpha^2}\right) e^{\alpha T},
\]

(62)

for each \( t \in [0, T] \) and \( n \geq 1 \), because \( 0 \leq D(p^n(\cdot, t)) \leq \alpha \). Also, integrating (61) over \((0, T)\) and using (62) we obtain

\[
\int_0^T \left(D(p^n(\cdot, t)) + \varepsilon_n\right) \int_{\mathbb{R}} \left(\frac{\partial p^n(x, t)}{\partial x}\right)^2 dx dt \leq \int_{\mathbb{R}} (p^n_0(x))^2 dx + \frac{Tc^2}{\alpha} + \alpha T \left(\int_{\mathbb{R}} (p^n_0(x))^2 dx + \frac{c^2}{\alpha^2}\right) e^{\alpha T}.
\]

(63)

Inequalities (62)-(63) and equalities (7)-(8) yield the existence of \( \rho \in L^\infty(0, T; L^2) \) with \( \frac{\partial \rho}{\partial t} \in L^2(0, T; H^{-1}) \) and \( \xi \in L^2(0, T; H^1) \) such that up to a subsequence the following convergences hold:

\[
\begin{align*}
p^n & \to p & \text{weakly-star in} & L^\infty(0, T; L^2), \\
(D(p^n) + \varepsilon_n)p^n & \to \xi & \text{weakly in} & L^2(0, T; H^1), \\
\frac{\partial p^n}{\partial t} & \to \frac{\partial p}{\partial t} & \text{weakly in} & L^2(0, T; H^{-1}),
\end{align*}
\]

(64)

Repeating several lines from the proof of Lemma 2.5 from Amigó et al. [1] we obtain that \( p \in C([0, T]; L^2_w) \) and the following convergence:

\[
p^n \to p \text{ in } C([0, T]; L^2_w),
\]

(65)
that is,
\[
\sup_{t \in [0,T]} |(p^n(t) - p(t), \cdot)_L^2| \to 0, \quad n \to \infty,
\]
for each \( z \in L^2 \).

Since for a fixed \( \eta \in L^2(0,T;L^2) \) the function \( \eta(t,\cdot) \) belongs to \( L^2 \) for a.a. \( t \in (0,T) \), (65) implies that
\[
\int_R p^n(x,t) \eta(x,t) dx \to \int_R p(x,t) \eta(x,t) dx \quad \text{for a.a. } t \in (0,T).
\]

Also, the first convergence in (64) gives
\[
| \int_R p^n(x,t) \eta(x,t) dx | \leq \sup_k \| p^k \|_{L^\infty(0,T;L^2)} \| \eta(t) \|_{L^2} \quad \text{for a.a. } t \in (0,T).
\]

Using the Dominated Convergence Theorem we obtain that
\[
\int_R p^n(x,t) \eta(x,t) dx \to \int_R p(x,t) \eta(x,t) dx \quad \text{strongly in } L^1(0,T).
\]

Combining this result with the property \( b_n \to b_0 \), weakly-star in \( L^\infty(0,T) \), we get
\[
\int_0^T b_n(t) \int_R p^n(x,t) \eta(x,t) dx dt \to \int_0^T b_0(t) \int_R p(x,t) \eta(x,t) dx dt,
\]
for each \( \eta \in L^2(0,T;L^2) \). Moreover, since \( p^n(t) \in H \) for each \( t \in [0,T] \), and \( H \) is a convex set, we get
\[
p(t) \in cl_{L^2} H = cl_{L^2} H,
\]
for each \( t \in [0,T] \). In particular, \( p(t) \geq 0 \), and, according to (13) and (65), arguing as in [1, p.214] we obtain that
\[
\int_R p(x,t) dx = 1,
\]
\[
D(p^n) \to D(p) \quad \text{in } C([0,T]),
\]
\[
(D(p^n) + \varepsilon_n) p^n \to (D(p) + \varepsilon_0) p \quad \text{weakly in } L^2(0,T;L^2),
\]
and
\[
\xi = (D(p) + \varepsilon_0) p.
\]

If we pass to the limit in (8), we obtain that \( p \) satisfies equality (8) with parameters \( \tau = 0, b = b_0 \) and \( \varepsilon = \varepsilon_0 \). Then, similarly to Amigó et al. [1, p.214], we obtain that \( p^n \to p \) in \( C([0,T]; X_w) \). In particular, \( p(t) \in H \) for each \( t \geq 0 \) and \( p \in C([0,T]; H_w) \) is a solution of problem (2)–(5), (9) on \( [0,T] \) with parameters \( p_0, \varepsilon_0, b_0 \), and convergence (14) holds.

**Step 2.** Additionally assume that \( \varepsilon_0 > 0 \) and \( p^n_0 \to p_0 \) in \( H \). Properties (64), (67) and (68) imply that for the entire sequence \( \{p^n\}_{n \geq 1} \) the following convergences hold:
\[
p^n \to p \quad \text{weakly in } L^2(0,T;H^1),
\]
\[
\frac{\partial p^n}{\partial t} \to \frac{\partial p}{\partial t} \quad \text{weakly in } L^2(0,T;H^{-1}).
\]

Moreover, we know from (67) that
\[
D(p^n) \to D(p) \quad \text{in } C([0,T]).
\]

Let us prove that (15) is true. The main idea of the proof is given from Zgurovsky et al. [18] and Zgurovsky and Kasyanov [20]; see also Gorban et al. [10]. If we
integrate equality (60) over the time interval \((0, t)\), then we obtain the following equality:

\[
\frac{1}{2} \int_\mathbb{R} (p^n(x, t))^2 dx + \int_0^t (D(p^n(\cdot, s)) + \varepsilon_n) \int_\mathbb{R} \left( \frac{\partial p^n(x, s)}{\partial x} \right)^2 dx ds \\
+ \int_0^t \int_{|x| > 1} (p^n(x, s))^2 dx ds \\
= \frac{1}{2} \int_\mathbb{R} (p^n_0(x))^2 dx + \int_0^t \frac{D(p^n(\cdot, s))}{\alpha} \langle \delta_0, p^n(s) \rangle ds, \tag{71}
\]

for each \(t \in [0, T]\) and \(n \geq 1\). According to (64), (69) and (70) the following inequalities and equalities hold:

\[
\begin{align*}
\lim_{n \to \infty} \int_\mathbb{R} (p^n(x, t))^2 dx &\geq \int_\mathbb{R} (p(x, t))^2 dx, \\
\lim_{n \to \infty} \int_0^t (D(p^n(\cdot, s)) + \varepsilon_n) \int_\mathbb{R} \left( \frac{\partial p^n(x, s)}{\partial x} \right)^2 dx ds \\
&\geq \int_0^t (D(p(\cdot, s)) + \varepsilon_0) \int_\mathbb{R} \left( \frac{\partial p(x, s)}{\partial x} \right)^2 dx ds, \\
\lim_{n \to \infty} \int_0^t \int_{|x| > 1} (p^n(x, s))^2 dx ds &\geq \int_0^t \int_{|x| > 1} (p(x, s))^2 dx ds, \\
\lim_{n \to \infty} \int_\mathbb{R} (p^n_0(x))^2 dx &\geq \int_\mathbb{R} (p_0(x))^2 dx, \\
\lim_{n \to \infty} \int_0^t \frac{D(p^n(\cdot, s))}{\alpha} \langle \delta_0, p^n(s) \rangle ds &= \int_0^t \frac{D(p(\cdot, s))}{\alpha} \langle \delta_0, p(s) \rangle ds,
\end{align*}
\]

for each \(t \in [0, T]\). On the other hand, multiplying (2) by \(p^n\) with parameters \(\tau = 0\), \(\varepsilon = \varepsilon_0\) and \(b = b_0\) for each \(n \geq 1\), according to Gajewski et al. [8, Chapter IV, Theorem 1.17] and the equality \(\int_\mathbb{R} \frac{\partial p^n(x, t)}{\partial x} p^n(x, t) dx = 0\) for a.e. \(t \in (0, T)\), we obtain the following equality:

\[
\begin{align*}
\frac{1}{2} \int_\mathbb{R} (p^n(x, t))^2 dx + \int_0^t (D(p^n(\cdot, s)) + \varepsilon_0) \int_\mathbb{R} \left( \frac{\partial p^n(x, s)}{\partial x} \right)^2 dx ds \\
+ \int_0^t \int_{|x| > 1} (p^n(x, s))^2 dx ds \\
= \frac{1}{2} \int_\mathbb{R} (p^n_0(x))^2 dx + \int_0^t \frac{D(p^n(\cdot, s))}{\alpha} \langle \delta_0, p^n(\cdot, s) \rangle ds, \tag{73}
\end{align*}
\]

for each \(t \in [0, T]\). The equalities and inequalities in (72) and (73) yield that

\[
\begin{align*}
\int_\mathbb{R} (p^n(x, t))^2 dx &\to \int_\mathbb{R} (p(x, t))^2 dx, \\
\int_0^t (D(p^n(\cdot, s)) + \varepsilon_n) \int_\mathbb{R} \left( \frac{\partial p^n(x, s)}{\partial x} \right)^2 dx ds \\
&\to \int_0^t (D(p(\cdot, s)) + \varepsilon_0) \int_\mathbb{R} \left( \frac{\partial p(x, s)}{\partial x} \right)^2 dx ds, \tag{74}
\end{align*}
\]

\[
\begin{align*}
\int_0^t \int_{|x| > 1} (p^n(x, s))^2 dx ds &\to \int_0^t \int_{|x| > 1} (p(x, s))^2 dx ds,
\end{align*}
\]
for each \( t \in [0, T] \). Since the following inequality holds:
\[
\left| \int_0^t (D(p^n(t, \cdot)) + \epsilon_n) \right| \left( \frac{\partial p^n(x, s)}{\partial x} \right)^2 \, dx \, ds - \int_0^t (D(p(t, s)) + \epsilon_0) \right| \left( \frac{\partial p^n(x, s)}{\partial x} \right)^2 \, dx \, ds \\
\leq \left( \epsilon_n - \epsilon_0 + \max_{s \in [0,1]} |D(p^n(s) - D(p(s, s)) | \right| \right| p^n \right| _{L^1(0,T; H^1)}^2,
\]
for each \( n \geq 1 \) and \( t \in [0, T] \), \( \epsilon_n \to \epsilon_0 > 0 \), and \( D(p^n) \to D(p) \) in \( C([0, T]) \), we have
\[
\left| \int_0^t \frac{\partial p^n(x, s)}{\partial x} \right| \left( \frac{\partial p^n(x, s)}{\partial x} \right)^2 \, dx \, ds \rightarrow \int_0^t \frac{\partial p(x, s)}{\partial x} \right| \left( \frac{\partial p(x, s)}{\partial x} \right)^2 \, dx \, ds,
\]
for each \( t \in [0, T] \). On the other hand, in light of (69) and using the Aubin-Lions theorem [14], we have that
\[
p^n \to p \ \text{in} \ L^2(0, T; L^2(-k, k)), \text{ for any } k > 0,
\]
which together with (74) gives
\[
\left| \int_0^t \int_0^T (p^n(x, s)) \right| \left( \frac{\partial p^n(x, s)}{\partial x} \right)^2 \, dx \, ds \\
\rightarrow \int_0^t \int_0^T (p(x, s)) \left( \frac{\partial p(x, s)}{\partial x} \right)^2 \, dx \, ds,
\]
for each \( t \in [0, T], \) and then joining (75) and (76) we have
\[
\left| p^n \right| _{L^2(0, T; H^1)} \rightarrow \left| p \right| _{L^2(0, T; H^1)}.
\]
Since \( L^2(0, T; H^1) \) is a Hilbert space, the first statement from (69) and (77) imply that \( p^n \to p \) in \( L^2(0, T; H^1) \), that is, statement (15) holds.

**Step 3.** Let us prove that \( p, p^n \in C([0, T]; H) \) for each \( n \geq 1 \), and that statement (16) holds. Since \( p^n, p \in L^2(0, T; H) \) and \( \frac{d}{dt} p^n, \frac{d}{dt} p \in L^2(0, T; H^{-1}) \) for each \( n \geq 1 \), Theorem 1.17 in [8, Chapter IV] implies that \( p^n, p \in C([0, T]; L^2). \) Moreover, \( p^n \to p \) in \( C([0, T]; L^2). \) Indeed, as \( L^2 \) is a Hilbert space, it is sufficient to prove that \( \left| p^n \right| _{L^2} \to \left| p \right| _{L^2} \) in \( C([0, T]). \) This statement follows from (71), (73), (15) and (67). Therefore, in order to prove that \( p, p^n \in C([0, T]; H) \) for each \( n \geq 1 \) and that statement (16) holds it is sufficient to establish that
\[
p, p^n \in C([0, T]; \mathcal{T}^1) \text{ and } \left| p^n - p \right| _{C([0, T]; \mathcal{T}^1)} \rightarrow 0.
\]

**Step 3.1.** Let us set
\[
K := \{ p^n, p : n = 1, 2, \ldots \} \quad \text{and} \quad c = c(K) := (\alpha + c_1 + 2B \max \{|\theta'(s)|(\sup_{g \in K} |g(0)|_{\mathcal{T}^1} + C) > 0, \]
where \( \theta \) and \( \eta \) are defined in (32) and \( c_1 \) is the positive constant from (35), that depends only on \( \theta \), whereas \( C > 0 \) is the universal constant from (12). The following inequality holds:
\[
\int_{|x| \geq k} g(x, t) |x| \, dx \leq \int_{|x| \geq k} g(x, 0) |x| \, dx - t \frac{2c_1}{k},
\]
for each \( k \geq 2B, g \in K, \) and \( t \in [0, T]. \) Indeed, let us fix arbitrary \( k \geq 2B \) and \( R \geq 2k. \) If we follow the proof of Lemma 3.1, then, according to inequality (57), we obtain that
\[
\frac{d}{dt} \int_0^T g(x, t) |x| \, dx + \frac{1}{2} \int_0^T g(x, t) |x| \, dx \leq \frac{c_1}{k},
\]
for each each \( g \in K \) and a.e. \( t \in (0, T). \) Applying Gronwall’s lemma to inequality (80) we have
\[
\int_0^T g(x, t) |x| \, dx \leq \int_0^T g(x, 0) |x| \, dx - t \frac{2c_1}{k} + \frac{2c_1}{k},
\]
Moreover, since statement (17) holds, because \( C \) inequality:
\[

g(x,t)|x|dx \leq \int_{|x| \geq k} g(x,0)|x|dx e^{-\frac{2c}{k}|x|} + \frac{2c}{k}, \tag{82}
\]
for each \( g \in K \) and \( t \in [0,T] \). Therefore, since the right hand-side of (82) does not depend on \( R \), then inequality (79) holds for each \( k \geq 2B \), \( g \in K \), and \( t \in [0,T] \).

**Step 3.2.** Since \( p^n_0 \rightarrow p_0 \) in \( H \), we get \( p^n_0 \rightarrow p_0 \) in \( L^1 \). Therefore,
\[
\lim_{k \to \infty} \sup_{g \in K} \int_{|x| \geq k} g(x,0)|x|dx = 0. \tag{83}
\]
Moreover, since \( p^n \rightarrow p \) in \( C([0,T];L^2) \), (79) and (83) imply (78). Thus, \( p, p^n \in C([0,T];H) \) for each \( n \geq 1 \) and statement (16) holds.

**Step 4.** Additionally assume that \( b_n \rightarrow b_0 \) in the Lebesgue measure on \([0,T]\). Then statement (17) holds, because
\[
\frac{\partial p^n}{\partial t} = -b_n(t) \frac{\partial p^n}{\partial x} + (D(p^n) + \varepsilon_n) \frac{\partial^2 p^n}{\partial x^2} - \chi_{B\setminus[-1,1]}(x)p^n + \frac{D(p^n)}{\alpha} g_0
\]
in the sense of distributions, for each \( n = 1, 2, \ldots, p^n \rightarrow p \) in \( L^2(0,T;H^1) \subset L^2(0,T;L^2) \), \( D(p^n) \rightarrow D(p) \) in \( C([0,T]) \), and \( b_n \rightarrow b_0 \) in the Lebesgue measure on \([0,T]\). Here we note that
\[
\int_0^T \|b_n(t)\|_{L^2} \|\frac{\partial p^n}{\partial x}\|^2_{H^{-1}} dt \to 0,
\]
if
\[
\int_0^T \|b_n(t)\|^2_{L^2} \|\frac{\partial p^n}{\partial x}\|^2_{H^{-1}} dt \to \int_0^T \|b(t)\|^2_{L^2} \|\frac{\partial p}{\partial x}\|^2_{H^{-1}} dt, \tag{84}
\]
because \( L^2(0,T;H^{-1}) \) is a Hilbert space and (66) holds. Since \( \|\frac{\partial p^n}{\partial x}\|_{H^{-1}} \rightarrow \|\frac{\partial p}{\partial x}\|_{H^{-1}} \) in \( L^1(0,T) \), \( b_n \rightarrow b_0 \) in the Lebesgue measure on \([0,T]\) (and hence passing to a subsequence \( b_n(t) \rightarrow b_0(t) \) for a.e. \( t \in (0,T) \)), and \( |b_n(t)| \leq B \) for a.e. \( t \in (0,T) \), the direct and converse Fatou’s lemmas (see [17], [7]) imply (84).

**Proof of Lemma 3.2.** Existence. Let \( 0 < \varepsilon \ll 1, 0 < \tau < T < \infty \), and \( p_\varepsilon \in H \) be arbitrary fixed. Then, there exists \( \{p^n_\varepsilon\}_{n \geq 1} \subset E \) such that \( p^n_\varepsilon \rightarrow p_\varepsilon \) in \( H \). Conditions (10) imply that for each \( n \geq 1 \) there exists a unique solution \( p^n_\varepsilon \) of problem (2)–(5), (9) on \([\tau,T]\) with initial data \( p^n_\varepsilon \). Then, according to Lemma 3.3, problem (2)–(5), (9) on \([\tau,T]\) with initial data \( p_\varepsilon \) has a solution \( p \). Moreover, \( p \in C([\tau,T];H) \).

**Uniqueness.** Lemma 2.1 in Cancès et al. [4] implies that for each \( 0 < \varepsilon \ll 1, 0 < \tau < T < \infty \), and \( p_\varepsilon \in H \) there exists no more than one solution \( p \) of the problem (2)–(4), (9) on \([\tau,T]\).

**Proof of Lemma 3.4.** Let \( K \subset H \) be a bounded set in \( L^2 \) and \( \varepsilon \in (0,1) \) be arbitrary. Fix \( p \in D_{0,\varepsilon} \) with \( p(0) \in K \). The following inequalities hold:
\[
\frac{d}{dt} \|p(t)\|_{L^2} + 2(D(p) + \varepsilon) \left\| \frac{\partial p}{\partial x} \right\|_{L^2}^2 + 2 \int_{|x| > 1} p^2(x,t)dx \leq 2 \frac{\alpha}{\alpha} D(p) \|p\|_{L^\infty(-1,1)} \leq CD(p) \left( \|p\|_{L^2(-1,1)} + \left\| \frac{\partial p}{\partial x} \right\|_{L^2(-1,1)} \right), \tag{85}
\]
where $C = \frac{2d}{\alpha}$, and $d$ is the constant from the embedding $H^1(-1,1) \subset L^\infty(-1,1)$.

Further we shall use the Poincaré-Wirtinger inequality:

$$
\|p - \overline{p}\|_{L^2(-1,1)} \leq d_1 \left\| \frac{\partial p}{\partial x} \right\|_{L^2(-1,1)},
$$

where $\overline{p} = \frac{1}{2} \int_{-1}^1 p(x)dx$. As $\|\overline{p}\|_{L^2(-1,1)} \leq \frac{1}{\sqrt{2}}$, we have

$$
\left\| \frac{\partial p}{\partial x} \right\|_{L^2(-1,1)} \geq C_1 (\|p\|_{L^2(-1,1)}^2 - 1), \tag{86}
$$

where $C_1 = \frac{d_1}{\sqrt{2}}$. Then

$$
\varepsilon \left\| \frac{\partial p}{\partial x} \right\|_{L^2(-1,1)} \geq \varepsilon C_1 (\|p\|_{L^2(-1,1)}^2 - 1),
$$

$$
D(p) \left\| \frac{\partial p}{\partial x} \right\|_{L^2(-1,1)} \geq D(p) C_1 (\|p\|_{L^2(-1,1)}^2 - 1).
$$

Thus

$$
\frac{d}{dt} \|p\|_{L^2}^2 + (D(p) + \varepsilon) \left\| \frac{\partial p}{\partial x} \right\|_{L^2}^2 + \varepsilon C_1 \|p\|_{L^2(-1,1)}^2 + D(p) C_1 \|p\|_{L^2(-1,1)}^2 + 2 \int_{|x| > 1} p^2 dx
$$

$$
\leq C D(p) \left( \|p\|_{L^2(-1,1)}^2 + \left\| \frac{\partial p}{\partial x} \right\|_{L^2(-1,1)}^2 \right) + C_1 \varepsilon + C_1 D(p). \tag{87}
$$

Using the inequalities

$$
C D(p) \|p\|_{L^2(-1,1)}^2 \leq \frac{C_1}{2} D(p) \|p\|_{L^2(-1,1)}^2 + \frac{C^2}{2C_1} D(p),
$$

$$
C D(p) \left\| \frac{\partial p}{\partial x} \right\|_{L^2(-1,1)}^2 \leq \frac{1}{2} D(p) \left\| \frac{\partial p}{\partial x} \right\|_{L^2(-1,1)}^2 + \frac{C^2}{2} D(p),
$$

we obtain

$$
\frac{d}{dt} \|p\|_{L^2}^2 + \frac{1}{2} (D(p) + \varepsilon) \left\| \frac{\partial p}{\partial x} \right\|_{L^2}^2 + \varepsilon C_1 \|p\|_{L^2(-1,1)}^2
$$

$$
+ \frac{C_1}{2} D(p) \|p\|_{L^2(-1,1)}^2 + 2 \int_{|x| > 1} p^2 dx \leq C_2 D(p) + C_1 \varepsilon, \tag{88}
$$

where $C_2 = C_1 + \frac{C^2}{2} + \frac{C^2}{2C_1}$. We assume that $\varepsilon C_1 < 1$ and denote

$$
C_3 = \min\left\{ \frac{C_1}{2}, 2 \right\}, \quad C_4 = C_1 + C_2, \quad C_5 = \frac{C_4}{C_3}.
$$

In all further arguments we will essentially use the fact that every open set in $(0, +\infty)$ is the union of no more than a countable number of open intervals.

Let us consider the open set

$$
A^0_1 = \{ t > 0 \mid D(p(t)) < \varepsilon \}.
$$

From (88) for any $t \in A^0_1$,

$$
\frac{d}{dt} \|p\|_{L^2}^2 + \varepsilon C_1 \|p\|_{L^2}^2 \leq \varepsilon C_4. \tag{89}
$$

Since

$$
\{ t > 0 \mid D(p(t)) = \varepsilon \} \subset \{ t > 0 \mid D(p(t)) < 2\varepsilon \},
$$

where
we can state that
\[ \frac{d}{dt} \|p\|_{L^2}^2 + \varepsilon C_1 \|p\|_{L^2}^2 \leq 2\varepsilon C_4, \forall t \in A_1 := \{ t > 0 \mid D(p(t)) \leq \varepsilon \}. \] (90)

Let us consider the open set
\[ A_2 = \{ t > 0 \mid D(p(t)) > \varepsilon \}. \]

From (88) we have
\[ \frac{d}{dt} \|p\|_{L^2}^2 + C_3 D(p) \|p\|_{L^2}^2 \leq D(p) C_4, \forall t \in A_2, \] (91)
or, equivalently,
\[ \frac{d}{dt} (C_3 \|p\|_{L^2}^2 - C_4) + C_3 D(p) (C_3 \|p\|_{L^2}^2 - C_4) \leq 0. \] (92)

Considering the open set
\[ \hat{A}_2 = \{ t \in A_2 \mid C_3 \|p(t)\|_{L^2}^2 - C_4 > 0 \} = \{ t \in A_2 \mid \|p(t)\|_{L^2}^2 > C_3 \}, \]
we deduce from (92) that
\[ \frac{d}{dt} \|p\|_{L^2}^2 + \varepsilon C_3 \|p\|_{L^2}^2 \leq \varepsilon C_4, \forall t \in \hat{A}_2. \] (93)

Combining (89)-(93) we deduce
\[ \frac{d}{dt} \|p\|_{L^2}^2 + \varepsilon C_3 \|p\|_{L^2}^2 \leq 2\varepsilon C_4, \forall t \in A := \{ t > 0 \mid \|p(t)\|_{L^2}^2 > C_5 \}. \] (94)

Now let \( \|p_0\|_{L^2}^2 > C_5 \). From the continuity of the map \( t \mapsto \|p(t)\|_{L^2} \) there exists a maximal point \( t_1 \in (0, +\infty) \) such that
\[ \|p(t_1)\|_{L^2}^2 > C_5, \forall t \in [0, t_1). \]

Then from (94)
\[ \|p(t)\|_{L^2}^2 \leq \|p_0\|_{L^2}^2 e^{-\varepsilon C_3 t} + 2C_5, \forall t \in [0, t_1). \] (95)

If \( t_1 = +\infty \), we immediately obtain (18).
Otherwise, \( \|p(t_1)\|_{L^2}^2 = C_5 \). If
\[ \|p(t)\|_{L^2}^2 \leq C_5, \forall t \geq t_1, \]
then we get (18). In other case there exists \( t_2 \geq t_1 \) such that
\[ \|p(t)\|_{L^2}^2 \leq C_5, \forall t \in [t_1, t_2], \]
and for some \( t_3 \in (0, +\infty) \) we have \( \|p(t)\|_{L^2}^2 > C_5 \), for all \( t \in (t_2, t_2 + t_3) \). Then from (94)
\[ \|p(t)\|_{L^2}^2 \leq C_5 e^{-\varepsilon C_3 (t-t_2)} + 2C_5, \forall t \in [t_2, t_2 + t_3). \] (96)

We can treat \( t_3 \) in the same way as \( t_1 \). Repeating this procedure, we obtain (18) in this case.

If \( \|p_0\|_{L^2}^2 \leq C_5 \), then we can repeat the previous arguments with \( t_1 = 0 \) and obtain the required result. \( \square \)

**Proof of Lemma 3.5.** Let \( K = \{ p_0 \in H : \|p_0\|_X \leq R \} \subset H \) be a bounded set, where \( R > 0 \) be an arbitrary fixed. The first statement of Lemma 3.1 (see inequality (12)) yields that for each \( p \in D_{r,\varepsilon} \) with \( p(\tau) \in K \) the following inequality holds:
\[ \|p(t)\|_{\hat{X}} \leq C + \sup_{g \in K} \|g\|_{\hat{X}}, \] (97)
for each $t \geq \tau$, where the positive constant $C$ depends only on parameters of problem (2) and does not depend on $K$, $\tau$, and $\varepsilon$. Therefore, it is sufficient to justify the a priori estimate in the space $L^2$.

Using (85) and the Poincaré-Wirtinger inequality we deduce that the following inequality holds in the sense of distributions on $(\tau, \infty)$:

$$
\frac{d}{dt} \int_\mathbb{R} (p(x,t))^2 \, dx + D(p(\cdot,t)) \int_\mathbb{R} \left( \frac{\partial p(x,t)}{\partial x} \right)^2 \, dx + \int_{|x|>1} (p(x,t))^2 \, dx \leq c_3 D(p(\cdot,t)),
$$

where the positive constants $c_2$ and $c_3$ do not depend on $\tau, \varepsilon$, and $p \in K$. Inequalities (86) and (98) imply the following inequality, that holds in the sense of distributions on $(\tau, \infty)$:

$$
\frac{d}{dt} \int_\mathbb{R} (p(x,t))^2 \, dx + c_4 D(p(\cdot,t)) \int_\mathbb{R} \left( \frac{\partial p(x,t)}{\partial x} \right)^2 \, dx + c_4 D(p(\cdot,t)) \int_\mathbb{R} (p(x,t))^2 \, dx + \int_{|x|>1} (p(x,t))^2 \, dx \leq D(p(\cdot,t)) c_5,
$$

where the positive constants $c_4, c_5$ do not depend on $\tau, \varepsilon$, and $p \in K$. Gronwall’s inequality implies that

$$
\|p(t)\|_{L^2}^2 + \int_\tau^t \exp \left\{ -c_4 \int_s^t D(p(\cdot,\xi)) \, d\xi \right\} \int_{|x|>1} (p(x,s))^2 \, dx \, ds \leq \|p(\tau)\|_{L^2}^2 \exp \left\{ -c_4 \int_\tau^t D(p(\cdot,\xi)) \, d\xi \right\} + \frac{c_5}{c_4},
$$

for each $t \geq \tau$. Therefore, inequality (19) holds, that is, the first statement of the lemma holds. Combining (19) and (97) we obtain the second statement of the lemma.

\[\square\]

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