The capacity of the noisy quantum channel†

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Abstract: An upper limit is given to the amount of quantum information that can be transmitted reliably down a noisy, decoherent quantum channel. A class of quantum error-correcting codes is presented that allow the information transmitted to attain this limit. The result is the quantum analog of Shannon’s bound and code for the noisy classical channel.

The ‘quantum’ in quantum mechanics means ‘how much’ — in quantum mechanics, classically continuous variables such as energy, angular momentum and charge come in discrete units called quanta. This discrete character of quantum-mechanical systems such as photons, atoms, and spins allows them to register ordinary digital information. A left-circularly polarized photon can encode a 0, for example, while a right-circularly polarized photon can encode a 1. Quantum systems can also register information in ways that classical digital systems cannot: a transversely polarized photon is in a quantum superposition of left and right polarization, and in some sense encodes both 0 and 1 at the same time. Even more surprising from the classical perspective are so-called entangled states, in which two or more quantum systems are in superpositions of correlated states, so that two photons can encode, for example, 00 and 11 at once. Such entangled states behave in ways

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that apparently violate classical intuitions about locality and causality (without, of course, actually violating physical laws).

Information stored on quantum systems that can exist in superpositions and entangled states is called quantum information. The unit of quantum information is the quantum bit, or qubit (pronounced ‘Q-bit’),\(^1\) the amount of quantum information that can be registered on a single two-state variable such as a photon’s polarization or a neutron’s spin. This paper puts fundamental limits on the amount of quantum information that can be transmitted reliably along a noisy communication channel such as an optical fiber. Theorems are presented that limit the rate at which arbitrary superpositions of qubits can be sent down a channel with given noise characteristics, and encoding schemes are presented that attain that limit.

It is important to compare the results presented here—the use of a quantum channel to transmit quantum information—with schemes that use quantum channels to transmit classical information, as in Caves and Drummond’s comprehensive review of quantum limits on bosonic communication rates.\(^2\) The limit to the rate at which arbitrary sequences of ordinary classical bits, suitably encoded as quantum states, can be transmitted down a quantum channel such as an optical fiber is given by Holevo’s theorem. In contrast, the results presented here limit the rate at which arbitrary superpositions of sequences of quantum bits can be sent reliably down a noisy, decoherent quantum channel. As such, the theorems presented in this paper are complementary to the results of Schumacher\(^1,3\) and Josza\(^3\) on the noiseless quantum channel. Any channel that can transmit quantum information can be used to transmit classical information as well. It is possible, however, for a channel to be able to transmit classical information without being able to transmit quantum information: examples of such completely decoherent channels will be discussed below.

The difference between quantum and classical information does not arise from a fundamental physical distinction between the systems that register, process, and transmit that information. As just noted, quantum channels can be used to transmit classical information. And after all, ‘classical’ information-registering systems such as capacitors and neurons are at bottom quantum-mechanical. The difference arises from the conditions under which such systems operate. When properly isolated from their environment, photons and atoms can exist in superpositions and entangled states for long periods of time, with experimentally measurable results. Capacitors and neurons, in contrast, inter-
act strongly with a thermal environment, which prevents them from exhibiting coherent quantum effects. As a result, quantum information can be used to perform tasks that classical information cannot.

A full theory of quantum information and its properties does not yet exist. However, the ability to transmit and process quantum information reliably provides the solution to problems to which no classical solution is known: if entangled quantum bits can be transmitted and received, quantum cryptographic techniques can be used to create provably secure shared keywords for unbreakable codes; while the ability to process quantum information allows quantum computers efficiently to factorize large numbers and to simulate local quantum systems.

For quantum information to prove useful, it must be transmitted and processed reliably. Quantum superpositions and entangled states tend to be easily disrupted by noise and by interactions with their environment, a process called decoherence. Until recently, decoherence and noise seemed insurmountable obstacles to reliable quantum information transmission and processing. However, in 1995, Shor exhibited the first quantum error-correcting routine. Since then, several such routines have been proposed. All of these routines have the feature, common to many classical error-correcting codes as well, that the rate of transmission of quantum information goes to zero as the reliability of transmission goes to one. This paper shows that arbitrarily complicated quantum states can in principle be encoded, subjected to high levels of noise and decoherence, then decoded to give a state arbitrarily close to the original state, all with a finite rate of transmission of quantum information. The paper states and outlines the proof of theorems that put an upper bound to the capacity of noisy, decoherent quantum channels to transmit quantum information reliably, and exhibits a class of quantum codes that attain that bound.

1. Quantum Sources

A quantum channel has a source that emits systems in quantum states, (the signal) to the channel, and a receiver that receives the noisy, decohered signal emitted by the channel. For example, the source could be a highly attenuated laser that emits individual monochromatic photons, the channel could be an optical fiber, and the receiver could be a photocell. Or the source could be a set of ions in an ion-trap quantum computer that have been prepared by a sequence of laser pulses in an entangled state, the channel could be the ion trap in which the ions evolve over time, and the receiver could be a microscope to read out the states of the ions via laser-induced fluorescence. This second
example indicates that a quantum channel can transmit quantum information from one time to another as well as from one place to another. As Shannon emphasized, a computer memory is a communications channel.

A more complete picture of a quantum channel is as follows (Figure 1): the input signal is some unknown quantum state; the input is fed into an encoder that transforms it into a redundant form; the encoded signal is sent down the channel, subjected to noise and decoherence; the noisy, decohered signal is then fed into a decoder that attempts to restore the original signal. Quantum encoding and decoding requires the ability to manipulate quantum states in a systematic fashion, for example, by using Kimble’s photonic quantum logic gates or Wineland’s realization of the ion-trap quantum computer proposed by Cirac and Zoller. From a practical point of view, such decoding and encoding may prove the most difficult part of reliable quantum information transmission and processing. This paper will simply exhibit coding and decoding schemes that attain the channel capacity: it will not address how such schemes can be carried out in practice.

In order to demonstrate the quantum analog of Shannon’s noisy coding theorem, it’s helpful to set up a quantum formalism that corresponds closely to the classical picture of a noisy channel. Quantum systems and quantum signals are described by states $|\psi\rangle$ in a Hilbert space $\mathcal{H}$, or more generally, by density matrices $\rho \in \mathcal{H}^* \otimes \mathcal{H}$. A quantum ensemble $\mathcal{E} = \{(|\psi_i\rangle, p_i)\}$ is a set of quantum states $|\psi_i\rangle$ belonging to the same Hilbert space $\mathcal{H}$, together with their probabilities $p_i$. The expectation value of a measurement on the ensemble corresponding to a Hermitian operator $M$ is $\langle M \rangle_{\mathcal{E}} = \sum_i p_i \langle \psi_i | M | \psi_i \rangle = \text{tr} M \rho_{\mathcal{E}}$, where $\rho_{\mathcal{E}} = \sum_i p_i |\psi_i\rangle \langle \psi_i |$ is the density matrix corresponding to the ensemble. The states $|\psi_i\rangle$ need be neither orthonormal nor normalized, as long as $\sum_i p_i \langle \psi_i | \psi_i \rangle = \text{tr} \rho_{\mathcal{E}} = 1$. That is, a quantum ensemble is just the quantum analog of a classical ensemble, where care has been taken to take into account the inherently statistical nature of quantum mechanics.

Two ensembles that have the same density matrix are statistically indistinguishable: no set of measurements can distinguish whether a sequence of states is drawn from one ensemble rather than the other. An example of statistically indistinguishable ensembles is $\mathcal{E}_1 = \{(|\uparrow\rangle, 1/2), (|\downarrow\rangle, 1/2)\}$, and

$$\mathcal{E}_2 = \{(|\uparrow\rangle, 1/3), (1/2|\uparrow\rangle + \sqrt{3}/2|\downarrow\rangle, 1/3), (1/2|\uparrow\rangle - \sqrt{3}/2|\downarrow\rangle, 1/3)\}$$

both with density matrices $\rho = 1/2 |\uparrow\rangle \langle \uparrow | + 1/2 |\downarrow\rangle \langle \downarrow |$. Note that an ensemble over a finite dimensional Hilbert space can contain an infinite number of states, e.g., $\mathcal{E} = \{(e^{i\phi} |\uparrow\rangle, p(\phi) = 1/2\pi)\}$, in which case each state is paired with a continuous probability
density, \( p(\phi) \), and \( \rho = \int_0^{2\pi} (1/2\pi)e^{i\phi} |\uparrow\rangle\langle\uparrow| e^{-i\phi} d\phi = |\uparrow\rangle\langle\uparrow| \). Because of the inherently statistical nature of quantum mechanics, different quantum ensembles can be statistically indistinguishable, while two classical ensembles are statistically indistinguishable if and only if they are identical.

A particularly interesting type of continuous quantum ensemble is the uniform ensemble over a Hilbert space \( \mathcal{H} \), \( \mathcal{E}_\mathcal{H} = \{ (|\phi\rangle \in \mathcal{H}, p_\phi = 1/\text{vol}\mathcal{H}) \} \), where \( \text{vol}\mathcal{H} \) is the volume of the unit sphere in \( \mathcal{H} \). This ensemble contains every possible state and superposition of states in \( \mathcal{H} \), all with equal probabilities. The corresponding density matrix is \( \rho_\mathcal{H} = (1/d) \sum_{i=1}^{d} |\phi_i\rangle\langle\phi_i| \), where \( d \) is the dimension of \( \mathcal{H} \) and \( \{|\phi_i\rangle\} \) is an orthonormal basis for \( \mathcal{H} \). If we wish to transmit arbitrary superpositions of states down quantum channels, the sources of interest are of the form \( \mathcal{E}_\mathcal{H} \) for some \( \mathcal{H} \).

Like Shannon, we will restrict our attention to stationary, ergodic sources.\(^{17}\) A stationary source is one for which the probabilities for emitting states doesn’t change over time; an ergodic source is one in which each sub-sequence of states appears in longer sequences with a frequency equal to its probability. (These assumptions are made for convenience of analysis only: in fact, the inherently statistical nature of quantum mechanics makes them less necessary in the quantum than in the classical case, and the results derived can be generalized to non-stationary, non-ergodic sources.) We define a stationary, ergodic ensemble over \( N \) time steps as one whose density matrix is the tensor product of \( N \) times its density matrix over a single time step: \( \rho^N = \rho \otimes \rho \otimes \cdots \otimes \rho \).

There are many different quantum ensembles with density matrix \( \rho \otimes \cdots \otimes \rho \). But as noted by Schumacher\(^1,3\) and Josza\(^3\), there is one ensemble in particular that effectively contains all such ensembles. Let \( \rho = \sum_{i} p_i |\phi_i\rangle\langle\phi_i| \), where the \( \phi_i \) are orthonormal. Consider the subspace \( \tilde{\mathcal{H}}^N \) spanned by the ‘high-probability’ product states \( |\phi_{i_1}\rangle \cdots |\phi_{i_N}\rangle \), where each \( |\phi_i\rangle \) occurs in the product \( \approx p_i N \) times. These states are the analog of high-probability sequences of symbols for a classical source. The following theorem then follows as an immediate corollary to the noiseless quantum channel source theorem of Schumacher\(^1,3\) and Josza\(^3\):

**Theorem 1.** (Quantum source theorem.) Let \( |\psi\rangle \) be selected from any ensemble with density matrix \( \rho \otimes \cdots \otimes \rho \). Then as \( N \to \infty \), \( |\psi\rangle \) is to be found in the high-probability subspace \( \tilde{\mathcal{H}}_N \) with probability \( 1 \). \( \tilde{\mathcal{H}}_N \) is a minimal subspace with this property, in the sense that any other such subspace contains \( \tilde{\mathcal{H}}_N \).
That is, as \( N \to \infty \), the ensemble \( \mathcal{E}_{\tilde{H}_N} \) contains with probability one the members of any ensemble with density matrix \( \rho \otimes \ldots \otimes \rho \). A more precise statement of theorem 1 is that as \( N \to \infty \), 
\[
\sum_{|\psi\rangle} p_{|\psi\rangle} \langle \psi | P_{\tilde{H}_N} | \psi \rangle \to 1,
\]
where \( P_{\tilde{H}_N} \) is the projection operator onto \( \tilde{H}_N \). By Shannon’s source theorem, the dimension of \( \mathcal{E}_{\tilde{H}_N} \) is approximately \( e^{NS} \) where \( S = -\text{tr} \rho \ln \rho \). As with Shannon’s theorems for classical sources, which simplify the analysis of the classical noisy channel by focusing on high-probability inputs, and as with the use of high-probability subspaces in the noiseless quantum channel theorem in references (1) and (3), the quantum source theorem simplifies the analysis of the noisy quantum channel by focusing on a particular subspace of inputs. A coding scheme that works for any ensemble with density matrix \( \rho \otimes \ldots \otimes \rho \) works for the states in the high probability subspace. Conversely, a coding scheme that works for the high-probability subspace works for any of the ensembles that it contains. Accordingly, from this point on, quantum sources will be taken to be ensembles over high-probability subspaces unless otherwise stated.

2. The Quantum Channel

A quantum communications channel takes quantum information as input and produces quantum information as output. An optical fiber is an example of a quantum channel: a photon in some quantum state goes in, suffers noise and distortion in passing through the fiber, and if it is not absorbed and does not tunnel out, emerges in a transformed quantum state. In the normal formulation of quantum mechanics, the ingoing system that carries quantum information is described by a density matrix \( \rho_{\text{in}} \), and the outgoing system is described by a density matrix \( \rho_{\text{out}} = S(\rho_{\text{in}}) \), where \( S \) is a trace-preserving linear operator called a super-scattering operator. For simplicity, the channel will be assumed to be time-independent and memoryless, so that it has the same effect on each quantum bit that goes through. (The generalization to time-dependent channels with memory is straightforward.)

An equivalent method of formulating the channel’s dynamics specify its effect on each of an orthonormal basis \( \{|\phi_i\rangle\} \) of input states: the output of the channel for input \( |\phi_i\rangle \) is then given by the ensemble \( \mathcal{E}_{|\phi_i\rangle} = \{|\psi_j(i)\rangle, p_{j(i)}\} \) of output states into which \( |\phi_i\rangle \) can evolve, together with the probabilities \( p_{j(i)} \) that \( |\phi_i\rangle \) evolves into the state \( |\psi_j(i)\rangle \). The density matrix and ensemble pictures of the effect of the channel are related as follows: 
\[
S(|\phi_i\rangle \langle \phi_i|) = \sum_{j(i)} \sqrt{p_{j(i)}} |\psi_j(i)\rangle \langle \psi_j(i)|, \quad \text{which for } i = i' \text{ gives } S(|\phi_i\rangle \langle \phi_i|) \sum_{j(i)} |\psi_j(i)\rangle \langle \psi_j(i)|.
\]

For example, if the channel is noiseless and distortion-free, then \( S \) is the identity opera-
tor, and \( \mathcal{E}_{|\phi_i\rangle} = \{(|\phi_i\rangle, 1)\} \). This channel transmits both classical and quantum information perfectly. Another example is the completely decohering channel, which can be thought of as the channel that destroys off-diagonal terms in the density matrix: 
\[
S(\sum_{ij} \alpha_{ij} |\phi_i\rangle\langle\phi_j|) = \sum_i \alpha_{ii} |\phi_i\rangle\langle\phi_i|,
\]
equivalently and perhaps more intuitively, as the channel that randomizes the phases of input states: 
\[
|\phi_i\rangle \longrightarrow \mathcal{E}_{|\phi_i\rangle} = \{(e^{i\lambda}|\phi_i\rangle, p(\lambda) = 1/2\pi\}\}.
\]

The completely decohering channel highlights the difference between the use of quantum channels to carry classical information and their use in carrying quantum information: it transmits classical information perfectly, but transmits no quantum information at all — no superpositions or entanglements survive transmission.

Most quantum channels are neither noiseless nor completely decohering. The next theorem quantifies just how much quantum information can be sent down a noisy, decohering channel. As above, we restrict our attention to stationary ergodic sources with density matrix \( \rho_{in} = \sum_i p_i |\phi_i\rangle\langle\phi_i| \). The inputs to the channel are then described by a density matrix \( \rho_{in}^N = \rho_{in} \otimes \ldots \otimes \rho_{in} \), and the output is described by a density matrix \( \rho_{out}^N = \rho_{out} \otimes \ldots \otimes \rho_{out} \), where \( \rho_{out} = S(\rho_{in}) = \sum_{i,j(i)} p_i p_{j(i)} |\psi_{j(i)}\rangle\langle\psi_{j(i)}| \).

As \( N \rightarrow \infty \), input states come from the subspace \( \tilde{H}_\text{in}^N \) with probability 1, and output states lie in the subspace \( \tilde{H}_\text{out}^N \) spanned by high-probability sequences of outputs, \( |\psi_{j_1(i_1)}\rangle \ldots |\psi_{j_N(i_N)}\rangle \), where each \( |\psi_{j(i)}\rangle \) appears in the sequence \( \approx p_i p_{j(i)} N \) times. The dimension of \( \tilde{H}_\text{out}^N \) is \( \approx 2^{-N \text{tr} \rho_{out} \log_2 \rho_{out}} \). To gauge the quantity of quantum information sent down the channel, look at the effect of the channel on a typical input state \( |\alpha_N\rangle = \sum_{i_1 \ldots i_N} \alpha_{i_1 \ldots i_N} |\phi_{i_1}\rangle \ldots |\phi_{i_N}\rangle \in \tilde{H}_\text{in}^N \), where the sum is over high-probability input sequences in which \( |\phi_i\rangle \) appears \( \approx p_i N \) times. We have,

**Theorem 2:** (Quantum channel theorem.) As \( N \rightarrow \infty \), when \( |\alpha_N\rangle \) is input to the channel, the output lies with probability 1 in a minimal subspace \( \tilde{H}_{\alpha}^N \) whose average dimension over \( \alpha_N \) is the minimum of \( e^{NS_{out}}, e^{NS_{\tilde{\alpha}}} \), where \( S_{\tilde{\alpha}} = -\text{tr} \rho_{\tilde{\alpha}} \ln \rho_{\tilde{\alpha}} \) and \( \rho_{\tilde{\alpha}} = \sum_{i,i'} \sqrt{p_i p_{i'}} S(|\phi_i\rangle\langle\phi_i|) \otimes |\phi_{i'}\rangle\langle\phi_{i'}|. \)

The proof of theorem 2 is somewhat involved, but the form of \( \rho_{\tilde{\alpha}} \) can be understood simply. One of the primary uses of a quantum channel is the distribution of entangled quantum states for the purpose of quantum cryptography or teleportation. Take a two-variable entangled state of the form \( \sum_i \sqrt{p_i} |\phi_i\rangle\langle\phi_i| \). Like the state \((1/\sqrt{2})(|0\rangle|0\rangle + |1\rangle|1\rangle)\) described in the introduction, this state is a maximally entangled state that registers all the states \( |\phi_i\rangle\langle\phi_i| \) at once; the factors of \( \sqrt{p_i} \) insure that each of the two quantum variables
take on its own is described by a density matrix \( \rho_{in} \). Now send the first variable down the channel. The result is a partially entangled state for the two variables described by density matrix \( \rho_\alpha \). That is, \( S_\alpha \) is the entropy increase when one of two fully entangled variables is sent down the channel. A thorough treatment of the effect of noisy channels on entangled states can be found in reference (18). The effect of the channel on an \( N \)-variable state \( |\alpha_N\rangle \) can be understood as follows: almost all input states \( |\alpha_N\rangle \) are fully entangled, with density matrix \( \rho_{in} \) describing each variable on its own.\(^{19}\) Sending \( n \) of the variables through the channel then increases the entropy by \( nS_\alpha \), which is in turn the logarithm of the dimension of the minimal subspace that can encompass the channel’s possible outputs. If \( S_\alpha > S_{out} \), then sending all the variables through completely randomizes the output as \( N \rightarrow \infty \), and no coherent quantum information survives the transmission through the channel.

Theorem 2 suggests that the amount of quantum information transmitted down the channel from a stationary, ergodic source with density matrix \( \rho_{in} \) be defined as \( I_Q(\rho_{in}) = -\text{tr}\rho_{out}\log_2\rho_{out} + \text{tr}\rho_\alpha\log_2\rho_\alpha = S_{out} - S_\alpha \) if \( S_{out} > S_\alpha \), = 0 otherwise. This definition of quantum information transmitted is the quantum analog of mutual information between channel inputs and outputs: when pure states are sent down the channel, \( I_Q \) tells how much information one gets about which pure state \( \in \hat{\mathcal{H}}_\text{in}^N \) went in by looking at the noisy mixed state \( \in \hat{\mathcal{H}}_\text{out}^N \) that comes out.\(^{20}\)

The full justification of \( I_Q \) as the quantum information transmitted down a quantum channel will be presented in the next section, in which quantum coding schemes will be presented that allow the reliable transmission of quantum information at a rate \( I_Q \), and in which it will be noted that no coding schemes exist for stationary, ergodic sources that can surpass this rate. For the moment, consider three examples of quantum channels, each with source described by \( \rho_{in} = (1/2)|0\rangle\langle 0| + |1\rangle\langle 1|) \). (i) In the noiseless quantum channel, \(-\text{tr}\rho_{out}\log_2\rho_{out} = 1, -\text{tr}\rho_\alpha\log_2\rho_\alpha = 0, \) and \( I_Q = 1 \) qubit, reflecting the fact that each qubit is received as sent. (ii) In the completely decohering/dephasing channel, \(-\text{tr}\rho_{out}\log_2\rho_{out} = 1, \rho_\alpha = (1/2)(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|), -\text{tr}\rho_\alpha\log_2\rho_\alpha = 1, \) and \( I_Q = 0 \) qubits, so that no quantum information is sent. (iii) Consider a partly dephasing channel in which \( |0\rangle\langle 0| \rightarrow |0\rangle\langle 0|, |1\rangle\langle 1| \rightarrow |1\rangle\langle 1|, \) and \( |0\rangle\langle 1| \rightarrow (1 - \epsilon)|0\rangle\langle 1|, |1\rangle\langle 0| \rightarrow (1 - \epsilon)|1\rangle\langle 0|. \) Here,

\[
\rho_\alpha = (1/2)(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|) + (1 - \epsilon)(|1\rangle\langle 0| \otimes |1\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1|)
\]

and \(-\text{tr}\rho_\alpha\log_2\rho_\alpha = -(1 - \epsilon/2)\log(1 - \epsilon/2) - (\epsilon/2)\log_2(\epsilon/2)\), giving an \( I_Q \) that ranges continuously from 1 for \( \epsilon = 0 \) (no decoherence) to 0 for \( \epsilon = 1 \) (complete decoherence).
3. Optimal codes for the noisy quantum channel

Define the capacity of a quantum channel to carry quantum information to be $C_Q = \max_{\rho_{in}} I_Q(\rho_{in})$. $C_Q$ is the maximum over all sources $\rho_{in}$ of the quantum information $I_Q$ transmitted down the channel. We then have the following

**Theorem 3.** (Noisy quantum channel coding theorem.) Consider a quantum channel with capacity $C_Q$. The output of a stationary, ergodic source with density matrix $\rho$ can be encoded, sent down the channel, and decoded with reliability $\to 1$ as $N \to \infty$ if and only if $-\text{tr} \rho \log_2 \rho \leq C_Q$.

Like Shannon’s noisy coding theorem, theorem 3 comes with the caveat that it applies to high-probability sources. The proof to theorem 3 will be given elsewhere: but the idea behind the proof, as well as the theorem’s meaning and implications can be understood as follows. The noisy, decohering quantum channel has two effects on the quantum information that it transmits. First, like the classical channel, it adds noise to the signal, flipping qubits and adding random information. Second, it decoheres the signal by randomizing phases and acquiring information about the quantum information transmitted. Decoherence is an effect with no classical analog: classical signals do not have phases, and acquiring information about a classical signal is harmless as long as the signal is not altered in the process. In quantum mechanics, however, acquiring information about the signal means effectively making a measurement on it, and quantum measurement unavoidably alters most quantum systems.

The problem of decoherence implies that signal must be encoded in such a way that any information the channel gets about the encoded state reveals nothing about which state of the source was sent. Otherwise, the channel can effectively ‘measure’ the output of the source, irretrievably disturbing it in the process. As noted by Shor, this may be accomplished by encoding the signal as an entangled state. In fact, each encoded signal must have the same density matrix $\rho_{in}$ as each other encoded signal for each qubit sent down the channel: otherwise the channel can distinguish between different signals and decohere them. If the signals are encoded as entangled states in this fashion, the channel can decohere the codeword, but it cannot decohere the original signal.

Suppose someone hands you a quantum system in some unknown state selected from an ensemble with density matrix $\rho \otimes \ldots \otimes \rho$, and asks you to transmit it reliably down a noisy, decoherent quantum channel. What do you do? (If someone hands you a system
in a known quantum state, no quantum channel is necessary: you can just use a classical channel to transmit instructions for recreating the state using a quantum computer.) The following encoding attains the channel capacity: First, identify a source for the channel that attains the channel capacity, so that \( I_Q(\rho_{in}) = C_Q \). Next, encode the state to be transmitted by applying a transformation that maps an orthonormal basis for the input high-probability subspace to a \emph{randomly chosen} set of orthogonal states taken from the high-probability subspace of the source that attains the channel capacity. Such random states have the desired property that they are fully entangled, and each qubit in the encoded signal has density matrix \( \rho_{in} \). Now send the encoded signal down the channel. Because the states are fully entangled, the channel cannot get any information about the original pre-encoded state: all the channel can do to disrupt the encoded state is add entropy \( S_{out} - C_Q \) per symbol transmitted. That is, the encoding protects the original state from decoherence; and as long as \( -\text{tr} \rho \log_2 \rho \leq C_Q \) there is enough redundancy in the encoded state to recreate the original state, just as in the classical case. This method works equally well if the initial state is pure, mixed, or entangled with some other system.

\textbf{Examples:} In the three cases discussed in the previous section, the channel capacity is just \( I_Q \), as calculated. The important fact to note is that even very high levels of decoherence (\( \epsilon \to 1 \)) can be tolerated in principle. A case of considerable interest is that in which each qubit system sent down the channel has a probability \( \eta \) of being decohered and randomized. In this case,

\[
\rho_{\bar{\alpha}} = \sum_{i,i'=0,1} \left( (1 - \eta)/2 |i\rangle\langle i'| \otimes |i\rangle\langle i'| + (\eta/4)|i\rangle\langle i| \otimes |i\rangle\langle i'| \right)
\]

\( S_{\bar{\alpha}} \) can be calculated for this case and is equal to \(- (3\eta/4) \log_2 (\eta/4) - (1 - 3\eta/4) \log_2 (1 - 3\eta/4)\), which is equal to 1 for \( \eta \approx .252 \). The highest rate of errors that can be corrected by an optimal coding procedure is just above 1/4 (see also reference (12)). This example contrasts with the classical channel, in which arbitrarily high levels of noise can be tolerated in principle: quantum coding can correct for arbitrarily high levels either of noise, or of decoherence, but not of both together.

\textbf{Discussion}

In practice, even if the channel capacity is not exceeded, the amount of noise and decoherence that can be tolerated is limited by the ability to encode and decode: as \( N \to \infty \), the error in the transmitted state goes to zero, but the amount of quantum
information processing that must be done to encode and decode becomes large. The encoding and decoding itself must be performed reliably.

The usefulness of the classical noisy coding theorem is also limited by coding difficulties: in particular, random codes are hard to encode and decode. In this respect, however, the quantum theorem has a considerable advantage. As Shannon noted, random codes are effective because the bits that make up the signal have no apparent order. In the classical case, this implies that sequences of bits must appear random. In the quantum case, however, as long as the encoded signal is fully entangled, each qubit in the signal taken on its own appears to be completely random. As a result, the code words themselves may be highly regular: a simple example of a set of codewords that are easy to encode and decode, but are sufficiently random to attain the channel capacity are $N$ qubit analogs of the familiar two-qubit entangled states

$$(1/\sqrt{2})(|01\rangle - |10\rangle), (1/\sqrt{2})(|01\rangle + |10\rangle), (1/\sqrt{2})(|00\rangle - |11\rangle), (1/\sqrt{2})(|00\rangle + |11\rangle).$$

In the classical case, random codes are hard to construct. In the quantum case, codes that are sufficiently random to attain the channel capacity can be constructed by a brief quantum computation.

In conclusion, this paper has derived fundamental limits on the amount of quantum information that can be sent reliably down a quantum channel, and has exhibited codes that attain those limits. In fact, almost all codes attain those limits. As with Shannon’s classical noisy coding theorem, the rate of transmission of quantum information remains finite as the probability of error goes to zero.
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2 and 3 (the codes do not obey the conditions of the theorems and so do not contradict them) has recently been suggested by P.W. Shor and J.A. Smolin.
Appendix 0

Properties of ensembles of states. The idea behind the the ensemble picture of quantum mechanics is to deal with mixtures and superpositions in the same formalism. Accordingly, a primary purpose of the ensemble picture is to make an explicit distinction between quantum states that can interfere with each other, and quantum states that can’t. The ensemble picture is constructed so that different members of an ensemble cannot interfere with each other, while corresponding members of different ensembles can interfere. The second purpose of the ensemble picture is to keep track explicitly of the normalization of states, so that high-probability sets of states can be identified correctly.

As noted on page 4, a quantum ensemble $\mathcal{E}_\psi = \{(|\psi_j\rangle, p_j)\}$ is a set of quantum states together with their probabilities. Ensembles are collections of vectors, and share many properties of vectors. For example, if $\mathcal{E}_\psi$ and $\mathcal{E}_\phi = \{(|\phi_i\rangle, q_i)\}$, we can define a scalar product $\mathcal{E}_\psi \cdot \mathcal{E}_\phi = \sum_j \sqrt{p_j q_j} \langle \psi_j | \phi_j \rangle$. If $\mathcal{E}$ is normalized, then $\mathcal{E} \cdot \mathcal{E} = \text{tr} \rho_\mathcal{E} = 1$. (Note that the rule for obtaining the proper statistics is to associate a factor of $\sqrt{p_j}$ with each occurrence of $|\psi_j\rangle$.) This vector-like character of ensembles allows the straightforward characterization of properties of quantum operators. For example, the trace-preserving character of the super-scattering operator (page 7) can be summarized by the requirement that $\mathcal{E}|\phi_j\rangle \cdot \mathcal{E}|\phi'_j\rangle = \delta_{jj'}$.

A type of ensemble that will prove useful below is one that is obtained by superposing corresponding states from two ensembles. If corresponding states have the same probability, for example if $p_j = q_j$ for the ensembles $\mathcal{E}_\phi$, $\mathcal{E}_\psi$ above, then the ensemble of superpositions of $\alpha$ times the states of $\mathcal{E}_\phi$ plus $\beta$ times the corresponding states of $\mathcal{E}_\psi$ is just $\{(|\alpha|\phi_j\rangle + \beta|\psi_j\rangle, p_j\}$, with density matrix $\rho$ as above. In fact, because we will work with ensembles of high-probability states, which have equal probabilities, this is the type of ensemble that we will have occasion to use below. If the corresponding states from the different ensembles do not have the same probabilities, then we write the ensemble of superposed states as $\mathcal{E}_{\alpha|\phi_i\rangle + \beta|\psi_j\rangle} = \{(\alpha|\phi_i\rangle + \beta|\psi_j\rangle, p_j q_j)\}$ to indicate the ensemble obtained by superposing $\alpha$ times the states of $\mathcal{E}_\phi$ plus $\beta$ times the corresponding states of $\mathcal{E}_\psi$, together with a list $p_j q_j$ of the probabilities of the individual states in the superposition. We specify superposition ensembles in this fashion to keep track explicitly of the normalization of the individual states in the superposition. The proper overall normalization of such ensembles is obtained as above by associating a factor of $\sqrt{p_j}$ with each $|\psi_j\rangle$ and a factor of $\sqrt{q_j}$ with
each $|φ_j⟩$, so that

$$\rho_{E_αφ + βψ} = \sum_j \alpha\bar{α}q_j|φ_j⟩⟨φ_j| + \alpha\bar{β}\sqrt{q_jp_j}|ψ_j⟩⟨ψ_j| + \beta\bar{β}\sqrt{p_jq_j}|ψ_j⟩⟨ψ_j|.$$  

Note that the superposition ensemble has the same density matrix as the ensemble of unnormalized states, $\{(α\sqrt{q_j}|φ_j⟩ + β\sqrt{p_j}|ψ_j⟩, 1)\}$. If we wish to superpose many ensembles, $E_i = \{|ψ_{j(i)}⟩, p_{j(i)}⟩\}$, we will use $i$ to index the different members, and $j$ to index the different members of each ensemble: e.g., $E_{β} = \{(\sum_i β_i|ψ_{j(i)}⟩, p_{j(i)}⟩)\}$ is the ensemble got by superposing the $j$’th members of each of the ensembles with probability $p_{j(i)}$ associated with the $j$’th member of the $i$’th ensemble. $E_{β}$ has density matrix $ρ_{β} = \sum_{j(i), j(i')} \sqrt{p_{j(i)}p_{j(i')}}|ψ_{j(i)}⟩⟨ψ_{j(i')}|$. In this notation, states with different $j$ cannot interfere, but states with the same $j$ but different $i$ can interfere.

This definition of superpositions of ensembles allows us to complete the identification of ensembles with vectors by defining $αE_{φ} + βE_{ψ} = E_{αφ + βψ}$. In addition, this definition of superposition makes a self-consistent connection between the ensemble and superscattering pictures of time evolution, a fact that will prove useful below. The ensemble picture is related to the operator sum representation of superscattering operators described, e.g., in reference (20).

**Appendix 1**

Outline of the proof of theorem 1. Theorem 1 follows from directly from the results of references (1) and (3), where a detailed treatment of high-probability subspaces may be found. The proof goes as follows. If $|ψ⟩$ is selected, with probability $p_{|ψ⟩}$, then

$$\sum_{|ψ⟩} p_{|ψ⟩}⟨ψ|P_{H^N} |ψ⟩ = \text{tr}P_{H^N} ρ^N$$

is just the classical probability of the set of high-probability sequences, and $→ 1$ as $N → ∞$. As a result, for any $ε > 0$, $N$ can be picked sufficiently large so that a state picked from any stationary, ergodic ensemble with density matrix $ρ$ has overlap $≥ 1 − ε$ with some state in $H^N$, with probability $≥ 1 − ε$. Minimality follows since $E_{H^N}$ is itself an ensemble with density matrix $ρ \otimes \ldots \otimes ρ$ as $N → ∞$. Minimality is a relatively weak property: $H^N$ need not be the only minimal subspace; but all other such minimal subspaces have approximately the same dimension as $N → ∞$.

**Appendix 2**
Outline of the proof of theorem 2. There are several ways to prove the noisy channel theorem. One way is to follow along the lines suggested in the text and analyze the channel’s effect on entangled states. The following method of proof is closer in spirit to the classical derivation of channel capacity.

In the density matrix picture of the channel, the channel has the effect,

$$|\alpha\rangle\langle\alpha| \rightarrow \rho_\alpha = \sum_{i_1\ldots,i_N} \alpha_{i_1\ldots,i_N} \bar{\alpha}_{i'_1\ldots,i'_{N}} S(|\phi_{i_1}\rangle\langle\phi_{i'_1}|) \otimes \cdots \otimes S(|\phi_{i_N}\rangle\langle\phi_{i'_{N}}|)$$

(2.1)

where the sum is taken over high-probability sequences in which $i$ appears $\approx p_i N$ times.

Equivalently, in the ensemble picture,

$$|\alpha\rangle \rightarrow \mathcal{E}_\alpha = \{(\sum_{i_1\ldots,i_N} \alpha_{i_1\ldots,i_N} |\psi_{j_1(i_1)}\rangle \cdots |\psi_{j_N(i_N)}\rangle, p_{j_1(i_1)} \cdots p_{j_N(i_N)}\} \equiv \{(\sum_i \alpha_i |\psi_{j(i)}\rangle, p_j(i))\}$$

(2.2)

where the superposition ensemble is defined as in appendix 0 and has density matrix $\rho_\alpha$. Theorem 1 implies that as $N \rightarrow \infty$, then with probability 1, the states of $\mathcal{E}_\alpha$ are to be found in the Hilbert space $\mathcal{H}_\alpha^N$ spanned by high-probability states of the form

$$\sum_{i_1\ldots,i_N} \alpha_{i_1\ldots,i_N} |\psi_{j_1(i_1)}\rangle \cdots |\psi_{j_N(i_N)}\rangle$$

where in each term of the superposition, $|\psi_{j(i)}\rangle$ appears $\approx p_i p_j(i) N$ times. The minimality of $\mathcal{H}_\alpha$ follows as in theorem 1. This proves the first part of theorem 2.

The dimension of the output Hilbert space $\mathcal{H}_\alpha^N$ is equal to one over the average overlap of two members of that space: $\dim \mathcal{H}_\alpha^N = (\text{tr}_{hp} \rho_\alpha^2)^{-1}$, where the trace $\text{tr}_{hp}$ is taken over high-probability sequences only. We wish to calculate the average dimension of the output Hilbert space over $\alpha$. Using the fact that $\langle \alpha_{i_1\ldots,i_N} |\bar{\alpha}_{i'_1\ldots,i'_{N}} \rangle = p_{i_1} \cdots p_{i_N} \delta_{i_1,i'_1} \cdots \delta_{i_N,i'_{N}}$, after some algebra, we obtain

$$\langle \text{tr}_{hp} \rho_\alpha^2 \rangle_\alpha = \text{tr}_{hp} (\rho_{out}^2)^N + \text{tr}_{hp} (\rho_\alpha^2)^N - \text{tr}_{hp} (\rho_{\bar{\alpha}}^2)^N$$

(2.4)

where $\rho_{out}$ and $\rho_{\bar{\alpha}}$ are defined as above, $\rho_{i/o} = \sum_i p_i S(|\phi_i\rangle\langle\phi_i|) \otimes |\phi_i\rangle\langle\phi_i|$, and $(\rho^2)^N = \rho^2 \otimes \cdots \otimes \rho^2$. We can now use the fact that $\text{tr}_{hp} (\rho^2)^N = 2^{N \text{tr} \log_2 \rho}$, which can be simply verified in a basis in which $\rho$ is diagonal. We then have

$$\text{tr}_{hp} (\rho_{out}^2)^N = 2^{N \text{tr} \rho_{out} \log_2 \rho_{out}} = 2^{-NS_{out}}$$

(2.5)

$$\text{tr}_{hp} (\rho_\alpha^2)^N = 2^{N \text{tr} \rho_{\alpha} \log_2 \rho_{\alpha}} = 2^{-NS_{\alpha}}$$

(2.6)

$$\text{tr}_{hp} (\rho_{i/o}^2)^N = 2^{N \text{tr} \rho_{i/o} \log_2 \rho_{i/o}} = 2^{-N(\sum_i S_{out}(i)+S_{in})}$$

(2.7)
As $N \to \infty$, $\langle \dim \tilde{H}_\alpha^N \rangle^{-1}$ goes to the largest of these three terms of which the third is less than or equal to either of the first two. We have actually calculated the average of the inverse of the dimension of the output subspace: however, the standard deviation

$$\sqrt{<\langle \text{tr}_{\text{hp}} \rho^2 \rangle_\alpha^2 - \text{tr}_{\text{hp}} \rho^2 >_\alpha^2}$$

is proportional to $(<\langle \text{tr}_{\text{hp}} \rho^2 >_\alpha^2 <\langle \text{tr}_{\text{hp}} \rho^2 >_\alpha^2 >)^{N/2}$ and so goes to zero exponentially faster in $N$ than $<\langle \text{tr}_{\text{hp}} \rho^2 >_\alpha^2$ except when $S_\alpha = S_{\text{out}}$, in which case $C_Q = 0$. As a result, the average of the inverse is the inverse of the average, and the average dimension of $\dim \tilde{H}_\alpha^N$ is the smaller of $2^{-N \text{tr} \rho_\alpha \log_2 \rho_\alpha}$ and $2^{-N \text{tr} \rho_{\text{out}} \log_2 \rho_{\text{out}}}$, proving the second half of theorem 2. Note also that the standard deviation of the dimension of $\tilde{H}_\alpha^N$ as a fraction of the average dimension also goes to zero as $N \to \infty$, showing that almost all $\alpha$ correspond to an output space of the same dimension.

### Appendix 3.

Outline of the proof of theorem 3. The high probability subspace for this source has dimension $2^{-N \text{tr} \rho \log_2 \rho}$. Encode the basis states $|\chi_i^N\rangle$ for the source as randomly chosen orthogonal states $|\alpha_i^N\rangle$ in the high-probability subspace of a source that attains the channel capacity. The channel takes each $|\alpha_i^N\rangle$ to some state in the ensemble $E_{\alpha_i}$ with minimal subspace $\tilde{H}_{\alpha_i}^N$. The average over $\alpha_i$ of the overlap $|\langle \psi_{\alpha_i} | \psi_{\alpha_j} \rangle|$ of states $|\psi_{\alpha_i}\rangle \in \tilde{H}_{\alpha_i}^N$, $|\psi_{\alpha_j}\rangle \in \tilde{H}_{\alpha_j}^N$, for $i \neq j$ can be calculated as in appendix 2, and is equal to $1/\dim \tilde{H}_{\text{out}}^N = 2^{N \text{tr} \rho_{\text{out}} \log_2 \rho_{\text{out}}}$. If $P_{\alpha_i}$ is the projection operator onto $\tilde{H}_{\alpha_i}^N$, we have

\[
\text{tr} P_{\alpha_i}^N P_{\alpha_j}^N = 2^{-N(\text{tr} \rho_{\text{out}} \log_2 \rho_{\text{out}} + \text{tr} \rho_\alpha \log_2 \rho_\alpha)} = 2^{-NC_Q} \quad .
\]

That is, as $N \to \infty$, the overlap between any two individual output subspaces $\to 0$ as long as the quantum channel capacity is not zero. The dimension of the direct sum of the output subspaces remains less than or equal to the dimension of $H_{\text{out}}^N$ if and only if $-\text{tr} \rho \log_2 \rho \leq C_Q$:

\[
\dim \oplus \sum_i \tilde{H}_{\alpha_i}^N \to 2^{-N(\text{tr} \rho \log_2 \rho + \text{tr} \rho_\alpha \log_2 \rho_\alpha)} = 2^{-N(C_Q - \zeta)} \quad ,
\]

where $\zeta = C_Q - (-\text{tr} \rho \log_2 \rho)$. So if $\zeta \geq 0$, the source entropy does not exceed the channel capacity, and the output states corresponding to different input basis states all fall in distinct subspaces. The overlap of any one output subspace with the direct sum of all the remaining subspaces goes as $2^{-N\zeta}$. If $\zeta < 0$, the output subspaces overlap and no unique decoding is possible. This proves that $C_Q$ is an upper limit on the channel capacity for
‘typical’ codewords belonging to the high-probability subspace (i.e., for a set of measure
1 as \( N \to \infty \)), but it does not rule out the possibility of the use of a set of codewords of
measure 0.

In the case \( \zeta \geq 0 \), a unitary decoding transformation can now be applied to the output
states to put each vector \( |\psi_{\alpha i}^N\rangle \in \tilde{H}_{\alpha i} \) into the form \( |\chi_i^N\rangle \otimes |\psi^N\rangle \), in which vectors in differ-
ent output subspaces but with the same \( |\psi_{j(i)}^N\rangle \) in (2.3) give the same \( |\psi^N\rangle \). Because of the
asymptotic orthogonality of the output spaces, this decoding recreates \( |\chi_i^N\rangle \) with fidelity
arbitrarily close to 1 as \( N \to \infty \). The crucial point is that this decoding also recreates su-
perpositions of input states with fidelity \( \to 1 \) as \( N \to \infty \): by going to the ensemble picture,
it can be verified that \( \sum_k \gamma_k |\chi_k^N\rangle \) is mapped to an ensemble \( \{(\sum_k \gamma_k |\chi_k^N\rangle \otimes |\psi^N\rangle, p_\psi)\} \).

The steps are as follows. First, encoding:

\[
\sum_k \gamma_k |\chi_k^N\rangle \rightarrow \sum_k \gamma_k \sum_{i_1...i_N} \alpha_{i_1...i_N} |\phi_{i_1}\rangle \cdots |\phi_{i_N}\rangle \quad (3.3a)
\]

Next, the effect of the channel:

\[
\rightarrow \{ \left( \sum_k \gamma_k \sum_{i_1...i_N} \alpha_{i_1...i_N} |\psi_{j1(i_1)}\rangle \cdots |\psi_{jN(i_N)}\rangle \right), p_{j1(i_1)} \cdots p_{jN(i_N)} \} \quad (3.3b)
\]

Finally, decoding:

\[
\rightarrow \{ \left( \sum_k \gamma_k |\chi_k\rangle \otimes \sum_{i_1...i_N} \beta_{i_1...i_N} |\psi_{j1(i_1)}\rangle \cdots |\psi_{jN(i_N)}\rangle \right), p_{j1(i_1)} \cdots p_{jN(i_N)} \} \\
= \{ (\sum_k \gamma_k |\chi_k\rangle \otimes |\psi^N\rangle, p_\psi) \} \quad (3.3d)
\]

The fact that the decoding process faithfully recreates superpositions can also be verified in
the density matrix picture by using the correspondence in appendix 2. Since the encoding
and decoding preserves pure states with their phases, it also preserves mixed states and
any entanglement between the input state and another quantum system.

This proves the if part of the theorem. The only if part for codewords from the
high-probability subspace was proved above. This proves the theorem as stated.

The limits set by theorems 2 and 3 hold only for codewords from the high-probability
set: by using codewords taken from the set of measure zero, it may be possible to improve
on these limits.\(^{21}\) A simple example of how this may be done is given by the method of
theorem 3 itself: block together the quantum symbols (e.g., qubits) in groups of \( \ell \), and
regard each group of \( \ell \) as a new, composite symbol. The minimization procedure used
for finding the quantum channel capacity in general yields a different, potentially higher
channel capacity for codes composed of the composite symbols.
Figure 1: Diagram of the noisy, decoherent quantum channel. To send an arbitrary quantum state $|\psi\rangle$ down the channel, first encode it in a redundant form $C(|\psi\rangle)$. The encoded state is sent down the channel, where it is subjected to noise and decoherence. The arrows indicate that noise is added to the signal, while decoherence arises from the environment getting information about the signal. The noisy, decoherent signal $N(C(|\psi\rangle))$ is then fed through a decoder that recreates the original state together with extra random information that depends on what errors occurred.