The Cosmology of M-Theory and Type II Superstrings

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Abstract

We review cosmological solutions of type II superstrings and M-theory, emphasizing the role of non-vanishing Ramond form backgrounds. Compactifications on flat and, more generally, maximally symmetric spatial subspaces are presented. We give a physical discussion of both inflating and subluminally expanding cosmological solutions of such theories and explore their singularity structure. An explicit example, in the context of the type IIA superstring, is constructed. We then analyze compactifications of M-theory on Ricci flat manifolds. The external part of U–duality and its relation to cosmological solutions is studied in the low energy theory. In particular, we investigate the behaviour of important cosmological properties, such as the Hubble parameters and the transition time between two asymptotic regions, under U-duality transformations. Motivated by Horava-Witten theory, we present an explicit example of manifestly U-duality covariant M–theory cosmology in a five-dimensional model resulting from compactification on a Calabi-Yau three-fold.

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1 Introduction

An important constraint on string theory or any generalization of string theory, such as M–theory, is that it should be compatible with the standard model of early universe cosmology. In the past, most focus has been on the weakly coupled heterotic string as the best model of low-energy particle physics. However, with the discovery of string dualities and the existence of D–brane states, the nature of string theory has changed dramatically. Strong-weak coupling duality symmetries connect each of the five consistent supersymmetric string theories together with eleven-dimensional supergravity. As a consequence, type II and eleven-dimensional supergravities may now be directly relevant to low-energy particle physics and cosmology. Both theories contain form fields, namely a three form in M-theory and Ramond-Ramond (RR) forms of various degrees in type II theories. Given this change of perspective, it clearly becomes important to study the cosmological solutions of type II superstrings and M-theory with non-trivial form fields excited. The present paper discusses a general framework for analyzing cosmologies of such theories. This framework has been presented in detail in \\

Various aspects of M–theory cosmology have been studied over the past two years such as cosmological solutions with nontrivial form fields, the possibility of singularity free solutions, and moduli and dilaton stabilization. Applications of T–duality and S–duality to string cosmology have recently been discussed in the refs. In particular, the $SL(2)$ symmetry of type IIB has been used to generate cosmological solutions with Ramond–Ramond fields. Cosmological solutions with Ramond forms obtained from black hole solutions have been constructed.

In reference, we considered compactifications on maximally symmetric subspaces. The first part of the present paper is a review of this formalism. For such compactifications, our general result is that the moduli-space potential which arises when non-trivial form-fields are excited, significantly affects the structure of the solution. When all the subspaces are flat, the potential is operative over a particular, finite part of the evolution. At the extremes we return to the simple Kaluza-Klein Kasner-type solutions with some subspaces expanding, some contracting. Consequently, there is always either an initial or a final curvature singularity. Thus the effect of the form fields is to interpolate between two different Kasner-type solutions. In general, for solutions with an initial singularity, one finds that the rate of expansion is always sub-luminal and so there is no inflation. On the other hand, those solutions with a final singularity, just as in the pre-big-bang models, may exhibit superinflation but are unphysical as stands because the inflation ends in a curvature singularity. The solution can be very different, however, when we allow for subspaces of non-vanishing constant curvature. The effect of the curvature is to introduce new terms into the moduli space potential. This can significantly change the singularity structure, in some cases.
giving solutions where the curvature always remains finite. This suggest the interesting possibility
that there may exist inflating solutions which are not forced to end in a singularity.

In reference [8], we analyzed U-duality covariant compactifications on Ricci flat manifolds in
general and Calabi-Yau manifolds in particular. In the second part of this paper we discuss the
results of [8]. Specifically, we will present a manifestly U-duality covariant formulation of M-theory
cosmology. Since U-duality rotates metric degrees of freedom and degrees of freedom from the 3-
form of 11-dimensional supergravity into each other, we will be naturally dealing with cosmologies
which have nontrivial Ramond–Ramond fields. The most interesting part of duality transformations
in cosmology is the one which acts non-trivially on the space–time metric and it is this part on which
we will concentrate. Therefore, we are going to reduce 11–dimensional supergravity on a Ricci–flat
manifold to D dimensions, thereby keeping the breathing mode of the internal space as the only
modulus [29] and focusing on the D-dimensional “external” part of U-duality. As a general rule,
this part of the U–duality group acting on cosmological solutions in D space–time dimensions is the
same as the U–duality group of 12 – D–dimensional maximal supergravity. As an explicit example,
we will study the case D = 5, corresponding to the U–duality group G = SL(5). This example
is motivated by the Horava–Witten construction of M–theory on S^1/Z_2 [30] which represents the
effective theory of strongly coupled heterotic string theory. This theory turns out to be effectively
5–dimensional for phenomenological values of the coupling constants in some intermediate energy
range [31].

As a result, we find large classes of cosmological solutions which are mapped into each other by
U–duality transformations. The most characteristic properties of these solutions are very similar to
the ones found for maximally symmetric subspaces. They usually have two different branches, one
with a future and the other with a past curvature singularity. Each branch consists of asymptotic
Kaluza–Klein regions which evolve into each other under the influence of the form field. We find that
U–duality relates to this structure in an interesting way. In the asymptotic Kaluza–Klein regions
the basic expansion properties such as Hubble parameters turn out to be U–duality invariants.
A related phenomenon, in the context of S–duality, has been observed in ref. [20, 21, 25]. U–
duality transformations, however, do change the transition time between asymptotic regions and
influence the details of the transition. Using the knowledge on how U–duality acts explicitly on the
cosmological solutions we, moreover, show that a U–duality version of the pre–big–bang scenario [32]
can be realized within our setting.

2 Solutions with Maximally Symmetric Subspaces – General Frame-
work

In this section, we present a general framework for finding cosmological solutions with non-trivial
form fields compactified on subspaces of constant curvature. The starting point for our investigation

is the following effective action

$$S = \int d^Dx \sqrt{g} \left[ e^{-2\phi}(R + 4(\partial\phi)^2 - \frac{1}{12}H^2) - \sum_{r} \frac{1}{2(\delta_r + 1)!}F_r^2 - \Lambda \right]$$

with the $D$–dimensional string frame metric $\bar{g}_{MN}$, the dilaton $\phi$, the NS 2–form $H$ and a number of RR $\delta_r$–forms $F_r = dA_r$. We also allow for a cosmological constant $\Lambda$ which appears in the massive extension of IIA supergravity [33]. Assuming this origin, it is restricted to be positive, $\Lambda > 0$.

The above action can account for a wide range of cosmological solutions in type II theories (where we usually have $D = 10$ in mind) and, if the dilaton is set to zero, in $D = 11$ supergravity. For simplicity we have kept only the kinetic terms for the form fields. In general, in both type II theories and eleven dimensional supergravity there are additional terms involving the coupling of form fields. We shall assume throughout this paper that for the configurations with which we are concerned, these terms do not contribute to the equations of motion.

In order to give a physical description of our solutions, we prefer to work in the canonical Einstein frame metric $g_{MN}$ which is related to the string frame metric by a conformal rescaling $g_{MN} = \exp(-4/(D - 2)\phi)\bar{g}_{MN}$. The corresponding action reads

$$S = \int d^Dx \sqrt{-g} \left[ R - \frac{4}{D - 2}(\partial\phi)^2 - \sum_{r} \frac{1}{2(\delta_r + 1)!}e^{-a(\delta_r)\phi}F_r^2 - \Lambda e^{-a_\Lambda\phi} \right]$$

where the NS field $H$ has been included in the sum over $F_r$. It is distinguished from the other forms by the dilaton couplings $a(\delta_r)$ given by

$$a(\delta_r) = \begin{cases} \frac{8}{D - 2} & \text{NS 2–form} \\ \frac{4\delta_r - 2(D - 2)}{D - 2} & \text{RR $\delta_r$–form} \end{cases}$$

The coupling $a_\Lambda$ for the cosmological constant

$$a_\Lambda = -\frac{2D}{D - 2}$$

equals the negative of the one for a RR $(D - 1)$–form.

The type of solutions we consider are characterized by a space split into $n$ flat subspaces, each of them characterized by a scale factor $\alpha_i$. We concentrate on flat subspaces in this paper. The effect of curved maximally symmetric subspaces can easily be incorporated into the formalism. In the flat case, the corresponding metric is given by

$$ds^2 = -N^2(\tau)d\tau^2 + \sum_{i=0}^{n-1} e^{2\alpha_i(\tau)}d\bar{x}_i^2$$

where $d\bar{x}_i^2$ is the measure of a $d_i$–dimensional flat maximally symmetric subspace and $\sum_{i=0}^{n-1} d_i = D - 1$. For solutions with this structure, the dilaton should depend on time only, $\phi = \phi(\tau)$. This
Kaluza–Klein-type Ansatz is about the simplest allowing for the cosmologically key properties of homogeneity and isotropy as well as for an “external” and “internal” space.

For a ten dimensional theory, the simplest choice is to split up the space into two subspaces \((n = 2)\) with \(d_0 = 3\) and \(d_1 = 6\). Then the \(d_0 = 3\) part could be interpreted as the spatial part of “our” 4-dimensional space–time with an evolution described by the scale factor \(\alpha_0\). The other six dimensions would form an internal space with a modulus \(\alpha_1\). Clearly, one is free to split these six dimensions even further or even allow for a further split of the 3-dimensional space.

The choice of subspaces is important in fixing the possible structure of the antisymmetric tensors fields \([34, 35]\). The symmetry of the above metric allows for two different Ans¨ atze for the form fields which we call “elementary” and “solitonic” in analogy to the two types of \(p\)-brane solutions. They are characterized by the following nonvanishing components of the field strength.

- **elementary**: if \(\sum_i d_i = \delta_r\) for some of the spatial subspaces \(i\) we may set

\[
(F_r)^{\mu_0 \mu_1 \ldots \mu_{\delta_r}} = A_r(\alpha) f_r^{\prime}(\tau) \epsilon_{\mu_0 \mu_1 \ldots \mu_{\delta_r}} , \quad A_r(\alpha) = e^{-2 \sum_i d_i \alpha_i}
\]

where \(\mu_1 \ldots \mu_{\delta_r}\) refer to the coordinates of these subspaces, \(f_r(\tau)\) is an arbitrary function to be fixed by the form field equation of motion, and the prime denotes the derivative with respect to \(\tau\). With raised indices, the symbol \(\epsilon^{\mu_1 \ldots \mu_{\delta_r}}\) takes the values 0 or 1 and is completely antisymmetric on all \(\delta_r\) indices. Note that the sum over \(i\) in the exponent runs only over those subspaces which are spanned by the form. An elementary form can therefore extend over one or more of the subspaces only if its degree matches the total dimension of these spaces. Consider for example the RR three-form of type IIA supergravity. If the space is split up as \((d_0, d_1) = (3,6)\) it fits into the 3-dimensional subspace and the above general Ansatz specializes to

\[
F^{\mu_0 \mu_1 \mu_2 \mu_3} = \exp(-6\alpha_0) f^{\prime}(\tau) \epsilon_{\mu_1 \mu_2 \mu_3} \quad \text{where } \mu_1, \mu_2, \mu_3 \text{ refer to the coordinates of this subspace.}
\]

- **solitonic**: if \(\sum_i d_i = \delta_r + 1\) for some of the spatial subspaces \(i\) we may allow for

\[
(F_r)^{\mu_0 \ldots \mu_{\delta_r+1}} = B_r(\alpha) \, w_r \, \epsilon_{\mu_0 \ldots \mu_{\delta_r+1}} , \quad B_r(\alpha) = e^{-2 \sum_i d_i \alpha_i}
\]

where \(\mu_1 \ldots \mu_{\delta_r+1}\) refer to the coordinates of these subspaces and \(w_r\) is an arbitrary constant. As for the elementary Ansatz, the sum over \(i\) in the exponent runs over the subspaces spanned by the form. It is easy to check that this Ansatz, already solves the form equation of motion.

Note that in contrast to the elementary Ansatz, the solitonic field strength does not have a time index. Therefore the matching condition for the dimensions differs. Given, for example, a split \((d_0, d_1) = (3,6)\) one has to use a 2 form instead of a three-form to fit into the 3-dimensional subspace. The above Ansatz then reads

\[
F_{\mu_1 \mu_2 \mu_3} = \exp(-6\alpha_0) w \epsilon_{\mu_1 \mu_2 \mu_3}.
\]
Having specified the form of our Ansatz, we now look to solve the equations of motion derived from the action (2). However, it is in fact easy to show that, under a very mild restriction, the resulting equations of motion can be obtained from a reduced Lagrangian which depends only on \( \alpha_i, \phi, N \) and \( f_r \), each as functions of \( \tau \). The Lagrangian is given by

\[
L = E \left[ \sum_{i=0}^{n-1} d_i \alpha'_i - \sum_{i,j=0}^{n-1} d_i d_j \alpha'_i \alpha'_j + \frac{4}{D-2} \phi'^2 + V_e - N^2 V_s \right]
\] (8)

with

\[
V_e = \frac{1}{2} \sum_r A_r (\alpha) e^{-a(\delta_r)\phi} f_r'^2
\]

\[
V_s = \frac{1}{2} \sum_r B_r (\alpha) w_r^2 e^{-a(\delta_r)\phi} + \Lambda e^{-a\Lambda\phi}
\]

\[
E = \frac{1}{N} \sum_{i=0}^{n-1} d_i \alpha_i .
\]

In the definitions of the potentials \( V_e \) and \( V_s \), the sum over \( r \) is understood to run over all the elementary and solitonic configurations which have been chosen according to the given rules. The equations of motion for the functions \( f_r \) to be derived from eq. (8) read

\[
\frac{d}{d\tau} \left( EA_r e^{-a(\delta_r)\phi} f_r' \right) = 0 .
\] (10)

The first integrals are

\[
f_r' = v_r E^{-1} A_r^{-1} e^{a(\delta_r)\phi}
\] (11)

where \( v_r \) are integration constants. Equation (11) can be used to eliminate \( f_r' \) from the elementary potential \( V_e \). This then reduces the problem to solving the remaining equations of motion now given purely in terms of \( \alpha_i, \phi \) and \( N \).

In fact, we find that the remaining equations can also be derived from a simple reduced Lagrangian. First introduce the notation \( \alpha = (\alpha_I) = (\alpha_i, \phi) \) for a general point in the moduli space. We also define a particular metric on the moduli space \( G_{IJ} \) by

\[
G_{ij} = 2(d_i \delta_{ij} - d_i d_j)
\]

\[
G_{in} = G_{ni} = 0
\]

\[
G_{nn} = \frac{8}{D-2} .
\] (12)

The equations of motion for \( \alpha \) and \( N \) following from eq. (8) then take the simple form

\[
\frac{d}{d\tau} (EG\alpha') + E^{-1} \frac{\partial U}{\partial \alpha} = 0
\]

\[
\frac{1}{2} E\alpha'^T G\alpha' + E^{-1} U = 0 .
\] (13)
The quantity $U$ is given by
\[ U = e^{2 \sum_{i=0}^{n-1} d_i \alpha_i \left( \frac{1}{N^2 V_e + V_s} \right)} \]  \hspace{1cm} (15)
where $f'$ in $V_e$ has been replaced using eq. (11). Clearly, these equations of motion can be derived from the simple Lagrangian (it is convenient to make the change of variables from $N$ to $E$),
\[ \mathcal{L} = \frac{1}{2} E \alpha'^T G \alpha' - E^{-1} U \]  \hspace{1cm} (16)
The first term is kinetic, while $U$ defines a potential in the moduli space. Further, $E$ is the metric on the particle worldline.

It is useful to rewrite the potential $U$ in a more systematic way as
\[ U = \frac{1}{2} \sum_{r=1}^{m} u_r^2 \exp(q_r \cdot \alpha) \]  \hspace{1cm} (17)
where the sum runs over all elementary and solitonic configurations as well as a possible cosmological constant term. The constants $u_r$ represent the integration constants $v_r$ in eq. (11) for elementary forms, the constants $w_r$ in the Ansatz (7) for solitonic forms or a cosmological constant. The type of each term is specified by the vectors $q_r$ which can be read off from eqs. (6), (7), (9) and the definition (15). For an elementary $\delta$–form they are given by
\[ q^{(el)} = (2\epsilon_i d_i, a(\delta)) , \quad \epsilon_i = 0, 1 , \quad \delta = \sum_{i=0}^{n-1} \epsilon_i d_i \]  \hspace{1cm} (18)
with $\epsilon_i = 1$ if the form is nonvanishing in the subspace $i$ and $\epsilon_i = 0$ otherwise. For type II theories the dilaton couplings $a(\delta)$ are given in eq. (3) and (4). To account for the $D=11$ case (or constant dilaton solutions) we may just set $a(\delta) = 0$.

Let us give an example at this point. An elementary IIA RR 3 form, put into the first subspace of a $D=10$ space split with $(d_0, d_1) = (3, 6)$, is characterized by a vector $q = (6, 0, -1/2)$. It generates a potential term in (17) which depends on $\alpha_0$ and the dilaton but not on $\alpha_1$. More generally, a potential term describing the effect of an elementary form depends only on those scale factors which correspond to subspaces spanned by the form. Since the entries $q_i$ of $q$ are always positive, the potential tends to drive the scale factors for these subspaces to smaller values. Therefore these subspaces tend to be contracting or at least less generically expanding than others.

The situation for a solitonic $\delta$–form is in some sense complementary. It is specified by a vector
\[ q^{(sol)} = (2\tilde{\epsilon}_i d_i, -a(\delta)) , \quad \tilde{\epsilon}_i \equiv 1 - \epsilon_i = 0, 1 , \quad \tilde{\delta} \equiv D - 2 - \delta = \sum_{i=0}^{n-1} \tilde{\epsilon}_i d_i \]  \hspace{1cm} (19)
with $\tilde{\epsilon}_i = 1$ if the form vanishes in the subspace $i$ and $\tilde{\epsilon}_i = 0$ otherwise. For example, a solitonic IIB RR 2 form in the first subspace of a space split as $(d_0, d_1) = (3, 6)$ is specified by $q = (0, 12, 1)$. It generates a potential term in (17) which depends on $\alpha_1$ and the dilaton but not on $\alpha_0$. More
generally, in contrast to the elementary case, the potential term now depends on those scale factors corresponding to subspaces not spanned by the form.

Finally, a cosmological constant is characterized by

\[ q^{(A)} = \left( 2d_i, \frac{2D}{D-2} \right). \]  

(20)

Note that for all these vectors \( \sum_{i=0}^{n-1} q_i > 0 \), a fact which we will use later on. The moduli space metric allows us to define a natural scalar product on the space of vectors \( q \)

\[ < q_1, q_2 > = q_1^T G^{-1} q_2 \]  

(21)

with the inverse of \( G \) given by

\[
\begin{align*}
(G^{-1})_{ij} &= -\frac{1}{2(D-2)} + \frac{1}{2d_i} \delta_{ij} \\
(G^{-1})_{in} &= (G^{-1})_{ni} = 0 \\
(G^{-1})_{nn} &= \frac{D-2}{8}.
\end{align*}
\]  

(22)

Since the metric \( G \) has Minkowskian signature, we can distinguish between space- and time-like vectors \( q \). As we will see, the structure of the solutions depends crucially on this distinction.

Generically, the models specified by the eqs. (13), (14) and (17) cannot be solved. A complete solution, however, can be found if the potential \( U \) consists of one exponential term only or if contact with Toda theory can be made. Here, we will discuss only the first of these two possibilities. For examples related to Toda theory see ref. [5, 6].

3 Solutions with One Potential Term

In this section, we will analyze models with just one form turned on (or a non-zero cosmological constant). The form may be elementary or fundamental and there may be any number of subspaces. All of these cases correspond to a potential

\[ U = \frac{1}{2} u^2 \exp(q \cdot \alpha) \]  

(23)

where \( u^2 > 0 \). We will start by giving the general form of the solution and then give a simple example in section 3.2.

One way of solving the equations of motion for a potential (23) is to use the gauge freedom in the definition of the time coordinate. We can always choose a gauge such that

\[ N = \exp((d - q) \cdot \alpha) \]  

(24)
where we introduce a vector giving the subspace dimensions $\mathbf{d} = (d_i, 0)$. This implies $E = \exp(\mathbf{q} \cdot \alpha)$ and the following set of equations for $\alpha$

$$
\frac{d}{d\tau}(G \alpha') + \frac{1}{2} u^2 \mathbf{q} = 0
$$

$$
\frac{E}{2} \alpha' G \alpha' + \frac{1}{2} u^2 = 0 .
$$

(25)

In this form they can be integrated immediately, leading to the general solution

$$
\alpha = c \ln |\tau_1 - \tau| + w \ln \left( \frac{s\tau}{\tau_1 - \tau} \right) + k
$$

(26)

where

$$
c = \frac{2G^{-1} \mathbf{q}}{< \mathbf{q}, \mathbf{q}>} .
$$

(27)

The sign $s = \pm 1$ is determined by $s = \text{sign}(< \mathbf{q}, \mathbf{q}>)$ and $w, k$ are integration constants subject to the constraints

$$
\mathbf{q} \cdot \mathbf{w} = 1
$$

$$
\mathbf{w}^T G \mathbf{w} = 0
$$

(28)

$$
\mathbf{q} \cdot \mathbf{k} = \ln \left( \frac{u^2 |< \mathbf{q}, \mathbf{q}>|}{4} \right) .
$$

$\tau_1$ is a free parameter which we can take to be positive. The range of $\tau$ should be chosen to ensure a positive argument of the second logarithm in eq. (26). This depends on the sign of $< \mathbf{q}, \mathbf{q}>$ and we have the two cases

$$
0 < \tau < \tau_1 \quad \text{for} \quad < \mathbf{q}, \mathbf{q}> > 0
$$

$$
\tau < 0 \text{ or } \tau > \tau_1 \quad \text{for} \quad < \mathbf{q}, \mathbf{q}> < 0 .
$$

(29)

Which of these cases is actually realized in type II models? Using the vectors $\mathbf{q}$ given in the end of section 2, we find for a solitonic or elementary $\delta$ form (or a cosmological constant which is similar to a RR $(D - 1)$-form)

$$
< \mathbf{q}, \mathbf{q}> = \frac{D - 2}{8} a(\delta)^2 + \frac{2}{D - 2} \delta \tilde{\delta} = \begin{cases} 4 & \text{NS} \\ \frac{D - 2}{2} & \text{R} \end{cases}
$$

(30)

which is always positive. Also the $D = 11$ 3 form leads to a positive result, as can be seen from the above formula by setting $a(\delta) = 0$. We conclude that, in the present context, we are dealing with spacelike vectors $\mathbf{q}$ only, and we have $0 < \tau < \tau_1$.

So far, our solutions have been expressed in terms of the time parameter $\tau$ which is defined by the gauge choice (24) for $N(\tau)$. For a discussion of the cosmological properties of our models, however, we should reexpress them in terms of the comoving time $t$, that is in the gauge where the

\footnote{Here we disregarded the somewhat marginal possibility that $\mathbf{q}$ is a null vector, i.e. $< \mathbf{q}, \mathbf{q}> = 0$.}
N = 1. This can be done by integrating the defining relation \( dt = N(\tau) d\tau \). The explicit expression for \( N(\tau) \) is given by inserting the solution (26) into the gauge fixing equation (24) for \( N(\tau) \), which gives
\[
N = \exp((d - q) \cdot k)|\tau_1 - \tau|^{-(x + \Delta - 1)}|\tau|^{x - 1}
\] (31)
with
\[
x = d \cdot w \quad \text{(32)}
\]
\[
\Delta = d \cdot c = 2 \frac{\langle d, q \rangle}{\langle q, q \rangle} \quad \text{(33)}
\]
Depending on the values of \( x \) and \( \Delta \), the gauge parameter \( N \) may have singularities at \( \tau = 0 \) and \( \tau = \tau_1 \). This determines the allowed range in the comoving time \( t \) as we will discuss in detail in the next subsection.

Another quantity which is of importance in discussing the physical content of our solutions is the scalar curvature \( R \). For the solution (26) it is given by
\[
R \sim |\tau_1 - \tau|^{2(x - \Delta)}|\tau|^{-2x} P_2(w, \alpha, \tau)
\] (34)
where \( P_2 \) is a second order polynomial in \( \tau \) which we will not need explicitly. The first two factors in this equation indicate potential singularities at \( \tau = 0 \) and \( \tau = \tau_1 \), depending on \( x \) and \( \Delta \) as in the case of the gauge parameter \( N \). However, in contrast to singularities in \( N \), such singularities are true coordinate independent curvature singularities. They will be further discussed in the next subsection.

### 3.1 Cosmology of Solutions with Spacelike \( q \)-Vectors

As already mentioned, the case of spacelike \( q \)-vectors is the most important in our context since all vectors \( q \) appearing within the \( D = 10 \) type II theories and M-theory are spacelike. In this section, we will discuss the cosmological properties which can be extracted from these models in general. A concrete illustrating example will be given in the next subsection.

Recall that the singularity structure of the solution (24) is determined by the quantities \( x \) and \( \Delta \) defined in eq. (32) and (33). What values are actually allowed for these quantities? From \( \langle d, q \rangle = -\sum_{j=0}^{n-1} q_j/2(D - 2) \) and \( \sum_{j=0}^{n-1} q_j > 0 \) it follows from eq. (33) that \( \Delta < 0 \). The parameter \( x \), which unlike \( \Delta \) depends on the parameters of the solution, turns out to be either \( x < \Delta \) or \( x > 0 \) in all specific examples we considered. This divides the set of initial conditions into two disconnected subsets corresponding to two classes of solutions with different properties.

We begin our discussion of these properties by analyzing the allowed ranges in comoving time \( t \). Recall from eq. (24) that the time parameter \( \tau \) is always in the range \( 0 < \tau < \tau_1 \) since we have \( \langle q, q \rangle > 0 \). The singularity structure of the gauge parameter \( N \) in eq. (31) then shows that this
range is mapped to the following ranges in $t$

$$\tau \rightarrow t \in \begin{cases} 
[-\infty, t_1] & \text{for } x < \Delta < 0, \quad (-) \text{ branch} \\
[t_0, +\infty] & \text{for } x > 0 > \Delta, \quad (+) \text{ branch}
\end{cases}.$$  \hspace{1cm} (35)

Here $t_0$ and $t_1$ are two finite unrelated values that appear as integration constants from integrating $dt = N(\tau)d\tau$. Thus we have found two disconnected branches corresponding to asymptotically positive and negative time ranges.

Let us next discuss the scalar curvature in each branch. We start with the $(−)$ branch. As inspection of eq. (34) shows, the curvature vanishes as $t \to -\infty$ ($\tau \simeq 0$) since $x < \Delta < 0$. With increasing time $R$ grows and, finally, the system runs into the curvature singularity at $t = t_1$ ($\tau = \tau_1$) since the power $2(x - \Delta)$ of the first term in eq. (34) is negative. Therefore, classically the $(−)$ solution cannot be continued beyond this point.

In the $(+)\text{ branch}$ the situation is similar but reversed in time. At $t = t_0$ ($\tau = 0$) we find a singularity since $x > 0$ in this branch. The solution cannot be extended into the past. As $t \to \infty$ ($\tau \simeq \tau_1$) the curvature behaves smoothly and approaches zero.

Though generically correct, the above argument has a loophole. For very specific values of the initial parameters $w$, the polynomial $P_2$ is proportional to $\tau$ or $\tau_1 - \tau$ so that it can cancel against one of the first two factors in eq. (34) which cause the singularity. If $|x|$ is sufficiently small, the singularity may disappear completely. For the $(−)$ branch this is realized if $w_n = c_n$ and $x \geq -1/2$. For the $(+)$ branch it occurs if $w_n = 0$ and $x \leq 1/2$. This phenomenon is quite similar to what happens in the curvature singularity free WZW model of ref. [36]

So far, we have considered quantities which provide information about the behaviour of the total $D$ dimensional space only. Let us now turn to the individual subspaces of dimension $d_i$. To analyze their behaviour, we should calculate their respective Hubble parameters $H_i$ in terms of the comoving time. In fact, it is possible to explicitly express the comoving time $t(\tau)$ in terms of hypergeometric functions. It is, however, more instructive to look at the asymptotic regions $\tau \simeq 0$ (corresponding to $t \to -\infty$ for the $(−)$ branch and $t \simeq t_0$ for the $(+)$ branch) and $\tau \simeq \tau_1$ (corresponding to $t \simeq t_1$ for the $(−)$ branch and $t \to \infty$ for the $(+)$ branch). In these regions the Hubble parameters can be written as\footnote{The dot denotes the derivative with respect to the comoving time $t$.}

$$H \equiv \dot{\alpha} = \frac{P}{t - t_s}$$  \hspace{1cm} (36)

with the constant expansion coefficients $p$ satisfying

$$pGp = 0, \quad d \cdot p = 1.$$  \hspace{1cm} (37)

The time shift $t_s$ depends on the asymptotic region and branch under consideration. The sign of $t - t_s$, however, is always well defined: It is negative in the $(−)$ branch and positive in the $(+)$
branch. If we combine the two equations (37) and use the explicit form of the metric \( G \) in (12) we find

\[
\frac{4}{D-2} \phi^2 \sum_{i=0}^{n-1} d_i p_i^2 = 1. \tag{38}
\]

The explicit expressions for \( p \) in terms of the integration constants are

\[
p = \begin{cases} \frac{w}{x} & \text{at } \tau \simeq 0 \\ \frac{w-c}{x-\Delta} & \text{at } \tau \simeq \tau_1 \end{cases} \tag{39}
\]

They have been calculated using the general solution (26) and the asymptotic limits of \( N(\tau) \) to be read off from eq. (31). The behavior of the Hubble parameters (36) along with eq. (37) indicates that the solutions behave asymptotically like those of pure Kaluza–Klein theory with a dilaton. This can be seen by a comparison with the solutions of ref. [37]. Therefore, one expects that the potential \( U \) provided by the form is effectively turned off in these limits. In fact, inserting the general solution (26) into the potential (23), we find

\[
U \sim (\tau_1 - \tau) \tau
\]

which implies that \( U \) is effectively zero near \( \tau \simeq 0 \) and \( \tau \simeq \tau_1 \). The effect of the form is therefore to generate a mapping

\[
p(\tau \simeq 0) \rightarrow p(\tau \simeq \tau_1)
\]

between two pure Kaluza–Klein states.

What do the above results mean for the evolution of the subspaces? We consider the (+) branch first. Remember that \( t - t_s > 0 \) in this branch so that from eq. (36) a positive \( p_i \) results in expansion and a negative \( p_i \) in contraction. Moreover, the equation \( \mathbf{d} \cdot \mathbf{p} = 1 \) shows that at least one of the \( p_i \) has to be positive. Consequently, at least one of the subspaces has to be expanding. From eq. (38) we conclude that \(|p_i| < 1\) always. The expansion is therefore subluminal. This behaviour is similar to a radiation or matter dominated universe corresponding to \( p_i = 1/2 \) and \( p_i = 2/3 \), respectively.

The situation is completely different in the (−) branch. Since \( t - t_s < 0 \), a positive \( p_i \) results in contraction and a negative \( p_i \) in expansion. Now we conclude from \( \mathbf{d} \cdot \mathbf{p} = 1 \) that at least one subspace must be contracting. Since we are in the negative time range, eq. (36) shows that expansion (\( H_i > 0 \)) goes along with an increasing \( H_i \), that is, a shrinking horizon size. Such a behaviour is also called superinflation since scales are stretched across the horizon even more rapidly than in “ordinary” inflation where the horizon size is approximately constant.

Our solutions allow various patterns of expanding and contracting spatial subspaces. The details of the evolution depend on the partition \( \{d_i\} \), the form and the subspaces it occupies and the initial conditions. Examples with 3 expanding and 6 contracting spatial dimensions as \( t \to \infty \) exist, as were given in ref. [3]. The effect of the form can be quite dramatic. For example, it can reverse expansion and contraction of two subspaces during the early asymptotic period into its converse during the late period.

A “preferred” cosmological scenario suggested by these solutions consists of a combination of the (−) and the (+) branch to account for inflation as well as for a postinflationary subluminal expansion. The apparent shortcoming of this scenario is that the (−) and the (+) branch constitute
two different \textit{a priori} unrelated solutions. As for string frame pre–big–bang models, one might argue \cite{28} that scale factor (T) duality between the branches provides the correct transition.

\section{3.2 An Example}

Up to this point our discussion has been rather general. Let us now illustrate the steps in our solution by giving a simple example. We consider the following situation: 10-dimensional spacetime is split into two subspaces with \( d = (d_0, d_1, 0) = (3, 6, 0) \) and an elementary IIA RR 3 form occupies the 3–dimensional subspace. This implies the Ansatz

\[
\begin{align*}
 ds^2 &= -N^2(\tau)d\tau^2 + e^{2\alpha_0} dx_0^2 + e^{2\alpha_1} dx_1^2 \\
 F_{0\mu_1\mu_2\mu_3} &= e^{-6\alpha_0} f'(\tau)e_{\mu_1\mu_2\mu_3} \\
 \phi &= \phi(\tau)
\end{align*}
\]

in accordance with the eqs. (3), (4). The equations of motion for this example can be derived from the Lagrangian

\[
\mathcal{L} = E \left[ -6\alpha_0'{}^2 - 30\alpha_1'{}^2 - 36\alpha_0'\alpha_1' + \frac{1}{2}\phi'^2 + V_e \right]
\]

which come from the general equations (8) and (9). The equation of motion for \( f \) can be integrated to give the first integral

\[
f' = uE^{-1} e^{6\alpha_0 + \frac{1}{2}\phi}
\]

with an integration constant \( u \). From the Lagrangian (11) we can compute the equations of motion for \( \alpha_0, \alpha_1 \) and \( \phi \). Using eq. (43) to replace \( f' \) in these equations, we arrive at

\[
\begin{align*}
 \frac{d}{d\tau}(E(-12\alpha_0' - 36\alpha_1')) + 3u^2 E^{-1} e^{6\alpha_0 - \phi/2} &= 0 \\
 \frac{d}{d\tau} (E(-36\alpha_0' - 60\alpha_1')) &= 0 \\
 \frac{d}{d\tau} (E\phi') - \frac{1}{4} u^2 E^{-1} e^{6\alpha_0 - \phi/2} &= 0 .
\end{align*}
\]

Let us compare these equations with the general ones given in the moduli space formalism in (13), (14) and (17). We see that they can be indeed written in this form if we set

\[
G = \begin{pmatrix} -12 & -36 & 0 \\ -36 & -60 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

and

\[
U = \frac{1}{2} u^2 e^{6\alpha_0 - \frac{1}{2}\phi} .
\]
The matrix $G$ above is consistent with the general formula (12) with $d_0 = 3$, $d_1 = 6$ and $D = 10$. In eq. (17) we introduced a systematic way of writing the effective potential by introducing a characteristic vector $q_f$ for each form. From eq. (16) we read off that this vector is given by $q = (6, 0, -1/2)$ for our example. This coincides with what one gets by applying the general rule (1) to the breakup $(d_0, d_1) = (3, 6)$ and a $\delta = 3$ form in the 3-dimensional subspace. The dilaton coupling $a(\delta)$ for a RR 3–form needed in eq. (6) follows from eq. (3) to be $a(\delta) = -1/2$.

In section 2 we also defined a scalar product (21) on the space of vectors $q$ using the inverse of $G$. From eq. (45) $G^{-1}$ is given by

$$G^{-1} = \begin{pmatrix} \frac{5}{48} & -\frac{1}{16} & 0 \\ -\frac{1}{16} & \frac{1}{48} & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

which agrees with the general formula (22) for $(d_0, d_1) = (3, 6)$ and $D = 10$. One can easily verify that $\langle q, q \rangle = q G^{-1} q = 4$. Therefore $q$ is indeed a spacelike vector, in agreement with the general result (23) which showed that this is true for all vectors obtained from type II forms.

The main problem in solving the system of equations (44) is the existence of two different exponentials, one coming from $E$, eq. (42), the other coming from the effective potential $U$, eq. (46). Fortunately, we have a gauge freedom (time reparameterization invariance) encoded in $N$ which we can use to get rid of one of the exponentials. Here, we choose the possibility of gauging away the potential by setting $E = \exp(6\alpha_0 - \phi/2)$. Given the definition of $E$ in eq. (42), this implies

$$N = \exp(-3\alpha_0 + 6\alpha_1 + \phi/2)$$

in accordance with the general formula (24) for $d = (3, 6, 0)$ and $q = (6, 0, -1/2)$. With this choice, the equations of motion (44) turn into

$$\frac{d}{d\tau} \left( e^{6\alpha_0 - \phi/2}(2\alpha_0' + 6\alpha_1') \right) = \frac{u^2}{2}$$

$$\frac{d}{d\tau} \left( e^{6\alpha_0 - \phi/2}(3\alpha_0' + 5\alpha_1') \right) = 0$$

$$\frac{d}{d\tau} \left( e^{6\alpha_0 - \phi/2}\phi' \right) = \frac{u^2}{2}$$

$$e^{6\alpha_0 - \phi/2}(6\alpha_0'^2 + 36\alpha_0'\alpha_1' + 30\alpha_1'^2 - \phi'^2) = \frac{u^2}{2} .$$

This is consistent with the general form (23) found for models with one term in the potential. Taking appropriate linear combinations of the first three equations we can derive an equation for the remaining exponent $6\alpha_0 - \phi/2$, which can be solved. Then $\alpha_0, \alpha_1, \phi$ can be expressed in terms of this solution. In this way one arrives at a general solution of the form (26) with coefficients $c$ given by

$$c = \left( \frac{5}{16}, -\frac{3}{16}, -\frac{1}{4} \right) .$$

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and the following constraints on the integration constants

\[
\begin{align*}
6w_0 - \frac{1}{2}w_2 &= 1 \\
12w_0^2 + 72w_0w_1 + 60w_1^2 &= w_2^2 \\
6k_0 - \frac{1}{2}k_2 &= \ln(u^2). 
\end{align*}
\]

(51)

Recall that the time parameter \(\tau\) is restricted by \(0 < \tau < \tau_1\).

To discuss the cosmology of these solutions we must perform a transformation to comoving time \(t\). To do this, we need the explicit form of the gauge parameter \(N\) which we find by inserting the solution (26) with (50), (51) into eq. (48)

\[
N = e^{-3k_0 + 6k_1 + k_2/2}\left|\frac{\tau}{\tau_1} - \tau\right|^{-x + \Delta - 1}\tau^{x - 1}.
\]

(52)

Here \(x = 3w_0 + 6w_1\) and \(\Delta = -3/16\). The quantities \(x, \Delta\) have been generally defined in eq. (12), (13) and their values can be easily reproduced by inserting \(d = (3, 6, 0)\) and \(c\) from eq. (50). The range of comoving time obtained by integrating \(dt = N(\tau)d\tau\) over \(0 < \tau < \tau_1\) crucially depends on the singularities in \(N\). Eq. (52) shows that there are potential singularities at \(\tau = 0\) and \(\tau = \tau_1\). Their appearance is controlled by the value of \(x\).

\[Fig 1: \text{Expansion coefficients for the (+) branch at } t \simeq t_0.\]
Let us therefore analyze which values of $x$ are allowed. The first two constraints (51) may be solved to express, for example, $w_0$ and $w_1$ as a function of $w_2$. This shows that $x$ depends on one free parameter only. Furthermore, since the second constraint (51) is quadratic in $w_1$, we find two branches satisfying $x < \Delta = -3/16$ and $x > 0$, respectively. We refer to these two branches as the $(-)$ and the $(+)$ branch. From eq. (52) we see that $0 < \tau < \tau_1$ is indeed mapped to the comoving time ranges given in eq. (29); that is to $[-\infty, t_1]$ for the $(-)$ branch and to $[t_0, \infty]$ for the $(+)$ branch ($t_0, t_1$ are integration constants). Moreover, the scalar curvature (34) has a future timelike singularity in the $(-)$ branch and a past timelike singularity in the $(+)$ branch. Both types of solutions are therefore not extendible.

![Graph](image)

**Fig 2: Expansion coefficients for the $(+)$ branch at $t \to \infty$.**

Information about the evolution of the two subspaces and the dilaton can be obtained form the respective Hubble parameters $H = \dot{\alpha}$ written as a function of comoving time. They can be calculated if $\tau \simeq 0$ or $\tau \simeq \tau_1$ since $N$ in eq. (52) can be integrated in these limits. Doing this for our example by using eq. (26), (50), (51) and $N, \tau(t)$ calculated from eq. (52), we find $H$ to be of the Kaluza–Klein form (36), (37). The expansion coefficients $p$ depend on the integration constants $w$ as in eq. (32).
In fact, using the first two constraints (51) we may rewrite \( p \) as a function of \( w_2 \) only, as we did for the parameter \( x \) before. The asymptotic expansion properties of our example therefore depend on one free parameter only. Instead of giving the explicit formulae, which are not particularly enlightening, let us give a graphical representation of \( p = p(w_2) \). We concentrate on the (+) branch (the expansion coefficients in the (−) branch can be worked out analogously) where \( t_0 < t < \infty \) and the asymptotic regions are characterized by \( t \approx t_0 (\tau \approx 0) \) and \( t \rightarrow \infty (\tau \approx \tau_1) \). The results are given in fig. 1 \((t \approx t_0)\) and fig. 2 \((t \rightarrow \infty)\). In both figures \(|p_0|, |p_1| < 1\) always, which illustrates our general result that the expansion in the (+) branch is always subluminal. We see that an early expansion of the 3–dimensional subspace \((p_0 > 0 \text{ in fig. 1})\) is turned into a contraction as \( t \rightarrow \infty (p_0 < 0 \text{ in fig. 2})\) for a wide range in \( w_2 \). This can be understood from the \( \alpha_0 \) dependence of the effective potential \([40]\). Moreover, the 6–dimensional space is always expanding as \( t \rightarrow \infty (p_1 > 0 \text{ in fig. 2})\). In a more realistic model, such an expansion should be stopped by, for instance, a nonperturbative potential for the modulus \( \alpha_1 \).

4 Ricci Flat Compactifications of 11–Dimensional Supergravity

So far, we have considered cosmological M–theory solutions which correspond to products of maximally symmetric spaces. We would now like to shift our attention to a different, partially overlapping class of configurations, namely those with internal Ricci–flat spaces. In particular, this allows for more realistic models with the internal space being a Calabi–Yau manifold. We will find that the characteristic features of such solutions agree with the ones we discovered previously for the space being a product of maximally symmetric subspaces. Especially, we confirm the rôle of the form field as mapping different Kaluza–Klein type expansions into each other. In addition, we study an important aspect of M–theory which we have not addressed earlier, namely U–duality symmetries and their relation to cosmological solutions. For a related approach to cosmology using T–duality see [39, 40].

Let us start by reducing the bosonic part of 11–dimensional supergravity to \( D \) space–time dimensions on a Ricci–flat manifold. The resulting low energy action will be the starting point for our discussion of cosmological solutions.

The bosonic part of the 11–dimensional supergravity Lagrangian reads

\[
\mathcal{L} = \sqrt{-g} \left[ \bar{R} - \frac{1}{2 \cdot 4!} F_{MNPQ} F^{MNPQ} \right] + \frac{1}{3 \cdot 3! (4!)^2} \varepsilon^{M_1...M_11} F_{M_1...M_4} F_{M_5...M_8} A_{M_9M_{10}M_{11}}. \tag{53}
\]

We are using the conventions of ref. [41]. The 11–dimensional metric and curvature are given by \( \bar{g}_{MN} \) and \( \bar{R} \), respectively, where uppercase letters are used to index the full space, that is, \( M, N, ... = 0, ..., 10 \). The 4–form field strength \( F_{MNPQ} \) is expressed in terms of the 3–form gauge field \( A_{NPQ} \) as \( F_{MNPQ} = 4 \partial_M A_{NPQ} \). For the class of compactifications we will be interested in, the Chern–Simons term in eq. (53) vanishes. Therefore, we drop this term from now on and
consider the non–topological part of the Lagrangian

\[ \mathcal{L} = \sqrt{-g} \left[ \tilde{\mathcal{R}} - \frac{1}{2 \cdot 4!} F_{MNPQ} F^{MNPQ} \right] \]  

only. Our main purpose is to investigate the relation of cosmological solutions and U–duality symmetries for the action (54). The focus in this paper will be on the external part of U–duality which acts non–trivially on the space–time metric rather than on the part which transforms moduli. In our reduction to \( D \) dimensions we will therefore keep a minimal moduli content only, that is, the breathing mode of the Ricci–flat manifold. Though formulae in this section are kept general, the most interesting cases are the ones for \( D = 4, 5 \). While the case \( D = 4 \) is of obvious relevance, the importance of \( D = 5 \) is motivated by the construction of Horava and Witten [30] for the effective action of the strongly coupled heterotic string.

Let us now be specific. We are using indices \( \mu, \nu... = 0, ..., d \equiv D - 1 \) for the external space–time, indices \( m, n, ... = 1, ..., d \) for the external spatial directions and indices \( a, b, ... = d + 1, ..., 10 \) for the \( \delta \)–dimensional internal space, where \( \delta = 11 - D \). Our Ansatz for the 11–dimensional fields is as follows

\[ \begin{align*}
\bar{g}_{\mu\nu} &= \bar{g}_{\mu\nu}(x^\rho) \\
\bar{g}_{\mu b} &= 0 \\
\bar{g}_{ab} &= \bar{b}^2(x^\rho) \Omega_{ab}(x^c) \\
A_{\mu\nu\rho} &= B_{\mu\nu\rho}(x^\sigma).
\end{align*} \]

All other components of \( A_{NPQ} \) are set to zero. Here \( \Omega_{ab} \) is the metric of a \( \delta \)–dimensional Ricci–flat manifold and \( \bar{b} \) is its breathing mode. Depending on the dimension, this manifold can be a Calabi–Yau space, a torus or even a product of both. The \( D \)–dimensional metric and 3–form are denoted by \( \bar{g}_{\mu\nu} \) and \( B_{\mu\nu\rho} \), respectively. As already discussed, we have considered the minimal moduli content represented by \( \bar{b} \) only. As a further simplification, we have neglected \( D \)–dimensional vector fields (graviphotons as well as those arising from \( A_{NPQ} \)) and \( D \)–dimensional 2 forms. With the above truncation, we arrive at a low energy theory independent on the details of the compactification, but we keep the characteristic 3–form as a low energy field. Furthermore, as we will see, the Ansatz (55) is consistent with the external part of U–duality, so that it provides a “minimal” setting for our purpose.

Using the truncation (55) the action (54) turns into

\[ \mathcal{L} = \sqrt{\bar{\Omega}} \sqrt{-\bar{g}} \bar{b}^\delta \left[ \tilde{\mathcal{R}} + \delta(\delta - 1) \bar{b}^{-2} (\partial_\mu \bar{b})^2 - \frac{1}{2 \cdot 4!} F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} \right], \]

where \( F_{\mu\nu\rho\sigma} = 4 \partial_{[\mu} B_{\nu\rho\sigma]} \). To get a canonical curvature term we perform the Weyl rotation

\[ \bar{g}_{\mu\nu} = \bar{b}^{-\frac{\delta}{D-2}} g_{\mu\nu}, \]

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to the Einstein frame metric $g_{\mu \nu}$. In this frame, eq. (56) reads

$$\mathcal{L} = \sqrt{-g} \left[ R - k^2 \bar{b}^{-2} (\partial_{\mu} \bar{b})^2 - b^{\frac{6(11-D)}{2}} \frac{1}{2 \cdot 4!} F_{\mu \nu \rho \sigma} F^{\mu \nu \rho \sigma} \right]$$

(58)

with $k^2 = \frac{D}{D-2} \delta^2 - \delta(\delta - 1)$. For a study of cosmological solutions of this Lagrangian we consider the Ansatz

$$g_{\mu \nu} = \begin{pmatrix} -\bar{N}^2(\tau) & 0 \\ 0 & \bar{G}_{mn}(\tau) \end{pmatrix}$$

$$B_{mnr} = B_{mnr}(\tau)$$

$$\bar{b} = \bar{b}(\tau) .$$

(59)

Here, time has been denoted by $\tau$. The equations of motion with these specialized fields inserted can be derived from a Lagrangian which is related to eq. (58) by a formal dimensional reduction to one dimension. This 1–dimensional Lagrangian is given by

$$\mathcal{L} = \bar{N}^{-1} \sqrt{\Phi} \left[ k^2 \bar{b}^{-2} \dot{\bar{b}}^2 - \frac{1}{4} \Phi^{\frac{2}{3}} \dot{\Phi}^2 - \frac{1}{4} \bar{G}_{mn} \ddot{\bar{G}}^{mn} + b^{\frac{6(11-D)}{2}} \bar{G}^{mn'} \bar{G}^{n'r} \dot{B}_{mnr} \dot{B}_{m'n'r'} \right]$$

(60)

where $\Phi = \det(\bar{G})$. Unlike in the first sections of this paper, here the dot denotes the derivative with respect to the time $\tau$.

In the last step we have dimensionally reduced $d = D - 1$ dimensions of a theory which by itself has been obtained reducing 11–dimensional supergravity. Therefore, one should expect the U–duality group of $(11 - d)$–dimensional maximal supergravity as a symmetry group of the Lagrangian (60). For example, for $D = 4$ ($d = 3$) the expected group is the one of 8–dimensional supergravity, that is, $G = SL(2) \times SL(3)$. For $D = 5$ ($d = 4$) one expects $G = SL(5)$, the U–duality group of 7–dimensional supergravity. As we will show, this is indeed the case. It is, however, hard to see directly from the Lagrangian in the form (60). The reason is that we have performed a Weyl rotation (57) which is different from the one that leads to $(11 - d)$–dimensional supergravity with a canonical Einstein term. We can compensate for this by the following nonlinear field redefinitions

$$\bar{b} = \Phi^{-\frac{1}{2(11-D)}} b$$

$$\bar{N} = \Phi^{-\frac{3}{2(11-D)(D-2)}} b^{\frac{11-d}{D-2}} N$$

$$\bar{G}_{mn} = \Phi^{-\frac{11-D}{(11-D)(D-2)}} b^{\frac{2(11-D)}{D-2}} G_{mn}$$

$$\bar{\Phi} = \Phi^{-\frac{(11-D)(D-1)}{(11-D)(D-2)}+1} b^{\frac{2(11-D)(D-1)}{D-2}} ,$$

(61)

which express the physical Einstein frame fields $\bar{G}_{mn}, \bar{N}, \bar{b}, \bar{\Phi}$ in terms of the new fields $G_{mn}, N, b, \Phi$ with

$$\Phi = \det(G) .$$

(62)

We drop the factor $\sqrt{\Omega}$ since it turns into a constant upon integration over the internal manifold.
Written in terms of these new variables the Lagrangian finally reads

\[ L = N^{-1} b^\delta \left[ -\delta(\delta - 1) b^{-2} \dot{b}^2 + \frac{1}{4(10 - D)} \Phi^{-2} \dot{\Phi}^2 - \frac{1}{4} \dot{G}_{mn} \dot{G}^{mn} 
+ \frac{1}{2} \cdot 3! G_m^{mn} G_{n^r}^{n^t} \dot{B}_{mnr} \dot{B}_{n^r n^t} \right]. \]  

(63)

This is the form of \( L \) we are going to use in our discussion of U–duality and cosmological solutions. For a physical interpretation of solutions one should, of course, transform back to the Einstein frame fields via eq. (61).

5 U–Duality Covariant Formulation

In this section, we will find the manifestly U–duality invariant form of the Lagrangian (63) and discuss its general solution.

The cases \( D = 4, 5 \) can be treated uniformly by considering an \( SL(n)/SO(n) \) sigma model (where \( n = 2, 3 \) for \( D = 4 \) and \( n = 5 \) for \( D = 5 \), for example) written in terms of the coset parameterization \( M \in SL(n)/SO(n) \).

Without reference to a specific parameterization, the coset \( M \) can be characterized by the conditions \( \det(M) = 1 \) and \( M = M^T \) which can be implemented via Lagrange multipliers. We are therefore considering the Lagrangian

\[ \mathcal{L}_1 = N^{-1} b^\delta \left[ -\delta(\delta - 1) b^{-2} \dot{b}^2 + \frac{1}{4} \mathrm{tr} \left( M^{-1} \dot{M} M^{-1} \dot{M} \right) \right] + \lambda (\det(M) - 1) + \mathrm{tr} \left( \gamma (M - M^T) \right) \]  

(64)

with the Lagrange multipliers \( \lambda, \gamma \). The \( SL(n) \) symmetry transformations are given by

\[ b \to b, \quad \lambda \to \lambda \]
\[ M \to PMP^T, \quad \gamma \to P^{T^{-1}} \gamma P^{-1}, \]

(65)

where \( P \in SL(n) \). After eliminating the Lagrange multipliers, we find as the \( SL(n) \) covariant equations of motion for \( M, b \) and \( N \) (in the gauge \( N = 1 \) which we can always choose by a suitable reparameterization of the time \( \tau \))

\[ \frac{d}{d\tau} \left( M^{-1} \dot{M} \right) + \delta H M^{-1} \dot{M} = 0 \]
\[ (\delta - 1) \ddot{H} + \frac{1}{2} \delta(\delta - 1) H^2 = -\rho \]
\[ \frac{1}{2} \delta(\delta - 1) H^2 = \rho, \]  

(66)

respectively. Clearly, the matrix \( M \) in these equations is restricted to be symmetric and unimodular. The Hubble constant \( H \) and the energy density \( \rho \) are defined by

\[ H = \frac{\dot{b}}{b}, \quad \rho = \frac{1}{8} \mathrm{tr} \left( M^{-1} \dot{M} M^{-1} \dot{M} \right). \]  

(67)
The second and third equation in (66) can be combined to find the following solution for the breathing mode

\[ H = \frac{1}{\delta \tau}, \quad b = b_0 |\tau|^{1/\delta}, \]  

(68)

where \( b_0 \) is an arbitrary constant. Inserting this into the first equation (66) we find for the coset \( M \)

\[ M = M_0 e^{I \ln |\tau|}, \]  

(69)

where the constant matrices \( M_0, I \) satisfy

\[
\begin{align*}
\det(M_0) &= 1, \\
\text{tr}(I) &= 0, \\
M_0 &= M_0^T, \\
M_0 I &= I^T M_0.
\end{align*}
\]

(70)

Furthermore, from the last equation (66) one obtains the zero energy constraint

\[ \text{tr}(I^2) = 4 \frac{\delta - 1}{\delta}. \]  

(71)

Eqs. (68)–(71) represent the complete solution of the Lagrangian (64) written in a manifestly \( SL(n) \) covariant form. The \( SL(n) \) transformation (65) on the coset \( M \) acts on the integration constants encoded in \( M_0, I \) by

\[
\begin{align*}
M_0 &\rightarrow P M_0 P^T \\
I &\rightarrow P^T^{-1} I P^T.
\end{align*}
\]

(72)

At this point, it is instructive to count the number of integration constants in our general solution. The matrices \( M_0, I \) satisfying the constraints (71) contain \( n^2 + n - 2 \) independent parameters. The zero energy condition (71) eliminates one of them so that we remain with \( n^2 + n - 3 \) independent integration constants. This is just about the correct number to describe the general solution for all degrees of freedom in the coset \( M \in SL(n)/SO(n) \). On the other hand, the group \( SL(n) \) consists of \( n^2 - 1 \) parameters which implies that for \( n > 2 \) not all solutions can be connected to each other by \( SL(n) \) transformations. More precisely, the total \( n^2 + n - 3 \)-dimensional solution space splits into \( n^2 - 1 \)-dimensional equivalence classes, each consisting of solutions related to each other by \( SL(n) \) transformations via eq. (72). The remaining \( n - 2 \) integration constants label different equivalence classes, that is, classes of solutions which cannot be connected by \( SL(n) \) transformations. It is useful in the following to make this structure more explicit in the solution (68). Diagonalizing \( M_0 \) and \( I \) using eq. (72) with appropriate matrices \( P \), it is straightforward to prove that eqs. (68), (69), (71) can equivalently be written in the form

\[ M = P \text{diag}(|\tau|^{p_1}, \ldots, |\tau|^{p_n}) P^T \]  

(73)

with

\[
\begin{align*}
\sum_{i=1}^n p_i &= 0, \\
\sum_{i=1}^n p_i^2 &= 4 \frac{\delta - 1}{\delta}.
\end{align*}
\]

(74)
and $P \in SL(n)$. The equivalence classes of $SL(n)$ unrelated solutions are parameterized by the $n-2$ constants $\{p_i\}$ subject to the constraints \((72)\). In addition, since $SL(n)$ contains permutations of the $n$ directions we should pick a definite order, say $p_i \geq p_j$ if $i < j$, for the $\{p_i\}$ to describe $SL(n)$ inequivalent solution. On the other hand, a specific class, characterized by a fixed set $\{p_i\}$, is generated by the matrices $P \in SL(n)$ in eq. \((73)\). In the next section, we will apply these general results to the example $D = 5$ and discuss the physical implications.

6 The Example $D = 5$

The U–duality group in the $D = 5$ case is $G = SL(5)$. Let us define the vector $B = (B^s)$ by

$$B_{mnr} = \frac{1}{\Phi} \epsilon_{mnr}B^s . \quad (75)$$

Then the $SL(5)/SO(5)$ coset $\mathcal{M}$ can be parameterized by \([12]\)

$$\mathcal{M} = \Phi^{-2/5} \begin{pmatrix} G & -GB \\ -B^T G & \Phi + B^T GB \end{pmatrix} , \quad (76)$$

where we have used a matrix notation $G = (G_{mn})$ for the metric. With the internal dimension $\delta = 6$, Lagrangian \([33]\) can then be put into the form

$$\mathcal{L} = N^{-1}b^6 \left[ -30b^{-2}b^2 - \frac{1}{4}\text{tr} \left( \dot{\mathcal{M}}\dot{\mathcal{M}}^{-1} \right) \right] , \quad (77)$$

which has manifest $SL(5)$ invariance. The explicit transformations are given by

$$b \rightarrow b , \quad \mathcal{M} \rightarrow P\mathcal{M}P^T \quad (78)$$

with $P \in SL(5)$.

We are now dealing with a $\delta = 6$–dimensional internal manifold which can be a torus $T^6$ or a Calabi–Yau 3–fold. The equations of motion for the Lagrangian \([74]\) are given by the general expressions \([66], [57]\) with $\delta = 6$ and $M = \mathcal{M}$ inserted. Here $\mathcal{M}$ is the $SL(5)/SO(5)$ coset explicitly given in terms of the metric and the 3–form in eq. \((76)\). From eq. \((61)\) the physical fields can be written as

$$\bar{b} = \Phi^{-1/10} b , \quad \bar{N} = \Phi^{-3/10} b^2 , \quad \bar{G}_{mn} = \Phi^{-2/5} b^4 G_{mn} . \quad (79)$$

The solution for the breathing mode can be read off from eq. \((68)\)

$$H = \frac{1}{6\tau} , \quad b = b_0 |\tau|^{1/6} . \quad (80)$$
For the coset $\mathcal{M}$ we have from eq. (73) and (74)

$$\mathcal{M} = P \text{diag}(|\tau|^{p_1}, ..., |\tau|^{p_5})P^T,$$

with

$$\sum_{i=1}^{5} p_i = 0, \quad \sum_{i=1}^{5} p_i^2 = \frac{10}{3}$$

and $P \in SL(5)$. As before, we require $p_i \geq p_j$ for $i < j$. The solution (81) contains 27 integration constants. Three of them are given by the parameters $\{p_i\}$ subject to the constraints (82), labeling the $SL(5)$ equivalence classes. The remaining 24 integration constants parameterize the $SL(5)$ matrix $P$ in eq. (81).

What is the general physical picture emerging from the solution (81)? As we will see explicitly below, our solutions have two different branches which correspond to the two different signs of $\tau$ in eq. (81). These branches are the $(-)$ and $(+)$ branches we have found earlier. From the structure of eqs. (81) and (79) it is clear that, depending on the choice of the $SL(5)$ matrix $P$, physically relevant quantities like Hubble expansion parameters will in general depend on linear combinations of the various exponents $|\tau|^{p_i}$. In certain asymptotic regions, however, one of these exponents will usually dominate and the expansion is described by simple power laws. These asymptotic regions correspond to the Kaluza–Klein regions found previously. The interesting observation here is, that the Hubble expansion rates in those Kaluza–Klein regions are U–duality invariant since this is the case for the powers $p_i$. A related observation for S–duality transformations has been made in ref. [20, 21, 25]. The choice of the $SL(5)$ matrix $P$, on the other hand, determines the time range for the asymptotic regions and the details of the transition.

Though generically clear, this picture is rather complex to analyze in detail for the general solution (81). Therefore we concentrate on the physically interesting case of FRW universes in the following. Consequently, we require a 3–dimensional spatial subspace of our 5–dimensional space to be isotropic. For the metric and the form field this implies

$$G = \begin{pmatrix} c13 & 0 \\ 0 & \phi \end{pmatrix}, \quad B = \begin{pmatrix} 03 \\ B \end{pmatrix},$$

where $c, \phi, B$ are time dependent scalars. Inserting (83) into the coset parameterization (74) for $\mathcal{M}$ results in

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_3 & 0 \\ 0 & \mathcal{M}_2 \end{pmatrix}, \quad \mathcal{M}_3 = c\Phi^{-2/5}1_3, \quad \mathcal{M}_2 = \Phi^{-2/5}\begin{pmatrix} \phi & -\phi B \\ -\phi B & \Phi + B^2\phi \end{pmatrix},$$

with $\Phi = c^3\phi$. Eq. (84) shows that (unlike in the case $D = 4$, see ref. [3]) the property “FRW universe” is not invariant under the full U–duality group $SL(5)$. In particular, FRW universes can be mapped into anisotropic solutions and vice versa using appropriate $SL(5)$ transformations.
we wish to stay within the class of FRW solutions, we should restrict ourselves to the subgroup
\( H \equiv SL(3) \times SL(2) \times U(1) \subset SL(5) \) which leaves the structure of \( \mathcal{M} \) in eq. (84) invariant. Explicitly, this subgroup acts as
\[
\mathcal{M}_{2,3} \to P_{2,3} \mathcal{M}_{2,3} P_{2,3}^T
\]
with \( P_2 \in GL(2) \), \( P_3 \in GL(3) \) and \( \det(P_2) \det(P_3) = 1 \). Let us consider equivalence classes of solutions with respect to this subgroup \( H \) instead of the full group \( SL(5) \). Then FRW universes are specified by
\[
p \equiv p_1 = p_2 = p_3
\]
in eq. (81). From eq. (82) we derive
\[
p_{4,5} = \frac{3p}{2} \pm \sqrt{\frac{5}{3} - \frac{15}{4} p^2}, \quad |p| \leq \frac{2}{3}.
\]
We have therefore found a one parameter set (with parameter \( p \)) of \( H \)-inequivalent classes of FRW universes, each equivalence class for a fixed \( p \) spanned by the action of the group \( H \) in eq. (83). How does \( H \) act explicitly? First of all, \( GL(3) \subset H \) is again part of the global coordinate transformations and therefore trivial. We concentrate on the \( SL(2) \) part and write
\[
P_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha \delta - \beta \gamma = 1.
\]
Then, \( \mathcal{M}_2, \mathcal{M}_3 \) take the form
\[
\mathcal{M}_2 = \begin{pmatrix} \alpha^2 |\tau|^{p_4} + \beta^2 |\tau|^{p_5} & \alpha \gamma |\tau|^{p_4} + \beta \delta |\tau|^{p_5} \\ \alpha \gamma |\tau|^{p_4} + \beta \delta |\tau|^{p_5} & \gamma^2 |\tau|^{p_4} + \delta^2 |\tau|^{p_5} \end{pmatrix}, \quad \mathcal{M}_3 = |\tau|^{p_1},
\]
where \( p_{4,5} \) are given by (85) in terms of the free parameter \( p \). By comparison with eq. (84) we can read off the expressions for \( c, \Phi, B, \phi \) and convert them to the physical fields via eq. (84). The result is
\[
\begin{align*}
\bar{b} &= b_0 (\alpha^2 |\tau|^{p_4} + \beta^2 |\tau|^{p_5})^{1/2} |\tau|^{p_4 + 1/6} |\tau|^{p_5 + 1/6} \\
\bar{N} &= b_0^2 (\alpha^2 |\tau|^{p_4} + \beta^2 |\tau|^{p_5})^{1/2} |\tau|^{3p_4 + 1/3} \\
\bar{G}_{mn} &= a^2 \delta_{mn}, \quad a = b_0 (\alpha^2 |\tau|^{p_4} + \beta^2 |\tau|^{p_5})^{1/2} |\tau|^{p_4 + 1/3} \\
B &= -\frac{a \gamma |\tau|^{p_4} + \beta \delta |\tau|^{p_5}}{a^2 |\tau|^{p_4} + \beta^2 |\tau|^{p_5}}.
\end{align*}
\]
We can solve for the comoving time \( t \) in two asymptotic regions leading to
\[
\begin{align*}
t &= \frac{2b_0^2 |\alpha|}{3p + p_4 + 8/3} |\tau|^{3p/2 + p_4 + 4/3} \text{sgn}(\tau), \quad \text{for } |\tau| \gg \tau_{\text{form}} \\
t &= \frac{b_0^2 |\beta|}{3p + p_5 + 8/3} \frac{2}{3} |\tau|^{3p/2 + p_5 + 4/3} \text{sgn}(\tau), \quad \text{for } |\tau| \ll \tau_{\text{form}}.
\end{align*}
\]
where
\[ \tau_{\text{form}} = \left( \frac{\beta}{\alpha} \right)^{\frac{2}{p_4-p_5}}. \] (92)

In these regions, we find for the Hubble parameter \( H_a \)
\[ H_a = \frac{P(p)}{t}, \quad P(p) = \begin{cases} 
\frac{2}{3} \frac{3p + p_4 + 1}{3p + p_4 + 8/3} & \text{for } |\tau| \gg \tau_{\text{form}} \\
\frac{2}{3} \frac{3p + p_5 + 1}{3p + p_5 + 8/3} & \text{for } |\tau| \ll \tau_{\text{form}} 
\end{cases}. \] (93)

The expansion coefficient \( P(p) \) depends on the free parameter \( p \) and is generically different in the two asymptotic regions. As can be seen from fig. 1, it is always positive for large \( |\tau| \gg \tau_{\text{form}} \) and can have both signs for \( |\tau| \ll \tau_{\text{form}} \). For the positive branch \( t > 0 \) this implies a universe which is expanding or contracting at early time and is turned into an expanding universe at late time. The situation for the negative branch is reversed; the universe is always contracting at early time (\( |t| \) large and \( t < 0 \)) and can be contracting or expanding later. As before, the Hubble parameter (93) and hence the aforementioned properties are \( SL(2) \) invariant. The transition time given by
\[ |\tau| \sim \tau_{\text{form}} = \left( \frac{\beta}{\alpha} \right)^{\frac{2}{p_4-p_5}}, \] (94)
on the other hand, depends on \( SL(2) \) parameters along with the details of the transition.

Fig 1: Expansion coefficient \( P(p) \) for \( |\tau| \gg \tau_{\text{form}} \) (solid curve) and \( |\tau| \ll \tau_{\text{form}} \) (dashed curve).

Suppose now, we choose a solution in the positive branch which is contracting for a short period of time and then turns into expansion. By applying appropriate \( SL(2) \) transformations to this solution the contraction period can be made arbitrarily long. An additional time reversal \( t \to -t \) leads to a negative branch solution with an expansion period that can be arbitrarily long. The extreme limits are possible. By choosing \( P_2 = 1 \) (\( \beta = 0 \) in particular) we have a positive
branch solution which is always expanding. As the other extreme we may set

\[ P_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

(\(\alpha = 0\) in particular) which generates an expanding negative time branch solution. We have therefore shown that a combination of U–duality and time reversal can map expanding negative and positive time branch solutions into each other. The expansion coefficients in the negative and positive branch then correspond to the lower and upper part of the curve in fig. 1. An analog mapping, carried out by a T–duality transformation combined with a time reversal, is the starting point of the pre–big–bang scenario \cite{32} of weakly coupled heterotic string cosmology.

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