Nonholonomic Clifford Structures and
Noncommutative Riemann–Finsler Geometry

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July 19, 2018

Abstract

We survey the geometry of Lagrange and Finsler spaces and discuss
the issues related to the definition of curvature of nonholonomic mani-
folds enabled with nonlinear connection structure. It is proved that
any commutative Riemannian geometry (in general, any Riemann–
Cartan space) defined by a generic off–diagonal metric structure (with
an additional affine connection possessing nontrivial torsion) is equi-
valent to a generalized Lagrange, or Finsler, geometry modeled on non-
holonomic manifolds. This results in the problem of constructing non-
commutative geometries with local anisotropy, in particular, related to
geometrization of classical and quantum mechanical and field theories,
even if we restrict our considerations only to commutative and non-
commutative Riemannian spaces. We elaborate a geometric approach
to the Clifford modules adapted to nonlinear connections, to the the-
ory of spinors and the Dirac operators on nonholonomic spaces and
consider possible generalizations to noncommutative geometry. We
argue that any commutative Riemann–Finsler geometry and general-
izations may be derived from noncommutative geometry by applying

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certain methods elaborated for Riemannian spaces but extended to nonholonomic frame transforms and manifolds provided with nonlinear connection structure.

**AMS Subject Classification:**
46L87, 51P05, 53B20, 53B40, 70G45, 83C65

**Keywords:** Noncommutative geometry, Lagrange and Finsler geometry, nonlinear connection, nonholonomic manifolds, Clifford modules, spinors, Dirac operator, off–diagonal metrics and gravity.

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1 Introduction

The goal of this work is to provide a better understanding of the relationship between the theory of nonholonomic manifolds with associated nonlinear connection structure, locally anisotropic spin configurations and Dirac operators on such manifolds and noncommutative Riemann–Finsler and Lagrange geometry. The latter approach is based on geometrical modelling of mechanical and classical field theories (defined, for simplicity, by regular Lagrangians in analytic mechanics and Finsler like anisotropic structures) and gravitational, gauge and spinor field interactions in low energy limits of string theory. This allows to apply the Serre–Swan theorem and think of vector bundles as projective modules, which, for our purposes, are provided with nonlinear connection (in brief, N–connection) structure and can be defined as a nonintegrable (nonholonomic) distribution into conventional horizontal and vertical submodules. We relay on the theory of Clifford and spinor structures adapted to N–connections which results in locally anisotropic (Finsler like, or more general ones defined by more general nonholonomic frame structures) Dirac operators. In the former item, it is the machinery of noncommutative geometry to derive distance formulas and to consider noncommutative extensions of Riemann–Finsler and Lagrange geometry and related off–diagonal metrics in gravity theories.

In [76] it was proposed that an equivalent reformulation of the general relativity theory as a gauge model with nonlinear realizations of the affine, Poincare and/or de Sitter groups allows a standard extension of gravity theories in the language of noncommutative gauge fields. The approach was developed in [77] as an attempt to generalize the A. Connes’ noncommutative geometry [17] to spaces with generic local anisotropy. The nonlinear connection formalism was elaborated for projective module spaces and the Dirac operator associated to metrics in Finsler geometry and some generalizations [69, 72] (such as Sasaki type lifts of metrics to the tangent bundles and vector bundle analogs) were considered as certain examples of noncommutative Finsler geometry. The constructions were synthesized and revised in connection to ideas about appearance of both noncommutative and Finsler geometry in string theory with nonvanishing B–field and/or anholonomic (super) frame structures [66, 18, 11, 65, 78, 70, 73] and in supergravity and gauge gravity [8, 30, 89, 90, 22]. In particular, one has considered hidden noncommutative and Finsler like structures in general relativity and extra
In this work, we confine ourselves to the classical aspects of Lagrange–Finsler geometry (sprais, nonlinear connections, metric and linear connection structures and almost complex structure derived from from a Lagrange or Finsler fundamental form) in order to generalize the doctrine of the ”spectral action” and the theory of distance in noncommutative geometry which is an extension of the previous results [17]. For a complete information on modern noncommutative geometry and physics, we refer the reader to [39, 41, 29, 20, 38], see a historical sketch in Ref. [31] as well the aspects related to quantum group theory [47, 45, 32] (here we note that the first quantum group Finsler structure was considered in [92]). The theory of Dirac operators and related Clifford analysis is a subject of various investigations in modern mathematics and mathematical physics [48, 49, 59, 61, 26, 11, 12, 14, 15, 67, 9] (see also a relation to Finsler geometry [91] and an off–diagonal ”non” Kaluza–Klein compactified ansatz, but without N–connection constructions [13]).

For an exposition spelling out all the details of proofs and important concepts preliminary undertaken on the subjects elaborated in our works, we refer to proofs and quotations in Refs. [56, 57, 80, 83, 82, 77, 78, 29, 63, 43, 50].

This paper consists of two heterogeneous parts:

The first (commutative) contains an overview of the Lagrange and Finsler geometry and the off–diagonal metric and nonholonomic frame geometry in gravity theories. In Section 2, we formulate the N–connection geometry for arbitrary manifolds with tangent bundles admitting splitting into conventional horizontal and vertical subspaces. We illustrate how regular Lagrangians induce natural semispray, N–connection, metric and almost complex structures on tangent bundles and discuss the relation between Lagrange and Finsler geometry and theirs generalizations. We formulate six most important Results 2.1–2.6 demonstrating that the geometrization of Lagrange mechanics and the geometric models in gravity theories with generic off–diagonal metrics and nonholonomic frame structures are rigorously described by certain generalized Finsler geometries, which can be modeled equivalently both on Riemannian manifold and Riemann–Cartan nonholonomic manifolds. This give rise to the Conclusion 2.1 stating that a rigorous geometric study of nonholonomic frame and related metric, linear connection and spin structures in both commutative and noncommutative Riemann geometries requests the elaboration of noncommutative Lagrange–Finsler geometry. Then, in Section 3, we consider the theory of linear connections on N–anholonomic

\[1\] The theory of N–connections should not be confused with nonlinear gauge theories and nonlinear relaxations of gauge groups.
manifolds (i.e. on manifolds with nonholonomic structure defined by $N$-connections). We construct in explicit form the curvature tensor of such spaces and define the Einstein equations for $N$-adapted linear connection and metric structures.

The second (noncommutative) part starts with Section 4 where we define noncommutative $N$-anholonomic spaces. We consider the example of noncommutative gauge theories adapted to the $N$-connection structure. Section 5 is devoted to the geometry of nonholonomic Clifford–Lagrange structures. We define the Clifford–Lagrange modules and Clifford $N$-anholonomic bundles being induced by the Lagrange quadratic form and adapted to the corresponding $N$-connection. Then we prove the Main Result 1, of this work, stating that any regular Lagrangian and/or $N$-connection structure define naturally the fundamental geometric objects and structures (such as the Clifford–Lagrange module and Clifford $d$-modules, the Lagrange spin structure and $d$-spinors) for the corresponding Lagrange spin manifold and/or $N$-anholonomic spinor ($d$-spinor) manifold. We conclude that the Lagrange mechanics and off–diagonal gravitational interactions (in general, with nontrivial torsion and nonholonomic constraints) can be adequately geometrized as certain Lagrange spin ($N$-anholonomic) manifolds.

In Section 6, we link up the theory of Dirac operators to nonholonomic structures and spectral triples. We prove that there is a canonical spin $d$-connection on the $N$-anholonomic manifolds generalizing that induced by the Levi–Civita to the naturally ones induced by regular Lagrangians and off–diagonal metrics. We define the Dirac $d$-operator and the Dirac–Lagrange operator and formulate the Main Result 2 (Theorem 6.1) arguing that such $N$-adapted operators can be induced canonically by almost Hermitian spin operators. The concept of distinguished spin triple is introduced in order to adapt the constructions to the $N$-connection structure. Finally, the Main Result 3, Theorem 6.1 is devoted to the definition, main properties and computation of distance in noncommutative spaces defined by $N$-anholonomic spin varieties. In these lecture notes, we only sketch in brief the ideas of proofs of the Main Results: the details will be published in our further works.

2 Lagrange–Finsler Geometry and Nonholonomic Manifolds

This section presents some basic facts from the geometry of nonholonomic manifolds provided with nonlinear connection structure [69, 53, 54, 19, 55].
The constructions and methods are inspired from the Lagrange–Finsler geometry and generalizations \[27, 10, 64, 56, 3, 52, 6, 57, 71, 73, 4, 58, 94\] and gravity models on metric–affine spaces provided with generic off–diagonal metric, nonholonomic frame and affine connection structures \[74, 87, 85, 81, 80, 83\] (such spaces, in general, possess nontrivial torsion and nonmetricity).

### 2.1 Preliminaries: Lagrange–Finsler metrics

Let us consider a nondegenerate bilinear symmetric form \(q(u, v)\) on a \(n\)-dimensional real vector space \(V^n\). With respect to a basis \(\{e_i\}_{i=1}^n\) for \(V^n\), we express

\[
q(u, v) = q_{ij} u^i v^j
\]

for any vectors \(u = u^i e_i, \ v = v^i e_i \in V^n\) and \(q_{ij}\) being a nondegenerate symmetric matrix (the Einstein’s convention on summing on repeating indices is adopted). This gives rise to the Euclidean inner product

\[
u | v \doteq q_E(u, v),
\]

if \(q_{ij}\) is positive definite, and to the Euclidean norm

\[
| \cdot | \doteq \sqrt{q_E(u, u)}
\]

defining an Euclidean space \((V^n, | \cdot |)\). Every Euclidean space is linearly isometric to the standard Euclidean space \(\mathbb{R}^n = (R^n, | \cdot |)\) if \(q_{ij} = \text{diag}(1, 1, ..., 1)\) with standard Euclidean norm, \(| y | \doteq \sqrt{\sum_{i=1}^n |y_i|^2}\), for any \(y = (y^i) \in R^n\), where \(R^n\) denotes the \(n\)-dimensional canonical real vector space.

There are also different types of quadratic forms/norms then the Euclidean one:

**Definition 2.1** A Lagrange fundamental form \(q_L(u, v)\) on vector space \(V^n\) is defined by a Lagrange functional \(L: V^n \to \mathbb{R}\), with

\[
q_L(y)(u, v) \doteq \frac{1}{2} \frac{\partial^2}{\partial s \partial t} [L(y + su + tu)]_{s=t=0}
\]

which is a \(C^\infty\)–function on \(V^n \setminus \{0\}\) and nondegenerate for any nonzero vector \(y \in V^n\) and real parameters \(s\) and \(t\).

Having taken a basis \(\{e_i\}_{i=1}^n\) for \(V^n\), we transform \(L = L(y^i e_i)\) in a function of \((y^i) \in R^n\).

The Lagrange norm is \(| \cdot |_L \doteq \sqrt{q_L(u, u)}\).
Definition 2.2 A Minkowski space is a pair \((V^n, F)\) where the Minkowski functional \(F\) is a positively homogeneous of degree two Lagrange functional with the fundamental form \([1]\) defined for \(L = F^2\) satisfying \(F(\lambda y) = \lambda F(y)\) for any \(\lambda > 0\) and \(y \in V^n\).

The Minkowski norm is defined by \(|\cdot|_F = \sqrt{q_F(u,u)}\).

Definition 2.3 The Lagrange (or Minkowski) metric fundamental function is defined
\[
g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j}(y) \tag{2}
\]

(or)
\[
g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}(y) \tag{3}
\]

Remark 2.1 If \(L\) is a Lagrange functional on \(R^n\) (it could be also any functional of class \(C^\infty\)) with local coordinates \((y^2, y^3, ..., y^n)\), it also defines a singular Minkowski functional
\[
F(y) = [y^1L(y^2, ..., y^n)]^2 \tag{4}
\]

which is of class \(C^\infty\) on \(R^n \setminus \{y^1 = 0\}\).

The Remark 2.1 states that the Lagrange functionals are not essentially more general than the Minkowski functionals \([94]\). Nevertheless, we must introduce more general functionals if we extend our considerations in relativistic optics, string models of gravity and the theory of locally anisotropic stochastic and/or kinetic processes \([57, 74, 87, 85, 81, 75]\).

Let us consider a base manifold \(M\), \(dimM = n\), and its tangent bundle \((TM, \pi, M)\) with natural surjective projection \(\pi : TM \rightarrow M\). From now on, all manifolds and geometric objects are supposed to be of class \(C^\infty\). We write \(\tilde{TM} = TM \setminus \{0\}\) where \(\{0\}\) means the null section of the map \(\pi\).

A differentiable Lagrangian \(L(x, y)\) is defined by a map \(L : (x, y) \in TM \rightarrow L(x, y) \in \mathbb{R}\) of class \(C^\infty\) on \(TM\) and continous on the null section \(0 : M \rightarrow TM\) of \(\pi\). For any point \(x \in M\), the restriction \(L_x = L|_{T_x M}\) is a Lagrange functional on \(T_x M\) (see Definition 2.3). For simplicity, in this work we shall consider only regular Lagrangians with nondegenerated Hessians,
\[
(L)g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L(x, y)}{\partial y^i \partial y^j} \tag{5}
\]
when \( \text{rank} |g_{ij}| = n \) on \( \tilde{T}M \), which is a Lagrange fundamental quadratic form \((2)\) on \( T_xM \). In our further considerations, we shall write \( M(L) \) if would be necessary to emphasize that the manifold \( M \) is provided in any its points with a quadratic form \((3)\).

**Definition 2.4** A Lagrange space is a pair \( L^n = [M, L(x, y)] \) with the metric form \((b)g_{ij}(x, y)\) being of constant signature over \( \tilde{T}M \).

**Definition 2.5** A Finsler space is a pair \( F^n = [M, F(x, y)] \) where \( F_x(y) \) defines a Minkowski space with metric fundamental function of type \((3)\).

The notion of Lagrange space was introduced by J. Kern \([37]\) and elaborated in details by the R. Miron’s school on Finsler and Lagrange geometry, see Refs. \([56, 57]\), as a natural extension of Finsler geometry \([27, 10, 64, 3, 52, 6, 41, 94]\) (see also Refs. \([71, 73]\), on Lagrange–Finsler supergeometry, and Refs. \([76, 77, 78]\), on some examples of noncommutative locally anisotropic gravity and string theory).

### 2.2 Nonlinear connection geometry

We consider two important examples when the nonlinear connection (in brief, \( N \)-connection) is naturally defined in Lagrange mechanics and in gravity theories with generic off-diagonal metrics and nonholonomic frames.

#### 2.2.1 Geometrization of mechanics: some important results

The Lagrange mechanics was geometrized by using methods of Finsler geometry \([56, 57]\) on tangent bundles enabled with a corresponding nonholonomic structure (nonintegrable distribution) defining a canonical \( N \)-connection. By straightforward calculations, one proved the results:

**Result 2.1** The Euler–Lagrange equations

\[
\frac{d}{d\tau} \left( \frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} = 0
\]  

\( ^2 \)We cite a recent review \([41]\) on alternative approaches to geometric mechanics and geometry of classical fields related to investigation of the geometric properties of Euler–Lagrange equations for various type of nonholonomic, singular or higher order systems. In the approach developed by R. Miron’s school \([56, 57, 58]\), the nonlinear connection and fundamental geometric structures are derived in general form from the Lagrangian and/or Hamiltonian: the basic geometric constructions are not related to the particular properties of certain systems of partial differential equations, symmetries and constraints of mechanical and field models.
where $y^i = \frac{dx^i}{d\tau}$ for $x^i(\tau)$ depending on parameter $\tau$, are equivalent to the "nonlinear" geodesic equations

$$\frac{d^2 x^i}{d\tau^2} + 2G^i_k(\frac{dx^j}{d\tau}) = 0 \quad (7)$$

where

$$2G^i(x,y) = \frac{1}{2} (L)^{ij} \left( \frac{\partial^2 L}{\partial y^i \partial x^j} y^k - \frac{\partial L}{\partial x^i} \right) \quad (8)$$

with $(L)^{ij}$ being inverse to $(L)$. 

**Result 2.2** The coefficients $G^i(x,y)$ from (8) define the solutions of both type of equations (6) and (7) as paths of the canonical semispray

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y) \frac{\partial}{\partial y^i}$$

and a canonical $N$–connection structure on $\tilde{T}M$,

$$(L)N^i_j = \frac{\partial G^i(x,y)}{\partial y^i} \quad (9)$$

induced by the fundamental Lagrange function $L(x,y)$ (see Section 2.3 on exact definitions and main properties).

**Result 2.3** The coefficients $(L)N^i_j$ defined by a Lagrange (Finsler) fundamental function induce a global splitting on $TTM$, a Whitney sum,

$$TTM = hTM \oplus vTM$$

as a nonintegrable distribution (nonholonomic, or equivalently, anholonomic structure) into horizontal ($h$) and vertical ($v$) subspaces parametrized locally by frames (vielbeins) $e_v = (e_i, e_a)$, where

$$e_i = \frac{\partial}{\partial x^i} - N^a_i(u) \frac{\partial}{\partial y^a} \quad \text{and} \quad e_a = \frac{\partial}{\partial y^a} \quad (10)$$

and the dual frames (coframes) $\vartheta^\mu = (\vartheta^i, \vartheta^a)$, where

$$\vartheta^i = dx^i \quad \text{and} \quad \vartheta^a = dy^a + N^a_i(u) dx^i \quad (11)$$

The vielbeins (10) and (11) are called $N$–adapted (co) frames. We omitted the label $(L)$ and used vertical indices $a, b, c, ...$ for the $N$–connection coefficients in order to be able to use the formulas for arbitrary $N$–connections).
We also note that we shall use 'boldfaced' symbols for the geometric objects and spaces adapted/enabled to N–connection structure. For instance, we shall write in brief \( e = (e, *e) \) and \( \vartheta = (\vartheta, *\vartheta) \), respectively, for

\[
e_\nu = (e_i, *e_k) = (e_i, e_a) \quad \text{and} \quad \vartheta^\mu = (\vartheta^i, *\vartheta^k) = (\vartheta^i, \vartheta^a).
\]

The vielbeins \((10)\) satisfy the nonholonomy relations

\[
[e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha = W^\gamma_{\alpha\beta} e_\gamma
\]

with (antisymmetric) nontrivial anholonomy coefficients \( W^b_{ia} = \frac{\partial}{\partial x^a} N^i_b \) and \( W^a_{ji} = \Omega^a_{ji} \) where

\[
\Omega^a_{ij} = \delta^a_{[j} N^i_{]b} = \frac{\partial N^a_i}{\partial x^j} - \frac{\partial N^a_j}{\partial x^i} + N^b_i \frac{\partial N^a_j}{\partial y^b} - N^b_j \frac{\partial N^a_i}{\partial y^b}.
\]

In order to preserve a relation with our previous denotations \([74, 69, 73]\), we note that \( e_\nu = (e_i, e_a) \) and \( \vartheta^\mu = (\vartheta^i, \vartheta^a) \) are, respectively, the former \( \delta_\nu = \frac{\partial}{\partial u^\nu} = (\delta_i, \partial^a) \) and \( \delta^\mu = \delta u^\mu = (dx^i, \delta y^a) \) which emphasize that the operators \((10)\) and \((11)\) define, correspondingly, certain 'N–elongated' partial derivatives and differentials which are more convenient for calculations on spaces provided with nonholonomic structure.

**Result 2.4** On \( \tilde{TM} \), there is a canonical metric structure \((L) g = [g, *g]\),

\[
(L) g = (L) g_{ij}(x, y) \vartheta^i \otimes \vartheta^j + (L) g_{ij}(x, y) *\vartheta^i \otimes *\vartheta^j
\]

constructed as a Sasaki type lift from \( M \).\(^3\)

We note that a complete geometrical model of Lagrange mechanics or a well defined Finsler geometry can be elaborated only by additional assumptions about a linear connection structure, which can be adapted, or not, to a defined N–connection (see Section 3.1).

**Result 2.5** The canonical N–connection \((9)\) induces naturally an almost complex structure \( F : \chi(\tilde{TM}) \rightarrow \chi(\tilde{TM}) \), where \( \chi \) denotes the module of vector fields on \( \tilde{TM} \);

\[
F(e_i) = *e_i \quad \text{and} \quad F(*e_i) = -e_i,
\]

\(^3\)In Refs. \([94, 58]\), it was suggested to use lifts with h- and v–components of type \((L) g = (g_{ij}, g_{ij} a / \parallel y \parallel) \) where \( a = \text{const} \) and \( \parallel y \parallel = g_{ij} y^i y^j \) in order to elaborate more physical extensions of the general relativity to the tangent bundles of manifolds. In another turn, such modifications are not necessary if we model Lagrange–Finsler structures by exact solutions with generic off–diagonal metrics in Einstein and/or gravity \([74, 87, 85, 81, 83, 82, 75]\). For simplicity, in this work, we consider only lifts of metrics of type \((14)\).
when
\[ F = \star e_i \otimes \psi^i - e_i \otimes \star \psi^i \] (15)
satisfies the condition \( F \mid F = -I \), i.e. \( F_{\alpha}^\beta F_{\beta}^\gamma = -\delta_{\gamma}^\alpha \), where \( \delta_{\gamma}^\alpha \) is the Kronecker symbol and "\( \mid \)" denotes the interior product.

The last result is important for elaborating an approach to geometric quantization of mechanical systems modeled on nonholonomic manifolds [25] as well for definition of almost complex structures derived from the real N-connection geometry related to nonholonomic (anisotropic) Clifford structures and spinors in commutative [69, 72, 86, 88, 84] and noncommutative spaces [76, 77, 78].

2.2.2 N-connections in gravity theories

For nonholonomic geometric models of gravity and string theories, one does not consider the bundle \( \tilde{TM} \) but a general manifold \( V \), \( dim V = n + m \), which is a (pseudo) Riemannian space or a certain generalization with possible torsion and nonmetricity fields. A metric structure is defined on \( V \), with the coefficients stated with respect to a local coordinate basis \( du^\alpha = (dx^i, dy^a) \), where

\[ g = g_{\alpha\beta}(u)du^\alpha \otimes du^\beta \]

A metric, for instance, parametrized in the form (16), is generic off-diagonal if it cannot be diagonalized by any coordinate transforms. Performing a frame transform with the coefficients

\[ e_{\alpha}^\alpha(u) = \begin{bmatrix} e_1 \frac{1}{2}(u) & N^b_i(u)e_b \frac{1}{2}(u) & N^e_i(u)h_{ae} \ N^e_i(u)h_{ae} & e_a^i(u) \ N^e_i(u)h_{ae} \ h_{ab} \end{bmatrix}, \] (17)

\[ e_{\beta}^\beta(u) = \begin{bmatrix} e_1 \frac{1}{2}(u) & -N_k^b(u)e_k \frac{1}{2}(u) & e_k \frac{1}{2}(u) \ N_k^b(u)e_k \frac{1}{2}(u) & e_a \frac{1}{2}(u) \ 0 \ e_a \frac{1}{2}(u) \ 0 \ e_a \frac{1}{2}(u) \end{bmatrix}, \] (18)

we write equivalently the metric in the form

\[ g = g_{\alpha\beta}(u) \varphi^\alpha \otimes \varphi^\beta = g_{ij}(u) \varphi^i \otimes \varphi^j + h_{ab}(u) \varphi^a \otimes \varphi^b, \] (19)

where \( g_{ij} \equiv \sum_i \varphi(i, e_j) \) and \( h_{ab} \equiv \sum \varphi(e_a, e_b) \) and

\[ e_\alpha = e_\alpha^\alpha \partial_\alpha \text{ and } \varphi^\beta = e_\beta^\beta du^\beta. \]

\[ ^4\text{the indices run correspondingly the values } i, j, k, ... = 1, 2, ..., n \text{ and } a, b, c, ... = n + 1, n + 2, ..., n + m. \]
are vielbeins of type (10) and (11) defined for arbitrary $N^b_i(u)$. We can consider a special class of manifolds provided with a global splitting into conventional "horizontal" and "vertical" subspaces (20) induced by the "off-diagonal" terms $N^b_i(u)$ and prescribed type of nonholonomic frame structure.

If the manifold $V$ is (pseudo) Riemannian, there is a unique linear connection (the Levi–Civita connection) $\nabla$, which is metric, $\nabla g = 0$, and torsionless, $\nabla T = 0$. Nevertheless, the connection $\nabla$ is not adapted to the nonintegrable distribution induced by $N^b_i(u)$. In this case, for instance, in order to construct exact solutions parametrized by generic off–diagonal metrics, or for investigating nonholonomic frame structures in gravity models with nontrivial torsion, it is more convenient to work with more general classes of linear connections which are $N$–adapted but contain nontrivial torsion coefficients because of nontrivial nonholonomy coefficients $W^\gamma_{\alpha\beta} (12)$.

For a splitting of a (pseudo) Riemannian–Cartan space of dimension $(n+m)$ (under certain constraints, we can consider (pseudo) Riemannian configurations), the Lagrange and Finsler type geometries were modeled by $N$–anholonomic structures as exact solutions of gravitational field equations [74, 87, 85, 81], see also Refs. [83, 82] for exact solutions with nonmetricity. One holds [80] the

**Result 2.6** The geometry of any Riemannian space of dimension $n+m$ where $n,m \geq 2$ (we can consider $n,m = 1$ as special degenerated cases), provided with off–diagonal metric structure of type (14) can be equivalently modeled, by vielbein transforms of type (17) and (18) as a geometry of nonholonomic manifold enabled with $N$–connection structure $N^b_i(u)$ and 'more diagonalized' metric (19).

For particular cases, we present the

**Remark 2.2** For certain special conditions when $n = m$, $N^b_i = (L)N^b_i$, and the metric (19) is of type (14), a such Riemann space of even dimension is 'nonholonomically' equivalent to a Lagrange space (for the corresponding homogeneity conditions, see Definition 2.2, one obtains the equivalence to a Finsler space).

Roughly speaking, by prescribing corresponding nonholonomic frame structures, we can model a Lagrange, or Finsler, geometry on a Riemannian manifold and, inversely, a Riemannian geometry is 'not only a Riemannian one’ but also could be a generalized Finsler one. It is possible to define similar constructions for the (pseudo) Riemannian spaces. This is a quite surprising result if to compare it with the "superficial" interpretation of the Finsler geometry as a nonlinear extension, 'more sophisticate' on the tangent bundle, of the Riemannian geometry.
It is known the fact that the first example of Finsler geometry was considered in 1854 in the famous B. Riemann’s hability thesis (see historical details and discussion in Refs. [94, 4, 57, 80]) who, for simplicity, restricted his considerations only to the curvatures defined by quadratic forms on hypersurfaces. Sure, for B. Riemann, it was unknown the fact that if we consider general (nonholonomic) frames with associated nonlinear connections (the E. Cartan’s geometry, see Refs. in [10]) and off–diagonal metrics, the Finsler geometry may be derived naturally even from quadratic metric forms being adapted to the N–connection structure.

More rigorous geometric constructions involving the Cartan–Miron metric connections and, respectively, the Berwald and Chern–Rund nonmetric connections in Finsler geometry and generalizations, see more details in subsection 3.1 result in equivalence theorems to certain types of Riemann–Cartan nonholonomic manifolds (with nontrivial N–connection and torsion) and metric–affine nonholonomic manifolds (with additional nontrivial nonmetricity structures) [80].

This Result 2.6 give rise to an important:

**Conclusion 2.1** To study generalized Finsler spinor and noncommutative geometries is necessary even if we restrict our considerations only to (non) commutative Riemannian geometries.

For simplicity, in this work we restrict our considerations only to certain Riemannian commutative and noncommutative geometries when the N–connection and torsion are defined by corresponding nonholonomic frames.

### 2.3 N–anholonomic manifolds

Now we shall demonstrate how general N–connection structures define a certain class of nonholonomic geometries. In this case, it is convenient to work on a general manifold \( V \), \( \dim V = n + m \), with global splitting, instead of the tangent bundle \( TM \). The constructions will contain those from geometric mechanics and gravity theories, as certain particular cases.

Let \( V \) be a \((n + m)\)-dimensional manifold. It is supposed that in any point \( u \in V \) there is a local distribution (splitting) \( V_u = M_u \oplus V_u \), where \( M \) is a \( n \)–dimensional subspace and \( V \) is a \( m \)–dimensional subspace. The local coordinates (in general, abstract ones both for holonomic and nonholonomic variables) may be written in the form \( u = (x, y) \), or \( u^\alpha = (x^i, y^a) \). We denote by \( \pi^\top : TV \to TM \) the differential of a map \( \pi : V^{n+m} \to V^n \) defined by fiber preserving morphisms of the tangent bundles \( TV \) and \( TM \). The kernel of \( \pi^\top \) is just the vertical subspace \( vV \) with a related inclusion mapping \( i : vV \to TV \).
Definition 2.6 A nonlinear connection (N–connection) \( N \) on a manifold \( V \) is defined by the splitting on the left of an exact sequence

\[
0 \to vV \to TV \to TV/vV \to 0,
\]

i. e. by a morphism of submanifolds \( N : TV \to vV \) such that \( N \circ i \) is the unity in \( vV \).

In an equivalent form, we can say that a N–connection is defined by a splitting to subspaces with a Whitney sum of conventional \( h \)-subspace, \((hV)\), and \( v \)-subspace, \((vV)\),

\[
TV = hV \oplus vV
\]

(20)

where \( hV \) is isomorphic to \( M \). This generalizes the splitting considered in Result 2.3.

Locally, a N–connection is defined by its coefficients \( N^a_i(u) \),

\[
N = N^a_i(u)dx^i \otimes \frac{\partial}{\partial y^a}.
\]

The well known class of linear connections consists a particular subclass with the coefficients being linear on \( y^a \), i. e. \( N^a_i(u) = \Gamma^a_{bj}(x)y^b \).

Any N–connection also defines a N–connection curvature

\[
\Omega = \frac{1}{2} \Omega^a_{ij}d^i \wedge d^j \otimes \partial_a,
\]

with N–connection curvature coefficients given by formula (12).

Definition 2.7 A manifold \( V \) is called N–anholonomic if on the tangent space \( TV \) it is defined a local (nonintegrable) distribution \((22)\), i. e. \( TV \) is enabled with a N–connection \((21)\) inducing a vielbein structure \((10)\) satisfying the nonholonomy relations \((12)\) (such N–connections and associated vielbeins may be general ones or any derived from a Lagrange/ Finsler fundamental function).

We note that the boldfaced symbols are used for the spaces and geometric objects provided/adapted to a N–connection structure. For instance, a vector field \( X \in TV \) is expressed \( X = (X \equiv -X, *X) \), or \( X = X^ae_a = X^ie_i + X^ae_a \), where \( X = -X = X^i e_i \) and \( *X = X^ae_a \) state, respectively, the irreducible (adapted to the N–connection structure) \( h \)- and \( v \)-components of the vector (which following Refs. 56 57 is called a distinguished vectors, in brief, d–vector). In a similar fashion, the geometric objects on \( V \) like
tensors, spinors, connections, ... are respectively defined and called d–tensors, d–spinors, d–connections if they are adapted to the N–connection splitting.\(^5\)

**Definition 2.8** A d–metric structure on N–anholonomic manifold \(V\) is defined by a symmetric d–tensor field of type \(g = [g, h]^{(19)}\).

For any fixed values of coordinates \(u = (x, y) \in V\) a d–metric it defines a symmetric quadratic d–metric form,

\[
q(x, y) = g_{ij}x^ix^j + h_{ab}y^ay^b, \tag{22}
\]

where the \(n + m\)-splitting is defined by the N–connection structure and \(x = x^i e_i + x^a e_a, y = y^i e_i + y^a e_a \in V^{n+m}\).

Any d–metric is parametrized by a generic off–diagonal matrix \(\begin{bmatrix} 16 \end{bmatrix}\) if the coefficients are redefined with respect to a local coordinate basis (for corresponding parametrizations of the the data \([g, h, N]\) such ansatz model a geometry of mechanics, or a Finsler like structure, in a Riemann–Cartan–Weyl space provided with N–connection structure \([80, 83]\); for certain constraints, there are possible models derived as exact solutions in Einstein gravity and noncommutative generalizations \([74, 81, 82]\)).

**Remark 2.3** There is a special case when \(\dim V = n + n, h_{ab} \to g_{ij}\) and \(N^a_i \to N^j_i\) in \((19)\), which models locally, on \(V\), a tangent bundle structure. We denote a such space by \(\tilde{V}_{(n,n)}\). If the N–connection and d–metric coefficients are just the canonical ones for the Lagrange (Finsler) geometry (see, respectively, formulas \((9)\) and \((14)\)), we model such locally anisotropic structures not on a tangent bundle \(TM\) but on a N–anholonomic manifold of dimension \(2n\).

We present some historical remarks on N–connections and related subjects: The geometrical aspects of the N–connection formalism has been studied since the first papers of E. Cartan \([10]\) and A. Kawaguchi \([35, 36]\) (who used it in component form for Finsler geometry). Then one should be mentioned the so called Ehresman connection \([23]\) and the work of W. Barthel \([5]\) where the global definition of N–connection was given. In monographs \([56, 57, 58]\), the N–connection formalism was elaborated in details and applied to the geometry of generalized Finsler–Lagrange and Cartan–Hamilton spaces, see also the approaches \([42, 40, 24]\). It should be noted that the works related to nonholonomic geometry and N–connections have appeared many

\(^5\)In order to emphasize h– and v–splitting of any d–objects \(Y, g, \ldots\) we shall write the irreducible components as \(Y = (\sim Y, \sim Y), g = (\sim g, \sim g)\) but we shall omit “–” or “∗” if the simplified denotations will not result in ambiguities.
times in a rather dispersive way when different schools of authors from geometry, mechanics and physics have worked many times not having relation with another. We cite only some our recent results with explicit applications in modern mathematical physics and particle and string theories: N–connection structures were modeled on Clifford and spinor bundles [69, 72, 88, 86], on superbundles and in some directions of (super) string theory [71, 73], as well in noncommutative geometry and gravity [76, 77, 78]. The idea to apply the N–connections formalism as a new geometric method of constructing exact solutions in gravity theories was suggested in Refs. [74, 75] and developed in a number of works, see for instance, Ref. [87, 85, 81]).

3 Curvature of N–anholonomic Manifolds

The geometry of nonholonomic manifolds has a long time history of yet unfinished elaboration: For instance, in the review [93] it is stated that it is probably impossible to construct an analog of the Riemannian tensor for the general nonholonomic manifold. In a more recent review [55], it is emphasized that in the past there were proposed well defined Riemannian tensors for a number of spaces provided with nonholonomic distributions, like Finsler and Lagrange spaces and various type of theirs higher order generalizations, i. e. for nonholonomic manifolds possessing corresponding N–connection structures. As some examples of first such investigations, we cite the works [54, 53, 19]. In this section we shall construct in explicit form the curvature tensor for the N–anholonomic manifolds.

3.1 Distinguished connections

On N–anholonomic manifolds, the geometric constructions can be adapted to the N–connection structure:

**Definition 3.1** A distinguished connection (d–connection) $D$ on a manifold $V$ is a linear connection conserving under parallelism the Whitney sum (20) defining a general N–connection.

The N–adapted components $\Gamma_{\alpha\gamma}^\beta$ of a d-connection $D = (\delta_{\alpha}|D)$ are defined by the equations $D_\alpha \delta_\beta = \Gamma_{\alpha\beta}^\gamma \delta_\gamma$, or

$$\Gamma_{\alpha\beta}^\gamma (u) = (D_{\alpha} \delta_\beta)|\delta_\gamma.$$  \hspace{1cm} (23)

In its turn, this defines a N–adapted splitting into h– and v–covariant derivatives, $D = D + \ast D$, where $D_k = (L^i_{jk}, L^a_{bk})$ and $\ast D_c = (C^c_{jk}, C^a_{bc})$ are
introduced as corresponding h- and v–parametrizations of (23),

\[ L^i_{jk} = (D_k e_j) \hat{\partial}^i, \quad L^a_{bk} = (D_k e_b) \hat{\partial}^a, \quad C^i_{jc} = (D_c e_j) \hat{\partial}^i, \quad C^a_{bc} = (D_c e_b) \hat{\partial}^a. \]

The components \( \Gamma^\gamma_{\alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}) \) completely define a d–connection \( \mathcal{D} \) on a N–anholonomic manifold \( V \).

The simplest way to perform computations with d–connections is to use N–adapted differential forms like \( \Gamma^\alpha_{\beta} = \Gamma^\alpha_{\beta\gamma} \hat{\partial}^\gamma \) with the coefficients defined with respect to N–elongate bases (11) and (10).

The torsion of d–connection \( \mathcal{D} \) is defined by the usual formula

\[ T(X, Y) = \mathcal{D}_X \mathcal{D}_Y - \mathcal{D}_Y \mathcal{D}_X - [X, Y]. \]

**Theorem 3.1** The torsion \( T^\alpha = \mathcal{D} \hat{\partial}^\alpha = d \hat{\partial}^\alpha + \Gamma^\alpha_{\beta} \wedge \hat{\partial}^\beta \) of a d–connection has the irreducible h– v– components (d–torsions) with N–adapted coefficients

\[
T^i_{jk} = L^i_{[jk]}, \quad T^i_{ja} = -T^i_{aj} = C^i_{ja}, \quad T^a_{ji} = \Omega^a_{ji}, \\
T^a_{bi} = T^a_{ib} = \frac{\partial N^a_i}{\partial y^b} - L^a_{bi}, \quad T^a_{bc} = C^a_{[bc]},
\]

(24)

**Proof.** By a straightforward calculation we can verify the formulas.

The Levi–Civita linear connection \( \nabla = \{\nabla_{\Gamma^\alpha_{\beta\gamma}}\} \), with vanishing both torsion and nonmetricity, is not adapted to the global splitting (20). One holds:

**Proposition 3.1** There is a preferred, canonical d–connection structure, \( \hat{\mathcal{D}} \), on N–anholonomic manifold \( V \) constructed only from the metric and N–connection coefficients \( [g_{ij}, h_{ab}, N^a_i] \) and satisfying the metricity conditions \( \hat{\mathcal{D}} g = 0 \) and \( \hat{T}^i_{jk} = 0 \) and \( \hat{T}^a_{bc} = 0 \).

**Proof.** By straightforward calculations with respect to the N–adapted bases (11) and (10), we can verify that the connection

\[ \hat{\Gamma}^\alpha_{\beta\gamma} = \nabla_{\Gamma^\alpha_{\beta\gamma}} + \hat{P}^\alpha_{\beta\gamma} \]

(25)

with the deformation d–tensor

\[ \hat{P}^a_{\beta\gamma} = (P^i_{jk} = 0, P^a_{bk} = \frac{\partial N^a_i}{\partial y^b}, P^i_{jc} = -\frac{1}{2} g^{ik} \Omega^a_{kj} h_{ca}, P^a_{bc} = 0) \]

satisfies the conditions of this Proposition. It should be noted that, in general, the components \( \hat{T}^i_{ja}, \hat{T}^a_{ji} \) and \( \hat{T}^a_{bi} \) are not zero. This is an anholonomic frame (or, equivalently, off–diagonal metric) effect.

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With respect to the N–adapted frames, the coefficients
\[ \hat{\Gamma}^\gamma_{\alpha \beta} = (\hat{L}^i_{jk}, \hat{L}^a_{bk}, \hat{C}^i_{jc}, \hat{C}^a_{bc}) \]
are computed:
\[ \hat{L}^i_{jk} = \frac{1}{2} g^{ir} \left( \frac{\delta g_{jr}}{\partial x^k} + \frac{\delta g_{kr}}{\partial x^j} - \frac{\delta g_{jk}}{\partial x^r} \right), \] (26)
\[ \hat{L}^a_{bk} = \frac{\partial N_k^a}{\partial y^b} + \frac{1}{2} h^{ac} \left( \frac{\delta h_{bc}}{\partial x^k} - \frac{\delta N_k^b}{\partial y^a} h_{dc} - \frac{\delta N_k^d}{\partial y^c} h_{db} \right), \]
\[ \hat{C}^i_{jc} = \frac{1}{2} g^{ij} \frac{\partial g_{jk}}{\partial y^c}, \]
\[ \hat{C}^a_{bc} = \frac{1}{2} h^{ad} \left( \frac{\partial h_{bd}}{\partial y^c} - \frac{\partial h_{cd}}{\partial y^b} - \frac{\partial h_{bc}}{\partial y^d} \right). \]

The d–connection (26) defines the 'most minimal' N–adapted extension of the Levi–Civita connection in order to preserve the metricity condition and to have zero torsions on the h– and v–subspaces (the rest of nonzero torsion coefficients are defined by the condition of compatibility with the N–connection splitting).

**Remark 3.1** The canonical d–connection \( \hat{\mathbf{D}} \) (27) for a local modelling of a \( \tilde{T}M \) space on \( \tilde{V}_{(n,n)} \) is defined by the coefficients \( \hat{\Gamma}^\gamma_{\alpha \beta} = (\hat{L}^i_{jk}, \hat{C}^i_{jk}) \) with
\[ \hat{L}^i_{jk} = \frac{1}{2} g^{ir} \left( \frac{\delta g_{jr}}{\partial x^k} + \frac{\delta g_{kr}}{\partial x^j} - \frac{\delta g_{jk}}{\partial x^r} \right), \]
\[ \hat{C}^i_{jk} = \frac{1}{2} g^{ir} \left( \frac{\partial g_{jr}}{\partial y^k} + \frac{\partial g_{kr}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^r} \right) \] (27)
computed with respect to N–adapted bases (10) and (11) when \( \hat{L}^i_{jk} \) and \( \hat{C}^i_{jk} \) define respectively the canonical h– and v–covariant derivations.

Various models of Finsler geometry and generalizations were elaborated by using different types of d–connections which satisfy, or not, the compatibility conditions with a fixed d–metric structure (for instance, with a Sasaki type one). Let us consider the main examples:

**Example 3.1** The Cartan’s d–connection (27) with the coefficients (27) was defined by some generalized Christoffel symbols with the aim to have a 'minimal' torsion and to preserve the metricity condition. This approach was developed for generalized Lagrange spaces and on vector bundles provided with N–connection structure (56, 57) by introducing the canonical d–connection (26). The direction emphasized metric compatible and N–adapted geometric constructions.

An alternative class of Finsler geometries is concluded in monographs [4, 94]:
Example 3.2 It was the idea of C. C. Chern \[10\] (latter also proposed by H. Rund \[63\]) to consider a d–connection \[\Gamma^\gamma_{\alpha\beta} = (\hat{L}^i_{jk}, C^i_{jk} = 0)\] and to work not on a tangent bundle \(TM\) but to try to ‘keep maximally’ the constructions on the base manifold \(M\). The Chern d–connection, as well the Berwald d–connection \[\Gamma^\gamma_{\alpha\beta} = (L^i_{jk} = \frac{\partial N^i_j}{\partial y^k}, C^i_{jk} = 0)\], are not subjected to the metricity conditions.

We note that the constructions mentioned in the last example define certain ‘nonmetric geometries’ (a Finsler modification of the Riemann–Cartan–Weyl spaces). For the Chern’s connection, the torsion vanishes but there is a nontrivial nonmetricity. A detailed study and classification of Finsler–affine spaces with general nontrivial N–connection, torsion and nonmetricity was recently performed in Refs. \[80, 83, 82\]. Here we also note that we may consider any linear connection can be generated by deformations of type

\[
\Gamma^\alpha_{\beta\gamma} = \hat{\Gamma}^\alpha_{\beta\gamma} + P^\alpha_{\beta\gamma},
\]

This splits all geometric objects into canonical and post-canonical pieces which results in N–adapted geometric constructions.

In order to define spinors on generalized Lagrange and Finsler spaces \[69, 72, 80, 88\] the canonical d–connection and Cartan’s d–connection were used because the metric compatibility allows the simplest definition of Clifford structures locally adapted to the N–connection. This is also the simplest way to define the Dirac operator for generalized Finsler spaces and to extend the constructions to noncommutative Finsler geometry \[76, 77, 78\]. The geometric constructions with general metric compatible affine connection (with torsion) are preferred in modern gravity and string theories. Nevertheless, the geometrical and physical models with generic nonmetricity also present certain interest \[28, 80, 83, 82\] (see also \[16\] where nonmetricity is considered to be important in quantum group co gravity). In such cases, we can use deformations of connection \[28\] in order to ‘deform’, for instance, the spinorial geometric constructions defined by the canonical d–connection and to transform them into certain ‘nonmetric’ configurations.

3.2 Curvature of d–connections

The curvature of a d–connection \(D\) on an N–anholonomic manifold is defined by the usual formula

\[
R(X, Y)Z \doteq D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z.
\]

By straightforward calculations we prove:
Theorem 3.2 The curvature $R^\alpha_{\beta\gamma\delta} \hat{=} D\Gamma^\alpha_\beta = d\Gamma^\alpha_\beta - \Gamma^\alpha_\gamma \wedge \Gamma^\gamma_\delta$ of a d–connection $\mathcal{D} = \Gamma^\alpha_\gamma$ has the irreducible h– v– components (d–curvatures) of $R^\alpha_{\beta\gamma\delta}$.

$$
\begin{align*}
R^i_{hjk} &= e_k L^i_{hj} - e_j L^i_{hk} + L^m_{hj} L^i_{mk} - L^m_{hk} L^i_{mj} - C^i_{h\alpha} \Omega^\alpha_{kj}, \\
R^a_{bjk} &= e_k L^a_{bj} + L^c_{bj} L^a_{ck} - L^c_{bk} L^a_{cj} - C^a_{bc} \Omega^c_{kj}, \\
R^i_{jka} &= e_a L^i_{jk} - D_k C^i_{ja} + C^i_{jb} T^b_{ka}, \\
R^c_{bka} &= e_a L^c_{bk} - D_k C^c_{ba} + C^c_{bd} T^c_{ka}, \\
R^i_{jbc} &= e_c C^i_{jb} - e_b C^i_{jc} + C^b_{jh} C^i_{hc} - C^j_{jc} C^i_{hb}, \\
R^a_{bcd} &= e_d C^a_{bc} - e_c C^a_{bd} + C^b_{bc} C^a_{ed} - C^e_{bd} C^a_{ec}.
\end{align*}
$$

(29)

Remark 3.2 For an N–anholonomic manifold $\tilde{\mathcal{V}}_{(n,n)}$ provided with N–symplectic canonical d–connection $\hat{\Gamma}^\tau_{\alpha\beta}$, see (27), the d–curvatures (29) reduces to three irreducible components

$$
\begin{align*}
R^i_{hjk} &= e_k L^i_{hj} - e_j L^i_{hk} + L^m_{hj} L^i_{mk} - L^m_{hk} L^i_{mj} - C^i_{h\alpha} \Omega^\alpha_{kj}, \\
R^i_{jka} &= e_a L^i_{jk} - D_k C^i_{ja} + C^i_{jb} T^b_{ka}, \\
R^a_{bcd} &= e_d C^a_{bc} - e_c C^a_{bd} + C^b_{bc} C^a_{ed} - C^e_{bd} C^a_{ec}.
\end{align*}
$$

(30)

where all indices $i, j, k...$ and $a, b...$ run the same values but label the components with respect to different h– or v–frames.

Contracting respectively the components of (29) and (30) we prove:

Corollary 3.1 The Ricci d–tensor $R_{\alpha\beta} \hat{=} R^\tau_{\alpha\beta\tau}$ has the irreducible h– v– components

$$
\begin{align*}
R_{ij} \hat{=} R^k_{ijk}, & \quad R_{ia} \hat{=} -R^k_{ika}, & \quad R_{ai} \hat{=} R^b_{aib}, & \quad R_{ab} \hat{=} R^c_{abc},
\end{align*}
$$

(31)

for a general N–holonomic manifold $\mathcal{V}$, and

$$
\begin{align*}
R_{ij} \hat{=} R^k_{ijk}, & \quad R_{ia} \hat{=} -R^k_{ika}, & \quad R_{ab} \hat{=} R^c_{abc},
\end{align*}
$$

(32)

for an N–anholonomic manifold $\tilde{\mathcal{V}}_{(n,n)}$.

Corollary 3.2 The scalar curvature of a d–connection is

$$
\hat{R} \hat{=} g^{\alpha\beta} R_{\alpha\beta} = g^{ij} R_{ij} + h^{ab} R_{ab},
$$

(33)

defined by the "pure" h– and v–components of (24).

Corollary 3.3 The Einstein d–densor is computed $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \hat{R}$.
For physical applications, the Riemann, Ricci and Einstein d–tensors can be computed for the canonical d–connection. We can redefine the constructions for arbitrary d–connections by using the corresponding deformation tensors like in (28), for instance,

\[ R^\alpha_\beta = \hat{R}^\alpha_\beta + D \mathcal{P}^\alpha_\beta + \mathcal{P}^\alpha_\gamma \wedge \mathcal{P}^\gamma_\beta \]  

(34)

for \( \mathcal{P}^\alpha_\beta = \mathcal{P}^\alpha_\beta_\gamma \varphi_\gamma \). A set of examples of such deformations are analyzed in Refs. [80, 83, 82].

4 Noncommutative N–Anholonomic Spaces

In this section, we outline how the analogs of basic objects in commutative geometry of N–anholonomic manifolds, such as vector/tangent bundles, N– and d–connections can be defined in noncommutative geometry [77, 78]. We note that the A. Connes’ functional analytic approach to the noncommutative topology and geometry is based on the theory of noncommutative C∗–algebras. Any commutative C∗–algebra can be realized as the C∗–algebra of complex valued functions over locally compact Hausdorff space. A noncommutative C∗–algebra can be thought of as the algebra of continuous functions on some ’noncommutative space’ (see main definitions and results in Refs. [17, 29, 39, 44]).

The starting idea of noncommutative geometry is to derive the geometric properties of “commutative” spaces from their algebras of functions characterized by involutive algebras of operators by dropping the condition of commutativity (see the Gelfand and Naimark theorem [31]). A space topology is defined by the algebra of commutative continuous functions, but the geometric constructions request a differentiable structure. Usually, one considers a differentiable and compact manifold \( M, \dim N = n \) (there is an open problem how to include in noncommutative geometry spaces with indefinite metric signature like pseudo–Euclidean and pseudo–Riemannian ones). In order to construct models of commutative and noncommutative differential geometries it is more or less obvious that the class of algebras of smooth functions, \( \mathcal{C} = C^\infty(M) \) is more appropriate. If \( M \) is a smooth manifold, it is possible to reconstruct this manifold with its smooth structure and the attached objects (differential forms, etc...) by starting from \( \mathcal{C} \) considered as an abstract (commutative) unity ∗–algebra with involution. As a set \( M \) can be identified with the set of characters of \( \mathcal{C} \), but its differential structure is connected with the abundance of derivations of \( \mathcal{C} \) which identify with the smooth vector fields on \( M \). There are two standard constructions: 1) when the vector fields are considered to be the derivations of \( \mathcal{C} \) (into itself) or 2)
one considers a generalization of the calculus of differential forms which is
the Kahler differential calculus (see, details in Lectures [21]). The noncom-
mutative versions of differential geometry may be elaborated if the algebra
of smooth complex functions on a smooth manifold is replaced by a noncom-
mutative associative unity complex \( \ast \)-algebra \( A \).

The geometry of commutative gauge and gravity theories is derived from
the notions of connections (linear and nonlinear ones), metrics and frames of
references on manifolds and vector bundle spaces. The possibility of extend-
ing such theories to some noncommutative models is based on the Serre–Swan
theorem [68] stating that there is a complete equivalence between the cate-
gory of (smooth) vector bundles over a smooth compact space (with bundle
maps) and the category of porjective modules of finite type over commutative
algebras and module morphisms. So, the space \( \Gamma (E) \) of smooth sections of
a vector bundle \( E \) over a compact space is a projective module of finite type
over the algebra \( C (M) \) of smooth functions over \( M \) and any finite projective
\( C (M) \)-module can be realized as the module of sections of some vector bu-
ndle over \( M \). This construction may be extended if a noncommutative algebra
\( A \) is taken as the starting ingredient: the noncommutative analogue of vector
bundles are projective modules of finite type over \( A \). This way one developed
a theory of linear connections which culminates in the definition of Yang–
Mills type actions or, by some much more general settings, one reproduced
Lagrangians for the Standard model with its Higgs sector or different type
of gravity and Kaluza–Klein models (see, for instance, Refs [17, 44]).

4.1 Modules as bundles

A vector space \( E \) over the complex number field \( \mathbb{C} \) can be defined also as a
right module of an algebra \( A \) over \( \mathbb{C} \) which carries a right representation of
\( A \), when for every map of elements \( E \times A \ni (\eta, a) \to \eta a \in E \) one hold the properties

\[
\lambda(ab) = (\lambda a)b, \quad \lambda(a + b) = \lambda a + \lambda b, \quad (\lambda + \mu)a = \lambda a + \mu a
\]

for every \( \lambda, \mu \in \mathbb{C} \) and \( a, b \in A \).

Having two \( A \)-modules \( E \) and \( F \), a morphism of \( E \) into \( F \) is any linear
map \( \rho : E \to F \) which is also \( A \)-linear, i.e. \( \rho(\eta a) = \rho(\eta)a \) for every \( \eta \in E \)
and \( a \in A \).

We can define in a similar (dual) manner the left modules and theirs mor-
phisms which are distinct from the right ones for noncommutative algebras
\( A \). A bimodule over an algebra \( A \) is a vector space \( E \) which carries both a
left and right module structures. The bimodule structure is important for
modeling of real geometries starting from complex structures. We may define the opposite algebra \( A^o \) with elements \( a^o \) being in bijective correspondence with the elements \( a \in A \) while the multiplication is given by \( a^o b^o = (ba)^o \). A right (respectively, left) \( A \)-module \( E \) is connected to a left (respectively right) \( A^o \)-module via relations \( a^o \eta = \eta a^o \) (respectively, \( a \eta = \eta a \)). One introduces the enveloping algebra \( A^e = A \otimes \mathbb{K} A^o \); any \( A \)-bimodule \( E \) can be regarded as a right [left] \( A^e \)-module by setting \( \eta (a \otimes b^o) = b^o \eta a \) [\( (a \otimes b^o) \eta = a \eta b \)].

For a (for instance, right) module \( E \), we may introduce a family of elements \( (e_t)_{t \in T} \) parametrized by any (finite or infinite) directed set \( T \) for which any element \( \eta \in E \) is expressed as a combination (in general, in more than one manner) \( \eta = \sum_{t \in T} e_t a_t \) with \( a_t \in A \) and only a finite number of non vanishing terms in the sum. A family \( (e_t)_{t \in T} \) is free if it consists from linearly independent elements and defines a basis if any element \( \eta \in E \) can be written as a unique combination (sum). One says a module to be free if it admits a basis. The module \( E \) is said to be of finite type if it is finitely generated, i.e. it admits a generating family of finite cardinality.

Let us consider the module \( A^K = A \otimes \mathbb{K} A \). The elements of this module can be thought as \( K \)-dimensional vectors with entries in \( A \) and written uniquely as a linear combination \( \eta = \sum_{t=1}^K e_t a_t \) were the basis \( e_t \) identified with the canonical basis of \( \mathbb{K}^K \). This is a free and finite type module. In general, we can have bases of different cardinality. However, if a module \( E \) is of finite type there is always an integer \( K \) and a module surjection \( \rho : A^K \rightarrow E \) with a base being a image of a free basis, \( \epsilon_j = \rho(e_j); j = 1, 2, ..., K \).

We say that a right \( A \)-module \( E \) is projective if for every surjective module morphism \( \rho : M \rightarrow N \) splits, i.e. there exists a module morphism \( s : E \rightarrow M \) such that \( \rho \circ s = id_E \). There are different definitions of porjective modules (see Ref. [39] on properties of such modules). Here we note the property that if a \( A \)-module \( E \) is projective of finite type over \( A \) if and only if there exists an idempotent \( p \in \text{End}_A A^K = M_K(A) \), \( p^2 = p \), the \( M_K(A) \) denoting the algebra of \( K \times K \) matrices with entry in \( A \), such that \( E = pA^K \). We may associate the elements of \( E \) to \( K \)-dimensional column vectors whose elements are in \( A \), the collection of which are invariant under the map \( p \), \( E = \{ \xi = (\xi_1, \ldots, \xi_K); \xi_j \in A, p\xi = \xi \} \). For simplicity, we shall use the term finite projective to mean projective of finite type.
4.2 Nonlinear connections in projective modules

The nonlinear connection (N–connection) for noncommutative spaces can be defined similarly to commutative spaces by considering instead of usual vector bundles their noncommutative analogs defined as finite projective modules over noncommutative algebras \[77\]. The explicit constructions depend on the type of differential calculus we use for definition of tangent structures and their maps. In this subsection, we shall consider such projective modules provided with N–connection which define noncommutative analogs both of vector bundles and of N–anholonomic manifolds (see Definition 2.7).

In general, one can be defined several differential calculi over a given algebra \(A\) (for a more detailed discussion within the context of noncommutative geometry, see Refs. \[17, 44\]). For simplicity, in this work we consider that a differential calculus on \(A\) is fixed, which means that we choose a (graded) algebra \(\Omega^*(A) = \bigcup_p \Omega^p(A)\) giving a differential structure to \(A\). The elements of \(\Omega^p(A)\) are called \(p\)–forms. There is a linear map \(d\) which takes \(p\)–forms into \((p+1)\)–forms and which satisfies a graded Leibniz rule as well the condition \(d^2 = 0\). By definition \(\Omega^0(A) = A\).

The differential \(df\) of a real or complex variable on a N–anholonomic manifold \(V\)

\[
df = \delta_t f \, dx^i + \partial_a f \, \delta y^a,
\]

\[
\delta_t f = \partial_t f - N_i^a \, \partial_a f , \quad \delta y^a = dy^a + N_i^a \, dx^i,
\]

where the N–elongated derivatives and differentials are defined respectively by formulas \(\Pi\) and \(\Pi\), in the noncommutative case is replaced by a distinguished commutator (d–commutator)

\[
\tilde{df} = [F, f] = [F^{[h]}, f] + [F^{[v]}, f]
\]

where the operator \(F^{[h]} \ (F^{[v]})\) acts on the horizontal (vertical) projective sub-module and this operator is defined by a fixed differential calculus \(\Omega^*(A^{[h]}) \ (\Omega^*(A^{[v]}))\) on the so–called horizontal (vertical) \(A^{[h]} \ (A^{[v]})\) algebras. We conclude that in order to elaborated noncommutative versions of N–anholonomic manifolds we need couples of ’horizontal’ and ’vertical’ operators which reflects the nonholonomic splitting given by the N–connection structure.

Let us consider instead of a N–anholonomic manifold \(V\) an \(A\)–module \(E\) being projective and of finite type. For a fixed differential calculus on \(E\) we define the tangent structures \(TE\).

**Definition 4.1** A nonlinear connection (N–connection) \(N\) on an \(A\)–module \(E\) is defined by the splitting on the left of an exact sequence of finite projective \(A\)–moduli

\[
0 \rightarrow vE \xrightarrow{i} TE \rightarrow TE/vE \rightarrow 0,
\]

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i. e. by a morphism of submanifolds \( \mathbf{N} : T\mathcal{E} \to v\mathcal{E} \) such that \( \mathbf{N} \circ \mathbf{i} \) is the unity in \( v\mathcal{E} \).

In an equivalent form, we can say that a N–connection is defined by a splitting to projective submodules with a Whitney sum of conventional \( h\)–submodule, \( (h\mathcal{E}) \), and \( v\)–submodule, \( (v\mathcal{E}) \).

\[
T\mathcal{E} = h\mathcal{E} \oplus v\mathcal{E}.
\] (35)

We note that locally \( h\mathcal{E} \) is isomorphic to \( TM \) where \( M \) is a differential compact manifold of dimension \( n \).

The Definition 4.1 reconsiders for noncommutative spaces the Definition 2.6. In result, we may generalize the concept of 'commutative' N–anholonomic space:

**Definition 4.2** A N–anholonomic noncommutative space \( \mathcal{E}_N \) is an \( \mathcal{A} \)–module \( \mathcal{E} \) possessing a tangent structure \( T\mathcal{E} \) defined by a Whitney sum of projective submodules (35).

Such geometric constructions depend on the type of fixed differential calculus, i. e. on the procedure how the tangent spaces are defined.

**Remark 4.1** Locally always N–connections exist, but it is not obvious if they could be glued together. In the classical case of vector bundles over paracompact manifolds this is possible \cite{50}. If there is an appropriate partition of unity, a similar result can be proved for finite projective modules. For certain applications, it is more convenient to use the Dirac operator already defined on N–anholonomic manifolds, see Section 7.

In order to understand how the N–connection structure may be taken into account on noncommutative spaces but distinguished from the class of linear gauge fields, we analyze an example:

### 4.3 Commutative and noncommutative gauge d–fields

Let us consider a N–anholonomic manifold \( \mathbf{V} \) and a vector bundle \( \beta = (B, \pi, \mathbf{V}) \) with \( \pi : B \to \mathbf{V} \) with a typical \( k \)-dimensional vector fiber. In local coordinates a linear connection (a gauge field) in \( \beta \) is given by a collection of differential operators

\[
\nabla_a = D_a + B_a(u),
\]
acting on $T\xi_N$ where

$$D_\alpha = \delta_\alpha \pm \Gamma_\alpha$$

with $D_i = \delta_i \pm \Gamma_i$ and $D_\alpha = \partial_\alpha \pm \Gamma_\alpha$

is a d–connection in $V$ ($\alpha = 1, 2, \ldots, n + m$), with the operator $\delta_\alpha$, being

$N$–elongated as in (10), $u = (x, y) \in \xi_N$ and $B_\alpha$ are $k \times k$–matrix valued functions. For every vector field

$$X = X^\alpha(u)\delta_\alpha = X^i(u)\delta_i + X^\alpha(u)\partial_\alpha \in T V$$

we can consider the operator

$$X^\alpha(u)\nabla_\alpha(f \cdot s) = f \cdot \nabla_X s + \delta_X f \cdot s$$

for any section $s \in \mathcal{B}$ and function $f \in C^\infty(V)$, where

$$\delta_X f = X^\alpha \delta_\alpha$$

and $\nabla_X f = f \nabla_X$.

In the simplest definition we assume that there is a Lie algebra $\mathcal{GL} B$ that acts

on associative algebra $B$ by means of infinitesimal automorphisms (derivations). This means that we have linear operators $\delta_X : B \to B$ which linearly depend on $X$ and satisfy

$$\delta_X(a \cdot b) = (\delta_X a) \cdot b + a \cdot (\delta_X b)$$

for any $a, b \in B$. The mapping $X \to \delta_X$ is a Lie algebra homomorphism, i.e. $[\delta_X, \delta_Y] = \delta_{[X,Y]}$.

Now we consider respectively instead of commutative spaces $V$ and $\beta$ the

finite projective $A$–module $E_N$, provided with $N$–connection structure, and the finite projective $B$–module $E_\beta$.

A d–connection $\nabla_X$ on $E_\beta$ is by definition a set of linear d–operators, adapted to the $N$–connection structure, depending linearly on $X$ and satisfying the Leibniz rule

$$\nabla_X(b \cdot e) = b \cdot \nabla_X(e) + \delta_X b \cdot e$$

for any $e \in E_\beta$ and $b \in \mathcal{B}$. The rule (37) is a noncommutative generalization of (36). We emphasize that both operators $\nabla_X$ and $\delta_X$ are distinguished by the $N$–connection structure and that the difference of two such linear d–operators, $\nabla_X - \nabla'_X$ commutes with action of $B$ on $E_\beta$, which is an endomorphism of $E_\beta$. Hence, if we fix some fiducial connection $\nabla'_X$ (for instance, $\nabla'_X = D_X$) on $E_\beta$ an arbitrary connection has the form

$$\nabla_X = D_X + B_X.$$
where $B_X \in \text{End}_B \mathcal{E}_\beta$ depend linearly on $X$.

The curvature of connection $\nabla_X$ is a two–form $F_{XY}$ which values linear operator in $\mathcal{B}$ and measures a deviation of mapping $X \to \nabla_X$ from being a Lie algebra homomorphism,

$$F_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$  

The usual curvature d–tensor is defined as

$$F_{\alpha\beta} = [\nabla_\alpha, \nabla_\beta] - \nabla_{[\alpha,\beta]}.$$  

The simplest connection on a finite projective $\mathcal{B}$–module $\mathcal{E}_\beta$ is to be specified by a projector $P : \mathcal{B}^k \otimes \mathcal{B}^k$ when the d–operator $\delta_X$ acts naturally on the free module $\mathcal{B}^k$. The operator $\nabla^L_X = P \cdot \delta_X \cdot P$ is called the Levi–Civita operator and satisfy the condition $\text{Tr}[\nabla^L_X, \phi] = 0$ for any endomorphism $\phi \in \text{End}_B \mathcal{E}_\beta$. From this identity, and from the fact that any two connections differ by an endomorphism that $\text{Tr}[\nabla_X, \phi] = 0$ for an arbitrary connection $\nabla_X$ and an arbitrary endomorphism $\phi$, that instead of $\nabla^L_X$ we may consider equivalently the canonical d–connection, constructed only from d–metric and N–connection coefficients.

5 Nonholonomic Clifford–Lagrange Structures

The geometry of spinors on generalized Lagrange and Finsler spaces was elaborated in Refs. [69, 72, 86, 88]. It was applied for definition of noncommutative extensions of the Finsler geometry related to certain models of Einstein, gauge and string gravity [71, 77, 78, 74, 87, 84]. Recently, it is was proposed an extended Clifford approach to relativity, strings and noncommutativity based on the concept of "C–space" [11, 12, 14, 15].

The aim of this section is to formulate the geometry of nonholonomic Clifford–Lagrange structures in a form adapted to generalizations for noncommutative spaces.

5.1 Clifford d–module

Let $V$ be a compact N–anholonomic manifold. We denote, respectively, by $T_xV$ and $T^*_xV$ the tangent and cotangent spaces in a point $x \in V$. We consider a complex vector bundle $\tau : E \to V$ where, in general, both the base $V$ and the total space $E$ may be provided with N–connection structure, and denote by $\Gamma^\infty(E)$ (respectively, $\Gamma(E)$) the set of differentiable (continous)
sections of \( E \). The symbols \( \chi(M) = \Gamma^\infty(TM) \) and \( \Omega^1(M) \cong \Gamma^\infty(T^*M) \) are used respectively for the set of d–vectors and one d–forms on \( TM \).

### 5.1.1 Clifford–Lagrange functionals

In the simplest case, a generic nonholonomic Clifford structure can be associated to a Lagrange metric on a \( n \)–dimensional real vector space \( V^n \) provided with a Lagrange quadratic form \( L(y) = q_L(y, y) \), see subsection 2.1. We consider the exterior algebra \( \wedge V^n \) defined by the identity element \( \mathbb{I} \) and antisymmetric products \( v_{[1]} \wedge ... \wedge v_{[k]} \) with \( v_{[1]}, ..., v_{[k]} \in V^n \) for \( k \leq \text{dim} V^n \) where \( \mathbb{I} \wedge v = v, v_{[1]} \wedge v_{[2]} = -v_{[2]} \wedge v_{[1]}, ... \)

**Definition 5.1** The Clifford–Lagrange (or Clifford–Minkowski) algebra is a \( \wedge V^n \) algebra provided with a product

\[
uv + vu = 2^{(L)} g(u, v) \mathbb{I}
\]

(38)

\[
(\text{or } uv + vu = 2^{(F)} g(u, v) \mathbb{I})
\]

(39)

for any \( u, v \in V^n \) and \( ^{(L)} g(u, v) \) (or \( ^{(F)} g(u, v) \)) defined by formulas (2) (or (3)).

For simplicity, hereafter we shall prefer to write down the formulas for the Lagrange configurations instead of dubbing of similar formulas for the Finsler configurations.

We can introduce the complex Clifford–Lagrange algebra \( \text{Cl}(L)(V^n) \) structure by using the complex unity “\( i \)”, \( V[\mathbb{C}] = V^n + iV^n \), enabled with complex metric

\[
^{(L)} g[\mathbb{C}](u, v + iw) = ^{(L)} g(u, v) + i (L) g(u, w),
\]

which results in certain isomorphisms of matix algebras (see, for instance, [29]),

\[
\text{Cl}(\mathbb{R}^{2m}) \cong M_{2^m}(\mathbb{C}),
\]

\[
\text{Cl}(\mathbb{R}^{2m+1}) \cong M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C}).
\]

We omitted the label \( (L) \) because such isomorphisms hold true for any quadratic forms.

The Clifford–Lagrange algebra possesses usual properties:

1. On \( \text{Cl}(L)(V^n) \), it is linearly defined the involution “\( ^* \)”,

\[
(\lambda v_{[1]}...v_{[k]})^* = \overline{\lambda} v_{[1]}...v_{[k]}, \ \forall \lambda \in \mathbb{C}.
\]
2. There is a $\mathbb{Z}_2$ graduation,

$$\mathbb{C}l_L(V^n) = \mathbb{C}l^+_L(V^n) \oplus \mathbb{C}l^-_L(V^n)$$

with $\chi_L(a) = \pm 1$ for $a \in \mathbb{C}l^+_L(V^n)$, where $\mathbb{C}l^+_L(V^n)$, or $\mathbb{C}l^-_L(V^n)$, are defined by products of an odd, or even, number of vectors.

3. For positive definite forms $q_L(u, v)$, one defines the chirality of $\mathbb{C}l_L(V^n)$,

$$\gamma_L = (-i)^n e_1 e_2 \ldots e_n, \quad \gamma^2 = \gamma^\ast \gamma = \mathbb{I}$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis of $V^n$ and $n = 2n'$, or $= 2n' + 1$.

In a more general case, a nonholonomic Clifford structure is defined by quadratic $d$–metric form $q(x, y)$ (22) on a $n + m$–dimensional real $d$–vector space $V^{n+m}$ with the $(n+m)$–splitting defined by the $N$–connection structure.

**Definition 5.2** The Clifford $d$–algebra is a $\wedge V^{n+m}$ algebra provided with a product

$$uv + vu = 2g(u, v) \mathbb{I} \quad (40)$$

or, equivalently, distinguished into $h$– and $v$–products

$$uv + vu = 2g(u, v) \mathbb{I}$$

and

$$^\ast u ^\ast v + ^\ast v ^\ast u = 2 ^\ast h( ^\ast u, ^\ast v) \mathbb{I}$$

for any $u = (u, ^\ast u), \ v = (v, ^\ast v) \in V^{n+m}$.

Such Clifford $d$–algebras have similar properties on the irreducible $h$– and $v$–components as the Clifford–Lagrange algebras. We may define a standard complexification but it should be emphasized that for $n = m$ the $N$–connection (in particular, the canonical Lagrange $N$–connection) induces naturally an almost complex structure (15) which gives the possibility to define almost complex Clifford $d$–algebras (see details in [69, 88]).

5.1.2 Clifford–Lagrange and Clifford $N$–anholonomic structures

A metric on a manifold $M$ is defined by sections of the tangent bundle $TM$ provided with a bilinear symmetric form on continuous sections $\Gamma(TM)$. In Lagrange geometry, the metric structure is of type $^{(L)}g_{ij}(x, y)$ (2) which allows us to define Clifford–Lagrange algebras $\mathbb{C}l_L(T_x M)$, in any point $x \in TM$,

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2 ^{(L)}g_{ij} \mathbb{I}.$$
For any point \( x \in M \) and fixed \( y = y_0 \), one exists a standard complexification, \( T_x M^{\mathbb{C}} = T_x M + i T_x M \), which can be used for definition of the 'involution' operator on sections of \( T_x M^{\mathbb{C}} \),

\[
\sigma_1 \sigma_2(x) \doteq \sigma_2(x) \sigma_1(x), \quad \sigma^*(x) \doteq \sigma(x)^* , \forall x \in M ,
\]

where "*" denotes the involution on every \( \mathbb{C}l(T_x M) \). The norm is defined by using the Lagrange norm, see Definition 2.1,

\[
\| \sigma \|_L = \sup_{x \in M} \{ | \sigma(x) |_L \},
\]

which defines a \( C^*_L \)-algebra instead of the usual \( C^* \)-algebra of \( \mathbb{C}l(T_x M) \).

Such constructions can be also performed on the cotangent space \( T^*_x M \), or for any vector bundle \( E \) on \( M \) enabled with a symmetric bilinear form of class \( C^\infty \) on \( \Gamma^\infty(E) \times \Gamma^\infty(E) \).

For Lagrange spaces modeled on \( \tilde{T}M \), there is a natural almost complex structure \( F \) induced by the canonical \( N \)-connection \( (L)N \), see the Results 2.2 and 2.5, which allows also to construct an almost Kahler model of Lagrange geometry, see details in Refs. 56, 57, and to define an Clifford–Kahler d–algebra \( \mathbb{C}l_{(KL)}(T_x M) \) for \( y = y_0 \), being provided with the norm,

\[
\| \sigma \|_{KL} = \sup_{x \in M} \{ | \sigma(x) |_{KL} \},
\]

which on \( T_x M \) is defined by projecting on \( x \) the d–metric \( (L)g \).

In order to model Clifford–Lagrange structures on \( \tilde{T}M \) and \( \tilde{T^*}M \) it is necessary to consider d–metrics induced by Lagrangians:

**Definition 5.3** A Clifford–Lagrange space on a manifold \( M \) enabled with a fundamental metric \( (L)g_{ij}(x,y) \) and canonical \( N \)-connection \( (L)N^i_j \) inducing a Sasaki type d–metric \( (L)g \) is defined as a Clifford bundle \( \mathbb{C}l_{(L)}(M) = \mathbb{C}l_{(L)}(T^*M) \).

For a general \( N \)-anholonomic manifold \( V \) of dimension \( n + m \) provided with a general d–metric structure \( g \) (for instance, in a gravitational model, or constructed by conformal transforms and embeddings into higher dimensions of a Lagrange (or Finsler) d–metrics), we introduce

**Definition 5.4** A Clifford \( N \)-anholonomic bundle on \( V \) is defined as \( \mathbb{C}l_{(N)}(V) = \mathbb{C}l_{(N)}(T^*V) \).

Let us consider a complex vector bundle \( \pi : E \to M \) provided with \( N \)-connection structure which can be defined by a corresponding exact chain of subbundles, or nonintegrable distributions, like for real vector bundles, see
Denoting by $V_m^\mathbb{C}$ the typical fiber (a complex vector space), we can define the usual Clifford map

$$c : \mathbb{C}l(T^*M) \rightarrow \text{End}(V_m^\mathbb{C})$$

via (by convention, left) action on sections $c(\sigma)\sigma^\dagger(x) = c(\sigma(x))\sigma^\dagger(x)$.

**Definition 5.5** The Clifford d–module (distinguished by a N–connection) of a N–anholonomic vector bundle $E$ is defined by the $C(M)$–module $\Gamma(E)$ of continuous sections in $E$,

$$c : \Gamma(\mathbb{C}l(M)) \rightarrow \text{End}(\Gamma(E)).$$

In an alternative case, one considers a complex vector bundle $\pi : E \rightarrow V$ on an N–anholonomic space $V$ when the N–connection structure is given for the base manifold.

**Definition 5.6** The Clifford d–module of a vector bundle $E$ is defined by the $C(V)$–module $\Gamma(E)$ of continuous sections in $E$,

$$c : \Gamma(\mathbb{C}l(N)(V)) \rightarrow \text{End}(\Gamma(E)).$$

A Clifford d–module with both N–anholonomic total space $E$ and base space $V$ with corresponding N–connections (in general, two independent ones, but the N–connection in the distinguished complex vector bundle must be adapted to the N–connection on the base) has to be defined by an ”interference” of Definitions 5.5 and 5.6.

### 5.2 N–anholonomic spin structures

Usually, the spinor bundle on a manifold $M$, $\text{dim}M = n$, is constructed on the tangent bundle by substituting the group $SO(n)$ by its universal covering $\text{Spin}(n)$. If a Lagrange fundamental quadratic form $(L)g_{ij}(x,y)$ is defined on $T_xM$ we can consider Lagrange–spinor spaces in every point $x \in M$. The constructions can be completed on $\tilde{T}M$ by using the Sasaki type metric $(\tilde{L})g$ being similar for any type of N–connection and d–metric structure on $TM$. On general N–anholonomic manifolds $V$, $\text{dim}V = n + m$, the distinguished spinor space (in brief, d–spinor space) is to be derived from the d–metric and adapted to the N–connection structure. In this case, the group $SO(n+m)$ is not only substituted by $\text{Spin}(n+m)$ but with respect to N–adapted frames and one defines irreducible decompositions to $\text{Spin}(n) \oplus \text{Spin}(m)$. 

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5.2.1 Lagrange spin groups

Let us consider a vector space $V^n$ provided with Clifford–Lagrange structures as in subsection 5.1. We denote a such space as $V^n_{(L)}$ in order to emphasize that its tangent space is provided with a Lagrange type quadratic form $(L)g$. In a similar form, we shall write $\mathbb{C}l(V^n_{(L)}) \equiv \mathbb{C}l(V^n_{(L)})$ if this will be more convenient. A vector $u \in V^n_{(L)}$ has a unity length on the Lagrange quadratic form if $(L)g(u, u) = 1$, or $u^2 = 1$, as an element of corresponding Clifford algebra, which follows from (38). We define an endomorphism of $V^n$:

$$\phi^{(L)}(u) \equiv \chi_{(L)}(u)uu^{-1} = -uu = (uv - 2(L)g(u, v))u = u - 2(L)g(u, v)u$$

where $\chi_{(L)}$ is the $\mathbb{Z}_2$ graduation defined by $(L)g$. By multiplication,

$$\phi^{u_1u_2}_{(L)}(v) \equiv u_2^{-1}u_1^{-1}vu_1u_2 = \phi^{u_2}_{(L)} \circ \phi^{u_1}_{(L)}(v),$$

which defines the subgroup $SO(V^n_{(L)}) \subset O(V^n_{(L)})$. Now we can define [69, 88]

**Definition 5.7** The space of complex Lagrange spins is defined by the subgroup $Spin_{(L)}^c(n) \equiv Spin^c(V^n_{(L)}) \subset \mathbb{C}l(V^n_{(L)})$, determined by the products of pairs of vectors $w \in V^n_{(L)}$ when $w \equiv \lambda u$ where $\lambda$ is a complex number of module 1 and $u$ is of unity length in $V^n_{(L)}$.

We note that $\ker \phi_{(L)} \cong U(1)$. We can define a homomorphism $\nu_{(L)}$ with values in $U(1)$,

$$\nu_{(L)}(w) = w_{2k}...w_1w_1...w_{2k} = \lambda_1...\lambda_{2k},$$

where $w = w_1...w_{2k} \in Spin^c(V^n_{(L)})$ and $\lambda_i = w_i^2 \in U(1)$.

**Definition 5.8** The group of real Lagrange spins $Spin_{(L)}^e(n) \equiv Spin(V^n_{(L)})$ is defined by $\ker \nu_{(L)}$.

The complex conjugation on $\mathbb{C}l(V^n_{(L)})$ is usually defined as $\overline{\lambda}v = \overline{\lambda}v$ for $\lambda \in \mathbb{C}$, $v \in V^n_{(L)}$. So, any element $w \in Spin(V^n_{(L)})$ satisfies the conditions $\overline{w}w = w^*w = 1$ and $\overline{w}w = w$. If we take $V^n_{(L)} = \mathbb{R}^n$ provided with a (pseudo) Euclidean quadratic form instead of the Lagrange norm, we obtain the usual spin–group constructions from the (pseudo) Euclidean geometry.

5.2.2 Lagrange spinors and d–spinors: Main Result 1

A usual spinor is a section of a vector bundle $S$ on a manifold $M$ when an irreducible representation of the group $Spin(M) \cong Spin(T^*_xM)$ is defined on the typical fiber. The set of sections $\Gamma(S)$ is an irreducible Clifford module.
If the base manifold of type $M(L)$, or is a general N–anholonomic manifold $V$, we have to define the spinors on such spaces as to be adapted to the respective N–connection structure.

In the case when the base space is of even dimension (the geometric constructions in in this subsection will be considered for even dimensions both for the base and typical fiber spaces), one should consider the so–called Morita equivalence (see details in [29, 50] for a such equivalence between $C(M)$ and $\Gamma(\mathcal{Kl}(M))$). One says that two algebras $A$ and $B$ are Morita–equivalent if

$$\mathcal{E} \otimes_A \mathcal{F} \simeq B \text{ and } \mathcal{F} \otimes_B \mathcal{F} \simeq A,$$

respectively, for $B$– and $A$–bimodules and $B$ – $A$–bimodule $\mathcal{E}$ and $A$ – $B$–bimodule $\mathcal{F}$. If we study algebras through theirs representations, we also have to consider various algebras related by the Morita equivalence.

**Definition 5.9** A Lagrange spinor bundle $S(L)$ on a manifold $M$, $\dim M = n$, is a complex vector bundle with both defined action of the spin group $\text{Spin}(V_n(L))$ on the typical fiber and an irreducible representation of the group $\text{Spin}(L(M)) \equiv \text{Spin}(M(L)) \triangleq \text{Spin}(T^*_x M(L))$. The set of sections $\Gamma(S(L))$ defines an irreducible Clifford–Lagrange module.

The so–called ”d–spinors” have been introduced for the spaces provided with N–connection structure [69, 72, 73]:

**Definition 5.10** A distinguished spinor (d–spinor) bundle $S \doteq (S, \star S)$ on an N–anholonomic manifold $V$, $\dim V = n + m$, is a complex vector bundle with a defined action of the spin d–group $\text{Spin}(V) \doteq \text{Spin}(V^n) \oplus \text{Spin}(V^m)$ with the splitting adapted to the N–connection structure which results in an irreducible representation $\text{Spin}(V) \doteq \text{Spin}(T^*V)$. The set of sections $\Gamma(S) = \Gamma(S) \oplus \Gamma(\star S)$ is an irreducible Clifford d–module.

The fact that $C(V)$ and $\Gamma(\mathcal{Kl}(V))$ are Morita equivalent can be analyzed by applying in N–adapted form, both on the base and fiber spaces, the consequences of the Plymen’s theorem (see Theorem 9.3 in Ref. [29]). This is connected with the possibility to distinguish the $\text{Spin}(n)$ (or, correspondingly $\text{Spin}(M(L))$, $\text{Spin}(V^n) \oplus \text{Spin}(V^m)$) an antilinear bijection $J : S \rightarrow S$ (or $J : S(L) \rightarrow S(L)$ and $J : S \rightarrow S$) with the properties:

$$J(\psi f) = (J\psi)f \text{ for } f \in C(M)( \text{ or } C(M(L)), C(V));$$
$$J(a \psi) = \chi(a)J\psi, \text{ for } a \in \Gamma(\mathcal{Kl}(M))( \text{ or } \Gamma(\mathcal{Kl}(M(L))), \Gamma(\mathcal{Kl}(V));$$
$$(J\phi|J\psi) = (\psi|\phi) \text{ for } \phi, \psi \in S( \text{ or } S(L), S).$$

(41)
Definition 5.11  The spin structure on a manifold $M$ (respectively, on $M_{(L)}$, or on $N$–anholonomic manifold $V$) with even dimensions for the corresponding base and typical fiber spaces is defined by a bimodule $S$ (respectively, $M_{(L)}$, or $V$) obeying the Morita equivalence $C(M) - \Gamma(\mathcal{Cl}(M))$ (respectively, $C(M_{(L)}) - \Gamma(\mathcal{Cl}(M_{(L)}))$, or $C(V) - \Gamma(\mathcal{Cl}(V))$) by a corresponding bijections and a fixed orientation on $M$ (respectively, on $M_{(L)}$, or $V$).

In brief, we may call $M$ ($M_{(L)}$, or $V$) as a spin manifold (Lagrange spin manifold, or $N$–anholonomic spin manifold). If any of the base or typical fiber spaces is of odd dimension, we may perform similar constructions by considering $\mathcal{Cl}^+$ instead of $\mathcal{Cl}$.

The considerations presented in this Section consists the proof of the first main Result of this paper (let us conventionally say that it is the 7th one after the Results 2.1–2.6:

Theorem 5.1 (Main Result 1) Any regular Lagrangian and/or $N$–connection structure define naturally the fundamental geometric objects and structures (such as the Clifford–Lagrange module and Clifford $d$–modules, the Lagrange spin structure and $d$–spinors) for the corresponding Lagrange spin manifold and/or $N$–anholonomic spinor ($d$–spinor) manifold.

We note that similar results were obtained in Refs. [69, 72, 88] for the standard Finsler and Lagrange geometries and theirs higher order generalizations. In a more restricted form, the idea of Theorem 5.1 can be found in Ref. [77], where the first models of noncommutative Finsler geometry and related gravity were considered (in a more rough form, for instance, with constructions not reflecting the Morita equivalence).

Finally, in this Section, we can make the

Conclusion 5.1 Any regular Lagrange and/or $N$–connection structure (the last one being any admissible $N$–connection in Lagrange–Finsler geometry and their generalizations, or induced by any generic off–diagonal and/or nonholonomic frame structure) define certain, corresponding, Clifford–Lagrange module and/or Clifford $d$–module and related Lagrange spinor and/or $d$–spinor structures.

It is a bit surprising that a Lagrangian may define not only the fundamental geometric objects of a nonholonomic Lagrange space but also the structure of a naturally associated Lagrange spin manifold. The Lagrange mechanics and off–diagonal gravitational interactions (in general, with nontrivial torsion and nonholonomic constraints) can be completely geometrized on Lagrange spin ($N$–anholonomic) manifolds.
6 The Dirac Operator, Nonholonomy, and Spectral Triples

The Dirac operator for a certain class of (non) commutative Finsler spaces provided with compatible metric structure was introduced in Ref. [77] following previous constructions for the Dirac equations on locally anisotropic spaces [69, 72, 73, 86, 88]. The aim of this Section is to elucidate the possibility of definition of Dirac operators for general N–anholonomic manifolds and Lagrange–Finsler spaces. It should be noted that such geometric constructions depend on the type of linear connections which are used for the complete definition of the Dirac operator. They are metric compatible and N–adapted if the canonical d–connection is used, see Proposition 3.1 (we can also use any its deformation which results in a metric compatible d–connection). The constructions can be more sophisticated and nonmetric (with some geometric objects not completely defined on the tangent spaces) if the Chern, or the Berwald d–connection, is considered, see Example 3.2.

6.1 N–anholonomic Dirac operators

We introduce the basic definitions and formulas with respect to N–adapted frames of type (10) and (11). Then we shall present the main results in a global form.

6.1.1 Noholonomic vielbeins and spin d–connections

Let us consider a Hilbert space of finite dimension. For a local dual coordinate basis $e^i = dx^i$ on a manifold $M$, $\dim M = n$, we may respectively introduce certain classes of orthonormalized vielbeins and the N–adapted vielbeins, \(^{6}\)

$$e^{i \downarrow}(x, y) e^{i \downarrow}(x, y) = \delta^{ij} \quad \text{and} \quad g^{ij}(x, y) e^{i \downarrow}(x, y) e^{j \downarrow}(x, y) = g^{ij}(x, y).$$

We define the the algebra of Dirac’s gamma matrices (in brief, h–gamma matrices defined by self–adjoints matrices $M_k(\mathbb{C})$ where $k = 2^{n/2}$ is the

\(^6 \) (depending both on the base coordinates $x = x^i$ and some "fiber" coordinates $y = y^a$, the status of $y^a$ depends on what kind of models we shall consider: elongated on $TM$, for a Lagrange space, for a vector bundle, or on a N–anholonomic manifold)
dimension of the irreducible representation of $\text{Cl}(M)$ for even dimensions, or of $\text{Cl}(M)^+$ for odd dimensions) from the relation

$$\gamma^i \gamma^j + \gamma^j \gamma^i = 2\delta^i_j \mathbb{I}.$$  

(43)

We can consider the action of $dx^i \in \text{Cl}(M)$ on a spinor $\psi \in S$ via representations

$$-c(dx^i) \doteqdot \gamma^i \text{ and } -c(dx^i)\psi \doteqdot \gamma^i \psi \equiv e^i_{\hat{i}} \gamma^i \psi.$$  

(44)

For any type of spaces $T_x M, TM, V$ possessing a local (in any point) or global fibered structure and, in general, enabled with a N–connection structure, we can introduce similar definitions of the gamma matrices following algebraic relations and metric structures on fiber subspaces,

$$e^a \doteqdot e^a_{\underline{a}}(x, y) e^a \text{ and } e^a \doteqdot e^a_{\underline{a}}(x, y) e^a,$$  

(45)

where

$$g^{ab}(x, y) e^a_{\underline{a}}(x, y) e^b_{\underline{b}}(x, y) = \delta^{\underline{a} \underline{b}} \text{ and } g^{ab}(x, y) e^a_{\underline{a}}(x, y) e^b_{\underline{a}}(x, y) = h^{ab}(x, y).$$

Similarly, we define the algebra of Dirac’s matrices related to typical fibers (in brief, $v$–gamma matrices defined by self–adjoints matrices $M_k'(\mathbb{C})$ where $k' = 2^{m/2}$ is the dimension of the irreducible representation of $\text{Cl}(F)$ for even dimensions, or of $\text{Cl}(F)^+$ for odd dimensions, of the typical fiber) from the relation

$$\gamma^\cdot_a \gamma^\cdot_b + \gamma^\cdot_b \gamma^\cdot_a = 2\delta^{\cdot a}_b \mathbb{I}.$$  

(46)

The action of $dy^a \in \text{Cl}(F)$ on a spinor $^*\psi \in ^*S$ is considered via representations

$$^*c(dy^\cdot a) \doteqdot \gamma^\cdot_a \text{ and } ^*c(dy^a) \doteqdot \gamma^a \doteqdot e^a_{\hat{a}} \gamma^\cdot a \doteqdot \psi.$$  

(47)

We note that additionally to formulas (44) and (47) we may write respectively

$$c(dx^\cdot a) \psi \doteqdot e^\cdot_i \gamma^i \psi \text{ and } c(dx^a) \doteqdot e^a_{\hat{a}} \gamma^\cdot a \doteqdot \psi.$$

but such operators are not adapted to the N–connection structure.

A more general gamma matrix calculus with distinguished gamma matrices (in brief, $d$–gamma matrices\footnote{In our previous works \cite{69, 72, 73, 80, 88} we wrote $\sigma$ instead of $\gamma$}) can be elaborated for N–anholonomic manifolds $V$ provided with $d$–metric structure $g = [g, ^*g]$ and for $d$–spinors $\tilde{\psi} \doteqdot (\psi, \ ^*\psi) \in S \doteqdot (S, \ ^*S)$, see the corresponding Definitions 2.4, 2.8 and
Firstly, we should write in a unified form, related to a d–metric (19), the formulas (42) and (45),

\[ e_{\hat{\alpha}} \triangleq e_{\hat{\alpha}}(u) e_{\hat{\alpha}} \quad \text{and} \quad e_{\hat{\alpha}} = e_{\hat{\alpha}}(u) e_{\hat{\alpha}}, \tag{48} \]

where

\[ g_{\hat{\alpha}\hat{\beta}}(u) e_{\hat{\alpha}}(u) e_{\hat{\beta}}(u) = \delta_{\hat{\alpha}\hat{\beta}} \quad \text{and} \quad g_{\hat{\alpha}\hat{\beta}}(u) e_{\hat{\alpha}}(u) e_{\hat{\beta}}(u) = g_{\hat{\alpha}\hat{\beta}}(u). \]

The second step, is to consider d–gamma matrix relations (unifying (43) and (46))

\[ \gamma_{\hat{\alpha}} \gamma_{\hat{\beta}} + \gamma_{\hat{\beta}} \gamma_{\hat{\alpha}} = 2 \delta_{\hat{\alpha}\hat{\beta}} I, \tag{49} \]

with the action of \( du^\hat{\alpha} \in \text{Cl}(V) \) on a d–spinor \( \tilde{\psi} \in S \) resulting in distinguished irreducible representations (unifying (44) and (47))

\[ c(du^\hat{\alpha}) \triangleq \gamma_{\hat{\alpha}} \quad \text{and} \quad c = (du^\alpha) \tilde{\psi} \triangleq \gamma_{\alpha} \tilde{\psi} \equiv e_{\hat{\alpha}} \gamma_{\hat{\alpha}} \tilde{\psi} \equiv \gamma_{\alpha} \tilde{\psi} \tag{50} \]

which allows to write

\[ \gamma^\alpha(u) \gamma^\beta(u) + \gamma^\beta(u) \gamma^\alpha(u) = 2 g^{\alpha\beta}(u) I. \tag{51} \]

In the canonical representation we can write in irreducible form \( \tilde{\gamma} \triangleq \gamma \oplus *\gamma \) and \( \tilde{\psi} \triangleq \psi \oplus *\psi \), for instance, by using block type of h– and v–matrices, or, writing alternatively as couples of gamma and/or h– and v–spinor objects written in N–adapted form,

\[ \gamma^\alpha \triangleq (\gamma^i, \gamma^a) \quad \text{and} \quad \tilde{\psi} \triangleq (\psi, *\psi). \tag{52} \]

The decomposition (51) holds with respect to a N–adapted vielbein (10). We also note that for a spinor calculus, the indices of spinor objects should be treated as abstract spinorial ones possessing certain reducible, or irreducible, properties depending on the space dimension (see details in Refs. 69, 72, 73, 86, 88). For simplicity, we shall consider that spinors like \( \tilde{\psi}, \psi, *\psi \) and all type of gamma objects can be enabled with corresponding spinor indices running certain values which are different from the usual coordinate space indices. In a ”rough” but brief form we can use the same indices \( i, j, a, b, \alpha, \beta, \ldots \) both for d–spinor and d–tensor objects.

The spin connection \( \nabla^S \) for the Riemannian manifolds is induced by the Levi–Civita connection \( \nabla^\Gamma \)

\[ \nabla^S \triangleq d - \frac{1}{4} \nabla_{jk}^\Gamma \gamma_{ij} \psi^k dx^k. \tag{53} \]

On N–anholonomic spaces, it is possible to define spin connections which are N–adapted by replacing the Levi–Civita connection by any d–connection (see Definition 3.1).
Definition 6.1 The canonical spin $d$–connection is defined by the canonical $d$–connection (25) as
\[ \hat{\nabla}^S \doteq \delta - \frac{1}{4} \hat{\Gamma}^{\alpha}_{\beta \mu} \gamma^\alpha \gamma^\beta \delta u^\mu, \] (54)
where the absolute differential $\delta$ acts in $N$–adapted form resulting in 1–forms decomposed with respect to $N$–elongated differentials like $\delta u^\mu = (dx^i, \delta y^a)$.

We note that the canonical spin $d$–connection $\hat{\nabla}^S$ is metric compatible and contains nontrivial $d$–torsion coefficients induced by the $N$–anholonomy relations (see the formulas (24) proved for arbitrary $d$–connection). It is possible to introduce more general spin $d$–connections $D^S$ by using the same formula (54) but for arbitrary metric compatible $d$–connection $\Gamma^{\alpha}_{\beta \mu}$.

In a particular case, we can define, for instance, the canonical spin $d$–connections for a local modelling of a $\tilde{T}M$ space on $\tilde{V}_{(n,n)}$ with the canonical $d$–connection $\hat{\Gamma}^{\gamma}_{\alpha \beta} = (\hat{L}^i_{jk}, \hat{C}^i_{jk})$, see formulas (27). This allows us to prove (by considering $d$–connection and $d$–metric structure defined by the fundamental Lagrange, or Finsler, functions, we put formulas (9) and (14) into (27)):

Proposition 6.1 On Lagrange spaces, there is a canonical spin $d$–connection (the canonical spin–Lagrange connection),
\[ \hat{\nabla}^{(SL)} \doteq \delta - \frac{1}{4} (L) \Gamma^{\alpha}_{\beta \mu} \gamma^\alpha \gamma^\beta \delta u^\mu, \] (55)
where $\delta u^\mu = (dx^i, \delta y^k = dy^k + (L) N^k_i \, dx^i)$.

We emphasize that even regular Lagrangians of classical mechanics without spin particles induce in a canonical (but nonholonomic) form certain classes of spin $d$–connections like (55).

For the spaces provided with generic off–diagonal metric structure (16) (in particular, for such Riemannian manifolds) resulting in equivalent $N$–anholonomic manifolds, it is possible to prove a result being similar to Proposition 6.1:

Remark 6.1 There is a canonical spin $d$–connection (54) induced by the off–diagonal metric coefficients with nontrivial $N_i^a$ and associated nonholonomic frames in gravity theories.

The $N$–connection structure also states a global $h$– and $v$–splitting of spin $d$–connection operators, for instance,
\[ \hat{\nabla}^{(SL)} \doteq \delta - \frac{1}{4} (L) \hat{L}^i_{jk} \gamma^\gamma dx^k - \frac{1}{4} (L) \hat{C}^{a}_{bc} \gamma^a \gamma^b dy^c. \] (56)
So, any spin $d$–connection is a $d$–operator with conventional splitting of action like $\nabla^{(S)} \equiv (-\nabla^{(S)}, \star \nabla^{(S)})$, or $\nabla^{(SL)} \equiv (-\nabla^{(SL)}, \star \nabla^{(SL)})$. For instance, for $\nabla^{(SL)} \equiv (-\nabla^{(SL)}, \star \nabla^{(SL)})$, the operators $-\nabla^{(SL)}$ and $\star \nabla^{(SL)}$ act respectively on a $h$–spinor $\psi$ as

$$\nabla^{(SL)} \psi \doteq dx^i \frac{\delta \psi}{dx^i} - dx^k \frac{1}{4} (L) \hat{L}_{jk} \gamma_i \gamma^j \psi$$

and

$$\star \nabla^{(SL)} \psi \doteq \delta y^a \frac{\partial \psi}{\delta y^a} - \delta y^c \frac{1}{4} (L) \hat{C}_{bc} \gamma_a \gamma^b \psi$$

being defined by the canonical $d$–connection (57).

**Remark 6.2** We can consider that the $h$–operator (57) defines a spin generalization of the Chern’s $d$–connection $[\text{Chern}] \Gamma_{\gamma \alpha \beta} = (\hat{L}_{jk}, \hat{C}_{jk} = 0)$, see Example 6.2, which may be introduced as a minimal extension, with Finsler structure, of the spin connection defined by the Levi–Civita connection $[\text{Finsler}]$ preserving the torsionless condition. This is an example of nonmetric spin connection operator because $[\text{Chern}] \Gamma_{\gamma \alpha \beta}$ does not satisfy the condition of metric compatibility.

We can define spin Chern–Finsler structures, considered in the Remark 6.2 for any point of an $N$–anholonomic manifold. There are necessary some additional assumptions in order to completely define such structures (for instance, on the tangent bundle). We can say that this is a deformed nonholonomic spin structure derived from a $d$–spinor one provided with the canonical spin $d$–connection by deforming the canonical $d$–connection in a manner that the horizontal torsion vanishes transforming into a nonmetricity $d$–tensor. The ”nonspinor” aspects of such generalizations of the Riemann–Finsler spaces and gravity models with nontrivial nonmetricity are analyzed in Refs. [83].

**6.1.2 Dirac $d$–operators: Main Result 2**

We consider a vector bundle $E$ on an $N$–anholonomic manifold $M$ (with two compatible $N$–connections defined as $h$– and $v$–splittings of $TE$ and $TM$)). A $d$–connection

$$D : \Gamma^\infty(E) \rightarrow \Gamma^\infty(E) \otimes \Omega^1(M)$$

preserves by parallelism splitting of the tangent total and base spaces and satisfy the Leibniz condition

$$D(f \sigma) = f(D \sigma) + \delta f \otimes \sigma$$

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for any \( f \in C^\infty(M) \), and \( \sigma \in \Gamma^\infty(E) \) and \( \delta \) defining an \( N \)-adapted exterior calculus by using \( N \)-elongated operators \((10)\) and \((11)\) which emphasize \( d \)-forms instead of usual forms on \( M \), with the coefficients taking values in \( E \).

The metricity and Leibniz conditions for \( \mathcal{D} \) are written respectively
\[
g(\mathcal{D}X, Y) + g(X, \mathcal{D}Y) = \delta[g(X, Y)],
\]
for any \( X, Y \in \chi(M) \), and
\[
\mathcal{D}(\sigma \beta) = \mathcal{D}(\sigma) \beta + \sigma \mathcal{D}(\beta),
\]
for any \( \sigma, \beta \in \Gamma^\infty(E) \).

For local computations, we may define the corresponding coefficients of the geometric \( d \)-objects and write
\[
\mathcal{D}\sigma \beta = \Gamma^\alpha_{\beta \mu} \sigma_\alpha \otimes \delta u^\mu = \Gamma^\alpha_{\beta i} \sigma_\alpha \otimes dx^i + \Gamma^\alpha_{\beta a} \sigma_\alpha \otimes \delta y^a,
\]
where fiber "acute" indices, in their turn, may split \( \alpha \equiv (\acute{i}, \acute{a}) \) if any \( N \)-connection structure is defined on \( TE \). For some particular constructions of particular interest, we can take \( E = T^*V, = T^*V(L) \) and/or any Clifford \( d \)-algebra \( E = \text{Cl}(V), \text{Cl}(V(L)) \),... with a corresponding treating of "acute" indices to of \( d \)-tensor and/or \( d \)-spinor type as well when the \( d \)-operator \( \mathcal{D} \) transforms into respective \( d \)-connection \( D \) and spin \( d \)-connections \( \hat{\nabla}^S \), \( \hat{\nabla}^{(SL)} \).... All such, adapted to the \( N \)-connections, computations are similar for both \( N \)-anholonomic (co) vector and spinor bundles.

The respective actions of the Clifford \( d \)-algebra and Clifford–Lagrange algebra (see Definitions 5.2 and 5.2) can be transformed into maps \( \Gamma^\infty(S) \otimes \Gamma^i\text{Cl}(V) \) and \( \Gamma^\infty(S(L)) \otimes \Gamma^i\text{Cl}(V(L)) \) to \( \Gamma^\infty(S) \) and, respectively, \( \Gamma^\infty(S(L)) \) by considering maps of type \((14)\) and \((30)\)

\[
\hat{c}(\psi \otimes a) \doteq c(a)\psi \quad \text{and} \quad \hat{c}(\psi \otimes a) \doteq c(a)\psi.
\]

**Definition 6.2** The Dirac \( d \)-operator (Dirac–Lagrange operator) on a spin \( N \)-anholonomic manifold \((V, S, J)\) (on a Lagrange spin manifold \((M(L), S(L), J)\)) is defined
\[
\text{ID} \doteq -i \ (\hat{c} \circ \nabla^S)
\]
\[
= (L) \text{ID} = -i \ (\hat{c} \circ \nabla^S), \quad (L)^* \text{ID} = -i \ (\hat{c} \circ \nabla^S(L))
\]
\[
= (L)^* \text{ID} = -i \ (\hat{c} \circ \nabla^{(SL)}), \quad (L) \text{ID} = -i \ (\hat{c} \circ \nabla^{(SL)})
\]

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Such $N$–adapted Dirac $d$–operators are called canonical and denoted $\hat{D} = (\hat{D}, \hat{\ast}D) \quad (\hat{L}_\ast \hat{D} = (\hat{L}_\ast \hat{D}, \hat{L}_\ast \hat{\ast}D))$ if they are defined for the canonical $d$–connection (26) (27) and respective spin $d$–connection (54) (55).

Now we can formulate the

**Theorem 6.1 (Main Result 2)** Let $(V, S, J) \quad ((M(L), S(L), J)$ be a spin $N$–anholonomic manifold (spin Lagrange space). There is the canonical Dirac $d$–operator (Dirac–Lagrange operator) defined by the almost Hermitian spin $d$–operator

$$\hat{\nabla}^S : \Gamma^\infty(S) \rightarrow \Gamma^\infty(S) \otimes \Omega^1(V)$$

(spin Lagrange operator)

$$\hat{\nabla}^{(SL)} : \Gamma^\infty(S(L)) \rightarrow \Gamma^\infty(S(L)) \otimes \Omega^1(M(L))$$

commuting with $J (41)$ and satisfying the conditions

$$(\hat{\nabla}^S \bar{\psi} \mid \bar{\phi}) + (\bar{\psi} \mid \hat{\nabla}^S \bar{\phi}) = \delta(\bar{\psi} \mid \bar{\phi}) \quad (62)$$

and

$$\hat{\nabla}^S(c(a)\bar{\psi}) = c(\hat{D}a)\bar{\psi} + c(a)\hat{\nabla}^S \bar{\psi}$$

for $a \in Cl(V)$ and $\bar{\psi} \in \Gamma^\infty(S)$

$$(\hat{\nabla}^{(SL)} \bar{\psi} \mid \bar{\phi}) + (\bar{\psi} \mid \hat{\nabla}^{(SL)} \bar{\phi}) = \delta(\bar{\psi} \mid \bar{\phi}) \quad (63)$$

and

$$\hat{\nabla}^{(SL)}(c(a)\bar{\psi}) = c(\hat{D}a)\bar{\psi} + c(a)\hat{\nabla}^{(SL)} \bar{\psi}$$

for $a \in Cl(M(L))$ and $\bar{\psi} \in \Gamma^\infty(S(L))$ determined by the metricity (58) and Leibnitz (59) conditions.

**Proof.** We sketch the main ideas of such Proofs. There two ways:

The first one is similar to that given in Ref. [29], Theorem 9.8, for the Levi–Civita connection, see similar considerations in [67]. In our case, we have to extend the constructions for $d$–metrics and canonical $d$–connections by applying $N$–elongated operators for differentials and partial derivatives. The formulas have to be distinguished into $h$– and $v$–irreducible components. We are going to present the related technical details in our further publications.

In other turn, the second way, is to argue a such proof is a straightforward consequence of the Result 2.6 stating that any Riemannian manifold can be modeled as a $N$–anholonomic manifold induced by the generic off–diagonal metric structure. If the results from [29] hold true for any Riemannian space,
the formulas may be rewritten with respect to any local frame system, as well
with respect to (10) and (11). Nevertheless, on N–anholonomic manifolds the
canonical d–connection is not just the Levi–Civita connection but a deforma-
tion of type (25): we must verify that such deformations results in N–adapted
constructions satisfying the metricity and Leibnitz conditions. The existence
of such configurations was proven from the properties of the canonical d–
connection completely defined from the d–metric and N–connection coeffi-
cients. The main difference from the case of the Levi–Civita configuration is
that we have a nontrivial torsion induced by the frame nonholonomy. But it
is not a problem to define the Dirac operator with nontrivial torsion if the
metricity conditions are satisfied. □

The canonical Dirac d–operator has very similar properties for spin N–
anholonomic manifolds and spin Lagrange spaces. Nevertheless, theirs geo-
metric and physical meaning may be completely different and that why we
have written the corresponding formulas with different labels and emphasized
the existing differences. With respect to the Main Result 2, one holds three
important remarks:

**Remark 6.3** The first type of canonical Dirac d–operators may be associ-
ated to Riemannian–Cartan (in particular, Riemann) off–diagonal metric
and nonholonomic frame structures and the second type of canonical Dirac–
Lagrange operators are completely induced by a regular Lagrange mechanics.
In both cases, such d–operators are completely determined by the coefficients
of the corresponding Sasaki type d–metric and the N–connection structure.

**Remark 6.4** The conditions of the Theorem 6.1 may be revised for any d–
connection and induced spin d–connection satisfying the metricity condition.
But, for such cases, the corresponding Dirac d–operators are not completely
defined by the d–metric and N–connection structures. We can prescribe cer-
tain type of torsions of d–connections and, via such ‘noncanonical’ Dirac
operators, we are able to define noncommutative geometries with prescribed
d–torsions.

**Remark 6.5** The properties (62) and (63) hold if and only if the metricity
conditions are satisfied (58). So, for the Chern or Berwald type d–connections
which are nonmetric (see Example 3.2 and Remark 6.2), the conditions of
Theorem 6.1 do not hold.

It is a more sophisticate problem to find applications in physics for such
nonmetric constructions but they define positively some examples of non-
metric d–spinor and noncommutative structures minimally deformed from

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8See Refs. [28] and [80, 83, 82] for details on elaborated geometrical and physical
models being, respectively, locally isotropic and locally anisotropic.
the Riemannian (non) commutative geometry to certain Finsler type (non) commutative geometries.

6.2 Distinguished spectral triples

The geometric information of a spin manifold (in particular, the metric) is contained in the Dirac operator. For nonholonomic manifolds, the canonical Dirac d–operator has h– and v–irreducible parts related to off–diagonal metric terms and nonholonomic frames with associated structure. In a more special case, the canonical Dirac–Lagrange operator is defined by a regular Lagrangian. So, such Driac d–operators contain more information than the usual, holonomic, ones.

For simplicity, hereafter, we shall formulate the results for the general N–anholonomic spaces, by omitting the explicit formulas and proofs for Lagrange and Finsler spaces, which can be derived by imposing certain conditions that the N–connection, d–connection and d–metric are just those defined canonically by a Lagrangian. We shall only present the Main Result and some important Remarks concerning Lagrange mechanics and/or Finsler structures.

**Proposition 6.2** If \( \hat{\mathbf{D}} = (\hat{\mathbf{D}}, \star \hat{\mathbf{D}}) \) is the canonical Dirac d–operator then

\[
\left[ \hat{\mathbf{D}}, f \right] = i \mathbf{c}(\delta f), \text{ equivalently, }
\]
\[
\left[ -\hat{\mathbf{D}}, f \right] + \left[ \star \hat{\mathbf{D}}, f \right] = i -c(dx^i \delta f) + i \star c(dy^a \partial f),
\]

for all \( f \in C^\infty(V) \).

**Proof.** It is a straightforward computation following from Definition 6.2.

The canonical Dirac d–operator and its irreversible h– and v–components have all the properties of the usual Dirac operators (for instance, they are self–adjoint but unbounded). It is possible to define a scalar product on \( \Gamma^\infty(S) \),

\[
\langle \psi, \phi \rangle = \int_V (\psi^* | \phi^* ) \mid \nu_g \mid
\]

where

\[
\nu_g = \sqrt{\text{det} g} \sqrt{\text{det} h} \ dx^1 \ldots dx^n \ dy^{n+1} \ldots dy^{n+m}
\]

is the volume d–form on the N–anholonomic manifold \( V \).

We denote by

\[
\mathcal{H}_N = L_2(V, S) = [ -\mathcal{H} = L_2(V, -S), \star \mathcal{H} = L_2(V, \star S)]
\]

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the Hilbert d–space obtained by completing $\Gamma^\infty(S)$ with the norm defined by the scalar product (64).

Similarly to the holonomic spaces, by using formulas (60) and (54), one may prove that there is a self–adjoint unitary endomorphism $\Gamma^{[cr]}$ of $\mathcal{H}_N$, called ”chirality”, being a $\mathbb{Z}_2$ graduation of $\mathcal{H}_N$, ⁹ which satisfies the condition

$$\widehat{D} \Gamma^{[cr]} = -\Gamma^{[cr]} \widehat{D}. \tag{66}$$

We note that the condition (66) may be written also for the irreducible components $\widehat{D}$ and $\cdot \widehat{D}$.

**Definition 6.3** A distinguished canonical spectral triple (canonical spectral d–triple) $(\mathcal{A}, \mathcal{H}_N, \widehat{D})$ for an algebra $\mathcal{A}$ is defined by a Hilbert d–space $\mathcal{H}_N$, a representation of $\mathcal{A}$ in the algebra $\mathcal{B}(\mathcal{H})$ of d–operators bounded on $\mathcal{H}_N$, and by a self–adjoint d–operator $\widehat{D}$, of compact resolution, ¹⁰ such that $[\widehat{D}, a] \in \mathcal{B}(\mathcal{H})$ for any $a \in \mathcal{A}$.

Roughly speaking, every canonical spectral d–triple is defined by two usual spectral triples which in our case corresponds to certain h– and v–irreducible components induced by the corresponding h– and v–components of the Dirac d–operator. For such spectral h(v)–triples we can define the notion of $KR^n$–cycle ($KR^m$–cycle) and consider respective Hochschild complexes. We note that in order to define a noncommutative geometry the h– and v–components of a canonical spectral d–triples must satisfy some well defined Conditions ¹⁷ ²⁹ (Conditions 1 - 7, enumerated in ²⁰, section II.4) which states: 1) the spectral dimension, being of order $1/(n + m)$ for a Dirac d–operator, and of order $1/n$ (or $1/m$) for its h– (or v)–components; 2) regularity; 3) finitness; 4) reality; 5) representation of 1st order; 6) orientability; 7) Poincaré duality. Such conditions can be satisfied by any Dirac operators and canonical Dirac d–operators (in the last case we have to work with d–objects). ¹¹

**Definition 6.4** A spectral d–triple satisfying the mentioned seven Conditions for his h– and v–irreversible components is a real one which defines

⁹We use the label [cr] in order to avoid misunderstanding with the symbol $\Gamma$ used for the connections.

¹⁰An operator $D$ is of compact resolution if for any $\lambda \in sp(D)$ the operator $(D - \lambda \mathbb{I})^{-1}$ is compact, see details in ²⁰ ²⁹.

¹¹We omit in this paper the details on axiomatics and related proofs for such considerations: we shall present details and proofs in our further works. Roughly speaking, we are in right to do this because the canonical d–connection and the Sasaki type d–metric for $N$–anholonomic spaces satisfy the bulk of properties of the metric and connection on the Riemannian space but ”slightly” nonholonomically modified).
a (d–spinor) N–anholonomic noncommutative geometry defined by the data \( (\mathcal{A}, \mathcal{H}_N, \widehat{\text{D}}, J, \Gamma^{[cr]} ) \) and derived for the Dirac d–operator (60).

For a particular case, when the N–distinguished structures are of Lagrange (Finsler) type, we can consider:

**Definition 6.5** A spectral d–triple satisfying the mentioned seven Conditions for his h– and v–irreversible components is a real one which defines a Lagrange, or Finsler, (spinor) noncommutative geometry defined by the data \( (\mathcal{A}, \mathcal{H}(\text{SL}), (\mathcal{L})\widehat{\text{D}}, J, \Gamma^{[cr]} ) \) and derived for the Dirac d–operator (61).

In Ref. [77], we used the concept of d–algebra \( \mathcal{A}_N = (−\mathcal{A}, \star \mathcal{A}) \) which we introduced as a ”couple” of algebras for respective h– and v–irreducible decomposition of constructions defined by the N–connection. This is possible if \( \mathcal{A}_N = −\mathcal{A} \oplus \star \mathcal{A} ) \), but we can consider arbitrary noncommutative associative algebras \( \mathcal{A} \) if the splitting is defined by the Dirac d–operator.

### 6.3 Distance in d–spinor spaces: Main Result 3

We can select N–anholonomic and Lagrange commutative geometries from the corresponding Definitions 6.4 and 6.5 if we put respectively \( \mathcal{A} \cong \mathcal{C}^\infty(\mathcal{V}) \) and \( \mathcal{A} \cong \mathcal{C}^\infty(\mathcal{V}(\mathcal{L})) \) and consider real spectral d–triples. One holds:

**Theorem 6.2 (Main Result 3)** Let \((\mathcal{A}, \mathcal{H}_N, \widehat{\text{D}}, J, \Gamma^{[cr]} ) \)
(or \((\mathcal{A}, \mathcal{H}(\text{SL}), (\mathcal{L})\widehat{\text{D}}, J, \Gamma^{[cr]} ) \)) defines a noncommutative geometry being irreducible for \( \mathcal{A} \cong \mathcal{C}^\infty(\mathcal{V}) \) (or \( \mathcal{A} \cong \mathcal{C}^\infty(\mathcal{V}(\mathcal{L})) \)), where \( \mathcal{V} \) (or \( \mathcal{V}(\mathcal{L}) \)) is a compact, connected and oriented manifold without boundaries, of spectral dimension \( \dim \mathcal{V} = n + m \) (or \( \dim \mathcal{V}(\mathcal{L}) = n + n \)). In this case, there are satisfied the conditions:

1. There is a unique d–metric \( g(\widehat{\text{D}}) = (g, \star g) \) of type (14) on \( \mathcal{V} \) (or of type (14) on \( \mathcal{V}(\mathcal{L}) \)) with the ”nonlinear” geodesic distance defined by

\[
d(u_1, u_2) = \sup_{f \in C(\mathcal{V})} \{ f(u_1, u_2)/ \parallel \parallel [\mathcal{D}, f] \parallel \leq 1 \}
\]

(we have to consider \( f \in C(\mathcal{V}(\mathcal{L})) \) and \((\mathcal{L})\widehat{\text{D}} \) if we compute \( d(u_1, u_2) \) for Lagrange configurations).  

2. The N–anholonomic manifold \( \mathcal{V} \) (or Lagrange space \( \mathcal{V}(\mathcal{L}) \)) is a spin N–anholonomic space (or a spin Lagrange manifold) for which the operators \( \mathcal{D} \) satisfying \( g(\mathcal{D}) = g(\widehat{\text{D}}) \) define an union of affine spaces identified by the d–spinor structures on \( \mathcal{V} \) (we should consider the operators \((\mathcal{L})\mathcal{D} \) satisfying \((\mathcal{L})g(\mathcal{D}) = (\mathcal{L})g(\widehat{\text{D}}) \) for the space \( \mathcal{V}(\mathcal{L}) \)).
3. The functional $S(\tilde{D}) = \int |\tilde{D}|^{-n-m+2}$ defines a quadratic $d$-form with $(n+m)$-splitting for every affine spaces which is minimal for $\tilde{D} = \tilde{D}$ as the Dirac $d$-operator corresponding to the $d$-spin structure with the minimum proportional to the Einstein–Hilbert action constructed for the canonical $d$-connection with the $d$-scalar curvature $\tilde{R}$ \[33\], \[12\]

$$S(\tilde{D}) = -\frac{n + m - 2}{24} \int_V \tilde{R} \sqrt{g} \sqrt{h} \ dx^1 \cdots dx^n \delta y^{n+1} \cdots \delta y^{n+k}.$$ 

**Proof.** In this work, we sketch only the idea and the key points of such a proof. The Theorem is a generalization for $N$-anholonomic spaces of a similar one, formulated in Ref. [17], with a detailed proof presented in [29], which seems to be a final almost generally accepted result. There are also alternative considerations, with useful details, in Refs. [63, 43]. For the Dirac $d$-operators, we have to start with the Proposition 6.2 and then to repeat all constructions from [17, 29], both on $h$- and $v$-subspaces, in $N$-adapted form.

The existence of a canonical $d$-connection structure which is metric compatible and constructed from the coefficients of the $d$-metric and $N$-connection structure is a crucial result allowing the formulation and proof of the Main Results 1-3 of this work. Roughly speaking, if the commutative Riemannian geometry can be extracted from a noncommutative geometry, we can also generate (in a similar, but technically more sophisticated form) Finsler like geometries and generalizations. To do this, we have to consider the corresponding parametrizations of the nonholonomic frame structure, off-diagonal metrics and deformations of the linear connection structure, all constructions being adapted to the $N$-connection splitting. If a fixed $d$-connection satisfies the metricity conditions, the resulting Lagrange–Finsler geometry belongs to a class of nonholonomic Riemann–Cartan geometries, which (in their turns) are equivalents, related by nonholonomic maps, of Riemannian spaces, see [80, 82]. However, it is not yet clear how to perform such a general proof for nonmetric $d$-connections (of Berwald or Chern type). We shall present the technical details of such considerations in our further works.

Finally, we emphasize that for the Main Result 3 there is the possibility to elaborate an alternative proof (like for the Main Result 2) by verifying that the basic formulas proved for the Riemannian geometry hold

\[12\] The integral for the usual Dirac operator related to the Levi–Civita connection $D$ is computed: $\int |D|^{-n+2} = \frac{1}{2\pi n!} \text{Res} |D|^{-n+2}$, where $\Omega_n$ is the integral of the volume on the sphere $S^n$ and $\text{Res}$ is the Wodzicki residu, see details in Theorem 7.5 [29]. On $N$-anholonomic manifolds, we may consider similar definitions and computations but applying $N$-elongated partial derivatives and differentials.
true on N–anholonomic manifolds by a corresponding substitution of the N–elongated differential and partial derivatives operators acting on canonical d–connections and d–metrics. All such constructions are elaborated in N–adapted form by preserving the respective h- and v–irreducible decompositions.

Finally, we can formulate three important conclusions:

**Conclusion 6.1** The formula (67) defines the distance in a manner as to be satisfied all necessary properties (finiteness, positivity conditions, ...) discussed in details in Ref. [29]. It allows to generalize the constructions for discrete spaces with anisotropies and to consider anisotropic fluctuations of noncommutative geometries [50, 51] (of Finsler type, and more general ones, we omit such constructions in this work). For the nonholonomic configurations we have to work with canonical d–connection and d–metric structures.

Following the N–connection formalism originally elaborated in the framework of Finsler geometry, we may state:

**Conclusion 6.2** In the particular case of the canonical N–connection, d–connection and d–metrics defined by a regular Lagrangian, it is possible a noncommutative geometrization of Lagrange mechanics related to corresponding classes of noncommutative Lagrange–Finsler geometry.

Such geometric methods have a number of applications in modern gravity:

**Conclusion 6.3** By anholonomic frame transforms, we can generate noncommutative Riemann–Cartan and Lagrange–Finsler spaces, in particular exact solutions of the Einstein equations with noncommutative variables 13, by considering N–anholonomic deformations of the Dirac operator.

**Acknowledgment**

The work summarizes the results communicated in a series of Lectures and Seminars:

1. S. Vacaru, A Survey of (Non) Commutative Lagrange and Finsler Geometry, lecture 2: Noncommutative Lagrange and Finsler Geometry. Inst. Sup. Tecnico, Dep. Math., Lisboa, Portugal, May 19, 2004 (host: P. Almeida).

13see examples in Refs. [79, 85, 83, 82, 78]
2. S. Vacaru, A Survey of (Non) Commutative Lagrange and Finsler Geometry, lecture 1: Commutative Lagrange and Finsler Spaces and Spinors. Inst. Sup. Tecnico, Dep. Math., Lisboa, Portugal, May 19, 2004 (host: P. Almeida).

3. S. Vacaru, Geometric Models in Mechanics and Field Theory, lecture at the Dep. Mathematics, University of Cantabria, Santander, Spain, March 16, 2004 (host: F. Etayo).

4. S. Vacaru, Noncommutative Symmetries Associated to the Lagrange and Hamilton Geometry, seminar at the Instituto de Matematicas y Fisica Fundamental, Consejo Superior de Investigationes, Ministerio de Ciencia y Tecnologia, Madrid, Spain, March 10, 2004 (host: M. de Leon).

5. S. Vacaru, Commutative and Noncommutative Gauge Models, lecture at the Department of Experimental Sciences, University of Huelva, Spain, March 5, 2004 (host: M.E. Gomez).

6. S. Vacaru, Noncommutative Finsler Geometry, Gauge Fields and Gravity, seminar at the Dep. Theoretical Physics, University of Zaragoza, Spain, November 19, 2003 (host: L. Boya).

The author is grateful to the Organizes and hosts for financial support and collaboration. He also thanks R. Picken for help and P. Martinetti for useful discussions.

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