Entropic inequalities for noncomposite quantum systems realized by superconducting circuits

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We study a class of entropic inequalities obtained for noncomposite quantum systems realized by a superconducting circuit with the Josephson junction. By using a mapping on a bipartite quantum state, we discuss possible realizations of various quantum logic gates for noncomposite quantum systems. In this framework, we consider log \( N \) entropic inequalities for Shannon and Rényi entropies based on the quantum Fourier transform. Implementation of the quantum Fourier transform algorithm on a quantum processor based on superconducting circuits opens a way for experimental verification of these inequalities.

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Introduction. During last decades, quantum correlations and the entanglement phenomenon in composite quantum systems, i.e., a system containing subsystems, have been considered as a potential resource for technologies such as information processing, long-distance secure communications, sensitive metrology, and other applications. Remarkable progress on efficient control for quantum systems on the level of single quantum objects, e.g., in experiments with photons, electrons, nuclear spins, NV-centers, quantum dots, superconducting circuits, ultracold atoms, ions and molecules in optical lattices, has been performed [1].

Fundamental results on generalizations of the Shannon information theory on a quantum domain have been obtained [2, 3]. Quantum correlations for states of composite systems are characterized by various information and entropic characteristics, including the von Neumann entropy and quantum mutual information [4], discord related measures [5, 6], entropic inequalities [7], contextuality [8], causality [9], subadditivity and strong subadditivity conditions [10–12]. Entropic properties of quantum states have been widely studied [11–13] in the framework of \( q \)-deformed entropic functions like the Rényi entropy [14] and the Tsallis entropy [15], depending on one extra parameter, as well as larger number of parameters [16].

Recently, possibility to use noncomposite quantum systems as a potential platform for implementation of quantum technologies has been discussed [17]. As it has been shown, the required properties of quantum systems, like correlations of their subsystems, are available not only in composite systems but in noncomposite systems as well [18–20]. In particular, for \( j = 3/2 \) spin system, information and entropic characteristics have been analyzed [17, 21]. The information properties of quantum states and their characteristics could be associated with arbitrary quantum systems. Thus, all mentioned above information and entropic measures for composite quantum systems indeed can be naturally mapped on the case of a noncomposite quantum system [17–21]. This paradigm opens new perspectives for implementation of quantum technologies, e.g., realizing of quantum algorithms by using noncomposite quantum systems.

The aim of the present work is to investigate the log \( N \) entropic inequalities, with \( N \) being the dimension of the Hilbert space, for a noncomposite quantum system [22]. Important feature of this class of entropic inequalities is a relation with the quantum Fourier transform [13], which plays a crucial role for several quantum algorithms, e.g., the Shor’s algorithm [23]. In our consideration, the noncomposite system is realized by a single superconducting circuit with the Josephson junction. Quantum nanoelectric circuits have been considered in early studies [24–29] as well as in recent theoretical [30, 31] and experimental works [32–37]. Quantum correlations and quantum discord phenomena for coupled \( LC \)-nanoelectric circuits [38] and superconducting circuits with the Josephson junction [39] have been discussed. The details of entropic inequalities as well as the uncertainty relations for \( q \)-deformed entropy need for a clarification, and it is a subject of the presented work.

Quantum states of superconducting circuits. Superconducting circuits with the Josephson junction are a highly promising setup for quantum information processing [35]. A performance of quantum algorithms, including the quantum Fourier transform, has been recently demonstrated [36, 37].

Here, we use a model of a superconducting nanoelectric circuit with the Josephson junction. The Hamiltonian of the system has the following form:

\[
\hat{H} = \frac{\hat{Q}}{2C} + \frac{I_c}{2} \left(1 - \cos \phi \right) - \frac{I_k \phi}{2},
\]

where \( \hat{Q} \) is the charge operator, \( C \) is the capacitance of the junction, \( I_c \) is the critical current, \( \phi \) is the phase
operator, \( I_k \) is the external classical current. Here, we use the units \( e = \hbar = 1 \), where \( e \) is the electron charge and \( \hbar \) is the Planck constant.

In this work, we are in particular interest in realization of a four-level quantum system. One can introduce the following mapping of an “original” four-level system on an “artificial” bipartite (two-qubit) system:

\[
|0\rangle \leftrightarrow |00\rangle, \quad |1\rangle \leftrightarrow |01\rangle, \quad |2\rangle \leftrightarrow |10\rangle, \quad |3\rangle \leftrightarrow |11\rangle,
\]

where we use the following notation:

\[
|ij\rangle \equiv |i\rangle_A \otimes |j\rangle_B.
\]

Here, \( A \) and \( B \) stand for “qubits” of a two-qubit quantum system.

For our original quantum system, we have the density matrix in quanta number basis in the following form:

\[
\rho = \rho_{AB} = \begin{pmatrix}
\rho_{00} & \rho_{01} & \rho_{02} & \rho_{03} \\
\rho_{10} & \rho_{11} & \rho_{12} & \rho_{13} \\
\rho_{20} & \rho_{21}^* & \rho_{22} & \rho_{23} \\
\rho_{30} & \rho_{31}^* & \rho_{32} & \rho_{33}
\end{pmatrix}.
\]

From density matrix (3), we obtain the following density matrices of qubits in their computational basis:

\[
\rho_A = \begin{pmatrix}
\rho_{00} + \rho_{11} & \rho_{02} + \rho_{13} \\
\rho_{02} + \rho_{13}^* & \rho_{22} + \rho_{33}
\end{pmatrix},
\rho_B = \begin{pmatrix}
\rho_{00} + \rho_{22} & \rho_{01} + \rho_{23} \\
\rho_{02} + \rho_{13}^* & \rho_{11} + \rho_{33}
\end{pmatrix}.
\]

It is crucial that the diagonal elements of qubits matrices (4) are composed of the diagonal elements of density matrix (3). This results in the fact that measurements of states of the original system in the quanta number basis is equivalent to the simultaneous measurements of the both virtual subsystems in the computational basis. Combined with implementation of unitary rotation operators (see below), this opens a way to the tomographic characterization of qubit states.

We consider a possible realizations of a set of various quantum logic gates for the system. We assume that we are able to apply \( \theta \)-pulses described by operator

\[
R_x(\theta) = \exp \frac{i\sigma_x\theta}{2} = \begin{pmatrix}
\cos(\theta/2) & i\sin(\theta/2) \\
i\sin(\theta/2) & \cos(\theta/2)
\end{pmatrix},
\]

on a transition between any two levels of our original four-level system. In principle, it is realizable by a coupling of a superconducting circuit to an external resonant field [35]. Non-linearity of the potential, which comes from the Josephson junction (1), makes it possible to address a particular transition in the considered four-level system at least at theoretical level of consideration.

On Fig. 1, we present a possible realizations of CNOT gates (with different controlling qubits), SWAP gate (an exchange of states between \( A \) and \( B \)), and \( \theta \)-pulse gates (5) acting on particular qubits.

**Entropic inequalities.** For a quantum system with the density matrix \( \rho \), the Shannon entropy, obtained after a measurement in the computational basis, has the form:

\[
H(\rho) = -\sum_m \rho_{mm} \log \rho_{mm}.
\]

The Rényi entropy depends on the extra parameter,

\[
\mathcal{R}_\alpha(\rho) = \frac{1}{1-\alpha} \log \sum_m (\rho_{mm})^\alpha, \quad \alpha \geq 0,
\]

where in the limit \( \alpha \to 1 \), Rényi entropy (7) transforms to Shannon entropy (6).
FIG. 2. Verification of log $N$ entropic inequalities (8) and (9) for noncomposite quantum systems realized by a superconducting circuits with the Josephson junction. The left hand side of entropic inequality (9) applied for state (4) at different values of extra parameter $\alpha$ of Rényi entropy (7): (a) $\alpha = 0.6$, (b) $\alpha = 1$, and (c) $\alpha = 10$. At $\alpha = 1$, (9) reduces to inequality (8).

There exists a specific class of inequalities for Shannon entropy (6) and Rényi entropy (7) related with the dimension $N$ of the Hilbert space, so-called log $N$ entropic inequalities [13, 22]. For Shannon entropy (6), the log $N$ inequality holds in the following form:

$$H(\rho) + H(F\rho F^\dagger) \geq \log N,$$

(8)

where $F$ is the quantum Fourier transform operator and we take log to base 2. The similar inequality takes place for Rényi entropy (7):

$$R_\alpha(\rho) + R_\beta(F\rho F^\dagger) \geq \log N, \quad \alpha^{-1} + \beta^{-1} = 2.$$  

(9)

Again, for $\alpha = \beta = 1$ inequality (9) reduces to (8).

In this work, we suggest a way to verify entropic inequalities (8) and (9) for states of a noncomposite quantum system realized by a superconducting circuits with the Josephson junction. We assume that initially the system is in the ground state $\rho = |0\rangle\langle 0|$. Then, we apply the following sequence of pulses. First, $\theta_1$-pulse on the transition $|0\rangle \leftrightarrow |3\rangle$. Second, we apply $\theta_2$-pulse on the transition $|0\rangle \leftrightarrow |1\rangle$. Finally, we apply $\pi/2$-pulse on the transition $|1\rangle \leftrightarrow |2\rangle$, where the unitary transformation of $\theta$ pulse is described by operator (5). In the result, we obtain the state $\rho = |\psi\rangle\langle \psi|$, where:

$$|\psi\rangle = \frac{\theta_2}{\sqrt{2}} \frac{\theta_1}{\sqrt{2}} |0\rangle + \frac{i}{\sqrt{2}} \frac{\theta_2}{\sqrt{2}} \sin \frac{\theta_2}{2} |1\rangle - \frac{1}{\sqrt{2}} \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} |2\rangle + i \sin \frac{\theta_1}{2} |3\rangle.$$  

(10)

State (10) can be mapped on two-qubit state via (2). By using the following notations:

$$a = \frac{1}{2} \cos^2 \frac{\theta_1}{2} \left( \cos^2 \frac{\theta_2}{2} + 1 \right), \quad c = 1 - a,$$

$$b = -\frac{1}{\sqrt{2}} \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \left( \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} - \sin \frac{\theta_1}{2} \right),$$

density matrices of qubits (4) can be written in the form:

$$\rho_A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad \rho_B = \begin{pmatrix} a & ib \\ -ib & c \end{pmatrix}.$$  

(11)

Further, by applying $\pi/2$-pulse on the qubit $B$ (see Fig. 1e), we obtain its state in the following form:

$$\widetilde{\rho}_B = \frac{1}{2} \begin{pmatrix} 1 + 2b & ic - a \\ ic + a & 1 - 2b \end{pmatrix}.$$  

(12)

On the other hand, one can see that the applying of the quantum one-qubit Fourier transform,

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

which is also the Hadamard gate on the state $\rho_A$, gives:

$$F\rho_A F^\dagger = \begin{pmatrix} 1 + 2b & a - c \\ a + c & 1 - 2b \end{pmatrix}.$$  

(13)

The diagonal elements of (12) and (13) coincides. Thus, we can use the state $\widetilde{\rho}_B$ instead of $F\rho_A F^\dagger$ to check entropic inequalities (8) and (9) for $\rho_A$.

In Fig. 2, we present the left-hand sides of Eq. (9) as function of parameters the $\theta_1$-pulse and $\theta_2$-pulse for different values of the extra parameter $\alpha$. The equality holds for a point $\theta_1 = \theta_2 = 0$ and the region $\theta_1 = \pi$. One can see that the increase of the parameter $\alpha$ makes the surfaces of left hand side of inequality to be sharper.

Conclusions. We summarize main results of this work. Implementation of various quantum logic gates for noncomposite systems realized by a superconducting circuit with the Josephson junction is studied. For this noncomposite quantum system, log $N$ entropic inequalities for Shannon and Rényi entropies based on the quantum Fourier transform are considered.

The framework of highly controllable and easily implementable setup results in opportunities of the experimental verification of these entropic inequalities.

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