Bayesian methods of vector autoregressions with tensor decompositions

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ABSTRACT

Vector autoregressions (VARs) are popular in analyzing economic time series. However, VARs can be over-parameterized if the numbers of variables and lags are moderately large. Tensor VAR, a recent solution to overparameterization, treats the coefficient matrix as a third-order tensor and estimates the corresponding tensor decomposition to achieve parsimony. In this paper, the inference of Tensor VARs is inspired by the literature on factor models. Firstly, we determine the rank by imposing the Multiplicative Gamma Prior to margins, i.e. elements in the decomposition, and accelerate the computation with an adaptive inferential scheme. Secondly, to obtain interpretable margins, we propose an interweaving algorithm to improve the mixing of margins and introduce a post-processing procedure to solve column permutations and sign-switching issues. In the application of the US macroeconomic data, our models outperform standard VARs in point and density forecasting and yield interpretable results consistent with the US economic history.

Keywords: Ancillarity-sufficiency interweaving strategy (ASIS), High-dimensional data, Markov chain Monte Carlo (MCMC), Increasing shrinkage prior, Factor model

1 Introduction

Vector autoregression (VAR) is a widely used tool for macroeconomic data since the advocacy of Sims (1980). However, due to over-parameterization, applying VARs to a large set of information is challenging. Over-parameterization is especially an issue for macroeconomic data, because the number of observations is usually smaller than the number of parameters. Without further assumption, inferring this kind of VARs is computationally intensive and leads to inaccurate forecasts as well as misleading impulse responses.

Several approaches have been proposed to solve over-parameterization in VARs. From the frequen-

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tist viewpoint, the most straightforward approach is the penalization strategy. For example, Hsu et al. (2008) and Basu and Michailidis (2015) provided theoretical and empirical support for Lasso (Tibshirani, 1996) in the VAR framework. Several variants (Song and Bickel, 2011; Shojaie and Michailidis, 2010; Nicholson et al., 2020) took account of the temporal effect to distinguish VARs from standard linear regressions.

From the Bayesian perspective, many shrinkage priors have been introduced to encourage parameters to concentrate on zero. The Minnesota prior (Litterman et al., 1979; Litterman, 1986; Doan et al., 1984) is the most prominent prior in the early work of Bayesian VARs. This prior assumes that the own lags of a variable are more informative than lags of other dependent variables, and lower lags explain more variation of times series than higher lags. Barabara et al. (2010); Koop (2013); Carriero et al. (2012); Giannone et al. (2015) extended Minnesota prior and achieved good predicting results. While Minnesota-type priors restrict coefficients in a certain pattern, Stochastic Search Variable Selection (SSVS) (George et al., 2008) requires less prior knowledge than Minnesota-type priors since the SSVS selects coefficients automatically, in the spirit of the spike-and-slab prior class. In contrast to Minnesota and SSVS, a more flexible class of priors called continuous shrinkage priors (Park and Casella, 2008; Brown and Griffin, 2010; Carvalho et al., 2009; Bhattacharya et al., 2015, to name a few) are well-studied and gain attention to VARs (Huber and Feldkircher, 2019; Follett and Yu, 2019; Huber et al., 2019; Gruber and Kastner, 2022).

Alternatively, VARs can achieve parsimony by assuming their coefficient matrices have low ranks. This method is called reduced-rank VAR (Velu et al., 1986; Carriero et al., 2011). A more recent study treated the coefficient matrix as a third-order tensor and inferred this tensor by its decomposition. We refer to this technique as Tensor VAR. The number of parameters in a Tensor VAR with an appropriate decomposition is fewer than that in the standard VAR, thus the Tensor VAR is parsimonious without imposing penalization or shrinkage prior. The Tensor VAR was first introduced by Wang et al. (2021) who decomposed the tensor by a Tucker decomposition (Tucker, 1966) so that parameter space can be shrunk in three dimensions, instead of one in the original reduced-rank VAR idea. In addition to Tensor VARs, Wang et al. (2021) also added regularization terms to induce more sparsity to the model. Fan et al. (2022) extended Tensor VARs to have random effects on each subject in the neuroimaging data. Apart from Tucker decomposition, CANDECOMP/ PARAFAC (CP) decomposition (Kiers, 2000) has also been applied in Zhang et al. (2021) with a time-varying VAR model. Tensor structures also gained popularity in time series models apart from VARs. Related work includes Autoregressive Tensor Processes (ART) (Billio et al., 2022) and time-varying networks (Billio et al., 2022).

Two questions need to be answered when adopting a tensor structure to the parameter space: firstly, how to choose rank(s); secondly, how to interpret margins, i.e. elements in the decomposition. To answer the first question in a frequentist setting, ridge-type ratio estimator (Xia et al., 2015) chooses ranks by minimizing the ratio of ordered eigenvalues of OLS coefficient matrices, while information criteria is an alternative (Zhou et al., 2013; Brandi and Di Matteo, 2021). In a Bayesian setting, authors initialize a large rank value
as the truncation level and shrink the rank to some smaller values by imposing shrinkage priors, such as multiway Dirichlet generalized double Pareto (M-DGDP) prior (Guhaniyogi et al., 2017) and multiway stick breaking shrinkage prior (Guhaniyogi and Spencer, 2021). To answer the second question, Wang et al. (2021) used a variant of Tucker decomposition, called the higher-order singular value decomposition (HOSVD) (De Lathauwer et al., 2000), to ensure identifiability. Chen et al. (2022) applied Varimax (Kaiser, 1958) and normalization to interpret import-export transport networks. Authors including Billio et al. (2022); Guhaniyogi et al. (2017); Zhang et al. (2021); Zhou et al. (2013) did not interpret margins because their emphasis is not on margins, but on the tensor itself.

In this paper, we employ the Tensor VAR structure with the CP decomposition, as is common in Bayesian tensor literature. The aforementioned questions are answered via two contributions, respectively, inspired by Bayesian methods in factor models. Our first contribution is to infer the rank through the help of an increasing shrinkage prior. Motivated by inferring the number of factors in a factor model, we apply the Multiplicative Gamma prior (MGP) (Bhattacharya and Dunson, 2011) to margins and infer them via an adaptive inferential scheme. This idea is closely related to the recent work in Fan et al. (2022), but our prior and the criterion in the adaptive inference are different from theirs. In the simulation study, we show the MGP infers ranks close to their true values and illustrate its merits in accuracy and computational efficiency by comparing it with the M-DGDP, which is one of the most commonly used priors in tensor-structured models.

Our second contribution is to propose two algorithms to achieve better mixing of margins so that they can be interpreted in later analysis. To the best of our knowledge, we are the first to suggest methods to improve the mixing of margins in tensor literature from a Bayesian perspective. The motivation of this contribution is twofold: (1) Wang et al. (2021); Chen et al. (2022) showed that margins are interpretable; (2) Bayesian tensor-structured models (not just Tensor VARs) need a solid foundation for mixing as models are getting more and more complex, e.g. building models with time-varying margins. The first algorithm proposed is a variant of Ancillarity-Sufficiency Interweaving Strategy (ASIS) (Yu and Meng, 2011) with three interweaving steps, inspired by the ASIS algorithm for factor models (Kastner et al., 2017). Simulation results show that our algorithm yields smaller inefficiency factors (IFs), suggesting this algorithm is more efficient in exploring posteriors than its counterpart without interweaving steps. Besides interweaving, margins are divided into three groups during inference to reduce dependence in Gibbs sampling. The second algorithm solves column permutations and sign-switching issues, which the interweaving strategy cannot completely solve. Following a similar work in factor models (Poworoznek et al., 2021), we propose a post-processing procedure that first chooses a pivot from our MCMC samples and then matches the sample in each iteration to this pivot. In addition to these two algorithms, we also clarify why Varimax can be used in tensor-structured models by showing that the tensor is still invariant to Varimax rotations, followed by discussing the difficulty of applying Varimax in Tensor VARs.

We examine the utility of Tensor VARs through two US macroeconomic data sets with medium
and large sizes. Tensor VARs treat the coefficient matrix in two ways: (1) the matricization of a third-order tensor; (2) a sum of the matricization of a third-order tensor and a matrix with only non-zero entries for own lags. The first one corresponds to the original Tensor VAR idea (Wang et al., 2021), and the second one accommodates the feature of Minnesota-type priors, the difference between the own-lag and cross-lag effects, due to the popularity of these priors. In point and density forecasting, these two Tensor VARs are competitive to standard VARs with different prior choices and obtain the best results for joint forecasts. We demonstrate how to interpret margins by inferring the whole large-scale data and constructing factors from inferred margins together with the lagged data. Both Tensor VARs can effectively reduce the number of parameters and construct factors consistent with economic history. Although the additional own-lag matrix in the second Tensor VAR introduces more parameters than the first one, this matrix allows us to infer a lower rank in general and encourages the tensor to explore the cross-lag effect.

The paper is organized as follows. Section 2 explains the Tensor VAR and its connection with factor models. Section 3 and 4 provide the MCMC schemes and post-processing algorithm. Section 5 shows results from simulation experiments. Section 6 presents the forecasting performance and interpretation of Tensor VARs. Section 7 concludes the paper.

2 Tensor VAR

2.1 Basic Notations and Operations

We follow the convention in tensor literature to introduce some basic notations and operations. See Kolda and Bader (2009) for a review.

A tensor $\mathbf{A}$ is a $J$th-order tensor if $\mathbf{A} \in \mathbb{R}^{I_1 \times \cdots \times I_J}$ with entries $A_{i_1,\ldots,i_J}$ for $i_j = 1, \ldots, I_j$. First- and second-order tensors are simply vectors and matrices. We will denote scalars, vectors, matrices and tensors as lower-case letters, lower-case bold letters, bold capital letters and bold Euler cursive capital letters, respectively. Similar to the definition of a diagonal matrix, a tensor is called a $J$th-order superdiagonal tensor if $I_1 = \cdots = I_J = I$ and only its $(i, \ldots, i)$ entries are non-zero for $i = 1, \ldots, I$. Extracting entries by a selected index from a matrix $\mathbf{A}$ and a tensor $\mathbf{A}$ is akin. We will denote $\mathbf{A}(i_1,:)$, $\mathbf{A}(:,i_2)$ and $\mathbf{A}(\ldots,i_J,:)$ as the $i_1$-th row, the $i_2$-th column in $\mathbf{A}$ and a $(J-1)$th-order tensor with entries having the index $i_J$ on the $j$-th dimension in $\mathbf{A}$.

Matricization of a $J$th-order tensor ($J>2$) is an operation that transforms the tensor into a matrix. There are $J$ possible matricizations for a $J$th-order tensor $\mathbf{A}$, and the matricization to the $j$-th dimension is called the mode-$j$ matricization with notation $\mathbf{A}_{(j)} \in \mathbb{R}^{I_j \times \prod_{j' \neq j} I_{j'}}$, where the $i_j$-th row corresponds to the vectorization of $\mathbf{A}_{(\ldots,i_j,:)}$. The $j$-mode product of $\mathbf{A}$ and a matrix $\mathbf{B} \in \mathbb{R}^{K \times I_j}$ is denoted as $\mathbf{A} \times_j \mathbf{B}$, which gives a $J$th-order tensor $\mathbf{C} \in \mathbb{R}^{I_1 \times \cdots \times I_{j-1} \times K \times I_{j+1} \times \cdots \times I_J}$ with the $(i_1, \ldots, i_{j-1}, k, i_{j+1}, \ldots, i_J)$ entry
as

\[ C_{i_1 \ldots i_{J-1}, j, i_{J+1} \ldots i_J} = \sum_{i_J = 1}^{I_J} A_{i_1 \ldots i_{J-1}, i_{J+1} \ldots i_J} B_{k, i_J}. \]

The following are some preliminaries for tensor decompositions. The vector outer product of two vectors \( \beta_1 \in \mathbb{R}^{I_1} \) and \( \beta_2 \in \mathbb{R}^{I_2} \) is \( \beta_1 \circ \beta_2 \), yielding an \( I_1 \)-by-\( I_2 \) matrix with its \( (i_1, i_2) \) entry as \( \beta_{1,i_1} \beta_{2,i_2} \). Assume we have \( J \) vectors with \( \beta_j \in \mathbb{R}^{I_j} \) denoting the \( j \)-th vector, then the \( J \)-way outer product of these vectors, \( \beta_1 \circ \cdots \circ \beta_J \), is a \( I_1 \times \cdots \times I_J \) tensor, with \( (i_1, \ldots, i_J) \) entry as \( \beta_{1,i_1} \cdots \beta_{J,i_J} \).

The most prominent tensor decompositions are CANDECOMP/PARAFAC (CP) decomposition (Kiers, 2000) and Tucker decomposition (Tucker, 1966). A rank-\( R \) CP decomposition of a \( J \)-th-order tensor is

\[
\mathbf{A} = \sum_{r=1}^{R} \beta_1^{(r)} \circ \cdots \circ \beta_J^{(r)} = \sum_{r=1}^{R} \mathbf{A}^{(r)},
\]

where entries in \( \beta_j^{(r)} \in \mathbb{R}^{I_j} \) for \( j = 1, \ldots, J \) and \( r = 1, \ldots, R \), are margins of the tensor, \( \mathbf{A}^{(r)} = \beta_1^{(r)} \circ \cdots \circ \beta_J^{(r)} \in \mathbb{R}^{I_1 \times \cdots \times I_J} \).

Let \( \mathbf{B}_j = (\beta_1^{(1)}, \ldots, \beta_J^{(R)}) \in \mathbb{R}^{I_j \times R} \), for \( j = 1, \ldots, J \), we denote the tensor \( \mathbf{A} \) decomposed by \( \mathbf{B}_1, \ldots, \mathbf{B}_J \) as \( \mathbf{A} = [\mathbf{B}_1, \ldots, \mathbf{B}_J]_{\text{CP}} \), for the sake of brevity. Another useful representation of margins is \( \mathbf{B} = (\mathbf{B}_1', \ldots, \mathbf{B}_J') \in \mathbb{R}^{(I_1+\cdots+I_J) \times R} \) to which we refer as a tensor matrix, then \( \mathbf{A}^{(r)} \) is constructed by margins in the \( r \)-th column of \( \mathbf{B} \).

Instead of having only one rank, Tucker decomposition has \( J \) ranks, \( R_1, \ldots, R_J \), to decompose a \( J \)-th-order tensor as

\[
\mathbf{A} = \sum_{r_1=1}^{R_1} \sum_{r_2=1}^{R_2} \cdots \sum_{r_J=1}^{R_J} \mathbf{G}_{1 \ldots, r_J} \beta_1^{(r_1)} \circ \cdots \circ \beta_J^{(r_J)},
\]

where \( \mathbf{G} \) is a \( J \)-th-order tensor with dimension \( R_1 \times \cdots \times R_J \).

Similar to notations related to the CP decomposition, we can group margins to \( J \) matrices with \( \mathbf{B}_j = (\beta_1^{(1)}, \ldots, \beta_J^{(R_j)}) \in \mathbb{R}^{I_j \times R_j} \), for \( j = 1, \ldots, J \), and denote the tensor \( \mathbf{A} \) decomposed by \( \mathbf{G}, \mathbf{B}_1, \ldots, \mathbf{B}_J \) as \( \mathbf{A} = [\mathbf{G}, \mathbf{B}_1, \ldots, \mathbf{B}_J]_{\text{Tucker}} \). Note that we can only construct a tensor matrix \( \mathbf{B} \) with a Tucker decomposition when \( R_1 = \cdots = R_J = R \). If a Tucker decomposition suffices this condition, and \( \mathbf{G} \) is a superdiagonal tensor, then this decomposition will simplify to a CP decomposition. When writing a CP decomposition in terms of a Tucker decomposition, a superdiagonal tensor with ones on non-zero entries will be useful in this paper, so we denote \( \mathcal{I} \) as this specific type of tensors to distinguish it from other tensors.
Though Tucker decomposition is more flexible than CP decomposition, we use CP decomposition due to its simplicity and leave Tucker decomposition for future work.

### 2.2 Tensor VAR

Let $y_t \in \mathbb{R}^N$ be the $t$-th observation in a multivariate time series. A $P$-order VAR model, $\text{VAR}(P)$, describes the linear relation between $y_t$ and its lags with transition matrices $A_1, \ldots, A_P \in \mathbb{R}^{N \times N}$,

$$
y_t = A_1 y_{t-1} + \ldots + A_P y_{t-P} + \epsilon_t
= Ax_t + \epsilon_t,
$$

where $t = 1 \ldots T$, $A = (A_1, \ldots, A_P)$ is an $N$-by-$NP$ coefficient matrix linearly connecting $y_t$ and its lags, $x_t = (y_{t-1}', \ldots, y_{t-P}')' \in \mathbb{R}^{NP}$. The error term $\epsilon_t$ follows a multivariate normal distribution with zero mean and a time-varying covariance matrix $\Omega_t$. In this paper, we factorize $\Omega_t$ according to Cogley and Sargent (2005),

$$
\Omega_t = H^{-1} S_t (H^{-1})',
$$

where $H^{-1}$ is a lower triangular matrix with ones as diagonal entries, and $S_t$ is a time-varying diagonal matrix with diagonal terms $(s_{t,1}, \ldots, s_{t,N})$. Priors for $H^{-1}$ and $S_t$ are presented in Section 3.

To fit the VAR model, we must estimate the $N^2P$ parameters in $A$ and parameters for the covariance matrix $\Omega_t$. This leads to the problem that the number of coefficients grows quadratically as the number of time series increases, thus VARs can become easily overparameterized. We address this problem by achieving parsimony of $A$ through tensor decomposition, in the spirit of Wang et al. (2021). Specifically, rather than treating $A$ as a matrix, we treat it as a third-order tensor $\mathcal{A} \in \mathbb{R}^{N \times N \times P}$, where $\mathcal{A}_{i_1,i_2,p}$ corresponds to the $(i_1, i_2)$ entry in $A_p$. So far, the number of entries in $\mathcal{A}$ is the same as that in $A$, but we can decompose $\mathcal{A}$ via a rank-$R$ CP decomposition,

$$
\mathcal{A} = [B_1, B_2, B_3]_{\text{CP}},
$$

where $B_1, B_2 \in \mathbb{R}^{N \times R}$, $B_3 \in \mathbb{R}^{P \times R}$. By choosing or inferring an appropriate value of $R$, the number of parameters in the coefficient matrix reduces from $N^2P$ to $(2N+P)R$, so the decomposition alleviates over-parameterization.

After explaining how to decompose the tensor corresponding to the coefficient matrix, we demonstrate how a VAR transforms into a Tensor VAR. Note that the mode-1 matricization of $\mathcal{A}$, $\mathcal{A}(1)$, is equivalent to the coefficient matrix $A$, so equation 2.3 can be changed to

$$
y_t = \mathcal{A}(1)x_t + \epsilon_t.
$$
The Tensor VAR connects \( y_t \) with its lags through margins in \( B_1, B_2, B_3 \),

\[
y_t = B_1 I_{(1)} \text{vec}(B'_{2} X_t B_3) + \epsilon_t
\]

(2.6)

\[
= \sum_{r=1}^{R} B_{1,(r)} \sum_{i_2=1}^{N} \sum_{i_3=1}^{P} \beta_{2,i_2}^{(r)} \beta_{3,i_3}^{(r)} y_{t-i_3,i_2} + \epsilon_t,
\]

(2.7)

where \( I_{(1)} \in \mathbb{R}^{R \times R^2} \) is the mode-1 matricization of a third-order \( I \) (see 2.1 for a detailed description), \( X_t = (y_{t-1}, \ldots, y_{t-P}) \), \( \text{vec}(\cdot) \) is the vectorization operation which transforms \( B'_{2} X_t B_3 \in \mathbb{R}^{R \times R} \) to an \( R^2 \)-dimensional vector, \( B_{1,(r)} \) is the \( r \)-th column of \( B_1 \), \( \beta_{2,i_2}^{(r)} \), \( \beta_{3,i_3}^{(r)} \) are the \((i_2,r)\) and \((i_3,r)\) entries of \( B_2 \) and \( B_3 \), respectively, \( y_{t-i_3,i_2} \) is the \( i_2 \)-th entry in \( y_{t-i_3} \).

We can relate 2.6 to a factor model (FM) (Stock and Watson, 2005, 2011), where \( B_1 \) is the factor loading and \( I_{(1)} \text{vec}(B'_{2} X_t B_3) \) contains \( R \) factors. Since the \( i_1 \)-th row in \( B_1 \) describes the linear relationship between \( y_{t,i_1} \) and factors, \( i_1 = 1, \ldots, N \), we refer to \( B_1 \) as "response loading" following Wang et al. (2021). The formation of factors describes how past information is combined. We look at the last three terms in 2.7 to understand this formation. If \( \beta_{2,i_2}^{(r)} = 0 \), the \( r \)-th factor will not contain information from any lagged values of \( y_{t,i_2} \). Similarly, \( \beta_{3,i_3}^{(r)} = 0 \) results to no information about the \( i_3 \)-th lag of \( y_{t} \) in the \( r \)-th factor. Therefore, the \( i_2 \)-th row of \( B_2 \) contains the effect from the \( i_2 \)-th variable to \( y_t \), and the \( i_3 \)-th row of \( B_3 \) is related to the effect from the \( i_3 \)-th lag to \( y_t \). This interpretation was also discussed in Wang et al. (2021), who called \( B_2 \) and \( B_3 \) "predictor loading" and "temporal loading", respectively.

Another way to explain the CP decomposition in the Tensor VAR is that it separates the lag effect from the variable-wise effect because it decomposes \( A_p \) as

\[
A_p = \sum_{r=1}^{R} (\beta_{1}^{(r)} \circ \beta_{2}^{(r)}) \beta_{3,p}^{(r)}
\]

The first two vectors \( \beta_{1}^{(r)} \) and \( \beta_{2}^{(r)} \) (for \( r = 1, \ldots, R \)) do not depend on the index of \( A_p \), which suggests that all transition matrices share these vectors. The only difference among these transition matrices reflects on the different entries in \( \beta_{3}^{(r)} \). By constructing models in this way, we can investigate the overall effect from a predictor to a response without taking account of lags and analyze the lag effect by comparing \( \beta_{3,p}^{(r)} \) for \( p = 1, \ldots, P \). Although we do not assume the effect from lower lags is more powerful than that from higher lags, it is feasible to impose shrinkage priors to gradually shrink \( \beta_{3,p}^{(r)} \) to zero as \( p \) increases. For example, Zhang et al. (2021) applied the Cumulative Shrinkage Prior (CUSP) (Legramanti et al., 2020) to \( B_3 \); another potential prior is the lag-wise specification in Huber and Feldkircher (2019), which could be an extension of inducing sparsity in Tensor VARs.

Note that 2.5 does not distinguish between the own-lag and cross-lag effects, so we build an extension of 2.5 with the assumption that the own-lag effect is more powerful than the cross-lag effect, which is a
part of the assumption of Minnesota-type priors. In particular, we add a matrix with the same size as the coefficient matrix, and only entries corresponding to the own-lag effect are assumed to be non-zero,

\[ y_t = \mathcal{A}_t(1)x_t + Dx_t + \epsilon_t. \] (2.8)

Empirically, we discover that a large magnitude of an own-lag coefficient (e.g., the magnitude is greater than 0.4), may result in a large estimate of rank and undesirably large coefficient magnitudes for this particular predictor (a column of the coefficient matrix). This discovery is unfavorable to both forecasting and parsimony. The additional term \( D \) solves this problem by separating the own-lag and cross-lag effects and actually induce sparsity by shrinking the rank to a smaller value, albeit we add more parameters in 2.8.

### 2.3 Inspiration from Factor Models

As shown in Wang et al. (2021) and 2.6, Tensor VARs are related to FMs in terms of model construction, but the connection between tensor-structured models (not just Tensor VARs) and FMs is not limited to model construction. We point out two popular questions in both FMs and tensor-structured models and present how related work in FMs inspired us to answer these two questions for Tensor VARs.

Let \( y_t \) still be the \( N \)-dimensional multivariate time series, and let \( f_t = (f_{t,1}, \ldots, f_{t,R})' \) be an \( R \)-dimensional (\( R < N \)) time-varying vector containing latent factors. Then the FM is written as

\[ y_t = \Lambda f_t + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(\mathbf{0}, \Omega_t), \]

where \( \Lambda \in \mathbb{R}^{N \times R} \) is the factor loading.

The first question is: how to determine an appropriate value of \( R \), i.e., the number of factors in the FM or the rank in the tensor-structured model? Note that both \( \Lambda \) and \( B \) have \( R \) columns, so the aforementioned question is equivalent to determining the number of columns in \( \Lambda \) and \( B \). Unsurprisingly, similar techniques have been proposed to answer this question in both fields, including information criteria (see Bai and Ng (2002); Alessi et al. (2010) for FMs and Zhou et al. (2013); Brandi and Di Matteo (2021) for tensors), ridge-type ratio estimator (see Xia et al. (2015) for FMs and Wang et al. (2021) for tensors) and imposing shrinkage priors (see Bhattacharya and Dunson (2011); Legramanti et al. (2020) for FMs and Guhaniyogi et al. (2017); Guhaniyogi and Spencer (2021) for tensors). However, the design of shrinkage priors in these two fields is different. Both priors for FMs are increasing shrinkage priors, which means the larger the column index of \( \Lambda \) is, the sparser the entries of that column will be induced; whereas two priors for tensor-structured models are based on overfitted mixture models (Rousseau and Mengersen, 2011) and are related to the Generalized Double Pareto prior (Armagan et al., 2013). More recently, Fan et al. (2022) filled the gap in determining \( R \) between FMs and tensor-structured models by adopting the MGP (Bhattacharya and Dunson, 2011) to infer ranks in Tensor VARs. Our idea coincides with theirs since the MGP is the prior for margins, but one difference is that we use the original MGP to directly relate inferring ranks to the
literature in FMs, whereas Fan et al. (2022) constructed a variant of the MGP by combining it with horseshoe prior. Another difference between these two ideas is in the adaptive inferential scheme, we refer readers to Section 5.2 for more details.

Rather than inferring factor loadings as in Bhattacharya and Dunson (2011), we impose the MGP to tensor matrix $B$. A margin $\beta_{j,i,j}^{(r)}$ (the $(i,j,r)$ entry of $B_j$) follows the prior below for $j = 1, 2, 3, r = 1, \ldots, R, i_1, i_2 = 1, \ldots, N$ and $i_3 = 1, \ldots, P$:

$$\beta_{j,i,j}^{(r)} \sim N(0, (\sigma_{j,i,j}^{(r)})^2), (\sigma_{j,i,j}^{(r)})^2 = \phi_{(r,j,i,j)}^{-1} \tau_r^{-1},$$

$$\phi_{(r,j,i,j)} \sim \text{Gamma}(\nu/2, \nu/2), \tau_r = \prod_{l=1}^r \delta_l,$$

$$\delta_1 \sim \text{Gamma}(a_1, 1), \delta_l \sim \text{Gamma}(a_2, 1), 1 < l < R,$$

where $\phi_{(r,j,i,j)}$ is a local parameter for the margin with the same index. We store all these local parameters in a matrix $\Phi$ in which each entry corresponds to the local parameter of an entry in $B$ with the same indices. The increasing shrinkage property is induced by $\tau_r$ since $\mathbb{E}(\tau_r) = \prod_{l=1}^r \mathbb{E}(\delta_l) = a_1 a_2^{r-1}$ increases with $r$, when $a_2 > 1$. Hyperparameter $\nu$ is set to be known, and $a_1$ and $a_2$ will be inferred with Gamma priors.

Durante (2017) showed that both $\mathbb{E}(\tau_r)$ and $\mathbb{E}(\tau_r^{-1})$ increase across the column indices when $1 < a_2 < 2$. This result means that the MGP has increasing shrinkage property only when $a_2 > 2$. Thus, we set priors for $a_1$ and $a_2$ as $\text{Gamma}(5,1)$ to have the increasing shrinkage property with a high probability. In addition, $a_2$ also controls the trade-off between inducing shrinkage and maintaining the scales of margins, which motivated the CUSP as an alternative to the MGP. However, in Tensor VARs, we find that the inference is sensitive to hyperparameters in CUSP.

The second question is: how to interpret results from FMs and tensor-structured models? Two issues obstruct the interpretability of these two types of models. The first issue is indeterminacy. It is well known that $\Lambda$ and $f_\ell$ are identified up to an invertible matrix $R$, since $\Lambda f_\ell$ is the same as $\Lambda_0 f_\ell$, if $\Lambda = \Lambda R^0$ and $\tilde{f}_\ell = R^{-1} f_\ell$. Similarly, margins in the CP decomposition for a $J$th-order tensor ($J > 2$) are identified up to scaling and permutation. In detail, $\mathcal{A} = [B_1, \ldots, B_J]_{\text{CP}} = [\tilde{B}_1, \ldots, \tilde{B}_J]_{\text{CP}}$, if $\tilde{B}_j$ comes from the following transformations for $j = 1, \ldots, J$:

1. Scaling: $\tilde{B}_j = B_j R_j$, and $R_j$ is an $R$-by-$R$ diagonal matrix satisfying $\prod_{j=1}^J R_{j,(r,r)} = 1$ for $r = 1, \ldots, R$, where $R_{j,(r,r)}$ is the $r$-th diagonal term in $R_j$.
2. Permutation: $\tilde{B}_j = B_j \Pi$ for an arbitrary $R$-by-$R$ column-wise permutation matrix $\Pi$.

To solve indeterminacy, imposing restrictions is the most straightforward remedy in both fields. For an FM, the factor loading can be restricted to a lower triangular matrix (Geweke and Zhou, 1996) or to have diagonal terms as ones (Aguilar and West, 2000). If we have a $J$-order tensor ($J > 2$), one restriction is that $\beta_{j,1}^{(r)} = 1$, for $j = 1, \ldots, J - 1$ and $r = 1, \ldots, R$, and set $\beta_{j,1}^{(r)}$ in descending order for $r = 1, \ldots, R$ (Zhou et al.,
An alternative remedy for FMs is to exploit indeterminacy by rotating the factor loading to a more interpretable form. Possible rotation includes Varimax (Kaiser, 1958) or Quartimin (Carroll, 1953; Jennrich and Sampson, 1966). An example of using Varimax in a tensor-structured model is Chen et al. (2022), who interpreted the trade network by describing different columns in $B$ as export and import hubs.

Varimax can be used in tensor-structured models because $A$ is invariant to Varimax rotations. Specifically, after using in- and post-processing algorithms to achieve satisfactory mixing in Markov chains and inferring the value of $R$, the tensor sampled in each iteration is $A = [B_1, \ldots, B_J]_{\text{CP}} = [\mathcal{I}, B_1, \ldots, B_J]_{\text{Tucker}}$ (the iteration index is omitted), where $\mathcal{I} \in \mathbb{R}^{R \times \cdots \times R}$ has order $J$. The second equation is valid because the CP decomposition is a specification of the Tucker decomposition. Assume Varimax is applied to all matrices by $\tilde{B}_j = B_j R_j$, where $R_j$ is the rotation matrix optimized from Varimax, for $j = 1, \ldots, J$. $A$ is still identified if it has the decomposition $A = [\tilde{G} = \mathcal{I} \times_1 R_1 \times_2 \cdots \times_J R_J, \tilde{B}_1, \ldots, \tilde{B}_J]_{\text{Tucker}}$. However, Varimax is not a suitable tool for interpreting Tensor VARs from a certain perspective. Since we can write a Tensor VAR like a factor model, see 2.6, it is natural to consider how these margins form factors. In the decomposition before using Varimax, the $r$-th factor is just made up of the $r$-th columns in $B_2$ and $B_3$, because $\mathcal{I}$ is a superdiagonal tensor. By replacing the $\mathcal{I}^{(1)}$ to $\tilde{G}^{(1)}$ and $B_1, \ldots, B_3$ to $\tilde{B}_1, \ldots, \tilde{B}_3$, the equation in 2.6 still holds for the decomposition after using Varimax. Yet the interpretation of this formation of factors will be difficult, because $\tilde{G}$ may not be a superdiagonal tensor, then each factor is related to all entries in $\tilde{B}_2$ and $\tilde{B}_3$. Apart from this difficulty, we do not gain much in interpretability ($B_j$ and $\tilde{B}_j$ are similar) from Varimax when imposing the MGP on margins in practice. Therefore, we do not use Varimax in this paper.

The second issue obstructing the interpretation of FMs or tensor-structured models is the poor mixing of parameters in the MCMC scheme. Kastner et al. (2017) discussed that the usual Gibbs sampler of FMs suffers slow convergence and poor mixing, so the interpretation is prohibitive. This issue is also found in our simulation study for Tensor VARs, see Section 5 for details. For FMs, Kastner et al. (2017) proposed a variant of Ancillarity-Sufficiency Interweaving Strategy (ASIS) (Yu and Meng, 2011) to improve mixing and solved sign-switching issues with a post-processing procedure. Other post-processing methods have been applied to solve column permutations and sign-switching issues in factor loadings (Poworoznek et al., 2021; Papastamoulis and Ntzoufras, 2022, to name a few). However, current literature on Bayesian inference for tensor-structured models neglects this issue because authors are often more interested in the tensor itself, so they pay more attention to the mixing of tensor entries, rather than the mixing of margins. We are motivated by the interpretability of tensors to consider the mixing of margins. Another motivation is that when the model becomes more complex, for example, a time-varying Tensor VAR similar to a dynamic factor model (Sargent et al., 1977; Stock and Watson, 1989) with a time-varying factor loading, we may not guarantee the good mixing of tensor itself. To the best of our knowledge, this is the first paper that discusses the mixing issue of margins. Solutions to alleviate the mixing issue in FMs pique our interest in ASIS and post-processing methods with application in tensor margins. More details can be found in Section 3.
3 Bayesian Inference

3.1 Prior

Apart from the shrinkage prior for $B$, we follow priors in Huber and Feldkircher (2019) for $H$ and $S_t$. Each non-zero off-diagonal entry in $H$ follows a variant of the normal-gamma distribution (Brown and Griffin, 2010),

$$H_{i,j} \sim \mathcal{N}\left(0, \left(\frac{2}{\lambda_h^2}\right)\psi_h^{(i,j)}\right), \psi_h^{(i,j)} \sim \text{Gamma}(a_h, a_h), \text{ for } i = 1, \ldots, N \text{ and } j < i,$$

where $\lambda_h^2$ is the global parameter which controls the overall shrinkage and follows Gamma(0.01, 0.01) prior, $\psi_h^{(i,j)}$ allows flexibility locally, hyperparameter $a_h$ follows an exponential prior with parameter 1.

The sequence $s_{1,n}, \ldots, s_{T,n}$ evolves with a stochastic volatility model (Jacquier et al., 2002; Kim et al., 1998). The logarithm of $s_{t,n}$ follows

$$\ln(s_{t,n}) | \ln(s_{t-1,n}), \mu_n, \psi_n, \sigma_n \sim \mathcal{N}(\mu_n + \psi_n(\ln(s_{t-1,n}) - \mu_n), \sigma_n^2),$$

$$\ln(s_{0,n}) | \mu_n, \psi_n, \sigma_n \sim \mathcal{N}(\mu_n, \sigma_n^2/(1 - \psi_n^2)).$$

Priors of hyperparameters, $\mu_n, \psi_n, \sigma_n^2$, are the same as those in Kastner and Frühwirth-Schnatter (2014), for $n = 1, \ldots, N$. We impose $\mathcal{N}(0, 100)$ to $\mu_n$, Beta(5, 1.5) to $\frac{1 + \psi_n}{\sigma_n^2}$ and Gamma(1/2, 1/2) to $\sigma_n^2$. The prior for $\sigma_n^2$ implies that $\sigma_n$ follows a standard normal distribution.

In the case of 2.8, we impose the same normal-gamma distribution to each non-zero entry in $D$. Let $d_{i,p}$ denote the own-lag coefficient for the $p$-th lag of the $i$-th response, then its prior is written as

$$d_{i,p} \sim \mathcal{N}\left(0, \left(\frac{2}{\lambda_d^2}\right)\psi_d^{(i,p)}\right), \psi_d^{(i,p)} \sim \text{Gamma}(a_d, a_d), \text{ for } i = 1, \ldots, N \text{ and } p = 1, \ldots, P.$$

Priors of hyperparameters are the same as those for lower triangular matrix $H$. The full conditional distributions and their derivations are given in Appendix A.

3.2 An Overview of MCMC Scheme

In this subsection, we compare the widely-used MCMC scheme for a tensor-structured model with our MCMC scheme. In the former scheme (Guhaniyogi et al., 2017; Billio et al., 2022), $\beta_{j}^{(r)}$ is sampled from $p(\beta_{j}^{(r)} | \beta_{<j}^{(r)}, B_{(-r)}, y_{1:T}, (\sigma_{j}^{(r)})^2)$, for $r = 1, \ldots, R$ and $j = 1, 2, 3$, where $\beta_{<j}^{(r)}$ contains all $\beta_{j}^{(r)}$ with $j' \neq j$, $B_{(-r)}$ is $B$ discarding its $r$-th column, $(\sigma_{j}^{(r)})^2$ has all prior variance corresponding to $\beta_{j}^{(r)}$. These full conditional distributions are then incorporated into a usual Gibbs sampler, so each $\beta_{j}^{(r)}$ sampled depends on other margins, and in turn, other margins are sampled given $\beta_{j}^{(r)}$ and other parameters. Note that this MCMC scheme fixes the rank $R$ during the inference, and $R$ can be determined $a posteriori$, in the spirit of the overfitted mixture model. We differentiate our MCMC scheme and the above one in three aspects.
Firstly, we infer tensor margins with three blocks separated by response, predictor and temporal loadings, because a Tensor VAR can be written as

\[ y_t = (x_t'(B_3 \otimes B_2)\mathcal{I}_1 \otimes I_N)\text{vec}(B_1) + \epsilon_t \]  
\[ = B_1\mathcal{I}_1((B_3'X_t') \otimes I_R)\text{vec}(B_2') + \epsilon_t \]  
\[ = B_1\mathcal{I}_1(I_R \otimes (B_2'X_t))\text{vec}(B_3) + \epsilon_t, \]  

where \( \mathcal{I} \in \mathbb{R}^{R \times R \times R} \), \( I_R \in \mathbb{R}^{R \times R} \) is an identity matrix, \( \otimes \) is for the Kronecker product. Thus, the sampling of a loading is conditional on the other two loadings.

Secondly, we do not use a usual Gibbs sampler to sample loadings. Instead, we introduce a variant of ASIS to reduce the parameter autocorrelation during the sampling. This variant contains four parameterizations for tensor margins and interweaves between full conditional distributions under a base parameterization and the other three. We refer readers to Section 3.3 for more detail about parameterizations, algorithms and full conditional distributions.

Lastly, the rank \( R \) in our case is adaptively inferred similarly to Bhattacharya and Dunson (2011), the detailed algorithm can be found in Section 3.4. In the following two subsections, we introduce the interweaving strategy and the adaptive inferential scheme.

### 3.3 Interweaving Strategy

In principle, we could run a standard Gibbs sampler to infer margins and other parameters, but in practice, Markov chains of margins suffer from poor mixing since these chains are highly autocorrelated. The poor mixing is an issue when interpreting margins as discussed in Section 2.3. We circumvent margins with poor mixing by introducing a variant of ASIS.

ASIS unfolds its strategy from its name: sampling the same block of parameters by interweaving two data augmentation schemes, ancillary statistic and sufficient statistic. The benefit of ASIS is that the sampling will be at least as good as the sampling from only one data augmentation; and a low correlation between these two augmentations leads to faster convergence and better mixing, compared to using either augmentation alone. Because of these benefits, ASIS has been applied to many models, including stochastic volatility (Kastner and Frühwirth-Schnatter, 2014), FMs (Kastner et al., 2017) and dynamic linear models (Simpson, 2015; Simpson et al., 2017).

While the original ASIS employed centred and non-centred parameterizations for these two data augmentations, our parameterizations are more related to those in Kastner et al. (2017) for sampling factor loadings and factors, due to the tensor structure. The tensor structure in the Tensor VAR leads to four parameterizations instead of two in Kastner et al. (2017). The first parameterization is simply \( B_1, B_2 \) and
\( B_3 \) described in Section 2. We call this parameterization the base one. The remaining three parameterizations come from specifications of scaling indeterminacy. In particular, \( \mathcal{A} = [B_1, B_2, B_3]_{\text{CP}} = [B_1^*, B_2^*, B_3^*]_{\text{CP}} \) when \( B_1^*, B_2^* \) are transformed from

\[
B_1^* = B_1 D_1^{-1}, B_2^* = B_2 D_1,
\]

where \( D_1 \) is a diagonal matrix with non-zero, non-infinite diagonal entries.

There are infinite choices of \( D_1 \) to get this equivalence, but since our objective is boosting the mixing of margins, we restrict \( D_1 \) to be related to \( B_1 \) and \( B_2 \). After narrowing down the choices of \( D_1 \), there are still multiple available choices, including but not limited to \( D_1 = \text{diag}(\beta_{j,1}^{(1)}, \ldots, \beta_{j,R}^{(R)}) \) (diagonal entries in \( B_1 \)), or \( D_1 = \text{diag}(\beta_{j,i}^{(1)}, \ldots, \beta_{j,i}^{(R)}) \) (the \( i \)-th row in \( B_j \), for \( j=1,2 \)). We choose \( D_1 = \text{diag}(\beta_{1,1}^{(1)}, \ldots, \beta_{1,1}^{(R)}) \) for further demonstration. This choice constrains the first row of \( B_1^* \) to be ones. Other choices of \( D_1 \) will be investigated in future work.

After the transformation, we are able to write the model in terms of \( B_1^*, B_2^* \) and \( D_1 \) for the second parameterization. For \( i_1, i_2 = 1, \ldots, N \), we have

\[
\beta_{1,1}^{(r)} = 1, \beta_{1,i_1}^{(r)} \sim \mathcal{N} \left( 0, \frac{\sigma_{1,i_1}^{(r)}}{\beta_{1,1}^{(r)}}^2 \right), \beta_{2,i_2}^{(r)} \sim \mathcal{N} \left( 0, \left( \frac{\sigma_{2,i_2}^{(r)}}{\beta_{1,1}^{(r)}} \right)^2 \right). \quad (3.4)
\]

The above parameterization only improves the mixing of margins in \( B_1 \) and \( B_2 \), so we also need a parameterization to improve the mixing of margins in \( B_3 \). An obvious choice is to pair \( B_2 \) and \( B_3 \). At this point, each \( B_j \) has been paired at least once, but we conjecture that an additional pair of \( B_1 \) and \( B_3 \) would provide better mixing than just considering three parameterizations because the mixing would be improved across margins in each pair of \( B_j \)'s. Transformations of these two pairs are similar to the one for \( B_1 \) and \( B_2 \),

\[
B_2^{**} = B_2 D_2^{-1}, B_3^{**} = B_3 D_2, \\
B_3^{***} = B_3 D_3^{-1}, B_1^{***} = B_1 D_3,
\]

where \( D_2 \) and \( D_3 \) are diagonal matrices with non-zero, non-infinite diagonal entries.

Similarly, diagonal entries in \( D_2 \) is the first row of \( B_2 \), and likewise for those in \( D_3 \) (as the first row of \( B_3 \)), then the last two parameterizations are presented in terms of \( B_2^{**}, B_3^{**}, D_2 \) and \( B_3^{***}, B_1^{***}, D_3 \), respectively. For \( i_1, i_2 = 1, \ldots, N, i_3 = 1, \ldots, P \), we have

\[
\beta_{2,1}^{**(r)} = 1, \beta_{2,i_2}^{**(r)} \sim \mathcal{N} \left( 0, \frac{\sigma_{2,i_2}^{**(r)}}{\beta_{2,1}^{**(r)}}^2 \right), \beta_{3,i_3}^{**(r)} \sim \mathcal{N} \left( 0, \left( \frac{\sigma_{3,i_3}^{**(r)}}{\beta_{2,1}^{**(r)}} \right)^2 \right), \quad (3.5)
\]

\[
\beta_{3,1}^{***(r)} = 1, \beta_{3,i_3}^{***(r)} \sim \mathcal{N} \left( 0, \frac{\sigma_{3,i_3}^{***(r)}}{\beta_{3,1}^{***(r)}}^2 \right), \beta_{1,i_1}^{***(r)} \sim \mathcal{N} \left( 0, \left( \frac{\sigma_{1,i_1}^{***(r)}}{\beta_{3,1}^{***(r)}} \right)^2 \right). \quad (3.6)
\]
We need to sample margins under the four parameterizations described in each iteration. The sampling using the base parameterization is stated in Appendix A, so we focus on sampling margins under the rest three parameterizations in this subsection. For $\beta_{1,1}^{(r)}$, its normal prior implies that $(\beta_{1,1}^{(r)})^2$ has a gamma prior, Gamma $\left(\frac{1}{2}, \frac{1}{2\sigma_{1,1}^2}\right)$. The conditional posterior of $(\beta_{1,1}^{(r)})^2$ under 3.4 is a Generalized Inverse Gaussian (GIG),

$$
(\beta_{1,1}^{(r)})^2 | B_{1,(r)}^*, B_{2,(r)}^* \sim \text{GIG} \left(0, \sum_{i_2=1}^{M} \left(\frac{\beta_{2,i_2}^{(r)}}{\sigma_{2,i_2}^2}\right)^2, \sum_{i_1=2}^{M} \left(\frac{\beta_{1,i_1}^{(r)}}{\sigma_{1,i_1}^2}\right)^2 + \left(\frac{1}{\sigma_{1,1}^2}\right)^2\right), \quad (3.7)
$$

where a variable $x \sim \text{GIG}(\lambda, \chi, \psi)$ has probability density function $p(x) = x^{\lambda-1} \exp\left(-\left(\frac{\chi}{x} + \frac{\psi}{x}\right)\right)$. Similarly, we can get conditional distributions of $(\beta_{2,1}^{(r)})^2$ under 3.5 and $(\beta_{3,1}^{(r)})^2$ under 3.6:

$$
(\beta_{2,1}^{(r)})^2 | B_{2,(r)}^{**}, B_{3,(r)}^{**} \sim \text{GIG} \left(\frac{M - P}{2}, \sum_{i_3=1}^{P} \left(\frac{\beta_{3,i_3}^{(r)}}{\sigma_{3,i_3}^2}\right)^2, \sum_{i_2=2}^{M} \left(\frac{\beta_{2,i_2}^{(r)}}{\sigma_{2,i_2}^2}\right)^2 + \left(\frac{1}{\sigma_{2,1}^2}\right)^2\right), \quad (3.8)
$$

$$
(\beta_{3,1}^{(r)})^2 | B_{3,(r)}^{***}, B_{1,(r)}^{***} \sim \text{GIG} \left(\frac{P - M}{2}, \sum_{i_1=1}^{M} \left(\frac{\beta_{1,i_1}^{(r)}}{\sigma_{1,i_1}^2}\right)^2, \sum_{i_3=2}^{P} \left(\frac{\beta_{3,i_3}^{(r)}}{\sigma_{3,i_3}^2}\right)^2 + \left(\frac{1}{\sigma_{3,1}^2}\right)^2\right). \quad (3.9)
$$

Four parameterizations and their corresponding conditional posteriors have been discussed. We then outline Algorithm 1 to explain how to interweave sampling under the base parameterization to the other three.

**Algorithm 1** Initialize unknown parameters and repeat the following steps in each iteration:

1. **Step (a):** Update $B_{1}^{\text{old}}$ under the base parameterization.
2. **Step (b*):** Store the first row of $B_{1}^{\text{old}}$ into $D_1$ and determine $B_{1}^{*}, B_{2}^{*}$.
3. **Step (b**):** Sample $(\beta_{1,1}^{\text{new}(r)})^2$ for $r = 1, \ldots, R$ using 3.7 under the second parameterization and store corresponding values in $D_1$.
4. **Step (b***): Update $B_{1}^{\text{new}}$ and $B_{2}^{\text{new}}$ with transformation

   $$
   B_{1}^{\text{new}} = B_{1}^{*}D_1, \quad B_{2}^{\text{new}} = B_{2}^{*}D_{1}^{-1}.
   $$

5. **Step (c):** Update $B_{2}^{\text{old}}$ under the base parameterization.
6. **Step (d*):** Store the first row of $B_{2}^{\text{old}}$ into $D_2$ and determine $B_{2}^{**}, B_{3}^{**}$.
7. **Step (d**):** Sample $(\beta_{2,1}^{\text{new}(r)})^2$ for $r = 1, \ldots, R$ using 3.8 under the third parameterization and store corresponding values in $D_2$.
8. **Step (d***): Update $B_{2}^{\text{new}}$ and $B_{3}^{\text{new}}$ with transformation

   $$
   B_{2}^{\text{new}} = B_{2}^{**}D_2, \quad B_{3}^{\text{new}} = B_{3}^{**}D_{2}^{-1}.
   $$
Step (e): Update $B_3^{\text{old}}$ under the base parameterization.

Step (*f*): Store the first row of $B_3^{\text{old}}$ into $D_3$ and determine $B_3^{***}$, $B_1^{***}$.

Step (**f**): Sample $(\beta_{3,1}^{\text{new}(r)})^2$ for $r = 1, \ldots, R$ using 3.9 under the fourth parameterization and store corresponding values in $D_3$.

Step (**f***): Update $B_3^{\text{new}}$ and $B_1^{\text{new}}$ with transformation

$$B_3^{\text{new}} = B_3^{***} D_3, \quad B_1^{\text{new}} = B_1^{***} D_3^{-1}.$$ 

Step (g): Sample other unknown parameters from their full conditional posteriors.

If we discard steps with *’s, this sampling is just Gibbs sampling using the base parameterization. The first interweaving (Step (a) to Step (c)) alternates samplings under the base and the second parameterization to update $B_1$ and $B_2$. Since $B_3$ is unchanged in the second parameterization, this interweaving does not involve $B_3$. Similarly, the second (third) interweaving, corresponding to Step (c) to Step (e) (Step (e) to Step (f)), alternates samplings under the base and the third (fourth) parameterization to update $B_2$ ($B_3$) and $B_3$ ($B_1$), while $B_1$ ($B_2$) is not involved. Every interweaving step starts at the base parameterization, then switches to an alternative parameterization and swaps back to the base one. It is noteworthy that some $B_j$’s in the algorithm have superscript new. This is because each $B_j$ is included in two interweaving steps, but we only store one sample for each $B_j$ in each iteration. It will be easier to distinguish between the one stored (with superscript "new") and the one left (with superscript new).

### 3.4 Adaptive Inference of Rank

We infer the rank using an adaptive inferential scheme, inspired by Bhattacharya and Dunson (2011) and Legramanti et al. (2020), who implemented an adaptive algorithm to infer the number of factors in an FM. Instead of finding inactive factors in their case, we aim to find inactive columns in $B$, i.e. those columns do not contribute much to the tensor $\mathcal{A}$. The algorithm is displayed in Algorithm 2.

In this algorithm, $R^*$ is the rank initialization set to be $\lceil 5 \log N \rceil$, which is the same as for the number of factors in Bhattacharya and Dunson (2011). Empirically, this initialization is large enough to estimate the coefficient matrix. In order to suffice diminishing adaptation condition (Roberts and Rosenthal, 2007) for the weak law of large number in adaptive MCMC, we discard inactive columns in the $m$-th iteration with probability $p(m) = \exp(\alpha_0 + \alpha_1 m)$, where $\alpha_0 \leq 0$, $\alpha_1 < 0$. This is why we need to initialize $\alpha_0$ and $\alpha_1$ at the beginning of the algorithm. $p(m)$ is getting smaller as the index of iteration $m$ becomes larger, then it will be less likely to change $R$ during the inference. Lastly, we need to set a criterion to decide whether a column in $B$ is active or not. In this paper, this criterion is related to the proportion of small magnitudes in $\mathcal{A}^{(r)}$ (see 2.1 for the definition), for $r = 1, \ldots, R$. For ease of explanation, we omit $m$ here. We regard an entry in $\mathcal{A}^{(r)}$ has a small magnitude if its absolute value is smaller than a threshold, e.g. $10^{-3}$. If the
Algorithm 2 Adaptive Inference of Rank

Initialize $R^*$, $\alpha_0$, $\alpha_1$ and set a criterion

while iteration $\tilde{m} < m \leq m_{\text{burn-in}}$ do

Sample $u$ from Uniform(0,1) and let $R^{(m)}$ be the rank at iteration $m$

if $p(m) \geq u$ then

$k = \# \{\text{inactive columns}\}$

if $k > 0$ then

Remove inactive columns in $B$ and related parameters

$R^{(m)} = R^{(m-1)} - k$

else

Add one column to $B$ and add values to related parameters

$R^{(m)} = R^{(m-1)} + 1$

end

end

The proportion of small magnitudes in $A^{(r)}$ is larger than a value set a-priori, e.g. 0.9, then we regard the $r$-th column in $B$ as inactive. Note that this criterion is different from that in Fan et al. (2022), who treated a column as inactive if it has an $l^2$ norm close to a neighborhood of zero.

Adaptive inference begins after the $\tilde{m}$-th iteration to stabilize Markov chains and stops at the last iteration during the burn-in period to allow easy interpretation of margins. If the number of inactive columns is greater than 0, we remove these columns in $B$ and remove corresponding parameters in $\Phi$, $\delta = (\delta_1, \ldots, \delta_R)$, $\tau = (\tau_1, \ldots, \tau_R)$. The rank will then be shrunk to a smaller number of active columns. If the algorithm does not detect any inactive column, we first sample a new column to $\Phi$, a new entry to $\delta$ and subsequently compute the new entry in $\tau$. A new column in $B$ will also be sampled using these newly-sampled hyperparameters.

4 Post-processing Procedure

The interweaving strategy only improves the mixing of entries in $B$ up to column permutations and sign-switching issues. The method proposed to solve these problems is inspired by the Match-Sign-Factor (MSF) algorithm in the R package infinitefactor (Poworoznek et al., 2021). The MSF performs a greedy search to rotate factor loadings and factors in FMs, and we apply a variant of this algorithm to Tensor VARs. Our algorithm is presented in 3, along with a detailed explanation divided into two parts: (1) solve column permutations by the label-matching method; (2) solve sign-switching issues by the sign-matching method. The following paragraph describes the first part.

Note that column permutations in $B$ are equivalent to those in $B_3$, so if we solve the equivalent issue in $B_3$, we will automatically solve column permutations in $B$. There are analogous equivalences related to $B_1$ and $B_2$, but the label matching related to $B_3$ gives the best mixing results of margins. We will
This rule guarantees that the tensor after sign matching is identified. After matching labels and signs, we can add a step to shrink further the rank determined in Algorithm 2. In each iteration, we compute proportions of small magnitudes of margins in the same way as in Section 3.4. If an average proportion of a particular column is greater than the threshold, this column is deleted.

After choosing the pivot, we match each column of $B_3$ in each iteration to the column in $(B_3^{(\text{pivot})}, -B_3^{(\text{pivot})})$ with the smallest Euclidean distance between these two columns. To avoid matching multiple columns of $B_3$ to the same column in $B_3^{(\text{pivot})}$ or $-B_3^{(\text{pivot})}$, a greedy search is applied. In particular, all distances are computed and stored in an $R$-by-$2R$ matrix, with $(r_1, r_2)$ entry as the Euclidean distance between $B_{3,(\cdot,r_1)}$ and $B_{3,(\cdot,r_2)}^{(\text{pivot})}$ if $r_2 \leq R$ or $-B_{3,(\cdot,r_2-R)}^{(\text{pivot})}$ otherwise. We first select the smallest entry in this distance matrix, say $(r_1^*, r_2^*)$ entry, then align $B_{3,(\cdot,r_1^*)}$ to $B_{3,(\cdot,r_2^*)}^{(\text{pivot})}$ or $-B_{3,(\cdot,r_2^*-R)}^{(\text{pivot})}$ so their column indices are the same. Since the order of columns in $B$ is the same as that in $B_3$, conducting this alignment to $B$ allows $B_{(:,r_1^*)}$ to match to $B_{(:,r_2^*)}^{(\text{pivot})}$ or $-B_{(:,r_2^*-R)}^{(\text{pivot})}$, where $B^{(\text{pivot})}$ is the tensor matrix corresponding to $B_3^{(\text{pivot})}$. Entries in the $r_1^*$-th row as well as $r_2^*$-th and $(r_2^*+R)$-th (or $(r_2^*-R)$-th and $r_2^*$-th) columns in the distance matrix are then set to be infinite later, so these entries will not be selected in the next matching. We repeat the above procedure for $R$ times, and the label matching is finished.

As mentioned earlier, selecting samples of $B_3$ as candidates for the pivot matrix, rather than $B_1$, $B_2$ and $B$, leads to the best mixing results of margins. Note that numbers of rows in $B_1$, $B_2$, $B_3$ and $B$ are $N$, $N$, $P$, $2N+P$, respectively. One possible reason for this best performance of using $B_3$ is that $N$ and $2N+P$ are greater than $P$ in our simulation and real data experiments, so it is easier to correctly match columns in $B_3$ to those in $B_3^{\text{pivot}}$, compared to similar procedures using $B_1$, $B_2$ and $B$.

Next, we explain the second part about the sign-matching method. It requires us to use $B^{(\text{pivot})}$, instead of $B_3^{(\text{pivot})}$ in this part, because all signs of entries in $B$ will be matched. For $j = 1, 2, r = 1, \ldots, R$, we compute the distances between $B_{j,(\cdot,r)}$ and $B_{j,(\cdot,r)}^{(\text{pivot})}$, $-B_{j,(\cdot,r)}^{(\text{pivot})}$. The smaller distance indicates whether to flip sign in $B_{j,(\cdot,r)}$ or not, and we record these two indicators (1 for keeping signs and -1 for flipping signs). After matching signs for $B_{1,(\cdot,r)}$ and $B_{2,(\cdot,r)}$, we decide whether to flip signs in $B_{3,(\cdot,r)}$ based on the former indicators. The rule of thumb is that the product of two indicators and the indicator for $B_{3,(\cdot,r)}$ should equal 1. This rule guarantees that the tensor after sign matching is identified.
Algorithm 3 Match Labels and Signs

Find a pivot matrix $B^{(\text{pivot})}_3$ and its corresponding tensor matrix $B^{(\text{pivot})}$

for each iteration do

Compute the $R$-by-$2R$ distance matrix $\Theta$

for $r = 1, \ldots, R$ do

Find $(r_1^*, r_2^*) = \arg\min_{r_1, r_2} \Theta_{r_1, r_2}$

if $r_2^* \leq R$ then

Match the $r_1^*$-th column in $B$ to the $r_2^*$-th column in $B^{(\text{pivot})}$

else

Match the $r_1^*$-th column in $B$ to the $(r_2^*-R)$-th column in $B^{(\text{pivot})}$

for $j = 1, 2$ do

Compute distance $d_1 = d(B_{j,(.,r)}^{(\text{pivot})}, B_{j,(.,r)}^{(\text{pivot})})$ and $d_2 = d(B_{j,(.,r)}, -B_{j,(.,r)}^{(\text{pivot})})$

if $d_1 \leq d_2$ then

Keep signs in $B_{j,(.,r)}$. Record $\text{ind}_{j,r} = 1$

else

flip signs in $B_{j,(.,r)}$. Record $\text{ind}_{j,r} = -1$

if $\text{ind}_{1,r} \text{ind}_{2,r} = 1$ then

Keep the signs in $B_{3,(.,r)}$

else

flip the signs in $B_{3,(.,r)}$

end

end

end

end

5 Simulation Results

5.1 Data and Implementation

We assess the merits of inferring ranks using the MGP and the adaptive inferential scheme in 5.2 and show that the interweaving strategy and the post-processing procedure can improve the mixing of margins in 5.3. The following two subsections use the same simulation data, which includes three scenarios with different combinations of the number of time series and rank $(N, R)$: $(10, 3)$, $(20, 5)$ and $(50, 10)$. In each scenario, we generate 10 data sets following VAR(3) models with independently generated parameters. The coefficient matrix of each model is the 1-mode matricization of a tensor from a CP decomposition, and the covariance matrix is an identity matrix. Margins of the CP decomposition follow uniform distributions with different parameters; see Table 1 for more detail. All time series are then checked by Dickey-Fuller test and Kwiatkowski–Phillips–Schmidt–Shin (KPSS) tests for stationarity.

We apply the MGP to both simulation experiments by setting $\nu = 3$, the threshold to be $10^{-3}$, and the threshold of the proportion of small magnitudes to be 0.9. Apart from the MGP, we briefly introduce
Table 1: Uniform distributions of margins in different locations indicated by rows and different scenarios indicated by columns.

|    | (10,3)       | (20,5)       | (50,10)      |
|----|--------------|--------------|--------------|
| $B_1$ | $U(-1,1)$    | $U(-1,1)$    | $U(-1,1)$    |
| $B_2$ | $U(-1,1)$    | $U(-1,1)$    | $U(-0.6,0.6)$|
| $B_{3,1}$ | $U(-1,1)$    | $U(-1,1)$    | $U(-0.6,0.6)$|
| $B_{3,2}$ | $U(-0.5,0.5)$ | $U(-0.2,0.2)$ | $U(-0.2,0.2)$ |
| $B_{3,3}$ | $U(-0.1,0.1)$ | $U(-0.1,0.1)$ | $U(-0.1,0.1)$ |

M-DGDP (Guhaniyogi et al., 2017) which is another prior applied in this section. M-DGDP is a global-local shrinkage prior proposed for a tensor-structured model with the following expression

$$\beta_j^{(r)} \sim \mathcal{N}(0, (\phi_r \tau) W_{jr}), \; w_{jr,k} \sim \text{Exp}(\lambda_{jr}/2), \; \lambda_{jr} \sim \text{Gamma}(a_\lambda, b_\lambda),$$

$$\Phi = (\phi_1, \ldots, \phi_R)' \sim \text{Dirichlet}(\alpha, \ldots, \alpha), \; \tau \sim \text{Gamma}(a_\tau, b_\tau),$$

where $W_{jr} = \text{diag}(w_{jr,1}, \ldots, w_{jr,I_j})$, $I_j = N$ when $j = 1, 2$ and $I_j = P$ when $j = 3$ in our case. $\alpha$ is uniformly distributed on a grid with values equally placed on $[R^{-J}, R^{-0.01}]$, $J$ is the order of tensor, which is 3 in our case, and $R$ is the rank set in advance.

We follow the same setting of hyperparameters as in Guhaniyogi et al. (2017), i.e. $a_\lambda = 3$ and $b_\lambda = \sqrt[3]{a_\lambda}, \; a_\tau = R\alpha, \; b_\tau = \alpha \sqrt[3]{R}$. For both priors, we set the rank as $\lceil 5 \log(N) \rceil$, but the adaptive inferential scheme is only applied when using the MGP after iteration reaches 200 in the burn-in period. We implement all simulations with Intel(R) Xeon(R) Gold 6140 CPU 2.30GHzr and R 4.2.0.

5.2 Rank Selection

The first simulation assesses the performance of our approach in inferring the rank $R$. Both samplers with MGP and M-DGDP were run for 10,000 iterations after 10,000 burn-in and incorporated the interweaving strategy. After running all simulation experiments, both MGP and M-DGDP have labels and signs matched, and ranks are determined by the procedure in Section 4. We also assess the performance through accuracy (mean squared error of coefficient matrix), efficiency (averaged effective sample size of coefficients) and approximate running time.

Table 2 shows simulation results from MGP and M-DGDP. Both models estimate coefficient matrices with similar accuracy, with MGP results being slightly better than M-DGDP for all three scenarios. Ranks inferred by both priors are overestimated, but the MGP is able to infer ranks closer to the true ranks. The MGP also explores coefficient posteriors more efficiently as suggested by ESS results from the first two scenarios. The adaptive shrinkage algorithm accelerates computation since the running time of the MGP grows more slowly with $N$ and $R$ in comparison to the growth rate of the M-DGDP.
Table 2: Performance of MGP and M-DGDP in 10 simulations for different dimensionality combinations.

| $(N, R)$ | method | MSE   | $R$ | ESS         | running time (hr) |
|----------|--------|-------|-----|-------------|-------------------|
| (10,3)   | MGP    | 0.054 | 4.8 | 4303.71     | 0.5               |
|          | M-DGDP | 0.055 | 4.8 | 4000.62     | 1.1               |
| (20, 5)  | MGP    | 0.044 | 7.6 | 2532.11     | 0.8               |
|          | M-DGDP | 0.046 | 9.3 | 2352.77     | 2.3               |
| (50, 10) | MGP    | 0.044 | 11.1| 918.13      | 1.1               |
|          | M-DGDP | 0.045 | 15.3| 929.69      | 13.4              |

5.3 Mixing

The second simulation focuses on the effect of the interweaving strategy and the post-processing procedure. We choose three prior settings (standard normal, MGP, M-DGDP) to infer margins with/without interweaving. The burn-in period still has 10,000 iterations, but we change the number of iterations after burn-in to 100,000 to demonstrate results with longer chains. To show the necessity of both the interweaving strategy and the post-processing procedure, we apply the label- and sign-matching methods to both sets of results with/without interweaving, and the effect of the post-processing procedure will be illustrated using results with interweaving.

Figure 1: Trace plots of the first 10,000 draws of $\beta_{1,1}^{(1)}$ in $N = 10$, $R = 3$ scenario after burn-in period. The inferential scheme adopts standard normal (top), MGP (middle) and M-DGDP (bottom) as priors and applies with (left panel) and without (right panel) interweaving strategy.

To give an insight to the effect of interweaving, Figure 1 shows trace plots of the margin $\beta_{1,1}^{(1)}$ when $N = 10$ and $R = 3$ based on different prior settings with/without interweaving. Even though we used the label- and sign-matching method, trace plots without interweaving still suffer from the mixing problem, while the interweaving strategy improves mixing. If we compare rows in Figure 1a, they present different behaviour about how the same inferential scheme explores posteriors using different prior settings. Draws inferred from the standard normal prior with interweaving jump between the maximum and minimum of a certain
range frequently and do not have any extreme values. Whereas those inferred from MGP and M-DGDP with interweaving explore posteriors by staying around the mean with occasional jumps to some extreme values. MGP and M-DGDP also show the effect of shrinkage. Draws of M-DGDP without interweaving in Figure 1b also occasionally jump to extreme values, but these draws stay around these extreme values for several iterations, unlike its interweaving counterpart which returns immediately. Figure 2 displays autocorrelations (acfs) of all draws of $\beta_{1,1}^{(1)}$ after the burn-in period. The acfs from the interweaving strategy decay quickly, with only the one from the standard normal prior showing non-negligible values by 100 lags. All of these three acfs without interweaving remain large for many lags.

Figure 2: Autocorrelations of $\beta_{1,1}^{(1)}$ in $N = 10$, $R = 3$ scenario after the burn-in period. The inferential scheme adopts standard normal (top), MGP (middle) and M-DGDP (bottom) as priors and applies with (left panel) and without (right panel) interweaving strategy.

We follow the procedure in Kastner et al. (2017) to compute the inefficiency factor (IF) of each margin in different scenarios and prior settings. A small IF means that the sampling of a parameter is efficient. Figure 3 displays boxplots of IFs where each panel corresponds to a scenario with a combination of $(N, R)$ and $B_j$. Each boxplot is constructed by average IFs of different rows in $B_j$ inferred from different data sets, so there are 100, 200 and 500 (for different scenarios) average IFs in the top and middle panels and 30 average IFs in the bottom panels. Overall, average IFs with interweaving have lower median values and less variation, compared to their counterparts without interweaving. Although some of these boxplots corresponding to inference with interweaving have outliers, these outliers are lower than the upper whiskers of boxplots from the non-interwoven algorithm. While the number of outliers increases with $N$ for both algorithms with/without interweaving, the median in each boxplot usually stays at the same level invariant to $N$.

Next, we demonstrate the necessity of matching labels and signs, even using an interweaving strategy. Figure 4 displays trace plots of the whole draws (with thinning as 10) of two selected margins, and we exclude the post-processing procedure at this time. All three panels in Figure 4a and the middle panel in Figure 4b have sign-switching issues. If we do not match signs, the interpretation of margins will be infeasible because the posterior mode or mean of some margins would be zero, but they should be non-zero.
The top panel in Figure 4b provides evidence of column permutations, with the sample mean moving from 0 to 0.5. The bottom panel in Figure 4b has neither sign switching nor column permutations, but the M-DGDP does not guarantee good mixing only with the interweaving strategy due to the evidence provided in Figure 4a.

Figure 3: Boxplots of average inefficiency factor of $B_1$ (top), $B_2$ (middle) and $B_3$ (bottom) from different scenarios: $N = 10, R = 3$ (left), $N = 20, R = 5$ (middle), $N = 50, R = 10$ (right). Inferential schemes with and without interweaving are represented as "I-" and "N-", respectively, followed by a prior setting.

Figure 4: Trace plots of $\beta_{1,1}^{(1)}$ in $N = 10, R = 3$ scenario (left) and $\beta_{1,2}^{(1)}$ in $N = 20, R = 5$ scenario (right) after burn-in period. The inferential scheme adopts standard normal (top), MGP (middle) and M-DGDP (bottom) as priors and applies with the interweaving strategy.
6 Real Data Application

6.1 Data and Implementation

We use the US macroeconomic data described in Korobilis and Pettenuzzo (2019) to assess the utility of Tensor VARs. The data contains 124 quarterly variables from Federal Reserve Economic Data (FRED) (McCracken and Ng, 2020) and spans from 1959Q1 to 2015Q4. All time series are transformed to stationarity and standardized to have mean zero and variance one, to avoid any scaling issues. We extract 20 and 40 variables to construct medium-scale (N=20) and large-scale (N=40) data sets. The selected variables can be divided into 8 categories: (i) output and income, (ii) consumption, orders and inventories, (iii) labour market, (iv) prices, (v) interest rate, (vi) money and credit, (vii) stock market and (vii) exchange rate. Since we have a lower triangular matrix $H$ in the model, the order of time series matters. We follow Carriero et al. (2019) and Bernanke et al. (2005) by splitting time series to slow, fast groups and Federal Funds Rate (FEDFUNDS). The slow group contains variables that respond to a shock of FEDFUNDS with a lag, and variables in the fast group respond to it contemporaneously. The order is slow variables, FEDFUNDS, and fast variables. A full description of the variables selected and their transformations can be found in Appendix C.

For each data set, we estimate various VAR models with 5 lags. We consider both Tensor VARs with and without the additional own-lag matrix $D$, denoted as Tensor MGP Own-Lag and Tensor MGP, respectively. For these two Tensor VARs, the threshold of small magnitude is $5 \times 10^{-4}$ and its proportion threshold is 0.95. Implementation of the MGP is the same as in Section 5 and the prior for $D$ is described in Section 3.1. For competitors, we include standard VARs with hierarchical Minnesota (Giannone et al., 2015), SSVS (George et al., 2008) and a specification of normal-gamma (NG) prior introduced to VARs by Huber and Feldkircher (2019). All of these three priors can be written as $A_{p,(i,j)} \sim \mathcal{N}(0, V_{p,(i,j)})$ for $(i, j)$ entry in $A_p$, where $i, j = 1, \ldots, N$ and $p = 1, \ldots, 5$.

For hierarchical Minnesota, $V_{p,(i,j)} = \begin{cases} \lambda_1^2 \hat{\sigma}_i^2, & \text{if } i = j \\ \lambda_1^2 \lambda_2 \hat{\sigma}_i \hat{\sigma}_j, & \text{if } i \neq j \end{cases}$, where $\hat{\sigma}_i^2$ is the variance estimate of $y_{t,i}$ sequence modelled by an AR(5) process. $\lambda_1$ and $\lambda_2$ have prior Gamma(0.01,0.01) and are inferred using a random walk Metropolis Hasting step. SSVS assigns either $\kappa_0^2 = 0.0001$ and $\kappa_1^2 = 4$ to $V_{p,(i,j)}$ with a prior probability 0.5. We apply the NG described for $H$ and $D$ in Section 3.1 to the coefficient matrix. Priors for $H$ and stochastic volatility $S_t$, for $t = 1, \ldots, T$, are the same for all models. The MCMC sampler runs 10,000 iterations after the 10,000 burn-in period.

6.2 Forecasting Results

We follow the expanding window procedure to assess the forecasting performance of our models. Specifically, we first fit each VAR model with the historical data from 1959Q1 to 1984Q4, then get 1-, 2- and 4-step-ahead forecasts for 1985Q1, 1985Q2 and 1985Q4, respectively. Next, we expand the historical data with the
endpoint at 1985Q1 and conduct the multi-step-ahead forecasting again. This procedure is repeated iteratively and stops after conducting the 1-step-ahead forecast of 2015Q4.

We use relative mean squared forecast error (RMSFE) and average log predictive likelihood (ALPL) to assess the point and density forecasting performance. We select 7 variables to evaluate the marginal forecasting performance of our models. These variables are Real Personal Income (RPI), CPI: All Items (CPIAUCSL), Real Gross Domestic Product (GDP), Federal Funds Rate (FEDFUNDS), Civilian Unemployment Rate (UNRATE), GDP deflator (GDPDEFL), 10-Year T-bond (GS10). For joint results, all 20 or 40 variables are considered. Marginal RMSFE for $i$-th variable from model $m$ is defined as

$$\text{RMSFE}_{m,i} = \frac{\text{MSFE}_{m,i}}{\text{MSFE}_{\text{benchmark},i}}, \text{ with } \text{MSFE}_{m,i} = \frac{1}{T-h-T+1} \sum_{t=T}^{T-h} (y_{t+h,i} - \mathbb{E}(y_{t+h,i} | y_{1:t}, m))^2.$$

The benchmark model is a standard VAR with coefficients following a flat prior $N(0, 10)$. $T$ is the total number of time points in the data set, and $h$ is the horizon. $\mathbb{E}(y_{t+h,i} | y_{1:t}, m)$ is the Monte Carlo estimate of posterior predictive mean. The joint RMSFE for model $m$ is formed with a similar fraction of MSFE between model $m$ and the benchmark model, but the joint MSFE of model $m$ is

$$\text{MSFE}_{m,\text{joint}} = \frac{1}{(T-h-T+1)N} \sum_{n=1}^{N} \sum_{t=T}^{T-h} (y_{t+h,i} - \mathbb{E}(y_{t+h,i} | y_{1:t}, m))^2.$$

The marginal ALPL for the $i$-th variable is

$$\text{ALPL}_{i} = \frac{1}{(T-h-T+1)} \sum_{t=T}^{T-h} \log p(y_{t+h,i} | y_{1:t}) = \frac{1}{(T-h-T+1)} \sum_{t=T}^{T-h} \text{LPDS}_{i,t+h}.$$

We follow Billio et al. (2022) to compute LPDS$_{i,t+h}$ by randomly generating a sequence of stochastic volatility $s_{t,i} = (s_{t+1,i}, \ldots, s_{t+h,i})$ with parameters sampled in each iteration during inference. Then LPDS$_{i,t+h}$ is approximated by its Monte Carlo estimate,

$$\text{LPDS}_{i,t+h} = \log \int p(y_{t+h,i} | y_{1:t}, s_{t,i})p(s_{t,i} | y_{1:t}) ds_{t,i} \approx \log \left( \frac{1}{L} \sum_{l=1}^{L} p(y_{t,i} | y_{1:t}, s_{t,i}^{(l)}) \right).$$

Similarly, the joint ALPL is approximated by its Monte Carlo estimate in terms of stochastic volatility and the lower triangular matrix sampled in each iteration,

$$\text{ALPL}_{\text{joint}} \approx \frac{1}{(T-h-T+1)} \sum_{t=T}^{T-h} \log \left( \frac{1}{L} \sum_{l=1}^{L} p(y_{t} | y_{1:t}, S_{(t+1):(t+h)}, H^{(l)}) \right).$$

Table 3 and Table 4 show the performance of joint and marginal point forecasts inferred from data sets with different sizes. Overall, Tensor VARs achieve the best joint performance for most horizons in both cases. For the marginal performance of specific variables, Tensor VARs are better relative to standard VARs in 13 out of 21 cases for the medium data set and 17 out of 21 cases for the large data set. Forecasts of RPI,
CPIAUCSL, GDPDEFL and GS10 are more favourable when using Tensor VARs, while standard VARs have better performance in forecasting GDP and UNRATE. For the two models considered in Tensor VARs, the performance of Tensor MGP and Tensor MGP Own-Lag is similar. If we move to individual models in standard VARs, the Minnesota prior achieves the best forecasting performance in GDP and UNRATE for all horizons when fitting the model with the medium-scale data. Forecasts of FEDFUNDS and UNRATE from SSVS prior are superior when the large-scale data is fitted to the model. Most RMSFEs decrease as the horizon gets longer, suggesting that the forecasting performance of flat prior deteriorates faster than other priors considered in this application. If we compare each evaluation in these two tables, most results inferred from the large data set are smaller than those inferred from the medium data set. This is because of the more severe overfitting issue of the flat prior when considering more information in VARs. This improvement is not from the inclusion of more variables, because if we consider MSFE rather than the relative one, the point forecasting performance is slightly better in the medium case as shown in Table 7 and Table 8 in Appendix B.

Table 5 and Table 6 present density forecasting performance from the medium and large data sets. Note that the ALPL is not relative to the benchmark model because some ALPLs from the benchmark model have extremely small values (e.g. values smaller than $-10^{-8}$), so non-relative ALPL provides better presentation. Similar to the point forecast result, Tensor VARs get better performance when making joint density forecasts. They also outperform standard VARs in marginal cases since they are the best models in 17 out of 21 cases in both tables. We notice several differences between the performance evaluated by RMSFE and ALPL. Tensor MGP Own-Lag is more competitive in GDP density forecasts than point forecasts. The dominant model of forecasting GS10 changes from Tensor VARs to the standard VAR with NG prior if we switch the evaluation from RMSFE to ALPL. Although RMSFEs of joint forecasts from SSVS have similar scales as those from other models in Table 4, SSVS performs much worse than other models in the joint density forecasting as shown in Table 6.

### 6.3 Interpretation

We demonstrate how to interpret Tensor VARs by fitting Tensor MGP Own-Lag and Tensor MGP with the whole large-scale data set ($N=40$). Both models infer the rank as 3, which means the Tensor VAR structure reduces the number of parameters in the coefficient matrix from 8,000 (standard VAR(5)) to 455 (Tensor MGP Own-Lag) and 255 (Tensor MGP). Implemented with the same processor mentioned in Section 5, the runtime of Tensor MGP Own-Lag and Tensor MGP is around 2.9 hours and 1.8 hours, respectively, compared to 11 hours by using a standard VAR with NG.

We first take a look at the result of Tensor MGP Own-Lag. Since the Tensor VAR can be written in the form of 2.6, inferred matrices $B_1$, $B_2$ and $B_3$ can construct a factor model, so it is plausible to get time series plots of factors after the inference. Figure 5a shows these plots for Tensor MGP Own-Lag. These factors are consistent with recession periods indicated by the National Bureau of Economic Research (NBER) (available on https://fred.stlouisfed.org/series/USRECQ). The first factor peaks during
Table 3: RMSFE of joint and marginal variables using the medium-scale data set. The best forecasts are in bold.

| Model       | Horizon | RMSFE          |
|-------------|---------|----------------|
| Tensor MGP  |         |                |
| 1           | 0.634   | 0.536, 0.839   |
| 2           | 0.615   | 0.567, 0.710   |
| 4           | 0.531   | 0.412, 0.796   |
| Tensor MPG Own-Lag | |                |
| 1           | 0.617   | 0.548, 0.816   |
| 2           | 0.614   | 0.565, 0.710   |
| 4           | 0.531   | 0.412, 0.796   |
| Minnesota   |         |                |
| 1           | 0.625   | 0.576, 0.817   |
| 2           | 0.629   | 0.586, 0.716   |
| 4           | 0.546   | 0.425, 0.794   |
| SSVS        |         |                |
| 2           | 0.647   | 0.608, 0.746   |
| 4           | 0.559   | 0.450, 0.793   |
| NG          |         |                |
| 1           | 0.562   | 0.335, 0.736   |
| 2           | 0.628   | 0.582, 0.714   |
| 4           | 0.537   | 0.420, 0.795   |

Table 4: RMSFE of joint and marginal variables using the large-scale data set. The best forecasts are in bold.

| Model       | Horizon | RMSFE          |
|-------------|---------|----------------|
| Tensor MGP  |         |                |
| 1           | 0.546   | 0.311, 0.722   |
| 2           | 0.461   | 0.221, 0.526   |
| 4           | 0.246   | 0.068, 0.229   |
| Tensor MPG Own-Lag | |                |
| 1           | 0.519   | 0.320, 0.714   |
| 2           | 0.457   | 0.221, 0.528   |
| 4           | 0.245   | 0.068, 0.228   |
| Minnesota   |         |                |
| 2           | 0.526   | 0.232, 0.536   |
| 4           | 0.312   | 0.074, 0.230   |
| SSVS        |         |                |
| 2           | 0.458   | 0.259, 0.532   |
| 4           | 0.247   | 0.086, 0.229   |
| NG          |         |                |
| 1           | 0.559   | 0.318, 0.792   |
| 2           | 0.466   | 0.225, 0.534   |
| 4           | 0.247   | 0.068, 0.229   |
### Table 5: ALPL of joint and marginal variables using the medium-scale data set. The best forecasts are in bold.

| Model   | Horizon | ALPL          | Joint | RPI    | CPIAUCSL | GDP     | FEDFUNDS | UNRATE  | GDPDEFL | GS10   |
|---------|---------|---------------|-------|--------|-----------|---------|-----------|---------|---------|--------|
|          |         |               |       | -18.760 | -1.206    | -1.205  | -0.998    | -0.418  | -0.926  | -1.160  | -1.310  |
| tensor MGP | 2       |               |       | -19.899 | -1.257    | -1.374  | -1.057    | -0.519  | -1.027  | -1.192  | -1.309  |
|          | 4       |               |       | -20.892 | -1.302    | -1.404  | -1.141    | -0.612  | -1.138  | -1.182  | -1.296  |
|          |         | Tensor MPG Own-Lag |       | -17.921 | -1.213    | -1.160  | -0.964    | -0.311  | -0.913  | -1.056  | -1.305  |
|          | 2       |               |       | -19.690 | -1.253    | -1.369  | -1.009    | -0.490  | -1.042  | -1.179  | -1.298  |
|          | 4       |               |       | -21.136 | -1.304    | -1.398  | -1.097    | -0.576  | -1.185  | -1.166  | -1.288  |
| Minnesota | 1       |               |       | -18.683 | -1.260    | -1.177  | -0.946    | -0.440  | -0.897  | -1.078  | -1.307  |
|          | 2       |               |       | -20.263 | -1.286    | -1.372  | -1.015    | -0.621  | -1.002  | -1.216  | -1.347  |
|          | 4       |               |       | -21.542 | -1.343    | -1.405  | -1.113    | -0.695  | -1.140  | -1.227  | -1.344  |
| SSVS    | 1       |               |       | -18.427 | -1.224    | -1.153  | -0.971    | -0.300  | -0.899  | -1.063  | -1.323  |
|          | 2       |               |       | -20.999 | -1.267    | -1.376  | -1.051    | -0.534  | -1.033  | -1.224  | -1.309  |
|          | 4       |               |       | -21.391 | -1.312    | -1.395  | -1.128    | -0.623  | -1.176  | -1.213  | -1.291  |
|          |         | Minnesota      |       | -18.427 | -1.233    | -1.190  | -0.967    | -0.332  | -0.891  | -1.120  | -1.306  |
|          | 2       |               |       | -20.999 | -1.267    | -1.376  | -1.051    | -0.534  | -1.033  | -1.224  | -1.309  |
|          | 4       |               |       | -21.391 | -1.312    | -1.395  | -1.128    | -0.623  | -1.176  | -1.213  | -1.291  |
|          |         | SSVS           |       | -18.683 | -1.260    | -1.177  | -0.946    | -0.440  | -0.897  | -1.078  | -1.307  |
|          | 2       |               |       | -20.263 | -1.286    | -1.372  | -1.015    | -0.621  | -1.002  | -1.216  | -1.347  |
|          | 4       |               |       | -21.542 | -1.343    | -1.405  | -1.113    | -0.695  | -1.140  | -1.227  | -1.344  |
|          |         | NG             |       | -18.683 | -1.260    | -1.177  | -0.946    | -0.440  | -0.897  | -1.078  | -1.307  |
|          | 2       |               |       | -20.263 | -1.286    | -1.372  | -1.015    | -0.621  | -1.002  | -1.216  | -1.347  |
|          | 4       |               |       | -21.542 | -1.343    | -1.405  | -1.113    | -0.695  | -1.140  | -1.227  | -1.344  |

### Table 6: ALPL of joint and marginal variables using the large-scale data set. The best forecasts are in bold.

| Model   | Horizon | ALPL          | Joint | RPI    | CPIAUCSL | GDP     | FEDFUNDS | UNRATE  | GDPDEFL | GS10   |
|---------|---------|---------------|-------|--------|-----------|---------|-----------|---------|---------|--------|
|          |         |               |       | -36.809 | -1.195    | -1.183  | -1.022    | -0.370  | -0.995  | -1.172  | -1.310  |
| tensor MGP | 2       |               |       | -39.325 | -1.263    | -1.348  | -1.060    | -0.474  | -1.083  | -1.187  | -1.300  |
|          | 4       |               |       | -41.590 | -1.303    | -1.369  | -1.132    | -0.576  | -1.189  | -1.187  | -1.290  |
|          |         | Tensor MPG Own-Lag |       | -34.596 | -1.224    | -1.153  | -0.971    | -0.300  | -0.899  | -1.063  | -1.323  |
|          | 2       |               |       | -38.578 | -1.267    | -1.376  | -1.051    | -0.534  | -1.033  | -1.224  | -1.309  |
|          | 4       |               |       | -40.949 | -1.314    | -1.392  | -1.113    | -0.561  | -1.167  | -1.166  | -1.292  |
| Minnesota | 1       |               |       | -39.416 | -1.254    | -1.176  | -0.984    | -0.487  | -0.919  | -1.126  | -1.482  |
|          | 2       |               |       | -42.514 | -1.289    | -1.385  | -1.065    | -0.697  | -1.002  | -1.238  | -1.525  |
|          | 4       |               |       | -45.142 | -1.341    | -1.421  | -1.162    | -0.805  | -1.153  | -1.262  | -1.587  |
| SSVS    | 1       |               |       | -66.780 | -26.497   | -1.186  | -1.022    | -0.242  | -0.877  | -1.230  | -1.311  |
|          | 2       |               |       | -55.001 | -2.355    | -1.365  | -1.097    | -0.497  | -1.001  | -1.306  | -1.304  |
|          | 4       |               |       | -58.560 | -7.303    | -1.388  | -1.194    | -0.592  | -1.149  | -1.309  | -1.297  |
|          |         | Minnesota      |       | -39.760 | -1.224    | -1.272  | -1.094    | -0.438  | -1.056  | -1.169  | -1.309  |
|          | 2       |               |       | -41.137 | -1.274    | -1.354  | -1.134    | -0.554  | -1.107  | -1.187  | -1.300  |
|          | 4       |               |       | -42.318 | -1.318    | -1.364  | -1.183    | -0.621  | -1.192  | -1.187  | -1.290  |
the recession of 1969–1970, the stagflation in the 1970s, the early 1990s recession, the dot-com bubble in the early 2000s and the financial crisis of 2008. Although the median of the second factor is flat, it is more uncertain during the above recession periods. The "double-dip" recession in 1980 is highlighted by the more volatile behaviour in the third factor. It is unsurprising for this consistency because these factors are linear functions of lagged data.

When interpreting a standard factor model, Despois and Doz (2022) and Kastner et al. (2017) focus on factor loading to illustrate the meaning of each factor. Because of the expression in 2.6, not only can we interpret results by looking at factor loading (response loading $B_1$ in our case), but also by explaining how these factors are formed. Figure 7 depicts the posterior mean of response, predictor and temporal loadings with important margins highlighted. We first focus on response loading. The first column in this loading is the densest, induced by the increasing shrinkage property. All categories except the exchange rate have variables corresponding to important margins, but margins for output and income have larger magnitudes, which means the first factor emphasizes this category more than others. The second column in the response loading is a sparse one. The second factor is only important to two variables in the labour market. Important margins in the third column correspond to variables in the interest rate category, suggesting that the third factor has an effect on interest rates.

Next, we interpret results from Tensor MGP Own-Lag by the formation of factors. The first column in predictor loading explains how a combination of variables forms the first factor. Similar to the first column in the response loading, it is the densest. This factor compressed variables from most categories (except prices and exchange rate), meaning that this factor represents the economy in general. Among those important margins, margins for HW1, TB6MS and M2REAL are relatively large. If we take a look at the first column in temporal loading, only the margin for the first lag is regarded as important, so the first factor is formed by the first lagged values of economic variables. The second factor has only one important margin, which corresponds to TB6MS, but the magnitude of this margin (0.15) is relatively small, compared to other important margins in other columns. This fact, along with small magnitudes in the second column of the temporal loading, explains why the median of the second factor is close to 0. The third factor is formed by variables in the interest rate category, so this factor mainly describes the interrelationship within this category.

The second model considered is Tensor MGP. Figure 5b shows time series plots of factors. It is noteworthy that the first factors in Tensor MGP and Tensor MGP Own-Lag are highly correlated. The median of the second factor has a similar trend as the lower bound of its counterpart in Figure 5a. The dynamic of these three factors is also consistent with US economic history; only the "double-dip" recession in 1980 is hard to find. Figure 8 presents posterior means of three loadings inferred from Tensor MGP. These loadings are denser than those inferred from Tensor MGP Own-Lag, and important margins have larger magnitudes because Tensor MGP needs to explore both the own-lag and cross-lag effects without the help of the own-lag matrix. The first columns of response and predictor loadings are the densest in their respective loadings.
Among important margins in these two columns, margins for PAYEMS have the largest magnitudes, so the first factor is mainly driven by the own-lag effect of PAYEMS. Most of the important margins in the second factor are for variables from the money and credit category, and only the first lag is considered important. Similar to PAYEMS in the first factor, the own-lag effect of M2REAL is the main driving force in the second factor. The third factor focuses on the linear relations within the money and credit category. To sum up, it is easier to detect the own-lag effect from loadings inferred by Tensor MGP than by Tensor MGP Own-Lag, suggesting that the additional own-lag matrix in the latter model allows the tensor to explore more cross-lag effect.

Since ranks inferred in these two models are the same, Tensor MGP Own-Lag introduces extra parameters in $D$ (200 parameters in this case), but we only fit models with a single data set in this subsection, so it is unclear how many extra parameters Tensor MGP Own-Lag introduces if we fit models to more data sets. To understand this, we use the expanding window procedure in Section 6.2 to obtain time series plots of ranks inferred from these two Tensor VARs as shown in Figure 6. The x-axis indicates the last time point used when fitting the model. The average numbers of parameters inferred by Tensor MGP Own-Lag and Tensor MGP are 484.92 and 370.60, respectively, so Tensor MGP Own-Lag introduces about 30% more parameters than Tensor MGP. Tensor MGP Own-Lag infers most of the ranks to be 3 or 4, except in one case when the inferred rank is 5. In contrast, Tensor MGP starts by inferring ranks like results from Tensor MGP Own-Lag, but ranks jump to larger values after the data set spans to 1996. The difference between these two plots shows that Tensor MGP Own-Lag tends to explain the data in a more stable way than the other model does.

Figure 5: Time series plots of factors with median (solid line) and 80% confidence interval (dashed line).
Figure 6: Time series plots of rank inferred by Tensor MGP Own-Lag and Tensor MGP.

Figure 7: Posterior mean of response, predictor and temporal loadings inferred from Tensor MGP Own-Lag. Important margins have magnitudes greater than 0.1.
### Figure 8: Posterior mean of response, predictor and temporal loadings inferred from Tensor MGP. Important margins have magnitudes greater than 0.1.

| **Response** | **Predictor** | **Temporal** |
|--------------|--------------|-------------|
| -0.39 -0.12 | 0.08 -0.00 | -0.11 -0.31 |
| -0.45 -0.11 | -0.07 0.01 | -0.04 -0.02 |
| -0.33 -0.11 | 0.18 0.03  | -0.00 0.00 |
| -0.26 0.21  | 0.08 0.03  | -0.00 0.00 |
| -0.07 -0.04 | -0.10 -0.04| -0.04 -0.02 |
| -0.27 -0.18 | 0.29 0.05  | -0.00 0.00 |
| -0.18 -0.17  | 0.19 -0.03 | -0.00 0.00 |
| -0.07 0.08  | 0.19 0.03  | -0.00 0.00 |
| -0.31 -0.15 | 0.26 0.03  | -0.00 0.00 |
| -0.35 -0.16 | 0.88 0.08  | -0.00 0.00 |
| -0.33 0.15  | 0.15 0.04  | -0.00 0.00 |
| 0.31 0.12  | 0.05 0.02  | -0.00 0.00 |
| -0.78 -0.04 | 1.52 0.07  | -0.13 -0.01 |
| 0.07 -0.06 | 0.04 0.02  | 0.01 -0.00 |
| 0.04 0.05  | -0.08 0.01 | 0.02 -0.00 |
| -0.10 -0.27 | 0.02 -0.10 | -0.05 -0.00 |
| -0.24 -0.00 | -0.17 -0.02| 0.03 -0.00 |
| -0.19 -0.00 | -0.08 0.02 | 0.02 -0.00 |
| -0.16 -0.01 | 0.12 0.03  | 0.03 -0.00 |
| -0.15 -0.01 | 0.09 0.03  | 0.02 -0.00 |
| -0.12 -0.01 | 0.09 -0.01 | 0.02 -0.00 |
| -0.03 -0.02 | 0.08 -0.00 | 0.00 -0.00 |
| -0.01 -0.02 | 0.16 0.02  | 0.01 -0.00 |
| -0.04 0.02  | 0.07 0.02  | 0.00 -0.00 |
| -0.02 0.07  | -0.05 -0.01| 0.01 -0.00 |
| 0.38 -0.17 | 0.00 -0.10| -0.30 -0.00 |
| -0.06 0.02 | -0.00 -1.75| -0.71 -0.00 |
| 0.07 -0.33 | 0.29 2.48 | 1.00 1.00 |
| -0.36 0.09 | 0.07 -0.22| -0.20 -0.00 |
| -0.46 -0.02 | 0.11 -0.12| -0.05 -0.00 |
| 0.24 -0.02 | -0.02 0.01| 0.01 -0.00 |
| -0.18 -0.16 | -0.02 -0.04| -0.01 -0.00 |
| 0.07 0.00  | 0.03 0.01 | -0.01 -0.00 |
| 0.08 -0.00 | -0.03 -0.01| 0.00 -0.00 |
| -0.11 -0.00 | -0.15 -0.09| -0.02 -0.00 |
| 0.18 -0.04 | 0.03 0.06 | 0.04 -0.00 |
| -0.01 -0.03 | 0.06 -0.02| -0.00 -0.00 |
| -0.01 0.02 | -0.16 -0.05| -0.01 -0.00 |
| -0.03 -0.02 | -0.12 -0.01| 0.01 -0.00 |
| 0.05 -0.00 | 0.13 0.02 | -0.01 -0.00 |

#### Conclusion

In this paper, we propose in- and post-processing algorithms to infer ranks of Tensor VARs and alleviate the mixing issue of margins. In particular, we applied the Multiplicative Gamma Prior (MGP) to margins and use an adaptive inferential scheme to infer the rank. To overcome the mixing issue, we introduce a variant of ASIS algorithm to allow better mixing of Markov chains and match labels and signs after the inference. In the simulation experiment, Tensor VARs with the MGP can infer ranks more accurately and efficiently than models with M-DGDP. The ASIS algorithm improves mixing evaluated by inefficiency factors, and we demonstrate the necessity of label and sign matching by trace plots of margins.
In the real data application, we conduct multi-step-ahead forecasting and interpret results from Tensor VARs with and without the own-lag matrix. The point and density forecasting performance of medium- and large-scale data sets show Tensor VARs are competitive to standard VARs with different prior settings, and they achieve the best joint results in both point and density evaluations. Apart from competitive forecasting performance, Tensor VARs can effectively reduce the number of parameters and speed up the inference. We interpret Tensor VARs as factor models with explainable formation and behaviour consistent with economic history. The additional own-lag matrix allows the tensor part to emphasize the cross-lag effect.

The expanding window estimates of the rank in Figure 6b show changes in the model structure around 1996 and after the financial crisis. It will be interesting to investigate the reason for these changes. Since the model structure is not static, a Tensor VAR with time-varying margins and ranks is a potential model to detect changes in the model structure. Note that this idea is different to Zhang et al. (2021), who kept margins time-invariant and switched each column of the tensor matrix $B$ on or off with a prior. Aside from the time-varying variant, we can incorporate Tucker decomposition into Tensor VARs to gain more flexibility, compared to the CP decomposition. The current MCMC scheme for inferring the rank and overcoming the mixing issue is for the CP decomposition. If we move to Tucker decomposition, $J$ ranks will be inferred for a $J$th-order tensor, and the ASIS algorithm needs to consider the base tensor $G$ in 2.2. For Tensor VARs, interpreting the formation of factors will be challenging if the coefficient matrix is treated as a tensor with Tucker decomposition. The reason is the same as the problem of Varimax in Tensor VARs discussed in 2.3. The $G$ is no longer a superdiagonal tensor as in 2.6, all entries in $G$ are potentially non-zero. Another interesting extension is to construct Tensor VARs with exogenous variables. The model structure simply changes $B_2$ from a $N$-by-$R$ matrix to a $M$-by-$R$ matrix, where $M$ is the number of times series as responses plus the number of exogenous variables. In Kastner et al. (2017), a choice for the diagonal matrix in their interweaving method is to select elements with the maximal absolute values in their corresponding columns. An analogy can be applied to our interweaving method. We leave these ideas for future investigation.

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35
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A Conditional Posterior Distributions and Derivations

A.1 Conditional Posteriors of $B_1$, $B_2$, $B_3$ and $D$

In this subsection, we consider a Tensor VAR with the additional own-lag matrix $D$. The inference of a Tensor VAR without $D$ is simply to treat $D$ as a zero matrix. Recall equation 2.8, 3.1 - 3.3, the Tensor VAR is written as

$$y_t = (x_t'(B_3 \otimes B_2)\mathcal{I}(1) \otimes I_N)\text{vec}(B_1) + Dx_t + \epsilon_t \quad \text{(A.1)}$$

$$= B_1\mathcal{I}(1)((B_3'X_t') \otimes I_R)\text{vec}(B_2') + Dx_t + \epsilon_t \quad \text{(A.2)}$$

$$= B_1\mathcal{I}(1)(I_R \otimes (B_2'X_t))\text{vec}(B_3) + Dx_t + \epsilon_t. \quad \text{(A.3)}$$

Assume the terms before vectorizations of $B_1$, $B_2'$ and $B_3$ as $Z_{t,1}$, $Z_{t,2}$ and $Z_{t,3}$, respectively. Given other parameters, the full conditional posterior of $\text{vec}(B_j)$ for $j = 1, 3$ or that of $\text{vec}(B'_j)$ for $j = 2$ is $\mathcal{N}(\mu_j, \Sigma_j)$ with

$$\Sigma^{-1}_j = \Sigma^{-1}_j + \sum_{t=1}^{T} Z_{t,j}'H'S_t^{-1}HZ_{t,j},$$

$$\mu_j = \Sigma^{-1}_j \sum_{t=1}^{T} Z_{t,j}'H'S_t^{-1}y_t^*,$$

where $y^*_t = H(y_t - Dx_t)$, $\Sigma_j$ is the prior covariance matrix of $\text{vec}(B_j)$ for $j = 1, 3$ or that of $\text{vec}(B'_j)$ for $j = 2$.

Given $B_1$, $B_2$, $B_3$ and other parameters, we can infer $D$ in a similar way as Carriero et al. (2022). Assume that $D = (\text{diag}(d_{1,1}, \ldots, d_{N,1}), \ldots, \text{diag}(d_{1,P}, \ldots, d_{N,P}))$, and $y_t^{**} = y_t - \mathcal{A}(1)x_t = Dx_t + \epsilon_t$, if we multiply both sides of the equation aforementioned with $H$, we get $\tilde{y}_t^{**} = Hy_t^{**} = HDx_t + u_t$, where $u_t \sim \mathcal{N}(0, S_t)$. This equation can be expanded to

$$\tilde{y}_{t,1}^{**} = (y_t^{(1)})'d_1 + u_{t,1},$$

$$\tilde{y}_{t,2}^{**} = h_{2,1}(y_t^{(1)})'d_1 + (y_t^{(2)})'d_2 + u_{t,2},$$

$$\vdots$$

$$\tilde{y}_{t,N}^{**} = h_{N,1}(y_t^{(1)})'d_1 + \cdots + h_{N,N-1}(y_t^{(N-1)})'d_{N-1} + (y_t^{(N)})'d_N + u_{t,N}, \quad \text{(A.4)}$$

where $y_t^{(j)}$ is a vector that contains the $P$ lagged values of $y_{t,j}$, $d_j = (d_{j,1}, \ldots, d_{j,P})'$, for $j = 1, \ldots, N$, $h_{i,j}$ is the $(i, j)$ entry of $H$.

It is noteworthy that A.4 is similar to (12) in Carriero et al. (2022). An important difference is that they multiplied the same $x_t$ to each row of the coefficient matrix, whereas we multiply $(y_t^{(j)})'$ to each $d_j$. After slightly modifying (13) - (15) in Carriero et al. (2022), we get the full conditional posterior
\(d_j \mid B_1, B_2, B_3, d_{-j}, H, S_{1:T} \sim N\left(\bar{\mu}_{d_j}, \bar{\Sigma}_{d_j}\right)\), with

\[
\Sigma_{d_j}^{-1} = \Sigma_{d_j}^{-1} + \sum_{i=j}^{N} h_{i,j}^2 \sum_{t=1}^{T} s_{t,i} y_t^{(j)} (y_t^{(j)})', \tag{A.5}
\]

\[
\bar{\mu}_{d_j} = \Sigma_{d_j}^{-1} \sum_{i=j}^{N} h_{i,j}^2 \sum_{t=1}^{T} s_{t,i} z_{t,i}^{(j)} y_t^{(j)}, \tag{A.6}
\]

where \(z_{t,j+t}^{(j)} = \tilde{y}_{t,j+t}^* - \sum_{i\neq j,i=1}^{j+l} h_{j,t,i} (y_t^{(i)})' d_i, d_{-j}\) represents \(D\) without \(d_j\), \(\Sigma_{d_j}\) is the prior covariance matrix of \(d_j\).

A more efficient way is to rewrite the system in A.4 to

\[(Y^{**} - X (D^{[j=0]})) H_{j:N,1:N}^T = Y^{(j)} d_j H_{j:N,k}^T + U_{j:N}, \tag{A.7}\]

where \(Y^{**} = (y_1^{**}, \ldots, y_T^{**})', X = (x_1, \ldots, x_T)', H_{j:N,1:N}^T\) is the block of \(H\) composed of \(j\)-th to \(N\)-th rows and all \(N\) columns, \(Y^{(j)} = (y_1^{(j)}, \ldots, y_T^{(j)})', D^{[j=0]}\) is the same as \(D\) except the \(j\)-th row as zeros, \(U_{j:N} = (u_{1,j:N}, \ldots, u_{T,j:N})'.\)

If we vectorize both sides of A.7, the new equation is

\[
\text{vec}\left((Y^{**} - X (D^{[j=0]})) H_{j:N,1:N}^T\right) = \left(H_{j:N,k} \otimes Y^{(j)}\right) d_j + \text{vec}(U_{j:N}).
\]

Let

\[
\tilde{Y}^{(j)} = \text{vec}\left((Y^{**} - X (D^{[j=0]})) H_{j:N,1:N}^T\right) / \text{vec}(S_{1:T,k:N}^{1/2})
\]

\[
\tilde{X}^{(j)} = \left(H_{j:N,k} \otimes Y^{(j)}\right) / \text{vec}(S_{1:T,j:N}^{1/2}),
\]

where \(./\) is Matlab element-by-element division operation, \(S_{1:T,j:N}\) is a \(T\)-by-(\(N-J+1\)) matrix with the \(t\)-th row has entries \((s_{t,j}, \ldots, s_{t,N})\).

Then A.5 and A.6 are modified to

\[
\Sigma_{d_j}^{-1} = \Sigma_{d_j}^{-1} + (\tilde{X}^{(j)})' \tilde{X}^{(j)} \tag{A.8}
\]

\[
\bar{\mu}_{d_j} = \Sigma_{d_j} (\tilde{X}^{(j)})' \tilde{Y}^{(j)}. \tag{A.9}
\]
A.2 Conditional Posteriors Related to Multiplicative Gamma Prior

Posteriors of hyperparameters in the MGP are similar to those in Bhattacharya and Dunson (2011). Since \(\phi_{(r,j,i)}\) is a local hyperparameter of \(\beta_{j,i,j}\), the derivation of its conditional posterior given \(\beta_{j,i,j}\) and \(\tau_{r}\) is

\[
p(\phi_{(r,j,i)} \mid \beta_{j,i,j}, \tau_{r}) \propto (\phi_{(r,j,i)})^{-1/2} \exp \left( -\frac{(\beta_{j,i,j})^2}{2\phi_{(r,j,i)}} \right) (\phi_{(r,j,i)})^{\tau_{r}-1} \exp \left( -\frac{\nu}{2} \phi_{(r,j,i)} \right)
\]

\[
= (\phi_{(r,j,i)})^{\nu+1} \exp \left( -\frac{\tau_{r}(\beta_{j,i,j})^2 + \nu}{2} \phi_{(r,j,i)} \right).
\]

Thus, the conditional posterior of \(\phi_{(r,j,i)}\) is a Gamma distribution

\[
\phi_{(r,j,i)} \mid \beta_{j,i,j}, \tau_{r} \sim \text{Gamma} \left( \frac{\nu + 1}{2}, \frac{\nu + \tau_{r}(\beta_{j,i,j})^2}{2} \right).
\]

\(\delta_{1}\) involves in all \(\tau_{r}\)'s, for \(r = 1, \ldots, R\), so the sampling \(\delta_{1}\) is conditional to all margins and corresponding hyperparameters, denoted as \(\cdot\). Combining likelihood and prior, we get

\[
p(\delta_{1} \mid \cdot) \propto \delta_{1}^{a_{1}-1} \exp(-\delta_{1}) \prod_{r=1}^{R} \prod_{j=1}^{3} \prod_{i_{j}=1}^{l_{j}} (\delta_{1})^{3} \exp \left( -\phi_{(r,j,i)} \tau_{r}^{(1)} \frac{(\beta_{j,i,j})^2}{2} \right)
\]

\[
= \delta_{1}^{a_{1}+\frac{(2N+P)R}{2} - 1} \exp \left( -1 + \sum_{r=1}^{R} \tau_{r}^{(1)} \sum_{j=1}^{3} \sum_{i_{j}=1}^{l_{j}} \phi_{(r,j,i)} (\beta_{j,i,j})^2 \right) \delta_{1},
\]

where \(\tau_{r}^{(1)} = \prod_{i_{j}=1, i_{j} \neq r} \delta_{i_{j}}\), and \(p_{j}\) is the number of rows in \(B_{j}\).

The derivation leads to a Gamma conditional posterior of \(\delta_{1}\)

\[
\delta_{1} \mid \cdot \sim \text{Gamma} \left( a_{1} + \frac{(2N+P)R}{2}, 1 + \frac{1}{2} \sum_{r=1}^{R} \tau_{r}^{(1)} \sum_{d=1}^{3} \sum_{k=1}^{p_{j}} \phi_{(r,j,i)} (\beta_{j,i,j}^{(l)})^2 \right).
\]

The derivation of the conditional posterior of \(\delta_{r}\), for \(r > 1\), is similar to the above derivation, but the prior and likelihood are slightly different. We first need to change \(a_{1}\) in (A.10) to \(a_{2}\), and since \(\delta_{r}\) is only related to \(\beta_{j,i,j}^{(l)}\)'s and their corresponding hyperparameters, where \(l \geq r\), the starting value of \(l\) is \(r\) rather than 1, and we amend \(\tau_{r}^{(1)}\) to \(\tau_{r}^{(r)}\). This results to a Gamma conditional posterior of \(\delta_{r}\)

\[
\delta_{r} \mid \cdot \sim \text{Gamma} \left( a_{2} + \frac{(2N+P)(R-r+1)}{2}, 1 + \frac{1}{2} \sum_{l=r}^{R} \tau_{l}^{(r)} \sum_{d=1}^{3} \sum_{i_{j}=1}^{p_{j}} \phi_{(r,j,i)} (\beta_{j,i,j}^{(l)})^2 \right),
\]

where we keep \(\cdot\) as conditions for brevity. \(\tau_{r}\) is updated as the product of \(\delta_{1}, \ldots, \delta_{r}\) in each iteration.
A.3 Details for Other Conditional Posteriors

Conditional posteriors related to the normal-gamma prior (hyperparameters of \( D \) and \( H \)) are almost identical to those in Huber and Feldkircher (2019). The only difference is that these posteriors are conditional on entries of \( D \) and \( H \), instead of the coefficient matrix.

The conditional posterior of \( H \) can also be found in Huber and Feldkircher (2019). For stochastic volatility, we use an ASIS algorithm proposed in Kastner and Frühwirth-Schnatter (2014) and implement it with an R package called \texttt{stochvol} (Kastner, 2016).
## B Additional Results

| Model     | Horizon | MSFE   |
|-----------|---------|--------|
| Tensor MGP | 1       | 0.710  |
|           | 2       | 0.771  |
|           | 4       | 0.834  |
| tensor MPG Own-Lag | 1       | 0.672  |
|           | 2       | 0.768  |
| Minnesota | 1       | 0.689  |
|           | 2       | 0.806  |
|           | 4       | 0.833  |
| SSVS      | 1       | 0.746  |
|           | 2       | 0.853  |
|           | 4       | 0.925  |
| NG        | 1       | 0.803  |
|           | 2       | 0.852  |

Table 7: MSFE of joint and marginal variables using the medium-scale data set. The best forecasts are in bold.

| Model     | Horizon | MSFE   |
|-----------|---------|--------|
| Tensor MGP | 1       | 0.753  |
|           | 2       | 0.795  |
|           | 4       | 0.829  |
| tensor MPG Own-Lag | 1       | 0.681  |
|           | 2       | 0.781  |
| Minnesota | 1       | 0.800  |
|           | 2       | 1.036  |
|           | 4       | 1.334  |
| SSVS      | 1       | 0.800  |
|           | 2       | 0.878  |
|           | 4       | 0.836  |
| NG        | 1       | 0.791  |
|           | 2       | 0.815  |
|           | 4       | 0.834  |

Table 8: MSFE of joint and marginal variables using the large-scale data set. The best forecasts are in bold.

## C Data

### Slow Variables

| Name | Description | Medium | Large | Category | Transformation |
|------|-------------|--------|-------|----------|----------------|
| RPI  |             |        |       |          |                |
| CPIAUCSL |        |        |       |          |                |
| GDP  |             |        |       |          |                |
| FEDFUNDS |       |        |       |          |                |
| UNRATE |         |        |       |          |                |
| GDPDEFL |        |        |       |          |                |
| GS10 |             |        |       |          |                |
| Name              | Description                              | Medium | Large | Category | Transformation |
|-------------------|------------------------------------------|--------|-------|----------|----------------|
| RPI               | Real Personal Income                     | x      | x     | 1        | 5              |
| W875RX1           | RPI ex. Transfers                         | x      | x     | 1        | 5              |
| INDPRO            | IP Index                                 | x      |       | 1        | 5              |
| GDP               | Real Gross Domestic Product               | x      |       | 1        | 5              |
| GDPDEFL           | GDP deflator                              | x      | x     | 1        | 6              |
| DPCERA3M086SBEA   | Real PCE                                 | x      | x     | 2        | 5              |
| CMRMTPSLx         | Real M& T Sales                           | x      | x     | 2        | 5              |
| RETAILx           | Retail and Food Services Sales            | x      | x     | 2        | 5              |
| HWI               | Help-Wanted Index for US                 | x      |       | 3        | 2              |
| HWIURATIO         | Help Wanted to Unemployed ratio           | x      |       | 3        | 2              |
| CLF160V           | Civilian Labor Force                     | x      |       | 3        | 5              |
| UNRATE           | Civilian Unemployment Rate               | x      | x     | 3        | 2              |
| PAYEMS"           | All Employees: Total nonfarm             | x      | x     | 3        | 5              |
| CES0600000007     | Hours: Goods-Producing                    | x      |       | 3        | 5              |
| OILPRICEx         | Crude Oil Prices: WTI                     | x      |       | 4        | 5              |
| CPIAUCSL          | CPI: All Items                            | x      | x     | 4        | 6              |
| FEDFUNDS          | Effective Federal Funds Rate              | x      | x     | 5        | 2              |
| CP3Mx             | 3-Month AA Comm. Paper Rate              | x      | x     | 5        | 2              |
| TB3MS             | 3-Month T-bill                           | x      |       | 5        | 2              |
| TB6MS             | 6-Month T-bill                           | x      |       | 5        | 2              |
| GS1               | 1-Year T-bond                             | x      |       | 5        | 2              |
| GS5               | 5-Year T-bond                             | x      |       | 5        | 2              |
| GS10              | 10-Year T-bond                            | x      | x     | 5        | 2              |
| AAA               | Aaa Corporate Bond Yield                 | x      |       | 5        | 2              |
| BAA               | Baa Corporate Bond Yield                 | x      |       | 5        | 2              |
| M1SL              | M1 Money Stock                            | x      |       | 6        | 5              |
| M2SL              | M2 Money Stock                            | x      |       | 6        | 5              |
| M2REAL            | Real M2 Money Stock                      | x      |       | 6        | 5              |
| BUSLOANS          | Commercial and Industrial Loans           | x      | x     | 6        | 5              |
| NONREVSLS         | Total Nonrevolving Credit                | x      | x     | 6        | 5              |
| INVEST            | Securities in Bank Credit                | x      |       | 6        | 5              |
| CONSPI            | Credit to PI ratio                       | x      | x     | 6        | 2              |
| S&P 500           | S&P 500                                  | x      |       | 7        | 5              |
| S&P: industrials  | S&P Industrial                           | x      |       | 7        | 5              |
| S&P div yield     | S&P Dividend yield                       | x      |       | 7        | 2              |
| S&P PE ratio      | S&P Price/Earnings ratio                 | x      |       | 7        | 5              |
| EXSZUSx           | Switzerland / U.S. FX Rate               | x      | x     | 8        | 5              |
| EXJPUSx           | Japan / U.S. FX Rate                     | x      | x     | 8        | 5              |
| EXUSUKx           | U.S. / U.K. FX Rate                      | x      | x     | 8        | 5              |

42
| 40 | EXCAUSx | Canada / U.S. FX Rate | x | x | 8 | 5 |

Transformation code: 2 - first differences; 5 - first differences of logarithms; 6 - second differences of logarithms.

Table 9: Data description.