Performance assessment of orthogonal space-time block codes in Nakagami-\(m\)/inverse Gaussian fading MIMO channels

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Abstract
This paper evaluates the performance of multiple-input multiple-output systems with orthogonal space-time block code transmit diversity technique experiencing over \(\mathcal{G}\) composite fading channels. Particularly, closed form expressions for ergodic capacity, average symbol error rate and outage probability are presented in high SNR regime. The ergodic capacity of multiple-input multiple-output systems with orthogonal space-time block code is also measured in low signal-to-noise ratio regime. Other essential statistical properties such as variance and probability density function of capacity are also explored for comprehensive analysis. Compared to the exact expressions, the derived analytical formulations are found to be involved with considerably less mathematical exercise and provide additional insights into the implications of model parameters on system performance. The accuracy of \(\mathcal{G}\) distribution in approximating Nakagami-\(m\)/lognormal over several realistic scenario are examined and simultaneously the efficiency of distribution is compared with that of Nakagami-\(m\)/Gamma distribution. The obtained analytical expressions are corroborated using Monte-Carlo simulations where it is observed that the approximated results remain notably tight in asymptotically high signal-to-noise ratio regime.

1 INTRODUCTION

Orthogonal space-time block code (OSTBC) system which is inherently a diversity oriented transmission scheme has gained enormous research interest due to its relative simplicity and reliability [1, 2]. Fundamentally, most of the related studies reported in the literature assume Rayleigh or Rician distribution that characterizes the small-scale fading statistics of communication channels. However, in practice wireless channels experience mixture of both small and large scale fading that constitute composite distributions [3].

In the existing composite distributions Nakagami-\(m\)/lognormal (NLN) is considered to be the most common distribution that can accurately model the wireless channels [4]. However, the primary drawback in NLN model is that the probability density function (PDF) of composite distribution is not in closed form and hence essential performance measures cannot be figured out. Therefore, limited research has been carried out on Nakagami-\(m\) faded MIMO wireless channels, e.g. [4]. In order to circumvent the mathematical intractability, researchers substitute Gamma to lognormal resulting into Nakagami-\(m\)/Gamma, often referred to as \(\mathcal{K}_G\) distribution. Due to analytically simple PDF structure, \(\mathcal{K}_G\) distributions are widely employed, e.g. [5–7] in OSTBC MIMO wireless fading scenarios. However, it is verified that Gamma distribution does not offer good approximation to lognormal distribution for large variance cases and the approximation accuracy significantly deteriorates in the tails of the distribution or low outage region.

Motivated by the above fact, the authors in [8, 9] proposed to replace Gamma by inverse Gaussian (IG) distribution resulting into Nakagami-\(m\)/IG composite model which is also known as \(\mathcal{G}\) fading model. Due to heavy tail behaviour, IG is proved to be more effective in characterizing higher shadowing levels compared to the Gamma which corresponds to \(\mathcal{K}_G\) distribution. The superiority is also measured in terms of Kullback–Leibler (KL) divergence in [8]. It is shown in [9] that \(\mathcal{G}\) distribution produce high accuracy in approximating NLN compared to \(\mathcal{K}_G\) in various environmental conditions. The authors have demonstrated that the performance in terms of capacity, SER and OP over \(\mathcal{G}\) distribution shows accurate agreement with the NLN distribution even in the heavy shadowing which
corresponds to a dense tree coverage area as opposed to the $\mathcal{K}_G$ distribution that deviates significantly in the observed scenario.

In MIMO antenna scenarios, authors in [10, 11] assess the performance of spatial multiplexing systems over Rayleigh/inverse Gaussian (RIG) composite fading channel. The channel statistics of Rayleigh distributions are jointly Gaussian which is suitable in using Wishart matrix theory for mathematical formulation of metrics needed to analyse MIMO system performance. On the other hand, Nakagami-$m$ distribution is considered to be superior model which encompasses Rayleigh model as special case. Moreover, Nakagami-$m$ distribution is found to be better portrayor of empirical data. However, the major problem with this model is the inferior scope in manipulating the joint eigenvalue statistics.

In numerous applications OSTBC transmit diversity technique have been incorporated where the MIMO channel can be transformed into an equivalent scalar Gaussian channel that significantly simplifies the mathematical analysis [12]. OSTBC scheme does not require any channel state information (CSI) at the transmitter and provides full diversity gain, due to which, it has been extensively used in modern wireless communication systems. Performance of OSTBC MIMO systems over Rayleigh fading channels is well known and some of the recent works can be found in [13], survey paper [14] and references therein. However, the effect of shadowing on system performance is ignored in aforementioned literature.

Composite fading is crucial in modelling wireless channels, especially for systems incorporating advanced communication techniques including co-operative relay networks [13], distributed antenna systems [15], handover algorithms, cochannel interferences and adaptive modulations. The study on performance evaluation of OSTBC MIMO systems over $\mathcal{K}_G$ composite fading channel is presented in [5, 6, 16]. However, $\mathcal{K}_G$ distribution is found to be inadequate while approximating NLN distribution in environments having coefficient of variation, $CV \sim 1$ [17] and frequent heavy shadowing [9]. We may note here that the generalised distributions such as $\kappa - \mu$ shadowed [18], $\eta - \mu$ shadowed [19], Fisher–Snedecor $F$ fading [20] can be used to model fading channels that include NLN and $\mathcal{K}_G$ distributions as their special cases. Recently, $\kappa - \mu$ distribution has been used in coverage analysis of user-centric wireless networks [21] and in the performance analysis of PPP-based heterogeneous cellular networks [22]. However, the major drawback in such generalised models is that the PDFs contain modified Bessel function of first kind or second kind. Thus the resulting composite PDF expression, specifically the $\kappa - \mu / IG$ does not end in closed form [23], and hence, further analytical investigation is impaired. Although few works followed series representation approaches such as Laguerre polynomial series in [22], such methods cannot render closed form analytical expressions and require complex numerical evaluation. Therefore taking both accuracy of channel modelling and mathematical complexity into account, in this paper the $\mathcal{G}$ distribution is considered as transmission channel model to obtain insightful results of essential performance measures.

In OSTBC transmit diversity technique, the effective instantaneous SNR of MIMO system becomes equivalent to single-input single-output (SISO) channel statistics. Consequently, for $\mathcal{G}$ distributed OSTBC MIMO channel, most of the analysis are similar to the work pertaining to SISO system presented in [9]. However, the exact results formulated in [9] led to complicated mathematical expressions with several special functions. As a result, straightforward interpretation on important measures of performances are not possible from the expressions formulated in [9]. The difficulty arises as the PDF expression of IG distribution is relatively in more complex form compared to Gamma and hence straightforward mathematical analysis cannot be performed like $\mathcal{K}_G$ distribution. Accordingly, for $\mathcal{G}$ distribution, performance evaluation in simple analytical form renders a challenging mathematical problem and to the best of the authors knowledge, this remains unsolved. Moreover, literature on $\mathcal{G}$ distribution in multiple antenna scenario has not been addressed so far.

In this paper, accurate approximations of essential performance measures are formulated in asymptotically high SNR regime for OSTBC MIMO systems. In particular, the expressions are formulated in high SNR ($\gamma = \infty$) and low SNR regime ($\gamma = 0$) to approximate the exact analytical results. The obtained expressions are rather informative, providing useful insights on the impact of shadowing and fading parameters on system performance. For instance, it is illustrated that the fading parameter, $m$ of $\mathcal{G}$ distribution has beneficial impact on capacity, symbol error rate (SER) and outage probability (OP). The main contribution of this paper are summarized as follows:

- At the outset, SNR distribution of OSTBC MIMO systems experiencing over $\mathcal{G}$ fading channel is derived and thereafter, the similarity with the corresponding PDF of SISO system is observed. Subsequently, the ergodic capacity in high and low SNR regime is presented in simple mathematical form. The PDF and asymptotic variance of MIMO channel capacity is also derived where the dependence of fading and shadowing parameters on capacity are analysed.

- In further analysis, we derive the asymptotic expressions of average SER and OP while formulating the above in high SNR regime. The resulting formulations are remarkably simple compared to the corresponding exact expressions presented in [9]. In this regard, the performance of OSTBC MIMO system is quantified in terms of diversity order and array gain.

- Finally, the derived analytical results are validated through Monte-Carlo simulations where the asymptotic expressions are found to be sufficiently tight in high SNR regime.

- In the last part of the analysis the accuracy of $\mathcal{G}$ distribution in approximating NLN is examined for OSTBC systems. Furthermore, the efficiency of the model is investigated in comparison to $\mathcal{K}_G$ distribution for several typical environmental scenario including light shadowing, average shadowing, frequent heavy shadowing and environments with various coefficient of variations.
2 | OSTBC MIMO SYSTEM MODEL

We consider a point-to-point MIMO system with \(N_t\) transmitting and \(N_r\) receiving antennas employing OSTBC transmission scheme. It is assumed that CSI is perfectly known to the receiver while unknown at the transmitting end. Furthermore, the channel is supposed to be quasi-static over \(T\) symbol periods. Accordingly, assuming a uniform power allocation across all transmitting antennas the input-output symbol periods. Consequently, the channel is supposed to be quasi-static over the receiver while unknown at the transmitting end. Further, block decoding can be expressed as the effective instantaneous SNR per symbol after space-time mon shape and scale parameter \(\kappa\) statistically independent Gamma random variable with com-

\[
Y = \sqrt{\frac{P}{N_t}}H\Xi^{1/2}S + N
\]

given as

\[
f_y(y) = \frac{y^{d-1}}{(\sqrt{\rho}\gamma + b)^{d+1/2}} K_{d+1/2}(\sqrt{\rho}\gamma + b)
\]

where, \(K(\cdot)\) is the \(r\)th order modified Bessel function of second kind[24]. Here, the associated constant terms can be defined as

\[
G = \sqrt{\frac{2\lambda}{\pi}} \exp\left(\frac{\lambda}{\mu}\right) \left(\frac{D^2 R_s N_r m^d}{\rho\Omega}\right), \quad d = N_r N_t
\]

\[
a = \frac{2\mu^2 D^2 R_s N_r m}{\rho\lambda\Omega}, \quad b = \mu^2, \quad c = \frac{\lambda}{\mu^2}
\]

As expected, the structure of PDF of OSTBC MIMO is analogous to [9, Equation (6)]. Consequently, the expressions of exact measures of performances are also similar that include complicated mathematical equations and special functions. In the following, we formulate approximated expressions in high SNR regime with relatively much simple mathematics. Noticeably, the resulting PDF involve additional additive term \(c\), compared to the corresponding PDF expression of \(K(\cdot)\) fading channel for OSTBC MIMO system presented in [7, Equation (6)]. Furthermore, \(c\) is also reflected in the argument of modified Bessel function \(K(\cdot)\). As a result, necessary performance measures cannot be accomplished in a straightforward manner and the mathematical analysis becomes complicated.

Now in the following we utilize the PDF expression (5) to obtain \(s\)th order moment of \(G\) distribution. To this end, by using the relation [24, Equation (6.596.3)] we obtain

\[
E[\gamma^s] = \frac{\Gamma(d + s)}{\Gamma(d)} \ exp\left(\frac{\lambda}{\mu}\right) \sqrt{\frac{2\lambda}{\pi\mu}} \times \left(\frac{\rho\Omega\mu}{D^2 R_s N_r m}\right)^s K_{s-1/2}(\lambda/\mu)
\]

The above equation is used to calculate the amount of fading (AF) as follows.

\[
AF = \frac{E[\gamma^2]}{E[\gamma]^2} - 1 = \left(1 + \frac{1}{\lambda}\right)\left(1 + \frac{\mu}{\lambda}\right) - 1
\]

Here it can be inferred that, \(AF\) ranges from values \(\mu/\lambda\) when \((m = +\infty)\) to \(\frac{m\lambda}{\lambda} + 2\) when \((N_r = N_t = 1, m = 1/2)\).

The aforementioned analysis revealed that \(G\) distribution results into closed-form PDF and moment expressions as opposed to NLN distribution. As such, it is required to measure the degree of accuracy of \(G\) distribution in approximating NLN for various performance metrics. In this respect, the
performance of $K_C$ fading channel is also evaluated and its accuracy of approximation is compared with that of $G$ distribution. In order to perform a fair comparison, we need to establish the relation amongst the parameters of NLN, $G$ and $K_C$ distributions. The moment matching method is incorporated, where the first and second order moments of lognormal, IG and Gamma distributions are equated to obtain the corresponding relationship. To this end, the mean ($\mu^i$) and standard deviation ($\lambda^i$) of lognormal shadowing can be estimated as

$$\mu^i = \ln(\mu) - \lambda^i/2, \quad \lambda^i = \sqrt{\ln(\mu/\lambda + 1)}$$  \hspace{1cm} (9)

while the shape ($\mu^s$) and rate ($\lambda^s$) parameters of Gamma shadowing can be

$$\mu^s = \lambda/\mu \quad \text{and} \quad \lambda^s = \lambda/\mu^2$$  \hspace{1cm} (10)

Similarly, the parameters of IG and Gamma distributions can be expressed as a function of lognormal parameters as follows

$$\mu = \exp(\mu^i + (\lambda^i)^2/2), \quad \lambda = \frac{\exp(\mu^i)}{2\sinh((\lambda^i)^2/2)}$$  \hspace{1cm} (11)

$$\mu^s = \frac{1}{\exp((\lambda^i)^2) - 1}, \quad \lambda^s = \frac{\mu^s}{\exp(\mu^i + (\lambda^i)^2/2)}$$  \hspace{1cm} (12)

By using the expressions (11) and (12) we measure the degree of accuracy of $G$ distribution in approximating NLN distribution as compared to $K_C$ distribution, where the detail analysis is presented in Section 6.

3 | CHANNEL CAPACITY

The ergodic capacity of OSTBC MIMO system is given by [25]

$$C = \frac{R_s}{\ln 2} \mathbb{E}[\ln (1 + \gamma)]$$  \hspace{1cm} (13)

where, $\mathbb{E}[\cdot]$ denotes the expectation operator. In this work, we find the exact ergodic capacity in high SNR regime ($\rho \to \infty$). Accordingly, by combining (2), (3) and (4) the asymptotic expression of ergodic capacity can be obtained as

$$C^\infty = \frac{R_s}{\ln 2} \ln \left(\frac{\rho \mu \Omega}{R_s N_t D^\varphi_m}\right) + \psi(d)$$

$$+ \exp\left(\frac{2A}{\mu}\right) E_i\left(-\frac{2A}{\mu}\right)$$  \hspace{1cm} (14)

where, $E_i(t) = \int_{t}^{\infty} \exp(-t/z) dt$ is the exponential integral function [24, Equation (8.360.1)] and $\psi(\cdot)$ denotes digamma function. The similar expression was also obtained by Gopal et al. in [10, Equations (9), (21)] for distributed MIMO RIG fading channels. However, [10, Equation (9)] corresponds to high SNR approximation of ergodic capacity upper bound, and in [10, Figure 2], significant performance gap can be observed between exact and upper bound plots. On the other hand, [10, Equation (21)] approximates to exact ergodic capacity in high SNR regime. However, the same cannot be used to evaluate the ergodic capacity for $G$ distribution due to the difficulty in manipulating the joint eigenvalue distribution of Nakagami fading channel.

3.1 | Low SNR analysis

In recent years, characterizing the performance in low SNR region has become an important area of research mostly due to the proliferation of low-power wireless devices like internet of things. In this work, we have incorporated different efficient techniques to quantify the low SNR sum rate. Accordingly, performing first order expansion of (13) using the property $\ln(1 + x) \approx x$, when $x \to 0$, is a standard procedure. However, such approach may lead to misleading results on the impact of channel in low SNR regime [26].

In order to achieve accurate results, we express the low SNR capacity in terms of normalized transmit energy per information bit ($E_b/N_0$) [26] as follows

$$C\left(\frac{E_b}{N_0}\right) \approx S_0 \log_2\left(\frac{E_b}{N_0}/N_0\right)$$  \hspace{1cm} (15)

where $E_b/N_0$ and $S_0$ represent minimum $E_b/N_0$ required to convey any positive rate reliably and the wide band slope, respectively and can be defined as

$$\frac{E_b}{N_0\min} = \frac{1}{C(0)} \quad \text{and} \quad S_0 = -\frac{2 \ln 2 \left[\dot{C}(0)\right]^2}{\ddot{C}(0)}$$  \hspace{1cm} (16)

where $C(0)$ and $\ddot{C}(0)$ are first and second order derivatives of $C$ with respect to $\rho$, given by the expression

$$C(0) = \frac{1}{\ln 2 N_t D^\varphi} \mathbb{E}[\xi x]$$  \hspace{1cm} (17)

$$\ddot{C}(0) = \frac{-R_s}{\ln 2 N_t D^\varphi} \mathbb{E}[\xi^2 x^2]$$  \hspace{1cm} (18)

In order to solve (16) we require $n$th order moment of $x$ and $\xi$. By utilizing the integral [24, Equation (3.381.4)] we obtain

$$\mathbb{E}[x^n] = \frac{\Gamma(n + d)}{\Gamma(d)} \left(\frac{\Omega}{\mu}\right)^n$$  \hspace{1cm} (19)
Similarly, positive moments of \( \xi \) can be given by [27]

\[
E[\xi^x] = \mu^x \sum_{k=0}^{x-1} \frac{(u+k-1)!}{k!(u-k-1)!} \left( \frac{\mu}{2\lambda} \right)^k
\]  

(20)

By substituting the expressions of (19) and (20) on (16), we obtain the expressions of \( E_b/N_0_{\text{min}} \) and \( s_0 \) for OSTBC systems over \( \mathcal{G} \) fading MIMO channel as

\[
E_b/N_0_{\text{min}} = \frac{\ln 2 D_v}{N_s \Omega \mu}
\]  

(21)

\[
s_0 = \frac{2R_s N_r N_v}{(N_s N_r + 1/m)(1 + \mu/\lambda)}
\]  

(22)

Recently, the authors in [28] have proposed two simple and accurate low SNR approximations of ergodic capacity. These techniques are based on Padé approximation that utilizes channel moments, the expression of which is earlier obtained in (7). Accordingly, we can represent these approximations as

(Approx.1) : \( C \approx \frac{E[y]}{\ln 2} \frac{1}{1 + \frac{1}{2} B_2} + R_1 \)  

(23)

(Approx.II) : \( C \approx \frac{E[y]}{\ln 2} \frac{1 + A_1}{1 + A_2 + A_3} + R_2 \)  

(24)

where, \( R_1 \) and \( R_2 \) are approximation errors and can be neglected as

\[
\rho \to 0, A_1 = \frac{18B_1 - 24B_1B_2 + 9(B_2)^2}{24B_3 - 18(B_2)^2}.
\]

\[
A_2 = \frac{18B_1 - 12B_2B_3}{24B_3 - 18(B_2)^2}, A_3 = \frac{9B_2B_4 - 8(B_3)^2}{24B_3 - 18(B_2)^2}
\]

and

\[
B_2 = \frac{E[y^4]}{E[y^2]}.
\]

We can note here that, Approx.I and Approx.II are expressed as a function of \( \rho \), as opposed to the case of Corollary 2 where the ergodic capacity is expressed as a function of \( E_b/N_0 \).

### 3.2 Variance of capacity

Variance of channel capacity, \( R \) is also an essential figure of merit in evaluating performance of MIMO communication systems. It provides better characterization of capacity behaviour of MIMO systems [5, 17, 18]. Specifically, the amount of fading (AoF), amount of dispersion (AoD), skewness and kurtosis can be expressed in terms of variance and higher order moments of channel capacity [18]. By utilising (3), the variance of capacity at high SNR can be expressed as

\[
\text{Var}(R^\infty) = \text{Var}\left[ \frac{R_s}{\ln 2} \left( \ln \left( \frac{\rho}{N_s R_s D_v^\infty} \right) + \ln (x) \right) \right]
\]  

(25)

As variance of constant is zero, (25) is further simplified to

\[
\text{Var}(R^\infty) = \left( \frac{R_s}{\ln 2} \right)^2 \left[ \text{Var}(\ln (x)) + \text{Var}(\ln (\xi)) \right]
\]  

(26)

In order to determine (26), it is required to evaluate the first and second order logarithmic moments of \( \xi \) and \( x \). The first log moments of \( \xi \) and \( x \) can be evaluated by using [24, Equation (4.352.1)] and [29, Equation (2.6.22.8)], respectively. Now we combine (4) with [24, Equation (4.358.2)] to calculate \( \mathbb{E}[\ln (x)^2] \) as

\[
\mathbb{E}[\ln (x)^2] = \left\{ \psi(d) - \ln \left( \frac{m}{\mu} \right) \right\}^2 + \zeta(2, d)
\]  

(27)

where, \( \zeta(\cdot, \cdot) \) is Riemann’s zeta function [24]. Similarly, for \( \xi \) we use (2) and then change the variable \( t = \frac{\lambda \xi}{(2\mu^2)} \) to obtain

\[
\mathbb{E}[\ln (\xi)^2] = \sqrt{\frac{\lambda}{2\pi}} \exp \left( \frac{\lambda}{\mu^2} \right) \frac{2\mu^2}{\lambda} \int_0^{\infty} \exp (-t) W(t) dt
\]  

(28)

where,

\[
W(t) = \exp \left( -\frac{\lambda^2}{4\mu^2 t} \right) \left( \frac{2\mu^2 t}{\lambda} \right)^{-\frac{1}{2}} \left\{ \ln \left( \frac{2\mu^2 t}{\lambda} \right) \right\}^2
\]  

(29)

The above equation can be solved by utilising Laguerre polynomial [30] and is represented in the following form

\[
\mathbb{E}[\ln (\xi)^2] = \sqrt{\frac{\lambda}{2\pi}} \exp \left( \frac{\lambda}{\mu^2} \right) \frac{2\mu^2}{\lambda} \sum_{j=1}^{N} u_j W(t_j)
\]  

(30)

where, \( t_j, u_j, j = 1, \ldots, N \) are zeros and weight factors of Laguerre polynomial tabulated in [30, Table 25.9]. Finally, combining the above results followed by some basic algebraic simplification we obtain the high SNR equivalent of capacity variance as

\[
\text{Var}(R^\infty) = \left( \frac{R_s}{\ln 2} \right)^2 \left[ \zeta(2, d) - (\ln \mu + \exp (2\lambda/\mu) \right.

\]

\[
\times \text{Ei} (-2\lambda/\mu)^2

\]

\[
+ \sqrt{\frac{\lambda}{2\pi}} \exp \left( \frac{\lambda}{\mu^2} \right) \frac{2\mu^2}{\lambda} \sum_{j=1}^{N} u_j W(t_j)
\]  

(31)
From the asymptotic expressions (14) and (31), we observe that the small and large scale fading parameters are decoupled in the high SNR regime. Furthermore, $\zeta(2, d)$ being a monotonically decreasing function of $d$, it can be stated that variance of capacity decreases with increase in $m$ (or channel condition).

### 3.3 PDF of channel capacity

In this subsection we evaluate the PDF of capacity for MIMO systems where the behaviour is examined for different scenario imposed by the effect of shadowing and severity of fading. In order to obtain the expression for PDF of capacity, we define $\gamma$ using channel capacity $R$ as

$$\gamma = 2^{R/R_s} - 1$$

where

$$R = \frac{R_s}{\ln 2} \ln (1 + \gamma)$$

(32)

(33)

Now we perform the mapping of one random process to another given by

$$f(R) = f_\gamma(\gamma) \frac{1}{R}$$

(34)

By utilizing (32) and (33), (34) can be expressed as

$$f(R) = \frac{2^{R/R_s} \ln 2}{R_s} f_\gamma(2^{R/R_s} - 1)$$

(35)

Finally, by plugging the PDF expression given in (5) on (35), the PDF of channel capacity can be obtained as

$$f(R) = \frac{2^{R/R_s} \ln 2}{R_s} G \left( \frac{2^{R/R_s} - 1}{\sqrt{a(2^{R/R_s} - 1) + b}} \right)^{d+1/2} K_{d+1/2}$$

(36)

$$\times \left( e^{\sqrt{a(2^{R/R_s} - 1) + b}} \right)$$

### 4 AVERAGE SER ANALYSIS

The average SER of $M$-QAM signalling scheme can be expressed in terms of moment generating function (MGF) given by [31]

$$\text{SER} = \frac{4\Delta_M}{\pi} \int_0^{\pi/2} I_{\text{SER}} d\theta - \frac{4\Delta_M^2}{\pi} \int_0^{\pi/4} I_{\text{SER}} d\theta$$

(37)

where $I_{\text{SER}} = \mathbb{M}_f(\frac{-\Delta_M}{\sin^2 \theta})$ and $\mathbb{M}_f(\cdot)$ is defined as $\mathbb{E}[\exp(-\gamma)]$ symbolises MGF. Additionally, $\Delta_M = (1 - 1/\sqrt{M})$ and $\Delta_M = 3/(2(M-1))$ are constants, specifically depend upon modulation order, $M$. In order to evaluate (37) in high SNR regime, we define

$$\frac{\rho \xi}{R_s N_s L_p \sin^2 \theta} = \frac{\Delta_M}{2} X = RX$$

(38)

such that

$$I_{\text{SER}} = \int_0^\infty \frac{1}{(1 + R\Omega/m)^\mu} f_R(r) dr$$

(39)

By plugging the PDF expression given in (4) and subsequently applying the relation [24, Equation (3.351.3)], the above integral can be solved as

$$I_{\text{SER}} = \int_0^\infty \frac{1}{(1 + R\Omega/m)^\mu} f_R(r) dr$$

(40)

We start the asymptotic analysis hereafter by considering $\rho \to \infty$ such that the term inside integral can be approximated to

$$\frac{1}{(1 + R\Omega/m)^\mu} \sim m + o(\rho^{-(d+1)})$$

(41)

and hence

$$I_{\text{SER}} = \int_0^\infty \left( \frac{mR_s N_s L_p \sin^2 \theta}{\Omega \rho \Delta_M} \right)^d \frac{1}{\xi^d} f_\xi(\xi) d\xi + o(\rho^{-(d+1)})$$

(42)

In this point we evaluate the $d$th order negative moment of $\xi$ by using (2) and [24, Equation (3.471.9)] as $E[1/\xi^d] = \mu^{-d} C_{\text{asy}}$ where, $C_{\text{asy}}$ is the parameter dependent constant defined by

$$C_{\text{asy}} = \sum_{k=0}^{d} \frac{(d + k)! (2\lambda/\mu)^{-k}}{(d - k)!}$$

(43)

so that

$$I_{\text{SER}} = \left( \frac{mR_s N_s L_p \sin^2 \theta}{\Omega \rho \Delta_M} \right)^d C_{\text{asy}} + o(\rho^{-(d+1)})$$

(44)

Furthermore, by using [29, Equation (1.5.2.3)] we evaluate

$$\phi_1 = \int_0^{\pi/2} \sin^{2d} \theta d\theta = \frac{\pi}{2} \prod_{i=1}^{d} \frac{2n - 1}{2n}$$

(45)
and
\[ \phi_2 = \frac{\pi/4}{2d+1} \int_0^{2d} (\theta) d\theta = \frac{\pi/2}{2d+1} \left( \frac{2d}{2d+1} \right) \]
\[ + \frac{(-1)^d}{2^{2d-1}} \sum_{n=1}^{d-1} (-1)^n \frac{(2d)}{2(d-n)}. \]

The expression (37) can thus be evaluated by replacing the results of (44), \( \phi_1 \) and \( \phi_2 \) to generate the following high SNR approximation
\[ \text{SER} \approx \left( \frac{4}{\pi} \Delta_M \phi_1 - \frac{4}{\pi} \Delta_M \phi_2 \right) \left( \frac{mR_N \text{D}^p}{\Omega \mu \rho_m} \right)^d \]
\[ \times \mathcal{C}_{\text{asy}} + d(\rho)^{-(d+1)} \] (45)

Hence, from (45) it can be asserted that the array gain and diversity order of \( G \) faded OSTBC MIMO system is
\[ \frac{\Omega \mu \rho_m}{mR_N \text{D}^p} \left( \frac{4}{\pi} \Delta_M \phi_1 - \frac{4}{\pi} \Delta_M \phi_2 \right) \] and \( d \), respectively. Consequently, diversity order depends only on MIMO system configuration and small-scale fading parameter \( m \) while it is independent of large-scale shadowing parameters and transmitter-receiver distance, \( D \).

5 | OUTAGE PROBABILITY ANALYSIS

OP is defined to be the probability that instantaneous output SNR falls below a given threshold level and is expressed as [5]
\[ P_{\text{out}} = \Pr (\gamma \leq \gamma_{th}) = F_{\gamma} (\gamma_{th}) \] (46)

where, \( \gamma_{th} \) is the predefined threshold level and \( F_{\gamma} (\cdot) \) denotes cumulative density function (CDF). In order to evaluate \( F_{\gamma} (\gamma_{th}) \), we factorize \( \gamma \) given in (3) as
\[ \gamma = UX \] (47)

where, \( U = \frac{\rho}{\rho_m R_N \text{D}^p} \) so that we can define
\[ F_{\gamma} (\gamma_{th}) = \int_0^{\infty} F_X \left( \frac{\gamma_{th}}{U} \right) f_U (u) du \] (48)

where \( F_X (x) = \int_0^x f_X (x) dx = \int_0^x \exp(-mx/\Omega) \Gamma(d+1)/\Omega^d/m^d (x)^{d-1} dx \). The above integral can be expressed in terms of lower in complete Gamma function, \( \gamma (\cdot, \cdot) \) given by
\[ F_X (x) = \frac{\gamma (d, mx/\Omega)}{\Gamma (d)} \] (49)

We expand \( \gamma (\cdot, \cdot) \) in series form according to [24, Equation (8.354.1)] and subsequently replace \( x \) by \( \gamma_{th}/U \) so that (49) can be expanded to the form
\[ F_X \left( \frac{\gamma_{th}}{U} \right) = \frac{1}{\Gamma (d)} \sum_{n=1}^{\infty} \frac{(-1)^n}{m (d+n)} \left( \frac{m \gamma_{th}}{U \Omega} \right)^{d+n} \] (50)

For asymptotic analysis and hence large value of average SNR, we consider only the smallest exponent of \( n \) (i.e. \( n = 0 \)) such that (50) can be approximated to
\[ F_X \left( \frac{\gamma_{th}}{U} \right) = \frac{(m \gamma_{th})^d}{\Omega \Gamma (d+1) U^d} + d(\rho)^{-(d+1)} \] (51)

Thus, by plugging the value of \( F_X \left( \frac{\gamma_{th}}{U} \right) \) in (48) and then following similar approach as presented in Section 4, the resulting expression for OP in high SNR regime can be obtained as
\[ P_{\text{out}} \approx \left( \frac{m \gamma_{th}}{\Omega \mu \rho_m} \right)^d \frac{1}{\Gamma (d+1)} \mathcal{C}_{\text{asy}} + d(\rho)^{-(d+1)} \] (52)

The OP expression for OSTBC MIMO system is similar to the analytical results presented in [17] and [9]. However, it is noteworthy that in comparison to [17, Equation 28] and [9, Equation 15], the expression obtained in (52) is significantly simplified that does not contain any summation operator and Bessel function, \( K_{\nu} (\cdot) \). Moreover, the OP performance with respect to \( \rho \) is apparently insightful in (52), the tightness of which is investigated in Section 6.

6 | NUMERICAL RESULT ANALYSIS

In this section, we validate the mathematical analysis presented in Sections 3–5 using Monte-Carlo simulations. In the simulation, one-rate code scheme \( (R_s = 1 \text{ b/s/Hz}) \) and \( \Omega = 1 \) are assumed throughout the experiment for simplicity. Furthermore, we consider the following default numerical values of the parameters, namely \( N_s = N_r = 2, \mu = 1, \lambda = 10, m = 0.5, D = 1.5 \) and \( e = 4 \) to signify practical scenario in our simulation. The plot for exact results are based on OSTBC MIMO equivalent of [9] in all figures.

In Figure 1, ergodic capacity of \( G \) faded MIMO channel with various shadowing parameters are presented. The analytical results are plotted according to the expressions (14). As the figure shows, analytical curves are matching well with the simulated results. Furthermore, ergodic capacity is increasing with the increase in number of receiving antennas and this happens so because of the improvement of receiver spatial diversity, thereby reducing the probability of noise enhancement effect [32]. The result also indicates that ergodic capacity increases with the increase of either \( \mu \) or \( \lambda \) and the increment is large for simultaneous increase in both the shadowing parameter values. The high SNR approximation shown in the figure is reasonably tight even for moderate range of SNR and thus implies that the
relatively much simpler expression presented in (14) can accurately predict the exact ergodic capacity.

The results obtained for low SNR approximations are shown in Figure 2 to illustrate the ergodic capacity with respect to $E_b/N_0$ where the approximated output is plotted using (15). It can be observed that the slope of ergodic capacity curve is higher for larger values of $\mu$ in the low SNR regime. Furthermore, the approximations are accurate for low SNR region and seem to be tighter for smaller values of $\mu$. On the similar ground, Figure 3 shows the low SNR ergodic capacity with respect to $\rho$ based on the analysis presented in (23) and (24). Response for various values of $\mu$ are shown here and it exhibits that the trend is exactly analogous to the one presented in Figure 2. Additionally, Approx.II is found to be tighter than Approx.I, since it utilizes more number of moments.

Figure 4 shows the variance of capacity for different combinations of small and large-scale fading parameters. It is observed that, the approximated results are reasonably tight above $\rho = 20$ dB in all circumstances. As we can see, the variance of capacity is decreasing with the rise of $m$ and $\lambda$ values while it is increasing with the shadowing parameter $\mu$. Moreover, the influence of $\lambda$ on capacity variance is more than that of $\mu$.

In Figures 5 and 6, the statistical properties of ergodic capacity are demonstrated through its PDF, the expression of which is presented in (36). From both Figures 5 and 6, we find that the mean of channel capacity increases with the increase in receiving antenna. Subsequently, both Figures 5 and 6 agree with the fact that the average channel capacity increases with the increase of $\mu$ and $\lambda$ values. From Figure 6 it can be verified that for higher values of $m$, ergodic capacity increases and the influence of fading is overcome by increasing the number of receiving antenna. Furthermore, in both figures we observe that the spectral range of capacity increases for higher values of $\mu$ whereas it decreases
FIGURE 5 The PDF of $f_C(c)$ versus Capacity (b/s/Hz) for various combinations of $\mu$ and $N_r$

FIGURE 6 The PDF of $f_C(c)$ versus Capacity (b/s/Hz) for various combinations of $m, \lambda$ and $N_r$

for larger $m$ and $\lambda$. We recall here that analogous observations were found in Figure 4 for analysis on variance of capacity.

The results obtained for average SER analysis is presented in Figure 7 with respect to $\rho$ for $M$−QAM signalling scheme. The 4-QAM constellation has lower average SER compared to the corresponding 16-QAM, as expected. Furthermore, the SER performance improves with the higher values of $\mu$. The results in Figure 7 also shows the SER with respect to $\nu$. As expected, the performance degrades with increase in path loss exponent. Above all, the approximated results are found to be accurate even for moderate range of SNR and thus implies that the relatively much simpler expression presented in (45) can accurately predict the exact average SER.

The outage probability with respect to $\rho$ is illustrated in Figure 8 for $\gamma_{th} = 0.5$ dB. It is found that, the asymptotic results are sufficiently tight in the high SNR regime. We can see here that the OP decreases for smaller values of $\lambda$. Moreover, the impact of $\lambda$ is more pronounced for larger multipath severity.

The performance of MIMO OSTBC systems for varying number of transmitting antennas, $N_t = 2$ and $N_t = 4$ is illustrated in Figures 9 and 10. For $N_t = 4$, four different symbols are transmitted in eight time slots in order to maintain orthogonality [33], and hence, the data rate becomes half. This results into lower spectral efficiency, as illustrated in Figure 9. On the other hand, the diversity of four transmitting antennas based OSTBC system is found to be significantly improved compared to two transmitting antennas based OSTBC system, and is demonstrated in Figure 10. Therefore, in OSTBC MIMO systems higher diversity is achieved at the cost of lower data rate and more hardware requirement, and vice versa.

In the subsequent discussions, the efficacy of IG distribution is illustrated for diverse environmental scenarios and the same is compared with the other distribution, $K_G$ which is widely used in current research works. The Figure 11 depicts here the
FIGURE 9  Ergodic capacity with $\mu = 1, \lambda = 10$; performance evaluation for varying number of transmitting antennas and fixed number of receiving antennas, $N_r = 2$

FIGURE 10  SER with $\mu = 10, \lambda = 10$; performance evaluation for varying number of transmitting antennas and fixed number of receiving antennas, $N_r = 2$

FIGURE 11  Performance of ergodic capacity over NLN $\mathcal{G}$ and $\mathcal{K}_G$ distributions with $\mu = 10, \lambda = 3$ ($CV > 1$) and $\mu = 1, \lambda = 30$ ($CV < 1$)

FIGURE 12  Performance evaluation over NLN, $\mathcal{G}$ and $\mathcal{K}_G$ distributions for frequent light shadowing, average shadowing and infrequent heavy shadowing in terms of Average SER

accuracy of IG and $\mathcal{K}_G$ model in approximating NLN distribution where the results pertaining to ergodic capacity of $\mathcal{G}$ and $\mathcal{K}_G$ distributions are obtained using high SNR expression, (14) and exact expression, [7, Equation (17)], respectively. In this regard, two distinct environmental conditions (similar to [17]) are being considered including (i) $\mu = 10, \lambda = 3$ and (ii) $\mu = 1, \lambda = 30$. The corresponding parameters of lognormal and Gamma distribution are obtained from (9) and (10), respectively. We note here that the case (i) and case (ii), respectively corresponds to $CV > 1$ and $CV < 1$ where $CV$ denotes coefficient of variation. $CV$ essentially determines the dispersion of a distribution and can be calculated as $\sqrt{\mu/\lambda}$. It is preferred to mention here that, the environmental scenarios namely, urban and suburban areas correspond to $CV \leq 1$. Figure 11 shows that for $CV < 1$ both $\mathcal{K}_G$ and $\mathcal{G}$ distributions are accurately approximating NLN distribution. On the other hand, ergodic capacity of $\mathcal{K}_G$ distributed channel deviates noticeably from NLN distribution over the environment with $CV > 1$ whereas $\mathcal{G}$ distribution shows satisfactory agreement with the NLN model. We note here that, the expression (14) of $\mathcal{G}$ distribution accurately approximates the NLN distribution in high SNR regime. The other performance measures such as SER and OP can be corroborated in an analogous fashion like capacity, as discussed recently. However, to intensify the analysis a different set of surrounding environment is considered as introduced by Loo et al. in [34]. The environmental conditions are based on lognormal distribution and can be classified as infrequent light shadowing ($\mu = 0.115, \lambda = 0.115$), average shadowing ($\mu = -0.115, \lambda = 0.161$) and frequent heavy shadowing ($\mu = -3.914, \lambda = 0.806$). As such, the shadowing parameters of IG and Gamma distributions can be calculated from (11) and (12), respectively. In Figures 12 and 13, comparative assessment of SER and OP are depicted, where the curves of $\mathcal{G}$ distributions are plotted according to (45), (52) and $\mathcal{K}_G$ distribution according to [7, Equations (25) and (10)]. Both Figures 12 and 13 indicate that NLN distribution can be effectively approximated by $\mathcal{K}_G$ and $\mathcal{G}$
distribution for infrequent light shadowing and average shadowing. However, for frequent heavy shadowing, performance curve of $K_G$ distribution deviates considerably, particularly after 25 dB. Contrary to this, $G$ distribution is still able to produce faithful results and thus indicates that it can be considered as an accurate substitute to NLN for diverse shadowing conditions.

7 | CONCLUSION

In this paper, $G$ distribution was considered to characterize the SNR statistics of radio wave propagation where a detailed investigation on the performance of OSTBC MIMO systems was presented in high SNR regime. In the approximate expression of ergodic capacity, the small and large scale fading parameters were found to be decoupled. In the analysis of low SNR, ergodic capacity performance was measured with respect to minimum bit energy per symbol as well as average SNR. Apart from that, the variance and PDF of capacity were also evaluated for in-depth analysis. Furthermore, tractable closed form expressions of average SER and OP were also formulated in high SNR regime. To this end, the performance of OSTBC MIMO systems were quantified in terms of diversity order and array gain.

In all instances we found that the derived analytical approximations yield useful practical insights with substantially less mathematical complexity compared to the corresponding exact expressions. Additionally, we compare the accuracy of $G$ and $K_G$ distributions in approximating the prevalent NLN model in terms of aforementioned performance metrics. It is verified that the mathematical analysis of $G$ distribution becomes more challenging compared to $K_G$ fading, yet able to provide better approximation accuracy. Finally, through Monte-Carlo simulations all the approximated expressions were observed to be sufficiently tight in high SNR regime and hence can be prescribed as suitable alternatives to exact expressions in various applications.
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How to cite this article: Pradhan, B.B., Roy, L.P., Mahapatra, D.K.: Performance assessment of orthogonal space-time block codes in Nakagami-$m$/inverse Gaussian fading MIMO channels. IET Commun. 15, 1518–1529 (2021). https://doi.org/10.1049/cmu2.12166