Random Schreier graphs and expanders

Luca Sabatini

Received: 9 June 2021 / Accepted: 5 April 2022 / Published online: 9 May 2022
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2022

Abstract
Let the group $G$ act transitively on the finite set $\Omega$, and let $S \subseteq G$ be closed under taking inverses. The Schreier graph $Sch(G, S)$ is the graph with vertex set $\Omega$ and edge set $\{ (\omega, \omega^s) : \omega \in \Omega, s \in S \}$. In this paper, we show that random Schreier graphs on $C \log |\Omega|$ elements exhibit a (two-sided) spectral gap with high probability, magnifying a well-known theorem of Alon and Roichman for Cayley graphs. On the other hand, depending on the particular action of $G$ on $\Omega$, we give a lower bound on the number of elements which are necessary to provide a spectral gap. We use this method to estimate the spectral gap when $G$ is nilpotent.

Keywords Expander graphs · Schreier graphs · Nilpotent groups

Mathematics Subject Classification 05C48 · 20P05

1 Introduction
Let $d \geq 2$, and let $\Gamma$ be an undirected, $d$-regular finite multigraph. For every vector $v \in \ell^2(\Gamma)$ and every vertex $x \in \Gamma$, the averaging operator (or random walk operator) $M = M(\Gamma)$ is defined by

$$Mv(x) := \frac{1}{d} \sum_{(x,y) \in E(\Gamma)} v(y).$$

We stress that the edges are counted with multiplicity. It is well known that the (real) spectrum of $M$ satisfies $\{1 = \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{|\Gamma|} \geq -1\}$. The magnitudes of the gaps $|1 - \lambda_2|$ and $|1 - \lambda_{|\Gamma|}|$ are linked to relevant properties of $\Gamma$, namely its isoperimetric number and pseudorandomness (for a rich introduction on these arguments, we refer
In the extremal cases, $\lambda_2 = 1$ if and only if $\Gamma$ is disconnected, and $\lambda|\Gamma| = -1$ if and only if $\Gamma$ has a bipartite connected component. Thus, it is natural to define

$$\text{gap}(\Gamma) := 1 - \lambda_2 \quad \text{and} \quad \lambda(\Gamma) := \max(|\lambda_2|, |\lambda|\Gamma|).$$

When $\epsilon > 0$, we say that $\Gamma$ is an $\epsilon$-expander if $\text{gap}(\Gamma) \geq \epsilon$, and a two-sided $\epsilon$-expander if $\lambda(\Gamma) \leq 1 - \epsilon$. If $G$ is a finite group and $S \subseteq G$ is a symmetric multiset\(^1\), the Cayley graph $Cay(G, S)$ is the $|S|$-regular graph having the elements of $G$ as vertices, and the edges are the couples $(g, gs)$ for all $g \in G$ and $s \in S$. In an influential paper, Alon and Roichman proved the following result:

**Theorem** (Alon-Roichman [2]). For every $\epsilon > 0$ there exists a constant $C_\epsilon$ such that the following holds. For every finite group $G$ one has

$$\mathbb{E}[\lambda(Cay(G, S \sqcup S^{-1}))] \leq \epsilon,$$

where $S \subseteq G$ is a random multiset of $\lceil C_\epsilon \log |G| \rceil$ elements.\(^2\)

In this article, we deal with what is perhaps the most natural generalization of Cayley graphs, Schreier graphs, which are defined as follows. When a group $G$ acts on a finite set $\Omega$ and $S \subseteq G$ is a finite symmetric multiset, $Sch(G \circlearrowright \Omega, S)$ is a $|S|$-regular graph which has the elements of $\Omega$ as vertices, and the edges are the couples $(\omega, \omega s)$, for all $\omega \in \Omega$ and $s \in S$. To lighten the notation, we immediately write

$$\lambda(G \circlearrowright \Omega, S) := \lambda(Sch(G \circlearrowright \Omega, S)) \quad \text{and} \quad \text{gap}(G \circlearrowright \Omega, S) := \text{gap}(Sch(G \circlearrowright \Omega, S)).$$

An important example is $G = \text{Sym}(n)$ in its natural action on $\{1, \ldots, n\}$: when a random multiset $S \subseteq G$ of size $d \geq 1$ is chosen, $Sch(\text{Sym}(n) \circlearrowright \{1, \ldots, n\}, S \sqcup S^{-1})$ is a “good” model for a random $2d$-regular graph [12]. In contrast, here we work with a generic transitive action of a generic group $G$. In this case, the Schreier graph is a quotient of the corresponding Cayley graph, and it is not hard to see that the aforementioned result from [2] implies that $C_\epsilon \log |G|$ elements are sufficient to gain $\lambda(G \circlearrowright \Omega, S \sqcup S^{-1}) \leq \epsilon$ with high probability. Our first main result is that $C_\epsilon \log |\Omega|$ elements are already enough.

**Theorem 1** (Random Schreier graphs are two-sided expanders). For every $\epsilon > 0$ there exists a constant $C_\epsilon$ such that the following holds. For every finite group $G$ and every transitive action of $G$ on $\Omega$, one has

$$\mathbb{E}[\lambda(G \circlearrowright \Omega, S \sqcup S^{-1})] \leq \epsilon,$$

where $S \subseteq G$ is a random multiset of $\lceil C_\epsilon \log |\Omega| \rceil$ elements. Moreover, we may choose $C_\epsilon \leq 2\epsilon^{-2} + o(1)$ when $|\Omega| \to +\infty$.

---

1 We work in the general context of multisets, which is useful in random arguments. A multiset is symmetric if each element appears with the same multiplicity as its inverse. Moreover, we write $S^{-1} := \{s^{-1} : s \in S\}$.

2 Unless explicitly stated otherwise, all logarithms are natural.
With the turn of the millennium, other mathematicians found different proofs of Alon-Roichman result, namely Landau and Russell [9], and Christofides and Markström [6], which proved an analogous of Theorem 1 for vertex-transitive graphs. These proofs, as ours, are based on Chernoff-type bounds on operator valued random variables [1]. The generality of Theorem 1 highlights a major difference between the original technique of [2], the trace method, and the subsequent approach. On the opposite side, it would be very interesting to understand how many elements are really needed for expansion. It is well known that abelian groups require many elements: more generally, a large abelian section of small index slows down the random walk, and make $\lambda$ quite close to 1. With a generic transitive action, we measure the abelian sections above the stabilizer of a point, and we weight them with respect to their index in $G$.

**Definition 2** Let $G$ act transitively on $\Omega$, and fix $\omega \in \Omega$. Let $Y := \text{Stab}(\omega)$. We define

$$\Theta(G \circ \Omega) := \max_{Y \leq H \leq G} |H/H'Y|^{1/[G:H]},$$

where $H' := [H, H]$ is the commutator subgroup of $H$.

Of course, the value of $\Theta$ does not depend on the particular $\omega$. Moreover, $1 \leq \Theta(G \circ \Omega) \leq |\Omega|$. The interest in $\Theta$ comes from the following theorem, which is proved in Sect. 4.

**Theorem 3** (Lower bound for expanding sets) Let $G$ act transitively on $\Omega$, and let $S \subseteq G$ be a symmetric multiset. If $\text{gap}(G \circ \Omega, S) \geq \varepsilon > 0$, then

$$|S| \geq \frac{2 \log \Theta(G \circ \Omega)}{\log(5/\varepsilon)}.$$

Therefore, the number of elements which are necessary to gain $\text{gap}(G \circ \Omega, S) \geq \varepsilon$ lies always between $C_\varepsilon \log \Theta(G \circ \Omega)$ and $C'_\varepsilon \log |\Omega|$ (the lower bound is always true, and the upper bound is true with high probability). Providing a lower bound on $\Theta(G \circ \Omega)$, we show that nilpotent Schreier graphs of bounded class require many elements:

**Theorem 4** (Spectral gap of nilpotent Schreier graphs) Let $G$ be a nilpotent group of class $c \geq 1$, and let $G$ act transitively on $\Omega$. If $S \subseteq G$ is any finite symmetric multiset, then

$$\text{gap}(G \circ \Omega, S) \leq 5 |\Omega|^{-f(|S|, c)},$$

where $f(|S|, c) := \frac{2(|S| - 1)}{|S|(|S|^c - |S|^c + |S| - 1)} \geq |S|^{-c-1}$. In particular, if $\text{gap}(G \circ \Omega, S) \geq \varepsilon > 0$, then $|S| \geq \left(\frac{\log |\Omega|}{\log(5/\varepsilon)}\right)^{1/(c+1)}$.

It is easy to see that if $G$ is a nilpotent group and $M < G$ is a maximal subgroup, then $G' \leq M$. In particular, $YG' \neq G$ for every $Y < G$. We will obtain
Theorem 4 via an improved version of this fact (Proposition 18), which can be of independent interest. The paper is organized as follows: in Sect. 2 we give all the preliminaries, and in Sect. 3 we prove Theorem 1. Section 4 is more algebraic, and contains the proofs of the Theorems 3 and 4.

2 Preliminaries

Schreier graphs may contain loops and multiple edges, even when \( S \) is a genuine set. Moreover, we see immediately that transitive actions are the only ones worthy of attention: if \( G \) acts on \( A \sqcup B \) where \( A \) and \( B \) are disjoint orbits, then

\[
\text{Sch}(G \odot (A \sqcup B), S) \cong \text{Sch}(G \odot A, S) \sqcup \text{Sch}(G \odot B, S)
\]

is a disjoint union of graphs. A transitive action of \( G \) is determined by the stabilizer of a point, i.e. by a subgroup of finite index. We reserve the capital letter \( Y \) for this subgroup. Given a symmetric multiset \( S \subseteq G \), following this notation we write

\[
\text{Sch}(G, Y, S) := \text{Sch}(G \odot \Omega, S).
\]

The quantities \( \lambda(G, Y, S) \) and \( \text{gap}(G, Y, S) \) are defined similarly.

Remark 5 [Reduction to faithful actions] Let \( N \leq Y \) be a normal subgroup of \( G \). It is easy to see that

\[
\text{Sch}(G, Y, S) \cong \text{Sch}(G/N, Y/N, \varphi(S)),
\]

where \( \varphi : G \to G/N \) is the natural projection, and \( \varphi(S) \) is considered with multiplicities. In particular, up to replacing \( G \) with the quotient by the normal core of \( Y \), we can always assume that \( G \) is finite.

We now describe the , as it is done implicitly in [8, Proposition 11.17]. To give a more precise description, when \( Y \leq G \), we denote by \( \text{Irr}(G, Y) \) the set of the irreducible complex representations (up to equivalence) of \( G \) having a nonzero \( Y \)-invariant vector.

Proposition 6 (Spectrum of a Schreier graph) Let \( G \) act transitively on \( \Omega \), and let \( Y \leq G \) be the stabilizer of a point. Then, as sets,

\[
\text{Spec(Sch}(G \odot \Omega, S)) = \bigcup_{\pi \in \text{Irr}(G, Y)} \text{Spec}(M_\pi),
\]

where, for each representation \( \pi \), the matrix \( M_\pi \) is defined as

\[
M_\pi := \frac{1}{|S|} \sum_{s \in S} \pi(s).
\]
Proof Let $\rho$ be the permutation representation of $G$ on the right-cosets of $Y$. The crucial observation is that the averaging operator $M$ of $Sch(G \circlearrowright \Omega, S)$ satisfies

$$M = M_\rho = \frac{1}{|S|} \sum_{s \in S} \rho(s).$$

Now, by a standard result in representation theory, $\rho$ is a nonnegative linear combination of distinct irreducible representations $\pi_1 = 1, ..., \pi_j$. Moreover, these are precisely the irreducible representations in $Irr(G, Y)$. Thus $M$ is equivalent to a block-type matrix

$$\begin{pmatrix}
M_{\pi_1} & M_{\pi_2} & \cdots \\
M_{\pi_2} & \ddots & \cdots \\
\vdots & \ddots & \ddots \\
M_{\pi_3} & \cdots & & M_{\pi_j}
\end{pmatrix},$$

and the proof follows. We stress that, by Frobenius reciprocity, the trivial representation has multiplicity one in a transitive permutation representation. □

**Lemma 7** (Bipartite Criterion for Schreier graphs) A connected Schreier graph $Sch(G \circlearrowright \Omega, S)$ is bipartite if and only if $S$ is disjoint from some subgroup of index 2 which contains the stabilizers.

Proof Let $Sch(G \circlearrowright \Omega, S)$ be connected and bipartite, and let $\omega \in \Omega$. Let $N := \{g \in G : \omega^g \in A\}$, where $A$ is the part containing $\omega$ in the bipartition. Then $N$ is a subgroup of $G$ containing the stabilizer of $\omega$, $S \cap N = \emptyset$ and $|G : N| = 2$. Since $N \triangleleft G$, it actually contains all the stabilizers. Now let $N \triangleleft G$ be of index 2 with $S \cap N = \emptyset$, and assume that $N$ contains the stabilizer of $\omega \in \Omega$. Fix $s \in S$, and let $A := \{\omega^n : n \in N\}$ and $B := \{\omega^{ns} : n \in N\}$. Then $|A| = |B| = |\Omega|/2$, and $A \cap B = \emptyset$, because $\omega^{n_1} = \omega^{n_2}$ would imply $s \in N$. □

As we have said in the introduction, the essential tool for [6, 9] is contained in [1, Theorem 19]. Here is a very special case of that result:

**Theorem 8** (Ahlswede-Winter) Let $X_1, ..., X_n$ be independent and identically distributed Hermitian random matrices in $M_m(\mathbb{C})$. Moreover, suppose that the eigenvalues of $X_i$ lie in $[-1, 1]$, and that every entry of $X_i$ has expectation zero, for all $i = 1, ..., n$. Then, for every $\varepsilon \in (0, 1/2)$,

$$\text{Prob} \left( \text{Spec} \left( \frac{1}{n} \sum_{i=1}^n X_i \right) \not\subset [-\varepsilon, \varepsilon] \right) \leq 2m \cdot \exp\left( -\frac{n\varepsilon^2}{\log 4} \right).$$

The original theorem of Ahlswede and Winter involves the weighted entropy function $H_{1/2}(x) := x \log(2x) + (1 - x) \log(2 - 2x)$. The version we stated appears in
Lemma 10 Let $G$ be a finite abelian group and let $S \subseteq G$ be a symmetric multiset. Then
\[
gap(G, S) \leq S \left|G\right|^{-2/|S|}.
\]

Proof If $\gap(G, S) = 0$, then there is nothing to prove. Let $r$ be a positive integer, and divide the unit circle into $r$ sections of the type $[e^{2\pi i k/r}, e^{2\pi i (k+1)/r}]$. This induces a partition of the $|S|$-dimensional torus into $r^{|S|}$ pieces. Let $\pi$ and $\sigma$ be two distinct irreducible characters of $G$ such that $(\pi(s))_{s \in S}$ and $(\sigma(s))_{s \in S}$ lie in the same class of this partition. So for every $s \in S$ one has $(\pi^{-1}\sigma)(s) \in [e^{-\pi i/r}, e^{\pi i/r}]$, and then $\sum_{s \in S}(\pi^{-1}\sigma)(s) \geq |S|\cos(\pi/r) \geq |S|(1 - \frac{\pi^2}{2r^2})$. Since $\pi^{-1}\sigma$ is a non-trivial irreducible character, from Proposition 6 we obtain
\[
\sum_{s \in S}(\pi^{-1}\sigma)(s) \leq |S|(1 - \gap(G, S)),
\]
and then \( r \leq \sqrt{\frac{\pi^2}{2 \text{gap}(G, S)}} \). If \( r \) is chosen larger than this quantity, then every irreducible character sends \( S \) to a different class of the partition, and the number of the irreducible characters (which is the cardinality of \( G \)) is bounded from above by \( r^{|S|/2} \). In particular,

\[
|G| \leq \left( \frac{5}{\text{gap}(G, S)} \right)^{|S|/2}.
\]

\( \square \)

**Remark 11** The paper [7] achieves the optimal exponent \(-4/|S|\) in two inequalities similar to that of Lemma 10. However, [7, Theorem 2] may not be effective when \( S \) contains many involutions (but provides \( \text{gap}(G, S) \leq 20|G|^{-4/|S|} \) when every involution appears with even multiplicity), and it seems hard to express [7, Theorem 6] in the form \( \text{gap}(G, S) \leq C|G|^{-4/|S|} \) which is required for Theorem 3 (we remark that \(|S|\) is assumed to be fixed in [7, Theorem 6]).

**Lemma 12** (Rayleigh–Ritz). Let \( M \) be a complex Hermitian matrix and \( v \) be a non-trivial vector. Let \( v = c_1v_1 + \ldots + c_jv_j \) be a decomposition of \( v \) in eigenvectors, and let \( \lambda_{\min}, \lambda_{\max} \) be, respectively, the minimum and the maximum of the eigenvalues associated with \( v_1, \ldots, v_j \). Then

\[
\frac{\langle Mv, v \rangle}{\langle v, v \rangle} \in [\lambda_{\min}, \lambda_{\max}].
\]

**Proof** Without loss of generality, we can assume that \( v_1, \ldots, v_j \) are unit vectors with distinct eigenvalues. In particular, they are pairwise orthogonal. Let \( \lambda_1, \ldots, \lambda_j \) be the eigenvalues associated with these vectors. We have \( \langle v, v \rangle = |c_1|^2 + \ldots + |c_j|^2 \), and

\[
\langle Mv, v \rangle = \langle c_1\lambda_1v_1, c_1v_1 \rangle + \ldots + \langle c_j\lambda_jv_j, c_jv_j \rangle = |c_1|^2\lambda_1 + \ldots + |c_j|^2\lambda_j.
\]

Hence \( \langle Mv, v \rangle/\langle v, v \rangle \) is a weighted average of the eigenvalues, and so it lies in the interval \([\lambda_{\min}, \lambda_{\max}]\). \( \square \)

**3 Random schreier graphs**

The following proposition contains most of the proof of Theorem 1. We work as in [6, Proof of Theorem 5].

**Proposition 13** Let the finite group \( G \) act transitively on \( \Omega \), and \( \epsilon, \delta > 0 \). Then, for a random multiset \( S \subseteq G \) of size \( \left\lceil \frac{\log 4}{\epsilon^2} \cdot \log \left( \frac{2|\Omega|}{\delta} \right) \right\rceil \), one has

\[
\text{Prob}(\lambda(G \cap \Omega, S \cup S^{-1}) \geq \epsilon) \leq \delta.
\]
Proof Let \( \varepsilon > 0 \), \( S := \{s_1, \ldots, s_d\} \subseteq G \) be a random multiset of size \( d \geq 1 \), and \( Y \leq G \) be the stabilizer of a point. Using the notation of Proposition 6, since the trivial representation has multiplicity one in a transitive permutation representation,

\[
\text{Prob}(\lambda(G, Y, S \sqcup S^{-1}) \geq \varepsilon) = \text{Prob}\left(\max_{1 \neq \pi \in \text{Irr}(G, Y)} \lambda(M_{\pi}) \geq \varepsilon\right),
\]

where \( \lambda(M_{\pi}) \) is the largest absolute value of an eigenvalue of \( M_{\pi} \). By standard probability theory, the previous quantity is at most \( \sum_{1 \neq \pi \in \text{Irr}(G, Y)} \text{Prob}(\lambda(M_{\pi}) \geq \varepsilon) \).

For every non-trivial \( \pi \in \text{Irr}(G, Y) \) we have

\[
M_{\pi} = \frac{1}{2d} \sum_{i=1}^{d} \pi(s_i) + \pi(s_i^{-1}) = \frac{1}{d} \sum_{i=1}^{d} X_i,
\]

where \( X_i := \frac{1}{2}(\pi(s_i) + \pi(s_i^{-1})) \). Now the random matrices \( X_i \)'s satisfy the hypotheses of Theorem 8. First, the eigenvalues of \( X_i \) lie in \([-1, 1]\), because the eigenvalues of \( \pi(s_i) \) are roots of unity. Second, if \( \rho \) is the regular representation of \( G \), then \( \sum_{g \in G} \rho(g) \) is the all-1 matrix, and it has rank 1. By the decomposition of \( \rho \), \( \sum_{g \in G} \rho(g) \) is equivalent to a block-type matrix having the matrices \( \sum_{g \in G} \pi(g) \) as blocks. Since \( \rho \) contains the trivial representation once, we have that \( \sum_{g \in G} \pi(g) \) is the zero operator, for every non-trivial irreducible representation \( \pi \). Hence

\[
\text{Prob}(\lambda(G \circ \Omega, S \sqcup S^{-1}) \geq \varepsilon) \leq 2e^{-d \varepsilon^2 / \log 4} \cdot \left( \sum_{1 \neq \pi \in \text{Irr}(G, Y)} \text{dim}(\pi) \right) \leq 2|\Omega| \cdot e^{-d \varepsilon^2 / \log 4},
\]

and the proof follows.

\( \square \)

Proof of Theorem 1 Let \( \lambda := \lambda(G \circ \Omega, S \sqcup S^{-1}) \), and fix \( \varepsilon > 0 \). Let \( \delta(|\Omega|) \) be a function tending to zero to be chosen later. Since \( \lambda \) takes value in \([0, 1]\), applying Proposition 13 with \( \varepsilon' := \varepsilon(1 - \delta) \) and \( \delta' := \delta \varepsilon \), we obtain

\[
\mathbb{E}[\lambda] \leq 1 \cdot \text{Prob}(\lambda \geq \varepsilon') + \varepsilon' \cdot \text{Prob}(\lambda < \varepsilon') \leq \delta' + \varepsilon' = \varepsilon.
\]

With these parameters, \( S \) is a random multiset of size

\[
\left\lceil \frac{\log 4}{\varepsilon^2(1 - \delta)^2} \cdot \log \left( \frac{2|\Omega|}{\delta \varepsilon} \right) \right\rceil.
\]

The desired bound on \(|S|\) follows choosing a slow function for \( \delta \), so that \( \log(1/\delta(|\Omega|)) = o(\log |\Omega|) \). In this way we obtain

\[
\log \left( \frac{2|\Omega|}{\delta \varepsilon} \right) = \log 2 + \log |\Omega| + \log(1/\delta) + \log(1/\varepsilon) \leq 2(1 - \delta)^2 \cdot \log 4 \cdot \log |\Omega|
\]

\( \square \) Springer
for every sufficiently large set \( \Omega \), where the last inequality follows because \( \delta \to 0 \) when \( |\Omega| \) grows, and \( \log 4 < 2 \).

We stress that, in the statements of Theorem 1 and Proposition 13, there is no substantial difference with the setting where repetitions are not allowed: in fact, \( O(\log |\Omega|) \) randomly chosen elements in \( G \) are almost surely made of distinct members.

### 4 Spectral gap and abelian sections

Given a finite group \( G \), the study of the algebraic conditions of \( G \) which are necessary (or sufficient) for the existence of a Cayley graph on \( G \) with good expansion has a long history, which goes back to [10, Section 3]. However, that article does not involve the spectral gap directly (but the so-called Kazhdan constant), and, as we said, only concerns the Cayley graph case. With this in mind, it is easy to see that our Theorem 3 is equivalent to the following.

**Proposition 14** (Improved Lubotzky–Weiss inequality). Let \( G \) act transitively on \( \Omega \), and let \( S \subseteq G \) be a symmetric multiset. Let \( Y \leq G \) be the stabilizer of a point. Then, for every intermediate subgroup \( Y \leq H \leq G \), one has

\[
gap(G \circ \Omega, S) \leq 5 |H/H'|^{-\frac{1}{2}}|G/H|^{\frac{1}{2}}.
\]

The next lemma is the main new ingredient required for the proof of Proposition 14. It is a generalized and improved version of [10, Proposition 3.9], and recalls the standard fact that \( \overline{S} \), as defined in (1), generates \( H \) if \( S \) generates \( G \).

**Lemma 15** Let \( Y \leq H \leq G \) and let \( S \subseteq G \) be a symmetric multiset. If \( \overline{S} \subseteq H \) is any multiset obtained via the Reidemeister–Schreier method, then

\[
gap(H, Y, \overline{S}) \geq \gap(G, Y, S).
\]

Moreover

\[
\lambda(H, Y, \overline{S}) \leq \lambda(G, Y, S).
\]

**Proof** Let \( T \subseteq G \) be the transversal of \( H \) which is used to provide \( \overline{S} \), and let \( R \subseteq H \) be any transversal of \( Y \) in \( H \). Moreover, let \( M \) and \( M^H \) be the averaging operators of \( \text{Sch}(G, Y, S) \) and \( \text{Sch}(H, Y, \overline{S}) \) respectively, and let \( M^H f = \tau f \) for some eigenvalue \( \tau \neq 1 \) and eigenfunction \( f \in \ell^2(H/Y) \), where \( H/Y \) denotes the set of the right-cosets of \( Y \) in \( H \). Let \( Z(H/Y) \subseteq \ell^2(H/Y) \) denote the (one-dimensional) subspace of constant functions. Since \( Z(H/Y) \) is the eigenspace of the trivial eigenvalue 1, and \( M^H \) is symmetric, we have \( f \in Z(H/Y)^\perp \), i.e. \( f \) is a function which sums to zero. We extend \( f \) to \( \tilde{f} \in \ell^2(G/Y) \) in the following way: \( \tilde{f}(Yrt) = f(Yr) \) for every \( r \in R \) and \( t \in T \). We notice that \( \tilde{f} \in Z(G/Y)^\perp \), in fact

\[
\langle \tilde{f}, 1 \rangle = \sum_{r \in R, t \in T} \tilde{f}(Yrt) = \sum_{r \in R} f(Yr) = |G : H| \sum_{r \in R} f(Yr) = |G : H| \langle f, 1 \rangle = 0.
\]
Moreover,
\[
\langle \mathbf{f}, \mathbf{f} \rangle = \sum_{r \in R} \tilde{f}(Yrt)^2 = \sum_{t \in T} \langle f, f \rangle = |G : H| \langle f, f \rangle
\]
and
\[
(M \mathbf{f}, \mathbf{f}) = \frac{1}{|S|} \sum_{r \in R} \sum_{s \in S} \tilde{f}(Yrts) \tilde{f}(Yrt) = \frac{1}{|S|} \sum_{r \in R} \sum_{s \in S} \tilde{f}(Yrts) f(Yr).
\]
Write \( rts = rts(\bar{s})^{-1} \bar{s} \), so that \( \tilde{f}(Yrts) = f(Yrts(\bar{s})^{-1}) \). It follows that
\[
(M \mathbf{f}, \mathbf{f}) = \frac{|G : H|}{|S|} \sum_{r \in R} \sum_{s \in S} f(Yrs) f(Yr) = |G : H| \langle M_H f, f \rangle.
\]
Finally, we have
\[
\tau = \frac{\tau \langle f, f \rangle}{\langle f, f \rangle} = \frac{\langle \mathbf{f}, \mathbf{f} \rangle}{\langle f, f \rangle} = \frac{\langle M_H \mathbf{f}, \mathbf{f} \rangle}{\langle f, f \rangle} = \frac{\langle M \mathbf{f}, \mathbf{f} \rangle}{\langle f, f \rangle}.
\]
Using the Rayleigh–Ritz Lemma 12, we see that the last ratio lies in \([-1, 1 - \text{gap}(G, Y, \bar{S})]\). Similarly, it lies in \([-\lambda(G, Y, S), \lambda(G, Y, S)]\).

**Remark 16** When looking at \( \bar{S} \) as a set (i.e. without the multiplicities induced by the Reidemeister–Schreier method), the inequality \( \text{gap}(H, Y, \bar{S}) \geq \text{gap}(G, Y, S) \) is not true in general. In fact, we found many counterexamples with the help of the GAP System [13]; the easiest are Cayley graphs on the dihedral group of order 8 and on the cyclic subgroup of order 4. We remark that the Kazhdan constant [10] (as well as the diameter of a Schreier graph) does not distinguish between sets and multisets by definition.

**Remark 17** Let \( Y \leq H \leq G \), and \( S \subseteq G \). When combined with Lemma 7, the second part of Lemma 15 implies that if \( S \) avoids some subgroup of \( H \) of index 2 which contains \( Y \), then \( S \) avoids some subgroup of \( G \) of index 2 which contains \( Y \). In particular, if \( G \) has no subgroups of index 2, then all Schreier graphs of the type \( \text{Sch}(H, Y, \bar{S}) \) are not bipartite.

**Proof of Proposition 14** From Lemma 15 we have \( \text{gap}(G, Y, S) \leq \text{gap}(H, Y, \bar{S}) \).
Since \( \text{Spec}(\text{Sch}(H, H'Y, \bar{S})) \) is contained in \( \text{Spec}(\text{Sch}(H, Y, \bar{S})) \) from Proposition 6, we gain \( \text{gap}(H, Y, \bar{S}) \leq \text{gap}(H, H'Y, \bar{S}) \). Since \( H'Y \) is normal in \( H \), from Remark 5 we have \( \text{Sch}(H, H'Y, \bar{S}) \cong \text{Cay}(H/H'Y, \varphi(\bar{S})) \), where we look at \( \varphi(\bar{S}) \) with multiplicities. Now we can apply Lemma 10 to this last (abelian) Cayley graph, to obtain
\[
\text{gap}(G, Y, S) \leq \text{gap}(H/H'Y, \varphi(\bar{S})) \leq 5 |H/H'Y|^{-2/|\varphi(\bar{S})|}.
\]
The fact that \( |\varphi(\bar{S})| = |\bar{S}| = |G : H||S| \) concludes the proof.
We move to study nilpotent groups more in detail. For every \( i \geq 1 \), let us denote by \( \gamma_i(G) \) the \( i \)-th term of the lower central series of \( G = \gamma_1(G) \). In particular, \( \frac{\gamma_i(G)}{\gamma_{i+1}(G)} \) is the center of \( \frac{G}{\gamma_{i+1}(G)} \). Moreover, we write \( x^y := y^{-1}xy \) and \([x, y] := x^{-1}x^y \). For every \( x, y, z \in G \), from basic properties of commutators we have

\[
[x, y, z] = [x, z]^y[y, z] \quad \text{and} \quad [x, yz] = [x, z][x, y]^z.
\]

Finally, we remark that, if \( x \in \gamma_i(G) \) and \( y \in \gamma_j(G) \), then \([x, y] \in \gamma_{i+j}(G) \).

**Proposition 18** Let \( G \) be a group, and let \( Y \leq G \) be a subgroup of finite index such that \( (Y, S) = G \) for some set \( S \subseteq G \) of size \( d \geq 2 \). Moreover, suppose that \( \gamma_{c+1}(G) \subseteq Y \) for some \( c \geq 1 \). Then

\[
|G : G'Y| \geq |G : Y|^{\beta(d, c)},
\]

where \( \beta(d, c) := \frac{d-1}{d^{c+1} - d^2 - d - 1} \geq \frac{1}{2d^2} \).

**Proof** For every \( i \geq 1 \) one has \( \gamma_i(G)Y = \gamma_i(G)(\gamma_{i+1}(G)Y) \). Then \( \gamma_{i+1}(G)Y \triangleleft \gamma_i(G)Y \), and the quotients \( \gamma_i(G)Y/\gamma_{i+1}(G)Y \) are all abelian. Now we have

\[
Y = \gamma_{c+1}(G)Y \triangleleft \gamma_{c}(G)Y \triangleleft \ldots \triangleleft \gamma_{3}(G)Y \triangleleft G'Y \triangleleft G.
\]

We write \( x \equiv_i z \) to say that \( x\gamma_i(G)Y = z\gamma_i(G)Y \). From (2), for every \( i \), the commutator mapping defines a surjective bilinear form

\[
\frac{\gamma_i(G)Y}{\gamma_{i+1}(G)Y} \times \frac{G}{G'Y} \to \frac{\gamma_{i+1}(G)Y}{\gamma_{i+2}(G)Y},
\]

by setting \((x\gamma_{i+1}(G)Y, zG'Y) \to [x, z]\gamma_{i+2}(G)Y \) for every \( x \in \gamma_i(G) \) and \( z \in G \). To check that this is well-defined, let \( \gamma_1 \in \gamma_{i+1}(G), \gamma_2 \in G', \gamma_1, \gamma_2 \in Y \). Working in \( \frac{\gamma_{i+1}(G)Y}{\gamma_{i+2}(G)Y} \) we have

\[
[x\gamma_1, z\gamma_2] \equiv_{i+2} [x, z\gamma_2] = [x, \gamma_2][x, z]^\gamma_2 \equiv_{i+2} [x, \gamma_2] \equiv_{i+2} [x, z],
\]

because \([x, \gamma_2] \in \gamma_{i+2}(G) \) and \([x, z] \in \gamma_{i+1}(G) \). From the universal property of tensor product, there exists a surjective group homomorphism

\[
\frac{\gamma_i(G)Y}{\gamma_{i+1}(G)Y} \otimes \frac{G}{G'Y} \to \frac{\gamma_{i+1}(G)Y}{\gamma_{i+2}(G)Y}
\]

for every \( 1 \leq i \leq c - 1 \). If \( d(G) \) denotes the minimal size of a generating set of \( G \), we have \( d(G/G'Y) \leq d \). The group on the left side of (3) can be generated by \( d(\gamma_i(G)Y/\gamma_{i+1}(G)Y) \cdot d \) elements. It follows from (3) and induction that
$d(\gamma_i(Y)Y/\gamma_{i+1}(Y)Y) \leq d^i$ for every $i \geq 1$. Similarly, the order of an elementary element

$$x \otimes z \in \frac{\gamma_i(Y)Y}{\gamma_{i+1}(Y)Y} \otimes \frac{G}{G'Y}$$

is the least common multiple between the orders of $x$ and $z$. By induction, the order of every element in $\gamma_i(Y)Y/\gamma_{i+1}(Y)Y$ is at most $|G/G'Y|$, for every $i \geq 1$. For all $i \geq 2$, this implies that

$$\left| \frac{\gamma_i(Y)Y}{\gamma_{i+1}(Y)Y} \right| \leq |G/G'Y|^{d^i},$$

while for $i = 1$ we leave $|G/G'Y|$ in the computation. Finally, we have

$$|G : Y| = \prod_{i=1}^c |\gamma_i(Y)Y/\gamma_{i+1}(Y)Y| \leq |G/G'Y|^{1+d^2+\ldots+d^c} = |G/G'Y|^{\frac{d^{c+1}-1}{d-1}-d}.$$ 

We remark that a similar inequality of the type $|G/G'| \geq |G|^{e(d,c)}$ is given in [5, Lemma 4.13]. Of course, the hypotheses of Proposition 18 are verified when $G$ is a $d$-generated group of class $c$. The exponent $\beta(d,c)$ is not far from the best possible: the next example provides arbitrarily large groups where $|G/G'|$ is roughly $|G|^c/d^{c-1}$.

**Example 19** Let $d \geq 3$ and define $W := \{x_1, \ldots, x_d : (x_1)^2 = \ldots = (x_d)^2 = 1\}$. If $G_n := W/\gamma_n(W)$ for every $n \geq 1$, then $G_n$ is a $d$-generated 2-group of class $n$, and

$$\log_2 |G_n/(G_n)'| = \log_2 |W/W'| = d.$$ 

We are about to show that $G_n$ is quite large. Let $F$ be the free subgroup of $W$ which is generated by the $d-1$ elements $x_1x_2, x_1x_3, \ldots, x_1x_d$. We have $|W : F| = 2$ and $W \cong F \rtimes \langle x_1 \rangle$. Let $(P_n)_{n \geq 1}$ be the exponent-2 central series of $F = P_1$. By [11, Lemma 6.2] we have $\gamma_n(W) = P_n$ for every $n \geq 2$. Now the behavior of $(P_n)_{n \geq 1}$ is well known: by [4, Lemma 20.7] we obtain

$$\log_2 |G_n| = 1 + \log_2 |F : P_{n+1}| \sim \frac{(d-1)^{n+2}}{n(d-2)^2}$$

when $n \to +\infty$.

**Proof of Theorem 4** Let $Y \leq G$ be the stabilizer of a point. If $Y \langle S \rangle \neq G$, then there is nothing to prove, because $\text{gap}(G \cap \Omega, S) = 0$. Otherwise $S$ satisfies the hypotheses of Proposition 18, and so $|G/G'Y| \geq |\Omega|^{\beta(|S|,c)}$. Applying Proposition 14 with $H = G$ we obtain the first inequality in the statement. The second inequality follows arranging the terms. 

\[\square\]
Remark 20 A result similar to Theorem 4 can be obtained via a purely combinatorial method. Indeed, a nilpotent group of bounded class has “polynomial growth,” as explained in [3]. This property is inherited by the Schreier graph, and so it is easy to see that this graph has very large diameter. Then, it is well known that a large diameter implies a small spectral gap (see [8, Sect. 2.4], for example). However, the bound we obtain along this road is weaker than Theorem 4 itself, and so it is not worth to state it precisely.

Acknowledgements The author thanks Pablo Spiga and the two anonymous referees for many useful comments and remarks.

Funding The Funding was provided by Università degli Studi di Firenze, Istituto Nazionale di Alta Matematica "Francesco Severi”.

Data availability All data generated or analyzed during this study are included in this published article.

References

1. Ahlswede, R., Winter, A.: Strong converse for identification via quantum channels. IEEE Trans. Inf. Theory 48, 569–579 (2002)
2. Alon, N., Roichman, Y.: Random Cayley graphs and expanders. Random Struct. Algorithms 5(2), 271–284 (1994)
3. Black, S.: Asymptotic growth of finite groups. J. Algebra 209, 402–426 (1998)
4. Blackburn, S., Neumann, P., Venkataraman, G.: Enumeration of Finite Groups, Cambridge Tracts in Mathematics 173 (2007)
5. Breuillard, E., Tointon, M.: Nilprogressions and groups with moderate growth. Adv. Math. 289, 1008–1055 (2016)
6. Christofides, D., Markström, K.: Expansion properties of random Cayley graphs and vertex-transitive graphs via matrix martingales. Random Struct. Algorithms 32(1), 88–100 (2008)
7. Friedman, J., Murty, R., Tillich, J.-P.: Spectral estimates for abelian Cayley graphs. J. Combin. Theory Ser. B 96, 111–121 (2006)
8. Hoory, S., Linial, N., Widgerson, A.: Expander graphs and their applications, Bulletin (New Series) of the American Mathematical Society 43 (4), 439–561 (2006)
9. Landau, Z., Russel, A.: Random Cayley graphs are expanders: a simple proof of the Alon-Roichman theorem, The Electronic Journal of Combinatorics 11 (2004), Research Paper 62
10. Lubotzky, A., Weiss, B.: Groups and expanders, DIMACS Series in Discrete Mathematics and Theoretical Computer. Science 10, 95–109 (1993)
11. Potočnik, P., Spiga, P., Verret, G.: Asymptotic enumeration of vertex-transitive graphs of fixed valency, J. Combin. Theory Ser. B 122, 221–240 (2017)
12. Puder, D.: Expansion of random graphs: new proofs, new results. Inventiones Mathematicae 201, 845–908 (2015)
13. The GAP Group, GAP—Groups, Algorithms, and Programming, Version 4.11.1, 2021 (https://www.gap-system.org)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.