Recursive Quantum Approximate Optimization Algorithm for the MAX-CUT problem on Complete graphs

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(Dated: November 30, 2022)

Quantum approximate optimization algorithms are hybrid quantum-classical variational algorithms designed to approximately solve combinatorial optimization problems such as the MAX-CUT problem. In spite of its potential for near-term quantum applications, it has been known that quantum approximate optimization algorithms have limitations for certain instances to solve the MAX-CUT problem, at any constant level $p$. Recently, the recursive quantum approximate optimization algorithm, which is a non-local version of quantum approximate optimization algorithm, has been proposed to overcome these limitations. However, it has been shown by mostly numerical evidences that the recursive quantum approximate optimization algorithm outperforms the original quantum approximate optimization algorithm for specific instances. In this work, we analytically prove that the recursive quantum approximate optimization algorithm is more competitive than the original one to solve the MAX-CUT problem for complete graphs with respect to the approximation ratio.

I. INTRODUCTION

There has been a growing interest in practical quantum computing in the noisy intermediate-scale quantum (NISQ) era. The NISQ devices have several restrictions due to noisy in quantum gates and limited quantum resources [1]. Diverse disciplines, for instances, combinatorial optimization, quantum chemistry, and machine learning, are regarded as potential areas of application to demonstrate a quantum advantage over the best known classical methods in the NISQ devices.

Quantum approximate optimization algorithm (QAOA) was designed to solve hard combinatorial optimization problems such as the MAX-CUT problem [2]. QAOA is a hybrid quantum-classical algorithm consisting of a parametrized quantum circuit and a classical optimizer to train it, and it has been proposed as one of the principal approaches to address the restrictions of the NISQ devices since the parameters such as the circuit depth can be handled [2]. Even though QAOA is one of the promising candidates of the NISQ algorithms, it has been known that QAOA has limited performance to solve the MAX-CUT problem on several instances for any constant depth [3–9].

The recursive quantum approximate optimization algorithm, the RQAOA for short, has been recently proposed to overcome the limitations of QAOA [4]. There have been known very few results on the RQAOA [4, 10, 11]. Only one of them analytically proves that the level-1 RQAOA performs better than any constant level QAOA for solving the MAX-CUT problem on cycle graphs [4] while the others give only numerical evidences to claim similar arguments for graph coloring problem [11] and for finding the largest energy of Ising Hamiltonian [10].

In this paper, we compare the performance of the level-1 QAOA and the level-1 RQAOA for solving the MAX-CUT problem on complete graphs, and show that the approximation ratio of the level-1 RQAOA is exactly one whereas that of the level-1 QAOA is strictly less than one for any complete graphs with $2n$ vertices. This implies that the RQAOA could be a better algorithm than the QAOA for the NISQ devices since higher level algorithms can produce uncorrectable errors on the NISQ devices.

This paper is organized as follows. In Sec. II we briefly review the MAX-CUT problem and QAOA to solve it. In Sec. III we introduce the RQAOA which is the non-local variant of QAOA, and prove that the level-1 RQAOA outperforms the level-1 QAOA for solving the MAX-CUT problem on complete groups. In Sec. IV we summarize our result, and discuss future works.

II. QAOA FOR THE MAX-CUT PROBLEM

Let $G = (V, E)$ be a graph with the set of vertices $V = \{1, 2, \ldots, n\}$ and the set of edges $E = \{(i, j) : i, j \in V\}$. The MAX-CUT problem is a well-known combinatorial optimization problem which aims to split $V$ into two disjoint parts such that the number of edges spanning two parts is maximized. The MAX-CUT problem can be formulated...
by maximizing the cost function

\[ C(x) = \frac{1}{2} \sum_{(i,j) \in E} (1 - x_i x_j) \]

for \( x = (x_1, x_2, \ldots, x_n) \in \{-1, 1\}^n \).

The Quantum approximate optimization algorithm (QAOA) is a quantum algorithm to deal with combinatorial optimization problems such as the MAX-CUT problem [2]. In this work, we focus on its application to the MAX-CUT problem. In this case, the classical cost function can be converted to a quantum problem Hamiltonian

\[ H_C = \frac{1}{2} \sum_{(i,j) \in E} (I - Z_i Z_j) , \]

where \( Z_i \) is the Pauli operator \( Z \) acting on the \( i \)-th qubit.

The approximation ratio of QAOA \( p \) is defined as \( r = \frac{F_p(\gamma^*, \beta^*)}{C_{\text{max}}} \), where \( C_{\text{max}} = \max_{x \in \{-1, 1\}^n} C(x) \). It has been shown that the QAOA has limitations for certain instances [4, 12, 13]. Bravyi et al. [4] stated that the locality and symmetry of the QAOA cause these limitations, and they proposed that a non-local version of QAOA called the recursive quantum approximate optimization algorithm (RQAOA).

III. RQAOA

In this section, we first briefly review the concept of the RQAOA. For the level-\( p \) RQAOA, denoted by RQAOA\( p \), we consider an Ising-like Hamiltonian

\[ H_n = \sum_{(i,j) \in E} J_{i,j} Z_i Z_j \]

which is defined on a graph \( G_n = (V, E) \) with \( |V| = n \), where \( J_{i,j} \in \mathbb{R} \) are arbitrary. The RQAOA\( p \) attempts to approximate

\[ \max_{x \in \{-1, 1\}^n} \langle x | H_n | x \rangle, \]

and it can be described by the following steps.

1. Apply the original QAOA to find the optimal state \( |\psi_p(\gamma^*, \beta^*)\rangle \) which maximizes \( \langle \psi_p(\gamma, \beta) | H_n | \psi_p(\gamma, \beta) \rangle \).

2. Compute

\[ M_{i,j} = \langle \psi_p(\gamma^*, \beta^*) | Z_i Z_j | \psi_p(\gamma^*, \beta^*) \rangle \]

for every edges \( (i, j) \in E \).

3. Choose a pair \((k, l)\) which maximizes the magnitude of \( M_{i,j} \).
4. Impose the constraint \( Z_k = \text{sgn}(M_{k,l})Z_l \), and replace it into the Hamiltonian \( H_n \).

\[
H_n = \sum_{(i,k) \in E} J_{i,k} Z_i Z_k + \sum_{i,j \neq k} J_{i,j} Z_i Z_j = \text{sgn}(M_{k,l}) \left[ \sum_{(i,k) \in E} J_{i,k} Z_i Z_l \right] + \sum_{i,j \neq k} J_{i,j} Z_i Z_j
\]

5. Call RQAOA recursively to maximize the expected value of a new Ising Hamiltonian \( H_{n-1} \) depending on \( n-1 \) variables:

\[
H_{n-1} = \sum_{(i,l) \in E_0} J'_{i,l} Z_i Z_l + \sum_{(i,j) \in E_1} J'_{i,j} Z_i Z_j,
\]

where

\[
E_0 = \{(i,l) : (i,k) \in E\},
E_1 = \{(i,j) : i,j \neq k\},
\]

and

\[
J'_{i,j} = \begin{cases} 
\text{sgn}(M_{k,l})J_{i,k} & \text{if} \ (i,l) \in E_0, \\
J_{i,j} & \text{if} \ (i,j) \in E_1.
\end{cases}
\]

6. The recursion stops when the number of variables reaches some suitable threshold value \( n_c \ll n \), and find \( x^* = \arg\max_{x \in \{-1,1\}^n} \langle x | H_{n_c} | x \rangle \) by a classical algorithm.

7. Reconstruct the original (approximate) solution \( \hat{x} \in \{-1,1\}^n \) from \( x^* \) using the constraints.

IV. OUR RESULT

Now, we investigate the performance of the original QAOA\(_1\) and RQAOA\(_1\) for the MAX-CUT problem on complete graphs, and we have the following theorem.

**Theorem 1.** Let \( K_{2n} \) be the complete graph with \( 2n \) vertices for \( n \geq 2 \) and let \( H_C = \frac{1}{2} \sum_{(i,j) \in E} (I - Z_i Z_j) \) be the problem Hamiltonian for the MAX-CUT problem. Then

1. RQAOA\(_1\) achieves the approximation ratio 1.

2. The approximation ratio of QAOA\(_1\) < 1. In particular, for \( n \geq 4 \), the approximation ratio of QAOA\(_1\) < \( 1 - \frac{1}{8n^2} \).

**Proof.**

1. Let

\[
H_{2n} = \frac{1}{2} \sum_{(i,j) \in E} (I - Z_i Z_j),
\]

where \( i, j \) are vertices of \( K_{2n} \). Consider a cost function of the form

\[
C_{2n}(x) = \sum_{(i,j) \in E} (I - x_i x_j)
\]

Suppose that \((\beta^*, \gamma^*) = \arg\max_{\beta, \gamma} \langle \psi_p(\gamma, \beta) | H_{2n} | \psi_p(\gamma, \beta) \rangle\). The exact form for the expectation value for QAOA with \( p = 1 \) has been known in [14] and it allows us to calculate \( M_{ij} \) as follows. For each edge \((i,j)\),

\[
M_{ij} = \langle \psi_p(\gamma^*, \beta^*) | Z_i Z_j | \psi_p(\gamma^*, \beta^*) \rangle
\]

\[
= \frac{1}{4} \left[ \sin 4\beta^* \cdot \sin \gamma^* \cdot 2 \cos^{2n-2} \gamma^* - \sin^2 2\beta^* \cdot (1 - \cos^{2n-2} 2\gamma^*) \right]
\]

\[
= \frac{1}{2} \sin 4\beta^* \cdot \sin \gamma^* \cdot \cos^{2n-2} \gamma^* - \frac{1}{8} \left( 1 - \cos 4\beta^* \right) \left[ 1 - (2 \cos 2\gamma^* - 1)^{n-1} \right].
\]
For the recursion step, we can pick a pair \((k, l)\) in \(E\) randomly since all \(M_{ij}\)'s coincide. Without loss of generality, assume that \((k, l) = (2n - 1, 2n)\). It can be easily shown that \(M_{i,j} < 0\) for all edges \((i, j)\) and thus, by imposing the constraint

\[ x_{2n} = -x_{2n-1}, \]

the RQAOA removes the variable \(x_{2n}\) from the cost function \(C_{2n}(x)\), we obtain the new cost function of the form

\[
\frac{1}{2}|E_{2n}| + \frac{1}{2} \left( x_{2n-1} + \cdots + x_{2n-2}x_{2n-1} + x_{2n-1}x_{2n-2} - \sum_{(i,j) \in E_{2n-1}} x_ix_j \right) 
\]

\[
= \frac{1}{2}|E_{2n}| + \frac{1}{2} - \frac{1}{2} \sum_{(i,j) \in E_{2n-2}} x_ix_j, 
\]

\[
= \frac{1}{2}|E_{2n}| + \frac{1}{2} - \frac{1}{2}|E_{2n-2}| + C_{2n-2}(x'). \tag{2}
\]

where \(x' \in \{-1, 1\}^{2n-2}\).}

![Diagram](image)

**FIG. 1:** This schematic diagram shows the change of the cost function after one iteration of the RQAOA through the graph. The RQAOA eliminates the variable \(x_6\) by imposing the constraint \(x_5 = -x_6\) on the cost function \(C_6(x)\). In the middle graph, the red dashed edges indicate the terms including \(x_6\) in \(C_6(x)\) and these terms will be canceled out after substituting \(-x_5\) for \(x_6\) due to the different sign. As a consequence, we obtain the new cost function in terms of \(C_4(x')\) with additional terms as we can see in Eq. (3).

Similarly, the RQAOA eliminates the variable \(x_{2n-2}\) by imposing the constraint

\[ x_{2n-2} = -x_{2n-3} \]

on the cost function \(C_{2n}(x)\), and we have the next cost function of the form

\[
\frac{1}{2}|E_{2n}| + \frac{1}{2} - \frac{1}{2} \sum_{(i,j) \in E_{2n-2}} x_ix_j 
\]

\[
= \frac{1}{2}|E_{2n}| + \frac{1}{2} - \frac{1}{2} \left( 1 - \sum_{(i,j) \in E_{2n-4}} x_ix_j \right) 
\]

\[
= \frac{1}{2}|E_{2n}| + \frac{1}{2} + \frac{1}{2} \left( 1 - \sum_{(i,j) \in E_{2n-4}} x_ix_j \right) 
\]

\[
= \frac{1}{2}|E_{2n}| + \frac{1}{2} + \frac{1}{2} - \frac{1}{2}|E_{2n-4}| + C_{2n-4}(x''), \tag{3}
\]

where \(x'' \in \{-1, 1\}^{2n-4}\). By imposing the constraints inductively,

\[
x_{2n} = -x_{2n-1} \\
x_{2n-2} = -x_{2n-3} \\
\vdots \\
x_{2n-(2k-2)} = -x_{2n-(2k-1)}, \tag{5}
\]
the cost function $C_{2n}(\mathbf{x})$ after eliminating variables $x_{2n}, x_{2n-2}, \ldots, x_{2n-(2k-2)}$ becomes

$$\frac{1}{2} |E_{2n}| + \frac{k}{2} - \frac{1}{2} |E_{2n-2k}| + C_{2n-2k}(\tilde{\mathbf{x}}),$$

where $\tilde{\mathbf{x}} \in \{-1, 1\}^{2n-2k}$. Now, we observe that

$$\max_{\mathbf{x} \in \{-1, 1\}^{2n}} C_{2n}(\mathbf{x}) \geq \max_{\mathbf{x} \in \{-1, 1\}^{2n} \text{with } 5} C_{2n}(\mathbf{x})$$

$$= \frac{1}{2} n(2n - 1) - \frac{1}{4}(2n - 2k)(2n - 2k - 1) + \frac{k}{2} + \max_{\tilde{\mathbf{x}} \in \{-1, 1\}^{2n-2k}} C_{2n-2k}(\tilde{\mathbf{x}})$$

$$= \frac{1}{2} n(2n - 1) - \frac{1}{2}(n - k)(2n - 2k - 1) + \frac{k}{2} + (n - k)^2$$

Thus, this completes the proof.

2. In order to obtain the bounds for the approximation ratio of QAOA1, we take the exact formula in (13) once again. For a complete graph with $2n$ vertices and $n \geq 2$, we have

$$\langle C_{ij} \rangle = \frac{1}{2} - \frac{1}{4} \sin^2(2\beta) \left(1 - \cos^{2n-2}(2\gamma)\right) + \frac{1}{2} \sin(4\beta) \sin \gamma \cos^{2n-2}(\gamma),$$

where $C_{ij} = \frac{1}{2} I - Z_i Z_j$ and $\langle C_{ij} \rangle := \langle \psi_1(\gamma, \beta) | C_{ij} | \psi_1(\gamma, \beta) \rangle$. The QAOA1 for the MAX-CUT problem on the complete graph $K_{2n}$ maximizes the expectation value

$$F_1(\gamma, \beta) = \langle \psi_1(\gamma, \beta) | H_C | \psi_1(\gamma, \beta) \rangle$$

$$= |E_{2n}| \langle C_{ij} \rangle,$$

or, equivalently, it maximizes the following function with respect to the parameters $\gamma, \beta$

$$f(\gamma, \beta) := \frac{1}{2} \sin(4\beta) \sin \gamma \cos^{2n-2}(\gamma) - \frac{1}{4} \sin^2(2\beta) \left(1 - \cos^{2n-2}(2\gamma)\right)$$

$$= \frac{1}{2} \sin(4\beta) \sin \gamma \cos^{2n-2}(\gamma) - \frac{1}{8} \left(1 - \cos(4\beta)\right) \left(1 - \cos^{2n-2}(2\gamma)\right).$$

Let us first differentiate the function $f$ by $\beta$ to obtain the optimal $\beta$ as a function of $\gamma$.

$$\frac{\partial f}{\partial \beta} = 2 \cos(4\beta) \sin \gamma \cos^{2n-2}(\gamma) - \frac{1}{2} \sin(4\beta) \left(1 - \cos^{2n-2}(2\gamma)\right)$$

Thus, we have

$$\frac{\partial f}{\partial \beta} = 0 \iff \tan(4\beta) = \frac{\frac{4}{2} \sin \gamma \cos^{2n-2}(\gamma)}{1 - \cos^{2n-2}(2\gamma)},$$

and hence the optimal parameter $\beta^*$ is $\arctan \left(\frac{4 \sin \gamma \cos^{2n-2}(\gamma)}{1 - \cos^{2n-2}(2\gamma)}\right)$. Using the trigonometric identities

$$\sin \left(\arctan(x)\right) = \frac{x}{\sqrt{1 + x^2}}$$

and $\cos \left(\arctan(x)\right) = \frac{1}{\sqrt{1 + x^2}}$

for $x > 0$, we obtain

$$f(\beta^*, \gamma) = \frac{1}{2} \sin \left(\arctan \left(\frac{4 \sin \gamma \cos^{2n-2}(\gamma)}{1 - \cos^{2n-2}(2\gamma)}\right)\right) \sin \gamma \cos^{2n-2}(\gamma)$$

$$- \frac{1}{8} \left[1 - \cos \left(\arctan \left(\frac{4 \sin \gamma \cos^{2n-2}(\gamma)}{1 - \cos^{2n-2}(2\gamma)}\right)\right)\right] \left(1 - \cos^{2n-2}(2\gamma)\right).$$

For the simplicity of calculation, let $d = 2n - 2$, $s_\gamma = \sin \gamma$, $c_\gamma = \cos \gamma$, and

$$x(\gamma) := \frac{4 \sin \gamma \cos^{2n-2}(\gamma)}{1 - \cos^{2n-2}(2\gamma)} = \frac{4 s_\gamma c_\gamma^d}{1 - (2 c_\gamma^2 - 1)^d}.$$
Then the function $f$ can be rewritten and simplified by using the constraint in Eq. (7) as

$$f(\gamma) := f(\beta^*, \gamma) = \frac{1}{2} \frac{x(\gamma)}{\sqrt{1 + x(\gamma)^2}} s^d - \frac{1}{8} \left(1 - \frac{1}{\sqrt{1 + x(\gamma)^2}}\right) \left(1 - (2c^2 - 1)^d\right)$$

$$= \frac{1}{8} \frac{x(\gamma)}{\sqrt{1 + x(\gamma)^2}} \left(1 - (2c^2 - 1)^d\right) x(\gamma) - \frac{1}{8} \left(1 - \frac{1}{\sqrt{1 + x(\gamma)^2}}\right) \left(1 - (2c^2 - 1)^d\right)$$

$$= \frac{1}{8} \left(1 - (2c^2 - 1)^d\right) \left(\sqrt{1 + x(\gamma)^2} - 1\right)$$

$$= \frac{1}{8} \left(\sqrt{(1 - (2c^2 - 1)^d)^2 + 16s^2 c^2 d} - (1 - (2c^2 - 1)^d)\right)$$

We want to show that

$$f(\gamma) = \frac{1}{8} \left(1 - (2c^2 - 1)^d\right) \left(\sqrt{1 + x(\gamma)^2} - 1\right) < \frac{1}{4n - 1}$$

for all $\gamma$. Then the approximation ratio of QAOA$_1$ for the MAX-CUT problem on complete graphs $K_{2n}$ is

$$\frac{F_p(\gamma^*, \beta^*)}{\max_x C_{2n}(x)} = \frac{\max_{\gamma, \beta} F_1(\gamma, \beta)}{n^2} = \frac{|E_{2n}| \left(\frac{1}{2} + \max_{\gamma} f(\gamma)\right)}{n^2}$$

$$= \frac{(2n - 1) \left(\frac{1}{2} + \max_{\gamma} f(\gamma)\right)}{n}$$

$$< 1 - \frac{1}{2n(4n - 1)}$$

$$< 1 - \frac{1}{8n^2}.$$ 

By substituting the definition of $x(\gamma)$, we can see that the inequality in Eq. (8) holds if and only if

$$\frac{4}{(4n - 1)^2} + \frac{1}{4n - 1} \left(1 - (2c^2 - 1)^d\right) - (1 - c^2) c^d > 0$$

for all $\gamma$. Now, let us define

$$g(t) := \frac{4}{(4n - 1)^2} + \frac{1}{4n - 1} \left(1 - (2t - 1)^d\right) - (1 - t) t^d$$

for $t := c^2 \in [0, 1]$. Then we can prove that $g(t) > 0$ for all $0 \leq t \leq 1$ (See Appendix A for the details)

\[ \square \]

V. CONCLUSION

In this work, we have analyzed the performance of the level-1 RAOA and the level-1 QAOA to solve the MAX-CUT problem on complete graphs with $2n$. In this case, we have proved that the level-1 RQAOA achieves the approximation ratio exactly one which means that it can always find the exact solution. On the other hand, we have also showed that the approximation ration of the level-1 QAOA is strictly less than one for any $n$. We expect that a similar result for complete graphs with $2n - 1$ vertices by exploiting the same argument.

There have been known not many results on the RQAOA. For the next step, it would be interesting to analyze the performance of the level-1 RAOA for solving the MAX-CUT problem on other graphs. Furthermore, it would be also considered the same argument for other combinatorial optimization problems such as the MAX-clique problem.

Appendix A: The positivity of the function $g(t)$

In this section, we want to show that for all $t := c^2 \in [0, 1]$,

$$g(t) := \frac{4}{(4n - 1)^2} + \frac{1}{4n - 1} \left(1 - (2t - 1)^{2n - 2}\right) - (1 - t)^{2n - 2} > 0.$$
To find the minimum of \( g(t) \), we observe that the necessary condition for the equation
\[
g'(t) = -\frac{4n-4}{4n-1} (2t-1)^{2n-3} + t^{2n-3}(-2n-2) + (2n-1)t = 0. \tag{A1}
\]

Since \( g \) is continuous, we need to see that \( g(0) > 0, g(1) > 0, \) and \( g(t^*) > 0 \) for all critical points \( t^* \in [0,1] \). Observe that
\[
g(0) = g(1) = \frac{1}{(4n-1)^2} > 0.
\]

We consider the critical points \( t^* \in (0,1) \).

\[
g'(t^*) = -\frac{4n-4}{4n-1} (2t^* - 1)^{2n-3} + t^{2n-3}(-2n-2) + (2n-1)t^* = 0, \tag{A2}
\]
or, equivalently,
\[
\frac{4n-4}{4n-1} (2t^* - 1)^{2n-3} = t^{2n-3}(-2n-2) + (2n-1)t^*.
\tag{A3}
\]

Now, by imposing the condition in Eq. (A3) on the function \( g \), we have
\[
g(t^*) = \frac{4}{(4n-1)^2} + \frac{1}{4n-1} \left( 1 - (2t^* - 1)^{2n-2} \right) - (1 - t^*)t^{2n-2}
\]
\[
= \frac{4}{(4n-1)^2} + \frac{1}{4n-1} \left[ 1 - \frac{4n-1}{4n-4} (2t^*-1)t^{2n-3}(-2n-2) + (2n-1)t^* \right] - (1 - t^*)t^{2n-2}
\]
\[
= \frac{4}{(4n-1)^2} + \frac{1}{4n-1} - t^{2n-3} \left( -\frac{1}{2n-2} + \frac{2n-1}{4n-4} t^* - \frac{1}{2} \right)
\]
\[
= \frac{4}{(4n-1)^2} - \frac{1}{4n-1} + \frac{t^{2n-3}}{2n-2} \left( \left( t^* - \frac{2n-1}{4} \right)^2 + \frac{(2n-1)^2}{16} - (n-1) \right).
\]

If we regard the third term in the last equation as a function of \( t \), we can easily see that it is decreasing on \((0,1)\).

Therefore,
\[
g(t^*) = \frac{4}{(4n-1)^2} + \frac{1}{4n-1} + \frac{t^{2n-3}}{2n-2} \left( -\left( t^* - \frac{2n-1}{4} \right)^2 + \frac{(2n-1)^2}{16} - (n-1) \right)
\]
\[
> \frac{4}{(4n-1)^2} + \frac{1}{4n-1} + \frac{1}{2n-2} \left( -\left( 1 - \frac{2n-1}{4} \right)^2 + \frac{(2n-1)^2}{16} - (n-1) \right)
\]
\[
= \frac{4}{(4n-1)^2} + \frac{1}{4n-1} - \frac{1}{2n-2} - \frac{1}{2}
\]
\[
= \frac{4(n-1)(4n-1)^2}{4n-13}
\]
\[
> 0
\]

for all \( n \geq 4 \) and hence, \( g(t) > 0 \) for all \( t \in [0,1] \).

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