Modeling the Variability of Active Galactic Nuclei by an Infinite Mixture of Ornstein–Uhlenbeck (OU) Processes

Tadafumi Takata1,2, Yusuke Mukuta3, and Yoshikiko Mizumoto4

1 Astronomy Data Center, National Astronomical Observatory of Japan, National Institutes of Natural Science (NINS), Osaka 2-21-1 Tokyo, 181-8588, Japan
2 The Graduate University for Advanced Study (SOKENDAI), Japan
3 Graduate School of Information Science and Technology, The University of Tokyo, Japan
4 National Astronomical Observatory of Japan, National Institutes of Natural Science (NINS), Japan

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Abstract

We develop an infinite mixture model of Ornstein–Uhlenbeck (OU) processes for describing the optical variability of QSOs based on treating the variability as a stochastic process. This enables us to get the parameters of the power spectral densities (PSDs) on their brightness variations by providing more flexible description of PSDs than the models based on a single OU process (damped random walk (DRW)). We apply this model to 67,507 variable objects extracted from Sloan Digital Sky Survey (SDSS) Stripe 82 photometric data and succeed in showing very high precision in identifying QSOs (~99% levels in completeness and purity) among variable objects based only on their variability, by investigating on 9855 spectroscopically confirmed objects (7714 QSOs and 2141 stars) in the data of SDSS Data Release 12 (DR12), with sufficient and accurate multiple measurements of their brightness. By comparing our results with the values based on other models that are used in previous research, it is revealed that our model can be used as the most effective method for selecting QSOs from a variable object catalog, especially regarding completeness and purity. The main reason for improved identification rates is the ability of our model to separate clearly QSOs and stars, especially on the small fraction of QSOs with variabilities that can be described better than with the simple DRW model.

Key words: galaxies: active – galaxies: nuclei – methods: statistical – quasars: general

1. Introduction

One of the remarkable properties of active galactic nuclei (AGNs), frequently represented by quasi-stellar objects (QSOs), is their variability seen in a wide range of the electromagnetic wave region. Just after the discovery of AGNs (Matthews & Sandage 1963; Schmidt 1963), their variability became well known (Greenstein 1963; Smith & Hoffleit 1963) and studied by many authors writing about theory and observational data analysis. Since Sasar et al. (2007) demonstrated that at least 90% of QSOs show the signature of variability with rms more than 0.03 mag, it is believed that most QSO/AGN populations have a feature of variability. As AGNs are among the most luminous celestial objects in the universe, their variability with an order of 10% of their total light in various timescales (less than one hour to many years) (Gaskell & Klimek 2003; Uttley & Casella 2014) is one of the largest fluctuations in energy, and discovering their originating physical processes is an attractive quest. Because the optical continuum radiation is believed to be predominantly coming from the accretion disk, it is straightforward to consider that some processes intrinsic to the disk are the origin of the brightness fluctuations. There is a lot of theoretical work discussing the origin of the variability, in the short and long term, based on the change of global accretion rate (e.g., Pereyra et al. 2006) and disk inhomogeneities propagating inward (e.g., Dexter & Agol 2011), and these fluctuations may arise from thermal or magnetorotational instabilities in a turbulent accretion flow (Hirose et al. 2009; Jiang et al. 2013), though it is still unknown what physical processes control their variability.

On the data analysis side, there is still a lot of research describing their behaviors by using structure functions (SFs) and power spectral densities (PSDs). In recent years, there has been much progress in the modeling of their variability (light curves) by means of massive data with long duration and/or dense cadences. There are many extensive studies on QSO/AGN variability, as a powerful tool for QSO selection using legacy and/or newly acquired data, and also as a probe for physical models of AGNs (Kelly et al. 2009, 2011, 2013; Kozlowski et al. 2010b, 2011, 2012, 2016; MacLeod et al. 2010, 2011, 2012; Schmidt et al. 2010, 2012; Butler & Bloom 2011; Kim et al. 2011; Palanque-Delabrouille et al. 2011; Ruan et al. 2012; Zuo et al. 2012; Andrae et al. 2013; Zu et al. 2013; Graham et al. 2014; Morganson et al. 2014; Cartier et al. 2015; De Cicco et al. 2015; Falocco et al. 2015; Caplar et al. 2017.)

A model with particularly successful results is the damped random walk (DRW) model. This model was first introduced by Press et al. (1992) and Rybicki & Press (1992), and the fast computational implementation was described by Rybicki & Press (1995), for inferring the time lag of variability among multiply imaged gravitationally lensed QSOs. With the detailed analysis by Kelly et al. (2009), Kozlowski et al. (2010b), and MacLeod et al. (2010) on various time series data, it is established that a DRW model can statistically explain the observed light curves of QSOs at a high enough fidelity level (0.01–0.02 mag). Although the DRW model is relatively simple in description, it successfully describes most of QSOs’ light curves, leading to very high rates in identifying QSOs from the variable sources, especially in optical wavelength (e.g., Kozlowski et al. 2010b; MacLeod et al. 2010, 2011; Choi et al. 2014), although it is revealed to be applicable also to mid-infrared data (Kozlowski et al. 2010a, 2016). It should also be noted that the confidence level and completeness of QSO selection will be increased by the combination of colors and...
variability (Peters et al. 2015), and/or with support by the image subtraction technique (Choi et al. 2014). The model calculation is relatively fast; it needs computational times only scaling to the number of data points \(O(N)\) in case \(N\) is the number of data points), by the implementation of SFs for the DRW model by using the autocorrelation function (MacLeod et al. 2010; Butler & Bloom 2011), although the other implementation using Markov Chain Monte Carlo (MCMC) for reproducing the continuous light curves and exponential covariance matrix takes longer in \(O(N^2)\) (Zu et al. 2011, 2013). On the other hand, there are claims that DRW is too simple to describe the “real” variability of QSOs/AGNs, because the possibility of the degeneracies cannot be eliminated (Kozlowski 2016). Kashiwal et al. (2015a) investigated the optical light curves of 20 QSOs observed with the Kepler satellite and found that fewer than half of them can be explained by the DRW model.

Thus, there are some alternatives for describing the variability in Fourier space with PSDs, by the model with mixture of Ornstein-Uhlenbeck (OU) processes (Kelly et al. 2011), Slepian wavelet variance (Graham et al. 2014), the continuous autoregression and moving average (CARMA) model (Kelly et al. 2014; Simm et al. 2016; Kashiwal et al. 2017), the broken power-law model (Zhu & Xue 2016), and so on.

The OU mixture model by Kelly et al. (2011) is one of the powerful candidates of QSO/AGN variability modeling because it is very natural to try to express the fluctuation phenomena by the combination of the simple random processes. The method of the OU mixture is based on the Long-term accumulation of experience learned in many studies of QSOs/AGNs and also of galactic objects’ variability. As this model treats the variability in Fourier space, we can learn their PSDs, which provide us with the information on how they vary. This should be very useful for investigating the physical processes that control the light variations. It is also revealed to be possible to apply to some wavelength data such as X-ray and optical data (Kelly et al. 2011). However, the model has some weak points; the first one is their time-consuming calculation, and the second one is the arbitrariness in the selection of the number of OU processes to be mixed. As suggested in Kelly et al. (2011), the number of mixed OU processes (expressed as \(M\) hereafter) should be larger than 30 to describe the sufficiently observed AGN’s light curves, and the calculation is too massive for large sample data, because the calculation time is on the order of \(O(MN^2)\), which is too massive for those for DRW model calculations.

We have tried to overcome these weak points by extending the mixture numbers of OU processes to infinity, and we succeeded in finishing the calculation in much less time than the original model and without the arbitrariness in the OU mixture. By applying our infinite OU mixture model to 67,507 variable sources extracted from the photometric data of Stripe 82 in the Sloan Digital Sky Survey (SDSS), we confirm the ability of our model to select QSOs from various variable objects, and we compare our results with those by other models previously suggested as successful. By our model we can show the variation of the PSDs of numerous (~8000) spectroscopically confirmed QSOs with sufficient multiple photometric measurements over ~10 yr and provide some implication for the power of PSDs in classifying QSOs/AGNs into subclasses and finding some rare-type QSOs/AGNs.

It should be emphasized here that we know that it is very difficult to describe the details on intrinsic variability and the underlying physical processes of individual QSOs in this study, as the data of SDSS Stripe 82 suffer from irregular and sparse time sampling, and our model uses the approximate PSD model. Therefore, our main purpose in this paper is to know where is the limit in distinguishing QSOs from stars only using their variability, with superpositions of OU processes, which is more flexible in describing variability of QSOs/AGNs than those based on a single OU process.

In Section 2, we describe the methodology by which we construct our model, and we describe the data used for our test in Section 4. In Section 3, we show the results by our analysis of Stripe 82 data. We show the methods on classification and identification rates based on variability in Section 5, discuss the features of our model in Section 6, and provide the conclusions in Section 7.

## 2. Methodologies

In this section, we describe existing models and our proposed models as their limit. We first describe the estimation method common to each model in Section 2.1. We describe the OU process in Section 2.2 and the model with finite mixture of the OU process proposed by Kelly et al. (2011) in Section 2.3. Afterward, we describe our model with infinite mixture of OU processes in Section 2.4 and practical implementation and model calculation in Section 2.5.

### 2.1. Estimation Method

For estimating the best model parameters, by which the physical process is described, we usually perform fitting by using maximum likelihood or Bayesian techniques. There is an advantage in estimating the parameters directly from the light curve by using all of the information in the data. It is especially useful to estimate the parameters of the power spectrum model, as they are not heavily biased by measurement errors, irregular sampling, or other windowing effects caused by the finite time span of the light curve, such as red-noise leak (Kelly et al. 2011). Though the likelihood function may be used to calculate a maximum likelihood estimate, we employ a Bayesian approach that calculates the posterior probability distribution of the model parameters, in order to reliably estimate the uncertainties of the model parameters. The probability distribution of the parameters for the given light curve by the observation is

\[
p(\theta|x_t, ..., x_N) \propto p(x_t, ..., x_N|\theta)p(\theta),
\]

where \(p(\theta)\) and \(p(\theta|x_t, ..., x_N)\) are the prior and the posterior distributions, respectively.

The likelihood function \(p(x_t, ..., x_N|\theta)\) can be written in the following form:

\[
p(x_t, ..., x_N|\theta) = \frac{1}{\sqrt{(2\pi)^N|C|}} \times \exp \left( -\frac{x^T C^{-1} x}{2} \right),
\]

where \(C\) is covariance and the vector \(x = (x_t, x_{t+1}, ..., x_{N-1}, x_N)\). Here we denote the model parameter as \(\theta\) and observed data as \(\{x_t\}_{t=0}^{N-1}\), where \(x_t\) is a brightness (apparent magnitude) at \(t_o\) and \(N\) is the number of observations (measurements). \(N\) is assumed to be about 10–100 in the data set. Given prior distribution \(p(\theta)\) and generation model \(p(x_t, ..., x_N|\theta)\), we use maximum a posteriori
(MAP) estimation as described below:

$$\theta_{\text{MAP}} = \arg\max_{\theta} p(\theta | x_1, \ldots, x_n)$$

(3)

$$= \arg\max_{\theta} p(x_1, \ldots, x_n | \theta)p(\theta),$$

(4)

to get the model parameter, where $\theta_{\text{MAP}}$ means the parameter set giving the maximum of likelihood for the model. In fact, it is hard to compute the exact $\theta_{\text{MAP}}$. Thus, we use MCMC to randomly sample $\{\theta_i\}_{i=1}^I$ from $p(\theta | x_1, \ldots, x_n)$ and use the median of $\{\theta_i\}_{i=1}^I$ instead, where $I$ is a number of samples. In this work, we use the Affine Invariant MCMC Ensemble Sampler (Foreman-Mackey et al. 2013) and set $I = 20,000$ for getting statistically stable results by numerous sampling. We tested the cases of $I = 5000, 10,000$, and $20,000$ and confirmed that there is little difference among the inferred parameter values in most cases, as mentioned by Zu et al. (2011). However, as our model will use four parameters, which is twice as many as used by Zu et al. (2011), and the model is more complex, we decided to use $I = 20,000$ for our calculation.

This requires that we can compute $p(x_1, \ldots, x_n | \theta)$ with small complexity. All models we handle in this paper are Gaussian processes. This means that the joint distribution of $\{x_i\}_{i=1}^n$ is an $N$-dimensional Gaussian distribution with $i, j$-element of covariance $C_{ij} = \text{cov}(x_i, x_j)$, where $\text{cov}$ is a covariance function. When $C$ is computed, it requires $O(N^3)$ complexity to compute $p(x_1, \ldots, x_n | \theta)$. This does not take much computation because $N$ is small. Thus, our motivation is to propose a covariance function with high representation ability and requires small computational complexity.

2.2. DRW Model (OU Process)

The most basic model is the OU process. This is a continuous autoregression model. The model parameter $\theta$ consists of time constant $\tau$ and noise magnitude $\zeta$. The OU process is a solution of the following stochastic differential equation:

$$dX_t = \frac{1}{\tau}X_t \, dt + \zeta \, dW_t,$$

(5)

where $W_t$ is white noise.

The covariance function of this model is

$$\mathcal{R}_{\text{OU}} = \frac{\pi \zeta^2}{2} e^{-\frac{t}{\tau}}.$$  

(6)

The most important advantage of the OU model is that we need only two parameters for the model ($\zeta$ and $\tau$). Also, it does not require much computation, as the covariance function is simple. Additionally, there is a faster implementation for the model computation by using the structure function that is derived from the autocorrelation function of the DRW model (Hughes et al. 1992; MacLeod et al. 2010).

The PSD can be derived by Fourier transformation of Equation (6):

$$\mathcal{P}_{\text{OU}}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \mathcal{R}_{\text{OU}} dt$$

(7)

$$= \frac{\zeta^2}{2\pi} \frac{1}{\omega^2 + \omega^2},$$

(8)

where $\omega_0$ is the reciprocal of damping timescale $\tau$, and it is a good approximation of the majority of QSOs with brightness variation (Kelly et al. 2009; Kozłowski et al. 2010b; MacLeod et al. 2010).

However, since the model is so simple (Kasliwal et al. 2015a; Kozłowski 2016), the scope of the method is limited. The actual/real process is considered to consist of stochastic processes with various scales. Thus, it is natural to consider the mixture model for describing the actual and relatively complex processes.

2.3. Mixed OU Model

Kelly et al. (2011) proposed a model for superpositions of OU processes by changing the scale and magnitude of the process, especially for the X-ray light curves. When the number of OU processes is $M$, the model parameters are the lowest and the highest frequencies $\omega_L$ and $\omega_H$, noise magnitude $\zeta$, and $\alpha$, which determines the magnitudes of each process (weight of mixture). It is noted that the model parameters and the method of mixing the OU process are designed to be able to produce the double-bending PSDs, which can be seen in X-ray variability around black holes and also in recent results of QSO variability studies based on the data with denser cadence, such as Kepler, and ground-based telescope data (Mushotzky et al. 2011; Carini & Ryle 2012; Wehrle et al. 2013; Edelson et al. 2014; Graham et al. 2014; Revalski et al. 2014; Chen & Wang 2015; Kasliwal et al. 2015a, 2015b; Shaya et al. 2015; Kasliwal et al. 2017).

When we denote $\omega_k$ as the $k$th value of $M$ with equal ratio division from $\omega_L$ to $\omega_H$ and the OU process in Equation (5) with $\tau = \frac{1}{\omega_k}$ as $X_k$, the mixture model is written as

$$X = \sum_{k=1}^{M} c_k X_k,$$

(9)

where $c_k$ are magnitudes of $X_k$ that are proportional to $\omega_k^{-\frac{\alpha}{2}}$ and $\sum_{k=1}^{M} c_k^2 = 1$. This model is regarded as the summation of various OU processes, with magnitudes depending on the frequency.

Then, the covariance function can be expressed in the following way:

$$\mathcal{R}_{\text{mixOU}} = \sum_{k=1}^{M} \frac{c_k^2 \zeta^2}{2\omega_k} e^{-\omega_k \tau |t|}.$$  

(10)

Since this is a mixture model of OU processes, it can model a wider variety of processes. The sum of exponentials in Equation (10) falls more slowly than a single exponential function. This means that the mixed OU process exhibits a longer range dependency than a single OU process and is better suited for modeling the light curves with long-timescale dependencies. The PSD can also be written in the relatively simple form

$$\mathcal{P}_{\text{mixOU}} = \sum_{k=1}^{M} \frac{c_k^2 \zeta^2}{2\pi} \frac{1}{\omega_k^2 + \omega^2},$$

(11)

where $\omega_k = 1/\tau_k$ and $\tau_k$ is the damping timescale of the $k$th OU process. It should be noted that Kelly et al. (2011) assume that all of the individual OU processes have the same value of $\zeta$, for avoiding the degeneracy coming from $c_k^2 \zeta^2$, in considering $\zeta_k$ as noise magnitude for the $k$th process. Also, it does not require much complexity when $M$ is small compared to $N$. Kelly et al. (2011) succeeded in implementing the efficient calculation by obtaining a state-space representation of the light curve with variation.
about 3000 data points and then using the Kalman recursions (Brockwell & Davis 2002).

However, it is not easy to determine the appropriate $M$ beforehand. It is unnatural to assume that only $M$ processes contribute to the variability. Considering $M$ as a hyperparameter and trying to search appropriate $M$ by changing $M$ is an option; however, it is not realistic because we use the data set with $10^{10}$ measurements, so it requires a too huge amount of time when trying many $M$ values. As it requires a time on the order of $O(\mathcal{N} M^2)$ to compute the covariance, it is inefficient in the case of large $M$ relative to $N$. If we can compute the covariance of the mixture OU process model with the same time spent for a single OU process model, it is very good for applying to a large data set. As described in the following sections, it is the case for the data set we are using this time.

2.4. Our Model

In the previous section, we denoted that it is unnatural to determine the number of OU processes $M$ beforehand and that computation is not efficient when $M$ is large. Thus, we try to set $M \to \infty$ in Equation (10).

The covariance function of the mixture model described in Equation (10) can also be written as

$$
\int_{\log \omega_\ell}^{\infty} \frac{c(\omega)^2 \zeta^2}{2 \omega} e^{-\omega \vert \delta \vert} d \log \omega,
$$

(12)

if written in the form of an integral using $\log(\omega)$. As $c(\omega)$ is proportional to $\omega^{\frac{1}{2}} \hat{\tau}$, the equation can be written as

$$
\int_{\omega_\ell}^{\infty} \frac{A_2 \omega^{2-\alpha} e^{-\omega \vert \delta \vert}}{2 \omega} d \omega = \int_{\omega_\ell}^{\infty} \frac{A_2 \zeta^2 \omega^{2-\alpha} e^{-\omega \vert \delta \vert}}{2} d \omega,
$$

(13)

by setting $A$ as a constant of proportionality. If we define $x = \omega \vert \delta \vert$,

$$
\frac{A_2 \zeta^2}{2} \int_{\omega_\ell \vert \delta \vert}^{\infty} \frac{x}{\vert \delta \vert} \eta^{-\alpha} d \eta = \frac{A_2 \zeta^2 \vert \delta \vert^{-1}}{2} \Gamma(1 - \alpha, \omega_\ell \vert \delta \vert, \omega_\ell \vert \delta \vert),
$$

(14)

and $A$ can be written as

$$
1 = \int_{\log \omega_\ell}^{\infty} A_2 \omega^{2-\alpha} d \log \omega,
$$

$$
= \int_{\omega_\ell}^{\infty} A_2 \omega^{2-\alpha} d \omega = A_2 \frac{\omega_\ell^{1-\alpha} - \omega_{\ell_{\alpha}}^{1-\alpha}}{2 - \alpha},
$$

(15)

because $\sum_{k=1}^{\alpha} = 1$.

Additionally, the integrand of Equation (12) is bounded in the range of $\omega_\ell < \omega < \omega_{\ell}$, $\vert \delta \vert \geq 0$, and we can exchange the integral and $\vert \delta \vert \to 0$. Therefore, we can analytically derive the equation by setting $e^{-\omega \vert \delta \vert} = 1$ in the integrand as follows:

$$
\begin{align*}
\mathcal{R}_{\text{infou}} &= \left\{ \begin{array}{ll}
\frac{(2 - \alpha) \zeta^2}{2(\omega_{\ell_{\alpha}}^{1-\alpha} - \omega_{\ell_{\alpha}}^{1-\alpha})} \vert \delta \vert^{\alpha-1} \Gamma(1 - \alpha, \omega_\ell \vert \delta \vert, \omega_\ell \vert \delta \vert) & (\vert \delta \vert > 0) \\
\frac{\zeta^2 (2 - \alpha) (\omega_{\ell_{\alpha}}^{1-\alpha} - \omega_{\ell_{\alpha}}^{1-\alpha})}{2(1 - \alpha)(\omega_\ell^{1-\alpha} - \omega_{\ell_{\alpha}}^{1-\alpha})} & (\vert \delta \vert = 0)
\end{array} \right.
\end{align*}
$$

(16)

as a covariance function, where $\Gamma(1 - \alpha, \omega_\ell \vert \delta \vert, \omega_\ell \vert \delta \vert)$ is an incomplete gamma function.

We can also derive the PSD by using the hypergeometric function. The power spectrum of the single OU process with a frequency $\omega$ can be written as

$$
\frac{\zeta^2}{2 \pi} \frac{1}{\omega^2 + \omega^2}.
$$

(17)

Then, the power spectrum of our model can be derived by integrating it with $\omega$:

$$
\frac{\zeta^2}{2 \pi} \int_{\log \omega_\ell}^{\infty} \frac{c(\omega)^2}{\omega^{2-\alpha}} d \log \omega = \frac{\zeta^2 A_2^2}{2 \pi} \int_{\omega_\ell}^{\infty} \frac{\omega^{2-\alpha}}{\omega^2 + \omega^2} d \omega.
$$

(18)

If we define $\omega = x \tan \theta$, the integration can be written as

$$
\frac{\zeta^2 A_2^2}{2 \pi} \int_{\arctan(\frac{\omega_\ell}{x})}^{\infty} x^{1-\alpha} \tan^{-\alpha} \theta \ x d \theta.
$$

(19)

Since the integrand of Equation (19) can be written as

$$
\int \tan^{-\alpha} (x) dx = \frac{1}{2 - \alpha} \text{Hyper}_{2F1} \left( \begin{array}{cc}
1, 1 - \frac{\alpha}{2} & \frac{\alpha}{2} \\
2 & -\frac{\alpha}{2}, & -\tan^2 (x) \end{array} \right) \tan^{2-\alpha} (x),
$$

(20)

the spectrum of our model can be described as

$$
\mathcal{P}_{\text{infou}} = \frac{\zeta^2 A_2^2 \tan^{-\alpha}}{2 \pi (2 - \alpha)} \text{Hyper}_{2F1} \left( \begin{array}{cc}
1, 1 - \frac{\alpha}{2} & \frac{\alpha}{2} \\
2 & -\frac{\alpha}{2}, & -\tan^2 (x) \end{array} \right) \tan^{2-\alpha} (x).
$$

(21)

The hypergeometric function $\text{Hyper}_{2F1}(a, b, c, z)$ is defined as

$$
\text{Hyper}_{2F1}(a, b, c, z) = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},
$$

(22)

where $(q)_n$ is the Pochhammer symbol, which is defined by

$$
(q)_n = \begin{cases} 1 & (n = 0) \\
q(q+1) \cdots (q+n-1) & (n > 0).
\end{cases}
$$

(23)

This model is regarded as the infinite superpositions of the OU process with scales from $\omega_\ell$ to $\omega_{\ell}$, thus, it is more natural than the Kelly et al. (2011) model, which assumes that only finite discrete OU processes contribute to the variability. Since we can omit $M$ in the covariance function as shown in Equation (16), it does not require $M$ times computation in the covariance calculation, which is leading the faster calculation.

2.5. Practical Implementation and Model Calculation

We infer the model parameters by the following procedures. First, we estimate covariances for all measurement sets, by using each measurement and errors for sampling the data, to produce the covariance matrix, as described in Section 2.4. After calculating the likelihood by the covariance, we try to get the posterior distribution of model parameters using the
MCMC sampler, and we infer the best parameter values by getting medians of each parameter distribution. We use a similar prior distribution to the prior used for the analysis by Kelly et al. (2011), but there is a difference in the range of \( \alpha \), which is \(-2 < \alpha < 0\) in Kelly et al. (2011). In our calculation for MAP estimation, for the prior distribution we assume the uniform distribution for \( \alpha \) in the range of \(-3 < \alpha < 3\). This is because the likelihood values are systematically larger than the case of the Kelly et al. (2011) definition, for the objects that seem to have different PSDs inferred by the DRW model. For \( \zeta \) we use a uniform prior in the range of \( \zeta > 0 \) and also assume the uniform prior on \( \omega_L \) and \( \omega_H \), which satisfy the following relations: \( \omega_{\text{min}} \ll \omega_L \ll \omega_{\text{max}} \) and \( \omega_H \ll \omega_{\text{max}} \). The upper and lower limits on the characteristic timescales, \( \tau_{\text{max}} = 1/\omega_{\text{min}} \) and \( \tau_{\text{min}} = 1/\omega_{\text{max}} \), are chosen to be \( 10^2 \) and \( 10^3 \) days, respectively. We set the range of \( \omega \) at about one order larger and smaller than the range of time intervals of the measurements, in order to avoid the effects coming from boundary conditions in time for our analysis, and we must only see the parameter ranges that can be covered by the observations, if we want to interpret the resultant PSDs.

We use a part of the JAVELIN package\(^5\) (Zu et al. 2011, 2013) for producing the light curve of the object with the given data points (usually 40–120 in our analysis), which is based on MCMC processing. In this package MCMC library \texttt{emcee}\(^6\) (Foreman-Mackey et al. 2013) is used for random sampling.

As the covariance function is an incomplete gamma function, we use the GNU Science Library for the faster calculation. As the JAVELIN package is written in the \texttt{python} language, we use \texttt{python} as the language for our software development.

As the calculation for each object can be processed in parallel, we implemented the multiprocess (parallel) calculation in our model analysis. Our model calculation takes about 2–3 minutes per object in a single thread (depends on the number of data points), and we complete our model calculation in about 5 days for all 67,507 objects in our sample with 20 threads processing by Intel Xeon E5507 (2.27 GHz) CPU.

### 3. Data

#### 3.1. SDSS Stripe 82 Photometric Catalog

We use the \texttt{gri} photometric information from the Ivezić et al. (2007) variable source catalog, downloaded from the site.\(^7\) The catalog contains light-curve information of 67,507 variable source candidates identified in the SDSS Stripe 82 area, based on SDSS Data Release 7 (DR7; Abazajian et al. 2009). Their criteria for selecting the candidates are

1. unresolved source in imaging data, at least one band with photometric error below 0.05 mag;
2. processing flags \texttt{BRIGHT}, \texttt{SATUR}, \texttt{BLENDED}, or \texttt{EDGE} are not set, meaning that the measurement was performed successfully;
3. at least 10 observations in the \textit{g} and \textit{r} bands;
4. the median \textit{g}-band magnitude is brighter than 20.5; 
5. \textit{rms} scatter >0.05 mag and \( \chi^2 \) per degree of freedom larger than 3 in both \textit{g} and \textit{r} bands, which means that the object is “statistically” variable in brightness (see the discussions in Sasar et al. 2007 for details).

This catalog is more extensive than the catalog of Sasar et al. (2007), because it includes both SDSS-I and SDSS-II, while the Sasar et al. (2007) catalog was based only on SDSS-I, and therefore we decided to use this catalog for our research. Though this catalog also contains \( \mu\)- and \( \varepsilon\)-band data, we did not use it for our study because of their lower signal-to-noise ratio (S/N) than those of three \( (g, r, \text{and} \ i) \) bands’ data. The number of measurements in \( r \)-band is distributed in the range between \( \sim10 \) and 140, with a maximum peak at around 60.

#### 3.2. SDSS DR12 Spectroscopic Catalog

We used SDSS Data Release 12 (DR12; Alam et al. 2015) data for identifying the spectroscopically confirmed variable source candidates in the photometric catalog mentioned in Section 3.1. On the SDSS CasJobs system for DR12 data,\(^8\) we used the “dbo.SpecPhoto” table to select the spectroscopically confirmed objects in the area of Stripe 82. We specified the area by \( |\theta| < 1\degree266 \) and R.A. in the range of \( 20\degree34\text{m} \) to \( 4\degree0\text{m} \). The number of objects extracted from the table is 221,656, including 25,674 QSOs, 78,761, stars and 117,221 galaxies.

Spatial matching with the photometric catalog, based on \( r\)-band data, is performed for the coordinates (photora, photodec) in the table, by the criterion of the difference of the coordinates between photometric and spectroscopic catalogs within 1 arcsec. We identified 11,908 matched objects, including 8105 QSOs, 3379 stars, and 424 galaxies, based on the spectroscopic classification on the table “dbo.SpecPhoto,” described as “QSO,” “STAR,” and “GALAXY” in column “Class,” respectively. There are 17 sources with multiple sources in the searched area, and we selected the nearest one as the counterpart in the cases.

It is noted that most spectroscopically confirmed objects in the photometric catalog have measurements over 40. For the validation of selection ability of QSOs from stars based only on their variability, we need a sample with enough photometric measurements and spectroscopic identifications. Thus, we decide to use the objects that are spectroscopically confirmed and have over 40 measurements with good accuracy, with the error of each measurement less than 0.05 mag, for our test of QSO selection, which is called a “good sample” hereafter, although we calculate the model parameters on all objects to investigate the effects of measurement numbers on our analysis. The total number of objects in the “good sample” is 9855, with 7714 QSOs and 2141 stars.

One important feature that affects the time series analysis using irregularly and spuriously sampled data is the cadence of each object and its similarity. We consider that the cadences in the sample are similar to each other, if the distributions of the sets of time intervals for each object, which are the histograms of time intervals, are similar for the whole sample. To check the cadence of the “good sample,” we calculated time intervals of multiple measurements for each object in the sample and produced the histogram as shown in Figure 1(a). We used 100 bins equally divided between \( -1 \) and 4 on log-scaled time intervals in days, and we stacked all the time intervals of each object to produce a stacked cadence template histogram. It is clear that the minimum time intervals are less than \( \sim1 \) day, and

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\(^5\) http://www.astronomy.ohio-state.edu/~yingzu/codes.html#javelin

\(^6\) http://dan.iel.fm/emcee/current/

\(^7\) http://www.astro.washington.edu/users/ivezic/sdss/catalogs/S82variables.html

\(^8\) https://skyserver.sdss.org/CasJobs/
objects with decrease as dissimilarity increases. We calculate the cadence template and for a single object, respectively. If both histograms. In this case they are histograms for a stacked sample, we used the Bhattacharyya coefficient (hereafter BC; Bhattacharyya 1943), which is frequently used for checking the similarity of two histograms. The definition of BC can be written as

$$
BC(p, q) = \sum_{i=1}^{n} \sqrt{p_i q_i},
$$

where $p_i$ and $q_i$ are normalized counts in the $i$th bin of two histograms. In this case they are histograms for a stacked cadence template and for a single object, respectively. If both histograms are the same, the BC value will be 1 and will decrease as dissimilarity increases.

The distribution of BCs for the “good sample,” shown in Figure 1(b), means that the cadences are very similar, as almost all objects show values of more than 0.95. This feature of the data set is suitable for detecting different types of intrinsic variability, as we can decrease the effect coming from irregular and sparse sampling. It should be noted that the results are the same for a histogram with a 1-day bin size, which means that cadences greater than 1 day long are very similar for the whole sample.

The similarity of the cadences is easily imagined because SDSS Stripe 82 data are taken through the time-delay-and-integrate (TDI) scanning mode (Gunn et al. 1998), which enables homogeneous scanning of the sky in each observation night.

As already mentioned, we cannot investigate the details of intrinsic variability since the data suffer from irregular and sparse time sampling. Considering the similarity of the cadence, however, the data provide enough information for detecting the relative differences of the photometric variations, and they can be powerful in their ability to separate QSOs from stars. This means that the effects coming from the irregular and sparse time sampling will emerge in the same way as the results for all sample objects and intrinsic features can be distinguished relatively.

### 4. Results

#### 4.1. Fitted Light Curves and Derived Parameters

By applying our model to the 67,507 variable sources, we derived four parameters ($\omega_H$, $\omega_L$, $\zeta$, $\alpha$) for all sources. These parameters are estimated by taking the median of posterior distributions, and lower and higher errors are estimated by getting the difference between the median and 16% or 84% quantile values, respectively. We calculated them on $g$, $r$, and $i$-band data independently. As the ability to distinguish QSOs from stars is very similar among the data sets on the three filters, we limit our description to only $r$-band data in the following.

In Figures 2 and 3, we show the inferred light curves for one of the spectroscopically confirmed QSOs and stars, respectively. These figures also show their posterior distributions of inferred parameters (histograms), the data sets of posteriors (contours), and their inferred PSDs, which will be described in detail in Section 4.2. The shaded part of the light curves shows the range of the 68% confidence level of our calculation based on MCMC, and the solid lines mean the inferred light curves, derived from the median of the 20,000 chain results.

On the most spectroscopically confirmed QSO, we can get well-fitted inferred light curves, which are characterized by variation with a long timescale (e.g., small frequency; Figure 2(a)). They are characterized by small $\omega_L$ and $\omega_H$.

On the other hand, we cannot have a good fit on the star’s light curve, which has a flat light curve without tracing the measurements, as shown in Figure 3(a). It is, however, not the failure of our model fitting because our model uses only a type of PSD assuming the QSO/AGN-like variability. In fact, the inferred PSD has larger $\zeta$ compared to QSOs/AGNs and a flat shape in the small $\omega$ range, which means that the variability is like white noise. As most variable stars are periodically varying their brightness, the results of our model fitting to stars...
Figure 2. One result of our model calculation for a QSO (ID number 7095). (a) Top: light curve and 68% confidence range for the inferred light curve (shaded area). The results of photometric measurements are shown by filled circles with error bars. Bottom: inferred PSD with 90% confidence range (shaded area). The horizontal dashed lines show the median (red) and average (green) values of errors on photometric measurements. The vertical blue dashed lines show inferred $\omega_L$ and $\omega_H$, respectively. (b) Pair plot showing the posterior distributions of our model parameters, log $\zeta$, $\alpha$, log $\omega_L$, and log $\omega_H$, from left to right, respectively. The histograms show the posterior distributions of each parameter, with median values indicated by the red dashed lines. Contour maps show the distributions of the posteriors on each set of our model parameters ($\zeta$, $\alpha$, $\omega_L$, $\omega_H$) about the object in MCMC chains.

Figure 3. Same as Figure 2, but for a star (ID No. 3271).
are reasonable. Most importantly, inferred parameters are good for the purpose of separating stars from QSOs in the parameter space, as shown in the following sections.

There are also 37 spectroscopically confirmed QSOs in the “good sample” without successful light-curve fits. They cannot be distinguished from most stars in optical variability using our model. On the other hand, we identify 199 QSO-like light curves on the objects classified as stars on the SDSS spectra. We investigated carefully their optical spectra by visual inspection and determined that 16 of them are possibly misidentified as “STAR.” Many misidentified objects have relatively weak emission lines in their spectra and are possibly categorized as weak emission line QSOs (Plotkin et al. 2010).

We should note that the inferred parameter $\omega_L$ for many QSOs is less than $\sim 1/4000$ days$^{-1}$, the maximum duration limit of the data (~10 yr). This is because we set the prior of $\omega_L$ as a uniform distribution in the range of $-5 < \log \omega_L < 2$. The values are thus the results of fitting our model to the data with a timescale shorter than ~4000 days, and $\omega_L$ smaller than 4000$^{-1}$ day$^{-1}$ is only the extrapolation based on our model. We should emphasize here that the values of $\omega_L$ are meaningful, as our purpose is only to identify QSOs, not to infer PSDs correctly, by using a model with an approximate PSD expression for describing variability of QSOs. We also mention briefly the interpretation of the inferred PSDs in Section 6.3.

For confirming the consistency of our model with the Kelly et al. (2011) finite OU mixture model, we checked the value of information criteria on the results. We use the Bayesian Information Criterion (BIC; Schwarz 1978) for this check. It is defined as follows:

$$BIC = k \ln(n) - 2 \ln(L),$$

where $L$ is the maximum value of the likelihood function, $n$ is the number of measurements, and $k$ is the number of estimated parameters in the model.

Information criteria are a common and useful mechanism for ranking a set of models. In time series analysis BIC and/or the Akaike Information Criterion (AIC; Akaike 1973) are used frequently, which are on the maximum likelihood estimate of the parameters. The BIC and AIC provide estimates of the relative information lost in using a model to represent the underlying process that generated the data. As we used Bayesian inference for our analysis, we use BIC for the check.

In Figure 4, we show three samples of BIC distributions on $M$ about spectroscopically confirmed QSOs. There are various tendencies on the distribution; however, we can see that our infinite OU mixture model provides the same level of the goodness of fit compared to Kelly et al. (2011) finite OU mixture model with large mixture numbers. It is also consistent with the suggestion by Kelly et al. (2011) that the number of mixed OU processes should be larger than 30 to describe observed AGNs’ light curves sufficiently. We should note here that the numbers of mixed OU processes ($\hat{M}$) that provide the best BIC on a mixed OU model for the spectroscopically confirmed QSOs in a “good sample” change from object to object, although most of them are $\hat{M} = 32, 64, 128, \infty$.

For checking the flexibility of our model to AGN/QSO variability, we show the covariance functions in Figure 5(a). The plus signs are covariance values calculated by the observation data set. We select five objects that are representative of our model parameter space as shown in Figure 5(b). We named them “Long,” “Middle,” “Short,” “DblBend,” and “PowerLaw.”

“Long,” “Middle,” and “Short” represent the difference of variability timescales, and they show a smaller difference between $\omega_{BP}$ and $\omega_L$. “DblBend” and “PowerLaw” represent the shapes of PSDs with double bending and power law that have much difference between $\omega_{BP}$ and $\omega_L$. The difference between “DblBend” and “PowerLaw” is whether $\omega_{BP}$ is lower or higher than 1 day$^{-1}$, which corresponds to the minimum of time intervals of the data.

It is clear that the covariance functions describe well the difference of variabilities by changing their height and shape, especially in the short-timescale range.

### 4.2. PSD Calculation and Derived Parameters

For investigating the details of the variability, we derive the PSD for each object. The method to calculate the PSD is described in Section 2.4, and PSDs for a sample of QSOs/AGNs and stars are shown in Figures 2 and 3, respectively. We can see some typical features in PSDs for QSOs and stars. For example, many QSOs show the PSDs with smoothly changing slopes, and the turning point appears at the border of white noise (slope $\beta = 0$) and red noise ($\beta = -2$), when we express the PSD $\propto \omega^\beta$. On the other hand, stars’ PSDs inferred by our model have a bending point at a larger $\omega$ range (log $\omega > 0$), with flat spectra in a smaller $\omega$ range, which is consistent with white noise. Since our model originally needs four parameters for describing PSDs for the variability of QSOs, it is not easy to visualize the behavior of inferred PSDs among different celestial objects, especially QSOs and stars. We therefore...
introduce the parameters, which show the typical frequency and height of PSDs. As the PSD derived from our model has the features that the slope will be asymptotic to 0 in \( \omega \to -\infty \) and to \( -2i \) in \( \omega \to \infty \), we define two asymptotic lines for the PSD and derive the \((\omega, P(\omega))\) at the crossing point of these lines, as \((\omega_c, P_c)\), and then try to describe our model with the three parameters \( \omega_c, P_c, \) and \( \alpha \) (Figure 6). This means that \( \omega_c, P_c, \) and \( \alpha \) are expressing the features of PSDs, which are corresponding to the absolute position of a typical timescale, the amplitude, and the difference of maximum and minimum timescales.

We define the feature point on the power spectrum derived from our model as the crossing point of the asymptotes in the cases of \( \omega \to 0 \) and \( \omega \to \infty \). When \( \omega \to 0 \), the power spectrum can be written as

\[
\frac{\zeta^2 A^2}{2}\int_{\omega_l}^{\omega_H} \frac{\omega^{1-\alpha}}{\omega^2} d\omega = \frac{\zeta^2 A^2}{2\pi} \left[ \int_{\omega_l}^{\omega_H} \omega^{-1-\alpha} d\omega \right] = \frac{\zeta^2 A^2 \omega_H^\alpha - \omega_L^{-\alpha}}{2\pi}. \tag{26}
\]

On the other hand, in the case of \( \omega \to \infty \), we can derive the spectrum as

\[
\frac{\zeta^2 A^2}{2\pi} \int_{\omega_l}^{\omega_H} \frac{\omega^{1-\alpha}}{x^2} d\omega = \frac{\zeta^2 A^2}{2\pi} \left[ \frac{1}{x^2} \int_{\omega_l}^{\omega_H} \omega^{1-\alpha} d\omega \right] = \frac{\zeta^2 A^2}{2\pi} \frac{\omega_H^{-2\alpha} - \omega_L^{-2\alpha}}{2 - \alpha}. \tag{27}
\]

By satisfying both formulae simultaneously, the coordinate of the crossing point is derived as

\[
x = \sqrt{-\alpha \omega_H^{-2\alpha} - \omega_L^{-2\alpha}} = \omega_c \quad (28)
\]

and we can define the point as the feature point.

We show the distribution of these three parameters \((\alpha, \log \omega_c, \) and \( \log P_c)\) on 67,507 variable sources by the 3D contour plot in Figure 7(a), which indicates at least three clusters and two extensions in the distribution. The clusters are characterized by large \( P_c \) and small \( \omega_c \) (Group1), small \( P_c \) and large \( \omega_c \) (Group2), and a small size cluster neighboring to Group2 with smaller \( \alpha \) (Group3). There are two extensions: one is the narrow one connecting Group1 and Group2 (Ext1), and the other extends to a smaller \( \alpha \) direction (Ext2).

For checking the correspondence of the object types (especially on QSOs and stars) to these structures in Figure 7(a), we plot the same figure except for the objects with spectroscopic confirmations.
as “QSO” or “STAR” in Figure 7(b). The gap between Group1 and Group2 is clear in Figure 7(b), which is not visible in Figure 7(a). Additionally, we can barely see Group3 and Ext2, and we can see Ext1 in the range of a smaller ω range.

It should be noted that the main contributors of these disappearing structures are objects with small numbers of measurements, which suffer from the noisy inference of the parameters. We confirm that the objects labeled “GALAXY” in the spectroscopic catalog of SDSS DR12 also belong to any of these structures.

We also plot these three parameters of each spectroscopically confirmed QSO and star on the same parameter space in Figure 8, to see the correlation between spectroscopic classification and variability features. These figures show that the majority of QSOs and stars belong to Group1 and Group2, respectively, and Group3 consists of stars. Stars are also the main contributors of two extensions.

On the QSOs and stars in the “good sample,” the effect caused by a small number of photometric measurements is reduced, and we can see a clearer separation of QSOs and stars in the parameter space. This means that distinguishing QSOs from stars is probably possible in very high identification rates, by using our model parameters, as probed by many previous similar works based only on optical variability (see, e.g., Kozłowski et al. 2010b; MacLeod et al. 2010, 2011; Butler & Bloom 2011; Andrae et al. 2013). The accuracy of the distinction should be higher in the case in which the number of photometric measurements increases.

Another common clear feature is that the ωc is larger when Pc is smaller in QSOs (Group1). This means that the power of variability is higher the longer the damping timescale is. This is the same feature revealed by the DRW model analysis (MacLeod et al. 2011).

5. Classification and Identification Rate of Our Model

It is a well-known fact that the optical variability information can be used as a powerful tool to distinguish a variable QSO population from other ones, such as variable stars, with high accuracy. It is therefore very interesting to investigate the identification rate of variable QSOs by the results based on our model calculation. As our method is based on the modeling of QSO/AGN variability, which comes from the long-term observational experience of many researchers, in the Fourier space, it is simply considered that the identification rate, especially completeness, may be higher than those by other models and calculations. This is because our model is customized to describe well the AGN-like variability, as well as the DRW model, which is not suitable to other types of variability like periodically variable stars.

We investigate the identification rate of the spectroscopically confirmed QSOs and stars, as performed in other studies for QSO classifications, because we can know the objects’ types independently from variability. We compare the discrimination ability of the estimated model parameters or their subsets. In this case, we use log ω1, log ωH, α, log ζ, log ωc, log Pc and some of their subsets.

As a classifier, we use the linear Support Vector Machine (SVM). We use the linear classifier because the linear SVM scales well when the number of samples becomes large and it is easy to interpret the learned model. We use the library of the linear SVM in the scikit-learn package9 for our classification.

In each trial, we randomly choose 2000 samples, half of which are QSOs and another half from stars, as test data and the rest as training data. We then decide the regularization parameter C from 2−10, 2−9, …, 2−10 by applying fivefold cross-validation on the chosen training data. Then, we learn the classifier on the training data with determined C and apply the learned model to the test data to get the results. We conducted 1000 trials and evaluated the completeness, purity (precision), average of accuracy, and recall. The definitions of these values in our analysis are shown in Figure 9.

9 http://scikit-learn.org/
In Figure 10 we show the box plot of the identification rates on our model based on two-, three-, and four-parameter space, and we list the evaluated identification rates in Table 1. The identification rates shown here are based on the results by using the “good sample.” Since we want to clarify the capability of optical variability to select QSOs, we constrain the discussions only to the results based on the “good sample” hereafter if there is no notification.

It is clear that the identification rates based on our optical variability modeling are very high; completeness is at about 97%–99% for all parameter sets, purities at 98%–99%, 93%–95% in accuracies, and 89%–92% in recall, respectively. There are very small differences among the identification rates by the different parameter sets. In completeness, the highest value is achieved by “Inf4,” “Inf3oz,” and “Inf2zl” data sets with 99.3%. “Inf4” also provides the highest in purity (99.0%), in accuracy (95.3%), and “Inf3cr” in recall (92.2%). It is noted that the rates are slightly decreased by adding the photometrically less measured sample (not the “good sample”), as represented by the data sets “InfAll4” and “InfAll3cr” in Table 1, which are based on the sample of all spectroscopically confirmed 8105 QSOs and 3379 stars.

**Figure 8.** Distribution of spectroscopically identified QSOs (blue) and stars (red) in derived parameter space ($\omega_c, \alpha, P_c$). The top left panel shows all sample data (8105 QSOs and 3379 stars). The top right panel is based on the “good sample” data. There are 7714 QSOs and 2141 stars. The bottom left and right panels show only QSOs and stars, respectively.

**Figure 9.** Definitions of values for describing identification rates used in our analysis.
By 1000 trials of the classification using the linear SVM, we can calculate the identification rates of our classification on each source, by counting up the numbers of successful classification on sources in the test sample. As the test data are randomly selected in each trial, all QSOs and stars in the “good sample” are selected into test data homogeneously. Therefore, we can calculate the identification rate for the whole sample without systematic biases. The range on the end of the whiskers. The box extends from the lower to upper quartile values of the data, with red lines at the median. The whiskers extend from the box to show the range of the data, Q3 + 1.5 × IQR. IQR is the interquartile value (Q3–Q1), where Q1 and Q3 are the first and third quartile values. Flier points are for those past the end of the whiskers.

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6. Discussions

6.1. Comparison with Other Models on Identification Rates

For confirming the performance of our model in QSO selection, we compare the identification rates (discrimination ability of “QSO” and “STAR”) to several models. The models we use for the comparison are Butler’s model (Butler & Bloom 2011), Zu’s single OU model (Zu et al. 2011, 2013), and Kelly’s OU mixture models (Kelly et al. 2011).

Butler & Bloom (2011) use the DRW model by Kelly et al. (2009) and implement the two parameters $\omega_1/\nu$ and $\omega_2/\nu$ for well separating QSOs and stars in the parameter space, and it is believed to be one of the most powerful tools for selecting QSOs. We use their software released on their web page.10 For computation using a single OU process, we use the JAVELIN package developed and maintained by Zu et al. (2011, 2013). Their program outputs $\tau$ and $\sigma$ ($\zeta$ in Section 2.2) and well separates QSOs and stars in 2D parameter space. We develop the program for calculation based on Kelly’s OU mixture model (Kelly et al. 2011). We use the same MCMC program as our model calculation, and we get the results in the cases of mixture number of OU process, $M$, are equal to 2, 4, 8, 16, 32, 64 and 128. As the time consumed for the OU mixture model calculation is too long to calculate on all sample objects, we limit the calculation only to those with spectroscopic confirmation (11,908 objects). On the other hand, we calculate 67,507 objects for Butler and single OU models. The distributions of QSOs and stars in the parameter spaces of each model are shown in Figure 12, and they show good performance on the separation of QSOs from stars.

The classifications are performed in the same manner as for our model, and we derive the same values for estimating their discrimination abilities. Per each sample, we have $\log \tau$, $\log \sigma$ for the single OU model, $X_{\text{qso}}^2/\nu$, $X_{\text{false}}^2/\nu$ for Butler’s model, and

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10 http://butler.lab.asu.edu/qso_selection/index.html
log $\omega_1$, log $\omega_2$, $\alpha$, log $\zeta$, log $\omega_3$, log $P_0$ for Kelly’s mixture model. We then apply the classifiers to the subsets of these parameters and determine whether the samples are QSOs or not.

In Figure 13, we show the box plots for the comparison of identification rates among these models, and we list their identification rate values in Table 2. It is clear that our model has the highest rate in completeness, especially 99.3% in the cases of “Inf4,” “Inf3oz,” and “Inf2zl,” and we have confirmed that those of Kelly’s model are also high compared to other models. It is very plausible that ours and Kelly’s models are more flexible models for describing QSO/AGN variability. The significance of the difference in completeness from those of other models such as the Butler & Bloom (2011) and Zu et al. (2011) models is verified by the pairwise multiple comparison test (Demsar 2006).\footnote{http://finzi.psych.upenn.edu/library/PMCMR/html/PMCMR-package.html} In terms of recall, the single OU model by Zu et al. (2011) shows the highest score, but the score of our model is higher than those of other models. It should be noted that the improvement of identification rates of QSOs, especially in relation to completeness, by our model compared to others comes from the discrimination ability on the objects at the edge of Group1.

In Figure 14, we show the 3D distribution of our model parameters for spectroscopically confirmed QSOs with the success rates by the Butler & Bloom (2011) model and JAVELIN. By comparing the success rate distribution with our model, shown in the middle of Figure 11, it is clear that...
most objects in the group (Group1) are classified successfully as “QSO” in our model, although some misclassifications can be seen in the results by other models. It is therefore reasonable to explain this improvement as being caused by the introduction of the mixed OU processes in our model.

We should, however, be cautious regarding the bad success rate on “Inf2zh,” which is classified based on \( \omega_H \) and \( \zeta \) instead of the very high performance on “Inf2zl.” This implies that QSOs with small damping timescales are indistinguishable from stars on the parameter plane defined by \( \omega_H \) and \( \zeta \). It should be noted that the larger number of mixtures of OU processes provides better success rates in completeness, although saturated around \( M = 8 \).

In Figure 15, we show the distribution of success rates of QSOs and stars in our 1000 trials of classification using linear SVM. We compare the performance of the “Inf4” model to others (“Inf3cr,” “JAVELIN,” and “Butler”) in this figure. The identification rates are computed for the whole “good sample,” on 9855 sources. In this figure we can see the clear advantage the “Inf4” model has over the “Butler” model in QSO and star selection and over “JAVELIN” in QSO selection in the high identification rate region. On the other hand, we cannot see any significant difference from “Inf3cr.”

As we use the linear SVM for our classification, the hyperplane separating the parameter zones for QSOs and stars can be expressed as follows:

\[
A\alpha + B\log_{10}(\omega_H) + C\log_{10}(\omega_L) + D\log_{10}(\zeta) + E\begin{cases} 0 & \text{(QSO)} \\ > 0 & \text{(star).} \end{cases}
\]  

Figure 13. Box plot for comparisons of identification rates on the different models in the same manner as Figure 10.

Table 2

Comparison of Identification Rates among Models

| Model    | Abbreviation | Parameters | Completeness | Precision (Purity) | Accuracy | Recall |
|----------|--------------|------------|--------------|--------------------|----------|--------|
| Our model | Inf4         | \( \omega_L, \omega_H, \zeta, \alpha \) | 0.993 \pm 0.005 | 0.990 \pm 0.001 | 0.956 \pm 0.004 | 0.905 \pm 0.012 |
|          | Inf3cr       | \( \omega_L, P, \zeta \) | 0.989 \pm 0.001 | 0.989 \pm 0.002 | 0.944 \pm 0.007 | 0.927 \pm 0.006 |
|          | Inf3oz       | \( \omega_L, \omega_H, \zeta \) | 0.993 \pm 0.005 | 0.986 \pm 0.003 | 0.940 \pm 0.006 | 0.899 \pm 0.010 |
|          | Inf2zh       | \( \omega_L, \zeta \) | 0.974 \pm 0.016 | 0.981 \pm 0.003 | 0.926 \pm 0.013 | 0.887 \pm 0.026 |
|          | Inf2zl       | \( \omega_L, \gamma \) | 0.993 \pm 0.002 | 0.985 \pm 0.003 | 0.939 \pm 0.005 | 0.896 \pm 0.007 |
|          | JAVELIN      | Jav        | \( \tau, \sigma \) | 0.980 \pm 0.007 | 0.986 \pm 0.002 | 0.956 \pm 0.006 | 0.938 \pm 0.012 |
|          | BUTLER       | But        | \( \lambda_q \lambda_p / \nu, \lambda_t / \nu \) | 0.980 \pm 0.004 | 0.988 \pm 0.001 | 0.943 \pm 0.004 | 0.913 \pm 0.005 |
|          | Mix(M = 2)   | KM2,4      | \( \omega_L, \omega_H, \zeta, \alpha \) | 0.985 \pm 0.030 | 0.985 \pm 0.003 | 0.926 \pm 0.013 | 0.887 \pm 0.026 |
|          | Mix(M = 4)   | KM4,4      | \( \omega_L, \omega_H, \zeta, \alpha \) | 0.989 \pm 0.001 | 0.985 \pm 0.002 | 0.925 \pm 0.011 | 0.876 \pm 0.019 |
|          | Mix(M = 8)   | KM8,4      | \( \omega_L, \omega_H, \zeta, \alpha \) | 0.989 \pm 0.001 | 0.985 \pm 0.002 | 0.927 \pm 0.010 | 0.882 \pm 0.017 |
|          | Mix(M = 16)  | KM16,4     | \( \omega_L, \omega_H, \zeta, \alpha \) | 0.989 \pm 0.009 | 0.985 \pm 0.002 | 0.928 \pm 0.011 | 0.882 \pm 0.018 |
|          | Mix(M = 32)  | KM32,4     | \( \omega_L, \omega_H, \zeta, \alpha \) | 0.989 \pm 0.008 | 0.986 \pm 0.002 | 0.929 \pm 0.011 | 0.883 \pm 0.017 |
|          | Mix(M = 64)  | KM64,4     | \( \omega_L, \omega_H, \zeta, \alpha \) | 0.990 \pm 0.001 | 0.986 \pm 0.002 | 0.928 \pm 0.011 | 0.882 \pm 0.017 |
|          | Mix(M = 128) | KM128,4    | \( \omega_L, \omega_H, \zeta, \alpha \) | 0.990 \pm 0.006 | 0.986 \pm 0.002 | 0.929 \pm 0.010 | 0.884 \pm 0.021 |
|          | Mix(M = 2)   | KM2,3cr    | \( \omega_L, \omega_H, \zeta, \alpha \) | 0.959 \pm 0.004 | 0.917 \pm 0.008 | 0.938 \pm 0.005 | 0.919 \pm 0.007 |
|          | Mix(M = 64)  | KM64,3cr   | \( \omega_L, \omega_H, \zeta, \alpha \) | 0.965 \pm 0.004 | 0.928 \pm 0.007 | 0.934 \pm 0.004 | 0.908 \pm 0.006 |
As the number fraction of QSOs to stars depends on the sky region and/or depth of the imaging data/catalog, it is convenient to use different hyperplanes for selecting QSOs from the variable object samples, as the “total” success rates, defined as the weighted mean of success rates for QSO and star selections. The information may be one of the important issues to decide the strategy for QSO selection using the given data set. For example, we should use conservative criteria to eliminate contamination by stars, if the purity of QSO selection is the highest priority on the selection.

We show some sets of hyperplane coefficients for separating QSOs and stars in four-parameter space with maximum total success rates, defined as the weighted mean of success rates for QSO and star selections. The information may be one of the important issues to decide the strategy for QSO selection using the given data set. For example, we should use conservative criteria to eliminate contamination by stars, if the purity of QSO selection is the highest priority on the selection.
success rates (weighted mean of QSO and star success rates) in Table 3 with an assumed number fraction of QSOs to stars in the sample. Note that in our photometric sample the fraction varies from about 0.1 to 2.2. For example, in case in which we apply the coefficients for $N_{\text{QSO}}/N_{\text{star}} = 1.5$, which is the rough average of the data set, we select 9077 QSO candidates out of 23,677 variable sources, which have more than 40 measurements with errors less than 0.05 mag, based only on photometric data. In this case it will provide an additional $\sim 1300$ QSO candidates to the current spectroscopically confirmed QSOs in the data set.

### 6.2. Failure Rates and Their Meanings

In our analysis, we identified 83 spectroscopically confirmed QSOs with the “false” classification by their optical variability. The “false” means that the success rate is lower than 50% in our analysis. It is about 1% of the spectroscopically confirmed QSOs, and it is interesting to know the reason for this failure in the classification. We visually inspect all the spectra of 83 QSOs and confirm that 79 objects can be reasonably classified as QSO/AGN. The remaining four objects are three stars and one object with failure in taking the spectrum (fiber allocation might have failed). It should be noted that the misclassification occurs on the common objects by the analysis based on other models, such as Butler’s model, single OU, and Kelly’s mixture model, and it is not a characteristic feature of our model. This means that their variability cannot be distinguished from those of stars.

There are several possibilities for explaining the failure of classification. The first is the effect caused by the irregularly taken time series data, which hides or dulls the emergence of the QSO/AGN-like variability features in their light curves. It may be coming from the limitation of the MCMC procedure for producing the light curves for our stochastic model analysis. It is, however, difficult to consider this a plausible reason, as most of the objects are successfully classified as “QSO” in spite of being under the conditions of very similar cadence, as described in Section 3.2.

The second possibility is that this discrepancy comes from the time difference between photometric and spectroscopic observations. As we based the SDSS DR12 data on the spectroscopic data, although the photometric data are coming from DR7, there is a possibility of a change of physical condition in these objects. We investigate the MIDs of the data taking on photometry and spectroscopy, and we confirm that 40 and 43 spectral data are taken in and out of the duration on the photometric data, respectively. Therefore, it is difficult to consider the time difference of photometry and spectroscopy as the main reason for the discrepancy.

The third and most plausible is that the objects are rare-type QSOs/AGNs whose variability features differ from those of most QSOs. There may be signatures of (quasi-)periodic features in their light curves, and/or they cannot be well described by the MCMC procedure as mentioned above.

As shown in the left panel of Figure 16(a), objects classified as not “QSO” (categorized in Group2 as described in Section 4.2) have single-bending PSDs with very large $\omega$ at the bending points, which is very similar to those for stars in our analysis. We also show the light curve of the same object in Figure 16(b), and it is clear that the variation in the short timescale is very large, and we cannot see the damping of brightness, which is seen in typical QSOs, as shown in Figure 2. This implies the rarity (they contribute only 1% of spectroscopically confirmed QSOs) of this QSO in terms of optical variability, and there might be a different physical process controlling the variation of brightness, at least in a short timescale.

We cannot see any significant difference in the spectra of the QSOs with “false” classification, compared to the typical variable QSOs, and it is plausible to consider the discrepancies as coming from the physical processes, which are effective in optical variability but not optical spectral features.

As the fractions of QSOs and stars spectroscopically misidentified are very small (4/83 for QSOs and 16/199 for stars), it is clear that the results of identification rates discussed in previous sections are not affected seriously.

### 6.3. Features of Our Model and Interpretation of QSOs’ PSD Parameters

We compare the goodness of the fit of our model to the single OU process (DRW) model by using AIC and BIC (Schwarz 1978).

AIC is derived by the following equation:

$$\text{AIC} = -2 \ln(L) + 2k,$$

where $L$ is the maximum value of the likelihood function and $k$ is the number of estimated parameters in the model. In the case

| $N_{\text{QSO}}/N_{\text{star}}$ | A | B | C | D | E | $R_{\text{QSO}}^a$ | $R_{\text{star}}^b$ | $R_{\text{QSO}}^b$ | $R_{\text{star}}^b$ |
|-----------------------------|---|---|---|---|---|----------------|----------------|----------------|----------------|
| 0.1                        | −0.728 | 0.063 | 0.227 | −0.069 | 1.297 | 0.971 | 0.870 | 0.981 |
| 0.5                        | −0.664 | 0.139 | 0.211 | −0.019 | 1.209 | 0.965 | 0.957 | 0.970 |
| 1.0                        | −0.775 | 0.146 | 0.201 | 0.151 | 1.560 | 0.964 | 0.966 | 0.961 |
| 1.2                        | −0.695 | 0.119 | 0.255 | −0.073 | 0.945 | 0.966 | 0.977 | 0.950 |
| 1.5                        | −0.695 | 0.119 | 0.255 | −0.073 | 0.945 | 0.966 | 0.977 | 0.950 |
| 2.0                        | −0.774 | 0.189 | 0.222 | 0.172 | 1.551 | 0.968 | 0.983 | 0.939 |
| 2.5                        | −0.768 | 0.210 | 0.250 | 0.111 | 1.467 | 0.971 | 0.987 | 0.931 |
| 3.0                        | −0.866 | 0.425 | 0.395 | −0.410 | 0.949 | 0.972 | 0.988 | 0.924 |
| 3.5                        | −0.768 | 0.210 | 0.250 | 0.111 | 1.467 | 0.973 | 0.987 | 0.931 |
| 5.0                        | −0.575 | 0.285 | 0.253 | −0.101 | 1.159 | 0.978 | 0.991 | 0.913 |

Notes.

a Total identification (success) rates are calculated by weighted mean of those for QSOs ($R_{\text{QSO}}$) and stars ($R_{\text{star}}$).

b QSOs ($R_{\text{QSO}}$) and stars ($R_{\text{star}}$) are values of completeness.
of our model, \( k = 4 \), and \( k = 2 \) for the DRW model. BIC is derived by Equation (25) as described in Section 4.1.

In Figure 17, we show the comparison of AICs and BICs for QSOs in the “good sample.” It is clear that our model works better for QSOs with large differences between \( \omega_L \) and \( \omega_H \), which correspond to variability different from DRW, since the smaller AIC/BIC values mean better fitness.

The trends seen about AICs are the same as for BICs, that our model works better for QSOs with larger discrepancies between \( \omega_L \) and \( \omega_H \), for example, \( \log \frac{\omega_L}{\omega_H} \leq -3 \).

It is therefore reasonable to think that our model provides a better fit than those based on the DRW model for QSOs with larger differences between \( \omega_L \) and \( \omega_H \). It should be noted that many QSOs, whose variability can be well described by the DRW model, show worse fitting results in our model. However, this causes no significant effect for selection of QSOs/AGNs from variable stars. The differences are only coming from the number of model parameters, and the likelihood values are equivalent about both models.

We also note here that the AIC in its original form is strictly only valid asymptotically, and the correction to AIC for finite sample sizes suggested by Sugiura (1978) and Hurvich & Tsai (1989), commonly denoted as c-AIC, may be used for the discussions. We also check the c-AIC and confirm that there is no difference from the result by using AIC.

The PSDs for the QSOs/AGNs with smaller AIC/BIC in our model show the power-law-like features, which have flatter slopes than \( \beta = -2 \), when we express the PSD \( \propto \omega^\beta \). In most cases they have slopes \( \beta \sim -1.8 \) or so. They have very small \( \omega_L \) (log \( \omega_L \sim -4 \)) and large \( \omega_H \) (log \( \omega_H \sim 1–2 \)) as shown by “PowerLaw” type variability in Figure 5(b). In our model such a variability can be expressed better than in the DRW model by selecting a large difference in \( \omega_L \) and \( \omega_H \).
On the other hand, most QSOs have small differences between $\omega_L$ and $\omega_H$, typically $\log \omega_L/\omega_H \sim -1$. In our model there is no constraint about the values of $\omega_L$ and $\omega_H$, except for $\omega_L \leq \omega_H$. One of the reasons for this small difference is our method for estimating model parameters. As we use median values of posteriors ($\omega_L$ and $\omega_H$), which have distributions in a certain level, and we use the constraint of $\omega_L \leq \omega_H$ in our calculation, it is reasonable to get some difference between $\omega_L$ and $\omega_H$. Since the typical errors of parameter inferences are 0.4 dex in this case, it is plausible to consider that this is the main reason for the small differences. This guess can be supported by the fact that the slope of the distribution of these objects in Figure 5(b) is well aligned to that of the “ideal” DRW model one as shown by green dotted lines. Since the data we used for the analysis are not dense in time, it is very difficult to discuss the hypothesis of their intrinsic origin.

6.4. Issues for Analyzing More Massive Data with Our Model

In our model the calculation time is on the order of $O(N^3)$, when $N$ is the number of observed data points. Compared to Kelly’s OU mixture model, which needs time with the order of $O(MN^2)$ ($M$ is the number of mixed OU processes), our model has an advantage in calculation time. However, it is still very massive for the data with numerous measurements, or with many observed objects. If we limit our analysis to those based on the stationary process, we may implement the methods based on phase-space description or Kalman filter for reducing the calculation time. Well-known time series analysis methods like the ARMA (Auto-Regression Moving Average) model, as suggested by Kelly et al. (2014), Simm et al. (2016), and Kasliwal et al. (2017), are useful candidates for the massive data time series analysis in the near future. It is, however, plausible to consider that some nonstationary processes also affect the optical variability of QSOs/AGNs. For analyzing time series data of QSOs/AGNs with a more complicated model, we should continue to strive for the effective algorithms using powerful computational equipment like many CPU cores and supportive database systems.

7. Conclusions

We develop the infinite mixture model of the OU process for describing the optical variability of QSOs/AGNs, which enables us to get the results within an appropriate time. The main reason for this faster calculation is our analytical derivation of the covariance function. It is based on the consideration for treating the variability as a stochastic process, and it enables us to get the parameters on their PSDs for their brightness variations.

We apply our model to 67,507 objects extracted from SDSS Stripe 82 photometric data with sufficient multiple measurements, and we succeed in showing very high precision in selecting QSOs among variable objects based only on their variability, by investigating \(~10,000\) spectroscopically confirmed objects. This can be a good first step for enabling the classification and investigating the physical processes among various types of QSOs/AGNs using their variability.

We find out that QSO/AGN variability can be well described in a certain regular manner in our model parameter space on their PSDs on QSO/AGN variability, with very few outliers. In general, the smaller the amplitude of variation, the shorter the damping timescale, which is consistent with the previous studies. The majority (95%) of spectroscopically confirmed QSOs may be well described by the DRW model. However, 4% of QSOs belong to the group with large differences between $\omega_L$ and $\omega_H$. The remaining 1% of QSOs are indistinguishable from stars in our model.

Our model is more flexible than other previously suggested models based on a single OU process (DRW), as we can describe their PSDs of variability with more accuracy. We eliminate the arbitrariness in the number of mixed OU processes from the original model, and our implementation enables calculations to finish within tolerable time. We show that this type of analysis is feasible in an era with more massive data, though more calculation speed is desirable.

Based on our model, we try the separation of QSOs and stars based only on the variability, and we succeed in identifying QSOs with 99.3% completeness and 99.0% purity. These numbers show that our model can be used as the most effective method for selecting QSOs from a variable object catalog and is superior to other models in terms of completeness. Other evaluations of the identification capabilities, such as accuracy and recall, show mostly the same levels as those for other competitive models, which are widely used and discussed for identifying QSOs based only on the variability. The main reason supporting our model’s superiority in terms of QSO selection comes from our introduction of the mixture of OU processes, which provides more flexibility on the description of PSDs and enables us to separate rare-type QSOs/AGNs, which cannot be well described by a simple DRW model, from variable stars in the parameter space.

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12 https://www.ism.ac.jp/shikoin/training/dsm/index_e.html
Erratum: “Modeling the Variability of Active Galactic Nuclei by an Infinite Mixture of Ornstein–Uhlenbeck (OU) Processes” (2018, ApJ, 869, 178)

Tadafumi Takata1,2, Yusuke Mukuta3, and Yoshikiko Mizumoto4

1 Astronomy Data Center, National Astronomical Observatory of Japan, National Institutes of Natural Science (NINS) Osaka 2-21-1 Tokyo, 181-8588, Japan
2 The Graduate University for Advanced Study (SOKENDAI), Japan
3 National Astronomical Observatory of Japan, National Institutes of Natural Science (NINS), Japan

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In the section “Classification and Identification Rate of our Model,” there are some errors on the definitions of identification rates in Figure 9, and the results shown in Tables 1 and 2, and the box plots for these results in Figures 10 and 13 must be replaced by the following figures and tables. It is also required that the purity and the recall in the main text should be read as average precision and

Table 1

Comparison of the Identification Rates on the Sets of Our Model Parameters

| Abbreviation | Parameters | Accuracy | Precision (Purity) | Recall (Completeness) | Average Precision |
|--------------|------------|----------|--------------------|------------------------|-------------------|
| Inf4         | ω2, ω1, ζ, α | 0.956±0.004 | 0.905±0.012 | 0.993±0.005 | 0.991±0.001 |
| Inf3cr       | ω2, P, α | 0.944±0.007 | 0.919±0.016 | 0.989±0.003 | 0.989±0.002 |
| Inf3oz       | ω2, ω1, ζ | 0.946±0.006 | 0.899±0.010 | 0.993±0.005 | 0.986±0.003 |
| Inf2zh       | ω2, ζ | 0.926±0.013 | 0.887±0.026 | 0.974±0.016 | 0.981±0.002 |
| Inf2zl       | ω2, ζ | 0.939±0.005 | 0.896±0.007 | 0.993±0.002 | 0.985±0.003 |
| InfAll4      | ω2, ω1, ζ, α | 0.944±0.004 | 0.924±0.007 | 0.974±0.004 | 0.982±0.002 |
| InfAll3cr    | ω2, P, α | 0.946±0.007 | 0.919±0.016 | 0.993±0.005 | 0.989±0.002 |

Table 2

Comparison of Identification Rates among Models

| Model       | Abbreviation | Parameters | Accuracy | Precision (Purity) | Recall (Completeness) | Average Precision |
|-------------|--------------|------------|----------|--------------------|------------------------|-------------------|
| Our Model   | Inf4         | ω2, ω1, ζ, α | 0.956±0.004 | 0.905±0.012 | 0.993±0.005 | 0.991±0.001 |
| Inf3cr      | ω2, P, α | 0.944±0.007 | 0.919±0.016 | 0.989±0.003 | 0.989±0.002 |
| Inf3oz      | ω2, ω1, ζ | 0.946±0.006 | 0.899±0.010 | 0.993±0.005 | 0.986±0.003 |
| Inf2zh      | ω2, ζ | 0.926±0.013 | 0.887±0.026 | 0.974±0.016 | 0.981±0.002 |
| Inf2zl      | ω2, ζ | 0.939±0.005 | 0.896±0.007 | 0.993±0.002 | 0.985±0.003 |
| JAVELIN     | Jav         | τ, σ | 0.956±0.006 | 0.938±0.012 | 0.980±0.007 | 0.986±0.002 |
| BUTLER      | But         | θ^2/h^2, σ^2 | 0.943±0.004 | 0.913±0.005 | 0.980±0.004 | 0.986±0.001 |
| Mix (M = 2) | KM2_4       | ω2, ω1, ζ, α | 0.926±0.015 | 0.882±0.026 | 0.995±0.010 | 0.985±0.001 |
| Mix (M = 4) | KM4_4       | ω2, ω1, ζ, α | 0.925±0.010 | 0.882±0.019 | 0.996±0.010 | 0.985±0.002 |
| Mix (M = 8) | KM8_4       | ω2, ω1, ζ, α | 0.927±0.010 | 0.882±0.021 | 0.996±0.009 | 0.986±0.003 |
| Mix (M = 16) | KM16_4    | ω2, ω1, ζ, α | 0.928±0.010 | 0.882±0.020 | 0.998±0.004 | 0.986±0.002 |
| Mix (M = 32) | KM32_4     | ω2, ω1, ζ, α | 0.929±0.010 | 0.883±0.021 | 0.997±0.008 | 0.987±0.002 |
| Mix (M = 64) | KM64_4     | ω2, ω1, ζ, α | 0.932±0.010 | 0.882±0.022 | 0.997±0.010 | 0.986±0.002 |
| Mix (M = 128) | KM128_4  | ω2, ω1, ζ, α | 0.932±0.010 | 0.884±0.021 | 0.999±0.008 | 0.986±0.002 |
| Mix (M = 2) | KM2_3cr    | ω2, P, α | 0.938±0.005 | 0.919±0.007 | 0.959±0.004 | 0.917±0.008 |
| Mix (M = 64) | KM64_3cr  | ω2, P, α | 0.934±0.004 | 0.908±0.005 | 0.965±0.004 | 0.928±0.007 |
Table 1. Definitions of values for describing the identification rates used in our analysis.

| Variability | Spectroscopy | QSO in spectroscopy | Star in spectroscopy |
|-------------|--------------|---------------------|----------------------|
| QSO in variability | TP (True Positive) | FP (False Positive) |
| Star in variability | FN (False Negative) | TN (True Negative) |

$$\text{Recall (Completeness)} = \frac{TP}{TP + FN}$$

$$\text{Accuracy} = \frac{TP + TN}{TP + TN + FP + FN}$$

$$\text{Precision (Purity)} = \frac{TP}{TP + FP}$$

Figure 9. (a) Definitions of values for describing the identification rates used in our analysis. (b) Schematic expression for calculating average precision, which is available with the area under the precision–recall curve shown as shaded.

Figure 10. Box plot of the identification rates of our models. This figure shows the results for the accuracy, purity (precision), completeness, and average precision of the $r$-band data, respectively. The box extends from lower to upper quartile values of the data (blue lines), with red lines at the median. The whiskers extend from the box to show the range of the data, $Q_3 + 1.5 \times IQR$. IQR is an interquartile value ($Q_3 - Q_1$), and $Q_1$ and $Q_3$ are the first and third quartile values. The flier points are for data beyond the end of the whiskers.

Figure 13. Box plot for comparisons of the identification rates of the different models in the same manner as Figure 10.
purity (precision), respectively. We also added a panel in Figure 9 to explain the definition of average precision.

Sorry for the confusion caused by these errors. We thank Daming Yang for pointing some of them out.

ORCID iDs

Tadafumi Takata © https://orcid.org/0000-0002-6592-4250