ON THE ALEKSEEV-GRÖBNER
FORMULA IN BANACH SPACES

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This paper is dedicated to Peter Kloeden on the occasion of his 70th birthday

Abstract. The Alekseev-Gröbner formula is a well known tool in numerical analysis for describing the effect that a perturbation of an ordinary differential equation (ODE) has on its solution. In this article we provide an extension of the Alekseev-Gröbner formula for Banach space valued ODEs under, loosely speaking, mild conditions on the perturbation of the considered ODEs.

1. Introduction. The Alekseev-Gröbner formula (see, e.g., Alekseev [1], Gröbner [6], and Hairer et al. [7, Theorem 14.5 in Chapter I]) is a well known tool in deterministic numerical analysis for describing the effect that a perturbation of an ordinary differential equation (ODE) has on its solution. Considering numerical methods for ODEs as appropriate perturbations of the underlying equations makes the Alekseev-Gröbner formula applicable for estimating errors of numerical methods (see, e.g., Hairer et al. [7, Theorem 7.9 in Chapter II], Iserles [8, Theorem 3.7], Iserles [9, Theorem 1], and Niesen [12, Theorem 1]). It is the main contribution of this work to provide an extension of the Alekseev-Gröbner formula for Banach space valued ODEs under, loosely speaking, mild conditions on the perturbation of the

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considered ODEs (see Corollary 5.2 in Section 5 and Theorem 1.1 below). As a consequence, our main result is well suited for the analysis of pathwise approximation errors between exact solutions of stochastic partial differential equations (SPDEs) of evolutionary type and their numerical approximations. In particular, it can be used as a key ingredient for establishing strong convergence rates for numerical approximations of SPDEs with a non-globally Lipschitz continuous, non-globally monotone nonlinearity, and additive trace-class noise. The precise result will be the subject of a future research article. In this introductory section we now present in the following theorem, Theorem 1.1 below, the main result of this article.

**Theorem 1.1.** Let \((V, \| \cdot \|_V)\) be a nontrivial \(\mathbb{R}\)-Banach space, let \(T \in (0, \infty)\), let \(f: [0, T] \times V \to V\) be a continuous function, assume for every \(t \in [0, T]\) that \(V \ni x \mapsto f(t, x) \in V\) is Fréchet differentiable, assume that \([0, T] \times V \ni (t, x) \mapsto \left( \frac{\partial}{\partial x} f \right)(t, x) \in L(V)\) is continuous, for every \(x \in V\), \(s \in [0, T]\) let \(X_{x, t}^s: [s, T] \to V\) be a continuous function which satisfies for every \(t \in [s, T]\) that \(X_{x, t}^s = x + \int_s^t f(\tau, X_{x, \tau}^s) \, d\tau\), and let \(Y, E: [0, T] \to V\) be strongly measurable functions which satisfy for every \(t \in [0, T]\) that \(\int_0^T \| f(\tau, Y_\tau) \|_V + \| E_\tau \|_V \, d\tau < \infty\) and \(Y_t = Y_0 + \int_0^t (f(\tau, Y_\tau) + E_\tau) \, d\tau\) for every \(t \in [0, T]\).

(i) it holds for every \(s \in [0, T]\), \(t \in [s, T]\) that \(V \ni x \mapsto X_{x, t}^s \in V\) is Fréchet differentiable,

(ii) it holds for every \(t \in [0, T]\) that \([0, t] \ni \tau \mapsto \left( \frac{\partial}{\partial x} X_{x, t}^\tau \right) E_\tau \in V\) is strongly measurable,

(iii) it holds for every \(t \in [0, T]\) that \(\int_0^t \| \left( \frac{\partial}{\partial x} X_{x, t}^\tau \right) E_\tau \|_V \, d\tau < \infty\), and

(iv) it holds for every \(s \in [0, T]\), \(t \in [s, T]\) that

\[
Y_t = X_{x, t}^s + \int_s^t \left( \frac{\partial}{\partial x} X_{x, t}^\tau \right) E_\tau \, d\tau. \quad (1)
\]

Theorem 1.1 is proven as Corollary 5.2 in Section 5 below. In the case where the Banach space \(V\) in Theorem 1.1 is finite-dimensional and under suitable additional regularity assumptions the identity (1) is referred to as the Alexeev-Gröbner formula in the literature (see, e.g., Alexeev [1], Gröbner [6], and Hairer et al. [7, Theorem 14.5 in Chapter I]). In this context we also refer to the article of Ladas et al. [11] in which an extension of the Alexeev-Gröbner formula to the case of Banach space valued ODEs has been derived under more restrictive regularity assumptions on the ODE and its perturbation than in Theorem 1.1.

The rest of this article is structured as follows. In Section 2 we recall some elementary and well known properties for Banach space valued functions (see Lemmas 2.1–2.7, Corollary 2.8, and Lemma 2.9). Thereafter we combine these elementary results to prove an abstract version of the Alexeev-Gröbner formula for Banach space valued ODEs under, roughly speaking, restrictive conditions on the solution as well as on the perturbation of the considered ODE, see Proposition 2.10 in Section 2 below for details. Sections 3 and 4 are devoted to presenting in detail some partially well known results on continuous differentiability of solutions to a class of Banach space valued ODEs with respect to initial value, initial time, and current time (see Lemma 4.8 in Section 4 below). Finally, we combine Proposition 2.10, Lemma 3.7 (the flow property of solutions to ODEs), and Lemma 4.8 to establish in Corollary 5.2 in Section 5 below the main result of this article.
2. Extended chain rule property for Banach space valued functions. In this section we prove an abstract version of the Alekseev-Gröbner formula for Banach space valued ODEs under, loosely speaking, restrictive conditions in Proposition 2.10. This will be used in Section 5 to prove in Corollary 5.2 an extension of the Alekseev-Gröbner formula (cf., e.g., Hairer et al. [7, Theorem 14.5 in Chapter I]) for Banach space valued functions. In order to prove Proposition 2.10 we first recall some elementary auxiliary lemmas; see Lemmas 2.1–2.7, Corollary 2.8, and Lemma 2.9. In particular, we recall the fundamental theorem of calculus for Banach space valued functions in Lemmas 2.2–2.4 (cf., e.g., Prévôt & Röckner [13, Proposition A.2.3]) and we derive suitable extensions thereof in Lemma 2.7 (cf., e.g., [10, Lemma 2.1]) and Corollary 2.8. Thereafter, we combine Lemma 2.7, Corollary 2.8, and Lemma 2.9 (cf., e.g., Rudin [14, Theorem 7.17 and the remark thereafter]) to establish Proposition 2.10.

**Lemma 2.1.** Let \((X, \mathcal{X})\) be a separable topological space, let \((Y, \mathcal{Y})\) be a topological space, and let \(f \in C(X, Y)\). Then \(f\) is strongly measurable.

**Proof of Lemma 2.1.** Throughout this proof let \(A \subseteq X\) be a countable dense subset of \(X\). Note that the assumption that \(f\) is continuous ensures that for every \(V \in \mathcal{Y}\) with \(f(X) \cap V \neq \emptyset\) it holds that

\[ \emptyset \neq \{ x \in X : f(x) \in V \} \in \mathcal{X}. \tag{2} \]

This and the fact that \(A \subseteq X\) is dense imply that for every \(V \in \mathcal{Y}\) with \(f(X) \cap V \neq \emptyset\) there exists \(a \in A\) such that \(f(a) \in V\). The fact that \(A \subseteq X\) is countable therefore implies that \(f(A) \subseteq f(X)\) is a countable dense subset of \(f(X)\). Hence, we obtain that \(f(X) \subseteq Y\) is separable. This and the fact that \(f\) is \(\mathcal{B}(X)/\mathcal{B}(Y)\)-measurable complete the proof of Lemma 2.1.

Observe that for every measurable space \((\Omega, \mathcal{F})\), every topological space \((Y, \mathcal{Y})\), and every function \(f : \Omega \to Y\) it holds that \(f\) is strongly measurable w.r.t. \(\mathcal{F}\) and \(\mathcal{Y}\) (\(f\) is strongly measurable) if \(f\) is \(\mathcal{F}/\mathcal{B}(Y)\)-measurable and \(f(\Omega) \subseteq Y\) is separable.

**Lemma 2.2.** Let \((V, \|\cdot\|_V)\) be an \(\mathbb{R}\)-Banach space, let \(a \in \mathbb{R}, b \in (a, \infty)\), and let \(f \in C([a, b], V)\). Then it holds for every \(t \in [a, b]\) that

\[ \limsup_{\tau \to t} \int_{\tau - h}^{\tau} f(s) \, ds - f(t) \|_V = 0. \]

**Lemma 2.3.** Let \((V, \|\cdot\|_V)\) be an \(\mathbb{R}\)-Banach space and let \(a \in \mathbb{R}, b \in (a, \infty), f \in C([a, b], V)\), \(F : [a, b] \to V\) satisfy for every \(t \in [a, b]\) that

\[ F(t) = F(a) + \int_{a}^{t} f(s) \, ds. \]

Then it holds for every \(t \in [a, b]\) that

\[ \limsup_{\tau \to t} \int_{\tau - h}^{\tau} f(s) \, ds - f(t) \|_V = 0. \]

**Lemma 2.4.** Let \((V, \|\cdot\|_V)\) be an \(\mathbb{R}\)-Banach space and let \(a \in \mathbb{R}, b \in (a, \infty), F \in C^1([a, b], V)\). Then it holds for every \(t \in [a, b]\) that

\[ F(t) - F(a) = \int_{a}^{t} F'(s) \, ds. \]

**Lemma 2.5.** Let \((V, dv)\) and \((W, dw)\) be metric spaces and let \(a \in \mathbb{R}, b \in (a, \infty), g \in C([a, b] \times V, W)\). Then it holds that the function

\[ V \ni v \mapsto ([a, b] \ni t \mapsto g(t, v) \in W) \in C([a, b], W) \tag{3} \]

is continuous.

**Proof of Lemma 2.5.** Throughout this proof let \(\varepsilon \in (0, \infty)\). Observe that the assumption that \(g \in C([a, b] \times V, W)\) implies that there exists a function \(\delta = \)
(\delta_{t,x})(t,x) \in [a, b] \times V : [a, b] \times V \to (0, \infty) \) such that for every \( s, t \in [a, b], x, x_0 \in V \) with \( \max\{s - t, d_V(x, x_0)\} < \delta_{t,x_0} \) it holds that
\[
d_{W}(g(s, x), g(t, x_0)) < \frac{\varepsilon}{2}.
\]
Moreover, note that the fact that \([a, b]\) is compact ensures that for every \( x_0 \in V \) there exist \( n \in \mathbb{N}, t_1, \ldots, t_n \in [a, b] \) such that \( a = t_1 < \ldots < t_n = b \) and
\[
[a, b] = \bigcup_{i=1}^{n} \{ r \in [a, b] : |r - t_i| < \delta_{t_i,x_0} \}.
\]
This and (4) demonstrate that for every \( x_0 \in V \) there exist \( n \in \mathbb{N}, t_1, \ldots, t_n \in [a, b] \) such that for every \( t \in [a, b], x \in V \) with \( d_V(x, x_0) < \min\{\delta_{t_1,x_0}, \ldots, \delta_{t_n,x_0}\} \) it holds that
\[
d_{W}(g(t, x), g(t, x_0)) \leq \min_{i \in [0, n) \cap \mathbb{N}} \left| d_{W}(g(t, x), g(t_i, x_0)) + d_{W}(g(t_i, x_0), g(t, x_0)) \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
As \( \varepsilon \in (0, \infty) \) was arbitrary, the proof of Lemma 2.5 is completed. \( \square \)

**Lemma 2.6.** Let \((V, \| \cdot \|_V)\) be a nontrivial \(\mathbb{R}\)-Banach space, let \(a, b \in \mathbb{R}, b \in (a, \infty)\), and let \(f : [a, b] \to V\) be a strongly measurable function such that \(\int_{a}^{b} \| f(s) \|_V \, ds < \infty\).
Then there exist functions \((f_n)_{n \in \mathbb{N}} \subseteq C([a, b], V)\) which satisfy that
\[
\limsup_{n \to \infty} \int_{a}^{b} \| f_n(s) - f(s) \|_V \, ds = 0.
\]
**Proof.** Throughout this proof let \(\eta \in C^\infty(\mathbb{R}, \mathbb{R})\) satisfy for every \( s \in (-1, 1), t \in (-\infty, -1) \cup [1, \infty) \) that \(\eta(s) = \exp((1/s^2-1))(\int_{-1}^{s} \exp((1/(u^2-1)) \, du)^{-1}\) and \(\eta(t) = 0\) and let \((\eta_\varepsilon)_{\varepsilon \in (0, \infty)} \subseteq C^\infty(\mathbb{R}, \mathbb{R})\) satisfy for every \(\varepsilon \in (0, \infty), s \in \mathbb{R}\) that \(\eta_\varepsilon(s) = \varepsilon^{-1}\eta(s/\varepsilon)\). Note that Step 2.b in Section A.1 in Prévôt & Röckner [13] ensures that there exist \((k_n)_{n \in \mathbb{N}} \subseteq \mathbb{N}, (v_{m,n})_{m,n \in \mathbb{N}} \subseteq \mathcal{B}([a, b])\) such that
\[
\limsup_{n \to \infty} \int_{a}^{b} \left\| \left( \sum_{m=1}^{k_n} v_{m,n}(1_B)_{B_{m,n}}(s) \right) - f(s) \right\|_V \, ds = 0.
\]
Next observe that, e.g., Theorem 6 in Section C.4 in Evans [5] shows that for every \(B \in \mathcal{B}([a, b])\) it holds that
\[
\limsup_{(0, \infty) \ni \varepsilon \to 0} \int_{a}^{b} \| 1_B(s) - (\eta_\varepsilon * 1_B)(s) \| \, ds = 0.
\]
This implies that for every \(m, n \in \mathbb{N}\) there exists \(\varepsilon_{m,n} \in (0, \infty)\) such that
\[
\int_{a}^{b} \| 1_{B_{m,n}}(s) - (\eta_{\varepsilon_{m,n}} * 1_{B_{m,n}})(s) \| \, ds < \frac{1}{m^2 n^2 \| v_{m,n} \|_V}.
\]
Combining this, the triangle inequality, and (8) proves that
\[
\limsup_{n \to \infty} \int_{a}^{b} \left\| \left( \sum_{m=1}^{k_n} v_{m,n}(\eta_{\varepsilon_{m,n}} * 1_{B_{m,n}})(s) \right) - f(s) \right\|_V \, ds
\leq \limsup_{n \to \infty} \int_{a}^{b} \left\| \left( \sum_{m=1}^{k_n} v_{m,n}(\eta_{\varepsilon_{m,n}} * 1_{B_{m,n}})(s) \right) - 1_{B_{m,n}}(s) \right\|_V \, ds.
The fact that for every $\varepsilon \in (0, \infty)$, $B \in \mathcal{B}([a, b])$ it holds that $[a, b] \ni s \mapsto (\eta_s \ast 1_B)(s) \in \mathbb{R}$ is continuous therefore implies that there exist $(f_n) \subseteq C([a, b], V)$ such that $\limsup_{n \to \infty} \int_a^b \|f(s) - f_n(s)\|_V \, ds = 0$. The proof of Lemma 2.6 is thus completed. \hfill \square

**Lemma 2.7.** Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space, let $(W, \|\cdot\|_W)$ be an $\mathbb{R}$-Banach space, let $a \in \mathbb{R}$, $b \in (a, \infty)$, $\phi \in C^1(V, W)$, $F: [a, b] \to V$, and let $f: [a, b] \to V$ be a strongly measurable function which satisfies for every $t \in [a, b]$ that $\int_a^t \|f(s)\|_V \, ds < \infty$ and $F(t) - F(a) = \int_a^t f(s) \, ds$. Then

(i) it holds that $[a, b] \ni s \mapsto \phi(F(s))f(s) \in W$ is strongly measurable,

(ii) it holds that $\int_a^b \|\phi'(F(s))f(s)\|_W \, ds < \infty$, and

(iii) it holds for every $t \in [a, b]$ that $\phi(F(t)) - \phi(F(a)) = \int_a^t \phi'(F(s))f(s) \, ds$.

**Proof of Lemma 2.7.** Throughout this proof let $(f_n) \subseteq C([a, b], V)$ be functions which satisfy $\limsup_{n \to \infty} \int_a^b \|f_n(s) - f(s)\|_V \, ds = 0$ (see Lemma 2.6) and let $F_n: [a, b] \to V$, $n \in \mathbb{N}$, be the functions which satisfy for every $n \in \mathbb{N}$, $t \in [a, b]$ that $F_n(t) = F(a) + \int_a^t f_n(s) \, ds$. Observe that the fact that $f$ is strongly measurable, the fact that the function $[a, b] \ni s \mapsto \phi(F(s)) \in L(V, W)$ is continuous, and the fact that the function $V \times L(V, W) \ni (v, A) \mapsto Av \in W$ is continuous implies that $[a, b] \ni s \mapsto \phi'(F(s))f(s) \in W$ is strongly measurable. The fact that the function $[a, b] \ni s \mapsto \phi'(F(s)) \in L(V, W)$ is continuous and the assumption that $f$ is strongly measurable and integrable hence establish items (i) and (ii). Next observe that Lemma 2.3 (with $V = V$, $a = a$, $b = b$, $f = f_n$, $F = F_n$ for $n \in \mathbb{N}$ in the notation of Lemma 2.3) ensures that for every $n \in \mathbb{N}$ it holds that $F_n \subseteq C^1([a, b], V)$ and $F_n^* = f_n$. Lemma 2.4 (with $V = V$, $a = a$, $b = b$, $F = \phi \circ F_n$ for $n \in \mathbb{N}$ in the notation of Lemma 2.4) and the chain rule for Fréchet derivatives therefore prove that for every $n \in \mathbb{N}$, $t \in [a, b]$ it holds that

$$
\phi(F_n(t)) - \phi(F_n(a)) = \int_a^t \phi'(F_n(s))f_n(s) \, ds.
$$

(12)

Moreover, note that for every $n \in \mathbb{N}$, $t \in [a, b]$ it holds that

$$
\|F_n(t) - F(t)\|_V = \left\| \int_a^t [f_n(s) - f(s)] \, ds \right\|_V \leq \int_a^b \|f_n(s) - f(s)\|_V \, ds.
$$

(13)

This assures that

$$
\limsup_{n \to \infty} \sup_{s \in [a, b]} \|F_n(s) - F(s)\|_V = 0.
$$

(14)

The fact that $\phi \in C^1(V, W)$ hence shows that for every $t \in [a, b]$ it holds that

$$
\limsup_{n \to \infty} \|\phi'(F_n(t)) - \phi'(F(t))\|_{L(V, W)} = 0.
$$

(15)

and

$$
\limsup_{n \to \infty} \|\phi'(F_n(t)) - \phi'(F(t))\|_{L(V, W)} = 0.
$$

(16)
Next observe that for every $n \in \mathbb{N}$, $t \in [a, b]$ it holds that
\[
\int_a^t \|\phi'(F_n(s))f_n(s) - \phi'(F(s))f(s)\|_W \ ds \\
\leq \int_a^t \|\phi'(F_n(s))\|_{L(V,W)} \|f_n(s) - f(s)\|_V \ ds \\
+ \int_a^t \|\phi'(F_n(s)) - \phi'(F(s))\|_{L(V,W)} \|f(s)\|_V \ ds \\
\leq \left[ \sup_{r \in [a,b]} \|\phi'(F(r) + (F_n(r) - F(r)))\|_{L(V,W)} \right] \int_a^b \|f_n(s) - f(s)\|_V \ ds \\
+ \left[ \sup_{r \in [a,b]} \|\phi'(F(r)) - \phi(F(r))\|_{L(V,W)} \right] \int_a^b \|f(s)\|_V \ ds.
\]

Combining (17)–(19) and the fact that \( \lim sup_{n \to \infty} \|f_n(s) - f(s)\|_V \ ds \) implies that for every $\varepsilon \in (0, \infty)$ there exists $N \in \mathbb{N}$ such that for every $n \in [N, \infty) \cap \mathbb{N}$ it holds that
\[
\sup_{r \in [a,b]} \|\phi'(F(r) + (F_n(r) - F(r)))\|_{L(V,W)} < \varepsilon. \tag{18}
\]

In particular, this and the fact that $\phi' \circ F \in C([a,b], L(V,W))$ imply that
\[
\sup_{n \in \mathbb{N}} \sup_{r \in [a,b]} \|\phi'(F(r)) - \phi(F(r))\|_{L(V,W)} < \infty. \tag{19}
\]

Combining (17)–(19) and the fact that $\lim sup_{n \to \infty} \int_a^b \|f_n(s) - f(s)\|_V \ ds = 0$ ensures that for every $t \in [a, b]$ it holds that
\[
\lim sup_{n \to \infty} \left| \int_a^t \phi'(F_n(s))f_n(s) \ ds - \int_a^t \phi'(F(s))f(s) \ ds \right|_W = 0. \tag{20}
\]

Moreover, observe that (15) shows that for every $t \in [a, b]$ it holds that
\[
\lim sup_{n \to \infty} \|\phi(F_n(t)) - \phi(F_n(a)) - [\phi(F(t)) - \phi(F(a))]\|_W = 0. \tag{21}
\]

This, (12), and (20) establish item (iii). The proof of Lemma 2.7 is thus completed.

\[\square\]

**Corollary 2.8.** Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space, let $(W, \|\cdot\|_W)$ be an $\mathbb{R}$-Banach space, let $a \in \mathbb{R}$, $b \in (a, \infty)$, $F: [a, b] \to V$, $\phi \in C^1([a,b] \times V, W)$, $\phi_{1,0} : [a,b] \times V \to W$, $\phi_{0,1} : [a,b] \times V \to L(V,W)$ satisfy for every $t \in [a,b]$, $x \in V$ that $\phi_{1,0}(t,x) = (\frac{\partial}{\partial x}) \phi(t,x)$, $\phi_{0,1}(t,x) = (\frac{\partial}{\partial t}) \phi(t,x)$, and let $f : [a, b] \to V$ be a strongly measurable function which satisfies for every $t \in [a,b]$ that $\int_a^b \|f(s)\|_V \ ds < \infty$ and $F(t) - F(a) = \int_a^t f(s) \ ds$. Then

(i) it holds that $[a,b] \ni s \mapsto [\phi_{1,0}(s,F(s)) + \phi_{0,1}(s,F(s))f(s)] \in W$ is strongly measurable,

(ii) it holds that $\int_a^b \left\| \phi_{1,0}(s,F(s)) + \phi_{0,1}(s,F(s))f(s) \right\|_W \ ds < \infty$, and

(iii) it holds for every $t \in [a,b]$ that
\[
\phi(t,F(t)) - \phi(a,F(a)) = \int_a^t \left[ \phi_{1,0}(s,F(s)) + \phi_{0,1}(s,F(s))f(s) \right] \ ds. \tag{22}
\]
Proof of Corollary 2.8. Throughout this proof let $\Phi \in C^1(\mathbb{R} \times V, W)$ be a function which satisfies for every $t \in [a, b]$, $x \in V$ that $\Phi(t, x) = \phi(t, x)$. Note that Lemma 2.7 (with $V = \mathbb{R} \times V$, $W = W$, $a = a$, $b = b$, $\phi = \Phi$, $F = ([a, b] \ni s \mapsto (s, F(s))) \in \mathbb{R} \times V$), $f = ([a, b] \ni s \mapsto (1, f(s))) \in \mathbb{R} \times V$ in the notation of Lemma 2.7 establishes items (i)–(iii). The proof of Corollary 2.8 is thus completed.

Lemma 2.9. Let $(V, \|\cdot\|_V)$ be an $\mathbb{R}$-Banach space, let $a \in \mathbb{R}$, $b \in (a, \infty)$, $t_0 \in [a, b]$, $(f_n)_{n \in N} \subseteq C^1([a, b], V)$, and assume that $(f_n(t_0))_{n \in N} \subseteq V$ and $((f_n)')_{n \in N} \subseteq C([a, b], V)$ are convergent. Then there exists $F \in C^1([a, b], V)$ such that for every $t \in [a, b]$ it holds that

$$\lim_{n \to \infty} \sup \left(\|f_n(t) - F(t)\|_V + \|(f_n)')(t) - F'(t)\|_V\right) = 0.$$ (23)

Proof of Lemma 2.9. Throughout this proof let $F, g : [a, b] \to V$ be the functions which satisfy for every $t \in [a, b]$ that $\limsup_{n \to \infty} \sup_{s \in [0, T]} \|(f_n)'(s) - g(s)\|_V = 0$, $F(t_0) = \lim_{n \to \infty} f_n(t_0)$, and

$$F(t) = F(t_0) + \int_{t_0}^t g(s) \, ds.$$ (24)

Observe that Lemma 2.4 (with $V = V$, $a = a$, $b = b$, $F = f_n$ for $n \in N$ in the notation of Lemma 2.4) shows that for every $n \in N$, $t \in [a, b]$ it holds that

$$f_n(t) = f_n(t_0) + \int_{t_0}^t (f_n)'(s) \, ds.$$ (25)

The assumption that $((f_n)')_{n \in N} \subseteq C([a, b], V)$ converges ensures that

$$\sup_{n \in N} \sup_{s \in [a, b]} \|(f_n)'(s)\|_V < \infty.$$ (26)

The dominated convergence theorem therefore proves that for every $t \in [a, b]$ it holds that

$$\lim_{n \to \infty} \sup \left(\int_{t_0}^t g(s) \, ds - \int_{t_0}^t (f_n)'(s) \, ds\right) = 0.$$ (27)

This and (25) imply that for every $t \in [a, b]$ it holds that

$$\lim_{n \to \infty} f_n(t) = F(t_0) + \int_{t_0}^t g(s) \, ds.$$ (28)

Equation (24) hence assures for every $t \in [a, b]$ that

$$F(t) = \lim_{n \to \infty} f_n(t).$$ (29)

The fact that $g \in C([a, b], V)$, (24), and Lemma 2.3 (with $V = V$, $a = a$, $b = b$, $f = g$, $F = F$ in the notation of Lemma 2.3) establish that for every $t \in [a, b]$ it holds that $F \in C^1([a, b], V)$ and $F'(t) = g(t)$. The proof of Lemma 2.9 is thus completed.

Proposition 2.10. Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space, let $t_0 \in \mathbb{R}$, $t \in (t_0, \infty)$, $\phi \in C^1(V, V)$, $F : [t_0, t] \to V$, $F : \{(u, r) \in [t_0, t]^2 : u \leq r\} \times V \to V$, $F : \{(u, r) \in [t_0, t]^2 : u \leq r\} \times V \to V$, $\Phi : \{(u, r) \in [t_0, t]^2 : u \leq r\} \times V \to L(V)$, let $f : [t_0, t] \to V$ be a strongly measurable function, assume that $([t_0, t] \ni x \ni \Phi_{u, t}(x) \in V) \subseteq C^1([t_0, t] \times V, W)$, assume for every $x \in V$, $t_1 \in (t_0, t)$ that $([t_0, t_1] \ni u \ni \Phi_{u, t_1}(x) \in V) \subseteq C^1([t_0, t_1], V)$, $([t_0, t_1] \ni u \ni \Phi_{u, t_1}(x) \in V) \subseteq C^1([t_0, t_1], V)$, $\{(u, r) \in (t_0, t)^2 : u < r \} \ni (s, \tau) \ni \Phi_{s, \tau}(x) \in V \subseteq C^1((t_0, t)^2 : u < r \}, V)$, assume for every $t_1 \in [t_0, t], t_2 \in [t_1, t], t_3 \in [t_2, t], x \in V$
that \( \int_0^t \|f(u)\|_V \, du < \infty \), \( F(t_1) = F(t_0) + \int_{t_0}^{t_1} f(u) \, du \), \( \Phi_{t_1,t_1}(x) = x \), \( \Phi_{t_1,t_2}(x) = \Phi_{t_2,t_3}(\Phi_{t_1,t_2}(x)) \), assume for every \( t_1 \in (t_0, t) \), \( t_2 \in (t_1, t) \), \( x \in V \) that \( \Phi_{t_1,t_2}(x) = \frac{\partial}{\partial t} \Phi_{t_1,t_2}(x) \), and assume for every \( t_1 \in [t_0, t] \), \( x \in V \) that \( \Phi_{t_1,t_1}(x) = \frac{\partial}{\partial t} \Phi_{t_1,t_1}(x) \).

Then

(i) it holds that \( [t_0, t] \ni s \mapsto \phi'(\Phi_{s,t}(F(s)))\Phi^*_{s,t}(F(s))[\hat{\Phi}_{s,s}(F(s)) - f(s)] \in V \) is strongly measurable,

(ii) it holds that \( \int_{t_0}^t \|\phi'(\Phi_{s,t}(F(s)))\Phi^*_{s,t}(F(s))[\hat{\Phi}_{s,s}(F(s)) - f(s)]\|_V \, ds < \infty \), and

(iii) it holds that

\[
\phi(\Phi_{t_0,t}(F(t_0))) - \phi(F(t)) = \int_{t_0}^t \phi'(\Phi_{s,t}(F(s)))\Phi^*_{s,t}(F(s))[\hat{\Phi}_{s,s}(F(s)) - f(s)] \, ds.
\]

Proof of Proposition 2.10. Throughout this proof let

\[
\Phi = (\Phi_{t_1,t_2}(x))_{(t_1,t_2) \in [t_0,t]^2 : u \leq r} \times V : \{ (u, r) \in [t_0, t]^2 : u \leq r \} \times V \to V
\]

be a function which satisfies for every \( t_1 \in (t_0, t) \), \( t_2 \in [t_0, t_2] \), \( x \in V \) that \( \Phi_{t_1,t_2}(x) = \frac{\partial}{\partial t} \Phi_{t_1,t_2}(x) \) and let \( \varphi : [t_0, t] \times V \to V \) be the function which satisfies for every \( s \in [t_0, t], v \in V \) that \( \varphi(s, v) = \Phi_{s,t}(v) \). Note that the assumption that \( ([t_0, t] \times V \ni (s, x) \mapsto \Phi_{s,t}(x) \in V) \in \mathcal{C}^1([t_0, t] \times V, V) \) shows that for every \( \tau \in [t_0, t] \) it holds that \( \varphi|_{[\tau, t] \times V} \in \mathcal{C}^1([\tau, t] \times V, V) \). Corollary 2.8 (with \( V = V, W = V, a = t, b = t \), \( \phi = \varphi|_{[\tau, t] \times V} \), \( F = F|_{[\tau, t]} \), \( f = f|_{[\tau, t]} \) for \( \tau \in [t_0, t] \) in the notation of Corollary 2.8) therefore implies

(a) that \( [t_0, t] \ni s \mapsto [\Phi^*_{s,t}(F(s))f(s) + \hat{\Phi}_{s,t}(F(s))] \in V \) is strongly measurable,

(b) that \( \int_{t_0}^t \|\Phi^*_{s,t}(F(s))f(s) + \hat{\Phi}_{s,t}(F(s))\|_V \, ds < \infty \), and

(c) that for every \( \tau \in [t_0, t] \) it holds that

\[
\Phi_{t_1,t}(F(t)) - \Phi_{\tau,t}(F(\tau)) = F(t) - \Phi_{\tau,t}(F(\tau))
\]

\[
= \int_{\tau}^t \left[ \Phi^*_{s,t}(F(s))f(s) + \hat{\Phi}_{s,t}(F(s)) \right] \, ds.
\]

Lemma 2.7 (with \( V = V, W = V, a = t_0, b = t, \phi = \varphi, F = ([t_0, t] \ni s \mapsto \Phi_{s,t}(F(s)) \in V), f = ([t_0, t] \ni s \mapsto \Phi^*_{s,t}(F(s))f(s) + \hat{\Phi}_{s,t}(F(s)) \in V) \) in the notation of Lemma 2.7) hence shows

(A) that \( [t_0, t] \ni s \mapsto \phi'(\Phi_{s,t}(F(s)))\Phi^*_{s,t}(F(s))f(s) + \hat{\Phi}_{s,t}(F(s)) \in V \) is strongly measurable,

(B) that \( \int_{t_0}^t \|\phi'(\Phi_{s,t}(F(s)))\Phi^*_{s,t}(F(s))f(s) + \hat{\Phi}_{s,t}(F(s))\|_V \, ds < \infty \), and

(C) that

\[
\phi(F(t)) - \phi(\Phi_{t_0,t}(F(t_0))) = \phi(\Phi_{t_1,t}(F(t))) - \phi(\Phi_{t_0,t}(F(t_0)))
\]

\[
= \int_{t_0}^t \phi'(\Phi_{s,t}(F(s)))\Phi^*_{s,t}(F(s))f(s) + \hat{\Phi}_{s,t}(F(s)) \, ds.
\]

Next observe that the assumption that \( ([t_0, t] \times V \ni (s, x) \mapsto \Phi_{s,t}(x) \in V) \in \mathcal{C}^1([t_0, t] \times V, V) \) and the chain rule ensure that for every \( u \in (t_0, t), s \in (t_0, u], x \in V \) it holds that

\[
\Phi_{s,t}(x) = \frac{\partial}{\partial s} \Phi_{s,t}(x) = \frac{\partial}{\partial s} \Phi_{u,t}(\Phi_{s,u}(x)) = \Phi_{u,t}(\Phi_{s,u}(x)) \hat{\Phi}_{s,u}(x).
\]

Moreover, note that the assumption that for every \( x \in V \) it holds that \( \{(u, r) \in (t_0, t)^2 : u < r\} \ni (s, \tau) \mapsto \Phi_{s,\tau}(x) \in V \) is continuously differentiable implies that
for every $s \in (t_0, t)$, $n \in [2, \infty) \cap \mathbb{N}$, $x \in V$ it holds that

$$\frac{\partial}{\partial s} \Phi_{s-(s-t_0)/n,s}(x) = (1 - 1/n) \Phi_{s-(s-t_0)/n,s}(x) + \Phi_{s-(s-t_0)/n,s}(x).$$ (35)

Combining the fact that for every $\varepsilon \in (0, (t-t_0)/6)$, $n \in [2, \infty) \cap \mathbb{N}$, $x \in V$ it holds that $[t_0 + \varepsilon, t - \varepsilon] \ni s \mapsto \Phi_{s-(s-t_0)/n,s}(x) \in V$ is continuously differentiable, Lemma 2.9 (with $V = V$, $a = t_0 + \varepsilon$, $b = t - \varepsilon$, $t_0 = (t_0 + \varepsilon)/2$), $f_n = ([t_0 + \varepsilon, t - \varepsilon] \ni s \mapsto \Phi_{s-(s-t_0)/n,s}(x) \in V$) for $\varepsilon \in (0, (t-t_0)/6)$, $n \in [2, \infty) \cap \mathbb{N}$, $x \in V$ in the notation of Lemma 2.9), and the assumptions that $\forall x \in V$, $t_1 \in (t_0, t)$: $([t_0, t_1] \ni u \mapsto \Phi_{u,t_1}^{x}(V) \in \mathcal{C}([t_0, t_1], V)$ therefore proves that for every $s \in (t_0, t)$, $x \in V$ it holds that

$$\frac{\partial}{\partial s} \Phi_{s,s}(x) = \Phi_{s,s}(x) + \Phi_{s,s}(x).$$ (36)

Hence, we obtain that for every $s \in (t_0, t)$, $x \in V$ it holds that

$$\Phi_{s,s}(x) + \Phi_{s,s}(x) = 0.$$ (37)

This and (34) imply that for every $s \in (t_0, t)$, $x \in V$ it holds that

$$\Phi_{s,t}(x) = \Phi_{s,t}(x) \Phi_{s,s}(x) = \Phi_{s,t}(x) \Phi_{s,s}(x) = -\Phi_{s,t}(x) \Phi_{s,s}(x).$$ (38)

Combining this with items (A)–(C) completes the proof of Proposition 2.10.

3. Continuity of solutions to initial value problems. In this section we prove in Corollary 3.8 joint continuity of the solution to a Banach space valued ODE with respect to initial value, initial time, and current time. More precisely, we first apply Lemma 3.1 to prove a local existence and uniqueness result for initial value problems in Lemma 3.2. Then we combine Lemma 3.2, Lemma 3.3, Corollary 3.4, and Lemmas 3.5–3.7 to establish Corollary 3.8.

Lemma 3.1. Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space and let $a \in \mathbb{R}$, $b \in [a, \infty)$, $s \in [a, b]$, $f \in \mathcal{C}([a,b] \times V, V)$, $X, Y \in \mathcal{C}([a, b], V)$ satisfy for every $t \in [a, b]$, $x \in V$ that $X_t - \int_a^t f(\tau, X_\tau) \, d\tau = Y_t - \int_a^t f(\tau, Y_\tau) \, d\tau$ and

$$\inf_{r \in (0, \infty)} \sup_{\tau \in [a,b]} \sup_{y \in V, \|y-x\|_V \leq r} \sup_{z \in V \setminus \{y\}, \|z-x\|_V \leq r} \frac{\|f(\tau, y) - f(\tau, z)\|_V}{\|y-z\|_V} < \infty.$$ (39)

Then it holds for every $t \in [a, b]$ that $X_t = Y_t$.

Proof of Lemma 3.1. Throughout this proof let $L_{x,r} \in [0, \infty)$, $x \in V$, $r \in (0, \infty)$, be the extended real numbers which satisfy for every $r \in (0, \infty)$, $x \in V$ that

$$L_{x,r} = \sup_{\tau \in [a,b]} \sup_{y \in V, \|y-x\|_V \leq r} \sup_{z \in V \setminus \{y\}, \|z-x\|_V \leq r} \frac{\|f(\tau, y) - f(\tau, z)\|_V}{\|y-z\|_V},$$ (40)

let $\alpha = \sup\{\{a\} \cup \{u \in [a, s]: X_u \neq Y_u\}\}$, and let $\beta = \inf\{\{b\} \cup \{u \in [s, b]: X_u \neq Y_u\}\}$. Observe that the hypothesis that $X: [a, b] \to V$ and $Y: [a, b] \to V$ are continuous functions ensures that there exists a function $\delta = (\delta_{a,x})_{x \in [a, b] \times (0, \infty)}: [a, b] \times (0, \infty) \to (0, \infty)$ such that for every $u \in [a, b]$, $\varepsilon \in (0, \infty)$, $t \in [u-\delta_{u,\varepsilon}, u+\delta_{u,\varepsilon}]$, it holds that

$$\|X_t - X_u\|_V < \varepsilon \quad \text{and} \quad \|Y_t - Y_u\|_V < \varepsilon.$$ (41)

This implies that for every $u \in [a, b]$ with $X_u = Y_u$ there exists $\varepsilon \in (0, \infty)$ with $L_{X_u, \varepsilon} < \infty$ such that for every $t \in [u - \min\{\delta_{u,\varepsilon}, 1/(1+2L_{X_u, \varepsilon})\}, u + \min\{\delta_{u,\varepsilon}, 1/(1+2L_{X_u, \varepsilon})\}$, $u + \min\{\delta_{u,\varepsilon}, 1/(1+2L_{X_u, \varepsilon})\}$.
\[ \|X_t - Y_t\|_V = \|(X_t - Y_t) - (X_u - Y_u)\|_V \]

\[ = \left\| \left[ \int_s^t f(\tau, X_\tau) \, d\tau - \int_s^u f(\tau, X_\tau) \, d\tau \right] - \left[ \int_s^u f(\tau, X_\tau) \, d\tau - \int_s^t f(\tau, Y_\tau) \, d\tau \right] \right\|_V \]

\[ = \left\| \int_u^{\max(u,t)} f(\tau, X_\tau) \, d\tau - \int_u^{\max(u,t)} f(\tau, Y_\tau) \, d\tau \right\|_V \]

\[ \leq \int_{\min(u,t)}^{\max(u,t)} \|f(\tau, X_\tau) - f(\tau, Y_\tau)\|_V \, d\tau \leq L_{X,u,\varepsilon} \int_{\min(u,t)}^{\max(u,t)} \|X_\tau - Y_\tau\|_V \, d\tau \]

\[ \leq L_{X,u,\varepsilon} \|t - u\| \left[ \sup_{\tau \in [\min(u,t), \max(u,t)]} \|X_\tau - Y_\tau\|_V \right] \]

\[ \leq L_{X,u,\varepsilon} \|t - u\| \left[ \sup_{\tau \in [u - \min(\delta_{u,\varepsilon}, 1/(1 + 2L_{X,u,\varepsilon})), u + \min(\delta_{u,\varepsilon}, 1/(1 + 2L_{X,u,\varepsilon}))]} \|X_\tau - Y_\tau\|_V \right] \]

\[ \leq \frac{L_{X,u,\varepsilon}}{1 + 2L_{X,u,\varepsilon}} \left[ \sup_{\tau \in [u - \min(\delta_{u,\varepsilon}, 1/(1 + 2L_{X,u,\varepsilon})), u + \min(\delta_{u,\varepsilon}, 1/(1 + 2L_{X,u,\varepsilon}))]} \|X_\tau - Y_\tau\|_V \right] \]

\[ \leq \frac{1}{2} \left[ \sup_{\tau \in [u - \min(\delta_{u,\varepsilon}, 1/(1 + 2L_{X,u,\varepsilon})), u + \min(\delta_{u,\varepsilon}, 1/(1 + 2L_{X,u,\varepsilon}))]} \|X_\tau - Y_\tau\|_V \right]. \]

Hence, we obtain that for every \( u \in [a, b] \) with \( X_u = Y_u \) there exists \( \varepsilon \in (0, \infty) \) with \( L_{X,u,\varepsilon} < \infty \) such that

\[ \left[ \sup_{\tau \in [u - \min(\delta_{u,\varepsilon}, 1/(1 + 2L_{X,u,\varepsilon})), u + \min(\delta_{u,\varepsilon}, 1/(1 + 2L_{X,u,\varepsilon}))]} \|X_\tau - Y_\tau\|_V \right] \]

\[ \leq \frac{1}{2} \left[ \sup_{\tau \in [u - \min(\delta_{u,\varepsilon}, 1/(1 + 2L_{X,u,\varepsilon})), u + \min(\delta_{u,\varepsilon}, 1/(1 + 2L_{X,u,\varepsilon}))]} \|X_\tau - Y_\tau\|_V \right]. \]

This shows that for every \( u \in [a, b] \) with \( X_u = Y_u \) there exists \( \varepsilon \in (0, \infty) \) with \( L_{X,u,\varepsilon} < \infty \) such that

\[ \left[ \sup_{\tau \in [u - \min(\delta_{u,\varepsilon}, 1/(1 + 2L_{X,u,\varepsilon})), u + \min(\delta_{u,\varepsilon}, 1/(1 + 2L_{X,u,\varepsilon}))]} \|X_\tau - Y_\tau\|_V \right] = 0. \]

Therefore, we obtain for every \( u \in [0, T] \) with \( X_u = Y_u \) that there exists \( \Delta \in (0, \infty) \) such that for every \( t \in [u - \Delta, u + \Delta] \cap [a, b] \) it holds that

\[ X_t = Y_t. \]

Moreover, observe that the fact that \( X \) and \( Y \) are continuous ensures that \( X_\alpha = Y_\alpha \) and \( X_\beta = Y_\beta \). Combining this with (45) demonstrates that \( \alpha = a \) and \( \beta = b \). The proof of Lemma 3.1 is thus completed.

**Lemma 3.2.** Let \((V, \|\cdot\|_V)\) be a nontrivial \( \mathbb{R} \)-Banach space and let \( R, h, \varepsilon \in (0, \infty), \) \( s_0 \in \mathbb{R}, \) \( L, M, \delta \in [0, \infty), \) \( x_0 \in V, f \in C(\mathbb{R} \times V, V) \) satisfy for every \( x \in V \) that

\[ \inf_{\tau \in [0, \infty)} \sup_{\tau \in [s_0 - h, s_0 + h]} \sup_{y \in V, \|x - y\|_V \leq r} \sup_{z \in V \setminus \{y\}, \|x - z\|_V \leq r} \frac{\|f(\tau, y) - f(\tau, z)\|_V}{\|y - z\|_V} < \infty, \]
L = \sup_{\tau \in [s_0-h,s_0+h]} \sup_{y \in V, ||x_0-y||_V \leq R+\varepsilon} \frac{||f(\tau, y) - f(\tau, z)||_V}{||y - z||_V}, \quad (47)

M = \sup_{\tau \in [s_0-h,s_0+h]} \sup_{y \in V, ||x_0-y||_V \leq R+\varepsilon} ||f(\tau, y)||_V, \quad (48)

and \( \delta = \min \{ \varepsilon/(2M+1), 1/(4L+1), h \} \).

Then

(i) it holds for every \( s \in [s_0 - \delta, s_0 + \delta] \), \( x \in \{ v \in V : ||x_0 - v||_V \leq R \} \) that there exists a unique continuous function \( X^x_{s, t} \in \mathbb{R} \):

\[
X^x_{s, t} = x + \int_s^t f(\tau, X^x_{s, \tau}) \, d\tau
\]  \( (49) \)

and

(ii) it holds that

\[
\sup_{s, t \in [s_0 - \delta, s_0 + \delta]} \sup_{x \in V, ||x_0 - x||_V \leq R} ||X^x_{s, t} - x_0||_V \leq R + \varepsilon. \quad (50)
\]

Proof of Lemma 3.2. Throughout this proof let \( A \) be the set given by

\( A = \{ \psi \in C([s_0 - \delta, s_0 + \delta], V) : \sup_{t \in [s_0 - \delta, s_0 + \delta]} ||\psi(t) - x_0||_V \leq R + \varepsilon \}. \quad (51) \)

Note that for every \( s, t \in [s_0 - \delta, s_0 + \delta] \), \( x \in \{ v \in V : ||x_0 - v||_V \leq R \} \), \( \psi \in A \) it holds that

\[
\left\| x + \int_s^t f(\tau, \psi(\tau)) \, d\tau - x_0 \right\|_V \leq ||x - x_0||_V + \int_{\min\{s, t \}}^{\max\{s, t \}} ||f(\tau, \psi(\tau))||_V \, d\tau
\]  \( (52) \)

\[ \leq R + M|t - s| \leq R + 2M\delta \leq R + (2M + 1)\delta \leq R + \varepsilon. \]

This ensures that there exist functions \( B_{s, x} : A \to A \), \( s \in [s_0 - \delta, s_0 + \delta] \), \( x \in \{ v \in V : ||x_0 - v||_V \leq R \} \), such that for every \( s, t \in [s_0 - \delta, s_0 + \delta] \), \( x \in \{ v \in V : ||x_0 - v||_V \leq R \} \), \( \psi_1, \psi_2 \in A \) it holds that

\[
(B_{s, x}(\psi))(t) = x + \int_s^t f(\tau, \psi(\tau)) \, d\tau.
\]  \( (53) \)

Next observe that \( (47) \) and \( (51) \) demonstrate that for every \( s, t \in [s_0 - \delta, s_0 + \delta] \), \( x \in \{ v \in V : ||x_0 - v||_V \leq R \} \), \( \psi_1, \psi_2 \in A \) it holds that

\[
\left\| (B_{s, x}(\psi_1))(t) - (B_{s, x}(\psi_2))(t) \right\|_V \leq \left\| \int_s^t [f(\tau, \psi_1(\tau)) - f(\tau, \psi_2(\tau))] \, d\tau \right\|_V
\]  \( (54) \)

\[ \leq \int_{\min\{s, t \}}^{\max\{s, t \}} ||\psi_1(\tau) - \psi_2(\tau)||_V \, d\tau
\]

\[ \leq L \int_{\min\{s, t \}}^{\max\{s, t \}} ||\psi_1(\tau) - \psi_2(\tau)||_V \, d\tau
\]

\[ \leq L(t - s)\left[ \sup_{\tau \in [s_0 - \delta, s_0 + \delta]} ||\psi_1(\tau) - \psi_2(\tau)||_V \right]
\]

\[ \leq 2L\delta \left[ \sup_{\tau \in [s_0 - \delta, s_0 + \delta]} ||\psi_1(\tau) - \psi_2(\tau)||_V \right]
\]

\[ \leq \frac{(4L + 1)\delta}{2} \left[ \sup_{\tau \in [s_0 - \delta, s_0 + \delta]} ||\psi_1(\tau) - \psi_2(\tau)||_V \right]
\]

\[ \leq \frac{1}{2} \left[ \sup_{\tau \in [s_0 - \delta, s_0 + \delta]} ||\psi_1(\tau) - \psi_2(\tau)||_V \right]. \]
This shows that for every \( s \in [s_0 - \delta, s_0 + \delta] \), \( x \in \{ v \in V : \|x_0 - v\|_V \leq R \} \), \( \psi_1, \psi_2 \in \mathcal{A} \) it holds that
\[
\sup_{t \in [s_0 - \delta, s_0 + \delta]} \|(B_{s,x}(\psi_1))(t) - (B_{s,x}(\psi_2))(t)\|_V \\
\leq \frac{1}{2} \left[ \sup_{t \in [s_0 - \delta, s_0 + \delta]} \|\psi_1(t) - \psi_2(t)\|_V \right].
\]

Banach’s fixed point theorem hence proves that there exist continuous functions \( X^x_{s,t} = (X^x_{s,t})_{t \in [s_0 - \delta, s_0 + \delta]} : [s_0 - \delta, s_0 + \delta] \to V \), \( s \in [s_0 - \delta, s_0 + \delta] \), \( x \in \{ v \in V : \|x_0 - v\|_V \leq R \} \), such that for every \( s, t \in [s_0 - \delta, s_0 + \delta] \), \( x \in \{ v \in V : \|x_0 - v\|_V \leq R \} \) it holds that
\[
X^x_{s,t} = x + \int_s^t f(\tau, X^x_{s,\tau}) \, d\tau
\]
and
\[
\|X^x_{s,t} - x_0\|_V \leq R + \varepsilon.
\]
Combining this and Lemma 3.1 (with \( V = V \), \( a = s_0 - \delta, b = s_0 + \delta, s = s \), \( f = ([s_0 - \delta, s_0 + \delta] \times V \ni (\tau, y) \mapsto f(\tau, y) \in V) \), \( X = ([s_0 - \delta, s_0 + \delta] \ni t \mapsto X^x_{s,t} \in V) \) for \( s \in [s_0 - \delta, s_0 + \delta] \), \( x \in \{ v \in V : \|x_0 - v\|_V \leq R \} \) in the notation of Lemma 3.1) completes the proof of Lemma 3.2.

Corollary 3.4 below is a modification of Lemma 3.2 for \( f \in \mathcal{C}([0, T] \times V, V) \) instead of \( f \in \mathcal{C}(\mathbb{R} \times V, V) \).

**Lemma 3.3.** Let \((V, \|\cdot\|_V)\) be a nontrivial \( \mathbb{R} \)-Banach space and let \( a \in \mathbb{R}, b \in (a, \infty), r \in (0, \infty), x \in V, f \in \mathcal{C}([a, b] \times V, V) \) satisfy that
\[
\sup_{\tau \in [a, b]} \sup_{y \in V, \|x-y\|_V \leq r} \sup_{z \in V \setminus \{y\}, \|x-z\|_V \leq r} \frac{\|f(\tau, y) - f(\tau, z)\|_V}{\|y-z\|_V} < \infty.
\]
Then
\[
\sup_{\tau \in [a, b]} \sup_{y \in V, \|x-y\|_V \leq r} \|f(\tau, y)\|_V < \infty.
\]

The proof of Lemma 3.3 is clear and therefore omitted.

**Corollary 3.4.** Let \((V, \|\cdot\|_V)\) be a nontrivial \( \mathbb{R} \)-Banach space, let \( T, R, \varepsilon \in (0, \infty), s_0 \in [0, T], x_0 \in V, f \in \mathcal{C}([0, T] \times V, V) \), for every \( x \in V, s \in [0, T] \) let \( X^x_{s,t} = (X^x_{s,t})_{t \in [s, T]} : [s, T] \to V \) be a continuous function which satisfies for every \( t \in [s, T] \) that \( X^x_{s,t} = x + \int_s^t f(\tau, X^x_{s,\tau}) \, d\tau \), assume that
\[
\sup_{\tau \in [0, T]} \sup_{y \in V, \|x_0-y\|_V \leq R+\varepsilon} \sup_{z \in V \setminus \{y\}, \|x_0-z\|_V \leq R+\varepsilon} \frac{\|f(\tau, y) - f(\tau, z)\|_V}{\|y-z\|_V} < \infty,
\]
and assume for every \( x \in V \) that
\[
\inf_{\tau \in (0, \infty)} \sup_{\tau \in [0, T]} \sup_{y \in V, \|x-y\|_V \leq r} \frac{\|f(\tau, y) - f(\tau, z)\|_V}{\|y-z\|_V} < \infty.
\]
Then
(i) there exists \( \delta \in (0, \infty) \) such that for every \( s \in [s_0 - \delta, s_0 + \delta] \cap [0, T] \), \( x \in \{ v \in V : \|x_0 - v\|_V \leq R \} \) there exists a unique continuous function \( Y^x_{s,t} = (Y^x_{s,t})_{t \in [s_0 - \delta, s_0 + \delta] \cap [0, T]} : [s_0 - \delta, s_0 + \delta] \cap [0, T] \to V \) such that for every \( t \in [s_0 - \delta, s_0 + \delta] \cap [0, T] \) it holds that
\[
Y^x_{s,t} = x + \int_s^t f(\tau, Y^x_{s,\tau}) \, d\tau,
\]
(ii) for every \( s, t \in [s_0 - \delta, s_0 + \delta] \cap [0, T], x \in \{ v \in V : \| x_0 - v \|_V \leq R \} \) with \( s \leq t \) it holds that \( X_{s,t}^x = Y_{s,t}^x \), and

(iii) for every \( s \in [s_0 - \delta, s_0 + \delta] \cap [0, T], x \in \{ v \in V : \| x_0 - v \|_V \leq R \} \) it holds that

\[
\sup_{t \in [s_0 - \delta, s_0 + \delta]} \| Y_{s,t}^x - x_0 \|_V \leq R + \varepsilon. \tag{63}
\]

**Proof of Corollary 3.4.** Throughout this proof let \( F \in \mathcal{C}(\mathbb{R} \times V, V) \) be the function which satisfies for every \( t \in \mathbb{R}, x \in V \) that

\[
F(t, x) = f(\min\{T, \max\{0, t\}\}, x), \tag{64}
\]

let \( L \in [0, \infty) \) be the real number given by

\[
L = \sup_{\tau \in [0, T]} \max_{y \in V, \| x_0 - y \|_V \leq R + \varepsilon} \sup_{z \in V \setminus \{y\}} \frac{\| F(\tau, y) - F(\tau, z) \|_V}{\| y - z \|_V}, \tag{65}
\]

let \( M \in [0, \infty) \) be the extended real number given by

\[
M = \sup_{\tau \in [0, T]} \max_{y \in V, \| x_0 - y \|_V \leq R + \varepsilon} \| F(\tau, y) \|_V, \tag{66}
\]

and let \( \delta \in [0, \infty) \) be the real number given by \( \delta = \min\{\varepsilon/(2M + 1), \varepsilon/(4L + 1), 1\} \). Note that Lemma 3.3 (with \( V = \mathbb{R} \), \( a = 0 \), \( b = T \), \( r = R + \varepsilon \), \( x = x_0 \), \( f = F \) in the notation of Lemma 3.3) proves that \( M < \infty \). This ensures that \( \delta \in (0, \infty) \). Combining this, the fact that \( M < \infty \), and item (i) of Lemma 3.2 (with \( V = \mathbb{R} \), \( R = R \), \( h = 1 \), \( \varepsilon = \varepsilon \), \( s_0 = s_0 \), \( L = L \), \( M = M \), \( \delta = \delta \), \( x_0 = x_0 \), \( f = F \) in the notation of item (i) of Lemma 3.2) establishes item (i). The fact that \( V \in [s_0 - \delta, s_0 + \delta] \cap [0, T], x \in \{ v \in V : \| x_0 - v \|_V \leq R \} : Y_{s,s}^x = X_{s,s}^x \) and Lemma 3.1 (with \( V = \mathbb{R} \), \( a = s \), \( b = \min\{s_0 + \delta, T\} \), \( f = (s, \min\{s_0 + \delta, T\}) \times V \ni (t, y) \rightarrow F(t, y) \in V \), \( X = (s, \min\{s_0 + \delta, T\}) \ni t \rightarrow X_{s,t}^x \in V \), \( Y = (s, \min\{s_0 + \delta, T\}) \ni t \rightarrow Y_{s,t}^x \in V \) for \( s \in [s_0 - \delta, s_0 + \delta] \cap [0, T], x \in \{ v \in V : \| x_0 - v \|_V \leq R \} \) in the notation of Lemma 3.1) hence show that item (ii) holds. In addition, item (ii) of Lemma 3.2 (with \( V = \mathbb{R} \), \( R = R \), \( h = 1 \), \( \varepsilon = \varepsilon \), \( s_0 = s_0 \), \( L = L \), \( M = M \), \( \delta = \delta \), \( x_0 = x_0 \), \( f = F \) in the notation of item (ii) of Lemma 3.2) establishes item (iii). This completes the proof of Corollary 3.4. \( \square \)

**Lemma 3.5.** Let \( (V, \| \cdot \|_V) \) be a nontrivial \( \mathbb{R} \)-Banach space, let \( T, R \in (0, \infty), s_0 \in [0, T], x_0 \in V, f \in \mathcal{C}([0, T] \times V, V), \) for every \( x \in V, s \in [0, T] \) let \( (X_{s,t}^x)_{y \in [s, T]} : [s, T] \rightarrow V \) be a continuous function which satisfies for every \( t \in [s, T] \)

\[
\inf_{r \in (0, \infty)} \sup_{\tau \in [0, T]} \max_{y \in V, \| x_0 - y \|_V \leq R + r} \sup_{z \in V \setminus \{y\}} \frac{\| f(\tau, y) - f(\tau, z) \|_V}{\| y - z \|_V} < \infty, \tag{67}
\]

and assume for every \( x \in V \) that

\[
\inf_{r \in (0, \infty)} \sup_{\tau \in [0, T]} \max_{y \in V, \| x_0 - y \|_V \leq r} \sup_{z \in V \setminus \{y\}, \| x_0 - z \|_V \leq r} \frac{\| f(\tau, y) - f(\tau, z) \|_V}{\| y - z \|_V} < \infty. \tag{68}
\]

Then there exists \( \delta \in (0, \infty) \) such that

\[
\{ (u, v) \in ([s_0 - \delta, s_0 + \delta] \cap [0, T])^2 : u \leq v \} \times \{ v \in V : \| x_0 - v \|_V \leq R \} \ni (s, t, x) \rightarrow X_{s,t}^x \in V \tag{69}
\]

is uniformly continuous.
Proof of Lemma 3.5. Throughout this proof let \( \varepsilon, L \in \mathbb{R} \) be real numbers which satisfy that

\[
L = \sup_{\tau \in [0, T]} \sup_{y \in V} \|x_0 - y\|_V \leq R + \varepsilon \sup_{z \in V \setminus \{y\}} \|x_0 - z\|_V \leq R + \varepsilon \frac{\|f(\tau, y) - f(\tau, z)\|_V}{\|y - z\|_V} \tag{70}
\]

and let \( M \in [0, \infty) \) be the extended real number given by

\[
M = \sup_{\tau \in [0, T]} \sup_{y \in V} \|x_0 - y\|_V \leq R + \varepsilon \|f(\tau, y)\|_V. \tag{71}
\]

Note that Lemma 3.3 (with \( V = V, a = 0, b = T, r = R + \varepsilon, x = x_0, f = f \) in the notation of Lemma 3.3) shows that \( M < \infty \). Next observe that Corollary 3.4 (with \( V = V, T = T, R = R, \varepsilon = \varepsilon, s_0 = s_0, x_0 = x_0, f = f, X_{s,t} = X_{s,t}^x \) for \( x \in V \), \( (s, t) \in [0, T] \) with \( s \leq t \) in the notation of Corollary 3.4) ensures that there exists \( \delta \in (0, \infty) \) such that for every \( s \in \left[s_0 - \delta, s_0 + \delta\right] \cap [0, T], t \in [s, \min\{s_0 + \delta, T\}] \), \( x \in \{v \in V : \|x_0 - v\|_V \leq R\} \) it holds that

\[
\|X_{s,t}^x - x_0\|_V \leq R + \varepsilon. \tag{72}
\]

Moreover, note that for every \( s, t, u \in \left[\max\{s_0 - \delta, 0\}, \min\{s_0 + \delta, T\}\right], x, y \in \{v \in V : \|x_0 - v\|_V \leq R\} \) with \( s, u \in [0, t] \) it holds that

\[
\|X_{s,t}^x - X_{u,t}^y\|_V \leq \|x - y\|_V + \left\| \int_s^t f(\tau, X_{s,\tau}^x) d\tau - \int_u^t f(\tau, X_{u,\tau}^y) d\tau \right\|_V \leq \|x - y\|_V + \max\left\{ \left\| \int_s^t f(\tau, X_{s,\tau}^x) d\tau \right\|_V, \left\| \int_u^t f(\tau, X_{u,\tau}^y) d\tau \right\|_V \right\} \tag{73}
\]

Combining this with (72) proves that for every \( s, t, u \in \left[\max\{s_0 - \delta, 0\}, \min\{s_0 + \delta, T\}\right], x, y \in \{v \in V : \|x_0 - v\|_V \leq R\} \) with \( s, u \in [0, t] \) it holds that

\[
\|X_{s,t}^x - X_{u,t}^y\|_V \leq \|x - y\|_V + M|u - s| + L \int_{\max\{s,u\}}^t \|X_{s,T}^x - X_{u,T}^y\|_V d\tau. \tag{74}
\]

The fact that \( M < \infty \) and Gronwall’s lemma therefore imply that for every \( s, t, u \in \left[\max\{s_0 - \delta, 0\}, \min\{s_0 + \delta, T\}\right], x, y \in \{v \in V : \|x_0 - v\|_V \leq R\} \) with \( s, u \in [0, t] \) it holds that

\[
\|X_{s,t}^x - X_{u,t}^y\|_V \leq (\|x - y\|_V + M|u - s|)e^{L|t-u|}. \tag{75}
\]

In addition, note that (72) shows that for every \( s, t, \tau \in \left[\max\{s_0 - \delta, 0\}, \min\{s_0 + \delta, T\}\right], x \in \{v \in V : \|x_0 - v\|_V \leq R\} \) with \( s \leq \min\{t, \tau\} \) it holds that

\[
\|X_{s,t}^x - X_{s,\tau}^x\|_V \leq \int_{\min\{\tau,t\}}^{\max\{\tau,t\}} \|f(\tau, X_{s,\tau}^x)\|_V d\tau \leq M|t - \tau|. \tag{76}
\]
Combining this with (75) assures that for every \( s, t, u, \tau \in [\max\{s_0 - \delta, 0\}, \min\{s_0 + \delta, T\}] \); \( x, y \in \{v \in V: \|x_0 - v\|_V \leq R\} \) with \( s \leq t, u \leq \tau \leq t \) it holds that
\[
\|X_{s,t}^x - X_{u,\tau}^y\|_V \leq \|X_{s,t}^x - X_{u,t}^0\|_V + \|X_{u,t}^0 - X_{u,\tau}^y\|_V \\[3mm] \\
\leq (\|x - y\|_V + M|u - s|)e^{LT} + M|t - \tau|.
\] (77)

The fact that \( M < \infty \) establishes (69). The proof of Lemma 3.5 is thus completed.

\( \blacksquare \)

Observe that for every \( T \in (0, \infty) \) and for every nontrivial \( \mathbb{R} \)-Banach space \((V, \|\cdot\|_V)\) it holds that \( C^{0,1}([0, T] \times V, V) \) is the set of all continuous functions \( f: [0, T] \times V \to V \) which satisfy that for every \( t \in [0, T] \) it holds that \( V \ni x \mapsto f(t, x) \in V \) is Fréchet differentiable and that \( [0, T] \times V \ni (t, x) \mapsto \left( \frac{\partial f}{\partial t}\right)(t, x) \in L(V) \) is continuous.

**Lemma 3.6.** Let \((V, \|\cdot\|_V)\) be a nontrivial \( \mathbb{R} \)-Banach space and let \( T \in (0, \infty) \). Then
\[
\inf_{r \in (0, \infty)} \sup_{\tau \in [0, T]} \sup_{y \in V, \|x_0 - y\|_V \leq r, x_0 \in V} \frac{\|f(\tau, y) - f(\tau, z)\|_V}{\|y - z\|_V} < \infty.
\] (78)

**Proof of Lemma 3.6.** Throughout this proof let \( f_{0,1}: [0, T] \times V \to L(V) \) be the function which satisfies that for every \( t \in [0, T] \), \( x \in V \) that \( f_{0,1}(t, x) = \left( \frac{\partial f}{\partial t}\right)(t, x) \). Note that the assumption that \( f \in C^{0,1}([0, T] \times V, V) \) implies that there exists a function \( \delta: [0, T] \to (0, \infty) \) such that for every \( x \in V \), \( s, t \in [0, T] \) with \( \max\{|s - t|, \|x - x_0\|_V\} < \delta \), it holds that
\[
\|f_{0,1}(s, x) - f_{0,1}(t, x_0)\|_{L(V)} < \frac{1}{2}.
\] (79)

Moreover, observe that the fact that \([0, T]\) is compact ensures that there exist \( n \in \mathbb{N}, t_1, \ldots, t_n \in [0, T] \) such that
\[
0 = t_1 < \ldots < t_n = T \quad \text{and} \quad [0, T] = \bigcup_{i=1}^n \{r \in [0, T]: |r - t_i| < \delta_i\}.
\] (80)

This and (79) demonstrate that there exist \( n \in \mathbb{N}, t_1, \ldots, t_n \in [a, b] \) such that for every \( x \in V \), \( t \in [0, T] \) with \( \|x - x_0\|_V < \min\{\delta_1, \ldots, \delta_n\} \) it holds that
\[
\|f_{0,1}(t, x) - f_{0,1}(t, x_0)\|_V \\
\leq \min_{i \in [0, n] \cap \mathbb{N}} \|f_{0,1}(t, x) - f_{0,1}(t_i, x_0)\|_V + \|f_{0,1}(t_i, x_0) - f_{0,1}(t, x_0)\|_V \\
< \frac{1}{2} + \frac{1}{2} = 1.
\] (81)

Hence, we obtain that for every \( t \in [0, T], y, z \in V \) with \( \max\{|y - x_0\|_V, \|z - x_0\|_V\} < \min\{\delta_1, \ldots, \delta_n\} \) it holds that
\[
\|f(t, y) - f(t, z)\|_V = \left\| \int_0^1 f_{0,1}(t, z + s(y - z))(y - z) \, ds \right\|_V \\
\leq \int_0^1 \|f_{0,1}(t, z + s(y - z))\|_{L(V)} \|y - z\|_V \, ds \\
\leq \|f_{0,1}(t, x_0)\|_V \|y - z\|_V \\
+ \int_0^1 \|f_{0,1}(t, z + s(y - z)) - f_{0,1}(t, x_0)\|_{L(V)} \|y - z\|_V \, ds \\
< (\|f_{0,1}(t, x_0)\|_V + 1) \|y - z\|_V < \infty.
\] (82)
The proof of Lemma 3.6 is thus completed.

**Lemma 3.7.** Let \( (V, \| \cdot \|_V) \) be a nontrivial \( \mathbb{R} \)-Banach space, let \( T \in (0, \infty) \), \( f \in \mathcal{C}^{0,1}(0, T] \times V, V) \), and for every \( x \in V \), \( s \in [0, T] \) let \( X^x_{s,t} = (X^x_{s,t})_{t \in [s, T]} : [s, T] \to V \) be a continuous function which satisfies for every \( t \in \mathbb{R}^+ \) that \( X^x_{s,t} = x + \int_s^t f(\tau, X^x_{s,\tau}) \, d\tau \). Then it holds that for every \( x \in V \), \( t_1 \in [0, T] \), \( t_2 \in [t_1, T] \), \( t_3 \in [t_2, T] \) that \( X^x_{t_2,t_3} = X^x_{t_1,t_3} \).

**Proof of Lemma 3.7.** Throughout this proof let \( f_{0,1} : [0, T] \times V \to L(V) \) be the function which satisfies for every \( t \in [0, T] \), \( x \in V \) that \( f_{0,1}(t, x) = (\frac{d}{dt} f)(t, x) \).

Observe that for every \( x \in V \), \( t_1 \in [0, T] \), \( t_2 \in [t_1, T] \), \( t_3 \in [t_2, T] \) it holds that
\[
X^x_{t_2,t_3} = X^x_{t_1,t_3} + \int_{t_2}^{t_3} f(\tau, X^x_{t_1,\tau}) \, d\tau.
\]

This, the assumption that \( \forall x \in V, s \in [0, T] : ([s, T] \ni t \mapsto X^x_{s,t} \in V) \in \mathcal{C}([s, T], V) \), and the assumption that \( f \in \mathcal{C}^{0,1}([0, T] \times V, V) \) imply that for every \( x \in V \), \( t_1 \in [0, T] \), \( t_2 \in [t_1, T] \), \( t_3 \in [t_2, T] \) it holds that
\[
\| X^x_{t_2,t_3} - X^x_{t_1,t_3} \|_V = \left\| \int_{t_2}^{t_3} f(\tau, X^x_{t_1,\tau}) \, d\tau + X^x_{t_1,t_2} - X^x_{t_1,t_3} \right\|_V
\]
\[
= \left\| \int_{t_2}^{t_3} f(\tau, X^x_{t_1,\tau}) \, d\tau \right\|_V \leq \int_{t_2}^{t_3} \left\| f(\tau, X^x_{t_1,\tau}) \right\|_V \, d\tau
\]
\[
= \int_{t_2}^{t_3} \left\| f_{0,1}(\tau, X^x_{t_1,\tau}) - f(\tau, X^x_{t_1,\tau}) \right\|_V \, d\tau
\]
\[
\leq \sup_{r \in [0, 1]} \left\| f_{0,1}(t, X^x_{t_1,\tau}) + (X^x_{t_1,\tau} - X^x_{t_1,\tau}) r \right\|_{L(V)} \right\| X^x_{t_2,t_3} - X^x_{t_1,t_3} \|_V \, d\tau
\]
\[
\leq \left( \int_{t_2}^{t_3} \right) \left( \sup_{r \in [0, 1]} \left\| f_{0,1}(t, X^x_{t_1,\tau}) + (X^x_{t_2,t_3} - X^x_{t_1,t_3}) r \right\|_{L(V)} \right) \| X^x_{t_2,t_3} - X^x_{t_1,t_3} \|_V \, d\tau
\]
\[
\leq \left( \int_{t_2}^{t_3} \right) \left( \sup_{r \in [0, 1]} \left\| f_{0,1}(t, X^x_{t_1,\tau}) + (X^x_{t_2,t_3} - X^x_{t_1,t_3}) r \right\|_{L(V)} \right) \| X^x_{t_2,t_3} - X^x_{t_1,t_3} \|_V \, d\tau < \infty.
\]

Gronwall’s lemma hence shows for every \( x \in V \), \( t_1 \in [0, T] \), \( t_2 \in [t_1, T] \), \( t_3 \in [t_2, T] \) that \( X^x_{t_1,t_3} = X^x_{t_2,t_3} \). The proof of Lemma 3.7 is thus completed.

**Corollary 3.8.** Let \( (V, \| \cdot \|_V) \) be a nontrivial \( \mathbb{R} \)-Banach space, let \( T \in (0, \infty) \), \( f \in \mathcal{C}^{0,1}(0, T] \times V, V) \), and for every \( x \in V \), \( s \in [0, T] \) let \( X^x_{s,t} = (X^x_{s,t})_{t \in [s, T]} : [s, T] \to V \) be a continuous function which satisfies for every \( t \in \mathbb{R}^+ \) that \( X^x_{s,t} = x + \int_s^t f(\tau, X^x_{s,\tau}) \, d\tau \).

Then it holds that \( \{(u, v) \in [0, T]^2 : u \leq v \} \times V \ni (s, t, x) \mapsto X^x_{s,t} \in V \) is a continuous function.

**Proof of Corollary 3.8.** Throughout this proof we denote by \( \angle_T \subseteq [0, T]^2 \) the set given by \( \angle_T = \{(s, t) \in [0, T]^2 : s \leq t \} \), let \((s_0, t_0, x_0) \in \angle_T \times V \), and let \( \varepsilon \in (0, \infty) \).

Note that Lemma 3.6 (with \( V = V, T = T, x_0 = x, f = f \) for \( x \in V \) in the notation
of Lemma 3.6) shows that there exists a function $r: V \to (0, \infty)$ such that for every $x \in V$ it holds that

$$\sup_{x \in [0,1]} \sup_{y \in V} \|x - y\| \|x\| \leq 2r_x \sup_{x \in V} \|x - z\| \|y - z\| < \infty. \quad (85)$$

Lemma 3.5 (with $V = V$, $T = T$, $R = r_x$, $s_0 = s$, $x_0 = x$, $f = f$, $X_{s,t}^x = X_{s,t}^x$) for $(s,t) \in \mathcal{T}$, $x \in V$ in the notation of Lemma 3.5 hence ensures that there exists a function $\delta = (\delta_x, s) \in [0,1] \times V: [0, T] \times [0, T] \times V \to (0, \infty)$ such that for every $s \in [0, T]$, $x \in V$ it holds that

$$\left(\mathcal{T} \cap \{(s - \delta_x, s + \delta_x, x) \mid (s, t) \in [0, T] \times V \} \right) \ni (u, t, y) \mapsto X_{u,t}^y \in V \quad (86)$$

is a uniformly continuous function. Furthermore, note that the fact that $\forall x \in V, s \in [0, T]: \{(s, T) \supseteq t \mapsto X_{s,t}^x \in V \} \subset \mathcal{C}(s, T, V)$ ensures that there exists a function $\Delta: [0, T] \to (0, \infty)$ such that for every $t, \tau \in [s_0, T]$ with $|t - \tau| < \Delta$, it holds that

$$\|X_{s_0,t}^x - X_{s_0,\tau}^x\|_V \leq \frac{1}{2} r_{X_{s_0,t}^x}. \quad (87)$$

This implies that there exist non-empty intervals $I_{s,t} \subseteq [0, T], s \in [0, T], x \in V$, such that for every $s \in (0, T)$, $x \in V$ it holds that $I_{s,t}$ is a uniformly continuous function. Furthermore, note that the function $\forall x \in V, s \in [0, T]: \{s \mapsto X_{s,t}^x \in V \} \subset \mathcal{C}(s, T, V)$ ensures that there exists a function $\Delta: [0, T] \to (0, \infty)$ such that for every $t, \tau \in [s_0, T]$ with $|t - \tau| < \Delta$, it holds that

$$\|X_{s_0,t}^x - X_{s_0,\tau}^x\|_V \leq \frac{1}{2} r_{X_{s_0,t}^x}. \quad (88)$$

This demonstrates that there exist real numbers $\tau_j \in (s_0, t_0), j \in [1, n] \cap \mathbb{N}$, such that for every $j \in [1, n] \cap \mathbb{N}$ it holds that

$$\tau_j \in I_{s_j, s_{j+1}} \cap (I_{s_j, I_{s_j, s_{j+1}}}). \quad (89)$$

Moreover, note that (87) (with $t = s_n, \tau = \tau_n$) ensures that $\|X_{s_0,t_0}^x - X_{s_0,\tau}^x\|_V \leq \frac{1}{2} r_{X_{s_0,t_0}^x}$. Combining this with (86) (with $s = s_n, x = X_{s_0,t_0}^x$) and Lemma 3.7 (with $V = V$, $T = T$, $f = f$, $X_{s,t}^x = X_{s,t}^x$) for $(s, t) \in \mathcal{T}$, $x \in V$ in the notation of Lemma 3.7) proves that there exists $\varepsilon_n \in (0, \min \{1, \delta, x_{s_0,t_0}^x, \frac{1}{2} r_{X_{s_0,t_0}^x} \})$ such that for every $s \in [0, \tau_n], t \in [\tau_n, T], x \in V$ with $\max \{\|X_{s_0,t_0}^x - X_{s_0,\tau}^x\|_V, |t_0 - t|\} < \varepsilon_n$ it holds that $\|X_{s_0,t_0}^x - X_{s_0,\tau}^x\|_V \leq r_{X_{s_0,t_0}^x}$ and

$$\|X_{s_0,t_0}^x - X_{s_0,t}^x\|_V = \|X_{s_0,t_0}^x - X_{s_0,\tau}^x\|_V = \frac{1}{2} r_{X_{s_0,t_0}^x} \quad (90)$$

Furthermore, observe that (87) (with $t = s_j, \tau = \tau_j$ for $j \in [1, n-1] \cap \mathbb{N}$) ensures that $\|X_{s_0,t_j}^x - X_{s_0,\tau}^x\|_V \leq \frac{1}{2} r_{X_{s_0,t_j}^x}$. This, (86) (with $s = s_j, x = X_{s_0,t_j}^x$) for $j \in [1, n-1] \cap \mathbb{N}$, and Lemma 3.7 (with $V = V, T = T, f = f, X_{s,t}^x = X_{s,t}^x$ for $(s, t) \in \mathcal{T}$, $x \in V$ in the notation of Lemma 3.7) ensure that there exists $\varepsilon_j \in (0, \min \{1, \delta, x_{s_0,t}^x, \frac{1}{2} r_{X_{s_0,t}^x} \})$, $j \in [1, n-1] \cap \mathbb{N}$, such that for every $j \in [1, n-1] \cap \mathbb{N}$, $s \in [0, \tau_j], x \in V$ with $\|X_{s_0,t}^x - X_{s,t}^x\|_V < \varepsilon_j$ it holds that $\|X_{s_0,t}^x - X_{s,t}^x\|_V \leq r_{X_{s_0,t}^x} \quad (91)$

In addition, note that (86) (with $s = s_0, x = x_0$) shows that there exists $\varepsilon_0 \in (0, \min \{1, \delta_{s_0,x_0}, \frac{1}{2} r_{x_0} \})$ such that for every $s \in [0, \tau], x \in V$ with $\max \{|x_0 -
Lemma 4.1. \( x \| V, |s_0 - s| < \varepsilon_0 \) it holds that
\[
\| X_{s_0}^{x_0} - X_{s,t}^{x} \|_V < \varepsilon_1.
\] Combining this with (90) and (91) implies that for every \( s \in [0,\tau_1], t \in [s,T], x \in V \) with \( \max\{\|x_0 - x\|_V, |s_0 - s|\} < \varepsilon_0 \) and \( |t_0 - t| < \min\{\varepsilon_n, |t_0 - \tau_n|\} \) it holds that
\[
\| X_{s_0,t_0}^x - X_{s,t}^{x} \|_V < \varepsilon.
\] Hence, we obtain that there exists \( \delta \in (0,\infty) \) such that for every \( (s,t,x) \in \mathcal{L}_T \times V \) with \( \max\{|s - s_0|, |t - t_0|, \|x - x_0\|_V\} < \delta \) it holds that
\[
\| X_{s_0,t_0}^x - X_{s,t}^{x} \|_V < \varepsilon.
\] The proof of Corollary 3.8 is thus completed.

4. Continuous differentiability of solutions to initial value problems. In this section we prove in Lemma 4.8 (cf., e.g., Driver [4, Theorem 19.13]) differentiability properties of solutions to initial value problems. In order to do so, we recall a few elementary auxiliary results in Lemmas 4.1–4.3 (cf., e.g., Driver [4, Theorem 19.14]), Lemma 4.4 (cf., e.g., Driver [4, Theorem 19.7]), and Lemmas 4.5–4.6. Then we combine them to establish continuous differentiability of the solution to the considered initial value problem with respect to the initial data in Lemma 4.7. In addition, we establish in Lemma 4.8 continuous differentiability of the solution to the considered initial value problem with respect to the initial time as well as the current time.

Lemma 4.1. Let \((U, \|\cdot\|_U), (V, \|\cdot\|_V),\) and \((W, \|\cdot\|_W)\) be \( \mathbb{R} \)-Banach spaces and let \( T \in (0,\infty), \mathcal{L}_T = \{(s,t) \in [0,T]^2; s \leq t\}, f \in \mathcal{C}([0,T] \times U, L(V,W)), y \in \mathcal{C}(\mathcal{L}_T, U), h \in \mathcal{C}(\mathcal{L}_T, V). \) Then \( \mathcal{L}_T \ni (s,t) \mapsto \int_s^t f(\tau, y(s,\tau))h(s,\tau)\,d\tau \in W \) is continuous.

Proof of Lemma 4.1. Throughout this proof let \( X : \mathcal{L}_T \to W \) be the function which satisfies for every \((s,t) \in \mathcal{L}_T\) that \( X_{s,t} = \int_s^t f(\tau, y(s,\tau))h(s,\tau)\,d\tau \). Observe that for every \( s \in [0,T], t, u \in [s,T] \) with \( t \leq u \) it holds that
\[
\| X_{s,u} - X_{s,t} \|_W = \left\| \int_t^u f(\tau, y(s,\tau))h(s,\tau)\,d\tau \right\|_W \\
\leq |t - u| \left[ \sup_{(r,\tau) \in \mathcal{L}_T} \| f(\tau, y(r,\tau))h(r,\tau) \|_W \right].
\] Hence, we obtain that for every \( s \in [0,T], t, u \in [s,T] \) it holds that
\[
\| X_{s,t} - X_{s,u} \|_W \leq |t - u| \left[ \sup_{(r,\tau) \in \mathcal{L}_T} \| f(\tau, y(r,\tau))h(r,\tau) \|_W \right].
\] In addition, note that for every \( s, u, t \in [0,T] \) with \( s, u \in [0,t] \) it holds that
\[
\| X_{s,t} - X_{s,u,t} \|_W = \left\| \int_s^t f(\tau, y(s,\tau))h(s,\tau)\,d\tau - \int_u^t f(\tau, y(u,\tau))h(u,\tau)\,d\tau \right\|_W \\
\leq \left\| \int_{\max\{s,u\}}^t f(\tau, y(\max\{s,u\},\tau))h(\max\{s,u\},\tau)\,d\tau \right\|_W \\
- \left\| \int_{\min\{s,u\}}^t f(\tau, y(\min\{s,u\},\tau))h(\min\{s,u\},\tau)\,d\tau \right\|_W \\
= \left\| \int_{\max\{s,u\}}^t f(\tau, y(\max\{s,u\},\tau))h(\max\{s,u\},\tau)\,d\tau \right\|_W \\
- \left\| \int_{\min\{s,u\}}^t f(\tau, y(\min\{s,u\},\tau))h(\min\{s,u\},\tau)\,d\tau \right\|_W.
\]
that for every \( n \in \mathbb{N} \), \( t_1, x_1), \ldots, (t_n, x_n) \in [0, T] \times C \) such that

\[
[0, T] \times C \subseteq \bigcup_{i=1}^{n} \{(\tau, \xi) \in [0, T] \times V : |t_i - \tau| + \|x_i - \xi\|_V < \frac{1}{2} \delta_{t_i}, x_i\}. \tag{103}
\]

This and (101) show that there exist \( n \in \mathbb{N}, (t_1, x_1), \ldots, (t_n, x_n) \in [0, T] \times C \) such that for every \( s \in \mathbb{N}, t, \xi \in [0, T] \times V, h \in C(\mathbb{R}, \mathbb{R}) \) with \( \sup_{(u, \tau) \in \mathbb{R}} \|h(u, \tau)\|_V < \frac{1}{2} \delta_{t, x} \), it holds that

\[
\int_{0}^{t} \left( \left\| f(t, y(s, t))h(s, t) - f(t, y(s, t))h(s, t) \right\|_V d\tau \right) = \int_{0}^{t} \left( \left\| f(t, y(s, t))h(s, t) - f(t, y(s, t))h(s, t) \right\|_V d\tau \right) = 0.
\]

The dominated convergence theorem hence completes the proof of Lemma 4.1. \( \square \)
which satisfies for every $t$
holds for every $y,h$

\begin{equation}
\min_{i \in [1,n]} \max_{s,t} \delta_{i,s-t} \text{ it holds that }
\begin{align*}
\|f_{0,1}(t,y(s,t) + rh(s,t)) - f_{0,1}(t,y(s,t))\|_{L(V)} \\
\leq \min_{i \in [1,n]} \max_{s,t} \left[ \|f_{0,1}(t,y(s,t) + rh(s,t)) - f_{0,1}(t_i,x_i)\|_{L(V)} \\
+ \|f_{0,1}(t_i,x_i) - f_{0,1}(t,y(s,t))\|_{L(V)} \right]
\end{align*}
\end{equation}

(104)

The proof of Lemma 4.2 is thus completed.

Lemma 4.3. Let $(V, \| \cdot \|_V)$ be a nontrivial $\mathbb{R}$-Banach space, let $T \in (0, \infty)$, $\angle_T = \{(s,t) \in [0,T]^2 \mid s \leq t\}$, $f = (f(t,x))(t,x) \in C^{0,1}([0,T] \times V, V)$, $F: C(\angle_T, V) \to C(\angle_T, V)$ satisfy for every $(s,t) \in \angle_T$, $y \in C(\angle_T, V)$ that

\begin{equation}
(F(y))(s,t) = \int_s^t f(\tau, y(s,\tau)) d\tau,
\end{equation}

and let $f_{0,1}: [0,T] \times V \to \mathbb{L}(V)$ be the function which satisfies for every $t \in [0,T]$, $x \in V$ that $f_{0,1}(t,x) = (\frac{\partial}{\partial x} f)(t,x)$. Then it holds for every $y, h \in C(\angle_T, V)$, $(s,t) \in \angle_T$ that $F \in C^1(C(\angle_T, V), C(\angle_T, V))$ and

\begin{equation}
(F'(y)h)(s,t) = \int_s^t f_{0,1}(\tau, y(s,\tau))h(s,\tau) d\tau.
\end{equation}

Proof of Lemma 4.3. Note that for every $y, h \in C(\angle_T, V)$, $(s,t) \in \angle_T$ it holds that

\begin{equation}
f(\tau, y(s,\tau) + h(s,\tau)) - f(\tau, y(s,\tau)) = \int_0^1 f_{0,1}(\tau, y(s,\tau) + rh(s,\tau))h(s,\tau) dr.
\end{equation}

(106)

This ensures that for every $y, h \in C(\angle_T, V)$, $(s,t) \in \angle_T$ it holds that

\begin{align*}
(F(y + h))(s,t) - (F(y))(s,t) - \int_s^t f_{0,1}(\tau, y(s,\tau))h(s,\tau) d\tau
\end{align*}

\begin{align*}
= \int_s^t \left[ f(\tau, y(s,\tau) + h(s,\tau)) - f(\tau, y(s,\tau)) - f_{0,1}(\tau, y(s,\tau))h(s,\tau) \right] d\tau
\end{align*}

\begin{align*}
= \int_s^t \left[ \int_0^1 \left[ f_{0,1}(\tau, y(s,\tau) + rh(s,\tau)) - f_{0,1}(\tau, y(s,\tau)) \right] h(s,\tau) dr \right] d\tau.
\end{align*}

(107)

Hence, we obtain for every $y, h \in C(\angle_T, V)$ that

\begin{align*}
\sup_{(s,t) \in \angle_T} \left| (F(y + h))(s,t) - (F(y))(s,t) - \int_s^t f_{0,1}(\tau, y(s,\tau))h(s,\tau) d\tau \right|_V
\end{align*}

\begin{align*}
\leq \left[ \sup_{(s,t) \in \angle_T} \|h(s,\tau)\|_V \right]
\cdot \left[ \sup_{(s,t) \in \angle_T} \left[ \int_s^t \left( \int_0^1 \|f_{0,1}(\tau, y(s,\tau) + rh(s,\tau)) - f_{0,1}(\tau, y(s,\tau))\|_{L(V)} dr \right) d\tau \right] \right]
\leq T \left[ \sup_{(s,t) \in \angle_T} \|h(s,\tau)\|_V \right]
\cdot \sup_{r \in [0,\infty]} \|f_{0,1}(\tau, y(s,\tau) + rh(s,\tau)) - f_{0,1}(\tau, y(s,\tau))\|_{L(V)}.
\end{align*}

(108)

Moreover, observe that Lemma 4.2 (with $V = V$, $\varepsilon = \varepsilon$, $T = T$, $y = y$, $f = f$ for $y \in C(\angle_T, V)$, $\varepsilon \in (0, \infty)$ in the notation of Lemma 4.2) shows that for every
y ∈ \mathcal{C}(\mathcal{L}_T, V), \varepsilon \in (0, \infty) \) there exists δ ∈ (0, \infty) such that for every h ∈ \mathcal{C}(\mathcal{L}_T, V)

\sup_{r \in [0, 1]} \sup_{(s, \tau) \in \mathcal{L}_T} \|h(u, \tau)\|_V < \delta \n
\text{it holds that}

\sup_{r \in [0, 1]} \sup_{(s, \tau) \in \mathcal{L}_T} \|f_0(\tau, y(s, \tau) + rh(s, \tau)) - f_0(\tau, y(s, \tau))\|_{L(V)} < \varepsilon. \quad (109)

Combining this with (108) implies that for every y ∈ \mathcal{C}(\mathcal{L}_T, V)

\limsup_{(C(\mathcal{L}_T, V) \setminus \{0\}) \ni h \to 0} \frac{\sup_{(s, \tau) \in \mathcal{L}_T} \|(F(y + h))(s, t) - (F(y))(s, t) - f_0'(\tau, y(s, \tau))(h(s, \tau))\|_V}{\sup_{(s, \tau) \in \mathcal{L}_T} \|h(s, \tau)\|_V} = 0. \quad (110)

Lemma 4.1 (with U = V, V = V, W = V, T = T, f = ([0, T] × V) \ni (t, x) \mapsto f_0(t, x) \in L(V)), y = y, h = h in the notation of Lemma 4.1 therefore proves that f is Fréchet differentiable and that for every y, h ∈ \mathcal{C}(\mathcal{L}_T, V), (s, t) ∈ \mathcal{L}_T it holds that

\left(F'(y)h\right)(s, t) = \int_s^t f_0(\tau, y(s, \tau)) h(s, \tau) \, d\tau. \quad (111)

This ensures that for every y, g, h ∈ \mathcal{C}(\mathcal{L}_T, V) it holds that

\sup_{(s, t) \in \mathcal{L}_T} \|(F'(y + h)g)(s, t) - (F'(y)g)(s, t)\|_V

\leq \sup_{(s, t) \in \mathcal{L}_T} \int_s^t \|f_0(\tau, y(s, \tau) + h(s, \tau)) g(s, \tau) - f_0(\tau, y(s, \tau)) g(s, \tau)\|_V \, d\tau

\leq T \sup_{(s, \tau) \in \mathcal{L}_T} \|g(s, \tau)\|_V \sup_{(s, \tau) \in \mathcal{L}_T} \|f_0(\tau, y(s, \tau) + h(s, \tau)) - f_0(\tau, y(s, \tau))\|_{L(V)}. \quad (112)

Combining this with (109) shows that for every y ∈ \mathcal{C}(\mathcal{L}_T, V)

\limsup_{(C(\mathcal{L}_T, V) \setminus \{0\}) \ni h \to 0} \|F'(y + h) - F'(y)\|_{L(C(\mathcal{L}_T, V), C(\mathcal{L}_T, V))} = 0. \quad (113)

The proof of Lemma 4.3 is thus completed. \hfill \Box

Lemma 4.4. Let \((X, \|\cdot\|_X), (Y, \|\cdot\|_Y), \text{ and } (Z, \|\cdot\|_Z)\) be nontrivial \(\mathbb{R}\)-Banach spaces, let \(f \in C^1(Y, Z), g \in C(X, Y)\), assume that \(f \circ g \in C^1(X, Z)\), and assume for every \(x \in X\) that \(f'(g(x))\) is bijective and that \([f'(g(x))]^{-1} \in L(Z, Y)\). Then

(i) it holds that \(g \in C^1(X, Y)\)

(ii) it holds for every \(x \in X\) that \(g'(x) = [f'(g(x))]^{-1}(f \circ g)'(x)\).

Proof of Lemma 4.4. Throughout this proof let \(o_1, o_2 : X \times X \to Z\) be the functions which satisfy for every \(x, h \in X\) that

\(o_1(x, h) = f(g(x + h)) - f(g(x)) - f'(g(x))(g(x + h) - g(x)) \quad (114)\)

and

\(o_2(x, h) = f(g(x + h)) - f(g(x)) - (f \circ g)'(x)h. \quad (115)\)

Observe that (114) and (115) imply for every \(x, h \in X\) that

\(g(x + h) - g(x) = [f'(g(x))]^{-1}(f \circ g)'(x)h + o_2(x, h) - o_1(x, h) \quad (116)\)

\(= [f'(g(x))]^{-1}(f \circ g)'(x)h + [f'(g(x))]^{-1}o_2(x, h) - [f'(g(x))]^{-1}o_1(x, h). \)
Moreover, note that the fact that \( g \) is continuous and the assumption that \( f \) is differentiable assure that for every \( x \in X \) there exists a function \( w: X \to [0, \infty) \) such that for every \( h \in X \) it holds that \( \limsup_{u \to 0} w(u) = 0 \) and

\[
\|o_1(x, h)\|_Z = w(h) \cdot \|g(x + h) - g(x)\|_Y.
\]

This shows that for every \( x \in X \) there exists \( \delta \in (0, \infty) \) such that for every \( h \in X \) with \( \|h\|_X < \delta \) it holds that

\[
\|(f'(g(x)))^{-1}||_{L(Z,Y)}\|o_1(x, h)\|_Z \leq \frac{1}{2}\|g(x + h) - g(x)\|_Y.
\]

Equation (116) hence proves that for every \( x \in X \) there exists \( \delta \in (0, \infty) \) such that for every \( h \in X \) with \( \|h\|_X < \delta \) it holds that

\[
\|g(x + h) - g(x)\|_Y \leq \|(f'(g(x)))^{-1}(f \circ g)'(x)\|_{L(X,Y)}\|h\|_X + \|(f'(g(x)))^{-1}\|_{L(Z,Y)}\|o_2(x, h)\|_Z + \frac{1}{2}\|g(x + h) - g(x)\|_Y.
\]

Therefore, we establish that for every \( x \in X \) there exists \( \delta \in (0, \infty) \) such that for every \( h \in X \) with \( \|h\|_X < \delta \) it holds that

\[
\frac{\|g(x+h)-g(x)\|_Y}{\|h\|_X} \leq \|(f'(g(x)))^{-1}(f \circ g)'(x)\|_{L(X,Y)}\|h\|_X + \|(f'(g(x)))^{-1}\|_{L(Z,Y)}\|o_2(x, h)\|_Z.
\]

Furthermore, note that the fact that

\[
\forall x \in X: \limsup_{(X \backslash \{0\}) \ni h \to 0} \|o_2(x, h)\|_Z = 0
\]

ensures that for every \( x \in X \) there exists \( \delta \in (0, \infty) \) such that for every \( h \in X \) with \( 0 < \|h\|_X < \delta \) it holds that

\[
\|(f'(g(x)))^{-1}\|_{L(Z,Y)}\|o_2(x, h)\|_Z \leq \frac{1}{2}.
\]

Equation (120) hence proves that for every \( x \in X \) there exists \( \delta \in (0, \infty) \) such that for every \( h \in X \) with \( 0 < \|h\|_X < \delta \) it holds that

\[
\frac{\|g(x+h)-g(x)\|_Y}{\|h\|_X} \leq 2\|(f'(g(x)))^{-1}(f \circ g)'(x)\|_{L(X,Y)} + \frac{2\|(f'(g(x)))^{-1}\|_{L(Z,Y)}\|o_2(x, h)\|_Z}{\|h\|_X} + 2\|(f'(g(x)))^{-1}(f \circ g)'(x)\|_{L(X,Y)} + 1.
\]

This shows that for every \( x \in X \) there exists \( \delta \in (0, \infty) \) such that for every \( h \in \{y \in X: g(x + y) \neq g(x)\} \) with \( 0 < \|h\|_X < \delta \) it holds that

\[
\frac{\|o_1(x, h)\|_Z}{\|g(x+h)-g(x)\|_Y} = \frac{\|o_1(x, h)\|_Z}{\|h\|_X} \cdot \frac{\|h\|_X}{\|g(x+h)-g(x)\|_Y} \geq \frac{1}{2\|(f'(g(x)))^{-1}(f \circ g)'(x)\|_{L(X,Y)} + 1}.
\]

Combining this with (117) implies that for every \( x \in X \) it holds that

\[
\limsup_{(X \backslash \{0\}) \ni h \to 0} \|o_1(x, h)\|_Z = 0.
\]

Next note that (116) implies that for every \( x \in X \) there exists \( \delta \in (0, \infty) \) such that for every \( h \in X \) with \( 0 < \|h\|_X < \delta \) it holds that

\[
\frac{\|g(x+h)-g(x)\|_Y}{\|h\|_X} \leq \|(f'(g(x)))^{-1}\|_{L(Z,Y)}\|o_2(x, h)\|_Z + \|o_1(x, h)\|_Z \leq \|(f'(g(x)))^{-1}\|_{L(Z,Y)}\|o_2(x, h)\|_Z + \frac{1}{2}\|o_1(x, h)\|_Z.
\]
Lemma 4.5. Let $(V,||\cdot||_V)$ be an $\mathbb{R}$-Banach space and let $T \in (0,\infty)$, $\mathcal{L}_T = \{(s,t) \in [0,T]^2 : s \leq t\}$, $\phi \in \mathcal{C}(\mathcal{L}_T, V)$, $A \in \mathcal{C}(\mathcal{L}_T, L(V))$. Then there exists a unique function $y \in \mathcal{C}(\mathcal{L}_T, V)$ such that for every $(s,t) \in \mathcal{L}_T$ it holds that

$$y(s,t) = \phi(s,t) + \int_s^t A(s,\tau)y(s,\tau)\,d\tau.$$  \hfill (127)

**Proof of Lemma 4.5.** Throughout this proof let $\mathcal{A} : \mathbb{R} \times [0,T] \to L(V)$ be a continuous function which satisfies for every $(s,\tau) \in \mathcal{L}_T$ that $A(s,\tau) = A(s,\tau)$, let $f_n : \mathcal{C}(\mathcal{L}_T, V) \to \mathcal{C}(\mathcal{L}_T, V)$, $n \in \mathbb{N}$, be the functions which satisfy for every $n \in \mathbb{N}$, $x \in \mathcal{C}(\mathcal{L}_T, V)$, $(s,t) \in \mathcal{L}_T$ that $(f_1(x))(s,t) = \phi(s,t) + \int_s^t A(s,\tau)x(s,\tau)\,d\tau$ (see Lemma 4.1) and let $\mathcal{N} \subseteq \mathbb{N}$ satisfy that

$$\mathcal{N} = \left\{ n \in \mathbb{N} : \left[ \sup_{(u,\tau) \in \mathcal{L}_T} \|A(u,\tau)\|_{L(V)} \right]^{n+1} \\sup_{(u,\tau) \in \mathcal{L}_T} \|x(u,\tau) - y(u,\tau)\|_V \right\}. \hfill (128)$$

Note that for every $x, y \in \mathcal{C}(\mathcal{L}_T, V)$, $(s,t) \in \mathcal{L}_T$ it holds that

$$(f_1(x))(s,t) - (f_1(y))(s,t) = \int_s^t A(s,\tau)[x(s,\tau) - y(s,\tau)]\,d\tau \hfill (129)$$

This implies that $1 \in \mathcal{N}$ and that $f_1$ is continuous. Moreover, observe that for every $n \in \mathcal{N}$, $x, y \in \mathcal{C}(\mathcal{L}_T, V)$, $(s,t) \in \mathcal{L}_T$ it holds that

$$\|f_{n+1}(x))(s,t) - (f_{n+1}(y))(s,t)\|_V \hfill (130)$$

This proof is thus completed.

This, (121), and (125) demonstrate that $g$ is differentiable and that item (ii) holds. The fact that $\{B \in L(Y,Z) : B$ is invertible $\} \ni C \mapsto C^{-1} \in \{B \in L(Z,Y) : B$ is invertible $\}$ is continuous (cf., e.g., Deitmar & Echterhoff [3, Lemma 2.1.5]) and the fact that $X \ni x \mapsto f'(g(x)) \in L(Y,Z)$ is continuous assure that $X \ni x \mapsto [f'(g(x))]^{-1} \in L(Z,Y)$ is continuous. Item (ii) hence establishes item (i). The proof of Lemma 4.4 is thus completed.
The fact that $1 \in N$ hence shows that $N = N$. This proves that for every $n \in N$, $x, y \in C(\mathcal{L}_T, V)$, $(s, t) \in \mathcal{L}_T$ it holds that

$$
\| (f_n(x))(s, t) - (f_n(y))(s, t) \|_V
\leq \frac{\sup_{(u, \tau) \in \mathcal{L}_T} \|A(u, \tau)\|_{L(V)} \|t-s\|}{n!} \left[ \sup_{(u, \tau) \in \mathcal{L}_T} \|x(u, \tau) - y(u, \tau)\|_V \right].
$$

(131)

In particular, for every $n \in N$, $x \in C(\mathcal{L}_T, V)$, $(s, t) \in \mathcal{L}_T$ it holds that

$$
\| (f_n(x))(s, t) - (f_{n+1}(x))(s, t) \|_V
\leq \frac{\sup_{(u, \tau) \in \mathcal{L}_T} \|A(u, \tau)\|_{L(V)} \|t-s\|}{n!} \left[ \sup_{(u, \tau) \in \mathcal{L}_T} \|x(u, \tau) - (f_1(x))(u, \tau)\|_V \right].
$$

(132)

Therefore, we establish that for every $m, n \in N$, $x, y \in C(\mathcal{L}_T, V)$ it holds that

$$
\sup_{(s, t) \in \mathcal{L}_T} \| (f_n(x))(s, t) - (f_{m+n}(y))(s, t) \|_V
\leq \sum_{k=0}^{m-1} \sup_{(s, t) \in \mathcal{L}_T} \| (f_{n+k}(x))(s, t) - (f_{n+k+1}(y))(s, t) \|_V
$$

$$
\leq \left[ \sup_{(u, \tau) \in \mathcal{L}_T} \|x(u, \tau) - (f_1(x))(u, \tau)\|_V \right] \sum_{k=0}^{m-1} \frac{\sup_{(u, \tau) \in \mathcal{L}_T} \|A(u, \tau)\|_{L(V)} \|t-s\|}{(n+k)!} \left[ \sup_{(u, \tau) \in \mathcal{L}_T} \|A(u, \tau)\|_{L(V)} T \right]^{n+k}
$$

$$
\leq \frac{\left[ \sup_{(u, \tau) \in \mathcal{L}_T} \|x(u, \tau) - (f_1(x))(u, \tau)\|_V \right] \sup_{(u, \tau) \in \mathcal{L}_T} \|A(u, \tau)\|_{L(V)} T}{n!} \left[ \sup_{(u, \tau) \in \mathcal{L}_T} \|A(u, \tau)\|_{L(V)} T \right]^{n+k}
$$

$$
\cdot \exp \left( \sup_{(u, \tau) \in \mathcal{L}_T} \|A(u, \tau)\|_{L(V)} T \right).
$$

(133)

This ensures that for every $x \in C(\mathcal{L}_T, V)$ the sequence $(f_n(x))_{n \in N} \subseteq C(\mathcal{L}_T, V)$ is Cauchy. The fact that $C(\mathcal{L}_T, V)$ is a Banach space hence shows that for every $x \in C(\mathcal{L}_T, V)$ there exists $z \in C(\mathcal{L}_T, V)$ such that

$$
\limsup_{n \to \infty} \left( \sup_{(s, t) \in \mathcal{L}_T} \| (f_n(x))(s, t) - z(s, t) \|_V \right) = 0.
$$

(134)

In the next step note that for every $x, z \in C(\mathcal{L}_T, V)$ with $\limsup_{n \to \infty} (\sup_{(s, t) \in \mathcal{L}_T} \| f_n(x)(s, t) - z(s, t) \|_V) = 0$ it holds that

$$
\sup_{(s, t) \in \mathcal{L}_T} \| (f_1(z))(s, t) - z(s, t) \|_V
\leq \limsup_{n \to \infty} \left( \sup_{(s, t) \in \mathcal{L}_T} \| (f_1(z))(s, t) - (f_1(f_n(x)))(s, t) \|_V \right)
\leq \limsup_{n \to \infty} \left( \sup_{(s, t) \in \mathcal{L}_T} \| (f_1(f_n(x)))(s, t) - z(s, t) \|_V \right)
\leq \limsup_{n \to \infty} \left( \sup_{(s, t) \in \mathcal{L}_T} \| (f_1(z))(s, t) - (f_1(f_n(x)))(s, t) \|_V \right)
\leq \limsup_{n \to \infty} \left( \sup_{(s, t) \in \mathcal{L}_T} \| (f_1(z))(s, t) - (f_1(f_n(x)))(s, t) \|_V \right) \leq \limsup_{n \to \infty} \left( \sup_{(s, t) \in \mathcal{L}_T} \| (f_1(z))(s, t) - (f_1(f_n(x)))(s, t) \|_V \right).$$

(135)

Combining this with (134) and the fact that $f_1 : C(\mathcal{L}_T, V) \to C(\mathcal{L}_T, V)$ is continuous demonstrates that there exists $z \in C(\mathcal{L}_T, V)$ such that

$$
f_1(z) = z.
$$

(136)
In addition, note that (131) implies that for every \( N \in \mathbb{N} \), \( x, y \in C(\mathbb{T}_t, V) \) with \( f_1(x) = x, f_1(y) = y \) it holds that
\[
\sum_{n=1}^{N} \sup_{(s,t) \in \mathbb{T}_t} \| x(s,t) - y(s,t) \|_V
= \sum_{n=1}^{N} \sup_{(s,t) \in \mathbb{T}_t} \| (f_n(x))_n(s,t) - (f_n(y))_n(s,t) \|_V
\leq \sum_{n=1}^{N} [\sup_{(u,\tau) \in \mathbb{T}_t} \| A(u,\tau) \|_{L(V)}(t-s)]^n
[\sup_{(u,\tau) \in \mathbb{T}_t} \| x(u,\tau) - y(u,\tau) \|_V]
\leq [\sup_{(u,\tau) \in \mathbb{T}_t} \| x(u,\tau) - y(u,\tau) \|_V] \exp(\sup_{(u,\tau) \in \mathbb{T}_t} \| A(u,\tau) \|_{L(V)}T) < \infty.
\]
(137)

This and (136) show that there exists a unique \( z \in C(\mathbb{T}_t, V) \) such that \( f_1(z) = z \). The proof of Lemma 4.5 is thus completed.

**Lemma 4.6.** Let \( (V, \| \cdot \|_V) \) be an \( \mathbb{R} \)-Banach space and let \( T \in (0,\infty) \), \( \mathbb{T}_t = \{ (s, t) \in [0,T]^2 : s \leq t \} \), \( \phi \in C(\mathbb{T}_t, V) \), \( A \in C(\mathbb{T}_t, L(V)) \), \( y \in C(\mathbb{T}_t, V) \) satisfy for every \((s,t) \in \mathbb{T}_t\) that
\[
y(s,t) = \phi(s,t) + \int_s^t A(s,\tau)y(s,\tau) d\tau.
\]
(138)

Then
\[
\sup_{(s,t) \in \mathbb{T}_t} \| y(s,t) \|_V \leq \left[ \sup_{(s,t) \in \mathbb{T}_t} \| \phi(s,t) \|_V \right] \exp \left( T \sup_{(s,t) \in \mathbb{T}_t} \| A(s,t) \|_{L(V)} \right). \tag{139}
\]

**Proof of Lemma 4.6.** Note that for every \((s,t) \in \mathbb{T}_t\) it holds that
\[
\| y(s,t) \|_V \leq \| \phi(s,t) \|_V + \int_s^t \| A(s,\tau)y(s,\tau) \|_V d\tau
\leq \| \phi(s,t) \|_V + \left[ \sup_{(u,v) \in \mathbb{T}_t} \| A(u,v) \|_{L(V)} \right] \int_s^t \| y(s,\tau) \|_V d\tau
\leq \sup_{(u,v) \in \mathbb{T}_t} \| \phi(u,v) \|_V + \left[ \sup_{(u,v) \in \mathbb{T}_t} \| A(u,v) \|_{L(V)} \right] \int_s^t \| y(s,\tau) \|_V d\tau < \infty.
\]
(140)

Gronwall’s lemma hence shows that for every \((s,t) \in \mathbb{T}_t\) it holds that
\[
\| y(s,t) \|_V \leq \left[ \sup_{(u,v) \in \mathbb{T}_t} \| \phi(u,v) \|_V \right] \exp \left( (t-s) \sup_{(u,v) \in \mathbb{T}_t} \| A(u,v) \|_{L(V)} \right). \tag{141}
\]

The proof of Lemma 4.6 is thus completed.

**Lemma 4.7.** Let \( (V, \| \cdot \|_V) \) be a nontrivial \( \mathbb{R} \)-Banach space, let \( T \in (0,\infty) \), \( \mathbb{T}_t = \{ (s, t) \in [0,T]^2 : s \leq t \} \), \( f \in C^{0,1}([0,T] \times V, V) \), let \( f_{0,1} : [0,T] \times V \rightarrow L(V) \) be the function which satisfies for every \( t \in [0,T] \), \( x \in V \) that \( f_{0,1}(t,x) = \left( \frac{\partial}{\partial \tau} f \right)(t,x) \), and for every \( x \in V \) let \( \{ X^x_{s,t} \}_{(s,t) \in \mathbb{T}_t} : \mathbb{T}_t \rightarrow V \) be a continuous function which satisfies for every \( (s,t) \in \mathbb{T}_t \) that \( X^x_{s,t} = x + \int_s^t f(\tau, X^x_{s,\tau}) \, d\tau \). Then
(i) it holds that \( \mathbb{T}_t \times V \ni (s,t,x) \mapsto X^x_{s,t} \in V \) is \( C^{0,0,1}(\mathbb{T}_t \times V, V) \) and
(ii) it holds for every \( x, y \in V \), \((s,t) \in \mathbb{T}_t\) that
\[
(\frac{\partial}{\partial x} X^x_{s,t}) y = y + \int_s^t f_{0,1}(\tau, X^x_{s,\tau})(\frac{\partial}{\partial x} X^x_{s,\tau}) y \, d\tau. \tag{142}
\]
Proof of Lemma 4.7. Throughout this proof let \( F: \mathcal{C}([t,T], V) \to \mathcal{C}([t,T], V) \) and \( G, H: V \to \mathcal{C}([t,T], V) \) be the functions which satisfy for every \( v \in V \), \( (s,t) \in \mathcal{T} \), \( z \in \mathcal{C}([t,T], V) \) that \( (F(z))(s,t) = \int_t^s f(\tau, z(\tau)) \, d\tau \), \( (G(v))(s,t) = X_{s,t}^v \), and \( (H(v))(s,t) = v \). Note that Lemma 4.3 (with \( V = V, T = T, f = f, F = (\mathcal{C}([t,T], V) \ni z \mapsto (\mathcal{T} \ni (s,t) \mapsto \int_0^s f(\tau, z(\tau)) \, d\tau) \in (\mathcal{C}([t,T], V)), f_{0,1} = f_{0,1} \) in the notation of Lemma 4.3) proves that for every \( y, h \in \mathcal{C}([t,T], V), (s,t) \in \mathcal{T} \) it holds that \( F \in C^1(\mathcal{C}([t,T], V), \mathcal{C}([t,T], V)) \) and
\[
(F'(y)h)(s,t) = h(s,t) - \int_s^t f_{0,1}(\tau, y(s,\tau))h(s,\tau) \, d\tau.
\] (143)
Therefore, we obtain that for every \( v \in V, h \in \mathcal{C}([t,T], V), (s,t) \in \mathcal{T} \) it holds that
\[
h(s,t) = (F'(G(v))h)(s,t) + \int_s^t f_{0,1}(\tau, G(v)(s,\tau))h(s,\tau) \, d\tau.
\] (144)
Combining this with Lemma 4.5 (with \( V = V, T = T, \phi = (\mathcal{T} \ni (s,t) \mapsto (F'(G(v))h)(s,t) \in V), A = (\mathcal{T} \ni (s,t) \mapsto f_{0,1}(\tau, G(v)(s,\tau)) \in L(V)) \) for every \( h \in \mathcal{C}([t,T], V), v \in V \) in the notation of Lemma 4.5) shows that for every \( v \in V \) it holds that \( F'(G(v)) \in L(\mathcal{C}([t,T], V), \mathcal{C}([t,T], V)) \) is invertible. In addition, Lemma 4.6 (with \( V = V, T = T, \phi = (\mathcal{T} \ni (s,t) \mapsto (F'(G(v))h)(s,t) \in V), A = (\mathcal{T} \ni (s,t) \mapsto f_{0,1}(\tau, G(v)(s,\tau)) \in L(V)) \) for every \( h \in \mathcal{C}([t,T], V), v \in V \) in the notation of Lemma 4.6) ensures that for every \( v \in V \) it holds that
\[
[F'(G(v))]^{-1} \in L(\mathcal{C}(\mathcal{T}, V), \mathcal{C}(\mathcal{T}, V)).
\] (145)
Moreover, observe that for every \( v \in V, (s,t) \in \mathcal{T} \) it holds that
\[
(F(G(v)))(s,t) = (G(v))(s,t) - \int_s^t f(\tau, (G(v))(s,\tau)) \, d\tau = X_{s,t}^v - \int_s^t f(\tau, X_{s,\tau}^v) \, d\tau = v = (H(v))(s,t).
\] (146)
Next we intend to prove that \( G \in C(V, \mathcal{C}([t,T], V)) \).
\] (147)
For this note that Corollary 3.8 (with \( V = V, T = T, f = f, X_{s,t}^x = X_{s,t}^x \) for \( (s,t) \in \mathcal{T}, x \in V \) in the notation of Corollary 3.8) shows that \( \mathcal{T} \times V \ni (s,t,x) \mapsto X_{s,t}^x \in V \) is continuous. This implies that there exists a function \( \delta: (0, \infty) \times \mathcal{T} \times V \to (0, \infty) \) such that for every \( \varepsilon \in (0, \infty), (s_0, t_0), (s, t) \in \mathcal{T}, v_0, v \in V \) with \( |s_0 - s| + |t_0 - t| + ||v_0 - v||_V < \delta_{s_0, t_0, v_0} \) it holds that
\[
\|X_{s_0,t_0}^{v_0} - X_{s,t}^{v}\|_V < \varepsilon/2.
\] (148)
The fact that for every \( v_0 \in V, \varepsilon \in (0, \infty) \) it holds that
\[
\mathcal{T} \subseteq \bigcup_{(s_0,t_0)\in\mathcal{T}} \{ (s,t) \in \mathcal{T} : |s_0 - s| + |t_0 - t| < \delta_{s_0,t_0,v_0}/2 \}
\] (149)
and the compactness of \( \mathcal{T} \) prove that for every \( v_0 \in V, \varepsilon \in (0, \infty) \) there exist \( N \in \mathbb{N}, (s_1, t_1), \ldots, (s_N, t_N) \in \mathcal{T} \) such that
\[
\mathcal{T} = \bigcup_{n=1}^N \{ (s,t) \in \mathcal{T} : |s_n - s| + |t_n - t| < \delta_{s_n,t_n,v_0}/2 \}.
\] (150)
Combining this with (148) demonstrates that for every \( v_0 \in V, \varepsilon \in (0, \infty) \) there exist \( N \in \mathbb{N}, (s_1, t_1), \ldots, (s_N, t_N) \in \mathcal{T} \) such that for every \( v \in V, (s,t) \in \mathcal{T} \) with
there exists a continuous function $C$ in Lemma 4.8 below asserts that there exists a continuous function $C$

The proof of Lemma 4.7 is thus completed.

Moreover, observe that (154) proves that for every $\Delta \in \mathbb{R}$, it holds that

$$H' \in C \big( \mathbb{R} \big) \quad \text{and} \quad (147) \text{ implies that } \sup_{x \in [a,b]} |f(x)| < \infty.$$ 

Equation (146) hence shows that $F \circ G$ is continuously differentiable. This, (145), (147), the facts that for every $v \in V$ it holds that $F$ is continuously differentiable and $F'(G(v)) \in L(C(\mathbb{R},V),\mathbb{R})$ is invertible, and Lemma 4.4 (with $X = V$, $Y = C(\mathbb{R},V)$, $Z = C(\mathbb{R},V)$, $f = F$, $g = G$ for $v \in V$ in the notation of Lemma 4.4) imply that $G$ is continuously differentiable and that for every $v \in V$ it holds that

$$G'(v) = [F'(G(v))]^{-1}H'(v).$$

Hence, we obtain that

$$(V \ni x \mapsto X^x \in C(\mathbb{R},V)) \in C^1(V,C(\mathbb{R},V)).$$

This implies that for every $x \in V$, $\varepsilon \in (0,\infty)$ there exists $\Delta \in (0,\infty)$ such that for every $s \in [a,b]$, $t \in [a,b]$, $y \in V$ with $|s - u| + |t - \tau| + \|x - y\| < \varepsilon$ it holds that

$$||X^x_{s,t} - X^u_{s,\tau}, V \leq ||X^x_{s,t} - X^y_{s,\tau}, V + ||X^x_{s,\tau} - X^y_{s,\tau}, V + \sup_{(v,r) \in \mathbb{R}^2} ||X^x_{v,r} - X^y_{v,r}, V < \varepsilon.$$ 

Moreover, observe that (154) proves that for every $x \in V$, $\varepsilon \in (0,\infty)$ there exists $\Delta \in (0,\infty)$ such that for every $s \in [a,b]$, $t \in [a,b]$, $y \in V$ with $|s - u| + |t - \tau| + \|x - y\| < \varepsilon$ it holds that

$$\frac{\partial}{\partial x} X^x_{s,t} - \frac{\partial}{\partial x} X^y_{s,\tau}, V \leq \frac{\partial}{\partial x} X^x_{s,t} - \frac{\partial}{\partial x} X^y_{s,\tau}, V + \sup_{(v,r) \in \mathbb{R}^2} ||X^x_{v,r} - X^y_{v,r}, V < \varepsilon.$$ 

This and (155) establish item (i). In addition, note that (152) and (153) ensure that for every $v \in V$, $(s,t) \in \mathbb{R}$ it holds that $(F'(G(v)))G'(v)(w)(s,t) = w$. Display (143) hence demonstrates that for every $(s,t) \in \mathbb{R}$, $v \in V$ it holds that

$$w = (F'(G(v)))G'(v)(w)(s,t)$$

$$= (G'(v)(w)(s,t) - \int_s^t f_{0,1}(\tau, G(v)(\tau))(G'(v)(w))(s,\tau) d\tau$$

$$= \frac{\partial}{\partial x} X^x_{s,t} w - \int_s^t f_{0,1}(\tau, X^y_{s,\tau})(\frac{\partial}{\partial x} X^y_{s,\tau} w) d\tau.$$ 

The proof of Lemma 4.7 is thus completed.

Observe that item (ii) in Lemma 4.8 below is equivalent to the statement that there exists a continuous function $C$ with $(r, u) \in (0, T) \times (0, u) \times V \to V$ which satisfies for every $t \in (0, T)$, $s \in [0, t]$, $x \in V$ that $C^x_{s,t} = \frac{\partial}{\partial x} X^x_{s,t}$, while item (iii) in Lemma 4.8 below asserts that there exists a continuous function $C$ with $(r, u) \in (0, T) \times (0, u) \times V \to V$ which satisfies for every $t \in (0, T)$, $s \in [0, t]$, $x \in V$ that $C^x_{s,t} = \frac{\partial}{\partial x} X^x_{s,t}$.

This concludes the proof of Lemma 4.4.
Lemma 4.8. Let $(V, \|\cdot\|_V)$ be a nontrivial $\mathbb{R}$-Banach space, let $T \in (0, \infty)$, $\mathcal{L}_T = \{(s, t) \in [0, T]^2 : s \leq t\}$, $f \in C^{0,1}(\mathcal{L}_T \times V, V)$, let $f_{0,1} : [0, T] \times V \to L(V)$ be the function which satisfies for every $t \in [0, T)$, $x \in V$ that $f_{0,1}(t, x) = (\frac{\partial}{\partial t} f)(t, x)$, and for every $x \in V$, $s \in [0, t]$ let $X_{s,t}^x = \langle X_s^x \rangle_{t \in [s, T]} : [s, T] \to V$ be a continuous function which satisfies for every $t \in [s, T]$ that $X_{s,t}^x = x + \int_s^t f(\tau, X_{s,\tau}^x) \, d\tau$. Then

(i) it holds for every $x \in V$, $t \in (0, T)$ that $((0, t] \ni s \mapsto X_{s,t}^x \in V) \in C^1((0, t], V)$,
(ii) it holds that $\{(r, u) \in [0, T]^2 : r \leq u\} \times V \ni (s, t, x) \mapsto \frac{\partial}{\partial t} X_{s,t}^x \in V$ is continuous,
(iii) it holds that there exists a unique continuous function $C : \{(r, u) \in [0, T]^2 : r \leq u\} \times V \to V$ which satisfies for every $t \in (0, T]$ that $C^x_{s,t} = \frac{\partial}{\partial t} X_{s,t}^x$,
(iv) it holds for every $x \in V$, $t \in (0, T)$, $s \in [0, t]$ that

$$\frac{\partial}{\partial s} X_{s,t}^x = -f(s, x) + \int_s^t f_{0,1}(\tau, X_{s,\tau}^x)(\frac{\partial}{\partial s} X_{s,\tau}^x) \, d\tau, \quad (158)$$

(v) it holds that $(\mathcal{L}_T \times V \ni (s, t, x) \mapsto X_{s,t}^x \in V) \in C^{0,0,1}(\mathcal{L}_T \times V, V)$,
(vi) it holds for every $x, y \in V$, $(s, t) \in \mathcal{L}_T$ that

$$\left(\frac{\partial}{\partial t} X_{s,t}^x\right)y = y + \int_s^t f_{0,1}(\tau, X_{s,\tau}^x)(\frac{\partial}{\partial t} X_{s,\tau}^x)y \, d\tau, \quad (159)$$

(vii) it holds for every $x \in V$, $t \in (0, T)$, $s \in [0, t]$ that

$$\frac{\partial}{\partial s} X_{s,t}^x = -(\frac{\partial}{\partial s} X_{s,t}^x)f(s, x), \quad (160)$$

(viii) it holds for every $x \in V$, $s \in [0, t]$ that $\{(s, t) \ni t \mapsto X_{s,t}^x \in V\} \in C^1([s, T], V)$,
(ix) it holds that $\{(r, u) \in [0, T]^2 : r \leq u\} \times V \ni (s, t, x) \mapsto \frac{\partial}{\partial t} X_{s,t}^x \in V$ is continuous,
(x) it holds that there exists a unique continuous function $D : \{(r, u) \in [0, T]^2 : r \leq u\} \times V \to V$ which satisfies for every $s \in [0, T)$, $t \in [s, T]$, $x \in V$ that $D^x_{s,t} = \frac{\partial}{\partial s} X_{s,t}^x$, and
(xi) it holds for every $x \in V$, $s \in [0, T)$, $t \in [s, T]$ that $\frac{\partial}{\partial t} X_{s,t}^x = f(t, X_{s,t}^x)$.

Proof of Lemma 4.8. Throughout this proof let $g : [-T, T]^3 \times V \to \mathbb{R}$ be a function which satisfies for every $s \in [0, T)$, $h \in [-s, T-s] \setminus \{0\}$, $\tau \in \max\{s, s+h\} \setminus T$, $x \in V$ with $X_{s+h,\tau}^x - X_{s,\tau}^x \neq 0$ that

$$g(s, h, \tau, x) = \frac{\|f(\tau, X_{s+h,\tau}^x) - f(\tau, X_{s,\tau}^x) - f_{0,1}(\tau, X_{s,\tau}^x)(X_{s+h,\tau}^x - X_{s,\tau}^x)\|_V}{\|X_{s+h,\tau}^x - X_{s,\tau}^x\|_V}, \quad (161)$$

which satisfies for every $s \in [0, T)$, $h \in [-s, T-s]$, $\tau \in \max\{s, s+h\} \setminus T$, $x \in V$ with $X_{s+h,\tau}^x - X_{s,\tau}^x = 0$ that $g(s, h, \tau, x) = 0$, and which satisfies for every $s, h \in [-T, T]$, $\tau \in [-T, T-s]$, $x \in V$ that $g(s, h, \tau, x) = 0$. Observe that Corollary 3.8 (with $V = V$, $T = T$, $f = f$, $X_{s,t}^x = X_{s,t}^x$ for $(s, t) \in \mathcal{L}_T$, $x \in V$ in the notation of Corollary 3.8) ensures that

$$(\mathcal{L}_T \times V \ni (s, t, x) \mapsto X_{s,t}^x \in V) \in C(\mathcal{L}_T \times V, V). \quad (162)$$

This and the assumption that $f \in C^{0,1}([0, T] \times V, V)$ show that for every $x \in V$ it holds that $(\mathcal{L}_T \ni (s, t) \mapsto f_{0,1}(s, X_{s,t}^x) \in L(V)) \in C(\mathcal{L}_T, L(V))$. Lemma 4.5
Moreover, the triangle inequality proves that for every $x \in V$ there exists a unique function $Y^x \in C(\mathcal{L}_T, V)$ such that for every $(s, t) \in \mathcal{L}_T$ it holds that

$$Y^x_{s,t} = -f(s, x) + \int_s^t f_{0,1}(\tau, X^x_{s,\tau})Y^x_{s,\tau} \, d\tau. \quad (163)$$

This ensures that there exists a function $q: [-T, T]^3 \times V \to \mathbb{R}$ which satisfies for every $s \in [0, T]$, $h \in [-s, T-s] \setminus \{0\}$, $\tau \in [\max\{s, s+h\}, T]$, $x \in V$ that $q(s, h, \tau, x) = \|(X^x_{s+h, \tau} - X^x_{s, \tau})/h - Y^x_{s, \tau}\|_V$ and which satisfies for every $s \in [0, T]$, $h \in [-T, T]$, $\tau \in [-T, \max\{s, s+h\})$, $x \in V$ that $q(s, h, \tau, x) = 0$. In the next step we note that the triangle inequality implies that for every $x \in V$, $s \in [0, T]$, $h \in [-s, T-s] \setminus \{0\}$, $t \in [\max\{s+h, s\}, T]$ it holds that

$$\left\| \frac{X^x_{s+h, \tau} - X^x_{s, \tau}}{h} - Y^x_{s,t} \right\|_V = \left\| -\frac{1}{|h|} \int_{\min\{s, s+h\}}^{\max\{s, s+h\}} f(\tau, X^x_{s,\tau}) \, d\tau + f(s, x) + \int_s^t \frac{f(\tau, X^x_{s+h, \tau} - f(\tau, X^x_{s, \tau})}{h} \, d\tau - \int_s^t f_{0,1}(\tau, X^x_{s,\tau})Y^x_{s,\tau} \, d\tau \right\|_V$$

$$\leq \left\| -\frac{1}{|h|} \int_{\min\{s, s+h\}}^{\max\{s, s+h\}} f(\tau, X^x_{s,\tau}) \, d\tau + f(s, x) \right\|_V + \int_s^t \left\| f(\tau, X^x_{s+h, \tau} - f(\tau, X^x_{s, \tau}) \right\|_V \, d\tau$$

$$+ \int_s^t \left\| f_{0,1}(\tau, X^x_{s,\tau})Y^x_{s,\tau} \right\|_V \, d\tau. \quad (164)$$

Next we intend to prove an upper bound for the l.h.s. of (164). To do so, we will estimate the terms on the r.h.s. of (164) separately. For this note that the fact that $\forall x \in V: (\mathcal{L}_T \ni (s, t) \mapsto X^x_{s,t} \in V) \in C(\mathcal{L}_T, V)$ shows that for every $x \in V$, $s \in [0, T]$ it holds that

$$\sup_{h \in [-s, T-s] \setminus \{0\}} \left\| \int_{\min\{s, s+h\}}^{\max\{s, s+h\}} \frac{1}{|h|} f(\tau, X^x_{s,\tau}) \, d\tau - f(s, x) \right\|_V < \infty. \quad (165)$$

In addition, the assumption that $\forall x \in V, s \in [0, T]: ([s, T] \ni t \mapsto X^x_{s,t} \in V) \in C([s, T], V)$ and the fact that $\forall x \in V: (\mathcal{L}_T \ni (s, t) \mapsto Y^x_{s,t} \in V) \in C(\mathcal{L}_T, V)$ ensure for every $x \in V$, $s \in [0, T]$ that

$$\sup_{h \in [-s, T-s] \setminus \{0\}} \left\| \int_s^{\max\{s, s+h\}} f_{0,1}(\tau, X^x_{s,\tau})Y^x_{s,\tau} \, d\tau \right\|_V < \infty. \quad (166)$$

Moreover, the triangle inequality proves that for every $x \in V$, $s \in [0, T]$, $h \in [-s, T-s] \setminus \{0\}$, $t \in [\max\{s, s+h\}, T]$ it holds that

$$\int_s^t \left\| f(\tau, X^x_{s+h, \tau} - f(\tau, X^x_{s, \tau})\right\|_V \, d\tau$$

$$\leq \int_s^t \left\| f(\tau, X^x_{s+h, \tau}) - f(\tau, X^x_{s, \tau}) - f_0(\tau, X^x_{s,\tau})(X^x_{s+h, \tau} - X^x_{s, \tau}) \right\|_V \, d\tau$$

$$\leq \int_s^t \left\| f(\tau, X^x_{s+h, \tau}) - f(\tau, X^x_{s, \tau}) \right\|_V \, d\tau.$$
Therefore, it holds for every $\varepsilon \in (0, \infty)$, $x \in V$, $s \in [0, T]$, $h \in [-s, T - s] \setminus \{0\}$, $t \in \max\{s, s + h\}, T$ with $|h| < \delta \varepsilon$ it holds that
\[
\|X_{s,t}^x - X_{s,t}^x\| \leq \varepsilon.
\] (168)

This implies that for every $\varepsilon \in (0, \infty)$, $x \in V$, $s \in [0, T]$, $h \in [-s, T - s] \setminus \{0\}$, $t \in \max\{s, s + h\}, T$ with $|h| < \delta \varepsilon$ it holds that
\[
\int_{\max\{s,s+h\}}^{t} \left\| f_{0,1}(\tau, X_{s,\tau}^x)(X_{s+h,\tau}^x - X_{s,\tau}^x) \right\|_V \, d\tau.
\] (167)

Furthermore, note that (162) assures that there exists $\delta : (0, \infty) \times V \to (0, \infty)$ such that for every $(s_1, t_1), (s_2, t_2) \in \mathcal{L}_T$, $\varepsilon \in (0, \infty)$, $x \in V$ with $\max\{|s_1 - s_2|, |t_1 - t_2|\} < \delta \varepsilon$ it holds that
\[
\|X_{s,t}^x - X_{s,t}^x\| \leq \varepsilon.
\] (168)

This implies that for every $\varepsilon \in (0, \infty)$, $x \in V$, $s \in [0, T]$, $h \in [-s, T - s] \setminus \{0\}$, $t \in \max\{s, s + h\}, T$ with $|h| < \delta \varepsilon$ it holds that
\[
\int_{\max\{s,s+h\}}^{t} \left\| f_{0,1}(\tau, X_{s,\tau}^x)(X_{s+h,\tau}^x - X_{s,\tau}^x) \right\|_V \, d\tau.
\] (167)

Therefore, it holds for every $\varepsilon \in (0, \infty)$, $x \in V$, $s \in [0, T]$, $h \in [-s, T - s] \setminus \{0\}$, $t \in \max\{s, s + h\}, T$ with $|h| < \delta \varepsilon$ that
\[
\int_{\max\{s,s+h\}}^{t} \left\| f_{0,1}(\tau, X_{s,\tau}^x)(X_{s+h,\tau}^x - X_{s,\tau}^x) \right\|_V \, d\tau.
\] (167)

Furthermore, observe that for every $x \in V$, $s \in [0, T]$, $h \in [-s, T - s] \setminus \{0\}$, $t \in \max\{s, s + h\}, T$ it holds that
\[
\int_{\max\{s,s+h\}}^{t} \left\| f_{0,1}(\tau, X_{s,\tau}^x)(X_{s+h,\tau}^x - X_{s,\tau}^x) \right\|_V \, d\tau.
\] (167)

Combining (164), (167), and (170) hence proves that for every $\varepsilon \in (0, \infty)$, $x \in V$, $s \in [0, T]$, $h \in [-s, T - s] \setminus \{0\}$, $t \in \max\{s, s + h\}, T$ with $|h| < \delta \varepsilon$ it holds that
\[
\left\| X_{s+h,\tau}^x - X_{s,\tau}^x \right\|_V \leq \int_{\min\{s,s+h\}}^{t} \left\| f_{\min}(\tau, X_{s,\tau}^x) \, d\tau - f(s, x) \right\|_V.
\]
This and (176) prove for every $x$. In the next step we intend to establish that for every $s$ for $V$ the fact that Lemma 3.6 (with $V$) $x$ $h$ for every $\in$,

\[
\left(\int_0^\infty \|X_{s+h,t}^x - X_s^x\|_V d\tau + \int_0^t \|Y_{s,t}^x\|_V d\tau\right) = 0.
\]

(172)

Grouwall’s lemma hence ensures that for every $x \in V$, $s \in [0,T]$ there exist $\varepsilon, C \in (0, \infty)$ such that for every $h \in [-s, T-s \setminus \{0\}, t \in [\max\{s, s+h\}, T]$ with $|h| < \delta_\varepsilon$ it holds that

\[
\left\|X_{s+h,t}^x - X_s^x\right\|_V \leq C + C \int_0^t \left\|X_{s+h,t}^x - X_s^x\right\|_V d\tau < \infty.
\]

(173)

In the next step we intend to establish that for every $x \in V$, $t \in (0,T]$, $s \in [0, t]$ it holds that

\[
\limsup_{\{0,T-s\}\ni \delta h \rightarrow 0} \left\|X_{s+h,t}^x - X_s^x\right\|_V = 0.
\]

(175)

For this we will analyze the terms on the r.h.s. of (164) separately. First, note that Lemma 2.2 (with $V = V$, $a = s$, $b = T$, $f = ([s,T] \ni \tau \mapsto f(\tau, X_{s,\tau}^x) \in V$) for $s \in [0,T)$, $x \in V$ in the notation of Lemma 2.2) shows that for every $x \in V$, $s \in [0, T]$ it holds that

\[
\limsup_{\{0,T-s\}\ni \delta h \rightarrow 0} \left\|\int_s^{s+h} \frac{1}{h} \left( f(\tau, X_{s,\tau}^x) d\tau - f(s, x) \right) \right\|_V = 0.
\]

(176)

In addition, (162) implies for every $x \in V$, $s \in (0, T]$ that

\[
\limsup_{\{s,0\}\ni \delta h \rightarrow 0} \left\|\int_s^{s+h} \frac{1}{h} f(\tau, X_{s+h,\tau}^x) d\tau + f(s, x) \right\|_V
\]

\[
= \limsup_{\{s,0\}\ni \delta h \rightarrow 0} \left\|\int_s^{s+h} \frac{1}{h} (f(\tau, X_{s+h,\tau}^x) - f(s, x)) d\tau \right\|_V
\]

\[
\leq \limsup_{\{s,0\}\ni \delta h \rightarrow 0} \left[ \frac{1}{h} \int_s^{s+h} d\tau \cdot \sup_{\tau \in [s+h, s]} \| f(\tau, X_{s+h,\tau}^x) - f(s, x) \|_V \right]
\]

\[
= \limsup_{\{s,0\}\ni \delta h \rightarrow 0} \left[ \sup_{\tau \in [s+h, s]} \| f(\tau, X_{s+h,\tau}^x) - f(s, x) \|_V \right] = 0.
\]

(177)

This and (176) prove for every $x \in V$, $s \in [0, T]$ that

\[
\limsup_{\{s,0\}\ni \delta h \rightarrow 0} \left\|\int_{\max\{s, s+h\}}^{\min\{s, s+h\}} \frac{1}{h} f(\tau, X_{s+n(s, s+h),\tau}^x) d\tau - f(s, x) \right\|_V = 0.
\]

(178)
Furthermore, observe that the fact that $\forall x \in V: (\mathcal{L}_T \ni (s,t) \mapsto Y^x_{s,t} \in V) \in \mathcal{C}(\mathcal{L}_T, V)$ and (163) prove for every $x \in V$, $s \in [0,T]$ that

\[
\limsup_{(\{s,\tau\} : \tau \in (\{0\}, h) \ni \mathcal{H})} \left\| Y^x_{s,\tau} \right\|_V = 0.
\]

Moreover, note that the triangle inequality, (168), and Lemma 3.6 (with $V = V$, $T = T$, $x_0 = x$, $f = f$ for $x \in V$ in the notation of Lemma 3.6) demonstrate that for every $s \in [0,T]$, $x \in V$ there exists $\varepsilon \in (0,\infty)$ such that for every $h \in [-s,T-s]$, $t \in [s,T]$ with $|h| < \delta_x^e$ it holds that

\[
\left| g(s, h, t, x) \right| \leq \left( \sup_{\tau \in [0,T]} \sup_{y \in V, \|x-y\|_V \leq \varepsilon} \frac{\|f(\tau,y) - f(\tau,z)\|_V}{\|y-z\|_V} + \sup_{\tau \in [s,T]} \|f_{0,1}(\tau, X^x_{s,\tau})\|_{L(V)} \right)^{\infty} < \infty.
\]

Fatou’s lemma and (174) therefore ensure that for every $x \in V$, $s \in [0,T]$ there exist $C \in (0,\infty)$ such that for every $t \in [s,T]$ with $|t| + |s| > 0$ it holds that

\[
\limsup_{(\{s,\tau\} : \tau \in (\{0\}, h) \ni \mathcal{H})} \int_{\max\{s, \tau\}}^{t} \left| g(s, h, \tau, x) \right| d\tau \leq \left( \sup_{\tau \in [0,T]} \sup_{v \in [\max\{s, \tau\}, \min\{s, \tau\}], \tau \in [\max\{s, v\}, T]} \|X^x_{\max\{s, \tau\} - X^x_{\min\{s, \tau\}}\|_V \right)^{\infty} \cdot \left( C \varepsilon^CT + \sup_{\tau \in [s,T]} \|Y^x_{s,\tau}\|_V \right) = 0.
\]

Combining (164), (167), (171), (174), (178), (179), and Fatou’s lemma hence proves that for every $x \in V$, $s \in [0,T]$, $t \in [s,T]$ with $|t| + |s| > 0$ it holds that

\[
\limsup_{(\{s,\tau\} : \tau \in (\{0\}, h) \ni \mathcal{H})} q(s, h, t, x) \leq \left( \sup_{\tau \in [s,T]} \|f_{0,1}(\tau, X^x_{s,\tau})\|_{L(V)} \right)^{\infty} \cdot \left( \limsup_{(\{s,\tau\} : \tau \in (\{0\}, h) \ni \mathcal{H})} \int_{\max\{s, \tau\}}^{t} \frac{\|X^x_{s+h,\tau} - X^x_{s,\tau} - Y^x_{s,\tau}\|_V}{\|h\|} d\tau \right)^{\infty} \leq \left( \sup_{\tau \in [s,T]} \|f_{0,1}(\tau, X^x_{s,\tau})\|_{L(V)} \right)^{\infty} \cdot \limsup_{(\{s,\tau\} : \tau \in (\{0\}, h) \ni \mathcal{H})} \int_{s}^{t} q(s, h, \tau, x) d\tau \leq \left( \sup_{\tau \in [s,T]} \|f_{0,1}(\tau, X^x_{s,\tau})\|_{L(V)} \right)^{\infty} \cdot \limsup_{(\{s,\tau\} : \tau \in (\{0\}, h) \ni \mathcal{H})} \int_{s}^{t} q(s, h, \tau, x) d\tau \leq \infty.
\]
Gronwall’s lemma hence establishes (175). In particular, we obtain that for every \( x \in V, t \in (0,T), s \in [0,t] \) it holds that \([0,t] \ni u \mapsto X^x_{u,t} \in V\) is differentiable and that
\[
\frac{\partial}{\partial s} X^x_{s,t} = Y^x_{s,t}. \tag{183}
\]
This and (163) establish items (i) and (iv). Moreover, note that (162) and Lemma 4.7 (with \( V = V, T = T, f = f, X^x_{s,t} = X^x_{s,t}, y = -f(s,x) \) for \( x \in V, (s,t) \in \mathcal{L}_T \) in the notation of Lemma 4.7) prove that items (v) and (vi) hold and that for every \( x \in V, (s,t) \in \mathcal{L}_T \) it holds that
\[
-\left(\frac{\partial}{\partial s} X^x_{s,t}\right)f(s,x) = -f(s,x) + \int_s^t f_{0,1}(\tau, X^x_{s,\tau})(-\left(\frac{\partial}{\partial s} X^x_{s,\tau}\right)f(s,x)) \, d\tau. \tag{184}
\]
Furthermore, observe that the fact that \((\mathcal{L}_T \times V \ni (s,t,x) \mapsto \frac{\partial}{\partial s} X^x_{s,t} \in L(V)) \in \mathcal{C}(\mathcal{L}_T \times V, L(V))\), the fact that \((\mathcal{L}_T \times V \ni (s,t,x) \mapsto f(t,x) \in V) \in \mathcal{C}(\mathcal{L}_T \times V, V)\), and the fact that \((L(V) \times V \ni (A,x) \mapsto Ax \in V) \in \mathcal{C}(L(V) \times V, V)\) ensure that
\[
(\mathcal{L}_T \times V \ni (s,t,x) \mapsto -\left(\frac{\partial}{\partial s} X^x_{s,t}\right)f(s,x)) \in \mathcal{C}(\mathcal{L}_T \times V, V). \tag{185}
\]
Combining this with (163) and (184) demonstrates that for every \( x \in V, (s,t) \in \mathcal{L}_T \) it holds that
\[
Y^x_{s,t} = -\left(\frac{\partial}{\partial s} X^x_{s,t}\right)f(s,x). \tag{186}
\]
Equations (183) and (185) therefore establish items (ii), (iii), and (vii). Next observe that Lemma 2.3 (with \( V = V, a = s, b = T, f = ([s,T] \ni t \mapsto f(t,X^x_{s,t}) \in V), F = ([s,T] \ni t \mapsto X^x_{s,t} \in V) \) for \( s \in [0,T), x \in V \) in the notation of Lemma 2.3) proves that for every \( x \in V, s \in [0,T), t \in [s,T] \) it holds that
\[
\limsup_{(\{s-T,t\} \setminus \{0\}) \ni h \to 0} \frac{\|X^x_{s,t+h} - X^x_{s,t} - f(t,X^x_{s,t})h\|_V}{|h|} = 0. \tag{187}
\]
This ensures that for every \( x \in V, s \in [0,T) \) it holds that \([s,T] \ni t \mapsto X^x_{s,t} \in V\) is differentiable and that for every \( x \in V, s \in [0,T), t \in [s,T] \) it holds that
\[
\frac{\partial}{\partial t} X^x_{s,t} = f(t,X^x_{s,t}). \tag{188}
\]
Combining this with (162) establishes items (viii), (ix), (x), and (xi). The proof of Lemma 4.8 is thus completed. \(\square\)

5. Alekseev-Gröbner formula. In this section we combine Proposition 2.10, Lemma 3.7, and Lemma 4.8 to establish in Corollary 5.2 below an extension of the Alekseev-Gröbner formula (cf., e.g., Hairer et al. [7, Theorem 14.5 in Chapter I]) for Banach space valued functions. Our proof of Corollary 5.2 in turn employs the following auxiliary result, Lemma 5.1 below. Observe that for every Banach space \((V, \|\cdot\|_V)\), for every Banach space \((W, \|\cdot\|_W)\), and for every \( a \in \mathbb{R}, b \in (a, \infty) \) it holds that \(C^1([a,b] \times V, W) = \{ \phi \in C^1([a,b] \times V, W) : \exists \Phi \in C^1(\mathbb{R} \times W) : \Phi|[a,b] \times V = \phi \}\).

Lemma 5.1. Let \((V, \|\cdot\|_V)\) be a nontrivial \(\mathbb{R}\)-Banach space, let \((W, \|\cdot\|_W)\) be an \(\mathbb{R}\)-Banach space, and let \( a \in \mathbb{R}, b \in (a, \infty) \), \( \phi \in C^0([a,b] \times V, W) \) satisfy for every \( x \in V \) that \([a,b] \ni t \mapsto \phi(t,x) \in W \in C^1([a,b], W)\) and \([a,b] \times V \ni (t,y) \mapsto (\frac{\partial}{\partial t} \phi)(t,y) \in W \in C([a,b] \times V, W)\). Then there exists \( \Phi \in C^1(\mathbb{R} \times W) \) such that for every \( t \in [a,b], x \in V \) it holds that \( \Phi(t,x) = \phi(t,x) \).
Proof of Lemma 5.1. Throughout this proof let \( \Phi: \mathbb{R} \times V \to W \) be the function which satisfies for every \( t \in \mathbb{R}, x \in V, k \in \mathbb{N} \) that

\[
\Phi(t, x) = \begin{cases} 
\phi(t, x) & : (t, x) \in [a, b] \times V \\
2\phi(a, x) - \phi(2a - t, x) & : (t, x) \in [a - (b - a), a) \times V \\
2\phi(b, x) - \phi(2b - t, x) & : (t, x) \in (b, b + (b - a)] \times V
\end{cases}
\] (189)

and

\[
\Phi(t, x) = \begin{cases} 
\phi(t, x) & : (t, x) \in [a, b] \times V \\
2\Phi(b - k(b - a), x) - \Phi(2(b - k(b - a)) - t, x) & : (t, x) \in [a - k(b - a), b - k(b - a)] \times V \\
2\Phi(2a + k(b - a), x) - \Phi(2a + k(b - a)) - t, x) & : (t, x) \in (a + k(b - a), b + k(b - a)] \times V \\
\end{cases}
\] (190)

let \( \Phi_{1,0}: \mathbb{R} \times V \to W \) be the function which satisfies for every \( t \in \mathbb{R}, x \in V, k \in \mathbb{N} \) that

\[
\Phi_{1,0}(t, x) = \begin{cases} 
\left(\frac{\partial}{\partial t}\phi\right)(t, x) & : (t, x) \in [a, b] \times V \\
\Phi_{1,0}(2(b - k(b - a)) - t, x) & : (t, x) \in [a - k(b - a), b - k(b - a)] \times V \\
\Phi_{1,0}(2a + k(b - a)) - t, x) & : (t, x) \in (a + k(b - a), b + k(b - a)] \times V
\end{cases}
\] (191)

and let \( \Phi_{0,1}: \mathbb{R} \times V \to L(V, W) \) be the function which satisfies for every \( t \in \mathbb{R}, x \in V, k \in \mathbb{N} \) that

\[
\Phi_{0,1}(t, x) = \begin{cases} 
\left(\frac{\partial}{\partial x}\phi\right)(t, x) & : (t, x) \in [a, b] \times V \\
\Phi_{0,1}(b - k(b - a), x) - \Phi_{0,1}(2(b - k(b - a)) - t, x) & : (t, x) \in [a - k(b - a), b - k(b - a)] \times V \\
\Phi_{0,1}(2a + k(b - a), x) - \Phi_{0,1}(2a + k(b - a)) - t, x) & : (t, x) \in (a + k(b - a), b + k(b - a)] \times V
\end{cases}
\] (192)

Note that the fact that \([a, b] \times V \ni (t, x) \mapsto \phi(t, x) \in W\) is continuous ensures that

\[
\Phi \in \mathcal{C}(\mathbb{R} \times V, W).
\] (193)

Next observe that for every \( t \in \mathbb{R}, x \in V \) it holds that \( \mathbb{R} \ni s \mapsto \Phi(s, x) \in W \) is differentiable and that

\[
\Phi_{1,0}(t, x) = \left(\frac{\partial}{\partial t}\Phi\right)(t, x).
\] (194)

Furthermore, note that the fact that \((a, b] \times V \ni (t, x) \mapsto \left(\frac{\partial}{\partial t}\phi\right)(t, x) \in W \) \( \in \mathcal{C}([a, b] \times V, W) \) assures that

\[
\Phi_{1,0} \in \mathcal{C}(\mathbb{R} \times V, W).
\] (195)

Combining this and (194) proves that

\[
\mathbb{R} \times V \ni (t, x) \mapsto \left(\frac{\partial}{\partial t}\Phi\right)(t, x) \in W \) \( \in \mathcal{C}(\mathbb{R} \times V, W).
\] (196)

In addition, note that for every \( t \in \mathbb{R}, x \in V \) it holds that \( V \ni v \mapsto \Phi(t, v) \in W \) is differentiable and that

\[
\Phi_{0,1}(t, x) = \left(\frac{\partial}{\partial x}\Phi\right)(t, x).
\] (197)
Moreover, observe that the fact that \([a, b] \times V \ni (t, x) \mapsto (\frac{\partial}{\partial s}) \phi(t, x) \in L(V, W)\) implies that 
\[
\Phi_{0,1} \in \mathcal{C}(\mathbb{R} \times V, L(V, W)). \tag{198}
\]
This and (197) show that 
\[
(\mathbb{R} \times V \ni (t, x) \mapsto (\frac{\partial}{\partial s}) \Phi(t, x) \in L(V, W)) \in \mathcal{C}(\mathbb{R} \times V, L(V, W)). \tag{199}
\]
Combining (193), (196), (199), and, e.g., Coleman [2, Corollary 3.4] completes the proof of Lemma 5.1. 

**Corollary 5.2.** Let \((V, \|\cdot\|_V)\) be a nontrivial \(\mathbb{R}\)-Banach space, let \(T \in (0, \infty), f \in \mathcal{C}^0, 1([0, T] \times V, V)\), let \(Y, E : [0, T) \to V\) be strongly measurable functions, for every \(x \in V, s \in [0, T]\) let \(X_{s,t}^x = (X_{s,t}^x)_{t \in [s, T]} : [s, T] \to V\) be a continuous function which satisfies for every \(t \in [s, T]\) that \(X_{s,t}^x = x + \int_s^t f(\tau, X_{\tau,t}^x) \, d\tau, \) and assume for every \(t \in [0, T]\) that \(\int_0^t \|\phi(\tau, Y_{\tau})\|_V + \|E_{\tau}\|_V \, d\tau < \infty\) and \(Y_t = Y_0 + \int_0^t f(\tau, Y_{\tau}) + E_{\tau} \, d\tau.\) Then

(i) it holds that \(((u, r) \in [0, T]^2 : u \leq r) \times V \ni (s, t, x) \mapsto X_{s,t}^x \in V \in \mathcal{C}^0, 1([0, T]^2 : u \leq r) \times V, V),\)

(ii) it holds that \(((u, r) \in [0, T]^2 : u < r) \times V \ni (s, t, x) \mapsto X_{s,t}^x \in V \) is continuously differentiable,

(iii) it holds for every \(t \in [0, T]\) that \([0, t] \ni \tau \mapsto \frac{\partial}{\partial \tau} X_{\tau,t}^x \in V\) is strongly measurable,

(iv) it holds for every \(t \in [0, T]\) that \(\int_0^t \|\frac{\partial}{\partial \tau} X_{\tau,t}^x\|_V \, d\tau < \infty,\) and

(v) it holds for every \(s \in [0, T], t \in [s, T]\) that

\[
Y_t = X_{s,t}^x + \int_s^t \frac{\partial}{\partial \tau} X_{\tau,t}^x \, d\tau. \tag{200}
\]

**Proof of Corollary 5.2.** Throughout this proof let \(\angle_T \subseteq [0, T]^2\) be the set given by 
\(\angle_T = \{(s, t) \in [0, T]^2 : s \leq t\}\) and let \(\Phi : \angle_T \times V \to V\) be the function which satisfies for every \((s, t) \in \angle_T, x \in V\) that \(\Phi_{s,t}(x) = X_{s,t}^x.\) Note that, e.g., Coleman [2, Corollary 3.4] and items (i), (ii), (v), (viii), and (ix) of Lemma 4.8 (with \(V = V, T = \infty, f = f, X_{s,t}^x = X_{s,t}^x\) for \((s, t) \in \angle_T, x \in V\) in the notation of items (i), (ii), (v), (viii), and (ix) of Lemma 4.8) establish items (i) and (ii). It thus remains to prove items (iii)-(v). For this observe that item (x) of Lemma 4.8 (with \(V = V, T = \infty, f = f, X_{s,t}^x = X_{s,t}^x\) for \((s, t) \in \angle_T, x \in V\) in the notation of item (x) of Lemma 4.8) ensures that there exists a unique continuous function \(\Phi : \angle_T \times V \to V\) which satisfies for every \(s \in [0, T], t \in [s, T], x \in V\) that 
\[
\Phi_{s,t}(x) = \frac{\partial}{\partial t}(\Phi_{s,t}(x)). \tag{201}
\]
Next observe that item (i) ensures that there exists a function \(\Phi^* : \angle_T \times V \to L(V)\) which satisfies for every \((s, t) \in \angle_T, x \in V\) that 
\[
\Phi^*_{s,t}(x) = \frac{\partial}{\partial t}(\Phi_{s,t}(x)). \tag{202}
\]
Moreover, note that items (i), (ii), and (v) of Lemma 4.8 (with \(V = V, T = \infty, f = f, X_{s,t}^x = X_{s,t}^x\) for \((s, t) \in \angle_T, x \in V\) in the notation of items (i), (ii), and (v) of Lemma 4.8 and Lemma 5.1 (with \(V = V, W = V, a = s, b = t, \) \(\phi = ((s, t) \times V \ni (u, x) \mapsto X_{s,t}^x \in V)\) for \(s \in [0, T], t \in (s, T)\) in the notation of Lemma 5.1) ensure that for every \(s \in [0, T], t \in (s, T)\) it holds that 
\[
((s, t) \times V \ni (u, x) \mapsto X_{s,t}^x \in V) \in \mathcal{C}^1([s, t] \times V, V). \tag{203}
\]
Combining item (ii), Lemma 3.7 (with $V = V, T = T, f = f, X^x_{s,t} = X^x_{s,t}$ for $(s,t) \in \angle_T, x \in V$ in the notation of Lemma 3.7), items (i) and (viii) of Lemma 4.8 (with $V = V, T = T, f = f, X^x_{s,t} = X^x_{s,t}$ for $(s,t) \in \angle_T, x \in V$ in the notation of items (i) and (viii) of Lemma 4.8), and Proposition 2.10 (with $V = V, t_0 = s, t = t, \phi = \text{Id}_V, F = Y^t_{[s,t]}, \Phi = \Phi_4((u,\tau) \in [s,t]; u \leq \tau) \times V, \Phi_{v,x} = \Phi_{v,x'}, \Phi^*_{v,x} = \Phi^*_{v,x'}, f = ([s,t] \ni \tau \mapsto f(\tau, Y_{\tau}) + E_{\tau} \in V)$ for $s \in [0,T], t \in (s,T), v \in [s,t], w \in [v,t]$ in the notation of Proposition 2.10) hence demonstrates that for every $(s,t) \in \angle_T$ it holds that

$$[s,t] \ni \tau \mapsto \Phi^*_{\tau,t}(Y_{\tau})\left[\Phi_{\tau,t}(Y_{\tau}) - f(\tau, Y_{\tau}) - E_{\tau}\right] \in V$$

is strongly measurable, \quad (204)

$$\int_s^t \left\| \Phi^*_{\tau,t}(Y_{\tau})\left[\Phi_{\tau,t}(Y_{\tau}) - f(\tau, Y_{\tau}) - E_{\tau}\right]\right\|_V d\tau < \infty, \quad \text{and} \quad (205)$$

$$X^Y_{s,t} - Y_t = \int_s^t \Phi^*_{\tau,t}(Y_{\tau})\left[\Phi_{\tau,t}(Y_{\tau}) - f(\tau, Y_{\tau}) - E_{\tau}\right] d\tau. \quad (206)$$

Moreover, note that Lemma 2.3 (with $V = V, a = 0, b = t, f = ([0, t] \ni s \mapsto f(s, X^x_{s,t}) \in V), F = ([0, t] \ni s \mapsto X^x_{s,t} \in V)$ for $t \in [0,T], x \in V$ in the notation of Lemma 2.3) shows that for every $\tau \in [0,T), t \in [\tau,T], x \in V$ it holds that

$$\Phi_{\tau,t}(x) = f(t, X^x_{\tau,t}). \quad (207)$$

Combining this, (204)–(206), and the fact that $\forall \tau \in [0,T), t \in [\tau,T], x \in V: \Phi^*_{\tau,t}(x) = \frac{\partial}{\partial x} X^x_{\tau,t}$ establishes items (iii)–(vi). The proof of Corollary 5.2 is thus completed.

\[ \square \]

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