Irreducible Representations of an Algebra underlying Hidden Symmetries of a class of Quasi Exactly Solvable Systems of Equations

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Abstract

The set of linear, differential operators preserving the vector space of couples of polynomials of degrees \( n \) and \( n - 2 \) in one real variable leads to an abstract associative graded algebra \( \mathcal{A}(2) \). The irreducible, finite dimensional representations of this algebra are classified into five infinite discrete sets and one exceptional case. Their matrix elements are given explicitly. The results are related to the theory of quasi exactly solvable equations.
1 Introduction

The number of quantum mechanical problems which possess a complete algebraic solution is rather limited. Some years ago [1, 2, 3], several Hamiltonians were exhibited which enjoy the property of having a finite number of algebraic eigenvalues. That is to say that a part of their spectrum can be obtained by solving an algebraic (rather than a differential) eigenvalue equation. Such equations and the corresponding linear operators are called “Quasi Exactly Solvable” (QES) (see [4] for a recent review).

In a one dimensional real space, the QES scalar operators [5] can be written (after a suitable redefinition of the space variable and of the unknown function) in the form of operators that preserve the vector space, say \( P(n) \), of polynomials \( p_n(x) \) of maximal degree \( n \) in the redefined space variable. The set of linear operators leaving \( P(n) \) globally invariant coincides with the envelopping algebra generated by the spin \( s = n/2 \) representation of the Lie group \( SL(2, R) \) [5]. This crucial observation allows a classification of the QES operators and reveals the fact that they possess a hidden symmetry.

The notion of Quasi Exactly Solvable systems of equations, first addressed in [3], was extended recently in [6, 7, 8]. In the case of two equations in one real variable, the relevant operators [6] (again after suitable redefinitions) are those preserving the vector space \( P(m, n) \) of couples of polynomials \( p_m(x) \) and \( p_n(x) \) of maximal degree \( m \) and \( n \) respectively in the redefined variable. The question of identifying a hidden algebra behind this QES system arises naturally. In [6], it was shown that the set of all linear operators preserving the space \( P(m, n) \) coincides with the envelopping algebra of a representation of a particular graded algebra. The structure of this algebra is strongly dependent on \( \Delta = |m - n| \). Indeed, the natural composition law of the algebra is such that the anticommutator of the “fermionic” generators is a polynomial of degree \( \Delta \) in the bosonic generators. Therefore, the relevant algebra (which we denote \( \mathcal{A}(\Delta) \)) is, in general, not of the Lie type. In the case \( \Delta = 1 \), it is \( \mathcal{A}(1) \) and isomorphic to \( osp(2, 2) \).

Up to now, the abstract algebra \( \mathcal{A}(\Delta) \) is obtained from a particular representation: the representation given by the linear operators preserving \( P(m, n) \). The structure constants of the algebra were obtained by writing commutators and anticommutators among the generators within this representation and imposing the Jacobi identities or equivalently the associativity or the braiding relations.
The problem of finding all the irreducible representations associated with the algebra of fixed $\Delta$ then arises naturally. Apart from its direct algebraic interest, the explicit construction of the representations also provides the classification of all the QES systems (consisting of an arbitrary number of equations) which possess the same underlying symmetry as the system of two equations we started with. The logical path is as follows. Once the abstract algebra originating from a system of $2 \times 2$ matrix operators preserving the space $P(m, n)$ has been obtained, one classifies all the representations of the algebra and realizes them in terms of matrix differential operators.

In this paper we reconsider and generalize the algebra $\mathcal{A}(2)$ and we classify all its irreducible representations. It appears that the abstract algebra has a rich set of irreducible representations (in fact several inequivalent infinite families of them). Each of these representations can be associated to a set of operators preserving a vector space $P(n_1, n_2, \ldots, n_k)$ of $k$-tuple of polynomials in one real variable and with maximal degrees $n_i$.

The paper is organized as follows. In section 2 we present the algebra $\mathcal{A}(2)$. We show that it admits a generalisation parametrized by three constants. We point out its symmetries, its automorphisms and compute its two Casimir operators. In section 3, we discuss the general properties of its representations and present the tensorial operators relevant for their construction. In section 4, we give the arguments leading to the classification of the allowed representations. Section 5 then presents explicitly all the representations and in particular the generic one. Finally section 6 indicates the way to map the various representations in the formalism of QES equations.

2 The abstract algebra

2.1 The algebra

The algebra $\mathcal{A}(2)$ contains the algebra $so(3) \otimes u(1)$ as a 4-generators Lie subalgebra together with six more generators which behave as two vectors under $so(3)$ and satisfy among themselves generalized anticommutation relations. Let us define the algebra more precisely.

Denote by $T_i$ the three generators of the $so(3)$ Lie subalgebra with the commutation relations

$$[T_i, T_j] = f_{i,j}^k T_k$$ (1)
We work in a complex basis for \( \mathfrak{so}(3) \) where the indices take the values \(+1, 0 \) and \(-1\). In this basis the \( f_{i,j}^k \) are antisymmetric in \( i \) and \( j \) and zero except for
\[
\begin{align*}
f_{0,+1}^+ &= 1 \\
f_{0,-1}^- &= -1 \\
f_{-1,+1}^0 &= 1
\end{align*}
\]
(2)
The corresponding metric \( g_{i,j} \)
\[
g_{i,j} = \frac{1}{2} f_{i,k}^l f_{j,l}^k
\]
(3) is symmetric and its non-zero elements are
\[
\begin{align*}
g_{0,0} &= g_{0,0}^0 = 1 \\
g_{-1,+1} &= g_{-1,+1}^{-1} = -1
\end{align*}
\]
(4) This metric is used to raise and lower the \( \mathfrak{so}(3) \) indices and allows us define
\[
\begin{align*}
f_{i}^{j,k} &= g_{i,m}^{j,m} f_{i,m}^k \\
f_{i,j}^{k} &= g_{i,m}^{j,n} f_{m,n}^k
\end{align*}
\]
(5) Hence, the \( \mathfrak{so}(3) \) Casimir operator \( T^2 \) is
\[
T^2 = g^{i,j} T_i T_j = T_0^2 - T_{-1} T_{+1} - T_{+1} T_{-1}
\]
(6) and takes the eigenvalues \( s(s+1) \) where \( s \) is an integer or a half-integer for selfadjoint representations.

The \( u(1) \) operator \( J \) commutes with the \( T_i \)
\[
[J, T_i] = 0
\]
(7) The six extra generators, the \( Q_i \) and the \( \overline{Q}_i \), vectors under \( \mathfrak{so}(3) \)
\[
\begin{align*}
[T_i, Q_j] &= f_{i,j}^k Q_k \\
[T_i, \overline{Q}_j] &= f_{i,j}^k \overline{Q}_k
\end{align*}
\]
(8,9)
have $J_{\text{value}} = -1$ and $+1$ respectively i.e.

\[
\begin{align*}
[J, Q_i] &= -Q_i \\
[J, \overline{Q}_i] &= \overline{Q}_i
\end{align*}
\] (10) (11)

The anticommutator of two $Q_i$ has $J_{\text{value}} = -2$. Since there exists in the algebra no generator with this $J_{\text{value}}$, the only realistic possibility which fulfills the $so(3) \times u(1)$ invariance is

\[
\{Q_i, Q_j\} = \frac{2}{3}g_{i,j}Q^2
\] (12)

where the factor $2/3$ is fixed by consistency and

\[
Q^2 = g^{i,j}Q_iQ_j = 3Q_0^2
\] (13)

Analogously the anticommutator of two $\overline{Q}_i$ is

\[
\{\overline{Q}_i, \overline{Q}_j\} = \frac{2}{3}g_{i,j}\overline{Q}^2
\] (14)

with

\[
\overline{Q}^2 = g^{i,j}\overline{Q}_i\overline{Q}_j = 3\overline{Q}_0^2
\] (15)

For the anticommutator of a $Q_i$ with a $\overline{Q}_j$, which has $J_{\text{value}}$ zero, we write, a priori, the most general expression quadratic in the operators $J$ and $T_i$ and with the correct $so(3)$ behaviour. We then impose the associativity relations and find that there are three remaining free parameters only

\[
\{Q_i, \overline{Q}_j\} = \alpha \left( (T_iT_j + T_jT_i) - g_{i,j}(J^2 + T^2) - 2f_{i,j}^kJT_k \right) - \beta(g_{i,j}J + f_{i,j}^kT_k) - \gamma g_{i,j}
\] (16)

The full set of generalised commutation relations is then given in (4), (7), (8), (9), (10), (11), (12), (14), (16). From now on, we will refer to this algebra as $\mathcal{A}$.

If $\alpha$ is non-zero, as we will suppose henceforth, it can be renormalised to any value by rescaling the operators $Q_i$ and/or $\overline{Q}_i$ by an appropriate multiplicative factor. In section 3, we have chosen to normalise $\alpha$ to

\[
\alpha = \frac{1}{2}
\] (17)
2.2 Symmetries of the algebra $\mathcal{A}$

In this section we present the reparametrization which the free parameters $(\alpha, \beta, \gamma)$ undergo when the operator $J$ is subjected to a translation. Two automorphisms of the algebra $\mathcal{A}$ are also given.

By a translation of $J$

$$ J \rightarrow J' = J - c $$

by a constant $c$, one obtains an algebra equivalent to the original algebra through the reparametrization

$$ \alpha' = \alpha $$
$$ \beta' = \beta + 2\alpha c $$
$$ \gamma' = \gamma + \alpha c^2 + \beta c $$

(18)

Obviously, by choosing $c$ suitably $\beta'$ or $\gamma'$ can be made zero. Hence, there is essentially only one free parameter in the algebra. The choice $\beta' = 0$ is particularly interesting, as the algebra with the constant term $\gamma'$ in the right hand side can simply be interpreted as the central extension of the algebra with both $\beta$ and $\gamma$ zero.

The two following quantities

$$ I_1 = 2\alpha J + \beta $$
$$ I_2 = \beta^2 - 4\alpha \gamma $$

(19) (20)

are obviously invariant under the reparametrisation (18) of the algebra. Note that the first quantity is an operator. We will show that the discussion of the representation can be best carried out in terms of the scalar invariant $I_2$ and of the eigenvalues of the operator $I_1$.

Another way of presenting the algebra is as follows. Using the freedom in the definitions of the $Q_i$’s, the $\overline{Q}_i$’s and the $J$, we have concluded that our algebra depends on one significant parameter only (say $\gamma$ when $\beta = 0$). As far as the representations are concerned, we can write a single algebra possessing the same set of irreducible representations as the family of algebras parametrized by $\gamma$, at the expense of introducing an additional generator $\Gamma$. Namely, we write

$$ \{Q_i, \overline{Q}_j\} = \alpha \left( (T_i T_j + T_j T_i) - g_{i,j} (J^2 + T^2) - 2f_{i,j}^k J T_k \right) - \Gamma g_{i,j} $$

(21)
and assume that $\Gamma$ commutes with all the generators. Then, within irreducible representations, $\gamma$ becomes an eigenvalue of $\Gamma$

$$\Gamma \rightarrow \gamma I$$

(22)

where $I$ is the identity matrix.

The algebra which was obtained originally in [6] corresponds to the particular case

$$\begin{align*}
\alpha &= 1/2 \\
\beta &= -1/2 \\
\gamma &= 0
\end{align*}$$

(23)

Let us also introduce two automorphisms of the algebra.

1. The interchange of $Q_i \leftrightarrow \overline{Q}_i$ made simultaneously with the replacement $J \rightarrow -J - \beta/\alpha$ is an automorphism of the algebra. Precisely

$$\begin{align*}
Q_i &= \overline{Q}_i \\
\overline{Q}_j &= Q_j \\
J &= -J' - \beta/\alpha
\end{align*}$$

(24)

Under this automorphism (see (19))

$$I_1 = -I'_1$$

(25)

2. A second more trivial automorphism is given by the multiplicative rescaling of $Q_i$ and $\overline{Q}_j$ by reciprocal factors and more precisely

$$\begin{align*}
Q_i &= \kappa Q'_i \\
\overline{Q}_j &= \frac{Q_j}{\kappa}
\end{align*}$$

(26)

2.3 Casimir Operators of the algebra $\mathcal{A}$

The two Casimir operators of the algebra $\mathcal{A}$ can also be computed. The first one is of maximal fourth degree in the generators while the second one is of sixth degree.
The Casimir operators can be constructed with $so(3)$ invariant operators. Hence they can be constructed with the operator $J$ and with scalars obtained from the vectors $T_i, Q_i$ and $\overline{Q}_i$. We adopt the following notation. Let $E_i, F_i$ and $G_i$ be arbitrary vectors. We define in general

$$J_{EF} = g^{ij} E_i F_j$$
$$J_{EFG} = f^{ijk} E_i F_j G_k$$

When there are $\overline{Q}_i$ vectors we have chosen, without loss of generality, to put them to the right. Among these invariants the two following identities are satisfied

$$3J_{TQQ} - 2J_{TT}J_{QQ} + 6J_{TQ}J_{TQ} = 0$$
$$3J_{T\overline{Q}\overline{Q}} - 2J_{TT}J_{\overline{Q}\overline{Q}} + 6J_{T\overline{Q}J_{T\overline{Q}}} = 0$$

Taking the most general form of the $so(3) \times u(1)$ invariant polynomial of maximal degree six in the generators and imposing commutation with the $Q$’s and the $\overline{Q}$’s one finds that there are two independent Casimir operators of degree four and six respectively. The Casimir operator of degree four $K_4$ is

$$K_4 = R_0 + R_1 J_{Q\overline{Q}} + R_2 J_{QQ}J_{\overline{Q}\overline{Q}} + R_3 J_{TQQ\overline{Q}}$$

with

$$R_0 = -\frac{(I_2 - I_1^2)(I_2 - (I_1 - 2\alpha)^2)}{64\alpha^2} + \frac{I_2 + I_1^2 - 2\alpha I_1}{8} J_{TT} - \alpha^2 4J_{TT}^2$$
$$R_1 = \frac{I_1 + 2\alpha}{2}$$
$$R_2 = \frac{1}{9}$$
$$R_3 = \alpha$$

We have chosen to collect in $R_0$ all the terms which do not involve the $Q$ or $\overline{Q}$ operators. It is amusing to note that if one acts on a state of given spin $s$, which means that $J_{TT}$ can be replaced by $s(s + 1)$, $R_0$ factorizes as

$$R_0 = -\frac{(I_2 - (2\alpha(s + 1) - I_1)^2)(I_2 - (2\alpha s + I_1)^2)}{64\alpha^2}$$
Use of this property will be made later in the text.

The Casimir operator of degree six $K_6$ (which is defined up to an arbitrary contribution of the fourth order one) can conveniently be chosen as

$$K_6 = S_0 + S_1 J_{Q\overline{Q}} + S_2 J_{TT} + S_3 J_{TQ} J_{T\overline{Q}} + S_4 J_{\overline{Q}Q}^2 + S_5 J_{QQ} J_{T\overline{Q}} + S_6 (J_{QQ} J_{T\overline{Q}}^2 + J_{TT} J_{Q\overline{Q}})$$

where we have grouped together in $S_i$ all operators which don’t depend on $Q$ or $\overline{Q}$

$$S_0 = 4 J_{TT} R_0$$
$$S_1 = \frac{52 \alpha^2 J_{TT} + 36 \alpha^2 + 16 \alpha J_{TT} I_1 + 12 \alpha I_1 - 3 I_2 + 3 I_1^2}{4 \alpha}$$
$$S_2 = \frac{12 \alpha^2 J_{TT} + 18 \alpha^2 + 6 \alpha I_1 - I_2 + I_1^2}{2 \alpha}$$
$$S_3 = -2 (4 \alpha + I_1)$$
$$S_4 = 1$$
$$S_5 = \frac{4 J_{TT} + 3}{3}$$
$$S_6 = -\frac{4}{3}$$

The fact that $S_0$ is proportional to $R_0$ is important and will be used later.

3 The Representations

In this section we want to describe the irreducible finite dimensional representations of the algebra $\mathcal{A}$. In the process of constructing explicitly the representations, we have come across finite dimensional representations which are reducible without being completely reducible. We have however not tried to classify all such representations.

3.1 The $J,s$ structure of the representations

From the commutation (and anticommutation relations) of the algebra $\mathcal{A}$ we infer the classification of the $J,s$ structure of the representations which is given in the theorem below.

Let us start by fixing some notations
1. The eigenvalues \( j \) of the operator \( J \) within a representation are discrete and range from \( j_t \), the eigenvalue with highest real part, to \( j_b \), the eigenvalue with lowest real part, in unit steps.

2. We define the “levels” as the subspaces corresponding to a given eigenvalue of \( J \). The total number of levels \( L \) is thus given by

\[
L = j_t - j_b + 1
\]  

(43)

3. The eigenspace corresponding to a given level is the space of a finite (possibly reducible) hermitian representation of the subalgebra \( so(3) \). It splits into a finite direct sum of spaces corresponding to a certain set of \( so(3) - \text{spins} \). Moreover its states can be classified according to the eigenvalues of \( T_0 \). The general basic states \( | s, s_0, j > \) (which may have multiplicity higher than one) thus satisfy

\[
T^2 | s, s_0, j > = s(s + 1) | s, s_0, j >
\]  

(44)

\[
J | s, s_0, j > = j | s, s_0, j >
\]  

(45)

\[
T_0 | s, s_0, j > = s_0 | s, s_0, j_t >
\]  

(46)

Classification Theorem

The complete set of finite dimensional irreducible representations of the algebra \( \mathcal{A} \) consists of five main series and one exceptional case.

G. The generic series of representations has \( L \) levels (with \( L \geq 4, s_t \geq 1 \)), acts on a space of \( 4(L - 2)(2s_t + 1) \) dimensions and has the following \( J, s \) hierarchy

\[
\begin{array}{ll}
J_{\text{value}} & \text{allowed } s_{\text{values}} \\
\hline
j_t & (s_t) \\
\hline
j_t - 1 & (s_t - 1, s_t, s_t + 1) \\
\hline
j_t - 2 & (s_t - 1, s_t, s_t, s_t + 1) \\
\vdots & \vdots \\
\hline
j_b + 2 & (s_t - 1, s_t, s_t, s_t - 1) \\
\hline
j_b + 1 & (s_t + 1, s_t, s_t - 1) \\
\hline
j_b & (s_t)
\end{array}
\]  

(47)
$H_+$. The $H_+$ series has $L$ levels (with $L \geq 2$), a dimension $4(L-1)(s_t+1)-2$ and the following $J, s$ hierarchy

\[
\begin{array}{ll}
J_{\text{value}} & \text{allowed } s_{\text{values}} \\
\dot{j}_t & (s_t) \\
\dot{j}_t - 1 & (s_t, s_t + 1) \\
\vdots & \vdots \\
\dot{j}_b + 1 & (s_t, s_t + 1) \\
\dot{j}_b & (s_t)
\end{array}
\]  

(48)

When $s_t = s_b = 0$ the $s = 0$ states at levels $\dot{j}_t - 1$ and $\dot{j}_b + 1$ are absent and the dimension is decreased by 2.

$H_-$. The $H_-$ series has $L$ levels (with $L \geq 2$, $L$ even, $s_t \geq 1$), a dimension $4(L-1)s_t + 2$ and the following $J, s$ hierarchy

\[
\begin{array}{ll}
J_{\text{value}} & \text{allowed } s_{\text{values}} \\
\dot{j}_t & (s_t) \\
\dot{j}_t - 1 & (s_t - 1, s_t) \\
\vdots & \vdots \\
\dot{j}_b + 2 & (s_t - 1, s_t) \\
\dot{j}_b + 1 & (s_t - 1, s_t) \\
\dot{j}_b & (s_t)
\end{array}
\]  

(49)

$T_+$. The $T_+$ series has $L$ levels (with $L \geq 2$, $L$ even, $s_t \geq 1/2$), a dimension $4(L-1)(s_t + 1)$ and the following $J, s$ hierarchy

\[
\begin{array}{ll}
J_{\text{value}} & \text{allowed } s_{\text{values}} \\
\dot{j}_t & (s_t) \\
\dot{j}_t - 1 & (s_t, s_t + 1) \\
\vdots & \vdots \\
\dot{j}_b + 1 & (s_t, s_t + 1) \\
\dot{j}_b & (s_t + 1)
\end{array}
\]  

(50)

$T_-$. The $T_-$ series has $L$ levels (with $L \geq 2$, $L$ even, $s_t \geq 3/2$), a dimension
4(L − 1)s_t and the following J, s hierarchy

\[
\begin{array}{ll}
J_{\text{value}} & \text{allowed } s_{\text{values}} \\
\hat{j}_t & (s_t) \\
\hat{j}_t - 1 & (s_t - 1, s_t) \\
\vdots & \vdots \\
\hat{j}_b + 1 & (s_t - 1, s_t) \\
\hat{j}_b & (s_t - 1)
\end{array}
\]  

E. The E exceptional case has 3 levels, a dimension 4(2s_t + 1), s_t ≥ 1, and
the following J, s hierarchy

\[
\begin{array}{ll}
J_{\text{value}} & \text{allowed } s_{\text{values}} \\
\hat{j}_t & (s_t) \\
\hat{j}_t - 1 & (s_t - 1, s_t + 1) \\
\hat{j}_t - 2 & (s_t)
\end{array}
\]  

In the next section we discuss the arguments leading to a proof of the classification theorem.

3.2 General Properties of Representations

Let us now sketch the first part of the proof of the classification theorem. This will be done in a few simple steps. The arguments will then be completed in the following sections.

1. A finite dimensional representation of the algebra A provides obviously a finite dimensional representation (possibly reducible) of its so(3) sub-algebra. Since so(3) is simple, we conclude at once that the finite dimensional representations of A are direct sums of irreducible representations of so(3), themselves equivalent to hermitian representations.

2. The operator J is diagonalizable and its spectrum has the form :

\[
j_w = j_t - (w - 1), \ w = 1, \ldots, L.
\]  

To see that this is the case, we first note that J, acting on a finite dimensional space, possesses at least one eigenvalue (real or complex)
and its corresponding eigenspace. It follows from the commutation rules \([10,11]\) that the operators \(\overrightarrow{Q}_i\) (resp. \(Q_i\)) raise (resp. lower) the \(J\) eigenvalue by one. Consider the eigenspace of \(J\) corresponding to the eigenvalue \(j_t\) with highest real part. This eigenspace (which we will call highest) is annihilated by all \(\overrightarrow{Q}_i\)'s. Let us call normally ordered the product of operators in which all the \(\overrightarrow{Q}_i\)’s stand rightmost. It follows from the commutation rules that any polynomial in the generators can be normally ordered. Therefore, due to the irreducibility, the whole representation space is obtained by acting on the highest eigenspace (highest level) with all polynomials in the operators \(J, T_i\) and \(Q_i\) and of degree in the \(Q_i\)’s not exceeding some non negative integer \(L - 1\).

The diagonalizability of \(J\) and the spectrum \([53]\) follow from the last statement and the commutation rules for the generators.

3. The eigenspace corresponding to the highest eigenvalue \(j_t\) carries a representation of \(so(3)\). It is easy to see that this must be an irreducible representation corresponding to a single spin \(s_t\), conveniently labeled \(s_t\).

In order to show this, it is sufficient to note that the space obtained by the action of all (normally ordered) polynomials in the generators (which do not decrease \(J\)) on the single spin subspace of the highest level is an invariant subspace.

4. Let us call anti-normally ordered the product of generators in which all the \(Q_i\)’s stand rightmost. The lowest level \(j_b\) is annihilated by the \(Q_i\)’s and the whole representation space is obtained by the action on the lowest space of (anti-normally ordered) polynomials in the generators. Therefore the lowest level carries also a single spin representation, say \(s_b\), of \(so(3)\).

5. It is easy to see that the representation space is the linear span of all the vectors obtained by the action of products of the \(Q_i\) operators only (resp. the \(\overrightarrow{Q}_i\)’s only) on all the states of the highest (resp. lowest) level. In particular, the set of all monomials of degree \(w\) (resp. \(L - w - 1\)) in \(Q_i\)'s (resp. \(\overrightarrow{Q}_i\)'s) generates the level corresponding to the eigenvalue \(j_t - w\).

6. The polynomials of degree \(w\) in the \(Q_i\) operators can be classified ac-
cording to their \(so(3)\) behaviour. Apart from the scalar \(Q^2\) operator
\[ Q^2 \quad J_{\text{value}} = -2 \quad (54) \]
which was defined in (13), it is useful to define a \(so(3)\) vector (axial vector) operator \(A_i\) of \(J_{\text{value}} = -2\) and a scalar operator (pseudoscalar) \(P\) of \(J_{\text{value}} = -3\)
\[ A_i = f^i_{j,k} Q_j Q_k \quad J_{\text{value}} = -2 \quad (55) \]
\[ P = f^{i,j,k} Q_i Q_j Q_k \quad J_{\text{value}} = -3. \quad (56) \]
It is not difficult to see that, as far as products of the \(Q_i\) operators are concerned, for even, say \(J_{\text{value}} = -2p, \ p \geq 1\), there are four independent operators
\[ (Q^2)^p \quad J_{\text{value}} = -2p, \ p \geq 1 \quad (57) \]
\[ (Q^2)^{p-1} A_i \quad J_{\text{value}} = -2p, \ p \geq 1 \quad (58) \]
one being an \(so(3)\) scalar and three others forming an \(so(3)\) vector. For odd, say \(J_{\text{value}} = -(2p + 1), \ p \geq 1\), there are also four independent operators
\[ (Q^2)^{p-1} P \quad J_{\text{value}} = -(2p + 1), \ p \geq 1 \quad (59) \]
\[ (Q^2)^p Q_i \quad J_{\text{value}} = -(2p + 1), \ p \geq 1 \quad (60) \]
Again there is an \(so(3)\) scalar and an \(so(3)\) vector. The case of first order monomials is exceptional: there is only one vector operator
\[ Q_i \quad J_{\text{value}} = -1 \quad (61) \]
Obviously, analogous results are valid for monomials in the \(\bar{Q_i}\) operators.

7. Using the above classification and the spin-addition theorem, we conclude that the following spin structure emerges: with \(s_t\) the (unique) spin corresponding to the level \(j_t\), the level \(j_t - 1\) consists at most of the spins \((s_t - 1, s_t, s_t + 1)\), while those corresponding to \(j_t - w\), \(w \geq 2\), consist at most of the spins \((s_t - 1, s_t, s_t, s_t + 1)\). Remark however that if \(s_t = 0\) the only spin which can be reached at level \(j_t - 1\) is 1 and only 0 and 1 at lower levels. If \(s_t = 1/2\) the only spins which can be reached are 1/2 and 3/2.
8. Starting from the eigenspace corresponding to the lowest level with \( J \) eigenvalue \( j_b \) and applying a similar reasoning with \( j_t \) replaced by \( j_b \), \( s_t \) by \( s_b \), and \( Q_i \) by \( \overline{Q}_i \), we conclude that the level \( j_b + 1 \) consists at most of the \textit{spins} \( (s_b - 1, s_b, s_b + 1) \), while the levels \( j_b + w, w \geq 2 \), consist at most of the \textit{spins} \( (s_b - 1, s_b, s_b, s_b + 1) \). Again \( s_b = 0 \) or \( s_b = 1/2 \) are special.

9. The four basic \( Q \)-monomials (of degree \( w + 1 \)) corresponding to the \( J_{\text{value}} = -(w+1) \) can be obtained from those (of degree \( w \)) corresponding to the \( J_{\text{value}} = -w \) by a multiplication by the sole operator \( Q_0 \). Hence, the dimension of the space corresponding to the \( j_t - w - 1, w \geq 2 \) level cannot be larger than the dimension of the space corresponding to \( j_t - w \) level. Following the same reasoning with \( j_t \) replaced by \( j_b \) and the \( Q_i \)'s replaced by \( \overline{Q}_i \)'s, we conclude that all levels corresponding to \( j_t - 2 \geq j \geq j_b + 2 \) have the same dimensionality. As \( Q_0 \) does not change the \( T_0 \)-content of the space the \textit{spin} content of all these levels is the same.

10. It is obvious from the preceding discussion that \( s_t \) and \( s_b \) can differ at most by one. Moreover, if they do differ, there can be only at most two \textit{spins} in the intermediate levels. This is the case for the representations \( T_+ \) \( [50] \) and \( T_- \) \( [51] \).

11. If at any level \( j \) the \textit{spin} structure is reduced to \((s_t + 1, s_t - 1)\) only, the next level (and the preceding level) has a single \textit{spin} state \( s_t \) only. Indeed the scalar (or pseudosalar) which generates the states at level \( j \) from the top level \( j_t \) must give zero acting on the \(|s_t, s_0, j_t\rangle \) states. It follows that the next vector operator (generating the states at level \( j - 1 \)) is also zero. The states at level \( j - 1 \) are reached only by the scalar operator. Hence the conclusion. The same holds for the preceding level by using the \( \overline{Q} \) operators. This shows that the exceptional case \( [52] \) is the only one of its kind.

12. Finally, let us mention that we can also exclude, on general grounds, the \textit{spin} patterns \((s_t - 1, s_t, s_t)\) or \((s_t, s_t, s_t + 1)\) or \((s_t - 1, s_t, s_t + 1)\) for the levels \( j_t - 2 \geq j \geq j_b + 2 \). We will only roughly sketch the arguments because the above patterns are excluded by actual calculations in the following sections. As it was stated above, all states of a given level
are obtained by the action of the operators (54-61) on all the states of the \( j_t \) level. Knowing what is, a priori, the spin content of the representation, we can construct out of \( T_i \)'s the projection operators on the spins \( s_t-1, s_t \) and \( s_t+1 \). Therefore we can construct explicitly all spin representations. Some algebra allows us then to show that the two spin \( s_t \) representations at any level \( j_t - 2 \geq j \geq j_b + 2 \) can be chosen in such a way that the \( Q_i \) operators do not mix the subspaces corresponding to spin \( s_t - 1 \) and one of the subspaces with spin \( s_t \) with the subspaces corresponding to spin \( s_t + 1 \) and the second spin \( s_t \). It is then a matter of little effort to show that no representation with three spin subspaces on the levels \( j_t - 2 \geq j \geq j_b + 2 \) is allowed.

### 3.3 Eigenvalues of the Casimir operator

Applied on the highest state \( |s_t, s_0, j_t> \) the two Casimir operators defined above take on the values (remember our choice \( \alpha = 1/2 \))

\[
K_4 = -\frac{1}{16}(I_2 - (I_t + s_t)^2)(I_2 - (I_t - (s_t + 1)^2) \tag{62}
\]

and

\[
K_6 = 4s_t(s_t + 1)K_4 \tag{63}
\]

where \( I_t \) is the value taken by the invariant \( I_1 \) (19) on the highest state.

\[
I_t = j_t + \beta \tag{64}
\]

It is interesting to note that the ratio \( K_6/4K_4 \) is simply the value of the \( so(3) \) Casimir for the highest state.

### 3.4 The Up, Down and Level tensorial operators. Identities

Since the states allowed at every level \( j_t - k \) (0 \( \leq k \leq q \)) have spin \( s_t - 1, s_t \) and \( s_t+1 \) and since the \( Q_i \) which map these states on those of \( J_{\text{value}} = j_t - k - 1 \) can move the \( s_{\text{value}} \) by at most 1 unit we are lead to define the following three obviously relevant tensorial operators
1. The up operator \((U^s_i)_{m,n}\) which maps the \(n\)th state \(-s \leq n \leq s\) of the representations \(s\) to the \(m\)th state \(-(s+1) \leq m \leq (s+1)\) of the representation \(s+1\) and which is defined by

\[
(U^s_i)_{m,n} = (s + 1)^{\frac{1}{2}}(2s + 3)^{\frac{3}{2}}C(s + 1, m; 1, i, s, n) \tag{65}
\]

where \(C(j3, m3; j1, m1, j2, m2)\) is the Clebsh-Gordon coefficient coupling \(j1\) and \(j2\) to make \(j3\). (We use the Condon-Shortley phase convention and normalisations.) Obviously this coefficient is non-zero only if \(m3 = m1 + m2\). For every \(i\), \(U^s_i\) is a \((2s + 3) \times (2s + 1)\) matrix.

2. The level operator \((L^s_i)_{n2,n1}\) which maps the \(n1\)th state \(-s \leq n1 \leq s\) of the representations \(s\) to the \(n2\)th state \(-s \leq n2 \leq s\) of the representation \(s\) and which is defined by

\[
(L^s_i)_{n2,n1} = s^\frac{1}{2}(s + 1)^{\frac{1}{2}}C(s, n2; 1, i, s, n1) \tag{66}
\]

For every \(i\), \(L^s_i\) is a \((2s + 1) \times (2s + 1)\) matrix.

3. The down operator \((D^s_i)_{n,m}\) which maps the \(m\)th state \(-(s+1) \leq m \leq (s+1)\) of the representations \(s+1\) to the \(n\)th state \(-s \leq n \leq s\) of the representation \(s\) and which is defined by

\[
(D^s_i)_{n,m} = (s + 1)^{\frac{1}{2}}(2s + 1)^{\frac{1}{2}}C(s, n; 1, i, s+1, m) \tag{67}
\]

For every \(i\), \(D^s_i\) is a \((2s + 1) \times (2s + 3)\) matrix.

These operators satisfy useful identities which we now list and which can be checked by explicit computations. In these identities we have suppressed the obvious matrix indices.

1. The down-up identity (for \(s \geq 1\)) is

\[
D^s_i U^s_j = \frac{1}{2}(L^s_i L^s_j + L^s_j L^s_i) + \frac{2s + 3}{2} \sum_{k} f_{i,j}^k L^s_k - (s + 1)^2 g_{i,j} I_{2s+1} \tag{68}
\]
where $I_{2s+1}$ is the unit matrix is the $2s + 1$-dimensional space of the representation $s$. Extra identities are valid for $s = 0$ and $s = 1/2$ leading to

$$D_i^{[0]} U_j^{[0]} = -g_{i,j} I_1$$

(69)

and

$$D_i^{[1/2]} U_j^{[1/2]} = 2f_{i,j}^k L^{[1/2]}_k - 2g_{i,j} I_2$$

(70)

2. The up-down identity (for $s \geq 0$) is

$$U_i^{[s]} D_j^{[s]} = \frac{1}{2}(L_i^{[s+1]} L_j^{[s+1]} + L_j^{[s+1]} L_i^{[s+1]}) - \frac{2s + 1}{2} f_{i,j}^k L^{[s+1]}_k$$

$$-(s + 1)^2 g_{i,j} I_{2s+3}$$

(71)

3. The non trivial level-level identity (for $s \geq 1/2$) is

$$L_i^{[s]} L_j^{[s]} = \frac{1}{2}(L_i^{[s]} L_j^{[s]} + L_j^{[s]} L_i^{[s]}) + \frac{1}{2} f_{i,j}^k L^{[s]}_k$$

(72)

This is nothing else than the commutation relations of the generators of $so(3)$. In other words, the factor in front of the right-hand side in (66) has been chosen in such a way that the $L_i$ satisfy exactly the commutation relations of the abstract $T_i$ operators (1). For $s = 1/2$, the identity can also be written more simply

$$L_i^{[1/2]} L_j^{[1/2]} = \frac{1}{2} f_{i,j}^k L^{[1/2]}_k + \frac{1}{4} g_{i,j} I_2$$

(73)

4. The up-up identity (for $s \geq 0$) is

$$U_i^{[s+1]} U_j^{[s]} = U_j^{[s+1]} U_i^{[s]}$$

(74)

5. The down-down identity (for $s \geq 0$) is

$$D_i^{[s]} D_j^{[s+1]} = D_j^{[s]} D_i^{[s+1]}$$

(75)

6. The level-up identities are two : namely (for $s \geq 1/2$)

$$L_i^{[s+1]} U_j^{[s]} = -\frac{1}{s} U_i^{[s]} L_j^{[s]} + \frac{s + 1}{s} U_j^{[s]} L_i^{[s]}$$

(76)
and the relation (again for $s \geq 1/2$) which certifies that the up operator $U_i$ behaves as a spin 1 operator

$$L_i^{[s+1]} U_j^{[s]} = U_j^{[s]} L_i^{[s]} + f_{i,j,k} U_k^{[s]}$$  \hspace{1cm} (77)$$

For $s = 0$, there is only one identity, namely

$$L_{i}^{[1]} U_{j}^{[0]} = f_{i,j,k} U_{k}^{[0]}$$  \hspace{1cm} (78)$$

7. The level-down identities are again two: namely (for $s \geq 1/2$)

$$L_i^{[s]} D_j^{[s]} = \frac{1}{s+2} D_i^{[s]} L_j^{[s+1]} + \frac{s+1}{s+2} D_j^{[s]} L_i^{[s+1]}$$  \hspace{1cm} (79)$$

and the relation (for $s \geq 1/2$) which certifies that the down operator $D_i$ behaves as a spin 1 operator

$$L_i^{[s]} D_j^{[s]} = D_j^{[s]} L_i^{[s+1]} + f_{i,j,k} D_k^{[s]}$$  \hspace{1cm} (80)$$

For $s = 0$, we have one identity

$$D_i^{[0]} L_j^{[1]} = f_{i,j,k} D_k^{[0]}$$  \hspace{1cm} (81)$$

These are all the identities we need to try to construct the representations.

4 The generic case

4.1 Form of the operators

We present here the form of the different operators in the generic case, i.e. when the number of levels, say $L \equiv j_t - j_b + 1$, is greater or equal to four and all the states of the (17) are present.

1. Without losing generality, we can assume that the operator $J$ can be diagonalized in blocks:

$$J = \begin{pmatrix} j_t & I_{(2s_t+1)} & 0 & 0 & \cdots \\ 0 & (j_t-1) & I_{(6s_t+3)} & 0 & \cdots \\ 0 & 0 & (j_t-2) & I_{(8s_t+4)} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$  \hspace{1cm} (82)$$
where the “levels” correspond to the subspaces of given $J$ (precisely to $J_{\text{value}} = j_t$, $j_t - 1$, \ldots, $j_b$) and where $I_m$ is a diagonal unit matrix of dimension $m$. The value of $m$ is $8s_t + 4$ from the $J_{\text{value}} = j_t - 2$ down, except for the two last $J_{\text{value}}$, namely for $j_b + 1$ and for $j_b$ for which we have respectively $m = 6s_t + 3$ and $m = 2s_t + 1$.

2. The $T_i$ operator assumes a block diagonal form made of $L_i^{[s_t]}$ type matrices inside the diagonal blocks of given $J_{\text{value}}$. More precisely in the block $[j_t, j_t]$ one has

$$T_i^{[j_t, j_t]} = L_i^{[s_t]}$$

in the block $[j_t - 1, j_t - 1]$

$$T_i^{[j_{t-1}, j_t-1]} = \begin{pmatrix} L_i^{[s_t-1]} & 0 & 0 \\ 0 & L_i^{[s_t]} & 0 \\ 0 & 0 & L_i^{[s_t+1]} \end{pmatrix}$$

Then starting from the block $[j_t - 2, j_t - 2]$ one has

$$T_i^{[j_{t-p}, j_{t-p}]} = \begin{pmatrix} L_i^{[s_{t-1}]} & 0 & 0 & 0 \\ 0 & L_i^{[s_t]} & 0 & 0 \\ 0 & 0 & L_i^{[s_{t+1}]} & 0 \\ 0 & 0 & 0 & L_i^{[s_{t+1}]} \end{pmatrix}$$

where $2 \leq p \leq j_b + 2$. For the two last blocks, namely $[j_b + 1, j_b + 1]$ and $[j_b, j_b]$, the content can be inferred from (83) $T_i^{[j_{b+1}, j_{b+1}]} \equiv T_i^{[j_{t-1}, j_t-1]}$ and $T_i^{[j_b, j_b]} \equiv T_i^{[j_t, j_t]}$.

3. The operators $Q_i$ have their representations in terms of the blocks situated exactly one step below the diagonal blocks (S2). In the block $[j_t - 1, j_t]$ one has a $(6s_t + 3) \times (2s_t + 1)$ matrix of the form

$$Q_i^{[j_{t-1}, j_t]} \equiv Q_i^{[1]} = \begin{pmatrix} c_1^{[1]} L_i^{[s_{t-1}]} \\ c_2^{[1]} L_i^{[s_t]} \\ c_3^{[1]} U_i^{[s_t]} \end{pmatrix}$$

where $c_1^{[1]}$, $c_2^{[1]}$ and $c_3^{[1]}$ are three arbitrary constants. The $[j_t - 2, j_t - 1]$
4. The form of the operators $\overline{Q}_i$ is obviously analogous to that of the $Q_i$ but the blocks are situated one step above the diagonal blocks i.e. in the positions $[j_i - p, j_i - p - 1]$, again in terms of the tensor operators $U_i^{[s]}, D_i^{[s]}$ and $L_i^{[s]}$, and the corresponding constants are labelled $\overline{c}$.

More precisely, the specific form of the operators $\overline{Q}_i$ can be obtained from
the form of the \( Q_i \) above by transposition and the interchange of the tensor operators \( D_i \) and \( U_i \) for the same svalue, with no change on the \( L_i \).

For later convenience, it is useful to define the following matrices with the coefficients \( c \) and \( \overline{c} \):

\[
C^{[1]} = \begin{pmatrix} c_{11}^{[1]} \\ c_{21}^{[1]} \\ c_{31}^{[1]} \end{pmatrix}, \quad \overline{C}^{[1]} = \begin{pmatrix} \overline{c}_{11}^{[1]} \\ \overline{c}_{12}^{[1]} \\ \overline{c}_{13}^{[1]} \end{pmatrix}
\]

and so on for \( C^{[2]} \) (4×3 matrix), \( \overline{C}^{[2]} \) (3×4 matrix) . . . , i.e. the matrices obtained from the \( Q \)'s and the \( \overline{Q} \)'s by replacing the operators \( U, D \) and \( L \) by the number one.

4.2 The equations

With the forms of the operators given above in (82,90) the equations (1,7,8,9,10,11) are automatically fulfilled. What remains to be imposed are the anticommutation relations of the \( Q_i \) and \( \overline{Q}_i \) among themselves (12, 14,16).

Obviously these relations are used to determine the parameters \( c \) and \( \overline{c} \). Using the identities of section (3.4), it appears that (12) results in the following constraints on the matrices \( C^{[k]} \)

\[
C^{[k]} C^{[k+1]} = 0 \quad , \quad k = 1, \cdots , L - 2 \quad (92)
\]

Similarly, using (14) together with the identities, we obtain

\[
\overline{C}^{[k]} \overline{C}^{[k+1]} = 0 \quad , \quad k = 1, \cdots , L - 2 \quad (93)
\]

The equations on \( c, \overline{c} \) obtained by imposing the relations (16) cannot be written is such a compact way. We observe that the anticommutator of the left hand side take a block diagonal form:

\[
\{ Q_a, Q_b \}_{j^1 j^1} = \overline{Q}_b^{[1]} Q_a^{[1]} \\
\{ Q_a, \overline{Q}_b \}_{j^1 p j^1 - p} = Q_a^{[p]} \overline{Q}_b^{[p]} + \overline{Q}_b^{[p+1]} Q_a^{[p+1]} \quad , \quad 1 \leq p \leq L - 2 \quad (94)
\]

\[
\{ Q_a, \overline{Q}_b \}_{j^2 j^2} = Q_a^{[L-1]} \overline{Q}_b^{[L-1]} \quad (95)
\]
Within each block, the identities of section (3.4) can be used to put the expressions as combinations of linearly independent operators. The identification of the coefficients of the independent operators of (96) with those of the right hand side of (16) then leads to a system of equations for products of parameters \( c \) with parameters \( \tau \).

### 4.3 Similarity transformations

It is not difficult to see that there remains some freedom in the definition of the operators \( Q \) and \( \overline{Q} \). This is related to the fact that we can rescale independently the vectors in the different spin representations and mix in an arbitrary way the two spin representations within one level. This freedom results in the following redefinition of the matrices \( C^{[k]} \) and \( \overline{C}^{[k]} \):

\[
C^{[k]} = U_{k+1}C^{[k]}U_k^{-1}
\]

\[
\overline{C}^{[k]} = U_kC^{[k]}U_{k+1}^{-1}
\]

for \( k = 1, \ldots, L-1 \). Here \( U_1 \) and \( U_L \) are (non zero) numbers, \( U_2 \) and \( U_{L-1} \) are \( 3 \times 3 \) diagonal invertible matrices and \( U_k \) are invertible matrices of the form

\[
U_k = \begin{pmatrix}
\mu_k & 0 & 0 & 0 \\
0 & \nu_k & \lambda_k & 0 \\
0 & \theta_k & \rho_k & 0 \\
0 & 0 & 0 & \sigma_k
\end{pmatrix}, \quad k = 3, \ldots, L-2.
\]

All parameters \( \mu_k, \nu_k, \ldots \) appearing in \( U_1, \ldots, U_L \) are complex numbers. We shall use this freedom to put the matrices \( C^{[k]} \) in a particularly simple form.

### 4.4 Canonical form of the \( C \) matrices

We now determine the parameters \( c \) and \( \tau \) in the case when \( L \geq 4 \) (lower dimensional cases are treated later) and assuming that the \( Q \) and \( \overline{Q} \) operators connect all, a priori possible, pairs of spins between consecutive levels (i.e. with the pattern of (17)).

We first concentrate on equations (92). These equations, together with the similarity transformations freedom (97,98,99) allows us to determine all
the matrices $C$ in function of only one parameter. The canonical forms of them read as follows

$$C^{[1]} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad C^{[2]} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$ (100)

$$C^{[3]} = C^{[4]} = \cdots = C^{[L-3]} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$ (101)

$$C^{[L-2]} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & -X & X \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad C^{[L-1]} = (1, -1, X)$$ (102)

This parametrisation greatly simplifies the solution of the other equations. In particular it leads, for (96), to linear constraints in the $c^{[k]}$'s. Moreover, the form of eq.(96) allows one to solve these linear equations recursively in $[k]$.

Let us now discuss how these equations are solved. Imposing the relation for the first block leads to a self consistent linear system for the parameter $c^{[1]}_1(k = 1, 2, 3)$. The solution of this system reads

$$\bar{c}^{[1]}_1 \equiv A = \frac{(I_t - s_t - 1)^2 - I_t}{2s_t(2s_t + 1)}$$ (103)

$$\bar{c}^{[1]}_2 \equiv B = \frac{s_t(s_t + 1) - (I_t - 1)^2 + I_t}{2s_t(s_t + 1)}$$ (104)

$$\bar{c}^{[1]}_3 \equiv C = \frac{(I_t + s_t)^2 - I_t}{2(2s_t + 1)(s_t + 1)}$$ (105)

where we define $I_t \equiv j_t + \beta$, i.e. the value of the operator $I_1$ for $J = j_t$ (remember we have normalized $\alpha = 1/2$). Remark that the Casimir value $K_4$ is nothing else but $-AC/16$. If we hadn’t normalized $C^{[1]}$ to unit values (100), the unique solution for $A, B$ and $C$ would have corresponded in general to $A = \bar{c}^{[1]}_1 c^{[1]}_1, B = \bar{c}^{[1]}_2 c^{[2]}_1$ an to $C = \bar{c}^{[1]}_3 c^{[3]}_1$. Hence the the restriction of the eigenspace of $J_{\text{value}} = j_t - 1$ to a space with two $so(3)$ spins instead of three leads to the vanishing of one of the functions $A, B$ or $C$.  

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It is useful to note that under the involution \( \{ s_t \leftrightarrow -(s_t + 1) \} \),

\[
\begin{align*}
B &= B \bigg|_{s_t \rightarrow -(s_t + 1)} \\
C &= A \bigg|_{s_t \rightarrow -(s_t + 1)}
\end{align*}
\]  

(106)

which means that \( B \) is invariant while \( A \) and \( C \) are interchanged. Analogous involutions will occur at higher levels.

Considering the second block in (106) we obtain an (apparently) overdetermined system of 19 linear equations in 10 variables. The solution nevertheless exists, is unique and reads

\[
\begin{pmatrix}
\frac{s_t+1}{s_t} B & C - \frac{1}{s_t} B & -C & 0 \\
-\frac{s_t+1}{s_t} A & \frac{1}{s_t} A & -\frac{1}{s_t+1} C & -\frac{s_t}{s_t+1} B \\
0 & -A & A + \frac{1}{s_t+1} B & -\frac{s_t+1}{s_t} B
\end{pmatrix}
\]  

(107)

Under the involution \( \{ s_t \leftrightarrow -(1 + s_t) \} \) (see (106)) the elements of \( \overline{C}^{[2]} \) are interchanged as follows \( \overline{C}^{[2]}_{i,j} \leftrightarrow \overline{C}^{[2]}_{4-i,5-j} \).

For the next blocks, the number of equations and of variables are respectively 24 and 14. Again the equations are compatible with each other and provide a unique solution.

The structure of the matrices \( C^{[k]} \) \( (k > 2) \) suggests to use a similar block decomposition for the \( \overline{C}^{[k]} \), i.e.

\[
\overline{C}^{[k]} = \begin{pmatrix} \overline{C}^{[k]}_{11} & \overline{C}^{[k]}_{12} \\ \overline{C}^{[k]}_{21} & \overline{C}^{[k]}_{22} \end{pmatrix}
\]  

(108)

The solution for \( \overline{C}^{[3]} \) reads

\[
\overline{C}^{[3]}_{11} = \frac{1}{s_t} \begin{pmatrix} -A(s_t+1) - B + C s_t & A + B(1 - s_t) + C s_t (s_t - 1) \\ -A(s_t+1)^2 - B(s_t+1) & A(s_t+1) - B(s_t^2 - 1) \end{pmatrix}
\]

(109)

\[
\overline{C}^{[3]}_{22} = \frac{1}{(s_t + 1)^2} \times
\]

24
\[
\begin{pmatrix}
-Bs_t(s_t + 2) - Cs_t & s_tB - s_t^2C \\
A(s_t + 1)(s_t + 2) + B(s_t + 2) + C & -A(s_t + 1) - B + Cs_t
\end{pmatrix}
\]

(110)

\[
C_{12}^{[3]} = -C \frac{s_t}{(s_t + 1)^2} \begin{pmatrix}
 s_t + 1 & 0 \\
 1 & s_t
\end{pmatrix}
\]

(111)

\[
C_{21}^{[3]} = -A \frac{(s_t + 1)}{s_t^2} \begin{pmatrix}
 s_t + 1 & -1 \\
 0 & s_t
\end{pmatrix}
\]

(112)

We remark that the blocks \(C_{12}^{[3]}\) and \(C_{21}^{[3]}\) are respectively proportional to the matrix element \(c_{[1]}^{[1]}\) and \(c_{[1]}^{[1]}\). Moreover, it is easy to check that \(C_{21}^{[3]}\) and \(C_{12}^{[3]}\) are related by the involution \(\{s_t \leftrightarrow -(1 + s_t)\}\)

\[
C_{21}^{[3]} = \sigma_1 C_{12}^{[3]} \sigma_1 \bigg|_{s_t \to -(s_t + 1)}
\]

(113)

and that

\[
C_{22}^{[3]} = \sigma_1 C_{11}^{[3]} \sigma_1 \bigg|_{s_t \to -(s_t + 1)}
\]

(114)

where \(\sigma_1\) is the first Pauli matrix.

The conditions coming from the next blocks determine a set of recursive relations for the elements of the 4\(\times\)4 matrices \(C_{[k]}^{[k]}\), \([k] > 3\). The structure of the matrices \(C_{[k]}^{[k]}\) is such that the equations relative to the four 2\(\times\)2 blocks defined above decouple.

First, the blocks \(C_{12}^{[k]}\) and \(C_{21}^{[k]}\) satisfy the recurrence relations

\[
C_{12}^{[k+1]} = \frac{s_t}{s_t + 1} C_{12}^{[k]}
\]

(115)

Hence, for all \([k] \geq 3\)

\[
C_{12}^{[k]} = -C \frac{s_t^{k-2}}{(s_t + 1)^{k-1}} \begin{pmatrix}
 s_t + 1 & 0 \\
 1 & s_t
\end{pmatrix}
\]

(116)

The matrices \(C_{21}^{[k]}\) are obtained analogously to eq. (113) by

\[
C_{21}^{[k]} = \sigma_1 C_{12}^{[k]} \sigma_1 \bigg|_{s_t \to -(s_t + 1)}
\]

(117)
The recursive equations for the block $C_{11}^{[k]}$ are

$$C_{11}^{[k+1]} + C_{11}^{[k]} = k^2 M_2 + k M_1 + M_0$$

(118)

where

$$M_2 = \frac{1}{2s_t^2} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$

(119)

$$M_1 = \frac{1}{s_t^2} \begin{pmatrix} I_t & -(s_t + I_t) \\ -(s_t - I_t) & -I_t \end{pmatrix}$$

(120)

$$M_0 = \frac{1}{2s_t^2} \begin{pmatrix} s_t(s_t + 1) + I_2 - I_t^2 & (s_t + I_t)^2 - I_2 \\ -(s_t - I_t)^2 - I_2 & -(s_t(s_t - 1) + I_2 - I_t^2) \end{pmatrix}$$

(121)

The solution, with the appropriate boundary condition $C_{11}^{[3]}$ (109), can easily be obtained

$$C_{11}^{[k]} = \frac{-(-1)^k C_{11}^{[3]} + (k^2 + 9(-1)^k) M_2}{2} + \frac{(k + 3(-1)^k) M_1 - M_2}{2}$$

(122)

The recursion equation for the matrix $C_{22}^{[k]}$ is completely analogous to the equation for $C_{11}^{[k]}$. The solution is simply obtained, for $k \geq 3$ by the involution $\{s_t \leftrightarrow -(1 + s_t)\}$

$$C_{22}^{[k]} = \sigma_1 C_{11}^{[k]} \sigma_1 \bigg|_{s_t \rightarrow -(s_t+1)}$$

(123)

If we define the matrices $N_i$ from the matrices $M_i$ by the involution

$$N_i = \sigma_1 M_i \sigma_1 \bigg|_{s_t \rightarrow -(s_t+1)}$$

(124)

the $C_{22}^{[k]}$ satisfy an equation of the form (118) with the $M_i$ replaced by the $N_i$.

In this approach, we have solved (96) starting from the highest value $j_t$ of $J$ and going down. Alternatively, these equations can be solved by starting from the lowest value $j_b$ of $J$ and proceeding by going up. This procedure
gives for instance the following values for $C^{[L-1]}$

$$
C_{11}^{[L-1]} \equiv \tilde{A} = \frac{(I_b + s_t + 1)^2 - I_2}{2s_t(2s_t + 1)} \quad (125)
$$

$$
C_{21}^{[L-1]} \equiv \tilde{B} = \frac{(I_b + 1)^2 - I_2 - s_t(s_t + 1)}{2s_t(s_t + 1)} \quad (126)
$$

$$
C_{31}^{[L-1]} \equiv \tilde{C} = \frac{(I_b - s_t)^2 - I_2}{2(s_t + 1)(2s_t + 1)X} \quad (127)
$$

with

$$
I_b = j_b + \beta = I_t - L + 1 \quad (128)
$$

One condition for the representation to be irreducible is that $\tilde{A}\tilde{B}\tilde{C} \neq 0$. The special values of the parameters which annihilate $\tilde{A}\tilde{B}\tilde{C}$ are discussed in the next section. It should also be remarked that, due to the presence of the, yet undetermined, parameter $X$ in eq. (102), the involution, analogous to (106), has to be changed slightly to

$$
\tilde{B} = \tilde{B} \bigg|_{s_t \rightarrow -(s_t + 1)} \quad (129)
$$

$$
\tilde{C} = \frac{1}{X} \left( \tilde{A} \bigg|_{s_t \rightarrow -(s_t + 1)} \right) \quad (130)
$$

For $C^{[L-2]}$ one obtains

$$
C^{[L-2]} = \begin{pmatrix}
-\frac{s_t + 1}{s_t} \tilde{B} & \frac{s_t + 1}{s_t} \tilde{A} & 0 \\
-\frac{1}{s_t} \tilde{B} - XC & \frac{1}{s_t} \tilde{A} & X \tilde{A} \\
0 & -\frac{1}{s_t + 1} \tilde{C} & -\tilde{A} + \frac{1}{s_t + 1} \tilde{B}
\end{pmatrix} \quad (131)
$$

and, using the block form (108) for $C^{[L-3]}$

$$
C_{11}^{[L-3]} = \frac{1}{s_t^2} \begin{pmatrix}
-A(s_t + 1) + B - CXs_t & (s_t + 1)(A(s_t + 1) - B) \\
-A + (s_t - 1)(-\tilde{B} + \tilde{C}Xs_t) & (s_t + 1)(A + B(s_t - 1))
\end{pmatrix} \quad (132)
$$

$$
C_{22}^{[L-3]} = \frac{1}{(s_t + 1)^2} \begin{pmatrix}
\tilde{B}s_t(s_t + 2) + \tilde{C}Xs_t & (s_t + 2)(-\tilde{A}(s_t + 1) + \tilde{B}) + CXs_t \\
s_t(B - CXs_t) & -\tilde{A}(s_t + 1) + B - CXs_t
\end{pmatrix} \quad (133)
$$
\[ C_{12}^{[L-3]} = -\tilde{A}X \frac{s_t + 1}{s_t^2} \begin{pmatrix} s_t + 1 & 0 \\ 1 & s_t \end{pmatrix} \]  
(134)

\[ C_{21}^{[L-3]} = -\tilde{C} \frac{s_t}{(s_t + 1)^2} \begin{pmatrix} s_t + 1 & -1 \\ 0 & s_t \end{pmatrix} \]  
(135)

The matrices \( C^{[L-k]} \) (for \( k > 3 \)) can then be determined recursively.

Remarkably, the value of the matrix \( C^{[L-3]} \) predicted from the recurrence relations, starting from the top, match with the value obtained by solving the equation from below provided only one relation among the parameters \( j_t, L, s_t \) is imposed. The value of the parameter \( X \) (see (102)) is also uniquely predicted by this procedure. The expressions of the constraint and of \( X \) appear to be quite different according to the parity of \( L \).

1. For \( L \) even, \( X \) is uniquely determined to be

\[ X = -\left( \frac{s_t}{s_t + 1} \right)^{L-2} \frac{(2s_t + L - 1)^2 - 4I_2}{(2s_t - L + 3)^2 - 4I_2} \]  
(136)

At the same time, the consistency of all equations fixes uniquely \( I_t \) as a function of \( L \):

\[ I_t = \frac{L - 1}{2} \]  
(137)

As a consequence, the spectrum of the operator \( I_1 \) is \( I_t, I_t - 1, \ldots, I_b = -I_t \), symmetric around zero. The spectrum of \( J \) is, obviously, the spectrum of \( I_1 \) shifted by \( -\beta \). In (136) we have to exclude the limiting cases \( X = 0 \) and \( X = \infty \)

\[ X = 0 \rightarrow I_2 = \frac{(2s_t + L - 1)^2}{4} \]  
(138)

which corresponds to a special limit (148), or

\[ X = \infty \rightarrow I_2 = \frac{(2s_t - L + 3)^2}{4} \]  
(139)

which corresponds to (157).

2. For \( L \) odd, the parameter \( X \) is fixed as

\[ X = -\left( \frac{s_t}{s_t + 1} \right)^{L-2} \frac{2s_t + L - 1}{2s_t - L + 3} \]  
(140)
while the quantity $I_t$ is determined by the equation

$$(I_t - \frac{L - 1}{2})^2 = I_2 + \frac{(2s_t + L - 1)(2s_t - L + 3)}{4}$$

which allows for two values of $I_t$ and, hence, of the spectrum of $I_1$. The two corresponding representations transform into each other under the automorphism (24). In (140) the limiting value $X = \infty$ has to be excluded, i.e.

$$X = \infty \rightarrow L = 2s_t + 3$$

These values correspond to the special case (143).

Hence, for fixed values of $L$ and of $s_t$, all the matrix elements of $C^{[k]}$ and of $\overline{C}^{[k]}$ are uniquely determined.

We further checked that the equations (14) (which leads to quadratic equations among the $C^{[k]}$) are automatically obeyed.

The discussion above demonstrates that the algebra $\mathcal{A}$ admits an infinite tower of irreducible representations labelled by the integers $L$ and $2s_t$. Their dimensions $d = 4(2s_t + 1)(L - 2)$ can be arbitrarily large and the spectrum of the operator $J$ is quantized. We refer them to as to the generic representations.

This result contrasts in many respects with its counterpart for the graded Lie algebra $osp(2, 2)$. In this case, the generic irreducible and finite dimensional representations consist of three levels only, with the following spin content (3)

$$(s_t), (s_t - 1, s_t + 1), (s_t)$$

analogous to the exceptional case (52).

4.5 Special limits

The $H_+$ and $H_-$ series of representations

We will now discuss the way to obtain the representations of type $H_+$ and $H_-$ of the theorem. We have constructed all the matrix elements of these representations by solving all the equations restricted, at the start, to the relevant eigenspaces of given $J$ (see (13), (13)). We have then realized that all the necessary information can be extracted from the generic representation extensively described in the previous section.
Let us start with the representation of type $H_+$. In this case, the equations corresponding to the bloc (14) lead to a same system of three equations in two variables (for instance $B = C_{12}^{[1]}C_{21}^{[1]}$ and $C = C_{13}^{[1]}c_{31}^{[1]}$). These equations are obviously identical to those determining $A, B$ and $C$ (103,104,105) when $A$ is put to zero. Since the solution in the generic case was unique, the new equations are compatible with each under the condition that the missing variable $A = c_{11}^{[1]}$ vanishes. Similarly, the equations associated to the last bloc (15) leads to the condition $\tilde{A} = 0$ (see (44)). Therefore, two necessary conditions for a representation of type $H_+$ to occur read

$$A = 0 \implies I_2 = (I_t - s_t - 1)^2$$
$$\tilde{A} = 0 \implies I_2 = (I_b + s_t + 1)^2$$ (144)

In this case, many elements of the matrices $C^{[k]}$ ($3 \leq k \leq L - 3$) (in particular $C_{12}^{[k]}$ and $C_{21}^{[k]}$) vanish and the generic representation can consistently be restricted to the subspace

$$(s_t), (s_t, s_t + 1), (s_t, s_t + 1), \ldots , (s_t, s_t + 1), (s_t), L \text{ levels}$$ (145)

The restriction of the matrices $C^{[k]}$ and $\tilde{C}^{[k]}$ to the lower-right $2 \times 2$ block provides the relevant matrix elements. The matrices $\tilde{C}^{[k]}_{22}$ obey the recurrence relation of the generic case but the initial condition if fixed already by $\tilde{C}^{[2]}_{22}$ (see (124)) i.e.

$$\tilde{C}^{[k]}_{22} = (-1)^k\tilde{C}^{[2]}_{22} + (k^2 - 4(-1)^k)N_0^2 + (k - 2(-1)^k)N_1 - N_2$$
$$+ (1 - (-1)^k)\frac{2N_0 - N_1}{4}$$ (146)

with

$$\tilde{C}^{[2]}_{22} = \frac{1}{s + 1} \begin{pmatrix} -C & -sC \\ B & sB \end{pmatrix}$$ (147)

We have checked that all the commutation relations are then satisfied provided that the consistency relations (137) (for $L$ even) or (141) (for $L$ odd) are also satisfied.

Summarising the results, we conclude that
1. The representation $H_+$ exists, when the number of levels $L$ is even, if
\[ I_t = \frac{L-1}{2} \]
\[ I_2 = \frac{1}{4}(L-2s_t-3)^2 \] (148)

Since $L$ and $2s_t$ are integers, the parameter $I_2 \equiv \beta^2 - 2\gamma$ is restricted to a discrete set of special values.

2. The representation $H_+$ exists, when the number of levels $L$ is odd, if
\[
L = 2s_t + 3 \\
I_2 = (I_t - s_t - 1)^2
\] (149)

and we see in particular that $s_t$ has to be restricted to be an integer.

Though the identities among the tensorial operators, which were written in section 3.4, take a different form when $s = 0$ or $s = 1/2$, about all the cases which involve these spins in one of the spaces behave in a normal way with one important exception pertaining to the representation $H_+$ when $s_t = 0$. Indeed, the $J, s$ hierarchy reduces to
\[
(0), (1), (0,1), \ldots, (0,1), (1), (0)
\] (150)

The spin $s = 0$ space is missing at levels $j_t - 1$ and $j_b + 1$. The equations which have to be satisfied are less numerous and we have obtained only one restriction instead of two both for even and odd $L$.

a. For $L$ even, the restriction is
\[ I_t = \frac{L-1}{2} \text{ for } L \text{ even and } s_t = 0 \] (151)

Moreover, for example, the matrix $\overline{C}^{(1)}$ has only one entry which should be non zero. Other matrix elements $\overline{C}^{(3)}(1,2), \ldots$ cannot be zero also. This excludes some values for $I_2$. Precisely
\[ I_2 \neq \frac{(L-(2p-1))^2}{4} + (p-1)(p-2) \text{ for } p = 1, 2, \ldots, \left\lfloor \frac{L}{4} \right\rfloor + 1 \] (152)
b. For \( L \) odd, the restriction reads

\[
I_2 = I_t^2 - (L-1)I_t + \frac{(L-1)(L-2)}{2} \quad \text{for } L \text{ odd and } s_t = 0 \tag{153}
\]

Again the matrix element of \( C^{[1]} \) and some other matrix elements have to be non zero. This excludes some values for \( I_t \). Precisely

\[
I_t \neq (2p + 1)/2 \quad \text{for } p = 1, 2, \ldots, L - 3 \tag{154}
\]

Let us now discuss, in the same way, the conditions of occurrence of the representation \( H_- \). Following the same reasoning as above, one shows that the necessary conditions for this representation to exist are \( C = \tilde{C} = 0 \), i.e.

\[
C = 0 \quad \rightarrow \quad I_2 = (I_t + s_t)^2 \\
\tilde{C} = 0 \quad \rightarrow \quad I_2 = (I_b - s_t)^2 \tag{155}
\]

Then, the generic representation can consistently be restricted to the subspace

\[
(s_t), (s_t - 1, s_t), (s_t - 1, s_t), \ldots, (s_t - 1, s_t), (s_t) \quad \text{, } L \text{ levels} \tag{156}
\]

Compatibility of the equations (153) with the consistency relations (137) (for \( L \) even) or (141) (for \( L \) odd) lead to the conclusion.

3. The representation \( H_- \) exists only when the number of levels \( L \) is even and if

\[
I_t = \frac{L-1}{2} \\
I_2 = \frac{1}{4}(L + 2s_t - 1)^2 \tag{157}
\]

The parameter \( I_2 \) is again restricted to a discrete set of special values.
4.6 Special limit

The exceptional representation

The exceptional representation can be obtained as a special limit of the generic series of representations. In complete analogy with the arguments given in the preceding section, it is obtained by putting to zero the parameter

\[ B = c_1 c_2 \]

(104)

which means that the spin states at level \( j_t - 1 \) have to be restricted to the values \( s_t - 1 \) and \( s_t + 1 \) and thus that the states corresponding to spin \( s_t \) at that level, have to be discarded. It is then easy to see that the representation closes by the addition of the next and last level \( j_t - 2 = j_b \) containing one set of \( s_t \) states only.

Given \( s_t \) there are two \( j_t \) fulfilling (158). These two cases are related by the first automorphism of the algebra (24,25) which transforms \( I_t \) into \(-I_b\) and thus \( I_t - 1 \) into \(-I_b - 1 = -(I_t - 1)\).

In obvious notation, the representation is completely determined by

\[ C^{[1]} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad C^{[2]} = \begin{pmatrix} 1 & -1 \end{pmatrix} \]

(159)

\[ \overline{C}^{[1]} = \frac{1}{2s_t + 1} \begin{pmatrix} -(I_t - s_t - \frac{3}{2}) & (I_t + s_t - \frac{1}{2}) \end{pmatrix} \]

(160)

and

\[ \overline{C}^{[2]} = \frac{1}{2s_t + 1} \begin{pmatrix} I_t + s_t - \frac{1}{2} \\ I_t - s_t - \frac{3}{2} \end{pmatrix} \]

(161)

4.7 Gluing of representations

The \( T_+ \) and \( T_- \) representations

We have also constructed explicitly the representations corresponding to the cases \( T_+ \) of the classification (50). The case \( T_- \) (51) can be obtained from the case \( T_+ \) by the automorphism (24) of the algebra.

a. The representation \( T_+ \) exists only if \( L \), the number of levels, is \textbf{even} and if the invariants \( I_1 \) and \( I_2 \) are fixed as follows :

\[ I_t = s_t + \frac{L + 1}{2} \]
\[ I_2 = \frac{(L - 1)^2}{4} \]  

(162)

Note the restricted values of the \( I_2 \) invariant. The representation \( T_+ \), however, does not exist if \( s_t = 0, s_b = 1 \).

b. The representation \( T_- \) exists only if \( L \), the number of levels, is \textbf{even} and if the invariants \( I_1 \) and \( I_2 \) are fixed as follows:

\[ I_t = -s_t + \frac{L - 1}{2} \]

\[ I_2 = \frac{(L - 1)^2}{4} \]  

(163)

Note the restricted values of the \( I_2 \) invariant. The representation \( T_- \), however, does not exist if \( s_t = 1, s_b = 0 \).

Again these representations are strongly related to the generic representations. In order to perceive the connection let us first remark that the representations \( H_- \) (when the levels \( j_t \) and \( j_b \) have spin \( s \)) and \( H_+ \) (when the levels \( j_t \) and \( j_b \) have spin \( s - 1 \)) have the same spin pattern as far as their “internal” part is concerned:

\[ H_- (j_t \text{ with spin } s) \]

\[ (s), (s - 1, s), \ldots, (s - 1, s), (s) \]  

(164)

\[ H_+ (j_t \text{ with spin } s - 1) \]

\[ (s - 1), (s - 1, s), \ldots, (s - 1, s), (s - 1) \]  

(165)

It is therefore tempting to try to match the upper part of the first of these representations with the lower part of the second one to produce a \( T_- \) (\( j_t \) with spin \( s_t = s, j_b \) with spin \( s_b = s_t - 1 \)) representation. The alternative matching would produce an \( T_+ \) (\( j_t \) with spin \( s_t = s - 1, j_b \) with spin \( s_b = s_t + 1 = s \)) representation.

For the representation \( H_- \) (upper part of \( T_- \)) the relevant part of the operators \( Q_i \) is given by the matrix (see [101])

\[ C^{[k]}_{11} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \]  

(166)

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and the relevant part of the operator \( \overline{Q}_i \) is parametrized by \( \overline{C}^{[k]}_{11} \) (see (122)). For the representation \( H_+ \) (lower part of \( T_- \)) the relevant matrices are respectively

\[
C^{[k]}_{22} = \begin{pmatrix}
-1 & 1 \\
-1 & 1
\end{pmatrix}
\] (167)

and \( \overline{C}^{[k]}_{22} \) (see (123)) with however \( s_t \) replaced by \( s_t - 1 \). We see at once that a smooth matching requires to change the sign of the operators \( Q_i \) and \( \overline{Q}_i \) of one of the two representations involved (using e.g. the automorphism (24) with \( \kappa = -1 \)). Imposing the equality between the blocks \( C^{[k]}_{11} \) appearing in the \( H_- \) representation with the \( -\overline{C}^{[k]}_{22} \) block appearing in the \( H_+ \) representation (with, remember, \( s_t \) shifted into \( s_t - 1 \)) at any level \( k \) implies the conditions (162) for the representation \( T_+ \) can be seen as coming from \( A(s_t) = 0 \) and \( C(s_t \to s_t + 1) = 0 \). In an analogous way, the conditions (163) for the representation \( T_- \) come from \( C(s_t) = 0 \) and \( \overline{A}(s_t \to s_t - 1) = 0 \).

5 Relations with QES operators

All the operators constructed above can be represented by linear differential operators preserving some vector space, say \( P(n_1, n_2, \ldots, n_k) \), whose vectors are

\[
p_{n_1}(x), \ p_{n_2}(x), \ldots, \ p_{n_k}(x)
\] (168)

(where \( p_n(x) \) are polynomials of degree at most \( n \) in \( x \)) for suitable values of \( k \) and \( n_k \).

For \( N = 1 \), this can be achieved by means of the following correspondence \((n \equiv 2s)\)

\[
L_a^{[s]} \rightarrow J_a(n) \equiv \begin{pmatrix}
\frac{d}{dx}, \ x \frac{d}{dx} - \frac{n}{2}, \ x^2 \frac{d}{dx} - nx
\end{pmatrix}
\]

\[
U_a^{[s]} \rightarrow q_a \equiv \begin{pmatrix}
1, \ x, \ x^2
\end{pmatrix}
\]

\[
D_a^{[s]} \rightarrow T_a(n) \equiv \begin{pmatrix}
\frac{d^2}{dx^2}, \ (x \frac{d}{dx} - n - 1) \frac{d}{dx}, \\
(x \frac{d}{dx} - n - 1)(x \frac{d}{dx} - n - 2)
\end{pmatrix}
\] (169)
The operators $J_a(n)$ are the ones introduced by Turbiner [1]. They preserve the vector space $P(n)$. The operators $q_a$ transform $P(n)$ into $P(n + 2)$ and the operators $\tilde{q}_a(n)$ transform $P(n + 2)$ into $P(n)$.

The equations (169) provide a correspondence between the tensorial operators (65, 66, 67) and linear differential operators. All the identities of section (3.4) are also obeyed by the differential operators. Only the metric, say $\tilde{g}$, is different from our metric (3) because the choice (169) corresponds to the normalisation: $\tilde{g}_{+,-} = -2, \tilde{g}_{0,0} = 1$.

The operators preserving $P(n, n - 2)$, which are at the root of this work, correspond to the representation $T_+$ for two levels and $s_l = n/2$.

The classification of linear differential operators preserving $P(n_1, \ldots, n_k)$ [8] involves a number of generators which quickly grows with $N$. The generators close under an appropriate choice for the commutators and anticommutators. The underlying algebraic structure is, in this respect, still rather obscure. The sets of ten differential operators obtained by applying the correspondence (169) to the representations constructed in the previous sections allows one to write (considering the elements of their enveloping algebra) the set of all differential operators underlying the algebra $\mathcal{A}$ as a hidden symmetry.

6 Conclusions

During the last years, many different algebras appeared to be relevant in several domains of theoretical physics: graded algebras, Virasoro and Kac-Moody algebras, W-algebras, ... Some of these mathematical structures can further be generalized and considered as deformed algebras in the framework of quantum algebras.

The study of the hidden symmetries underlying the quasi exactly solvable equations has revealed the occurrence of yet other types of graded (but not Lie) algebras, the ones called $\mathcal{A}(\Delta)$ in this paper. Given the integer $\Delta$, $\mathcal{A}(\Delta)$ contains $so(3) \times u(1)$ as a bosonic subalgebra and two sets of fermionic generators, each of them transforming as a spin $s = \Delta/2$ multiplet. In this respect, the $\mathcal{A}(\Delta)$ algebras extend the well known $N = 2$ supersymmetric algebra $osp(2, 2)$ with which it coincides for $\Delta = 1$.

In this paper we have studied and classified the irreducible, finite dimensional representations of $\mathcal{A}(2)$. It appears that the representations of this
algebra possess a rich structure. Namely, they assemble into five independant families plus one exceptional representation.

Many new computations could be carried out in relation with the algebras $\mathcal{A}(\Delta)$ for $\Delta > 2$. For example: a concise formulation of their structure constants and the classification of their representations. More challenging is the construction of physical systems admitting $\mathcal{A}(\Delta)$ as a hidden symmetry. The interesting examples, known so far, are related to $\mathcal{A}(1) \ [3, 6, 7, 10]$. In absence of any real physical example related to $\mathcal{A}(2)$, we simply mention a mathematical application which is discussed in $[11]$: the finite dimensional representations of the Lie superalgebra $osp(3,2)$ can be formulated in terms of some of the operators (169).

The algebraic structure $\mathcal{A}(\Delta)$ could also be looked at from the point of view of quantum deformations. Indeed, considering finite difference QES equations (rather than differential QES equations), it was recognized that the hidden algebra becomes $sl(2)_q$, a deformation of $sl(2)$. Therefore, we can hope that some deformations of the algebra $\mathcal{A}(\Delta)$ will emerge from the study of finite difference QES systems.
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