The transient case of the quenched trap model

Stanislav Burov
Physics Department, Bar-Ilan University, Ramat Gan 5290002, Israel
E-mail: stasbur@gmail.com

Received 17 March 2020
Accepted for publication 11 May 2020
Published 14 July 2020

Abstract. In this work the anomalous diffusion in the quenched trap model with diverging mean waiting times is examined. The approach of randomly stopped time is extensively applied in order to obtain asymptotically exact representation of the disorder averaged positional probability density function. We establish that the dimensionality and the geometric properties of the lattice, on top of which the disorder is imposed, dictate the plausibility of the approximation that only includes annealed disorder. Specifically, for any case when the probability to return to the origin ($Q_0$) is less than 1, i.e. the transient case, the quenched trap model can be mapped on to the continuous time random walk. The explicit form of the mapping is provided. In the case when an external force is applied on a tracer particle in a media described by the quenched trap model, the response to such force is calculated and a non-linear response for sufficiently low dimensionality is observed.

Keywords: Brownian motion, diffusion, driven diffusive systems, transport properties
Brownian motion is probably the simplest manifestation of a transport in random environment. In this case the particle path is constantly modified by collisions with molecules that compose the surrounding media. The trajectory will appear as if the direction of motion is randomly changes as a function of time and a simple random walk (RW) is quite useful to describe the motion. The continuum representation of an RW is a regular diffusion [1]. When the motion of the particle occurs in a complex media, the simple RW might be insufficient for proper description of the transport. In many materials the basic linear dependence of the mean squared displacement (MSD), $\langle x^2(t) \rangle$, is missing and instead $\langle x^2(t) \rangle \sim t^\alpha$ while $0 < \alpha < 1$. Such behavior is termed anomalous subdiffusion and materials where it appears include living cells [2–5], blinking quantum dots [6], plasma membrane [7], filamentous networks [8] and many more [9]. The modeling of transport in these systems is quite complicated, when compared to the original RW. In the works of Scher and Montroll [10] the continuous time random walk
The transient case of the quenched trap model (CTRW) approach for transport in amorphous materials was developed. The idea behind CTRW is the existence of regions of local arrest, i.e. traps, where the traced particle waits for some random time before it continues its motion inside the media. When the expected random waiting times diverge the behavior is non-ergodic [11, 12] and CTRW will produce the mentioned subdiffusive scaling of the MSD. While CTRW became extremely popular and applicative [13–15], this approach treats the disorder in the media as annealed and uncorrelated. Quenchness of the disorder in the media is more physically appealing in many situations but it implies existence of strong correlations that in their turn introduce significant difficulties in calculating basic properties of the transport [16]. When the local dwell times of CTRW are fixed the model is known as the quenched trap model (QTM).

The QTM was found to be an important model that describes glassy behavior such as aging, weak-ergodicity breaking and non self-averaging [17–22]. Beyond the applications of the QTM, the difficulty of untangling the behavior dictated by quenched disorder, that is associated with QTM, posed this model and methods for its solution as a fundamental problem of anomalous transport [13]. The presence of the mentioned correlations, imposed by the quenched disorder, make the treatment of the QTM highly non-trivial task. Over the years many theoretical methods were devised to advance the general understanding of the QTM. The method of semi-equilibration [23] allowed to determine the average velocity and diffusion constant in the one-dimensional ($d=1$) case for the non-anomalous transport. Description of the QTM in terms of master equation and their representation in the Fourier space produced the scaling behavior of the QTM propagator at the origin [24, 25]. Renormalization Group approach [26], and scaling arguments [27], provided the existence of a critical dimension, $d=2$, for the QTM and the scaling behavior of the MSD. Based on these works a qualitative understanding that for sufficient high dimension ($d>2$) the behavior of the QTM can be mapped on-to the representation that includes only annealed disorder, i.e. CTRW. Further, the behavior of the QTM was studied for various lattices under the simplification of directed walk, i.e. without returns to previously visited traps [28]. The decimation of disorder allowed Monthus to calculate (among other quantities) the behavior of the positional probability density function (PDF) in $d=1$ case in the limit of very low temperatures [29, 30]. Rigorous probabilistic approach to the QTM led to mathematically exact scaling theorems [31, 32] and further generalization of the QTM to such models as the randomly trapped random walk [33, 34]. The effect of fractal structures for QTM [35] and behavior of the QTM under influence of a bias [36] are part of a current research.

The previously obtained results suggest that for any dimension $d>2$ the behavior of QTM converges to the one of CTRW. A simple hand-waving argument that support this qualitative result is that in sufficiently high dimensions the traced particle rarely returns to the same lattice point, thus reducing the effect of strong correlations imposed by the quenched disorder. The Pólya’s [1] theorem states that the probability to return to the origin (or any previously occupied position) is less then 1 for any dimension above $d=2$. For the one-dimensional biased case the scaling behavior of the mean position is known [13] and was also computed in the limit of very low temperatures and very small forces by decimation of the disorder [29, 30]. In the high dimensional cases, with or without a bias, the behavior is only obtained qualitatively, i.e. without precise quan-
The transient case of the quenched trap model

...tities that will allow a quantitative utilization of the mapping between CTRW and the QTM. A quantitative form of the mapping between the biased case, or the general transient case, of the QTM to CTRW is still missing. A valid question then is what is the quantitative representation of the mapping between QTM and CTRW? Can one extend this mapping to the cases where dimensionality is low but the formerly raised hand-waiving argument still holds, i.e. the biased case? In this manuscript we will provide an explicit form of the mapping between QTM and CTRW for any transient case in any dimension. Here we generalize the randomly stopped time approach, that was originally developed for the unbiased case in one-dimension [37, 38], to any transient case in any dimension and the biased cases. Unlike the unbiased one-dimensional case, for the transient cases the randomly stopped time approach allows to obtain explicit formula for the mapping of QTM to CTRW. We manage to obtain a subordination of the spatial process to the temporal $\alpha$-stable process for all transient cases. Unlike the CTRW where the subordinated spatial process advances as a function of the number of jumps [13, 39, 40], for QTM the local time of the spatial process is quite different. A brief summary of part of our results was published in a letter [41] without proofs. The current manuscript introduces the explicit behavior in the presence of external force and also the derivations that are crucial not only for the current results but also for further development of the randomly stopped time approach.

This paper is organized as follows. In section 2 the QTM is defined together with local time, measurement time and the subordination approach. In section 3 the local time $S_\alpha$ as a function of performed jumps $N$ is explored and the mean value of the local time is computed in section 3.1 and the second moment in section 3.2. In section 3.3 we summarize the calculations of the first and second moments (obtained in two previous subsections) and show that the local time convergences to the number of jumps that the process has performed. In section 4 the previously established convergence of the local time is exploited in order to establish an explicit mapping between the CTRW and QTM, by the means of double subordination. The formulas are applied to the one-dimensional case of biased QTM. In section 4.1 we obtain analytic expressions for the moments of the transient case of the QTM and show how the quenched disorder gives rise to the non-linear response of externally applied force. The summary is provided in section 5. Several appendices supply specific technical calculations and are referenced in the manuscript.

2. The quenched trap model and subordination

The QTM is defined as a random jump process of a particle on top of a lattice of dimension $d$. For every lattice point $x$ a quenched random variable $\tau_x$ is defined. This quenched variable $\tau_x$ defines the time that the particle is going to spend at $x$ before jumping to some other site $x'$, i.e. $\tau_x$ is the local dwell time. The probability to jump from $x$ to $x'$ is provided by $p(x', x)$. In the following we will assume translational invariance of the lattice that leads to $p(x', x)$ of the form $p(x' - x)$. The quenched dwell times $\{\tau_x\}$ are, real, positive and independently distributed random variables with

https://doi.org/10.1088/1742-5468/ab99bd
The transient case of the quenched trap model

$$\psi(\tau_x) \sim \tau_x^{-(1+\alpha)} A/|\Gamma(-\alpha)| \quad (\tau_x \to \infty) \quad (1)$$

as the PDF ($A > 0$). The value of the exponent $\alpha$ is bounded to $0 < \alpha < 1$. For such values of $\alpha$ the average dwell time is diverging, $\int_0^\infty \tau \psi(\tau) \, d\tau \to \infty$ and the model gives rise to anomalous subdiffusion and aging [17]. The physical picture behind this definition of QTM is a thermally activated particle that is jumping between various energetic traps. When a particle is in a trap, the average escape time is provided by the Arrhenius law $\tau \propto \exp(E_x/T)$, where $E_x > 0$ is the depth of the trap $x$ and $T$ is the temperature. When the distribution of $E_x$s is $f(E) = \frac{1}{T^g} \exp(-E_x/T)$, the average escape time is distributed according to equation (1), and $\alpha = T/T_g$. For low temperatures $T < T_g$ and glassy behavior, i.e. aging and non-self-averaging, is observed [21]. The QTM is thus a version of a transport on top of a random energetic landscape with exponential distribution of trap depths.

When the dwell times $\tau_x$ do have mean values and specifically distributed according to exponential distribution, the PDF to observe the particle on lattice site $x$ at time $t$, $P(x, t)$ is provided by the following master equation [13]

$$\frac{dP(x, t)}{dt} = \sum_{x'} \left\{ \frac{1}{\tau_x} p(x, x') P(x', t) - \frac{1}{\tau_x} p(x', x) P(x', t) \right\}. \quad (2)$$

For the case when the distribution of the dwell times does not have a finite first moment, like in the case of equation (1), the dynamics is not Markovian and the master equation (that is local in time) must be replaced by a generalized master equation [42]. Specifically the integro-diifferential equation that replaces equation (2) is

$$\frac{dP(x, t)}{dt} = \int_0^t dt' \sum_{x'} \{ K_{x'x}(t-t') P(x', t') - K_{x'x}(t-t') P(x', t') \}, \quad (3)$$

where the kernels $K_{x'x}(t')$ are provided by

$$K_{x'x}(t') = \phi_x(t') p(x', x). \quad (4)$$

The temporal part of the kernel, i.e. $\phi_x(t')$, is determined by the distribution of waiting times at different sites, $\psi_x(\tau)$. The Laplace transform of $\phi_x(t')$, $\hat{\phi}_x(u) = \int_0^\infty \phi(t') \exp(-ut') \, dt'$ is [42]

$$\hat{\phi}_x(u) = \frac{u\hat{\psi}_x(u)}{1 - \hat{\psi}_x(u)}, \quad (5)$$

where $\hat{\psi}_x(u)$ is the Laplace transform of $\psi_x(\tau_x)$. The QTM, as described in the previous paragraph, is determined by the generalized master equation (equation (3)) with quenched waiting time at each site. This means that for each site a waiting time $\tau_x$ is generated according to PDF in equation (1) and it is not varied during the process. The appropriate distribution of waiting time at site $x$ is then simply $\psi_x(\tau) = \delta(\tau - \tau_x)$, i.e. the same waiting time for different visits of the site, for a given disorder. The CTRW model is a version of a trap model which is similar to QTM but includes annealed disor-
order. In CTRW each time a given site is visited the waiting time at the site is generated from scratch and the distribution of the waiting time is independent of the specific site $x$. This means that $\psi_x(\tau) = \psi(\tau)$, that is provided by equation (1). The positional PDF $P(x, t)$ in CTRW is also determined by the generalized master equation equation (3), similarly to QTM.

We wish to perform a separation of the QTM into two processes. The first one is a spatial process on top of the lattice. This process is defined by the jump probabilities $p(x' - x)$ with some local time and in general will be governed by a local master equation like equation (2). The other process is a temporal process that transforms the local time into the measurement time $t$, that is defined by the dwell times. How exactly the measurement time and the local time are defined and related to each other is crucial for the solution of the QTM.

2.1. Measurement time and local time

During the measurement time $t$, the particle has visited several lattice points and stayed exactly $\tau_x$ at each lattice $x$. The measurement time $t$ is then simply given by

$$t = \sum_x n_x(t)\tau_x - \tau_{\text{Last}}$$  \hspace{1cm} (6)

where $n_x(t)$ is the number of times the particle visited site $x$ during $t$ and the summation is over all the lattice points. $\tau_{\text{Last}}$ is the additional time that the particle would have spent in the last site before performing a jump, after $t$ has passed. $n_x(t)$ are correlated since the number of times the particle visited site $x$ should not be very different from the number of times the particle has visited adjacent sites. While the visitation numbers $n_x(t)$ in equation (6) were generated by the random walk during time $t$, we are interested in the behavior of the sum $\sum_x$ on the right-hand side of equation (6), when the process is terminated after a specific local time has passed. For example we can count the total number of jumps performed by the particle and terminate the process when it reaches a specific threshold $N$. We can call $N$ the local time. The purpose of this section is to define a useful local time for the QTM. Before doing so we will show how the number of jumps is a proper local time for CTRW. When addressing CTRW the equation that defines the measurement time for QTM, i.e. equation (6), is replaced by

$$t = \sum_{i=0}^{N(t)} \tau_i - \tau_{\text{Last}},$$ \hspace{1cm} (7)

where $\tau_i$ is the waiting time that the particle spent in a site before performing the $(i + 1)$ jump. $\tau_{\text{Last}}$ is defined similarly to the definition for QTM. $N(t)$ is the total number of jumps that the particle performed during time $t$. Unlike the QTM, for CTRW when the particle returns to the same site it attains a new waiting time. So all the different $\tau_i$s in equation (7) are independent (for simplicity we assume that the process starts as the particle starts its waiting period at the origin). While the sum on the right-hand side of equation (7) depends on time by the virtue of $N(t)$, we wish to examine the behavior of almost identical sum

https://doi.org/10.1088/1742-5468/ab99bd
The transient case of the quenched trap model

\[ \tilde{t}(N) = \sum_{i=0}^{N} \tau_i \]  

(8)

that is independent of \( t \), \( N > 0 \) is a parameter and \( \tau_i \) are distributed according to equation (1). \( \tilde{t}(N) \) is a random variable that is defined by \( N \) and the distribution of \( \tau_i \)s that are independent, identically distributed random variables. It is known [13] that in the large \( N \) limit

\[ \frac{\tilde{t}(N)}{N^{1/\alpha}} \sim \eta \quad (N \to \infty) \]  

(9)

where \( \eta \) is distributed according to one sided Lévy distribution \( l_{\alpha,1} \). So we know the distribution of \( \tilde{t}(N) \) for a given \( N \) and we want to obtain the distribution of \( N(\tilde{t}) \) for a given \( \tilde{t} \), i.e., to switch the roles of the random variable and the parameter. The connection between the PDF of \( \tilde{t}(N) \), and the PDF \( N(\tilde{t}) \left( \mathcal{P}_\tau(N(\tilde{t})) \right) \) is constructed in the following manner (see also [43]). The random variable \( \tilde{t}(N) \) is a growing function of \( N \), in the sense that \( \sum_{i=0}^{N_2} \tau_i > \sum_{i=0}^{N_1} \tau_i \) if \( N_2 > N_1 \), i.e., a given realization of \( \tilde{t}(N) \) is a monotonically growing function of \( N \) and there is a one-to-one correspondence between \( N \) and \( \tilde{t}(N) \). This means that for a given \( \tilde{t} \) and \( N \), if \( \tilde{t} > \tilde{t}(N) \) then \( N < N(\tilde{t}) \), due to the one-to-one correspondence between \( N \) and \( \tilde{t}(N) \) that dictates that \( \tilde{t}(N(\tilde{t})) = \tilde{t} \) (for a given realization of \( \tilde{t}(N) \)). Then, by the means of the Heaviside step function [44]

\[ \Theta (N - N(\tilde{t})) = 1 - \Theta (\tilde{t} - \tilde{t}(N)) \]  

(10)

\( \Theta(y) = 1 \) if \( y > 0 \), it is zero if \( y < 0 \) and \( \Theta(0) = 1/2 \). When averaging over disorder, i.e. different realizations of \( \tau_i \), \( \Theta(N - N(\tilde{t})) \) transforms to \( \langle \Theta(N - N(\tilde{t})) \rangle \), which is the cumulative distribution of observing \( N(\tilde{t}) \) in time \( \tilde{t} \). Then taking the derivative of both sides with respect to \( N \) yields

\[ \mathcal{P}_\tilde{t}(N) = -\frac{\partial \langle \Theta (\tilde{t} - \tilde{t}(N)) \rangle}{\partial N} \]  

(11)

where \( \mathcal{P}_\tilde{t}(N) \) is the probability to observe \( N \) jumps during time \( \tilde{t} \) and \( \langle \Theta (\tilde{t} - \tilde{t}(N)) \rangle \) is the cumulative distribution of obtaining time \( \tilde{t} \) if \( N \) jumps were performed. Due to equation (9), the cumulative distribution of \( \tilde{t}(N) \) is similar (up to an additive constant) to the cumulative distribution of \( \eta \). We use equation (11) and the fact that no jumps were performed between measurement time \( t \) and \( \tilde{t} \) and obtain

\[ \mathcal{P}_\tilde{t}(N) = \frac{t}{\alpha} N^{-1-1/\alpha} l_{\alpha,1} \left( \frac{t}{N^{1/\alpha}} \right) \quad (t \to \infty). \]  

(12)

\( \mathcal{P}_\tilde{t}(N) \) is the probability to observe \( N \) jumps during a given measurement time \( t \). The standard renewal theory tools provide expression for \( \mathcal{P}_\tilde{t}(N) \) [12] that is identical to the presented formula, i.e., equation (12). In equation (12) we omitted the explicit \( N(\tilde{t}) \) form, \( N \) in this formula is the random variable and \( t \) is the parameter.
The transient case of the quenched trap model

The number of jumps performed by the particle during time \( t \) is a natural choice for the local time for CTRW. When the probability to observe \( N \) jumps is known, the positional PDF is easily obtained (see [13] and discussion in section 4). The usefulness of the number of jumps as a local time stems from the ability to obtain an explicit formula for the probability to observe a specific value for this local time for a given measurement time \( t \), i.e., equation (12). It is straightforward to derive an equation similar to equation (12) if the measurement time satisfies a relation that is mathematically resembles equation (9), i.e., \( t = G(t) \eta \) where \( \eta \) is a random variable that has \( l_{\alpha, A, 1} \) as a PDF. We will now show that the quantity

\[
S_\alpha(t) = \sum_x (n_x(t))^\alpha.
\] (13)

can serve as a local time for QTMM in the same sense that the number of jumps (performed during time \( t \)) is the local time for CTRW. Specifically we will show that for a given \( S_\alpha \) the random variable \( t(S_\alpha) \) satisfies the relation

\[
\eta = t(S_\alpha)/(S_\alpha)^{1/\alpha}
\] (14)
in the \( t \to \infty \) and \( S_\alpha \to \infty \) limit. We will show that \( \eta \) is distributed according to one-sided Lévy stable distribution \( l_{\alpha, A, 1} \). Thus we will term \( S_\alpha \) as the local time of QTM.

We start with an equation, analogous to equation (9), that gives the expression for the measurement time as a function of a given \( S_\alpha \),

\[
t(S_\alpha) = \sum_n n_x \tau_x,
\] (15)

where for simplicity of presentation we omit the explicit dependence of \( n_x \) on \( S_\alpha \) and simply write \( n_x \). We consider that \( \{n_x\} \) are fixed (an outcome of a given experiment with specific \( S_\alpha = \sum_x n_x^\alpha \)) then \( \eta \) depends on the realization of the disorder, i.e. \( \{\tau_x\} \). The PDF of \( \eta \) is found by examining disorder averaged \( \exp(-u \eta) \), i.e. \( \langle \exp(-u \eta) \rangle \), that is given by

\[
\langle e^{-u \eta} \rangle = \langle \exp \left( -u \sum_x \frac{n_x \tau_x}{(S_\alpha)^{1/\alpha}} \right) \rangle.
\] (16)

Since the \( \{\tau_x\} \) are independent and distributed according to equation (1), equation (16) takes the form

\[
\langle e^{-u \eta} \rangle = \prod_x \hat{\psi} \left[ \frac{n_x u}{(S_\alpha)^{1/\alpha}} \right],
\] (17)

where the product is over all the lattice sites and \( \hat{\psi}(u) = \int_0^\infty \exp(-\tau_x u) \psi(\tau_x) d\tau_x \). Due to equation (1) the small \( u \to 0 \) limit of \( \hat{\psi}(u) \) is \( \hat{\psi}(u) \sim 1 - Au^\alpha \) and equation (17) takes the form

https://doi.org/10.1088/1742-5468/ab99bd
The transient case of the quenched trap model

\[ \langle e^{-u\eta} \rangle = \prod_x \left( 1 - \frac{n_x^\alpha}{S_\alpha} Au^\alpha \right) . \]  

(18)

in the \( u \to 0 \) limit. By taking log of both sides of equation (18), the multiplication on the right-hand side is transformed into summation. Expansion of \( \log \left( 1 - Au^\alpha n_x^\alpha / S_\alpha \right) \) in Taylor series and summation over all \( n_x \) provides the leading term, \(-Au^\alpha\). In appendix A.1 it is shown that in the limit of \( S_\alpha \to \infty \) all the additional terms in the expansion of the logarithm converge to 0 as \( S_\alpha \to \infty \). The same holds true also if higher orders of \( u \) in the expansion of \( \tilde{\psi}(u) = 1 - Au^\alpha + Bu^\beta + \ldots \) are taken into account. Finally we can state that in the large \( S_\alpha \) limit

\[ \langle e^{-u\eta} \rangle = e^{-Au^\alpha} \]  

(19)

which means that the PDF of \( \eta \) is one sided Lévy stable distribution \( l_{\alpha,A,1} \) [14, 40]. We managed to obtain the distribution of \( \eta \) and the distribution of the measurement time \( t \) for a given local time \( S_\alpha \), since \( t(S_\alpha) = S_\alpha^{1/\alpha} \eta \). Because \( S_\alpha \) is positive and strictly growing, as we let the particle jump from one lattice point to another, the probability of observing a specific \( S_\alpha(t) \) for a given measurement time \( t \) (i.e., \( \mathcal{P}_t(S_\alpha) \)) is obtained in the similar fashion as equation (12) was obtained for CTRW, i.e.,

\[ \mathcal{P}_t(S_\alpha) \sim \frac{t}{\alpha} S_\alpha^{-1/\alpha-1} l_{\alpha,A,1} \left( \frac{t}{S_\alpha^{1/\alpha}} \right) \quad (t \to \infty) \]  

(20)

As in equation (12) we omit the explicit dependence of the random variable \( (S_\alpha(t)) \) on \( t \), \( t \) is the parameter and \( S_\alpha \) is the values of random variable. The measurement time \( t \) is the quantity that is set in any experiment or calculation. Equation (20) describes the probability to obtain various \( S_\alpha \) when averaging over disorder and letting the process to evolve up to time \( t \). We use this disorder-averaged relation between local time \( S_\alpha \) and \( t \) in the next subsection while constructing the representation of the QTM propagator in terms of the two processes.

2.2. Subordination

The subordination technique that is extensively applied in this work is based on the property of conditional probability. Instead of computing directly the PDF \( P(x, t) \), the conditional PDF to reach \( x \), given that a random variable \( G \) has occurred (i.e. \( P_G(x) \)), is computed. The probability of the random variable \( G \) (i.e., \( \mathcal{P}_t(G(t)) \)), for a given measurement time \( t \), is computed as well. When the particle reaches \( x \) at time \( t \), one of all possible \( G \) has to occur and this means that \( P(x, t) \) is provided by summation over all possible outcomes of \( G \), i.e., \( P(x, t) = \sum_G P_G(x) \mathcal{P}_t(G) \). This summation over conditional probabilities is termed subordination. In the following we choose \( S_\alpha \) as the random variable \( G \) for the subordination. The quantity \( P_{S_\alpha}(x) \) is the probability to find the particle at position \( x \) (starting from the origin) after local time \( S_\alpha \) has passed. Due to dependence of \( S_\alpha \) on local visitation numbers \( n_x \) (equation (13)), the local time is a function of both the number of jumps and the trajectory that was traced
by the particle. When the process starts, $S_\alpha$ equals to zero and its value is updated each time the particle performs a jump. As $S_\alpha$ crosses a specific value, the process is terminated.

The PDF $P(x, t)$ to find the particle at position $x$ after measurement time $t$ is calculated by conditioning on all the possible $S_\alpha$ that can occur during the process. One needs to sum over all the possible $P_{S_\alpha}(x)$ multiplied by the appropriate probability to observe such $S_\alpha$ at time $t$, for a given disorder. After averaging over disorder the PDF takes the form

$$
\langle P(x, t) \rangle = \sum_{S_\alpha} P_{S_\alpha}(x) P_t(S_\alpha) \tag{21}
$$

and due to equation (20) in the $t \to \infty$ limit we obtain

$$
\langle P(x, t) \rangle \sim \int_0^\infty P_{S_\alpha}(x) \frac{t}{\alpha} S_\alpha^{-1/\alpha-1} l_{\alpha, A, 1} \left( \frac{t}{S_\alpha^{1/\alpha}} \right) dS_\alpha. \tag{22}
$$

while we replaced the summation by integral [13]. Equation (22) represents the propagator of the QTM as a subordination of two processes. The spatial process that is terminated at random local time $S_\alpha$ and the temporal process that involves the disorder and make the mapping between local time and measurement time. We must notice that $P_{S_\alpha}(x)$ is independent of the disorder since it depends on the visitation numbers and not on the actual time that the particle spent at a given site. While the function $l_{\alpha, A, 1}(\ldots)$ is known, the missing part is the probability $P_{S_\alpha}$ that is obtained in the following for the case of a transient spatial process.

### 3. Local time $S_\alpha$

The propagator $P_{S_\alpha}(x)$ is independent of the measurement time $t$ and lacks the disorder that is present in the QTM and is a simple jump process on a lattice, but nevertheless it is highly non-trivial. The main complication is the local time $S_\alpha$ that is dependent on the path traced by the particle. If the local time is simply the number of jumps $N$, the probability to find the particle at $x$ after $N$ jumps is completely defined by the corresponding probabilities after $(N - 1)$ jumps. This is not the case for $P_{S_\alpha}(x)$. The arrival to $x$ do not increases $S_\alpha$ by one as happens with the number of jumps, but rather the increase of $S_\alpha$ depends on the total number of times that $x$ was previously visited.

In the case of one-dimensional simple random walk (RW) the shape of $P_{S_\alpha}(x)$ was previously [37] computed in the limit of $\alpha \to 0$. In this example $P_{S_\alpha}(x)$ has a very distinctive $V$ shape (with a minimum at the origin) and is quite different from the regular Gaussian propagator of the random walk.

Before the actual calculation of $P_{S_\alpha}(x)$, the study of the properties of $S_\alpha$ is in place. Specifically the first two moments of $S_\alpha$, i.e. $\overline{S_\alpha}$ and $\overline{S_\alpha^2}$. The averaging is with respects to many trajectories of the RW walk on a lattice without traps. The results of sections 3.1 and 3.2 are summarized in section 3.3.
3.1. $\overline{S}_\alpha(N)$

The mean value of $S_{\alpha}$ is obtained from equation (13), $\overline{S}_\alpha = \sum_x n_x^{\alpha}$. The lattice is translationally invariant and for simplicity we assume that the process starts at the origin. Defining $\beta_N(x; k)$ to be the probability for the RW to visit lattice site $x$ exactly $k$ times after $N$ steps, we write the average local time after $N$ steps as

$$\overline{S}_\alpha(N) = \sum_x \sum_{k=0}^{\infty} k^\alpha \beta_N(x; k).$$

(23)

The probability $\beta_N(x; k)$ is the probability to arrive to $x$ at-least $k$ times minus the probability to arrive at-least $k+1$ times during $N$ jumps. Since the $k$th arrival must occur during these $N$ jumps, $\beta_N(x; k)$ is expressed as

$$\beta_N(x; k) = \sum_{m=1}^{N} f_m(x; k) - \sum_{m=1}^{N} f_m(x; k + 1) \quad x \neq 0$$

(24)

where $f_N(x; k)$ is the probability to reach site $x$ for the $k$th time after $N$ steps. By defining $f_N(0)$ to be the probability of first return to the origin ($x = 0$) after $N$ steps, we write the recursive form for $f_N(x; k)$

$$f_N(x; k + 1) = \sum_{m=0}^{N} f_m(x; k) f_{N-m}(0).$$

(25)

The generating function $\hat{f}_z(x; k) = \sum_{N=0}^{\infty} z^N f_N(x; k)$ is then

$$\hat{f}_z(x; k) = \left[ \hat{f}_z(0) \right]^{k-1} \hat{f}_z(x)$$

(26)

where $\hat{f}_z(0)$ is the generating function of the probability of first return to $0$ and $\hat{f}_z(x)$ is the generating function of the probability of first arrival to $x$. Equations (24) and (26) provide the generating function of $\beta_N(x; k)$

$$\hat{\beta}_z(x; k) = \frac{1}{1-z} \left[ 1 - \hat{f}_z(0) \right] \left[ \hat{f}_z(0) \right]^{k-1} \hat{f}_z(x) \quad x \neq 0$$

$$\hat{\beta}_z(0; k) = \frac{1}{1-z} \left[ 1 - \hat{f}_z(0) \right] \left[ \hat{f}_z(0) \right]^{k-1} \hat{f}_z(0)$$

(27)

Equation (27) allows us to compute the generating function of $\overline{S}_\alpha(N)$, while the summation $\sum_x \hat{f}_z(x)$ can be obtained by the means of $c_N(x)$, the probability to find the particle at position $x$ after $N$ steps (started at $0$). Since $c_N(x)$ is related to $f_N(x)$ by

$$c_N(x) = \delta_{N,0} \delta_{x,0} + \sum_{m=1}^{N} f_m(x) c_{N-m}(0)$$

(28)

https://doi.org/10.1088/1742-5468/ab99bd
the generating functions \( \hat{f}_z(x) \) and \( \hat{c}_z(x) \) are connected by

\[
\hat{f}_z(x \neq 0) = \hat{c}_z(x \neq 0) / \hat{c}_z(0),
\]

\( \hat{f}_z(0) = 1 - 1 / \hat{c}_z(0). \) (29)

Together with the fact that \( \sum_x c_N(x) = 1 \) and consequently \( \sum_x \hat{c}_z(x) = 1 / (1 - z) \), equations (23), (27) and (29) result in

\[
\overline{S}_\alpha(z) = \left[ \frac{1 - \hat{f}_z(0)}{1 - z} \right]^2 \sum_{k=0}^{\infty} k^\alpha \hat{f}_z(0)^{k-1}. \]

(30)

For the case when the spatial process is transient and the probability of eventually returning to the origin \( Q_0 = \sum_{N=0}^{\infty} f_N(0) \), is less than 1, the asymptotic \((N \to \infty)\) is readily obtained from equation (30). For \( z \to 1, \hat{f}_z(0) \to Q_0 < 1 \). The fact that \( \sum_{N=0}^{\infty} N z^N = z / (1 - z)^2 \) and Tauberian theorem [1] implies that

\[
\overline{S}_\alpha(N) \sim \Lambda N \quad (N \to \infty)
\]

(31)

where

\[
\Lambda = \frac{(1 - Q_0)^2}{Q_0} Li_{-\alpha}(Q_0)
\]

(32)

and \( Li_{\alpha}(b) = \sum_{k=1}^{\infty} b^k / k^\alpha \) is the Polylogarithm function [44].

The form of average \( S_\alpha \) as expressed in equation (31) will be essential in the following for asymptotic representation of \( \langle P(x, t) \rangle \) for the transient case by the means of \( PS_\alpha(x) \). The average behavior of \( S_\alpha \) suggests that the local time \( S_\alpha \) is not very much different from the regular local time, i.e. the number of jumps \( N \), at-least for the transient case \( Q_0 < 1 \). The behavior of the second moment of \( S_\alpha \) should indicate if one indeed can exchange the local time \( S_\alpha \) by a linear function of \( N \).

3.2. \( \overline{S}_\alpha^2(N) \)

The goal of this section is to provide the conditions for a plausible substitution of \( S_\alpha \) by its average value \( \langle S_\alpha \rangle \). The second moment of \( S_\alpha \) is computed in a similar fashion as the first moment was computed in section 3.1, and the first moment of \( S_\alpha \) (equation (23)) is generalized to

\[
\overline{S}_\alpha^2(N) = \sum_x \sum_{x'} \sum_{k_1} \sum_{k_2} k_1^\alpha k_2^\beta N(x; k_1, x'; k_2)
\]

(33)

where \( \beta_N(x; k_1, x'; k_2) \) is the probability that in \( N \) steps the RW will visit site \( x \) exactly \( k_1 \) times and the site \( x' \) exactly \( k_2 \) times. This probability is calculated in the terms of \( f_N(x, k_1, x', k_2) \), the probability to arrive to \( x \) after \( N \) steps for the \( k_1 \)th time while visiting \( x' \) exactly \( k_2 \) times. \( \beta_N(x; k_1, x'; k_2) \) is the probability that the \( k_1 \)th arrival was performed but not the \( (k_1 + 1) \)th, i.e.
The transient case of the quenched trap model

\[ \beta_N(x; k_1, x'; k_2) = \sum_{l=0}^{N} \left\{ [f_l(x, k_1; x', k_2) - f_l(x, k_1 + 1; x', k_2)] + [f_l(x', k_2; x, k_1) - f_l(x', k_2 + 1; x, k_1)] \right\} \quad (k_1 + k_2 \geq 2). \] (34)

The range \( k_1 > 0 \) and \( k_2 > 0 \) is sufficient since \( \beta_N(x, k_1; x', k_2) \) is multiplied by \( k^\alpha \) in equation (33). We define the probability to start at \( x \) and after \( N \) steps to reach \( x' \), without visiting \( x \) or \( x' \) on the way, as \( M_N(x, x') \) and the probability to start at \( x \) and return to the same site after \( N \) steps, without visiting \( x \) or \( x' \) on the way, as \( T_N(x, x') \). The probability \( f_N(x, k_1; x'; k_2) \) is recursively expressed in terms of \( M_N(x, x') \) and \( T_N(x, x') \)

\[ f_N(x, k_1 + 1; x', k_2) = \sum_{l=0}^{N} f_l(x, k_1; x', k_2)T_{N-l}(x, x') + f_l(x', k_2; x, k_1)M_{N-l}(x', x) \] (35)

where \( f_N(x, 0; x', k_2) = 0 \). Equation (35) leads to the following expression in \( z \) space

\[ \hat{f}_z(x, k_1 + 1; x', k_2) = \hat{f}_z(x, k_1; x', k_2)\hat{T}_z(x, x') + \hat{f}_z(x', k_2; x, k_1)\hat{M}_z(x', x). \] (36)

Application of additional transformation \( k_1 \to \xi_1 \) and \( k_2 \to \xi_2 \), by performing a double summation \( \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \xi_1^k \xi_2^l \) on both sides of equation (36) delivers

\[ \left[ 1 - \xi_1 \hat{T}_z(x, x') \right] \hat{f}_z(x, \xi_1; x', \xi_2) - \xi_1 \hat{M}_z(x', x)\hat{f}_z(x', \xi_2; x, \xi_1) = \xi_1 \hat{f}_z(x, 1; x', \xi_2) \] (37)

where \( \hat{f}_z(x, \xi_1; x', \xi_2) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \xi_1^k \xi_2^l \hat{f}_z(x, k_1; x', k_2) \) and \( \hat{f}_z(x, 1; x', \xi_2) = \sum_{k_2=1}^{\infty} \xi_2^l \hat{f}_z(x, 1; x', k_2) \). In a similar fashion we obtain

\[ \left[ 1 - \xi_2 \hat{T}_z(x', x) \right] \hat{f}_z(x', \xi_2; x, \xi_1) - \xi_2 \hat{M}_z(x, x')\hat{f}_z(x, \xi_1; x', \xi_2) = \xi_2 \hat{f}_z(x', 1; x, \xi_1). \] (38)

Equations (37) and (38) are linear equations in terms of \( \hat{f}_z(x, \xi_1; x', \xi_2) \) and \( \hat{f}_z(x', \xi_2; x, \xi_1) \) that attain the solution

\[ \hat{f}_z(x, \xi_1; x', \xi_2) = \frac{\xi_1 \left[ 1 - \xi_2 \hat{T}_z(x', x) \right] f'_z(x, 1; x', \xi_2) + \xi_1 \xi_2 \hat{M}_z(x', x)\hat{f}_z(x, 1; x, \xi_1)}{[1 - \xi_1 \hat{T}_z(x, x')]\left[ 1 - \xi_2 \hat{T}_z(x', x) \right] - \xi_1 \xi_2 \hat{M}_z(x, x')\hat{M}_z(x', x)}. \] (39)

Since \( f_N(x', k + 1; x, 0) = \sum_{l=0}^{N} f_l(x', k; x, 0)T_{N-l}(x', x) \), the transform \( f'_z(x', \xi_2; x, 0) = \sum_{k=1}^{\infty} \xi_2^k \hat{f}_z(x', k; x, 0) \) is

\[ f'_z(x', \xi_2; x, 0) = \frac{\xi_2 \hat{f}_z(x', 1; x, 0)}{1 - \xi_2 \hat{T}_z(x', x)}. \] (40)

https://doi.org/10.1088/1742-5468/ab99bd
The transient case of the quenched trap model

By using the expression \( f_N(x, 1; x', k_2) = \sum_{l=0}^N f_l(x', k_2; x, 0) \) and equation (40) we obtain

\[
\hat{f}_z(x, 1; x', \xi_2) = \frac{\xi_2 \hat{f}_z(x', 1; x, 0) \hat{M}_z(x', x)}{1 - \xi_2 \hat{T}_z(x', x)}, \tag{41}
\]

and then by substitution of equations (39) and (41) in equation (34), and using equation (36), we obtain for \( \hat{\beta}_z(x; \xi_1, x'; \xi_2) = \sum_{N=0}^\infty \sum_{k_1=1}^\infty \sum_{k_2=1}^\infty z^N \xi_1^{k_1} \xi_2^{k_2} \beta_N(x; k_1, x'; k_2) \)

\[
\hat{\beta}_z(x; \xi_1, x'; \xi_2) = \frac{1}{1 - z} \left\{ \left( 1 - \hat{T}_z(x, x') - \hat{M}_z(x', x) \right) \right. \\
\times \frac{\xi_1 \hat{f}_z(x', 1; x, 0) \hat{M}_z(x', x) + \xi_2^2 \hat{f}_z(x', x) \hat{f}_z(x', 0) \hat{M}_z(x', x)}{[1 - \xi_1 \hat{T}_z(x', x')][1 - \xi_2 \hat{T}_z(x', x)] - \xi_1 \xi_2 \hat{M}_z(x, x') \hat{M}_z(x', x) + \left. \left( 1 - \hat{\hat{T}}_z(x', x) - \hat{\hat{M}}_z(x', x') \right) \right\} \\
\times \frac{\xi_2 \xi_1 \hat{f}_z(x, 1; x', 0) \hat{M}_z(x, x') + \xi_2 \hat{f}_z(x, x') \hat{f}_z(x, 0) \hat{M}_z(x, x')}{[1 - \xi_1 \hat{T}_z(x', x)][1 - \xi_2 \hat{T}_z(x', x')] - \xi_2 \xi_1 \hat{M}_z(x', x) \hat{M}_z(x, x')} \right\} . \tag{42}
\]

The generating functions of the two-point probabilities \( T_N(x, x'), M_N(x, x') \) and \( f_N(x, 1; x', 0) \) that define the behavior of \( \hat{\beta}_z(x; \xi_1, x'; \xi_2) \) are expressed in terms of the generating function of the probability of first arrival \( f_N(x) \), which is provided by equation (29). In appendix A.2 we show that

\[
\hat{f}_z(x, 1; x', 0) = \frac{\hat{f}_z(x) - \hat{f}_z(x') \hat{f}_z(x - x')}{1 - \hat{f}_z(x' - x) \hat{f}_z(x - x')} \\
\hat{M}_z(x, x') = \frac{\hat{f}_z(x' - x) - \hat{f}_z(x' - x') \hat{f}_z(x - x')}{1 - \hat{f}_z(x' - x) \hat{f}_z(x - x')} \tag{43} \\
\hat{T}_z(x, x') = \frac{\hat{f}_z(x') - \hat{f}_z(x - x') \hat{f}_z(x - x')} \]

Since the generating function of \( \beta_N(x, k_1; x', k_2) \) is represented in terms of \( \hat{f}_z(x), \hat{f}_z(x'), \hat{f}_z(x - x') \) and \( \hat{f}_z(x' - x) \), the summation over \( x \) and \( x' \) can be achieved in the \( t \to \infty \) limit. Due to equation (29) and the already mentioned fact that \( \sum_x \hat{c}_z(x) = 1/(1 - z) \), the summation over all possible \( x \) and \( x' \) on the right-hand side of equation (42) can be expanded in a power series over \( 1/(1 - z) \). The Tauberian theorem [1] states that the leading order in \( t \) space is provided by the leading order of \( 1/(1 - z) \) in the \( z \to 1 \) limit in \( z \) space. It is clear that \( \sum_x \sum_x \hat{c}_z(x) \hat{c}_z(x') = 1/(1 - z)^2 \), but in equation (42) all the multiplications of generating functions of single point probabilities are of mixed origin,
Substituting the expression in equations (44) and (29) in equation (46) lead to
\[
\sum_x \sum_{x'} \hat{c}_z(x) \hat{c}_z(x' - x) = \frac{1}{(1 - z)^2}
\] (44)
for any case of transient RW (the roles of x and x' can be interchanged). Any other terms of the form \(\sum_x \sum_{x'} \hat{c}_z(x - x') \hat{c}_z(x' - x)\) or \(\sum_x \sum_{x'} \hat{c}_z(x) \hat{c}_z(x' - x)\) (or generally multiplication of any number of terms greater than 2) grow slower than \(1/(1 - z)^2\) when \(z \to 1\) (see appendix A.3). This means that when expanding the denominator in equation (42) and utilizing equation (43), all the terms in the expansion, except the zero order, i.e. \(1/[1 - \xi_1 \hat{T}_z(x', x)] [1 - \xi_1 \hat{T}_z(x, x')]\), will grow slower than \(1/(1 - z)^2\) after summation over x and x'. Then in the \(z \to 1\) limit we use
\[
\hat{f}_z(x; 1 : x', 0) = \hat{f}_z(x)
\]
\[
M_z(x, x') = \hat{f}_z(x' - x) - \hat{f}_z(0) \hat{f}_z(x' - x) \quad (z \to 1)
\] (50)
and the only relevant terms in the summation over x and x' are
\[
\sum_x \sum_{x'} \hat{\beta}_z(x; \xi_1, x'; \xi_2) \to \sum_x \sum_{x'} \frac{(1 - \hat{f}_z(0))^2}{1 - z} \frac{\xi_1 \xi_2}{[1 - \xi_1 \hat{f}_z(0)][1 - \xi_2 \hat{f}_z(0)]}
\times \left\{ \hat{f}_z(x') \hat{f}_z(x - x') + \hat{f}_z(x) \hat{f}_z(x' - x) \right\}.
\] (46)
Substituting the expression in equations (44) and (29) into equation (46) leads to
\[
\sum_x \sum_{x'} \hat{\beta}_z(x; \xi_1, x'; \xi_2) \to \frac{2(1 - \hat{f}_z(0))^4}{(1 - z)^3} \frac{\xi_1 \xi_2}{[1 - \xi_1 \hat{f}_z(0)][1 - \xi_2 \hat{f}_z(0)]},
\] (47)
and since \(\sum_{k=1}^{\infty} \xi^k (\hat{f}_z(0))^{k-1} = \xi/[1 - \xi \hat{f}_z(0)]\)
\[
\sum_x \sum_{x'} \hat{\beta}_z(x; k_1, x'; k_2) \to \frac{2(1 - \hat{f}_z(0))^4}{(1 - z)^3} \hat{f}_z(0)^{k_1-1} \hat{f}_z(0)^{k_2-1}.
\] (48)
Eventually from equations (48) and (33) we obtain
\[
\overline{S}_\alpha^2(z) \to \frac{(1 - \hat{f}_z(0))^4}{\hat{f}_z(0)^2} \left\{ Li_{-\alpha} (\hat{f}_z(0)) \right\}^2 \frac{2}{(1 - z)^3}
\] (49)
then according to the identity \(\sum_{N=0}^{\infty} z^N N(N - 1) = 2z^2/(1 - z)^3\), and the Tauberian theorem, the asymptotic behavior of \(\overline{S}_\alpha^2(N)\) is
\[
\overline{S}_\alpha^2(N) \sim \frac{(1 - Q_0)^4}{Q_0^2} \left\{ Li_{-\alpha}(Q_0) \right\}^2 N^2 \quad N \to \infty.
\] (50)

https://doi.org/10.1088/1742-5468/ab99bd
This relation shows that for any transient RW, in the large $N$ limit the second moment of $S_\alpha(N)$ converges to a square of the mean of $S_\alpha(N)$, i.e.

$$\frac{S_\alpha(N)^2}{S_\alpha(N)^2} \xrightarrow{N \to \infty} 1. \quad (51)$$

### 3.3. Convergence to a $\delta$-function

We had shown that the distribution of $S_\alpha$ is such that in the $N \to \infty$ limit the square of the first moment converges to the second moment. The minimal value of $S_\alpha/N$ is $2(N/2)^\alpha/N$ that is achieved if the RW performed $N/2$ back and forward jumps between two sites. The maximal value of $S_\alpha/N$ is 1, that is achieved if the RW never visited any site twice. Since those two limits are achieved for a very specific trajectories of the RW the probability of the minimal and maximal values of $S_\alpha$ converges to 0 in the $N \to \infty$ limit. For the random variable

$$s = S_\alpha/N, \quad (52)$$

the PDF $\lambda(s)$ is defined for $0 \leq s \leq 1$ and $\lambda(s) \to 0$ when $s \to 0$, or $s \to 1$. Moreover the proven equivalence of $S_\alpha^2$ and $\overline{S_\alpha}^2$ in the $N \to \infty$ limit means that

$$\left(\int_0^1 s\lambda(s) \, ds\right)^2 = \int_0^1 (s)^2\lambda(s) \, ds \quad N \to \infty. \quad (53)$$

Since $\lambda(s)$ is a PDF and $(\ldots)^2$ is a strictly convex function, Jensen inequality [45] states that $\left(\int_0^1 s\lambda(s) \, ds\right)^2 \leq \int_0^1 (s)^2\lambda(s) \, ds$ and the equality is achieved only when $s$ is constant, i.e. $\lambda(s)$ is a $\delta$-function. Then from equation (31) we obtain that

$$\lambda(s) \xrightarrow{N \to \infty} \delta(s - \Lambda), \quad (54)$$

where the constant $\Lambda$ is provided in equation (32). This result means that in the large $N$ limit the local time $S_\alpha$ and the number of jumps $N$ are equivalent up to a transformation $S_\alpha \to \Lambda N$. This result is presented in figure 1, where the random variable $S_\alpha(N)/N$ (obtained from numerical simulation) converges to a non-zero constant for large $N$. In the next section we utilize this result to establish the form of $P_{S_\alpha}(x)$ and a simplified representation of the positional probability density function, i.e. $P(x, t)$.

### 4. Double subordination and the equivalence of CTRW and the transient QTM

The PDF $P(x, t)$, as it is presented by equation (22), depends on $P_{S_\alpha}(x)$. The form of $P_{S_\alpha}(x)$ is obtained by using again the subordination approach where the local time $S_\alpha$ is subordinated to $N$, number of jumps performed, and the spatial process is given provided by $W_Nx$—the PDF of regular RW, i.e.
Figure 1. Convergence of $S_\alpha(N)$ to $\Lambda N$. Both panels describe the behavior of $S_\alpha$ for a one dimensional RW with probability 0.7 to make a step $+1$ and probability 0.3 to make a step $-1$. The thick line in both panels are simulation results while the dashed line is the theoretical prediction of equation (54) with $\Lambda$ provided by equation (32). For both panels $Q_0 = 0.6$. Panel (a) presents the case with $\alpha = 0.5$ while panel (b) the case with $\alpha = 0.25$.

$$P_{S_\alpha}(x) = \sum_{N=0}^{\infty} W_N(x) G_{S_\alpha}(N, x). \quad (55)$$

$G_{S_\alpha}(N, x)$ is the probability to perform $N$ steps before reaching $x$ provided that the value of $S_\alpha$ is known. In the previous section we have shown that in the $N \to \infty$ limit the PDF of $s = S_\alpha/N$, i.e. $\lambda(s)$, is converging to $\delta(s - \Lambda)$. For $\lambda(s)$, $S_\alpha$ is the random variable and $N$ is the parameter. For $G_{S_\alpha}$, $N$ is the random variable and $S_\alpha$ is the parameter. The convergence of $\lambda(s)$ to a $\delta$-function shows that in the $N \to \infty$ limit these two quantities are interchangeable and then for a transient RW

$$G_{S_\alpha}(N, x) \overset{S_\alpha \to \infty}{\longrightarrow} \delta(S_\alpha - \Lambda N), \quad (56)$$

independent of the value of $x$. The double subordination approach prescribes the disorder averaged PDF $\langle P(x, t) \rangle$ the form

$$\langle P(x, t) \rangle = \sum_{S_\alpha} \sum_{N=0}^{\infty} W_N(x) G_{S_\alpha}(N, x) P_t(S_\alpha) \quad (57)$$

where we used equations (21) and (55). When taking the limit $t \to \infty$ the form of $P_t(S_\alpha)$ in equation (20) dictates that only large $S_\alpha$ need to be considered, and then according to equation (56) only large $N$ are of interest, finally we obtain that

$$\langle P(x, t) \rangle \sim \int_{0}^{\infty} W_N(x) \frac{t/\Lambda^{1/\alpha}}{\alpha} N^{-1/\alpha-1} l_{\alpha, A, 1} \left( \frac{t/\Lambda^{1/\alpha}}{N^{1/\alpha}} \right) dN \quad t \to \infty, \quad (58)$$

where the transition to integration is the regular practice of the subordination technique [13]. It is important to notice that in the case of continuous time random walk (CTRW) [1] the particle experience each jump a new waiting time $\tau$, independent of the previous

https://doi.org/10.1088/1742-5468/ab99bd
visitation even if it is currently located in a previously visited site. This makes the CTRW a kind of mean-field approximation of the QTM and specifically, according to the discussion in section 2.1 the local time for CTRW is simply \( N \) and the probability to observe \( N \) jumps during time \( t \) is provided by equation (12). Accordingly, only one level of subordination is needed, i.e. \( \langle P(x,t) \rangle_{\text{CTRW}} = \sum_N W_N(x) P_t(N) \), that leads to

\[
\langle P(x,t) \rangle_{\text{CTRW}} \sim \int_0^\infty W_N(x) \frac{t}{\alpha} N^{-1/\alpha - 1} l_{\alpha,A,1} \left( \frac{t}{N^{1/\alpha}} \right) \, dN \quad t \to \infty. \tag{59}
\]

Comparison of equations (59) and (58) leads to

\[
\langle P(x,t) \rangle_{\text{QTM}} \sim \langle P(x,t/\Lambda^{1/\alpha}) \rangle_{\text{CTRW}} \quad t \to \infty, \tag{60}
\]

or simply said: the disorder averaged propagator of a transient QTM is equivalent to the propagator of CTRW taken at time \( t/\Lambda^{1/\alpha} \). Eventually we proved that a simple transformation of time for CTRW

\[
t \to t/\Lambda^{1/\alpha} \tag{61}
\]

makes this model the asymptotic representation of the transient case of the QTM. Equation (60) states that for every situation that the propagator of CTRW can be computed \([40]\), the propagator of QTM can be computed as well. The constant \( \Lambda^{-1/\alpha} \) is provided by equation (32) and displayed in figure 2 for \( 0 \leq Q_0 < 1 \). This constant is positive and > 1 for any \( Q_0 \). In the limit when \( Q_0 \to 1 \), i.e. the approach to the recurrent case, \( L_{\alpha,A}(Q_0) \sim (1 - Q_0)^{-1-\alpha} \) \([44]\) and \( \Lambda^{-1/\alpha} \sim (1 - Q_0)^{-(1-\alpha)/\alpha} \) diverges as \( Q_0 \to 1 \). This divergence signifies the limitation of the presented result to the transient case \( 0 \leq Q_0 < 1 \). When \( Q_0 = 0 \) the QTM is exactly described by the CTW since the particle never returns to previously visited site, indeed in this case \( \Lambda^{-1/\alpha} = 1 \). For any \( 0 < Q_0 < 1 \) the constant is greater than one. This means that the QTM process is faster than CTRW, i.e. the two models attain the same PDF’s but for QTM it is achieved on
shorter time-scales. Such behavior can be attributed to the fact that CTRW never re samples previously visited traps (the disorder is annealed), while it is not true for QTM. Since CTRW never re samples previously visited traps it has a higher probability (when compared to QTM) to find deeper traps, which means that its propagation is going to be slower than QTM, on average.

For the one-dimensional case of a biased RW on a simple lattice with constant spacing the $W_N(x)$ is a binomial distribution that is very well approximated by the Gaussian approximation

$$W_N(x) = \frac{1}{\sqrt{2\pi 4q(1-q)N}} e^{-\frac{(x-(2q-1)N)^2}{8q(1-q)N}} \quad (N \gg 1), \quad (62)$$

where $q$ is the probability to jump to the right one step on the lattice and $1-q$ is the probability to jump to the left. The return probability for this process is $Q_0 = 2(1-q)$, as proven in the next section. For several values of $\alpha$ the form of $l_{\alpha,A,1}$ is explicitly known [40], specifically for $\alpha = 1/2$,

$$l_{1/2,1,1}(\eta) = \frac{1}{2\sqrt{\pi}} \eta^{-3/2} e^{-\frac{\eta}{4}}. \quad (63)$$

Then according to equation (58), for the one-dimensional case the PDF is provided by

$$\langle P(x,t) \rangle \sim \int_0^\infty \sqrt{t} e^{-\frac{(x-(2q-1)N)^2}{16(1-q)N}} \left( \frac{2(1-q)}{(2q-1)^2 Li_{-1/2}(2(1-q))} \right)^{-1} \times \exp \left[ -N^2 \frac{2(1-q)}{(2q-1)^2 Li_{-1/2}(2(1-q))} \right]^2 dN \quad (t \to \infty). \quad (64)$$

In figure 3 we perform a comparison between a numerical simulation of the QTM and the theoretical result of equation (64). The comparison is performed for $t = 10^3$ and it is excellent for this finite time.
4.1. Moments of the QTM and non-linear response

The explicit form of the disorder averaged PDF, expressed by equation (58), permits evaluation of different moments \( \langle x^\mu \rangle \). Indeed, the approximation works for the regime when the measurement time is sufficiently large and many jumps have been performed. In this limit the probability density \( W_N(x) \) attains the Gaussian form and all the moments \( \int |x|^\mu W_N(x) dx \) can be easily computed [46]. Generally we can say that

\[
\int |x|^\mu W_N(x) dx = B_\mu N^{\gamma_\mu}.
\]

The constant \( B_\mu \) depends on the power \( \mu \) and the lattice that determine the properties of the Gaussian approximation, i.e. second moment and the mean of the Gaussian distribution. Then according to equation (58) the \( \mu \)th moment \( \langle |x|^\mu \rangle \) is provided by

\[
\langle |x|^\mu \rangle = \frac{\Gamma[1 + \gamma_\mu]}{A^{\gamma_\mu} \Gamma[1 + \alpha \gamma_\mu]} \frac{B_\mu}{A^{\gamma_\mu} \Lambda^{\gamma_\mu}}.
\]

The constants \( \gamma_\mu, B_\mu \) and \( Q_0 \) depend only on the lattice dimension and the type of the RW on top of this lattice.

Of a special interest is the behavior of the first moment when an external force is applied, i.e. response of the system to a bias. In the QTM model the force is applied in such a way that it is not affecting the dwell times \( \tau_x \) but rather determines the transition probabilities between different locations [30, 47, 48]. When the imposed external force \( F_0 \) is sufficiently weak the transition probabilities \( p(x - x') \) should be proportional to \( \exp(F_0(x - x')/2k_BT) \) for transition from \( x' \) to \( x \), and to \( \exp(-F_0(x - x')/2k_BT) \) for the reverse transition. Here we assume that the force is constant and is applied in the direction of \( x - x' \), otherwise one needs to use the projection of the force in the \( x - x' \) direction. Since we are interested only in the limit of weak force it is possible to expand the exponential up to first order in \( F_0 \). In the case of a simple binomial RW on top of a one-dimensional lattice the probability \( q \) to perform a jump to the right will be \( q = \frac{1}{2}(1 + F_0a/2k_BT) \) and the probability to jump to the left \( 1 - q = \frac{1}{2}(1 - F_0a/2k_BT) \), where \( a \) is the lattice spacing. For dimensions \( d \geq 2 \) similar expansion will take place, the only difference is that \( F_0 \) will be multiplied by some \( \cos(\theta) \) where \( \theta \) is the appropriate angle between the direction of the force and local axis of the lattice. The presence of the force affects not only the constant \( B_\mu \) in equation (66) but also the constant \( \Lambda \) by the means of \( Q_0 \). Of special interest is the one-dimensional case. For \( d = 1Q_0 \), without the presence of external force, is one [1]. When external small external force \( F_0 \) is added \( Q_0 \) is decreased but still attains values in the vicinity of one and consequently [due to the form of \( \Lambda \) in equation (32)] contributes to a non-trivial dependence on the force of the first moment.

We calculate the first moment of the one dimensional QTM with a presence of a weak force \( F_0 \), i.e., the case of traps on a simple one-dimensional lattice with probabilities \( q = \frac{1}{2}(1 + F_0a/2k_BT) \) to jump to the right and \( 1 - q \) to jump to the left. For the spatial
process $W_N(x)$ this is the case of a binomial random walk and thus for sufficiently large $N$ the Gaussian limit is attained

$$W_N(x) \sim \exp\left[ -\frac{(x-(2q-1)N)^2}{8q(1-q)N} \right] \sqrt{8\pi q(1-q)N}$$

and

$$\int_{-\infty}^{\infty} x W_N(x) \, dx = (2q - 1) N \quad (68)$$

meaning that $B_1 = 2q - 1$ and $\gamma_1 = 1$. Equation (68) describes the linear response to the external force for the spatial part of the QTM. The return probability $Q_0 \to \sum_{N=0}^{\infty} f_N(0) = \hat{f}_z(0)$ is provided by equation (29) while the Fourier transform of the jump probability $p(x)$ is $\overline{p}(k) = \sum_{x} e^{i(kx)} p(\dot{x})$ dictates the form of $\hat{c}_z(0)$ for dimension $d$ [1]

$$\hat{c}_z(0) = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{1}{1 - z \overline{p}(k)} \, d^d k. \quad (69)$$

For $d = 1$, $\overline{p}(k) = q \exp(i k) + (1 - q) \exp(-i k)$ and equation (69) is

$$\hat{c}_z(0) = \frac{1}{2\pi i} \oint_{|y|=1} \frac{1}{y - z q y^2 - z(1-q)} \, dy, \quad (70)$$

by changing the variable to $y = \exp(i k)$ the integral in equation (70) is transformed into

$$\hat{c}_z(0) = \frac{1}{2\pi i} \oint_{|y|=1} \frac{1}{y - z(1-q)/q} \, dy. \quad (71)$$

For any $z < 1$ the two solutions of $-z q y^2 + y - z(1-q) = 0$ are located on the real line while one of them is for $y > 1$ and the other is is for $y < 1$. This means that the integral depends on the presence of one single pole $\forall z < 1$. This pole is located at $y = (1-q)/q$ for $z = 1$ and the integral in equation (71) in the $z \to 1$ limit is

$$\hat{c}_0(0) = \frac{1}{2q - 1}. \quad (72)$$

Then according to equation (29) for $d = 1$ the probability to return to the starting point, given the process is biased (i.e. $q > 1/2$), is

$$Q_0 = 2(1-q). \quad (73)$$

Finally, according to equations (66), (68), (73) and (32) we obtain that

$$\langle x(t) \rangle \sim \frac{1}{\Gamma[1 + \alpha]} \frac{2(1-q)}{(2q-1) Li_{-\alpha} [2(1-q)]} t^\alpha, \quad (74)$$

and when explicitly writing the probability $q = \frac{1}{2} (1 + F_0 a / 2 k_B T)$ and the fact that the spacing of the lattice is $a$, $\langle x(T) \rangle$ is transformed into
The transient case of the quenched trap model

Figure 4. Comparison of the numerical simulation of the first moment $\langle x(t) \rangle$ for a $1d$ QTM with a presence of external force $F_0$ and theoretical prediction of equations (75) and (76). The symbols describe the results of the numerical simulation with $t = 10^8$, $\alpha = 1/2$, $a = 1$ and $A = 1$. The thick line is theory as described by equation (75) without fitting parameters and the dashed line is the prediction of equation (76). For sufficiently small $F_0 a / k_B T$ the two theoretical results coincide.

$$\langle x(t) \rangle \sim \frac{a}{t^{\alpha/2}} \left( 1 - \frac{F_0 a}{2k_B T} \right) t^{\alpha/2}. \tag{75}$$

For small $F_0 \to 0$ we use the asymptotic relation $Li_{-\alpha}(1-y) \sim \Gamma[1+\alpha]y^{-\alpha-1} \tag{44}$ and obtain the non-linear response to externally applied small force

$$\langle x(t) \rangle \sim \frac{a}{t^{\alpha/2}} \left( \frac{F_0 a}{2k_B T} \right) \frac{1}{\Gamma[1+\alpha]} t^{\alpha/2}. \tag{76}$$

A convincing comparison between the analytical results of equations (75) and (76) and numerical simulation is presented in figure 4. It is clear from the figure that both theoretical result due to equations (75) and (76) coincide for sufficiently small external force $F_0$.

The behavior of the first moment for small forces, as described by Eq, (76) does not satisfy linear response. The response to external force is anomalous and the force enters the equation with an exponent $\alpha < 1$. This behavior for a one-dimensional biased QTM was previously predicted by using scaling analysis [13, 47] and also obtained by exploitation of the Renormalization Group techniques in the limit of low temperatures, i.e. $\alpha \to 0$ [30]. The non-linear response is present only due to the strong disorder and the quenched nature of the disorder. For the annealed case with power-law waiting times the response is linear [13]. From the treatment of the one-dimensional case it becomes clear that the non-linearity appears solely due to presence of $\Lambda$ in the denominator of equation (66). According to equation (32) $\Lambda$ depends on $Q_0$ in a non-trivial fashion. When a small external force is present it alters the probability of return $Q_0$. Of special interest are the cases where $Q_0 = 1$ when $F_0 = 0$. Addition of small $|F_0|$ will decrease $Q_0$ and introduce a non-linear contribution due to the divergence $\Lambda$ in the limit of $Q_0 \to 1$. For the cases where $Q_0 < 1$ when the external force is non-present, addition of a non-zero external force slightly decreases $Q_0$ that is translated to a small change in $\Lambda$ and the linear response is not affected. It is then according to classical result of

https://doi.org/10.1088/1742-5468/ab99bd
Pólya [1], the non-linear response is to be expected for \( d = 1, 2 \) while for any higher dimension the strong quenched disorder will not alter the linear response to external field.

5. Summary

The properties of transport in the QTM have been extensively explored over the years. In this manuscript we provided an explicit mapping between the transient cases of QTM and the widely used CTRW. This result allows to generalize any result that is known for the CTRW to the case of QTM. Immediate applications include, first-passage properties [49], super-diffusive fluctuations for anomalous transport [50, 51], representation by the means of fractional equations [14], large deviation properties [52] and many more. The non trivial dependence of the mapping on the probability to return to the origin, \( Q_0 \), implies that we should expect very important differences between the QTM and CTRW for low dimensions even when the process is transient. Like the existence of non-linear response to externally applied field that was calculated for the QTM and is absent for CTRW. The developed theoretical framework of double subordination and two-point probabilities have merit on their own. We hope that these methods will help in addressing the recurrent case of QTM. Finally we would like to notice that existence of explicit mappings between the QTM and other models of transport in disordered media, such as the barrier model [53], can allow to address the general case of transport in a random-potential landscape [54].

Acknowledgments

This work was supported by the Pazy foundation grant 61139927. I thank D.A. Kessler for fruitful discussions and the anonymous referee for suggestion to replace \( \psi(u) \) by \( \log(\psi(u)) \).

Appendix A

A.1. Additional terms of \( \hat{\psi}(u) \)

In section 2.1 it was shown that when the expansion of \( \hat{\psi}(u) \) is of the form \( \hat{\psi}(u) \sim 1 - Au^\alpha \), equation (19) holds. Here we show that additional terms in the expansion, i.e. \( \hat{\psi}(u) \sim 1 - Au^\alpha + Bu^\beta \) with \( \beta > \alpha \), will not change this equation when \( S_\alpha \to \infty \). In such a case

\[
\langle e^{-u\eta} \rangle = \prod_x \left( 1 - \frac{n_x^\alpha}{S_\alpha} Au^\alpha + \frac{n_x^\beta}{S_\alpha^{\beta/\alpha}} Bu^\beta \right)
\]

and the multiplication will produce the terms mentioned in section 2.1 and also terms of the form \( \sum_x n_x^\beta Bu^\beta / S_\alpha^{\beta/\alpha} \), \( \sum_x \sum_x n_x^\alpha n_x^\beta ABu^{\alpha+\beta} / S_\alpha^{1+\beta/\alpha} \), \( \sum_x \sum_x n_x^\beta n_x^\beta B^2 u^{2\beta} / S_\alpha^{2\beta/\alpha} \) etc. Since \( \sum_x n_x^\beta = S_\beta \), the behavior of the term \( \sum_x n_x^\beta Bu^\beta / S_\alpha^{\beta/\alpha} \) is dictated by the ratio
\( S_\beta / S_\alpha^\beta / \alpha \). For the transient case, i.e. presence of bias or \( d > 2 \), we have shown in section 3 that \( S_{\alpha} \sim \Lambda N \) when \( N \rightarrow \infty \). This means that in the limit of many jumps, \( N \rightarrow \infty \), the ratio \( S_\beta / S_\alpha^\beta / \alpha \) is decaying like \( N^{-\frac{d+1}{2}} \), \( (\beta > \alpha) \). Therefore, all the terms that are not of the form \( \left( \sum_{x} \frac{M_z}{M_z^\alpha} \psi_\alpha \right)^j \) will decay to 0 in the \( N \rightarrow \infty \) limit. We can then state that only the two first terms in the expansion of \( \hat{\psi}(u) \) \((1 - Au^\alpha)\) are needed.

A.2. Generating functions of two-point probabilities

In section 3.2 three two-point probabilities were crucial for the behavior of \( \beta_N(x, k_1 : x', k_2) \): \( I \) \( f_N(x, 1 : x', 0) \), \( II \) \( M_N(x, x') \) and \( III \) \( T_N(x, x') \).

The probability \( f_N(x, 1 : x', 0) \) is the probability to start at point 0 and after \( N \) steps to reach the point \( x \) for the first time, without visiting \( x' \) even once. So from all the possibilities to reach \( x \) for the first time after \( N \) we must subtract those where the point \( x' \) was visited at-least once (before reaching \( x \)), i.e.

\[
f_N(x, 1 : x', 0) = f_N(x) - \sum_{i=0}^{N} f_i(x', 1 : x, 0) f_{N-i}(x - x'),
\]

(78)

where \( f_N(x) \) is the first-passage probability defined in equation (29). The translational invariance of the lattice was utilized. According to equation (78) the \( z \)-transform of \( f_N(x, 1 : x', 0) \)

\[
\hat{f}_z(x, 1 : x', 0) = \hat{f}_z(x) - \hat{f}_z(x', 1 : x, 0) \hat{f}_z(x - x').
\]

(79)

By switching the places of \( x \) and \( x' \) in equation (78) and performing a \( z \)-transform we obtain

\[
\hat{f}_z(x', 1 : x, 0) = \hat{f}_z(x') - \hat{f}_z(x, 1 : x', 0) \hat{f}_z(x' - x).
\]

(80)

Substitution of equation (80) into equation (79) leads to an expression for \( \hat{f}_z(x, 1 : x', 0) \) in terms of a generating function of \( f_N(x) \)

\[
\hat{f}_z(x, 1 : x', 0) = \frac{\hat{f}_z(x) - \hat{f}_z(x') \hat{f}_z(x - x')}{1 - \hat{f}_z(x' - x) \hat{f}_z(x - x')},
\]

(81)

The probability \( M_N(x, x') \) is the probability to start at \( x \) and after \( N \) steps to reach \( x' \) for the first time, without returning to \( x \) on the way. Due to translational invariance of the lattice \( M_N(x, x') \) is expressible in terms of \( f_N(x, 1 : x', 0) \), i.e. \( M_N(x, x') = f_N(x' - x, 1 : 0, 0) \). Then according to equation (81) the generating function of \( M_N(x, x') \) is

\[
\hat{M}_z(x, x') = \frac{\hat{f}_z(x' - x) - \hat{f}_z(0) \hat{f}_z(x' - x)}{1 - \hat{f}_z(x' - x) \hat{f}_z(x - x')}.
\]

(82)

The probability \( T_N(x, x') \) is the probability to return to \( x \) after \( N \) steps without visiting \( x' \) on the way. Once again the translational invariance of the lattice allows to utilize
Due to translational invariance the right-hand side of equation (88) equals to

\[ T_z(x, x') = \frac{\hat{f}_z(0) - \hat{f}_z(x' - x)f_z(x - x')}{1 - \hat{f}_z(x' - x)f_z(x - x')} \] (83)

### A.3. Properties of \( c_N(x) \) and summation over all lattice points

The probability to find the particle at position \( x \) after \( N \) steps (when starting at 0), \( c_N(x) \) is normalized, i.e. \( \sum_x c_N(x) = 1 \), where the summation is over all possible lattice points. This leads to the following relation

\[ \sum_x c_N(x)e^{ia\cdot x} \underset{a \to 0}{\longrightarrow} 1 \] (84)

and consequently for the generating function \( \hat{c}_z(x) = \sum_{N=0}^{\infty} z^N c_N(x) \)

\[ \sum_x \hat{c}_z(x)e^{ia\cdot x} \underset{a \to 0}{\longrightarrow} \frac{1}{1 - z}. \] (85)

For the single jump probability \( p(x) \) the characteristic function is defined as \( \hat{p}(a) = \sum_{x_1, x_2, \ldots, x_d} p(x)e^{ia\cdot x} \), where \( x = (x_1, x_2, \ldots, x_d) \) are all possible single steps on the lattice. Since all the jumps of the RW on the lattice are independent, \( \sum_x c_N(x)e^{ia\cdot x} = (\hat{p}(a))^N \) and according to equation (85)

\[ \sum_x \hat{c}_z(x)e^{ia\cdot x} = \frac{1}{1 - z\hat{p}(a)} \underset{a \to 0}{\longrightarrow} \frac{1}{1 - z}. \] (86)

According to equation (86) the double sum \( \sum_x \sum_{x'} \hat{c}_z(x)\hat{c}_z(x') \) is simply

\[ \sum_x \sum_{x'} \hat{c}_z(x)\hat{c}_z(x') = \lim_{a \to 0} \sum_x \sum_{x'} \hat{c}_z(x)\hat{c}_z(x')e^{ia\cdot x}e^{ia\cdot x'} = \lim_{a \to 0} \frac{1}{(1 - z\hat{p}(a))^2} = \frac{1}{(1 - z)^2}. \] (87)

This result is simply extended to the case of \( \sum_x \sum_{x'} \hat{c}_z(x)\hat{c}_z(x' - x) \). Indeed,

\[ \sum_x \sum_{x'} \hat{c}_z(x)\hat{c}_z(x' - x)e^{ia\cdot x}e^{ia\cdot x'} = \sum_x \hat{c}_z(x)e^{ia\cdot x} \sum_{x'} \hat{c}_z(x' - x)e^{ia(x' - x)}, \] (88)

due to translational invariance the right-hand side of equation (88) equals to \( \frac{1}{1 - z\hat{p}(2a)} \) and we obtain

\[ \sum_x \sum_{x'} \hat{c}_z(x)\hat{c}_z(x' - x)e^{ia\cdot x}e^{ia\cdot x'} = \lim_{a \to 0} \frac{1}{1 - z\hat{p}(2a)} = \frac{1}{1 - z}. \] (89)

Sums of terms of the form \( \hat{c}_z(x')\hat{c}_z(x - x') \) produce similar result. Generally speaking, when the arguments of \( \hat{c}_z(...)\hat{c}_z(...) \) cover all possible points \( (x, x') \) of the 2d lattice, the double summation will provide the result \( 1/(1 - z)^2 \).

https://doi.org/10.1088/1742-5468/ab99bd
We turn now to calculation of sums of the form \( \sum_x \sum_{x'} \hat{c}_z(x) \hat{c}_z(x'-x) e^{iax'} \) must be inspected. According to the convolution theorem

\[
\sum_{x'} \hat{c}_z(x') \hat{c}_z(-x') e^{iax'} = \left( \frac{1}{2\pi} \right)^d \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} \frac{1}{1 - z \hat{p}(b)} \frac{1}{1 - z \hat{p}(b - a)} d^d b, \tag{90}
\]

where \( d^d b = db_1 db_2 \ldots db_d \). When the \( a \to 0 \) limit is taken, the integrand on the right-hand side of equation (90) is simply \( 1/(1 - z \hat{p}(b))^2 \). Moreover, the asymptotic limit of \( N \to \infty \) is translated as the \( z \to 1 \) limit in the \( z \) space. In this limit the main contribution to the integral in equation (90) is from the values of \( b \) that are in the vicinity of \( 0 \), since \( \hat{p}(0) = 1 \) and the integrand converges to \( 1/(1 - z)^2 \). We concentrate on two types of \( \hat{p}(b) \) expansions in the vicinity of \( b = 0 \). The first type is a linear case

\[
\hat{p}(b) \sim 1 + ib \cdot B \quad b \to 0. \tag{91}
\]

This is the case of an RW with a bias in the \( B \) direction. Then

\[
\left( \frac{1}{2\pi} \right)^d \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} \frac{1}{1 - z \hat{p}(b)} d^d b \sim \left( \frac{1}{2\pi} \right)^d \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} \frac{1}{1 - z(1 + i b \cdot B)} d^d b, \tag{92}
\]

and since \( 1/(1 - z(1 + i b \cdot B))^2 = (1 - z)^{-2} \left[ 1 + (1 - z)^2 (b \cdot B)^2/\left[ 1 + (1 - z)^2 (b \cdot B)^2 \right] \right]^2 \) we obtain for equation (92) (after making \( z \to 1 \) substitution)

\[
(1 - z)^{-d-2} \left( \frac{1}{2\pi} \right)^d \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} \frac{[1 + i b \cdot B]^2}{[1 + (b \cdot B)^2]^2} d^d b'. \tag{93}
\]

We see that in the \( z \to 1 \) limit the \( z \) dependence arrives from the \( (1 - z)^{d-2} \) pre-factor and the fact that the range of integration diverges as \( 1/(1 - z) \). For \( d = 1 \) extra caution is needed since the pre-factor \( 1/(1 - z) \) diverges while the integral \( \int_{-\infty}^{\infty} [1 + i bB]^2/[1 + (bB)^2] db' = 0 \). Exact calculation of the integral in equation (93) for \( d = 1 \) shows that

\[
\frac{1}{2\pi (1 - z)} \int_{-\pi}^{\pi} \frac{[1 + i bB]^2}{[1 + (bB)^2]^2} db' = \frac{1}{1 + z(z - 2 + B^2 \pi^2)} \xrightarrow{z \to 1} \frac{1}{B^2 \pi^2} \tag{94}
\]

a constant and is not diverging in the \( z \to 1 \) limit. This proofs that for \( d = 1 \) and the case of a present bias \( (B \neq 0) \) the sum \( \sum_x \sum_{x'} \hat{c}_z(x') \hat{c}_z(-x') \) converges to a constant when \( z \to 1 \) so the double sum \( \sum_x \sum_{x'} \ldots \) diverges as \( 1/(1 - z) \) (and not as \( 1/(1 - z)^2 \)) in the \( z \to 1 \) limit. For any \( d \geq 2 \) the pre-factor \( 1/(1 - z)^{d-2} \) in equation (93) is not diverging and the only divergences are possible from the range of the integration when \( z \to 1 \). Inspection of the function \( [1 + i \sum_{j=1}^{d} b_j B_j]^2/[1 + (\sum_{j=1}^{d} b_j B_j)^2]^2 \) shows that when the \( |b_j| \to \infty \) the leading order of this function is \( \sim 1/(b_1 B_1 + \sum_{j=2}^{d} b_j B_j)^2 \). Integration over \( b_1 \) provides a leading order of \( 1/(b_2 B_2 + \sum_{j=3}^{d} b_j B_j)^2 \) for \( |b_2| \to \infty \). Next integration over \( b_2 \) will provide

https://doi.org/10.1088/1742-5468/ab99bd
The transient case of the quenched trap model

a leading order of $\log \left( \sum_{j=1}^{d} b'_j B_j \right)$ for the other $b'_j$s. By continuing the integration over all the different $b'_j$ (d integrals in total) we obtain that the integrals in equation (93) are diverging as $| (1 - z)^{2-d} \log (1 - z) |$ when $z \to 1$. Then from equations (94), (93) and (90) it is established that

$$\sum_{x'} \hat{c}_z(x') \hat{c}_z(-x') \sim \begin{cases} \frac{1}{B^2 \pi^2} & d = 1 \\ \frac{1}{| \log (1 - z) |} & d \geq 2 \end{cases}$$ (95)

Finally we have shown that for any dimension of the lattice $d$, when the RW has a bias (i.e. $B \neq 0$), the double sum $\sum_x \sum_{x'} \hat{c}_z(x) \hat{c}_z(x' - x) \hat{c}_z(x - x')$ is growing as $| \log (1 - z) |/(1 - z) |$ in the $z \to 1$ limit.

The second type of behavior is the case without bias, i.e.

$$\hat{p}(b) \sim 1 - (b \cdot B)^2 \quad b \to 0.$$ (96)

In a similar fashion as equation (93) was derived, we obtain that

$$\sum_{x'} \hat{c}_z(x') \hat{c}_z(-x') \sim \begin{cases} (1 - z)^{-1/2} & d = 1 \\ (1 - z)^{-1} & d = 2 \\ | \log ((1 - z)) | & d > 2 \end{cases}$$ (97)

The integral on the right-hand side of equation (97) is always positive and the integration coordinates can be transformed into generalized polar coordinates. In this case the only non-constant integration is of the form $\int_0^{\sqrt{1/(1 - z)}} r^{d-1}/(1 + r^2)^2$ that is diverges as $(1 - z)^{2-d/2} \log(1 - z)$ for $d \geq 4$ and converges for any $d < 4$. Eventually in the $z \to 1$ limit

$$\sum_{x'} \hat{c}_z(x') \hat{c}_z(-x') \sim \begin{cases} (1 - z)^{-3/2} & d = 1 \\ (1 - z)^{-1} & d = 2 \\ | \log((1 - z)) | & d > 2 \end{cases}$$ (98)

We have shown that for any dimension $d > 2$, when the RW has no bias (i.e. $B = 0$), the double sum $\sum_x \sum_{x'} \hat{c}_z(x) \hat{c}_z(x' - x) \hat{c}_z(x - x')$ is growing as $| \log (1 - z) |/(1 - z) |$ in the $z \to 1$ limit.

We have proven that for the specific case of $\sum_x \sum_{x'} \hat{c}_z(x) \hat{c}_z(x' - x) \hat{c}_z(x - x')$ and transient RW the double sum diverges slower than $1/(1 - z)^2$ in the $z \to 1$ limit. This result holds also for any double summation over $x$ and $x'$ and triple multiplications of the probability densities $\hat{c}_z(x - x') \hat{c}_z(x' - x) \hat{c}_z(x')$ (or any permutation of the positions).

Again, due to the properties of the convolution integrals that lead to equations (95) and (98). When the double summation is performed over multiplication of more than three $\hat{c}_z(x)$s the result will be equivalent to several convolutions integral. Since each convolution reduces the order of divergence of $1/(1 - z)$, additional convolutions will only reduce the divergences that appear in equations (95) and (98). This means that the results of this section show that any double summation over $x$ and $x'$ and $n$th
The transient case of the quenched trap model

multiplication of positional PDFs diverges slower than $1/(1 - z)^2$ when $z \to 1$, if the RW is transient.

References

[1] Weiss G H 1994 Aspects and Applications of the Random Walk (Amsterdam: North-Holland)
[2] Jeon J-H, Tejedor V, Burov S, Barkai E, Selhuber-Unkel C, Berg-Sorensen K, Oddershede L and Metzler R 2011 Phys. Rev. Lett. 106 048103
[3] Novikov D S, Fiems E, Jensen J H and Helpern J A 2011 Proc. Natl Acad. Sci. USA 110 4911
[4] Bouchaud J P and Georges A 1990 J. Physique 48 1855
[5] Jensen J L 1906 Acta Math. 30 175
[6] Abramowitz M and Stegun I A 1972 Handbook of Mathematical Functions (New York: Dover Publications)
[7] Redner S 2001 A Guide to First-Passage Processes (Cambridge: Cambridge University Press)
The transient case of the quenched trap model

[50] Khoury M, Lacasta A M, Sancho J M and Lindenberg K 2011 Phys. Rev. Lett. 106 090602
[51] Bénichou O et al 2013 Phys. Rev. Lett. 111 260601
[52] Barkai E and Burov S 2020 Phys. Rev. Lett. 124 060603
[53] Jack R L and Sollich P 2008 J. Phys. A: Math. Theor. 41 1
[54] Camboni F and Sokolov I M 2012 Phys. Rev. E 85 050104