Monotonicity of quantum relative entropy revisited

This paper is dedicated to Elliott Lieb and Huzihiro Araki
on the occasion of their 70th birthday

Dénes Petz

Department for Mathematical Analysis
Budapest University of Technology and Economics
H-1521 Budapest XI., Hungary

Monotonicity under coarse-graining is a crucial property of the quantum relative entropy. The aim of this paper is to investigate the condition of equality in the monotonicity theorem and in its consequences as the strong sub-additivity of von Neumann entropy, the Golden-Thompson trace inequality and the monotonicity of the Holevo quantity. The relation to quantum Markov states is briefly indicated.

Key words: quantum states, relative entropy, strong sub-additivity, coarse-graining, Uhlmann’s theorem, $\alpha$-entropy.

1 Introduction

Quantum relative entropy was introduced by Umegaki [24] as a formal generalization of the Kullback-Leibler information (in the setting of finite von Neumann algebras). Its real importance was understood much later and the monograph [14] already deduced most information quantities from the relative entropy.

One of the fundamental results of quantum information theory is the monotonicity of relative entropy under completely positive mappings. After the discussion of some particular cases by Araki [3] and by Lindblad [12], this result was proven by Uhlmann [23] in full generality and nowadays it is referred as Uhlmann’s theorem. The strong sub-additivity property of entropy can be obtained easily from Uhlmann’s theorem (see [14] about this point and as a general reference as well) and Ruskai discussed
the relation of several basic entropy inequalities in details \[21, 22\]. The aim of this paper is to investigate the condition of equality in the monotonicity theorem and in its consequences. The motivation to do this comes from the needs of quantum information theory developed in the setting of matrix algebras in the last ten years \[13\], on the other hand work \[22\] has given also some stimulation.

The paper is written entirely in a finite dimensional setting but some remarks are made about the possible more general scenario.

2 Uhlmann’s theorem

Let \( H \) be a finite dimensional Hilbert space and \( D_i \) be statistical operators on \( H \) \( (i = 1, 2) \). Their relative entropy is defined as

\[
S(D_1, D_2) = \begin{cases} 
\text{Tr} D_1(\log D_1 - \log D_2) & \text{if supp} \ D_1 \subset \text{supp} \ D_2, \\
+\infty & \text{otherwise}.
\end{cases}
\]

(1)

If \( \lambda > 0 \) is the smallest eigenvalue of \( D_2 \), then \( S(D_1, D_2) \) is always finite and \( S(D_1, D_2) \leq \log n - \log \lambda \), where \( n \) is the dimension of \( H \).

Let \( K \) be another finite dimensional Hilbert space. We call a linear mapping \( T : B(H) \to B(K) \) coarse graining if \( T \) is trace preserving and 2-positive, that is

\[
\begin{bmatrix} T(A) & T(B) \\ T(C) & T(D) \end{bmatrix} \geq 0 \quad \text{if} \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \geq 0.
\]

(2)

Such a \( T \) sends a statistical operator to statistical operator and satisfies the Schwarz inequality \( T(a^*a) \geq T(a)a^* \). The concept of coarse graining is the quantum version of the Markovian mapping in probability theory. All the important examples are actually completely positive. We work in this more general framework because the proofs require only the Schwarz inequality.

\( B(H) \) and \( B(K) \) are Hilbert spaces with respect to the Hilbert-Schmidt inner product and the adjoint of \( T : B(H) \to B(K) \) is defined:

\[
\text{Tr} \ A T(B) = \text{Tr} \ T^*(A) B \quad (A \in B(K), B \in B(H)).
\]

The adjoint of a coarse graining \( T \) is 2-positive again and \( T^*(I) = I \). It follows that \( T^* \) satisfies the Schwarz inequality as well.

The following result is known as Uhlmann’s theorem.

**Theorem 1** (\[23, 14\]) For a coarse graining \( T : B(H) \to B(K) \) the monotonicity

\[
S(D_1, D_2) \geq S(T(D_1), T(D_2))
\]

holds.
It should be noted that relative entropy was defined in the setting of von Neumann algebras first by Umegaki [24] and extended by Araki [3]. Uhlmann’s monotonicity result is more general than the above statement.

To the best knowledge of the author, it is not known if the monotonicity theorem holds without the hypothesis of 2-positivity.

3 The proof of Uhlmann’s theorem and its analysis

The simplest way to analyse the equality in the monotonicity theorem is to have a close look at the proof of the inequality. Therefore we present a proof which is based on the relative modular operator method. The concept of relative modular operator was developed by Araki in the modular theory of operator algebras [4], however it could be used very well in finite dimensional settings. For example, Lieb’s concavity theorem gets a natural proof by this method [15].

Let $D_1$ and $D_2$ be density matrices acting on the Hilbert space $\mathcal{H}$ and assume that they are invertible. On the Hilbert space $\mathcal{B}(\mathcal{H})$ one can define an operator $\Delta$ as

$$\Delta a = D_2 a D_1^{-1} \quad (a \in \mathcal{B}(\mathcal{H})).$$

This is the so-called relative modular operator and it is the product of two commuting positive operators: $\Delta = LR$, where

$$La = D_2 a \quad \text{and} \quad Ra = a D_1^{-1} \quad (a \in \mathcal{B}(\mathcal{H})).$$

Since $\log \Delta = \log L + \log R$, we have

$$S(D_1, D_2) = \langle D_1^{1/2}, (\log D_1 - \log D_2) D_1^{1/2} \rangle = -\langle D_1^{1/2}, (\log \Delta) D_1^{1/2} \rangle.$$

The relative entropy $S(D_1, D_2)$ is expressed by the quadratic form of the logarithm of the relative modular operator. This is the fundamental formula what we use (and actually this is nothing else but Araki’s definition of the relative entropy in a general von Neumann algebra [3]).

Let $T$ be a coarse graining as in Theorem [1]. We assume that $D_1$ and $T(D_1)$ are invertible matrices and set

$$\Delta a = D_2 a D_1^{-1} \quad (a \in \mathcal{B}(\mathcal{H})) \quad \text{and} \quad \Delta_0 x = T(D_2) x T(D_1)^{-1} \quad (x \in \mathcal{B}(\mathcal{K})).$$

$\Delta$ and $\Delta_0$ are operators on the spaces $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$, respectively. Both become a Hilbert space with the Hilbert-Schmidt inner product. The relative entropies in the theorem are expressed by the resolvent of relative modular operators:

$$S(D_1, D_2) = -\langle D_1^{1/2}, (\log \Delta) D_1^{1/2} \rangle = \int_0^\infty \langle D_1^{1/2}, (\Delta + t)^{-1} D_1^{1/2} \rangle - (1 + t)^{-1} dt.$$
\[ S(T(D_1), T(D_2)) = -(T(D_1)^{1/2}, (\log \Delta_0)T(D_1)^{1/2}) \]
\[ = \int_0^\infty \langle T(D_1)^{1/2}, (\Delta_0 + t)^{-1}T(D_1)^{1/2} \rangle - (1 + t)^{-1} dt, \]

where the identity
\[ \log x = \int_0^\infty (1 + t)^{-1} - (x + t)^{-1} dt \]
is used. The operator
\[ VxT(D_1)^{1/2} = T^*(x)D_1^{1/2} \]
is a contraction:
\[ \|T^*(x)D_1^{1/2}\|^2 = \text{Tr} D_1 T^*(x^*)T^*(x) \leq \text{Tr} D_1 T^*(x^*)x = \text{Tr} T(D_1)x x = \|xT(D_1)^{1/2}\|^2 \]
since the Schwarz inequality is applicable to \( T^* \). A similar simple computation gives that
\[ V^*\Delta V \leq \Delta_0. \] (4)
The function \( y \mapsto (y + t)^{-1} \) is operator monotone (decreasing) and operator convex, hence
\[ (\Delta_0 + t)^{-1} \leq (V^*\Delta V + t)^{-1} \leq V^*(\Delta + t)^{-1}V \] (5)
(see \[8\]). Since \( VT(D_1)^{1/2} = D_1^{1/2} \), this implies
\[ \langle D_1^{1/2}, (\Delta + t)^{-1}D_1^{1/2} \rangle \geq \langle T(D_1)^{1/2}, (\Delta_0 + t)^{-1}T(D_1)^{1/2} \rangle . \] (6)
By integrating this inequality we have the monotonicity theorem from the above integral formulas.

Now we are in the position to analyse the case of equality. If
\[ S(D_1, D_2) = S(T(D_1), T(D_2)), \]
then
\[ \langle T(D_1)^{1/2}, V^*(\Delta + t)^{-1}VT(D_1)^{1/2} \rangle = \langle T(D_1)^{1/2}, (\Delta_0 + t)^{-1}T(D_1)^{1/2} \rangle . \] (7)
for all \( t > 0 \). This equality together with the operator inequality \[8\] gives
\[ V^*(\Delta + t)^{-1}D_1^{1/2} = (\Delta_0 + t)^{-1}T(D_1)^{1/2} \] (8)
for all \( t > 0 \). Differentiating by \( t \) we have
\[ V^*(\Delta + t)^{-2}D_1^{1/2} = (\Delta_0 + t)^{-2}T(D_1)^{1/2} \] (9)
and we infer
\[ \|V^*(\Delta + t)^{-1}D_1^{1/2}\|^2 = \langle (\Delta_0 + t)^{-2}T(D_1)^{1/2}, T(D_1)^{1/2} \rangle = \langle V^*(\Delta + t)^{-2}D_1^{1/2}, T(D_1)^{1/2} \rangle = \|T(D_1)^{1/2}\|^2 \]
When $\|V^*\xi\| = \|\xi\|$ holds for a contraction $V$, it follows that $VV^*\xi = \xi$. In the light of this remark we arrive at the condition

$$VV^*(\Delta + t)^{-1}D_1^{1/2} = (\Delta + t)^{-1}D_1^{1/2}$$

and

$$V(\Delta_0 + t)^{-1}T(D_1)^{1/2} = VV^*(\Delta + t)^{-1}D_1^{1/2} = (\Delta + t)^{-1}D_1^{1/2}$$

By Stone-Weierstrass approximation we have

$$Vf(\Delta_0)T(D_1)^{1/2} = f(\Delta)D_1^{1/2}$$

for continuous functions. In particular for $f(x) = x^{it}$ we have

$$T^*(T(D_2)^{it}T(D_1)^{-it}) = D_2^{it}D_1^{-it}.$$  \hfill (11)

This condition is necessary and sufficient for the equality.

**Theorem 2** Let $T : B(\mathcal{H}) \to B(\mathcal{K})$ be a 2-positive trace preserving mapping and let $D_1, D_2 \in B(\mathcal{H}), T(D_1), T(D_2) \in B(\mathcal{K})$ be invertible density matrices. Then the equality $S(D_1, D_2) = S(T(D_1), T(D_2))$ holds if and only if the following equivalent conditions are satisfied:

1. $T^*(T(D_1)^{it}T(D_2)^{-it}) = D_1^{it}D_2^{-it}$ for all real $t$.
2. $T^*(\log T(D_1) - \log T(D_2)) = \log D_1 - \log D_2$.

The equality implies (11) which is equivalent to (1) in the Theorem. Differentiating (1) at $t = 0$ we have the second condition which obviously applies the equalities of the relative entropies. \hfill □

The above proof follows the lines of [7]. The original paper is in the setting of arbitrary von Neumann algebras and hence slightly more technical (due to the unbounded feature of the relative modular operators). Condition (2) of Theorem 2 appears also in the paper [22] in which different methods are used.

Next we recall a property of 2-positive mappings. When $T$ is assumed to be 2-positive, the set

$$\mathcal{A}_T := \{X \in B(\mathcal{H}) : T(X^*X) = T(X)T(X^*) \text{ and } T(X^*X) = T(X^*)T(X)\}.$$  \hfill (12)

is a *-sub-algebra of $B(\mathcal{H})$ and

$$T(XY) = T(X)T(Y) \quad \text{for all } X \in \mathcal{A}_T \text{ and } Y \in B(\mathcal{H}).$$
Corollary 1 Let \( T : B(\mathcal{H}) \to B(\mathcal{K}) \) be a 2-positive trace preserving mapping and let \( D_1, D_2 \in B(\mathcal{H}), T(D_1), T(D_2) \in B(\mathcal{K}) \) be invertible density matrices. Assume that \( T(D_1) \) and \( T(D_2) \) commute. Then the equality \( S(D_1, D_2) = S(T(D_1), T(D_2)) \) implies that \( D_1 \) and \( D_2 \) commute.

Under the hypothesis \( u_t := T(D_1)T(D_2)^{-1} \) and \( w_t := D_1T(D_2)^{-1} \) are unitaries. Since \( T^\star \) is unital \( u_t \in A_{T^\star} \) for every \( t \in \mathbb{R} \). We have

\[
    w_{t+s} = T^\star(u_{t+s}) = T^\star(u_tu_s) = T^\star(u_t)T^\star(u_s) = w_tw_s
\]

which shows that \( w_t \) and \( w_s \) commute and so do \( D_1 \) and \( D_2 \). \( \square \)

4 Consequences and related inequalities

3.1. The Golden-Thompson inequality. The Golden-Thompson inequality tells that

\[
    \text{Tr } e^{A+B} \leq \text{Tr } e^A e^B
\]

holds for self-adjoint matrices \( A \) and \( B \). It was shown in [18] that this inequality can be reformulated as a particular case of monotonicity when \( e^A/\text{Tr} e^A \) is considered as a density matrix and \( e^{A+B}/\text{Tr} e^{A+B} \) is the so-called perturbation by \( B \). Corollary 5 of the original paper is formulated in the context of von Neumann algebras but the argument was adapted to the finite dimensional case in [19], see also p. 128 in [14]. The equality holds in the Golden-Thompson inequality if and only if \( AB = BA \).

One of the possible extensions of the Golden-Thompson inequality is the statement that the function

\[
    p \mapsto \text{Tr } (e^{pB/2}e^{pA}e^{pB/2})^{1/p}
\]

is increasing for \( p > 0 \). The limit at \( p = 0 \) is \( \text{Tr } e^{A+B} \) [1]. It was proved by Friedland and So that the function ([13]) is strictly monotone or constant [4]. The latter case corresponds to the commutativity of \( A \) and \( B \).

3.2. A posteriori relative entropy. Let \( E_j \) (\( 1 \leq j \leq m \)) be a partition of unity in \( B(\mathcal{H})_+ \), that is \( \sum E_j = I \). (The operators \( E_j \) could describe a measurement giving finitely many possible outcomes.) Any density matrix \( D_i \in B(\mathcal{H}) \) determines a probability distribution

\[
    \mu_i = (\text{Tr } D_i E_1, \text{Tr } D_i E_2, \ldots, \text{Tr } D_i E_m).
\]

It follows from Uhlmann’s theorem that

\[
    S(\mu_1, \mu_2) \leq S(D_1, D_2). \quad (14)
\]

We give an example that the equality in (14) may appear non-trivially.
**Example 1** Let $D_2 = \text{Diag}(1/3, 1/3, 1/3), D_1 = \text{Diag}(1-2\mu, \mu, \mu), E_1 = \text{Diag}(1, 0, 0)$ and

$$E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & x & z \\ 0 & \bar{z} & 1-x \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1-x & -z \\ 0 & -\bar{z} & x \end{bmatrix}$$

When $0 < \mu < 1/2$, $0 < x < 1$ and for the complex $z$ the modulus of $z$ is small enough we have a partition of unity and $S(\mu_1, \mu_2) = S(D_1, D_2)$ holds.

First we prove a lemma.

**Lemma 1** If $D_2$ is an invertible density then the equality in (14) implies that $D_2$ commutes with $D_1, E_1, E_2, \ldots, E_m$.

The linear operator $T$ associates a diagonal matrix

$$\text{Diag} (\text{Tr} DE_1, \text{Tr} DE_2, \ldots, \text{Tr} DE_m)$$

to the density $D$ acting on $\mathcal{H}$ and under the hypothesis (11) is at our disposal. We have

$$\langle D_1^{1/2}, T^*(T(D_1)^{it}T(D_2)^{-it})D_2^{1/2} \rangle = \langle D_1^{1/2}, D_1^{it}D_2^{-it}D_2^{1/2} \rangle .$$

Actually we benefit from the analytic continuation and we put $-i/2$ in place of $t$. Hence

$$\sum_{j=1}^{m} (\text{Tr} E_j D_1)^{1/2} (\text{Tr} E_j D_1)^{1/2} = \text{Tr} D_1^{1/2} D_2^{1/2} . \quad (15)$$

The Schwarz inequality tells us that

$$\text{Tr} D_1^{1/2} D_2^{1/2} = \langle D_1^{1/2}, D_2^{1/2} \rangle = \sum_{j=1}^{m} \langle D_1^{1/2} E_j^{1/2}, D_2^{1/2} E_j^{1/2} \rangle$$

$$\leq \sum_{j=1}^{m} \sqrt{\langle D_1^{1/2} E_j^{1/2}, D_1^{1/2} E_j^{1/2} \rangle} \sqrt{\langle D_2^{1/2} E_j^{1/2}, D_2^{1/2} E_j^{1/2} \rangle}$$

$$= \sum_{j=1}^{m} (\text{Tr} E_j D_1)^{1/2} (\text{Tr} E_j D_2)^{1/2} .$$

The condition for equality in the Schwarz inequality is well-known: There are some complex numbers $\lambda_j \in \mathbb{C}$ such that

$$D_1^{1/2} E_j^{1/2} = \lambda_j D_2^{1/2} E_j^{1/2} . \quad (16)$$

(Since both sides have positive trace, $\lambda_j$ are actually positive. ) The operators $E_j$ and $E_j^{1/2}$ have the same range, therefore

$$D_1^{1/2} E_j = \lambda_j D_2^{1/2} E_j . \quad (17)$$
Summing over $j$ we obtain
\[ D_2^{-1/2} D_1^{1/2} = \sum_{j=1}^{m} \lambda_j E_j. \]

Here the right hand side is self-adjoint, so $D_2^{-1/2} D_1^{1/2} = D_1^{1/2} D_2^{-1/2}$ and $D_1 D_2 = D_2 D_1$. Now it follows from (14) that $E_j$ commutes with $D_2$. □

Next we analyse the equality in (14). If $D_2$ is invertible, then the previous lemma tells us that $D_1$ and $D_2$ are diagonal in an appropriate basis. In this case $S(\mu_1, \mu_2)$ is determined by the diagonal elements of the matrices $E_j$. Let $E(A)$ denote the diagonal matrix whose diagonal coincides with that of $A$. If $E_j$ is a partition of unity, then so is $E(E_j)$. However, given a partition of unity $E_j$ of diagonal matrices, there could be many choice of a partition of unity $E_j$ such that $E(E_j) = F_j$, in general. In the moment we do not want to deal with this ambiguity, and we assume that we have a basis $e_1, e_2, \ldots, e_n$ consisting of common eigenvectors of the operators $D_1, D_2, E(E_1), E(E_2), \ldots, E(E_n)$:

\[ D_j e_k = v_k^{j} e_k \quad \text{and} \quad E(E_j) e_k = w_{kj} e_k \quad (i = 1, 2, j = 1, 2, \ldots, m, k = 1, 2, \ldots, n). \]

The matrix $[w_{kj}]_{kj}$ is (raw) stochastic and condition (17) gives

\[ \frac{v_k^1}{v_k^2} w_{kj} = (\lambda_j)^2 w_{kj} \]

This means that $w_{kj} \neq 0$ implies that $v_k^1/v_k^2$ does not depend on $k$. In other words, $D_1 D_2^{-1}$ is constant on the support of any $E_j$.

Let $j$ be equivalent with $k$, if the support of $E(E_j)$ intersects the support of $E(E_k)$. We denote by $[j]$ the equivalence class of $j$ and let $J$ be the set of equivalence classes.

\[ P_{[j]} := \sum_{k \in [j]} E(E_k) \]

must be a projection and $\{P_{[j]} : [j] \in J\}$ is a partition of unity. We deduced above that

\[ D_1 D_2^{-1} P_{[j]} = \lambda_j P_{[j]} \]

One cannot say more about the condition for equality. All these extracted conditions hold in the above example and $E(E_k)$’s do not determine $E_k$’s, see the freedom for the variable $z$ in the example.

We can summarise our analysis as follows. The case of equality in (14) implies some commutation relation and the whole problem is reduced to the commutative case. It is not necessary that the positive-operator-valued measure $E_j$ should have projection values.

3.3. The Holevo bound. Let $E_j (1 \leq j \leq m)$ be a partition of unity in $B(\mathcal{K})_+$, $\sum_j E_j = I$. We assume that the density matrix $D \in B(\mathcal{H})$ is in the form of a convex...
combination \( D = \sum_i p_i D_i \) of other densities \( D_i \). Given a coarse graining \( T : B(\mathcal{H}) \to B(\mathcal{K}) \) we can say that our signal \( i \) appears with probability \( p_i \), it is encoded by the density matrix \( D_i \), after transmission the density \( T(D_i) \) appears in the output and the receiver decides that the signal \( j \) was sent with the probability \( \text{Tr} T(D_i) E_j \). This is the standard scheme of quantum information transmission. Any density matrix \( D_i \in B(\mathcal{H}) \) determines a probability distribution \( \mu_i = (\text{Tr} T(D_i) E_1, \text{Tr} T(D_i) E_2, \ldots, \text{Tr} T(D_i) E_m) \).

On the output. The inequality

\[
S(\mu) - \sum_i p_i S(\mu_i) \leq S(D) - \sum_i p_i S(D_i)
\]

(where \( \mu := \sum_i p_i \mu_i \) and \( D := \sum_i p_i D_i \)) is the so-called Holevo bound for the amount of information passing through the communication channel. Note that the Holevo bound appeared before the use of quantum relative entropy and the first proof was more complicated.

\( \mu_i \) is a coarse-graining of \( T(D_i) \), therefore inequality (18) is of the form

\[
\sum_i p_i S(R(D_i), R(D)) \leq \sum_i p_i S(D_i, D).
\]

On the one hand, this form shows that the bound (18) is a consequence of the monotonicity, on the other hand, we can make an analysis of the equality. Since the states \( D_i \) are the codes of the messages to be transmitted, it would be too much to assume that all of them are invertible. However, we may assume that \( D \) and \( T(D) \) are invertible. Under this hypothesis Lemma 1 applies and tells us that the equality in (18) implies that all the operators \( T(D_i), T(D_i) \) and \( E_j \) commute.

3.4. \( \alpha \)-entropies. The \( \alpha \)-divergence of the densities \( D_1 \) and \( D_2 \) is

\[
S_\alpha(D_1, D_2) = \frac{4}{1 - \alpha^2} \text{Tr} \left( D_1 - D_1^{\frac{1+\alpha}{2}} D_2^{\frac{1-\alpha}{2}} \right),
\]

which is essentially

\[
\langle D_2^{1/2}, \Delta^{\frac{1+\alpha}{2}} D_2^{1/2} \rangle
\]

up to constants in the notation of Sect. 2. The proof of the monotonicity works for this more general quantity with a small alteration. What we need is

\[
\langle D_2^{1/2}, \Delta^\beta D_2^{1/2} \rangle = \frac{\sin \pi \beta}{\pi} \int_0^\infty -t^\beta \langle D_2^{1/2}, (\Delta + t)^{-1} D_2^{1/2} \rangle + t^{\beta-1} dt
\]

for \( 0 < \beta < 1 \). Therefore for \( 0 < \alpha < 2 \) the proof of the above Theorem 2 goes through for the \( \alpha \)-entropies. The monotonicity holds for the \( \alpha \)-entropies, moreover (1) and (2) from Theorem 2 are necessary and sufficient for the equality.

The role of the \( \alpha \)-entropies is smaller than that of the relative entropy but they are used for approximation of the relative entropy and for some other purposes (see [9], for example).
5 Strong subadditivity of entropy and the Markov property

The strong subadditivity is a crucial property of the von Neumann entropy it follows easily from the monotonicity of the relative entropy. (The first proof of this property of entropy was given by Lieb and Ruskai [11] before the Uhlmann’s monotonicity theorem.) The strong subadditivity property is related to the composition of three different systems. It is used, for example, in the analysis of the translation invariant states of quantum lattice systems: The proof of the existence of the global entropy density functional is based on the subadditivity and a monotonicity property of local entropies is obtained by the strong subadditivity [20].

Consider three Hilbert spaces, \( \mathcal{H}_j, j = 1, 2, 3 \) and a statistical operator \( D_{123} \) on the tensor product \( \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \). This statistical operator has marginals on all subproducts, let \( D_{12}, D_2 \) and \( D_{23} \) be the marginals on \( \mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{H}_2 \otimes \mathcal{H}_1, \mathcal{H}_2 \otimes \mathcal{H}_3 \), respectively. (For example, \( D_{12} \) is determined by the requirement \( \text{Tr} \ D_{123}(A_{12} \otimes I_3) = \text{Tr} \ D_{12} A_{12} \) for every operator \( A_{12} \) acting on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \); \( D_2 \) and \( D_{23} \) are similarly defined.) The strong subadditivity asserts the following:

\[
S(D_{123}) + S(D_2) \leq S(D_{12}) + S(D_{23}) \tag{20}
\]

In order to prove the strong subadditivity, one can start with the identities

\[
S(D_{123}, \text{tr}_{123}) = S(D_{12}, \text{tr}_{12}) + S(D_{123}, D_{12} \otimes \text{tr}_3)
\]

\[
S(D_2, \text{tr}_2) + S(D_{23}, D_2 \otimes \text{tr}_3) = S(D_{23}, \text{tr}_{23}),
\]

where \( \text{tr} \) with a subscript denotes the density of the corresponding tracial state, for example \( \text{tr}_{12} = I_{12} / \dim(\mathcal{H}_1 \otimes \mathcal{H}_2) \). From these equalities we arrive at a new one,

\[
S(D_{123}, \text{tr}_{123}) + S(D_2, \text{tr}_2) = S(D_{12}, \text{tr}_{12}) + S(D_{23}, \text{tr}_{23})
\]

\[
+ S(D_{123}, D_{12} \otimes \text{tr}_3) - S(D_{23}, D_2 \otimes \text{tr}_3).
\]

If we know that

\[
S(D_{123}, D_{12} \otimes \text{tr}_3) \geq S(D_{23}, D_2 \otimes \text{tr}_3) \tag{21}
\]

then the strong subadditivity follows. Set a linear transformation \( B(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3) \rightarrow B(\mathcal{H}_2 \otimes \mathcal{H}_3) \) as follows:

\[
T(A \otimes B \otimes C) := B \otimes C(\text{Tr} \ A) \tag{22}
\]

T is completely positive and trace preserving. On the other hand, \( T(D_{123}) = D_{23} \) and \( T(D_{12} \otimes \text{tr}_3) = D_2 \otimes \text{tr}_3 \). Hence the monotonicity theorem gives [21].

This proof is very transparent and makes the equality case visible. The equality in the strong subadditivity holds if and only if we have equality in (21). Note that \( T \) is the partial trace over the third system and

\[
T^*(B \otimes C) = I \otimes B \otimes C. \tag{23}
\]
Theorem 3 Assume that $D_{123}$ is invertible. The equality holds in the strong subadditivity (20) if and only if the following equivalent conditions hold:

1. $D_{123}^t D_{12}^{-t} = D_{23}^t D_{2}^{-t}$ for all real $t$.
2. $\log D_{123} - \log D_{12} = \log D_{23} - \log D_2$.

Note that both condition (1) and (2) contain implicitly tensor products, all operators should be viewed in the three-fold-product. Theorem 2 applies due to (23) and this is the proof. □

It is not obvious the meaning of conditions (1) and (2) in Theorem 3. The easy choice is

$$
\log D_{12} = H_1 + H_2 + H_{12}, \quad \log D_{23} = H_2 + H_3 + H_{23}, \quad \log D_2 = H_2
$$

for a commutative family of self-adjoint operators $H_1, H_2, H_3, H_{12}, H_{23}$ and to define $\log D_{123}$ by condition (2) itself. This example lives in an abelian subalgebra of $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ and a probabilistic representation can be given.

$D_{123}$ may be regarded as the joint probability distribution of some random variables $\xi_1, \xi_2$ and $\xi_3$. In this language we can rewrite (1) in the form

$$
\frac{\text{Prob}(\xi_1 = x_1, \xi_2 = x_2, \xi_3 = x_3)}{\text{Prob}(\xi_1 = x_1, \xi_2 = x_2)} = \frac{\text{Prob}(\xi_2 = x_2, \xi_3 = x_3)}{\text{Prob}(\xi_2 = t_2)}
$$

or in terms of conditional probabilities

$$
\text{Prob}(\xi_3 = x_3|\xi_1 = x_1, \xi_2 = x_2) = \text{Prob}(\xi_3 = x_3|\xi_2 = x_2).
$$

In this form one recognizes the Markov property for the variables $\xi_1, \xi_2$ and $\xi_3$; subscripts 1, 2 and 3 stand for “past”, “present” and “future”. It must be well-known that for classical random variables the equality case in the strong subadditivity of the entropy is equivalent to the Markov property. The equality

$$
S(D_{123}) - S(D_{12}) = S(D_{23}) - S(D_2)
$$

means an equality of entropy increments. Concerning the Markov property, see [2] or pp. 200–203 in [14].

Theorem 4 Assume that $D_{123}$ is invertible. The equality holds in the strong subadditivity (20) if and only if there exists a completely positive unital mapping $\gamma : B(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3) \to B(\mathcal{H}_2 \otimes \mathcal{H}_3)$ such that

1. $\text{Tr}(D_{123} \gamma(x)) = \text{Tr}(D_{123}x)$ for all $x$.
2. $\gamma|B(\mathcal{H}_2) \equiv \text{identity}$. 

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If $\gamma$ has properties (1) and (2), then $\gamma^*(D_{23}) = D_{123}$ and $\gamma^*(D_2 \otimes \text{Tr}_3) = D_{12} \otimes \text{Tr}_3$ for its dual and we have equality in (21).

To prove the converse let

$$E(A \otimes B \otimes C) := B \otimes C(\text{Tr} A/ \dim \mathcal{H}_1)$$

which is completely positive and unital. Set

$$\gamma(\cdot) := D_{23}^{-1/2} E(D_{123}^{1/2} \cdot D_{123}^{1/2}) D_{23}^{-1/2}$$

If the equality holds in the strong subadditivity, then property (1) from Theorem 3 is at our disposal and it gives $\gamma(x) = x$ for $x \in B(\mathcal{H}_2)$. □

In a probabilistic interpretation $E$ and $\gamma$ are conditional expectations. $E$ preserves the tracial state and it is a projection of norm one. $\gamma$ leaves the state with density $D_{123}$ invariant, however it is not a projection. (Accardi and Cecchini called this $\gamma$ generalised conditional expectation, [1].)

It is interesting to construct translation invariant states on the infinite tensor product of matrix algebras (that is, quantum spin chain over $\mathbb{Z}$) such that condition (26) holds for all ordered subsystems 1, 2 and 3.

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