THE SPARSE T1 THEOREM

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Abstract. We impose standard \( T1 \)-type assumptions on a Calderón-Zygmund operator \( T \), and deduce that for bounded compactly supported functions \( f, g \) there is a sparse bilinear form \( \Lambda \) so that

\[
|\langle Tf, g \rangle| \lesssim \Lambda(f, g).
\]

The proof is short and elementary. The sparse bound quickly implies all the standard mapping properties of a Calderón-Zygmund on a (weighted) \( L^p \) space.

1. Introduction

We recast the statement of the \( T1 \) theorem of David and Journé [6], replacing the conclusion that the operator \( T \) admits a quantitative bound on its \( L^2 \)-norm, with the conclusion that \( T \) admits a quantitative sparse bound. From the sparse bound, one can quickly derive a wide range of (weighted) \( L^p \) type inequalities for \( T \). That is, the theory devoted to deriving these properties for \( T \) can be replaced by the much simpler approach via sparse operators.

We say that an operator \( T \) is a Calderón-Zygmund operator on \( \mathbb{R}^d \) if (a) it is bounded on \( L^2 \), (b) there is a kernel \( K(x, y) : \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, x) : x \in \mathbb{R}^d\} \to \mathbb{R} \) so that for functions \( f, g \) smooth, compactly supported, have disjoint closed supports,

\[
B_T(f, g) = \langle Tf, g \rangle = \int \int K(x, y)f(y)g(x) \, dy \, dx.
\]

(c) For some constant \( \mathcal{K}_T \), the kernel \( K(x, y) \) satisfies

\[
|K(x, y)| \leq \frac{\mathcal{K}_T}{|x-y|}, \quad x \neq y \in \mathbb{R}^d,
\]

\[
|K(x, y) - K(x', y)| < \mathcal{K}_T \frac{|x-x'|^\eta}{|x-y|^{d+\eta}}, \quad 0 < 2|x-x'| < |x-y|.
\]

And, the same condition with the roles of \( x \) and \( y \) reversed. Above, \( \eta > 0 \) is a fixed small constant.

A sparse bilinear form \( \Lambda(f, g) \) is defined this way: There is a collection of cubes \( S \), so that for each \( S \in S \), there is an \( E_S \subset S \) so that (a) \( |E_S| > c|S| \), and (b) \( \|\sum_{S \in S} 1_{E_S}\|_\infty \leq c^{-1} \). Then, set

\[
\Lambda(f, g) = \sum_{S \in S} \langle f \rangle_S \langle g \rangle_S |S|,
\]

where \( \langle f \rangle_S = |S|^{-1} \int_S f(x) \, dx \). Here, we will not focus on the role of the constant \( 0 < c < 1 \), and remark that many times it is assumed that the sets \( E_S \) being pairwise disjoint, that is \( \|\sum_{S \in S} 1_{E_S}\|_\infty = 1 \).

Research supported in part by grant NSF-DMS-1600693.
Our generalization does not affect the outlines of the theory, and makes some arguments somewhat simpler.

It is very useful to think of \( \Lambda(f, g) \) as a \textit{positive bilinear Calderón-Zygmund form}. In particular, all the standard inequalities can be quickly proved for \( \Lambda \). And, for weighted inequalities, it is easy to derive bounds that are sharp in the \( A_p \) characteristic.

Our formulation of the \( T1 \) theorem considers the usual \( L^1 \) testing condition on \( T \), phrased in bilinear language.

\textbf{Theorem 1.1.} Suppose that \( T \) is a Calderón-Zygmund operator on \( \mathbb{R}^d \), and moreover there is a constant \( T \) so that for all cubes \( Q \) and functions \( |\phi| < 1_Q \), there holds
\begin{equation}
|B_T(1_Q, \phi)| + |B_T(\phi, 1_Q)| \leq T|Q|.
\end{equation}
Then there is a constant \( C = C(\mathcal{X}_T, T, d, \eta) \) so that for all bounded compactly supported functions \( f, g \), there is a sparse operator \( \Lambda \) so that
\begin{equation}
|B_T(f, g)| < CA(|f|, |g|).
\end{equation}

The proof is elementary, using (a) facts about averages and conditional expectations; (b) random dyadic grids as a convenient tool to reduce the complexity of the argument; (c) orthogonality of martingale transforms, and the most sophisticated fact (d) a sparse bound for a certain bilinear square function, with complexity, detailed in Lemma 4.5. In addition, the testing condition (1.2) appears solely in the construction of the stopping times. The proof is carried out in \$3\$. There are many terms, organized so that there is one crucial term, in \$3.2\$. Almost all of the remaining cases use standard off-diagonal considerations, and the simple argument to prove the sparse bound for a martingale transform. This is detailed in \$4\).

The consequences of the sparse bound (1.3) are:

1. The weak type \((1, 1)\) inequality, and the \( L^p \) inequalities, for \( 1 < p < \infty \). These hold with the sharp dependence upon \( p \). To wit, using \( \|M : L^p \to L^p\| \lesssim p' = \frac{p}{p-1} \), we have

\[
\Lambda(f, g) = \int \sum_{S \in \mathcal{S}} \langle f \rangle_S \langle g \rangle_S 1_S \, dx \lesssim \int \sum_{S \in \mathcal{S}} \langle f \rangle_S \langle g \rangle_S 1_{E_S} \, dx \\
\leq \int Mf \cdot Mg \, dx \lesssim \|Mf\|_p \|Mg\|_{p'} \lesssim p \cdot p' \|f\|_p \|g\|_{p'}.
\]

2. The weighted version of the same, relative to \( A_p \) weights. The dependence upon the \( A_p \) characteristic is sharp, for \( 1 < p < \infty \), and the best known for the case of \( p = 1 \). See the arguments in [17].

3. The exponential integrability results of Karagulyan [12, 19].

Our statement of the \( T1 \) theorem follows the ‘testing inequality’ approach of the Sawyer two weight theorems [20, 21], and the statement in Stein’s monograph [22]. Our approach is a descendant of the radically dyadic approach of Figuel [8], further influenced by the martingale approach of Nazarov-Treil-Volberg [18]. (Also see [10,]). Our use of the stopping cubes follows that of the proof of the two weight Hilbert transform estimate [15].
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The bound by sparse operators has been an active and varied recent research topic. It had a remarkable success in Lerner’s approach to the $A_2$ bound [17], which cleverly bounded on the weighted norm of a Calderón-Zygmund by a the norm of a sparse operator. The pointwise approach first established in [3], with a somewhat different approach in [13]. The latter approach has been studied from several different points of view [2, 7, 16, 23]. The form approach used here, is however successful in settings where the pointwise approach would fail, most notably the setting of the bilinear Hilbert transform [4], Bochner Riesz multipliers [1], and oscillatory singular integrals [14]. The interested reader can consult the papers above for more information and references.

This paper proves the sparse bound without appealing to any structural theory of Calderón-Zygmund operators such as boundedness of maximal truncations, which is the approach started in [13]. The other prominent structural fact one could use is the Hytönen structure theorem [9]. This is the approach followed by Culiuc-Di Plinio-Ou [5] also using bilinear forms. They show that this approach has further applications to the matricial setting, avoiding difficulties for the pointwise approach in this setting.

2. Random Grids

All the proofs here will use Hytönen’s random dyadic grids from [9]. Recall again, that the standard dyadic grid in $\mathbb{R}^d$ is

$$\mathcal{D}^0 := \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k, \quad \mathcal{D}_k := \{ 2^k (0,1)^d + m : m \in \mathbb{Z}^d \}.$$ 

For a binary sequence $\omega := (\omega_j)_{j \in \mathbb{Z}} \in (\{0,1\}^d)^\mathbb{Z}$ we define a general dyadic system by

$$\mathcal{D}^\omega := \{ Q + \omega : Q \in \mathcal{D}^0 \},$$

where $Q + \omega = Q + \sum_{j:2^{-j}<\ell Q} 2^{-j} \omega_j$. We consider the standard uniform probability measure on $\{0,1\}^d$, that is, it assigns $2^{-d}$ to every point. We place on $\omega$, the probability measure $P$, the corresponding product measure on $(\{0,1\}^d)^\mathbb{Z}$. This way, we can see $(\mathcal{D}^\omega)$ as a collection of grids with a random set of parameters $\omega$. For every $\omega$, these dyadic grids satisfy the required properties, namely

1. For $P,Q \in \mathcal{D}^\omega$, $P \cap Q \in \{ P, Q, \emptyset \}$.
2. For fixed $k \in \mathbb{Z}$, the collection $\mathcal{D}^\omega_k = \{ Q \in \mathcal{D}^\omega : \ell Q = 2^{-k} \}$ partitions $\mathbb{R}^d$.

**Definition 2.1** (Good-bad intervals). Let $0 < \gamma < 1$ and a positive integer $r$ such that $r \geq (1-\gamma)^{-1}$. We say that $Q \in \mathcal{D}^\omega_k$ is $r$-bad, if there is an integer $s \geq r$, and a choice of coordinate, so that the vectors

$$\omega_k + (1-\gamma)s, \omega_k + (1-\gamma)s + 1, \ldots, \omega_k + s \in \{0,1\}^d,$$

all agree in that one coordinate. If $Q$ is not $r$-bad, then it is called $r$-good.

From now on, we are going to omit the dependence on $r$, and we will refer to the cubes as only good or bad. The following lemmas are well known.

**Lemma 2.1.** If $Q$ is good, then for any cube $P$ with $2^r \ell Q < \ell P$ we have

$$\text{dist}(Q, \partial P) \gtrsim (\ell Q)^\gamma (\ell P)^{1-\gamma},$$

where the implied constant is absolute.
Lemma 2.2. Fix $0 < \gamma < 1$ and $r > \gamma^{-1}$, then, there is a constant $C_d$ such that
\[ P(Q \text{ is good}) \geq 1 - C_d \gamma^{-1}2^{-\gamma r}. \]

For an arbitrary dyadic grid $D^\omega$, every function $f \in L^2(\mathbb{R}^d)$ admits an orthogonal decomposition
\[ f = \sum_{Q \in D^\omega} \Delta_Q f. \]

Given a dyadic grid $D^\omega$, we define the good and bad projections as
\[ P_{\omega}^{\text{bad}} f := \sum_{Q \in D^\omega} \Delta_Q f, \quad P_{\omega}^{\text{good}} f := \sum_{Q \in D^\omega} \Delta_Q f. \]

The following lemma says that in average, the bad projections tend to be small.

Lemma 2.3. For all $1 < p < \infty$ there is an $\varepsilon_p > 0$ such that for all $0 < \gamma < 1$ and $r > \gamma^{-1}$ we have
\[ \mathbb{E}_\omega \| P_{\omega}^{\text{bad}} f \|^p_{L^p} \lesssim 2^{-\varepsilon_p r} \| f \|^p_{L^p}. \]

Using this lemma, we can prove that it is enough to estimate bounds only for good functions, in the following sense

Lemma 2.4. Let $1 < p < \infty$. If $T : L^p \mapsto L^p$ is a bounded operator. If $0 < \gamma < 1$ is fixed and $r > C(1 + \log 1/\gamma)$, then
\[ \| T : L^p \mapsto L^p \| \leq 4M, \]
where $M$ is the best constant in the inequality
\[ \mathbb{E}_\omega | \langle T P_{\omega}^{\text{good}} f, P_{\omega}^{\text{good}} g \rangle | \leq M \| f \|_{L^p} \| g \|_{L^{p'}}. \]

3. The Proof of the Sparse Bound

As a consequence of Lemma 2.4, it is enough for the remainder of the argument to show this: There is a choice of constant $C > 1$, so that for all $f$ and $g$ compactly supported, and almost all grids $D^\omega$, there is a sparse operator $\Lambda = \Lambda_{f,g,D^\omega}$, so that
\[ | \langle T P_{\omega}^{\text{good}} f, P_{\omega}^{\text{good}} g \rangle | \leq C \Lambda(\| f \|, \| g \|). \]

In view of the Lemma 4.7, the random sparse operator above can be replaced by a deterministic one. Averaging over choices of grid will complete the proof.

Almost all random dyadic grids have the property that the functions $f, g$ are supported on a single good dyadic cube. And, hence, on a sequence of dyadic cubes which exhaust $\mathbb{R}^n$. This fact and goodness are the only facts about random grids utilized, so we suppress the $\omega$ dependence below. The inner product in (3.1) is expanded
\[ \langle T P_{\omega}^{\text{good}} f, P_{\omega}^{\text{good}} g \rangle = \sum_{P \in D} \sum_{Q \in D} \langle T \Delta_P f, \Delta_Q g \rangle. \]

We will further only consider the case of $\ell P \geq \ell Q$, the reverse case being addressed by duality. The fact that $P$ and $Q$ are good will be suppressed, but always referenced when it is used. And, by $Q \in P$ we will mean that $Q \subset P$ and $2^r \ell Q \leq \ell P$. Goodness of $Q$ then implies that
\[ \text{dist}(Q, \text{skel} P) \geq (\ell Q)^{(\ell P)^{1-\epsilon}}, \]
where \( \text{skel}P \) is the union of \( \partial P' \), where \( P' \) is a child of \( P \). We will likewise suppress the role of the dyadic grid in our notation.

As just mentioned, the two functions \( f, g \) are supported on a single good cube \( P_0 \in \mathcal{D} \), which we can take to be very large. Therefore, we can restrict the sum in (3.2) to only cubes \( P, Q \subset P_0 \). The bound we obtain will be independent of the choice of \( P_0 \). The sum we consider is then broken into several subcases.

\[
\sum_{P \subset P_0, Q : Q \subset P_0} \sum_{tP \geq tQ} \langle T \Delta_P f, \Delta_Q g \rangle
\]

(3.4)

\[
= \sum_{P \subset P_0} \sum_{Q \in P} \langle T \Delta_P f, \Delta_Q g \rangle \quad \text{(inside)}
\]

(3.5)

\[
+ \sum_{P \subset P_0} \sum_{Q : 2^i Q \lesssim tP} \sum_{Q \subset 3P \setminus P} \langle T \Delta_P f, \Delta_Q g \rangle \quad \text{(near)}
\]

(3.6)

\[
+ \sum_{P \subset P_0} \sum_{Q : 4Q \lesssim tP} \langle T \Delta_P f, \Delta_Q g \rangle \quad \text{(far)}
\]

(3.7)

\[
+ \sum_{P \subset P_0} \sum_{Q : 2^i Q \lesssim 2^i tQ} \langle T \Delta_P f, \Delta_Q g \rangle \quad \text{(neighbors)}
\]

3.1. Stopping Cubes. We define a sparse collection \( S \) of stopping cubes, and associated stopping values in the following way: Add \( P_0 \) to the collection \( S \), and set \( \sigma_f(P_0) = \langle |f| \rangle_{P_0} \), and similarly for \( g \). In the recursive stage of the construction, for minimal \( S \in \mathcal{S} \), define three sets,

\[
\begin{align*}
F_S^1 &= \bigcup \{ S' \in \mathcal{D}(S) : \langle |f| \rangle_{S'} > C_0 \sigma_f(S), \ S' \text{ maximal} \}.
F_S^2 &= \bigcup \{ S' \in \mathcal{D}(S) : \langle |g| \rangle_{S'} > C_0 \sigma_g(S), \ S' \text{ maximal} \}.
F_S^3 &= \bigcup \{ S' \in \mathcal{D}(S) : \langle |T1_S| \rangle_{S'} > C_0 |T, \ S' \text{ maximal} \}.
\end{align*}
\]

Let \( F_S = F_S^1 \cup F_S^2 \cup F_S^3 \), and \( F_S \) be the family of dyadic components of \( F_S \). The weak-type bound for the dyadic maximal function and the testing condition (1.2) implies that there exists \( C_0 \) big enough, such that \(|F_S| < \frac{1}{2} |S|\). Recursively, add, every \( F_S \) to the collection \( S \) to form a sparse collection.

We set \( P^* \) to be the smallest stopping cube \( S \) that contains \( P \). And we set \( Q^* \) to be the smallest stopping cube \( S \) such that \( Q \in \mathcal{S} \). The Haar projection associated to \( S \) is \( \Pi_{Sg} = \sum_{Q \subset P, Q \cap S^c} \Delta_Q g \).

3.2. The Inside Terms. We turn our attention to the main term, that of (3.4), for which there are three subcases. The argument of \( T \) is \( \Delta_P f \), which we write as

\[
\Delta_P f = \Delta_P f \mathbf{1}_{P \setminus P_Q} + \mathbf{1}_{P_Q} \Delta_P f
\]

(3.8)

\[
= \Delta_P f \mathbf{1}_{P \setminus P_Q} + \langle \Delta_P f \rangle_{P_Q} \cdot \begin{cases} S - S_{P_Q} & S = Q^c \cap P_Q \\ S + 1_{P_Q \setminus S} & S = Q^c \subseteq P_Q \end{cases}
\]

where \( Q \in P \), and \( P_Q \) is the child of \( P \) that contains \( Q \).
First Subcase. Control the first term on the right in (3.8) by off-diagonal considerations. Central to all of these off-diagonal arguments are the class of forms $B_{u,v}$ defined in (4.1), which are in turn bounded by Lemma 4.5.

Since $Q$ is a good cube, the inequality (3.3) holds: That is $Q$ is a long way from the skeleton of $P$. By (4.10), we have
\[ |\langle T(\Delta_P f \cdot 1_{P \setminus P_Q}), \Delta_Q g \rangle| \lesssim P_\eta \langle \Delta_P f \cdot 1_{P \setminus P_Q} \rangle(Q) \| \Delta_Q g \|_1 \lesssim [\ell(Q/\ell_P)]^{\eta'} \langle \Delta_P f \rangle_{P} \| \Delta_Q g \|_1. \]

Using the notation of (4.1), for integers $v \geq r$, we have
\[ \sum \sum_{P \subset Q} |\langle T(\Delta_P f \cdot 1_{P \setminus P_Q}), \Delta_Q g \rangle| \lesssim 2^{-(\eta'')v} B_{0,v}^{P}(f, g) \]
and by Lemma 4.5, this is in turn dominated by a choice of sparse form. Sparse forms are again dominated by a fixed form. We can sum this estimate over $v \geq r$, so this case is complete.

Second Subcase. We turn attention to the second term in (3.8), in which we have $\langle \Delta_P f \rangle_{P_Q} 1_S$. This is the most intricate step, in that we combine several elementary steps. The bound we prove is uniform over a choice of $S \in S$. Namely,
\[ \left| \sum_{Q : Q^r = S} \sum_{P : Q \subset P} \langle T(\Delta_P f \cdot 1_S), \Delta_Q g \rangle \right| \lesssim \langle \| f \|_S \langle \| g \|_S \rangle \rangle_S |S| \]
This is the one point in the argument in which the implied constant depends upon the testing constant $T$ in (1.2).

For each cube $Q$ with $Q^r = S$, define $\epsilon_Q$ by
\[ \epsilon_Q \| f \|_S := \sum_{P \in P, Q \subset P} \langle \Delta_P f \rangle_{P_Q} . \]
By the first stopping condition, corresponding to the control of the averages of $f$, $\{ \epsilon_Q \}_{Q \in \mathcal{D}}$ is uniformly bounded. In particular, this operator is a martingale transform.
\[ \Pi_S g = \sum_{Q : Q^r = S} \epsilon_Q \Delta_Q g. \]

We make the following observation about the second stopping condition, corresponding to the control of the averages of $g$. Setting a conditional expectation on $S$ to be
\[ \mathbb{E}(\phi \mid \mathcal{F}_S) = \begin{cases} \phi(x) & x \in S \setminus F_s \\ \langle \phi \rangle_{S'} & x \in S', \ S' \in \mathcal{F}_S \end{cases} \]
Then, $\| \mathbb{E}(g1_S \mid \mathcal{F}_S) \|_\infty \lesssim \langle \| g \|_S \rangle_S$. We also have $\Pi_S^c g = \Pi_S^c \mathbb{E}(g1_S \mid \mathcal{F}_S)$. Therefore, by the $L^2$ bound for martingale transforms,
\[ \| \Pi_S^c g \|_2 \leq \| \mathbb{E}(g1_S \mid \mathcal{F}_S) \|_2 \lesssim \langle \| g \|_S \rangle_S |S|^{1/2}. \]

The point of our third stopping condition, corresponding to the control of the average of $T1_S$, is that $\mathbb{E}(T1_S \mid \mathcal{F}_S)$ is bounded in $L^\infty$ by a constant multiple of $T$. Collecting these observations, we
can rewrite our sum as below, in which in the first step we use the definition (3.10) to collapse the sum over $P$.
\[
\text{LHS of (3.9)} = |\langle f \rangle_s(T1_{s}, \Pi_s g)| \\
= |\langle f \rangle_s(T1_{s}, E(\Pi_s g | \mathcal{F}_s))| \\
= |\langle f \rangle_s(E(T1_{s} | \mathcal{F}_s), \Pi_s g)| \\
\lesssim \|\langle f \rangle_s\|_{\mathcal{F}_s}\|\Pi_s g\|_{\mathcal{F}_s} \lesssim \|\langle f \rangle_s\|_{\mathcal{F}_s}\|S\|.
\]
This completes this case.

**Third Subcase.** We address the top alternative in (3.8), namely we bound
\[
\sum_{S} \sum_{Q: Q' = S} \sum_{P : \Pi_{Q} \subseteq S} \langle \Delta P f \rangle_{P_{Q}} \langle T1_{S \setminus P_{Q}}, \Delta Q g \rangle
\]
This is similar to the first subcase, since $1_{S \setminus P_{Q}}$ is supported in $(2Q)^c$, then the off-diagonal estimates also imply
\[
|\langle T1_{S \setminus P_{Q}}, \Delta Q g \rangle| \lesssim P_{Q}(1_{S \setminus P_{Q}})(Q)\|\Delta Q g\|_{1} \lesssim \left(\frac{Q}{TP}\right)^{\gamma'}\|\Delta Q g\|_{1}.
\]
Holding the relative lengths of $Q$ and $P$ fixed, we then have for integers $v \geq r$,
\[
\sum_{S} \sum_{Q: Q' = S} \sum_{P : \Pi_{Q} \subseteq S} |\langle T(\Delta P f 1_{S \setminus P_{Q}}), \Delta Q g \rangle| \lesssim 2^{-\gamma v} B^{0,v}(f, g).
\]
We use the notation (4.1), and Lemma 4.5 to complete this case.

**Fourth Subcase.** We address the bottom alternative in (3.8), namely the case in which $S = Q^c \subseteq P_{Q}$. The point here is to gain geometric decay in the degree to which $Q$ and $P_{Q}$ are separated in the stopping tree $S$.

Given $S \in \mathcal{S}$, let $S = S^{(0)} \subseteq S^{(1)} \subseteq \cdots \subseteq P_{0}$ be the maximal chain of stopping cubes which contain $S$, and continue up to $P_{0}$. For each $S^{(i)} \in \mathcal{S}$, and integer $t \geq 1$, we bound
\[
(3.11) \quad |\sum_{S : S^{(i)} = S_{0}} \sum_{P : S^{(i-1)} \in P_{Q} \subseteq S_{0}} \langle \Delta P f \rangle_{P_{Q}} \langle T1_{P_{Q} \setminus S^{(i-1)}}, \Pi_S g \rangle| \lesssim 2^{-ct} \|\langle f \rangle_{S_{0}}\|_{S_{0}}\|g\|_{S_{0}}\|S_{0}|.
\]
The point is to use the off-diagonal estimates, but there is a complication in that the stopping cubes are not good. To address this, we let $Q(S)$ be the maximal good cubes with $Q^c = S$, and set
\[
\Pi_{Q^c} g = \sum_{Q : Q^c = S, Q \subseteq Q^c} \Delta Q g, \quad Q^c \in Q(S).
\]
The goodness of the cubes implies that $\text{dist}(Q^c, \partial S^{(i-1)}) \geq (\ell Q^c)^{c} (\ell S^{(i-1)})^{1-\epsilon} \geq 2^{i/2} \ell Q^c$, by (3.3). The second point is that we have
\[
\left\|\sum_{P : S^{(i-1)} \in P_{Q} \subseteq S_{0}} \langle \Delta P f \rangle_{P_{Q}} 1_{P_{Q} \setminus S^{(i-1)}}\right\|_{\infty} \lesssim \|\langle f \rangle s\|.
\]
Combining these last two observations with (4.12), we see that for each \( Q' \in \mathcal{Q}(S) \),
\[
\left| \sum_{S' : S' \in S_0} \sum_{P' : S' \in P \subset S_0} \langle \Delta_P f, \Delta_Q g \rangle \right| \lesssim 2^{-t/2} \langle |f| \rangle_{S_0} \| \tilde{\Pi}_{Q'} g \|_1 \\
\lesssim 2^{-t/2} \langle |f| \rangle_{S_0} \langle |g| \rangle_s |Q'|
\]
Here we have used the stopping condition to dominate \( \tilde{\Pi}_{Q'} g \). To conclude, we simply observe that
\[
\sum_{S : S' \in S_0} \langle |g| \rangle_{S} \sum_{Q' \in \mathcal{Q}(S)} |Q'| \lesssim \sum_{S : S' \in S_0} \langle |g| \rangle_{S} \lesssim \langle |g| \rangle_{S_0} |S_0|.
\]
Our proof of (3.11) is complete.

3.3. The Near Terms. We address the term in (3.5). Fix an integer \( v \geq r \), and consider \( Q \subset 3P \setminus P \) with \( 2^v \ell Q = \ell P \). The cube \( Q \) is good, so that by (3.3) and (4.10), we have
\[
\| \langle T \Delta_P f, \Delta_Q g \rangle \| \lesssim 2^{-v} \| \Delta_P f \|_P \| \Delta_Q g \|_1.
\]
But, then, we have
\[
\| (3.5) \| \lesssim 2^{-v} B^{0,v}(f, g),
\]
where the latter bilinear form is defined in (4.1). It follows from (4.1) that the near term is dominated by a sparse bilinear form.

3.4. The Neighbors. We bound the term in (3.7). For \( P \), let \( P', P'' \) be choices children of \( P \). There are at most \( O(1) \) such choices. For integers \( 0 \leq v \leq r \), we bound
\[
\sum_{P' : \ell P' \leq \ell P < 2^v \ell P, \ Q \cap 3P \neq \emptyset} \langle T (\Delta_P f \cdot 1_{P'}), 1_{P''} \Delta_Q g \rangle.
\]
The case of \( P' \neq P'' \) is straight forward. The function \( \Delta_P f \cdot 1_{P'} \) is constant, so that the Hardy inequality immediately implies that
\[
\| \langle T (\Delta_P f \cdot 1_{P'}), 1_{P''} \Delta_Q g \rangle \| \lesssim \| \langle \Delta_P f \rangle_P^1 \|_{P'}^{1/2} \| 1_{P''} \Delta_Q g \|_2 \\
\lesssim \| \langle \Delta_P f \rangle_P \| \cdot \| \Delta_Q g \|_1.
\]
And this can be summed to the bound we want. Namely, it is dominated by \( B^{0,v}(f, g) \), where the last term is defined in (4.1).

The case of \( P' = P'' \) reduces to the testing inequality, and we have the same bound as above.

3.5. The Far Term. We address the terms in (3.6). For integers \( u, v \geq 1 \), we impose additional restrictions on \( P \) and \( Q \), and obtain a sparse bound with geometric decay in these parameters. From this, the required bound follows. Namely, we have for
\[
\ell P = \ell P', \ P' \subset 3^{u-1} P, \ 2^v \ell Q = \ell P, \ Q \subset 3^{u+1} P \setminus 3^n P,
\]
we have from (4.10) the estimate below.
\[
\| \langle T \Delta_P f, \Delta_Q g \rangle \| \lesssim 2^{-v(u+v)} \| \Delta_P f \|_{P'} \| \Delta_Q g \|_1.
\]
Therefore, appealing to the definition in (4.1)
\[
\sum_{P' : P' \cap 3P \neq \emptyset} \langle T (\Delta_P f \cdot 1_{P'}), 1_{P''} \Delta_Q g \rangle \lesssim 2^{-v(u+v)} B^{u,v}(f, g).
\]
By Lemma 4.5, this case is complete.

4. Lemmas

We collect three separate groups of Lemma, (a) the sparse domination of a class of dyadic forms; (b) standard off-diagonal estimates; and (c) a Hardy inequality.

Sparse Domination. We define a class of (sub) bilinear forms that are basic to the proof. For a cube $P$, let $i_P = \log_2(tP)$. Let $D_k f = \sum_{P : i_P = 2^k} \Delta_P f$, and define

$$B_{u,v}^n(f, g) = \sum_P \langle |D_{i_P - u} f| \rangle_{3P} \langle |D_{i_P - v} g| \rangle_{3P} |P|$$

Above, $u, v \geq 0$ are fixed integers, so that we are taking the martingale differences that are somewhat smaller, over the triple of $P$. We comment that this is a dyadic operator of complexity $u + v$, in the language of [9].

We remark that a standard argument would write

$$B_{u,v}^n(f, g) = \int \sum_P \langle |D_{i_P - u} f| \rangle_{3P} \langle |D_{i_P - v} g| \rangle_{3P} 1_P(x) \, dx$$

It is clear that we would dominate this last integral by a product of square functions

$$\int (S_{u} f \cdot S_{v} g) \, dx,$$

with the square functions defined by

$$(S_{u} f)^2 = \sum_P \langle |D_{i_P - u} f| \rangle_{3P}^2 1_P.$$

The deepest fact needed in our proof of the $T1$ theorem is this: The square functions $S_{u}$ are weakly bounded, with constant linear in $u$.

**Lemma 4.4.** We have the inequality below, valid for all integers $u \geq 0$

$$\|S_{u} f : L^1 \to L^{1,\infty}\| \lesssim (1 + u).$$

**Proof.** The square function $S_{u}$ is bounded on $L^2$, with constant independent of $u$, by the orthogonality of martingale differences. To prove the weak-type inequality, we take $f \in L^1$, and apply the Calderón-Zygmund decomposition at height 1. Thus, $f = g + b$, where $\|g\|_2 \lesssim \|f\|_{1}^{1/2}$, and we have

$$b = \sum_{B \in \mathcal{B}} b_B,$$

where $\mathcal{B}$ consists of disjoint dyadic cubes with $\sum_{B \in \mathcal{B}} |B| \lesssim \|f\|_1$, and $b_B$ is supported on $B$, has integral zero, and $\|b_B\|_1 \lesssim |B|$.

We do not estimate $S_{u} f$ on the set $E = \bigcup_{B \in \mathcal{B}} 3B$. And estimate

$$|\{x \notin E : S_{u} f(x) > 2\}| \leq |\{S_u g > 1\}| + |\{x \notin E : S_u b(x) > 1\}|$$

The first term is controlled by the $L^2$ bound and the fact that $\|g\|_2^2 \leq \|f\|_1$. 


Concerning the function \( b \), observe that for \( P \not\subset E \), that we have \( \langle |D_{i_P-u}f| \rangle_{3P} \neq 0 \) only if there is some \( B \in \mathcal{B} \) with \( B \subset 3P \), and \( 2^u \ell B \geq \ell P \). For a fixed \( B \), there are only 3\(^d(1+u) \) such choices of \( P \). Therefore, we will estimate

\[
|\{ x \notin E : S_\nu b(x) > 1 \}| \lesssim \sum_{P : P \not\subset E} \int_{P} |\Delta_b| dx
\]

\[
\lesssim \sum_{v=1}^{u} \sum_{P : P \not\subset E B \in \mathcal{B} : B \subset P} \int_{P} |\Delta_b_B| dx
\]

\[
\lesssim \sum_{v=1}^{u} \sum_{P : P \not\subset E B \in \mathcal{B} : B \subset P} \sum_{2^u \ell B = \ell P} |B| \lesssim u \sum_{B \in \mathcal{B}} |B| \lesssim u \| f \|_1.
\]

Our proof is complete. \( \square \)

The previous estimate is the principal tool in this sparse bound, which we use repeatedly in our proof of the sparse result.

**Lemma 4.5.** For all \( u, v \geq 0 \), all bounded compactly supported functions \( f, g \), there is a sparse collection \( S \) so that

\[
B^{u,v}(f,g) \lesssim (1 + u)(1 + v)\Lambda(f,g).
\]

It is an easy corollary from the conclusion above for \( u, v = 0 \) that martingale transforms satisfy a sparse bound. And, we also comment that the linear dependence of the constant above presents no difficulty in application, as we will always have a term that decreases geometrically in \( u + v \).

**Proof.** Note that from the equality for \( B^{u,v} \) in (4.2), we have

\[
B^{u,v}(f,g) \lesssim \int S_u f \cdot S_v g \ dx
\]

with the square functions defined by (4.3). But, we localize this familiar argument. Define

\[
(S_{u,P_0} f)^2 = \sum_{P : P \subset P_0} \langle |D_{i_P-u}f| \rangle_{3P}^2 \mathbf{1}_P,
\]

we have for an absolute constant \( C \), and all choices of \( u \geq 0 \),

\[
(4.6) \quad |\{ x \in 3P_0 : S_{u,P_0} f > C(1 + u)\langle |f| \rangle_{3P_0} \}| \leq \frac{1}{8} |P_0|.
\]

Moreover, the set on the left is contained in \( P_0 \).

We construct the sparse bound this way. Fix a large (non-dyadic) cube \( P_0 \) that \( \frac{1}{3}P_0 \) contains the support of both \( f \) and \( g \). The sparse cubes outside of \( P_0 \) can be taken to \( 3^k P_0 \), for \( k \in \mathbb{N} \). We need to construct the sparse collection inside of \( P_0 \). Consider the restricted sum

\[
I(P_0) := \int P : P \subset P_0 \langle |D_{i_P-u}f| \rangle_{3P} \langle |D_{i_P-v}g| \rangle_{3P} \mathbf{1}_P dx.
\]

Using (4.6), set

\[
E_0 = \{ S_{u,P_0} f > C(1 + u)\langle |f| \rangle_{3P_0} \} \cup \{ S_{v,P_0} g > C(1 + v)\langle |g| \rangle_{3P_0} \}.
\]
This set is contained in $P_0$, and has measure at most $\frac{4}{3}|P_0|$. Let $E_0$ be the maximal dyadic components of $E_0$. We have by Cauchy-Schwartz and construction,

$$ I(P_0) \leq C^2 \|f\|_{3P_0}\|g\|_{3P_0}|P_0| + \sum_{Q \in E_0} I(Q). $$

The first term on the right is the first term in our sparse bound. We recurse on the second terms. This completes the proof. □

A very general fact about sparse forms is that they admit a ‘universal domination.’

**Lemma 4.7.** Given $f,g$, there is a sparse operator $\Lambda_0$, and constant $C > 1$ so that for any other sparse operator $\Lambda$, we have $\Lambda(f,g) < C\Lambda_0(f,g)$.

**Proof.** Recall that shifted dyadic grids are a collection $G$ of at most $3^d$ dyadic grids $\mathcal{G} \in G$, so that every cube $Q \subset \mathbb{R}^d$ can be approximated by some cube in a dyadic grid $\mathcal{G} \in G$. Namely, for each cube $Q$, there is a $\mathcal{G}$ and a cube $P \in \mathcal{G}$ so that $\frac{1}{8} Q \subset P \subset Q$ and $Q \subset 6P$. See [11, Lemma 2.5] for an explicit proof.

Shifted grids permit us to construct a universal sparse operator for each grid $\mathcal{G} \in G$. We show this: For any dyadic grid $\mathcal{G}$, let $S \subset \mathcal{G}$ be such that for $S \subset \mathcal{G}$, there is a set $E_S \subset S$ so that $|E_S| > c|S|$ and $\|\sum_{S \in S} 1_{E_S}\|_\infty \leq c^{-1}$. Given non-negative $f,g$ bounded and compactly supported, we construct $\mathcal{U}_G \subset \mathcal{G}$ so that there are pairwise disjoint exceptional sets $\{E_Q : Q \in \mathcal{U}_G\}$ so that $E_Q \subset Q$ and $|E_Q| \geq \frac{1}{2}|Q|$, and moreover,

$$ \sum_{S \in \mathcal{G}} \langle f \rangle_S \langle g \rangle_S 1_S \leq 16^d e^{-2} \sum_{U \in \mathcal{U}_G} \langle f \rangle_U \langle g \rangle_U 1_U. \quad (4.8) $$

To complete the proof of the Lemma, we remark that the collection $\{\mathcal{U}_G : \mathcal{G} \in G\}$ is sparse. It dominates every sparse operator formed from some $\mathcal{G} \in G$, hence is universal for all sparse operators.

For integers $k$, let $U_k$ be the maximal cubes $Q \in \mathcal{G}$ so that $\langle f \rangle_Q \langle g \rangle_Q \geq 8^{2k}$. Then, the product is at most $8^{2k+2d/3}$. The cubes $Q \in U_k$ are pairwise disjoint, by maximality. We check that the children are small in measure. Setting $\mathcal{C}(Q) = \{P \in U_{k+1} : P \subsetneq Q\}$, we can write $\mathcal{C}(Q) = \mathcal{C}_f(Q) \cup \mathcal{C}_g(Q)$, where $P \in \mathcal{C}_f(Q)$ if $P \in \mathcal{C}(Q)$ and $\langle f \rangle_P > 4^d \langle f \rangle_Q$, and similarly for $\mathcal{C}_g(Q)$. But, then it is clear that

$$ \sum_{P \in \mathcal{C}(Q)} |P| \leq 4^{-d}|Q| \leq \frac{1}{4}|Q|. $$

We set $E_Q = Q \setminus \bigcup_{P \in \mathcal{C}(Q)} P$. This set has measure at least $\frac{1}{2}|Q|$.

Set $U_G = \bigcup_k U_k$. The sets $\{E_Q : Q \in U\}$ are pairwise disjoint. Now, given the sparse collection as above, provided $\langle f \rangle_S \langle g \rangle_S \neq 0$, each $S \in \mathcal{G}$ has a parent $S^u \in U$, namely the smallest element of $U$ that contains $S$. Then,

$$ \sum_{S \in \mathcal{G}} \langle f \rangle_S \langle g \rangle_S 1_S = \sum_{U \in U_G} \sum_{S \in S^u} \langle f \rangle_S \langle g \rangle_S 1_S \leq 16^d \sum_{U \in U_G} \langle f \rangle_U \langle g \rangle_U \sum_{S \in S^u} 1_S \leq 16^d e^{-2} \sum_{U \in U_G} \langle f \rangle_U \langle g \rangle_U 1_U. $$

This verifies (4.8), so completes the proof.
Off-Diagonal Estimates. We begin with the very common off-diagonal estimate. For \( \eta > 0 \) consider the Poisson-like operator

\[
P_\eta \Phi(Q) := \int_{\mathbb{R}^d} \frac{(\ell Q)^\eta \Phi(y)}{(\ell Q)^{d+\eta} + \text{dist}(y, Q)^{d+\eta}} dy.
\]

Lemma 4.9 (Off-diagonal estimate). Let \( g \) be a function with \( \int g \, dx = 0 \), supported on a cube \( Q \), and \( f \in L^2 \) supported on \( (2Q)^c \), then we have

\[
| (T f, g) | \lesssim P_\eta |f|(Q) \|g\|_1 \leq P_\eta |f|(Q)^{1/2} \|g\|_2.
\]

Proof: Let \( x_Q \) be the center of \( Q \), then we have

\[
| (T f, g) | = \left| \int_Q \int_{(2Q)^c} K(x, y) f(y) g(x) \, dy \, dx \right| = \left| \int_{(2Q)^c} \int_Q (K(x, y) - K(x_Q, y)) f(y) g(x) \, dx \, dy \right|
\]

\[
\leq K_T \int_{(2Q)^c} \int_Q \frac{|x - x_Q|^\eta}{|x - y|^{d+\eta}} |f(y)| g(x) \, dx \, dy \lesssim K_T P_\eta |f|(Q) \|g\|_1.
\]

And the second inequality follows from Cauchy-Schwarz. \( \square \)

Lemma 4.11. Suppose that \( Q \subset P \) and \( Q \) is good, then there is \( \eta' = \eta'(\eta, \gamma) > 0 \), such that

\[
P_\eta 1_{2^d \setminus P} (Q) \lesssim \left[ \frac{\ell Q}{\ell P} \right]^{\eta'}.
\]

Proof. Let \( \lambda = (\ell P/\ell Q)^{1-\gamma} \). By goodness of \( Q \), Lemma 2.1 implies

\[
P_\eta 1_{2^d \setminus P} (Q) = \int_{2^d \setminus P} \frac{(\ell Q)^\eta}{(\ell Q)^{d+\eta} + \text{dist}(y, Q)^{d+\eta}} dy \leq \int_{\mathbb{R}^d} \frac{(\ell Q)^\eta}{(\ell Q)^\gamma (\ell P)^{1-\gamma})^{d+\eta} + \text{dist}(y, Q)^{d+\eta}} dy \leq \left[ \frac{\ell Q}{\ell P} \right]^{\eta(1-\gamma)} P_\eta 1_{\mathbb{R}^d} (\lambda Q).
\]

So, the result follows. \( \square \)

Hardy’s Inequality. This is the version of Hardy’s inequality that we need. It can be proved from the one dimensional version. In point of fact, we only need this in the case where the function \( f \) below is constant.

Lemma 4.13. For any cube, \( P \), and \( 1 < p < \infty \), we have

\[
\int_{3P \setminus P} \int_P \frac{|f(x)g(y)|}{|x - y|^n} \, dx \, dy \lesssim \|f\|_p \|g\|_{p'}.
\]
References

[1] C. Benea, F. Bernicot, and T. Luque, Sparse bilinear forms for Bochner Riesz multipliers and applications, ArXiv e-prints (May 2016), available at http://arxiv.org/abs/1605.06401.

[2] F. Bernicot, D. Frey, and S. Petermichl, Sharp weighted norm estimates beyond Calderón-Zygmund theory, Anal. PDE 9 (2016), no. 5, 1079–1113.

[3] J. M. Conde-Alonso and G. Rey, A pointwise estimate for positive dyadic shifts and some applications, Math. Ann. 365 (2016), no. 3-4, 1111–1135. MR3521084

[4] A. Culiuc, F. Di Plinio, and Y. Ou, Domination of multilinear singular integrals by positive sparse forms, ArXiv e-prints (March 2016), available at http://arxiv.org/abs/1603.05317.

[5] A. Culiuc, F. Di Plinio, and Y. Ou, Uniform sparse domination of singular integrals via dyadic shifts, ArXiv e-prints (October 2016), available at http://arxiv.org/abs/1610.01958.

[6] G. David and J.-L. Journé, A boundedness criterion for generalized Calderón-Zygmund operators, Ann. of Math. (2) 120 (1984), no. 2, 371–397. MR763911

[7] K. Domelevo and S. Petermichl, A sharp maximal inequality for differentially subordinate martingales under a change of law, ArXiv e-prints (July 2016), available at http://arxiv.org/abs/1607.06319.

[8] T. Figiel, Singular integral operators: a martingale approach, Geometry of Banach spaces (Strobl, 1989), 1990, pp. 95–110. MR1110189

[9] T. P. Hytönen, The sharp weighted bound for general Calderón-Zygmund operators, Ann. of Math. (2) 175 (2012), no. 3, 1473–1506. MR2912709

[10] M. T. Lacey, Sparse Bounds for Oscillatory and Random Singular Integrals, ArXiv e-prints (September 2016), available at http://arxiv.org/abs/1609.06364.

[11] M. T. Lacey, E. T. Sawyer, C.-Y. Shen, and I. Uriarte-Tuero, Two-weight inequality for the Hilbert transform: a real variable characterization, I, Duke Math. J. 163 (2014), no. 15, 2795–2820. MR3285857

[12] A. K. Lerner, On pointwise estimates involving sparse operators, New York J. Math. 22 (2016), 341–349.

[13] A. K. Lerner, A simple proof of the $A_2$ conjecture, Int. Math. Res. Not. IMRN 14 (2013), 3159–3170. MR3085756

[14] F. Nazarov, S. Treil, and A. Volberg, The $Tb$-theorem on non-homogeneous spaces, Acta Math. 190 (2003), no. 2, 151–239. MR1998349

[15] C. Ortiz-Caraballo, C. Pérez, and E. Rela, Exponential decay estimates for singular integral operators, Math. Ann. 357 (2013), no. 4, 1217–1243. MR3124931

[16] E. T. Sawyer, A characterization of a two-weight norm inequality for maximal operators, Studia Math. 75 (1982), no. 1, 1–11. MR676801

[17] E. M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. MR1232192

[18] A. Volberg and P. Zorin-Kranich, Sparse domination on non-homogeneous spaces with an application to $A_p$ weights, ArXiv e-prints (June 2016), available at http://arxiv.org/abs/1606.03340.

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