CROSS RATIOS ASSOCIATED WITH MAXIMAL REPRESENTATIONS

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Abstract. We define a generalization of the classical four-point cross ratio of hyperbolic geometry on the unit circle given by invariant functions on Shilov boundaries of arbitrary bounded symmetric domains of tube type. This generalization is functorial and well-behaved under products. In fact, these two properties determine our extension uniquely. Any maximal representation of a closed surface group can be used to pull back our generalized cross ratios to functions on the circle; these pullbacks turn out to be strict cross ratios in the sense of Labourie and can be used to estimate the translation length of an element under the corresponding representation. The corresponding estimates show that maximal representations are well-displacing. This implies in particular that the action of the mapping class group on the moduli space of maximal representations into a Hermitian Lie group is proper.

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1. Introduction

Let \( \Gamma \) be the fundamental group of a closed surface \( \Sigma \) of genus \( g \geq 2 \) and \( \Gamma \hookrightarrow PU(1, 1) \) a fixed hyperbolization of \( \Sigma \), i.e. an isometric action of \( \Gamma \) on the Poincare disc \( \mathbb{D} \) with \( \Sigma = \Gamma \backslash \mathbb{D} \). Every \( \gamma \in \Gamma \) is hyperbolic, i.e. fixes a unique unit speed geodesic \( \sigma_\gamma \). We have
\[
\gamma \cdot \sigma_\gamma (t) = \sigma_\gamma (t + \tau_\mathbb{D} (\gamma)) ,
\]
where
\[
\tau_\mathbb{D} (\gamma) := \inf_{x \in \mathbb{D}} d(x, \gamma \cdot x)
\]

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denotes the \textit{translation length} of \( \gamma \). The action of \( \Gamma \) on \( D \) extends to its boundary \( S^1 \), and it turns out that the translation length of each \( \gamma \in \Gamma \) can be recovered from the boundary action: Indeed, denote by
\[
[a : b : c : d] := \frac{(a - d)(c - b)}{(c - d)(a - b)}
\]
the four point cross ratio on \( \mathbb{CP}^1 \) and abbreviate \( \gamma^{\pm} := \sigma_{\gamma}(\pm \infty) \in S^1 \). Then for any \( \xi \in S^1 \setminus \{\gamma^{\pm}\} \) we have
\[
\tau_D(\gamma) = \tau_D^\infty(\gamma) := \log[\gamma^{-} : \xi : \gamma^{+} : \gamma \cdot \xi].
\]
In particular, \( \tau_D^\infty(\gamma) \) does not depend on the choice of \( \xi \in S^1 \setminus \{\gamma^{\pm}\} \). We refer to \( \tau_D^\infty(\gamma) \) as the \textit{virtual translation length} of \( \gamma \).

Now consider the case of a representation \( \rho : \Gamma \to G \) into a semisimple Lie group (always assumed without compact factors and with finite center) with associated symmetric space \( X \). Although \( \rho(\Gamma) \) need not consist of hyperbolic elements anymore, the translation length
\[
\tau_X(\rho(\gamma)) := \inf_{x \in X} d(x, \rho(\gamma) \cdot x)
\]
can still be defined for every \( \gamma \in \Gamma \). We can thus ask the similar question, whether \( \tau_X(\rho(\gamma)) \) can be computed from the action of \( \rho(\Gamma) \) on some suitable limit set in some boundary of \( X \). In this article we deal with the case, where the associated symmetric space of \( G \) is isomorphic to a bounded symmetric domain of tube type and \( \rho : \Gamma \to G \) is a representation with maximal Toledo invariant, or \textit{maximal representation} for short. In this case we denote by \( \bar{S} \) the Shilov boundary of the bounded symmetric domain \( D \) isomorphic to \( X \). We then need to define a generalized cross ratio on \( \bar{S} \) in order to define a virtual translation length for the \( \Gamma \) action. Such generalized cross ratios have been defined by various people in various degrees of generality. In the symplectic case the construction is quite classical (see e.g. [17, Subsec. 4.2.6]). Beyond that case there does not seem to be a consensus about the definition. Our key observation is that there is actually only one real-valued extension of the classical cross ratio to higher rank Shilov boundaries, provided one insists on the right kind of functoriality. (On the contrary, there are various natural generalizations to vector valued or operator valued cross ratios, see e.g. [19], [4] and [3]). This is the content of the following theorem:

\textbf{Theorem 1.1.} For each bounded symmetric domain \( D \) of tube type with Shilov boundary \( \bar{S} \) there exists a subset \( \bar{S}^{(4+)} \) of \( \bar{S}^4 \) (defined in Definition 4.1 below) and a function \( B_\bar{S} : \bar{S}^{(4+)} \to \mathbb{R}^\times \) called the \textit{generalized cross ratio} of \( D \), such that the family of functions \( \{B_\bar{S}\} \) is characterized uniquely by the following properties:

(i) \( B_\bar{S} \) is invariant under the group of orientation-preserving biholomorphic automorphisms of \( D \).

(ii) If \( f : D_1 \to D_2 \) is an affine holomorphic map of symmetric spaces which is induced from a balanced Jordan algebra homomorphism (see Definition 3.11 below), then the corresponding generalized cross ratios \( B_{\bar{S}_1}, B_{\bar{S}_2} \) satisfy
\[
B_{\bar{S}_2}(\tilde{f}(v_1), \ldots, \tilde{f}(v_4)) = B_{\bar{S}_1}(v_1, \ldots, v_4),
\]
where \( (v_1, \ldots, v_4) \in \bar{S}_1^{(4+)} \) and \( \tilde{f} \) is the boundary extension of \( f \).
(iii) If $D = D_1 \times D_2$ is a direct product of symmetric spaces of ranks $r_1, r_2$ with projections $p_j : D \to D_j$ then

$$B_S(v_1, \ldots, v_4)^{r_1+r_2} = B_{S_1}(p_1(v_1), \ldots, p_1(v_4))^{r_1} B_{S_2}(p_2(v_1), \ldots, p_2(v_4))^{r_2}. $$

(iv) $B_{S_1}$ is the restriction of the classical four point cross ratio.

(Theorem 1.1 will be proved in Subsection 4.5 below.)

Generalizing the classical cross ratio we will define

$$B_S(a, b, c, d) := \frac{k(d, a)k(b, c)}{k(d, c)k(b, a)}$$

for a kernel function $k : \tilde{S} \times \tilde{S} \to \mathbb{C}$ (see Section 3).

Returning to our maximal representation $\rho : \Gamma \to G$, the limit set of $\Gamma$ in $\tilde{S}$ is homeomorphic to a circle and can be parameterized by a continuous limit curve $\varphi : S^1 \to \tilde{S}$. For $\gamma \in \Gamma$ we may thus define the virtual translation length of $\rho(\gamma)$ to be

$$\tau^\infty_D(\rho(\gamma)) := \log B_S(\varphi(\gamma^-), \varphi(\xi), \varphi(\gamma^+), \rho(\gamma), \varphi(\xi)),$$

where $\xi \in S^1 \setminus \{\gamma \pm \}$ is arbitrary. Again, this does not depend on the choice of the basepoint $\xi$. It is not quite true that the translation length and the virtual translation length of $\rho(\gamma)$ always coincide. However, at least the virtual translation length can be used to bound the actual translation length from below. More precisely, if $\text{rk} D$ denotes the rank of the bounded symmetric domain $D$ and $\dim_C D$ its complex dimension, then we obtain for all $\gamma \in \Gamma$ the estimate

$$\tau_D(\rho(\gamma)) \geq \frac{\text{rk} D}{\dim_C D} \cdot \tau^\infty_D(\rho(\gamma)).$$

In order to apply this fact we observe that the pullback of a generalized cross ratios $B_S : \tilde{S}^{(4+)} \to \mathbb{R}^\times$ to $S^1$ under a limit curve of a maximal representation happens to be a strict cross ratio in the sense of Labourie [17]. As a consequence, any two virtual translation length functions are at bounded distance. This together with the estimate then implies the following results:

**Theorem 1.2.** Let $\rho : \Gamma \to G$ be a maximal representation.

(i) For any finite generating set $S$ of $\Gamma$ there exist $A, B > 0$ such that

$$\tau_D(\rho(\gamma)) \geq A \cdot l_S(\gamma) - B$$

for all $\gamma \in \Gamma$.

(ii) For every $x \in D$ and every finite generating set $S$ of $\Gamma$ the map

$$(\Gamma, d_S) \to (D, d_D), \quad \gamma \mapsto \gamma.x$$

is a quasi-isometric embedding.

(Theorem 1.2 will be proved in Theorem 6.20 and Theorem 6.21 below.)

In the language of [13] the first part of the theorem says that maximal representations are well-displacing. This has the following well-known consequence:
Corollary 1.3. Let $G$ be a semisimple Lie group without compact factors and with finite center and assume that the symmetric space of $G$ is a bounded symmetric space of tube type. Let $\Gamma$ be the fundamental group of a closed surface $\Sigma_g$ of genus $g \geq 2$ and denote by $\text{Mod}_g$ the corresponding mapping class group. Finally, write $\mathcal{M}_{\text{max}}(\Gamma, G)$ for the space of maximal representations of $\Gamma$ into $G$ up to conjugation. Then the action of $\text{Mod}_g$ on $\mathcal{M}_{\text{max}}(\Gamma, G)$ is proper.

(Corollary 1.3 will be proved in Subsection 6.6 below.)

For symplectic, respectively, classical simple groups Corollary 1.3 was proved by Labourie [17] and Wienhard [20]. Since $\mathcal{M}_{\text{max}}(\Gamma, \text{PSL}_2(\mathbb{R}))$ is canonically identified with the Teichmüller space of $\Sigma$, we can think of the spaces $\mathcal{M}_{\text{max}}(\Gamma, G)$ as higher Teichmüller spaces. The quotients $\text{Mod}_g \backslash \mathcal{M}_{\text{max}}(\Gamma, G)$ should then be considered as higher analogs of the moduli space of hyperbolic structures on $\Sigma$. Our results imply that these spaces are orbifolds.

Let us briefly summarize the structure of this article; for a more detailed overview over its content see also the introductions to the individual sections. The generalized cross ratios, whose existence is claimed in Theorem 1.1, are constructed in Section 4, where it is also proved that these functions have the desired properties. The preceding two sections contain preparatory material, which is needed for the construction of these cross ratios. We suggest to the hasty reader to start from Section 4 and to refer backwards as necessary. In order to support this approach let us briefly summarize the content of the preparatory sections:

Section 2 contains mostly well-known material, albeit sometimes in a somewhat reorganized form. In the first subsection we relate bounded symmetric domains to Lie groups on the one hand and Jordan algebras on the other hand; in particular, we introduce notations for the various groups and spaces occurring in the context of bounded symmetric domains. We also provide a short dictionary on how to translate between the different pictures. The second subsection has the more specific purpose to relate maximal polydiscs to Jordan frames; the main result here is Proposition 2.10. The final subsection is devoted to orbits of pairwise transverse triples and quadruples. The Maslov index and the corresponding notion of maximality are introduced; these are important tools throughout. The most important result in this subsection is Proposition 2.15.

The purpose of Section 3 is to establish the existence of suitable kernel functions on boundaries of bounded symmetric domains of tube types. These kernel functions are a crucial ingredient in our construction of the generalized cross ratios and their construction is probably one of the most technical parts in the whole article. Their main properties are summarized in Theorem 3.12 and Corollary 3.15. The reader who is willing to take these results on faith can skip the whole section on first reading. The Section is divided into four subsections, the first of which explains the relation between kernel functions and transversality. Based on this, the normalized kernel functions are then constructed in two steps, first for simple Jordan algebras and then in the general case. Finally, we compare our normalized kernel to the
The final two sections provide applications of the results of Section 4. In the first subsection of Section 5, it is explained how our generalized cross ratios can be used in order to associate a strict cross ratios in the sense of [17] to any maximal representation. The second subsection is devoted to the proof of the crucial Proposition 5.8 which allows one to compare to arbitrary strict cross ratios. This result is implicitly contained in [17], but unfortunately not stated in the present generality. For the convenience of the reader we included a full proof. Section 6 is finally devoted to the proof of Inequality (1) and its consequences. We first deal with a simple special case, which demonstrates the power of our functorial approach. The next two subsections establish our main inequality by first estimating the translation length from below and then estimating the virtual translation length from above. The final three subsections establish Theorem 1.2 and Corollary 1.3. In fact, both follow easily once (1) is established.

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2. The algebraic structure of bounded symmetric domains of tube type

Bounded symmetric domains can be studied using geometric, Lie theoretic and Jordan algebraic methods. While our point of view in this article is mostly Jordan algebraic, we still need to be able to translate between the various approaches and in particular to express geometric notions in algebraic terms. The purpose of this section is to provide all the basic definitions and notations related to either of the three approaches and to sketch the relations between them. We will need this relations for example for the proof of Theorem 1.1 (ii). With the exception of Proposition 2.15 we do not present any new results here, but only collect and reformulate various foundational material for ease of later reference. We thus recommend readers familiar with the structure theory of bounded symmetric domains to skip this section and to return to it when required.

In the first subsection we recall the equivalence of categories between Euclidean Jordan algebras and marked bounded symmetric domains. The main reference for this is [2]. We use this opportunity to fix our notations. The main result of this subsection is Proposition 2.2 which provides a linear action for the Levi factor of a certain maximal parabolic. The second subsection relates the geometric notion of a maximal polydisc to the algebraic notion of a Jordan frame. This correspondence can be deduced from the results in [15] quite easily; lacking a suitable reference, we decided to include the proofs to keep the exposition self-contained. In the last subsection we recall the classification of pairwise transverse triples in the Shilov boundary of a bounded symmetric domain of tube type due to Clerc and Ørsted [13]. Besides being important in their own right, these results finally allows us to prove Proposition 2.15 which provides sufficient conditions for quadruples to be
2.1. Algebraic categories related to bounded symmetric domains. A bounded symmetric domain (or bsd for short) is a connected bounded open subset $D$ of a complex vector space together with a family of biholomorphic involutions $\{\sigma_x : D \to D \mid x \in D\}$ such that $x$ is an isolated fix point of $\sigma_x$. We denote by $G := G(D)_0$ denotes the group of orientation preserving biholomorphic automorphisms of $D$. It is well-known that $G$ is transitive on $D$, in particular any bsd is homogeneous. A tube in $\mathbb{C}^n$ is a subset of the form $\mathbb{R}^n + i\Omega$, where $\Omega$ is an open convex cone in $\mathbb{R}^n$ and a bsd is of tube type if it is biholomorphic to a tube. A pointed bounded symmetric domain is a triple $(D, \{\sigma_x\}, o)$, where $(D, \{\sigma_x\})$ is a bsd and $o \in D$ is a fixed basepoint. The topological boundary $D \setminus D$ of a bounded symmetric domain contains a unique closed $G$-orbit $\hat{S}$ referred to as the Shilov-boundary of $D$. A marked bounded symmetric domain is a quadruple $(D, \{\sigma_x\}, o, \xi)$, where $(D, \{\sigma_x\}, o)$ is a pointed bsd and $\xi \in \hat{S}$ is a point in the Shilov boundary. Together with the obvious morphisms, i.e. holomorphic maps preserving the pointwise involutions and fixing both base points, marked bsds form a category, which we denote by $mBSD$. The full subcategory of marked bsds of tube type is denoted $mBSD_T$. The purpose of this subsection is to exhibit algebraic categories which are equivalent to $mBSD_T$. This will allow us to reduce geometric statements on bounded symmetric domains to algebraic problems.

Let $G$ be a connected semisimple Lie group with universal cover $\tilde{G}$, Lie algebra $\mathfrak{g}$ and Killing form $\kappa$. An involution $\sigma : \tilde{G} \to \tilde{G}$ is called a Cartan involution if $-\kappa(\cdot, d\sigma(\cdot))$ is positive-definite. In this case $(\tilde{G}, \sigma)$ and $(\mathfrak{g}, d\sigma)$ are called globally symmetric and infinitesimally symmetric pair respectively. Every globally symmetric pair defines a homogeneous space $X = \tilde{G}/\tilde{G}_\sigma$, which can be equipped with the structure of a Riemannian symmetric space. If there exists a $\tilde{G}$-invariant complex structure $J$ on $X$, then the triple $(\tilde{G}, \sigma, J)$ is called a global Hermitian symmetric triple. In this case, a theorem of Harish-Chandra ensures that $(X, J)$ can be realized as a bsd $D$; the triple $(\tilde{G}, \sigma, J)$ is called of tube type, if $D$ is. The action of $\tilde{G}$ extends to the Shilov boundary $\hat{S}$ of $D$. This action is in fact transitive and the point stabilizers form a conjugacy class of maximal parabolic subgroups of $\tilde{G}$. A maximal parabolic in this conjugacy class will be referred to as a Shilov parabolic for $\tilde{G}$. A marked global Hermitian symmetric triple is a quadruple $(\tilde{G}, \sigma, J, Q_+)$ consisting of a global Hermitian symmetric triple together with a Shilov parabolic. Morphisms of such triples are smooth group homomorphisms preserving the additional structure involved. We denote by $mHST$ and $mHST_T$ the categories of all marked global Hermitian symmetric triples respectively those of tube types. Every marked global Hermitian symmetric triple defines a marked infinitesimal Hermitian symmetric triple $(\mathfrak{g}, d\sigma, J, q_+)$, where $J = J_{\mathfrak{g}^*}^{\sigma}$ and $q_+$ is the Lie algebra of $Q_+$. We denote by $mhst_T$ the category of all marked infinitesimal Hermitian symmetric triples of tube type.
Let $V$ be a finite-dimensional (real or complex) vector space, equipped with a (real- or complex-) bilinear mapping

$$m : V \times V \to V, \quad (x, y) \mapsto xy.$$  

We call $V$ (or rather the pair $(V, m)$) a (real or complex) Jordan algebra if the following three axioms are satisfied:

(i) $m$ is commutative, i.e. $xy = yx$  
(ii) There exists a unit element $e \in V$ with $ex = xe = x$ for all $x \in V$.  
(iii) For all $x, y \in V$ we have $x((xy)y) = (xx)(xy)$.

A real Jordan algebra is called Euclidean if there exists a Euclidean inner product $(\cdot, \cdot)$ on $V$ such that left multiplication by elements of $V$ defines a self-adjoint operator, i.e.

$$\forall x, y, z \in V : \quad (xy, z) = (y, xz).$$

In fact, the inner product can always be chosen to be $(x, y) = \text{tr}(xy)$, where tr is the Jordan algebra trace in the sense of [15, p. 29]. Moreover [15, Prop. VIII.4.2], V is Euclidean iff it is formally real, i.e.

$$\forall x, y \in V : \quad x^2 + y^2 = 0 \Rightarrow x = y = 0.$$ 

A morphism of Jordan algebras is a (real or complex) linear map preserving both the Jordan product and the unit element. We denote by \( \mathfrak{Jor}_+ \) the category of Euclidean Jordan algebras. Then we have:

**Proposition 2.1.** There are equivalences of categories

$$\mathfrak{mBSD}_T \cong \mathfrak{mJST}_T \cong \mathfrak{mhst}_T \cong \mathfrak{Jor}_+.$$ 

This follows immediately from the results in [2]. Since we need the explicit correspondences in the sequel, let us briefly recall them: Let us start from a Euclidean Jordan algebra $V$ and denote by $V^\mathbb{C} := V \otimes \mathbb{C}$ its complexification. The interior of the set \( \{ x^2 \mid x \in V \} \) in $V$ is an open cone $\Omega$ and thus $T_\Omega := V + i\Omega \subset V^\mathbb{C}$ is a tube. We refer to $\Omega$ as the symmetric cone associated with $V$. Put

$$D(c) := \{ w \in V^\mathbb{C} \mid e - w \text{ invertible} \}, \quad D(p) := \{ z \in V^\mathbb{C} \mid z + ie \text{ invertible} \}.$$ 

Then the Cayley transform

$$c : D(c) \to D(p), \quad c(w) = i(e + w)(e - w)^{-1}$$

is biholomorphic with inverse

$$p : D(p) \to D(c), \quad p(z) = (z - ie)(z + ie)^{-1}.$$ 

By [15, Theorem X.1.1], we have $T_\Omega \subset D(p)$ and hence we obtain a bounded domain $D_V := p(T_\Omega)$ of tube type. This bounded domain is in fact symmetric; since we do not need the symmetric structure \( \{ \sigma_x \} \) explicitly, we refer the reader to [2] Thm. X.3.2] for a proof of this fact. A marking of $D_V$ is obtained by $0 \in D_V$ and $e \in \hat{S}$. Since $0$ and $e$ are preserved by Jordan algebra homomorphisms, the assignment

$$\mathfrak{Jor}_+ \to \mathfrak{mBSD}_T, \quad V \mapsto (D_V, \{ \sigma_x \}, 0, e)$$

is functorial. Now, given a marked bsd $(D, \{ \sigma_x \}, \alpha, \xi)$ we define $G = G(D)_0$, $K := \text{Stab}_G(G)$, $Q_+ := \text{Stab}_G(G)$ and denote by $g, \mathfrak{g}, q_+$ the respective Lie algebras. Denote by $p$ a Killing-orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Then the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus p$ defines a Cartan involution $d\sigma$ with $\pm 1$-eigenspaces $\mathfrak{k}, p$.  

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**Cross Ratios**

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**Equivalence of Categories**

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**Functorial Assignment**

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**Cartan Decomposition**

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Denoting by $J$ the restriction of the complex structure of $D$ to $p = T_oD$ we obtain an assignment
\[ \text{mBSD} \to \text{mhst}, \quad (D, \{\sigma_x\}, \alpha, \xi) \to (g, d\sigma, J, q_+), \]
which clearly preserves tube type. For a proof that this assignment is functorial and defines in fact an equivalence of categories one uses [2, Thm. V.1.9]. On the other hand, the equivalence of categories $\text{mHST}_T \cong \text{mhst}_T$ is given simply by integration. It thus remains to prove the equivalence of categories $\text{mhst}_T \equiv \text{Jor}_+$. Given any infinitesimal Hermitian symmetric triple $(g, d\sigma, J)$ (not necessarily of tube type) we define a Lie triple product $[\cdot, \cdot, \cdot] : p \to p$ by
\[ [X, Y, Z] := [[X, Y], Z] \quad (X, Y, Z \in p). \]
The invariance of the complex structure $J$ then yields
\[ [JX, Y, Z] = -[X, JY, Z], \]
i.e. the triple $(p, [\cdot, \cdot, \cdot], J)$ is a twisted complex Lie triple system in the sense of [2, Def. III.2.1]. Its associated Hermitian Jordan triple system $(p, \{\cdot, \cdot, \cdot\}, J)$ is given by
\[ \{X, Y, Z\} := \frac{1}{2}([X, Y, Z] - [JX, Y, JZ]). \]
Since the Lie triple product can be reconstructed from the Jordan triple product by the formula
\[ [X, Y, Z] = \{X, Y, Z\} - \{Z, X, Y\}, \]
the categories of twisted complex Lie triple system and Hermitian Jordan triple products (morphisms being complex linear maps preserving the respective triple product) are equivalent. (This is a special case of Bertram's Jordan-Lie functor, see [2, Prop. III.2.7]). Moreover, by [2, Thm. V.1.9] both categories are equivalent to the category $\text{hst}$ of unmarked Hermitian symmetric triples. Now assume that the Jordan triple system $(p, \{\cdot, \cdot, \cdot\}, J)$ arises from a marked Hermitian symmetric triple system of tube type. Then there exists a unitary tripotent, i.e. an element $e \in p$ such that $\{e, e, v\} = v$ for all $v \in p$. Any such tripotent define a Jordan algebra structure on $p$ via
\[ (2) \quad v \cdot w := \{v, e, w\}. \]
Now the set of unitary tripotents in $p$ can be canonically identified with the Shilov boundary $\tilde{S}$ considered before. The marking therefore determines a unique unitary tripotent $e$ by demanding $q_+ = \text{Stab}_e(g)$. This choice of unitary tripotent then determines a specific Jordan algebra structure on $p$ via (2). It remains to find the correct Euclidean real form $V$ of the complex Jordan algebra $p$. For this we define an involution $z \mapsto \bar{z}$ of $p$ by
\[ \bar{z} = \{e, z, e\}. \]
Then $V := \{z \in p \mid z = \bar{z}\}$ is the desired real form and the assignment
\[ \text{mhst} \to \text{Jor}_+, \quad (g, d\sigma, J, q_+) \to V \]
provides the last functor in the circle
\[ \text{Jor}_+ \to \text{mBSD}_T \to \text{mHST}_T \to \text{mhst}_T \to \text{Jor}_+. \]
Given a Euclidean Jordan algebra $V$ we will denote by $D_V$ the associated bounded symmetric domain. We also abbreviate $G := G(D_V)_0$ and $Q_+ = \text{Stab}_e(G)$ as above. The explicit form of the equivalence $\text{mHST}_T \cong \text{Jor}_+$ exhibits a close relation
between the Shilov parabolic $Q_+$ and the symmetric cone $\Omega$ associated with $V$.

Write

$$G(\Omega) := \{ g \in GL(V) \mid g\Omega \subset \Omega \}$$

for the group of linear transformations preserving $\Omega$ and denote by

$$L := G(\Omega)_0$$

its identity component. (We warn the reader that in the literature on Hermitian Jordan algebras including [15] this group is often denoted $G$, which conflicts with our notation for $G(\mathcal{D})_0$.) $L$ is reductive with one-dimensional center and the maximal compact subgroup of $L$ is

$$M := \text{Stab}_L(e).$$

Now we have:

**Proposition 2.2.** Denote by $L(Q_+)$ the Levi factor of $Q_+$. Then the Cayley transform $c$ defines an isomorphism

$$L(Q_+)_0 \cong L, \quad h \mapsto c \circ h \circ p.$$ 

**Proof.** We denote by $Q_-$ the stabilizer of $-e$ in $G$. Then $Q_{\pm}$ are opposite maximal parabolics and thus $L(Q_+) = Q_+ \cap Q_-$. Now let $g \in L$ and consider $h := p \circ g \circ c \in G$. Since $g$ is linear, it fixes 0, whence $h$ fixes $p(0) = -e$. Moreover, since $g$ acts linearly we obtain

$$h.e = \lim_{\epsilon \to 0} h.((1 - \epsilon)e)$$

$$= \lim_{\epsilon \to 0} (p \circ g)((2 - \epsilon)ie)(ee)^{-1} = \lim_{\epsilon \to 0} (p \circ g)\left(\frac{2 - \epsilon}{e} \cdot ie\right)$$

$$= \lim_{\epsilon \to 0} p\left(\frac{2 - \epsilon}{e} \cdot ig.e\right) = \lim_{\epsilon \to 0} \left(g.e - \frac{\epsilon}{2 - \epsilon} e, e + \frac{\epsilon}{2 - \epsilon} e\right)^{-1}$$

$$= (g.e)(g.e)^{-1} = e.$$ 

Thus $h$ fixes $\pm e$ and since $g \in L$ was arbitrary we obtain the inclusion $p \circ L \circ c \subset (Q_+ \cap Q_-)_0$. It remains to show that both groups have the same real dimension. For this we observe that by [15] Thm. III.5.1 we have $M = \text{Aut}(V)_0$ and by [15] Prop. X.3.1 and Thm. X.5.3 we have $K/M = \hat{S}$, whence

$$\dim \hat{S} = \dim K - \dim M.$$ 

Since $L/M = \Omega$ is open in $V$ and $V$ is homeomorphic to an open subset of $\hat{S}$ we also have

$$\dim \hat{S} = \dim V = \dim \Omega = \dim L - \dim M,$$

whence

$$\dim L = \dim K. \quad (3)$$

On the other hand, since $G/Q_{\pm} = \hat{S}$ and $G/K = \mathcal{D}$ we have

$$\dim G - \dim K = \dim \mathcal{D} = 2 \dim V = 2 \dim \hat{S} = 2(\dim G - \dim Q_{\pm}),$$

i.e.

$$\dim Q_{\pm} = \frac{1}{2}(\dim G - \dim K).$$
Using (3) and the fact that \( Q_+ Q_- \) is open in \( G \) we obtain
\[
\dim(Q_+ \cap Q_-) = \dim Q_+ + \dim Q_- - \dim(G) = \dim K = \dim L,
\]
finishing the proof. \( \square \)

The first part of the proof of the Proposition above actually shows that
\[
c \circ L(Q_+) \circ p \supset G(\Omega).
\]
Since \( L(Q_+) \) need not be connected, the opposite inclusion is not obvious. However, since \( L(Q_+) \) is algebraic, the possible failure of this opposite inclusion can easily be corrected:

**Corollary 2.3.** There exists \( M \in \mathbb{N} \) (depending only on \( G \)) such that for all \( g \in c \circ L(Q_+) \circ p \) the power \( g^M \) is contained in \( G(\Omega) \).

*Proof.* Since \( L(Q_+) \) is algebraic and irreducible, \( \pi_0(L(Q_+)) \) is finite. Denote by \( M \) the order of this quotient. Now let \( g \in c \circ L(Q_+) \circ p \), say \( g = c \circ h \circ p \) with \( h \in L(Q_+) \). Then we have \( h^M \in L(Q_+)_0 \) and thus \( g^M = c \circ h^M \circ p \in L \) by Proposition 2.2. \( \square \)

Another application of Proposition 2.2 concerns the following result:

**Corollary 2.4.** Let \( \alpha : V_1 \to V_2 \) be a morphism of Euclidean Jordan algebras and \( \alpha^C : V_1^C \to V_2^C \) its complexification. Then the following hold:

(i) The corresponding bounded symmetric domains satisfy \( \alpha^C(D_1) \subset D_2 \).

(ii) The corresponding Shilov boundaries satisfy \( \alpha^C(S_1) \subset S_2 \).

(iii) The corresponding symmetric cones satisfy \( \alpha(\Omega_1) \subset \Omega_2 \).

*Proof.* (i) is an immediate consequence of Proposition 2.1 (ii) follows from the fact that
\[
\tilde{S} = \{ z \in V_1^C \mid z \text{ invertible, } z^{-1} = \bar{z} \}.
\]
(iii) Denote by \( \alpha^\downarrow : (g_1, \sigma_1, J_1, q_{+,1}) \to (g_2, \sigma_2, J_2, q_{+,2}) \) the image of \( \alpha \) under the equivalence of categories \( 3\mathfrak{or}_{+} \to \mathfrak{msh}_{+} \). Then \( \alpha^\downarrow \) maps the Levi factor of \( q_{+,1} \) to the Levi factor of \( q_{+,2} \). Since Jordan algebra homomorphisms preserve the Cayley transform, we derive from Proposition 2.2 that \( \alpha^\downarrow \) maps the Lie algebra of \( L_1 := G(\Omega_1)_0 \) to that of \( L_2 \). Since \( \Omega_j = L_j e \) and \( \alpha(e) = e \) the result follows. \( \square \)

We will use the notations \( G, K, L, M, Q_\pm \) throughout this article. If we want to stress the corresponding Euclidean Jordan algebra \( V \) then we will add a subscript \( -V \), e.g. writing \( G_V \) instead of \( G \). We will use the corresponding small gothic letters to denote the associated Lie algebras.

### 2.2. Idempotents, Jordan frames and maximal polydiscs.

Throughout this subsection \( V \) denotes a Euclidean Jordan algebra. A basic reference on such Jordan algebras suitable for our purposes is \([15]\). In view of the equivalence of categories provided by Proposition 2.1 all geometric properties of bsds of tube type are reflected by algebraic properties of the corresponding Euclidean Jordan algebras. Here we shall provide an algebraic description of maximal polydiscs. We start from the observation that a Jordan algebra while not associative in general, is always power-associative in the sense that the subalgebra of \( V \) generated by a single element \( x \) is associative (see \([15\text{ II.1.2}]\)). In particular, the powers \( x^n \) of an element are
well-defined. An element $x \in V^C$ with $x^2 = x$ is called idempotent. The following definition is fundamental for the whole theory of Euclidean Jordan algebras:

**Definition 2.5.** A Jordan frame of a Euclidean Jordan algebra $V$ is a family $(c_1, \ldots, c_r)$ of non-zero idempotents satisfying the following conditions:

(i) each $c_i$ is primitive, i.e. it cannot be written as the sum of two non-zero idempotents,

(ii) the idempotents are orthogonal, i.e. $c_ic_j = 0$ for $i \neq j$,

(iii) $\sum_{i=1}^r c_i = e$, where $e$ is the unit of $V$.

The cardinality $r$ of a Jordan frame is independent of the choice of Jordan frame and called the rank of the Jordan algebra $V$; it coincides with the rank of the associated bsd. The following well-known result is a version of the spectral theorem for Euclidean Jordan algebras:

**Proposition 2.6.** ([15, III.1.2]) Suppose $V$ has rank $r$. Then for $x$ in $V$ there exists a Jordan frame $(c_1, \ldots, c_r)$ and real numbers $\lambda_1, \ldots, \lambda_r$ such that

$$x = \sum_{i=1}^r \lambda_ic_i.$$ 

The following lemma will be used in Section 3 below:

**Lemma 2.7.** Let $d_1, \ldots, d_m$ be a collection of pairwise orthogonal idempotents in an Euclidean Jordan algebra $V$ with $d_1 + \cdots + d_m = e$. Then there exists a Jordan frame $c_1, \ldots, c_r$ of $V$ and numbers $i_1 < \cdots < i_m < i_{m+1} = r$ such that

$$d_j = \sum_{i_j+1}^{i_{j+1}} c_j.$$ 

**Proof.** If each $d_i$ is primitive, then we are done. Otherwise, there is at least one non-primitive idempotent among the $d_j$, say $d_1$. We can then write $d_1 = f_1 + f_2$ with $f_1, f_2$ idempotents. Then $f_1 + f_2 + d_2 + \cdots + d_m = e$. Moreover we have

$$f_1 + f_2 = d_1 = d_1^2 = (f_1 + f_2)^2 = f_1^2 + 2f_1f_2 + f_2^2 = f_1 + f_2 + 2f_1f_2,$$

whence $f_1f_2 = 0$. This implies in particular that

$$d_1f_j = (f_1 + f_2)f_j = f_j^2 = f_j,$$

whence $f_1, f_2$ are in the 1-eigenspace of $d_1$, while $d_2, \ldots, d_m$ are in the 0-eigenspace of $d_1$. We deduce from [2 Satz 1.12.3 a)] that $f_1, f_2, d_2, \ldots, d_m$ is a collection of pairwise orthogonal idempotents whose sum equals $e$. Proceeding inductively, we either end up with a Jordan frame or produce larger and larger systems of pairwise orthogonal idempotents. By finite-dimensionality of $V$, the latter is impossible. □

We now relate the notion of Jordan frame to the following geometric notion:

**Definition 2.8.** Let $D$ be a bounded symmetric domain of rank $r$. An isometrically embedded copy of $D^s$ in $D$ with $s \leq r$ is called polydisc. If $r = s$ it is called maximal. A maximal polydisc is called centered, if it contains $0 \in V^C$. 
Since the embedding is isometric, every polydisc contains a flat of dimension $r$. In fact, every polydisc is the complexification of a maximal flat. There are lots of maximal polydiscs in $\mathcal{D}$:

**Proposition 2.9.** Let $x, y \in \mathcal{D}$. Then there exists a maximal polydisc containing them.

**Proof.** There exists a geodesic joining $x$ and $y$. This geodesic is a flat and lies therefore in a maximal flat. The complexification of this maximal flat is a maximal polydisc. $\square$

Now the relation between Jordan frames and polydiscs is provided by the following proposition:

**Proposition 2.10.**

(i) If $c = (c_1, \ldots, c_r)$ is a Jordan frame, then the map

$$\varphi_c : \mathbb{D}^r \to \mathcal{D}$$

$$\left(\lambda_1, \ldots, \lambda_r\right) \mapsto \sum_{i=1}^{r} \lambda_i c_i$$

defines a centered maximal polydisc in $\mathcal{D}$.

(ii) For any polydisc $P$ and any Jordan frame $(c_1, \ldots, c_r)$ there exists $g \in G$ such that

$$g \cdot P = \left\{ \sum_{i} \lambda_i c_i | \lambda_i \in \mathbb{D} \right\}.$$

**Proof.** (i) The map which maps $(\mu_1, \ldots, \mu_r) \in \mathbb{C}^r$ to $\sum \mu_i c_i$ is an injective Jordan algebra morphism $\alpha : \mathbb{R}^r \to V$ and thus induces $\alpha_0 : \mathbb{D}^r \to \mathcal{D}$. The metric structures on both sides come from the Jordan algebra structures. Since they are equal on the image of this map, $\varphi$ is an isometry. It is clearly injective, hence it defines an isometric copy of $\mathbb{D}^r$ in $\mathcal{D}$.

(ii) Let $A \subset P$ be a maximal flat and $\sigma$ a regular geodesic in $A$. Then $A$ is the only flat containing $\sigma$. There exists $h \in G(\mathcal{D})$ such that $0 = h \cdot \sigma(0)$. The point $h \cdot \sigma(1)$ can be written in the form $u \cdot \sum \lambda_i c_i$ for some $u \in K$ (see [15 X.3.2]). Now put $g := u^{-1} h$. The geodesic $g \cdot \sigma$ is contained in the flats $g \cdot A$ and $\left\{ \sum \lambda_i c_i | \lambda_i \in (-1, 1) \right\}$. But being regular, $g \cdot \sigma$ lies in only one unique flat, hence these two flats have to be equal. Therefore their complexifications have to be equal and we obtain (ii). $\square$

Notice that the image of the Shilov boundary under the maximal polydisc embedding associated to the Jordan frame $(c_1, \ldots, c_r)$ is given by

$$\left\{ \sum_{j=1}^{r} \lambda_j c_j | |\lambda_j| = 1 \right\}.$$

**Proposition 2.11.** Let $z$ be in the Shilov boundary. Then there exists a Jordan frame $(c_1, \ldots, c_r)$ and complex numbers $\lambda_i$ with $|\lambda_i| = 1$ such that

$$z = \sum_{i=1}^{r} \lambda_i c_i.$$

**Proof.** Write $z = x + iy$ with $x, y \in V$. By [15 Proposition X.2.3] we have

$$x^2 + y^2 = e$$
and \([L(x), L(y)] = 0\). In view of [15, Lemma X.2.2] the latter yields the existence of a Jordan frame \((c_1, \ldots, c_r)\) and real numbers \(\mu_1, \ldots, \mu_r, \nu_1, \ldots, \nu_r\) such that

\[
x = \sum_{j=1}^r \mu_j c_j, \quad y = \sum_{j=1}^r \nu_j c_j
\]

and hence

\[
z = \sum_{j=1}^r \lambda_j c_j,
\]

where \(\lambda_j = \mu_j + iv_j\). Now (7) implies \(|\lambda_j|^2 = \mu_j^2 + \nu_j^2 = 1\). Hence \(z\) is contained in the Shilov boundary of the polydisc embedding \(\iota : \mathbb{D}^r \to \mathcal{D}\) associated with the Jordan frame \((c_1, \ldots, c_r)\).

\[\square\]

2.3. Orbits of transverse triples and quadruples. Let \(V\) be a Euclidean Jordan algebra, \(\mathcal{D}\) its bounded symmetric domain and \(\hat{S}\) the corresponding Shilov boundary. We use the notations \(G, K, L, M, Q_{\pm}\) for the associated groups as introduced in Subsection 2.1. Since \(Q_+\) is a maximal parabolic in \(G\), the Shilov boundary \(\hat{S} = G(\mathcal{D})_0/Q_+\) carries the structure of a generalized flag manifold. In particular, we can define transversality for points in \(\hat{S}\):

**Definition 2.12.** \(gQ_+, hQ_+ \in \hat{S}\) are transverse, denoted \(z \triangleleft w\), if \(Q_+ - 1 hQ_+\) coincides with the unique cell of maximal dimension in the Bruhat decomposition of \(\hat{S}\) with respect to \(Q_+\).

For the definition of Bruhat decomposition see e.g. [16, Thm. 7.40]. We write

\[
\hat{S}^{(n)} := \{(z_1, \ldots, z_n) \in \hat{S}^n | \forall i \neq j : z_i \triangleleft z_j\}
\]

for the set of pairwise transversal \(n\)-tuples in \(\hat{S}\). Since the \(G\)-action preserves transversality, each \(\hat{S}^{(n)}\) is a union of \(G\)-orbits. For \(n = 2\) we see from the definition that \(\hat{S}^{(2)}\) is the unique \(G\)-orbit in \(\hat{S}^2\) of maximal dimension This characterization can be used to identify \(\hat{S}^{(2)}\) in concrete examples. In the case of \(G = \text{Sp}(2n, \mathbb{R})\) the Shilov boundary is identified with the set \(\mathcal{L}(\mathbb{R}^{2n})\) of Lagrangian subspaces of \(\mathbb{R}^{2n}\). Classically, two Lagrangian subspaces \(V, W\) of \(\mathbb{R}^{2n}\) are called transverse if \(V \oplus W = \mathbb{R}^{2n}\). Clearly,

\[
\mathcal{L}(\mathbb{R}^{2n})^{(2)} = \{(V, W) \in \mathcal{L}(\mathbb{R}^{2n})^2 | V \oplus W = \mathbb{R}^{2n}\}
\]

is an open \(\text{Sp}(2n, \mathbb{R})\)-orbit, hence \(\hat{S}^{(2)} = \mathcal{L}(\mathbb{R}^{2n})^{(2)}\) in this case.

For \(n = 3\) orbits in \(\hat{S}^{(3)}\) can be classified in terms of the Maslov index \(\mu_\hat{S}\) of \(\hat{S}\) as defined in [13]. Rather than repeating the definition, we explain a simple way to compute the Maslov index based on the following result:

**Proposition 2.13** (Clerc-Neeb, [12, Theorem 3.1]). Any triple \((z_1, z_2, z_3) \in \hat{S}^3\) is contained in the Shilov boundary of an embedded maximal polydisc.

Combining Proposition 2.13 and Proposition 2.10 we find for every \((z_1, z_2, z_3) \in \hat{S}^3\) a Jordan frame \((c_1, \cdots, c_r)\) and \(g \in G(\mathcal{D})\) such that

\[
g \cdot z_i = \sum_{j=1}^r \lambda_{ij} c_j \quad (i = 1, 2, 3)
\]
for some $\lambda_{ij} \in S^1$. If $(z_1, z_2, z_3) \in \tilde{S}^{(3)}$ then all the triples $(\lambda_{1j}, \lambda_{2j}, \lambda_{3j})$ are pairwise transverse, hence contained in the domain of the orientation cocycle $o : (S^1)^{(3)} \to \{\pm 1\}$, which takes value $\pm 1$ depending on whether the given triple is positively or negatively oriented. The results of Clerc and Ørsted in [13] then imply that

$$
\mu_{\tilde{S}}(z_1, z_2, z_3) = \mu_{\tilde{S}}(g \cdot z_1, g \cdot z_2, g \cdot z_3) = \sum_{j=1}^{r} o(\lambda_{1j}, \lambda_{2j}, \lambda_{3j}).
$$

From this description we see immediately that

$$
\mu_{\tilde{S}}(\tilde{S}^{(3)}) = \{-r, -r + 2, \ldots, r - 2, r\}.
$$

A triple $(z_1, z_2, z_3) \in \tilde{S}^{(3)}$ with $\mu_{\tilde{S}}(z_1, z_2, z_3) = r$ is hence called maximal. As mentioned above, the Maslov index classifies triples of pairwise transverse points:

**Proposition 2.14** (Clerc-Ørsted, [13 Theorem 4.3]). Two triples $(z_1, z_2, z_3), (w_1, w_2, w_3) \in \tilde{S}^{(3)}$ lie in the same $G(D)_0$ orbit iff

$$
\mu_{\tilde{S}}(z_1, z_2, z_3) = \mu_{\tilde{S}}(w_1, w_2, w_3).
$$

The classification of orbits in $\tilde{S}^3$ is much more involved; see [12]. Here we are interested in generalized cross ratios, which are defined on a subset of $\tilde{S}^4$. Unfortunately, the analog of Proposition 2.13 fails in this case, i.e. not every orbit $\tilde{S}^{(4)}$ contains a representative in the boundary of a maximal polydisc. We shall use the following proposition as a substitute:

**Proposition 2.15.** Let $(z_1, \ldots, z_4) \in \tilde{S}^{(4)}$, and suppose $(z_i, z_j, z_k)$ is maximal for some $\{i, j, k\} \subset \{1, \ldots, 4\}$. Then $z_1, \ldots, z_4$ are contained in the boundary of a common maximal polydisc.

**Proof.** We may assume w.l.o.g. that $(z_1, z_2, z_3)$ is maximal. Since

$$
\mu_{\tilde{S}}(z_1, z_2, z_3) = r = \mu_{\tilde{S}}(-e, -ie, e),
$$

it follows from Proposition 2.14 that there exists $g \in G$ with $g.(z_1, z_2, z_3) = (-e, -ie, e)$. Let $z = g. z_4$. By Proposition 2.11 there exists a Jordan frame $(c_1, \ldots, c_r)$ and $\lambda_i \in \mathbb{C}$ with $|\lambda_i| = 1$ such that

$$
z = \sum_{i=1}^{r} \lambda_i c_i.
$$

We deduce that

$$
g.(z_1, \ldots, z_4) = \left(\sum (-1) \cdot c_j, \sum (-i) \cdot c_j, \sum 1 \cdot c_j, \sum \lambda_j c_j\right)
$$

is contained in the Shilov boundary of the polydisc embedding $\iota : \mathbb{D}^r \to D$ associated with the Jordan frame $(c_1, \ldots, c_r)$. Correspondingly, $(z_1, \ldots, z_4)$ is contained in the Shilov boundary of the embedded maximal polydisc $g^{-1} \circ \iota$. $\square$

### 3. Normalized kernels of Euclidean Jordan algebras

After the preliminary results in Section 2 we now turn to the construction of generalized cross ratios. In view of the above equivalence of categories, we will work with Jordan algebras rather than bounded symmetric domains. Thus our problem is to construct a generalized cross ratio $B_V$ on 4-tuples on the Shilov boundary.
Cross ratios associated with a Euclidean Jordan algebra $V$. In view of the various cocycle relations expected from a generalized cross ratio, a natural ansatz is to define

$$B_V(a,b,c,d) := \frac{k_V(d,a)k_V(b,c)}{k_V(d,c)k_V(b,a)}$$

for some function $k_V : \mathcal{S} \times \mathcal{S} \to \mathbb{C}$. In order to obtain $G$-invariance this function should have some nice equivariance properties. If $B_V$ is supposed to be functorial, then also the normalization of $k_V$ has to be chosen carefully. Since this is a delicate issue we devote a whole section to the construction of the kernels $k_V$ and their basic properties. The actual study of the corresponding cross ratios will then be undertaken in the next section. The reader willing to take the main properties of the kernels $k_V$ on faith can skip this section except for the definition of balanced Jordan algebra homomorphisms (Definition 3.11) which will be used throughout.

The first subsection introduces automorphy kernels and their relation to various notions of transversality. Based on this notion we construct in subsequent subsections the kernels $k_V$ first for simple Euclidean Jordan algebras, and then for general Euclidean Jordan algebras. Finally, we compare the our normalized kernels $k_V$ to the Bergman kernels of the corresponding bounded symmetric domains.

### 3.1. The automorphy kernel and transversality

Let $V$ be a Euclidean Jordan algebra. Given $z \in V^C$ we denote by $L(z)$ the left-multiplication by $z$. Then for all $z, w \in V^C$ the box operator and the quadratic representation are defined by

$$z \square w := L(zw) + [L(z), L(w)],$$

and

$$P(z) := 2L(z)^2 - L(z^2)$$

respectively. Following [13] (see also [14] and [18]) we define the automorphy kernel

$$K : V^C \times V^C \to \text{End}(V^C),$$

by

$$K(z, w) := I - 2z \square \overline{w} + P(z)P(\overline{w}).$$

**Example 3.1.** Let $V = \mathbb{R}$ so that $V^C = \mathbb{R}_C = \mathbb{C}$. For $x, w, z \in \mathbb{C}$ we then have

$$(z \square w)x = L(zw)x + [L(z), L(w)]x = (zw)x,$$

whence $z \square w$ is multiplication with $zw$. Similarly,

$$P(z)x = (2L(z)^2 - L(z^2))x = z^2 x,$$

hence $P(z)$ is multiplication by $z^2$. Combining these two observations we can calculate $K(z, w)$: We obtain

$$K(z, w)x = x - 2zwx + z^2w^2 = (1 - zw)^2 x$$

and thus $K(z, w)$ is multiplication by $(1 - zw)^2$.

We also use the quadratic representation to define the structure group of $V^C$ to be

$$\text{Str}(V^C) := \{g \in GL(V^C) \mid P(gx) = gP(x)g^\top\}.$$

If $z, w \in D$ then $K(z, w) \in \text{Str}(V^C)$ [13, p. 315]. Let us briefly describe the behavior of $K$ under automorphisms. Since we do not want to digress to deeply into the structure theory of bounded symmetric domain, we refer the reader to [18]...
Chapter II, Section 5] for the definition of the canonical automorphy factor $J$ of a bounded symmetric domain $\mathcal{D}$. All we have to know at this place is that $J$ restricts to a map $J : \mathcal{D} \times V^C \to \text{Str}(V^C)$ satisfying

\[(8) \quad K(gz, gw) = J(g, z)K(z, w)J(g, w)^* \]

for $g \in G(\mathcal{D})$, $z, w \in \mathcal{D}$. The automorphy kernel allows us to detect transversality on the Shilov boundary $\tilde{\mathcal{S}}$.

**Proposition 3.2.** Let $V$ be a Euclidean Jordan algebra, $\mathcal{D}$ the associated bounded symmetric domain and $\tilde{\mathcal{S}}$ its Shilov boundary. Then $z, w \in \tilde{\mathcal{S}}$ are transverse iff one of the following conditions holds true:

1. $\det(z - w) \neq 0$.
2. $K(z, w)$ is invertible.
3. $K(z, w) \in \text{Str}(V^C)$.
4. $\det K(z, w) \neq 0$.

**Proof.** Let us first prove that $z, w$ are transverse iff (iv) holds. Indeed, it follows from [8] that

\[
\{(z, w) \in \tilde{\mathcal{S}} | \det K(z, w) \neq 0\}
\]

is a $G$-orbit. By continuity of $\det K(\cdot, \cdot)$ this orbit is open, hence coincides with $\tilde{\mathcal{S}}(2)$. It thus remains to prove equivalence of (i)-(iv). The implication (i) $\Rightarrow$ (ii) is provided in [13, Lemma 5.1]. The implication (ii) $\Rightarrow$ (iii) follows from the fact that $K(z_0, w_0) \in \text{Str}(V^C)$ for $z_0, w_0 \in \mathcal{D}$ together with the continuity of $K$ and the fact that $\text{Str}(V^C)$ is closed in $GL(V^C)$. Finally, the implication (iii) $\Rightarrow$ (iv) is obvious.

Thus it remains to show (iv) $\Rightarrow$ (i). Thus let $w, z \in \tilde{\mathcal{S}}$ be arbitrary and assume $\det K(z, w) \neq 0$. We first claim that there exists $g \in G$ such that $e - g \cdot w$ and $e - g \cdot z$ are invertible. Indeed, if $\mathcal{D}$ is a polydisc with $w = (w_1, \ldots, w_r)$ and $z = (z_1, \ldots, z_r)$, then one can clearly find an element $g = (g_1, \ldots, g_r) \in SO(2r) \subset G$ such that $g_i \cdot w_i$ and $g_i \cdot z_i$ are both not equal to 1. The general case is reduced to this case by applying Proposition 2.13 to the triple $(e, w, z)$, thereby finishing the proof of the claim. Next observe that (8) implies

\[
\det K(gz, gw) = \det(J(g, z)) \det K(z, w) \det(J(g, w)) \neq 0
\]

and thus our assumption yields $\det K(gz, gw) \neq 0$. Now note that with $e - gw$ also $e - \overline{gw} = \overline{e - gw}$ is invertible, hence [13, Lemma X.4.4 ii] applies and yields

\[
K(gz, gw) = P(e - gz)P(c(gz) + c(gw))P(e - gw),
\]

whence

\[
\det(P(c(gz) + c(gw))) \neq 0.
\]

A simple calculation shows that $c(gw) = -c(gw)$. Since $gw \in \mathcal{S} \cap D(e)$, the image $c(gw)$ is contained in $V$, whence $c(gw) = c(gw)$. We thus obtain

\[
\det(P(c(gz) - c(gw))) \neq 0.
\]

Using the definition of the Cayley transform and [13, p.190] we obtain

\[
c(gz) - c(gw) = i((e + gz)(e - z)^{-1} - (e + gw)(e - gw)^{-1})
= i(-ie + 2i(e - gz)^{-1} + ie - 2i(e - gw)^{-1})
= -2((e - gz)^{-1} - (e - gw)^{-1}).
\]
We thus obtain

\[ \det P(-2((e - g)z^{-1} - (e - gw)^{-1})) \neq 0. \]  

Now we can apply Hua’s formula \cite[Lemma X.4.4]{15} to obtain

\[
P(-2((e - g)z^{-1} - (e - gw)^{-1}))
\]

\[= P(e - g)z^{-1}P(-2((e - g) - (e - gw)))P(e - gw)^{-1}
\]

\[= P(e - g)z^{-1}P(-2(gw - g))P(e - gw)^{-1}. \]

Combining this with (9) and using that \(P(e - g)z^{-1}\) and \(P(e - gw)^{-1}\) are invertible, we obtain

\[ \det P(-2(gw - g)) \neq 0. \]

Thus \(P(-2(gw - g))\) is invertible. By \cite[Prop. II.3.1]{15} this implies that \(-2(gw - g)\) and hence \(gz - gw\) is invertible. Thus \(\det(gz - gw) \neq 0\), which by \cite[Prop. 3.2]{13} implies \(\det(z - w) \neq 0\). This finishes the proof of Proposition 3.2. \(\square\)

Proposition 3.2 implies immediately:

**Corollary 3.3.** The image \(p(V)\) of \(V\) under the inverse Cayley transform is precisely the subset of points in \(\tilde{S}\), which are transverse to \(e\).

### 3.2. The normalized kernel function I: The simple case.

The aim of this subsection is to define a suitably normalized kernel functions on the bounded symmetric domain \(D\) of a given simple Euclidean Jordan algebra \(V\), whose extension to the Shilov boundary reflects transversality. We have seen in Proposition 3.2 that transversality is characterized by non-singularity of the automorphy kernel \(K : V^C \times V^C \to \text{End}(V^C)\), which when restricted to \(\mathcal{D}^{(2)} := D \cup \tilde{S}^{(2)}\) takes values in the structure group \(\text{Str}(V^C)\). In order to obtain a numerical kernel function, one can compose \(K\) with an arbitrary character \(\chi\) of \(\text{Str}(V^C)\). We denote the resulting function by

\[
k_\chi : \mathcal{D}^{(2)} \to \mathbb{C}^\times, \quad (z, w) \mapsto \chi(K(z, w)).
\]

Since we would like our kernel functions to be compatible under (certain) Jordan algebra homomorphisms, we will have to choose the character \(\chi\) carefully. Recall our assumption that \(V\) is supposed to be simple and denote by \(n := \dim V\) its dimension. We then choose \(\chi := \det^\frac{1}{n}\) and denote the associated kernel function by \(k_\chi := k_{\det^\frac{1}{n}}\). Here, the fractional exponent has to be interpreted as follows: If \(k_{\det} : \mathcal{D}^{(2)} \to \mathbb{C}^\times\) denotes the kernel function \(\det(K(z, w))\), then \(k_\chi\) is defined as the unique function \(\tilde{D}^{(2)} \to \mathbb{C}^\times\) satisfying \(k_\chi^n = k_{\det}\) and \(k_\chi(0, 0) = 1\). The kernel \(k_\chi\) extends in fact continuously to all of \(\tilde{S}^{(2)}\) due to the following general lemma:

**Lemma 3.4.** Let \(X\) be a manifold, \(f : X \to \mathbb{C}\) be a continuous function, \(X' := f^{-1}(\mathbb{C} \setminus \{0\})\). Let \(\tilde{f} : X' \to \mathbb{C} \setminus \{0\}\) be any continuous function with \(\tilde{f}^n = f\big|_{X'}\). Then \(\tilde{f}\) extends continuously by 0 to all of \(X\).

**Proof.** Extend \(\tilde{f}\) to all of \(X\) by 0. We show that this extension is continuous. For this let \(x_k \in X'\) with \(x_k \to x\), where \(x \in X \setminus X'\). Then \(f(x_k) \to f(x) = 0\) by continuity of \(f\), hence \(\tilde{f}(x_k)^n \to 0\). This, however, implies already \(\tilde{f}(x_k) \to 0 = \tilde{f}(x)\), which yields continuity of the extended function. \(\square\)
Applying this to the continuous function \( k_{\det} \) we obtain the desired extension of \( k_V \) to \( \mathbb{S}^2 \). In view of Proposition 3.2 and (8) this extension has the following properties:

**Corollary 3.5.** Let \( V \) be a simple Euclidean Jordan algebra. Then the normalized kernel \( k_V : \mathbb{S}^2 \to \mathbb{C} \) satisfies

\[
k_V(z, w) \neq 0 \iff z \parallel w
\]

and

\[
k_V(gz, gw) = j_V(g, z) k_V(z, w) j_V(g, w), \quad (g \in G, z, w \in \mathbb{S}),
\]

where \( j_V := \det \frac{1}{2} J \).

The factor \( \frac{1}{2} \) in the normalization exponent \( \frac{1}{2n} \) was added to achieve the following normalization:

**Example 3.6.** Let \( V = (\mathbb{R}, \cdot) \) so that \( D = \mathbb{D} \) is the unit disc. Since \( \text{dim} \ V = \text{rk}(V) = 1 \) it follows from Example 5.1 that \( k_{\det}(z, w) = (1 - zw)^2 \). Now our normalization is chosen in such a way that

\[
k_{\det}(z, w) = 1 - zw.
\]

This additional normalization factor does not affect the proof of the following desired invariance property:

**Lemma 3.7.** Let \( \alpha : V_1 \to V_2 \) be an injective morphism of simple Jordan algebras. Denote by \( D_1 \) and \( D_2 \) respectively the corresponding bounded symmetric domains and by \( \mathbb{S}_1 \) and \( \mathbb{S}_2 \) the respective Shilov boundaries. Then for all \( z, w \in D_1 \cup \mathbb{S}_1 \) we have

\[
(10) \quad k_{V_2}(\alpha^C(z), \alpha^C(w)) = k_{V_1}(z, w).
\]

**Proof.** By continuity it suffices to prove (10) for \( z, w \in D \). We would like to apply [13, Prop. 6.2]; for this we have to translate this proposition into our language. Define a character \( \chi_j \) of \( \text{Str}(V_j^C) \) by

\[
det(gz) = \chi_j(g) \det(z) \quad (g \in \text{Str}(V_j^C), z \in D_j),
\]

where \( \det \) is the Jordan algebra determinant. Denote by \( r_j \) and \( n_j \) respectively the rank and dimension of \( V_j \). Then the proposition of Clerc and Ørsted states that

\[
k_{\chi_j}(\alpha^C(z), \alpha^C(w)) = k_{\chi_j}(z, w)^{\frac{r_j}{n_j}}.
\]

On the other hand, since the \( V_j \) are simple, [13, Prop. III.4.3] applies and shows

\[
\chi_j = \det \frac{r_j}{n_j}.
\]

We thus obtain

\[
k_{V_2}(\alpha^C(z), \alpha^C(w)) = \det \frac{r_2}{n_2} (K(\alpha^C(z), \alpha^C(w)))
\]

\[
= \left[ \det \frac{n_2}{r_2} (K(\alpha^C(z), \alpha^C(w))) \right]^{\frac{r_2}{n_2}}
\]

\[
= \left[ k_{\chi_2}(\alpha^C(z), \alpha^C(w)) \right]^{\frac{r_2}{n_2}}
\]

\[
= \left[ k_{\chi_1}(z, w)^{\frac{r_2}{n_1}} \right]^{\frac{r_2}{n_2}}
\]

\[
= \left[ \det \frac{n_1}{r_1} K(z, w) \right]^{\frac{r_1}{n_1}}
\]
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\[ = \det \frac{1}{\sqrt{n}} K(z, w) \]

\[ = k_{V_1}(z, w). \]

We have been slightly sloppy here by not specifying the arc of the various roots. Still, it is easy to make the above argument precise. Strictly speaking we have only shown that

\[ k_{V_2}(\alpha C(z), \alpha C(w))^{2n_1} = k_{V_1}(z, w)^{2n_1} \]

rather than

\[ k_{V_2}(\alpha C(z), \alpha C(w)) = k_{V_1}(z, w). \]

However, since \( k_{V_2}(\alpha C(0), \alpha C(0)) = k_{V_1}(0, 0) \) the former implies the latter by uniqueness. We will allow ourselves this kind of sloppyness regarding roots, whenever it is clear how to make the arguments precise.

### 3.3. The normalized kernel function II: The general case.

We would like to define a normalized kernel with invariance properties generalizing those of Lemma 3.7 also in the case where the involved Jordan algebras are not simple. For this we first observe that any Euclidean Jordan algebra \( V \) is semisimple ([15 Prop. III.4.4]) and thus decomposes into simple ideals, say \( V = V_1 \oplus \cdots \oplus V_n \). Accordingly we identify elements \( z \in V \) with vectors \( z = (z_1, \ldots, z_n)^T \) with \( z_j \in V_j \) and define \( k_V : D^2 \to \mathbb{C}^\times \) by the formula

\[ (11) \quad k_V(z, w) = \prod_{i=1}^n (k_{V_i}(z_i, w_i))^{rk_{V_i}}. \]

More precisely, if \( \tilde{k}_V : D^2 \to \mathbb{C}^\times \) is given by the formula

\[ \tilde{k}_V(z, w) := \prod_{i=1}^n (k_{V_i}(z_i, w_i))^{rk_{V_i}}, \]

then \( k_V \) is defined to be the unique function satisfying

\[ k_V(z, w)^{rk(V)} = \tilde{k}_V(z, w), \quad k_V(0, 0) = 1. \]

As before Lemma [3.4] implies that \( k_V \) extends to a function \( D^2 \cup \tilde{S}^2 \to \mathbb{C} \), which vanishes precisely on non-transversal pairs. Observe that for \( V \) simple the function \( k_V \) coincides with the normalized kernel \( k_V \) defined earlier so that there it is justified to use the same notation and to refer to \( k_V \) in general as the normalized kernel function of \( V \). Moreover, the following property follows right from the definition:

**Proposition 3.8.** Let \( V \) be a Euclidean Jordan algebra. If \( V = V_1 \oplus V_2 \) is the sum of two (not necessarily simple) ideals of ranks \( r_1, r_2 \) with projections \( p_j : V \to V_j \) then

\[ k_V(z, w)^{r_1 + r_2} = k_{V_1}(p_1(z), p_1(w))^{r_1} k_{V_2}(p_2(z), p_2(w))^{r_2}. \]

Let us spell this out in the case of a polydisc:
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Example 3.9. Let \( V = (\mathbb{R}, \cdot)^r \) so that \( \mathcal{D} = \mathbb{D}^r \) is the standard polydisc. In our sloppy notation the normalized kernel function \( k_{\mathbb{R}^r} \) of the polydisc \( \mathbb{D}^r \) is

\[
k_{\mathbb{R}^r}((z_1, \ldots, z_r), (w_1, \ldots, w_r)) = \prod_{j=1}^{r} (1 - z_j \overline{w}_j)^{\frac{1}{2}}.
\]

We can see already from the case of polydiscs that our normalized kernel is not invariant under arbitrary Jordan algebra homomorphisms:

Example 3.10. Consider the Jordan algebra embedding \( \alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) given by \((\lambda_1, \lambda_2) \mapsto (\lambda_1, \lambda_1, \lambda_2)\). Then

\[
k_{\mathbb{R}^2}(\lambda, \mu) = (1 - \lambda_1 \overline{\mu}_1)^{\frac{1}{2}} (1 - \lambda_2 \overline{\mu}_2)^{\frac{1}{2}} \neq (1 - \lambda_1 \overline{\mu}_1)^{\frac{1}{4}} (1 - \lambda_2 \overline{\mu}_2)^{\frac{1}{4}} = k_{\mathbb{R}^3}(\alpha(C)(\lambda), \alpha(C)(\mu)).
\]

We want to exclude bad behavior as in the last example. Denote by \( \text{tr}_V \) the Jordan algebra trace of \( V \). Thus if \((c_1, \ldots, c_r)\) is a Jordan frame for \( V \) then

\[
x = \sum_{j=1}^{r} \lambda_j c_j \Rightarrow \text{tr}_V(x) = \sum_{j=1}^{r} \lambda_j.
\]

Now we define:

Definition 3.11. A Jordan algebra homomorphism \( \alpha_V : V \rightarrow W \) is called balanced if for all \( v \in V \)

\[
\frac{1}{\text{rk}_V} \text{tr}_V(v) = \frac{1}{\text{rk}_W} \text{tr}_W(\alpha(v)).
\]

The notion is clearly invariant under complexification. Moreover, we have the following characterization of balanced Jordan algebra homomorphisms: Let \((c_1, \ldots, c_r)\) be a Jordan frame in \( V \) and \( \alpha : V \rightarrow W \) a Jordan algebra homomorphism. Then \( \alpha(c_1), \ldots, \alpha(c_r) \) is a family of idempotents with \( \alpha(c_i)\alpha(c_j) = 0 \) and \( \sum \alpha(c_i) = e \). Thus Proposition 2.7 applies and there exists a Jordan frame \((c_{11}, \ldots, c_{1l_1}, \ldots, c_{r1}, \ldots, c_{rl_r})\) of \( W \) such that

\[
\alpha(c_j) = \sum_{k=1}^{l_j} c_{jk}.
\]

We have \( \text{tr}_W(\alpha(c_j)) = l_j \) and thus \( \alpha \) is balanced if and only if

\[
l_1 = \cdots = l_r.
\]

Conversely, if the latter condition is true for any Jordan frame \((c_1, \ldots, c_r)\) of \( V \), then \( \alpha \) is balanced. Note that we obtain in particular

\[
\text{rk}_W = l_j \cdot \text{rk}_V \quad (j = 1, \ldots, r),
\]

so that \( \text{rk}_W \) is divisible by \( \text{rk}_V \). In particular, the morphism in Example 3.10 is not balanced. Also note that a balanced Jordan algebra homomorphism is injective unless it is trivial. In our attempts to prove an invariance theorem we will restrict attention to balanced Jordan algebra homomorphisms. Even in this case we cannot quite obtain the same kind of invariance as in Lemma 3.7. In order to formulate our (slightly weaker) result, we introduce the following terminology: Two elements
v_1, v_2 are called co-diagonalizable if there exists a Jordan frame \((c_1, \ldots, c_r)\) and elements \(\lambda_j \in \mathbb{D}, \mu_j \in \mathbb{D}\) such that

\[
v_1 = \sum_{j=1}^{r} \lambda_j c_j \in \mathcal{D}_V, \quad v_2 = \sum_{j=1}^{r} \mu_j c_j \in \mathcal{D}_V.
\]

Then we have:

**Theorem 3.12.** Let \(\alpha : V \rightarrow W\) be a non-trivial Jordan algebra homomorphism. If \(\alpha\) is balanced, then for every pair of co-diagonalizable elements \(v_1, v_2 \in \mathcal{D}\) we have

\[
k_W(\alpha^c(v_1), \alpha^c(v_2)) = k_V(v_1, v_2).
\]

Conversely, if (12) holds for all co-diagonalizable \(v_1, v_2 \in \mathcal{D}_V\), then \(\alpha\) is balanced.

For the proof we consider first the case, where \(V\) is a maximal polydisc in \(W\). In this case we have the following version of Theorem 3.12, which is a slight extension of the results of Clerc and Ørsted in [13]:

**Lemma 3.13.** If \((c_1, \ldots, c_r)\) is a Jordan frame in a Jordan algebra \(W\) and \(\lambda_j \in \mathbb{D}, \mu_j \in \mathbb{D}\), then

\[
k_W(\sum_{j=1}^{r} \lambda_j c_j, \sum_{j=1}^{r} \mu_j c_j) = \prod_{j=1}^{r} (1 - \lambda_j \mu_j)^2.
\]

**Proof.** Assume first that \(W\) is simple. Then [13, Lemma 5.4] applies directly and we obtain

\[
k_XW(\sum_{j=1}^{r} \lambda_j c_j, \sum_{j=1}^{r} \mu_j c_j) = \prod_{j=1}^{r} (1 - \lambda_j \mu_j)^2,
\]

whence

\[
k_XW(\sum_{j=1}^{r} \lambda_j c_j, \sum_{j=1}^{r} \mu_j c_j)^r = \left(\prod_{j=1}^{r} (1 - \lambda_j \mu_j)^2\right)^{2r}.
\]

Since

\[
\prod_{j=1}^{r} (1 - 0 \cdot 0) = 1
\]

this settles the simple case. For the general case, consider a decomposition \(W = W_1 \oplus \cdots \oplus W_n\) into simple ideals. Let \(r_i := \text{rk}(W_i)\) and \((c_{l_1}, \ldots, c_{l_{r_i}})\) be a Jordan frame for \(W_i\). Then \((c_{1_1}, \ldots, c_{n_{r_n}})\) is a Jordan frame for \(W\) and, in fact, any Jordan frame for \(W\) is of this form (as follows e.g. from [15, Prop. X.3.2]). Let

\[
z := \sum_{l=1}^{n} \sum_{j=1}^{r_i} \lambda_{l_j} c_{l_j}, \quad w := \sum_{l=1}^{n} \sum_{j=1}^{r_i} \mu_{l_j} c_{l_j}.
\]

By definition we have

\[
k_W(z, w)^{\text{rk}W} = \prod_{l=1}^{n} (k_{W_l}(z_l, w_l))^{\text{rk}W_l},
\]
where
\[ z_l = \sum_{j=1}^{r_i} \lambda_{lj}c_{lj}, \quad w_l := \sum_{j=1}^{r_i} \mu_{lj}c_{lj}. \]

By the simple case we have
\[ (k_{W_l}(z_l, w_l))^{rk_{W_l}} = \prod_{j=1}^{r_i} (1 - \lambda_{lj}\mu_{lj}), \]
and thus
\[ k_W(z, w)^{rk_W} = \prod_{l=1}^{n} \prod_{j=1}^{r_i} (1 - \lambda_{lj}\mu_{lj}). \]

\[ \square \]

From this the general case follows easily:

**Proof of Theorem 3.12.** If \( \alpha \) is balanced, then \( r_V := rk(V) \) and \( r_W := rk(W) \) are related by \( r_W = \mu_{\alpha}r_V \) for some constant \( \mu_{\alpha} \). Given a Jordan frame \((c_1, \ldots, c_r)\) in \( V \) and elements
\[ v_1 = \sum_{j=1}^{r} \lambda_j c_j \in D_V, \quad v_2 = \sum_{j=1}^{r} \mu_j c_j \in D_V \]
with \( \lambda_j \in \mathbb{D}, \mu_j \in \mathbb{D} \) we have
\[ \alpha^C(v_1) = \sum_{j=1}^{r} \lambda_j \alpha(c_j), \quad \alpha^C(v_2) = \sum_{j=1}^{r} \mu_j \alpha(c_j). \]

Now each \( \alpha(c_j) \) decomposes as
\[ \alpha(c_j) = d_{j1} + \cdots + d_{j\mu_{\alpha}}, \]
where the \( d_{ji} \) are primitive idempotents. Now we obtain
\[ k_V(v_1, v_2)^{r_V} = \prod_{j=1}^{r} (1 - \lambda_j\mu_j), \]
whence
\[ k_V(v_1, v_2)^{r_W} = \left( \prod_{j=1}^{r} (1 - \lambda_j\mu_j) \right)^{\mu_{\alpha}} = \prod_{j=1}^{r} (1 - \lambda_j\mu_j)^{\mu_{\alpha}}. \]

Similarly,
\[ k_W(\alpha^C(v_1), \alpha^C(v_2))^{r_W} = \prod_{j=1}^{r} (1 - \lambda_j\mu_j)^{\mu_{\alpha}}. \]

As \( k_V(0, 0) = k_W(\alpha^C(0), \alpha^C(0))^{r_W} \), this implies (12). On the other hand, if \( \alpha \) is not balanced and \( v_1, v_2 \) are as above, then
\[ k_V(v_1, v_2) = \prod_{j=1}^{r} (1 - \lambda_j\mu_j)^{\frac{1}{r_V}} \]
\[ \neq \prod_{j=1}^{r} (1 - \lambda_j\mu_j)^{\mu_{\alpha}(c_j)} \]
\[ = k_W(\alpha^C(v_1), \alpha^C(v_2)). \]
Example 3.14. The following are examples of balanced Jordan algebra homomorphisms:

- Jordan algebra homomorphisms $\alpha : V \to W$ between simple Jordan algebras are balanced by Lemma 3.7.
- If $\text{rk}(V) = \text{rk}(W)$ then every injective Jordan algebra homomorphism $\alpha : V \to W$ is balanced.
- In particular, maximal polydisc embeddings are balanced.
- Any Jordan algebra homomorphism $\alpha : \mathbb{R} \to W$ is balanced.
- Compositions of balanced Jordan algebra homomorphisms are balanced.

So far we have been working on the interior of a bounded symmetric domain. Of course, by continuity our discussion extends the Shilov boundary. In particular we have:

Corollary 3.15. Let $V$ be a Euclidean Jordan algebra and $k_V : (\mathcal{D} \cup \mathcal{S})^2 \to \mathbb{C}$ the associated normalized kernel. Then

(i) If $z, w \in \mathcal{S}$, then $k_V(z, w) \neq 0 \iff z \parallel w$.

(ii) There exists a continuous function $j_V : G \times V \to \mathbb{C}^\times$ such that

$$k_V(gz, gw) = j_V(g, z)k_V(z, w)j_V(g, w), \quad (g \in G, z, w \in \mathcal{S}).$$

(iii) If $(c_1, \ldots, c_r)$ is a Jordan frame for $V$ and $\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_r \in \overline{D}$, then

$$(14) \quad k_V(\sum_{j=1}^r \lambda_j c_j, \sum_{j=1}^r \mu_j c_j) = \prod_{j=1}^r (1 - \lambda_j \mu_j)^{\frac{1}{r}}.$$

3.4. Comparison to the Bergman kernel. Given a Euclidean Jordan algebra $V$ with associated bounded symmetric domain $\mathcal{D}$, we can forget about the algebraic structure of $V$ and consider $\mathcal{D} \subset V$ simply as a bounded domain. From this data we can then define the Bergman kernel

$$K_\mathcal{D} : \mathcal{D} \times \mathcal{D} \to \mathbb{C}^\times$$

as the reproducing kernel of the Bergman space $\mathcal{H}^2(\mathcal{D})$. It turns out that our normalized kernel $k_V$ is uniquely determined by $K_\mathcal{D}$. More precisely we have:

Proposition 3.16. The restriction $k_V : \mathcal{D} \times \mathcal{D} \to \mathbb{C}^\times$ of the normalized kernel and the Bergman kernel $K_\mathcal{D}$ are related by the formula

$$k_V(z, w)^r = C \cdot K_\mathcal{D}(z, w)^{-1},$$

where $C$ is some constant depending only on the domain $\mathcal{D}$.

Proof. By [15] Prop. X.4.5] the Bergman kernel on $\mathcal{D}$ is of the form

$$K_\mathcal{D}(z, w) = \frac{1}{\text{Vol}(\mathcal{D})} \cdot \det(K(z, w))^{-1}.$$

Let us first assume that $V$ is simple of rank $r$. Then

$$k_V(z, w) = \det(K(z, w))^{\frac{1}{r}} = \text{Vol}(\mathcal{D}) \cdot K_\mathcal{D}(z, w)^{-\frac{1}{r}}.$$
In the general case we decompose \( V = V_1 \oplus \cdots \oplus V_n \) into simple ideals with \( r_i := \text{rk}(V_i) \) so that \( r := \text{rk}(V) = \sum r_i \). Now the Bergman kernel is multiplicative, i.e. we have

\[
K_D = \prod_{i=1}^{n} K_{D_i}.
\]

Now the simple case yields

\[
k_{V_i}^{r_i} = \text{Vol}(D_i)^{r_i} : K_{D_i}^{-\frac{1}{r_i}},
\]

and thus by definition of \( k_V \) we have

\[
k_V = \prod_{i=1}^{n} k_{V_i}^{r_i} = \left( \prod_{i=1}^{n} k_{V_i}^{r_i} \right)^{\frac{1}{r}}
\]

\[
= \left( \prod_{i=1}^{n} \text{Vol}(D_i)^{r_i} : K_{D_i}^{-\frac{1}{r_i}} \right)^{\frac{1}{r}}
\]

\[
= \left( \prod_{i=1}^{n} \text{Vol}(D_i)^{\frac{1}{2}} \right) : K_D^{-\frac{1}{2}}.
\]

Note that the relation to the Bergman kernel together with the normalization \( k_V(0,0) = 1 \) determines \( k_V \) uniquely. Thus, while the definition of \( k_V \) involves the Jordan algebra \( V \) and thus the structure of \( D \) as a marked bounded symmetric domain, the above proposition shows that the result depends only on \( D \) as a bounded domain.

4. **Generalized cross ratios on Shilov boundaries**

In this section we use the normalized kernel functions defined in the last section in order to construct a family of generalized cross ratios on Shilov boundaries of bounded symmetric domains of tube type. In the first subsection we provide an axiomatic characterization of these generalized cross ratios. In the second subsection we give an explicit construction and show that this construction satisfies the desired axioms. The third subsection explains how to compute generalized cross ratios using maximal polydiscs. In the fourth subsection we deduce a number of cocycle properties of the generalized cross ratios. In the final subsection we express the generalized cross ratio in terms of Bergman kernels, thereby showing that it is independent of the marking of the bounded symmetric domain.

4.1. **The axiomatic approach.** Before we can define generalized cross ratios, we have to clarify, which properties these functions should have. We will thus collect in this subsection a number of axioms that our functions are supposed to satisfy. We will then see in the subsequent subsections that such functions exist and are in fact uniquely determined by the axioms. The object that we want to generalize is the classical four point cross ratio of hyperbolic geometry. Recall that the latter is defined on quadruples of mutually distinct points in \( \mathbb{C} \mathbb{P}^1 \) by the formula
\[ [a : b : c : d] := \frac{(a - d)(c - b)}{(c - d)(a - b)}. \]

This cross ratio restricts to a real-valued function on \((S^1)^{(4)}\), which is invariant under \(PU(1, 1)\). We would like to generalize this cross ratio from \(S^1\) to the Shilov boundary \(\tilde{S}_V\) of an arbitrary Euclidean Jordan algebra, that is, we would like to define a family of functions

\[ B_V : \tilde{S}_V^{(4)} \to \mathbb{R}^x, \]

which are invariant under the respective groups \(G_V\). Moreover, these generalized cross ratios should be related to each other. E.g. it would be nice to have

\[ B_W(\alpha(v_1), \alpha(v_2), \alpha(v_3), \alpha(v_4)) = B_V(v_1, v_2, v_3, v_4) \]

for any Jordan algebra homomorphism \(\alpha : V \to W\) and \(v_1, v_2, v_3, v_4 \in \tilde{S}_V\). This, however, is too much of wishful thinking. Firstly, we should not ask (16) for arbitrary Jordan algebra homomorphism \(\alpha : V \to W\) since these can be rather badly behaved, as we have seen before. Rather, as suggested by Theorem 3.12, we should restrict to balanced homomorphism. Secondly, if we insist that our generalized cross ratio should be real-valued then we cannot define it on all of \(\tilde{S}_V^{(4)}\). A posteriori, the following domain turns out to be the correct one:

**Definition 4.1.** Let \(V\) be a Euclidean Jordan algebra, \(D\) its bounded symmetric domain and \(\tilde{S}\) the associated Shilov boundary. A quadruple \((a, b, c, d)\) \(\in \tilde{S}^{(4)}\) is called extremal if any triple \((x, y, z)\) \(\in \tilde{S}^{(3)}\) of pairwise distinct points with \(x, y, z \in \{a, b, c, d\}\) is either maximal or minimal. We denote the set of extremal quadruples in \(\tilde{S}^{(4)}\) by \(\tilde{S}^{(4+)}\).

Note that an extremal quadruple is contained in the Shilov boundary of a maximal polydisk by Proposition 2.15. Now we can formulate the axioms, which the desired extension of the classical cross ratio should satisfy:

**Theorem 4.2.** There exists a unique system of functions

\[ \{B_V : \tilde{S}_V^{(4+)} \to \mathbb{R}^x \mid V \text{ Euclidean Jordan algebra} \} \]

with the following properties:

(i) If \(D_V\) is the bounded domain associated with \(V\), then \(B_V\) is \(G(D_V)\)-invariant.

(ii) If \(\alpha : V \to W\) is a balanced Jordan algebra homomorphism, \((v_1, \ldots, v_4) \in \tilde{S}_V^{(4+)}\) and \((\alpha(v_1), \ldots, \alpha(v_4)) \in \tilde{S}_W^{(4+)}\), then

\[ B_W(\alpha(v_1), \ldots, \alpha(v_4)) = B_V(v_1, \ldots, v_4). \]

(iii) If \(V = V_1 \oplus V_2\) is the sum of two ideals of ranks \(r_1, r_2\) with projections \(p_j : V \to V_j\) then

\[ B_V(v_1, \ldots, v_4)^{r_1 + r_2} = B_{V_1}(p_1(v_1), \ldots, p_1(v_4))^{r_1}B_{V_2}(p_2(v_1), \ldots, p_2(v_4))^{r_2}. \]

(iv) \(B_\mathbb{R}\) is the restriction of the classical four point cross ratio.

Condition (iii) was added here, to rigidify the situation and to obtain uniqueness. It is unclear to us, whether it follows from the other three axioms.
4.2. Explicit construction of generalized cross ratios. The proof of Theorem 4.2 is constructive, i.e. we will construct the functions \( B_V \) explicitly. The functions \( B_V \) extend to all of \( \tilde{S}^{(4)} \) and these extensions are given as follows:

**Definition 4.3.** Let \( V \) be a Euclidean Jordan algebra. Then the cross ratio of \( V \) is the function

\[
B_V : \tilde{S}^{(4)} \to \mathbb{C}^\times
\]

given by

\[
B_V(a, b, c, d) := \frac{k_V(d, a)k_V(b, c)}{k_V(d, c)k_V(b, a)},
\]

We claim that the cross ratios satisfy the conditions of Theorem 4.2. The proof of this fact will occupy the remainder of this section. We start by checking condition (iv):

**Example 4.4.** Let \( V = (\mathbb{R}, -) \) so that \( \tilde{S} = S^1 \). By Example 3.6 we have \( k_\mathbb{R}(a, b) = 1 - ab \). Then for four mutually distinct points \( a, b, c, d \in S^1 \) we have

\[
B_\mathbb{R}(a, b, c, d) = \frac{(1 - da)(1 - bc)}{(1 - dc)(1 - ba)} = \frac{(a - d)(c - b)}{(c - d)(a - b)} = [a : b : c : d].
\]

Condition (iii) follows directly from Proposition 3.8. For the proofs of the remaining properties it will be useful to extend \( B_V \) to a continuous function

\[
\bar{B}_V : \mathcal{D}^{(4)} \to \mathbb{C}^\times, \quad (a, b, c, d) \mapsto \frac{k_V(d, a)k_V(b, c)}{k_V(d, c)k_V(b, a)}.
\]

where \( \mathcal{D}^{(4)} := \mathcal{D}^4 \cup \tilde{S}^{(4)} \). If \( V \) is simple then by definition \( \bar{B}_V \) is the unique continuous function satisfying

\[
\bar{B}_V(a, b, c, d)^{2\dim V} = B_{V, \det}(a, b, c, d), \quad \bar{B}_V(0, 0, 0, 0) = 1,
\]

where

\[
B_{V, \det} : \mathcal{D}^{(4)} \to \mathbb{C}^\times, \quad (a, b, c, d) \mapsto \frac{k_{\det}(d, a)k_{\det}(b, c)}{k_{\det}(d, c)k_{\det}(b, a)}.
\]

This characterized \( \bar{B}_V \) and hence the restriction \( B_V \) in the simple case, and the general case can be reduced to this situation using property (iii). Using this characterization we now prove:

**Proposition 4.5.** Let \( V \) be a Euclidean Jordan algebra. Then the cross ratio \( B_V \) is \( G \)-invariant, i.e. for all \( (a, b, c, d) \in \tilde{S}^{(4)} \) we have

\[
B_V(a, b, c, d) = B_V(ga, gb, gc, gd) \quad (g \in G).
\]

**Proof.** Let us first assume that \( V \) is simple. By Corollary 3.5 we have

\[
k_{\det}(gz, gw) = j_{\det}(g, z)k_{\det}(z, w)j_{\det}(g, w)
\]

for all \( g \in G, \ z, w \in \mathcal{D}^{(2)} \). From this we obtain for all \( g \in G \) and \( (a, b, c, d) \in \mathcal{D}^4 \cup \tilde{S}^{(4)} \) the following relation:

\[
B_{V, \det}(ga, gb, gc, gd) = \frac{j_{\det}(g, d)k_{\det}(d, a)j_{\det}(g, a)j_{\det}(g, b)k_{\det}(b, c)j_{\det}(g, c)}{j_{\det}(g, d)k_{\det}(d, c)j_{\det}(g, c)j_{\det}(g, b)k_{\det}(b, a)j_{\det}(g, a)}.
\]
observed in Proposition 2.1: Given a Jordan algebra homomorphisms
In order to deduce condition (ii) we make use of the equivalence of categories
the simple case using Condition (iii).

Since \( \tilde{B} \) is uniquely determined by (18), this implies \( \tilde{B}(ga, gb, gc, gd) = \tilde{B}(a, b, c, d) \). Restricting to the Shilov boundary, we obtain the statement of the proposition for simple \( V \). The general case is easily reduced to the simple case using Condition (iii).

In order to deduce condition (ii) we make use of the equivalence of categories observed in Proposition 2.1. Given a Jordan algebra homomorphisms \( \alpha : V \to W \), there exists a group homomorphism \( \alpha^1 : \tilde{G}_V \to \tilde{G}_W \) which is equivariant with respect to \( \alpha^C : D_V \to D_W \). (Here \( \tilde{G}_V, \tilde{G}_W \) denote the universal coverings of \( G_V \) and \( G_W \).) In particular, given \( g \in \tilde{G}_V \) there exists \( h \in \tilde{G}_W \) such that for all \( v \in D_V \)

\[
\alpha^C(gv) = h \alpha^C(v).
\]

Since the actions of \( \tilde{G}_V, \tilde{G}_W \) factor through the actions of \( G_V \) and \( G_W \) respectively that for every \( g \in G_V \) there exists \( h \in G_W \) such that (19) holds for all \( v \in V \). Now we can prove Condition (ii): If \( (v_1, \ldots, v_4) \) is extremal then by Proposition 2.15 we find \( g \in G_V \) such that \( gv_1, \ldots, gv_4 \) are diagonalized by a common Jordan frame \( (c_1, \ldots, c_r) \). Let \( h \in G_W \) be an element such that (19) holds for all \( v \in V \). Using Proposition 4.5 and Theorem 3.12 we now obtain

\[
B_V(v_1, \ldots, v_4) = B_V(gv_1, \ldots, gv_4) = k_V(gv_1, gv_2, gv_3) k_V(gv_2, gv_4) k_V(gv_4, gv_1) = k_W(\alpha^C(gv_1), \alpha^C(gv_2), \alpha^C(gv_3), \alpha^C(gv_4)) k_W(\alpha^C(gv_2), \alpha^C(gv_3), \alpha^C(gv_4)) k_W(\alpha^C(gv_3), \alpha^C(gv_4), \alpha^C(gv_1)) k_W(\alpha^C(gv_4), \alpha^C(gv_1), \alpha^C(gv_2)) k_W(\alpha^C(gv_1), \alpha^C(gv_2), \alpha^C(gv_3)) k_W(\alpha^C(gv_2), \alpha^C(gv_3), \alpha^C(gv_4)) k_W(\alpha^C(gv_3), \alpha^C(gv_4), \alpha^C(gv_1)) k_W(\alpha^C(gv_4), \alpha^C(gv_1), \alpha^C(gv_2)) = B_W(ha(v_1), \ldots, ha(v_4)) = B_W(\alpha(v_1), \ldots, \alpha(v_4)).
\]

This shows that the cross ratios satisfy Conditions (i) - (iv) of Theorem 4.2. Conversely, these conditions determine the family \( \{B_V\} \): Any \( (a, b, c, d) \in S^{14+} \) is contained in the boundary of a maximal polydisc by Proposition 2.15. Since the embedding of a maximal polydisc is balanced, the family \( \{B_V\} \) is uniquely determined by the family \( \{B_{rV}\} \). Finally, condition (iii) implies that

\[
B_{rV}(a, b, c, d) = \prod B_{r}(a_j, b_j, c_j, d_j).
\]

The right hand side is uniquely determined by (iv). If we denote by \( \alpha : \mathbb{R} \to \mathbb{R}^r \) the diagonal embedding, then \( \alpha \) is balancing and therefore

\[
B_{rV}(0, 0, 0, 0) = B_{rC}(\alpha(0), \ldots, \alpha(0)) = B_{r}(0, 0, 0, 0),
\]
and again the latter is determined by (iv). The last two equations determine \( \{ B_{R^r} \} \) uniquely. The uniqueness of the whole system \( \{ B_{V} \} \) follows.

Thus we have proved Theorem 4.2 up to the claim that the cross ratio is real-valued on \( \tilde{S}^{(4+)} \). This latter fact will be proved in the next subsection, where we also provide techniques to compute the cross ratio effectively.

### 4.3. Computation of generalized cross ratios

Let \( V \) be a Euclidean Jordan algebra and \( (a,b,c,d) \in \tilde{S}^{(4+)} \). The aim of this subsection is to provide an effective way to compute \( B_{V}(a,b,c,d) \). As a byproduct we will see that \( B_{V}(a,b,c,d) \in \mathbb{R} \), thereby finishing the proof of Theorem 4.2. As a first step we apply Proposition 2.15 in order to find a polydisc whose Shilov boundary contains \( (a,b,c,d) \). Actually, the proof of the proposition shows slightly more: Let us assume that \( (a,b,c) \) is maximal. Then we can find \( g \in G \) and a Jordan frame \( (c_1,\ldots,c_r) \) of \( V \) such that

\[
g.(a,b,c,d) = (-e, -ie, e, \sum_{j=1}^{r} \lambda_j c_j).
\]

By Property (iii) of Theorem 4.2 we thus have

\[
B_{V}(a,b,c,d) = \tilde{B}_{R^r}(-e, -ie, e, \lambda).
\]

We may therefore assume \( V = \mathbb{R}^r \) and \( (a,b,c) = (-e, -ie, e) \) provided \( (a,b,c) \) is maximal. If \( (a,b,c) \) is minimal rather than maximal, then we may assume \( (a,b,c) = (e, -ie, -e) \) by the same argument. In the remainder of this section we will always assume \( (a,b,c) = (-e, -ie, e) \); the minimal case can be treated accordingly. Since \( (-e, -ie, e, \lambda) \) is assumed extremal, the possible values of \( \lambda \) are seriously restricted: Indeed, \( (-1, \lambda_j, 1) \) is positive iff \( \lambda_j \) is contained in the lower half-circle and negative, iff \( \lambda_j \) is contained in the upper half-circle. Since \( (-e, -ie, e, \lambda) \) is either maximal or minimal we see that either \( \lambda_j \) is contained in the lower half-circle for all \( j = 1,\ldots,r \) or in the upper half-circle for all \( j = 1,\ldots,r \). Correspondingly, let us call \( \lambda \) positive or negative. In the positive case, all the \( \lambda_j \) are contained in a fixed quarter circle. This special position of \( \lambda \) allows us now to compute \( B_{V}(-e, -ie, e, \lambda) \) as follows: Let \( \tilde{B}_{V} \) as in (17). Given \( a \in \mathbb{D}^r \) we write \( a_j \) for its \( j \)-th component. Now let \( (a,b,c,d) \in \mathbb{C}^4 \). We have

\[
\tilde{B}_{V}(a,b,c,d)^r = \frac{k_V(d,a)^r k_V(b,c)^r}{k_V(d,c)^r k_V(b,a)^r} = \prod_{j=1}^{r} [a_j : b_j : c_j : d_j]
\]

and

\[
\tilde{B}_{V}(0,0,0,0) = 1.
\]

Moreover, \( \tilde{B}_{V} \) is uniquely determined by (20) and (21). From this we obtain the following result almost immediately:

**Lemma 4.6.** If \( \lambda_1 = \cdots = \lambda_r \), then

\[
B_{R^r}(-e, -ie, e, \lambda) = [-1 : -i : 1 : \lambda_1].
\]

**Proof.** The curve \( \varphi(t) := (-te, -ite, te, t\lambda) \) connects \((0, 0, 0, 0)\) and \((-e, -ie, e, \lambda)\). Since

\[
\tilde{B}_{V}(\varphi(t))^r = B_{R^r}(-t : -it : t : \lambda_1 t)^r
\]
and both functions agree at 0, we see that
\[
B_V(-e, -ie, e, \lambda) = \tilde{B}_V(\phi(1)) = B_R(-1 : -i : 1 : \lambda) = [-1 : -i : 1 : \lambda_1].
\]

\[\square\]

Although the situation of Lemma 4.6 is very special, the general case can now be reduced to it. Indeed we have:

**Proposition 4.7.** The cross-ratio \(B_R\) is real-valued on \(((S^1)^r)^{(4+)}\). More precisely, \(B_R(-e, -ie, e, \lambda)\) is positive/negative iff \(\lambda\) is positive/negative.

**Proof.** Consider the function \(f : (S^1 \setminus \{-1, -i, 1\})^r \to S^1\) given by
\[
f(\lambda) := \frac{B_R(-e, -ie, e, \lambda)}{|B_R(-e, -ie, e, \lambda)|}.
\]
We have \(f(\lambda)^r \in \mathbb{R} \cap S^1 = \{\pm 1\}\), hence \(f\) takes values in the set \(R_{2r}\) of \(2r\)-th roots of unity. Since \(R_{2r}\) is discrete and \(f\) is continuous, \(f\) must be locally constant. In particular, if \(\lambda\) and \(\mu\) are contained in the same connected component of \((S^1 \setminus \{-1, -i, 1\})^r\) and \(B_R(-e, -ie, e, \mu)\) is a positive/negative real number, then the same is true for \(B_R(-e, -ie, e, \lambda)\). Combining this with Lemma 4.6 we obtain the proposition. \[\square\]

Since \(20\) admits precisely one positive/negative \(r\)-th real root, this determines the cross ratio. To summarize our discussion, let us call an extremal quadruple \((a, b, c, d)\) positive/negative if it is conjugate to \(B_R(-e, -ie, e, \lambda)\) for some positive/negative \(\lambda\). Then we obtain the following formula for \(B_R\):

**Corollary 4.8.** Suppose \((a, b, c)\) is maximal and \((a, b, c, d) \in ((S^1)^r)^{(4+)}\). Then
\[
B_R(a, b, c, d) = \epsilon(a, b, c, d) \cdot \left( \prod_{j=1}^{r} [a_j : b_j : c_j : d_j] \right)^{\frac{1}{r}},
\]
where
\[
\epsilon(a, b, c, d) = \begin{cases} +1 & (a, b, c, d) \text{ positive} \\ -1 & (a, b, c, d) \text{ negative} \end{cases}
\]

In the case, where \((a, b, c, d)\) is positive, there are two possibilities for \(d\): Either, each \(d_j\) lies in between \(a_j\) and \(b_j\) or between \(b_j\) and \(c_j\). This corresponds to the cases of \((a, d, b)\) or \((b, d, c)\) being maximal. These two cases can be distinguished by the cross ratio as follows:

**Lemma 4.9.** If \((a, b, c)\) and \((a, d, b)\) are maximal, then \(0 < B_R(a, b, c, d) < 1\). If \((a, b, c)\) and \((b, d, c)\) are maximal, then \(B_R(a, b, c, d) > 1\).

**Proof.** The assumptions imply \(0 < [a_j : b_j : c_j : d_j] < 1\), respectively \([a_j : b_j : c_j : d_j] > 1\) for each \(j\), hence the lemma follows from the explicit formula in Corollary 4.8. \[\square\]

In particular we obtain:

**Corollary 4.10.** For every Euclidean Jordan algebra \(V\) with associated Shilov boundary \(\tilde{S}\) we have \(B_V((\tilde{S}^{4+})) = \mathbb{R} \setminus \{0, 1\}\).
Proof. Let \((a, b, c, d) \in \hat{S}^{(4+)}\). If \((a, b, c)\) is maximal, then depending on \(d\) we have either \(B_{\mathbb{R}}(a, b, c, d) < 0\) (if \((a, b, c, d)\) is negative) or \(B_{\mathbb{R}}(a, b, c, d) \geq 1\) (if \((a, b, c, d)\) is maximal). If \((a, b, c)\) is minimal one may argue similarly (or reduce to the former case by means of the cocycle properties to be proved below). This shows the inclusion \(\subset\). For the converse inclusion, it suffices to see that \(B_{\mathbb{R}}\) is onto \(\mathbb{R} \setminus \{0, 1\}\) and \(\mathbb{R}\) has a balanced embedding into every Euclidean Jordan algebra. □

The reason that the cross ratio is not onto \(\mathbb{R}\) is due to our choice of domain. In fact, we can extend \(B_{\mathbb{V}}\) continuously to the slightly larger domain \(\hat{S}^{4*}\) which is defined as follows: \(\hat{S}^{4*}\) contains \(\hat{S}(4+)\) and \((a, b, c, d)\) is contained in \(\hat{S}^{4*} \setminus \hat{S}(4+)\) iff
\[
(a = c) \lor (b = d) \lor (b = a) \lor (c = b)
\]
and moreover \(\{a, b, c, d\}\) contains a maximal triple. This implies in particular that at most two of \(a, b, c, d\) coincide and the non-coinciding pairs are transverse. Recall that the normalized kernel function \(k_{\mathbb{V}}\) extends to all of \(\hat{S}\) and satisfies \(k_{\mathbb{V}}(a, b) \neq 0\) iff \(a \not\sim b\). Thus
\[
B_{\mathbb{V}}(a, b, c, d) := \frac{k_{\mathbb{V}}(d, a)k_{\mathbb{V}}(b, c)}{k_{\mathbb{V}}(d, c)k_{\mathbb{V}}(b, a)}
\]
is well-defined as long as \(d \not\sim c\) and \(b \not\sim a\). In particular, \(B_{\mathbb{V}}\) extends continuously to \(\hat{S}^{4*}\). We denote this extension by the same letter. Then we have:

**Proposition 4.11.** The cross ratio \(B_{\mathbb{V}} : \hat{S}^{4*} \to \mathbb{R}\) is onto. Moreover,
\[
\begin{align*}
x = z \text{ or } y = t & \iff B_{\mathbb{V}}(x, y, z, t) = 1 \\
t = x \text{ or } z = y & \iff B_{\mathbb{V}}(x, y, z, t) = 0.
\end{align*}
\]

**Proof.** If \(x = z \text{ or } y = t\) then enumerator and denominator coincide, whence \(B_{\mathbb{V}}(x, y, z, t) = 1\). If \(y = x \text{ or } z = t\) then one of the two terms in the numerator vanishes. In any other case we have \((x, y, z, t) \in \hat{S}^{(4+)}\), hence \(B_{\mathbb{V}}(x, y, z, t) \notin \{0, 1\}\). This proves the converse direction. □

### 4.4. Cocycle properties.

Generalized cross ratios satisfy various cocycle properties. The key observation for the proof of this fact is the following lemma:

**Lemma 4.12.** If \(X\) is a set and \(k : X^2 \to \mathbb{C}^*\) is an arbitrary function then
\[
b : \left\{ \begin{array}{c}
X^4 \to \mathbb{C}^* \\
(a, b, c, d) \mapsto \frac{k(d, a)k(b, c)}{k(d, c)k(b, a)}
\end{array} \right.
\]
has the following properties:

\[
\begin{align*}
(22) & \quad b(a, b, c, d) = b(c, d, a, b) \\
(23) & \quad b(a, b, c, d) = b(a, b, c, e)b(a, e, c, d) \\
(24) & \quad b(a, b, c, d) = b(a, b, e, d)b(e, b, c, d)
\end{align*}
\]

**Proof.** We have
\[
b(c, d, a, b) = \frac{k(b, c)k(d, a)}{k(b, a)k(d, c)} = \frac{k(d, a)k(b, c)}{k(d, c)k(b, a)}
\]
Moreover,
\[ b(a, b, c, e)k(b, a) = b(c, d)k(b, a) \]
and
\[ b(a, b, e, d)k(b, e) = b(a, c)k(b, e) \]

Since the normalized kernel \( k \) is only partially defined, this does not directly apply. Still we have:

**Corollary 4.13.** The cross ratio \( B_V : S_{4^*} \to \mathbb{R} \) satisfies (22) \( \cdots \) (24) above, whenever both sides of the equation are well-defined.

**Proof.** Again we can easily reduce to the case, where \( V \) is simple. In this case we can apply Lemma 4.12 to the function \( k_V|_{D^2} \). By continuity, the relations extend to \( S^{(4)} \).

We also have the following property:

**Corollary 4.14.** For all \((a, b, c, d)\) \( \in S^{(4^*)} \) we have
\[ B_V(a, b, c, d) = B_V(b, a, d, c). \]

**Proof.** We first do a number of reduction steps. Since \( B_V(a, b, c, d) \) \( \in \mathbb{R} \) for \((a, b, c, d) \in S^{(4^*)}\) it suffices to show that
\[ \overline{B_V(a, b, c, d)} = B_V(b, a, d, c). \]

We shall prove the latter statement for \( \overline{B_V} \) and \((a, b, c, d) \in D^4 \). The statement for \((a, b, c, d) \in \overline{S}^{(4^+)} \) will then follow by continuity. Using Property (iii) of Theorem 4.2 we may assume that \( V \) is simple. Moreover, since the desired symmetry is preserved by passing to a power, it suffices to show that
\[ \overline{B_V,det}(a, b, c, d) = B_V,det(b, a, d, c) \quad ((a, b, c, d) \in D^4), \]

By [13] we have
\[ K^*(z, w) = K(w, z) \]
and thus
\[ \overline{k_{det}(z, w)} = k_{det}(w, z) \]
for all \( w, z \in D \). In particular, for \((a, b, c, d) \in D^4 \) we deduce
\[ \overline{B_V,det}(a, b, c, d) = \frac{k_{det}(d, a) \cdot k_{det}(b, c)}{k_{det}(d, c) \cdot k_{det}(b, a)}. \]
4.5. The Bergman cross ratio and the proof of Theorem 1.1.\footnote{\[1.1\]} Given any bounded symmetric domain we can find a Euclidean Jordan algebra $V$ such that $D = D_V$. We then obtain a generalized cross ratio $B_V$ on the Shilov boundary $\hat{S}$ of $D$. The Jordan algebra $V$, however, is not unique; it corresponds to a choice of a marking on the bounded symmetric domain $D$. In order to deduce Theorem 1.1 from Theorem 4.2 we need to show that $B_V$ is actually independent of this choice and depends only on $D$. Now for any complex domain $\mathcal{C}$ the corresponding Bergman kernel $K_C$ can be used in order to define a Bergman cross ratio

$$B_C : \mathcal{C}^4 \rightarrow \mathcal{C}^\times, \quad (x, y, z, t) \mapsto \frac{K_C(t, x)^{-1}K_C(y, z)^{-1}K_C(t, x)K_C(y, z)}{K_C(t, z)^{-1}K_C(y, x)^{-1}K_C(t, x)K_C(y, z)};$$

in the case of a bounded symmetric domain $D$ associated with a Euclidean Jordan algebra $V$ Proposition 3.10 yields $B_V = B_D$, where $B_V$ is the extension of $B_V$ as in \[17\]. In particular, $B_D$ determines $B_V$ on $D^4$, and thus by continuity also $B_V$. This shows that $B_V$ depends only on $D$ and thus proves Theorem 1.1. We can also use the above description of $B_V$ in terms of the Bergman kernel in order to obtain the following invariance property of $B_V$:

**Proposition 4.15.** Let $\mathcal{C}$ be a complex domain and $c : D \rightarrow \mathcal{C}$ be a biholomorphism. Then for all $(x, y, z, t) \in D^4$ we have

$$\tilde{B}_V(x, y, z, t)^{rk V} = B_C(c(x), c(y), c(z), c(t)).$$

**Proof.** Since $\tilde{B}_V(x, y, z, t)^{rk V} = B_D(x, y, z, t)$ it suffices to show that $B_D(x, y, z, t) = B_C(c(x), c(y), c(z), c(t))$.

For this we apply \[14\] Prop. IX.2.4\] in order to relate the Bergman kernels on $D$ and $\mathcal{C}$. Denote by $J_c$ the complex Jacobi matrix of $c$. Then

$$K_D(z, w) = K_C(c(z), c(w)) \det c(J_c(z)) \det c(J_c(w)).$$

Hence

$$B_D(x, y, z, t) = \frac{K_D(t, z)K_D(y, x)}{K_D(t, x)K_D(y, z)} = \frac{K_C(c(t), c(z)) \det c(J_c(t)) \det c(J_c(z))}{K_C(c(t), c(x)) \det c(J_c(t)) \det c(J_c(x))} = \frac{K_C(c(y), c(z)) \det c(J_c(y)) \det c(J_c(z))}{K_C(c(y), c(x)) \det c(J_c(y)) \det c(J_c(x))}.$$
5. Maximal representations, limit curves and strict cross ratios

In this section we explain how the generalized cross ratio functions defined above can be used to associate with any maximal representation into a Hermitian group of tube type a strict cross ratio in the sense of [17]. We then prove that all strict cross ratios are equivalent in a sense made precise in Proposition 5.8. The latter result is essentially contained, but not stated in [17], where it is presented in the context of a more general theory of adapted flows. The proof we give here avoids this machinery and tries to be more elementary.

5.1. The cross ratio of a maximal representation. Returning to the problem of the introduction, let Σ be a closed, oriented surface of genus \( g \geq 2 \) with fundamental group \( \Gamma \). We fix a hyperbolization of Σ, i.e. a faithful homomorphism \( \Gamma \to PU(1,1) \) with discrete image so that \( \Sigma = \Gamma\backslash \mathbb{D} \). We also fix a Euclidean Jordan algebra \( V \) and denote by \( D \) and \( \check{S} \), respectively the associated bounded symmetric domain and Shilov boundary. The groups \( G, K, L, Q_+ \) are defined as before. The aim of this section is to construct an invariant associated with a maximal representation \( \varrho : \Gamma \to G \). Our basic references concerning maximal representations are [10] and [8]. Let us briefly recall the main definitions: Denote by \( \omega_D \) the Kähler form on \( D \) associated with the metric of minimal holomorphic sectional curvature \(-1\).

Given an arbitrary \( \rho \)-equivariant map \( f : D \to D \) we define the Toledo invariant \( T_\varrho \) of \( \rho \) by

\[
T_\varrho := \frac{1}{2\pi} \int_{\Sigma} f^* \omega_D.
\]

This does not depend on the choice of \( f \). The Toledo invariant satisfies a generalized Milnor-Wood inequality, in the present normalization given by

\[
|T_\varrho| \leq |\chi(\Sigma)| \cdot \text{rk}(V).
\]

Accordingly, the representation \( \varrho \) is called maximal if \( T_\varrho = |\chi(\Sigma)| \cdot \text{rk}(V) \). Every maximal representation \( \varrho : \Gamma \to G \) defines a canonical limit curve, i.e. an injective continuous \( \varrho \)-equivariant map \( \varphi : S^1 \to \check{S} \). Since these limit curves play a crucial role in our construction and for lack of suitable references, we briefly digress to explain their construction:

First consider a maximal representation \( \varrho : \Gamma \to G \) with Zariski-dense image. In this case, a \( \varrho \)-equivariant injective map \( \varphi : S^1 \to \check{S} \) is constructed in [7, Ch.7], and by [8] this map is continuous. We claim that \( \varphi \) is in fact the unique limit curve \( \varphi : S^1 \to \check{S} \) for \( \varrho \). Indeed, \( S^1 \) is the closure of any of its \( \Gamma \)-orbit, and since \( \varphi \) is continuous and \( \varrho \)-equivariant, this implies that \( \varphi(S^1) \) is a minimal closed \( \Gamma \)-invariant subset of \( \check{S} \). Now the first lemma in Section 3.6 of [1] applies to show that \( \varphi(S^1) = \Lambda \Gamma \), where the latter denotes the limit set of \( \Gamma \) in \( \check{S} \). This implies in particular, that any other limit curve \( S^1 \to \check{S} \) has to intersect \( \varphi \). The uniqueness of \( \varphi \) then follows from the following lemma:

**Lemma 5.1.** Let \( \varphi_1, \varphi_2 : S^1 \to \check{S} \) be two limit curves for the same maximal representation \( \varrho \) with \( \varphi_1(S^1) \cap \varphi_2(S^1) \neq \emptyset \). Then \( \varphi_1 = \varphi_2 \).
Proof. By equivariance of the \( \varphi_j \) the intersection contains a \( \Gamma \)-orbits, but since the \( \Gamma \)-action on \( S^1 \) is minimal this implies that this preimage is the full circle and thus \( \varphi_1(S^1) = \varphi_2(S^1) \). Every \( \gamma \in \Gamma \) has a unique attractive fixed point \( \gamma^+ \) in \( S^1 \). By equivariance, this is mapped under both \( \varphi_j \) to the unique attractive fixed point of \( \rho(\gamma) \) in \( \varphi_1(S^1) = \varphi_2(S^1) \). We deduce \( \varphi_1(\gamma^+) = \varphi_2(\gamma^+) \) for all \( \gamma \in \Gamma \) and since \( \{ \gamma^+ | \gamma \in \Gamma \} \) is dense in \( S^1 \) we deduce that \( \varphi_1 = \varphi_2 \).

Thus we have proved:

**Corollary 5.2.** If \( g : \Gamma \to G \) is a Zariski-dense maximal representation, then there exists a unique limit curve \( \varphi : S^1 \to \hat{S} \) for \( g \).

If we drop the assumption of Zariski-dense image, then Corollary 5.2 fails. Indeed, if \( \varphi \) is a limit curve for \( g \) and \( g \in C_G(\varrho(\Gamma)) \), then \( g \varphi \) is again a limit curve for \( g \). We therefore cannot hope for a unique limit curve in general. Still, there is an explicit construction for a canonical limit curve, which we now describe: For this let \( \varrho : \Gamma \to G \) be an arbitrary maximal representation, and denote by \( H \) the Zariski-closure of \( \varrho(\Gamma) \) in \( G \). It follows from [10] Thm. 5] that \( H \) is of tube type. Then \( g \) factors as \( g = \iota \circ g' \), where \( \iota : H \to G \) is the inclusion and \( g' : \Gamma \to H \) is a maximal representation with Zariski dense image. In particular, there is a unique limit curve \( \varphi' : S^1 \to \hat{S}_H \) for \( g' \). In order to extend this limit curve to a limit curve for \( g \) we observe that by the general structure theory of maximal representations developed in [10], the embedding \( \iota \) is tight in the sense of [11]. Then [11] Thm 4.1 implies that a \( \iota \)-equivariant map \( \tilde{\gamma} : \hat{S}_H \to \hat{S}_G \) can be constructed as follows: Denote by \( D_H \) and \( D_G \) the symmetric spaces of \( H \) and \( G \) respectively and choose a basepoint \( p \in D_G \). Then \( \text{Stab}_H(p) \) and hence \( \iota(\text{Stab}_H(p)) \) are compact, whence \( \iota(\text{Stab}_H(p)) \) is contained in a maximal compact subgroup of \( G \). We thus find \( x \in G \) with \( \text{Stab}_G(x) \supset \iota(\text{Stab}_H(p)) \). Given \( p \) and \( x \) we define a \( \iota \)-equivariant continuous, totally geodesic map

\[
\varphi : S^1 \to \hat{S}, \quad g \cdot p \mapsto \iota(g)x.
\]

Now every point \( \xi \in \hat{S}_H \) can be represented uniquely by a geodesic ray \( \sigma \) emanating from \( p \). Then \( \varphi \circ \sigma \) is a geodesic ray in \( D_G \), which by the results in [11] Chapter 4] ends in some point \( \tilde{\gamma}(\xi) \in \hat{S}_G \). This yields the desired continuous \( \iota \)-equivariant embedding \( \tilde{\gamma} : \hat{S}_H \to \hat{S}_G \), and we can define the limit curve \( \varphi \) of \( g \) by \( \varphi := \tilde{\gamma} \circ \varphi' \).

A priori the above construction of \( \tilde{\gamma} \) depends on the choices of basepoints \( p \) and \( x \). In order to obtain a canonical limit curve \( \varphi \) we have to show that \( \tilde{\gamma} \) is a posteriori independent of these choices. We show the independence of \( x \) first: Suppose \( y \in D_G \) is another base point with \( \text{Stab}_G(y) \supset \iota(\text{Stab}_H(p)) \). Then for any \( g \in G \) we have

\[
d(\iota(g)x, \iota(g)y) = d(x, y).
\]

Since the geodesics through \( p \) in \( D_H \) are of the form \( \sigma(t) = \exp(tX)p \) this implies that the images of a geodesic in \( D_H \) under the two maps \( f_x \) and \( f_y \) corresponding to \( x \) and \( y \) respectively are at bounded distance, hence define the same point in the Shilov boundary. This shows that \( \tilde{\gamma} \) is independent of the choice of \( x \) once \( p \) is given. Now let us prove independence of \( p \): For this we suppose that two points \( p, p' \in D_H \) are given. We then find \( h_0 \in H \) such that \( p' = h_0p \). Now let \( x \in D_G \) such that \( \text{Stab}_G(x) \supset \iota(\text{Stab}_H(p)) \). Then \( p \) and \( x \) determine a map \( f : D_H \to D_G \)
by (26). Similarly, given $x' \in D_G$ with $\text{Stab}_G(x') \supset \iota(\text{Stab}_H(p'))$ we obtain a map $f' : D_H \to D_G$ by $f'(h_{p'}) := hx'$. We have to show that $f$ and $f'$ induce the same map $\hat{\iota}$ of Shilov boundaries. By the previous considerations we may choose $x' \in D_G$ arbitrary subject to the condition $\text{Stab}_G(x') \supset \iota(\text{Stab}_H(p'))$; in particular we may chose $x' = \rho(h_0).x$. But with this choice of $x'$ the maps $f$ and $f'$ coincide. This proves that $\hat{\iota}$ and consequently $\varphi$ do not depend on any of the choices made to define them. We may thus refer to $\varphi$ as the canonical limit curve of $\varphi$. This canonical limit curve has the following key property:

**Proposition 5.3** (Burger-Iozzi-Wienhard, [9]). Let $\varphi : \Gamma \to G$ be a maximal representation and

$$
\varphi : S^1 \to \hat{S}
$$

the canonical limit curve of $\varphi$. Then for every positively/negatively oriented triple $(x, y, z) \in (S^1)^3$, the triple $(\varphi(x), \varphi(y), \varphi(z))$ is maximal/minimal.

We deduce from the proposition that two distinct points $x \neq y \in S^1$ are mapped to transverse points under every limit curve. As another consequence of Proposition 5.3 we deduce that $\varphi$ induces an injective map

$$
\varphi^{(4)} : (S^1)^{4*} \to S^4_V, \quad (x, y, z, t) \mapsto (\varphi(x), \varphi(y), \varphi(z), \varphi(t)),
$$

where

$$(S^1)^{4*} := \{(x, y, z, t) \in (S^1)^4 \mid x \neq t, y \neq z\}.$$

**Definition 5.4.** Let $\rho : \Gamma \to G$ be a maximal representation and $\varphi : S^1 \to \hat{S}$ the associated canonical limit curve. Then the function

$$
b_\rho := (\varphi^{(4)})^* B_V : (S^1)^{4*} \to \mathbb{R}, \quad (x, y, z, t) \mapsto B_V(\varphi(x), \varphi(y), \varphi(z), \varphi(t)).
$$

is called the cross ratio of the maximal representation $\varphi$.

The main properties of cross ratios of maximal representations are collected in the following theorem:

**Theorem 5.5.** The cross ratio $b_\rho : (S^1)^{4*} \to \mathbb{R}$ is a continuous $\Gamma$-invariant function satisfying the following properties:

\begin{align*}
(27) \quad b_\rho(x, y, z, t) &= b_\rho(z, t, x, y) \\
(28) \quad b_\rho(x, y, z, t) &= b_\rho(x, y, z, w)b_\rho(x, w, z, t) \\
(29) \quad b_\rho(x, y, z, t) &= b_\rho(x, y, w, t)b_\rho(w, y, z, t) \\
(30) \quad x = z \text{ or } y = t &\quad \iff \quad b_\rho(x, y, z, t) = 1 \\
(31) \quad t = x \text{ or } z = y &\quad \iff \quad b_\rho(x, y, z, t) = 0
\end{align*}

**Proof.** $\Gamma$-invariance on $(S^1)^{(4)}$ follows from $G$-invariance of $B_V$ on $S^{(4+)}$ (i.e. Property (i) of Theorem 4.2). By continuity we obtain $\Gamma$-invariance on all of $(S^1)^{4*}$. By a similar extension argument, Properties (27)-(29) follow from Corollary 4.13. Finally, Properties (30)-(31) follow from Proposition 4.11. \qed

In fact, it follows from Corollary 4.14 that $b_\rho$ also satisfies

$$
b_\rho(x, y, z, t) = b_\rho(y, x, t, z).
$$

However, we are not going to use this property in the sequel. Instead let us focus on the properties mentioned in Theorem 5.5. In the language of [17] the theorem
says precisely that \( b_+ \) is a strict cross ratio. In the next subsection we shall recall some general properties of such strict cross ratios.

### 5.2. Strict cross ratios

In this subsection we collect basic properties of strict cross ratios, i.e. continuous \( \Gamma \)-invariant functions on \((S^1)^4^*\) satisfying (27)-(31) above. The material is adapted from [17]. Let \( b : (S^1)^4^* \to \mathbb{R} \) be any strict cross ratio. We will occasionally use the following two cocycle identities:

\[
\begin{align*}
(32) & \quad b(x, y, x, t) = b(x, y, z, t) b(z, y, x, t) = 1 \\
(33) & \quad \log b(x, y, z, t) = -\log b(z, y, x, t)
\end{align*}
\]

The former is an immediate consequence of (30) and (29) and the latter follows by applying the logarithm. We will usually consider \( x, y \) fixed and study \( g(t) = b(x, y, z, t) \) as a function of \( t \). Let us assume that \((x, y, z)\) is positively oriented. We then divide the circle into three open disjoint intervals \( I_1 = (x, y), I_2 = (y, z) \) and \( I_3 = (z, x) \) so that

\[
S^1 = \{ x \} \cup I_1 \cup \{ y \} \cup I_2 \cup \{ z \} \cup I_3.
\]

The function \( g \) is then defined on \((S^1 \setminus \{ z \})\). By Axiom (31), \( x \) is the only zero of \( g \). Since \( g(y) = 1 \) is positive, \( g \) is positive on \( I_u := I_1 \cup \{ y \} \cup I_2 \). Let \( a, b \in I_u \) such that \( g(a) = g(b) \). Then we have:

\[
1 = g(a)g(b)^{-1} = b(x, y, z, a)b(x, y, z, b)^{-1} = b(-1, a, 1, b),
\]

but by Axiom (30) this can only be the case if \( a = b \). Hence \( g|_{I_u} \) is injective. Furthermore, by Axiom (31) and (32) we have

\[
\lim_{t \to z} g(t) = \lim_{t \to z} b(z, y, x, t)^{-1} = +\infty.
\]

Since \( g(x) = 0 \) the intermediate value theorem implies that \( g|_{I_u} \) is surjective and therefore defines a homeomorphism between \( I_u \) and \((0, \infty)\). Now consider the case \( t \in I_3 \). We claim that \( g(t) \) is negative on \( I_3 \). For this we first observe that as above

\[
\lim_{t \to z} g(t) \in \{ \pm \infty \}.
\]

Again, since \( g(x) = 0 \) the intermediate value theorem implies that \( g \) maps \( I_3 \) homeomorphically to either \((0, \infty)\) or \((-\infty, 0)\). However, the former would imply the existence of \( t_0 \in I_3 \) with \( g(t_0) = 1 \), which contradicts Axiom (30). Hence

\[
\lim_{t \to z} g(t) = -\infty
\]

and \( g \) maps \( I_3 \) homeomorphically to \((-\infty, 0)\). We have proved:

**Proposition 5.6.** If \((x, y, z)\) is a positively oriented triple on \(S^1\) and \( b : (S^1)^4^* \to \mathbb{R} \) is a strict cross ratio, then

\[
g : S^1 \setminus \{ z \} \to \mathbb{R}, \quad t \mapsto b(x, y, z, t)
\]

is a homeomorphism with \( g(x) = 0 \).

Notice that as a homeomorphism \( S^1 \setminus \{ z \} \to \mathbb{R} \) the function \( g \) is automatically monotonous. In the sequel we denote by

\[
(S^1)^3^* := \{(x, y, z) \in (S^1)^3 \mid (x, y, z) \text{ positively ordered}\}
\]

the set of positively ordered triples.
Corollary 5.7. For every strict cross ratio there exists a $\Gamma$-equivariant continuous map
\[ \psi : \mathbb{R} \times (S^1)^3+ \to S^1 \]
such that $\log b(x, y, z, \psi_s(x, y, z)) = s$. 

Proof. Put $\psi_s(x, y, z) := (\log g)^{-1}(s)$. \qed

Proposition 5.8 (Labourie [17]). Let $b_1$ and $b_2$ be strict cross ratios. Then there exists a $C > 0$ such that
\[ C^{-1} \leq \frac{|\log b_1|}{|\log b_2|} \leq C \]

Proof. We adapt an argument of Labourie [17], Chapter 2: Let $\psi^1_s$ and $\psi^2_s$ be maps associated to $b_1$ and $b_2$ by means of Corollary 5.7. Define a function $T : (S^1)^3+ \to \mathbb{R}$ by
\[ T(x, y, z) := b_1(x, y, z, \psi^1_s(x, y, z)) \]
This map is positive and continuous. Since $\psi^2_s$ is $\Gamma$-equivariant, $T$ is $\Gamma$-invariant. Furthermore it satisfies $\psi^2_s(x, y, z) = \psi^1_s T(x, y, z)(x, y, z)$. Since $(S^1)^3+ / \Gamma$ is compact, $T$ has a global maximum $A$. For $t \in S^1$ there exists $s, \tilde{s} \in \mathbb{R}$ such that
\[ t = \psi^1_s(x, y, z) = \psi^2_s(x, y, z). \]
If $|s| \in [n, n+1)$ for $n \in \mathbb{N}$, then we have by definition of $A$ that $|\tilde{s}| \in (0, A(n+1)) \subset (0, 2A|s|)$, hence $|\tilde{s}| \leq 2A|s|$. We can calculate:
\[ |\log b_1(x, y, z, t)| = |\log b_1(x, y, z, \psi^1_s(x, y, z))| = |\tilde{s}| \leq 2A|s| = 2A|\log b_2(x, y, z, \psi^2_s(x, y, z))| = 2A|\log b_2(x, y, z, t)|. \]
This proves the upper bound for $(x, y, z) \in (S^1)^3+$. If $(x, y, z)$ is negatively oriented, then $(z, y, x)$ is in $(S^1)^3+$. The upper bound for this case follows from the fact that:
\[ |\log b(x, y, z, t)| = |\log b(z, y, x, t)|. \]
The lower bound is obtained by reversing the roles of $b_1$ and $b_2$. \qed

We will apply this in the following form:

Corollary 5.9. Let $\rho : \Gamma \to G$ be a maximal representation with associated cross ratio $b_\rho$. Then there exists $C > 0$ such that for all $(x, y, z, t) \in (S^1)^4$,
\[ |\log b_\rho(x, y, z, t)| \geq C \cdot |\log[x : y : z : t]|. \]

6. Well-displacing and quasi-isometry property

We keep the notation introduced in the last section. In particular, $\Sigma$ denotes a closed, oriented surface of genus $g \geq 2$ with fundamental group $\Gamma$ and fixed hyperbolization $\Gamma \to PU(1, 1)$, while $V$ denotes a Euclidean Jordan algebra with bounded symmetric domain $\mathcal{D}$, Shilov boundary $\check{\mathcal{S}}$ and associated group $G$. We fix a maximal representation $\varphi : \Gamma \to G$ and denote by $\varphi : S^1 \to \check{\mathcal{S}}$ its canonical limit curve. Recall from the introduction that for $\gamma \in \Gamma$ the translation length of $\rho(\gamma)$ is defined by
\[ \tau_{\mathcal{D}}(\rho(\gamma)) = \inf_{x \in \mathcal{D}} d(\rho(\gamma)x, x). \]
Since $\Sigma$ is smooth and closed, $\gamma$ cannot be elliptic or parabolic respectively, hence must be hyperbolic. We denote by $\gamma^+ \in S^1$ the attractive, and by $\gamma^-$ the repulsive fixed point of $\gamma$ in $S^1$. The virtual translation length of $\rho(\gamma)$ is defined by

$$\tau_D^\infty(\rho(\gamma)) = \log b_{\rho}(\gamma^-,\xi,\gamma^+,\gamma\xi)$$

for $\xi \in S^1 \setminus \{\gamma^\pm\}$. We will see below that this does not depend on the choice of $\xi$. The main result in this section is Theorem 6.17 which says that there exists a constant $C > 0$ such that

$$\tau_D(\rho(\gamma)) \geq C \tau_D^\infty(\rho(\gamma)).$$

From the resulting inequality we deduce that $\rho$ is well-displacing and that the corresponding action on $D$ is quasi-isometric. This implies in particular the desired properness of the mapping class group on Hermitian higher Teichmüller spaces.

Before we prove Inequality (34) we discuss a special case, where we have equality and $C = 1$.

6.1. **Fuchsian representations.** A maximal representation $\varrho : \Gamma \to G$ is called **Fuchsian** if it factors as $\varrho : \Gamma \xrightarrow{\hat{\varrho}} PU(1,1) \xrightarrow{t} G,$

where $\hat{\varrho} : \Gamma \to PU(1,1)$ is some fixed hyperbolization and $t : PU(1,1) \to G$ is some embedding. It then follows automatically from maximality of $\varrho$ that the embedding $t$ is tight in the sense of [11]. Now we claim:

**Proposition 6.1.** If $\varrho : \Gamma \to G$ is a Fuchsian representation then

$$\tau_D^\infty(\varrho(\gamma)) = \tau_D(\varrho(\gamma)).$$

The proposition will be an immediate consequence of the following three lemmas:

**Lemma 6.2.** For all $\gamma \in \Gamma$ we have

$$\tau_D(\gamma) = \tau_D^\infty(\gamma).$$

**Proof.** Observe that both sides of the claimed equality are invariant under $PSL_2(\mathbb{R})$ and Cayley transform. Conjugating $\gamma$ with the Cayley transform and some $g \in PSL(2,\mathbb{R})$, we can transform the whole situation to the upper half plane model $\mathbb{H}$ and we may assume without loss of generality that $\gamma$ acts by translation on the imaginary axis, i.e. the attractive fixpoint $\gamma^+$ is $\infty$, while the repulsive fix point $\gamma^-$ is 0. Then $\gamma$ can be written as

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

with $a > 1$ and we have

$$\tau_D(\gamma) = d(i, \gamma i).$$

Using the relation between the hyperbolic distance and the classical cross ratio on $\mathbb{H}$ we obtain

$$\tau_D(\gamma) = d(i, \gamma i) = \log[i : 0 : \gamma i : \infty] = \log \frac{\gamma i - 0}{i - 0} = \log \frac{ia^2}{i} = \log a^2.$$
through $\gamma i$ and intersects the real axis in the points $\gamma(1) = a^2$ and $\gamma(-1) = -a^2$. Now we have a quadruple of points in $\mathbb{R} \cup \{\infty\}$ and we calculate their cross ratio:

$$\tau_{\mathbb{D}^\infty}(\gamma) = \log[0 : 1 : \infty : \gamma 1] = \log \frac{a^2}{1} = \log a^2 = \tau_{\mathbb{D}}(\gamma).$$

Lemma 6.3. For all $\gamma \in \Gamma$ we have

$$\tau_{\mathbb{D}^\infty}(\varphi(\gamma)) = \tau_{\mathbb{D}^\infty}(\hat{\varphi}(\gamma))$$

Proof. The Lie group homomorphism $t : PU(1,1) \to G$ corresponds to a morphism $\alpha : \mathbb{R} \to V$ of the associated Jordan algebras. By Example 3.14 this morphism is balanced. Now denote by $\varphi$ and $\hat{\varphi}$ the canonical limit curve of $\varphi$ and $\hat{\varphi}$ respectively. Then $\varphi = \alpha C \circ \hat{\varphi}$ and thus Property (ii) of Theorem 4.2 implies that the generalized cross ratios associated to $\varphi$ and $\hat{\varphi}$ are the same. This in turn shows that the corresponding virtual translation lengths are the same.

Lemma 6.4. For all $\gamma \in \Gamma$ we have

$$\tau_{\mathbb{D}}(\varphi(\gamma)) = \tau_{\mathbb{D}}(\hat{\varphi}(\gamma))$$

Proof. Let $\gamma \in \Gamma$. Then $\gamma$ is hyperbolic and thus there exists a geodesic $\sigma$ in $\mathbb{D}$ such that $\gamma$ fixes $\sigma$ and

$$\hat{\varphi}(\gamma) \cdot \sigma(s) = \sigma(s + \tau_{\mathbb{D}}(\hat{\varphi}(\gamma))) \quad (s \in \mathbb{R}).$$

Now there exists a $t$-equivariant totally geodesic embedding $t^1 : \mathbb{D} \to \mathbb{D}$. If $\sigma^1$ denotes the image of $\sigma$ under $t^1$ then $\varphi(\gamma)$ fixes $\sigma^1$ and

$$\varphi(\gamma) \cdot \sigma^1(s) = \sigma^1(s + \tau_{\mathbb{D}}(\hat{\varphi}(\gamma))) \quad (s \in \mathbb{R}).$$

By [6] Thm II.6.8] this implies the lemma.

6.2. Bounding the translation length from below. In this subsection we return to the general case of an arbitrary maximal representation $\varphi : \Gamma \to G$ of $\Gamma$ into a Hermitian group of tube type. As a first step towards our main inequality we will bounded the translation length of $\rho(\gamma)$ from below in terms of some auxiliary data. This auxiliary data will then be related to the virtual translation length in the next subsection. We will use the fact that for $\gamma \in \Gamma$ the element $\varphi(\gamma)$ is contained in the Levi factor of a certain Shilov parabolic. We will transport the whole situation into the Levi factor $L$, which has a linear action on the symmetric cone $\Omega$ of the underlying Jordan algebra $V$. Using this linear action, we can estimate the dynamics of $\varphi(\gamma)$ using the reductive symmetric space of $GL(V)$.

We introduce the notations $g_0 := \rho(\gamma)$ and $g_0^\pm := \varphi(\gamma^\pm)$. Since $G$ is transitive on $S^{(2)}$, there exists $h_1 \in G$ such that $h_1 g_0^\pm = \pm e$. Set $g_1 := h_1 g_0 h_1^{-1}$. Since $g_0$ fixes $g_0^\pm \in S$, we see that $g_1$ fixes $\pm e$, whence Corollary 2.3 applies and we find $M \in \mathbb{N}$ such that $g_2 := (c \circ g_1 \circ c^{-1})^M \in G(\Omega)$. We will use $g_2$ to estimate $\tau_{\mathbb{D}}(\varphi(\gamma))$; for this we need the following lemma:

Lemma 6.5. Let $X$ be a metric space and $g \in Is(X)$ an isometry. Then for any $M \in \mathbb{N}$:

$$\tau_X(g) \geq \frac{1}{M} \tau_X(g^M).$$
Proof. We have:
\[ d(g^M x, x) \leq \sum_{i=1}^{M} d(g^i x, g^{i-1} x) = M \cdot d(gx, x). \]
Taking the infimum on both sides finishes the proof.

Since the Cayley transform is isometric we obtain
\[ \tau_D(\rho(\gamma)) = \tau_D(g_0) = \inf_{x \in D} d(g_0 x, x) = \inf_{x \in D} d(g_0 h_1^{-1} x, h_1^{-1} x) \]
\[ = \inf_{x \in D} d(h_1 g_0 h_1^{-1} x, x) = \tau_D(g_1) = \tau_{T\Omega}(c \circ g_1 \circ c^{-1}) \]
\[ \geq \frac{1}{M} \tau_{T\Omega}(g_2). \]
and thus
\[ (37) \quad \tau_D(\rho(\gamma)) \geq \frac{1}{M} \cdot \tau_{T\Omega}(g_2). \]

Lemma 6.6. Let \( h \in L \). Then for all \( z \in T\Omega \) we have
\[ d(z, h z) \geq d(i\text{Im}(z), h(i\text{Im}(z))). \]

Proof. Since \( T\Omega \) is an open subset of \( V^C \), we can identify the tangent space of \( T\Omega \) at any point \( z \in T\Omega \) canonically with \( V^C \). Using this identification, the Hermitian metric \( H \) on \( T\Omega \) admits the following description (see [15, Prop. X.1.3]): Let \( n := \dim V \), \( r := \text{rk}(V) \). Then given \( z \in T\Omega \) and \( a, b \in V^C \) we have
\[ H_z(a, b) = \left( \frac{2n}{r} P \left( \frac{z - \bar{z}}{i} \right)^{-1} a \bigg| b \right) = \left( \frac{2n}{r} P (2\text{Im}(z))^{-1} a \bigg| b \right) = H_{\text{Im}(z)}(a, b). \]
In other words, translation in the direction of the real axis is isometric for \( H \). We have \( H = g + i\omega \), where \( g \) is the Riemannian metric on \( T\Omega \) and \( \omega \) is the Kähler form. In particular, since \( \omega \) is skew-symmetric, we have for all \( z \in T\Omega \) and all \( a \in V^C \) the equality
\[ g_z(a, a) = H_z(a, a) = H_z(\text{Re}(a), \text{Re}(a)) + H_z(i\text{Im}(a), i\text{Im}(a)) \]
\[ = g_z(\text{Re}(a), \text{Re}(a)) + g_z(i\text{Im}(a), i\text{Im}(a)). \]
In particular,
\[ g_z(a, a) \geq g_z(i\text{Im}(a), i\text{Im}(a)) = g_{i\text{Im}(z)}(i\text{Im}(a), i\text{Im}(a)). \]
Thus, given any path \( \sigma : [0, 1] \to T\Omega \) with \( \sigma(0) = z \), \( \sigma(1) = h z \) we have
\[ l(\sigma) = \int_0^1 \sqrt{g_{\sigma(t)}(\dot{\sigma}(t), \dot{\sigma}(t))} dt \]
\[ \geq \int_0^1 \sqrt{g_{i\text{Im}(\sigma(t))}(i\text{Im}(\dot{\sigma}(t)), i\text{Im}(\dot{\sigma}(t)))} dt \]
\[ = l(i\text{Im}(\sigma(t))). \]
Observe that \( i\text{Im}(\sigma(t)) \) is a path joining \( i\text{Im}(z) \) and \( i\text{Im}(h z) = h(i\text{Im}(z)) \). (The latter equality is the only place where we use the special form of \( h \in L \).) Passing to the infimum over all \( \sigma \), we obtain the lemma.

Specializing to \( h = g_2 \) and passing to the infimum over all \( z \in T\Omega \) we deduce:
Corollary 6.7. Let $g_2 \in G(\Omega)$. Then
\[ \tau_{G(\Omega)}(g_2) = \tau_{\Omega}(g_2). \]

Observe that $\Omega$ is the reductive symmetric space associated with $L$. In particular, the inclusion $L \to GL(V)$ induces an isometric embedding of $\Omega$ into the reductive symmetric space $\mathcal{P}(V)$ of $GL(V)$, which happens to be the space of all positive definite symmetric matrices of size $\dim(V)$. A detailed exposition of $\mathcal{P}(V)$ is provided in [6, Ch. II.10]. We will use the results described therein without further reference. We will need the following result concerning $\mathcal{P}(V)$:

Lemma 6.8. Let $g \in GL(V)$. Then
\[ \tau_{\mathcal{P}(V)}(g) \geq \frac{1}{\sqrt{\dim V}} |\log \det(g)|^2. \]

Proof. Let $p \in \mathcal{P}(V)$ and $c : [0, d(p, gp)] \to \mathcal{P}(V)$ a unit speed geodesic joining $p$ with $gp$. Then there exists $h \in GL(V)$ such that $p = hh^\top$ and an element $X$ of norm 1 in the Lie algebra of $GL(V)$ such that $c(t) = h \exp(tX)h^\top$. Moreover, $gp = ghh^\top g^\top$. Since the $d(p, gp) = gp$ we have
\[
\begin{align*}
\exp(d(p, gp) \cdot X)h^\top &= ghh^\top g^\top \\
\Rightarrow \quad \det(h \exp(d(p, gp) \cdot X)h^\top) &= \det(ghh^\top g^\top) \\
\Rightarrow \quad \exp(d(p, gp) \cdot \tr(X)) &= \det(g)^2 \\
\Rightarrow \quad \exp(d(p, gp) \cdot \tr(X)) &= \exp(\log \det(g)^2)
\end{align*}
\]

Since both $d(p, gp) \cdot \tr(X)$ and $\log \det(g)^2$ are real this implies
\[ d(p, gp) \cdot \tr(X) = \log \det(g)^2, \]
whence
\[ d(p, gp) \cdot |\tr(X)| = |\log \det(g)|^2. \]

Now observe that
\[ |\tr(X)| = |(X|1)| \leq \|X\||1\| = 1 \cdot \sqrt{\dim V} = \sqrt{\dim V}. \]

Inserting into (38) we obtain
\[ d(p, gp) \geq \frac{1}{\sqrt{\dim V}} |\log \det(g)|^2. \]

Passing to the infimum over all $p \in \mathcal{P}(V)$ we obtain the lemma. \qed

Remark 6.9. We want to apply Lemma 6.8 to the element $g_2$ defined above. We recall that by construction $g_2$ has 0 as a repulsive fixed point (since $g_1$ has $-e$ as a repulsive fixed point) at the boundary at infinity. This implies that $\det(g_2) \geq 1$ and hence $\log \det(g)^2 \geq 0$. In particular, the estimate in Lemma 6.8 becomes
\[ \tau_{\mathcal{P}(V)}(g_2) \geq \frac{1}{\sqrt{\dim V}} \log \det(g_2)^2, \]
when specialized to $g_2$. 
Corollary 6.10. For all \( \gamma \in \Gamma \),
\[
\tau_\mathcal{D}(\rho(\gamma)) \geq \frac{1}{M \cdot \sqrt{\dim V}} \cdot \log \det(g_2)^2.
\]

Proof. Combining (37), Corollary 6.7 and Lemma 6.8 (in combination with Remark 6.9) we obtain
\[
\tau_\mathcal{D}(\rho(\gamma)) = \tau_\mathcal{D}(g_0) \geq \frac{1}{M} \tau_{\mathcal{V}}(g_2) = \frac{1}{M} \tau_{\Omega}(g_2) \geq \frac{1}{M} \tau_{\mathcal{V}}(V(g_2)) \geq \frac{1}{M \cdot \sqrt{\dim V}} \cdot \log \det(g_2)^2.
\]

6.3. Bounding the virtual translation length from above. We keep the notations from the last subsection. In particular, given \( \gamma \in \Gamma \) the elements \( g_0, g_1, g_2, h_1, h_2 \) are defined as before. In order to prove the desired inequality between translation length and virtual translation length (Theorem 6.17) we have to estimate \( \tau_\mathcal{D}^\infty(\rho(\gamma)) \) from above. As before we denote \( \varphi^\pm := \varphi(\gamma^\pm) \).

Lemma 6.11. We have \( \tau_\mathcal{D}^\infty(\rho(\gamma)) > 0 \).

Proof. By definition \( \tau_\mathcal{D}^\infty(\rho(\gamma)) = \log b_\gamma(\gamma^-, \gamma^+, \gamma) \), where \( \xi \neq \gamma^\pm \). Since \( \gamma^+ \) is the attractive fixed point of \( \gamma \), the point \( \gamma\xi \) is between \( \xi \) and \( \gamma^+ \). More precisely: \( (\gamma^-, \xi, \gamma^+) \) and \( (\xi, \gamma\xi, \gamma^+) \) are either both positive or both negative. In both cases \( \tau_\mathcal{D}^\infty(\rho(\gamma)) \) is positive.

Our next observation is that \( \tau^\infty(\rho(\gamma)) \) does not depend on the choice of \( \xi \). We shall prove a slightly more general statement: Given \( x, z \in \bar{S} \) we define
\[
\bar{S}^{(4)}_{x,z} := \{(y,t) \in S^2 | (x,y,z) \in \bar{S}^{(4)}_{x,z}\}.
\]

With this notation we have
\[
(\varphi(\xi), \varphi(\gamma\xi)) = (\varphi(\xi), g_0\varphi(\xi)) \in \bar{S}^{(4)}_{g_0^+, g_0^-}
\]
for all \( \xi \in \Sigma_1 \setminus \{\gamma^\pm\} \). Now we claim:

Lemma 6.12. Suppose that \( y \in \bar{S} \setminus \{g_0^\pm\} \) and \( (y,g_0,y) \in \bar{S}^{(4)}_{g_0^+, g_0^-} \) then
\[
\tau_\mathcal{D}^\infty(\rho(\gamma)) = \log B_V(g_0^-, y, g_0^+, g_0y)
\]

Proof. Let \( F(y) := B_V(g_0^-, y, g_0^+, g_0y) \). We have to show that \( F(y) = F(z) \) for all \( y, z \in \bar{S} \) satisfying the conditions of the lemma. If \( (y,z) \in \bar{S}^{(4)}_{g_0^-, g_0^+} \). Then we can compute
\[
F(z) = B_V(g_0^-, z, g_0^+, g_0z) = B_V(g_0^-, z, g_0^+, y) \cdot B_V(g_0^-, y, g_0^+, g_0z) = B_V(g_0^-, y, g_0z, g_0g_0^+, g_0y) \cdot B_V(g_0^-, y, g_0^+, g_0z) = B_V(g_0^-, y, g_0^+, g_0z) \cdot B_V(g_0^-, g_0z, g_0^+, g_0y) = B_V(g_0^-, y, g_0^+, g_0y) = F(y).
\]

Otherwise we can find \( w \in \bar{S} \) satisfying the conditions of the lemma such that both \( (y, w) \in \bar{S}^{(4)}_{g_0^-, g_0^-} \) and \( (z, w) \in \bar{S}^{(4)}_{g_0^+, g_0^-} \). Then \( F(y) = F(w) = F(z) \), finishing the proof of the lemma in the general case.

\( \square \)
Corollary 6.13. Suppose that $z \in \tilde{S} \setminus \{\pm e\}$ and that $(z, g_1 z) \in \tilde{S}^{(4)}_{-e, e}$. Then
\[ \tau^{\infty}_V (\rho(\gamma)) = \log B_V (-e, z, e, g_1 z). \]

Proof. Apply Lemma 6.12 to $y := h_1^{-1} z$ and use $G$-invariance of $B_V$.

Recall that $g_2 = (c \circ g_1 \circ p)^M$. We thus need the following lemma:

Lemma 6.14. If $g \in G$ and $g^\pm$ are fixed points of $g$, then for all $M \in \mathbb{N}$,
\[ \log B_V (g^-, x, g^+, g^M x) = M \cdot \log B_V (g^-, x, g^+, gx). \]

Proof. We have
\[ \log B_V (g^-, x, g^+, g^M x) = \sum_{i=1}^M \log B_V (g^-, g^i x, g^+, g^{i+1} x) = M \cdot \log B_V (g^-, x, g^+, gx). \]

For the next step we observe that by Proposition 3.16 and the transformation behavior of the Bergman kernel under biholomorphic maps [15, Prop. IX.2.4] that the Bergman kernel $K_{T_0}$ of the tube $T_0$ extends continuously to transversal pairs on $V$. Denoting this extension by the same letter $T_0$ we prove:

Proposition 6.15. Let $w \in V$ such that $0, w$ and $g_2 w$ are pairwise transversal. Then
\[ \tau^{\infty}_V (\rho(\gamma)) = \frac{1}{M \cdot \text{rk} V} \cdot \log \left( \frac{K_{T_0} (w, 0)}{K_{T_0} (g_2 w, 0)} \cdot \lim_{n \to \infty} \frac{K_{T_0} (g_2 w, c(x_n))}{K_{T_0} (w, c(x_n))} \right), \]
where $x_n$ is a sequence in $D$ converging to $e$.

Proof. Combining Corollary 6.13 and Lemma 6.14 with Proposition 4.15 applied to the Cayley transform $c : D \to T_0$.

Now the right hand side can be computed explicitly:

Proposition 6.16.
\[ \tau^{\infty}_V (\rho(\gamma)) = \frac{1}{M \cdot \text{rk} V} \cdot \log \det (g_2)^2 \]

Proof. We first show that
\[ \lim_{n \to \infty} \frac{K_{T_0} (g_2 w, c(x_n))}{K_{T_0} (w, c(x_n))} = 1. \]

Indeed, let $\lambda \in [0, 1)$. Then
\[ c(\lambda \cdot e) = -\frac{1 + \lambda}{1 - \lambda} e. \]

Using [15, X.1.3] we obtain
\[ \lim_{n \to \infty} \frac{K_{T_0} (g_2 w, c(x_n))}{K_{T_0} (w, c(x_n))} = \lim_{\lambda \to 1} \left( \frac{\det (g_2 w - \frac{1 + \lambda}{1 - \lambda} e)}{\det (w - \frac{1 + \lambda}{1 - \lambda} e)} \right)^{-\frac{2n}{M}}. \]
= \lim_{\lambda \to 1} \left( \frac{\det \left( \frac{1-\lambda}{1+\lambda} g_2 w - e \right)}{\det \left( \frac{1-\lambda}{1+\lambda} w - e \right)} \right)^{-\frac{2n}{n}} = 1.

In view of Proposition 6.15 it now suffices to show that $K_{T_\Omega}(w,0) = K_{T_\Omega}(g_2 w,0)\det C J_{g_2}(w)\det C J_{g_2}(0)$, where $J_{g_2}$ denotes the complex Jacobi matrix of $g_2$. Note that $g_2$ is a real matrix, because it is in $G(\Omega) \subset GL(V)$. Since it is linear, we have $J_{g_2} \equiv g_2$ and $g_2 0 = 0$, whence

$$K_{T_\Omega}(w,0) = K_{T_\Omega}(g_2 w,0)\det C J_{g_2}(w)\det C J_{g_2}(0) = K_{T_\Omega}(g_2 w,0)\det (g_2)^{2}.$$ 

Dividing both sides by $K_{T_\Omega}(g_2 w,0)$ the proposition follows.

Comparing with (6.10) we ultimately obtain:

**Theorem 6.17.** For all $\gamma \in \Gamma$ and all $\xi \in S^1 \setminus \{\gamma^\pm\}$ the inequality

$$\tau_D(\rho(\gamma)) \geq \frac{\rk V}{\sqrt{\dim V}} \cdot \tau_D^\infty(\rho(\gamma))$$

holds.

**Proof.** By Corollary 6.10 and Proposition 6.10 we have

$$\tau_D(\rho(\gamma)) \geq \frac{1}{M \cdot \sqrt{\dim V}} \log \det (g_2)^2$$

$$= \frac{\rk V}{\sqrt{\dim V}} \cdot \left( \frac{1}{M \cdot \rk V} \cdot \log \det (g_2)^2 \right)$$

$$= \frac{\rk V}{\sqrt{\dim V}} \cdot \tau_D^\infty(\rho(\gamma)).$$

6.4. **Well-displacing.** Before we can deduce well-displacing of maximal representations from our main inequality, we need one more ingredient: If $\Gamma$ is a finitely generated group and $S$ a finite set of generators, then we denote by $\|\cdot\|_S$ the word length with respect to $S$. We then define the word metric $d_S$ by

$$d_S(\gamma_1, \gamma_2) = l_S(\gamma_1^{-1} \gamma_2),$$

where

$$l_S(\gamma_1) := \inf_{\eta \in \gamma} \|\eta \gamma \eta^{-1}\|_S.$$

Then the Švarc-Milnor lemma reads as follows:

**Lemma 6.18** (Švarc-Milnor, [6 Prop. I.8.19]). Let $(X,d)$ be a length space. If a group $\Gamma$ acts properly and cocompactly by isometries on $X$, then $\Gamma$ is finitely generated and for every finite generating set $S$ with associated word metric $l_S$ on $\Gamma$ and every basepoint $x_0 \in X$ the map

$$\Gamma, d_S) \rightarrow (X,d), \quad \gamma \mapsto \gamma x_0$$

is a quasi-isometry.
Note that the constants appearing in the quasi-isometry inequality may depend on \(x_0\). Nevertheless, applying this to the \(\Gamma\) action on the disc chosen in the introduction, we obtain:

**Corollary 6.19.** Let \(S\) be an arbitrary finite generating set for \(\Gamma\) and \(l_S\) the associated word length. Then there exist constants \(A, B > 0\) such that for every \(\gamma \in \Gamma\)

\[
\tau_D(\gamma) \geq A \cdot l_S(\gamma) - B.
\]

**Proof.** We fix a compact fundamental domain \(F\) for the \(\Gamma\)-action on \(\mathbb{D}\). We know that every \(\gamma \in \Gamma\) is hyperbolic, i.e. there exists a geodesic \(\sigma\) on which \(\gamma\) acts by translation and we have \(\gamma \cdot \sigma(t) = \sigma(t + \tau_D(\gamma))\) for all \(t\). There exists \(\eta \in \Gamma\) such that \(\eta \sigma\) intersects \(F\), say \(y := \eta \sigma(t_0) \in F\). Then we have for any \(x \in F\):

\[
d(x, \gamma x) = d(ex, \gamma x) \geq A \cdot d_S(e, \gamma) - B' = A \cdot l_S(\gamma) - B'
\]

for all \(\gamma \in \Gamma\). We deduce

\[
\tau_D(\gamma) = \tau_D(\eta \gamma \eta^{-1}) \geq d(x, \eta \gamma \eta^{-1} x) - 2\text{diam}(F)
\]

\[
\geq A \cdot l_S(\eta \gamma \eta^{-1}) - B' - 2\text{diam}(F) = A \cdot l_S(\gamma) - (B' + 2\text{diam}(F)).
\]

\[
\square
\]

Now we can finally prove:

**Theorem 6.20** (Well-displacing). For any finite generating set \(S\) of \(\Gamma\) there exist \(A, B > 0\) such that

\[
\tau_D(\rho(\gamma)) \geq A \cdot l_S(\gamma) - B
\]

for all \(\gamma \in \Gamma\).

**Proof.** Using Corollary 5.9, Theorem 6.17, Lemma 6.2 and Corollary 6.19 we find positive constants \(C_1, \cdots, C_4\) such that

\[
\tau_D(\rho(\gamma)) \geq C_1 \cdot \tau_D(\rho(\gamma)) \geq C_1 \cdot \log b_\rho(\gamma^- : \xi : \gamma^+ : \gamma \xi)
\]

\[
\geq C_2 \cdot \log[\gamma^- : \xi : \gamma^+ : \gamma \xi] = C_2 \cdot \tau_D(\gamma) = C_2 C_3 \ell_S(\gamma) - C_2 C_4.
\]

\[
\square
\]

### 6.5. Quasi-isometry property

As a simple consequence of Theorem 6.20 we obtain the following result:

**Theorem 6.21** (Quasi-isometry). Let \(\rho : \Gamma \to G\) be a maximal representation. Then for every \(x \in \mathcal{D}\) and every finite generating set \(S\) of \(\Gamma\) the map

\[
(\Gamma, d_S) \to (\mathbb{D}, d_D), \quad \gamma \mapsto \gamma x
\]

is a quasi-isometric embedding.
Proof. Since for $\gamma_1, \gamma_2 \in \Gamma$ we have both $d_S(\gamma_1, \gamma_2) = l_S(\gamma_2^{-1}\gamma_1)$ and $d_D(\gamma_1, x, \gamma_2, x) = d(\gamma_2^{-1}\gamma_1, x)$ it suffices to show that there exist $A, B > 0$ such that for all $\gamma \in \Gamma$

$$A^{-1} \cdot l_S(\gamma) - B \leq d_D(\gamma, x, x) \leq A \cdot l_S(\gamma) + B.$$  

The left inequality follows immediately from Theorem 6.20. Indeed, we have

$$d_D(\gamma, x, x) \geq \tau_D(\gamma) \geq A \cdot l_S(\gamma) - B$$

for some constants $A, B > 0$ independent of $\gamma$. On the other hand, the right inequality is elementary: Write $\gamma = s_1 \cdots s_l S(\gamma)$ with $s_i \in S$. Then

$$d(x, \gamma x) \leq d(x, s_1 x) + d(s_1 x, s_1 s_2 x) + \ldots + d(s_1 \cdots s_l(\gamma) x, \gamma x)$$

$$= d(x, s_1 x) + d(x, s_2 x) + \ldots + d(x, s_l(\gamma) x)$$

$$\leq \max_{s \in S} d(x, sx) \cdot l_S(\gamma).$$

□

6.6. Properness of the mapping class group. It is now easy to deduce Corollary 1.3 from the introduction. Indeed, it suffices to establish Inequality (2.1) of [20, Prop. 2.4]. Now the upper bound is already established in [20, Lemma 2.7], and the lower bound was established within the proof of Theorem 6.20.

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