Transport signatures of electron-tunneling-assisted non-Abelian braiding

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Majorana bound states (MBSs) are non-Abelian quasiparticles in vortices of topological superconductors. They have been identified as building blocks for topologically protected qubits where quantum state evolution within a degenerate ground state manifold proceeds via braiding of distant MBSs. We present a theory of coherent time-dependent electron tunneling from a metal tip into a Corbino geometry topological Josephson junction where four MBSs rotate. The time averaged tunneling conductance exhibits, as a function of bias voltage between the tip and the Josephson junction, distinctive conductance peaks that are separated by $\hbar/(4T_{J})$ (where $T_{J}$ is the time period of the system Hamiltonian). This separation is a result of interference between processes where electron tunneling between the tip and the junction interrupts the rotation of the MBS after different number of round trips. The interference effect is shown to be a direct consequence of two non-commuting braiding operations—a rotation of the four MBSs along the Josephson junction and a tunneling assisted rotation—reflecting the non-Abelian nature of MBSs. This mechanism of non-Abelian state evolution actively utilizes electron tunneling that changes the fermion occupation number parity of the system rather than avoiding it while the MBSs are spatially decoupled from each other and hence are not fused physically. The proposed scheme would provide an alternative route for detecting the non-Abelian statistics.

I. INTRODUCTION

A braiding operation reveals the quantum statistics of identical particles [1–3]. Majorana zero-energy states bound to certain defects (e.g., vortices or edges) in topological superconductors are quasiparticles obeying non-Abelian statistics [4–7]. In an isolated system with $2N$ decoupled Majorana states, there is a $2^N$-fold degenerate ground state manifold $\{|\Psi\rangle\}$, and adiabatically moving one Majorana state around another acts as a unitary matrix on the manifold. Such unitary matrices of different braiding operations, $A$ and $B$, are in general non-commutative, so that the order of operations matter,

$$AB|\Psi\rangle \neq BA|\Psi\rangle \quad \text{or} \quad (AB - BA)|\Psi\rangle \neq 0. \quad (1)$$

Non-Abelian braiding is one of the hallmarks of topological quantum phases associated with non-Abelian statistics appearing in many contexts [3, 8, 9] and also represents the basic resource for executing topologically protected gates for quantum computing [1, 10].

The essence of the present work is to provide transport signatures of Majorana bound states (MBSs) induced by Eq. (1). The envisioned system is a Corbino geometry topological Josephson junction (JJ), formed by two $s$-wave superconductors on a topological insulator (TI) surface. Four vortices, each hosting a MBS, rotate along the junction, and the time-dependent tunneling conductance between the junction and a metallic tip is measured [11]. A ground state of the system evolves in the fourfold degenerate ground state manifold, governed by the rotation and the coherent electron tunneling processes. The evolution can be cast into two braiding operators (corresponding to $A$ and $B$ in Eq. (1)) which do not commute: one is a parity-conserving rotation and the other is a tunneling-assisted braiding. In the low bias voltage regime, the time-averaged conductance exhibits unusual peak positions, which we interpret as a direct signature of non-commutativity of two braiding operators. This result is distinguished from the result of the same transport experiment but with Majorana braidings that commute.

Tremendous amounts of proposals and experiments have been made great achievements in the realization [12–18], manipulation [19–24] and detection [25–36] of MBSs in superconducting hybrid structures. In particular, a recent experiment exploiting a quantum anomalous Hall insulator-superconductor structure [37] boosts interest in searches for transport signatures of non-Abelian braiding [38, 39]. Based on such hybrid structures, the authors of Refs. [38, 39] theoretically investigated transport properties of Mach-Zehnder-like interferometers of chiral Majorana modes. The overlap or fusion of two paths of Majorana modes whose relative dynamics is determined by braiding with the other Majoranas signals a unitary evolution (which is not a phase factor) of Majorana modes.

Different to these recent studies in Refs. [38, 39], we demonstrate interference involving four rotating MBSs whose braiding operations are assisted by tunneling of
electrons into or out of the MBSs and thus in which the number parity of the fermion occupation states formed by the MBSs is not conserved. Moreover, the resulting conductance which is averaged over a time interval much larger than the time taken for a single braiding operation shows non-Abelian effects not relying on a specific choice of fusing pairs of the MBSs or on a specific initial ground state. The experiment we propose could serve as an unambiguous diagnostic tool to probe non-commuting braiding operations of MBSs via certain periods of conductance peaks.

The paper is organized as follows. Section II introduces the model Hamiltonian for a Corbino JJ on a TI surface threaded by a magnetic flux. By solving the Hamiltonian, we show that the Corbino JJ hosts four decoupled MBSs at zero energy whose positions can be moved adiabatically along the circle by changing the superconducting phase difference across the junction. The operators of parity-conserving rotation and tunneling-assisted braiding are constructed. In Section III, we obtain the time-averaged differential conductance in the weak tunneling regime shows non-Abelian effects not relying on a specific choice of fusing pairs of the MBSs or on a specific initial ground state. The experiment we propose could serve as an unambiguous diagnostic tool to probe non-commuting braiding operations of MBSs via certain periods of conductance peaks.

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II. ROTATING MAJORANA BOUND STATES

A. Parity-conserving Majorana rotation

We consider a Corbino JJ deposited on the surface (x-y plane) of a 3-dimensional TI [Fig. 1(a)]. The junction is formed by thin films of inner (S₁) and outer (S₂) s-wave superconductors and contains four magnetic flux quanta, 4Φ₀ with Φ₀ = h/(2e), inducing a phase difference across the junction (see Eq. (4)). The model Hamiltonian for the TI surface proximity coupled to the Corbino JJ in polar coordinates (r, θ) is given by [40]

\[ H_C = \frac{1}{2} \int d^2r \Phi^\dagger(r) H_C \Phi(r), \]

\[ H_C = \begin{pmatrix} \mathcal{H}_0 - \mu & \Delta(r) \\ \Delta^\ast(r) & -\mu - \mathcal{H}_0 \end{pmatrix}, \]

where \( \Phi = (\Phi_1, \Phi_2, \Phi_1^\dagger, -\Phi_2^\dagger)^T \) is the Nambu spinor, \( \mathcal{H}_0 = v_F (\sigma_x p_x + \sigma_y p_y) \) with Pauli spin matrices \( \sigma_x, \sigma_y \) describes the surface states and \( \mu \) is the chemical potential. The proximity-induced superconducting gap \( \Delta(r) \) is

\[ \Delta(r) = \begin{cases} \Delta_0 e^{i\phi_1} & 0 \leq r < R, \\ \Delta_0 e^{-i\phi_1 + i\phi_2} & r > R, \end{cases} \]

where \( \phi_1 \) and \( \phi_2 \) are spatially uniform phases in each superconducting region, and the polar-angle-dependent phase \( -4\theta \) at \( r > R \) is due to the presence of the four flux quanta [41].

We solve the Bogoliubov-de Gennes (BdG) equation \( \mathcal{H}_C \Psi(r) = E \Psi(r) \) for \( \mu = 0 \), and obtain four Majorana bound states, \( \Psi_{Mj}(r) \) with \( j \in \{1, 2, 3, 4\} \), at exactly zero energy \( E = 0 \). They are spatially separated and orthogonal to each other. We refer to Appendix A for the analytical calculation of the states. Note that the MBSs remain at the zero energy for small \( \mu \) and weak disorder (see Appendix in Ref. [11]). Their probability densities
are shown in Fig. 1(b) for different values of $\phi_1 - \phi_2$. The positions of the MBSs $\theta_j$ are determined by the condition $\Delta \phi = \pi \mod 2\pi$ where $\Delta \phi = \phi_1 - \phi_2 + 4\theta$ is the local phase difference across the junction. They are given by $\theta_j = (3\pi - 2\pi j)/4 - (\phi_1 - \phi_2)/4$. The phases $\phi_1$ and $\phi_2$ are gauge dependent, however, their difference is gauge invariant and determines the position of the MBSs. Since all positions along the circle are equally likely in terms of energy due to the rotation symmetry, only their relative position is fixed by the number of fluxes. Since $\phi_1$ and $\phi_2$ are the single-valued phases of the superconducting order parameters, their initial values, and thus the initial positions $\theta_j$, are chosen spontaneously. Our proposal does not depend on the initial configuration of the MBSs (see Eq. (25) below) and the only requirement is that the phase difference will be evolving in time due to the presence of a dc-bias voltage $V_J$ across the junction.

If we change $\phi_1 - \phi_2$ by $2\pi$, the four MBSs rotate by $\pi/2$ in a clockwise direction maintaining their relative distances, as plotted in Fig. 1(b), leading to a transformation

$$
\begin{align*}
\gamma_1 &\rightarrow -s\gamma_2, \\
\gamma_2 &\rightarrow -s\gamma_3, \\
\gamma_3 &\rightarrow s\gamma_4, \\
\gamma_4 &\rightarrow -s\gamma_1,
\end{align*}
$$

(5)

FIG. 2. World lines of rotating MBSs in the Corbino JJ. (a) Rotation operator $U_i$ defined in Eq. (6). (b) Tunneling-assisted rotation $\hat{U}_c$ defined in Eq. (13). (c) Illustration of $U_cU_i^\dagger \neq U_i^\dagger \hat{U}_c$ which gives rise to a peculiar interference effect described in Eq. (30). Here, the case of $s = -1$ in Eq. (5) is considered.

where $\gamma_j = \int d^2 r \Psi_{Mj}(r)\Phi(r)$ and $s = 1(-1)$ corresponds to the change of $\phi_1(\phi_2)$ by $2\pi(-2\pi)$. Graphical representation of the transformation is given in Fig. 2(a) for the $s = -1$ case.

A rotation operator $U_c$ for the transformation such that $\gamma_j \rightarrow U_c\gamma_j U_c^\dagger$ can be constructed as

$$
U_c = U_{11}U_{12}U_{23},
$$

(6)

where $U_{ij}$ is the braiding exchange operator of $\gamma_i$ and $\gamma_j$ given by $U_{ij} = \exp(2\pi \gamma_{ij}/4)$ [42]. We describe the transformation in terms of fermionic operators defined as $f_1 = 2\gamma_{11} + \gamma_{12}$ and $f_2 = 2\gamma_{21} + \gamma_{22}$. With the state $|00\rangle$ defined by $f_1^\dagger|00\rangle = f_2^\dagger|00\rangle = 0$, we specify the occupation number states,

$$
\begin{align*}
|00\rangle, \\
|10\rangle &= f_1^\dagger|00\rangle, \\
|11\rangle &= f_1^\dagger f_2^\dagger|00\rangle, \\
|01\rangle &= f_2^\dagger|00\rangle.
\end{align*}
$$

(7)

The two states in each fermion-occupation-number parity subspace form a qubit. In the basis $|00\rangle, |11\rangle, |10\rangle, |01\rangle$, $U_c$ in Eq. (6) is represented as

$$
U_c = \begin{pmatrix}
U_{ee} & 0 \\
0 & U_{oo}
\end{pmatrix} = \begin{pmatrix}
e^{-i\frac{\pi}{2}\hat{n}_{ee}} & 0 \\
0 & e^{-i\frac{\pi}{2}\hat{n}_{oo}}
\end{pmatrix},
$$

(8)

where $U_{ee}(U_{oo})$ is the evolution operator acting on the even (odd) parity space that rotates the qubit by $\pi/2(\pi)$ about the direction of $\hat{n}_{ee}(\hat{n}_{oo})$ given by

$$
\hat{n}_{ee} = (0, 1, 0), \quad \hat{n}_{oo} = \frac{s}{\sqrt{2}}(-1, 0, 1).
$$

(9)

$\hat{\tau} = (\tau_x, \tau_y, \tau_z)$ are Pauli matrices acting on the qubit, and $0$ is $2 \times 2$ null matrix. The qubit rotations induced by $U_{ee}$ and $U_{oo}$ on the Bloch sphere are illustrated in Fig. 3. The parity of the fermion occupation number is conserved in the transformation $U_c$.

The adiabatic rotation can be achieved if the voltage $V_J$ across the junction is much smaller than the excitation energy of the junction, which will be discussed in Sec. V. For a finite $V_J$, $\phi_1 - \phi_2$ varies in time $t$ as $\phi_1 - \phi_2 = \phi_0 + 2eV_J t/h$ where $\phi_0$ is a spontaneously chosen constant, as discussed above. The states $\Psi_{Mj}(r, \phi_1(t), \phi_2(t))$ then become instantaneous eigenstates of $H_C[\phi_1(t), \phi_2(t)]$ at zero energy, and $U_c$ in Eq. (6) can be considered as the
time evolution operator of the MBSs from \( t \) to \( t + T_J \), where
\[
T_J = \frac{\pi \hbar}{eV_J}
\]
is the time taken for the \( \pi/2 \)-rotation, and, equivalently, it is the time period of the system Hamiltonian Eq. (17). The Majorana operators have a time dependence as 
\[
\gamma_j(\phi_1(t), \phi_2(t)) = \gamma_j(t) obeying \gamma_j(t + T_J) = U_c(t)\gamma_j(t)U_c^\dagger.
\]
Note that the rotation operator has no non-topological corrections as long as the adiabaticity is fulfilled because the states are exactly at zero energy [43, 44].

**B. Tunneling-assisted Majorana braiding**

To explore the effect of electron tunneling, we connect a metal tip to the Corbino JJ, as depicted in Fig. 1(a). The tip is located such that an electron can tunnel onto or off the Corbino JJ through \( \gamma_1(t_0) \) at \( t = t_0 \), and we assume that the tunnel coupling is switched on at \( t = t_0 \). A coherent time-dependent tunneling event between the tip and adiabatically rotating Majorana states can occur at discrete times \( t_q = t_0 + qT_J \), where \( q = 0, 1, 2, \ldots \). Creation or annihilation of an electron via a Majorana state at \( t = t_q \) is described by \( \gamma_1(t_0)|\Psi_M(t_q)\rangle \) where \( |\Psi_M(t_q)\rangle = U^q_c|\Psi_M(t_0)\rangle \) is the state obtained by linearly combining the occupation number states given in Eq. (7) evolving from \( t_0 \) to \( t_q \) with preserving its parity. Other choices of Majorana states coupled to the tip at \( t = t_0 \) (here \( \gamma_1(t_0) \)) and different initial preparations of \( |\Psi_M(t_0)\rangle \) do not change the result in Eq. (25). Hereafter, we will denote \( \gamma_1(t_0) \) by \( \gamma_1 \).

The time evolution of a Majorana state from \( t = t_q \) to \( t_q^\prime \) at which tunneling events occur is described by the Majorana Green’s function,
\[
M(t_q, t_q^\prime) = -i\langle \Psi_M(t_q)|\gamma_1(t_q^\prime)\gamma_1(t_q)|\Psi_M(t_q)\rangle,
\]
where \( \gamma_1(t_q^\prime) = (U_c^\dagger)^q\gamma_1 U_c^q \). It can be considered as an overlap of two time-evolved states with tunneling occurring at different times, \( |\Psi_M(t_q)\rangle = \gamma_1 U^q_c|\Psi_M(t_q^\prime)\rangle \) and \( |\Psi_M^\dagger(t_q)\rangle = U^q_c\gamma_1|\Psi_M(t_q^\prime)\rangle \) where \( n = q - q' \). Note that \( \gamma_1 \) and \( U_c \) contained in their evolution do not commute, and thus the overlap can be less than one. For example, the overlap for \( |\Psi_M(t_q)\rangle \) is \( \cos \eta(00) + \sin \eta|11\rangle \) and \( n = 1 \) is given by
\[
|\langle \Psi_M(t_q)|\Psi_M^\dagger(t_q)\rangle| = |\sin 2\eta|,
\]
in contrast to Abelian statistics where the time evolution of a state is described by a simple phase factor. For a more comprehensive description of the tunneling effect, we introduce a new parity conserving operator
\[
\tilde{U}_c = \gamma_1 U_c \gamma_1 \gamma_1,
\]
so that the Majorana Green’s function can be written as
\[
M(t_q, t_q^\prime) = -i\langle \Psi_M(t_q^\prime)|U_c^\dagger U_c \gamma_1|\Psi_M(t_q)\rangle.
\]
\( \tilde{U}_c \) consists of three events: changing fermion-spin-number parity due to the tunneling at \( t = t_q \), followed by an evolution for a time \( T_J \) with \( U_c \), and then changing the parity again at \( t = t_q + T_J \). Due to the occurrence of tunneling twice, \( \tilde{U}_c \) can be regarded as an effective parity conserving rotation operator of four MBSs over discrete time steps \( t_q \), besides \( U_c \). The transformation governed by \( \tilde{U}_c \) is drawn in Fig. 2(b). In terms of braiding exchange operators, it is expressed as \( \tilde{U}_c = \gamma_1 U^1_c U^{12} U^{23} \gamma_1 = U^{14} U^{21} U^{23} \), where we have used
\[
\gamma_1 U_{ij} \gamma_1 = \begin{cases} U_{ji} & \text{if } i = 1 \text{ or } j = 1, \\ U_{ij} & \text{if } i, j \neq 1. \end{cases}
\]

The expression implies that the tunneling effectively reverses the direction of the braiding \( U_{ij} \) if \( U_{ij} \) involves \( \gamma_1 \), as shown in Fig. 2(b). In the occupation number basis defined in Eq. (7), \( \tilde{U}_c \) is represented by the interchange of \( U_{co} \) and \( U_{co} \) in Eq. (8),
\[
\tilde{U}_c = \begin{pmatrix} U_{co} & 0 \\ 0 & U_{co} \end{pmatrix}.
\]
We find that \( U_{i}^\dagger \) and \( \tilde{U}_c \) (or \( U_c \) and \( \tilde{U}_c \)) do not commute,
\[
[U_{i}^\dagger, \tilde{U}_c] = \begin{pmatrix} 0 & 0 \\ 0 & [U_{co}, U_{co}] \end{pmatrix} = si(-\tau_x - \tau_z \tau_x + \tau_z) \neq 0.
\]
This is indicative of the different braiding evolutions of world lines corresponding to the two operator products \( U_{i}^\dagger \tilde{U}_c \) and \( \tilde{U}_c U_{i}^\dagger \) in Fig. 2(c).

We emphasize that basically, it is the tunneling of electrons reversing the braiding directions of pairwise exchanges of MBSs involving the MBS coupled to the tip (cf. Eq. (14) and Fig. 2(b)) that governs the change \( U_c \rightarrow \tilde{U}_c \) and \( [U_{co}, U_{co}] \neq 0 \) [Fig. 3] which is responsible for the non-Abelian effect. Interestingly, the non-commutativity occurs in each state space of a given parity, although the tunneling changes the parity. This shows that electron tunneling can be a part of braiding operations.

In addition, the non-commutativity is independent of the choice of the computational basis states such as those in Eq. (7). Choosing (or fusing) different pairs of MBSs to form Dirac fermions defines another occupation number basis. Since this corresponds to a unitary transformation, it cannot render the commutator to be zero, indicating that the non-commuting property arises from the non-Abelian nature of MBSs.

We show below that the non-commuting braiding \( U_c \) and \( \tilde{U}_c \) result in observable interference signatures (Sec. III A) free of the necessity of physically fusing MBSs. The interference signatures are clearly distinguishable from those for four MBSs whose evolutions with and without tunneling would commute (Sec. III B).
III. TRANSPORT SIGNATURES OF ROTATING MAJORANA BOUND STATES

A. Tunneling conductance

In order to obtain the tunneling current between the tip and the Corbino JJ in the weak coupling limit, we extend the formalism of Ref. [11] to four MBSs with a time-evolution operator presented by $U_c$. The total Hamiltonian of our experimental setup is

$$H(t) = H_N + H_C(t) + H_T(t).$$

$$H_N = \sum_{k\sigma} \epsilon_k c_k^{\dagger} c_{k\sigma}$$ is the tip Hamiltonian of electrons with momentum $k$ and spin $\sigma$. Since we are interested in the low-energy sector of the theory, tunneling between the tip and the four MBSs is the only relevant process,

$$H_T(t) = \sum_{j\sigma} [V_{j\sigma}(t) c_{k\sigma}^\dagger \gamma_j(t) + \text{H.c.}],$$

where $V_{j\sigma}(t)$ is the coupling between the tip and $\gamma_j(t)$. Around $t = t_q$ where the coupling strength to $\gamma_1$ is maximal, we model $V_{j\sigma}(t)$ as a function which increases and decreases exponentially as $\gamma_1$ approaches to and leaves from the tip, respectively, while its phase does not change significantly, and we also assume that $V_{j\sigma}(t_0)$ for $j \neq 1$ are zero (see Appendix B). Moreover, since the Majorana states are spin polarized, we only consider electrons of the tip with their spin parallel to that of the Majorana states, and drop the notation $\sigma$ in what follows; electrons with opposite spin are reflected at the junction between the tip and the Corbino JJ and do not contribute to the tunneling current. Then the tunneling Hamiltonian becomes

$$H_T(t) = \sum_{kq} e^{-\lambda|t-t_q|} V_{jk}(t_0) c_k^{\dagger} \gamma_1 + \text{H.c.},$$

where $\lambda^{-1}$ is the tunneling duration. Here we have assumed the condition $\lambda^{-1} \ll T_j$, implying that only nearest-neighbor coupling between the tip and the MBSs is taken into account. The variation of the coupling strength with time in an exponential manner can be justified by the fact that the Majorana states shown in Fig. 1(b) are exponentially localized along the azimuthal direction. Specifically, the low-energy Hamiltonian for a linear topological Josephson junction [12] incorporating the spatial variation of a superconducting phase difference due to the four flux quanta provides an approximate form for the Majorana state, whose angle dependence is given by $\frac{\pi}{2} e^{i\phi_{Mij}}(R, \theta)|^2 \propto \exp(-2R/\xi)(\bar{\theta} - \theta)^2$.

Using lowest order perturbation theory in $H_T(t)$ given in Eq. (19), the differential conductance of the time-averaged current measured after many rotation cycles of MBSs, $\bar{I} = \frac{1}{T_j} \int_{t-T_j}^{t} dt I(t)$ with $t-t_0 \gg T_j$, is obtained by

$$\frac{d\bar{I}}{dV} = \frac{e}{h} \int_{-\infty}^{\infty} d\epsilon T(\epsilon) [S(\epsilon) + S(-\epsilon)] \frac{dI(\epsilon - eV)}{d\epsilon}.$$ (20)

The time-averaged tunneling probability $T(\epsilon)$ and the term $S(\epsilon)$ which describes interference between the tunneling processes at different times are given by

$$T(\epsilon) = \frac{2\Gamma T_j}{\hbar} \left( \frac{2\Delta T_j}{\lambda^2 T_j^2 + \xi^2} \right)$$

$$S(\epsilon) = \text{Re} \left\{ \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{Q} e^{in\pi} M(t_Q, t_{Q-n}) \right\}. $$ (21)

Here $\xi = \epsilon/\epsilon(T_j)$ and the integer $Q \gg 1$, which will go to infinity later, is the greatest integer among possible $q$’s satisfying $t_q = t_l$. $\Gamma = 2\pi\rho|V_{1k}(t_0)|^2$ where $\rho$ is the tip density of states. We assumed a wide-band approximation where $\rho$ and $V_{1k}(t_0)$ are energy independent and we neglected the contributions proportional to $e^{-\lambda T_j/2}$; note that these small contributions do not change the positions of conductance peaks. The details for the calculation of $\bar{I}$ are given in Appendix B.

The term $S(\epsilon) + S(-\epsilon)$ in Eq. (20) is written in terms of the Majorana Green’s function $M(t_0, t_0')$ in Eq. (11), and contains information of the non-commuting braiding operations. It yields

$$S(\epsilon) + S(-\epsilon) = \sum_{m=-[Q/4]}^{[Q/4]} e^{im(4\xi + \alpha)}. $$ (22)

The notation $[Q/4]$ denotes the integer part of the number $Q/4$ and we have used anti-commutation relations

$$\left\{ \gamma_1(t_l), \gamma_1(t_{l'}) \right\} = \left\{ 2e^{i\alpha} \text{for } q - q' = 4m, 0 \text{ otherwise.} \right\}$$ (23)

The phase factor $e^{i\alpha}$ with $\alpha = \pi$ comes from a $2\pi m$-rotation of the four MBSs, $(U^\dagger)^{4m} \gamma_j(U_c)^{4m} = (-1)^m \gamma_j$, and is physically due to crossing branch cuts emanating from the MBSs. In the limit $Q \to \infty$ (or $t - t_0 \to \infty$), we obtain

$$\frac{d\bar{I}}{dV} = \frac{e}{\hbar} \frac{\pi}{8T_j k_B T} \sum_l T(\epsilon_l) \text{sech}^2 \left( \frac{\epsilon V - \epsilon_l}{2k_B T} \right),$$ (24)

where $\epsilon_l = \hbar T_j(2\pi l - \alpha)$ with integer $l$. This perturbative calculation is valid for $T(\epsilon_l, \hbar/(8T_j)) \ll k_B T \ll E_g$ where
$E_q$ is the excitation energy of the junction. The $dI/dV$ in Eq. (25) is plotted in Fig. 4 for realistic parameters. It shows peaks at

$$ eV = \varepsilon_t = \frac{\hbar}{4T_J}(2\pi l - \alpha). \quad (26) $$

This is our main result. The peak positions are determined by $T_J$ and $\alpha$, but are independent of system details such as the initial Majorana state at $t = t_0$ and the tunneling strength $\Gamma$. The peaks are separated by $\hbar/(4T_J)$ despite that the system Hamiltonian $H(t)$ in Eq. (17) is periodic with periodicity $T_J$. The results are the same for the case of an anticlockwise rotation of four MBSs.

In order to underpin that the unusual conductance peaks stem from the non-commuting property of $U_c$ and $\tilde{U}_c$, we express the Majorana Green’s function as (cf. Section II B)

$$ M(t_Q, t_Q - n) = -i \text{Tr} \left[ \rho_0 (U_c^n)^Q (\tilde{U}_c)^{Q-n} \right] = -i \text{Tr} \left[ \rho'_0 (\tilde{U}_c)^n (U_c^n)^{n} \right], \quad (27) $$

where we used the cyclic property of the trace. $\rho_0$ is a density matrix of the initial Majorana state at $t = t_0$ and $\rho'_0 = (U_c^n)^Q \rho_0 (U_c^n)^Q$. As Eq. (25) is independent of the initial state, the specific form of $\rho_0$ or $\rho'_0$ is unimportant. Then $S(\varepsilon)$ in Eq. (22) can be rewritten as

$$ S(\varepsilon) = \text{Re} \left\{ \text{Tr} \left[ \rho'_0 \tilde{S}(\varepsilon) \right] \right\}, \quad \tilde{S}(\varepsilon) = \frac{1}{2} + \sum_{n=1}^{Q} e^{in\varepsilon} (U_c^n)^{n} (U_c^n)^{n}. \quad (28) $$

$\tilde{S}(\varepsilon)$ is a sum of composite operations of the parity-preserving rotation $U_c$ and the tunneling-assisted braiding $\tilde{U}_c$. As illustrated in Fig. 2(c), the operations $U_c^n$ and $\tilde{U}_c$ do not commute, and thus the sum cannot be treated as a simple geometric series. As explained already in Section II B around Eq. (12), the term $\text{Re}\{\text{Tr}[\rho'_0 e^{in\varepsilon} (U_c^n)^{n} (U_c^n)^{n}]\}$ comes from the overlap between the following two processes of temporal length $QT_J$: In process I, an electron tunnel from the tip to $\gamma_1$ at $t_0 + (Q - n)T_J$, and in process II, the tunneling happens at $t_0 + QT_J$. Here $e^{in\varepsilon}$ is the dynamical phase factor gained for the time interval $nT_J$. The interference between terms of different $n$ determines the peak positions of the conductance.

Since the positions of the conductance peaks are independent of the parity, let us assume that an even parity state is prepared at $t = t_0$; the case of an odd parity state is obtained in a similar way. In the limit $Q \to \infty$, $\tilde{S}(\varepsilon)$ for an even parity is given by

$$ \tilde{S}(\varepsilon)|_{\text{even}} = \frac{1}{2} + \sum_{n=1}^{\infty} e^{in\varepsilon} (U_{co})^n (U_{ce}^\dagger)^n = \frac{1}{2} + \left( -\sin\varepsilon e^{i\varepsilon} - i\tau_\varepsilon e^{i2\varepsilon} + \sin\varepsilon e^{i3\varepsilon} - e^{i4\varepsilon} \right) \times \sum_{m=0}^{\infty} e^{im(4\varepsilon + \pi)}. \quad (30) $$

Here $n$ is a number counting how many $\pi/2$ rotations of MBSs are performed before tunneling back to the tip with operations $(U_{co})^n (U_{ce})^n$. Obviously, this expression produces the same coherent interference effect shown in Eq. (23), as it should be. It elucidates the role of non-commuting braiding operations. In the second line in Eq. (30), the summation is classified into four categories in each of which the Pauli matrix (including the identity matrix) is factored out, manifesting the interference with the period $4T_J$. Since the matrices are linearly independent, there exists no basis transformation, or equivalently, a way of fusing MBSs, in which the interfering terms are diagonalized simultaneously,

$$ \sum_{n=1}^{\infty} e^{in\varepsilon} (U_{co})^n (U_{ce})^n \leftrightarrow \sum_{n=1}^{\infty} e^{in\varepsilon} e^{in\varphi} \quad (31) $$

with any phase $\varphi$. The absence of such a transformation is characteristic of the non-commuting braiding operations. We argue in Sec. III B that an interference effect of four MBSs where such a phase $\varphi$ can be found (see Eq. (33)) yields conductance peaks which are clearly distinguishable from those in Eq. (26).

We note that the peak positions found in the weak tunneling limit at finite temperature remain the same independent of the tunneling strength. We show in Sec. IV that the Floquet analysis taking into account the tunneling strength to all orders at zero temperature reproduces the same peak positions as given in Eq. (26).

### B. Case of commuting braiding operators

For an unambiguous demonstration of the relation of the conductance peak positions to the presence of non-Abelian operations, we consider the same tunneling experiment as above, that is, a tip is coupled to $\gamma_1$ at $t = t_0$ and the system Hamiltonian is periodic $H(t) = H(t + T_J)$, but with commutative operations of four MBSs. We explicitly show that the resulting conductance peak positions are different from those in Eq. (26).

Let us introduce two different evolution operators, $W$ and $\tilde{W} = \gamma_1 W \gamma_1$, corresponding to the parity-conserving braiding operator $U_c$ and the tunneling-assisted braiding operator $\tilde{U}_c$, respectively. The only difference compared to $U_c$ and $\tilde{U}_c$ is that $W$ and $\tilde{W}$ commute such that $[W, \tilde{W}] = 0$. This commutativity condition allows us to
find the generic form of $W$ (and thus of $\bar{W}$) to be,

$$W = \begin{pmatrix} W_x & 0 \\ 0 & W_0 \end{pmatrix} = \left( e^{i\beta \hat{n}_w \cdot \vec{\tau}} 0 \\ 0 e^{i\beta' \hat{n}_w \cdot \vec{\tau}} \right),$$

(32)

up to an overall phase factor which does not affect the tunneling current. The rigorous derivation of this form of $W$ is given in Appendix C. $W$ is obtained by interchanging $W_x$ and $W_0$ in the $W$ matrix. Here the general commuting braiding operations are characterized by the unit vector $\hat{n}_w$ and angles $\beta$ and $\beta'$. Two related situations are drawn in Fig. 5.

Different to Eq. (31), the interfering terms in this case commute and thus can be expressed as

$$\sum_{n=1}^{\infty} e^{in\tilde{\epsilon}W_{\alpha}^n(W_{\alpha}^0)^n} = \sum_{n=1}^{\infty} e^{in\tilde{\epsilon}e^{n(\beta'-\beta)\hat{n}_w \cdot \vec{\tau}}},$$

(33)

indicating that the relative dynamics between $n$ and $n+1$ cycles adds a phase factor to the eigenstate of $\hat{n}_w \cdot \vec{\tau}$. In order to find its consequence, we calculate the anti-commutation relation,

$$\{\gamma_i(t_q), \gamma_j(t_{q'})\} = (\bar{W}W^t)^{(q-q')} + (WW^t)^{(q-q')} = 2 \cos [(q - q')(\beta - \beta')],$$

(34)

where $\gamma_i(t_q) = (W^t)^{\gamma_i}W^q$. By substituting this into $S(\varepsilon) + S(-\varepsilon)$ of Eq. (20), the peak positions of the conductance in the low-bias voltage regime are found as

$$eV = \frac{\hbar}{T_j} [2\pi j \pm (\beta - \beta')],$$

(35)

where $j$ is an integer. If $0 \leq |\beta - \beta'| < \pi$, the peak separations $2|\beta - \beta'|\hbar/T_j$ and $(2\pi - 2|\beta - \beta'|)\hbar/T_j$ appear alternately. If $\pi \leq |\beta - \beta'| < 2\pi$, then the separations of $(2|\beta - \beta'| - 2\pi)\hbar/T_j$ and $(4\pi - 2|\beta - \beta'|)\hbar/T_j$ are seen alternately. Note that for any value of $|\beta - \beta'|$, these peak configurations cannot give rise to the results shown in Eq. (26), manifesting the noncommutative structure of non-Abelian statistics.

IV. FLOQUET ANALYSIS

We confirm the result in Eq. (26) by using a Floquet analysis for four rotating MBSs including all orders in tunneling at zero temperature [11]. This Floquet description is applicable as the time-dependent Hamiltonian in Eq. (17) is periodic in time with periodicity $T_j$, $H(t) = H(t + T_j)$ [45].

The Floquet Hamiltonian $H_F$ is defined by the time-independent Hamiltonian that would yield the same evolution as with $U_c$ in Eq. (6) after one period $T_j$,

$$U_c = e^{-\frac{\pi}{4} H_F T_j},$$

(36)

$$H_F = \frac{\hbar}{T_j} \left( \begin{array}{cc} \frac{\pi}{4} \hat{n}_{ce} \cdot \vec{\tau} & 0 \\ 0 & \frac{\pi}{4} \hat{n}_{co} \cdot \vec{\tau} \end{array} \right) - \frac{2\pi l \hbar}{T_j} \mathbb{I},$$

where $\hat{n}_{ce}$ and $\hat{n}_{co}$ are given in Eq. (9), and $l$ is an integer. The last term $-2\pi l \hbar/T_j$ in $H_F$ only shifts the energy levels and can be ignored for the moment. At the end of the calculation, we will restore this term. The representation of $H_F$ in terms of Majorana operators is

$$H_F = \frac{i}{\sqrt{2}} E_0 \left( s_1 \gamma_2 + \frac{1}{\sqrt{2}} \gamma_1 \gamma_3 - s_1 \gamma_4 + s_2 \gamma_3 - \frac{1}{\sqrt{2}} \gamma_2 \gamma_4 - s_3 \gamma_4 \right)$$

(37)

$$= \frac{i}{2} \sum_{i \neq j} t_{ij} \gamma_i \gamma_j,$$

where $E_0 = \pi \hbar/(4T_j)$ and the subindices $i$ and $j$ range from 1 to 4. $t_{ij}$ describing the effective coupling between MBSs caused by the rotation is the $(i,j)$ component of the matrix $t$ in the basis $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$.

$$t = \frac{E_0}{\sqrt{2}} \left( \begin{array}{cccc} 0 & s & \frac{1}{\sqrt{2}} & -s \\ -s & 0 & s & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -s & 0 & -s \\ s & \frac{1}{\sqrt{2}} & s & 0 \end{array} \right)$$

(38)

We calculate the differential conductance of the metal tip coupled to the Majorana network described by $H_F$ following the Keldysh technique calculation used in Ref. [61]. We consider the case where the tip is coupled only to $\gamma_1$. The differential conductance then is given by the formula

$$\frac{dI}{dV} = \frac{2e^2}{h} \int d\omega \Gamma \text{Im}[G^{\text{R}}_1(\omega)] \frac{d}{d\omega} n_F(\omega - eV),$$

(39)
where \( G^{R}_{11}(\omega) \) is the \((1,1)\) component of the \(4 \times 4\) matrix \(G^{R}(\omega)\) given by

\[
G^{R}(\omega) = 2[\omega - 2i\Gamma]^{-1},
\]

where \( \Gamma \) describes the tip-MBS tunneling and it is the \(4 \times 4\) matrix whose components are given by \((\Gamma)_{ij} = \Gamma \delta_{i1} \delta_{j1}\). \(G^{R}_{11}(\omega)\) is then computed as

\[
G^{R}_{11}(\omega) = \frac{2\omega(\omega^2 - 5E_0^2)}{(\omega^2 - E_0^2)(\omega^2 - 9E_0^2) + 2i\omega \Gamma(\omega^2 - 5E_0^2)}.
\]

(41)

Substituting this into Eq. (39) gives the differential conductance at zero temperature,

\[
\frac{dI}{dV} = \frac{2e^2}{\hbar} \left[ 1 + \frac{(eV)^2 - E_0^2)^2((eV)^2 - 9E_0^2)^2}{4(eV)^2\Gamma^2((eV)^2 - 5E_0^2)^2} \right]^{-1}
\]

(42)

which exhibits peaks at \(eV = \pm E_0\) and \(\pm 3E_0\). If we restore the term \(-2\pi i\hbar/T_J\) in Eq. (36), the peaks would be at

\[
eV = \pm \frac{\pi \hbar}{4T_J} - \frac{2\pi \hbar}{T_J}, \quad \pm \frac{3\pi \hbar}{4T_J} - \frac{2\pi \hbar}{T_J}.
\]

(43)

Therefore, we conclude that the Floquet theory gives a consistent result with the time-averaged differential conductance shown in Fig. 4.

V. CONCLUSION

We have demonstrated that a non-Abelian state evolution can be identified in tunneling conductance measurements between four rotating MBSSs in a Corbino geometry topological Josephson junction and a metal tip. Unitary evolutions of the MBSSs acting on even and odd parity subspaces, which are separable if the fermion parity is conserved, are intertwined by electron tunneling, inducing parity-conserving and tunneling-assisted braiding operators. Coherent interference between different orders of round trips of Majorana states governed by these braiding operators yields a time-averaged conductance exhibiting peaks with a period of \(\hbar/(4T_J)\) as a function of bias voltage between the metal tip and the Josephson junction, whereas the period of the Hamiltonian is \(T_J\). This constitutes a clear signature of non-Abelian state evolution of four MBSSs.

We explicitly showed that these results have their origin in the non-commutativity of the parity-conserving and tunneling-assisted braiding operators and are therefore independent on the way we fuse the MBSSs into fermions. The noncommuting nature of braiding operators acting within a degenerate ground state manifold is one of the hallmarks of non-Abelian quasiparticles. We contrasted this scenario with a generic tunneling experiment of four MBSSs whose unitary evolutions of even and odd parity sectors commute to rule out accidental observations that could lead to the same results. We expect that other kinds of exotic zero modes such as MBSSs in time-reversal invariant topological superconductors [46–52] and parafermions [53–60] could be analyzed with our time-dependent tunneling scheme to manifest the quantum statistics of the corresponding modes.

The peak positions obtained at finite temperature in the weak tunneling limit are shown to be independent of initial conditions of the setup, such as fermion parity and fermion occupation numbers in which the Majorana states are initialized. This property is plausible since the tunneling current is averaged over a long time after the Corbino Josephson junction has been coupled to the metal tip, so that initial conditions become irrelevant. By using Floquet theory, the parity-preserved rotating Majorana states are transformed to effectively static Majorana states. The transformation encodes the rotation dynamics in the coupling between the static Majorana states. It allows us to map the time-dependent tunneling problem to tunneling between a metal tip and static coupled Majorana states. This Floquet analysis confirms that the peak positions remain the same at zero temperature and arbitrary tunneling strength. The height of the peak is \(2e^2/h\) at zero temperature like for tunneling into a Majorana zero mode localized at the end of a topological superconducting nanowire [61, 62], and it is proportional to \(\Gamma/(k_B T)\) at finite temperature for weak tunneling rates \(\Gamma/h\). An interesting direction for further study would be to investigate if the non-Abelian signature persists in tunneling currents measured at a time which is not far from the moment when the Corbino JJ is suddenly connected to the metal tip where the initial condition of the system may be important [63, 64].

The experimental realization may be challenging, but within reach of current experiments. Assuming the proximity-induced superconducting gap \(\Delta_0 = 1\) meV that can be achieved, for example, in thin-films of Nb or NbN [65, 66], the excitation energy gap at the junction can be estimated by \(E_g = \Delta_0 \sqrt{4\xi/R} \sim 0.9\) meV for the radius of the junction \(R = 5\) [11, 67]. We require a coherent and adiabatic rotation of the MBSSs so that \(T_J\) (the time taken for the \(\pi/2\) rotation) should satisfy \(\hbar/E_g ( = 0.7\) ps) \(\ll T_J \ll t_{qp} (\gtrsim \mu s)\) where \(t_{qp}\) is the quasiparticle poisoning time [68, 69]. At the same time, the temperature should be much smaller than the separation between the conductance peaks \(\hbar/(4T_J)\). We believe that the Corbino geometry topological Josephson junction can also be realized in heterostructures of a thin-film topological insulator and a superconductor [70] or Pb/Co/Si(111) two-dimensional topological superconductor [71].

Our proposal provides a blueprint for an experimentally feasible way to test the non-Abelian character of MBSSs via a time-averaged conductance experiment. We thereby utilize a combination of parity switching tunneling events with parity-conserving braiding cycles to build two non-commuting braiding operators leading to peculiar interference effects in transport. Our findings provide a new way of looking at braiding experiments,
by actively using parity switching events by tunneling, instead of avoiding them.

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Appendix A: Majorana wave functions

In this appendix, we provide the details of the calculation of Majorana wave functions \( \Psi_{Mj}(r) \) with \( j \in \{1, 2, 3, 4\} \) in a Corbino geometry topological Josephson junction shown in Fig. 1(b). We solve the BdG equation \( \mathcal{H}_C \Psi(r, \phi_1, \phi_2) = E\Psi(r, \phi_1, \phi_2) \) for \( E = 0 \) and \( \mu = 0 \). Hereafter, we use the dimensionless length scale \( r \) normalized by \( \xi = \hbar v_F / \Delta_0 \).

For \( r < R \), the wave function \( \Psi_{r<R}(r, \theta) \) is given by

\[
\Psi_{r<R}(r, \theta) = \sum_{m=-\infty}^{\infty} a_m \begin{pmatrix} e^{im\theta} e^{i\phi_1/2} I_m(r) \\ 0 \\ e^{i(m+1)\theta} e^{-i\phi_1/2} I_{m+1}(r) \end{pmatrix} + b_m \begin{pmatrix} 0 \\ 0 \\ -e^{i(m+1)\theta} e^{-i\phi_1/2} I_{m+1}(r) \end{pmatrix},
\]

where \( I_m(r) \) is the modified Bessel function of the first kind, and \( a_m \) and \( b_m \) are coefficients. The wave function \( \Psi_{r>R}(r, \theta) \) at \( r > R \) is given by

\[
\Psi_{r>R}(r, \theta) = \sum_{n=-\infty}^{\infty} c_n \begin{pmatrix} e^{i(n+1)\theta} e^{i\phi_2/2} r^{-2} K_{n+2}(r) \\ 0 \\ e^{i(n+5)\theta} e^{-i\phi_2/2} r^{-2} K_{n+3}(r) \end{pmatrix} + d_n \begin{pmatrix} 0 \\ 0 \\ e^{i(n+4)\theta} e^{-i\phi_2/2} r^{-2} K_{n+2}(r) \end{pmatrix},
\]

where \( K_n(r) \) is the modified Bessel function of the second kind, and \( c_n \) and \( d_n \) are coefficients. We consider only the wave functions with spin down as those for spin up become non-normalizable solutions, and hence the coefficients \( a_m \) and \( c_n \) should be zero for all \( m \) and \( n \). In order to get the coefficients \( b_m \) and \( d_n \) we match the spin-down components at \( r = R \),

\[
\sum_{m=-\infty}^{\infty} b_m \begin{pmatrix} e^{i(m+1)\theta} e^{i\phi_1/2} I_{m+1}(R) \\ -e^{i(m+1)\theta} e^{-i\phi_1/2} I_{m+1}(R) \end{pmatrix} = \sum_{n=-\infty}^{\infty} d_n \begin{pmatrix} 0 \\ 0 \\ e^{i(n+4)\theta} e^{-i\phi_2/2} r^{-2} K_{n+2}(R) \end{pmatrix},
\]

leading to

\[
\begin{align*}
    b_1 e^{i\phi_1/2} I_{l+1}(R) &= d_1 e^{i\phi_2/2} R^2 K_{l+3}(R), \\
    -b_{l+4} e^{-i\phi_2/2} I_{l+4}(R) &= d_l e^{-i\phi_2/2} R^2 K_{l+2}(R),
\end{align*}
\]

and the following recurrence relations,

\[
\begin{align*}
    b_{l+4} &= -e^{i(\phi_1-\phi_2)} I_{l+1}(R) K_{l+2}(R) b_l, \\
    d_{l+4} &= -e^{i(\phi_1-\phi_2)} I_{l+5}(R) K_{l+2}(R) d_l,
\end{align*}
\]

where \( l \) and \( l' \) are integers. From these recurrence relations we can construct four linearly independent solutions,

\[
\Psi_{ij}(r, \theta) = \Theta(R-r) \sum_{m=-\infty}^{\infty} b_{4m+y} \Psi_{4m+y}^<(r, \theta) + \Theta(r-R) \sum_{n=-\infty}^{\infty} d_{4n+y} \Psi_{4n+y}^>(r, \theta)
= b_2 \Psi_1'(r, \theta),
\]

\( a_m \) and \( b_m \) are coefficients. We consider only the wave functions with spin down as those for spin up become non-normalizable solutions, and hence the coefficients \( a_m \) and \( c_n \) should be zero for all \( m \) and \( n \). In order to get the coefficients \( b_m \) and \( d_n \) we match the spin-down components at \( r = R \),

\[
\sum_{m=-\infty}^{\infty} b_m \begin{pmatrix} e^{i(m+1)\theta} e^{i\phi_1/2} I_{m+1}(R) \\ -e^{i(m+1)\theta} e^{-i\phi_1/2} I_{m+1}(R) \end{pmatrix} = \sum_{n=-\infty}^{\infty} d_n \begin{pmatrix} 0 \\ 0 \\ e^{i(n+4)\theta} e^{-i\phi_2/2} r^{-2} K_{n+2}(R) \end{pmatrix},
\]

leading to

\[
\begin{align*}
    b_1 e^{i\phi_1/2} I_{l+1}(R) &= d_1 e^{i\phi_2/2} R^2 K_{l+3}(R), \\
    -b_{l+4} e^{-i\phi_2/2} I_{l+4}(R) &= d_l e^{-i\phi_2/2} R^2 K_{l+2}(R),
\end{align*}
\]

and the following recurrence relations,

\[
\begin{align*}
    b_{l+4} &= -e^{i(\phi_1-\phi_2)} I_{l+1}(R) K_{l+2}(R) b_l, \\
    d_{l+4} &= -e^{i(\phi_1-\phi_2)} I_{l+5}(R) K_{l+2}(R) d_l,
\end{align*}
\]

where \( l \) and \( l' \) are integers. From these recurrence relations we can construct four linearly independent solutions,
where \( \eta \in \{-2, -1, 0, 1\} \) and \( \Psi'_\eta(r, \theta) \) are

\[
\Psi'_\eta(r, \theta) = \Theta(R - r) \sum_{m = -\infty}^{\infty} B_{m\eta} \Psi_{4m+\eta}^<(r, \theta) + \Theta(r - R) \sum_{n = -\infty}^{\infty} D_{n\eta} \Psi_{4n+\eta}^>(r, \theta). \tag{A7}
\]

Here the wave functions \( \Psi_{4m+\eta}^<(r, \theta) \) at \( r < R \) and \( \Psi_{4n+\eta}^>(r, \theta) \) at \( r > R \) are given by

\[
\Psi_{4m+\eta}^<(r, \theta) = \begin{pmatrix}
0 \\
-e^{i(4m+\eta+1)\theta} e^{i\phi_1/2} I_{4m+\eta+1}(r) \\
-e^{i(4m+\eta)\theta} e^{-i\phi_1/2} I_{4m+\eta}(r) \\
0
\end{pmatrix},
\]

\[
\Psi_{4n+\eta}^>(r, \theta) = \begin{pmatrix}
0 \\
i e^{i(4n+\eta+1)\theta} e^{i\phi_2/2} I_{4n+\eta+1}(r) \\
i e^{i(4n+\eta+4)\theta} e^{-i\phi_2/2} I_{4n+\eta+4}(r) \\
0
\end{pmatrix}, \tag{A8}
\]

and coefficients \( B_{m\eta} \) and \( D_{n\eta} \) are

\[
B_{m\eta} = \begin{cases}
(-1)^m e^{i m (\phi_1 - \phi_2)} & \text{for } m \geq 1, \\
1 & \text{for } m = 0, \\
(-1)^m e^{i m (\phi_1 - \phi_2)} & \text{for } m \leq -1,
\end{cases}
\]

\[
D_{n\eta} = \begin{cases}
(-1)^n e^{i (n+1/2) (\phi_1 - \phi_2)} & \text{for } n \geq 1, \\
e^{i (\phi_1 - \phi_2)/2} & \text{for } n = 0, \\
(-1)^n e^{i (n+1/2) (\phi_1 - \phi_2)} & \text{for } n \leq -1.
\end{cases}
\]

By superposing the solutions \( \Psi'_\eta(r, \theta) \) in Eq. (A7) and using particle-hole symmetry, we find four Majorana states \( \Psi_{Mj} \) satisfying \( \Xi \Psi_{Mj} = \Psi_{Mj} \) where \( \Xi = \sigma_y \tau_y \mathcal{C} \) is the particle-hole operator and \( \mathcal{C} \) is the operator for complex conjugation. They are given by

\[
\Psi_{Mj}(r, \theta) = \sum_{\eta = -2}^{2} \frac{1}{\sqrt{4N_\eta}} \exp \left[ i \frac{\pi}{4} - i \left( \eta + \frac{1}{2} \right) \theta_j \right] \Psi'_\eta(r, \theta), \tag{A9}
\]

where the azimuthal angles \( \theta = \theta_j \) at which \( \Psi_{Mj} \) are localized are given by \( \theta_j = (3\pi - 2\pi j)/4 - (\phi_1 - \phi_2)/4 \) and \( N_\eta \) are normalization constants such that

\[
N_\eta = \int d^2 r \Psi_{Mj}'(r) \Psi_{Mj}(r). \tag{A10}
\]

Appendix B: Time-averaged tunneling current

The time-dependent tunneling current between a metal tip and a Corbino geometry topological Josephson junction in the weak tunneling limit can be obtained using lowest order perturbation theory. To lowest order in \( H_T(t) \), we find
the tunneling current \( \langle I(t) \rangle = -e(dN_T(t)/dt) \),
\[
\langle I(t) \rangle = \frac{1}{i\hbar} \int_{t_0}^{t} dt' \langle [\hat{I}(t), \hat{H}_T(t')] \rangle
\]
\[
= \frac{2e}{\hbar^2} \text{Re} \left\{ \int_{t_0}^{t} dt' \sum_{kq} \Gamma_{kqq}(t,t')[G_k(t,t') - \bar{G}_k(t,t')]M(t,t') \right\}, \quad (B1)
\]
where \( N_T(t) = \sum_k c_k^\dagger(t) c_k(t) \) is the metal tip number operator. \( \hat{H}_T(t') \) and \( \hat{I}(t) \) which are expressed in the interaction picture are given by
\[
\hat{H}_T(t') = \sum_{kq} e^{-\lambda|t'-t_0|/\hbar} V_{kq}(t_0) \hat{c}_k \hat{\gamma}_1(t_0) + \text{H.c.}, \quad (B2)
\]
\[
\hat{I}(t) = \frac{e}{\hbar} \sum_{kq} [ie^{-\lambda|t'-t_0|/\hbar} V_{kq}(t_0) \hat{c}_k \hat{\gamma}_1(t_0) + \text{H.c.}]. \quad (B3)
\]
The tunneling Hamiltonian \( \hat{H}_T(t') \) switched on at time \( t_0 \) is valid in the low energy regime where MBSs are the only relevant states for the tunneling current and for \( \lambda^{-1} \ll T_J \). The coupling coefficient \( V_{kq}(t_0) \) between the tip and \( \hat{\gamma}_1(t_0) \) is
\[
V_{kq}(t_0) = \int d^2 r \ t_{k}(r) \Psi_{M1\downarrow}(r,t_0), \quad (B4)
\]
where \( t_{k}(r) \) is the tunneling coefficient between the tip and the junction and \( \Psi_{M1\downarrow}(r,t_0) \) is the electron spin-down component of the Majorana wave function \( \Psi_{M1}(r) \) in Eq. (A9). In Eq. (B1), the time-dependent tunneling parameter \( \Gamma_{kqq}(t,t') \) and the tip-electron Green’s functions \( \hat{G}_k(t,t') \) are given by
\[
\Gamma_{kqq}(t,t') = |V_{kq}(t_0)|^2 e^{-\lambda|t-t_0|/\hbar} e^{-\lambda|t'-t_0'|/\hbar},
\]
\[
G_k(t,t') = i\langle \hat{c}_k(t) \hat{c}_k^\dagger(t') \rangle = -ie^{i(\varepsilon_k + eV)(t-t')/\hbar} \left[ 1 - n_F(\varepsilon_k) \right],
\]
\[
\bar{G}_k(t,t') = -ie^{i(\varepsilon_k + eV)(t-t')/\hbar} n_F(\varepsilon_k), \quad (B5)
\]
where \( \langle \cdot \rangle \) is the expectation value over a thermal ensemble of initial states at \( t = t_0 \), and \( n_F(\varepsilon_k) = 1/[1 + e^{\varepsilon_k/(k_B T)}] \) is the Fermi-Dirac distribution at \( t = t_0 \) with the temperature \( T \). Since the tunneling current is exponentially small except for \( t = t_q \) and \( t' = t_{q'} \) due to the presence of the exponential factor of \( \Gamma_{kqq}(t,t') \), we can approximate the Majorana Green’s function,
\[
M(t,t') \approx M(t_q,t_{q'}) = -i\langle \Psi_M(t_0) | \hat{\gamma}_1(t_q) \hat{\gamma}_1(t_{q'}) | \Psi_M(t_0) \rangle. \quad (B6)
\]
If \( t \) is very far from \( t_0 \), we can find that the difference between \( \langle I(t) \rangle \) and \( \langle I(t - T_J) \rangle \) is negligibly small,
\[
\langle I(t) \rangle - \langle I(t - T_J) \rangle \sim \sum_k e^{-i(\varepsilon_k + eV)(t-t_0)} n_F(\varepsilon_k) \text{c.c.} \sim 0, \quad (B7)
\]
yielding a time-periodic behavior of the tunneling current \( \langle I(t) \rangle = \langle I(t - T_J) \rangle \). Without loss of generality, we assume that \( t \) is in the interval \([Q - 1/2]T_J, (Q + 1/2]T_J \) where \( Q \) is a very large integer, \( Q \gg 1 \). Then the time-averaged tunneling current over an interval \([t - T_J, t] \) is
\[
\bar{I} = \frac{1}{T_J} \int_{t-T_J}^{t} dt \langle I(t) \rangle. \quad (B8)
\]
Let us change the variable in Eq. (B1) from \( \varepsilon_k \) to \( \varepsilon_k = eV \). After some algebra, we find \( \bar{I} \) as
\[
\bar{I} = \frac{e}{\hbar} \int_{-\infty}^{\infty} d\varepsilon \ T(\varepsilon)S(\varepsilon)[n_F(\varepsilon - eV) - n_F(\varepsilon + eV)]
\]
\[
= \frac{e}{\hbar} \int_{-\infty}^{\infty} d\varepsilon \ T(\varepsilon)S(\varepsilon)n_F(\varepsilon - eV) + \frac{e}{\hbar} \int_{-\infty}^{\infty} d\varepsilon \ T(\varepsilon)S(\varepsilon)n_F(\varepsilon + eV) - \frac{e}{\hbar} \int_{-\infty}^{\infty} d\varepsilon \ T(\varepsilon)S(\varepsilon)
\]
\[
= \frac{e}{\hbar} \int_{-\infty}^{\infty} d\varepsilon \ T(\varepsilon)[S(\varepsilon) + S(-\varepsilon)]n_F(\varepsilon - eV) - \frac{e}{\hbar} \int_{-\infty}^{\infty} d\varepsilon \ T(\varepsilon)S(\varepsilon), \quad (B9)
\]
where \( T(\varepsilon) \) and \( S(\varepsilon) \) were defined in Eqs. (21) and (22). The second term in the third line can be disregarded because it is independent of the bias voltage and does not contribute to the tunneling conductance.
Appendix C: Derivation of a generic form of $W$

In Sec. III B, we argue that the generic form of the matrix $W$ satisfying $[W, \bar{W}] = 0$ is

$$W = \begin{pmatrix} W_e & 0 \\ 0 & W_o \end{pmatrix} = \begin{pmatrix} e^{i\beta \hat{n}_w \cdot \vec{\tau}} & 0 \\ 0 & e^{i\beta' \hat{n}_w \cdot \vec{\tau}} \end{pmatrix}$$

(C1)

where

$$\bar{W} = \gamma_1(t_0)W \gamma_1(t_0) = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} \begin{pmatrix} W_e & 0 \\ 0 & W_o \end{pmatrix} \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} = \begin{pmatrix} W_o & 0 \\ 0 & W_e \end{pmatrix}.$$  (C2)

Here $\sigma_0$ is the $2 \times 2$ identity matrix and $0$ is the null matrix. In this appendix, we prove this argument. Let us define $W$ as

$$W = \begin{pmatrix} W_e & 0 \\ 0 & W_o \end{pmatrix} = \begin{pmatrix} e^{i\beta \hat{n}_e \cdot \vec{\tau}} & 0 \\ 0 & e^{i\beta' \hat{n}_o \cdot \vec{\tau}} \end{pmatrix}.$$  (C3)

Case 1. $\beta$ or $\beta'$ is equal to a multiple of $\pi$.

We show that the form of $W$ in Eq. (C1) holds for the following cases

- $\beta = m\pi$ and $\beta' \neq n\pi$
- $\beta \neq m\pi$ and $\beta' = n\pi$
- $\beta = m\pi$ and $\beta' = n\pi$,

where $m$ and $n$ are integers. It is enough to consider the first case where $\beta = m\pi$ and $\beta' \neq n\pi$ as the proofs for the other two cases are similar. The matrix $W_e$ in this case is

$$W_e = \cos\beta + i \hat{n}_e \cdot \vec{\tau} \sin\beta = (-1)^m \sigma_0,$$  (C6)

independent of $\hat{n}_e$ and the commutation relation between $W$ and $\bar{W}$ is zero,

$$[W, \bar{W}] = \begin{pmatrix} [W_e, W_o] & 0 \\ 0 & [W_o, W_e] \end{pmatrix} = 0.$$  (C7)

We can rewrite $W_e$ as

$$W_e = \cos\beta + i \hat{n}_o \cdot \vec{\tau} \sin\beta = e^{i\beta \hat{n}_o \cdot \vec{\tau}},$$  (C8)

which is a valid expression if $\beta = m\pi$. Therefore, $W$ is written as

$$W = \begin{pmatrix} W_e & 0 \\ 0 & W_o \end{pmatrix} = \begin{pmatrix} e^{i\beta \hat{n}_o \cdot \vec{\tau}} & 0 \\ 0 & e^{i\beta' \hat{n}_o \cdot \vec{\tau}} \end{pmatrix},$$  (C9)

which completes the proof in this case by changing the notation $\hat{n}_e$ by $\hat{n}_o$.

Case 2. $\beta \neq m\pi$ and $\beta' \neq n\pi$.

In this case, we solve the problem

$$[W, \bar{W}] = \begin{pmatrix} [W_e, W_o] & 0 \\ 0 & [W_o, W_e] \end{pmatrix} = 0.$$  (C10)
Specifically, we need to solve
\[ [W, \hat{W}] = [\cos \beta + i \hat{n}_e \cdot \vec{\tau} \sin \beta, \cos \beta' + i \hat{n}_o \cdot \vec{\tau} \sin \beta'] \]
\[ = -\sin \beta \sin \beta' [\hat{n}_e \cdot \vec{\tau}, \hat{n}_o \cdot \vec{\tau}] \]
\[ = 0. \]  
(C11)

Because \( \sin \beta \sin \beta' \neq 0 \) in this case, \( [\hat{n}_e \cdot \vec{\tau}, \hat{n}_o \cdot \vec{\tau}] \) should be zero, which leads to
\[ [\hat{n}_e \cdot \vec{\tau}, \hat{n}_o \cdot \vec{\tau}] = 2i [(e_y o_z - e_z o_y) \tau_x + (e_z o_x - e_x o_z) \tau_y + (e_x o_y - e_y o_x) \tau_z] \]
\[ = 0. \]  
(C12)

As the Pauli matrices \( \tau_{x,y,z} \) form an orthogonal basis, we have
\[ e_y o_z - e_z o_y = 0, \]
\[ e_z o_x - e_x o_z = 0, \]
\[ e_x o_y - e_y o_x = 0, \]  
(C13)

and
\[ \frac{o_x}{e_x} = \frac{o_y}{e_y} = \frac{o_z}{e_z}, \]  
(C14)

which allows us to find
\[ (o_x, o_y, o_z) = s(e_x, e_y, e_z), \]  
(C15)

where \( s = 1 \) or \( -1 \). Therefore, \( \hat{n}_o = s\hat{n}_e \) and \( W \) in this case is obtained by
\[ W = \begin{pmatrix}
\hat{e}^{i\beta\hat{n}_e \cdot \vec{\tau}} & 0 \\
0 & \hat{e}^{i\beta\hat{n}_o \cdot \vec{\tau}}
\end{pmatrix} = \begin{pmatrix}
\hat{e}^{i\beta\hat{n}_e \cdot \vec{\tau}} & 0 \\
0 & \hat{e}^{i\beta\hat{n}_w \cdot \vec{\tau}}
\end{pmatrix}. \]  
(C16)

By redefining notations \( s\beta' \) and \( \hat{n}_e \) by \( \beta' \) and \( \hat{n}_w \), we obtain Eq. (C1).
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