Multiple Zeta Values and Ideles

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Abstract

In this paper we give two idelic representations of the multiple zeta values - one using iterated integrals over the finite ideles and the other using iterated integrals over the idele class group. Each of the representations leads to a shuffle relation. Thus, we recover in a unified way the two types of shuffle relations of multiple zeta values via the iterated integrals over finite ideles and via iterated integrals over the idele class group.

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0 Introduction

Multiple zeta values are the values of a multiple zeta functions at the positive integers. Let us recall the Riemann zeta values:

$$\zeta(k) = \sum_{n>0} \frac{1}{n^k}.$$  

They were examined first by Euler.

Multiple zeta values of depth $d$ is

$$\zeta(k_1, \ldots, k_d) = \sum_{0<n_1<\cdots<n_d} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}}.$$  

These values were also examined by Euler.

Kontsevich expressed the multiple zeta values as iterated integrals.

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Theorem 0.1 ([G]) Let $k_1, \ldots, k_d$ be $d$ positive integers with $k_d > 1$. Then

$$
\zeta(k_1, \ldots, k_d) = \int_0^1 \cdots \int_0^{x_3} \left( \int_0^{x_2} \frac{dx_1}{1-x_1} \right) \left( \int_0^{x_1} \frac{dx_2}{x_2} \right) \cdots \left( \int_0^{x_1} \frac{dx_k}{k-1} \right) \left( \int_0^{x_k} \frac{dx_{k_1+1}}{1-x_{k_1+1}} \right) \cdots \left( \int_0^{x_{k_1+d}} \frac{dx_{k_1+d}}{x_{k_1+d}} \right).
$$

The iterated integral on the right hand side can be written as

$$
\int \cdots \int_{0 < x_1 < x_2 < \ldots < x_{k_1+d} < 1} \frac{dx_1}{1-x_1} \wedge \frac{dx_2}{x_2} \wedge \cdots \wedge \frac{dx_{k_1+1}}{x_{k_1+1}} \wedge \cdots \wedge \frac{dx_{k_1+d}}{x_{k_1+d}}.
$$

In this paper we give idelic interpretation of multiple zeta values (MZVs). The advantage of this approach is that the double shuffle relations for MZVs follow directly from the two idelic representations. One representation of a MZV is as an iterated integral over the finite ideles, which we define in this paper. The shuffle relations for MZV follows directly from that representation. The other representation of MZV is as an iterated integral over the idele class group. The shuffle relations for MZV follows directly from this representation. We prove the following:

Theorem 0.2 (Double shuffle relations via ideles)

(a) $\zeta(k_1, \ldots, k_d) = I(k_1, \ldots, k_d)$, where $I(k_1, \ldots, k_d)$ is an iterated integral over the finite ideles $A^\times$. Moreover, this this representation proves the shuffle relations among multiple zeta values.

(b) $\zeta(k_1, \ldots, k_d) = J(k_1, \ldots, k_d)$, where $J(k_1, \ldots, k_d)$ is an iterated integral over the ideles $A^\times$. Moreover, this this representation proves the shuffle relations among multiple zeta values.

In Section 1, we recall particular Haar measures over local fields. We relate them to the Riemann zeta function following Tate [T]. Then we define iterated integrals over the finite ideles and prove a stuffle relation formula for them. We also define iterated integrals over the idele class group and prove a shuffle relation for them.

In Sections 2, we express any MZV as an iterated integral over the finite ideles by applying the results from Section 1. Similarly, we express any MZV as an iterated integral over the idele class group. These two representations together with the stuffle and the shuffle relations from Section 1 allows us to prove the double shuffle relations for MZVs, Theorem 0.2 using ideles.

This approach has many useful features. First, it generalizes Tate’s results to definition multiple zeta functions via idelic integration (Theorem 2.1). Second, we expect that similar results to hold for function fields over a finite field. That could lead to double shuffle relations for analogues of MZV over function fields over a finite field. Finally, we expect similar idelic representations hold for multiple Dedekind zeta values, (defined in [H2]). They should lead to the two types of shuffle relations among multiple Dedekind zeta values [H3].

Acknowledgment: I would like to thank also to Professor Goncharov for the interest in this work and for the encouragement he gave me.
This work was initiated at Max-Planck Institute für Mathematik. I am very grateful for the stimulating atmosphere, created there. Many thanks are due to the University of Durham for the kind hospitality during the academic year 2005-2006, when part of this work was done, and to the Arithmetic Algebraic Geometry Marie Curie Network for the financial support.

1 Iterated integrals over the finite ideles or over the idele class group

According to Tate’s thesis [1] the Riemann zeta function can be written as a product of \( p \)-adic integrals. We want to expand this representation to multiple zeta functions as iterated adelic integrals. More precisely, iterated integrals over the finite ideles and over the idele class group.

Let \( x_p \) be a \( p \)-adic number. Let \( |x_p|_p \) be the normalized \( p \)-adic norm so that \( |p|_p = p^{-1} \). Let \( d_p x_p \) be the additive \( p \)-adic Haar measure so that

\[
\int_{\mathbb{Z}_p} d_p x_p = 1.
\]

Let \( d_p^\times x_p \) be the Haar measure on \( \mathbb{Q}_p^\times \) normalized so that

\[
\int_{\mathbb{Z}_p^\times} d_p^\times x_p = 1.
\]

The relation between the two measures is the following

\[
d_p^\times x_p = \frac{p - 1}{p} \frac{dx_p}{|x_p|_p}.
\]

Indeed, \( \frac{dx_p}{|x_p|_p} \) is a multiplicative Haar measure. Also,

\[
\mathbb{Z}_p - \{0\} = \bigcup_{k=0}^\infty p^k \mathbb{Z}_p^\times,
\]

and

\[
\int_{p^k \mathbb{Z}_p^\times} dx_p = p^{-k} \int_{\mathbb{Z}_p^\times} dx_p.
\]

We have,

\[
1 = \int_{\mathbb{Z}_p} dx = \sum_{k=0}^\infty \int_{p^k \mathbb{Z}_p^\times} dx_p = \sum_{k=0}^\infty p^{-k} \int_{\mathbb{Z}_p^\times} dx_p = \frac{p}{p - 1} \int_{\mathbb{Z}_p^\times} dx_p.
\]

Then

\[
\int_{\mathbb{Z}_p^\times} dx_p = \frac{p - 1}{p}.
\]
Therefore,
\[
\int_{\mathbb{Z}_p^\times} dx_p = \int_{\mathbb{Q}_p^\times} dx_p = \frac{p-1}{p} = \frac{p-1}{p} \int_{\mathbb{Z}_p^\times} d_p^\times x_p.
\]

Let \( E_p(x_p) \) be a function defined on \( \mathbb{Q}_p^\times \) by
\[
E_p(x_p) = \begin{cases} 
1 & x \in \mathbb{Z}_p - \{0\}, \\
0 & \text{otherwise}
\end{cases}
\]
The local factor of the Riemann zeta function is given by
\[
\frac{1}{1 - p^{-s}} = \int_{\mathbb{Q}_p^\times} E_p(x_p) |x_p|^s d_p^\times x_p.
\]
Indeed, for fixed value of \( k \) the integrand \( E_p(x_p) |x_p|^s \) is constant on the set \( p^k \mathbb{Z}_p^\times \). For \( k < 0 \) we have \( E_p(x_p) = 0 \). For \( k \geq 0 \) we have
\[
\int_{p^k \mathbb{Z}_p^\times} E_p(x_p) |x_p|^s d_p^\times x_p = \int_{p^k \mathbb{Z}_p^\times} |x_p|^s d_p^\times x_p = p^{-ks}.
\]
Also, \( \mathbb{Z}_p - \{0\} = \bigcup_{k=0}^{\infty} p^k \mathbb{Z}_p^\times \).
\[
\int_{\mathbb{Q}_p^\times} E_p(x_p) |x_p|^s d_p^\times x_p = \int_{\mathbb{Z}_p - \{0\}} |x_p|^s d_p^\times x_p = \sum_{k=0}^{\infty} \int_{p^k \mathbb{Z}_p^\times} |x_p|^s d_p^\times x_p = \sum_{k=0}^{\infty} p^{-ks} = \frac{1}{1 - p^{-s}}
\]

Denote by \( x_\infty \) an element of \( \mathbb{R} \). Let \( |x_\infty|_\infty \) be the norm which by definition is the absolute value of the real number. (We save the notation \( |x| \) for a norm of an idele.) Let \( dx_\infty \) be the Haar measure on the additive group of the real numbers. Consider the multiplicative Haar measure on \( \mathbb{R}^\times \), namely,
\[
\frac{dx_\infty}{|x_\infty|_\infty}
\]
Let
\[
E_\infty(x_\infty) = \begin{cases} 
e^{-x_\infty} & \text{for } x_\infty > 0, \\
0 & \text{for } x_\infty < 0.
\end{cases} \tag{1.1}
\]
We are going to integrate \( E_\infty(x_\infty) \) with respect to the multiplicative measure. The Mellin transform of \( E_\infty(x_\infty) \) gives the Gamma function
\[
\int_{\mathbb{R} - \{0\}} E_\infty(x)|x|^s \frac{dx}{|x|} = \Gamma(s).
\]
Let \( x \in \hat{\mathbb{Z}} \). Denote by \( |x|_f \) the product of all the \( p \)-adic norms. Namely,
\[
|x|_f = \prod_{p: \text{finite}} |x_p|.
\]
Let also
\[
E_f(x) = \prod_{p: \text{finite}} E_p(x_p).
\]
Denote by 
\[ d_f x = \prod_p d_p x_p, \]
the multiplicative measure on the finite ideles given as product of all local multiplicative measures over \( \mathbb{Q}_p \) for all primes \( p \).

**Definition 1.1** Let \( G_1, \ldots, G_n \) be integrable functions on the finite ideles \( \mathbb{A}_f^\times \) with support on \( \hat{\mathbb{Z}} \setminus \{0\} \). Assume that the function \( G_i \) is constant on each set with a fixed norm, that is \( G_i \) is constant on the set \( \{ x_f \in \mathbb{A}_f^\times \mid ||x_f|| = q \} \) for a chosen \( q \in \mathbb{Q} \). We define an iterated integral over the finite ideles in the following way
\[
\int_{\mathbb{A}_f^\times} G_1 \circ \cdots \circ G_n = \int_{|x_1|_f > \cdots > |x_n|_f > 0} G_1(x_1) \cdots G_n(x_n) dx_1 dx_2 \cdots dx_n.
\]

**Definition 1.2** We define the set \( \text{St}(i,j) \) of stuffles \( \sigma \) between an ordered set with \( i \) elements of \( \mathbb{Z} \)
\[ 0 < k_1 < \cdots < k_i \]
and an ordered set with \( j \) elements of \( \mathbb{Z} \)
\[ 0 < k_{i+1} < \cdots < k_{i+j} \]
to be all possible choices of ordered sets with \( l \) elements of \( \mathbb{Z} \)
\[ 0 < m_1 < \cdots < m_l \]
such that
\[ \{m_1, \ldots, m_l\} = \{k_1, \ldots, k_i\} \cup \{k_{i+1}, \ldots, k_{i+j}\} \]
where the two sets on the right hand side might not be disjoint. If \( \sigma \) is in \( \text{St}(i,j) \) we denote by \( \sigma \) both the map of inclusion
\[ \{k_1, \ldots, k_i\} \to \{m_1, \ldots, m_l\} \]
and the map of inclusion
\[ \{k_{i+1}, \ldots, k_{i+j}\} \to \{m_1, \ldots, m_l\}. \]

**Theorem 1.3** (Stuffle relations for iterated integrals over the finite ideles) Let \( G_1, \ldots, G_{i+j} \) be integrable functions on the finite ideles such that each of them is constant on each set of fixed norm, that is \( G_i \) is constant on the set \( \{ x_f \in \mathbb{A}_f^\times \mid ||x_f|| = q \} \) for a chosen \( q \in \mathbb{Q} \). Then
\[
\int_{\mathbb{A}_f^\times} G_1 \circ \cdots \circ G_i \times \int_{\mathbb{A}_f^\times} G_{i+1} \circ \cdots \circ G_{i+j} = \sum_{\sigma \in \text{St}(i,j)} G_1^\sigma \circ \cdots \circ G_i^\sigma
\]
where the sum is over all the stuffles \( \sigma \in \text{St}(i,j) \) and
\[
G_n^\sigma = \begin{cases} 
G_s & \text{if } \sigma(k_s) = m_n \text{ for only one value of } s; \\
G_s G_t & \text{if } \sigma(k_s) = m_n \text{ and } \sigma(k_t) = m_n \text{ for different values of } s \text{ and } t.
\end{cases}
\]
Proof. Define \( g_i(k) \) to be the integral of \( G_i \) over the subset of the finite ideles of norm \( 1/k \), that is,
\[
g_i(k) = \int_{|x| = \frac{1}{k}} G_i(x) \, dx.
\]
Since each of the functions \( G_i \) is constant on subsets of the finite ideles with fixed norm, we obtain that
\[
\int_{\mathbb{A}_f^\times} G_1 \circ \cdots \circ G_i = \sum_{0 < k_1 < \ldots < k_i} g_1(k_1) \cdots g_i(k_i).
\]
Then the stuffle relations among iterated integrals over the finite ideles are reduced to the stuffle relations of infinite series, namely,
\[
\sum_{0 < k_1 < \ldots < k_i} g_1(k_1) \cdots g_i(k_i) \sum_{0 < k_{i+1} < \ldots < k_{i+j}} g_{i+1}(k_{i+1}) \cdots g_i(k_{i+j}) = \sum_{\sigma \in St(i, j)} \sum_{0 < m_1 < \ldots < m_l} g_{\sigma}(m_1) \cdots g_{\sigma}(m_l),
\]
where the first sum is over all the stuffles \( \sigma \in St(i, j) \) and
\[
g_{\sigma} = \begin{cases} 
g_s & \text{if } \sigma(k_s) = m_n \text{ for only one value of } s; 
g_s g_t & \text{if } \sigma(k_s) = m_n \text{ and } \sigma(k_t) = m_n \text{ for different values of } s \text{ and } t. 
\end{cases}
\]
\( \Box \)

Definition 1.4 Let \( G_1, \ldots, G_n \) be integrable functions on the idele classes \( \mathbb{A}^\times / \mathbb{Q}^\times \). Assume that the function \( G_i \) is constant on each set with a fixed norm, that is, \( G_i \) is constant on \( \{ x \in \mathbb{A}^\times \mid |x| = r \} \) for a chosen \( r \in \mathbb{R}_{>0} \). We define an iterated integral over the ideles in the following way
\[
\int_{\mathbb{A}^\times / \mathbb{Q}^\times} G_1 \circ \cdots \circ G_n = \int_{|x_1| > \ldots > |x_n| > 0} G_1(x_1) \cdots G_n(x_n) \, dx_1 \, dx_2 \cdots dx_n.
\]

Definition 1.5 We define a shuffle of two ordered sets \( \{1, \ldots, i\} \) and \( \{i + 1, \ldots, i + j\} \) as a permutation \( \sigma \) of \( i + j \) elements such that \( \sigma \) satisfies the conditions
\[
\sigma(1) < \cdots < \sigma(i)
\]
and
\[
\sigma(i + 1) < \cdots < \sigma(i + j).
\]
We denote by \( Sh(i, j) \) the set of all shuffles between an ordered set of \( i \) elements and an ordered set of \( j \) elements.

The key difference between a shuffle and a stuffle is that in the shuffle we have two disjoint sets while in the stuffle the two sets might have common elements.
Theorem 1.6 (Shuffle relations for iterated integrals over the idele class group) Let $G_1, \ldots, G_{i+j}$ be integrable functions on the finite ideles such that each of them is constant on each set of fixed norm, that is $G_i$ is constant on $\{x_f \in \mathbb{A}_f^\times \mid |x_f|_f = q \}$ for a chosen $q \in \mathbb{Q}$. Then

$$
\int_{\mathbb{A}_f^\times} G_1 \circ \cdots \circ G_i \times \int_{\mathbb{A}_f^\times} G_{i+1} \circ \cdots \circ G_{i+j} = \sum_{\sigma \in \text{Sh}(i,j)} G_{\sigma(1)} \circ \cdots \circ G_{\sigma(i+j)}
$$

where the sum is over all the shuffles $\sigma \in \text{Sh}(i,j)$.

Proof. Let $\mathbb{A}^\times / \mathbb{Q}^\times \to \mathbb{R}_{>0}$ be the map under the norm of an idele. Since $G_i$ for $l = 1, \ldots, i + j$ are constant on subsets of the idele class group of fixed norm, we can define

$$
g_i(r) = G_i(x)
$$

for $|x| = r$. Since each of the functions $G_i$ are constant on subsets of the finite ideles with constant norm, we obtain that

$$
\int_{\mathbb{A}_f^\times} G_1 \circ \cdots \circ G_i = \sum_{0 < r_1 < \cdots < r_i} g_1(r_1) \frac{dr_1}{r_1} \cdots g_i(r_i) \frac{dr_i}{r_i}.
$$

Then the shuffle relation among iterated integrals over the idele class group reduces to the shuffle relation of iterated path integrals

$$
\int_{k_1 > \cdots > k_i > 0} \frac{g_1(r_1) \cdots g_i(r_i) \frac{dr_1}{r_1} \cdots \frac{dr_i}{r_i}}{k_1 \cdots k_i} \times 
\int_{r_{i+1} > \cdots > r_{i+j} > 0} \frac{g_{i+1}(r_{i+1}) \cdots g_{i+j}(r_{i+j}) \frac{dr_{i+1}}{r_{i+1}} \cdots \frac{dr_{i+j}}{r_{i+j}}}{r_{i+1} \cdots r_{i+j}}
$$

$$
= \sum_{\sigma \in \text{Sh}(i,j)} \int_{r_1 > \cdots > r_{i+j} > 0} g_{\sigma(1)}(r_1) \cdots g_{\sigma(i+j)}(r_{i+j}) \frac{dr_1}{r_1} \cdots \frac{dr_{i+j}}{r_{i+j}}.
$$

$\square$

2 Double shuffle relations for multiple zeta values via ideles

Let us iterate the function $E_f(x_f)|x_f|^s$ over the finite ideles. We define

$$
I(s_1, s_2) = \int_{|x_1|_f > |x_2|_f} E_f(x_1)|x_1|^s d_f^\times x_1 E_f(x_2)|x_2|^s d_f^\times x_2.
$$

If we iterate $d$ times, we obtain a multiple zeta function of depth $d$. Namely, if we set

$$
I(s_1, \ldots, s_d) = \int_{|x_1|_f > \cdots > |x_d|_f} E_f(x_1)|x_1|^{s_1} d_f^\times x_1 \cdots E_f(x_d)|x_d|^{s_d} d_f^\times x_d
$$

(2.2)

Recall the definition of multiple zeta function,

$$
\zeta(s_1, \ldots, s_d) = \sum_{0 < n_1 < \cdots < n_d} \frac{1}{n_1^{s_1} \cdots n_d^{s_d}}.
$$
Theorem 2.1 Multiple zeta functions can be represented as iterated integrals over the finite ideles, namely,
\[ \zeta(s_1, \ldots, s_d) = I(s_1, \ldots, s_d). \]

Proof. We are going to prove the theorem for \( d = 2 \). For larger values of \( d \) the proof is essentially the same.

Restrict the integral to the domain where the support of \( f_f \) is not zero. For any such idele we have \(|x|_f = 1/n\) for some \( n \in \mathbb{N} \). Therefore,
\[
I(s_1, s_2) = \\
= \int_{|x_1|_f > |x_2|_f} E_f(x_1) |x_1|_f^{s_1} d_f^x x_1 E_f(x_2) |x_2|_f^{s_2} d_f^x x_2 = \\
= \sum_{0 < n_1 < n_2} \int_{n_1 \hat{\mathbb{Z}} \times n_2 \hat{\mathbb{Z}}} |x_1|_f^{s_1} d_f^x x_1 |x_2|_f^{s_2} d_f^x x_2 = \\
= \sum_{0 < n_1 < n_2} \frac{1}{n_1^{s_1} n_2^{s_2}} = \\
= \zeta(s_1, s_2)
\]
is a double zeta function. \( \square \)

Let \( x \) be an element of the adeles over \( \mathbb{Q} \). We are going to write \( x_{\infty} \) for the infinite coordinate of the adele \( x \), and \( x_p \) for the \( p \)-adic coordinate. Consider the function
\[ E(x) = E_{\infty}(x_{\infty}) \prod_p E_p(x_p) \]

Let \( d^x x \) be a multiplicative measure on the ideles given by the product of local multiplicative measures considered above for all of the local fields. Let \( \mathbb{A} \) be the adels over the rational numbers \( \mathbb{Q} \). We are going to integrate over the idele class group \( \mathbb{A}^\times / \mathbb{Q}^\times \).

For an idele \( x \in \mathbb{A}^\times \) let
\[ |x| = |x_{\infty}|_\infty \prod_p |x_p|_p \]
be the product of all the local valuations and let
\[ e(x) = \sum_{q \in \mathbb{Q}^\times} E(qx). \]

Define also
\[ w_1(x) = e(x) d^x x \]

and let
\[ w_0(x) = d^x x \]
be measures on \( \mathbb{A}^\times \).

We define the following integral
\[ J(m, n) = \int_{|x_1| > |x_2| > \ldots > |x_{m+n}|} w_1(x_1)w_0(x_2) \ldots w_0(x_m)w_1(x_{m+1})w_0(x_{m+2}) \ldots w_0(x_{m+n}) \]
More generally, we define
\[
J(n_1, \ldots, n_d) = \int \prod_{|x_1|>|x_2|>\cdots>|x_{n_1+\cdots+n_d}|} d \left( w_1(x_{n_1+\cdots+n_i}) \prod_{j=2}^{n_i} w_0(x_{n_1+\cdots+n_{i-1}+j}) \right).
\]

(2.3)

**Theorem 2.2** The multiple zeta values can be represented as iterated integrals over the idele class group, namely,
\[
\zeta(n_1, \ldots, n_d) = J(n_1, \ldots, n_d).
\]

**Proof.** We are going to prove the Theorem for \(d = 2\). For larger \(d\) the proof is essentially the same.

We want to modify the function \(E(x)\) in the definition of \(J(m, n)\) so that the two new functions are defined over \(\mathbb{A}^\times/\mathbb{Q}^\times\) and over \(\mathbb{R}_0^\times\), respectively.

Denote by \(\bar{x}\) the projection of an idele \(x\) to an element of \(\mathbb{A}^\times/\mathbb{Q}^\times\). Let
\[
e(\bar{x}) = \sum_{q \in \mathbb{Q}^\times} E(qx).
\]

Note that \(|qx| = |x|\) for \(q \in \mathbb{Q}\) and \(x \in \mathbb{A}\). Denote by \(|\bar{x}| := |x|\) the norm of an element in \(\mathbb{A}^\times/\mathbb{Q}^\times\). Recall that \(\mathbb{Q}^\times\) is a discrete subgroup of \(\mathbb{A}^\times\). For that reason we can take the same measure \(d^\times x\) on the set \(\mathbb{A}^\times/\mathbb{Q}^\times\).

Now we define the corresponding function \(c(t)\) for \(t \in \mathbb{R}\) and \(t > 0\). The function \(e\) is constant on each set \((qt, q\hat{\mathbb{Z}}^\times)\), where \(t \in \mathbb{R}_{>0}\) is fixed, \(\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p\) and \(q\) varies in \(\mathbb{Q}^\times\). We can put \(c(t)\) to be the value of \(e\) on any of the elements in the set \((qt, q\hat{\mathbb{Z}}^\times)\). Let \(x\) be an idele and \(t = |x|\) be positive real number. Let us examine more carefully the relation between \(e(x)\) and \(c(t)\). For some \(q \in \mathbb{Q}^\times\) and \(t \in \mathbb{R}_0^\times\) we have \(x \in (qt, q\hat{\mathbb{Z}}^\times)\). If \(q\) is not an integer then \(e(x) = 0\). Also, \(E_{\infty}(\text{negative } t) = 0\). For these reasons we can sum over all positive integers. Let \(x_f\) be the finite idele of \(x\). That is, \(x_f\) consists of all coordinates of \(x\) except the coordinate corresponding to the infinite place. Let, also
\[
E_f(x_f) = \prod_p E_p(x_p),
\]

where the product is over all primes (finite places) \(p\). Denote by \(d^\times x_f\) the product of multiplicative Haar measure of \(\mathbb{Q}_p^\times\) over all primes \(p\). Denote, also, by \(\hat{\mathbb{Z}}\) the product of the \(p\)-adic integers over all primes \(p\). Then
\[
c(|x|) = \sum_{n \in \mathbb{N}} E_{\infty}(nr) \times \int_{n\hat{\mathbb{Z}}^\times} E_f(x_f) d^\times x_f.
\]

For \(n \in \mathbb{N}\) the integral becomes
\[
\int_{n\hat{\mathbb{Z}}^\times} E_f(x_f) d^\times x_f = 1.
\]
For the infinite place we have
\[ E_\infty(nt) = e^{-nt}. \]
Therefore,
\[ c(t) = \sum_{n=1}^{\infty} e^{-nt}. \]
Iteration of the measures \( w_1 \)'s and \( w_0 \)'s can be written as iteration of \( c(t) \):
\[ J(m, n) = \int_{|x_1|>|x_2|>...>|x_{m+n}|} w_1(x_1)w_0(x_2)\ldots w_0(x_m)w_1(x_{m+1})\ldots w_0(x_{m+n}) = \]
\[ = \int_{t_1>t_2>...>t_{m+n}>0} c(t_1)c(t_{m+1})dt_1\ldots dt_{m+n} = \zeta(m, n) \quad (2.4) \]

Proof. (of the main Theorem 0.2) Part (a) follows from the explicit formulas for MZV in terms of the finite ideles from Theorem 2.1 and from the corresponding stuffle relations from Theorem 1.3. Part (b) follows from the explicit formulas for MZV in terms of idele class group from Theorem 2.2 and from the corresponding shuffle relations from Theorem 1.6.

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